General Coordinate Representation Rule

$$
\begin{aligned}
|\Psi\rangle &= \int_{-\infty}^{+\infty} dx\ \Psi(x)|x\rangle &\doteq&\quad \Psi(x)\\
\hat{A} &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx\, dx'\ g(x,x')|x\rangle\langle x'| &\doteq&\quad g(x,x')\\
\langle\Phi| &= \int_{-\infty}^{+\infty} dx\ \Phi^*(x)\langle x| &\doteq&\quad \Phi^*
\end{aligned}
$$

$$
\begin{aligned}
\Psi(x) &= \langle x|\Psi\rangle\\
g(x,x') &= \langle x|\hat{A}|x'\rangle\\
\Phi^*(x) &= \langle \Phi|x\rangle
\end{aligned}
$$

$$
\begin{aligned}
\langle\Phi|\Psi\rangle &= \int_{-\infty}^{+\infty} dx\ \ \Phi^*(x)\Psi(x)\\
\hat{A}|\Psi\rangle &\doteq \int_{-\infty}^{+\infty} dx' g(x,x')\Psi(x')\\
\langle\Phi|\hat{A}|\Psi\rangle &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx\, dx'\ \Phi^*(x)g(x,x')\Psi(x')
\end{aligned}
$$

6.2.3 Coordinate representation of Position operator (continued)

Recap:

We introduced the position operator as

$$\hat{X} = \int_{-\infty}^{+\infty} dx\ \ x\ \ |x\rangle\langle x|$$

but we would like to find the canonical representation in the form

$$\hat{X} = \iint_{-\infty}^{+\infty} dx\, dx'\ \ g(x,x')\ \ |x\rangle\langle x'|$$

So we start as

$$\langle x|\hat{X}|x'\rangle = \langle x|\left[\int_{-\infty}^{+\infty} dx''\ x''\ \ |x''\rangle\langle x''|\right]|x'\rangle$$

$$= \int_{-\infty}^{+\infty} dx''\ \ x''\ \ \ \langle x|x''\rangle\ \langle x''|x'\rangle$$

6.2.3.1 Overlap of position states

To continue from here, we need to know what the overlap between two position states are:

$$\langle x | x'' \rangle = \begin{cases} 0 & x \neq x'' \\ ? & x = x'' \end{cases}$$

It turns out, that we cannot choose

$$\langle x | x'' \rangle = 1 \quad \text{for } x = x''$$

The integration would always give zero, as the overlap functions are different from zero only in single points! Instead, we make the following assignment

Definition: $\langle x | x' \rangle = \delta(x - x')$

uses Dirac δ function

6.2.3.2 Background: Dirac δ function

Operational definition

$$\int_a^b dx \; \delta(x - \alpha) f(x) = \begin{cases} f(\alpha) & x \in (a, b) \\ 0 & x \notin [a, b] \end{cases}$$

for any continuous function f(x)

analogue in the finite dimensional case:

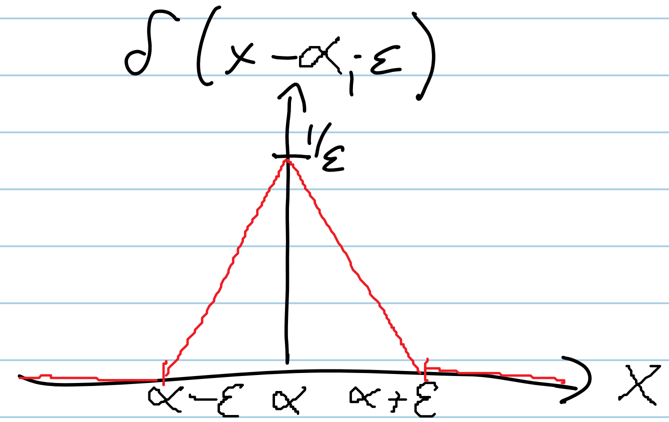
$$\sum_k \delta_{ik} f_k = f_i \quad \text{(Kronecker Delta } \delta_{ik}\text{)}$$

δ-function is not a proper function, it is a so-called "*distribution*"

- it takes a definite meaning only under an integral (integral kernel)

Representation of δ-function as a limit of ordinary functions:

Example: (other representations possible and used!)



check: $$\int_{-\infty}^{+\infty} dx\, \delta(x-\alpha;\varepsilon) = 1 \quad \forall \varepsilon$$

$$\lim_{\varepsilon \to 0} \delta(x-\alpha;\varepsilon) = \delta(x-\alpha)$$

Properties of Dirac delta-function:

1) $\delta(x-\alpha) = \delta(\alpha - x)$ (symmetric)

2) variable substitution of the delta-function apply in integration as with any other function (think of it as the limit of ordinary functions as shown above!)

6.2.3.3 Coordinate Representation

$$\langle x|\hat{X}|x'\rangle = \int_{-\infty}^{+\infty} dx''\, x''\, \delta(x-x'')\,\delta(x''-x')$$

$$= x\,\delta(x-x')$$

Alternative:

$$\hat{X} = \int_{-\infty}^{+\infty} dx\; x\,|x\rangle\langle x| = \int dx\,dx'\; x\,\delta(x-x')\,|x\rangle\langle x'|$$

Example of usage:

$$\hat{X}^2 = \hat{X}\hat{X} \doteq \underbrace{\int_{-\infty}^{+\infty} dx'}\ \overbrace{\underbrace{x\,\delta(x-x')}_{\doteq \hat{X}}}^{g(x,x')}\ \overbrace{\underbrace{x'\,\delta(x'-\cancel{x''})}_{\doteq \hat{X}}}^{g(x',x'')}$$

Inner product
(corresponding to sum in matrix multiplication)

$$= X \qquad\qquad x\,\delta(x - \langle x'|$$

$$= x^2\,\delta(x-x')$$

$$\doteq \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dx'\ x^2\delta(x-x')\ |x\rangle$$

$$= \int_{\infty}^{+\infty} dx\ x^2\,|x\rangle\langle x|$$

Note: Don't shy away from substituting a variable that is part of the label of a ket when performing the integral over a delta-function: That lable turns into the argument of a function whenever you apply a bra-vector to that ket!

Alternative:

$$\hat{X}^2 = \hat{X}\hat{X} = \left(\underbrace{\int_{-\infty}^{+\infty} dx\ \ x\,|x\rangle\langle x|}_{\hat{X}}\right)\left(\underbrace{\int_{-\infty}^{+\infty} dx'\ |x'\rangle\langle x'|}_{\hat{X}}\right)$$

$$= \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dx'\ \ x\,x'\ |x\rangle\underbrace{\langle x|x'\rangle}_{\delta(x-x')}\langle x'|$$

$$= \int_{-\infty}^{+\infty} dx\ \ x^2\,|x\rangle\langle x'|$$

6.2.3.5 Eigenstates of Position Operator

$$\hat{X}|x\rangle = \int dx'\, x' |x'\rangle\langle x'|x\rangle$$

$$= \int dx\, x' \;\delta(x'-x)\, |x'\rangle = x|x\rangle$$

The states |x> are eigenstates of the position operator to eigenvalues x.

6.2.3.6 Functions of Position operators

Later on, we will use the Hamilton operator that describes the dynamics of a point particle in a potential (1-dimensional)

The classical energy can be written as

$$E = \frac{p^2}{2m} + V(x)$$

where

p is the momentum of the particle
x its position
m its mass

We will turn this into a Hamilton operator of the form

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + V(\hat{x})$$

where

$\hat{p}$ is the momentum operator (to be defined in the next section)
$V(\hat{x})$ is the operator form of the potential (defined below)

We need to make clear what the symbol $V(\hat{x})$ is supposed to mean, and similar to the exponentiaton of the Hamilton operator to find the unitary time evolution operator, this meaning is delivered by doing a Taylor series expansion of the classical potential V(x) first:

$$V(x) = \sum_{n=0}^{\infty} \frac{1}{n!} c_n x^n$$ with some coefficients c_n

We can then define

$$V(\hat{x}) := \sum_{n=0}^{\infty} \frac{1}{n!} c_n \hat{x}^n$$

which is well defined as we can find readily

$$\hat{x}^n = \int_{-\infty}^{+\infty} dx \; x^n \, |x\rangle\langle x|$$

This actually helps to simplify the operator expression as follows:

$$V(\hat{x}) = \sum_{n=0}^{\infty} \frac{1}{n!} c_n \int_{-\infty}^{+\infty} dx \; x^n \, |x\rangle\langle x|$$

$$= \int_{-\infty}^{+\infty} dx \; \underbrace{\sum_{n=0}^{\infty} \frac{1}{n!} c_n x^n}_{= V(x)} \; |x\rangle\langle x|$$

$$= \int_{-\infty}^{+\infty} dx \; V(x) \; |x\rangle\langle x|$$

If we were to look for the full canonical coordinate representation, we can find it as

$$V(\hat{x}) \doteq V(x) \, \delta(x - x')$$