PHYS 234: Quantum Physics 1

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These notes are based off of the electronic notes created by Professor Norbert Lutkenhaus. These notes were LaTeX'd by Dakota St. Laurent.

If two sets of lecture slides overlap, I will not include the overlapped material. That would be a giant waste of my time. I use the symbol $\surd$ to note that I've finished an example.

These notes are considerably fancier than the notes I made for PHYS 256. When I reference an equation, figure or a section number inside a paragraph, you can click on the number itself to take you to that section! Even in the table of contents, you can click on the section you want to take you there. Happy reading!

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1 The Photoelectric and Compton Effects

1.1 Classical Physics

Physics at the end of the 19th century was described by two distinct concepts: particles and waves. Particles follow the rules of classical mechanics, such as laws relating to energy, momentum, and what not. Light as a wave followed Maxwell's theory of light. As you may have learned in PHYS 256, light can carry energy and momentum. A series of experiments challenged the separation of particles and waves:

- Blackbody radiation (waves act like particles)
- Photoelectric effect (waves $\rightarrow$ particles)
- Compton effect (waves $\rightarrow$ particles)
- Davisson-Germer experiment (particles $\rightarrow$ waves)
- Double-slit experiment (particles $\rightarrow$ waves)

Point particles can be described by position and momentum, and have an associated energy (kinetic and/or potential). We can also describe collisions (the exchange of energy and momentum) accurately.

Waves can be described by amplitude over large regions. There is an energy distributed over the entire wave, and there are interference effects found between two waves.

1.2 The Photoelectric Effect

Check out this website for simulations of some of the experiments we talk about here. The following diagram shows an apparatus for the display of the photoelectric effect. However, note that the battery is an external voltage source and is not necessary for the effect to take place; this can be seen using the website's simulations.

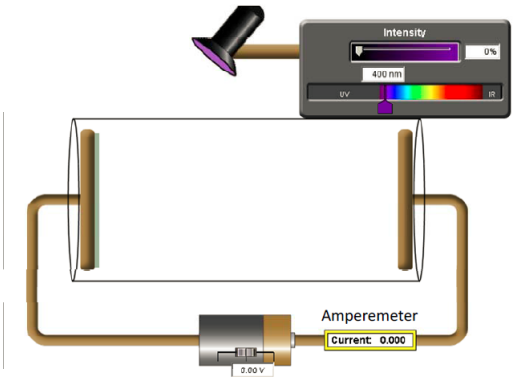


Figure 1: An example of a set-up for the photoelectric effect. As soon as light falls on the metal surface, a current flows and is measured by the ampmeter.

The classical explanation: the light (as a wave) is absorbed by the electrons in the metal. This energy transfer allows the electrons to escape the metal. The electrons are eventually captured by the opposing metal plate and return. This is measured as current. More detailed experimental observations noticed that:

- The experiment depends on the frequency of the light source: if the frequency falls below some value f_0 then no current flows
- The frequency cut-off exists independent of the intensity of the light; even very weak light sources will result in some current
- The effect is instantaneous; there is no delay between switching on the light and the emergence of current

Of course, these observations brought up some problems for the approach of classical explanations:

- Energy can be transmitted by light (waves) independent of the frequency, so there is no reason why a specific cut-off frequency should exist
- A higher intensity means more energy is deposited, regardless of the frequency

- As the electrons absorb the incoming light, there should be a delay between the light being switched on and the current starting to flow, especially for weak sources

If we add an external voltage source into the circuit, as seen in Figure 1, we find that a change of voltage leads to a change of the cut-off frequency, so we can rewrite the cut-off frequency as a function of voltage: $f_0(V)$. For any frequency, there is a voltage V_0 that just about stops the current (called the stopping voltage).

1.2.1 Einstein's Theory of the Photoelectric Effect (1905)

This is the postulate:

- Light comes in quantized units (the photon), rather than in a continuous form
- Each photon of frequency f carries an energy E_f given by the equation:

$$E_f = hf \tag{1.1}$$

Note that for visible light, $f \approx 10^{15}$ Hz so $E_f \approx 10^{-19}$ J.

1.2.2 The explanation of the photoelectric effect

One photon interacts with one electron. In the interaction, the photon is absorbed and transfers its energy to the electron. The electron uses this energy to free itself from the metal, and the remaining energy is used as kinetic energy for the electron to fly to the opposite metal plate:

$$E_f = W + E_k \tag{1.2}$$

where
W = the "work function"
E_k = electron's kinetic energy

The work function is simply what we have been discussing previously: the minimum energy required for the electron to escape. This equation can be rewritten, however. Since energy conservation tells us that the photon must deposit a minimum energy of W, so then $hf_0 = W \Leftrightarrow f_0 = \frac{W}{h}$.

In the case that the frequency of the photons are too low, the electron cannot leave the metallic surface and no current flows, regardless of how many photons hit the surface. Even if an electron absorbs a photon below the cut-off frequency, its energy will quickly dissipate through collisions, typically before it has a chance to absorb a second photon.

The additional voltage applied to the circuit can (but not always) make it more difficult for the electron to escape and reach the other plate. After escaping the surface, they must have sufficient kinetic energy to pass through the electrostatic potential eV, where e is the electric charge of the electron and V is the applied voltage. We then find equation (1.2) to become, after rearranging some terms:

$$V_0 = \frac{hf_0 - W}{e} \tag{1.3}$$

1.3 The Compton Effect

In the photoelectric effect, we treated light as being composed of individual light particles, called photons, that carry some energy. It then makes sense to think that the photons also have momentum.

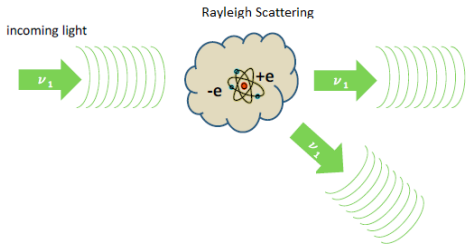


Figure 2: Rayleigh scattering. Incoming light from oscillating EM fields creates oscillating charges in atoms and molecules. This oscillation creates light at the same frequency, and can be radiated in new directions. The intensity of the scattering is proportional to f^4.

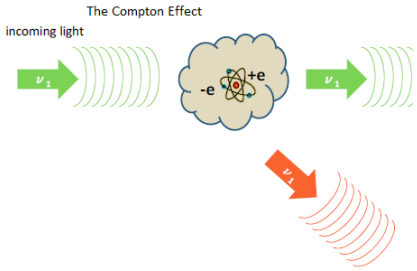


Figure 3: The Compton Effect. The scattered light has a different frequency; the frequency depends on the direction. A bigger deflection causes a bigger change in frequency.

Electromagnetic radiation is scattered by a target object. In classical theory, the charges in the target object will respond to the incoming wave and start to oscillate. All oscillating charges emit radiation at the frequency of oscillation, and this newly generated set of waves can also be detected at an angle θ with respect to the incoming wave. This classical model explains why the sky is blue and all that jazz. The scattering process itself, though, does not change the frequency of incoming and outgoing radiation.

However, an experimental problem occurred. In experiments with X-ray radiation on a graphene target, one observes that two separate frequencies at an angle θ result in different intensities. This effect is independent of the material, though intensities may vary.

1.3.1 Explanation via Quantum Theory

Some photons of the incoming light collide with electrons in the material as particles. For this, we need particles that are loosely bound to the nuclei and can be pushed around; graphene is a good material for this kind of thing. The process is similar to that found in classical mechanics: a game of pool. The moving ball (photon) hits a "resting" ball (electron); the electron gains energy and the photon loses energy. This causes the photon's frequency to change.

The exact amount of energy difference depends on the angle of the photon's deflection. We can derive it using relativistic forms of energy and what not, but we will not do that here. We find that:

$$p_f = \frac{hf}{c} \tag{1.4}$$

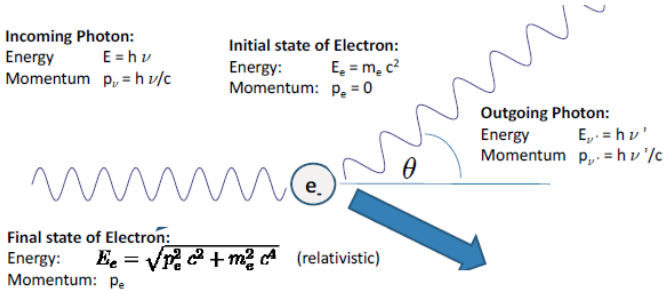


Figure 4: A photon bouncing off of an electron, as described previously.

We of course apply conservation laws and the fact that $\lambda = \frac{c}{f}$. We arrive at the following equations:

$$\Delta\lambda = \frac{h}{m_e c}(1 - cos(\theta)) = \lambda_c(1 - cos(\theta)) \tag{1.5}$$

$$\Delta f = f - f' \tag{1.6}$$

where

$\lambda_c = \frac{h}{m_e c} \approx 2.43 \times 10^{-12}$ m

e_m = electron's rest mass

The Compton Effect is negligible in the optical domain (visible wavelengths) but very relevant for the X-ray domain.

2 De Broglie Wavelength and the Davisson-Germer Experiment

We have shown that wave phenomena can exhibit particle features. We can rewrite the momentum instead as $p = \frac{h}{\lambda}$ using a simple wave relationship. There is nothing in this reformed equation that has to do with light. This led to the following postulate.

2.1 The De Broglie postulate (1924)

Particles can exhibit wave-like properties with an assigned wavelength, called the De Broglie wavelength:

$$\lambda = \frac{h}{p} \tag{2.1}$$

2.2 The Davisson-Germer Experiment

We must first understand the Bragg Grating; it is an optical filter that reflects particular wavelengths and transmits all others. Note that reflection, however, is common to both waves and particles.

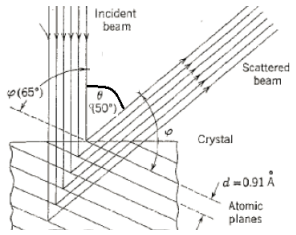


Figure 5: A set of Bragg gratings.

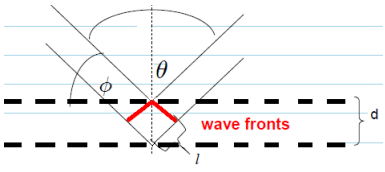


Figure 6: Another diagram.

From Figure 6 we can find that $\Delta = 2l = 2d\sin(\phi)$. Due to constructive interference being of the form $\Delta = n\lambda$, we find the Bragg condition for scattering maxima to be:

$$n\lambda = 2d\sin(\phi) \tag{2.2}$$

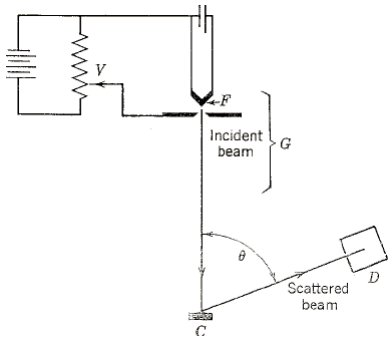


Figure 7: The set-up of the Davisson-Germer experiment. F is the source of the beams of electrons, C is a nickel Braff grating and D is a movable detector.

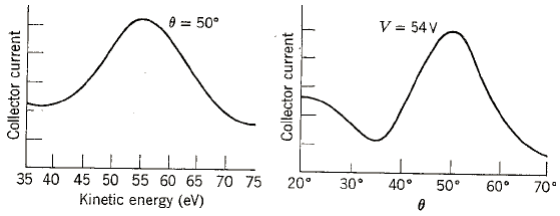


Figure 8: The results of the Davisson-Germer experiment.

Although it may not be obvious, the Davisson-Germer experiment confirmed that particles, such as electrons, exhibited wave-like properties, such as diffraction. (Wikipedia is always your friend!)

3 Young's and Stern-Gerlach Experiments

3.1 Optics Review

This is going to be very brief, but consider two light beams hitting the same spot on the screen. We find that the total electric field at that spot is simply the sum of the individual beams: $\vec{E_{tot}} = \vec{E_1} + \vec{E_2}$.

The intensity of the light at the same spot is proportional to the square of the electric field: $I \propto |\vec{E_1} + \vec{E_2}|^2 = |\vec{E_1}|^2 + |\vec{E_2}|^2 + 2\vec{E_1} \cdot \vec{E_2}$. The final term of this equation can be negative or positive, as it represents the interference of the two beams.

3.2 Young's Double Slit Experiment - Light

A double-slit set-up causes an interesting interference pattern, as seen below.

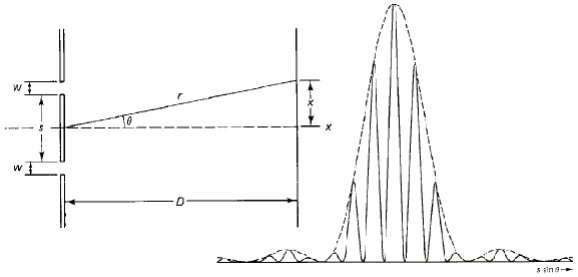


Figure 9: A double-slit experiment set-up and its resulting interference pattern. w is the slit width, s is the slit separation (center-to-center), D is the distance between the slits and the screen, r is the path length and x is the "height" of the image.

Please refer to the URL given in section 1.2 for simulations of this experiment. One of the most interesting things about this experiment is that it showed interference even with one individual photon; photons can interfere with themselves. The wave nature of the fields give rise to a probability distribution of where to find the photon.

3.3 Young's Double Slit Experiment - Particles

Electrons, when put through the same apparatus, display the same interference pattern. This even occurs when the number of electrons is increased by one every single time. A single electron can interfere with itself, as well.

However, attempting to monitor which slit the electron goes through destroys the interference pattern.

Interference is due to us knowing which paths the particles take. If we learn which paths they take (by using detectors at the slits), we no longer add amplitude but just the intensity on the screen for the two slits individually. (This will be explained later - it is due to observing the particles and thus changing the system.)

3.4 The Stern-Gerlach Experiment

This section deals with the build-up to the mathematical description of quantum mechanics. Imagine a compass needle in a homogeneous magnetic field: the needle will be parallel to the magnetic field lines. It may rotate, but there is no net-force. Thus, a moving compass needle will be unaffected by the field and follow its normal path. A theoretical explanation is that forces are generated by a gradient of energy for different configurations. We come to understand that the energy of a dipole in a magnetic field is:

$$E_{mag} = -\vec{\mu} \cdot \vec{B} \tag{3.1}$$

where
μ = magnetic dipole moment
B = magnetic field strength

However, if $\vec{B}$ is constant everywhere in space, then the total energy does not depend on the location of the dipole, but only on the orientation of the magnetic dipole.

Of course, if we are dealing with an inhomogeneous magnetic field, the energy typically does depend on the location of the dipole. In addition to the torque found within the homogeneous field, there will be a net-force on the magnetic dipole towards the locations in space with lower energy:

$$\vec{F} = -\vec{\nabla} E_{mag} \tag{3.2}$$

Depending on the relative orientation between the dipole moment and the magnetic field, the dipole is either:

- drawn towards stronger fields (parallel)

- drawn towards weaker fields (antiparallel)
- unaffected by any force (orthogonal)
- a combination of these three things (other orientations)

The Stern-Gerlach apparatus looks like this:

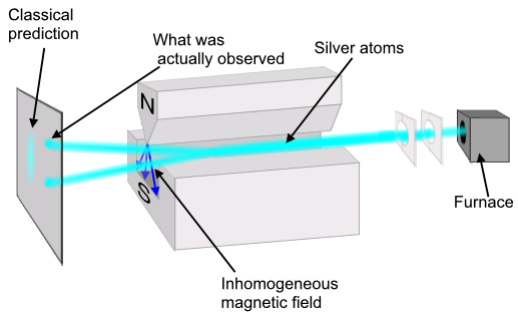


Figure 10: The set-up of the Stern-Gerlach experiment. Fascinating.

Classically, we would predict that all the (neutral) silver atoms to be distributed randomly but continuous. However, the atoms were either "oriented" up or down, as seen above. We can say that the Stern-Gerlach device is an instrument used to measure the components of the magnetic dipole moment with respect to the orientation of the device itself (the measurement direction).

3.5 Basic Probability Theory

Consider a fair coin toss with values h (heads) and t (tails). We of course know that the probability of either of these values is 50%. In other terms:

$$\lim_{n\to\infty} \frac{\text{number of heads}}{n} = \frac{1}{2}$$
$$\lim_{n\to\infty} \frac{\text{number of tails}}{n} = \frac{1}{2}$$

The definition of probability distribution comprises of the set of events (the event space), such as $X = \{x_1, \ldots, x_m\}$ and the probability of an event $x_i \epsilon X$ to occur, $p(x)$. A probability distribution has the following properties:

$$0 \leq p(x) \leq 1 \tag{3.3}$$

$$\sum_{x_i \epsilon X} p(x_i) = 1 \tag{3.4}$$

3.5.1 Joint and Marginal Probability Distribution

As an example, consider picking objects out of a hat: either balls (b) or cubes (c) that are either yellow (y) or green (g). Thus there are four possibilities: a green ball, a yellow ball, a green cube, or a yellow cube. Let us define the probabilities as such:

$$\mathcal{P}(b, y) = \frac{1}{3} \qquad \mathcal{P}(b, g) = \frac{1}{4}$$
$$\mathcal{P}(c, y) = \frac{1}{6} \qquad \mathcal{P}(c, g) = \frac{1}{4}$$

Now let us define three sets: $Y = \{\text{ball}, \text{cube}\}$, $Z = \{\text{yellow}, \text{green}\}$ and the combinations $X = \{\text{yellow ball}, \text{green ball}, \text{yellow cube}, \text{green cube}\}$. Each event x_i consists of a combination of an event in Y and an event in Z. There is a joint probability distribution $p_{YZ}(y, z)$ for $y_i \epsilon Y$ and $z_i \epsilon Z$. There is the normalization:

$$\sum_{y_i \epsilon Y} \sum_{z_i \epsilon Z} \mathcal{P}_{YZ}(y, z) = 1 \tag{3.5}$$

The marginal distribution is defined as:

$$\mathcal{P}_Y(y_i) = \sum_{z_i \epsilon Z} \mathcal{P}_{YZ}(y_i, z_i) = 1 \tag{3.6}$$

The probability to grab a ball is simply:

$$\mathcal{P}_Y(ball) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

Each marginal distribution has a normalization:

$$\sum_{y_i \epsilon Y} = 1 \tag{3.7}$$

3.5.2 Conditional Probability Distribution

The conditional probability is the probability that an object, for example, will be a ball, given that you know it is yellow. The equation for this is:

$$\mathcal{P}_{Y|Z}(y_i|z_i) := \frac{\mathcal{P}_{YZ}(y_i, z_i)}{\mathcal{P}_Z(z_i)} \tag{3.8}$$

There is also the normalization:

$$\sum_{y_i \epsilon Y} \mathcal{P}_{Y|Z}(y_i|z_j) = 1 \tag{3.9}$$

$\forall z_i \epsilon Z$.

4 The Stern-Gerlach Experiment

There are several observations that can be made using the Stern-Gerlach apparatus.

4.1 Experiment A (Observation 1)

Experiment A is a normal set-up, as seen in Figure 10, where the device is directed along the z "axis". On average, we find that 50% of the atoms are oriented up and oriented down. We could then describe the magnetic moment as:

$$\vec{\mu} = \begin{bmatrix} 0 & 0 & \pm A \end{bmatrix} = \pm A\hat{k} \tag{4.1}$$

Observation 1 shows that atoms are either pulled up or down (two discrete values) with equal probability.

4.2 Experiment B (Observation 2)

Experiment B is the exact same as Experiment A, except the device is directed for (along) the x "axis":

$$\vec{\mu} = \begin{bmatrix} \pm A & 0 & 0 \end{bmatrix} = \pm A\,\hat{i} \tag{4.2}$$

Observation 2 shows that the two discrete values (up or down) persists, independent of the orientation of the magnets. The atoms behave as if their magnetic dipole were either parallel or antiparallel to whatever measurement axis the device is oriented in.

4.3 Experiment C (Observation 3)

Experiment C is a bit different. It is best described using the figure below:

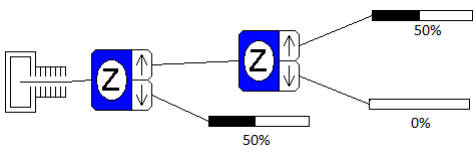


Figure 11: The apparatus for what we call Experiment C.

Observation 3 shows that two identically oriented Stern-Gerlach devices in sequence give the same result. We introduce the beginning of bra-ket notation here.

Postulate 1: The state of a quantum mechanical system, including all information that can be known about it, is represented mathematically by a normalized ket $|\psi\rangle$.

4.4 Experiment D (Observation 4)

Experiment D is identical to Experiment C, but the second z-oriented device is replaced by an x-oriented device.

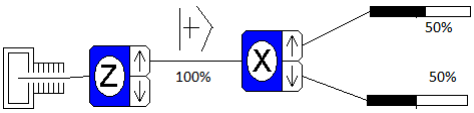


Figure 12: The apparatus for what we call experiment D. Here, we only consider the up-oriented atoms from the z-oriented device.

Observation 4 shows that, if there is a sequence of two Stern-Gerlach devices and second device is rotated with respect to the first one, we will still see the same splitting events (with respect to the second measurement direction), even if the devices are perpendicular.

4.5 Experiment E (Observation 5)

This last experiment (E) is arguably the most significant. In experiment D, we could still say that the X and Z measurements reveal independent properties. However, this is not the case:

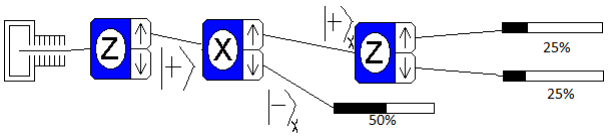


Figure 13: Let the mindfuckery sink in.

Classically, we would expect that all of the atoms are oriented upwards for their z-component. However, this does not happen. The measurement from the x-device makes the atoms "forget" what z-direction they were once

oriented. The history of the system is then irrelevant.

The measurements of x and z are incompatible; they do not correspond to independent properties but instead access the same system in an incompatible way. They are then called **incompatible observables**. Not being able to measure both components is a fundamental "problem" in quantum mechanics.

We should note here that the kets $|+\rangle$ and $|-\rangle$ are 2D unit vectors, similar to how $\hat{i}$, $\hat{j}$, and $\hat{k}$ are 3D unit vectors. $|+\rangle$ represents the state after an atom comes out of the top exit, whereas $|-\rangle$ represents the state for the bottom exit.

5 The Stern-Gerlach Experiment and Vector Spaces

5.1 The Stern-Gerlach Experiment (continued)

Notice that, in the set-up of Experiment E, we have only considered a single direction for the atoms coming out of the x-oriented device. Now, consider the following (BILL NYE SWAG):

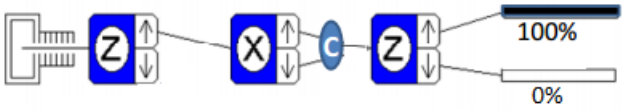


Figure 14: A sequence of Stern-Gerlach devices.

Classically, instead of what we see above, we would predict a total of 50% of the atoms ending in the up z-component: double of what we see in Figure 13 as both x-components are being considered. However, this is not the case. The end result looks identical to Figure 11 (if we only consider the top atoms in Experiment C). It is as if the x-oriented device is not even there; the atoms seem to remember which z-component they were assigned after the first-device.

Another interesting thing to note is that, when we compare Figure 13 and Figure 14, we see that there are more atoms in the down z-component in the former, even though there are more paths in the set-up of the latter. In other words, we have a situation where allowing more paths to reach a specific target results in less things reaching that target. Classical probability theory cannot explain this aspect of quantum mechanics.

Imagine what kind of procedure where combining two effects led to a negation, instead of an enhancement. This is identical to an idea found in optics: wave interference. Similar to what we did in section 3.1, we add amplitudes and take the square of them to find a resultant probability. This introduces interference effects. **In quantum mechanics, measurement is an interaction that enables someone to determine which path a system took.**

5.2 An Intro to Quantum Mechanics

For the motivation behind all of this, refer to section 3.1. Recall the ket $|\psi\rangle$. The generic label for the ket is the "state vector". We give it a linear vector space structure to be able to sum up possible paths. We make sure that observed quantities are quadratic in the state vector to get interference terms (as seen in section 3.1).

5.3 Dirac Notation and Linear Algebra

There are several things to discuss! We will also go through a few definitions; however, I will not go through the most basic definitions. If no direction (subscript) is given on a ket, we assume it is in the z-direction.

Dimensions: for a number of dimensions d, it is ideal to choose the number of mutually exclusive outcomes in a measurement (after checking that for each outcome, there exists a source which always triggers that outcome). In the previous Stern-Gerlach experiments discussed, we should choose $d = 2$.

Dirac Notation: Kets: Mathematically, the ket $|\psi\rangle$ is simply a column vector. Instead of writing $\vec{A} = A_x e_x + A_y e_y + A_z e_z$, where e_i are unit vectors for the x, y and z directions respectively, we write:

$$|A\rangle = A_x|e_x\rangle + A_y|e_y\rangle + A_z|e_z\rangle = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

Dirac Notation: Bras: The ket is not alone in its silly notation. We define the bra, denoted $\langle\phi|$, as a row vector of complex conjugates. We must first assume the components of A are complex ($A_i \, \epsilon \, \mathbb{C}$). Then,

$$\langle A| = A_x^*\langle e_x| + A_y^*\langle e_y| + A_z^*\langle e_z| = \begin{pmatrix} A_x^*, & A_y^*, & A_z^* \end{pmatrix}$$

Dirac Notation: Bra-kets: We can combine the bras and kets. Let $B = B_x e_x + B_y e_y + B_z e_z$, where A_i, $B_i \, \epsilon \, \mathbb{C}$. Then,

$$\langle A|B\rangle = \begin{pmatrix} A_x^*, & A_y^*, & A_z^* \end{pmatrix} \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = A_x^* B_x + A_y^* B_y + A_z^* B_z \qquad (5.1)$$

This, of course, can generalize to i elements for each vector A and B. There are a few interesting properties of bras and kets.

Basis: a basis is a set of linearly independent vectors that can represent any vector within a vector space via a linear combination.

Orthonormality: two vectors are orthonormal if they are unit vectors and if they are orthogonal (their dot product is 0).

Orthonormal basis: A basis for a vector space whose vectors are orthonormal. If the basis is, for example, the set of kets $\{|\psi_i\rangle\}_{i=1}^{d}$, then:

$$\langle\psi_i|\psi_j\rangle = \delta_{ij}$$

(Note that we make use of the Kronecker Delta, which can be read about here). We can expand any state vector as a combination of orthonormal basis vectors:

$$|\psi\rangle = \sum_{i}^{d} \alpha_i|\psi_i\rangle = \sum_{i}^{d} \langle\psi_i|\psi\rangle|\psi_i\rangle$$

(After skipping some steps, of course). Where $\alpha_i \, \epsilon \, \mathbb{C}$.

Normalization: We must have:

$$\sum_{i=1}^{d} |\langle\psi_i|\psi\rangle|^2 = 1$$

$$\langle\psi|\psi\rangle = \sum_{i=1}^{d} |\alpha_i|^2 = |\alpha_1|^2 + |\alpha_2|^2 + \dots$$

Where $\langle\psi|\psi\rangle = 0 \iff \alpha_i = 0 \; \forall i$. From here on, we will only be considering the 2D case (as seen in the Stern-Gerlach experiments).

Dual vector spaces: The vector space of kets has an associated dual vector space with the set of bras. This means that each element of the ket vector space is one-to-one with the set of bras. We must note that:

$$|+\rangle \longleftrightarrow \langle+| \quad \text{and} \quad |-\rangle \longleftrightarrow \langle-|$$

In other words, the unit vectors map to themselves. Other properties are:

$$\left.\begin{aligned}\langle+|+\rangle = 1\\ \langle-|-\rangle = 1\end{aligned}\right\} \quad \text{Normalization}$$

$$\left.\begin{aligned}\langle +|-\rangle &= 0\\ \langle -|+\rangle &= 0\end{aligned}\right\} \quad \text{Orthogonality}$$

And also that:

$$\begin{aligned}\langle +|\psi\rangle &= \langle +|(a|+\rangle + b|-\rangle)\\ &= \langle +|a|+\rangle + \langle +|b|+\rangle\\ &= a\langle +|+\rangle + b\langle +|-\rangle\\ &= a\end{aligned}$$

Similarly, $\langle -|\psi\rangle = b$. Using this method, we can also find that $\langle \psi|+\rangle = a^*$ and that $\langle \psi|-\rangle = b^*$. We can generalize this:

$$\langle \phi|\psi\rangle = \langle \psi|\phi\rangle^* \tag{5.2}$$

5.4 Measurement Probabilities

We can apply bra-ket notation to the Stern-Gerlach experiment:

Postulate: For a given input state $|\psi\rangle$ and a measurement with output states $|+\rangle$ and $|-\rangle$, the measurement results occur within these probabilities:

$$\mathcal{P}_{up} = |\langle +|\psi\rangle|^2 \tag{5.3a}$$
$$\mathcal{P}_{down} = |\langle -|\psi\rangle|^2 \tag{5.3b}$$

This can be extended to a general case with output states $|a_i\rangle$.

5.5 Quantum Mechanical Postulates and Formalism

Consider a simple Stern-Gerlach set-up (as seen in Figure 10). The incoming (input) state can be expanded as such:

$$|\psi\rangle = a|+\rangle + b|-\rangle \tag{5.4}$$

In this equation, a, b are probability amplitudes for the events $+$ and $-$ respectively. The probability for each respective event is simply $|a|^2$ and $|b|^2$. This is what we call postulate 4:

5.5.1 Postulate 4

Postulate 4 is defined as such: For an input state described by a (normalized) ket $|\psi\rangle$ and a measurement with mutually exclusive events i described by elements of an orthonormal basis $\{|\psi_i\rangle\}$ where $i = 1, \ldots, d$, then the probability $\mathcal{P}(i)$ to observe the outcome i is given by:

$$\mathcal{P}(i) = |\langle\phi_i|\psi\rangle|^2 \tag{5.5}$$

5.5.2 Application to Experiment C

Consider Experiment C, as seen in Figure 11, but let us ignore the down atoms from the first device. Then, for the second device, there is only an input $|+\rangle$, with possible outcomes $|+\rangle, |-\rangle$. It is "obvious" that the probabilities are $\mathcal{P}(+) = |\langle+|+\rangle|^2 = 1$ and $\mathcal{P}(-) = |\langle-|+\rangle|^2 = 0$. These are calculated by using the mathematical definition of the bra-ket, and remembering that these kets are unit vectors.

5.5.3 Application to Experiment D

Now we consider Experiment D, as seen in Figure 12. We have:

$$\begin{gathered}\text{input: } |+\rangle = |+\rangle_z \\ \text{outputs: } |+\rangle_x \text{ or } |-\rangle_x\end{gathered}$$

We assume that we can expand the kets as such (and it works out!):

$$\begin{aligned}|+\rangle_x &= a|+\rangle + b|-\rangle \\ |-\rangle_x &= c|+\rangle + d|-\rangle\end{aligned}$$

where $a, b, c, d \,\epsilon\, \mathbb{C}$. Then:

$$\begin{aligned}\mathcal{P}(+_x) &= |{}_x\langle+|+\rangle|^2 = |\langle+|+\rangle_x|^2 \\ &= |\langle+|(a|+\rangle + b|-\rangle)|^2 \\ &= |a\langle+|+\rangle + b\langle+|-\rangle|^2 \\ &= |a|^2 \\ &= \frac{1}{2} \qquad \text{(experimentally)}\end{aligned} \tag{5.6}$$

Similarly, we find $\mathcal{P}(-_x) = |d|^2 = \frac{1}{2}$. Since we must satisfy the normalization condition ${}_x\langle +|+\rangle_x = 1$, we find that $|b|^2 = \frac{1}{2}$. We also find $|c|^2 = \frac{1}{2}$. Since a, b, c, d are all complex numbers, they may have an imaginary component. This encourages us to set:

$$\begin{aligned} a &= \frac{1}{\sqrt{2}} e^{i\delta} \\ b &= \frac{1}{\sqrt{2}} e^{i\alpha} \\ c &= \frac{1}{\sqrt{2}} e^{i\gamma} \\ d &= \frac{1}{\sqrt{2}} e^{i\beta} \end{aligned}$$

where $\alpha, \beta, \gamma, \delta \, \epsilon \, [0, 2\pi]$, so then:

$$\begin{aligned} |+\rangle_x &= \frac{1}{\sqrt{2}} e^{i\delta} |+\rangle + \frac{1}{\sqrt{2}} e^{i\alpha} |-\rangle \\ |-\rangle_x &= \frac{1}{\sqrt{2}} e^{i\gamma} |+\rangle + \frac{1}{\sqrt{2}} e^{i\beta} |-\rangle \end{aligned}$$

Because quantum mechanics is some serious black voodoo shit, we can multiply any incoming state vector (input) by an arbitrary phase factor $e^{i\Delta}$ without changing the predictions. These are called **global phases**. This can be shown mathematically:

$$\begin{aligned} \mathcal{P}(i) &= |\langle \phi_i | e^{i\Delta} | \psi \rangle|^2 \\ &= |\langle \phi_i | \psi \rangle e^{i\Delta}|^2 \\ &= |\langle \phi_i | \psi \rangle|^2 |e^{i\Delta}|^2 \\ &= |\langle \phi_i | \psi \rangle|^2 \end{aligned} \tag{5.7}$$

We can rewrite:

$$|+\rangle_x = \underbrace{e^{i\delta}}_{\text{globalphase}} \frac{1}{\sqrt{2}} (|+\rangle + e^{i(\alpha-\delta)} |-\rangle$$

So we can omit the term $e^{i\delta}$, which implies we can set $\delta = 0$. Then (by applying a similar method to the other state) we find:

$$|+\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle + e^{i\alpha}|-\rangle)$$
$$|-\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle + e^{i\beta}|-\rangle)$$

We also know that ${}_x\langle +|-\rangle_x = 0$ so then:

$$\begin{aligned}
&\frac{1}{\sqrt{2}}(\langle +| + e^{-i\alpha}\langle -|)\frac{1}{\sqrt{2}}(|+\rangle + e^{i\beta}|-\rangle) \\
&= \frac{1}{2}(\langle +|+\rangle + e^{i(\beta-\alpha)}\langle -|-\rangle) \\
&= \frac{1}{2}(1 + e^{i(\beta-\alpha)} = 0 \\
&\iff e^{i(\beta-\alpha)} = -1 \\
&\Rightarrow \beta - \alpha = \pi \\
&\iff \beta = \alpha + \pi
\end{aligned}$$

So then:

$$e^{i\beta} = e^{i(\alpha+\pi)} = e^{i\alpha}e^{i\pi} = -e^{i\alpha}$$

Which gives us:

$$|+\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle + e^{i\alpha}|-\rangle)$$
$$|-\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle - e^{i\alpha}|-\rangle)$$

We can now pick any value for α as it will always be consistent with our observations. Of course, we choose $\alpha = 0$. Then, FINALLY:

$$|+\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \tag{5.8}$$
$$|-\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \tag{5.9}$$

We can then finally calculate the probabilities by evaluating $|{}_x\langle +|+\rangle|^2$ and $|{}_x\langle -|+\rangle|^2$. We of course find each of their values to be $\frac{1}{2}$. So then:

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |+\rangle_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |-\rangle_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

6 Superpositions, Mixtures and Spin

6.1 Postulates (Review)

Let's review our postulates.

6.1.1 Postulate 1

The state of a quantum mechanical system is represented mathematically by a normalized vector, a ket, $|\psi\rangle$. This symbol summarizes everything you can know about the system, everything you need to know to predict measurement results, and corresponds to an element of a complex vector space of a suitable dimension.

6.1.2 Postulate 3

A measurement of mutually exclusive outcomes can be described by a set of orthonormal basis vectors $\{|\phi_i\rangle\}$ where $i = 1, \ldots, d$.

6.1.3 Postulate 4

For an input state described by a normalized ket $|\psi\rangle$ and a measurement with mutually exclusive events i described by elements of an orthonormal basis $\{|\phi_i\rangle\}$, where $i = 1, \ldots, d$, the probability $\mathcal{P}(i)$ to observe outcome i is given by $\mathcal{P}(i) = |\langle\phi_i|\psi\rangle|^2$.

6.2 The Update-Postulate

Postulate 5 (the update rule): After a measurement on an input state described by the state vector $|\psi\rangle$ with outcome i associated with the ket $|\phi_i\rangle$, the outgoing state is described by the state vector ket $|\phi_i\rangle$.

The measurement gives us new knowledge. The update rule expresses the fact that for our simple measurements, only the measurement result of the preceding step characterizes all future predictions, not the steps before that measurement.

6.3 Superposition States

Recall Experiment E, as seen in Figure 14. We can easily describe the output of the first device as $|+\rangle_z = \frac{1}{\sqrt{2}}(|+\rangle_x + |-\rangle_x)$. Before the combiner, the state is represented by $\frac{1}{\sqrt{2}}(|+, \text{upper path}\rangle_x + |-, \text{lower path}\rangle_x)$. After the combiner, the state is represented by $\frac{1}{\sqrt{2}}(|+\rangle_x + |-\rangle_x) = |+\rangle_z$.

We must remember that amplitudes remain amplitudes until measurements are take; this is the idea of the superposition of possibilities. In addition, one set of amplitudes for one set of events can be converted into different amplitudes for another set of events (example: z-basis to x-basis). Measurements turn amplitudes into probabilities.

For the atoms after the Z-oriented device, only the internal state is important, with amplitudes for different possible events. The X-oriented device introduces a new degree of freedom: the atoms can be located in the upper path or the lower path. Without measurement (finding which path was taken), the inner degree of freedom (the amplitude for the x-direction) and outer degree of freedom (which path) appear now in only two combinations; they are still only possibilities, not events. The combiner quantum mechanically removes the outer degree of freedom, leaving the amplitudes as possibilities. The atoms then only have their internal degree of freedom. To summarize, without measurement of the location after the X-oriented device: the two independent paths from source to detector interfere (the amplitude contributions from each path add up), and the particle is in a superposition of being from one path and the other path.

6.4 Mixture versus Superposition

Consider a source that emits the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle_x + |-\rangle_x) = |+\rangle_z$, a superposition. That seems like this would be identical to the source emitting states $|+\rangle_x$ and $|-\rangle_x$, each with a probability of $\frac{1}{2}$ (a mixed state source). We know that $\mathcal{P}(+_z) = |_z\langle +|\psi\rangle|^2 = 1$ and $\mathcal{P}(-_z) = |_z\langle -|\psi\rangle|^2 = 0$. In other words, we know that the prediction rule for superposition states is represented by Postulate 4 (Section 6.1.3).

We say that if the information for which path the particle took is available, then the state is a mixture. If the information is unavailable, it is a superposition of paths. We must realize that mixtures have two random vari-

ables, s (the signal, describing which state was prepared) and i (describing the measurement outcome). Quantum mechanics gives the conditional probability that an outcome i is triggered given the state $|\psi_s\rangle$ has been prepared:

$$\mathcal{P}(i|s) = |\langle\phi_i|\psi_s\rangle|^2 \tag{6.1}$$

while the joint probability of having a prepared signal s and obtained outcome i is:

$$\mathcal{P}(i,s) = \mathcal{P}(s)|\langle\phi_i|\psi_s\rangle|^2 \tag{6.2}$$

In our experiments, we do not know which signal is being prepared in each run, but we can observe the overall probability that outcome i is being triggered. This is given by the marginal distribution:

$$\mathcal{P}(i) = \sum_{s=1}^{n} \mathcal{P}(s)|\langle\phi_i|\psi_s\rangle|^2 \tag{6.3}$$

We then find that:

$$\mathcal{P}(+_z) = \frac{1}{2}|_z\langle+|+\rangle_x|^2 + \frac{1}{2}|_z\langle+|-\rangle_x|^2 = \frac{1}{2}$$
$$\mathcal{P}(-_z) = \frac{1}{2}|_z\langle-|+\rangle_x|^2 + \frac{1}{2}|_z\langle-|-\rangle_x|^2 = \frac{1}{2}$$

The above equations are saying that the probability of finding the outcome $|+\rangle_z$ (or $|-\rangle_z$) with a source that emits a probability of $\frac{1}{2}$ of having the state $|+\rangle_x$ and a probability of $\frac{1}{2}$ of having the state $|-\rangle_x$ is given by finding the probability of the outcome using one of the two original states, multiplying it by the probability of having the original state, and summing.

Mixed sources allow many different descriptions which are all equivalent. However, if the source is described by only one ket, characterized by a measurement where it will always give one specific outcome, then this is the only way to describe the source. These are called **pure state sources**. An example is the output state of a Stern-Gerlach device where one output is blocked.

6.5 Spin and Dipole Moments

Recall Equation (3.1). Let us only consider the magnetic field to be in the z-direction, so then $\vec{B} = B_z\hat{\mathbf{e}}_z$. We say $\vec{\mu} = \mu_x\hat{\mathbf{e}}_x + \mu_y\hat{\mathbf{e}}_y + \mu_z\hat{\mathbf{e}}_z$, so then the magnetic energy is $E_{mag}(z) = -\mu_z B_z(z)$. Now recall Equation (3.2). We then of course find $\vec{F} = -\mu_z(\frac{d}{dz}B_z(z))\hat{\mathbf{e}}_z$. From classical considerations, we can split up the dipole moment of an electron as such:

$$\vec{\mu} = -g\frac{e}{m_e}\vec{S}$$

where

$g =$ some geometric factor

$e =$ elementary charge

$m_e =$ electron mass

$S =$ spin

We do not know what spin is yet, so do not fret if you are confused. What makes more sense is to write:

$$\mu_z = -g\frac{e}{m_e}S_z \tag{6.4}$$

where this can be done for the x and y directions as well. Our systems do not have pre-described values for all three directions, however. Instead, the system has amplitudes to manifest particular values of the dipole moments in each separate experiment, with probabilities related to the probability amplitudes of the corresponding ket-vector description of the systems. Experimentally, we find that:

$$S_z = \pm\frac{\hbar}{2} \tag{6.5}$$

where $\hbar$ is Planck's reduced constant. We find this value for the other measurement directions as well. (This value leads to the name of spin-1/2 particles!)

6.5.1 Spin in Higher Dimensional Spaces

If we use different kinds of atoms, we may find different splitting patterns, such as:

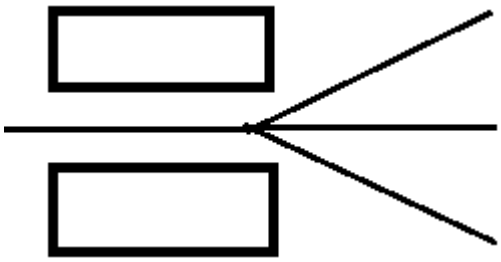


Figure 15: Hey, a diagram.

Which MAY lead us to $S_z \, \epsilon \, \{-\hbar, 0, +\hbar\}$. These systems correspond to what we call Spin-1 particles, whose internal states need to be described by 3D complex vector spaces. Systems with more splitting, which leads to more complicated things.

7 Operators

7.1 Expectation Values

We determined there to be two values of spin from experiments: $\pm\frac{\hbar}{2}$. The average spin value is the difference divided by two, so we find $\frac{\hbar}{2}$ to be the average. Average (mean) values are often referred to as expectation values, and we will refer to them by that name from now on. Using an outcome i with an assigned value a_i, the expectation value is denoted and defined as such:

$$\langle A \rangle := \sum_{i=1} a_i \mathcal{P}(i) \tag{7.1}$$

7.2 The Outer Product and Operators

One of the things that the Stern-Gerlach experiment showed was the existence of spin. The two values (up and down) that we have "seen" thus far are spin values of $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$, corresponding to up and down values respectively. Let us only consider a single device as the set-up. Here, we show that the expectation value of S_z can be rewritten:

$$\begin{aligned} \langle S_z \rangle &= \frac{\hbar}{2}\mathcal{P}_{up} + (-\frac{\hbar}{2})\mathcal{P}_{down} \\ &= \frac{\hbar}{2}|\langle +|\psi\rangle_z|^2 - \frac{\hbar}{2}|\langle -|\psi\rangle_z|^2 \\ &= \frac{\hbar}{2}\langle\psi|+\rangle_z\langle +|\psi\rangle_z - \frac{\hbar}{2}\langle\psi|+\rangle_z\langle +|\psi\rangle_z \end{aligned} \tag{7.2}$$

Now consider:

$$\begin{aligned} \langle\psi|+\rangle\langle +|\psi\rangle &= \left[\begin{pmatrix} a^*, & b^* \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]\left[\begin{pmatrix} 1, & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}\right] \\ &= \begin{pmatrix} a^*, & b^* \end{pmatrix}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1, & 0 \end{pmatrix}\right]\begin{pmatrix} a \\ b \end{pmatrix} \\ &= \begin{pmatrix} a^*, & b^* \end{pmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \end{aligned} \tag{7.3}$$

In other words, $|+\rangle\langle +|$ (denoted P_+) is an **operator**. In general, this notation is also known as the **outer product**:

$$|A\rangle\langle B| = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix} \begin{pmatrix} B_1^* & B_2^* & \cdots & B_n^* \end{pmatrix} = \begin{bmatrix} A_1B_1^* & A_1B_2^* & \cdots & A_1B_n^* \\ A_2B_1^* & A_2B_2^* & \cdots & A_2B_n^* \\ \vdots & \vdots & \ddots & \vdots \\ A_nB_1^* & A_nB_2^* & \cdots & A_nB_n^* \end{bmatrix} \tag{7.4}$$

Operators map vectors to vectors:

$$P_+|\psi\rangle = |\psi'\rangle \tag{7.5}$$

Generally, the resulting state is not normalized. However:

$$P_+|\psi\rangle = |+\rangle\langle+|\psi\rangle = \langle+|\psi\rangle|+\rangle$$

and we know that $|+\rangle$ is a normalized vector, so the resulting state is in fact normalized for this case. Going back to the expectation value of S_z, we find that:

$$\langle S_z\rangle = \langle\psi| \underbrace{\left(\frac{\hbar}{2}|+\rangle\langle+| \; - \; \frac{\hbar}{2}|-\rangle\langle-|\right)}_{:=S_z} |\psi\rangle = \langle\psi|S_z|\psi\rangle \tag{7.6}$$

We can directly calculate S_z:

$$\begin{aligned} S_z &= \frac{\hbar}{2}|+\rangle\langle+| \; - \; \frac{\hbar}{2}|-\rangle\langle-| \\ &= \frac{\hbar}{2}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) \\ &= \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned} \tag{7.7}$$

If we did some calculations, we would find that:

$$S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{7.8}$$

We can write any operator as such:

$$A = \sum_i a_i|a_i\rangle\langle a_i| \tag{7.9}$$

(where a_i are the coefficients in the state and $|a_i\rangle$ are the states corresponding to those coefficients) which allows us to write the expectation value as such:

$$\langle A \rangle = \langle \psi_{in} | A | \psi_{in} \rangle \tag{7.10}$$

7.3 Coordinate Representation

Consider a state $|\psi\rangle$, a possibly non-normalized state $|\psi'\rangle$, and an operator A. We can write:

$$|\psi'\rangle = \sum_{i=1}^{d} a'_i |a'_i\rangle \qquad A = \sum_{i=1}^{d} \sum_{j=1}^{d} a_{ij} |a_i\rangle\langle a_j| \qquad |\psi\rangle = \sum_{i=1}^{d} a_i |a_i\rangle \tag{7.11}$$

Which can be expanded:

$$\begin{pmatrix} a'_1 \\ \vdots \\ a'_d \end{pmatrix} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,d} \\ \vdots & \ddots & \vdots \\ a_{d,1} & \cdots & a_{d,d} \end{bmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix} \tag{7.12}$$

Basically we are saying $|\psi'\rangle = A|\psi\rangle$. We should note a few additional properties of operators.

Conjugation: $\langle \phi | \psi \rangle = \langle \psi | \phi \rangle^*$.

Associativity: $|\psi\rangle\langle\phi|\eta\rangle = \langle\phi|\eta\rangle|\psi\rangle$. We can interpret this as a ket $|\psi\rangle$ multiplied by a number $\langle\phi|\eta\rangle$, or an operator $|\psi\rangle\langle\phi|$ acting on a ket $|\eta\rangle$.

7.3.1 Y-oriented Stern-Gerlach Device

One can do measurements in the y-direction. The details do not matter too much right now, but atoms would move back and forth (up and down in the y-direction). We can express $|+\rangle_y$ and $|-\rangle_y$ in terms of the z-basis like the way we did with x, however, there are more constraints. Let's skip to the coordinate representation summary:

$$|+\rangle = \begin{pmatrix}1\\0\end{pmatrix} \qquad |+\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad |+\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$$
$$|-\rangle = \begin{pmatrix}0\\1\end{pmatrix} \quad |-\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \qquad |-\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$
$$S_z = \frac{\hbar}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \qquad S_x = \frac{\hbar}{2}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \qquad S_y = \frac{\hbar}{2}\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}$$

The polarization of a single photon is:

- z-direction: horizontal/vertical linear polarization
- x-direction: $+45/-45$ degree linear polarization
- y-direction: right/left circular polarization

8 Eigensystems

8.1 Eigenvalue Problems

Recall the definition of S_z. Now consider:

$$\begin{aligned} S_z|+\rangle &= \frac{\hbar}{2}(|+\rangle\langle+| - |-\rangle\langle-|)\ |+\rangle \\ &= \frac{\hbar}{2}(|+\rangle\langle+|+\rangle - |-\rangle\langle-|+\rangle) \\ &= \frac{\hbar}{2}|+\rangle \end{aligned}$$

Now let us apply the definition:

$$\frac{\hbar}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1\\0\end{pmatrix}$$

This looks awfully similar to eigenvectors and eigenvalues! The ket $|+\rangle$ is an eigenvector of the operator S_z corresponding to the eigenvalue $\frac{\hbar}{2}$. It turns out that the measurement vectors (such as $|a_i\rangle$) are always eigenvectors of the operator (such as A). In other words, for the general case,

$$A|\psi\rangle = \lambda|\psi\rangle \tag{8.1}$$

8.2 Finding Eigenvalues

Just like good ol' linear algebra. To find the eigenvalues of the system, we solve the characteristic polynomial, which is defined as such:

$$\det(A - \lambda I) = 0 \tag{8.2}$$

When we find these eigenvalues, we plug them into the following equation and solve the system of equations:

$$(A - \lambda I)|\psi_\lambda\rangle = \vec{0} \tag{8.3}$$

$$\begin{bmatrix} a_{1,1}-\lambda & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2}-\lambda & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n}-\lambda \end{bmatrix}\begin{pmatrix}\alpha_1\\ \vdots\\ \vdots\\ \alpha_n\end{pmatrix} = \begin{pmatrix}0\\ \vdots\\ \vdots\\ 0\end{pmatrix}$$

Non-degenerate eigenvalues are eigenvalues that occur only once. When we plug this into Equation (8.3), we will find a unique solution. **Degenerate eigenvalues** are repeated eigenvalues. When we plug this into the same equation, we will find a whole subset of vectors.

Example 1: Find the eigenvalues and eigenvectors of the operator S_y.

Solution: First, we find the eigenvalues by solving Equation (8.2).

$$\det\left(\frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} - \lambda I\right) = \begin{vmatrix}-\lambda & -i\frac{\hbar}{2}\\ i\frac{\hbar}{2} & -\lambda\end{vmatrix} = \lambda^2 - \left(\frac{\hbar}{2}\right)^2 = 0$$

This of course leads us to find eigenvalues of $\pm\frac{\hbar}{2}$. Let $|\psi_\lambda\rangle = \begin{pmatrix}a\\ b\end{pmatrix}$. To find these values "specifically", we must solve Equation (8.3). Set $\lambda = \frac{\hbar}{2}$:

$$\begin{bmatrix}-\frac{\hbar}{2} & -i\frac{\hbar}{2}\\ i\frac{\hbar}{2} & -\frac{\hbar}{2}\end{bmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$
$$\frac{\hbar}{2}\begin{bmatrix}-1 & -i\\ i & -1\end{bmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$
$$\frac{\hbar}{2}\begin{bmatrix}-i & 1\\ i & -1\end{bmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$
$$\frac{\hbar}{2}\begin{bmatrix}-i & 1\\ 0 & 0\end{bmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$
$$\frac{\hbar}{2}\begin{pmatrix}-ia+b\\ 0\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

This implies that $b = ia$, so the eigenvector corresponding to $\lambda = \frac{\hbar}{2}$ is $\begin{pmatrix}a\\ ia\end{pmatrix}$. However, we require all states to be normalized!

$$|a|^2 + |ia|^2 = 1$$
$$a^2 + a^2 = 1$$
$$a = \frac{1}{\sqrt{2}}$$

Then the eigenvector becomes $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$. Due to a theorem in linear algebra

dealing with complex eigenvectors, the other eigenvector is $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$. $\surd$

8.3 General Spin Direction

We denote the general direction as $\vec{m}$, a unit vector in 3D space:

$$\vec{m} = \begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix} = \begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix} \tag{8.4}$$

(These coordinates are NOT based off of what was taught in MATH 227. Please click here for a reference.) We go on to define S, whose components are operators:

$$S = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} \tag{8.5}$$

We also define another variable S_m, which can be represented by a dot product:

$$S_m = \vec{m} \cdot \vec{S} = m_x S_x + m_y S_y + m_z S_z \tag{8.6}$$

Or using coordinate representation:

$$S_m = \frac{\hbar}{2}\begin{bmatrix} \cos(\theta) & e^{-i\phi}\sin(\theta) \\ e^{i\phi}\sin(\theta) & \cos(\theta) \end{bmatrix} \tag{8.7}$$

S_m has eigenvalues $\pm\frac{\hbar}{2}$ and two eigenvectors:

$$|+\rangle_m = \cos\left(\frac{\theta}{2}\right)|+\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|-\rangle \tag{8.8a}$$

$$|-\rangle_m = \sin\left(\frac{\theta}{2}\right)|+\rangle - \cos\left(\frac{\theta}{2}\right)e^{i\phi}|-\rangle \tag{8.8b}$$

The assignment of the eigenvectors to the outcome of a generalized Stern-Gerlach measurement leads to the correct predictions of outcome probabilities.

Example 2: Given $m_0 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1 & 0\end{pmatrix}^T$, calculate S_m, its eigenvectors, and

its expectation value.

Solution: By looking at the value of m_0, we can see that $\theta = \frac{\pi}{2}$ and $\phi = \frac{\pi}{4}$. Then,

$$S_{m_0} = \frac{\hbar}{2}\begin{bmatrix} 0 & e^{-i\frac{\pi}{4}} \\ e^{i\frac{\pi}{4}} & 0 \end{bmatrix}$$
$$|+\rangle_{m_0} = \frac{1}{\sqrt{2}}(|+\rangle \ + e^{i\frac{\pi}{4}}|-\rangle)$$
$$|-\rangle_{m_0} = \frac{1}{\sqrt{2}}(|+\rangle \ - e^{i\frac{\pi}{4}}|-\rangle)$$

While the expectation value is calculated as such (I will update this when I understand it):

$$|\psi\rangle = |+\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$
$$\Rightarrow \ \langle S_{m_0}\rangle = \langle\psi|S_{m_0}|\psi\rangle$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1, & 1\end{pmatrix}\frac{\hbar}{2}\begin{bmatrix} 0 & e^{-i\frac{\pi}{4}} \\ e^{i\frac{\pi}{4}} & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$
$$= \frac{\hbar}{4}\begin{pmatrix}1, & 1\end{pmatrix}\begin{pmatrix}e^{-i\frac{\pi}{4}}\\ e^{i\frac{\pi}{4}}\end{pmatrix} = \frac{\hbar}{4}(e^{-i\frac{\pi}{4}} + e^{i\frac{\pi}{4}})$$
$$= \frac{\hbar}{2}\cos(\frac{\pi}{4}) = \frac{\hbar}{2\sqrt{2}} \ \surd$$

9 Hermitian and Normal Operators

9.1 Hermitian (Adjoint) Operators

Let's jump right into it. We denote the **Hermitian adjoint** using the superscript †. The Hermitian adjoint of a matrix is simply the conjugate transpose of the matrix (take the conjugate and then the transpose, or the other way around - both work). It turns out that:

$$|\phi\rangle = A|\psi\rangle \iff \langle\phi| = \langle\psi|A^\dagger \tag{9.1}$$

It also turns out that:

$$\langle\phi|A|\psi\rangle^* = \langle\psi|A^\dagger|\phi\rangle \tag{9.2}$$

$$(|\psi\rangle\langle\phi|)^\dagger = |\phi\rangle\langle\psi| \tag{9.3}$$

An operator A is called a **Hermitian operator** or a **self-adjoint operator** if $A = A^\dagger$. A Hermitian operator always has real eigenvalues and an orthonormal set of eigenvectors.

9.2 Normal Operators

An operator B is called a **normal operator** if $BB^\dagger = B^\dagger B$. All Hermitian operators are normal. If we consider a normal operator B with eigenvalues λ_i and corresponding eigenvectors $|v_i\rangle$, then eigenvectors belonging to different eigenvalues are orthogonal. That is to say:

$$\langle v_i|v_j\rangle = 0 \qquad \text{for } \lambda_i \neq \lambda_j \tag{9.4}$$

(This following part is stated in the notes but in a different notation; I will use the standard linear algebra notation first.) All normal matrices are diagonalizable, such that:

$$B = UEU^\dagger \tag{9.5}$$

E is a diagonal matrix whose entries are the eigenvalues of B, and U is a matrix whose column vectors are the eigenvectors of B. In terms of Dirac notation:

$$B = \sum_i \lambda_i |v_i\rangle\langle v_i| \tag{9.6}$$

9.3 Observables and Measurements

After doing theory on the level of observables, one would like to find an experimental evaluation procedure for a given observable. This is done by :

- finding the eigensystem of the observable
- finding a measurement setting that corresponds to the eigenvectors of the observable
- assigning the eigenvalues of the observables to the outcomes that correspond to the respective eigenvector
- performing the experiment and finding the experimental value of the expectation value

Given an observable operator A with a distribution of outcomes $\mathcal{P}(i)$ we know that the operator has an expectation value of $\sum_i a_i \mathcal{P}(i)$. Then, the operator has a variance of distribution (in three forms, so I'm gonna skip steps here!):

$$\Delta A = \sqrt{\sum_i \mathcal{P}(i)|a_i - \langle A\rangle|^2} \tag{9.7}$$

$$\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2} \tag{9.8}$$

$$\Delta A = \sqrt{\langle (A - \langle A\rangle I)^2\rangle} \tag{9.9}$$

where I is the identity matrix.

9.4 Commutators

Frequently in quantum mechanics, what we call the **commutator** of two observable operators plays a role. We define the commutator as:

$$[A, B] := AB - BA \tag{9.10}$$

We say that two operators commute if $[A, B] = 0$. If A and B commute and are both normal operators, then the operators share the same eigenvectors, although the eigenvalues may be different. If A has eigenvalues λ_{a_i} and B has eigenvalues λ_{b_i} but both have eigenvectors $|v_i\rangle$, then:

$$\begin{aligned} A &= \sum_i \lambda_{a_i} |v_i\rangle\langle v_i| \\ B &= \sum_i \lambda_{b_i} |v_i\rangle\langle v_i| \end{aligned} \tag{9.11}$$

10 Heisenberg Uncertainty Relation and Operators

10.1 The Uncertainty Relation

Consider two Hermitian observable operators A and B and a source of pure states $|\psi\rangle$. It turns out that:

$$\Delta A \Delta B \geq \frac{1}{2}|\langle\psi|[A,B]|\psi\rangle| \tag{10.1}$$

You may have seen this before in the following form:

$$\Delta x \Delta p \geq \frac{1}{2}\hbar \tag{10.2}$$

where x is the position operator (not just the position!) and p is the momentum operator. These operators satisfy the relation $[x,p] = i\hbar$. We will talk about this later in the course. However you most likely know that this relationship says that you cannot make an accurate measurement for position and an accurate measurement for momentum together.

Example: Let $A = S_x$ and $B = S_y$. Then:

$$\begin{aligned}
[A,B] &= \frac{\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} - \frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\frac{\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \\
&= \left(\frac{\hbar}{2}\right)^2\left(\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix} - \begin{bmatrix}-i & 0\\ 0 & i\end{bmatrix}\right) \\
&= \frac{\hbar^2}{4}(2i)\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \\
&= \hbar i S_z
\end{aligned}$$

which then gives us:

$$\Delta S_x \Delta S_y \geq \frac{\hbar}{2}|\langle\psi|S_z|\psi\rangle|$$

So unless the right side is zero, neither a measurement of S_x or S_y can give a sharp value. If one of the measurements is accurate, then the left hand side would vanish, but that means the right side must as well! For $\Delta S_x \neq 0$:

$$\Delta S_y \geq \frac{\frac{\hbar}{2}|\langle\psi|S_z|\psi\rangle|}{\Delta S_x}$$

The uncertainty relation is a warning that we are not allowed to think of any outcome as something that is predetermined and simply needs to be uncovered.

10.2 Projection Operators

Let's get to it.

$$\begin{gathered}|\psi\rangle = a|+\rangle + b|-\rangle = |+\rangle a + |-\rangle b \\ \Rightarrow a = \langle+|\psi\rangle,\ b = \langle-|\psi\rangle\end{gathered}$$

$$\begin{gathered}|\psi\rangle = |+\rangle\langle+|\psi\rangle\ +\ |-\rangle\langle-|\psi\rangle \\ = \underbrace{(|+\rangle\langle+|)}_{:=P_+}|\psi\rangle + \underbrace{(|-\rangle\langle-|)}_{:=P_-}|\psi\rangle \\ = (P_+ + P_-)|\psi\rangle \\ \Rightarrow P_+ + P_- = I\end{gathered}$$

P_+ is the projection onto state $|+\rangle$ and P_- is the projection onto state $|-\rangle$. Both are what we call **projection operators**. I is the identity matrix. The arguments of the ketbra represent what state we are projecting onto. In a general case, given an orthornormal basis a (consisting of vectors $|a_i\rangle$), the projection is defined as:

$$P_i = |a_i\rangle\langle a_i| \tag{10.3}$$

And we then find the property of **closure**, also called the **completeness relaton** or the **resolution of identity**:

$$I = \sum_i P_i \tag{10.4}$$

Projection operators are Hermitian, and they also satisfy the equation $P_i^2 = P_i$. We should also note that a projection acting on its own state is

equal to the state, and a projection acting on an orthogonal state is zero:

$$P_+|+\rangle = |+\rangle \tag{10.5a}$$
$$P_+|-\rangle = 0 \tag{10.5b}$$

10.2.1 Application of Projection Operators to Selective Operations and Postulate 5

Consider a z-oriented Stern-Gerlach device with the down $(-)$ outcome disabled. Recall Equation (5.3a):

$$\begin{aligned} \mathcal{P}_+ &= |\langle +|\psi\rangle|^2 \\ &= \langle +|\psi\rangle^* \langle +|\psi\rangle \\ &= \langle \psi|+\rangle\langle +|\psi\rangle \\ &= \langle \psi|P_+|\psi\rangle \end{aligned} \tag{10.6}$$

We can then say that the probability of the measurement (up for a z-oriented device, in this case) can be calculated using the input state $|\psi\rangle$ and the projector P_+ that corresponds to the result.

We learned previously that measurements disturb systems. In other words, as you should know, if the input state $|\psi\rangle$ is measured to have a value $S_z = \frac{\hbar}{2}$, then the new input state (the output state) is $|+\rangle$. We can describe this change of the input state as:

$$|\psi'\rangle = \frac{P_+|\psi\rangle}{\sqrt{\mathcal{P}_+}} = |+\rangle \tag{10.7}$$

Postulate 5: After a measurement of A, that yields the result a_i, the quantum system is in a new state that is the normalized projection of the original system ket onto the ket, or kets, corresponding to the system of the measurement. This is mathematically written as:

$$|\psi'\rangle = \frac{P_n|\psi\rangle}{\sqrt{\mathcal{P}_n}} \tag{10.8}$$

If the output is not normalized, the process still keeps track (internally) of the probability of the system passing the selector via its norm:

$$\mathcal{P}_+ = (\langle \psi|P_+)(P_+|\psi\rangle) \tag{10.9}$$

10.3 Experiment E Revisited

Consider Experiment E from Section 4.5:

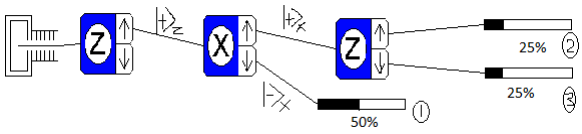


Figure 16: Experiment E as seen previously.

After we pass through the first device, we have a normalized output state $|+\rangle$. The second device selects either $|+\rangle_x$, corresponding to selector P_{+x}, or $|-\rangle_x$, corresponding to selector P_{-x}. At the third device, we have selectors P_{+z} and P_{-z}.

10.3.1 Detector #1

At Detector # 1 (labeled on the figure), we have an unnormalized state:

$$P_{-x}|+\rangle_z = |-\rangle_x \ {}_x\langle -|+\rangle \approx |-\rangle_x$$

The probability of an atom arriving there is:

$$\begin{aligned}\langle +|P_{-x}^{+}P_{-x}|+\rangle &= \langle +|P_{-x}|+\rangle \\ &= |\langle +|-\rangle_x|^2 = \frac{1}{2}\end{aligned}$$

10.3.2 Detector #2

The unnormalized state that arrives at Detector # 2 is:

$$P_{+z}P_{+x}|+\rangle = |+\rangle_z \ ({}_z\langle +|+\rangle_x \ {}_x\langle +|+\rangle_z)$$

Note that the term in brackets is actually the probability:

$$\begin{aligned}{}_z\langle +|+\rangle_x \ {}_x\langle +|+\rangle_z &= |\langle +|_z|+\rangle_x \ \langle +|_x+\rangle_z|^2 \\ &= |{}_z\langle +|+\rangle_x|^4 = \frac{1}{4}\end{aligned}$$

10.3.3 Detector #3

The unnormalized state that arrives at Detector # 3 is:

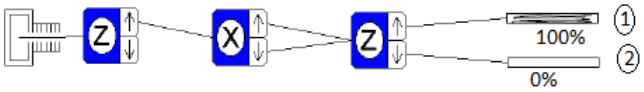
$$P_{-z}P_{+x}|+\rangle = |-\rangle_z \ ({}_z\langle -|+\rangle_x \ {}_x\langle +|+\rangle_z)$$

Once again, the term in brackets is the probability, which turns out to be:

$${}_z\langle -|+\rangle_x \ {}_x\langle +|+\rangle_z = |{}_z\langle -|+\rangle_x|^4 = \frac{1}{4}$$

10.4 Experiment "F"

Although we did not refer to it as Experiment F, that's what we'll do now. This is the experiment seen in Section 5.1 (Figure 14), but I'll post it here for convenience:



We pass through the first detector and have a state $|+\rangle$. At the "collision" after the second detector, we have a state defined as $P_{+x}|+\rangle + P_{-x}|+\rangle$.

10.4.1 Detector #1

The output state at Detector # 1 is:

$$P_+(P_{+x}|+\rangle + P_{-x}|+\rangle) = P_+(P_{+x} + P_{-x})|x\rangle = P_+ I|+\rangle = P_+|+\rangle = |+\rangle$$

The probability is then simply 1.

10.4.2 Detector #2

The output state is undefined, as the probability is equal to 0.

10.5 Definition: Interference

For interference paths, **the state vectors are added; the probabilities are not added.** We define interference as such: paths interfere if it is in principle impossible for an outside observer to tell afterwards which path had been taken.

11 Unitary Operators and Bombs!

11.1 Unitary Operations

Consider the following figure:

Figure 17: The unitary operator within a system.

U is an operator which changes the state of the system:

Input	Output
$\|+\rangle$	$\|-\rangle$
$\|-\rangle$	$\|+\rangle$
$\|+\rangle_x$	$\|+\rangle_x$
$\|-\rangle_x$	$-\|-\rangle_x$
$\|+\rangle_y$	$i\|-\rangle_y$
$\|-\rangle_y$	$i\|+\rangle_y$

Unitary operators do not change the magnitude of a vector:

$$\|Ux\| = \|x\| \tag{11.1}$$

They represent non-selective operations (as opposed to projection operators, which represent selective operations). Unitary operators map pairs of orthogonal states into new pairs of orthogonal states. They also have this property:

$$UU^\dagger = U^\dagger U = I \tag{11.2}$$

That is to say, the unitary operator times its conjugate transpose is the identity matrix. In a 2D case, U can also be represented as such:

$$U = |-\rangle\langle+| \ + \ |+\rangle\langle-|$$

We can extend this to a general case. If U maps the states $|a_i\rangle$ of one basis to states $|b_i\rangle$ of another basis then U can be expressed as:

$$U = \sum_i |b_i\rangle\langle a_i| \tag{11.3}$$

All eigenvalues of a unitary operator have a magnitude of 1 ($\|\lambda_i\| = 1$). Somehow, we can represent eigenvalues as seen below, and thus rewrite the unitary operator as such:

$$U = \sum_i \lambda_i |\varphi_i\rangle\langle\varphi_i| = \sum_i e^{i\varphi_i} |\varphi_i\rangle\langle\varphi_i| \tag{11.4}$$

In Equation 11.4, φ_i are the eigenstates of the system.

11.1.1 Pauli Operators

The **Pauli Operators** are defined as follows:

$$\sigma_x := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_y := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{11.5}$$

11.2 Bomb Detection (an example)

Consider a bomb that has an "optical fuse": a fuse that will set off if a single photon hits it. A manufacturing company produces bombs but only 70% of them work. The goal is to find a process that tells you what bombs will work with certainty.

How can we find such a process? If you shine light on the fuse, either the bomb blows up, so there is no bomb, or it does not blow up. In either case, we are left with a non-working bomb. With quantum mechanics, however, we can apply methods that we applied to the Stern-Gerlach experiment. We can apply an optical interferometer to this problem:

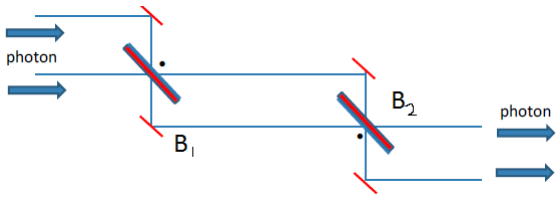


Figure 18: An optical interferometer. The thin red lines are mirrors, while the red/blue lines are 50% beam-splitters that split half of the photons up and the other half down.

Because the beam splitters are 50-50, we can easily argue that the states are $|up\rangle \to \frac{1}{\sqrt{2}}|up\rangle + \frac{1}{\sqrt{2}}|down\rangle$ and $|down\rangle \to \frac{1}{\sqrt{2}}|up\rangle - \frac{1}{2}|down\rangle$. Then, we can say that the beamsplitters are operators B_1 and B_2 respectively, such that:

$$B_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad B_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Consider sending a photon through the bottom path at the start. Mathematically, we would find that:

$$\begin{aligned} |out\rangle &= B_2 B_1 |down\rangle \\ &= B_2 \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned} \tag{11.6}$$

This means that the photon comes out of the upper path at the end. Now consider adding the bomb into the system. There are three possibilities:

1. Bomb blows up, so neither detector receives a photon
2. Bomb doesn't blow up and detector 1 receives a photon
3. Bomb doesn't blow up and detector 2 receives a photon

Let us break this into two cases:

11.2.1 Non-Working Bomb

If we send the photon through the bottom path at the beginning, as we did before, we will find that Detector 1 will always receive the photon.

11.2.2 Working Bomb

If we send the photon through the bottom path at the beginning, we will find that the bomb will go off 50% of the time, Detector 1 will receive a photon 25% of the time, and Detector 2 will receive a photon 25% of the time.

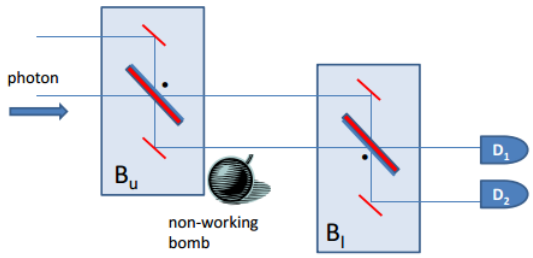
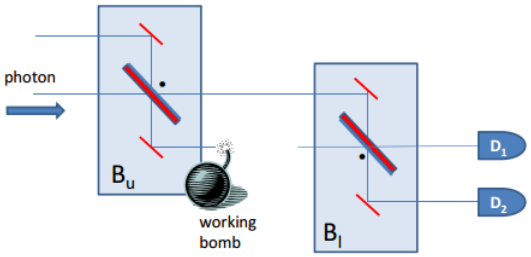


Figure 19: Figure 18, with a bomb added in. Awesome.



11.2.3 Combining Both Results

Basically, we have this:

Bomb Status	**"Boom"**	D_1	D_2
Defective	-	100%	-
Working	50%	25%	25%

Let's say that, initially, 70% of bombs are working. Then we have 17.5% guaranteed working bombs ($0.25 \cdot 70\%$), 47.5% undetermined bombs ($1 \cdot 30\% + 0.25 \cdot 70\%$) and 35% exploded bombs ($0.5 \cdot 70\%$).

12 The Quantum Zeno Effect

12.1 Preparation and Observations

Recall Figure 17. What exactly is this operator, U? If we were to look inside this box, we would see that it contains a homogeneous magnetic field with an x-orientation and a certain strength:

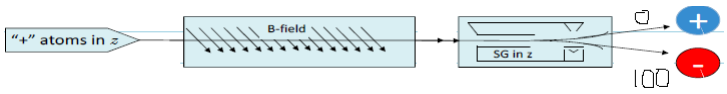


Figure 20: A box!

The unitary operator can expressed as:

$$U = i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(We have a global phase i in front due to consistency). Now let us split up this box (of length L) into n different sections, each with length ℓ:

Figure 21: Boxes!

Each box can be expressed using:

$$U_\ell = \begin{bmatrix} \cos\left(\frac{\ell}{L}\frac{\pi}{2}\right) & i\sin\left(\frac{\ell}{L}\frac{\pi}{2}\right) \\ i\sin\left(\frac{\ell}{L}\frac{\pi}{2}\right) & \cos\left(\frac{\ell}{L}\frac{\pi}{2}\right) \end{bmatrix} \tag{12.1}$$

Note that $\ell = 0$ gives $U_\ell = I$ and $\ell = L$ gives $U_\ell = U$. There is also the interesting property that:

$$U_\ell U_{\ell'} = U_{\ell+\ell'} \tag{12.2}$$

Dividing L into n equal parts gives:

$$U_{\frac{L}{n}} = \begin{bmatrix} \cos\left(\frac{\pi}{2n}\right) & \sin\left(\frac{\pi}{2n}\right) \\ i\sin\left(\frac{\pi}{2n}\right) & \cos\left(\frac{\pi}{2n}\right) \end{bmatrix} \tag{12.3}$$

which is equal to U_ℓ if we set $L = n\ell$. Then:

$$\left(U_{\frac{L}{n}}\right)^n = U \tag{12.4}$$

Now let us separate each individual box by an equal distance between each. Note that this does not change the outcome of the experiment!

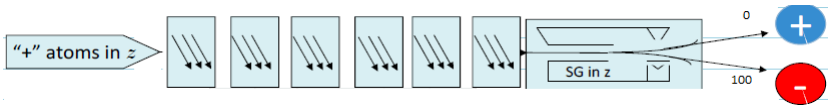


Figure 22: Separated boxes!

Now let us insert a z-oriented Stern-Gerlach device in each spacing, where the z-down component is blocked:

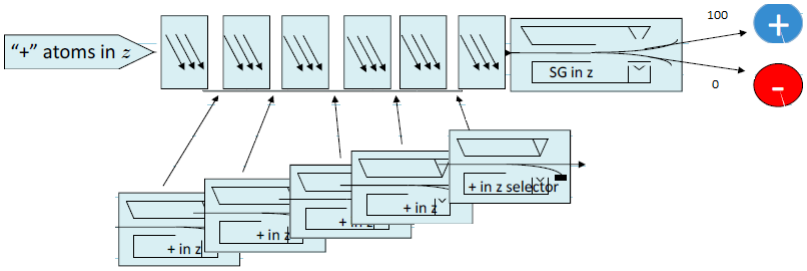


Figure 23: Yo dawg, I heard you like boxes, so I split your box into boxes and put some Stern-Gerlach boxes in between those boxes!

We find that all the atoms that emerge at the end are up-z oriented, and as the number of boxes (n) increases, more atoms make it to the end. The first part is more clear than the second part, so let's do a bit of math!

12.2 Quantitative Analysis

When we send the set of up-z atoms through the U_l operator and then through a Stern-Gerlach device (ie: one iteration), we find that:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\pi}{2n}\right) & \sin\left(\frac{\pi}{2n}\right) \\ -\sin\left(\frac{\pi}{2n}\right) & \cos\left(\frac{\pi}{2n}\right) \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \cos\left(\frac{\pi}{2n}\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{12.5}$$

So the probability is:

$$\mathcal{P}(+) = \cos^2\left(\frac{\pi}{2n}\right) \tag{12.6}$$

For n iterations, we raise the product of matrices to the power of n which results in:

$$\left(\cos\left(\frac{\pi}{2n}\right)\right)^n \begin{pmatrix}1\\0\end{pmatrix} \tag{12.7}$$

The probability is then:

$$\left(\cos^2\left(\frac{\pi}{2n}\right)\right)^n \approx 1 - \frac{\pi^2}{4n} \tag{12.8}$$

(This is a Taylor series expansion at $n = \infty$). We can then see, as $n \to \infty$, the probability is 1, so the end state is always z_+.

12.3 Physical Interpretation

For the physical case, we should recall that measurement is interaction. If we had someone observing a z-oriented Stern-Gerlach device, all atoms would always pass but we would see what path they take. This interaction causes a change to the system.

The magnetic field tries to change the spin but it is constantly interrupted by a measurement which turns the probability amplitude into a probability. This freezes the evolution of the system in the initial state. This is called the **quantum Zeno effect**: the constant observation of a system causes the system to behave differently than it would if it was not being observed.

13 Density Matrices and Bloch Vectors

13.1 Trace

The **trace** of a matrix can be represented in bra-ket notation and in coordinate notation:

$$Tr(A) = \sum_{i=1}^{d} \langle \phi_i | A | \phi_i \rangle \tag{13.1}$$

$$Tr(A) = \sum_{i=1}^{d} a_{i,i} \tag{13.2}$$

The trace is linear and has a cyclic property:

$$Tr(\alpha A + \beta B) = \alpha Tr(A) + \beta Tr(B) \tag{13.3}$$

$$Tr(ABC) = Tr(BCA) = Tr(CAB) \tag{13.4}$$

Also, in quantum mechanics, there is the property that:

$$Tr(|\psi\rangle\langle\phi|) = \langle\phi|\psi\rangle \tag{13.5}$$

And if A is a normal operator with eigenvalues λ_i:

$$Tr(A) = \sum_{i=1}^{d} \lambda_i \tag{13.6}$$

13.2 Density Matrix

13.2.1 Introduction: Mixed State vs. Pure State

A pure state source $|\psi\rangle$ follows the prediction rule:

$$\mathcal{P}(i) = |\langle \phi_i | \psi \rangle|^2$$

as we have seen before. A mixed state source $\{\mathcal{P}(\text{signal } s), |\psi_s\rangle\}_{s=1}^{n}$ follows the prediction rule:

$$\mathcal{P}(i) = \sum_{s=1}^{n} \mathcal{P}(\text{signal } s)|\langle\phi_i|\psi_s\rangle|^2$$

while the expectation value prediction for an observable A for a mixed source state is given by:

$$\begin{aligned}\langle A\rangle &= \sum_s \mathcal{P}(s)\langle\psi_s|A|\psi_s\rangle \\ &= \sum_s \mathcal{P}(s) Tr\left(A|\psi_s\rangle\langle\psi_s|\right) \\ &= Tr\left(A\underbrace{\left(\sum_s \mathcal{P}(s)|\psi_s\rangle\langle\psi_s|\right)}_{\text{density matrix}}\right)\end{aligned} \tag{13.7}$$

Wow, those brackets are obnoxious.

13.2.2 Definition

The **density matrix** is seen below in a less messy way:

$$\rho := \sum_s \mathcal{P}(s)|\psi_s\rangle\langle\psi_s| \tag{13.8}$$

It has the following properties:

$$\begin{gathered}\rho = \rho^\dagger \\ \langle\psi|\rho|\psi\rangle \geq 0 \quad \forall|\psi\rangle \\ Tr(\rho) = 1\end{gathered}$$

The density matrix completely describes the source. If several source preparations yield the same density matrix, then the sources are completely equivalent. For a given density matrix, one can always find one particular source using spectral decomposition. For a mixed state source:

$$\langle A\rangle = Tr(\rho A) \tag{13.9}$$

Example: Consider a source that first emits 50% $|+\rangle$ and 50% $|-\rangle$, and later emits 50% $|+\rangle_x$ and 50% $|-\rangle_x$. What are the density matrices?

Solution: Let ρ_1 be the first situation and let ρ_2 be the second situation. Then:

$$\rho_1 = \frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$
$$\rho_2 = \frac{1}{2}\cdot\frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} + \frac{1}{2}\cdot\frac{1}{2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

The density matrices are identical, so all predictions in both situations will be identical. $\surd$

It does not matter how one prepares a source by mixing pure state sources; if the density matrices of two different preparation methods are identical, then the mixed state sources are equivalent.

13.3 The Bloch Vector

13.3.1 Expansion of the Density Matrix

Recall the Pauli operators. Any density matrix can be expanded using the Pauli operators as such:

$$\rho = c_0 I + c_x\sigma_x + c_y\sigma_y + c_z\sigma_z \tag{13.10}$$

Note that $Tr(I) = 2, Tr(\sigma_x) = Tr(\sigma_y) = Tr(\sigma_z) = 0, Tr(\rho) = 1$. This leads us to find that $c_0 = \frac{1}{2}$. It is then more convenient to write:

$$\rho = \frac{1}{2}(I + \vec{v}\cdot\vec{\sigma}) \tag{13.11}$$

where $\vec{v}\cdot\vec{\sigma} = v_x\sigma_x + v_y\sigma_y + v_z\sigma_z$. Since we are apparently dealing with a mixed source state,

$$\begin{aligned}\langle\sigma_x\rangle &= Tr\left(\frac{1}{2}(I+\vec{v}\cdot\vec{\sigma})\sigma_x\right)\\ &= \frac{1}{2}Tr((I+v_x\sigma_x+v_y\sigma_y+v_z\sigma_z))\\ &= \frac{1}{2}Tr(\sigma_x+v_xI-iv_y\sigma_z+iv_z\sigma_y)\\ &= v_x\end{aligned} \tag{13.12}$$

Similarly,

$$\langle\sigma_y\rangle = v_y \tag{13.13}$$

$$\langle\sigma_z\rangle = v_z \tag{13.14}$$

$\vec{v}$ is what we call the **Bloch Vector**.

13.3.2 Properties

$\vec{v}$ is a 3D vector over the real numbers. Since the density matrix and the Pauli operators are Hermitian, we find:

$$\begin{aligned} & p = p^\dagger\\ \iff \quad & \frac{1}{2}(I+\vec{v}\cdot\vec{\sigma}) = \frac{1}{2}(I+(\vec{v})^*\cdot\vec{\sigma})\\ \iff \quad & \vec{v} = (\vec{v})^*\end{aligned}$$

Another property is that $\parallel \vec{v} \parallel \leq 1$. This can be shown by first writing:

$$\vec{v} = \parallel \vec{v} \parallel \hat{\mathbf{e}}_v$$

where $\hat{\mathbf{e}}_v$ is the unit vector in the direction of $\vec{v}$. Then:

$$\rho = \frac{1}{2}(I+\sqrt{\parallel \vec{v} \parallel}\hat{\mathbf{e}}_v\cdot\vec{\sigma})$$

but $\hat{\mathbf{e}}_v\cdot\sigma = \pm 1$, so:

$$\rho = \frac{1}{2}(1\pm\sqrt{\parallel \vec{v} \parallel})$$

But $\rho \geq 0$ so then $1-\sqrt{\parallel \vec{v} \parallel} \geq 0$, which of course leads us to our desired property. In a nutshell:

1. The components of $\vec{v}$ are real

2. $\| \vec{v} \| \leq 1$

And:

$$\vec{v} = (v_x, v_y, v_z) = (\langle \sigma_x \rangle, \langle \sigma_y \rangle, \langle \sigma_z \rangle) \tag{13.15}$$

13.3.3 Bloch Sphere

Any density matrix of a two-dimensional quantum mechanical system is completely characterized by its Bloch vector $\vec{v}$ and can therefore be represented as a vector within a 3D sphere:

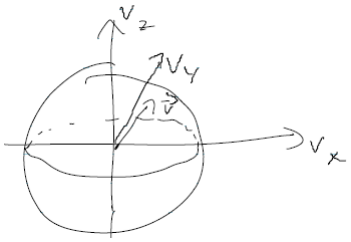


Figure 24: Riveting.

Pure states are always on the surface. If $\vec{v} \cdot \vec{v} = 1$ then ρ has eigenvalues 0 and 1, and can be represented as:

$$\rho = |\psi\rangle\langle\psi| \tag{13.16}$$

for some vector $|\psi\rangle$ which is the eigenvector to eigenvalue 1.

14 Dynamics

14.1 The Schrödinger Equation

Postulate 6: The time evolution of a quantum system is determined by the **Hamiltonian** (total energy) operator $H(t)$ through the **Schrödinger Equation**:

$$i\hbar\frac{d}{dt}|\Psi(t)\rangle = H(t)|\Psi(t)\rangle \tag{14.1}$$

Reconsider Figure 20. Note that $|\psi(t=0)\rangle = |+\rangle$ and $|\psi(t_f)\rangle = |-\rangle$. This means that the system is time-dependent!

14.1.1 The Hamiltonian

The **Hamiltonian**, also called the **Hamilton Operator**, denoted H, defines the dynamics of the system. The units of this operator are Joules (energy!).

14.2 Derivation of the S.E.

Consider the state $|\Psi(t)\rangle$ evolving over time. Well, if a state is changing at all, it must be acted upon by an operator. That makes sense. However, this operator itself depends on time, so then:

$$|\Psi(t+\Delta t)\rangle = U(t,\Delta t)|\psi(t)\rangle \tag{14.2}$$

For small time steps Δt:

$$U(t,\Delta t) = 1 + M(t)\Delta t + O(\Delta t^2) \tag{14.3}$$

For some operator $M(t)$. First, note that $U(t,0) = I$. The above step is a Taylor series expansion; we assume that the unitary operators depend smoothly on the time difference Δt, so we can perform a Taylor expansion of U at its value for $\Delta t = 0$. To do this, let us use coordinate representation:

$$U(\Delta t) := \begin{bmatrix} U_{1,1}(\Delta t) & \cdots & U_{1,n}(\Delta t) \\ \vdots & \ddots & \vdots \\ U_{n,1}(\Delta t) & \cdots & U_{n,n}(\Delta t) \end{bmatrix} \tag{14.4}$$

where we Taylor expand each element with respect to Δt. However, since $U(\Delta t = 0) = I$, we find that $U_{i,j}|_{\Delta t=0} = \delta_{i,j}$, the Kronecker-Delta. Then:

$$\begin{aligned} U(\Delta t) &= I + \begin{bmatrix} \frac{d}{d\Delta t}(U_{1,1}(\Delta t))|_{\Delta t=0} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{d}{d\Delta t}(U_{n,n}(\Delta t))|_{\Delta t=0} \end{bmatrix} \Delta t + O(\Delta t^2) \\ &= M \end{aligned} \tag{14.5}$$

Plugging this into Equation (14.2), rearranging and using properties of the series $O(\Delta t^2)$ gives:

$$\frac{|\Psi(t+\Delta t)\rangle - |\Psi(t)\rangle}{\Delta t} = M(t)|\Psi(t)\rangle + O(\Delta t) \tag{14.6}$$

Since U is Hermitian, we first find that:

$$U^\dagger = I + M^\dagger \Delta t + O(\Delta t^2) \tag{14.7}$$

where the operators being a function of time is implied. Then:

$$U^\dagger U = I + (M + M^\dagger)\Delta t + O(\Delta t^2) = I \tag{14.8}$$

Then it follows that:

$$M(t) + M^\dagger(t) = 0 \tag{14.9}$$

We want to parameterization M to introduce H, which is also Hermitian. So:

$$M(t) = -\frac{i}{\hbar} H(t) \tag{14.10}$$

So we take the limit as $\Delta t \to 0$ and rearrange:

$$i\hbar \frac{d}{dt}|\Psi(t)\rangle = H(t)|\Psi(t)\rangle \tag{14.11}$$

The structure of Schrödinger's equation is not quantum mechanical, but just a true statement about smoothly changing time dependent vectors. Note that:

$$|\psi(t)\rangle = \sum_n c_n(t)|\phi_n\rangle$$

$$\Rightarrow \qquad \frac{d}{dt}|\psi(t)\rangle = \sum_n \frac{d}{dt}c_n(t)|\phi_n\rangle$$

14.3 Energy Eigenbasis

We will only consider the Hamiltonian to be time-independent, so that:

$$i\hbar\frac{d}{dt}|\Psi(t)\rangle = H|\Psi(t)\rangle \tag{14.12}$$

The state can be represented using an expansion of the eigenstates of H:

$$|\Psi(t)\rangle = \sum_n c_n(t)|E_n\rangle \tag{14.13}$$

So then the left hand side and right hand side respectively are:

$$i\hbar\frac{d}{dt}|\Psi(t)\rangle = \sum_n i\hbar\left(\frac{d}{dt}c_n(t)\right)|E_n\rangle \tag{14.14}$$

$$H|\Psi(t)\rangle = \sum_n c_n(t)H|E_n\rangle = \sum_n E_n c_n(t)|E_n\rangle \tag{14.15}$$

This implies that:

$$i\hbar\frac{d}{dt}c_n(t) = E_n c_n(t) \tag{14.16}$$

with an initial value $c_n(0) = c_n$. The solution, of course, is trivial:

$$c_n(t) = c_n e^{-\frac{i}{\hbar}E_n t} \tag{14.17}$$

To find the state, we must find the eigenvectors ($|E_n\rangle$) and eigenvalues (E_n) of the Hamiltonian, expand the initial state in the eigenbasis:

$$|\Psi(0)\rangle = \sum_n c_n|E_n\rangle \tag{14.18}$$

then we write down the solution, DUH:

$$|\Psi(t)\rangle = \sum_n c_n e^{-\frac{i}{\hbar}E_n t}|E_n\rangle \tag{14.19}$$

15 Half-Spin Particle Dynamics

15.1 Half-Spin Particle in an External Magnetic Field

Recall Equation (3.1). The Hamiltonian for a half-spin particle can be written as:

$$H = \frac{e}{m_e}\vec{B}\cdot\vec{S} = \frac{e}{m_e}(B_xS_x + B_yS_y + B_zS_z) \tag{15.1}$$

If we consider the magnetic field to point in the z-direction, without a loss of generality, we find:

$$\begin{aligned} H &= \frac{e}{m_e}B_zS_z \\ &= \omega S_z \\ &= \frac{1}{2}\omega\hbar|+\rangle\langle+| - \frac{1}{2}\omega\hbar|-\rangle\langle-| \end{aligned} \tag{15.2}$$

This gives us eigenvalues of $\pm\frac{\hbar\omega}{2}$ with corresponding eigenvectors $|+\rangle$ and $|-\rangle$.

15.1.1 General Hamiltonian

A quick side note: any Hamiltonian can be written as:

$$H = \sum_n E_n|E_n\rangle\langle E_n| \tag{15.3}$$

where E_n are eigenvalues, $|E_n\rangle$ are eigenstates.

15.2 Time Evolution

15.2.1 Time Evolution for Energy Eigenstates

Consider $|\psi(0)\rangle = |+\rangle$. Then:

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\frac{\hbar\omega}{2}t}|+\rangle = e^{-\frac{i\omega}{2}t}|+\rangle \tag{15.4}$$

This means that if a system is prepared initially in an eigenstate of the Hamiltonian, then time evolution results in a time-dependent global phase (I think this should say time-independent; refer to the equations below for

my reasoning). Expectation values of all observables remain unchanged! An initial state as an eigenstate evolves as seen here:

$$|\psi(0)\rangle = |E_n\rangle \tag{15.5}$$
$$|\psi(t)\rangle = e^{-\frac{iE_n}{\hbar}t}|E_n\rangle \tag{15.6}$$

So then:

$$\begin{aligned} \langle A\rangle_{t=0} &= \langle\psi(0)|A|\psi(0)\rangle && (15.7)\\ &= \langle E_n|A|E_n\rangle \\ \langle A\rangle_t &= \langle\psi(t)|A|\psi(t)\rangle && (15.8)\\ &= e^{\frac{iE_n}{\hbar}t}\langle\psi(0)|A|\psi(0)\rangle e^{-\frac{iE_n}{\hbar}t} \\ &= \langle\psi(0)|A|\psi(0)\rangle \\ &= \langle A\rangle_{t=0} && (15.9) \end{aligned}$$

15.2.2 Time Evolution for General States

Consider $|\psi(0)\rangle = \cos\left(\frac{\theta}{2}\right)|+\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|-\rangle$. Then:

$$\begin{aligned} c_+(0) &= \cos\left(\frac{\theta}{2}\right) \\ c_-(0) &= \sin\left(\frac{\theta}{2}\right)e^{i\phi} \end{aligned} \tag{15.10}$$

So then:

$$\begin{aligned} c_+(t) &= e^{-\frac{i\omega}{2}t}\cos\left(\frac{\theta}{2}\right) \\ c_-(t) &= e^{\frac{i\omega}{2}t}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{aligned} \tag{15.11}$$

which leads to:

$$|\psi(t)\rangle = e^{-\frac{i\omega}{2}t}\cos\left(\frac{\theta}{2}\right)|+\rangle + e^{\frac{i\omega}{2}t}e^{i\phi}\sin\left(\frac{\theta}{2}\right)|-\rangle \tag{15.12}$$

But this can be rewritten:

$$|\psi(t)\rangle = \cos\left(\frac{\theta}{2}\right)|+\rangle + e^{i(\omega t+\phi)}\sin\left(\frac{\theta}{2}\right)|-\rangle \tag{15.13}$$

due to $e^{-\frac{i\omega}{2}t}$ being a common factor, and thus a global phase. So, the time evolution of a general state causes a relative phase between the two energy contributions. This phase oscillates with frequency ω and has an initial phase ϕ.

15.3 Spin Components for General States

I'm going to skip the math because it looks horrifying to type up, but if you use Equation (7.1) or Equation (7.10), you would find that:

$$\langle S_z\rangle = \frac{\hbar}{2}\cos(\theta) \tag{15.14}$$

$$\langle S_y\rangle = \frac{\hbar}{2}\sin(\theta)\sin(\omega t+\phi) \tag{15.15}$$

$$\langle S_x\rangle = \frac{\hbar}{2}\sin(\theta)\cos(\omega t+\phi) \tag{15.16}$$

16 Bloch Vector: A General Case

Recall that the Bloch vector is:

$$\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} \langle \sigma_x \rangle \\ \langle \sigma_y \rangle \\ \langle \sigma_z \rangle \end{pmatrix} \tag{16.1}$$

It turns out that:

$$\vec{v} = \frac{2}{\hbar} \begin{pmatrix} \langle S_x \rangle \\ \langle S_y \rangle \\ \langle S_z \rangle \end{pmatrix} \tag{16.2}$$

For a general spin direction S_n,

$$\langle S_n \rangle = \left\langle \vec{n} \cdot \vec{S} \right\rangle \tag{16.3}$$

$$= \left\langle \frac{\hbar}{2} \left({}_n|+\rangle\langle +|_n - {}_n|-\rangle\langle -|_n \right) \right\rangle \tag{16.4}$$

$$= \frac{\hbar}{2} \left(\mathcal{P}(+_n) - \mathcal{P}(-_n) \right) \tag{16.5}$$

$$= \frac{\hbar}{2} (2\mathcal{P}(+_n) - 1) \tag{16.6}$$

which leads to:

$$\mathcal{P}(+_n) = \frac{1}{2} \left(\frac{2}{\hbar} \langle S_n \rangle + 1 \right) \tag{16.7}$$

Since $S_n = \vec{n}\vec{\sigma}$,

$$\frac{2}{\hbar} \langle S_n \rangle = Tr \left(\rho \frac{2}{\hbar} \frac{\hbar}{2} \vec{n} \cdot \vec{\sigma} \right) = Tr(\rho \vec{n} \cdot \vec{\sigma}) = \langle \vec{n} \cdot \vec{\sigma} \rangle \tag{16.8}$$

Then:

$$\langle \vec{n} \cdot \vec{\sigma} \rangle = Tr\left(\frac{1}{2}(I + \vec{v} \cdot \vec{\sigma})\vec{n} \cdot \vec{\sigma}\right) \tag{16.9}$$

$$= Tr\left(\vec{n} \cdot \vec{\sigma}\right) + Tr\left(\vec{v} \cdot \vec{\sigma}\vec{n} \cdot \vec{\sigma}\right) \tag{16.10}$$

$$= 0 + \frac{1}{2}Tr\left(\sum_k v_k \sigma_k\right)\left(\sum_l n_l \sigma_l\right) \tag{16.11}$$

$$= \frac{1}{2}\sum_{k,l} v_k n_l Tr(\underbrace{\sigma_k \sigma_l}_{\text{see Wiki}}) \tag{16.12}$$

$$= \frac{1}{2}\sum_k v_k n_k Tr(I) \tag{16.13}$$

$$= \sum_k v_k n_k = \vec{v} \cdot \vec{n} \tag{16.14}$$

Plugging this into our equation for the probability gives:

$$\mathcal{P}(+_n) = \frac{1}{2}\left(1 + \vec{v} \cdot \vec{n}\right) \tag{16.15}$$

$$\mathcal{P}(-_n) = \frac{1}{2}\left(1 - \vec{v} \cdot \vec{n}\right) \tag{16.16}$$

If you do not understand the idea of $\vec{\sigma}$, see the Wikipedia article on Pauli matrices. Also: $\vec{n}$ is an arbitrary direction and $\vec{S_n}$ is the spin in that direction.

17 Applications of Time Evolution

Let us do a recap, first. The Hamiltonian of a system without outer influence is:

$$H = \sum_n E_n |E_n\rangle\langle E_n| \tag{17.1}$$

The initial state of a system can be described as:

$$|\psi(0)\rangle = \sum_n c_n |E_n\rangle \tag{17.2}$$

The state at a time t can be described as:

$$|\psi(t)\rangle = \sum_n c_n e^{-\frac{iE_n}{\hbar}t} |E_n\rangle \tag{17.3}$$

17.1 Constant Energy Shift

We shift all eigen-energies by Δ, so then:

$$H' = \sum_n (E_n + \Delta)|E_n\rangle\langle E_n| \tag{17.4}$$

$$|\psi'(0)\rangle = \sum_n |E_n\rangle \tag{17.5}$$

$$|\psi'(t)\rangle = e^{-\frac{i\Delta}{\hbar}t}|\psi(t)\rangle \tag{17.6}$$

Note that the energy shift leads to a time dependent global phase, so it is physically irrelevant.

17.2 Neutrino Oscillations

Neutrinos are relativistic leptons that interact via the weak interaction. The electron neutrino is found when the proton decays into the neutron, positron and the electron neutrino:

$$p \to n + e^+ + v_e$$

While the pion's decay process (pion to muon and muon neutrino) gives us the muon neutrino:

$$\pi^+ \to \mu^+ + v_\mu$$

There is also the tau neutrino (ν_τ). The three neutrino types can be said to correspond to states in a 3D vector state ($|v_e\rangle, |v_\mu\rangle, |v_\tau\rangle$). Then, the general state is given as:

$$|\psi\rangle = \alpha|v_e\rangle + \beta|v_\mu\rangle + \gamma|v_\tau\rangle \tag{17.7}$$

However, when considering the sun as a source, we can neglect tau neutrinos. The predicted fraction of muon and electron neutrinos does not match observation. We can explain this using quantum mechanics! Because the weak interaction is, y'know, weak, it can be neglected so neutrinos behave like free particles. We then need to consider the free Hamiltonian.

Neutrinos have mass and mass corresponds to energy. They also have kinetic energy due to momentum. Unfortunately, the Hamiltonian is relativistic. The eigenstates are not $|v_e\rangle, |v_\mu\rangle$ but some random states $|v_1\rangle, |v_2\rangle$. Then:

$$H = \sum_j E_j |v_j\rangle\langle v_j| \tag{17.8}$$

where $E_j = \sqrt{(pc)^2 + (m_i c^2)^2}$. We choose the mass eigenstates as a basis. So then:

$$|v_e\rangle = \cos\left(\frac{\theta}{2}\right)|v_1\rangle + \sin\left(\frac{\theta}{2}\right)|v_2\rangle = \begin{pmatrix}\cos\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right)\end{pmatrix} \tag{17.9}$$

$$|v_\mu\rangle = \sin\left(\frac{\theta}{2}\right)|v_1\rangle - \cos\left(\frac{\theta}{2}\right)|v_2\rangle = \begin{pmatrix}\sin\left(\frac{\theta}{2}\right)\\ -\cos\left(\frac{\theta}{2}\right)\end{pmatrix} \tag{17.10}$$

The time evolution of the electron neutrino in a 2D subspace is:

$$|\psi(0)\rangle = |v_e\rangle = \begin{pmatrix}\cos\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right)\end{pmatrix} \tag{17.11}$$

$$|\psi(t)\rangle = \begin{pmatrix}e^{-\frac{iE_1}{\hbar}t}\cos\left(\frac{\theta}{2}\right)\\ e^{-\frac{iE_2}{\hbar}t}\sin\left(\frac{\theta}{2}\right)\end{pmatrix} \tag{17.12}$$

We can apply this for the muon neutrino as well. We find that the chance of the electron neutrino "turning into" (?) the muon neutrino is:

$$\mathcal{P}(v_e \to v_\mu) = |\langle v_\mu | \psi(t) \rangle|^2 = ... = \sin^2(\theta) \sin^2\left(\frac{(E_1 - E_2)t}{2\hbar}\right) \tag{17.13}$$

We use the classical approximation $E \approx m_i c^2$ and we use the fact that $t = \frac{L}{c}$ (neutrinos travel at approximately the speed of light) so then:

$$\mathcal{P}(v_e \to v_\mu) = \sin^2\left(\frac{\theta}{2}\right) \sin^2\left(\frac{(m_1^2 - m_2^2)Lc^3}{4E\hbar}\right) \tag{17.14}$$

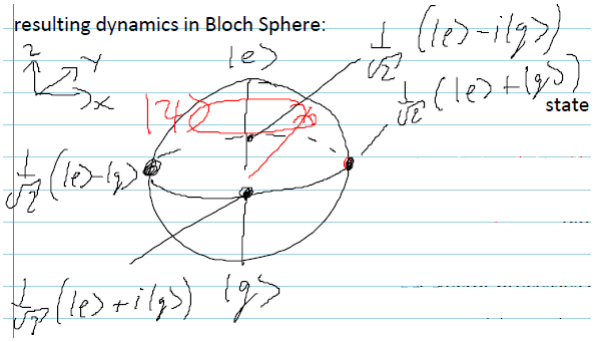
Using experimental parameters $m_1^2 - m_2^2 = 8 \times 10^{-}5eV$ we find that $\theta = 69°$.

17.3 Quantum Clocks

We can measure time with a mechanical pendulum. We place it in the right point, start a signal, let the pendulum go, stop the signal later and determine the new position. The components of a quantum clock consist of the two internal states of an atom: the ground state $|g\rangle$ and the excited state $|e\rangle$. We then find the Hamiltonian to be:

$$H = E_g|g\rangle\langle g| + E_e|e\rangle\langle e| \tag{17.15}$$

The state is $|\psi\rangle = \frac{1}{\sqrt{2}}(|e\rangle + |g\rangle)$ while the corresponding Bloch sphere is:



The Bloch vector precesses around the z-axis with frequency $\Omega = \frac{E_e - E_g}{\hbar}$. The state at a time t is:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{\frac{iE_e}{\hbar}t}|e\rangle + e^{\frac{iE_g}{\hbar}t}|g\rangle\right) \tag{17.16}$$

$$= \frac{1}{\sqrt{2}}e^{\frac{iE_g}{\hbar}t}\left(e^{\frac{i(E_e-E_g)}{\hbar}t}|e\rangle + |g\rangle\right) \tag{17.17}$$

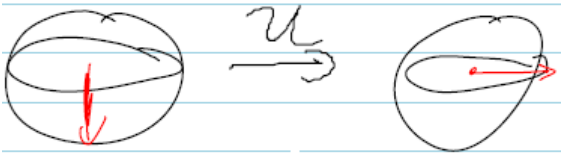
We are able to perform a unitary operation on the state because why not? If we turn on an additional strong magnetic field along the y-axis, it dominates over the free evolution (the Hamiltonian). We find that a rotation of 90 degree is:

$$U_{\frac{\pi}{2}} = \frac{1}{\sqrt{2}}(|e\rangle + |g\rangle)\langle g| + \frac{1}{\sqrt{2}}(|e\rangle - |g\rangle)\langle e| = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \tag{17.18}$$

The inverse operator is given by:

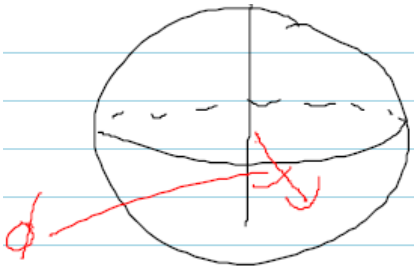
$$U_{-\frac{\pi}{2}} = |g\rangle\left(\frac{1}{\sqrt{2}}(\langle e| + \langle g|)\right) + |e\rangle\left(\frac{1}{\sqrt{2}}(\langle e| - \langle g|)\right) \tag{17.19}$$

So if we start in the state $|g\rangle$ and rotate it, we get the state $\frac{1}{\sqrt{2}}(|e\rangle + |g\rangle)$:

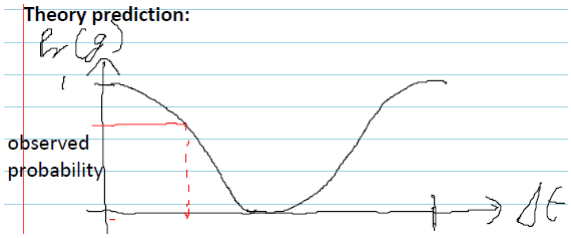


If we let the system freely evolve (under the Hamiltonian) for a time Δt, the Bloch vector will rotate in the xy-plane by an angle $\phi = \Omega\Delta t$:

And then if we apply the inverse operation, the Bloch vector will end up in the yz-plane:



We then perform a measurement in our basis $\{|e\rangle, |g\rangle\}$. We repeat these steps many times (with the same value of Δt each time) to find the probability of the system in a specific state. It turns out that:



The probability of the ground state is $\mathcal{P}(g) = \cos^2\left(\frac{\Omega}{2}\Delta t\right)$, where the period of the function is $T = \frac{2\pi}{\Omega}$.

18 Dynamics with Unitary Operators

Recall the solution to Schrödinger's Equation:

$$|\psi(t)\rangle = \sum_n c_n e^{-i\frac{E_n}{\hbar}t}|E_n\rangle \tag{18.1}$$

with initial conditions:

$$|\psi(0)\rangle = \sum_n c_n|E_n\rangle \tag{18.2}$$

Although we technically discussed it, we never formally wrote that:

$$|\psi(t)\rangle = U_t|\psi(0)\rangle \tag{18.3}$$

which can unfortunately be written as:

$$\begin{pmatrix} c_1(t) \\ \vdots \\ c_n(t) \end{pmatrix} = \begin{bmatrix} e^{-i\frac{E_1}{\hbar}t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{-i\frac{E_n}{\hbar}t} \end{bmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \tag{18.4}$$

so then U can be written as:

$$U = \sum_{k=1}^{n} e^{-i\frac{E_n}{\hbar}t}|E_k\rangle\langle E_k| \tag{18.5}$$

which looks a lot like the Hamiltonian:

$$H = \sum_{k=1}^{n} E_k|E_k\rangle\langle E_k| \tag{18.6}$$

Using the fact that e raised to a matrix (operators, in our case) can still be expanded in the same way with a Taylor series, we can write:

$$e^{-i\frac{H}{\hbar}t} = \sum_{l=0}^{\infty} \left(-\frac{i}{\hbar}t\right)^l H^l \tag{18.7}$$

$$= \sum_{l=0}^{\infty} \left(-\frac{i}{\hbar}t\right)^l \sum_{k=0}^{n} E_k^l |E_k\rangle\langle E_k| \tag{18.8}$$

$$= \sum_{k=0}^{n} \left(\sum_{l=0}^{\infty} \left(-\frac{i}{\hbar}t\right)^l E_k^l \right) |E_k\rangle\langle E_k| \tag{18.9}$$

$$= \sum_{k=0}^{n} e^{-i\frac{E_k}{\hbar}t} |E_k\rangle\langle E_k| \tag{18.10}$$

$$= U \tag{18.11}$$

Example: Given a Hamiltonian $H = wS_y$, we would find eigenvalues $\pm\frac{\hbar w}{2}$ with eigenvectors $|\pm\rangle_y$. We would then calculate:

$$U_t = e^{-i\frac{w}{2}t}{}_y|+\rangle\langle+|_y + e^{i\frac{w}{2}t}{}_y|-\rangle\langle-|_y$$

Changing this to our z-basis eventually spits out:

$$U_t = \begin{bmatrix} \cos\left(\frac{w}{2}t\right) & -\sin\left(\frac{w}{2}t\right) \\ \sin\left(\frac{w}{2}t\right) & \cos\left(\frac{w}{2}t\right) \end{bmatrix}$$

The alternative approach is to use the fact that, for any operator M where $M^2 = I$:

$$e^{iM\alpha} = \cos(\alpha)I + i\sin(\alpha)M \tag{18.12}$$

Since $H = wS_y = \frac{\hbar}{2}w\sigma_y$ where $\sigma_y^2 = I$, we find that:

$$\begin{aligned} U_t &= e^{-i\frac{w}{2}t\sigma_y} \\ &= \cos\left(-\frac{w}{2}t\right) I + i\sin\left(-\frac{w}{2}t\right)\sigma_y \\ &= \cos\left(\frac{w}{2}t\right) I - i\sin\left(\frac{w}{2}t\right)\sigma_y \\ &= \begin{bmatrix} \cos\left(\frac{w}{2}t\right) & -\sin\left(\frac{w}{2}t\right) \\ \sin\left(\frac{w}{2}t\right) & \cos\left(\frac{w}{2}t\right) \end{bmatrix} \end{aligned}$$

19 Introduction to Continuous Systems

Consider a measurement of a particle's position, x. We can write an associated ket $|x\rangle$ which characterizes the measurement outcome to detect the particle at this position. The position measurement gives mutually exclusive results, so that:

$$\langle x|x'\rangle = 0 \qquad \text{for } x \neq x'$$

Because I'm incredibly lazy, I'm going to post a picture of a table instead of writing it out myself:

	Discrete	Continuous
State	$\|\Psi\rangle$	$\|\Psi\rangle$
coordinate representation	$\|\Psi\rangle \doteq \begin{pmatrix} \Psi_1 \\ \vdots \\ \Psi_n \end{pmatrix}$	$\|\Psi\rangle \doteq \langle x\|\Psi\rangle =: \Psi(x)$ wavefunction (complex valued!) (position representation)
completeness relations	$\mathbb{1} = \sum_{k=1}^{n} \|\phi_k\rangle\langle\phi_k\|$ orthonormal basis	$\mathbb{1} = \int_{-\infty}^{+\infty} dx\ \|x\rangle\langle x\|$ position states
dual vector	$\langle\Psi\| \doteq (\ \Psi_1^*, \ \dots\ \Psi_n^*\ )$	$\langle\Psi\| \doteq \langle\Psi\|x\rangle = \Psi(x)^*$
scalar product	$\langle\Phi\|\Psi\rangle = \sum_{k=1}^{n} \Phi_k^* \Psi_k$	$\langle\Phi\|\Psi\rangle = \int_{-\infty}^{+\infty} dx\ \Phi(x)^* \Psi(x)$
normalization	$1 \stackrel{!}{=} \langle\Psi\|\Psi\rangle = \sum_{k=1}^{n} \|\Psi_k\|^2$	$1 \stackrel{!}{=} \langle\Psi\|\Psi\rangle = \int_{-\infty}^{+\infty} dx\ \|\Psi(x)\|^2$
probability prediction	$\Pr(\text{"k"}) = \|\langle\phi_k\|\Psi\rangle\|^2 = \|\Psi_k\|^2$ probability	$p(x)\,dx = \|\langle x\|\Psi\rangle\|^2\,dx = \|\Psi(x)\|^2\,dx$ probability density

So, the idea here is that the x in $\psi(x)$ plays the same role as the index k in $|\psi\rangle = \psi_k$. In addition, we cannot turn any function into a position wave-function just by normalization. If the integral for the normalization condition fails to converge, then the function doesn't work.

19.1 Probability

The next thing to take away from this is that our equation for probability is different. If we attempt to find the particle in one exact position, the

probability of us finding it is 0. Magic. What we want to do instead is ask what the probability is of finding the particle in a certain region of space, such as inside the interval $[x_1, x_2]$. We denote probability densities, as seen above, as:

$$p(x) = |\psi(x)|^2 \tag{19.1}$$

which can sort of be thought of as an infinitesimal probability. Or, instead, we can write the probability to find the particle in the interval $[x, x + dx]$ as:

$$p(x)dx = |\psi(x)|^2 \, dx \tag{19.2}$$

19.2 Position Operator

The mean value of the position is given by:

$$\langle x \rangle := \int x p(x) \, dx = \int x |\psi(x)|^2 \, dx \tag{19.3}$$

However, this can be rewritten:

$$\begin{aligned}
\langle x \rangle &= \int x |\langle x|\psi\rangle|^2 \, dx && (19.4)\\
&= \langle \psi | \left(\underbrace{\int x|x\rangle\langle x| \, dx}_{:=\hat{x}} \right) |\psi\rangle && (19.5)\\
&= \langle \psi | \hat{x} | \psi \rangle && (19.6)\\
&= \langle \hat{x} \rangle && (19.7)
\end{aligned}$$

We have to be careful here because of the amount of x's we have going around. The hat on the x represents the operator nature. It is used in these notes whenever it is ambiguous whether or not we are referring to an operator or a scalar, or something else. So, the operator for x is:

$$\hat{x} = \int_{-\infty}^{\infty} x|x\rangle\langle x| \, dx \tag{19.8}$$

19.2.1 Coordinate Representation

We apparently need to get the proper coordinate representations of our operators so that we can translate all knowledge from the finite dimensional case to the new infinite dimension case. For the former case:

$$\hat{A} = I\hat{A}I \tag{19.9}$$

$$= \left(\sum_{k=1}^{d} |\phi_k\rangle\langle\phi_k|\right) \hat{A} \left(\sum_{l=1}^{d} |\phi_k\rangle\langle\phi_k|\right) \tag{19.10}$$

$$= \sum_{k=1}^{d}\sum_{l=1}^{d} |\phi_k\rangle \underbrace{\langle\phi_k|\hat{A}|\phi_l\rangle}_{:=\alpha_{k,l}} \langle\phi_l| \tag{19.11}$$

$$= \sum_{k=1}^{d}\sum_{l=1}^{d} \alpha_{k,l}|\phi_k\rangle\langle\phi_l| \tag{19.12}$$

so that $\alpha_{k,l}$ itself is the coordinate representation of the operator. In infinite dimensions,

$$\hat{B} = I\hat{B}I \tag{19.13}$$

$$= \left(\int_{-\infty}^{\infty} |x\rangle\langle x|\, dx\right) \hat{B} \left(\int_{-\infty}^{\infty} |x'\rangle\langle x'|\, dx'\right) \tag{19.14}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x\rangle \underbrace{\langle x|\hat{B}|x'\rangle}_{:=f(x,x')} \langle x'|\, dx\, dx' \tag{19.15}$$

where the function $f(x, x')$ is now the coordinate representation of the operator B with respect to the position basis $|x\rangle$. We can use this to calculate the expectation value:

$$\langle\psi|\hat{B}|\psi\rangle = \langle\psi|I\hat{B}I|\psi\rangle \tag{19.16}$$

$$= \langle\psi| \int_{-\infty}^{\infty} |x\rangle\langle x|\, dx \hat{B} \int_{-\infty}^{\infty} |x'\rangle\langle x'|\, dx'|\psi\rangle \tag{19.17}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle\psi|x\rangle\langle x|\hat{B}|x'\rangle\langle x'|\psi\rangle\, dx\, dx' \tag{19.18}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi^*(x) f(x,x')\psi(x)\, dx\, dx' \tag{19.19}$$

which is the analogue of the finite dimension case. The form of the position operator (Equation (19.3)) looks similar to the representation of an operator in the eigenbasis of the operator. Consider:

$$\langle x|A|x'\rangle = \langle x| \left(\int_{-\infty}^{\infty} x''|x''\rangle\langle x''|\, dx'' \right) |x'\rangle \tag{19.20}$$

$$= \int_{-\infty}^{\infty} x''\langle x|x''\rangle\langle x''|x'\rangle\, dx'' \tag{19.21}$$

Where:

$$\langle x|x''\rangle = \begin{cases} 0 & x \neq x'' \\ ? & x = x'' \end{cases} \tag{19.22}$$

It turns out that we cannot choose $\langle x|x''\rangle = 1$ for the second case because the integration would always give zero, as the overlap functions are different from zero only in single points. What? Okay. The answer is given via the Dirac delta function. This special role is actually a consequence of the states not being proper physical state, so they do not have normalizable wave functions.

$$|\psi\rangle \neq |x\rangle \tag{19.23}$$

In other words, we will never prepare a system in the state of the position operator. Still, they form a convenient basis to work with. We don't need to worry about the fact that we cannot normalize them.

20 The Position Operator

We introduced the position operator as

$$\hat{x} = \int_{-\infty}^{\infty} x|x\rangle\langle x|\, dx \tag{20.1}$$

where

$$\langle \hat{x} \rangle = \int x|\psi(x)|^2\, dx \tag{20.2}$$

and we got to a point where

$$\langle x|\hat{x}|x'\rangle = \int_{-\infty}^{\infty} x''\langle x|x''\rangle\langle x''|x'\rangle\, dx'' \tag{20.3}$$

Etc. So, we use the Dirac delta function and write:

$$\langle x|x'\rangle = \delta(x - x') \tag{20.4}$$

20.1 The Dirac Delta Function

So, the DDF is pretty interesting. For a continuous function $f(x)$,

$$\int_a^b \delta(x-c)f(x)dx = \begin{cases} f(c) & x \in (a,b) \\ 0 & \text{otherwise} \end{cases} \tag{20.5}$$

The DDF isn't really a function. It's kinda weird. Look at some of the properties on Wiki. One we should point out though is:

$$\delta(x - c) = \delta(c - x) \tag{20.6}$$

So then we can write:

$$\langle x|\hat{x}|x'\rangle = x\delta(x - x') \tag{20.7}$$

which can lead us to:

$$\hat{x} = \int_{-\infty}^{\infty} x|x\rangle\langle x|\, dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\delta(x-x')|x\rangle\langle x|\, dx\, dx' \tag{20.8}$$

20.2 Eigenstates of the Position Operator

Well. I think we're just going to assume the same bounds we've been using the entire time, sorry if that's a bit confusing.

$$\hat{x}|x\rangle = \int x'|x'\rangle\langle x'|x\rangle\, dx' \tag{20.9}$$

$$= \int x'\delta(x'-x)|x'\rangle\, dx \tag{20.10}$$

$$= x|x\rangle \tag{20.11}$$

The states $|x\rangle$ are eigenstates of the position operator, corresponding to eigenvalues x.

20.3 Functions of Position Operators

Later on, we will use the Hamiltonian that describes the dynamics of a point particle in a potential. The classical energy can be written as:

$$E = \frac{p^2}{2m} + V(x) \tag{20.12}$$

(momentum, mass, position). We will turn this into a Hamilton operator of the form:

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + V(\hat{x}) \tag{20.13}$$

where $\hat{p}$ is the momentum operator (not yet defined) and $V(\hat{x})$ is the operator form of the potential. To define the latter, we write the classical potential as a Taylor series with arbitrary coefficients c_n:

$$V(x) = \sum_{n=0}^{\infty} \frac{1}{n!} c_n x^n \tag{20.14}$$

so then:

$$V(\hat{x}) := \sum_{n=0}^{\infty} \frac{1}{n!} c_n \hat{x}^n \tag{20.15}$$

where:

$$\hat{x}^n = \int_{-\infty}^{\infty} x^n |x\rangle\langle x| \, dx \tag{20.16}$$

which actually simplifies things!

$$V(\hat{x}) = \sum_{n=0}^{\infty} \frac{1}{n!} c_n \int_{-\infty}^{\infty} x^n |x\rangle\langle x| \, dx \tag{20.17}$$

$$= \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{1}{n!} c_n x^n |x\rangle\langle x| \, dx \tag{20.18}$$

$$= \int_{-\infty}^{\infty} V(x) |x\rangle\langle x| \, dx \tag{20.19}$$

Using coordinate representation:

$$V(\hat{x}) = V(x)\delta(x - x') \tag{20.20}$$

21 Momentum Operator

21.1 The Translation Operator

Momentum is linked to velocity, which means it's linked to a shift in position. Let us first consider a shift in position: a translation. We define the translation operator by its action onto position eigenstates:

$$\hat{T}(a)|x\rangle = |x + a\rangle \tag{21.1}$$

where $\hat{T}(0) = I$ and it is a unitary operator. Similar to our discussion of the time evolution operator, we can do an expansion of the operator as such:

$$\hat{T}(dx) = I - \frac{i}{\hbar}\hat{p}dx \tag{21.2}$$

where $\hat{p}$ is a self-adjoint operator. The physical dimension of this operator is that of momentum. We define the momentum as the operator affecting the infinitesimal change of a quantum state in an infinitesimal translation.

21.2 The Momentum Operator

The time-independent momentum operator can be given by

$$\hat{T}(a) = e^{-\frac{i}{\hbar}\hat{P}a} \tag{21.3}$$

21.2.1 Coordinate Representation

Consider

$$\begin{aligned}
\langle x|\hat{P}|x'\rangle &= \langle x| - \frac{i\hbar}{dx}(I - \hat{T}(dx)|x'\rangle \\
&= -\frac{i\hbar}{dx}(-\langle x|x'\rangle + \langle x|x' + dx\rangle) \\
&= i\hbar\frac{\langle x|x' + dx\rangle - \langle x|x'\rangle}{dx} \\
&= \frac{\hbar}{i}\frac{\delta(x - x' - dx) - \delta(x - x')}{-dx} \\
&= \frac{\hbar}{i}\frac{\delta(x - x' + (-dx)) - \delta(x - x')}{-dx} \\
&= \frac{\hbar}{i}\frac{d}{dx}(\delta(x - x')) \qquad (21.4)
\end{aligned}$$

If we skip some steps in the next process (refer to the introduction of continuous systems for the steps we skip), we can find that

$$\begin{aligned}
\langle\hat{p}\rangle &= \langle\psi|\hat{p}|\psi\rangle \qquad (21.5) \\
&= \int\int \psi^*(x)\frac{\hbar}{i}\frac{d}{dx}(\delta(x - x'))\psi(x')\,dx\,dx' \qquad (21.6) \\
&= \left(\int \psi^*(x)\frac{\hbar}{i}\,dx\right)\frac{d}{dx}\int \delta(x - x')\psi(x')\,dx' \qquad (21.7) \\
&= \int \psi^*(x)\frac{\hbar}{i}\frac{d}{dx}(\psi(x))\,dx \qquad (21.8)
\end{aligned}$$

21.2.2 Eigenstates of the Momentum Operator

We are, by definition, looking for states $|p\rangle$ that have the property

$$\hat{p}|p\rangle = p|p\rangle$$

To determine $\langle x|p\rangle$, consider

$$\langle x|\hat{p}|p\rangle = p\langle x|p\rangle \qquad (21.9)$$

(this is a viable step due to the fact that $|p\rangle$ is an eigenstate). However, we also know

$$\langle x|\hat{p}|p\rangle = \int \delta(x-x')\frac{d}{dx'}\left(\frac{\hbar}{i}\langle x'|p\rangle\right)\,dx' \tag{21.10}$$

$$= \frac{\hbar}{i}\frac{d}{dx}(\langle x|p\rangle) \tag{21.11}$$

which gives us

$$p\langle x|p\rangle = \frac{\hbar}{i}\frac{d}{dx}(\langle x|p\rangle) \tag{21.12}$$

which has the solution

$$\langle x|p\rangle = Ne^{\frac{ipx}{\hbar}} \tag{21.13}$$

This is the wave function of a momentum eigenstate. It cannot be normalized, since

$$|\langle x|p\rangle|^2 = |N|^2 \tag{21.14}$$

So momentum eigenstates are examples of kets that do not correspond directly to physical states (that we are used to). We can show that (but we won't because we're lazy),

$$\langle p'|p\rangle = |N|^2 2\pi\hbar\delta(p-p') \tag{21.15}$$

so we choose $N = \frac{1}{\sqrt{2\pi\hbar}}$. Then,

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{ipx}{\hbar}} \tag{21.16}$$

Momentum has the completeness property

$$I = \int_{-\infty}^{\infty} |p\rangle\langle p|\,dp \tag{21.17}$$

and we can represent it using the operator, as such

$$\hat{p} = \int_{-\infty}^{\infty} p|p\rangle\langle p|\,dp \tag{21.18}$$

It is quite similar to the position operator.

21.3 Heisenberg Uncertainty Revisited

Consider

$$\hat{x}|\psi\rangle = \int |x\rangle\langle x|\hat{x}|\psi\rangle \, dx \tag{21.19}$$

$$= \int x\psi(x)|\psi\rangle \, dx \tag{21.20}$$

$$= x\psi(x) \tag{21.21}$$

and also

$$\hat{p}|\psi\rangle = \left(\int \int |x\rangle\langle x|\hat{p}|x'\rangle\langle x'| \, dx \, dx' \right) |\psi\rangle \tag{21.22}$$

$$= \int \int \frac{h}{i}\frac{d}{dx}(\delta(x-x'))\psi(x')|x\rangle \, dx \, dx' \tag{21.23}$$

$$= \int \frac{\hbar}{i}\frac{d\psi(x)}{dx}|x\rangle \, dx \tag{21.24}$$

$$= \frac{\hbar}{i}\frac{d\psi(x)}{dx} \tag{21.25}$$

So then

$$\hat{x}\hat{p}|\psi\rangle = x\frac{\hbar}{i}\frac{d\psi(x)}{dx} \tag{21.26}$$

$$\hat{p}\hat{x}|\psi\rangle = \frac{\hbar}{i}\frac{d}{dx}(x\psi(x)) \tag{21.27}$$

$$= \frac{\hbar}{i}\psi(x) + \frac{\hbar}{i}x\frac{d\psi(x)}{dx} \tag{21.28}$$

which implies

$$(\hat{x}\hat{p} - \hat{p}\hat{x})|\psi\rangle = i\hbar\psi(x) \tag{21.29}$$

which itself implies, if you recall the commutator:

$$[\hat{x}, \hat{p}] = i\hbar \tag{21.30}$$

We're going to skip ALL the steps, but if we did the math, we would find that

$$\Delta\hat{x}\Delta\hat{p} \geq \frac{\hbar}{2} \tag{21.31}$$

You should know the physical implications of this, I think. What we should take away from this is that position and momentum eigenstates are not physical states, but are limits of a family of physical states.

22 Time Evolution of a Free Particle

A particle of mass m that can move free along a line (in 1D) is described by the Hamilton operator

$$H = \frac{1}{2m}\hat{p}^2 \tag{22.1}$$

So the Schrödinger Equation is

$$i\hbar\frac{\partial\phi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\phi(x,t)}{\partial x^2} \tag{22.2}$$

In the discrete case, we just looked for the characteristic polynomial of the Hamiltonian. However, we do not have this tool in the continuous case.

22.1 Eigenstates and Eigenvalues of H

The momentum states $|p\rangle$ are the eigenstates of the Hamiltonian with degenerate eigenvalues given by $E = \frac{p^2}{2m}$, so we associate $|+p\rangle$ with $p = +\sqrt{2mE}$ and $|-p\rangle$ with $p = -\sqrt{2mE}$. We usually denote $k = \frac{\sqrt{2mE}}{\hbar}$ so then $p = \pm k\hbar$. A general eigenstate of H of a free particle with energy E can be written as

$$|\psi_E\rangle = a|+p\rangle + b|-p\rangle \tag{22.3}$$

$$\Rightarrow \psi_E(x) = \frac{1}{2\pi k}\left(a\underbrace{e^{ikx}}_{\langle x|p\rangle} + b\underbrace{e^{-ikx}}_{\langle x|-p\rangle}\right) \tag{22.4}$$

Because of the degenerate eigenvalues, we will use the momentum to characterize the eigenstates instead of the eigenenergy.

22.2 Time Evolution of Energy Eigenstates

So, we find that

$$|\psi_E(t)\rangle = ae^{-i\frac{p^2}{2m\hbar}t}|p\rangle + be^{-i\frac{p^2}{2m\hbar}t}|-\rangle \tag{22.5}$$

which implies

$$\psi_E(x,t) = \frac{1}{\sqrt{2\pi\hbar}} a e^{-i\frac{p^2}{2m\hbar}t} e^{ikx} + \frac{1}{\sqrt{2\pi\hbar}} b e^{-i\frac{p^2}{2m\hbar}t} e^{-ikx} \tag{22.6}$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \left(a e^{-i(wt-kx)} + b e^{-i(wt+kx)} \right) \tag{22.7}$$

where $w = \frac{E}{\hbar} = \frac{p^2}{2m\hbar}$ and $k = \frac{\sqrt{2mE}}{\hbar} = \frac{p}{\hbar}$. Oh dear, it's optics all over again. The first exponential term is a wave moving right, while the second exponential term is a wave moving left. The phase velocity is

$$v_{ph} = \frac{p^2}{m} = \frac{v_c}{2} \tag{22.8}$$

where v_c is the classical speed. The group velocity is

$$v_{group} = \left.\frac{dw}{dk}\right|_{k_0} = \frac{d}{dk}\left(\frac{k^2\hbar}{2m}\right) = \frac{k\hbar}{m} = \frac{p}{m} = v_c \tag{22.9}$$

22.3 Time Evolution for General States

The most general solution is then given by writing the initial state as a linear combination of the momentum eigenstates. All integrals seen below are from $-\infty$ to ∞:

$$|\psi(0)\rangle = I|\psi(0)\rangle \tag{22.10}$$

$$= \left(\int |p\rangle\langle p| \, dp \right) |\psi(0)\rangle \tag{22.11}$$

$$= \int |p\rangle\langle p|\psi(0)\rangle \, dp \tag{22.12}$$

$$= \int \psi_p(p,0)|p\rangle \, dp \tag{22.13}$$

$$\tag{22.14}$$

where

$$\psi_p(p,0) = \langle p|\psi(0)\rangle \tag{22.15}$$
$$= \langle p|I|\psi(0)\rangle \tag{22.16}$$
$$= \langle p|\left(\int |x\rangle\langle x|\,dx\right)|\psi(0)\rangle \tag{22.17}$$
$$= \int \langle p|x\rangle\langle x|\psi(0)\rangle\,dx \tag{22.18}$$
$$= \int \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{-ipx}{\hbar}}\psi(x,0)\,dx \tag{22.19}$$

Then, somehow,

$$|\psi(t)\rangle = \int \psi_p(p,0)e^{-\frac{ip^2}{2m\hbar}t}|p\rangle\,dp \tag{22.20}$$

note that $e^{-\frac{ip^2}{2m\hbar}t} = e^{-\frac{iE}{\hbar}t}$. We then find

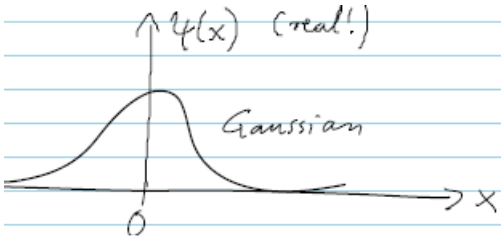
$$\langle x|\psi(t)\rangle = \psi(x,t) = \int \psi_p(p,0)e^{-\frac{ip^2}{2m\hbar}t}\left(\frac{1}{\sqrt{2\pi\hbar}}e^{\frac{ipx}{\hbar}}\right)\,dp \tag{22.21}$$

22.4 Example of the Wave Function

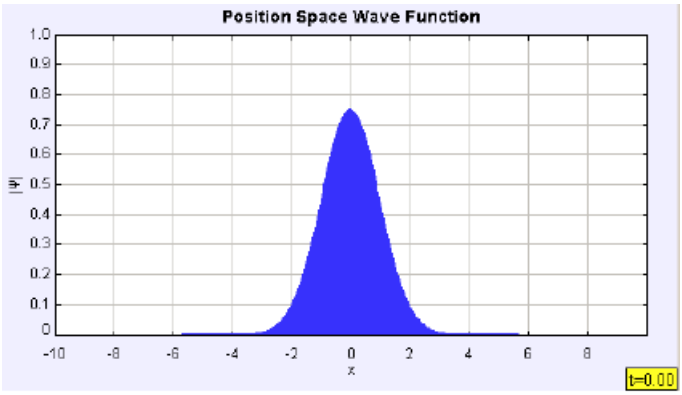
Consider the state

$$\psi(x) = \frac{1}{(2\pi)^{\frac{1}{4}}\sqrt{\Delta\hat{x}}}e^{-\left(\frac{x}{2\Delta\hat{x}}\right)^2} \tag{22.22}$$

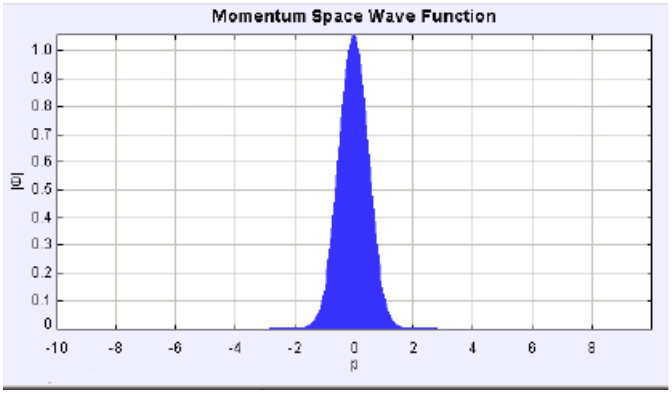
This is a Gaussian distribution:



We know that, from the Heisenberg uncertainty relation, the more we contain the initial wave-packet to a narrow region, a higher variance of momentum will come about. For $\psi(x,0)$ we find



which leads to $\psi(p,0)$ showing



where

$$\psi_p(p,0) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-\frac{ipx}{\hbar}} \psi(x,0)\, dx \tag{22.23}$$

At a later time, we find

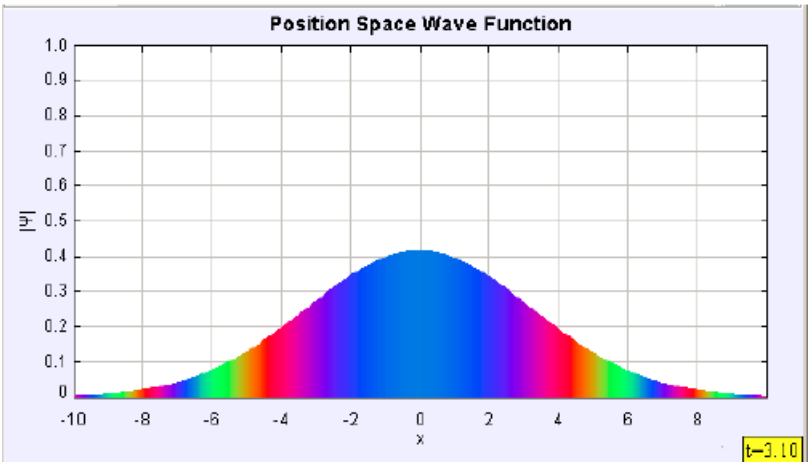


Figure 25: The colour corresponds to the complex phase of the wavefunction.

where the state is given by

$$\psi(x,t) = \int_{-\infty}^{\infty} \psi_p(p,0) e^{-\frac{ip^2 t}{2m\hbar}} \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{ipx}{\hbar}}\, dp \tag{22.24}$$

The wave function spreads out. Cool.

22.5 Time Evolution and Initial Values

In quantum mechanics, we have the initial condition of $\psi(x,0)$, which leads to another initial condition of $\psi_p(p,0)$. But,

$$\psi(x,0) \Rightarrow |\langle x|\psi(0)\rangle|^2 dx = |\psi(x,0)|^2\, dx \tag{22.25}$$

which itself implies

$$|\langle p|\psi(0)\rangle|^2\, dp = |\psi_p(p,0)|^2\, dp \tag{22.26}$$

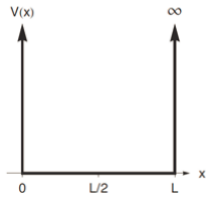
I don't know why we wrote that. Basically, since the wavefunction is complex, we can write it as

$$\psi(x,0) = |\psi(x,0)| e^{if(x)} \tag{22.27}$$

where $f(x) \in \mathbb{R}$. This implies that all information about the initial momentum distribution sits in the complex phase $e^{if(x)}$ of the initial wave function $\psi(x,0)$.

23 The Infinite Potential Well

Consider the infinite potential energy well:



The Hamiltonian is given by

$$\hat{H} = \frac{\hat{P}^2}{2m} + V(\hat{X}) \tag{23.1}$$

(we are now using capitals for all of our operators because... well, you know why.) The potential is given by

$$V(x) = \begin{cases} \infty & -\infty < x < 0 \\ 0 & 0 \leq x \leq L \\ \infty & L < x < \infty \end{cases} \tag{23.2}$$

Just like we did once upon a time (in the finite dimensional cases), we find the eigensystem of H, decompose the initial state into the eigenstates of H and write down the final solution in order to solve the Schrödinger Equation. However, the Schrödinger Equation for a non-relativistic particle independent of time is:

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi_E(x) = E\psi_E(x) \tag{23.3}$$

Decomposing the eigenstates is analogous to

$$\langle E_n|\psi(0)\rangle = \int \psi(x,0)\psi_{E_n}(x)^*\,dx \tag{23.4}$$

So then the final solution is

$$\psi(x,t) = \sum_n \langle E_n|\psi(0)\rangle e^{-\frac{iE_n}{\hbar}t}\psi_{E_n}(x) \tag{23.5}$$

(just a note: $\psi_E(x)$ is an eigenstate). For sections I and III (outside of the well), $\psi_E(x) = 0$ and the potential is infinite. The state is zero because there is no probability of the particle existing outside of the well. Then, ignoring the zero part (so to speak),

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \infty\right)\psi_E(x) = E\psi_E(x) \tag{23.6}$$

For section II (inside the well), we have

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + 0\right)\psi_E(x) = E\psi_E(x) \tag{23.7}$$

We assume a solution is

$$\psi_E(x) = Ae^{\frac{ipx}{\hbar}} + Be^{-\frac{ipx}{\hbar}} \tag{23.8}$$

Since wavefunctions are continuous, we can use the boundary conditions $\psi(0) = 0$ and $\psi(L) = 0$. We reformulate the solution to be

$$\psi_E(x) = \begin{cases} 0 & \text{outside} \\ A\cos\left(\frac{px}{\hbar}\right) + B\sin\left(\frac{px}{\hbar}\right) & \text{inside} \end{cases} \tag{23.9}$$

(We really shouldn't, but it does end up working out). These boundary conditions give $A = 0$ so then

$$\frac{pL}{\hbar} = n\pi \tag{23.10}$$

where n is a positive integer. Since $E = \frac{p^2}{2m}$, we find

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} \tag{23.11}$$

So we are now at

$$\psi_E(x) = \begin{cases} 0 & \text{outside} \\ B\sin\left(\frac{\sqrt{2mE_n}x}{\hbar}\right) & \text{inside} \end{cases} \tag{23.12}$$

I'm going to skip the steps, but if you use the normalization condition (where the bounds on the integral are now $[0, L]$) we would find that $B^2 = \frac{2}{L}$, so then, finally

$$\psi_{E_n}(x) = \begin{cases} 0 & \text{outside} \\ \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) & \text{inside} \end{cases} \tag{23.13}$$

We must note that $n \neq 0$ because the particle will always have some energy. Also, just as a reminder

$$\int_{-\infty}^{\infty} \psi^*_{E_n}(x)\psi_{E_m}(x)\,dx = \delta_{m,n} \tag{23.14}$$

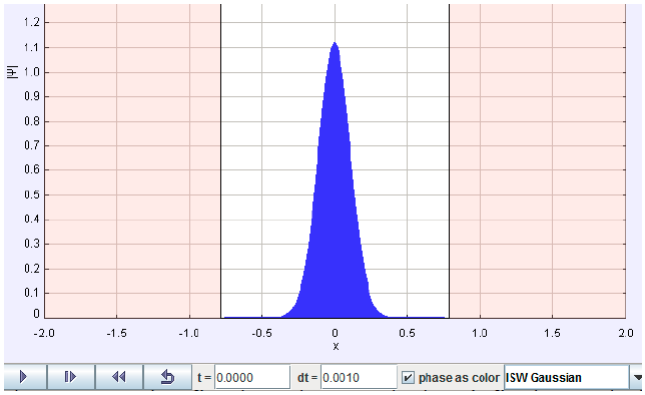
as different energy states are orthogonal. Since they also form an orthonormal basis on an interval $x \in [0, a]$, any square integrable function $f(x)$ with zeros at the boundaries can be expanded as

$$f(x) = \sum_{n=1}^{\infty} c_n \psi_E(x) \tag{23.15}$$

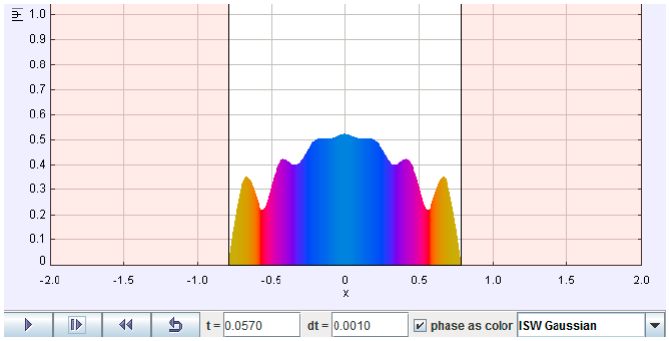
$$= \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \tag{23.16}$$

23.1 Example of Time Evolution

Here is a starting configuration for a position wave function:



As time goes on, the wave packet first broadens, and when it reaches the boundary, interference effects show (the wave components are reflected from the wall):



What is interesting is that, after some time, we get the original wave packet back. This has to do with the fact that each energy eigenstate has a phase that oscillates with some frequency, thus coming back to the original value. If there is a finite number of frequencies, then there is a point where all phases of the energy eigenstates are back at their original value at the same time.

23.2 Discussion

23.2.1 Discrete vs Continuous Energies

Only discrete values of energy are possible for eigenstates. In general, a discrete spectrum gives rise to special effects, such as the spectroscopy of atoms. In classical mechanics, energy states are continuous. How can we reconcile these differences? Well,

$$\lim_{n\to\infty} \frac{E_{n+1} - E_n}{E_n} = \lim_{n\to\infty} \frac{(n+1)^2 - n^2}{n^2} = \lim_{n\to\infty} \frac{2n+1}{n^2} = 0 \tag{23.17}$$

So for high energies in an infinite potential well, the energy values are basically continuous. High energies are defined with respect to the depth of the well.

23.2.2 Bound States

Quantum mechanical states that are energetically confined to some region are called bound states. These bound states always have a discrete energy spectrum. An example of this is an electron in a Coulomb potential.

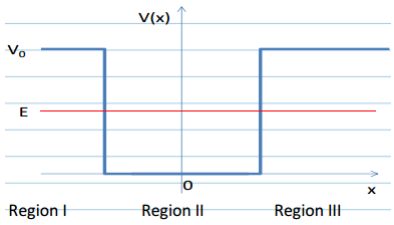
23.2.3 Ground State

The ground state energy (the state of the lowest energy) is non-zero. A minimum energy leads to vacuum fluctuations, somehow.

24 Finite Potential Well

24.1 Some Solutions

We are now dealing with a finite potential well, as seen here:



For now, $0 \leq E \leq V_0$. For a brief moment, consider the solution to

$$\frac{\partial^2 f(x)}{\partial x^2} = Cf(x) \tag{24.1}$$

Explicitly, we find

$$f(x) = \begin{cases} A_+ e^{\sqrt{C}x} + A_- e^{-\sqrt{C}} & C \geq 0 \\ A_+ e^{i\sqrt{-C}} + A_- e^{-i\sqrt{-C}} & C < 0 \end{cases} \tag{24.2}$$

So for Regions I and III, skipping some steps, the Schrödinger Equation gives

$$\frac{\partial^2}{\partial x^2}(\psi_E(x)) = \frac{2m(V_0 - E)}{\hbar^2}\psi_E(x) \tag{24.3}$$

The solution to this, for Region I, is,

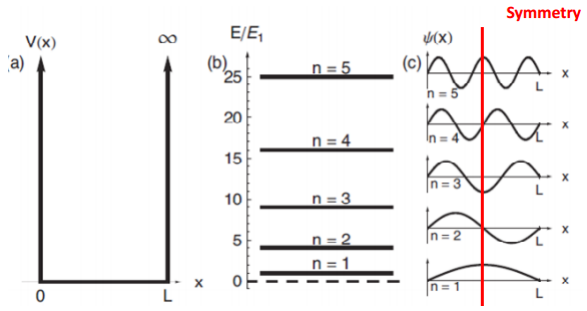
$$\psi_E(x) = A_+ e^{\kappa x} + A_- e^{-\kappa x} \tag{24.4}$$

and for Region III, the solution is

$$\psi_E(x) = C_+ e^{\kappa x} + C_- e^{-\kappa x} \tag{24.5}$$

where $\kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$. Note that this is greater than 0, which is why we don't have a complex solution. For Region II, the Schrödinger Equation gives

$$\frac{\partial^2}{\partial x^2}(\psi_E(x)) = -\frac{2mE}{\hbar^2}\psi_E(x) \tag{24.6}$$



while the solution is

$$\psi_E(x) = B_+ e^{ikx} + B_- e^{-ikx} \tag{24.7}$$

which can be rewritten

$$\psi_E(x) = B_c \cos(kx) + B_s \sin(kx) \tag{24.8}$$

where $k = \frac{\sqrt{2mE}}{\hbar}$. This is greater than zero, but it has a negative sign in front of it in the differential equation, which gives us our complex solution. Since the wave function cannot grow to positive or negative infinity, we lose a term in our Region I and III solutions ($A_-, C_+ = 0$). So then we have:

$$\psi_E(x) = \begin{cases} A_+ e^{\kappa x} & I \\ B_c \cos(kx) + B_s \sin(kx) & II \\ C_- e^{-\kappa x} & III \end{cases} \tag{24.9}$$

24.2 Parity Operator

Recall the infinite potential well. We can find a symmetry of eigenfunctions:

The parity operator is defined as

$$\hat{\Pi}|x\rangle = |-x\rangle \tag{24.10}$$

If the parity operator commutes with the Hamiltonian, all eigenfunctions are either symmetric or antisymmetric. The parity operator will always commute with the Hamiltonian if the potential is symmetric ($V(-x) = V(x)$). However, symmetry is dependent on the choice of the coordinate system.

24.3 Symmetry Simplifications of the Solutions

If we have a symmetric solution, then $B_s = 0$ and $A_+ = C_-$. If we have an antisymmetric solution, then $B_c = 0$ and $A_+ = -C_-$. In other words, for a symmetric solution

$$\psi_E(x) = \begin{cases} A_+ e^{\kappa x} & I \\ B_c \cos(kx) & II \\ A_+ e^{-\kappa x} & III \end{cases} \tag{24.11}$$

and for an antisymmetric solution

$$\psi_E(x) = \begin{cases} A_+ e^{\kappa x} & I \\ B_s \sin(kx) & II \\ -A_- e^{-\kappa x} & III \end{cases} \tag{24.12}$$

Because the wavefunction and its derivative must be continuous, we can match conditions at the boundaries like we did with the infinite potential well. However, I don't want to write this out, so I'll leave you to it to find that

$$\kappa = \begin{cases} k \tan(ka) & \text{symmetric} \\ -k \cot(ka) & \text{antisymmetric} \end{cases} \tag{24.13}$$

This is only solvable using numeric methods.

25 Eigenspectrum of Bound States in Step Potentials

For the start, we will only concentrate on symmetric solutions. We let $E \to E_n$ so then we have:

$$\kappa_n = \frac{\sqrt{2m(V_0 - E_n)}}{\hbar} \tag{25.1}$$

$$k_n = \frac{\sqrt{2mE_n}}{\hbar} \tag{25.2}$$

So then, since we have κ in terms of k (from the previous lecture), we can somehow find that

$$\psi_{E_n}(x) = A_+ \begin{cases} e^{\kappa_n x} & x \leq -a \\ e^{-\kappa_n x} \frac{\cos(k_n x)}{\cos(k_n a)} & |x| < a \\ e^{-\kappa_n x} & x \geq a \end{cases} \tag{25.3}$$

Using the property of normalization, we can find A_+. But whatever.

25.1 Graphical Solutions

Before we get on with it, to make our lives easier, we define

$$z := ka = \frac{\sqrt{2mE}a}{\hbar} \tag{25.4}$$

and

$$z_0 := \frac{\sqrt{2mV_0}a}{\hbar} \tag{25.5}$$

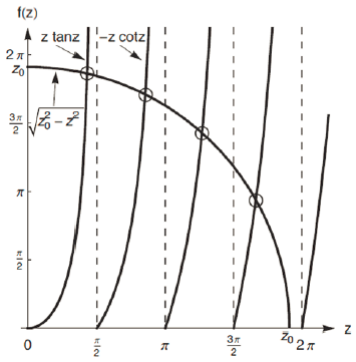
so then

$$\kappa a = \sqrt{z_0^2 - z^2} \tag{25.6}$$

which leads to

$$\sqrt{z_0^2 - z^2} = \begin{cases} z\tan(z) & \text{symmetric} \\ -z\cot(z) & \text{antisymmetric} \end{cases} \tag{25.7}$$

Graphing this for $z_0 = 6$ gives



25.1.1 Discussion of Eigenstates

There is at least one eigenstate, no matter which value z_0 takes, because the circle intersects at least with the first branch of $z\tan(z)$. This is true even for very small potentials or very small widths. Note that the second part of that statement may seem to contradict the uncertainty relation: a small position leads to a high momentum spread, leading to a large kinetic energy, which should let the particle escape. However, the math tells us that this doesn't happen.

This doesn't happen because the particle can "leak" into the wall, and the probability of finding the particle within the wall becomes more substantial as the well's size decreases. In addition, if the eigenenergy is close to the potential, more of the wavefunction goes into the wall.

25.1.2 Additional Material: Limit of the Infinite Well

We apparently expect to recover the analytic solution. We can do this using the graphs. For $V_0 \to \infty$ we find that $z_n = \frac{n\pi}{2}$ where z_n are intersection points. This corresponds to the derived energy levels (E_n). Setting $x = \frac{a}{2}$ gives

$$\cos(ka) = \cos\left(\frac{n\pi}{2}\right) = 0 \tag{25.8}$$

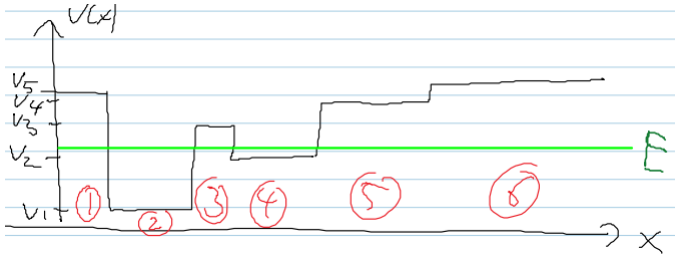
(where n is odd) for a symmetric solution. For an antisymmetric solution, we get

$$\sin(ka) = \sin\left(\frac{n\pi}{2}\right) = 0 \tag{25.9}$$

(where n is even).

25.2 Bound States in Step Potentials

Consider a potential well that looks like



In principle, we have all the tools to find the energy eigenstates of a Hamiltonian with a step-potential like the one above. Whenever the energy eigenstate is above the potential, we get oscillator behaviour

$$A_+ e^{ikx} + A_- e^{-ikx} \tag{25.10}$$

with wave number $k = \frac{1}{\hbar}\sqrt{2m(E - V_j)}$. When the energy is below the potential, there is a superposition of exponential growth and decay

$$A_+ e^{\kappa x} + A_- e^{-\kappa x} \tag{25.11}$$

with length scale $\kappa = \frac{1}{\hbar}\sqrt{2m(V_j - E)}$. When $x \to \pm\infty$, the wave function has to decay exponentially. We also know that the wavefunction is continuous, as is its derivative (except in the case of infinite potential steps). If the potential is symmetric around some coordinate, then all energy eigenfunctions are either symmetric or antisymmetric with respect to that point.

25.2.1 Discrete Energy Spectra

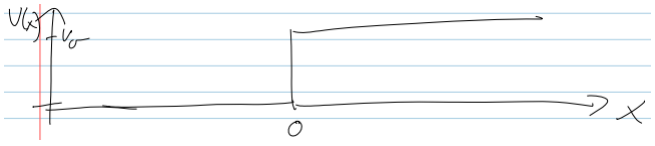
Consider a potential with N sections. We call an energy eigenstate a bound state if its energy is below the step potential for $x \to \pm\infty$. The mathematical ansatz gives $2N$ amplitudes (2 for each section). The requirement that there is no exponential growth at infinity means that there will be 2 constraints

(amplitudes go to zero). At each interface, we have two linear constraints on the amplitudes due to continuity, so we end up having $2N-4$ constraints. So we have $2N$ amplitudes AND $2N$ constraints. This means that, in general, the constraints have to be linearly independent for non-trivial solutions. This is the determinant condition.

As the constraints depend on the wavenumber and the decay/growth in each section, which are energy dependent, this determinant condition will impose constraints on the value of energy E where the determinant condition reveals linear dependency of constraints. This gives us a discrete energy spectrum. Once the constraints are linear independent, we can eliminate amplitudes. For bound states, we then only have one amplitude, which can be found by the normalization condition.

26 Potential Step

A potential step, to explain briefly, is an increase in the potential that continues forever, as seen below



26.1 $E < V_0$

An ansatz for the wavefunction is

$$\psi(x) = \begin{cases} A_+ e^{ikx} + A_- e^{-ikx} & x < 0 \\ B_+ e^{\kappa x} + B_- e^{-\kappa x} & x > 0 \end{cases} \tag{26.1}$$

where $k = \frac{\sqrt{2mE}}{\hbar}$ and $\kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$. Because we can't have exponentially blahblahblah, we get $B_+ = 0$. Note that the total wavefunction cannot be normalized. Using matching conditions, we also find

$$A_+ + A_- = B_- \tag{26.2}$$

$$ikA_+ - ikA_- = -\kappa B_- \tag{26.3}$$

So there are three amplitudes with two linear constraints. This means that we get a family of non-trivial solutions for any value of E. Multiplying the first equation by ik and adding it to the second one (and skipping some steps) results in

$$B_- = \frac{2k}{k + i\kappa} A_+ \tag{26.4}$$

Doing something similar (subtracting two from one), we get

$$A_- = \frac{k - i\kappa}{k + i\kappa} A_+ \tag{26.5}$$

Note that

$$\frac{k - i\kappa}{k + i\kappa} = e^{i\phi} \tag{26.6}$$

(this can be proven by taking the modulus). So, we end up with

$$\psi_E(x) = \begin{cases} A_+ \left(e^{ikx} + \frac{k-i\kappa}{k+i\kappa} e^{-ikx}\right) & x < 0 \\ A_+ \left(\frac{2k}{k+i\kappa} e^{-\kappa x}\right) & x > 0 \end{cases} \tag{26.7}$$

The normalization is important in order to get the time evolution of initial states correctly. The incoming wave gets reflected, but some particles can penetrate the barrier. For each value of E, there is only one eigenfunction, so the eigenspaces are nondegenerate. If you forgot how to get a time-dependent solution, just do the following steps! Calculate

$$|\psi(0)\rangle = \int \langle \psi_E | \psi(0) \rangle |\psi_E\rangle \, dE \tag{26.8}$$

where

$$\langle \psi_E | \psi(0) \rangle = \int \psi_E^*(x) \psi(x, 0) \, dx \tag{26.9}$$

so then

$$|\psi(t)\rangle = \int \langle \psi_E | \psi(0) \rangle e^{-\frac{iEt}{\hbar}} |\psi_E\rangle \, dE \tag{26.10}$$

which gives us

$$\psi(x,t) = \int \langle \psi_E | \psi(0) \rangle e^{-\frac{iEt}{\hbar}} \psi_E(x) \, dE \tag{26.11}$$

26.2 $E > V_0$

An ansatz:

$$\psi(x) = \begin{cases} A_+ e^{ikx} + A_- e^{-ikx} & x < 0 \\ B_+ e^{i\kappa x} + B_- e^{-i\kappa x} & x > 0 \end{cases} \tag{26.12}$$

where $k = \frac{\sqrt{2mE}}{\hbar}$ and $\kappa = \frac{\sqrt{2m(E-V_0)}}{\hbar}$. There are no physical constraints to help us out. We then have four amplitudes and two constraints. This gives us two free parameters, so we can expect a continuous eigenspectrum. Let us choose the free parameters to be A_+ and B_-, which are the amplitudes of the waves coming from the left and right respectively. We must split this

up into two solution sets (setting one of them to zero) which will eventually spit out one amplitude, which we can find via normalization.

26.2.1 $B_- = 0$

We find that, due to continuity,

$$A_+ + A_- = B_+ \tag{26.13}$$
$$ikA_+ - ikA_- = \kappa B_+ \tag{26.14}$$

We then find

$$A_- = \frac{k-\kappa}{k+\kappa}A_+ \qquad B_+ = \frac{2k}{k+\kappa}A_+ \tag{26.15}$$

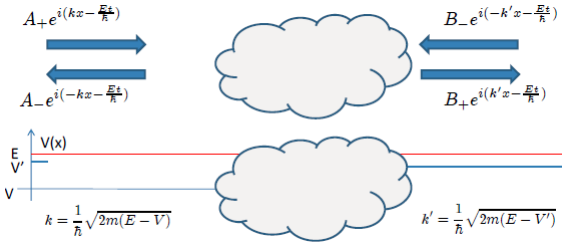
which gives us

$$\psi(x) = A_+ \begin{cases} e^{ikx} + \frac{k-\kappa}{k+\kappa}e^{-ikx} & x \leq 0 \\ \frac{2k}{k+\kappa}e^{-i\kappa x} & x > 0 \end{cases} \tag{26.16}$$

We can do this for $A_+ = 0$ but we didn't. Anyways, from this solution, we can find A_+ via normalization.

27 Probability Flux

There is an image of a cloud that I could put in here first but it might confuse you and I have no idea what it is asking so I am going to simply tell you that **probability flux** is the probability per unit time per unit area. I think. That's what Wikipedia says at least. Now consider the cloud!



The probability flux for each component (for $E > V(x)$) is

$$S_{comp} = \frac{\hbar}{m} k|A|^2 \tag{27.1}$$

For wave packets, probability flux turns into probabilities of transmittance and reflectance of particles. For the system above, the probability flux from the left side is $\frac{\hbar}{m}k|A_+|^2$ while the probability flux from the right side is $\frac{\hbar}{m}k|B_-|^2$. However, in experiments, we typically only have particles coming from one side.

In the above diagram, the particles are only coming from the left side. The reflection probability is

$$R = \frac{S_r}{S_i} = \frac{|A_-|^2}{|A_+|^2} \tag{27.2}$$

while the transmission probability is

$$T = \frac{S_t}{S_i} = \frac{k'|B_+|^2}{k|A_+|^2} \tag{27.3}$$

where the subscripts i, r, t denote input, reflected, and transmitted, respectively. If we apply this to the potential step as seen in the previous section (but now using $k_1 = k$ and $k_2 = \kappa$, we find

$$R = \left(\frac{k_1 - k_2}{k_1 + k_2}\right) \tag{27.4}$$

$$T = \frac{4k_1k_2}{(k_1 + k_2)^2} \tag{27.5}$$

with the identity

$$R + T = 1 \tag{27.6}$$

27.1 "Additional Material"

I doubt this is bonus stuff. If we write the probability density as $w(x,t)\,dx$ where

$$w(x,t) = |\psi(x,t)|^2 \tag{27.7}$$

and the probability flux as $S(x,t)\,dx$ where

$$S(x,t) = \frac{\hbar}{2mi}\left(\psi^*(x,t)\frac{\partial}{\partial x}(\psi(x,t)) - \psi(x,t)\frac{\partial}{\partial x}(\psi^*(x,t))\right) \tag{27.8}$$

then the **continuity equation for probability** is

$$\frac{\partial}{\partial t}(w(x,t)) + \frac{\partial}{\partial x}(S(x,t)) = 0 \tag{27.9}$$

Using the fundamental theorem of calculus,

$$\frac{d}{dt}\int_a^b w(x,t)\,dx = -\int_a^b \left(\frac{\partial}{\partial x}(S(x,t))\right)\,dx = S(a) - S(b) \tag{27.10}$$

27.1.1 Special Cases

For energy eigenstates, time evolution can be ignored as it cancels out, so $S_{\psi(x,t)} = S_{\psi_E(x)}$. Using $\psi_E(x) = A_+ e^{ikx} + A_- e^{-ikx}$, we find that

$$S(x) = \frac{\hbar k}{m}(|A_+|^2 - |A_-|^2) \tag{27.11}$$

(the incoming flux minus the reflected flux). Using $\psi_E(x) = |\psi_E(x)|e^{i\phi}$, we find that

$$S_\psi = 0$$

Now if we consider the case of the potential step with $E < V$, we find that

$$|A_-|^2 = |\left(\frac{k - i\kappa}{k + i\kappa}\right) A_+|^2 = |A_+|^2$$

which implies

$$S(x) = 0$$

The flux of incoming particles from the left must equal the flux of reflected particles:

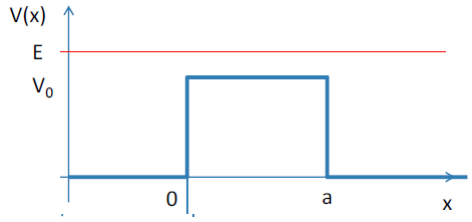
$$S_i(x) = \frac{\hbar k}{m}|A_+|^2 \tag{27.12}$$

$$S_o(x) = -\frac{\hbar k}{m}|A_+|^2 \tag{27.13}$$

28 Potential Barrier

28.1 $E > V_0$

Here's what the potential barrier looks like:



The potential barrier has partial reflectance and transmittance, a continuous eigenspectrum, and degenerate eigenvalues. An ansatz for this case is

$$\psi_E(x) = \begin{cases} A_+ e^{ik_1 x} + A_- e^{-ik_1 x} & x \leq 0 \\ B_+ e^{ik_2 x} + B_- e^{-ik_2 x} & 0 \leq x \leq a \\ C_+ e^{ik_1 x} + C_- e^{-ik_1 x} & x < a \end{cases} \tag{28.1}$$

where $k_1 = \frac{\sqrt{2mE}}{\hbar}$ and $k_2 = \frac{\sqrt{2m(E-V_0)}}{\hbar}$. Choosing waves from the left gives us $C_{=}0$. Using boundary conditions, we will eventually end up with

$$C_+ = \frac{4k_1 k_2 e^{-ik_1 a}}{4k_1 k_2 \cos(k_2 a) - 2i(k_1^2 + k_1^2)\sin(k_2 a)} A_+ \tag{28.2}$$

Because $T = \frac{S_t}{S_i} = \frac{|C_+|^2}{|A_+|^2}$, we find

$$T = \frac{16k_1^2 k_2^2}{16k_1^2 k_2^2 \cos^2(k_2 a) + 4(k_1^2 + k_2^2)^2 \sin^2(k_2 a)} \tag{28.3}$$

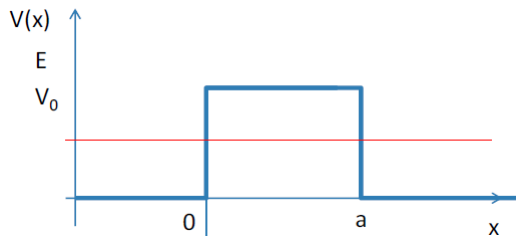
Introduce $\epsilon = \frac{E}{V_0}$ and $K_0 = \frac{1}{\hbar}\sqrt{2mV_0}$ to get

$$T = \frac{1}{1 + \frac{1}{4\epsilon(\epsilon-1)} \sin^2(K_0 a\sqrt{\epsilon - 1})} \tag{28.4}$$

and we use $1 = T + R$ to find R. For some energy values, we have perfect transmittance. For high energies, $T \to 1$.

28.2 $E < V_0$

Here's what the potential barrier looks like:



The potential barrier has partial reflectance and transmittance, a continuous eigenspectrum, and degenerate eigenvalues. An ansatz for this case is

$$\psi_E(x) = \begin{cases} A_+ e^{ikx} + A_- e^{-ikx} & x \leq 0 \\ B_+ e^{\kappa x} + B_- e^{\kappa x} & 0 \leq x \leq a \\ C_+ e^{ikx} + C_- e^{-ikx} & x < a \end{cases} \tag{28.5}$$

where $k = \frac{\sqrt{2mE}}{\hbar}$ and $\kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$. Choosing waves from the left gives us $C_- = 0$. Using boundary conditions, we will eventually end up with

$$T = \frac{1}{1 + \frac{1}{4\epsilon(1-\epsilon)} \sinh^2(K_0 a \sqrt{\epsilon - 1})} \tag{28.6}$$

and we use $1 = T + R$ to find R. For some energy values, we have perfect transmittance. For high energies, $T \to 1$.

28.2.1 Approximation

Somehow, we can neglect the 1 in the denominator of T and, using $ka \gg 1$ (for wide barriers), we can approximate the hyperbolic sine term as $(\frac{1}{2}e^{\kappa a})^2$. This gives us

$$T \approx 16 \frac{E}{V_0} \left(1 - \frac{E}{V_0}\right) e^{-2\kappa a} \tag{28.7}$$

28.3 Scanning Tunneling Microscopy

The scanning tunneling microscope is used for imaging surfaces at the atomic level. When a conducting tip is brought close to the desired surface, a voltage difference applied between the tip and the surface can allow electrons to tunnel through the gap. For our case, we will say the gap is a nanometer wide and the difference $V_0 - E$ is about 4 eV. Then,

$$\kappa \approx 10^{10}$$

so then $\kappa a \approx 10 \gg 1$ which is dumb but sure. For positions a_1, a_2 (I think $a_1 = 10^{-9}$ and $a_2 = 1.3 \cdot 10^{-9}$) we find that

$$T(a_1) = e^{-2\kappa a_1} = e^{-20}$$
$$T(a_2) = e^{-2\kappa a_2} = e^{-26}$$

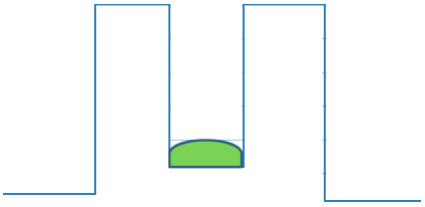
which gives the ratio

$$\frac{T(a_1)}{T(a_2)} = e^6 \approx 400 \tag{28.8}$$

and since the current is proportional to the transmission coefficient, we can see that the current required is very sensitive to the distance.

28.4 Radioactive Decay Application

Tunnel can also explain the behaviour of radioactive decay. We use the following step potential as a toy model of what is going on in a nucleus:

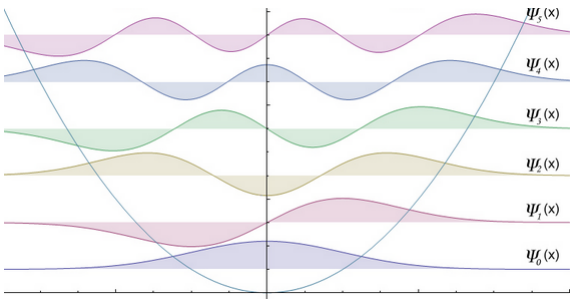


Initially, the nucleus particles are trapped between two potential barriers. The system is set up so that it would be energetically beneficial for the particle to be outside the barrier. The energy eigenstates corresponding to the initial state have components that have waves going to plus or minus

infinity, so at any time there is a probability for the particle to leave the central well and move outside. Over time, the presence of the particle in the potential well decays exponentially, as expected from our observation of radioactive decay. Also, directly from the start there is a probability for the decay to happen. The time scale of the exponential decay depends on the barrier height and width, and also on the difference between the potential in and outside the potential well.

29 Harmonic Oscillator

The potential graph for the harmonic oscillator looks like this



The equation for the potential is simply

$$V(x) = \frac{1}{2}mw^2x^2 \tag{29.1}$$

so the Hamiltonian is

$$\hat{H} = \frac{\hat{P_x}^2}{2m} + \frac{1}{2}mw^2x^2 \tag{29.2}$$

We expect there to be bound states with a discrete energy spectrum, vacuum fluctuations, and either symmetric or asymmetric eigenstates, depending on the symmetry of the potential. The Schrödinger Equation gives

$$\left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}mw^2x^2\right)\psi_E(x) = E\psi_E(x) \tag{29.3}$$

We use the ansatz

$$\psi_E(x) = \sum_{n=0}^{\infty} a_n x^n \tag{29.4}$$

This will lead us to find

$$E_n = \left(n + \frac{1}{2}\right)\hbar w \tag{29.5}$$

and the wavefunction is

$$\psi_{E_n} = \frac{1}{\sqrt{\sqrt{\pi} 2^n n! x_0}} e^{\frac{x^2}{2x_0^2}} H_n\left(\frac{x}{x_0}\right) \tag{29.6}$$

where $x_0 = \sqrt{\frac{\hbar}{mw}}$ and $H_n(y)$ is the nth Hermite polynomial, defined as such:

$$H_n(y) = (-1)^n e^{y^2} \frac{d^n}{dy^n}\left(e^{-y^2}\right) \tag{29.7}$$

29.1 Operator Approach to Solutions

Let us introduce a non-Hermitian operator

$$\hat{a} = \sqrt{\frac{mw}{2\hbar}}\left(\hat{x} + \frac{i}{mw}\hat{p_x}\right) \tag{29.8}$$

and its Hermitian conjugate

$$\hat{a}^\dagger = \sqrt{\frac{mw}{2\hbar}}\left(\hat{x} - \frac{i}{mw}\hat{p_x}\right) \tag{29.9}$$

so the Hamiltonian can be written as

$$\hat{H} = \frac{1}{2}\hbar w(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) \tag{29.10}$$

where

$$\begin{aligned}
\hat{a}\hat{a}^\dagger &= \frac{mw}{2\hbar}\left(\hat{x}^2 + \frac{\hat{p_x}^2}{m^2w^2} - \frac{i}{mw}(\hat{x}\hat{p_x} - \hat{p_x}\hat{x})\right) \\
&= \frac{mw}{2\hbar}\left(\hat{x}^2 + \frac{\hat{p_x}^2}{m^2w^2} - \frac{i}{mw}[\hat{x}, \hat{p}]\right) \\
&= \frac{1}{\hbar w}\hat{H} + \frac{1}{2}I
\end{aligned} \tag{29.11}$$

so then

$$\hat{a}^\dagger\hat{a} = \frac{1}{\hbar w}\hat{H} - \frac{1}{2}I \tag{29.12}$$

which gives us

$$\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} = \frac{2}{\hbar w}\hat{H} \tag{29.13}$$

and also

$$[\hat{a}, \hat{a}^\dagger] = I \tag{29.14}$$

FINALLY, we can express the Hamiltonian as

$$\hat{H} = \hbar w \left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \tag{29.15}$$

30 Harmonic Oscillator Solutions

The operators we defined in the previous section are called the **ladder operators**:

$$\hat{a} = \sqrt{\frac{mw}{2\hbar}}\left(\hat{X} + \frac{i}{mw}\hat{P}\right) \tag{30.1}$$

$$\hat{a}^\dagger = \sqrt{\frac{mw}{2\hbar}}\left(\hat{X} - \frac{i}{mw}\hat{P}\right) \tag{30.2}$$

The first of which is the **lowering operator** and the latter of which is the **raising operator**. For convenience, we say

$$\hat{N} = \hat{a}^\dagger\hat{a} \tag{30.3}$$

so then we write the Hamiltonian as

$$\hat{H} = \hbar w\left(\hat{N} + \frac{1}{2}I\right) \tag{30.4}$$

30.1 The Eigenstates of N

30.1.1 Relationships

The eigenstates of the Hamiltonian are the eigenstates of $\hat{N}$. If $|\lambda\rangle$ is an eigenstate of $\hat{N}$ corresponding to the eigenvalue λ then it turns out that $\hat{a}^\dagger|\lambda\rangle$ is also an eigenstate of $\hat{N}$ corresponding to the eigenvalue $(\lambda + 1)$ and $\hat{a}|\lambda\rangle$ is an eigenstate with eigenvalue $(\lambda - 1)$.

30.1.2 Normalization

All of the states $|\lambda\rangle$ are thought of as being normalized. We start with

$$\hat{a}|\lambda\rangle = c_\lambda|\lambda - 1\rangle$$

which somehow gives us

$$\langle\lambda|\hat{a}^\dagger\hat{a}|\lambda\rangle = \langle\lambda|\hat{N}|\lambda\rangle = \lambda$$

so then $c_\lambda = \sqrt{\lambda}$, giving us

$$\hat{a}|\lambda\rangle = \sqrt{\lambda}|\lambda - 1\rangle \tag{30.5}$$

Similarly,

$$\hat{a}^\dagger|\lambda\rangle = \sqrt{\lambda + 1}|\lambda + 1\rangle \tag{30.6}$$

For each value of $\lambda \in [0, 1)$ we could have a ladder of connected eigenstates $|\lambda+m\rangle$ where $m \in \mathbb{Z}$. However, this assumes that we were given an eigenstate in this ladder, which allows us to construct the other elements of the ladder. Are we able to construct eigenstates for all ladders? We can use a physical argument to find the answer.

30.2 Existence of Ground States

If $|\lambda\rangle$ is the eigenstate of $\hat{N}$ with eigenvalue λ, then the same state is the eigenstate to the Hamiltonian with an energy eigenvalue $\hbar w(\lambda + \frac{1}{2})$. There must be a lowest eigenstate. Let us denote this as $|\underline{\lambda}\rangle$. How does this correspond to the lowering operator, which brings us to lower energy eigenstates?

$$\hat{N}\hat{a}|\underline{\lambda}\rangle = (\underline{\lambda} - 1)\hat{a}|\underline{\lambda}\rangle \tag{30.7}$$

For some value of m, we must have

$$\hat{a}|\underline{\lambda}\rangle = 0 \tag{30.8}$$

For such a state (well, for any state), we know that on one hand,

$$\hat{N}|\underline{\lambda}\rangle = \underline{\lambda}|\underline{\lambda}\rangle \tag{30.9}$$

But on the other hand, we also know

$$\hat{N}|\underline{\lambda}\rangle = \hat{a}^\dagger\hat{a}|\underline{\lambda}\rangle = 0|\underline{\lambda}\rangle \tag{30.10}$$

which somehow all implies that $m = 0$. So the ground state is the state with the label $|0\rangle$ and satisfies

$$\hat{a}|0\rangle = 0 \tag{30.11}$$

and

$$\hat{N}|0\rangle = 0|0\rangle \tag{30.12}$$

I have no idea what's going on anymore. The ground state is the eigenvector of $\hat{N}$ with an eigenvalue of 0. This means that there is only one ladder of eigenstates.

30.3 Eigensystem of the Hamiltonian

What we should probably take away from this crap is this:

$$\hat{H}|n\rangle = \hbar w \left(n + \frac{1}{2}\right)|n\rangle \tag{30.13}$$

We have eigenstates $|n\rangle$, eigenenergies $E_n = \hbar w \left(n + \frac{1}{2}\right)$ and the ground state $|0\rangle$ with an energy of $E_0 = \frac{1}{2}\hbar w$.

30.4 Wavefunction of Eigenstates

30.4.1 Ground State

In this section, we derive the coordinate representation of the eigenstate. We begin with the ground state. We know that $\hat{a}|0\rangle = 0$. Then we have this set of equations:

$$\int \langle x|\hat{a}|x'\rangle\langle x'|0\rangle \, dx' = 0 \tag{30.14}$$

$$\langle x|\hat{x}|x'\rangle = x\delta(x - x') \tag{30.15}$$

$$\langle x|\hat{p}|x'\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x}(\delta(x - x')) \tag{30.16}$$

$$\langle x'|0\rangle = \psi_0(x) \tag{30.17}$$

Plugging this into our equation for the ground state ($\hat{a}|0\rangle = 0$) by doing a shit tonne of rearranging eventually gives us

$$\sqrt{\frac{mw}{2\hbar}}\left(x + \frac{\hbar}{mw}\frac{\partial}{\partial x}\right)\psi_0(x) = 0 \tag{30.18}$$

which we can rearrange into a simple first order ODE and get the solution:

$$\psi_0(x) = \left(\frac{mw}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{mwx^2}{2\hbar}} \tag{30.19}$$

where the constant at the front is our normalization constant.

30.4.2 Excited States

We now use the relationship $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. For the first excited state $n = 0$,

$$\int \langle x|\hat{a}^\dagger|x'\rangle\langle x'|0\rangle = \langle x|1\rangle \tag{30.20}$$

which leads us to

$$\sqrt{\frac{mw}{2\hbar}}\left(x - \frac{\hbar}{mw}\frac{\partial}{\partial x}\right)\psi_0(x) = \psi_1(x) \tag{30.21}$$

which in turns leads to

$$\psi_1(x) = \left(\frac{4}{\pi}\left(\frac{mw}{\hbar}\right)^3\right)^{\frac{1}{4}} x e^{-\frac{mwx^2}{2\hbar}} \tag{30.22}$$

In general,

$$\frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle = |n\rangle \tag{30.23}$$

In coordinate representation,

$$\frac{1}{\sqrt{n!}}\langle x|(\hat{a}^\dagger)^n|0\rangle = \psi_n(x) \tag{30.24}$$

$$\frac{1}{\sqrt{n!}}\left(\sqrt{\frac{mw}{2\hbar}}\left(x - \frac{\hbar}{mw}\frac{\partial}{\partial x}\right)\right)^n \psi_0(x) = \psi_n(x) \tag{30.25}$$

31 Appendix A: Course Stuff

hello Liz!! and Dylan!!

Nick is a giant douchenozzle :)

Two marking schemes:

- 20% Assignments, 5% Clicker, 30% Midterm, 45% Final
- 20% Assignments, 5% Clicker, 20% Midterm, 55% Final

32 Appendix B: Pre-Lecture 13 Recap

Consider making a measurement in an experiment, an observable operator A, with an input state $|\psi_{in}\rangle$ with n assigned outputs. We have n kets with labels $|a_i\rangle$, which are the eigenvectors of A. Each of these kets refer to an assigned value of a_i, which are the eigenvalues of A.

The probability for any assigned value occuring is given by:

$$\mathcal{P}(i) = |\langle a_i|\psi_{in}\rangle|^2$$

The expectation value for A is given by:

$$\langle A\rangle = \sum_i a_i \mathcal{P}(i) = \langle \psi_{in}|A|\psi_{in}\rangle$$

The value of A itself is given by:

$$A = \sum_i a_i |a_i\rangle\langle a_i|$$

33 Appendix C: Acknowledgements

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Figure 26: What a faggot.