First-Order Logic Part1

Dr. Igor Ivkovic

iivkovic@uwaterloo.ca

[with material from “Mathematical Logic for Computer Science”, by Zhongwan, published by World Scientific]

Objectives

- Introduction to First-Order Logic
- Terms and WFFs for First-Order Logic
- First-Order Interpretations and Satisfiability

Alphabet

- The **alphabet** of the language of first-order logic consists of the following symbols.
  - **Constant Symbols**: c, d, c1, c2, … , d1, d2, ...
  - **Function Symbols**: f, g, h, f1, f2, ... , g1, g2, ...
  - **Variables**: x, y, z, x1, x2, ... , y1, y2, ...
  - **Predicate (Relational) Symbols**: P, Q, P1, P2, ... ,Q1,Q2, ...
  - **Logical Connectives**: $\neg, \wedge, \vee, \rightarrow$
  - **Quantifiers**: $\forall$ ("for all" or "for each"), $\exists$ ("there exists")
  - **Punctuation**: "(", ")", "," and ".".

Arity

- Each predicate symbol P and each function symbol f is associated with a natural number called its **arity**
  - *ar(P)* and *ar(f)*
- Predicate and function symbols with arity 1 (2, 3) are called unary (binary, ternary) respectively
- The constant, functional, and predicate symbols are called the non-logical symbols (or parameters)
- Predicate symbols with arity 0 are essentially **propositional symbols**, and function symbols with arity 0 are essentially **constants**

Terms

- Let CS be a set of constant symbols, FS a set of function symbols, and VS a set of variables
- Define the **set of terms** TS inductively as:

1. $\mathsf{CS} \subseteq \mathsf{TS}$,
2. $\mathsf{VS} \subseteq \mathsf{TS}$, and
3. if $f \in \mathsf{FS}$ and $t_1, \ldots, t_n \in \mathsf{TS}$, then $f(t_1, \ldots, t_n) \in \mathsf{TS}$, where $n = \mathsf{ar}(f)$;

- No other strings are terms

Well-Formed Formulæ (WFFs)

- Let PS be a set of predicate symbols, TS a set of terms, and VS a set of variables
- Define the **set of formulæ** of first-order logic (WFF) inductively as follows:

1. if $P \in \mathsf{PS}$ and $t_1, \ldots, t_n \in \mathsf{TS}$, where $n = \mathsf{ar}(P)$, then $P(t_1, \ldots, t_n) \in \mathsf{WFF}$;
2. if $\varphi \in \mathsf{WFF}$, then $(\neg\varphi) \in \mathsf{WFF}$;
3. if $\varphi, \psi \in \mathsf{WFF}$, then $(\varphi \star \psi) \in \mathsf{WFF}$ for each $\star \in \{\wedge, \vee, \rightarrow\}$;
4. if $x \in \mathsf{VS}$ and $\varphi \in \mathsf{WFF}$, then $(\forall x.\varphi) \in \mathsf{WFF}$ and $(\exists x.\varphi) \in \mathsf{WFF}$;

- No other strings are elements of WFF

Free and Bound Variables

- Assume the following shorthands:

$\varphi \vee \psi$ for $(\neg\varphi) \to \psi$, $\varphi \wedge \psi$ for $\neg(\varphi \to (\neg\psi))$, and $(\exists x.\varphi)$ for $\neg(\forall x.(\neg\varphi))$

- Let $\varphi \in$ WFF
- Define the **set of free variables** of φ, denoted $\mathsf{FV}(\varphi)$

1. If $\varphi = P(t_1, \ldots, t_{\mathsf{ar}(P)})$, then $\mathsf{FV}(\varphi) = \{x \mid x$ appears in t_i for some $0 < i \leq \mathsf{ar}(P)\}$;
2. if $\varphi = (\neg\psi)$, then $\mathsf{FV}(\varphi) = \mathsf{FV}(\psi)$;
3. if $\varphi = (\psi \to \eta)$, then $\mathsf{FV}(\varphi) = \mathsf{FV}(\psi) \cup \mathsf{FV}(\eta)$; and
4. if $\varphi = (\forall x.\psi)$, then $\mathsf{FV}(\varphi) = \mathsf{FV}(\psi) - \{x\}$.

- Variables in the set $\mathsf{FV}(\varphi)$ are called *free (in φ)*; other variables that occur in φ are called *bound (in φ)*.
- For a set of formulæ Σ, can define $\mathsf{FV}(\Sigma) = \bigcup_{\varphi \in \Sigma} \mathsf{FV}(\varphi)$

Closed Formulæ (Sentences)

- A first-order formula $\varphi \in \mathsf{WFF}$ is **closed** (or a **sentence**) iff $\mathsf{FV}(\varphi) = \emptyset$

First-Order Interpretations (Structures)

- A **first-order interpretation** (or **structure**) *I* is a pair $(D, (.)^I)$ where:
  - *D* is a non-empty set, called the domain (or universe) and
  - $(.)^I$ is an interpretation function that maps

    constant symbols $c \in \mathsf{CS}$ to individuals $(c)^I \in D$,

    function symbols $f \in \mathsf{FS}$ to functions $(f)^I : D^{\mathsf{ar}(f)} \to D$, and

    predicate symbols $P \in \mathsf{PS}$ to relations $(P)^I \subseteq D^{\mathsf{ar}(P)}$.

- For a fixed selection *L* of constant, function, and predicate symbols, called the vocabulary (or the signature (or the language)), can define L-structures to be those interpretations restricted to symbols in *L*

Valuation

- Let *D* be a domain and VS a set of variables. A **valuation** (or **assignment**) is a mapping $\theta : \mathsf{VS} \to D$.
- For a valuation θ, a variable *x* and a term *v*, the valuation $\theta[x/v]$ is defined by

$$\theta[x/v](y) = \begin{cases} v & \text{if } x = y, \\ \theta(y) & \text{otherwise.} \end{cases}$$

Meaning of Terms

- Let *I* be a first-order interpretation and θ valuation. For a term $t \in \mathsf{TS}$, we define the interpretation of *t*, denoted $(t)^{I,\theta}$ as follows

1. $(c)^{I,\theta} = (c)^I$ for $t \in \mathsf{CS}$ (i.e., t is a constant),
2. $(x)^{I,\theta} = \theta(x)$ for $t \in \mathsf{VS}$ (i.e., t is a variable), and
3. $(f(t_1, \ldots, t_{\mathsf{ar}(f)}))^{I,\theta} = (f)^I((t_1)^{I,\theta}, \ldots, (t_{\mathsf{ar}(f)})^{I,\theta})$ otherwise (i.e., for t a functional term).

Satisfaction Relation

- The **satisfaction relation** $\models$ between a first-order interpretation *I*, a valuation θ, and a formula $\varphi \in \mathsf{WFF}$ written $I, \theta \models \varphi$ is defined as follows
  - $I, \theta \models P(t_1, \ldots, t_{\mathsf{ar}(P)})$ iff $((t_1)^{I,\theta}, \ldots, (t_{\mathsf{ar}(P)})^{I,\theta}) \in (P)^I$ for $P \in \mathsf{PS}$;
  - $I, \theta \models (\neg\varphi)$ iff $I, \theta \not\models \varphi$;
  - $I, \theta \models (\varphi \to \psi)$ iff whenever $I, \theta \models \varphi$ then also $I, \theta \models \psi$.
  - $I, \theta \models (\forall x.\varphi)$ iff $I, \theta[x/v] \models \varphi$ for all $v \in D$.

  A pair (I, θ) such that $I, \theta \models \varphi$ is called a *(pointed) model of* φ.

- Can define

  $\mathsf{mod}(\varphi)$ to be the set of models of φ: $\mathsf{mod}(\varphi) = \{\, (I, \theta) \mid (I, \theta) \models \varphi \,\}$.

Satisfiability and Validity

- A modal formula φ is

- *valid* iff $I, \theta \models \varphi$ for all interpretations I and all valuations θ (i.e., true in all models),
- *satisfiable* iff $I, \theta \models \varphi$ for some interpretation I and some valuation θ (i.e., has a model), and
- *unsatisfiable* otherwise.

- Definitions of **logical implication** $(\Sigma \models \varphi)$ and **equivalence** and their properties are now the same as for propositional logic

Relevance

- The Relevance lemma allows us to consider only L-structures for an appropriately chosen set *L* of non-logical parameters

Let L be the set of all non-logical symbols in $\varphi \in \mathsf{WFF}$, and let

1. I_1 and I_2 be two interpretations such that $I_1(s) = I_2(s)$ for all $s \in L$ and
2. θ_1 and θ_2 be two valuations such that $\theta_1(x) = \theta_2(x)$ for all $x \in \mathsf{FV}(\varphi)$.

Then $I_1, \theta_1 \models \varphi$ if and only if $I_2, \theta_2 \models \varphi$.

Food for Thought

- **Read:**
  - Chapter 3, Sections 3.1, 3.2 and 3.3 from Zhongwan
    - Read the material discussed in class in more detail
    - Follow the notation conventions discussed in class
  - Handout on “First-Order Logic”
    - Available from the course schedule web page or through LEARN
- **Answer Assignment #4 questions**
  - Assignment #4 includes several practice exercises related to First-Order Logic