4. INHOMOGENEOUS EQUATIONS

FROM THE VARIATION-OF-PARAMETERS FORMULA, WE DERIVED PREVIOUSLY THE FULL SOLUTION TO THE INHOMOGENEOUS DIFFERENTIAL EQUATION:

$$\frac{d\vec{y}}{dx} = \mathbb{A}(x)\vec{y}(x) + \vec{b}(x) \;;\; \vec{y}(x_0) = \vec{y}^{\,0}$$

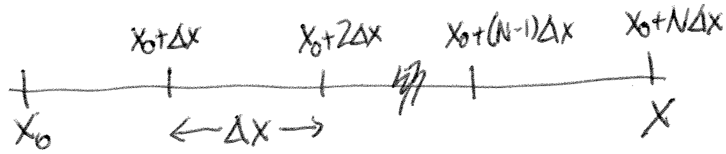
AS,

$$\vec{y}(x) = \underline{\Phi}(x,x_0)\cdot\vec{y}^{\,0} + \int_{x_0}^{x} \underline{\Phi}(x,x')\cdot\vec{b}(x')\,dx' \xrightarrow[\text{IF } \mathbb{A}(x)=\mathbb{A}]{} e^{\mathbb{A}x}\cdot\vec{y}^{\,0} + \int_0^x e^{\mathbb{A}(x-x')}\cdot\vec{b}(x')\,dx'$$

ARE YOU PUZZLED BY THIS?

WHY SHOULD THE SAME FUNCTION APPEAR TO PROPAGATE THE INITIAL CONDITIONS AND THE INHOMOGENEOUS 'FORCING'? WHAT IS THIS EXPRESSION SAYING?

IT MAY HELP TO WRITE THE INTEGRAL OUT AS A RIEMANN SUM, WITH THE DOMAIN OF INTEGRATION DIVIDED INTO N PIECES:



THEN,

$$\vec{y}(x) \approx \underline{\Phi}(x,x_0)\cdot\vec{y}^{\,0} + \sum_{n=0}^{N-1} \underline{\Phi}(x,x_0+n\Delta x)\cdot\vec{b}(x_0+n\Delta x)\,\Delta x$$

$$= \underbrace{\underline{\Phi}(x,x_0)\cdot\vec{y}^{\,0}}\, + \underline{\Phi}(x,x_0+(N-1)\Delta x)\,\vec{b}(x_0+(N-1)\Delta x)\Delta x + \dots + \underline{\Phi}(x,x_0+\Delta x)\,\vec{b}(x_0+\Delta x)\Delta x + \underbrace{\underline{\Phi}(x,x_0)\,\vec{b}(x_0)\Delta x}$$

THIS IS HOW THE SOLUTION BEGINS

A MOMENT LATER, ADD THIS PIECE
A MOMENT LATER, ADD $\underline{\Phi}(x,x_0+2\Delta x)\,\vec{b}(x_0+2\Delta x)\Delta x$,
AND SO ON...

THE INHOMOGENEOUS PART $\vec{b}(x_0+n\Delta x)\Delta x$ ACTS LIKE A 'MOVING' INITIAL CONDITION. EACH PIECE CONTRIBUTES TO THE SOLUTION FOR ALL $x' \in [x_0+n\Delta x, x]$ AND THE FULL SOLUTION IS A SUPERPOSITION OF ALL OF THESE CONTRIBUTIONS!

EXPRESSING THE FULL SOLUTION AS A LINEAR SUPERPOSITION IS AN INHERENTLY LINEAR CHARACTERISTIC OF THE EQUATION. THAT IS TO SAY, THIS VARIATION-OF-PARAMETERS SOLUTION, ALONG WITH THE INTUITIVE INTERPRETATION,

ONLY HOLDS FOR SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS.

ARE THERE OTHER WAYS TO UNDERSTAND THE VARIATION-OF-PARAMETERS? ENGINEERS WOULD CALL THE VARIATION-OF-PARAMETERS FORMULA A 'CONVOLUTION WITH THE IMPULSE RESPONSE' TO UNDERSTAND WHAT THIS MEANS, AND GAIN A VALUABLE PERSPECTIVE ON THE FUNDAMENTAL MATRIX $\Phi(x, x_0)$, LET'S TAKE A LOOK AT THE CASE OF CONSTANT COEFFICIENT MATRIX

$$\frac{d\vec{y}}{dx} = \mathbb{A}\cdot\vec{y} + \vec{b}(x) \; ; \; \vec{y}(0) = \vec{y}^{\,0}$$

WITH FULL SOLUTION:

$$\vec{y}(x) = e^{\mathbb{A}x}\cdot\vec{y}^{\,0} + \int_0^x e^{\mathbb{A}(x-x')}\,\vec{b}(x')\,dx'$$

AND ASK: HOW COULD WE SOLVE THE DIFFERENTIAL EQUATION USING LAPLACE TRANSFORMS?

LAPLACE TRANSFORM OF A SYSTEM OF LINEAR DIFF. EQs

RECALL THAT FOR A SCALAR FUNCTION $y(x)$, THE LAPLACE TRANSFORM IS DEFINED BY THE INTEGRAL:

$$\mathcal{L}[y(x)](s) = \int_0^\infty y(x)e^{-sx}\,dx \equiv Y(s)$$

FOR EXAMPLE, $\mathcal{L}[e^{ax}] = \frac{1}{s-a}$.

THE LAPLACE TRANSFORM HAS A NUMBER OF USEFUL PROPERTIES - MOST USEFUL IS THAT IT CONVERTS DIFFERENTIAL EQs OF $y(x)$ INTO ALGEBRAIC EQs OF $Y(s)$ VIA THE PROPERTY:

$$\mathcal{L}\left[\frac{dy}{dx}\right] = \int_0^\infty \frac{dy}{dx}e^{-sx}\,dx = \ldots \text{INTEGRATION-BY-PARTS} \ldots$$
$$= sY(s) - y(0)$$

WE CAN DEFINE THE MATRIX-VECTOR LAPLACE TRANSFORM SIMILARLY:

$$\mathcal{L}[\vec{y}(x)] = \int_0^\infty \underbrace{\vec{y}(x)\, e^{-\mathbb{I}sx}}_{\text{A VECTOR OF SCALAR LAPLACE TRANSFORMS.}} dx = \vec{Y}(s)$$

IT IS STRAIGHTFORWARD TO SHOW THAT THE DERIVATIVE PROPERTY HOLDS:

$$\mathcal{L}\left[\frac{d\vec{y}}{dx}\right] = s\,\mathbb{I}\,\vec{Y}(s) - \vec{y}(0)$$

SO THAT THE LAPLACE TRANSFORM OF THE SYSTEM OF DIFFERENTIAL EQS,

$$\frac{d\vec{y}}{dx} = \mathbb{A}\cdot\vec{y}\ ;\ \vec{y}(0) = \vec{y}^{\,0} \quad \underset{\text{TRANSFORMS TO}}{\Longleftrightarrow} \quad s\,\mathbb{I}\,\vec{Y}(s) - \vec{y}^{\,0} = \mathbb{A}\,\vec{Y}(s)$$

OR, SOLVING FOR $\vec{Y}(s)$: $\quad \vec{Y}(s) = [\mathbb{I}s - \mathbb{A}]^{-1}\cdot\vec{y}^{\,0} \quad (*)$

$[s\mathbb{I} - \mathbb{A}]^{-1}$ IS THE MATRIX ANALOGUE OF $1/s-a$ AND IS EXACTLY THE LAPLACE TRANSFORM OF THE MATRIX EXPONENTIAL!

$$\mathcal{L}[e^{\mathbb{A}x}] = [\mathbb{I}s - \mathbb{A}]^{-1}$$

SO, THE INVERSE-LAPLACE TRANSFORM OF $(*)$ IS:

$$\mathcal{L}[\vec{Y}(s)] = \vec{y}(x) = e^{\mathbb{A}x}\cdot\vec{y}^{\,0}$$

NOTHING NEW HERE, REALLY. BUT WHAT ABOUT AN <u>INHOMOGENEOUS</u> SYSTEM?

$$\text{eg.}\quad \frac{d\vec{y}}{dx} = \mathbb{A}\cdot\vec{y} + \vec{b}(x)\ ;\ \vec{y}(0) = \vec{y}^{\,0}$$

AGAIN, TAKING THE LAPLACE TRANSFORM,

$$s\,\mathbb{I}\,\vec{Y}(s) - \vec{y}^{\,0} = \mathbb{A}\vec{Y}(s) + \vec{B}(s) \quad \vec{B}(s) = \mathcal{L}[\vec{b}(x)] = \int_0^\infty \vec{b}(x)\, e^{-\mathbb{I}sx}\, dx$$

OR, $\quad [s\mathbb{I} - \mathbb{A}]\,\vec{Y}(s) = \vec{y}^{\,0} + \vec{B}(s)$ $\quad$ OR, $\quad \vec{Y}(s) = \underbrace{[s\mathbb{I} - \mathbb{A}]^{-1}\vec{y}^{\,0}}_{\text{THIS WE CAN INVERT}} + \underbrace{[s\mathbb{I} - \mathbb{A}]^{-1}\cdot\vec{B}(s)}_{\text{HOW DO WE INVERT THIS?}}$

CONVOLUTION INTEGRALS

IN ADDITION TO THE DERIVATIVE PROPERTY $\mathcal{L}[\frac{d\vec{y}}{dx}] = s\,\mathbb{I}\,\vec{Y}(s) - \vec{y}(0)$, ONE OF THE MOST USEFUL PROPERTIES OF THE LAPLACE TRANSFORM IS HOW IT WORKS ON CONVOLUTION INTEGRALS

A 'CONVOLUTION' IS AN INTEGRAL OF THE TYPE $\int_0^{x} f(x-x')g(x')\,dx'$ [ALSO OVER INFINITE DOMAIN: $\int_{-\infty}^{\infty} f(x-x')g(x')\,dx'$]

A PRODUCT OF FUNCTIONS, ONE OF WHICH IS BEING INTEGRATED 'BACKWARDS' FROM $x \to 0$.

SOME PROPERTIES OF THE CONVOLUTION INTEGRAL.

NOTATION: $f*g = \int_0^x f(x-x')g(x')\,dx'$

1. COMMUTIVITY: $f*g = g*f$

PROOF: $f*g = \int_0^x f(x-x')g(x')\,dx'$ CHANGE OF VARIABLE $\rho = x-x'$, $d\rho = -dx'$

$$= \int_{\rho=x}^{\rho=0} f(\rho)g(x-\rho)(-d\rho)$$

$$= \int_0^x g(x-\rho)f(\rho)\,d\rho = g*f$$

2. DISTRIBUTIVITY: $f*(g_1+g_2) = f*g_1 + f*g_2$

3. ASSOCIATIVITY: $f*(g*h) = (f*g)*h$

4. ZERO ELEMENT: $f*0 = 0*f = 0$

5. IDENTITY ELEMENT? NOT $g(x)=1$

$$f*1 = \int_0^x f(x-x')\,dx' = \int_0^x f(x')\,dx'$$ = AREA UNDER $f(x')$ FROM $x'=0$ TO $x'=x$.

IMPULSE FORCING FUNCTION: THE DIRAC DELTA-FUNCTION $\delta(x-x')$

IN ENGINEERING & PHYSICS, THE PERSPECTIVE THAT IS OFTEN MOST USEFUL IS TO CONSIDER THE HOMOGENEOUS EQUATION AS THE INTRINSIC DYNAMICS OF THE SYSTEM (ELECTRICAL CIRCUIT, CHEMICAL PLANT, AIRPLANE..) AND THE INHOMOGENEOUS AS A 'FORCING' ON THAT SYSTEM; EITHER AS INPUT OR AS SOME KIND OF CONTROL:

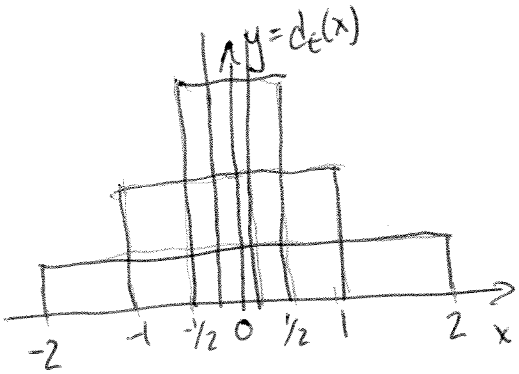
$$\underbrace{\frac{d\vec{y}}{dx} - A(x)\vec{y}}_{\text{INTRINSIC DYNAMICS}} = \vec{b}(x)$$

INPUT TO THE SYSTEM; 'FORCING' OR 'CONTROL'

WE WILL SEE THAT AN IMPORTANT INPUT $\vec{b}(x)$, THAT HELPS CHARACTERIZE THE SYSTEM DYNAMICS, IS A FORCE THAT IS APPLIED FOR ONLY AN INSTANT, SO-CALLED 'IMPULSIVE FORCING'.

TO APPROXIMATE THIS BEHAVIOUR, CONSIDER THE FUNCTION:

$$d_\epsilon(x) = \begin{cases} \frac{1}{2\epsilon} & -\epsilon < x < \epsilon \\ 0 & \text{OTHERWISE} \end{cases}$$



DEFINE $I(\epsilon) = \int_{-\infty}^{\infty} d_\epsilon(x')dx'$

THEN,

- $\lim_{\epsilon \to 0} d_\epsilon(x) = 0$ FOR ALL $x \neq 0$
- $\lim_{\epsilon \to 0} I(\epsilon) = \lim_{\epsilon \to 0} 1 = 1.$

IN THE LIMIT $\epsilon \to 0$, WE ARRIVE AT THE DIRAC DELTA 'FUNCTION' $\delta(x)$ DEFINED BY:

$$\text{i)}\ \delta(x) = 0 \quad x \neq 0 \qquad \text{ii)}\ \int_{-\infty}^{\infty} \delta(x')\,dx' = 1$$

$\delta(x)$ IS NOT A 'FUNCTION' REALLY - IT IS A GENERALIZED FUNCTION OR 'DISTRIBUTION'. MEANINGFUL AS AN INTEGRAND, BUT NOT POINT-WISE.

FOR A CONTINUOUS FUNCTION $f(x)$,

$$\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} d_\epsilon(x) f(x)\, dx = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} f(x)\, dx$$

RECALL THE MEAN-VALUE THEOREM FOR INTEGRALS,

GIVEN ϵ, THERE EXISTS x^* SO THAT $\displaystyle\int_{-\epsilon}^{\epsilon} f(x)\,dx = \underbrace{2\epsilon}_{\text{LENGTH OF DOMAIN}} \cdot \underbrace{f(x^*)}_{\text{AVERAGE VALUE.}}$

SO, $\displaystyle\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = \ldots = \lim_{\epsilon \to 0} \frac{1}{2\epsilon}\left[2\epsilon f(x^*)\right]$ WITH $x^* \in [-\epsilon, \epsilon]$

$$= f(0).$$

IN GENERAL, $\displaystyle\int_{-\infty}^{\infty} \delta(x - x_0) f(x)\, dx = f(x_0)$ ['SIFTING PROPERTY']

NOTICE THAT $\delta(x)$ IS THE IDENTITY ELEMENT FOR CONVOLUTION!

$$\delta * f = \int_0^{x} \delta(x - x') f(x')\, dx' = f(x).$$

WHAT ABOUT THE LAPLACE TRANSFORM OF $\delta(x)$? NOT STRICTLY ~~DEFINED~~ POSSIBLE, BUT DEFINE IT AS THE LIMIT

$$\mathcal{L}[\delta(x - x_0)] = \int_0^{\infty} \delta(x - x_0) e^{-sx}\, dx = \lim_{\epsilon \to 0} \int_0^{\infty} d_\epsilon(x - x_0) e^{-sx}\, dx$$

$$= \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{x_0 - \epsilon}^{x_0 + \epsilon} e^{-sx}\, dx = \lim_{\epsilon \to 0} -\frac{1}{2\epsilon s} e^{-sx} \Big|_{x_0 - \epsilon}^{x_0 + \epsilon}$$

$$= \lim_{\epsilon \to 0} \frac{1}{2\epsilon s} e^{-sx_0}\left(e^{s\epsilon} - e^{-s\epsilon}\right) = \lim_{\epsilon \to 0} \frac{\sinh(s\epsilon)}{s\epsilon} e^{-sx_0}$$

$$= e^{-sx_0}$$

L'HÔPITAL'S RULE

SO: $\boxed{\begin{array}{l} \mathcal{L}[\delta(x - x_0)] = e^{-sx_0} \\ \mathcal{L}[\delta(x)] = \lim_{x_0 \to 0} e^{-sx_0} = 1 \end{array}}$