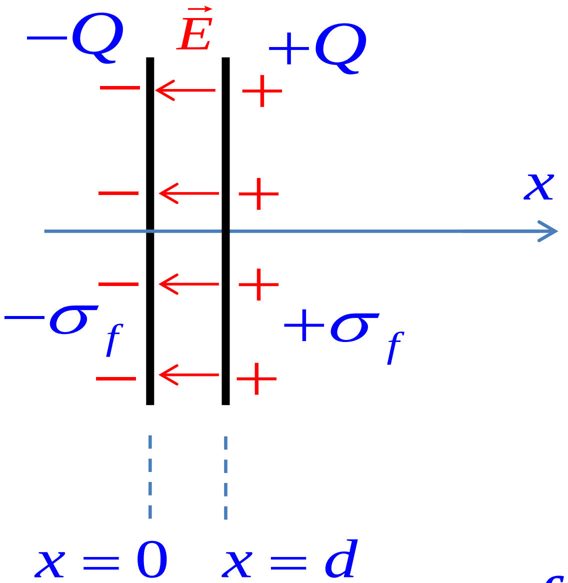
III Capacitors and Review of DC Circuits

1 Capacitors

Parallel plate capacitor :
(plate dimensions $\ll d$*)*

$$C = \frac{Q}{V} = \frac{\varepsilon \oint \vec{E} \bullet d\vec{A}}{\int \vec{E} \bullet d\vec{l}}$$



- *assume Q (charge it up)*
- *find V in terms of Q using Gauss's Law*
- *obtain C from* $C = \frac{Q}{V}$

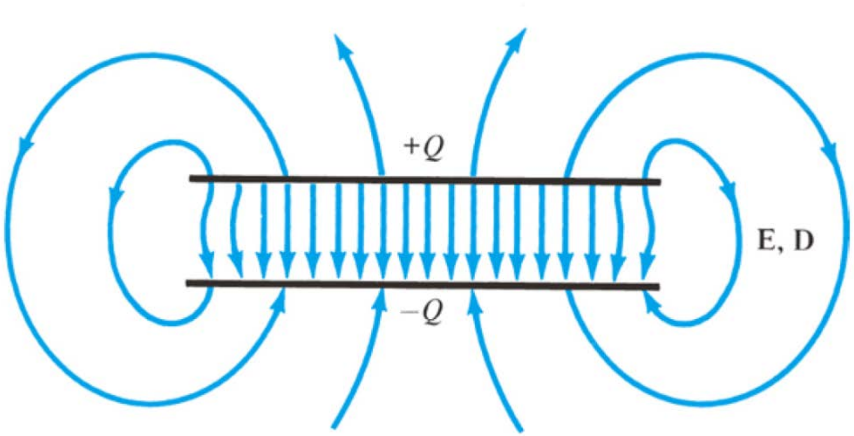
$-$ *for infinite sheet :* $\vec{D} = \frac{\sigma_f}{2}\hat{n}$

$-$*then* $\vec{E}$ *between plates :* $\vec{E} = \frac{\sigma_f}{\varepsilon_0}(-\hat{x}) = \frac{Q_f}{A\varepsilon_0}(-\hat{x})$

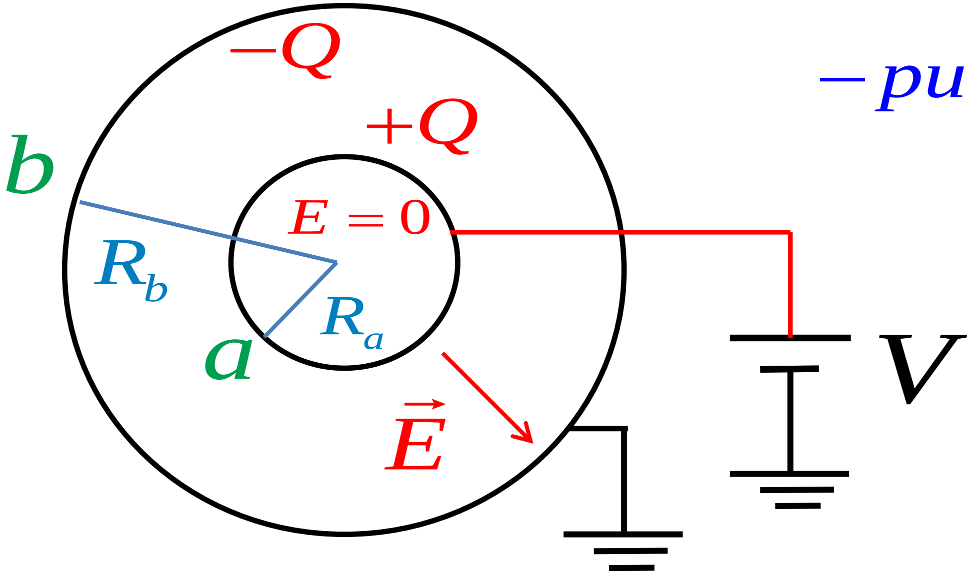
$$\therefore \Delta V = V_d - V_0 = -\int_0^d \left(-\frac{Q_f}{A\varepsilon_0} \right)\hat{x} = \frac{Q_f d}{A\varepsilon_0}$$

$$\Delta V = \frac{Q_f d}{A\varepsilon_0} \qquad C = \frac{Q}{V} \qquad \Rightarrow C = \frac{A\varepsilon_0}{d}$$

-remember that it is asssumed that any fringe field is negligible



$$Example: Capacitance\ of\ a\ pair\ of\ concentric,\ conducting\ spherical\ shells\ of\ radii\ R_a\ and\ R_b.$$



$-put\ +Q\ on\ inner\ sphere$

$-\ \vec{E}\ is\ set\ up$

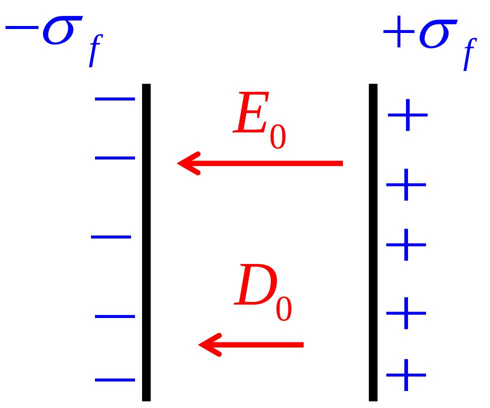
$-outer\ shell\ acquires\ -Q$

- Start by finding E from Gauss's Law

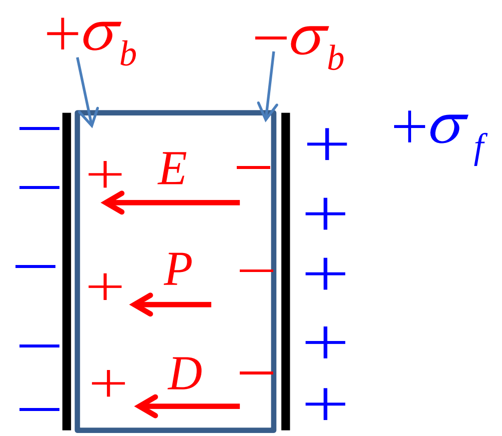
- then find V_{BA} and finally C from C = Q/V

$$-get\ C = \frac{ab}{4\pi\varepsilon_0(b-a)}$$

- **C without dielectric:**   - **C with dielectric:**



$-\sigma_f$



$$\vec{D} = \sigma_f = \vec{D}_0$$

$$\vec{D}_0 = \varepsilon_0 \vec{E}_0 = \sigma_f$$

$$\vec{E} = \frac{\vec{D}}{\varepsilon} = \frac{\vec{D}_0}{\varepsilon_r \varepsilon_0} = \frac{\vec{E}_0}{\varepsilon_r} \quad \Rightarrow E < E_0$$

$$\Delta V_0 = E_0 d$$

$$C_0 = \frac{Q}{\Delta V_0}$$

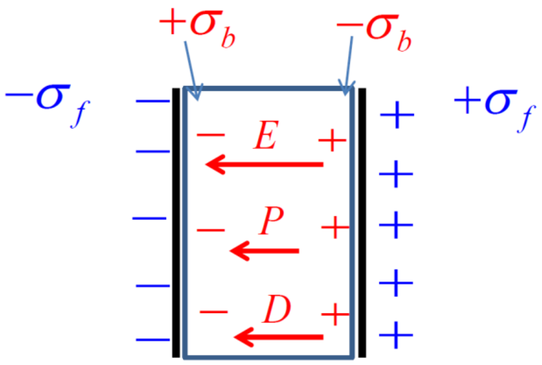
$$\vec{P} = \chi_e \varepsilon_0 \vec{E} = (\varepsilon_r - 1)\varepsilon_0 \vec{E} = \frac{(\varepsilon_r - 1)}{\varepsilon_r} \varepsilon_0 \vec{E}_0 = \varepsilon_0 (\vec{E}_0 - \vec{E})$$

$$\Rightarrow E = E_0 - \frac{P}{\varepsilon_0} \quad or \ E = E_0 - E_b \ where \ E_b = \frac{P}{\varepsilon_0} \equiv local\ field$$

$$-also\ |\sigma_b| = |P_n| = P = \frac{(\varepsilon_r - 1)}{\varepsilon_r}\varepsilon_0 E_0 = \frac{(\varepsilon_r - 1)}{\varepsilon_r} D_0 = \frac{(\varepsilon_r - 1)}{\varepsilon_r}|\sigma_f| \qquad or\ \sigma_b = -\frac{(\varepsilon_r - 1)}{\varepsilon_r}\sigma_f$$

$$and\ E_b = \frac{|\sigma_b|}{2\varepsilon_0} + \frac{|\sigma_b|}{2\varepsilon_0} = \frac{|\sigma_b|}{\varepsilon_0} \qquad (as\ for\ parallel\ plate\ capacitors\ with\ \sigma_b)$$

$- C$ *with dielectric continued* $--$



$$\Delta V = \int \vec{E} \bullet d\vec{l} \;= Ed \;= \frac{E_0}{\varepsilon_r} d \;= \frac{\Delta V_0}{\varepsilon_r}$$

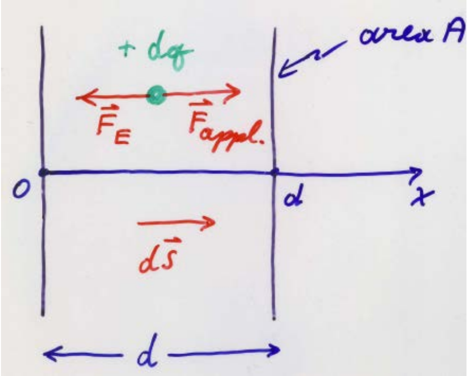
$$\Rightarrow \Delta V \text{ is less than } \Delta V_0$$

$$C = \frac{Q}{\Delta V} = \frac{Q}{\left(\dfrac{\Delta V_0}{\varepsilon_r}\right)} \; ; \quad \therefore \frac{C}{C_0} = \varepsilon_r \qquad or \quad C = \varepsilon_r C_0$$

$$\text{In addition,} \quad W_E = U_C = \frac{1}{2} C V^2 = \frac{1}{2} Q V = \frac{1}{2} \frac{Q^2}{C}$$

$(--$ *see the following aside re development of* $W_E)$

Aside:



$$At\ t=0,\ \sigma=0,\ \vec{E}=0,\ \vec{F}_E=0$$

$\Rightarrow$ *can move 1st dq across without doing work*

Let $+q$ *be on the RH plate at time t.*

$$Then\ \ \sigma=\frac{q}{A},\ \vec{E}=\frac{q}{A\varepsilon_0}(-\hat{i})$$

$$and\ \ \vec{F}_E=|\vec{E}|dq(-\hat{i})=\frac{q}{A\varepsilon_0}dq(-\hat{i})$$

dW required to move $+dq$ *across:*

$$dW=\int_{x=0}^{d}\vec{F}_{ext}\bullet d\vec{s}=-\int_{x=0}^{d}\vec{F}_E\bullet d\vec{s} \qquad (note:\ |F_E|(-\hat{i})\bullet|dx|\,\hat{i}=-|F_E|dx\ )$$

$$=\frac{qdq}{A\varepsilon_0}\int_0^d dx=\frac{qdq\ d}{A\varepsilon_0} \qquad ,\ but\ C=\frac{A\varepsilon_0}{d}\ \ so\ that\ \ dW=\frac{qdq}{C}$$

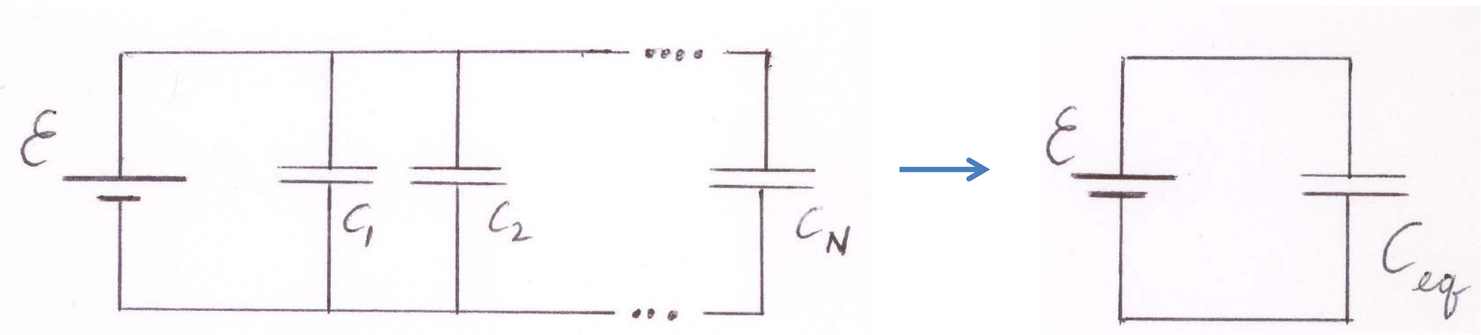
$$\therefore W=\int_{q=0}^{Q}\frac{qdq}{C}=\frac{Q^2}{2C}=U_C \quad (=potential\ energy\ stored\ in\ C)$$

$$(=work\ you\ or\ battery\ had\ to\ do)$$

End Aside

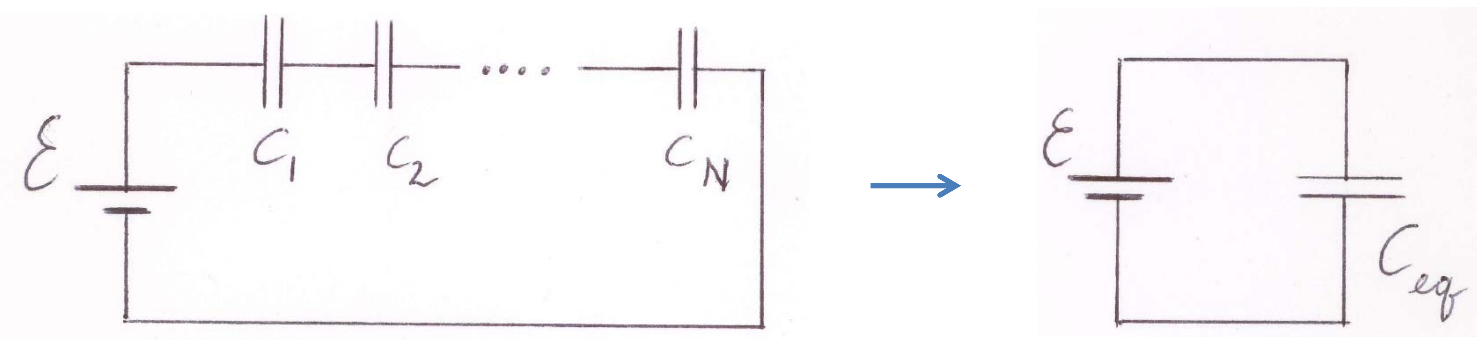
Capacitors in parallel and in series:

Parallel:



$$C_{eq} = \sum_{i=1}^{N} C_i$$

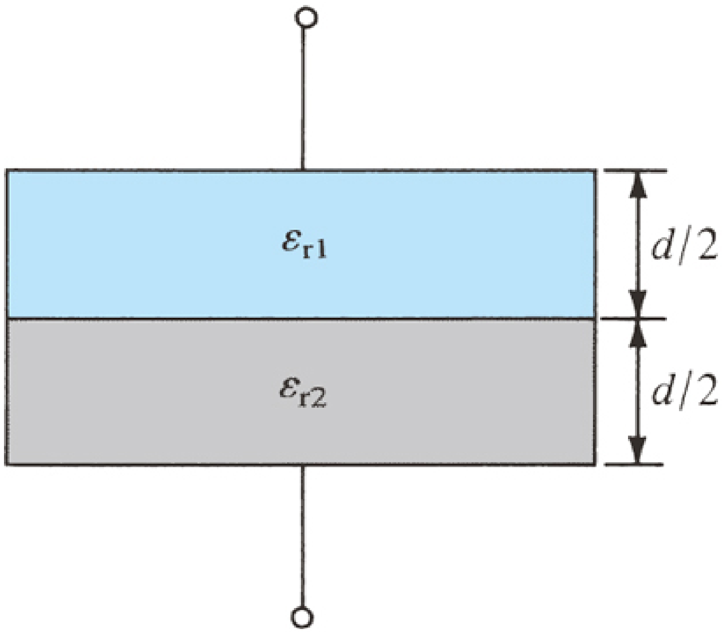
Series:



$$\frac{1}{C_{eq}} = \sum_{i=1}^{N} \frac{1}{C_i}$$

Example

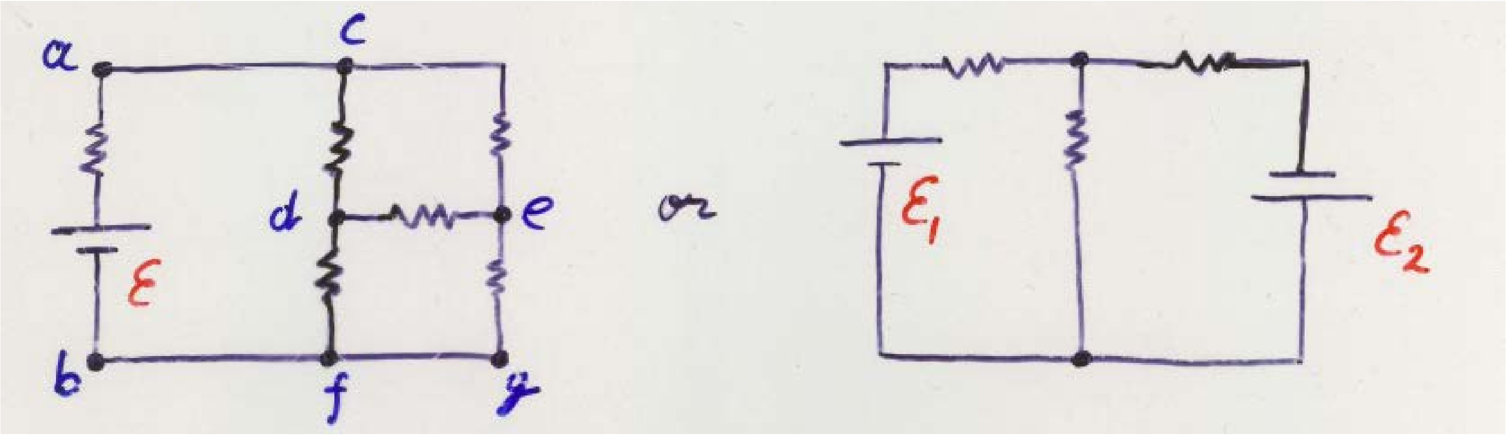
Determine the capacitance of the following:



2 Review of DC Circuits

OHM's Law: $V = I\, R$

and KIRCHHOFF's Rules (*for more complicated circuits*):

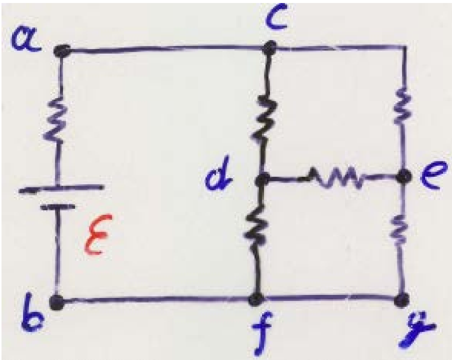


Branch point or node point:

a point where 3 or more conductors are joined; eg., c, d, f, e

Closed Loop:

closed conducting path; eg., acdfba, acegfba



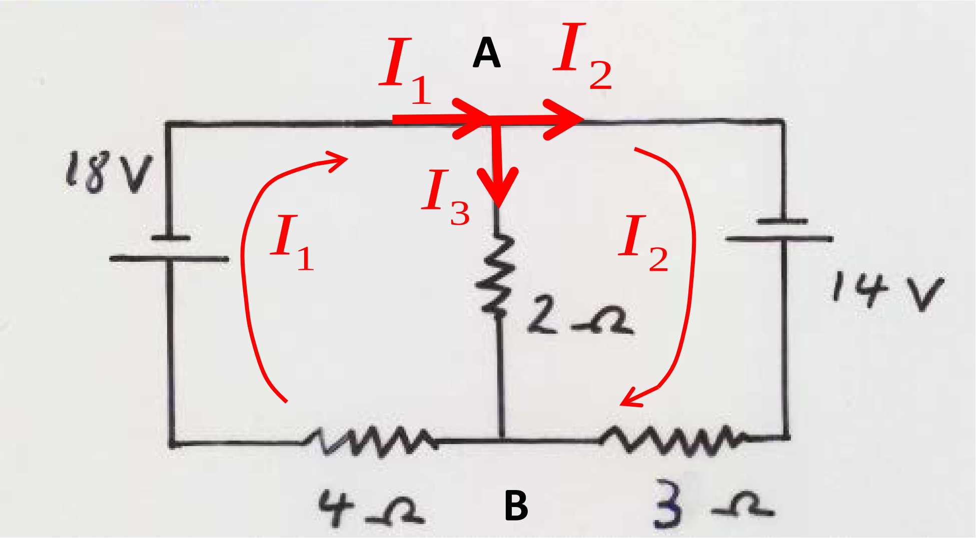
Point or Current *rule* : *Algebraic sum of currents meeting at a branch point is zero.*

$$\sum_i I_i = 0 \quad (\textit{currents entering } +ve, \textit{ currents leaving } -ve)$$

Voltage or Loop *rule* : *Algebraic sum of potential differences in any loop must equal zero.*
(independent (non − crossing) loops)

$$\sum_i \mathcal{E}_i - \sum_k (IR)_k = 0$$

Example *Solve for all currents. Find* V_{AB}.



Kirchhoff's current, point, or node rule:

- *point* A: $\sum I_{in} = \sum I_{out} \;\; \Rightarrow I_1 - I_2 - I_3 = 0 \;\; or \; I_1 = I_2 + I_3 \qquad (i)$
- *point* B: $\quad I_2 + I_3 = I_1 \;;\; nothing \; new$

Kirchhoff's loop rule:

- *loop* 1: $-18 - 2I_3 - 4I_1 = 0 \qquad (ii)$
- *loop* 2: $14 - 3I_2 + 2I_3 = 0 \qquad (iii)$

- *3 equations*
(linear, nonhomogeneous)
- *3 unknowns*
–use Cramer's rule

Solve to get $I_1 = -2.39\ A,\ I_2 = 1.84\ A,\ I_3 = -4.23\ A$

To solve a set of linear, nonhomogeneous equations:

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2$$
$$.$$
$$.$$
$$.$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots + a_{nn}x_n = b_n$$

By Cramer's Rule:

$$x_1 = \frac{D_1}{D},\ \ x_2 = \frac{D_2}{D},\ \ldots\ldots\ \ , x_n = \frac{D_n}{D}$$

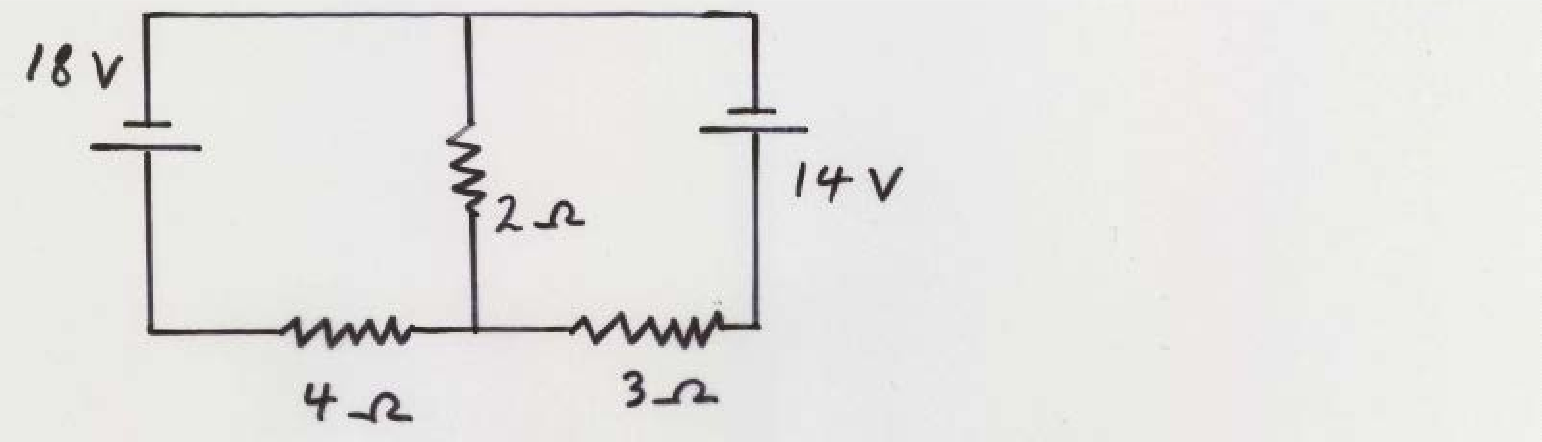
where

$$D = \begin{vmatrix} a_{11} & \ldots\ldots\ldots\ldots & a_{1n} \\ a_{21} & \ldots\ldots\ldots\ldots & a_{2n} \\ . & & \\ . & & \\ . & & \\ a_{n1} & \ldots\ldots\ldots & a_{nn} \end{vmatrix}$$

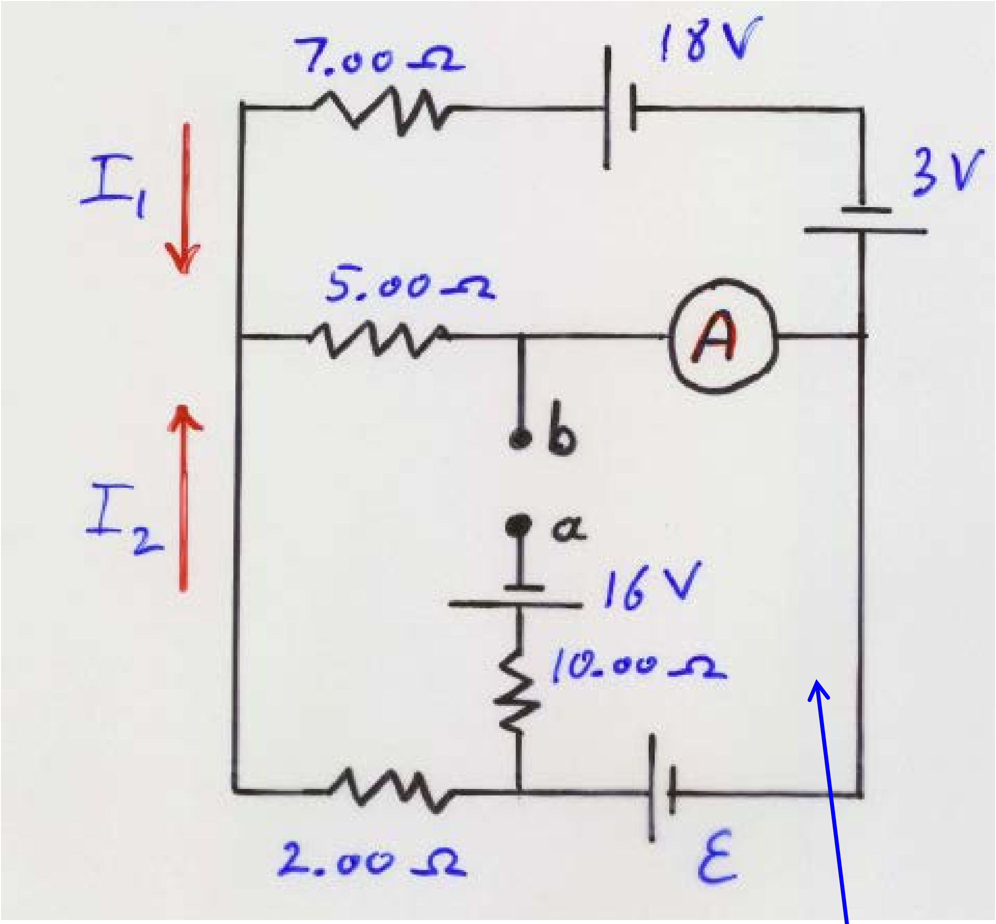
and where $D_1, \ldots.., D_n$ are determinants obtained by replacing in D the k^{th} column by the column with elements $b_1,\ b_2,\ b_n$.

Example (same as previous example)

Solve for all currents using only the loop rules. Find $I_{2\Omega}$.



Example



The ammeter reading is 2.00 A.

Find I1, I2, ε and V_{ab}.

$-can\ get\ I_1 = 0.714\ A$

$I_2 = 1.29\ A$

$\mathcal{E} = 12.6\ V$

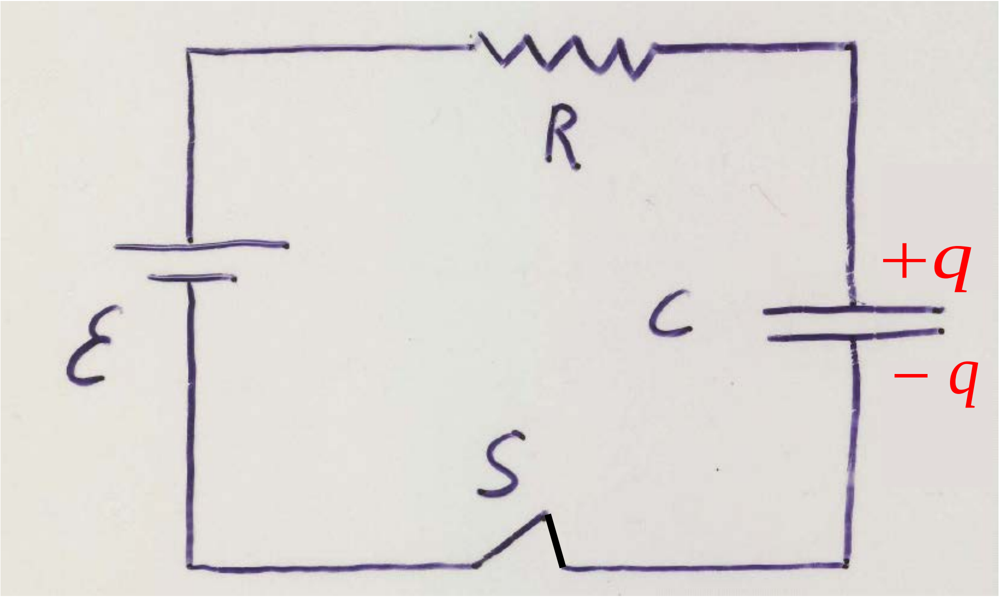
Although I is 0 can still write down a loop equation:

$$12.6 - 10.00 \text{ x } 0 - 16 - V_{ab} = 0$$

$(V_{ab} = -\ 3.4\ V)$

RC circuits (series, DC)

Charging:



$t < 0:$

$Q_C = 0;\ V_C = 0;\ V_R = 0$

$-$ at $t = 0$ the switch is closed

$-$ at $t = 0^+:$

$Q_C = 0;\ V_C = 0;\ V_R = \mathcal{E}$

$$I = \frac{V_R}{R} = \frac{\mathcal{E}}{R}$$

$-$ as $t \to \infty$ C, is fully charged: $I = 0;\ V_R = 0;\ V_C = \mathcal{E}$ and $Q = C\mathcal{E}$

$-$ at some time t:

q on C is $q(t)$; current is $i(t)$

K's loop rule:

$$\mathcal{E} - iR - \frac{q}{C} = 0$$

$$\mathcal{E} - iR - \frac{q}{C} = 0$$

but $i = \dfrac{dq}{dt}$ *so that* $\mathcal{E} - \dfrac{dq}{dt}R - \dfrac{q}{C} = 0$

or $(\mathcal{E} - \dfrac{q}{C})dt - Rdq = 0$

Separate variables $\dfrac{dq}{C\mathcal{E} - q} = \dfrac{1}{RC}dt$

or $\displaystyle\int_{q_i}^{q_{final}} \frac{dq}{C\mathcal{E} - q} = \frac{1}{RC}\int_{t_i}^{t_f} dt$

Here $q_i = 0, \quad q_{final} = q, \quad t_i = 0, \quad t_f = t$

$$\therefore \quad \int_0^q \frac{dq'}{C\mathcal{E} - q'} = \frac{1}{RC}\int_0^t dt'$$

$$\int_0^q \frac{dq'}{C\mathcal{E} - q'} = \frac{1}{RC}\int_0^t dt'$$

$$But \int \frac{dx}{a+bx} = \frac{1}{b}\ln(a+bx)$$

$$\text{- } here\ a = C\mathcal{E}\ and\ b = -1$$

$$\therefore -\ln(C\mathcal{E} - q')\Big|_0^q = \frac{1}{RC}t'\Big|_0^t$$

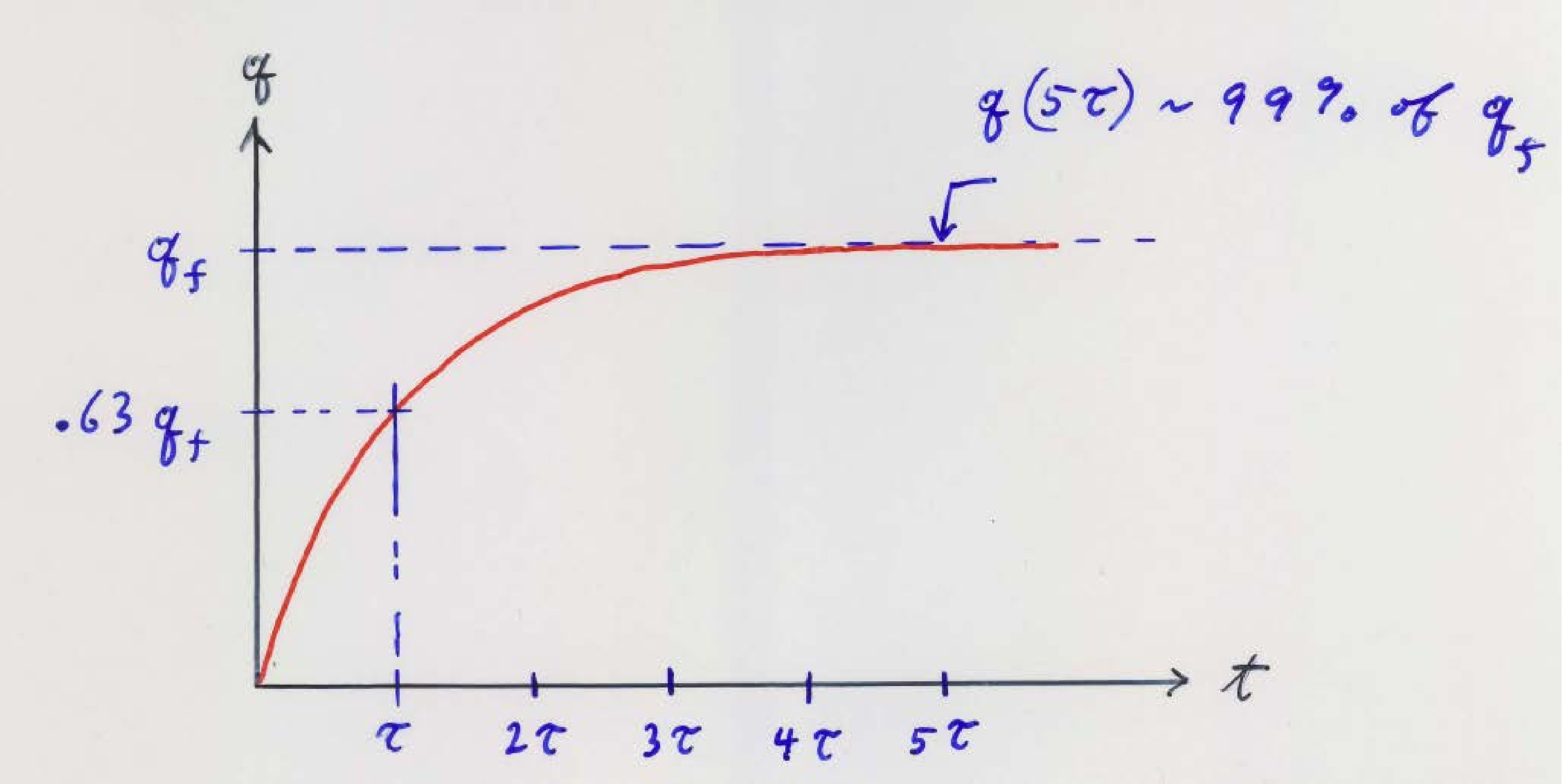
$$or\quad \ln\left(\frac{C\mathcal{E} - q}{C\mathcal{E}}\right) = -\frac{t}{RC}$$

$$\therefore\ q(t) = C\mathcal{E}\left(1 - e^{-\frac{t}{RC}}\right)$$

$$or\quad q(t) = q_{final}\left(1 - e^{-\frac{t}{RC}}\right)$$

$$RC \equiv time\ constant\ of\ the\ circuit = \tau$$

$$q(t) = q_{final}\left(1 - e^{-\frac{t}{RC}}\right)$$



$$q(t) = q_{final}\left(1 - e^{-\frac{t}{RC}}\right) \qquad \therefore v_C(t) = \frac{q(t)}{C} = \mathcal{E}\left(1 - e^{-\frac{t}{RC}}\right)$$

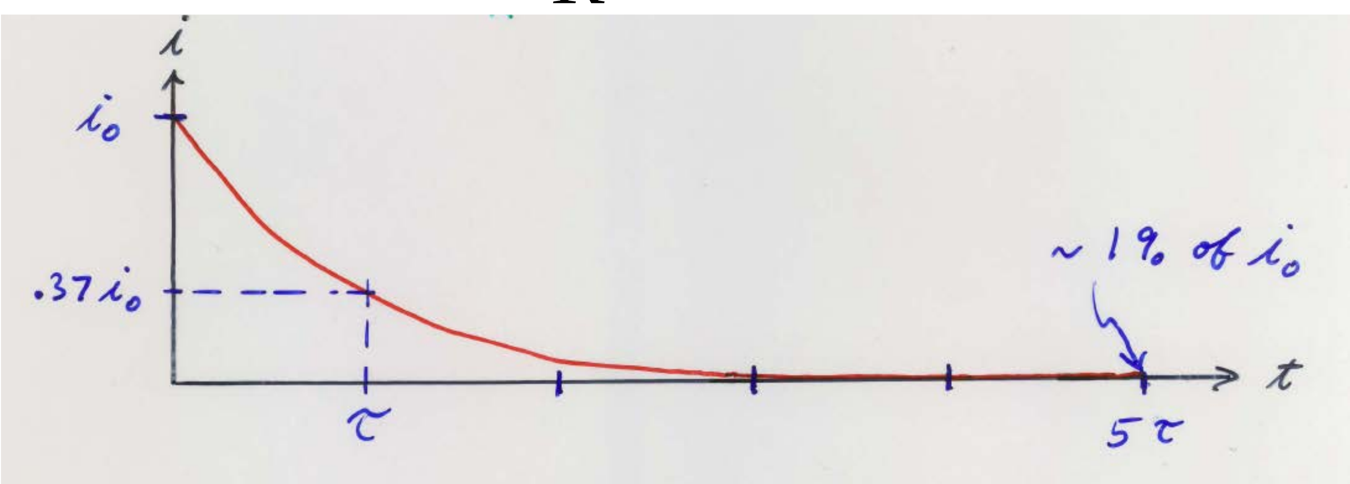
$$\text{Note that when } t = 0,\ v_C = 0$$

$$t \to \infty,\ v_C = \mathcal{E}$$

$v_C(t)$ *follows the same behaviour with* t *as* $q(t)$.

$$i(t) = \frac{dq(t)}{dt} = -C\mathcal{E}e^{-\frac{t}{RC}}\left(\frac{-1}{RC}\right)$$

$$i(t) = \frac{\mathcal{E}}{R}e^{-\frac{t}{RC}} = i_0 e^{-\frac{t}{RC}}$$



Example

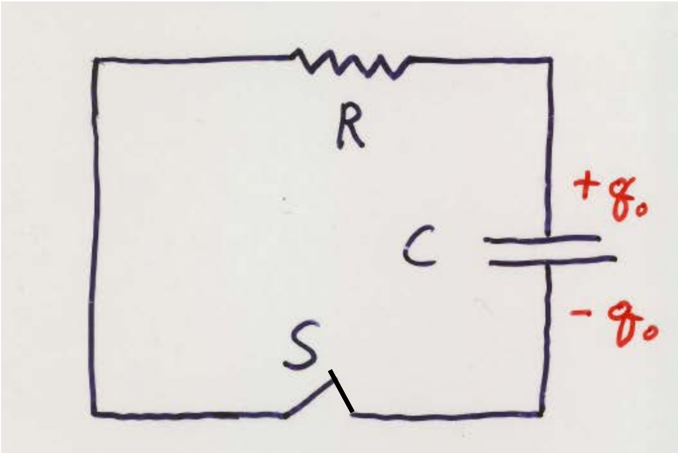
A 15 $k\Omega$ *resistor and capacitor are connected in series and a* 12 *V potential is suddenly applied. The potential across the capacitor rises to* 5.0 *V in* 1.3 μs.

a) *Calculate the time constant.* (2.4 x $10^{-6}\,s$)

b) *Find the capacitance of the capacitor.* (160 *pF*)

c) *When will* v_C *be* $\frac{1}{2}\mathcal{E}$? (1.66 μs)

Discharging:



Let the charge initially be

$$q_0 \ or \ q_i = \mathcal{E}C$$

With the switch open:

$$I = 0,\ V_R = 0,\ V_C = \mathcal{E} = \frac{q_0}{C}$$

$$At\ t = 0\ close\ the\ switch: \quad i_0 = \frac{V_C}{R} = \frac{q_0}{CR}$$

$$V_R + V_C = 0,\ so\ that\ at\ time\ t$$

$$-i(t)R + \frac{q(t)}{C} = 0$$

$$or \quad \frac{dq(t)}{dt}R + \frac{q(t)}{C} = 0$$

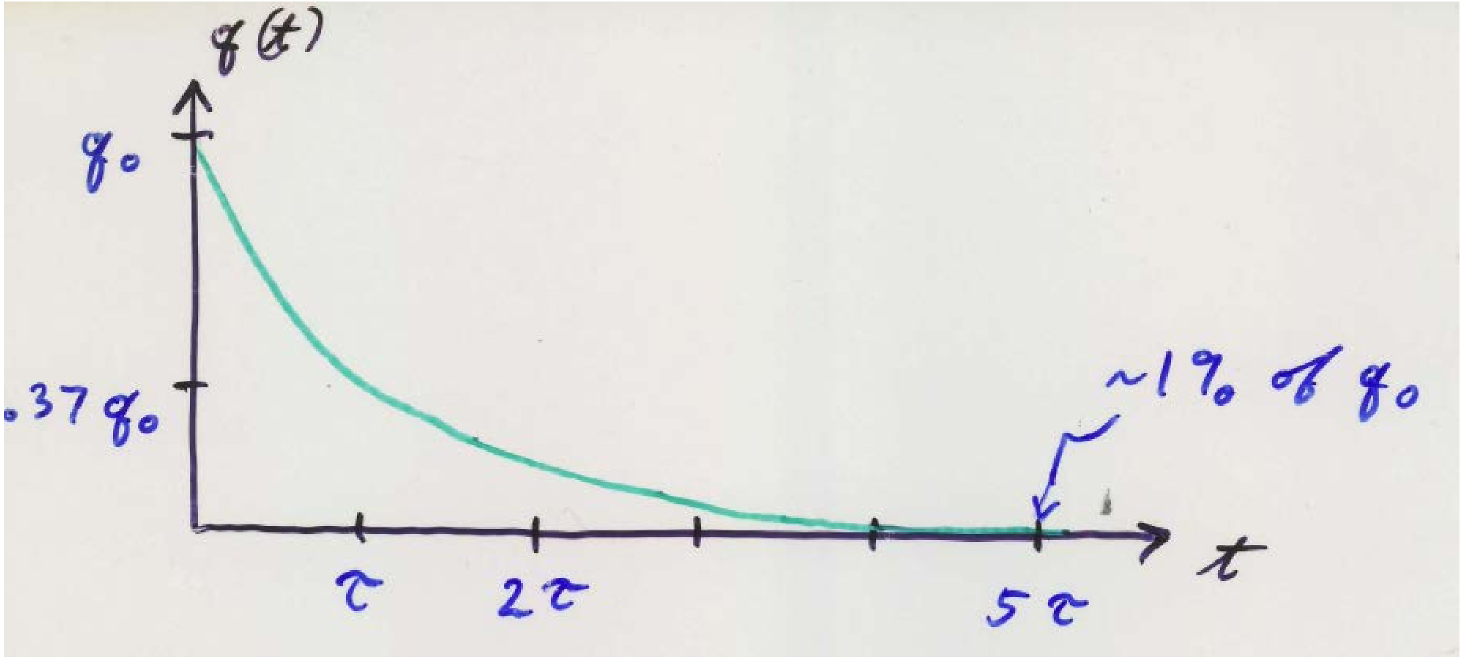
where we have set
i(t) = - dq/dt

$$\frac{dq(t)}{dt}R + \frac{q(t)}{C} = 0$$

$$\int_{q_0}^{q} \frac{dq'}{q'} = -\frac{1}{RC}\int_0^t dt'$$

$$\ln\frac{q}{q_0} = -\frac{t}{RC}$$

$$q(t) = q_0 e^{-\frac{t}{RC}}$$



$$q(t) = q_0 e^{-\frac{t}{RC}}$$

$$then\ v_C(t) = \frac{q(t)}{C} = \frac{q_0}{C} e^{-\frac{t}{RC}}$$

$$= \frac{\mathcal{E}C}{C} e^{-\frac{t}{RC}} = \mathcal{E} e^{-\frac{t}{RC}}$$

$$here\ i(t) = -\frac{dq}{dt} = \frac{q_0}{RC} e^{-\frac{t}{RC}}$$

$$= \frac{\mathcal{E}C}{RC} e^{-\frac{t}{RC}} = i_0 e^{-\frac{t}{RC}}$$

$$here\ v_R(t) = -i(t)R = -\frac{\mathcal{E}}{R} e^{-\frac{t}{RC}} R = -\mathcal{E} e^{-\frac{t}{RC}}$$

$$Note\ that\ at\ t = 0$$

$$q = q_0, \quad i = \frac{\mathcal{E}}{R}, \quad v_R = -\mathcal{E}, \quad v_C = +\mathcal{E}$$