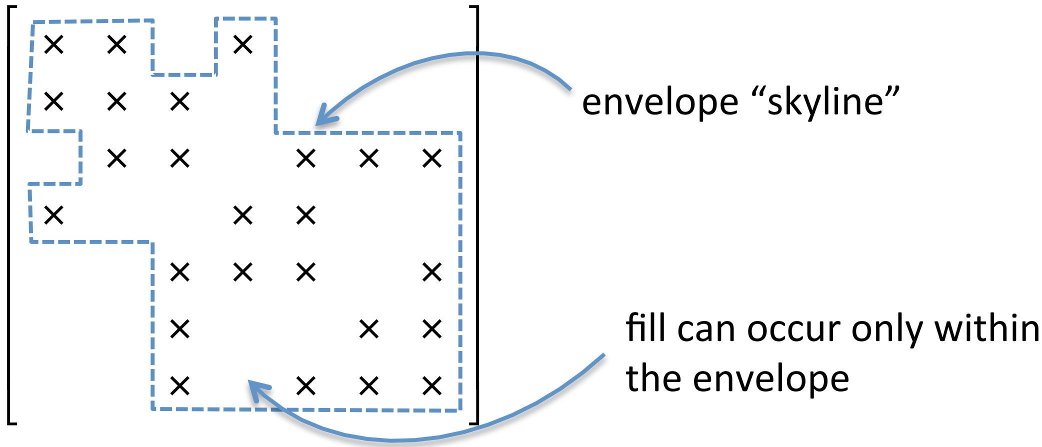
May 21, 2014

Envelope methods

- In general, bandwidth is not the same for each row

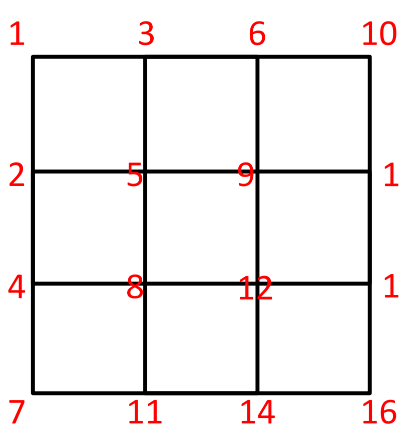


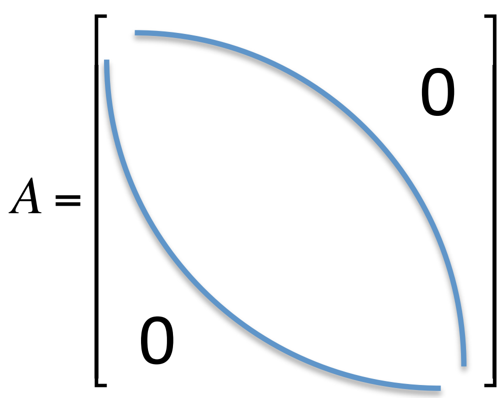
- In each row of L , fill can occur only between the 1st nonzero in a row and the diagonal.

- To limit the amount of fill, keep the envelope as close to the diagonal as possible.

- Try to number nodes so that graph neighbours have numbers as close together as possible.

e.g.

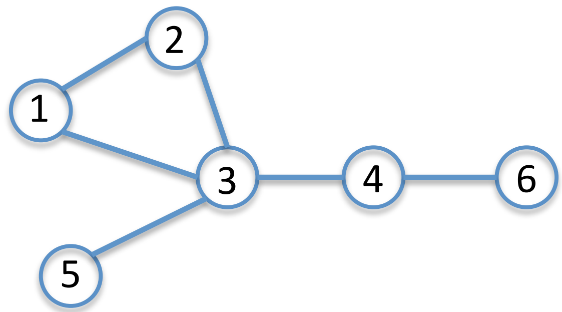




max bandwidth = 4, average = 2

Def: Degree of a node = number of nodes adjacent to a given node.

e.g.



deg(node 3) = 4

Envelope ordering strategies based on level set S_i :

S_1 -> consists of a single node, the starting node

S_2 -> all (graph) neighbours of the node in S_1

S_3 -> all neighbours of nodes in S_2 that are not in S_1, S_2.

In general, S_i consists of all neighbours of S_{i-1} that are not in S_1, S_2, ..., S_{i-2} .

Ordering: nodes in S_1, nodes in S_2, etc.

Cuthill-McKee ordering (1969)

1) Determine starting node

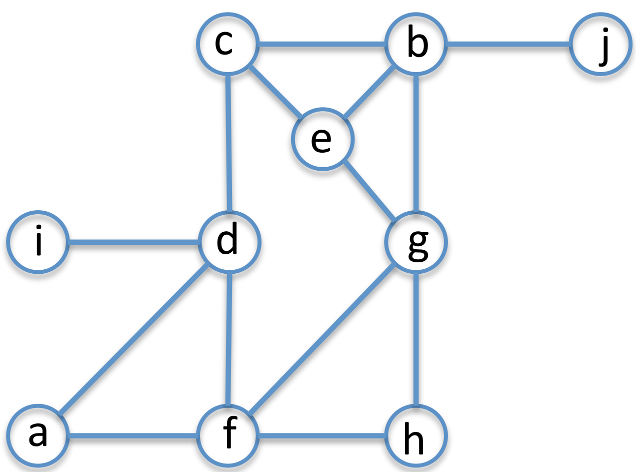
2) For i = 1, …, n, find all unnumbered neighbours of node i and number them in order of degree (smallest first)

   Surprisingly, the reverse ordering is better, so add 3)

3) Reverse Cuthill-McKee (RCM, 1971, George)

$$\text{node}_i^{RCM} = \text{node}_{n-i+1}^{CM} \qquad i = 1, 2, \ldots, n$$

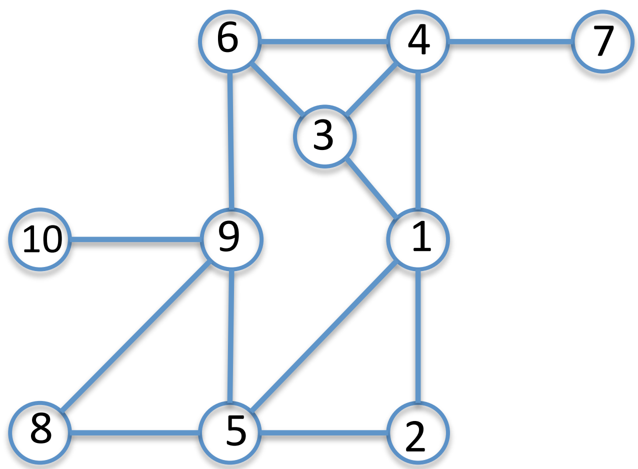
Example 1:



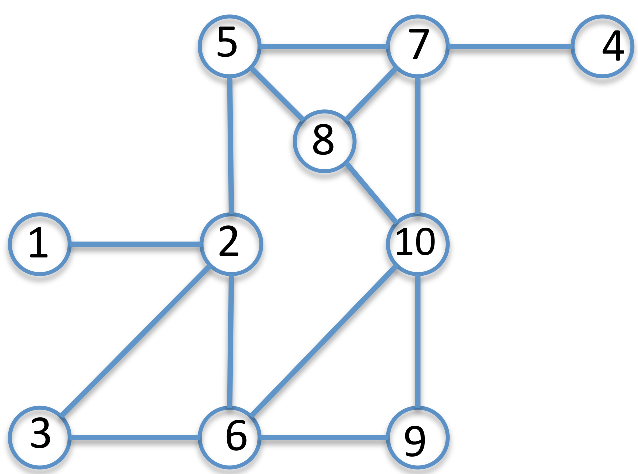
node #	node	unnumbered neighbours
1	g	h, e, b, f
		deg(h)=2, deg(e)=3, deg(b)=4, deg(f)=4
2	h	
3	e	c deg(c) = 3
4	b	j deg(j) = 1
5	f	a, d deg(a) = 2, deg(d) = 4
6	c	

node #	node	unnumbered neighbours
7	j	
8	a	
9	d	i deg(i) = 1
10	i	

CM ordering

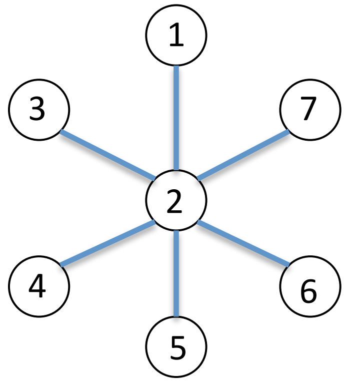


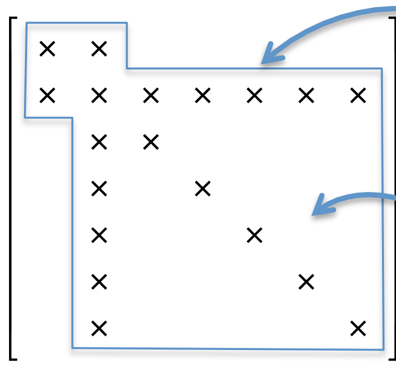
RCM ordering



Example 2

CM ordering

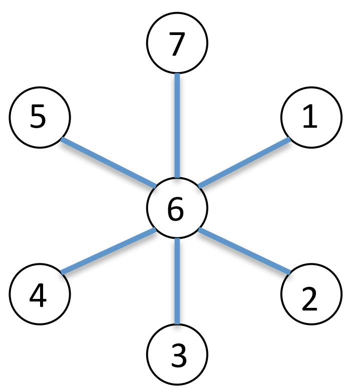


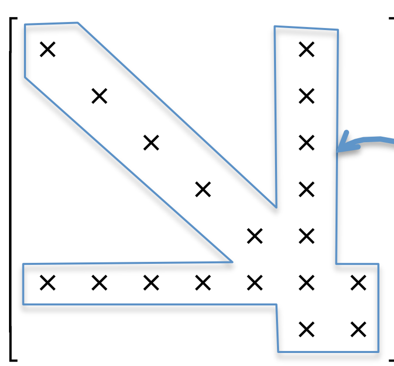


envelope

fills in completely

RCM ordering



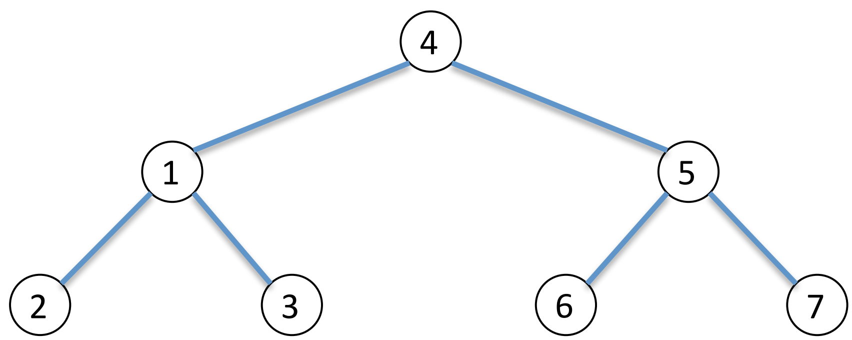


much smaller envelope
(no fill)

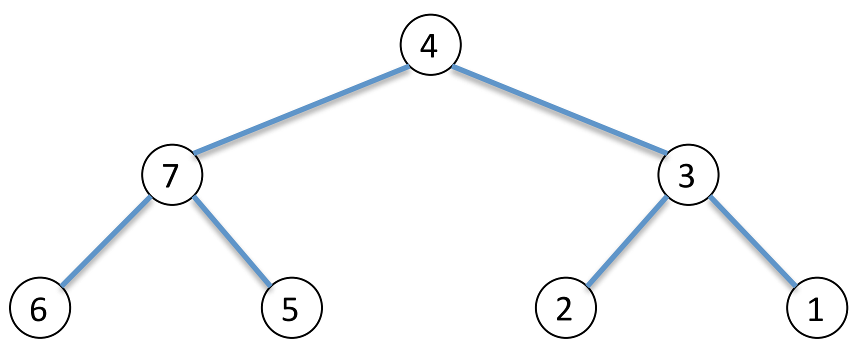
Interesting property of RCM:

If a graph is a tree, then no matter what node you start with, RCM ordering produced no fill (not true for CM)

Example 3



CM ordering



RCM ordering

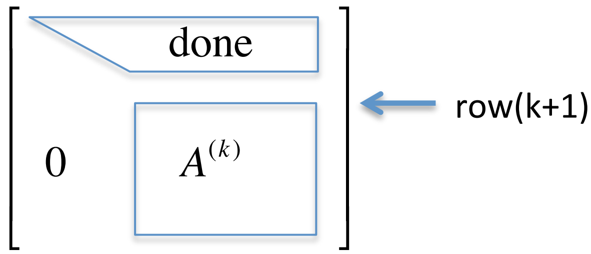
Notes

1) RCM does not necessary produce an optimal ordering (i.e. ordering which introduces least amount of fills)
2) In general, NP-complete problem to find optimal ordering.

<u>Local strategy</u> (Markowitz 1957)

- min fill-in only for the current step of GE

E.g. after k steps of GE:

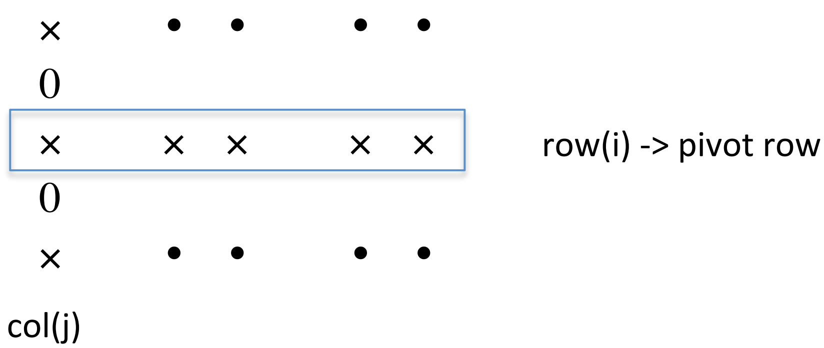


Let $r_i^{(k)}$ = number of entries in row i of $A^{(k)}$

$c_j^{(k)}$ = number of entries in col j of $A^{(k)}$

Then the max possible amount of fill = $(r_i^{(k)} - 1)(c_j^{(k)} - 1)$

E.g.



Markowitz strategy: select $a_{ij}^{(k)}$ that min:

$$(r_i^{(k)} - 1)(c_j^{(k)} - 1)$$

Note: different from $r_i^{(k)} c_j^{(k)}$, prefer $r_i = 1$ or $c_j = 1$

For symmetric structure, $\min r_i^{(k)} = \min c_j^{(k)}$.

Thus, we find node i, $k+1 \le i \le n$ such that

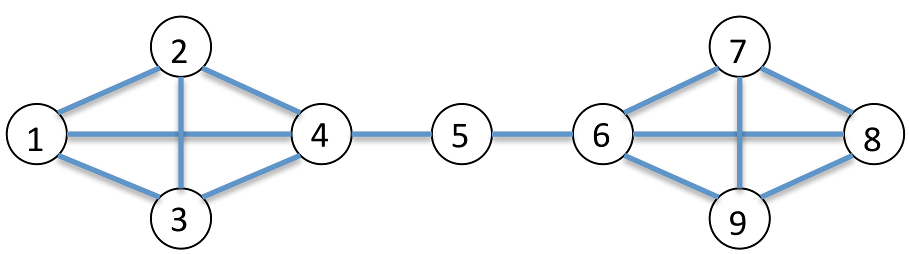
$$\min \; r_i^{(k)} - 1$$

Then we use $a_{ii}^{(k)}$ as the pivot.

Minimum degree ordering

- In the matrix graph, choose the node with least edges connected to it.
- A tree graph will be ordered correctly.
- It is a local strategy; no guarantee it will produce the least amount of total fill.

Counter example:



The given order produces no fill.

Minimum degree ordering will start with node 5:

$$\begin{array}{cccc} & 5 & 6 & 4 \\ 5 & \times & \times & \times \\ 6 & \times & \times & \bullet \\ 4 & \times & \bullet & \times \end{array}$$

fill in

Produces fill at the first step -> not optimal

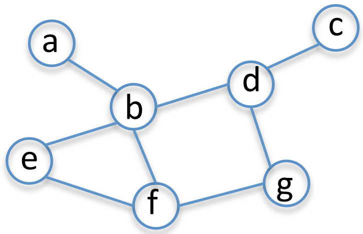
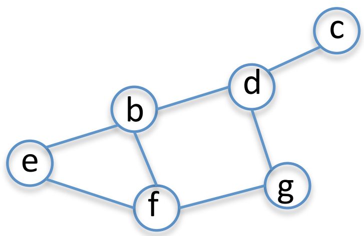
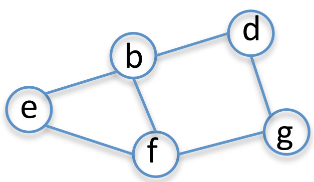
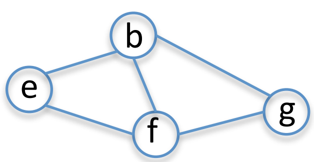
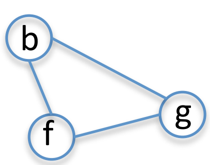
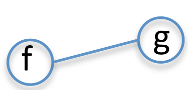
<u>Tie-breaking</u>

1) Select the node that had the smallest node number in the original order.

2) RCM preordering -> min. deg.

   Tie broken by selecting earlier RCM ordered node.

Example

k	Elimination graph $G(A^{(k-1)})$	node	min deg
1		a	1
2		c	1
3		d	2
4		e	2
5		b	2
6		f	1
7		g	0