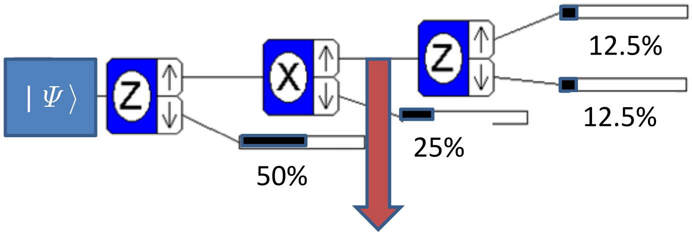
Projectors and Selective measurements



unnormalized state $P_{|+\rangle_x} P_{|+\rangle_z} |\Psi\rangle = |+\rangle_{xx}\langle +|+\rangle\langle +|\Psi\rangle$

normalized state: $|+\rangle_x$

probability amplitude $_x\langle +|+\rangle\langle +|\Psi\rangle$

for prediction of last z-measurement:
(renormalize state)

$$|\Psi\rangle \to |\Psi'\rangle = \frac{P_{|+\rangle_x} P_{|+\rangle} |\Psi\rangle}{\sqrt{\langle \Psi | P_{|+\rangle} P_{|+\rangle_x} P_{|+\rangle} |\Psi\rangle}} = |+\rangle_x$$

Conditional probabilities:

use $|\Psi'\rangle$: Prob("+") = $|\langle + | \Psi' \rangle|^2 = 1/2$

Prob("-") = $|\langle - | \Psi' \rangle|^2 = 1/2$

Postulate 1:
The state of a quantum mechanical system is represented mathematically by a *normalized* vector, a symbol ket $|\Psi\rangle$. This symbol
- summarizes everything you can know about the system
- and everything you need to know to predict measurement results
- corresponds to an element of a complex vector space of suitable dimension.

Postulate 3:
A measurement with mutually exclusive outcomes can be described by a set of orthonormal basis vectors $\{ |\phi_i\rangle \}$, i = 1, … , d

Postulate 4
For an input state described by (normalized) ket $|\Psi\rangle$
and a measurement with mutually exclusive events "i" described by elements of an orthonormal basis $\{ |\phi_i\rangle \}$, i = 1, … , d
the probability Pr("i") to observe outcome "i" is given by

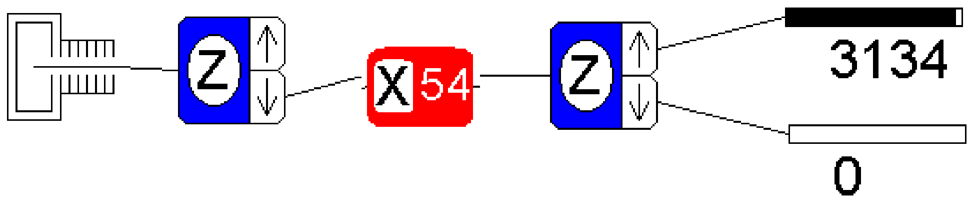
$$\Pr(\text{"i"}) = |\langle \phi_i | \Psi \rangle|^2$$

Postulate 5 (selective measurement)
After a measurement on an input state associated with state vector $|\Psi\rangle$
with outcome "i" associated with ket $|\phi_i\rangle$ and corresponding projector $P_{|\phi_i\rangle}$
the outgoing state is described by state vector $\frac{P_{|\Phi_i\rangle}|\Psi\rangle}{\sqrt{\langle\Psi|P_{|\Phi_i\rangle}|\Psi\rangle}}$

4.6 Unitary operations

4.6.1 Basic observations

Exploring a new device: Box X54



input		output
$\vert +\rangle$	➔	$\vert -\rangle$
$\vert -\rangle$	➔	$\vert +\rangle$
$\vert +\rangle_x$	➔	$\vert +\rangle_x$
$\vert -\rangle_x$	➔	$-\vert -\rangle_x$
$\vert +\rangle_y$	➔	$i\,\vert -\rangle_y$
$\vert -\rangle_y$	➔	$-i\,\vert +\rangle_y$

(phase factors i not observable, inserted for consistency)

U is an operator which changes the state of the system

our example:

$$\mathcal{U}|+\rangle = |-\rangle$$
$$\mathcal{U}|-\rangle = |+\rangle$$

These operators are called ***Unitary Operators***!

M8 Mathematical Properties of unitary operators

Unitary operators do for complex vector spaces what rotations/reflections do for real vector spaces.

norm conservation:

$$|\psi'\rangle = \mathcal{U}|\psi\rangle$$

$$\Rightarrow \langle\psi'|\psi'\rangle = \langle\psi|\mathcal{U}^\dagger\mathcal{U}|\psi\rangle \quad \forall\psi$$

$$\Rightarrow \mathcal{U}^\dagger\mathcal{U} = \mathbb{1}$$

That means the hermitian conjugate of a unitary matrix is its inverse!

$$\mathcal{U}^+|\psi\rangle = \mathcal{U}^+\mathcal{U}|\psi\rangle = |\psi\rangle$$

and one can also show

$$\mathcal{U}\mathcal{U}^+ = \mathbb{1}$$

more general: conservation of scalar product

$$|\psi'\rangle = \mathcal{U}|\psi\rangle$$

$$|\varphi'\rangle = \mathcal{U}|\varphi\rangle$$

$$\Rightarrow \langle\psi'|\varphi'\rangle = \langle\psi|\mathcal{U}^+\mathcal{U}|\varphi\rangle$$

$$= \langle\psi|\varphi\rangle$$

U is a linear operator, and it is completely described by the action on a set of basis vectors:

Our Example:

$$\mathcal{U}|+\rangle = |-\rangle$$

$$\mathcal{U}|-\rangle = |+\rangle$$

$$\Rightarrow \mathcal{U}(\alpha|+\rangle + \beta|-\rangle) = \alpha|-\rangle + \beta|+\rangle$$

$$\Rightarrow \mathcal{U}|+\rangle_x = |+\rangle_x$$

$$\mathcal{U}|-\rangle_x = \mathcal{U}\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) = \frac{1}{\sqrt{2}}(|-\rangle - |+\rangle)$$

$$= -|-\rangle_x$$

<u>coordinate representation:</u>

matrix coefficients: $\langle a_i|\mathcal{U}|a_j\rangle$

Our example: $\mathcal{U} \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (\sim S_x)$

Pauli operators

$$\sigma_x \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y \doteq \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

braket representation

simple representation:

$$U = \underbrace{|-\rangle}_{\text{out}}\overbrace{\langle +|}^{\text{in}} + \underbrace{|+\rangle}_{\text{out}}\overbrace{\langle -|}^{\text{in}}$$

Generalization:

if U maps states of one basis $|a_i\rangle$

to states of a different basis $|b_i\rangle$

then U can be expressed as

$$\boxed{U = \sum_i |b_i\rangle\langle a_i|}$$

U is a normal operator (See M7, Lecture 11)!

$$U^\dagger U = U U^\dagger = \mathbb{1}$$

==> **spectral decomposition:** (look at similarity of spin operator σ_x)

(diagonal representation)

Our Example:

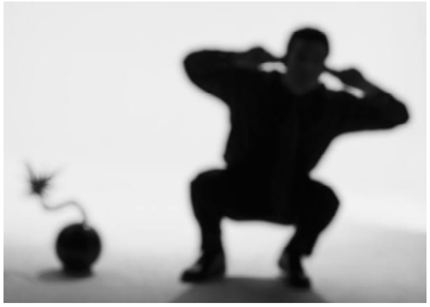
$$U = |+\rangle_x\langle +|_x - |-\rangle\langle -|_x$$

General: Unitary operators are normal and have a spectral decomposition with eigenvalues λ_k that satisfy $|\lambda_k| = 1 \Rightarrow \lambda_i = e^{i\varphi_k}$

$$U = \sum_k e^{i\varphi_k} |\varphi_k\rangle\langle\varphi_k|$$

eigenstates

4.6.2 Applications Bomb Detection (See Schumacher/Westmoreland, Chapter 2)
4.6.2.1. Problem Descriptions



Bomb description
optical fuse ➔ single photon will blow up bomb

Problem:
manufacturing process produces only 70% of working bombs, the other fuses don't work

Task:
find a process that gives at least some bombs whose fuse will work with certainty ...

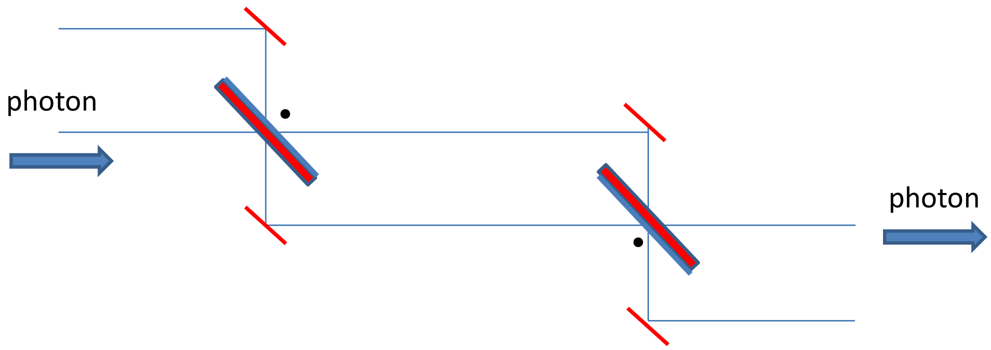
How could that possibly work?
If you shine light on the fuse, it
- either blows up ==> there is no bomb left
- it does not blow up ==> it is an non-working bomb
In either case, you are not left with a working bomb

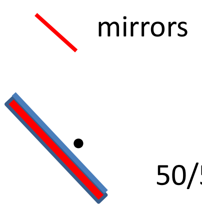
Can we still do it? YES! With quantum mechanics it works,
but we need better systems than Stern-Gerlach devices. But the new systems will work with the same theoretical framework!

The Stern-Gerlach device is just a generic device using systems that have two mutually exclusive states.

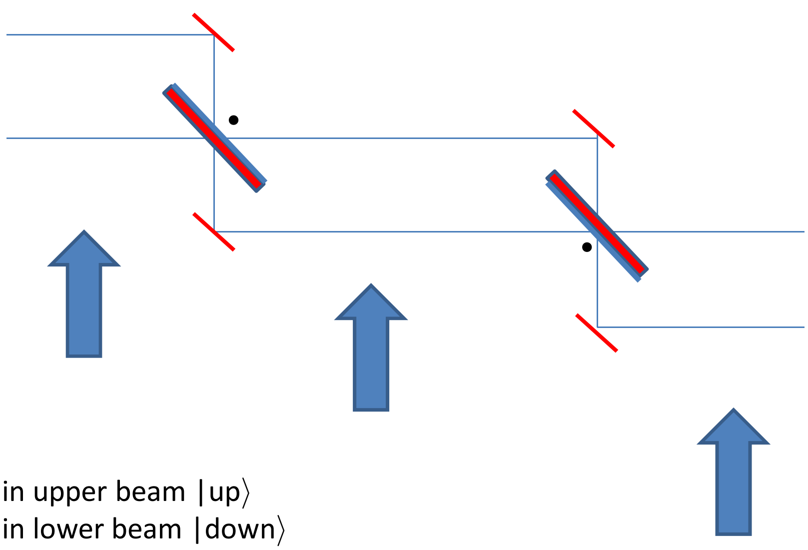
4.6.2.2 Optical Interferometers with single photon input

Optical Interferometer





Quantum Description

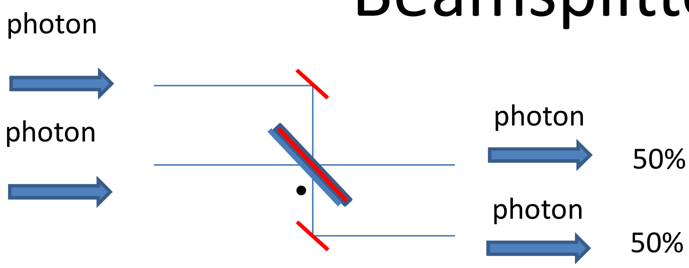


one photon in
➔ one photon out

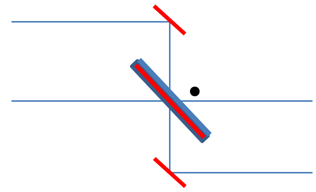
state space:
single photon in upper beam $|up\rangle$
single photon in lower beam $|down\rangle$

set as (orthonormal) standard basis

Beamsplitter



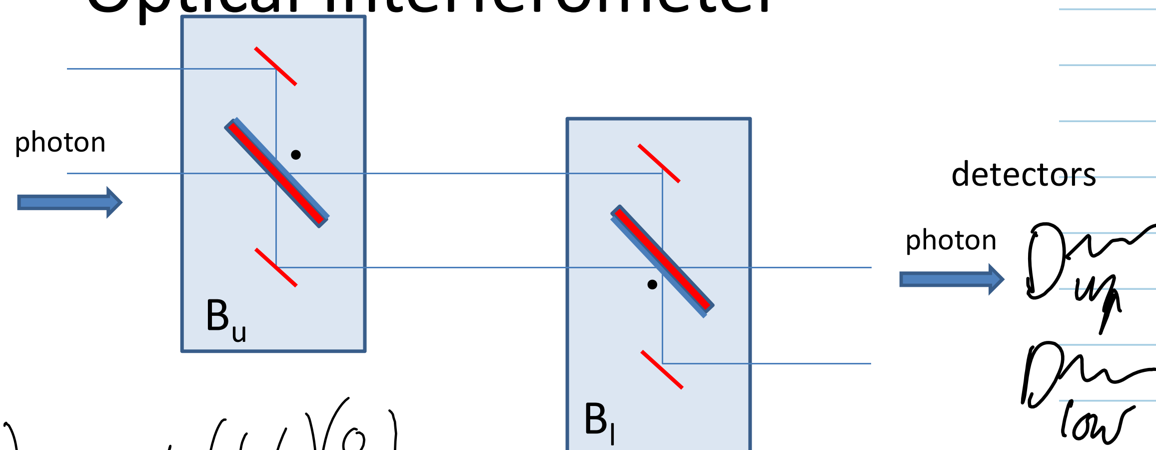
$$|up\rangle \Longrightarrow \frac{1}{\sqrt{2}}|up\rangle + \frac{1}{\sqrt{2}}|low\rangle$$
$$|low\rangle \Longrightarrow \frac{1}{\sqrt{2}}|up\rangle - \frac{1}{\sqrt{2}}|low\rangle$$



$$B_l \leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$B_u \leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

Optical Interferometer



$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$