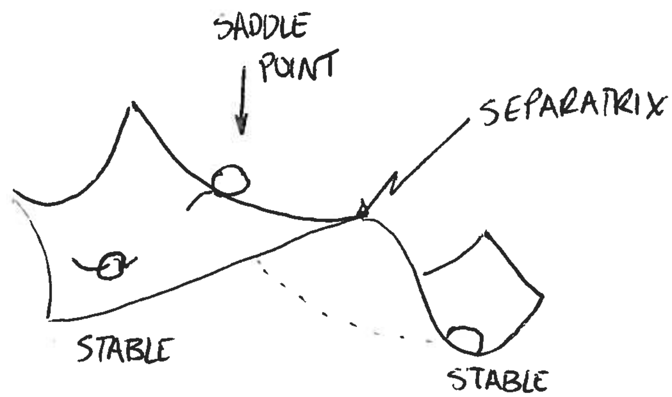
THE REGION WITHIN WHICH INITIAL CONDITIONS LEAD TO A STEADY-STATE IS CALLED ITS '<u>BASIN OF ATTRACTION</u>'. THE LINE SEPARATING THE BASINS OF ATTRACTION (i.e. TRAJECTORIES LEADING TO THE SADDLE POINT) IS CALLED A <u>SEPARATRIX</u>.

IN ONE-DIMENSION:

BISTABLE POTENTIAL WELL

IN TWO-DIMENSIONS:



SO FAR, WE HAVE LOOKED AT FAIRLY TAME SYSTEMS - BUT UNSTABLE POINTS CAN GIVE RISE TO VERY INTERESTING BEHAVIOUR.

STABLE ORBITS - CENTRES, LIMIT CYCLES & STRANGE ATTRACTORS

ONE VARIETY OF PERIODIC BEHAVIOUR AVAILABLE IN LINEAR & NONLINEAR SYSTEMS. (eg. THE HARMONIC OSCILLATOR).

A NONLINEAR EXAMPLE IS THE LOTKA-VOLTERRA SYSTEM OF EQUATIONS

$x(t)$ = PREY POPULATION $\quad y(t)$ = PREDATOR POPULATION

$$\frac{dx}{dt} = \underbrace{a x}_{\text{GROWTH OF PREY}} - \underbrace{\alpha \cdot x \cdot y}_{\text{DEATH BY PREDATION}} \qquad \frac{dy}{dt} = \underbrace{\beta \cdot x \cdot y}_{\text{GROWTH BY PREDATION}} - \underbrace{b \cdot y}_{\text{'NATURAL DEATH'}}$$

EQUILIBRIA: $(0,0)$ & $(b/\beta, a/\alpha)$

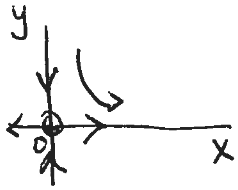
JACOBIAN $J(x,y) = \begin{bmatrix} a - \alpha y & -\alpha x \\ \beta \cdot y & -b + \beta x \end{bmatrix}$

$$J(0,0) = \begin{bmatrix} a & 0 \\ 0 & -b \end{bmatrix}$$

EIGENVALUES $a, -b$

EIGENVECTORS $\begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}0\\1\end{pmatrix}$

SADDLE POINT



$$J(b/\beta, a/\alpha) = \begin{bmatrix} 0 & -\frac{\alpha b}{\beta} \\ \frac{\beta a}{\alpha} & 0 \end{bmatrix}$$

EIGENVALUES $\pm i\sqrt{ab}$

CENTRE- LINEARIZED BEHAVIOUR IS INCONCLUSIVE

NOTICE -

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y(-b+\beta x)}{x(a-\alpha y)} \quad \text{SEPARABLE}$$

$$a \ln y - \alpha y + b \ln x - \beta x = \text{Constant} \quad (*)$$

PROOF THAT LOTKA-VOLTERRA CYCLES ARE CLOSED ORBITS:

LET $H(x) = \frac{b}{\beta} \ln x - x$, $G(y) = \frac{a}{\alpha} \ln y - y$

THEN THE SOLUTION CURVES (*) ARE LEVEL CURVES OF THE SURFACE:

$$V(x,y) = \beta \cdot H(x) + \alpha G(y)$$

SINCE

$$\frac{dH}{dx} = \frac{(b/\beta)}{x} - 1 \quad \text{AND} \quad \frac{d^2H}{dx^2} = -\frac{(b/\beta)}{x^2} < 0$$

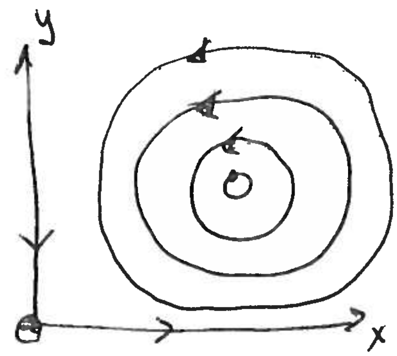
$H(x)$ ACHEIVES A MAX. AT $x = b/\beta$.

SIMILARLY, $G(y)$ IS A MAX AT $y = a/\alpha$.

SO THE SURFACE $V(x,y)$ HAS A SINGLE MAX. AT $(b/\beta, a/\alpha)$ WHICH MEANS THE LEVEL CURVES ARE CLOSED CURVES.

THEREFORE, ALL TRAJECTORIES OF THE LOTKA-VOLTERRA SYSTEM ARE PERIODIC.

YOU CAN SHOW THAT THESE ARE CLOSED CURVES AROUND $(b/\beta, a/\alpha)$



THESE CONCENTRIC PERIODIC TRAJECTORIES ARE NOT REPRESENTATIVE OF OSCILLATING NONLINEAR SYSTEMS.

LIMIT CYCLES

IN CARTESIAN COORDINATES, THE NONLINEAR SYSTEM,

$$\frac{d}{dt}x = y + x - x(x^2+y^2) \qquad \frac{d}{dt}y = -x + y - y(x^2+y^2)$$

LOOKS COMPLICATED - IT HAS ONE EQUILIBRIUM POINT $(x^*, y^*) = (0,0)$ AND THE JACOBIAN IS: $J(0,0) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ WITH $\lambda = 1 \pm i$ EIGENVALUES

i.e. UNSTABLE SPIRAL.

A CHANGE TO POLAR COORDINATES REVEALS A NEW KIND OF EQUILIBRIUM: $x = r\cos\theta \quad y = r\sin\theta \quad (r^2 = x^2+y^2,\ \theta = \arctan(\frac{y}{x}))$

$$r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} = (x^2+y^2) - (x^2+y^2)^2 = r^2 - (r^2)^2 = r^2(1-r^2)$$

OR, $\boxed{\frac{dr}{dt} = r(1-r^2)}$

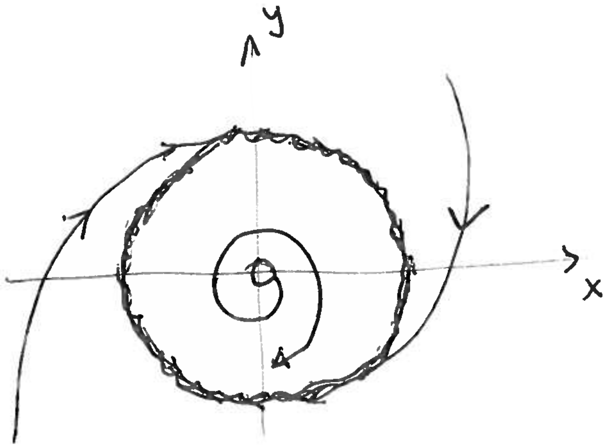
FOR θ: $-r^2\frac{d\theta}{dt} = y\frac{dx}{dt} - x\frac{dy}{dt} = x^2+y^2$ OR $\boxed{\frac{d\theta}{dt} = -1}$ ALL TRAJECTORIES MOVE ~~COUN~~ CLOCKWISE.

WHAT DO THE EQUILIBRIA LOOK LIKE? FOR $r^* = 0, 1$.

WE ALREADY KNOW $r=0$ (i.e. $(x,y) = (0,0)$) IS UNSTABLE.

WHAT ABOUT $r=1$? $\frac{dr}{dt} > 0$ FOR $r<1$ & $\frac{dr}{dt} < 0$ FOR $r>1$

SO IT IS STABLE



$r=1$ IS AN ASYMPTOTICALLY STABLE CLOSED CURVE!

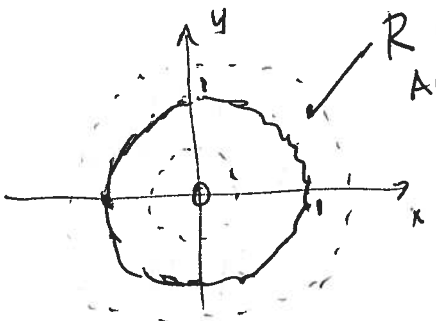
CALLED A 'LIMIT CYCLE'

UNSTABLE & SEMISTABLE LIMIT CYCLES ARE ALSO POSSIBLE.

THIS EXAMPLE WAS OF COURSE CONSTRUCTED TO FACILITATE THE ANALYSIS. IN GENERAL, IT IS DIFFICULT TO CONFIRM THE EXISTENCE OF A LIMIT CYCLE. A VERY USEFUL RESULT IS:

POINCARÉ - BENDIXSON THEOREM

LET R BE A CLOSED ANNULAR REGION IN 2D PHASE SPACE. IF THERE ARE TRAJECTORIES THAT REMAIN IN R FOR ALL $t > 0$, AND THERE ARE NO EQUILIBRIA IN R, THEN THERE IS A PERIODIC SOLUTION IN R.



R IN OUR EXAMPLE ABOVE EXCLUDES $(x,y) = (0,0)$.

CAN REPHRASE: LET D BE A CLOSED REGION IN PHASE SPACE THAT CONTAINS AN UNSTABLE EQUILIBRIUM POINT...

THE POINCARÉ - BENDIXSON THEOREM DOES NOT APPLY TO HIGHER-ORDER SYSTEMS. IN DIMENSION 3 OR HIGHER, BOUNDED TRAJECTORIES NEED NOT SETTLE TO AN EQUILIBRIUM OR A LIMIT CYCLE: THEY CAN APPROACH A STRANGE ATTRACTOR

STRANGE ATTRACTORS

IN DIMENSIONS 3 OR HIGHER, SYSTEMS HAVE A LOT OF FREEDOM TO EXPLORE PHASE SPACE, AND THIS CAN LEAD TO INTERESTING BEHAVIOUR:

EX. LORENTZ EQUATIONS (FROM WEATHER PREDICTION/FLUID DYN.)

$$\left.\begin{aligned} \frac{dx}{dt} &= \sigma(-x+y) \\ \frac{dy}{dt} &= rx - y - xz \\ \frac{dz}{dt} &= -bz + xy \end{aligned}\right\}$$

WITH CONSTANT σ, r & b.

THREE UNSTABLE EQUILIBRIA: $(r > 1)$ $(\sigma > 1 + b)$

$(x^*, y^*, z^*) =$

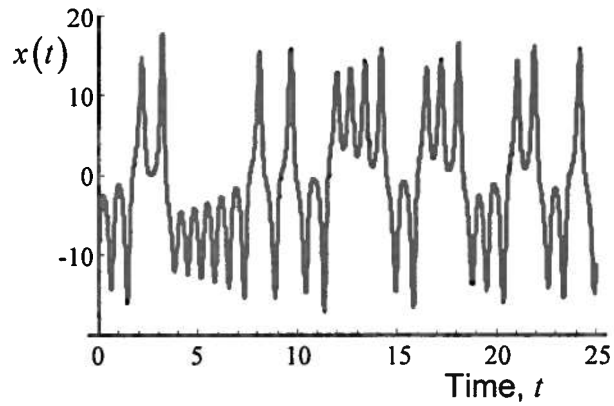
$(0,0,0)$

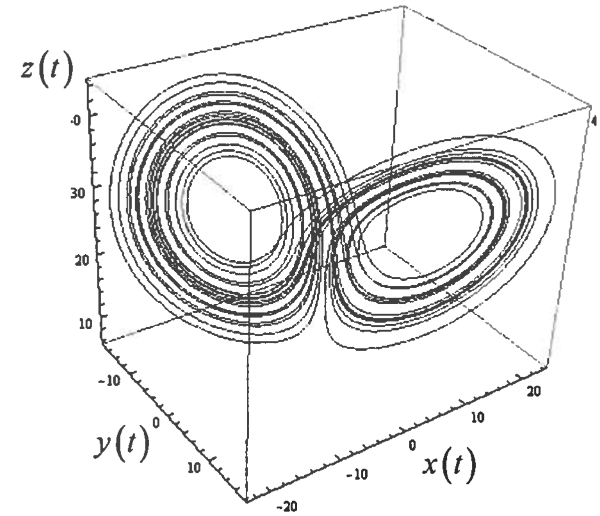
$(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$

$(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$

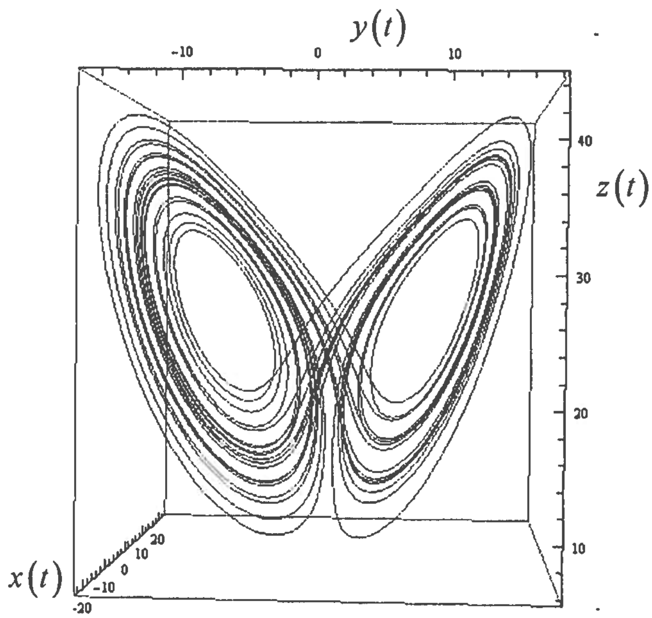
& SOLUTIONS ARE BOUNDED

LORENTZ USED THE VALUES $\sigma = 10$, $b = 8/3$ & $r = 28$; HE NOTICED AN EXTREME SENSITIVITY TO INITIAL CONDITIONS AND SEEMINGLY-CHAOTIC BEHAVIOUR IN THE SOLUTIONS.





BUT PLOTTED AS A PHASE PLOT, WITH $(x(t), y(t), z(t))$ AS A PATH IN CARTESIAN SPACE, THERE IS BEAUTIFUL REGULARITY IN THE SOLUTION.



LORENTZ "BUTTERFLY ATTRACTOR"

THE UNSTABLE EQUILIBRIA ARE AT THE CENTERS OF THE BUTTERFLY WINGS, AND THE TRAJECTORY CYCLES ABOUT THEM, THOUGH IT NEVER CROSSES ITSELF NOR BEGINS TO REPEAT.

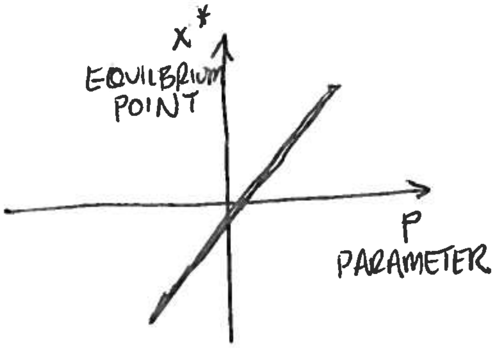
THIS IS AN EXAMPLE OF A '<u>STRANGE ATTRACTOR</u>'
IT IS A DETERMINISTIC SYSTEM, BUT FOR ALL INTENTS & PURPOSES ITS LONG-TERM BEHAVIOUR IS UNPREDICTABLE!

BIFURCATIONS

THE EQUILIBRIUM POINTS OF A SYSTEM GENERALLY DEPEND UPON THE PARAMETERS IN THE MODEL; THE STABILITY OF THE EQUILIBRIA LIKEWISE DEPENDS UPON THE PARAMETER VALUES.

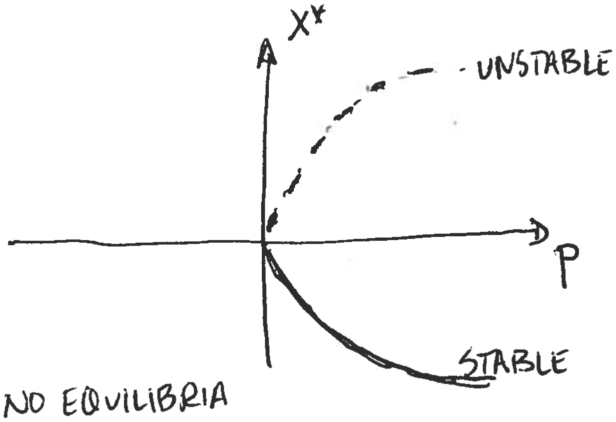
A PLOT OF THE EQUILIBRIA AS A FUNCTION OF PARAMETER VALUES IS CALLED A CONTINUATION DIAGRAM

EX. $\frac{dx}{dt} = p - x$; $x^* = p$.



FOR SOME SYSTEMS, THE NUMBER OR STABILITY-TYPE OF THE EQUILIBRIA CHANGES AT A PARTICULAR PARAMETER VALUE - THESE PARAMETER VALUES ARE CALLED BIFURCATION POINTS

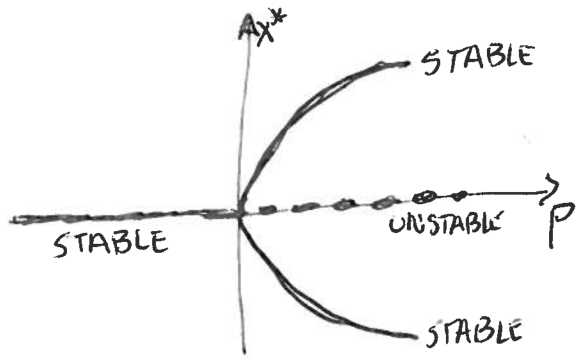
EX. $\frac{dx}{dt} = x^2 - p$. ; $x^* = \pm\sqrt{p}$ $(p \geq 0)$.



p=0 IS A BIFURCATION POINT. THIS PARTICULAR TYPE IS A SADDLE-NODE BIFURCATION

EX. $\frac{dx}{dt} = px - x^3 = x(p - x^2)$

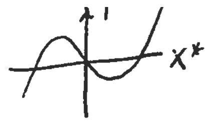
$x^* = 0, \pm\sqrt{p}$.

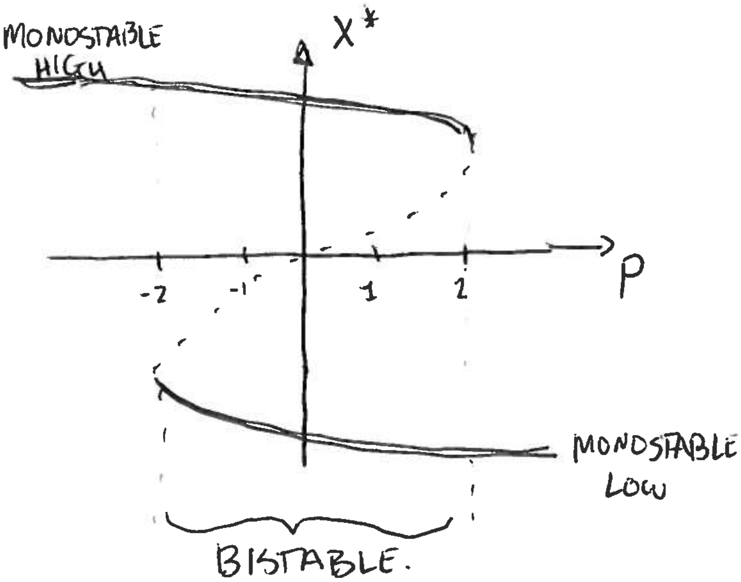


p=0 IS A PITCHFORK BISTABILITY.

SYSTEM CHANGES FROM MONOSTABLE → BISTABLE.

EX. $\frac{dx}{dt} = 3x - x^3 - p$; EQUILIBRIA: $p = 3x^* - (x^*)^3$





TWO SADDLE NODE BIFURCATIONS, AT $p = -2$ & $p = 2$

MONOSTABLE $p < -2$ OR $p > 2$

BISTABLE $-2 < p < 2$

IMAGINE YOU CAN CONTROL THE PARAMETER 'p' LIKE A SWITCH