EXAMPLE: CRITICALLY-DAMPED HARMONIC OSCILLATOR: FIND THE GENERAL SOLUTION OF THE HOMOGENEOUS EQ. $y''+2y'+y=0$.

SOLUTION: ASSUMING A SOLUTION OF THE FORM $y=Ce^{rx}$, THE CHARACTERISTIC EQUATION $r^2+2r+1=0$ HAS A DOUBLE-ROOT AT $r=-1$.

ONE OF THE HOMEGENEOUS SOLUTIONS IS $y_1(x)=C_1e^{-x}$; WHAT ABOUT THE OTHER? IDEA: USE THE WRONSKIAN!

WE WANT THE OTHER LINEARLY-INDEPENDENT SOLUTION. USE ABEL'S IDENTITY WITH $W_0 \neq 0$:

$$W[y_1,y_2](x) = y_1y_2' - y_2y_1' = W_0 \exp\left[-\int_{x_0}^{x} P(x')\,dx'\right]$$

WITH $y_1 = C_1e^{-x}$ AND $P(x')=2$;

$$y_2' = y_2\frac{(-C_1e^{-x})}{C_1e^{-x}} + \frac{W_0}{C_1}e^{x}\underbrace{e^{-\int_{x_0}^{x}2\,dx'}}_{e^{-2(x-x_0)}}$$

FIRST-ORDER LINEAR D.E. FOR y_2 $\longrightarrow$ $y_2' = -y_2 + C_3e^{-x}$

THE INTEGRATING FACTOR $I(x) = e^{\int dx} = e^{x}$ AND THE FULL SOLUTION IS

$$y_2 = e^{-x}\left[\int e^{x}C_3e^{-x}\,dx + C_4\right] = e^{-x}\left[C_3x + C_4\right]$$

$$= C_3xe^{-x} + \underbrace{C_4e^{-x}}$$

CAN BE COMBINED WITH y_1

ALTOGETHER,

$$\boxed{y_H(x) = \underbrace{C_1e^{-x}}_{\text{EASY}} + \underbrace{C_2xe^{-x}}_{\text{USE THE WRONSKIAN}}}$$

THIS STRATEGY OF USING THE WRONSKIAN (& ABEL'S IDENTITY) TO FIND A SECOND LINEARLY-INDEPENDENT SOLUTION BY SOLVING A DIFFERENTIAL EQUATION OF ~~SE~~ LOWER-ORDER IS CALLED: '<u>REDUCTION-OF-ORDER</u>'. IT HAS SOME ANALOGIES WITH THE GRAM-SCHMIDT PROCESS FOR TO PRODUCE AN ORTHONORMAL BASIS IN LINEAR ALGEBRA

IN THE EXAMPLE ABOVE, WE USED A SPECIFIC $y_1(x)$; BUT THE METHOD CAN BE GENERALLY APPLIED. START WITH ABEL'S IDENTITY TO GENERATE A LINEAR FIRST-ORDER DIFFERENTIAL EQUATION, THEN USE AN INTEGRATING FACTOR TO ARRIVE AT THE FULL SOLUTION (SHOW THIS.)

$$y_2(x) = y_1(x) \int_{x_0}^{x} \frac{1}{y_1^2(x')} \exp\left[-\int_{x_0}^{x'} P(x'')\,dx''\right] dx'$$

REDUCTION-OF-ORDER SOLUTION FOR A SECOND-ORDER HOMOGENEOUS EQUATION

VARIATION OF PARAMETERS

IT IS USEFUL TO THINK OF THE DIFFERENTIAL EQUATION AS A LINEAR OPERATOR $\mathcal{L}$ ACTING ON THE FUNCTION $y(x)$:

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x) \iff \mathcal{L}(x)\cdot y(x) = R(x)$$

WHERE
$$\mathcal{L}(x) \equiv \left[\frac{d^2}{dx^2} + P(x)\frac{d}{dx} + Q(x)\right]$$

AND THE LINEARLY-INDEPENDENT HOMOGENEOUS SOLUTIONS $y_1(x)$ & $y_2(x)$ AS THE BASIS VECTORS ASSOCIATED WITH THE LINEAR OPERATOR $\mathcal{L}(x)$. IT SHOULD COME AS NO SURPRISE THAT WE CAN CONSTRUCT A PARTICULAR SOLUTION $y_p(x)$ IN TERMS OF THE HOMOGENEOUS SOLUTIONS- IN ANALOGY WITH LINEAR ALGEBRA, WE WRITE,

$$y_p(x) = V_1(x)\,y_1(x) + V_2(x)\,y_2(x)$$

AND DETERMINE THE WEIGHTING FUNCTIONS V_1 & V_2. THIS APPROACH IS CALLED 'VARIATION OF PARAMETERS'.

TAKING A DERIVATIVE,

$$y_p' = V_1 y_1' + V_1' y_1 + V_2 y_2' + V_2' y_2$$

WE CAN SIMPLIFY THE ALGEBRA BY SETTING $V_1' y_1 + V_2' y_2 = \varphi(x)$. IN FACT, WE CAN SIMPLIFY THE ALGEBRA A LOT BY SETTING

$$\boxed{V_1' y_1 + V_2' y_2 = 0} \quad (1)$$

THEN, THE FIRST TWO DERIVATIVES OF y_p ARE,

$$y_p' = v_1 y_1' + v_2 y_2' \quad \text{AND} \quad y_p'' = v_1 y_1'' + v_2 y_2'' + v_1' y_1' + v_2' y_2'$$

SUBSTITUTING INTO THE ORIGINAL DIFFERENTIAL EQ., AND USING THE FACT THAT $y_1(x)$ & $y_2(x)$ SOLVE THE HOMOGENEOUS EQ,

$$y_p'' + P(x)\,y_p' + Q(x)\,y_p = R(x)$$

$$v_1 \underbrace{\left[y_1'' + P y_1' + Q y_1\right]}_{0} + v_2 \underbrace{\left[y_2'' + P y_2' + Q y_2\right]}_{0} + v_1' y_1' + v_2' y_2' = R(x)$$

$$\boxed{v_1' y_1' + v_2' y_2' = R(x)} \quad (2)$$

~~COMBING~~ COMBINING EQS (1) & (2) IN MATRIX-VECTOR FORM:

BECAUSE y_1 & y_2 ARE LINEARLY INDEPENDENT, $W[y_1,y_2] \neq 0$, AND THIS MATRIX IS INVERTIBLE!

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ R(x) \end{bmatrix}$$

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \underbrace{\frac{1}{W[y_1,y_2](x)} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}}_{\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1}} \begin{bmatrix} 0 \\ R(x) \end{bmatrix}$$

OR,

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \frac{R(x)}{W[y_1,y_2](x)} \begin{bmatrix} -y_2 \\ y_1 \end{bmatrix}$$

INTEGRATING EACH COMPONENT TO DETERMINE $v_1(x)$ & $v_2(x)$, THE FULL PARTICULAR SOLUTION IS:

$$y_p(x) = y_1(x) \underbrace{\int_{x_0}^{x} \frac{-y_2(x')\,R(x')}{W[y_1,y_2](x')}\,dx'}_{v_1(x)} + y_2(x) \underbrace{\int_{x_0}^{x} \frac{y_1(x')\,R(x')}{W[y_1,y_2](x')}\,dx'}_{v_2(x)}$$

NOTICE THAT WE CAN WRITE THE PARTICULAR SOLUTION AS

$$y_p(x) = \int_{x_0}^{x} \underbrace{\frac{y_2(x)y_1(x') - y_1(x)y_2(x')}{W[y_1,y_2](x')}} R(x')\,dx' \quad (*)$$

THE KERNEL OF THIS INTEGRAL OPERATOR ON $R(x)$ IS CALLED THE 'GREEN'S FUNCTION' ASSOCIATED WITH THE LINEAR DIFFERENTIAL OPERATOR,

$$G(x,x') = \frac{y_2(x)y_1(x') - y_1(x)y_2(x')}{W[y_1,y_2](x')}$$

THE INTEGRAL $(*)$ CAN THEN BE THOUGHT OF AS THE 'INVERSE' OF THE LINEAR DIFFERENTIAL OPERATOR, ACTING ON THE INHOMOGENEOUS FUNCTION $R(x)$:

$$\underbrace{\left[\frac{d^2}{dx^2} + P(x)\frac{d}{dx} + Q(x)\right]}_{\mathcal{L}(x)} y_p(x) = R(x)$$

$$\mathcal{L}(x)y_p(x) = R(x) \quad \Longleftrightarrow \quad y_p(x) = \int_{x}^{x_0} G(x,x')R(x')\,dx'.$$

THE CONNECTION BETWEEN LINEAR ALGEBRA & DIFFERENTIAL EQS RUNS DEEP; YOU WILL SEE MORE IN COURSES ON PARTIAL DIFFERENTIAL EQS (eg. AMATH 353), QUANTUM MECHANICS (eg. AM 373 / AM 473), AND FUNCTIONAL ANALYSIS (eg. AMATH 731).

EXERCISE • SHOW THAT $\left[\frac{d^2}{dx^2} + P(x)\frac{d}{dx} + Q(x)\right]\int_{x_0}^{x} G(x,x')R(x')\,dx' = R(x).$

• SHOW THAT $\int_{x_0}^{x} G(x,x')\left[\frac{d^2}{dx'^2}y_p + P(x')\frac{d}{dx'}y_p + Q(x')y_p\right]dx' = y_p(x).$

//.

EXAMPLE $x^2y'' - 2xy' + 2y = x^3\sin x$

$y_1 = x$ AND $y_2 = x^2$ ARE INDEPENDENT SOLUTIONS OF THE HOMOGENEOUS EQ ON AN INTERVAL NOT CONTAINING $x=0$.

RE-WRITING IN STANDARD FORM:

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = x\sin x$$

AND $W[y_1,y_2](x) = 2x^2 - x^2 = x^2$. USING OUR EXPRESSION FOR THE PARTICULAR SOLUTION:

$$y_p = \underbrace{x}_{y_1} \cdot \underbrace{\int_{x_0}^{x} -x'^2 \frac{(x'\sin x')}{x'^2}\,dx'}_{v_1} + \underbrace{x^2}_{y_2} \underbrace{\int_{x_0}^{x} \frac{x' \cdot (x'\sin x')}{x'^2}\,dx'}_{v_2}$$

~~AT THE LOWER LIMIT OF INTEGRATION,~~

$$= x \int_{x_0}^{x} -x'\sin x'\,dx' + x^2 \int_{x_0}^{x} \sin x'\,dx'$$

$$= x\,[x\cos x - \sin x \underbrace{- x_0\cos x_0 + \sin x_0}_{\text{CONSTANT } C_1}] + x^2[-\cos x + \underbrace{\cos x_0}_{\text{CONSTANT } C_2}]$$

$$= x^2\cos x - x\sin x - C_1 x - x^2\cos x + C_2 x^2$$

$$= -x\sin x \underbrace{- C_1 x + C_2 x^2}_{\text{HOMOGENEOUS SOLUTIONS.}}$$

THE LOWER-LIMIT ($x' = x_0$) WILL ALWAYS PRODUCE A LINEAR COMBINATION OF THE HOMOGENEOUS SOLUTIONS. THESE CAN SIMPLY BE COMBINED WITH THE GENERAL SOLUTION. ALTOGETHER, THE PARTICULAR SOLUTION IS:

$$y_p(x) = -x\sin x.$$

TRY IT: $y_p' = -x\cos x - \sin x$ & $y_p'' = -2\cos x + x\sin x$.

SO, $x^2 y_p'' - 2x y_p' + 2y_p = \cancel{[-x^3\cos x - x^2\sin x]}$

$$= (\cancel{-2x^2\cos x} + x^3\sin x) + (\cancel{2x^2\cos x} + \cancel{2x\sin x}) - \cancel{2x\sin x}$$

$$= x^3 \sin x \checkmark$$

EXERCISE: $y'' + y = \sec x$ $(0 \le x < \pi/2)$. GIVEN $y_1 = \sin x$ & $y_2 = \cos x$

SOLUTION: $y_p = x\sin x + \cos x \ln(\cos x)$

POWER SERIES SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

RECALL ONE OF THE MOST IMPORTANT RESULTS FROM FRESHMAN CALCULUS - POWER SERIES REPRESENTATION OF FUNCTIONS. THESE ARE INFINITE-ORDER POLYNOMIALS,

$$\sum_{n=0}^{\infty} \underbrace{C_n}_{\text{CONSTANTS.}} (x-x_0)^n = C_0 + C_1(x-x_0) + \ldots$$

WE SAY THE SERIES CONVERGES IF THE SEQUENCE OF PARTIAL SUMS $S_N = \sum_{n=0}^{N} C_n (x-x_0)^n$ CONVERGES AS $N \to \infty$,

$$\lim_{N\to\infty} S_N = \sum_{n=0}^{\infty} C_n (x-x_0)^n$$

THE INTERVAL ABOUT x_0 FOR WHICH THE SERIES CONVERGES IS CALLED THE 'RADIUS OF CONVERGENCE' ρ.

ie.
FOR $|x-x_0|<\rho$, THE POWER SERIES CONVERGES ABSOLUTELY, AND BEHAVES, FOR ALL INTENTS & PURPOSES, LIKE AN ORDINARY FUNCTION. IN PARTICULAR,

IF $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ & $\sum_{n=0}^{\infty} b_n(x-x_0)^n$ CONVERGE TO $f(x)$ & $g(x)$, RESPECTIVELY, FOR $|x-x_0|<\rho$, THEN (FOR $|x-x_0|<\rho$),

i) $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x-x_0)^n$ [LIKEWISE, MULTIPLICATION AND DIVISION]

ii) SERIES CAN BE DIFFERENTIATED & INTEGRATED TERM-WISE,

eg/ $\frac{df}{dx} = \sum_{n=1}^{\infty} n \cdot a_n (x-x_0)^{n-1}$ & $\int f(x)dx = C + \sum_{n=0}^{\infty} \frac{a_n}{(n+1)}(x-x_0)^{n+1}$

A FUNCTION THAT IS EQUAL TO A POWER SERIES IN A NEIGHBOURHOOD OF x_0 IS SAID TO BE <u>ANALYTIC</u> AT x_0, AND THE COEFFICIENTS OF THE POWER SERIES ARE GIVEN BY TAYLOR'S FORMULA:

$$C_n = \frac{f^{(n)}(x_0)}{n!}$$

BEING ANALYTIC IMPLIES BEING INFINITELY-DIFFERENTIABLE, BUT THE OPPOSITE IS NOT TRUE. NEVERTHELESS, POLYNOMIALS & RATIONAL FUNCTIONS ARE ANALYTIC ON POINTS FOR WHICH THEY ARE DEFINED.

SOME FAMOUS EXAMPLES-

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \dots \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots \qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \dots$$

A VERY CLEVER APPROACH TO THE SOLUTION OF ORDINARY DIFFERENTIAL EQUATION IS TO ASSUME A POWER-SERIES FORM FOR THE SOLUTION, THEN SOLVE FOR THE COEFFICIENTS. THAT IS, MAKE A TAYLOR SERIES EXPANSION BEFORE YOU KNOW WHAT THE FUNCTION IS. IN SOME CASES, THE SERIES IS RECOGNIZABLE AND CAN BE CONVERTED INTO A CLOSED-FORM, BUT OFTEN THE POWER-SERIES ITSELF DEFINES A NEW 'SPECIAL FUNCTION'.

EX. $y' = y \ ; \ y(0) = 1$

ASSUME $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$ AND USE THE DIFFERENTIAL EQ. TO FIND CONSTRAINTS THAT DETERMINE a_n. DIFFERENTIATING TERM-WISE,

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

TO 'SOLVE' THE DIFFERENTIAL EQUATION, WE NEED TO CHOOSE COEFFICIENTS $a_0, a_1, a_2, \dots$ SO THAT

$$\underbrace{a_1 + 2a_2 x + 3a_3 x^2 + \dots + (n+1) a_{n+1} x^n + \dots}_{y'} = \underbrace{a_0 + a_1 x + \dots + a_n x^n + \dots}_{y}$$

THIS EQUATION IS SATISFIED FOR ALL X IF THE COEFFICIENTS OF X^n MATCH; THAT ~~DE~~ GENERATES AN INFINITE SYSTEM OF EQUATIONS,

$$\begin{aligned} a_1 &= a_0 \\ 2a_2 &= a_1 \\ 3a_2 &= a_2 \\ &\vdots \\ (n+1) a_{n+1} &= a_n \end{aligned} \qquad \text{OR,} \quad \boxed{a_{n+1} = \frac{a_n}{(n+1)} \quad n = 0, 1, 2 \dots}$$

APPLY THE RECURSIVE CONSTRAINT: $a_{n+1} = \frac{a_n}{(n+1)} = \frac{1}{(n+1)} \frac{a_{n-1}}{n} = \dots = \frac{a_0}{(n+1)!}$

SO THE SOLUTION IS:

$$y = \sum_{n=0}^{\infty} a_0 \frac{x^n}{n!}$$

FROM THE INITIAL CONDITION, $y(0) = 1 \Rightarrow a_0 = 1$