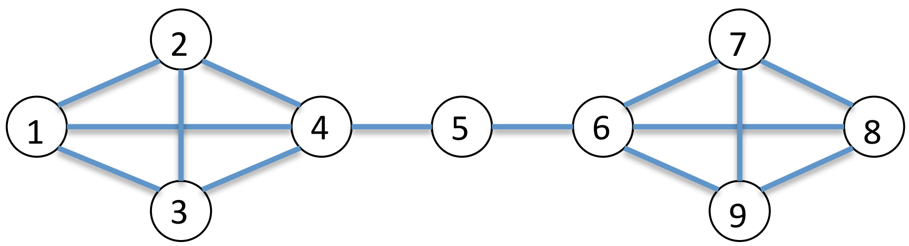
May 28, 2014

Minimum degree ordering

- In the matrix graph, choose the node with least edges connected to it.
- A tree graph will be ordered correctly.
- It is a local strategy; no guarantee it will produce the least amount of total fill.

Counter example:



The given order produces no fill.

Minimum degree ordering will start with node 5.

Produces fill at the first step -> not optimal

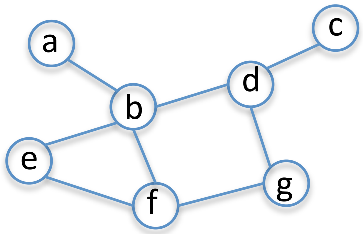
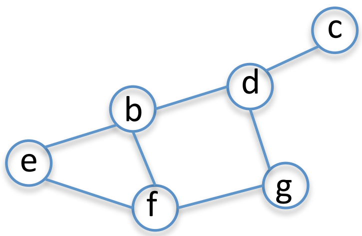
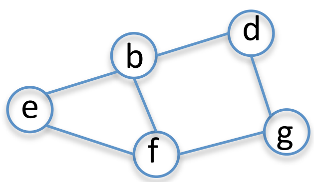
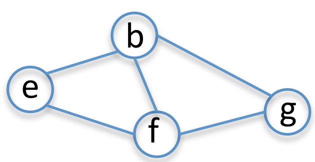
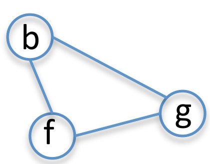
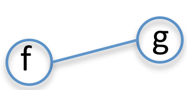
Tie-breaking

1) Select the node that had the smallest node number in the original order.

2) RCM preordering -> min. deg.

   Tie broken by selecting earlier RCM ordered node.

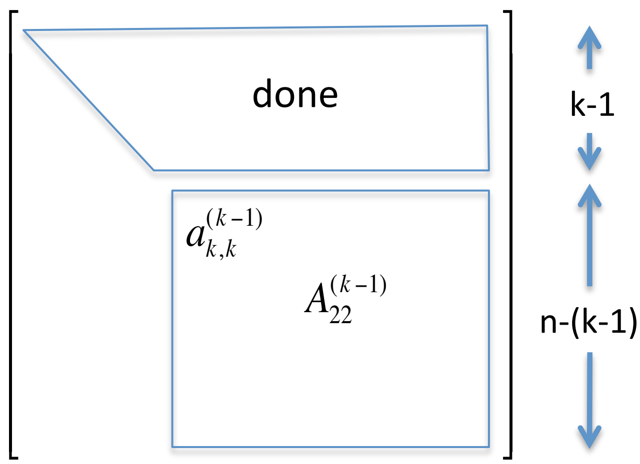
Example

k	Elimination graph $G(A^{(k-1)})$	node	min deg
1		a	1
2		c	1
3		d	2
4		e	2
5		b	2
6		f	1
7		g	0

Stability of factorization

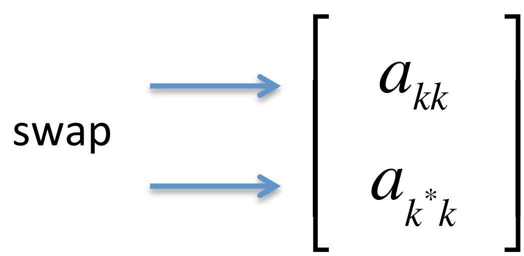
- Problem arises: 1) zero pivot: $a_{kk}^{(k-1)} = 0$, or 2) small pivot: $a_{kk}^{(k-1)} \approx 0$

- Pivoting



Complete: search the largest element in A_{22}.

Partial: search the largest element in col k.



- Modified LU factorization: PA = LU, P = permutation obtained from swapping during partial pivoting.

- Pivoting is stable but bad news for sparse matrices -> destroy sparsity

- Several conditions on A will ensure pivoting is not necessary.

Theorem: Suppose A is SPD. Then during GE, $a_{kk}^{(k-1)} > 0$.

Pf: For n = 1, obviously true. In general, use inductive argument.
Write

$$A = \begin{bmatrix} a_{11} & v^T \\ v & A_{22} \end{bmatrix}$$

Since A is SPD, $a_{11} > 0$.

Eliminate v using a_{11} as pivot:

$$\begin{bmatrix} a_{11} & v^T \\ v & A_{22} \end{bmatrix} \xrightarrow{\text{GE}} \begin{bmatrix} a_{11} & v^T \\ & A_{22} - \frac{v}{a_{11}} v^T \end{bmatrix}$$

Let $A_{22}^{(1)} = A_{22} - vv^T/a_{11}$. Note $A_{22}^{(1)}$ is symmetric.

Want to show $A_{22}^{(1)}$ is SPD.

Let x in R^{n-1}, $x \neq 0$. Consider y defined as

$$y = \begin{bmatrix} -\frac{x^T v}{a_{11}} \\ x \end{bmatrix} \in \Re^n$$

A SPD => $y^T A y > 0$

i.e.

$$0 < \begin{bmatrix} -\frac{x^T v}{a_{11}} & x^T \end{bmatrix} \begin{bmatrix} a_{11} & v^T \\ v & A_{22} \end{bmatrix} \begin{bmatrix} -\frac{x^T v}{a_{11}} \\ x \end{bmatrix}$$

$$= \begin{bmatrix} 0 & x^T A_{22} - \frac{x^T v v^T}{a_{11}} \end{bmatrix} \begin{bmatrix} -\frac{x^T v}{a_{11}} \\ x \end{bmatrix}$$

$$= x^T A_{22} x - \frac{1}{a_{11}} x^T (vv^T) x$$

$$= x^T (A_{22} - \frac{vv^T}{a_{11}}) x = x^T A_{22}^{(1)} x$$

Thus $A_{22}^{(1)}$ is SPD => $a_{22}^{(1)} > 0$ => pivot is positive.

Continuing this process, one can show that

$$a_{kk}^{(k-1)} > 0 \qquad \forall k$$

Other matrices that pivoting is not necessary:

- Row diagonally dominant

  i.e. $$\left|a_{k,k}\right| > \sum_{j \neq k} \left|a_{k,j}\right| \qquad k = 1, \ldots, n$$

- Column diagonally dominant

  i.e. $$\left|a_{k,k}\right| > \sum_{i \neq k} \left|a_{i,k}\right| \qquad k = 1, \ldots, n$$

Image Denoising

- Images are treated as 2D functions:

  $u_{i,j}$ = pixel value at row i, col j

- Inverse problem: Given
  1) the observed image: $u^0 = u^* + n$ (n = noise)
  2) estimate of variance of noise:

     $||n||^2 = \sigma^2$ σ = known parameter

  find an approximation to the original image u^*

PDE approach

min "fluctuation of pixel values"

subject to "noise constraint level"

- The objective function is to get rid of noise
- The constraint is necessary so that you don't get an image of constant pixel values

  i.e. $|| u - u^0 ||^2 \approx \sigma^2 = || u^* - u^0 ||^2$

  The noise level of u^0 is the same w.r.t u and w.r.t. u^*

- Ill-posed problem: many images u satisfy the same constraint.
- Need a selection criterion -> regularization

Regularization models

Noise level constraint optimization problem:

$$\min_{u} \quad R(u)$$
$$\text{subject to} \quad \left\| u - u^0 \right\|^2 = \sigma^2$$

Equivalently:

$$\min_{u} \quad f(u) \equiv \alpha R(u) + \left\| u - u^0 \right\|^2$$

α measures tradeoff between fit and regularity

- If $\alpha \approx 0$, $u \approx u^0$
- If $\alpha \approx \infty$, $u \approx$ constant

What is R(u)? The idea is to min the fluctuation of u