Propositional Logic Proofs Part1

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[with material from "Mathematical Logic for Computer Science", by Zhongwan, published by World Scientific]

Objectives

- Axiomatic Deduction Systems
- The Hilbert System
- Proofs using Axiomatic Deduction

Axiomatic Deduction Systems /1

- **Deduction Rule:**
  - A tuple $\langle A_1, \ldots A_k \rangle$ where $k \geq 1$ and $A_i \in$ Form ($\mathcal{L}^p$) for all k
- **Axioms:**
  - A deduction rule where k = 1; formally provable formula
- **Inference Rule:**
  - A deduction rule where k > 1
- **Deduction System S for Form ($\mathcal{L}^p$) :**
  - A set of deduction rules; that is, a set of axioms and inference rules
  - **The Hilbert System (H)** is an example of a deduction system for the set of propositional logic formulas

Axiomatic Deduction Systems /2

- **Formal Deduction Proof (of a Deduction System S):**
  - A sequence $\langle B_0, \ldots B_n \rangle$ where $n \geq 1$ and $B_i \in$ Form $(\mathcal{L}^p)$ for all i, such that for every B_i there is a rule $\langle A_1, \ldots A_{k-1}, B_i \rangle \in S$ where $\{A_1, \ldots A_{k-1}\} \subseteq \{B_0, \ldots B_{i-1}\}$
- **Theorem (of a Deduction System S):**
  - A formula B for which a proof $\langle B_0, \ldots B_n, B \rangle$ exists in S
  - Denoted as $\vdash_S B$ (denoted as $\vdash$ in the textbook)
- **Deducible Theorem:**
  - Given $B \in$ Form $(\mathcal{L}^p)$ and a set $\Sigma \subseteq$ Form $(\mathcal{L}^p)$, B is deducible from Σ, denoted as $\Sigma \vdash_S B$, if B is a theorem of $S \cup \{\langle A \rangle \mid A \in \Sigma\}$

Axiomatic Deduction Systems /3

- **Definition 3.1. The Hilbert System (H):**
  - A deduction system for the propositional logic formulas
  - Ax1: $\langle(A \Rightarrow (B \Rightarrow A))\rangle$
  - Ax2: $\langle((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))\rangle$
  - Ax3: $\langle(((\neg A) \Rightarrow (\neg B)) \Rightarrow (B \Rightarrow A))\rangle$;
  - R1: From $A \Rightarrow B$ and A infer B
    - Also known as *Modus Ponens (MP)*
  - (A WFF) A_1 is formally provable by H iff $\vdash_H A_1$ holds
- **Textbook Deduction System (T):** (Section 4.1)
  - Ax1 – Ax12 and R1
  - (A WFF) A_2 is formally provable by T iff $\vdash_T A_2$ holds
    - Also written as $\varnothing \vdash A_2$ in the textbook

Axiomatic Deduction Systems /4

- **Prove that $\vdash_H (A \Rightarrow A)$ holds**
  1. $(A \Rightarrow ((A \Rightarrow A) \Rightarrow A))$ *(by Ax1)*
  2. $((A \Rightarrow ((A \Rightarrow A) \Rightarrow A)) \Rightarrow ((A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A)))$ *(by Ax2)*
  3. $((A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A))$ *(by R1, (1), (2))*
  4. $(A \Rightarrow (A \Rightarrow A))$ *(by Ax1)*
  5. $(A \Rightarrow A)$ *(by R1, (3), (4))*
- **The proof is similar for $\vdash_T A \Rightarrow A$**

Axiomatic Deduction Systems /5

- **Theorem 3.1. $\{(A \Rightarrow B), (B \Rightarrow C)\} \vdash_H (A \Rightarrow C)$**
  Proof:
  1. $(B \Rightarrow C)$ *(by Assumptions)*
  2. $((B \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C)))$ *(by Ax1)*
  3. $(A \Rightarrow (B \Rightarrow C))$ *(by R1, (1), (2))*
  4. $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$ *(by Ax2)*
  5. $((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$ *(by R1, (3), (4))*
  6. $(A \Rightarrow B)$ *(by Assumptions)*
  7. $(A \Rightarrow C)$ *(by R1, (5), (6))*
- The theorem can be used as an extra deduction rule

Axiomatic Deduction Systems /6

- **Prove that $S \cup \{Th3\text{-}1\} \vdash_H ((\neg A) \Rightarrow (A \Rightarrow B))$ holds:**
  1. $((\neg A) \Rightarrow ((\neg B) \Rightarrow (\neg A)))$ *(by Ax1)*
  2. $(((\neg B) \Rightarrow (\neg A)) \Rightarrow (A \Rightarrow B))$ *(by Ax3)*
  3. $((\neg A) \Rightarrow (A \Rightarrow B))$ *(by Theorem 3.1)*

- **Prove: If $\Sigma \vdash_H A$ and $\Sigma \vdash_H (\neg A)$ then $\Sigma \vdash_H B$ for any B**
  1. $(\neg A)$ *(by Assumptions)*
  2. $((\neg A) \Rightarrow ((\neg B) \Rightarrow (\neg A)))$ *(by Ax1)*
  3. $((\neg B) \Rightarrow (\neg A))$ *(by R1, (1), (2))*
  4. $(((\neg B) \Rightarrow (\neg A)) \Rightarrow (A \Rightarrow B))$ *(by Ax3)*
  5. $(A \Rightarrow B)$ *(by R1, (3), (4))*
  6. A *(by Assumptions)*
  7. B *(by R1, (5), (6))*

Axiomatic Deduction Systems /7

- **Theorem 3.2. Deduction Theorem:**
  - For A, B $\in$ Form $(\mathcal{L}^p)$ and $\Sigma \subseteq$ Form $(\mathcal{L}^p)$,
    $\Sigma \vdash_H A \Rightarrow B$ iff $\Sigma \cup \{A\} \vdash_H B$
- **Prove that $\vdash_H ((B \Rightarrow C) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$:**
  - First, prove $\{(B \Rightarrow C), (A \Rightarrow B), A\} \vdash_H C$ using
    $\langle (B \Rightarrow C), (A \Rightarrow B), A, B, C \rangle$
  - Then, use the Deduction Theorem to derive
    (1) $\{(B \Rightarrow C), (A \Rightarrow B)\} \vdash_H (A \Rightarrow C)$
    (2) $\{(B \Rightarrow C)\} \vdash_H ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$
    (3) $\vdash_H ((B \Rightarrow C) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$

Axiomatic Deduction Systems /8

- **Definition 3.2. Extensions to the Hilbert System:**
  - $\langle (A \Rightarrow (B \Rightarrow (A \wedge B))) \rangle$ ($\wedge$ introduction)
  - $\langle ((A \wedge B) \Rightarrow A) \rangle$, $\langle ((A \wedge B) \Rightarrow B) \rangle$
  - $\langle (A \Rightarrow (A \vee B)) \rangle$, $\langle (A \Rightarrow (B \vee A)) \rangle$ ($\vee$ introduction)
  - $\langle ((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \vee B \Rightarrow C))) \rangle$
- **Definition 3.3. Formula Consistency:** (Definition 5.2.2)
  - A set $\Sigma \subseteq$ Form (L^p) is consistent iff there is no $A \in$ Form (L^p) such that $\Sigma \vdash_H A$ and $\Sigma \vdash_H (\neg A)$
- **Theorem 3.3. Formula Consistency:** (Exercise 5.2.2)
  - A set $\Sigma \subseteq$ Form (L^p) is consistent iff there is $A \in$ Form (L^p) such that $\Sigma \vdash_H A$ does not hold

Axiomatic Deduction Systems /9

- **Soundness of the Hilbert System:**
  - For $A \in$ Form ($\mathcal{L}^p$) and $\Sigma \subseteq$ Form ($\mathcal{L}^p$), $\Sigma \vdash_H A \Rightarrow \Sigma \vDash A$
- **Completeness of the Hilbert System:**
  - For $A \in$ Form ($\mathcal{L}^p$) and $\Sigma \subseteq$ Form ($\mathcal{L}^p$), $\Sigma \vDash A \Rightarrow \Sigma \vdash_H A$
- More details about the above in the next lecture

Food for Thought

- **Read:**
  - Chapters 4 and 5 (Section 5.2) from Zhongwan
    - Read proofs discussed in class in more detail
    - Skip the material not related to propositional logic
- Answer the following exercises:
  - Exercises 5.2.1, 5.2.2 and 5.2.4
- (Optional) Read:
  - Chapter 5 from Nissanke (applied to the Hilbert System)
    - Complete at least a few exercises from each section