Discrete Fourier Transform (DFT)

- f(t) not usually known exactly
- Periodic data, period T, $f(t \pm T) = f(t)$
  - If not periodic, we can "make" periodic by repeating the information
- N evenly spaced observations at
  - $t_j = j\ (T/n)$, for j=0:N-1
- Let $f_j = f(t_j)$, j = 0:N-1

Discrete Fourier Transform (DFT)

- Trignometric Interpolation:
  - Approximate f(t) by fitting a combination of sine and cosine waves at the points t_j

- Find coefficients a_k, b_k to approximate *f(t)* with

$$a_0 + \sum_{k=1}^{???} a_k \cos\left(\frac{2\pi kt}{T}\right) + \sum_{k=1}^{???} b_k \sin\left(\frac{2\pi kt}{T}\right)$$

- Or, use the complex coefficients form:

$$\sum_{k \to -\frac{N}{2}+1}^{N/2} F_k e^{\frac{i2\pi kt}{T}}$$

- Recall, $tj = jT/N$, so

$$f_j = \sum_{k=-\frac{N}{2}+1}^{N/2} F_k e^{\frac{i2\pi k t_j}{T}} = \sum_{k=-\frac{N}{2}+1}^{N/2} F_k e^{\frac{i2\pi k j}{N}}$$

- Would like the sum over k = 0:N:

$$f_j = S_- + S_+ = \sum_{k=-\frac{N}{2}+1}^{-1} F_k e^{\frac{i2\pi k j}{N}} + \sum_{k=0}^{N/2} F_k e^{\frac{i2\pi k j}{N}}$$

- Concentrate on S_-

- Translate k= -N/2+1, -N/2+2, …, -2, -1

  into p= N/2+1, N/2+2, …, N-1, N-1
- i.e. p = k+N → replace k with p-N

$$S_{-} = \sum_{p=\frac{N}{2}+1}^{N-1} F_{(p-N)} e^{\frac{i2\pi(p-N)j}{N}}$$

- Rewriting the exponent leads to:

$$S_{-} = \sum_{p=\frac{N}{2}+1}^{N-1} F_{(p-N)} e^{\frac{i2\pi pj}{N}}$$

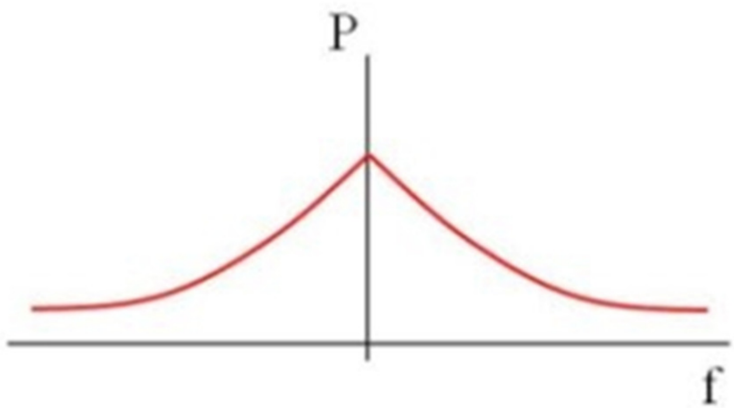
- Make the coefficients periodic as well, with period N, i.e.
- Set $F_p = F_{p-N}$, for p = N/2+1: N-1:

$$S_- = \sum_{p=\frac{N}{2}+1}^{N-1} F_p e^{\frac{i2\pi pj}{N}}$$

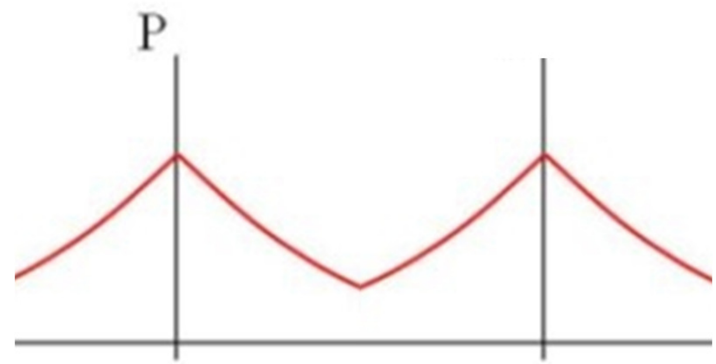
- Re-combining, we get

$$f_j = \sum_{k=0}^{N-1} F_k e^{\frac{i2\pi kj}{N}}$$

"Shifting" the Power Spectrum
Plot $|F_k|^2$ = power of k^{th} harmonic



HIGH) (Low) (HIGH



Low) (HIGH) (Low

Note the location of the coefficients of the high and low frequency harmonics.

$$f_n = \sum_{j=0}^{N-1} F_j e^{i\frac{2j\pi n}{N}}$$

- There are two sets of values:
  - f_n (given observations)
    - function values over time (Time Domain)
  - F_j (coefficients – still unknown)
    - coefficients of harmonics of different frequencies (interpolation coefficients)
    - Transformed data
    - Frequency Domain
- Move between domains using FOURIER TRANSFORM

Finding the values of F_k, k=0:N-1 Needed terminology

- The N^{th} roots of unity are the integer powers of:

$$W_N = e^{i2\pi/N}, \text{ k = 0:N-1}$$

- Properties of W_N:
  - ${W_N}^N = 1$
  - ${W_N}^{-k} = {W_N}^{N-k}$
  - ${W_N}^k = {W_N}^{k \bmod N}$

Discrete Fourier Transform: W = W_N

$$F_n = \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-i\frac{2\pi nk}{N}} = \frac{1}{N}\sum_{k=0}^{N-1} f_k W^{-kn}$$

(transformed data)

$$f_n = \sum_{k=0}^{N-1} F_k e^{i\frac{2\pi nk}{N}} = \sum_{k=0}^{N-1} F_k W^{nk}$$

(original data - inverse transformation)

Example: Consider the data

$$(0,1),\ (\frac{1}{6},\frac{1}{2}),\ (\frac{2}{6},\frac{-1}{2}),\ (\frac{3}{6},-1),\ (\frac{4}{6},\frac{-1}{2}),\ (\frac{5}{6},\frac{1}{2})$$

- $T = 1$, $t_n = (1/6)n$, $f = 1, \frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2})$
- $W = e^{i\pi/3} = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2}$
- $W^2 = e^{i2\pi/3} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$
- $W^3 = e^{i3\pi/3} = \cos(\pi) + i\sin(\pi) = -1 + i0$
- $W^4 = e^{i4\pi/3} = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$
- $W^5 = e^{i5\pi/3} = \cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right) = \frac{1}{2} - i\frac{\sqrt{3}}{2}$
- $W^6 = W^0 = 1$

$$F_n = \frac{1}{N}\sum_{k=0}^{N-1} f_k W^{-kn}, \text{ for n=0:5}$$

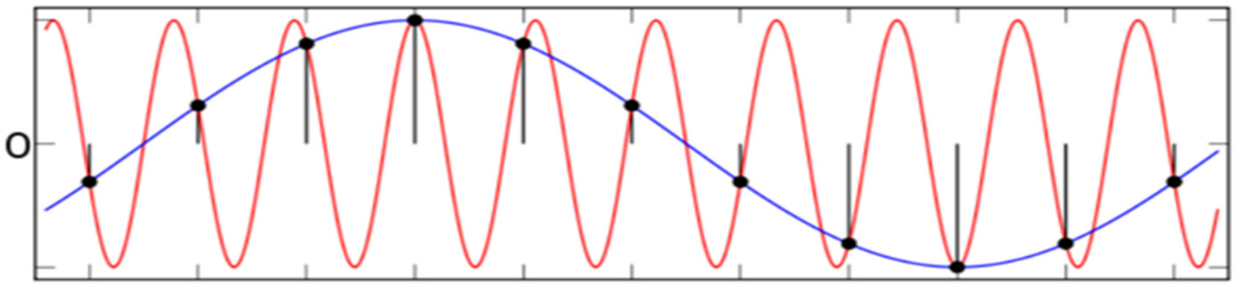
- $F_0 = \frac{1}{6}(f_0W^0 + f_1W^{0\cdot 1} + f_2W^{0\cdot 2} + f_3W^{0\cdot 3} + f_4W^{0\cdot 4} + f_4W^{0\cdot 5}) = 0$
- $F_1 = \frac{1}{6}(f_0W^0 + f_1W^{-1\cdot 1} + f_2W^{-1\cdot 2} + f_3W^{-1\cdot 3} + f_4W^{-1\cdot 4} + f_4W^{-1\cdot 5}) = \frac{1}{2}$
- $F_2 = \frac{1}{6}(f_0W^0 + f_1W^{-2\cdot 1} + f_2W^{-2\cdot 2} + f_3W^{-2\cdot 3} + f_4W^{-2\cdot 4} + f_4W^{-2\cdot 5}) = 0$
- $F_3 = \frac{1}{6}(f_0W^0 + f_1W^{-3\cdot 1} + f_2W^{-3\cdot 2} + f_3W^{-3\cdot 3} + f_4W^{-3\cdot 4} + f_4W^{-3\cdot 5}) = 0$
- $F_4 = \frac{1}{6}(f_0W^0 + f_1W^{-4\cdot 1} + f_2W^{-4\cdot 2} + f_3W^{-4\cdot 3} + f_4W^{-4\cdot 4} + f_4W^{-4\cdot 5}) = 0$
- $F_5 = \frac{1}{6}(f_0W^0 + f_1W^{-5\cdot 1} + f_2W^{-5\cdot 2} + f_3W^{-5\cdot 3} + f_4W^{-5\cdot 4} + f_4W^{-5\cdot 5}) = \frac{1}{2}$

Interpolating function with period T=1:

$$f(t) = \sum_{k=0}^{N-1} F_k e^{i2\pi kt} = F_1 e^{i2\pi kt} + F_5 e^{i10\pi kt}$$
$$= F_1 e^{i2\pi kt} + F_{-1} e^{-i2\pi kt} = \cos 2\pi t$$

Aliasing

- Consider the following situation – multiple harmonics "fit" the observations



- The coefficients in the DFT may be affected by some higher order frequencies from the CFT (frequencies above N/(2T) – Nyquist frequency)
- May cause poor digital images or "echos" on radio signals.
- Solution: Sample at a higher rate (i.e. more often) or filter the data before digitizing.