II Electric Fields in Material Space

–so far have looked at electrostatic electric fields in what is effectively free space ("vacuum field theory)

- *look at electric fields in nonconductors (insulator or dielectric) and conductors (also metals)*

- *classified according to conductivity* (σ) [*units: mhos per meter or siemens per meter* (S/m)]

- *metal*: $\sigma \gg 1$

1 or 2 electrons per atom / molecule not bound to specific atom / molecule $\Rightarrow$ *abundance of free electrons*

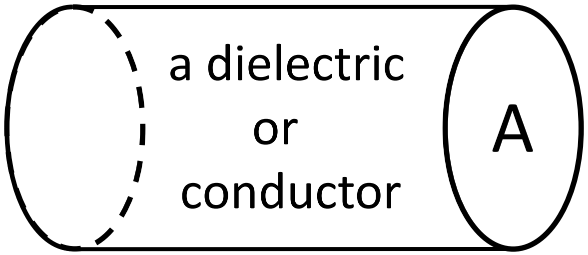
- *insulator*: $\sigma \ll 1$

electrons bound to specific atoms / molecules

- *also semiconductor and superconductors*

1 Convection and Conduction Currents

Current Density :



Let net +ve charge dq flow through A in time dt

- *instantaneous current* $I = \dfrac{dq}{dt}$ $[\dfrac{coulomb}{second}$, or amperes (A)]

Consider an element ΔA:

- *current density* $J \equiv \dfrac{I_{\Delta A}}{\Delta A}$ $[amperes\ / m^2]$

  $(assuming\ J \perp \Delta A)$

- *in general* $I_{\Delta \vec{A}} = \vec{J} \bullet \Delta \vec{A}$

  and $I = \int_S \vec{J} \bullet d\vec{A}$

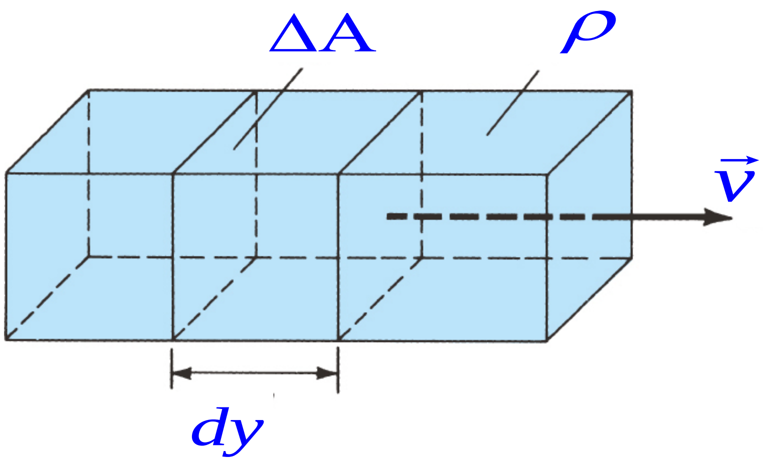
Note that effectively I through A is the flux of the current density $\vec{J}$.

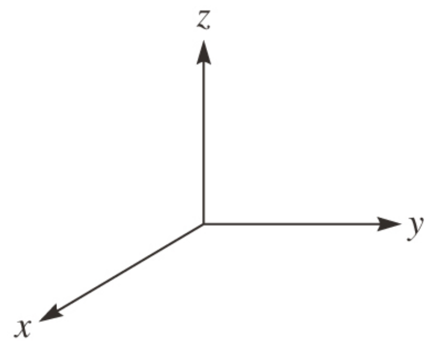
Convection Current :

- *current through insulating medium :*

  e.g. liquid, rarefied gas, vacuum

–a filament of material :





$$I = \frac{dq}{dt}$$

$$I_{\Delta A} = \frac{\rho \Delta A dy}{dt}$$

$$I_{\Delta A} = \rho \Delta A v_y$$

$\equiv$ convection current

$$\therefore \; J_y = \frac{I_{\Delta A}}{\Delta A} = \rho v_y$$

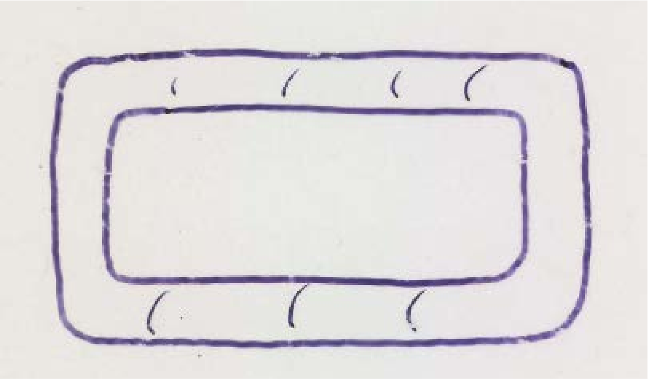
$$\text{or } \vec{J} = \rho \vec{v}$$

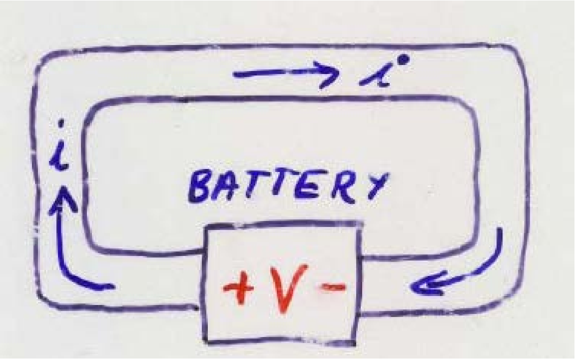
$J \equiv$ convection current density

Conduction Current :

- *current in conductor*

Consider loop of copper wire:



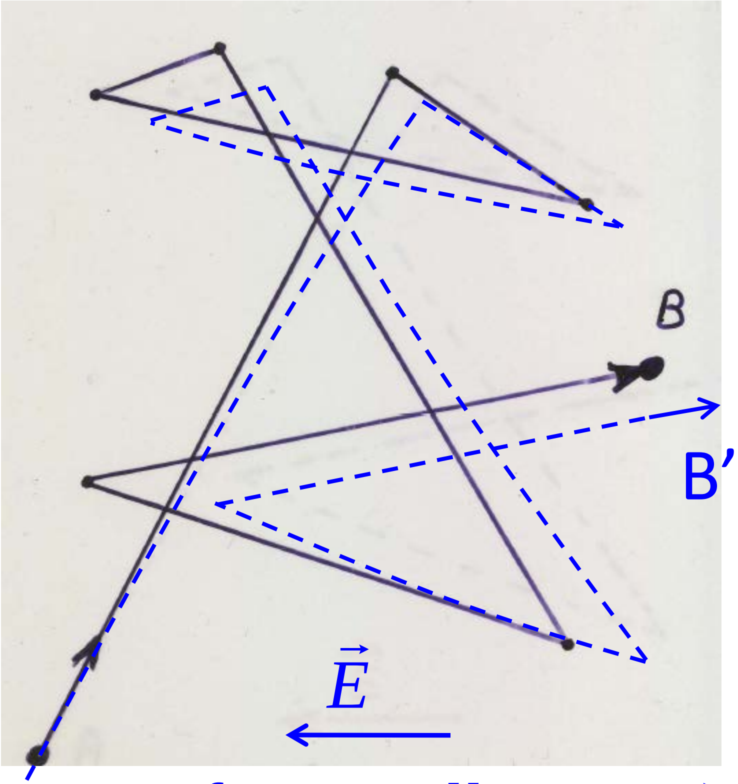


$-$ *loop is in electrostatic equilibrium*

$-$ *entire loop is at same potential*

ΔV *appears across ends of wire*

- $\vec{E}$ *set up inside wire*
- *free* e^- *accelerate in CCW sense*



—— without E - - - - with E

u = average thermal velocity, fn(T)

λ = mean free path = ave. distance between collisions

τ = mean free time (between collisions)

$$\vec{v}_d = drift\ velocity = ave.\ \vec{v}\ due\ to\ \vec{E}$$

$$\vec{F} = -e\vec{E} \qquad \therefore \vec{a} = \frac{-e\vec{E}}{m}$$

- $just\ after\ a\ collision: \vec{v}_i$
- $just\ before\ the\ next\ collision: \vec{v}_f = \vec{v}_i + \vec{a}t = \vec{v}_i - \frac{e\vec{E}}{m}t$

$-consider\ average\ \vec{v}_{f,ave}\ for\ all\ electrons\ over$

$all\ collision\ times\ (average\ is\ \tau)\ and\ all\ possible,\ random\ \vec{v}_i:$

$-average\ for\ \vec{v}_i\ is\ zero;\ average\ for\ 2^{nd}\ term\ is\ -\frac{e\vec{E}}{m}\tau$

- $this\ is\ the\ drift\ velocity,\ \vec{v}_d = -\frac{e\vec{E}}{m}\tau$ (of the whole electron cloud) (typically in mm/s)

$$\vec{v}_d = -\frac{e\vec{E}}{m}\tau$$

$-define\ \rho \equiv volume\ charge\ density\ (n\ electrons\ per\ unit\ volume)$

$$\rho = -ne$$

$\bullet conduction\ current\ density:$

$$\vec{J} = \rho\vec{v}_d = \frac{ne^2\tau}{m}\vec{E}$$

$$\vec{J} = \sigma\vec{E} \quad (\sigma = \frac{ne^2\tau}{m}) \quad -point\ form\ of\ Ohm's\ Law$$

$e.g.$

$gold: \quad \sigma = 5.8\text{x}10^7\ siemens/m\ \ (or\ S/m)\ \ (or\ mhos/m)$

$silicon: \sigma = 5.8\text{x}10^7\ siemens/m$

$mica: \quad \sigma = 10^{15}\ siemens/m$

Note :

– *from Gauss's Law had that for a conductor at electrostatic equilibrium* $E = 0$ *inside*

Now consider Ohm's Law instead :

- *in perfect conductor* $\sigma = \infty$

$\Rightarrow$ *to maintain finite* $\vec{J}$ *in a perfect conductor* $E = 0$ *inside*

– – *as before :* $E = 0,\ \rho_e = 0,\ V_{ab} = 0$ *inside a conductor*

Note : *also* $\vec{J} = \dfrac{1}{\rho}\vec{E}$

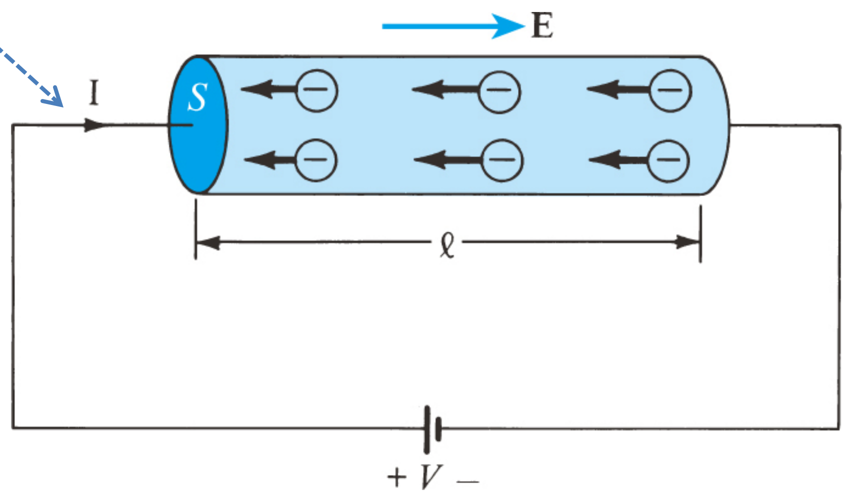
where $\rho = \dfrac{1}{\sigma} =$ *resistivity* [*ohm* · *meter*]

$\left(i.e.\ \ \rho = \dfrac{|E|}{|J|} \right)$ (ρ *depends on properties of the material and temperature*)

Ohm's Law:

- *useful to recast the above ideas in terms of I and ΔV across wire*

A



$$E = \frac{V_{ab}}{l} = \frac{V}{l}$$

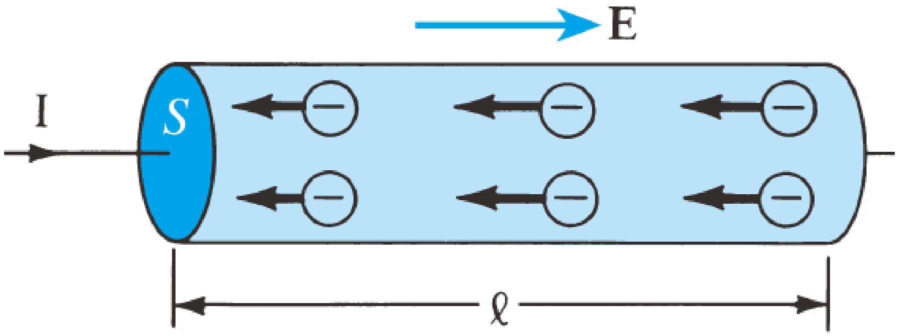
$$J = \frac{I}{A} = \sigma E = \frac{E}{\rho}$$

$$\therefore \quad V = lE = l\rho J = \frac{\rho l}{A} I$$

$or \quad V = I\,R$ OHM's Law

where $R = \dfrac{\rho l}{A}$ ohms or $[\Omega] \equiv$ *electrical resistance*

Note on sign of I :



$$"\vec{I}" = \vec{J}A \qquad \vec{J} = \rho\vec{v}_d$$

$$\rho = nq$$

$$\therefore \ \vec{I} = qn\vec{v}_d A$$

- *for $-ve$ charge carriers: q is $-ve$ and v_d is $-ve$ w.r.t. $\vec{E}$*
- *for $+ve$ charge carriers: q is $+ve$ and v_d is $+ve$ w.r.t. $\vec{E}$*

i.e., I is in same direction

Power :

$$P = \frac{dW}{dt} = \vec{F} \bullet \vec{v}$$

$$\therefore P = \int_V \rho_{charge} dv \; \vec{E} \bullet \vec{v}_d$$

$$P = \int_V \vec{E} \bullet \vec{J} \; dv \qquad (\text{Joule's Law})$$

also $w_P \equiv$ power density $[W / m^3]$

$$w_P \equiv \frac{dP}{dv} = \vec{E} \bullet \vec{J} = \sigma \left| \vec{E} \right|^2$$

– for conductor of uniform cross section $(dv = dAdl)$:

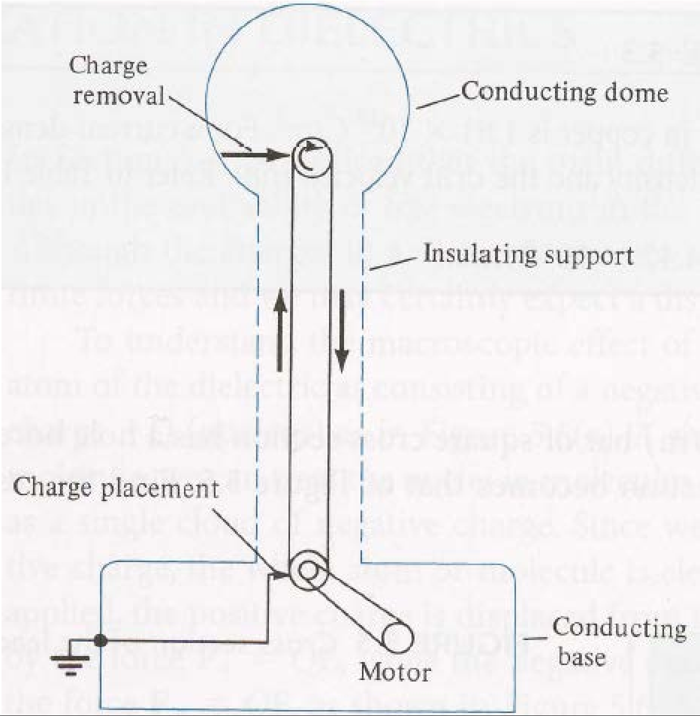
$$P = \int_L Edl \int_A JdA = VI$$

$$\Rightarrow P = I^2 R$$

PRACTICE EXERCISE 5.2

In a Van de Graaff generator, $w = 0.1$ m, $u = 10$ m/s, and from the dome to the ground there are leakage paths having a total resistance of 10^{14} Ω. If the belt carries charge 0.5 μC/m^2, find the potential difference between the dome and the base. *Note:* In the steady state, the current through the leakage path is equal to the charge transported per unit time by the belt.

Answer: 50 MV.



$$w = width\ of\ \ belt = 0.1\ m$$

$$\sigma = belt's\ surface\ \ charge\ density$$
$$= 0.5\ \mu C / m^2$$

$$u = belt's\ velocity = 10\ m/s$$

$$R_{leakage} = 10^{14}\ \Omega$$

Example

A lightning bolt strikes an elm tree located at $(0,0,0)$, with the long axis of the tree's trunk coinciding with the positive $z-$axis. The xy - plane coincides with the surface of the earth. As a consequence a significant current passes into the ground. Assume that the damp ground has $\rho = 50\ \Omega \cdot m$. It turns out that given the nature of the sub - surface soil condition that the current spreads out according to

$$\vec{J} = \frac{50000}{r^3}(2\sin\theta\hat{r} + \sin\theta\hat{\theta})$$

a) Find the current entering the elm tree by calculating the current passing through a hemispherical shell of radius 15 m, $\frac{\pi}{2} < \theta < \pi, 0 < \phi < 2\pi$.

Is this a correctrepresentation of total current entering / leaving the elm tree?

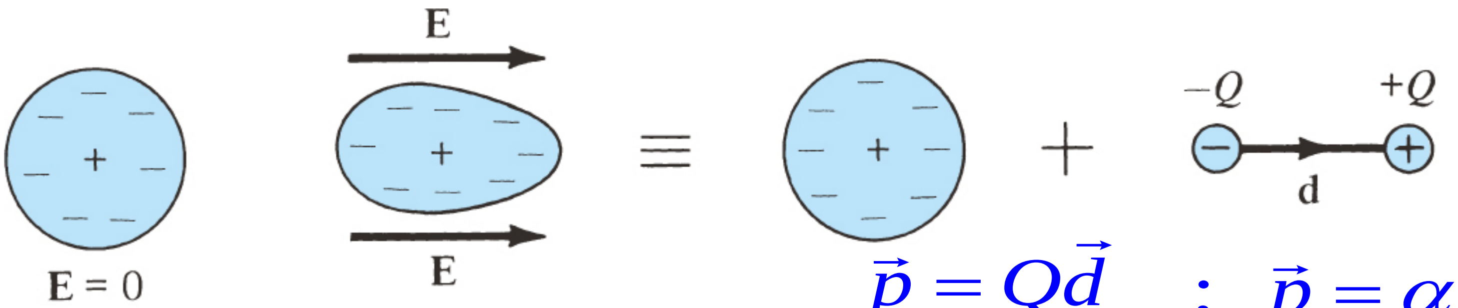
b) *A cow, facing the tree, stands a distance of* 15 *m away from the tree. Her front feet are positioned side by side (assume they make contact with the earth as a point* 15 *m from the origin) and her hind feet are positioned side by side effectively making contact with the earth at a point* 2 *meters further away from the origin, with both points lying along a radial line in the xy - plane at* $\phi = 52^o$. *Although the cow is not hit by the lightning, it is killed. Quantify why.*

Answer: a) 32.89 kA; b) 2461 V

2 Polarization in Dielectrics

Induced Dipoles:

–*neutral*, *non – polar atom*



(*effect of* $\vec{F} = Q\vec{E}$)

atom has been polarized

$\vec{p} = Q\vec{d}$; $\vec{p} = \alpha\vec{E}$

$\alpha \equiv$ *atomic polarizability*

For N dipoles in volume Δv:

$$\vec{p}_{total} = Q_1\vec{d}_1 + Q_2\vec{d}_2 + \ldots + Q_N\vec{d}_N = \sum_{k=1}^{N} Q_k\vec{d}_k \quad [C/m^2]$$

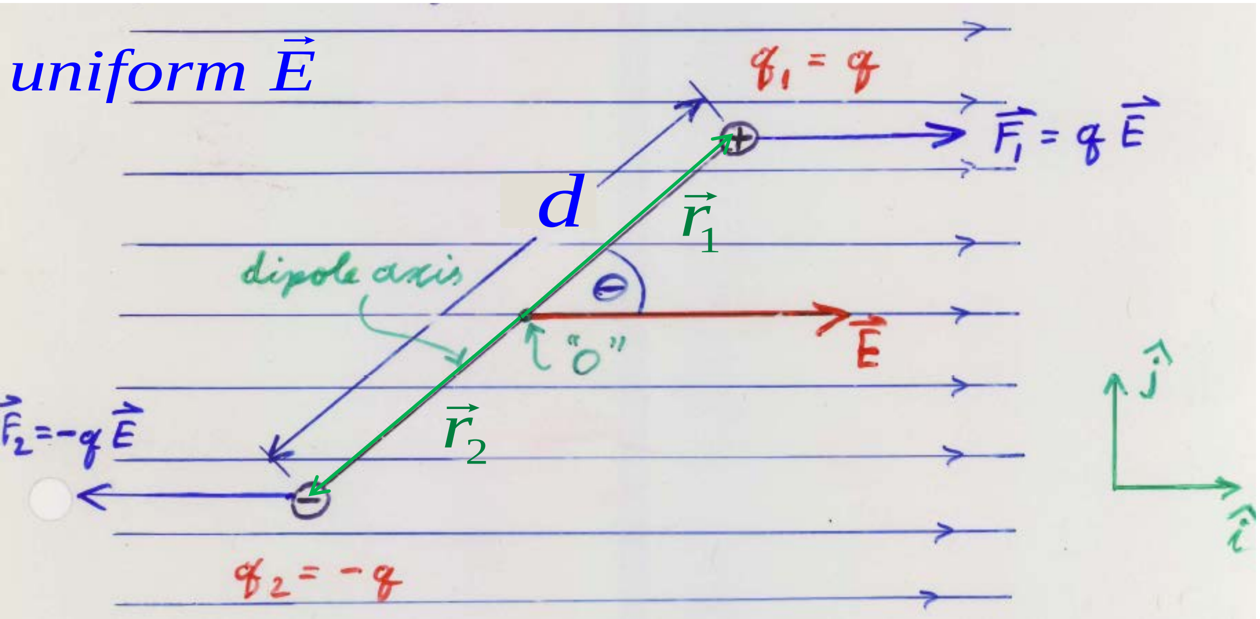
–*define polarization* $\vec{P} = \lim_{\Delta v \to 0} \dfrac{\sum_{k=1}^{N} Q_k\vec{d}_k}{\Delta v}$ $\begin{pmatrix} \text{dipole moment} \\ \text{per unit volume} \end{pmatrix}$

i.e. $E-$ *field applied to a* nonpolar dielectric :

$\rightarrow$ *creation of dipoles that align with* $\vec{E}$

Polar molecules – Permanent Dipoles :

(e.g. the water molecule)



$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 = 0$$

$$\vec{\tau}_{net} = \vec{r}_1 \mathrm{x} \vec{F}_1 + \vec{r}_2 \mathrm{x} \vec{F}_2$$

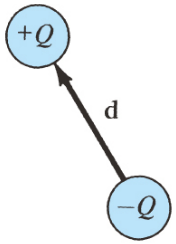
$$\vec{\tau}_{net} = (\vec{d}/2)\mathrm{x}(q\vec{E}) + (-\vec{d}/2)\mathrm{x}(-q\vec{E})$$

$$\vec{\tau}_{net} = q\vec{d}\mathrm{x}\vec{E}$$

• $\vec{p}$ *in uniform* $\vec{E}$:

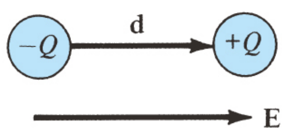
$$\vec{\tau} = \vec{p}\mathrm{x}\vec{E}$$

i.e. a polar molecule will tend to line up with the field :



$\vec{E} = 0$

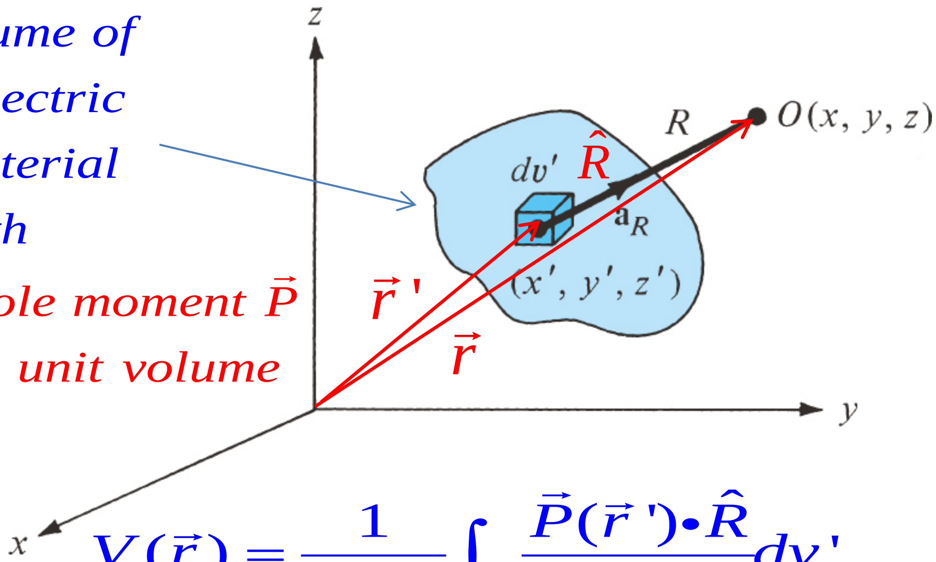
$\vec{E} \neq 0$



Field due to polarized dielectric:

$-had\ V(\vec{r}) = \dfrac{\vec{p}\bullet(\hat{r}-\vec{r}\,')}{4\pi\varepsilon_0 |\vec{r}-\vec{r}\,'|^3}$

volume of dielectric material with dipole moment $\vec{P}$ per unit volume



Find $\vec{E}$ at external point O by first finding V:

–due to dipole moment $\vec{P}dv'$

$$dV = \frac{\vec{P}(\vec{r}\,')\bullet\hat{R}dv'}{R^2}$$

$$\begin{pmatrix} \vec{R} = \vec{r} - \vec{r}\,' \\ R^2 = (x-x')^2 + (y-y')^2 + (z-z')^2 \end{pmatrix}$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\vec{P}(\vec{r}\,')\bullet\hat{R}}{R^2}dv'$$

Can shown (see section 7.7) $\vec{\nabla}'\left(\dfrac{1}{R}\right) = \dfrac{\hat{R}}{R^2}$ $\quad$ (i.e. the gradient of $(1/R)$ w. r. t (x', y', z'))

$$\therefore\ \frac{\vec{P}\bullet\hat{R}}{R^2} = \vec{P}\bullet\vec{\nabla}'\left(\frac{1}{R}\right) \quad \text{and} \quad V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \vec{P}\bullet\vec{\nabla}'\left(\frac{1}{R}\right)dv'$$

Noting the vector identity $\vec{\nabla}'\bullet f\vec{A} = f\vec{\nabla}'\bullet\vec{A} + \vec{A}\bullet\vec{\nabla}'f$, the integrand becomes

$$\vec{\nabla}'\bullet\left(\frac{\vec{P}}{R}\right) - \frac{\vec{\nabla}'\bullet\vec{P}}{R} \quad \text{and} \quad V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\int_V \vec{\nabla}'\bullet\left(\frac{\vec{P}}{R}\right)dv' - \int_V \frac{\vec{\nabla}'\bullet\vec{P}}{R}dv'\right]$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\int_V \vec{\nabla}'\bullet\left(\frac{\vec{P}}{R}\right)dv' - \int_V \frac{\vec{\nabla}'\bullet\vec{P}}{R}dv'\right]$$

Now apply divergence theorem to 1^{st} term:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\oint_S \frac{1}{R}\vec{P}\bullet d\vec{A}' - \frac{1}{4\pi\varepsilon_0}\int_V \frac{1}{R}(\vec{\nabla}'\bullet\vec{P})dv'$$

–integrand of 1^{st} term: $\dfrac{(\vec{P}\bullet\hat{n}')dA'}{R}$

–compare 1^{st} term to $V(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int_S \frac{\sigma(\vec{r}\,')dA'}{|\vec{r}-\vec{r}\,'|}$

$\Rightarrow 1^{st}$ *term = potential of surface charge distribution with*

$\sigma_b = \vec{P}\bullet\hat{n}$ *(primes have been droped) (understood that $\vec{P}(\vec{r}\,')$ and $\hat{n}'$ are involved)*

–compare 2^{nd} term to $V(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int_V \frac{\rho(\vec{r}\,')dV'}{|\vec{r}-\vec{r}\,'|}$

$\Rightarrow 2^{nd}$ *term = potential of volume charge distribution with*

$\rho_b = -\vec{\nabla}\bullet\vec{P}$ *(understood that source point coordinates are involved)*

σ_b and ρ_b are bound charge densities *- not free to move in the dielectric - formed due to charge displacement on molecular scale*

Then

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\oint_S \frac{\sigma_b}{R}\,dA' - \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho_b}{R}\,dv'$$

i.e. to find the potential (and field) of a polarized object can add up all contributions from the infinitesmal dipoles

OR can find the bound charges σ_b, ρ_b and then find the fields using the usual methods (e.g. Gauss's Law, etc.)

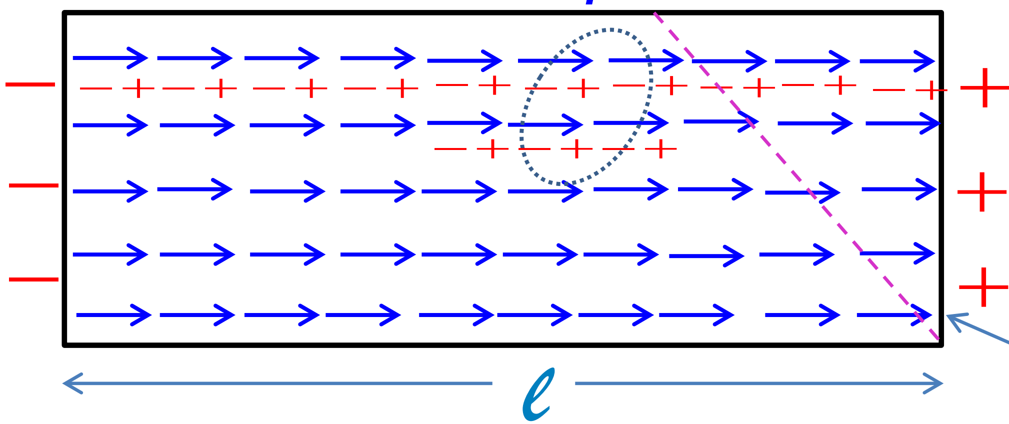
Note that this a completely general result – applies to any object of any shape

Note that σ_b *and* ρ_b *are also referred to as the* polarization charge densities, or induced charge densities

Physical interpretation of bound charge densities:

-the bound charge densities result from actual charge built-up in the dielectric

–dielectric with uniform $\vec{P}$:



- for any particular small volume element net q is zero $\Rightarrow \rho_b = 0$

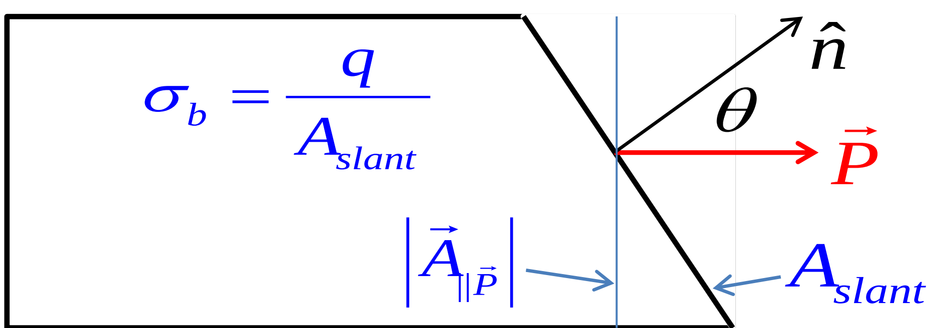
(consistent with $\rho_b = -\vec{\nabla}\bullet\vec{P}$*)*

A (bound charge +q)

–such cancellation does not occur at the ends, faces:

$$p = P(A\ell) \;\; ; \;\; \text{also } p = q\ell \quad \Rightarrow q = PA \text{ and } \sigma_b = \frac{q}{A} = P$$

If end is cut at angle w.r.t. $\vec{P}$:



$$\left|\vec{A}_{\|\vec{P}}\right| = A_{slant}\cos\theta ; \;\; \sigma_b = \frac{q}{\left|\vec{A}_{\|\vec{P}}\right|}\cos\theta$$

but $\left|\vec{A}_{\|\vec{P}}\right| = \text{above } A; \; q = \text{above } q$

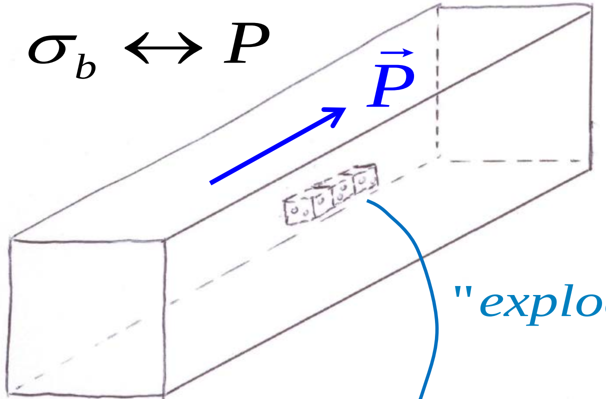
$$\therefore \sigma_b = P\cos\theta = \vec{P}\bullet\hat{n}$$

–as from earlier math derivation

Another view of $\sigma_b \leftrightarrow P$

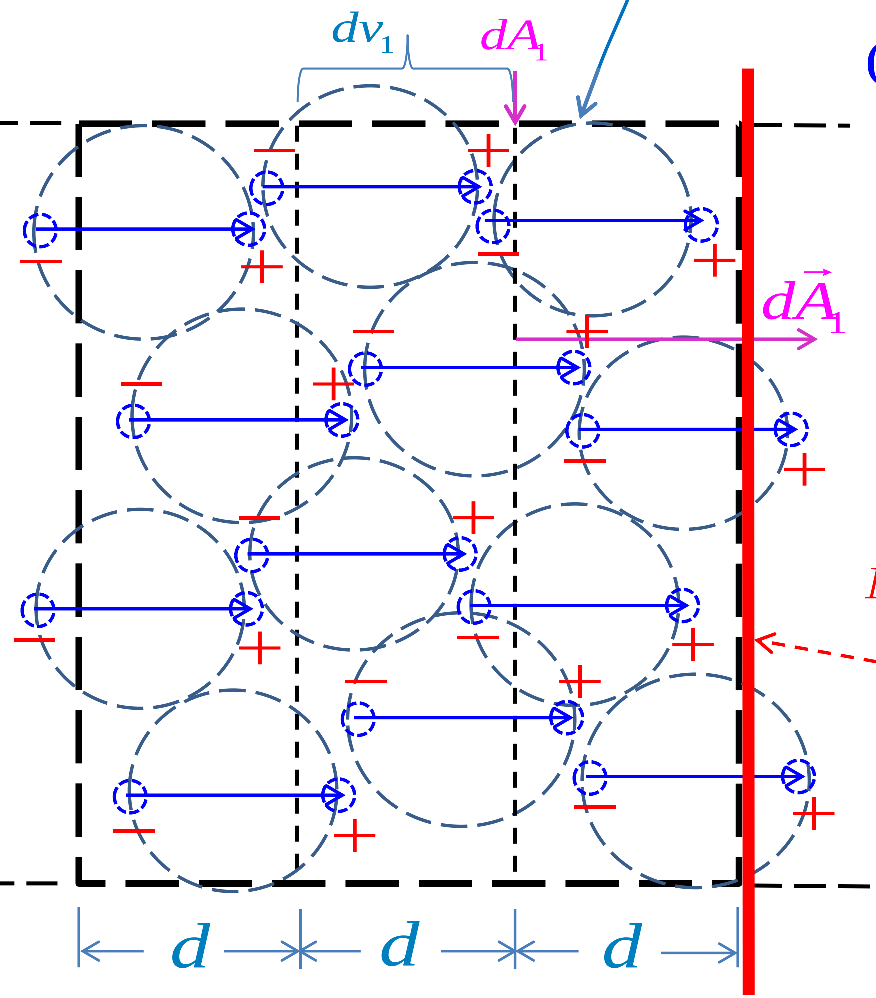
– a slab of dielectric:

– uniform $\vec{P}$

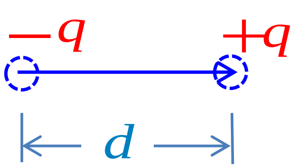


"exploded view" of a volume element

$\vec{P}$



– represent each polarized molecule by "stick & ball dipole":



N molecules / unit volume (uniform density)

– total $+ve$ charge crossing dA_1, left to right = positive charge $(4q)$ inside volume dv_1

$$dv_1 = \vec{d} \bullet d\vec{A}_1$$

$$dq = Nq\vec{d} \bullet d\vec{A} = N\vec{p} \bullet d\vec{A}$$

$$= \vec{P} \bullet d\vec{A} = \vec{P} \bullet \hat{n} dA$$

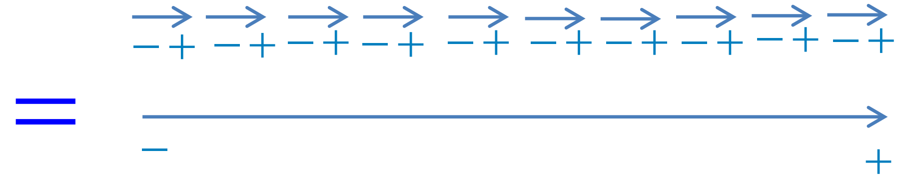
NOTE: zero net Q crosses dA_1

– Now cut the material:

$\Rightarrow +4q$ in the surface molecular layer

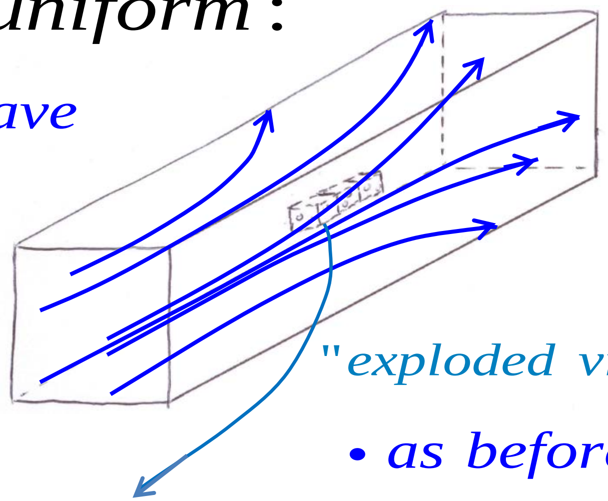
$$\therefore \frac{dq}{dA} = \sigma_b = \vec{P} \bullet \hat{n}$$ (as before)

In above discussion can think of strings of dipoles spanning the material :



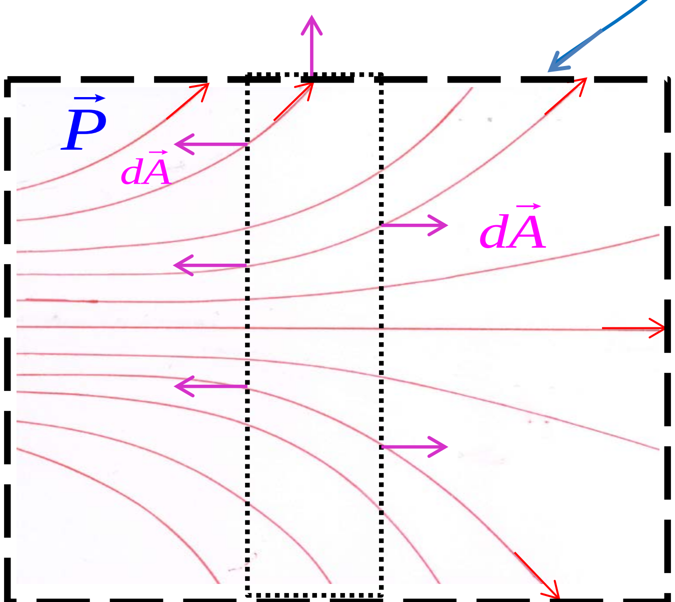
If $\vec{P}$ is nonuniform :

(dipoles may have different magnitudes and directions)



$\vec{P}$ vectors

"exploded view" of a volume element



- *as before, some charges are displaced into / out of a particular volume*
- *however now volume may end up with more +ve / −ve charge than of the opposite sign*

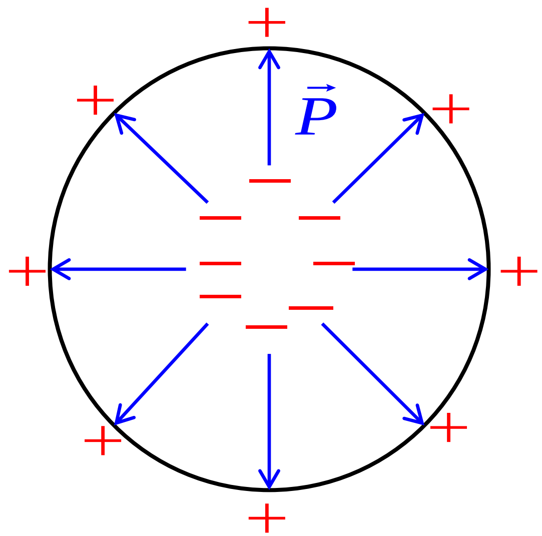
$(\sigma_b = \vec{P} \bullet \hat{n}$ *still applies to vol element*$)$

$\Rightarrow$ *volume charge density has built up*

- *must have* $Q_{\rho_b} = -Q_{\sigma_b}$

– more explicitly –

– spherical dielectric, radially polarized :



– divergence of $\vec{P}$ produces –ve charge build – up in the interior

$\Rightarrow$ *bound volume charge*

- *in addition, get bound surface charge* $(\sigma_b = \vec{P} \bullet \hat{n})$

$$Q_{surf} = -Q_{vol}$$

$$\therefore \int_V \rho_b dv = -\oint_S \sigma_b dA = -\oint_S \vec{P} \bullet d\vec{A} = -\int_V (\vec{\nabla} \bullet \vec{P}) dv$$

- *applies to any volume* $\quad \therefore \rho_b = -\vec{\nabla} \bullet \vec{P}$ $\quad$ *(as before)*

- remember that for electrically neutral dielectric:

$$Q_T = \oint_S \sigma_b dA + \int_S \rho_b dv = 0$$

PRACTICE EXERCISE 5.5

A thin rod of cross-sectional area A extends along the x-axis from $x = 0$ to $x = L$. The polarization of the rod is along its length and is given by $P_x = ax^2 + b$. Calculate ρ_{pv} and ρ_{ps} at each end. Show explicitly that the total bound charge vanishes in this case.

Answer: $0, -2aL, -b, aL^2 + b$, proof.

Example

Consider a uniformly polarized sphere of radius a with polarization P. Calculate the electric field due to the sphere. (i.e. inside and outside the sphere)

Hint: You may use the result that for a charged shell with surface charge density $P\cos\theta$:

$$V(r,\theta)=\begin{cases}\dfrac{P}{3\varepsilon_0}r\cos\theta, & \text{for } r\le R\\[2ex] \dfrac{P}{3\varepsilon_0}\dfrac{R^3}{r^2}\cos\theta, & \text{for } r\ge R\end{cases}$$

Answer: $\vec{E}_{inside}=\dfrac{-P}{3\varepsilon_0}(\cos\theta\hat{r}-\sin\theta\hat{\theta})$; $\vec{E}_{outside}=\dfrac{PR^3}{3\varepsilon_0 r^3}(2\cos\theta\hat{r}+\sin\theta\hat{\theta})$

Comment re problem on Assignment #4:

Consider the uniformly polarized sphere of radius a with polarization $\vec{P}$ discussed in class. Take the polarization to be directed along the (+ve) $z-$axis.

a) Find the potential and electric field along the z-axis, for positive values of z. (i.e. for positive z, inside and outside the sphere)

Solve this problem by first working out the potential on the $z-$axis (for $+ve\ z$); i.e., do not simply use the result for $V(\vec{r})$ that was provided in class.

b) Express the potential and electric field for $z > a$ in terms of the total dipole moment of the sphere.

Field inside a dielectric:

a) Assumed that $\vec{P}$ *(and* $\vec{E}$, $\vec{D}$*) inside a dielectric is a continuous (smooth) vector field (function of* $\vec{r}$*)*

b) However, the dielectric is made up of atoms, molecules (nuclei, electrons) $\Rightarrow$ *on the microscopic scale it must be wildly varying*

–in fact, not just with $\vec{r}$*, but also with time*

c) Often the "pure" or "perfect" dipole field (i.e. $r \gg d$*) is used as a basis for calculating the fields – starting with V*

–in fact, were quantifying with $\vec{P}$*, starting with* $V(\vec{r})$ *of a pure dipole*

$$\sigma_b = \vec{P} \bullet \hat{n} \qquad\qquad \rho_b = -\vec{\nabla} \bullet \vec{P}$$

–in principle the "physical" dipole field should be used

d) Rather than these microscopic fields*,*

here we are interested in the macroscopic fields

$\rightarrow$ *average of the microscopic fields over time and over a volume of sufficient size to encompass a large number of molecules (dipoles), but small enough to be an infinitesmal volume element*

e) In any case

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\oint_S \frac{1}{R}\vec{P}\bullet\hat{n}'\,dA' - \frac{1}{4\pi\varepsilon_0}\int_V \frac{1}{R}(\vec{\nabla}'\bullet\vec{P})\,dv'$$

$$and\ /\ or\quad V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\oint_S \frac{\sigma_b}{R}\,dA' - \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho_b}{R}\,dv'$$

from which the electric field can be calculated.

The Electric Displacement and Gauss's Law in presence of dielectrics:

- as we polarize the dielectric a polarization field is set up
  - the field due to the bound charges

- there may be other charge density present
  - which is not a consequence of polarization
  - e.g. electrons in a conductor; ions in the dielectric

• such charge is called "free charge"

with free charge density ρ_f

$\therefore$ within the dielectric the total volume charge density becomes

$$\rho = \rho_b + \rho_f$$

• then Gauss's Law becomes

$$\rho = \rho_b + \rho_f = \varepsilon_0 \vec{\nabla} \bullet \vec{E} = -\vec{\nabla} \bullet \vec{P} + \rho_f$$

i.e. $\vec{E}$ is the total field (due to polarization and free charge)

$$\Rightarrow \quad \vec{\nabla} \bullet (\varepsilon_0 \vec{E} + \vec{P}) = \rho_f$$

$$\text{or} \quad \vec{\nabla} \bullet \vec{D} = \rho_f$$

where $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ $\equiv$ electric displacement

i.e. application of $\vec{E}$ to dielectric increases the flux density $\vec{D}$ in the dielectric by an amount $\vec{P}$

(earlier definition of flux density $\vec{D} = \varepsilon_0 \vec{E}$ was special a case)

• thus $\quad \vec{\nabla} \bullet \vec{D} = \rho_f$ –differential form of Gauss's Law

$\oint_S \vec{D} \bullet d\vec{A} = Q_{f_{enc}}$ –integral form of Gauss's Law

–useful form as only free charge, which we can control, is involved

Example – the bar electret

Consider a cylindrical dielectric (*radius* a *and length* L) *that has a "frozen-in", uniform polarization* $\vec{P} \parallel$ *to axis of cylinder.*

(*only certain materials can be permanently polarized;*
e.g. barium titanate)

a) *Find the bound charge.*

b) *Sketch the electric field for* i) $L \gg a$ *and* ii) $L \ll a$.

Example *A spherical shell with inner and outer radii a and b, respectively, is made of dielectric material and has* "frozen-in" $\vec{P}(\vec{r}) = \frac{c}{r}\hat{r}$, *where c is a constant. Find* $\vec{E}$ *everywhere, as follows:*

i) Find all bound charge and then use our earlier form of Gauss's Law, $\phi = \oint_S \vec{E} \bullet d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$, *to find* $\vec{E}$ *it produces.*

ii) Use the equation we just developed $\left(\oint_S \vec{D} \bullet d\vec{A} = Q_{f_{enc}} \right)$ *to find* $\vec{D}$ *and then find* $\vec{E}$ *from* $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$.

Answer:

for $r < a, r > b$:

$\vec{E} = 0$

for $a < r < b$:

$\vec{E} = -\frac{c}{\varepsilon_0 r}\hat{r}$

$Comment\ re\ \vec{D}\ versus\ \vec{E}:$

$-had\ Gauss's\ Law: \vec{\nabla}\bullet\vec{D} = \rho_f$

$and\ earlier\ \vec{\nabla}\bullet\vec{E} = \dfrac{\rho}{\varepsilon_0}\ (\rho = total\ charge\ density)$

$-does\ this \Rightarrow that\ \vec{D} \leftrightarrow \vec{E},\ except\ for\ factor\ \varepsilon_0$

$and\ that\ the\ source\ for\ \vec{D}\ is\ only\ \rho_f\ (instead\ of\ \rho)??$

$NO:$

$a)\ no\ Coulomb's\ Law\ for\ \vec{D}:\ \ \vec{D}(r) \neq \dfrac{1}{4\pi}\int \dfrac{\rho_f(\vec{r}\,')}{R^2}\hat{R}dv$

b) the reason for $\vec{D}$ and $\vec{E}$ not being as simply comparable is connected with the fact that the divergence $\vec{\nabla}\bullet\vec{D}=\rho_f$ is not enough to define a vector field; also need to know the curl

$-specifically, in$ electrostatics $curl\ of\ \vec{E}=0\ always,$

$$but\ \vec{\nabla}\mathrm{x}\vec{D}\ is\ not\ always\ zero:\vec{\nabla}\mathrm{x}\vec{D}=\varepsilon_0(\vec{\nabla}\mathrm{x}\vec{E})+(\vec{\nabla}\mathrm{x}\vec{P})=\vec{\nabla}\mathrm{x}\vec{P}$$

$$where\ in\ general\ \vec{\nabla}\mathrm{x}\vec{P}\ is\ not\ always\ zero$$

$$e.g.\ the\ bar\ electret:\rho_f=0\Rightarrow D=0\ ??\ everywhere??$$

$$would\Rightarrow\vec{E}=-\vec{P}/\varepsilon_0\ inside\ and\ E=0\ outside!?!\ (since\ \vec{D}=\varepsilon_0\vec{E}+\vec{P})$$

$$-but\ this\ is\ not\ the\ case\ here!$$

$$(-assignment\ problem: show\ where\ \vec{\nabla}\mathrm{x}\vec{P}\neq 0\ in\ bar\ electret\ problem)$$

- *If the appropriate symmetry is present can find* $\vec{D}$ *from* $\oint_S\vec{D}\bullet d\vec{A}=Q_{f_{enc}}$.
- *If not, need another approach, but cannot ASSUME that* $\vec{D}$ *is only determined by* ρ_f.

Electric susceptibility, dielectric constant and strength

– *had* $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

• *for many dielectrics the polarization* $\propto E$ *(E not too large):*

$\vec{P} = \varepsilon_0 \chi_e \vec{E}$ *where* $\chi_e \equiv$ *electric susceptibility (dimensionless)*

–*substitution into above* $\vec{D}$: $\vec{D} = \varepsilon_0 (1+\chi_e)\vec{E} = \varepsilon_0 \varepsilon_r \vec{E}$

or $\vec{D} = \varepsilon \vec{E}$

linear, isotropic dielectrics

where $\varepsilon = \varepsilon_0 (1+\chi_e) \equiv$ *permittivity*

$$\varepsilon_r = 1 + \chi_e = \frac{\varepsilon}{\varepsilon_0} \equiv \text{relative permittivity or dielectric constant}$$

ε_r *for vacuum* $= 1$; ε_r *for air (dry)* $= 1.000536$; ε_r *for water* $= 80.1$

• *dielectric strength* $\equiv$ *maximum electric field that the dielectric can withstand without electric breakdown*

$E_{\max}$ *for air*: $3\text{x}10^6 \ V/m$

Classification of dielectrics:

LINEAR:

- if $\vec{P} = \varepsilon_0 \chi_e \vec{E}$
- if $\vec{D}$ varies linearly with $\vec{E}$ $\left[\vec{D} = \varepsilon_0(1+\chi_e)\vec{E} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E}\right]$
- if ε does not change with applied $\vec{E}$

– otherwise nonlinear

HOMOGENEOUS:

- if ε, or χ_e, do not vary from point to point in the dielectric

(no coordinate dependence)

– otherwise inhomogeneous or nonhomogeneous

ISOTROPIC:

- if $\vec{D}$ and $\vec{E}$ in same direction
- if ε does not change with direction

– otherwise anisotropic or nonisotropic

$-$ *for anisotropic dielectrics* $\vec{D}$, $\vec{E}$ *and* $\vec{P}$ *are not parallel:*

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yz} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

permittivity tensor

$-$*can also be written in terms of a susceptibilty tensor* $\chi_{e\ ij}$ $(note: \varepsilon = \varepsilon_0(1+\chi_e))$

$Comment\ re\ \vec{D}\ versus\ \vec{E}$ – continued:

$-in\ linear\ dielectrics\ \vec{P}\ and\ \vec{D}\ are\ proportional\ to\ \vec{E}$

$-does\ this \Rightarrow since\ curl\ of\ \vec{E} = 0,$

$curl\ of\ \vec{P} = 0\ and\ curl\ of\ \vec{D} = 0\ as\ well?$

$-YES,\ IF\ the\ volume\ is\ filled\ only\ with\ one\ homogeneous$

$linear\ dielectric:$

$\vec{\nabla}\bullet\vec{D} = \rho_f\ and\ \vec{\nabla}\mathrm{x}\vec{D} = 0,\ so\ that\ \vec{D}\ is\ found\ from\ q_f$

($as\ if\ dielectric$ is not there)

$$Then\ \ \vec{D} = \varepsilon_0 \vec{E}_{vac}$$

$$and\quad \vec{E} = \frac{1}{\varepsilon}\vec{D} = \frac{1}{\varepsilon_r}\vec{E}_{vac}$$

$--however\ .....$

$--$ *however .....*

$-$*if you have two linear dielectrics joined :*

$$\oint \vec{P} \bullet d\vec{l} \neq 0 \text{ in spite of the fact that } \oint \vec{E} \bullet d\vec{l} = 0$$

$-$*but by Stokes's theorem* $\oint_L \vec{P} \bullet d\vec{l} = \int_S (\vec{\nabla} \mathrm{x} \vec{P}) \bullet d\vec{A}$

$\Rightarrow$ *in this case* $\vec{\nabla} \mathrm{x} \vec{P}$ *cannot be zero everywhere within the loop*

PRACTICE EXERCISE 5.6

A parallel-plate capacitor with plate separation of 2 mm has a 1 kV voltage applied to its plates. If the space between its plates is filled with polystyrene ($\varepsilon_r = 2.55$), find **E**, **P**, and ρ_{ps}. Assume that the plates are located at $x = 0$ and $x = 2$ mm.

and $\vec{D}$

Answer: $500\mathbf{a}_x$ kV/m, $6.853\mathbf{a}_x$ μC/m^2, 6.853 μC/m^2. $\vec{D} = 11.27\hat{x}\ \mu C/m^2$

How does the capacitance of the above capacitor change if the dielectric is replaced by vacuum? $C = \varepsilon_r C_{vac}$

Example

A metal sphere of radius a carries a charge Q.

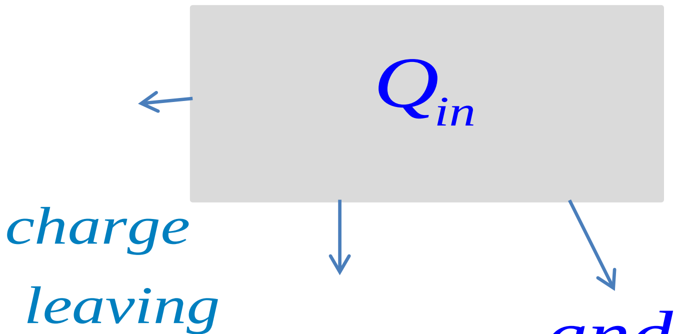
It is surrounded by a linear dielectric (ε) out to a radius b.

Find the potential at the center (relative to infinity).

$$Answer:\ V_{center} = \frac{Q}{4\pi}\left(\frac{1}{\varepsilon_0 b} + \frac{1}{\varepsilon a} - \frac{1}{\varepsilon b}\right)$$

3 Continuity Equation and Relaxation Time

q *inside* V :



$$I_{out} = -\frac{dQ_{in}}{dt} = \oint_S \vec{J} \bullet d\vec{A} = \int_V \vec{\nabla} \bullet \vec{J} dv$$

$$\text{and} \quad -\frac{dQ_{in}}{dt} = -\frac{d}{dt}\int_V \rho dv = -\int_V \frac{\partial \rho}{\partial t} dv$$

$$\therefore \int_V \vec{\nabla} \bullet \vec{J} dv = -\int_V \frac{\partial \rho}{\partial t} dv$$

$$\text{or} \quad \vec{\nabla} \bullet \vec{J} = -\frac{\partial \rho}{\partial t} \quad \rightarrow \text{continuity (of current) equation}$$

$-$*no accumulation of charge at any point*

$$-\text{ for steady currents} \Rightarrow \frac{\partial \rho}{\partial t} = 0;\ \vec{\nabla} \bullet \vec{J} = 0$$

$$\Rightarrow (q \text{ leaving } V) = (q \text{ entering } V) \text{ (Kirchhoff's current law)}$$

Now introduce charge at an interior point of the dielectric or conductor:

–sub Ohm's Law ($\vec{J} = \sigma\vec{E}$) *and Gauss's Law* ($\vec{\nabla}\bullet\vec{E} = \frac{\rho}{\varepsilon}$)

into continuity equation: $\vec{\nabla}\bullet\vec{J} = \vec{\nabla}\bullet\sigma\vec{E} = \frac{\sigma\rho}{\varepsilon} = -\frac{\partial\rho}{\partial t}$

$$\therefore \quad \frac{\partial\rho}{\partial t} + \frac{\sigma}{\varepsilon}\rho = 0 \quad \xrightarrow{\text{sep variables}} \quad \frac{\partial\rho}{\rho} = -\frac{\sigma}{\varepsilon}\partial t$$

$$\therefore \quad \ln\rho = -\frac{\sigma t}{\varepsilon} + C; \text{ at } t = 0 \ \rho = \rho_0, \quad \therefore \quad \ln\rho = -\frac{\sigma t}{\varepsilon} + \ln\rho_0$$

$$\therefore \quad \rho = \rho_0 e^{-\frac{t}{T_r}}$$

where $T_r = \frac{\varepsilon}{\sigma} \equiv$ *relaxation time (rearrangement time)*

e.g. for copper: $T_r \sim 10^{-19}\,s$; *for quartz* $T_r \sim 50$ *days*

Example – Problem 5.30

Given that $\vec{J} = \dfrac{5e^{-10^4 t}}{r}\hat{r}\ A/m^2$, *at* $t = 0.1\ ms$, *find*

a) the current passing surface $r = 2\ m$, *and*

b) the charge density ρ *on that surface.*

Answer : $a)\ 46.23\ A;\ \ b)\ 45.98\ \mu C/m^3$

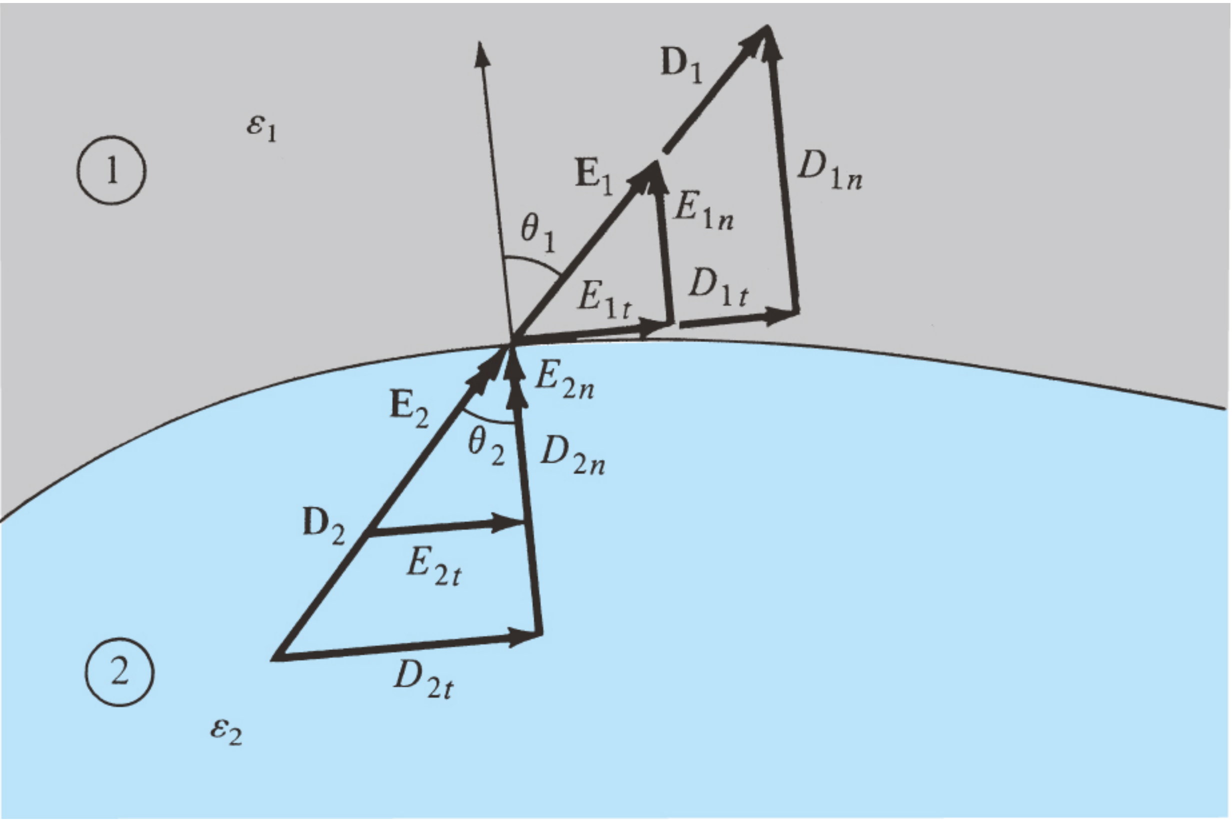
4 Boundary Conditions

$-at\ boundary\ decompose\ \vec{E} = \vec{E}_{tangential} + \vec{E}_{normal} = \vec{E}_t + \vec{E}_n$

$-use\ Maxwell's\ equations\ \left(\oint_L \vec{E}\bullet d\vec{l} = 0\ and\ \oint_S \vec{D}\bullet d\vec{A} = Q_{f_{enc}}\right)$

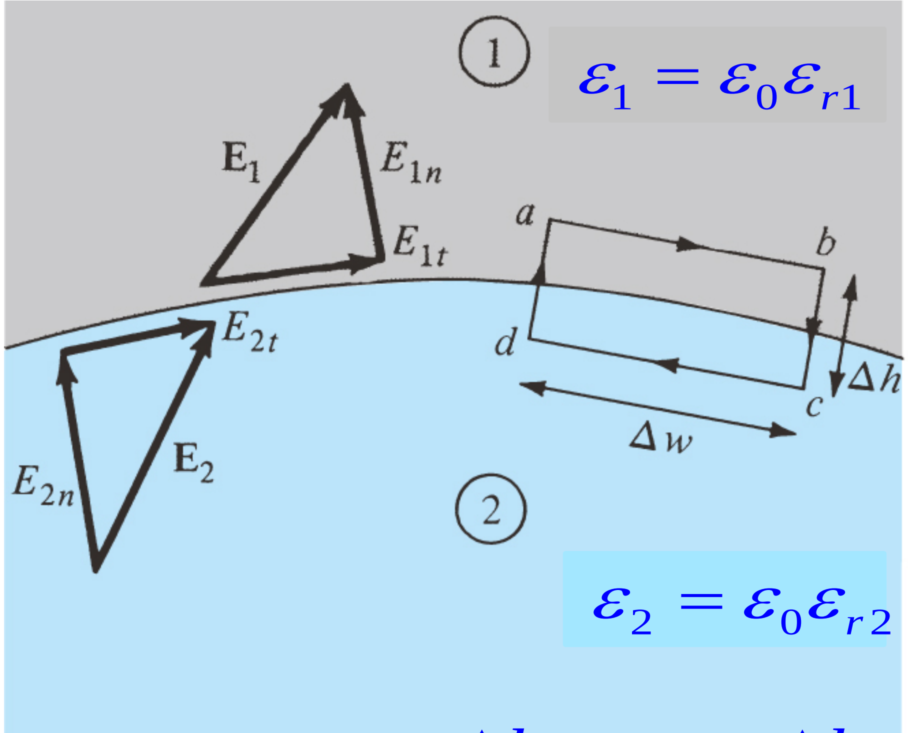
$to\ find\ boundary\ conditions$

$A.\ Dielectric - dielectric$



$-expect\ refraction$
$of\ the\ electric\ field$
$across\ the\ interface$

$Note:\ from\ 2 \rightarrow 1$



$$\vec{E}_1 = \vec{E}_{1t} + \vec{E}_{1n}$$

$$\vec{E}_2 = \vec{E}_{2t} + \vec{E}_{2n}$$

–apply Maxwell's 2nd eqn to closed loop abcd :(assuming loop dimensions very small w.r.t. spatial variation of $\vec{E}$ *i.e.* $\Delta h,\ \Delta w \to$ *small)*

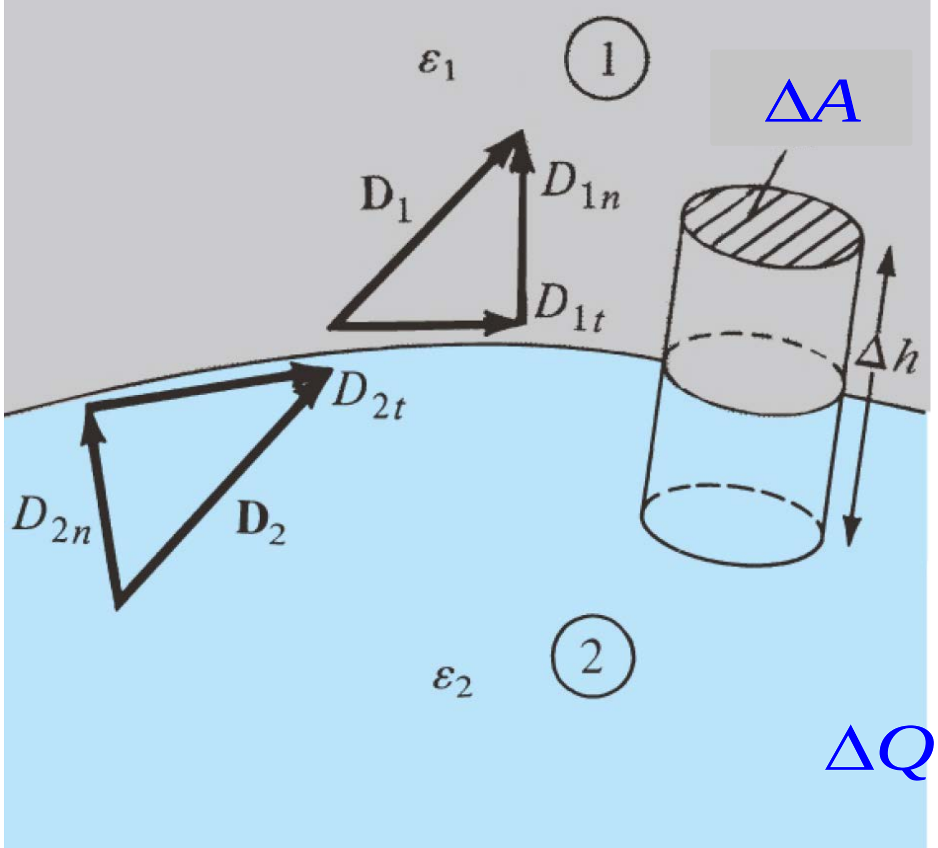
$$E_{1t}\Delta w - E_{1n}\frac{\Delta h}{2} - E_{2n}\frac{\Delta h}{2} - E_{2t}\Delta w + E_{2n}\frac{\Delta h}{2} + E_{1n}\frac{\Delta h}{2} = 0$$

$$0 = (E_{1t} - E_{2t})\Delta w$$

or $E_{1t} = E_{2t} \Rightarrow \vec{E}_t$ continuous *across boundary*

–also $\vec{D} = \varepsilon\vec{E} = \vec{D}_t + \vec{D}_n \Rightarrow \dfrac{D_{1t}}{\varepsilon_1} = E_{1t} = E_{2t} = \dfrac{D_{2t}}{\varepsilon_2}$

or $\dfrac{D_{1t}}{\varepsilon_1} = \dfrac{D_{2t}}{\varepsilon_2} \Rightarrow \vec{D}_t$ discontinuous *across boundary*



–apply Maxwell's 1^{st} eqn to pillbox: (assuming pillbox dimensions very small w.r.t. spacial variation of $\vec{E}$ i.e. ΔA, $\Delta h \to$ small)

–on sides: let $\Delta h \to 0$

$\Rightarrow$ no contribution to flux

$$\Delta Q_{f_{enc}} = \sigma_f \Delta A = D_{1n}\Delta A - D_{2n}\Delta A$$

or $D_{1n} - D_{2n} = \sigma_f \quad \Rightarrow$ if $\sigma_f \neq 0$, D_n is discontinuous

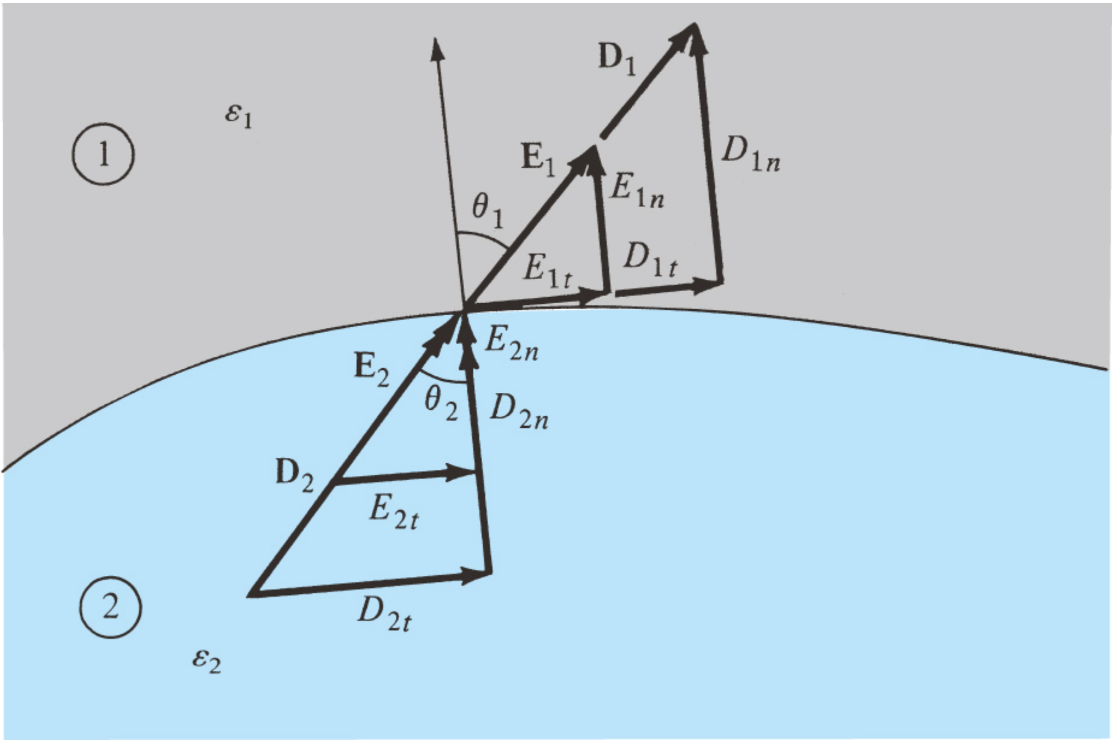
–if NO FREE CHARGE:

$$D_{1n} = D_{2n} \qquad \Rightarrow D_n \text{ is continuous across boundary}$$

and

$$\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n} \qquad \Rightarrow E_n \text{ is discontinous across boundary}$$

Reconsider :



$-had\ \ E_{1t} = E_{2t}$

$$\Rightarrow \ \ E_1 \sin\theta_1 = E_2 \sin\theta_2$$

$-had\ \ D_{1n} = D_{2n}$

$$\Rightarrow \ \ D_1 \cos\theta_1 = D_2 \cos\theta_2$$

or

$$\varepsilon_1 E_1 \cos\theta_1 = \varepsilon_2 E_2 \cos\theta_2$$

$$\therefore \frac{\tan\theta_1}{\varepsilon_1} = \frac{\tan\theta_2}{\varepsilon_2}$$

$--and\ \ since\ \ \varepsilon = \varepsilon_0 \varepsilon_r :$

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}$$

, law of refraction of the electric field at a boundary free of charge

$Example \sim 5.9$

Two extensive homogeneous isotropic dielectrics meet on the plane $z = 0$. For $z > 0$, $\varepsilon_{r1} = 4$ and for $z < 0, \varepsilon_{r2} = 3$. A uniform electric field $\vec{E}_1 = 5\hat{x} - 2\hat{y} + 3\hat{z}\ kV/m$ exists for $z \geq 0$. Find

a) $\vec{E}_2$ for $z \leq 0$

b) The angles θ_1 and θ_2.

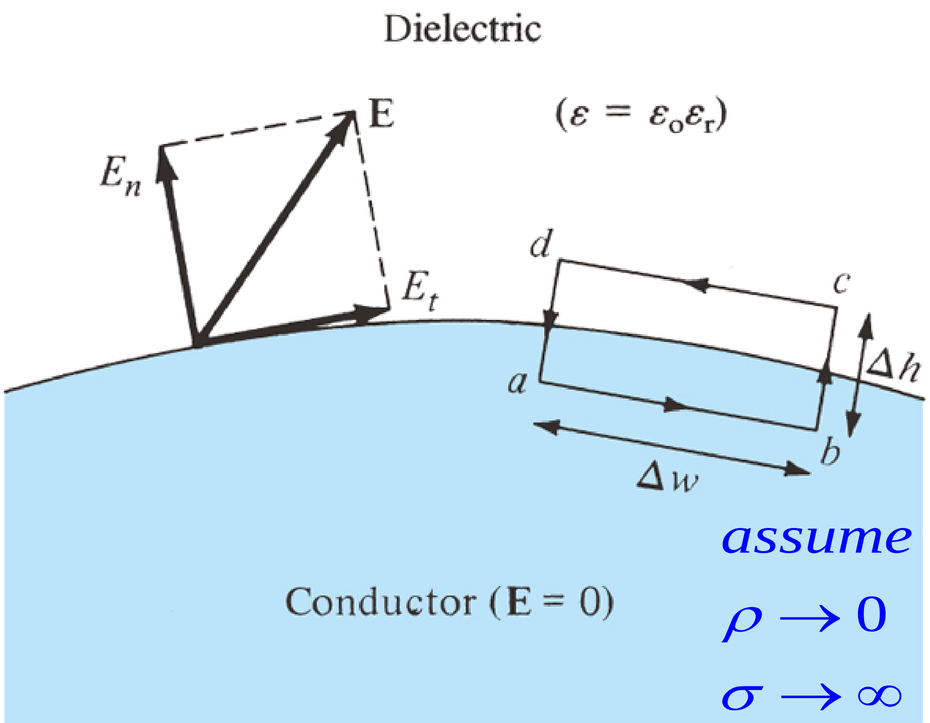
c) Find the energy density (J/m^3) in region 1.

a) $(\vec{E}_2 = 5\hat{x} - 2\hat{y} + 4\hat{z}\ kV/m)$

b) $(\theta_1 = 60.9^\circ, \theta_2 = 53.4^o)$

c) $(672\ \mu J/m^3)$

B. Conductor – dielectric



assume

$\rho \to 0$

$\sigma \to \infty$

– *from Maxwell's* 2^{nd} *eqn*:

$$0 = 0 \cdot \Delta w + 0 \cdot \frac{\Delta h}{2} + E_n \frac{\Delta h}{2}$$

$$- E_t \Delta w - E_n \frac{\Delta h}{2} - 0 \cdot \frac{\Delta h}{2}$$

$$0 = -E_t \Delta w \quad \Longrightarrow E_t = 0$$

– *from Maxwell's* 1^{st} *eqn*:

$$\Delta Q = D_n \cdot \Delta A - 0 \cdot \Delta A$$

$$\Rightarrow \quad D_n = \frac{\Delta Q}{\Delta A} = \sigma$$

$$\therefore D_t = \varepsilon_0 \varepsilon_r E_t = 0$$

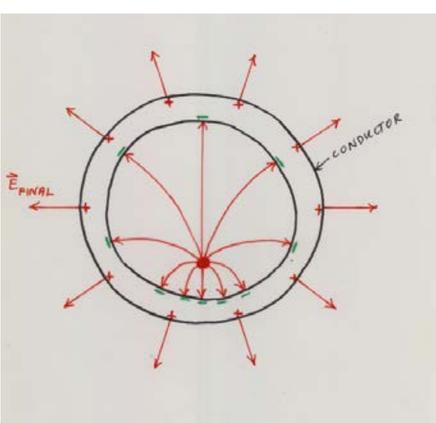
$$D_n = \varepsilon_0 \varepsilon_r E_n = \sigma$$

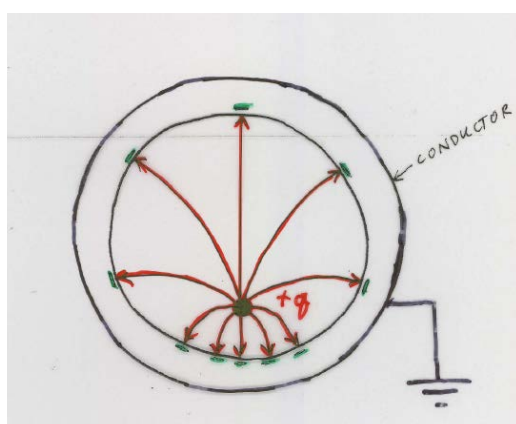
- *for conductor at electrostatic equilibrium:*
  - *within:* $\rho = 0,\ \vec{E} = 0$
  - *within: from* $\vec{E} = -\vec{\nabla} V = 0$ *have that* $\Delta V = 0$ *inside*

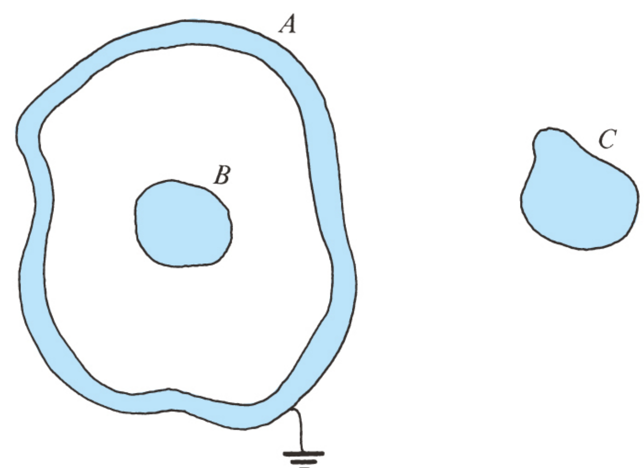
    $\Rightarrow$ *equipotential volume*
  - *external:* $D_t = \varepsilon_0 \varepsilon_r E_t = 0$ *and* $D_n = \varepsilon_0 \varepsilon_r E_n = \sigma$

    i.e. $\vec{E}$ *normal to surface*

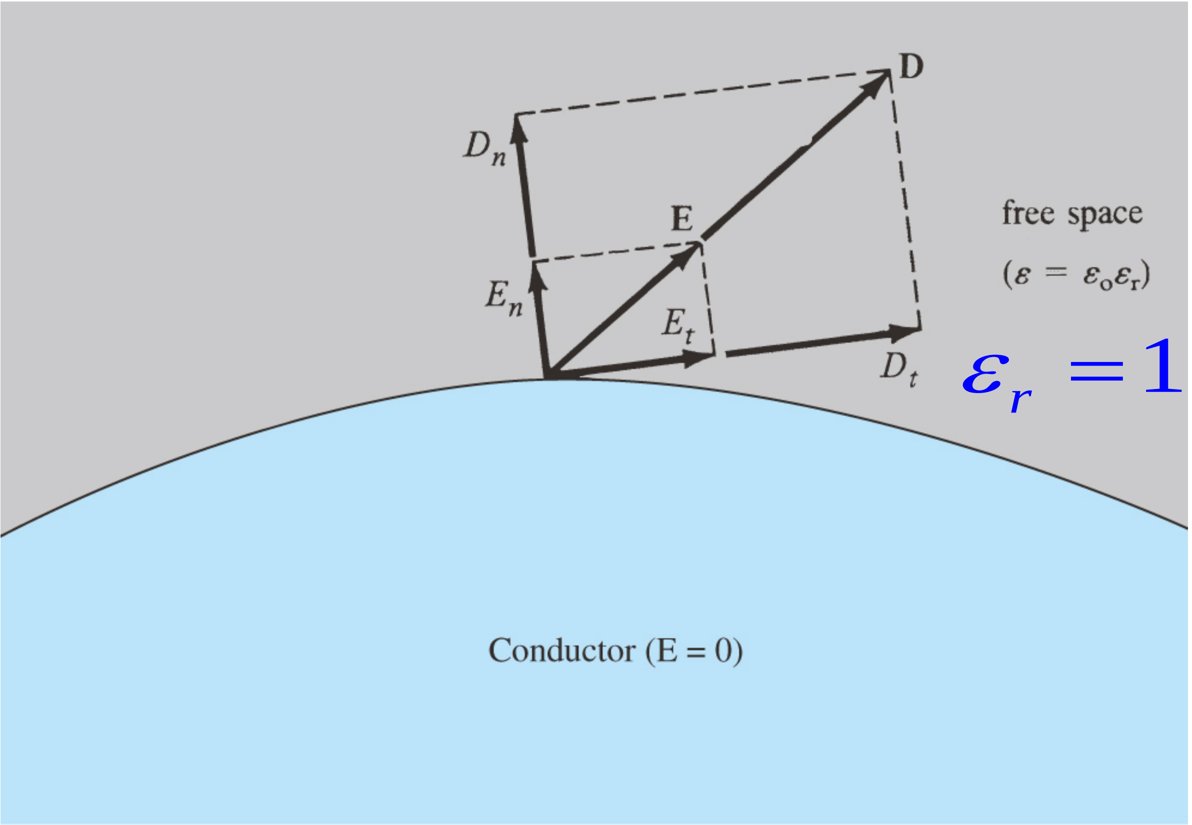
- *one practical application of* $\vec{E} = 0$
  - *electrostatic screening or shielding:*







$C.$ Conductor – free space



$\varepsilon_r = 1$

–conductor – dielectric boundary conditions become:

$$D_t = \varepsilon_0 E_t = 0 \quad and \quad D_n = \varepsilon_0 E_n = \sigma$$

$Example - Problem\ 5.36$

$A\ dielectric\ interface\ is\ defined\ by\ 4x+3y=10\ m.$

$The\ region\ including\ the\ origin\ is\ free\ space,$

$where\ \vec{D}_1 = 2\hat{x} - 4\hat{y} + 6.5\hat{z}\ nC/m^2.$

$In\ the\ other\ region,\ \varepsilon_{r2} = 2.5.$

$Find\ \vec{D}_2\ and\ the\ angle\ \theta_2\ that\ \vec{D}_2\ makes\ with\ the\ normal.$

$\vec{D}_2 = 5.96\hat{x} - 9.28\hat{y} + 16.25\hat{z}\ nC/m^2;\quad \theta_2 = 87.66^o$