# APPROXIMATION OF DIFFERENTIAL EQUATIONS — INTRODUCTION TO PERTURBATION EXPANSIONS

TAYLOR SERIES APPROXIMATION IS USED EVERYWHERE IN SCIENCE & ENGINEERING, TYPICALLY IN THE CONTEXT OF <u>PERTURBATION APPROXIMATIONS</u> OF DIFFERENTIAL EQUATIONS.

THE IDEA IS SIMPLY DEMONSTRATED BY LOOKING AT <u>PERTURBATION APPROXIMATIONS</u> OF ALGEBRAIC EQUATIONS.

<u>EX.</u> SUPPOSE WE WANT TO SOLVE $x^2 + \epsilon x - 1 = 0$ FOR <u>SMALL</u> $\epsilon$.
THE EXACT SOLUTION IS:

$$x = -\tfrac{1}{2}\epsilon \pm \sqrt{1 + \tfrac{1}{4}\epsilon^2}$$

USING THE BINOMIAL EXPANSION, WE CAN EXPAND THESE SOLUTIONS AS A POWER SERIES IN $\epsilon$:

$$x^{(1)} = 1 - \tfrac{1}{2}\epsilon + \tfrac{1}{8}\epsilon^2 - \tfrac{1}{128}\epsilon^4 + \cdots$$

$$x^{(2)} = -1 - \tfrac{1}{2}\epsilon - \tfrac{1}{8}\epsilon^2 + \tfrac{1}{128}\epsilon^4 + \cdots$$

WE KNOW FROM THE BINOMIAL THEOREM THAT THESE SERIES CONVERGE IF, AND ONLY IF, $|\epsilon| < 2$.

BUT SUPPOSE WE DIDN'T KNOW THE QUADRATIC FORMULA. WE COULD ASSUME THAT THE SOLUTION '$x$' CAN BE WRITTEN AS A POWER SERIES:

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$$ AND SEE WHAT HAPPENS...

SUBSTITUTING INTO $x^2 + \epsilon x - 1$, WE GET:

$$(-1 + x_0^2) + (x_0 + 2x_0 x_1)\epsilon + (x_1 + x_1^2 + 2x_0 x_2)\epsilon^2 + \cdots = 0$$

TO SATISFY THIS EQUATION, ALL OF THE COEFFICIENTS OF $\epsilon^n$ MUST VANISH. LOOKING AT EACH COEFFICIENT,

$\epsilon^0$: $\quad x_0^2 - 1 = 0 \quad$ OR $\quad x_0 = \pm 1.$ $\quad$ EASY.

LET'S LOOK AT THE SOLUTION THAT BEGINS $x^{(1)} = 1 + \ldots$

AT $\epsilon^1$: $1 + 2x_1 = 0$ OR $x_1 = -\frac{1}{2}$

AT $\epsilon^2$: $-\frac{1}{4} + 2x_2 = 0$ OR $x_2 = \frac{1}{8}$

AT $\epsilon^3$: $2x_3 = 0$ OR $x_3 = 0$

AT $\epsilon^4$: $\frac{1}{64} + 2x_4 = 0$ OR $x_4 = -\frac{1}{128}$

$\vdots$

ALTOGETHER, $x^{(1)} = 1 - \frac{1}{2}\epsilon + \frac{1}{8}\epsilon^2 - \frac{1}{128}\epsilon^4 + \ldots$

IDEA: WE CAN ASSUME A TAYLOR SERIES SOLUTION TO SIMPLIFY THE PROBLEM!

WE CAN USE THIS SAME STRATEGY TO SOLVE NONLINEAR EQUATIONS.
eg. SHOW THAT ONE SOLUTION TO:

$$\cos[x] = \epsilon \cdot x$$

IS: $x = \frac{\pi}{2} - \frac{\pi}{2}\epsilon + \frac{\pi}{2}\epsilon^2 - \frac{\pi}{48}(\pi^2 + 24)\epsilon^3 + \ldots$

eg. SHOW THAT THE SOLUTIONS TO $x^2 + e^{\epsilon x} = 5$ BEGIN:

$$x = \pm 2 - \epsilon/2 + \ldots$$

## PERTURBATION APPROXIMATION OF DIFFERENTIAL EQUATIONS

THE REAL POWER OF THIS APPROACH COMES IN SOLVING DIFFERENTIAL EQUATIONS.

EX. MOTION OF AN OBJECT PROJECTED UPWARD FROM THE SURFACE OF THE EARTH. LET $x(t)$ DENOTE THE HEIGHT ABOVE THE SURFACE. APPLYING NEWTON'S $2^{nd}$ LAW:

$$\frac{d^2x}{dt^2} = -\frac{gR^2}{(x+R)^2} \quad \text{FOR } t > 0$$

$x(0) = 0 \quad x'(0) = V_0.$ RADIUS OF THE EARTH.

IF WE ARE INTERESTED IN KEEPING THE ~~HIGH~~ HEIGHT SMALL (SMALL COMPARED TO WHAT), THEN WE CAN ARGUE $x+R \approx R$ AND,

$$\frac{d^2x}{dt^2} = -\frac{gR^2}{(x+R)^2} \approx -g.$$

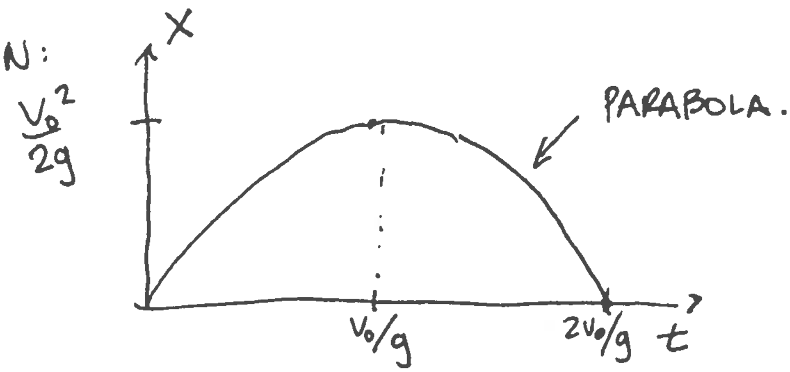
INTEGRATING ONCE: $\frac{dx}{dt} = -gt + C_1$

USING $x'(0) = V_0$ TO SET $C_1$, $\frac{dx}{dt} = -gt + V_0$

INTEGRATING AGAIN: $x = -\frac{g}{2}t^2 + V_0 t + C_2$

USING $x(0)=0$ TO SET $C_2$, $x = \boxed{-\frac{1}{2}gt^2 + V_0 t}$ (*)

PLOTTING THIS SOLUTION:




WE COMPUTED (*) BY ASSUMING THE HEIGHT WAS SMALL — THIS AMOUNTS TO KEEPING $V_0$ SMALL COMPARED WITH THE RADIUS OF THE EARTH. THE UNITS DON'T MAKE SENSE — REALLY WANT $\frac{V_0^2}{g} \ll R$.

HOW CAN WE MAKE A SYSTEMATIC APPROXIMATION? FIRST, NEED TO GET RID OF UNITS.

MEASURE DISTANCE IN UNITS OF $V_0^2/2g$ SO THAT

$$x = (V_0^2/2g)\,\hat{x}$$

UNITS OF DISTANCE

UNITLESS FUNCTION

AND MEASURE TIME IN UNITS OF $V_0/g$ SO THAT

$$t = (V_0/g)\,\hat{t}.$$

THEN:

$$\frac{d^2x}{dt^2} = \frac{(V_0^2/2g)}{(V_0/g)^2}\frac{d^2\hat{x}}{d\hat{t}^2} = \frac{-gR^2}{\left(\frac{V_0^2}{2g}\hat{x}+R\right)^2} = \frac{-g}{\left(\frac{V_0^2}{2gR}\hat{x}+1\right)^2}$$

CLEAN THIS UP, $\quad \dfrac{d^2\hat{x}}{d\hat{t}^2} = \dfrac{-2}{\left(\frac{V_0^2}{2gR}\hat{x}+1\right)^2}$

THE GROUP $(V_0^2/2gR)$ IS A UNITLESS RATIO BETWEEN THE HEIGHT OF THE PARABOLIC SOLUTION & THE RADIUS OF THE EARTH. CALL THIS:

$$\epsilon = \frac{V_0^2}{2gR} \qquad \text{AND FIND } \hat{x} = \hat{x}_0 + \hat{x}_1\epsilon + \cdots$$

THAT IS, TAKE THE DIFFERENTIAL EQUATION:

$$\frac{d^2\hat{x}}{d\hat{t}^2} = \frac{-2}{(\epsilon\hat{x}+1)^2} \qquad \hat{x}(0)=0 \qquad \frac{V_0}{2}\frac{d\hat{x}}{d\hat{t}}(0) = \frac{dx}{dt}(0) = V_0 \quad \text{OR} \quad \frac{d\hat{x}}{d\hat{t}}(0) = 2.$$

AND SUBSTITUTE $\hat{x} = \hat{x}_0(\hat{t}) + \epsilon\,\hat{x}_1(\hat{t}) + \epsilon^2\hat{x}_2(\hat{t}) + \cdots$ AND SEE WHAT HAPPENS.

AT $\epsilon^0$: $\quad \dfrac{d^2\hat{x}_0}{d\hat{t}^2} = -2 \quad \hat{x}_0(0)=0,\ \hat{x}_0'(0)=2$

$$\frac{d\hat{x}_0}{d\hat{t}} = -2\hat{t} + C = -2\hat{t}+2$$

$$\hat{x}_0 = -\hat{t}^2 + 2\hat{t} + C$$

$$\boxed{\hat{x}_0 = -\hat{t}^2 + 2\hat{t}}$$

AT $\epsilon^1$: $\quad \dfrac{d^2\hat{x}_1}{dt^2} = 4\hat{x}_0 = (-t^2+2t)4 \quad \hat{x}_1(0)=0 \quad \hat{x}_1'(0)=0$

$$\frac{d\hat{x}_1}{d\hat{t}} = \left(-\frac{\hat{t}^3}{3}+\hat{t}^2\right)4 + C = \left(-\frac{\hat{t}^3}{3}+\hat{t}^2\right)4$$

$$\boxed{\hat{x}_1 = \left(-\frac{\hat{t}^4}{12}+\frac{\hat{t}^3}{3}\right)4 + C = -\frac{\hat{t}^4}{3} + \frac{\hat{t}^3}{3}4}$$

KEEPING TWO TERMS:

$$\hat{x}(\hat{t}) = (-\hat{t}^2 + 2\hat{t}) + \epsilon\left(-\frac{\hat{t}^4}{12} + \frac{\hat{t}^3}{3}\right)4 + \cdots$$

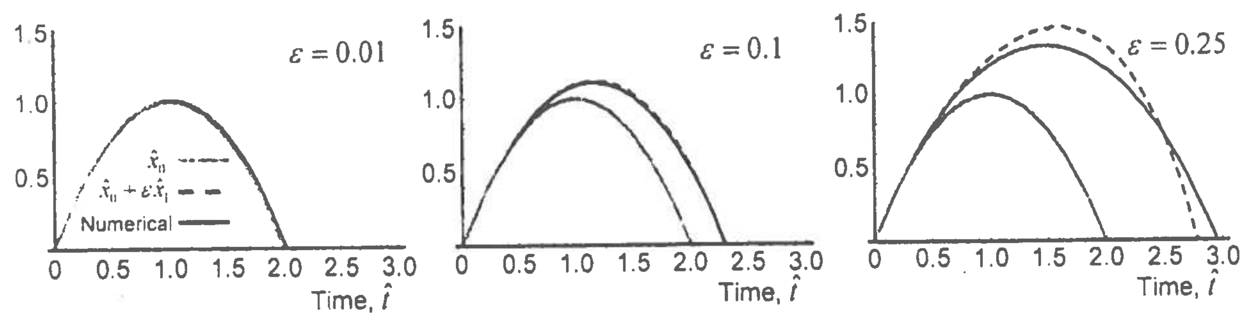
$$= (2\hat{t} - \hat{t}^2) + \epsilon\frac{4\hat{t}^3}{3}\left(1 - \frac{\hat{t}}{4}\right) + \cdots$$

RESTORE THE UNITS: $\hat{x} = \left(\frac{2g}{V_0^2}\right)x$ AND $\hat{t} = \left(\frac{g}{V_0}\right)t$

$$x(t) \simeq \left(\frac{V_0^2}{2g}\right)\left(-\left(\frac{g}{V_0}\right)^2 t^2 + 2\left(\frac{g}{V_0}\right)t\right) + \frac{\epsilon}{3}\left(\frac{g}{V_0}\right)^3 t^3\left(1 - \left(\frac{g}{V_0}\right)\frac{t}{4}\right) + \cdots$$

$$= \left[-\frac{1}{2}gt^2 + V_0 t\right] + \epsilon\left[\frac{g^2 t^3}{6V_0}\left(1 - \frac{gt}{4V_0}\right)\right] + \cdots$$

THE DIFFERENTIAL EQUATION CAN BE APPROXIMATED NUMERICALLY USING A VARIATION ON EULER'S METHOD. COMPARING THE NUMERICAL SOLUTION TO A FEW TERMS OF THE PERTURBATION SERIES,




AS A POINT OF REFERENCE, $\epsilon \simeq 1$ IS APPROXIMATELY EQUAL TO THE ESCAPE VELOCITY $V_{ESCAPE} = 11.2$ km/s; $\epsilon \simeq 0.3$ CORRESPONDS ROUGHLY TO THE SPEED OF A BULLET 1200 km/h

[$g \simeq 10\,m/s^2$, $R \simeq 6400$ km].

## POINCARÉ - LINDSTEDT METHOD

THERE ARE MANY EXAMPLES WHERE THE NAIVE PERTURBATION SERIES DOESN'T WORK; AND MANY METHODS DESIGNED TO ADDRESS THESE DIFFICULTIES.

CLASSIC TEXT: 'ASYMPTOTIC METHODS & PERTURBATION THEORY' BY BENDER & ORSZAG.

COURSE: AMATH 732 - PERTURBATION METHODS & ASYMPTOTIC ANALYSIS.

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EX. DUFFING EQUATION $x''(t) + x(t) - \epsilon x^3(t) = 0$; $x(0) = 1$ & $x'(0) = 0$.

AS BEFORE, TRY $x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \ldots$

COLLECTING LIKE-POWERS OF $\epsilon$, WE END UP WITH THE SET OF EQUATIONS:

$\epsilon^0$: $x_0'' + x_0 = 0$; $x_0(0) = 1$ & $x_0'(0) = 0$ $\Rightarrow$ $x_0(t) = \cos t$.

AT $\epsilon^1$: $$x_1'' + x_1 = x_0^3 = \cos^3 t = \underbrace{\frac{3}{4}\cos t}_{\text{THIS TERM WILL CAUSE PROBLEMS...}} - \frac{1}{4}\cos 3t; \quad x_1(0) = 0 \ \& \ x_1'(0) = 0$$

$$x_1(t) = \frac{1}{32}(\cos t - \cos 3t) + \frac{3}{8} t \sin t.$$

HERE IS THE PROBLEM: THIS TERM GROWS WITH '$t$'. FOR $t > \epsilon$, THE ORDERING IMPLIED BY OUR PERTURBATION SERIES BREAKS-DOWN.

~~WHAT'S THE~~ WHERE IS THIS COMING FROM? THE NAIVE EXPANSION RESULTS IN A SET OF RECURSIVE EQUATIONS:

$$x_n'' + x_n = x_{n-1}^3$$

WHERE $x_{n-1}(t)$ OSCILLATES AT THE SAME FREQUENCY AS $x_n$, IRRESPECTIVE OF $\epsilon$! LIKE A DOUBLE-BOUNCE ON A TRAMPOLINE, THIS SO-CALLED 'RESONANT FORCING' LEADS TO DIVERGENCE.

POINCARÉ'S IDEA: LET THE FREQUENCY OF THE OSCILLATIONS CHANGE WITH $\epsilon$!

ie:   $x(\tau) = x_0(\tau) + \epsilon x_1(\tau) + \epsilon^2 x_2(\tau) + \ldots$

WHERE   $\tau = t(1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \ldots)$

THESE ARE CONSTANTS THAT WILL BE CHOSEN TO ELIMINATE THE DIVERGENT TERMS.

USING THE CHAIN RULE -

$$\frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \frac{dx}{d\tau}(1+\epsilon\omega_1+\epsilon^2\omega_2+\ldots) \;\&\; \frac{d^2x}{dt^2} = \frac{d^2x}{d\tau^2}(1+\omega_1\epsilon+\omega_2\epsilon^2+\ldots)^2 \simeq \frac{d^2x}{d\tau^2}(1+2\epsilon\omega_1+\ldots)$$

WITH SUBSTITUTION INTO THE ORIGINAL DIFF. EQ:

$$(1+\epsilon\omega_1+\epsilon^2\omega_2+\ldots)^2 x''(\tau) + x(\tau) - \epsilon x^3(\tau) = 0 \;;\; x(0) = 1 \;\&\; x'(0) = 0.$$

COLLECTING POWERS OF $\epsilon$:

$\epsilon^0$: $x_0''(\tau) + x_0(\tau) = 0$; $x(0)=1$ & $x'(0)=0$   $\Rightarrow$ $x_0(\tau) = \cos\tau$

$\epsilon^1$: $x_1''(\tau) + x_1(\tau) = -2\omega_1 x_0''(\tau) + x_0^3(\tau)$; $x_1(0) = x_1'(0) = 0$.

$$= 2\omega_1\cos\tau + \cos^3\tau$$

$$= \underbrace{(2\omega_1 + 3/4)\cos\tau}_{} - \tfrac{1}{4}\cos 3\tau$$

IT IS THIS TERM THAT CAUSES ALL THE PROBLEMS. SO CHOOSE $\omega_1 = -3/8$ TO MAKE IT DISAPPEAR!

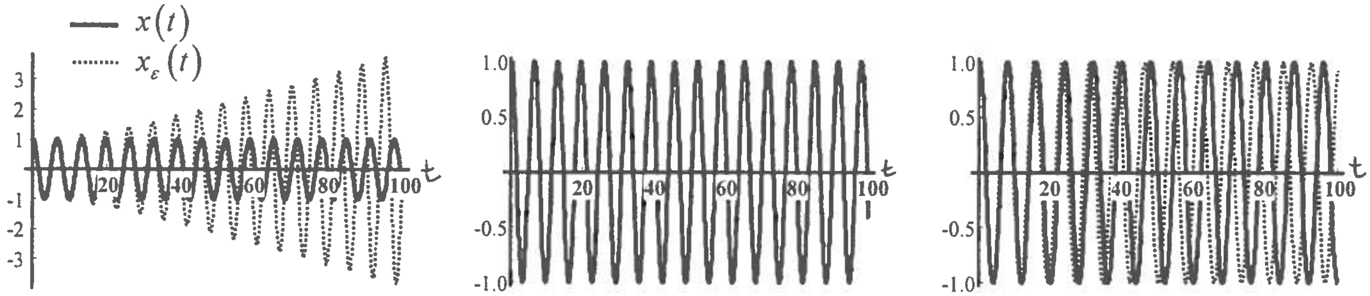
$$\Rightarrow x_1(\tau) = \frac{1}{32}(\cos\tau - \cos 3\tau)$$

ALTOGETHER,

$$x(\tau) \simeq \cos\tau + \frac{\epsilon}{32}(\cos\tau - \cos 3\tau) \quad \text{WITH} \quad \tau = t(1 - 3/8\,\epsilon).$$

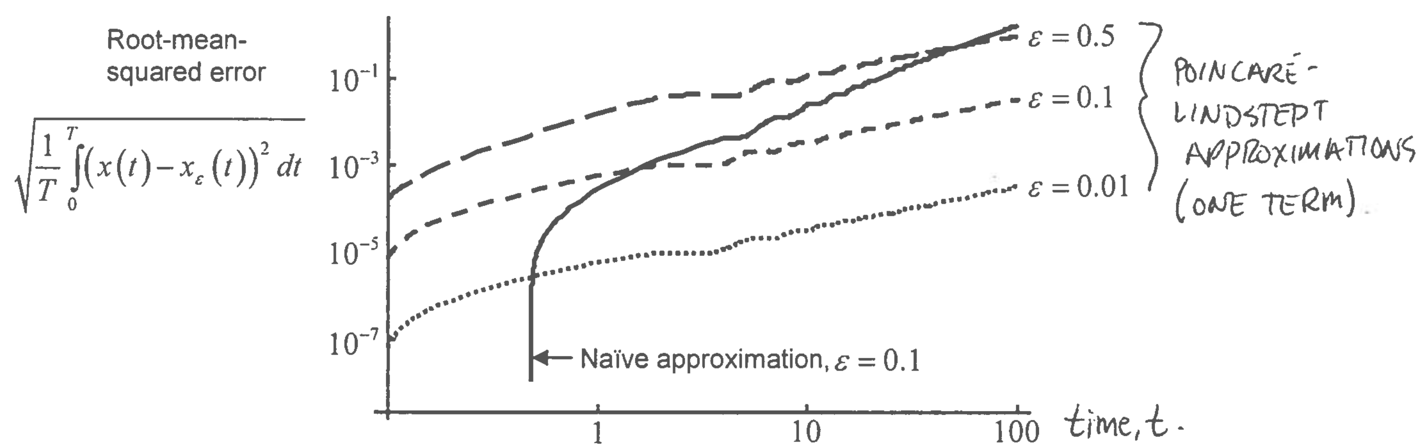
HOW DOES IT ~~WORK~~ ~~IN~~ COMPARE WITH THE EXACT (i.e. NUMERICAL) SOLUTION? VERY WELL.

WE CAN COMPARE THE ONE-TERM APPROXIMATIONS, i.e., $x_\epsilon(t) \sim x_0 + \epsilon x_1$ TO THE NUMERICAL SOLUTION FOR DIFFERENT VALUES OF $\epsilon$.
FIRST THING TO NOTICE IS THE DIVERGENCE OF THE NAIVE APPROXIMATION & HOW THE POINCARÉ APPROXIMATION (BY DESIGN) REMAINS BOUNDED:




Naïve approximation, $\varepsilon = 0.1$ Poincare-Lindstedt, $\varepsilon = 0.1$ Poincare-Lindstedt, $\varepsilon = 0.5$

WE CAN BE MORE QUANTITATIVE BY COMPARING THE ROOT-MEAN-SQUARE ERROR BETWEEN THE NUMERICAL & PERTURBATION SOLUTIONS (ROOT-MEAN-SQUARE IS RELATED TO THE $\mathcal{L}^2$-NORM).




EVENTUALLY, THE POINCARÉ APPROXIMATION IS COMPLETELY OUT-OF-PHASE WITH THE NUMERICAL SOLUTION, BUT STILL BOUNDED IN AMPLITUDE.