MATRIX EXPONENTIAL. RECALL THAT FOR AN AUTONOMOUS SYSTEM,

$$\frac{d\vec{y}}{dx} = \mathbb{A}\cdot\vec{y}$$

THE FUNDAMENTAL MATRIX $\Phi(x,0) \equiv \Phi(x)$ HAS THE FOLLOWING PROPERTIES:

i) $\Phi(0) = \mathbb{I}_n$

ii) $\dfrac{d\Phi}{dx} = \mathbb{A}\cdot\Phi$

iii) $\Phi(x_1+x_2) = \Phi(x_1)\Phi(x_2)$

iv) $\Phi(-x) = \left[\Phi(x)\right]^{-1}$.

THESE ARE ALL PROPERTIES OF THE EXPONENTIAL. SUGGEST WE DEFINE A NEW FUNCTION CALLED THE MATRIX EXPONENTIAL

$$e^{\mathbb{A}} = \sum_{n=0}^{\infty}\frac{\mathbb{A}^n}{n!} = \mathbb{I} + \mathbb{A} + \frac{\mathbb{A}^2}{2!} + \frac{\mathbb{A}^3}{3!} + \cdots + \frac{\mathbb{A}^n}{n!} + \cdots$$

CONVERGES FOR ALL $\mathbb{A}$.

NOTICE:

i) $e^{0} = \mathbb{I}$

ii) $$\frac{d}{dx}e^{\mathbb{A}x} = \frac{d}{dx}\sum_{n=0}^{\infty}\frac{\mathbb{A}^n x^n}{n!} = \sum_{n=1}^{\infty}\frac{\mathbb{A}^n x^{n-1}}{(n-1)!} = \mathbb{A}\sum_{n=1}^{\infty}\frac{\mathbb{A}^{n-1}x^{n-1}}{(n-1)!}$$

$$= \mathbb{A}\cdot\sum_{n=0}^{\infty}\frac{\mathbb{A}^n x^n}{n!} = \mathbb{A}\,e^{\mathbb{A}x}$$

HAVE TO PAY ATTENTION TO COMMUTIVITY

iii) $$e^{\mathbb{A}+\mathbb{B}} = \mathbb{I} + (\mathbb{A}+\mathbb{B}) + \frac{(\mathbb{A}+\mathbb{B})^2}{2} + \cdots = \mathbb{I} + (\mathbb{A}+\mathbb{B}) + \frac{1}{2}\left(\mathbb{A}^2 + \underline{\mathbb{A}\mathbb{B} + \mathbb{B}\mathbb{A}} + \mathbb{B}^2\right) + \cdots$$

FOR THE PRODUCT:

$$e^{\mathbb{A}}e^{\mathbb{B}} = \left(\sum_{n=0}^{\infty}\frac{\mathbb{A}^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{\mathbb{B}^n}{n!}\right) = \left(\mathbb{I} + \mathbb{A} + \frac{\mathbb{A}^2}{2} + \cdots\right)\left(\mathbb{I} + \mathbb{B} + \frac{\mathbb{B}^2}{2} + \cdots\right)$$

$$= \mathbb{I} + (\mathbb{A}+\mathbb{B}) + \left(\frac{\mathbb{A}^2}{2} + \underline{\mathbb{A}\mathbb{B}} + \frac{\mathbb{B}^2}{2}\right) + \cdots$$

IN GENERAL,

$$e^{\mathbb{A}+\mathbb{B}} = e^{\mathbb{A}}e^{\mathbb{B}}$$ (AND ONLY IF) IF $\mathbb{A}$ & $\mathbb{B}$ COMMUTE i.e. $\mathbb{A}\mathbb{B} = \mathbb{B}\mathbb{A}$.

iv) $\mathbb{A}$ COMMUTES WITH $-\mathbb{A}$ SO, $e^{\mathbb{A}}e^{-\mathbb{A}} = e^{-\mathbb{A}}e^{\mathbb{A}} = e^{(\mathbb{A}-\mathbb{A})} = e^{0} = \mathbb{I}$

AND $e^{-\mathbb{A}} = \left[e^{\mathbb{A}}\right]^{-1}$

HOW DOES THIS MATRIX EXPONENTIAL CONNECT TO THE SOLUTION OF OUR DIFFERENTIAL EQ. $\frac{d\vec{y}}{dx} = \mathbb{A}\cdot\vec{y}$ & $\vec{y}(0) = \vec{y}^{\,0}$?

WE CAN INTEGRATE THIS FIRST-ORDER EQUATION:

$$\vec{y} = \vec{y}^{\,0} + \int_0^x \mathbb{A}\cdot\vec{y}(x')\,dx'$$

THIS IS AN EQUIVALENT INTEGRAL EQUATION FOR $\vec{y}(x)$. ALTHOUGH NO EASIER TO SOLVE THAN THE ORIGINAL DE., IT DOES SUGGEST AN APPROXIMATION SCHEME: MAKE A GUESS FOR $\vec{y}(x)$ AND SUBSTITUTE INTO THE RIGHT-HAND SIDE. USE THE RESULTING EXPRESSION AS AN UPDATED-GUESS, THEN ITERATE...

i.e.
$$\vec{y}^{(n)}(x) = \vec{y}^{\,0} + \int_0^x \mathbb{A}\cdot\vec{y}^{(n-1)}(x')\,dx'$$

START WITH $\vec{y}^{(0)} = \vec{y}^{\,0}$:

$$\vec{y}^{(1)} = \vec{y}^{\,0} + \int_0^x \mathbb{A}\cdot\vec{y}^{\,0}\,dx'$$
$$= \vec{y}^{\,0} + (\mathbb{A}x)\cdot\vec{y}^{\,0} = (\mathbb{I} + \mathbb{A}x)\cdot\vec{y}^{\,0}$$

AGAIN:

$$y^{(2)} = \vec{y}^{\,0} + \int_0^x \mathbb{A}\cdot\vec{y}^{(1)}(x')\,dx' = \vec{y}^{\,0} + \int_0^x \mathbb{A}\cdot(\mathbb{I} + \mathbb{A}x')\,\vec{y}^{\,0}\,dx'$$
$$= \vec{y}^{\,0} + (\mathbb{A}x)\vec{y}^{\,0} + \left(\frac{\mathbb{A}^2x^2}{2}\right)\vec{y}^{\,0} = \left(\mathbb{I} + \mathbb{A}x + \tfrac{1}{2}(\mathbb{A}x)^2\right)\cdot\vec{y}^{\,0}$$

KEEP GOING: $\lim\limits_{n\to\infty} \vec{y}^{(n)} = \left(\sum\limits_{n=0}^{\infty} \frac{\mathbb{A}^n x^n}{n!}\right)\cdot\vec{y}^{\,0} = e^{\mathbb{A}x}\cdot\vec{y}^{\,0}$

AND SO, $\Phi(x) = e^{\mathbb{A}x}$.

WE CAN USE THIS SAME IDEA (AT LEAST FORMALLY) EVEN IF $\mathbb{A}(x)$ IS NOT CONSTANT, WHICH WE'LL CONSIDER SHORTLY.

FIRST, HOW DO WE ACTUALLY CALCULATE $e^{\mathbb{A}x}$?

SEVERAL CASES.

1. IF A IS DIAGONAL: e.g. $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ THEN

$$e^{Ax} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} ax & 0 \\ 0 & bx \end{bmatrix} + \frac{1}{2}\begin{bmatrix} a^2x^2 & 0 \\ 0 & b^2x^2 \end{bmatrix} + \dots + \frac{1}{n!}\begin{bmatrix} (ax)^n & 0 \\ 0 & (bx)^n \end{bmatrix} + \dots$$

$$= \begin{bmatrix} 1+(ax)+\frac{1}{2}(ax)^2+\dots+\frac{1}{n!}(ax)^n+\dots & 0 \\ 0 & 1+(bx)+\frac{1}{2}(bx)^2+\dots+\frac{1}{n!}(bx)^n+\dots \end{bmatrix} = \begin{bmatrix} e^{ax} & 0 \\ 0 & e^{bx} \end{bmatrix}$$

2. IF A IS <u>DIAGONALIZABLE</u>, THEN $\overbrace{P^{-1} \cdot A \cdot P}^{\text{'SIMILARITY' TRANSFORM.}} = D$ ← DIAGONAL

↑ COLUMNS ARE EIGENVECTORS OF A.

THEN,

$$e^{A} = e^{P \cdot D \cdot P^{-1}} = I + (P \cdot D \cdot P^{-1}) + \frac{1}{2}(P \cdot D \cdot P^{-1})^2 + \dots + \frac{1}{n!}(P \cdot D \cdot P^{-1})^n + \dots$$

$$= I + P \cdot D \cdot P^{-1} + \frac{1}{2} P \cdot D^2 \cdot P^{-1} + \dots + \frac{1}{n!} P \cdot D^n \cdot P^{-1} + \dots$$

$$= P\left[I + D + \frac{1}{2}D^2 + \dots + \frac{1}{n!}D^n + \dots\right]P^{-1}$$

$$= P \cdot e^{D} \cdot P^{-1}$$

TO DEAL WITH <u>NONDIAGONIZABLE</u> MATRICES, WE CONSIDER POSSIBLE JORDAN (OR CANNONICAL OR NORMAL) - FORM DECOMPOSITIONS

JORDAN FORM: SIMILARITY TRANSFORM $A = P \cdot J \cdot P^{-1}$ RETURNS A MATRIX J OF JORDAN CANNONICAL FORM:

$$J = \begin{bmatrix} B_1 & & & 0 \\ & B_2 & & \\ & & \ddots & \\ 0 & & & B_n \end{bmatrix} \text{ WHERE}$$

$B = [\lambda]$ DISTINCT EIGENVALUE

$= \underbrace{\begin{bmatrix} \lambda & 1 & 0 & \\ 0 & \lambda & 1 & \\ & & \ddots & 1 \\ & & 0 & \lambda \end{bmatrix}}_{k \times k}$ FOR EIGENVALUE REPEATED 'k' TIMES

$= \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ COMPLEX EIGENVALUES.

1. REPEATED REAL EIGENVALUES LEAD TO JORDAN BLOCKS OF THE FORM

$$B = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}}_{aI} + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{N}$$

N IS <u>NILPOTENT</u>. FOR 'k' REPEATED EIGENVALUES, $N^n = 0$ FOR $n \geq k$.

IN THIS 2×2 EXAMPLE,

$$e^{\mathbb{B}x} = e^{a\mathbb{I}x} e^{\mathbb{N}x} = \begin{bmatrix} e^{ax} & 0 \\ 0 & e^{ax} \end{bmatrix} \Big[\mathbb{I} + \mathbb{N}x + \underbrace{\tfrac{1}{2}\mathbb{N}^2x^2 + \dots + \tfrac{1}{n!}\mathbb{N}^n x^n + \dots}_{\text{ALL VANISH } \mathbb{N}^n = \mathbb{0},\ n \geq 2} \Big]$$

$$= \begin{bmatrix} e^{ax} & 0 \\ 0 & e^{ax} \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{ax} & xe^{ax} \\ 0 & e^{ax} \end{bmatrix}$$

2. COMPLEX EIGENVALUES GIVE RISE TO JORDAN BLOCKS OF THE FORM

$$\mathbb{B} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}}_{a\mathbb{I}} + \underbrace{\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}}_{\mathbb{M}} \quad \text{so } e^{\mathbb{B}x} = \begin{bmatrix} e^{ax} & 0 \\ 0 & e^{ax} \end{bmatrix} \Big[\mathbb{I} + \mathbb{M}x + \frac{\mathbb{M}^2x^2}{2} + \dots + \frac{\mathbb{M}^n x^n}{n!} + \dots \Big]$$

NOTICE $\mathbb{M}^n$ HAS VERY REGULAR STRUCTURE:

$$\mathbb{M} = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \quad \mathbb{M}^2 = \begin{bmatrix} -b^2 & 0 \\ 0 & -b^2 \end{bmatrix} \quad \mathbb{M}^3 = \begin{bmatrix} 0 & -b^3 \\ b^3 & 0 \end{bmatrix} \quad \mathbb{M}^4 = \begin{bmatrix} b^4 & 0 \\ 0 & b^4 \end{bmatrix} \quad \mathbb{M}^5 = \begin{bmatrix} 0 & b^5 \\ -b^5 & 0 \end{bmatrix} \dots$$

SO

$$e^{\mathbb{M}x} = \begin{bmatrix} 1 - \frac{b^2x^2}{2} + \frac{b^4x^4}{4!} - \dots & b - \frac{b^3x^3}{3!} + \dots \\ -bx + \frac{b^3x^3}{3!} - \dots & 1 - \frac{b^2x^2}{2} + \frac{b^4x^4}{4!} - \dots \end{bmatrix} = \begin{bmatrix} \cos(bx) & \sin(bx) \\ -\sin(bx) & \cos(bx) \end{bmatrix}$$

ALTOGETHER,

$$e^{\mathbb{B}x} = \begin{bmatrix} e^{ax}\cos(bx) & e^{ax}\sin(bx) \\ -e^{ax}\sin(bx) & e^{ax}\cos(bx) \end{bmatrix}$$

IN SUMMARY, FOR $\frac{d}{dx}\vec{y} = \mathbb{A}\cdot\vec{y}$ WITH $\vec{y}(0) = \vec{y}^{\,0}$

THE SOLUTION IS: $\vec{y}(x) = e^{\mathbb{A}x}\cdot\vec{y}^{\,0}$

TO COMPUTE $e^{\mathbb{A}x}$, FIND EIGENVALUE/EIGENVECTOR PAIRS & WRITE $\mathbb{A}$ IN JORDAN FORM $\mathbb{A} = \mathbb{P}^{-1}\mathbb{J}\cdot\mathbb{P}$

SO THAT

$$\boxed{\vec{y}(x) = \mathbb{P}^{-1} e^{\mathbb{J}x}\, \mathbb{P}\, \vec{y}^{\,0}}$$

NON-AUTONOMOUS SYSTEMS

WE CAN USE THE SAME ITERATIVE METHOD AS ABOVE TO OBTAIN A FORMAL (i.e. NOT-SO-USEFUL) SOLUTION TO THE CASE WHERE THE COEFFICIENT MATRIX $\mathbb{A}(x)$ IS NON-CONSTANT. RE-WRITING THE DIFF. EQ AS AN INTEGRAL:

$$\underbrace{\frac{d\vec{y}}{dx} = \mathbb{A}(x)\cdot\vec{y}\,;\ \vec{y}(x_0) = \vec{y}^{\,0}}_{\text{DIFF. EQ. FOR } \vec{y}(x).} \iff \underbrace{\vec{y}(x) = \vec{y}^{\,0} + \int_{x_0}^{x} \mathbb{A}(x_1)\vec{y}(x_1)\,dx_1}_{\text{INTEGRAL EQ. FOR } \vec{y}(x).}$$

ITERATE THE INTEGRAL EQ, STARTING WITH $\vec{y}^{(0)} = \vec{y}^{\,0}$:

$$\vec{y}^{(1)}(x) = \vec{y}^{\,0} + \int_{x_0}^{x} \mathbb{A}(x_1)\vec{y}^{\,0}\,dx_1 = \vec{y}^{\,0} + \underbrace{\int_{x_0}^{x} dx_1}\,\mathbb{A}(x_1)\vec{y}^{\,0}$$

SIMPLIFIES THE NOTATION IF I KEEP THE VARIABLE OF INTEGRATION & ITS LIMITS GROUPED TOGETHER.

ITERATING AGAIN:

$$\vec{y}^{(2)}(x) = \vec{y}^{\,0} + \int_{x_0}^{x} dx_2\left[\mathbb{A}(x_2)\,\vec{y}^{(1)}(x_2)\right]$$

$$= \vec{y}^{\,0} + \int_{x_0}^{x} dx_2\,\mathbb{A}(x_2)\left[\vec{y}^{\,0} + \int_{x_0}^{x_2} dx_1\,\mathbb{A}(x_1)\vec{y}^{\,0}\right]$$

$$= \vec{y}^{\,0} + \int_{x_0}^{x} dx_2\,\mathbb{A}(x_2)\vec{y}^{\,0} + \int_{x_0}^{x} dx_2 \int_{x_0}^{x_2} dx_1\,\mathbb{A}(x_2)\mathbb{A}(x_1)\,\vec{y}^{\,0}$$

NOTICE:
$x_1 \in [x_0, x_2]$
$x_2 \in [x_0, x]$
SO, $x \geq x_2 \geq x_1$

KEEP GOING:

$$\lim_{N\to\infty} \vec{y}^{(N)}(x) = \lim_{N\to\infty}\left[\mathbb{I} + \sum_{n=1}^{N} \int_{x_0}^{x} dx_n \int_{x_0}^{x_n} dx_{n-1} \cdots \mathbb{A}(x_n)\mathbb{A}(x_{n-1})\cdots\mathbb{A}(x_1)\right]\cdot\vec{y}^{\,0}$$

IN PHYSICS, THIS IS CALLED THE 'ORDERED EXPONENTIAL' AND GIVEN THE SYMBOL:

$$\vec{y}(x) = \underbrace{\Big\lceil \exp\Big[\int_{x_0}^{x} \mathbb{A}(x')\,dx'\Big]\Big\rceil}\cdot\vec{y}^{\,0}$$

THESE OUTER BRACKETS MEAN FORM THE INFINITE SERIES IN THE PRECEDING LINE, WITH $x \geq x_n \geq x_{n-1} \geq \ldots \geq x_1$ ORDERING OF THE INTEGRALS.

IT IS $\underline{\text{VERY}}$ IMPORTANT TO NOTICE THAT, IN GENERAL,

$$\Gamma \exp\left[\int_{x_0}^{x} \mathbb{A}(x')dx'\right] \neq \exp\left[\int_{x_0}^{x} \mathbb{A}(x')dx'\right]$$

$\underline{\text{UNLESS}}$ THE MATRIX $\mathbb{A}(x_1)\mathbb{A}(x_2) = \mathbb{A}(x_2)\mathbb{A}(x_1)$ COMMUTES FOR ALL x_1 & x_2 OR, EQUIVALENTLY,

$$\mathbb{A}(x)\cdot\left[\int_{x_0}^{x} \mathbb{A}(x')dx'\right] = \left[\int_{x_0}^{x} \mathbb{A}(x')dx'\right]\cdot\mathbb{A}(x). \quad (*)$$

$\underline{\text{IF}}$ THAT COMMUTIVITY RELATION IS OBEYED, THEN:

$$\vec{y}(x) = \left[\exp\left[\int_{x_0}^{x} \mathbb{A}(x')dx'\right]\right]\cdot\vec{y}^{\,0} \quad (**)$$

$\underline{\text{EXERCISE}}$: 1. SHOW THAT $(**)$ SOLVES THE DIFF. EQ. $\frac{d\vec{y}}{dx} = \mathbb{A}(x)\vec{y}$; $\vec{y}(x_0) = \vec{y}^{\,0}$ IF $(*)$ HOLDS. $\underline{\text{HINT}}$: LET $\mathbb{B}(x) = \int_{x_0}^{x} \mathbb{A}(x')dx'$ AND USE THE SERIES REP. FOR $\exp \mathbb{B}(x)$.

2. SHOW THAT FOR $\mathbb{A}(x) = \begin{bmatrix} f(x) & g(x) \\ g(x) & f(x) \end{bmatrix}$, THE MATRIX AND ITS INTEGRAL $(*)$ COMMUTE.

3. RE-WRITE MATHIEU'S EQUATION: $y'' + [a + 2q\cos(x)]y = 0$ $\quad y(0)=1,\ y'(0)=0$ AS A SYSTEM OF 1^{ST} ORDER DIFF. EQS, ~~AND SHOW THAT~~ WITH COEFFICIENT MATRIX $\mathbb{A}(x)$, THE SOLUTION IS $\underline{\text{NOT}}$ GIVEN BY $\exp\left[\int_0^x \mathbb{A}(x')dx'\right]\vec{y}^{\,0}$

—//.