QUALITATIVE ANALYSIS OF 2nd ORDER DIFFERENTIAL EQUATIONS

THE FROBENIUS METHOD IS GUARANTEED TO PRODUCE SOLUTIONS TO LINEAR DIFFERENTIAL EQUATIONS, BUT IT IS DIFFICULT TO GET A SENSE OF HOW THE INFINITE SERIES BEHAVES. THIS IS WHERE QUALITATIVE ANALYSIS CAN HELP.

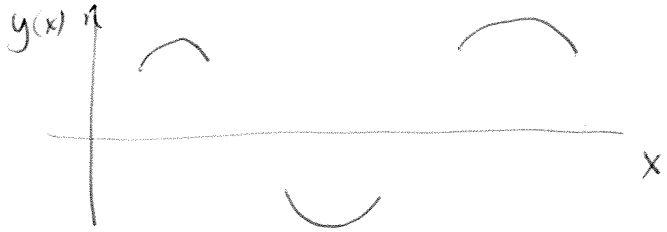
EX. $y''+y=0$ HAS HOMOGENEOUS SOLUTIONS $y_1 = \sin x$ & $y_2 = \cos x$

WITHOUT KNOWING ANYTHING ABOUT THE FUNCTIONS $\sin x$ & $\cos x$, THE STRUCTURE OF THE DIFFERENTIAL EQUATION TELLS YOU THE SOLUTIONS MUST OSCILLATE:

$$\underbrace{\frac{d^2y}{dx^2}}_{\text{CONCAVITY!}} = -y$$

IF A SOLUTION $y(x)$ IS POSITIVE, THEN IT IS CONCAVE DOWN

SIMILARLY, IF $y(x)$ IS NEGATIVE, THEN IT IS CONCAVE UP.



FURTHERMORE, AGAIN WITHOUT KNOWING ANYTHING ABOUT $\sin x$ & $\cos x$, YOU CAN USE THE WRONSKIAN TO SHOW THAT $\sin^2 x + \cos^2 x =$ CONSTANT (AND SHOW CONSTANT $=1$ IF YOU ADD TO THE DIFFERENTIAL EQ. THE INITIAL CONDITIONS $y_1(0)=0$, $y_1'(0)=1$ & $y_2(0)=1$, $y_2'(0)=0$ - SEE COURSE NOTES)

IN A SIMILAR WAY, WE CAN DEFINE THE EXPONENTIAL FUNCTION AS THE SOLUTION TO: $y'-y=0 \quad y(0)=1.$

THESE FUNCTIONS - $\sin x$, $\cos x$, e^x - ARE ALL FUNCTIONS WE HAVE ENCOUNTERED IN OTHER CONTEXTS, BUT THEY CAN BE DEFINED IN TERMS OF THE DIFFERENTIAL EQUATIONS THAT GENERATE THEM.

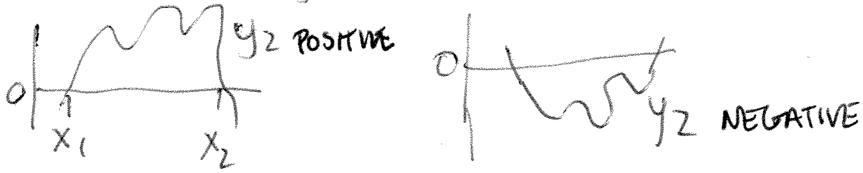
WHAT ABOUT OTHER 'SPECIAL FUNCTIONS'?

- STURM SEPARATION THEOREM: IF $y_1(x)$ AND $y_2(x)$ ARE LINEARLY-INDEPENDENT SOLUTIONS OF $y'' + P(x)y' + Q(x)y = 0$, THEN THE ZEROS OF THESE FUNCTIONS ARE DISTINCT, AND OCCUR ALTERNATELY.

PROOF: AWAY FROM SINGULAR POINTS, THE WRONSKIAN
$W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$ IS NONZERO, AND SO HAS CONSTANT SIGN.

I) SUPPOSE THE ZEROS OF y_1 & y_2 WERE NOT DISTINCT,
i.e. $y_1(x_0) = y_2(x_0) = 0$, THEN $W(x_0) = 0$. CONTRADICTION.
SO THE ZEROS MUST BE DISTINCT.

II) SUPPOSE x_1 AND x_2 ARE SUCCESSIVE ZEROS OF y_2: $y_2(x_1) = y(x_2) = 0$ AND $y_2(x) \neq 0$ FOR $x \in (x_1, x_2)$. MUST SHOW y_1 NECESSARILY VANISHES BETWEEN x_1 & x_2. BECAUSE $y_2(x) \neq 0$ FOR $x \in (x_1, x_2)$, WE HAVE TWO POSSIBILITIES:



NOTICE $W(x_1) = y_1(x_1)y_2'(x_1)$ AND $W(x_2) = y_1(x_2)y_2'(x_2)$ HAVE THE SAME SIGN BECAUSE $W(x)$ IS NONZERO. BUT $y_2'(x_1)$ AND $y_2'(x_2)$ HAVE OPPOSITE SIGNS! THAT MEANS $y_1(x_1)$ AND $y_1(x_2)$ HAVE OPPOSITE SIGNS. BY THE INTERMEDIATE VALUE THEOREM (OR SIMPLY AS A CONSEQUENCE OF CONTINUITY) THAT IMPLIES $y_1(x) = 0$ FOR SOME $x \in (x_1, x_2)$.

~~BUT~~ IF WE APPLY THIS SAME REASONING TO y_1, WE CONCLUDE THAT IT CAN ONLY HAVE ONE ZERO $x \in (x_1, x_2)$ BECAUSE OTHERWISE y_2 WOULD HAVE TO HAVE A THIRD ZERO IN (x_1, x_2) WHICH IS A CONTRADICTION.

SO, THE ZEROS OF y_1 & y_2 OCCUR ALTERNATELY.

WHAT MADE OUR ANALYSIS OF THE HARMONIC OSCILLATOR SO SIMPLE WAS THAT y'' WAS PROPORTIONAL TO y. OBVIOUSLY, THIS IS ~~RARE~~. BUT WE CAN CONVERT THE EQUATION $y'' + P(x)y' + Q(x)y = 0$ INTO NORMAL FORM TO MAKE IT LOOK MORE LIKE THE HARMONIC OSCILLATOR & FACILITATE THE ANALYSIS.

RECALL THAT IF WE RE-WRITE $y(x) = u(x)v(x)$, WE CAN ALWAYS CHOOSE $v(x) = \exp[-\frac{1}{2}\int P(x')dx']$ SO THAT THE FIRST-DERIVATIVE VANISHES -

$$y'' + P(x)y' + Q(x)y = 0 \xrightarrow{y = \exp[-\frac{1}{2}\int P(x')dx'] \cdot u(x)} \boxed{u'' + q(x)u = 0.}$$

NORMAL FORM

WITH $q(x) = Q(x) - \frac{1}{4}P^2(x) - \frac{1}{2}P'(x)$.

IF $q(x)$ HAS CONSTANT SIGN, SOLUTIONS $u(x)$ BEHAVE AS WE WOULD EXPECT: FOR $q(x) > 0$, $u(x)$ OSCILLATES (eg. $\cos x$ & $\sin x$); FOR $q(x) < 0$, $u(x)$ DIVERGES (eg. $\sinh x$ & $\cosh x$). WE WILL PROVE THESE IN DETAIL. FIRST, NOTE THAT IF $u(x_0) = 0$ & $u'(x_0) = 0$, THEN THE SOLUTION MUST BE $u(x) = 0$ [BY THE UNIQUENESS THEOREM].

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LOOK AT THE DIVERGENCE CASE, $q(x) < 0$, FIRST:

THEOREM: IF $q(x) < 0$ FOR ALL x, THEN A NON-TRIVIAL SOLUTION $u(x)$ HAS AT MOST ONE ZERO [OR $u(x) = 0$ FOR ALL x.]

PROOF: WE'RE NOT INTERESTED IN THE TRIVIAL SOLUTION $u(x) \equiv 0$. SUPPOSE $u(x_0) = 0$, THEN $u'(x_0) \neq 0$ [OR ELSE $u(x) \equiv 0$ BY UNIQUENESS]. SUPPOSE $u'(x_0) > 0$ [SIMILAR FOR $u'(x_0) < 0$], THEN $u''(x) = -q(x)u(x) > 0$ FOR $x > x_0$ AND $u(x)$ CAN ONLY INCREASE; $u''(x) = -q(x)u(x) < 0$ FOR $x < x_0$ AND $u(x)$ CAN ONLY DECREASE. ∎

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THE PROOF FOR OSCILLATORY SOLUTIONS, $q(x) > 0$, IS SLIGHTLY MORE COMPLICATED.

THEOREM: IF $q(x)>0$ FOR $x>0$, ~~THEN IF~~ $\int_0 q(x)\,dx = \infty$, THEN A NON-TRIVIAL SOLUTION $u(x)$ HAS INFINITELY MANY ZEROS ON THE POSITIVE X-AXIS, $x\in[0,\infty)$.

PROOF: PROOF BY CONTRADICTION - SUPPOSE THERE IS SOME POINT x_0 AFTER WHICH $u(x)$ HAS NO MORE ZEROS: $u(x)\neq 0$ FOR $x\in[x_0,\infty)$. SAY $u(x)>0$ ON THIS INTERVAL $[x_0,\infty)$ [PROOF IS IDENTICAL IF $u(x)<0$].

THEN, ON $[x_0,\infty)$ $u''=-q(x)u(x)<0$, SO CONCAVE DOWN $\cap$ AND SO WILL HIT ZERO IF $u'<0$. ← THAT IS THE CONTRADICTION WE'RE AFTER. TO SHOW THAT EVENTUALLY $u'<0$, WE INTRODUCE A QUOTIENT:

$$V(x) = -\frac{u'(x)}{u(x)} \qquad x>x_0$$

← $u(x)$ IS POSITIVE, SO IF WE CAN SHOW $V(x)$ IS EVENTUALLY POSITIVE, THEN $u'(x)<0$ AND WE'RE DONE.

DIFFERENTIATING $V(x)$:

$$V'(x) = -\left[\frac{u''u-(u')^2}{u^2}\right] = -\frac{u''}{u} + \left(\frac{u'}{u}\right)^2 = q(x) + [V(x)]^2$$

INTEGRATING BOTH SIDES, FROM $x=x_0$ TO SOME POINT $x=x_1$,

$$V(x_1)-V(x_0) = \int_{x_0}^{x_1} q(x)\,dx + \int_{x_0}^{x_1} V^2(x)\,dx$$

OR

$$V(x_1) = \underbrace{V(x_0)}_{\text{NEGATIVE NUMBER}} + \underbrace{\int_{x_0}^{x_1} V^2(x)\,dx}_{\text{POSITIVE NUMBER}} + \int_{x_0}^{x_1} q(x)\,dx$$

↑ WE KNOW THAT $\int_{x_0}^{\infty} q(x)\,dx \to \infty$ SO WE CAN CHOOSE x_1 TO MAKE THIS TERM LARGER THAN ANY NUMBER.

THEREFORE, THERE IS AN x_1 AT WHICH $V(x_1)>0$, AND $u'(x_1)$ BECOMES NEGATIVE SO BY CONCAVITY $u''<0$, ~~AND~~ $u(x)$ MUST THEN CROSS THE AXIS AT SOME POINT $x>x_0$.

NO MATTER HOW LARGE WE CHOOSE x_0, WE CAN ALWAYS FIND A LARGER ZERO. THAT IS THE CONTRADICTION WE NEEDED TO CONCLUDE THAT THERE IS NO LARGEST ZERO. ∎

EX. AIRY'S EQUATION: $y'' - xy = 0$.

ASSUME A SERIES SOLUTION $y = \sum_{n=0}^{\infty} a_n x^n$. HIERARCHY FOR THE COEFFICIENTS:

$$\left.\begin{array}{lll} 2a_2 = 0 & \Rightarrow & a_2 = 0 \\ -a_0 + 6a_3 & \Rightarrow & a_3 = a_0/6 \\ -a_1 + 12a_4 & \Rightarrow & a_4 = a_1/12 \\ -a_2 + 20a_5 & \Rightarrow & a_5 = 0 \end{array}\right\}$$

SO IT GOES...

$a_2, a_5, a_8, a_{11}, \ldots = 0$

$a_3, a_6, a_9, a_{12}, \ldots \propto a_0$

$a_4, a_7, a_{10}, a_{13}, \ldots \propto a_1$

IN GENERAL,

$$2a_2 = 0$$

$$(n+2)(n+1)a_{n+2} - a_{n-1} = 0 \qquad n = 1, 2, 3, \ldots$$

THE TWO LINEARLY INDEPENDENT SOLUTIONS ARE,

$$y_1(x) = 1 + \frac{x^3}{6} + \frac{x^6}{180} + \frac{x^9}{12960} + \cdots = 1 + \sum_{n=1}^{\infty} \frac{x^{3n} \prod_{i=1}^{n}(3i+1)}{(3n+1)!}$$

$$y_2(x) = x + \frac{x^4}{12} + \frac{x^7}{504} + \frac{x^{10}}{45360} + \cdots = x + 2\sum_{n=1}^{\infty} \frac{x^{3n+1} \prod_{i=1}^{n}(3i+2)}{(3n+2)!}$$

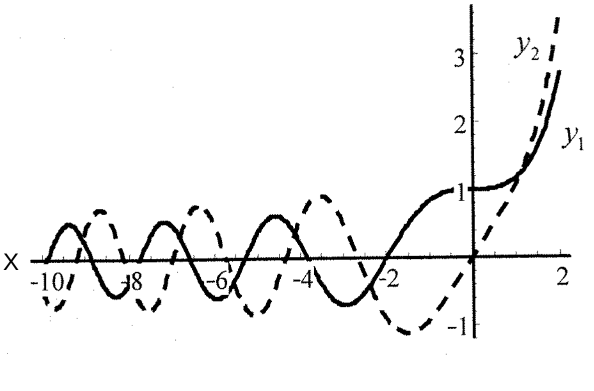
BUT WHAT DO THESE FUNCTIONS LOOK LIKE? THE EQUATION IS ALREADY IN NORMAL FORM, WITH $q(x) = -x$. OBVIOUS THE SIGN OF $q(x)$ WILL CHANGE AT $x=0$ [THIS IS CALLED A 'TURNING POINT'.] WHAT CAN WE SAY ABOUT THE SOLUTIONS?

i) FOR $x > 0$, $q(x) < 0$ AND y_1 & y_2 WILL HAVE AT MOST ONE ZERO.

ii) FOR $x < 0$, $q(x) = -x > 0$ AND $\int_0^{-\infty} (-x)\,dx \to \infty$

SO y_1 & y_2 WILL HAVE INFINITELY MANY ZEROS.

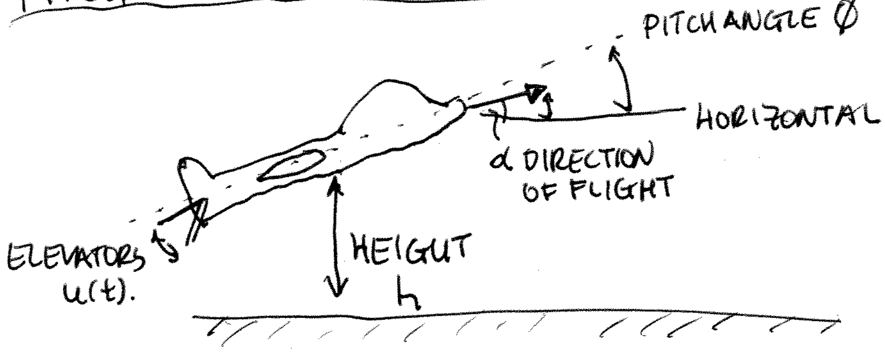
MOREOVER, BY THE STURM SEPARATION THEOREM, THE ZEROS ARE DISTINCT & ALTERNATING.



SYSTEMS OF (FIRST-ORDER) LINEAR DIFFERENTIAL EQUATIONS

SO FAR WE'VE DISCUSSED 2nd ORDER DIFFERENTIAL EQUATIONS IN A SINGLE UNKNOWN $y(x)$. IN ~~A~~ MORE GENERAL SETTING, DIFF. EQS. ~~DES~~ CHARACTERIZE MULTIPLE INTERACTING COMPONENTS (VOLTAGES, POPULATIONS, MASSES, CHEMICALS, ...) DES MODELED BY COUPLED SYSTEMS OF EQUATIONS.

EX. PITCH OF A FIXED-WING AIRCRAFT.



FOR SMALL ANGLES,

$$\dot{h} = c \cdot \alpha(t)$$
$$\ddot{\phi} = -\omega^2 [\phi(t) - \alpha(t) - b \cdot u(t)]$$
$$\dot{\alpha} = a[\phi(t) - \alpha(t)]$$

$a, b, \& \omega$ ARE CONSTANT PARAMETERS

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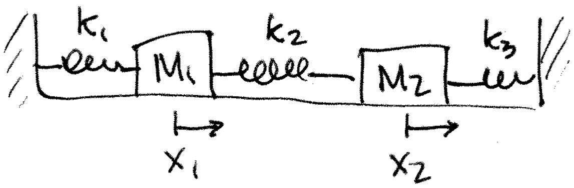
EX. COUPLED MASS-SPRING SYSTEM



$$m\ddot{x}_1 = -k_1 x_1 + k_2 \underbrace{(x_2 - x_1)}_{\text{COMPRESSION ON SPRING 2}}$$
$$m\ddot{x}_2 = k_2(x_1 - x_2) - k_3 x_2$$

START, FOR EXAMPLE, WITH $x_1(0)=0$, $x_1'(0)=0$ AND $x_2(0) = x_2^0 \neq 0$, $x_2'(0)=0$.

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THESE EXAMPLES LOOK COMPLICATED BUT ANY SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS CAN BE WRITTEN AS A SYSTEM OF FIRST-ORDER DIFFERENTIAL EQUATIONS

EX.

$$\frac{d^n y}{dx^n} = F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^{n-1}y}{dx^{n-1}}\right) \qquad (*)$$

DEFINE:

$$y_1 = y(x)$$
$$y_2 = dy/dx$$
$$\vdots$$
$$y_n = d^{n-1}y/dx^{n-1}$$
$$\frac{dy_n}{dx} = d^n y/dx^n$$

THEN (*) CAN BE WRITTEN:

$$\frac{dy_1}{dx} = y_2$$
$$\frac{dy_2}{dx} = y_3$$
$$\vdots$$
$$\frac{dy_n}{dx} = F(x, y_1, y_2, \ldots, y_n)$$

FOR LINEAR ORDINARY DIFFERENTIAL EQUATIONS, WE CAN WRITE THE SYSTEM EVEN MORE COMPACTLY USING VECTOR-MATRIX NOTATION.

EX: OUR 2^{nd}-ORDER EQUATION $y''+P(x)y'+Q(x)y=R(x)$, UPON DEFINING $y_1=y$, $y_2=y'$,

$$y' \to \frac{dy_1}{dx}=y_2$$

$$y'' \to \frac{dy_2}{dx}=-P(x)y_2-Q(x)y_1+R(x)$$

OR, AS MATRICES:

$$\frac{d}{dx}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ -Q(x) & -P(x) \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}+\begin{bmatrix} 0 \\ R(x) \end{bmatrix}$$

$$\frac{d}{dx}\vec{y}=\underline{\underline{A}}\cdot\vec{y}+\vec{b}$$

$\vec{y}$ & $\vec{b}$ ARE 1x2 COLUMN VECTORS
$\underline{\underline{A}}$ IS A 2x2 MATRIX.

EX: MASS-SPRING SYSTEM CALL $X_1=x_1$, $X_2=x_2$, $\frac{dx_1}{dt}=X_3$ AND $\frac{dx_2}{dt}=X_4$

THEN,

$$\frac{dX_1}{dt}=X_3$$

$$\frac{dX_2}{dt}=X_4$$

$$\frac{d^2x_1}{dt^2}=\frac{dX_3}{dt}=\frac{1}{m}\left[-(k_1+k_2)X_1+k_2X_2\right]$$

$$\frac{d^2x_2}{dt^2}=\frac{dX_4}{dt}=\frac{1}{m}\left[-(k_2+k_3)X_2+k_2X_1\right]$$

OR,

$$\frac{d}{dt}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{(k_1+k_2)}{m} & \frac{k_2}{m} & 0 & 0 \\ \frac{k_2}{m} & -\frac{(k_2+k_3)}{m} & 0 & 0 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$$

TRY: WRITE THE AIRCRAFT SYSTEM AS VECTOR-MATRIX FIRST-ORDER SYSTEM.

THE GENERAL FORM OF A LINEAR SYSTEM IS:

$$\frac{d}{dx}\vec{y}=\underline{\underline{A}}(x)\vec{y}+\vec{b}(x).$$

- IF $\vec{b}(x)=\vec{0}$, THE SYSTEM IS HOMOGENEOUS
- IF $\underline{\underline{A}}(x)=\underline{\underline{A}}$ [CONSTANT], THE SYSTEM IS 'AUTONOMOUS' OR HAS CONSTANT COEFFICIENTS.

USING OUR 2×2 EXAMPLE FOR $y'' + P(x)y' + Q(x)y = R(x)$, AS A GUIDE; EXISTENCE & UNIQUENESS CRITERIA CARRY OVER TO THE FIRST-ORDER SYSTEMS: IF $\underline{\underline{A}}(\cdot)$ AND $\vec{b}(\cdot)$ ARE CONTINUOUS, THEN EVERY INITIAL-VALUE PROBLEM HAS A UNIQUE SOLUTION.

HOMOGENEOUS SOLUTIONS

$$\frac{d\vec{y}}{dx} = \underline{\underline{A}}(x)\vec{y} \quad ; \quad \vec{y}(\cancel{x}) = \vec{y}_0$$

MANY OF THE IDEAS FROM THE FIRST COUPLE OF WEEKS OF LECTURES CARRY OVER DIRECTLY TO FIRST-ORDER SYSTEMS.

$\boxed{1}$ BECAUSE d/dx AND MATRIX-MULTIPLICATION ARE LINEAR OPERATIONS, WE KNOW THAT IF ~~$\vec{x}_1(x)$ AND $\vec{x}_2$~~ $\vec{y}_1(x)$ AND $\vec{y}_2(x)$ ARE SOLUTIONS OF THE HOMOGENEOUS SYSTEM, THEN

$$\vec{y}(x) = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) \text{ IS ALSO A SOLUTION.}$$

$\boxed{2}$ FOR A HOMOGENEOUS SYSTEM CHARACTERIZED BY AN $n \times n$ MATRIX $\underline{\underline{A}}(x)$, ~~WE CAN HAVE~~ THERE ARE n LINEARLY-INDEPENDENT SOLUTIONS $\{\vec{y}_1(x), \vec{y}_2(x), \ldots, \vec{y}_n(x)\}$ AND THE GENERAL SOLUTION IS WRITTEN:

$$\vec{y}_H(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + \cdots + c_n\vec{y}_n = \overbrace{\begin{bmatrix} | & | & & | \\ \vec{y}_1 & \vec{y}_2 & \cdots & \vec{y}_n \\ | & | & & | \end{bmatrix}}^{n\times n} \cdot \overbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}}^{n\times 1}$$

WE REQUIRE n LINEARLY-INDEPENDENT SOLUTIONS IN ORDER TO SATISFY AN ARBITRARY INITIAL CONDITION $\vec{y}_0^{\,0}$ $n\times 1$ VECTOR,

ie.

$$\begin{bmatrix} | & | & & | \\ \vec{y}_1(0) & \vec{y}_2(0) & \cdots & \vec{y}_n(0) \\ | & | & & | \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \underbrace{\begin{bmatrix} [\vec{y}_0^{\,0}]_1 \\ [\vec{y}_0^{\,0}]_2 \\ \vdots \\ [\vec{y}_0^{\,0}]_n \end{bmatrix}}_{\vec{y}_0}$$

IN ORDER FOR THE MATRIX TO BE INVERTIBLE, WE MUST HAVE:

$$\det\left[\vec{y}_1, \vec{y}_2\ \vec{y}_3, \ldots, \vec{y}_n\right] \neq 0$$; THIS DETERMINANT

IS THE 'n'-DIMENSIONAL 'WRONSKIAN.

EXERCISE: DOES THIS DEFINITION CORRESPOND TO OUR PREVIOUS DEFINITION FOR $y'' + P(x)y' + Q(x)y = 0$? HOW ARE $\vec{y}_1$ & $\vec{y}_2$ DEFINED?

THE FUNDAMENTAL MATRIX

THE HOMOGENEOUS & INHOMOGENEOUS SOLUTIONS TO A SYSTEM OF FIRST-ORDER DIFFERENTIAL EQUATIONS IS CONVENIENTLY EXPRESSED IN TERMS OF THE LINEARLY-INDEPENDENT SOLUTIONS THAT SOLVE THE HOMOGENEOUS INITIAL VALUE PROBLEM SUCH THAT

$$\vec{Y}_i(x_0) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$$ ONE IN THE i^{th} ROW.

THINK OF THE HOMOGENEOUS SOLUTIONS $\vec{y}_n$ AS ORTHOGONAL BASIS FUNCTIONS; $\vec{Y}_n$ PLAYS THE ROLE OF AN ORTHO-NORMAL BASIS FUNCTION.

WITH THIS CANONICAL SET $\vec{Y}_n(x)$ WE CAN CONSTRUCT:

$$\underline{\underline{\Phi}}(x, x_0) = \begin{bmatrix} | & | & & | \\ \vec{Y}_1(x) & \vec{Y}_2(x) & \cdots & \vec{Y}_n(x) \\ | & | & & | \end{bmatrix}$$

WHICH IS CALLED THE FUNDAMENTAL MATRIX (OR PROPAGATOR) FOR THE SYSTEM.

SOME PROPERTIES OF THE FUNDAMENTAL MATRIX:

1. BY THE DEFINITION OF $\vec{Y}_n(x)$:

$$\underline{\underline{\Phi}}(x_0, x_0) = \begin{bmatrix} | & | & & | \\ \vec{Y}_1(x_0) & \vec{Y}_2(x_0) & \cdots & \vec{Y}_n(x_0) \\ | & | & & | \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \equiv \underbrace{\mathbb{I}_n}_{n \times n \text{ IDENTITY MATRIX}}$$