Convergence of Fixed Point Methods

- Let g be continuous over interval $[a, b]$,
- g(x) $\in [a, b]$ for all $x \in [a, b]$,
- x* is a fixed point of g $\in [a, b]$,
- $\exists \delta\ s.t.\ g'(x)$ is continuous on $[x^* - \delta, x^* + \delta]$
- Define $x_k = g(x_{k+1})$

Then

- If $|g'(x^*)| < 1$, $\exists \varepsilon$ s.t. $\{x_k\}$ converges to x* for $|x_0 - x^*| < \varepsilon$
- If $|g'(x^*)| > 1$ then $\{x_k\}$ diverges for any x_0.

- Background: Suppose g is a real-valued function, defined and continuous on bounded, closed interval [a,b].
- g is a contraction on [a,b] if ∃L ∈ $(0,1)$, such that$|g(x) - g(y)| \le L|x - y|, \forall x, y \in [a, b]$.
- Note: g is a contraction →
  - Interval [a,b] is "contracted" to interval [g(a), g(b)]
  - Slope of secant line is bounded by L
  - |g'(x)| is bounded by L

Contraction Mapping Theorem

Suppose:

- g is continuous over $[a,b]$
- $a \leq g(x) \leq b$ for all $x \leq [a,b]$
- g is a contraction on $[a,b]$

Then

- g has a fixed point x^* on $[a,b]$
- g has only one fixed point on $[a,b]$
- $\{x_k\} \rightarrow x^*$, where $x_{k+1}=g(x_k)$ and $x_0 \in [a, b]$

Our convergence result follows from the contraction mapping theorem

- Previous assumptions apply
- If $|g'(x^*)| < 1$, then g is a contraction
  - $|x_i - x^*| \leq L|x_{i-1} - x^*|$ with $L < 1$

    $$\rightarrow \frac{|x_i - x^*|}{|x_{i-1} - x^*|} = \frac{|e_i|}{|e_{i-1}|} \leq L < 1$$
  - Linear convergence
- If $|g'(x^*)| > 1$, then the errors are get larger, not smaller → iterates diverge.
- What happens if $|g'(x^*)| = 1$?

Overview of root finding

Method	Converge?	Speed of conv.	Need f'?
Bisection	Yes	Linear	No
Regula Falsi	Yes	Linear	No
Newton	Depends on f and x_0	Quadratic	Yes
Secant	Depends on f and x_0 and x_1	~1.6	No
Fixed point	Depends on g and x_0	Linear	No