Power in AC circuits revisited :

-- had looked at $P_R,\ P_L,\ P_C$

For the RLC circuit : $P(t) = i(t)v(t) = [I_0 \sin(\omega t - \phi)]V_0 \sin \omega t$

$$= I_0 V_0 \sin \omega t \sin(\omega t - \phi)$$

but $\sin(\omega t - \phi) = \sin \omega t \cos \phi - \cos \omega t \sin \phi$

$$\therefore P(t) = I_0 V_0 \sin^2 \omega t \cos \phi - I_0 V_0 \sin \omega t \cos \omega t \sin \phi$$

-- want $P_{ave} = \dfrac{1}{T}\int_0^T P(t)dt$

1^{st} *the* $\sin^2 \omega t$ *term :*

let $\omega t = \theta,\ \omega dt = d\theta;$ *upper limit :* $\theta = \omega T = 2\pi;$ *also* $\dfrac{1}{T} = \dfrac{\omega}{2\pi}$

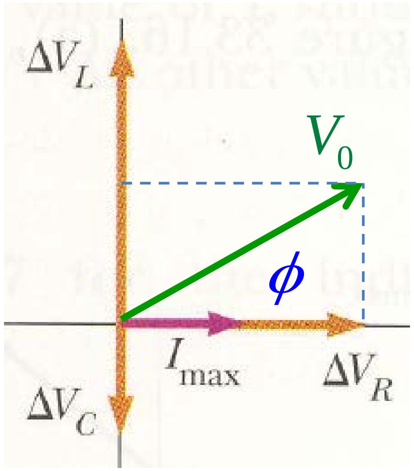
and $\dfrac{\omega}{2\pi}\int_0^{2\pi} \sin^2 \theta \dfrac{d\theta}{\omega} = \dfrac{1}{2\pi}\left(\dfrac{1}{2}\theta - \dfrac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi} = \dfrac{1}{2}$

In 2^{nd} *term* $\sin \omega t \cos \omega t = \dfrac{1}{2}\sin 2\omega t \to 0$ *when averaged over a cycle*

$$\therefore P_{ave} = \frac{1}{2} I_0 V_0 \cos\phi \quad or \quad P_{ave} = \frac{1}{2}\sqrt{2} I_{rms} \sqrt{2} V_{rms} \cos\phi = I_{rms} V_{rms} \cos\phi$$

$$\cos\phi \equiv power\ factor$$

Also :



$$V_0 \cos\phi = V_R = I_0 R$$

$$or \quad \frac{V_0 \cos\phi}{\sqrt{2}} = \frac{I_0 R}{\sqrt{2}} \qquad or \quad V_{rms} \cos\phi = I_{rms} R$$

$$so\ that\ or\ P_{ave} = I_{rms} V_{rms} \cos\phi$$

$$can\ also\ be\ written\ P_{ave} = I_{rms}^2 R$$

$$\Rightarrow on\ average\ all\ power\ is\ dissipated\ in\ R$$

Example

Calculate the power factor in the previous example.

$$Power\ factor = \cos\phi = \cos(-46.8^o) = 0.685$$

i.e. this indicates how much power is actually being used.

For example:

$-if\ \phi = 90^o\ then\ P_{ave} = 0,\ even\ though\ a\ voltage\ is$
$being\ applied\ and\ a\ curent\ flows$

$\rightarrow implications\ re\ hydro\ \cos t\,??$

Additional points re use of phasor approach:

1) *represent* $v(t)$, $i(t)$ *as complex numbers*

$$(j = \sqrt{-1} \text{ and } e^{j\theta} = \cos\theta + j\sin\theta\ )$$

$$v^C(t) = V_0(\cos\omega t + j\sin\omega t) = V_0 e^{j\omega t}$$

$$\text{and} \quad v(t) = V_0\cos\omega t = \mathrm{Re}[v^C(t)] = \mathrm{Re}[V_0 e^{j\omega t}]$$

$$\text{if} \quad v(t) = V_0\cos(\omega t + \phi)$$

$$v^C(t) = V_0[\cos(\omega t + \phi) + j\sin(\omega t + \phi)] = V_0 e^{j(\omega t + \phi)}$$

$$v(t) = V_0\cos(\omega t + \phi) = \mathrm{Re}[v^C(t)]$$

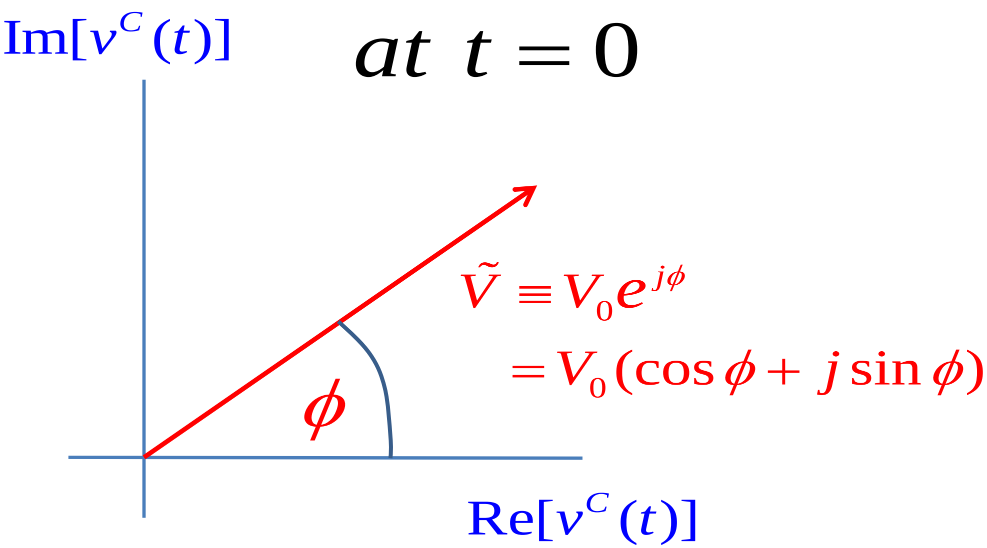
$$= \mathrm{Re}[V_0 e^{j(\omega t + \phi)}]$$

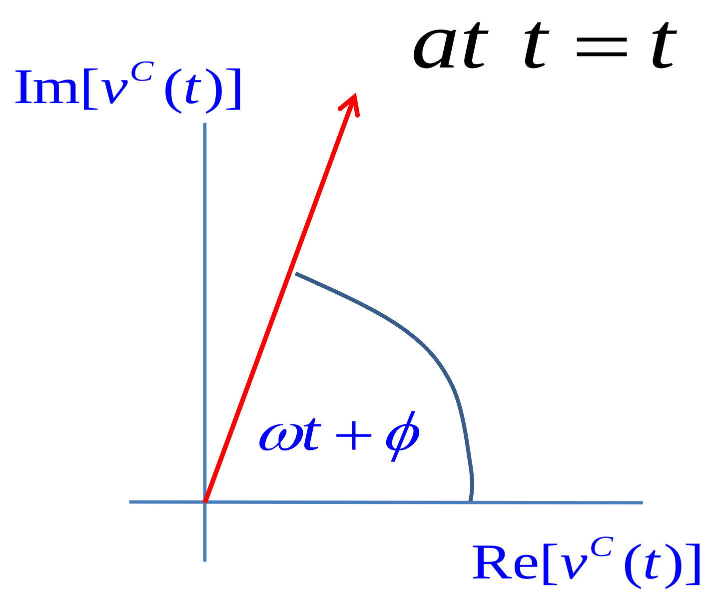
$$= \mathrm{Re}[V_0 e^{j\phi} e^{j\omega t}]$$

$$= \mathrm{Re}[\tilde{V} e^{j\omega t}]$$

where (V-tilde): $\tilde{V} \equiv V_0 e^{j\phi} \equiv$ *the phasor*

- *rotating vector or phasor diagram:*





$(V_0 e^{j\phi} \equiv \textit{the phasor})$

$--\textit{same for } i(t)$

2) *the real time voltages and currents are obtained by multiplying* $\tilde{V}$ *by* e^{jwt} *and taking the real part*

$$v(t) = \mathrm{Re}[\tilde{V} e^{j\omega t}] = \mathrm{Re}[V_0 e^{j\phi} e^{j\omega t}]$$

$$= \mathrm{Re}[V_0(\cos(\omega t + \phi) + j\sin(\omega t + \phi))]$$

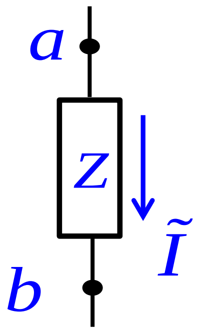
$$= V_0(\cos(\omega t + \phi)$$

3) *capacitors, inductors and resistors have impedances*

$$Z_C = \frac{1}{j\omega C} \qquad Z_L = j\omega L \qquad Z_R = R$$

j gives the phase shift that the element introduces

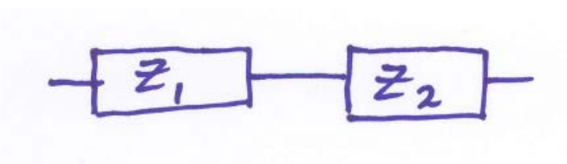
4) *can apply a generalized Ohm's Law:*



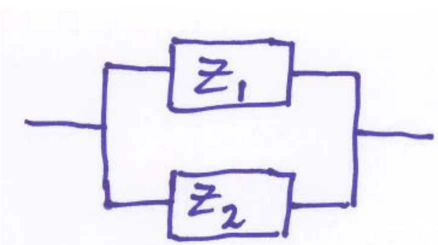
$$\tilde{V}_{AB} = Z\ \tilde{I}$$

also $\left|\tilde{V}_{ab}\right| = |Z|\left|\tilde{I}\right|$

5) *rules for combing impedances are the same as for resistors*



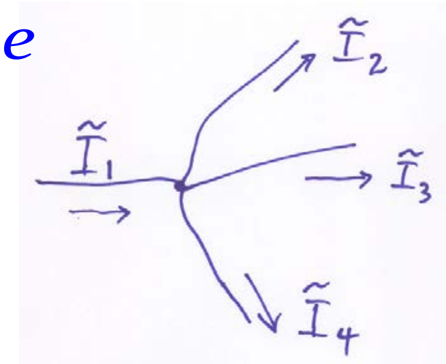
$Z = Z_1 + Z_2$



$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2}$

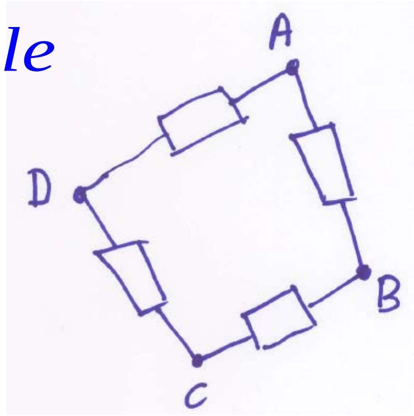
6) *Kirchhoff's rules apply to phasors*

KC rule



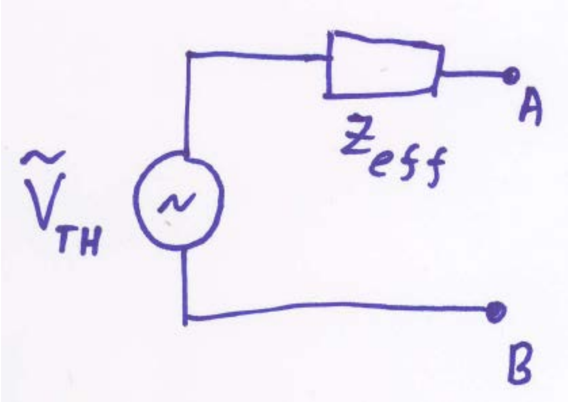
$$\tilde{I}_1 = \tilde{I}_2 + \tilde{I}_3 + \tilde{I}_4$$

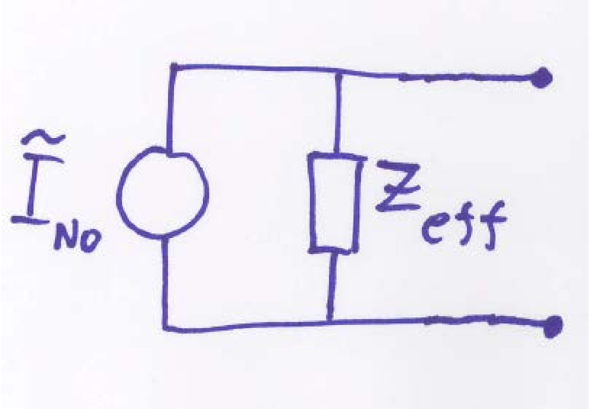
KV rule



$$\tilde{V}_1 + \tilde{V}_2 + \tilde{V}_3 + \tilde{V}_4 = 0$$

7) *generalized Thevenin's and Norton's equivalent circuits*





- *Ohms's Law always applies but subscripts cannot be mixed :*

  $$e.g.\ \ v_i^c(t) = i_i^c(t) Z_i^c;\ \ V_{rms} = I_{rms}\left|Z\right|;\ \ \tilde{V}_k = \tilde{I}_k Z_k^c\ \ (in\ this\ case\ put\ Z\ in\ polar\ form)$$

- *If addition or subtraction of complex numbers is involved use component or cartesian form;*

  $$e.g.\ \ \vec{Z} = R + jX$$

- *For multiplication or division of complex numbers use polar or exponential form;*

  $$e.g.\ \ \vec{Z} = Ze^{j\phi},\ \ v^c(t) = V_0 e^{j(\omega t+\phi)}$$

$Brief\ aside\ on\ Decibels\ (dB):$

$-used\ in\ electronics\ to\ characterize\ properties\ of\ filters,\ amplifiers$

$-used\ in\ audio\ to\ characterize\ sound\ intensity$

$dB\ compares\ power\ logarithmically:$

$$power\ ratio\ P_1\ to\ P_2\ in\ dB = 10\log_{10}(\frac{P_1}{P_2})$$

$$-say, P_1 = 2P_2,\ then\ C = 10\log_{10}(2) \simeq 3\ dB$$

$$-say, P_1 = \frac{P_2}{2},\ then\ C = 10\log_{10}(\frac{1}{2}) \simeq -3\ dB \qquad (the\ 3\ dB\ point)$$

- $dB\ in\ audio:$

$-\ sound\ intensity\ measured\ relative\ to$

$threshold\ of\ hearing\ intensity\ (P_2 = 10^{-12}\ W/m^2)$

$$Thus, \quad \frac{P_1\ (threshold\ of\ pain)}{P_2 (reference\ intensity)} = \frac{10\ W/m^2}{10^{-12}\ W/m^2} = 10^{13}$$

$$or\quad C = 130\ dB \qquad (13\ decades!)$$

- dB *in electronics*:

– *power gain or attenuation conveniently measured in dB*

– *useful since often* output power $\propto$ input power

- *can also be used to compare signal amplitudes*:

$$Since\ \ P = I^2 R \ or\ P = \frac{V^2}{R}$$

$$C = 10\log_{10}\left|\frac{P_1}{P_2}\right| = 10\log_{10}\left|\frac{\tilde{V}_1}{\tilde{V}_2}\right|^2 = 20\log_{10}\left|\frac{\tilde{V}_1}{\tilde{V}_2}\right|$$

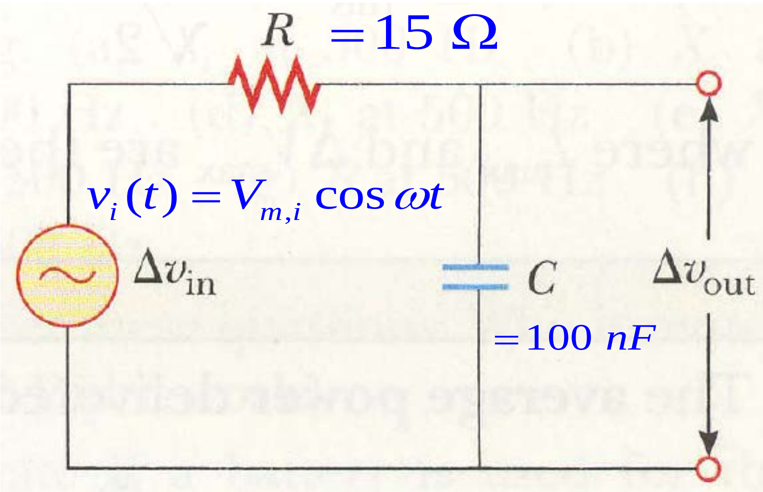
$$C_{voltage} = 20\log_{10}\left|\frac{\tilde{V}_1}{\tilde{V}_2}\right|$$

$$C = 10\log_{10}\left|\frac{P_1}{P_2}\right| = 20\log_{10}\left|\frac{\tilde{I}_1}{\tilde{I}_2}\right|$$

$$C_{current} = 20\log_{10}\left|\frac{\tilde{I}_1}{\tilde{I}_2}\right|$$

– – *end aside.*

Example $\quad Low - Pass\ Filter\ (RC\ low - pass\ filter)$

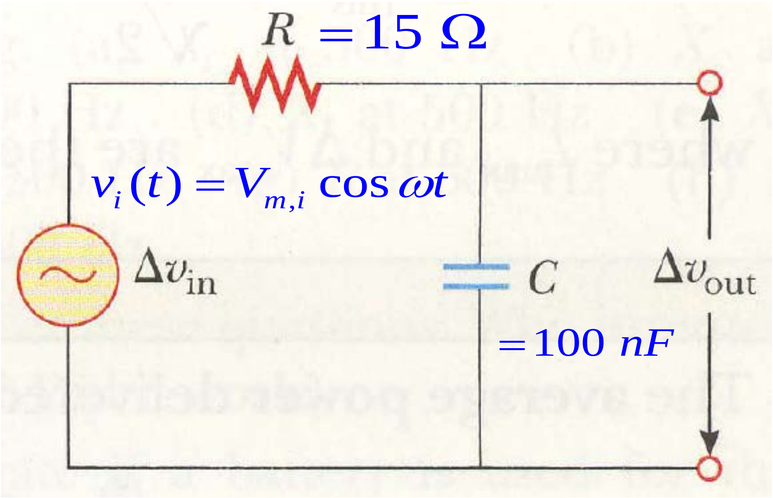


$V_{m,i} = 8\ V,\ \ f = 25\ kHz$

$Find\ \ a)\ Z_i^c(no\ load) = R + jX$

$b)\ draw\ the\ impedance\ diagram$

$and\ find\ \ \phi,\ Z_i = \left|Z_i^c\right|$



$V_{m,i} = 8\ V,\ \ f = 25\ kHz$

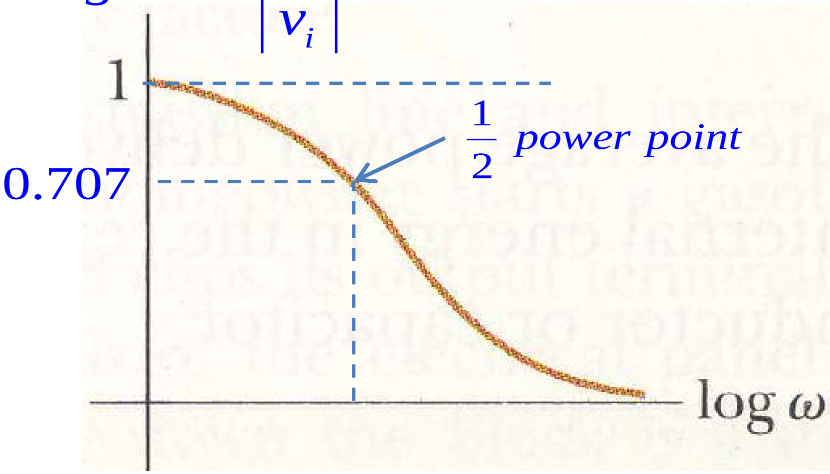
c) express Z_i^c in polar form $Z_i = \left|Z_i^c\right| e^{j\phi}$

d) find $i^c(t)$ (in phasor / exponential notation), and $i(t)$ for the circuit

e) find the gain $= \dfrac{v_o^{peak}}{v_i^{peak}} = \dfrac{V_{m,o}}{V_{m,i}} = \left|\dfrac{v_o}{v_i}\right|$;

plot gain versus ω

$$gain = \left|\frac{v_o}{v_i}\right|$$



$$gain = \frac{1}{\sqrt{1+(R\omega C)^2}}$$

$$dB = 10\log\frac{P_o}{P_i} \quad or \quad dB = 20\log\frac{V_o}{V_i}$$

$-$ for $-3dB$ point, the half $-$ power point:

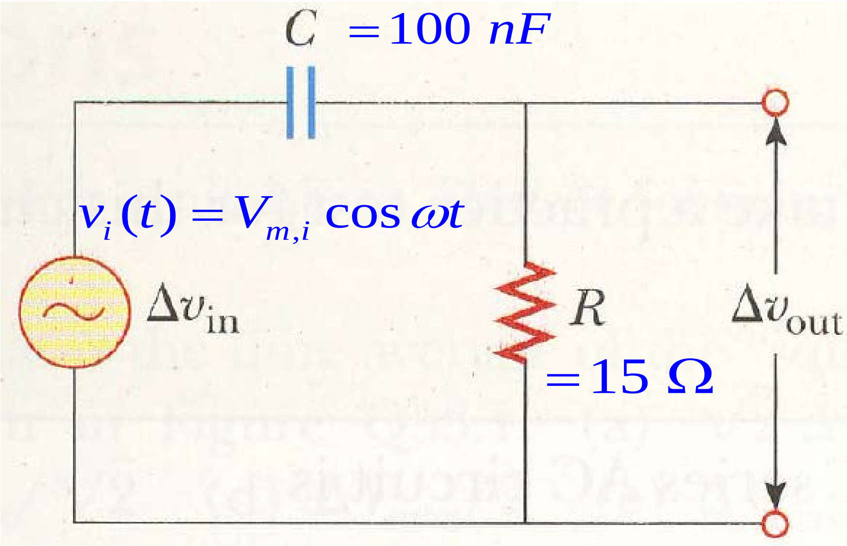
$$\left|\frac{v_o}{v_i}\right| = .707 \ at \ \omega = \frac{1}{RC}$$

$$bandwidth \equiv \omega_B = \frac{1}{RC} \quad (f_B =$$

f) find $v_o^c(t)$

Example

$High-Pass\ Filter\ (RC\ high-pass\ filter)$



$Find\ \ a)\ Z_i^c(no\ load) = R + jX$

$b)\ draw\ the\ impedance\ diagram$

$and\ find\ \phi,\ Z_i = \left|Z_i^c\right|$

$c)\ express\ Z_i^c\ in\ polar\ form\ Z_i = \left|Z_i^c\right| e^{j\phi}$

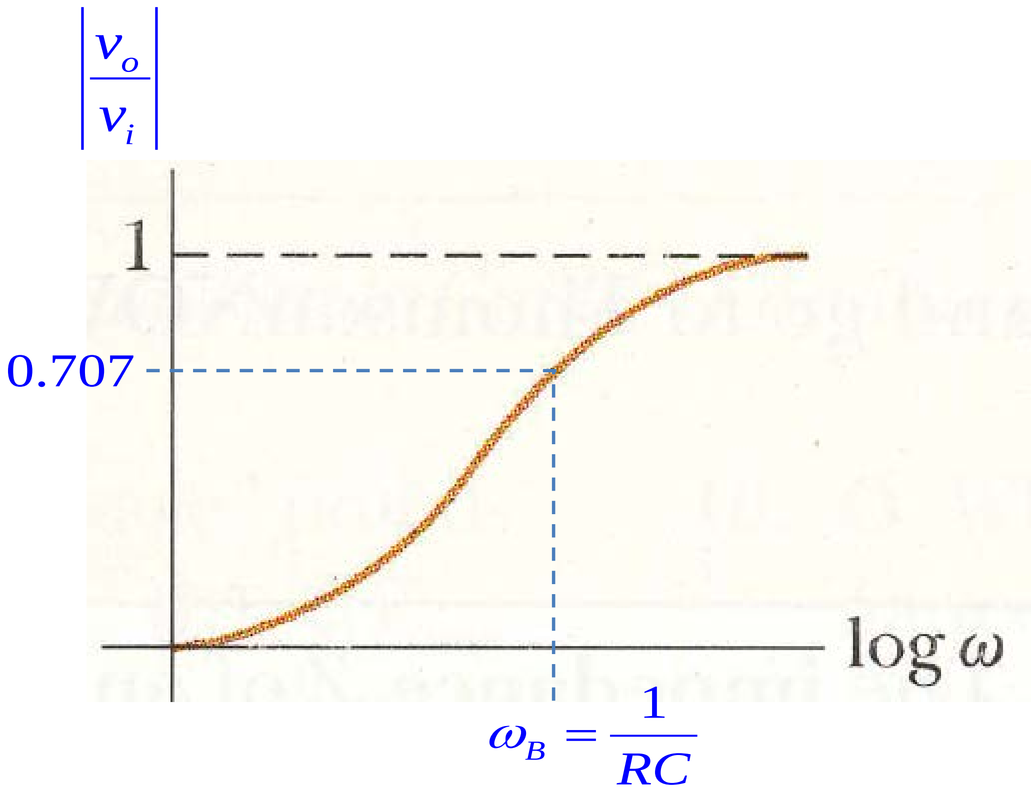
$d)\ find\ i^c(t)\ (in\ phasor/exponential$

$notation),\ and\ i(t)\ for\ the\ circuit$

$Parts\ a-d\ will\ be\ identical\ to\ those\ for\ the\ low-pass\ filter.$

$$e)\ find\ the\ gain = \frac{v_o^{peak}}{v_i^{peak}} = \frac{V_{m,o}}{V_{m,i}} = \left|\frac{v_o}{v_i}\right|;\ \ plot\ gain\ versus\ \omega$$

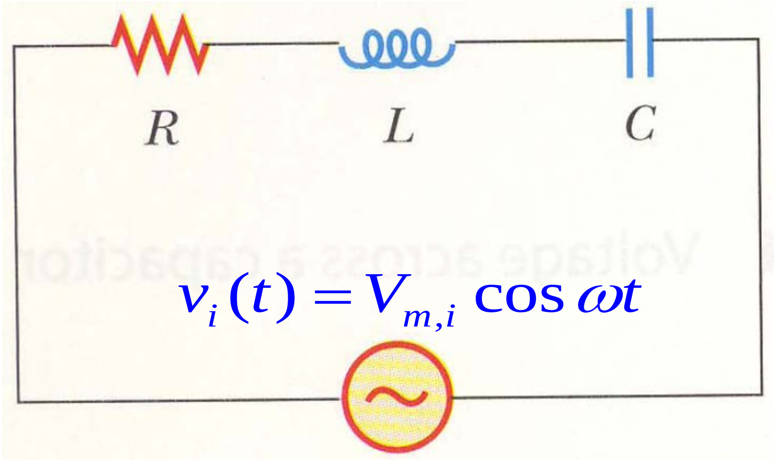
$$-assignment?\quad \left(\left|\frac{v_o}{v_i}\right| = \frac{\omega RC}{\sqrt{1+\omega^2R^2C^2}}\right)$$



$$\left|\frac{v_o}{v_i}\right| = \frac{\omega RC}{\sqrt{1+\omega^2 R^2 C^2}}$$

$f)$ *find* $v_o^c(t)$ $\quad$ (*assignment*?)

Resonance in a RLC Circuit



$v_i(t) = V_{m,i}\cos\omega t$

$$Z_i^c(no\ load) = R + jX$$

$$= R + j\omega L + \frac{1}{j\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

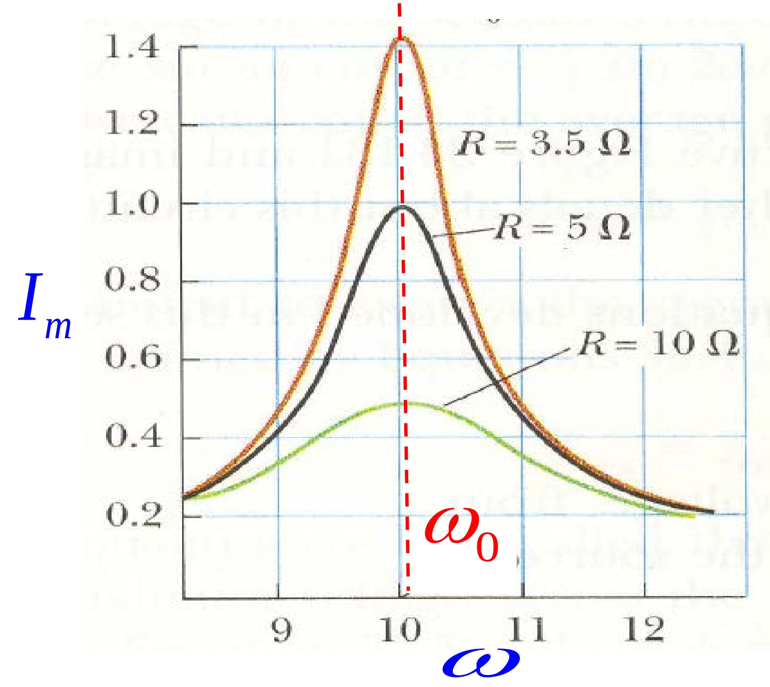
$$I_{m,i} = \frac{|\tilde{V}_i|}{|Z_i|} = \frac{V_{m,i}}{Z_i} = \frac{V_{m,i}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

$\therefore I_m$ *goes through a maximum*

for $\omega L - \dfrac{1}{\omega C} = 0$ *or*

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$\equiv$ *Resonance frequency*



–different "levels of resonance" depend on resistive losses

–the "quality" of the circuit

$$\text{Quality factor, } Q \equiv 2\pi \frac{E_S}{E_L} = 2\pi \frac{\text{Maximum energy stored per cycle}}{\text{Maximum energy lost per cycle}}$$

$$-\text{ for RLC circuit}: Q = 2\pi \frac{\frac{1}{2} L I_m^2}{\int_0^T i^2(t) R dt} = \frac{\pi L I_m^2}{R \int_0^T I_m^2 \cos^2(\omega t) dt} = \frac{\omega L}{R}$$

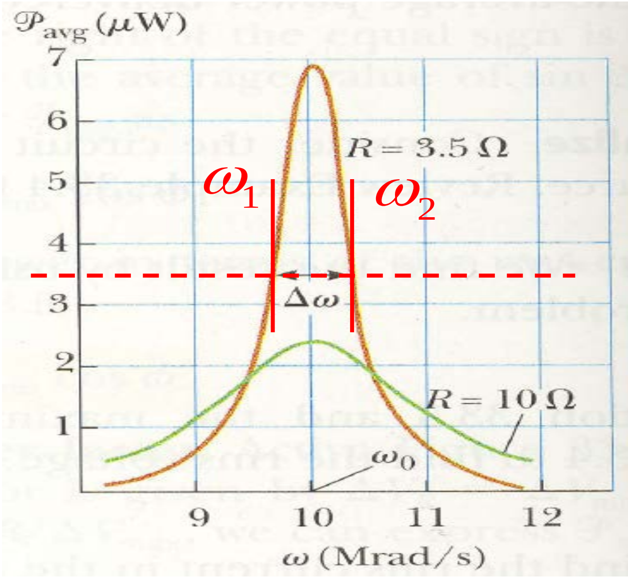
$$-\text{also } Q = \frac{\omega_0}{\Delta\omega} \text{ where } \Delta\omega \equiv \text{full width at half}-\text{power points}$$

$$\Delta\omega = \omega_2 - \omega_1 = \text{bandwidth}$$

$$\text{Consider } P_{ave} = I_{rms}^2 R = \frac{V_{rms}^2}{|Z|} R$$

$$= \frac{V_{rms}^2 R}{R^2 + (X_L - X_C)^2} = \frac{V_{rms}^2 R \omega^2}{R^2 \omega^2 + L^2(\omega^2 - \omega_0^2)^2}$$

$$\left(\text{Note } (P_{ave})_{max} = \frac{1}{2} \frac{V_{rms}^2}{R} \right)$$



Example

A certain receiving circuit in a radio consists of a series RLC circuit where the signal is detected across a resistor and for which $L = 5\ \mu H$*. Signals at various frequencies are being transmitted by a transmitter. We are interested in tuning in on a* $102\ MHz$ *signal in the presence of a* $103\ MHz$ *signal of equal power. In order to isolate the desired signal a variable capacitor is used and it is required that the detected power of the* $103\ MHz$ *signal, at* $102\ MHz$*, is no greater than* 50% *of the maximum of incoming power of the* $103\ MHz$ *signal.*

Find the minimum Q needed and the maximum R allowed.

(*Note that this does not represent good selectivity.*)

$Note:$

$$a)\ for\ a\ series\ LC\ circuit\ at\ resonance\ \omega_0 = \frac{1}{\sqrt{LC}}:$$

$$-then\ \ Z_{resonance} = \left|XX^*\right| = \omega_0 L - \frac{1}{\omega_0 C}\ \ = 0\ when\ \ \omega_0 = \frac{1}{\sqrt{LC}}$$

$$b)\ for\ a\ parallel\ LC\ circuit\ at\ resonance\ \omega_0 = \frac{1}{\sqrt{LC}}:$$

$$-then\ \ \frac{1}{Z_{resonance}} = \frac{1}{j\omega_0 L} + \frac{1}{(1/\ j\omega_0 C)} = j\omega_0 C + \ \frac{1}{j\omega_0 L} = \frac{-\omega_0^2 LC + 1}{j\omega_0 L}$$

$$or\ Z_{resonance} = \frac{j\omega_0 L}{1-\omega_0^2 LC}$$

$$\Rightarrow Z_{resonance} = \infty\ when\ \ \omega_0^2 LC = 1\ \ or\ when\ \ \omega_0 = \frac{1}{\sqrt{LC}},\ as\ before$$

Related uses of RLC circuits:

–selectively attenuate (at certain ω):

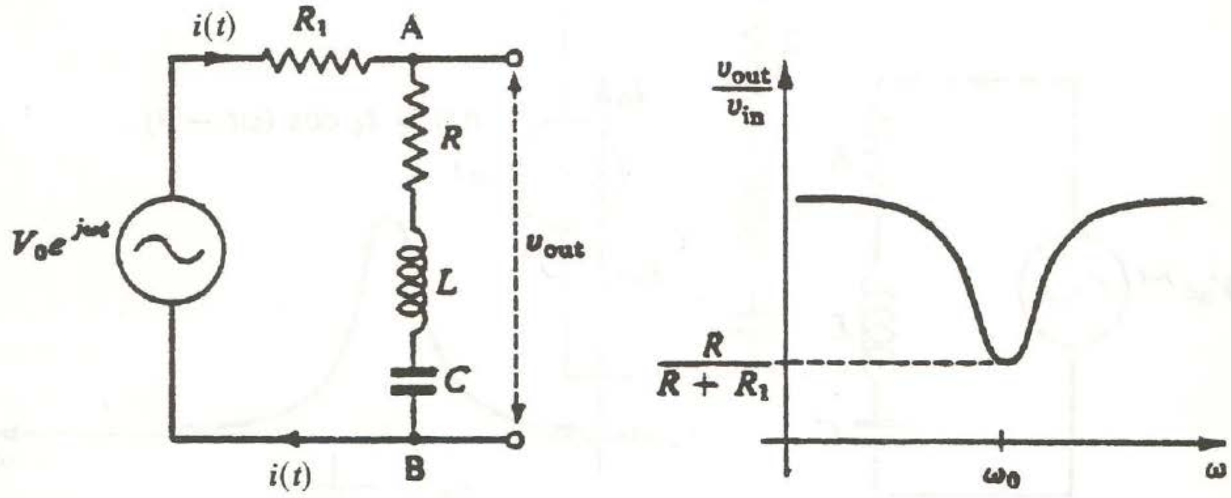


FIGURE 2.37 Series *RLC* circuit to attenuate frequencies near ω_0.

–select (at certain ω):

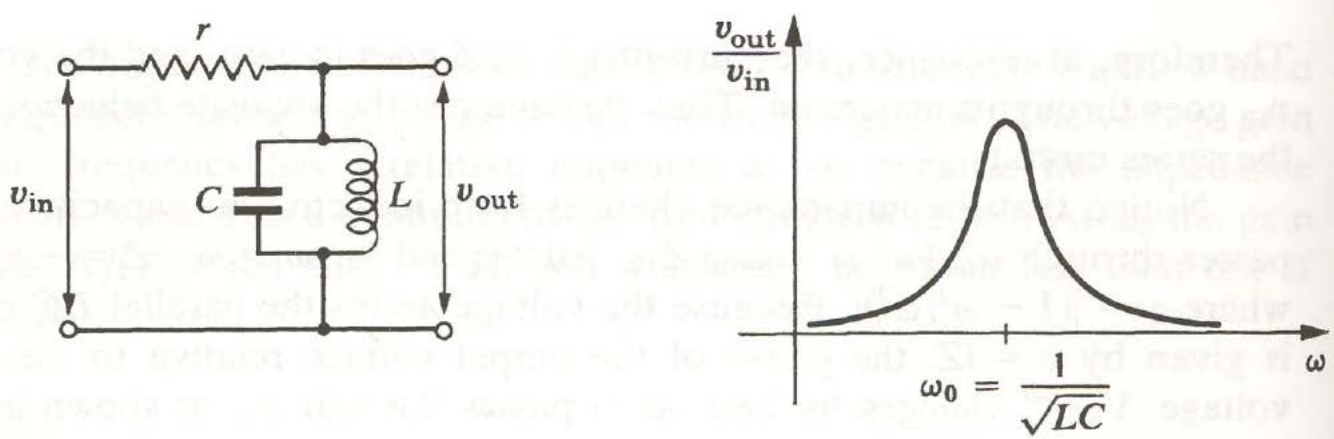


FIGURE 2.40 Parallel *LC* circuit used to select frequencies near ω_0.