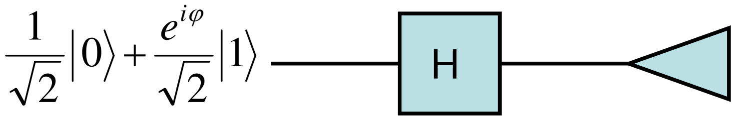
CO481/CS467/PHYS467 ASSIGNMENT 1 Solutions
(will constitute 10% out of the 50% assignment marks)

We denote the Hadamard gate (with respect to the computational basis $\{|0\rangle, |1\rangle\}$) by

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

1. **5 marks**



(a) i. Compute the state of the qubit after a Hadamard transformation H is applied to the state $\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\phi}}{\sqrt{2}}|1\rangle\right)$, for the value $\phi = \pi/4$.
Write this state (in Dirac notation) in the first row and second column of the table below. Express the qubit state in the form $e^{i\alpha}(\cos(\theta)|0\rangle + e^{i\beta}\sin(\theta)|1\rangle)$.

Solution:

Applying the Hadamard matrix gives

$$\frac{1}{2}\left((1 + e^{i\frac{\pi}{4}})|0\rangle + (1 - e^{i\frac{\pi}{4}})|1\rangle\right)$$

Note that $1 + e^{i\phi} = e^{i\phi/2}(e^{-i\phi/2} + e^{i\phi/2})$ and $e^{i\theta} = \cos\theta + i\sin\theta$. Hence

$$\frac{1 + e^{i\phi}}{2} = e^{i\phi/2}\cos(\phi/2)$$

Similarly

$$\frac{1 - e^{i\phi}}{2} = e^{i(\phi/2 - \pi/2)}\sin(\phi/2)$$

This gives us $\alpha = \frac{\pi}{8}$, $\beta = \frac{-3\pi}{8}$ and $\theta = \frac{\pi}{8}$. The final state is

$$e^{i\frac{\pi}{8}}\cos\frac{\pi}{8}|0\rangle + e^{-i\frac{3\pi}{8}}\sin\frac{\pi}{8}|1\rangle = e^{i\frac{\pi}{8}}\left(\cos\frac{\pi}{8}|0\rangle + e^{-i\frac{\pi}{2}}\sin\frac{\pi}{8}|1\rangle\right)$$

ii. Compute the probability of obtaining $|0\rangle$ if the resulting state is measured in the computational basis. Write this value (both in closed form and a 3-digit approximation) in the first row and third column of table below.

Solution:

The probability of a result "0" when measuring in the computational basis is the square of the modulus of the amplitude of $|0\rangle$, which is

$$\cos^2 \frac{\pi}{8} \approx 0.854$$

iii. Also complete the table below for the values of $\phi = 3\pi/4, 5\pi/4, 7\pi/4$.

Solution:

Using analogous arguments we can do the calculations for $3\pi/4$. We then note that $5\pi/4$ and $7\pi/4$ give the complex conjugate of the first two calculations, which changes a few signs. The results are given in the table below

ϕ	output state	p_0	
$\pi/4$	$e^{i\frac{\pi}{8}}(\cos\frac{\pi}{8}\|0\rangle + e^{-i\frac{\pi}{2}}\sin\frac{\pi}{8}\|1\rangle)$	$\cos^2\frac{\pi}{8}$	0.854
$3\pi/4$	$e^{i\frac{3\pi}{8}}(\cos\frac{3\pi}{8}\|0\rangle + e^{-i\frac{\pi}{2}}\sin\frac{3\pi}{8}\|1\rangle)$	$\cos^2\frac{3\pi}{8}$	0.146
$5\pi/4$	$e^{-i\frac{3\pi}{8}}(\cos\frac{3\pi}{8}\|0\rangle + e^{i\frac{\pi}{2}}\sin\frac{3\pi}{8}\|1\rangle)$	$\cos^2\frac{3\pi}{8}$	0.146
$7\pi/4$	$e^{-i\frac{\pi}{8}}(\cos\frac{\pi}{8}\|0\rangle + e^{i\frac{\pi}{2}}\sin\frac{\pi}{8}\|1\rangle)$	$\cos^2\frac{\pi}{8}$	0.854

(b) Note that question 1 shows how to distinguish either of the states $\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\pi/4}}{\sqrt{2}}|1\rangle\right)$, $\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i7\pi/4}}{\sqrt{2}}|1\rangle\right)$ from either of the states $\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i3\pi/4}}{\sqrt{2}}|1\rangle\right)$, $\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i5\pi/4}}{\sqrt{2}}|1\rangle\right)$ with high probability (i.e. guessing based on whether one measures 0 or 1 gives the correct answer with probability $> 85\%$).

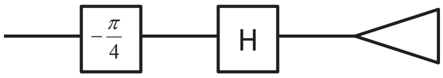
Describe an experiment (provide a figure and explanation) that would allow you to distinguish $\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\pi/4}}{\sqrt{2}}|1\rangle\right)$ from $\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i5\pi/4}}{\sqrt{2}}|1\rangle\right)$ with certainty.

Hint: Use a $-\pi/4$ phase shifter :

$$P_{-\pi/4} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} = |0\rangle\langle 0| + e^{-i\pi/4}|1\rangle\langle 1|.$$

Solution:

We apply the $-\pi/4$ phase shifter to the state. This takes the state $\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\pi/4}}{\sqrt{2}}|1\rangle\right)$ to $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i5\pi/4}}{\sqrt{2}}|1\rangle\right)$ to $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. Applying the Hadamard gate, we see that the resulting states are $|0\rangle$ and $|1\rangle$, respectively, which when



measured in the computational basis distinguishes between the two states with certainty.

2. **2 marks**

   Find a 1-qubit state vector $|+_H\rangle$ such that $H|+_H\rangle = |+_H\rangle$.

   Find a 1-qubit state vector $|-_H\rangle$ such that $H|-_H\rangle = -|-_H\rangle$.

   Solution:

   We are asked to find the eigenvectors of H. In order to do this, we can follow the standard method learned in any introductory linear algebra course.

   We are already given that the eigenvectors are $+1$ and -1 (which follows easily since $H^2 = I$, which implies that an eigenvalue λ satisfies $H^2|\psi\rangle = \lambda^2|\psi\rangle = \pm|\psi\rangle$).

   We solve two linear systems (one for each eigenvalue) to find the eigenvectors. One can use matrix notation or Dirac notation in order to determine the linear equations.

   For the eigenvector with eigenvalue $+1$, let $|+_H\rangle = a|0\rangle + b|1\rangle$. Then $H|+_H\rangle = \frac{a+b}{\sqrt{2}}|0\rangle + \frac{a-b}{\sqrt{2}}|1\rangle$. This gives the equations $a = \frac{a+b}{\sqrt{2}}$ and $b = \frac{a-b}{\sqrt{2}}$, both of which lead to $b = (\sqrt{2}-1)a$ (equivalently $a = (\sqrt{2}+1)b$; note that $(\sqrt{2}-1)(\sqrt{2}+1) = 1$).

   To get a solution with norm 1, we substitute this equation in to $1 = |a|^2 + |b|^2 = a^2 + b^2 = a^2(4 - 2\sqrt{2})$.

   This implies $a^2 = \frac{1}{2(2-\sqrt{2})} = \frac{2+\sqrt{2}}{4}$ and thus $a = \pm\frac{\sqrt{2+\sqrt{2}}}{2}$. Without loss of generality, we can choose $a = \frac{\sqrt{2+\sqrt{2}}}{2}$, and thus $b = \frac{(\sqrt{2}-1)\sqrt{(2+\sqrt{2})}}{2}$.

So

$$|+_H\rangle = \frac{\sqrt{2+\sqrt{2}}}{2}|0\rangle + \frac{(\sqrt{2}-1)\sqrt{2+\sqrt{2}}}{2}|1\rangle$$

(which is roughly $0.924|0\rangle + 0.383|1\rangle$). Alternative equivalent expressions are possible. You may have noticed that $|+_H\rangle = \cos(\pi/8)|0\rangle + \sin(\pi/8)|1\rangle$, for example.

Similarly, we can compute (or alternatively, find by inspection a vector orthogonal to $|+_H\rangle$ since there is only one basis vector left to find):

$$|-_H\rangle = \frac{(\sqrt{2}-1)\sqrt{2+\sqrt{2}}}{2}|0\rangle - \frac{\sqrt{2+\sqrt{2}}}{2}|1\rangle.$$

3. **3 marks**

   Prove that for $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|1\rangle$, and for any unitary one qubit gate G we have:

   $$G \otimes I|\beta_{00}\rangle = I \otimes G^t|\beta_{00}\rangle$$

   where G^t is the transpose of G.

   Solution 1:

   Using matrix notation,

   $$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

   Let

   $$G = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

   $$G \otimes I = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}$$

   $$I \otimes G^t = \begin{pmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{pmatrix}$$

   Multiplying by $|\beta_{00}\rangle$ just sums the first and last columns (and multiplies by a scalar), which we can easily see results in

$$\frac{1}{\sqrt{2}}\left(\begin{array}{c} a \\ b \\ c \\ d \end{array}\right)$$

for both matrices, thus the two sides are equal.

Solution 2:

Let

$$G = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)$$

Then we apply G to the first qubit, giving

$$\begin{aligned}(G \otimes \mathbb{I})|\beta_{00}\rangle &= (a|0\rangle + c|1\rangle)|0\rangle + (b|0\rangle + d|1\rangle)|1\rangle \\ &= a|00\rangle + b|10\rangle + c|01\rangle + d|11\rangle\end{aligned}$$

Meanwhile, applying G^t to the second qubit, we get

$$\begin{aligned}(\mathbb{I} \otimes G^t)|\beta_{00}\rangle &= |0\rangle(a|0\rangle + b|1\rangle) + |1\rangle(c|0\rangle + d|1\rangle) \\ &= a|00\rangle + b|10\rangle + c|01\rangle + d|11\rangle \\ &= (G \otimes \mathbb{I})|\beta_{00}\rangle\end{aligned}$$

Note that we do not need to use the fact that G is unitary in our solutions.

4. **3 marks**

Give a protocol that wins the following game with probability $\frac{2}{3}$.

An adversary does one of the following (with no restriction on the probability with which she chooses option 1 or 2):

1) Gives you either $|0\rangle$ or $|1\rangle$, with equal probability.

OR

2) Gives you $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$.

You must guess whether the adversary performed 1) or 2).

(Hint: First come up with a protocol that has 'one-sided' error. Then add an extra final step that makes the error probability equal in both cases.)

Solution:

We first propose a solution with one-sided error, as suggested.

Following the pattern in question 1, we calculate that if the state given was $\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$ then after applying a Hadamard and measuring, the result will always be "0".

For the other case the result will be "0" or "1" with probability 1/2 each.

If we measure a "1", we know that we are in *option* 1, so if we guess "1" we will always guess correctly in this case.

Suppose that whenever we measure a "0", we guess *option* 2. Then we will be correct if the adversary had given us *option* 2, and in the other case we lose.

Overall, with this strategy, whenever the adversary picks *option* 1 we win with probability 1/2, and whenever the adversary picks *option* 2 we win with probability 1. Note that, with this strategy, whenever we guess *option 1* we will always be correct (so no error in this case).

To turn this protocol into the required protocol we do some classical post-processing. If we measure and get a "1", we know we are in *option* 1, so we should always guess this.

However, if we measure a "0", then we could be in either case. The idea is that when we obtain "0" to guess *option* 2 with some probability $p < 1$ so that we boost our chances of winning for option 1. Thus we guess *option* 1 with probability $1 - p$.

With such a post-processing protocol, if we are in *option* 2 then we win with probability p. In *option* "1" we win with probability $\frac{1}{2} + \frac{1}{2}(1-p) = 1 - \frac{p}{2}$, since we measure a 0 or 1 each with probability $\frac{1}{2}$ and the probabilities of winning given the measurement outcome are $(1-p)$ and 1, respectively.

By setting $p = \frac{2}{3}$, we obtained the desired balance of probabilities.

5. **2 marks**

(a) Draw a classical circuit, with 3 input bits and 2 output bits, that consists of XOR gates and $FANOUT$, and maps

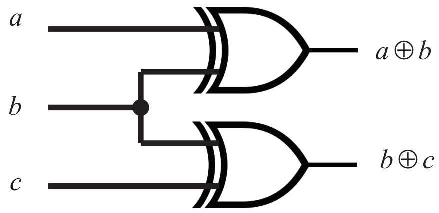
$000 \mapsto 00$	$001 \mapsto 01$	$100 \mapsto 10$	$010 \mapsto 11$
$111 \mapsto 00$	$110 \mapsto 01$	$011 \mapsto 10$	$101 \mapsto 11$

(Aside: Note that the above map may be used to identify where a single bit flip (or no bit flip) has occurred, assuming the input state was either 000 or 111).

Solution:

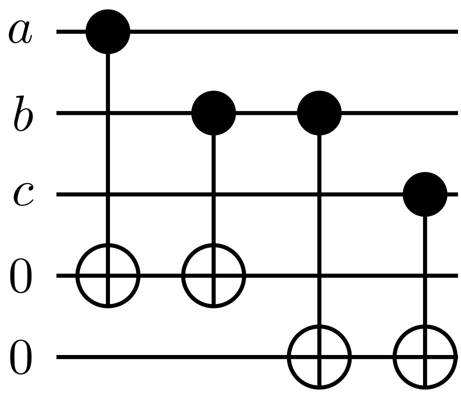
Two XOR gates and one fanout (other notation is fine, as long as it's clear) suffice:

(b) Give a reversible 5-bit circuit for computing the above function while keeping the input intact. E.g. the circuit should map $100\ 00 \mapsto 100\ 10$. Use only CNOT gates.



Solution:

One can reversibly compute the same XOR values using CNOT gates. The reversible CNOT preserves the value of the control bit, so the middle bit value can be reused:



6. **3 marks**

   One may express the action of unitary operations using several notations. E.g. the X gate can be represented using

- matrix notation:

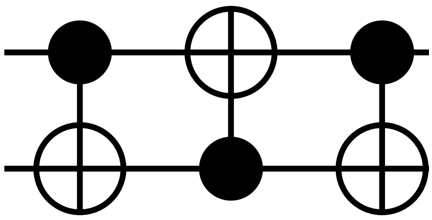
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- Dirac notation for the operator:

$$|0\rangle\langle 1| + |1\rangle\langle 0|$$

- element-wise action for a basis, using Dirac notation:
  $|0\rangle \mapsto |1\rangle$
  $|1\rangle \mapsto |0\rangle$.

For the following circuit, calculate the unitary it computes, and describe the unitary using each of the above three notations.



Solution:

One may calculate the final answer using various notations. For example, in matrix notation, we first note that the $CNOT$ with the second bit as control bit and first bit as target bit is represented as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and thus multiplying out the three corresponding matrices gives

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This can then be translated into the other two notations.

Alternatively, when describing the element-wise action on a basis using Dirac notation, we get (each of the arrows denotes the action of one of the CNOTs):

$|00\rangle \mapsto |00\rangle \mapsto |00\rangle \mapsto |00\rangle$

$|01\rangle \mapsto |01\rangle \mapsto |11\rangle \mapsto |10\rangle$

$|10\rangle \mapsto |11\rangle \mapsto |01\rangle \mapsto |01\rangle$

$|11\rangle \mapsto |10\rangle \mapsto |10\rangle \mapsto |11\rangle$.

In summary $|00\rangle \mapsto |00\rangle \qquad |01\rangle \mapsto |10\rangle \qquad |10\rangle \mapsto |01\rangle \qquad |11\rangle \mapsto |11\rangle$, which can be translated into the matrix notation above, or as an operator in Dirac notation:

$$|00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11|.$$

More compactly, one could also write $|ab\rangle \mapsto |ba\rangle$ for all $a, b \in \{0, 1\}$ and $\sum_{a,b\in\{0,1\}} |ba\rangle\langle ab|$.

Note that this circuit implements the two-qubit SWAP gate.