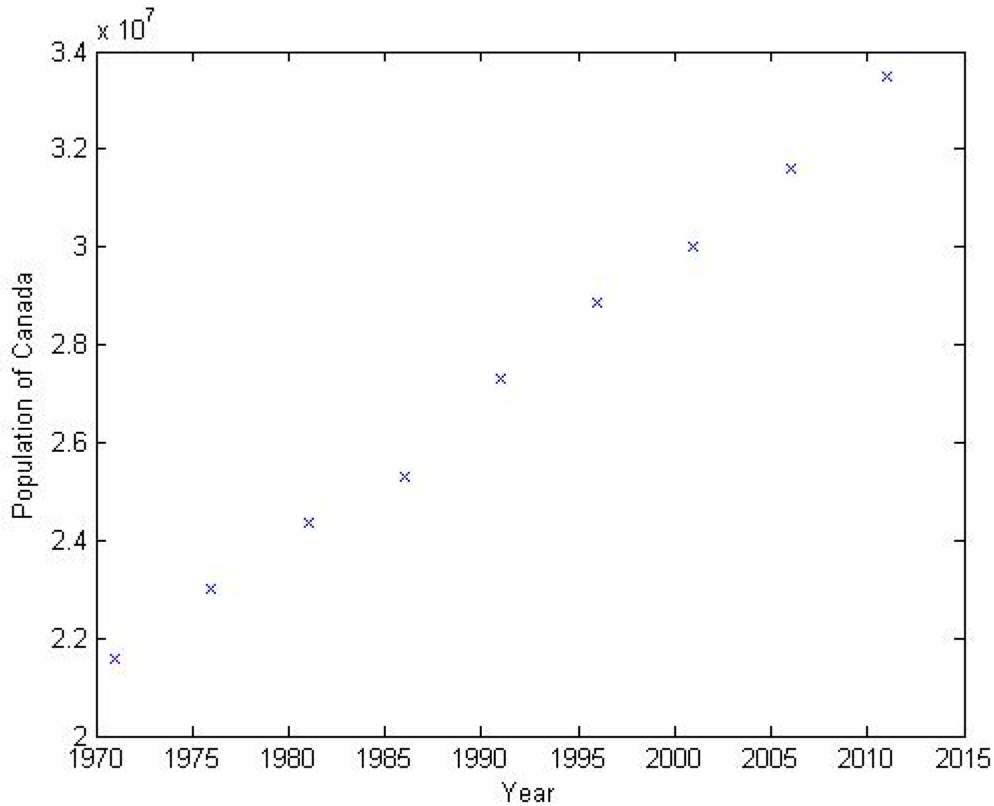
Module 04: Interpolation

Starting Monday, February 10, 2014

Consider the population of Canada (in millions):

Year	1971	1976	1981	1986	1991	1996	2001	2006	2011
Pop	21.57	22.99	24.34	25.31	27.30	28.85	30.01	31.61	33.48



Interpolation

- Given points (x_i, y_i), for $i=0:n$, find a function $f(x)$ such that $f(x_i) = y_i$.
- Why?
  - Answer questions about values in the range
  - Answer questions about derivatives

Interpolation

- Issues to consider regarding f(x)?
  – Cost of finding f
  – Difficulty of evaluation
  – Cost of evaluation of f
  – Continuity
  – Estimation of derivatives

→We will use polynomials of degree:

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n$$

Vandermonde Matrix

$$\begin{bmatrix} 1 & x_0 & {x_0}^2 & \cdots & {x_0}^n \\ 1 & x_1 & {x_1}^2 & \cdots & {x_1}^n \\ 1 & x_2 & {x_2}^2 & \cdots & {x_2}^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & {x_n}^2 & \cdots & {x_n}^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$ → Va = y

It can be shown that

$$\det(V) = \prod_{0 \le i \le j \le n} \left(x_i - x_j \right)$$

Note: det(V) = 0 ↔ $x_i = x_j$ for some i≠j

So, this system has a solution for distinct x_j values.

Revisiting population

Consider the data:

Years since 1995	1	6	11	16
Pop (millions)	28.85	30.01	31.61	33.48

Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 6 & 36 & 216 \\ 1 & 11 & 121 & 1331 \\ 1 & 16 & 256 & 4096 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 28.85 \\ 30.01 \\ 31.61 \\ 33.48 \end{bmatrix}$$

$$p(x) = -0.000227x^3 + \mathbf{0.01288}x^2 + 0.1516x + 28.6858$$

Solving for coefficients a

- System has n+1 equations and n+1 unknowns
- Can solve using GEPP or find QR factors
- However,
  - Either approach required $O(n^3)$ operations
  - $\kappa_2(V)$ grows exponentially → Ill-conditioned system
- Let's consider a faster approach which is not as sensitive to condition of V.

Motivating example

- Suppose we have points (x_0,y_0) and (x_1,y_1)
- Consider $p(x) = a_0 + a_1x$
  - $p(x_0) = a_0 + a_1x_0 = y_0$
  - $P(x_1) = a_0 + a_1x_1 = y_1$
- Solve the 2x2 system:
  - $a_0 = (y_1-y_0)/(x_1-x_0)$
  - $a_1 = (x_1y_0-x_0y_1)/(x_1-x_0)$

Rewriting p(x) gives

$$p(x) = \frac{(x - x_1)}{(x_0 - x_1)} y_0 + \frac{(x - x_0)}{(x_1 - x_0)} y_1$$

$$p(x) = \ell_0(x) y_0 + \ell_1(x) y_1$$

where $\ell_{\mathbf{0}}$, $\ell_{\mathbf{1}}$ (degree-1 polynomials) are called the Lagrange polynomials

→p(x) is a sum of degree-1 polynomial

→In this format, we don't need to solve a system of equations – just calculate the coefficients directly.

Lagrange polynomials for larger n

For k=0:n, define

$$\ell_k(\boldsymbol{x}) = \prod_{j=0, j\neq k}^{n} \frac{x - x_j}{x_k - x_j}$$

The interpolating polynomial of degree n+1 is given by

$$p_n(x) = \sum_{k=0}^{n} \ell_k(x) y_k$$

Lagrange basis

- Let $P_n(x) = \{y_n(x): y_n\ (x)$ is a poly of deg $\leq n\}$
- $P_n(x)$ is a vector space with the basis

  $B = \{1,x,x_2,\ldots,x_n\}$.
- It is not the only basis

  $B_\ell = \{\ell_0(x), \ell_1(x), \ldots, \ell_n(x)\}$ is also a basis
- Any polynomial in $P_n(x)$ can be written as a linear combination of $\ell_k(x)$ in B_ℓ
- The interpolating polynomial is unique – same polynomial regardless of way it is calculated.
- How many flops are required?