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Givens rotation

- Zero elements more selectively.
- Givens rotations have the form:

$$G(i,k,\theta)^T = \begin{array}{cc} \begin{bmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & c & & & -s & & \\ & & & 1 & & & & \\ & & & & \ddots & & & \\ & & & & & 1 & & \\ & & s & & & c & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{bmatrix} & \begin{array}{l} \\ \\ \text{row(i)} \\ \\ \\ \\ \text{row(k)} \\ \\ \\ \end{array} \\ \begin{array}{ccccc} & \text{col(i)} & & \text{col(k)} & \end{array} & \end{array}$$

c = cos θ

s = sin θ

- Easy to check that G(i,k,θ) is orthogonal.
- Consider $y = G(i,k,\theta)^T x$. Then

$$y_j = \begin{cases} c x_i - s x_k & j = i \\ s x_i + c x_k & j = k \\ x_j & j \neq i,k \end{cases}$$

To make $y_k = 0$, let $c = x_i \Big/ \sqrt{x_i^2 + x_k^2}$, $\quad s = -x_k \Big/ \sqrt{x_i^2 + x_k^2}$

Notes

1) 5 flops to compute c & s.
2) θ is not needed.
3) When computing $G^T(i,k,\theta)$ A, only row(i) & row(k) are affected.

<u>Example</u>

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad G(2,4,\theta)^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{-2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$G^T x = \begin{bmatrix} 1 \\ \sqrt{20} \\ 3 \\ 0 \end{bmatrix}$$

<u>Givens QR method</u>

e.g.
$$\begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix} \xrightarrow{(3,4)} \begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \\ 0 & \times & \times \end{bmatrix} \xrightarrow{(2,3)} \begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \end{bmatrix} \xrightarrow{(1,2)}$$

$$\begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \end{bmatrix} \xrightarrow{(3,4)} \begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \end{bmatrix} \xrightarrow{(2,3)} \begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \\ 0 & 0 & \times \end{bmatrix} \xrightarrow{(3,4)} R$$

Let G_j = j-th Givens rotation. Then

$$G_k^T \ldots G_2^T G_1^T A = R$$

$$A = QR \qquad Q = G_1 G_2 \ldots G_k$$

flops(Givens QR) = $3mn^2 - 3n^3$ = 1.5 × flops(Householder QR)

<u>Hessenberg QR via Givens</u>

A Hessenberg matrix has the form:

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & & \times & \times & \times \\ & & & \times & \times \end{bmatrix} \xrightarrow{(1,2)} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & & \times & \times & \times \\ & & & \times & \times \end{bmatrix} \xrightarrow{(2,3)} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ & 0 & \times & \times & \times \\ & & \times & \times & \times \\ & & & \times & \times \end{bmatrix} \xrightarrow{(3,4)}$$

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ & 0 & \times & \times & \times \\ & & 0 & \times & \times \\ & & & \times & \times \end{bmatrix} \xrightarrow{(4,5)} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ & 0 & \times & \times & \times \\ & & 0 & \times & \times \\ & & & 0 & \times \end{bmatrix}$$

i.e. $\quad G_{n-1}^T G_{n-2}^T \ldots G_1^T A = R$

$A = QR \qquad Q = G_1 \ldots G_{n-1}$

flops(QR) ~ $3n^2$

Eigenvalue Problems

Def: Let $A \in R^{n\times n}$. A nonzero vector $x \in R^n$ is an eigenvector and $\lambda \in C$ is its corresponding eigenvalue if

$$A x = \lambda x$$

- If x is an eig. vector, then αx, $\alpha \neq 0$, is also an eig. vector.

Def: The set $\Lambda(A) = \{ \lambda : \lambda \text{ eig. value of } A \}$ is the spectrum of A.

An eigendecomposition of A is:

$$A = X \Lambda X^{-1}$$

where

$$X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

and $\quad A x_i = \lambda_i x_i \qquad i = 1, 2, \ldots, n$

i.e. $\quad A X = X \Lambda$

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

Def: The characteristic polynomial of A, $p_A(x)$, is the degree n polynomial defined by

$$p_A(z) = \det(zI - A)$$

Theorem: λ is an eigenvalue of A iff $p_A(\lambda) = 0$.

Pf: λ is eigenvalue

$\Leftrightarrow$ $\lambda x - A x = 0$ for some $x \neq 0$

$\Leftrightarrow$ $\lambda I - A$ is singular

$\Leftrightarrow$ $\det(\lambda I - A) = 0$

Notes

1) By fundamental theorem of algebra, $p_A(z)$ has n (complex) roots. So A has n (complex) eigenvalues.

2) Given a monic polynomial of degree n,

$$p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$$

Consider

$$A = \begin{bmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ & 1 & \ddots & & \vdots \\ & & \ddots & 0 & -a_{n-2} \\ & & & 1 & -a_{n-1} \end{bmatrix}$$

Then $\Lambda(A) = \{$ roots of $p(z) \}$

3) No analytic formula for roots of polynomial of degree 5 or above $\rightarrow$ numerical approximation.

Example:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 2 \\ 4 & -4 & 5 \end{bmatrix}$$

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 1 & 0 & 0 \\ -2 & \lambda + 1 & -2 \\ -4 & 4 & \lambda - 5 \end{bmatrix} = (\lambda - 1) \begin{vmatrix} \lambda + 1 & -2 \\ 4 & \lambda - 5 \end{vmatrix}$$

$$= (\lambda - 1)[(\lambda + 1)(\lambda - 5) - (4)(-2)]$$

$$= (\lambda - 1)(\lambda^2 - 4\lambda - 5 + 8)$$

$$= (\lambda - 1)(\lambda - 1)(\lambda - 3)$$

$\therefore$ eigenvalues are: 1, 3.

Eigenvector for $\lambda = 1$:

$$(\lambda I - A)x = 0 \quad \Rightarrow \quad \begin{bmatrix} 0 & 0 & 0 \\ -2 & 2 & -2 \\ -4 & 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2 x_1 + 2 x_2 - 2 x_3 = 0$$

e.g. (1, 0, -1) is an eigenvector for $\lambda = 1$.

Since $\lambda = 1$ has multiplicity = 2, there exists another linearly independent eigenvector, for example, (1, 1, 0).

Eigenvector for $\lambda = 3$:

$$(3I - A)x = 0 \quad \Rightarrow \quad \begin{bmatrix} 2 & 0 & 0 \\ -2 & 4 & -2 \\ -4 & 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = 0$, $4 x_2 - 2 x_3 = 0$

e.g. (0, 1, 2)

Finally,

$$AX = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 2 \\ 4 & -4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ -1 & 0 & 6 \end{bmatrix}$$

$$X\Lambda = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ -1 & 0 & 6 \end{bmatrix}$$

$\therefore \qquad A X = X \Lambda$

<u>Note</u>

We never compute eigenvalues by finding the roots of the characteristic polynomial.