# Contents

# Chapter 1

# Curves & Vector Fields

The notion of a *curve* is of fundamental importance in Applied Mathematics courses because it arises in many physical and mathematical contexts. The path or trajectory of a particle moving in space, e.g. a satellite orbiting the earth, is a curve in $\mathbb{R}^3$. A section of a power line suspended between two pylons is another example of a curve in $\mathbb{R}^3$. Looking ahead in this course, the field lines of a vector field $\mathbf{F}(x, y, z)$ are a family of curves in $\mathbb{R}^3$ (more on this in Section 1.2). More generally, if one considers a physical system whose state at time $t$ is a vector $\mathbf{x}(t) \in \mathbb{R}^n$, then the evolution in time of the system will be described by a curve in $\mathbb{R}^n$ (the value of $n$ will depend on the complexity of the system).

The first mathematical description of a curve that one encounters is the graph of a function $f : \mathbb{R} \to \mathbb{R}$, described by an equation

$$y = f(x), \qquad a \leq x \leq b.$$

This way of describing curves has a number of limitations: it is only valid for curves in $\mathbb{R}^2$, and it doesn't give a simple description of the motion of particles (e.g. a particle moving in a circle). So one introduces vector-valued functions and uses a parametric description of curves (MATH 138),

$$\mathbf{x} = \mathbf{g}(t), \qquad a \leq t \leq b,$$

which provides a sufficiently general description of curves for most purposes.

In Section 1.1 we review vector-valued functions and their use in describing curves in $\mathbb{R}^2$ and $\mathbb{R}^3$, discussing the topic in greater depth than in MATH 138.

## 1.1 Curves in $\mathbb{R}^n$

### 1.1.1 Curves as vector-valued functions

Let $\mathbf{g} : [a, b] \to \mathbb{R}^n$ be a *vector-valued function*, i.e. a function which associates with each real number $t \in [a, b]$ a unique vector $\mathbf{g}(t) \in \mathbb{R}^n$. We can express $\mathbf{g}(t)$ in terms of its components relative to the standard basis in $\mathbb{R}^n$:

$$\mathbf{g}(t) = (g_1(t), g_2(t), \ldots, g_n(t)), \tag{1.1}$$

where the $g_i$, $i = 1, \ldots, n$, are scalar functions called the *component functions of* $\mathbf{g}$. We shall work with functions $\mathbf{g}$ which are (at least) *continuous*, which means that the component functions are continuous.

Consider a (continuous) vector-valued function $\mathbf{g} : [a, b] \to \mathbb{R}^n$. As $t$ runs from $a$ to $b$, the function $\mathbf{g}$ determines an ordered succession of points $\mathbf{g}(t)$ in $\mathbb{R}^n$, which we shall refer to as a *curve* $\mathcal{C}$, in $\mathbb{R}^n$. The curve $\mathcal{C}$ is described by the equations

$$\mathbf{x} = \mathbf{g}(t), \qquad a \le t \le b. \tag{1.2}$$

We shall refer to the function $\mathbf{g}$ as a *parametrization of* $\mathcal{C}$, and to $t$ as a *parameter on* $\mathcal{C}$. We can think of the curve $\mathcal{C}$ as the image of the interval $[a, b]$ under the function $\mathbf{g}$, represented pictorially as in Figure 1.1.

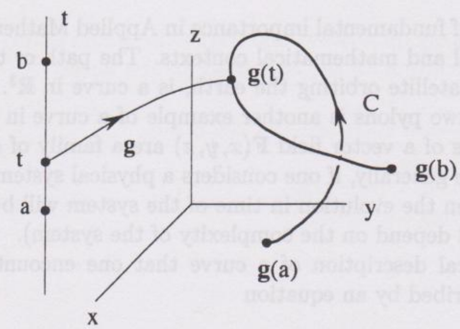



Figure 1.1: A vector-valued function $\mathbf{g}$ defines a curve $\mathcal{C}$ in $\mathbb{R}^3$.

One can think of the function $\mathbf{g}$ bending and stretching the line segment $a \le t \le b$ so as to create the curve $\mathcal{C}$. The ordering of points in the interval $[a, b]$ determines an ordering of points in $\mathcal{C}$, called *the orientation of* $\mathcal{C}$. The points $\mathbf{g}(a)$ and $\mathbf{g}(b)$ are called *the endpoints of* $\mathcal{C}$.

**Example 1.1:**
The vector-valued function $\mathbf{g} : [0, a] \to \mathbb{R}^3$ defined by

$$\mathbf{g}(t) = \mathbf{r}_0 + t\mathbf{u}, \qquad 0 \le t \le a, \tag{1.3}$$

where $\mathbf{r}_0$ and $\mathbf{u}$ are given vectors in $\mathbb{R}^3$, determines a curve $\mathbf{x} = \mathbf{g}(t)$, which is the straight line segment in $\mathbb{R}^3$, joining the points $\mathbf{g}(0) = \mathbf{r}_0$ and
$\mathbf{g}(a) = \mathbf{r}_0 + a\mathbf{u}$, traversed in the direction of $\mathbf{u}$.

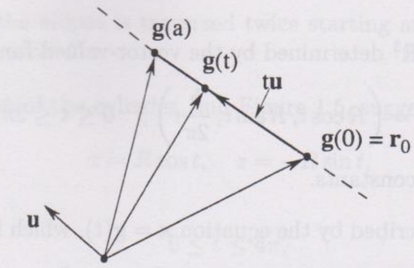



Figure 1.2: The curve $\mathbf{x} = \mathbf{r}_0 + t\mathbf{u}$, $0 \le t \le a$. □

*Comment:*

One can interpret an element $\mathbf{a} = (a_1, a_2, \dots, a_n)$ of $\mathbb{R}^n$ in two ways, either as the position vector of a point in $\mathbb{R}^n$ (e.g. the vector $\mathbf{r}_0$ in Example 1.1), or as a vector attached to a point in $\mathbb{R}^n$ (e.g. the vector $\mathbf{u}$ in Example 1.1). In the second interpretation the vector would represent some physical quantity such as the velocity or acceleration of a particle, or a force acting on a particle.

**Example 1.2:**

The vector-valued function $\mathbf{g} : [0, \pi] \to \mathbb{R}^2$ defined by

$$\mathbf{g}(t) = (b\cos t, b\sin t), \quad 0 \le t \le \pi,$$

where $b$ is a positive constant, determines a curve $\mathbf{x} = \mathbf{g}(t)$ which is half the circle $x^2+y^2 = b^2$ of radius $b$, traversed counterclockwise from $(b, 0)$ to $(-b, 0)$.

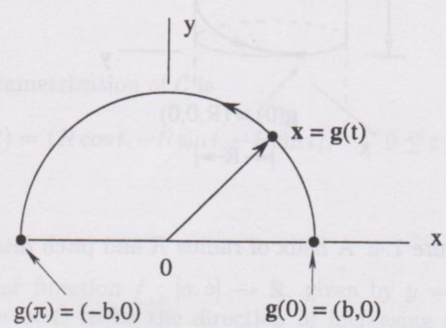



Figure 1.3: The curve $\mathbf{x} = (b\cos t, b\sin t)$, $0 \le t \le \pi$.

Here is a more complicated example, which arises in many scientific contexts.

**Example 1.3:**

Describe the curve $C$ in $\mathbb{R}^3$ determined by the vector-valued function

$$\mathbf{g}(t) = \left( R\cos t, R\sin t, \frac{h}{2\pi}t \right), \quad 0 \le t \le 2\pi, \tag{1.4}$$

where $R$ and $h$ are positive constants.

*Solution:* The curve is described by the equation $\mathbf{x} = \mathbf{g}(t)$, which in component form reads

$$x = R\cos t, \quad y = R\sin t, \quad z = \frac{h}{2\pi}t.$$

Since

$$x^2 + y^2 = R^2,$$

and $0 \le t \le 2\pi$, the curve rotates once around the cylinder of radius $R$, whose axis is the $z$-axis. Since $z$ increases linearly with $t$, the curve rises uniformly as it moves around the cylinder, giving one revolution of a *helix*. The constant $R$ is the *radius* of the helix, and the constant $h$, the change in $z$ during one revolution, is called the *pitch* of the helix.

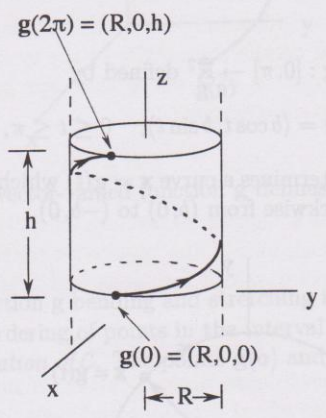



Figure 1.4: A helix of radius $R$ and pitch $h$.

In some situations one is given a curve $C$ described geometrically, and it is necessary to find a parametrization $\mathbf{g}(t)$ of $C$. Here's an example.

**Example 1.4:**

The cylinder $x^2 + z^2 = R^2$ intersects the plane $z = y$ in a closed curve (an ellipse). We specify the orientation of the curve $C$ by stating that the ellipse is traversed counter-clockwise when viewed from the positive $y$-direction. We complete the description of the

*Solution:* The equation of the cylinder, and Figure 1.5, suggest that we choose

$$x = R\cos t, \quad z = -R\sin t,$$

with

$$0 \leq t \leq 4\pi,$$

to give two revolutions. The equation of the plane, $z = y$, then determines $y$ in terms of $t$, i.e.

$$y = -R\sin t.$$

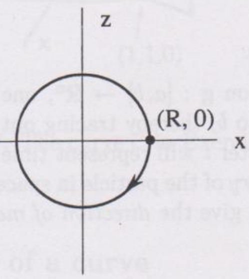



Figure 1.5: Projection of the ellipse into the $xz$-plane. The positive $y$-axis points into the page.

Thus, a possible parametrization of $\mathcal{C}$ is

$$\mathbf{g}(t) = (R\cos t, -R\sin t, -R\sin t), \qquad 0 \leq t \leq 4\pi.$$

*An important special case:*

The graph of a scalar function $f : [a, b] \to \mathbb{R}$, given by $y = f(x)$ is a curve $\mathcal{C}$ in $\mathbb{R}^2$. We take the orientation of $\mathcal{C}$ to be the direction of increasing $x$. It is easy to obtain a parametrization $\mathbf{g}(t)$ of this curve. Just let $x = t$, and then $y$ is given by $y = f(t)$, with $a \leq t \leq b$. Thus a parametrization of $\mathcal{C}$ is

$$\mathbf{g}(t) = (t, f(t)), \qquad a \leq t \leq b. \tag{1.5}$$

□

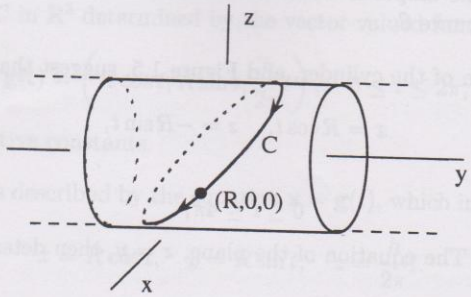



Figure 1.6: The curve $C$ in Example 1.4.

*A curve as the path of a particle:*

Given a vector-valued function $\mathbf{g} : [a, b] \to \mathbb{R}^n$, one can imagine the image point $\mathbf{g}(t)$ moving in $\mathbb{R}^n$ as $t$ runs from $a$ to $b$, thereby tracing out the curve $C$. When describing the motion of a particle, the parameter $t$ will represent time and the curve $C \in \mathbb{R}^3$, (i.e. $n = 3$) will represent the *path* or *trajectory* of the particle in space during the time interval $a \leq t \leq b$. The orientation of the curve will give the *direction of motion* along the curve.

*Comment:*

Given a vector-valued function $\mathbf{g} : [a, b] \to \mathbb{R}^n$, it is important to make a distinction between the *range of* $\mathbf{g}$, i.e. the subset $\{\mathbf{g}(t) | a \leq t \leq b\} \subset \mathbb{R}^n$, and *the curve $C$ determined by* $\mathbf{g}$, i.e. the ordered succession of points $\mathbf{g}(t) \in \mathbb{R}^n$, as $t$ runs from $a$ to $b$. For example, first consider $\mathbf{g}_1 : [0, 2\pi] \to \mathbb{R}^2$ defined by

$$\mathbf{g}_1(t) = (\cos t, \sin t).$$

The curve $C_1$ given by $\mathbf{x} = \mathbf{g}_1(t)$, $0 \leq t \leq 2\pi$, is the circle $x^2 + y^2 = 1$, *traversed once* in a counterclockwise direction,starting at the point $(1, 0)$. Second, consider $\mathbf{g}_2 : [0, 2\pi] \to \mathbb{R}^2$ defined by

$$\mathbf{g}_2(t) = (\cos 2t, \sin 2t).$$

The curve $C_2$ given by $\mathbf{x} = \mathbf{g}_2(t)$, $0 \leq t \leq 2\pi$, is the circle $x^2 + y^2 = 1$ *traversed twice* in counterclockwise direction, starting at the point $(1, 0)$.

The point is that $\mathbf{g}_1$ and $\mathbf{g}_2$ *have the same range in* $\mathbb{R}^2$, namely the points on the circle $x^2 + y^2 = 1$, but *the curves $C_1$ and $C_2$ that they determine are not the same.* For example, a particle whose path is $C_1$ will travel a distance of $2\pi$, while for $C_2$ the distance travelled will be $4\pi$. $\square$

**Exercise 1.1:**

Describe the curve $C$ determined by the vector-valued function

$$\mathbf{g}(t) = (3 \sin t, 4 \cos t), \qquad 0 \leq t \leq \pi.$$

are as shown in Figure 1.7. Find a parametrization g(·)

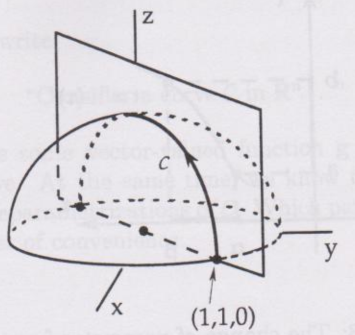



Figure 1.7: The curve $C$ in Exercise 1.2.

### 1.1.2 Reparametrization of a curve

An essential aspect of describing curves is that *infinitely many different vector-valued functions describe the same curve $C$ in $\mathbb{R}^n$.*

For example, consider $\mathbf{g} : [0, \pi] \to \mathbb{R}^2$ defined by

$$\mathbf{g}(t) = (\cos t, \sin t), \quad 0 \leq t \leq \pi,$$

and also $\hat{\mathbf{g}} : [0, \frac{\pi}{2}] \to \mathbb{R}^2$ defined by

$$\hat{\mathbf{g}}(\tau) = (\cos 2\tau, \sin 2\tau), \qquad 0 \leq \tau \leq \tfrac{\pi}{2},$$

where we use a different letter $\tau$ for the parameter in the second case. The point is that $\mathbf{g}$ and $\hat{\mathbf{g}}$ are different functions, but determine the same curve in $\mathbb{R}^2$, namely the circle $x^2 + y^2 = 1$ traversed halfway around counterclockwise from $(1, 0)$ to $(-1, 0)$. We say that $\mathbf{g}$ *and* $\hat{\mathbf{g}}$ *are different parametrizations of the same curve $C$.* Note that the two functions are related by

$$\hat{\mathbf{g}}(\tau) = \mathbf{g}(h(\tau)),$$

where

$$h(\tau) = 2\tau.$$

In general, let $h : [\alpha, \beta] \to [a, b]$ be a continuous, increasing (and hence one-to-one) function. Given $\mathbf{g} : [a, b] \to \mathbb{R}^n$, we define $\hat{\mathbf{g}} : [\alpha, \beta] \to \mathbb{R}^n$ by

$$\hat{\mathbf{g}}(\tau) = \mathbf{g}(h(\tau)), \qquad \alpha \leq \tau \leq \beta. \tag{1.6}$$

As $\tau$ runs from $\alpha$ to $\beta$, $t = h(\tau)$ runs from $a$ to $b$ (see Figure 1.8). Thus by equation (1.6), $\hat{g}(\tau)$ *follows the same ordered succession of points in* $\mathbb{R}^n$ *as does* $\mathrm{g}(t)$, and hence *determines the same curve* $C$ in $\mathbb{R}^n$

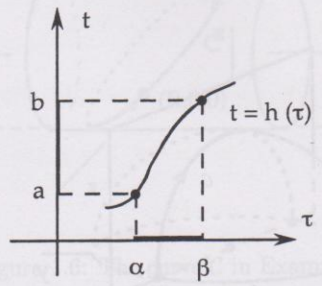



Figure 1.8: The change of parameter function $h$.

We shall refer to $\hat{g}$ as a *reparametrization of the curve* $C$. the situation is represented pictorially in Figure 1.9.

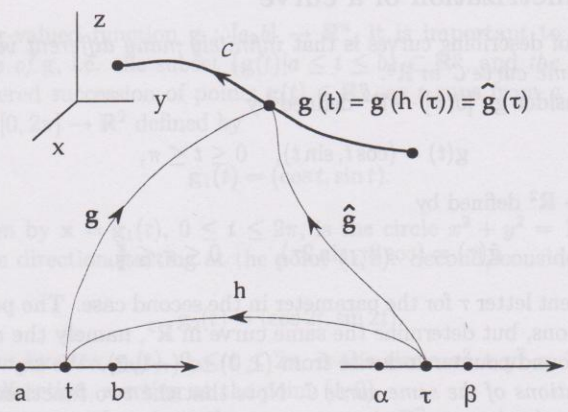



Figure 1.9: Two parametrizations g and ĝ of a curve $C$.

In terms of the motion of a particle, reparametrizations have the following interpretation. Consider a particle that moves on a curve in 3-space between two given points. One can imagine the particle speeding up or slowing down as it moves along the curve. Thus given one curve, there are infinitely many ways a particle can move along the curve, and each one corresponds to a different parametrization. We will shed more light on this matter when we discuss tangent vectors and velocity vectors in the next subsection.

*Comment:*

In these notes, when we write

"Consider a curve $C$ in $\mathbb{R}^n$ ..."

we shall mean that there is some vector-valued function $\mathbf{g} : [a, b] \to \mathbb{R}^n$ which gives a parametrization of this curve. At the same time, we know that there are infinitely many other functions $\tilde{\mathbf{g}}$ that give reparametrizations of $C$. Which parametrization we use for doing a calculation will be a matter of convenience.

### 1.1.3 Limits

Given a vector-valued function $\mathbf{g} : [a, b] \to \mathbb{R}^n$, the Newton quotient at $t$ is

$$\frac{1}{\Delta t}\left[\mathbf{g}(t + \Delta t) - \mathbf{g}(t)\right] \tag{1.7}$$

for a non-zero change $\Delta t$. The derivative of $\mathbf{g}$ at $t$, denoted $\mathbf{g}'(t)$, is defined to be the vector that is the limit of the Newton quotient as $\Delta t \to 0$. So we need to think about the limit of a vector-valued function.

**Definition:**

Given a vector-valued function $\mathbf{g}$ defined in a neighbourhood of $t_0$ and a constant vector $\mathbf{L}$, the statement $\lim\limits_{t\to t_0}\mathbf{g}(t) = \mathbf{L}$ means that

$$\lim_{t\to t_0} \| \mathbf{g}(t) - \mathbf{L} \| = 0.$$

*Comments:*

1) Since $\| \mathbf{g}(t) - \mathbf{L} \|$ is Euclidean distance between the vectors $\mathbf{g}(t)$ and $\mathbf{L}$, the definition captures the usual idea of "limit", namely that the vector $\mathbf{g}(t)$ gets arbitrarily close to the unique vector $\mathbf{L}$ as $t$ approaches $t_0$.

2) When doing calculations with vector-valued functions one works with components. So we need to express the basic concepts in terms of components. Let

$$\mathbf{g}(t) = (g_1(t), \dots, g_n(t)), \qquad \mathbf{L} = (L_1, \dots, L_n).$$

Then

$$\lim_{t\to t_0} \mathbf{g}(t) = \mathbf{L} \iff \lim_{t\to t_0} g_i(t) = L_i, \quad i = 1, 2, \dots, n. \tag{1.8}$$

This equivalence follows on noting that

$$\| \mathbf{g}(t) - \mathbf{L} \| = \left[ (g_1(t) - L_1)^2 + \cdots + (g_n(t) - L_n)^2 \right]^{1/2},$$

which implies

$$|g_i(t) - L_i| \leq \| \mathbf{g}(t) - \mathbf{L} \|, \quad \text{for} \quad i = 1, 2, \ldots, n,$$

so that

$$\lim_{t \to t_0} \| \mathbf{g}(t) - \mathbf{L} \| = 0 \quad \text{implies} \quad \lim_{t \to t_0} g_i(t) = L_i, \quad i = 1, \ldots, n. \qquad \square$$

We can now talk about a vector-valued function being continuous.

**Definition:**

$\mathbf{g}$ *is continuous at* $t_0$ means that $\mathbf{g}(t_0)$ is defined, $\lim\limits_{t \to t_0} \mathbf{g}(t)$ exists, and $\lim\limits_{t \to t_0} \mathbf{g}(t) = \mathbf{g}(t_0)$.

*Comments:*

i) In terms of component functions

$$\mathbf{g} \quad \text{is continuous at} \quad t_0 \;\Leftrightarrow\; g_i \quad \text{is continuous at} \quad t_0, \quad i = 1, 2, \ldots, n.$$

ii) The curves one encounters in physics are invariably defined by continuous functions. If the function $\mathbf{g}$ was discontinuous at $t_0$, say $\lim\limits_{t \to t_0^+} \mathbf{g}(t) = \mathbf{L}^+$ and $\lim\limits_{t \to t_0^-} \mathbf{g}(t) = \mathbf{L}^-$ in terms of one-sided limits, with $\mathbf{L}^+ \neq \mathbf{L}^-$, then the curve would have a "break" in it. $\square$

### 1.1.4 Derivatives

**Definition:**

Given a vector-valued function $\mathbf{g} : [a, b] \to \mathbb{R}^n$, *the derivative of* $\mathbf{g}$ *at* $t$, denoted $\mathbf{g}'(t)$, is defined by

$$\mathbf{g}'(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[ \mathbf{g}(t + \Delta t) - \mathbf{g}(t) \right], \tag{1.9}$$

provided this limit exists.

*Comments:*

i) If $\mathbf{g}'(t)$ exists we say that $\mathbf{g}$ is *differentiable at* $t$.

ii) In terms of component functions

$$\mathbf{g}(t) = (g_1(t), \ldots, g_n(t)),$$

the Newton quotient is given by

$$\frac{1}{\Delta t} [\mathbf{g}(t + \Delta t) - \mathbf{g}(t)] = \left( \frac{g_1(t + \Delta t) - g_1(t)}{\Delta t}, \ldots, \frac{g_n(t + \Delta t) - g_n(t)}{\Delta t} \right).$$

It follows from (1.8) and (1.9) that

$$\mathbf{g}'(t) \quad \text{exists} \quad \Longleftrightarrow \quad g_i'(t) \quad \text{exists}, \quad i = 1, 2, \ldots, n,$$

and

$$\mathbf{g}'(t) = (g_1'(t), \ldots, g_n'(t)). \tag{1.10}$$

In practice one calculates the derivative $\mathbf{g}'(t)$ using (1.10) and routine differentiation. $\square$

*Physical interpretation of the derivative:*

Consider a curve $C$ in $\mathbb{R}^3$, $\mathbf{x} = \mathbf{g}(t)$, which represents the path of a particle. We write

$$\Delta\mathbf{x} = \mathbf{g}(t + \Delta t) - \mathbf{g}(t).$$

Then the Newton quotient (1.7) is

$$\left(\frac{1}{\Delta t}\right)[\mathbf{g}(t + \Delta t) - \mathbf{g}(t)] = \left(\frac{1}{\Delta t}\right)\Delta\mathbf{x}.$$

Thus by (1.9), $\mathbf{g}'(t)$ represent the rate of change of position with respect to time $t$, i.e. $\mathbf{g}'(t)$ *is the velocity* $\mathbf{v}(t)$ *of the particle.* We write

$$\mathbf{v}(t) = \mathbf{g}'(t) \quad \text{or} \quad \mathbf{v}(t) = \frac{d\mathbf{x}}{dt}. \tag{1.11}$$

Writing the position function $\mathbf{g}(t)$ in terms of its component functions,

$$\mathbf{g}(t) = (x(t), y(t), z(t)),$$

equations (1.10) and (1.11) imply that

$$\mathbf{v}(t) = (x'(t), y'(t), z'(t)), \tag{1.12}$$

giving the components of the velocity vector. The magnitude of the velocity vector

$$\| \mathbf{v}(t) \| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \tag{1.13}$$

is of course *the speed of the particle.*

If $\mathbf{g}'(t)$ is itself differentiable, the function $\mathbf{g}$ will have a second derivative denoted $\mathbf{g}''(t)$, and

$$\mathbf{g}''(t) = (x''(t), y''(t), z''(t))$$

in terms of the component functions. The second derivative represents the *acceleration vector* of the particle, and we write

$$\mathbf{a}(t) = \mathbf{g}''(t), \quad \text{or} \quad \mathbf{a}(t) = \frac{d^2\mathbf{x}}{dt^2}. \tag{1.14}$$

*Geometrical interpretation of the derivative:*

Given a curve

$$\mathbf{x} = \mathbf{g}(t),$$

with $\mathbf{g}'(t_0) \neq \mathbf{0}$, then $\mathbf{g}'(t_0)$ *is a vector in the direction of the tangent line at the point* $\mathbf{x} = \mathbf{g}(t_0)$. This conclusion follows from Figure 1.10. Note that as $\Delta t \to 0$ the increment $\Delta \mathbf{g}$ tends to zero and the secant line approaches the position of the tangent line. This result can also be confirmed on physical grounds: the velocity vector $\mathbf{v}(t_0) = \mathbf{g}'(t_0)$ gives the instantaneous direction of motion and hence must be tangent to the path of the particle when $t = t_0$.

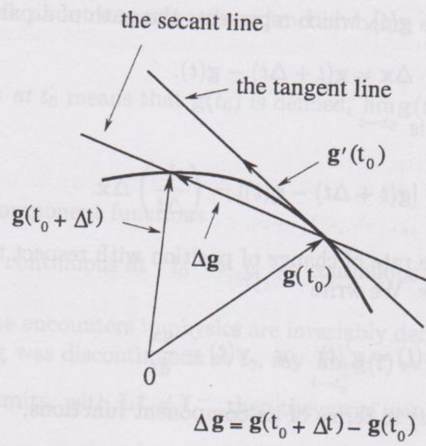



Figure 1.10: The secant line and tangent line to a curve.

*Equation of the tangent line:*

Given a curve $\mathbf{x} = \mathbf{g}(t)$ with $\mathbf{g}'(t_0) \neq \mathbf{0}$, the equation of the tangent line at $\mathbf{x} = \mathbf{g}(t_0)$ is

$$\mathbf{x} = \mathbf{g}(t_0) + (t - t_0)\mathbf{g}'(t_0), \tag{1.15}$$

as follows from Figure 1.11. Referring to Figure 1.11, we note that

$$\mathbf{OQ} = \mathbf{OP} + \mathbf{PQ},$$

and $\mathbf{PQ}$ is a multiple of $\mathbf{g}'(t_0)$ which we write as $(t - t_0)\mathbf{g}'(t_0)$. This choice of parameter on the tangent line ensures that $t = t_0$ gives the point of tangency.

In equation (1.15) we have defined a vector-valued function which we denote by $\mathbf{L}_{t_0}(t)$:

$$\mathbf{L}_{t_0}(t) = \mathbf{g}(t_0) + (t - t_0)\mathbf{g}'(t_0) \tag{1.16}$$

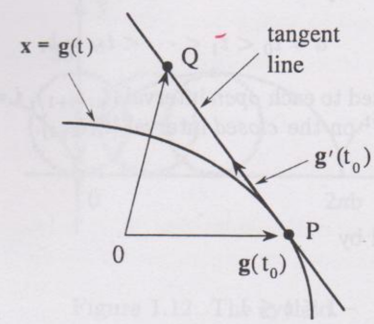



Figure 1.11: The tangent line to a curve $\mathbf{x} = \mathbf{g}(t)$.

called the *linearization of* $\mathbf{g}$ *at* $t_0$ Since the tangent line approximates the curve for $t$ sufficiently close to $t_0$, we obtain the *linear approximation*

$$\mathbf{g}(t) \approx \mathbf{L}_{t_0}(t),$$

or in full,

$$\mathbf{g}(t) \approx \mathbf{g}(t_0) + (t - t_0)\mathbf{g}'(t_0), \tag{1.17}$$

for $t$ sufficiently close to $t_0$. We can write (1.17) more concisely in terms of the increments

$$\Delta\mathbf{x} = \mathbf{g}(t) - \mathbf{g}(t_0), \quad \Delta t = t - t_0.$$

Rearranging (1.17) gives

$$\Delta\mathbf{x} \approx (\Delta t)\mathbf{g}'(t_0), \tag{1.18}$$

for $\Delta t$ sufficiently close to zero.

*Terminology:*

i) When we say that a vector-valued function $\mathbf{g} : [a, b] \to \mathbb{R}^n$ is *of class* $C^1$ (or more briefly, is $C^1$) we mean that $\mathbf{g}$ has a derivative function $\mathbf{g}'$ that is itself a continuous function: think of $C^1$ as "$C$ for continuous" and "1 for first derivative". We shall usually work with $C^1$ *curves*, i.e. curves described by $C^1$ functions, although one can imagine situations where the path of a particle would be a continuous but not a $C^1$ curve. There is a subtle point concerning $C^1$ curves. Based on experience with the graphs of scalar functions one might expect that a $C^1$ curve would look "smooth". A $C^1$ curve $\mathbf{x} = \mathbf{g}(t)$ can, however, have cusps (i.e. sharp spikes, or corners). See Examples 1.5 and 1.6 below. Physically, these occur when a particle stops, and changes its direction of motion.

$$a = t_0 < t_1 < \cdots < t_N = b,$$

such that $\mathbf{g}$, when restricted to each *open* interval $(t_i, t_{i+1})$, $i = 0, 1, \ldots, N-1$, coincides with a function that is $C^1$ on the *closed* interval $[t_i, t_{i+1}]$.

**Example 1.5:**

Consider the curves defined by

i) $\mathbf{x} = \mathbf{g}_1(t) = (t\,|t|, t^2), \quad -1 \le t \le 1$
ii) $\mathbf{x} = \mathbf{g}_2(t) = (t,\ |t|), \quad -1 \le t \le 1.$

In both cases the component functions $x(t)$, $y(t)$ satisfy $y = |x|$, so the curve is a $\vee$, i.e. the curve is not "smooth". The point we wish to illustrate is that $\mathbf{g}_1$ *is* $C^1$, *while* $\mathbf{g}_2$ *is piecewise* $C^1$ *but not* $C^1$. The difference is that $\mathbf{g}_1'(0) = \mathbf{0}$, while $\mathbf{g}_2'(0)$ does not exist (convince yourself of this). Physically, $\mathbf{g}_1$ represents the path of a particle that slows down and comes to rest at time $t = 0$ and then changes direction and speeds up again, while $\mathbf{g}_2$ represents the path of a particle that bounces off a hard surface without coming to rest (an idealized billiard ball).

**Example 1.6:**

The curve defined by

$$\mathbf{x} = \mathbf{g}(t) = b(t - \sin t, 1 - \cos t), \quad t \in \mathbb{R}, \tag{1.19}$$

where $b$ is a constant, is called a *cycloid.*

The derivative is

$$\mathbf{g}'(t) = b(1 - \cos t, \sin t),$$

so that

$$\mathbf{g}'(2n\pi) = \mathbf{0}, \quad n = 0, \pm 1, \pm 2, \ldots .$$

Note the properties of the component functions:

i) $x' \ge 0$, $x$ is non-decreasing,

ii) $y \ge 0$, $y$ is periodic.

This curve is $C^1$, but is not smooth. It is the path of a point on a uniformly rolling circle. (See Figure 1.12.) $\square$

*Comment:*

A curve $\mathbf{x} = \mathbf{g}(t)$, $a \le t \le b$, with $\mathbf{g}$ of class $C^1$ and $\mathbf{g}'(t) \ne \mathbf{0}$ for all $t \in [a, b]$, will be smooth, because it will have a unique non-zero tangent vector at each point. We shall refer to such a curve as a *smooth curve.*

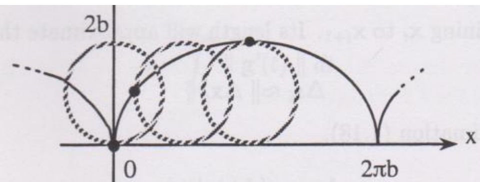



Figure 1.12: The cycloid.

### 1.1.5 Arclength

We want to calculate the *arclength $s$ of a curve $C$* in $\mathbb{R}^n$ given by

$$\mathbf{x} = \mathbf{g}(t), \qquad a \leq t \leq b. \tag{1.20}$$

We assume that the curve is of class $C^1$. If $C$ represents the path of a particle then the arclength $s$ represents the distance travelled by the particle in the time interval $a \leq t \leq b$. If $C$ represents a hanging cable (e.g. a section of a power line) then $s$ represents the actual length of the cable.

A partition of $[a, b]$,

$$a = t_0 < t_1 < \cdots < t_N = b$$

defines $N + 1$ points on the curve (1.20), given by $\mathbf{x}_i = \mathbf{g}(t_i)$, $i = 0, 1, \ldots, N$. Joining these points in order with straight lines yields a *polygonal arc* that can be regarded as approximating the curve $C$. See Figure 1.13 for the case $N = 4$.

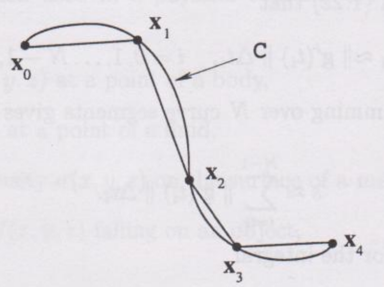



Figure 1.13: A polygonal arc approximating a curve $C$.

The i$^{\text{th}}$ increment

$$\Delta \mathbf{x}_i = \mathbf{x}_{i+1} - \mathbf{x}_i$$

represents the vector joining $\mathbf{x}_i$ to $\mathbf{x}_{i+1}$. Its length will approximate the arclength $\Delta s_i$ of the ith curve segment:

$$\Delta s_i \approx \| \Delta \mathbf{x}_i \| . \tag{1.21}$$

Using the linear approximation (1.18),

$$\Delta \mathbf{x}_i \approx (\Delta t_i)\mathbf{g}'(t_i), \tag{1.22}$$

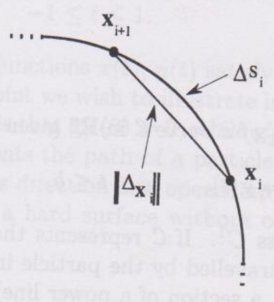



Figure 1.14: The length of the line segment approximates the arclength of the curve segment.

where

$$\Delta t_i = t_{i+1} - t_i,$$

for $\Delta t_i$ sufficiently small.

It follows from (1.21) and (1.22) that

$$\Delta s_i \approx \| \mathbf{g}'(t_i) \| \, \Delta t_i, \quad i = 0, 1, \ldots N-1, \tag{1.23}$$

for $\Delta t_i$ sufficiently small. Summing over $N$ curve segments gives an approximation for the total arclength $s$:

$$s \approx \sum_{i=0}^{N-1} \| \mathbf{g}'(t_i) \| \, \Delta t_i.$$

The sum is a Riemann sum for the integral

$$\int_a^b \| \mathbf{g}'(t) \| \, dt,$$

and will thus approximate the integral with increasing accuracy as the partition becomes increasingly fine. We thus expect that *the arclength $s$ of the curve $\mathbf{x} = \mathbf{g}(t)$, $a \leq t \leq b$ is given by*

$$s = \int_a^b \| \mathbf{g}'(t) \| \, dt. \tag{1.24}$$

*Comments:*

i) In practice we use equation (1.24) as the *definition* of arclength of a $C^1$ curve. The steps leading to (1.21) do not constitute a proof because of the approximations made, but should be viewed as a heuristic justification of (1.24). It is important to understand these steps, however, because they form the basis for the definition of the concept of *line integral* in Chapter 2.

ii) If the curve $C$ represents the path of a particle then the integrand $\| \mathbf{g}'(t) \|$ in (1.24) is the speed of the particle, and so the formula (1.24) has a simple physical interpretation:

*distance travelled is the integral of the*
*speed with respect to time.*

## 1.2 Vector fields

So far we have considered vector-valued functions $\mathbf{g} : [a, b] \to \mathbb{R}^n$, which define curves. In this section we consider a second type of vector-valued function of great importance in physics and engineering applications, namely *vector fields.*

### 1.2.1 Examples from physics

Many physical quantities are characterized by giving a *magnitude*, i.e. a real number, at each point of space. Such quantities are represented by real-valued functions $f : \mathbb{R}^3 \to \mathbb{R}$. Functions of this type, when used in a physical context, are called *scalar fields.* Some examples are the following:

- the temperature $T(x, y, z)$ at a point of a body,
- the pressure $p(x, y, z)$ at a point of a fluid,
- the electric charge density $\sigma(x, y, z)$ on the surface of a metallic object,
- the intensity of light $I(x, y, z)$ falling on an object,
- ... and so on.

On the other hand there are many physical quantities that are characterized by giving a *magnitude* and a *direction*, i.e. a *vector*, at each point of space. Such quantities are represented by a function $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ that associates with each point $(x, y, z) \in \mathbb{R}^3$ a unique vector $\mathbf{F}(x, y, z) \in \mathbb{R}^3$. Functions of this type, when used in a physical context, are called *vector fields*.

**Example 1.7:**

The gravitational field due to a spherical body of mass $M$, such as the earth, can be represented by the vector field $\mathbf{F}$ defined by

$$\mathbf{F}(x, y, z) = -\frac{GM}{r^3}\mathbf{r}, \tag{1.25}$$

where $\mathbf{r} = (x, y, z)$, $r = \| \mathbf{r} \| = \sqrt{x^2 + y^2 + z^2}$, and $G$ is the gravitational constant.

The vector $\mathbf{F}(x, y, z)$ represents the force exerted on a test body of unit mass placed at position $(x, y, z)$. The minus sign accounts for the fact that the test body will be attracted to the earth, i.e. $F(x, y, z)$ points towards the origin.

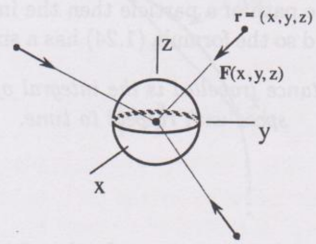



Figure 1.15: The gravitational field of a spherical body.

*Comment:*

In this example the domain of the vector field is the open set $\mathcal{U} = \mathbb{R}^3 - \{(0, 0, 0)\}$. □

**Example 1.8:**

Consider a distribution of fluid flowing in a steady state. Then at a given point $(x, y, z)$ the fluid has a uniquely defined velocity $\mathbf{v}(x, y, z)$ that is independent of the time. The vector field $\mathbf{v}$ is called the *velocity field of the fluid*. See Figure 1.16.

**Example 1.9:**

There is also a scalar field $\rho(x, y, z)$ associated with a fluid, namely the *mass density of the fluid*, which is independent of time for a steady state flow. One is interested in the rate at which mass is transferred by the fluid, and this rate is determined by another vector field formed from $\rho$ and $\mathbf{v}$, as follows.

Consider a plane surface element of area $\Delta S$, with unit normal vector $\mathbf{n}$, located at position $(x, y, z)$. In time $\Delta t$, the fluid in a column of length $\| \mathbf{v} \| \Delta t$ will flow through the surface element. Since the vertical height of the column is $\mathbf{v} \cdot \mathbf{n} \Delta t$ (see Figure 1.17) the volume of the column, and hence the volume of fluid transported, is

$$\Delta V \approx (\mathbf{v} \cdot \mathbf{n}) \Delta S \Delta t.$$

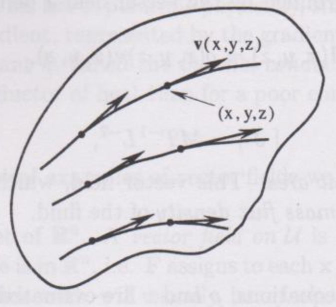



Figure 1.16: The velocity field of a fluid.

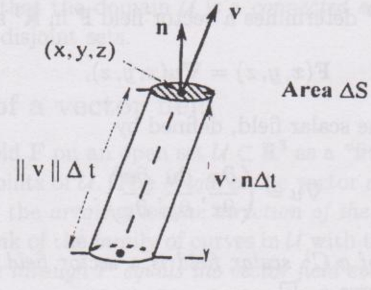



Figure 1.17: Flow of a fluid through a plane surface element.

It follows that mass of fluid transported across the surface element in time $\Delta t$ is

$$\Delta M \approx \rho \Delta V \approx \rho \mathbf{v} \cdot \mathbf{n} \Delta S \Delta t.$$

Thus, the *mass transferred per unit time across the surface element* is

$$(\rho \mathbf{v}) \cdot \mathbf{n} \Delta S. \tag{1.26}$$

This quantity is called the *mass flux across the surface element*:

"flux" means "rate of flow".

It is worth checking the physical dimensions of the mass flux (note that $\mathbf{n}$, being a unit normal, is dimensionless):

$$[(\rho \mathbf{v}) \cdot \mathbf{n} \Delta S] = (ML^{-3})(LT^{-1})(L^2) = MT^{-1},$$

i.e. *mass per unit time.*

The mass flux (1.26) is determined by the vector field **J** defined by

$$\mathbf{J}(x,y,z) = \rho(x,y,z)\mathbf{v}(x,y,z), \tag{1.27}$$

which has physical dimensions

$$[\,\mathbf{J}\,] = MT^{-1}L^{-2},$$

i.e. *mass per unit time per unit area.* This vector field, which describes the transport of mass by the fluid, is called the *mass flux density* of the fluid.

*Comment:*

In (1.26) and the preceding equations, $\rho$ and $v$ are evaluated at the given point $(x,y,z)$. We are assuming that $\|\,\mathbf{v}\,\|\,\Delta t$ and $\Delta S$ are sufficiently small that $\rho$ and $\mathbf{v}$ can be regarded as constant throughout the cylinder in Figure 1.17.

**Example 1.10:**

Any $C^1$ scalar field $u$ in $\mathbb{R}^3$ determines a vector field $\mathbf{F}$ in $\mathbb{R}^3$ according to

$$\mathbf{F}(x,y,z) = \nabla u(x,y,z), \tag{1.28}$$

where $\nabla u$ is the *gradient* of the scalar field, defined by

$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right). \tag{1.29}$$

In other words, *the gradient of a $C^1$ scalar field is a vector field.* This type of vector field will arise frequently in the course. $\square$

The concept of a *flux density vector field,* as in Example 1.3, where we studied the mass flux density, applies to any physical quantity that is transferred in space, e.g. heat energy, which is the next example.

**Example 1.11:**

Consider a piece of material that is heated on one side and cooled on the other in such a way that the temperature attains a steady state, i.e. the temperature $u(x,y,z)$ depends on position $(x,y,z)$ but not on time $t$. The temperature is a scalar field. From everyday experience we know that heat will flow (i.e. energy will be transferred) from hot regions to cold regions. As with mass flow, heat flow can be described mathematically by a vector field $\mathbf{j}(x,y,z)$, which gives the rate of heat transfer through a small surface element at $(x,y,z)$, called the *heat flux density,* i.e.

$$(\mathbf{j}\cdot\mathbf{n})\Delta S$$

is the rate of transfer of heat through a small plane surface element of area $\Delta S$ and having unit normal $\mathbf{n}$ (compare with equation (1.26)).

Experimental work shows that the heat flux density may be represented by *Fourier's law*

$$\mathbf{j}(x,y,z) = -k\nabla u(x,y,z), \tag{1.30}$$

where $k > 0$ is a constant. This law is plausible physically, since one expects that the larger is the spatial temperature gradient, represented by the gradient $\nabla u$, the larger will be the rate of heat transfer. The constant $k$, called the *thermal conductivity*, depends on the material, being bigger for a good conductor of heat than for a poor conductor (e.g. $k_{\text{copper}}/k_{\text{glass}} \gg 1$). $\square$

Having given some physical examples of vector fields we now introduce the terminology formally.

Let $\mathcal{U}$ be an open subset of $\mathbb{R}^n$. A *vector field on* $\mathcal{U}$ is a function $\mathbf{F} : \mathcal{U} \to \mathbb{R}^n$, whose domain is $\mathcal{U}$ and whose range is in $\mathbb{R}^n$, i.e. $\mathbf{F}$ assigns to each $\mathbf{x} \in \mathcal{U}$ a unique vector $\mathbf{F}(\mathbf{x}) \in \mathbb{R}^n$.

The vector fields we work with will usually be *of class* $C^1$, which means that the $n$ component functions,

$$\mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), \ldots, F_n(\mathbf{x}))$$

are $C^1$ functions.

We shall also assume that the domain $\mathcal{U}$ is a *connected set*, which means that $\mathcal{U}$ is not the union of two or more disjoint sets.

### 1.2.2 Field lines of a vector field

One visualizes a vector field $\mathbf{F}$ on an open set $\mathcal{U} \subset \mathbb{R}^3$ as a "field of vectors", represented by arrows, attached to the points of $\mathcal{U}$. The *length* of the vector at a point gives the *strength of the field* at the point, and the *arrow* gives *the direction of the field*.

It is also helpful to think of the family of curves in $\mathcal{U}$ with the property that at each point $P$ *the tangent to the curve through* $P$ *equals the vector field evaluated at* $P$. These curves are called the *field lines*[1] *of the vector field*. One thinks of a field line threading its way through the vector field, always following the direction of the vector field (see Figure 1.18).

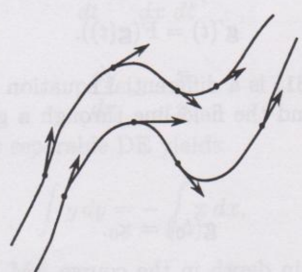


Figure 1.18: Two field lines of a vector field.

[1] These curves are also called *integral curves* of the vector field.

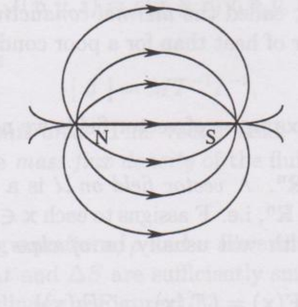



Figure 1.19: Field lines of a magnetic field.

In the case of the *velocity field of a fluid* the field lines are simply the paths of the fluid particles (see Figures 1.16 and 1.18). In the case of a magnetic field one can do a simple experiment to visualize the field lines. Take a sheet of paper and sprinkle it with iron filings. Then put a bar magnet under the paper and shake the paper slightly. You will observe the iron filings arranging themselves in lines going from one magnetic pole to the other. The strength of the magnetic field is revealed by the density of packing of the filings. In this way one obtains a picture of the field lines of the magnetic field.

We now consider the problem of determining the field lines of a given vector field $F(\mathbf{x})$.

Let the curve $\mathbf{x} = \mathbf{g}(t)$, assumed $C^1$, be a field line. Its tangent vector is $\mathbf{g}'(t)$, and thus the defining condition of a field line is written

$$\mathbf{g}'(t) = \mathbf{F}(\mathbf{g}(t)). \tag{1.31}$$

(see Figure 1.20). Equation (1.31) is a differential equation for the unknown vector-valued function $\mathbf{g}(t)$. If you want to find the field line through a given point $\mathbf{x}_0$ then you should impose the initial condition

$$\mathbf{g}(t_0) = \mathbf{x}_0. \tag{1.32}$$

DEs such as (1.31) are studied in depth in the course AM 451. In general they can only be solved numerically using a computer. For this course, however, it will be enough to concentrate on simple types that can be solved explicitly, as in the examples to follow.

**Example 1.12:**

Find the field lines of the vector field

$$\mathbf{F}(x, y) = (-y, x) \tag{1.33}$$

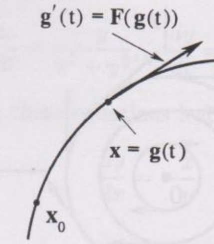



Figure 1.20: The field line of a vector field $\mathbf{F}$ through a given point $\mathbf{x}_0$.

in $\mathbb{R}^2$, and sketch the field portrait.

*Solution:* A field line $\mathbf{x} = \mathbf{g}(t)$ satisfies

$$\mathbf{g}'(t) = \mathbf{F}(\mathbf{g}(t)).$$

In terms of components $\mathbf{g}(t) = (x(t), y(t))$, this reads

$$(x'(t), y'(t)) = (-y(t), x(t)),$$

giving

$$\frac{dx}{dt} = -y(t), \quad \frac{dy}{dt} = x(t). \tag{1.34}$$

These two coupled DEs can be written as a single DE by using the chain rule:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt},$$

giving

$$\frac{dy}{dx} = -\frac{x}{y}$$

by equations (1.34). Solving this separable DE yields

$$\int y\,dy = -\int x\,dx,$$

giving

$$x^2 + y^2 = C,$$

where $C$ is a constant.

The conclusion is that *any field line of the given vector field is a circle centred on the origin.* Equations (1.34) show that the circles are traversed counterclockwise, giving Figure 1.21. We note that the field line through a given point $(x_0, y_0)$ is the circle of radius $\sqrt{x_0^2 + y_0^2}$ – specifying a point on the field line fixes the value of the constant $C$. $\square$

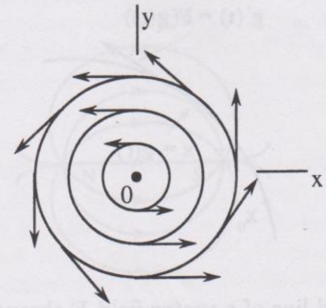



Figure 1.21: The field lines $x^2 + y^2 = C$ of the vector field $\mathbf{F} = (-y, x)$.

**Exercise 1.3:**
The vector field in Example 1.12 can be interpreted physically as the velocity field of a rigidly rotating disc with unit angular velocity. Verify that

$$\theta' \equiv \frac{d\theta}{dt} = 1.$$

?

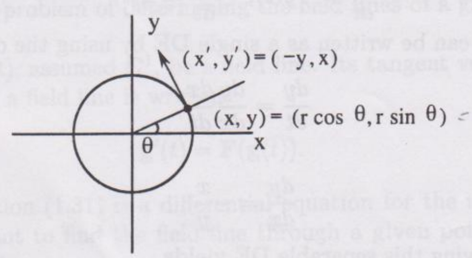



Figure 1.22: Velocity of a point on a rotating disc.

**Example 1.13:**
Find the field lines of the vector field

$$\mathbf{F}(x, y) = \left( \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right) \tag{1.35}$$

on the open set $\mathcal{U} = \mathbb{R}^2 - \{(0,0)\}$, and sketch the "portrait".

*Solution:* We have to solve the DEs

$$\frac{dx}{dt} = \frac{-y}{x^2+y^2}, \quad \frac{dy}{dt} = \frac{x}{x^2+y^2}.$$

Proceeding as in Example 1.12, these equations lead to the same DE

$$\frac{dy}{dx} = -\frac{x}{y},$$

giving

$$x^2 + y^2 = C,$$

where $C$ is a constant, as the field lines. $\square$

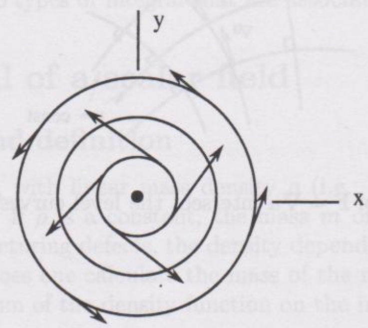



Figure 1.23: Field lines of the vector field (1.35).

*Comment:*

The vector field (1.35) could represent the velocity field of a fluid swirling down a drain. Note that for Example 1.12,

$$\| \mathbf{F}(x,y) \| = \sqrt{x^2+y^2},$$

i.e. the speed equals the distance from the origin, while for Example 1.13,

$$\| \mathbf{F}(x,y) \| = \frac{1}{\sqrt{x^2+y^2}}, \quad (x,y) \neq (0,0),$$

i.e. the speed equals the reciprocal of the distance from the origin. We have indicated this difference in Figures 1.21 and 1.23 by the size of the arrows. We note that Examples 1.12 and 1.13 illustrate that *different vector fields can have the same field lines.*

**Exercise 1.4:**

Find the field lines of the vector field $\mathbf{F}(x,y) = (x, 2y)$ in $\mathbb{R}^2$, and sketch the field portrait.

*Gradient vector fields:*

The field lines of a vector field $\mathbf{F}(\mathbf{x}) = \nabla u(\mathbf{x})$ in $\mathbb{R}^2$ that is the gradient of a scalar field can be drawn without solving a DE. We know (Calculus 3) that the gradient $\nabla u$ of a scalar field $u$ is orthogonal to the level curves $u =$ constant of the scalar field. It follows that *the field lines of the vector field* $\mathbf{F} = \nabla u$ *are the orthogonal trajectories of the family of level curves of* $u$.

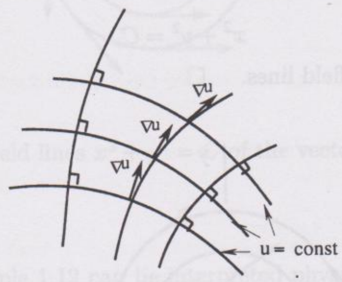



Figure 1.24: The field lines of $\mathbf{F} = \nabla u$ intersect the level curves $u =$ constant orthogonally.

# Chapter 2

# Line Integrals & Green's Theorem

In this chapter we define two types of integral that are associated with a curve in $\mathbb{R}^n$.

## 2.1 Line integral of a scalar field

### 2.1.1 Motivation and definition

Consider a nuclear fuel rod, with linear mass density $\rho$ (i.e. the physical dimensions are $[\rho] = ML^{-1}$) and length $\ell$. If $\rho$ is a constant, the mass $m$ of the rod is simply $m = \rho\ell$. Suppose that due to manufacturing defects, the density depends on position on the rod, say $\rho = \rho(x)$, $0 \le x \le \ell$. How does one calculate the mass of the rod? Well, one approximates the mass $m$ as a Riemann sum of the density function on the interval $0 \le x \le \ell$:

$$m \approx \sum_{i=1}^{n} \rho(x_i)\Delta x_i,$$

leading to the formula

$$m = \int_0^\ell \rho(x)dx. \tag{2.1}$$

A similar but more difficult problem is to find the mass of a suspended wire (say, part of a hydro line, see Figure 2.1) whose linear mass density depends on position. Evidently, we will need some sort of integral along the curve $C$ that represents the wire. This new type of integral, which we now introduce, is called the *line integral of a scalar field.*

Given a *curve* $C$ in $\mathbb{R}^n$, defined by $\mathbf{x} = \mathbf{g}(t)$, $a \le t \le b$, where $\mathbf{g}$ is a class $C^1$, and a *scalar field* $f$ continuous on $C$, the line integral of $f$ along $C$ is denoted by

$$\int_C f\,ds.$$

We first give a tentative definition, motivated by the problem of calculating the mass of the hanging wire. As in the calculation of arclength (Section 1.1.5) we introduce a partition of $[a, b]$,

$$a = t_0 < t_1 < \cdots < t_N = b,$$

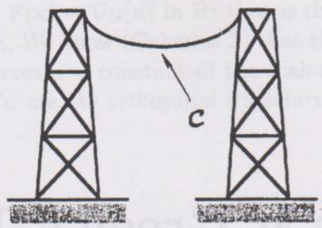



Figure 2.1: A curve $C$ representing a hanging wire.

which defines $N+1$ points $\mathbf{x}_i = \mathbf{g}(t_i)$ on the curve $C$. Let $\Delta s_i$ be the arclength of the i$^{\text{th}}$ curve segment. We make the *tentative definition*:

$$\int_C f\,ds = \lim_{\substack{N\to\infty\\ |\Delta t_i|\to 0}} \sum_{i=0}^{N-1} f(\mathbf{x}_i)\Delta s_i, \tag{2.2}$$

provided the limit exists ($\Delta t_i = t_i - t_{i-1}$ as before).

Thinking of the scalar field $f$ as representing the linear density of the hanging wire, *the term $f(\mathbf{x}_i)\Delta s_i$ approximates the mass of the $i^{th}$ segment of the wire*, and so we expect the limit of the sum to give the *total mass of the wire.*

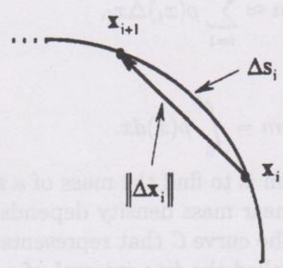



Figure 2.2: The i$^{\text{th}}$ segment of the hanging wire.

Substituting $\mathbf{x}_i = \mathbf{g}(t_i)$ and using the approximation $\Delta s_i \approx \| \mathbf{g}'(t_i) \| \Delta t_i$ (see equation (1.23)), we can approximate the sum in (2.2) as

$$\sum_{i=0}^{N-1} f(\mathbf{x}_i)\Delta s_i \approx \sum_{i=0}^{N-1} f(\mathbf{g}(t_i)) \| \mathbf{g}'(t_i) \| \Delta t_i, \tag{2.3}$$

for $\Delta t_i$ sufficiently small. In the limit as $N \to +\infty$ and $|\Delta t_i| \to 0$ the right side of (2.3) equals the Riemann integral $\displaystyle\int_a^b f(\mathbf{g}(t)) \parallel \mathbf{g}'(t) \parallel dt$, and we expect the approximation in (2.3) to become increasingly accurate. Comparison of (2.2) and (2.3) then motivates the definition to follow.

**Definition:**

Consider a curve $C$ in $\mathbb{R}^n$ given by $\mathbf{x} = \mathbf{g}(t)$, $a \le t \le b$, where $\mathbf{g}$ is of class $C^1$, and a scalar field $f$ continuous on $C$. The *line integral of* $f$ *along* $C$ is defined by

$$\int_C f\, ds = \int_a^b f(\mathbf{g}(t)) \parallel \mathbf{g}'(t) \parallel dt. \tag{2.4}$$

*Comment:*

We see that the line integral is defined in terms of an ordinary Riemann integral. The formula (2.4) can be remembered easily as follows:

> "$f$" is evaluated on the curve $C$ giving "$f(\mathbf{g}(t))$", and the symbol "$ds$" reminds one of $\Delta s$ in (2.2), which is approximated as $\Delta s \approx \parallel \mathbf{g}'(t) \parallel \Delta t$, leading to "$\parallel \mathbf{g}'(t) \parallel dt$".

**Example 2.1:**

Evaluate the line integral $\displaystyle\int_C f\, ds$ in $\mathbb{R}^2$, where the curve $C$ is the upper semi-circle of radius $b$ joining $(b, 0)$ and $(-b, 0)$, and $f$ is the scalar field defined by $f(x, y) = x^2 + 3y^2$.

*Solution:* We introduce the standard parametrization for the semi-circle,

$$\mathbf{x} = \mathbf{g}(t) = (b\cos t, b\sin t), \quad 0 \le t \le \pi.$$

Then

$$\mathbf{g}'(t) = (-b\sin t, b\cos t),$$

and the magnitude is

$$\parallel \mathbf{g}'(t) \parallel = \sqrt{(-b\sin t)^2 + (b\cos t)^2} = b,$$

after simplifying. Evaluating the scalar field on the curve $C$ gives

$$f(\mathbf{g}(t)) = (b\cos t)^2 + 3(b\sin t)^2 = b^2(3 - 2\cos^2 t),$$

after simplifying. By the definition (2.4) of the line integral,

$$\int_C f\,ds = \int_0^\pi f(\mathbf{g}(t))\ \|\ \mathbf{g}'(t)\ \|\ dt$$

$$= \int_0^\pi b^2(3 - 2\cos^2 t) b\,dt \qquad \textit{Aside: } 2\cos^2 t = 1 + \cos 2t.$$

$$= \cdots = 2\pi b^3. \quad \square$$

**Exercise 2.1:**

Evaluate the line integral $\int_C f\,ds$ in $\mathbb{R}^3$, where $C$ is the helix $\mathbf{x} = \mathbf{g}(t) = (R\cos t, R\sin t, t)$, $0 \le t \le 4\pi$, and $f$ is the scalar field $f(x,y,z) = z$.

*An important consistency requirement:*

Given a hanging wire represented by a curve $C$ and a scalar field $f$ representing the linear mass density of the wire, we have seen that the line integral $\int_C f\,ds$ represents the mass of the wire, which can be calculated using the definition (2.4). We know, however, that a curve $C$ has infinitely many different parametrizations (Section 1.1.2). Clearly, *the value of the line integral* (the mass of the wire) *should not depend on which parametrization we use.* Consider a different parametrization for the same curve $C$:

$$\mathbf{x} = \hat{\mathbf{g}}(\tau), \qquad \alpha \le \tau \le \beta,$$

with $\hat{\mathbf{g}}(\tau) = \mathbf{g}(h(\tau))$ and $t = h(\tau)$ (see equation (1.6)). The definition (2.4) becomes

$$\int_C f\,ds = \int_\alpha^\beta f(\hat{g}(\tau))\ \|\ \hat{g}'(\tau)\ \|\ d\tau. \tag{2.5}$$

The consistency requirement is that *the Riemann integrals in* (2.4) *and* (2.5) *must be equal.* The proposition to follow establishes the consistency.

**Proposition 2.1:**

Consider a curve $C$ given by

$$\mathbf{x} = \mathbf{g}(t), \qquad a \le t \le b,$$

with g of class $C^1$. Under a change of parameter $t = h(\tau)$, with $h$ of class $C^1$ and $h'(\tau) > 0$ for $\tau \in [\alpha, \beta]$, the curve is described by

$$\mathbf{x} = \hat{\mathbf{g}}(\tau), \qquad \alpha \le \tau \le \beta,$$

with

$$\hat{\mathbf{g}}(\tau) = \mathbf{g}(h(\tau)). \tag{2.6}$$

Then

$$\int_a^b f(\mathbf{g}(t)) \parallel \mathbf{g}'(t) \parallel dt = \int_\alpha^\beta f(\hat{\mathbf{g}}(\tau)) \parallel \hat{\mathbf{g}}'(\tau) \parallel d\tau. \tag{2.7}$$

*Proof:* (outline)

Differentiate (2.6) with respect to $\tau$ and use the Chain Rule to get

$$\hat{\mathbf{g}}'(\tau) = h'(\tau)\mathbf{g}'(h(\tau)).$$

*Aside:* See Problem Set 1, #20.

Since $h'(\tau) > 0$ it follows that

$$\parallel \hat{\mathbf{g}}'(\tau) \parallel = \parallel \mathbf{g}'(h(\tau)) \parallel h'(\tau). \tag{2.8}$$

Substitute (2.6) and (2.8) into the integral on the right in (2.7). Since $t = h(\tau)$ and $a = h(\alpha)$, $b = h(\beta)$, the Change of Variable Theorem now implies the integral on the right equals the integral on the left.

**Exercise 2.2:**

Repeat Example 2.1 using a different parametrization of $\mathcal{C}$, for example

$$\mathbf{x} = \hat{\mathbf{g}}(\tau) = (b\cos 2\tau, b\sin 2\tau), \quad 0 \le \tau \le \tfrac{\pi}{2},$$

and confirm that you get the same value for the line integral.

### 2.1.2 Applications

When working with a line integral in a physical context it is essential to keep in mind that

*an integral is the limit of a sum*[1]

(as in equation (2.2)). An important special case arises if $f(\mathbf{x}) = 1$ for all $\mathbf{x} \in \mathcal{C}$. Then the definition (2.4) becomes

$$\int_{\mathcal{C}} ds = \int_a^b \parallel \mathbf{g}'(t) \parallel dt,$$

which by equation (1.24) equals the arclength of the curve $\mathcal{C}$. In words, *the line integral of the constant scalar field $f(\mathbf{x}) = 1$ along a curve $\mathcal{C}$ equals the arclength of $\mathcal{C}$*, i.e. one thinks of the line integral $\int_{\mathcal{C}} ds$ as summing the elements of arclength along the curve $\mathcal{C}$ to give the total arclength.

We now give a glimpse of some other applications of the line integral of a scalar field

i) *An "everyday" example:*

The base of a vertical curved fence is a curve $\mathcal{C}$ in the $xy$-plane, and its height at position $(x, y)$ is $h(x, y)$. What is the total area of the fence?

[1]This statement applies to ANY integral.

Consideration of the preliminary definition (2.2) leads to the conclusion that the area $A$ is given by

$$A = \int_C h\, ds.$$

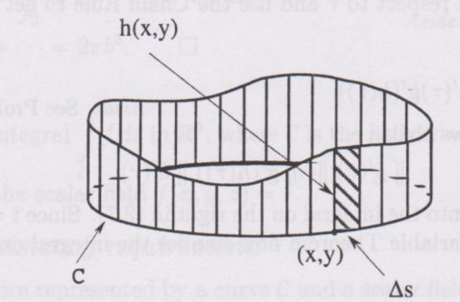



Figure 2.3: A curved fence whose base is a plane curve $C$.

One thinks of the line integral as summing the product of height $h$ and element of arclength $\Delta s$. □

**Exercise 2.3:**

The base of a vertical fence is given by $\mathbf{x} = \mathbf{g}(t) = (b\cos^3 t, b\sin^3 t)$, $0 \le t \le \pi$, where $b$ is a positive constant, and the height at position $\mathbf{x} = (x, y)$ is $h(x, y) = b + \frac{1}{3}y$. Show that the area of the fence is $\frac{17}{5}b^2$. □

*Reference:* Marsden & Tromba, page 417, example 2.

**ii)** *Line integral of a linear density function*

It is helpful to think of the physical dimensions of quantities when interpreting a line integral

$$I = \int_C f\, ds. \tag{2.9}$$

In terms of the preliminary definition (2.2), we have

$$I = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(x)\Delta s_i,$$

where $\Delta s_i$ is the arclength of the i[th] segment of $C$. Since[2]

$$[f(\mathbf{x}_i)\Delta s_i] = [f(\mathbf{x}_i)][\Delta s_i] = [f(\mathbf{x}_i)]L,$$

[2]We use the symbol $[I]$ to denote the dimensions of a physical quantity $I$.

it follows that the dimensions of the integral $I$ are

$$[I] = [f]L. \tag{2.10}$$

In the hanging wire problem, $f$ represents the *linear mass density*, i.e. $[f] = ML^{-1}$, and the line integral $I$ in (2.9) represents the *total mass* of the wire. Equation (2.10) gives

$$[I] = ML^{-1}L = M,$$

which is consistent with the interpretation of $I$.

As another example, think of the curve $C$ as representing a conductor (e.g. a copper wire) which is charged, with $f$ being the *linear charge density*, i.e. $[f] = [\text{charge}]L^{-1}$. The line integral $I$ will give the *total charge* on the conductor. Equation (2.10) implies

$$[I] = [\text{charge}]L\,L^{-1} = [\text{charge}],$$

which is again consistent with the interpretation of $I$.

In general we can think of the scalar field $f$ as representing the *linear density* of some "physical stuff", which is distributed on a curve $C$. Then the line integral $I$ in (2.9) gives the *total amount* of "physical stuff" on the curve. As before, equation (2.10) guarantees dimensional consistency.

iii) *Average value of a scalar field on a curve*

For a continuous function $f : [a, b] \to \mathbb{R}$, the *average value of $f$ over the interval* $[a, b]$ is defined by

$$\langle f \rangle = \frac{\int_a^b f(x)dx}{\int_a^b dx},$$

i.e. the average value is the integral of $f$ over $[a, b]$ divided by the length of the interval. We can generalize the concept of "average of a function" to the case of a scalar field $f$ that is continuous on a curve $C$ in $\mathbb{R}^n$. In this case we define the average value by

$$\langle f \rangle = \frac{\int\limits_C f\,ds}{\int\limits_C ds}, \tag{2.11}$$

i.e. the average value of $f$ is the line integral of $f$ along $C$ divided by the arclength of $C$.

*Comment:*

If we choose a partition so that the $N$ curve segments of $C$ are of equal length (i.e. $\Delta s_i = \Delta s$, $i = 0, 1, \ldots, N-1$), it follows from (2.11) and (2.2) that

$$\langle f \rangle \approx \frac{1}{N} \sum_{i=0}^{N-1} f(\mathbf{x}_i),$$

i.e. the "continuous average of $f$" is approximated by the "discrete average" of $N$ values of the scalar field on $C$.

**Exercise 2.4:**

The steady state temperature of a circular metal plate of radius $b$ centred on the origin in the $xy$-plane is given by

$$u(x,y) = \frac{u_0}{b^2}(x^2 - y^2),$$

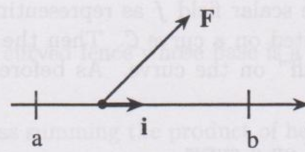
where $u_0$ is a constant. Show that the average temperature along the diameter $y = (\tan\theta)x$ is given by

$$\langle u \rangle = \tfrac{1}{3}u_0 \cos 2\theta.$$

## 2.2 Line integral of a vector field

### 2.2.1 Motivation and definition

Consider a particle moving along the $x$-axis from $x = a$ to $x = b$ under the action of a force $\mathbf{F}$. If $\mathbf{F}$ is constant the *work done* on the particle is simply




$$W = (\mathbf{F}\cdot\mathbf{i})(b-a),$$

where $\mathbf{F}\cdot\mathbf{i}$ is the component of $\mathbf{F}$ in the direction of motion ($\mathbf{i}$ is a unit vector in the $x$-direction). If $\mathbf{F} = \mathbf{F}(x)$ then the work done can be calculated as an integral (the limit of a Riemann sum),

$$W = \int_a^b (\mathbf{F}\cdot\mathbf{i})dx.$$

A similar but more difficult problem is to calculate the work done by a force field $\mathbf{F}(\mathbf{x})$ acting on a particle moving in a complicated way in space. Evidently we will need some sort of integral along the curve $C$ that represents the path of the particle in space. This new type of integral is called the *line integral of a vector field.*

Given a *curve* $C$ in $\mathbb{R}^n$, defined by $\mathbf{x} = \mathbf{g}(t)$, $a \le t \le b$, where $\mathbf{g}$ is of class $C^1$, and a *vector field* $\mathbf{F}$ continuous on $C$, the line integral of $\mathbf{F}$ along $C$ is denoted by

$$\int_C \mathbf{F}\cdot d\mathbf{x}.$$

We first give a tentative definition, motivated by the problem of calculating the work done by a force field. The development parallels that of Section 2.1, 2.1.1 very closely. We introduce a partition of $[a,b]$,

$$a = t_0 < t_1 < \cdots < t_N = b,$$

which defines $N+1$ points $\mathbf{x}_i = \mathbf{g}(t_i)$ on the curve $C$. Let

$$\Delta \mathbf{x}_i = \mathbf{x}_{i+1} - \mathbf{x}_i$$

be the increment vector associated with the i[th] curve segment. We make the *tentative definition:*

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \lim_{\substack{N \to \infty \\ |\Delta t_i| \to 0}} \sum_{i=0}^{N-1} \mathbf{F}(\mathbf{x}_i) \cdot \Delta \mathbf{x}_i, \tag{2.12}$$

provided the limit exists. Thinking of the vector field as the force field acting on the moving particle, *the term* $\mathbf{F}(\mathbf{x}_i) \cdot \Delta \mathbf{x}_i$ *approximates the work done on the particle while it moves from* $\mathbf{x}_i$ to $\mathbf{x}_{i+1}$, and so we expect the limit of the sum to give the *total work done on the particle.*

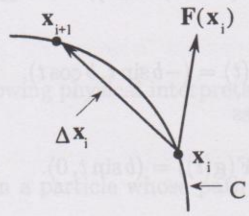



Figure 2.4: The i[th] segment of the path of a particle and the increment vector $\Delta \mathbf{x}_i$.

Substituting $\mathbf{x}_i = \mathbf{g}(t_i)$ and using the approximation

$$\Delta \mathbf{x}_i \approx (\Delta t_i)\mathbf{g}'(t_i),$$

(see equation (1.22)), we can approximate the sum in (2.12) as

$$\sum_{i=0}^{N-1} \mathbf{F}(\mathbf{x}_i) \cdot \Delta \mathbf{x}_i \approx \sum_{i=0}^{N-1} \mathbf{F}(\mathbf{g}(t_i)) \cdot \mathbf{g}'(t_i) \Delta t_i \tag{2.13}$$

for $\Delta t_i$ sufficiently small. In the limit as $N \to \infty$ and $|\Delta t_i| \to 0$ the right side of (2.13) equals the Riemann integral $\displaystyle\int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t)dt$, and we expect the approximation in (2.13) to become increasingly accurate. Comparison of (2.13) and (2.12) then motivates the definition to follow.

**Definition:**

Consider a curve $C$ in $\mathbb{R}^n$ given by $\mathbf{x} = \mathbf{g}(t)$, $a \le t \le b$, where $\mathbf{g}$ is of class $C^1$, and a vector field $\mathbf{F}$ continuous on $C$. The *line integral of* $\mathbf{F}$ *along* $C$ is defined by

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t)\, dt. \tag{2.14}$$

*Comment:*

The line integral of a vector field is defined in terms of an ordinary Riemann integral. The formula (2.14) can be remembered easily, because $\mathbf{F}\cdot d\mathbf{x}$ reminds one of $\mathbf{F}(\mathbf{g}(t)) \cdot \Delta\mathbf{x}$, which is approximated by $\mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t)\Delta t$.

**Example 2.2:**

Evaluate the line integral $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{x}$, where $C$ is the quarter circle of radius $b$, $x^2 + y^2 = b^2$, joining $(b, 0)$ to $(0, b)$, and $\mathbf{F}$ is the vector field defined by $\mathbf{F}(x, y) = (y, 0)$.

*Solution:* We use the usual parametrization for the circle:

$$\mathbf{x} = \mathbf{g}(t) = (b\cos t, b\sin t), \qquad 0 \le t \le \tfrac{\pi}{2}.$$

Then

$$\mathbf{g}'(t) = (-b\sin t, b\cos t).$$

Evaluating the vector field on $C$ gives

$$\mathbf{F}(\mathbf{g}(t)) = (b\sin t, 0).$$

By the definition (2.14),

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{x} &= \int_0^{\pi/2} (b\sin t, 0)\cdot(-b\sin t, b\cos t)dt \\ &= -b^2\int_0^{\pi/2} \sin^2 t\, dt \qquad\qquad \textit{Aside: } \sin^2 t = \tfrac{1}{2}(1-\cos 2t) \\ &= -\tfrac{1}{4}\pi b^2. \quad \square\end{aligned}$$

*Comment:*

The line integral $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{x}$ is the limit of the sum of scalar products $\mathbf{F}\cdot\Delta\mathbf{x}$. One can thus predict the *sign* of the line integral by referring to Figure 2.5, which shows that $\mathbf{F}\cdot\Delta\mathbf{x} < 0$ except at the point $(b, 0)$, since the angle $\phi$ between $\mathbf{F}$ and $\Delta\mathbf{x}$ satisfies $\frac{\pi}{2} < \phi \le \pi$.

**Exercise 2.5:**

Let $C$ be the straight line joining $(b, 0)$ to $(0, b)$ and let $\mathbf{F}$ be the vector field $\mathbf{F}(x, y) = (y, 0)$, as in Example 2.2. Show that $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{x} = -\tfrac{1}{2}b^2$.

**Exercise 2.6:**

Compute the line integral of the vector field $\mathbf{F}(\mathbf{x}) = (x, y, z)$ along the helix $C$ defined by $\mathbf{x} = \mathbf{g}(t) = (\cos t, \sin t, t)$, $0 \le t \le 2\pi$.

*Answer:* $2\pi^2$.

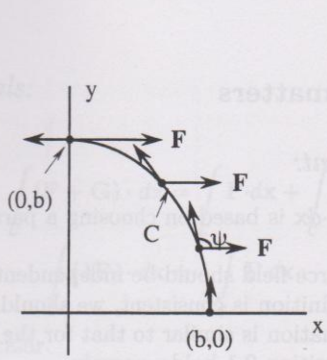


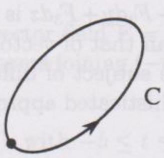

Figure 2.5: The vector field $\mathbf{F}(x, y) = (y, 0)$ in Example 2.2.

### 2.2.2 Applications

So far we have considered the following physical interpretation of the line integral of a vector field along a curve $C$:

> if $\mathbf{F}$ is a force field acting on a particle whose path is the curve $C$, then $\int_C \mathbf{F}\cdot d\mathbf{x}$
>
> represents the work done by the force field on the particle.

In this course a major role is played by line integrals $\int_C \mathbf{F}\cdot d\mathbf{x}$, where $C$ is a *closed curve*, i.e. the end point coincides with the initial point. Line integrals of this type enter into two of the principal theorems, namely Green's theorem (Section 2.4) and Stokes' theorem (Section 4.3). The physical interpretation of the line integral depends on the interpretation of the vector field, two important cases being where $\mathbf{F}$ is a *velocity field* (in fluid dynamics), and where $\mathbf{F}$ is an *electric field* (in electromagnetic theory).



### 2.2.3 Some technical matters

*The consistency requirement:*

The definition (2.14) of $\int_C \mathbf{F}\cdot d\mathbf{x}$ is based on choosing a parametrization for the curve $C$. However, the work done by a force field should be independent of the parametrization, and so in order to show that the definition is consistent, we should verify that it is independent of the parametrization. The situation is similar to that for the line integral of a scalar field, and a result analogous to Proposition 2.1 holds, namely

$$\int_a^b \mathbf{F}(\mathbf{g}(t))\cdot \mathbf{g}'(t)dt = \int_\alpha^\beta \mathbf{F}(\hat{\mathbf{g}}(\tau))\cdot \hat{\mathbf{g}}'(\tau)d\tau, \tag{2.15}$$

where the notation is defined in Proposition 2.1.

*Other notation for the line integral:*

There is another notation for $\int_C \mathbf{F}\cdot d\mathbf{x}$ that is popular in physics and engineering, namely

$$\int_C \mathbf{F}\cdot d\mathbf{x} \equiv \int_C F_1 dx + F_2 dy + F_3 dz,$$

where $\mathbf{F} = (F_1, F_2, F_3)$. In a purely formal sense, one expands the "scalar product" $\mathbf{F}\cdot d\mathbf{x}$ with "$d\mathbf{x} = (dx, dy, dz)$". In terms of this alternate notation, the definition reads

$$\int_C F_1 dx + F_2 dy + F_3 dz = \int_a^b \left[ F_1(\,)\frac{dx}{dt} + F_2(\,)\frac{dy}{dt} + F_3(\,)\frac{dz}{dt}\right] dt, \tag{2.16}$$

where $\mathbf{x} = (x(t), y(t), z(t))$ is a parametrization of $C$ and $F_i(\,) = F_i(x(t), y(t), z(t))$, $i = 1, 2, 3$.

Mathematically, the quantity $F_1dx + F_2dy + F_3dz$ is called a *differential form*. The theory of differential forms is more modern than that of vector fields, and is useful, for instance, in generalizing vector calculus, and in the subject of differential geometry. For our purposes, however, we do not need this more sophisticated approach.

**Exercise 2.7:**

Evaluate the line integral

$$I = \int_C \cos z\, dx + e^x dy + e^y dz,$$

where the curve $C$ is given by $\mathbf{x}(t) = (1, t, e^t)$, $0 \le t \le 2$.

*Answer:* $2e + \frac{1}{2}e^4 - \frac{1}{2}$.

*Properties of line integrals:*

i) *Linearity:*

$$\int_{\mathcal{C}} (\mathbf{F}+\mathbf{G})\cdot d\mathbf{x} = \int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{x} + \int_{\mathcal{C}} \mathbf{G}\cdot d\mathbf{x}, \tag{2.17}$$

$$\int_{\mathcal{C}} (\lambda\mathbf{F})\cdot d\mathbf{x} = \lambda \int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{x}, \tag{2.18}$$

where $\lambda$ is a constant scalar.

These properties follow immediately from the definition (2.14) and the corresponding properties of the Riemann integral.

ii) *Additivity:*

If $\mathcal{C}$ is a $C^1$ curve that is the union of two curves $\mathcal{C}_1$ and $\mathcal{C}_2$ joined end-to-end and consistently oriented ($\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$), then

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{x} = \int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{x} + \int_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{x} \tag{2.19}$$

*Line integral along a piecewise $C^1$ curve:*

Let $\mathcal{C}$ be a continuous curve which is piecewise of class $C^1$ i.e. $\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_n$, where the individual pieces $\mathcal{C}_i$, $i = 1, \ldots, n$ are of class $C^1$. Motivated by equation (2.19), we define the line integral of a vector field $\mathbf{F}$ along $\mathcal{C}$ by

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{x} = \int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{x} + \cdots + \int_{\mathcal{C}_n} \mathbf{F}\cdot d\mathbf{x}. \tag{2.20}$$

Each line integral on the right is, of course, defined as a Riemann integral by equation (2.14).

**Example 2.3:**

Compute the line integral of the vector field $\mathbf{F} = (y, -2x)$ along the piecewise $C^1$ curve consisting of the two straight line segments joining $(-b, b)$ to $(0,0)$ and $(0,0)$ to $(2b, b)$, where $b$ is a positive constant.

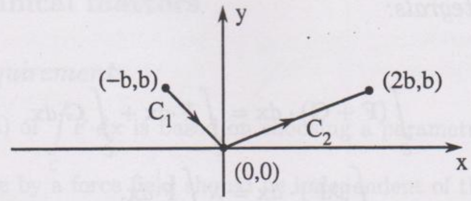
*Solution:* For $\mathcal{C}_1$, $\mathbf{x} = \mathbf{g}_1(t) = (t, -t)$, with $-b \le t \le 0$, giving

$$\mathbf{g}_1'(t) = (1, -1), \quad \text{and} \quad \mathbf{F}(\mathbf{g}_1(t)) = (-t, -2t).$$

By the definition (2.14),

$$\int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{x} = \int_{-b}^{0} (-t, -2t)\cdot(1,-1)\,dt$$

$$= \int_{-b}^{0} t\,dt = -\tfrac{1}{2}b^2.$$




For $C_2$, $\mathbf{x} = \mathbf{g}_2(t) = (2t, t)$, $0 \le t \le b$, and a similar calculation yields

$$\int_{C_2} \mathbf{F}\cdot d\mathbf{x} = -b^2.$$

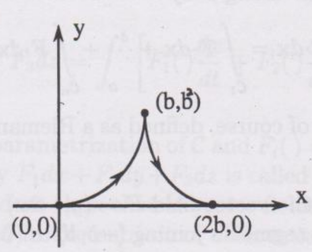
Finally, by (2.20),

$$\int_{C} \mathbf{F}\cdot d\mathbf{x} = \int_{C_1} \mathbf{F}\cdot d\mathbf{x} + \int_{C_2} \mathbf{F}\cdot d\mathbf{x}$$
$$= -\tfrac{1}{2}b^2 - b^2 = -\tfrac{3}{2}b^2.$$

**Exercise 2.8:**

Calculate the line integral of the vector field $\mathbf{F} = (y, -2x)$ along the piecewise smooth curve consisting of two parabolic segments as drawn.

*Answer:* $2b^2$. □




## *Reversal of orientation:*

If $C$ is a curve in $\mathbb{R}^n$ with a specific orientation, we denote by $-C$ the curve that is obtained by reversing the orientation. Specifically if $C$ is given by

$$\mathbf{x} = \mathbf{g}(t), \qquad a \le t \le b,$$

then $-C$ is given by

$$\mathbf{x} = \hat{\mathbf{g}}(\tau), \qquad a \le \tau \le b,$$

where

$$\hat{\mathbf{g}}(\tau) = \mathbf{g}(a + b - \tau).$$

Observe that $\hat{\mathbf{g}}(a) = \mathbf{g}(b)$ and $\hat{\mathbf{g}}(b) = \mathbf{g}(a)$, which reverses the orientation.

It is an important result that *reversing the orientation of a curve changes the sign of the line integral of a vector field along the curve.*

**Proposition 2.2:**

If $\mathcal{C}$ is a piecewise-smooth curve and $\mathbf{F}$ is a vector field continuous on $\mathcal{C}$, then

$$\int_{-\mathcal{C}} \mathbf{F}\cdot d\mathbf{x} = -\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{x}.$$

*Proof:*

By definition of line integral

$$\begin{aligned}
\int_{-\mathcal{C}} \mathbf{F}\cdot d\mathbf{x} &= \int_{\tau=a}^{b} \mathbf{F}(\hat{g}(\tau))\cdot \hat{\mathbf{g}}'(\tau)d\tau \\
&= -\int_{\tau=a}^{b} \mathbf{F}(\mathbf{g}(a+b-\tau))\cdot \mathbf{g}'(a+b-\tau)d\tau \quad \text{(by the equation for } \hat{\mathbf{g}}(\tau)) \\
&= -\int_{t=b}^{a} \mathbf{F}(\mathbf{g}(t))\cdot \mathbf{g}'(t)(-1)dt \quad \text{(by the change of variable } t = a+b-\tau) \\
&= -\int_{t=a}^{b} \mathbf{F}(\mathbf{g}(t))\cdot \mathbf{g}'(t)dt \quad \text{(reverse the limits of integration)} \\
&= -\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{x}. \quad \text{(by definition of line integral)} \quad \square
\end{aligned}$$

*Comment:*

The change in sign is physically reasonable if one thinks in terms of work done by a force field $\mathbf{F}$; for example, if the height of an object above the earth's surface is increased, the work done by the gravitational field is *negative*, whereas if an object falls, the work done is *positive*.

## 2.3 Path-independent line integrals

Let $\mathbf{F} : \mathcal{U} \to \mathbb{R}^n$ be a continuous vector field on the connected open set $\mathcal{U} \in \mathbb{R}^n$. Consider two points $\mathbf{x}_1, \mathbf{x}_2$ in $\mathcal{U}$, and imagine all possible piecewise smooth curves in $\mathcal{U}$ joining $\mathbf{x}_1$ to $\mathbf{x}_2$. In general, the value of the line integral $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{x}$ will depend on the particular curve $\mathcal{C}$ joining $\mathbf{x}_1$ to $\mathbf{x}_2$. In physical terms, thinking of $\mathbf{F}$ as a force field, the work done on the particle as it moves from $\mathbf{x}_1$ to $\mathbf{x}_2$ in general depends on the path followed by the particle. There are, however, certain special vector fields (force fields) with the property that the line integral (the work done) depends only on the endpoints of the curve and not on the particular curve joining the two points.

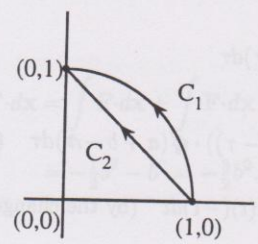
As an example, consider the vector field (see Problem Set 1, #14):

$$\mathbf{E}(x,y) = -kq\left(\frac{x}{(x^2+y^2)^{3/2}}, \frac{y}{(x^2+y^2)^{3/2}}\right). \tag{2.21}$$

As endpoints, consider $\mathbf{x}_1 = (1,0)$ and $\mathbf{x}_2 = (0,1)$. You will find that for any piecewise smooth curve $\mathcal{C}$ joining $\mathbf{x}_1$ to $\mathbf{x}_2$, $\int_{\mathcal{C}} \mathbf{E}\cdot d\mathbf{x}$ has the same value, namely zero.

**Exercise 2.9:**

Show that the line integral of the vector field (2.21) equals zero for each of the curves $\mathcal{C}_1$ and $\mathcal{C}_2$ joining $(1,0)$ to $(0,1)$, where $\mathcal{C}_1$ is the quarter circle.




**Definition:**

Let $\mathbf{F}$ be a continuous vector field on a connected open set in $\mathbb{R}^n$. The line integral $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{x}$ is *path-independent in $\mathcal{U}$* mean that given any two points $\mathbf{x}_1, \mathbf{x}_2$ in $\mathcal{U}$, the line integral has the same value for all piecewise smooth curves in $\mathcal{U}$ that join $\mathbf{x}_1$ to $\mathbf{x}_2$. $\square$

Exercise 2.9 gives a hint that the line integral of the vector field (2.21) is path-independent in $\mathcal{U} = \mathbb{R}^2 - \{(0,0)\}$. Of course we cannot prove that a line integral is path-independent by calculating its value all different curves joining different pairs of points because there are infinitely many possibilities! So an important question is: *how can we tell whether a given line integral is path-independent or not*? To find out, let us be guided by one of the most important results in elementary calculus, the first Fundamental Theorem.

### 2.3.1 First Fundamental Theorem for Line Integrals

Recall that if $f$ is continuous on an interval $[a,b]$, then the new function $g$ defined by

$$g(x) = \int_a^x f(t)\,dt, \qquad a \le x \le b, \tag{2.22}$$

is such that

$$g'(x) = f(x). \tag{2.23}$$

This result is the first Fundamental Theorem of Calculus (FTC).

**Q:** Can we extent this theorem to line integrals?

**A:** Yes, provided the line-integral is *path-independent*.

If the line integral $\int_C \mathbf{F}\cdot d\mathbf{x}$ is path-independent and $C$ joins $\mathbf{x}_0$ to $\mathbf{x}$, we denote the line-integral by

$$\int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{F}\cdot d\mathbf{x}.$$

In this case we can define a new function – a scalar field $\phi$ – by

$$\phi(x) = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{F}\cdot d\mathbf{x},$$

in analogy with (2.22). We can now state the *first Fundamental Theorem for line integrals.*

**Theorem 2.1:**

Let $\mathcal{U}$ be a connected open subset of $\mathbb{R}^n$, and let $\mathbf{F} : U \to \mathbb{R}^n$ be a continuous vector field whose line integral is path independent in $\mathcal{U}$.

If

$$\phi(\mathbf{x}) = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{F}\cdot d\mathbf{x}, \tag{2.24}$$

where $\mathbf{x}_0$ is a specified point, then,

$$\nabla\phi(\mathbf{x}) = \mathbf{F}(\mathbf{x}) \tag{2.25}$$

for all $\mathbf{x} \in \mathcal{U}$.

*Proof:*

For simplicity we give the proof in $\mathbb{R}^2$. With $\phi$ defined by (2.24), we have to prove that

$$\frac{\partial\phi}{\partial x} = F_1, \quad \frac{\partial\phi}{\partial y} = F_2,$$

where $F_1$ and $F_2$ are the components of $\mathbf{F}$.

The key idea is this: since the line integral is path-independent, we are free to make a special choice of the curve joining $\mathbf{x}_0 = (x_0, y_0)$ to $\mathbf{x} = (x, y)$, i.e. to choose a "custom-designed" curve. Figure 2.6 shows the curve we need. Suitable parametrizations for $C_1$ and $C_2$ are

$$\mathbf{x} = \mathbf{g}_1(t) = (x_0, t), \quad y_0 \le t \le y,$$
$$\mathbf{x} = \mathbf{g}_2(t) = (t, y), \quad x_0 \le t \le x.$$

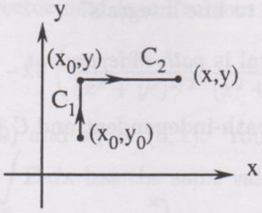



Figure 2.6: A piecewise smooth curve $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ joining $(x_0, y_0)$ to $(x, y)$.

Using these equations and the definition of line integral,

$$\phi(x,y) = \int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{x} = \int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{x} + \int_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{x}$$

$$= \int_{y_0}^{y} F_2(x_0,t)\,dt + \int_{x_0}^{x} F_1(t,y)\,dt.$$

It now follows from the first FTC for Riemann integrals (see equations (2.22) and (2.23)) that

$$\frac{\partial \phi}{\partial x} = 0 + F_1(x,y),$$

since we are treating $y$ as a constant.

Similarly we get the result for $\frac{\partial \phi}{\partial y}$ by choosing a different path (do it!). $\square$

*Terminology:*

The significance of Theorem 2.1 is this: *any vector field* $\mathbf{F}$ *whose line integral is path-independent can be written as the gradient of a* $C^1$ *scalar field.* Such a vector field is called a *gradient field.* The scalar field $\psi$ is called a *potential* for $\mathbf{F}$, for physical reasons that we'll soon see. The level sets $\psi(\mathbf{x}) = C$ of the potential $\psi$ are called *equipotentials.* In $\mathbb{R}^2$, we have *equipotential lines* $\psi(x,y) = C$ and in $\mathbb{R}^3$ we have *equipotential surfaces* $\psi(x,y,z) = C$.

As an example, we note that the vector field $\mathbf{F}(x,y)$ given by (2.21) (the electric field due to a point of charge $q$ at the origin) is derivable from the potential

$$\psi(x,y) = \frac{kq}{\sqrt{x^2+y^2}} \tag{2.26}$$

i.e. $\mathbf{E}(x,y) = \nabla\psi(x,y)$ (verify this!). The equipotential lines are given by $\psi(x,y) = \text{constant}$, i.e.

$$x^2 + y^2 = \text{const.}$$

### 2.3.2 Second Fundamental Theorem for Line Integrals

Continuing the train of thought from the previous subsection we ask

**Q:** In elementary calculus we learned that if $G, g : [a, b] \to \mathbb{R}$ are such that $g$ is continuous and $G' = g$, then

$$\int_a^b g(x)dx = G(b) - G(a), \tag{2.27}$$

(the second FTC). Is there a way to extend this result to line integrals?

**A:** Yes, provided the vector field $\mathbf{F}$ is a gradient field, i.e. $\mathbf{F} = \nabla\phi$.

This generalization is *the Second Fundamental Theorem for line integrals.*

**Theorem 2.2:**

Let $\mathbf{F} : \mathcal{U} \to \mathbb{R}^n$ be a continuous vector field on a connected open set $\mathcal{U} \subset \mathbb{R}^n$, and let $\mathbf{x}_1, \mathbf{x}_2$ be two points in $\mathcal{U}$.

If $\mathbf{F} = \nabla\phi$, where $\phi : \mathcal{U} \to \mathbb{R}$ is a $C^1$ scalar field, and $\mathcal{C}$ is any curve in $\mathcal{U}$ joining $\mathbf{x}_1$ to $\mathbf{x}_2$, then

$$\int_{\mathcal{C}} \mathbf{F}{\cdot}d\mathbf{x} = \phi(\mathbf{x}_2) - \phi(\mathbf{x}_1). \tag{2.28}$$

*Proof:*

Let $\mathcal{C}$ be given by

$$\mathbf{x} = \mathbf{g}(t), \quad t_1 \le t \le t_2,$$

so that

$$\mathbf{x}_1 = \mathbf{g}(t_1), \quad \mathbf{x}_2 = \mathbf{g}(t_2). \tag{2.29}$$

By the hypothesis,

$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F}{\cdot}d\mathbf{x} &= \int_{\mathcal{C}} (\nabla\phi) \cdot d\mathbf{x} \\
&= \int_{t_1}^{t_2} \nabla\phi(\mathbf{g}(t)) \cdot \mathbf{g}'(t)dt \quad \text{(by definition of line integral)} \\
&= \int_{t_1}^{t_2} \frac{d}{dt}[\phi(\mathbf{g}(t))]dt \quad \text{(by the Chain Rule)} \\
&= \phi(\mathbf{g}(t_2)) - \phi(\mathbf{g}(t_1)) \quad \text{(by the second FTC)} \\
&= \phi(\mathbf{x}_2) - \phi(\mathbf{x}_1). \quad \text{(by (2.29))} \quad \square
\end{aligned}$$

*Comment:*

The significance of Theorem 2.2 is two-fold:

i) if $\mathbf{F}$ is a *gradient field*, the line integral $\int_C \mathbf{F}\cdot d\mathbf{x}$ only depends on the end points of the curve $C$, and hence is *path-independent*,

ii) if the potential $\phi$ is known, then equation (2.28) gives the value of the line integral immediately.

**Exercise 2.10:**

We have seen that the vector field

$$\mathbf{E}(x,y) = -kq\left(\frac{x}{(x^2+y^2)^{3/2}}, \frac{y}{(x^2+y^2)^{3/2}}\right)$$

on $\mathcal{U} = \mathbb{R}^2 - \{(0,0)\}$ is a gradient field with potential

$$\phi(x,y) = \frac{kq}{\sqrt{x^2+y^2}}.$$

In exercise 9 you showed that the line integral of $\mathbf{E}$ along each of 2 curves joining $(1,0)$ to $(0,1)$ equalled zero. Use Theorem 2.2 to verify this result. □

*Looking ahead:*

So far (Theorems 2.1 and 2.2) we have established that $\int_C \mathbf{F}\cdot d\mathbf{x}$ is path-independent if and only if $\mathbf{F}$ is a gradient field. Moreover, if $\mathbf{F}$ is a gradient field and we can find a potential $\phi$, then the line integral $\int_C \mathbf{F}\cdot d\mathbf{x}$ can be quickly evaluated. Now we are faced with two problems:

i) if we are given a vector field $\mathbf{F}$, how can we tell quickly whether it is a gradient field?

ii) if we know $\mathbf{F}$ is a gradient field, how do we find a potential $\phi$?

Answering the first question requires the famous *Green's theorem*, while the second is more straightforward. But before dealing with these questions we first discuss the physical significance of the Second Fundamental Theorem for line integrals.

### 2.3.3 Conservative (i.e. gradient) vector fields

Thinking of the vector field $\mathbf{F}$ in Theorem 2.2 as a force field, the line integral $\int_C \mathbf{F}\cdot d\mathbf{x}$ equals the work done by the force field on a particle as the particle moves along the curve $C$ from $\mathbf{x}_1$ to $\mathbf{x}_2$. The theorem asserts that if the force field is a gradient field, $\mathbf{F} = \nabla\phi$, then the work done depends only on the potential at the end points $\mathbf{x}_1$ and $\mathbf{x}_2$.

In this physical context, it is customary, to define a scalar field $V = -\phi$, so that

$$\mathbf{F} = -\nabla V. \tag{2.30}$$

Theorem 2.2 then has the form

$$\int_C \mathbf{F}\cdot d\mathbf{x} = -V(\mathbf{x}_2) + V(\mathbf{x}_1). \tag{2.31}$$

Since "work" is the same as "energy", physicists call $V(\mathbf{x})$ *the potential energy of the particle at position* $\mathbf{x}$, when moving under the action of $\mathbf{F}$.

*Comment:*

The minus sign in equation (2.30) becomes appropriate when one thinks, for example of the force field due to the earth's gravitational field. The potential energy of a particle increases if its distance from the earth's centre increases i.e. $\nabla V$ *points radially outwards*, while *the gravitational force field* $\mathbf{F}$ *acts radially inwards.*

One can also relate the work done to the *kinetic energy* $K$ of the particle, defined by

$$K = \tfrac{1}{2}m \parallel \mathbf{v} \parallel^2. \tag{2.32}$$

Describing the path $C$ of the particle by $\mathbf{x} = \mathbf{r}(t)$, $t_1 \leq t \leq t_2$, the work done can be written

$$\int_C \mathbf{F}\cdot d\mathbf{x} = \int_{t_1}^{t_2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt. \tag{2.33}$$

But Newton's second law tells us that

$$m\mathbf{r}''(t) = \mathbf{F}(\mathbf{r}(t)).$$

It follows that

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= m\mathbf{v}'(t) \cdot \mathbf{v}(t) \quad (\text{since } \mathbf{r}'(t) = \mathbf{v}(t)) \\ &= \tfrac{1}{2}m[\mathbf{v}(t) \cdot \mathbf{v}(t)]' \quad (\text{property of the derivative}) \\ &= \frac{d}{dt}K(t) \quad (\text{by (2.32)}). \end{aligned}$$

Thus, by (2.33) and the FTCII,

$$\int_C \mathbf{F}\cdot d\mathbf{x} = K(t_2) - K(t_1). \tag{2.34}$$

Equation (2.34) and (2.31) gives

$$K(t_1) + V(\mathbf{r}(t_1)) = K(t_2) + V(\mathbf{r}(t_2)), \tag{2.35}$$

for any two times $t_1$ and $t_2$. In words, *for a gradient force field one can define a potential energy* $V$ *of a particle in such a way that the sum of the potential energy and kinetic energy* $K$ *is constant*, i.e. *conservation of energy holds.*

It is for this reason that when vector fields are thought of a force fields, gradient fields are also called *conservative fields.*

*Comment:*

In many applications in the real world, conservation of energy does not have the simple form of (2.35), because, for example, of energy losses due to friction – think of the space shuttle re-entering the atmosphere. Dissipative, i.e. non-conservative forces have also to be considered. It is nevertheless important to be able to find out whether a given force field is conservative, and this is the problem we now consider.

## 2.4 Green's Theorem

In this section we introduce Green's theorem, and discuss a number of applications, including how to spot conservative/gradient vector fields in $\mathbb{R}^2$.

### 2.4.1 The theorem

We need some additional terminology related to curves.

Consider a curve $C$ in $\mathbb{R}^n$ given by

$$\mathbf{x} = \mathbf{g}(t), \quad a \leq t \leq b,$$

with $\mathbf{g}$ continuous.

i) $C$ is a *closed curve* means that $\mathbf{g}(a) = \mathbf{g}(b)$.

ii) $C$ is a *simple closed curve* means that $\mathbf{g}(a) = \mathbf{g}(b)$ and $\mathbf{g}$ is a one-to-one function on the interval $a \leq t < b$. (Note the strict inequality $t < b$.) In geometric terms a simple closed curve has no self-intersections.

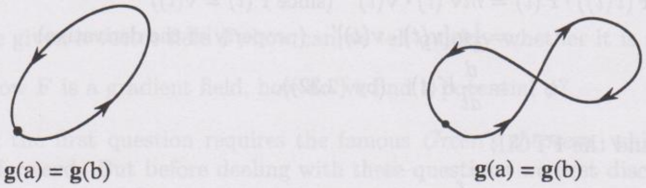



Figure 2.7: A simple closed curve. A non-simple closed curve.

iii) In what follows we shall consider a bounded open subset $\mathcal{D}$ of $\mathbb{R}^2$, whose *boundary*, denoted by $\partial\mathcal{D}$, is a simple closed curve. In this situation we assume that the curve $\partial\mathcal{D}$ is oriented *counter-clockwise*, so that if you walk around the boundary, the region $\mathcal{D}$ is on your left.

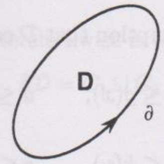



Figure 2.8: A bounded open subset $\mathcal{D}$ and its boundary $\partial\mathcal{D}$ oriented counter-clockwise.

**Theorem 2.3 (Green's theorem):**

Let $\mathcal{D}$ be a bounded subset of $\mathbb{R}^2$ whose boundary $\partial\mathcal{D}$ is a piecewise $C^1$ simple closed curve oriented counter-clockwise. If $\mathbf{F} = (F_1, F_2)$ is of class $C^1$ on $\mathcal{D} \cup \partial\mathcal{D}$ then

$$\int_{\partial\mathcal{D}} \mathbf{F}\cdot d\mathbf{x} = \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy. \tag{2.36}$$

*Digression on iterated integrals:*

If $\mathcal{D}$ is described by inequalities of the form

$$f(x) \le y \le g(x),$$
$$a \le x \le b,$$

and $H(x,y)$ is continuous on $\mathcal{D} \cup \partial\mathcal{D}$, then the double integral $\iint_{\mathcal{D}} H(x,y)dx\,dy$ can be expressed as an iterated integral:

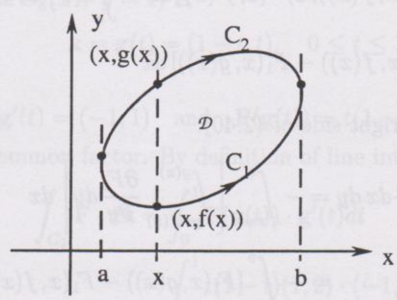



Figure 2.9: A region $\mathcal{D}$ and its boundary $\partial\mathcal{D} = \mathcal{C}_1 \cup (-\mathcal{C}_2)$.

$$\iint_{\mathcal{D}} H(x,y)dx\,dy = \int_{x=a}^{b} \left[\int_{y=f(x)}^{g(x)} H(x,y)dy\right] dx. \tag{2.37}$$

*Proof of Green's theorem:*

We give a proof subject to the assumption that $\mathcal{D}$ can be described by inequalities of the form

$$f(x) \leq y \leq g(x), \qquad a \leq x \leq b, \tag{2.38}$$

and

$$h(y) \leq x \leq k(y), \qquad c \leq y \leq d, \tag{2.39}$$

where $f, g, h$ and $k$ are $C^1$ functions.

It is sufficient to prove two special cases of (2.36), namely

$$\int_{\partial\mathcal{D}} (F_1, 0) \cdot d\mathbf{x} = \iint_{\mathcal{D}} -\frac{\partial F_1}{\partial y} dx\, dy, \tag{2.40}$$

and

$$\int_{\partial\mathcal{D}} (0, F_2) \cdot d\mathbf{x} = \iint_{\mathcal{D}} \frac{\partial F_2}{\partial x} dx\, dy. \tag{2.41}$$

The sum of (2.40) and (2.41) gives (2.36).

We prove (2.40), using the inequalities (2.38). Consider the curves $\mathcal{C}_1, \mathcal{C}_2$ (see Figure 2.9) given by

$$\mathbf{x} = (x, f(x)) \quad \text{and} \quad \mathbf{x} = (x, g(x))$$

respectively, with $a \leq x \leq b$, i.e. we use $x$ as parameter. The boundary $\partial\mathcal{D}$ is then the union $\partial\mathcal{D} = \mathcal{C}_1 \cup (-\mathcal{C}_2)$. By definition of the line integral,

$$\begin{aligned}
\int_{\partial\mathcal{D}} (F_1, 0) \cdot d\mathbf{x} &= \int_{\mathcal{C}_1} (F_1, 0) \cdot d\mathbf{x} - \int_{\mathcal{C}_2} (F_1, 0) \cdot d\mathbf{x} \quad \left( \text{since } \int_{-\mathcal{C}_2} = -\int_{\mathcal{C}_2} \right) \\
&= \int_a^b \{F_1(x, f(x)), 0) \cdot (1, f'(x)\} dx - \int_a^b \{F_1(x, g(x)), 0) \cdot (1, g'(x)\} dx \\
&= \int_a^b [F_1(x, f(x)) - F_1(x, g(x))]\, dx
\end{aligned} \tag{2.42}$$

We now apply (2.37) to the right side of (2.40):

$$\begin{aligned}
\iint_{\mathcal{D}} -\frac{\partial F_1}{\partial y} dx\, dy &= -\int_{x=a}^{b} \left[ \int_{y=f(x)}^{g(x)} \frac{\partial F_1}{\partial y} dy \right] dx \\
&= -\int_a^b [F_1(x, g(x)) - F_1(x, f(x))]\, dx,
\end{aligned} \tag{2.43}$$

by the FTCII. Equation (2.40) follows, on comparing (2.42) and (2.43). A similar argument based on (2.39) yields (2.41) (do it!), which completes the proof. □

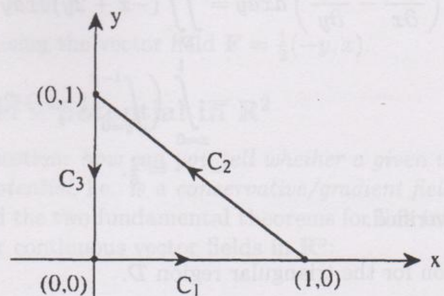
**Example 2.4:**

Verify Green's theorem for the vector field $\mathbf{F}(\mathbf{x}) = (xy, 2xy)$ and the triangluar region $\mathcal{D}$ with vertices $(0,0), (1,0)$ and $(0,1)$.

*Solution:*

a) The boundary of $\mathcal{D}$ taken counterclockwise is the piecewise $C^1$ curve

$$\partial\mathcal{D} = C_1 \cup C_2 \cup C_3.$$




$$\int_{\partial\mathcal{D}} \mathbf{F}\cdot d\mathbf{x} = \int_{C_1} \mathbf{F}\cdot d\mathbf{x} + \int_{C_2} \mathbf{F}\cdot \mathbf{x} + \int_{C_3} \mathbf{F}\cdot d\mathbf{x}. \tag{2.44}$$

Observe that $\mathbf{F} = \mathbf{0}$ on $C_1 (y = 0)$ and on $C_3 (x = 0)$, which implies that

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{x} = 0 = \int_{C_3} \mathbf{F}\cdot d\mathbf{x}. \tag{2.45}$$

A vector function for $C_2 (x + y = 1)$ is

$$\mathbf{x} = \mathbf{g}(t) = (1-t, t), \quad 0 \le t \le 1.$$

It follows that

$$\mathbf{g}'(t) = (-1, 1) \quad \text{and} \quad \mathbf{F}(\mathbf{g}(t)) = t(1-t)(1,2),$$

after taking out a common factor. By definition of line integral,

$$\begin{aligned}\int_{C_2} \mathbf{F}\cdot d\mathbf{x} &= \int_0^1 \mathbf{F}(\mathbf{g}(t))\cdot \mathbf{g}'(t)dt \\ &= \int_0^1 t(1-t)(1,2)\cdot(-1,1)dt \\ &= \int_0^1 t(1-t)dt = \cdots = \tfrac{1}{6}\end{aligned} \tag{2.46}$$

Substituting (2.45) and (2.46) in (2.44) gives

$$\int_{\partial\mathcal{D}} \mathbf{F}\cdot d\mathbf{x} = \tfrac{1}{6}.$$

b) We calculate

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = -x + 2y.$$

The right side of Green's theorem is

$$\iint_{\mathcal{D}} \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \iint_{\mathcal{D}} (-x + 2y) dxdy$$

$$= \int_{x=0}^{1} \left( \int_{y=0}^{1-x} (-x + 2y) dy \right) dx$$

$$= \cdots = \tfrac{1}{6}.$$

Green's theorem is verified. □

The limits of integration for the triangular region $\mathcal{D}$.

*Aside:* $\mathcal{D}$ is defined by the inequalities

$$0 \le y \le 1 - x,$$

$$0 \le x \le 1,$$

which give the limits of integration.

Green's theorem can be used to express a given line integral as a double integral, or conversely, to express a given double integral as a line integral, provided you can choose a suitable vector field. Here is an example of the latter.

**Example 2.5:**

Use a line integral to calculate the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

*Solution:* The area $A$ of a plane region $\mathcal{D}$ can be expressed as a double integral:

$$A = \iint_{\mathcal{D}} (1) dx\, dy.$$

The vector field $\mathbf{F} = (0, x)$ satisfies $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$, so that Green's theorem (2.36) gives

$$A = \int_{\partial \mathcal{D}} (0, x) \cdot d\mathbf{x},$$

where $\partial \mathcal{D}$ is the ellipse (a simple closed curve) oriented counter-clockwise. Using the standard parametrization,

$$\mathbf{x} = (a \cos t, b \sin t), \quad 0 \le t \le 2\pi,$$

the definition of line integral gives

$$
\begin{aligned}
A &= \int_0^{2\pi} (0, a\cos t)\cdot(-a\sin t, b\cos t)dt \qquad\qquad \textit{Aside: } \cos 2t = 2\cos^2 t - 1\\
&= \cdots = \pi\, ab.
\end{aligned}
$$

**Exercise 2.11:**

Repeat example 2.5 using the vector field $\mathbf{F} = \frac{1}{2}(-y, x)$.

### 2.4.2 Existence of a potential in $\mathbb{R}^2$

We now return to the question: *how can you tell whether a given vector field* $\mathbf{F} : \mathcal{U} \to \mathbb{R}^2$ *in* $\mathbb{R}^2$ *is derivable from a potential,* i.e. *is a conservative/gradient field?*

So far we have proved the two fundamental theorems for line integrals (theorems 2.1 and 2.2), which show that for continuous vector fields in $\mathbb{R}^n$:

$$
\int_C \mathbf{F}\cdot d\mathbf{x} \text{ is path-independent in } \mathcal{U}\subset\mathbb{R}^n \iff \mathbf{F} \text{ is a gradient field in } \mathcal{U}\subset\mathbb{R}^n \tag{2.47}
$$

We begin the final stage of the discussion by establishing that

$$\text{“}\int_C \mathbf{F}\cdot d\mathbf{x} \quad \text{is path-independent in} \quad \mathcal{U}\text{”}$$

is equivalent to

$$\text{“}\int_C \mathbf{F}\cdot d\mathbf{x} = 0 \quad \text{for all simple closed curves in} \quad \mathcal{U}\text{”}.$$

The reason for doing this is that Green's theorem deals with line integrals around simple closed curves.

**Proposition 2.3:**

Let $\mathcal{U}$ be an open subset of $\mathbb{R}^n$. A continuous vector field $\mathbf{F} : \mathcal{U} \to \mathbb{R}^n$ is path independent in $\mathcal{U}$ if and only if $\int_C \mathbf{F}\cdot d\mathbf{x} = 0$ for every simple closed curve in $\mathcal{U}$.

*Proof:*

1) Suppose $\int_C \mathbf{F}\cdot d\mathbf{x}$ is path-independent in $\mathcal{U}$. Let $C$ be a simple closed curve in $\mathcal{U}$.

Decompose $C$ into $C_1$ and $C_2$ as in figure 2.10. Then

$$
\begin{aligned}
\int_C = \int_{C_1} + \int_{C_2} &= \int_{C_1} - \int_{-C_2}\\
&= 0,
\end{aligned}
$$

since the line integral is path independent.

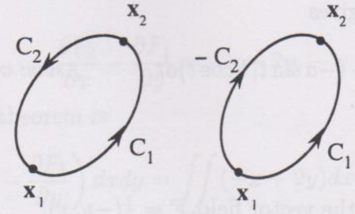



Figure 2.10: Decomposing a simple closed curve $\mathcal{C}$ into $\mathcal{C}_1$ and $\mathcal{C}_2$

2) Suppose $\displaystyle\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{x} = 0$ for every simple closed curve in $\mathcal{U}$. Let $\mathbf{x}_1, \mathbf{x}_2$ be any two points in $\mathcal{U}$, and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two curves joining $\mathbf{x}_1$ to $\mathbf{x}_2$ which do not intersect each other. Then $\mathcal{C}_1 \cup (-\mathcal{C}_2)$ is a simple closed curve $\mathcal{C}$. It follows that

$$\int_{\mathcal{C}_1} - \int_{\mathcal{C}_2} = \int_{\mathcal{C}_1} + \int_{-\mathcal{C}_2} = \int_{\mathcal{C}} = 0.$$

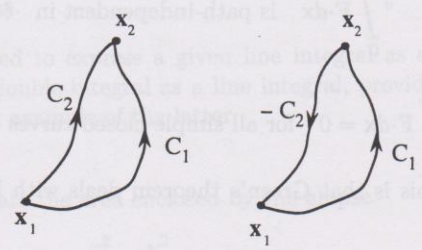



Figure 2.11: Two curves $\mathcal{C}_1, \mathcal{C}_2$ joining $\mathbf{x}_1$ to $\mathbf{x}_2$ form a simple closed curve $\mathcal{C} = \mathcal{C}_1 \cup (-\mathcal{C}_2)$.

If $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect each other, introduce a third curve $\mathcal{C}_3$ that does not intersect either $\mathcal{C}_1$ or $\mathcal{C}_2$. It follows as before that

$$\int_{\mathcal{C}_1} = \int_{\mathcal{C}_3} \quad \text{and} \quad \int_{\mathcal{C}_2} = \int_{\mathcal{C}_3}.$$

We have thus shown that $\displaystyle\int_{\mathcal{C}_1} = \int_{\mathcal{C}_2}$ for any two curves joining $\mathbf{x}_1$ to $\mathbf{x}_2$, i.e. the line integral is path-independent. $\square$

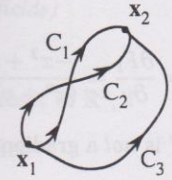



Figure 2.12: A third curve $C_3$ avoids intersections.

Combining Proposition 2.3 with the result (2.47) gives the following:

$$\int_C \mathbf{F}\cdot d\mathbf{x} = 0 \quad \text{for every simple closed curve in} \quad \mathcal{U} \subset \mathbb{R}^n \iff \mathbf{F} \quad \text{is a gradient field in} \quad \mathcal{U} \subset \mathbb{R}^n \tag{2.48}$$

We now *restrict our considerations to* $\mathbb{R}^2$. Suppose that the $C^1$ vector field $\mathbf{F} : \mathcal{U} \to \mathbb{R}^2$ is a gradient field, i.e. $\mathbf{F} = \nabla\phi$, or in component form,

$$(F_1, F_2) = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right).$$

It follows that

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x} = 0,$$

since $\phi$ is of class $C^2$. This result, with (2.48), means that if $\mathbf{F}$ is a gradient field in $\mathbb{R}^2$, the formula in Green's theorem,namely

$$\int_{\partial\mathcal{D}} \mathbf{F}\cdot d\mathbf{x} = \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy$$

is identically satisfied. This formula also suggests that if $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$ in $\mathcal{U}$, then $\int_{\partial\mathcal{D}} \mathbf{F}\cdot d\mathbf{x} = 0$ for any simple closed curve $\partial\mathcal{D}$ in $\mathcal{U}$, so that by (2.48), $\mathbf{F}$ is a gradient field. The following example, however, shows that the situation is not as simple as this.

**Example 2.6:**

Let $\mathcal{U} = \mathbb{R}^2 - \{(0,0)\}$. Show that the vector field $\mathbf{F} : \mathcal{U} \to \mathbb{R}^2$ defined by

$$\mathbf{F}(\mathbf{x}) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right) \tag{2.49}$$

satisfies

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0 \quad \text{in} \quad \mathcal{U},$$

but that $\mathbf{F}$ is not a gradient field in $\mathcal{U}$.

*Solution:* A simple calculation gives

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y} = \frac{-x^2+y^2}{(x^2+y^2)^2}.$$

According to (2.48) we can show that $\mathbf{F}$ is *not* a gradient field *by giving a simple closed curve* $\mathcal{C}$ *in* $\mathcal{U}$ *such that* $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{x} \neq 0$.

Consider the unit circle $\mathcal{C}$ given by

$$\mathbf{x} = (\cos t, \sin t), \qquad 0 \leq t \leq 2\pi.$$

A routine calculation gives $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{x} = 2\pi \neq 0$. (do it!) $\square$

*Comment:*

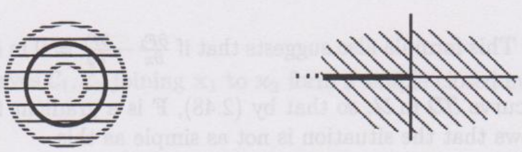
The essential point is that Green's theorem cannot be applied to the vector field (2.49) on the set $\mathcal{D} = \{(x,y) \mid x^2+y^2 \leq 1\}$ whose boundary is the circle $\mathcal{C}$, $x^2+y^2=1$, *since* $\mathbf{F}$ *is not* $C^1$ *on* $\mathcal{D}$ – it is not even defined at $(0,0)$. Another way of looking at the difficulty is that the set $\mathcal{U} = \mathbb{R}^2 - \{(0,0)\}$ on which $\mathbf{F}$ is $C^1$ has a "hole" in it – the point $(0,0)$ has been deleted. So we need to introduce a "no holes" restriction on the set $\mathcal{U}$ on which $\mathbf{F}$ is $C^1$.

**Definition:**

A connected open set $\mathcal{U} \subset \mathbb{R}^n$ is *simply-connected* means that every simple close curve in $\mathcal{U}$ can be shrunk continuously to a point while remaining in $\mathcal{U}$.

e.g. i) $\mathcal{D} = \{(x,y) \mid 1 < x^2+y^2 < 4\}$ is not simply-connected in $\mathbb{R}^2$. The circle drawn cannot be shrunk continuously to a point.

ii) $\mathcal{D} = \mathbb{R}^2 - \{(x,0) \mid x \leq 0\}$ is simply-connected in $\mathbb{R}^2$.



iii) $\mathbb{R}^3$ minus a finite number of points is simply-connected.

iv) $\mathbb{R}^3$ minus an infinite line is not simply-connected.

We are now ready to state the theorem on detecting gradient vector fields in $\mathbb{R}^2$. Having done the preparatory work, the proof is short!

**Theorem 2.4** *(test for conservative fields)*
If

i) $\mathcal{U}$ is a simply-connected open subset of $\mathbb{R}^2$,

ii) $\mathbf{F}$ is a $C^1$ vector field that satisfies

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0 \tag{2.50}$$

in $\mathcal{U}$, then there exists a single-valued $C^2$ potential $\phi$ in $\mathcal{U}$, i.e.

$$\mathbf{F} = \nabla\phi \quad \text{in} \quad \mathcal{U}.$$

*Proof:*
Let $\mathcal{C}$ be any simple closed curve in $\mathcal{U}$. Since $\mathcal{U}$ is simply-connected the interior $\mathcal{D}$ of $\mathcal{C}$ belongs to $\mathcal{U}$ and thus (2.50) holds in $\mathcal{D} \cup \mathcal{C}$. By Green's theorem,

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{x} = 0.$$

Since $\mathcal{C}$ is arbitrary it follows from (2.48) that $\mathbf{F}$ is a gradient field in $\mathcal{U}$. □

**Example 2.7:**
Test the vector field

$$\mathbf{F}(\mathbf{x}) = (ye^{xy}, xe^{xy} + 2y) \tag{2.51}$$

for being conservative, and if it is, find a potential $\phi$.

*Solution:* A routine calculation shows that $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$ on $\mathbb{R}^2$. Since $\mathbb{R}^2$ is simply-connected, $\mathbf{F}$ is conservative by Theorem 2.4.
To find a potential, we have to integrate the equations

$$\frac{\partial \phi}{\partial x} = F_1 = ye^{xy}, \quad \frac{\partial \phi}{\partial y} = F_2 = xe^{xy} + 2y. \tag{2.52}$$

The first gives

$$\phi(x, y) = e^{xy} + K(y), \tag{2.53}$$

where the "constant of integration" depends on $y$. Differentiate (2.53) with respect to $y$ and use the second equation in (2.52):

$$xe^{xy} + 2y = xe^{xy} + K'(y),$$

giving $K'(y) = 2y$, and hence

$$K(y) = y^2 + C,$$

where $C$ is constant. By (2.53) the potential is

$$\phi(x, y) = e^{xy} + y^2 + C,$$

unique up to an additive constant $C$. $\square$

**Exercise 2.12:**

Test whether the vector field $\mathbf{F} : \mathcal{U} \to \mathbb{R}^2$ is conservative on the set $\mathcal{U} \subset \mathbb{R}^2$, and if so, find a potential $\phi$:

i) $\mathbf{F} = \left(\frac{1}{y}, -\frac{x}{y^2}\right), \quad \mathcal{U} = \{(x, y) \mid y > 0\}$,

ii) $\mathbf{F} = (y\cos(xy), -x\cos(xy)), \quad \mathcal{U} = \mathbb{R}^2$,

iii) $\mathbf{F} = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right), \quad \mathcal{U} = \mathbb{R}^2 - \{(0,0)\}$.

**Answers:**
i) Yes; $\phi = \frac{x}{y} + C$ ii) NO iii) Yes; $\phi = \frac{1}{2}\ln(x^2 + y^2) + C$.

*Comment:*
Exercise 2.12 iii) shows that even if $\mathcal{U}$ is *not* simply-connected the vector field *may* be conservative. $\square$

## 2.5 Vorticity and circulation

Theorem 2.4 (test for conservative fields) shows that given a vector field $\mathbf{F} = (F_1, F_2)$ in $\mathbb{R}^2$, the quantity

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

is of fundamental importance: if it is zero on a simply-connected set $\mathcal{U} \subset \mathbb{R}^2$, then $\mathbf{F}$ is a gradient/conservative field in $\mathcal{U}$.

This quantity also plays an important role when the vector field is the velocity field $\mathbf{v} = (v_1, v_2)$ of a fluid flow in two dimensions[3] and in this context it is called the *vorticity of the fluid*, denoted by $\Omega$:

$$\Omega = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}. \tag{2.54}$$

We now describe the physical significance of the vorticity. Think of a small wooden paddle wheel that is carried along by the fluid. The question is: *will the motion of the fluid cause the paddle wheel to rotate about its axis?*

In order to illustrate the problem we consider two simple vector fields

$$\mathbf{u} = (u, 0),$$

[3]By a fluid flow in two dimensions we mean a three dimensional flow which is "stratified", i.e. the fluid velocity is the same in each of a family of planes. Without loss of generality $\mathbf{v}(\mathbf{x}) = (v_1(x, y), v_2(x, y), 0)$.

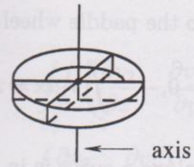



Figure 2.13: A paddle wheel.

where $u$ is a positive constant with $[u] = LT^{-1}$, and

$$\mathbf{v} = (\alpha y, 0),$$

where $\alpha$ is a positive constant with $[\alpha] = T^{-1}$. It follows from figure 2.14 that the velocity field $\mathbf{u}$ will not cause the paddle wheel to rotate, while figure 2.15 shows that the paddle wheel will rotate under the action of $\mathbf{v}$.

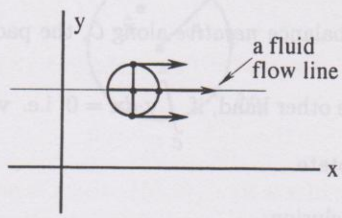



Figure 2.14: The velocity field $\mathbf{u} = (u, 0)$ acting on a paddle wheel.

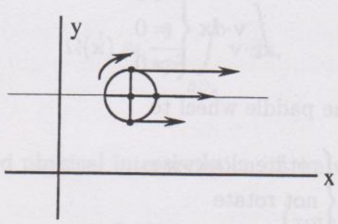



Figure 2.15: The velocity field $\mathbf{v} = (\alpha y, 0)$ acting on a paddle wheel.

This difference between $\mathbf{u}$ and $\mathbf{v}$ can be characterized by considering the line integral of the velocity fields around the circle $C$ of radius $\rho$ that represents the circumference of the paddle wheel. By (2.54) the vorticity for each velocity field is

$$\Omega_{\mathbf{u}} = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} = 0, \qquad \Omega_{\mathbf{v}} = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = -\alpha,$$

and hence Green's theorem applied to the paddle wheel disc gives

$$\int_C \mathbf{u}\cdot d\mathbf{x} = 0, \qquad \int_C \mathbf{v}\cdot d\mathbf{x} = -\pi\rho^2\alpha.$$

The line integral of a velocity field $\mathbf{v}$ along a curve is in fact equal to the line integral of the tangential component $\mathbf{v}\cdot\mathbf{T}$ along the curve, where $\mathbf{T}$ is the *unit tangent vector* to the curve. This result is seen as follows:

$$\begin{aligned}
\int_C \mathbf{v}\cdot d\mathbf{x} &= \int_a^b \mathbf{v}(\mathbf{g}(t))\cdot \mathbf{g}'(t)dt \qquad\qquad \textit{Aside:}\,\mathbf{g}'(t) = \mathbf{T}\parallel \mathbf{g}'(t)\parallel \\
&= \int_a^b \mathbf{v}(\mathbf{g}(t))\cdot \mathbf{T}\parallel \mathbf{g}'(t)\parallel dt \qquad\qquad (2.55)\\
&= \int_C (\mathbf{v}\cdot\mathbf{T})ds.
\end{aligned}$$

If $\int_C \mathbf{v}\cdot d\mathbf{x} < 0$, i.e. $\mathbf{v}\cdot\mathbf{T}$ is on balance *negative* along $C$, the paddle wheel will rotate in the opposite direction to $\mathbf{T}$. On the other hand, if $\int_C \mathbf{v}\cdot d\mathbf{x} = 0$, i.e. $\mathbf{v}\cdot\mathbf{T}$ is on balance *zero* along $C$, the paddle wheel will not rotate.

We now summarize the conclusion.

Let the curve $C$, oriented counter-clockwise as usual, be the circumference of a paddle wheel. If

$$\int_C \mathbf{v}\cdot d\mathbf{x}\begin{cases} < 0\\ = 0\\ > 0\end{cases},$$

then the fluid flow will cause the paddle wheel to

$$\begin{cases}\text{rotate clockwise}\\ \text{not rotate}\\ \text{rotate counter-clockwise.}\end{cases}$$

The quantity $\int_C \mathbf{v}\cdot d\mathbf{x}$ is called *the circulation of the fluid around the simple closed curve $C$.*

Green's theorem leads to a relation between the vorticity $\Omega$ and the circulation $\int_C \mathbf{v}\cdot d\mathbf{x}$.

Let $\mathcal{D}_\rho$ be the disc of radius $\rho$ centred at $\mathbf{x}$ with boundary $\partial\mathcal{D}_\rho$ oriented counter-clockwise.

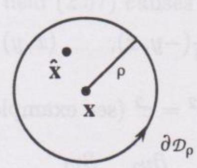
By Green's theorem

$$\int_{\partial\mathcal{D}_\rho} \mathbf{v}\cdot d\mathbf{x} = \iint_{\mathcal{D}_\rho} \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) dx\, dy$$

$$= \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)(\hat{\mathbf{x}})\pi\rho^2,$$

where $\hat{\mathbf{x}}$ is some point in $\mathcal{D}_\rho$. The last step follows from the Mean Value Theorem for integrals. Divide by the area $\pi\rho^2$ and let $\rho \to 0$ giving

$$\underbrace{\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)(\mathbf{x})}_{\Omega(\mathbf{x})} = \lim_{\rho\to 0}\left[\frac{1}{\pi\rho^2}\int_{\partial\mathcal{D}_\rho} \mathbf{v}\cdot d\mathbf{x}\right]. \tag{2.56}$$




In words,

the vorticity at $\mathbf{x}$ equals the circulation
per unit area at $\mathbf{x}$.

For our purposes it is helpful to write an approximation,

$$\Omega(\mathbf{x}) \approx \frac{1}{\pi\rho^2}\int_{\partial\mathcal{D}_\rho} \mathbf{v}\cdot d\mathbf{x},$$

for $\rho$ sufficiently close to 0.

We thus obtain the desired physical interpretation of the vorticity:

$$\Omega(\mathbf{x}) \begin{cases} < 0 \\ = 0 \\ > 0 \end{cases} \Rightarrow \text{ a paddle wheel at } \mathbf{x} \text{ will } \begin{cases} \text{rotate clockwise} \\ \text{not rotate} \\ \text{rotate counter-clockwise.} \end{cases}$$

We conclude with two classic vector fields in $\mathbb{R}^2$ that illustrate vorticity and circulation.

**Example 2.8:**

Consider the vector field

$$\mathbf{v} = \alpha(-y, x), \tag{2.57}$$

where $\alpha$ is a positive constant with $[\alpha] = T^{-1}$. The flow lines are circles $x^2 + y^2 = c^2$, traversed counter-clockwise (see example 1.12 in Chapter 1). The vorticity $\Omega$ is non-zero,

$$\Omega = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 2\alpha.$$

Since the domain of $\mathbf{v}$ is $\mathbb{R}^2$, Green's theorem can be applied to *any* circle of radius $\rho$, giving

$$\int_\rho \mathbf{v}\cdot d\mathbf{x} = 2\alpha(\pi\rho^2)$$

for the circulation. It follows that a paddle wheel will rotate counter-clockwise (see Figure 2.16).

**Example 2.9:**

Consider the vector field (a "vortex field")

$$\mathbf{v} = \frac{\alpha}{x^2 + y^2}(-y, x), \qquad (x, y) \neq (0, 0). \tag{2.58}$$

The flow lines are again circles $x^2 + y^2 = c^2$ (see example 1.13 in Chapter 1). The vorticity $\Omega$ is, however, zero:

$$\Omega = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 0,$$

(verify). In this case the domain of $\mathbf{v}$ is $\mathbb{R}_2 - \{(0, 0)\}$, which is not simply-connected. Green's theorem can be applied to any circle *that does not enclose or pass through the origin*, giving

$$\int_C \mathbf{v}\cdot d\mathbf{x} = 0$$

(since $\Omega = 0$). Thus a paddle wheel will *not rotate* as it moves with the fluid (see Figure 2.17). However, for a circle of radius $b$ that encloses the origin the circulation is non-zero:

$$\int_C \mathbf{v}\cdot d\mathbf{x} = 2\pi\alpha b.$$

(Verify using the definition of line integral.) Thus in this case, although the fluid is *locally non-rotating* (i.e. the paddle wheel does not rotate), it *does rotate globally*, i.e. there are simple closed curves with non-zero circulation.

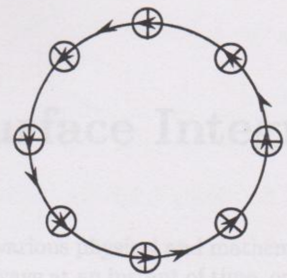


Figure 2.16: The velocity field (2.57) causes the paddle wheel to rotate.

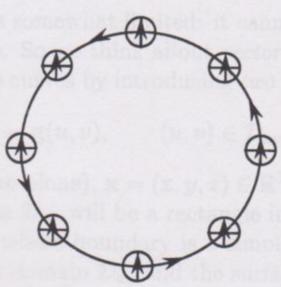


Figure 2.17: The velocity field (2.58) does not cause the paddle wheel to rotate.

# Chapter 3

# Surfaces & Surface Integrals

The notion of a *surface* arises in various physical and mathematical contexts, e.g. the wing or fuselage of an aircraft, an ocean wave at an instant of time, or a soap film formed by dipping a closed wire loop into a soap solution. In a mathematical context, when deriving the equation that governs heat transfer one considers an arbitrary finite chunk of the conducting medium that is bounded by a surface $\Sigma$, and one writes the heat flux through $\Sigma$ as a surface integral.

## 3.1 Parametrized surfaces

### 3.1.1 Surfaces as vector-valued functions

The first mathematical description of a surface that one encounters is the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$. The graph is a surface in $\mathbb{R}^3$, described by the equation

$$z = f(x, y), \qquad (x, y) \in \mathcal{D} \subset \mathbb{R}^2.$$

This way of describing surfaces is somewhat limited: it cannot describe a surface that "folds over" such as a sphere or a torus. So we think about *vector-valued functions* and generalize the way they are used to describe curves by introducing *two* parameters $u$ and $v$, and writing

$$\mathbf{x} = \mathbf{g}(u, v), \qquad (u, v) \in \mathcal{D}_{uv}. \tag{3.1}$$

Here $\mathcal{D}_{uv}$ is a subset of $\mathbb{R}^2$ (the $uv$-plane), $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ ($xyz$-space), and $\mathbf{g} : \mathcal{D}_{uv} \to \mathbb{R}^3$ is a vector-valued function. Often $\mathcal{D}_{uv}$ will be a rectangle in the $uv$-plane, but in general it will be a bounded subset of $\mathbb{R}^2$, whose boundary is a simple closed curve. Figure 3.1 gives a schematic representation of the domain $\mathcal{D}_{uv}$ and the surface $\Sigma$ : as the point $(u, v)$ moves through the set $\mathcal{D}_{uv}$ the image point $\mathbf{g}(u, v)$ sweeps out the surface $\Sigma$ in $\mathbb{R}^3$.

One thinks of the function $\mathbf{g}$ as a map from $\mathbb{R}^2 \to \mathbb{R}^3$ that acts on the rectangle $\mathcal{D}_{uv}$, bending and stretching it so as to form the surface $\Sigma$. It is helpful to think of the surface $\Sigma$ as being generated by two families of curves, namely the images of the two families of straight lines $u$ = constant and $v$ = constant. The two families of curves form a "grid" or "fish-net" that covers the surface. We think of $u$ and $v$ as being coordinates on the surface $\Sigma$ and we refer to the curves on $\Sigma$ that are defined by $u$ = constant and $v$ = constant as *coordinate curves* which form a *coordinate grid.*

# Chapter 3

# Surfaces & Surface Integrals

The notion of a *surface* arises in various physical and mathematical contexts, e.g. the wing or fuselage of an aircraft, an ocean wave at an instant of time, or a soap film formed by dipping a closed wire loop into a soap solution. In a mathematical context, when deriving the equation that governs heat transfer one considers an arbitrary finite chunk of the conducting medium that is bounded by a surface $\Sigma$, and one writes the heat flux through $\Sigma$ as a surface integral.

## 3.1 Parametrized surfaces

### 3.1.1 Surfaces as vector-valued functions

The first mathematical description of a surface that one encounters is the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$. The graph is a surface in $\mathbb{R}^3$, described by the equation

$$z = f(x, y), \qquad (x, y) \in \mathcal{D} \subset \mathbb{R}^2.$$

This way of describing surfaces is somewhat limited: it cannot describe a surface that "folds over" such as a sphere or a torus. So we think about *vector-valued functions* and generalize the way they are used to describe curves by introducing *two* parameters $u$ and $v$, and writing

$$\mathbf{x} = \mathbf{g}(u, v), \qquad (u, v) \in \mathcal{D}_{uv}. \tag{3.1}$$

Here $\mathcal{D}_{uv}$ is a subset of $\mathbb{R}^2$ (the $uv$-plane), $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ ($xyz$-space), and $\mathbf{g} : \mathcal{D}_{uv} \to \mathbb{R}^3$ is a vector-valued function. Often $\mathcal{D}_{uv}$ will be a rectangle in the $uv$-plane, but in general it will be a bounded subset of $\mathbb{R}^2$, whose boundary is a simple closed curve. Figure 3.1 gives a schematic representation of the domain $\mathcal{D}_{uv}$ and the surface $\Sigma$ : as the point $(u, v)$ moves through the set $\mathcal{D}_{uv}$ the image point $\mathbf{g}(u, v)$ sweeps out the surface $\Sigma$ in $\mathbb{R}^3$.

One thinks of the function $\mathbf{g}$ as a map from $\mathbb{R}^2 \to \mathbb{R}^3$ that acts on the rectangle $\mathcal{D}_{uv}$, bending and stretching it so as to form the surface $\Sigma$. It is helpful to think of the surface $\Sigma$ as being generated by two families of curves, namely the images of the two families of straight lines $u =$ constant and $v =$ constant. The two families of curves form a "grid" or "fish-net" that covers the surface. We think of $u$ and $v$ as being coordinates on the surface $\Sigma$ and we refer to the curves on $\Sigma$ that are defined by $u =$ constant and $v =$ constant as *coordinate curves* which form a *coordinate grid*.

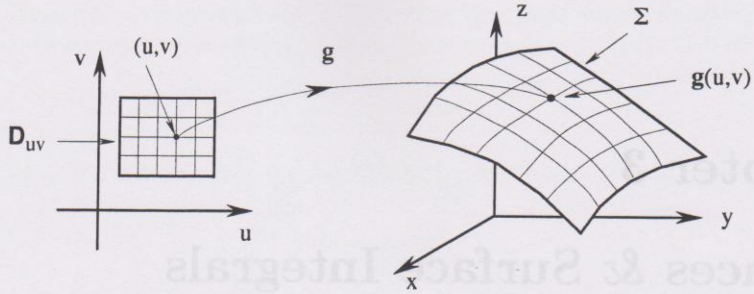



Figure 3.1: The vector-valued function $\mathbf{g}$ maps the domain $\mathcal{D}_{uv}$ onto the surface $\Sigma$.

**Example 3.1:**

The equation

$$\mathbf{x} = \mathbf{g}(u,v) = \mathbf{a} + u\mathbf{e}_1 + v\mathbf{e}_2, \quad (u,v) \in \mathcal{D}_{uv}, \tag{3.2}$$

where $\mathcal{D}_{uv} = \{(u,v) \mid -1 \le u, v \le 1\}$, and $\mathbf{e}_1, \mathbf{e}_2$ are two linearly independent vectors in $\mathbb{R}^3$, describes a surface which is a piece of the plane through the point $\mathbf{a}$, and containing the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. Referring to Figure 3.2, the vector $\mathbf{x} - \mathbf{a}$ lies in the plane and hence is a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$.

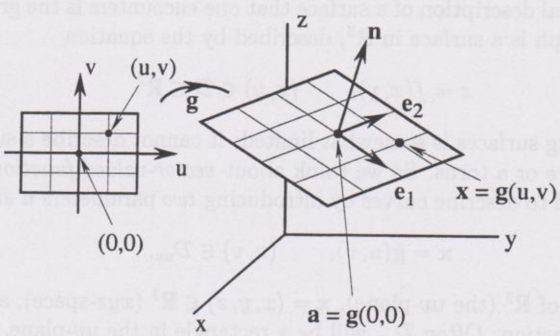



Figure 3.2: Parametric representation of a plane.

One can obtain the equation of the plane in standard form

$$n_1(x-a) + n_2(y-b) + n_3(z-c) = 0 \tag{3.3}$$

by calculating a normal vector $\mathbf{n}$. Since $\mathbf{e}_1$ and $\mathbf{e}_2$ lie in the plane the vector product $\mathbf{e}_1 \times \mathbf{e}_2$ is a vector normal to the plane:

$$\mathbf{n} = \mathbf{e}_1 \times \mathbf{e}_2.$$

Then, taking the scalar product of equation (3.2) with $\mathbf{n}$ gives

$$\mathbf{n}\cdot(\mathbf{x}-\mathbf{a}) = 0,$$

which is the standard form (3.3).

*Recall:* The vector product $\mathbf{a} \times \mathbf{b}$ can be evaluated using a "symbolic determinant":

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, \tag{3.4}$$

which, when formally expanded by the first row, gives

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}. \quad \square$$

**Exercise 3.1:**

A plane in $\mathbb{R}^3$ is given by

$$\mathbf{x} = (1,2,3) + u(1,1,0) + v(1,-1,1),$$

with $(u,v) \in \mathbb{R}^2$. Find the equation of the plane in standard form.

*Answer:* $(x-1) - (y-2) - 2(z-3) = 0.$

**Exercise 3.2:**

Find a vector-valued function to describe the plane $x - 3y + z = 2$.
*Hint:* Let $u = x$, $v = y$.

*Answer:* $\mathbf{x} = \mathbf{g}(u,v) = (0,0,2) + u(1,0,-1) + v(0,1,3). \quad \square$

**Example 3.2:**

The vector-valued function $\mathbf{g} : \mathcal{D}_{uv} \to \mathbb{R}^3$ defined by

$$\mathbf{g}(u,v) = (\sin u \cos v, \sin u \sin v, \cos u), \tag{3.5}$$

with

$$\mathcal{D}_{uv} = \{(u,v) \mid 0 \le u \le \pi, \quad 0 \le v \le 2\pi\},$$

describes the surface of the unit sphere in $\mathbb{R}^3$. This can be verified by writing $\mathbf{x} = \mathbf{g}(u,v)$ and verifying that

$$\| \mathbf{x} \|^2 = x^2 + y^2 + z^2 = 1.$$

The vector-valued function (3.5) is obtained from the formulas that relate spherical polar coordinates to Cartesian coordinates:

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta. \end{aligned}$$

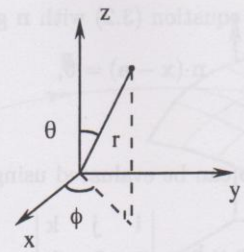



Figure 3.3: Spherical polar coordinates.

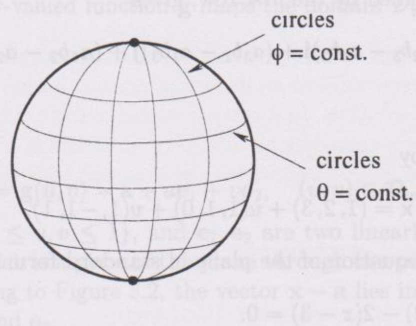



Figure 3.4: The coordinate grid on the sphere created by the spherical polar angles $\theta$ and $\phi$.

To obtain a unit sphere we choose $r = 1$, and then let $u = \theta$, $v = \phi$. The coordinate lines $u$ = constant on the sphere are *circles of constant latitude*, and the coordinate lines $v$ = constant are circles of constant longitude, as shown in Figure 3.4.

*Comment:*

Consider the surface $\Sigma$ defined by

$$z = f(x, y), \qquad (x, y) \in \mathcal{D} \subset \mathbb{R}^2.$$

One can write a vector equation for $\Sigma$ by introducing $u = x$ and $v = y$ as parameters. Then $z = f(u, v)$, and the vector equation for the surface is

$$\mathbf{x} = \mathbf{g}(u, v) = (u, v, f(u, v)), \qquad (u, v) \in \mathcal{D}. \tag{3.6}$$

**Exercise 3.3:**

Give a vector-valued function to describe the cone $x^2 + y^2 = z^2$ with $0 \leq z \leq 1$.

*Answer:* $\mathbf{x} = \mathbf{g}(u, v) = (u \cos v, u \sin v, u)$,
with $(u, v) \in \mathcal{D}_{uv} = \{(u, v) \mid 0 \leq u \leq 1, 0 \leq v \leq 2\pi\}$.

### 3.1.2 The tangent plane

Consider a surface in $\mathbb{R}^3$ given by

$$\mathbf{x} = \mathbf{g}(u, v), \qquad (u, v) \in \mathcal{D}_{uv},$$

where $\mathbf{g}$ is of class $C^1$. The coordinate curves on the surface, obtained by setting either $u$ or $v$ to be constant, are given by

$$\mathbf{x} = \mathbf{g}(u, v_0), \quad \mathbf{x} = \mathbf{g}(u_0, v).$$

The vectors

$$\frac{\partial \mathbf{g}}{\partial u}(u_0, v_0) \quad \text{and} \quad \frac{\partial \mathbf{g}}{\partial v}(u_0, v_0) \tag{3.7}$$

are tangent to these curves at the point $\mathbf{g}(u_0, v_0)$, and hence are tangent to the surface at that point. We can thus regard these vectors, *provided that they are linearly independent*, as defining *the tangent plane of the surface at the point* $\mathbf{g}(u_0, v_0)$.

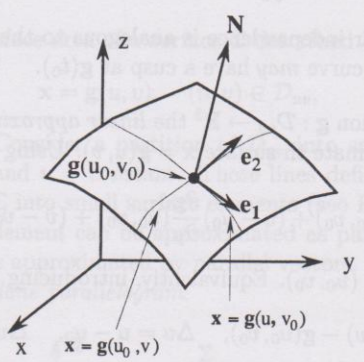



Figure 3.5: The surface $\mathbf{x} = \mathbf{g}(u, v)$, and tangent vectors $\mathbf{e}_1 = \frac{\partial \mathbf{g}}{\partial u}(u_0, v_0)$ and $\mathbf{e}_2 = \frac{\partial \mathbf{g}}{\partial v}(u_0, v_0)$.

Referring to Example 3.1, the equation of the tangent plane can thus be written by using the vectors (3.7) as $\mathbf{e}_1$ and $\mathbf{e}_2$, in the form

$$\mathbf{x} = \mathbf{g}(u_0, v_0) + (u - u_0)\frac{\partial \mathbf{g}}{\partial u}(u_0, v_0) + (v - v_0)\frac{\partial \mathbf{g}}{\partial v}(u_0, v_0). \tag{3.8}$$

Furthermore, the vectors (3.7) define a vector $\mathbf{N}$ that is *normal to the surface*,

$$\mathbf{N} = \frac{\partial \mathbf{g}}{\partial u}(u_0, v_0) \times \frac{\partial \mathbf{g}}{\partial v}(u_0, v_0), \tag{3.9}$$

where the vector product is defined by equation (3.4). Knowing $\mathbf{N}$, the tangent plane at the point $\mathbf{g}(u_0, v_0)$ can also be written in the form

$$\mathbf{N} \cdot (\mathbf{x} - \mathbf{g}(u_0, v_0)) = 0. \tag{3.10}$$

*Comment:*

The requirement that the tangent vectors (3.7) be linearly independent is essential: if they are linearly dependent (in particular if one or both is the zero vector), then the surface *may* not have a tangent plane at the point in question. A classic example is the cone

$$\mathbf{x} = \mathbf{g}(u, v) = (u\cos v, u\sin v, u),$$

with $0 \leq v \leq 2\pi$ and $u \geq 0$. The function g is of class $C^1$, but *the surface does not have a tangent plane at the point* $\mathbf{g}(0, v) = (0, 0, 0)$. The tangent vectors are

$$\frac{\partial \mathbf{g}}{\partial u} = (\cos v, \sin v, 1), \quad \frac{\partial \mathbf{g}}{\partial v} = (-u\sin v, u\cos v, 0),$$

and are linearly dependent when $u = 0$, since

$$\frac{\partial \mathbf{g}}{\partial v}(0, v) = (0, 0, 0).$$

This requirement of linear independence is analogous to the requirement that $\mathbf{g}'(t) \neq \mathbf{0}$ for a curve: if $\mathbf{g}'(t_0) = \mathbf{0}$ the curve *may* have a cusp at $\mathbf{g}(t_0)$. □

For a vector-valued function $\mathbf{g} : \mathcal{D}_{uv} \to \mathbb{R}^3$ the *linear approximation* corresponds to using the tangent plane to approximate th surface $\mathbf{x} = \mathbf{g}(u, v)$. Using (3.8) we have

$$\mathbf{g}(u, v) \approx \mathbf{g}(u_0, v_0) + (u - u_0)\frac{\partial \mathbf{g}}{\partial u}(u_0, v_0) + (v - v_0)\frac{\partial \mathbf{g}}{\partial v}(u_0, v_0),$$

for $(u, v)$ sufficiently close to $(u_0, v_0)$. Equivalently, introducing the increments

$$\Delta\mathbf{g} = \mathbf{g}(u, v) - \mathbf{g}(u_0, v_0), \quad \Delta u = u - u_0, \quad \Delta v = v - v_0,$$

we obtain

$$\Delta\mathbf{g} \approx \Delta u \frac{\partial \mathbf{g}}{\partial u}(u_0, v_0) + \Delta v \frac{\partial \mathbf{g}}{\partial v}(u_0, v_0), \tag{3.11}$$

for $\Delta u$, $\Delta v$ sufficiently close to zero.

**Example 3.3:**

The surface $S$ defined by

$$z = f(x, y), \qquad (x, y) \in \mathcal{D}, \tag{3.12}$$

can be described equivalently by

$$\mathbf{x} = \mathbf{g}(u, v) = (u, v, f(u, v)), \quad (u, v) \in \mathcal{D}$$

(see (3.6)). The tangent vectors (3.7) are given by

$$\frac{\partial \mathbf{g}}{\partial u} = \left(1, 0, \frac{\partial f}{\partial u}\right), \quad \frac{\partial \mathbf{g}}{\partial v} = \left(0, 1, \frac{\partial f}{\partial v}\right). \tag{3.13}$$

The normal vector (3.9) is

$$\mathbf{N} = \left( -\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1 \right), \tag{3.14}$$

(verify, using (3.4)). As a check, one can also calculate a normal vector by writing (3.12) in the form $h(x, y, z) = 0$, where

$$h(x, y, z) = z - f(x, y). \tag{3.15}$$

We know that the gradient $\nabla h$ is orthogonal to the surface $h(x, y, z) = 0$. Using (3.15)

$$\nabla h = \left( -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right)$$

in agreement with (3.14). $\square$

### 3.1.3 Surface area

We want to calculate the surface area of a surface $\Sigma$ described by

$$\mathbf{x} = \mathbf{g}(u, v), \quad (u, v) \in \mathcal{D}_{uv},$$

where g is a $C^1$ function. Consider a partition of $\mathcal{D}_{uv}$ into small rectangles by means of parallel lines $u =$ constant and $v =$ constant. These lines define a coordinate grid on the surface $\Sigma$ that decomposes $\Sigma$ into small surface elements (see Figure 3.1). For $\Delta u$ and $\Delta v$ sufficiently small a surface element can be approximated as part of a plane, and the edges of the surface element can be approximated by parallel vectors. Thus *we can approximate a surface element as a small plane parallelogram.*

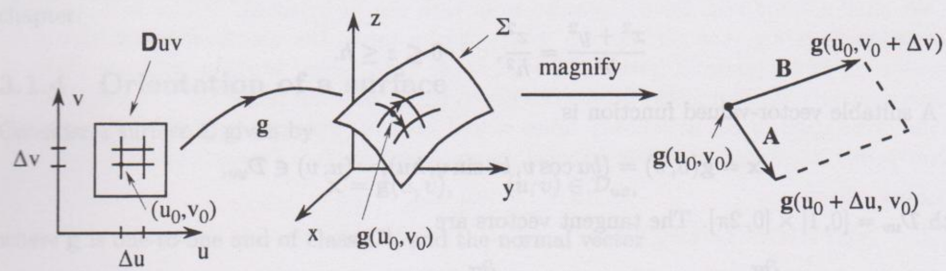



Figure 3.6: A surface element on a surface $\Sigma$.

Recall that the area of a parallelogram defined by two vectors $\mathbf{A}$ and $\mathbf{B}$ is $\| \mathbf{A} \times \mathbf{B} \|$, the magnitude of the vector product. This result follows from the formula

$$\| \mathbf{A} \times \mathbf{B} \| = \| \mathbf{A} \| \| \mathbf{B} \| \sin \theta,$$

where $\theta$ is the angle between the vectors.

In order to approximate the area $\Delta S$ of a surface element we need to approximate the vectors $\mathbf{A}$ and $\mathbf{B}$ that define the sides of the surface element in Figure 3.6. Using the linear approximation (3.11) we obtain

$$\mathbf{A} \approx (\Delta u)\frac{\partial \mathbf{g}}{\partial u}(u_0, v_0), \quad \mathbf{B} \approx (\Delta v)\frac{\partial \mathbf{g}}{\partial v}(u_0, v_0).$$

Thus, simplifying $\Delta S \approx \| \mathbf{A} \times \mathbf{B} \|$ gives

$$\Delta S \approx \left\| \frac{\partial \mathbf{g}}{\partial u}(u_0, v_0) \times \frac{\partial \mathbf{g}}{\partial v}(u_0, v_0) \right\| \Delta u \Delta v. \tag{3.16}$$

To obtain the total area $S$ we have to sum over all surface elements determined by the partition of $\mathcal{D}_{uv}$ and take the limit as $N \to \infty$ and $\max(\Delta u)$, $\max(\Delta v) \to 0$. This process leads to a double integral over the set $\mathcal{D}_{uv}$ in the $uv$-plane. With the above as motivation we make the following.

**Definition:**

The *surface area* of the surface by $\mathbf{x} = \mathbf{g}(u, v)$, $(u, v) \in \mathcal{D}_{uv}$, where $\mathbf{g}$ is $C^1$, is defined by

$$S = \iint\limits_{\mathcal{D}_{uv}} \left\| \frac{\partial \mathbf{g}}{\partial u} \times \frac{\partial \mathbf{g}}{\partial v} \right\| du\, dv. \tag{3.17}$$

**Example 3.4:**

Calculate the surface area of a cone of radius $b$ and height $h$.

*Solution:* The cone is given by

$$\frac{x^2 + y^2}{b^2} = \frac{z^2}{h^2}, \qquad 0 \le z \le h.$$

A suitable vector-valued function is

$$\mathbf{x} = \mathbf{g}(u, v) = (bu\cos v, bu\sin v, hu), \qquad (u, v) \in \mathcal{D}_{uv},$$

with $\mathcal{D}_{uv} = [0, 1] \times [0, 2\pi]$. The tangent vectors are

$$\frac{\partial \mathbf{g}}{\partial u} = (b\cos v, b\sin v, h), \quad \frac{\partial \mathbf{g}}{\partial v} = (-bu\sin v, bu\cos v, 0),$$

giving

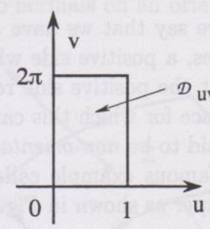
$$\frac{\partial \mathbf{g}}{\partial u} \times \frac{\partial \mathbf{g}}{\partial v} = \begin{vmatrix} i & \mathbf{j} & \mathbf{k} \\ b\cos v & b\sin v & h \\ -bu\sin v & bu\cos v & 0 \end{vmatrix} = (-bhu\cos v, -bhu\sin v, b^2 u),$$

and

$$\left\| \frac{\partial \mathbf{g}}{\partial u} \times \frac{\partial \mathbf{g}}{\partial v} \right\| = \sqrt{b^2 + h^2}\, bu.$$

Thus by (3.17) the surface area is

$$\begin{aligned} S &= \iint_{\mathcal{D}_{uv}} \sqrt{b^2+h^2}\, bu\, du\, dv \\ &= \sqrt{b^2+h^2}\, b \int_{u=0}^{1} \left( \int_{v=0}^{2\pi} u\, dv \right) du \\ &= \cdots = \pi b \sqrt{b^2+h^2}. \end{aligned}$$




*Comment:*

This example is simply intended to show you how the "machinery" works. The surface area of a cone can be calculated by simple geometry. □

The most important part of this subsection is the approximation (3.16) for the area of a surface element, which we shall use when introducing *surface integrals*, the main goal of this chapter.

### 3.1.4 Orientation of a surface

Consider a surface $\Sigma$ given by

$$\mathbf{x} = \mathbf{g}(u, v), \qquad (u, v) \in \mathcal{D}_{uv},$$

where $\mathbf{g}$ is one-to-one and of class $C^1$, and the normal vector

$$\mathbf{N} = \frac{\partial \mathbf{g}}{\partial u} \times \frac{\partial \mathbf{g}}{\partial v}$$

is non-zero for all $(u, v) \in \mathcal{D}_{uv}$. Since $\mathbf{g}$ is one-to-one the surface does not intersect itself (see Figure 3.7), and since $\mathbf{g}$ is of class $C^1$, $\mathbf{N}$ varies continuously over $\Sigma$. In the sequel we shall need to work with a unit normal $\mathbf{n}$ on $\Sigma$. There are two choices for $\mathbf{n}$ at each point, namely

$$\mathbf{n} = \pm \frac{1}{\| \mathbf{N} \|} \mathbf{N}.$$

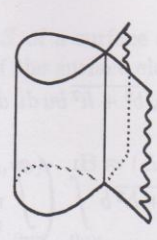


Figure 3.7: A surface which intersects itself.

Once we have made a choice of $\mathbf{n}$ we say that we have *assigned an orientation to* $\Sigma$. An oriented surface has two distinct sides, a positive side which is the side on which $\mathbf{n}$ points and a negative side. You could paint the positive side red and the negative side blue and the colours would not merge. A surface for which this cannot be done, i.e. a surface which does not have two distinct sides, is said to be *non-orientable*. The existence of such surfaces is counter-intuitive, but there is a famous example called *the Moëbius band* that can be constructed easily using a piece of paper as shown in Figure 3.8.


Figure 3.8: Identifying $AB$ and $CD$ creates a cylinder. Identifying $BA$ and $CD$ creates a Möbius band.

We shall assume that the surfaces we work with can be oriented. When working with a *closed surface*, e.g. the surface of a sphere or of a torus, the *standard orientation is to choose* $\mathbf{n}$ *to be the outward normal.* On the other hand, when working with a surface $\Sigma$ whose boundary $\partial\Sigma$ is a piecewise smooth closed curve, we shall relate the orientation of $\Sigma$ to the orientation of $\partial\Sigma$ as follows: *when viewed from the side of* $\Sigma$ *on which the normal vector points, the boundary* $\partial\Sigma$ *is oriented counter-clockwise* (see Figure 3.10).

## 3.2 Surface Integrals

### 3.2.1 Scalar fields

As motivation, think of a surface $\Sigma$ coated with a thin film of silver of surface density $\rho$ (mass per unit area), that varies over the surface. In order to calculate the total mass of silver on the surface we need to define a *surface integral* over $\Sigma$. This concept is analogous to the line integral of a scalar field, as defined in Section 2.1, but with surface area replacing arclength.

Consider a surface $\Sigma$ given by

$$\mathbf{x} = \mathbf{g}(u, v), \qquad (u, v) \in \mathcal{D}_{uv},$$

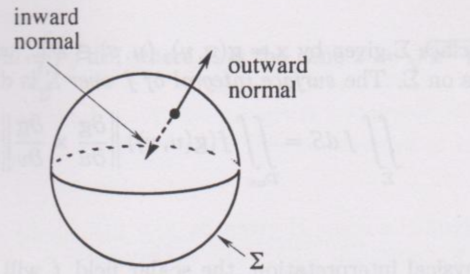



Figure 3.9: The two normals on an orientable closed surface.

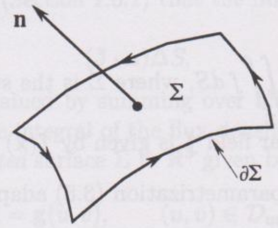



Figure 3.10: Orientation of the boundary of a surface.

and a scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ continuous on $\Sigma$. Consider a partition $\mathcal{P}$ of $\Sigma$ into $N$ surface elements as in Section 3.1.3. The surface integral of $f$ over $\Sigma$ is denoted by

$$\iint_{\Sigma} f\, dS.$$

As a tentative definition we write

$$\iint_{\Sigma} f\, dS = \lim_{|\mathcal{P}|\to 0} \sum_{\mathcal{P}} f(\mathbf{x})\Delta S, \tag{3.18}$$

where the summation is taken over all surface elements of the partition $\mathcal{P}$, $\mathbf{x}$ is a point in the surface element and $\Delta S$ is the area of the surface element. The symbol $|\mathcal{P}|$ denotes the maximum of the areas of the surface elements.

Using the approximation (3.16) we can write

$$f(\mathbf{x})\Delta S \approx f(\mathbf{g}(u,v)) \left\| \frac{\partial \mathbf{g}}{\partial u}(u,v) \times \frac{\partial \mathbf{g}}{\partial v}(u,v) \right\| \Delta u\, \Delta v.$$

The sum (3.18) then becomes a Riemann sum for a *double integral* over the set $\mathcal{D}_{uv}$ in the $uv$-plane. These considerations lead to the working definition below.

**Definition:**

Consider a surface $\Sigma$ given by $\mathbf{x} = \mathbf{g}(u, v)$, $(u, v) \in \mathcal{D}_{uv}$, with g of class $C^1$, and a scalar field $f$ continuous on $\Sigma$. The *surface integral of $f$ over $\Sigma$* is defined by

$$\iint_{\Sigma} f\, dS = \iint_{\mathcal{D}_{uv}} f(\mathbf{g}(u, v)) \left\| \frac{\partial \mathbf{g}}{\partial u} \times \frac{\partial \mathbf{g}}{\partial v} \right\| du\, dv. \tag{3.19}$$

*Comment:*

As regards physical interpretation, the scalar field $f$ will usually represent the *surface density* of some physical quantity and the surface integral $\iint_{\Sigma} f\, dS$ gives the *total amount of the physical quantity on the surface $\Sigma$.*

**Example 3.5:**

Evaluate the surface integral $\iint_{\Sigma} f\, dS$, where $\Sigma$ is the surface of the sphere of radius $b$ centred on the origin and the scalar field $f$ is given by $f(\mathbf{x}) = z^2$.

*Solution:* We use the standard parametrization (3.5) adapted to a sphere of radius $b$:

$$\mathbf{x} = \mathbf{g}(u, v) = b(\sin u \cos v, \sin u \sin v, \cos u),$$

with $(u, v) \in \mathcal{D}_{uv} = \{(u, v) \mid 0 \leq u \leq \pi,\ 0 \leq v \leq 2\pi\}$. The tangent vectors are

$$\frac{\partial \mathbf{g}}{\partial u} = b(\cos u \cos v,\ \cos u\ \sin v, -\sin u),$$
$$\frac{\partial \mathbf{g}}{\partial v} = b(-\sin u\ \sin v,\ \sin u\ \cos v, 0),$$

giving

$$\begin{aligned}\frac{\partial \mathbf{g}}{\partial u} \times \frac{\partial \mathbf{g}}{\partial v} &= b^2(\sin^2 u \cos v,\ \sin^2 u \sin v,\ \sin u \cos u) \\ &= (b \sin u)\mathbf{x}.\end{aligned}$$

It follows that

$$\left\| \frac{\partial \mathbf{g}}{\partial u} \times \frac{\partial \mathbf{g}}{\partial v} \right\| = (b \sin u) \parallel \mathbf{x} \parallel = b^2 \sin u,$$

since $\sin u \geq 0$ on $[0, \pi]$. By definition of the surface integral (3.19)

$$\begin{aligned}\iint_{\Sigma} z^2 dS &= \iint_{\mathcal{D}_{uv}} (b \cos u)^2 (b^2 \sin u) du\, dv \\ &= b^4 \int_{v=0}^{2\pi} \int_{u=0}^{\pi} \cos^2 u \sin u\, du \quad dv \\ &= \cdots = \tfrac{4}{3}\pi b^4. \quad \square\end{aligned}$$

**Exercise 3.4:**

Evaluate the surface integral $\iint_\Sigma f\,dS$, where $\Sigma$ is the cone $z = \sqrt{x^2+y^2}$, $0 \le z \le 1$, and $f(\mathbf{x}) = z^2$.

*Answer:* $\frac{\pi}{\sqrt{2}}$.

### 3.2.2 Vector fields

As motivation, think of a vector field $\mathbf{J}$ that is a *flux density field*, for example mass flux density (example 3 in Section 1.6.1) or heat flux density (example 5 in Section 1.6.1). Let $\Sigma$ be a (piecewise) smooth oriented surface with unit normal $\mathbf{n}$. Consider a surface element of area $\Delta S$ on $\Sigma$. We have seen (Section 1.6.1) that the flux through the surface element is given by

$$(\mathbf{J}\cdot\mathbf{n})\Delta S. \tag{3.20}$$

The total flux through $\Sigma$ is obtained by summing over all surface elements. This process leads to the notion of the surface integral of the flux density field $\mathbf{J}$ over the surface $\Sigma$.

In general, consider an *oriented* surface $\Sigma$ in $\mathbb{R}^3$ given by

$$\mathbf{x} = \mathbf{g}(u,v), \qquad (u,v) \in \mathcal{D}_{uv},$$

with unit normal $\mathbf{n}$. Let $\mathbf{F}$ be a vector field on $\mathbb{R}^3$ that is continuous on $\Sigma$. The surface integral of $\mathbf{F}$ over $\Sigma$ is denoted by

$$\iint_\Sigma \mathbf{F}\cdot\mathbf{n}\,dS.$$

Consider a partition $\mathcal{P}$ of $\Sigma$ into $N$ surface elements, as in Section 3.1.3. Motivated by the form of the expression (3.20), we write the tentative definition of surface integral as follows:

$$\iint_\Sigma \mathbf{F}\cdot\mathbf{n}\,dS = \lim_{|\mathcal{P}|\to 0} \sum_{\mathcal{P}} \mathbf{F}(\mathbf{x})\cdot\mathbf{n}\,\Delta S, \tag{3.21}$$

where the notation has the same meaning as in (3.18). The unit normal $\mathbf{n}$ is obtained by normalizing[1] the normal vector (3.9):

$$\mathbf{n} = \frac{1}{\left\|\frac{\partial \mathbf{g}}{\partial u}\times\frac{\partial \mathbf{g}}{\partial v}\right\|}\left(\frac{\partial \mathbf{g}}{\partial u}\times\frac{\partial \mathbf{g}}{\partial v}\right). \tag{3.22}$$

Using (3.22) and the approximation (3.16) for $\Delta S$ i.e.

$$\Delta S \approx \left\|\frac{\partial \mathbf{g}}{\partial u}\times\frac{\partial \mathbf{g}}{\partial v}\right\|\Delta u \Delta v,$$

[1]One has to verify that this normal vector does agree with the assigned orientation of $\Sigma$. If not, one has to change the parametrization so as to reverse the direction of $\mathbf{n}$.

we obtain

$$\mathbf{F}(\mathbf{x})\cdot\mathbf{n}\,\Delta S \approx \mathbf{F}(\mathbf{g}(u,v))\cdot\left(\frac{\partial\mathbf{g}}{\partial u}\times\frac{\partial\mathbf{g}}{\partial v}\right)\Delta u\Delta v.$$

The sum (3.21) then becomes a Riemann sum for a double integral over the set $\mathcal{D}_{uv}$. These considerations lead to the working definition below.

**Definition:**

Consider an oriented surface $\Sigma$ given by $\mathbf{x} = \mathbf{g}(u,v)$, $(u,v) \in \mathcal{D}_{uv}$, with $\mathbf{g}$ of class $C^1$, and a vector field $\mathbf{F}$ continuous on $\Sigma$. The *surface integral of* $\mathbf{F}$ *over* $\Sigma$ is defined by

$$\iint\limits_{\Sigma} \mathbf{F}\cdot\mathbf{n}dS = \iint\limits_{\mathcal{D}_{uv}} \mathbf{F}(\mathbf{g}(u,v))\cdot\left(\frac{\partial\mathbf{g}}{\partial u}\times\frac{\partial\mathbf{g}}{\partial v}\right)dudv. \tag{3.23}$$

*Comment:*

If $\mathbf{F}$ is a flux density vector field, then the surface integral $\iint\limits_{\Sigma}\mathbf{F}\cdot\mathbf{n}dS$ equals the total flux of $\mathbf{F}$ through the surface $\Sigma$.

**Example 3.6:**

The surface $\Sigma$ is the piece of the plane $y + z = 1$ cut out by the cylinder $x^2 + y^2 = a^2$, oriented so that the unit normal $\mathbf{n}$ points upwards. The temperature in the vicinity of $\Sigma$ is given by $T(\mathbf{x}) = x^2 + y^2 + z^2$. Calculate the total heat flux through $\Sigma$. Denote the thermal conductivity by $k$.

*Solution:* We parametrize $\Sigma$ by writing

$$\mathbf{x} = \mathbf{g}(u,v) = (u, v, 1-v), \quad u^2 + v^2 \leq a^2,$$

i.e. we choose $x = u$, $y = v$. The tangent vectors are

$$\frac{\partial\mathbf{g}}{\partial u} = (1,0,0), \quad \frac{\partial\mathbf{g}}{\partial v} = (0,1,-1),$$

giving

$$\frac{\partial\mathbf{g}}{\partial u}\times\frac{\partial\mathbf{g}}{\partial v} = (0,1,1),$$

which has the correct orientation. By equation (1.30) the heat flux density vector is

$$\mathbf{J} = -k\nabla T = -2k(x,y,z).$$

The total heat flux $J(\Sigma)$ through $\Sigma$ is

$$J(\Sigma) = \iint\limits_{\Sigma}\mathbf{J}\cdot\mathbf{n}dS.$$

By definition of the surface integral

$$J(\Sigma) = \iint_{\mathcal{D}_{uv}} -(2k)(u, v, 1-v) \cdot (0, 1, 1)\,dudv$$

$$= \iint_{\mathcal{D}_{uv}} (-2k)\,dudv.$$

Since $\mathcal{D}_{uv}$ is the disc $u^2 + v^2 \leq a^2$ and the integrand is constant,

$$J(\Sigma) = -2\pi k a^2. \quad \square$$

*Comment:*

The temperature $T$ in Example 3.6 increases radially outwards, and thus we expect that heat will be transferred across $\Sigma$ in the direction of $-\mathbf{n}$. We thus expect that the heat flux across $\Sigma$ will be negative.

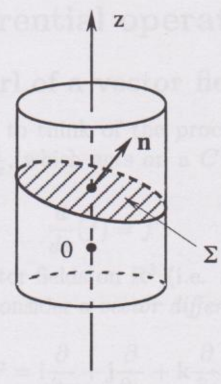



Figure 3.11: The surface $\Sigma$ in Example 3.6.

**Exercise 3.5:**

Evaluate the surface integral $\iint_{\Sigma} \mathbf{F} \cdot \mathbf{n} dS$ where $\Sigma$ is the cylinder $x^2 + z^2 = b^2$, $0 \leq y \leq h$, excluding the ends, and $\mathbf{F}$ is the vector field $\mathbf{F}(\mathbf{x}) = (0, 0, z)$. The surface $\Sigma$ is oriented so that the normal is in the outward radial direction.

*Answer:* $\pi b^2 h$.

### 3.2.3 Properties of surface integrals

As with any integral, surface integrals possess the properties of *linearity* and *additivity*. The formal statements are analogous to those for line integrals in Section 2.2.3. In addition, if a surface $\Sigma$ is *piecewise* $C^1$ instead of being $C^1$, i.e. the surface $\Sigma$ is the union of several surfaces whose defining functions are $C^1$, one can define the surface integral over $\Sigma$ as the sum of the integrals over the separate pieces.

# Chapter 4

# Gauss' and Stokes' Theorems

In this chapter we first introduce the *divergence* and *curl* of a vector field, which we then use in the formulation of Gauss' and Stokes' theorems.

## 4.1 The vector differential operator $\nabla$

### 4.1.1 Divergence and curl of a vector field

In elementary calculus it is useful to think of the process of differentiation as defining a *differential operator*, denoted by $\frac{d}{dx}$, which acts on a $C^1$ function $f$ to give the derivative function $f'$:

$$\frac{d}{dx}(f) = f'.$$

When working with scalar and vector fields on $\mathbb{R}^3$ (i.e. functions of $x, y$ and $z$) it is useful to generalize the operator $\frac{d}{dx}$, and consider a *vector differential operator* which we denote by $\nabla$:

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}, \tag{4.1}$$

or equivalently

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), \tag{4.2}$$

relative to Cartesian coordinates.

Firstly, $\nabla$ can act on a *scalar field* $f : \mathbb{R}^3 \to \mathbb{R}$,

$$\nabla f = \mathbf{i}\frac{\partial}{\partial x}(f) + \mathbf{j}\frac{\partial}{\partial y}(f) + \mathbf{k}\frac{\partial}{\partial z}(f),$$

which we rewrite as

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}, \tag{4.3}$$

and recognize as the *gradient of the scalar field* $f$. Secondly, $\nabla$ can act on a *vector field* $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ to give a *scalar field* $\nabla \cdot \mathbf{F}$:

$$\nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (F_1, F_2, F_3),$$

which, in analogy with the scalar product of two vectors, is written

$$\nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}. \tag{4.4}$$

This scalar field is called the *divergence* of $\mathbf{F}$, and is also written $\operatorname{div}\mathbf{F}$. We note that Gauss' Theorem leads to a physical interpretation of $\nabla\cdot\mathbf{F}$, which we shall discuss later.

Thirdly, in analogy with the vector product, $\nabla$ can act on a vector field $\mathbf{F}$ on $\mathbb{R}^3$ to give a *vector field* $\nabla\times\mathbf{F}$:

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}. \tag{4.5}$$

Written out in full this gives

$$\nabla\times\mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}. \tag{4.6}$$

This vector field is called the *curl* of $\mathbf{F}$, and is also written $\operatorname{curl}\mathbf{F}$. On recalling that the vorticity scalar of a vector field on $\mathbb{R}_2$ is given by

$$\Omega = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y},$$

(see equation (2.54)) we see that the curl is the generalization to three dimensions of the vorticity scalar in $\mathbb{R}^2$. Indeed in fluid mechanics, if $\mathbf{v}$ is the velocity field of a fluid then $\mathbf{w} = \nabla\times\mathbf{v}$ is called the *vorticity field* of the fluid. We thus expect $\nabla\times\mathbf{F}$ to describe rotational properties of the vector field $\mathbf{F}$. This interpretation will be clarified after we discuss Stokes' theorem.

**Exercise 4.1:**

In this exercise, $\mathbf{r} = (x, y, z)$, $r = \|\mathbf{r}\|$. By writing out components, verify that

$$\text{i) } \nabla r = \frac{1}{r}\mathbf{r}, \qquad \text{ii) } \nabla\cdot\mathbf{r} = 3, \qquad \text{iii) } \nabla\times\mathbf{r} = \mathbf{0}.$$

**Exercise 4.2:**

Consider the vector field $\mathbf{F}$ defined by

$$\mathbf{F}(\mathbf{x}) = A\mathbf{x},$$

where $A$ is a $3\times3$ constant matrix with entries $a_{ij}$. The matrix $A$ acts on the position vector $\mathbf{x}$ written as a column vector. Show that

$$\nabla\cdot\mathbf{F} = a_{11} + a_{22} + a_{33},$$
$$\nabla\times\mathbf{F} = (a_{32} - a_{23}, a_{13} - a_{31}, a_{21} - a_{12}). \qquad \square$$

As an example of the use of divergence and curl in physics, we now present *Maxwell's equations*, the fundamentalequations that describe *electromagnetic fields.* Our purpose is not to derive the equations or to study their physical implications – it is simply to present an important example of the use of the divergence and curl.

An electromagnetic field is composed of an electric field $\mathbf{E}(t,x,y,z)$ and a magnetic field $\mathbf{H}(t,x,y,z)$, which are *time-dependent* vector fields on $\mathbb{R}^3$. With appropriate choice of units, Maxwell's equations read

$$\frac{\partial \mathbf{E}}{\partial t} = c\nabla \times \mathbf{H} - 4\pi \mathbf{J} \tag{4.7}$$

$$\frac{\partial \mathbf{H}}{\partial t} = -c\nabla \times \mathbf{E} \tag{4.8}$$

$$\nabla{\cdot}\mathbf{E} = 4\pi\varepsilon \tag{4.9}$$

$$\nabla{\cdot}\mathbf{H} = 0, \tag{4.10}$$

where $\varepsilon$ is the charge density and $\mathbf{j}$ is the current vector. The constant $c$ has the dimensions of velocity, and is in fact the *speed of light* in a vacuum. Mathematically these equations form a system of linear partial differential equations in 6 unknowns, namely the components of $\mathbf{E}$ and $\mathbf{H}$.

One of the most remarkable predictions of Maxwell's equations is that electromagnetic fields can transport energy through space in the form of waves, called *electromagnetic waves* or *electromagnetic radiation* (light, radio waves, X-rays are all examples, differing in frequency). See #6 in Problem Set 4.

### 4.1.2 Identities involving $\nabla$

The differential operator $\nabla$ satisfies various identities which are used in deriving and working with the field equations of classical physics, e.g. Maxwell's equations, or the equations of fluid dynamics.

The first set of identities refers to the *gradient*.

$G_1$ *Sum* of two scalar fields:

$$\nabla(f+g) = \nabla f + \nabla g.$$

$G_2$ *Product* of two scalar fields:

$$\nabla(fg) = f\nabla g + g\nabla f.$$

$G_3$ *Scalar product* of two vector fields:

$$\nabla(\mathbf{F}\cdot\mathbf{G}) = (\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}{\cdot}\nabla)\mathbf{F} + \mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{G}\times(\nabla\times\mathbf{F}).$$

The second set refers to the *divergence*

$D_1$ *Sum* of two vector fields:

$$\nabla\cdot(\mathbf{F}+\mathbf{G}) = \nabla{\cdot}\mathbf{F} + \nabla{\cdot}\mathbf{G}.$$

$D_2$ *Product* of a scalar field and a vector field:

$$\nabla \cdot (f\mathbf{F}) = f\nabla\cdot\mathbf{F} + (\nabla f)\cdot\mathbf{F}.$$

$D_3$ *Vector product* of two vector fields:

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G}\cdot(\nabla \times \mathbf{F}) - \mathbf{F}\cdot(\nabla \times \mathbf{G}).$$

The third set refers to the *curl.*

$C_1$ *Sum* of two vector fields:

$$\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}.$$

$C_2$ *Product* of a scalar field and a vector field:

$$\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + \nabla f \times \mathbf{F}.$$

$C_3$ *Vector product* of two vector fields:

$$\nabla \times (\mathbf{F} \times \mathbf{G}) = (\nabla\cdot\mathbf{G})\mathbf{F} - (\nabla\cdot\mathbf{F})\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G}.$$

The fourth set can be thought of as "zero identities".

$Z_1$ *Curl* of a *gradient*:

$$\nabla \times (\nabla f) = 0, \tag{4.11}$$

for any $C^2$ scalar field $f$,

$Z_2$ *Divergence* of a *curl*:

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0, \tag{4.12}$$

for any $C^2$ vector field $\mathbf{F}$.

**Note:** In $G_3$ and $C_3$ there appear the terms $(\mathbf{F} \cdot \nabla)\mathbf{G}$ and $(\mathbf{G} \cdot \nabla)\mathbf{F}$. In component form the expression $\mathbf{F} \cdot \nabla$ reads

$$\begin{aligned}\mathbf{F} \cdot \nabla &= (F_1, F_2, F_3) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \\ &= F_1\frac{\partial}{\partial x} + F_2\frac{\partial}{\partial y} + F_3\frac{\partial}{\partial z}.\end{aligned}$$

Observe that $\mathbf{F} \cdot \nabla$ is a *scalar* differential operator. For example

$$\mathbf{i} \cdot \nabla = \frac{\partial}{\partial x},$$

where $\mathbf{i} = (1, 0, 0)$. Note that one cannot reverse the order of the symbols in the expression $\mathbf{F} \cdot \nabla$:

$$\underbrace{\mathbf{F} \cdot \nabla}_{\substack{\text{a scalar} \\ \text{differential operator}}} \neq \underbrace{\nabla \cdot \mathbf{F}}_{\substack{\text{divergence of } \mathbf{F}, \\ \text{a scalar field}}}$$

The scalar operator $\mathbf{F} \cdot \nabla$ can act on a vector field $\mathbf{G}$ to give a vector field $(\mathbf{F} \cdot \nabla)\mathbf{G}$.

**Exercise 4.3:**

Let $\mathbf{A} = (A_1, A_2, A_3)$ be a constant vector field and let $\mathbf{r} = (x, y, z)$. Show that

$$(\mathbf{A} \cdot \nabla)\mathbf{r} = \mathbf{A}.$$

One can also ask about the "curl of a curl", i.e.

$$\nabla \times (\nabla \times \mathbf{F}).$$

Unlike the other two expressions with repeated derivatives, namely $Z_1$ and $Z_2$, this one is not identically zero, and in order to give an expression for it, we need to form a second order differential operator from $\nabla$, called the *Laplacian*, and denoted by $\nabla^2$. This operator is defined by taking the divergence of a gradient field:

$$\nabla^2 f = \nabla \cdot (\nabla f) \tag{4.13}$$

Writing this out in terms of partial derivatives, i.e.

$$\nabla^2 f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right),$$

leads to

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \tag{4.14}$$

One can formally write

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{4.15}$$

The operator $\nabla^2$, which is a scalar differential operator, is called the *Laplacian*. Note that the Laplacian can act on a vector field $\mathbf{F}$,

$$\nabla^2 \mathbf{F} = \frac{\partial^2 \mathbf{F}}{\partial x^2} + \frac{\partial^2 \mathbf{F}}{\partial y^2} + \frac{\partial^2 \mathbf{F}}{\partial z^2} \tag{4.16}$$

to give a vector field.

We can now state the "curl of a curl" identity:

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}. \tag{4.17}$$

In words: *the curl of the curl of* $\mathbf{F}$ *equals the gradient of the divergence of* $\mathbf{F}$ *minus the Laplacian of* $\mathbf{F}$.

The Laplacian operator $\nabla^2$ commutes with the vector operator $\nabla$ in the following ways:

$$\nabla(\nabla^2 f) = \nabla^2(\nabla f), \tag{4.18}$$

$$\nabla \cdot (\nabla^2 \mathbf{F}) = \nabla^2(\nabla \cdot \mathbf{F}), \tag{4.19}$$

$$\nabla \times (\nabla^2 \mathbf{F}) = \nabla^2(\nabla \times \mathbf{F}), \tag{4.20}$$

where $f$ is a $C^3$ scalar field and $\mathbf{F}$ is a $C^3$ vector field. Note that in (4.18), $\nabla^2 f$ is a scalar field while $\nabla f$ is a vector field.

*Comment:*

Proving the identities listed in this section involves a straightforward but sometimes lengthy calculation – one simply expresses each side of an identity in terms of components, thereby showing that they are equal. The "zero identities" $Z_1$ and $Z_2$ are essentially a consequence of the equality of mixed second order partial derivatives e.g. $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$. See Problem Set 4, #3-7.

**Exercise 4.4:**

Use exercise 4.1 and the chain rule to show that

$$\nabla f(r) = \frac{f'(r)}{r}\mathbf{r},$$

where $f : \mathbb{R} \to \mathbb{R}$ is a $C^1$ function, $r = \| \mathbf{r} \|$ and $\mathbf{r} = (x, y, z)$.

**Exercise 4.5:**

Use exercises 4.1 and 4.4 and the identities involving $\nabla$ to verify the following results:

i) $\nabla \cdot (f(r)\mathbf{r}) = rf' + 3f$ ii) $\nabla \cdot (f(r)\mathbf{A}) = \dfrac{f'(r)}{r}\mathbf{r} \cdot \mathbf{A}$

iii) $\nabla \times (f(r)\mathbf{r}) = \mathbf{0}$ iv) $\nabla \times (f(r)\mathbf{A}) = \dfrac{f'(r)}{r}\mathbf{r} \times \mathbf{A}$

v) $\nabla(\mathbf{A} \cdot \mathbf{R}) = \mathbf{A}$ vi) $\nabla \times (\mathbf{A} \times \mathbf{r}) = 2\mathbf{A}$

vii) $\nabla^2 f(r) = f''(r) + \frac{2f'(r)}{r}$

Here $f(r)$ is a $C^2$ function of $r = \| \mathbf{r} \|$, and $\mathbf{A}$ is a constant vector field in $\mathbb{R}^3$.

### 4.1.3 Expressing $\nabla$ in curvilinear coordinates

In problems in which there is symmetry about a point it is usually helpful to introduce *polar coordinates* $(\rho, \phi)$ in $\mathbb{R}^2$ or *spherical coordinates* $(r, \theta, \phi)$ in $\mathbb{R}^3$. If there is symmetry about a line in $\mathbb{R}^3$, one thinks of *cylindrical coordinates* $(\rho, \phi, z)$. These three systems of coordinates are examples of curvilinear coordinates. Let's begin by reviewing the definitions of these coordinates.

**Polar coordinates:**

$$x = \rho \cos \phi$$
$$y = \rho \sin \phi, \tag{4.21}$$

with $\rho \geq 0$, $0 \leq \phi \leq 2\pi$.

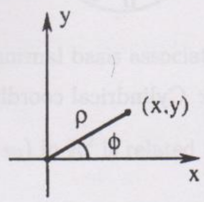



Figure 4.1: Polar coordinates.

**Spherical coordinates:**

$$x = r \sin \theta \cos \phi$$
$$y = r \sin \theta \sin \phi \tag{4.22}$$
$$z = r \cos \theta,$$

with $r \geq 0$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$.

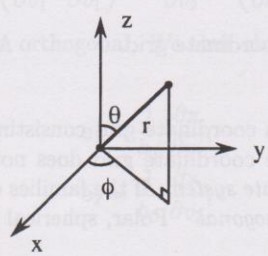



Figure 4.2: Spherical coordinates.

**Cylindrical coordinates:**

$$x = \rho \cos \phi$$
$$y = \rho \sin \phi \tag{4.23}$$
$$z = z,$$

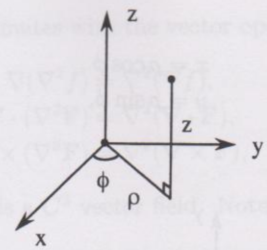



Figure 4.3: Cylindrical coordinates.

with $\rho \geq 0$, $0 \leq \phi \leq 2\pi$, $z \in \mathbb{R}$.

Cartesian coordinates $(x, y)$ in $\mathbb{R}^2$ define a coordinate grid consisting of two families of orthogonal straight lines $x$ = constant and $y$ = constant. Polar coordinates $(\rho, \phi)$ define a coordinate grid consisting of concentric circles $\rho$ = constant and radial half-lines $\phi$ = constant.

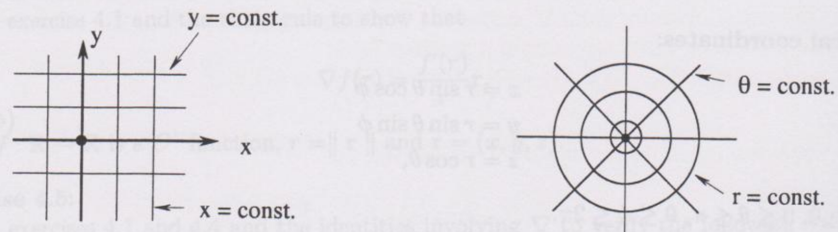



Figure 4.4: A Cartesian coordinate grid. A polar coordinate grid.

Coordinate systems in $\mathbb{R}^3$ define a coordinate grid consisting of *three* families of curves or lines. A coordinate system whose coordinate grid does not consist of families of parallel lines is called a *curvilinear coordinate system.* If the families of curves intersect orthogonally, we call the coordinate system *orthogonal.* Polar, spherical and cylindrical coordinates are orthogonal.

Our goal is to learn how to write the gradient $\nabla f$, the divergence $\nabla \cdot \mathbf{F}$ and the curl $\nabla \times \mathbf{F}$ in terms of spherical and cylindrical coordinates in $\mathbb{R}^3$. In Cartesian coordinates the operator $\nabla$ has the form

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

The key to writing this operator in curvilinear coordinate grid to construct an *orthonormal basis* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, the analogue of the Cartesian basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Figure 4.5 shows the orthonormal basis $\{\mathbf{e}_\rho, \mathbf{e}_\phi\}$ determined by polar coordinates $\{\rho, \phi\}$ in $\mathbb{R}^2$.

For simplicity we begin by introducing the orthonormal basis determined by an orthogonal curvilinear coordinate system in $\mathbb{R}^2$. The generalization to $\mathbb{R}^3$ then becomes clear. A general

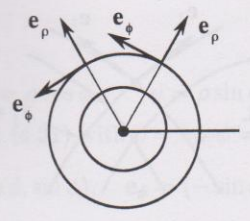



Figure 4.5: The orthonormal basis associated with polar coordinates.

curvilinear coordinate system $(v_1, v_2)$ in $\mathbb{R}^2$ is related to a Cartesian coordinate system by a pair of equations

$$\begin{aligned} x &= f(v_1, v_2) \\ y &= g(v_1, v_2), \end{aligned} \tag{4.24}$$

(see for example (4.21)). It is convenient to write these equations in vector form

$$\mathbf{x} = \mathbf{F}(v_1, v_2), \tag{4.25}$$

where $\mathbf{x} = (x, y)$, $\mathbf{F} = (f, g)$. The curves of the coordinate grid are then given by

$$\mathbf{x} = \mathbf{F}(v_1, \ell), \qquad \mathbf{x} = \mathbf{F}(k, v_2),$$

where $k$ and $\ell$ are constants, and the tangent vectors are

$$\frac{\partial \mathbf{x}}{\partial v_1} = \left(\frac{\partial x}{\partial v_1}, \frac{\partial y}{\partial v_1}\right), \quad \frac{\partial \mathbf{x}}{\partial v_2} = \left(\frac{\partial x}{\partial v_2}, \frac{\partial y}{\partial v_2}\right). \tag{4.26}$$

By assumption these vectors are orthogonal. We thus obtain an orthonormal basis by normalizing them

$$\begin{aligned} \mathbf{e}_1 &= \frac{1}{h_1}\frac{\partial \mathbf{x}}{\partial v_1}, \\ \mathbf{e}_2 &= \frac{1}{h_2}\frac{\partial \mathbf{x}}{\partial v_2}, \end{aligned} \tag{4.27}$$

where

$$\begin{aligned} h_1 &= \left\| \frac{\partial \mathbf{x}}{\partial v_1} \right\| = \sqrt{\left(\frac{\partial x}{\partial v_1}\right)^2 + \left(\frac{\partial y}{\partial v_1}\right)^2}, \\ h_2 &= \left\| \frac{\partial \mathbf{x}}{\partial v_2} \right\| = \sqrt{\left(\frac{\partial x}{\partial v_2}\right)^2 + \left(\frac{\partial y}{\partial v_2}\right)^2}. \end{aligned} \tag{4.28}$$

We can now express the gradient vector field $\nabla u$ of a given scalar field $u$ relative to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$:

$$\nabla u = (\nabla u \cdot \mathbf{e}_1)\mathbf{e}_1 + (\nabla u \cdot \mathbf{e}_2)\mathbf{e}_2. \tag{4.29}$$

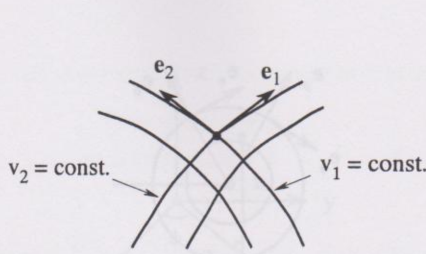



Figure 4.6: The orthonormal basis.

Using the Chain Rule and equation (4.26) we obtain

$$\frac{\partial u}{\partial v_1} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial v_1} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v_1} = \nabla u \cdot \frac{\partial \mathbf{x}}{\partial v_1}.$$

Dividing by $h_1$ and using (4.27) gives

$$\nabla u \cdot \mathbf{e}_1 = \frac{1}{h_1}\frac{\partial u}{\partial v_1}.$$

Similarly

$$\nabla u \cdot \mathbf{e}_2 = \frac{1}{h_2}\frac{\partial u}{\partial v_2}.$$

It now follows from (4.29) that

$$\nabla u = \frac{1}{h_1}\frac{\partial u}{\partial v_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial u}{\partial v_2}\mathbf{e}_2.$$

We can write this equation in the form

$$\nabla u = \left(\frac{1}{h_1}\mathbf{e}_1\frac{\partial}{\partial v_1} + \frac{1}{h_2}\mathbf{e}_2\frac{\partial}{\partial v_2}\right)u. \tag{4.30}$$

Thus, *relative to the curvilinear coordinate system* $(v_1, v_2)$, *the operator* $\nabla$ *has the form*

$$\nabla = \frac{1}{h_1}\mathbf{e}_1\frac{\partial}{\partial v_1} + \frac{1}{h_2}\mathbf{e}_2\frac{\partial}{\partial v_2}, \tag{4.31}$$

where the basis vectors $\mathbf{e}_1, \mathbf{e}_2$ are given by (4.27) and the scale factors $h_1, h_2$ by (4.28). Equations (4.27) (4.31) generalizes to $\mathbb{R}^3$ in an obvious way:

$$e_i = \frac{1}{h_i}\frac{\partial \mathbf{x}}{\partial v_i}, \quad i = 1, 2, 3 \tag{4.32}$$

$$\nabla = \frac{1}{h_1}\mathbf{e}_1\frac{\partial}{\partial v_1} + \frac{1}{h_2}\mathbf{e}_2\frac{\partial}{\partial v_2} + \frac{1}{h_3}\mathbf{e}_3\frac{\partial}{\partial v_3}, \tag{4.33}$$

where

$$h_i = \left\|\frac{\partial \mathbf{x}}{\partial v_i}\right\| \tag{4.34}$$

We now give the expression for $\nabla$ in each of the three classical coordinate systems.

**Polar coordinates in $\mathbb{R}^2$:**

The defining equations are

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi.$$

Using equations (4.27), (4.28) and (4.31) with $\rho = v_1$, $\phi = v_2$ we obtain

$$\mathbf{e}_\rho = (\cos\phi, \sin\phi), \quad \mathbf{e}_\phi = (-\sin\phi, \cos\phi), \tag{4.35}$$

$$h_\rho = 1, \qquad h_\phi = \rho,$$

and

$$\nabla = \mathbf{e}_\rho \frac{\partial}{\partial \rho} + \frac{1}{\rho}\mathbf{e}_\phi \frac{\partial}{\partial \phi}. \tag{4.36}$$

**Cylindrical coordinates in $\mathbb{R}^3$:**

The defining equations are

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi, \qquad z = z.$$

Using equations (4.32)-(4.34) we obtain with $(v_1, v_2, v_3) = (\rho, \phi, z)$,

$$\mathbf{e}_\rho = (\cos\phi, \sin\phi, 0), \quad \mathbf{e}_\phi = (-\sin\phi, \cos\phi, 0), \quad \mathbf{e}_z = (0, 0, 1), \tag{4.37}$$

$$h_\rho = 1, \qquad h_\phi = \rho, \qquad h_z = 1,$$

and

$$\nabla = \mathbf{e}_\rho \frac{\partial}{\partial \rho} + \frac{1}{\rho}\mathbf{e}_\phi \frac{\partial}{\partial \phi} + \mathbf{e}_z \frac{\partial}{\partial z}. \tag{4.38}$$

*Comment:* It is important to note that an orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ determined by a curvilinear coordinate system $(v_1, v_2, v_3)$ differs from the Cartesian basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ in a significant way – *the vector fields* $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ *vary from point to point* (see for example (4.35) and (4.37).

**Spherical coordinates in $\mathbb{R}^3$:**

The defining equations are

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

Using equations (4.32)-(4.34) with $(v_1, v_2, v_3) = (r, \theta, \phi)$, we obtain

$$\begin{aligned} \mathbf{e}_r &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), & h_r &= 1 \\ \mathbf{e}_\theta &= (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta), & h_\theta &= r \\ \mathbf{e}_\phi &= (-\sin\phi, \cos\phi, 0), & h_\phi &= r\sin\theta, \end{aligned} \tag{4.39}$$

and

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \frac{1}{r}\mathbf{e}_\theta \frac{\partial}{\partial \theta} + \frac{1}{r\sin\theta}\mathbf{e}_\phi \frac{\partial}{\partial \phi}. \qquad \square \tag{4.40}$$

We are now in a position to calculate the divergence $\nabla \cdot \mathbf{F}$ and the Laplacian $\nabla^2 f$ in the different coordinate systems. We will illustrate the procedure by calculating $\nabla \cdot \mathbf{F}$ in polar coordinates in $\mathbb{R}^2$.

**Example 4.1:** Calculate the divergence $\nabla \cdot \mathbf{F}$ in polar coordinates in $\mathbb{R}_2$.

*Solution:* We write $\mathbf{F}$ in terms of the basis (4.35):

$$\mathbf{F} = F_\rho \mathbf{e}_\rho + F_\phi \mathbf{e}_\phi.$$

Using (4.36), and taking into account that $\mathbf{e}_\rho$ and $\mathbf{e}_\phi$ are not constant vector fields,

$$\begin{aligned}
\nabla \cdot \mathbf{F} &= \mathbf{e}_\rho \frac{\partial}{\partial \rho} \cdot (F_\rho \mathbf{e}_\rho + F_\phi \mathbf{e}_\phi) + \frac{1}{\rho} \mathbf{e}_\phi \frac{\partial}{\partial \phi} \cdot (F_\rho \mathbf{e}_\rho + F_\phi \mathbf{e}_\rho) \\
&= \mathbf{e}_\rho \cdot \left( \frac{\partial F_\rho}{\partial \rho} \mathbf{e}_\rho + F_\rho \frac{\partial \mathbf{e}_\rho}{\partial \rho} \right) + \mathbf{e}_\rho \cdot \left( \frac{\partial F_\phi}{\partial \rho} \mathbf{e}_\phi + F_\phi \frac{\partial \mathbf{e}_\phi}{\partial \rho} \right) \\
&\quad + \frac{1}{\rho} \mathbf{e}_\phi \cdot \left( \frac{\partial F_\rho}{\partial \phi} \mathbf{e}_\rho + F_\rho \frac{\partial \mathbf{e}_\rho}{\partial \phi} \right) + \frac{1}{\rho} \mathbf{e}_\phi \cdot \left( \frac{\partial F_\phi}{\partial \phi} \mathbf{e}_\phi + F_\phi \frac{\partial \mathbf{e}_\phi}{\partial \phi} \right).
\end{aligned}$$

From (4.35) we obtain

$$\begin{aligned}
\frac{\partial \mathbf{e}_\rho}{\partial \rho} &= \mathbf{0}, \quad & \frac{\partial \mathbf{e}_\rho}{\partial \phi} &= \mathbf{e}_\phi, \\
\frac{\partial \mathbf{e}_\phi}{\partial \rho} &= \mathbf{0}, \quad & \frac{\partial \mathbf{e}_\phi}{\partial \phi} &= -\mathbf{e}_\rho.
\end{aligned} \tag{4.41}$$

Using these results and the fact that

$$\mathbf{e}_\rho \cdot \mathbf{e}_\rho = 1, \quad \mathbf{e}_\rho \cdot \mathbf{e}_\phi = 0, \quad \mathbf{e}_\phi \cdot \mathbf{e}_\phi = 1,$$

the expansion for $\nabla \cdot \mathbf{F}$ simplifies dramatically to give

$$\nabla \cdot \mathbf{F} = \frac{\partial F_\rho}{\partial \rho} + \frac{1}{\rho} F_\rho + \frac{1}{\rho} \frac{\partial F_\phi}{\partial \phi}. \qquad \square \tag{4.42}$$

*Comment:* The surprise is the appearance of the term $\frac{1}{\rho} F_\rho$, which is due to the fact that the basis vectors are not constant. Equation (4.42) can be written more concisely as

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho} \left[ \frac{\partial}{\partial \rho} (\rho F_\rho) + \frac{\partial F_\phi}{\partial \phi} \right]. \tag{4.43}$$

**Example 4.2:** Calculate the Laplacian $\nabla^2 f$ in polar coordinates in $\mathbb{R}^2$.

*Solution:* We use (4.13),

$$\nabla^2 f = \nabla \cdot (\nabla f) \tag{4.44}$$

Equation (4.36) implies that

$$\nabla f = \frac{\partial f}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi. \tag{4.45}$$

We apply (4.43) with $\mathbf{F} = \nabla f$, i.e.

$$F_\rho = \frac{\partial f}{\partial \rho}, \quad F_\phi = \frac{1}{\rho}\frac{\partial f}{\partial \phi}, \tag{4.46}$$

as follows from (4.45). Equation (4.44), in conjunction with (4.43) and (4.46) now gives

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} \qquad \square \tag{4.47}$$

In a similar fashion we can derive the expression for $\nabla\cdot\mathbf{F}$ in *cylindrical coordinates*:

$$\nabla\cdot\mathbf{F} = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho F_\rho) + \frac{\partial F_\phi}{\partial \phi}\right] + \frac{\partial F_z}{\partial z}, \tag{4.48}$$

and in *spherical coordinates*:

$$\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial(r^2 F_r)}{\partial r} + \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(\sin\theta F_\theta) + \frac{\partial F_\phi}{\partial \phi}\right). \tag{4.49}$$

The formulas for the Laplacian $\nabla^2 f = \nabla\cdot(\nabla f)$ are also useful in applications. For cylindrical coordinates it follows from (4.38) and (4.48) that

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}, \tag{4.50}$$

and for spherical coordinates it follows from (4.40) and (4.49) that

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}. \tag{4.51}$$

We leave the details as an exercise, (#26 & 27 in Problem Set 4).

## 4.2 Gauss' Theorem

In this section we state Gauss' theorem, give a proof subject to a simplifying assumption on the domain, and then discuss some applications in physics.

### 4.2.1 The theorem

Consider a bounded subset $\Omega$ of $\mathbb{R}^3$, whose boundary $\partial\Omega$ is a *single piecewise smooth oriented closed* surface. A solid sphere, a solid cube and a solid torus have this property. The region between two concentric spheres does not have this property because its boundary is the union of *two disjoint surfaces*. We note that a *closed surface* is one which has no boundary curve – it is the two-dimensional analogue of a *closed curve*, which has no boundary points (i.e. end points).

Gauss's theorem – also known as the Divergence theorem – relates the surface integral of a vector field over a surface $\partial\Omega$ to the triple integral of the divergence of $\mathbf{F}$ over the region $\Omega$ enclosed by $\partial\Omega$.

**Theorem 4.1** (Gauss' theorem):
Let $\Omega$ be a bounded subset of $\mathbb{R}^3$ whose boundary $\partial\Omega$ is a single piecewise smooth oriented closed surface. If the vector field $\mathbf{F}$ is of class $C^1$ on $\Omega \cup \partial\Omega$, then

$$\iiint_{\Omega} \nabla\cdot\mathbf{F}dV = \iint_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}dS,$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$.

In order to simplify the proof of the theorem we shall require $\Omega$ to be a *special domain*, defined by the following restriction:

*any line through an interior point of* $\Omega$
*intersects the boundary* $\partial\Omega$ *in two points.*

A solid sphere and a solid cube are both special domains, but a solid torus is not.

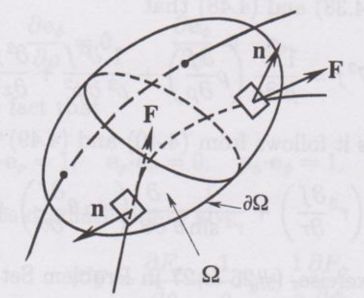



Figure 4.7: Gauss' theorem.

*Comment:*

If we project a special domain in the $z$-direction, as shown in Figure 4.8, we can describe it by inequalities of the form

$$f_\ell(x,y) \le z \le f_u(x,y),$$

where

$$(x,y) \in D_{xy}.$$

It follows that a triple integral over $\Omega$ can be written in the form

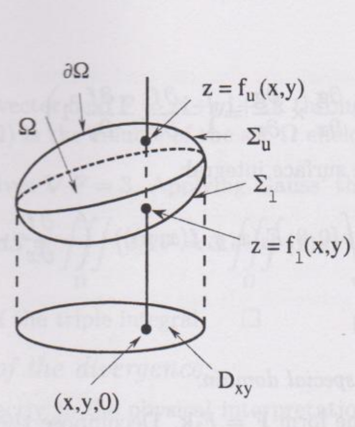


Figure 4.8: A special domain projected in the $z$-direction.

$$\iiint_{\Omega} h\,dV = \iint_{D_{xy}} \left( \int_{z=f_\ell(x,y)}^{f_u(x,y)} h\,dz \right) dx\,dy. \tag{4.52}$$

In addition we can represent the boundary $\partial\Omega$ as the union of an *upper surface*

$$\Sigma_u : z = f_u(x,y), \qquad (x,y) \in \mathcal{D}_{xy}, \tag{4.53}$$

and a lower surface

$$\Sigma_\ell : z = f_\ell(x,y), \qquad (x,y) \in \mathcal{D}_{xy}, \tag{4.54}$$

i.e.

$$\partial\Omega = \Sigma_u \cup \Sigma_\ell. \tag{4.55}$$

This decomposition forms the basis of the proof of Gauss' theorem. We shall also require an expression for the surface integral of a vector field over a surface of the form $\Sigma_u$ or $\Sigma_\ell$, which assumes a particularly simple form, given in the proposition to follow.

**Proposition 4.1:**

Consider a surface $\Sigma$ given by $z = f(x,y)$ with $(x,y) \in D_{xy}$, and with unit normal $\mathbf{n}$ oriented in the positive $z$-direction. Then for any vector field of the form $\mathbf{F} = F_3\mathbf{k}$,

$$\iint_{\Sigma} \mathbf{F} \cdot \mathbf{n} dS = \iint_{D_{xy}} F_3(x,y,f(x,y)) dx\,dy. \tag{4.56}$$

*Proof:*

We use the parametrization

$$\mathbf{x} = \mathbf{g}(x,y) = (x,y,f(x,y)),$$

which leads to

$$\frac{\partial \mathbf{g}}{\partial x} \times \frac{\partial \mathbf{g}}{\partial y} = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$$

(exercise). By definition of the surface integral:

$$\iint_{\Sigma} \mathbf{F} \cdot \mathbf{n} dS = \iint_{D_{xy}} (0, 0, F_3(x, y, f(x, y))) \cdot \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right) dx\, dy,$$

which simplifies to (4.56). $\square$

*Proof of Gauss' theorem for a special domain:*

Consider a vector field of the form $\mathbf{F} = F_3\mathbf{k}$. Decompose the boundary $\partial\Omega$ as in (4.55), noting that on $\Sigma_u$ the outward normal is in the *positive* $z$-direction while on $\Sigma_\ell$ it is in the *negative* $z$-direction. Thus by (4.56)

$$\begin{aligned}\iint_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} dS &= \iint_{\Sigma_u} \mathbf{F} \cdot \mathbf{n} dS + \iint_{\Sigma_\ell} \mathbf{F} \cdot \mathbf{n} dS \\ &= \iint_{D_{xy}} [F_3(x, y, f_u(x, y)) - F_3(x, y, f_\ell(x, y))]\, dx\, dy.\end{aligned}$$

On the other hand, using (4.52),

$$\begin{aligned}\iiint_{\Omega} \frac{\partial F_3}{\partial z} dV &= \iint_{D_{xy}} \left[\int_{z=f_\ell(x,y)}^{f_u(x,y)} \frac{\partial F_3}{\partial z} dz\right] dx\, dy \\ &= \iint_{D_{xy}} [F_3(x, y, f_u(x, y)) - F_3(x, y, f_\ell(x, y))]\, dx\, dy,\end{aligned}$$

after applying the second Fundamental Theorem of Calculus to the $z$-integral. We have thus shown that

$$\iiint_{\Omega} \frac{\partial F_3}{\partial z} dV = \iint_{\partial\Omega} (F_3\,\mathbf{k})\cdot\mathbf{n}\, dV. \tag{4.57}$$

By projecting in the $y$- and $x$-directions, one similarly obtains

$$\iiint_{\Omega} \frac{\partial F_2}{\partial y} dV = \iint_{\partial\Omega} (F_2\,\mathbf{j})\cdot\mathbf{n}\, dV, \tag{4.58}$$

$$\iiint_{\Omega} \frac{\partial F_1}{\partial x} dV = \iint_{\Omega} (F_1\,\mathbf{i})\cdot\mathbf{n}\, dV. \tag{4.59}$$

Summing (4.57)-(4.59) gives Gauss' theorem. $\square$

Here is a simple application of Gauss' theorem.

**Example 4.1:**

Show that the flux of the vector field $\mathbf{F} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ through any oriented closed surface $\partial\Omega$ equals $3V(\Omega)$, where $V(\Omega)$ is the volume of the set $\Omega$ enclosed by $\partial\Omega$.

*Solution:* Differentiation gives $\nabla\cdot\mathbf{F} = 3$. Applying Gauss' theorem to $\mathbf{F}$ gives

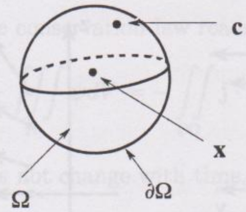
$$\iint_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_{\Omega} (3)\,dV = 3\iiint_{\Omega}(1)\,dV = 3V(\Omega),$$

by the usual interpretation of the triple integral. □

### *Physical interpretation of the divergence:*

Gauss' theorem leads directly to the physical interpretation of $\nabla\cdot\mathbf{F}$ at a given point $\mathbf{x}$. Apply Gauss' theorem to a sphere of radius $\varepsilon$ centred at $\mathbf{x}$:

$$\iiint_{\Omega(\varepsilon)} \nabla\cdot\mathbf{F}\,dV = \iint_{\partial\Omega(\varepsilon)} \mathbf{F}\cdot\mathbf{n}\,dS. \tag{4.60}$$




By the Mean Value Theorem for integrals there exists a point $\mathbf{c}\in\Omega$ such that

$$\iiint_{\Omega(\varepsilon)} (\nabla\cdot\mathbf{F})\,dV = [\nabla\cdot\mathbf{F}(\mathbf{c})]V(\varepsilon),$$

($\nabla\cdot\mathbf{F}$ is continuous since $\mathbf{F}$ is of class $C^1$), where $V(\varepsilon)$ is the volume of the sphere. By (4.60),

$$\nabla\cdot\mathbf{F}(\mathbf{c}) = \frac{1}{V(\varepsilon)}\iint_{\partial\Omega(\varepsilon)} \mathbf{F}\cdot\mathbf{n}\,dS.$$

Taking the limit as $\varepsilon\to 0^+$, so that $\mathbf{c}\to\mathbf{x}$, we get

$$\nabla\cdot\mathbf{F}(\mathbf{x}) = \lim_{\varepsilon\to 0^+}\frac{1}{V(\varepsilon)}\iint_{\partial\Omega(\varepsilon)} \mathbf{F}\cdot\mathbf{n}\,dS.$$

For $\varepsilon$ sufficiently close to 0 we can write the approximation

$$\nabla\cdot\mathbf{F}(\mathbf{x}) \approx \frac{1}{V(\varepsilon)}\iint_{\partial\Omega(\varepsilon)} \mathbf{F}\cdot\mathbf{n}\,dS. \tag{4.61}$$

Thus, *the divergence* $\nabla\cdot\mathbf{F}(x)$ *equals the flux per unit volume at* $\mathbf{x}$ *of the vector field* $\mathbf{F}$. $\square$

**Exercise 4.6:**

The figure shows four vector fields in $\mathbb{R}^3$ – the $x$-axis is out of the page and there is no dependence on $x$. For each vector field estimate geometrically whether the flux through the surface of a small cube is positive, negative or zero. Then equation (4.61) gives a prediction about $\nabla\cdot\mathbf{F}$. Verify your prediction by inventing a vector field for each picture and calculating $\nabla\cdot\mathbf{F}$.

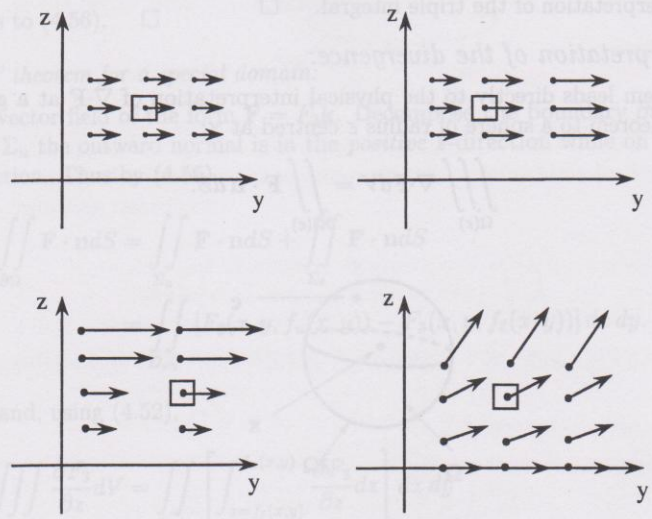



Figure 4.9: Four vector fields.

### 4.2.2 Conservation laws and PDEs

Gauss' theorem plays a fundamental role in deriving the partial differential equations (PDEs) that govern a variety of physical phenomena. We will illustrate how the theorem is used by considering the law of conservation for a physical quantity described by a density scalar and a flux density vector field (e.g. mass or heat energy).

Let $\psi(\mathbf{x}, t)$ be a $C^1$ scalar field that represents the *density* of some physical quantity $Q$ (amount of $Q$ *per unit volume.* It follows that the amount of $Q$ in a bounded subset $\Omega \subset \mathbb{R}^3$ is given by

$$\iiint_{\Omega} \psi dV.$$

Let $\mathbf{j}(\mathbf{x}, t)$ be a $C^1$ vector field that represents the *flux density* of the physical quantity $Q$ (rate of flow of $Q$ per unit area). It follows that the flux of $Q$ across the boundary surface

$\partial\Omega$ is

$$\iint\limits_{\partial\Omega} \mathbf{j}\cdot\mathbf{n}dS,$$

where $\mathbf{n}$ is the unit outward normal.

We now formulate the law of conservation in integral form for the physical quantity $Q$. Let $D\subset\mathbb{R}^3$ be the domain of $\psi$ and $\mathbf{j}$, and let $\Omega$ be an *arbitrary* bounded subset, as in the statement of Gauss' theorem. If the quantity $Q$ is neither created nor destroyed in $D$ (i.e. no sources or sinks), then conservation of $Q$ has the following form:

$$\left\{\begin{array}{c}\text{the rate at which the}\\ \text{amount of } Q \text{ in } \Omega\\ \text{increases}\end{array}\right\} = -\left\{\begin{array}{c}\text{the rate at which } Q\\ \text{leaves } \Omega \text{ across } \partial\Omega\end{array}\right\}.$$

The minus sign is needed because the amount of $Q$ in $\Omega$ *decreases* if the flux across $\partial\Omega$ is *positive.*

In terms of $\psi$ and $\mathbf{j}$ the above conservation law reads

$$\frac{d}{dt}\iiint\limits_{\Omega}\psi dV = -\iint\limits_{\partial\Omega}\mathbf{j}\cdot\mathbf{n}\,dS. \tag{4.62}$$

Since $\psi$ is of class $C^1$ and $\Omega$ does not change with time,

$$\frac{d}{dt}\iiint\limits_{\Omega}\psi dV = \iiint\limits_{\Omega}\frac{\partial\psi}{\partial t}dV.$$

In addition we can use Gauss' theorem to write the surface integral as a triple integral,

$$\iint\limits_{\partial\Omega}\mathbf{j}\cdot\mathbf{n}\,dS = \iiint\limits_{\Omega}\nabla\cdot\mathbf{j}\,dV.$$

Equation (4.62) then assumes the form

$$\iiint\limits_{\Omega}\left(\frac{\partial\psi}{\partial t}+\nabla\cdot\mathbf{j}\right)dV = 0. \tag{4.63}$$

We now invoke a famous lemma from analysis (that applies to any type of integral).

**Lemma 4.1:**

If the scalar field $h$ is continuous on $D\subset\mathbb{R}^3$ and

$$\iiint\limits_{\Omega} hdV = 0$$

for *any* subset of $\Omega \subset D$, then

$$h = 0 \quad \text{on} \quad D.$$

*Proof:*

Suppose $h(\mathbf{a}) > 0$ for some $\mathbf{a} \in D$. Since $h$ is continuous on $D, h(\mathbf{x}) > 0$ for all $\mathbf{x}$ in some neighbourhood $N$ of $\mathbf{a}$. Then $\iiint_N h dV > 0$, contradicting the hypothesis. Thus $h(\mathbf{x}) = 0$ for all $\mathbf{x} \in D$. $\square$

Since the integral in equation (4.63) is evaluated over an arbitrary subset $\Omega \in D$, the lemma implies that

$$\frac{\partial \psi}{\partial t} + \nabla \cdot \mathbf{j} = 0 \tag{4.64}$$

in $D$. This equation is the *differential form of the conservation law.*

We consider two special cases. In the first $\psi$ is mass density of a fluid and $\mathbf{j} = \rho \mathbf{v}$ is the mass flux density, where $\mathbf{v}$ is the velocity of the fluid. The conservation law (4.64) becomes

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{4.65}$$

which is called the *equation of continuity* for the fluid.

In the second case, we consider a conducting medium whose *temperature* $u(\mathbf{x}, t)$ is a function of position and time. The scalar field $\psi$ is the heat energy density given by

$$\psi = c\rho u,$$

where $c$ is the specific heat of the material (the amount of heat required to raise the temperature of unit mass by one degree), and $\rho$ is the density. The vector field $\mathbf{j}$ is the heat flux density, given by

$$\mathbf{j} = -k\nabla u,$$

where $k$ is the thermal conductivity (Fourier's law). The conservation law (4.64) assumes the form

$$\frac{\partial}{\partial t}(c\rho u) + \nabla \cdot (-k\nabla u) = 0. \tag{4.66}$$

We assume that the medium is homogeneous so that $k, c$ and $\rho$ are independent of position, and we also assume that they do not change with time. Recalling the definition of the Laplacian, $\nabla^2 u = \nabla \cdot (\nabla u)$ (see equation (4.13)), equation (4.66) assumes the form

$$\frac{\partial u}{\partial t} - a^2 \nabla^2 u = 0, \tag{4.67}$$

where $a^2 = \frac{k}{c\rho}$ is called the thermal diffusivity. Equation (4.67) is one of the fundamental equations of mathematical physics, and is called the *heat diffusion equation.*

The heat diffusion equation also governs other diffusive processes. If a chemical is dissolved in a solvent and the concentration is not uniform, then the chemical will diffuse (be transported) from regions of high concentration to regions of law concentration. If $C(\mathbf{x}, t)$ is the concentration, then the flux vector field will be given by *Fick's law*:

$$\mathbf{j} = -D\nabla C, \tag{4.68}$$

where $D$ is the diffusion coefficient. The conservation law (4.64), with $\psi = C$ gives

$$\frac{\partial C}{\partial t} - D\nabla^2 C = 0, \tag{4.69}$$

which has the same form as the heat diffusion equation.

### 4.2.3 The Generalized Divergence Theorem

Consider a subset $\Omega$ of $\mathbb{R}^3$ with boundary $\partial\Omega$, as in Gauss' theorem. Suppose that the vector field $\mathbf{F}$ is $C^1$ on $\Omega \cup \partial\Omega$ *except at one point* $\mathbf{a}$ *in the interior of* $\Omega$. In this situation Gauss' theorem cannot be applied. The theorem can, however, be generalized so as to apply in this case, as follows.

Surround the point $\mathbf{a}$ by a closed surface $\partial H$, lying entirely in $\Omega$ – a sufficiently small sphere centred at $\mathbf{a}$ will do. Remove the region $H$ inside $\partial H$ from $\Omega$. Then $\mathbf{F}$ is $C^1$ on the resulting set $\Omega - H$. In this situation Gauss' theorem holds in the following modified form:

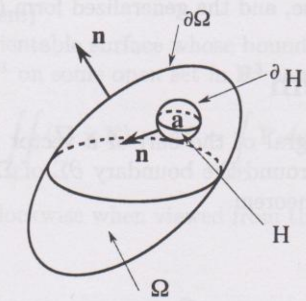



Figure 4.10: A set $\Omega \subset \mathbb{R}^3$ with a hole $H$ enclosing the point $\mathbf{a}$.

$$\iint_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}\, dS = \iiint_{\Omega - H} \nabla\cdot\mathbf{F} dV + \iint_{\partial H} \mathbf{F}\cdot\mathbf{n}\, dS, \tag{4.70}$$

where the normals on both $\partial\Omega$ and $H$ are outward.

Intuitively, you can think of the surface integral over $\partial H$ as compensating for the fact that you had to delete part of the original set $\Omega$, thereby creating a hole $H$, in order to obtain a set $\Omega - H$ on which $\mathbf{F}$ is $C^1$.

Equation (4.70) is proved by subdividing $\Omega - H$ into two regions on which the usual form of Gauss' theorem can be applied. The details are left as a challenging exercise.

We now briefly discuss an important application of the generalized divergence theorem (4.70).

**Gauss' Law:**

Let $\Omega$ be a bounded subset of $\mathbb{R}^3$ whose boundary $\partial\Omega$ is a single piecewise smooth oriented closed surface. Suppose that $(0,0,0) \notin \partial\Omega$. Then

$$\iint_{\partial\Omega} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS = \begin{cases} 4\pi & \text{if} \quad (0,0,0) \in \Omega \\ 0 & \text{if} \quad (0,0,0) \notin \Omega. \end{cases}$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $r = \| \mathbf{r} \|$ and $\mathbf{n}$ is the unit outward normal on $\partial\Omega$.

*Proof:*
The proof depends on two facts, which we leave as exercises:

i) $\nabla \cdot \left(\dfrac{\mathbf{r}}{r^3}\right) = 0$ for $r \neq 0$,

ii) $\displaystyle\iint_{\partial H_\varepsilon} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS = 4\pi,$
where $\partial H_\varepsilon$ is the sphere of radius $\varepsilon$ centred at $(0,0,0)$.

Apply Gauss' theorem in one case, and the generalized form (4.70) in the other case. $\square$

## 4.3 Stokes' Theorem

Stokes' theorem relates the integral of the curl of a vector field over a surface $\Sigma$ to the line integral of the vector field around the boundary $\partial\Sigma$ of $\Sigma$. The theorem is the natural generalization in $\mathbb{R}^3$ of Green's theorem.

### 4.3.1 The theorem

We motivate the form of the theorem by writing Green's theorem in a vector form in $\mathbb{R}^3$. The formula in Green's theorem is

$$\int_C \mathbf{F}\cdot d\mathbf{x} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy, \tag{4.71}$$

where $\mathbf{F} = (F_1, F_2)$. We can think of $\mathbf{F}$ as a vector field $\mathbf{F} = (F_1, F_2, 0)$ in $\mathbb{R}^3$ in which case we can write

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = (\nabla \times \mathbf{F})\cdot\mathbf{k}, \tag{4.72}$$

using the definition (4.6) of $\nabla \times \mathbf{F}$. We can also regard the plane region $D$ as a surface $\Sigma$ in $\mathbb{R}^3$ whose unit normal $\mathbf{n}$ is $\mathbf{n} = \mathbf{k}$. It follows that a surface integral over $\Sigma$ is simply a double integral over $D$:

$$\iint_\Sigma (\mathbf{G}\cdot\mathbf{n})dS = \iint_D (\mathbf{G}\cdot\mathbf{k})dxdy,$$

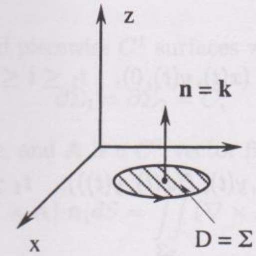



Figure 4.11: The plane region $D$ viewed as a surface $\Sigma$ in $\mathbb{R}^3$.

for any vector field $\mathbf{G}$. With these changes, (4.71) becomes

$$\int_C \mathbf{F}\cdot d\mathbf{x} = \iint_{\Sigma} (\nabla \times \mathbf{F})\cdot\mathbf{n}\, dS,$$

which gives the form of Stokes' theorem.

**Theorem 4.2:** (Stokes' theorem)
Let $\Sigma$ be a piecewise $C^1$ orientable surface whose boundary $\partial\Sigma$ is a simple piecewise $C^1$ closed curve. If $\mathbf{F}$ is of class $C^1$ on some open set in $\mathbb{R}^3$ containing $\Sigma \cup \partial\Sigma$, then

$$\iint_{\Sigma} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \int_{\partial\Sigma} \mathbf{F}\cdot d\mathbf{x}, \tag{4.73}$$

where $\partial\Sigma$ is oriented counter-clockwise when viewed from the side on which the unit normal $\mathbf{n}$ points.

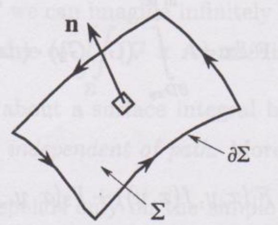



Figure 4.12: An oriented surface $\Sigma$ with piecewise $C^1$ boundary $\partial\Sigma$.

*Proof:* (outline)
Suppose that $\Sigma$ is given by

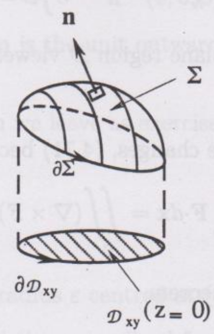
$$z = f(x,y), \quad (x,y) \in \mathcal{D}_{xy}.$$

Suppose that $\partial\mathcal{D}_{xy}$ is given by

$$\mathbf{h}(t) = (x(t), y(t), 0), \quad t_1 \le t \le t_2.$$

Then $\partial\Sigma$ is given by

$$\mathbf{g}(t) = (x(t), y(t), f(x(t), y(t))), \quad t_1 \le t \le t_2.$$




Step 1: Using $x, y$ as parameters on $\Sigma$, expand $\displaystyle\iint_{\Sigma} (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$ in terms of components, and write it as a double integral over $\mathcal{D}_{xy}$.

Step 2: Show that

$$\underbrace{\int_{\partial\Sigma}}_{\substack{\text{line integral}\\ \text{in } \mathbb{R}^3}} \mathbf{F}\cdot d\mathbf{x} = \underbrace{\int_{\partial\mathcal{D}_{xy}}}_{\substack{\text{line integral}\\ \text{in } \mathbb{R}^2}} (G_1, G_2)\cdot(dx, dy),$$

where

$$G_1(x,y) = F_1(x,y,f(x,y)) + F_3(x,y,f(x,y))\frac{\partial f}{\partial x},$$
$$G_2(x,y) = F_2(x,y,f(x,y)) + F_3(x,y,f(x,y))\frac{\partial f}{\partial y}.$$

Step 3: Apply Green's theorem to $\displaystyle\int_{\partial\mathcal{D}_{xy}} (G_1, G_2)\cdot(dx, dy)$, and compare the results to Step 1. □

**Exercise 4.7:** Do #13 in Problem Set 4. The common answer is 0.

**Corollary:**
If $\Sigma_1$ and $\Sigma_2$ are two oriented piecewise $C^1$ surfaces with

$$\partial\Sigma_1 = \partial\Sigma_2 = C,$$

where $C$ is a simple closed curve, and $\mathbf{A}$ is a $C^1$ vector field, then

$$\iint_{\Sigma_1} (\nabla \times \mathbf{A})\cdot\mathbf{n}_1 dS = \iint_{\Sigma_2} (\nabla \times \mathbf{A})\cdot\mathbf{n}_2 dS.$$

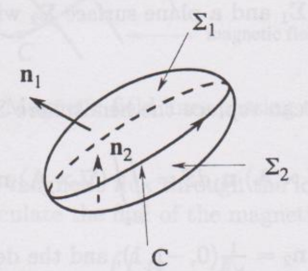



Figure 4.13: Two surfaces with the same boundary $C$.

*Proof:*
By Stokes' theorem both surface integrals equal $\displaystyle\int_C \mathbf{A}\cdot d\mathbf{x}$. □

*Comment:*
Given a simple closed curve $C$ we can imagine infinitely many surfaces $\Sigma$ with $\partial\Sigma = C$. The Corollary implies that the value of $\displaystyle\iint_\Sigma (\nabla \times \mathbf{A})\cdot\mathbf{n} dS$ is independent of which surface $\Sigma$ we choose. One can thus talk about a surface integral being *independent of surface*, in analogy with a line integral being *independent of path*. Moreover, since $\displaystyle\iint_\Sigma (\nabla \times \mathbf{A})\cdot\mathbf{n} dS$ is independent of the surface and depends only on the simple closed curve $C = \partial\Sigma$, one can talk about *the flux of the vector field* $\mathbf{F} = \nabla \times \mathbf{A}$ *through the simple closed curve* $C$.

**Example 4.2:** Calculate the flux of the vector field $\mathbf{F} = \nabla \times \mathbf{A}$, where

$$\mathbf{A} = (2z - y, x - z, y - x),$$

through the hemisphere

$$\Sigma_1 : x^2 + y^2 + z^2 = a^2,$$

with $z - y \geq 0$.

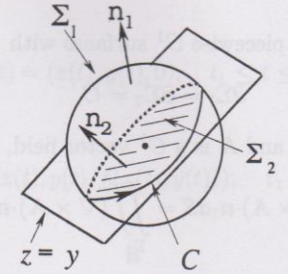



Figure 4.14: A hemisphere $\Sigma_1$ and a plane surface $\Sigma_2$ with the same boundary $C$.

*Solution:* By the corollary we can replace the hemisphere $\Sigma_1$ by the plane surface $\Sigma_2$, i.e. the equatorial disc:

$$\iint_{\Sigma_1} (\nabla \times \mathbf{A})\cdot\mathbf{n}_1 dS = \iint_{\Sigma_2} (\nabla \times \mathbf{A})\cdot\mathbf{n}_2 dS.$$

The unit normal to the plane is $\mathbf{n}_2 = \frac{1}{\sqrt{2}}(0, -1, 1)$, and the definition of $\nabla \times \mathbf{A}$ gives

$$\nabla \times \mathbf{A} = (2, 3, 2),$$

and hence

$$(\nabla \times \mathbf{A})\cdot\mathbf{n}_2 = -\tfrac{1}{\sqrt{2}}.$$

Thus

$$\begin{aligned}\iint_{\Sigma_1} (\nabla \times \mathbf{A})\cdot\mathbf{n}_1 d &= \iint_{\Sigma_2} \left(-\tfrac{1}{\sqrt{2}}\right) dS \\ &= -\tfrac{1}{\sqrt{2}}\pi a^2,\end{aligned}$$

since the disc $\Sigma_2$ has radius $a$. □

### 4.3.2 Faraday's law

As an illustration of the use of Stokes' theorem in deriving field equations in physics, we consider Faraday's law in the theory of electromagnetism, which states:

"*The voltage change around a loop*[1] *is proportional to the negative of the time rate of change of the magnetic flux through the loop.*"

The change in voltage $\Delta V$ across a curve segment $\Delta\mathbf{x}$ is approximated by

$$\Delta V \approx \mathbf{E}\cdot\Delta\mathbf{x}.$$

[1] "loop" is synonymous with "simple closed curve".

Thus, the change in voltage around the loop is

$$\int_C \mathbf{E}\cdot d\mathbf{x}.$$

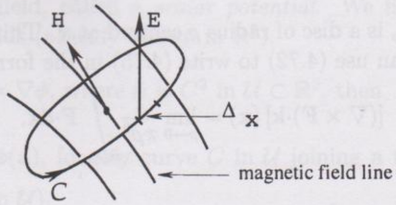



Figure 4.15: Magnetic field lines passing through a loop.

To write an expression for the magnetic flux through the loop we imagine a fixed surface $\Sigma$ whose boundary is $C$, and calculate the flux of the magnetic field $\mathbf{H}$ through $\Sigma$:

$$\iint_\Sigma \mathbf{H}\cdot\mathbf{n}\,dS.$$

*Note:* In calculating the flux in this way, we are tacitly assuming that the flux integral is independent of the surface $\Sigma$. That a magnetic field has this property is in fact a consequence of the fact that the magnetic field satisfies $\nabla\cdot\mathbf{H} = 0$. This surface-independence property of $\mathbf{H}$ will be established in the next section.

Faraday's law in integral form thus reads

$$\frac{d}{dt}\left(\iint_\Sigma \mathbf{H}\cdot\mathbf{n}dS\right) = -c\int_C \mathbf{E}\cdot d\mathbf{x}, \tag{4.74}$$

where $c$ is a constant.

We now use Stokes' theorem to write the line integral in (4.74) as a surface integral. In addition, since $\Sigma$ does not change with time, we can take the $t$-derivative inside the surface integral, giving

$$\iint_\Sigma \left(\frac{\partial\mathbf{H}}{\partial t} + c\nabla\times\mathbf{E}\right)\cdot\mathbf{n}\,dS = 0.$$

Since $\Sigma$ is an arbitrary surface in the domain $U$ in which we are working, and the integrand is continuous by assumption, it follows that

$$\frac{\partial\mathbf{H}}{\partial t} + c\nabla\times\mathbf{E} = \mathbf{0},$$

one of Maxwell's equations (see Section 4.1.1).

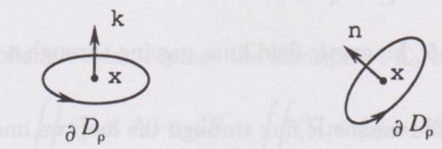
### 4.3.3 The physical interpretation of $\nabla \times \mathbf{F}$

In Section 2.5 (see equation (2.53)) we showed that

$$\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)(\mathbf{x}) = \lim_{\rho \to 0} \frac{1}{\pi \rho^2} \int_{\partial D_\rho} \mathbf{F} \cdot d\mathbf{x}, \tag{4.75}$$

where $\mathbf{F} = (F_1, F_2)$, and $D_\rho$ is a disc of radius $\rho$ centred at $\mathbf{x}$. Thinking of $\mathbf{F}$ as a vector field in $\mathbb{R}^3$, $\mathbf{F} = (F_1, F_2, 0)$ we can use (4.72) to write (4.75) in the form

$$[(\nabla \times \mathbf{F}) \cdot \mathbf{k}](\mathbf{x}) = \lim_{\rho \to 0} \frac{1}{\pi \rho^2} \int_{\partial D_\rho} \mathbf{F} \cdot d\mathbf{x}.$$




One can now use Stokes' theorem and the Mean Value Theorem for Integrals to generalize this result to a circle of radius $\rho$ lying in an inclined plane in $\mathbb{R}^3$:

$$[(\nabla \times \mathbf{F}) \cdot \mathbf{n}](\mathbf{x}) = \lim_{\rho \to 0} \frac{1}{\pi \rho^2} \int_{\partial D_\rho} \mathbf{F} \cdot d\mathbf{x},$$

leading to the following interpretation:

> The component of the curl $\nabla \times \mathbf{F}$ in the direction $\mathbf{n}$, is the circulation per unit area around a circle lying in the plane orthogonal to $\mathbf{n}$. Moreover, the direction of $\nabla \times \mathbf{F}$ gives the direction of $\mathbf{n}$ for which the circulation is a maximum.

Thus, we see that the curl of a vector field on $\mathbb{R}^3$ is the appropriate generalization of the vorticity $\Omega(\mathbf{F})$ for a vector field in $\mathbb{R}^2$, i.e. $\nabla \times \mathbf{F}$ describes the rotational aspects of the vector field $\mathbf{F}$ (see Section 2.5). In fluid dynamics, given a $C^1$ velocity field $\mathbf{v}$, the vector field

$$\mathbf{w} = \nabla \times \mathbf{v}$$

is called the *vorticity field* of the fluid. If $\mathbf{w} = \mathbf{0}$ the fluid is said to be *irrotational*.

## 4.4 The Potential Theorems

There are two classes of vector fields that are special from a mathematical point of view and important from a physical point of view. The first is the class of *irrotational* vector fields and the second is the class of *divergence-free* vector fields. The mathematical bond between them is that, *subject to a restriction on the domain*, both classes are derivable from a potential, a *scalar potential* in the first case ($\mathbf{F} = \nabla \phi$) and a *vector potential* in the second case ($\mathbf{F} = \nabla \times \mathbf{A}$).

### 4.4.1 Irrotational vector fields

Consider a *gradient vector field* (a.k.a. a *conservative* vector field) in $\mathbb{R}^3$

$$\mathbf{F} = \nabla\phi, \tag{4.76}$$

where $\phi$ is a $C^2$ scalar field, called a *scalar potential*. We first summarize the principal properties of gradient fields (already known in $\mathbb{R}^2$).

**Proposition 4.1:** If $\mathbf{F} = \nabla\phi$, where $\phi$ is $C^2$ in $\mathcal{U} \subset \mathbb{R}^3$, then

i) $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{x} = \phi(\mathbf{b}) - \phi(\mathbf{a})$, for any curve $C$ in $\mathcal{U}$ joining $\mathbf{a}$ to $\mathbf{b}$ (i.e. the line integral is path-independent in $\mathcal{U}$),

ii) $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{x} = 0$ for any simple closed curve in $\mathcal{U}$,

iii) $\nabla \times \mathbf{F} = \mathbf{0}$ in $\mathcal{U}$.

*Proof:*

i) is the second Fundamental Theorem for line integrals, valid in $\mathbb{R}^n$.

ii) is an immediate consequence of i), by choosing $\mathbf{b} = \mathbf{a}$.

iii) is a consequence of the "zero identity" $Z_1$. □

The key question is: how does one determine whether a given vector field $\mathbf{F}$ is a gradient field, i.e. has a scalar potential $\phi$?

In view of Proposition 4.1 iii) one might conjecture that if $\mathbf{F}$ is irrotational ($\nabla \times \mathbf{F} = \mathbf{0}$), then $\mathbf{F} = \nabla\phi$ for some $C^2$ scalar field. As in $\mathbb{R}^2$ (see Theorem 2.4) one needs a restriction on the domain $\mathcal{U} \subset \mathbb{R}^3$ in question. The following theorem generalizes Theorem 2.4.

**Theorem 4.3** (scalar potential):

If $\mathbf{F}$ is of class $C^1$ and $\nabla \times \mathbf{F} = \mathbf{0}$ in $\mathcal{U} \subset \mathbb{R}^3$, and $\mathcal{U}$ is simply-connected, then there exists a single-valued $C^2$ scalar field $\phi$ such that $\mathbf{F} = \nabla\phi$ in $\mathcal{U}$.

*Proof:*

Let $C$ be a simple closed curve in $\mathcal{U}$. Since $\mathcal{U}$ is simply-connected it is plausible[2] that there exists an oriented $C^1$ surface $\Sigma$ in $\mathcal{U}$ such that $C = \partial\Sigma$. Apply Stokes' theorem to $\mathbf{F}$ on $\Sigma$ and $C$:

$$\int_C \mathbf{F}\cdot d\mathbf{x} = \iint_\Sigma (\nabla \times \mathbf{F})\cdot\mathbf{n}\, dS = 0.$$

[2] See the comment at the end of the proof.

Since $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{x} = 0$ for any simple closed curve in $\mathcal{U}$, it follows that the line integral is independent of path in $\mathcal{U}$ and hence that there exists a potential function $\phi$ in $\mathcal{U}$ (see Proposition 2.3, Section 2.4.2). $\square$

*Comment:*

It is difficult to prove that for any simple closed curve $C$ in a simply-connected set $\mathcal{U} \subset \mathbb{R}^3$, there exists a surface $\Sigma$ such that $C = \partial\Sigma$. We followed Marsden & Tromba (page 520) in relying on geometrical intuition. Some authors (e.g. Corwin & Szczarba, page 343), *define* a simply-connected set $\mathcal{U}$ in $\mathbb{R}^3$ to be a set such that for any simple closed curve $C$ there is a surface $\Sigma$ with $C = \partial\Sigma$, thereby side-stepping the problem. If one wants to avoid this problem completely, one can give a direct proof of Theorem 4.3 by defining a potential $\phi$ as a line integral along a specific path and explicitly verifying that $\nabla\phi = \mathbf{F}$ (see Davis & Snider, page 206). We also refer to Flanigan & Kazdan for a simple proof of this theorem in $\mathbb{R}^n$ (see the final conclusion Theorem 10.42, proved on page 574). $\square$

Here is a classical counter-example to show that the requirement that $\mathcal{U}$ be simply-connected is essential.

**Example 4.3:**

Consider the vector field

$$\mathbf{F} = \left( \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right)$$

on the subset

$$\mathcal{U} = \mathbb{R}^3 - \{(x,y,z) \,|\, x = y = 0,\ z \in \mathbb{R}\}.$$

Then $\mathbf{F}$ is $C^1$ on $\mathcal{U}$, and it follows from the definition (4.6) of $\nabla \times \mathbf{F}$ that $\nabla \times \mathbf{F} = \mathbf{0}$ (do it!). Since a circle encircling the $z$-axis cannot be shrunk to a point in $\mathcal{U}$ the set $\mathcal{U}$ is *not* simply-connected.[3] Thus theorem 4.3 is not applicable, and leaves open the question as to whether a potential exists in $\mathcal{U}$. This question can be answered using Proposition 4.1 ii). A straight-forward calculation shows that for the circle $C : x^2 + y^2 = b^2,\ z = 0$,

$$\int_C \mathbf{F}\cdot d\mathbf{x} = 2\pi.$$

If $\mathbf{F} = \nabla\phi$, this integral would be zero. Hence a potential $\phi$ does not exist in $\mathcal{U}$. $\square$

**Exercise 4.8:**

Show that the vector field

$$\mathbf{F} = (y+z, z+x, x+y)$$

is a gradient field on $\mathbb{R}^3$, and find a potential $\phi$.

*Answer:* $\phi = xy + yz + zx$. $\square$

[3]Equivalently, for such a circle, there is no surface $\Sigma$ in $\mathcal{U}$ such that $C = \partial\Sigma$.

We now give a simple application of these ideas to fluid dynamics.

*An irrotational and incompressible fluid:*

Consider a vector field $\mathbf{v}(\mathbf{x},t)$ in $\mathbb{R}^3$ which represents the *velocity* of a fluid, and a scalar field $\rho(\mathbf{x},t)$ which represent its *density*. We have seen that conservation of mass leads to the equation of continuity (4.65):

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0. \tag{4.77}$$

We assume that the fluid is *incompressible* i.e. that its density does not depend on time or on position ($\rho =$ constant). Then (4.77) reduces to

$$\nabla\cdot\mathbf{v} = 0. \tag{4.78}$$

We now assume that the motion is *steady state* i.e. $\mathbf{v}$ is independent of time, and *irrotational*, $\nabla\times\mathbf{v} = \mathbf{0}$, in $\mathbb{R}^3$. By Theorem 4.3 there exists a potential $\phi$:

$$\mathbf{v} = \nabla\phi. \tag{4.79}$$

Substituting (4.79) in (4.78) gives $\nabla\cdot(\nabla\phi) = 0$, i.e.

$$\nabla^2\phi = 0,$$

where $\nabla^2$ is the Laplacian (see (4.13)). Thus, *the velocity potential $\phi$ of an incompressible fluid in steady state motion with zero vorticity satisfies Laplace's equation $\nabla^2\phi = 0$.* $\square$

### 4.4.2 Divergence-free vector fields

Consider a vector field $\mathbf{F}$ defined by

$$\mathbf{F} = \nabla\times\mathbf{A},$$

where $\mathbf{A}$ is a $C^2$ vector field, called a *vector potential* for $\mathbf{F}$. We first summarize the principal properties of such vector fields.

**Proposition 4.2:**
If $\mathbf{F} = \nabla\times\mathbf{A}$, where $\mathbf{A}$ is $C^2$ in $\mathcal{U}\subset\mathbb{R}^3$, then

i) $\displaystyle\iint_\Sigma \mathbf{F}\cdot\mathbf{n}dS = \int_C \mathbf{A}\cdot d\mathbf{x}$,
where $\Sigma$ is *any* surface such that $C = \partial\Sigma$, (i.e. the surface integral is surface-independent in $\mathcal{U}$).

ii) $\displaystyle\iint_\Sigma \mathbf{F}\cdot\mathbf{n}dS = 0$ for any closed surface in $\mathcal{U}$.

iii) $\nabla\cdot\mathbf{F} = 0$.

*Proof:*

i) is simply the statement of Stokes' Theorem.

ii) follows from Stokes' Theorem by subdividing the closed surface $\Sigma$ into two surfaces $\Sigma_1$ and $\Sigma_2$, with common boundary $C$ (see Figure 4.13).

iii) is a consequence of the zero identity $Z_2$ (see (4.12)).

The key question is: how does one determine whether a given vector field $\mathbf{F}$ has a vector potential $\mathbf{A}$ ($\mathbf{F} = \nabla \times \mathbf{A}$)?

In view of Proposition 4.2, it is natural to conjecture that if $\mathbf{F}$ is divergence-free ($\nabla \cdot \mathbf{F} = 0$), then there will exist a vector potential $\mathbf{A}$. This conjecture is in fact true, provided that we impose a restriction on the set $\mathcal{U}$.

**Definition:**

A subset $\mathcal{U} \subset \mathbb{R}^3$ (or $\mathbb{R}^2$) is *star-shaped* means that there is a point $P \in \mathcal{U}$ such that for any point $Q \in \mathcal{U}$ the line segment $PQ$ is contained in $\mathcal{U}$ (see for example, Davis & Snider, page 159).

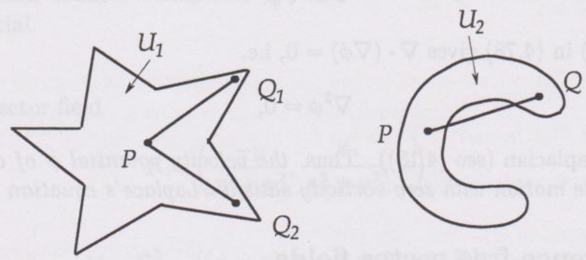



Figure 4.16: $\mathcal{U}_1$ is star-shaped, but $\mathcal{U}_2$ is not.

*Comment:*

i) The set $\mathcal{U} = \mathbb{R}^3 - \{(0,0,0)\}$ is *not* star-shaped, but is simply-connected. In fact if $\mathcal{U}$ is star-shaped, then $\mathcal{U}$ is simply-connected.

ii) If $\mathcal{U} \subset \mathbb{R}^3$ is star-shaped and $\Sigma$ is a closed surface in $\mathcal{U}$, then the interior of $\Sigma$ lies in $\mathcal{U}$.

**Theorem 4.4** (vector potential):

If $\mathbf{F}$ is of class $C^1$ and $\nabla \cdot \mathbf{F} = 0$ in $\mathcal{U} \subset \mathbb{R}^3$, and $\mathcal{U}$ is star-shaped, then there exists a vector field $\mathbf{A}$ such that $\mathbf{F} = \nabla \times \mathbf{A}$ in $\mathcal{U}$.

*Proof:*

We refer to Davis & Snider (pages 214-5). $\square$

**Example 4.4:**

Any constant vector field $\mathbf{B}$ satisfies $\nabla\cdot\mathbf{B} = 0$, and hence has a vector potential $\mathbf{A}$ in any star-shaped set $\mathcal{U}$. The identity

$$\nabla \times (\mathbf{B} \times \mathbf{r}) = 2\mathbf{B}, \tag{4.80}$$

where $\mathbf{B}$ is a constant vector field and $\mathbf{r} = (x, y, z)$ (see Exercise 1 in Section 4.1.1) shows that

$$\mathbf{A} = \tfrac{1}{2}\mathbf{B} \times \mathbf{r}$$

is a vector potential[4] for $\mathbf{B}$ i.e. $\mathbf{B} = \nabla \times \mathbf{A}$. $\square$

**Exercise 4.9:**

Show that the vector field $\mathbf{F} = (y, z, x)$ is divergence-free in $\mathbb{R}^3$, and find a vector potential $\mathbf{A}$.

*Answer:* $\mathbf{A} = \frac{1}{2}(z^2, x^2, y^2)$ $\square$

**Exercise 4.10:**

Show that any vector field of the form

$$\mathbf{F} = (\nabla f) \times \mathbf{B},$$

where $\mathbf{B}$ is a constant vector field and $f$ is a $C^2$ scalar field, is divergence-free on $\mathbb{R}^3$, and find a vector potential $\mathbf{A}$.

*Answer:* $\mathbf{A} = f\mathbf{B}$ (see identities $D_3$ and $C_2$ on/near page 84).

As a physical example of Theorem 4.4 we note that a magnetic field $\mathbf{H}$ is divergence-free ($\nabla\cdot\mathbf{H} = 0$), by Maxwell's equations. Hence *any magnetic field in a star-shaped set* $\mathcal{U} \subset \mathbb{R}^3$ *has a vector potential,* $\mathbf{H} = \nabla \times \mathbf{A}$. It follows that the magnetic flux through a surface $\Sigma$ bounded by a simple closed curve $C = \partial\Sigma$ is independent of $\Sigma$ (see the comment after the corollary to Stokes' theorem). Thus given a simple closed curve $C$, one can talk about *the magnetic flux through* $C$ (see the discussion of Faraday's law in Section 4.3.2). $\square$

*Note:* Because of the connection with magnetic fields, a vector field that satisfies $\nabla\cdot\mathbf{H} = 0$ is also called a *solenoidal field.*

We finally give a classical counter-example for Theorem 4.4, to show that the requirement that $\mathcal{U}$ be star-shaped is essential.

**Example 4.5:**

Consider the inverse square law vector field

$$\mathbf{F} = \frac{\mathbf{r}}{r^3},$$

with $\mathbf{r} = (x, y, z)$ and $r = \|\mathbf{r}\|$. Then $\mathbf{F}$ is $C^1$ on the set $\mathcal{U} = \mathbb{R}^3 - \{(0, 0, 0)\}$ and a standard calculation (do it!) shows that $\nabla\cdot\mathbf{F} = 0$. However, $\mathcal{U}$ is not star-shaped (missing a point!).

[4] Since (4.80) holds in any subset, $\mathbf{A}$ is a vector potential in any subset of $\mathbb{R}^3$.

Thus Theorem 4.4 is not applicable and leaves open the question of whether **F** has a vector potential **A** in $\mathcal{U}$. This question can be answered using Proposition 4.2 ii). We have seen that if $\Sigma$ is any sphere centred on $(0,0,0)$, then

$$\iint\limits_{\Sigma} \mathbf{F} \cdot \mathbf{n}\, dS = 4\pi,$$

(see Gauss' Law in Section 4.2.3). If $\mathbf{F} = \nabla \times \mathbf{A}$, then this surface integral would be zero. Hence a vector potential does not exist. □

# Chapter 5

# Fourier Series and Fourier Transforms

In the Chapter we give an introduction to the branch of mathematics called *Fourier analysis.* The essential idea is that one can write a given function $f(x)$ as a "sum" of sinusoidal functions $\sin \omega x$ and $\cos \omega x$ (or more concisely, $e^{i\omega x}$, thinking of Euler's formula). If the given function is periodic, the "sum" is an infinite series, the *Fourier series* of $f$, while if $f$ is not periodic, the sum is an improper integral, the *Fourier integral* of $f$. The Fourier series is obtained by calculating the *Fourier coefficients* of $f$, and the Fourier integral, by calculating the *Fourier transform* of $f$. These are the key concepts that we shall discuss in what follows.

Historically, Fourier analysis first gained prominence in the early part of the nineteenth century, through the work of Joseph Fourier concerning the diffusion of heat.[1] Earlier in this course we showed that this process is described by a partial differential equation (PDE), the so-called diffusion equation (see equation (4.67)). Fourier showed that the solutions of this PDE, subject to appropriate boundary and initial conditions, could be written as Fourier series. This development exemplifies one area of application of Fourier analysis, namely the solution of problems involving linear PDEs.[2]

In the twentieth century, Fourier analysis found application in a totally different area, namely signal processing. The idea is that the Fourier coefficients or the Fourier transform of a signal $f(t)$, where $t$ is time, represent the analysis of the signal into its constituent frequencies. This representation is of fundamental importance in connection with the problem of sampling a continuous signal in order to create a discretized version of it, which forms the theoretical foundation for technological developments such as CDs.

## 5.1 Fourier Series

In connection with his work on the diffusion of heat Fourier argued that any[3] function of period $2\pi$, even discontinuous ones, can be written as the sum of a series of sine and cosine

[1] Fourier's book on this subject, "Théorie Analytique de la Chaleur", was published in 1822, but he began his work in 1804. We refer to Körner 1988, pages 478-480, for a short biographical essay on Fourier.

[2] This topic forms a major part of AMATH 353.

[3] We know now that the function has to satisfy certain restrictions.

functions of the form:

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \tag{5.1}$$

where $a_0, a_n$ and $b_n$ are constants. The series (5.1), with the coefficients calculated in an appropriate way (see Section 5.1.1), is called *the Fourier series of the function f*.

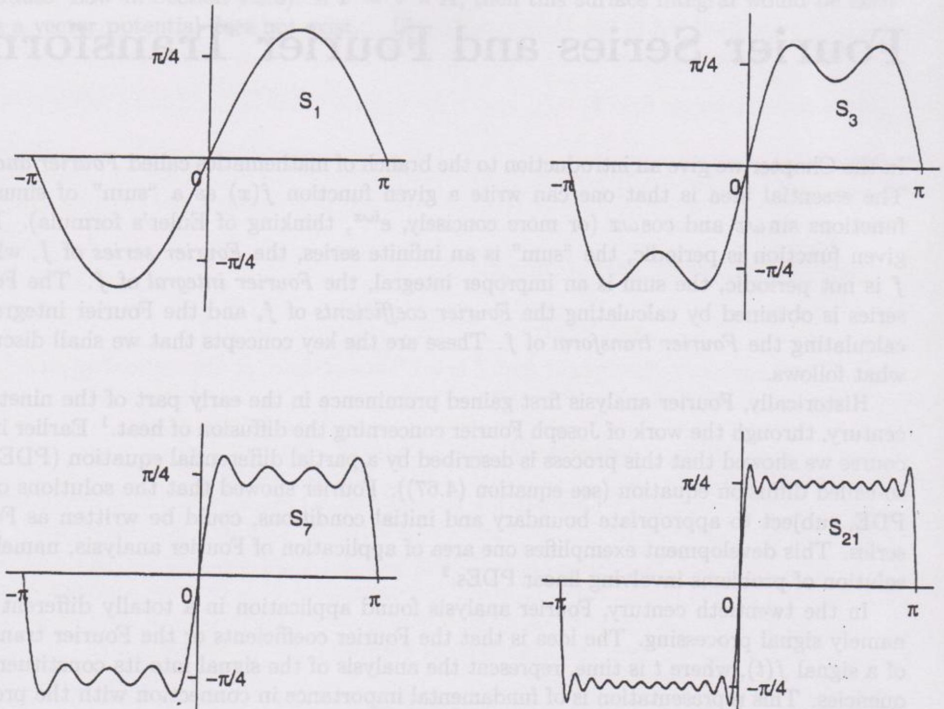



Figure 5.1:

Partial sums $S_1, S_3, S_7$ and $S_{21}$ of the Fourier series

$$\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots$$

of the function $f(x) = \begin{cases} \frac{\pi}{4}, & 0 < x < \pi \\ -\frac{\pi}{4}, & -\pi < x < 0. \end{cases}$

A remarkable fact about Fourier series is that *the sum function $f(x)$ is not necessarily continuous* even though the terms in the series are continuous. This behaviour is in strong contrast to the case of Taylor series, namely

$$f(x) = \sum_{n=0}^{\infty} a_n (x-b)^n,$$

for which the sum function has derivatives of all orders. Both Fourier series and Taylor series, although differing significantly in their properties, provide an *approximation of the given function by a finite sum of simpler terms*, if one truncates the series after $n$ terms. In Figure 5.1 we show a succession of partial sums for a Fourier series of a discontinuous function. Observe that the accuracy of the approximation increases as the number of terms increases.

The goal of this section is to give the reader a working knowledge of Fourier series. We first show how to find the Fourier coefficients of a given function, using symmetry to simplify the calculations where possible. We then discuss (but do not prove) a convergence theorem which enables one to find the sum of a Fourier series.

### 5.1.1 Calculating Fourier coefficients

We begin by assuming that the given function $f$ on the interval $-\pi < x < \pi$ can be written as the sum of a trigonometric series

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{5.2}$$

Whereas the coefficients of a Taylor series are determined by the *derivatives* of the function $f$, it turns out that the coefficients in a Fourier series can be expressed as *integrals*.

So we have to assume that the given function $f$ is *integrable* i.e. that its Riemann integral exists on the interval $-\pi \leq x \leq \pi$. To proceed we need certain trigonometric integrals:

$$\int_{-\pi}^{\pi} \cos mx \, \cos nx \, dx = \begin{cases} 0 & \text{if} \quad m \neq n \\ \pi & \text{if} \quad m = n \neq 0 \\ 2\pi & \text{if} \quad m = n = 0 \end{cases} \tag{5.3}$$

$$\int_{-\pi}^{\pi} \cos mx \, \sin \, nx \, dx = 0, \tag{5.4}$$

$$\int_{-\pi}^{\pi} \sin mx \, \sin nx \, dx = \begin{cases} 0, & \text{if} \quad m \neq n \\ \pi, & \text{if} \quad m = n \neq 0, \end{cases} \tag{5.5}$$

where $m$ and $n$ are non-negative integers.

These formulas can be verified by writing the products of the trigonometric functions as sums, e.g.,

$$\cos(m+n)x = \cos mx \, \cos nx - \sin mx \, \sin nx$$
$$\cos(m-n)x = \cos mx \, \cos nx + \sin mx \, \sin nx,$$

giving

$$\cos mx \, \cos nx = \tfrac{1}{2}[\cos(m+n)x + \cos(m-n)x],$$

etc.

**Exercise 5.1:**

Verify the formulae (5.3)-(5.5).

To find the coefficient $a_0$ in (5.2) we integrate (5.2) from $-\pi$ to $\pi$:

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} \tfrac{1}{2}a_0 dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)dx. \tag{5.6}$$

We now make the assumption (which we shall justify in Section 5.3.2) that we can integrate the series term-by-term i.e.

$$\int_{-\pi}^{\pi}\left[\sum_{1}^{\infty}(\quad)\right]dx = \sum_{1}^{\infty}\left[\int_{-\pi}^{\pi}(\quad)dx\right].$$

Since $\cos nx$ and $\sin nx$ are $2\pi$ periodic, the integral of the $n^{\text{th}}$ term of the series in (5.6) equals 0. Equation (5.6) thus gives

$$\int_{-\pi}^{\pi} f(x)dx = \pi a_0, \tag{5.7}$$

a formula for $a_0$.

We next multiply (5.2) by $\cos mx$ and integrate from $-\pi$ to $\pi$. Again assuming we can integrate the series term-by-term we get

$$\int_{-\pi}^{\pi} f(x)\cos mx \, dx = \int_{-\pi}^{\pi} \tfrac{1}{2}a_0 \cos mx \, dx$$
$$+ \sum_{n=1}^{\infty}\left[a_n \int_{-\pi}^{\pi} \cos mx \cos nx \, dx + b_n \int_{-\pi}^{\pi} \cos mx \sin nx \, dx\right].$$

On using (5.3) and (5.4) this equation simplifies to

$$\int_{-\pi}^{\pi} f(x)\cos mx \, dx = \pi a_m, \tag{5.8}$$

giving a formula for $a_m, m = 1, 2, \ldots$.

Finally, we multiply (5.2) by $\sin nx$ and integrate from $-\pi$ to $\pi$. Again assuming we can integrate the series term-by-term we get

$$\int_{-\pi}^{\pi} f(x)\sin mx\,dx = \int_{-\pi}^{\pi} \tfrac{1}{2}a_0 \sin mx\,dx$$
$$+\sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\sin mx\cos nx\,dx + b_n\int_{-\pi}^{\pi}\sin mx\sin nx\,dx\right].$$

On using (5.4) and (5.5) this equation simplifies to

$$\int_{-\pi}^{\pi} f(x)\sin mx\,dx = \pi b_m, \tag{5.9}$$

giving a formula for $b_m, m = 1, 2, \ldots$ .

We now summarize the results given by equations (5.7)-(5.9):

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \tag{5.10}$$

for $n = 0, 1, 2, \ldots$, and

$$b_n = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \tag{5.11}$$

for $n = 1, 2, 3, \ldots$. *Equations* (5.10) *and* (5.11) *are the formula for the Fourier coefficients of an integrable function* $f$.

**Example 5.1:** Calculate the Fourier series for the function

$$f(x) = \tfrac{1}{2}(\pi - |\,x\,|), \tag{5.12}$$

for $-\pi < x < \pi$.

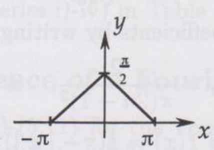



Figure 5.2: Graph of the function (5.12).

*Solution:* We begin by observing that $f(x)$ is an even function. It thus follows immediately from (5.11) that

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = 0,$$

since $f(x)\sin(nx)$ is an odd function[4] and

$$\int_{-\pi}^{\pi} (\quad) dx = 0$$

for any odd function.

By (5.10)

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\pi - \mid x \mid) \cos nx \, dx.$$

Observe that $(\pi - \mid x \mid)\cos nx$ is an even function[4] since both $\pi - \mid x \mid$ and $\cos nx$ are both even. For an even function

$$\int_{-\pi}^{\pi} (\quad) dx = 2\int_{0}^{\pi} (\quad) dx.$$

Thus the formula for $a_n$ can be written

$$a_n = \tfrac{1}{\pi}\int_0^{\pi} (\pi - x)\cos nx \, dx,$$

since $\mid x \mid = x$ on the interval $0 \le x \le \pi$. For $n = 0$ we obtain directly

$$a_0 = \tfrac{1}{\pi}\int_0^{\pi} (\pi - x) dx = \tfrac{1}{2}\pi.$$

For $n > 0$ we integrate by parts, obtaining

$$\begin{aligned} a_n &= \tfrac{1}{\pi}\left[ (\pi - x)\tfrac{1}{n}\sin nx \Big|_0^{\pi} - \int_0^{\pi} (-1)\tfrac{1}{n}\sin nx \, dx \right] \\ &= -\frac{1}{\pi n^2}\cos nx \Big|_0^{\pi} \\ &= \frac{1}{\pi n^2}(1 - \cos n\pi) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{2}{\pi n^2} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

It is more convenient to relabel the coefficients by writing

$$a_{2n} = 0, \quad a_{2n-1} = \frac{2}{\pi(2n-1)^2}, \quad n = 1, 2, \ldots .$$

Thus the Fourier series of the function $f(x) = \frac{1}{2}(\pi - \mid x \mid)$ is

$$\tfrac{1}{4}\pi + \tfrac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}. \tag{5.13}$$

We discuss the convergence of this series in Section 5.1.2. □

The following table gives some simple Fourier series and the functions from which they arise. The functions we have chosen are either even, in which case the Fourier series is a cosine series, or odd, in which case the Fourier series is a sine series. The symmetry determines $f(x)$ in the interval $-\pi < x < 0$.

---

[4] We discuss the use of symmetry to simplify the calculation of Fourier coefficients further in Section 5.1.3.

Table 5.1: Examples of Fourier series

| | $f(x)$ | $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ |
|---|---|---|
| i) | $\frac{1}{2}x, \quad 0 < x < \pi$ | $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$ |
| ii) | $\frac{1}{2}(\pi - x), \quad 0 < x < \pi$ | $\sum_{n=1}^{\infty}\frac{\sin nx}{n}$ |
| iii) | $\frac{1}{4}\pi, \quad 0 < x < \pi$ | $\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}$ |
| iv) | $\begin{cases} \frac{1}{4}\pi, & 0 < x < \frac{1}{2}\pi \\ -\frac{1}{4}\pi, & \frac{1}{2}\pi < x < \pi \end{cases}$ | $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos(2n-1)x}{2n-1}$ |
| v) | $\frac{1}{12}(\pi^2 - 3x^2), \quad 0 < x < \pi$ | $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos nx}{n^2}$ |
| vi) | $\frac{1}{4}(\pi - x)^2 - \frac{1}{12}\pi^2, \quad 0 < x < \pi$ | $\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$ |
| vii) | $\frac{1}{8}\pi(\pi - 2x), \quad 0 < x < \pi$ | $\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$ |
| viii) | $\begin{cases} \frac{1}{4}\pi x, & 0 < x < \frac{1}{2}\pi \\ \frac{1}{4}\pi(\pi - x), & \frac{1}{2}\pi < x < \pi \end{cases}$ | $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin(2n-1)x}{(2n-1)^2}$ |

Exercise 5.2: Verify the Fourier series i)-iv) in Table 5.1.

### 5.1.2 Pointwise convergence of a Fourier series

We were led to the formulae (5.10)-(5.11) for the Fourier coefficients $a_n, b_n$ by a heuristic argument starting with the assumed series (5.2). We now regard (5.10) and (5.11) as the *definition of the Fourier coefficients* of the given integrable function $f$. We are then faced with the question: if we use these coefficients to form the Fourier series

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

*does this series converge pointwise, and does its sum equal* $f(x)$? The answer is that if $f$ satisfies certain conditions the series does converge for all $x$. Whether the sum of the series equals $f(x)$ for $-\pi < x < \pi$ depends on whether $f$ is continuous at $x$.

In order to state the standard theorem on pointwise convergence of a Fourier series, we need to introduce the notion of a function $f$ being "piecewise $C^1$" on an interval. Geometrically this means that the graph of $f$ consists of a finite number of smooth sections. For later use we also define the related concept of "piecewise continuous".

**Definition 5.1:**

A function $f : [a, b] \to \mathbb{R}$ is *piecewise continuous* (respectively $C^1$) means that there is a partition

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

such that $f$, when restricted to each *open* interval $x_{i-1} < x < x_i$, coincides with a function that is continuous (respectively $C^1$) on the *closed* interval $x_{i-1} \leq x \leq x_i$. $\square$

Four examples are shown in Figure 5.3. The function $H(x)$ is the Heaviside function:

$$H(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0. \end{cases}$$

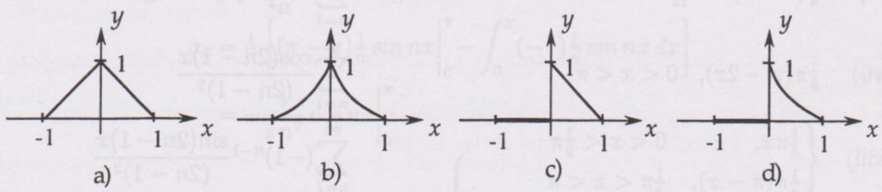



Figure 5.3:

a) $f(x) = 1- \mid x \mid$ is continuous and piecewise $C^1$.
b) $f(x) = 1 - x^{2/3}$ is continuous but *not* piecewise $C^1$.
c) $f(x) = H(x)(1- \mid x \mid)$ is piecewise continuous and piecewise $C^1$.
d) $f(x) = H(x)(1 - x^{2/3})$ is piecewise continuous but *not* piecewise $C^1$.

If a function is piecewise continuous but not continuous on a finite interval, it will have a finite number of *jump discontinuities*, characterized by the fact that

$$\lim_{x \to x_0^+} f(x) \neq \lim_{x \to x_0^-} f(x).$$

For convenience we introduce the notation

$$f(x_0^+) = \lim_{x \to x_0^+} f(x), \quad f(x_0^-) = \lim_{x \to x_0^-} f(x). \tag{5.14}$$

If a Fourier series converges pointwise on the interval $-\pi \le x \le \pi$, then the series will converge pointwise for all $x \in \mathbb{R}$ and its sum will be a *periodic function of period* $2\pi$ (since each term in the series is of period $2\pi$). It is thus necessary to use period $2\pi$ functions when stating theorems concerning convergence of Fourier series. This restriction does not limit the scope of the theorems, since given any function $f$ defined on the interval $-\pi \le x \le \pi$ we can introduce a *period* $2\pi$ *extension of* $f$: this is simply a function $f_p$ of period $2\pi$ that equals the given function $f$ on the *open* interval $-\pi < x < \pi$.

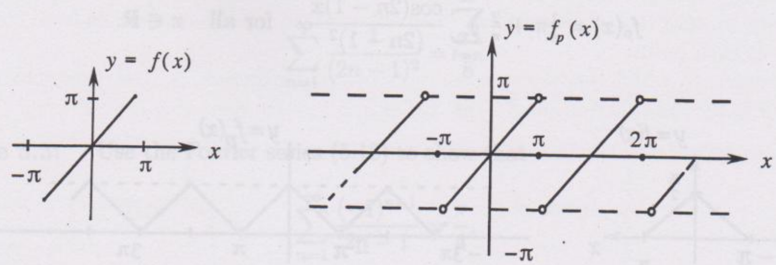



Figure 5.4: A period $2\pi$ extension of $f(x) = x$, $-\pi < x < \pi$.

If the given function $f$ satisfies $f(\pi) \neq f(-\pi)$, any period $2\pi$ extension $f_p$ will have jump discontinuities at $x = n\pi$. The value of $f_p(x)$ at $x = n\pi$ can be assigned arbitrarily, but it is natural to choose the average,

$$f_p(x) = \tfrac{1}{2}[f_p(x^+) + f_p(x^-)], \tag{5.15}$$

in terms of the notation (5.14), as in Figure 5.4.[5]

We can now state the convergence theorem.

**Theorem 5.1** (Pointwise convergence of a Fourier series):

If $f_p$ has period $2\pi$ and is piecewise $C^1$, then the Fourier series of $f_p$ converges pointwise for all $x$, and

i) if $f_p$ is continuous at $x$, the sum is $f_p(x)$,

ii) if $f_p$ is not continuous at $x$, the sum is $\frac{1}{2}[f_p(x_+) + f_p(x_-)]$.

*Proof:* See Churchill and Brown, pages 91-4, Theorem 1. The proof is lengthy. □

**Example 5.2:** The Fourier series of $f(x) = \frac{1}{2}(\pi - |x|)$ on the interval $-\pi \le x \le \pi$ is

$$\tfrac{1}{4}\pi + \tfrac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}. \tag{5.16}$$

[5] If $f$ satisfies $f(\pi) = f(-\pi)$ then it will have a *unique continuous* period $2\pi$ extension $f_p$.

(see Example 5.1).

Sketch the graph of the period $2\pi$ function to which the series (5.16) converges pointwise for all $x \in \mathbb{R}$.

*Solution:* The period $2\pi$ extension $f_p$ of the given function $f(x)$ is shown in Figure 5.5. Since $f(-\pi) = f(\pi)$, the extension $f_p$ is continuous,[6] as can be seen from the Figure. Since $f_p$ is also piecewise $C^1$ as can be seen by inspection, Theorem 5.1 implies that *the Fourier series* (5.16) *converges pointwise to* $f_p$ *for all* $x$, and we can write

$$f_p(x) = \tfrac{1}{4}\pi + \tfrac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}, \quad \text{for all} \quad x \in \mathbb{R}. \tag{5.17}$$

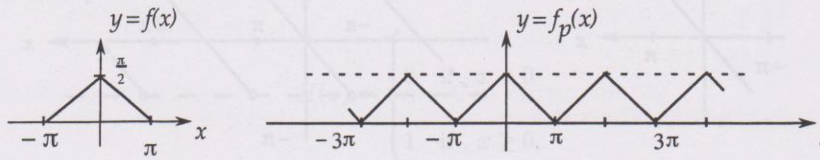



Figure 5.5: The period $2\pi$ extension of $f(x) = \frac{1}{2}(\pi - |x|)$.

**Example 5.3:** The Fourier series of the function

$$f(x) = \tfrac{1}{2}x, \qquad 0 < x < \pi,$$

is

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n} \tag{5.18}$$

(see #i) in Table 5.1). Sketch the graph of the period $2\pi$ function to which the series (5.18) converges pointwise for all $x \in \mathbb{R}$.

*Solution:* The sum of the series (5.18) on $\mathbb{R}$ is an odd function, since $\sin(nx)$, $n = 1, 2, \ldots$ is odd. So we first construct the odd extension of $f(x)$ to $0 < x < \pi$, shown in Figure 5.6(a). We then extend the function to have period $2\pi$. Since $f(-\pi) \neq f(\pi)$, the $2\pi-$ *periodic* extension $f_p$ has a jump discontinuity at $x = k\pi$, $k \in Z$. At these points we define $f_p(x)$ by (5.15), leading to Figure 5.6(b). Since $f_p$ is piecewise $C^1$ by inspection, Theorem 5.1 implies that the series (5.18) converges pointwise to $f_p(x)$ for all $x \in \mathbb{R}$, and we can write

$$f_p(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n}, \quad \text{for all} \quad x \in \mathbb{R}. \tag{5.19}$$

[6]Compare with Example 5.3 to follow, in which case $f(-\pi) \neq f(\pi)$, leading to an extended function that is discontinuous.

**Example 5.4:** Use the Fourier series (5.16) to sum $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

*Solution:* We can choose $x = 0$ in (5.17), obtaining

$$f_p(0) = \tfrac{1}{4}\pi + \tfrac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \tag{5.20}$$

But $f_p(0) = f(0) = \frac{\pi}{2}$. Hence equation (5.20) gives

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

**Exercise 5.3:** Use the Fourier series (5.18) to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4}.$$

*Hint:* Choose $x = \frac{\pi}{2}$.

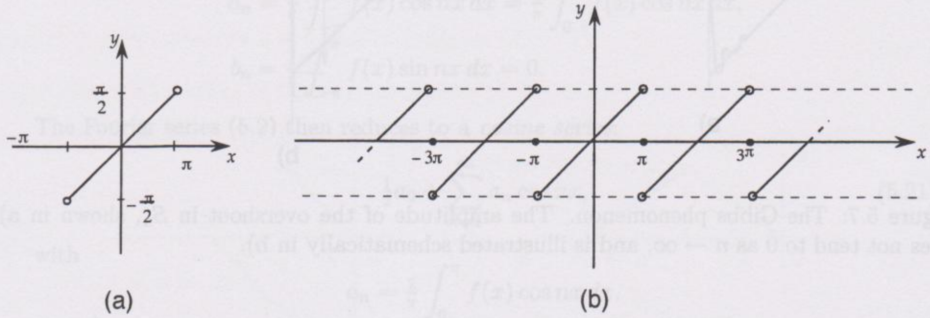



Figure 5.6: (a) shows the odd extension of $f(x) = \frac{1}{2}x$, $0 < x < \pi$. (b) shows the period $2\pi$ extension $f_p$ of the function (a).

**Exercise 5.4:** Sketch the graph of the period $2\pi$ function to which each of the Fourier series in Table 5.1 converges pointwise for all $x \in \mathbb{R}$. The previous example 5.3 gives the result for series i).

**The Gibbs phenomenon:**

The graphs of selected partial sums of the series #iii) in Table 5.1 are shown in Figure 5.1 (restricted to the interval $0 \le x \le 2\pi$). The graphs illustrate a special feature of the convergence of Fourier series at a point of discontinuity, namely that the partial sums "overshoot" on either side of a jump discontinuity. One might expect that as the number of terms in the partial sum increases the overshoot would tend to zero. The overshoot peak does become increasingly narrow as $n$ increases and it moves successively closer to the point of discontinuity, but its amplitude does not tend to zero, and in fact approaches a value of approximately $0.09H$, where $H$ is the jump in the function at the point of discontinuity. It is important to note that the overshoot does not prevent the Fourier series from converging pointwise at each point.

This behaviour, which is referred to as the Gibbs phenomenon, is illustrated schematically in Figure 5.7, for the function in # ii) in Table 5.1.

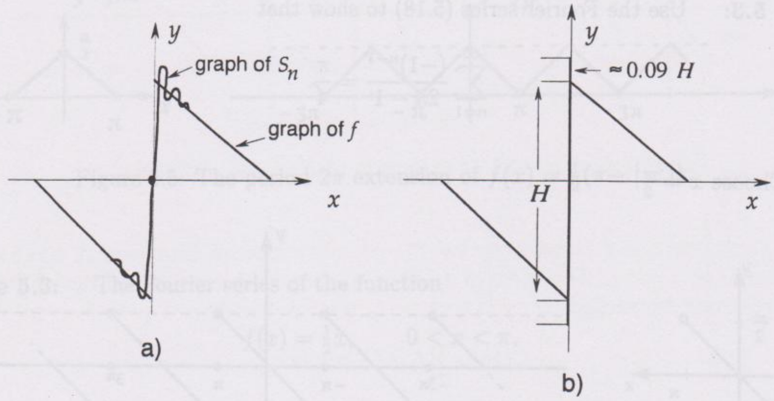



Figure 5.7: The Gibbs phenomenon. The amplitude of the overshoot in $S_n$, shown in a), does not tend to 0 as $n \to \infty$, and is illustrated schematically in b).

### 5.1.3 Symmetry properties

As illustrated in Example 5.1, symmetry properties of the given function can simplify the calculation of the Fourier coefficients. In this section we discuss symmetry properties in more detail.

We begin by briefly reviewing the properties of even and odd functions:

i) $f$ is *even* means that $f(-x) = f(x)$ for all $x$.

   $f$ is *odd* means that $f(-x) = -f(x)$ for all $x$.

ii) The product of two even or two odd functions is even.

   The product of an odd and an even function is odd.

iii) If $f$ is *even*, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$. If $f$ is *odd*, then $\int_{-a}^{a} f(x)dx = 0$.

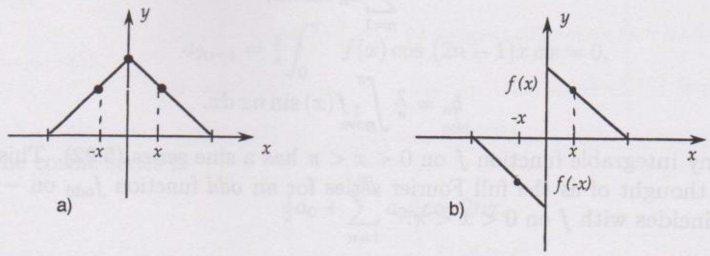



Figure 5.8: a) $f$ is even, $f(-x) = f(x)$. b) $f$ is odd, $f(-x) = -f(x)$.

*Special cases of Fourier series:*

i) *Cosine series:*

Suppose that $f$ is an *even* function. Equations (5.10) and (5.11) for the Fourier coefficients simplify to

$$a_n = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \tfrac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx,$$

$$b_n = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0.$$

The Fourier series (5.2) then reduces to a *cosine series*:

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx, \tag{5.21}$$

with

$$a_n = \tfrac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx.$$

Thus any integrable function $f$ on $0 < x < \pi$ has a cosine series (5.21). This cosine series can be thought of as the full Fourier series for an *even* function $f_{\text{even}}$ on $-\pi < x < \pi$ that coincides with $f$ on $0 < x < \pi$.

ii) *Sine series:*

Suppose $f$ is an *odd* function. Equations (5.10) and (5.11) for the Fourier coefficients simplify to

$$a_n = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0,$$

$$b_n = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \tfrac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx.$$

The Fourier series (5.2) then reduces to a *sine series*:

$$\sum_{n=1}^{\infty} b_n \sin nx, \tag{5.22}$$

with

$$b_n = \tfrac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$$

Thus any integrable function $f$ on $0 < x < \pi$ has a sine series (5.22). This sine series can be thought of as the full Fourier series for an *odd* function $f_{\text{odd}}$ on $-\pi < x < \pi$ that coincides with $f$ on $0 < x < \pi$.

*Symmetry about the line* $x = \frac{\pi}{2}$:

When working with cosine and sine series it is helpful to think of symmetry about the line $x = \frac{\pi}{2}$.

i) $f$ is *even with respect to the line* $x = \frac{\pi}{2}$ means

$$f(\pi - x) = f(x) \quad \text{for all} \quad x.$$

ii) $f$ is *odd with respect to the line* $x = \frac{\pi}{2}$ means

$$f(\pi - x) = -f(x) \quad \text{for all} \quad x.$$

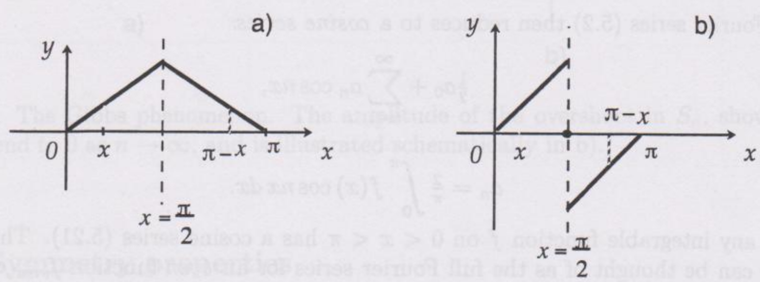



Figure 5.9: a) $f$ is even with respect to $x = \frac{\pi}{2}$. b) $f$ is odd with respect to $x = \frac{\pi}{2}$.

**Exercise 5.5:**
Show that

i) $\cos 2n\,x$ and $\sin(2n-1)x$ are even with respect to $x = \frac{\pi}{2}$.

ii) $\cos(2n-1)x$ and $\sin 2nx$ are odd with respect to $x = \frac{\pi}{2}$.

*Symmetry with respect to $x = \frac{\pi}{2}$ and the cosine series:*

i) If $f(\pi - x) = f(x)$ on $0 < x < \pi$, then by (5.21),

$$a_{2n-1} = \tfrac{2}{\pi}\int_0^\pi \underset{\overset{\uparrow}{\text{even}}}{f(x)}\, \underset{\overset{\uparrow}{\text{odd}}}{\cos\,(2n-1)x}\, dx = 0,$$

and the cosine series is

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_{2n} \cos 2nx. \tag{5.23}$$

ii) If $f(\pi - x) = -f(x)$ on $0 < x < \pi$, then

$$a_{2n} = \tfrac{2}{\pi}\int_0^\pi \underset{\overset{\uparrow}{\text{odd}}}{f(x)}\, \underset{\overset{\uparrow}{\text{even}}}{\cos\,2nx}\, dx = 0,$$

and the cosine series is

$$\sum_{n=1}^{\infty} a_{2n-1} \cos(2n-1)x. \tag{5.24}$$

*Symmetry with respect to $x = \frac{\pi}{2}$ and the sine series:*

i) If $f(\pi - x) = f(x)$ on $0 < x < \pi$, then by (5.22),

$$b_{2n} = \tfrac{2}{\pi}\int_0^\pi \underset{\overset{\uparrow}{\text{even}}}{f(x)}\, \underset{\overset{\uparrow}{\text{odd}}}{\sin\,2nx}\, dx = 0,$$

and the sine series is

$$\sum_{n=1}^{\infty} b_{2n-1} \sin(2n-1)x. \quad \Box \tag{5.25}$$

ii) If $f(\pi - x) = -f(x)$ on $0 < x < \pi$, then

$$b_{2n-1} = \tfrac{2}{\pi}\int_0^\pi \underset{\overset{\uparrow}{\text{odd}}}{f(x)}\, \underset{\overset{\uparrow}{\text{even}}}{\sin\,(2n-1)x}\, dx = 0,$$

and the sine series is

$$\sum_{n=1}^{\infty} b_{2n} \sin 2nx. \tag{5.26}$$

**Example 5.5:** Find the Fourier sine series of

$$f(x) = \tfrac{\pi}{4}, \quad 0 < x < \pi.$$

Use the fact that $f$ is even with respect to $x = \frac{\pi}{2}$.

(This series is # iii) in Table 5.1.)

*Solution:* Since we want the Fourier sine series, we first extend $f$ to be an odd function $f_{\text{odd}}$ on $-\pi < x < \pi$:

$$f_{\text{odd}}(x) = \begin{cases} \frac{\pi}{4}, & 0 < x < \pi \\ -\frac{\pi}{4}, & -\pi < x < 0. \end{cases}$$

The Fourier sine coefficients are then giving by (5.11):

$$b_n = \frac{1}{x}\int_{-\pi}^{\pi} f_{\text{odd}}(x)\sin(nx)dx.$$

Since $f_{\text{odd}}(x)\sin(nx)$ is even (the product of two odd functions), we can write

$$\begin{aligned} b_n &= \tfrac{2}{\pi}\int_0^{\pi} f_{\text{odd}}(x)\sin(nx)dx \\ &= \tfrac{2}{\pi}\int_0^{\pi} \left(\tfrac{\pi}{4}\right)\sin(nx)dx. \end{aligned}$$

Next, since the given function is symmetric about $x = \frac{\pi}{2}$, we have

$$b_{2n} = 0, \quad (\text{since} \quad \sin(2nx) \quad \text{is antisymmetric})$$

and

$$\begin{aligned} b_{2n-1} &= \tfrac{2}{\pi}\int_0^{\pi} \tfrac{\pi}{4}\sin(2n-1)x\,dx \\ &= \tfrac{4}{\pi}\int_0^{\frac{\pi}{2}} \tfrac{\pi}{4}\sin(2n-1)x\,dx \\ &= -\frac{\cos(2n-1)x}{(2n-1)}\Bigg|_0^{\frac{\pi}{2}} \\ &= \frac{1}{1n-2}. \end{aligned}$$

The Fourier sine series is thus

$$\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}.$$

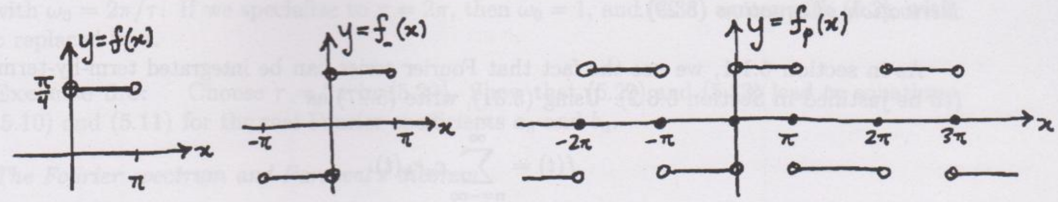



Figure 5.10: The odd, period $2\pi$ extension of $f(x) = \frac{\pi}{4}, 0 < x < \pi$.

### 5.1.4 Complex form of the Fourier series

The general Fourier series (5.2) can be written more concisely by using complex numbers, i.e. by writing the $n^{\text{th}}$ term in terms of $e^{inx}$. This complex representation leads naturally to the Fourier integral and Fourier transform, and is the most convenient representation for applications to signal processing.

For a $\tau$-periodic function $f$, the *complex Fourier series* is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}, \tag{5.27}$$

where

$$\omega_0 = \frac{2\pi}{\tau} \tag{5.28}$$

is the fundamental frequency. The *complex Fourier coefficients* $c_n$, $n = 0, \pm 1, \pm 2, \ldots$, are given by the formula

$$c_n = \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t)e^{-in\omega_0 t}dt, \tag{5.29}$$

and satisfy

$$c_{-n} = \bar{c}_n, \tag{5.30}$$

where the overbar denotes complex conjugation. This condition ensures that the function $f(t)$ in (5.27) is real-valued. The complex basis functions

$$e_n(t) = e^{in\omega_0 t} \tag{5.31}$$

satisfy the *orthogonality condition*

$$\int_{-\tau/2}^{\tau/2} e_n(t)\bar{e}_m(t)dt = \begin{cases} 0, & \text{if} \quad m \neq n \\ \tau & \text{if} \quad m = n \end{cases} \tag{5.32}$$

(verify as an exercise).

*Derivation of equation* (5.29):

As in section 5.1.1, we use the fact that Fourier series can be integrated term-by-term (to be justified in Section 5.3.2). Using (5.31), write (5.27) as

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e_n(t).$$

Multiply by $\overline{e_m(t)}$ and integrate from $-\frac{\tau}{2}$ to $\frac{\tau}{2}$:

$$\begin{aligned}
\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t)\overline{e_m(t)}dt &= \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \left( \sum_{n=-\infty}^{\infty} c_n e_n(t)\overline{e_m(t)} \right) dt \\
&= \sum_{n=-\infty}^{\infty} c_n \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e_n(t)\overline{e_m(t)}dt \qquad \text{(integrate term-by-term)} \\
&= \tau c_m \qquad \text{(by (5.32))}.
\end{aligned}$$

Since $\overline{e_m(t)} = e^{-im\omega_0 t}$ (see (5.31)), we obtain (5.29), with $n$ replaced by $m$.

*Relation between the real and complex forms*

Write $c_n$, $n = 0, 1, 2, \ldots$ in terms of real and imaginary parts:

$$c_n = \tfrac{1}{2}(a_n - ib_n). \tag{5.33}$$

Then by (5.30),

$$c_{-n} = \tfrac{1}{2}(a_n + ib_n). \tag{5.34}$$

Setting $n = 0$ in these equations shows that

$$b_0 = 0, \qquad c_0 = \tfrac{1}{2}a_0. \tag{5.35}$$

By Euler's formula ($e^{i\theta} = \cos\theta + i\sin\theta$), the basis vectors (5.31) have the form

$$e_n(t) = \cos(n\omega_0 t) + i\sin(n\omega_0 t). \tag{5.36}$$

We can write (5.27) as a series summed from 1 to $\infty$:

$$f(t) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} \left( c_n e^{in\omega_0 t} + c_{-n} e^{-in\omega_0 t} \right), \tag{5.37}$$

where we have made use of (5.35). It now follows, using (5.33), (5.34), (5.31) and (5.36) that

$$c_n e^{in\omega_0 t} + c_{-n} e^{-in\omega_0 t} = a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)$$

(verify as an exercise). Thus (5.37) becomes the real form of the Fourier series of a $\tau$-periodic function $f(t)$:

$$f(t) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[ a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t) \right], \tag{5.38}$$

with $\omega_0 = 2\pi/\tau$. If we specialize to $\tau = 2\pi$, then $\omega_0 = 1$, and (5.38) agrees with (5.2), with $x$ replaced by $t$.

**Exercise 5.6:** Choose $\tau = 2\pi$ in (5.29). Show that (5.29) and (5.33) lead to equations (5.10) and (5.11) for the real Fourier coefficients $a_n$ and $b_n$.

*The Fourier spectrum and Parseval's theorem*

We think of the Fourier series of a $\tau$-periodic function $f$ as the decomposition of a signal $f(t)$ into its constituent frequencies or *harmonics*

$$\omega_0, \quad 2\omega_0, \quad 3\omega_0, \ldots$$

where $\omega_0 = \frac{2\pi}{\tau}$ is the *fundamental frequency.* The Fourier coefficients $c_n$ describe the relative importance of the various harmonics. We write

$$c_n = |c_n|e^{i\psi_n}$$

where $|c_n|$ is the *amplitude* and $\psi_n$ is the *phase* of the $n$th harmonic. The Fourier coefficients $c_n$ essentially contain all information about the $\tau$-periodic function $f$, and the set of numbers

$$\{c_n\}_{n \in Z}$$

is called the *Fourier spectrum of $f$*. In particular, the real numbers $|c_n|$ form the *amplitude spectrum of $f$*. The upshot is that we can represent a $\tau$-periodic signal either directly as a function of time $t$ in the *time domain,* or through its spectrum in the *frequency domain.* We illustrate this duality schematically below.

$$f(t) \qquad \longleftrightarrow \qquad \{c_n\}_{n\in\mathbb{Z}} \qquad \text{Fourier spectrum}$$

One can measure the strength of a signal in two ways, either in the *time domain* using the *integral*

$$\frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} [f(t)]^2 dt,$$

or in the *frequency domain,* using the *series*

$$\sum_{n=-\infty}^{\infty} |c_n|^2.$$

Parseval's theorem states that these two quantities are equal.

**Theorem 5.2** (Parseval's formula for a $\tau$-periodic function):
If a $\tau$-period function $f$ has a complex Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}, \tag{5.39}$$

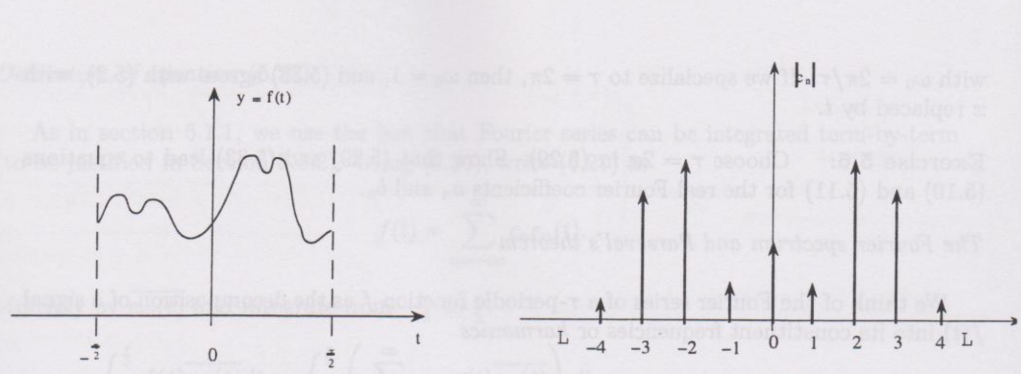



Figure 5.11: The amplitude spectrum of a $\tau$-periodic signal.

then

$$\frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t)^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2. \tag{5.40}$$

**Proof:** Multiply (5.39) by $f(t)$ and integrate from $-\frac{\tau}{2}$ to $\frac{\tau}{2}$:

$$\begin{aligned}
\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t)^2 dt &= \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \left(\sum_{n=-\infty}^{\infty} c_n f(t) e^{in\omega_0 t}\right) dt \\
&= \sum_{n=-\infty}^{\infty} \left(c_n \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t) e^{in\omega_0 t} dt\right) \quad \text{(integrate term-by-term)} \\
&= \tau \sum_{n=-\infty}^{\infty} c_n \bar{c}_n \quad \text{(using the complex conjugate of (5.29))} \\
&= \tau \sum_{n=-\infty}^{\infty} |c_n|^2 \quad (\text{since} \quad z\bar{z} = |z|^2 \quad \text{for a complex number}). \qquad \square
\end{aligned}$$

While the main significance of Parseval's formula is in connection with signal analysis, it is also useful for evaluating the sums of series that are impossible to do by direct means.

**Example 5.6:** Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{5.41}$$

*Solution:* We have seen (Table 5.1, ii) that the Fourier series of

$$f(t) = \tfrac{1}{2}(\pi - t), \quad 0 < t < \pi,$$

is

$$\sum_{n=1}^{\infty} \frac{\sin nt}{n}$$

By (5.33)-(5.34) the complex Fourier coefficients are

$$c_n = -\frac{i}{2n}, \qquad c_{-n} = \frac{i}{2n}, \qquad c_0 = 0,$$

which implies

$$|c_n| = |c_{-n}| = \frac{1}{2n}.$$

We apply Parseval's theorem (5.40) with $\tau = 2\pi$, writing the 2-sided series as a 1-sided series:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f_{\text{odd}}(t)]^2 dt = 2\sum_{n=1}^{\infty}|c_n|^2,$$

where $f_{\text{odd}}$ is the odd extension of $f$ to $-\pi < t < \pi$.

Using the fact that $f_{\text{odd}}(t)^2$ is *even* we obtain

$$\frac{1}{\pi}\int_{0}^{\pi}\tfrac{1}{4}(\pi - t)^2 dt = 2\sum_{n=1}^{\infty}\frac{1}{(2n)^2}.$$

Evaluating the integral gives the desired result.

*Comment:* The key to doing this problem is to pick a function whose Fourier coefficients have a suitable form. There is no unique choice. For example, you could also use example i) in Table 3,

$$f(x) = \tfrac{1}{2}x, \qquad -\pi < x < \pi.$$

whose Fourier series is

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}.$$

**Exercise 5.7:** Use this series to verify (5.41).

## 5.2 Convergence of series of functions

In this section we discuss the convergence of series of functions in more depth. We have seen in Section 5.1.2 that if a series of functions converges pointwise to a function $f$,

$$f(x) = \sum_{n=1}^{\infty} a_n(x),$$

the sum function may be discontinuous even though the terms of the series are continuous. In Section 5.1.1, when calculating Fourier coefficients we encountered the need to integrate a series of functions term-by-term. For these reasons it is necessary to discuss the following questions.

$Q_1$: Under what conditions is the sum of a series of continuous functions itself continuous?

$Q_2$: Under what conditions can a series of (integrable) functions be integrated term-by-term? i.e. under what conditions is it true that

$$\int_a^b \sum_{n=1}^{\infty} a_n(x)dx = \sum_{n=1}^{\infty} \int_a^b a_n(x)dx. \tag{5.42}$$

The key point as regards these questions is this: results that are valid for *finite sums* of functions are not necessarily valid for infinite series of functions. In order to answer the above questions, we have to introduce two new types of convergence for series of functions, namely

i) uniform convergence, and

ii) mean square convergence.

In studying the convergence of a series of functions $\sum_{n=1}^{\infty} a_n(x)$, one considers the $N^{\text{th}}$ partial sum

$$S_N(x) = \sum_{n=1}^{N} a_n(x).$$

Convergence of the series is then defined in terms of convergence of the sequence $\{S_N\}$ of partial sums, i.e. does this sequence have a limit as $N \to \infty$ in some suitably defined sense. Thus the discussion in this section is initially formulated in terms of sequences of functions.

### 5.2.1 A deficiency of pointwise convergence

Any definition of convergence of a sequence of functions $\{f_n\}$ to a limit function $f$ must require that the difference $\mid f_n(x) - f(x) \mid$ becomes small in some sense as $n \to \infty$. So far (in Section 5.1.2), we have worked with pointwise convergence. We now give the formal definition, and then illustrate the "deficiency" with an example.

**Definition 5.2:**

A sequence of functions $\{f_n\}$ *converges pointwise to* $f$ *on* $[a, b]$ means that[7]

$$\lim_{n\to\infty} \mid f_n(x) - f(x) \mid = 0, \tag{5.43}$$

for all $x \in [a, b]$.

[7]You could equivalently write $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in [a, b]$. Equation (5.43) is preferable, however, because it emphasizes that one is interested in the difference between $f_n(x)$ and $f(x)$.

**Definition 5.3:**

A series of functions $\sum_{n=1}^{\infty} a_n$ *converges pointwise to a function* $f$ *on* $[a, b]$ means that the sequence of partial sums, defined by

$$S_N(x) = \sum_{n=1}^{N} a_n(x),$$

converges pointwise to $f$ on $[a, b]$. The function $f$ is called the *sum of the series*, and we write

$$f(x) = \sum_{n=1}^{\infty} a_n(x) \quad \text{pointwise on} \quad [a, b]. \qquad \square$$

The intention in defining convergence of a sequence of functions $\{f_n\}$ is that one wants $f_n$, for $n$ sufficiently large, to approximate the limit function $f$ with specified accuracy

$$f(x) \approx f_n(x) \quad \text{for} \quad n > N.$$

Heuristically, one wants the graph of $f_n$ to approximate the graph of $f$ with specified accuracy *over the whole interval.* Surprisingly, the notion of pointwise convergence (5.30) does not guarantee this type of approximation, as we now show with a simple example.

**Example 5.7:** Consider the sequence of functions $\{f_n\}$ defined by

$$f_n(x) = \frac{2nx}{1 + n^2x^2}, \quad 0 \leq x \leq 1. \tag{5.44}$$

Show that

$$\lim_{n\to\infty} f_n(x) = 0 \quad \text{for all} \quad x \in [0, 1], \tag{5.45}$$

but that

$$\max_{0\leq x\leq 1} |f_n(x)| = 1, \quad \text{for all} \quad n \in \mathbb{N}. \tag{5.46}$$

Sketch the graphs of $f_2$ and $f_n$, for $n$ large.

*Solution:* Consider $x \in [0, 1]$, $x \neq 0$. Then

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \left(\tfrac{1}{n}\right) \left(\frac{2x}{x^2 + \frac{1}{n^2}}\right) = (0)\left(\tfrac{2}{x}\right) = 0.$$

Consider $x = 0$. Then

$$\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} (0) = 0.$$

We have thus established (5.45).

Next, using first year calculus, it follows that $f_n$ has a maximum value of 1 at $x = \frac{1}{n}$:

$$f_n\left(\tfrac{1}{n}\right) = 1, \quad \text{for all} \quad n \in \mathbb{N},$$

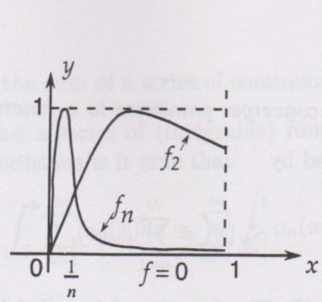



Figure 5.12: The graphs of the functions $f_2, f_n$.

which establishes (5.46). The graphs are shown in Figure 5.11. □

*Discussion:* The existence of the peaks may lead one to doubt (5.45) for $x$ very close to 0. But the definition of pointwise convergence says: *first pick an $x$-value and then let $n \to +\infty$.* So no matter how close to zero your $x$-value is, call it $x_*$, for $n$ large enough *the peaks will lie to the left of $x_*$*, and so $\lim_{n\to\infty} f_n(x_*) = 0$.

The main conclusion from Example 5.5 is this: if a sequence $\{f_n\}$ converges pointwise to $f$ on an interval $a \leq x \leq b$, *the graph of the function $f_n$ does not necessarily lie close to the graph of the limit function $f$ over the whole interval, no matter how large $n$ is.* We say colloquially that "pointwise convergence permits the function $f_n$ to have spikes", i.e. spike-like deviations from the graph of the limit function $f$. The situation can in fact be worse than that depicted in Figure 5.11, since *the height of the spikes can increase without bound as $n \to \infty$.* Consider, for example, $g_n(x) = \sqrt{n} f_n(x)$, where $f_n(x)$ is given by (5.44). The function $g_n$ will have a spike of height $\sqrt{n}$. Finally, we note that the Gibbs phenomenon for Fourier series, mentioned in Section 5.1.2, is reminiscent of the behaviour of $f_n$ in Example 5.6.

### 5.2.2 The maximum norm and mean square norm

In order to define uniform and mean square convergence we have to introduce a concept of *distance between two functions.* Instead of focusing on individual values of functions we have to simultaneously consider all values of the functions over the given interval, i.e. we have to think of a function $f$ as a single mathematical entity, as an element of a function space, and define a notion of distance in this space.

We begin by considering the space of all continuous functions on a finite interval $[a, b]$, denoted by

$$C[a, b].$$

We recognize that $C[a, b]$ is a vector space over the reals, with addition defined by

$$(f + g)(x) = f(x) + g(x) \quad \text{for all} \quad x \in [a, b], \tag{5.47}$$

and multiplication by a scalar defined by

$$(\lambda f)(x) = \lambda f(x) \quad \text{for all} \quad x \in [a, b]. \tag{5.48}$$

We now rely on the vector space $\mathbb{R}^n$ to provide an analogy. In $\mathbb{R}^n$ we introduce the Euclidean norm (i.e. the magnitude of a vector)

$$\| \mathbf{x} \|_2 = \sqrt{x_1^2 + \cdots + x_n^2}, \tag{5.49}$$

where

$$\mathbf{x} = (x_1, x_2, \ldots, x_n),$$

and then define the distance $d(\mathbf{x}, \mathbf{y})$ between two points $\mathbf{x}$ and $\mathbf{y}$ by

$$d(\mathbf{x}, \mathbf{y}) = \| \mathbf{x} - \mathbf{y} \|_2 . \tag{5.50}$$

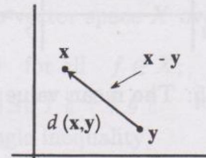



Figure 5.13: Euclidean distance in $\mathbb{R}^n$.

So we need to define a *norm* in $C[a, b]$, which will give the magnitude of a continuous function. There are two "reasonable" ways to do this:

i) use the *maximum value* of $|f(x)|$ over the interval, and

ii) use the *mean value* $\langle |f| \rangle$ of $|f(x)|$ over the interval, defined by

$$\langle |f| \rangle = \frac{1}{b-a} \int_a^b |f(x)|\, dx.$$

**Definition 5.4:**
The *maximum norm* $\| f \|_\infty$ on the space $C[a, b]$ is defined by

$$\| f \|_\infty = \max_{a \le x \le b} | f(x) | . \tag{5.51}$$

For the norm based on the mean of the function it turns out to be more convenient to consider the square root of the mean of the function squared,

$$\left[ \int_a^b f(x)^2 dx \right]^{\frac{1}{2}} , \tag{5.52}$$

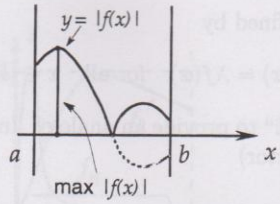



Figure 5.14: The Maximum value of $|\ f(x)\ |$.

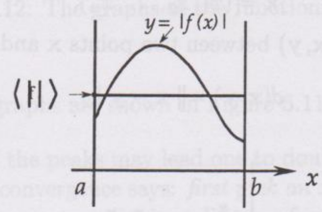



Figure 5.15: The mean value of $|f|$.

omitting the normalizing factor $(b-a)$. The reason for this preference will become clear later, but for the moment we point out the analogy between the expression (5.52) and the Euclidean norm (5.49) in $\mathbb{R}^n$, thinking of the integral as the limit of a sum.

**Definition 5.5:**

The *mean square norm* $\|\ f\ \|_2$ on the space $C[a, b]$ is defined by

$$\|\ f\ \|_2 = \left[\int_a^b f(x)^2\,dx\right]^{\frac{1}{2}}. \tag{5.53}$$

The two norms are related by an important inequality, which we shall refer to as *the basic inequality for function norms.*

**Proposition 5.1:**

$$\|\ f\ \|_2 \leq \sqrt{b-a}\ \|\ f\ \|_\infty, \tag{5.54}$$

for all functions $f$ in $C[a, b]$.

*Proof:*
By definition of $\| f \|_2$:

$$\begin{aligned}
\| f \|_2^2 &= \int_a^b f(x)^2 dx \\
&\leq \max_{a\leq x\leq b} [f(x)]^2 (b-a) \quad \text{(property of the integral)} \\
&= \left[\max_{a\leq x\leq b} | f(x) |\right]^2 (b-a) \\
&= (b-a) \| f \|_\infty^2 \quad \text{(by definition of } \| f \|_\infty). \qquad \square
\end{aligned}$$

*Technical digression:*

i) *Properties of norms*

It should be verified that $\| f \|_\infty$ and $\| f \|_2$, as defined by (5.39) and (5.42), do satisfy the properties of a norm on a vector space $X$ over $\mathbb{R}$:

$$N1 : \| f \| \geq 0 \quad \text{for all} \quad f \in X; \quad \| f \| = 0 \quad \text{iff} \quad f = 0.$$

$$N2 : \| f + g \| \leq \| f \| + \| g \|, \quad \text{for all} \quad f, g \in X.$$
(the triangle inequality)

$$N3 : \| \lambda f \| = | \lambda | \| f \|, \quad \text{for all} \quad f \in X, \quad \lambda \in \mathbb{R}.$$

We leave these details as an exercise.

ii) We have mentioned that the mean square norm $\| f \|_2$ on $C[a,b]$ is analogous to the Euclidean norm $\| \mathbf{x} \|_2$ on $\mathbb{R}^n$. One can define another norm $\| \mathbf{x} \|_\infty$ on $\mathbb{R}^n$, called the max norm, analogous to the max norm on $C[a,b]$:

$$\| \mathbf{x} \|_\infty = \max_{1\leq i\leq n} | x_i |,$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

This norm has computational advantages over $\| \mathbf{x} \|_2$, and is used in scientific computation.

iii) *Norms of piecewise continuous functions*

When working with Fourier series we encounter piecewise continuous functions. The set of all piecewise continuous functions on an interval $a \leq x \leq b$ forms a vector space, which we denote by $PC[a,b]$.

It is clear that the mean square norm $\| f \|_2$ can be defined for $f \in PC[a,b]$ using (5.52), since the integral of a piecewise continuous function is defined as a sum of integrals over subintervals.

On the other hand, in defining the max norm (5.51) we are making use of the theorem according to which a function continuous on a finite closed interval attains a maximum value on that interval. However, a piecewise continuous function does not necessarily attain a maximum value on a finite closed interval, as shown in Figure 5.16, and for such a function

$$\| f \|_\infty = \max | f(x) |$$

does not exist. But observe that in the example $f(x) < \frac{1}{2}$ for all $x$ and $f(x)$ attains values arbitrarily close to $\frac{1}{2}$. So one calls $\frac{1}{2}$ the *least upper bound* or *supremum* of $f$ on the interval $0 \le x \le 1$ and we write

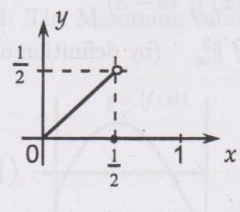



Figure 5.16: A piecewise continuous function that does not attain a maximum.

$$\sup_{0 \le x \le 1} | f(x) | = \tfrac{1}{2}.$$

More generally, for any piecewise continuous function on an interval $a \le x \le b$, $| f(x) |$ is bounded above and hence has a least upper bound. So for functions in $PC[a, b]$ we define the max norm (also called the sup norm) by

$$\| f \|_\infty = \sup_{a \le x \le b} | f(x) | . \tag{5.55}$$

More generally, (5.55) is defined for any *bounded* function on $[a, b]$.

### 5.2.3 Uniform and Mean Square Convergence

We now return to one of our main goals in this section, to define *uniform convergence and mean square convergence* (also called convergence in the mean). The definitions are formulated using the function norms $\| \quad \|_\infty$ and $\| \quad \|_2$, as defined by equations (5.51) and (5.53), respectively.

**Definition 5.6:**

A sequence $\{f_n\}$ in $C[a, b]$ *converges uniformly* to $f \in C[a, b]$ means that

$$\lim_{n \to \infty} \| f_n - f \|_\infty = 0,$$

i.e.

$$\lim_{n \to \infty} \left[ \max_{a \le x \le b} | f_n(x) - f(x) | \right] = 0. \tag{5.56}$$

**Definition 5.7:**

A sequence $\{f_n\}$ in $C[a,b]$ *converges in the mean* to $f \in C[a,b]$ means that

$$\lim_{n\to\infty} \| f_n - f \|_2 = 0,$$

i.e.

$$\lim_{n\to\infty} \left[ \int_a^b \{f_n(x) - f(x)\}^2 dx \right]^{\frac{1}{2}} = 0. \tag{5.57}$$

*Note:* These definitions are also valid on the set $PC[a,b]$ of *piecewise continuous* functions, with $\| f \|_\infty$ defined by (5.55).

*Discussion:*

In order to understand the difference between pointwise convergence (5.43) and uniform convergence (5.56) it is helpful to think in terms of an *$\varepsilon$-neighbourhood* defined by a norm.

In $\mathbb{R}^2$, an $\varepsilon$-neighbourhood of a point $\mathbf{x}_0$ is defined by

$$\{\mathbf{x} \in \mathbb{R}^2 | \ \| \mathbf{x} - \mathbf{x}_0 \|_2 < \varepsilon\},$$

and is a disc of radius $\varepsilon$, centred at $\mathbf{x}_0$.

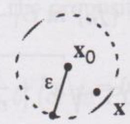



Figure 5.17: An $\varepsilon$-neighbourhood in $\mathbb{R}^2$.

In $C[a,b]$ an $\varepsilon$-neighbourhood of a function $f_0$, relative to the max norm, is similarly defined by

$$\{f \in C[a,b] | \ \| f - f_0 \|_\infty < \varepsilon\},$$

and is the set of functions whose graphs lie within a strip of width $2\varepsilon$, centred on the graph of $f_0$.

Thus, if $\{f_n\}$ converges uniformly to $f$, given any $\varepsilon > 0$ it follows that *for all $n$ sufficiently large, the graph of $f_n$ will lie within the strip of width $2\varepsilon$ centred on the graph of $f$.* In other words, for $n$ sufficiently large, the graph of $f_n$ approximates the graph of $f$ over the whole interval. This means that uniform convergence eliminates the possibility of "spikes" in the graph of $f_n$, as in example 5.6. On the other hand, convergence in the mean does not eliminate "spikes" in $f_n$, since it only requires that the mean of $| f_n(x) - f(x) |^2$ over $[a,b]$ tends to zero.

We note in passing that Figure 5.18 provides heuristic justification for the following result:

> If the sequence $\{f_n\}$ converges uniformly to $f$ on $[a,b]$ and $f_n$ is continuous for all $n$, then $f$ is continuous.

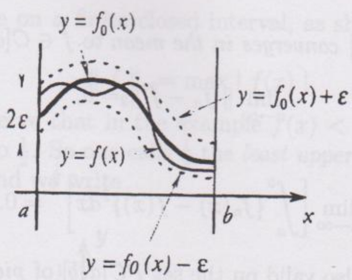



Figure 5.18: An $\varepsilon$-neighbourhood in $C[a, b]$.

The idea is that if the graph of each $f_n$ has no "breaks" and convergence is uniform, then the graph of $f$ is forced to have no "breaks". □

We now give an example in which we use the definition to prove that a sequence converges uniformly.

**Example 5.8:** Consider the sequence $\{f_n\}$ in $C[-1, 1]$ defined by

$$f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}. \tag{5.58}$$

Find the function $f$ to which the sequence $\{f_n\}$ converges pointwise. Prove that the sequence $\{f_n\}$ converges uniformly to $f$.

*Solution:* Observe that for any $x \in [-1, 1]$,

$$\lim_{n\to\infty} f_n(x) = \sqrt{x^2} = \mid x \mid,$$

i.e. $\{f_n\}$ converges pointwise to

$$f(x) = \mid x \mid .$$

We now calculate $\parallel f_n - f \parallel_\infty$. We have

$$\begin{aligned} \mid f_n(x) - f(x) \mid &= \sqrt{x^2 + \frac{1}{n^2}} - \sqrt{x^2} \\ &= \frac{\frac{1}{n^2}}{\sqrt{x^2 + \frac{1}{n^2}} + \sqrt{x^2}}. \end{aligned}$$

By inspection, the maximum occurs at $x = 0$ (when the denominator is a minimum). Thus

$$\parallel f_n - f \parallel_\infty = \max_{-1\le x\le 1} \mid f_n(x) - f(x) \mid = \tfrac{1}{n}.$$

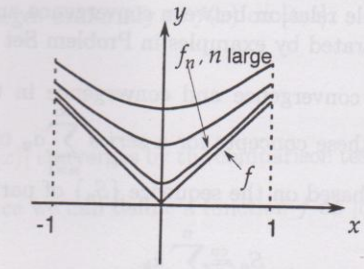



Figure 5.19: The graph of $f_n, n$ large, approximates the graph of $f$.

Since

$$\lim_{n\to\infty} \| f_n - f \|_\infty = \lim_{n\to\infty} \left(\tfrac{1}{n}\right) = 0.$$

the sequence $\{f_n\}$ converges uniformly to $f$ on the interval $-1 \leq x \leq 1$. $\square$

It is important to note that the three types of convergence in $C[a,b]$ are not equivalent. From the discussion so far one expects that *uniform convergence is the strongest of the three types.* This expectation is confirmed by the proposition to follow.

**Proposition 5.2:**
If $\{f_n\}$ converges uniformly to $f$ in $C[a,b]$ (or in $PC[a,b]$), then

i) $\{f_n\}$ converges in the mean to $f$, and
ii) $\{f_n\}$ converges pointwise to $f$.

*Proof:*

i) By Proposition 5.1,

$$\| f_n - f \|_2 \leq \sqrt{b-a}\ \| f_n - f \|_\infty,$$

for all $n$. Since $\lim_{n\to\infty} \| f_n - f \|_\infty = 0$ by hypothesis, the Squeeze theorem implies that $\lim_{n\to\infty} \| f_n - f \|_2 = 0$, i.e. $\{f_n\}$ converges to $f$ in the mean. $\square$

ii) It follows from the definition (5.51) that

$$| f_n(x) - f(x) | \leq \| f_n - f \|_\infty \tag{5.59}$$

for all $x \in [a,b]$. Thus, since $\lim_{n\to\infty} \| f_n - f \|_\infty = 0$ it follows that $\lim_{n\to\infty} | f_n(x) - f(x) | = 0$ for all $x \in [a,b]$, i.e. $\{f_n\}$ converges pointwise to $f$, by definition (see (5.43)). $\square$

*Comment:* There is no simple relation between convergence and the mean and pointwise convergence. This fact is illustrated by examples in Problem Set 5.

Having discussed uniform convergence and convergence in the mean for *sequences* of functions, we can now define these concepts for a *series* $\sum_{n=1}^{\infty} a_n$ of continuous functions, i.e. $a_n \in C[a,b]$. The definition is based on the sequence $\{S_n\}$ of partial sums, defined by

$$S_n = \sum_{k=1}^{n} a_k.$$

**Definition 5.8:**

A series $\sum_{k=1}^{\infty} a_k$ in $C[a,b]$ *converges uniformly to* $f$ means that the sequence $\{S_n\}$ of partial sums converges uniformly to $f$, i.e.

$$\lim_{n\to\infty} \left\| f - \sum_{k=1}^{n} a_k \right\|_{\infty} = 0. \tag{5.60}$$

**Definition 5.9:**

A series $\sum_{k=1}^{\infty} a_k$ in $C[a,b]$ *converges in the mean to* $f$ means that the sequence $\{S_n\}$ of partial sums converges in the mean to $f$, i.e.

$$\lim_{n\to\infty} \left\| f - \sum_{k=1}^{n} a_k \right\|_{2} = 0. \tag{5.61}$$

*Note:* These definitions are also valid in the space of piecewise continuous functions, $PC[a,b]$. □

We now discuss a test for proving that a series of functions converges uniformly on an interval.

**Proposition 5.3** (Weierstrass M-test):

If $H_1$: $\quad | a_n(x) | \le M_n$ for all $x \in [a,b]$,

$H_2$: $\quad \sum_{n=1}^{\infty} M_n$ converges,

then

i) $\sum_{n=1}^{\infty} a_n(x)$ converges absolutely for each $x \in [a,b]$, with sum $f(x)$, and

ii) $\sum_{n=1}^{\infty} a_n(x)$ converges uniformly to $f(x)$ on $[a, b]$.

*Proof:*

By $H_1$ and $H_2$, $\sum_{n=1}^{\infty} |a_n(x)|$ converges by the comparison test, for all $x \in [a, b]$, i.e. $\sum_{n=1}^{\infty} a_n(x)$ converges absolutely. Hence we can define a function $f$ on $[a, b]$ by

$$f(x) = \sum_{n=1}^{\infty} a_n(x).$$

We also let

$$S_n(x) = \sum_{k=1}^{n} a_k(x).$$

Then

$$\begin{aligned} | f(x) - S_n(x) | &= \left| \sum_{k=n+1}^{\infty} a_k(x) \right|, \\ &\leq \sum_{k=n+1}^{\infty} |a_k(x)| \quad \text{(since the series converges absolutely)}, \\ &\leq \sum_{k=n+1}^{\infty} M_k \quad \text{for all} \quad x \in [a, b] \quad \text{(by } H_1\text{)}. \end{aligned}$$

It follows that

$$\begin{aligned} \| f - S_n \|_\infty &\leq \sum_{k=n+1}^{\infty} M_k \\ &< \varepsilon \quad \text{for } n \text{ sufficiently large}, \end{aligned}$$

since $\sum_{k=1}^{\infty} M_k$ converges. Thus $\lim_{n\to\infty} \| f - S_n \|_\infty = 0$, and the series converges uniformly on $[a, b]$, by the definition. $\square$

**Example 5.9:** Prove that the Fourier series $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ converges uniformly on any finite closed interval.

*Solution:* Observe that $\left|\frac{\cos nx}{n^2}\right| \leq \frac{1}{n^2}$ for all $x$, and the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{converges.}$$

Thus by the Weierstrass M-test the result follows. $\square$

*Continuity of the sum function*

The continuity result for sequences stated in the discussion prior to Example 5.8 suggests an analogous result for series.

**Theorem 5.3:** If the series $\sum_{k=1}^{\infty} a_k(x)$ converges uniformly to $f(x)$ on $[a, b]$, and if each term of the series is continuous, then the sum function is continuous.

*Proof:* See Taylor and Mann, 1983, pg. 620, Theorem III. $\square$

This result and Example 5.9 shows that the function

$$f(x) = \sum_{n=1}^{\infty} \frac{\cos x}{n^2}, \quad -\pi \leq x \leq \pi,$$

is continuous. We find $f(x)$ explicitly later (see Example 5.12).

### 5.2.4 Termwise integration of series

The theorem that we prove in this section states that if a series of functions converges in the mean on some interval, then the series can be integrated term-by-term, and the integrated series converges uniformly.

**Theorem 5.4:** If $\sum_{k=1}^{\infty} a_k(t)$ converges in the mean on the interval $a \leq t \leq b$, then the integrated series $\sum_{k=1}^{\infty} \left( \int_a^x a_k(t)dt \right)$ converges uniformly on the interval $a \leq x \leq b$, and

$$\int_a^x \left( \sum_{k=1}^{\infty} a_k(t)\, dt \right) = \sum_{k=1}^{\infty} \left( \int_a^x a_k(t)dt \right).$$

The proof follows quickly from the following lemma, which states that if $f$ and $f_n$ are close in the mean square norm, then their integrals are close in the max norm.

**Lemma:** If

$$g(x) = \int_a^x f(t)dt, \quad g_n(x) = \int_a^x f_n(t)dt, \tag{5.62}$$

for $a \leq x \leq b$, then

$$\| g - g_n \|_\infty \leq \sqrt{b-a} \, \| f - f_n \|_2 .$$

*Proof:* For any $x \in [a, b]$,

$$|g(x) - g_n(x)| = \left| \int_a^x [f(t) - f_n(t)]dt \right| \quad \text{(property of the integral)}$$

$$\leq \int_a^x |f(t) - f_n(t)|dt \quad \text{(property of the integral)}$$

$$\leq \int_a^b |f(t) - f_n(t)|dt \quad \text{(since } x \leq b \text{ and the integrand is positive)}$$

$$\leq \left[ \int_a^b |f(t) - f_n(t)|^2 dt \right]^{\frac{1}{2}} \left[ \int_a^b 1 dt \right]^{\frac{1}{2}} \quad \text{(by the Cauchy-Schwarz inequality}^8\text{)}$$

$$= \sqrt{b-a} \parallel f - f_n \parallel_2 .$$

Since $x$ is arbitrary, it follows that

$$\parallel g - g_n \parallel_\infty \leq \sqrt{b-a} \parallel f - f_n \parallel_2 .$$

□

**Proof of the theorem:**

Choose

$$f(t) = \sum_{k=1}^{\infty} a_k(t), \qquad f_n(t) = \sum_{k=1}^{n} a_k(t)$$

in the lemma. Then by (5.62),

$$g(x) = \int_a^x \left( \sum_{k=1}^{\infty} a_k(t) \right) dt \tag{5.63}$$

and

$$g_n(x) = \int_a^x \left( \sum_{k=1}^{n} a_k(t) \right) dt = \sum_{k=1}^{n} \left( \int_a^x a_k(t) dt \right), \tag{5.64}$$

where we are integrating a *finite* sum. It follows from the Lemma and (5.64) that

$$\left\| g(t) - \sum_{k=1}^{n} \left( \int_a^x a_k(t) dt \right) \right\|_\infty \leq \sqrt{b-a} \left\| f(t) - \sum_{k=1}^{n} a_k(t) \right\|_2 . \tag{5.65}$$

Since $\sum_{k=1}^{\infty} a_k(t)$ converges in the mean to $f(t)$ on the interval $a \leq t \leq b$,

$$\lim_{n \to \infty} \left\| f(t) - \sum_{k=1}^{n} a_k(t) \right\|_2 = 0,$$

[8]The Cauchy-Schwarz inequality: $\left[ \int_a^b p(x) q(x) dx \right]^2 \leq \left[ \int_a^b p(x)^2 dx \right] \left[ \int_a^b q(x)^2 dx \right]$.

by definition (see (5.61)). It thus follows from (5.65) and the Squeeze Theorem that

$$\lim_{n\to\infty}\left\| g(x) - \sum_{k=1}^{n}\left(\int_a^x a_k(t)dt\right)\right\|_\infty = 0.$$

Hence, by definition of uniform convergence (see (5.61)), $\displaystyle\sum_{k=1}^{\infty}\left(\int_a^x a_k(t)dt\right)$ converges uniformly to $g(x)$, and

$$\sum_{k=1}^{\infty}\left(\int_a^x a_k(t)dt\right) = g(x) = \int_a^x \left(\sum_{k=1}^{\infty} a_k(t)\right) dt,$$

by (5.63). □

In Section 5.3.2 we shall apply this theorem to Fourier series.

## 5.3 A Second Look at Fourier Series

In this section we use the results from Section 5.2 to discuss the convergence of Fourier series in greater depth. We are then in a position to prove that Fourier series can be integrated termwise.

### 5.3.1 Uniform and mean square convergence

We now give conditions that guarantee convergence in the mean and uniform convergence for a Fourier series. For convenience we also repeat the conditions that guarantee pointwise convergence (Theorem 5.1).

Given an integrable function $f$ defined on the interval $-\pi \leq x \leq \pi$, we construct the Fourier series (5.2):

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

with $a_n, b_n$ given by (5.10) and (5.11). As in Section 5.1.2, we let $f_p$ denote the period $2\pi$ extension of $f$ that satisfies

$$f_p(x) = \tfrac{1}{2}\left[f_p(x^+) + f_p(x^-)\right] \tag{5.66}$$

at any point of discontinuity of $f_p$ (see (5.15)). We now assume that $f_p$ satisfies one of the following hypotheses[9]:

$H_1$ : $f_p$ is piecewise continuous.
$H_2$ : $f_p$ is piecewise $C^1$.
$H_3$ : $f_p$ is piecewise $C^1$ and continuous.

[9]The weakest hypothesis is that the function be *square integrable*, i.e. $\displaystyle\int_{-\pi}^{\pi}[f_p(x)]^2dx < \infty$. Such a function describes a signal of *finite energy*. If $f_p$ is square integrable, then its Fourier series converges in the mean to $f_p$. The proof of this result is advanced (see J.D. Pryce, Basic methods of linear functional analysis, Theorem 13.5(ii), p. 195).

**Theorem 5.5:** (Convergence of Fourier series)

i) If $f_p$ satisfies $H_1$, then the Fourier series of $f$ converges in the mean to $f_p$ on any finite interval,

ii) If $f_p$ satisfies $H_2$, then the Fourier series of $f$ converges pointwise to $f_p(x)$ for all $x \in \mathbb{R}$,

iii) If $f_p$ satisfies $H_3$, then the Fourier series of $f$ converges uniformly to $f_p$ on any finite interval.

*Proofs:*

i) Davis, 1989, page 145,

ii) Churchill & Brown 1978, pages 91-4, Theorem 1,

iii) Davis, 1989, page 142, Theorem 1.

We illustrate these theorems with some examples.

**Example 5.10:** We have seen (Example 5.1) that the Fourier series of the function

$$f(x) = \tfrac{1}{2}(\pi - \mid x \mid), \quad -\pi < x < \pi$$

is

$$\tfrac{1}{4}\pi + \tfrac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}.$$

The period $2\pi$ extension $f_p$ of $f$ is shown in Figure 5.5. Since $f_p$ is *continuous* and *piecewise* $C^1$, Theorem 5.6 implies that the Fourier series of $f$ converges uniformly to $f_p$ on any finite closed interval, and we write

$$f_p(x) = \tfrac{1}{4}\pi + \tfrac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2} \quad \text{uniformly.}$$

This result is in agreement with Theorem 5.3 that the sum of a uniformly convergent series of continuous functions is a continuous function. □

**Example 5.11:** We have seen (Example 5.5) that the Fourier series of the function

$$f(x) = \begin{cases} \tfrac{1}{4}\pi, & 0 < x < \pi \\ -\tfrac{1}{4}\pi, & -\pi < x < 0. \end{cases}$$

is

$$\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}$$

The period $2\pi$ extension $f_p$ of $f$ that satisfies

$$f_p(x) = \tfrac{1}{2}\left[f_p(x_+) + f_p(x_-)\right]$$

at any jump discontinuity (see (5.66)), is shown in Figure 5.10. Since $f_p$ is piecewise continuous, Theorem 5.5, part i), implies that the Fourier series of $f$ converges in the mean to $f_p$. In addition, since $f_p$ is piecewise $C^1$, Theorem 5.5, part ii), implies that the Fourier series converges pointwise to $f_p$ on $\mathbb{R}$. Hypothesis $H_3$ of Theorem 5.6 is not satisfied. One cannot infer from Theorem 5.5, however, that the Fourier series does not converge uniformly. This conclusion does in fact follow from the contrapositive of Theorem 5.3 (the Continuity Theorem). □

**Exercise 5.8:**

Discuss the convergence properties on $\mathbb{R}$ of the Fourier cosine series of the function $f$ shown in Figure 5.20. Sketch the graph of the sum function. It is not necessary to find the series.

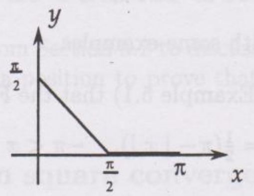



Figure 5.20:

### 5.3.2 Integration of Fourier series

We can now establish a useful property of Fourier series, namely that term-wise integration is permissible.

**Theorem 5.6:** The Fourier series of a period $2\pi$ piecewise continuous function can be integrated term-by-term, over any finite interval.

*Proof:* Let $f_p$ be a period $2\pi$ piecewise continuous function. Theorem 5.5 implies that the Fourier series of $f_p$ converges in the mean to $f_p$. We can thus apply Theorem 5.4 (the integration theorem) to conclude that the Fourier series of $f_p$ can be integrated term-by-term over any finite interval. □

For clarity we state the theorem explicitly:

If

$$f_p(t) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \quad \text{in the mean,}$$

then

$$\int_a^x f_p(t)dt = \int_a^x \tfrac{1}{2}a_0 dt + \sum_{n=1}^{\infty} \int_a^x (a_n \cos nt + b_n \sin nt)dt, \qquad (5.67)$$

and the convergence is uniform.

*Comment:* Theorem 5.6 provides the justification for the derivation of the formulae for the Fourier coefficients in Section 5.1 (see equation (5.6)), and for the proof of Parseval's theorem in Section 5.1.4 (see equation (5.40)).

We now give an example to show how term-by-term integration can be used to derive a new Fourier series expansion from a known one.

**Example 5.12:** Given that

$$\tfrac{1}{2}(\pi - x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}, \qquad 0 < x < \pi,$$

(see # ii), Table 5.1, show that

$$\tfrac{1}{4}(\pi - x)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}, \qquad (5.68)$$

uniformly on $0 < x < \pi$.

*Solution:* We have

$$\tfrac{1}{2}(\pi - t) = \sum_{n=1}^{\infty} \frac{\sin nt}{n}, \qquad 0 < t < \pi,$$

using $t$ as the independent variable. The given series converges in the mean by Theorem 5.5(i), since its odd period $2\pi$ extension is piecewise continuous. We can thus use Theorem 5.6 to integrate term-by-term from 0 to $x$, $x < \pi$:

$$\begin{aligned}\int_0^x \tfrac{1}{2}(\pi - t)dt &= \int_0^x \left( \sum_{n=1}^{\infty} \frac{\sin nt}{n} \right) dt \\ &= \sum_{n=1}^{\infty} \left( \int_0^x \frac{\sin nt}{n} dt \right) \qquad \text{(the key step!, using Theorem 5.6)}\end{aligned}$$

Evaluating the integrals leads to

$$-\tfrac{1}{4}(\pi - t)^2 \Big|_0^x = \sum_{n=1}^{\infty} \left( -\frac{\cos nt}{n^2} \Big|_0^x \right),$$

and hence

$$-\tfrac{1}{4}(\pi - x)^2 + \tfrac{1}{4}\pi^2 = -\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^2}. \qquad (5.69)$$

In Section 5.1.4 we used Parseval's theorem to show that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$, (see Example 5.5) which, when substituted in (5.69), leads to (5.68). $\square$

**Exercise 5.9:**

Sketch the graph of the sum function on $\mathbb{R}$ of the Fourier series (5.68).

**Exercise 5.10:**

Use the Fourier series (5.68) to show that

$$\tfrac{1}{12}x(\pi-x)(2\pi-x)=\sum_{n=1}^{\infty}\frac{\sin nx}{n^3},\quad 0<x<\pi.$$

Discuss convergence of the series on $\mathbb{R}$.

**Exercise 5.11:**

Verify the Fourier series # vii) in Table 5.1 by integrating the series # iii) in the table.

## 5.4 The Fourier transform & Fourier integral

In this section we introduce the Fourier transform and the Fourier integral of a non-periodic function $f(t)$. We begin with the Fourier coefficients and Fourier series for a $\tau$-periodic function and then let $\tau\to+\infty$.

### 5.4.1 The definition

Given a *$\tau$-periodic* function $f$ (a periodic signal), we have seen that the *Fourier coefficients* $c_n$ of $f$ are defined by

$$c_n=\frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}f(t)e^{-in\omega_0 t}dt,\tag{5.70}$$

where $\omega_0=\frac{2\pi}{\tau}$ is the fundamental frequency (see (5.29)). The function $f$ can be expressed in terms of the Fourier coefficients through the *Fourier series*

$$f(t)=\sum_{n=-\infty}^{\infty}c_n e^{in\omega_0 t},\tag{5.71}$$

(see (5.27)). We refer to (5.70) as the *analysis equation* (one is decomposing the periodic signal into pure sinusoids), while (5.71) is the *synthesis equation* (one is reconstructing the original signal from the pure sinusoids – the harmonics).

We now extend these ideas to *non-periodic* functions $f$, defined on the whole real line $\mathbb{R}$. We restrict our considerations to functions that are square-integrable[10] on $\mathbb{R}$, i.e.

$$\int_{-\infty}^{\infty}f(t)^2dt<\infty,\tag{5.72}$$

[10]Engineers refer to such a function as a finite-energy signal.

and satisfy

$$\lim_{t\to\pm\infty} f(t) = 0. \tag{5.73}$$

It turns out that the complex Fourier coefficients

$$\{c_n\}_{n\in\mathbb{Z}},$$

which constitute the *discrete Fourier spectrum* of $f$, have to be replaced by a complex-valued function

$$F(\omega), \qquad \omega \in \mathbb{R}.$$

This function, called the *Fourier transform of* $f$, is defined by

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt, \tag{5.74}$$

and is said to constitute the *continuous Fourier spectrum* of $f$. In this situation the Fourier series (5.71) is replaced by the *Fourier integral*, given by

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega. \tag{5.75}$$

*Derivation of equation* (5.75) *from equation* (5.74)

We give a heuristic derivation, by applying a limiting process to equations (5.70) and (5.71). Let $f(t)$ be a function which is zero outside a finite interval. Construct the $\tau$-periodic extension $f_\tau(t)$ of $f(t)$, with $\tau$ sufficiently large (see the figure).

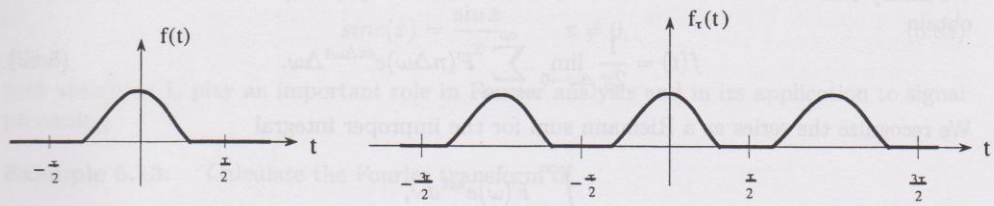



Figure 5.21: A function $f(t)$, zero outside a finite interval, and its $\tau$-periodic extension $f_\tau(t)$.

It follows that for all $t \in \mathbb{R}$,

$$\lim_{\tau\to\infty} f_\tau(t) = f(t) \tag{5.76}$$

(all humps except the central one get pushed out to infinity as $\tau \to \infty$).

We can represent $f_\tau(t)$ by its complex Fourier series

$$f_\tau(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}, \tag{5.77}$$

with

$$c_n = \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f_\tau(t)e^{-in\omega_0 t}dt,$$

and

$$\omega_0 = \frac{2\pi}{\tau}. \tag{5.78}$$

By definition of $f_\tau(t)$ (see the figure) we can write

$$c_n = \frac{1}{\tau}\int_{-\infty}^{\infty} f(t)e^{-in\omega_0 t}dt. \tag{5.79}$$

Comparing (5.74) and (5.79) yields

$$c_n = \frac{1}{\tau}F(n\omega_0). \tag{5.80}$$

In the limit as $\tau \to \infty$, the fundamental frequency $\omega_0$ tends to zero, by (5.78). We introduce the increment in the frequency domain:

$$\Delta\omega = \frac{2\pi}{\tau}. \tag{5.81}$$

We now substitute (5.80) into (5.77), and express $\omega_0$ and $\tau$ in terms of $\Delta\omega$ using (5.78) and (5.77). The result is

$$f_\tau(t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} F(n\Delta\omega)e^{in\Delta\omega t}\Delta\omega.$$

We finally take the limit as $\tau \to \infty$ and hence $\Delta\omega \to 0$ by (5.81). On account of (5.76) we obtain

$$f(t) = \frac{1}{2\pi}\lim_{\Delta\omega\to 0}\sum_{n=-\infty}^{\infty} F(n\Delta\omega)e^{in\Delta\omega t}\Delta\omega. \tag{5.82}$$

We recognize the series as a Riemann sum for the improper integral

$$\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega,$$

so that (5.82) becomes (5.75).

We conclude this section by summarizing in the following table the four key formulae of Fourier analysis

| $\tau$-periodic functions ($\omega_0 = 2\pi/\tau$) | square-integrable functions on $\mathbb{R}$ |
|---|---|
| *Fourier coefficients* $c_n = \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t)e^{-in\omega_0 t}dt$ | *Fourier transform* $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$ |
| *Fourier series* $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}$ | *Fourier integral* $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$ |

## 5.4.2 Calculating Fourier transforms using the definition

The *rectangular window function*, defined by

$$W(t) = \begin{cases} 1, & \text{if} \quad |t| < \frac{1}{2} \\ 0, & \text{if} \quad |t| > \frac{1}{2} \end{cases} \tag{5.83}$$

and the *sinc function*, defined by

$$sinc(x) = \frac{\sin x}{x}, \qquad x \neq 0, \tag{5.84}$$

with $sinc(0) = 1$, play an important role in Fourier analysis and in its application to signal processing.

**Example 5.13:** Calculate the Fourier transform of

$$f(t) = W(t).$$

*Solution:* By definition (5.74),

$$
\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} W(t)e^{-i\omega t}dt \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-i\omega t}dt && \text{(by (5.83))} \\
&= -\frac{1}{i\omega}(e^{-i\omega t})\Big|_{t=-\frac{1}{2}}^{t=\frac{1}{2}} && \text{(by the FTC)} \\
&= -\frac{1}{i\omega}(e^{-\frac{1}{2}i\omega} - e^{\frac{1}{2}i\omega}) \\
&= -\frac{1}{i\omega}\left[(\cos\tfrac{1}{2}\omega - i\sin\tfrac{1}{2}\omega) - (\cos\tfrac{1}{2}\omega + i\sin\tfrac{1}{2}\omega)\right] && \text{(by Euler's formula)} \\
&= \frac{2}{\omega}\sin\tfrac{1}{2}\omega \\
&= sinc\left(\frac{\omega}{2}\right) && \text{(by (5.84)).}
\end{aligned}
$$

To summarize,

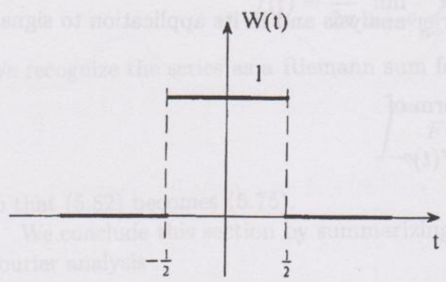
the Fourier transform of the window function $W(t)$ is $sinc\left(\frac{1}{2}\omega\right)$.

We write this result symbolically as

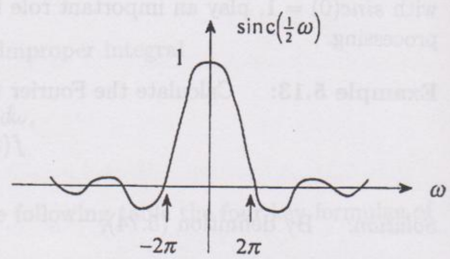
$$\mathcal{F}(W(t)) = sinc\left(\tfrac{1}{2}\omega\right), \tag{5.85}$$

where we think of $\mathcal{F}$ as the operation of taking the Fourier transform.

The graphs of $W(t)$ and its Fourier transform are shown in the figure.




a signal n the time domain




the Fourier transform in the frequency domain

Figure 5.22: The graph of $W(t)$ and its Fourier transform.

*Comment:* If $f(t)$ is even, the Fourier transform $F(\omega)$ is real, and the formula (5.74) can be written in a real form, as follows:

$$
\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \\
&= \int_{0}^{\infty} f(t)e^{-i\omega t}dt + \int_{-\infty}^{0} f(t)e^{-i\omega t}dt \\
&= \int_{0}^{\infty} f(t)e^{-i\omega t}dt - \int_{\infty}^{0} f(-\hat{t})e^{i\omega \hat{t}}d\hat{t} && \text{(make the change of variable } \hat{t} = -t) \\
&= \int_{0}^{\infty} f(t)(e^{i\omega t} + e^{-i\omega t})dt && \text{(use } f(-\hat{t}) = f(\hat{t}) \text{ and relabel } \hat{t} \text{ as } t) \\
&= 2\int_{0}^{\infty} f(t)\cos\omega t\, dt && \text{(by Euler's formula)}
\end{aligned}
$$

In summary, if $f(t)$ is even, the Fourier transform of $f$ can be written in the form

$$F(\omega) = 2\int_{0}^{\infty} f(t)\cos\omega t\, dt. \tag{5.86}$$

**Exercise 5.12:** Verify (5.85) using (5.86).

Similarly it follows that if $f$ is odd, then

$$\mathcal{F}(f(t)) = -2i\int_{0}^{\infty} f(t)\sin\omega t\, dt.$$

We now give a table containing a list of simple functions and their Fourier transforms. You are asked to verify some of these on Problem Set 5. Two of the functions are the rectangular window function $W(t)$ (see (5.83)) and the triangular window function $T(t)$:

$$W(t) = \begin{cases} 1, & \text{if } |t| < \frac{1}{2} \\ 0, & \text{if } |t| > \frac{1}{2} \end{cases} \qquad T(t) = \begin{cases} 1 - |t|, & \text{if } |t| < 1 \\ 0, & \text{if } |t| > 1. \end{cases} \tag{5.87}$$

Table 5.2: Functions $f(t)$ and their Fourier transforms $F(\omega)$

| $f(t)$ | $F(\omega)$ |
|---|---|
| $W(t)$ | $\text{sinc}\left(\frac{1}{2}\omega\right)$ |
| $\text{sinc}(t)$ | $\pi W\left(\frac{1}{2}\omega\right)$ |
| $e^{-\lvert t\rvert}$ | $\dfrac{2}{1+\omega^2}$ |
| $\dfrac{1}{1+t^2}$ | $\pi e^{-\lvert\omega\rvert}$ |
| $e^{-\frac{1}{2}t^2}$ | $\sqrt{2\pi}e^{-\frac{1}{2}\omega^2}$ |
| $T(t)$ | $\left[\text{sinc}\left(\frac{1}{2}\omega\right)\right]^2$ |

### 5.4.3 Properties of the Fourier transform

In this section we develop some general properties of the Fourier transform operator $\mathcal{F}$, which maps a signal $f(t)$ to its Fourier transform $F(\omega)$:

$$\mathcal{F}(f(t)) = F(\omega), \tag{5.88}$$

where $F(\omega)$ is defined by

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt. \tag{5.89}$$

These properties are needed when applying Fourier analysis (e.g., solving PDEs in AMATH 353), and are also useful for calculating specific Fourier transforms.

• **Linearity**

A fundamental property of $\mathcal{F}$ is that it is a *linear operator*:

$$\mathcal{F}(f+g) = \mathcal{F}(f) + \mathcal{F}(g)$$
$$\mathcal{F}(cf) = c\mathcal{F}(f),$$

where $c$ is a constant.

• **Scaling formula**

We know the Fourier transform of the unit gate function $W(t)$, defined by (5.83). How can we calculate the Fourier transform of the gate function $W(at)$ of width $1/a$? We can do this without resorting to the definition, by using the *dilation formula*.

$$\begin{aligned} \text{If} \quad \mathcal{F}(f(t)) &= F(\omega) \\ \text{then} \quad \mathcal{F}(f(at)) &= \frac{1}{a}F\left(\frac{\omega}{a}\right), \end{aligned} \tag{5.90}$$

where $a > 0$ is a constant.

*Derivation:* We use the definition (5.74) of the Fourier transform, and then make a change of variable $\hat{t} = at$:

$$\begin{aligned}\mathcal{F}(f(at)) &= \int_{-\infty}^{\infty} f(at)e^{-i\omega t}dt \\ &= \frac{1}{a}\int_{-\infty}^{\infty} f(\hat{t})e^{-i(\frac{\omega}{a})\hat{t}}d\hat{t} \\ &= \frac{1}{a}F\left(\frac{\omega}{a}\right) \qquad (\text{ replace } \omega \text{ by } \frac{\omega}{a} \text{ in (5.74)}) \quad \square\end{aligned}$$

*Comment:* In performing a change of variable in an improper integral one should, strictly speaking, use the definition, as follows. With $a > 0$, make a change of variable $\hat{t} = at$:

$$\int_0^{\infty} f(at)dt = \lim_{r\to\infty}\int_0^r f(at)dt = \lim_{r\to\infty}\frac{1}{a}\int_0^{ar} f(\hat{t})d\hat{t} = \frac{1}{a}\int_0^{\infty} f(\hat{t})d\hat{t}.$$

In these notes we will omit the intermediate steps.

**Example 5.14:** Calculate the Fourier transform of the gate function $f(t) = W(at)$, $a > 0$, and sketch both graphs. This example illustrates the effect of a scaling in time.

*Solution:* By (5.85)

$$\mathcal{F}(W(t)) = sinc\left(\tfrac{1}{2}\omega\right).$$

The dilation formula (5.90) gives

$$\mathcal{F}(W(at)) = \frac{1}{a}sinc\left(\frac{\omega}{2a}\right) \tag{5.91}$$

The graphs are

Thus, if $0 < a \ll 1$, the gate is wide and the sinc graph is sharply peaked, with rapid oscillations. On the other hand, if $a \gg 1$, the gate is narrow and the sinc graph is very flat.

**Exercise 5.13:** Show that

i)

$$\mathcal{F}(T(at)) = \frac{1}{a}\left[sinc\left(\frac{\omega}{2a}\right)\right]^2, \tag{5.92}$$

where $T$ is the triangular window function defined by (5.87).

ii) $\mathcal{F}(e^{-a|t|}) = \dfrac{2a}{a^2+\omega^2}$, (see Table 5.2).

In both cases, sketch the graphs and indicate how their shape changes as the positive constant $a$ changes.

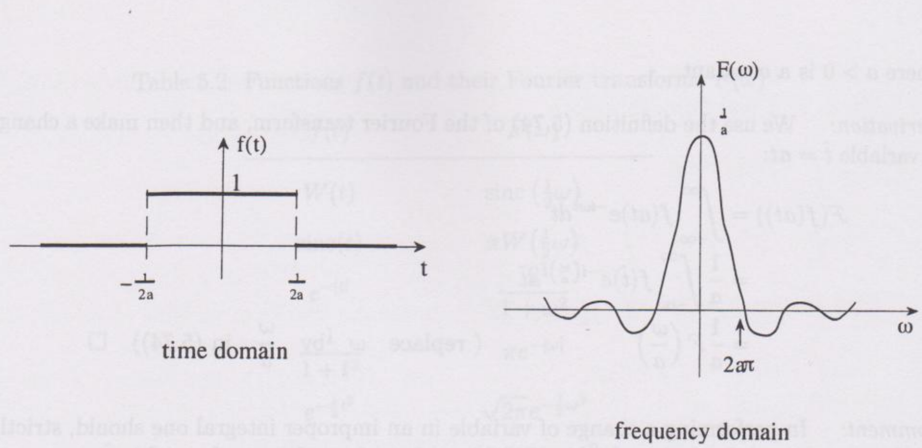



Figure 5.23: The graph of $W(t)$ and its Fourier transform.

• **Shift formula** (aka the *frequency shifting* property).

Suppose you know the Fourier transform of $f(t)$:

$$\mathcal{F}(f(t)) = F(\omega).$$

Is there an easy way to calculate the Fourier transform of

$$g(t) = f(t)\cos(\omega_0 t),$$

i.e., the given signal is multiplied[11] by a signal of frequency $\omega_0$?

We can calculate such a Fourier transform without resorting to the definition, by using the marvellous *shift formula*.

$$\begin{aligned} \text{If} \quad \mathcal{F}(f(t)) &= F(\omega), \\ \text{then} \quad \mathcal{F}(e^{i\omega_0 t} f(t)) &= F(\omega - \omega_0), \end{aligned} \tag{5.93}$$

i.e., *multiplying the given signal* $f(t)$ *by a sinusoid* $e^{i\omega_0 t}$ *in the time domain causes a translation* (aka a shift) *in the frequency domain.*

*Derivation:* By the definition (5.74) we have

$$\mathcal{F}(f(t)) = F(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t}\, dt. \tag{5.94}$$

[11]This process is called *amplitude modulation of the "carrier"* $\cos(\omega_0 t)$ *by the signal* $f(t)$.

Hence,

$$\begin{aligned}
\mathcal{F}(e^{i\omega_0 t}f(t)) &= \int_{-\infty}^{\infty} e^{i\omega_0 t}f(t)e^{-i\omega t}dt \qquad \text{(replace } f(t) \text{ by } e^{i\omega_0 t}f(t) \text{ in (5.94))} \\
&= \int_{-\infty}^{\infty} f(t)e^{-i(\omega-\omega_0)t}dt \qquad \text{(rearrange)} \\
&= F(\omega-\omega_0) \qquad \text{(replace } \omega \text{ by } \omega-\omega_0 \text{ in (5.94))} \qquad \square
\end{aligned}$$

**Example 5.15:** Calculate the Fourier transform of a gate-pulse modulated carrier $\cos(\omega_0 t)$

$$f(t) = W(at)\cos(\omega_0 t),$$

where $W(t)$ is the unit gate function, and $a > 0$, $\omega_0 > 0$ are constants. Sketch both graphs.

*Solution:* We write $f(t)$ in complex form by using Euler's formula:

$$f(t) = \tfrac{1}{2}W(at)\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right).$$

By (5.91)

$$\mathcal{F}(W(at)) = \frac{1}{a} sinc\left(\frac{\omega}{2a}\right).$$

By linearity and the shift formula (5.93),

$$\begin{aligned}
\mathcal{F}(W(at)\cos\omega_0 t) &= \tfrac{1}{2}\left[\mathcal{F}(W(at)e^{i\omega_0 t}) + \mathcal{F}(W(at)e^{-i\omega_0 t})\right] \\
&= \frac{1}{2a}\left[sinc\left(\frac{\omega-\omega_0}{2a}\right) + sinc\left(\frac{\omega+\omega_0}{2a}\right)\right].
\end{aligned}$$

The graphs are shown in Figure 5.24.

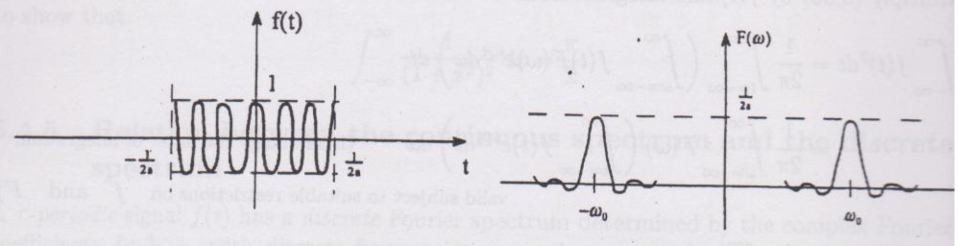



Figure 5.24: The graph of $f(t) = \frac{1}{2}W(at)\cos(\omega_0 t)$ and its Fourier transform.

**Exercise 5.14:** Calculate the Fourier transform of a triangular-pulse modulated carrier $\cos(\omega_0 t)$.

$$f(t) = T(at)\cos(\omega_0 t),$$

where $T(t)$ is the triangular window function defined by (5.87). Sketch the graphs of $f(t)$ and $F(\omega)$.

### 5.4.4 Parseval's formula for a non-periodic function

As you might imagine, Parseval's formula for a periodic function

$$\frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t)^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2,$$

can be generalized to non-periodic functions.

• **Parseval's formula:**

$$\text{If } \mathcal{F}(f(t)) = F(\omega),$$

$$\text{then } \int_{-\infty}^{\infty} f(t)^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega. \tag{5.95}$$

*Derivation:* We have the Fourier integral (5.75):

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega \tag{5.96}$$

and the Fourier transform (5.74):

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt. \tag{5.97}$$

Multiply (5.96) by $f(t)$ and integrate from $-\infty$ to $\infty$:

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)^2 dt &= \frac{1}{2\pi}\int_{t=-\infty}^{\infty}\left(\int_{\omega=-\infty}^{\infty} f(t)F(\omega)e^{i\omega t}d\omega\right)dt\\ &= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} F(\omega)\left(\int_{t=-\infty}^{\infty} f(t)e^{i\omega t}dt\right)d\omega \quad \text{(interchange the order of integration, valid subject to suitable restrictions on } f \text{ and } F)\\ &= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} F(\omega)\overline{F(\omega)}d\omega \quad \text{(by the complex conjugate of (5.97))}\\ &= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} |F(\omega)|^2 d\omega.\end{aligned}$$

□

**Example 5.16:** (an impossible integral!)

Show that

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}dx = \pi.$$

*Solution:* Recognizing the integrand as $[sinc(x)]^2$, which is related to the Fourier transform of the gate function $W(t)$, we think of using Parseval's formula.

Choose $a = \frac{1}{2}$ in (5.91):

$$\mathcal{F}\left(W(\tfrac{1}{2}t)\right) = 2sinc(\omega).$$

Thus, by Parseval's formula:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}[2sinc(\omega)]^2 d\omega = \int_{-\infty}^{\infty}\left[W\left(\tfrac{1}{2}t\right)\right]^2 dt. \tag{5.98}$$

On noting that

$$W\left(\tfrac{1}{2}t\right) = \begin{cases} 1 & \text{if} \quad |t| < 1 \\ 0 & \text{if} \quad |t| > 1 \end{cases}$$

we obtain

$$\int_{-\infty}^{\infty} W\left(\tfrac{1}{2}t\right)^2 dt = \int_{-1}^{1}(1)dt = 2.$$

Thus (5.98) leads to

$$\int_{-\infty}^{\infty}[sinc(\omega)]^2 d\omega = \pi.$$

**Exercise 5.15:** Use the Fourier transform

$$\mathcal{F}(e^{-|t|}) = \frac{2}{1+\omega^2}$$

to show that

$$\int_{-\infty}^{\infty}\frac{1}{(1+x^2)^2}\,dx = \frac{\pi}{2}.$$

### 5.4.5 Relation between the continuous spectrum and the discrete spectrum

A *$\tau$-periodic* signal $f(t)$ has a *discrete* Fourier spectrum determined by the complex Fourier coefficients $\{c_n\}_{n\in\mathbb{Z}}$, with discrete frequencies $n\omega_0$, where $\omega_0 = 2\pi/\tau$ is the fundamental frequency (see Figure 5.11). A *non-periodic* finite-energy signal $f(t)$ has a *continuous* Fourier spectrum determined by the complex Fourier transform $F(\omega)$, where the frequency $\omega$ assumes all real values (see figures 5.21 and 5.23). In this section we show that there is a simple relation between the discrete spectrum and the continuous spectrum.

For fixed $\tau > 0$ consider a signal $f(t)$ that is non-zero only on a subinterval of the interval $-\frac{\tau}{2} \le t \le \frac{\tau}{2}$ (see Figure 5.25 a)). The *Fourier transform* of $f(t)$, as given by equation (5.74), becomes

$$F(\omega) = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t)e^{-i\omega t}dt. \tag{5.99}$$

Let $g(t)$ be the $\tau$-periodic function such that

$$g(t) = f(t), \quad \text{if} \quad -\frac{\tau}{2} \le t \le \frac{\tau}{2}, \tag{5.100}$$

(see Figure 5.25 b)). The *Fourier coefficients* of $g(t)$ are given by

$$c_n = \frac{1}{\tau} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} g(t) e^{-in\omega_0 t} dt, \tag{5.101}$$

with $\omega_0 = 2\pi/\tau$ being the fundamental frequency (see (5.70)). We can substitute (5.100) in (5.101) and if we then compare (5.101) with (5.99), we obtain

$$\tau c_n = F(n\omega_0).$$

This equation gives the desired relation between the discrete spectrum and the continuous spectrum. Note that the continuous spectrum forms an "envelope" for the discrete spectrum (compare b) and d) in Figure 5.25). As $\tau \to \infty$, the discrete spectrum (d) becomes finer and finer (the spacing $\to 0$) and approaches the continuous spectrum b).

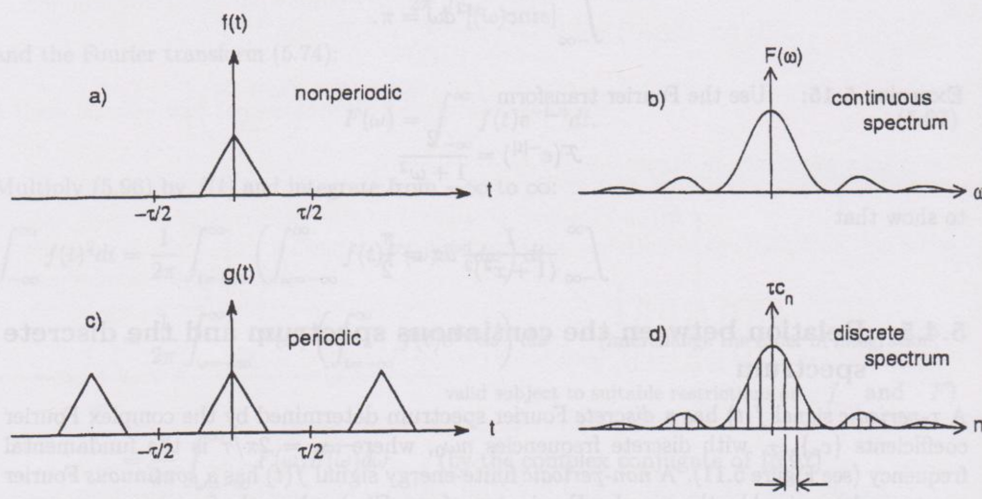



Figure 5.25: A non-periodic signal a) and its continuous spectrum b), and the associated periodic signal c) and its discrete spectrum d), scaled by the period $\tau$.

### 5.4.6 Things are simpler in the frequency domain

We give three brief illustrations of the claim made in the title of this section.

- **Signal separation**

Consider a signal of finite duration in the time domain consisting of the superposition of two high frequency components of widely differing frequencies:

$$f(t) = (A_1 \cos \omega_1 t + A_2 \cos \omega_2 t) W(at),$$

with $\omega_2 \gg \omega_1$. The graph of this function is complicated and it is not helpful to draw it. On the other hand, the Fourier transform of this signal is relatively simple. Using linearity and the shift formula we obtain

$$F(\omega) = \mathcal{F}(f(t)) = \frac{1}{2a} A_1 \left[ sinc\left(\frac{\omega - \omega_1}{2a}\right) + sinc\left(\frac{\omega + \omega_1}{2a}\right) \right] + \frac{1}{2a} A_2 \left[ sinc\left(\frac{\omega - \omega_2}{2a}\right) + sinc\left(\frac{\omega + \omega_2}{2a}\right) \right]$$

(exercise, similar to Example 5.15). The graph of $F(\omega)$ is shown in Figure 5.26.

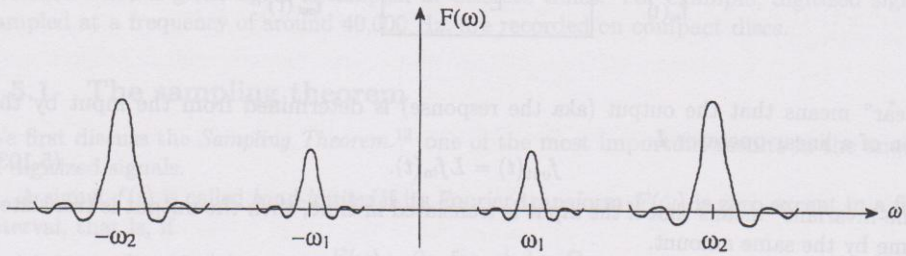



Figure 5.26: Signal separation in the frequency domain.

We say *the signals have been separated in the frequency domain.*

- **Differentiation**

Differentiation in the time domain becomes multiplication by $i\omega$ (i.e., a purely algebraic operation) in the frequency domain.

*Differentiation formula*

$$\begin{aligned} \text{If } \mathcal{F}(f(t)) &= F(\omega) \\ \text{then } \mathcal{F}(f'(t)) &= i\omega F(\omega), \end{aligned} \tag{5.102}$$

provided that $f$ is a suitably regular function on $\mathbb{R}$.

*Derivation:* We apply the definition (5.74) and then integrate by parts:

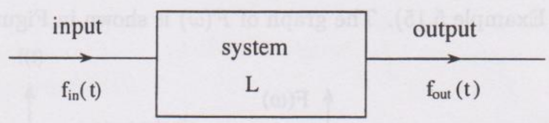
$$\begin{aligned}\mathcal{F}(f'(t)) &= \int_{-\infty}^{\infty} f'(t)e^{-i\omega t}dt \\ &= f(t)e^{-i\omega t}\Big|_{-\infty}^{\infty} + i\omega \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \\ &= 0 + i\omega F(\omega). \qquad \text{(by the definition (5.74)).}\end{aligned}$$

Note that the first term is zero since we require that $\lim_{t\to\pm\infty} f(t) = 0$ (see equation (5.73)).

**Exercise 5.16:** Use (5.102) and a result from Table 5.2 to find $\mathcal{F}(te^{-\frac{1}{2}t^2})$.

**• Response of a linear time-invariant system**

Many signal processing devices can be modelled by using a so-called *linear time-invariant* (LTI) *system*, illustrated schematically below.




"Linear" means that the output (aka the response) is determined from the input by the action of a linear operator $L$:

$$f_{\text{out}}(t) = Lf_{\text{in}}(t). \tag{5.103}$$

"Time-invariant" means that if the input is translated in time, then the output is translated in time by the same amount.

One can show that *if the input to an LTI system is a sinusoid* $e^{i\omega t}$ *then the output is a sinusoid of the same frequency*, but *the amplitude and phase may change, depending on the frequency*:

$$L(e^{i\omega t}) = \alpha(\omega)e^{i\omega t}, \tag{5.104}$$

where $\alpha(\omega)$ is a complex function of frequency $\omega$, called the *system function*.

The key point is that the system function $\alpha(\omega)$ determines the Fourier transform $F_{\text{out}}(\omega)$ of the output $f_{\text{out}}(t)$ in a simple algebraic way in terms of the Fourier transform of the $F_{\text{in}}(\omega)$ of the input $f_{\text{in}}(t)$:

$$F_{\text{out}}(\omega) = \alpha(\omega)F_{\text{in}}(\omega). \tag{5.105}$$

*Derivation:* Assuming that $f_{\text{in}}(t)$ has a Fourier integral representation

$$f_{\text{in}}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_{\text{in}}(\omega)e^{i\omega t}d\omega,$$

we obtain, using (5.103),

$$f_{\text{out}}(t) = Lf_{\text{in}}(t) = \frac{1}{2\pi}L\left(\int_{-\infty}^{\infty} F_{\text{in}}(\omega)e^{i\omega t}d\omega\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F_{\text{in}}(\omega)L(e^{i\omega t})d\omega \qquad \text{(by linearity)}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F_{\text{in}}(\omega)\alpha(\omega)e^{i\omega t}d\omega \qquad \text{(by (5.104))}.$$

But by definition the Fourier integral representation of $f_{\text{out}}(t)$ is

$$f_{\text{out}}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_{\text{out}}(\omega)e^{i\omega t}d\omega.$$

Comparing these two expressions yields (5.105).

## 5.5 Digitized signals – a glimpse

We end by briefly discussing two topics relating to *digitized signals*, that is, signals that are obtained when a given signal is sampled at discrete times. For example, digitized signals, sampled at a frequency of around 40,000 Hz, are recorded on compact discs.

### 5.5.1 The sampling theorem

We first discuss the *Sampling Theorem*.[12] one of the most important results in the analysis of digitized signals.

A signal $f(t)$ is called *band-limited* if its Fourier transform $F(\omega)$ is zero except in a finite interval, that is, if

$$F(\omega) = 0 \quad \text{for} \quad |\omega| > \Omega. \tag{5.106}$$

Then $f_B = \Omega/2\pi$ is called the *cut-off frequency*. The sampling theorem essentially shows that a band-limited signal $f(t)$ with cut-off frequency $\Omega/2\pi$ can be reconstructed exactly using only the values of $f(t)$ at the discrete sampling times $t = 0, \pm\frac{\pi}{\Omega}, \pm\frac{2\pi}{\Omega}, \ldots$ i.e. using a time spacing $\Delta t = \pi/\Omega$, or equivalently, a *sampling frequency* $f_s = \Omega/\pi$. Thus, provided the sampling frequency is twice the cut-off frequency ($f_s = 2f_B$), one can reconstruct the continuous signal $f(t)$ exactly. This is another situation in which the sinc function

$$\text{sinc}(x) = \frac{\sin x}{x}, \quad x \neq 0; \quad \text{sinc}(0) = 1,$$

comes into play.

---

[12] The origins of the Sampling Theorem can be traced to Cauchy and Borel in the nineteenth century, followed by E.T. Whittaker in 1915, motivated by interpolation theory. The theorem was rediscovered by Nyquist in 1928 in the context of signal processing, while working at Bell Labs, and was subsequently incorporated into information theory by Claude Shannon in the late 1940's.

**Theorem 5.7** (Sampling theorem):
If $f(t)$ is a band-limited signal with cut-off frequency $\Omega/2\pi$, then

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{\Omega}\right) \operatorname{sinc}(\Omega t - n\pi). \tag{5.107}$$

*Proof:* Since $f(t)$ is band-limited, i.e. (5.106) holds, equation (5.75) for the Fourier integral becomes

$$f(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} F(\omega) e^{i\omega t} d\omega. \tag{5.108}$$

We now take the unexpected step of writing the function $F(\omega)$ as a complex Fourier series on the interval $-\Omega < \omega < \Omega$. Replacing $t$ by $\omega$ in (5.71), and choosing the period $\tau$ to be $2\Omega$, we get

$$F(\omega) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi\omega}{\Omega}}, \quad -\Omega < \omega < \Omega. \tag{5.109}$$

The coefficients $c_n$ are given by (5.70), again with $t$ replaced by $\omega$, and $\tau = 2\Omega$:

$$c_n = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} F(\omega) e^{-\frac{in\pi\omega}{\Omega}} d\omega. \tag{5.110}$$

Comparing (5.108) with $t = -\frac{n\pi}{\Omega}$ and (5.110) leads to the conclusion that

$$c_n = \frac{\pi}{\Omega} f\left(-\frac{n\pi}{\Omega}\right).$$

We now substitute this result in (5.109), obtaining

$$\begin{aligned} F(\omega) &= \frac{\pi}{\Omega} \sum_{n=-\infty}^{\infty} f\left(-\frac{n\pi}{\Omega}\right) e^{\frac{in\pi\omega}{\Omega}} \\ &= \frac{\pi}{\Omega} \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{\Omega}\right) e^{-\frac{in\pi\omega}{\Omega}}, \end{aligned} \tag{5.111}$$

for $-\Omega < \omega < \Omega$. In the last step we simply replaced $n$ by $-n$. We finally obtain the desired expression for $f(t)$ by substituting (5.111) in (5.108) and rearranging:

$$f(t) = \frac{1}{2\Omega} \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{\Omega}\right) \int_{-\Omega}^{\Omega} e^{-\frac{in\pi\omega}{\Omega}} e^{i\omega t} d\omega.$$

Evaluating the integral and using Euler's identity gives

$$\begin{aligned} \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} e^{-\frac{in\pi\omega}{\Omega}} e^{i\omega t} d\omega &= \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} e^{\frac{i(\Omega t - n\pi)}{\Omega}\omega} d\omega \\ &= \operatorname{sinc}(\Omega t - n\pi) \end{aligned}$$

(fill in the details as an exercise), leading to the desired result (5.107). □

### 5.5.2 The discrete Fourier transform (DFT)

The problem of numerically computing the Fourier coefficients $c_n$ of a $\tau$-periodic function $f(t)$ leads to another aspect of Fourier analysis, the *discrete Fourier transform*, as follows. It is convenient to write the formula for $c_n$ (see (5.70)) as an integral over the interval $[0, \tau]$, instead of $[-\frac{\tau}{2}, \frac{\tau}{2}]$:

$$c_n = \frac{1}{\tau} \int_0^\tau f(t) e^{-2\pi i n t/\tau} dt. \tag{5.112}$$

Given $n$, choose $N > n$, and consider a uniform partition of the interval $[0, \tau]$ into $N$ subintervals of length

$$\Delta t = \frac{\tau}{N},$$

and with left end points

$$t_k = k\left(\frac{\tau}{N}\right), \qquad k = 0, 1, \ldots, N-1. \tag{5.113}$$

The integral is then approximated by the Riemann sum associated with this partition, and (5.112) becomes

$$c_n \approx \frac{1}{N} \sum_{k=0}^{N-1} f\left(\frac{k\tau}{N}\right) e^{-2\pi i k n/N}. \tag{5.114}$$

This equation approximates the Fourier coefficients $c_n$, $n = 0, 1, \ldots, N-1$ in terms of the values of $f(t)$ at the $N$ discrete points given by (5.113). We can think of the signal as having been sampled at these discrete time values. We introduce the notation

$$f[k] = f\left(\frac{k\tau}{N}\right), \qquad k = 0, 1, \ldots, N-1.$$

These $N$ numbers then represent a digitized signal. Motivated by (5.114) we define the *discrete Fourier transform of* $f[k]$ (DFT) by

$$F[n] = \frac{1}{N} \sum_{k=0}^{N-1} f[k] e^{-2\pi i k n/N}, \tag{5.115}$$

$n = 0, 1, \ldots, N-1$. We can think of this formula as approximating the first $N$ Fourier coefficients $c_n$ of a $\tau$-periodic signal $f(t)$, or as giving the exact Fourier coefficients $F[n]$ in the frequency domain of a digitized signal $f[k]$ in the time domain. It can be shown that the inverse of the DFT (5.115) is given by

$$f[n] = \sum_{k=0}^{N-1} F[k] e^{2\pi i k n/N}. \tag{5.116}$$

In mathematical terms, the DFT (5.115) is a linear transformation from $\mathbb{C}^N$ to $\mathbb{C}^N$, i.e. it maps a vector

$$\mathbf{f} = (f[0], \ldots, f[N-1])$$

onto a vector

$$\mathrm{F} = (F[0], \dots, F[N-1]).$$

We can then write (5.115) in matrix form as

$$\mathrm{F} = \frac{1}{N}\mathcal{F}_N \mathrm{f}, \tag{5.117}$$

where the $N \times N$ matrix $\mathcal{F}_N$ is given by

$$\mathcal{F}_N = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2N-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{N-1} & \omega^{2N-2} & \dots & \omega^{(N-1)(N-1)} \end{pmatrix},$$

with

$$\omega = e^{-2\pi i/N}.$$

This matrix is called the *Fourier matrix*.

In exploring large sets of data in many areas including digital signal analysis, it is necessary to calculate the Fourier coefficients $F[n]$, $n = 0, 1, \dots, N-1$, as given by (5.115), for large values of $N$. Doing the matrix multiplication in (5.117) directly requires $N^2$ operations, which is time consuming for large $N$. The *Fast Fourier Transform* (FFT), introduced in 1965 by Cooley and Tukey, exploits the symmetry properties of $\mathcal{F}_N$ to do the calculation using $O(N \log N)$ operations, a significant saving. The FFT is one of the topics studied in the numerical computation course AM 371/CM 271/CS 371.

*Reference* (a famous mathematical paper):

Cooley, J.W. and Tukey, J.W. (1965) An algorithm for the machine computation of complex Fourier series, *Math. Comp.* **19**, 297-301.

# Problem Set 1: Vector-valued functions

## (a) Curves

1. The motion of a particle is given by $\mathbf{x} = \mathbf{g}(t)$, where $t$ is time and $\mathbf{g} : \mathbb{R} \to \mathbb{R}^3$ is given by
$$\mathbf{g}(t) = (t\sin t, t\cos t, \sqrt{3}t), 0 \leq t \leq 4\pi.$$
   (a) Show that the motion occurs along a quadric surface. Use this to sketch the trajectory.
   (b) Find the velocity, speed, and acceleration of the particle.

2. The path of a particle is given by $\mathbf{x}(t) = t\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$.
   (a) Sketch the path by first eliminating the time $t$.
   (b) Suppose the particle flies off the path at $t = 1$ sec. Where will it be 2 seconds later assuming that it is subject to no further force?

3. The path of a particle is given by $\mathbf{x}(t) = (a\cos t, b\sin t, ct)$, where $a, b, c > 0$, and $t$ is time.
   (a) Show that the path lies on a quadric surface and sketch it.
   (b) Show that the acceleration always points towards the $z$-axis.

4. Sketch the curves in $\mathbb{R}^2$ defined by $\mathbf{x} = \mathbf{g}(t)$, where
   (i) $\mathbf{g}(t) = (\cos(t^3), \sin(t^3)), \quad 0 \leq t \leq (2\pi)^{\frac{1}{3}}$.
   (ii) $\mathbf{g}(t) = (\cos^3 t, \sin^3 t), \quad 0 \leq t \leq 2\pi$.

   Find all $t$ such that $\mathbf{g}'(t) = \mathbf{0}$. If possible, find a parameterization
$$\mathbf{x} = \hat{\mathbf{g}}(\tau), \quad \tau_1 \leq \tau \leq \tau_2,$$
   such that $\hat{\mathbf{g}}$ is $C^1$ and $\hat{\mathbf{g}}'(\tau) \neq \mathbf{0}$ for all $\tau \in [\tau_1, \tau_2]$.

5. Find parametric equations for the curve of intersection of $y + z = 1$ and $x^2 + z^2 = 4$.

6. Find a vector-valued function $\mathbf{g} : \mathbb{R} \to \mathbb{R}^3$ to describe the curve of intersection of $x^2 + y^2 + z^2 = 6$ and $x + y + z = 0$.

7. Consider the vector-valued function $\mathbf{g} : \mathbb{R} \to \mathbb{R}^3$, defined by
$$\mathbf{g}(t) = \mathbf{a} + (\sin t)\mathbf{b},$$
   where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ are constants. Describe the curve $\mathbf{x} = \mathbf{g}(t)$ in $\mathbb{R}^3$, and give a sketch. Interpret physically in terms of the motion of a particle.

8. The motion of a particle in the plane is described by $\mathbf{x} = (\cos g(t), \sin g(t))$, where $g : \mathbb{R} \to \mathbb{R}$ is a $C^1$ function. Describe the motion, first assuming that $g'(t) > 0$ for all $t$, and then assuming that $g'(t)$ changes sign at $t = t_1$.

9. Prove that if a particle moves (in 3-space) with constant speed, then its acceleration vector is either zero or orthogonal to its velocity vector.

10. Prove that if a particle moves on the surface of a sphere centred at $(0,0,0)$, then its velocity vector is orthogonal to its position vector. Is the converse true?

11. A curve joining two points $P_1$ and $P_2$ in $\mathbb{R}^3$ is given by

$$\mathbf{x} = \mathbf{g}(t), \quad t_1 \leq t \leq t_2,$$

and also by

$$\mathbf{x} = \hat{\mathbf{g}}(\tau), \quad \tau_1 \leq \tau \leq \tau_2,$$

where

$$\tau = h(t), \quad \text{with} \quad h'(t) > 0,$$

and $\mathbf{g}, \hat{\mathbf{g}}, h$ are $C^1$ functions.

a) Prove that

$$\int_{t_1}^{t_2} \| \mathbf{g}'(t) \| \, dt = \int_{\tau_1}^{\tau_2} \| \hat{\mathbf{g}}'(\tau) \| \, d\tau.$$

b) Interpret this result geometrically.

12. A disc rotates back and forth with angular velocity $(\cos t)$ rad/sec. An insect starting 1 cm from the center of the disc at time $t = 0$ crawls outward at a rate of $2t$ cm/sec. Find the position, velocity and speed of the insect after $2\pi$ seconds. Give the components of position and velocity relative to a fixed Cartesian coordinate system.

### (b) Vector fields

13. Consider the vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\mathbf{F}(x, y) = (x, x^2)$. Find the field lines and sketch the field portrait.

14. The electric field $\mathbf{E}$ due to an electric charge $q$ in the plane is given by

$$\mathbf{E}(x, y) = kq \left( \frac{x}{(x^2 + y^2)^{3/2}}, \frac{y}{(x^2 + y^2)^{3/2}} \right),$$

where $k$ is a constant. Find the field lines and sketch the field portrait.

15. A fluid flow in $\mathbb{R}^3$ has velocity field $\mathbf{v} = (2x, z, -z^2)$.

    (a) Find the equation of the flow line which passes through an arbitrary point $(x_0, y_0, z_0)$ at time $t = 0$.

(b) A fluid particle is at position $(e^2, 0, 1)$ at time $t = 0$. Find its position at time $t = 1$.

16. Given the vector field $\mathbf{F}(\mathbf{x}) = (-y, x, -z)$, find the equation of the field line through an arbitrary point $(x_0, y_0, z_0)$. In particular, give a vector-valued function $\mathbf{g}(t)$ that satisfies

$$\mathbf{g}(0) = (x_0, y_0, z_0).$$

Sketch the exceptional field lines and some typical field lines of the vector field.

17. Suppose a vector field $\mathbf{F}$ on $\mathbb{R}^3$ is given by the gradient of a scalar field $\psi$, according to $\mathbf{F} = -\nabla\psi$. Suppose also that $\mathbf{x} = \mathbf{g}(t)$ is a field line of $\mathbf{F}$. Show that $h(t) = \psi(\mathbf{g}(t))$ is a decreasing function of $t$.

18. Sketch the field lines of the vector field

$$\mathbf{F} = \mathbf{B} \times \mathbf{r} + \lambda\mathbf{B},$$

where $\mathbf{B}$ is a constant vector field and $\lambda$ is a constant. Consider the three cases $\lambda > 0$, $\lambda = 0$, $\lambda < 0$.

## (c) The Chain Rule

19. Suppose that $\mathbf{g} : \mathbb{R} \to \mathbb{R}^n$ is a vector-valued function and that $f : \mathbb{R}^n \to \mathbb{R}$ is a scalar field. Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(t) = f(\mathbf{g}(t)).$$

Show that if $f$ and $\mathbf{g}$ are of class $C^1$, then

$$h'(t) = \nabla f(\mathbf{g}(t)) \cdot \mathbf{g}'(t).$$

20. Suppose that $\mathbf{f}$ is a vector-valued function on $\mathbb{R}$ and $g$ is a real-valued function of one variable. Define $\mathbf{h} : \mathbb{R} \to \mathbb{R}^n$ by

$$\mathbf{h}(t) = \mathbf{f}(g(t)).$$

Show that if $\mathbf{f}$ and $g$ are of class $C^1$, then

$$\mathbf{h}'(t) = g'(t)\mathbf{f}'(g(t)).$$

21. Suppose that $g$ is a scalar field on $\mathbb{R}^n$ and $f$ is a real-valued function of one variable. Define $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(\mathbf{x}) = f(g(\mathbf{x})).$$

Show that if $f$ and $g$ are of class $C^1$, then

$$\nabla h(\mathbf{x}) = f'(g(\mathbf{x}))\nabla g(\mathbf{x}).$$

22. Let $\mathbf{u} = \mathbf{F}(\mathbf{x})$ be a $C^1$ map of the $xy$-plane into the $uv$-plane. Consider a smooth curve $\mathbf{x} = \mathbf{x}(t)$ in the $xy$-plane. Suppose that $\mathbf{F}$ maps this curve into the curve $\mathbf{u} = \mathbf{u}(t)$ in the $uv$-plane. Show that the tangent vectors are related according to

$$\mathbf{u}'(t) = D\mathbf{F}(\mathbf{x}(t))\mathbf{x}'(t).$$

where $D\mathbf{F}$ is the derivative matrix of $\mathbf{F}$.

## Bonus questions

B1. Consider the family of curves

$$\mathbf{x} = \mathbf{g}(t) = (\sin(t-\alpha), \sin t), \quad 0 \leq t \leq 2\pi.$$

Describe how the curve changes as the constant $\alpha$ assumes values from 0 to $2\pi$. Give a sketch.

*Comment:* These curves could be displayed on an oscilloscope by feeding in signals $\sin(t-\alpha)$ and $\sin t$ to the horizontal and vertical inputs respectively.

B2. Can the Mean Value Theorem be generalized to vector-valued functions, i.e. is the following statement true?

If $\mathbf{g} \in C^1[a,b]$, then there exists a number $c$ with $a < c < b$ such that

$$\mathbf{g}(b) - \mathbf{g}(a) = (b-a)\mathbf{g}'(c).$$

B3. Give a curve $\mathcal{C} : \mathbf{x} = \mathbf{g}(t)$, $a \leq t \leq b$ in $\mathbb{R}^2$, where $\mathbf{g} \in C[a,b]$ such that $\mathcal{C}$ does *not* have finite arclength.

B4. *The dipole vector field*

The electric potential $\phi_\ell$ due to a pair of charges, one positive and one negative, a distance $\ell$ apart as shown, is

$$\phi_\ell(\mathbf{x}) = \frac{e}{\sqrt{x^2+y^2+\left(z-\frac{\ell}{2}\right)^2}} - \frac{e}{\sqrt{x^2+y^2+\left(z+\frac{\ell}{2}\right)^2}}.$$

We assume that the charge $e$ and separation $\ell$ are related by

$$e\ell = \mu,$$

where $\mu$ is a give constant. The potential for *dipole of strength* $\mu$ *is* obtained by letting $\ell \to 0^+$ while keeping $\mu$ fixed.

i) By evaluating $\lim_{\ell \to 0^+} \phi_\ell(\mathbf{x})$, show that the potential for a dipole of strength $\mu$ is

$$\phi(\mathbf{x}) = \frac{\mu z}{r^3},$$

where $r = \sqrt{x^2 + y^2 + z^2}$.

ii) Describe the field lines of the dipole vector field $\mathbf{F} = -\nabla\phi$ qualitatively, and give a sketch. It is helpful to show that each field line lies in a vertical plane through the $z$-axis.

B5. Sketch the field lines of the vector field $\mathbf{F}(\mathbf{x}) = (\sin y, -\sin x)$ in $\mathbb{R}^2$. It is not necessary to find the field lines explicitly.

B6. Consider the family of curves

$$\mathbf{x} = \mathbf{g}(t) = (\sin \omega t, \sin t), \quad 0 \le t < +\infty,$$

where $\omega \in \mathbb{R}$ is given. Describe the curves qualitatively.

B7. Consider a charged particle of mass $m$ and charge $e$ moving in a constant electric field $\mathbf{E} = (0, E, 0)$ and constant magnetic field $\mathbf{H} = (0, 0, H)$. It is assumed that the speed $v$ of the particle satisfies $v/c \ll 1$, where $c$ is the speed of light (i.e. non-relativistic motion), and that $E/H \ll 1$, i.e. the electric field is much weaker than the magnetic field. If the initial velocity is in the $x$-direction it can be shown that the path of the particle is

$$\mathbf{x} = \mathbf{g}(t) = \frac{cE}{\omega H}(\omega t - \varepsilon \sin \omega t, \quad \varepsilon(1 - \cos \omega t), \; 0),$$

where $\omega = eH/mc$, and $\varepsilon$ is a dimensionless parameter that is determined by the initial velocity:

$$\mathbf{g}'(0) = c\frac{E}{H}(1 - \varepsilon, 0, 0).$$

Sketch the path of the particle, and show how it depends qualitatively on the parameter $\varepsilon$.

*Reference*: Landau L.D. and Lifshitz, E.M., The Classical Theory of Fields, Fourth Revised English Edition, Section 22.

B8. Let $m$ be the mass of a planet and $M$ that of the sun, both idealized to be point particles in space. Newton's law of gravitation states that the force between these bodies is given by the vector field

$$\mathbf{F} = -\frac{GmM}{r^3}\mathbf{r},$$

where $\mathbf{r}$ is the position vector of the planet relative to the sun, $r = \|\ \mathbf{r}\ \|$ and $G$ is Newton's gravitational constant. The motion of the planet relative to the sun is described by Newton's second law:

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}.$$

i) Show that the planet moves in a fixed plane through the centre of the sun.

*Hint:* Show that **r** and $\frac{d\mathbf{r}}{dt}$ lie in a fixed plane.

ii) Let $A(t)$ be the area swept out by the radius vector from time $t = 0$ to time $t$. Prove that that rate of change of area is constant i.e., the radius vector sweeps out equal areas in equal times. This result is known as *Kepler's second law.*

*Hint:* Show that $\frac{dA}{dt} = \frac{1}{2}\rho^2\frac{d\phi}{dt}$, where $\rho$ and $\phi$ are polar coordinates in the fixed plane referred to in i).

B9. The force exerted on a particle of charge $q$ and mass $m$ moving in a magnetic field **B** is

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}.$$

i) Assuming that the motion is non-relativistic ($v \ll c$) so that Newton's Second Law of Motion is valid, show that the kinetic energy $\frac{1}{2}mv^2$ of the particle is constant.

ii) Show that if the magnetic field is constant, then the component of the velocity of the particle in the direction of **B** is constant.

iii) Suppose that the magnetic field is constant. Choose coordinates so that **B** acts in the $z$-direction, i.e. $\mathbf{B} = B\mathbf{k}$, where $B$ is a constant. Let

$$\mathbf{R} = \mathbf{r} - (\mathbf{r} \cdot \mathbf{k})\mathbf{k},$$

i.e. **R** is the projection of the position of the particle into the $xy$-plane. Show that

$$\frac{d\mathbf{R}}{dt} = \omega(\mathbf{R} - \mathbf{R}_0) \times \mathbf{k},$$

where $\mathbf{R}_0$ is a constant vector and

$$\omega = \frac{qB}{m}.$$

Hence describe the motion of the charged particle.

*Hint:* Show that $\| \mathbf{R} - \mathbf{R}_0 \|$ is constant.

## Problem Set 2:
## Line Integrals and Green's Theorem

1. Consider the scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x,y,z) = 2x - y + z - 1$, and the curve $C$ given by $\mathbf{x} = \mathbf{g}(t) = (\cos t, \sin t, t)$. Compute $\int_C f\,ds$.

2. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{x}$ from $(1,0,0)$ to $(1,0,4)$, where $\mathbf{F} = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$,

   (a) along the line segment joining $(1,0,0)$ and $(1,0,4)$

   (b) along the helix $\mathbf{x}(t) = (\cos 2\pi t,\ \sin 2\pi t,\ 4t)$.

3. Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{x}$, where $\mathbf{F} = (x+y)\mathbf{i} + (y+z)\mathbf{j} + (z+x)\mathbf{k}$, and $C$ is the intersection of the plane $x + y + z = 1$ with the cylinder $x^2 + y^2 = 1$. Orient $C$ clockwise as viewed from above.

4. In each case calculate the work done by the force field $\mathbf{F}$ acting on a particle moving on the curve $C$.

   (a) $\mathbf{F} = -y^2\mathbf{i} + x^2\mathbf{j}$, $C$ is the upper half of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, from $(a,0)$ to $(-a,0)$,

   (b) $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$, $C$ is the curve $\mathbf{x} = (a\cos t, a\sin t, bt)$, $0 \le t \le 2\pi$, and $a, b$ are positive constants.

   (c) $\mathbf{F} = (y^2 - z^2)\mathbf{i} + 2yz\mathbf{j} - x^2\mathbf{k}$, $C$ is the curve of intersection of $y = x^2$ and $z = x^3$ joining the points $(0,0,0)$ and $(1,1,1)$.

   (d) $\mathbf{F} = 2x\mathbf{i} - 3y\mathbf{j} + z^2\mathbf{k}$, $C$ is the line segment from $(1,0,0)$ to $(0,1,\frac{\pi}{2})$.

5. A force field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $\mathbf{F}(\mathbf{x}) = (x^3, y, z)$. Compute the work done by $\mathbf{F}$ on a particle moving along the curve given by $\mathbf{x} = \mathbf{g}(t) = (0, b\cos t, b\sin t)$, $0 \le t \le \pi$. Explain the result geometrically.

6. Show by inspection that the velocity field
$$\mathbf{v}(\mathbf{x}) = \left(ye^{-2xy}, xe^{-2xy}\right)$$
is the gradient of a scalar field $\phi(\mathbf{x})$. Hence evaluate $\int_C \mathbf{v} \cdot d\mathbf{x}$ along the curve $C$ given by $\mathbf{x}(t) = (t^2 - t,\ t^2 + t)$, $1 \le t \le 2$.

7. (i) Consider the closed curve $C$, $\mathbf{x}(t) = (a\cos t, b\sin t)$, where $a$ and $b$ are positive constants, and the velocity field $\mathbf{v} = (-y, x)$. By sketching the velocity field at various points of $C$, predict whether the circulation of $\mathbf{v}$ around $C$ is positive, negative or zero. Verify your prediction.

(ii) Repeat (i) with $\mathbf{v} = (x, y)$.

8. A piece of wire has the shape of the circle $x^2 + y^2 = a^2$, and the mass density at $(x, y)$ is $|x| + |y|$ (mass per unit length).

   (i) Compute the mass of the wire.

   (ii) Compute the average mass density of the wire.

9. Imagine a wire located at the intersection of $x^2 + y^2 + z^2 = 1$ and $x + y + z = 0$, whose density depends on position according to $\phi(\mathbf{x}) = x^2$ gm/unit length. Show that the mass of the wire is $\frac{2}{3}\pi$ gm.

10. (i) Consider a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ and a curve $C$ parametrized in polar coordinates by $r = r(\theta)$, $\theta_1 \leq \theta \leq \theta_2$. Show that

$$\int_C f\,ds = \int_{\theta_1}^{\theta_2} f(r\cos\theta, r\sin\theta)\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

   (ii) Compute the arclength of the curve $r = 1 + \cos\theta$, $0 \leq \theta \leq 2\pi$.

11. Suppose that $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector field that satisfies $\| \mathbf{F}(\mathbf{x}) \| \leq M$ for all $\mathbf{x} \in \mathbb{R}^n$, where $M$ is a constant, and suppose that $C$ is a $C^1$ curve of length $\ell$. Prove that

$$\left| \int_C \mathbf{F} \cdot d\mathbf{x} \right| \leq M\ell.$$

   It is necessary to begin by using the definition of line integral.

12. Decide whether Green's theorem can be used to evaluate the line integral

$$\int_{\partial D} \ln(x + \sqrt{x^2 + y^2})\,dx\,,$$

   (see equation (2.16) for this notation) where $\partial D$ is the boundary of the region $D$:

   (i) $D$ is the disc $x^2 + y^2 \leq 1$, (ii) $D$ is the disc $(x - 2)^2 + y^2 \leq 1$,

   (iii) $D$ is the disc $(x + 2)^2 + y^2 \leq 1$.

13. Use Green's theorem to evaluate the line integral $\displaystyle\int_{\partial D} xy^2 dx + 2x^2 y\,dy$, where $\partial D$ is the ellipse $\frac{1}{9}x^2 + \frac{1}{4}y^2 = 1$, oriented suitably.

14. Show that the area of the region enclosed by the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$ is $\frac{3}{8}\pi a^2$.

15. Show that the area of the region enclosed by the curve $\mathbf{x} = (a\sin 2t, a\sin t)$, $0 \leq t \leq 2\pi$, is $\frac{8}{3}a^2$.

16. (a) The area of a region $\mathcal{D} \subset \mathbb{R}^2$ bounded by a simple closed curve $\partial\mathcal{D}$ is given by the line integral

$$A(\mathcal{D}) = \tfrac{1}{2} \int_{\partial\mathcal{D}} (xdy - ydx).$$

Derive this formula using Green's Theorem.

(b) A region $\mathcal{D}$ in the plane is defined in polar coordinates by $r = f(\theta)$, $\alpha \leq \theta \leq \beta$. Use (a) to show that

$$A(\mathcal{D}) = \tfrac{1}{2} \int_{\alpha}^{\beta} f(\theta)^2 d\theta.$$

17. (a) Verify, by evaluating the line integral , that

$$\int_C xdx + xy\,dy = 0$$

for every circle $C$ centered on the origin (see equation (2.16) for this notation).

(b) Explain this result geometrically by using Green's theorem.

(c) Prove that $\int_C xdx + xy\,dy$ is not independent of path in $\mathbb{R}^2$.

18. (a) Show that the vector field

$$\mathbf{F}(\mathbf{x}) = (\sin xy + xy \cos xy)\,\mathbf{i} + x^2 \cos xy\,\mathbf{j}$$

is conservative in $\mathbb{R}^2$, and find a potential function $\phi$.

(b) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{x}$,

(i) along the straight line from $(1, 0)$ to $(\pi, 1)$,

(ii) counterclockwise around the circle $x^2 + y^2 = 1$.

19. Determine whether the following vector fields are conservative or not; if they are, find the corresponding potential:

(i) $\mathbf{F}(\mathbf{x}) = (3x^2y, x^3)$ (ii) $\mathbf{F}(\mathbf{x}) = (2xe^y + y,\ x^2e^y + x - 2y)$.

20. a) In Newton's theory of gravity, the earth's gravitational field is given by

$$\mathbf{g}(\mathbf{r}) = -\frac{GM}{r^3}\mathbf{r},$$

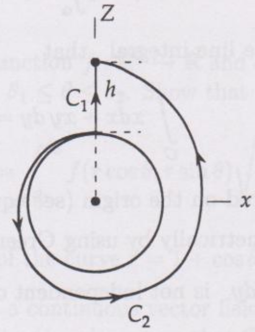
where $\mathbf{r} = (x, y, z)$ is the position vector relative to the centre of the earth, $r = \| \mathbf{r} \|$, $M$ is the mass of the earth and $G$ is the gravitational constant.

A rocket of mass $m$ is launched from position $(0, 0, R)$ and travels to $(0, 0, R+h)$ in a time $T$, along the straight line path $C_1$. A similar rocket travels along the spiral path $C_2$ given by

$$\mathbf{r} = \mathbf{h}(t) = \left(R + \frac{h}{T}t\right)\left(-\sin\frac{2\pi t}{T}, 0, \cos\frac{2\pi t}{T}\right),$$

with $0 \leq t \leq T$.

By evaluating the appropriate line integrals, show that the work done by the gravitational field along $C_1$ equals the work done along $C_2$.




b) Consider the gravitational field in a).

i) Verify that the field satisfies an inverse square law, i.e. $\| \mathbf{g}(x) \|$ is inversely proportional to the square of the distance from the centre.

ii) Show that the gravitational field is conservative by finding a potential $\phi$.

iii) Hence, verify the expression for the work done in a).

21. A "central" (or radial) force field in $\mathbb{R}^n$ has the form

$$\mathbf{F} = f(r)\mathbf{r}, \quad r \neq 0,$$

where $\mathbf{r} = (x_1, \ldots, x_n)$ and $r = \|\mathbf{r}\|$. Show that this force field is conservative in $\mathbb{R}^n - \{0\}$ by finding a potential function $\phi$.

22. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $g : \mathbb{R}^2 \to \mathbb{R}$ be of class $C^1$ in $D \subset \mathbb{R}^2$. Prove that

$$\int_C (f\nabla g) \cdot d\mathbf{x} = -\int_C (g\nabla f) \cdot d\mathbf{x}$$

for any piecewise smooth simple closed curve $C$ in $D$.

23. Let $D$ be a region bounded by a piecewise smooth simple closed curve $C$ oriented counterclockwise. Let $f, g$ be of class $C^2$ in an open set containing $D$. Prove that

$$\int_C f\nabla g \cdot d\mathbf{x} = \iint_D \frac{\partial(f, g)}{\partial(x, y)}\, dxdy\,.$$

24. Discuss whether or not the following sets are (i) connected (ii) simply-connected.

(a) $\mathbb{R}^2 - \{(x, y) \mid y = 0,\, x \leq 1\}$,
(b) $\mathbb{R}^2 - \{(x, y) \mid y = 0,\, |x| \leq 1\}$,
(c) $\{(x, y) \mid x^2 + y^2 < 4\} \cap \{(x, y) \mid x^2 + 4y^2 > 4\}$,
(d) $\mathbb{R}^3 - \{(x, y, z) \mid z = 0,\, x^2 + y^2 = 1\}$.

25. Consider a frictional force which is constant in magnitude, and acts to oppose the motion of a particle. Show that the work done against friction is proportional to the length of the path.

26. A particle moves according to Newton's Second Law of Motion under the influence of a conservative force field

$$\mathbf{F} = -\nabla V,$$

where scalar field $V$ is the potential of the field.
Prove that if the particle has constant speed, then it moves on an equipotential surface.

27. The force exerted on a particle of charge $q$ moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ is

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}.$$

Show that the work done by the magnetic field on the particle is zero.

## Bonus questions

B1. *The Generalized Green's Theorem*

Let $\partial\mathcal{D}$ be a simple closed curve in $\mathbb{R}^2$ enclosing a region $\mathcal{D}$. Suppose that $\mathcal{D}$ contains $n$ non-overlapping holes $H_1, \ldots, H_n$, with boundaries $\partial H_1, \ldots, \partial H_n$ oriented as shown. Let $\mathbf{F}$ be a vector field which is $C^1$ on $\mathcal{D} - \bigcup_{i=1}^n H_i$ and on the boundary of this set.

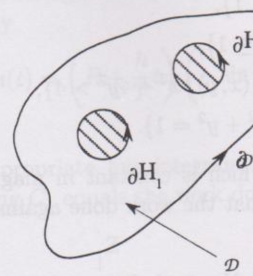
Prove that

$$\int_{\partial D} \mathbf{F} \bullet d\mathbf{x} = \iint_{\mathcal{D} - \bigcup_{i=1}^n H_i} \omega(\mathbf{F})\, dx\, dy + \sum_{i=1}^{n} \int_{\partial H_i} \mathbf{F} \bullet d\mathbf{x}$$

where

$$\omega(\mathbf{F}) = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}.$$

*Comment:* This theorem is useful when the vector field $\mathbf{F}$ is $C^1$ on $\mathcal{D}$ except at a finite number of points, called singularities of $\mathbf{F}$. One then excludes the singularities by deleting a neighbourhood of each singularity.




B2. *An application of Green's theorem to heat transfer*

Consider a sheet of metal, with both faces insulated, represented by the plane region $\mathcal{D}$. The boundary $\partial\mathcal{D}$ of the sheet is kept at a prescribed temperature (for example, part of the boundary is in contact with ice and part with boiling water). The sheet is allowed to reach *thermal equilibrium*, which means that the temperature $u(x, y)$ at position $(x, y)$ on the sheet no longer changes with time. Under these circumstances, it can be shown using Fourier's law of heat transfer (see page 18) that the temperature $u(x, y)$ satisfies Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

In order for this model to be physically reasonable, it should have the property that *if the temperature is constant all along the boundary, then, when the sheet reaches thermal equilibrium, the temperature at all points of the sheet must equal this constant value.* Prove that the model satisfies this property.

*Hint:* Argue that the constant value can be chosen to be zero, and then show that

$$\iint_{\mathcal{D}} \| \nabla u \|^2 \, dxdy = 0.$$

# Problem Set 3:

## Surfaces and Surface Integrals

1. Sketch each of the following surfaces $\mathbf{x} = \mathbf{g}(u, v)$, where $\mathbf{g} : \mathcal{D}_{uv} \to \mathbb{R}^3$ is given below . Find the tangent plane at the indicated point.

   i) $\mathbf{g}(u,v) = (\cos v \sin u, 1 + \sin v \sin u,\ \cos u)$, $\mathcal{D}_{uv} = [0,\pi] \times [0, 2\pi]$; at $\mathbf{g}\left(\frac{\pi}{4}, \frac{\pi}{3}\right)$

   ii) $\mathbf{g}(u,v) = (\sin v, u, \cos v)$, $\mathcal{D}_{uv} = [-1,3] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; at $\mathbf{g}\left(1, \frac{\pi}{3}\right)$.

2. Sketch each surface $\mathbf{x} = \mathbf{g}(u, v)$, where $\mathbf{g} : \mathcal{D}_{uv} \to \mathbb{R}^3$ is given below. Discuss whether the surface is smooth.

   i) $\mathbf{g}(u,v) = (u, |u|, v)$, $\quad \mathcal{D}_{uv} = \{(u,v)|\ |u| \leq 1, \quad 0 \leq v \leq 2\}$,

   ii) $\mathbf{g}(u,v) = (u \cos v, u \sin v, a e^{-|u-b|})$,

   where $\mathcal{D}_{uv} = \{(u,v)|u > 0, 0 \leq v \leq 2\pi\}$, and $a, b$ are positive constants.

3. Sketch the "helicoid" $\mathbf{x} = \mathbf{g}(u,v) = (u\cos v, u \sin v, v)$, $(u,v) \in [0,1] \times [0, 2\pi]$, and find its area.

4. Give the function $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^3$ that describes the cone $z = \sqrt{x^2 + y^2}$, using as parameters $(u,v) = (r, \theta)$,where $r$ and $\theta$ are polar coordinates in $\mathbb{R}^2$. Then determine the area of the portion of the cone lying:

   (i) below $z = 1$,

   (ii) inside the cylinder $x^2 + y^2 = 2ay$, $a > 0$.

5. Give a vector-valued function to describe the quadric surface $x^2 + y^2 - z^2 = 1$, with $|z| \leq \frac{3}{4}$.

   *Hint:* Use the identity
   $$\cosh^2 u - \sinh^2 u = 1.$$
   for the hyperbolic functions. Recall that
   $$\cosh u = \tfrac{1}{2}(e^u + e^{-u}), \quad \sinh = \tfrac{1}{2}(e^u - e^{-u}).$$

6. The surface of a torus is generated by revolving the circle
   $$(y-b)^2 + z^2 = a^2,\ x = 0,$$
   where $b > a > 0$, about the $z$-axis.

   (a) Show that the surface is given by $\mathbf{x} = \mathbf{g}(u, v)$, where
   $$\mathbf{g}(u,v) = ((b + a\cos u)\cos v,\ (b + a \cos u) \sin v,\ a \sin u),$$
   and give the domain in the $uv$-plane.

(b) Calculate the surface area of the torus.

7. A surface of revolution is created by rotating a curve $z = f(u), 0 \leq a \leq u \leq b$, about the $z$-axis through an angle $v$, with $0 \leq v \leq 2\pi$.

   (a) Find a function $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^3$ which describes the points $\mathbf{x} = \mathbf{g}(u, v)$ on this surface.

   (b) Find an expression for the surface area of this surface.

8. Evaluate the surface integral $\displaystyle\iint_\Sigma f dS$ where

   (i) $f(x, y, z) = xz$ and $S$ is the part of the cylinder $x^2 + y^2 = 1$ between the planes $z = 0$ and $z = x + 2$.

   (ii) $f(x, y, z) = x^2$ and $S$ is the boundary of the region $D$ in $\mathbb{R}^3$ inside the cone $z^2 = x^2 + y^2$ and between the planes $z = 1$ and $z = 2$.

9. a) A sphere of radius $b$ centred at the origin has a silver coating of surface density $\rho(x, y, z) = k\left[1 + \epsilon\left(\frac{z}{b}\right)^2\right]$ mass per unit area, where $k$ and $\epsilon$ are positive constants. Calculate the total mass of silver.

   b) Referring to part a), find the average density of silver. At what points on the sphere does the density equal its average value?

10. Using symmetry where possible, evaluate the surface integrals

    (i) $\displaystyle\iint_\Sigma x dS$, (ii) $\displaystyle\iint_\Sigma xz dS$, (iii) $\displaystyle\iint_\Sigma x^2 dS$, (iv) $\displaystyle\iint_\Sigma z^2 dS$,

    where $S$ is the paraboloid $y = x^2 + z^2, \quad y \leq 1$.

11. Calculate the flux $\displaystyle\iint_\Sigma \mathbf{F} \cdot \mathbf{n}\, dS$ and circulation $\displaystyle\int_{\partial\Sigma} \mathbf{F} \cdot d\mathbf{x}$ for each vector field $\mathbf{F}$, where $\Sigma$ is the triangular piece of the plane $x + 2y + 3z = 6$ with vertices on the coordinate axes, and $\partial\Sigma$ is the boundary of $\Sigma$, oriented counterclockwise as seen from $(0, 0, 4)$.

    (i) $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ (ii) $\mathbf{F} = y^2\mathbf{i} + \mathbf{j} + x^2\mathbf{k}$.

12. Evaluate $\displaystyle\iint_\Sigma \mathbf{r} \cdot \mathbf{n}\, dS$, where $\mathbf{r} = (x, y, z)$ and $\Sigma$ is the unit sphere centred on the origin.

13. Suppose $T(x, y, z) = x^2 + y^2 + z^2$ represents the temperature in a region of space containing the origin of the coordinate system. Compute the heat flux across the unit sphere.

14. Compute the flux of the vector field $\mathbf{F} = (x^3, y^3, z^3)$ across the unit sphere.

15. Compute the circulation of the vector field

$$\mathbf{F}(x, y, z) = (x, x + y, x + y + z)$$

around the curve $C$ defined by the equations

$$x^2 + y^2 = 1 \quad \text{and} \quad z = y.$$

16. Let $S$ be the unit sphere centred on the origin. In each case, by sketching the vector field $\mathbf{F}$, make a conjecture as to whether the flux of $\mathbf{F}$ across $S$ in the outward direction is positive, negative or zero. Verify your conjecture by evaluating the surface integral

(i) $\mathbf{F} = (0, 0, z)$ (ii) $\mathbf{F} = (0, 0, z^2)$ (iii) $\mathbf{F} = (-y, x, 0)$ .

17. Suppose that $\mathbf{v} = (0, y, 0)$ is the velocity field of a fluid and let $S$ be the surface defined by $y = x^2 + z^2$, with $y \leq 1$. Calculate the volume of fluid crossing $S$ in unit time, in the direction of increasing $y$.

18. Suppose that $\Sigma$ is a smooth surface with area $S(\Sigma)$ and suppose that $\mathbf{F}$ is a continuous vector field on $\mathbb{R}^3$ that satisfies $\| \mathbf{F}(\mathbf{x}) \| \leq M$ for all $\mathbf{x} \in \Sigma$. Prove that

$$\left| \iint_{\Sigma} \mathbf{F} \cdot \mathbf{n} \, dS \right| \leq M S(\Sigma).$$

It is necessary to begin by using the definition of surface integral.

## Bonus questions

B1. *Non-Euclidean geometry*

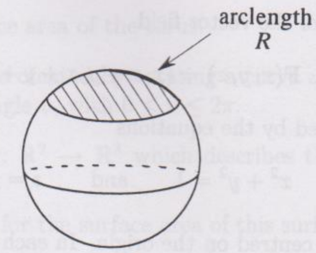
Let $\Sigma$ be a disc of 'radius' $R$ (see diagram) on the surface of a sphere of radius $b$, where $0 < R < \pi b$. Show that the surface area of the disc is

$$S = \pi R^2 f\left(\tfrac{R}{b}\right),$$

where $f$ is a function which you should determine. Show that $0 < S < \pi R^2$ for $0 < R < \pi b$.

*Comment*: This result shows that the intrinsic geometry on a sphere is different from that on the plane. The intrinsic geometry of surfaces is discussed in detail in AM 333.

B2. Let $\mathbf{x} = \mathbf{g}(u, v)$, $(u, v) \in \mathcal{D}_{uv}$ and $\mathbf{x} = \hat{\mathbf{g}}(p, q)$, $(p, q) \in \mathcal{D}_{pq}$ be parametrizations of a surface $\Sigma$.




i) What condition on the mapping $(u, v) = \mathbf{h}(p, q)$ that relates $p, q$ and $u, v$ will ensure that the orientation of the surface is the same for both parametrizations?

ii) Verify that the definition of the surface integral of a vector field is independent of the parametrization.

B3. Let $\mathbf{x} = \mathbf{h}(u)$, $a \le u \le b$, be a simple closed curve lying in a horizontal plane, and let $\mathbf{b}$ be a point not on the plane.

i) Describe the surface with parametrization

$$\mathbf{x} = \mathbf{g}(u, v) = \mathbf{b} + v\mathbf{h}(u),$$

and

$$a \le u \le b, \qquad 0 \le v \le 1.$$

ii) Describe the surface with parametrization

$$\mathbf{x} = \mathbf{g}(u, v) = \mathbf{h}(u) + v\mathbf{b},$$

and

$$a \le u \le b, \qquad 0 \le v \le \ell.$$

B4. Describe the surface corresponding to the following parametrization:

$$\mathbf{x} = \mathbf{g}(u, v) = \mathbf{p}(u) + v\mathbf{q}(u),$$

where

$$\mathbf{p}(u) = (\cos u, \sin u, 0),$$
$$\mathbf{q}(u) = \left(\cos \tfrac{u}{2}\right) \mathbf{p}(u) + \left(\sin \tfrac{u}{2}\right) \mathbf{k},$$

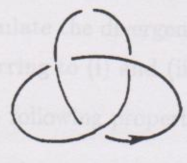
and

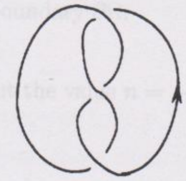
$$0 \le u \le 2\pi, \qquad -\tfrac{1}{2} \le v \le \tfrac{1}{2}.$$

B5. i) Find a parametrization for a trefoil knot in $\mathbb{R}^3$.

ii) Discuss whether the soap film surface bounded by the trefoil knot is orientable.

iii) The trefoil knot can be deformed continuously into the knot as shown. Discuss whether the soap film surface bounded by this knot is orientable.





## Problem Set 4:
## Gauss' Theorem and Stokes' Theorem

1. Let $\mathbf{F} = (x^2 + y^2 + z^2)^n (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$, and let $\Omega$ be the sphere of radius $b$ centered on the origin.

   (i) Calculate the outward flux of $\mathbf{F}$ across the boundary $\partial\Omega$.

   (ii) Calculate the divergence of $\mathbf{F}$, $\nabla \bullet \mathbf{F}$.

   (iii) Referring to (i) and (ii), what is special about the value $n = -\frac{3}{2}$?

2. Verify the following properties of the divergence.

   (i) *Sum* of two vector fields:
   $$\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}.$$

   (ii) *Product* of a scalar field and a vector field:
   $$\nabla \cdot (f\mathbf{F}) = f\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f$$

   (iii) *Vector product* of two vector fields:
   $$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}).$$

3. Verify the following properties of the curl.

   (i) *Sum* of two vector fields:
   $$\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}.$$

   (ii) *Product* of a scalar field and a vector field:
   $$\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + \nabla f \times \mathbf{F}.$$

   (iii) *Vector product* of two vector fields:
   $$\nabla \times (\mathbf{F} \times \mathbf{G}) = (\nabla \cdot \mathbf{G})\mathbf{F} - (\nabla \cdot \mathbf{F})\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$$

4. Verify the following "zero identities":

   (i) $\nabla \times (\nabla f) = \mathbf{0}$, for any $C^2$ scalar field $f$.

   (ii) $\nabla \cdot (\nabla \times \mathbf{F}) = 0$, for any $C^2$ vector field $\mathbf{F}$.

5. Let $f, g$ be $C^2$ scalar fields on $\mathbb{R}^3$.

   (i) Verify that $\nabla \times (f\nabla g) = \nabla f \times \nabla g$.

   (ii) Simplify $\quad \nabla \times (f\nabla g + g\nabla f)$.

   (iii) Verify that $\quad \nabla \cdot (f\nabla g) = f\nabla^2 g + \nabla f \cdot \nabla g$.

6. a) Verify the "curl of a curl" identity

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}.$$

   where $\nabla^2$ is the Laplacian operator.

   b) Use Maxwell's equations (see Section 4.1.1) to show that if the charge density $\varepsilon$ and current vector $\mathbf{J}$ are zero, the electric field vector $\mathbf{E}$ satisfies the "wave equation"

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} - c^2 \nabla^2 \mathbf{E} = 0.$$

   The magnetic field $\mathbf{H}$ satisfies a similar equation.

7. Calculate the flux of the vector field $\mathbf{F} = (x^3, y^3, z^3)$ outward across the unit sphere $x^2 + y^2 + z^2 = 1$ using Gauss' Theorem.

8. Evaluate $\displaystyle\iint_{\Sigma} \mathbf{F} \cdot \mathbf{n}\, dS$ in the following cases. Use Gauss' Theorem and/or symmetry whenever convenient.

   (a) $\mathbf{F} = (x^2, y^2, z^2)$, $\Sigma = \partial\Omega$, $\Omega = \{(x, y, z) : | x | \le 1, \ | y | \le 1, \ | z | \le 1\}$, $\mathbf{n}$ is the outward unit normal.

   (b) $\mathbf{F} = (y, x+2, z)$, $\Sigma$ is the part of the cylindrical surface $x^2+y^2 = 9$, $0 \le z \le 4$, $x \ge 0$, $y \ge 0$, $\mathbf{n}$ points away from the $z$-axis.

   (c) $\mathbf{F} = \nabla f$, $f(x, y, z) = xyz + 5$, $\Sigma$ is the sphere $x^2 + y^2 + z^2 = 9$, $\mathbf{n}$ is the outward unit normal.

9. Show that the volume $V$ of the region enclosed by a closed piecewise smooth surface $\Sigma$ can be expressed as

$$V = \tfrac{1}{3} \iint_{\Sigma} \mathbf{r} \cdot \mathbf{n}\, dS,$$

   where $\mathbf{n}$ is the unit outward normal and $\mathbf{r} = (x, y, z)$.

10. The electric potential $u(\mathbf{x})$ in $\mathbb{R}^3$ due to a point charge $q$ at $\mathbf{x_0}$ is given by

$$u(\mathbf{x}) = \frac{q}{\|\mathbf{x} - \mathbf{x_0}\|}.$$

   (i) Calculate the electric field $\mathbf{E} = -\nabla u$, and draw the field lines.

(ii) Verify that $\iint\limits_{\Sigma} \mathbf{E}\cdot\mathbf{n}\,dS = 4\pi q$, where $\Sigma$ is any piecewise smooth surface enclosing the charge. Verify that the above integral is zero if $\Sigma$ does not enclose the charge.

*Hint:* You are essentially being asked to prove Gauss' Law. See Sec. 4.2.3 for an outline of the proof.

11. One of Maxwell's equations reads
$$\nabla\cdot\mathbf{E} = 4\pi\varepsilon,$$
where $\varepsilon$ is the charge density (see equation (4.9)). Let $\mathcal{D}$ being a region of $\mathbb{R}^3$ bounded by a piecewise smooth closed surface $\partial\mathcal{D}$. Prove that the flux of the electric field through $\partial\mathcal{D}$ equals $4\pi$ times the total charge contained in $\mathcal{D}$ (i.e. the flux of the electric field across $\partial\mathcal{D}$ essentially measures the charge contained within $\partial\mathcal{D}$). Assume that $\mathbf{E}$ is a $C^1$ vector field in $\mathbb{R}^3$.

12. Consider a radial vector field
$$\mathbf{F} = f(r)\mathbf{r},$$
where $r = \parallel \mathbf{r} \parallel$ and $f$ is a $C^1$ function of $r$, for $r > 0$. Let $\Omega$ be the solid sphere of radius $b$ centred on the origin.

    i) Calculate the flux of $\mathbf{F}$ through the sphere $\partial\Omega$ of radius $b$.

    ii) Assuming that $f$ is $C^1$ for $r \geq 0$, calculate the integral of $\nabla\cdot\mathbf{F}$ over $\Omega$, and verify that Gauss' theorem is satisfied.

    iii) Show that if $\mathbf{F}$ satisfies $\nabla\cdot\mathbf{F} = 0$, then Gauss' theorem cannot be applied. What special property does the flux have in this case?

13. Verify Stokes' Theorem for the vector field $\mathbf{F} = (z, x, y)$ and the surface $\Sigma$ defined by $z = 1 - x^2 - y^2$, $z \geq 0$.

14. In each case, use Stokes' Theorem to evaluate the circulation $\int_C \mathbf{F}\cdot d\mathbf{x}$ of the vector field $\mathbf{F}$ around the closed curve $C$.

    (i) $\mathbf{F} = (x, x+y, , x+y+z)$, and $C$ is the curve defined by $x^2+y^2 = 1$, $z = y$, where the orientation of $C$ is counterclockwise when viewed from the point $(0, 0, 10)$.

    (ii) $\mathbf{F} = (x\sin y, -y\sin x, (x+y)z^2)$, and $C$ is the union of the line segments successively joining the points $(0,0,0)$, $(\frac{\pi}{2},0,0)$, $(\frac{\pi}{2},0,1)$, $(0,0,1)$, $(0,\frac{\pi}{2},1)$, $(0,\frac{\pi}{2},0)$ and $(0,0,0)$.

15. Use Stokes' Theorem to evaluate $\iint\limits_{\Sigma} \nabla\times\mathbf{F}\cdot\mathbf{n}\,dS$, where
$\mathbf{F} = (y-z, -x-z, x+y)$, $\Sigma$ is the portion of $z = 9 - x^2 - y^2$ with $z \geq 0$ and $\mathbf{n}$ is the upward pointing unit normal.

16. Determine whether the following vector fields $\mathbf{F}$ are conservative and if so, find a scalar potential $\phi$.

(i) $\mathbf{F} = xy(2yz\mathbf{i} + 2xz\mathbf{j} + xy\mathbf{k})$.

(ii) $\mathbf{F} = (y\cosh z, x\cosh z, xy\sinh z)$.

17. Consider the vector field

$$\mathbf{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, z\right), \quad x^2+y^2 \neq 0\,.$$

Verify that $\nabla \times \mathbf{F} = \mathbf{0}$ but that there does not exist a potential function on the domain of $\mathbf{F}$. Why does this not contradict the theorem on conservative vector fields?

18. Find the work done by the vector field

$$\mathbf{F} = e^{xy}(z + xyz,\ x^2 z,\ x)$$

in moving a particle along the curve

$$\mathbf{x} = \mathbf{g}(t) = (t\cos t,\ t\sin t,\ t), \quad 0 \le t \le 2\pi\,.$$

19. Any *constant* vector field $\mathbf{F}$ on $\mathbb{R}^3$ satisfies $\nabla \cdot \mathbf{F} = 0$ and hence has a vector potential, i.e.$\mathbf{F} = \nabla \times \mathbf{A}$. Find a suitable vector field $\mathbf{A}$. Is $\mathbf{A}$ uniquely determined?

20. A *collinear* vector field is a vector field of the form

$$\mathbf{F} = \lambda \mathbf{A},$$

where $\mathbf{A}$ is a constant vector field and $\lambda$ is an arbitrary $C^1$ scalar field. Find all collinear vector fields that satisfy

$$\nabla \times \mathbf{F} = 0 \quad \text{and} \quad \nabla \cdot \mathbf{F} = 0.$$

21. An *axially symmetric* vector field is a vector field of the form

$$\mathbf{F} = g(\rho, z)(-y, x, 0),$$

where $g$ is a $C^1$ function and $\rho = \sqrt{x^2+y^2}$.

i) Find all axially symmetric vectors that satisfy

$$\nabla \times \mathbf{F} = \mathbf{0} \quad \text{and} \quad \nabla \cdot \mathbf{F} = 0.$$

ii) Find all axially symmetric vector fields with the property that the circulation

$$\oint_C \mathbf{F} \cdot d\mathbf{x}$$

around a circle centred on the $z$-axis is independent of the radius of the circle.

iii) Show that any axially symmetric vector field has a vector potential of the form

$$\mathbf{A} = f(\rho, z)\mathbf{k}.$$

22. A *radial* vector field is a vector field of the form

$$\mathbf{F} = f(r)\mathbf{r},$$

where $f$ is a $C^1$ function of the radial distance $r = \| \mathbf{r} \|$, for $r > 0$.

i) Find all radial vector fields that satisfy

$$\nabla \times \mathbf{F} = \mathbf{0} \quad \text{and} \quad \nabla \cdot \mathbf{F} = 0.$$

ii) Find all radial vector fields $\mathbf{F} = f(r)\mathbf{r}$ with the property that the flux

$$\iint_{\Sigma} \mathbf{F} \cdot \mathbf{n} dS$$

through a sphere $\Sigma$ centred on the origin, is independent of the radius of the sphere.

23. The magnetic field due to a current $I$ flowing along the $z$-axis is given by

$$\mathbf{B} = \frac{2I}{c(x^2 + y^2)}(-y, x, 0), \quad x^2 + y^2 > 0.$$

i) Show that the circulation of the magnetic field around any simple closed curve that encircles the $z$-axis is given by

$$\oint_{C} \mathbf{B} \cdot d\mathbf{x} = \frac{2I}{c}.$$

*Hint:* First establish the result for any circle $x^2 + y^2 = b^2$, $z = h$.

ii) Show that

$$\oint_{C} \mathbf{B} \cdot d\mathbf{x} = 0$$

for any simple closed curve that does not encircle the $z$-axis.

24. The *wave equation* in three dimensions is the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0,$$

where $u = u(\mathbf{x}, t)$.

i) Verify that

$$u = \frac{1}{r} f(r - ct),$$

where $r = \| \mathbf{x} \| = \sqrt{x^2 + y^2 + z^2}$ and $f$ is an arbitrary $C^2$ function, is a solution of the wave equation. This solution is called a *spherical wave*, because the value of $u$ for fixed $t$ is constant on spheres centred on the origin.

ii) Verify that

$$u = f(\mathbf{k} \cdot \mathbf{x} - ct),$$

where $\mathbf{k}$ is a constant unit vector and $f$ is an arbitrary $C^2$ function, is a solution of the wave equation. This solution is called a *plane wave*, because the value of $u$ for fixed $t$ is constant on planes, the planes orthogonal to $\mathbf{k}$.

25. Consider the vector field $\mathbf{F} = xy\mathbf{e}_x + y^2\mathbf{e}_y$ in $\mathbb{R}^2$ in terms of Cartesian coordinates.

    i) Show that $\mathbf{F} = \rho^2 \sin\phi \mathbf{e}_\rho$ in terms of polar coordinates.

    ii) Calculate the divergence $\nabla \cdot \mathbf{F}$ first using Cartesian coordinates and then using polar coordinates. Verify that the expressions are equal.

26. i) Starting with the expression (4.38) for $\nabla$ in *cylindrical coordinates*, derive the expression (4.48) for the divergence $\nabla \bullet \mathbf{F}$ of a vector field.

    ii) Hence derive the expression (4.50) for the Laplacian $\nabla^2 f$ of a scalar field.

27. i) Starting with the expression (4.40) for $\nabla$ in *spherical coordinates*, derive the expression (4.49) for the divergence $\nabla \bullet \mathbf{F}$ of a vector field.

    ii) Hence derive the expression (4.51) for the Laplacian $\nabla^2 f$ of a scalar field.

28. Elliptic coordinates $(\eta, \phi)$ in the plane are defined by the equation

$$(x, y) = (a\cosh\eta\cos\phi,\ a\sinh\eta\sin\phi),$$

where $a$ is a positive constant.

    i) Describe the two families of coordinate curves $\eta =$ constant and $\phi =$ constant, and illustrate them with a sketch.

    ii) Calculate the coordinate basis vector fields $\mathbf{e}_\eta$ and $\mathbf{e}_\phi$, and show that they are mutually orthogonal.

## Bonus questions

B1. *Acoustics* (ref. W.A. Strauss, Partial Differential Equations)

The linearized equations for propagation of sound in a gas are

$$\frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot \mathbf{v} = 0, \quad \frac{\partial \mathbf{v}}{\partial t} = -\frac{c}{\rho_0} \nabla \rho,$$

where $\rho_0$ is the undisturbed density, a constant and $c$ is also a constant. The unknowns are the perturbed density $\rho$ of the gas and the velocity $\mathbf{v}$ of the disturbance.

i) Show that the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ is constant in time.

ii) Verify that the density $\rho$ satisfies the wave equation

$$\rho_{tt} = c^2 \nabla^2 \rho.$$

iii) Verify that if $\nabla \times \mathbf{v} = 0$, then the velocity $\mathbf{v}$ satisfies the wave equation

$$\mathbf{v}_{tt} = c^2 \nabla^2 \mathbf{v}.$$

B2. Let $\mathbf{r}$ be the position vector of a planet or satellite moving in a plane orbit. Choose coordinates so that the orbit lies in the $xy$-plane. Let $\mathbf{e}_r$ and $\mathbf{e}_\theta$ be the unit basis vector associated with polar coordinates in the plane. The planet's position is described by

$$r = r(t) \quad \text{and} \quad \theta = \theta(t),$$

where $t$ is time.

a) Starting with

$$\mathbf{r} = r\mathbf{e}_r,$$

derive the following equations:

i) $\mathbf{r} \times \mathbf{r}' = r^2 \theta' \mathbf{k}$

ii) $\mathbf{r} \cdot \mathbf{r}'' = r[r'' - r(\theta')^2]$,

where $'$ denotes differentiation with respect to time.

b) Newton's law of gravitation and Newton's second law of motion lead to

$$\mathbf{r}'' = -\frac{GM}{r^3}\mathbf{r}, \tag{1}$$

where $M$ is the mass of the sun and $G$ is the gravitational constant (see B8 in Problem Set 1). Equation (1) is a vector DE for the position vector $\mathbf{r}(t)$ of the planet at time $t$. One can solve it by using the results of a) to reduce it to a scalar DE as follows.

First show that $\mathbf{r} \times \mathbf{r}'$ is constant. Then use a) to show that

$$r^2 \theta' = H$$

$$r'' - r(\theta')^2 = -\frac{GM}{r^2},$$

where $H$ is a constant, the angular momentum per unit mass of the planet.

c) The orbit of the planet can be described by eliminating time $t$, thereby expressing $r$ as a function of $\theta$,

$$r = r(\theta).$$

The resulting DE is much simplified if one introduces the new variable

$$u = \frac{1}{r}.$$

Show that a planetary orbit satisfies

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{H^2}.$$

*Hint:* Show that $\dfrac{dr}{dt} = -H\dfrac{du}{d\theta}$.

Hence show that planetary orbits are ellipses.

B3. The motion of an incompressible fluid with constant density $\rho$ and constant viscosity $\mu$ is governed by the partial differential equation

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = \rho\mathbf{F} - \nabla p + \mu\nabla^2\mathbf{v}\,,$$

where $\mathbf{v}(\mathbf{x}, t)$ is the velocity field, $p(\mathbf{x}, t)$ the pressure and $\mathbf{F}$ the external force acting (e.g. gravity). The *vorticity field* of the fluid is defined by

$$\boldsymbol{\omega} = \nabla \times \mathbf{v}\,.$$

Derive the *vorticity equation*

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{v}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{v} + \frac{\mu}{\rho}\nabla^2\boldsymbol{\omega} + \nabla\times\mathbf{F}\,.$$

*Hint:* Use identity $G_3$ to calculate $(\mathbf{v}\cdot\nabla)\mathbf{v}$.

B4. Let $\Omega$ be a bounded subset in $\mathbb{R}^3$, whose boundary is a simple piecewise smooth surface $\partial\Omega$. Let $\mathbf{n}$ be the unit outward normal on $\partial\Omega$.

i) Prove that

$$\iiint\limits_{\Omega} \nabla f\, dV = \iint\limits_{\partial\Omega} f\mathbf{n}dS,$$

for any $C^1$ scalar field $f$.

*Hint:* Apply Gauss' theorem to $\mathbf{F} = f\mathbf{C}$, where $\mathbf{C}$ is a constant vector field.

ii) Prove that

$$\iiint_{\Omega} (\nabla \times \mathbf{F}) dV = \iint_{\partial\Omega} (\mathbf{n} \times \mathbf{F}) dS,$$

for any $C^1$ vector field $\mathbf{F}$.

*Comment:* In the notes we defined two types of surface integrals:

i) $\displaystyle\iint_{\Sigma} f dS$, where $f$ is a scalar field,

ii) $\displaystyle\iint_{\Sigma} \mathbf{F} \cdot \mathbf{n} dS$, where $\mathbf{F}$ is a vector field.

Both of these integrals are *scalar-valued.* Question B2 uses *vector-valued* surface integrals of the form

$$\iint_{\Sigma} \mathbf{F} dS.$$

As part of the solution you will need to define this concept.

B5. A body $\Omega$ with surface $\partial\Omega$ immersed in a fluid experiences a bouyancy force $\mathbf{B}$ given by

$$\mathbf{B} = -\iint_{\partial\Omega} p\mathbf{n} dS.$$

Here the pressure $p$ is a $C^1$ scalar field satisfying

$$\nabla p = \rho \mathbf{g},$$

where $\rho(\mathbf{x})$ is the mass density of the fluid and $\mathbf{g}$ the constant acceleration of gravity.

Show that $\mathbf{B}$ equals the weight of the displaced fluid in magnitude, and acts in the opposite direction to the gravitational force.

B6. Let $D$ be a bounded subset of $\mathbb{R}^3$ whose boundary is a piecewise smooth closed surface $\partial D$, with outward unit normal $\mathbf{n}$. Let $f$ be a $C^1$ scalar field and $\mathbf{G}$ be a $C^1$ vector field on $\mathbb{R}^3$. Prove the generalized *integration by parts formula:*

$$\iiint_{D} \nabla f \cdot \mathbf{G}\, dV = \iint_{\partial D} f\mathbf{G} \cdot \mathbf{n}\, dS - \iiint_{D} f \nabla \cdot \mathbf{G}\, dV\,.$$

## Problem Set 5:

## Fourier series and Fourier transforms

### (a) Fourier Series

1. Consider the full-wave rectification function $f$ defined by

$$f(x) = |\sin x|, \quad \text{for} \quad -\pi < x < \pi.$$

   a) Find the Fourier series of $f$.

   b) Sketch the graph of the function to which the series converges pointwise on $\mathbb{R}$. Use the pointwise convergence theorem to justify your answer.

   c) By choosing a suitable value of $x$ in the series that you have found, show that $\sum_{k=1}^{\infty} \frac{1}{4k^2 - 1} = \frac{1}{2}$.

2. Let $f(x) = \begin{cases} 0 & \text{if} \quad -\pi < x < 0 \\ x & \text{if} \quad 0 < x < \pi \end{cases}$.

   (a) Find the Fourier series of $f$.

   (b) Sketch the graph of function to which the series converges pointwise on $\mathbb{R}$. Justify your answer.

   (c) Show that $\displaystyle\sum_{m=1}^{\infty} \frac{1}{(2m-1)^2} = \frac{\pi^2}{8}$.

3. For each of the following functions defined on the interval $(0, \pi)$:

   (a) Find the Fourier sine series.

   (b) Find the Fourier cosine series.

   (c) Sketch the functions to which the Fourier sine and cosine series converge on $\mathbb{R}$.

   (i) $f(x) = 2x - \pi$ (ii) $f(x) = \begin{cases} 1 & \text{if} \quad 0 < x < \pi/2 \\ -1 & \text{if} \quad \pi/2 < x < \pi \end{cases}$ (iii) $f(x) = \sin x$.

4. (a) Show that the Fourier series for $f(x) = x^2$, $-\pi < x < \pi$, is

$$\frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos(nx)}{n^2}.$$

(b) Use the pointwise convergence theorem to find the pointwise sum function of the series on $\mathbb{R}$. Sketch its graph.

(c) Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

(d) Show that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$.

5. Let $f$ be defined on $0 < x < \pi$ by

$$f(x) = \begin{cases} \frac{1}{2\epsilon}, & \text{if} \quad |x - \frac{\pi}{2}| < \epsilon \\ 0 & \text{otherwise.} \end{cases}$$

a) Show that the Fourier sine series of $f$ is

$$\frac{2}{\pi} \sum_{k=1}^{\infty} (-1)^{k-1} \operatorname{sinc}\left[(2k-1)\varepsilon\right] \sin(2k-1)x.$$

b) Sketch the graph of the function to which the series converges on $\mathbb{R}$.

6. Consider the triangular window (or sawtooth pulse) function defined on the interval $-\frac{1}{2}\tau \le t \le \frac{1}{2}\tau$ by

$$T_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon}\left(1 - \frac{1}{\varepsilon\tau}|t|\right), & \text{if} \quad |t| \le \varepsilon\tau \\ 0, & \text{otherwise.} \end{cases}$$

Show that the Fourier cosine coefficients of $T_\varepsilon$ are given by

$$a_n = 2\left[\operatorname{sinc}(n\pi\varepsilon)\right]^2.$$

7. i) Show that the Fourier sine series of the function $f$ defined on the interval $0 < x < \pi$ by

$$f(x) = \begin{cases} x, & 0 < x < \pi - \varepsilon \\ \frac{(\pi-\varepsilon)}{\varepsilon}(\pi - x), & \pi - \varepsilon < x < \pi. \end{cases}$$

is

$$2\sum_{1}^{\infty} (-1)^{n-1} \operatorname{sinc}(n\varepsilon) \frac{\sin nx}{n}.$$

ii) Sketch the graph of the function $f_p$ to which this series converges on $\mathbb{R}$.

8. Let $f$ be a $C^1$ function on $[-\pi, \pi]$. Prove that the Fourier coefficients of $f$ satisfy

$$| a_n | \le \frac{K}{n}, \quad | b_n | \le \frac{L}{n}, \quad n = 1, 2, \cdots ,$$

for some constants $K$ and $L$.

9. a) Derive the identity

$$\tfrac{1}{2} + \cos\theta + \cdots + \cos n\theta = \frac{\sin\left(n + \frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}.$$

*Hint:* Sum $1 + z + z^2 + \cdots + z^n$ and let $z = e^{i\theta}$.

b) The Dirichlet kernel is defined by $D_n(x) = \dfrac{\sin(n + \frac{1}{2})x}{2\pi \sin\frac{1}{2}x}$ , $-\pi \le x \le \pi$. Give a qualitative sketch of the graph for large $n$.

10. The Fourier series for the function $f(x) = \frac{1}{2}x$, $-\pi < x < \pi$ is

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sin nx}{n}.$$

a) Sketch the graph of the function $f_p$ which is the pointwise sum of the series on $\mathbb{R}$.

b) Use Parseval's formula to show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

c) Use the integration theorem to show that

$$\tfrac{1}{12}(\pi^2 - 3x^2) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos nx}{n^2} \quad , \quad -\pi < x < \pi$$

and

$$\tfrac{1}{12}x(\pi^2 - x^2) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sin nx}{n^3} \quad , \quad -\pi < x < \pi$$

d) Discuss whether the sum of each Fourier series in c) is piecewise continuous, continuous, piecewise $C^1$ or $C^1$ on $\mathbb{R}$. Sketch the graphs of the sum functions.

11. Apply Parseval's theorem to one of the series in #10 to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

12. Use Parseval's formula and a suitable Fourier series to sum $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

*Hint:* Look at Table 5.1.

13. Consider the function $f$ defined by $f(x) = b, \quad 0 < x < L$.

    i) Extend $f$ as an odd function and find its Fourier series (the sine series).

    ii) Discuss convergence of the series on $\mathbb{R}$, and sketch the graph of its sum function.

14. i) Prove that the series
$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$$
converges pointwise on $\mathbb{R}$, and uniformly on any finite interval.. Call its sum $f(x)$.

    ii) Obtain the Fourier series of the function $g$ defined by
$$g(x) = \int_0^x f(t)dt,$$
justifying your method by referring to appropriate theorems.

    iii) What can you say about continuity and smoothness of the function $g$? Explain.

### (b) Fourier transforms

15. Show that the Fourier transform of
$$f(t) = e^{-|t|} \text{ is } F(\omega) = \frac{2}{1+\omega^2}.$$
Sketch the graphs in the time domain and in the frequency domain.

16. Show that
$$\mathcal{F}(T(t)) = \left[\text{sinc } \left(\tfrac{1}{2}\omega\right)\right]^2,$$
where $T(t)$ is the triangular gate function:
$$T(t) = \begin{cases} 1 - |t|, & \text{if } \quad |t| < 1 \\ 0, & \text{if } \quad |t| > 1. \end{cases}$$
Sketch the graphs of $T(t)$ and its Fourier transform.

17. Find the Fourier transform of
$$f(t) = \begin{cases} 1, & -1 < t < 0 \\ -1, & 0 < t < 1 \\ 0, & |t| > 1. \end{cases}$$
in two ways,

a) using the definition, and

b) using the fact that $f(t) = T'(t)$, where $T(t)$ is the triangular gate function.

18. i) Knowing $\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$, evaluate

$$I(\omega) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} \cos \omega x dx.$$

*Hint:* Use Leibniz's theorem to show that

$$I'(\omega) = -\omega I(\omega).$$

ii) Use i) to show that

$$\mathcal{F}(e^{-\frac{1}{2}x^2}) = \sqrt{2\pi}\, e^{-\frac{1}{2}\omega^2}.$$

19. a) Find the Fourier transform $F(\omega)$ of the normal density

$$f(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-\mu)^2}{2\sigma^2}},$$

where $\sigma, \mu$ are real parameters with $\sigma > 0$.

b) Sketch the graphs of $f(t)$ and of $Re\ F(w)$ and $Im\ F(w)$.

20. a) Find the Fourier transform $F(\omega)$ of

$$f(t) = \cos(2\pi\nu t) e^{-\pi(t/\alpha)^2},$$

where $\alpha$ and $\nu$ are positive real numbers.

b) Sketch the graphs of $f(t)$ and $F(\omega)$.

21. a) Verify that if

$$g(t) = tf(t), \quad \text{then} \quad G(\omega) = iF'(\omega),$$

where $F(\omega) = \mathcal{F}(f(t))$ and $G(\omega) = \mathcal{F}(g(t))$.

b) Find the Fourier transform of $f(t) = te^{-t^2}$ in two ways.

22. Show that if $f$ is odd then

$$\mathcal{F}(f(t)) = -2i \int_0^{\infty} f(t) \sin \omega t\, dt.$$

23. a) Show that if $\mathcal{F}(f(t)) = F(\omega)$, then

$$\mathcal{F}(f(t - c)) = e^{-ic\omega} F(\omega).$$

b) Hence find the Fourier transform of

$$f(t) = \tfrac{1}{2}\left[W(t-c) + W(t+c)\right],$$

where $W(t)$ is the window function (5.83). Give a qualitative sketch of the graphs of $f(t)$ and $\mathcal{F}(f(t))$.

24. a) Show that if $\mathcal{F}(f(t)) = F(\omega)$, then

$$\mathcal{F}(F(t)) = 2\pi f(-\omega).$$

b) Use a) and known results to show that

i) $\mathcal{F}\left(\dfrac{1}{1+x^2}\right) = \pi e^{-|\omega|}$

ii) $\mathcal{F}(\text{sinc }(x)) = \pi W\left(\tfrac{1}{2}\omega\right)$.

25. Find the Fourier transform of

$$g(t) = \begin{cases} \cos \pi t, & \text{if } \quad -\tfrac{1}{2} < t < \tfrac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

26. a) Verify that the Fourier transform of $f(t) = e^{-t}H(t)$ is $F(\omega) = \frac{1}{1+i\omega}$, where

$$H(t) = \begin{cases} 1 & \text{if } \; t > 0 \\ 0 & \text{if } \; t < 0 \end{cases}$$

is the Heaviside step function.

b) Verify that if $g(t) = f(-t)$, then the Fourier transforms are related by $G(\omega) = F(-\omega)$.

c) Use a) and b) to verify that

$$\mathcal{F}(e^{-|t|}) = \frac{2}{1+\omega^2}.$$

## (c) Convergence

27. Consider the sequence $\{f_n\}$ of functions defined by

$$f_n(x) = \frac{x^{2n}}{1+x^{2n}}, \quad \text{for} \quad |x| \le 2.$$

i) Find the pointwise limit function $f$. Sketch the graph of $f$ and a typical $f_n$ on the same axes.

ii) Sketch the graph of $f_n - f$ and evaluate $\| f_n - f \|_\infty$. Does $\{f_n\}$ converge uniformly to $f$ on $[-2, 2]$?

28. Consider the sequence of functions $\{f_n\}$ defined by

$$f_n(x) = nxe^{-\frac{1}{2}n^2x^2} - x, \quad -1 \le x \le 1.$$

Let $f$ denote the pointwise limit function.

i) Determine whether $\{f_n\}$ converges uniformly to $f$.

ii) Determine whether $\{f_n\}$ converges in the mean to $f$.

iii) Sketch graphs of $f$ and a typical function $f_n$ on the same axes.

29. (a) Let $f_n(x) = (\sin x)^n$, $0 \le x \le \pi$. Find $f(x) = \lim_{n\to\infty} f_n(x)$. Does $\{f_n\}$ converge uniformly to $f$? Does $\{f_n\}$ converge to $f$ in the mean?

(b) Repeat (a) for $f_n(x) = (\sin x)^{\frac{1}{n}}$.

30. Consider the sequence $\{f_n\}$ on the interval $0 \le x \le 1$ defined by

$$f_n(x) = \frac{2n\,x}{1 + n^2x^4}.$$

a) Find the pointwise limit function $f$.

b) Evaluate $\int_0^1 f(x)dx$ and $\lim_{n\to\infty} \int_0^1 f_n(x)dx$.

c) By referring to appropriate theorems, draw a conclusion about whether $\{f_n\}$ i) converges in the mean to $f$ and ii) converges uniformly to $f$.

## Bonus questions

B1. *Detective Work*

Jack calculates some Fourier coefficients and as a result, writes the formula

$$\frac{x}{2} = \sum_{1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n},$$

but forgets to specify the interval of $x$ values. Jill does a similar calculation and arrives at the formula

$$\frac{x}{2} = \frac{2}{\pi} \sum_{1}^{\infty} (-1)^{n+1} \frac{\sin(2n-1)x}{(2n-1)^2},$$

but likewise forgets to state the interval.

In each case, deduce the appropriate interval, and then sketch the graph of the function to which the series converges on $\mathbb{R}$.

B2. *Fourier gives Laplace a headache*

When Fourier first introduced, in 1807, the series that now bear his name, some of the leading mathematicians of the time, e.g., Laplace and Lagrange, were skeptical because some of the results were initially counterintuitive. Here's an example (see, T. Körner, 1988, page 478).

By calculating some Fourier coefficients, Fourier arrived at the amazing formula

$$\cos x = \frac{8}{\pi} \sum_{n=1}^{\infty} \left( \frac{n}{4n^2 - 1} \sin 2nx \right),$$

but forgot to write down the interval of validity. Laplace would not accept that this expansion was true because cos( ) is an *even* function, sin( ) is an odd function, and a sum of odd functions is odd.

i) Repeat Fourier's calculation.

ii) Deduce the interval of validity of the above expansion, showing that Laplace's concerns were unwarranted.

iii) Sketch the graph of the sum function of the series, valid for all $x \in \mathbb{R}$.

iv) By integrating, derive a cosine series for $\sin x$, and sketch the graph of the sum fucntion of the series, valid for all $x \in \mathbb{R}$.

B3. *Frequency separation*

a) Consider the rectangular window (or pulse) function $P_\varepsilon$ defined on the interval $-\frac{1}{2}\tau \leq t \leq \frac{1}{2}\tau$ by

$$P_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon}, & \text{if} \quad |t| \leq \frac{1}{2}\varepsilon\tau \\ 0, & \text{otherwise,} \end{cases}$$

where $\varepsilon$ is a constant satisfying $0 < \varepsilon \leq 1$. Show that the Fourier cosine coefficients of $P_\varepsilon$ are given by

$$a_n = 2\text{sinc}\,(n\pi\varepsilon).$$

b) Consider the pulse modulated carrier $\cos(N\omega_0 t)$, defined on the interval $-\frac{\tau}{2} \leq t \leq \frac{\tau}{2}$ by

$$f(t) = P_\varepsilon(t)\cos(N\omega_0 t),$$

where $\omega_0 = \frac{2\pi}{\tau}$ and $N$ is a fixed positive integer ($N\omega_0$ is called the *carrier frequency* of the signal). Show that the Fourier cosine coefficients of $f$ are given by

$$a_n = \text{sinc}\,[(n+N)\pi\varepsilon] + \text{sinc}\,[(n-N)\pi\varepsilon].$$

c) Suppose that $N\pi\varepsilon \gg 1$. Give a qualitative sketch of the Fourier amplitude spectrum of $f$.

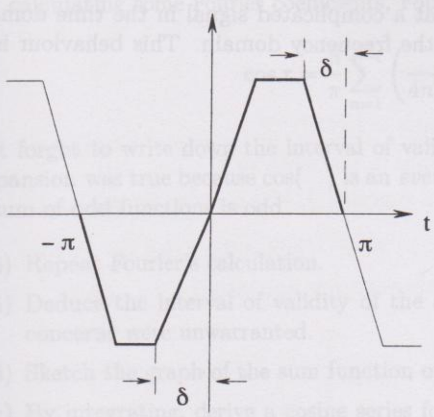
d) Consider a combined signal of the form

$$f(t) = A_1 P_\varepsilon(t)\cos(N_1\omega_0 t) + A_2 P_\varepsilon(t)\cos(N_2\omega_0 t),$$

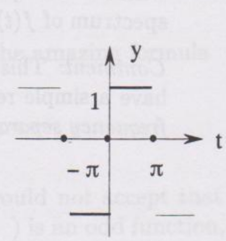
where $1 \ll N_1\pi\varepsilon \ll N_2\pi\varepsilon$. Give a qualitative sketch of the Fourier amplitude spectrum of $f(t)$.

*Comment:* This result shows that a complicated signal in the time domain can have a simple representation in the frequency domain. This behaviour is called *frequency separation.*

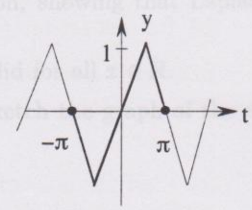
B4. *The trapezoidal wave*




A trapezoidal wave $\left(0 < \delta < \frac{\pi}{2}\right)$




A square wave $(\delta = 0)$




A saw-tooth wave $\left(\delta = \frac{\pi}{2}\right)$

i) Show that the Fourier series of the trapezoidal wave is

$$\frac{4}{\pi}\sum_{n=1}^{\infty} \operatorname{sinc}\left[(2n-1)\delta\right] \frac{\sin(2n-1)t}{(2n-1)}.$$

ii) Hence find the Fourier series of the square wave and triangle wave by taking suitable limits of the $n^{\text{th}}$ term.

*Note:* The goal is to do a complicated calculation with finesse.

B5. Prove that the trigonometric series

$$\sum_{n=2}^{\infty} \frac{\sin nx}{\ln n}$$

converges pointwise for all $x$, but that it is not the Fourier series for any square integrable function.

B6. i) Show that the function $f$ defined by

$$f(x) = -\ln\left|2\sin\tfrac{x}{2}\right|, \quad \text{for} \quad |x| \le \pi, \quad x \neq 0, \quad \text{and} \quad f(0) = 0,$$

is square integrable but not piecewise continuous on $[-\pi, \pi]$.

ii) Find the Fourier series of $f$, and show that the series does not converge pointwise at $x = 0$.

*Comment:* Since $f$ is square integrable the series converges to $f$ in the mean.

B7. It is known that the equilibrium temperature distribution $u(x, y)$ in a plane sheet of metal satisfies Laplace's equation:

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

(see B4. in Problem Set 2).

i) For a sheet in the shape of a disc, it is convenient to use polar coordinates $\rho$ and $\phi$. Consider the series representation of the temperature function:

$$u(\rho, \phi) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} \rho^n (a_n \cos n\phi + b_n \sin n\phi).$$

Show that this function satisfies Laplace's equation $\nabla^2 u = 0$ for $0 \le \rho \le b$, $0 \le \phi \le 2\pi$. Assume that term-by-term differentiation is permissible.

ii) Hence find the temperature of a metal disc of radius $b$, if the temperature on the boundary is $100°$ for $0 < \phi < \pi$ and $0°$ for $\pi < \phi < 2\pi$.