6.4 Momentum Operator

6.4.1 Momentum and Translation

Momentum is linked to velocity, that is, to a shift in position. So let us start with a position shift (translation). This type of relations has been studied and is well known in Classical Mechanics!
We define the Translation Operator which be define by its action onto position eigenstates:

$$\boxed{\hat{T}(a)|x\rangle = |x+a\rangle}$$

This operator has the following properties:

$$\hat{T}(0) = \mathbb{1} \quad \text{and} \quad \hat{T} \quad \text{unitary}$$

So, similar to our discussion of the time evolution operator, we can do an expansion of this operator:

$$\hat{T}(\delta_x) = \mathbb{1} - \frac{i}{\hbar}\hat{p}\ \delta_x$$

Where $\hat{p}$ is a self-adjoint operator, and the factors are chosen for convenience, as we did in the definition of the Hamilton operator.
The physical dimension of this operator is that of momentum!

So we define the momentum as the operator effecting the infinitesimal change of a quantum state in infinitesimal translation

Motivation Momentum Operator

time evolution:

$$\hat{U}(t)|\Psi(t_0)\rangle = |\Psi(t_0+t)\rangle$$

$$\hat{U}(\delta t) = \mathbb{1} - \frac{i}{\hbar}\hat{H}\ \delta t$$

time independent Hamilton operator:

$$\hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t}$$

Displacement:

$$\hat{T}(a)|x\rangle = |x+a\rangle$$

$$\hat{T}(\delta x) = \mathbb{1} - \frac{i}{\hbar}\hat{P}\ \delta x$$

momentum operator

momentum operator
(time independent)

$$\hat{T}(a) = e^{-\frac{i}{\hbar}\hat{P}a}$$

6.4.2 Coordinate representation of the momentum operator

$$\langle x|\hat{p}|x'\rangle = \langle x|\frac{-i\hbar}{\delta_x}(\mathbb{1}-T(\delta_x))|x'\rangle$$

$$= \frac{\hbar i}{\delta_x}\left(-\langle x|x'\rangle + \langle x|x'+\delta_x\rangle\right)$$

$$= \hbar i \,\cancel{\frac{\langle x|x'+\delta_x\rangle - \langle x|x'\rangle}{\delta_x}}$$

$$x+(-\delta_x)-x'$$

$$= \frac{\hbar}{i}\,\frac{\delta(x-x'-\delta x) - \delta(x-x')}{(-\delta_x)}$$

$$= \frac{\hbar}{i}\frac{d}{dx}\delta(x-x')$$

Usage: $\langle\hat{p}\rangle = \langle\psi|\hat{p}|\psi\rangle$

$$= \int dx \int dx'\, \psi^*(x)\frac{\hbar}{i}\frac{d}{dx}\delta(x-x')\,\psi(x')$$

$$= \int dx\, \psi^*(x)\frac{\hbar}{i}\frac{d}{dx}\int dx'\,\delta(x-x')\,\psi(x')$$

$$= \int dx\, \psi^*(x)\frac{\hbar}{i}\frac{d}{dx}\psi(x)$$

6.4.3 Eigenstates of the Momentum Operator

6.4.3.1 Basic Property

We are looking for states |p> that have the property

$$\hat{p}|p\rangle = p|p\rangle \qquad p\in\mathbb{R}$$

6.4.3.2 Coordinate representation of momentum eigenstates

To determine $\langle x|p\rangle$ we observe that

$$\langle x|\hat{p}|p\rangle = p\langle x|p\rangle$$ as |p> is an eigenstates

on the other hand, we have from the coordinate representation of the operator $\hat{p}$

$$\langle x|\hat{P}|p\rangle = \int dx' \, \delta(x-x') \frac{d}{dx'}\frac{\hbar}{i} \langle x'|p\rangle$$

$$= \frac{\hbar}{i}\frac{d}{dx}\langle x|p\rangle$$

therefore the position representation of |p> has to satisfy the differential equation

$$p\langle x|p\rangle = \frac{\hbar}{i}\frac{d}{dx}\langle x|p\rangle$$

which has the solution

$$\langle x|p\rangle = N \exp\left[\frac{ipx}{\hbar}\right]$$

This is the wave function of a momentum eigenstates!

Note that it cannot be normalized to something like $\int dx \, |\psi(x)|^2 dx = 1$
since we find

$$|\langle x|p\rangle|^2 = |N|^2$$

So momentum eigenstates are again examples of kets that do not correspond directly to physical states (they can do so only in some limit that we will discuss later. Still, we can fix the normalization constant by observing that

$$\langle p'|p\rangle = |N|^2 \, 2\pi\hbar \, \delta(p-p')$$

The proof of this is given at the end of this lecture notes, and involve a special representation of the Dirac delta-function. (Not part of lecture material.)
We choose to set

$$N = \frac{1}{\sqrt{2\pi\hbar}}$$

so that we then have we have

$$\boxed{\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left[\frac{ipx}{\hbar}\right]}$$

Note:

We can compare this wave function of the position eigenstate with the general form of a (spatial) wave

$$\exp\left[2\pi \frac{x}{\lambda}\right]$$

from which we find the wavelength relation

$$\lambda = \frac{h}{p}$$

which is exactly what de Broglie postulated!

Our choice of the normalization also results in the following relationships (without proof):

completeness:

$$\mathbb{1} = \int_{-\infty}^{+\infty} dp \; |p\rangle\langle p|$$

Operator representation:

$$\hat{p} = \int_{-\infty}^{+\infty} dp \; p \; |p\rangle\langle p|$$

We see that the operators P and X are very similar in structure!

Position Operator

$$\hat{X} = x|x\rangle\langle x|$$

eigenvector:

$$\hat{X}|x\rangle = x|x\rangle$$

(position) coordinate representation:

$$\hat{X} \doteq x\,\delta(x - x')$$

$$|x\rangle \doteq \Psi(x') = \langle x'|x\rangle = \delta(x - x')$$

Momentum Operator

$$\hat{P} = \int_{-\infty}^{+\infty} dp \;\; p|p\rangle\langle p|$$

eigenvector:

$$\hat{P}|p\rangle = p|p\rangle$$

(position) coordinate representation:

$$\hat{P} \doteq (-i)\hbar \frac{d}{dx}\,\delta(x - x')$$

$$|p\rangle \doteq \Psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \exp\left[\frac{ipx}{\hbar}\right]$$

normalization determined by overlap structure: (omitted in lectures)

$$\langle p'|p\rangle = \int dx \, \langle p'|x\rangle\langle x|p\rangle$$

$$= \int dx \, \mathcal{N}^* \exp\left[-\frac{ip'x}{\hbar}\right] \mathcal{N} \exp\left[\frac{ipx}{\hbar}\right]$$

$$= \int dx \, |\mathcal{N}|^2 \exp\left[i\frac{(p-p')x}{\hbar}\right]$$

$$\boxed{\int dx \, \exp\left[i\frac{(p-p')x}{\hbar}\right] = 2\pi\hbar \, \delta(p-p')}$$

(Representation of Dirac delta function)

$$= |\mathcal{N}|^2 \, 2\pi\hbar \, \delta(p-p')$$

$\Rightarrow$ choose: $\mathcal{N} = \frac{1}{\sqrt{2\pi\hbar}}$

$$\Rightarrow \langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left[\frac{ipx}{\hbar}\right]$$

6.5 Commutator and Heisenberg Uncertainty for Position and Momentum Operators

6.5.1 Commutator properties for Position and Momentum operator

$$\hat{X}|\psi\rangle = \int dx \, |x\rangle\langle x|\hat{X}|\psi\rangle$$

$$= \int dx \, x \, \psi(x) \, |x\rangle$$

$$\doteq \; x\psi(x)$$

$$\left[= \int dx' \; x\,\delta(x-x')\,\psi(x') \right]$$

$$\hat{p}|\psi\rangle = \int dx \int dx' \, |x\rangle\langle x|\hat{p}_x|x'\rangle\langle x'|\psi\rangle$$

$$= \int dx \int dx' \, \frac{\hbar}{i}\frac{d}{dx}\delta(x-x')\,\psi(x')\;|x\rangle$$

$$= \int dx \;\frac{\hbar}{i}\frac{d}{dx}\psi(x)\,|x\rangle$$

$$\doteq \frac{\hbar}{i}\frac{d}{dx}\psi(x)$$

$$\Rightarrow \hat{x}\hat{p}|\psi\rangle \doteq x\frac{\hbar}{i}\frac{d}{dx}\psi(x)$$

$$\hat{p}\,\hat{x}|\psi\rangle \doteq \frac{\hbar}{i}\frac{d}{dx}\left(x\psi(x)\right)$$

$$= \frac{\hbar}{i}\psi(x) + \frac{\hbar}{i}x\frac{d}{dx}\psi(x)$$

$$\Rightarrow (\hat{x}\hat{p} - \hat{p}\hat{x})|\psi\rangle \doteq i\hbar\,\psi(x) \quad \forall\,|\psi\rangle$$

Then using the definition of the translation operator in terms of the momentum operator, we find: $\boxed{[\hat{x},\hat{p}] = i\hbar}$

6.5.2 Uncertainty Relation

$$\boxed{\Delta A \, \Delta B \geq \frac{1}{2} \left| \langle \psi | [\hat{A}, \hat{B}] | \psi \rangle \right|}$$

Heisenberg Uncertainty Relation

Specialization: $\hat{A} \to \hat{x}$

$\hat{B} \to \hat{p}$

$$\Rightarrow \boxed{\Delta \hat{x} \, \Delta \hat{p} \geq \frac{1}{2} \hbar}$$ independent of the state!

Interpretation:

There is no source which in a position measurement would give a sharp value ($\Delta x = 0$) and at the same time gives a sharp value in a momentum measurement ($\Delta p = 0$).
(Quite in contrast to classical mechanics.)

Why can't we put a particle exactly at one spot? $|\psi\rangle \to |x\rangle$

In that case, $\Delta \hat{x} \to 0$ and therefore $\Delta \hat{p} \to \infty$

But: wide distribution of momentum will lead to diverging energy!
==> A particle at one point would have infinite energy
==> moreover, with large momentum, it could be anywhere a short time later

Position Eigenstates are not physical states, but are limits of a family of physical states.

Similarly, Momentum Eigenstates are not physical states, but limits of a family of physical states