L07 Stern Gerlach Explanation

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8:25 AM

Continuation of M2: Vector Spaces
M2.4: The Scalar Product

Definition Scalar Product:
combination of a dual vector and a vector, resulting in a complex number
Bilinear in the dual vector and the vector

$$\langle \phi | \psi \rangle \in \mathbb{C}$$

The value of the scalar product can be evaluated using the bi-linearity
and the action on elements of the orthonormal basis:

Evaluation of scalar product (Example, dimension 2)
The scalar product can be evaluation in a simple way

$$|\phi\rangle \doteq \begin{pmatrix} c \\ d \end{pmatrix} \qquad |\psi\rangle \doteq \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\updownarrow \qquad\qquad \downarrow$$

$$\langle \phi | \doteq (c^*, d^*) \longrightarrow (c^*, d^*)\begin{pmatrix} a \\ b \end{pmatrix} = ac^* + bd^*$$

$$= \langle \phi | \psi \rangle$$

Properties:
symmetry: $\langle \alpha | \beta \rangle = \langle \beta | \alpha \rangle^*$
positivity: $\langle \alpha | \alpha \rangle \geq 0$
and $\langle \alpha | \alpha \rangle = 0 \Leftrightarrow |\alpha\rangle = |0\rangle$
distributive: $\langle \alpha | \big(b|\beta\rangle + c\,|\gamma\rangle \big) = b\,\langle \alpha | \beta \rangle + c\,\langle \alpha | \gamma \rangle$

2.5 Uses of Scalar product

2.5.1 normalization of vectors

$$\| |\psi\rangle \| := \sqrt{\langle\psi|\psi\rangle}$$

2.5.2 The scalar product defines orthogonality relation:

$|\psi\rangle, |\varphi\rangle$ orthogonal

$$\Leftrightarrow \quad \langle\psi|\varphi\rangle = 0$$

==> can introduce orthonormal basis: $\{|\varphi_i\rangle\}_{i=1}^{d}$
set of vectors that span the whole vector space such that the

$$\langle\varphi_i|\varphi_i\rangle = 1 \qquad \forall\, i = 1, \dots, d$$

$$\langle\varphi_i|\varphi_j\rangle = 0 \quad \begin{cases} \forall\, i, j \in 1, \dots, d \\ i \neq j \end{cases}$$

$\Rightarrow$ Kronecker Delta $\quad \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

$$\Rightarrow \quad \langle\varphi_i|\varphi_j\rangle = \delta_{ij} \qquad i, j = 1, \dots, d$$

Notation for standard basis in dimension two:

$|+\rangle, |-\rangle$ orthonormal basis

2.5.3 Basis Expansion

Basis spans vector space, so each element of vector space can be expanded into basis states:

$$|\psi\rangle = \sum_{i=1}^{d} \alpha_i |\varphi_i\rangle$$

Calculation method of coefficients of basis expansion:

For orthonormal basis elements we find

$$\langle \varphi_j | \psi \rangle = \sum_{i=1}^{d} \alpha_i \langle \varphi_j | \varphi_i \rangle$$

$$= \sum_{i=1}^{d} \alpha_i \, \delta_{ij} = \alpha_j$$

$$\Rightarrow |\psi\rangle = \sum_{i=1}^{d} \underbrace{\langle \varphi_i | \psi \rangle}_{\in \mathbb{C}} |\varphi_i\rangle$$

Normalization condition (in terms of basis coefficients for *orthonormal basis*):

$$|\psi\rangle = \sum_{i=1}^{d} \alpha_i |\varphi_i\rangle$$

$$\Rightarrow \langle \psi | \psi \rangle = \left(\sum_{j=1}^{d} \alpha_j^* \langle \varphi_j | \right) \left(\sum_{i=1}^{d} \alpha_i |\varphi_i\rangle \right)$$

$$= \sum_{i=1}^{d} \sum_{j=1}^{d} \alpha_j^* \alpha_i \underbrace{\langle \varphi_j | \varphi_i \rangle}_{\delta_{ij}}$$

$$= \sum_{i=1}^{d} \sum_{j=1}^{d} \alpha_j^* \alpha_i \delta_{ij}$$

$$= \sum_{i=1}^{d} |\alpha_i|^2 \overset{!}{=} 1$$

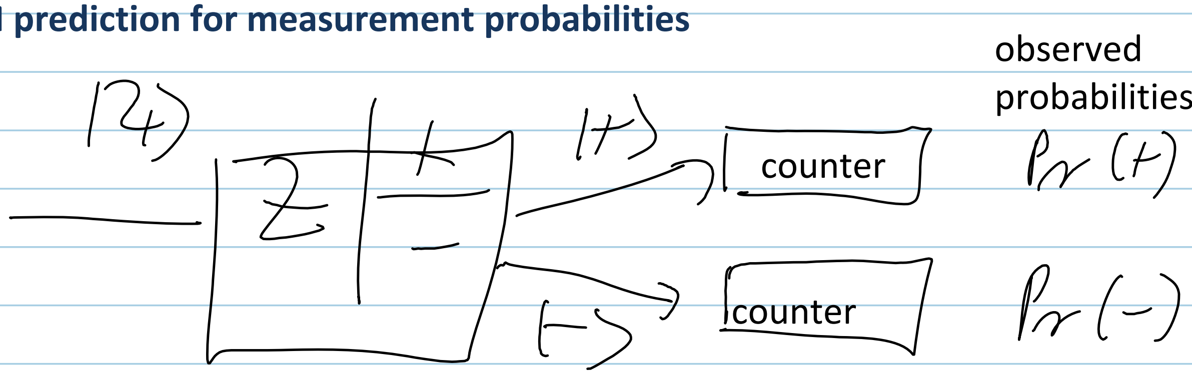
So we have for ~~an~~ a orthonormal basis for any normalized state $|\psi\rangle$

$$\boxed{\sum_{i=1} |\langle \varphi_i | \psi \rangle|^2 = 1}$$

Note: $|\langle \varphi_i | \psi \rangle|^2 = |\langle \psi | \varphi_i \rangle|^2$

3. Quantum Mechanical Postulates and Formalism

3.1 QM prediction for measurement probabilities



Motivation of rules: must have something to do with amplitudes & vectors!
==> expand normalized incoming state into the orthonormal basis states of the measurement

$$|\psi\rangle = \alpha |+\rangle + \beta |-\rangle$$

probability amplitudes

Light interference:

amplitudes for specific locations are squared to give the intensity:

==> the probability of an event is the modulus square of the amplitude

Quantum Physics

probability amplitude for event "+" α

probability for event "+" $|\alpha|^2$

Postulat 4

For an input state described by (normalized) ket

$$| \Psi \rangle$$

and a measurement with mutually exclusive events "i" described by elements of an orthonormal basis

$$\{ | \phi_i \rangle \}, i = 1, \dots , d$$

the probability Pr("i") to observe outcome "i" is given by

$$\Pr(\text{"i"}) = | \langle \phi_i | \Psi \rangle |^2$$

Notes:

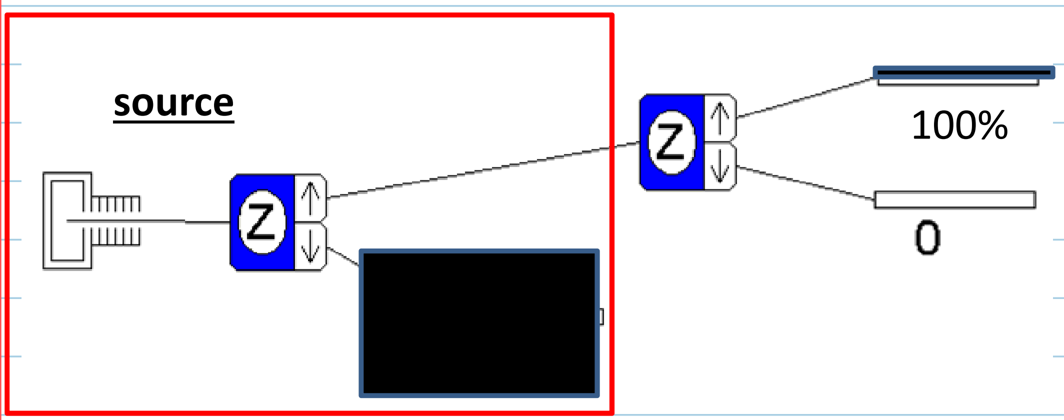
- as seen above in 2.2.2, we have automatically guaranteed that

a) $\sum_i \Pr(\text{"i"}) = 1$

b) $0 \leq \Pr(\text{"i"}) \leq 1$

3.2 Example: Application to Experiment 1

Experiment 1



Experiment 1:

input

$|+\rangle$

outcomes

$|+\rangle$ or $|-\rangle$

Pr("+") = $|\langle + | + \rangle|^2 = 1$

Pr("-") = $|\langle - | + \rangle|^2 = 0$

results of the prediction
can be directly argued from the
property that the states

form an orthonormal basis!

Alternative calculation using the coordinate representation

coordinate representation:

$$|+\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|-\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

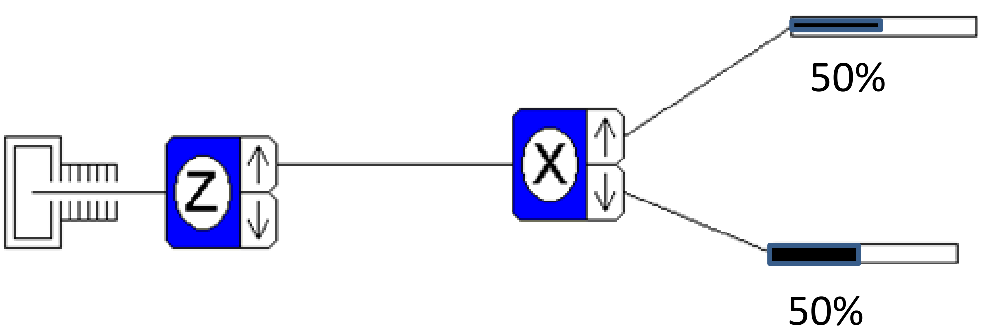
$$Pr(\text{"+"}) = \left| (1,0) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right|^2 = 1$$

$$Pr(\text{"-"}) = \left| (0,1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right|^2 = 0$$

3.3 Application to experiment 2

3.3.1 General Structure

Experiment 2



Experiment 2:

Prediction

input $|+\rangle = |+\rangle_z$

outcomes

$|+\rangle_x$ or $|-\rangle_x$

$\Pr(\text{“+”}) = |{}_x\langle + | + \rangle|^2 =$?

$\Pr(\text{“-”}) = |{}_x\langle - | + \rangle|^2 =$?

we cannot complete prediction if we don't know the connection between

$|+\rangle, |-\rangle$

and

$|+\rangle_x, |-\rangle_x$

3.3.2 Determine the Representation of the x-Basis

How to connect the basis in x-direction with the standard basis (z-direction)

3.3.2.1 Starting point: general ansatz

General Ansatz:

expansion in standard basis

$$
\begin{aligned}
|+\rangle_x &= a|+\rangle + b|-\rangle \\
|-\rangle_x &= c|+\rangle + d|-\rangle
\end{aligned}
$$

a,b,c,d $\in \mathcal{C}$

Any basis vector can be expanded in the standard basis (or any other basis)

3.3.2.2 Step 1: Include Observations

we can make use of the experimental observation that we find a 50/50 result in Experiment 2. The theoretical expression for this prediction are

measurement outcome
input state

$$Pr(\text{"+" is } x) = |{}_x\langle +|+\rangle|^2 = |\langle +|+\rangle_x|^2$$

$$= |\langle +|(a|+\rangle + b|-\rangle)|^2$$

$$= |a\cancel{\langle +|+\rangle}^{1} + b\cancel{\langle +|-\rangle}^{0}|^2$$

orthonormal basis!

$$= |a|^2$$

$$\Rightarrow |a|^2 = \frac{1}{2}$$

similarly:

measurement outcome
input state

$$Pr(\text{"-" is } x) = |{}_x\langle -|+\rangle|^2 = |\langle +|-\rangle_x|^2$$

$$= |\langle -|(c|+\rangle + d|-\rangle)|^2$$

$$= |c\cancel{\langle -|+\rangle}^{0} + d\cancel{\langle -|-\rangle}^{1}|^2$$

orthonormal basis!

$$= d^2$$

$$\Rightarrow |d|^2 = \frac{1}{2}$$

3.3.2.3. Step 2: normalization condition

$$1 = {}_x\langle +|+\rangle_x$$

$$\Rightarrow 1 = |a|^2 + |b|^2$$

$$|a|^2 = \frac{1}{2}$$

$$\Rightarrow |b|^2 = \frac{1}{2}$$

$$1 = {}_x\langle - | - \rangle_x \qquad |d|^2 = \tfrac{1}{2}$$

$$\Rightarrow 1 = |c|^2 + |d|^2 \qquad \Rightarrow |c|^2 = \tfrac{1}{2}$$

Both steps together:

$$|a|^2 = |b|^2 = |c|^2 + |d|^2 = 1/2 \; !$$

M3: Complex Numbers

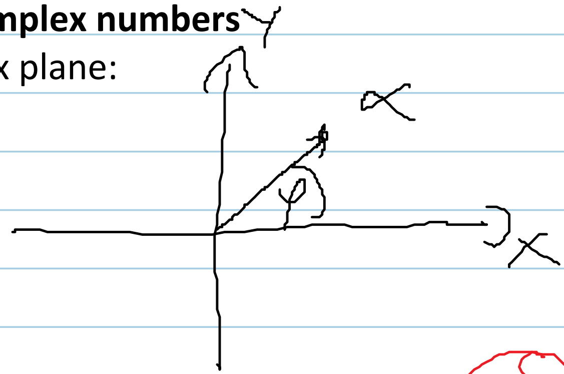
$$\alpha \in \mathbb{C}$$

complex number

$$\Rightarrow \alpha = x + iy \;, \quad \underbrace{x, y \in \mathbb{R}}_{\text{real numbers}}$$

M3: complex numbers

complex plane:



==> write

$$\alpha = \underbrace{|\alpha|}_{\text{amplitude}} \, e^{i\varphi} \quad \text{phase factor}$$

$$|\alpha| = \sqrt{x^2 + y^2}$$

connection:

$$e^{i\varphi} = \cos\varphi + i \sin\varphi$$

$$\Rightarrow \boxed{|e^{i\varphi}| = \sqrt{\cos^2\varphi + \sin^2\varphi} = 1}$$

$$\alpha = \sqrt{x^2 + y^2}\,(\cos\varphi + i \sin\varphi)$$

Step 1 & 2:
Set

$$a \longrightarrow \frac{1}{\sqrt{2}} e^{i\delta}$$

$$b \longrightarrow \frac{1}{\sqrt{2}} e^{i\alpha}$$

$$c \longrightarrow \frac{1}{\sqrt{2}} e^{i\gamma}$$

$$d \longrightarrow \frac{1}{\sqrt{2}} e^{i\beta}$$

phases unknown!

so we reduce
the problem
to finding
four real numbers
(the phases)

$$\boxed{\begin{aligned} |+\rangle_x &= \frac{1}{\sqrt{2}} e^{i\delta}|+\rangle + \frac{1}{\sqrt{2}} e^{i\alpha}|-\rangle \\ |-\rangle_x &= \frac{1}{\sqrt{2}} e^{i\gamma}|+\rangle + \frac{1}{\sqrt{2}} e^{i\beta}|-\rangle \\ &\alpha, \beta, \gamma, \delta \in [0, 2\pi) \end{aligned}}$$

3.3.2.4 Step 3: Freedom of Global phases

Note that we can multiply any incoming state vector with a phase factor
of the form $e^{i\Delta}$
without changing the predictions!
Such overall phase factors are called 'global phase'

$$|\psi\rangle \longrightarrow e^{i\Delta}|\psi\rangle$$

$$\Rightarrow Pr(\text{" , "}) \longrightarrow |\langle\phi_i|e^{i\varphi}|\psi\rangle|^2$$

$$= |\langle\phi_i|\psi\rangle e^{i\varphi}|^2$$

$$= |\langle\phi_i|\psi\rangle|^2 |e^{i\varphi}|^2$$

$$= |\langle\phi_i|\psi\rangle|^2$$

(as before)

So we can rewrite

$$|+\rangle_x = \overbrace{e^{i\delta}}^{\text{global phase}} \frac{1}{\sqrt{2}} \left(|+\rangle + \overbrace{e^{i(\alpha-\delta)}}^{\text{relative phase}} |-\rangle \right)$$

and notice that only the difference between α and δ is of relevance, as we can omit the leading global phase. Equivalently, we can simply choose to set $\delta = 0$

NOTE: in contrast to the irrelevant global phase, the relative phase (see above) is very important for the prediction of probabilities. Relative phases are phases between different terms in a sum.

$$\boxed{\begin{aligned} |+\rangle_x &= \frac{1}{\sqrt{2}}\left(|+\rangle + e^{i\alpha}|-\rangle\right) \\ |-\rangle_x &= \frac{1}{\sqrt{2}}\left(|+\rangle + e^{i\beta}|-\rangle\right) \\ &\alpha, \beta \in [0, 2\pi) \end{aligned}}$$

3.3.2.5 Step 4: Orthogonality Condition

we now explore the fact that "up in x" and "down in x" are mutually exclusive events in a measurement, and should be denoted by orthogonal vectors:

$${}_x\langle +|-\rangle_x \overset{!}{=} 0$$

$$\Rightarrow \frac{1}{\sqrt{2}}\left(\langle +| + e^{-i\alpha}\langle -|\right) \frac{1}{\sqrt{2}}\left(|+\rangle + e^{i\beta}|-\rangle\right)$$

$$= \frac{1}{2}\left(\langle +|+\rangle + e^{i(\beta-\alpha)}\langle -|-\rangle\right)$$ other two terms give zero!

$$= \frac{1}{2}\left(1 + e^{i(\beta-\alpha)}\right)$$

so we conclude that $e^{i(\beta-\alpha)} = 0$

$$\Rightarrow \beta = \alpha + \pi$$

$$
\begin{aligned}
|+\rangle_x &= \frac{1}{\sqrt{2}}\left(|+\rangle + e^{i\alpha}|-\rangle\right) \\
|-\rangle_x &= \frac{1}{\sqrt{2}}\left(|+\rangle - e^{i\alpha}|-\rangle\right) \\
&\alpha \in [0, 2\pi)
\end{aligned}
$$

$e^{i(\alpha+\pi)} = e^{i\alpha} e^{i\pi}$
$= -e^{i\alpha}$

3.3.2.6 Step 5: final choice, Result

We are now done with the constraints. We can now pick the value of alpha as we want, it will be always consistent with our observations. In view of future choices and further orientations of the Stern-Gerlach device, it turns out to be a smart choice to choose

$\alpha = 0$

$$
\begin{aligned}
|+\rangle_x &= \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right) \\
|-\rangle_x &= \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right)
\end{aligned}
$$

That means for the coordinate representation with respect to the standard basis (z-direction)

3.3.3 Coordinate representation and application to Experiment 2

coordinate representation:

$$
|+\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |+\rangle_x \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}
$$

$$
|-\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |-\rangle_x \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}
$$

with these derivations, our description of states and measurements of course agrees with the experiment 2! (Because we used the result!)

$$Pr(\text{"+" in } x) = |{}_x\langle +|+\rangle|^2$$

$$= \left| \frac{1}{\sqrt{2}}(1,1)\begin{pmatrix}1\\0\end{pmatrix}\right|^2 = \frac{1}{2}$$

$$Pr(\text{"-" in } x) = |{}_x\langle -|+\rangle|^2$$

$$= \left| \frac{1}{\sqrt{2}}(1,-1)\begin{pmatrix}1\\0\end{pmatrix}\right|^2 = \frac{1}{2}$$