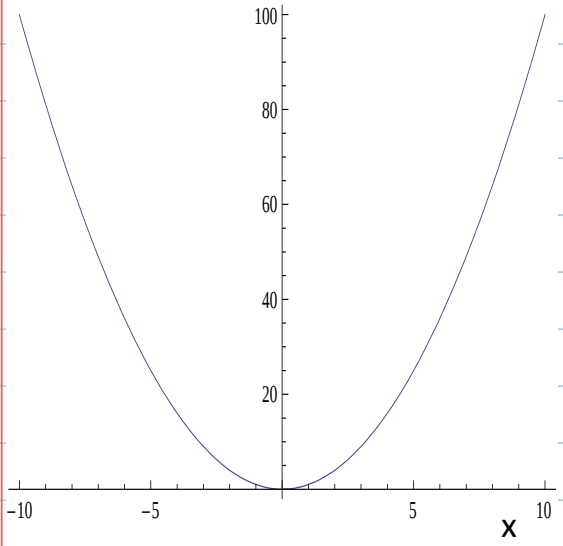
L33 Wave functions of the harmonic oscillator



$$\hat{H} = \frac{1}{2m}\hat{P}^2 + \frac{1}{2}m\omega^2\hat{X}^2 \qquad [\hat{X},\hat{P}] = i\hbar\mathbb{1}$$

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\mathbb{1}\right)$$

ladder operators:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{X} + \frac{i}{m\omega}\hat{P}\right)$$
$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{X} - \frac{i}{m\omega}\hat{P}\right)$$

commutator:

$$\left[\hat{a},\hat{a}^\dagger\right] = \mathbb{1}$$

For now:

use only structure ==> forget about the specific form of the operators

$\hat{N} := a^\dagger a$ & a & $a^\dagger$!

$[a, a^\dagger] = \mathbb{1}$

and discuss the eigenvalues and the relation of the eigenstates of N,
and thus also that of the Hamilton operator

$\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\mathbb{1}\right)$

8.3.2 Relations between eigenstates of N

8.3.2.1 Basic relation

The eigenstates of H are the eigenstates of the operator $\hat{N} := a^\dagger a$

Given some eigenstate of N:

$\hat{N}|\lambda\rangle = \lambda|\lambda\rangle, \lambda \in \mathbb{R}$

Then also $\hat{a}^+|\lambda\rangle$ will be an eigenstate of $\hat{N}$!

Proof:

$$\hat{N}\, a^+|\lambda\rangle = a^+ a\, a^+|\lambda\rangle$$

$$= a^+\left(a^+ a + \mathbb{1}\right)|\lambda\rangle$$

$$= a^+\left(\hat{N} + \mathbb{1}\right)|\lambda\rangle$$

$$= a^+ \hat{N}|\lambda\rangle + a^+|\lambda\rangle$$

$$= (\lambda + 1)\, a^+|\lambda\rangle$$

$\Rightarrow a^+|\lambda\rangle$ is an eigenstate to N with eigenvalue $\lambda + 1$

Similarly, we can prove

$$\hat{N}\,\hat{a}|\lambda\rangle = (\lambda - 1)\,\hat{a}|\lambda\rangle$$

For this reason we call

$\hat{a}$ lowering operator

$\hat{a}^+$ raising operator

} ladder operators

8.3.2.2 Normalization

Normalization of the states (all $|\lambda\rangle$ are thought of as being normalized)

$$a|\lambda\rangle = c_\lambda |\lambda - 1\rangle$$

gives

$$\langle\lambda|a^+ a|\lambda\rangle = \langle\lambda|\hat{N}|\lambda\rangle = \lambda$$

$$\Rightarrow \quad c_\lambda = \sqrt{\lambda}$$

$$a|\lambda\rangle = \sqrt{\lambda}\,|\lambda - 1\rangle$$

similarly $a^\dagger|\lambda\rangle = \sqrt{\lambda+1}\,|\lambda+1\rangle$

Harmonic Oscillator

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\mathbb{1}\right)$$

commutator: $\left[\hat{a}, \hat{a}^\dagger\right] = \mathbb{1}$

Essential part:

$$\hat{N} = \hat{a}^\dagger\hat{a}$$

Clicker Question:

It turns out, that

$$\hat{a}|\lambda\rangle \sim |\lambda - 1\rangle$$

What is the proper normalization constant?

A) λ
B) λ -1
C) $\sqrt{\lambda - 1}$
D) $\sqrt{\lambda}$
E) $\sqrt{\lambda + 1}$

Step 1: If $|\lambda\rangle$ is an eigenstate of operator N then also

$$a^\dagger|\lambda\rangle$$

is an (unnormalized) eigenstate of operator N with eigenvalue λ+1!

Step 2: Normalization: $a^\dagger|\lambda\rangle = \sqrt{\lambda+1}\,|\lambda+1\rangle$

Similar: $a|\lambda\rangle = \sqrt{\lambda}\,|\lambda-1\rangle$

So far, we have shown that there is a connection between eigenstates using the ladder operators.
In principle, for each value of $\lambda \in [0,1)$
we could have a ladder of connected eigenstates $|\lambda + m\rangle$

$m = -\infty \ldots -1, 0, +1, ? \ldots +\infty$

But remember: all of this assumed that we we were given an eigenstate in the ladder, from which we construct all other elements of the ladder!
But will we be able for find eigenstates for all ladders?
Of course, the answer lies in the full mathematical problem (including the form of the ladder operators themselves), but fortunately we can find an elegant short-cut using a physical argument:

8.3.3 Existence of Ground States: Consequences

If $|\lambda\rangle$ is the eigenstate of N with eigenvalue λ
then the same state is eigenstate to the Hamiltonian H with energy eigenvalue

$\hbar\omega\left(\lambda + \frac{1}{2}\right)$

There must a lowest energy eigenstate ... $|\lambda\rangle$
how does it fit with the fact that the lowering operator can bring us to lower eigenenergy states?

$$\hat{N}\,\hat{a}|\lambda\rangle = (\lambda - 1)\,\hat{a}|\lambda\rangle$$

Only way: for some value of n

$$\hat{a}|\lambda\rangle = 0$$

For such a state we find on one hand:

$$\hat{N}|\lambda\rangle = \lambda|\lambda\rangle$$

on the other hand $\hat{N}|\lambda\rangle = a^{\dagger}a|\lambda\rangle = 0\,|\lambda\rangle$

That is, we must have λ=0

The **ground state** is the state with label $|0\rangle$
and satisfies

$$\hat{a}|0\rangle = 0$$
$$\hat{N}|0\rangle = 0\,|0\rangle$$

==> eigenvector to N with eigenvalue 0

That means, that there is only one ladder of eigenstates that can indeed be a solution to the mathematical problem, and we found this constraint by physical reasoning.

8. 5 Eigensystem of Hamiltonian

Structure of eigensystem of Hamilton Operator of Harmonic Oscillator

$$\hat{H}|n\rangle = \underbrace{\hbar\omega\left(n+\tfrac{1}{2}\right)}_{E_n}|n\rangle$$

E_n Eigenenergies

Eigenfunctions $|n\rangle$, $n = 0, 1, 2, \ldots \infty$

Eigenvalues (Eigenenergies) $E_n = \hbar\omega\left(n+\tfrac{1}{2}\right)$

Ground state $|0\rangle$, $E_0 = \tfrac{1}{2}\hbar\omega$

Connection between eigenstates: ladder operator action

$$\hat{a}^+ |n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle \qquad n = 0, 1, 2, \ldots$$

includes $\hat{a}|0\rangle = 0$

8.3.5 Wave function of Eigenstates

8.3.5.1 Wave function of Ground State

So far we talked abstractly about the structure of the eigenstates and eigenvalues of the Hamiltonian that describes the Harmonic Oscillator.
We now derive the coordinate representations of the eigenstates.

For the ground state $|0\rangle$ we have

$$\hat{a}\,|0\rangle = 0$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}_x\right)$$

Then

$$\int dx' \langle x|\hat{a}|x'\rangle\langle x'|0\rangle = 0$$

$$\langle x|\hat{x}|x'\rangle = x\,\delta(x-x')$$

$$\langle x|\hat{p}|x'\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x}\,\delta(x-x')$$

$$\langle x'|0\rangle = \psi_0(x)$$

wave function of the ground state in position coordinate representation

$$\Rightarrow \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\psi_0(x) = 0$$

The ground state satisfies a simple first order differential equation!
==> solution

$$\Rightarrow \quad \frac{d}{dx}\psi_0(x) = -\frac{m\omega x}{\hbar}\psi_0(x)$$

$$\Rightarrow \quad \psi_0(x) = \underbrace{A}\, e^{-\frac{1}{2}\frac{m\omega x^2}{\hbar}}$$

normalization constant

$$\Rightarrow \quad \boxed{\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}}}$$

8.3.5.2 Wave functions of Excited states

The state with energy higher than the ground state (excited states) can now be simply generated using the relationship

$$a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

First exited state: $(n=0)$

$$\int dx' \langle x|\hat{a}^+|x'\rangle\langle x'|0\rangle = \langle x|1\rangle$$

$$\hat{a}^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}_x\right)$$

$$\sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\psi_0(x) = \psi_1(x)$$

So we can obtain the first excited state by simple operation (including differentiation) from the ground state wave function

$$\psi_1(x) = \left[\frac{4}{\pi}\left(\frac{m\omega}{\hbar}\right)^3\right]^{1/4} \times e^{-\frac{m\omega x^2}{2\hbar}}$$

In general:

$$\frac{1}{\sqrt{n!}}(a^+)^n|0\rangle = |n\rangle$$

Coordinate representation:

$$\frac{1}{\sqrt{n!}}\langle x|(a^+)^n|0\rangle = \psi_n(x)$$

$$\frac{1}{\sqrt{n!}}\left[\sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\right]^n \psi_0(x) = \psi_n(x)$$

e.g. $$\psi_2(x) = \left(\frac{m\omega}{4\pi\hbar}\right)^{1/4}\left(2\frac{m\omega}{\hbar}x^2 - 1\right)e^{-\frac{m\omega x^2}{2\hbar}}$$

For graphical representation, see simulation on PhET webpage ...
(See ACE system)

We note that indeed the wave function for each n is either symmetric or anti-symmetric, just as expected from the symmetry of the potential.