CO481/CS467/PHYS467 ASSIGNMENT 2 Solutions (will constitute 10% out of the 50% assignment marks)

1. **(3 marks)**

   Suppose Alice and Bob are playing the following game.

   A referee gives Alice a bit s and Bob a bit t. Alice outputs a bit a and Bob outputs a bit b. Alice and Bob win if $a \oplus b = s \wedge t$.

   Alice and Bob may meet in advance of the game and agree upon any strategy they wish, including sharing private random bits. They may not communicate during the execution of the game. The referee may pick s and t according to any strategy she wishes, although they cannot depend on the private random bits of Alice and Bob.

   Alice and Bob will execute many iterations of this game.

   Note that there are four possible functions from $\{0,1\}$ to $\{0,1\}$: identity ($b \mapsto b$), NOT ($b \mapsto \overline{b}$), constant-0 ($b \mapsto 0$) and constant-1 ($b \mapsto 1$).

   Note that if on one iteration of this game, Alice and Bob both output "0" regardless of the values of their input bits (i.e. they both implement the constant-0 function), then they win the game on all inputs except $(s,t) = (1,1)$ (i.e. they win on 3 out of 4 inputs, without using quantum mechanics).

   (a) Note that if the referee suspects they are using the above strategy, she might deliberately pick $s = t = 1$ and thwart their strategy.

   Give three other strategies, one for each of $(s', t') \in \{(0,0),(0,1),(1,0)\}$, so that Alice and Bob win the game if and only if $(s,t) \neq (s',t')$.

   Solution:

   If $(s',t') = (0,0)$, then Alice outputs $a = \overline{s}$ and Bob outputs $b = t$. $s \wedge t$ agrees with $\overline{s} \oplus t$ for all $(s,t) \in \{0,1\}^2$ except $(0,0)$, so this works.

   Similarly, if $(s',t') = (0,1)$, then Alice outputs $a = 0$ and Bob outputs $b = t$, and when $(s',t') = (1,0)$, Alice outputs $a = s$ while Bob outputs $b = 0$.

   (b) Give a randomized strategy for Alice and Bob whereby they win the game with probability 75% regardless of the strategy for choosing s and t used by the referee.

   Solution:

   Alice and Bob use two random shared bits, b_1, b_2. They look at their values (b_1, b_2), and use the protocol from the previous part corresponding to $(s',t') = (b_1, b_2)$. As the choice of questions s and t used by the referee cannot depend on the value of the private random bits of Alice and Bob, with 75% probability $(s,t) \neq (b_1,b_2)$, and the protocol succeeds.

2. **partial measurement (2 marks)**

   Suppose you are given the two qubit state

   $$\frac{3}{5\sqrt{2}}|00\rangle + \frac{4}{5\sqrt{2}}|01\rangle + \frac{4}{5\sqrt{2}}|10\rangle + \frac{3}{5\sqrt{2}}|11\rangle$$

   and you measure the first qubit.

   (a) What is the probability of measuring $|0\rangle$?

   Solution:

   The probability of measuring $|0\rangle$ is $|\frac{3}{5\sqrt{2}}|^2 + |\frac{4}{5\sqrt{2}}|^2 = \frac{9+16}{50} = \frac{1}{2}$ (i.e. the sum of square of the absolute value of the amplitudes corresponding to states with a "0" in the first qubit).

   (b) In this case (that you obtain $|0\rangle$), what is the resulting state of the second qubit?

   Solution:

   The resulting state will be

   $$\frac{1}{\sqrt{1/2}}\left(\frac{3}{5\sqrt{2}}|0\rangle + \frac{4}{5\sqrt{2}}|1\rangle\right) = \frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle$$

   An alternative solution is to express the state as

   $$\frac{1}{\sqrt{2}}|0\rangle\left(\frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle\right) + \frac{1}{\sqrt{2}}|1\rangle\left(\frac{4}{5}|0\rangle + \frac{3}{5}|1\rangle\right)$$

   (note that $\frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle$ and $\frac{4}{5}|0\rangle + \frac{3}{5}|1\rangle$ are normalized states).

   Now it is straightforward to see that measuring the first qubit yields $|0\rangle$ with probability $\frac{1}{2}$ and leaves the second qubit in state $\frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle$.

3. **Gates through teleportation (4 marks)** Alice wishes to implement a one-qubit unitary U. Suppose she possesses the entangled state $|\psi_U\rangle = \frac{1}{\sqrt{2}}|0\rangle U|0\rangle + \frac{1}{\sqrt{2}}|1\rangle U|1\rangle$. Furthermore, let us assume that $UX = \pm XU$ and $UZ = \pm ZU$.

   Suppose Alice has a qubit in some arbitrary state $|\phi\rangle = a|0\rangle + b|1\rangle$, and performs a Bell measurement on her qubit in the state $|\phi\rangle$ and the first qubit of $|\psi_U\rangle$.

   (a) For each of the four possible Bell measurement outcomes, what is the resulting state of the second qubit (i.e. the other qubit of $|\psi_U\rangle$)? (Express your answer simply in terms of U, $|\phi\rangle$, and the Pauli operators.)

Solution:

We can expand and then express the first two qubits in the Bell basis

$$\begin{aligned}
& (a|0\rangle + b|1\rangle)(\tfrac{1}{\sqrt{2}}|0\rangle U|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle U|1\rangle) \\
= \quad & \tfrac{1}{\sqrt{2}}|0\rangle|0\rangle aU|0\rangle) + \tfrac{1}{\sqrt{2}}|0\rangle|1\rangle aU|1\rangle) \\
& +\tfrac{1}{\sqrt{2}}|1\rangle|0\rangle bU|0\rangle) + \tfrac{1}{\sqrt{2}}|1\rangle|1\rangle bU|1\rangle) \\
= \quad & \tfrac{1}{\sqrt{2}}(\tfrac{1}{\sqrt{2}}(\tfrac{|00\rangle+|11\rangle}{\sqrt{2}}) + \tfrac{1}{\sqrt{2}}(\tfrac{|00\rangle-|11\rangle}{\sqrt{2}}))aU|0\rangle \\
& +\tfrac{1}{\sqrt{2}}(\tfrac{1}{\sqrt{2}}(\tfrac{|01\rangle+|10\rangle}{\sqrt{2}}) + \tfrac{1}{\sqrt{2}}(\tfrac{|01\rangle-|10\rangle}{\sqrt{2}}))aU|1\rangle \\
& +\tfrac{1}{\sqrt{2}}(\tfrac{1}{\sqrt{2}}(\tfrac{|01\rangle+|10\rangle}{\sqrt{2}}) - \tfrac{1}{\sqrt{2}}(\tfrac{|01\rangle-|10\rangle}{\sqrt{2}}))bU|0\rangle \\
& +\tfrac{1}{\sqrt{2}}(\tfrac{1}{\sqrt{2}}(\tfrac{|00\rangle+|11\rangle}{\sqrt{2}}) - \tfrac{1}{\sqrt{2}}(\tfrac{|00\rangle-|11\rangle}{\sqrt{2}}))bU|1\rangle
\end{aligned}$$

We can reorganize the terms as

$$\begin{aligned}
& \frac{1}{2}\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)(aU|0\rangle) + bU|1\rangle) + \frac{1}{2}\left(\frac{|00\rangle - |11\rangle}{\sqrt{2}})\right)(aU|0\rangle - bU|1\rangle) \\
+ & \frac{1}{2}\left(\frac{|01\rangle + |10\rangle}{\sqrt{2}}\right)(aU|1\rangle) + bU|0\rangle) + \frac{1}{2}\left(\frac{|01\rangle - |10\rangle}{\sqrt{2}}\right)(aU|1\rangle) - bU|0\rangle)
\end{aligned}$$

which equals

$$\begin{aligned}
& \frac{1}{2}\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)U(a|0\rangle + b|1\rangle) + \frac{1}{2}\left(\frac{|00\rangle - |11\rangle}{\sqrt{2}})\right)U(a|0\rangle - b|1\rangle) \\
+ & \frac{1}{2}\left(\frac{|01\rangle + |10\rangle}{\sqrt{2}}\right)U(a|1\rangle + b|0\rangle) + \frac{1}{2}\left(\frac{|01\rangle - |10\rangle}{\sqrt{2}}\right)U(a|1\rangle - b|0\rangle)
\end{aligned}$$

Thus, measurement outcome
$\frac{|00\rangle+|11\rangle}{\sqrt{2}}$ means the remaining qubit is in state $U(a|0\rangle + b|1\rangle)$.
$\frac{|00\rangle-|11\rangle}{\sqrt{2}}$ means the remaining qubit is in state $U(a|0\rangle - b|1\rangle) = UZ(a|0\rangle + b|1\rangle)$
$\frac{|01\rangle+|10\rangle}{\sqrt{2}}$ means the remaining qubit is in state $U(a|1\rangle + b|0\rangle) = UX(a|0\rangle + b|1\rangle)$
and
$\frac{|01\rangle-|10\rangle}{\sqrt{2}}$ means the remaining qubit is in state $U(a|1\rangle - b|0\rangle) = UXZ(a|0\rangle + b|1\rangle)$.

(b) How can Alice use the results of the Bell measurement to correct the output state of the remaining qubit to be the state $U|\phi\rangle$ (up to global phase)?

Solution:

On measurement outcome $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$ no correction is needed since she already has $U(a|0\rangle + b|1\rangle)$.
On measurement outcome $\frac{|00\rangle-|11\rangle}{\sqrt{2}}$, she can apply a Z gate (since $UZ = \pm ZU$) to obtain $\pm U(a|0\rangle + b|1\rangle)$

On measurement outcome $\frac{|01\rangle+|10\rangle}{\sqrt{2}}$ she can apply an X gate (since $UX = \pm XU$) to obtain $\pm U(a|0\rangle + b|1\rangle)$

On measurement outcome $\frac{|01\rangle-|10\rangle}{\sqrt{2}}$ she can apply ZX (since $UXZ = \pm XZU$) to obtain $\pm U(a|0\rangle + b|1\rangle)$.

4. (**4 marks**)

a) Find the density matrix of the state $\frac{1}{\sqrt{3}}|0\rangle + \frac{e^{i2\pi/3}}{\sqrt{3}}|1\rangle + \frac{e^{i4\pi/3}}{\sqrt{3}}|2\rangle$.

Solution:

$$\frac{1}{\sqrt{3}}\begin{pmatrix}1\\ e^{i\frac{2\pi}{3}}\\ e^{i\frac{4\pi}{3}}\end{pmatrix}\cdot\frac{1}{\sqrt{3}}\begin{pmatrix}1 & e^{-i\frac{2\pi}{3}} & e^{-i\frac{4\pi}{3}}\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1 & e^{-i\frac{2\pi}{3}} & e^{-i\frac{4\pi}{3}}\\ e^{i\frac{2\pi}{3}} & 1 & e^{-i\frac{2\pi}{3}}\\ e^{i\frac{4\pi}{3}} & e^{i\frac{2\pi}{3}} & 1\end{pmatrix}$$

b) Find the density matrix of the state $\left\{(|0\rangle, \frac{1}{4}), (|1\rangle, \frac{1}{4}), (\frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle, \frac{1}{2})\right\}$.

Solution:

$$\frac{1}{4}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \frac{1}{4}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} + \frac{1}{4}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}2 & i\\ -i & 2\end{pmatrix}$$

c) Find a different mixture, involving only two pure states, that gives the same mixed state as in part b).

Solution:

We can diagonalize

$$\frac{1}{4}\begin{pmatrix}2 & i\\ -i & 2\end{pmatrix}$$

as

$$\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{3}{4} & 0\\ 0 & \frac{1}{4}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}}\end{pmatrix}.$$

In other words, it has eigenvalues $\frac{3}{4}$ and $\frac{1}{4}$ with respective eigenvectors $\frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$ and $\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$. This corresponds to the mixture:

$\left\{(\frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle, \frac{3}{4}), (\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle, \frac{1}{4})\right\}$.

d) Give a two-qubit pure state $|\Psi\rangle$ such that $Tr_2|\Psi\rangle\langle\Psi|$ equals the state in part b).

Solution:

One solution is the state

$$\frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle\right)|0\rangle + \frac{1}{2}\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle\right)|1\rangle.$$

This has been obtained tensoring the first state in the spectral decomposition obtained in part c) with $|0\rangle$ and assigning this state probability amplitude $\sqrt{\frac{3}{4}}$, and the second one with $|1\rangle$ and assigning this state probability amplitude $\sqrt{\frac{1}{4}}$. By construction, tracing out the second qubit will give a state with the desired spectral decomposition.

5. **(4 marks)**

Consider $|\psi_1\rangle = \frac{3}{4}|00\rangle + \frac{1}{4}|01\rangle - \frac{\sqrt{3}}{4}|10\rangle + \frac{\sqrt{3}}{4}|11\rangle$ which has Schmidt decomposition $\left(\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle\right))\frac{\sqrt{3}}{2}|0\rangle + \left(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\right)\frac{1}{2}|1\rangle$.

(a) Compute the 2×2 matrix corresponding to the partial trace $Tr_2(|\psi_2\rangle\langle\psi_2|)$ of $|\psi_2\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2\sqrt{2}}|01\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|11\rangle$.

Solution:

There are various notations one could use to calculate the partial trace.
If we rewrite $|\psi_2\rangle = \frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}(\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle)|1\rangle$ it is easy to trace out the second system to get

$$\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}\left(\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle\right)\left(\frac{1}{2}\langle 0| - \frac{\sqrt{3}}{2}\langle 1|\right)$$

which in matrix form is

$$\begin{pmatrix} \frac{5}{8} & -\frac{\sqrt{3}}{8} \\ -\frac{\sqrt{3}}{8} & \frac{3}{8} \end{pmatrix}$$

(b) Find probabilities p_1 and p_2 and eigenvector states $|\phi_1\rangle$ and $|\phi_2\rangle$ such that $Tr_2(|\psi_2\rangle\langle\psi_2|) = p_1|\phi_1\rangle\langle\phi_1| + p_2|\phi_2\rangle\langle\phi_2|$.

Solution: If we diagonalize the above partial trace matrix we get:

$\frac{3}{4}\left(\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle\right)\left(\frac{\sqrt{3}}{2}\langle 0| - \frac{1}{2}\langle 1|\right) + \frac{1}{4}\left(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\right)\left(\frac{1}{2}\langle 0| + \frac{\sqrt{3}}{2}\langle 1|\right)$.

(c) Express the state $|\psi_2\rangle$ in its Schmidt decomposition form.

Solution:

We know the eigenvectors for the partial trace matrix from the previous part will be the Schmidt basis for the first system. Rewriting the first system in this new basis (i.e. substitute $|0\rangle = \frac{\sqrt{3}}{2}(\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle) + \frac{1}{2}(\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle)$, etc.) gives:

$$|\psi_2\rangle = \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) + \frac{1}{2}\left(\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right).$$

(d) Find a 1-qubit unitary U such that $I \otimes U(|\psi_2\rangle) = |\psi_1\rangle$.

Solution:

In order to map

$$|\psi_2\rangle = \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) + \frac{1}{2}\left(\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$$

to

$$|\psi_1\rangle = \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle\right)|0\rangle + \frac{1}{2}\left(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\right)|1\rangle$$

we simply need a unitary on the second qubit that maps

$$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \mapsto |0\rangle$$

and

$$\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \mapsto |1\rangle,$$

in other words, the Hadamard transformation.

6. **(1 mark)**

What is the Bloch vector of the following state?

$$\rho = \begin{pmatrix} \frac{1}{2} - \frac{1}{2\sqrt{2}} & -\frac{i}{2\sqrt{2}} \\ \frac{i}{2\sqrt{2}} & \frac{1}{2} + \frac{1}{2\sqrt{2}} \end{pmatrix}$$

Solution:

This state can be expressed as $\frac{1}{2}\left(I + \frac{1}{\sqrt{2}}Y - \frac{1}{\sqrt{2}}Z\right)$.

The Bloch vector is then $\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

7. **(1 mark)**

   Let $\phi \in (0, \pi/3)$. Prove that

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i\phi} & 0 & 0 \\ 0 & 0 & e^{i\phi} & 0 \\ 0 & 0 & 0 & e^{i3\phi} \end{pmatrix}$$

   is entangling.

   Solution:

   It suffices to find an input state which this operator maps to an entangled state.

   We apply the given unitary to the product state

$$\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

   This resulting state is given by

$$\frac{1}{2}\left(|00\rangle + e^{i\phi}|01\rangle + e^{i\phi}|10\rangle + e^{i3\phi}|11\rangle\right).$$

   To prove this state is entangled, we try now to match this with a expression of the form $(a|0\rangle + b|1\rangle)(c|0\rangle + d|1\rangle)$.

   Suppose

$$\frac{1}{2}\left(|00\rangle + e^{i\phi}|01\rangle + e^{i\phi}|10\rangle + e^{i3\phi}|11\rangle\right) = (a|0\rangle+b|1\rangle)(c|0\rangle+d|1\rangle) = ac|00\rangle+ad|01\rangle+bc|10\rangle+bd|11\rangle.$$

   Multiplying the coefficients of $|00\rangle$ and $|11\rangle$ in both expressions gives $e^{i3\phi} = abcd$, and similarly multiplying the coefficients of $|01\rangle$ and $|10\rangle$ in both expressions gives $e^{i2\phi} = abcd$.

   For this to hold, we need $e^{i\phi} = 1$, which does not hold for $\phi \in (0, \frac{\pi}{3})$. Thus the output state is not entangled, and consequently the gate is entangling.

8. **(1 mark)**

   Prove that

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

   is *not* entangling.

Solution:

It suffices to show than any product state will get mapped to a product state.

It is easy to verify that a input of the form $|\phi\rangle|\psi\rangle$ gets mapped to $Z|\psi\rangle Z|\phi\rangle$, and thus is still a product state. In other words, this operator is just the SWAP gate (which clearly maps product states to product states) times the $Z \otimes Z$ gate (which is certainly not entangling since it is a tensor product of two one-qubit operations).