L11 Operators, Matrices and Observables

September-29-11
10:18 PM

M7: Adjoint, Normal, Hermitian Operators and their Eigensystem

M7.1: Adjoint operators

M7.1.1 Summary Sheet

Adjoint Operators

operator A

$$\langle\Psi|\ A = \langle\tilde{\Psi}|$$

↕ dual vector space relation

$$A^\dagger\ |\Psi\rangle = |\tilde{\Psi}\rangle$$

adjoint operator $A^\dagger$

Rule:

Bra-ket notation:

$$\langle\phi|A|\Psi\rangle^* = \langle\Psi|A^\dagger|\phi\rangle$$

$$(|\Psi\rangle\langle\phi|)^\dagger = |\phi\rangle\langle\Psi|$$

Coordinate representation (finite dimension):

$(A^\dagger)_{i,j} = A^*_{j,i}$

➔ Transposition of matrix and complex conjugation of entries

$$A \doteq \begin{pmatrix} 1 & 1+2i \\ 1-3i & 4 \end{pmatrix}$$

$$A^\dagger \doteq \begin{pmatrix} 1 & 1+3i \\ 1-2i & 4 \end{pmatrix}$$

Special cases:

$A = A^\dagger$

➔ Hermitian (self-adjoint) operator

Note:
in infinite dimensional vector spaces the properties of being *hermitian* and *self-adjoint* are closely related, but distinct notions. The difference occurs when A and $A^\dagger$ have different domains on which they act.

More details (mostly omitted during lectures):

M7.1.2 Details

M7.1.2.1 Definition and motivation of adjoint operators

Operators map vectors to vectors

$$A|\psi\rangle = |\tilde{\psi}\rangle$$

We use this in expectation value calculations:

$$\langle A\rangle = \langle\psi|A|\psi\rangle$$

$$= \langle\psi|\tilde{\psi}\rangle$$

more generally:

$$\langle\psi|\underbrace{A|\psi\rangle}_{|\tilde{\psi}\rangle} = \langle\psi|\tilde{\psi}\rangle$$

$$(\alpha,\beta)\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = (\alpha,\beta)\begin{pmatrix} a' \\ b' \end{pmatrix}$$

in coordinate representation

$$(\alpha',\beta')\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\langle \tilde{\varphi} | \psi \rangle$$

$$\langle \tilde{\varphi}| = \langle \varphi | A$$

$$|\tilde{\varphi}\rangle = ???$$

Answer: $|\tilde{\varphi}\rangle = A^{+}|\varphi\rangle$ for all vectors $|\varphi\rangle$

we call A^{+} the adjoint operator (adjoint to operator A)

M7.1.2.2: Coordinate representation of adjoint operators

Operators A and A^{+} are defined by their action on the basis states
==> sufficient to look at their coordinate representation

$$A_{ij} = \langle a_i | A | a_j \rangle$$

$$(A^{+})_{ij} = \langle a_i | A | a_j \rangle$$

$$\langle a_i | A | a_j \rangle = \langle \tilde{a}_i | a_j \rangle$$

$$= \langle a_j | \tilde{a}_i \rangle^{*}$$

$$= \langle a_j | A^{+} | a_i \rangle^{*}$$

$$= (A^{+})^{*}_{ji}$$

$$A_{ij}$$

by reordering we find

$$(A^+)_{ji} = A^*_{ij}$$

complex conjugation

interchange of row and column index: transposition

Example:

Given operator A with coordinate representation

$$A \Leftrightarrow \begin{pmatrix} 1 & 1+2i \\ 1-3i & 4 \end{pmatrix}$$

then the corresponding adjoint operator is given by

$$A^+ \Leftrightarrow \begin{pmatrix} 1 & 1+3i \\ 1-2i & 4 \end{pmatrix}$$

The coordinate representation of adjoint operator A^+ is connected to the coordinate representation of the original operators A by:
- transposition of the matrix (exchange of column and row indices)
- complex conjugation

==> the two actions together are called **hermitian conjugation** and is denoted by the superscript +!

M7.1.2.3 Rules for Hermitian Conjugation

Note: hermitian conjugation is its own inverse!

$$(A^+)^+ = A$$

resulting in

$$\langle \tilde{\varphi}| = \langle \varphi | A^+$$

$$\Rightarrow |\tilde{\varphi}\rangle = A|\varphi\rangle$$

Note: from the coordinate representation discussion above we find:

$$\langle \varphi | A | \chi \rangle^* = \langle \varphi | (A|\chi\rangle)^+$$

$$= \langle \chi | A^+ | \varphi \rangle$$

which should be compared to the earlier relation

$$\langle \varphi | \chi \rangle^* = \langle \chi | \varphi \rangle$$

When writing down an operator in braket notation, we find

$$A = \sum_{i,j} a_{ij} |a_i\rangle\langle a_j|$$

$$A^\dagger = \sum_{i,j} a_{ij}^* |a_j\rangle\langle a_i|$$

In the simplest case we have

$$\left(|\psi\rangle\langle\varphi|\right)^\dagger = |\varphi\rangle\langle\psi|$$

We can extend the notion of hermitian conjugation to complex numbers, bras and kets as follows:

a) complex numbers: c

$$c^\dagger = c^* \quad \text{(complex conjugation)}$$

b) bras: $\langle\psi|$

$$\langle\psi|^\dagger = |\psi\rangle$$

c) kets: $|\psi\rangle$

$$|\psi\rangle^\dagger = \langle\psi|$$

General rule for bra-ket notation:
for any abstract bra-ket expression, we can obtain the hermitian conjugate by

- inverting the sequence of objects
- applying the hermitian conjugation to each object individually

Examples:

$$A|\psi\rangle = |\psi\rangle^\dagger A^\dagger$$

$$= \langle\psi| A^\dagger$$

$$\langle\varphi|A|\psi\rangle^\dagger = \langle\varphi|A|\psi\rangle^* = \langle\psi|A^\dagger|\varphi\rangle$$

$$|\varphi\rangle\langle\psi| = |\psi\rangle\langle\varphi|$$

The rule is compatible with the free association (grouping) rule of bra-ket objects!

M7.2 Normal Operators

M7.2.1 Definition

Definition:

Any operator A that has the property that

$$A^\dagger A = A A^\dagger$$

is called *normal*.

Remark: Any Hermitian operator is automatically normal.

M7.2.2 Eigensystem of Normal Operators and Spectral Decomposition

eigensystem of normal operators

Given a normal operator A
with eigenvalues λ_i
and corresponding eigenvectors $|\lambda_i\rangle$

Then the following holds:
eigenvectors belonging to different eigenvalues are orthogonal

$$\langle \lambda_i | \lambda_j \rangle = 0 \quad \text{for } \lambda_i \neq \lambda_j$$

For degenerate eigenvalues, we can always the eigenvectors to be orthogonal:
(degenerate case) $\lambda_i = \lambda_j$

we can always choose $\langle \lambda_i | \lambda_j \rangle = \sigma$

Spectral decomposition for normal operators

Given any normal operator A
with eigenvalues λ_i
and eigenvectors $|\lambda_i\rangle$

$$A = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$$

M7.3 Hermitian Operators

M7.3.1 Definition

An operator A is called Hermitian (**self-adjoint)** if $A = A^\dagger$

in general we have $A \neq A^\dagger$!

M7.3.2 Eigensystem of Hermitian Operators

Hermitian operators have real-valued eigenvalues

a hermitian operator H
has real valued eigenvalues λ_i
and orthonormal set of eigenvectors $|\lambda_i\rangle$

Proof (omitted in class)

for special case of Hermitian operators:

Part a: eigenvalues are real

$$\langle \lambda_i | A | \lambda_i \rangle = \lambda_i \langle \lambda_i | \lambda_i \rangle = \lambda_i$$

$$\Rightarrow \langle \lambda_i | A | \lambda_i \rangle^* = \lambda_i^*$$

on the other hand:

$$\langle \lambda_i | A | \lambda_i \rangle^* = \langle \lambda_i | A^\dagger | \lambda_i \rangle$$

A hermitian

$$= \langle \lambda_i | A | \lambda_i \rangle$$

$$= \lambda_i$$

therefore: $\lambda_i = \lambda_i^*$ eigenvectors are real valued!

Part b: eigenvectors orthogonal for $\lambda_i \neq \lambda_j$

$$\langle \lambda_i | A | \lambda_j \rangle$$

$$(i) = \langle \lambda_i | A | \lambda_j \rangle = \lambda_j \langle \lambda_i | \lambda_j \rangle$$

$$(ii) = \langle \lambda_i | A | \lambda_j \rangle = \lambda_i \langle \lambda_i | \lambda_j \rangle$$

Note $(A|\lambda_i\rangle)^\dagger = \langle \lambda_i | A^\dagger = \langle \lambda_i | A$

$(\lambda_i | \lambda_i \rangle)^\dagger$

$\langle \lambda_i | \lambda_i \rangle$

$\Rightarrow \langle \lambda_i | A = \lambda_i \langle \lambda_i |$

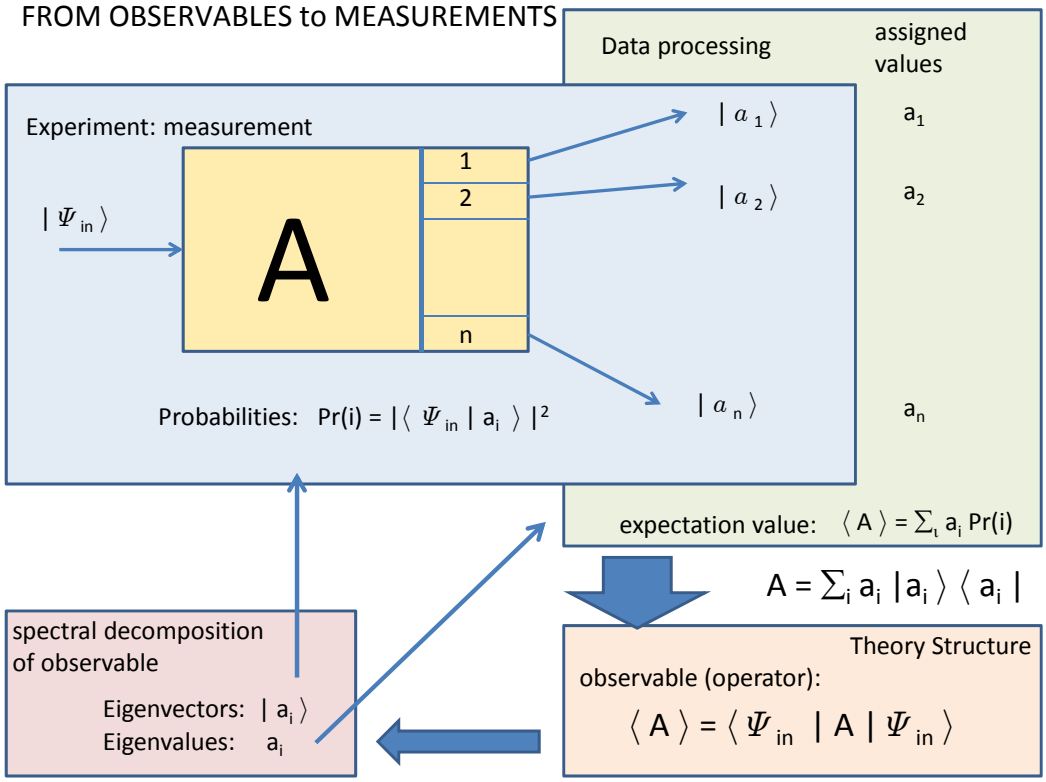
comparison: (i) & (ii)

$$\lambda_j \langle \lambda_i | \lambda_j \rangle = \lambda_i \langle \lambda_i | \lambda_j \rangle$$

$$\Rightarrow \langle \lambda_i | \lambda_j \rangle = 0$$

eigenvectors are orthogonal!

4.3 From Observables to Measurement Prescription



In section 4.1 we learned how we can go from an experimental evaluation of a mean value to a theoretical evaluation using observables.

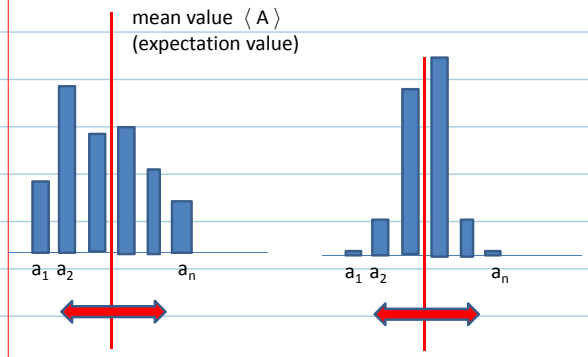
One can actually also go the other way round:
After doing theory on the level of observables, one would like to find an experimental evaluation procedure for a given observable. This is done easily by

- finding the eigensystem of the observable
- finding a measurement setting that corresponds to the eigenvectors of the observable
- assigning the eigenvalues of the observables to the outcomes that correspond to the respective eigenvector
  ==> perform the experiment and find the experimental value of the expectation value!

4.4 Uncertainties in Quantum Mechanics

4.4.1 Variances of Probability Distributions

4.4.1.1 Definition



General observable: A distribution of outcomes: $Pr(i)$

expectation value $\langle A \rangle = \sum_i a_i \, Pr(i)$

variance of distribution

$$\Delta A = \sqrt{\sum_i Pr(i) \, |a_i - \langle A \rangle|^2}$$

$$= \sqrt{\sum_i Pr(i) \left(a_i^2 - 2 a_i \langle A \rangle + \langle A \rangle^2 \right)}$$

$$= \sqrt{\sum_i Pr(i) \, a_i^2 - 2 \langle A \rangle^2 + \langle A \rangle^2}$$

$$\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$$

alternative form

$$\Delta A = \sqrt{\langle (A - \langle A \rangle \mathbb{1})^2 \rangle}$$

second alternative form

the symbol $\mathbb{1}$ denotes the identity operators

4.4.1.2 Examples

Example 1: deterministic outcome

Note: if quantum mechanical system is prepared in an eigenstate of A then we get always the same measurement outcome, which gives a shapr distribution with vanishing variance!

$$A|a\rangle = a|a\rangle$$

$$\Rightarrow \langle A \rangle = a$$

$$\langle A^2 \rangle = a^2$$

$$\Rightarrow \Delta A = 0$$

Example 2: Spin component measurements

Input state $|\psi\rangle = |+\rangle \left(= |+\rangle_z\right)$

Measurement of $S_z = \frac{\hbar}{2}\left(|+\rangle\langle+| - |-\rangle\langle-|\right)$

$$\langle S_z \rangle = \langle\psi| S_z |\psi\rangle$$
$$= \frac{\hbar}{2}$$

$$\langle S_z^2 \rangle = \langle\psi| S_z^2 |\psi\rangle$$
$$= \langle\psi| \left(\frac{\hbar^2}{4}\left(|+\rangle\langle+| + |-\rangle\langle-|\right)\right) |\psi\rangle$$
$$= \frac{\hbar^2}{4}$$

$= \mathbb{1}$ identity operator

$$\Rightarrow \Delta S_z = \sqrt{\langle S_z \rangle^2 - \langle S_z \rangle^2}$$
$$= 0$$

Measurement of $S_x = \frac{\hbar}{2}\left(|+\rangle_x\langle+| - |-\rangle_x\langle-|\right)$

$$\langle S_x \rangle = \langle\psi| S_x |\psi\rangle$$
$$= (1,0)\, \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= (1,0)\, \frac{\hbar}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$\langle S_x^2 \rangle = (1,0)\, \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \frac{\hbar^2}{4} (1,0) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \frac{\hbar^2}{4}$$

$$\Rightarrow \Delta S_x = \sqrt{\langle S_x^2 \rangle - \langle S_x \rangle^2}$$
$$= \frac{\hbar}{2}$$

4.4.2 Commutator of Operators

Frequently in quantum mechanics, the ***commutator*** of two observables play a role. For general operators A and B, we define the commutator as the operators

$$[A,B] := AB - BA$$

We say that *two operators A and B commute* if [A,B] = 0

Note 1: If A and B commute and are normal operators, then the two operators share the same eigenvectors, though the eigenvalues may be different.

$$[A,B] = 0$$

$$\Rightarrow A = \sum_i \lambda_i^{(A)} |\lambda_i\rangle\langle\lambda_i|$$

$$B = \sum_i \lambda_i^{(B)} |\lambda_i\rangle\langle\lambda_i|$$

4.4.3 Heisenberg Uncertainty Relation

Consider two (hermitian) observables: A and B

Consider a source of pure states $|\psi\rangle$

$$\Delta A \, \Delta B \geq \frac{1}{2} |\langle\psi| [A,B] |\psi\rangle|$$

Heisenberg Uncertainty Relation

Example: (uses concepts that will be develop in last parts of the course!!!)

$$\Delta x \, \Delta p \geq \frac{1}{2}\hbar$$

position operator X

momentum operator P

satisfies (see second part of course)

$$[X,P] = i\hbar$$

Cannot prepare a state that is sharp in position AND in momentum!

==> more later