Modal Logic Part2

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[with material from “Mathematical Logic for Computer Science”, by Zhongwan, published by World Scientific; and “Modal Logic Tutorial”, by V. Goranko, Indian School on Logic and Its Applications (ISLA) 2010, Online]

Objectives

- Introduction to Model Checking
- Modal Logic Proofs using the System K
- Soundness and Completeness of the System K

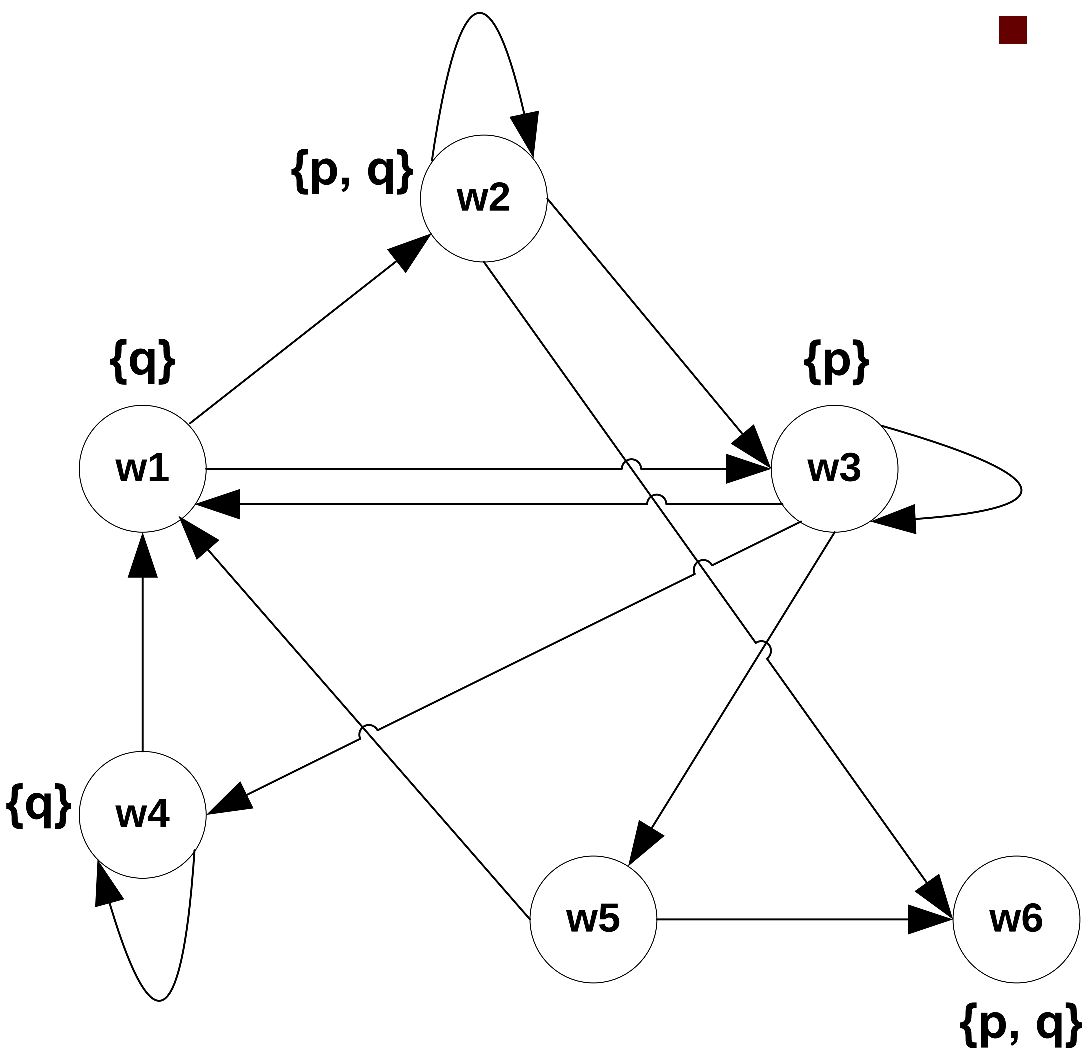
Introduction to Model Checking /1

- **Model Checking:**
  - Checking whether a given model satisfies a given property, where the model is usually specified in some logical language
  - The process may or may not be algorithmically decidable, depending on the logical formalism and the class of models
  - For modal logic, we distinguish three classes of model checking: Local, Global, and Satisfiability Checking
- **Local Model Checking:**
  - Given a finite Kripke Structure (Interpretation) M, a node $w \in W$ and a modal formula A, determine if $M, w \models A$
- **Global Model Checking:**
  - Given a finite Kripke Model (Interpretation) M and a modal formula A, determine all $w \in W$ such that $M, w \models A$
    - The set of nodes w where $M, w \models A$ is denoted as $||A||_M$

Introduction to Model Checking /2

- **Satisfiability Checking:**
  - Given a finite Kripke Model (Interpretation) M and a modal formula A, determine if $M, w \models A$ for any $w \in W$
    - That is, determine if $||A||_M \neq \emptyset$
- **Global Model Checking Approach:**
  - For atomic propositions p, compute $||p||_M$ for each p from the structure, by listing all nodes where p holds
  - For $\Diamond A$ formulas, compute $||\Diamond A||_M$ by listing all nodes w which have a successor s (i.e., wRs holds) in $||A||_M$
  - For $\Box A$ formulas, compute $||\Box A||_M$ by listing all nodes w which have all their successor s (i.e., wRs holds) in $||A||_M$
  - For $A \wedge B$, compute $||A \wedge B||_M$ by computing $||A||_m \cap ||B||_M$
  - For $A \vee B$, compute $||A \vee B||_M$ by computing $||A||_m \cup ||B||_M$

Introduction to Model Checking /3



- Example M (on the left):
  - $||p||_M = \{w2, w3, w6\}$
  - $||\Diamond p||_M = \{w1, w2, w3, w5\}$
    - **Why not w6?**
    - **Need ANY successor s such that I, s $\models$ p**
  - $||\Box p||_M = \{w1, w2, w6\}$
    - **Why w6?**
    - **Need ALL successors s (Ø) such that I, s $\models$ p**
  - $||p \wedge \Box p||_M = \{w2, w6\}$
  - $||p \vee \Box p||_M =$ $\{w1, w2, w3, w6\}$
  - $||\Diamond\Box p||_M =$ $\{w1, w2, w3, w4, w5\}$

Modal Logic Proofs /1

- **Definition 7.1. The System K:**
  - A deduction system for the modal logic formulas
  - Ax1: $\langle (A \Rightarrow (B \Rightarrow A)) \rangle$
  - Ax2: $\langle ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))) \rangle$
  - Ax3: $\langle (((\neg A) \Rightarrow (\neg B)) \Rightarrow (B \Rightarrow A)) \rangle$
  - Ax4: $\langle (\Box(A \Rightarrow B) \Rightarrow (\Box A \Rightarrow \Box B)) \rangle$
  - MP: $\langle A \Rightarrow B, A, B \rangle$
  - Nec: $\langle A, \Box A \rangle$
    - Where A cannot be deduced from the assumptions
    - Also referred to as $\Box$-Introduction
  - A WFMF A_1 is formally provable by K iff $\vdash_K A_1$ holds

Modal Logic Proofs /2

- **Prove that $\vdash_K \Box(A \Rightarrow A)$ holds**
  1. $(A \Rightarrow A)$ (by Theorem from Notes #3)
  2. $\Box(A \Rightarrow A)$ (by Nec, (1))
- **That is, we can use any previously proved theorem of H since K includes H axioms and MP rule**

Modal Logic Proofs /3

- **Prove that $\vdash_K \Box(A \wedge B) \Rightarrow \Box A$ holds**
  1. $A \wedge B \Rightarrow A$ (by $\wedge$-Elimination)
  2. $\Box(A \wedge B \Rightarrow A)$ (by Nec, (1))
  3. $\Box(A \wedge B \Rightarrow A) \Rightarrow (\Box(A \wedge B) \Rightarrow \Box A)$ (by Ax4)
  4. $\Box(A \wedge B) \Rightarrow \Box A$ (by MP, (3), (2))
- **Prove that $\vdash_K \Box(A \wedge B) \Rightarrow \Box B$ holds**
  1. $A \wedge B \Rightarrow B$ (by $\wedge$- Elimination)
  2. $\Box(A \wedge B \Rightarrow B)$ (by Nec, (1))
  3. $\Box(A \wedge B \Rightarrow B) \Rightarrow (\Box(A \wedge B) \Rightarrow \Box B)$ (by Ax4)
  4. $\Box(A \wedge B) \Rightarrow \Box B$ (by MP, (3), (2))

Modal Logic Proofs /4

- **Prove that $\vdash_K \Box(A \wedge B) \Rightarrow (\Box A \wedge \Box B)$ holds**
  - Enough to prove that $\Box(A \wedge B) \vdash_K (\Box A \wedge \Box B)$

1. $\Box(A \wedge B)$ (by Assumptions)
2. $\Box(A \wedge B) \Rightarrow \Box A$ (see Previous Slides)
3. $\Box(A \wedge B) \Rightarrow \Box B$ (see Previous Slides)
4. $\Box A$ (by MP, (2), (1))
5. $\Box B$ (by MP, (2), (1))
6. $\Box A \Rightarrow (\Box B \Rightarrow (\Box A \wedge \Box B))$ (by $\wedge$-Introduction)
7. $\Box B \Rightarrow (\Box A \wedge \Box B)$ (by MP, (6), (4))
8. $\Box A \wedge \Box B$ (by MP, (7), (5))

Modal Logic Proofs /5

- **Prove that $\vdash_K \Box(A \wedge B) \Rightarrow (\Box A \wedge \Box B)$ holds**
  - Alternatively without using the Deduction Theorem for K, first prove $(A \Rightarrow B) \Rightarrow (A \Rightarrow C) \Rightarrow (A \Rightarrow (B \wedge C))$ in H and then use that theorem instead

1. $\Box(A \wedge B) \Rightarrow \Box A$ (see Previous Slides)
2. $\Box(A \wedge B) \Rightarrow \Box B$ (see Previous Slides)
3. $(\Box(A \wedge B) \Rightarrow \Box A) \Rightarrow (\Box(A \wedge B) \Rightarrow \Box B) \Rightarrow$ $(\Box(A \wedge B) \Rightarrow (\Box A \wedge \Box B))$ (by Theorem above)
4. $(\Box(A \wedge B) \Rightarrow \Box B) \Rightarrow$ $(\Box(A \wedge B) \Rightarrow (\Box A \wedge \Box B))$ (by MP, (3), (1))
5. $\Box(A \wedge B) \Rightarrow (\Box A \wedge \Box B)$ (by MP, (4), (2))

Modal Logic Proofs /6

- **Deduction Theorem for the System K:**
  - For A, B $\in$ Form (L^{pm}) and $\Sigma \subseteq$ Form (L^{pm}), $\Sigma \vdash_K A \Rightarrow B$ iff $\Sigma \cup \{A\} \vdash_K B$
  - Proof: Based on the Deduction Theorem proof for H with the addition of the Nec inference rule (see A3.Q4)
- **Soundness of the System K:**
  - For A $\in$ Form (L^{pm}) and $\Sigma \subseteq$ Form (L^{pm}), $\Sigma \vdash_K A \Rightarrow \Sigma \vDash A$
  - Proof: By induction on the structure of A; similar to the Soundness proof for H with the addition of the Nec rule
- **Completeness of the System K:**
  - For A $\in$ Form (L^{pm}) and $\Sigma \subseteq$ Form (L^{pm}), $\Sigma \vDash A \Rightarrow \Sigma \vdash_K A$
  - Proof: Based on the Maximal Consistency property

Food for Thought

- **Read:**
  - Chapter 8, Sections 8.1 and 8.2 from Zhongwan
    - Read the material discussed in class in more detail
    - Skip the material not mentioned in class
  - Handout on “Modal Logic”
    - Available from the course schedule web page or through LEARN
- **Answer Assignment #3 questions**
  - Assignment #3 includes several practice exercises related to Modal Logic