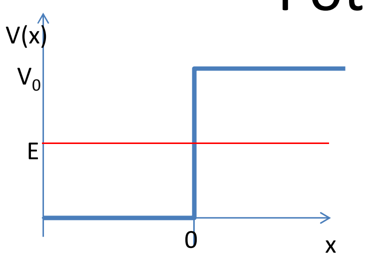
7.4.3 Overview of solutions for potential step:

Potential Step



Eigenstates of Hamiltonian

$$\hat{H} = \frac{\hat{P}_x^2}{2m} + V(\hat{X})$$

Eigenstates for energy eigenvalue range $0<E<V_0$
non-generate continuous eigenvalues:

$$\Psi_E(x) = A_+ \begin{cases} e^{ikx} + \frac{k-i\kappa}{k+i\kappa} e^{-ikx} & x \leq 0 \\ \frac{2k}{k+i\kappa} e^{-\kappa x} & x > 0 \end{cases} \qquad \kappa = \frac{1}{\hbar}\sqrt{2m(V_0-E)} \qquad k = \frac{1}{\hbar}\sqrt{2mE}$$

Eigenstates for energy eigenvalue range $E>V_0$
Degenerate continuous eigenvalues:

Flux ONLY from $-\infty$:

$$\Psi_E(x) = A_+ \begin{cases} e^{ik_1x} + \frac{k_1-k_2}{k_1+k_2} e^{-ik_1x} & x \leq 0 \\ \frac{2k_1}{k_1+k_2} e^{-ik_2x} & x > 0 \end{cases}$$

Flux ONLY from $+\infty$:

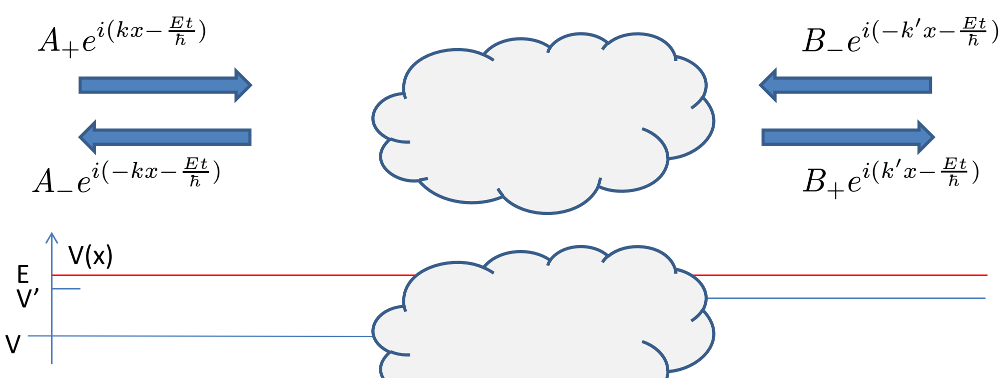
$$\Psi_E(x) = \begin{cases} \ldots & x \leq 0 \\ \ldots & x > 0 \end{cases} \qquad k_1 = \frac{1}{\hbar}\sqrt{2mE} \qquad k_2 = \frac{1}{\hbar}\sqrt{2m(E-V_0)}$$

Note that the second set of solutions (Flux only from the right) will be part of the assignment. As demonstrated in the lecture, also wave packets coming from the right will be partly reflected back, and partly transmitted.

7.5 Probability Flux and Scattering Experiments

7.5.1 Probability Flux

Probability Flux



$$k = \frac{1}{\hbar}\sqrt{2m(E-V)} \qquad k' = \frac{1}{\hbar}\sqrt{2m(E-V')}$$

Probability flux for each component:
(applies only for E > V(x)!)

$$S_{comp} = \frac{\hbar}{m} k |A|^2$$

Think about probability flux the same way you would think about a stream of particles (flux of particles).

For wave packets, this then turns over into probabilities of transmittance and reflectance of particles.

Flux and the Input to the system:
no physical constraints as such, most
But as seen above:
probability flux from left $(-\infty)$: $\frac{\hbar k}{m} |A_+|^2$

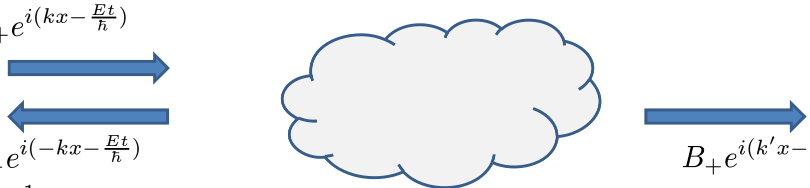
probability flux from right $(+\infty)$: $\frac{\hbar k}{m} |B_-|^2$

We already identified two orthogonal sets of solutions: 1) solution with inflow of particles only from the left $(B_- = 0)$ 2) solutions with inflow of particles only from the right $(A_+ = 0)$ The most general solution will be a superposition of both In experiments, typically one has inflow of particles only from one side! therefore the most general solution will have two open parameters

7.5.2 Transmittance and Reflectance

Probability Flux

$A_+ e^{i(kx - \frac{Et}{\hbar})}$



$A_- e^{i(-kx - \frac{Et}{\hbar})}$

$k = \frac{1}{\hbar}\sqrt{2m(E-V)}$

$B_+ e^{i(k'x - \frac{Et}{\hbar})}$

$k' = \frac{1}{\hbar}\sqrt{2m(E-V')}$

Probability flux for each component: (applies only for E > V(x)!) $S_{comp} = \frac{\hbar}{m} k |A|^2$

Reflection probability:

$$R = \frac{S_{reflect}}{S_{in}} = \frac{|A_-|^2}{|A_+|^2}$$

Transmission probability:

$$T = \frac{S_{transmit}}{S_{in}} = \frac{k'}{k}\frac{|B_+|^2}{|A_+|^2}$$

SCATTERING EXPERIMENTS:
These definitions hold for any potential configuration hidden in the 'cloud'!
==> in high energy physics, we learn something about particles and their interaction by throwing them at each other, and by learning from their reflection and transmittance behaviour, which then takes place in three dimensions, not just in one as in our example ...

7.5.3 Application to potential step:

Reflection coefficient:

$$R = \frac{S_R}{S_{in}} = \frac{k_1 |A_-|^2}{k_1 |A_+|^2} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2$$

Transmission coefficient:

$$T = \frac{S_T}{S_{in}} = \frac{k_2 |B_+|^2}{k_1 |A_+|^2} = \frac{4 k_1 k_2}{(k_1 + k_2)^2}$$

We always find

$$R + T = 1 \qquad k_1 = \frac{1}{\hbar}\sqrt{2mE} \qquad k_2 = \frac{1}{\hbar}\sqrt{2m(E - V_0)}$$

Note: for broad wave packets, all contributing energy eigenstates are close to one central value, and these wave packets behave close to states with definite energy corresponding to that central value of the energy distribution.

7.5.4 Additional Material: Probability Flux

Definition:

probability density $\omega(x,t)\,dx$

$$\omega(x,t) = |\psi(x,t)|^2$$

probability flux:

$$S(x,t)\,dx$$

$$S(x,t) = \frac{\hbar}{2mi}\left(\psi^* \frac{\partial}{\partial x}\psi - \left(\frac{\partial}{\partial x}\psi^*\right)\psi\right)$$

(arguments (x,t) omitted for clarity...)

continuity equation for probability:

$$\frac{\partial}{\partial t}\omega(x,t) + \frac{\partial}{\partial x}S(x,t) = 0$$

Proof (omitted in lecture)

$$\frac{\partial}{\partial t}\rho(x,t) = \left(\frac{\partial}{\partial t}\psi^*(x,t)\right)\psi(x,t) + \psi^*(x,t)\,\frac{\partial}{\partial t}\psi(x,t)$$

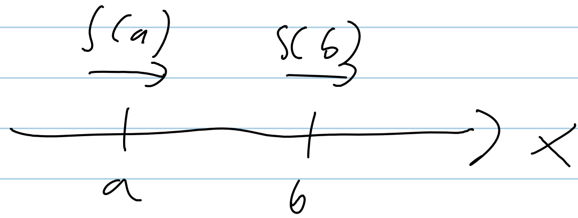
$$= \left[-\frac{1}{i\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi^*(x,t)\right]\psi(x,t) + \psi^*(x,t)\left[\frac{1}{i\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi(x,t)\right]$$

$$= -\frac{\hbar}{2mi}\left[\psi^*\frac{\partial^2}{\partial x^2}\psi - \left(\frac{\partial^2}{\partial x^2}\psi^*\right)\psi\right]$$ (omitting the arguments x and t)

$$= -\frac{\partial}{\partial x}\underbrace{\left[\frac{\hbar}{2mi}\left(\psi^*\frac{\partial}{\partial x}\psi - \left(\frac{\partial}{\partial x}\psi^*\right)\psi\right)\right]}_{=:\, S(x,t)}$$

interpretation:

$$\frac{d}{dt}\int_a^b dx\, \rho(x,t) = -\int_a^b dx\left(\frac{\partial}{\partial x}S(x,t)\right) = S(a) - S(b)$$



The probability flux describes the inflow of probability into some region ...

Special cases:

1) For energy eigenstates, time evolution can be ignored as it cancels out:

$$S_{\psi(x,t)} = S_{\psi_E(x)}$$

2) $\psi_E(x) = A_+ e^{ikx} + A_- e^{-ikx}$

$$\Rightarrow S(x) = \frac{\hbar k}{m}\left(|A_+|^2 - |A_-|^2\right)$$

incoming flux reflected flux

3) $S_\psi = 0 \quad \forall\ \psi_E(x) = e^{i\varphi}\,|\psi_E(x)|$

Then $\psi_E^*(x) = e^{-i\varphi}\,\psi_E(x)$

φ does not depend on x!

4) specialization to the case of potential step with E < V

$$|A_-|^2 = \left|\left(\frac{k-i\kappa}{k+i\kappa}\right)A_+\right|^2 = \frac{k^2+\kappa^2}{k^2+\kappa^2}\,|A_+|^2 = |A_+|^2$$

$$\Rightarrow S(x) = 0$$

$\underline{x>0:}\quad \psi(x) = e^{i\varphi}\,|\psi(x)|$

$$\Rightarrow S(x) = 0$$

flux of incoming particles (from left side) $\quad S_{in}(x) = \frac{\hbar k}{m}|A_+|^2$

equals flux of reflected particles (going to left) $\quad S_{out}(x) = -\frac{\hbar k}{m}|A_+|^2$

For x >0 we find a probability density of particles which is proportional to the flux of incoming particles ...

7.6 Potential Barrier

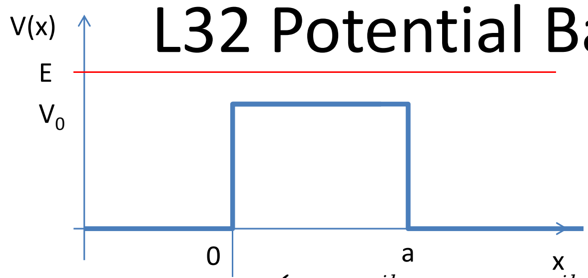
7.6.1 General Properties

Also the next system fits into the picture we set up for scattering experiments:

- partial reflectance/transmittance
- continuous eigenspectrum
- degenerate eigenvalues, which lead to our ability to choose two sets of eigenstates, with flux only from the right or only from the left ...

7.6.2 Solutions for E>V

L32 Potential Barrier E > V_0



$$k_1 = \frac{1}{\hbar}\sqrt{2mE}$$
$$k_2 = \frac{1}{\hbar}\sqrt{2m(E-V_0)}$$

1)Ansatz:
$$\Psi_E(x) = \begin{cases} A_+\, e^{ik_1x} + \; A_-e^{-ik_1x} & x \le 0 \\ B_+\, e^{ik_2x} + \; B_-e^{-ik_2x} & 0 \le x \le a \\ C_+\, e^{ik_1x} + \; C_-e^{-ik_1x} & a < x \end{cases}$$

2) Choice: only incoming waves from -∞ ➔ $C_- = 0$

3)Continuity of Ψ(x) and d/dx Ψ(x) at x=0 and x=a:

$$\begin{aligned} \Psi : (x=0) \quad & A_+ + A_- &=& \; B_+ + B_- \\ \frac{d}{dx}\Psi : (x=0) \quad & ik_1A_+ - ik_1A_- &=& \; ik_2B_+ - ik_2B_- \\ \Psi : (x=a) \quad & B_+\, e^{ik_2a} + B_-\, e^{-ik_2a} &=& \; C_+e^{ik_1a} \\ \frac{d}{dx}\Psi : (x=b) \quad & ik_2\; B_+e^{ik_2a} - ik_2\; B_-e^{-ik_2a} &=& \; ik_1C_+e^{ik_1a} \end{aligned}$$

4) Solve for C_+ as function of incoming amplitude A_+:

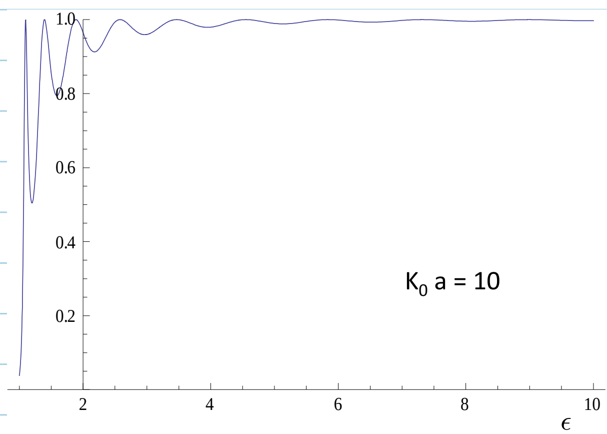
$$C_+ = \frac{4k_1k_2e^{-ik_1a}}{4k_1k_2\cos(k_2a) - 2i(k_1^2+k_2^2)\sin(k_2a)}A_+$$

7.6.3 Calculation of Transmission Probability for E> V

$$T = \frac{S_T}{S_{in}} = \frac{k_1|C_+|^2}{k_1|A_+|^2} = \frac{|C_+|^2}{|A_+|^2} = \frac{16k_1^2k_2^2}{16k_1^2k_2^2\cos^2(k_2a) + 4(k_1^2+k_2^2)^2\sin^2(k_2a)}$$

Introduce:

$$\epsilon = \frac{E}{V_0} \quad K_0 = \frac{1}{\hbar}\sqrt{2mV_0} \qquad\qquad T = \frac{1}{1 + \frac{1}{4\epsilon(\epsilon-1)}\sin^2(K_0a\sqrt{\epsilon-1})}$$



Reflection: R= 1-T

Observations:

1) for some energy values perfect transmittance!

$$K_0\sqrt{\epsilon-1}a = k_2a = n\,\pi$$

(standing wave in the domain of barrier)

2) For high energies: T→ 1

We see that for specific energies, we have perfect transmittance!