May 16, 2014

Example: n=4

$$
\begin{array}{ll}
i=1 & -\frac{1}{h^2}T_0+\frac{2}{h^2}T_1-\frac{1}{h^2}T_2+0\cdot T_3+0\cdot T_4=f_1\\
i=2 & -\frac{1}{h^2}T_1+\frac{2}{h^2}T_2-\frac{1}{h^2}T_3+0\cdot T_4=f_2\\
i=3 & 0\cdot T_1-\frac{1}{h^2}T_2+\frac{2}{h^2}T_3-\frac{1}{h^2}T_4=f_3\\
i=4 & 0\cdot T_1+0\cdot T_2-\frac{1}{h^2}T_3+\frac{2}{h^2}T_4-\frac{1}{h^2}T_5=f_4
\end{array}
$$

Matrix form:

$$
\begin{bmatrix}
\frac{2}{h^2} & \frac{-1}{h^2} & & \\
\frac{-1}{h^2} & \frac{2}{h^2} & \frac{-1}{h^2} & \\
 & \frac{-1}{h^2} & \frac{2}{h^2} & \frac{-1}{h^2} \\
 & & \frac{-1}{h^2} & \frac{2}{h^2}
\end{bmatrix}
\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix}
=
\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}
$$

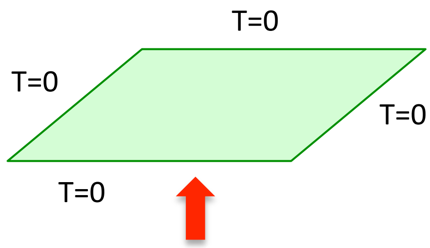
In general:

$$
\frac{1}{h^2}
\begin{bmatrix}
2 & -1 & & \\
-1 & \ddots & \ddots & \\
 & \ddots & \ddots & -1 \\
 & & -1 & 2
\end{bmatrix}
\begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix}
=
\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}
$$

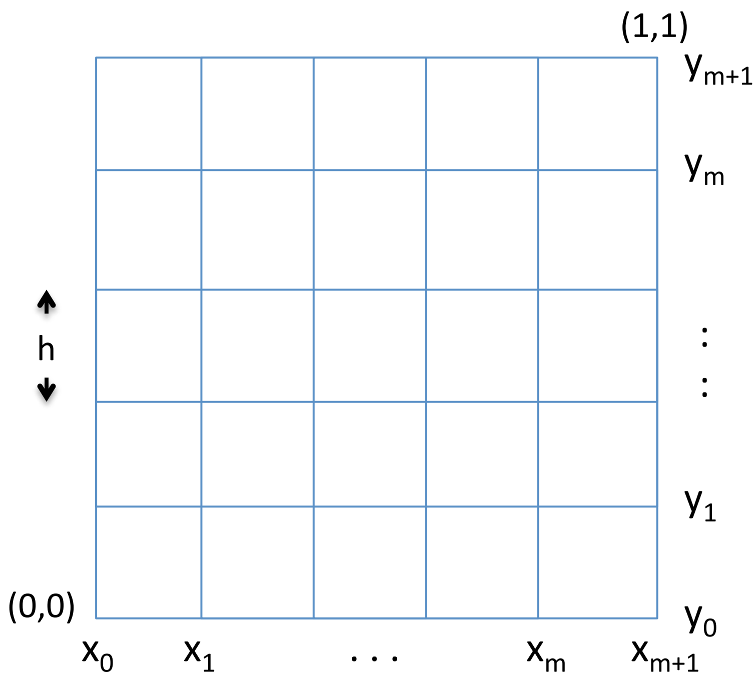
1D Laplacian matrix; it is tridiagonal

Two dimensions

$$-\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = f(x,y)$$



2D computational grid:



Approx. the temp T at (x_i, y_j) by

$$T_{ij} \approx T(x_i, y_j)$$

$$-\frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{h^2} - \frac{T_{i,j-1} - 2T_{i,j} + T_{i,j+1}}{h^2} = f_{i,j} \qquad i,j = 1,2,\ldots,m$$

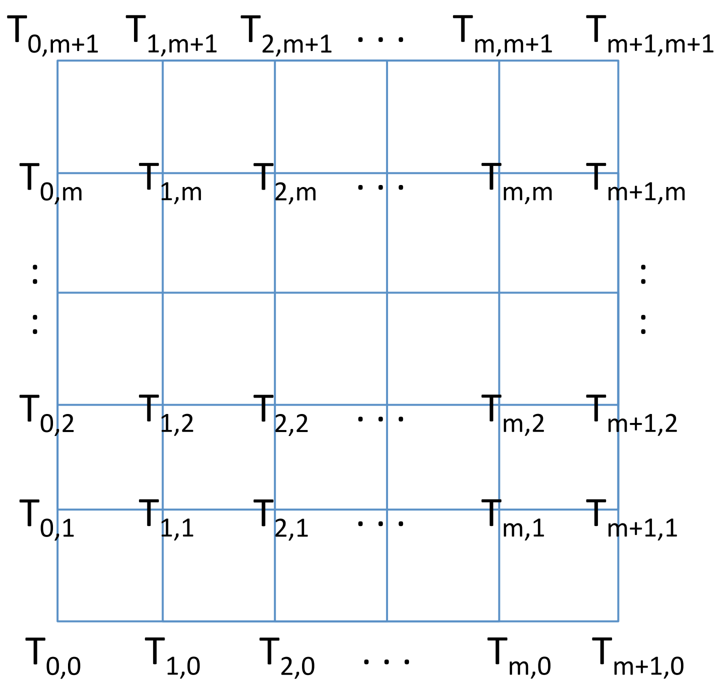
i.e. $$\frac{4}{h^2}T_{i,j} - \frac{1}{h^2}T_{i-1,j} - \frac{1}{h^2}T_{i+1,j} - \frac{1}{h^2}T_{i,j-1} - \frac{1}{h^2}T_{i,j+1} = f_{i,j}$$

5-pt stencil

An easy way to denote finite difference equations:

$$
\begin{array}{ccccc}
 & & T_{i,j+1} & & \\
 & & | & & \\
T_{i-1,j} & - & T_{i,j} & - & T_{i+1,j} \\
 & & | & & \\
 & & T_{i,j-1} & &
\end{array}
\qquad
\begin{bmatrix}
 & -\frac{1}{h^2} & \\
-\frac{1}{h^2} & \frac{4}{h^2} & -\frac{1}{h^2} \\
 & -\frac{1}{h^2} &
\end{bmatrix}
$$

Numbering of unknowns



Note: The values on the boundary are zero due to b.c.

The unknowns are:

$$
\begin{array}{cccc}
T_{1,1} & T_{2,1} & \cdots & T_{m,1} \\
T_{1,2} & T_{2,2} & \cdots & T_{m,2} \\
\vdots & \vdots & & \vdots \\
T_{1,m} & T_{2,m} & \cdots & T_{m,m}
\end{array}
$$

Total number = m × m = m² = n

<u>Natural ordering</u>: first in the x-direction, then y-direction.

i.e. $T_{1,1}$, $T_{2,1}$, . . . , $T_{m,1}$; $T_{1,2}$, $T_{2,2}$, . . .

The system of linear equations:

$$
\begin{aligned}
i=1, j=1: &\quad \frac{4}{h^2}T_{1,1} - \frac{1}{h^2}T_{2,1} - \frac{1}{h^2}T_{1,2} = f_{1,1} \\
i=2, j=1: &\quad -\frac{1}{h^2}T_{1,1} + \frac{4}{h^2}T_{2,1} - \frac{1}{h^2}T_{3,1} - \frac{1}{h^2}T_{2,2} = f_{2,1} \\
\vdots \quad &\quad \vdots \\
i=m, j=m: &\quad -\frac{1}{h^2}T_{m,m-1} - \frac{1}{h^2}T_{m-1,m} + \frac{4}{h^2}T_{m,m} = f_{m,m}
\end{aligned}
$$

Matrix form: (m=4, n=16)

$$
\frac{1}{h^2}
\left[
\begin{array}{cccc|cccc|cccc|cccc}
4 & -1 & & & -1 & & & & & & & & & & & \\
-1 & \ddots & \ddots & & & \ddots & & & & & & & & & & \\
 & \ddots & \ddots & -1 & & & \ddots & & & & & & & & & \\
 & & -1 & 4 & & & & -1 & & & & & & & & \\
\hline
-1 & & & & 4 & -1 & & & -1 & & & & & & & \\
 & \ddots & & & -1 & \ddots & \ddots & & & \ddots & & & & & & \\
 & & \ddots & & & \ddots & \ddots & -1 & & & \ddots & & & & & \\
 & & & -1 & & & -1 & 4 & & & & -1 & & & & \\
\hline
 & & & & -1 & & & & 4 & -1 & & & -1 & & & \\
 & & & & & \ddots & & & -1 & \ddots & \ddots & & & \ddots & & \\
 & & & & & & \ddots & & & \ddots & \ddots & -1 & & & \ddots & \\
 & & & & & & & -1 & & & -1 & 4 & & & & -1 \\
\hline
 & & & & & & & & -1 & & & & 4 & -1 & & \\
 & & & & & & & & & \ddots & & & -1 & \ddots & \ddots & \\
 & & & & & & & & & & \ddots & & & \ddots & \ddots & -1 \\
 & & & & & & & & & & & -1 & & & -1 & 4
\end{array}
\right]
\begin{array}{c} \uparrow \\ m \\ \downarrow \end{array}
$$

$\longleftarrow$ m $\longrightarrow$

$\longleftarrow$ m subblocks $\longrightarrow$

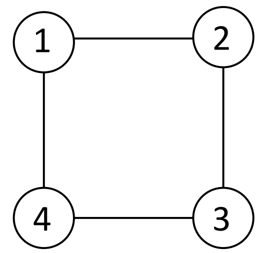
Graph representation of matrices

Given a sparse symmetric matrix A , a node is associated with each row. If $a_{ij} \neq 0$, there exists an edge from node i to j.

e.g.

$$A = \begin{array}{c} \begin{bmatrix} \times & \times & & \times \\ \times & \times & \times & \\ & \times & \times & \times \\ \times & & \times & \times \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$G(A):$



Graph of a matrix often has simple physical / geometric interpretation.

e.g. 1D Laplacian

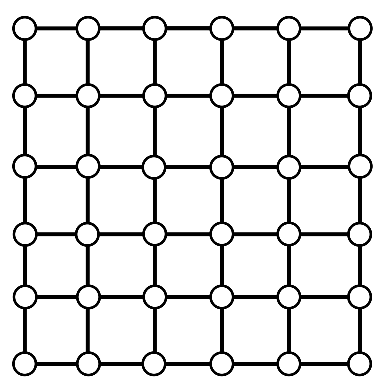
$$A = \begin{bmatrix} \times & \times & & \\ \times & \times & \times & \\ & \times & \times & \times \\ & & \times & \times \end{bmatrix}$$

$G(A):$

2D Laplacian

$$A = \begin{bmatrix}
4 & -1 & & & -1 & & & & & & & & & & & \\
-1 & \ddots & \ddots & & & \ddots & & & & & & & & & & \\
 & \ddots & \ddots & -1 & & & \ddots & & & & & & & & & \\
 & & -1 & 4 & & & & -1 & & & & & & & & \\
-1 & & & & 4 & -1 & & & -1 & & & & & & & \\
 & \ddots & & & -1 & \ddots & \ddots & & & \ddots & & & & & & \\
 & & \ddots & & & \ddots & \ddots & -1 & & & \ddots & & & & & \\
 & & & -1 & & & -1 & 4 & & & & -1 & & & & \\
 & & & & -1 & & & & 4 & -1 & & & -1 & & & \\
 & & & & & \ddots & & & -1 & \ddots & \ddots & & & \ddots & & \\
 & & & & & & \ddots & & & \ddots & \ddots & -1 & & & \ddots & \\
 & & & & & & & -1 & & & -1 & 4 & & & & -1 \\
 & & & & & & & & -1 & & & & 4 & -1 & & \\
 & & & & & & & & & \ddots & & & -1 & \ddots & \ddots & \\
 & & & & & & & & & & \ddots & & & \ddots & \ddots & -1 \\
 & & & & & & & & & & & -1 & & & -1 & 4
\end{bmatrix}$$

$G(A):$



GE and matrix graph

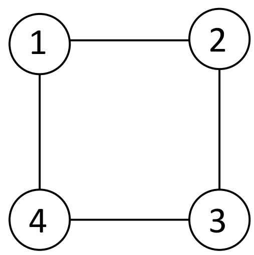
“Visualize” eliminations by matrix graph

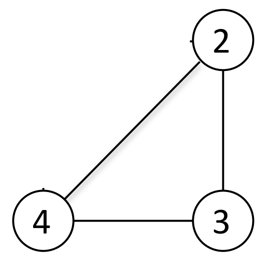
e.g.

$$\begin{bmatrix} \times & \times & & \times \\ \times & \times & \times & \\ & \times & \times & \times \\ \times & & \times & \times \end{bmatrix} \xrightarrow{\text{GE}} \begin{bmatrix} \times & \times & & \times \\ 0 & \times & \times & \bullet \\ & \times & \times & \times \\ \times & \bullet & \times & \times \end{bmatrix} \quad A^{(1)}$$

fill-in

G(A)





G($A^{(1)}$)

Elimination of node i produces a new graph with

1) node i deleted, all edges containing node i deleted
2) new edge (j, k) added (fill-in) if there was an edge (i, j) & (i, k) in the old graph

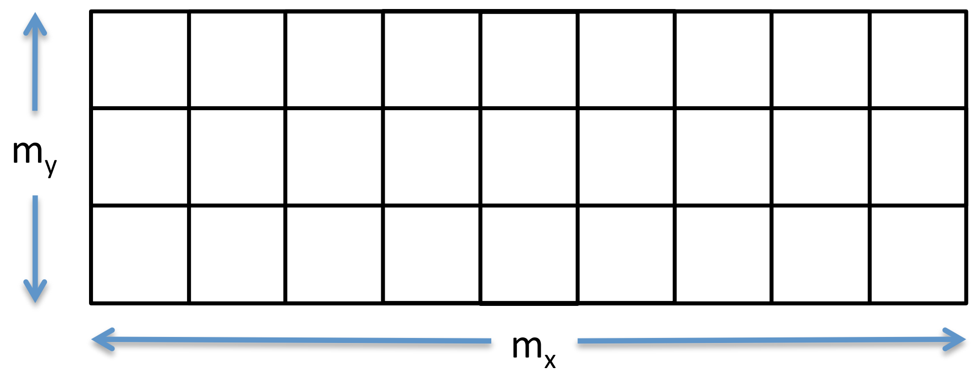
<u>Notes</u>

1) Matrix (with symmetric structure) graph is unchanged by renumbering of the nodes
2) But orderings (which nodes to be removed first) may result in much less fill during GE.

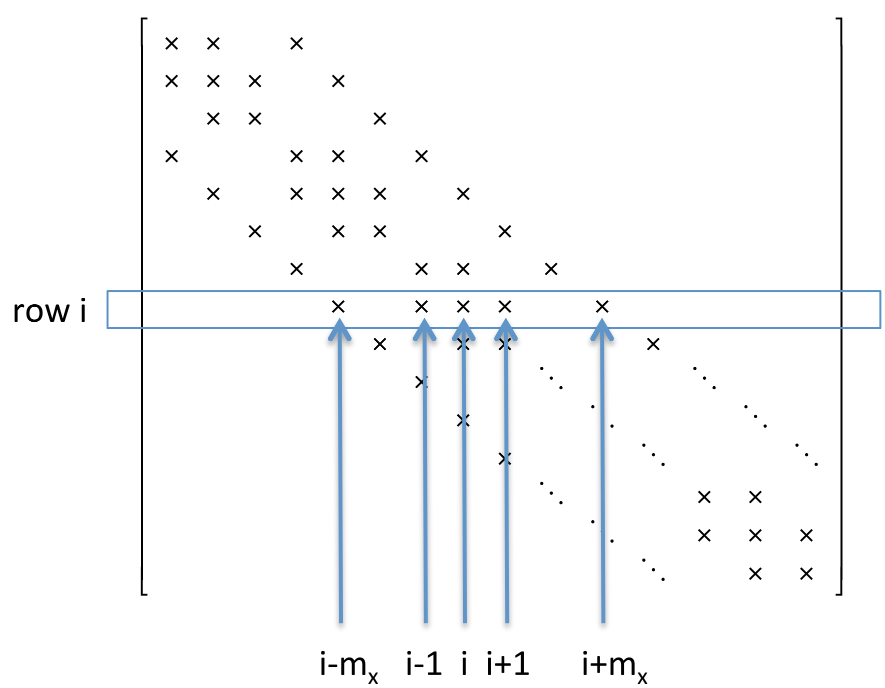
Ordering algorithms

Band matrices

Suppose we have a matrix with matrix graph:

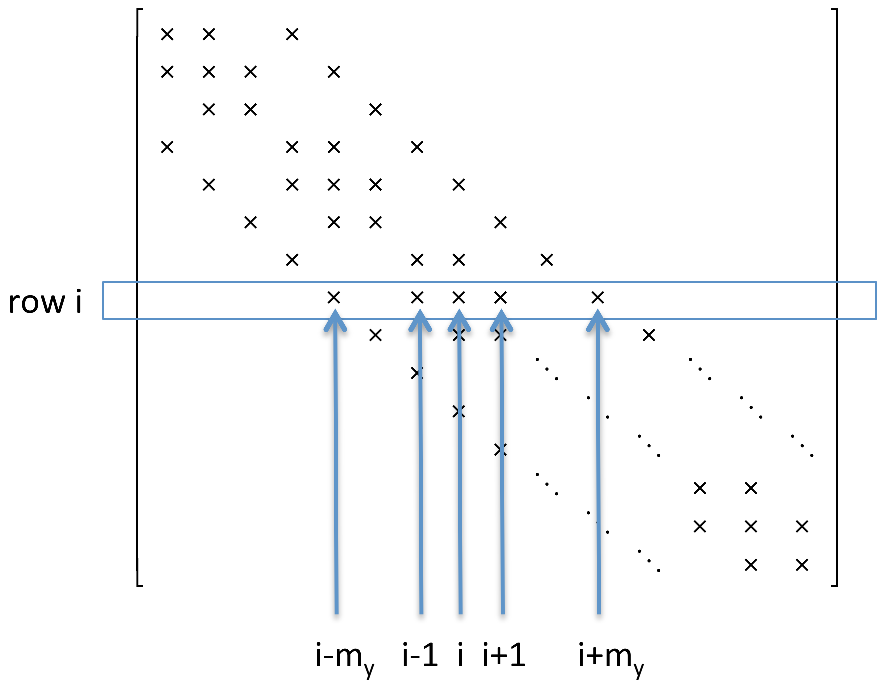


Assume $m_x >> m_y$. If we use natural ordering, what would the matrix look like?



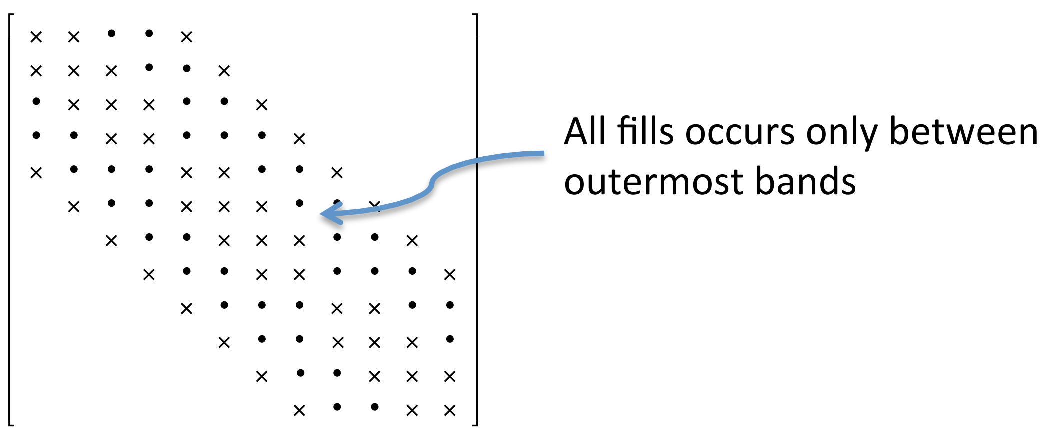
It has bandwidth m_x.

If we had numbered along y-direction first, the matrix would become:



Which ordering results in less fill? Why?

Note: GE with no pivoting preserves the band structure.



Amount of work for factoring a band matrix $= O(m^2 n)$, m = bandwidth

x-first ordering -> flops(GE) = $O(m_x^2\ n)$

y-first ordering -> flops(GE) = $O(m_y^2\ n)$