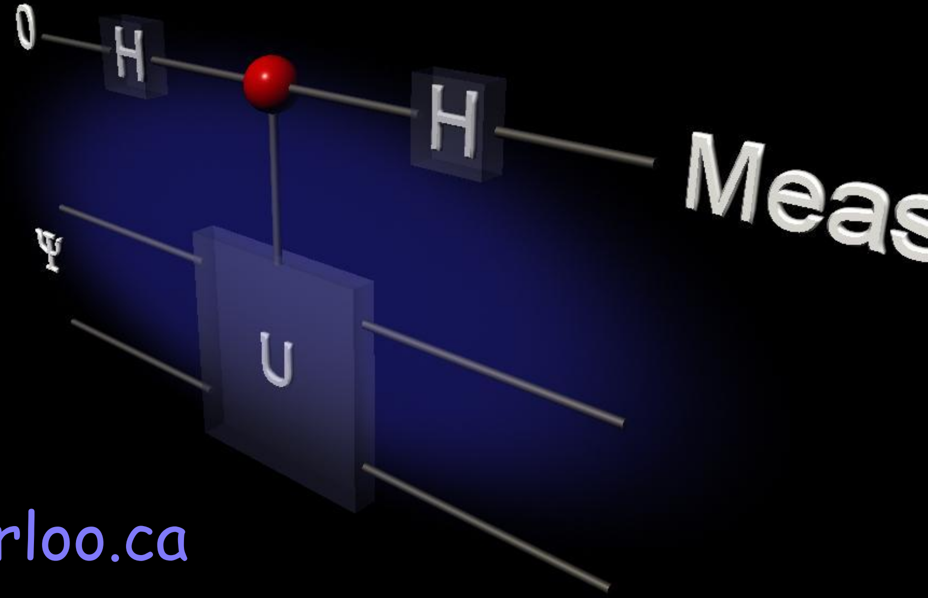
Introduction to Quantum Information Processing

CO481 CS467 PHYS467



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Tuesdays and Thursdays 10am-11:15am

Institute for Quantum Computing

UNIVERSITY OF WATERLOO

PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

Overview

Lecture 15

- Classical Error Correction
- Quantum Error Correction

Classical Error Correcting Codes

- Suppose errors in our physical system for storing 0 and 1 cause each physical bit to be toggled independently with probability p

- We can reduce the probability of error to be in $O(p^2)$ by using a "repetition code"

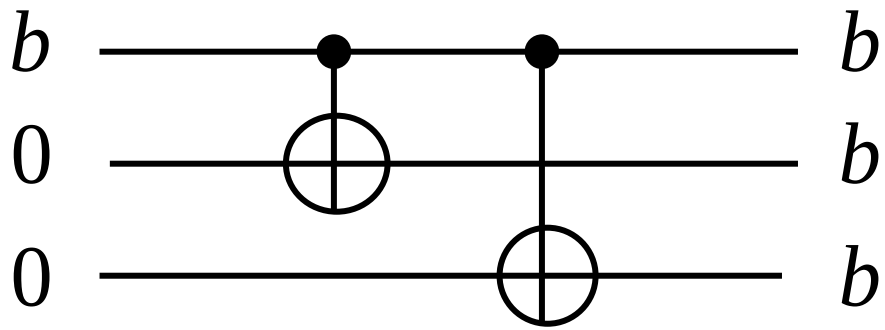
For example:

Encode logical 0 $\longrightarrow$ 000

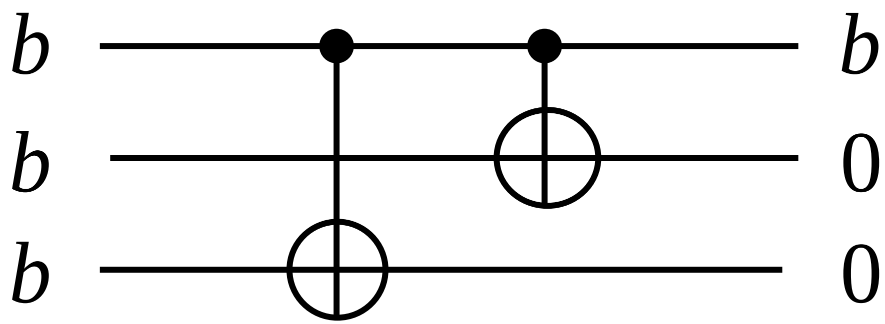
Encode logical 1 $\longrightarrow$ 111

Circuits for Encoding and Decoding

Encoding



Decoding



Classical Error Correcting Codes

- After the errors occur, decode the logical bits by taking the majority answer of the three bits and correct the encoded bits

- So

$$\boxed{\begin{array}{l} 000 \to 000 \\ 001 \to 000 \\ 010 \to 000 \\ 100 \to 000 \end{array}} \qquad \boxed{\begin{array}{l} 111 \to 111 \\ 011 \to 111 \\ 101 \to 111 \\ 110 \to 111 \end{array}}$$

Classical Error Correcting Codes

- As long as less than 2 errors occurred, we will keep the correct value of the logical bit

- The probability of 2 or more errors is

$$3p^2(1-p)+p^3=3p^2-2p^3\in O(p^2)$$

This is smaller than p provided $p<\frac{1}{2}$

Concatenation

- We have a procedure for reducing the effective error rate from

$$p \to cp^2$$

- If we then apply this error correction procedure to the logical bits, we can reduce the error rate from

$$p \to cp^2 \to c^3 p^4$$

- If we concatenate this procedure k times we reduce the effective error rate

$$p \to \frac{(cp)^{2^k}}{c}$$

Concatenation

- Thus, as long as $\mathrm{cp} < 1$, we can achieve an arbitrarily low error rate.

- Suppose we have a S-gate computation on n bits that we wish to perform with output error at most ε

- It suffices for each logical gate (at the top level of concatenation) to be implemented with effective error rate at most $\dfrac{\varepsilon}{S}$

Concatenation

- Thus we want $$\frac{(cp)^{2^k}}{c} \le \frac{\varepsilon}{S}$$

- This implies $$k \in O(\log\log(S/\varepsilon))$$

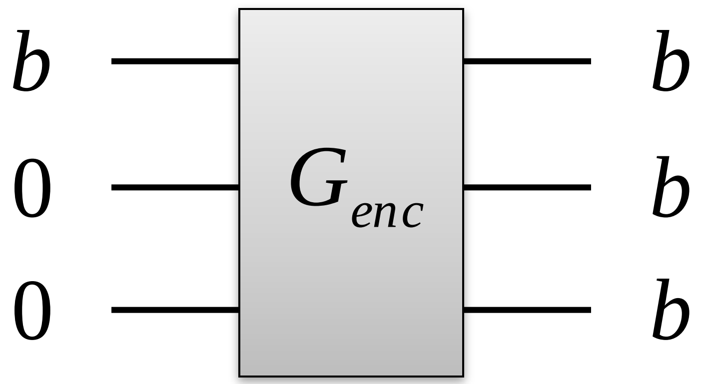
- The resulting encoded computation requires

$O(n3^k) = O(n\log^m(S/\varepsilon))$ **bits**, and

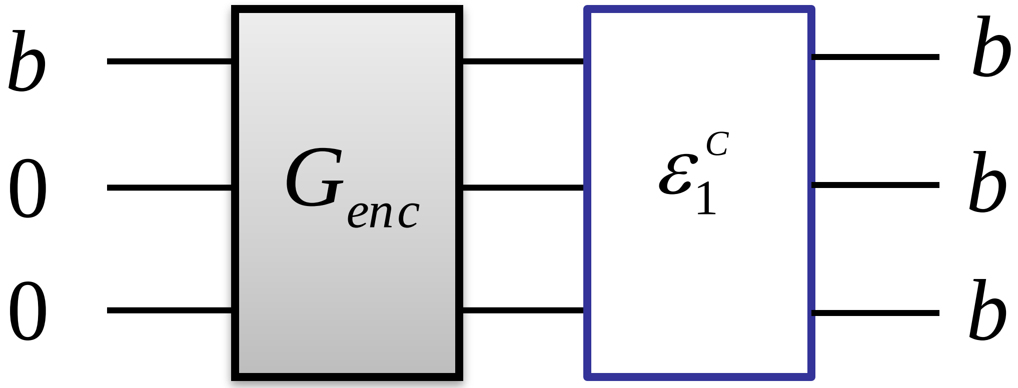
$O(S\cdot 3^k) = O(S\log^m(S/\varepsilon))$ **gates**

for some constant $m \ge 1$ *(This is a modest overhead)*

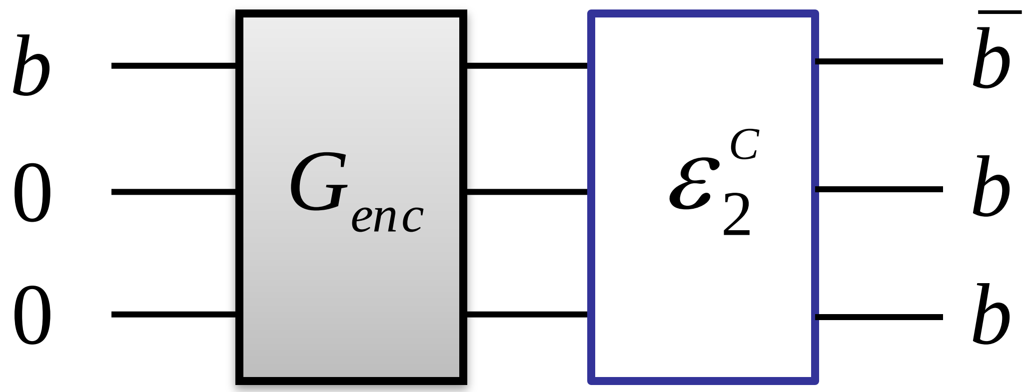
Circuits for Encoding and Decoding



Circuits for Encoding and Decoding

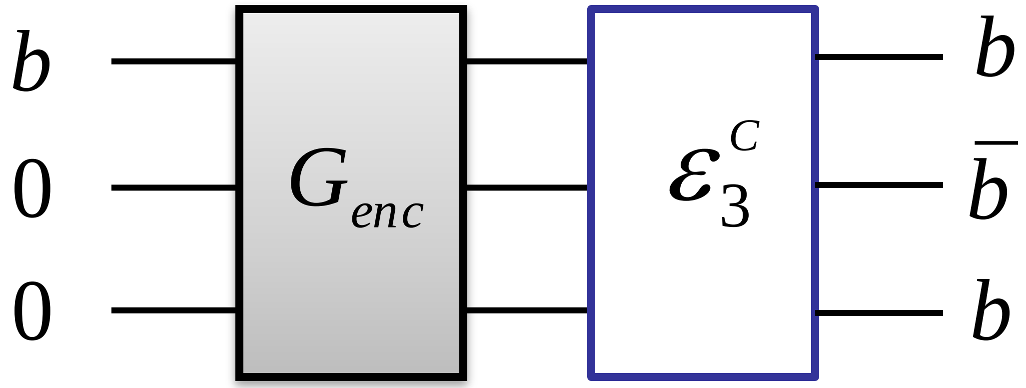


Circuits for Encoding and Decoding



$$\overline{0} = 1$$
$$\overline{1} = 0$$

Circuits for Encoding and Decoding



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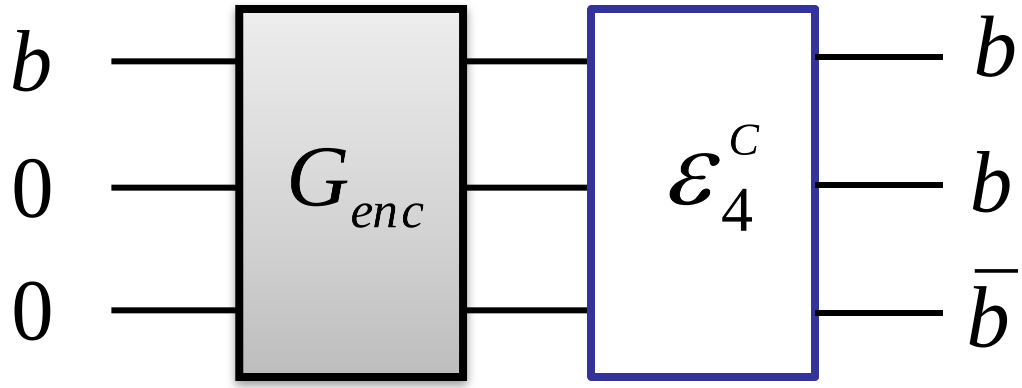
Circuits for Encoding and Decoding



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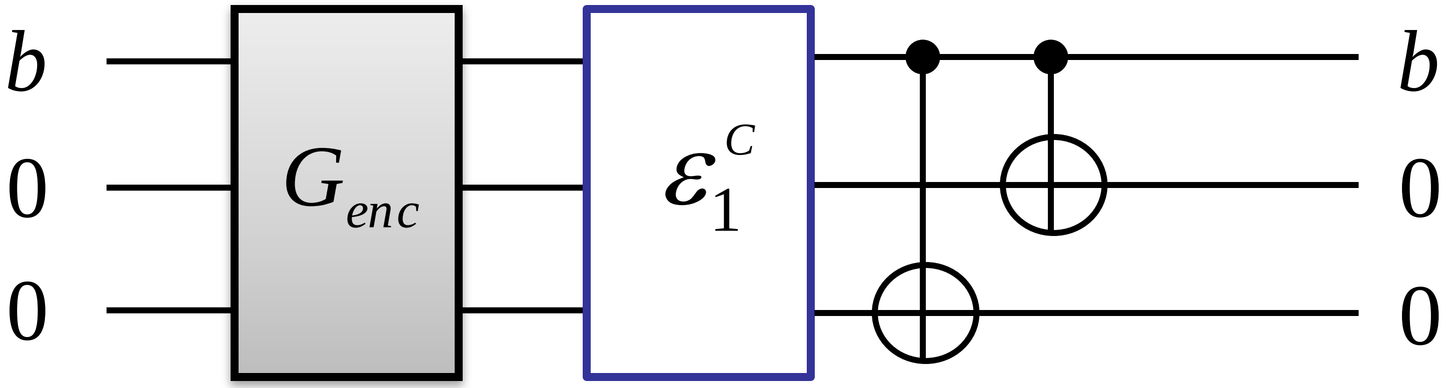
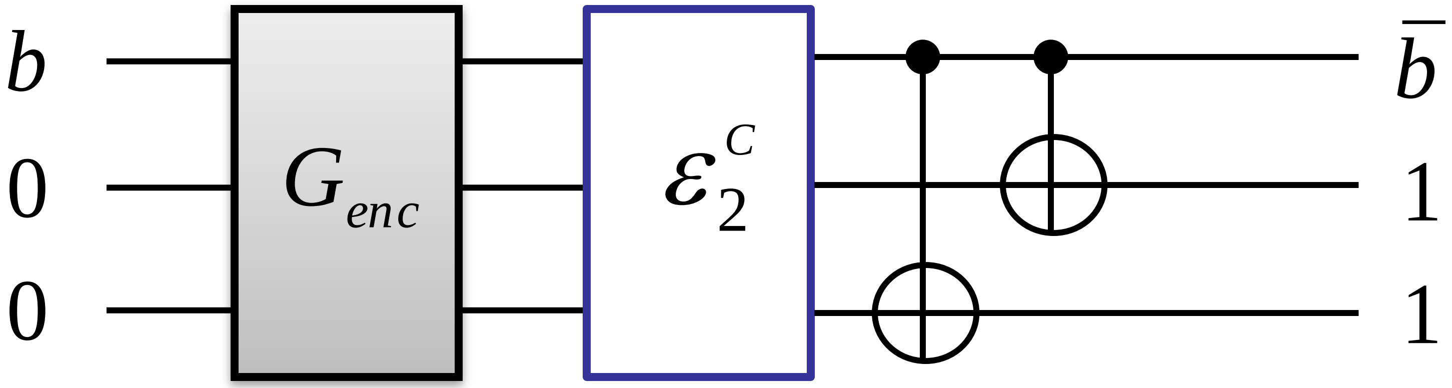
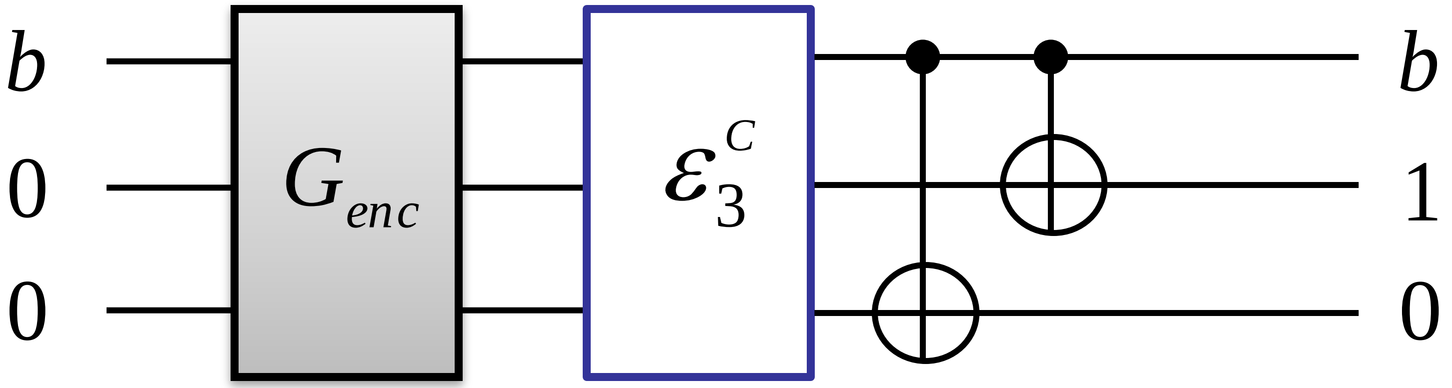
Circuits for Encoding and Decoding



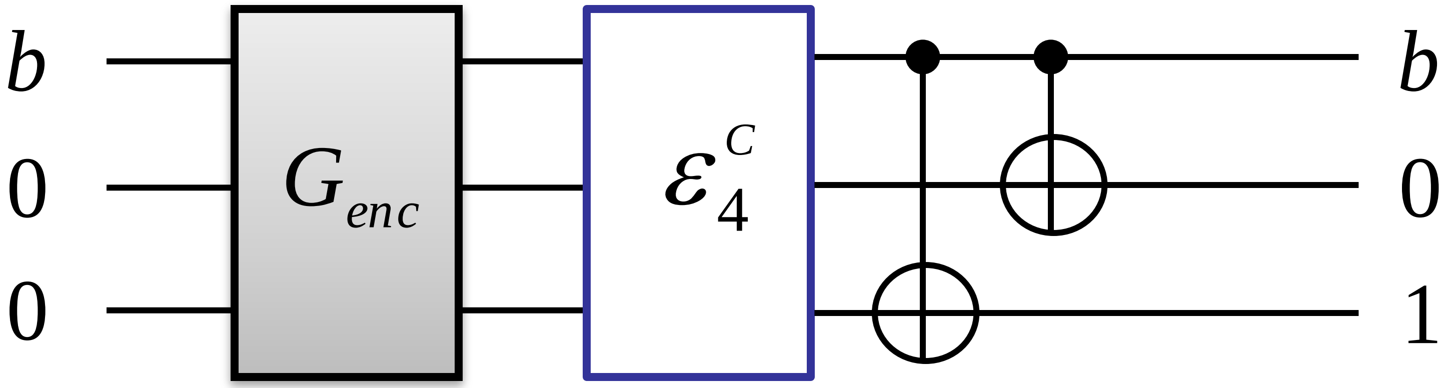
Circuits for Encoding and Decoding



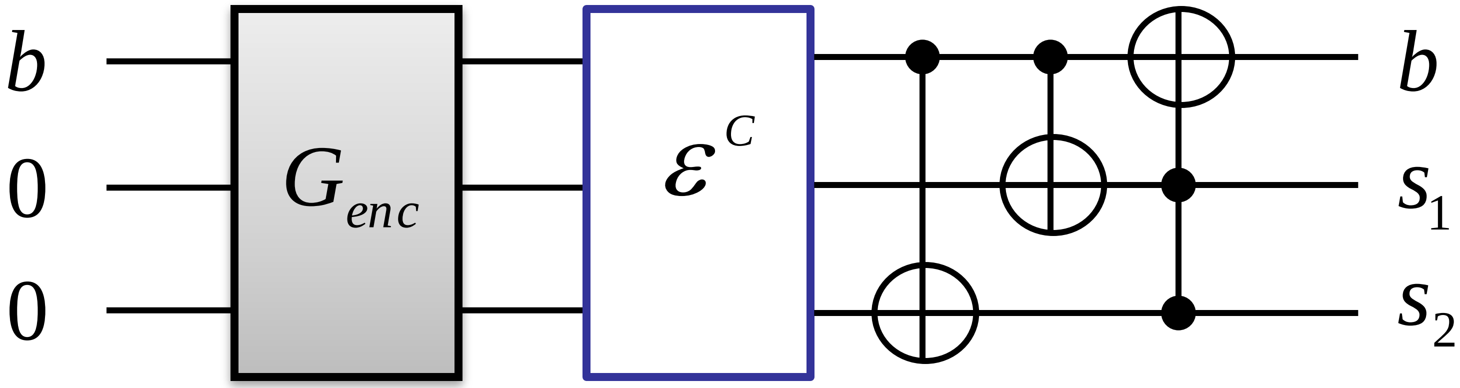
Circuits for Encoding and Decoding



Circuits for Encoding and Decoding

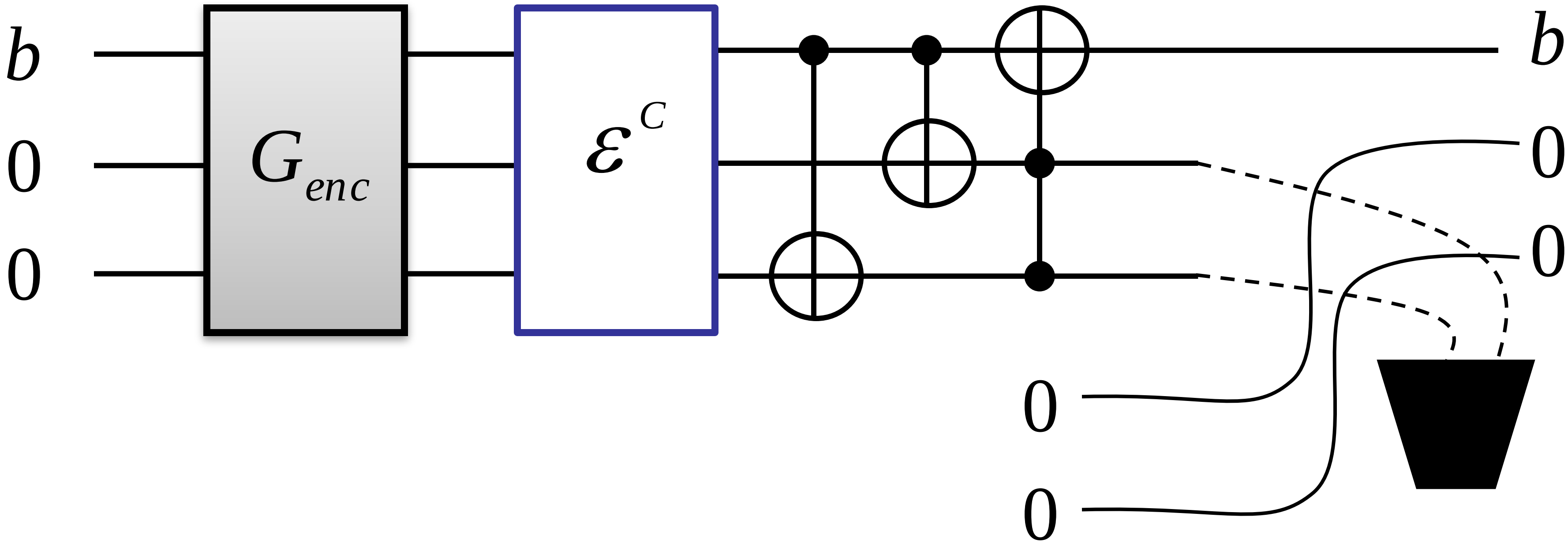


Circuits for Encoding and Decoding

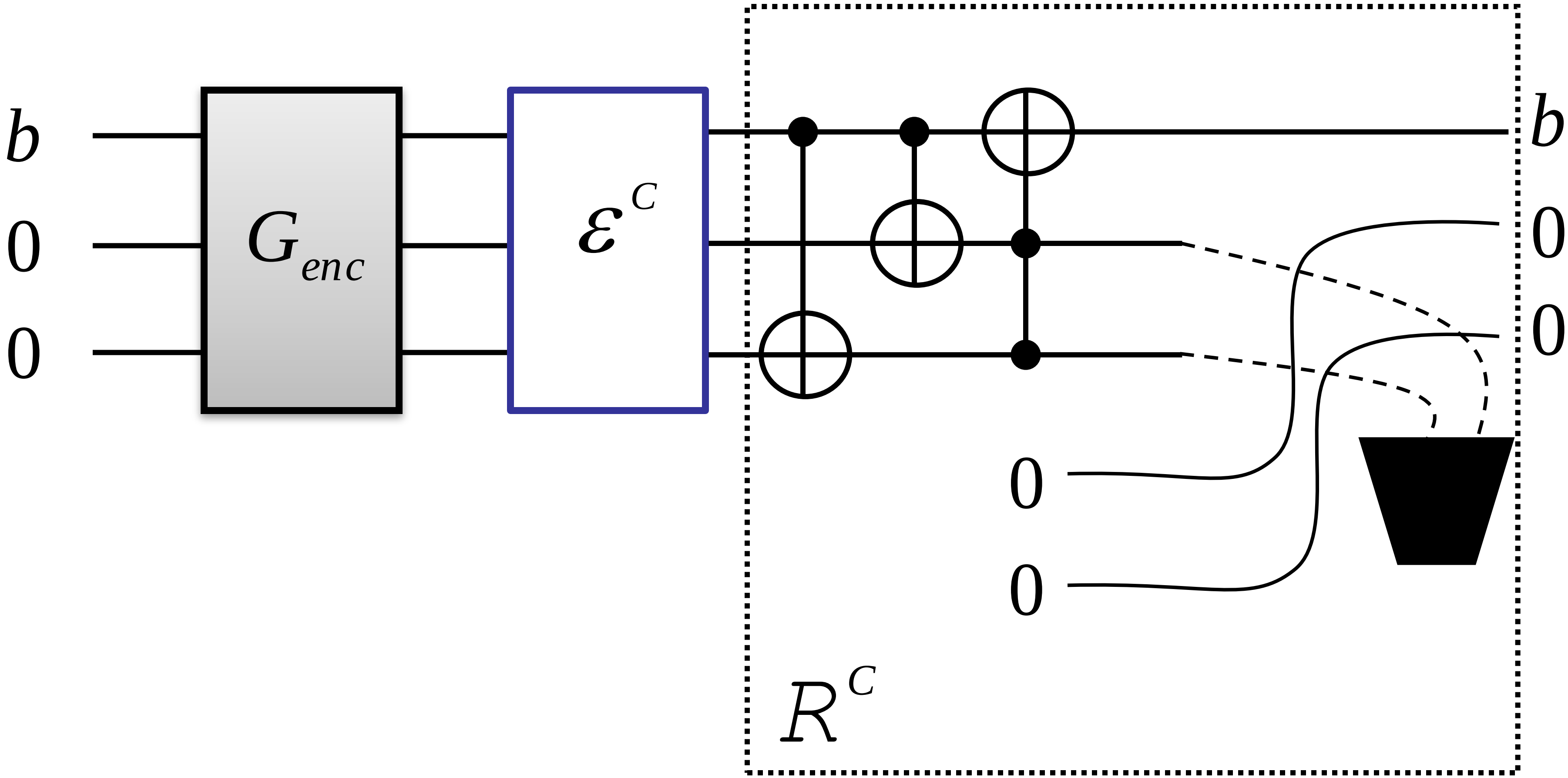


$$\mathcal{E}^C \in \{\mathcal{E}_1^C, \mathcal{E}_2^C, \mathcal{E}_3^C, \mathcal{E}_4^C\} = \{III, XII, IXI, IIX\}$$

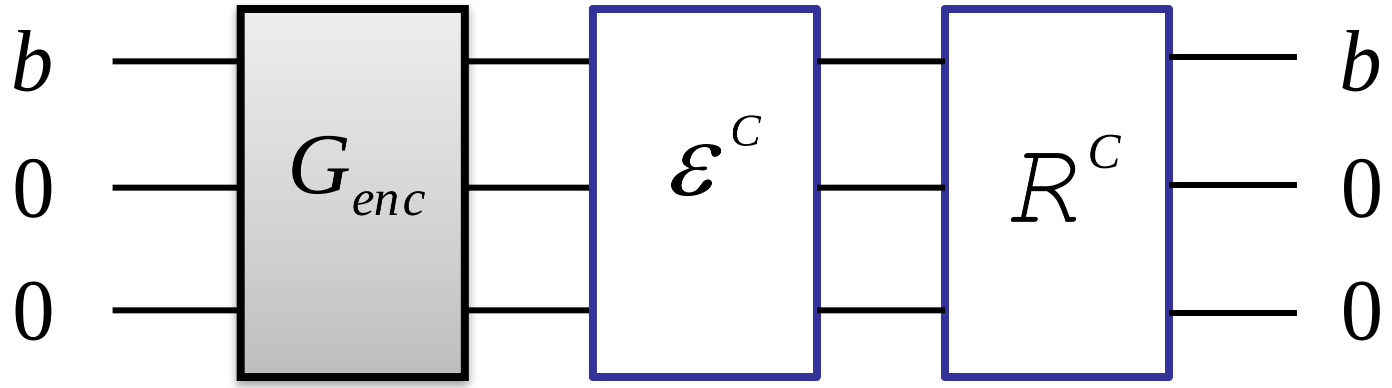
Circuits for Encoding and Decoding



Circuits for Encoding and Decoding



Circuits for Encoding and Decoding



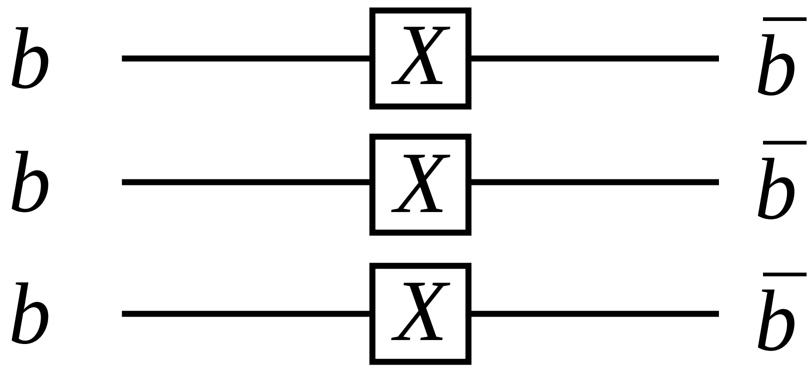
…towards fault-tolerance

- In order to perform a logic operation U, we cannot afford to decode, perform U, and then encode again (how would we correct any errors made during the encoding and decoding?)

- We need to perform the logic operations directly on the encoded states

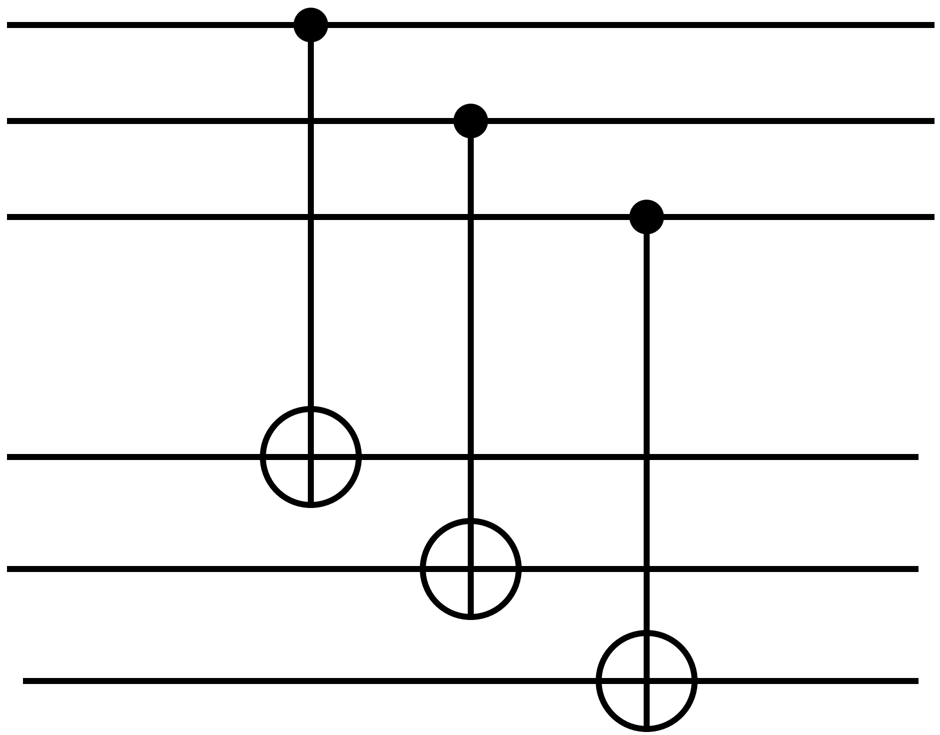
Perform operations on logical bits

- e.g. NOT gate



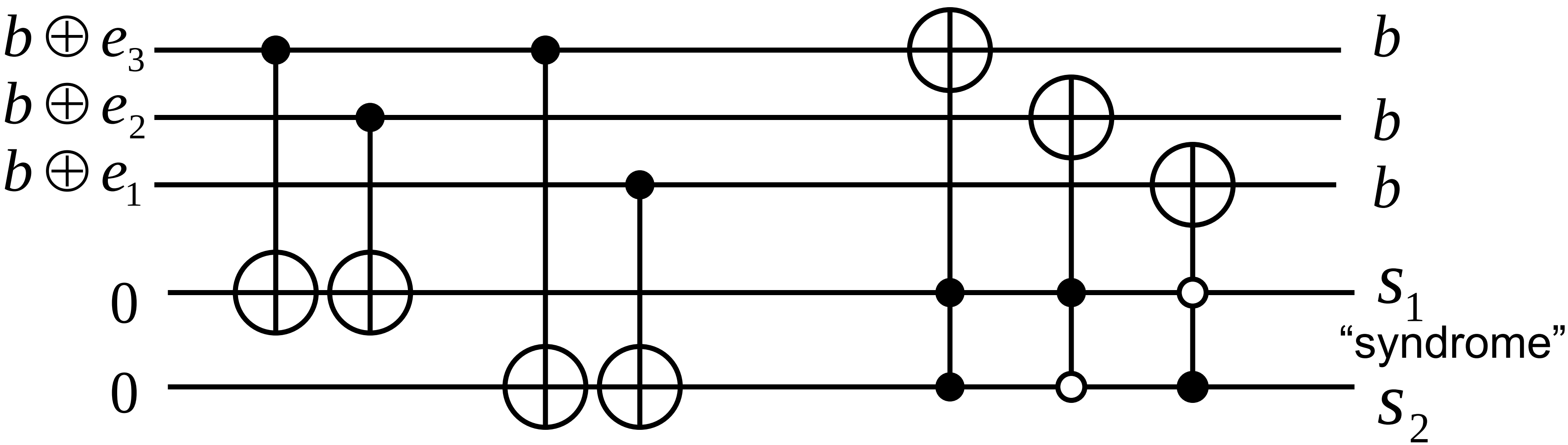
Perform operations on logical bits

- e.g. c-NOT gate



Reversible circuit for error correction on encoded bits

- Assume that at most one e_i equals 1



- If $s_1 s_2 = 00$ then no error occurred

- Otherwise, the error occurred in bit j where

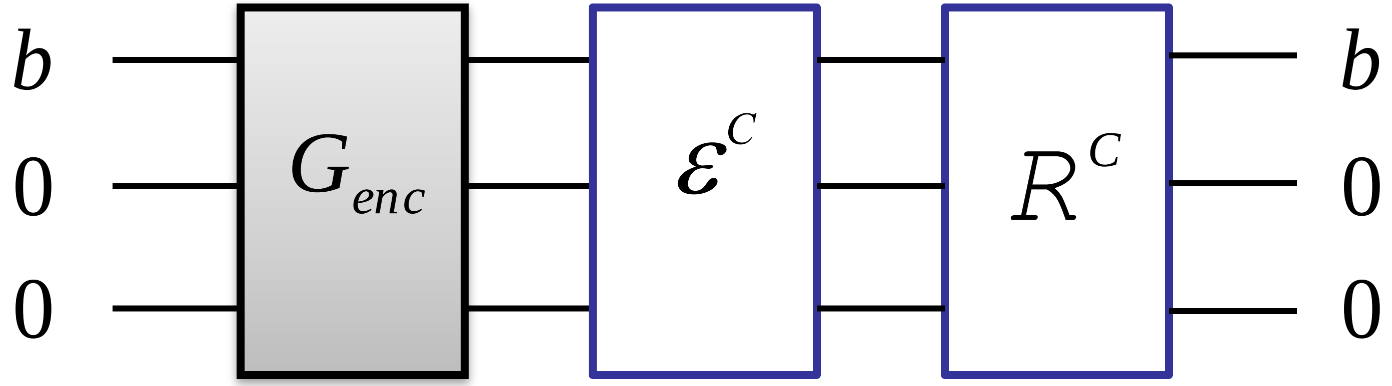
$$j = 2s_1 + s_2$$

More general perspective

- We have some encoding operation G_{enc} that maps the logical bit string b to the encoded string b_{enc}

- We have some error operation $\mathcal{E}^C$ acting on the bits encoding the logical string b_{enc}

- We have some recovery operation R^C



Necessary and sufficient condition for error correction to be possible

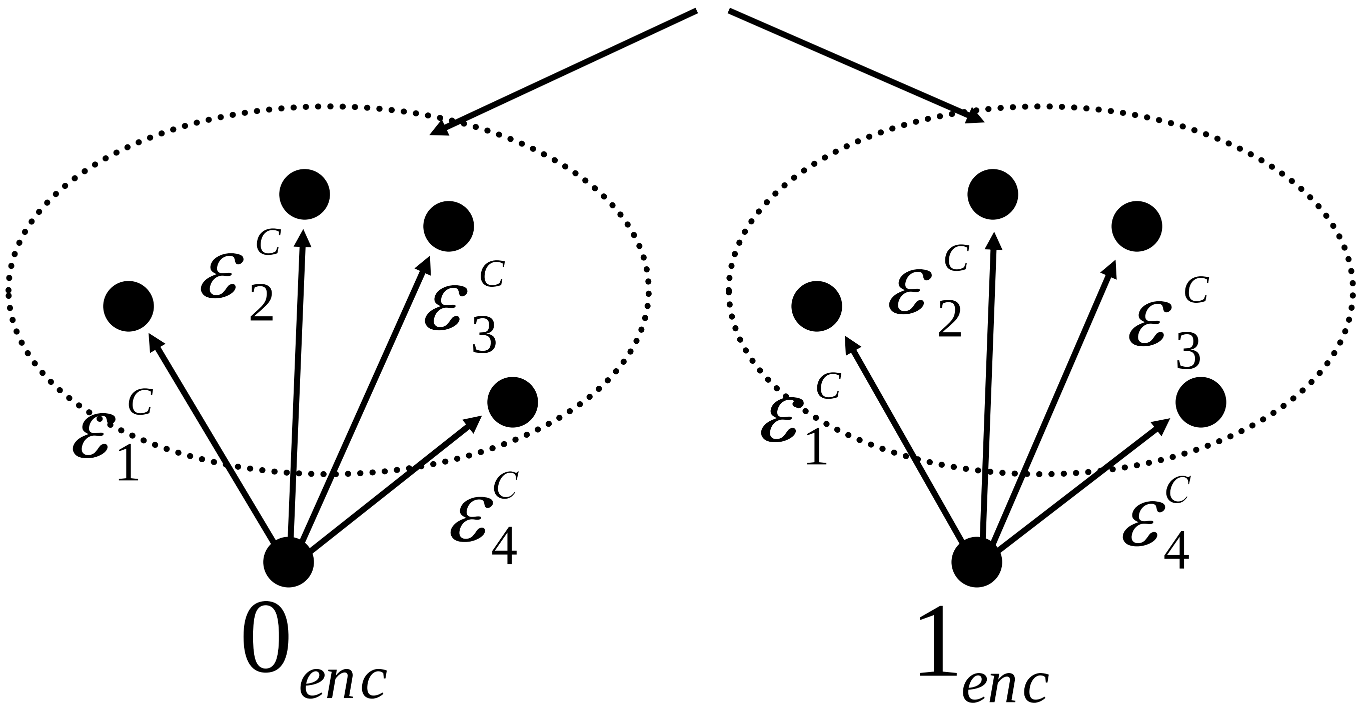
- We can extend the definitions to encoding strings of bits instead of just one bit
- For any string k let k_{enc} denote its encoding
- Let $\{\varepsilon_i^C\}$ denote the set of possible errors that occur to the strings
- For the encoding to correct all of the possible errors in $\{\varepsilon_i^C\}$ acting on any encoded string, we must have

$$\varepsilon_i^C\left(k_{enc}\right) \neq \varepsilon_j^C\left(l_{enc}\right), \quad \forall k \neq l$$

Necessary and sufficient condition for error correction to be possible

e.g.

These sets must be disjoint



$$\varepsilon_i^C\left(k_{enc}\right) \neq \varepsilon_j^C\left(l_{enc}\right), \quad \forall k \neq l$$

Reducing uncorrectable errors via concatenation

- Thus, if we restrict to errors $\mathcal{E}_i^C$ that flip at most one bit, the 3-bit repetition code corrects perfectly.

- The error model with independent bit flips leads to uncorrectable errors with probability in $O(p^2)$

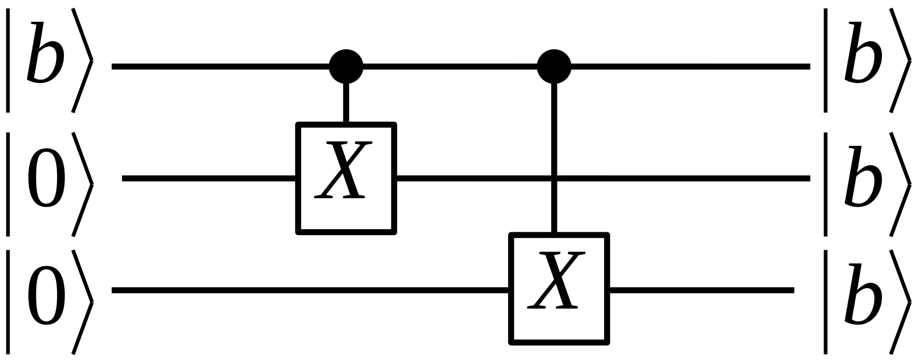
- The previous analysis shows that k levels of concatenation for an S gate computation will allow us to achieve a final error probability of ε with

$$k \in O(\log \log(S / \varepsilon))$$

Quantum Error Correcting Codes

- e.g. : encode a logical $|0\rangle$ with the state $|000\rangle$ and a logical $|1\rangle$ with the state $|111\rangle$

Quantum Network for encoding



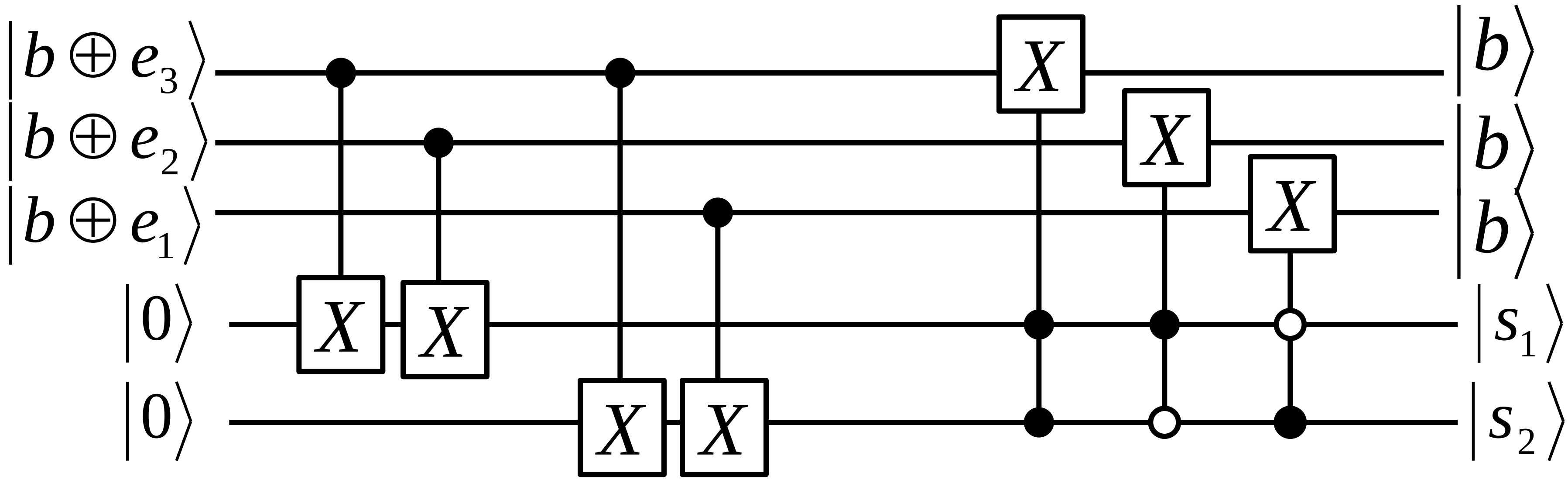
$$\left(\alpha|0\rangle+\beta|1\rangle\right)|0\rangle|0\rangle \rightarrow \alpha|0\rangle|0\rangle|0\rangle+\beta|1\rangle|1\rangle|1\rangle$$

Aside : This is not quantum cloning

$$\alpha|0\rangle|0\rangle|0\rangle + \beta|1\rangle|1\rangle|1\rangle \neq \left(\alpha|0\rangle + \beta|1\rangle\right)^{\otimes 3}$$

Quantum network for correcting errors

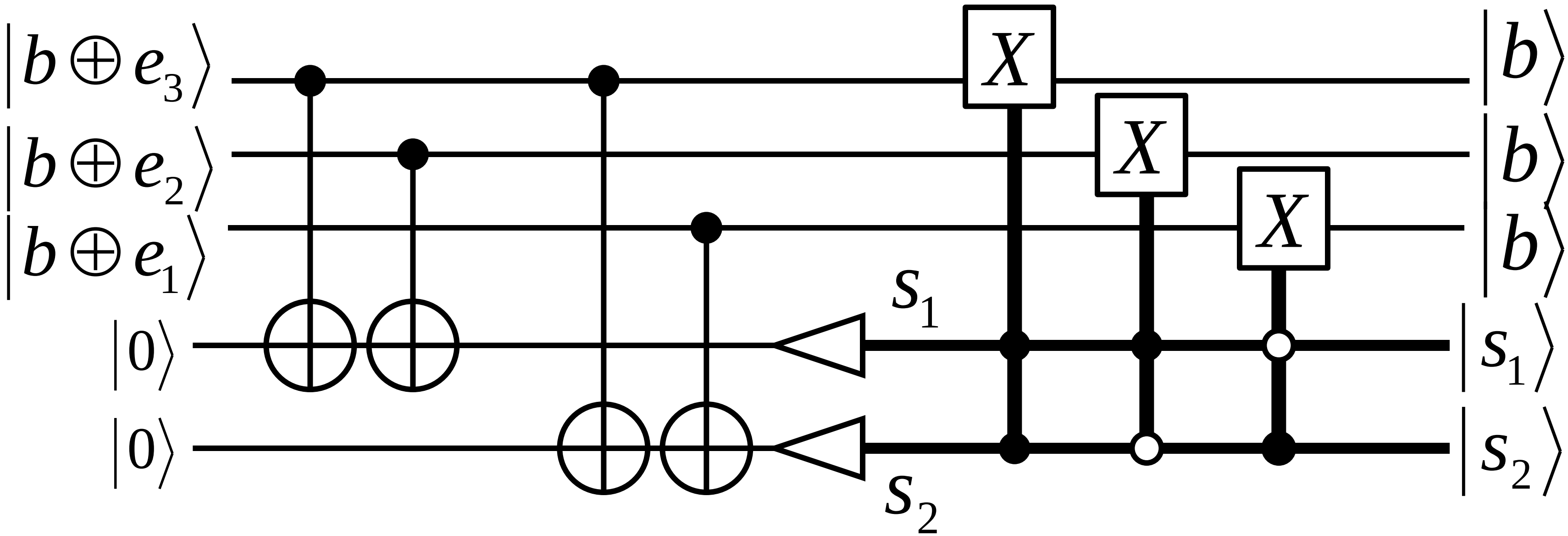
- Assume that at most one e_i equals 1



$$\alpha|e_3\rangle|e_2\rangle|e_1\rangle+\beta|1\oplus e_3\rangle|1\oplus e_2\rangle|1\oplus e_1\rangle\mapsto$$

$$\alpha|0\rangle|0\rangle|0\rangle+\beta|1\rangle|1\rangle|1\rangle$$

Equivalently



What about continuous errors??

- Suppose e.g. that instead of a full X error, we get some partial rotation on one of the qubits, e.g.

$$\alpha|0\rangle|0\rangle|0\rangle + \beta|1\rangle|1\rangle|1\rangle \mapsto$$
$$\alpha\big(\cos(\theta)|0\rangle + i\sin(\theta)|1\rangle\big)|0\rangle|0\rangle$$
$$+\beta\big(i\sin(\theta)|0\rangle + \cos(\theta)|1\rangle\big)|1\rangle|1\rangle$$

What about continuous errors??

- Note that

$$\alpha\big(\cos(\theta)|0\rangle + i\sin(\theta)|1\rangle\big)|0\rangle|0\rangle + \beta\big(i\sin(\theta)|0\rangle + \cos(\theta)|1\rangle\big)|1\rangle|1\rangle$$

$$= \cos(\theta)\big(\alpha|0\rangle|0\rangle|0\rangle + \beta|1\rangle|1\rangle|1\rangle\big) + i\sin(\theta)\big(\alpha|1\rangle|0\rangle|0\rangle + \beta|0\rangle|1\rangle|1\rangle\big)$$

The errors are discretized!

- We will compute one of two syndromes and can correct the X error in either case

$$\begin{pmatrix} \cos(\theta)\left(\alpha|0\rangle|0\rangle|0\rangle+\beta|1\rangle|1\rangle|1\rangle\right) \\ +i\sin(\theta)\left(\alpha|1\rangle|0\rangle|0\rangle+\beta|0\rangle|1\rangle|1\rangle\right) \end{pmatrix}|00\rangle$$

$$\begin{aligned} \mapsto \cos(\theta)\left(\alpha|0\rangle|0\rangle|0\rangle+\beta|1\rangle|1\rangle|1\rangle\right)|0\rangle|0\rangle \\ +i\sin(\theta)\left(\alpha|1\rangle|0\rangle|0\rangle+\beta|0\rangle|1\rangle|1\rangle\right)|1\rangle|1\rangle \end{aligned}$$

$$\begin{aligned} \mapsto \cos(\theta)\left(\alpha|0\rangle|0\rangle|0\rangle+\beta|1\rangle|1\rangle|1\rangle\right)|0\rangle|0\rangle \\ +i\sin(\theta)\left(\alpha|0\rangle|0\rangle|0\rangle+\beta|1\rangle|1\rangle|1\rangle\right)|1\rangle|1\rangle \end{aligned}$$

The errors are discretized!

- If we actually measure the syndromes and classically control the corrections then we get same outcome on the encoded bits:

$$\cos(\theta)\big(\alpha|0\rangle|0\rangle|0\rangle + \beta|1\rangle|1\rangle|1\rangle\big)|00\rangle$$
$$+i\sin(\theta)\big(\alpha|1\rangle|0\rangle|0\rangle + \beta|0\rangle|1\rangle|1\rangle\big)|11\rangle$$

$$\mapsto \big(\alpha|0\rangle|0\rangle|0\rangle + \beta|1\rangle|1\rangle|1\rangle\big)|0\rangle|0\rangle$$ with probability $\cos^2(\theta)$

$$\mapsto \big(\alpha|1\rangle|0\rangle|0\rangle + \beta|0\rangle|1\rangle|1\rangle\big)|0\rangle|0\rangle$$ with probability $\sin^2(\theta)$

- We have corrected the X error, but what if we get a Z error?

$$\cos(\theta)\Big(\alpha|0\rangle|0\rangle|0\rangle+\beta|1\rangle|1\rangle|1\rangle\Big)|00\rangle$$
$$+\sin(\theta)\Big(\alpha|0\rangle|0\rangle|0\rangle-\beta|1\rangle|1\rangle|1\rangle\Big)|11\rangle$$

Correcting Phase Errors

- Suppose the environment effects error

$$\left|x_1\right\rangle\left|x_2\right\rangle\left|x_3\right\rangle \rightarrow Z^{e_1} \otimes Z^{e_2} \otimes Z^{e_3}\left|x_1\right\rangle\left|x_2\right\rangle\left|x_3\right\rangle$$

on our quantum computer, where

$$e_j \in \{0,1\} \qquad e_1 + e_2 + e_3 \leq 1$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Quantum Error Correction

- We can encode

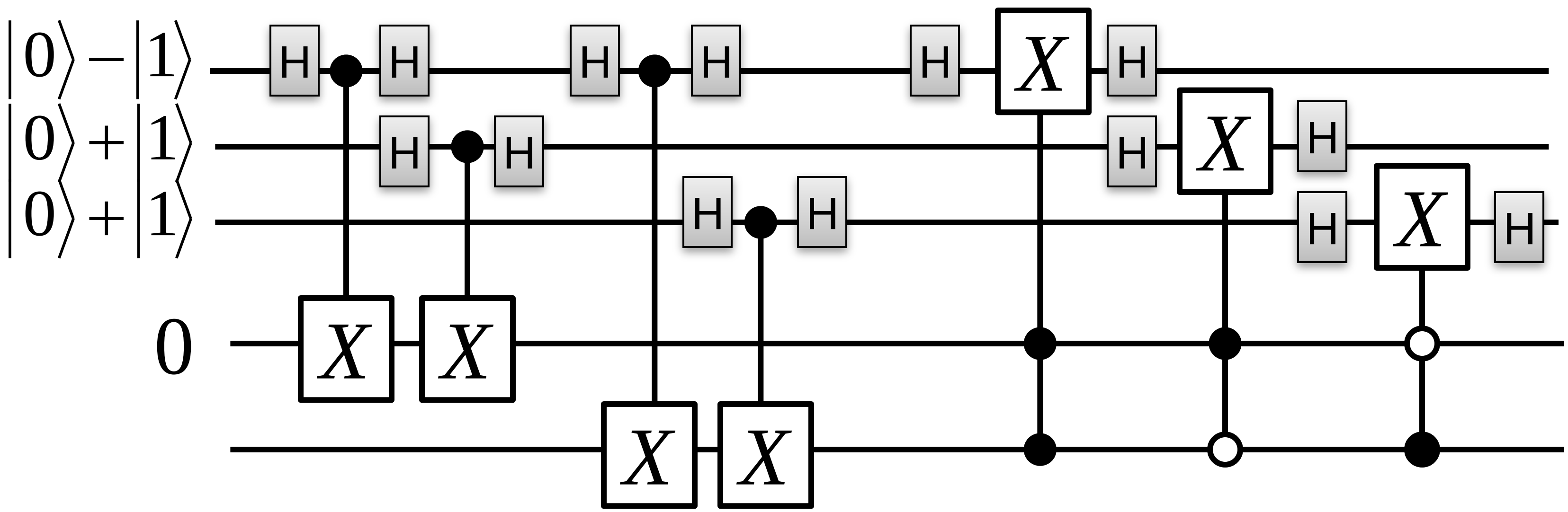
$$|0_L\rangle \rightarrow \frac{1}{2\sqrt{2}}\left(|0\rangle + |1\rangle\right)\left(|0\rangle + |1\rangle\right)\left(|0\rangle + |1\rangle\right)$$

$$|1_L\rangle \rightarrow \frac{1}{2\sqrt{2}}\left(|0\rangle - |1\rangle\right)\left(|0\rangle - |1\rangle\right)\left(|0\rangle - |1\rangle\right)$$

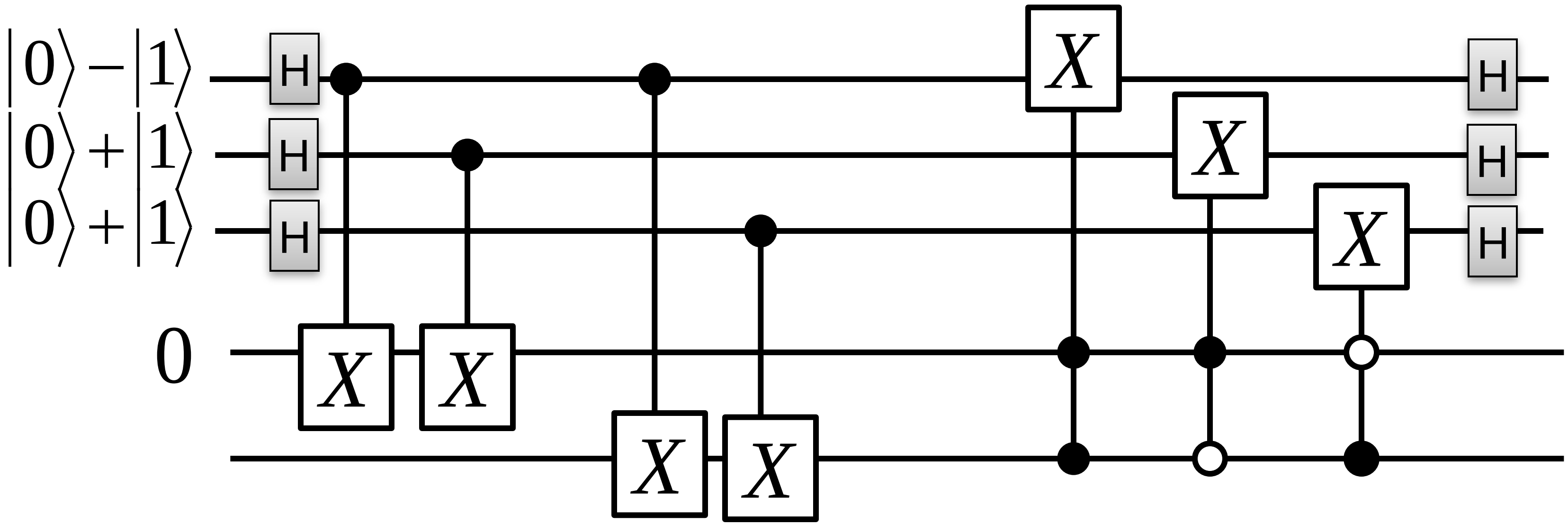
- Consider error term $Z \otimes I \otimes I$ acting on the logical 0 gives

$$\left(|0\rangle - |1\rangle\right)\left(|0\rangle + |1\rangle\right)\left(|0\rangle + |1\rangle\right)$$

Quantum Error Correction



Equivalently



Correcting both phase errors and bit flip errors

- Consider the codewords

$$|0_L\rangle = (|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$

$$|1_L\rangle = (|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$$

- We can easily correct any single X- error in one of the 3 three-bit parts

- We can then also correct a single Z- error on one of the 9 qubits.

- This means we can also correct Y-errors on one of the 9 qubits