Propositional Logic Proofs Part2

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[with material from “Mathematical Logic for Computer Science”, by Zhongwan, published by World Scientific]

Objectives

- Deduction Theorem Proof
- Soundness Proof
- Satisfiability and Consistency Proofs
- Completeness Proof
- Compactness Theorem and Applications (next time)

Deduction Theorem Revisited: Proof /1

- **Theorem 3.2. Deduction Theorem:**
  - For A, B $\in$ Form ($\mathcal{L}^p$) and $\Sigma \subseteq$ Form ($\mathcal{L}^p$), $\Sigma \vdash_H A \Rightarrow B$ iff $\Sigma \cup \{A\} \vdash_H B$
- **Proof (if $\Sigma \vdash_H A \Rightarrow B$ then $\Sigma \cup \{A\} \vdash_H B$):**
  - From $\Sigma \vdash_H A \Rightarrow B$ then (1) $\Sigma \cup \{A\} \vdash_H A \Rightarrow B$ (adding A as another assumption does not invalidate $A \Rightarrow B$)
  - Also, for $\Sigma \cup \{A\}$ it holds that (2) $\Sigma \cup \{A\} \vdash_H A$ (based on the definition of a deducible theorem) (Definition 4.1.1)
  - Hence, $\Sigma \cup \{A\} \vdash_H B$ *(by R1, (1), (2))*

Deduction Theorem Revisited: Proof /2

- **Proof (if $\Sigma \cup \{A\} \vdash_H B$ then $\Sigma \vdash_H A \Rightarrow B$):**
  - By induction on the structure of $\Sigma \cup \{A\} \vdash_H B$
- (Base Case)
  - If B is an axiom or $B \in \Sigma$ then $\{B, (B \Rightarrow (A \Rightarrow B)), (A \Rightarrow B)\}$ is the proof, so $\Sigma \vdash_H A \Rightarrow B$
    - For the above, if $B \in \Sigma$ then B is an assumption; else B is an axiom
  - If B equals A (i.e., $B \in \{A\}$) then $\vdash_H A \Rightarrow A$ is the proof for it, as shown in Lecture Notes #3, so $\Sigma \vdash_H A \Rightarrow B$
- (Inductive Case)
  - If B is not compliant with the base case then it must be obtained using R1 from some two formulas C and $C \Rightarrow B$
  - By the Induction Hypothesis:

    $\Sigma \cup \{A\} \vdash_H C$ so $\Sigma \vdash_H A \Rightarrow C$

    $\Sigma \cup \{A\} \vdash_H C \Rightarrow B$ so $\Sigma \vdash_H A \Rightarrow (C \Rightarrow B)$

Deduction Theorem Revisited: Proof /3

- (Inductive Case Continued)
  1. $\Sigma \cup \{A\} \vdash_H C$ so $\Sigma \vdash_H A \Rightarrow C$ *(derived)*
  2. $\Sigma \cup \{A\} \vdash_H C \Rightarrow B$ so $\Sigma \vdash_H A \Rightarrow (C \Rightarrow B)$ *(derived)*
  3. $\Sigma \vdash_H (A \Rightarrow (C \Rightarrow B)) \Rightarrow ((A \Rightarrow C) \Rightarrow (A \Rightarrow B)))$ *(by Ax2)*
  4. $\Sigma \vdash_H (A \Rightarrow C) \Rightarrow (A \Rightarrow B)$ *(by R1, (2), (3))*
  5. $\Sigma \vdash_H A \Rightarrow B$ *(by R1, (1), (4))*
- **Hence, $\Sigma \vdash_H A \Rightarrow B$**

Soundness of the Hilbert System: Proof

- **Soundness of the Hilbert System:**
  - For $A \in \text{Form}(\mathcal{L}^p)$ and $\Sigma \subseteq \text{Form}(\mathcal{L}^p)$, $\Sigma \vdash_H A \Rightarrow \Sigma \vDash A$
- Proof by induction on the structure of A
- (Base Case)
  - If A is an axiom then $\vDash A$ since all axioms are valid, so it follows that $\Sigma \vDash A$
  - If $A \in \Sigma$ then for some truth valuation t such that $\Sigma^t = 1$ it must hold that $A^t = 1$ (i.e., $mod(\Sigma) \subseteq mod(A)$)
- (Inductive Case)
  - If A is not compliant with the base case then it must be obtained using R1 from some two formulas $C \Rightarrow A$ and C
  - By the Induction Hypothesis: $\Sigma \vDash C \Rightarrow A$ and $\Sigma \vDash C$
  - Based on the truth table for $(C \Rightarrow A) \wedge C$, A must be true
  - Hence, it follows that $\Sigma \vDash A$

Satisfiability and Consistency Proofs /1

- **Theorem 4.1. Formula Inconsistency:**
  - If $\Sigma \subseteq$ Form ($\mathcal{L}^p$) is inconsistent iff for all A$\in$ Form ($\mathcal{L}^p$) it holds that $\Sigma \vdash_H A$
- Proof (LHS $\Rightarrow$ RHS):
  - Let A be an arbitrary formula
  - If Σ is inconsistent then for some B both (1) $\Sigma \vdash_H B$ and (2) $\Sigma \vdash_H \neg B$ must hold
  - Based on the theorem (3) $\vdash_H B \Rightarrow (\neg B \Rightarrow A)$ *(Notes #3)*, using R1 on (1) and (3) derives (4) $\Sigma \vdash_H \neg B \Rightarrow A$
  - Then, using R1 on (2) and (4) derives $\Sigma \vdash_H A$ as required

Satisfiability and Consistency Proofs /2

- **Theorem 4.1. Formula Inconsistency:**
  - If $\Sigma \subseteq$ Form (L^p) is inconsistent iff for all A$\in$ Form (L^p) it holds that $\Sigma \vdash_H A$
- Proof (RHS $\Rightarrow$ LHS):
  - Let us assume that for all A $\in$ Form (L^p) it holds that $\Sigma \vdash_H A$ but Σ is still consistent
  - Consider A and its negation $\neg$A
  - Based on the above, both $\Sigma \vdash_H A$ and $\Sigma \vdash_H \neg A$ must hold
  - But, based on the definition of the formula consistency, if Σ is consistent then $\Sigma \vdash_H A$ and $\Sigma \vdash_H \neg A$ cannot both hold for any A $\in$ Form (L^p); and this is a contradiction
  - Hence, if for all A $\in$ Form (L^p) it holds that $\Sigma \vdash_H A$ then Σ is inconsistent

Satisfiability and Consistency Proofs /3

- **Theorem 3.3. Formula Consistency:** (Exercise 5.2.2)
  - A set $\Sigma \subseteq$ Form $(\mathcal{L}^p)$ is consistent iff there is A $\in$ Form $(\mathcal{L}^p)$ such that $\Sigma \vdash_H$ A does not hold
- Proof:
  - Based on Theorem 4.1 that was just shown, it follows that if Σ is consistent there must exist A $\in$ Form $(\mathcal{L}^p)$ for which $\Sigma \vdash_H$ A does not hold (LHS $\Rightarrow$ RHS)
  - Similarly based on Theorem 4.1, if there exists A $\in$ Form $(\mathcal{L}^p)$ for which $\Sigma \vdash_H$ A does not hold then Σ is consistent (RHS $\Rightarrow$ LHS)
  - Hence, a set $\Sigma \subseteq$ Form $(\mathcal{L}^p)$ is consistent iff there is A $\in$ Form $(\mathcal{L}^p)$ such that $\Sigma \vdash_H$ A does not hold (LHS $\Leftrightarrow$ RHS)

Satisfiability and Consistency Proofs /4

- **Theorem 4.2. Soundness Applied:** (Theorem 5.2.3)
  - If $\Sigma \subseteq$ Form (L^p) is satisfiable then Σ is consistent
- Proof:
  - If $\Sigma \subseteq$ Form (L^p) is satisfiable then $\Sigma^t = 1$ for some truth valuation t (i.e., $t \vDash \Sigma$)
  - For the same t there must exist A for which $A^t \neq 1$, and from there $\Sigma \vDash A$ does not hold (i.e., mod(Σ) $\not\subset$ mod(A))
    - Choose any proposition or its negation for which t does not hold
  - Based on the contra positive of the Soundness Theorem, if $\Sigma \vDash A$ does not hold then $\Sigma \vdash_H A$ also cannot hold
  - Based on Theorem 4.1 since there exists A for which $\Sigma \vdash_H A$ does not hold Σ is consistent
  - Hence, if $\Sigma \subseteq$ Form (L^p) is satisfiable then Σ is consistent

Completeness of the Hilbert System /1

- **Completeness of the Hilbert System:**
  - For $A \in \text{Form}(\mathcal{L}^p)$ and $\Sigma \subseteq \text{Form}(\mathcal{L}^p)$, $\Sigma \models A \Rightarrow \Sigma \vdash_H A$
- The proof of this will require several steps
- **Theorem 4.3.**
  - For $\Sigma = \{A_1, A_2, \ldots A_n\}$, if $\Sigma \models A$ then
    (1) $\models A_1 \Rightarrow (A_2 \Rightarrow (\ldots (A_n \Rightarrow A)\ldots))$ (is a tautology)
- Proof:
  - Let us assume that there exists a valuation t for which (1) evaluates to false
  - Based on the truth table of "$\Rightarrow$" applied recursively to (1), it follows that the only way for (1) to be false is for A to be false and all A_i to be true
  - Since $\Sigma \models A$ was given (i.e., $mod(\Sigma) \subseteq mod(A)$) having A be false is a contradiction; hence, $\models A_1 \Rightarrow (A_2 \Rightarrow (\ldots (A_n \Rightarrow A)\ldots))$
- Consider $B = A_1 \Rightarrow (A_2 \Rightarrow (\ldots (A_n \Rightarrow A)\ldots))$ for the next step

Completeness of the Hilbert System /2

- **Theorem 4.4.**
  - **Used to prove that if $\models B$ then $\vdash_H B$ (i.e., if B is a tautology then B is a theorem)**
  - Let B be a formula such that $p_1, p_2, \ldots p_n$ are its only propositional atoms
  - Let k be any line in A's truth table for a valuation t
  - Let A_i equal p_i in line k if $p_i^t = 1$, or A_i equal $\neg p_i$ if $p_i^t = 0$, for all $1 \leq i \leq n$
  - It then follows that $\{A_1, A_2, \ldots A_n\} \vdash_H B$ is provable if the entry for B in line k valuates to true (i.e., $B^t \models 1$)
  - And that $\{A_1, A_2, \ldots A_n\} \vdash_H \neg B$ is provable if the entry for B in line k valuates to false (i.e., $B^t \models 0$)
- Proof:
  - By induction on the structure of B

Completeness of the Hilbert System /3

- **Example:**
  - If the truth table is a tautology then there are 2^n table entries for which $\{A_1, A_2, \ldots A_n\} \vdash_H B$
  - Consider propositions p and q and formula $p \wedge q \Rightarrow p$, which is a tautology and for which $\models p \wedge q \Rightarrow p$ holds
  - The theorem produces:

    $\{p, q\} \vdash_H p \wedge q \Rightarrow p$

    $\{\neg p, q\} \vdash_H p \wedge q \Rightarrow p$

    $\{p, \neg q\} \vdash_H p \wedge q \Rightarrow p$

    $\{\neg p, \neg q\} \vdash_H p \wedge q \Rightarrow p$
  - Using the Hilbert extensions, we can deduce that $\vdash_H p \wedge q \Rightarrow p$

Completeness of the Hilbert System /4

- **Finally, we need to prove that $\{A_1, A_2, \dots A_n\} \vdash_H A$**
  - Based on the previous step, it follows that
    (1) $\vdash_H A_1 \Rightarrow (A_2 \Rightarrow (\dots (A_n \Rightarrow A)\dots))$
  - Let us introduce (2) $\{A_1, A_2, \dots A_n\}$ as assumptions
  - If we recursively apply R1 on (1) and (2) it follows that
    $\{A_1, A_2, \dots A_n\} \vdash_H A$
  - Therefore, for $A \in$ Form ($\mathcal{L}^p$) and $\Sigma \subseteq$ Form ($\mathcal{L}^p$),
    $\Sigma \models A \Rightarrow \Sigma \vdash_H A$

Food for Thought

- **Read:**
  - Chapters 4 and 5 (Sections 5.2 and 5.3) from Zhongwan
    - Read proofs discussed in class in more detail
    - Skip the material not related to propositional logic
- Answer the following exercises (from Section 5.3):
  - Exercises 5.3.1, 5.3.3 and 5.3.4