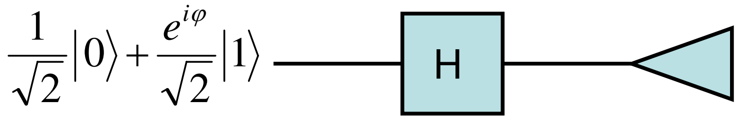
CO481/CS467/PHYS467 ASSIGNMENT 1
Due Wednesday, January 16 at 11:59pm, electronically using LEARN
(will constitute 10% out of the 50% assignment marks)

We denote the Hadamard gate (with respect to the computational basis $\{|0\rangle, |1\rangle\}$) by

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

1. **5 marks**



(a) i. Compute the state of the qubit after a Hadamard transformation H is applied to the state $\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\phi}}{\sqrt{2}}|1\rangle\right)$, for the value $\phi = \pi/4$.
Write this state (in Dirac notation) in the first row and second column of the table below. Express the qubit state in the form $e^{i\alpha}(\cos(\theta)|0\rangle + e^{i\beta}\sin(\theta)|1\rangle)$.

ii. Compute the probability of obtaining $|0\rangle$ if the resulting state is measured in the computational basis. Write this value (both in closed form and a 3-digit approximation) in the first row and third column of table below.

iii. Also complete the table below for the values of $\phi = 3\pi/4, 5\pi/4, 7\pi/4$.

ϕ	output state	p_0
$\pi/4$		
$3\pi/4$		
$5\pi/4$		
$7\pi/4$		

(b) Note that question 1 shows how to distinguish either of the states $\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\pi/4}}{\sqrt{2}}|1\rangle\right)$, $\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i7\pi/4}}{\sqrt{2}}|1\rangle\right)$ from either of the states $\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i3\pi/4}}{\sqrt{2}}|1\rangle\right)$, $\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i5\pi/4}}{\sqrt{2}}|1\rangle\right)$ with high probability (i.e. guessing based on whether one measures 0 or 1 gives the correct answer with probability $> 85\%$).

Describe an experiment (provide a figure and explanation) that would allow you to distinguish $\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\pi/4}}{\sqrt{2}}|1\rangle\right)$ from $\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i5\pi/4}}{\sqrt{2}}|1\rangle\right)$ with certainty.

Hint: Use a $-\pi/4$ phase shifter :

$$P_{\pi/4} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} = |0\rangle\langle 0| + e^{-i\pi/4}|1\rangle\langle 1|.$$

2. **4 marks**

Find a 1-qubit state vector $|+_H\rangle$ such that $H|+_H\rangle = |+_H\rangle$.

Find a 1-qubit state vector $|-_H\rangle$ such that $H|-_H\rangle = -|-_H\rangle$.

3. **3 marks**

Prove that for $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|1\rangle$, and for any unitary one qubit gate G we have:

$$G \otimes I|\beta_{00}\rangle = I \otimes G^t|\beta_{00}\rangle$$

where G^t is the transpose of G.

4. **3 marks**

Give a protocol that wins the following game with probability $\frac{2}{3}$.

An adversary does one of the following (with no restriction on the probability with which she chooses option 1 or 2):

1) Gives you either $|0\rangle$ or $|1\rangle$, with equal probability.

OR

2) Gives you $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$.

You must guess whether the adversary performed 1) or 2).

(Hint: First come up with a protocol that has 'one-sided' error. Then post-process so that the error probability is equal in both cases.)

5. **2 marks**

(a) Draw a classical circuit, with 3 input bits and 2 output bits, that consists of XOR gates and $FANOUT$, and maps

$000 \mapsto 00$	$001 \mapsto 01$	$100 \mapsto 10$	$010 \mapsto 11$
$111 \mapsto 00$	$110 \mapsto 01$	$011 \mapsto 10$	$101 \mapsto 11$

(Aside: Note that the above map may be used to identify where a single bit flip (or no bit flip) has occurred, assuming the input state was either 000 or 111).

(b) Give a reversible 5-bit circuit for computing the above function while keeping the input intact. E.g. the circuit should map 100 00 $\mapsto$ 100 10. Use only CNOT gates.

6. **3 marks**

One may express the action of unitary operations using several notations. E.g. the X gate can be represented using

- matrix notation:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- Dirac notation for the operator:

$$|0\rangle\langle 1| + |1\rangle\langle 0|$$

- element-wise action for a basis, using Dirac notation:
  $|0\rangle \mapsto |1\rangle$
  $|1\rangle \mapsto |0\rangle$.

For the following circuit, calculate the unitary it computes, and describe the unitary using each of the above three notations.