Lagrange Example: Population example

Years since 1995	1	6	11	16
Pop (millions)	28.85	30.01	31.61	33.48

$$l_0(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)}$$

$$= \frac{(x - 6)(x - 11)(x - 16)}{(-5)(-10)(-15)}$$

$$= \frac{(x-6)(x-11)(x-16)}{-750}$$

$$= \frac{x^3 - 33x^2 + 338x - 1056}{-750}$$

Lagrange Example: Population example

Years since 1995	1	6	11	16
Pop (millions)	28.85	30.01	31.61	33.48

$$l_1(x) = \frac{(x-1)(x-11)(x-16)}{250} = \frac{x^3-28x^2+203x-176}{250}$$

$$l_2(x) = \frac{(x-1)(x-6)(x-16)}{-250} = \frac{x^3-23x^2+118x-96}{-250}$$

$$l_3(x) = \frac{(x-1)(x-6)(x-11)}{750} = \frac{x^3-18x^2+83x-66}{750}$$

$$p(x) = 28.85l_0(x) + 30.01l_1(x) + 31.61l_2(x) + 33.48l_3(x)$$
$$p(x) = -0.000227x^3 + 0.01288x^2 + 0.1516x + 28.6858$$

Error bound on Lagrange Interpolation

Theorem: Assume that

- $x_0, x_1, x_2, \ldots, x_n$ are distinct values of [a,b]
- f is (n+1) times continuously differentiable over [a,b]

Then, for all x $\in$[a,b], $\exists\xi(x)$ $\in$[a,b] such that

$$f(x) = p(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$$

for the Lagrange interpolating polynomial $p_n(x) = \sum_{k=0}^{n} \ell_k(x) y_k$.

Background for proof

- Generalized Rolle's Theorem: Let f be n times continuously differentiable on (a,b). If f(x)=0 at the distinct values x_0, x_1, …,x_n, then $\exists c \in (a, b)$ such that $f^{(n)}(c)=0$.
- For any $x \in [a, b]$, define g(t) for $t \in [a, b]$:

$$g(t) = \big(f(t) - p(t)\big) - [f(x) - p(x)] \prod_{i=0}^{n} \frac{t - x_i}{x - x_i}$$

Starting with different information

- Suppose we have the following information:
  - n+1 points: $x_0, x_1, x_2, \ldots, x_n$
  - Function values: $y_0, y_1, y_2, \ldots, y_n$
  - First derivatives: $y'_0, y'_1, y'_2, \ldots, y'_n$
- We have 2(n+1) observations → we can determine 2(n+1) unknowns
- Consider a polynomial of degree 2n+1:

$$p(x) = a_0 + a_1x+a_2x^2+ \ldots + a_{2n+1}x^{2n+1}$$

$$p'(x) = a_1+2a_2x+ \ldots + (2n+1)a_{2n+1}x^{2n}$$

Try to solve directly

$$\begin{bmatrix} 1 & x_0 & {x^2}_0 & \cdots & {x^{2n+1}}_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & {x^2}_n & \cdots & {x^{2n+1}}_n \\ 0 & 1 & 2x_0 & \cdots & (2n+1){x^{2n}}_0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2x_n & \cdots & (2n+1){x^{2n}}_n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_{2n+1} \end{bmatrix} = \begin{bmatrix} y_0 \\ \vdots \\ y_n \\ {y'}_0 \\ \vdots \\ {y'}_n \end{bmatrix}$$

A full system solve I expensive, and the system is ill-conditioned.

→ Like with Lagrange polynomials, use an alternate basis related to Hermite polynomials to simplify the calculations

Hermite Interpolation

- If f is continuous and continuously differentiable over [a,b], the unique polynomial that agrees with y_k and y_k', for k=0:n, is the polynomial of degree at most 2n+1, given by

$$H(x) = \sum_{j=0}^{n} y_j H_j(x) + \sum_{j=0}^{n} y'_j \widehat{H}_j(x) \text{ where}$$

$$H_j(x) = \left(1 - 2\left(x - x_j\right) l'_j\left(x_j\right)\right) [l_j(x)]^2$$

$$\widehat{H}_j(x) = (x - x_j)\, [l_j(x)]^2$$

This works?

- Recall: $l_j(x_j)$ = 1, $l_j(x_k)$ = 0 for j≠k

- $H_j(x) = \left(1 - 2(x - x_j)l'_j(x_j)\right)[l_j(x)]^2$
  - $H_j(x_j) =$1, $H_j(x_k) = 0$ for j≠k

- $\widehat{H}_j(x) = (x - x_j)\,[l_j(x)]^2$
  - $\widehat{H}_j(x_j)$=0, $\widehat{H}_j(x_k)$=0 for j≠k

This works?

- $H(x) = \sum_{j=0}^{n} y_j H_j(x) + \sum_{j=0}^{n} y'_j \widehat{H}_j(x)$
  - $H(x_k) = \sum_{j=0}^{n} y_j H_j(x_k) + \sum_{j=0}^{n} y'_j \widehat{H}_j(x_k) = y_k$
- $H'(x) = \sum_{j=0}^{n} y_j H_j'(x) + \sum_{j=0}^{n} y'_j \widehat{H}_j'(x)$
  - $H_j'(x_k) = 0$ for all j, k
  - $\widehat{H}_j'(x_k)$ = 0 if j≠k, $\widehat{H}_k'(x_k)$ = 1
  - $H'(x_k) = \sum_{j=0}^{n} y_j H_j'(x_k) + \sum_{j=0}^{n} y'_j \widehat{H}_j'(x_k) = y'_k$