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Poisson equations ($- u_{xx} - u_{yy} = f$)

Theorem: Let A be the 2D Laplacian matrix. The eigenvalues of A are given by:

$$\lambda_{ij} = \frac{4}{h^2}[\sin^2(\tfrac{i\pi h}{2}) + \sin^2(\tfrac{j\pi h}{2})] \qquad 1 \le i, j \le m$$

- The smallest eigenvalue is attained with i = j = 1:

$$\lambda_{\min} = \frac{8}{h^2}\sin^2(\tfrac{\pi h}{2})$$

- The largest eigenvalue is attained with i = j = m:

$$\begin{aligned}\lambda_{\max} &= \frac{8}{h^2}\sin^2(\tfrac{m\pi h}{2}) \\ &= \frac{8}{h^2}\sin^2(\tfrac{\pi}{2}(1-h)) \qquad (h = \tfrac{1}{m+1},\ mh = 1-h) \\ &= \frac{8}{h^2}\cos^2(\tfrac{\pi h}{2})\end{aligned}$$

- A is SPD and an M-matrix.

Richardson

$$\rho(I-\theta A) = \max\{|1-\theta \tfrac{8}{h^2}\sin^2(\tfrac{\pi h}{2})|, |1-\theta \tfrac{8}{h^2}\cos^2(\tfrac{\pi h}{2})|\}$$

Convergence holds for $0 < \theta < h^2/(4\cos^2(\pi h/2))$ and

$$\theta_{opt} = \frac{2}{\lambda_{\min}+\lambda_{\max}} = \frac{h^2}{4} \qquad \rho_{opt} = \frac{\lambda_{\min}-\lambda_{\max}}{\lambda_{\min}+\lambda_{\max}} = 1-2\sin^2(\tfrac{\pi h}{2})$$

Jacobi

$D = 4/h^2 I = \theta_{opt}^{-1} I$ = optimal Richardson

$\therefore \quad \rho(I - D^{-1}A) = \rho_{opt} = 1 - 2\sin^2(\pi h/2) = \cos(\pi h)$

By Taylor expansion,

$$\cos(x) = 1 - x^2/2 + x^4/4! + \ldots$$

$\therefore \quad \rho(I - D^{-1}A) = \cos(\pi h) = 1 - \pi^2/2\, h^2 + O(h^4)$

For small mesh size h, $\rho(G^J) \approx 1$ -> slow convergence

GS & SOR

$$\begin{aligned}\rho(I - M_{GS}^{-1}A) = \rho(I - M_J^{-1}A)^2 &= \cos^2(\pi h) \\ &= 1 - \sin^2(\pi h) \\ &= 1 - \pi^2 h^2 + O(h^4)\end{aligned}$$

- For small h, slow convergence for GS.
- Convergence rate = 2 × convergence rate of Jacobi.
- For SOR,

$$\begin{aligned}\omega_{opt} &= \tfrac{2}{1+\sin(\pi h)} \\ \rho_{opt}^{SOR} &= \omega_{opt} - 1 = \tfrac{1-\sin(\pi h)}{1+\sin(\pi h)} \\ &= 1 - 2\pi h + O(h^2)\end{aligned}$$

- Optimal SOR is an order of magnitude better than GS and Jacobi.

Convergence analysis for CG

Recall: $F(x^k) = \min\{F(x^0 + \sum_{i=0}^{k-1} \alpha_i p^i) : \alpha_0, \ldots, \alpha_{k-1} \in R\}$

$$\Rightarrow \; F(x^k) - F(x) = \min\{F(x^0 + \sum_{i=0}^{k-1} \alpha_i p^i) - F(x) : \alpha_0, \ldots, \alpha_{k-1} \in R\}$$

$$\vdots$$

$$\text{algebra} \qquad (F(y) - F(x) = \tfrac{1}{2}\|y - x\|_A^2)$$

$$\vdots$$

$$\tfrac{1}{2}\|x^k - x\|_A^2 = \min\{\tfrac{1}{2}\left\|e^0 + \sum_{i=0}^{k-1} \alpha_i p^i\right\|_A^2 : \alpha_0, \ldots, \alpha_{k-1} \in R\}$$

Let $e^k = x^k - x$. Then

$$\|e^k\|_A = \min\{\left\|e^0 + \sum_{i=0}^{k-1} \alpha_i p^i\right\|_A : \alpha_0, \ldots, \alpha_{k-1} \in R\}$$

$$= \min\{\left\|e^0 + \sum_{i=0}^{k-1} \gamma_i A^i r^0\right\|_A : \alpha_0, \ldots, \alpha_{k-1} \in R\}$$

(since span $\{ p^0, \ldots, p^{k-1} \}$ = span $\{ r^0, \ldots, A^{k-1} r^0 \}$)

Let $Q_{k-1}(x) = \gamma_0 + \gamma_1 x + \ldots + \gamma_{k-1} x^{k-1}$. Then

$Q_{k-1}(A) = \gamma_0 I + \gamma_1 A + \ldots + \gamma_{k-1} A^{k-1}$ = sum $\gamma_i A^i$

$$\therefore \quad e^0 + \sum_{i=0}^{k-1} \gamma_i A^i r^0 = e^0 + Q_{k-1}(A) r^0$$

$$= e^0 + Q_{k-1}(A) A e^0 \qquad (r^0 = A(x - x^0) = A e^0)$$

$$= (I + Q_{k-1}(A) A) e^0$$

Let $P_k(x) = 1 + Q_{k-1}(x)x$. Then $\deg(P_k) \le k$, $P_k(0) = 1$, and

$$e^0 + \sum_{i=0}^{k-1} \gamma_i A^i r^0 = P_k(A) e^0$$

∴ $||e^k||_A = \min \{ ||P_k(A)e^0||_A : P_k(x) = \text{poly. of deg} \le k, P_k(0) = 1 \}$

In other words, let $\tilde{P}_k(x)$ be a poly. of deg ≤ k, $\tilde{P}_k(0) = 1$. Then

$$\left\| e^k \right\|_A \le \left\| \tilde{P}_k(A) e^0 \right\|_A$$

∴ CG finds the optimal poly. P_k to min the error in the A-norm.

By choosing $P_k(x)$ appropriately, one can show:

Theorem:

$$\left\| x^k - x \right\|_A^2 \le 2 \left(\frac{\sqrt{\kappa(A)} - 1}{\sqrt{\kappa(A)} + 1} \right)^k \left\| x^0 - x \right\|_A^2$$

where $\kappa(A) = \lambda_{max} / \lambda_{min}$.

Notes

1) This is upper bound only. CG convergence is usually better.

2) CG convergence depends only all $\{\lambda_j\}$, not just λ_{min}, λ_{max}.

   E.g. A has 3 distinct eigenvalues: $\lambda_1 < \lambda_2 < \lambda_3$

   Define Lagrange poly. $P_3(x)$ of deg ≤ 3 such that

   $$P_3(0) = 1, \quad P_3(\lambda_j) = 0 \qquad j = 1, 2, 3.$$

   Then $$\left\| e^3 \right\|_A^2 \le \left\| P_3(A) e^0 \right\|_A^2 = \sum_{j=1}^{3} \xi_j^2 P_3^2(\lambda_j) \lambda_j = 0$$

   $\Rightarrow$ CG converges in 3 iterations, independent of $\kappa(A)$.

3) For Poisson equation, the asymptotic convergence rates for SOR and CG are the same. However, no optimal parameter needed for CG.