June 4, 2014

Iterative Methods

Splitting

Let $\quad A = M - N \qquad M \approx A$

Then
$$A x = b$$
$$(M - N)\, x = b$$
$$M x = N x + b$$

Define an iterative method by:

$$M x^{k+1} = N x^k + b$$

Then
$$x^{k+1} = M^{-1} N x^k + M^{-1} b$$
$$= M^{-1} (M - A)\, x^k + M^{-1} b$$
$$= x^k + M^{-1} (b - A x^k)$$

Note: If $M = A$, then $\quad x^{k+1} = x^k + A^{-1} (b - A x^k)$
$$= x^k + x - x^k = x$$

$\rightarrow$ one step convergence

But one needs to compute $A^{-1} (b - Ax^k)$

Goals:

(1) $M \approx A$

(2) M^{-1} is easy to compute

Richardson iteration

- $M = 1/\theta\, I$ (θ is appropriately chosen)

Then $M^{-1} = \theta\, I$

Thus $x^{k+1} = x^k + \theta\, I\, (b - A\, x^k)$

Consider the i-th entry of x^{k+1} :

$$x_i^{k+1} = x_i^k + \theta(b_i - \sum_{j=1}^{n} a_{i,j} x_j^k)$$

Algorithm

x^0 = initial guess

for $k = 0, 1, 2, \ldots$

for $i = 1, 2, \ldots, n$

$$x_i^{k+1} = x_i^k + \theta(b_i - \sum_{j=1}^{n} a_{i,j} x_j^k)$$

end

end

Note

Need 2 separate vectors x^k, x^{k+1}.

Jacobi iteration

- M = D = diagonal of A = $\begin{bmatrix} a_{1,1} & & \\ & \ddots & \\ & & a_{n,n} \end{bmatrix}$

Thus
$$x^{k+1} = x^k + D^{-1}(b - Ax^k)$$
$$x_i^{k+1} = x_i^k + \tfrac{1}{a_{i,i}}(b_i - \sum_{j=1}^{n} a_{i,j} x_j^k)$$
$$= x_i^k + \tfrac{1}{a_{i,i}}(b_i - \sum_{j \neq i} a_{i,j} x_j^k - a_{i,i} x_i^k)$$
$$= \tfrac{1}{a_{i,i}}(b_i - \sum_{j \neq i} a_{i,j} x_j^k)$$

Interpretation

Let $r^k = b - A x^k$ (residual vector of x^k)

Then $x^k = x \Leftrightarrow r^k = 0$

Thus $||r^k||_2 \approx 0 \Rightarrow x^k \approx x$

Consider $r_i^k = b_i - \sum_{j=1}^{n} a_{i,j} x_j^k = b_i - \sum_{j \neq i} a_{i,j} x_j^k - a_{i,i} x_i^k$

In general, $r_i^k \neq 0$. Now modify x_i^k so that $r_i^k = 0$.

i.e.
$$b_i - \sum_{j \neq i} a_{i,j} x_j^k - a_{i,i} x_i^{k+1} = 0$$
$$x_i^{k+1} = \tfrac{1}{a_{i,i}}(b_i - \sum_{j \neq i} a_{i,j} x_j^k)$$

Gauss-Seidel iteration

Let $A = D - L - U$, where D = diagonal of A, L = strictly lower Δ part, U = strictly upper Δ part

i.e.

$$A = \begin{bmatrix} \ddots & & -U \\ & D & \\ -L & & \ddots \end{bmatrix}$$

- $M = D - L$ = lower Δ part of A.

i.e. $\quad x^{k+1} = x^k + (D-L)^{-1} (b - A x^k)$

Interpretation

Modify x_i^k so that $r_i^k = 0$. Use the new x_j^{k+1}, $j < i$, from the previous updates.

i.e. $$r_i^k = b_i - \sum_{j<i} a_{i,j} x_j^{k+1} - a_{i,i} x_i^{k+1} - \sum_{j>i} a_{i,j} x_j^k = 0$$

Thus $$x_i^{k+1} = \tfrac{1}{a_{i,i}} (b_i - \sum_{j<i} a_{i,j} x_j^{k+1} - \sum_{j>i} a_{i,j} x_j^k)$$

Backward GS

- $M = D - U$: $\quad x^{k+1} = x^k + (D-U)^{-1} (b - A x^k)$

Symmetric GS

A forward sweep followed by a backward sweep:

SOR iteration

- Successive over-relaxation
- Weighted average of x_i^k and $x_i^{k+1} = GS(x_i^k)$:

$$x_i^{k+1} = (1-\omega)x_i^k + \frac{\omega}{a_{i,i}}(b_i - \sum_{j<i} a_{i,j}x_j^{k+1} - \sum_{j>i} a_{i,j}x_j^k)$$

$$\text{SOR} = \begin{cases} \text{GS} & \omega = 1 \\ \text{Under-relaxation} & \omega < 1 \\ \text{Over-relaxation} & \omega > 1 \end{cases}$$

- $M = 1/\omega\ D - L$
- For a suitably chosen ω (>1), SOR can be much better than GS.

Convergence Analysis

Q1: Under what condition does the iteration converge?

Q2: If the iteration converges, how fast is it?

Def: λ is an eigenvalue and v an eigenvector of A if

$$A v = \lambda v \qquad\qquad v \neq 0$$

Def: The spectral radius of A, $\rho(A) = \{|\lambda|: \lambda \text{ eig of } A\}$, is the largest absolute value of the eigenvalues of A.

Theorem: The iterative method:

$$x^{k+1} = x^k + M^{-1} (b - A x^k)$$

is convergent for any x^0 and b if and only if

$$\rho(I - M^{-1} A) < 1$$

- $I - M^{-1} A$ is called the iteration matrix
- $\rho(I - M^{-1} A)$ is called the rate of convergence

Minimization formulation

Assume A is SPD. Consider the functional:

$$F(x) \equiv \tfrac{1}{2} x^T A x - b^T x \qquad x \in R^n$$

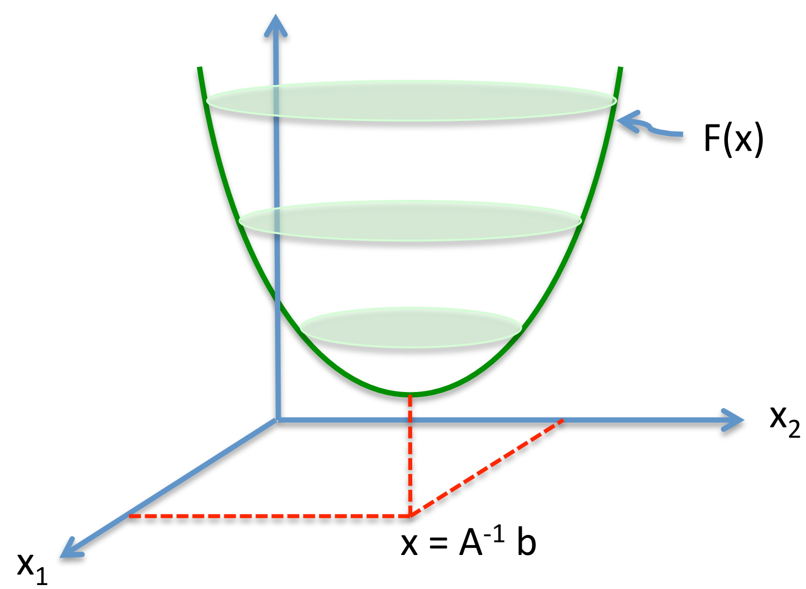
Theorem: The solution of $A x = b$ is equivalent to the solution of the minimization problem:

$$\min_x \; F(x)$$

Pf: The minimizer satisfies $F'(x) = 0$ i.e. $\frac{\partial F}{\partial x_k} = 0$ and $F'(x) = Ax-b$

Note: F(x) is convex. So a local min = global min.

e.g. n = 2



Search directions

Idea: min F(x) along a direction p ($p \neq 0$)

Let x^k = current approx. Define $x^{k+1} = x^k + \alpha\, p$

Determine α by min $F(x^{k+1})$ along p

Let $$f(\alpha) \equiv F(x^k + \alpha p)$$

$$= \tfrac{1}{2}(x^k + \alpha p)^T A(x^k + \alpha p) - (x^k + \alpha p)^T b$$

$$= \tfrac{1}{2}(x^k)^T A x^k + \tfrac{\alpha}{2} p^T A x^k + \tfrac{\alpha}{2}(x^k)^T A p + \tfrac{\alpha^2}{2} p^T A p$$

$$- (x^k)^T b - \alpha p^T b$$

$$= \tfrac{1}{2}(x^k)^T A x^k - (x^k)^T b + \alpha p^T A x^k - \alpha p^T b + \tfrac{\alpha^2}{2} p^T A p$$

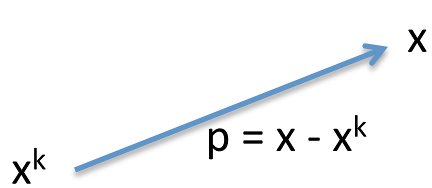
$$= F(x^k) - \alpha p^T (b - A x^k) + \tfrac{\alpha^2}{2} p^T A p$$

$$0 = f'(\alpha) = -p^T (b - A x^k) + \alpha p^T A p$$

$$\Rightarrow \qquad \alpha = \frac{p^T (b - A x^k)}{p^T A p} = \frac{p^T r^k}{p^T A p}$$

<u>Notes</u>

1) A is SPD $\Rightarrow$ $p^T A p > 0$
2) What is the optimal search direction p?



However x is unknown.

Steepest descent method

- Local optimal direction

Consider $f'(\alpha) = p^T (A x^k - b) + \alpha p^T A p$

Then $f'(0) = p^T F'(x^k)$ $\quad (F'(x) = A x - b)$

$= $ changes in F at x^k in the direction of p

Idea: make $f'(0)$ as negative as possible by varying p

Assume $||p|| = 1$. Then

$f'(0)$ max if $p = F'(x^k) / ||F'(x^k)||$ = steepest ascent

$f'(0)$ min if $p = - F'(x^k) / ||F'(x^k)||$ = steepest descent

$= r^k / ||r^k||$ $\quad (F'(x) = -r)$

Steepest descent method:

$x^{k+1} = x^k + \alpha^k r^k$ $\quad \alpha^k$ = step length

The optimal $\alpha^k = (r^k)^T(r^k) / (r^k)^T A r^k$ $\quad (\alpha = p^T r^k / p^T A p)$

Also $\quad r^{k+1} = b - A\,x^{k+1}$

$$= b - A\,(x^k + \alpha_k\, r^k)$$

$$= b - A\,x^k - \alpha_k\, A\, r^k$$

$$= r^k - \alpha_k\, A\, r^k$$

<u>Algorithm</u>

Given x^0, compute $r^0 = b - A\,x^0$

for $k = 0, 1, 2, \ldots$

$\quad \alpha_k = (r^k)^T(r^k) \,/\, (r^k)^T A\, r^k$

$\quad x^{k+1} = x^k + \alpha_k\, r^k$

$\quad r^{k+1} = r^k - \alpha_k\, A\, r^k$

end

<u>Notes</u>

1) Only 1 matrix-vector product ($A\,r^k$) per iteration

2) "Nonlinear" iterative method:

$$x^{k+1} = x^k + \alpha_k\,(b - A\,x^k)$$

i.e. $\quad M = M^k = 1/\alpha_k\, I$