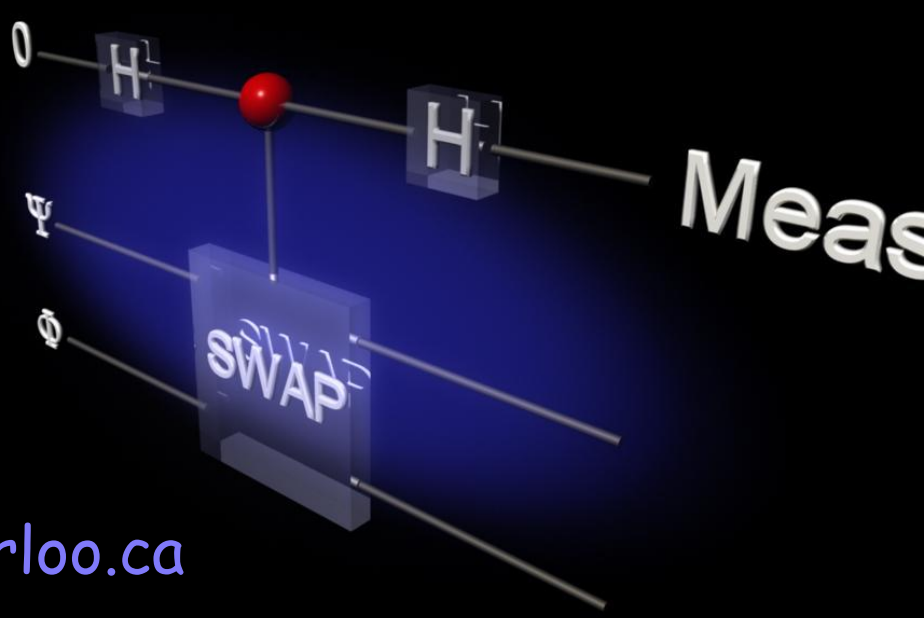
# Introduction to Quantum Information Processing




CS467 C&O481 PHYS467

*Lecture 6 (24 January 2013)*

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Tuesdays and Thursdays 10am-11:15am

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# Reading

- All of Chapters 1 and 2. From sections 3.1 till 3.5.2. Chapter 4. Sections 5.1 and 5.2.

# Projective measurements

- A Von Neumann measurement (also called  Lüders measurement) is a special kind of *projective measurement*: a complete projective measurement.

- Positive Operator Valued Measure (POVM) measurements are even more general.

- All of these more general notions of measurement can be derived using the 4 postulates we have presented.

- A formalism often used to talk about projective measurements is that of *measuring an "observable".*

# The observable formalism

We described measurements with respect to an orthonormal basis

$$\left\{\left|\psi_0\right\rangle,\left|\psi_1\right\rangle,\left|\psi_2\right\rangle,\cdots,\left|\psi_{N-1}\right\rangle\right\}$$

Given input state $\left|\Phi\right\rangle=\sum_{k=0}^{N-1}\beta_k\left|\psi_k\right\rangle$ $\qquad \beta_k=\left\langle\psi_k\middle|\Phi\right\rangle$

A Von Neumann measurement on this state wrt this basis yields outcome “k” with probability $\left|\beta_k\right|^2$ and in this case leaves the system in state $\left|\psi_k\right\rangle$

# The observable formalism

We can rephrase this statement in the observable formalism, by first adding a real-valued label $a_k$ corresponding to each outcome “k”.

We define the *observable* M, which will by construction be a Hermitian operator, to be $M = \sum_j a_j \left|\psi_j\right\rangle\left\langle\psi_j\right| = \sum_j a_j P_j$

For simplicity, let’s first consider the case that each eigenvalue occurs only once (i.e. no “degeneracies”).

# The observable formalism

- If there are no degeneracies, then to "measure the observable *M*" means to perform a Von Neumann measurement wrt the basis $\{|\psi_j\rangle\}$ and to output label $a_j$ if the outcome is $|\psi_j\rangle$.

Note that when measuring $|\Phi\rangle = \sum_j \beta_j |\psi_j\rangle$, we get outcome $a_j$ (or we could just say "j"), with probability

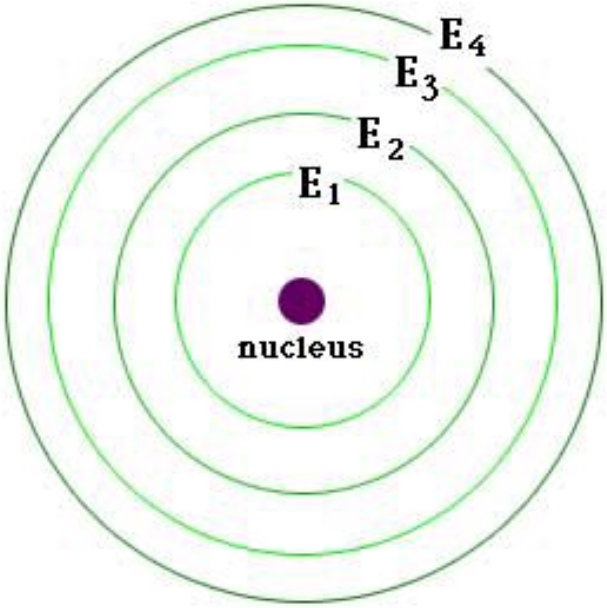
$$\Pr(j) = |\beta_j|^2 = |\langle\Phi|\psi_j\rangle|^2 = \langle\Phi|\psi_j\rangle\langle\psi_j|\Phi\rangle = \langle\Phi|P_j|\Phi\rangle$$

and the resulting state is .

$$\frac{P_j|\Phi\rangle}{\sqrt{\Pr(j)}} = \frac{\beta_j}{|\beta_j|}|\psi_j\rangle \cong |\psi_j\rangle$$

# The observable formalism

- Normally, the value $a_j$ corresponds to a relevant physical parameter. E.g. if one is measuring the energy level of an electron, the value $a_j$ could be the corresponding energy of the energy eigenstate .



# Example: 1-qubit measurement as measuring an "observable" (1/2)

- We have the projection operators $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$ satisfying

$$P_0 + P_1 = I$$

- We consider the projection operator or "observable"

$$M = 0P_0 + 1P_1 = P_1$$

What are the eigenvalues?

- When we measure this observable *M*, the probability of getting the outcome *b* is

$$\Pr(b) = |\alpha_b|^2 = |\langle \Phi | b \rangle|^2 = \langle \Phi | b \rangle \langle b | \Phi \rangle = \langle \Phi | P_b | \Phi \rangle$$

and we are in that case left with the state $\dfrac{P_b|\Phi\rangle}{\sqrt{\Pr(b)}} = \dfrac{\alpha_b}{|\alpha_b|}|b\rangle \approx |b\rangle$

# Example: 1-qubit measurement as measuring an “observable” (2/2)

- Equivalently, we could consider the observable

$$Z = P_0 - P_1 = |0\rangle\langle 0| - |1\rangle\langle 1|$$

which has eigenvalues +1 and -1, and the same eigenvectors as

$$M = P_1 = 0|0\rangle\langle 0| + 1|1\rangle\langle 1|$$

- Measuring the *Z* observable is also equivalent to a Von Neumann measurement in the computational basis.

- The only difference is that we associate a value of +1 to outcome $|0\rangle$ and a value of -1 to $|1\rangle$.

# Many observables correspond to equivalent Von Neumann measurements

Note that in this formalism, measuring the observable

$$M = \sum_k a_k P_k$$

where the $a_k$ are distinct, is equivalent to measuring the observable

$$M' = \sum_k b_k P_k$$

where the $b_k$ are distinct, up to a relabeling of the measurement outcomes.

- In many practical instances, the measuring apparatus outputs a sum (or average) of the eigenvalues of the results of many measurements.

  Therefore the actual values are important in these cases.

# The observable formalism

- In general, given the state $|\varphi\rangle$, measuring an observable $M = \sum_k a_k P_k$ (which might have degeneracies), we obtain outcome $a_j$ (or just "j") with probability

$$\Pr(j) = \langle \Phi | P_j | \Phi \rangle.$$

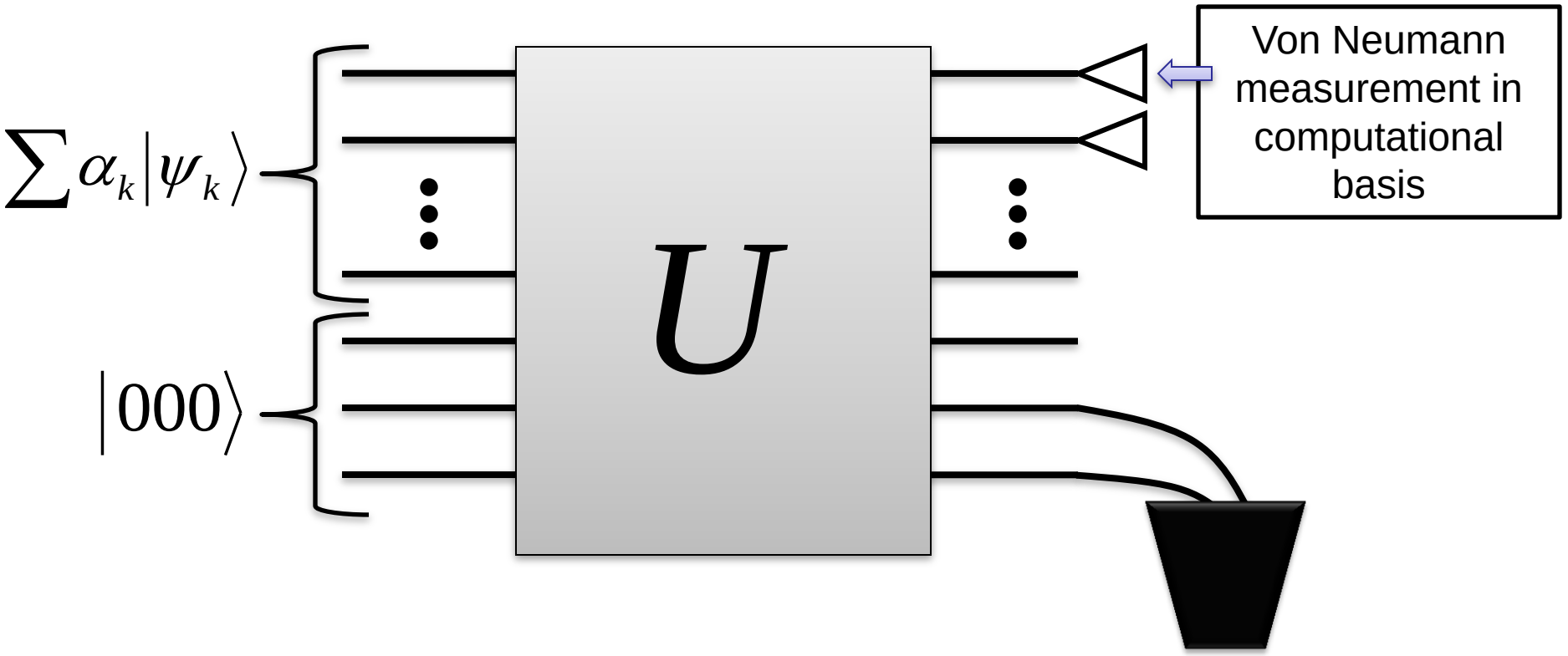
- The resulting state is

$$\frac{P_j |\Phi\rangle}{\sqrt{\Pr(j)}}.$$

- Note that one can conveniently express the expected value of the measurement of *M* as

$$\sum_k a_k \langle \Phi | P_k | \Phi \rangle = \langle \Phi | \left( \sum_k a_k P_k \right) | \Phi \rangle \quad = \langle \Phi | M | \Phi \rangle$$

# General measurement




More general notions of measurement can be derived from the simple Von Neumann measurement.

# Dealing with impure states …

When we measure part of a quantum state, the remaining state cannot in general be described as a “pure” state.

The resulting “mixed” states are best described in terms of a *density matrix*.

Example : Imagine we have a joint two-qubit system represented by a Bell state. Is the joint state completely known?

What is the state of the first qubit? Do we have any knowledge about it?

# Trace of a matrix

The trace of a matrix *A* is the sum of its diagonal elements

e.g.

$$Tr(A) = Tr\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} = a_{00} + a_{11} + a_{22}$$

Some properties:

$$Tr[xA + yB] = xTr[A] + yTr[B]$$

$$Tr[AB] = Tr[BA]$$

$$Tr[ABC] = Tr[CAB]$$

$$Tr[UAU^{\dagger}] = Tr[A]$$

Orthonormal basis $\{|\phi_i\rangle\}$

$$Tr[A] = \sum_i \langle \phi_i | A | \phi_i \rangle$$

# Density Matrices can describe pure states

$$\boxed{|\phi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle}$$

Notice that $\alpha_0 = \langle 0|\phi\rangle$, and $\alpha_1 = \langle 1|\phi\rangle$.

So the probability of getting 0 when measuring $|\phi\rangle$ is:

$$p(0) = |\alpha_0|^2 = |\langle 0|\phi\rangle|^2$$

$$= \langle 0|\phi\rangle\left(\langle 0|\phi\rangle\right)^* = \langle 0|\phi\rangle\langle\phi|0\rangle$$

$$= \langle 0\|\phi\rangle\langle\phi\|0\rangle = Tr\left(\langle 0\|\phi\rangle\langle\phi\|0\rangle\right)$$

$$= Tr\left(|0\rangle\langle 0\|\phi\rangle\langle\phi|\right) = Tr\left(|0\rangle\langle 0|\rho\right)$$

where $\rho = |\phi\rangle\langle\phi|$ is called the *density matrix* for the state $|\phi\rangle$

# Mixture of pure states

A state described by a state vector $|\phi\rangle$ is called a *pure state*.

What if we have a qubit which is known to be in the pure state $|\phi_1\rangle$ with probability $p_1$, and in $|\phi_2\rangle$ with probability $p_2$ ?

More generally, consider probabilistic mixtures of pure states (called *mixed states*):

$$\phi = \left\{ \left( |\phi_1\rangle, p_1 \right), \left( |\phi_2\rangle, p_2 \right), \ldots \right\}$$

State #1

Probability to be in state #1

# Density matrix can also describe mixed states

…then the probability of measuring 0 is given by conditional probability:

$$p(0) = \sum_i p_i \cdot \left(\text{prob.of measuring 0 given pure state} \left|\phi_i\right\rangle\right)$$

$$= \sum_i p_i \cdot Tr\left(\left|0\right\rangle\left\langle 0\right|\left|\phi_i\right\rangle\left\langle\phi_i\right|\right)$$

$$= Tr \sum_i p_i \left|0\right\rangle\left\langle 0\right|\left|\phi_i\right\rangle\left\langle\phi_i\right|$$

$$= Tr\left(\left|0\right\rangle\left\langle 0\right|\rho\right)$$

Density matrices contain all the useful information about an arbitrary quantum state.

where $\rho = \sum_i p_i \left|\phi_i\right\rangle\left\langle\phi_i\right|$ is the *density matrix* for the mixed state

# Density matrix

If we perform a Von Neumann measurement of the state $\rho = |\psi\rangle\langle\psi|$ wrt a basis containing $|\phi\rangle$, the probability of obtaining $|\phi\rangle$ is

$$|\langle\psi|\phi\rangle|^2 = Tr(\rho|\phi\rangle\langle\phi|)$$

# Density matrix

If we perform a Von Neumann measurement of the state $\{(|\psi_k\rangle, q_k)\}$ wrt a basis containing $|\phi\rangle$, the probability of obtaining $|\phi\rangle$ is

$$\sum_k q_k |\langle\psi_k|\phi\rangle|^2 = \sum_k q_k Tr(|\psi_k\rangle\langle\psi_k\|\phi\rangle\langle\phi|)$$

$$= Tr\left(\sum_k q_k |\psi_k\rangle\langle\psi_k\|\phi\rangle\langle\phi|\right)$$

$$= Tr(\rho|\phi\rangle\langle\phi|)$$

# Density Matrix

If we apply the unitary operation $U$ to $\left|\psi\right\rangle$ the resulting state is $U\left|\psi\right\rangle$ with density matrix

$$U\left|\psi\right\rangle\left(U\left|\psi\right\rangle\right)^{\dagger} = U\left|\psi\right\rangle\left\langle\psi\right|U^{\dagger}$$

# Density matrix

If we apply the unitary operation *U* to

$$\{(|\psi_k\rangle, q_k)\}$$

the resulting state is

$$\{(U|\psi_k\rangle, q_k)\}$$

with density matrix

$$\sum_k q_k U|\psi_k\rangle\langle\psi_k|U^\dagger = U\left(\sum_k q_k|\psi_k\rangle\langle\psi_k|\right)U^\dagger$$

$$= U\rho U^\dagger$$

# Density matrix

In other words, the density matrix contains all the information necessary to compute the probability of any outcome in any future measurement.

# Density matrix

Note that there are an infinite number of decompositions of a mixed state into a mixture of pure states.

These decompositions are all equivalent and indistinguishable.

Are there any “natural” or “special” decompositions? e.g. with a minimum number of terms?

# Spectral decomposition

- Often it is convenient to rewrite the density matrix as a mixture of its eigenvectors.

- Recall that eigenvectors with distinct eigenvalues are orthogonal; for the subspace of eigenvectors with a common eigenvalue (“degeneracies”), we can select an orthonormal basis.

# Spectral decomposition

- In other words, we can always “diagonalize” a density matrix so that it is written as

$$\rho = \sum_k p_k \left| \phi_k \right\rangle \left\langle \phi_k \right|$$

where $\left| \phi_k \right\rangle$ is an eigenvector with eigenvalue $p_k$ and forms an orthonormal basis $\left\{ \left| \phi_k \right\rangle \right\}$.

# Partial trace

- How can we compute probabilities for a partial system? Do we need to know the state of the whole system?

  Example:

$$\sum_{x,y} \alpha_{xy} |x\rangle |y\rangle$$

  - Suppose we are only able to interact with or measure the first system.
  - For convenience, denote $\boxed{|\Phi_y\rangle = \sum_x \frac{\alpha_{xy}}{\sqrt{p_y}} |x\rangle}$ $\quad p_y = \sum_x |\alpha_{xy}|^2$
  - So $\sum_{x,y} \alpha_{xy} |x\rangle |y\rangle = \sum_y \sqrt{p_y} |\Phi_y\rangle |y\rangle$

- E.g. suppose we wish to compute the probability of measuring $|w\rangle$ in the first register.

$$\Pr(w) = \sum_y \left|\alpha_{wy}\right|^2 = \sum_y p_y \left|\frac{\alpha_{wy}}{\sqrt{p_y}}\right|^2$$

$$= \sum_y p_y Tr\left(|w\rangle\langle w|\Phi_y\rangle\langle\Phi_y|\right)$$

$$= Tr\left(|w\rangle\langle w|\left(\sum_y p_y |\Phi_y\rangle\langle\Phi_y|\right)\right)$$

- So what really matters is $\sum_y p_y \left|\Phi_y\right\rangle\left\langle\Phi_y\right|$

- So we are interested in the map

$$\left(\sum_{x,y} \alpha_{xy} \left|x\right\rangle\left|y\right\rangle\right)\left(\sum_{v,z} \alpha^*_{vz} \left\langle v\right|\left\langle z\right|\right)$$

$$=\left(\sum_y \sqrt{p_y}\left|\Phi_y\right\rangle\left|y\right\rangle\right)\left(\sum_z \sqrt{p_z}\left\langle\Phi_z\right|\left\langle z\right|\right)$$

$$\mapsto \sum_y p_y \left|\Phi_y\right\rangle\left\langle\Phi_y\right|$$

- What is this map??

# Partial trace

- One way to describe this map is as the map

$$\sum_y \sqrt{p_y}\left|\Phi_y\right\rangle\left|y\right\rangle \mapsto \left\{\left(p_{y,}\left|\Phi_y\right\rangle\right)\right\}$$

*(can think of this as measuring the 2nd register, but not looking at the outcome)*

- Using density matrix representation for states:

$$\rho = \sum_{y,z} \sqrt{p_y p_z}\left|\Phi_y\right\rangle\left\langle\Phi_z\right| \otimes \left|y\right\rangle\left\langle z\right|$$

$$\mapsto \sum_y p_y \left|\Phi_y\right\rangle\left\langle\Phi_y\right| = Tr_2 \rho$$

# Partial trace

$\rho = Tr_2 \rho$ is in fact a linear map that takes bipartite states to single-system states

$$Tr_2\left(|i\rangle\langle k| \otimes |j\rangle\langle l|\right) = |i\rangle\langle k| \otimes Tr\left(|j\rangle\langle l|\right)$$

$$= |i\rangle\langle k| \otimes \langle l|j\rangle = \langle l|j\rangle |i\rangle\langle k|$$

Confirm that

$$\rho = \sum_{y,z} \sqrt{p_y p_z} |\Phi_y\rangle\langle\Phi_z| \otimes |y\rangle\langle z|$$

$$\mapsto \sum_{y,z} \sqrt{p_y p_z} |\Phi_y\rangle\langle\Phi_z| \otimes \langle z\|y\rangle = \sum_y p_y |\Phi_y\rangle\langle\Phi_y|$$

# Partial trace

- We can also trace out the first system.

$$Tr_1\left(|i\rangle\langle k| \otimes |j\rangle\langle l|\right) = Tr\left(|i\rangle\langle k|\right) \otimes |j\rangle\langle l|$$

- Back to our example:

$$\sum_{x,y} \alpha_{xy} |x\rangle|y\rangle = \sum_x \sqrt{p_x} |x\rangle|\Theta_x\rangle$$

# Partial trace using matrices

- Tracing out the 2nd system

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{Tr_2} \begin{bmatrix} Tr\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} & Tr\begin{bmatrix} a_{02} & a_{03} \\ a_{12} & a_{13} \end{bmatrix} \\ Tr\begin{bmatrix} a_{20} & a_{21} \\ a_{30} & a_{31} \end{bmatrix} & Tr\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} a_{00} + a_{11} & a_{02} + a_{13} \\ a_{20} + a_{31} & a_{22} + a_{33} \end{bmatrix}$$

# Distant transformations don't change the local density matrix

- Notice that a unitary transformation on the system that is traced out does not affect the result of the partial trace.

  i.e.

$$\sum_y \sqrt{p_y}\left|\Phi_y\right\rangle U\left|y\right\rangle \cong \left(I\otimes U\right)\rho\left(I\otimes U^{\dagger}\right)$$

$$\xrightarrow{Trace_2}\left\{\left(p_{y,}\left|\Phi_y\right\rangle\right)\right\}\cong \rho_2 = Tr_2\rho$$

*(can think of this as measuring the 2nd register **in any basis**, and not looking at the outcome)*

# Partial trace

- For example, consider tracing out by measuring the second qubit in the computational basis and ignoring the outcome

$$\alpha|00\rangle + \beta|11\rangle \xrightarrow{Tr_2} |\alpha|^2 |0\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1|$$

- In a different basis

$$\alpha|00\rangle + \beta|11\rangle = \frac{1}{\sqrt{2}}\left(\alpha|0\rangle + \beta|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$+ \frac{1}{\sqrt{2}}\left(\alpha|0\rangle - \beta|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$$

# Partial trace

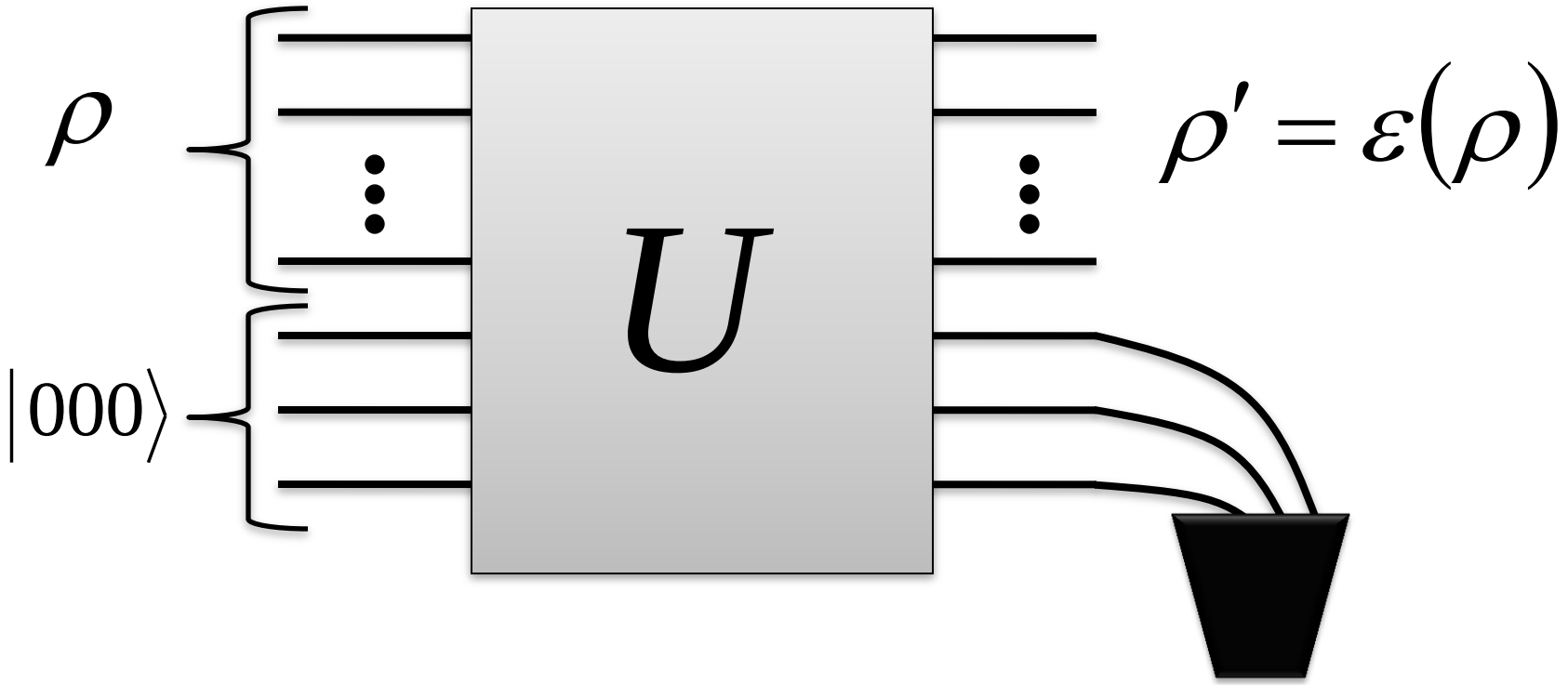
$$\frac{1}{\sqrt{2}}\left(\alpha|0\rangle+\beta|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle+\frac{1}{\sqrt{2}}|1\rangle\right)+\frac{1}{\sqrt{2}}\left(\alpha|0\rangle-\beta|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle-\frac{1}{\sqrt{2}}|1\rangle\right)$$

$$\xrightarrow{Tr_2}\frac{1}{2}\left(\alpha|0\rangle+\beta|1\rangle\right)\left(\alpha^*\langle 0|+\beta^*\langle 1|\right)$$

$$+\frac{1}{2}\left(\alpha|0\rangle-\beta|1\rangle\right)\left(\alpha^*\langle 0|-\beta^*\langle 1|\right)$$

$$=|\alpha|^2|0\rangle\langle 0|+|\beta|^2|1\rangle\langle 1|$$

# Aside:
# General operation




$$\rho \mapsto \rho' = Tr_2\left(\rho \otimes |000\rangle\langle 000|\right)$$

# Important

- Thus, any general quantum transformation on the traced out system, including measurement (without communicating back the answer) does not affect the partial trace.

- “Tracing out” the second system corresponds to discarding or ignoring the second system. Hypothetical operations, like measurements, on the second system might help with some mathematical or conceptual analysis, but they are not physically significant if the second system is truly isolated/discarded.

# Why?

- Operations on the 2nd system do not affect the statistics of any outcomes of measurements on the first system

- Note that if it were possible to affect the statistics non-locally, then a party in control of the 2nd system could instantaneously communicate information to a party controlling the 1st system.

# Schmidt decomposition theorem

- Consider a bipartite state

$$|\Psi\rangle = \sum_{x,y} \alpha_{xy} |x\rangle |y\rangle$$

- ***Theorem***: There exist orthonormal bases $\{|\phi_i\rangle\}$, $\{|\psi_j\rangle\}$ for the first and second system, respectively, and non-negative real numbers $p_0 \geq p_1 \geq \mathsf{L}$ such that

$$|\Psi\rangle = \sum_i \sqrt{p_i} |\phi_i\rangle |\psi_i\rangle$$

- The values $\sqrt{p_i}$ are called the *Schmidt coefficients*, and the number of non-zero values is called the *Schmidt number* of $|\Psi\rangle$.

# Schmidt decomposition theorem

- How do I find the Schmidt decomposition? How do I find the orthonormal bases?
- Note that if $|\Psi\rangle = \sum_i \sqrt{p_i}\,|\phi_i\rangle|\psi_i\rangle$ is the Schmidt decomposition, then

$$Tr_2|\Psi\rangle\langle\Psi| = \sum_i p_i |\phi_i\rangle\langle\phi_i|$$

will be diagonal in the Schmidt basis.

Thus diagonalizing $Tr_2|\Psi\rangle\langle\Psi|$ will reveal the Schmidt basis for the first system, as well as the Schmidt coefficients.

Once the first system is expressed in its Schmidt basis, the Schmidt basis for the second system is easily found.

A more efficient and elegant solution is given in N&C.

# Note that local transformations don't change the Schmidt coefficients

$$\sum_y \sqrt{p_y} U_A \left|\phi_y\right\rangle U_B \left|\psi_y\right\rangle = \sum_y \sqrt{p_y} \left|\phi_y'\right\rangle \left|\psi_y'\right\rangle$$

with new Schmidt bases:

$$\left\{\left|\phi_y'\right\rangle\right\} \left\{\left|\psi_y'\right\rangle\right\}$$

# Recall: Distant transformations don't change the local density matrix

- Recall that a unitary transformation on the system that is traced out does not affect the result of the partial trace
- I.e.

$$\sum_y \sqrt{p_y}\left|\phi_y\right\rangle U\left|\psi_y\right\rangle \cong \left(I\otimes U\right)\left|\Psi\right\rangle\left\langle\Psi\right|\left(I\otimes U^{\dagger}\right)$$

$$\xrightarrow{Trace_2} \sum_i \sqrt{p_i}\left|\phi_i\right\rangle\left\langle\phi_i\right| = Tr_2\left|\Psi\right\rangle\left\langle\Psi\right|$$

# Conversely:

- Any two purifications of the same dimension of the same state are equivalent up to a local unitary on the ancilla system.
- I.e. If

$$Tr_2 |\psi\rangle\langle\psi| = Tr_2 |\phi\rangle\langle\phi|$$

then

$$|\psi\rangle = (I \otimes U)|\phi\rangle$$

for some $U$

- Easy to prove with Schmidt decomposition
- Can be used to prove that “bit commitment” is impossible.

# Purification of a mixed state

- Suppose we have the mixed state

$$\left\{\left(\left|\phi_k\right\rangle, p_k\right) \middle| k = 1,2,\ldots,M\right\}, \left|\phi_k\right\rangle \in H = \mathbb{C}^N$$

- This state is described by the density matrix

$$\rho = \sum_{k=1}^{M} p_k \left|\phi_k\right\rangle\left\langle\phi_k\right| \qquad \rho \in L(H)$$

- A purification of this mixed state is a pure state $\left|\phi\right\rangle \in H_a \otimes H$ in some larger Hilbert space satisfying

$$\rho = Tr_a \left|\phi\right\rangle\left\langle\phi\right|$$

# Purification of a mixed state

$$\rho = \sum_{k=1}^{M} p_k \left|\phi_k\right\rangle\left\langle\phi_k\right|$$

- One example of a purification is

$$\left|\phi\right\rangle = \sum_{i=1}^{M} \sqrt{p_i}\left|i\right\rangle\left|\phi_i\right\rangle \qquad \left|\phi\right\rangle \in H_a \otimes H,$$

$$H_a = C^M$$

- How big does $H_a$ ***need*** to be??

# Recall: Spectral decomposition

We can diagonalize

$$\rho = \sum_{k=1}^{N} q_k |\psi_k\rangle\langle\psi_k|$$

where $|\psi_k\rangle$ is an eigenvector with eigenvalue $q_k$ and $\{|\psi_k\rangle\}$ forms an orthonormal basis.

We can assume, w.l.o.g. that

$$q_1 \geq q_2 \geq \cdots \geq q_N \geq 0$$

So

$$\rho = \sum_{k=1}^{s} q_k |\psi_k\rangle\langle\psi_k|$$

where $s$ is the number of non-zero eigenvalues (i.e. the rank) of $\rho$.

- Thus, a purification for is $\rho$

$$|\gamma\rangle = \sum_{i=0}^{s} \sqrt{q_i}\,|i\rangle|\psi_i\rangle$$

- Thus the dimension of $H_a$ does not need to be more than the rank of $\rho$.

- Exercise: the dimension of $H_a$ must be at least the rank of $\rho$

# Bit Commitment

**http://www.cs.uwaterloo.ca/~watrous/lecture-notes/519/19.ps**

- Alice has a bit $\mathbf{b}$ she wishes to commit to Bob
- But Alice does not want Bob to know what the bit is until she chooses to reveal it
- Bob wants to be assured that Alice doesn't change the value of the bit before it is revealed.
- Any protocol for bit commitment must be:
  - **Binding** (Alice cannot change the bit after she commits it)
  - **Concealing** (Bob cannot learn about the bit until it is revealed by Alice)

# Example

- Alice locks the one bit message in a box
- To reveal, she provides Bob a key to the box
- If Bob truly had no way to look inside the box without the key, this scheme would be concealing.
- If Alice could not remotely change the value of the bit, or via the key she provides Bob, this would be binding.

# No information theoretically secure classical bit commitment protocol

- No protocol can be both information theoretically concealing and binding

- It is possible to achieve information theoretically concealing, and computationally binding, or vice versa.

Can quantum mechanics help??

# Can quantum mechanics help?

A possible (incorrect) protocol

- To commit “0” Alice sends either $|0\rangle$ or $|1\rangle$ with equal probability

- To commit “1” Alice sends either $|+\rangle$ or $|-\rangle$ with equal probability

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \qquad |-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

# Can quantum mechanics help?

- To reveal, Alice sends Bob a classical description of which of the four states she sent; Bob checks her claim with a measurement

- Before the reveal phase, from Bob's perspective, he has

$$\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|$$

in either case, so the protocol is indeed concealing

# But is it binding?

- It might appear that if Alice sends Bob a description of a state different from what she originally sent, she will get caught with probability $1/2$
- This argument is flawed.
- Alice could implement her first step by sending Bob one half of the state

$$\sqrt{\frac{1}{2}}|0\rangle|0\rangle + \sqrt{\frac{1}{2}}|1\rangle|1\rangle \qquad \text{or} \qquad \sqrt{\frac{1}{2}}|0\rangle|+\rangle + \sqrt{\frac{1}{2}}|1\rangle|-\rangle$$

# But is it binding?

- Alice could implement her first step by sending Bob one half of the entangled state

$$\sqrt{\frac{1}{2}}|0\rangle|0\rangle + \sqrt{\frac{1}{2}}|1\rangle|1\rangle \qquad \text{or} \qquad \sqrt{\frac{1}{2}}|0\rangle|+\rangle + \sqrt{\frac{1}{2}}|1\rangle|-\rangle$$

- ***If*** she irreversibly measures her half of the state, the previous logic would apply.

- However, a dishonest Alice can maintain her half of the state without measuring it, and change it to the other entangled state at any time if she wishes, by a local operation. Then she can measure her half, and tell Bob she sent the state for the second half corresponding to the outcome of her measurement.

# But is it binding?

- Check

$$\sqrt{\frac{1}{2}}|0\rangle|0\rangle + \sqrt{\frac{1}{2}}|1\rangle|1\rangle = (H \otimes I)\sqrt{\frac{1}{2}}|0\rangle|+\rangle + \sqrt{\frac{1}{2}}|1\rangle|-\rangle$$

- Thus the given protocol is ***not*** binding.

- In fact, there cannot exist any quantum bit commitment scheme that is both information theoretically concealing and information theoretically binding.