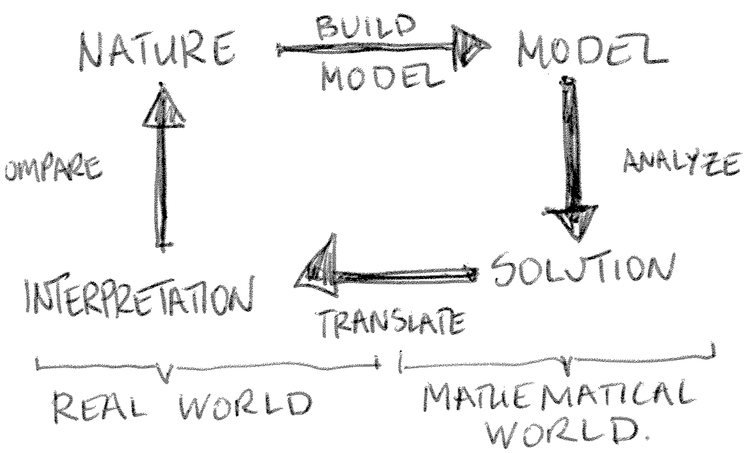
AMATH 351 - APPLIED ORDINARY DIFFERENTIAL EQUATIONS

MATHEMATICS IS THE LANGUAGE OF SCIENCE & ENGINEERING; IT IS USED TO ENCODE THE REGULARITY WE SEE IN NATURE, AND ALLOWS US TO GAIN SOME PREDICTIVE CONTROL OVER OUR ENVIRONMENT.

THE WAY THIS GOES IN PRACTICE IS TO IDENTIFY CONSERVATION PRINCIPLES OPERATING ON SOME VARIABLE OF INTEREST, THEN USE MATHEMATICS TO PREDICT HOW THE CONSERVED QUANTITY IS TRANSFORMED IN SPACE & TIME.

THAT IS VERY ABSTRACT, AND WE'LL LOOK AT CONCRETE EXAMPLES IN A MOMENT, BUT FIRST LETS LOOK AT MATHEMATICAL MODEL BUILDING IN THE ABSTRACT. THE PROCESS IS OFTEN DEPICTED AS A 'CYCLE'.



IN THIS COURSE, WE'RE GOING TO FOCUS PRIMARILY ON THE MATHEMATICAL SIDE OF THINGS

$$\text{MODEL} \xrightarrow{\text{ANALYSIS}} \text{SOLUTION} \xrightarrow{\text{TRANSLATE}} \text{INTERPRET}$$

BUT PROBABLY THE MOST IMPORTANT STEP IS THE INITIAL ABSTRACTION:

$$\text{NATURE} \xrightarrow[\text{MODEL}]{\text{BUILD}} \text{MODEL}$$

THE CYCLE, AS I'VE DRAWN IT, SEEMS TO IMPLY THAT THE PROCESS IS ALGORITHMIC, AND THAT A MACHINE COULD DO IT. FIT THIS OR THAT PARAMETER TO A UNIVERSAL TEMPLATE MODEL, THEN ITERATE UNTIL THE PHENOMENON IS REPRODUCED. BUT THERE IS SOMETHING MORE SUBTLE GOING ON HERE: TRANSLATING WHAT WE SEE, WHAT WE EXPERIENCE INTO LANGUAGE IS EXACTLY WHAT POETS, PAINTERS & COMPOSERS DO. IT IS A DISTILLATION OF AN EXPERIENCE COMMON TO EVERYONE, ALTHOUGH NEVER QUITE ARTICULATED. THE MOST GIFTED ARTISTS & SCIENTISTS ARE THOSE WHO CAN IDENTIFY PROFOUND EXPERIENCES & ARTICULATE THEM WELL.

AS A COUNTER POINT TO THIS MODEL BUILDING 'CYCLE' KEEP IN MIND NELSON GOODMAN'S DESCRIPTION OF SCIENCE:

WE AIM FOR SIMPLICITY, WE HOPE FOR TRUTH.	OR, AS APPLIED MATHEMATICIANS	WE AIM FOR SIMPLICITY, WE HOPE FOR UTILITY.

HAVING SAID ALL OF THAT, THEN WHY FOCUS ON THE MATHEMATICAL SIDE? TWO REASONS-

FIRST, WE MUST NOT BE LIMITED BY TECHNIQUE. IMAGINE THE DIFFICULTY IN WRITING A POEM IN A FOREIGN LANGUAGE. OR TRY TO IMAGINE A MONSTER; A TOTALLY FANTASTIC BEAST. WHAT DOES IT LOOK LIKE? IF YOU DECOMPOSE IT, YOU'LL FIND IT IS A COLLAGE OF ANIMALS YOU HAVE SEEN BEFORE. IN APPLIED MATHEMATICS, TOO, OUR MODELS ARE LIMITED BY OUR EXPERIENCE & BY THE TECHNICAL VOCABULARY WE DEVELOP IN THIS COURSE.

SECOND, IT'S EASIER TO TEACH MATHEMATICAL TECHNIQUE THAN IT IS TO TEACH MODEL BUILDING. HOW TO CHOOSE APPROPRIATE STATE VARIABLES, HOW TO IDENTIFY CONSERVATION PRINCIPLES, THESE ARE ACHEIVED BY TRIAL-AND-ERROR AND A GREAT DEAL OF LUCK. LET'S LOOK AT SOME HISTORICAL SUCCESSES.

NEWTONIAN MECHANICS: THE GENIUS OF NEWTON WAS TO IDENTIFY A CONSERVED QUANTITY (LINEAR MOMENTUM) AND A WAY TO QUANTIFY THE CHANGE IN LINEAR MOMENTUM (FORCES):

$$\text{RATE OF CHANGE OF LINEAR MOMENTUM} = \underbrace{\text{RATE OF LINEAR MOMENTUM INCREASE} - \text{RATE OF LINEAR MOMENTUM DECREASE}}_{\text{'FORCES' ACTING ON THE SYSTEM}}$$

LINEAR MOMENTUM IS MASS x VELOCITY.

FOR CONSTANT MASS 'm', THE RATE OF CHANGE IN LINEAR MOMENTUM IS MASS x ACCELERATION, OR $\underline{m \cdot a = F}$. DENOTING THE POSITION OF THE OBJECT BY $x(t)$, WE NATURALLY ARRIVE AT A DIFFERENTIAL EQ.

$x(t)$	dx/dt	d^2x/dt^2
POSITION	VELOCITY	ACCELERATION

ie.

$$\underbrace{\boxed{m\frac{d^2x}{dt^2} = \text{FORCES ACTING ON THE OBJECT.}}}_{\text{NEWTON'S 2}^{\text{nd}}\text{ LAW}}$$

FIGURE THESE OUT EMPIRICALLY (BY EXPERIMENT) FOR ANY CONTEXT YOU CAN IMAGINE.

EXAMPLE: ROBERT HOOKE (WHO HAD A MUTUAL HATRED OF NEWTON) MEASURED THE FORCE OF A SPRING ACTING ON A MASS DISPLACED FROM EQUILIBRIUM. SPECIFICALLY, BY HANGING WEIGHTS OF A SPRING HE NOTED TWO WEIGHTS PRODUCED TWICE THE STRETCH OF ONE WEIGHT; IN MODERN NOTATION WE WRITE: $F = -kx$ WHERE $x(t)$ IS THE DISTANCE OF THE MASS MEASURED FROM EQUILIBRIUM $x(t)=0$. COMBINING THIS EXPRESSION WITH NEWTON'S 2^{nd} LAW:

$$m\frac{d^2x}{dt^2} = -kx \quad \text{OR} \quad \frac{d^2x}{dt^2} + \omega^2 x = 0$$

WE CALL THIS MODEL THE 'HARMONIC OSCILLATOR'

FREQUENCY OF THE HARMONIC OSCILLATOR $\omega^2 = k/m$.

NOTICE, HOOKE COULD EMPIRICALLY MEASURE 'k' FOR A GIVEN SPRING, BUT WITH NEWTON'S LAW, YOU CAN CONCLUDE THAT THE FREQUENCY OF OSCILLATION WILL BE RELATED TO MASS AS: $\omega \propto 1/\sqrt{m}$.

EXAMPLE: NEWTON HIMSELF DEDUCED THE MATHEMATICAL FORM FOR THE FORCE OF GRAVITY. IF AN OBJECT OF MASS 'm' IS LAUNCHED FROM THE SURFACE OF THE EARTH AT VELOCITY v_0,

$$m\frac{d^2x}{dt^2} = \frac{-gmR^2}{(x+R)^2}$$

$$x(0)=0 \qquad \frac{dx}{dt}(0) = v_0$$

WHERE 'g' IS THE GRAVITATIONAL ACCELERATION, 'R' IS THE RADIUS OF THE EARTH & $x(t)$ IS THE HEIGHT ABOVE THE SURFACE.

THERE IS NO KNOWN SOLUTION TO THIS EQUATION! IF $x(t) \ll R$, THEN THE EQUATION REDUCES TO $d^2x/dt^2 = -g$ OR, $x(t) = -gt^2/2 + v_0 t$ WHICH IS A PARABOLIC PATH.

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EPIDEMIOLOGY: THE SPREAD OF DISEASE. HERE THE FUNDAMENTAL CONSTRAINT IS ON THE PARTITIONING OF THE POPULATION INTO 'TYPES': SUSCEPTIBLES 'S', INFECTED 'I' AND RECOVERED 'R'. SUPPOSE UPON RECOVERY, THE INDIVIDUAL BECOMES IMMUNE; THEN SUSCEPTIBLES BECOME INFECTED AT A RATE PROPORTIONAL TO THEIR MUTUAL CONTACT.

$$\frac{dS}{dt} = -\beta \cdot S \cdot I \qquad \frac{dI}{dt} = \beta \cdot S \cdot I - \gamma \cdot I \qquad \frac{dR}{dt} = \gamma \cdot I$$

RATE OF INFECTION

RATE OF RECOVERY

NOTICE THAT WE HAVE ASSUMED $S+I+R=N$ (CONSTANT) [Q: HOW IS THIS ASSUMPTION MANIFEST IN THE MODEL?] ANALYSIS OF THE MODEL REVEALS THAT $R_0 = N \cdot \frac{\beta}{\gamma}$ IS AN IMPORTANT PARAMETER. IF $R_0 > 1$, THEN THE MODEL EXHIBITS AN 'EPIDEMIC' STEADY STATE. THE PARAMETER R_0 HAS A STRAIGHTFORWARD INTERPRETATION: IT IS THE NUMBER OF SECONDARY INFECTIONS ONE CASE DEVELOPS OVER ITS INFECTIOUS PERIOD. [Q: IS '$R_0>1$ LEADS TO EPIDEMIC' A TAUTOLOGY?].

$I'(0)>0$

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CLASSIFICATION OF ORDINARY DIFFERENTIAL EQUATIONS

ORDINARY DIFFERENTIAL EQUATIONS CONTAIN DERIVATIVES, BUT ONLY OF A SINGLE VARIABLE FUNCTION, FOR EXAMPLE $y(x)$. THE HIGHEST-ORDER DERIVATIVE IN THE EQUATION IS CALLED THE 'ORDER' OF THE EQUATION. IN GENERAL, FOR AN n^{th}-ORDER ORDINARY DIFFERENTIAL EQUATION,

$$F\left(\frac{d^n y}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \ldots, \frac{dy}{dx}, y(x), x\right) = 0.$$

BUT IF $F(\cdot)$ IS LINEAR, WE CALL,

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

A 'LINEAR ORDINARY DIFFERENTIAL EQUATION.' THERE ARE SEVERAL ADDITIONAL QUALIFIERS:

i) IF $f(x)=0$, THE EQUATION IS 'HOMOGENEOUS'

ii) IF $a_n(x)=a_n$, $a_{n-1}(x)=a_{n-1}, \ldots, a_0(x)=a_0$ ARE ALL CONSTANT, THEN THE EQUATION IS 'AUTONOMOUS', OR SAID TO HAVE 'CONSTANT COEFFICIENTS'.

WE'LL START BY ANALYZING A GENERAL LINEAR SECOND-ORDER DIFFERENTIAL EQUATION:

$$\boxed{a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)}$$

YOU HAVE ALREADY SEEN SOLUTION METHODS IN THE CASE OF CONSTANT COEFFICIENTS (eg. LAPLACE TRANSFORMS) - HERE, WE'LL LOOK AT THE MORE GENERAL CASE. NOTICE WE CAN RE-WRITE THIS EQUATION IN A NUMBER OF WAYS -

1. GENERAL FORM: $a_2(x)\, y''(x) + a_1(x)\, y'(x) + a_0 y(x) = f(x)$

2. STANDARD FORM: THIS WILL BE OUR PREFERRED FORM FOR THE BEGINING OF THE COURSE -

$$\boxed{y''(x) + P(x)y'(x) + Q(x)y(x) = R(x)}$$

WHERE $P(x) = a_1(x)/a_2(x)$, $Q(x) = a_0(x)/a_2(x)$ & $R(x) = f(x)/a_2(x)$, AND $a_2(x) \neq 0$ [POINTS WHERE $a_2(x) = 0$ ARE CALLED 'SINGULAR' AND WE'LL TALK ABOUT THEM LATER IN THE COURSE...].

2b. ASSOCIATED HOMOGENEOUS EQUATION: SET $R(x) = 0$

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0 \quad (*)$$

2c. NORMAL FORM OF THE HOMOGENEOUS EQUATION: TRANSFORMATION OF $(*)$ TO NORMAL FORM IS VERY SIMILAR TO THE 'INTEGRATING FACTOR' YOU'VE USED TO SOLVE FIRST-ORDER LINEAR DIFFERENTIAL EQUATIONS. THE IDEA IS TO ELIMINATE THE FIRST-DERIVATIVE TERM.

WRITE ~~ASSUME THAT~~ $y(x) = u(x)v(x)$. SUBSTITUTING INTO THE HOMOGENEOUS EQUATION,

$$v \cdot u'' + [2v' + P(x)v]u' + [v'' + P(x)v' + Q(x)v]u = 0$$

WE CAN CHOOSE $v(x)$ TO ELIMINATE THE COEFFICIENT OF $u'(x)$:

$$2v' + P(x)v = 0$$

OR, $v' = -\frac{1}{2}P(x)v$ ← SEPARABLE!

SOLUTION IS:

$$v(x) = \exp\left[-\frac{1}{2}\int P(x')dx'\right]$$

WITH THIS CHOICE THE HOMOGENEOUS EQUATION REDUCES TO:

$$u''(x) + q(x)\,u(x) = 0 \quad \text{WITH} \quad q(x) = Q(x) - \frac{1}{4}P^2(x) - \frac{1}{2}P(x).$$

IF WE CAN SOLVE THIS EQUATION FOR $u(x)$, THEN THE FULL SOLUTION IS:

$$y(x) = \exp\left[-\frac{1}{2}\int P(x')\,dx'\right]\cdot u(x).$$

LINEAR SECOND-ORDER DIFFERENTIAL EQUATIONS CAN BE USEFULY BE CLASSIFIED WITH RESPECT TO HOW THE CONSTANTS OF INTEGRATION ARE SPECIFIED. THERE ARE TWO POSSIBILITIES:

1. <u>INITIAL VALUE PROBLEMS</u> (IVP) TWO CONDITIONS ARE IMPOSED AT THE <u>SAME</u> TIME (OR SAME LOCATION):

$$\text{eg.} \quad y(0) = \alpha \quad y'(0) = \beta.$$

2. <u>BOUNDARY VALUE PROBLEMS</u> (BVP): IN CONTRAST TO IVP, CONDITIONS ARE IMPOSED AT <u>DIFFERENT</u> TIMES (OR DIFFERENT LOCATIONS)

$$\text{eg.} \quad y(0) = \alpha \quad y(1) = \beta.$$

OUR FOCUS IN THIS COURSE WILL BE ON INITIAL VALUE PROBLEMS, ALTHOUGH BOUNDARY VALUE PROBLEMS ARISE OFTEN IN THE SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS (AMATH 353).

WE'VE LOOK AT SOME MATHEMATICAL MODELS; LET'S LOOK AT SOME EXAMPLE EQUATIONS.

1. <u>DAMPED HARMONIC OSCILLATOR</u>: AS A PHYSICAL MODEL, IT DESCRIBES THE MOTION OF A HARMONIC OSCILLATOR WITH ENERGY DISSIPATION (THROUGH $2\beta\frac{dx}{dt}$)

$$\frac{d^2x}{dt^2} + 2\beta\cdot\frac{dx}{dt} + \omega_0^2 x = 0$$

MATHEMATICALLY, SOMETHING INTERESTING HAPPENS WHEN $\beta = \omega_0$. (CRITICAL DAMPING).

2. EMDEN'S EQUATION: AS A PHYSICAL MODEL, IT DESCRIBES THE DENSITY & INTERNAL TEMPERATURE OF STARS. MATHEMATICALLY, WE'LL SEE IT AGAIN IN THE CONTEXT OF 'SERIES SOLUTIONS'.

$$xy'' + 2y' + xy = 0$$

$$y(0) = 1 \quad , \quad y'(0) = 0.$$

3. SEPARATION OF VARIABLES: ORDINARY DIFFERENTIAL EQUATIONS ARISE IN THE STUDY OF PARTIAL DIFFERENTIAL EQUATIONS, PARTICULARLY IN THE CONTEXT OF 'SEPARATION OF VARIABLES'.

FOR EXAMPLE, THE HEIGHT OF A ROUND DRUM SKIN $u(r,\theta,t)$ IS A FUNCTION OF TIME 't' AND THE POLAR COORDINATES r & θ. WHEN STRUCK, ITS MOTION APPROXIMATELY OBEYS THE 'WAVE EQUATION',

$$\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right]$$

IF WE TRY TO FIND SOLUTIONS OF THE FORM

$$u(r,\theta,t) = R(r)S(\theta)T(t),$$

THEN THE PARTIAL DIFFERENTIAL EQ. SEPARATES INTO A FAMILY OF ORDINARY DIFFERENTIAL EQUATIONS,

$$\underbrace{\begin{gathered} T'' + \omega^2 c^2 T = 0 \\ S'' + \alpha^2 S = 0 \end{gathered}}_{}$$

WHERE ω & α ARE CONSTANTS

AND

$$\boxed{r^2 R'' + rR' + (\omega^2 r^2 - \alpha^2)R = 0}$$

BESSEL'S EQUATION

NO SOLUTIONS IN TERMS OF ELEMENTARY FUNCTIONS – IN FACT, THIS EQUATION <u>DEFINES</u> A FAMILY OF 'SPECIAL FUNCTIONS' CALLED 'BESSEL FUNCTIONS'.

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LET'S GO BACK TO OUR GENERAL (NON-SINGULAR $a_2(x) \neq 0$) 2^{nd}-ORDER LINEAR DIFFERENTIAL EQUATION:

$$y'' + P(x)y' + Q(x)y = R(x)$$

OUR FOCUS WILL BE ON METHODS TO DETERMINE THE SOLUTION $y(x)$ [SUBJECT TO ~~BOUNDARY~~/INITIAL CONDITIONS] ON A FINITE INTERVAL $x \in [a,b]$.

THE CONDITIONS FOR EXISTENCE & UNIQUENESS OF THE SOLUTION $y(x)$ ARE STRAIGHTFORWARD:

IF $P(x)$, $Q(x)$ & $R(x)$ ARE CONTINUOUS ON $[a,b]$, THEN

$$y'' + P(x)y' + Q(x)y = R(x)$$

HAS ONE, AND ONLY ONE, SOLUTION $y(x)$ ON THE ENTIRE INTERVAL SATISFYING THE INITIAL CONDITIONS $y(x_0) = \alpha$ & $y'(x_0) = \beta$ [WHERE $x_0 \in [a,b]$ AND $\alpha, \beta \in \mathbb{R}$.]

BUT HOW DO WE FIND $y(x)$?

AS IN THE CASE OF CONSTANT COEFFICIENTS, WE BREAK THE SOLUTION UP AS A SUPERPOSITION OF

i) A GENERAL SOLUTION $y_H(x)$ OF THE HOMOGENEOUS EQUATION ($R(x)=0$) THAT SATISFIES THE INITIAL CONDITIONS, AND,

ii) A PARTICULAR SOLUTION $y_P(x)$ THAT SATISFIES THE INHOMOGENEOUS EQUATION ($R(x) \neq 0$).

ie. WE WRITE: $\boxed{y(x) = y_H(x) + y_P(x)}$

LET'S FIRST LOOK AT THE GENERAL SOLUTION $y_H(x)$:

SOLUTION OF THE ASSOCIATED HOMOGENEOUS EQUATION

THE GENERAL SOLUTION $y_H(x)$ FOR A 2^{nd}-ORDER LINEAR DIFFERENTIAL EQUATION IS COMPOSED OF A LINEAR COMBINATION OF LINEARLY-INDEPENDENT SOLUTIONS TO THE HOMOGENEOUS EQ:

$$y_H(x) = C_1 y_1(x) + C_2 y_2(x)$$

FOR CONSTANTS C_1 & C_2 CHOSEN TO SATISFY THE INITIAL CONDITIONS.

ASIDE: SUBSTITUTE $y_H(x)$ INTO THE HOMOGENEOUS EQ:

$$y_H'' + P(x)y_H' + Q(x)y_H = [C_1 y_1'' + C_2 y_2''] + P(x)[C_1 y_1' + C_2 y_2'] + Q(x)[C_1 y_1 + C_2 y_2]$$

$$= C_1[y_1'' + P(x)y_1' + Q(x)y_1] + C_2[y_2'' + P(x)y_2' + Q(x)y_2]$$

$$= 0 \quad \text{IF } y_1 \text{ \& } y_2 \text{ SATISFY THE HOMOGENEOUS D.E.}$$

WHY DO WE NEED LINEARLY-INDEPENDENT SOLUTIONS y_1 & y_2?
SO THAT WE CAN SATISFY ARBITRARY INITIAL CONDITIONS!
LINEAR-DEPENDENCE MEANS $y_1(x) = c \cdot y_2(x)$ FOR SOME $c \in \mathbb{R}$;
HOW DO WE CHECK FOR LINEAR INDEPENDENCE?

LINEAR INDEPENDENCE & THE WRONSKIAN

SUPPOSE WE WANT TO SATISFY THE INITIAL CONDITIONS $y_H(x_0) = \alpha$ AND $y_H'(x_0) = \beta$. THEN,

$$\left.\begin{aligned} C_1 y_1(x_0) + C_2 y_2(x_0) &= \alpha \\ C_1 y_1'(x_0) + C_2 y_2'(x_0) &= \beta \end{aligned}\right\} \text{SYSTEM OF TWO EQUATIONS WITH TWO UNKNOWNS } (C_1 \,\&\, C_2)$$

WE CAN SOLVE THIS SYSTEM IF THE MATRIX IS INVERTIBLE -

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

FOR INVERSION, THE MATRIX MUST BE NON-SINGULAR, i.e.

$$\det \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} = y_1(x_0) y_2'(x_0) - y_2(x_0) y_1'(x_0) \neq 0$$

AS A USEFUL SHORT-HAND, THIS DETERMINANT IS CALLED THE WRONSKIAN:

$$\boxed{W[y_1, y_2](x) = y_1(x) y_2'(x) - y_2(x) y_1'(x)}$$

TO SATISFY ARBITRARY INITIAL CONDITIONS, WE NEED $W[y_1, y_2](x_0) \neq 0$.
BUT IS $x = x_0$ A SPECIAL POINT? NO!

UNIFORMITY OF THE WRONSKIAN: IF $y_1(x)$ AND $y_2(x)$ ARE ANY TWO SOLUTIONS TO THE HOMOGENEOUS EQ: $y_H'' + P(x) y_H'(x) + Q(x) y_H = 0$ ON THE INTERVAL $x \in [a,b]$, THEN THEIR WRONSKIAN $W[y_1, y_2](x)$ IS EITHER ZERO EVERYWHERE OR ZERO NOWHERE ON $[a,b]$.

PROOF: DIFFERENTIATE THE WRONSKIAN,

$$\begin{aligned} W'(x) &= y_1 y_2'' + y_1' y_2' - y_2 y_1'' - y_2' y_1' \\ &= y_1 y_2'' - y_2 y_1'' \end{aligned}$$

SUBSTITUTE USING THE HOMOGENEOUS EQ: $y_i'' = -P(x)y_i' - Q(x)y_i \quad , i \in \{1,2\}$

$$W'(x) = y_1[-P(x)y_2' - Q(x)y_2] - y_2[-P(x)y_1' - Q(x)y_1]$$
$$= -P(x)[y_1y_2' - y_2y_1'] = -P(x)\cdot W(x)$$

BUT THIS IS A SEPARABLE FIRST-ORDER DIFFERENTIAL EQUATION FOR $W(x)$; THE SOLUTION IS:

$$\boxed{W(x) = W_0 \exp\left[-\int_{x_0}^{x} P(x')dx'\right]}$$

ABEL'S IDENTITY

THE EXPONENTIAL IS NEVER ZERO; SO THE WRONSKIAN $W(x)$ IS EITHER ZERO EVERYWHERE ($W_0 = 0$) OR NOWHERE (OTHERWISE). ∎

NOW WE CAN COME BACK TO A CLAIM MADE EARLIER & STATE AS A COROLLARY THAT TWO SOLUTIONS $y_1(x)$ AND $y_2(x)$ OF THE HOMOGENEOUS EQUATION ARE LINEARLY-DEPENDENT IF, AND ONLY IF, THEIR WRONSKIAN $W[y_1, y_2](x) = 0$.

AND, WE CAN PROVE THE MORE GENERAL CLAIM:

WITH INITIAL CONDITIONS $y(x_0) = \alpha$, $y'(x_0) = \beta$

> GIVEN A 2^{nd}-ORDER LINEAR DIFFERENTIAL EQ. $y'' + P(x)y' + Q(x)y = R(x)$, IF $y_H(x)$ IS THE <u>GENERAL SOLUTION</u> OF THE HOMOGENEOUS EQ. ($R(x) = 0$) AND $y_P(x)$ IS A <u>PARTICULAR SOLUTION</u> TO THE FULL EQ ($R(x) \neq 0$), THEN THE COMPLETE SOLUTION $y(x)$ IS A SUPERPOSITION OF THE TWO:
>
> $$y(x) = y_H(x) + y_P(x)$$

PROOF: THERE ARE TWO PARTS: SHOW THAT $y(x)$ SOLVES THE DIFFERENTIAL EQUATION AND THEN SHOW THAT $y(x)$ CAN SATISFY THE INITIAL CONDITIONS.

FIRST PART IS EASY - THE EQUATION IS LINEAR IN $y^{(n)}(x)$.

FOR THE SECOND PART, WRITE $y_H(x) = C_1 y_1(x) + C_2 y_2(x)$ WHERE $y_1(x)$ & $y_2(x)$ ARE LINEARLY-INDEPENDENT SOLUTIONS.

FOR INITIAL CONDITIONS $y(x_0) = \alpha$ & $y'(x_0) = \beta$, WE MUST SOLVE THE SYSTEM:

$$c_1 y_1(x_0) + c_2 y_2(x_0) = y(x_0) - y_p(x_0) \qquad c_1 y_1(x_0) + c_2 y_2(x_0) = \alpha - y_p(x_0)$$
$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = y'(x_0) - y_p'(x_0) \quad \text{OR,} \quad c_1 y_1'(x_0) + c_2 y_2'(x_0) = \beta - y_p'(x_0)$$

THE FORCING FUNCTION $R(x)$ [AND THE ASSOCIATED PARTICULAR SOLUTION $y_p(x)$] MODIFIES THE VALUES OF C_1 & C_2; BUT SO LONG AS y_1 & y_2 ARE LINEARLY-INDEPENDENT, THE SYSTEM IS SOLVABLE.

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{W[y_1,y_2](x_0)} \begin{bmatrix} y_2'(x_0) & -y_2(x_0) \\ -y_1'(x_0) & y_1(x_0) \end{bmatrix} \begin{bmatrix} \alpha - y_p(x_0) \\ \beta - y_p'(x_0) \end{bmatrix}$$

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LET'S LOOK AT SOME EXAMPLES OF HOW THE LINEARLY-INDEPENDENT SOLUTIONS $y_1(x)$ & $y_2(x)$ CAN BE USED TO SOLVE 2^{nd} ORDER DIFFERENTIAL EQUATIONS.

EXAMPLE. HARMONIC OSCILLATOR: SHOW THAT $y_H(x) = C_1 \sin x + C_2 \cos x$ IS THE SOLUTION TO THE HOMOGENEOUS HARMONIC OSCILLATOR

$y'' + y = 0$, AND FIND THE VALUES OF C_1 & C_2 SUCH THAT $y(0) = 2$ & $y'(0) = 3$.

SOLUTION: IT IS SIMPLE TO SHOW THAT BOTH $\sin x$ & $\cos x$ OBEY THE DIFFERENTIAL EQ. FURTHERMORE, THE WRONSKIAN,

$$W[y_1,y_2](x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1$$

IS NONZERO, SO $\sin x$ & $\cos x$ ARE LINEARLY-INDEPENDENT. TO FIND C_1 & C_2 FROM THE INITIAL CONDITIONS:

$$\begin{bmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{OR} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

THE MATRIX $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ IS ITS OWN INVERSE, SO

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{OR,} \quad \boxed{\begin{matrix} C_1 = 3 \\ C_2 = 2 \end{matrix}}$$

AND $y_H(x) = 3 \sin x + 2 \cos x$.