PHYS 256: Geometrical and Physical Optics (Lecture Notes v1.8)

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It's quite unfortunate you're reading this document. It's okay though. I was naive once and didn't think 256 would be so bad... then I learned my lesson. **Note:** f will always refer to frequency. I don't use ν because I'm not a giant douche!

A picture of O'Donovan because fuck you, that's why!

Contents

1 Introduction to Waves

Here's a few quick things you should know:

- Light is a 3D electromagnetic (EM) wave that has vibrating E and M fields
- Light is a wave (physical optics) and a particle (geometrical optics)
- Light is a transverse wave propagating in spacetime emitted by a source
- The sign convention of wave direction is set up as: negative means to the right, positive means to the left.

1.1 Propagating Waves

A wave is called a propagating wave (or a traveling wave) if it maintains its shape in space and time. Mathematically this can be described as:

$$f(x,0) = f(x')$$

$$\text{where}$$
$$x' = x - vt$$

There's another way to describe this but I'll come back to it when I actually know what it is.

1.2 The Wave Equation

The wave equation is a second order partial differential equation. Why would I put that in here? Maybe later.

1.3 Harmonic Sinusoidal Waves

A harmonic sinusoidal wave is of the form

$$\Psi(x,t) = A\sin(k(x \mp vt) + \varepsilon)$$

$$\text{where}$$
$$A = \text{amplitude}$$
$$k = \text{propagation constant}$$
$$\varepsilon = \text{initial phase}$$

1.4 Transverse Waves

A transverse wave repeats after a distance of one wavelength, a time period of the temporal period or an additional phase of 2π. Mathematically this is represented as:

- $\Psi(x,t) = \Psi(x+\lambda, t)$
- $\Psi(x,t) = \Psi(x, t+\tau)$
- $\Psi(x,t) = A\sin(k(x \mp vt) + \varepsilon) = A\sin(k(x \mp vt) + \varepsilon + 2\pi)$

$$\text{where}$$
$$\lambda = \text{wavelength}$$
$$\tau = \text{temporal period}$$

The transverse wave equation can be rewritten by fucking around with all those crazy damn relationships:

$$\Psi(x,t) = A\sin(kx \mp \omega t + \varepsilon)$$

$$\text{where}$$
$$\omega = \text{angular frequency}$$

The transverse wave satisfies the wave equation (durrrrrr).

1.5 The Principle of Superposition

If $\Psi_1(x,t)$ is a solution to the wave equation and $\Psi_2(x,t)$ is also a solution to the wave equation then the combination $(\Psi_1(x,t)+\Psi_2(x,t))$ is also a solution to the wave equation.

1.6 Adding Waves

Consider two waves,
$\Psi_1(x,t) = A_1\sin(kx - \omega t + \varepsilon_1)$ and $\Psi_2(x,t) = A_2\sin(kx - \omega t + \varepsilon_2)$. To add them together, we simply take $\Psi_1(0,0)$ and $\Psi_2(0,0)$ which yields $A_1\sin(\varepsilon_1)$ and $A_2\sin(\varepsilon_2)$ respectively, and use vector addition on a Cartesian plot. Note that the waves must have the same frequency.

The new amplitude A and the new initial phase ε is found as such:

$$A^2 = A_1^2 + A_2^2 - 2A_1A_2\cos(180 - (\varepsilon_2 - \varepsilon_1))$$

$$\tan(\varepsilon) = \frac{A_1\sin(\varepsilon_1)+A_2\sin(\varepsilon_2)}{A_1\cos(\varepsilon_1)+A_2\cos(\varepsilon_2)}$$

So the combination of the two waves can be written as:

$$\Psi(x,t) = A\sin(kx - \omega t + \varepsilon)$$

2 3D Waves

2.1 2D Waves

Here are some things you should know about **2D Waves**:

- The energy of the wave decreases with distance
- The propagation direction is perpendicular to the wavefront

That's all, I guess.

2.2 3D Planar Waves

The representation of a 3D Planar Wave is

$$\vec{\Psi}(\vec{r}, t) = \vec{A} \sin(\vec{k} \cdot \vec{r} - \omega t + \varepsilon)$$

Here are some things you should know about **3D Planar Waves**:

- Waves move in direction $\vec{k}$, which is the plane's normal vector
- If the plane is defined by $(\vec{r} - \vec{r_0})$, then $\vec{k} \cdot (\vec{r} - \vec{r_0}) = 0$ and $\vec{k} \cdot \vec{r} =$ constant
- At one time within one specific plane of the wave, $\Psi(\vec{k} \cdot \vec{r}) =$ constant vector
- Wavelength is measured along $\vec{k}$, perpendicular to the wavefront
- Planar wavefronts are due to distant sources

2.3 3D Planar Harmonic EM Wave

Even though this is blatantly not true in all realness simply due to Euler's formula, we can instead represent the 3D Planar Wave equation as

$$\vec{E}(\vec{r}, t) = \vec{E_0}\ e^{i(\vec{k} \cdot \vec{r} - \omega t + \varepsilon)}$$

$$\text{where}$$
$$\vec{k} = (k_x, k_y, k_z)$$

The 3D Planar Harmonic EM Wave satisfies the wave equation. Duh.

2.4 Polarization

If the source "vibrates" back and forth, two separate pulses (one E, one M) emerge:

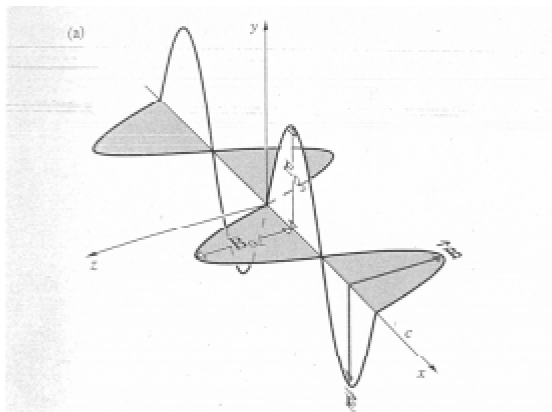


Figure 1: An example!

$\vec{E}$, $\vec{B}$ and $\vec{k}$ are all perpendicular to each other and $\vec{E} \times \vec{B} = |E||B|\widehat{k}$. In Figure 1 (note that the axes are not conventional), the wave is traveling along the x-axis ($\vec{k}$ is along x), the magnetic field is propagating in the xz-plane ($\vec{B}$ is along z) and the electric field is propagating in the xy-plane ($\vec{E}$ is along y). Mathematically, we can represent this as

$$E_y(x,t) = E_{0y} \sin(kx - \omega t + \varepsilon)$$
$$B_z(x,t) = B_{0z} \sin(kx - \omega t + \varepsilon)$$

2.5 Spherical Waves

Here are some things you should know about **spherical waves**:

- A point source produces spherical wavefronts
- Isotropic (definition): radiation is uniform in all directions

2.5.1 3D Harmonic EM Spherical Wave

I'm not sure how this differs from other spherical waves but we represent the 3D Harmonic EM Spherical Wave as:

$$\vec{E}(\vec{r},t) = \vec{E}_0(r)\ e^{i(\vec{k}\cdot\vec{r}\mp\omega t+\varepsilon)}$$

$$\text{where}$$
$$\vec{E}_0 = \frac{\vec{A}}{r}$$
$$\vec{A} = \text{source strength}$$

Physically, $\vec{E}_0$ represents the amplitude at radius r.

2.6 Spherical and Planar Waves

Here are a few things you should know that apply to **spherical and planar wavefronts**:

- The amplitude of an undamped planar EM wave does not change as it propagates
- The amplitude of a spherical EM wave changes like $1/r$
- A diverging wavefront becomes planar at a large radius
- I THINK a converging wavefront starts curved and becomes more curved
- Diverging: $\vec{E}(\vec{r},t) = \vec{E}_0(r)e^{i(\vec{k}\cdot\vec{r}-\omega t+\varepsilon)}$
- Converging: $\vec{E}(\vec{r},t) = \vec{E}_0(r)e^{i(\vec{k}\cdot\vec{r}+\omega t+\varepsilon)}$

3 Properties of EM Radiation

3.1 The Poynting Vector

Hey, another list:

- The Poynting Vector represents the energy flux density
- Energy is distributed equally in both the electric and magnetic components of the wave
- The average flow of energy is irradiance: $I = \langle \vec{S} \rangle_t = \frac{1}{T}\int_0^T S(t)dt$
- Irradiance is also referred to as radiant flux density

We represent electric energy density and magnetic energy density respectively as:

$$u_E = \frac{\varepsilon_0}{2}E^2 \quad \text{and} \quad u_B = \frac{1}{2\mu}B^2$$

$$\begin{gathered}\text{where}\\ \varepsilon_0 = 8.854 \times 10^{-12} \quad F \cdot m^{-1}\\ \mu_0 = 4\pi \times 10^{-7} \quad N \cdot A^{-2}\end{gathered}$$

So the total density of the EM wave is simply:

$$u = u_E + u_B$$

And then the Poynting vector is given by:

$$S = uc = \frac{1}{\mu_0}EB$$

3.2 Radiant Flux Density

We can also write irradiance (radiant flux density) as:

$$I = \frac{\Phi}{A}hf$$

$$\begin{gathered}\text{where}\\ \Phi = \text{photon flux}\\ A = \text{area}\\ hf = \text{photon energy}\end{gathered}$$

3.3 Pressure of an EM Wave

There are SEVERAL ways to write pressure. Here I will only refer to pressure as related to irradiance and energy density; refer to the equation sheet for other relationships. The average pressure measured for an absorbing surface can be written as:

$$P = u = \frac{I}{c}$$

While the average pressure measured for a reflecting surface is written as:

$$P = \frac{2I}{c}$$

where
u = energy density
I = irradiance

If the light is not in a vacuum, use the speed of light in that medium.

3.4 Force of an EM Wave

There are once again a million ways to represent this shit so I am going to use the relationship of force and pressure because yolo. The force of light for an absorbing surface is written as:

$$F = \frac{P}{c}$$

While the force of light for a reflecting surface is written as:

$$F = \frac{2P}{c}$$

where
P = power

3.5 Momentum of an EM Wave

Holy crap I am getting sick of writing the fact that there are many ways to write this as well. Why didn't I copy paste? The momentum is written as:

$$\Delta\vec{p} = \vec{F}\Delta t$$

The momentum carried per unit volume of wave (whatever that means) is:

$$\vec{p}_v = \frac{u}{c}$$

With the relationship:

$$\vec{p} = \vec{p}_v c \Delta t A$$

3.6 Index of Refraction

Simply put, the index of refraction is defined as:

$$n = \frac{c}{v}$$

$$\text{where}$$
$$v = \text{speed}$$

While the new wavelength (new as in second medium) is related to the old wavelength (old as in initial medium) by:

$$\lambda = \frac{\lambda_0}{n}$$

4 Light and Matter

4.1 Photoelectric Effect

The photoelectric effect: a photon with enough energy can free an electron from a metal, with any extra energy being converted to kinetic energy. The "fundamental equation" for the photon is:

$$E = hf$$

$$\text{where}$$
$$h = 6.626 \times 10^{-34} \quad J \cdot s$$

We usually write **momentum** as:

$$p = \frac{E}{c}$$

The **photon flux density** is defined as:

$$\frac{\Phi}{A} = \frac{I}{E}$$

where

Φ = photon flux

A = area

I = irradiance

E = energy

4.2 Huygens' Principle

"Every point on a propagating wavefront serves as the source of spherical secondary wavelets, such that the wavefronts at some later time is the envelope of these wavelets. These wavelets have the same frequency, speed and wavelength as the original wavefront."

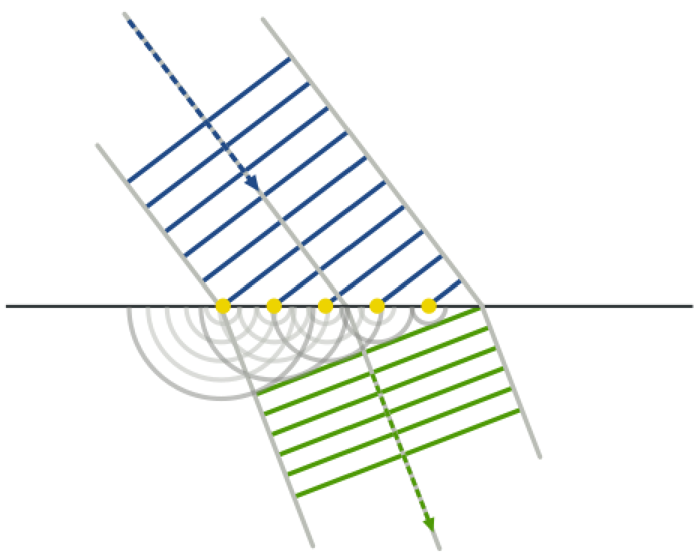


Figure 2: The effect of Huygens' Principle.

4.3 Light Interacting with Matter

Light interacts with matter through the following ways:

- Transmission (including refraction)
- Scattering (oscillation of dipoles)
- Reflection (a type of scattering)
- Absorption (with or without re-emission)

4.4 Dispersion

An important problem is that the index of refraction depends on wavelength. This means that light passing through a medium will act different depending on both the material and the wavelength. This is important if more than one wavelength is being used, such as white light; this results in chromatic aberration.

4.5 Non-Resonant Elastic Scattering

- An oscillating dipole is induced by a field because the electron cloud deforms and oscillates with the field which changes the wave phase (α)
- Transparent materials cause molecules and atoms to re-radiate in all directions at the same frequency (also known as a point source!)
- **Non-Resonant Elastic Scattering** is defined as photons elastically scattering in all directions at a frequency that is not the resonant frequency; the lifetime of re-radiation is very short

4.6 Scattering and Refractive Index

- Repetitive scattering (and re-scattering) slows down the wave and gives a slower phase
- The refractive index of gases is lower compared to solids (due to density)
- Secondary re-emitted waves are slower as ω, the angular frequency, increases

The relationship between refractive index and angular frequency is:

$$(n(\omega))^2 = 1 + \frac{A}{\omega_0^2 - \omega^2}$$

where
ω_0 = resonant frequency

4.7 Lateral Light Scattering (Rayleigh Scattering)

- More dipole vibration in one colour compared to other colours causes re-radiation, a lateral electric field vibration, and scattering
- **Rayleigh Scattering** is the scattering of light due to particles much smaller than the wavelength
- in a gas, most EM fields are unchanged (they propagate in the original direction)
- The sky is blue due to the fact that the gases in the atmosphere scatter blue light laterally
- The amount of scattering is $\propto \frac{1}{\lambda^4}$

4.8 Mie Scattering

- Larger particles scatter light more at longer wavelengths compared to small particles
- Particles in similar size to the wavelength have scattering independent of the wavelength (light is white)

4.9 Transparency

- Transparency is simply of a lack of scattering, or scattering in the forward direction
- Denser materials have less lateral scattering because the waves interfere destructively in the lateral direction but constructively in the forward direction

- Light is relatively unchanged in the forward direction and is in phase

4.10 Reflection and Refraction

- Re-radiated unpaired oscillations half a wavelength deep give reflected waves, whereas at any other distance there is destructive interference
- Reflection is mostly dependent of the wavelength
- If the wavelength is far larger than the spacing, there is both reflection and refraction

The **Law of Reflection** simply states that the incidence angle is equal to the reflection angle. **Snell's Law** relates the refractive indices of the initial and final mediums to the angle of incidence and the angle of refraction:

$$n_i \sin(\theta_i) = n_t \sin(\theta_t)$$

where
n_i = first refractive index
n_t = second refractive index
θ_i = angle of incidence
θ_t = angle of refraction

4.10.1 Pencils and Beams

- Recall that a ray is perpendicular to the wavefront
- A pencil is all rays from a point object
- A beam is all rays from an extended object

5 Introduction to Geometrical Optics

I'm going to start with apparent depth because I'm not sure where else to start.

5.1 Apparent Depth

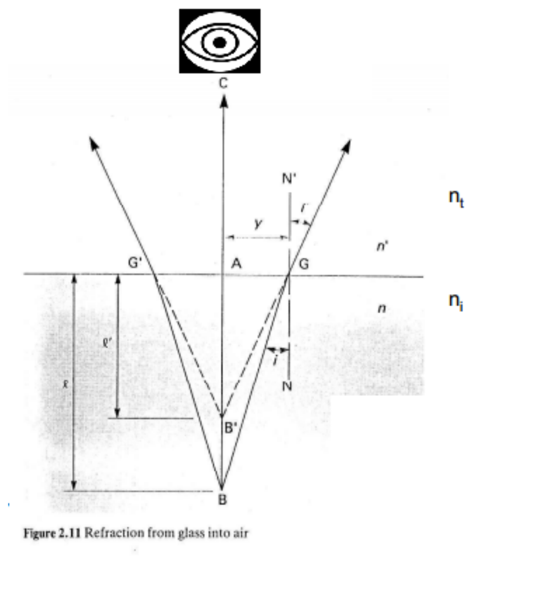


Figure 3: A diagram describing the effects of different mediums on apparent depth.

I suppose the derivation isn't that important, so here's the relationship between apparent depth and real depth:

$$l' = \frac{n_t}{n_i} l$$

where

l' = apparent depth

l = actual depth

n_i = initial refractive index

n_t = final refractive index

5.2 Total Internal Reflection (TIR), Fibre Optics

The **critical angle** is the smallest angle of incidence for which no light is transmitted. Once again, the derivation isn't important (but IS quite simple via Snell's Law):

$$\theta_c = \arcsin(\frac{n_t}{n_i})$$

We define n_c to be the refractive index for the cladding material (the material on the outside of the fibre) and n_f to be the refractive index for the fibre itself (the inside). In this case, n_0 is the refractive index of the medium outside of the fibre. The numerical aperture NA is given by:

$$NA = n_0 \sin(\theta_m) = n_f \sqrt{1 - (\frac{n_c}{n_f})^2}$$

where

$$\theta_m = \text{limiting (incidence) angle}$$

5.3 Refracting Prisms

Refracting prisms are designed to change the direction of incident light. They may be used to disperse light into its different wavelengths; the larger the apical angle of the prism, the more dispersion there is.

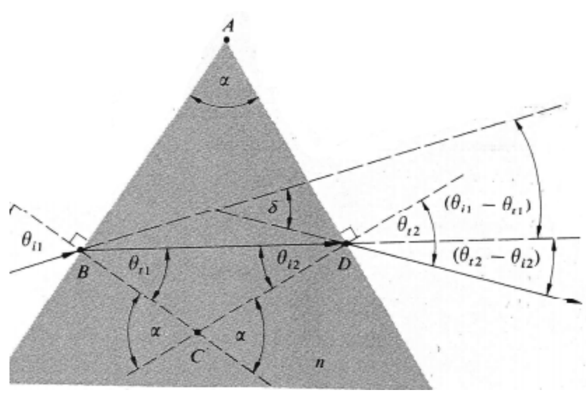


Figure 4: A prism with all of the lovely angles.

In the above diagram, we say the apical (apex) angle is α and the total angular deviation is δ. The first and second incidence and refraction angles are clear. Well, it turns out:

$$\alpha = \theta_{t1} + \theta_{i2}$$
$$\delta = \theta_{i1} + \theta_{t2} - \alpha$$

5.3.1 Angle of Minimum Deviation

The angle of minimum deviation (in air) is when:

$$\theta_{t1} = \theta_{i2} = \frac{\alpha}{2}$$
$$\theta_{i1} = \theta_{t2} = \frac{\delta_{min}+\alpha}{2}$$

So then by using Snell's Law we can find that:

$$\delta_{min} = 2\arcsin(n\sin(\tfrac{\alpha}{2})) - \alpha$$

$$n = \frac{\sin(\frac{\delta_{min}+\alpha}{2})}{\sin(\frac{\alpha}{2})}$$

For **small apical angles (thin prisms)** we use the approximation

$$\delta_{min} = n\alpha - \alpha = 2\theta_{i1} - \alpha$$

5.4 Energy Conversation

The power is conserved when light reflects and refracts; that is to say mathematically:

$$P_i\cos(\theta_i) = P_r\cos(\theta_r) + P_t\cos(\theta_t)$$

where
$i =$ incidence
$r =$ reflected
$t =$ refracted

5.5 Fresnel's Equations

Electric fields can be either perpendicular or parallel to the plane of incidence; we will denote these fields as $E_\perp$ and $E_\parallel$ respectively, and similarly any other perpendicular-or-parallel-dependent variables.

5.5.1 Reflectance, Transmittance, Amplitude Coefficients

Please refer to the equation sheet for further relationships.

$$R = r^2$$

$$T = \frac{n_t \cos(\theta_t)}{n_i \cos(\theta_i)} t^2$$

$$1 = T + R$$

where

$r =$ reflection amplitude coefficient

$t =$ transmission amplitude coefficient

$R =$ reflectance

$T =$ transmittance

In other words, r is the change of amplitude upon reflection and t is the change of amplitude upon transmission (refraction). These also have perpendicular and parallel components which are related to the electric field components:

$$r_\perp = \left(\frac{E_{or}}{E_{oi}}\right)_\perp \qquad r_\parallel = \left(\frac{E_{or}}{E_{oi}}\right)_\parallel$$

$$t_\perp = \left(\frac{E_{ot}}{E_{oi}}\right)_\perp \qquad t_\parallel = \left(\frac{E_{ot}}{E_{oi}}\right)_\parallel$$

5.5.2 Approximations

These approximations only apply to near-normal incidence angles (small θ_i).

$$r_\perp = -r_\parallel \approx \frac{\theta_i - \theta_t}{\theta_i + \theta_t} = \frac{n_t - n_i}{n_t + n_i}$$

$$t_\perp = t_\parallel \approx \frac{2n_i}{n_i + n_t}$$

$$T \approx \frac{n_t}{n_i} t^2 = \frac{4 n_i n_t}{(n_i + n_t)^2}$$

$$R = r^2 \approx \left(\frac{n_t - n_i}{n_t + n_i}\right)^2$$

$$1 = T + R$$

5.5.3 Exact Equations

These are horrifying, holy shit. I'm scared to write them because of how long it'll take me...

$$r_\parallel = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} \qquad r_\perp = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$$

$$t_\parallel = \frac{2\sin(\theta_t)\cos(\theta_i)}{\sin(\theta_i + \theta_t)\cos(\theta_i - \theta_t)} \qquad t_\perp = \frac{2\sin(\theta_i)\cos(\theta_t)}{\sin(\theta_i + \theta_t)}$$

$$t_\perp + (-r_\perp) = 1 \quad \text{for all angles}$$

$$t_\parallel + r_\parallel = 1 \quad \text{for small angles}$$

5.6 Polarization from Reflection

Polarization from reflection occurs at Brewster's Angle and when $n_t > n_i$. This occurs in external reflection but does not occur in the parallel electric field plane ($E_\parallel$ - there is no polarization). Reflected light is polarized and transmitted light is slightly polarized. Brewster's Angle is also called the polarization angle and is denoted as such:

$$\theta_B = \arctan\left(\frac{n_t}{n_i}\right)$$

6 Mirrors, Lenses and #swug

6.1 Flat Mirrors

Consider a mirror that is not 100% reflective where, when light is transmitted, it does not actually diffract. The time it takes for light to travel a distance d after being reflected is the same time it would take for the light to travel the distance if it were transmitted: What I'm trying to get at is the following

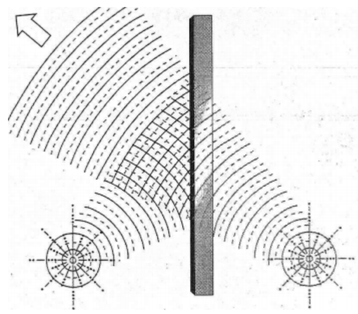


Figure 5: A flat mirror.

equation which applies to **planar reflecting surfaces in air**:

$$\frac{1}{s_o} + \frac{1}{s_i} = 0$$

where

s_o = distance of object

s_i = distance of image

We define transverse (vertical) magnification as:

$$M_T = \frac{y_i}{y_o} = \frac{-s_i}{s_o}$$

where

y_o = object height

y_i = image height

6.2 Spherical Mirrors

When we are dealing with spherical mirrors, the following equation is used:

$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{-2}{R} = \frac{1}{f}$$

where
$R =$ radius of curvature
$f =$ focal length

Here is a lovely table showing sign conventions because yes:

Quantity	$+$	$-$
s_o	Left of V, real object	Right of V, virtual object
s_i	Left of V, real image	Right of V, virtual image
f	Concave mirror	Convex mirror
R	Curvature center is right of V	Curvature center is left of V
y_o	Above axis, erect object	Below axis, inverted
y_i	Above axis, erect image	Below axis, inverted

Focal points can be described as such (I'll use a list 'cause I haven't in a while):

- F_o: the first focal point; the object point that produces an image at infinity
- F_i: the second focal point; the image point that is formed by an object at infinity (such as collimated light)
- f_o: the first focal length
- f_i: the second focal length

6.3 Single Surface Refraction

6.3.1 Optical Axis

Here are some things to know about the **optical axis** and other stuff:

- The optical axis is an axis of symmetry of an optical system, where if the system is rotated about the axis, it will not change
- The vertex of a surface is the intersection of the optical axis with the surface
- For a single spherical surface, any normal to the surface through the center of curvature is an optical axis

6.3.2 Spherical Surfaces

Here we use the paraxial approximation $\sin(\theta) \approx \theta$:

$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R} = D$$

$$\text{where}$$
$$D = \text{"power"}$$

For a spherical surface's first focal point, (recall that) we can say the image appears at infinity, so $s_i = \infty$ and $f_o = s_o$:

$$f_o = \frac{n_1 R}{n_2 - n_1}$$

Similarly, for the second focal point, we can say the object is at an infinite distance, so $s_o = \infty$ and $f_i = s_i$:

$$f_i = \frac{n_2 R}{n_2 - n_1}$$

We can further define transverse magnification as such:

$$M_T = \frac{-s_i n_1}{s_o n_2}$$

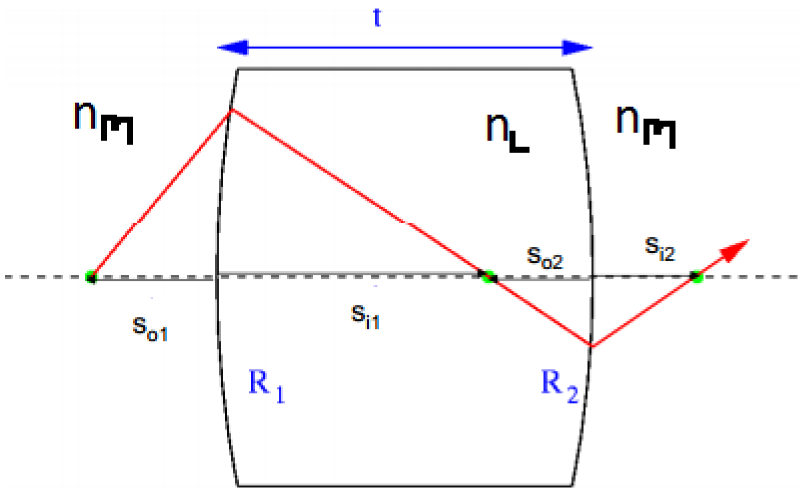
6.3.3 Flat Surfaces

When dealing with a flat surface (a "flat lens" for example), we let $R = \infty$:

$$s_i = \frac{-n_2}{n_1} s_o$$

6.4 Thin Lenses

Consider the following diagram of a lens and look at my unbelievable MS Paint skillz:



Clearly, $s_{o2} = t - s_{i1}$. When $t \approx 0$ we find $s_{o2} = -s_{i1}$. Using the previous equation for the power of a lens, we arrive at the **thin lens equation**:

$$\frac{n_m}{s_o} + \frac{n_m}{s_i} = (n_L - n_m)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = D_{tl}$$

where

$n_m =$ medium refractive index

$n_L =$ lens refractive index

$R_1 =$ first lens' radius

$R_2 =$ second lens' radius

$D_{tl} =$ thin lens power

Apparently, in all thin lens cases, $f = f_o = f_i$ and I have no idea why this is true but cool. Each of the focal lengths will be at equal distances from each other unless the medium in front of one lens is different from the medium behind the other. However, in air, the thin lens equation can then become as such:

$$\frac{1}{f} = \frac{1}{f_o} = \frac{1}{f_i} = \frac{1}{s_o} + \frac{1}{s_i} = (n_L - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = D_{tl}$$

6.4.1 Undeviated Rays

(I'm sorry if this has nothing to do with anything. Blame Heather if it doesn't make sense.) An undeviated ray does not go through a center of curvature but instead goes through the center of a thin lens, called a nodal point. Then, we define the lengths x_i and x_o and the transverse magnification can be rewritten:

$$s_o = f_o + x_o$$
$$s_i = f_i + x_i$$
$$M_T = \frac{y_t}{y_o} = \frac{-f}{x_o} = \frac{-x_i}{f}$$

We then define the **longitudinal magnification** as such when using the approximation $\Delta x_o \leq 10\%$ of Δx_i:

$$M_L = \frac{\Delta x_i}{\Delta x_o} \approx \frac{-f^2}{x_o^2} = -(M_T)^2$$

When we cannot approximate, we can instead use this relation of coordinates in the above equation:

$$\Delta x_i = x_{2i} - x_{1i} = f^2 \left(\frac{1}{x_{2o}} - \frac{1}{x_{1o}}\right)$$

6.5 Thick Lenses

Similarly to a set of two thin lenses, the image from the first lens is considered to be the object of the second lens. The transverse magnification then becomes defined as such for a pair of two lenses:

$$M_T = M_1 M_2$$

In the special case where the two lenses are touching, we can arrive at this familiar equation:

$$\frac{1}{s_{o1}} + \frac{1}{s_{i2}} = \frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{f}$$

If the two lenses are separated, then we arrive at what is called the thick lens equation:

$$\frac{n_m}{f_o} = \frac{n_m}{f_i} = \frac{n_L - n_m}{R_1} - \frac{n_L - n_m}{R_2} + \frac{(n_L - n_m)^2 d}{n_L R_1 R_2} = D$$

where

$d =$ separation

6.5.1 Cardinal Points and Ray Tracing

These are things not many people understood. Cardinal points refer to three pairs of points found in an optical system: focal points, principal points, and nodal points.

Focal Points I'm pretty sure we all know what a focal point is.

Principal Points and Planes The two principal planes have the property that a ray emerging from the lens appears to have crossed the rear principal plane at the same distance from the axis that that same ray appeared to cross the front principal plane, as viewed from the front of the lens. This means that the lens can be treated as if all of the refraction happened at the principal planes. Mathematically, we can represent stuff as (in air)

$$f = f_{eff} = H_2 F_i$$

$$h_1 = |V_1 \vec{H}_1| = \frac{f(n_L - 1)d}{R_2 n_L} = \frac{fd}{f_{o2}}$$

$$h_2 = |V_2 \vec{H}_2| = \frac{-f(n_L - 1)d}{R_1 n_L} = \frac{-fd}{f_{i1}}$$

where

$h =$ distance

$V =$ vertices

$H =$ principal planes

(refer to Figure 6)

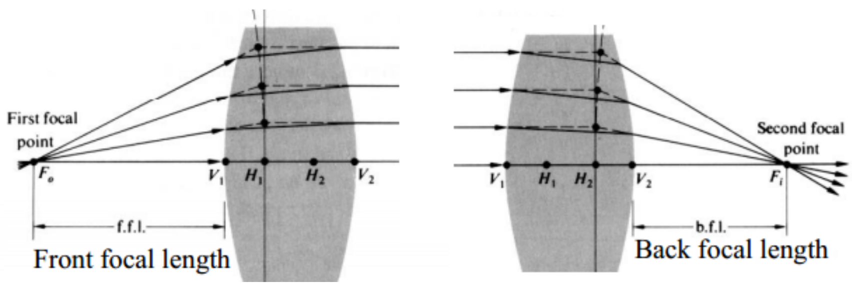


Figure 6: A thick lens with principal planes labelled.

The first principal plane causes light from F_o to refract to ∞ whereas the second principal plane causes collimated light incident to the plane to focus at F_i. Note that a pair of thin lens of the same power (as the thick lens) located at each principal plane can replace the thick lens. The equations above then become the following:

$$h_1 = |\vec{O_1 H_1}| \qquad h_2 = |\vec{O_2 H_2}|$$

where

$O =$ center of lens

Nodal Points The front and rear nodal points have the property that a ray aimed at one of them will be refracted by the lens such that it appears to have come from the other at the same angle. This means that the angular magnification of these two angles is 1.

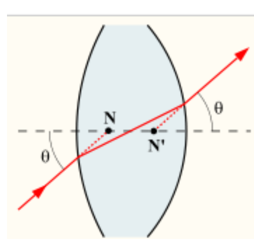


Figure 7: The effect of nodal points.

6.5.2 Focal Lengths, Undeviated Rays

Referring to Figure 6, we can define the front, back, and effective focal lengths as such:

$$ffl = f_o - h_1 = f_o(1 - \frac{d}{f_{o2}})$$

$$bfl = f_i + h_2 = f_i(1 - \frac{d}{f_{i1}})$$

$$f_{eff} = f = f_i = f_o$$

where
ffl = front focal length
bfl = back focal length
f_{eff} = effective focal length

Please refer to section 6.4.1 for undeviated rays of thick lenses, as they are equivalent to the undeviated rays for thin lenses.

7 Telescopes

Here's some stuff about telescopes. Lots of equations so prepare yourself.

7.1 Astronomical Telescope

When you see the equations below, refer to Figure 8 for geometrical definitions.

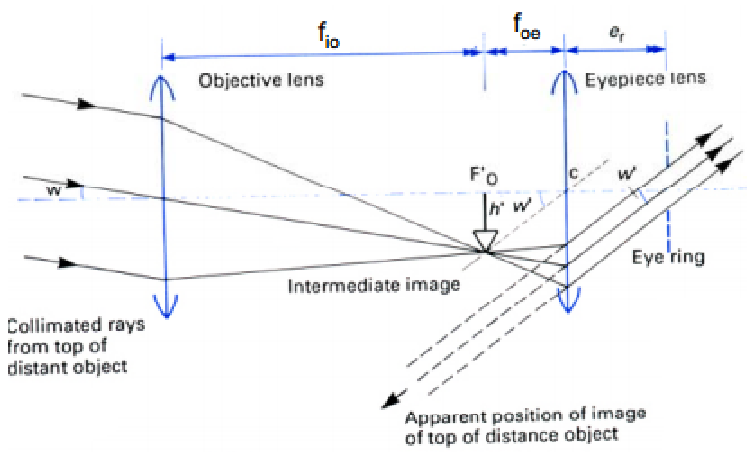


Figure 8: An astronomical telescope.

$$M_\alpha = \frac{w'}{w} = \frac{-f_{ob}}{f_e} = \frac{\alpha_a}{\alpha}$$

$$d_{tl} = f_{ob} + f_e$$

$$h' = -f_{io}\tan(w) = f_{oe}\tan(w')$$

where

M_α = angular magnification

f_{ob} = objective focal length

f_e = eyepiece focal length

w, α = incident angles

w', α_a = refracted angles

d_{tl} = tube length

We define the the center of the objective to be the entrance pupil and the center of the image to be the exit pupil (the person's pupil). All rays that go through the entrance pupil also go through the exit pupil. The pair of these pupils defines the size of the axial bundle as well as the angular magnification.

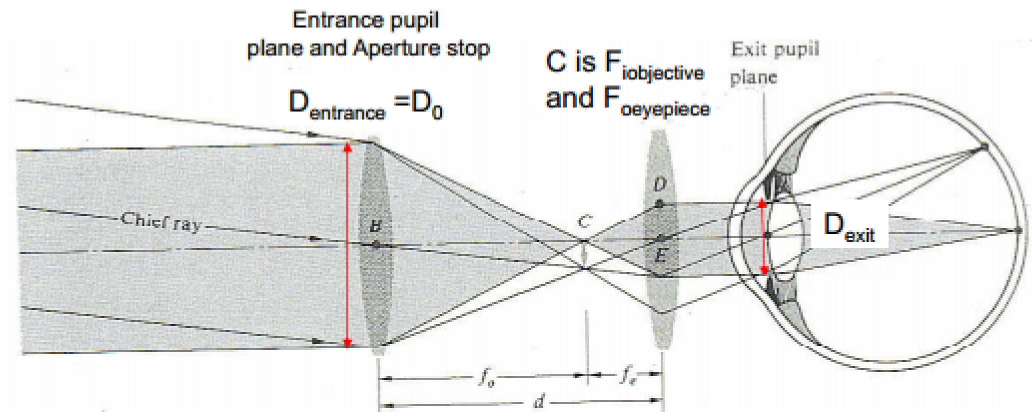


Figure 9: A pair of entrance and exit pupils for an astronomical telescope.

$$M_\alpha = \frac{D_o}{D_e}$$

where

D_o = entrance pupil diameter

D_e = exit pupil diameter

7.2 Galilean Telescope

I actually have no damn idea what the difference between an astronomical telescope and a Galilean telescope is but below is a diagram where you might be able to see it. I believe the difference is that the eyepiece focal length f_e is negative instead of positive, as found with an astronomical telescope. Don't quote me on it though. The equations appear to be identical, however, so please refer to the previous subsection (7.1) for the equations.

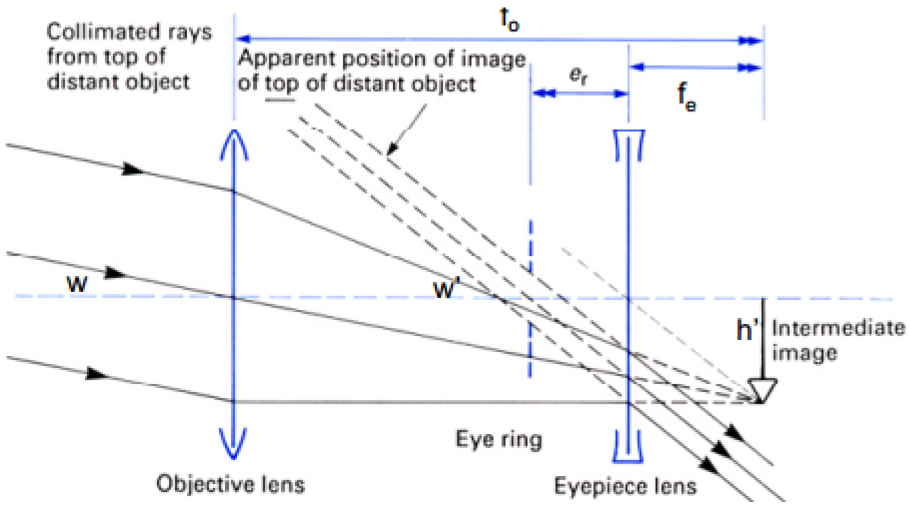


Figure 10: A Galilean telescope.

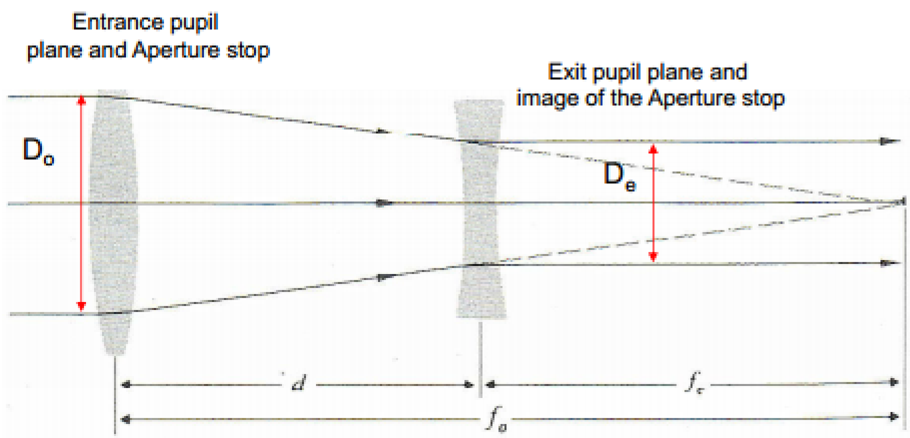


Figure 11: A pair of entrance and exit pupils of a Galilean telescope.

7.3 Telescope Resolution

Shortest subsection ever? Maybe so. This is for light through a telescope:

$$\theta_{res} = \frac{1.22\lambda}{D_o}$$

where

θ_{res} = angle of resolution

8 Apertures, Rays, f-numbers, Depths

8.1 Apertures

(For some of the equations that apply here, see Section 6.3.2 about spherical lenses). Apertures can be classified as an aperture stop or as a field stop. An aperture is a hole or opening through which light travels; specifically, it determines the cone angle of a group of rays that focus.

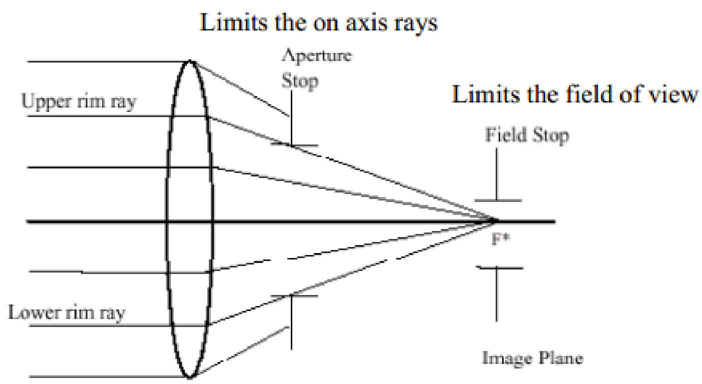


Figure 12: An example of an aperture stop and a field stop.

8.1.1 Aperture Stop

Modifying an aperture stop can change:

- the amount of aberrations (and thus change the amount of rays to those that are sharply focused)
- the blur due to diffraction
- the field depth or focus depth
- the illumination (sort of; see Section 8.3)

8.1.2 Field Stop

Modifying a field stop can change:

- the illumination
- the field of view

I believe the main purpose of the field stop is to manipulate the field of view. If the field stop is placed at the image plane (the plane containing the image), there will be a sharp cut-off in intensity at the edge of the image.

8.2 Pupils (revisited)

(For the equations that apply here, please refer to Section 6.3.2). The entrance and exit pupils are the conjugate points imaged throughout an optical system.

8.2.1 Entrance Pupil

The entrance pupil is the image of the aperture stop as seen from the object. The size of the entrance pupil limits the rays from an object and thus limiting the intensity and the amount of light reaching the image plane. If there are no lenses between the object and the aperture stop, the aperture stop itself serves as the entrance pupil.

8.2.2 Exit Pupil

The exit pupil is the image of the aperture stop as seen from the image. The exit pupil defines the cone of light that will intersect the image plane, as well as the illumination of the image.

8.3 Rays

Another short subsection! Praise Kwanzaabot!

8.3.1 Chief Ray

The chief ray is also called the principal ray. It is defined as any ray that comes from an off-axis object that passes through the center of the aperture stop; it enters the system directed toward the midpoint of the entrance pupil and leaves the system by passing through the center of the exit pupil.

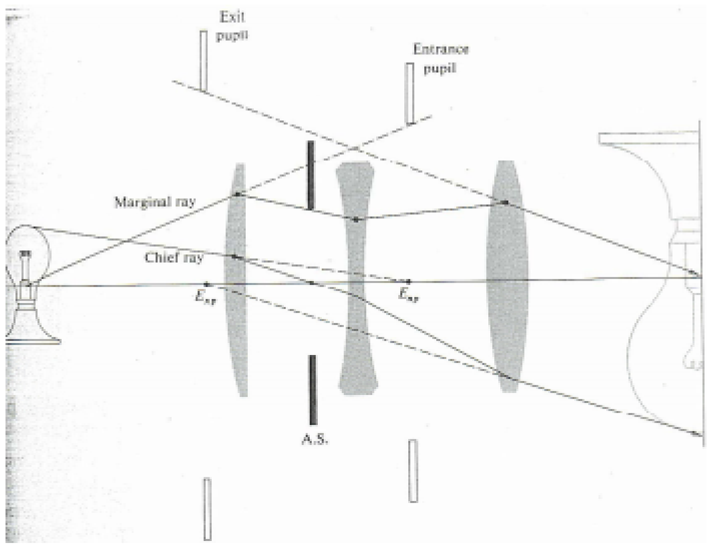


Figure 13: An optical system with chief and marginal rays.

8.3.2 Marginal Rays

The marginal ray is defined as a ray that comes from an object and goes towards the edge of the entrance pupil and the edge of the aperture stop, while appearing to have come from the edge of the exit pupil.

8.4 f-numbers

The f-number is also called the focal ratio. A smaller f-number gives a brighter image and a small diffraction blur. The notation for f-number is kinda really dumb but it's written as such:

$$f/\# = \frac{f}{D_o}$$

where

$f/\# =$ f-number

$D_o =$ entrance pupil diameter

$f =$ focal length

Note that a lens with a f-number of 2 is written as $f/2$. Weird notation. Why not subscripts? Wait, never mind, I know why. The irradiance is related to

f-number	Exposure time
f/1.4	1
f/2	2
f/2.8	4
f/4	8
f/5.6	16
f/8	32
f/11	64
f/16	128

Figure 14: The relationship between f-numbers and the exposure time.

the f-number:

$$I\alpha\ \frac{1}{(f/\#)^2}$$

There is also a relationship between the f-number (sometimes referred to as speed in this respect) and the photographic exposure time, as seen in this table:

The f-numbers go up by increments of $\sqrt{2}$ while the exposure times double. In my opinion, the easiest way to go about this relationship: if given the exposure time, take the square root and multiply by $\sqrt{2}$; if given the f-number, square it and then divide it by 2.

8.5 Diffraction

For small apertures, the image will be blurred due to diffraction, where the entrance pupil diameter determining the amount of blur. For large apertures, the paraxial approximation does not apply and all rays from an object point will not focus to a sharp image point (see Section...........) The smallest possible spot's diameter before blurring occurs is given by:

$$b = 2.44\lambda f/\#$$

where

$b =$ "blur diameter"

If two points have an angular separation θ_{res} as defined below, they are barely able to be resolved (but still can!):

$$\theta_{res} = 1.22\frac{\lambda}{D_o}$$

For a circular aperture, the blur due to refraction is given by an angle θ_b:

$$\theta_b = 2.44\frac{\lambda}{D_o}$$

As we saw above, $\theta_{res} = \frac{1}{2}\theta_b$. We define the linear extent of detail (resolved in images on the image plane) as h' using this equation:

$$h' = 1.22\lambda\frac{f}{D_o}$$

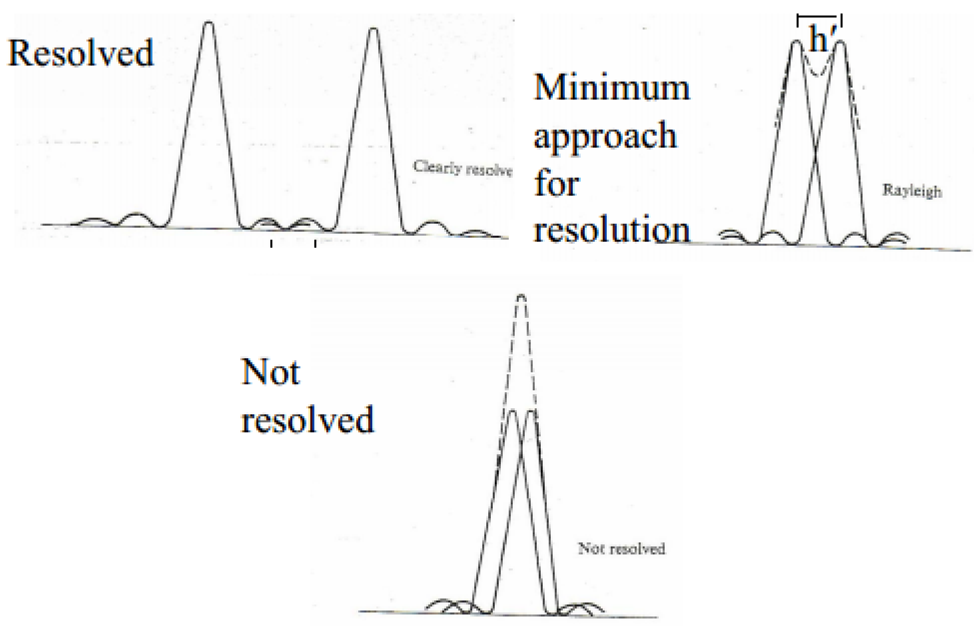


Figure 15: The physical description of the linear extent of detail h'.

8.6 Depths

8.6.1 Depth of Focus

The depth of focus is the range along the optical axis for which the image has acceptable sharpness for a fixed object; in other words, it is the range over which the image screen can be moved and leave the image in-focus. (This is ignoring diffraction, of course.) We define the depth of focus (linear defocus) as such:

$$\delta' = \pm b(f/\#)$$

where
δ' = depth of focus
b = blur diameter

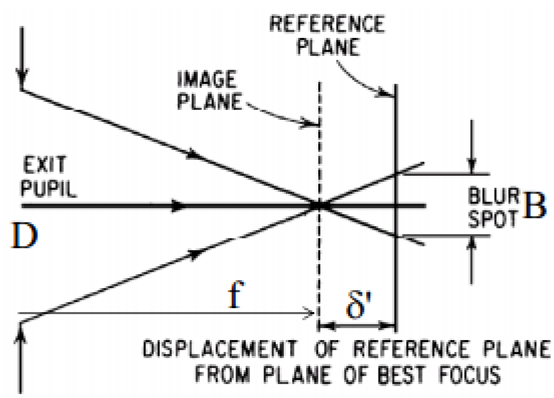


Figure 16: The linear defocus of a system.

8.6.2 Depth of Field

The depth of field is the range along the optical axis for which the object is allowed to move for which the image has acceptable sharpness for a fixed image plane. We define the depth of field as such:

$$\delta \approx \frac{\delta'}{M_T^2}$$

where
M_T = transverse magnification

8.7 Numerical Aperture

I think this is different from the numerical aperture discussed in Section 5.2. Just know the following stuff:

$$\theta_b \propto \frac{1}{NA}$$

$$f/\# = \frac{1}{2NA}$$

9 Aberrations

There are two main types of aberrations discussed below: chromatic and monochromatic. Note that aberrations are caused because the system is not ideal; it is actually realistic.

9.1 Chromatic Aberrations

A chromatic aberration is a type of distortion in which there is a failure in focusing all colours to the same convergence point within an optical system. This occurs due to the fact that lenses have different refractive indices for different wavelengths of light. I'm going to make a list here because I don't want to make more subsections for these two topics alone:

- **Lateral (transverse) chromatic aberration** is when all the colours focus in the same plane but not all of them focus along the optical axis
- **Longitudinal chromatic aberration** is when the foci of the different colours lay along the optical axis but not in the same plane

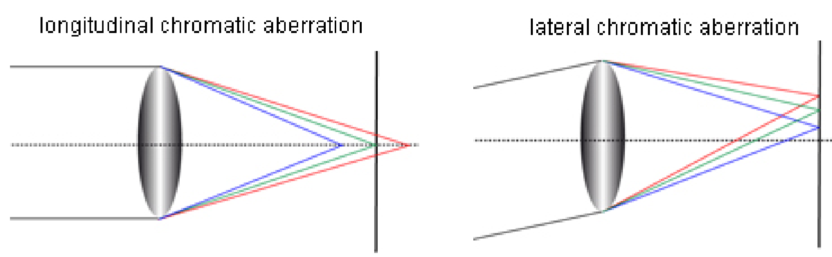


Figure 17: The two types of chromatic aberration.

We define the LCA (longitudinal chromatic aberration) as the following:

$$LCA = s_i\lambda_r - s_i\lambda_b$$

where
λ_r = red wavelength
λ_b = blue wavelength

We introduce the dispersive power, also known as the V-number or the Abbe number, which quantifies the dependence of the refractive index on the wavelength. It is defined as such:

$$V_d = \frac{n_d - 1}{n_F - n_C}$$

In this case, n_d is the refractive index for a wavelength of 587.6 nm, n_F is the refractive index for a wavelength of 486.1 nm, and n_C is the refractive index for a wavelength of 656.3 nm; these correspond to the Fraunhofer lines (sodium contains d while hydrogen contains C and F).

A higher dispersion means a lower V number and a higher dependence on the wavelength, and thus a bigger spread of focus. Positive lenses have lower dispersion whereas negative lenses have higher dispersion. This problem is fixed by an achromatic doublet, as discussed below.

9.2 Achromatic Doublet

The achromatic doublet is a set-up within an optical system to make $LCA = 0$. In other words, the red focal point is equal to the blue focal point: $f_R = f_B$. This is done by placing two lenses with different V-numbers beside each other. If the focal points are indeed the same ($f_R = f_B$), then we find this equation:

$$f_{1d}V_1 + f_{2d}V_2 = 0$$

$$\frac{1}{f_d} = \frac{1}{f_{1d}} + \frac{1}{f_{2d}}$$

where

$$f_d = \text{focal length in air}$$

9.3 Monochromatic Aberrations

(See Section 9.3.4 about the discussion of the Airy disc.) An actual wavefront (non-ideal) will give a larger image from a point object that is more blurred than an Airy disc. Monochromatic aberrations shift light out of a central peak, decreasing its irradiance but increasing the irradiance in the surrounding area.

9.3.1 Spherical Aberration

Spherical aberrations occur due to the increased refraction of rays when they pass through the edges of a lens. It results in an imperfection of the produced image (how clear, thank you Wikipedia).

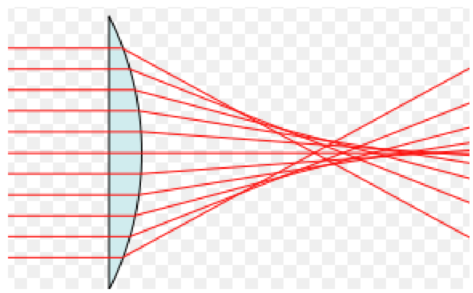


Figure 18: A spherical aberration in action.

When we look at the intensity of light from a spherical aberration, we can note that the intensity is perfectly symmetrical around a point at the very middle:

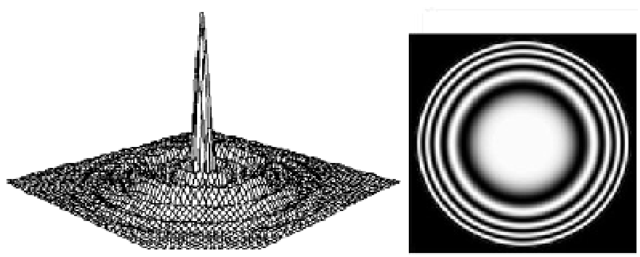


Figure 19: Two intensity diagrams of a spherical aberration.

9.3.2 Comatic Aberration

Comatic aberrations (also known as comas) occur due to a simple imperfection within the lens itself, and causes off-axis images to have something that looks like a tail. A coma is a function of lens shape and coma blur increases with an increasing off-axis distance.

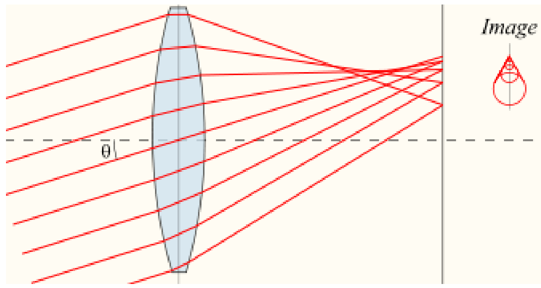


Figure 20: A comatic aberration, with the "tail" seen on the right side under the "Image" label. (Please ignore the jagged lines.)

When we look at the intensity diagrams of a comatic aberration, we see what we already described as the tail quite clearly. In addition, we see it is symmetrical only along one axis:

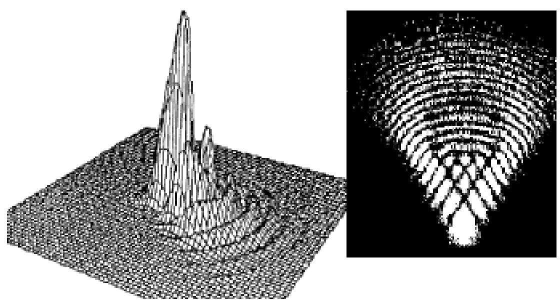


Figure 21: Intensity diagrams of a comatic aberration.

9.3.3 Astigmatic Aberration

An optical system with astigmatism is one where rays that propagate in two different planes from the same point have different foci. This usually occurs in asymmetric lenses (such as those that are elliptically-shaped). Transverse blur depends on the ray height of the lens and the object's distance off the axis.

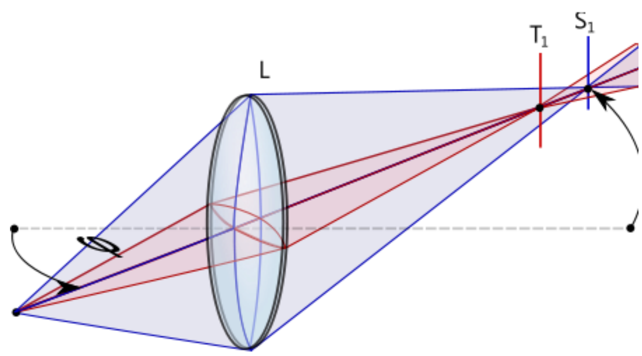


Figure 22: An optical system that is astigmatic.

When we look at the intensity diagrams of an astigmatic aberration, we can note that the intensity is not perfectly symmetrical all the way around a single point; instead, it is symmetrical in two different ways (each way is along a different axis), similar to that of an ellipse. An example is seen below, but remember that this is not what all astigmatisms look like:

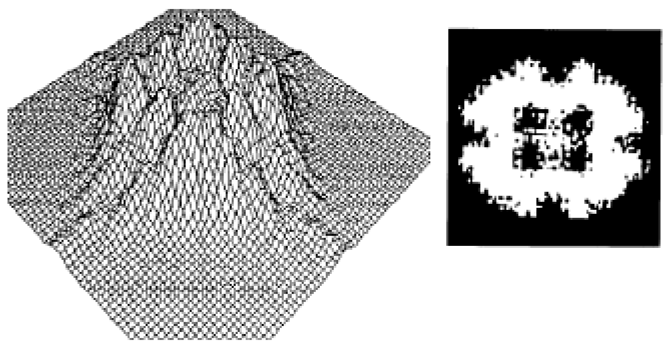


Figure 23: Intensity diagrams of an astigmatic aberration.

9.4 Airy Disc

An ideal wavefront gives an image of a point object blurred by diffraction; this is the Airy disc. In other words, the Airy disk is a description of the best focused spot of light that a perfect lens with a circular aperture can make, limited by the diffraction of light. The image of a point source is called the **point spread function** (PSF).

9.5 The Petzval Condition

Let's keep it simple. The Petzval condition for two lenses is the following:

$$n_1 f_1 + n_2 f_2 = 0$$

Using one of the many relationships found in a previous section, when we combine two thin lenses with opposite powers (so that $f_1 = -f_2$), we find that the focal length of the total system is:

$$f = \frac{f_1^2}{d}$$

where

f = focal length of system
f_1 = focal length of one lens
d = separation of lenses

(You can probably guess what the Petzval condition is for a given number of n lenses!)

9.6 Distortion

Another short subsection, how lovely. If the magnification of an optical system is constant for all angles ($M_T = \frac{y_i}{y_o}$) then there is no distortion. Distortion is often expressed as a % change in magnification.

10 Interference

We have seen before that you are able to add waves (Sections 1.5 and 1.6). This addition may lead to the enhancement of multiple wavelengths. We have covered this before but in a slightly different form:

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos(\delta)$$

where

$$\delta = \text{phase difference}$$

Constructive interference occurs when $\cos(\delta)$ is positive and is a maximum value when $\delta = 0, \pm 2\pi, \pm 4\pi$ ... (see Section 10.2)

and **destructive interference** occurs when $\cos(\delta)$ is negative and is a minimum value when $\delta = \pm\pi, \pm 3\pi$ ... (see Section 10.2)

10.1 Coherence

(This is regarded as bonus material but is useful in understanding the next section.) Coherence is a measure of the correlation between the phases measured at different points in space and time on a wave.

Temporal coherence is the measure of the interval over which the lightwave resembles a sinusoid. A similar definition is that it is the measure of the time interval over which waves generated by a source have the same temporal frequency, relative phase, and polarization. Points along a series of wavefronts have a predictable behaviour (such as where the next point is going to be, as seen below.)

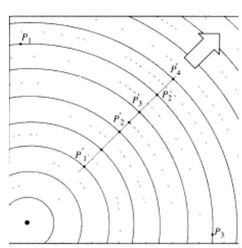


Figure 24: An example of temporal coherence; we can see that the points are following a pattern.

Spatial coherence is when the disturbance of multiple laterally separated points is in-phase and stays in-phase. If two points are not along the same wavefront but are laterally spaced, then the point source is not spatially coherent. This is why we use a pinhole, as it makes the resulting waves spatially coherent.

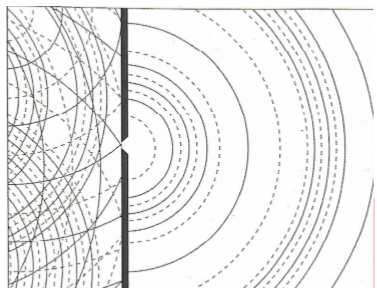


Figure 25: An example of a pinhole causing resulting waves to be spatially coherent, but not temporally coherent.

10.1.1 Producing Interference

We can produce interference in two different ways:

Wavefront division: we take a single point source that is spatially coherent and then use apertures to split the wavefront into two spatially coherent interfering sources that have identical temporal coherence.

Amplitude division: a single beam is partially reflected and transmitted in two portions but then recombine which create an interference pattern; this ensures the required spatial and temporal coherence between the two beams.

10.2 Phase and Path Difference

A quick part about this stuff. If two waves from the same or identical sources that are in phase travel different distances to reach a point where they interfere but one lags behind the other but a distance (Δx), they will be out of phase by $\frac{\Delta x}{\lambda}$ cycles (one cycle = $360\,\text{deg}$ or 2π radians)

We define the optical path length as the following:

$$OPL = \sum_i n_i x_i$$

where

$OPL =$ optical path length

$n_i =$ ith refractive index

$x =$ distance traveled in ith medium

In a similar fashion, we define the optical path difference of two waves (A and B) as the difference of the two optical path lengths, one for each separate wave:

$$\Lambda = OPD = OPL_B - OPL_A$$

where

$\Lambda =$ optical path difference

We can now write the phase difference as such:

$$\delta = k_0 \Lambda$$

where

$k_0 = \dfrac{2\pi}{\lambda_0}$

$\lambda_0 =$ initial wavelength

If we rewrote $\delta = \dfrac{\Delta x}{\lambda}$ for **maximum** and **minimum** interference values, we find that:

$$\Delta x = m\lambda \text{ (constructive interference)}$$

$$\Delta x = (m + \tfrac{1}{2})\lambda \text{ (destructive interference)}$$

where

m is an integer

10.3 Intensity for 2 Sources

Briefly, if we had two equal, coherent inputs, the total intensity would be:

$$I_{1+2} = 4I_0 \cos^2(\tfrac{\delta}{2})$$

where

$$I_0 = \text{initial intensity for one source}$$

Then we can easily see:

$$I_{max} = 4I_0 \text{ when } \delta = 0, 2m\pi$$
$$I_{min} = 0 \text{ when } \delta = \pi, (2m+1)\pi$$

While any other sum of I_1 and I_2 gives what we call incomplete interference: $I_{min} > 0$.

10.4 Superposition of Many Waves Revisited

In the slides, these apply to extended distant sources but in Hecht (Page 284 - 285) it does not specify. Recall that the addition of two waves results in $E = E_0 \cos(\alpha \pm \omega t)$, where E_0 is found by the cosine law (see Section 1.6). For N waves, the first equation remains the same but E_0 changes:

$$E_0^2 = \sum_{i=1}^{N} E_{0i}^2 + 2\sum_{j>i}^{N}\sum_{i=1}^{N} E_{0i}E_{0j}\cos(\alpha_i - \alpha_j)$$

If the sources are incoherent and all of the same amplitude (E_{01}), the above equation simply becomes

$$E_0^2 = NE_{01}^2$$

If instead the sources are coherent and all of the same amplitude, the equation becomes:

$$E_0^2 = N^2E_{01}^2$$

10.5 Two Slit/Source Interference

If you remember anything from high school optics: decreasing the separation of the slits results in an increase of the separation of the fringes; increasing the distance from the screen or increasing the wavelength results in a larger separation of the fringes; and minima fall half way between maxima.

In the following figure, we assume the light from each source and coherent and that the slits are the exact same. If the slits are narrow enough, they act like spatially coherent point sources.

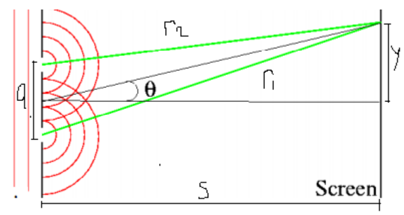


Figure 26: An example of a double slit set-up.

Using geometry, we can find that:

$$r_1 - r_2 = a\sin(\theta) \approx \frac{ay}{s}$$

where

$a =$ distance between slits

$s =$ length of set-up

$y =$ height

For maxima in the set-up seen in Figure 26, we must set $r_1 - r_2 = m\lambda$ where m is once again an integer. The height of a maximum and the separation between the maxima is then:

$$y_m = \frac{m\lambda s}{a}$$

$$\Delta y = \frac{\lambda s}{a}$$

where

$y_m =$ height of the maximum

$\Delta y =$ separation

10.6 Thin Films

(This is by far the most confusing section of this course in my opinion so I apologize greatly if it seems difficult, as I myself don't understand it too well.) A **thin film** is a thin piece of material (comparable to the wavelength of light) that permits both reflection and transmission of light where the reflected and transmitted light rays interfere.

Recall that waves reflect at the boundaries of different media. Partial reflection and the recombination of waves is a form of amplitude splitting, as discussed previously in Section 10.1.1.

External reflection: A reflection at a hard (or dense, or fixed) surface produces a reflected wave that is 180 deg out of phase with the incident wave; this also occurs if $n_2 > n_1$ at near-normal incidence angles.

Internal reflection: A reflection at a soft (or low-density, or "open") surface produces a reflected wave that is in-phase with the incident wave; this also occurs if $n_1 > n_2$ at near-normal incidence angles.

10.6.1 Anti-Reflection Coatings

We can coat a lens with a thin film to reduce the amount of reflection from the lens. The thin film has a different refractive index from the lens. We must then decide the thickness of the lens and the material (the refractive index) of the lens. Two conditions must be satisfied: the optical path condition and the amplitude condition.

Let n_0, n_f and n_s be the refractive indices of the initial medium, film and substrate (lens). We must have $n_s > n_f > n_0$ for the coating to work. Reflected light at the initial/film media boundary (A) will be in phase with light that was reflected at the film/substrate boundary (B) after passing through point C if the optical path difference is zero.

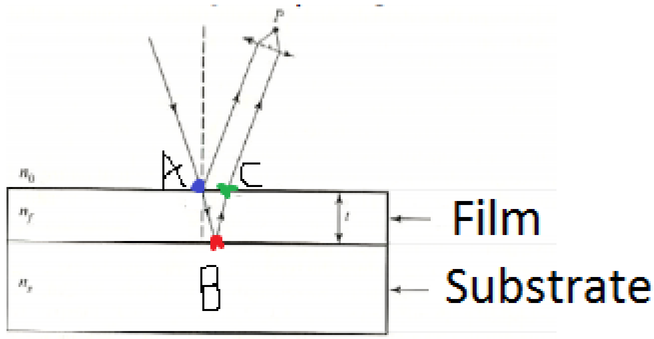


Figure 27: Reflection in a thin film system.

Optical Path Condition To maximize the amount of interference, an optical path difference is required. For an anti-reflection coating, we want destructive interference, of course, as it reduces or completely removes reflected light. This means that $\Lambda = OPD = (m + \frac{1}{2})\lambda_0$.

However, light normal to the surface must have an optical path difference of $\Lambda = 2n_f t = (m + \frac{1}{2})\lambda_0$. This means that, if we set $m = 0$ then the smallest possible thickness is given by

$$n_f t = \frac{\lambda_0}{4}$$

Amplitude Condition In order for complete destructive interference to occur, the irradiance of the beam exiting at point C must equal the irradiance of the reflected beam at point A (referring to Figure 27). This is done by making the proportion of light reflected at each interface equal; this in turn is controlled by choosing specific refractive indices.

For monochromatic light (a single wavelength) passing through a coated lens in air at near-normal incidence angles, we find that:

$$n_f = \sqrt{n_s} \text{ (in air)}$$
$$n_f = \sqrt{n_0 n_s} \text{ (not in air)}$$

10.7 Interferometers

An interferometer is used to split the amplitude of incident light. Recall that recombining the split light rays produces interference; the amount of interference is dependent on the optical path difference.

10.7.1 Michelson Interferometer

The Michelson interferometer was used to measure the speed of light and other relativistic things. It is the archetype of amplitude division devices.

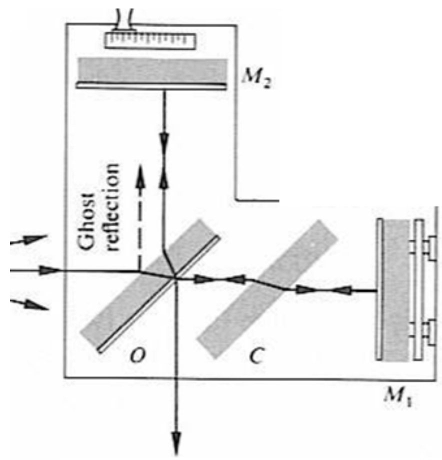


Figure 28: The set-up of the Michelson Interferometer.

Notice that the path between point O and Mirror 1 (M1) results in two external reflections, while the path between point O and M2 results in an external reflection and an internal reflection. We can further define the total phase difference as:

$$\delta = \frac{2\pi\Lambda}{\lambda} + \Delta r$$

where

$$\Delta r = \text{phase change from reflection}$$

Michelson Fringes Michelson fringes are fringes in a Michelson interferometer set-up, I suppose. Seems legit. Totally. In the slides we begin to use Λ as optical path length (OPL) and thus $\Delta\Lambda$ as optical path difference (OPD) but I refuse to do so, so I apologize. For an extended source, the path difference is $\Lambda = 2d\cos(\theta)$. Due to a phase difference of $\frac{\lambda}{2}$ from reflection, we find that:

$$2d\cos(\theta) = m\lambda \text{ (dark fringes)}$$
$$2d\cos(\theta) = (m + \tfrac{1}{2})\lambda \text{ (bright fringes)}$$

Note that the variables are defined as seen in Figure 29.

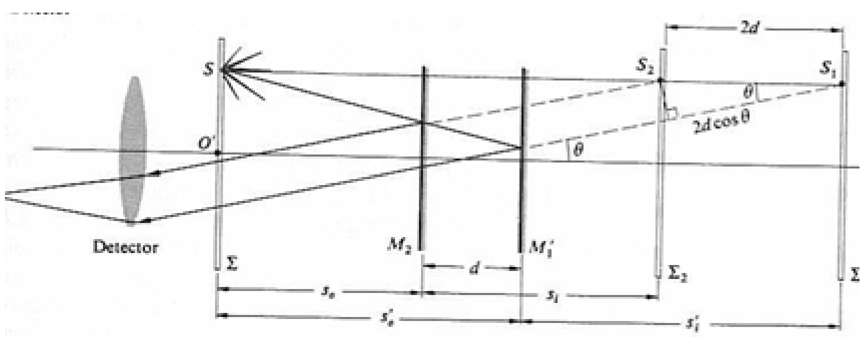


Figure 29: A set-up with Michelson fringes.

(Dark) Fringe Ordering As θ increases, $\cos(\theta)$ decreases, which then causes Λ to be smaller; we consider these values to correspond to destructive fringes of lower order. The highest order destructive fringe occurs at $\theta = 0$, so:

$$m_{max} = \frac{2d}{\lambda}$$
$$p = m_{max} - m$$

I'm not sure what p is supposed to really represent. However, consider a set of 100 fringes. I believe we would consider $p = 0$ (or $p = 1$) to be the highest order fringe and $p = 100$ to be the lowest order fringe.

11 Multiple Slit Interference

(Scratch what I said about thin films, that turned out to make sense. Unfortunately, this is where I get completely screwed.) I'm sure you have noticed that narrow slits result in circular wavefronts. In other words, narrow slits act like point sources. Recall that δ is the phase difference, as well. In addition, recall the first equation from Section 10. Consider the following figure:

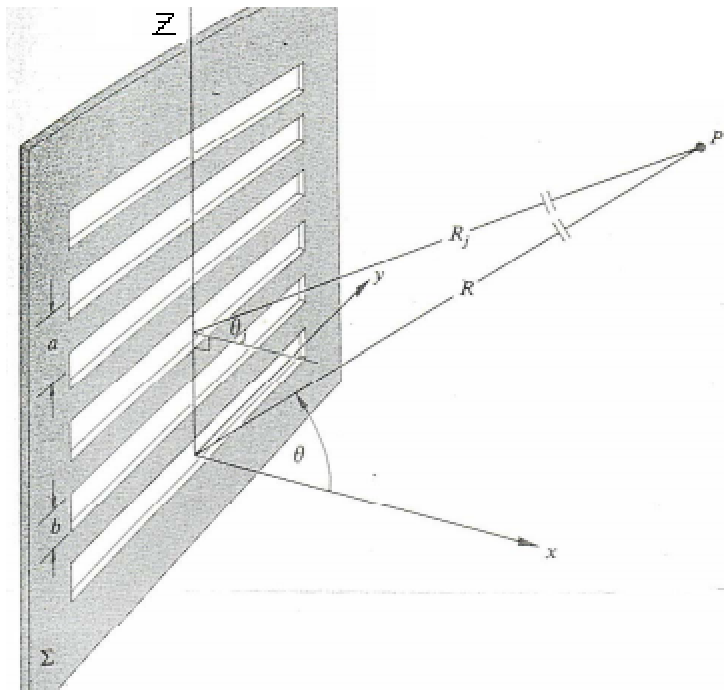


Figure 30: A multiple-slit set up with thin slits, with a as the distance between the centers of the slits, b as the width of one slit, and θ being the angle between R and a line perpendicular to the slits.

We are dealing with far-fields in the next little while, so we must satisfy the **Fraunhofer condition**: $R^2 > \dfrac{a^2}{\lambda}$ and $R^2 > \dfrac{b^2}{\lambda}$.

Constructive interference for N slits is the exact same as the 2-slit case. Note that $d = a$ from now on. Refer to Figure 30 for a better "definition" of θ.

$$d\sin(\theta) = m\lambda$$

where
$d =$ separation
$\theta =$ viewing angle

The phase difference between adjacent "pieces" (slits, I think) is the same but with slightly different notation:

$$\delta_N = \frac{2\pi}{\lambda}\ d\sin(\theta)$$

11.1 Constructive Interference

For N slits, the primary maxima occur when all of the phasors are aligned (pointing in the same direction). As discussed in Section 10.4, the total amplitude of added waves that are from a coherent source with the same amplitude is $E_0^2 = N^2 E_{01}^2$. Note that we can rewrite this as a proportionality:

$$I \propto E^2 = N^2 I_0$$

A set of two adjacent slits is simply a two-slit interference set-up, so the phase difference for constructive interference is $\delta_N = 2m\pi$, where $d\sin(\theta) = m\lambda$.

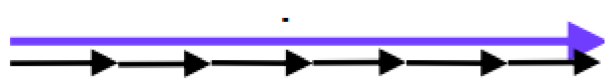


Figure 31: An example of a phasor diagram for a six-slit set-up that is constructively interfering.

11.2 Destructive Interference

Note that this ONLY applies to a system that has an even number of slits. Also note that we change notation once again for the phase difference.

We now instead have $\delta = (2m+1)\pi$ where $d\sin(\theta) = (m + \frac{1}{2})\lambda$.

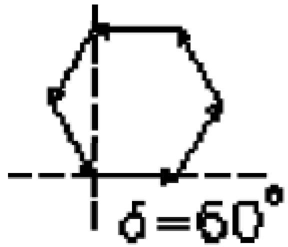


Figure 32: An example of a phasor diagram for a six-slit set-up that is destructively interfering. The phase difference here is 60 deg.

11.3 General Phase Difference

Consider the following diagram and then refer to the equations afterwards.

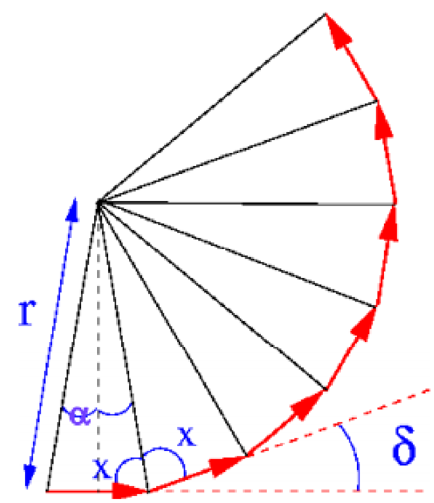


Figure 33: A phasor diagram.

I believe the idea for N slits is to simply add each vector tail-to-tail with an angle δ, the phase difference, in between. It will begin to form a circle, as seen above. We see that r is the radius of this circle, and x is an angle between two adjacent phasors and the arm between them. (Sorry, that was a shitty explanation.) Using geometry, we find that $\delta = 2\alpha$ and $\delta = 180 - 2x$.

We can go further and find even more equations after considering the next diagram:

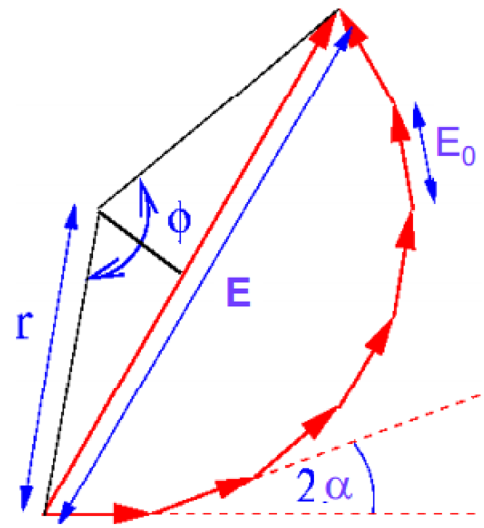


Figure 34: Another phasor diagram.

$$E_{0f} = E_0 \frac{\sin(N\alpha)}{\sin(\alpha)}$$

$$I = I_0 \Big(\frac{\sin(N\alpha)}{\sin(\alpha)} \Big)^2$$

From L'Hopital's rule, it follows that $I(0) = N^2 I_0$. We note that $E_0 f$ is the final wave's amplitude and I is its irradiance.

11.4 Maxima/Minima Diagrams

This is what we'll call them. That works. When we are dealing with $N > 2$ slits, we need to "introduce" some terms: primary maxima, second order maxima, and minima. Consider a three-slit diagram:

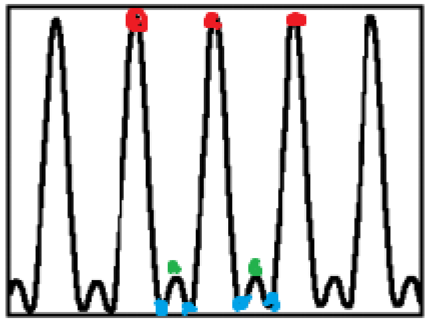


Figure 35: A three-slit maxima/minima diagram. Red dots represent primary (first-order) maxima, green dots represent second order maxima, and blue dots represent minima.

The number of second order maxima in a system with N slits is given by $N-2$. The number of minima in the same system is given by $N-1$. As N increases, the primary maxima get narrower, and the height of the second order maxima decreases.

Let us introduce p, where

$$\frac{\delta}{2} = \frac{p\pi}{N}$$

We have maxima when $\frac{p\pi}{N} = m\pi$ and minima for all other p.

To understand this more clearly, consider a three-slit system ($N = 3$). Then, the maxima occur at 0, $\frac{3}{3}\pi$, $\frac{6}{3}\pi$... ($p = 0$, 3, 6...) so then $\delta = 0$, 2π, 4π...

The minima occur at $\frac{1}{3}\pi$, $\frac{2}{3}\pi$, $\frac{4}{3}\pi$... ($p = 1$, 2, 4...) so $\delta = \frac{2\pi}{3}$, $\frac{4\pi}{3}$, $\frac{8\pi}{3}$...

11.5 Phasors on the Exam!

Okay, well, everyone I've talked to is pretty sure that they'll be on the exam so blame them if it's not.

11.5.1 Three-Slit Phasor Diagrams

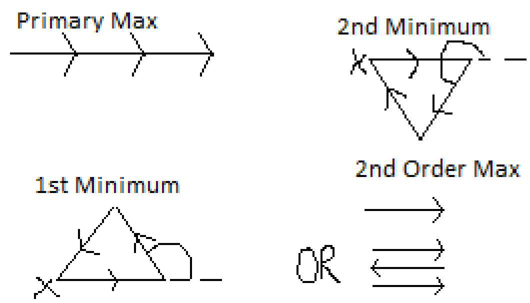


Figure 36: A set of phasor diagrams for a 3-slit set-up. Note that there are two versions of the only second-order maximum. X is the starting point in any case.

Through calculations as described in Section 11.4, the first minimum has a phase difference of $\frac{2\pi}{3}$, whereas the second minimum has a phase difference of $\frac{4\pi}{3}$.

11.5.2 Four-Slit Phasor Diagrams

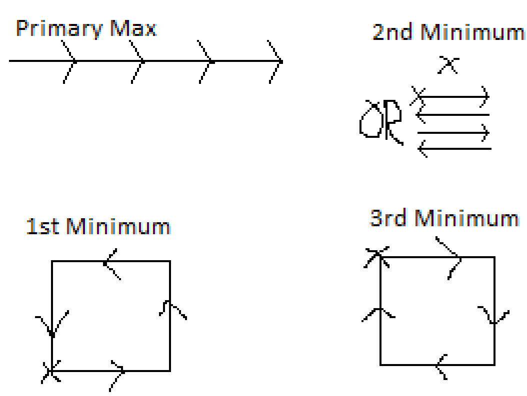


Figure 37: A set of phasor diagrams for a 4-slit set-up. Note that there are two versions of the second minimum, as seen in the figure. X is the starting point in any case. Also please note that we do NOT need to know the phasor diagrams for the second-order maxima, as apparently they are counter-intuitive.

Once again, through some calculations, the first minimum has a phase difference of $\frac{\pi}{2}$, the second minimum has one of π and the third has one of $\frac{3\pi}{2}$.

12 Diffraction

In this section, we consider apertures of width that are similar to the wavelength of light. This causes different diffraction patterns. We tend to treat the aperture as if it were made up of multiple thin slits. However, we will assume that the Fraunhofer condition holds (the second part at least - the part referring to slit widths). As we split the wide slit into N thin slits:

$$d = \frac{b}{N}$$

where
$d =$ thin slit separation
$b =$ wide slit separation
$N =$ number of thin slits

We then find the phase difference between adjacent thin slits to be:

$$\delta_N = \frac{2\pi b}{\lambda N}\ \sin(\theta)$$

I'm not sure why we care, but what is δ in the following diagram?

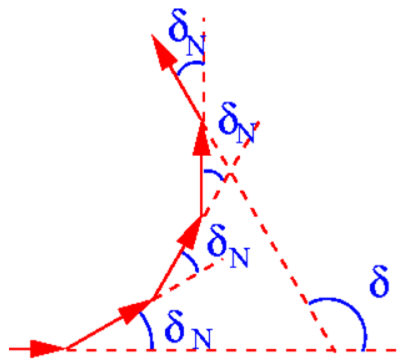
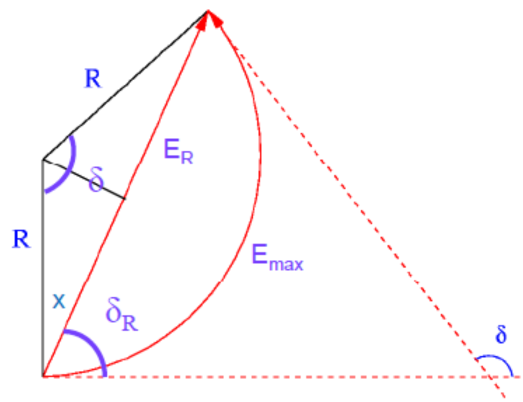


Figure 38: A phasor diagram for a wide slit divided into multiple thin slits.

Using simple geometry we see that $\delta = (N-1)\delta_N$. If we plug this into one of the previous equations, it turns out that:

$$\delta = \frac{2\pi}{\lambda}\ b\sin(\theta)$$

Intuitively, we can say that as $N \to \infty$, the phasors can actually be simply replaced by a curve. Then, we can draw the phasor diagram as such:



Let us define $\beta = \frac{\delta}{2}$ and $I(0) = I_0$. Through the power of mathemagic (such as L'Hopital's rule being applied to $I(0)$) we find that:

$$E_R = E_{max} \left(\frac{\sin(\beta)}{\beta}\right)$$

$$I_R = I_0 \left(\frac{\sin(\beta)}{\beta}\right)^2$$

In other words, $\beta = 0$ is a maximum.

Figure 39: About damn time.

13 Other Stuff

Thanks to the lovely **Heather Young** (bold font swag) for letting me borrow her notes because I was too much of an incompetent donkey to write my own for the second half of the course (and still am). I also appreciate the fact that she "let" me write all over her notes, which made this LaTeX-ing procedure more bearable. Actually most of this is not true because I only got parts of Sections 6 and 7 from her but whatever she can feel special I suppose.

Thanks to Jesse Laurin for pointing out that Figure 1 does not use conventional axes; that I used Ψ instead of Φ in section 4.1; and a general typo in section 6.3.2.

Thanks to Mark Bourgon the Bearded One for pointing out that one of my intensity diagrams for comatic aberrations in Section 9.3.2 was incorrect.

Thanks to Simon Hartwig and Ian McMullan for pointing out that one of the equations in Section 3.6 is blatantly untrue, as frequency should not change when changing media!