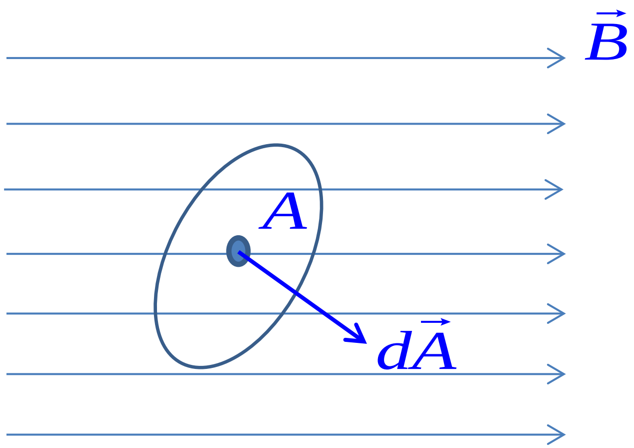
IV AC Circuits

1 Electromagnetic Induction

Magnetic Flux:

– for electrostatic case had $\phi_E = \int_S d\phi_E = \int_S \vec{E} \bullet d\vec{A}$

–consider a B– field: $\left(B \left[\frac{N}{C \cdot m/s} \right] \text{ or } \left[\frac{N}{A \cdot m} \right] \text{ or } T \text{ (the tesla)} \right)$



–define $\phi_B \equiv$ magnetic flux

$$\phi_B = \int_S d\phi_B = \int_S \vec{B} \bullet d\vec{A}$$

(units for ϕ_B : $[\frac{Nm}{A}]$ or weber $[Wb] = tesla \cdot m^2$)

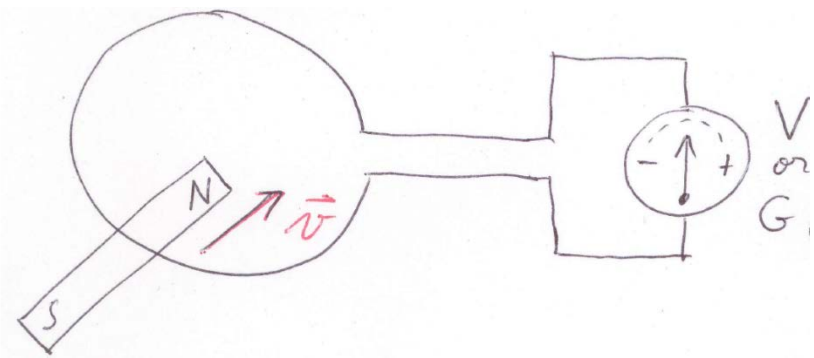
$$also \Rightarrow T(\text{the tesla}) = \frac{Wb}{m^2}$$

Electromagnetic Induction Phenomenon :

Michael Faraday's observations (1831-1832):

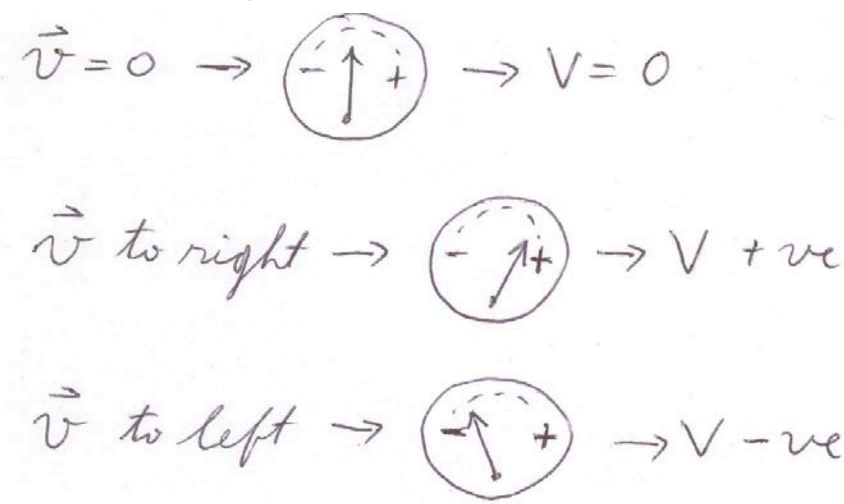
i) Relative motion causes changes in flux linkage

=> Induced electromotive force (emf) results



(Galvanometer)

Faraday found :



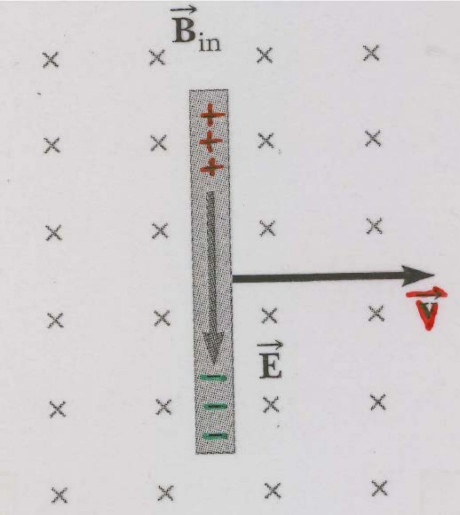
$\Rightarrow$ *Loop is linked to some of the flux of the magnet (flux is "cros sin g" loop)*

$-$*in addition if* $v \uparrow \Rightarrow V \uparrow$

$\Rightarrow$ *larger* $\dfrac{d\phi_B}{dt} \Rightarrow$ *larger* $emf_{induced}$

$(\vec{E}$ *must be induced*$)$

ii) "Cutting" of flux by a moving conductor:

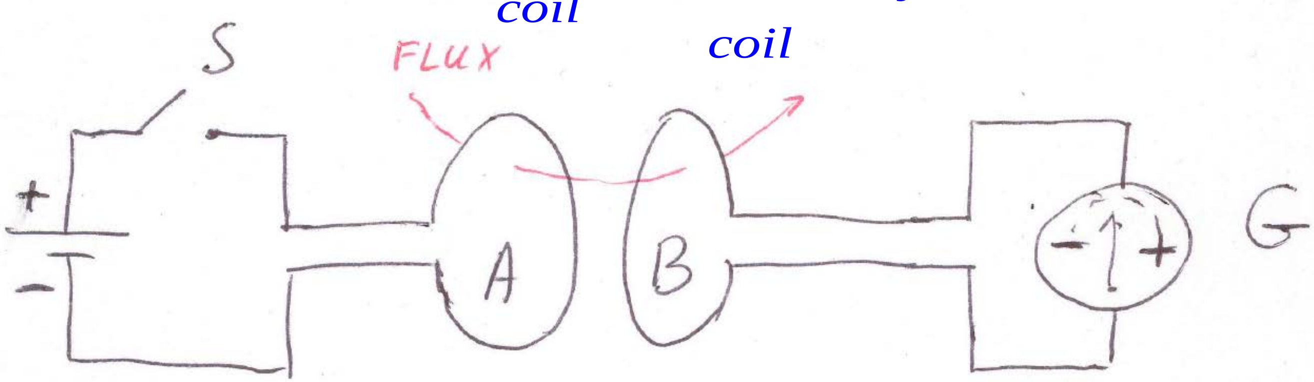


$-$ produces induce emf

iii) Mutual inductance:

primary coil

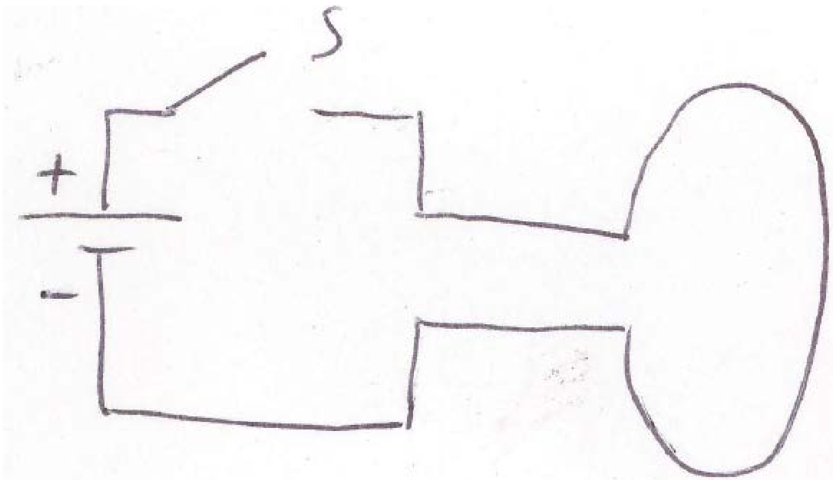
secondary coil



$-G$ deflects only when S is opened or closed $\Rightarrow$ changing ϕ_B

iv) Self – induction :

(first discovered by Joseph Henry (1832) and independently by Michael Faraday in 1834)



- as S is closed / opened get a self-induced emf in loop (circuit) due to change in its own current

–induced emf tends to oppose the change produced by applied emf

Faraday's Law

(Michael Faraday – British Physicist and Chemist (1791-1867)
- discovery of electromagnetic induction and laws of electrolysis
- invention of electric motor, electric generator, transformer)

- from his observations Michael Faraday concluded "the emf induced in a circuit is directly proportional to the time rate of change of the magnetic flux through the open surface defined by the circuit"

(circuit is same as closed path)

$$\varepsilon = -\frac{d\phi_B}{dt}$$ Faraday's Law of induction

- in SI units the factor -1 is the constant of proportionality

– for coil of N turns $\varepsilon = -N\frac{d\phi_B}{dt}$

$$\mathcal{E} = -N \frac{d\phi_B}{dt}$$

$-$ *the sign indicates how polarity of induced emf is related to sign of flux and its rate of change*

- *sign of flux:*
  - *proceed around path in proposed direction of emf*
  - *use RH rule to determine direction of* $\vec{A}$
  - *then* $\phi_B = \vec{B} \bullet \vec{A} = BA\cos\theta$

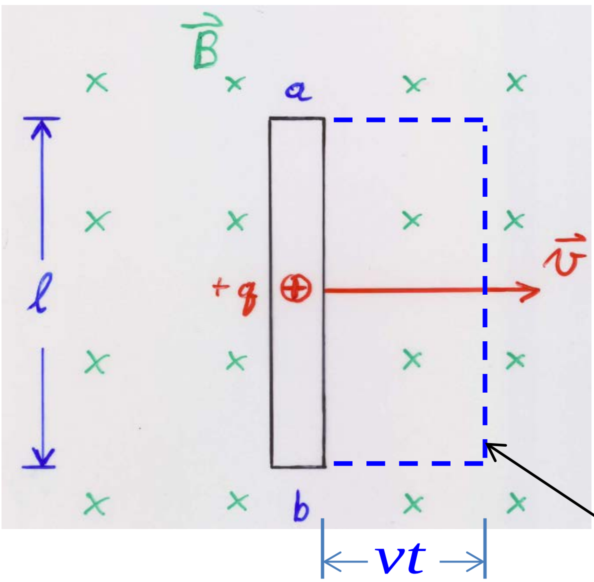
    $+ve$ *if* $\vec{B}$ *and* $\vec{A}$ *in same direction*

    $-ve$ *if in opposite directions*

- *sign of* $\frac{d\phi_B}{dt}$: *if* $\phi_B \uparrow$ *then* $\frac{d\phi_B}{dt}$ *is* $+ve$

  if $\phi_B \downarrow$ *then* $\frac{d\phi_B}{dt}$ *is* $-ve$

Faraday's Law applied to motional emf (conductor moving in B):



$$\mathcal{E} = -\frac{d\phi_B}{dt}$$

$\phi_B = \vec{B} \bullet \vec{A}$

$-$*in time t*

rod sweeps out an area $A = vtl$

the circuit

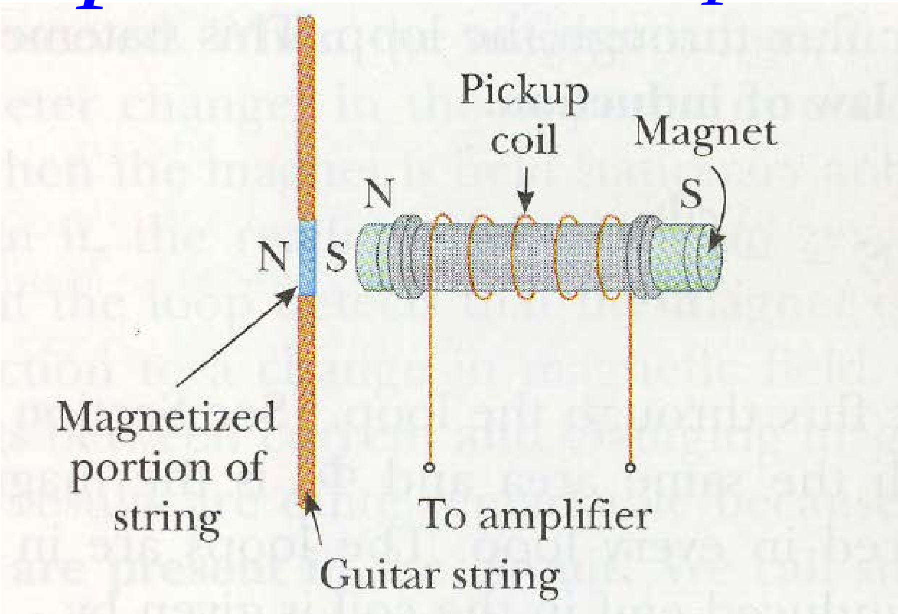
$-$*proceed around circuit in CW direction*: $\phi_B = \vec{B} \bullet \vec{A} = BA$

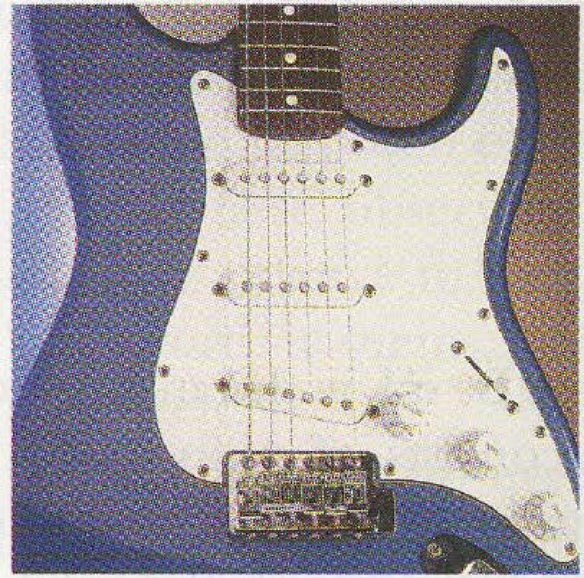
$$\mathcal{E} = -\frac{d(Bvtl)}{dt} = -vBl$$

But here flux through circuit is effectively decreasing

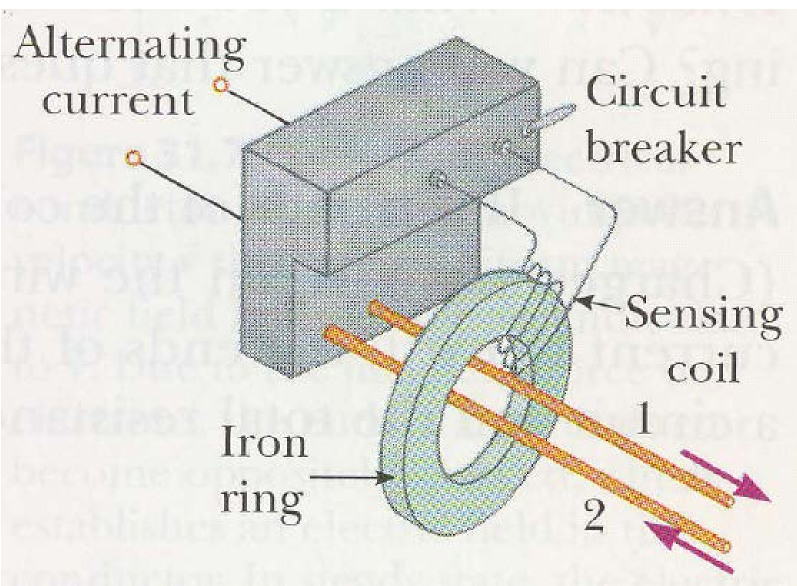
$\Rightarrow \dfrac{d\,|\phi_B|}{dt}$ *is negative* $\qquad \Rightarrow \varepsilon \uparrow$ *in CW direction*

Example *Guitar pick – up*





Example *Ground fault interuptor*

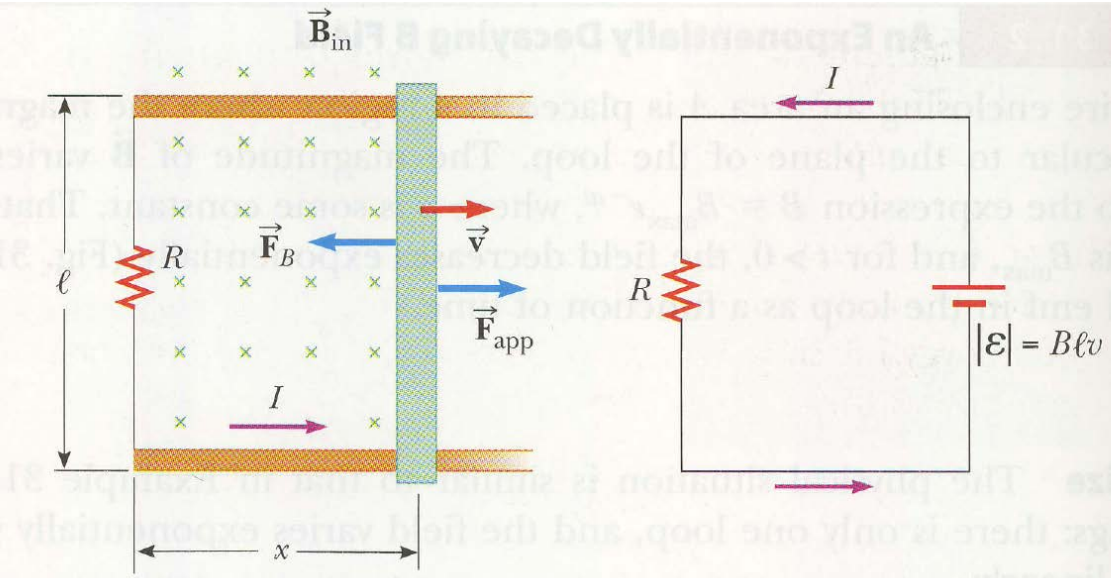


$Lenz's\ Law$ (Heinrich Lenz in 1834)

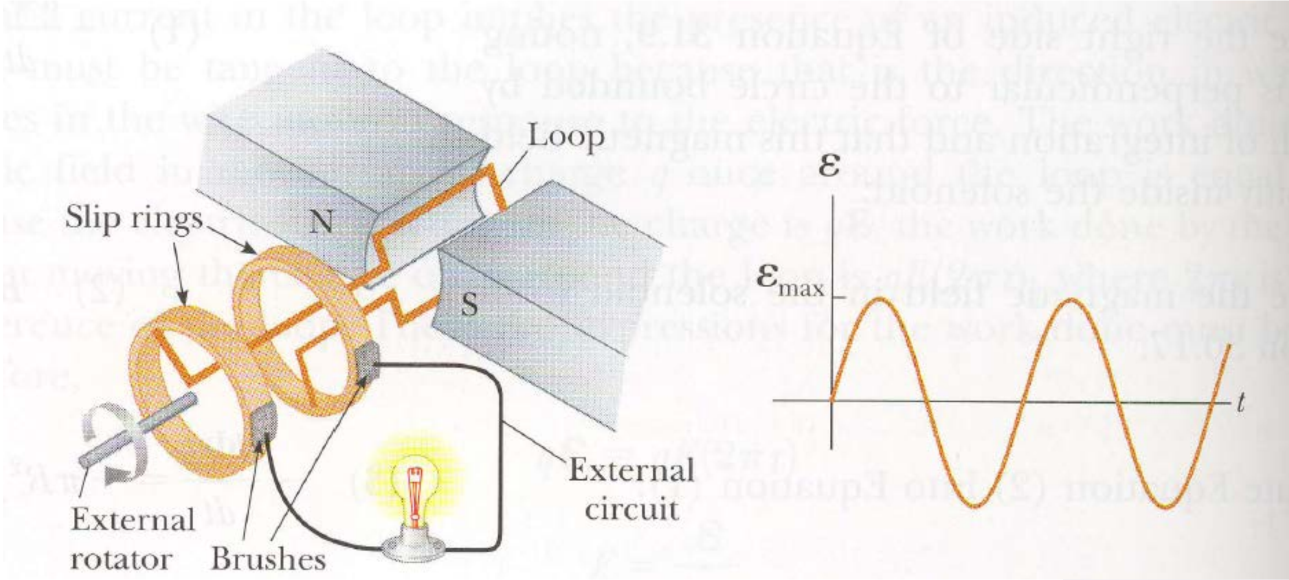
"*The direction of an induced current is such as to oppose the cause producing it.*"

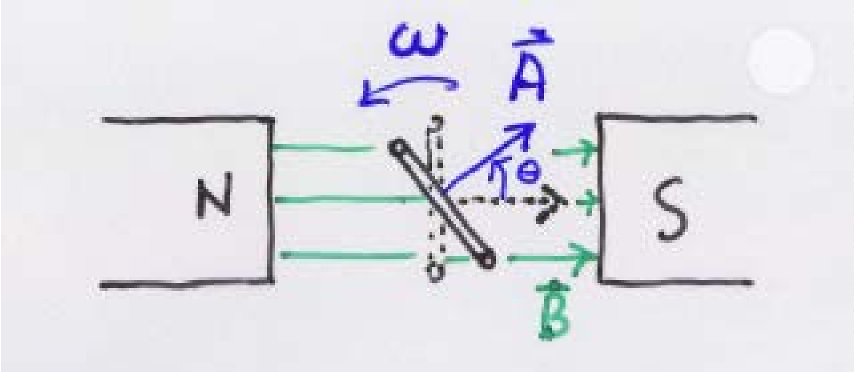
($I_{induced}$ *tends* to keep original flux from changing)

Example



Example *Generator*



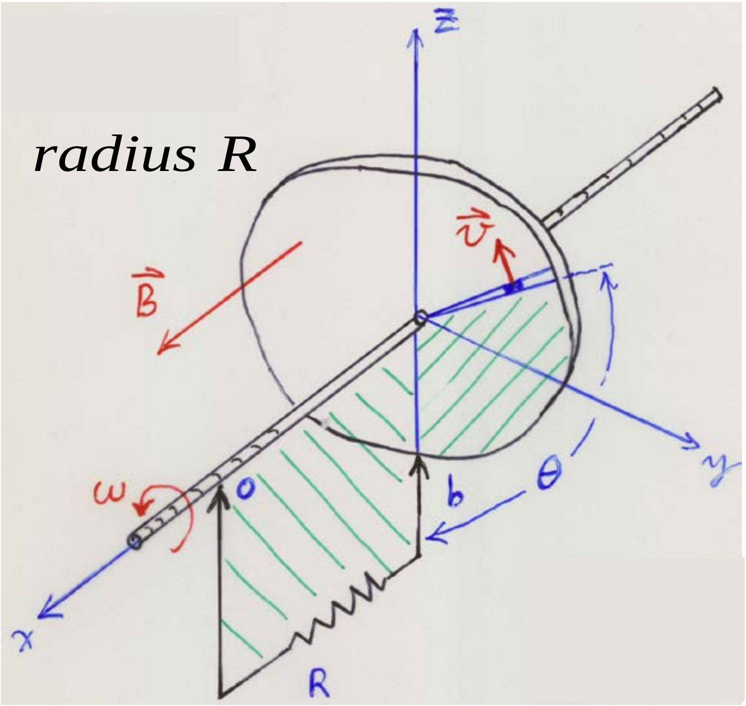


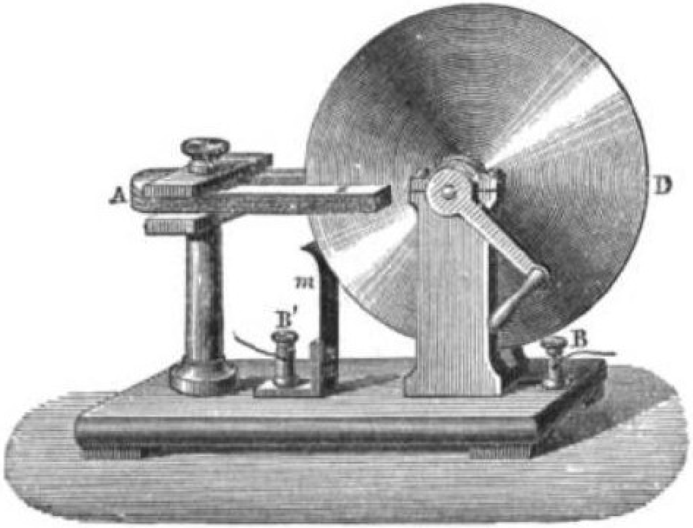
$$Let\ \ \theta = 0\ \ at\ \ t = 0$$

$$\phi_B = BA\cos\theta = BA\cos\omega t$$

$$\varepsilon = -N\frac{d\phi_B}{dt} = NAB\omega\sin\omega t$$

Example Faraday's Disk Dynamo





-can show $\mathcal{E}_{RO} = V_R - V_O = \frac{1}{2}\omega B R^2$

(low $\mathcal{E}$, large I)

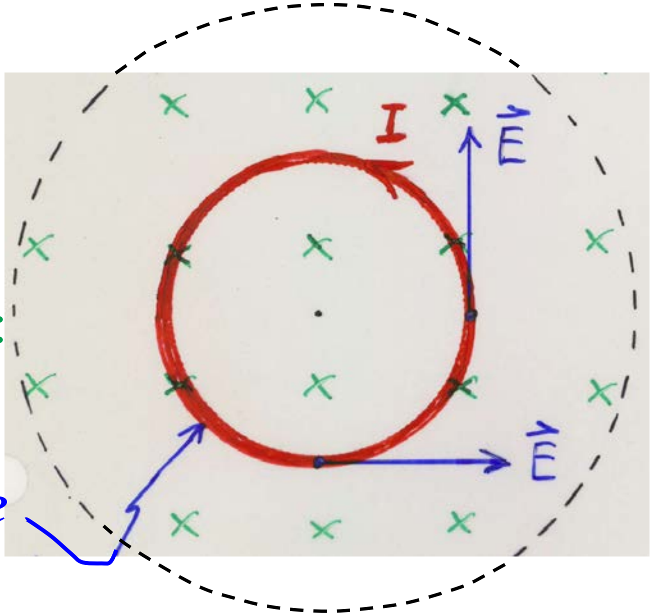
(e.g. electric welding)

Induced emf and Electric fields

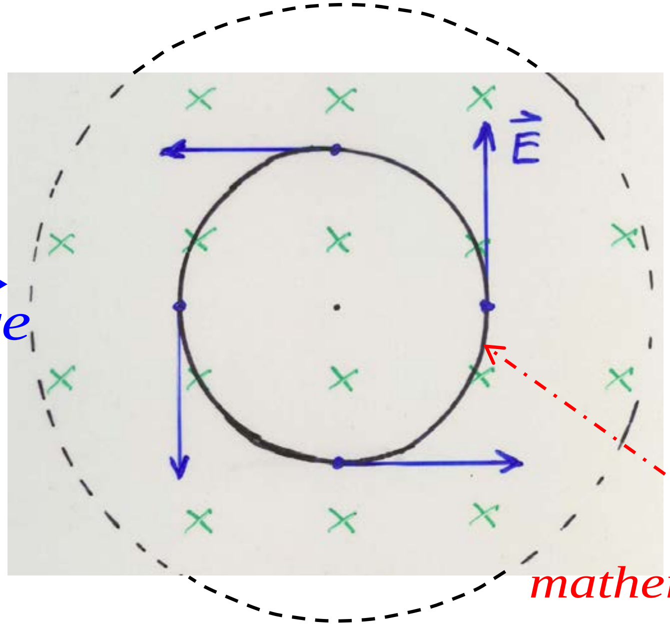
(Had considered induced emf due to conductor moving in $\vec{B}$ and due to changing flux threading circuit)

Consider $\vec{B}$ of an ideal solenoid :

loop of wire



No wire $\rightarrow$



mathematical loop

– let $\vec{B}\uparrow$ at constant rate

$$\mathcal{E} = -\frac{d\phi_B}{dt},$$ and I flows as shown

$\Rightarrow \vec{E}$ is induced by changing ϕ_B

– now $\vec{E}$ is still there

– connection between $\vec{E}_{induced}$ and emf ?

– move $q \rightarrow ds$: $dW = \vec{F}\bullet d\vec{s} = q\vec{E}\bullet d\vec{s}$

$$W = q\oint \vec{E}\bullet d\vec{s} \text{ for loop}$$

$$\therefore \quad \mathcal{E} = \oint \vec{E}\bullet d\vec{s} = -\frac{d\phi_B}{dt} \quad \left(= -\frac{d}{dt}\int_{Surf.} \vec{B}\bullet d\vec{A}\right)$$

$$\mathcal{E} = \oint \vec{E} \bullet d\vec{s} = -\frac{d\phi_B}{dt} \qquad \left(= -\frac{d}{dt}\int_{Surf.} \vec{B} \bullet d\vec{A}\right)$$

$-$ *this is a more general form of Faraday's Law*

- *this* $\vec{E}$ *is a non-electrostatic field*

$-$ *here* $\oint \vec{E} \bullet d\vec{s} \neq 0 \implies$ *non-conservative*

In this example the B– field has line symmetry

– we expect the induced E to be constant on a circle,

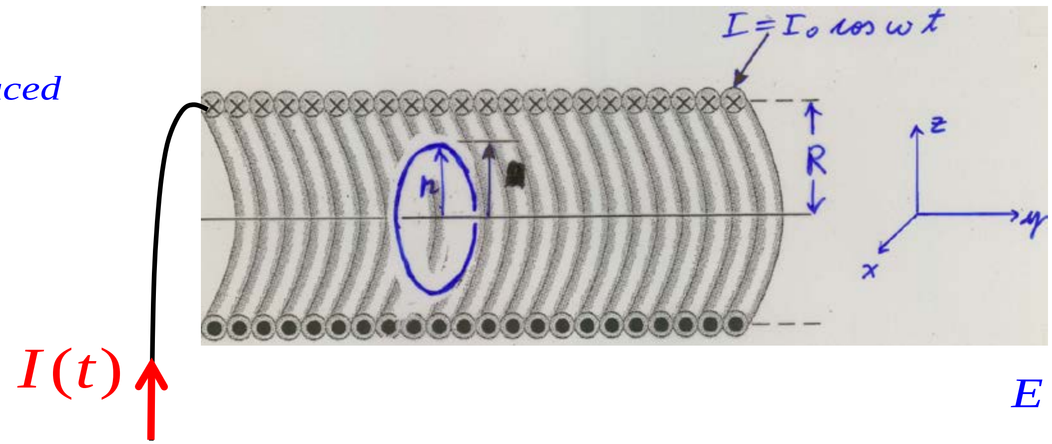
centered on the symmetry axis (central axis).

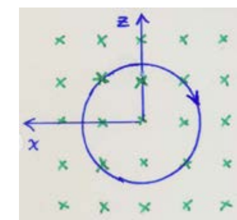
In general the E lines would not form circles.

Example *Electric field due to solenoid*

In an ideal solenoid (radius R, n turns / unit length)
$I(t) = I_0 \cos \omega t.$ $(B = \mu_0 n I(t)$; *only inside solenoid*)

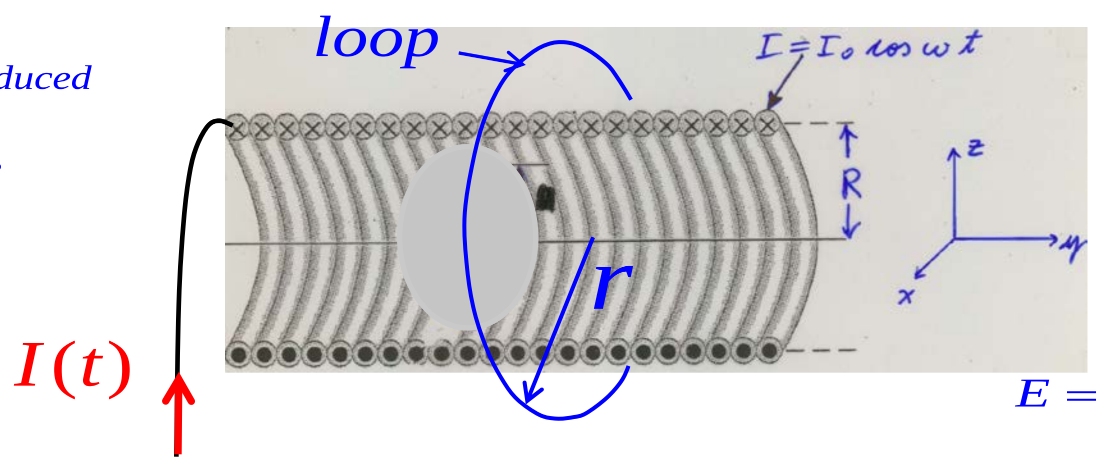
a) *Find* $\vec{E}_{induced}$
for $r < R$.

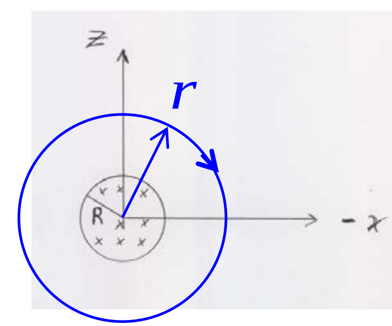




$$E = +\frac{\mu_0 n I_0 \omega}{2} r \sin \omega t$$

b) *Find* $\vec{E}_{induced}$
for $r > R$.

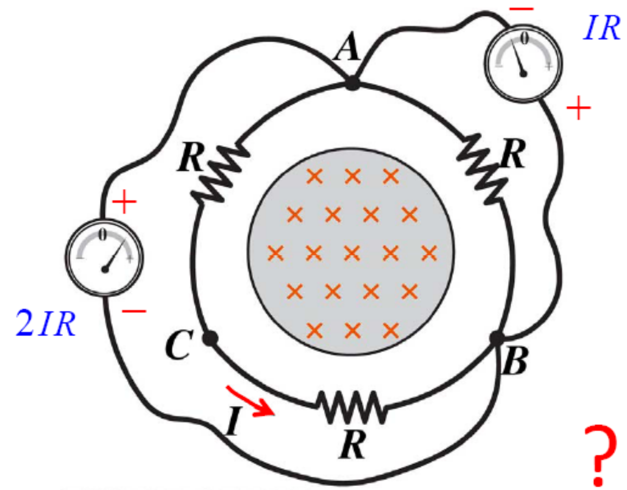


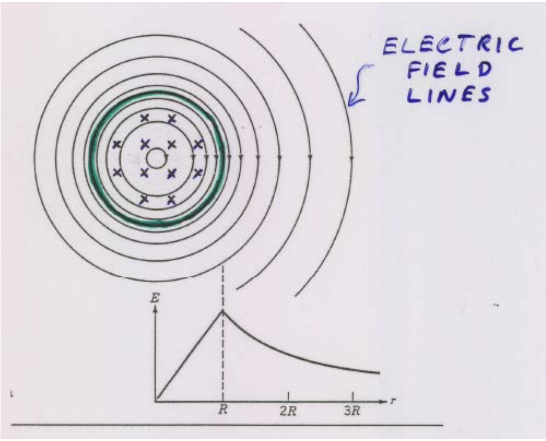


$$E = \frac{\mu_0 n I_0 \omega R^2}{2r} \sin \omega t$$

Induced E field is created at $r > R$ *even though* $B = 0$ *there!!*

Have pointed out that this E is nonconservative:





- *from electrostatics we had*

$$V_f - V_i = -\int_i^f \vec{E} \bullet d\vec{s}$$

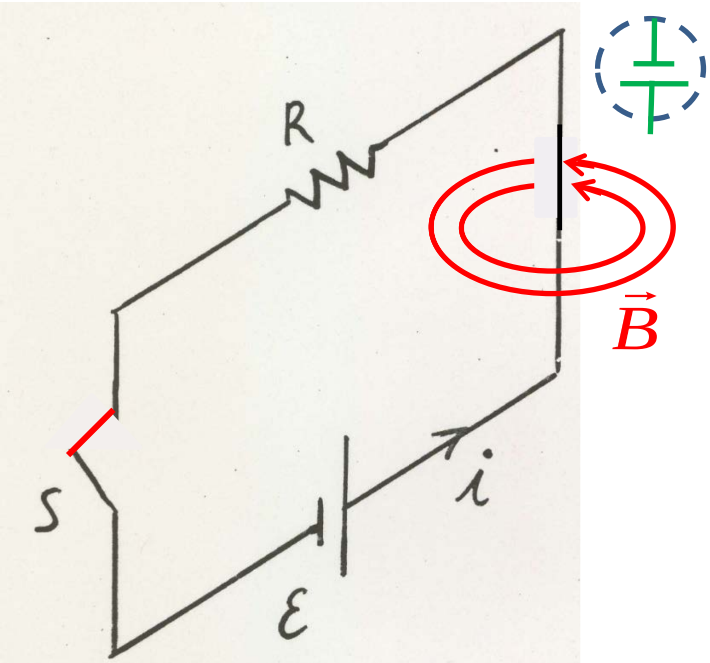
and $\oint \vec{E} \bullet d\vec{s} = 0$

But if changing ϕ_B *is present:* $\mathcal{E} = \oint \vec{E} \bullet d\vec{s} = -\dfrac{d\phi_B}{dt}$

$\Rightarrow$ *cannot assign an electric potential to induced* $\vec{E}$ *fields*

Inductance

- *Self – Inductance*



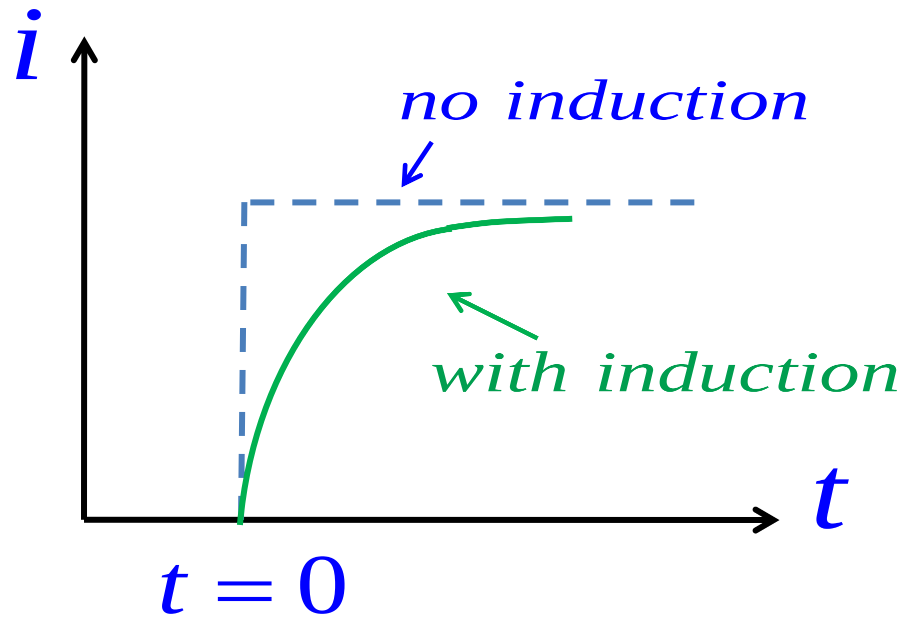
polarity of self – induced emf

- according to Lenz's Law induced flux opposes flux increase

$\vec{B}_{induced}$ *is in opposite direction to* $\vec{B}$

- and induced emf opposes applied emf

$-$effect of this on how current changes?



$-$ for capacitors we had

$q \propto V$ and $q = CV$

$-$ for a coil

$\phi_B \propto i$

or $\phi_B = \text{constant x } i$

$-$usually include number of turns:

$$\underbrace{N\phi_B}_{\text{flux linkage}} = \underbrace{Li}_{\text{inductance}} \quad \text{or} \quad L = \frac{N\phi_B}{i}$$

units: $T \cdot m^2 / A$ or $1 \text{ henry} = 1\,H = 1\,Tm^2 / A$

Connection between self-induction and self-inductance:

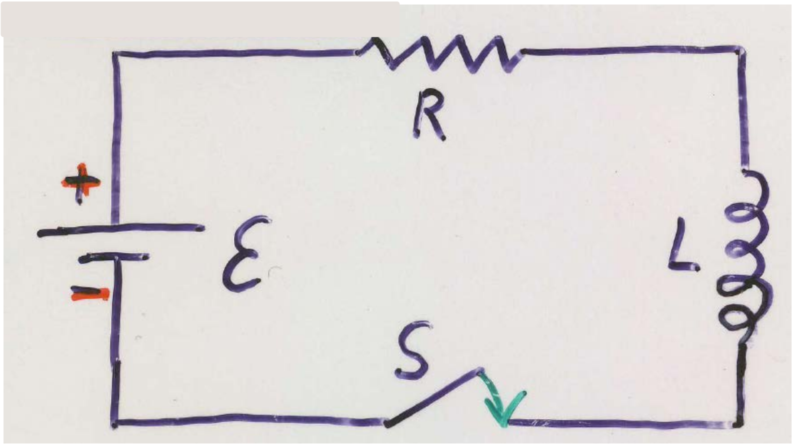
- from Faraday's Law $\mathcal{E} = -\dfrac{d(N\phi_B)}{dt}$

i.e., $\mathcal{E} = -\dfrac{d(Li)}{dt}$

$$\mathcal{E}_L = -L\frac{di}{dt} \qquad (self-induced\ emf)$$

- another defining eqn. for L: $L = -\dfrac{\mathcal{E}_L}{di/dt}$

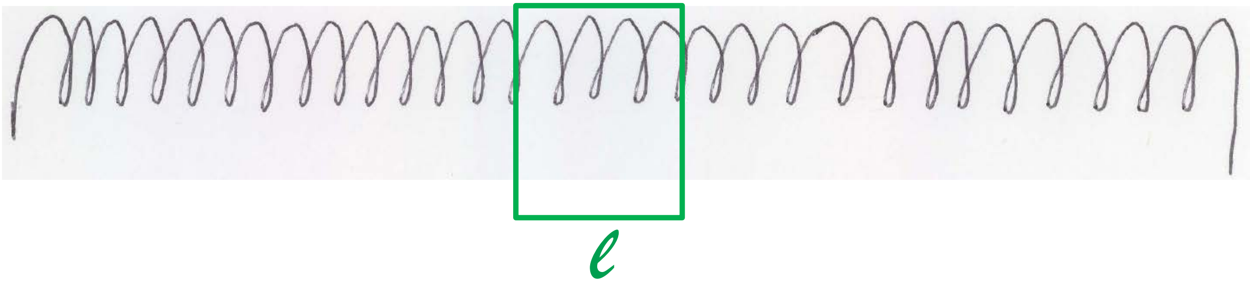
- circuit symbol :



$L \equiv inductor$

Example   Inductance of a solenoid

Consider a long solenoid of cross – sectional area A and n turns / unit length. Find L per unit length near its center.



Get $$\frac{L}{\ell} = \mu_0 n^2 A$$

Notes

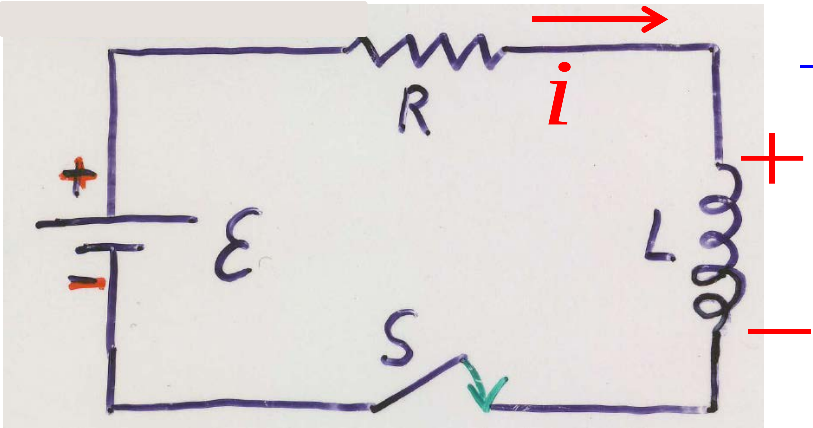
1) L only depends on geometric factors (compare with C).

2) $L \propto n^2$ (since $L = \dfrac{N\phi_B}{i}$ and $N = n \cdot \text{length}$ and $\phi_B \propto n$ (through B))

LR Circuits

Charging:

After S is closed, apply Kirchfoff's loop rule:



– *at some time t let current* $= i$ – *and* $\uparrow$

$$\mathcal{E} - iR - L\frac{di}{dt} = 0$$

i.e., $\frac{di}{dt} = \frac{R}{L}\left(\frac{\mathcal{E}}{R} - i\right)$ *or* $\frac{di}{\left(\frac{\mathcal{E}}{R} - i\right)} = \frac{R}{L}dt$

– *integrating both sides:* [*note* $\int \frac{dx}{a+bx} = \frac{1}{b}\ln(a+bx)$]

$$-\ln\left(\frac{\mathcal{E}}{R} - i\right) = \frac{R}{L}t + constant$$

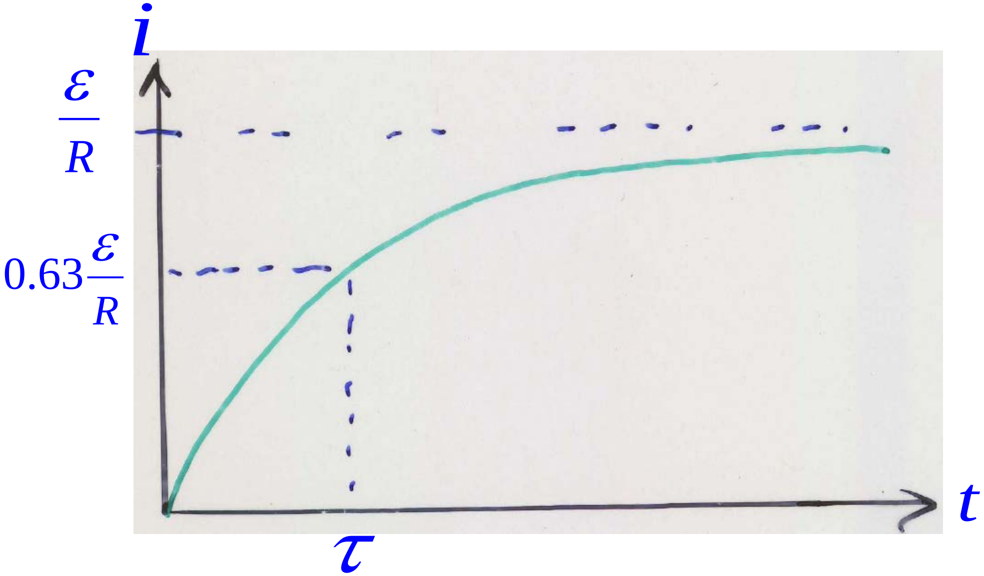
At $t = 0,\ i = 0 \ \Rightarrow constant = -\ln\frac{\mathcal{E}}{R}$

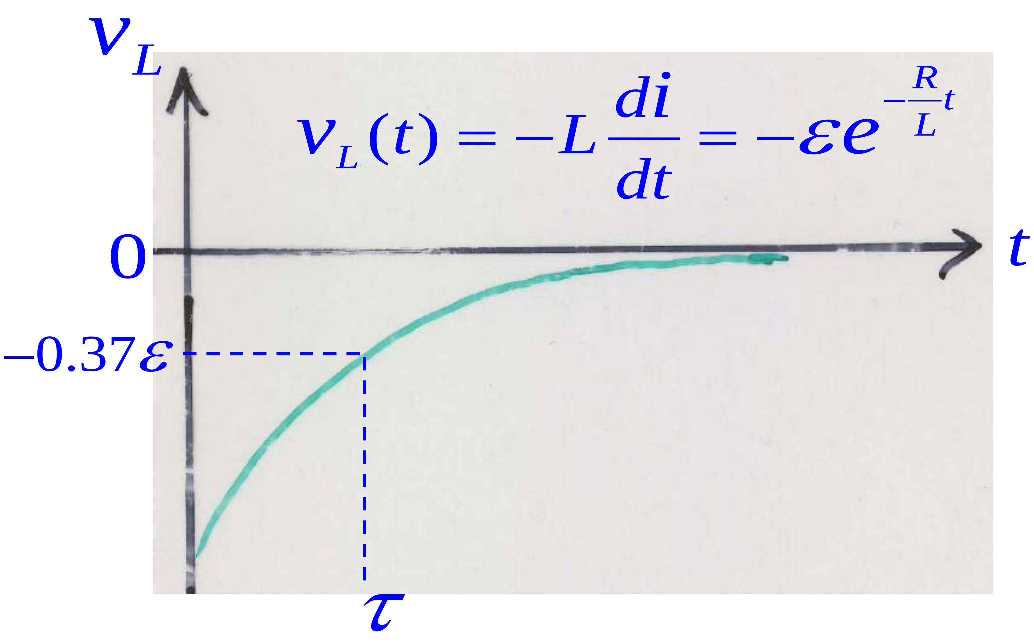
Then $\ln\left(\frac{\mathcal{E}}{R} - i\right) - \ln\frac{\mathcal{E}}{R} = -\frac{R}{L}t = \ln\left(1 - \frac{Ri}{\mathcal{E}}\right)$

or $i(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{R}{L}t}\right)$

$$i(t) = \frac{\varepsilon}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

define $\frac{L}{R} = \tau \equiv$ *time constant*





Example

13. Consider the circuit in Figure P32.12, taking $\mathcal{E} = 6.00$ V, $L = 8.00$ mH, and $R = 4.00\ \Omega$. (a) What is the inductive time constant of the circuit? (b) Calculate the current in the circuit 250 μs after the switch is closed. (c) What is the value of the final steady-state current? (d) After what time interval does the current reach 80.0% of its maximum value?

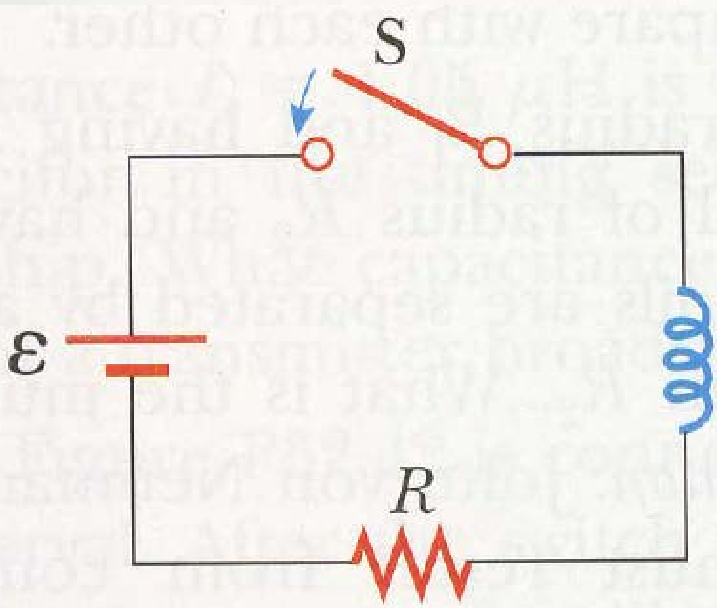


Figure P32.12 Problems 12, 13, 14, and 15.

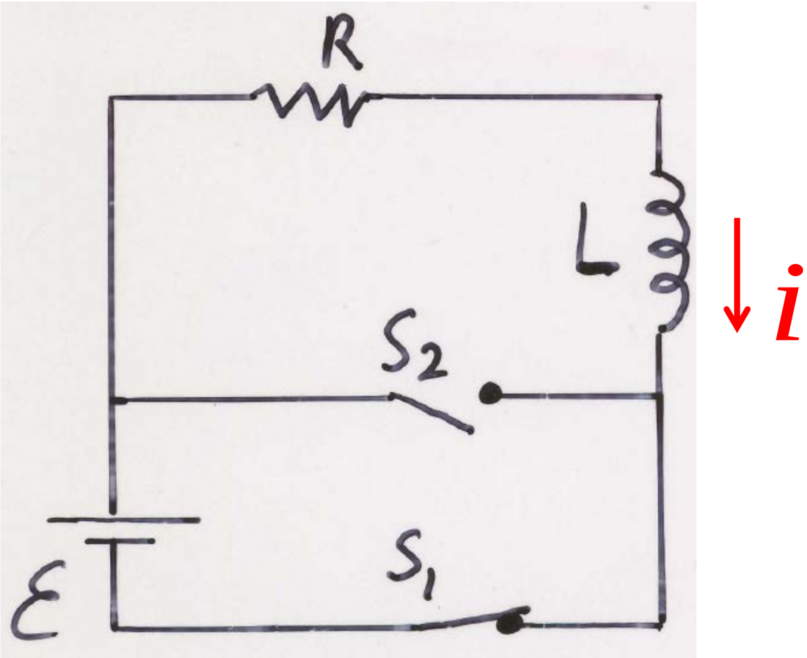
$a)\ \tau = 2.00\ ms$

$b)\ I = 0.176\ A$

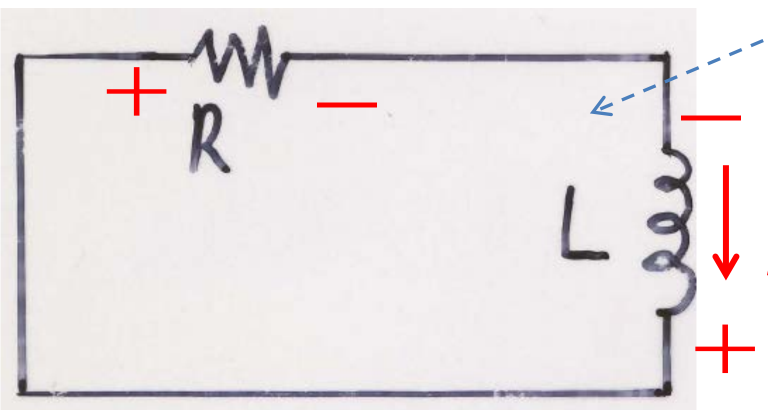
$c)\ \mathrm{I}_{max} = 1.50\ A$

$d)\ t = 3.22\ ms$

LR circuit *Discharging :*



Let S_1 *be closed for a long time* (S_2 *open*)

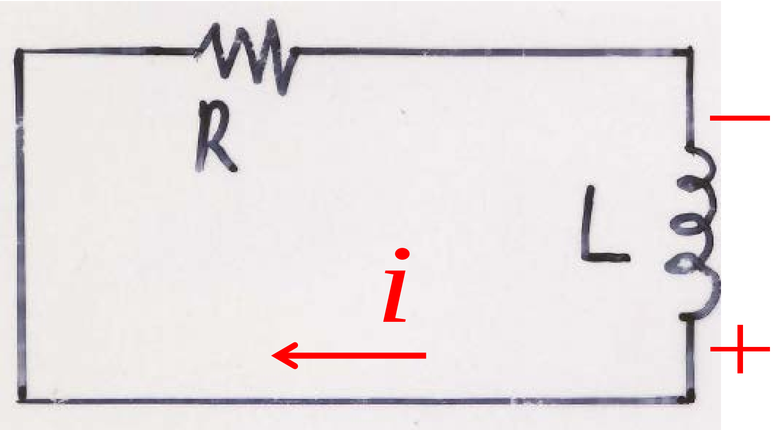


At $t = 0$ *open* S_1 *and close* S_2

$$iR + L\frac{di}{dt} = 0$$

$$\frac{di}{i} = -\frac{R}{L}dt$$

$$\ln i = -\frac{R}{L}t + constant$$



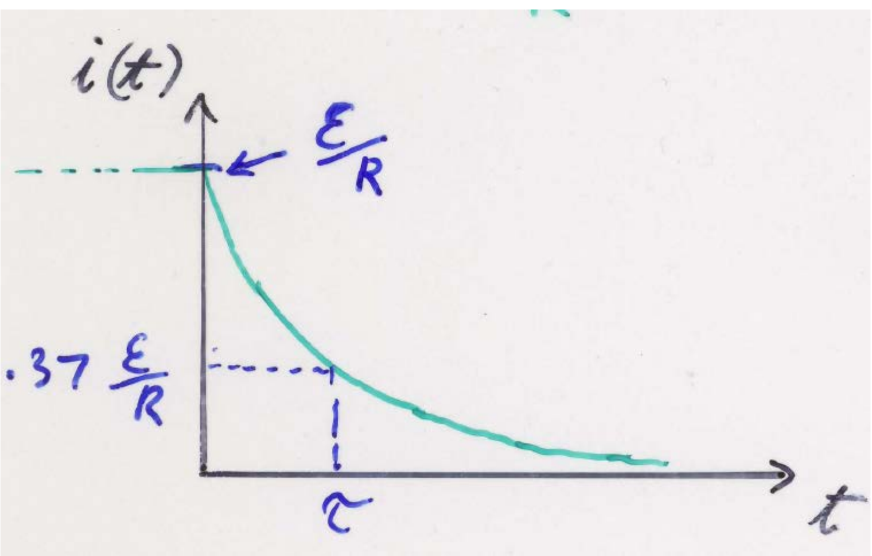
$$\ln i = -\frac{R}{L}t + constant$$

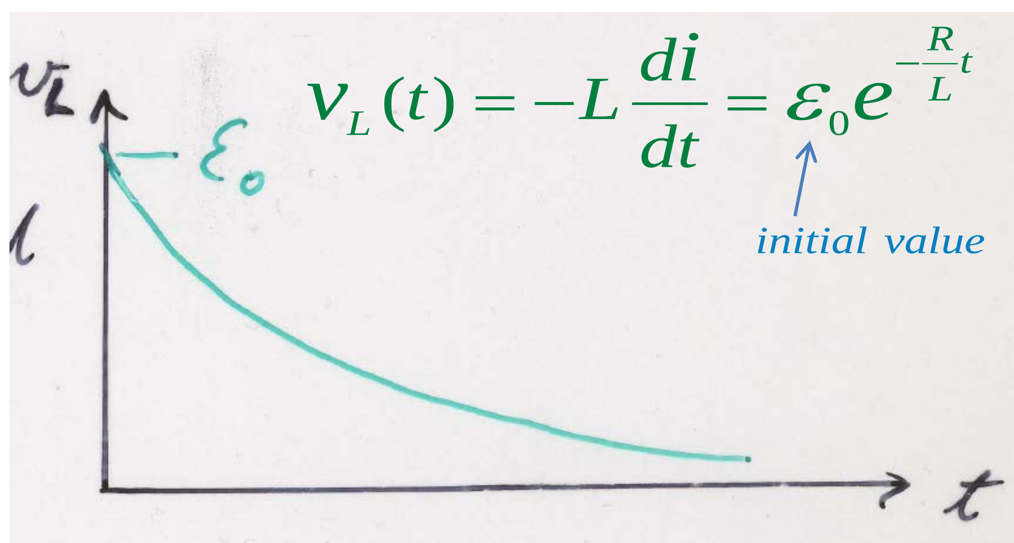
$$At\ t = 0,\ i = \frac{\mathcal{E}}{R} \quad \Rightarrow constant = \ln\frac{\mathcal{E}}{R}$$

$$Then\ \ \ln i - \ln\frac{\mathcal{E}}{R} = -\frac{R}{L}t$$

$$or\quad i(t) = \frac{\mathcal{E}}{R}e^{-\frac{R}{L}t} = i_0 e^{-\frac{R}{L}t} \qquad or\quad i(t) = i_0 e^{-\frac{t}{\tau}}$$

$$\tau \equiv time\ constant = \frac{L}{R}$$



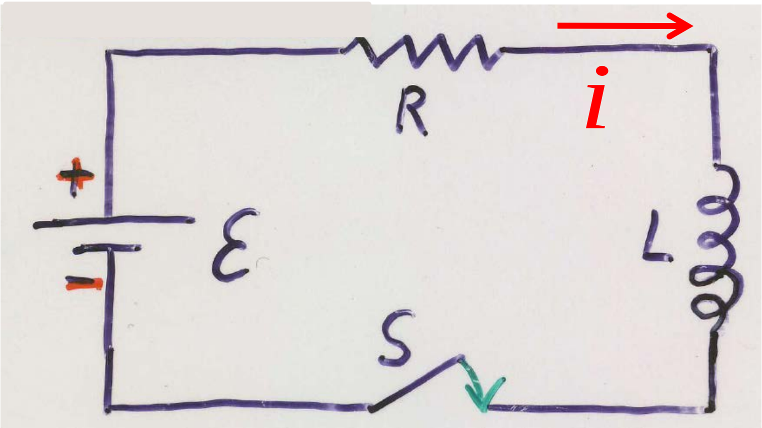


$$v_L(t) = -L\frac{di}{dt} = \mathcal{E}_0 e^{-\frac{R}{L}t}$$

initial value

Energy in magnetic field – in inductor

Consider LR circuit: –charging



$$\varepsilon - iR - L\frac{di}{dt} = 0$$

multiply by i : $\varepsilon i - i^2 R - Li\frac{di}{dt} = 0$

power delivered by source

rate at which energy dissipated in R

rate at which energy is stored in L (note $di/dt > 0$)

i.e., rate at which inductor stores energy is $P = Li\frac{di}{dt}$

Since $dW = dU = Pdt$, $dU = Lidi$

and $U = \int dU = L\int_0^I idi = \frac{1}{2}LI^2$

Magnetic Energy Density u_B

For the ideal (long) solenoid we had $L = \mu_0 n^2 A\ell$.

$$\therefore \quad U = \frac{1}{2}LI^2 = \frac{1}{2}\mu_0 n^2 A\ell I^2 = \frac{1}{2\mu_0}(\mu_0 nI)^2 A\ell = \frac{B^2}{2\mu_0}\underbrace{A\ell}_{V}$$

and the magnetic energy density (U/V)

$$u_B = \frac{B^2}{2\mu_0}$$

(This is a general results - not just for a solenoid)

(compare with $u_E = \frac{1}{2}\varepsilon_0 E^2$ *)*

Example *Magnetic energy and MRI scanners*

– *MRI scanners use large, liquid helium* $(4K)$ *cooled superconducting* $(R = 0)$ *solenoids to generate the B field.*

– *if the magnet quenches, the coil becomes normal* $(R \neq 0)$ *and a large amount of magnetic energy is explosively released.*

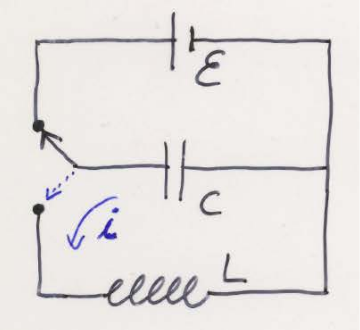
$I = 2.4\ kA,\ L = 0.53\ H,\ R_{normal} = 31\ m\Omega$

a) Find the stored energy.

b) Find the rate of energy release immediately after a quench.

c) Find the time constant. *d) Find* $\mathcal{E}_L$ (max).

LC circuit :



$-$1st set switch in its "up" position - let C fully charge :

$$Q_0 = C\mathcal{E}, \ V_C = \mathcal{E}, \ U_C = \frac{1}{2}\frac{Q_0^2}{C}$$

At $t = 0$ move switch to "down" position. Inductor reacts to sudden increase in i :

$-$at $t = 0^+$: $i = 0, \ q = Q_0$

$-$at time t : $i(t) = \dfrac{dq}{dt}, \ q = q(t)$

From K's rules, at any time, $v_L + v_C = 0$

$$\text{or} \quad v_L(t) + v_C(t) = -L\frac{di}{dt} + \frac{q}{C} = 0$$

At instant depicted, i is $+$ve. But q is decreasing $\Rightarrow \dfrac{dq}{dt}$ is $-$ve

$$\therefore \ \text{set } i = -\frac{dq}{dt}$$

$$\text{and} \quad \frac{d^2q}{dt^2} + \left(\frac{1}{LC}\right)q = 0$$

$$\frac{d^2q}{dt^2}+\left(\frac{1}{LC}\right)q=0$$

$$Let\ q(t)=Q_0\cos(\omega t+\phi)\ and\ sub\ into\ differential\ equation \rightarrow get\ \ \omega^2=\frac{1}{LC}$$

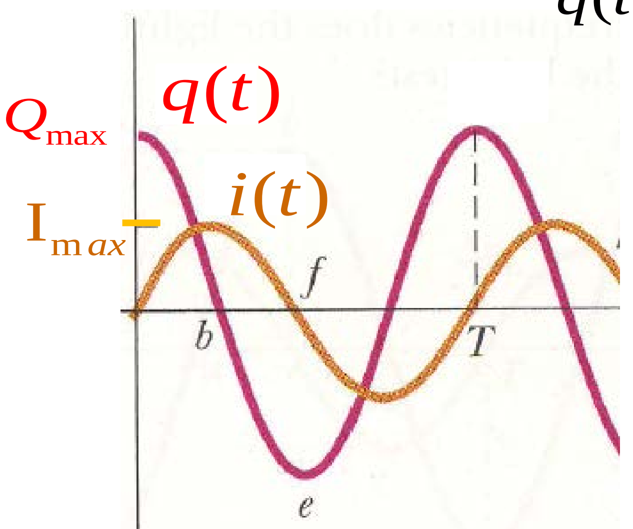
$$or\ \ \omega=\sqrt{\frac{1}{LC}}\ \Rightarrow harmonic\ oscillator\ motion\ \left(\frac{d^2x}{dt^2}+\frac{k}{m}x=0\right)$$

- $q(t)=Q_{\max}\cos(\omega t+\phi)$
- $i(t)=-\frac{dq}{dt}=-Q_{\max}(-\omega)\sin(\omega t+\phi)$

$$-get\ \phi\ from\ initial\ conditions: at\ t=0,\ i=0\ and\ q=Q_{\max}(=C\mathcal{E}):$$

$$0=-\omega Q_{\max}\sin\phi \quad \Rightarrow \ \phi=0$$

$$q(t)=Q_{\max}\cos\omega t\ ,\ \ i(t)=\omega Q_{\max}\sin\omega t=\mathrm{I}_{\max}\sin\omega t$$



Also note that

$$\sin\alpha=\cos(90^o-\alpha)=\cos(\alpha-90^o)$$

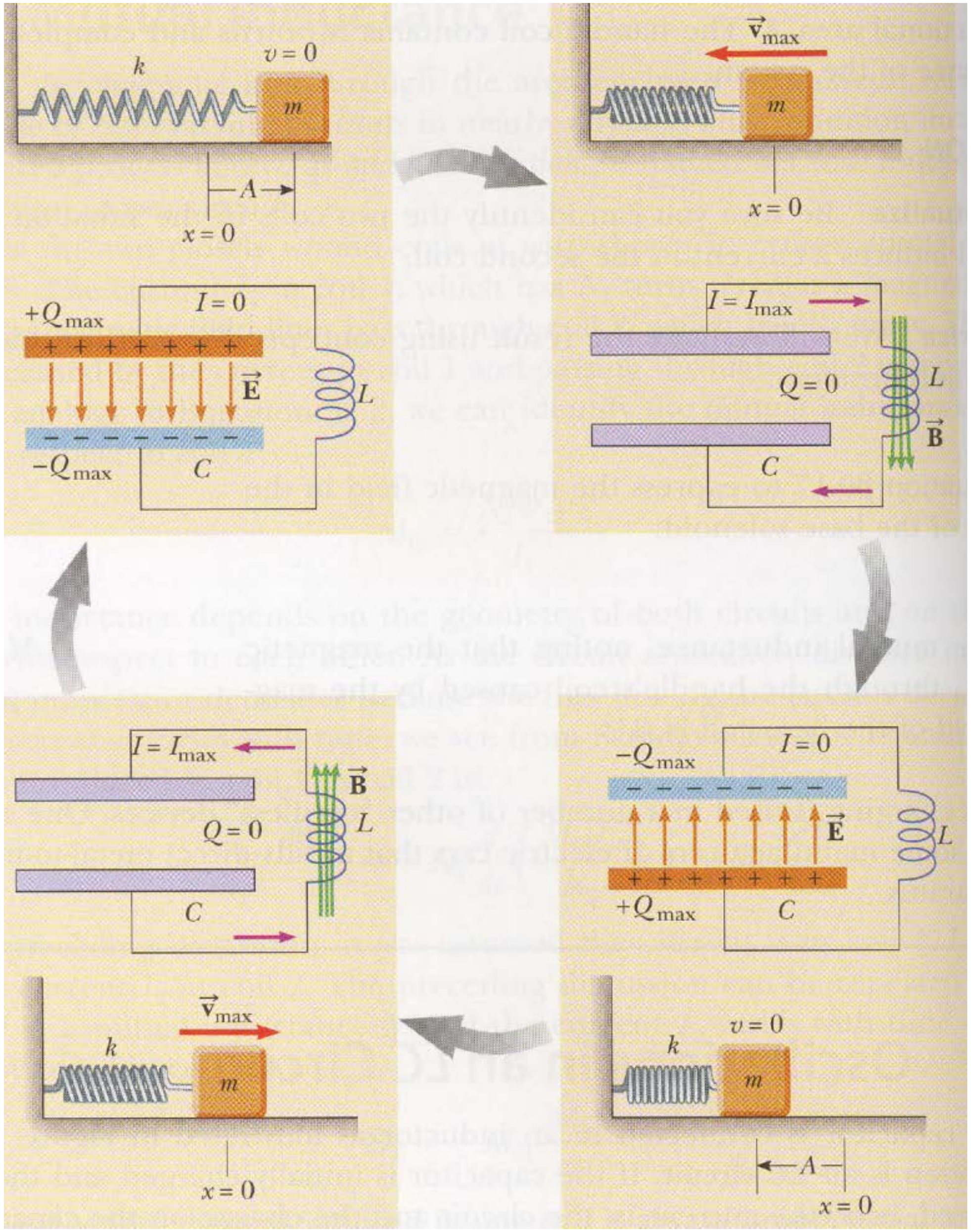
$$\therefore i(t)=\mathrm{I}_{\max}\cos(\omega t-90^o)$$

$$\Rightarrow i\ lags\ q$$

$$U = U_C + U_L = \frac{Q_{max}^2}{2C}\cos^2 \omega t + \frac{1}{2} L I_{max}^2 \sin^2 \omega t$$

a) $t = 0$

b) $t = \frac{T}{4}$



d) $t = \frac{3T}{4}$

c) $t = \frac{T}{2}$

TABLE 9.1 Generalized Forms of Maxwell's Equations

Differential Form	Integral Form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v \, dv$	Gauss's law *–how charges produce electric fields* *– field lines begin / end on q*
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	Nonexistence of isolated magnetic charge* *–magnetic field lines have neither end nor beginning*
$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$\oint_L \mathbf{E} \cdot d\mathbf{l} = -\dfrac{\partial}{\partial t}\int_S \mathbf{B} \cdot d\mathbf{S}$	Faraday's law *–changing ϕ_B produces $\vec{E}$*
$\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$	$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{S}$	Ampère's circuit law *–electric current and changing ϕ_E produces $\vec{B}$*

*This is also referred to as Gauss's law for magnetic fields.