Module 05:
Numerical Integration

Starting Friday, February 28, 2014

The Definite Integral

Given a continuous function f(x) and an interval [a,b], determine:

$$I = \int_a^b f(x)dx$$

An exact solution exists:

- Find F such that $\frac{d}{dx}F(x) = f(x)$
- $I = F(b) - F(a)$

Numerical Integration

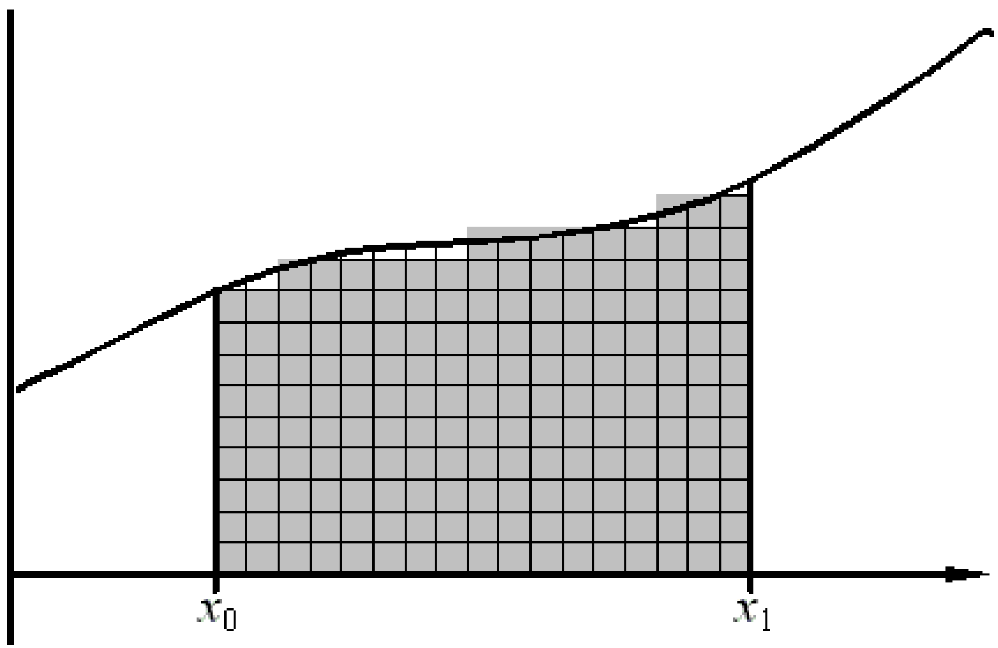
- Find a numerical approximation of

$$I = \int_a^b f(x)dx$$

- Why?
  - No closed form for F(x)
  - f(x) is not known, only $(x_k, f(x_k))$ is known

First Approach: A Visual Solution

- Integral of f(x) between a and b is the area under the graph of y = f(x) for a≤x≤b
- Lower bound: area of completely enclosed squares
- Upper bound: area of all squares



Next approach:

- Interpolate *f(x)* by a polynomial *p(x)*, and integrate *p(x)* over [a,b]

- Involves evaluating *f(x)* at points over [a,b]
  - Use Lagrange polynomials to interpolate f(x)
  - Using different number of points leads to different approximations and errors terms
  - Approximation has the form: $\sum a_k f(x_k)$
  - Called *Numerical Quadrature*

Recall: Error bound on Lagrange Interpolation

Theorem: Assume that

- $x_0, x_1, x_2, \ldots, x_n$ are distinct values of [a,b]
- f is (n+1) times continuously differentiable over [a,b]

Then, for all x Є[a,b], ∃$\xi(x)$ Є[a,b] such that

$$f(x) = \sum_{k=0}^{n} \ell_k(x) f(x_k) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_k)$$

for the Lagrange interpolating polynomial

Let's integrate!!!!

$$\int_a^b f(x)dx = \int_a^b \left(\sum_{k=0}^{n} \ell_k(x) f(x_k) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_k) \right) dx$$

$$\int_a^b f(x)dx = \sum_{k=0}^{n} f(x_k) \int_a^b \ell_k(x)dx + \frac{1}{(n+1)!} \int_a^b \left(f^{(n+1)}(\xi(x)) \prod_{i=0}^{n} (x - x_k) \right) dx$$

Rewriting gives us …

$\int_a^b f(x)dx = \sum_{k=0}^{n} \alpha_k f(x_k) + E(f),$

where

$$\alpha_k = \int_a^b \ell_k(x)dx$$

$$\mathrm{E(f)} = \frac{1}{(n+1)!}\int_a^b \left(\prod_{k=0}^{n}(x - x_k)\ \right) f^{(n+1)}(\xi(x))dx$$

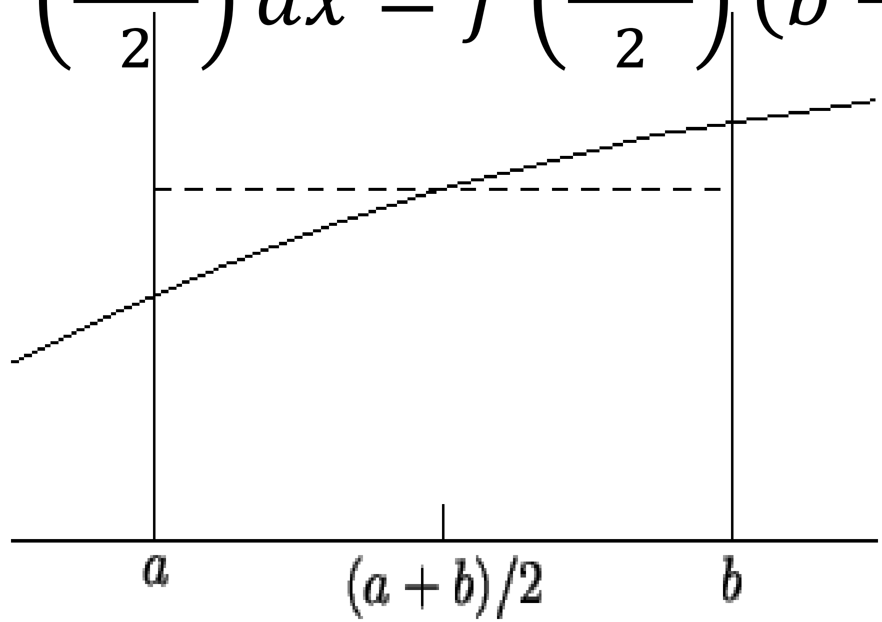
E(f) is the error associated with the given quadrature.

Some specifics …

- Let's consider different numbers of evaluation points …

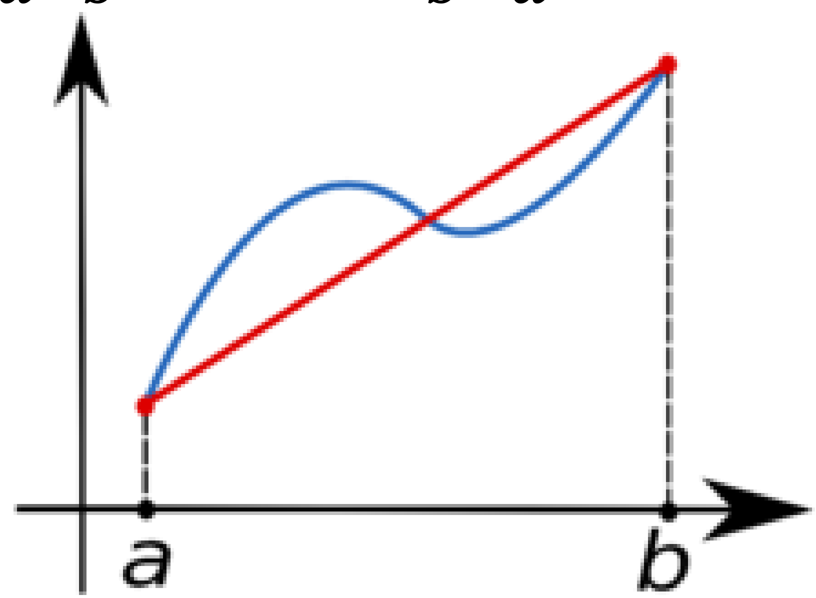
- Newton-Cotes Rules – points of evaluation are equally spaced

- Simplest approach: Midpoint rule
- Approximate f(x) by value at midpoint $p_0(x) = f\left(\frac{a+b}{2}\right)$, so
- $\widehat{I_o} = \int_a^b f\left(\frac{a+b}{2}\right) dx = f\left(\frac{a+b}{2}\right)(b-a)$



Use the endpoints: Trapezoid Rule

- Approximate f(x) by the straight line joining (a,f(a)) to (b,f(b)), i.e. the Lagrange polynomial of degree 1:
- $p_1(x) = \frac{x-b}{a-b}f(a) + \frac{x-a}{b-a}f(b)$

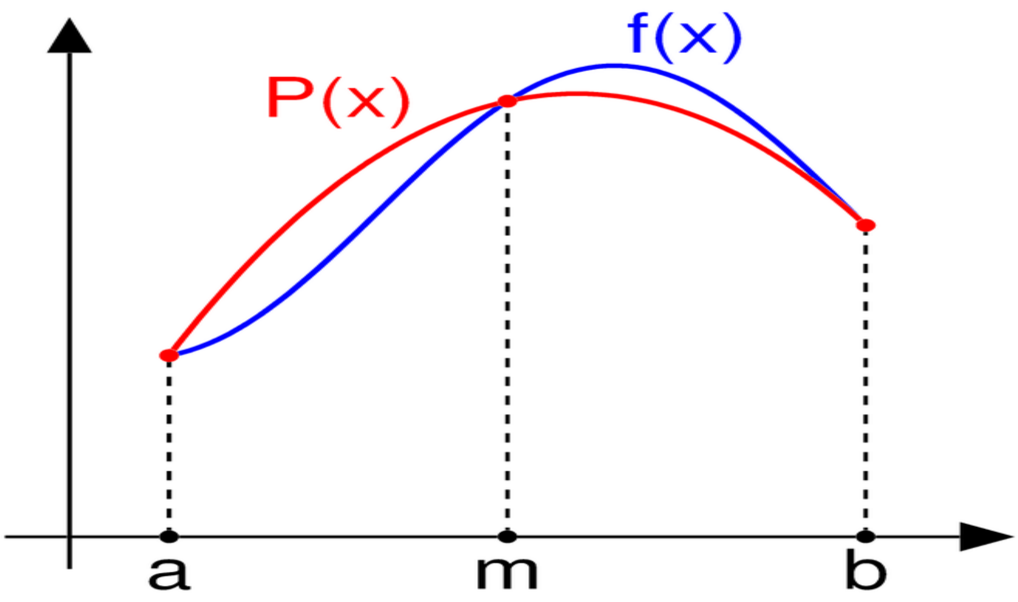


Use the endpoints: Trapezoid Rule

- $\widehat{I_1} = \int_a^b \left(\frac{x-b}{a-b} f(a) + \frac{x-a}{b-a} f(b) \right) dx$
- .
- .
- .
- $\boldsymbol{\widehat{I_1} = \frac{b-a}{2}(f(b) + f(a))}$
- **NOTE: Add function values**

Let's get more accurate ... Simpson's Rule

- Fit a Lagrange polynomial of degree 2 to *(a,f(a)), (m, f(m), (b, f(b))*, where $m = (a+b)/2$.



Let's get more accurate … Simpson's Rule

- $p_2(x) = \frac{(x-m)(x-b)}{(a-m)(a-b)} f(a) + \frac{(x-a)(x-b)}{(m-a)(m-b)} f(m) + \frac{(x-a)(x-m)}{(b-a)(b-m)} f(b)$
- $\widehat{I_2} = \int_a^b \left(\frac{(x-m)(x-b)}{(a-m)(a-b)} f(a) + \frac{(x-a)(x-b)}{(m-a)(m-b)} f(m) + \frac{(x-a)(x-m)}{(b-a)(b-m)} f(b) \right) dx$
- $= w_a f(a) + w_m f(m) + w_b f(b),$
- where
- $w_a = \int_a^b \left(\frac{(x-m)(x-b)}{(a-m)(a-b)} \right) dx = \frac{b-a}{6}$
- $w_m = \int_a^b \left(\frac{(x-a)(x-b)}{(m-a)(m-b)} \right) dx = \frac{4(b-a)}{6}$
- $w_b = \int_a^b \left(\frac{(x-a)(x-m)}{(b-a)(b-m)} \right) dx = \frac{b-a}{6}$

Let's get more accurate … Simpson's Rule

- $\widehat{I_2} = \frac{b-a}{6}(f(a) + 4f(m) + f(b))$

- $\widehat{I_2} = \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$

Composite Rules

- Divide the interval into n equal pieces
- $h = (b - a)/n$
- $x_k = a + hk$
- $I = \int_a^b f(x)dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)dx = \sum_{i=1}^{n} I_i$
- Each of the approaches can be used in this manner