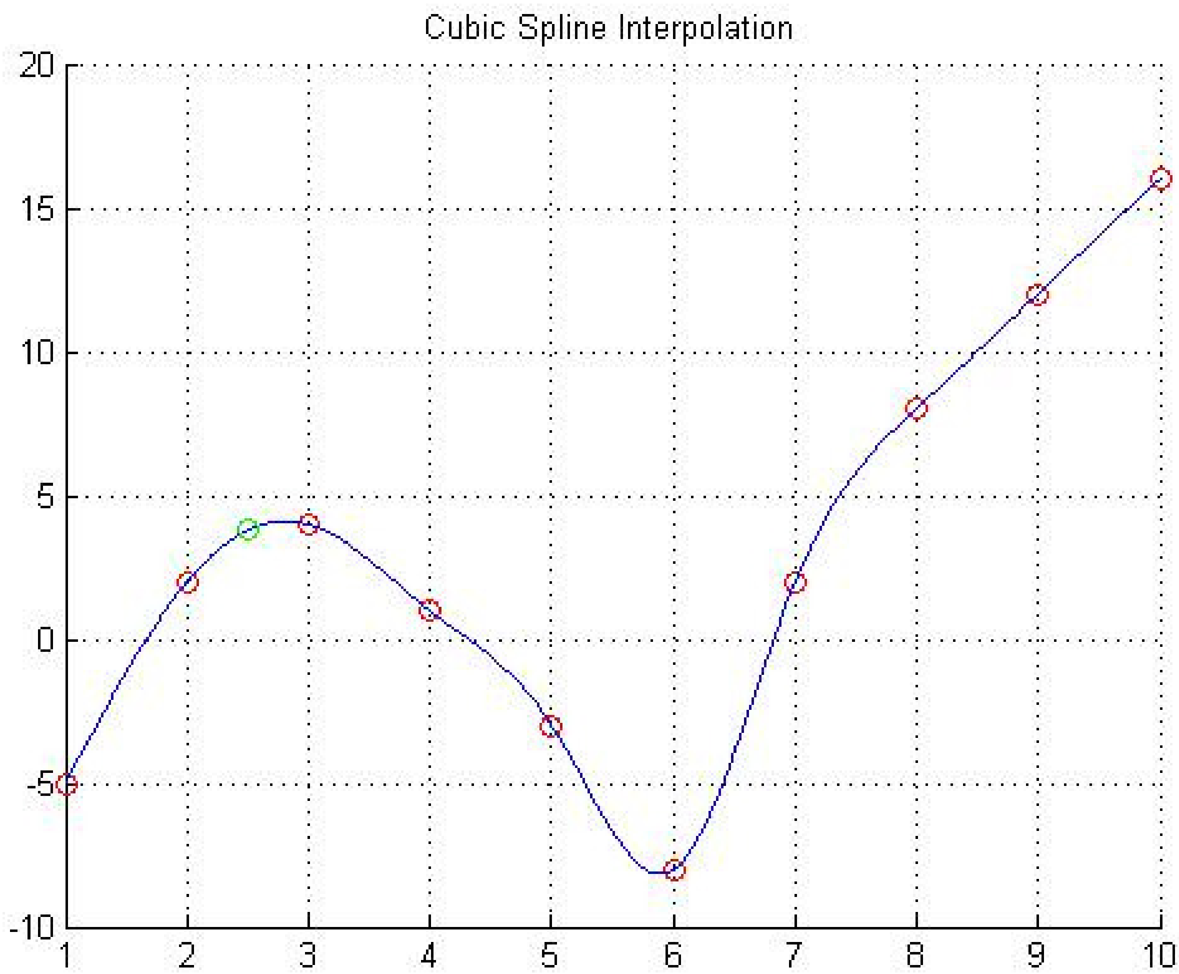
Cubic Spline Interpolation



Set the scene … spline *S(x)*

$S(x) = S_j(x)$ for $x \in [x_j, x_{j+1}]$, where

$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$$

Since j=0:n-1, we have 4n unknowns.

What "knowns" do we have?

Knowns: for k=0:n-1

The function values are known at the left endpoints of the interval $[x_k, x_{k+1}]$:

$$s_k(x_k) = y_k$$

$$s_k(x_k) = a_k + b_k(x_k - x_k) + c_k(x_k - x_k)^2 + d_k(x_k - x_k)^3$$

$\Rightarrow a_k = y_k$

(so, we need 3n more equations)

Knowns: for k=0:n-1

The function is continuous at the right endpoints of the intervals $[x_k, x_{k+1}]$:

$$s_k(x_{k+1}) = s_{k+1}(x_{k+1}) = y_{k+1}$$

$$y_{k+1} = a_k + b_k(x_{k+1} - x_k) + c_k(x_{k+1} - x_k)^2 + d_k(x_{k+1} - x_k)^3$$

$$y_{k+1} = a_k + b_k h_k + c_k {h_k}^2 + d_k {h_k}^3$$

where $h_k = x_{k+1} - x_k$

(we still need 2n more equations)

Knowns: for k=0:n-2

The first derivative is continuous at the right endpoints of the intervals $[x_k, x_{k+1}]$:

$$s'_k(x_{k+1}) = s'_{k+1}(x_{k+1})$$
$$b_k + 2c_k h_k + 3d_k {h_k}^2 = b_{k+1} + \ 2c_{k+1}0 + 3d_k 0$$
$$b_k + 2c_k h_k + 3d_k {h_k}^2 = b_{k+1}$$

(we still need n+1 equations)

Knowns: for k=0:n-2

The second derivative is continuous at the right endpoints of the intervals $[x_k, x_{k+1}]$:

$$s''_k(x_{k+1}) = s''_{k+1}(x_{k+1})$$
$$2c_k + 6d_k h_k = 2c_{k+1} + 6d_{\boldsymbol{k+1}} 0$$
$$\boldsymbol{2c_k + 6d_k h_k = 2c_{k+1}}$$
$$\boldsymbol{c_k + 3d_k h_k = c_{k+1}}$$

(we still need 2 more equations)

Multiple choices for extra restrictions

- Complete (clamped) spline:
  - Provide first derivative values at endpoints
  - $s'_0(x_0) = b_0 = y'_0$
  - $s'_{n-1}(x_n) = b_{n-1} + 2c_{n-1}h_{n-1} + 3d_{n-1}{h_{n-1}}^2 = y'_n$
- Natural cubic spline:
  - Set second derivative values at endpoints to 0
  - $s''_0(x_0) = 2c_0 = y''_0 = 0$
  - $s''_{n-1}(x_n) = 2c_{n-1} + 6d_{n-1}h_{n-1} = y''_n = 0$

Additional choices for extra restrictions

- Periodic (repeating) spline:
  - First and second derivatives equal at endpoints
  - $s'_0(x_0) = s'_{n-1}(x_n) \rightarrow b_0 = b_{n-1} + 2c_{n-1}h_{n-1} + 3d_{n-1}h_{n-1}{}^2$
  - $s''_0(x_0) = s''_{n-1}(x_n) \rightarrow 2c_0 = 2c_{n-1}h_{n-1} + 6d_{n-1}h_{n-1}$
- "Not a **knot**" spline:
  - Require continuous third derivatives at x_1 and x_{n-1}
  - $s'''_0(x_1) = s'''_1(x_1) \rightarrow d_0 = d_1$
  - $s'''_{n-2}(x_{n-1}) = s'''_{n-1}(x_{n-1}) \rightarrow d_{n-2} = d_{n-1}$

Requirements for a Natural Spline

1) For k=0:n-1: a_k=y_k
2) For k=0:n-1: $b_k h_k + \ c_k {h_k}^2 + d_k {h_k}^3 = y_{k+1} - a_k$
3) For i=0:n-2: $b_k + 2c_k h_k + 3d_k {h_k}^2 = b_{k+1}$
4) For i=0:n-2: $\boldsymbol{c_k + 3d_k h_k = c_{k+1}}$

5) c_0=0

6) $c_{n-1} + 3d_{n-1}h_{n-1} = 0$

Note: We define a_n=y_n and c_n=0 for simplicity.

After many simplifications ….

We can define the cubic splines by the defining the system: A x = b, where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ h_0 & 2(h_0+h_1) & h_1 & 0 & \cdots & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 & \vdots & 0 \\ \vdots & 0 & & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ 0 & 0 & & & 0 & 1 \end{bmatrix}$$

(Note: A is diagonally dominant)

$$x = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ \vdots \\ c_n \end{bmatrix} \text{and } b = \begin{bmatrix} 0 \\ z_1 \\ \vdots \\ z_{n-1} \\ 0 \end{bmatrix}, \text{ for } z_j = \frac{3}{h_j}\left(a_{j+1} - a_j\right) - \frac{3}{h_{j-1}}\left(a_j - a_{j-1}\right)$$

Solving this system

- Special Result: When a matrix A is strictly diagonally dominant, it can be solved with Gaussian Elimination without any pivoting.
- In addition, each column of A has only one entry below the diagonal, so only one elimination per column.
- The vector c can be found using fast, special GE.
- Then, find vector b using vector c.
- Then, find vector d using b and c.
- Finally, the vector a was known.

→ Piecewise cubic spline fully specified (and much faster than GE with partial pivoting on the 4n x 4n system)