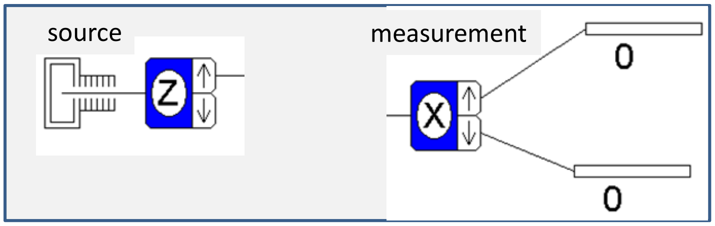
Chapter 3 Summary

preparation (system) | dynamics | measurement (apparatus)



$| \Psi \rangle$

(normalized) element of complex vector space dimension d

$\{|a_i\rangle\}_{i=1}^{d}$

orthonormal basis

Update Rule:
after measurement system is in state of measurement result

Prediction Rule

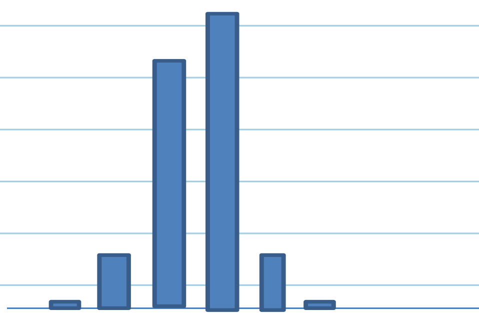
$\Pr(\text{"i"}) = |\langle a_i | \Psi \rangle|^2$

Superposition rule:
Unmeasured system remain in superposition of potential outcomes

Chapter 4 Operators in Quantum Mechanics

4.1 Observables

4.1.1 classical statistics and expectation values



Experiments in physics typically give some distribution.

Example:

look at the way we experimentally determine the value of the Spin of an electron.

==> on the screen you see atoms piling up with different splittings

==> each location can be related to one specific value of the spin

==> we use the mean value of the splitting between upper and lower curve

==> we obtain the value $\frac{\hbar}{2}$ for the spin

Note that the probability to appear in some interval on the screen is the actual experimental event, and we assign to each interval a number that relates either to the splitting, or directly to the corresponding spin.
So there are two steps: experimental probabilities and data processing where we assign to each outcome specific values. These values come from the context why we do experiment in the first place!

mean values are often referred to as ***expectation value***!

outcome "i" <==> assigned value a_i $\quad i = 1, \dots, n$

occuring with probability $Pr(\text{"i"})$

==> expectation value $\langle A \rangle := \sum_{i=1}^{n} a_i \, Pr(\text{"i"})$

Where do the assigned values come from? Context!
e.g. when we determined the VALUE of the spin of a silver atom, we used the distance between the spot where the atoms hit the screen from the x-asis (to determine the z-coordinate)
==> the expectation value of this distance linked the spin component

$$S_z = \pm \frac{\hbar}{2}$$

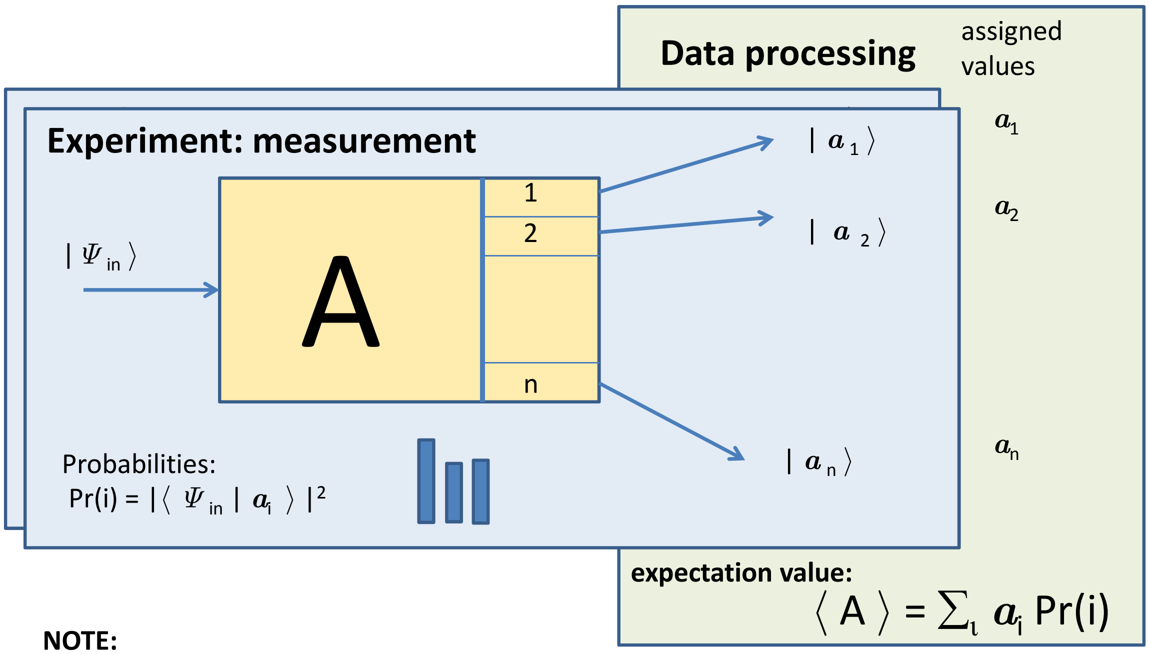
Everyday example:
Shooting at the hoot in Basketball:
two possible events
"ball goes in"
"ball does not go in " (out)

Experiment:
You observe the probabilities
Pr ("in") = 3/4
Pr ("out") = 1/4

Dataprocessing:
"in" <==> assigned value 2 points
"out" <==> assigned value 0 points

Expectation Value of points:
3/4 x 2 points + 1/4 x 0 points = 3/2 points

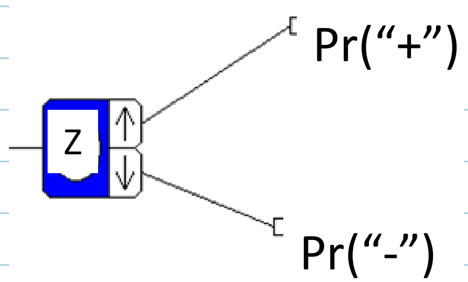
4.1.2 application to quantum mechanics experiments:
Step 1: Experiment (measurrement) where one observes the probability that event "i" occurs
Step 2: Data Processing: assign to each outcome some value and form expectation value of these values with the corresponding outcome probability:



NOTE:
sometimes we use the assigned value also as label of the ket describing the measurement outcome!
➔ convenience, whenever the assigned values are all different from each other
➔ but it does not mean that we cannot assign the same value to different outcomes

4.1.3 Introducing the Observable-Operators via Example of spin component

Observables



assigned value:
$+\hbar/2$

assigned value
$-\hbar/2$

Experimental evaluation

$$\langle S_z \rangle = \hbar/2 \Pr(\text{“+”}) - \hbar/2 \Pr(\text{“-”})$$

then we talk about the expectation value of the measurement denoted as

$$\langle S_z \rangle = \frac{\hbar}{2} P_{r+} + \left(-\frac{\hbar}{2}\right) P_{r-}$$

$$= \frac{\hbar}{2} |\langle +_z | \psi \rangle|^2 - \frac{\hbar}{2} |\langle -_z | \psi \rangle|^2$$

$$= \frac{\hbar}{2} \langle \psi | + \rangle \langle + | \psi \rangle - \frac{\hbar}{2} \langle \psi | - \rangle \langle - | \psi \rangle$$

Now look at the expression

$$|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$$
$$|\alpha|^2 + |\beta|^2 = 1$$

$$\langle \psi | + \rangle \langle + | \psi \rangle$$

$$\Updownarrow$$

$$\left[(\alpha^*, \beta^*)\begin{pmatrix}1\\0\end{pmatrix}\right]\left[(1,0)\begin{pmatrix}\alpha\\\beta\end{pmatrix}\right]$$ free association!

$$(\alpha^*, \beta^*)\left[\begin{pmatrix}1\\0\end{pmatrix}(1,0)\right]\begin{pmatrix}\alpha\\\beta\end{pmatrix}$$

outer product!

$$= (\alpha^*, \beta^*)\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix}$$

$$\langle \psi | \quad P_+ \quad | \psi \rangle$$

$$P_+ = |+\rangle\langle +|$$ is an operator!

operators map vectors to vectors!

$$P_+|\psi\rangle = |\tilde{\psi}\rangle$$ (in general scenario: resulting state not normalized!)

here: $P_+|\psi\rangle = |+\rangle\langle +|\psi\rangle = \underbrace{\langle +|\psi\rangle}_{\text{complex number}} \underbrace{|+\rangle}_{\text{normalized vector}}$

Then we can rewrite the expectation value as

$$\langle S_z \rangle = \langle \psi | \underbrace{\left(\frac{\hbar}{2} |+\rangle\langle +| - \frac{\hbar}{2} |-\rangle\langle -| \right)}_{=: S_z \ \text{(operator)}} |\psi\rangle$$

$$= \langle \psi | S_z | \psi \rangle$$

In coordinate representation (with respect to basis $|+\rangle_z, |-\rangle_z$)

$$S_z \Longleftrightarrow \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \frac{\hbar}{2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

similiar

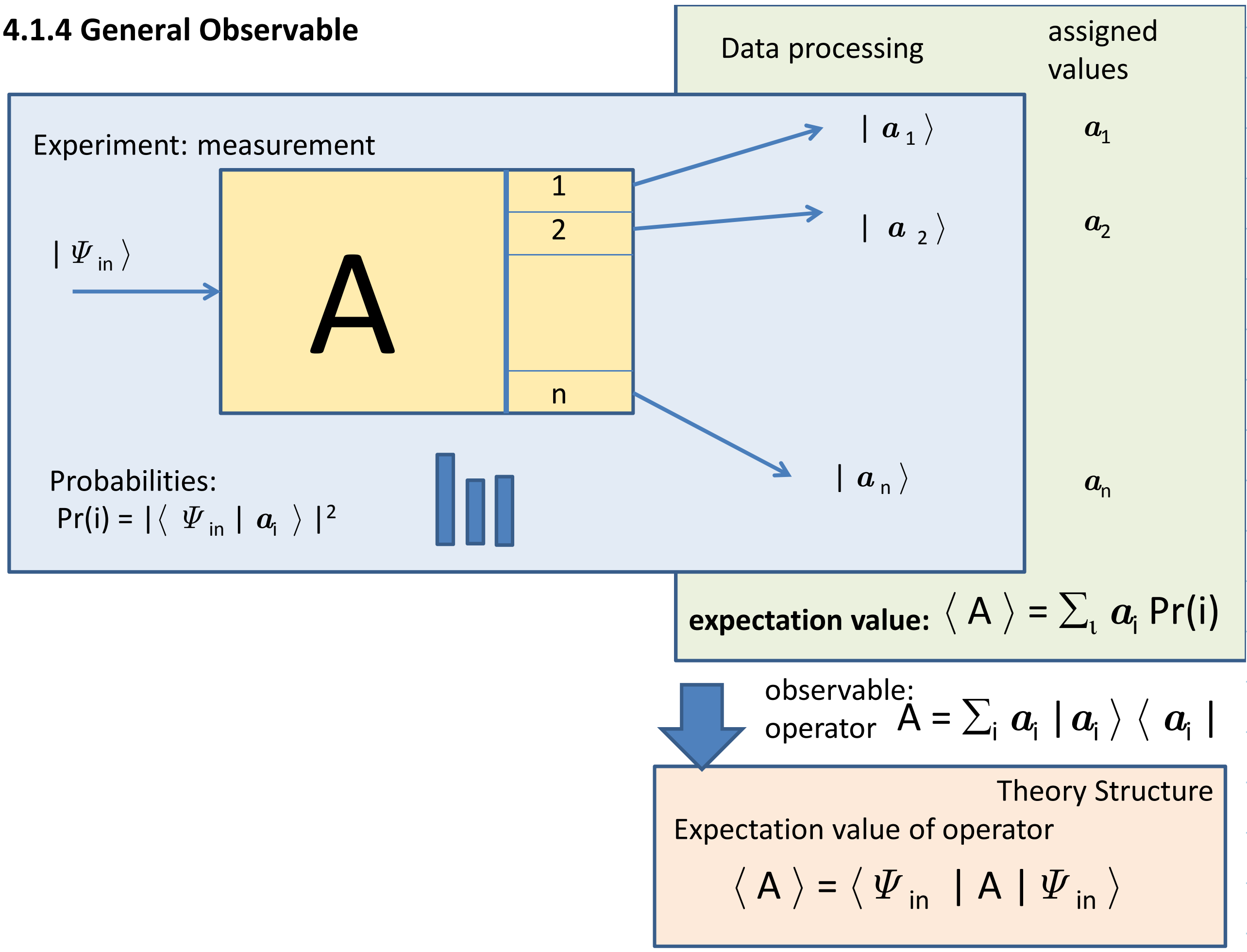
$$S_x = \frac{\hbar}{2} |+\rangle_x \langle +|_x - \frac{\hbar}{2} |-\rangle_x \langle -|_x$$

$$= \frac{\hbar}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} (1, 1) - \frac{\hbar}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{1}{\sqrt{2}} (1, -1)$$

$$= \frac{\hbar}{2} \left[\frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right]$$

$$= \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

4.1.4 General Observable



M4: Math Inset: outer product, operators and their coordinate representation

outer product:

$$|\psi\rangle \qquad\qquad \langle\varphi|$$
$$\shortparallel \qquad\qquad \shortparallel$$

$$|\psi\rangle\langle\varphi| \doteq \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix} \begin{pmatrix} \delta_1 & \dots & \delta_n \end{pmatrix} = \begin{pmatrix} \gamma_1\delta_1 & \gamma_1\delta_2 & \dots & \gamma_1\delta_n \\ \gamma_2\delta_1 & & & \\ \vdots & & & \\ \gamma_n\delta_1 & & \dots & \gamma_n\delta_n \end{pmatrix}$$

Operators and Coordinate Representation

$$a_{ij} = \langle a_i | A | a_j \rangle \qquad a_i = \langle a_i | \Psi \rangle$$

$$|\tilde{\Psi}\rangle = \sum_{i=1}^{d} \tilde{a}_i |\tilde{a}_i\rangle \qquad A = \sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij} |a_i\rangle\langle a_j| \qquad |\Psi\rangle = \sum_{i=1}^{d} a_i |a_i\rangle$$

$$\doteq \qquad \doteq \qquad \doteq$$

$$\begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \vdots \\ \tilde{a}_d \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1d} \\ a_{21} & a_{22} & \dots & a_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d1} & a_{d2} & \dots & a_{dd} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix}$$

$$\doteq$$

$$\boxed{|\tilde{\Psi}\rangle = A\, |\Psi\rangle}$$

as long as one uses for all abstract objects the same orthonormal basis …

M5: Training Unit in bra-ket notation manipulation

$| \Psi \rangle$ is a ket , an element of our basic vector space
$\langle \Psi |$ is a bra, an element of the dual vector space
$\langle \phi | \Psi \rangle$ is a complex number (result of scalar product)
Rule: $\langle \phi | \Psi \rangle = \langle \Psi | \phi \rangle^*$

A is an operator
A $| \Psi \rangle$ is a ket
$\langle \Psi |$ A is a bra
A B is an operator
$|\Psi\rangle\langle \phi|$ is an operator
Rule: freedom to group elements (associative)
$|\Psi\rangle \langle \phi | \eta \rangle$ is
- a ket $|\Psi\rangle$ multiplied by the number $\langle \phi | \eta \rangle$
- is an operator $|\Psi\rangle \langle \phi |$ acting on a ket $| \eta \rangle$

Rule: numbers commute with all other objects
$|\Psi\rangle \langle \phi | \eta \rangle = \langle \phi | \eta \rangle |\Psi\rangle$

clicker question:
The object
$\langle \Psi | A | \phi \rangle \langle \eta |$
is
A) a bra
B) a ket
C) an operator
D) a number
E) not a valid object

clicker question:
The object
$\langle \eta | \langle \Psi | A | \phi \rangle$
is
A) a bra
B) a ket
C) an operator
D) a number
E) not a valid object

clicker question:
The object
$\langle \eta | \langle \Psi | A$
is
A) a bra
B) a ket
C) an operator
D) a number
E) not a valid object

Answers: 1) A 2) A 3) E

4.2 Y-direction Stern Gerlach Experiment and summary Coordinate Representation Spin 1/2

Spin in Y-direction

One can also do Stern-Gerlach measurements in a y-direction. The experimental details don't matter too much right now (we can explain that much better in the third section), but it is clear:

Z-Stern-Gerlach

applies an inhomogeneous field to the atom in z-direction,
which moves it up/down

X-Stern-Gerlach

applies an inhomogeneous field to the atom in x-direction,
which moves is right/left (up/down in x-direction)

so

Y-Stern-Gerlach

applies an inhomogeneous field to the atom in y-direction,
which moves if forward/backward (up/down in y-direction)

We can use such a measurement in the SPINS demonstration (see weblink in earlier lecture notes) We notice

Input state					result Y-measurement
$\vert + \rangle_z$					50/50
$\vert + \rangle_x$					50/50

So how can we express the measurement vectors

$|+\rangle_y$ & $|-\rangle_y$

in the standard basis?

We have to do the same calculation as we did in 3.3.2 (Lecture 07), only we have even more constraints to play with!
More can be found in Example 1.3 (McIntyre, page 13)

We find the resulting vectors, and can use these also to calculate a spin component vector S_y to find:

Coordinate representations

$$|+\rangle \doteq \begin{pmatrix}1\\0\end{pmatrix} \qquad |-\rangle \doteq \begin{pmatrix}0\\1\end{pmatrix}$$

$$|+\rangle_x \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad |-\rangle_x \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$|+\rangle_y \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} \qquad |1\rangle_y \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$

$$S_z \doteq \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \qquad S_x \doteq \frac{\hbar}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \qquad S_y \doteq \frac{\hbar}{2}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}$$

Spin measurement

z-direction	x-direction	y-direction

Polarization of single photon

horizontal/vertical linear polarization	+ 45/ -45 degree linear polarization	right/left circular polarization