Modal Logic Part1

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[with material from “Mathematical Logic for Computer Science”, by Zhongwan, published by World Scientific; and “Modal Logic Tutorial”, by N. Alechina, Midlands Graduate School in Foundations of CS, Online]

Objectives

- Defining Modal Logic Formulas
- Satisfiability of Modal Logic
- Kripke Structures
- Using Modal Logic to Represent Programs

Introducing Modal Logic /1

- **In Classical Propositional Logic**
  - Propositions are true or false
- **In Modal Propositional Logic**
  - We distinguish between necessarily true and necessarily false propositions
  - **That is, each proposition is evaluated in a particular mode or context of interpretation**
  - Necessarily true propositions are said to be necessary □
  - Necessarily false propositions are said to be impossible
  - Not impossible propositions are said to be possible ◊
  - **Possible propositions are all true propositions, either necessary or not necessary**

Introducing Modal Logic /2

- **What We Can and What We Cannot Assert?**
  - From the truth of A, we can assert the truth of “A is possible” but we cannot assert if “A is necessary”
    - Hmmm, why is that?
  - From the falsehood of A, we can assert the falsehood of “A is necessary” but we cannot assert if “A is possible”
  - As shown above, we shall use □ (instead of L in your textbook) to mean “necessary” or “always”
  - And we shall use ◊ (instead of M in your textbook) to mean “possible” or “eventually”

Introducing Modal Logic /3

- **The Modal Propositional Language L^{pm}**
  - Created by adding the symbols □ and ◊ to L^{p}
    - Formally, only □ is added and ◊ is derived as ◊A ⇔ ¬□¬A
- **Definition 6.1. Form(L^{pm}):** (Section 8.1)
  - An expression of L^{pm} is a formula of L^{pm} iff it can be generated using the following (formation) rules:
    [1] Atom(L^{pm}) ⊆ Form(L^{pm}),
    [2] If A ∈ Form(L^{pm}) then (¬A) ∈ Form(L^{pm}), (□A) ∈ Form(L^{pn}) , and (◊A) ∈ Form(L^{pn})
    [3] If A, B ∈ Form(L^{pm}) then (A * B) ∈ Form(L^{pm}), where * stands for any of the standard binary connectives, from {∧, ∨, ⇒, ⇔}
  - Form(L^{pm}) can be abbreviated as WFMF

Introducing Modal Logic /4

- **(Modal Logic) World:**
  - A conceivable state of affairs
  - Also referred to as modal logic interpretation, or as valuation in the textbook
- **Definition 6.2. Values of Formulas:** (Definition 8.2.3)
  - Let K be a set of valuation for $\mathcal{L}^{\text{pm}}$ and let $t \in K$ be the valuation of a formula; and let R be an equivalence relation on K
  - $p^t \in \{1,0\}$ for atom p
  - $(\neg A)^t = \{1 \text{ if } A^t = 0; 0 \text{ otherwise}\}$
  - $(\Box A)^t = \{1 \text{ if for every } t_i \in K \text{ where } tRt_i, A^{t_i} = 1; 0 \text{ otherwise}\}$
  - $(\Diamond A)^t = \{1 \text{ if for some } t_j \in K \text{ where } tRt_j, A^{t_j} = 1; 0 \text{ otherwise}\}$
  - $(A \wedge B)^t = \{1 \text{ if } A^t = B^t = 1; 0 \text{ otherwise}\}$
  - $(A \vee B)^t = \{1 \text{ if } A^t = 1 \text{ or } B^t = 1; 0 \text{ otherwise}\}$
  - $(A \Rightarrow B)^t = \{1 \text{ if } A^t = 0 \text{ or } B^t = 1; 0 \text{ otherwise}\}$
  - $(A \Leftrightarrow B)^t = \{1 \text{ if } A^t = B^t; 0 \text{ otherwise}\}$

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- **Definition 6.3. The Kripke Structure:**
  - Kripke structure (or modal interpretation) is a triple I = (W, R, V) such that
  - **W is a non-empty set of possible Worlds**
  - **R $\subseteq$ W x W is an accessibility Relation**
  - **V: (Prop x W) $\rightarrow$ {0, 1} is a Valuation function that assigns a propositional interpretation to each world**
  - Prop is a set of propositional symbols
  - **The pair (W, R) is called the frame of I**
  - Kripke Structure is just a graph with W defining the graph nodes and R defining the graph edges

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- **Definition 6.4. Satisfiability and Validity:** (Def. 8.2.4)
  - A set of formulas $\Sigma \in$ Form(L^{pm}) is satisfiable if there are truth valuations K, an equivalence relation R, and some truth valuation $t \in$ K, such that $\Sigma^t = 1$ by Def. 6.2 above
  - A formula A $\in$ Form(L^{pm}) is valid if for all truth valuations K, an equivalence relation R, and all truth valuation $t \in$ K, it holds that $A^t = 1$ as defined in Definition 6.2 above
- Framed differently, for a modal interpretation I, a world w, and well-formed formula A
  - If I satisfies A in the world w, this is denoted as I, w $\vDash$ A
    - **I, w can then be referred to as a (pointed) model of A**
  - If I does not satisfy A in the world w, this is denoted as I, w $\nvDash$ A

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- **Satisfiability for the Kripke Structures:**
  - Given $I = (W, R, V)$ and $w \in W$, we define what it means for a formula to be satisfied in a world w of I
  - $I, w \models p$ iff $V(p, w) = 1$ for $p \in Prop$
  - $I, w \models \neg A$ iff $I, w \not\models A$
  - $I, w \models A \Rightarrow B$ iff either $I, w \not\models A$ or $I, w \models B$
  - $I, w \models (\Box A)$ iff for all worlds $v \in W$ accessible from w it holds that $I, v \models A$
  - $I, w \models (\Diamond A)$ iff for some world $v \in W$ accessible from w it holds that $I, v \models A$
- **In addition:**
  - Modal formula A is valid if $I, w \models A$ holds for all interpretations I and all $w \in W$
  - Modal formula A is satisfiable if $I, w \models A$ holds for some interpretation I and some $w \in W$
  - Modal formula A is unsatisfiable otherwise

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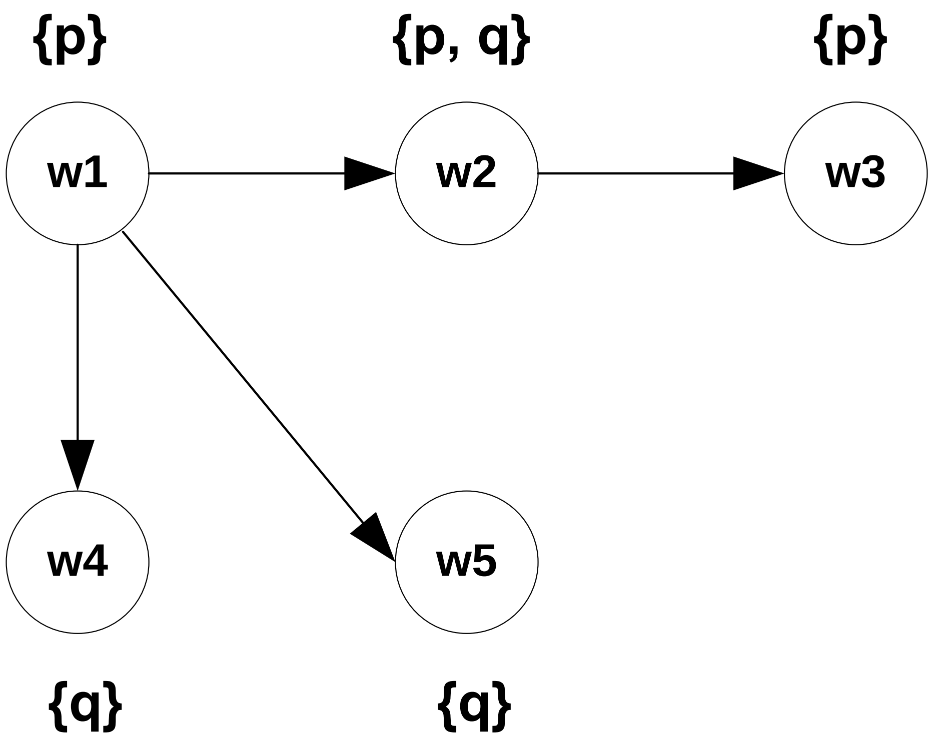


- Example I (on the left):
  - $I, w1 \models p$
  - $I, w1 \models \Box q$
  - $I, w1 \models \neg\Box p$
  - $I, w1 \models \neg\Box\neg p$
  - $I, w1 \models \Diamond p$
  - $I, w1 \models \Diamond\Box p$ (why?)
- Valid:
  - $\Box(p \Rightarrow p)$
  - $\Box p \Rightarrow \Box p$
- Satisfiable:
  - $\Box p \Rightarrow p$

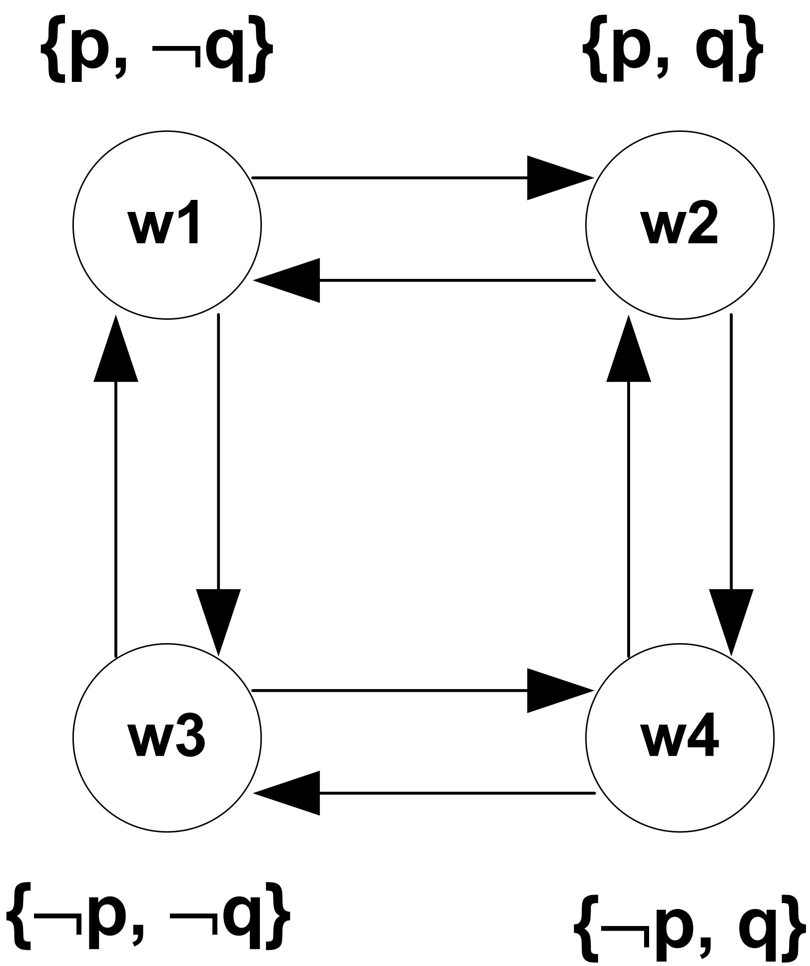
Introducing Modal Logic /9

- **Definition 6.5. Modal Logic Frames:**
  - A modal formula A characterizes a class of frames F if I, w $\models$ A for all I and w $\in$ W where the frame (W, R) $\in$ F
  - And J, w $\not\models$ A for some J and w $\in$ W where (W, R) $\notin$ F
- **Some important formula that characterize frames:**
  - $\Box p \Rightarrow p$; defines reflexive frames (T)
  - $\Box p \Rightarrow \Box\Box p$; defines transitive frames (4)
  - $p \Rightarrow \Box\Diamond p$; defines symmetric frames (B)
  - $\Box p \Rightarrow \Diamond p$; defines serial frames (D)
  - $\Box\Box p \Rightarrow \Box p$; defines dense frames
  - $\Diamond p \Rightarrow \Box p$; defines unique frames

Introducing Modal Logic /10

- **Using Modal Logic:**
  - One can define possible worlds as states in computation
  - R, the accessibility relation, can be viewed as a transition relation between states
  - V, the valuation function, tells us which variables are set in which state
  - **That is, we can define a program and its states as a Kripke structure, and use the modal logic to formally define the properties of the program**

Introducing Modal Logic /11



- **Expressing Program Structure Example:**
  - Let us assume that Boolean variables p and q are used in this program
  - As the program transitions from one state (world) to another, it flips the parity of one variable, as shown on the left
  - For instance, as it moves from w2 to w4 it makes p become not true

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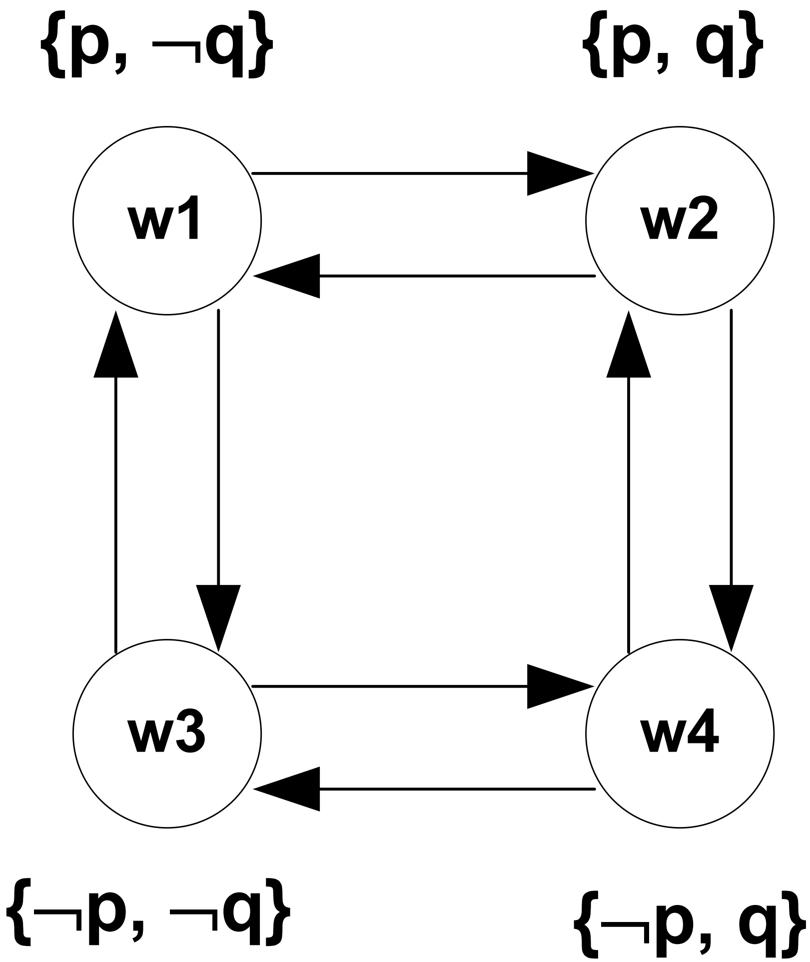

- What can you say about this program that holds true in all states?
  - $\Diamond\neg p \wedge \Diamond p$
  - $\Diamond\neg q \wedge \Diamond q$
  - $p \wedge q \Rightarrow \Box(\neg p \vee \neg q)$
  - $p \wedge q \Rightarrow \Diamond\Diamond(p \wedge q)$
  - **Can you think of anything else?**

Food for Thought

- **Read:**
  - Chapter 8, Sections 8.1 and 8.2 from Zhongwan
    - Read the material discussed in class in more detail
    - Skip the material not mentioned in class
  - Handout on “Modal Logic”
    - Available from the course schedule web page or through LEARN
- **Answer Assignment #3 questions**
  - Assignment #3 includes several practice exercises related to Modal Logic
  - The assignment will be posted by Thursday but it will be due after the midterm exam