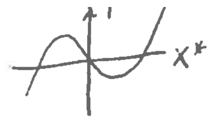
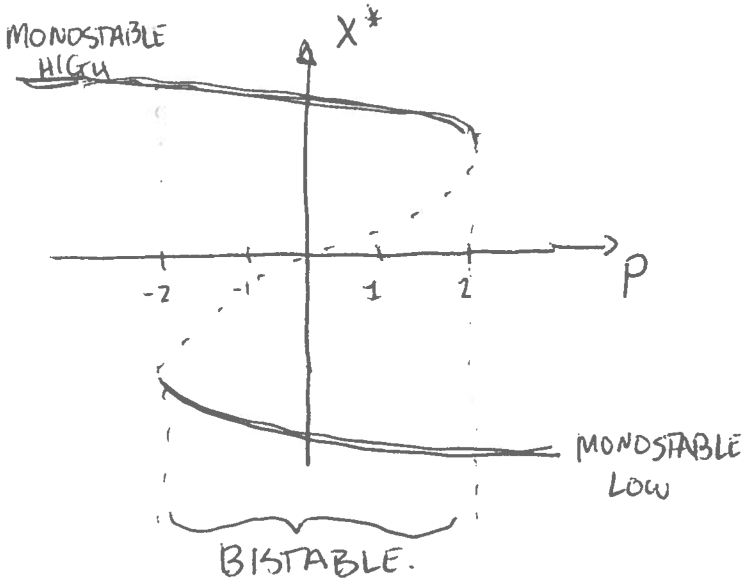
EX. $\frac{dx}{dt} = 3x - x^3 - p$; EQUILIBRIA: $p = 3x^* - (x^*)^3$



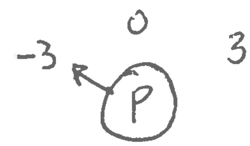


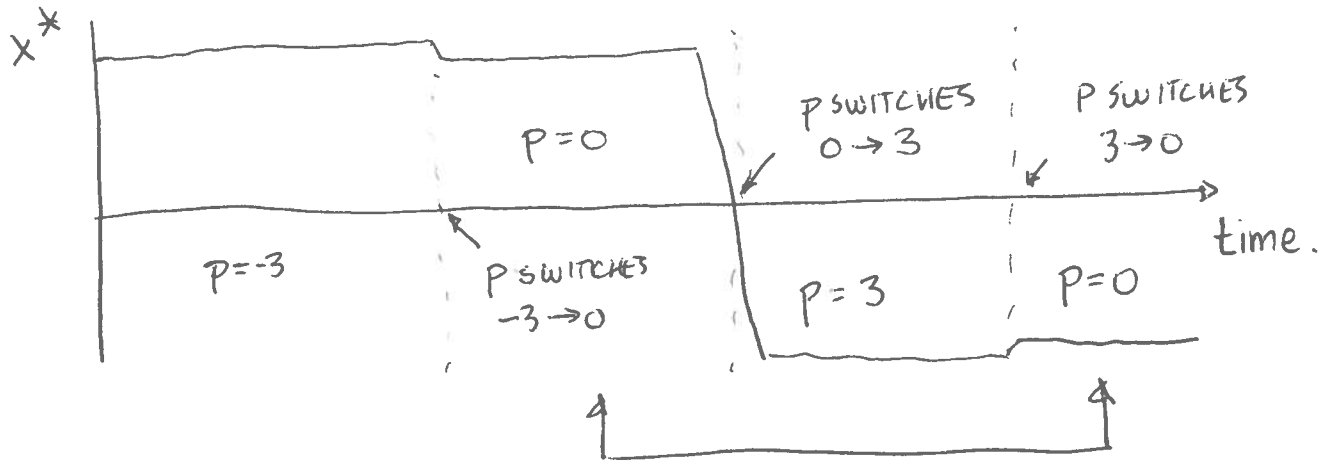
TWO SADDLE NODE BIFURCATIONS, AT $p = -2$ & $p = 2$.

MONOSTABLE $p < -2$ OR $p > 2$

BISTABLE $-2 < p < 2$

IMAGINE YOU CAN CONTROL THE PARAMETER 'p' LIKE A SWITCH





IN THESE TWO REGIONS, THE EQUATION IS IDENTICAL, BUT THE EQUILIBRIUM BEHAVIOUR IS VERY DIFFERENT. IN A SENSE, THE SYSTEM 'REMEMBERS' THE PARAMETER VALUE BEFORE IT IS SWITCHED TO 'p=0'. THIS TYPE OF HISTORY-DEPENDENCE IS ONLY POSSIBLE IN NONLINEAR SYSTEMS. IT IS CALLED

HYSTERESIS.

SO FAR, WE HAVE LOOKED AT 1D-BIFURCATIONS. THE PRINCIPLE IS SIMILAR IN HIGHER-DIMENSIONAL SYSTEMS, ALTOUGH OF COURSE, A MUCH RICHER VARIETY OF BEHAVIOUR IS POSSIBLE.

BIFURCATIONS IN TWO-DIMENSIONAL SYSTEMS

AS IN THE ONE-DIMENSIONAL CASE, THE TYPE AND MULTIPLICITY OF EQUILIBRIA ARE PARAMETER-DEPENDENT. PERHAPS THE MOST INTERESTING NEW BIFURCATIONS ARE ORBITAL BIFURCATIONS.

EX. SIMPLE LINEAR SYSTEM

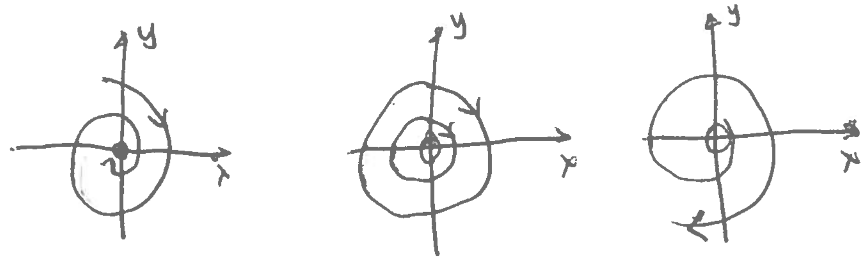
$$\frac{dx}{dt} = px + y \qquad \frac{dy}{dt} = -x + py$$

EQUILIBRIUM: $(x^*, y^*) = (0,0)$

JACOBIAN: $J(0,0) = \begin{bmatrix} p & 1 \\ -1 & p \end{bmatrix}$

EIGENVALUES: $p \pm i$

PHASE PORTRAITS



$p<0$ STABLE SPIRAL

$p=0$ CENTRE

$p>0$ UNSTABLE SPIRAL

THE MOST INTERESTING IS A BIFURCATION TO A LIMIT CYCLE.

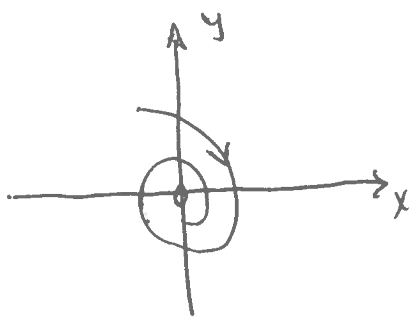
EX. $\frac{dx}{dt} = px + y - x^3 \qquad \frac{dy}{dt} = -x + py - y^3$

- VERY SIMILAR TO LINEAR EXAMPLE ABOVE...

EQUILIBRIUM: $(x^*, y^*) = (0,0)$

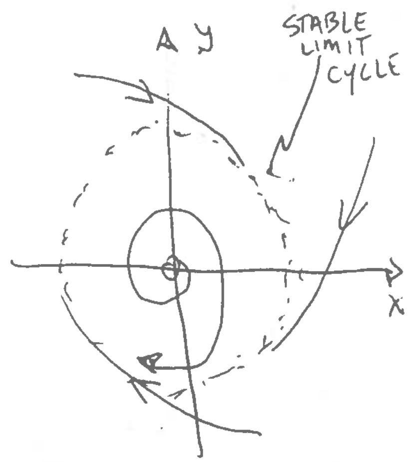
JACOBIAN: $J(0,0) = \begin{bmatrix} p & 1 \\ -1 & p \end{bmatrix}$

EIGENVALUES $= p \pm i$



$p<0$

STABLE SPIRAL



$p>0$, UNSTABLE SPIRAL TO A STABLE LIMIT CYCLE.

THIS IS A <u>HOPF BIFURCATION</u>: A PAIR OF COMPLEX-CONJUGATE EIGENVALUES CROSS THE IMAGINARY AXIS.

IF THE LIMIT CYCLE IS STABLE, WE CALL THIS A <u>SUPERCRITICAL</u> HOPF BIFURCATION.

OTHERWISE, IT IS <u>SUB-CRITICAL</u>.

GLOBAL STABILITY ANALYSIS OF NONLINEAR SYSTEMS

SO FAR, WE HAVE CONSIDERED LOCAL STABILITY; IE. LINEARIZED STABILITY LOCAL TO AN EQUILIBRIUM POINT. IN SOME CASES, IT IS POSSIBLE TO PROVIDE MORE GENERAL STATEMENTS ABOUT SYSTEM BEHAVIOUR.

EXCEPTION:
HOMOCLINIC ORBITS

IF EVERY TRAJECTORY SATISFIES $\lim_{t\to\infty} \vec{x}(t) = \vec{x}^*$, THEN THE EQUILIBRIUM $\vec{x}^*$ IS CALLED GLOBALLY ATTRACTING.
AN EQUILIBRIUM THAT IS STABLE & GLOBALLY ATTRACTING IS CALLED GLOBALLY ASYMPTOTICALLY STABLE.

IN GENERAL, ESTABLISHING GLOBAL STABILITY IS DIFFICULT, BUT FOR SOME SYSTEMS, WE ARE LUCKY & CAN FIND AN AUXILLARY FUNCTION THAT 'HEMS-IN' OUR TRAJECTORIES.

LYAPUNOV'S DIRECT METHOD

THE IDEA IS TO FIND A SCALAR FUNCTION THAT (i) DECREASES IN VALUE ALONG TRAJECTORIES OF THE MODEL & (ii) TAKES A UNIQUE MINIMUM AT THE EQUILIBRIUM $\vec{x}^*$.

EX. $\frac{dx}{dt} = -x$. THE FUNCTION $V(x) = x^2$ SATISFIES THE NECESSARY REQUIREMENTS:

i) $\frac{d}{dt}V(x(t)) = \frac{dV}{dx} \cdot \underbrace{\frac{dx}{dt}}_{\text{FROM THE MODEL}} = (2x)(-x) = -2x^2 < 0$ IE. DECREASES ALONG $x(t)$.

ii) $V(x^*)$ HAS A UNIQUE MINIMUM AT $x^* = 0$.

SO WE KNOW THAT $\lim_{t\to\infty} x(t) = 0$ WITHOUT KNOWING $x(t)$ EXPLICITLY.

IE. WITHOUT HAVING TO SOLVE THE DIFFERENTIAL EQ.

TO APPLY THE METHOD TO SYSTEMS REQUIRES SOME ADDITIONAL TECHNICAL DETAILS, BUT THE IDEA IS THE SAME.

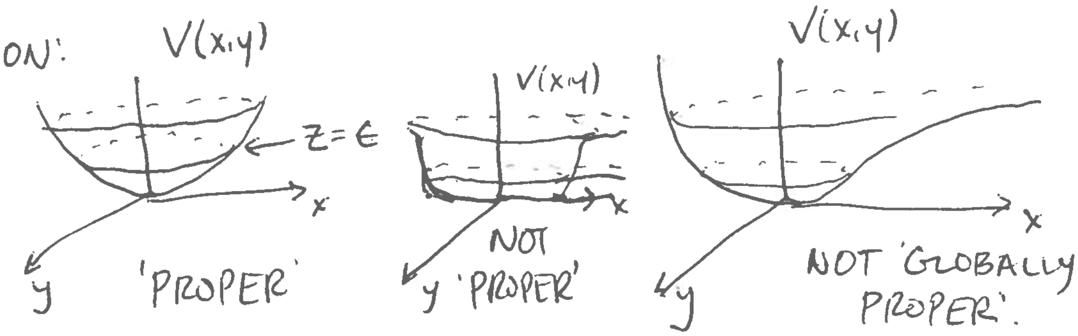
TECHNICAL DETAILS:

NON-NEGATIVE ~~POSITIVE~~ REALS

AS YOU GO 'OUT', MUST GO 'UP'. "RADIALLY UNBOUNDED"

1. A FUNCTION $V: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ IS CALLED <u>PROPER</u> IF THE SET $\{\vec{x} \in \mathbb{R}^n \mid V(\vec{x}) \leq \epsilon\}$ (CALLED A 'SUBLEVEL SET') IS BOUNDED FOR $\epsilon > 0$ SUFFICIENTLY SMALL; <u>GLOBALLY</u> <u>PROPER</u> IF BOUNDED FOR ALL $\epsilon > 0$.

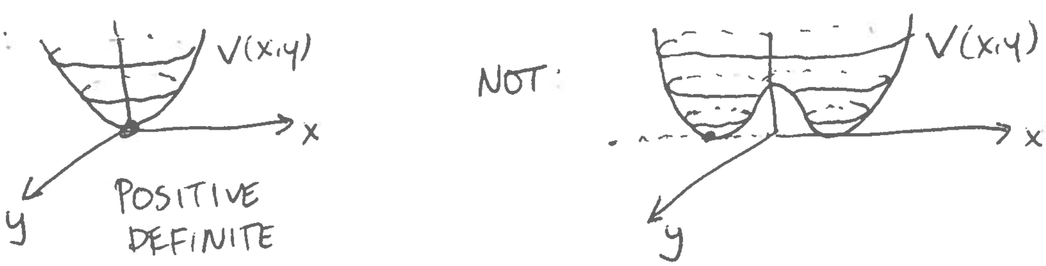
INTUITION:



2. A FUNCTION IS CALLED <u>POSITIVE DEFINITE</u> IF IT IS POSITIVE WITH A UNIQUE MINIMUM: i.e. THERE EXISTS A POINT $\vec{x}^*$ SO THAT

i) $V(\vec{x}^*) = 0$ AND ii) $V(\vec{x}) > 0$ FOR ALL $\vec{x} \neq \vec{x}^*$.

INTUITION:



NOW, WE CAN WRITE OUT A GENERAL DEFINITION FOR THE LYAPUNOV FUNCTION $V(\vec{x})$.

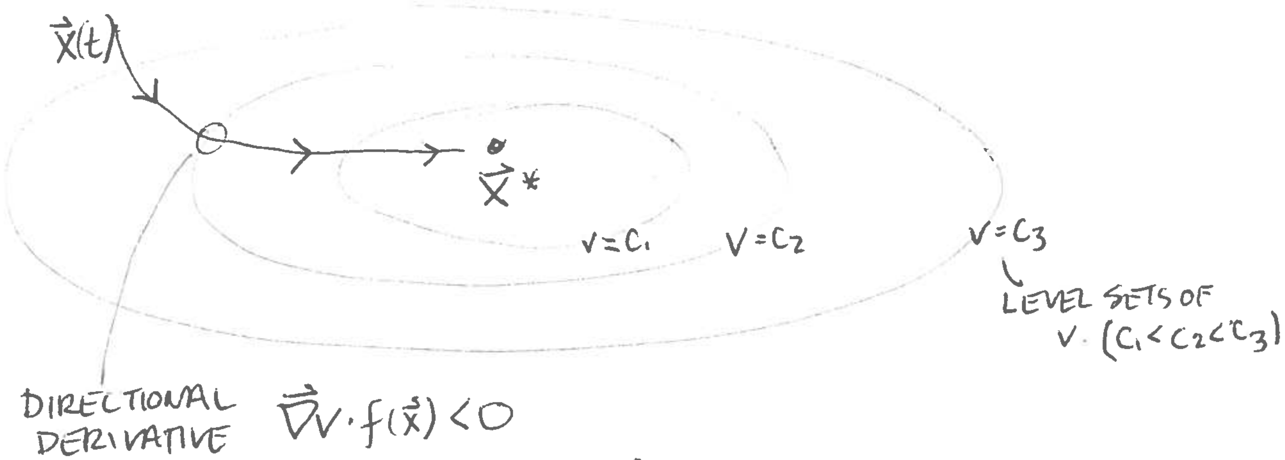
GIVEN A SYSTEM $\frac{d\vec{x}}{dt} = f(\vec{x})$, A FUNCTION $V(\vec{x})$ THAT IS i) <u>PROPER</u> ON A DOMAIN ii) <u>POSITIVE DEFINITE</u> AND iii) DECREASING ALONG TRAJECTORIES i.e. $\frac{d}{dt} V(\vec{x}(t)) = \frac{\partial V}{\partial \vec{x}} \cdot f(\vec{x}) = \vec{\nabla} V \cdot f(\vec{x}) < 0$ FOR ALL $\vec{x} \in D$ IS CALLED A <u>LYAPUNOV FUNCTION</u> ON D.

IF THERE IS A LYAPUNOV FUNCTION ON D, THEN ALL TRAJECTORIES IN D ARE ATTRACTED TO THE EQUILIBRIUM POINT $\vec{x}^*$: $\lim_{t\to\infty} \vec{x}(t) = \vec{x}^*$

i.e. D IS THE BASIN OF ATTRACTION FOR x^*.

IF D IS THE WHOLE PHASE-SPACE, $D=\mathbb{R}^n$, THEN $\vec{x}^*$ IS GLOBALLY ASYMPTOTICALLY STABLE.

INTUITION:



EX. VERIFY THAT (0,0) IS GLOBALLY ASYMPTOTICALLY STABLE FOR

$$\frac{dx}{dt} = -x - xy^2 \qquad \frac{dy}{dt} = -y - x^2 y.$$

HOW DO WE FIND $V(x,y)$? NO GENERAL METHOD! IN THIS EXAMPLE, $V(x,y) = x^2+y^2$ WORKS: PROPER & POSITIVE DEFINITE

AND,
$$\frac{dV}{dt} = \vec{\nabla}V\cdot\vec{f}(\vec{x}) = \begin{bmatrix}2x & 2y\end{bmatrix}\begin{bmatrix}-x-xy^2\\ -y-yx^2\end{bmatrix} = -(2x^2+2x^2y^2+2y^2+2x^2y^2) < 0$$

UNLESS $(x,y)=(0,0)$.

APPROXIMATION OF DIFFERENTIAL EQUATIONS — INTRODUCTION TO PERTURBATION EXPANSIONS

TAYLOR SERIES APPROXIMATION IS USED EVERYWHERE IN SCIENCE & ENGINEERING, TYPICALLY IN THE CONTEXT OF PERTURBATION APPROXIMATIONS OF DIFFERENTIAL EQUATIONS.

THE IDEA IS SIMPLY DEMONSTRATED BY LOOKING AT PERTURBATION APPROXIMATIONS OF ALGEBRAIC EQUATIONS.

EX. SUPPOSE WE WANT TO SOLVE $x^2 + \epsilon x - 1 = 0$ FOR SMALL ϵ.
THE EXACT SOLUTION IS:

$$x = -\frac{1}{2}\epsilon \pm \sqrt{1 + \frac{1}{4}\epsilon^2}$$

USING THE BINOMIAL EXPANSION, WE CAN EXPAND THESE SOLUTIONS AS A POWER SERIES IN ϵ:

$$x^{(1)} = 1 - \frac{1}{2}\epsilon + \frac{1}{8}\epsilon^2 - \frac{1}{128}\epsilon^4 + \dots$$

$$x^{(2)} = -1 - \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \frac{1}{128}\epsilon^4 + \dots$$

WE KNOW FROM THE BINOMIAL THEOREM THAT THESE SERIES CONVERGE IF, AND ONLY IF, $|\epsilon| < 2$.

BUT SUPPOSE WE DIDN'T KNOW THE QUADRATIC FORMULA. WE COULD ASSUME THAT THE SOLUTION 'x' CAN BE WRITTEN AS A POWER SERIES:

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots$$ AND SEE WHAT HAPPENS...

SUBSTITUTING INTO $x^2 + \epsilon x - 1$, WE GET:

$$(-1 + x_0^2) + (x_0 + 2x_0 x_1)\epsilon + (x_1 + x_1^2 + 2x_0 x_2)\epsilon^2 + \dots = 0$$

TO SATISFY THIS EQUATION, ALL OF THE COEFFICIENTS OF ϵ^n MUST VANISH. LOOKING AT EACH COEFFICIENT,

ϵ^0: $x_0^2 - 1 = 0$ OR $x_0 = \pm 1$. EASY.

LET'S LOOK AT THE SOLUTION THAT BEGINS $x^{(1)} = 1 + \ldots$

AT ϵ^1: $1 + 2x_1 = 0$ OR $x_1 = -\frac{1}{2}$

AT ϵ^2: $-\frac{1}{4} + 2x_2 = 0$ OR $x_2 = \frac{1}{8}$

AT ϵ^3: $2x_3 = 0$ OR $x_3 = 0$

AT ϵ^4: $\frac{1}{64} + 2x_4 = 0$ OR $x_4 = -\frac{1}{128}$

$\vdots$

ALTOGETHER, $x^{(1)} = 1 - \frac{1}{2}\epsilon + \frac{1}{8}\epsilon^2 - \frac{1}{128}\epsilon^4 + \ldots$

IDEA: WE CAN ASSUME A TAYLOR SERIES SOLUTION TO SIMPLIFY THE PROBLEM!

WE CAN USE THIS SAME STRATEGY TO SOLVE NONLINEAR EQUATIONS.
eg. SHOW THAT ONE SOLUTION TO:

$$\cos[x] = \epsilon \cdot x$$

IS: $$x = \frac{\pi}{2} - \frac{\pi}{2}\epsilon + \frac{\pi}{2}\epsilon^2 - \frac{\pi}{48}(\pi^2 + 24)\epsilon^3 + \ldots$$

eg. SHOW THAT THE SOLUTIONS TO $x^2 + e^{\epsilon x} = 5$ BEGIN:

$$x = \pm 2 - \epsilon/2 + \ldots$$

PERTURBATION APPROXIMATION OF DIFFERENTIAL EQUATIONS

THE REAL POWER OF THIS APPROACH COMES IN SOLVING DIFFERENTIAL EQUATIONS.

EX. MOTION OF AN OBJECT PROJECTED UPWARD FROM THE SURFACE OF THE EARTH. LET $x(t)$ DENOTE THE HEIGHT ABOVE THE SURFACE. APPLYING NEWTON'S 2nd LAW:

$$\frac{d^2x}{dt^2} = -\frac{gR^2}{(x+R)^2} \quad \text{FOR } t > 0$$

$x(0) = 0 \quad x'(0) = V_0.$ RADIUS OF THE EARTH.