L12 Heisenberg Uncertainty, Projection Operators

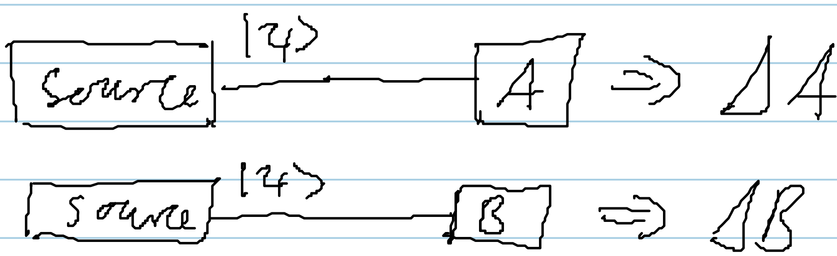
(4.4.3 continued)

$$\boxed{\Delta A \, \Delta B \geq \frac{1}{2} \left| \langle \psi | [A, B] | \psi \rangle \right|}$$

Heisenberg Uncertainty Relation

NOTE: Heisenberg's Uncertainty Relations is a statement about quantum mechanical states and observables, not about any back-reaction of measurements onto the state itself!

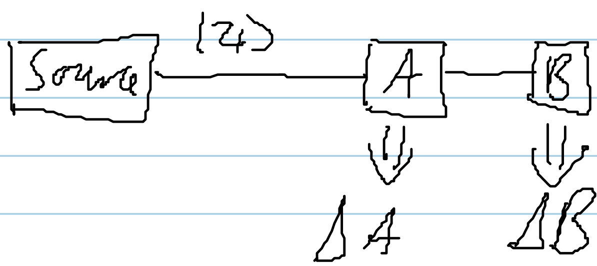
Uncertainty Relation Scenario:



Heisenberg

$$\Delta A \, \Delta B \geq \frac{1}{2} \left| \langle \psi | [A, B] | \psi \rangle \right|$$

Back-Reaction Scenario:



In the back-reaction Scenario one can formulate some uncertainty principles as well, but they quantitatively differ form Heisenbergs uncertainty relation.

4.4.4 Spin Example

$$A = S_x \qquad B = S_y$$

$$[A, B] \leftrightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$- \frac{\hbar^2}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \frac{\hbar^2}{4}\left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right]$$

$$= \frac{\hbar^2}{4} 2i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \hbar i S_z$$

$$\Delta S_x \quad \Delta S_y \geq \frac{\hbar}{2} |\langle \psi | S_z | \psi \rangle|$$

So unless $\langle \psi | S_z | \psi \rangle = 0$

neither an S_x measurement nor an S_y measurement can possibly give a sharp value!
If one of the two measurement would give sharp values,
then the left hand side would vanish. Then the right hand side has to vanish too!

For $\Delta S_x \neq 0$

$$\Rightarrow \quad \Delta S_y \geq \frac{\frac{\hbar}{2} |\langle \psi | S_z | \psi \rangle|}{\Delta S_x}$$

The uncertainty relation is again a warning that we are not allowed to think of the outcomes of x, y, z measurement as something that is predetermined and only needs to be uncovered.

4.4.5 Position/Momentum Example:
(uses things that we will learn in later part of the course)

$$\Delta x \, \Delta p \geq \frac{1}{2}\hbar$$

position operator X satisfies (see second part of course)

momentum operator P

$$[X, P] = i\hbar \mathbb{1}$$

Since the right hand side of the identity does not depend on the state $|\psi\rangle$ anymore, we get a somehow stronger statement:
one cannot prepare any quantum state so that the particle will be always found in one particular location (we cannot reach

$\Delta x = 0$

Same holds for the momentum of a particle. Asymptotically, we can narrow the position of a particle down (decrease Δx), but that means that the uncertainty about its momentum must increase!

4.5 Projection Operators and Selective Operations

4.5.1 Motivation and definition

$$|\psi\rangle = a|+\rangle + b|-\rangle$$
$$= |+\rangle a + |-\rangle b$$
$$a = \langle +|\psi\rangle$$
$$b = \langle -|\psi\rangle$$

$$|\psi\rangle = |+\rangle\langle +|\psi\rangle + |-\rangle\langle -|\psi\rangle$$
$$= \underbrace{\left(|+\rangle\langle +|\right)}_{=: P_+}|\psi\rangle + \underbrace{\left(|-\rangle\langle -|\right)}_{=: P_-}|\psi\rangle$$

P_+ projection operator, projection onto state $|+\rangle$
P_- projection operator, projection onto state $|-\rangle$

$$= (P_+ + P_-) |\psi\rangle$$

$$\Rightarrow \quad P_+ + P_- = \mathbb{1}$$

$$|+\rangle\langle +| + |-\rangle\langle -| = \mathbb{1}$$

Definition:

We call an operator of the form

$$P_{|\varphi\rangle} = |\varphi\rangle\langle\varphi|$$

a projection operator. Our notation indicates onto what state we project

4.5.2 Properties and Completeness Relations

More generally:
given orthonormal basis $|a_i\rangle$

we can defined projectors $P_{|a_i\rangle} = |a_i\rangle\langle a_i|$

and we have the

resolution of identity
(closure, completeness relation)

$$\mathbb{1} = \sum_i |a_i\rangle\langle a_i| = \sum_i P_i$$

Projection operators are hermitian operators

$$P_{a_i} = P_{a_i}^\dagger$$

Projection operators satisfy

$$P_{|\psi\rangle}^2 = P_{|\psi\rangle}$$

$$\left(|\psi\rangle\langle\psi|\right)\left(|\psi\rangle\langle\psi|\right) = |\psi\rangle\left(\langle\psi|\psi\rangle\right)\langle\psi| = |\psi\rangle\langle\psi|$$

We say that two projection operators are orthogonal if

$$P_{|\varphi\rangle} \, P_{|\psi\rangle} = 0$$

This happens whenever the states they project onto are orthogonal:

$$P_{|\varphi\rangle} P_{|\psi\rangle} = \left(|\varphi\rangle\langle\varphi|\right)\left(|\psi\rangle\langle\psi|\right)$$

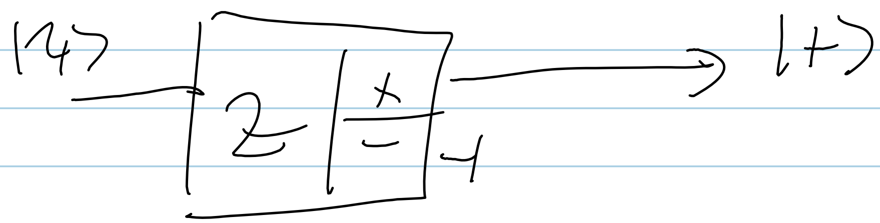
$$= |\varphi\rangle\big(\underbrace{\langle\varphi|\psi\rangle}_{\in\,\mathbb{C}}\big)\langle\psi|$$

$$= \langle\varphi|\psi\rangle \; |\varphi\rangle\langle\psi|$$

This operator vanishes if and only if $\langle\varphi|\psi\rangle = 0$

4.5.3 Application of Projection operators to selective operation

Selector Operation:



$$|+\rangle = \frac{P_+|\psi\rangle}{\sqrt{\langle\psi|P_+|\psi\rangle}}$$ normalization!

Note about normalization:

$$\langle\psi|P_+^\dagger \, P_+|\psi\rangle = \langle\psi|P_+^2|\psi\rangle = \langle\psi|P_+|\psi\rangle$$

This is actually the probability to have the atom pass!

$$|+\rangle = \frac{P_+|\psi\rangle}{}$$

$$|+\rangle = \frac{P_+|\psi\rangle}{\sqrt{Pr(+)}}$$

What happens if we do not normalize the output?

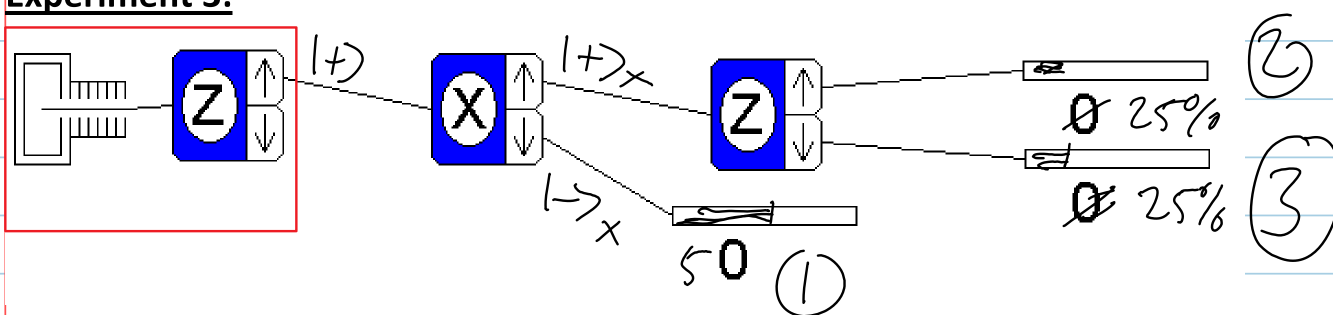
$$|\psi\rangle \longrightarrow \boxed{P_+} \longrightarrow P_+|\psi\rangle$$

the output vector is not normalized, but it keeps internally track of the probability that the system passes the selector via its norm!

$$Pr(+) = \left(\langle\psi|P_+\right)\left(P_+|\psi\rangle\right)$$

Example:

<u>Experiment 3:</u>



starting point: effective source with normalized output state

$|+\rangle$

measurement selecting either $|+\rangle_x$ or $|-\rangle_x$

Upper arm corresponds to selector $P_{|+\rangle_x}$

lower arm to $P_{|-\rangle_x}$

second measurement applied only to upper arm of x measurement, z- measurement applies $P_{|+\rangle_z}$ or $P_{|-\rangle_z}$

Detector 1:

At detector 1 the following unnormalized state arrives:

$$P_{|-\rangle_x}|+\rangle = |-\rangle_x \underbrace{{}_x\langle -|+\rangle}_{\in \mathbb{C}} \sim |-\rangle_x$$

The probability that an atom arrives is

$$\langle +|P^+_{|-\rangle_x} P_{|-\rangle_x}|+\rangle = \langle +|P_{|-\rangle_x}|+\rangle$$
$$= |\langle +|-\rangle_x|^2 = \frac{1}{2}$$

Detector 2:

Here the following unnormalized state arrives:

$$P_{|+\rangle_z} P_{|+\rangle_x}|+\rangle = |+\rangle_z \underbrace{{}_z\langle +|+\rangle_x}_{\in \mathbb{C}} \underbrace{{}_x\langle +|+\rangle_z}_{\in \mathbb{C}}$$

$$\sim |+\rangle_z$$

probability:

$$|{}_z\langle +|+\rangle_x \, {}_x\langle +|+\rangle_z|^2$$
$$= |{}_z\langle +|+\rangle_x|^4 = \frac{1}{4}$$

Detector 3:

unnormalized outcome

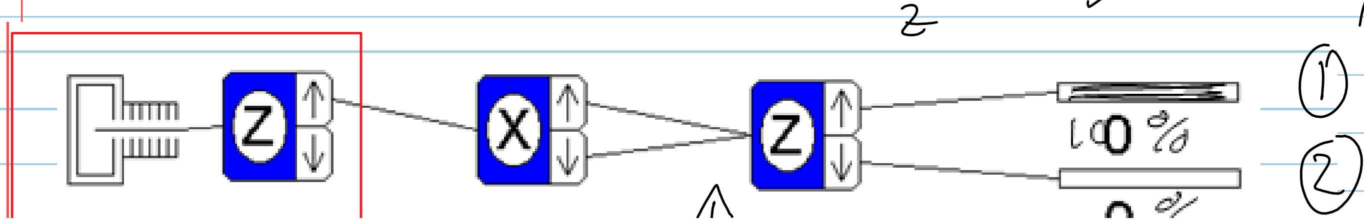
$$P_{|-\rangle_z} P_{|+\rangle_x}|+\rangle = |-\rangle_z \, {}_z\langle -|+\rangle_x \, {}_x\langle +|+\rangle_z$$

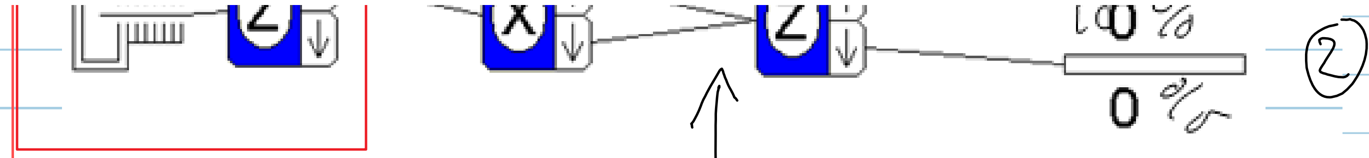
$$\sim |-\rangle_z$$

probability

$$|{}_z\langle -|+\rangle_x|^4 = \frac{1}{4}$$

<u>Experiment 4</u>





Source $|+\rangle$

$$\left(P_{|+\rangle_x}|+\rangle + P_{|-\rangle_x}|+\rangle\right)$$

Detector 1: upper path lower path
output state

$$\overbrace{P_{|+\rangle} P_{|+\rangle_x}|+\rangle}^{\text{upper path}} + \overbrace{P_{|+\rangle} P_{|-\rangle_x}|+\rangle}^{\text{lower path}}$$

$$= P_{|+\rangle}\underbrace{\left(P_{|+\rangle_x} + P_{|-\rangle_x}\right)}_{=\mathbb{1}}|+\rangle$$

$$= P_{|+\rangle}|+\rangle = |+\rangle$$

outcome state: $|+\rangle$
probability: 1

Detector 2:

$$P_{|-\rangle}|+\rangle = 0$$

outcome state undefined, because
probability: 0

For interference paths, the state vectors are added, not the probabilities!

When do paths interfere, and when do they behave like classical mixtures?
Pathes interfere if for an outsider observer it is in principle impossible to tell afterwards which path has been taken.