From last day

- $W_N = e^{i2\pi/N}, (W_N)^2 = W_{N/2}, (W_N)^{N/2} = -1$
- Assume N = 2^m:

$$F_k = \frac{1}{N}\left(f_0 + f_1 {W_N}^{-k} + f_2 {W_N}^{-2k} + \cdots f_{\frac{N}{2}-1} {W_N}^{-\left(\frac{N}{2}-1\right)k}\right)$$
$$+ \frac{1}{N}\left(f_{\frac{N}{2}} {W_N}^{-\left(\frac{N}{2}\right)k} + f_{\frac{N}{2}+1} {W_N}^{-\left(\frac{N}{2}+1\right)k} + f_{\frac{N}{2}+2} {W_N}^{-\left(\frac{N}{2}+2\right)k} + \cdots f_{\frac{N}{2}+\frac{N}{2}-1} {W_N}^{-\left(\frac{N}{2}+\frac{N}{2}-1\right)k}\right)$$

(develop rest of algorithm and running time on board)

Example of FFT in action for N=8 (1)

- $[f_0, f_1, f_2, f_3, f_4, f_5, f_6, f_7] = [1,2,3,4,5,6,7,8]$
- To calculate $[F_0, F_2, F_4, F_6]$ we need $[g_0, g_1, g_2, g_3]$, where
- $g_0 = \frac{1}{2}(\mathrm{f}_0 + \mathrm{f}_4) = 3$
- $g_1 = \frac{1}{2}(\mathrm{f}_1 + \mathrm{f}_5) = 4$
- $g_2 = \frac{1}{2}(\mathrm{f}_2 + \mathrm{f}_6) = 5$
- $g_3 = \frac{1}{2}(\mathrm{f}_3 + \mathrm{f}_7) = 6$

Example of FFT in action for N=8 (2)

- $[g_0, g_1, g_2, g_3\ ] = [3,4,5,6]$
- To calculate its FFT we need $[l_0, l_1], [m_0, m_1]$ where
- $l_0 = \frac{1}{2}(\mathrm{g}_0 + \mathrm{g}_2) = 4$
- $l_1 = \frac{1}{2}(\mathrm{g}_1 + \mathrm{g}_3) = 5$
- $m_0 = \frac{1}{2}(\mathrm{g}_0 - \mathrm{g}_2) = -1$
- $m_1 = \frac{1}{2}(\mathrm{g}_1 - \mathrm{g}_3){W_4}^{-1} = -1(-\mathrm{i}) = \mathrm{i}$

Example of FFT in action for N=8 (3)

- We can calculate the DFT of these 2-vectors directly, i.e.
- DFT($[l_0, l_1]$) $= \left[\frac{9}{2}, -\frac{1}{2}\right] = [L_0, L_1]$
- DFT($[m_0, m_1]$) $= \left[-\frac{1}{2} + \frac{1}{2}i, -\frac{1}{2} - \frac{1}{2}i\right] = [M_0, M_1]$
- We can combine these to get DFT($[g_0, g_1, g_2, g_3]$) $= [L_0, M_0, L_1, M_1]$

$$= [\frac{9}{2}, -\frac{1}{2} + \frac{1}{2}i, -\frac{1}{2}, -\frac{1}{2} - \frac{1}{2}i] = [G_0, G_1, G_2, G_3]$$

Example of FFT in action for N=8 (4)

- $[f_0, f_1, f_2, f_3, f_4, f_5, f_6, f_7] = [1,2,3,4,5,6,7,8]$
- To calculate $[F_1, F_3, F_5, F_7]$ we also need $[h_0, h_1, h_2, h_3]$ where
- $h_0 = \frac{1}{2}(\mathrm{f}_0 - \mathrm{f}_4) = \frac{1}{2}(1-5) = -2$
- $h_1 = \frac{1}{2}(\mathrm{f}_1 - \mathrm{f}_5(W_8)^{-1}) = \frac{1}{2}(2-6)\left(\frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}}\right) = -\sqrt{2} + i\sqrt{2}$
- $h_2 = \frac{1}{2}(\mathrm{f}_2 - \mathrm{f}_6(W_8)^{-2}) = \frac{1}{2}(3-7)(-i) = 2\mathrm{i}$
- $h_3 = \frac{1}{2}(\mathrm{f}_3 - \mathrm{f}_7(W_8)^{-3}) = \frac{1}{2}(4-8)\left(-\frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}}\right) = \sqrt{2} + i\sqrt{2}$

Example of FFT in action for N=8 (5)

- $[h_0, h_1, h_2, h_3\ ] = \left[-2, -\sqrt{2} + i\sqrt{2}, 2i, \sqrt{2} + i\sqrt{2}\right]$
- To calculate its FFT we need $[r_0, r_1], [s_0, s_1]$ where
- $r_0 = \frac{1}{2}(\mathrm{h}_0 + \mathrm{h}_2) = \frac{1}{2}(-2 + 2i) = -1 + i$
- $r_1 = \frac{1}{2}(\mathrm{h}_1 + \mathrm{h}_3) = \frac{1}{2}(-\sqrt{2} + i\sqrt{2} + \sqrt{2} + i\sqrt{2)} = i\sqrt{2}$
- $s_0 = \frac{1}{2}(\mathrm{h}_0 - \mathrm{h}_2) = \frac{1}{2}(-2 - 2\mathrm{i}) = -1 - \mathrm{i}$
- $s_1 = \frac{1}{2}(\mathrm{h}_1 - \mathrm{h}_3){W_4}^{-1} = \frac{1}{2}(-\sqrt{2} + i\sqrt{2} - \sqrt{2} - i\sqrt{2)} = \sqrt{2}i$

Example of FFT in action for N=8 (6)

- We can calculate the DFT of these 2-vectors directly, i.e.
- DFT($[r_0, r_1]$) $= \left[-\frac{1}{2} + \mathrm{i} * 1.2071, -\frac{1}{2} - \mathrm{i} * 0.2017\right] = [R_0, R_1]$
- DFT($[s_0, s_1]$) $= \left[-\frac{1}{2} + i * \ 0.2071, -\frac{1}{2} - i * 1.2071\right] = [S_0, S_1]$
- We can combine these to get

DFT($[h_0, h_1, h_2, h_3\ ]$) $= [R_0, S_0, R_1, S_1\ ]$

$= [-\frac{1}{2} + \mathrm{i} * 1.2071, -\frac{1}{2} + i * \ 0.2071, -\frac{1}{2} - \mathrm{i} * 0.2017, -\frac{1}{2} - i * 1.2071]$

$=[H_0, H_1, H_2, H_3\ ]$

Example of FFT in action for N=8 (7)

- This brings us back to the top level of the derivation:
  FFT([1,2,3,4,5,6,7,8]=
  $[G_0, H_0, G_1, H_1\ , G_2\ , H_2, G_3, H_3\ ]$

$$= [\frac{9}{2}, -\frac{1}{2} + \mathrm{i} * 1.2071, -\frac{1}{2} + \frac{1}{2} i, -\frac{1}{2} + i * \ 0.2071,$$

$$-\frac{1}{2}, -\frac{1}{2} - \mathrm{i} * 0.2017, -\frac{1}{2} - \frac{1}{2} i, -\frac{1}{2} - i * 1.2071]$$