L34 Wave functions of the harmonic oscillator

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8.7 Classical Limit

How do the quantum mechanical solutions for eigenenergy E_n correspond to the classical solutions in mechanics for the same energy?

Example:

Pendulum with length L = 10 cmin a gravity field

==> $\omega = \sqrt{\frac{g}{L}} \approx 10\ s^{-1}$ (radian/s)

mass $1\,g$

speed $1\,\frac{cm}{s}$

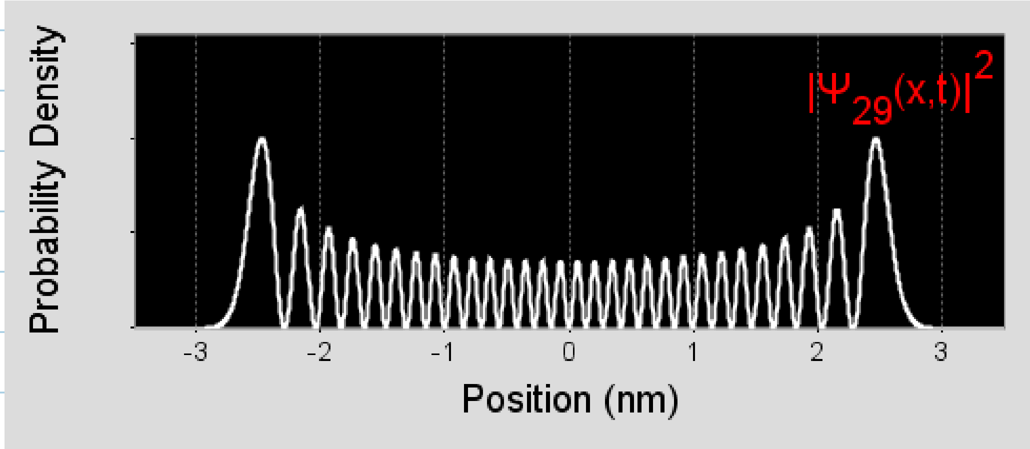
$$E_{kin} = \frac{1}{2}\, 10^{-3}\, 10^{-4} = 5 \cdot 10^{-8}\, J$$

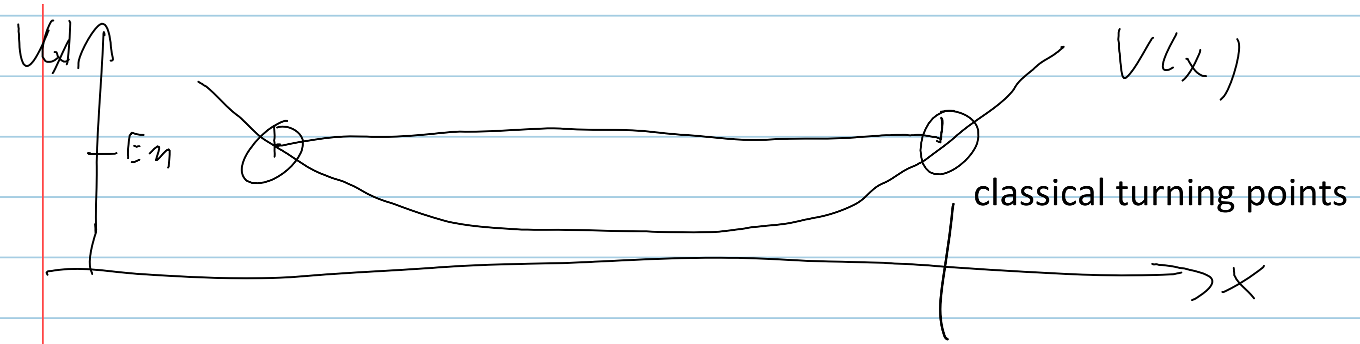
$$\hbar\omega\left(n + \frac{1}{2}\right) = E$$

$$\Rightarrow n \approx \frac{E}{\hbar\omega} \approx \frac{5 \cdot 10^{-8}}{10 \quad 10^{-34}}$$

$$\approx \frac{1}{2}\, 10^{26}$$

The steps between energy levels are very small compared to typical energies in our day-to-day world.





Classical probability (averaged over cycles!)

$$E_n = \frac{1}{2} m \omega^2 x_n^2$$

$$\frac{dx}{dt} = v(x)$$

$$dt = \frac{1}{v(x)} dx$$

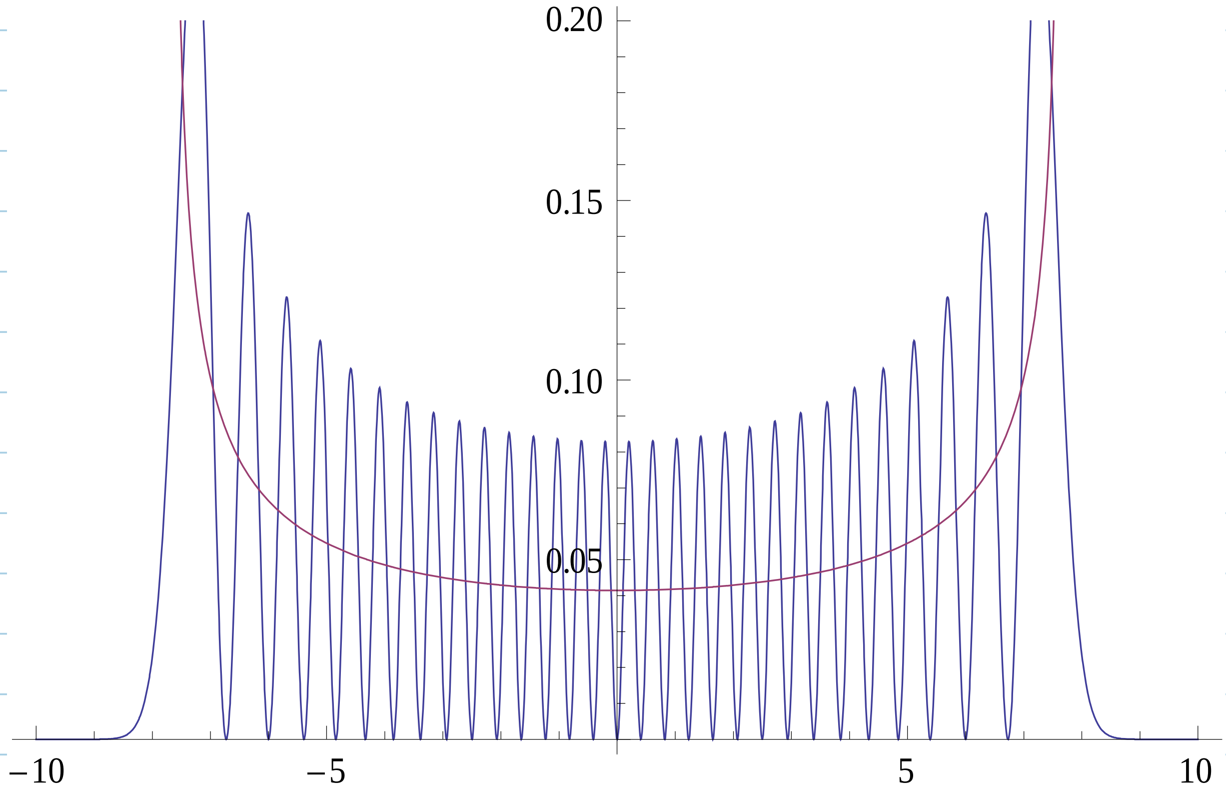
$$p(x)\,dx \sim \frac{1}{v(x)} dx$$

$$E_n = \underbrace{\frac{1}{2} m \omega^2 x^2}_{\text{potential}} + \underbrace{\frac{1}{2} m v^2}_{\text{kinetic}}$$

$$\Rightarrow p(x)\,dx \sim \frac{1}{\sqrt{\frac{2E_n}{m} - \omega^2 x^2}} dx$$

$$= \frac{1}{\omega \sqrt{x_n^2 - x^2}} dx$$

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \quad n = 2g$$



for large n:
- The energy eigenstates are basically confined to the interval between the classical turning points
- there is a very rapid spatial osciallation of the probability density (the number of zero's corresponds to the number n
- Doing a spatial averaging of this quantum mechanical probability distribution, we recover the probability distribution for the classical mechanics case in the limit of large n

8.8 Time evolution of wave packets

$$|\psi;0\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$$

$$|\psi;t\rangle = \sum_{n=0}^{\infty} c_n \exp\left[-\frac{i}{\hbar}(n+\tfrac{1}{2})\hbar\omega t\right]|n\rangle$$

$$= e^{-i\frac{1}{2}\omega t} \sum_{n=0}^{\infty} c_n \, e^{-in\omega t} |n\rangle$$

Example (coherent states):

$|\alpha\rangle$ with $\alpha \in \mathbb{C}$ amplitude

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$\Rightarrow$ time evolution

$$|\alpha, t\rangle = e^{-\frac{i\omega t}{2}} e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-in\omega t} |n\rangle$$

$$= e^{-\frac{i\omega t}{2}} e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\left(\alpha e^{-i\omega t}\right)^n}{\sqrt{n!}} |n\rangle$$

$$= \underbrace{e^{-\frac{i\omega t}{2}}}_{\text{global phase -> no physical relevance here}} \left|\alpha e^{-i\omega t}\right\rangle$$

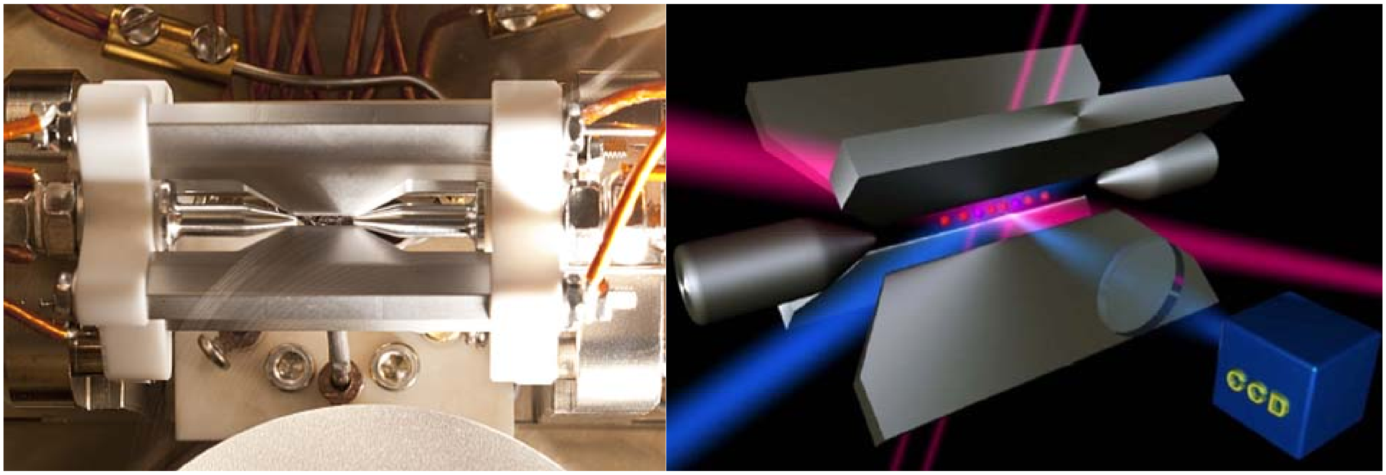
Under time evolution, a coherent states remains a coherent state, although with oscillating phase of its amplitude

The time evolution of coherent states shows behavior very similar to that of a classical mechanical system!

The "Ehrenfest theorem" shows that the expectation value of the position follows a Newton's law with the force being derived from the potential.

8.9

Ion trap



Rainer Blatt’s group (Innsbruck)
http://heart-c704.uibk.ac.at/research/qsim/index.html

Ion traps can confine a single ion. The constrain the ions in two directions, thereby effectively creating a one-dimensional system.
In this system, the ion can be cooled down to the ground state, and also be brought into higher Energy eigenstates, showing all the quantum mechanical features of a harmonic oscillator.