CO481/CS467/PHYS467 ASSIGNMENT 2
Due Wednesday, January 30th at 11:59pm, electronically using LEARN
(will constitute 10% out of the 50% assignment marks)

1. **(3 marks)**

   Suppose Alice and Bob are playing the following game.

   A referee gives Alice a bit s and Bob a bit t. Alice outputs a bit a and Bob outputs a bit b. Alice and Bob win if $a \oplus b = s \wedge t$.

   Alice and Bob may meet in advance of the game and agree upon any strategy they wish, including sharing private random bits. They may not communicate during the execution of the game. The referee may pick s and t according to any strategy she wishes, although they cannot depend on the private random bits of Alice and Bob.

   Alice and Bob will execute many iterations of this game.

   Note that there are four possible functions from $\{0,1\}$ to $\{0,1\}$: identity ($b \mapsto b$), NOT ($b \mapsto \overline{b}$), constant-0 ($b \mapsto 0$) and constant-1 ($b \mapsto 1$).

   Note that if on one iteration of this game, Alice and Bob both output "0" regardless of the values of their input bits (i.e. they both implement the constant-0 function), then they win the game on all inputs except $(s,t) = (1,1)$ (i.e. they win on 3 out of 4 inputs, without using quantum mechanics).

   (a) Note that if the referee suspects they are using the above strategy, she might deliberately pick $s = t = 1$ and thwart their strategy.

   Give three other strategies, one for each of $(s', t') \in \{(0,0), (0,1), (1,0)\}$, so that Alice and Bob win the game if and only if $(s,t) \neq (s',t')$.

   (b) Give a randomized strategy for Alice and Bob whereby they win the game with probability 75% regardless of the strategy for choosing s and t used by the referee.

2. **partial measurement (2 marks)**

   Suppose you are given the two qubit state

$$\frac{3}{5\sqrt{2}}|00\rangle + \frac{4}{5\sqrt{2}}|01\rangle + \frac{4}{5\sqrt{2}}|10\rangle + \frac{3}{5\sqrt{2}}|11\rangle$$

   and you measure the first qubit.

   (a) What is the probability of measuring $|0\rangle$?

   (b) In this case (that you obtain $|0\rangle$), what is the resulting state of the second qubit?

3. **Gates through teleportation (4 marks)** Alice wishes to implement a one-qubit unitary U. Suppose she possesses the entangled state $|\psi_U\rangle = \frac{1}{\sqrt{2}}|0\rangle U|0\rangle + \frac{1}{\sqrt{2}}|1\rangle U|1\rangle$. Furthermore, let us assume that $UX = \pm XU$ and $UZ = \pm ZU$.

   Suppose Alice has a qubit in some arbitrary state $|\phi\rangle = a|0\rangle + b|1\rangle$, and performs a Bell measurement on her qubit in the state $|\phi\rangle$ and the first qubit of $|\psi_U\rangle$.

   (a) For each of the four possible Bell measurement outcomes, what is the resulting state of the second qubit (i.e. the other qubit of $|\psi_U\rangle$)? (Express your answer simply in terms of U, $|\phi\rangle$, and the Pauli operators.)

   (b) How can Alice use the results of the Bell measurement to correct the output state of the remaining qubit to be the state $U|\phi\rangle$ (up to global phase)?

4. **(4 marks)**

   a) Find the density matrix of the state $\frac{1}{\sqrt{3}}|0\rangle + \frac{e^{i2\pi/3}}{\sqrt{3}}|1\rangle + \frac{e^{i4\pi/3}}{\sqrt{3}}|2\rangle$.

   b) Find the density matrix of the state $\left\{\left(|0\rangle, \frac{1}{4}\right), \left(|1\rangle, \frac{1}{4}\right), \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle, \frac{1}{2}\right)\right\}$.

   c) Find a different mixture, involving only two pure states, that gives the same mixed state as in part b).

   d) Give a two-qubit pure state $|\Psi\rangle$ such that $Tr_2|\Psi\rangle\langle\Psi|$ equals the state in part b).

5. **(4 marks)**

   Consider $|\psi_1\rangle = \frac{3}{4}|00\rangle + \frac{1}{4}|01\rangle - \frac{\sqrt{3}}{4}|10\rangle + \frac{\sqrt{3}}{4}|11\rangle$ which has Schmidt decomposition $\left(\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle\right))\frac{\sqrt{3}}{2}|0\rangle + \left(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\right)\frac{1}{2}|1\rangle$.

   (a) Compute the 2×2 matrix corresponding to the partial trace $Tr_2(|\psi_2\rangle\langle\psi_2|)$ of $|\psi_2\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2\sqrt{2}}|01\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|11\rangle$.

   (b) Find probabilities p_1 and p_2 and eigenvector states $|\phi_1\rangle$ and $|\phi_2\rangle$ such that $Tr_2(|\psi_2\rangle\langle\psi_2|) = p_1|\phi_1\rangle\langle\phi_1| + p_2|\phi_2\rangle\langle\phi_2|$.

   (c) Express the state $|\psi_2\rangle$ in its Schmidt decomposition form.

   (d) Find a 1-qubit unitary U such that $I \otimes U(|\psi_2\rangle) = |\psi_1\rangle$.

6. **(1 mark)**

   What is the Bloch vector of the following state?

$$\rho = \begin{pmatrix} \frac{1}{2} - \frac{1}{2\sqrt{2}} & -\frac{i}{2\sqrt{2}} \\ \frac{i}{2\sqrt{2}} & \frac{1}{2} + \frac{1}{2\sqrt{2}} \end{pmatrix}$$

7. **(1 mark)**

   Let $\phi \in (0, \pi/3)$. Prove that

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i\phi} & 0 & 0 \\ 0 & 0 & e^{i\phi} & 0 \\ 0 & 0 & 0 & e^{i3\phi} \end{pmatrix}$$

   is entangling.

8. **(1 mark)**

   Prove that

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

   is *not* entangling.