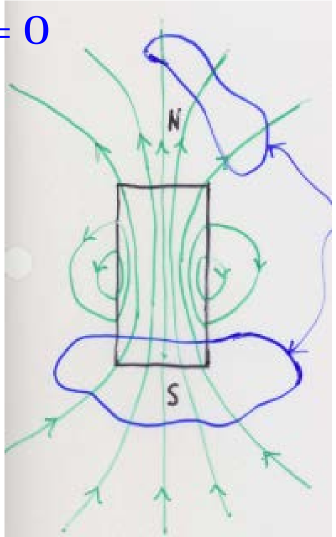
TABLE 9.1 Generalized Forms of Maxwell's Equations

Differential Form	Integral Form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v \, dv$	Gauss's law *–how charges produce electric fields* *– field lines begin / end on q*
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	Nonexistence of isolated magnetic charge* *–magnetic field lines have neither end nor beginning*
$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$\oint_L \mathbf{E} \cdot d\mathbf{l} = -\dfrac{\partial}{\partial t}\int_S \mathbf{B} \cdot d\mathbf{S}$	Faraday's law *–changing ϕ_B produces $\vec{E}$*
$\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$	$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{S}$	Ampère's circuit law *–electric current and changing ϕ_E produces $\vec{B}$*

*This is also referred to as Gauss's law for magnetic fields.

displacement current

Re $\vec{\nabla} \bullet \vec{B} = 0$



closed surfaces

– no magnetic monopoles : only dipoles

– since magnetic field lines are continuous, # of lines entering a closed surface = # lines leaving

or

$$\phi_B = \oint_{CS} \vec{B} \bullet d\vec{A} = 0$$

3 Alternating Current (AC) circuits

$- \textit{will consider sinusoidal voltages (e.g. emf, IR drops):}$

$$v(t) = V_{\max} \sin(\omega t + \phi)$$

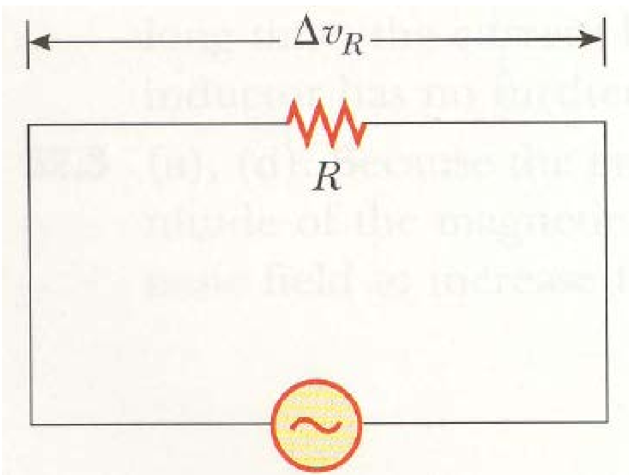
phase angle

$$\omega = 2\pi f = \frac{2\pi}{T}$$

instantaneous voltage

peak value or amplitude, also V_0

1) *R in AC circuits*



$$\varepsilon(t) = v(t) = V_m \sin(\omega t)$$

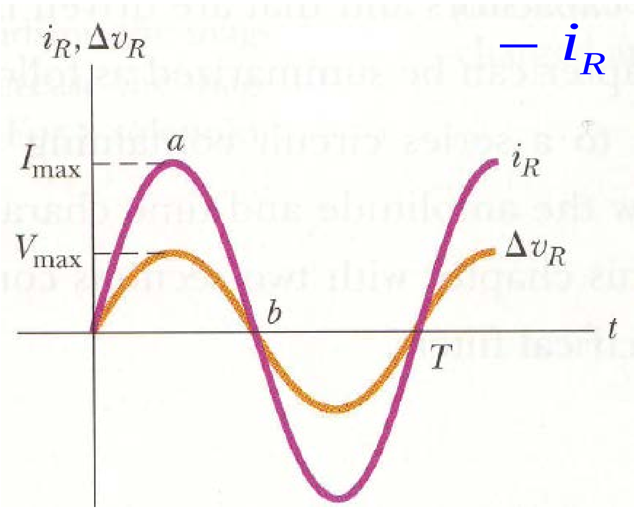
Kirchhoff's Voltage rule:

$$v + v_R = 0\ ;\quad \text{here } v - i_R R = 0$$

$$\therefore\ i_R(t) = \frac{v}{R} = \frac{V_m}{R}\sin\omega t = I_m \sin\omega t$$

$$\text{and } v_R(t) = i_R(t) R = I_m R \sin\omega t$$

Note re notation: the "Δ" from Δv, Δi has been dropped from all written text. Small letters ⇒ time dependent quantities

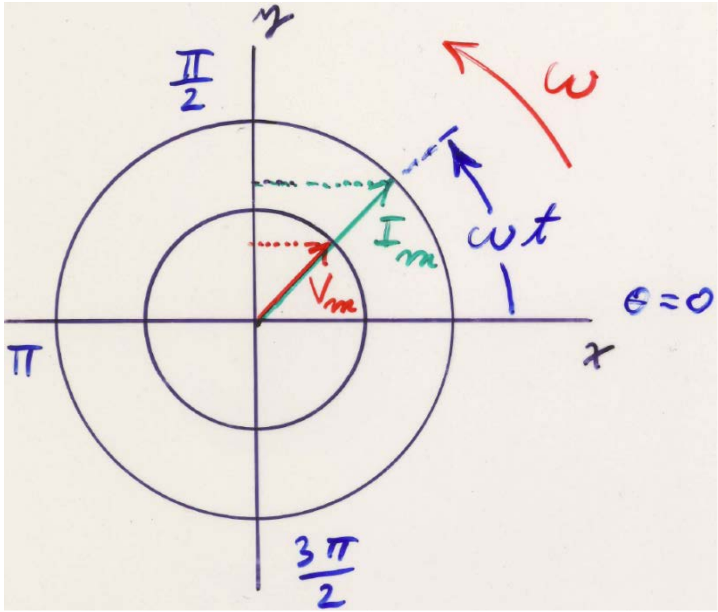


– i_R and v_R have identical behaviour with time

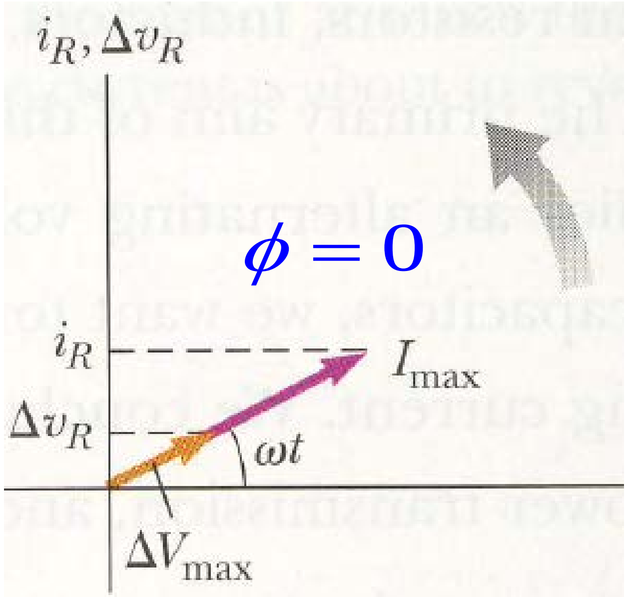
– they are in phase

– here this is true for the whole circuit emf $v(t)$ and circuit current $i(t)$ as well

• or as rotating vectors $\vec{V}_m$ and $\vec{I}_m$:



or simply :
PHASOR
diagram
(of phasors)



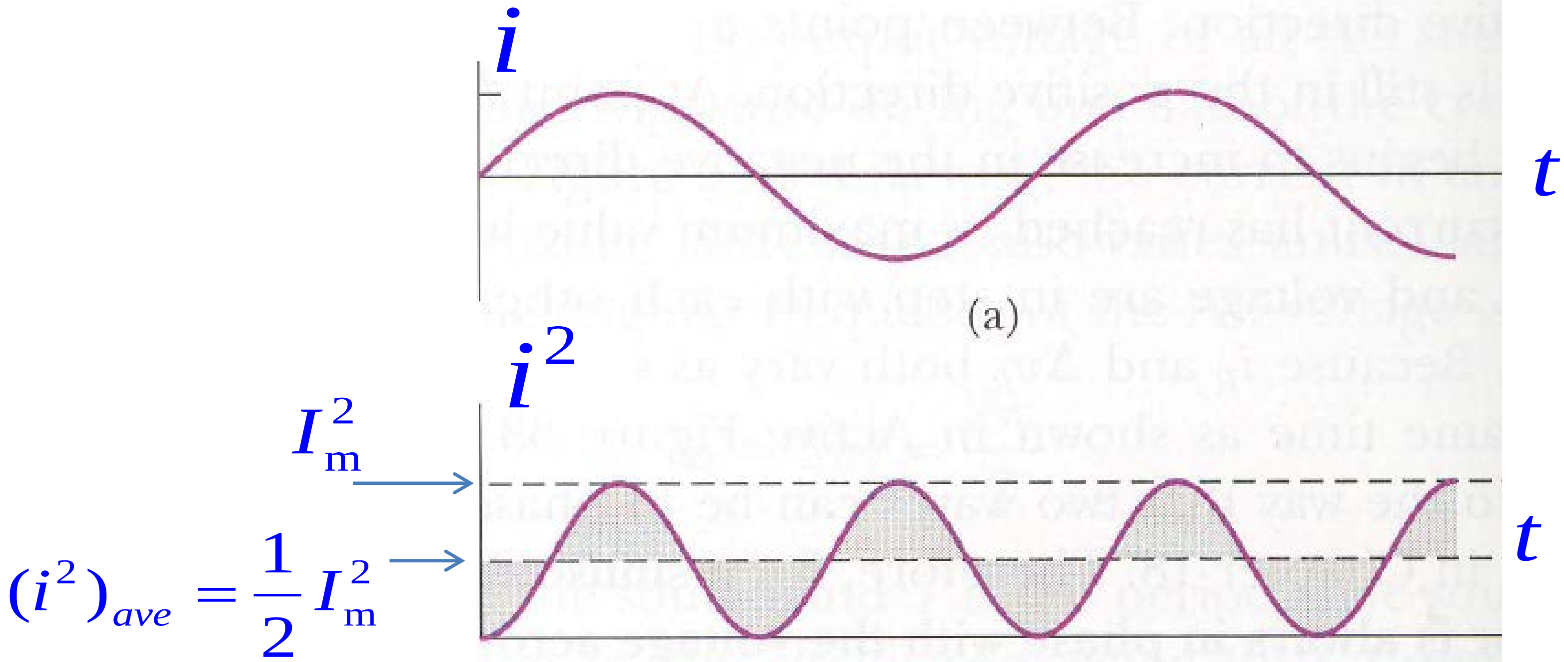
– projections of $\vec{V}_m$ and $\vec{I}_m$ onto y-axis give v_R and i_R

Power dissipated in AC circuits

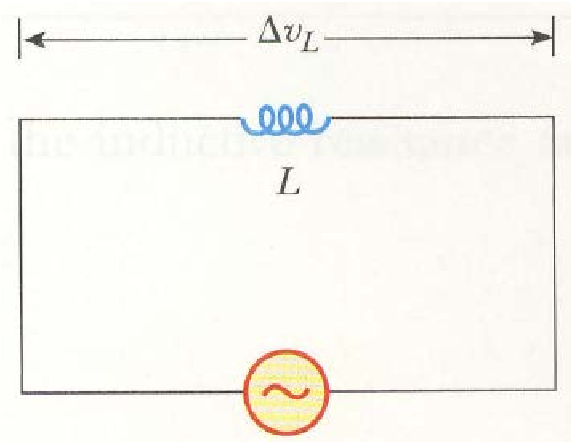
and root-mean-square values:

$$P_R = i^2(t)R \qquad P_R(ave) = \frac{1}{2} I_{\max}^2 R = I_{rms}^2 R$$

$$V_{rms} = \frac{V_m}{\sqrt{2}}, \; I_{rms} = \frac{I_m}{\sqrt{2}}$$


i
t
(a)
i^2
I_m^2
$(i^2)_{ave} = \frac{1}{2} I_m^2$
t

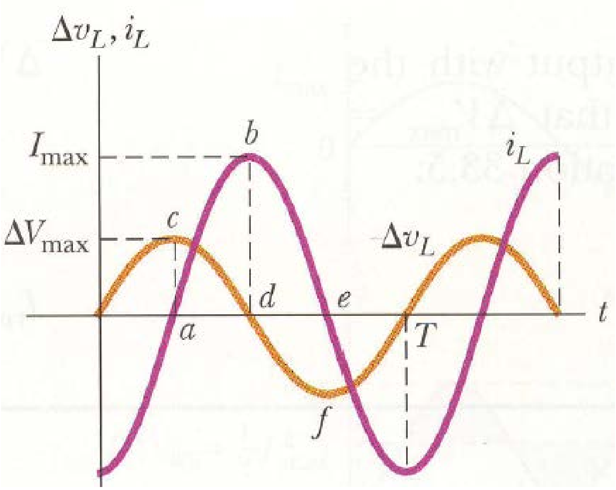
2) *L in AC circuits*



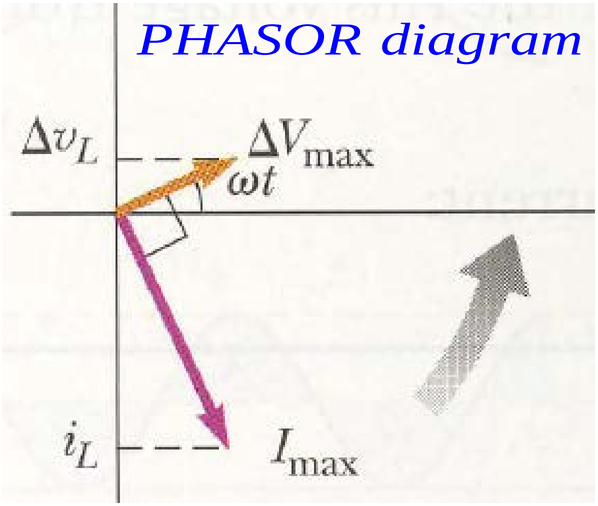
$$v(t) = V_m \sin(\omega t)$$

Can show:

$$i_L(t) = i(t) = -I_m \cos \omega t = I_m \sin\left(\omega t - \frac{\pi}{2}\right) \text{ with } I_m = \frac{V_m}{\omega L}$$



PHASOR diagram



$$Can\ write:\ i(t)=\frac{V_m}{\omega L}\sin\ (\omega t-\frac{\pi}{2})\ \ \text{as}\ \ i(t)=\frac{V_m\sin\ (\omega t-\frac{\pi}{2})}{X_L}$$

$$where\ X_L=\omega L\ \equiv\ inductive\ reactance$$

$$-\ a\ time\ independent\ "ac\ resistance"\ of\ the\ inductor$$

$$Then\ v_L(t)=I_m X_L\sin\omega t\quad (i.e.\ Ohm's\ Law;\ V_m=I_m X_L)$$

$$Note:\ when\ f\,(=\frac{\omega}{2\pi})\rightarrow 0;\ \ X_L\rightarrow 0$$

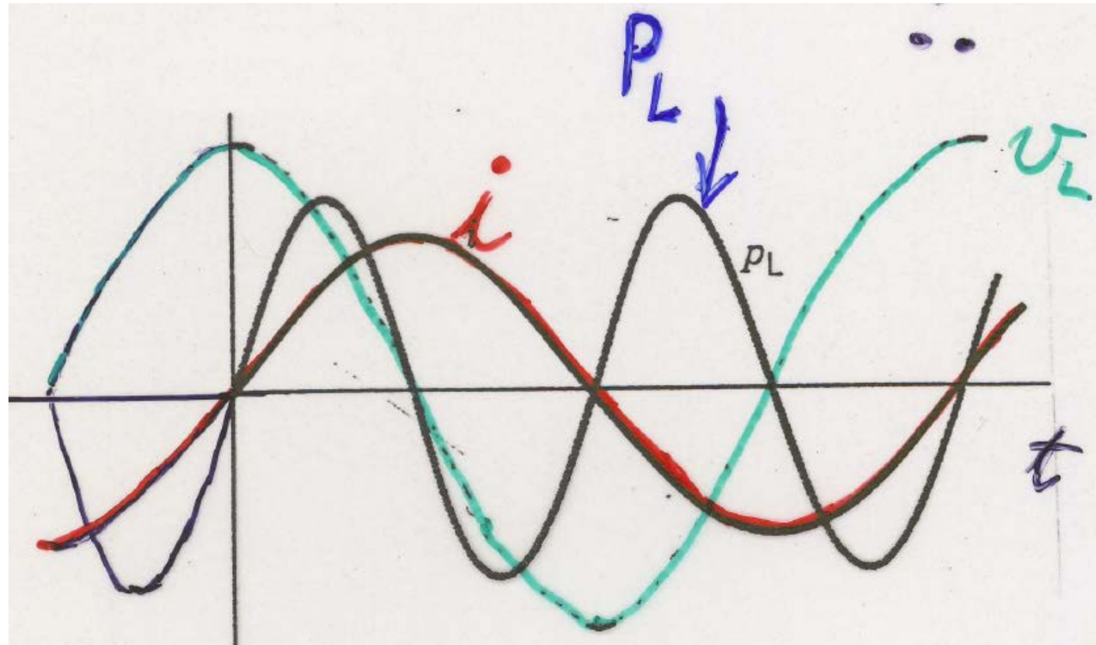
$$f\rightarrow\infty;\ \ X_L\rightarrow\infty\ \ (infinite\ \frac{di}{dt})$$

Instantaneous power supplied to inductor:

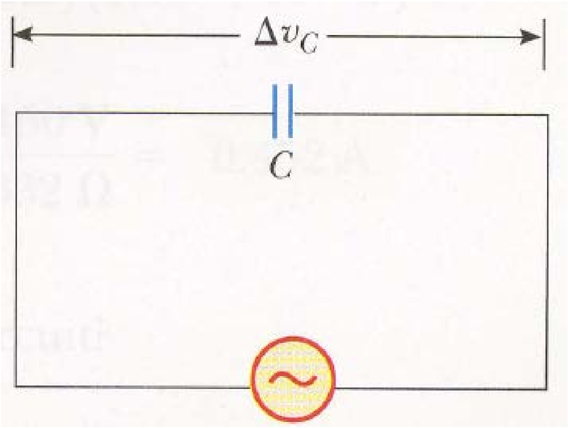
–can show:

$$P_L(t) = -I_m^2 X_L \frac{\sin 2\omega t}{2}$$

$$P_L(ave) = 0$$



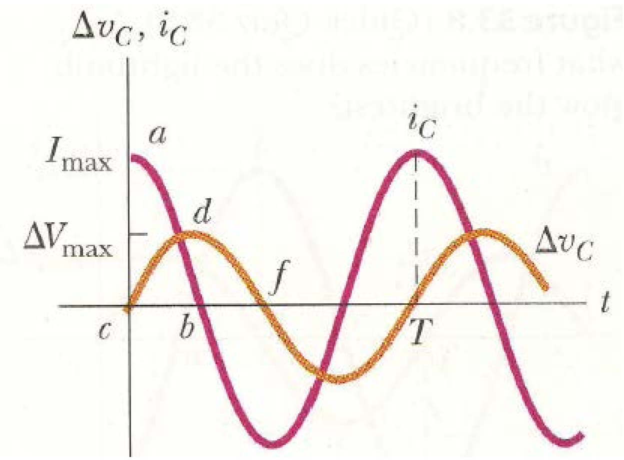
3) C in AC circuits



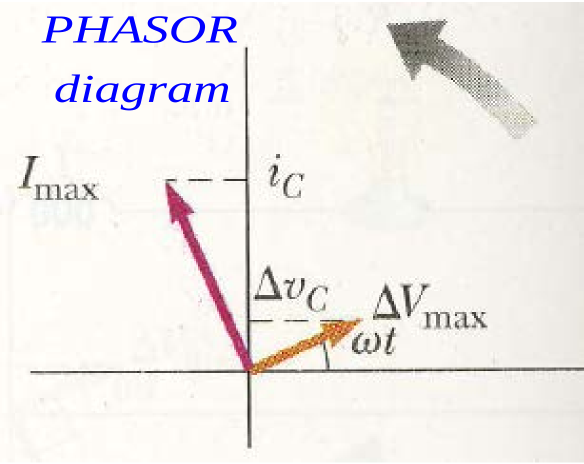
$$v(t) = V_m \sin(\omega t)$$

From Kirchhoff's voltage or loop rule get:

$$i_C(t) = i(t) = I_m \cos\omega t = I_m \sin\left(\omega t + \frac{\pi}{2}\right) \text{ with } I_m = \omega C V_m$$



PHASOR diagram



$Can\ write:\ i(t) = \omega C V_m \sin\ (\omega t + \frac{\pi}{2})$ as $i(t) = \frac{V_m \sin\ (\omega t + \frac{\pi}{2})}{X_C}$

$where\ X_C = \frac{1}{\omega C} \equiv\ capacitive\ reactance$

$-\ a\ time\ independent$ "$ac\ resistance$" $of\ the\ capacitor$

$Then\ v_C(t) = I_m X_C \sin \omega t \quad (i.e.\ Ohm's\ Law)$

$Note:\ when\ f\,(= \frac{\omega}{2\pi}) \to 0;\ \ X_C \to \infty$

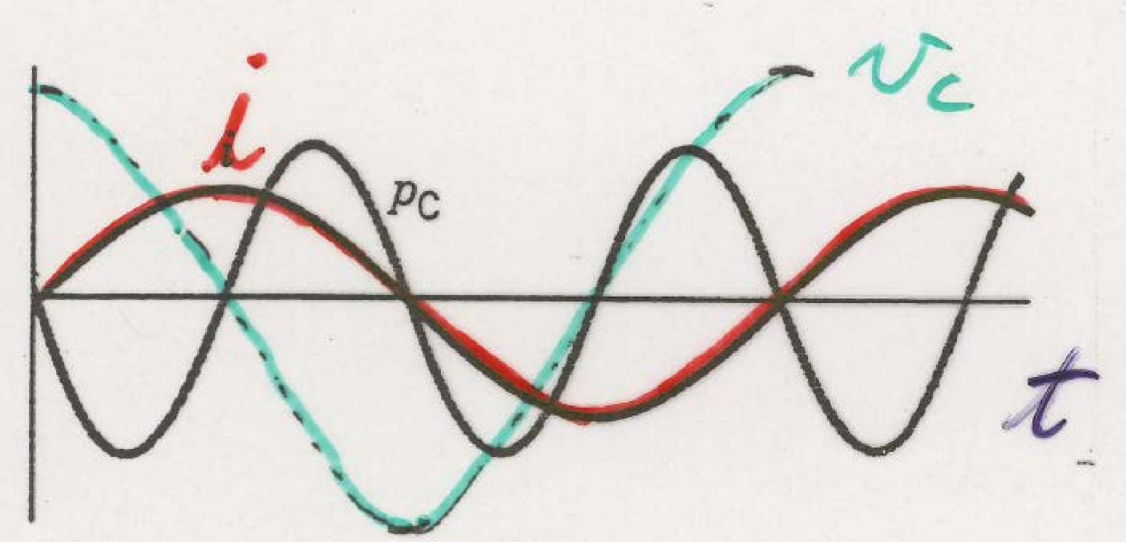
$f \to \infty;\ \ X_L \to 0 \quad (DC\ open)$

Instantaneous power supplied to capacitor:

–can show:

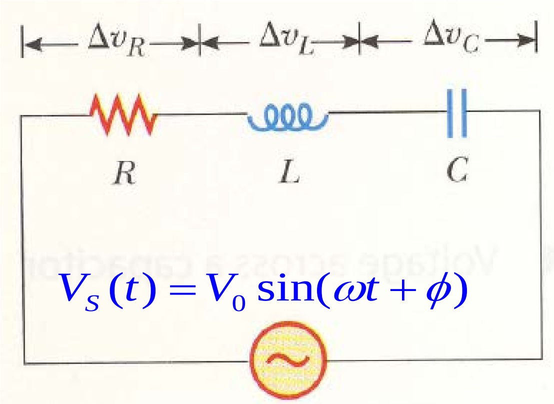
$$P_C(t) = I_m^2 X_C \frac{\sin 2\omega t}{2}$$

$$P_C(ave) = 0$$



4) *More complicated AC circuits*

e.g. series RLC circuit



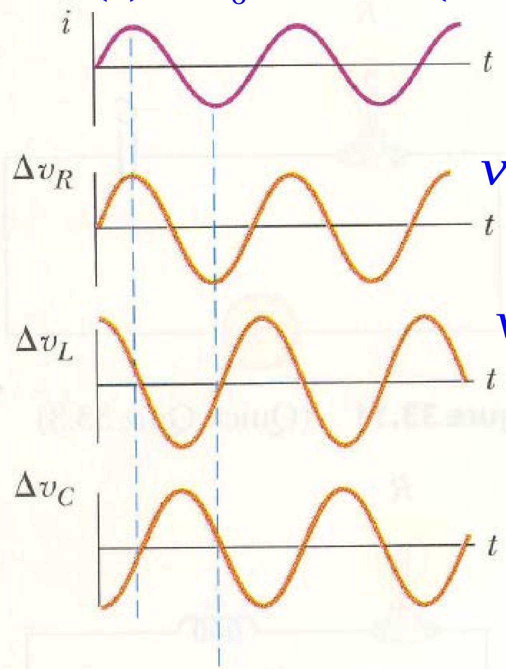
Letting $Q = V_C C$, *K's Loop Rule gives*

$$V_S(t) = \frac{Q}{C} + R\frac{dQ}{dt} + \frac{d^2Q}{dt^2}$$

–can solve this 2^{nd} *order DE given*

that $Q = 0,\ \frac{dQ}{dt} = 0$ *at* $t = 0$

Let $i(t) = I_0 \sin \omega t$ ("*reference* ϕ")



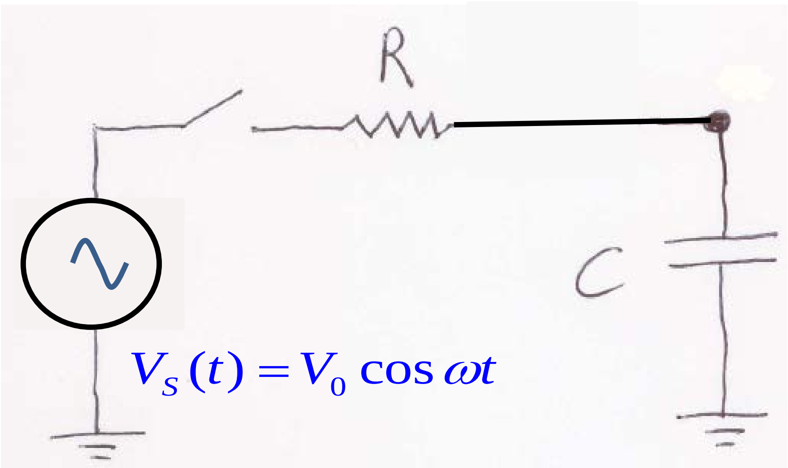
$$v_R = I_0 R \sin \omega t = V_R \sin \omega t$$

$$v_L = I_0 X_L \sin(\omega t + \frac{\pi}{2}) = V_L \cos \omega t$$

$$v_C = I_0 X_C \sin(\omega t - \frac{\pi}{2}) = -V_C \cos \omega t$$

- *Real circuits often are much more complicated and analytical solutions to the resulting DEs may not be straight forward*
- *use phasor approach*

$- \textit{to make this more evident, consider just RC in an AC circuit}:$



Letting $Q = V_C C$, *K's loop rule gives*

$$V_0 \cos \omega t = R\frac{dQ}{dt} + \frac{Q}{C}$$

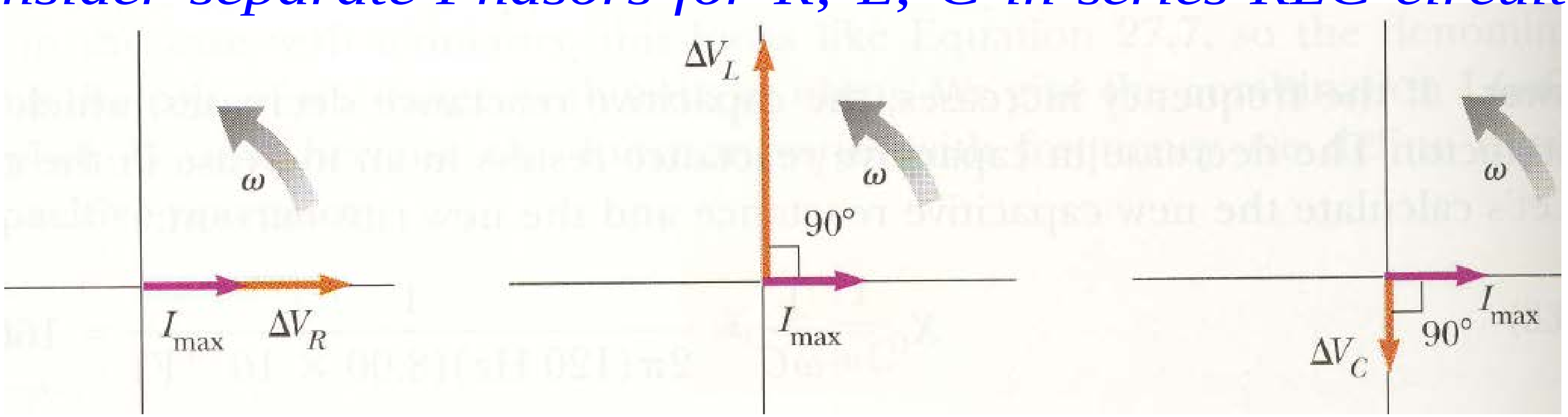
Letting $V_C = V_C^0$ *at* $t = 0$ *one can get (general form):*

$$Q = f_1(V_0, R, C, \omega)\cos \omega t + f_2(V_0, R, C, \omega, V_C^0)\sin \omega t + f_3(V_0, R, C, \omega, V_C^0)e^{\frac{-t}{RC}}$$

- *many practical RLC circuits* actually do *contain sinusoidal voltage and / or current sources.*

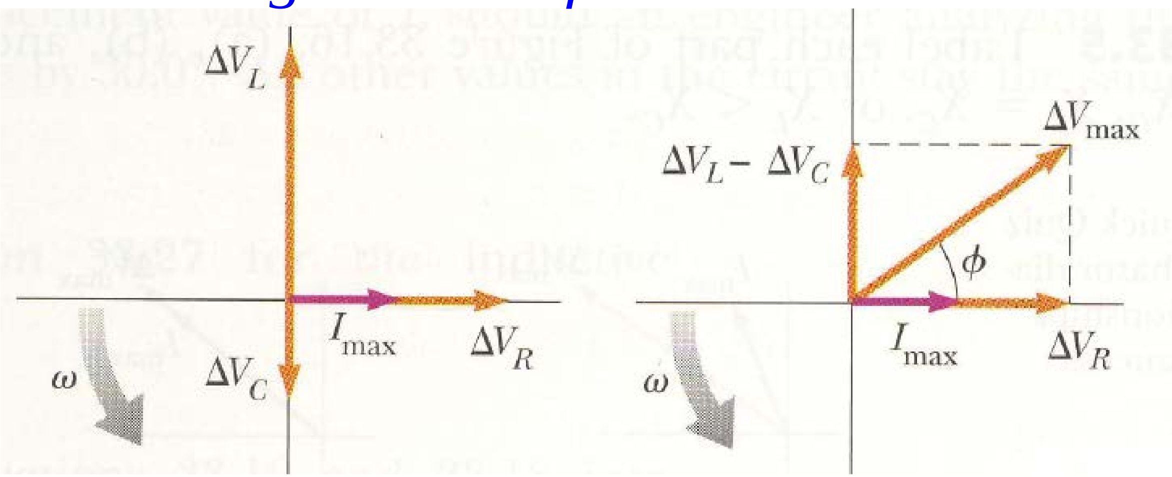
- *if, in addition, we can assume that we are looking for long-time behaviour only, the phasor approach is most effective*

Consider separate Phasors for R, L, C in series RLC circuit:



Resistor *Inductor* *Capacitor*

Combining Phasors for series RLC case:



$$V_{max} \rightarrow V_0;\ I_{max} \rightarrow I_0$$

$$V_0 = \sqrt{V_R{}^2 + (V_L - V_C)^2}$$

$$= \sqrt{(I_0 R)^2 + (I_0 X_L - I_0 X_C)^2}$$

$$V_0 = I_0 \sqrt{R^2 + (X_L - X_C)^2}$$

or $V_0 = I_0 Z$, *where* $Z \equiv$ *impedance*

$$\phi = \tan^{-1}\left(\frac{V_L - V_C}{V_R}\right) = \tan^{-1}\left(\frac{X_L - X_C}{V_R}\right)$$

$[X_L > X_C \Rightarrow$ *inductive; i lags v*$]$

$[X_L < X_C \Rightarrow$ *capacitive; i leads v*$]$

Example

A 1000 Ω *resistor is connected in series to a* 0.6 H *inductor and a* 2.5 μF *capacitor. This RLC combination is then connected across a voltage source that varies as*

$$v(t) = (80\ V)\sin\left(\frac{1000}{\pi}t\right)$$

a) Calculate Z *and show* X_L, X_C, R, Z *and* ϕ *in an impedance diagram.*

b) Calculate the peak current and write the expression for $i(t)$.

In the previous example calculate V_{rms} across the inductor, capacitor and resistor.

$$V_{rms} = I_{rms} X_L \;\; ; \;\; I_{rms} = \frac{I_0}{\sqrt{2}} = \frac{5.47\text{x}10^{-2}\,A}{\sqrt{2}} = 3.868\text{x}10^{-2}\,A$$

$$\therefore V_{rms,\,L} = (3.868\text{x}10^{-2}\,A)(191.0\;\Omega) = 7.39\;V$$

$$V_{rms,\,C} = I_{rms} X_C = (3.868\text{x}10^{-2}\,A)(1256.6\;\Omega) = 48.61\;V$$

$$V_{rms,\,\mathrm{R}} = I_{rms} R = (3.868\text{x}10^{-2}\,A)(1000\;\Omega) = 38.68\;V$$

Note $\sum V_{rms} = 94.68\;V$

but $V_{rms,\,applied} = \frac{V_0}{\sqrt{2}} = \frac{80\;V}{\sqrt{2}} = 56.57\;V$

In the previous example calculate V_{peak} across the inductor, capacitor and resistor.

$$V_L = I_0 X_L = (5.47\text{x}10^{-2}\,A)(191.0\ \Omega) = 10.45\ V$$

$$V_C = I_0 X_C = (5.47\text{x}10^{-2}\,A)(1256.6\ \Omega) = 68.74\ V$$

$$V_R = I_0 R = (5.47\text{x}10^{-2}\,A)(1000\ \Omega) = 54.7\ V$$

$\sum V_{peak} = 133.89\ V$ -- *also not equal to $v_{applied}$*

But for any t, $\sum v$'s must equal $v_{applied}$.