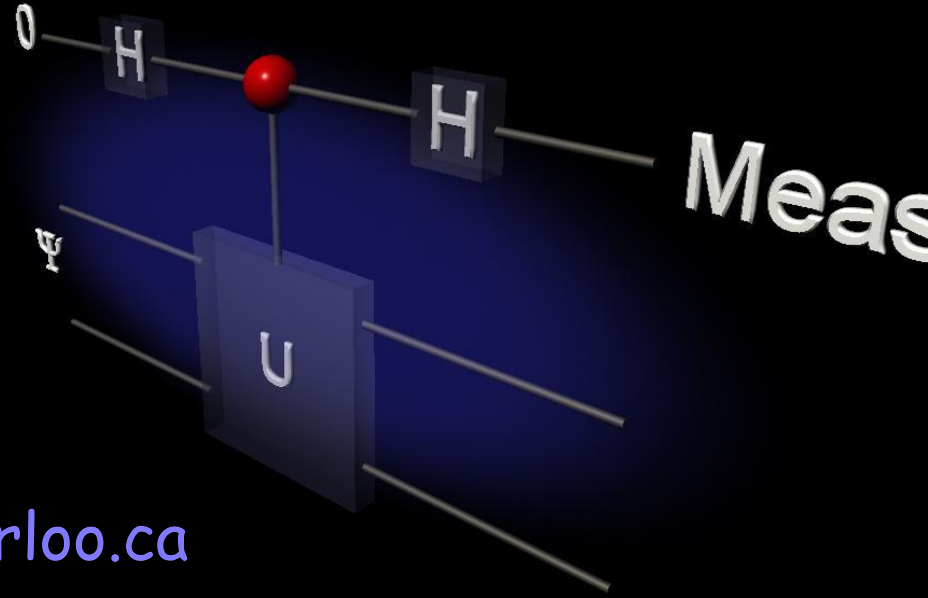
Introduction to Quantum Information Processing



CO481 CS467 PHYS467

Michele Mosca mmosca@iqc.uwaterloo.ca

Tuesdays and Thursdays 10am-11:15am

IQC Institute for Quantum Computing

UNIVERSITY OF WATERLOO

PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

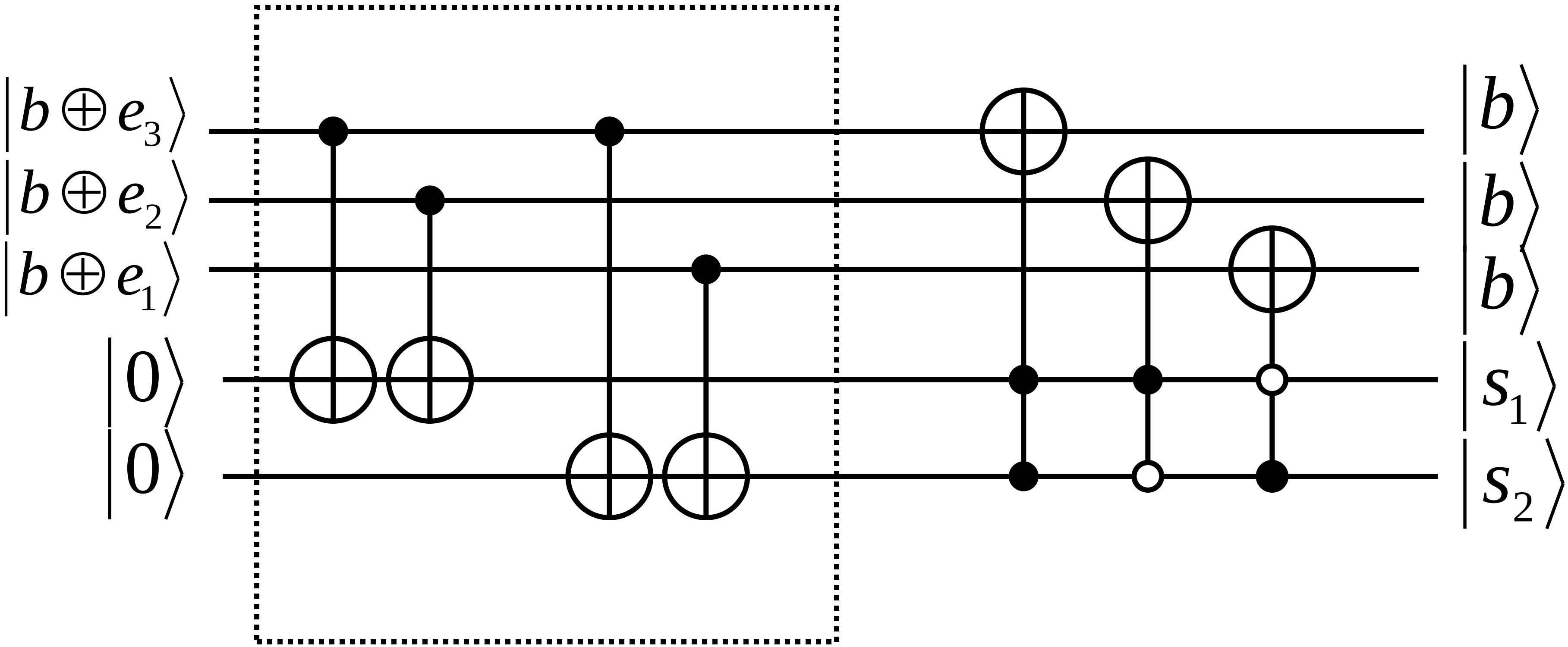
Overview

Lecture 16

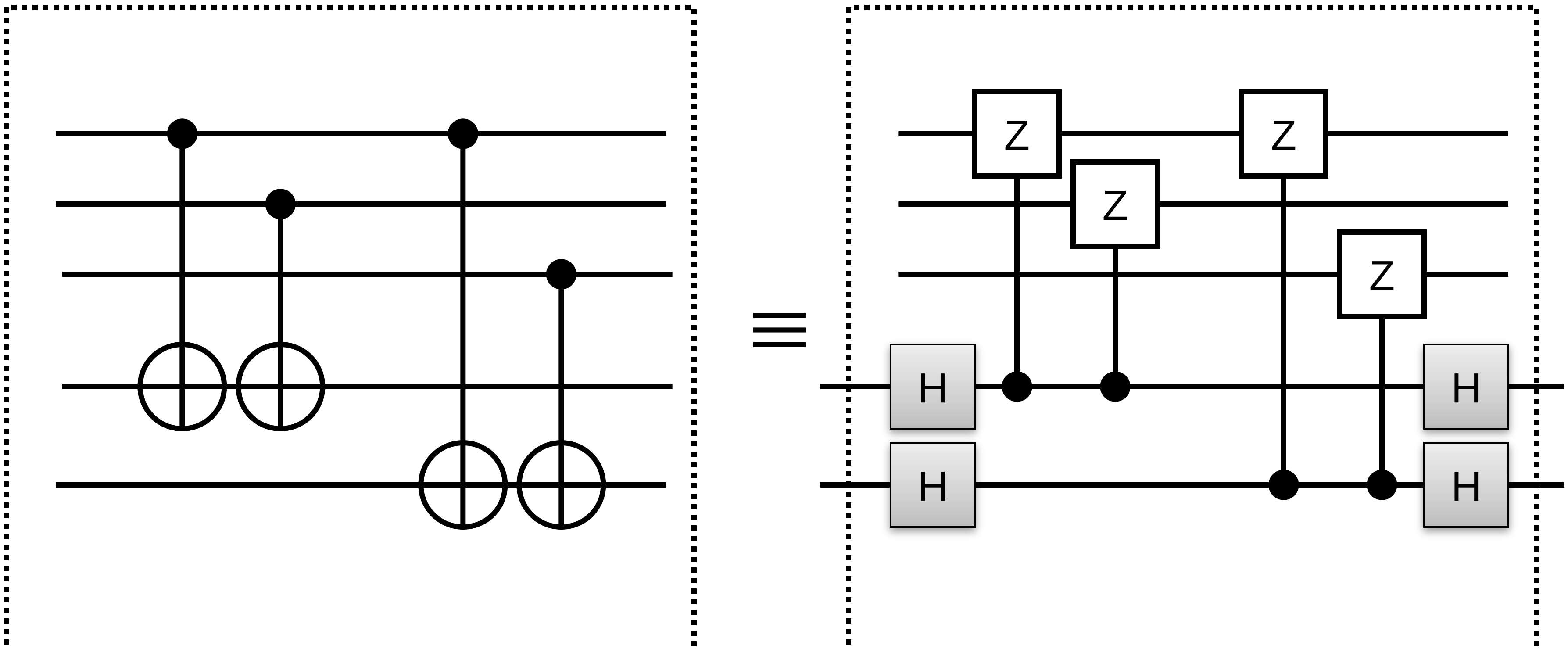
- More Quantum Error Correction
- The DiVincenzo Criteria and requirements for fault tolerance

Reading: Chapter 10, Section 3.5.3

Recall syndrome computation for 3-qubit bit flip code



Equivalent to measuring a Pauli observable



Measuring these parities is equivalent to measuring the observables ZZI and ZIZ

The codespace is "stabilized" by these operators

$$ZZI\left(\alpha|000\rangle+\beta|111\rangle\right)=\alpha|000\rangle+\beta|111\rangle$$

$$ZIZ\left(\alpha|000\rangle+\beta|111\rangle\right)=\alpha|000\rangle+\beta|111\rangle$$

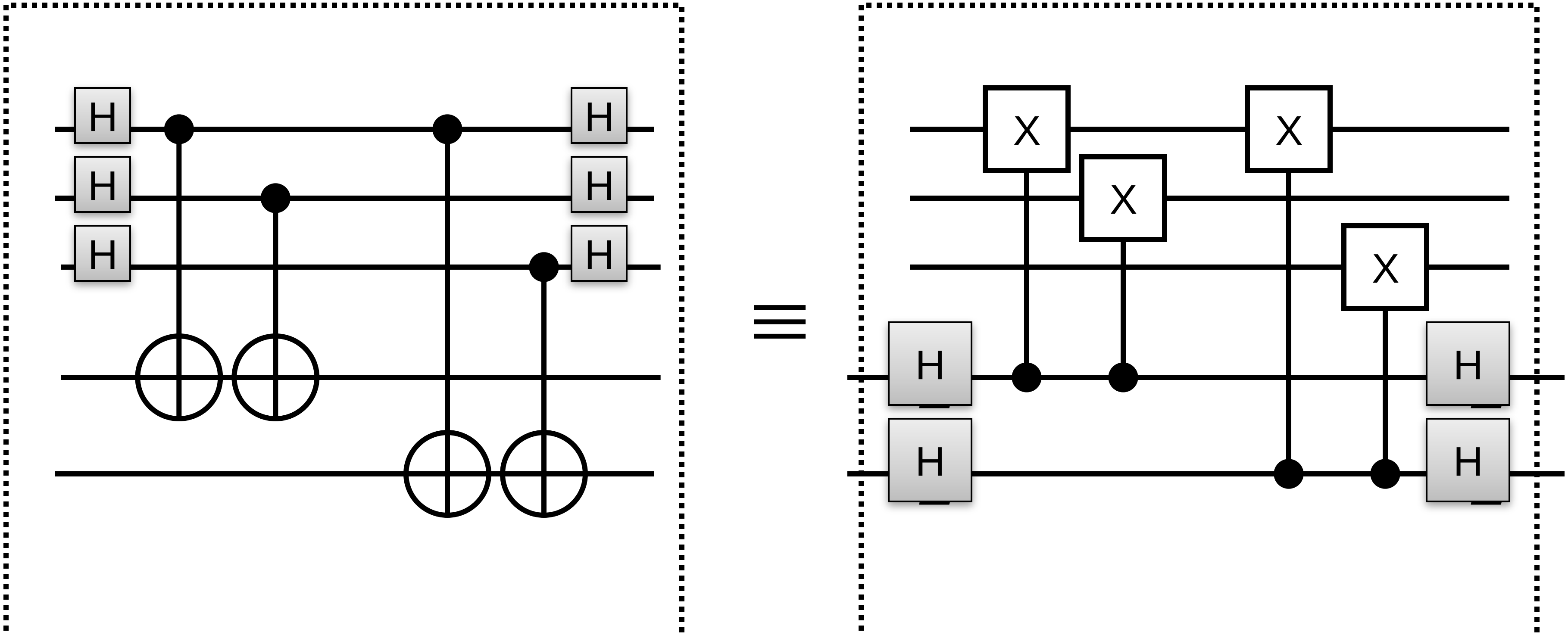
Note that the entire codespace consists of $+1$ eigenvectors of these two operators. In fact, we can define the codespace in this way.

The codespace is "stabilized" by these operators

$$
\begin{array}{lcc}
 & ZZI & ZIZ \\
III\left(\alpha|000\rangle + \beta|111\rangle\right) & +1 & +1 \\
XII\left(\alpha|000\rangle + \beta|111\rangle\right) & -1 & -1 \\
IXI\left(\alpha|000\rangle + \beta|111\rangle\right) & -1 & +1 \\
IIX\left(\alpha|000\rangle + \beta|111\rangle\right) & +1 & -1
\end{array}
$$

Applying an X error to any one of the qubits changes at least one of the eigenvalues to -1. The pattern of eigenvalues tells us where a single X error occurred.

Equivalent to measuring a Pauli observable



Measuring these "parities" is equivalent to measuring the observables XXI and XIX

The codespace is “stabilized” by these operators

$$
\begin{array}{lcc}
 & XXI & XIX \\
III\left(\alpha\left|+++\right\rangle+\beta\left|---\right\rangle\right) & +1 & +1 \\
ZII\left(\alpha\left|+++\right\rangle+\beta\left|---\right\rangle\right) & -1 & -1 \\
IZI\left(\alpha\left|+++\right\rangle+\beta\left|---\right\rangle\right) & -1 & +1 \\
IIZ\left(\alpha\left|+++\right\rangle+\beta\left|---\right\rangle\right) & +1 & -1
\end{array}
$$

Applying a Z error to any one of the qubits changes at least one of the eigenvalues to -1. The pattern of eigenvalues tells us where a single Z error occurred.

Correcting both phase errors and bit flip errors

- Consider the codewords

$$|0_L\rangle = \left(|000\rangle + |111\rangle\right)\left(|000\rangle + |111\rangle\right)\left(|000\rangle + |111\rangle\right)$$

$$|1_L\rangle = \left(|000\rangle - |111\rangle\right)\left(|000\rangle - |111\rangle\right)\left(|000\rangle - |111\rangle\right)$$

- We can easily correct any single X- error in one of the 3 three-bit parts

- We can then also correct a single Z- error on one of the 9 qubits.

- This means we can also correct Y-errors on one of the 9 qubits

The codespace is "stabilized" by these operators

$$|\psi\rangle = \alpha\big(|000\rangle + |111\rangle\big)\big(|000\rangle + |111\rangle\big)\big(|000\rangle + |111\rangle\big)$$
$$+ \beta\big(|000\rangle - |111\rangle\big)\big(|000\rangle - |111\rangle\big)\big(|000\rangle - |111\rangle\big)$$

	$XXXXXXIII$	$XXXIIIXXX$	$ZZIIIIIII$	$ZIZIIIIII$	$IIIZZIIII$	...
$IIIIIIIII\|\psi\rangle$	$+1$	$+1$	$+1$	$+1$	$+1$	
$ZIIIIIIII\|\psi\rangle$	-1	-1	$+1$	$+1$	$+1$	
$XIIIIIIII\|\psi\rangle$	$+1$	$+1$	-1	-1	$+1$	
$YIIIIIIII\|\psi\rangle$	-1	-1	-1	-1	$+1$	

Measuring the values of the observables XXXXXXIII, XXXIIIXXX, ZZIIIIIII, ZIZIIIIII, IIIZZIIII, IIIZIZIII, IIIIIIZZI, IIIIIIZIZ will tell us whether an X, Z or Y error effectively occurred on any one qubit.

Correcting an arbitrary error

$$U = aI + bX + cY + dZ$$

$$UIIIIIIII|\psi\rangle$$

$$\begin{aligned} &= a|\psi\rangle \\ &+ bXIIIIIIII|\psi\rangle \\ &+ cYIIIIIIII|\psi\rangle \\ &+ dZIIIIIIII|\psi\rangle \end{aligned} \quad \mapsto \quad \begin{aligned} &a|\psi\rangle|00000000\rangle \\ &+ bXIIIIIIII|\psi\rangle|00110000\rangle \\ &+ cYIIIIIIII|\psi\rangle|11110000\rangle \\ &+ dZIIIIIIII|\psi\rangle|11000000\rangle \end{aligned}$$

When computing the syndromes we get the following state.

Correcting an arbitrary error

$$a|\psi\rangle|00000000\rangle$$
$$+bXIIIIIIII|\psi\rangle|00110000\rangle$$
$$+cYIIIIIIII|\psi\rangle|11110000\rangle$$
$$+dZIIIIIIII|\psi\rangle|11000000\rangle$$

We can either quantumly apply the appropriate Pauli correction to get the following.

$$a|\psi\rangle|00000000\rangle$$
$$+b|\psi\rangle|00110000\rangle$$
$$+c|\psi\rangle|11110000\rangle$$
$$+d|\psi\rangle|11000000\rangle$$
$$=|\psi\rangle\begin{pmatrix} a|00000000\rangle \\ +b|00110000\rangle \\ +c|11110000\rangle \\ +d|11000000\rangle \end{pmatrix}$$

Correcting an arbitrary error

$$a|\psi\rangle|00000000\rangle$$
$$+bXIIIIIIII|\psi\rangle|00110000\rangle$$
$$+cYIIIIIIII|\psi\rangle|11110000\rangle$$
$$+dZIIIIIIII|\psi\rangle|11000000\rangle$$

$$\mapsto XIIIIIIII|\psi\rangle|00110000\rangle$$
$$(with\ prob.\ |b|^2)$$

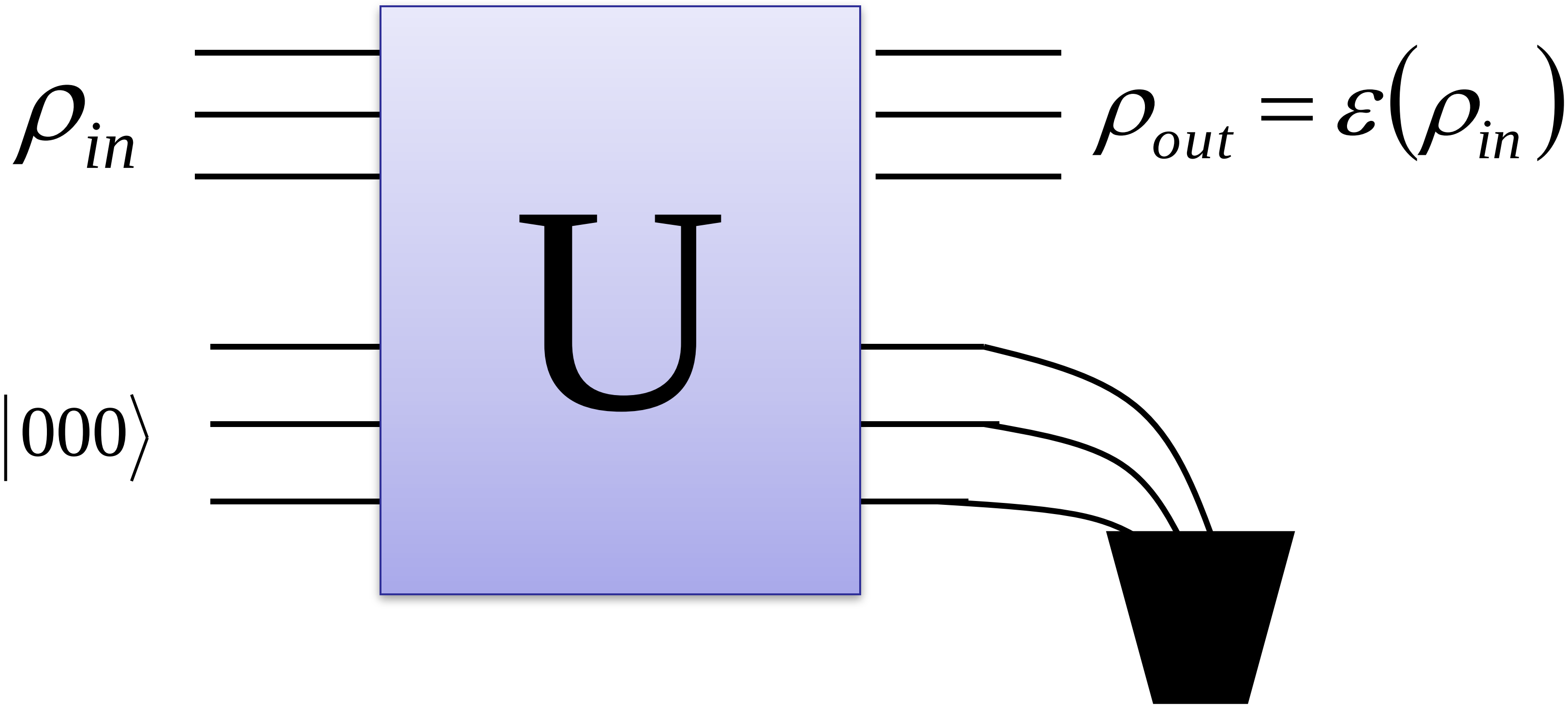
Or we can measure the syndrome, and then classically control which Pauli correction to apply.

$$\mapsto |\psi\rangle|00110000\rangle$$

A more general perspective

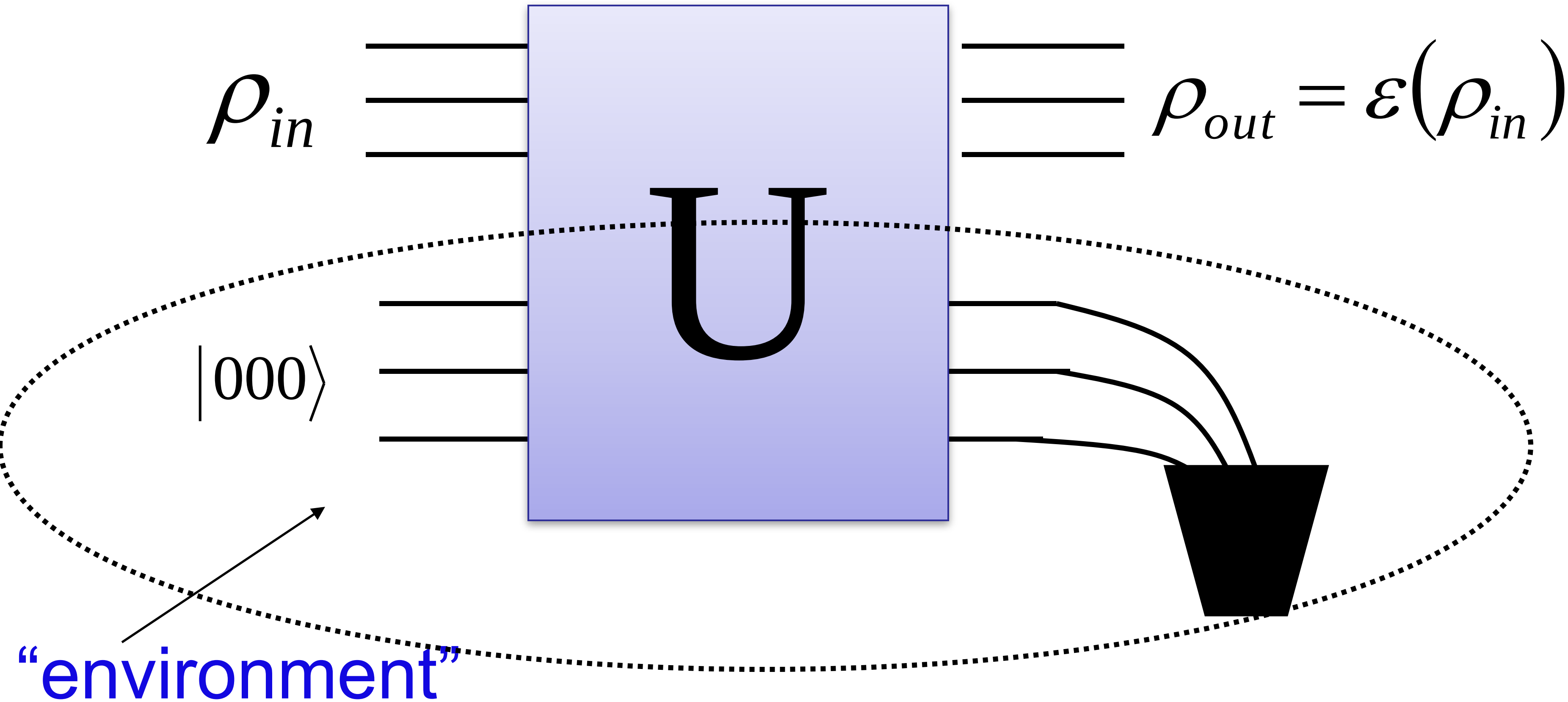
- General description of error operations
- Condition for error correction to be possible

General Operations



$$\rho_{in} \mapsto \rho_{out} = Tr_2\left(U\left(\rho_{in} \otimes |000\rangle\langle 000|\right)U^{\dagger}\right)$$

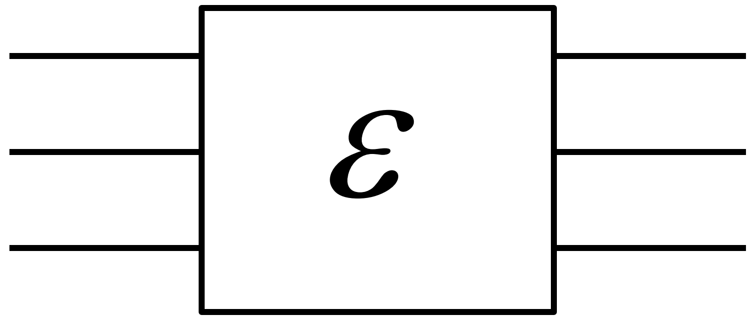
Error operation



$$\rho_{in} \mapsto \rho_{out} = Tr_2\left(U\left(\rho_{in} \otimes |000\rangle\langle 000|\right)U^{\dagger}\right)$$

Error operation (see 3.5.3)

ρ_{in}



$\rho_{out} = \varepsilon(\rho_{in})$

$$\rho_{in} \mapsto \rho_{out} = \sum_i \varepsilon_i \rho_{in} \varepsilon_i^{\dagger}$$

where

$$\sum_i \varepsilon_i^{\dagger} \varepsilon_i = I$$

ε_i are “Kraus” operators

Error operation

Thus the error operation acting on our encoded qubits can be described in terms of Kraus operators

$$\rho_{in} \mapsto \rho_{out} = \sum_i \varepsilon_i \rho_{in} \varepsilon_i^\dagger$$

$$\sum_i \varepsilon_i^\dagger \varepsilon_i = I$$

Necessary condition for quantum error correction

Let $|0_{enc}\rangle, |1_{enc}\rangle$ be the encodings of $|0\rangle, |1\rangle$

For an encoding to permit the perfect correction of the error operator $\mathcal{E}$ that maps

$$\rho_{in} \mapsto \rho_{out} = \sum_i \varepsilon_i \rho_{in} \varepsilon_i^\dagger$$

we must have

$$\langle l_{enc} | \varepsilon_i^\dagger \varepsilon_j | m_{enc} \rangle = c_{ij} \delta_{lm}$$

$$\delta_{lm} = 1 \; if \;\; l = m$$
$$\delta_{lm} = 0 \; if \;\; l \neq m$$

Necessary condition for quantum error correction

The same encoding and error correction procedure will also correct errors induced by any error operator $\mathfrak{F}$ that maps

$$\rho_{in} \mapsto \rho_{out} = \sum_i \mathsf{F}_i \rho_{in} \mathsf{F}_i^\dagger$$

where the F_i are linear combinations of $\mathcal{E}_i$

This explains why the Shor 9-qubit code will correct any 1-qubit error (and not just 1-qubit unitary errors)

Concatenation

An error operator will in general not be composed *only* of correctable operators.

However, we can define a more general notion of error probability, and for a given error model, a “good” code will have the property that if there is an error probability of p in the individual qubits, the probability of an uncorrectable error is in $O(p^2)$, and thus the effective error rate on encoded qubits is $O(p^2)$.

Concatenation

A “good” code will have the property that if there is an error probability of p in the individual qubits, the probability of an uncorrectable error is in $O(p^2)$, and thus the effective error rate on encoded qubits is $O(p^2)$.

If errors are “incoherent”, the same analysis as for classical error correction follows.

For general errors (which include “coherent” errors), a very similar analysis with similar asymptotics follows.

The DiVincenzo Criteria
(from D. Gottesman presentation on fault-tolerant QC*)

1. A scalable physical system with well-characterized qubits.

2. The ability to initialize the state of the qubits to a simple fiducial state, such as $|000\ldots\rangle$.

3. Long relevant decoherence times, much longer than the gate operation time.

4. A "universal" set of quantum gates.

5. A qubit-specific measurement capability.

6. The ability to interconvert stationary and flying qubits.
7. The ability to faithfully transmit flying qubits between specified locations.

*www.perimeterinstitute.ca/personal/dgottesman/FTreqs.ppt)

Requirements for Fault-Tolerance
(from D. Gottesman presentation on fault-tolerant QC*)

- Low gate error rates.
- Ability to perform operations in parallel.
- A way of remaining in, or returning to, the computational Hilbert space.
- A source of fresh initialized qubits during the computation.
- Benign error scaling: error rates that do not increase as the computer gets larger, and no large-scale correlated errors.

*www.perimeterinstitute.ca/personal/dgottesman/FTreqs.ppt