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Basic theory of linear algebra

Def: The range of A is defined as:

$$\text{range}(A) = \{\, y : y = Ax \text{ for some } x \,\}$$

Theorem: range(A) = space spanned by the columns of $A=[a_1 \dots a_n]$

$$= \{\, y = x_1 a_1 + x_2 a_2 + \dots + x_n a_n = \sum a_j x_j \,\}$$

Thus range(A) is also called the column space of A.

Def: column rank = dimension of column space

row rank = dimension of row space

Theorem: column rank = row rank

Thus we simply call it the rank of A, rank(A).

Def: An $m \times n$ matrix A is of full rank if

$$\text{rank}(A) = \min(m, n)$$

Thus, if $m \geq n$, ($A = \square$), then a full rank matrix has n independent column vectors.

Def: A nonsingular (invertible) matrix is a square matrix of full rank.

Def: The null space of A, null(A), is defined as:

$$\text{null}(A) = \{ x: Ax = 0 \}$$

Matrix inverse

- $(A\,B)^{-1} = B^{-1} A^{-1}$, $(A^{-1})^T = (A^T)^{-1} = A^{-T}$
- $B^{-1} = A^{-1} - B^{-1} (B - A) A^{-1}$

Pf: $B\,(A^{-1} - B^{-1} (B - A) A^{-1}) = B\,A^{-1} - (B - A)\,A^{-1}$

$$= B\,A^{-1} - B\,A^{-1} + I = I$$

- Sherman-Morrison-Woodbury formula:

$$(A + U V^T)^{-1} = A^{-1} - A^{-1} U (I + V^T A^{-1} U)^{-1} V^T A^{-1}$$

where U, V = $R^{n\mathrm{x}k}$ (U = □). Thus a rank k correction to A results to a rank k correction of the inverse.

e.g. k = 1.

$$u = \begin{bmatrix} u_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad v = \begin{bmatrix} v_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

$$uv^T = \begin{bmatrix} u_1 v_1 & 0 & \cdots & 0 \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \Rightarrow A + UV^T = \begin{cases} a_{ij} & i \neq 1 \text{ or } j \neq 1 \\ a_{11} + u_1 v_1 & \end{cases}$$

$$(A + uv^T)^{-1} = A^{-1} - \underbrace{A^{-1}u}_{\text{nx1}} (1 + \underbrace{v^T A^{-1} u}_{\text{1x1}})^{-1} \underbrace{v^T A^{-1}}_{\text{1xn}}$$