L20 - Unitaries from Time Evolution

5.10 Dynamics described by Unitary operators

5.10.1 Motivation

So far we look in dynamics to be able to determine a state at a later time t from its initial value following a very simple recipe:

Solution of Schrödinger's equation via Energy Eigenstates:

Step 1: find eigenvectors $|E_n\rangle$ and eigenvalue E_n of H

Step 2: Expand initial state in eigenbasis

$$|\Psi(0)\rangle = \sum_{n} c_n |E_n\rangle$$

Step 3: Write down solution

$$|\Psi(t)\rangle = \sum_{n} c_n e^{-i\frac{E_n t}{\hbar}} |E_n\rangle$$

On the other hand, in Lecture 16 when we introduced the time evolution of states we started with some unitary operator describing how to initial and final state are connected:

which is convenient in order to find out the effect of the same time evolution for many different input states.

5.10.2 Unitary operator

We can find this unitary operator by writing the connection between input and output state in the coordinate representation with respect to the eigenbasis of the Hamiltonian:

$$\begin{pmatrix}
c_{i}(t) \\
e^{-i\frac{E_{i}E_{i}}{E_{i}}}
\end{pmatrix}$$

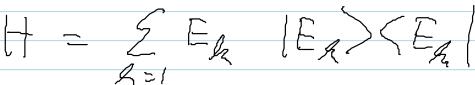
$$\begin{pmatrix}
c_{i}(t) \\
e^{-i\frac{E_{i}E_{i}}{E_{i}}}
\end{pmatrix}$$

$$\begin{pmatrix}
c_{n}(t) \\
c_{n}(t)
\end{pmatrix}
=
\begin{pmatrix}
c_{n}(t) \\
c_{n}(t)
\end{pmatrix}
=
\begin{pmatrix}$$

From this, we can read off the formal description of U:

5.10.3 Unitary Operator as Exponential of Hamilton Operator

Let us compare the unitary operator to the Hamilton Operator H:



Let us now introduce an mathematical tool to make the connection:

Exponentiation of Operators

The exponential function can be expanded in a Taylor series as

$$e^{\times \times} = \underbrace{\sum_{\ell = 0}^{\infty} \frac{1}{\ell!}}_{\ell!} (\times \times)^{\ell}$$

We can now use this notation to define the exponent of an operator as

$$e^{M} := \sum_{0=0}^{\infty} \frac{1}{2!} M^{\ell}$$

Using this notation, we can now verify that

Proof:
$$M = Q$$

$$= Q$$

$$-if$$

$$= Q$$

$$=$$

$$= \underbrace{2}_{A=0}^{M} \underbrace{2}_{A=0}^{\infty} \underbrace{F_{A}}_{A=0}^{\infty} \underbrace{F_{A}}_{A=0}^{\infty$$

5.10.4 Example:

Given an Hamilton Operator

5.10.4.1 Direct approach

Thus we have
$$\frac{1}{2}C$$

$$\frac{1}{2}$$

We can translate this now into a coordinate representation. I choose the representation with respect to the z-basis, $\{(+), (-)\}$

$$= \frac{1}{2} \left(\frac{1}{1} + \frac{1}{2} \right) \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{2} \right) \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right) \left(\frac{1}{1} + \frac{1}{2} + \frac{1$$

5.10.4.2 Alternative approach:

From the definition of the Taylor series expansion of the operator exponentiation, it is easy to find the following formula which holds for all operators M such that

In the above example we find

where the Pauli Operator ysatisfies:

so we have
$$\frac{c_{1}c_{2}}{\sqrt{2}} = \cos \left(-\frac{\omega t}{z}\right) 1 + i \sin \left(-\frac{\omega t}{z}\right) C_{y}$$

$$= \cos \left(\frac{c_{1}c_{2}}{z}\right) 1 + i \sin \left(\frac{\omega t}{z}\right) C_{y}$$

Inserting the coordinate representation of the operators with respect to the standard basis, we find

th respect to the standard basis, we find
$$\mathcal{L} = \left(\frac{\omega^{+}}{z} - \mathcal{L} \frac{\omega^{+}}{z} \right)$$

$$\mathcal{L} = \left(\frac{\omega^{+}}{z} - \mathcal{L} \frac{\omega^{+}}{z} \right)$$

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