Introduction to Quantum Information Processing
Assignment 5 Solutions

1. **8 marks** *Energy conservation and maintaining coherence*

   Consider a qubit where $|0\rangle$ and $|1\rangle$ have different energies E_0 and E_1, and one-qubit gates are effected using light fields containing photons with energy (roughly) equal to $E_1 - E_0$. If the light field interacting with the qubit consists of exactly n photons, then a transition from $|1\rangle$ to $|0\rangle$ will leave $n+1$ photons in the light field; a transition from $|0\rangle$ to $|1\rangle$ will leave $n-1$ photons in the light field, $|0\rangle \mapsto |0\rangle$ or $|1\rangle \mapsto |1\rangle$ leaves n photons in the light field. In other words, if we try to map $|0\rangle$ to $|0\rangle + |1\rangle$ starting with a light field with exactly n photons, the light field will be entangled with the qubit (i.e. $|0\rangle|n\rangle + |1\rangle|n-1\rangle$), preventing quantum interference in the qubit if the light is discarded.

   We generally model the light field used as being in a "coherent state", which is a special superposition of different numbers of photons. For simplicity ("coherent states" have coefficients proportional to $\alpha^n/\sqrt{n!}$), suppose our light field, before the interaction with the qubit, is in the state

   $$|\psi_k\rangle = \sum_{n=1}^{k} \frac{1}{\sqrt{k}} |n\rangle$$

   where $|n\rangle$ denotes the state with n identical photons.

   Let $\overline{H}$ be the (hypothetical) interaction (the natural interaction would have amplitudes that depend on n) between the light field and qubit that maps $|0\rangle|n\rangle \mapsto \frac{1}{\sqrt{2}}|0\rangle|n\rangle + \frac{1}{\sqrt{2}}|1\rangle|n-1\rangle$ and $|1\rangle|n\rangle \mapsto \frac{1}{\sqrt{2}}|0\rangle|n+1\rangle - \frac{1}{\sqrt{2}}|1\rangle|n\rangle$.

   (a) **3 marks** *Approximating the Hadamard gate*

   Suppose the qubit and light field start in the state $|0\rangle|\Psi_k\rangle$ and then they interact according to $\overline{H}$. Compute the density matrix of the qubit after discarding the light field. In other words, compute

   $$\rho = Tr_2(\overline{H}(|0\rangle\langle 0| \otimes |\psi_k\rangle\langle\psi_k|)\overline{H}^\dagger)$$

   (i.e. trace out the light field register after the interaction that maps $|0\rangle|\psi_k\rangle \mapsto \overline{H}|0\rangle|\psi_k\rangle$).
   Thus show that

   $$1 - \langle +|\rho|+\rangle \in O\left(\frac{1}{k}\right)$$

   where $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = H|0\rangle$ is the desired output (call this difference the "error").
   (*Hint: Note that when you compute $\overline{H}|0\rangle|\psi_k\rangle$, the state can be written in the form $\alpha|+\rangle|$ something $\rangle + \beta|junk\rangle$ where $|\alpha|^2 \in 1 - O(1/k)$.*)

It can similarly be shown that for any one-qubit state ρ_1, the map $\rho_1 \mapsto \rho_2 = Tr_2(\overline{H}(\rho_1 \otimes |\psi_k\rangle\langle\psi_k|)\overline{H}^\dagger)$ implements the ideal Hadamard gate with error in $O(\frac{1}{k})$. Thus if one has a large supply of copies of the state $|\psi_k\rangle$, for large k, one can perform many Hadamard gates with high precision.

Solution:

We have

$$\begin{aligned}
\overline{H}|0\rangle|\psi_k\rangle &= \sum_{n=1}^{k} \frac{1}{\sqrt{k}}\frac{1}{\sqrt{2}}\left(|0\rangle|n\rangle + |1\rangle|n-1\rangle\right) \\
&= \frac{1}{\sqrt{k}}\frac{1}{\sqrt{2}}\left(\sum_{n=1}^{k}|0\rangle|n\rangle + \sum_{n=0}^{k-1}|1\rangle|n\rangle\right) \\
&= \frac{1}{\sqrt{k}}\frac{1}{\sqrt{2}}\left(\sum_{n=1}^{k-1}|0\rangle|n\rangle + |1\rangle|n\rangle\right) + \frac{1}{\sqrt{k}}\frac{1}{\sqrt{2}}\left(|0\rangle|k\rangle + |1\rangle|0\rangle\right) \\
&= \sqrt{\frac{k-1}{k}}\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\left(\sum_{n=1}^{k-1}\frac{1}{\sqrt{k-1}}|n\rangle\right) + |junk\rangle \\
&= \sqrt{\frac{k-1}{k}}|+\rangle\left(\sum_{n=1}^{k-1}\frac{1}{\sqrt{k-1}}|n\rangle\right) + |junk\rangle
\end{aligned}$$

, where the norm of $|junk\rangle$ is $\frac{2}{2k} = \frac{1}{k}$
By the properties of the partial trace, we know that ρ is the mixed state that we obtain if we measure the number of photos of the light field in any orthogonal basis, and then ignore the result.
Letting the orthogonal basis be one that includes $\sum_{n=1}^{k-1}\frac{1}{\sqrt{k-1}}|n\rangle$ as an element, we have then

$$\rho = \frac{k-1}{k}|+\rangle\langle+| + \frac{1}{k}\rho_{junk}$$

, where ρ_{junk} is the density matrix corresponding to tracing out the light field for a normalized $|junk\rangle$.
We have then

$$1 - \langle+|\rho|+\rangle = 1 - \frac{k-1}{k}\langle+|+\rangle^2 - \frac{1}{k}\langle+|\rho_{junk}|+\rangle$$

Using that $\langle x|\rho_{junk}|x\rangle \geq 0$ for any x (as ρ_{junk} is a density matrix, so it is positive semidefinite) we obtain

$$1 - \langle + | \rho | + \rangle \leq 1 - \frac{k-1}{k} = \frac{1}{k}$$

so $1 - \langle + | \rho | + \rangle \in O\left(\frac{1}{k}\right)$, as desired

(b) **3 marks** *Phase reference*

Suppose that instead of $|\psi_k\rangle$ we are given a state we can denote by $|e^{i\phi}\psi_k\rangle$ and define as

$$|e^{i\phi}\psi_k\rangle = \frac{1}{\sqrt{k}} \sum_{n=1}^{k} e^{in\phi} |n\rangle.$$

Show that the map

$$\rho_1 \mapsto \rho_2 = Tr_2(\overline{H}(\rho_1 \otimes |e^{i\phi}\psi_k\rangle\langle e^{i\phi}\psi_k|)\overline{H}^\dagger)$$

approximates the map

$$|0\rangle \mapsto \frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\phi}}{\sqrt{2}}|1\rangle$$

$$e^{i\phi}|1\rangle \mapsto \frac{1}{\sqrt{2}}|0\rangle - \frac{e^{i\phi}}{\sqrt{2}}|1\rangle$$

with error in $O(\frac{1}{k})$ on inputs $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$.

(Again, let "error" denote the difference between 1 *and* $\langle$ *ideal state*$|$ *actual state* $|$ *ideal state*

(This error bound can be shown to be the case on arbitrary one-qubit inputs.)

Solution:

For input $|0\rangle\langle 0|$, in the same way as in part a) we have

$$\begin{aligned}
\overline{H}|0\rangle|e^{i\phi}\psi_k\rangle &= \sum_{n=1}^{k} e^{in\phi} \frac{1}{\sqrt{k}} \frac{1}{\sqrt{2}} (|0\rangle|n\rangle + |1\rangle|n-1\rangle) \\
&= \frac{1}{\sqrt{k}} \frac{1}{\sqrt{2}} \left(\sum_{n=1}^{k} e^{in\phi}|0\rangle|n\rangle + e^{i\phi} \sum_{n=0}^{k-1} e^{in\phi}|1\rangle|n\rangle \right) \\
&= \frac{1}{\sqrt{k}} \frac{1}{\sqrt{2}} (|0\rangle + e^{i\phi}|1\rangle) \sum_{n=1}^{k-1} e^{in\phi}|n\rangle + \frac{1}{\sqrt{k}} \frac{1}{\sqrt{2}} (|0\rangle|k\rangle + e^{i\phi}|1\rangle|0\rangle) \\
&= \sqrt{\frac{k-1}{k}} \left(\frac{|0\rangle + e^{i\phi}|1\rangle}{\sqrt{2}} \right) \left(\sum_{n=1}^{k-1} \frac{1}{\sqrt{k-1}} |n\rangle \right) + |junk\rangle
\end{aligned}$$

where as before, the norm of $|junk\rangle$ is $\frac{1}{k}$.

Doing the same analysis as before, we obtain that we can write the actual mixed state ρ that we obtain as

$$\rho = \frac{k-1}{k}|\phi\rangle\langle\phi| + \frac{1}{k}\rho_{junk}$$

, where $|\phi\rangle = \frac{|0\rangle + e^{i\phi}|1\rangle}{\sqrt{2}}$, with

$$1 - \langle\phi|\rho|\phi\rangle = 1 - \frac{k-1}{k}\langle\phi|\phi\rangle^2 - \frac{1}{k}\langle\phi|\rho_{junk}|\phi\rangle \leq 1 - \frac{k-1}{k} = \frac{1}{k} \in O\left(\frac{1}{k}\right)$$

For input $|1\rangle\langle 1|$, similarly to $|0\rangle\langle 0|$, we have

$$\begin{aligned}
\overline{H}e^{i\phi}|1\rangle|e^{i\phi}\psi_k\rangle &= \sum_{n=1}^{k} e^{i(n+1)\phi}\frac{1}{\sqrt{k}}\frac{1}{\sqrt{2}}\left(|0\rangle|n+1\rangle - |1\rangle|n\rangle\right) \\
&= \frac{1}{\sqrt{k}}\frac{1}{\sqrt{2}}\left(\sum_{n=2}^{k+1} e^{in\phi}|0\rangle|n\rangle - e^{i\phi}\sum_{n=1}^{k} e^{in\phi}|1\rangle|n\rangle\right) \\
&= \frac{1}{\sqrt{k}}\frac{1}{\sqrt{2}}\left(|0\rangle - e^{i\phi}|1\rangle\right)\sum_{n=2}^{k} e^{in\phi}|n\rangle + \frac{1}{\sqrt{k}}\frac{1}{\sqrt{2}}\left(|0\rangle|k+1\rangle - e^{i\phi}|1\rangle|1\rangle\right) \\
&= \sqrt{\frac{k-1}{k}}\left(\frac{|0\rangle - e^{i\phi}|1\rangle}{\sqrt{2}}\right)\left(\sum_{n=2}^{k}\frac{1}{\sqrt{k-1}}|n\rangle\right) + |junk\rangle
\end{aligned}$$

where as before, the norm of $|junk\rangle$ is $\frac{1}{k}$. The same analysis as before (note that $\left(\sum_{n=1}^{k-1}\frac{1}{\sqrt{k-1}}\langle n|\right)\left(\sum_{n=2}^{k}\frac{1}{\sqrt{k-1}}|n\rangle\right) \in 1 - O(\frac{1}{k})$), letting now $|\phi\rangle$ be $\frac{|0\rangle - e^{i\phi}|1\rangle}{\sqrt{2}}$, gives us the desired result.

(c) **2 marks** *Picking a random phase reference*
Prove that

$$\sum_j |e^{i\phi_j}\psi_k\rangle\langle e^{i\phi_j}\psi_k| = \sum_{n=1}^{k}|n\rangle\langle n|$$

where the first sum is over values $\phi_j \in \{0, 2\pi/k, 2\pi 2/k, \ldots, 2\pi(k-1)/k\}$, the kth roots of 1.
In other words, one can obtain a random $|e^{i\phi_j}\psi_k\rangle$ by simply picking a random $|n\rangle$. This also works if we integrate equally over all $\phi_k \in [0, 2\pi)$.

Solution:

If we use the definition of $|e^{i\phi_j}\psi_k\rangle$ and expand the left hand side, we obtain

$$\sum_{j=0}^{k-1} |e^{i\phi_j}\psi_k\rangle\langle e^{i\phi_j}\psi_k| = \sum_{j=0}^{k-1}\sum_{n_1=1}^{k}\sum_{n_2=1}^{k}\frac{1}{k}e^{in_1 2\pi j/k}e^{-in_2 2\pi j/k}|n_1\rangle\langle n_2|$$

We can then rewrite this as

$$\sum_{j=0}^{k-1}\sum_{n_1=1}^{k}\sum_{n_2=1}^{k}\frac{1}{k}e^{i(n_1-n_2)2\pi j/k}|n_1\rangle\langle n_2| = \sum_{n_1=1}^{k}\sum_{n_2=1}^{k}\frac{1}{k}\sum_{j=0}^{k-1}e^{i(n_1-n_2)2\pi j/k}|n_1\rangle\langle n_2|$$

As we saw when analyzing the QFT, the value of the innermost sum is $\delta_{n_1,n_2}k$, so the previous expression is equal to

$$\sum_{n_1=1}^{k}\sum_{n_2=1}^{k}\frac{1}{k}\delta_{n_1,n_2}k|n_1\rangle\langle n_2| = \sum_{n=1}^{k}|n\rangle\langle n|$$

2. *Error correction*

   7 marks

   Suppose we are using the 3-qubit phase-flip code and recovery operator, with error operation $\mathcal{E} : \rho \mapsto U_\theta \otimes U_\theta \otimes U_\theta \rho U_\theta^\dagger \otimes U_\theta^\dagger \otimes U_\theta^\dagger$, where $U_\theta = R_z(\theta)$ and $\sin^2(\theta/2) = p$. (In other words, a U_θ operation is applied to each qubit.)

   (a) **1.5 marks**

   Suppose we start with the encoded state $|+\rangle|+\rangle|+\rangle$, and then the above error occurs.

   Express the resulting state $U_\theta \otimes U_\theta \otimes U_\theta|+\rangle|+\rangle|+\rangle$ as a superposition of $|+\rangle|+\rangle|+\rangle, |+\rangle|+\rangle|-\rangle, |+\rangle|-\rangle|+\rangle, |-\rangle|+\rangle|+\rangle, |+\rangle|-\rangle|-\rangle, |-\rangle|+\rangle|-\rangle, |-\rangle|-\rangle|+\rangle, |-\rangle|-\rangle|-\rangle$.

   Solution:

   Remember that we can express $R_z(\theta)$ as $\cos(\theta/2)I - i\sin(\theta/2)Z$. Therefore, $U_\theta \otimes U_\theta \otimes U_\theta|+\rangle|+\rangle|+\rangle = (\cos(\theta/2)I - i\sin(\theta/2)Z)^{\otimes 3}|+\rangle|+\rangle|+\rangle$.

   Using the fact that Z maps $|+\rangle$ to $|-\rangle$ and viceversa, and computing all terms in the tensor product, we have that the answer is

   $\cos^3(\theta/2)|+\rangle|+\rangle|+\rangle - i\cos^2(\theta/2)\sin(\theta/2)(|+\rangle|+\rangle|-\rangle + |+\rangle|-\rangle|+\rangle + |-\rangle|+\rangle|+\rangle) - \cos(\theta/2)\sin^2(\theta/2)(|+\rangle|-\rangle|-\rangle + |-\rangle|+\rangle|-\rangle + |-\rangle|-\rangle|+\rangle) + i\sin^3(\theta/2)|-\rangle|-\rangle|-\rangle$.

   (b) **1.5 marks** Show that the probability of being in a state that will not be corrected by "majority voting" in the $\{|+\rangle, |-\rangle\}$ basis is in $O(p^2)$.

   Solution:

   The terms that do not get corrected are the ones that have at least two (i.e. a majority) of $|-\rangle$'s. Their total amplitude is $3\cos^2(\theta/2)(\sin^4(\theta/2)) + \sin^6(\theta/2) = 3(1-p)(p^2) + p^3 = 3p^2 - 2p^3 \in O(p^2)$.

(c) Suppose we measure the stabilizers XXI and XIX.

i. **0.5 marks each**

Show that the probability of obtaining +1 and +1 is in $1 - O(p)$.
What is the resulting state of the encoded qubit in this case?
What is the resulting state of the encoded qubit after the appropriate correction is done?
Show that the resulting error probability in this case is in $O(p^3)$.

Solution:

The terms in which we obtain +1 and +1 are the ones that have an even number of $|+\rangle$'s counting the first and second positions, and the same happens when we consider the first and third positions. Those are $\cos^3(\theta/2)|+\rangle|+\rangle|+\rangle$ and $i\sin^3(\theta/2)|-\rangle|-\rangle|-\rangle$.
Their total amplitude is $\cos^6(\theta/2)+\sin^6(\theta/2) = (1-p)^3+p^3 = 1+3p^2-3p \in 1-O(p)$, as $-3p^2+3p \leq 3p$.
When we measure +1 and +1, no correction is made. Therefore, the (normalized) resulting state is

$$\frac{\cos^3(\theta/2)}{\sqrt{\cos^6(\theta/2)+\sin^6(\theta/2)}}|+\rangle|+\rangle|+\rangle + i\frac{\sin^3(\theta/2)}{\sqrt{\cos^6(\theta/2)+\sin^6(\theta/2)}}|-\rangle|-\rangle|-\rangle$$

We have then that the error probability in this case is the amplitude in the result for $|-\rangle|-\rangle|-\rangle$, given by $\dfrac{sin^6(\theta/2)}{\cos^6(\theta/2)+\sin^6(\theta/2)} = \dfrac{p^3}{1-3p(1-p)} \leq p^3 \in O(p^3)$. This is because the denominator $1-3p(1-p)$ is always $\geq \frac{1}{4}$, as can be seen using elementary calculus.

ii. **0.5 marks each**

Show that the probability of obtaining +1 and −1 is in $O(p)$.
What is the resulting state of the encoded qubit in this case?
What is the resulting state of the encoded qubit after the appropriate correction is done?
Show that the resulting error probability in this case is in $O(p)$.

Solution:

The terms in which we obtain +1 and −1 are the ones that have an even number of $|+\rangle$'s counting the first and second positions, and an odd number when we consider the first and third positions. Those are $-i\cos^2(\theta/2)\sin(\theta/2)|+\rangle|+\rangle|-\rangle$ and $-\cos(\theta/2)\sin^2(\theta/2)|-\rangle|-\rangle|+\rangle$.
Their total amplitude is $\cos^4(\theta/2)\sin^2(\theta/2)+\cos^2(\theta/2)\sin^4(\theta/2) = \cos^2(\theta/2)\sin^2(\theta/2) = p-p^2 \in O(p)$.
The correction that is done in this case consists of a Z gate applied to the third qubit. The (normalized) resulting state is then

$$\frac{-i\cos^2(\theta/2)\sin(\theta/2)}{|\cos(\theta/2)\sin(\theta/2)|}|+\rangle|+\rangle|+\rangle - \frac{\cos(\theta/2)\sin^2(\theta/2)}{|\cos(\theta/2)\sin(\theta/2)|}|-\rangle|-\rangle|-\rangle$$
$$= -i|\cos(\theta/2)|\frac{\sin(\theta/2)}{|\sin(\theta/2)|}|+\rangle|+\rangle|+\rangle - |\sin(\theta/2)|\frac{\cos(\theta/2)}{|\cos(\theta/2)|}|-\rangle|-\rangle|-\rangle$$
$$= \cos(\theta/2)|+\rangle|+\rangle|+\rangle - i\sin(\theta/2)|-\rangle|-\rangle|-\rangle$$

, up to a global phase.
We have then that the error probability in this case is the amplitude in the result for $|-\rangle|-\rangle|-\rangle$, given by $\sin^2(\theta) = p \in O(p)$.
Note that we have determined that this case occurs with probability $O(p)$, and the error that occurs is $O(p)$, so the total error for the error correction protocol coming from this case is $O(p^2)$. Similarly, the one coming from case i is $O(p^3)$. We can repeat this analysis in the other two cases to prove that the total error is in $O(p^2)$.

(d) *Decoherence-free subspace* **1 marks**

Find a basis for a two-dimensional subspace of two qubits that is not affected by the phase noise operation $\rho \mapsto U_\theta \otimes U_\theta \rho U_\theta^\dagger \otimes U_\theta^\dagger$, where $U_\theta = R_z(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$.
In other words, find two orthogonal 2-qubit states $|\phi_0\rangle$ and $|\phi_1\rangle$ such that, for $i = 0, 1$,

$$U_\theta \otimes U_\theta |\phi_i\rangle = e^{i\alpha}|\phi_i\rangle$$

where α is a constant.

Solution:

We can see that the map $U_\theta \otimes U_\theta$ will take $|\phi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ to $e^{-i\theta}a|00\rangle + b|01\rangle + c|10\rangle + e^{i\theta}d|11\rangle$. We want to be able to take a global phase away, and get $|\phi\rangle$ back
Now, note that $|01\rangle$ and $|10\rangle$ have phase equal to 1. Therefore, we can just let $|\phi_0\rangle$ be $|01\rangle$ and $|\phi_1\rangle$ be $|10\rangle$.

3. $BQP \subseteq P^{\sharp P}$ **4 marks**

Consider the following circuit, with input state $|000\rangle$. Note that, with respect to the computational basis, there are 8 computational paths taken, each with amplitude $\pm\frac{1}{\sqrt{8}}$.
For example, the path

$$|000\rangle \mapsto |100\rangle \mapsto |110\rangle \mapsto |100\rangle \mapsto |100\rangle \mapsto |101\rangle$$

is taken with amplitude $\frac{1}{\sqrt{8}}$, since the first Hadamard maps $|000\rangle$ to $|100\rangle$ with amplitude $\frac{1}{\sqrt{2}}$. Then the CNOT maps $|100\rangle$ to $|110\rangle$ with amplitude 1. Then the second

Hadamard maps $|110\rangle$ to $|100\rangle$ with amplitude $\frac{1}{\sqrt{2}}$, and so on. The product of these transition amplitudes is $\frac{1}{\sqrt{8}}$.

(a) Write down the other 7 paths and the amplitude of each path.

Solution:

(b) NP-hard problems can be reduced to a question of the form: Given a function $f : \{0,1\}^n \to \{0,1\}$ (computable in time polynomial in n), does there exist an input string x such that $f(x) = 1$.

The counting version of this decision problem is to output the number of inputs x such that $f(x) = 1$. The class of problems of this form is denoted $\sharp P$. These are function problems (i.e. the output is not just 'accept' or 'reject'). The class of decision problems solvable in polynomial time given an oracle for solving any $\sharp P$ problem is denoted $P^{\sharp P}$.

Note that the class BQP is equivalent to the class of languages L for which there exists a family of polynomial sized quantum circuits A_n consisting of $CNOT$, Toffoli and Hadamard gates (and each A_n can be constructed in polynomial time by a classical computer) where A_n has n input qubits and with the property that for any positive integer n and any string x of length n

i. if $x \in L$ then $|\langle 00\ldots 0|A_n|x\rangle|^2 \geq \frac{2}{3}$
ii. if $x \notin L$ then $|\langle 00\ldots 0|A_n|x\rangle|^2 \leq \frac{1}{3}$

Prove that $BQP \subseteq P^{\sharp P}$. In other words, for any language $L \in BQP$, show how to decide membership in L in polynomial time using an oracle for some $\sharp P$ language.

Solution:

Let L be a language in BQP, and let $\{A_n\}$ be the family of circuits for deciding (with high probability) membership in L.

An n-bit string $x \in L$ if and only if $|\langle 00\ldots 0|A_n|x\rangle|^2 \geq \frac{2}{3}$.

Suppose A_n has t Hadamard gates. We can without loss of generality assume A_n is of the form $U_0 H_{m_1} U_1 H_{m_2} U_2 \ldots H_{m_t} U_t$, where each U_i consists of $CNOT$ and Toffoli gates, and H_{m_j} corresponds to a Hadamard gate on qubit m_j (for $1 \leq m_j \leq n$).

Thus this circuit induces 2^t computational paths starting from $|x\rangle$ and ending with some $|y\rangle$ for $y \in \{0,1\}^n$. This is true since each U_i simply permutes the n-bit strings, and each Hadamard gate gives two paths.

For each string $s = s_1 s_2 \ldots s_t \in \{0,1\}^t$, let $f(x)$ denote the end state of the computational path obtained by tracing through the circuit A_n and following the path where the jth Hadamard gate outputs the value $|s_j\rangle$.

For example, in part a), we have that $f(101) = 101$.

Let $g_+(s) = 1$ if $f(s) = 000$ and the amplitude of $|000\rangle$ when following path s is $\frac{1}{\sqrt{2^t}}$, and let $g_+(s) = 0$ otherwise.

Let $g_-(s) = 1$ if $f(s) = 000$ and the amplitude of $|000\rangle$ when following path s is $\frac{-1}{\sqrt{2^t}}$.

Note that $g_+(s)$ and $g_-(s)$ can be computed in time polynomial in n given the circuit A_n.

Thus using our $\sharp P$ oracle, we can count the number of paths m_+ that lead to $|00\ldots\rangle$ with amplitude $\frac{1}{\sqrt{2^t}}$ on input $|x\rangle$, and separate call to the oracle can tell us the number of paths m_- that lead to $|00\ldots\rangle$ with amplitude $\frac{-1}{\sqrt{2^t}}$ on input $|x\rangle$.

Thus, the final amplitude of $|00\ldots0\rangle$ is $\frac{(m_+ - m_-)}{\sqrt{2^t}}$. We can thus compute $\left|\frac{(m_+ - m_-)}{\sqrt{2^t}}\right|^2$. If this value is $\geq \frac{2}{3}$ then $x \in L$ and if it is $\leq \frac{1}{3}$ then $x \notin L$.