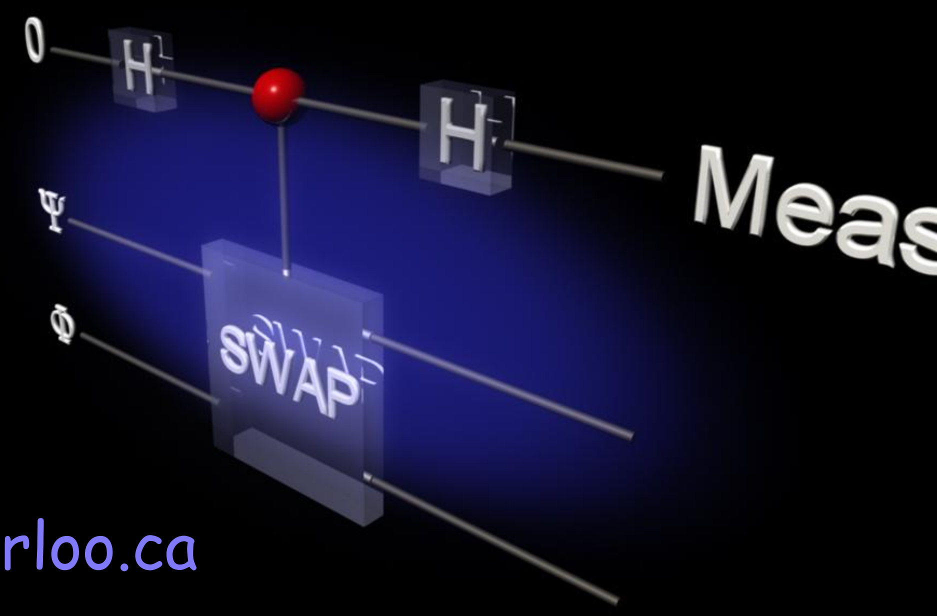
Introduction to Quantum Information Processing

CS467 C&O481 PHYS467

Lecture 5 (22 January 2013)

Michele Mosca mmosca@iqc.uwaterloo.ca

Tuesdays and Thursdays 10am-11:15am



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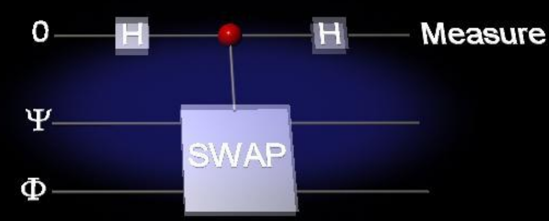
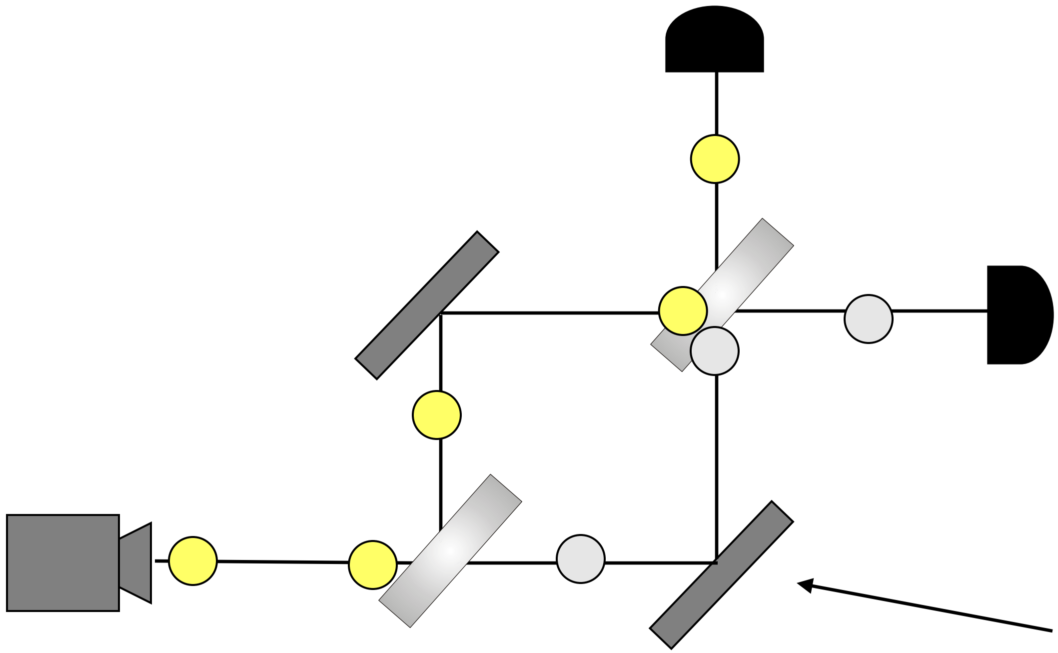
Alternative "classical" explanation





full mirror

Suppose that each particle has an invisible companion particle, such that whenever there are two paths to be chose, the real particle takes one path, and the invisible companion particle takes the other.

When the real particle goes through the second beamsplitter, the companion particle informs the real particle about what it would have experienced had it taken the other path.

Motivation for Bell´s inequality

- In lecture one we saw an experiment that gives evidence for quantum mechanics

- We can try to come up with classical explanations for the corresponding observations

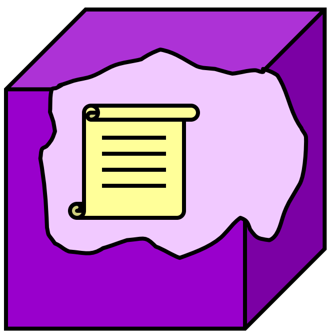
- Bell´s inequality, and the experimental tests of it, provides further evidence against these explanations

Bell's Inequality and its violation

Part I: physicist's view:

Can a physical state have ***pre-determined*** outcomes for each possible measurement that can be applied to it?

qubit:



where the "manuscript" is something like this:

if {|0⟩,|1⟩} measurement then output **0**

if {|+⟩,|−⟩} measurement then output **1**

if **...** (infinitely many cases)

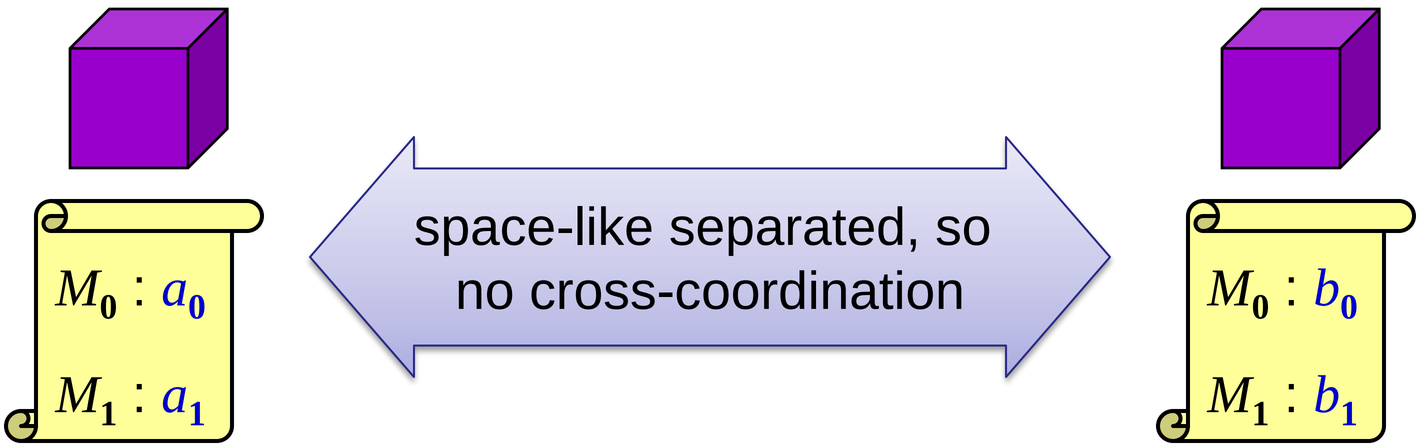
table could be implicitly given by some formula

called ***hidden variables***

[Bell, 1964]
[Clauser, Horne, Shimony, Holt, 1969]

Bell Inequality

Imagine a bipartite system, where one of two measurements, called M_0 and M_1, will be applied to each part:



Define:

$A_0 = (-1)^{a0}$

$A_1 = (-1)^{a1}$

$B_0 = (-1)^{b0}$

$B_1 = (-1)^{b1}$

Claim: $A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \leq 2$

Proof: $A_0 \underbrace{(B_0 + B_1)} + A_1 \underbrace{(B_0 - B_1)} \leq 2$

one is ± 2 and the other is 0

Bell Inequality

$A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \leq 2$ is called a ***Bell Inequality****

Question: could one, in principle, design an experiment to check if this Bell Inequality holds for a particular system?

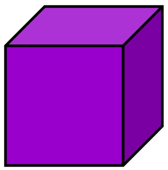
Answer 1: ***no, not directly***, because A_0, A_1, B_0, B_1 cannot all be measured (only ***one*** $A_s B_t$ term can be measured)

Answer 2: ***yes, indirectly***, by making many runs of this experiment: pick a random $st \in \{00, 01, 10, 11\}$ and then measure with M_s and M_t to get the value of $A_s B_t$

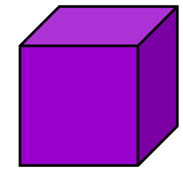
The ***average*** of $A_0 B_0$, $A_0 B_1$, $A_1 B_0$, $-A_1 B_1$ should be ≤ ½

* also called CHSH Inequality

Violating the Bell Inequality



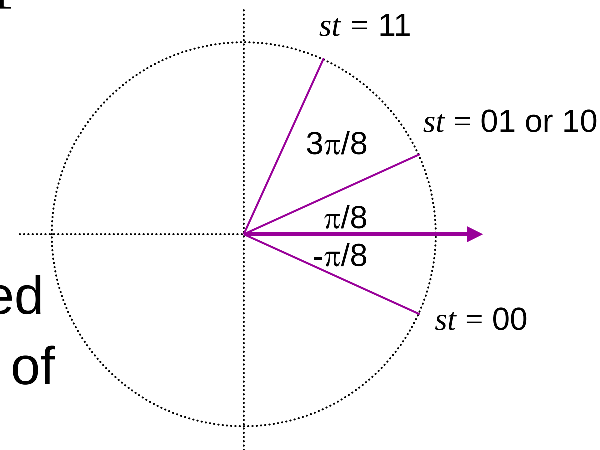
Two-qubit system in state
$|\phi\rangle = |00\rangle - |11\rangle$



Applying rotations θ_A and θ_B yields:

$$\cos(\theta_A + \theta_B)\,\underbrace{(|00\rangle - |11\rangle)}_{AB\,=\,+1} + \sin(\theta_A + \theta_B)\,\underbrace{(|01\rangle + |10\rangle)}_{AB\,=\,-1}$$

Define
M_0: rotate by $-\pi/16$ then measure
M_1: rotate by $+3\pi/16$ then measure



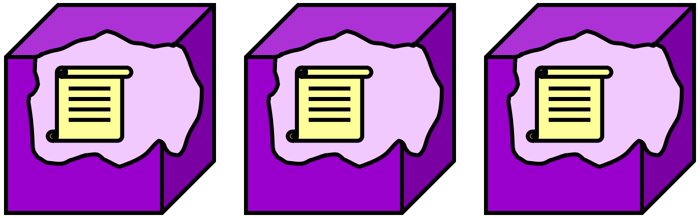
Then $A_0 B_0$, $A_0 B_1$, $A_1 B_0$, $-A_1 B_1$ all have expected value $\frac{1}{2}\sqrt{2}$, which ***contradicts*** the upper bound of ½

$\cos^2(\pi/8) = \frac{1}{2} + \frac{1}{4}\sqrt{2}$

Bell Inequality violation: summary

Assuming that physical systems are governed by ***local hidden variables*** leads to the Bell inequality

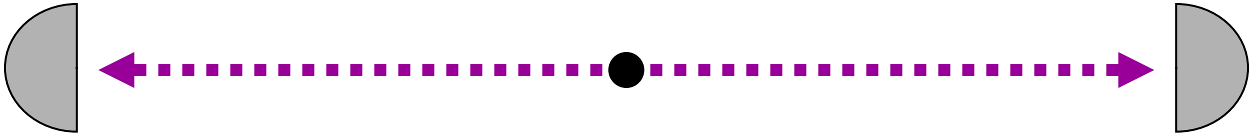
$$A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \le 2$$



But this is ***violated*** in the case of Bell states (by a factor of $\sqrt{2}$)

Therefore, no such local hidden variable theory can explain this.

This is, in principle, experimentally verifiable, and experiments along these lines have actually been conducted (first by Alain Aspect, 1982)



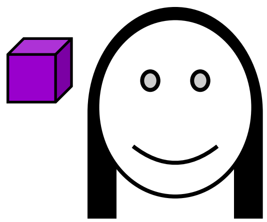
http://en.wikipedia.org/wiki/File:Alain_Aspect_in_Tel_Aviv_University.jpg

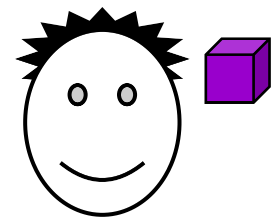
http://pirsa.org/06070050/

Bell's Inequality and its violation

Part II: computer scientist's view:

input: s t





output: a b

Rules:

1. No communication after inputs received
2. They ***win*** if $a \oplus b = s \wedge t$

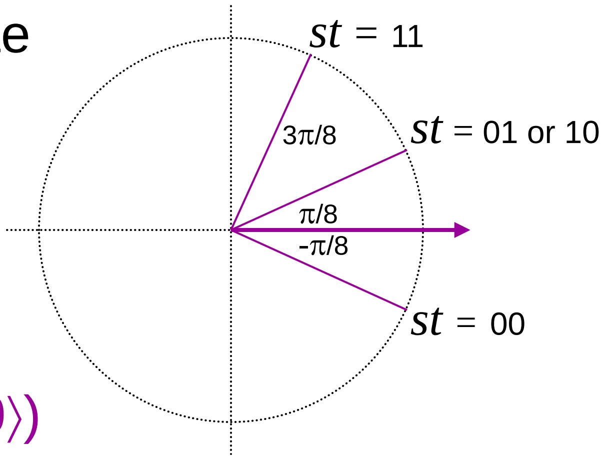
st	$a \oplus b$
00	0
01	0
10	0
11	1

With classical resources, $\Pr[a \oplus b = s \wedge t] \leq 0.75$

But, with prior entanglement state $|00\rangle - |11\rangle$, $\Pr[a \oplus b = s \wedge t]$ $= \cos^2(\pi/8) = ½ + ¼\sqrt{2} = 0.853\ldots$

The quantum strategy

- Alice and Bob start with entanglement $|\phi\rangle = |00\rangle - |11\rangle$
- **Alice:** if $s = 0$ then rotate by $\theta_A = -\pi/16$ else rotate by $\theta_A = +3\pi/16$ and measure
- **Bob:** if $t = 0$ then rotate by $\theta_B = -\pi/16$ else rotate by $\theta_B = +3\pi/16$ and measure



$$\cos(\theta_A + \theta_B)\,(|00\rangle - |11\rangle) + \sin(\theta_A + \theta_B)\,(|01\rangle + |10\rangle)$$

Success probability:

$\Pr[a \oplus b = s \wedge t] = \cos^2(\pi/8) = ½ + ¼\sqrt{2} = 0.853\ldots$

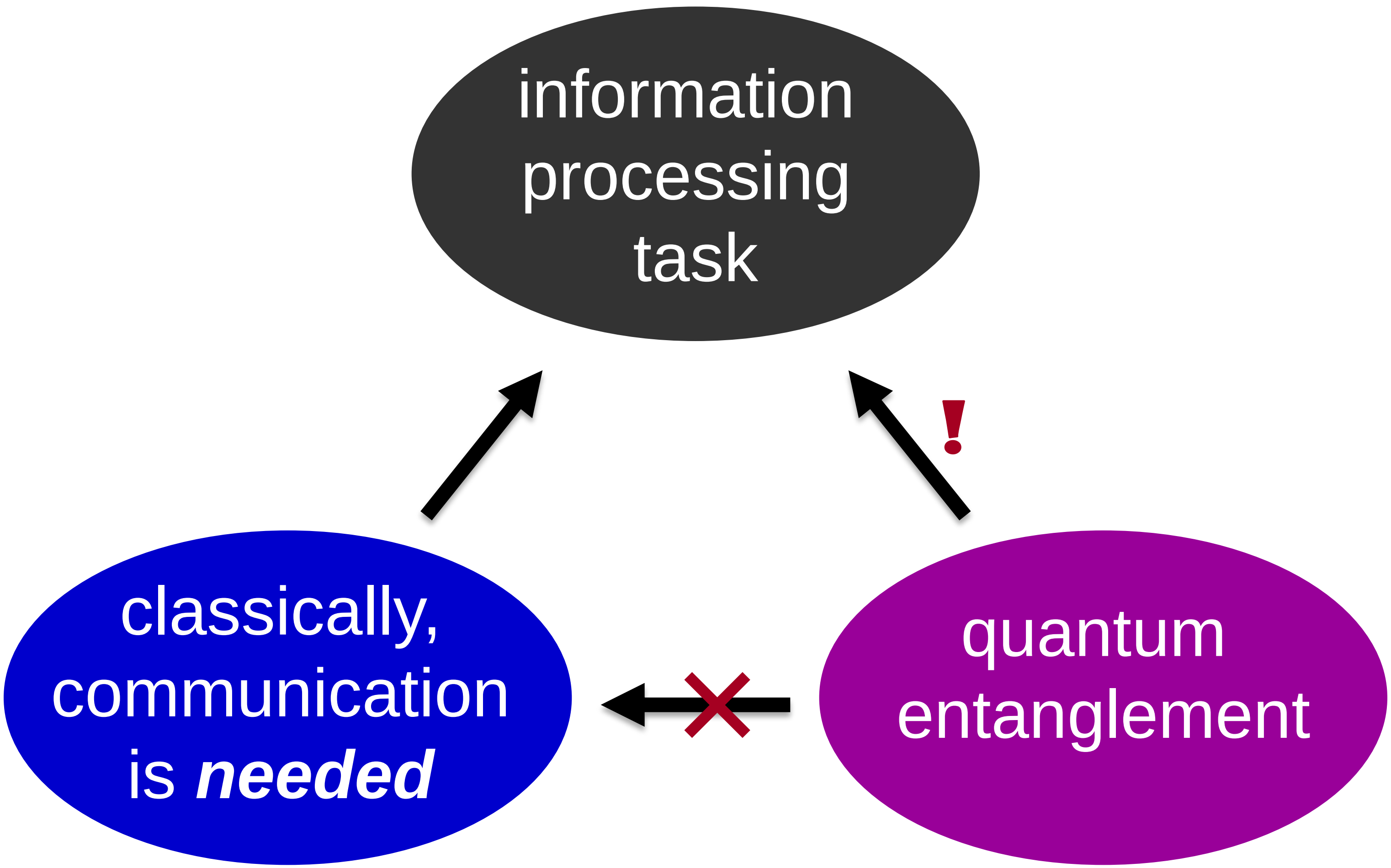
Is the quantum strategy optimal ?

One might wonder if there is a strategy that succeeds with probability higher than $0.853\ldots$

Maybe using more entanglement than one Bell state makes this possible?

Tsirelson [1980]**:** For ***any*** quantum strategy, the success probability is at most $\cos^2(\pi/8)$

Nonlocality in operational terms



- Measurements
- Traces and partial traces
- Density matrices

Von Neumann measurement in the computational basis

- We have repeatedly talked about the ability to measure a qubit in the computational basis

$$\{|0\rangle, |1\rangle\}$$

- With a state in the general form

$$|\Phi\rangle = (\alpha_0|0\rangle + \alpha_1|1\rangle),$$

if we perform a measurement, we get $|b\rangle$ with probability $|\alpha_b|^2$.

Comment about global phases

- With a state in the general form

$$|\Phi\rangle = (\alpha_0|0\rangle + \alpha_1|1\rangle),$$

if we perform a measurement, we get $|b\rangle$ with probability $|\alpha_b|^2$.

- Note that, for any β , measuring $e^{i\beta}|\Phi\rangle = (e^{i\beta}\alpha_0|0\rangle + e^{i\beta}\alpha_1|1\rangle),$ also outputs $|b\rangle$ with probability $\left|e^{i\beta}\alpha_b\right|^2 = |\alpha_b|^2$ for b = 0 or 1.

- Further, $U(e^{i\beta}|\Phi\rangle) = e^{i\beta}U|\Phi\rangle$, $(e^{i\beta}|\Phi\rangle)\otimes|\psi\rangle = e^{i\beta}|\Phi\rangle\otimes|\psi\rangle$

- In other words, such “global phases” have no physical significance. It’s usually easier to leave them in, and know we can ignore them, than to formally define things modulo global phase.

Projective measurements

- A Von Neumann measurement (also called Lüders measurement) is a special kind of *projective measurement*: a <u>complete</u> projective measurement.

- Positive Operator Valued Measure (POVM) measurements are even more general.

- All of these more general notions of measurement can be derived using the 4 postulates we have presented.

- A formalism often used to talk about projective measurements is that of *measuring an "observable".*

The observable formalism (1/3)

Consider a Hermitian operation *M* on the relevant state (Hilbert) space (i.e. $M = M^\dagger$)

- *M* has a spectral decomposition

$$M = \sum_j a_j |\psi_j\rangle\langle\psi_j|,$$

where the a_j are real numbers and the $|\psi_j\rangle$ form an orthonormal basis.

Why must the eigenvalues be real numbers?

- Since $\{|\psi_j\rangle\}$ is an orthonormal basis, we also have that

$$\sum_j |\psi_j\rangle\langle\psi_j| = I$$

Why?

The observable formalism (2/3)

- Let's further organize these terms into projectors with distinct eigenvalues

$$M = \sum_j a_j |\psi_j\rangle\langle\psi_j| = \sum_k a_k P_k$$

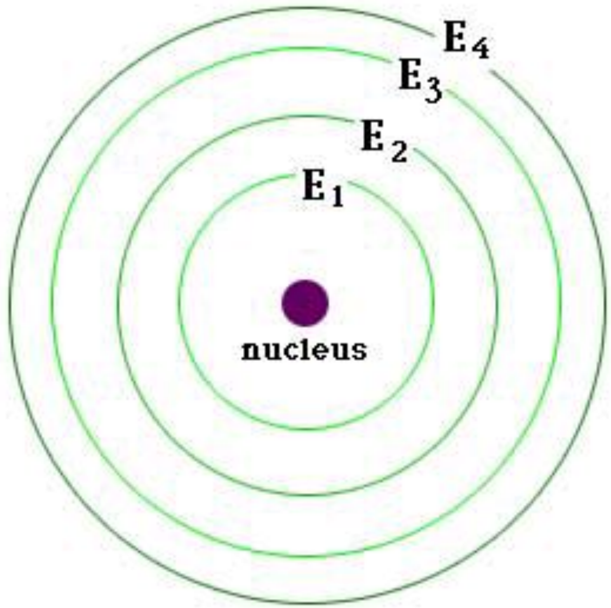
where

$$P_k = \sum_{\substack{j \\ a_j = a_k}} |\psi_j\rangle\langle\psi_j|$$

What does the projector P_k do to an arbitrary state?

The observable formalism (3/3)

- For simplicity, let's first consider the case that each eigenvalue occurs only once (i.e. no "degeneracies").

- If there are no degenera  ervable *M*" means to perform a Von Neumann $\psi_j \rangle \}$ and to output label a_j if the outcome is $|\psi_j\rangle$.

- Note that when measuri ome is a_j (or we could just say "j"), the resulting sta .

- Normally, the value a_j corresponds to a relevant physical parameter. E.g. if one is measuring the energy level of an electron, the value a_j could be the corresponding energy of the energy eigenstate $|\psi_j\rangle$.

Example: 1-qubit measurement as measuring an "observable" (1/2)

- We have the projection operators $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$ satisfying

$$P_0 + P_1 = I$$

- We consider the projection operator or "observable"

$$M = 0P_0 + 1P_1 = P_1$$

What are the eigenvalues?

- When we measure this observable *M*, the probability of getting the outcome *b* is

$$\Pr(b) = \langle \Phi | P_b | \Phi \rangle = |\alpha_b|^2$$

and we are in that case left with the state $\dfrac{P_b|\Phi\rangle}{\sqrt{\Pr(b)}} = \dfrac{\alpha_b}{|\alpha_b|}|b\rangle \approx |b\rangle$

Example: 1-qubit measurement as measuring an "observable" (2/2)

- Equivalently, we could consider the observable

$$Z = P_0 - P_1 = |0\rangle\langle 0| - |1\rangle\langle 1|$$

which has eigenvalues +1 and -1, and the same eigenvectors as

$$M = P_1 = 0|0\rangle\langle 0| + 1|1\rangle\langle 1|$$

- Measuring the *Z* observable is also equivalent to a Von Neumann measurement in the computational basis.

- The only difference is that we associate a value of +1 to outcome $|0\rangle$ and a value of -1 to $|1\rangle$.

Many observables correspond to equivalent Von Neumann measurements

Note that in this formalism, measuring the observable

$$M = \sum_k a_k P_k$$

where the a_k are distinct, is equivalent to measuring the observable

$$M' = \sum_k b_k P_k$$

where the b_k are distinct, up to a relabeling of the measurement outcomes.

- In many practical instances, the measuring apparatus outputs a sum (or average) of the eigenvalues of the results of many measurements.

  Therefore the actual values are important in these cases.

Von Neumann measurement in the computational basis

- Suppose we have the ability to measure <u>each qubit</u> in the basis

$$\{|0\rangle, |1\rangle\}$$

- Say we have the state

$$\sum_{x\in\{0,1\}^n} \alpha_x |x\rangle$$

- If we <u>measure all *n* qubits</u>, then we obtain $|x\rangle$ with probability $|\alpha_x|^2$.

- The probability of measuring a $|0\rangle$ in the first qubit is then equal to

$$\sum_{x\in 0\{0,1\}^{n-1}} |\alpha_x|^2$$

Partial Measurements

- If we only <u>measure the first qubit</u> and leave the rest alone, then we still get $|0\rangle$ with probability

$$p_0 = \sum_{x \in 0\{0,1\}^{n-1}} |\alpha_x|^2$$

- The remaining *(n-1)* qubits are then in the renormalized state

$$\sum_{y \in \{0,1\}^{n-1}} \frac{\alpha_{0y}}{\sqrt{p_0}} |y\rangle$$

- (This is similar to Bayes Theorem)

… in terms of the observables

This partial measurement corresponds to measuring the observable

$$M = 0|0\rangle\langle 0| \otimes I^{n-1} + 1|1\rangle\langle 1| \otimes I^{n-1}$$

$$= |1\rangle\langle 1| \otimes I^{n-1}$$

or equivalently

$$Z \otimes I^{n-1} = |0\rangle\langle 0| \otimes I^{n-1} - |1\rangle\langle 1| \otimes I^{n-1}$$

This is an example of a projective measurement that is <u>not complete</u>.
The observable has degeneracies.

The observable formalism

- In general, given the state $|\varphi\rangle$, measuring an observable $M = \sum_k a_k P_k$ (which might have degeneracies), we obtain outcome a_j (or just "j") with probability

$$\Pr(j) = \langle \Phi | P_j | \Phi \rangle .$$

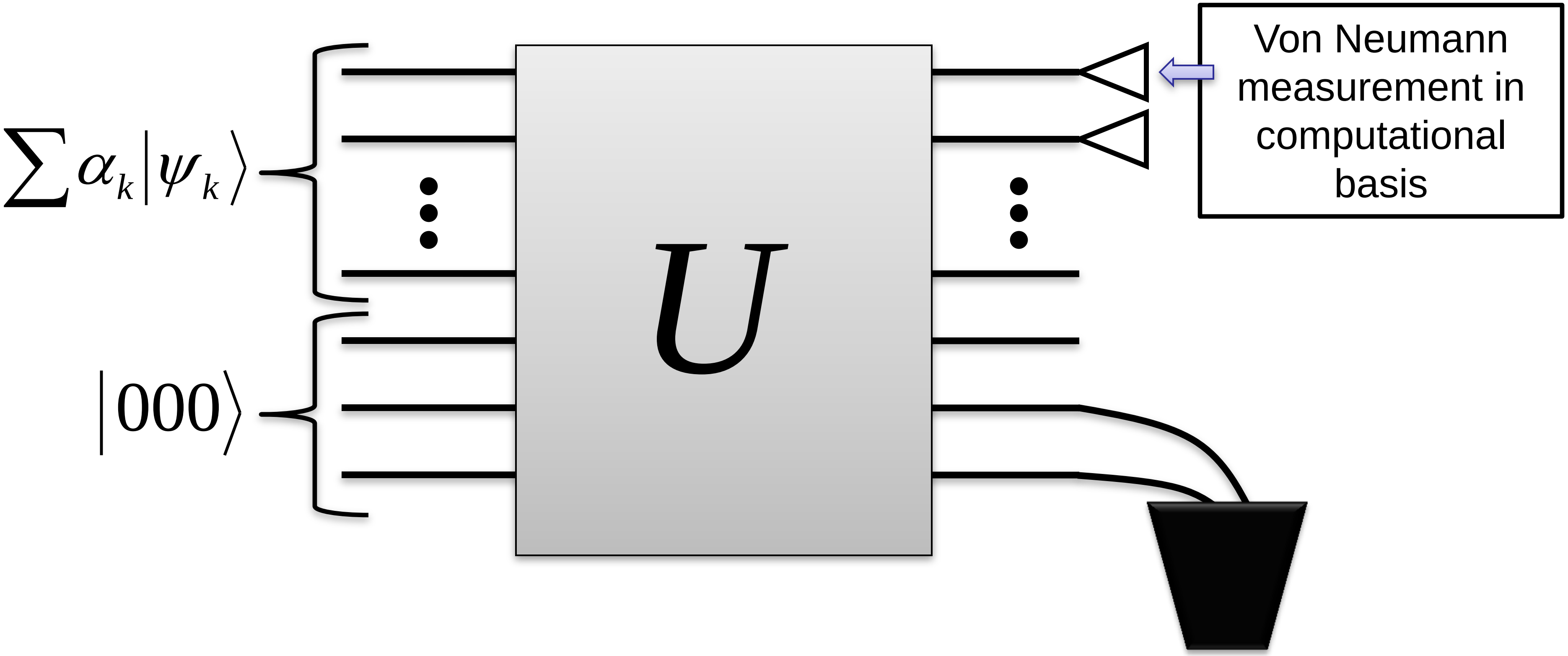
- The resulting state is

$$\frac{P_j | \Phi \rangle}{\sqrt{\Pr(j)}} .$$

- Note that one can conveniently express the expected value of the measurement of M as

$$\sum_k a_k \langle \Phi | P_k | \Phi \rangle = \langle \Phi | \left(\sum_k a_k P_k \right) | \Phi \rangle \quad = \langle \Phi | M | \Phi \rangle$$

General measurement



More general notions of measurement can be derived from the simple Von Neumann measurement.

Dealing with impure states …

When we measure part of a quantum state, the remaining state cannot in general be described as a “pure” state.

The resulting “mixed” states are best described in terms of a *density matrix*.

Example : Imagine we have a joint two-qubit system represented by a Bell state. Is the joint state completely known?

What is the state of the first qubit? Do we have any knowledge about it?

Trace of a matrix

The trace of a matrix *A* is the sum of its diagonal elements

e.g.

$$Tr(A) = Tr\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} = a_{00} + a_{11} + a_{22}$$

Some properties:

$$Tr[xA + yB] = xTr[A] + yTr[B]$$

$$Tr[AB] = Tr[BA]$$

$$Tr[ABC] = Tr[CAB]$$

$$Tr[UAU^{\dagger}] = Tr[A]$$

Orthonormal basis $\{\left|\phi_i\right\rangle\}$

$$Tr[A] = \sum_i \langle \phi_i | A | \phi_i \rangle$$

Density Matrices can describe pure states

$$\boxed{\left|\phi\right\rangle = \alpha_0\left|0\right\rangle + \alpha_1\left|1\right\rangle}$$

Notice that $\alpha_0 = \langle 0|\phi\rangle$, and $\alpha_1 = \langle 1|\phi\rangle$.

So the probability of getting 0 when measuring $|\phi\rangle$ is:

$$p(0) = \left|\alpha_0\right|^2 = \left|\left\langle 0|\phi\right\rangle\right|^2$$

$$= \left\langle 0|\phi\right\rangle\left(\left\langle 0|\phi\right\rangle\right)^* = \left\langle 0|\phi\right\rangle\left\langle \phi|0\right\rangle$$

where $\rho = |\phi\rangle\langle\phi|$ is called the *density matrix* for the state $|\phi\rangle$

$$= \left\langle 0\|\phi\right\rangle\left\langle \phi\|0\right\rangle = Tr\left(\left\langle 0\|\phi\right\rangle\left\langle \phi\|0\right\rangle\right)$$

$$= Tr\left(\left|0\right\rangle\left\langle 0\|\phi\right\rangle\left\langle \phi\right|\right) = Tr\left(\left|0\right\rangle\left\langle 0\right|\rho\right)$$

Mixture of pure states

A state described by a state vector $|\phi\rangle$ is called a *pure state*.

What if we have a qubit which is known to be in the pure state $|\phi_1\rangle$ with probability p_1, and in $|\phi_2\rangle$ with probability p_2 ?

More generally, consider probabilistic mixtures of pure states (called *mixed states*):

$$\phi = \left\{ \left(|\phi_1\rangle, p_1 \right), \left(|\phi_2\rangle, p_2 \right), \dots \right\}$$

State #1

Probability to be in state #1

Density matrix
can also describe mixed states

…then the probability of measuring 0 is given by conditional probability:

$$p(0) = \sum_i p_i \cdot \left(\text{prob. of measuring 0 given pure state } |\phi_i\rangle\right)$$

$$= \sum_i p_i \cdot Tr\left(|0\rangle\langle 0||\phi_i\rangle\langle\phi_i|\right)$$

$$= Tr \sum_i p_i |0\rangle\langle 0||\phi_i\rangle\langle\phi_i|$$

$$= Tr\left(|0\rangle\langle 0|\rho\right)$$

Density matrices contain all the useful information about an arbitrary quantum state.

where $\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i|$ is the *density matrix* for the mixed state

Density Matrix

If we apply the unitary operation *U* to $|\psi\rangle$ the resulting state is $U|\psi\rangle$ with density matrix

$$U|\psi\rangle\left(U|\psi\rangle\right)^{\dagger} = U|\psi\rangle\langle\psi|U^{\dagger}$$

Density matrix

If we apply the unitary operation *U* to

$$\{(q_k, |\psi_k\rangle)\}$$

the resulting state is

$$\{(q_k, U|\psi_k\rangle)\}$$

with density matrix

$$\sum_k q_k U|\psi_k\rangle\langle\psi_k|U^\dagger = U\left(\sum_k q_k |\psi_k\rangle\langle\psi_k|\right)U^\dagger$$

$$= U\rho U^\dagger$$

Density matrix

If we perform a Von Neumann measurement of the state $\rho = |\psi\rangle\langle\psi|$ wrt a basis containing $|\phi\rangle$, the probability of obtaining $|\phi\rangle$ is

$$\left|\langle\psi\,|\,\phi\rangle\right|^2 = Tr\left(\rho|\phi\rangle\langle\phi|\right)$$

Density matrix

If we perform a Von Neumann measurement of the state $\{(q_k, |\psi_k\rangle)\}$ wrt a basis containing $|\phi\rangle$, the probability of obtaining $|\phi\rangle$ is

$$\sum_k q_k |\langle\psi_k|\phi\rangle|^2 = \sum_k q_k Tr(|\psi_k\rangle\langle\psi_k||\phi\rangle\langle\phi|)$$

$$= Tr\left(\sum_k q_k |\psi_k\rangle\langle\psi_k||\phi\rangle\langle\phi|\right)$$

$$= Tr(\rho|\phi\rangle\langle\phi|)$$

Density matrix

In other words, the density matrix contains all the information necessary to compute the probability of any outcome in any future measurement.

Density matrix

Note that there are an infinite number of decompositions of a mixed state into a mixture of pure states.

These decompositions are all equivalent and indistinguishable.

Are there any “natural” or “special” decompositions? e.g. with a minimum number of terms?

Spectral decomposition

- Often it is convenient to rewrite the density matrix as a mixture of its eigenvectors.

- Recall that eigenvectors with distinct eigenvalues are orthogonal; for the subspace of eigenvectors with a common eigenvalue (“degeneracies”), we can select an orthonormal basis.

Spectral decomposition

- In other words, we can always “diagonalize” a density matrix so that it is written as

$$\rho = \sum_k p_k \left| \phi_k \right\rangle \left\langle \phi_k \right|$$

where $\left| \phi_k \right\rangle$ is an eigenvector with eigenvalue p_k and forms an orthonormal basis $\left\{ \left| \phi_k \right\rangle \right\}$.

Partial trace

- How can we compute probabilities for a partial system?

  Example:

$$\sum_{x,y} \alpha_{xy} |x\rangle |y\rangle$$

- Suppose we are only able to interact with or measure the first system.
- For convenience, denote $\boxed{|\Phi_y\rangle = \sum_x \frac{\alpha_{xy}}{\sqrt{p_y}} |x\rangle}$ $\qquad p_y = \sum_x |\alpha_{xy}|^2$
- So $$\sum_{x,y} \alpha_{xy} |x\rangle |y\rangle = \sum_y \sqrt{p_y} |\Phi_y\rangle |y\rangle$$

- E.g. suppose we wish to compute the probability of measuring $|w\rangle$ in the first register.

$$\Pr(w) = \sum_y \left|\alpha_{wy}\right|^2 = \sum_y p_y \left|\frac{\alpha_{wy}}{\sqrt{p_y}}\right|^2$$

$$= \sum_y p_y Tr\left(|w\rangle\langle w|\Phi_y\rangle\langle\Phi_y|\right)$$

$$= Tr\left(|w\rangle\langle w|\left(\sum_y p_y|\Phi_y\rangle\langle\Phi_y|\right)\right)$$

$$= \langle w|\left(\sum_y p_y|\Phi_y\rangle\langle\Phi_y|\right)|w\rangle$$

- So what really matters is $\sum_y p_y \left|\Phi_y\right\rangle\left\langle\Phi_y\right|$

- So we are interested in the map

$$\left(\sum_{x,y} \alpha_{xy} \left|x\right\rangle\left|y\right\rangle\right)\left(\sum_{v,z} \alpha^*_{vz} \left\langle v\right|\left\langle z\right|\right)$$

$$= \left(\sum_y \sqrt{p_y} \left|\Phi_y\right\rangle\left|y\right\rangle\right)\left(\sum_z \sqrt{p_z} \left\langle\Phi_z\right|\left\langle z\right|\right)$$

$$\mapsto \sum_y p_y \left|\Phi_y\right\rangle\left\langle\Phi_y\right|$$

- What is this map??

Partial trace

- One way to describe this map is as the map

$$\sum_y \sqrt{p_y}\left|\Phi_y\right\rangle\left|y\right\rangle \mapsto \left\{\left(p_{y,}\left|\Phi_y\right\rangle\right)\right\}$$

(can think of this as measuring the 2nd register, but not looking at the outcome)

- Using density matrix representation for states:

$$\rho = \sum_{y,z} \sqrt{p_y p_z}\left|\Phi_y\right\rangle\left\langle\Phi_z\right| \otimes \left|y\right\rangle\left\langle z\right|$$

$$\mapsto \sum_y p_y \left|\Phi_y\right\rangle\left\langle\Phi_y\right| = Tr_2 \rho$$

Partial trace

$$\rho = Tr_2 \rho$$

is in fact a linear map that takes bipartite states to single-system states

$$Tr_2\left(|i\rangle\langle k| \otimes |j\rangle\langle l|\right) = |i\rangle\langle k| \otimes Tr\left(|j\rangle\langle l|\right)$$

$$= |i\rangle\langle k| \otimes \langle l|j\rangle = \langle l|j\rangle |i\rangle\langle k|$$

Confirm that

$$\rho = \sum_{y,z} \sqrt{p_y p_z} |\Phi_y\rangle\langle\Phi_z| \otimes |y\rangle\langle z|$$

$$\mapsto \sum_{y,z} \sqrt{p_y p_z} |\Phi_y\rangle\langle\Phi_z| \otimes \langle z\|y\rangle = \sum_y p_y |\Phi_y\rangle\langle\Phi_y|$$

Partial trace

- We can also trace out the first system.

$$Tr_1\left(|i\rangle\langle k|\otimes|j\rangle\langle l|\right)=Tr\left(|i\rangle\langle k|\right)\otimes|j\rangle\langle l|$$

- Back to our example:

$$\sum_{x,y}\alpha_{xy}|x\rangle|y\rangle=\sum_{x}\sqrt{p_x}|x\rangle|\Theta_x\rangle$$

Partial trace using matrices

- Tracing out the 2nd system

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{Tr_2} \begin{bmatrix} Tr\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} & Tr\begin{bmatrix} a_{02} & a_{03} \\ a_{12} & a_{13} \end{bmatrix} \\ Tr\begin{bmatrix} a_{20} & a_{21} \\ a_{30} & a_{31} \end{bmatrix} & Tr\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} a_{00} + a_{11} & a_{02} + a_{13} \\ a_{20} + a_{31} & a_{22} + a_{33} \end{bmatrix}$$

Distant transformations don't change the local density matrix

- Notice that a unitary transformation on the system that is traced out does not affect the result of the partial trace.

i.e.

$$\sum_y \sqrt{p_y}\,|\Phi_y\rangle U|y\rangle \cong (I\otimes U)\rho(I\otimes U^\dagger)$$

$$\xrightarrow{Trace_2} \left\{\left(p_y, |\Phi_y\rangle\right)\right\} \cong \rho_2 = Tr_2\rho$$

*(can think of this as measuring the 2nd register **in any basis**, and not looking at the outcome)*

Partial trace

- For example, consider tracing out by measuring the second qubit in the computational basis and ignoring the outcome

$$\alpha|00\rangle+\beta|11\rangle \xrightarrow{Tr_2} |\alpha|^2|0\rangle\langle 0|+|\beta|^2|1\rangle\langle 1|$$

- In a different basis

$$\alpha|00\rangle+\beta|11\rangle=\frac{1}{\sqrt{2}}\left(\alpha|0\rangle+\beta|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle+\frac{1}{\sqrt{2}}|1\rangle\right)$$

$$+\frac{1}{\sqrt{2}}\left(\alpha|0\rangle-\beta|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle-\frac{1}{\sqrt{2}}|1\rangle\right)$$

Partial trace

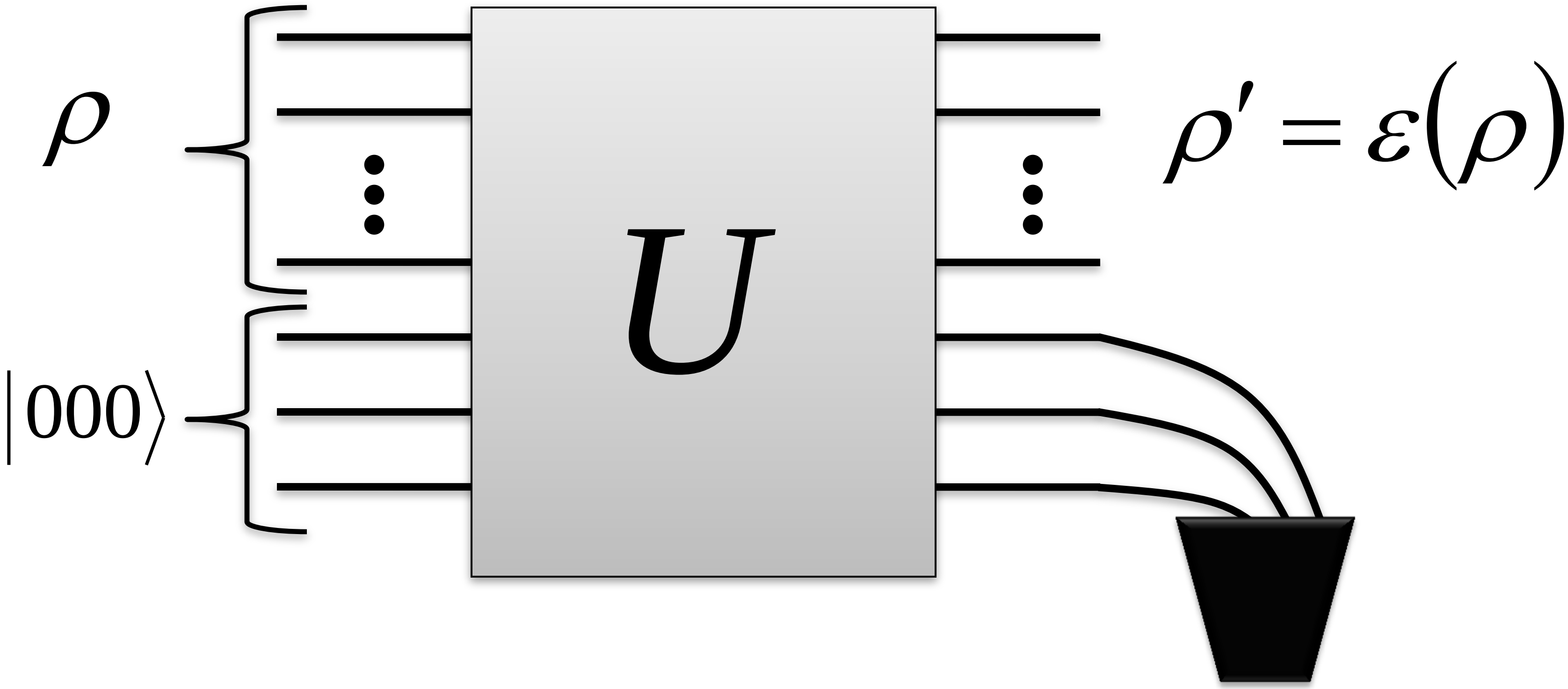
$$\frac{1}{\sqrt{2}}\left(\alpha|0\rangle+\beta|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle+\frac{1}{\sqrt{2}}|1\rangle\right)+\frac{1}{\sqrt{2}}\left(\alpha|0\rangle-\beta|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle-\frac{1}{\sqrt{2}}|1\rangle\right)$$

$$\xrightarrow{Tr_2}\frac{1}{2}\left(\alpha|0\rangle+\beta|1\rangle\right)\left(\alpha^*\langle 0|+\beta^*\langle 1|\right)$$

$$+\frac{1}{2}\left(\alpha|0\rangle-\beta|1\rangle\right)\left(\alpha^*\langle 0|-\beta^*\langle 1|\right)$$

$$=|\alpha|^2|0\rangle\langle 0|+|\beta|^2|1\rangle\langle 1|$$

Aside:
General operation



$$\rho \mapsto \rho' = Tr_2\left(\rho \otimes |000\rangle\langle 000|\right)$$

Important

- Thus, any general quantum transformation on the traced out system, including measurement (without communicating back the answer) does not affect the partial trace.

- “Tracing out” the second system corresponds to discarding or ignoring the second system. Hypothetical operations, like measurements, on the second system might help with some mathematical or conceptual analysis, but they are not physically significant if the second system is truly isolated/discarded.

Why?

- Operations on the 2nd system should not affect the statistics of any outcomes of measurements on the first system

- Note that if it were possible to affect the statistics non-locally, then a party in control of the 2nd system could instantaneously communicate information to a party controlling the 1st system.