Developing an error bound for Simpson's rule

- Expand f at m, using third Taylor's polynomial
- There exists some $\xi(x) \in$[a,b] such that
- $f(x) =$
  $f(m) + f'(m)(x-m) + \frac{1}{2}f''(m)(x-m)^2 +$
  $\frac{1}{6}f'''(m)(x-m)^3 + \frac{1}{24}f^{(4)}(\xi(x))(x-m)^4$
- Integrate both sides over [a,b]

$$I = f(m)(b-a) + G(x) + H(x)$$

Where

$$G(x) = \left[\frac{f'(m)(x-m)^2}{2} + \frac{f''(m)(x-m)^3}{6} + \frac{f'''(m)(x-m)^4}{24}\right]_a^b$$

$$H(x) = \frac{1}{24}\int_a^b f^{(4)}(\vartheta(x))(x-m)^4 dx$$

Recall: $I = f(m)(b-a) + G(x) + H(x)$

(b-m) = (b-a)/2 = h and *(a-m) = (a-b)/2 = -h.*

$$G(x) = \left[\frac{f'(m)(x-m)^2}{2} + \frac{f''(m)(x-m)^3}{6} + \frac{f'''(m)(x-m)^4}{24}\right]_a^b$$

$$= \frac{f'(m)}{2}((b-m)^2-(a-m)^2)$$

$$+ \frac{f''(m)}{6}((b-m)^3-(a-m)^3)$$

$$+ \frac{f'''(m)}{24}((b-m)^4-(a-m)^4)$$

$$= \frac{f''(m)}{6}((b-m)^3-(a-m)^3)$$

$$= \frac{f''(m)}{6}(2h^3) = \frac{f''(m)}{3}h^3$$

$$I = f(m)(b-a) + G(x) + H(x)$$

$$H(x) = \frac{1}{24}\int_a^b f^{(4)}(\vartheta(x))(x-m)^4 dx$$

By the MVT for Integrals, $\exists \gamma_S \epsilon [a,b]\ s.t.$

$$H(x) = \frac{f^{(4)}(\gamma_S)}{24}\int_a^b (x-m)^4 dx$$

$$= \frac{f^{(4)}(\gamma_S)}{24}\frac{1}{5}(\,(b-m)^5 - (a-m)^5)$$

$$= \frac{f^{(4)}(\gamma_S)}{120}(\,h^5 - (-h)^5)$$

$$= \frac{f^{(4)}(\gamma_S)}{60}(h^5)$$

$$I = 2hf(m) + \frac{h^3}{3} f''(m) + \frac{h^5}{60} f^{(4)}(\gamma_S)$$

It can be shown, using

- *Taylor's expansion for f(a)=f(m-h) and f(b)=f(m+h), and the IVT for a continuous* $f^{(4)}$ *over [a,b], that there exists* ρ_S *in [a,b] that*

$$f''(m) = \frac{1}{h^2}[f(a) - 2f(m) + f(b)] - \frac{h^2}{12} f^{(4)}(\rho_S)$$

Substituting this into I above, leads to

$$I = \frac{h}{3}[f(a) + 4f(m) + f(b)] - \frac{h^5}{12}\left(\frac{f^{(4)}(\rho_S)}{3} - \frac{f^{(4)}(\gamma_S)}{5}\right)$$

Replacing ρ_S *and* γ_S *with a common value* α *in [a,b]:*

$$I = \widehat{I_2} - \frac{h^5}{90} f^{(4)}(\alpha)$$

So,

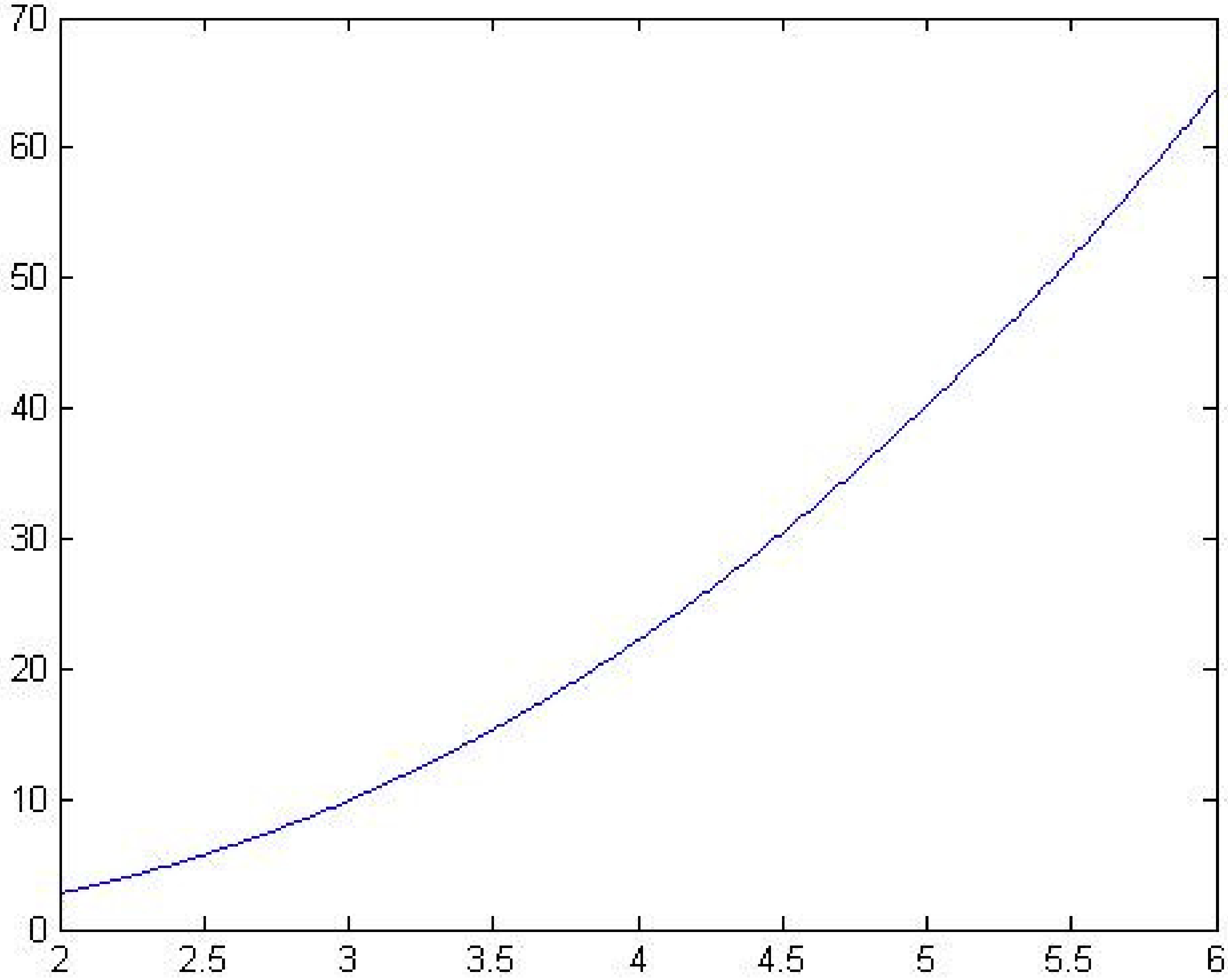
$$|E(f_S)| = \left|I - \widehat{I_2}\right| = \frac{h^5}{90}\left|f^{(4)}(\alpha)\right|$$

Note that the approximation is exact for all polynomials of degree <= 3.

Method	Formula for $\hat{I}$	Error	Precision
Mid-point	$(b-a)f\left(\frac{a+b}{2}\right)$	$\frac{(b-a)^3}{24} f''(\xi_M)$	1
Trape-zoid	$\frac{(b-a)}{2}[f(a)+f(b)]$	$-\frac{(b-a)^3}{12} f''(\xi_T)$	1
Simpson	$\frac{(b-a)}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right) + f(b)\right]$	$-\frac{(b-a)^5}{2880} f^{(4)}(\xi_S)$	3

Precision: degree of polynomials for which error is 0.

Exercise: Approximate $\int_2^6 x^2 \ln x \, dx$



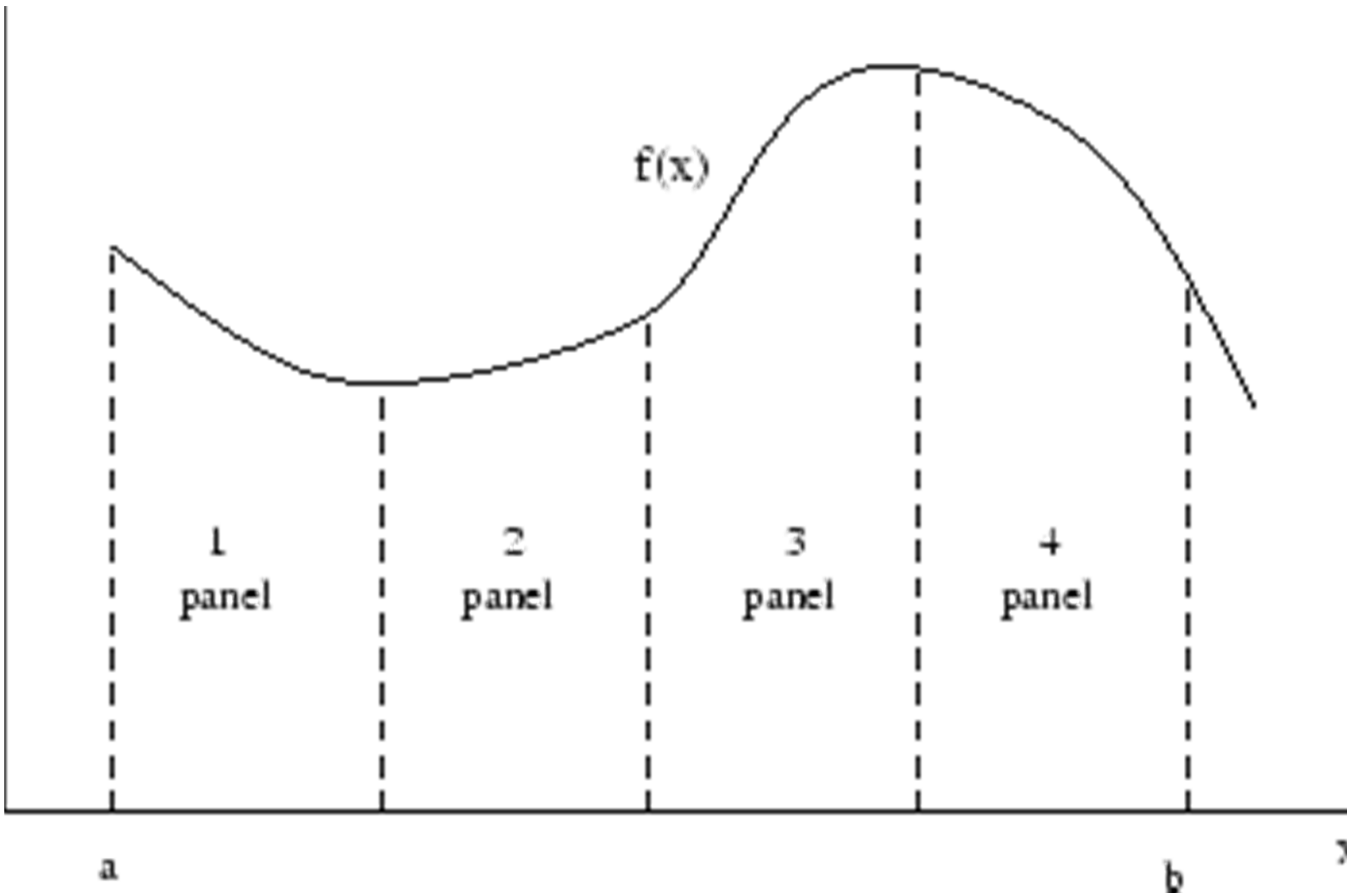
Method	Formula for $\hat{I}$	Error = Actual – Estimate
Midpoint	$$4f(4)$$ =4*22.1807 =	
Trapezoid	$$\frac{4}{2}[f(2)+f(6)]$$ =2(2.7726 + 64.5033) =	
Simpson	$$\frac{4}{6}[f(2)+4f(4)+f(6)]$$ =2/3(2.7726+4*22.7228+64.5033)/3 =	

Exact answer = $\left(\frac{x^3(3\ln(x)-1)}{9}\right)\Big|_2^6 =$

Composite Rules

- Divide the interval into n equal pieces
- $h = (b - a)/n$
- Set $x_0 = a, x_n = b,$
- $x_k = a + hk\ for\ k = 0{:}\,n$
- $\boldsymbol{I = \int_a^b f(x)dx = \sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} f(x)dx = \sum_{i=0}^{n-1} I_i}$
- Each of the formulas can be used in this manner

Composite Midterm Rule



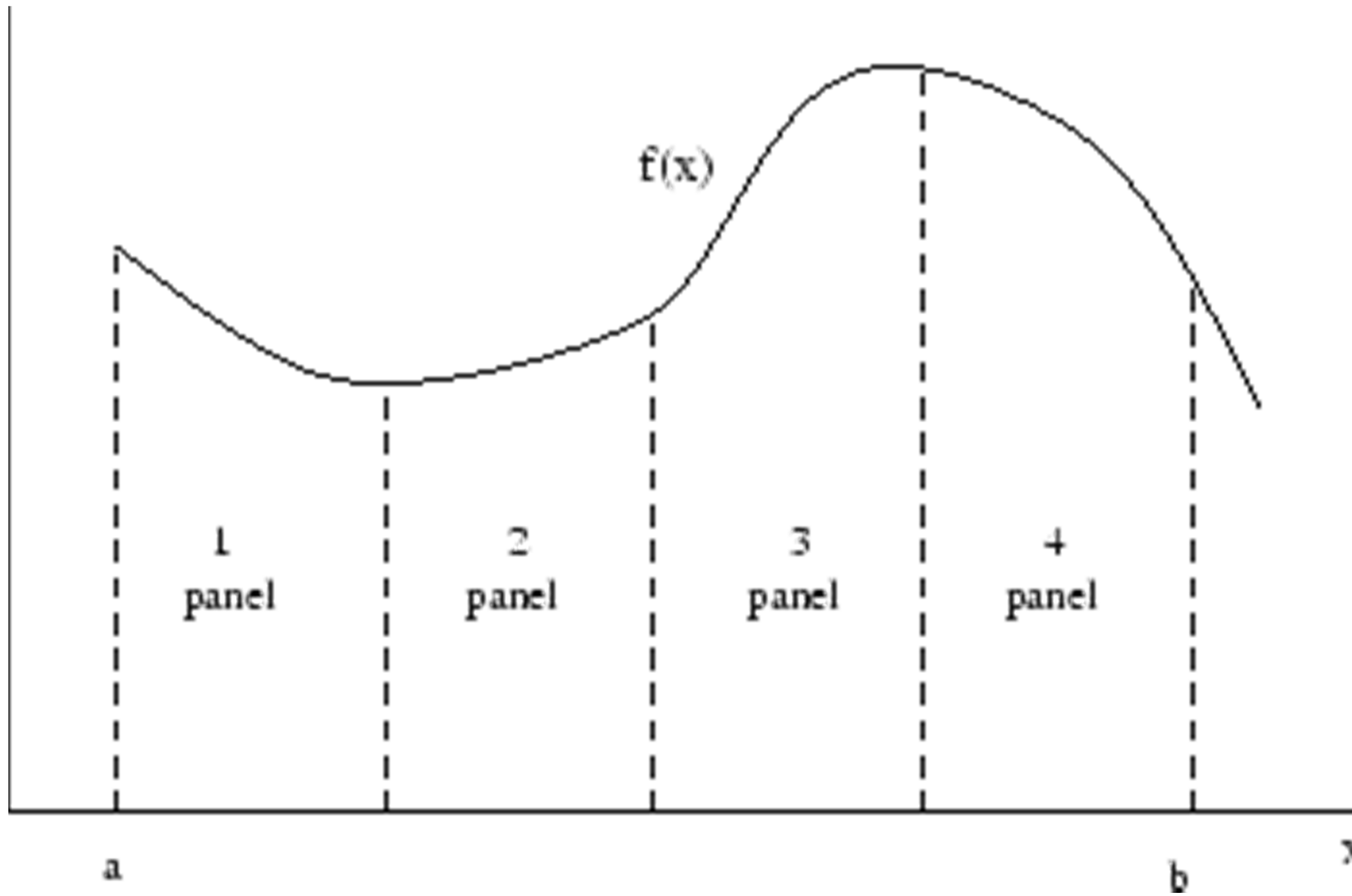
$$I = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)dx$$

$$I = \sum_{i=0}^{n-1} (x_{i+1} - x_i) f\left(\frac{x_i + x_{i+1}}{2}\right) + \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^3}{24} f''(\gamma_i)$$

$$I = \sum_{i=0}^{n-1} \widehat{I_{0,i}} + \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^3}{24} f''(\gamma_i)$$

Determine a bound on the global truncation error using the local truncation errors: $|I - \sum_{i=0}^{n-1} \widehat{I_{0,i}}|$

Composite Trapezoid Rule



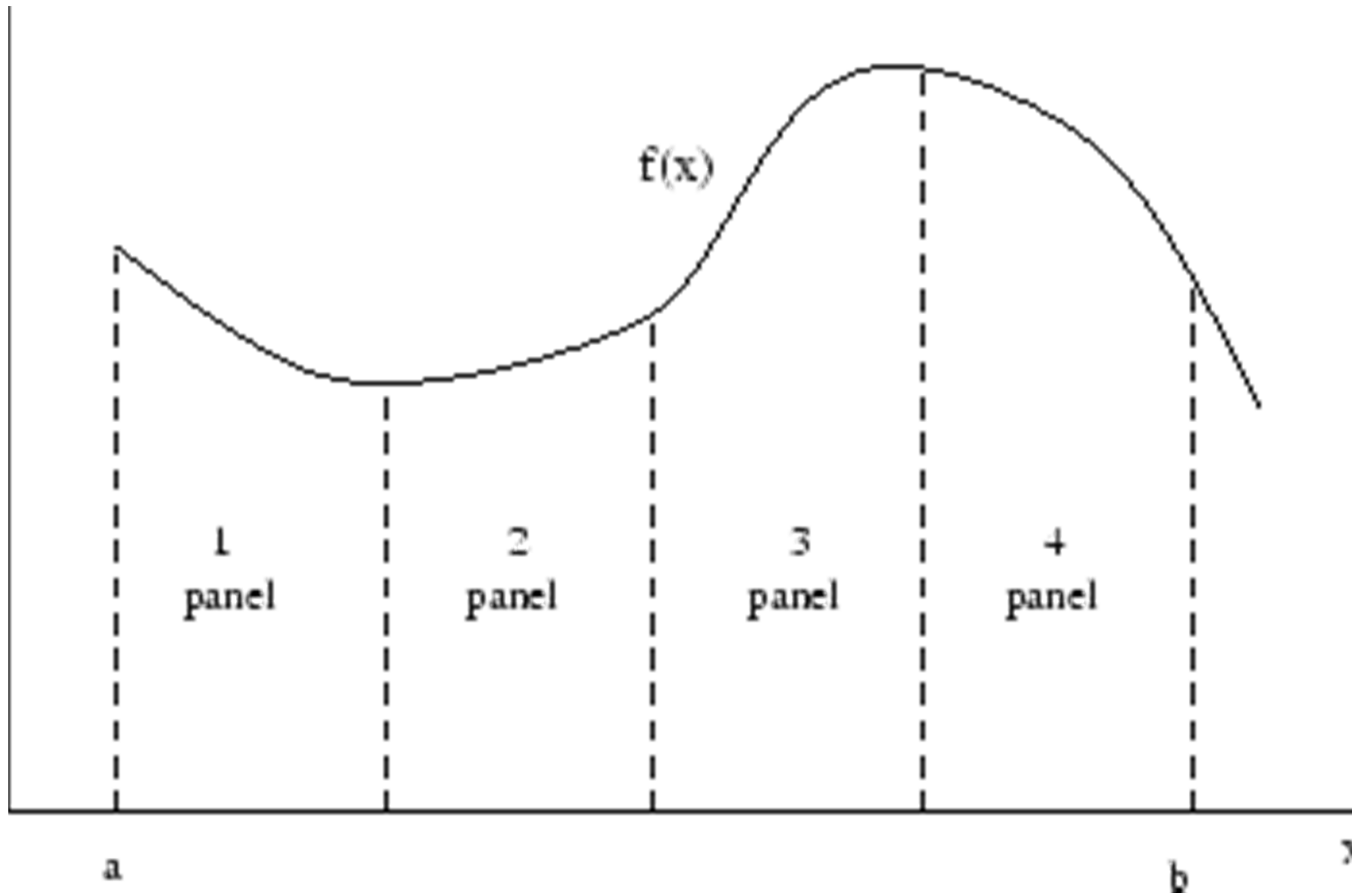
Formula and Global Truncation Error Composite Trapezoid Rule

- Using same approach as for Composite Midpoint Rule, can show Global Truncation error is $O(h^2)$.
- Formula:

$$\frac{h}{2}\sum_{i=0}^{n-1}(f(x_i)+f(x_{i+1}))$$

$$=\frac{h}{2}(f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n))$$

Composite Simpson's Rule



Formula and Global Truncation Error Composite Simpson's Rule

- Using same approach as for Composite Midpoint Rule, can show Global Truncation error is ~~O(h²).~~ $O(h^4)$.
- Formula:

$$\frac{h}{6}\sum_{i=0}^{n-1}\left(f(x_i) + 4\,f(m_i) + \,f(x_{i+1})\right)$$

$$= \frac{h}{6}\left(f(x_0) + 4\,f(m_0) + \,2f(x_1) + 4\,f(m_1) + \,2f(x_2)\right.$$
$$\left. + \,\ldots + \,2f(x_{n-1}) + 4\,f(m_{n-1}) + \,f(x_n)\right)$$