Propositional Logic Part2

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Objectives

- Propositional Semantics
- Logical Implications
- Adequate Sets of Connectives

Propositional Formulas: Additional Remarks /1

- **Based on Lemma 1 and 2 from Lecture Notes #1:**
  - Every formula in L^{p} has the same number of left and right parentheses (Lemma 2.3.1 from the textbook)
  - Any non-empty proper initial segment of a formula of L^{p} has more left than right parentheses, and
  - Any non-empty proper terminal segment of a formula of L^{p} has less left than right parentheses (Lemma 2.3.2)
  - **Corollary 1: Any non-empty proper initial or terminal segment of a formula of L^{p} cannot be a formula itself** (part of Lemma 2.3.2)

Propositional Formulas: Additional Remarks /2

- Use the corollary to better understand the previously discussed proofs
- For Theorem 1 from Lecture Notes #1 (Theorem 2.3.3)
  - Consider $(A \wedge B) = (A1 \vee B1)$
  - Remove the brackets on both sides and consider
    $A \wedge B = A1 \vee B1$
  - A and A1 have the same starting point (i.e., beginning of the expression for both left and right-hand sides)
  - If $A \neq A1$ then A is a proper initial segment of A1 or A1 is a proper initial segment of A
  - However, both A and A1 are formulas, so according to Corollary 1, this is a contradiction; hence, $A = A1$
  - It then follows that symbols $\wedge$ and $\vee$ are equal, which is impossible; ergo, $(A \wedge B)$ cannot be interpreted as $(A1 \vee B1)$

Propositional Formulas: Additional Remarks /3

- For Theorem 2 from Lecture Notes #1 (Theorem 2.3.6)
  - Consider that any ¬ in any A has a unique scope
  - Based on the definition of formulas (Definition 2.2.2) (¬B) must be a segment of A where B is a formula
  - Let us assume that B and B1 are different scopes of ¬
  - Then, if we consider (¬B) and (¬B1), both have the same starting point (i.e., expressions "(¬") but not necessarily the same ending point
  - So, B is a proper initial segment of B1 or B1 is a proper initial segment of B
  - However, both B and B1 are formulas, so according to Corollary 1, this is a contradiction
  - Hence, B = B1 and the scope of ¬ in any A is unique
  - Similar approach is applied to the premise that * in any A has unique left and right scopes; the starting point in that case is the middle of expressions (C1 * C2) and (C1' * C2')

Propositional Formulas: Additional Remarks /4

- For Theorem 3 from Lecture Notes #1 (Theorem 2.3.7)
  - Consider that if a formula A is a segment of (¬B) then A is a segment of B or
    A = (¬B); that is, if A is a proper segment of (¬B) then A is a segment of B
  - Consider three cases where A ≠ (¬B):
    (1) A contains the first bracket of (¬B)
    (2) A contains the last bracket of (¬B)
    (3) A contains the first ¬ symbol of (¬B)
  - Case 1. A is a proper initial segment of (¬B) so A is not a formula based on Corollary 1
  - Case 2. A is a proper terminal segment of (¬B) so A is not a formula based on Corollary 1
  - Case 3. A must also contain the first bracket (otherwise not a formula), and then A is a proper initial segment of (¬B) so A is not a formula based on Corollary 1
  - Since A must be a formula, all three cases end in a contradiction; hence, A is a segment of B or A = (¬B)

Propositional Formulas: Additional Remarks /5

- For Exercise 2.3.2 (Section 2.3)
  - Show that at most one of UV and VW is a formula where U, V, and W are non-empty propositional expressions
  - Assume that both UV and VW are formulas
  - Use Lemma 2 to deduce that since V is a non-empty proper terminal segment of UV, V must have more right brackets than left brackets
  - Use Lemma 2 also to deduce that since V is a non-empty proper initial segment of VW, V must have more left brackets than right brackets
  - This is a contradiction since V cannot have both more or less left brackets than right brackets at one time
  - Hence, both UV and VW cannot be formulas

Introduction

- **Truth Tables:**
  - Valuations of compound propositions based on the truth values of the simple propositions

A	$\neg$ A
0	1
1	0

A	B	A $\wedge$ B	A $\vee$ B	A $\Rightarrow$ B	A $\Leftrightarrow$ B
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

Propositional Semantics /1

- **Definition 2.1. Truth Valuation:** (Definition 2.4.1)
  - A function (mapping) with the set of all proposition symbols as its domain and the set $\{0,1\}$ as its range
  - Also referred to as the **Interpretation Function I**
- Interpretation function assigns a value to every propositional symbol simultaneously
  - We use small letter t to denote truth valuations; the value that t assigns to any formula A is written as A^t
- **Definition 2.2. Values of Formulas:** (Definition 2.4.2)
  - $p^t \in \{1,0\}$
  - $(\neg A)^t = \{1 \text{ if } A^t = 0; 0 \text{ otherwise}\}$
  - $(A \wedge B)^t = \{1 \text{ if } A^t = B^t = 1; 0 \text{ otherwise}\}$
  - $(A \vee B)^t = \{1 \text{ if } A^t = 1 \text{ or } B^t = 1; 0 \text{ otherwise}\}$
  - $(A \Rightarrow B)^t = \{1 \text{ if } A^t = 0 \text{ or } B^t = 1; 0 \text{ otherwise}\}$
  - $(A \Leftrightarrow B)^t = \{1 \text{ if } A^t = B^t; 0 \text{ otherwise}\}$

Propositional Semantics /2

- **Theorem 2.1. Truth Valuation:** (Theorem 2.4.3)
  - For any A $\in$ Form(L^p) and any truth valuation t, $A^t \in \{1,0\}$
- Example:
  - $A = p \vee q \Rightarrow q \wedge r$; and $p^{t1} = q^{t1} = r^{t1} = 1$
  - What is the value of A^{t1}?
  - What is the value of A^{t2} if $p^{t2} = q^{t2} = r^{t2} = 0$?
- Let Σ be a set of formulas; then
  $\Sigma^t = \{1$ if for each $A \in \Sigma$, $A^t = 1$; 0 otherwise$\}$
- **Definition 2.3. Satisfiability:** (Definition 2.4.4)
  - A set of formulas Σ is **satisfiable** iff there is some truth valuation t such that $\Sigma^t = 1$
  - When $\Sigma^t = 1$, t is said to satisfy Σ
  - **Similarly for A $\in$ Form(L^p) , if $A^t = 1$, t is said to satisfy A**

Propositional Semantics /3

- **Definition 2.4. The Satisfiability Relation:**
  - For every interpretation (i.e., truth valuation) I and well-formed formula A, either I satisfies A denoted as $I \models A$, or I does not satisfy A denoted as $I \not\models A$
  - Other textbooks denote formulas with lower-case Greek symbols, such as φ (phi), ψ (psi), and ω (omega)
- **Definition 2.5. The Model of a Formula:**
  - An interpretation I such that $I \models A$ is called a model of A
  - We define mod(A) to be the set of all models of A; that is, mod(A) = { all I | such that $I \models A$}
  - Framed differently, for a WFF A, t is a model of A iff $A^t = 1$, and mod(A) = { all t | such that $A^t = 1$}

Propositional Semantics /4

- **Definition 2.6. Tautology:** (Definition 2.4.5)
  - A well-formed formula A is a tautology iff for any truth valuation t, $A^t = 1$
  - Framed differently, if $I \models A$ for all interpretations I then and only then A is a valid formula or tautology
- **Definition 2.7. Contradiction:** (Definition 2.4.5)
  - A well-formed formula A is a contradiction iff for any truth valuation t, $A^t = 0$
  - Framed differently, if $I \not\models A$ for all interpretations I then and only then A is a contradiction (i.e., unsatisfiable)

Propositional Semantics /5

- **Simplifications:**
  - Replace formulas expressions with their simplified but correct equivalents
  - For instance, replace $A \wedge 1$ or $1 \wedge A$ with A, and $A \wedge 0$ or $0 \wedge A$ with 0
  - Start replacing symbols from one of the atoms by first setting that atom value to 1 then to 0, and then create a valuation tree based on the expression simplification
  - Stop when there are 1s or 0s in every branch of the tree
  - **If all leaf nodes equal to 1 then A is a tautology**
- **Example:**
  - Simplify $A = (p \wedge q \Rightarrow r) \wedge (p \Rightarrow q) \Rightarrow (p \Rightarrow r)$ and determine if it is a tautology (i.e., a valid formula)

Logical Implications /1

- **Definition 2.8. Logical Implication:** (Definition 2.5.1)
  - Let $\Sigma \subseteq \text{Form}(\mathcal{L}^p)$ and $A \in \text{Form}(\mathcal{L}^p)$
  - A is a logical implication of Σ iff for any truth valuation t, $\Sigma^t = 1$ implies $A^t = 1$
  - Σ logically implies A and is written as $\Sigma \vDash A$
  - Known as **tautological consequence** in the textbook
- Example: $A \Rightarrow B, B \Rightarrow C \vDash A \Rightarrow C$
  - Use Definition 2.2 specified earlier
- **Definition 2.9. Logical Equivalence:**
  - Two formulas A and B are logically equivalent iff $A \vDash B$ and $B \vDash A$, and are denoted with $A \equiv B$
  - Framed differently, $A \equiv B$ iff mod(A) = mod(B)
- Informally, two formulas are logically equivalent iff they are identical in meaning
  - Prove their logical equivalence by showing that equivalence $(A \Leftrightarrow B)$ of the two formulas is a tautology

Logical Implications /2

- **Logical Equivalences:**
  - $A \vee A \equiv A$ (idempotent)
  - $A \wedge A \equiv A$
  - $A \vee B \equiv B \vee A$ (commutative)
  - $A \wedge B \equiv B \wedge A$
  - $A \vee (B \vee C) \equiv (A \vee B) \vee C$ (associative)
  - $A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$
  - $A \vee (B \wedge C) \equiv$ (distributive) $(A \vee B) \wedge (A \vee C)$
  - $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$
  - $A \vee (A \wedge B) \equiv A$
  - $A \wedge (A \vee B) \equiv A$
- **Continued…**
  - $\neg(A \vee B) \equiv \neg A \wedge \neg B$
  - $\neg(A \wedge B) \equiv \neg A \vee \neg B$ (De Morgan's Laws)
  - $A \Rightarrow B \equiv \neg A \vee B$
  - $A \Rightarrow B \equiv \neg B \Rightarrow \neg A$ (contra positive)
  - $A \Rightarrow (B \Rightarrow C) \equiv (A \Rightarrow B) \Rightarrow (A \Rightarrow C) \equiv (A \Rightarrow B) \Rightarrow C$
  - $A \Leftrightarrow B \equiv (A \Rightarrow B) \wedge (B \Rightarrow A)$
  - $A \Leftrightarrow B \equiv B \Leftrightarrow A$
  - $(A \Leftrightarrow B) \Leftrightarrow C \equiv A \Leftrightarrow (B \Leftrightarrow C)$

Logical Implications /3

- **Theorem 2.2. Logical Implication and Satisfiability:**
  - Let $\Sigma \subseteq Form(\mathcal{L}^p)$ and $A \in Form(\mathcal{L}^p)$.
  - Then $\Sigma \vDash A$ iff $\Sigma \cup \{(\neg A)\}$ is a contradiction (i.e., unsatisfiable)
  - That is, there exists no truth valuation t that satisfies $\Sigma \cup \{(\neg A)\}$
- Example:
  $(A \Rightarrow \neg B) \vee C,\ B \wedge \neg C,\ A \Leftrightarrow C \nvDash \neg A \wedge (B \Rightarrow C)$
  - Again use Definition 2.2 specified earlier
- **Theorem 2.3. Replaceability:** (Theorem 2.5.4)
  - If $B \equiv C$ and A' results from A by replacing some occurrences of B by C in A then $A \equiv A'$

Logical Implications /4

- **Definition 2.10 Literals and Clauses:** (Definition 2.7.1)
  - Atoms and their negations are called literals
  - Disjunctions (conjunctions) with literals as disjuncts (conjuncts) are called disjunctive (conjunctive) clauses
- **Definition 2.11 Normal Forms:** (Definition 2.7.2)
  - A disjunction with conjunctive clauses as its disjuncts is called is a disjunctive normal form
  - A conjunction with disjunctive clauses as its conjuncts is called is a conjunctive normal form
- **Theorem 2.4:** (Theorem 2.7.3)
  - Any A $\in$ Form($\mathcal{L}^p$) is logically equivalent to some disjunctive normal form
- **Theorem 2.5:** (Theorem 2.7.4)
  - Any A $\in$ Form($\mathcal{L}^p$) is logically equivalent to some conjunctive normal form

Adequate Sets of Connectives /1

- Note that $A \Rightarrow B \equiv \neg A \vee B$ so $\Rightarrow$ can be defined in terms of $\neg$ and $\vee$
- Similarly $A \vee B \equiv \neg A \Rightarrow B$ so $\vee$ can be defined in terms of $\neg$ and $\Rightarrow$
- Other than the standard five connectives introduced so far, there are more unary, binary, and n-ary connectives
  - For any $n \geq 1$ there are $2^{2\wedge n}$ distinct n-ary connectives
  - We shall denote connectives with italic small Latin letters *f* and *g*; that is, denote them in application as *f*A1…An

A	f_1A	f_2A	f_3A	f_4A
0	1	1	0	0
1	1	0	1	0

Adequate Sets of Connectives /2

- **Definition 2.12 Adequate Connectives:**
  - A set of connectives is said to be adequate iff any n-ary connective, where $n \geq 1$, can be defined using the members of the set
- Emil Post [1921] proved that the standard set of connectives ($\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$) is adequate
  - However, some of the subsets of the standard set are also adequate
- **Theorem 2.6 Adequate Connectives1:** (Theorem 2.8.1)
  - $\{\neg, \wedge, \vee\}$ is an adequate set of connectives
  - Proof based on Theorem 2.4
- **Theorem 2.7 Adequate Connectives2:** (Theorem 2.8.2)
  - $\{\neg, \wedge\}$, $\{\neg, \vee\}$, $\{\neg, \Rightarrow\}$ are adequate sets of connectives

Food for Thought

- **Read:**
  - Chapter 2, Sections 2.4, 2.5, 2.7 and 2.8 from Zhongwan
    - Read proofs presented in class in more detail
    - Cursory reading of proofs omitted but mentioned in class
- Answer the following exercises:
  - Exercises 2.4.1 and 2.5.2
  - Exercises 2.7.2 and 2.8.1
- (Optional) Read:
  - Chapters 4, Sections 4.1 and 4.2 from Nissanke
    - Complete at least a few exercises from each section