Coordinate representation in Energy Eigenbasis (time independent Hamiltonian)

Schrödinger Equation

$$i\,\hbar\,\frac{d}{dt}\,|\Psi(t)\rangle = H\,|\Psi(t)\rangle$$

expansion in eigenstates of H:

$$|\Psi(t)\rangle = \sum_n c_n(t)\,|E_n\rangle$$

dynamical equation: $i\hbar\frac{d}{dt}c_n(t) = E_n c_n(t)$

initial values: $c_n(0) = c_n$

$$\rightarrow\quad c_n(t) = c_n\, e^{-i\frac{E_n t}{\hbar}}$$

Solution of Schrödinger's equation via Energy Eigenstates:

Step 1: find eigenvectors $|E_n\rangle$ and eigenvalue E_n of H

Step 2: Expand initial state in eigenbasis

$$|\Psi(0)\rangle = \sum_n c_n\,|E_n\rangle$$

Step 3: Write down solution

$$|\Psi(t)\rangle = \sum_n c_n\, e^{-i\frac{E_n t}{\hbar}}\,|E_n\rangle$$

Clicker Question:

Given the Hamilton Operator

$$H = \frac{1}{2}\hbar\omega|+\rangle\langle+| - \frac{1}{2}\hbar\omega|-\rangle\langle-|$$

and the initial state $|+\rangle$

What is the probability to find outcome "+ in z" as a function of t?

A) 1

B) $e^{-i\omega t/2}$

C) Something else

D) don't know

5.5 Spin 1/2 particle in external magnetic field

5.5.1 Motivation of Hamiltonian from Classical Energy Expression

$$E_{mag} = -\vec{\mu}\cdot\vec{B}$$

classical physics (non-quantum)

$$\vec{\mu} \xrightarrow{QM} -\frac{ge}{2m_e}\vec{S}$$

$$g \approx 2$$

General Hamiltonian for Spin 1/2 particle in homogeneous magnetic field:

$$H = \frac{e}{m_e}\vec{B}\cdot\vec{S} \iff -\vec{B}\cdot\vec{\mu}$$

$$= \frac{e}{m_e}\left(B_x S_x + B_y S_y + B_z S_z\right)$$

numbe

Operators

Special Case Hamiltonian (spin 1/2 in hom. Magnetic field in z-direction

Without loss of generality: B-field points into z-direction

$$H = \frac{e}{m_e} B_z S_z$$

$$= \omega S_z \qquad \boxed{\omega := \frac{eB_z}{m_e}}$$

$$= +\frac{1}{2}\omega\hbar|+\rangle\langle+| - \frac{1}{2}\omega\hbar|-\rangle\langle-|$$

NOTE: factor 1/2 up to what we showed last lecture!

5.5.2 Eigensystem of Hamiltonian

Eigensystem of Special Case Hamiltonian:

eigenvalues $\hbar\omega/2 \qquad -\hbar\omega/2$

corresponding eigenvectors $|+\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |-\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

General Hamiltonian: In general, any Hamiltonian can be represented using the eigensystem as $H = \sum_n E_n |E_n\rangle\langle E_n|$

Eigenvalues: E_n

Eigenstate: $|E_n\rangle$

5.5.3 Time Evolution for Energy Eigenstates:

$|\psi(0)\rangle = |+\rangle$ already expanded in eigenbasis!

$$\Rightarrow \quad |\psi(t)\rangle = e^{-\frac{i}{\hbar}\frac{\hbar\omega}{2}t}|+\rangle$$

$$= e^{-i\frac{\omega t}{2}}|+\rangle$$

Consequences:

If a system is prepared initially in an eigenstate of the Hamilton Operator, then time evolution results in a time dependent global phase

==> Expectation values of all observables remain unchanged!

observable A

initial state in energy eigenstate: $|\psi(0)\rangle = |E_n\rangle$

later time $|\psi(t)\rangle = e^{-\frac{iE_n t}{\hbar}}|E_n\rangle$

$$\langle A \rangle_{t=0} = \langle \psi(0) | A | \psi(0) \rangle$$

$$= \langle E_n | A | E_n \rangle$$

$$\langle A \rangle_t = \langle \psi(t) | A | \psi(t) \rangle$$

$$= e^{\frac{iE_n t}{\hbar}} \langle \psi(0) | A | \psi(0) \rangle e^{-\frac{iE_n t}{\hbar}}$$

$$= \langle \psi(0) | A | \psi(0) \rangle$$

$$= \langle A \rangle_{t=0}$$

5.5.4 Time Evolution for general initial state

$$|\psi(0)\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2} e^{i\phi} |-\rangle$$

$$\Rightarrow \quad c_+(0) = \cos\frac{\theta}{2}$$

$$c_-(0) = \sin\frac{\theta}{2} e^{i\phi}$$

Parameterization:
see 4.2.2 (Lecture 10)
General Spin direction

Then the time evolution gives

$$c_+(t) = e^{-i\frac{\omega t}{2}} \cos\frac{\theta}{2}$$

$$c_-(t) = e^{i\frac{\omega t}{2}} e^{i\phi} \sin\frac{\theta}{2}$$

$$|\psi(t)\rangle = e^{-i\frac{\omega t}{2}} \cos\frac{\theta}{2} |+\rangle + e^{i\frac{\omega t}{2}} e^{i\phi} \sin\frac{\theta}{2}$$

In coordinate representation (using standard basis (z))

$$\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \longrightarrow \begin{pmatrix} e^{-i\frac{\omega t}{2}} \cos\frac{\theta}{2} \\ e^{i\frac{\omega t}{2}} e^{i\phi} \sin\frac{\theta}{2} \end{pmatrix}$$

$$= e^{-i\frac{\omega t}{2}} \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i(\omega t + \phi)} \sin\frac{\theta}{2} \end{pmatrix}$$

the overall factor $e^{-i\frac{\omega t}{2}}$ is a global phase which will not influence any observation, including any expectation values.

So what time evolution does, is that it has the relative phase between the two energy contribution oscillate in time with a frequency ω, starting with the initial phase ϕ

5.5.5 Spin Component Expectation values for general state

We now calculate the expectation values of the spin components operators. To remind you of the different ways that can be use, I employ two ways to do so:

$$\langle S_z\rangle = \frac{\hbar}{2}\left(|\langle +|\psi(t)\rangle|^2 - |\langle -|\psi(t)\rangle|^2\right)$$
$$= \frac{\hbar}{2}\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)$$
$$= \frac{\hbar}{2}\cos\theta$$

$$\langle S_y\rangle = \frac{\hbar}{2}\langle\psi(t)|S_y|\psi(t)\rangle$$
$$= e^{i\frac{\omega t}{2}}\left(\cos\frac{\theta}{2},\ e^{-i(\omega t+\phi)}\sin\frac{\theta}{2}\right)\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}e^{-i\frac{\omega t}{2}}\begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i(\omega t+\phi)}\sin\frac{\theta}{2}\end{pmatrix}$$

$$\frac{\hbar}{2}\left(\cos\frac{\theta}{2},\ e^{-i(\omega t+\phi)}\sin\frac{\theta}{2}\right)\begin{pmatrix}-i e^{i(\omega t+\phi)}\sin\frac{\theta}{2}\\ i\cos\frac{\theta}{2}\end{pmatrix}$$
$$= \frac{\hbar}{2}\underbrace{\cos\frac{\theta}{2}\sin\frac{\theta}{2}}\ \underbrace{\left(-i e^{i(\omega t+\phi)} + i e^{-i(\omega t+\phi)}\right)}$$
$$= \frac{\hbar}{2}\ \frac{1}{2}\sin\theta\qquad 2\sin(\omega t+\phi)$$

$$2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = \sin\alpha$$
$$\sin\alpha = \frac{1}{2i}\left(e^{i\alpha} - e^{-i\alpha}\right)$$

$$= \frac{\hbar}{2} \sin\theta \ \sin(\omega t + \phi)$$

$$\langle S_x \rangle = \ldots$$

$$= \frac{\hbar}{2} \sin\theta \ \cos(\omega t + \phi)$$

5.5.6 Bloch Vector

The time evolution can be easily visualized by introducing a three dimensional real-valued vector as

$$\vec{V} = \frac{2}{\hbar} \begin{pmatrix} \langle S_x \rangle \\ \langle S_y \rangle \\ \langle S_z \rangle \end{pmatrix} = \begin{pmatrix} \sin\theta \ \cos(\omega t + \phi) \\ \sin\theta \ \sin(\omega t + \phi) \\ \cos\theta \end{pmatrix}$$