L10 Eigensystem

M6: Eigensystem of Operators

M6.1 Definition of eigenvectors/eigenvalues

$$S_z|+\rangle = \frac{\hbar}{2}\Big(|+\rangle\langle+| - |-\rangle\langle-|\Big)|+\rangle$$
$$= \frac{\hbar}{2}|+\rangle\langle+|+\rangle - \frac{\hbar}{2}|-\rangle\langle-|+\rangle$$
$$= \frac{\hbar}{2}|+\rangle$$

$\Rightarrow |+\rangle$ is an eigenvector of the operator S_z to the eigenvalue $\frac{\hbar}{2}$!

Calculation in coordinate representation: $\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1\\0\end{pmatrix}$

By construction, the measurement vectors $|a_i\rangle$ of a measurement are always eigenvectors of any operator A that one constructs using this measurement in calculating some expectation values.

M6.2 How to find systematically eigenvalues

$$A|\psi_\lambda\rangle = \lambda|\psi_\lambda\rangle$$

λ: Eigenvalue

$|\psi_\lambda\rangle$: corresponding eigenvector

$$\Rightarrow A|\psi_\lambda\rangle - \lambda|\psi_\lambda\rangle = 0$$
$$(A - \mathbb{1}\lambda)|\psi_\lambda\rangle = 0$$

Identity operator

Now go to some coordinate representation

$$\begin{pmatrix} A_{11}-\lambda & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22}-\lambda & & \vdots \\ \vdots & & & A_{n-1} \\ A_{n1} & \cdots & A_{n-1} & A_{n}-\lambda \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_n \end{pmatrix} = 0$$

==> Linear Algebra: rows of matrix must be linearly dependent for this to have a nontrivial solution!
(trivial solution: $\alpha_i = 0$ for all values of i)

==> $\det(A - \lambda \mathbb{1}) = 0$ **characteristic polynomial**

==> find values for λ that satisfy that condition!
Any solution is an eigenvalue!

This condition is a polynomial of degree d (= dimension of vector space)

Over complex numbers, there will be always d solutions!

$$\det(A - \lambda \mathbb{1}) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$$

Non-degenerate eigenvalues:
an eigenvalue that occurs only once $\lambda_i \neq \lambda_j$ for all $j \neq i$

degenerate eigenvalues: some eigenvalues might be the same

Example: seek eigenvalues of operator S_y

$$\Rightarrow \det\left(\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} -\lambda & -i\frac{\hbar}{2} \\ i\frac{\hbar}{2} & -\lambda \end{pmatrix} = \lambda^2 - \left(-i\frac{\hbar}{2}\right)\left(i\frac{\hbar}{2}\right)$$

$$= \lambda^2 - \frac{\hbar^2}{4}$$

eigenvalues need to satisfy:

$$\Rightarrow \quad \lambda^2 - \frac{\hbar^2}{4} = 0$$

$\Rightarrow$ two eigenvalues found:

$$\lambda_+ = \frac{\hbar}{2}$$

$$\lambda_- = -\frac{\hbar}{2}$$

M6.3 How to find systematically eigenvectors for given eigenvalue:

In a coordinate representation parameterize the eigenvector we are seeking as

$$|\psi_\lambda\rangle = \sum_i \alpha_i |a_i\rangle$$

$$|\psi_\lambda\rangle \Longleftrightarrow \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Solve explicitly the set of linear equation:

$$\left(A - \lambda \mathbb{1}\right) \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = 0$$

If eigenvalue is non-degenerate, you find unique solution (up to global phase)
if eigenvalue is degenerate, you find a whole subset of vectors
==> choose an orthormal basis of the subspace as eigenvectors

Example:

Eigenvectors of S_y belonging to eigenvalue $\lambda_+ = +\frac{\hbar}{2}$

$$\Rightarrow \quad \underline{|\psi_{\frac{\hbar}{2}}\rangle} \Longleftrightarrow \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\left(S_y - \frac{\hbar}{2}\mathbb{1}\right) \begin{pmatrix} a \\ b \end{pmatrix} \overset{!}{=} 0$$

$$\frac{\hbar}{2}\begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$\Rightarrow \frac{\hbar}{2}\begin{pmatrix} -a - ib \\ ia - b \end{pmatrix} = 0$$

$$\Rightarrow \quad b = ia$$

eigenvector $\begin{pmatrix} a \\ ia \end{pmatrix}$

normalization $|a|^2 + |ia|^2 \overset{!}{=} 1$

$$2|a|^2 = 1$$

$$|a| = \frac{1}{\sqrt{2}}$$

choice:
(up to global phase) $a = \frac{1}{\sqrt{2}}$

$$|+\rangle_y \longleftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

Similarly:

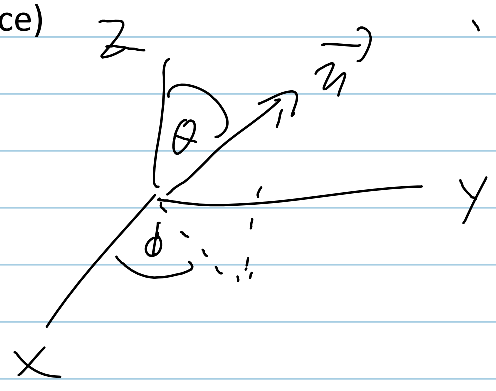
$$|\psi_{-\hbar/2}\rangle = |-\rangle_y \longleftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

4.2 Spin Component in general direction

4.2.1 Ansatz for general Spin component via directional unit vector

direction : $\vec{n}$ (unit vector in 3-dimensional space)

$$\vec{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} = \begin{pmatrix} \sin\theta \cos\phi \\ \sin\theta \sin\phi \\ \cos\theta \end{pmatrix}$$



$$S = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}$$ components are operators!

$$S_n = \vec{n} \cdot \vec{S}$$

The vector representation is here just a convenient abbreviation, the second line is the actual definition!

$$= n_x S_x + n_y S_y + n_z S_z$$

coordinate representation

$$S_n \longleftrightarrow \frac{\hbar}{2} \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{+i\phi} & \cos\theta \end{pmatrix}$$

4.2.2 Eigenvalues and Eigenvectors of general spin component operators

eigenvalues $\pm \frac{\hbar}{2}$

eigenvectors

$$|+\rangle_n = \cos\frac{\theta}{2}\,|+\rangle + \sin\frac{\theta}{2}\, e^{i\phi}\,|-\rangle$$

$$|-\rangle_n = \sin\frac{\theta}{2}\,|+\rangle - \cos\frac{\theta}{2}\, e^{i\phi}\,|-\rangle$$

Experimental observation:
The assignment of the eigenvectors to the outcome of a generalized Stern-Gerlach Measurement (orientation of inhomogeneous magnetic field) lead to the correct predictions of outcome probabilities!

4.2.3 Examples of use

Example: $\vec{n}_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} \theta = \frac{\pi}{2} \\ \phi = \frac{\pi}{4} \end{cases}$

$$\Rightarrow S_{n_0} = \frac{\hbar}{2} \begin{pmatrix} 0 & e^{-i\frac{\pi}{4}} \\ e^{i\frac{\pi}{4}} & 0 \end{pmatrix}$$

$$|+\rangle_{n_0} = \frac{1}{\sqrt{2}}\left(|+\rangle + e^{i\frac{\pi}{4}}|-\rangle\right)$$

$$|-\rangle_{n_0} = \frac{1}{\sqrt{2}}\left(|+\rangle - e^{i\frac{\pi}{4}}|-\rangle\right)$$

Example Application Expectation Value:

$$|\psi\rangle = |+\rangle_x \longleftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$\Rightarrow \langle S_{n_0}\rangle = \langle\psi|S_{n_0}|\psi\rangle$$

$$\longleftrightarrow \frac{1}{\sqrt{2}}(1,1)\,\frac{\hbar}{2}\begin{pmatrix}0 & e^{-i\pi/4}\\ e^{i\pi/4} & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$= \frac{\hbar}{4}(1,1)\begin{pmatrix}e^{i\pi/4}\\ e^{-i\pi/4}\end{pmatrix} = \frac{\hbar}{4}\left(e^{i\frac{\pi}{4}} + e^{-i\frac{\pi}{4}}\right)$$

$$= \frac{\hbar}{2}\cos\frac{\pi}{4} = \frac{\hbar}{2\sqrt{2}}$$

Example Probability:

Given a source with state

$$|\psi\rangle = |+\rangle_x \longleftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

What is the probability to find outcome "up" in a Stern-Gerlach Experiment with respect to the direction

$$\vec{n}_0 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$$

Answer:

$$\text{Prob}(+ \text{ in } \vec{n}_0) = \left| {}_x\langle + | + \rangle_{\vec{n}_0} \right|^2$$

$$= \left| \frac{1}{\sqrt{2}} (1, 1) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\pi/4} \end{pmatrix} \right|^2$$

$$= \frac{1}{4} \left| 1 + e^{i\pi/4} \right|^2$$

$$= \frac{1}{4} \left(1 + e^{i\frac{\pi}{4}}\right)\left(1 + e^{-i\frac{\pi}{4}}\right)$$

$$= \frac{1}{4} \left(1 + 1 + e^{i\frac{\pi}{4}} + e^{-i\pi/4}\right)$$

$$= \frac{1}{4} \left(2 + 2\cos\frac{\pi}{4}\right)$$

$$= \frac{1}{2} \left(1 + \frac{1}{\sqrt{2}}\right) \approx 0.85$$