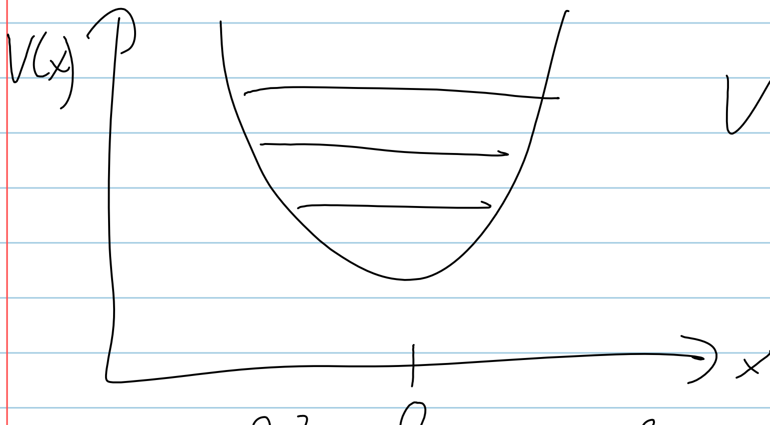
L32 Harmonic Oscillator

8 Harmonic Oscillator

8.1 Motivation



$$V(x) = \frac{1}{2} m \omega^2 x^2$$

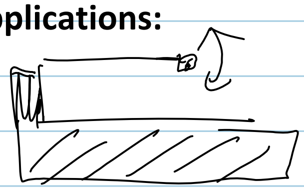
$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2$$

Expectation:

- bound states with discrete energy spectrum
- vacuum fluctuations (ground state at energy above minimum of potential!)
- eigenstates either symmetric or asymmetric (due to symmetry of potential)

Applications:

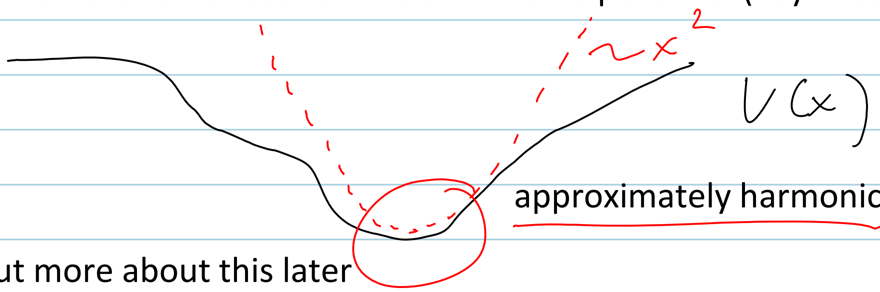
a)



any spring:
nano-engineering

b) Particles in traps:
ions, atoms ...

Any potential is for small oscillations a harmonic potential (Taylor series)



c) Light ... but more about this later

8.2 Analytic Approach

Search for eigenstates and eigenvalues:

$$\left(-\frac{\hbar}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m \omega^2 x^2\right) \psi_E(x) = E \psi_E(x)$$

Ansatz:

$$\psi_E(x) = \sum_{n=0}^{\infty} d_n x^n$$

+ physicality constraints ... (normalizable states)

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

$$\psi_{E_n} = \frac{1}{\sqrt{\sqrt{\pi}\, 2^n n!\, x_0}}\, e^{-\frac{x^2}{2x_0^2}} H_n\left(\frac{x}{x_0}\right)$$

$$x_0 = \sqrt{\frac{\hbar}{m\omega}}$$

Hermite polynomials:

$$H_n(y) = (-1)^n e^{y^2} \frac{d^n}{dy^n} e^{-y^2}$$

$$H_0(y) = 1$$

$$H_1(y) = 2y$$

$$H_2(y) = 2 - 4y^2$$

$$H_3(y) = 12y - 8y^3$$

$$H_4(y) = 12 - 48y^2 + 16y^4$$

...

8.2 Operator Approach to solutions: Definitions

The Hamiltonian of the harmonic oscillator is given by

Introduce a non-hermitian operator

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}_x\right)$$

and its hermitian conjugate

$$\hat{a}^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}_x\right)$$

then the Hamiltonian can be written as

$$\underline{\hat{H}} = \hbar\omega\,\frac{1}{2}\left(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a}\right)$$

$$\hat{a}\hat{a}^+ = \frac{m\omega}{2\hbar}\left(\hat{x}^2 + \frac{\hat{p}_x^2}{m^2\omega^2} - \frac{i}{m\omega}\underbrace{(\hat{x}\hat{p}_x - \hat{p}_x\hat{x})}_{[\hat{x},\hat{p}] = i\hbar}\right)$$

$$= \frac{1}{\hbar\omega}\hat{H} + \frac{1}{2}\mathbb{1}$$

$$\underline{\hat{a}^+\hat{a}} = \frac{1}{\hbar\omega}\hat{H} - \frac{1}{2}\mathbb{1}$$

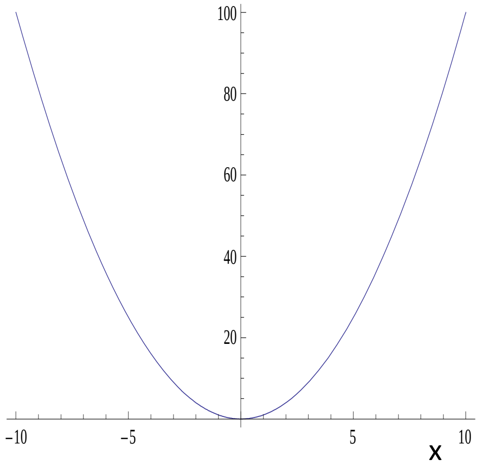
$$\Rightarrow 1)\ \hat{a}\hat{a}^+ + \hat{a}^+\hat{a} = \frac{2}{\hbar\omega}\hat{H}$$

$$2)\ [\hat{a},\hat{a}^+] = \hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = \mathbb{1}$$

$$\left(\Rightarrow \hat{a}\hat{a}^+ = 1 + \hat{a}^+\hat{a}\right)$$

$$\Rightarrow \boxed{\hat{H} = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)}$$

Harmonic Oscillator



$$\hat{H} = \frac{1}{2m}\hat{P}^2 + \frac{1}{2}m\omega^2\hat{X}^2$$

$$[\hat{X}, \hat{P}] = i\hbar\,\mathbb{1}$$

$$\boxed{\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\mathbb{1}\right)}$$

$$\hat{N} = \hat{a}^\dagger\hat{a} \qquad \hat{N}|n\rangle = n\,|n\rangle$$

$$\hat{H}|n\rangle = \hbar\omega(n+1/2)\,|n\rangle$$

n=0,1,2,3 ...

ladder operators:

$$\begin{aligned}\hat{a} &= \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{X} + \frac{i}{m\omega}\hat{P}\right)\\ \hat{a}^\dagger &= \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{X} - \frac{i}{m\omega}\hat{P}\right)\end{aligned}$$

commutator:

$$\left[\hat{a}, \hat{a}^\dagger\right] = \mathbb{1}$$