QUALITATIVE ANALYSIS OF NONLINEAR SYSTEMS

FORMULATING GENERAL n^{th}-ORDER SYSTEMS OF DIFFERENTIAL EQUATIONS INTO SYSTEMS OF FIRST-ORDER DIFFERENTIAL EQS ALLOWS INSIGHT INTO SYSTEM BEHAVIOUR <u>WITHOUT ACTUALLY SOLVING THE EQUATIONS</u>! THIS IS PARTICULARLY TRUE FOR 1- AND 2-DIMENSIONAL SYSTEMS, WHERE IT IS POSSIBLE TO SKETCH THE SOLUTIONS IN THEIR ENTIRETY ON A 2D-PIECE OF PAPER.

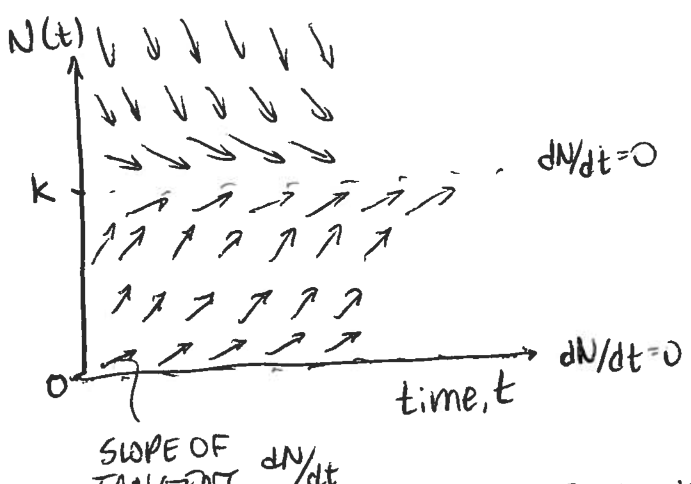
EX. <u>LOGISTIC GROWTH</u>: $\dfrac{dN}{dt} = \underbrace{rN}_{\text{EXPONENTIAL GROWTH}}\left(1 - \dfrac{N}{K}\right)$ MODELS A GROWING POPULATION $N(t)$.

$(r, K > 0)$

$-\dfrac{rN^2}{K}$ IS A TERM TO ACCOUNT FOR COMPETITION
i.e. WHEN $N \gtrsim K$, RESOURCES BECOME SCARCE.

THIS IS A SEPARABLE EQUATION, BUT WE CAN GET AS MUCH (OR MORE) INSIGHT FROM LOOKING AT QUALITATIVE SOLUTIONS.

THE IDEA (DUE TO EULER) IS TO RECOGNIZE THAT dN/dt IS THE SLOPE OF THE TANGENT LINE TO THE SOLUTION $N(t)$ AT ANY POINT $(t, N(t))$. A PLOT OF THESE TANGENT LINES IS CALLED A "<u>PHASE PORTRAIT</u>". FOR $N(t) \geq 0$, THE EQUILIBRIA ARE $N^* = 0, K$



FOR $N > K$, $rN(1 - N/K) < 0$

FOR $N < K$, $rN\left(1 - \frac{N}{K}\right) > 0$ SO $dN/dt > 0$

FOR INITIAL CONDITIONS $N(0) \neq 0$, ALL SOLUTIONS TEND TO K:

$$\lim_{t \to \infty} N(t) = K$$

APPLY THIS SAME INTUITION TO A SYSTEM: TWO POPULATIONS GROWING LOGISTICALLY & COMPETING WITH ONE ANOTHER:

$$\frac{dx}{dt} = \underbrace{\epsilon_1 x\left[1-\frac{\sigma_1}{\epsilon_1}x\right]}_{\text{LOGISTIC GROWTH}} - \alpha_1 xy \qquad \frac{dy}{dt} = \underbrace{\epsilon_2 y\left[1-\frac{\sigma_2}{\epsilon_2}y\right]}_{\text{LOGISTIC GROWTH}} - \alpha_2 xy$$

COMPETITION

FIXED POINTS OR EQUILIBRIA POINTS ARE:

$(0,0)$ ZERO POPULATION

$(\epsilon_1/\sigma_1, 0)$ POPULATION X WINS

$(0, \epsilon_2/\sigma_2)$ POPULATION Y WINS

$\left(\dfrac{\epsilon_1\sigma_2 - \epsilon_2\alpha_1}{\sigma_1\sigma_2-\alpha_1\alpha_2}, \dfrac{\sigma_1\epsilon_2 - \epsilon_1\alpha_2}{\sigma_1\sigma_2-\alpha_1\alpha_2}\right)$ POPULATIONS CO-EXIST.

ARE ANY OF THESE STABLE?

LOOK AT A COUPLE OF EXAMPLES -

CASE 1. $\dfrac{dx}{dt} = x(1-x-y) \qquad \dfrac{dy}{dt} = y\left(\frac{3}{4} - y - \frac{1}{2}x\right)$

EQUILIBRIA:

$(0,0) \quad (1,0) \quad (0,3/4) \quad (1/2, 1/2)$

THE JACOBIAN IS: $\dfrac{\partial \vec{f}}{\partial \vec{x}} = \begin{bmatrix} 1-2x-y & -x \\ -\frac{1}{2}y & \frac{3}{4}-2y-\frac{1}{2}x \end{bmatrix} = J(x,y)$

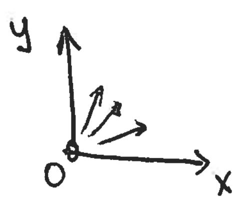
AT (0,0):

$J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 3/4 \end{bmatrix}$

EIGENVALUES $1, 3/4 > 0$

UNSTABLE NODE

EIGENVECTORS $\binom{1}{0}, \binom{0}{1}$



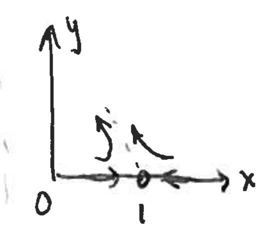
AT (1,0).

$J(1,0) = \begin{bmatrix} -1 & -1 \\ 0 & 1/4 \end{bmatrix}$

EIGENVALUES $-1, 1/4$

SADDLE POINT

EIGENVECTORS $\binom{1}{0}, \binom{4}{-5}$



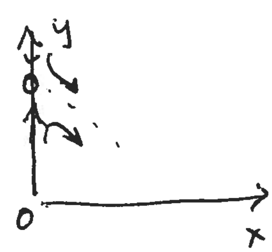
AT (0,3/4)

$J(0,3/4) = \begin{bmatrix} 1/4 & 0 \\ -3/8 & -3/4 \end{bmatrix}$

EIGENVALUES $1/4, -3/4$

SADDLE POINT

EIGENVECTORS $\binom{8}{-3}, \binom{0}{1}$



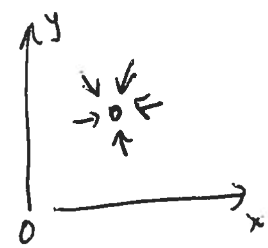
AT (1/2, 1/2)

$J(1/2,1/2) = \begin{bmatrix} -1/2 & -1/2 \\ -1/4 & -1/2 \end{bmatrix}$

E.VALUES $-\frac{1}{2} \pm \frac{1}{2\sqrt{2}}$

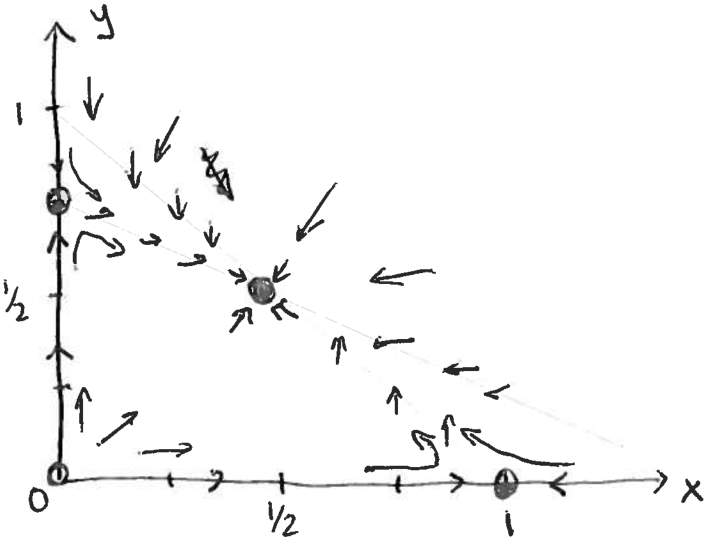
STABLE!

E. VECTORS $\binom{\sqrt{2}}{-1}, \binom{\sqrt{2}}{1}$



PUT IT ALL TOGETHER TO SKETCH A PHASE PORTRAIT (GLOBAL BEHAVIOUR)

O: UNSTABLE NODES
: SADDLE POINTS
: STABLE NODES



WE FIRST SOLVED THE ALGEBRAIC SYSTEM

$$\frac{dx}{dt} = 0 \qquad \frac{dy}{dt} = 0$$

TO FIND THE EQUILIBRIA, THEN CHARACTERIZED THE LOCAL STABILITY.

LETS SHIFT OUR THINKING FROM ALGEBRA TO GEOMETRY – WHAT WE'RE ABOUT TO TALK ABOUT IS A TECHNIQUE THAT ENDOWS YOU WITH AMAZING POWERS; USING CURVE SKETCHING IDEAS FROM FRESHMAN CALCULUS, YOU CAN GENERATE A PHASE PORTRAIT WITHOUT EVEN CALCULATING A SINGLE EIGENVALUE!

MAIN IDEA: FOR A SYSTEM $\frac{dx}{dt} = f(x,y)$ $\frac{dy}{dt} = g(x,y)$, THE EQUILIBRIA CORRESPOND TO POINTS THAT SIMULTANEOUSLY SATISFY $f(x,y)=0$ & $g(x,y)=0$. IF YOU PLOT THE CURVES $f(x,y)=0$ & $g(x,y)=0$, THEN THE EQUILIBRIA ARE SIMPLY THEIR POINTS OF INTERSECTION. THE CURVES $f(x,y)=0$ & $g(x,y)=0$ ARE CALLED 'NULLCLINES'

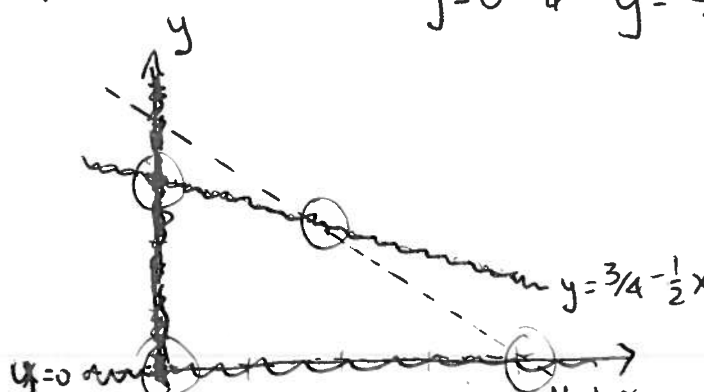
FOR OUR EXAMPLE SYSTEM: $\frac{dx}{dt} = x(1-x-y)$ & $\frac{dy}{dt} = y\left(\frac{3}{4} - y - \frac{1}{2}x\right)$

THE x-NULLCLINES ARE:

$x=0$ & $y = 1-x$

THE y-NULLCLINES ARE:

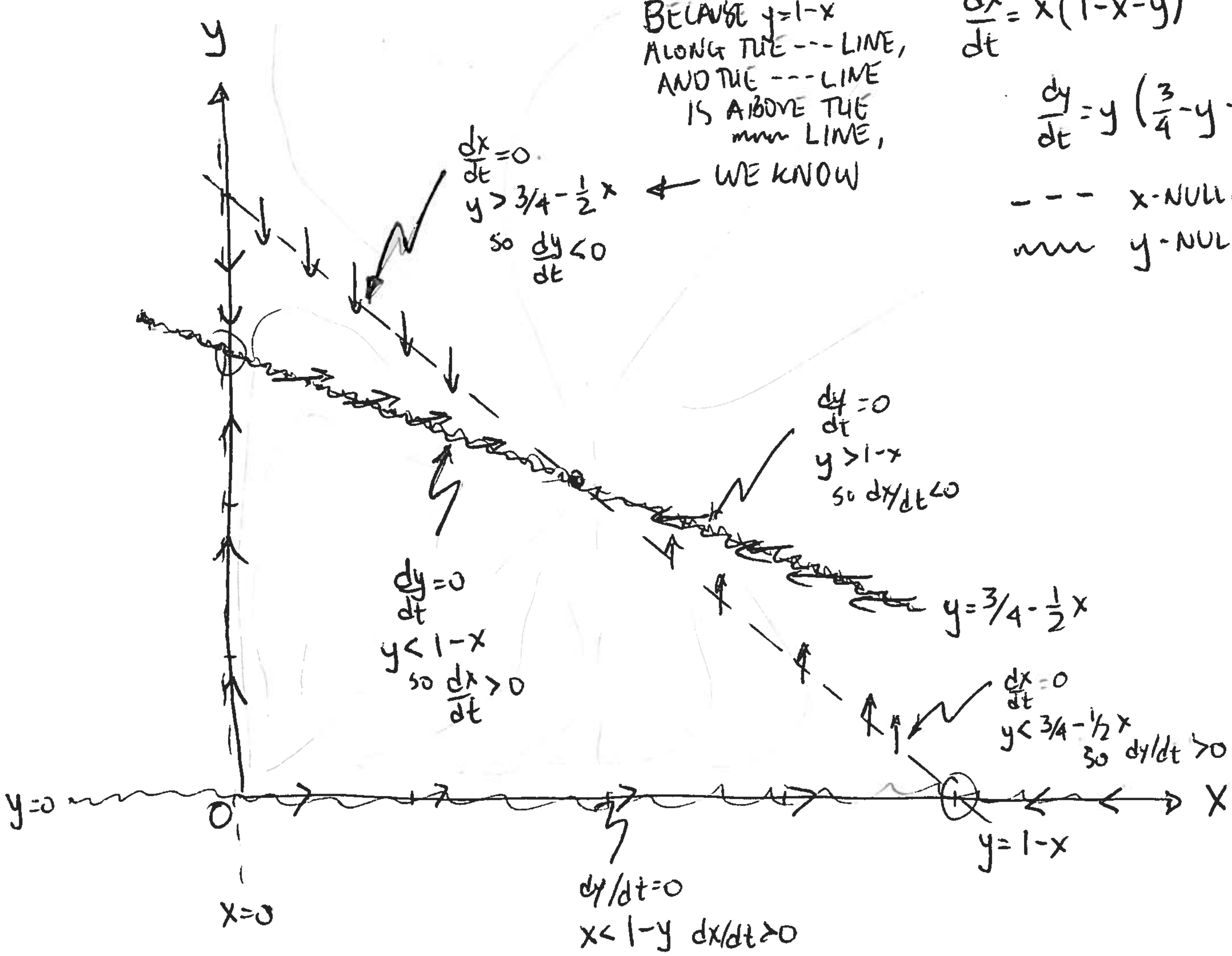
$y=0$ & $y = \frac{3}{4} - \frac{1}{2}x$



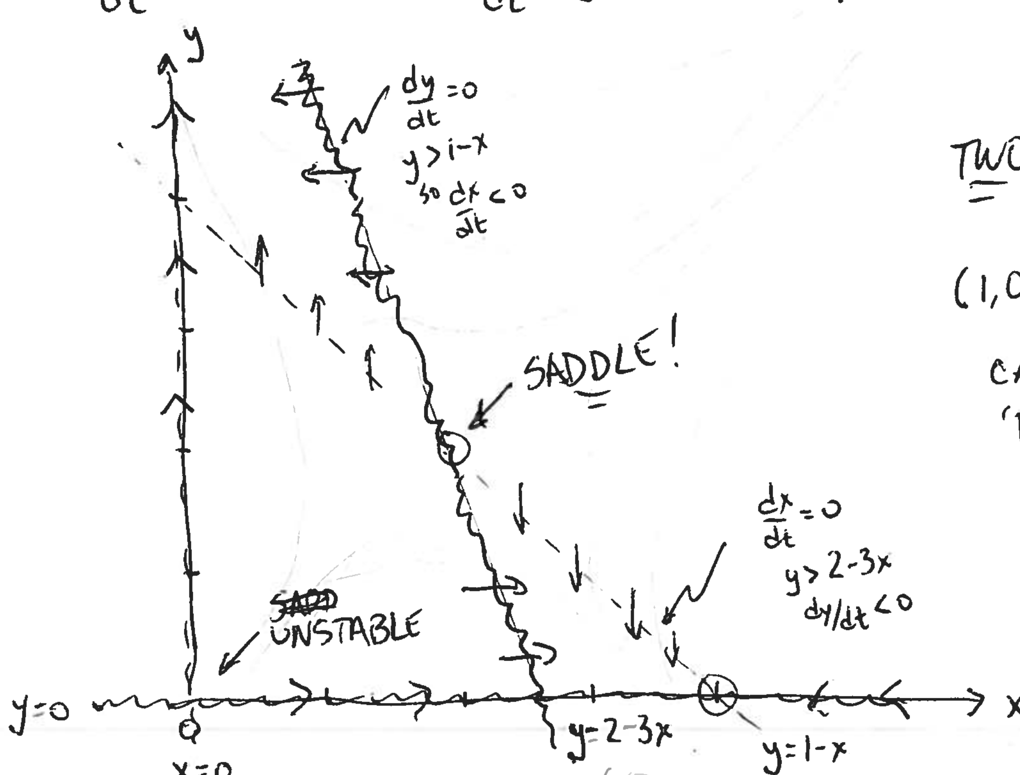
WHAT ABOUT PHASE PORTRAIT?

$$\frac{dx}{dt} = x(1-x-y)$$

$$\frac{dy}{dt} = y\left(\frac{3}{4} - y - \frac{1}{2}x\right)$$



CASE 2: $\dfrac{dx}{dt} = x(1-x-y) \qquad \dfrac{dy}{dt} = y\left(\frac{1}{2} - \frac{1}{4}y - \frac{3}{4}x\right)$



TWO STABLE POINTS:

(1,0) & (0,2).

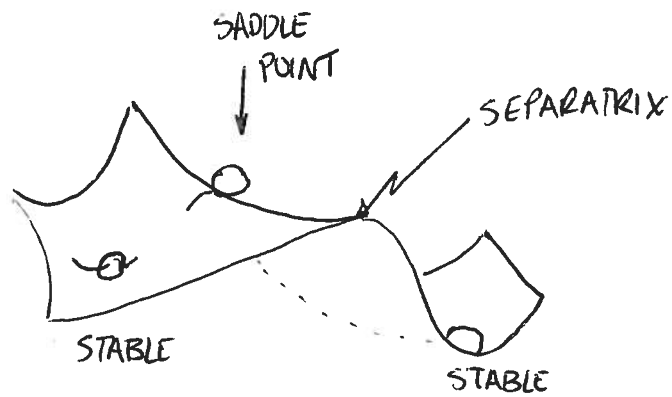
CALLED A 'BISTABLE SYSTEM'.

THE REGION WITHIN WHICH INITIAL CONDITIONS LEAD TO A STEADY-STATE IS CALLED ITS 'BASIN OF ATTRACTION'. THE LINE SEPARATING THE BASINS OF ATTRACTION (i.e. TRAJECTORIES LEADING TO THE SADDLE POINT) IS CALLED A SEPARATRIX.

IN ONE-DIMENSION:

BISTABLE POTENTIAL WELL

IN TWO-DIMENSIONS:



SO FAR, WE HAVE LOOKED AT FAIRLY TAME SYSTEMS - BUT UNSTABLE POINTS CAN GIVE RISE TO VERY INTERESTING BEHAVIOUR.

STABLE ORBITS - CENTRES, LIMIT CYCLES & STRANGE ATTRACTORS

ONE VARIETY OF PERIODIC BEHAVIOUR AVAILABLE IN LINEAR & NONLINEAR SYSTEMS. (eg. THE HARMONIC OSCILLATOR).

A NONLINEAR EXAMPLE IS THE LOTKA-VOLTERRA SYSTEM OF EQUATIONS

$x(t)$ = PREY POPULATION $y(t)$ = PREDATOR POPULATION

$$\frac{dx}{dt} = \underbrace{a x}_{\text{GROWTH OF PREY}} - \underbrace{\alpha \cdot x \cdot y}_{\text{DEATH BY PREDATION}} \qquad \frac{dy}{dt} = \underbrace{\beta \cdot x \cdot y}_{\text{GROWTH BY PREDATION}} - \underbrace{b \cdot y}_{\text{'NATURAL DEATH'}}$$

EQUILIBRIA: $(0,0)$ & $(b/\beta, a/\alpha)$

JACOBIAN $J(x,y) = \begin{bmatrix} a-\alpha y & -\alpha x \\ \beta \cdot y & -b + \beta x \end{bmatrix}$