LAPLACE TRANSFORM OF A CONVOLUTION & 'IMPULSE RESPONSE'

THE LAPLACE TRANSFORM IS PARTICULARLY WELL-SUITED TO DEALING WITH CONVOLUTION INTEGRALS, AND BY EXTENSION SOLVING NON-HOMOGENEOUS DIFF. EQ. WITH CONSTANT COEFFICIENTS.

THE MOST USEFUL PROPERTY IN THIS CONTEXT IS: THE LAPLACE TRANSFORM OF A CONVOLUTION IS SIMPLY THE PRODUCT OF THE INDIVIDUAL TRANSFORMS:

$$\mathcal{L}[f*g] = \mathcal{L}[f]\cdot\mathcal{L}[g]$$

PROOF: CONSIDER THE PRODUCT:

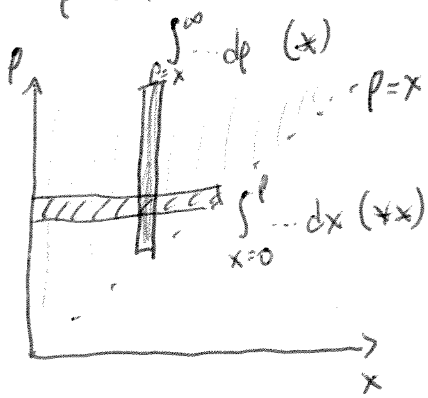
$$\mathcal{L}[f]\cdot\mathcal{L}[g] = \left(\int_0^\infty e^{-sx} f(x)\,dx\right)\Big(\int_0^\infty \underbrace{e^{-sr}\, g(r)\, dr}_{\text{INDEPENDENT OF } x}\Big)$$

$$= \int_{x=0}^{\infty}\int_{r=0}^{\infty} e^{-s(x+r)} g(r) f(x)\, dr\, dx$$

MAKE A CHANGE OF VARIABLES – $\rho = r + x$

$$= \int_{x=0}^{\infty}\int_{\rho=x}^{\infty} e^{-s\rho} g(\rho - x) f(x)\, d\rho\, dx \qquad (*)$$

REGION OF INTEGRATION:
$\rho \in [x,\infty]$ & $x \in [0,\infty]$



REVERSE THE ORDER OF INTEGRATION,

$$= \int_{\rho=0}^{\infty}\left[\int_{x=0}^{\rho} e^{-s\rho} g(\rho - x) f(x)\, dx\right] d\rho$$

$$= \int_{\rho=0}^{\infty} e^{-s\rho}\left[\int_{x=0}^{\rho} g(\rho - x) f(x)\, dx\right] d\rho$$

$$= \int_{\rho=0}^{\infty} e^{-s\rho} (f*g)(\rho)\, d\rho = \mathcal{L}[f*g] \blacksquare$$

EXAMPLE: SOLVE THE INITIAL VALUE PROBLEM

$$y''(x) + y(x) = f(x) \quad ; \quad y(0) = y'(0) = 0.$$

TAKING THE LAPLACE TRANSFORM, WITH $Y(s) = \mathcal{L}[y(x)]$ & $F(s) = \mathcal{L}[f(x)]$,

$$s^2 Y(s) - s y(0) - y'(0) + Y(s) = F(s)$$

OR,

$$Y(s) = \frac{F(s)}{1+s^2} = \mathcal{L}[f(x)] \cdot \mathcal{L}[\sin(x)] = \mathcal{L}[f * \sin(x)]$$

CAN CHECK THAT $\mathcal{L}[\sin x] = \frac{1}{1+s^2}$

OR,

$$y(x) = f * \sin = \int_0^x \sin(x - x') f(x') \, dx'$$

IN GENERAL, $Y(s) = F(s) \cdot G(s)$ CALLED 'THE TRANSFER FUNCTION'

IN THIS EXAMPLE, $G(s) = \frac{1}{1+s^2}$

SUPPOSE WE SOLVED THE EQUATION WITH AN 'IMPULSIVE FORCE'

$$y_I'' + y_I = \delta(x)$$

SAME AS ABOVE, BUT $F(s) = \mathcal{L}[\delta(x)] = 1$, SO,

$$\mathcal{L}[y_I] = Y_I(s) = \frac{1}{1+s^2} = G(s)$$

$$\text{AND} \quad y_I = \mathcal{L}^{-1}[G(s)] = \sin x$$

THE SOLUTION TO THE DIFFERENTIAL EQUATION SUBJECT TO IMPULSIVE FORCING $\delta(x)$ IS CALLED THE <u>IMPULSE RESPONSE</u> AND IT IS SYNONYMOUS WITH '<u>GREEN'S FUNCTIONS</u>' THAT WE ENCOUNTERED EARLIER, BECAUSE THE PARTICULAR SOLUTION IS ALWAYS WRITTEN AS A CONVOLUTION OF THE <u>IMPULSE RESPONSE</u> AND THE FORCING:

$$y_p = \int_0^x y_I(x - x') f(x') \, dx'$$

CLEARLY, THE 'IMPULSE RESPONSE' FUNCTION IS WHAT WE HAVE BEEN CALLING THE 'FUNDAMENTAL MATRIX'.

TO PHYSICISTS & ENGINEERS, THE NAMING CONVENTION IS:

$$\mathcal{L}\left[\text{IMPULSE RESPONSE}\right] = \text{TRANSFER FUNCTION}$$

RECAST THE NON-HOMOGENEOUS HARMONIC OSCILLATOR AS A SYSTEM:

$$y'' + y = f(x) \iff \frac{d}{dx}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} f(x)$$

$y_1 = y$ & $y_2 = y'$

THE FUNDAMENTAL MATRIX $\Phi(x)$ HAS COLUMNS MADE FROM THE UNFORCED INITIAL VALUE PROBLEM, WITH:

$$\left.\begin{array}{ccc} y_1(0) = 1 & & y_1(0) = 0 \\ y_2(0) = 0 & \& & y_2(0) = 1 \\ \Downarrow & & \Downarrow \\ y_1 = \cos x & & y_1 = \sin x \\ y_2 = -\sin x & & y_2 = \cos x \end{array}\right\} \quad \Phi(x) = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$$

BY THE VARIATION-OF-PARAMETERS:

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \Phi(x)\begin{bmatrix} y_1^0 \\ y_2^0 \end{bmatrix} + \int_0^x \Phi(x-x')\begin{bmatrix} 0 \\ 1 \end{bmatrix} f(x')\,dx'$$

$$= \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}\begin{bmatrix} y_1^0 \\ y_2^0 \end{bmatrix} + \int_0^x \begin{bmatrix} \sin(x-x') \\ \cos(x-x') \end{bmatrix} f(x')\,dx'$$

TRUE FOR ANY CONTINUOUS FORCE $f(x)$ - LOOK AT A PARTICULAR CHOICE:

$$\begin{bmatrix} y_1^0 \\ y_2^0 \end{bmatrix} = \begin{bmatrix} 0 \\ a \end{bmatrix} \quad \& \quad f(x) = b \cdot \delta(x)$$

CONSTANTS

THEN,

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = a\begin{bmatrix} \sin x \\ \cos x \end{bmatrix} + b\begin{bmatrix} \sin x \\ \cos x \end{bmatrix} = \begin{bmatrix} y(x) \\ y'(x) \end{bmatrix}$$

IDENTICAL CONTRIBUTIONS!

IMPULSIVE FORCING IS EQUIVALENT TO NON-ZERO INITIAL CONDITIONS.

CAN THINK OF AN ARBITRARY FORCING FUNCTION $f(x)$ AS A SET OF STRUNG-TOGETHER IMPULSES:

$$f(x) = \int \underbrace{\delta(x-x')\, f(x')}_{\text{IMPULSE } f(x') \text{ AT } x=x'} dx'$$

THIS IS THE SAME INTUITION WE ARRIVED AT USING THE RIEMANN SUM REPRESENTATION OF THE CONVOLUTION!

LAPLACE TRANSFORMS ARE OFTEN USED IN CONTROL THEORY (AMATH 455/655) TO ESTABLISH THE <u>STABILITY</u> OF A SYSTEM, OR TO FIND PARTICULAR INPUT FUNCTIONS $f(x)$ THAT WILL <u>STABILIZE</u> A SYSTEM.

FOR EXAMPLE, THE TRANSFER FUNCTION $G(s)$ CAN OFTEN BE WRITTEN AS A RATIONAL FUNCTION:

$$G(s) = \frac{N(s)}{D(s)}$$

THE ROOTS OF THE DENOMINATOR $D(s)$ ARE CALLED 'POLES', s^*

<u>THEOREM</u>: IF ALL POLES s^* LIE STRICTLY IN THE LEFT-COMPLEX PLANE, $Re(s^*) < 0$, THEN THE SYSTEM IS <u>STABLE</u>, AND ALL SOLUTIONS $\vec{y}(x)$ SATISFY: $\lim_{x\to\infty} \vec{y}(x) = \vec{0}$.

THIS THEOREM IS NOT SO MYSTERIOUS - FOR A ~~TIME-VAR~~ SYSTEM WITH CONSTANT COEFFICIENTS,

$$\frac{d\vec{y}}{dx} = A \cdot \vec{y}$$

THE TRANSFER FUNCTION IS $\vec{G}(s) = [\mathbb{I}s - A]^{-1}$

THE 'POLES' ARE EXACTLY WHAT WE CALL <u>EIGENVALUES</u>.

IF YOU THINK OF THE FUNDAMENTAL MATRIX, e^{Ax}, IT IS COMPOSED OF FUNCTIONS OF THE FORM $\underbrace{e^{\lambda x}}_{\text{REAL EIGENVALUES}}$, $\underbrace{e^{Re(\lambda)x}(\cos[Im(\lambda)x] + \sin[Im(\lambda)x])}_{\text{COMPLEX EIGENVALUES}}$

OR ~~$xe^{\lambda x}$~~ $\underbrace{xe^{\lambda x}, x^2e^{\lambda x} \ldots}_{\text{REPEATED EIGENVALUES WITH DEGENERATE EIGENVECTORS}}$

THE LONG-TERM ($x \to \infty$) BEHAVIOUR OF THESE FUNCTIONS IS DOMINATED BY THE REAL PART OF THE EIGENVALUE $Re(\lambda)$

i) IF ALL EIGENVALUES OF A HAVE NEGATIVE REAL PART, THEN <u>ALL</u> SOLUTIONS ARE <u>STABLE</u> $\lim_{x \to \infty} \vec{y}(x) = \vec{0}$.

ii) IF ANY EIGENVALUE OF A HAS POSITIVE REAL PART, THEN MOST SOLUTIONS WILL BE <u>UNSTABLE</u> $\lim_{x \to \infty} |\vec{y}(x)| \to \infty$

[UNLESS WE EXCLUDE THESE USING PARTICULAR INITIAL CONDITIONS]

iii) IF ALL EIGENVALUES HAVE ZERO REAL PART, THEN THE BEHAVIOUR DEPENDS ON POLYNOMIAL TERMS $xe^{\lambda x}, x^2e^{\lambda x} \ldots$

a) FOR DISTINCT EIGENVALUES, SOLUTIONS REMAIN BOUNDED (eg. $\sin x$, $\cos x$, CONSTANT) & THE SYSTEM IS 'NEUTRALLY STABLE'

b) IF JORDAN FORM OF A CONTAINS BLOCKS $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ ETC, THEN SOLUTIONS GROW A<u>LGEBRAICA</u>LLY (ie LIKE x, x^2, ETC.)

WE CAN CARRY OVER SOME OF THIS INSIGHT WHEN WE LOOK AT NONLINEAR DIFFERENTIAL EQUATIONS

NONLINEAR SYSTEMS & STABILITY

EXCELLENT BOOK BY STROGATZ "NONLINEAR DYNAMICS & CHAOS"

SO FAR, OUR FOCUS HAS BEEN ON LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS - AS WE'LL SEE, OUR INTUITION FOR LINEAR SYSTEMS WITH CONSTANT COEFFICIENTS CARRIES OVER TO THE ANALYSIS OF THE STABILITY OF NONLINEAR SYSTEMS.

WE'LL LOOK AT NONLINEAR SYSTEMS WITH CONSTANT COEFFICIENTS,

$$\frac{d}{dx}\vec{y}(x) = \vec{f}(\vec{y}(x))$$

$\vec{f}(\vec{y})$ DOES NOT EXPLICITLY DEPEND UPON 'x'.

DEFINITION: IF THE VECTOR $\vec{y}^*$ SATISFIES THE ALGEBRAIC SYSTEM OF EQUATIONS $\vec{f}(\vec{y}^*)=\vec{0}$, THEN WE CALL $\vec{y}(x)=\vec{y}^*$ AN EQUILIBRIUM OR STEADY-STATE SOLUTION OF THE SYSTEM BECAUSE AT $\vec{y}(x)=\vec{y}^*$, $\frac{d\vec{y}}{dx}=0$.

EX. THE 'BRUSSELATOR'

$$\frac{dx_1}{dt} = 1 + a x_1^2 x_2 - (1+b)x_1 \qquad \frac{dx_2}{dt} = -a x_1^2 x_2 + b x_1$$

WITH a,b CONSTANTS. CONVINCE YOURSELF THAT $(x_1^*, x_2^*) = (1, b/a)$ IS THE ONLY STEADY-STATE FOR THIS SYSTEM.

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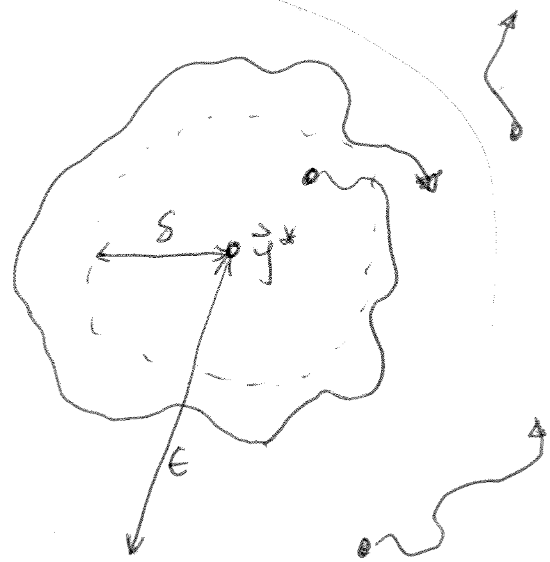
WE CAN RARELY SOLVE NONLINEAR SYSTEMS, BUT WE CAN GAIN SOME SENSE OF THEIR QUALITATIVE BEHAVIOUR BY LOOKING AT HOW SOLUTIONS BEHAVE CLOSE TO THEIR STEADY-STATES.

FOR EXAMPLE, WE SAY AN EQUILIBRIUM SOLUTION $\vec{y}^*$ IS STABLE IF NEARBY SOLUTIONS REMAIN NEARBY.

MORE SPECIFICALLY - IF, FOR ALL $\epsilon>0$, THERE EXISTS A $\delta>0$ SO THAT INITIAL CONDITIONS THAT ARE δ-CLOSE

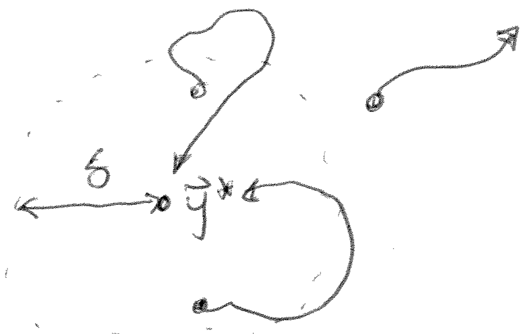
TO $\vec{y}^*$, i.e. $\|\vec{y}^0 - \vec{y}^*\| < \delta$, PRODUCE SOLUTIONS $\vec{y}(x)$ THAT REMAIN ϵ-CLOSE TO $\vec{y}^*$, $\|\vec{y}(x) - \vec{y}^*\| < \epsilon$ FOR ALL $x \geq 0$.



AN EQUILIBRIUM SOLUTION $\vec{y}^*$ THAT IS NOT STABLE IS CALLED '<u>UNSTABLE</u>'

WE CAN FURTHER REFINE OUR NOTION OF STABILITY BY DISTINGUISHING BETWEEN 'ATTRACTING' AND 'NONATTRACTIN EQUILIBRIA.

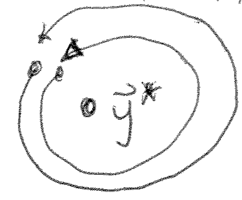
AN EQUILIBRIUM SOLUTION $\vec{y}^*$ IS '<u>ATTRACTING</u>' IF NEARBY TRAJECTORIES CONVERGE TO $\vec{y}^*$. FORMALLY, THE EXISTS A δ SO THAT ALL INITIAL CONDITIONS $\|\vec{y}^0 - \vec{y}^*\| < \delta$ PRODUCE SOLUTIONS THAT SATISFY: $\lim_{x \to \infty} \vec{y}(x) = \vec{y}^*$



AN EQUILIBRIUM THAT IS BOTH <u>STABLE</u> & <u>ATTRACTING</u> IS CALLED "ASYMPTOTICALLY STABLE"

AN EQUILIBRIUM THAT IS <u>STABLE</u>, BUT <u>NOT</u> ATTRACTING, IS CALLED "NEUTRALLY STABLE".

e.g.

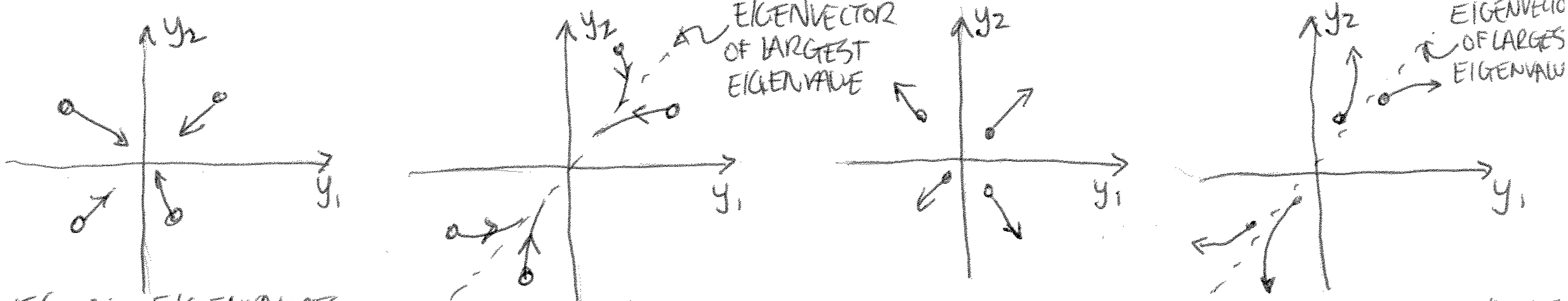


LET'S LOOK MORE CLOSELY AT THE STABILITY OF 2x2 <u>LINEAR</u> SYSTEM WITH CONSTANT COEFFICIENTS.

RECALL THAT THE STABILITY OF $\frac{d\vec{y}}{dx} = A \cdot \vec{y}$ DEPENDS ON THE EIGENSTRUCTURE OF A. WE CAN VISUALIZE THE DIFFERENT POSSIBILITIES BY PLOTTING THE TWO SOLUTIONS $\vec{y} = [y_1, y_2]$ SIMULTANEOUSLY IN A "PHASE PLOT".

NEGATIVE REAL EIGENVALUES

POSITIVE REAL EIGENVALUES



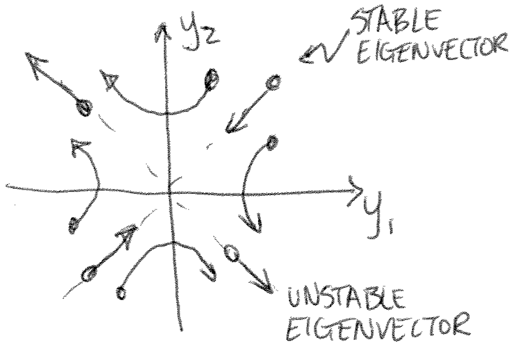
EQUAL EIGENVALUES — UNEQUAL EIGENVALUES

THE ORIGIN IS CALLED A "<u>STABLE</u> NODE"

EQUAL EIGENVALUES — UNEQUAL EIGENVALUES

ORIGIN IS CALLED AN "<u>UNSTABLE</u> NODE"

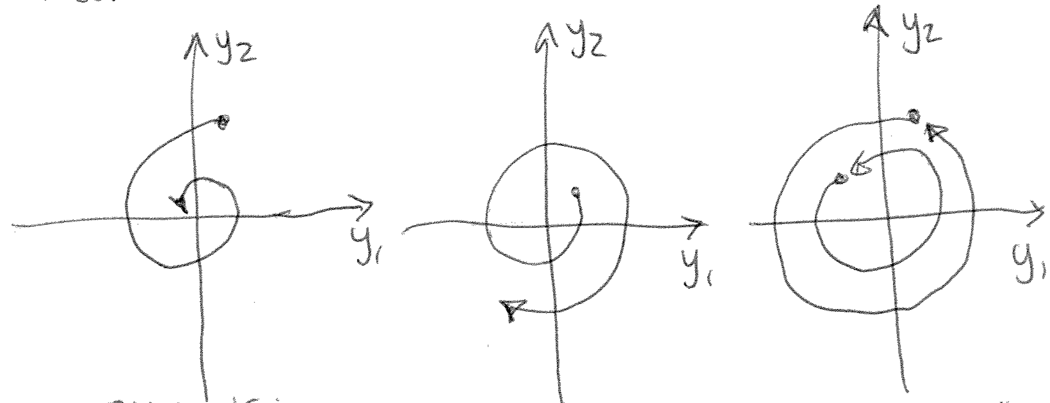
REAL EIGENVALUES, BUT OPPOSITE SIGN



ORIGIN CALLED "SADDLE"

COMPLEX EIGENVALUES

NEGATIVE REAL PART — POSITIVE REAL PART — ZERO REAL PART



ORIGIN IS: "STABLE SPIRAL" — "UNSTABLE SPIRAL" — "CENTRE"

<u>LINEAR STABILITY ANALYSIS</u>

THE STABILITY OF THE EQUILIBRIA OF <u>NONLINEAR</u> SYSTEMS CAN TYPICALLY BE DETERMINED BY "LINEARIZING" THE EQUATIONS ABOUT AN EQUILIBRIUM SOLUTION $\vec{y}^*$, AND EXAMING THE <u>LOCAL</u> EIGEN-STRUCTURE OF THE DYNAMICS.

FOR A NONLINEAR SYSTEM $\frac{d\vec{y}}{dx} = \vec{f}(\vec{y})$ WITH EQUILIBRIUM $\vec{y}^*$, TAKE A MULTIVARIABLE TAYLOR EXPANSION OF $\vec{f}(\vec{y})$ ABOUT $\vec{y}^*$:

$$\vec{f}(\vec{y}) = \vec{f}(\vec{y}^*) + \underbrace{\frac{\partial \vec{f}}{\partial \vec{y}}}_{\text{JACOBIAN MATRIX}}(\vec{y}^*)\,(\vec{y}-\vec{y}^*) + \mathcal{O}(\|\vec{y}-\vec{y}^*\|)$$

WHERE THE JACOBIAN MATRIX IS:

$$\frac{\partial \vec{f}}{\partial \vec{y}}(\vec{y}^*) = \begin{bmatrix} \frac{\partial f_1}{\partial y_1}(\vec{y}^*) & \frac{\partial f_1}{\partial y_2}(\vec{y}^*) & \cdots & \frac{\partial f_1}{\partial y_n}(\vec{y}^*) \\ \frac{\partial f_2}{\partial y_1}(\vec{y}^*) & \frac{\partial f_2}{\partial y_2}(\vec{y}^*) & & \vdots \\ \vdots & & \ddots & \\ \frac{\partial f_n}{\partial y_1}(\vec{y}^*) & \cdots & \cdots & \frac{\partial f_n}{\partial y_n}(\vec{y}^*) \end{bmatrix}$$

A MATRIX OF CONSTANTS

TAKE A LOOK AT THE BEHAVIOUR OF A TRAJECTORY 'CLOSE' TO THE EQUILIBRIUM: $\vec{z}(x) = \vec{y}(x) - \vec{y}^*$ [SOMETIMES CALLED 'NORMAL MODES']

THEN, $$\frac{d}{dx}\vec{z} = \frac{d\vec{y}}{dx} \approx \frac{\partial \vec{f}}{\partial \vec{y}}(\vec{y}^*)\left(\vec{y}(x) - \vec{y}^*\right) = \left[\frac{\partial \vec{f}}{\partial \vec{y}}(\vec{y}^*)\right] \cdot \vec{z}(x)$$

OR, $\frac{d}{dx}\vec{z} = \mathbb{A} \cdot \vec{z}$ WHERE $\mathbb{A}$ IS THE JACOBIAN MATRIX.

IF THE REAL PARTS OF THE EIGENVALUES OF THE JACOBIAN $\mathbb{A}$ ARE NONZERO, THEN THE STABILITY OF $\vec{y}^*$ IS THE SAME AS THE STABILITY OF $\vec{0}$ IN $\frac{d\vec{z}}{dx} = \mathbb{A}\vec{z}$. [FOR ZERO REAL PART, WE MUST INCLUDE HIGHER-ORDER TERMS IN THE TAYLOR SERIES...]

THE CONCLUSION IS: EIGENVALUE CRITERIA FOR STABILITY OF LINEAR CONSTANT COEFFICIENT MARICES CAN BE APPLIED AT EACH EQUILIBRIUM OF A NONLINEAR SYSTEM!

LET'S LOOK AT SOME EXAMPLES.

EX. BRUSSELATOR: $\frac{dx_1}{dt} = 1 + a x_1^2 x_2 - (1+b) x_1 \quad \frac{dx_2}{dt} = -a x_1^2 x_2 + b x_1$

IN MATRIX-VECTOR NOTATION:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 + a x_1^2 x_2 - (1+b) x_1 \\ -a x_1^2 x_2 + b x_1 \end{bmatrix} \equiv \begin{bmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{bmatrix}$$

THE EQUILIBRIUM SOLUTION OF ~~FOR~~ THIS SYSTEM IS $(x_1^*, x_2^*) = (1, b/a)$

THE JACOBIAN MATRIX EVALUATED AT THAT EQUILIBRIUM IS:

$$\frac{\partial \vec{f}}{\partial \vec{x}}(\vec{x}^*) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{\vec{x}=(x_1^*,x_2^*)} = \begin{bmatrix} 2a x_1 x_2 - (1+b) & a x_1^2 \\ b - 2a x_1 x_2 & -a x_1^2 \end{bmatrix}_{\vec{x}=\vec{x}^*}$$

$$= \begin{bmatrix} b-1 & a \\ -b & -a \end{bmatrix}$$

MATRIX OF CONSTANTS.

THE EIGENVALUES ARE: $\lambda_{1,2} = \frac{1}{2}\left([b-(1+a)] \pm \sqrt{(1+a-b)^2 - 4a}\right)$

THE EQUILIBRIUM IS STABLE FOR $b \leq 1+a$ [ASYMPTOTICALLY STABLE FOR $b < 1+a$; NEUTRALLY STABLE FOR $b = 1+a$.]

WHAT ABOUT $b > 1+a$? ...