Propositional Logic Proofs Part3 Sequent Calculus

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[with material from “Mathematical Logic for Computer Science”, by Zhongwan, published by World Scientific]

Objectives

- Completeness Theorem Revisited
- Compactness Theorem and Applications
- Introducing Sequent Calculus

Axiomatic Deduction Revisited

- **Prove that $\vdash_H (\neg \neg A \Rightarrow A)$**
  Proof: (enough to prove that $\{\neg \neg A\} \vdash_H A$)
  1. $(\neg\neg A)$ *(by Assumptions)*
  2. $(\neg\neg A) \Rightarrow ((\neg\neg\neg\neg A) \Rightarrow (\neg\neg A))$ *(by Ax1)*
  3. $(\neg\neg\neg\neg A) \Rightarrow (\neg\neg A)$ *(by R1, (2), (1))*
  4. $((\neg\neg\neg\neg A) \Rightarrow (\neg\neg A)) \Rightarrow ((\neg A) \Rightarrow (\neg\neg\neg A))$ *(by Ax3)*
  5. $(\neg A) \Rightarrow (\neg\neg\neg A)$ *(by R1, (4), (3))*
  6. $((\neg A) \Rightarrow (\neg\neg\neg A)) \Rightarrow ((\neg\neg A) \Rightarrow A)$ *(by Ax3)*
  7. $(\neg\neg A) \Rightarrow A$ *(by R1, (6), (5))*
  8. A *(by R1, (7), (1))*
- The theorem can be used as an extra deduction rule

Axiomatic Deduction Revisited

- **Prove that $\vdash_H (A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$**
  Proof: (enough to prove that $\{(A \Rightarrow B)\} \vdash_H (\neg B \Rightarrow \neg A)$)
  1. $(A \Rightarrow B)$ *(by Assumptions)*
  2. $(\neg\neg A) \Rightarrow A$ *(by Theorem from Notes #5)*
  3. $(\neg\neg A) \Rightarrow B$ *(by Theorem 3.1, (2), (1))*
  4. $B \Rightarrow (\neg\neg B)$ *(by Theorem from A2, Question 2c)*
  5. $(\neg\neg A) \Rightarrow (\neg\neg B)$ *(by Theorem 3.1, (3), (4))*
  6. $((\neg\neg A) \Rightarrow (\neg\neg B)) \Rightarrow ((\neg B) \Rightarrow (\neg A))$ *(by Ax3)*
  7. $(\neg B) \Rightarrow (\neg A)$ *(by R1, (6), (5))*
- The theorem can be used as an extra deduction rule

Structural Induction Revisited /1

- **Solution Sketch for Assignment #1 Question 3:**

[25 marks] Let φ be a valid formula, let θ be an arbitrary substitution, and let t be an arbitrary truth evaluation. We want to show that $\theta(\varphi)^t = 1$. [5 marks]

Define a new truth evaluation s by $p^s := \theta(p)^t$, where p is an arbitrary propositional variable. We will show that $A^s = \theta(A)^t$ for all formulas A by using induction on the structure of formulas:

[Base Case] For all atoms p, by definition we have $p^s = \theta(p)^t$, as required. [5 marks]

[Inductive Step 1] For all formulas A with $A^s = \theta(A)^t$ we have

$$\begin{aligned}(\neg A)^s &= 1 - A^s \\ &= 1 - \theta(A)^t \\ &= (\neg\theta(A))^t \\ &= \theta(\neg A)^t\end{aligned}$$

and $(\neg A)^s = \theta(\neg A)^t$, as required. [5 marks]

Structural Induction Revisited /2

- **Solution Sketch for Assignment #1 Question 3:**

[Inductive Step 2] For all formulas A, B with $A^s = \theta(A)^t$ and $B^s = \theta(B)^t$ we have

$$\begin{aligned}(A \wedge B)^s &= \begin{cases} 1 & \text{if } A^s = B^s = 1 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } \theta(A)^t = \theta(B)^t = 1 \\ 0 & \text{otherwise} \end{cases} \\ &= (\theta(A) \wedge \theta(B))^t \\ &= \theta(A \wedge B)^t \end{aligned}$$

and $(A \wedge B)^s = \theta(A \wedge B)^t$ as required, and similarly for the other connectives. [5 marks]

Therefore $\theta(\varphi)^t = \varphi^s$, however $\varphi^s = 1$ since φ is valid. [5 marks] Thus $\theta(\varphi)^t = 1$, and since t was arbitrary this shows that $\theta(\varphi)$ is valid, as required.

Completeness Theorem: Revisited /1

- **Theorem 4.4.**
  - Let B be a formula such that $p_1, p_2, \ldots p_n$ are its only propositional atoms
  - Let k be any line in A's truth table for a valuation t
  - Let A_i equal p_i in line k if $p_i^t = 1$, or A_i equal $\neg p_i$ if $p_i^t = 0$, for all $1 \leq i \leq n$
  - It then follows that $\{A_1, A_2, \ldots A_n\} \vdash_H B$ is provable if the entry for B in line k valuates to true (i.e., $B^t \vDash 1$)
  - And that $\{A_1, A_2, \ldots A_n\} \vdash_H \neg B$ is provable if the entry for B in line k valuates to false (i.e., $B^t \vDash 0$)
- Proof:
  - By induction on the structure of B

Completeness Theorem: Revisited /2

- **Theorem 4.4.**
  - Proof by induction on the structure of B
- **(Base Case)**
  - If B is a propositional atom p then it follows that $B \vdash_H B$ and $\neg B \vdash_H \neg B$
- **(Inductive Case 1: $\neg$)**
  - If B is of the form $(\neg B_1)$ where B and B_1 have the same atomic propositions then we have two cases to consider
  - (Case 1.1) If B evaluates to 1 then B_1 evaluates to 0; hence $\{A_1, A_2, \ldots A_n\} \vdash_H \neg B_1$ based on the induction hypothesis, and since $B = (\neg B_1)$ then $\{A_1, A_2, \ldots A_n\} \vdash_H B$
  - (Case 1.2) If B evaluates to 0 then B_1 evaluates to 1; hence $\{A_1, A_2, \ldots A_n\} \vdash_H B_1$ based on the induction hypothesis, and since $B_1 = (\neg(\neg B_1))$ then $\{A_1, A_2, \ldots A_n\} \vdash_H (\neg(\neg B_1))$ and that is equivalent to $\{A_1, A_2, \ldots A_n\} \vdash_H (\neg B)$ since $B = (\neg B_1)$

Completeness Theorem: Revisited /3

- **(Inductive Case 2: $\Rightarrow$)**
  - Let $B = (B_1 \Rightarrow B_2)$ where B, B_1 and B_2 have the corresponding atomic propositions
  - (Case 2.1) If B evaluates to 0 then B_1 evaluates to 1 and B_2 to 0; hence $\{A_1, A_2, \ldots A_n\} \vdash_H \neg(B_1 \Rightarrow B_2)$ based on the induction hypothesis and proof that $\{B_1, \neg B_2\} \vdash_H \neg(B_1 \Rightarrow B_2)$ holds
  - If B evaluates to 1 then we have three sub cases
  - (Case 2.2) If B_1 evaluates to 0 and B_2 evaluates to 0 then $\{A_1, A_2, \ldots A_n\} \vdash_H (B_1 \Rightarrow B_2)$ based on the induction hypothesis and proof that $\{\neg B_1, \neg B_2\} \vdash_H (B_1 \Rightarrow B_2)$
  - (Case 2.3) If B_1 evaluates to 0 and B_2 evaluates to 1 then $\{A_1, A_2, \ldots A_n\} \vdash_H (B_1 \Rightarrow B_2)$ based on the induction hypothesis and proof that $\{\neg B_1, B_2\} \vdash_H (B_1 \Rightarrow B_2)$
  - (Case 2.4) If B_1 evaluates to 1 and B_2 evaluates to 1 then $\{A_1, A_2, \ldots A_n\} \vdash_H (B_1 \Rightarrow B_2)$ based on the induction hypothesis and proof that $\{B_1, B_2\} \vdash_H (B_1 \Rightarrow B_2)$
- **(Inductive Case 3 & 4: $\wedge$ & $\vee$)**
  - Similar proofs to the above

Compactness Theorem /1

- **Theorem 5.1. Compactness Theorem** (Theorem 6.1.1)
  - $\Sigma \subseteq$ Form (L^p) is satisfiable iff every finite subset Σ_0 of Σ is also satisfiable
- **Proof (LHS $\Rightarrow$ RHS):**
  - If Σ be satisfiable then Σ is consistent *(by Theorem 4.2)*
  - Let us assume that there exists a finite subset Σ_0 of Σ that is inconsistent; this in turn will make Σ inconsistent
  - That is a contradiction; hence, every finite subset Σ_0 of Σ is consistent
  - By the Soundness Theorem, if every finite subset Σ_0 of Σ is consistent then it is also satisfiable

Compactness Theorem /2

- **Theorem 5.1. Compactness Theorem** (Theorem 6.1.1)
  - $\Sigma \subseteq$ Form (L^p) is satisfiable iff every finite subset Σ_0 of Σ is also satisfiable
- **Proof (RHS $\Rightarrow$ LHS):**
  - If every finite subset Σ_0 of Σ be satisfiable then every finite subset Σ_0 of Σ is consistent *(by Theorem 4.2)*
  - Let us assume that Σ is inconsistent; then there exists a finite subset Σ_0 of Σ that is inconsistent
  - That is a contradiction; hence Σ is consistent
  - By the Completeness Theorem, if Σ is consistent then Σ is also satisfiable

Compactness Theorem /3

- **Corollary 5.2. Compactness Applied 1**
  - $\Sigma \vDash A$ iff there exists a finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \vDash A$
- **Theorem 5.3. Compactness Applied 2**
  - $\Sigma \vDash A$ iff there exists a finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \vdash_H A$
- Proof (LHS $\Rightarrow$ RHS):
  - Assume $\Sigma \vDash A$ then there exists a finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \vDash A$
  - By the Completeness Theorem, since $\Sigma_0 \vDash A$ then $\Sigma_0 \vdash_H A$
- Proof (RHS $\Rightarrow$ LHS):
  - Assume that there exists a finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \vdash_H A$
  - By the Soundness Theorem, since $\Sigma_0 \vdash_H A$ then $\Sigma_0 \vDash A$
  - Since mod(Σ) is the intersection of model sets of each of its formulas and since Σ_0 is finite then $\Sigma \vDash A$

Introducing Sequent Calculus /1

- **Deductive Inference:**
  - Proceeds from premises to a conclusion (e.g., MP rule)

$$\frac{A \qquad A \Rightarrow B}{B}(MP)$$

- **Reductive Inference:**
  - Proceeds backwards, from a conclusion (or goal sequent) to sufficient set of premises

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}(Cut)$$

  - Also known as proof search (basis for logic programming)

Introducing Sequent Calculus /2

- **Definition 5.4. Sequent: $\Gamma \vdash \Delta$ is called a sequent**
  - In a sequent, Γ, A means $\Gamma \cup \{A\}$
  - Both Γ and Δ are sets of formulas (can be empty), so $\Gamma \vdash \Delta$ implies $\{A_1, A_2, \ldots A_n\} \vdash \{B_1, B_2, \ldots B_n\}$
  - This is interpreted as $A_1 \wedge A_2 \wedge \ldots A_n \Rightarrow B_1 \vee B_2 \vee \ldots B_n$
  - **That is, all of the Ai's being true implies that at least one of the Bi's is true**
  - Instead of writing $\Gamma \cup \{A\}$ every time, we use "," instead to simplify the notation as Γ, A; applies to both sides

- **The Sequent Calculus System (LK)**
  - Abbreviation for Calculus of "**L**ogic **K**lassical" in German
    - German: Logistischer Klassischer Kalkül
  - Formal system for reductive inference

Introducing Sequent Calculus /3

- **Axiom:** $$\frac{}{\Gamma, A \vdash A, \Delta}\,(Axiom)$$
  - Also called the basic sequent
  - It is always true since all formulas in Γ, A (i.e., LHS) are true by definition, so hence at least A is true in A, Δ
- **Cut Rule:** $$\frac{\Gamma \vdash A, \Delta \qquad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\,(Cut)$$
  - To better understand the cut rule, consider it in its simplified form where Δ = B
  - That is, Γ is used to prove A, and Γ and A are used to prove B, so Γ on its own can be used to prove B

$$\frac{\Gamma \vdash A, B \qquad \Gamma, A \vdash B}{\Gamma \vdash \mathrm{B}}\,(Cut\ Simplified)$$

Introducing Sequent Calculus /4

- **Negation Rules:**

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, (\neg A) \vdash \Delta}(\neg \mathrm{L}) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash (\neg A), \Delta}(\neg \mathrm{R})$$

- **Implication Rules:**

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, (A \Rightarrow B) \vdash \Delta}(\rightarrow L)$$

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash (A \Rightarrow B), \Delta}(\rightarrow R)$$

Introducing Sequent Calculus /5

- **Conjunction Rules (to handle Hilbert Extensions):**

$$\frac{\Gamma, \mathrm{A}, \mathrm{B} \vdash \Delta}{\Gamma, (A \wedge B) \vdash \Delta}(\wedge L)$$

$$\frac{\Gamma \vdash \Delta, \mathrm{A} \qquad \Gamma \vdash \Delta, \mathrm{B}}{\Gamma \vdash \Delta, (A \wedge B)}(\wedge R)$$

- **Disjunction Rules (to handle Hilbert Extensions):**

$$\frac{\Gamma, \mathrm{A} \vdash \Delta \qquad \Gamma, \mathrm{B} \vdash \Delta}{\Gamma, (A \vee B) \vdash \Delta}(\vee L)$$

$$\frac{\Gamma \vdash \Delta, \mathrm{A}, \mathrm{B}}{\Gamma \vdash \Delta, (A \vee B)}(\vee R)$$

Introducing Sequent Calculus /6

- **Structural Rules (not necessary; formally shown):**

$$\frac{\Gamma \vdash \Delta}{\mathrm{A}, \Gamma \vdash \Delta}(Weaken - L) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}(Weaken - R)$$

$$\frac{\mathrm{A}, \mathrm{A}, \Gamma \vdash \Delta}{\mathrm{A}, \Gamma \vdash \Delta}(Contract - L) \qquad \frac{\Gamma \vdash \mathrm{A}, \mathrm{A}, \Delta}{\Gamma \vdash \mathrm{A}, \Delta}(Contract - R)$$

$$\frac{\Gamma, \mathrm{A} \vdash \Delta}{\mathrm{A}, \Gamma \vdash \Delta}(Exchange - L) \qquad \frac{\Gamma \vdash \Delta, \mathrm{A}}{\Gamma \vdash \mathrm{A}, \Delta}(Exchange - R)$$

Introducing Sequent Calculus /7

- **Hilbert Axiom proofs using Sequent Calculus:**
  - Axiom1 Proof:

$$\dfrac{\dfrac{\dfrac{}{p, q \vdash p}\,(Ax)}{p \vdash q \to p}\,(\to R)}{\vdash p \to (q \to p)}\,(\to R)$$

  - Recall:

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash (A \Rightarrow B), \Delta}(\to R)$$

Introducing Sequent Calculus /8

- **Hilbert Axiom proofs using Sequent Calculus:**
  - Axiom2 Proof:

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{p \to (q \to r), p \vdash p, r}\,(Ax) \qquad \dfrac{\dfrac{}{p, q \vdash p, r}\,(Ax) \qquad \dfrac{\dfrac{}{p, q \vdash q, r}\,(Ax) \qquad \dfrac{}{p, q, r \vdash r}\,(Ax)}{q \to r, p, q \vdash r}\,(\to L)}{p \to (q \to r), p, q \vdash r}\,(\to L)}{p \to (q \to r), p \to q, p \vdash r}\,(\to L)}{p \to (q \to r), p \to q \vdash p \to r}\,(\to R)
$$

$$
\dfrac{\dfrac{p \to (q \to r), p \to q \vdash p \to r}{p \to (q \to r) \vdash (p \to q) \to (p \to r)}\,(\to R)}{\vdash (p \to (q \to r)) \to ((p \to q) \to (p \to r))}\,(\to R)
$$

  - Recall:

$$
\dfrac{\Gamma \vdash A, \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, (A \Rightarrow B) \vdash \Delta}\,(\to L)
$$

Introducing Sequent Calculus /9

- **Hilbert Axiom proofs using Sequent Calculus:**
  - Axiom3 Proof:

$$
\dfrac{\dfrac{\dfrac{}{q, p \vdash p}(Ax)}{q \vdash (\neg p), p}(\neg R) \qquad \dfrac{\dfrac{}{q \vdash q, p}(Ax)}{q, (\neg q) \vdash p}(\neg L)}{\dfrac{\dfrac{\neg p \to \neg q, q \vdash p}{\neg p \to \neg q \vdash q \to p}(\to R)}{\vdash (\neg p \to \neg q) \to (q \to p)}(\to R)}(\to L)
$$

  - Recall:

$$
\frac{\Gamma \vdash A, \Delta}{\Gamma, (\neg A) \vdash \Delta}(\neg \mathrm{L}) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash (\neg A), \Delta}(\neg \mathrm{R})
$$

Introducing Sequent Calculus /10

- **Soundness of the Sequent Calculus (LK):**
  - For $A \in$ Form (L^p) and $\sum \subseteq$ Form (L^p), $\sum \vdash A \Rightarrow \sum \vDash A$
  - Proof by induction on the derivation of $\Gamma \vdash \Delta$ using the inductive hypothesis $\Gamma \vdash \Delta \Rightarrow$ mod(Γ) $\subseteq$ MOD(Δ), where MOD(Δ) is the union of mod(A_i)'s for all $A_i \in \Delta$
- **Completeness of the Sequent Calculus (LK):**
  - For $A \in$ Form (L^p) and $\sum \subseteq$ Form (L^p), $\sum \vDash A \Rightarrow \sum \vdash A$
  - Proof by reduction of a Hilbert System proof to LK proof, and then using the completeness of the Hilbert system
- **Cut Elimination Theorem:**
  - For every proof of $\Gamma \vdash \Delta$ in LK there exists another proof of the same sequent in LK – {Cut}

Food for Thought

- **Read:**
  - Chapter 6, Section 6.1 from Zhongwan
    - Read proofs discussed in class in more detail
    - Skip the material not related to propositional logic
  - Handout on “Classical Sequent Calculus”
    - Available from the course schedule web page
    - Read proofs discussed in class in more detail
- **Answer Assignment #2 questions**
  - Assignment #2 includes several practice exercises related to Axiomatic Deduction