So far: Estimating $\mathrm{I} = \int_a^b f(x)dx$

$$I = \int_a^b f(x)dx = \widehat{I_n} + E(f),$$

$$E(\mathrm{f}) = \frac{1}{(n+1)!}\int_a^b \left(\prod_{k=0}^{n}(x - x_k)\right) f^{(n+1)}(\xi(x))dx$$

- Midpoint: $\widehat{I_o} = hf\left(\frac{a+b}{2}\right), h = (b-a)$
- Trapezoid: $\widehat{I_1} = \frac{h}{2}(f(b) + f(a)), h = (b-a)$
- Simpson's: $\widehat{I_2} = \frac{h}{3}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right),$

$$h = \frac{b-a}{2}$$

Useful Result: Weighted Mean Value Theorem for Integrals

Suppose G and φ are continuous over [a,b], and φ can be integrated over [a,b] and does not change sign over [a,b], then there exists a value τ Є[a,b] such that

$$\int_a^b G(t)\varphi(t)dt = G(\tau)\int_a^b \varphi(t)dt$$

error bound for trapezoid rule (n=1)

$$E(f_T) = \frac{1}{(1+1)!}\int_a^b \big((x-a)(x-b)\big)f''(\xi(x))dx$$

.

. (proof developed in class)

.

$|E(f_T)| = \frac{h^3}{12}|f''(\xi_T)|$ for some $\xi_T \in [a,b]$

Note: if f is linear, the error is 0.

Using similar approach, we can develop error bounds for …

- Midpoint Rule: involving f' and h^2
- Simpson's Rule: involving f''' and h^4
- However, for these rules, we can get even better results by taking a different approach

Developing an error bound for the midpoint rule

- Let m = (a+b)/2
- Expand f about m using Taylor's expansion

→ There exists some $\xi(x) \in [a,b]$ such that

$$f(x) = f(m) + f'(m)(x-m) + \frac{1}{2} f''(\xi(x))(x-m)^2$$

- Integrate over [a,b]
- … (prove in class)
- $|E(f_M)| = \frac{h^3}{24} |f''(\xi_M)|$ for some $\xi_M \in [a,b]$

Note: if f is linear, the error is 0.

Developing an error bound for Simpson's rule

- Expand f at m, using third Taylor's polynomial
- There exists some $\xi(x)\in[a,b]$ such that
- $f(x) = f(m) + f'(m)(x-m) + \frac{1}{2}f''(m)(x-m)^2 + \frac{1}{6}f'''(m)(x-m)^3 + \frac{1}{24}f^{(4)}(\xi(x))(x-m)^4$
- Integrate both sides over [a,b]

$$I = f(m)(b-a) + G(x) + H(x)$$

Where

$$G(x) = \left[\frac{f'(m)(x-m)^2}{2} + \frac{f''(m)(x-m)^3}{6} + \frac{f'''(m)(x-m)^4}{24}\right]_a^b$$

$$H(x) = \frac{1}{24}\int_a^b f^{(4)}(\vartheta(x))(x-m)^4 dx$$

Composite Rules

- Divide the interval into n equal pieces
- $h = (b - a)/n$
- $x_k = a + hk$
- $I = \int_a^b f(x)dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)dx = \sum_{i=1}^{n} I_i$
- Each of the approaches can be used in this manner