

AMATH 353: *Partial Differential Equations 1*  
or  
*Methods in Applied Mathematics 2*

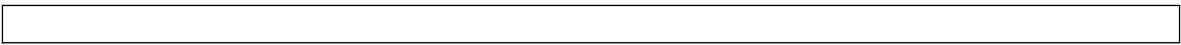
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# Preface

These lecture notes were developed for AMATH 353 : Partial Differential Equations 1 at the University of Waterloo. They were written in the Spring of 2012 Semester. I would like to thank Jenn Hernes for her helpful comments in fixing up the manuscript.



# Modelling with PDEs

## 1.1 Introduction

A *Partial Differential Equation*, or PDE, is an equation that relates a function, say  $u(x, t)$  with its partial derivatives,

$$F(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots) = 0.$$

These arises because often it is easier to use basic principles to build a model that describes a process using partial derivatives and relates them in some way. One example of a principle we will discuss shortly is a *conservation law*. Then, using various analytical (or numerical) techniques we can obtain exact or approximate solutions.

In the above equation the independent variables are  $x$ , for space, and  $t$ , for time. In general there can be more than two independent variables if we are in higher dimension and we could have two or more dependent variables, say  $u(x, t)$  and  $v(x, t)$ . If there is any hope of finding a unique solution we need as many equations as we have dependent variables otherwise the system is said to be underdetermined. Also to find a solution certain conditions that need to be imposed like at the boundary (if there is one) and/or the initial time (if we looking at solutions that change in time). It is for this reason that these equations are sometimes known as Initial Boundary Value Problems (IBVPs for short). Note that we do not need a time like variable and will sometimes have only spatial variables. In that case the PDE is more properly described as a Boundary Value Problem (BVP). This is to be contrasted with Initial Value Problems (IVPs) that you studied extensively in AMATH 250/251.

The PDE or system of PDEs is said to be linear if  $F(u(x, t))$  has a linear dependency on  $u(x, t)$  and its partial derivatives. If it depends nonlinearly on any of terms involving  $u(x, t)$ , then it is said to be nonlinear. The vast majority of PDEs that are derived to describe some aspect of nature are in fact nonlinear and are incredibly difficult if not impossible to solve exactly. Under certain assumptions we can find linear equations that approximate the true behaviour and these are more easily solved. Better to find an approximate solution than no solution at all! The reason why there is such a vast difference between linear and nonlinear equations is that if we have two linear solutions



to a homogeneous PDE then their superposition is also a solution. This is not the case with nonlinear equations. In this course we present mostly the linear theory because it is much more well established but we do delve into nonlinear equations where we can since they are of universal importance.

## 1.2 Conservation Laws

[This section is based primarily on section 1.2 of [2]]

Hundreds of years ago it was believed that matter could spontaneously generate itself. Today it is well accepted that you don't get something from nothing. That is to say that if you are measuring a field, say heat, mass of a chemical or pretty much anything else, that the the total amount in a given region can only change by the “stuff” moving out and/or in. In this section we derive a mathematical equation that says exactly this in one spatial dimension since it begins to illustrate how we use mathematics to describe the world. Then we show how we can derive a similar equation in higher dimension. This is more complicated in that it involves using our knowledge of vector calculus. If you are feeling uncomfortable with this then I suggest you brush up on the material from AMATH 231 (Vector Calculus) right away.

### 1.2.1 One-Dimension

A *conservation law* is a physical principle that when written mathematically can be expressed either as an *integral (global) or differential (local) equation*. This is also called a *governing equation* since it forces a constraint on the field that perhaps let us determine how the solution changes in time and space. That is if we can solve the equation, which is not always the case. In this course I want you to learn not only how to solve PDEs but how to derive them since equations don't simply fall from the sky. Often we must model the phenomenon that we are interested in before we can begin to solve the equations.



Figure 1.1: A cylindrical domain.

To begin consider a long but very thin cylindrical container that is pointed in the  $x$ -direction. The thinness of the rod is necessary to guarantee that the properties of the rod do not vary in the cross section but can vary along the rod. Suppose there is a quantity (or field)  $u(x, t)$  that can change along the rod or with time but does not change with distance from the centre. For concreteness let us say that  $u(x, t)$  is the density or concentration that measures the amount of something per unit volume. If we are talking about a chemical then the units would be mass/length<sup>3</sup> for example. If the

cross section of the tube is uniform, say  $A$  (units of  $\text{length}^2$ ), then the amount of the substance in the interval  $I$ , between  $x = a$  and  $x = b$ , can be written as the following integral,

$$\text{Total amount of quantity in } I \equiv \int_a^b u(x, t) A \, dx.$$

Note that the integrand is a *global property* because it sums up a property of the material over a length. We assume that  $u(x, t)A$  that has units of  $(\text{mass}/\text{length}^3) \cdot \text{length}^2 = \text{mass}/\text{length}$ . More generally this can be written as,

$$[uA] = \frac{\text{amount}}{\text{volume}} \cdot \text{area} = \frac{\text{amount}}{\text{length}}.$$

If we integrate with respect to  $x$  then we are multiplying by a distance and therefore the final units in the amounts of  $u$ .

In the absence of any flux in or out of the domain the total mass (or amount in general) over the entire domain must necessarily be conserved with respect to time. However, if there is a connection between the rod and the exterior world then the amount of the substance will increase if there is some material coming in from the left or the right ends of the interval. Conversely, the amount of the substance will decrease if the substance is moving out from either the left or right ends of the interval. In these two cases we say there is a *flux* in and out of the system, respectively.

To write this down mathematically we need to define *the flux*, what we denote with  $\phi(x, t)$ , of  $u$  at position  $x$  and time  $t$ . The dimensions of the flux are equal to the dimensions of  $u$  per time, which is  $\text{mass}/(\text{length}^2 \cdot \text{time})$  in our example but in general using the standard convention,

$$[\phi] = \frac{\text{amount}}{\text{area} \cdot \text{time}}.$$

That is to say the flux is the amount of the substance per unit area and per time. We assume the convention that  $\phi(x, t)$  is positive (negative) means that the flow at  $x$  is in the positive (negative)  $x$ -direction. Therefore, at a particular time  $t$  the amount of the rate at which the quantity is flowing into the interval is,

$$\text{Net rate that the quantity flows into } I \equiv A\phi(a, t) - A\phi(b, t).$$

This quantity has units of amount/time.

In addition to the quantity changing because of transport there is also the possibility that it is being added (there is a source) or it is being removed (there is a sink) at a given location at a given time. To describe this mathematically we define  $f(x, t, u)$  to be the *source function* that describes how much of  $u$  is generated per volume per time,

$$[f] = \frac{\text{amount}}{\text{volume} \cdot \text{time}}.$$

One can also imagine a case where the source depends on the derivatives but we assume the simpler case here. Therefore, the rate at which  $u$  is created or destroyed at  $x$  and time  $t$  can be written as,

$$\text{Rate that the quantity is produced in } I \equiv \int_a^b f(x, t, u(x, t)) A \, dx,$$

which also must have units of amount per time.

In words, the fundamental conservation law states that

$$\boxed{\begin{array}{c} \text{Net rate of change} \\ \text{of the amount in } I \end{array}} = \boxed{\begin{array}{c} \text{Net increase by} \\ \text{transport into } I \end{array}} + \boxed{\begin{array}{c} \text{Rate at which the amount} \\ \text{is produced in } I \end{array}}.$$

Intuitively this should be easy to understand. Given all the work we have done we are now able to state this rather succinctly in the language of mathematics as,

$$(1.2.1) \quad \boxed{\frac{d}{dt} \int_a^b u(x, t) \, dx = \phi(a, t) - \phi(b, t) + \int_a^b f(x, t, u(x, t)) \, dx},$$

after dividing by  $A$  since it is constant throughout. This equation says that the rate of change of the total amount of  $u$  in the interval  $I$  in time is equal to the sum of the rate that the amount of  $u$  produced by transport and sources. This equation is an example of a *global conservation law* in one spatial dimension. Note that this equation does not require that  $u(x, t)$  is differentiable in space since we only integrate it with respect to  $x$ .

Global properties are very helpful to learn about the dynamics of the whole system. However, if we want to know what is going on at every position we need to obtain a local or differential version. If we make certain assumptions on  $u(x, t)$  then we can take this integral equation into a differential equation. In particular, we require that its first order partial derivatives are continuous. Also, we recall the two parts of the Fundamental Theorem of Calculus (see MATH 127/137/147 for details) which state,

$$(1.2.2) \quad \int_a^b \frac{\partial \phi}{\partial x} \, dx = \phi(b, t) - \phi(a, t),$$

$$(1.2.3) \quad \frac{d}{dt} \int_a^b u(x, t) \, dx = \int_a^b \frac{\partial u}{\partial t} \, dx.$$

If we use the second part of the theorem on the left of equation (1.2.1) and the first part of the theorem on the right hand side of the equation, we get after moving everything to the same side

$$\begin{aligned} \int_a^b \frac{\partial u}{\partial t}(x, t) \, dx &= - \int_a^b \frac{\partial \phi}{\partial x}(x, t) \, dx + \int_a^b f(x, t, u(x, t)) \, dx, \quad \text{or} \\ \int_a^b \left[ \frac{\partial u}{\partial t}(x, t) + \frac{\partial \phi}{\partial x}(x, t) - f(x, t, u(x, t)) \right] \, dx &= 0. \end{aligned}$$

But if this is true for all intervals  $I = [a, b]$  then we require that the integrand is exactly zero everywhere and for all time.

To see why this is true you can argue this by contradiction. If it is non-zero somewhere then we can pick an interval that would violate the above equation. Therefore, using our *global conservation law* we deduce a *local conservation law* which is in the form of a differential equation,

$$(1.2.4) \quad \boxed{\frac{\partial u}{\partial t}(x, t) + \frac{\partial \phi}{\partial x}(x, t) = f(x, t, u(x, t))},$$

for all  $x$  and  $t$  in our domain and interval of time in question. This equation shows that the local flux of  $u$  in time must be balanced by the rate of change of the flux in space and the source term. Often  $\partial \phi / \partial x$  is referred to as the flux in the equation since it is due to the flux across the domain. Also, the source term  $f$  is called a *reaction term* (in chemical contexts) or growth or interaction (in biological contexts).

Observe that this is an example of a PDE but it is not closed in the sense that we do not know the relationship between the flux,  $\phi$ , and the dependent variable,  $u$ . This relationship is referred to as a *constitutive relation* and when specified it then yields a closed system that perhaps we can solve.

If our solution is smooth then equations (1.2.1) and (1.2.4) are equivalent. However, if there is a discontinuity in the solution (as we will see can occur in some problems), the second equation is no longer valid but the first is. Therefore, the global equation is more general but it is harder to solve and that is why we usually prefer the local expression. Regardless, the latter are the ones that are almost always considered in courses and research.

## 1.2.2 Higher-Dimensions

If we want to consider a higher-dimensional problem we can still begin with equation (1.2.1) but now we must figure out how to describe each term mathematically. To do this in say three-dimensions let us suppose that any point in space is described as  $(x, y, z)$ . Then, the density must be written as  $u(x, y, z, t)$ , or  $u(\vec{x}, t)$  for brevity. Let the volume in question be denoted by  $V$  that has a smooth boundary  $\partial V$ . Using our knowledge of multivariable calculus we can write the total amount of the substance as,

$$(1.2.5) \quad \text{Total amount in } V \equiv \iiint_V u(x, y, z, t) \, dV.$$

Note that  $dV = dx dy dz$  is the volume element in three dimensions.

A second term is the source function,  $f(x, y, z, t, u)$ . It is basically the same as in the one-dimensional example but here we must also integrate over the entire volume,

$$(1.2.6) \quad \text{Rate that } u \text{ is produced by source in } V \equiv \iiint_V f(x, y, z, t, u) \, dV.$$

The third and final term that we must deal with is the flux. This is more complicated because now the flux must have a magnitude and a direction,  $\vec{\phi}(\vec{x}, t)$ . Following convention we use  $\hat{n}(\vec{x})$  to denote the unit outwards normal of the boundary at the position  $\vec{x}$  and time  $t$ . Therefore, the outward flux through the boundary is,

$$(1.2.7) \quad \text{Net outward flux through } \partial V \equiv \iint_{\partial V} \vec{\phi}(\vec{x}, t) \cdot \hat{n}(\vec{x}, t) dA.$$

If we combine these three equations we can rewrite equation (1.2.1) as,

$$(1.2.8) \quad \frac{d}{dt} \iiint_V u(x, y, z, t) dV = - \iint_{\partial V} \vec{\phi}(\vec{x}, t) \cdot \hat{n}(\vec{x}, t) dA + \iiint_V f(x, y, z, t, u) dV.$$

This is the *global conservation* law that governs our system and again does not require that the solution or the flux are smooth. If we furthermore assume that the solution is smooth then we can derive a PDE. To do this in a useful manner we must first recall the *Divergence theorem* which allows us to rewrite the flux as,

$$\iiint_V \vec{\nabla} \cdot \vec{\phi} dV = \iint_{\partial V} \vec{\phi}(\vec{x}, t) \cdot \hat{n}(\vec{x}, t) dA.$$

Using this we can rewrite the global conservation law as

$$(1.2.9) \quad \frac{d}{dt} \iiint_V u(x, y, z, t) dV = - \iiint_V \vec{\nabla} \cdot \vec{\phi} dV + \iiint_V f(x, y, z, t, u) dV.$$

These integrals can be combined to yield,

$$(1.2.10) \quad \iiint_V \left[ \frac{\partial u}{\partial t}(x, y, z, t) + \vec{\nabla} \cdot \vec{\phi} - f(x, y, z, t, u) \right] dV = 0.$$

In this form we can apply the same argument as before that for this to be true for any arbitrary volume we need that the integrand is identically equal to zero. This yields our local or differential form of the conservation law,

$$(1.2.11) \quad \boxed{\frac{\partial u}{\partial t}(x, y, z, t) + \vec{\nabla} \cdot \vec{\phi} = f(x, y, z, t, u)}.$$

If we compare this with equation (1.2.4) then we see how this is the natural higher-dimensional extension of what we saw previously.

### 1.3 Constitutive Relations

[This section is based primarily on section 1.3 of [2]]

Equations (1.2.4) and (1.2.11) are each examples of PDEs where the unknowns are the state variable  $u$  and the flux  $\phi$ , which must be expressed in terms of  $u$  and possibly

space and time. The *constitutive relation* or *equation of state* is an relationship that specifies this precisely. The most general way to write it for our purposes would be,

$$\phi(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}).$$

In some sense it is the backbone of the model in that it gives the equation it's personality and its form. The fundamental law of conservation is true for many cases and is correct given the assumptions that we made. The constitutive relation is something obtained by empirical observations as to how things tend to flow for a particular problem. In this section we discuss several classical choices that gives rise to famous equations that we will solve throughout the course.

### 1.3.1 Diffusion Equation

In the one-dimensional case, in the absence of any sources the conservation law becomes,

$$\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} = 0,$$

for all  $x$  in the domain and  $t > 0$ . We observe that if we have a concentration of a chemical, molecular motion tends to redistribute the chemical and eventually yield a uniform distribution of the substance. The rate at which things are transported depends on the gradient of the density and it always goes from high density to low density. This idea is well encompassed in *Fick's law* which assumes that,

$$(1.3.1) \quad \phi(x, t) = -D \frac{\partial u}{\partial x}(x, t),$$

where  $D$  is the diffusion constant that must have units of,

$$[D] = \frac{\text{length}^2}{\text{time}}.$$

This law is not something that we can derive but is something that is observed to be true in many physical, biological and chemical processes.

If we substitute the above constitutive law into the conservation principle than we get a PDE that is only in terms of one unknown variable, namely  $u(x, t)$ ,

$$(1.3.2) \quad \boxed{\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}}.$$

This shows that for the solution to evolve it is necessary that the concavity is the solution is non-zero. For example if we have a minimum (maximum) the concavity is positive (negative) and locally the solution will tend to increase (decrease). It does so until  $u$  is a linear function of space everywhere. When this occurs we have realized a

steady solution.

**Question:** What would the equation look like if diffusion varied in space, i.e.  $D(x)$ ?

Even though we don't know how the solution is going to change by looking at the equation we can learn about the behaviour of the solution. If the solution changes in time then it is the diffusion constant and a typical length scale that will determine the rate at which things change. This can be obtained by asking the rather simple question, how do we get the units of time by combining diffusion,  $D$ , and length,  $L$ ? The only way to do this is as follows,

$$T = \frac{L^2}{D}.$$

Thus, we expect the time scale of the solution to be the length scale squared divided by the diffusion rate. The larger the length the larger the time scale. The larger the diffusion, the shorter the time scale. When we find an analytical solution to the diffusion equation we will verify what we have established here through a heuristic argument. Furthermore, we will explain why diffusion acts different on different length scales.

### 1.3.2 Heat Equation

Suppose that  $u = u(x, t)$  is the thermal energy density (units of energy per volume) in a heat-conducting object. From thermodynamics we have that,

$$u = \rho CT,$$

where  $\rho$  is the mass density (mass per unit volume),  $C$  is the specific heat of the object (energy per unit mass per degree) and  $T = T(x, t)$  is the temperature in degrees. Again, we neglect sources and then find that the conservation law in local form is,

$$\frac{\partial}{\partial t}(\rho CT) + \frac{\partial \phi}{\partial x} = 0,$$

where  $\phi(x, t)$  is the flux of energy. For heat it is typical to use something similar to Fick's law that is called *Fourier's law of heat conduction* that supposes,

$$\phi = -K \frac{\partial T}{\partial x}(x, t),$$

where  $K$  is the *thermal conductivity*. Mathematically, this is the same as Fick's law but we give it a different name because it is a different physical principle and it does give credit to Fourier who stated this for heat.

We substitute Fourier's law into the conservation principle and get,

$$\frac{\partial}{\partial t}(\rho CT) = K \frac{\partial^2 T}{\partial x^2}.$$

If we assume that  $\rho$  and  $C$  are constant then we can define  $k = K/(\rho C)$  and get the famous *heat equation*

$$\boxed{\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}}.$$

The only difference between this equation and equation (1.3.2) is the name of the variable and the diffusion coefficient. It is for this reason that we often refer to this as the diffusion equation.

Observe that in this case the time scale is,

$$T = \frac{L^2 \rho C}{K}.$$

### 1.3.3 Reaction-Diffusion Equation

To include sources in the context of Fick's law is relatively easy. The conservation principle is

$$\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} = f(x, t, u),$$

which when substituted in the constitutive law yields

$$(1.3.3) \quad \boxed{\frac{\partial u}{\partial t} - D \frac{\partial^2 u}{\partial x^2} = f(x, t, u)}.$$

This is called a *reaction-diffusion equation* since it has both diffusion, set by  $D$ , and reactions that can occur because of the sources on the right-hand-side (RHS). This equation is only nonlinear in the case where the RHS is nonlinear. In the next subsection we will look at a particular example of a biological system that is a nonlinear reaction-diffusion equation.

### 1.3.4 Fisher's Equation

In early studies of DE the first equation that is typically studied is the equation for exponential growth,

$$\frac{du}{dt} = ru,$$

where  $r$  is the growth (or death) rate of the population. We know that the general family of solutions to this equation is the exponential,  $u = ae^{rt}$ , where the constant  $a$  is set by the initial conditions. If the growth rate is positive the solution grows exponentially, which might be realistic in the case of small population sizes but ceases to be when the population gets large enough that its food supply can no longer sustain the population.

This limitation motivates considering a slightly more realistic model, namely the *logistic equation*,

$$\frac{du}{dt} = ru \left(1 - \frac{u}{K}\right),$$



where  $K$  is not the thermal constant but the *carrying capacity* of the population. The right-hand-side is positive for  $0 < u < K$  and therefore if the population size is relatively small the solution will grow. However, if the population exceeds the carrying capacity, i.e.  $u > K$  the right-hand-side is negative and the solution will decrease. This is because the environment cannot sustain a population size that is too large.

If we make the logistic assumption for the source in the reaction-diffusion equation we get that the conservation principle is

$$\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} = ru \left(1 - \frac{u}{K}\right).$$

After we insert Fick's law then we get what is known as *Fisher's equation*,

$$(1.3.4) \quad \boxed{\frac{\partial u}{\partial t} - D \frac{\partial^2 u}{\partial x^2} = ru \left(1 - \frac{u}{K}\right)}.$$

This was derived first by R.A. Fisher in the study of genetic.

### 1.3.5 Burger's Equation

We again neglect source terms and consider a constitutive relation that is slightly more general than Fick's law,

$$\phi = -D \frac{\partial u}{\partial x} + Q(u).$$

The first part on the RHS is exactly Fick's law but in addition we allow for the possibility of the flux depends on the magnitude of the solution as well through  $Q(u)$ .

If we substitute this into the conservation principle we get

$$(1.3.5) \quad \frac{\partial u}{\partial t} - D \frac{\partial^2 u}{\partial x^2} + \frac{\partial Q}{\partial x}(u) = 0.$$

This shows that the evolution of  $u(x, t)$  is set by two terms: 1) diffusion by Fick's law and 2) an extra term that is due to the transport that we will call *advection*. There are many choices for  $Q(u)$  but perhaps the most famous choice is  $Q(u) = u^2/2$ . In this case the above equation becomes,

$$(1.3.6) \quad \boxed{\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = D \frac{\partial^2 u}{\partial x^2}}.$$

This is *Burger's equation* and is pivotal in the study of fluid mechanics. It includes both diffusion, RHS, and nonlinear advection, LHS and consequently is known as a *nonlinear-advection-diffusion* equation. Note that advection is just a fancy word for transport. In the special case with  $D = 0$  we recover what is called the *inviscid Burger's equation*,

$$(1.3.7) \quad \boxed{\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0}.$$

This could be described as a *nonlinear advection* equation.

### 1.3.6 Advection Equation

The simple assumption that the flux depends linearly on the solution,  $u$ , has a constitutive law of,

$$\phi = cu,$$

where  $c$  has units of speed. With this the conservation law is,

$$(1.3.8) \quad \boxed{\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0}.$$

This is the *linear advection equation*. It is a simplified version of the inviscid Burger's equation in that it is the linear analogue.

**Question:** How do you construct a nonlinear-advection-diffusion-reaction equation?

### 1.3.7 Diffusion in Three-Dimensions

The higher-dimension version of the conservation principle is

$$(1.3.9) \quad \frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{\phi} = f(x, y, z, t, u).$$

The higher-dimension version of Fick's law is,

$$(1.3.10) \quad \vec{\phi} = -D\vec{\nabla}u,$$

that is to say the flux is in the opposite direction of where  $u$  increases most rapidly. By convention we define the Laplacian operator to be the divergence of the gradient,

$$\nabla^2 = \vec{\nabla} \cdot \vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Therefore, when we substitute Fick's law into the conservation principle above we get,

$$(1.3.11) \quad \frac{\partial u}{\partial t} - D\nabla^2 u = f(x, y, z, t, u).$$

This is the three-dimensional version of the reaction-diffusion equation. If there are no sources,  $f = 0$ , then we get the diffusion equation,

$$(1.3.12) \quad \frac{\partial u}{\partial t} = D\nabla^2 u.$$

Note that steady solutions of this equation are governed by the equation,

$$(1.3.13) \quad \boxed{\nabla^2 u = 0},$$

which is referred to as *Laplace's equation*.

## 1.4 Wave Equation

[This section is based primarily on section 4.2 of [1]]

An alternative way to derive PDEs is through using Newton's law which says that the mass times the acceleration of a particle is equal to the sum of the forces. Mathematically it can be written as,

$$F = ma,$$

where  $m$  is a mass,  $a$  is the acceleration of a particle and  $F$  is the sum of the forces.

To make things concrete consider a string that is stretched tightly in the horizontal. Define,

$$y = u(x, t),$$

to be the displacement of the string at position  $x$  and time  $t$  away from the equilibrium position. Consider a subset of the string that has end points of  $x_0 = x - \Delta x$  and  $x_1 = x + \Delta x$  and the centre is therefore at  $x$ . The parameters that describe the string are listed below,

$\rho(x)$  = density of the string with units of mass/length<sup>3</sup>,

$A$  = cross-sectional area of the string with units of length<sup>2</sup>,

$T$  = string tension  $\left(\frac{F}{A}\right)$  with units of Newton/length<sup>2</sup>,

$u(x, t)$  = vertical deflection about the state of no test with units of length.

Therefore, the mass of the string is,

$$\text{Mass} \equiv \rho(x)A\Delta x.$$

For simplicity we assume that  $A$  is constant.

The vertical component of the acceleration can be written as,

$$\text{Acceleration} \equiv \frac{\partial^2 u}{\partial t^2}.$$

The tension at the left point of the interval is parallel to the string and acts outwards, which we denote with  $T(x - \Delta x, t)$ . Similarly, at the right end of the interval the tension acts parallel to the end of string with a magnitude of  $T(x + \Delta x, t)$ . As well, we define the angle that the string makes with the horizontal at these two endpoints is  $\theta(x - \Delta x, t)$  and  $\theta(x + \Delta x, t)$ , respectively. To find the components of tension that are in the vertical we use trigonometry to find that at the left and right ends of the interval we have,

$$\begin{aligned} & -T(x - \Delta x, t)A \sin \theta(x - \Delta x, t), \\ & T(x + \Delta x, t)A \sin \theta(x + \Delta x, t). \end{aligned}$$

Notice that the first is negative because it acts to the left.

For the sake of simplicity, not because it is realistic, we assume that the string is perfectly flexible so that there is no resistance or friction present. If the only two forces that act are gravity and tension, they must balance in both the horizontal and vertical directions. Newton's second law can be written as

$$ma = \text{Force of Tension} + \text{Force of Gravity}.$$

If we substitute into this equation we get

$$\rho(x)A\Delta x \frac{\partial^2 u}{\partial t^2} = AT(x + \Delta x, t) \sin \theta(x + \Delta x, t) - AT(x, t) \sin \theta(x, t) - \rho(x)A\Delta x Q(x, t).$$

We used  $Q(x, t)$  to represent the force of gravity. In general it can be any body force.

If we divide by  $A\Delta x$  and then take the limit as  $\Delta x \rightarrow 0$  we get,

$$\begin{aligned} \rho(x) \frac{\partial^2 u}{\partial t^2} &= \lim_{\Delta x \rightarrow 0} \frac{T(x + \Delta x, t) \sin \theta(x + \Delta x, t) - T(x, t) \sin \theta(x, t)}{\Delta x} - \rho(x)Q(x, t), \\ \rho(x) \frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial x} [T(x, t) \sin \theta(x, t)] - \rho(x)Q(x, t). \end{aligned}$$

Note that we used the definition of the derivative to rewrite the RHS.

The above equation is not closed because it involves the angle and the displacement. To rewrite everything in terms of one variable, preferably  $u$ , we need to relate these two. For any displacement we can say that,

$$\frac{\partial u}{\partial x} = \tan \theta = \frac{\sin \theta}{\cos \theta}$$

From this we deduce that,

$$\sin \theta = \frac{\frac{\partial u}{\partial x}}{\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}}.$$

Therefore, Newton's second law yields the following equation

$$\rho(x) \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \left[ T(x, t) \frac{\frac{\partial u}{\partial x}}{\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}} \right] - \rho(x)Q(x, t).$$

This is a rather complicated and nonlinear PDE that is both second order in space and time.

If we assume that the tension is constant and expand out the derivative

$$\begin{aligned} \frac{\partial}{\partial x} \left( \frac{u_x}{\sqrt{1 + u_x^2}} \right) &= \frac{\sqrt{1 + u_x^2} u_{xx} - 2u_{xx} u_x^2 (1 + u_x^2)^{-1/2}}{1 + u_x^2}, \\ &= \frac{u_{xx}}{(1 + u_x^2)^{3/2}}. \end{aligned}$$

which means that the PDE can be written as,

$$\rho(x) \frac{\partial^2 u}{\partial t^2} = \frac{T}{\left(1 + \left(\frac{\partial u}{\partial x}\right)^2\right)^{3/2}} \frac{\partial^2 u}{\partial x^2} - \rho(x)Q(x, t).$$

If the displacement is small, then  $\theta$  is also small and we can use our Taylor expansions for trigonometric functions to approximate the above to get  $\tan \theta \approx \theta$ . Therefore, in the case of small amplitude we get that Newton's law yields the following governing equation,

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho(x)} \frac{\partial^2 u}{\partial x^2} - Q(x, t).$$

Finally, if we assume that the density is constant and that gravity is much smaller than the tension we get the *wave equation*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

where  $c^2 = T/\rho$  has the units of speed squared.

If we assume that the density is constant in the nonlinear case we get a similar equation but with a nonlinear coefficient,

$$c^2 \left( \frac{\partial u}{\partial x} \right) = \frac{T}{\rho \left( 1 + \left( \frac{\partial u}{\partial x} \right)^2 \right)^{3/2}}$$

It is important to note that this is a linear equation because we assumed that the amplitude of the displacement is small. In general this is not the case and you would get a nonlinear evolution equation. Also, you can easily include resistance to get an equation that is dissipative. This is something that you will explore on the first assignment.

### 1.4.1 Boundary Conditions

To find a unique solution we must impose boundary and initial conditions. Here we discuss four different types that are popular choices.

- 1) If the string has ends that are clamped down then the appropriate boundary conditions are of the *Dirichlet type*,

$$u(0, t) = 0, \quad \text{and} \quad u(L, t) = 0,$$

if the ends are located at  $x = 0, L$ . If the left-end is moving up and down at a prescribed rate this can be prescribed as,

$$u(0, t) = f(t),$$

where  $f(t)$  is the height of the left end-point at time  $t$ . This is a non-homogeneous Dirichlet condition

- 2) It is possible the left-end is free to move but always stays horizontal,

$$\frac{\partial u}{\partial x}(0, t) = 0.$$

This is a *Neuman boundary condition*. In the context of the diffusion or heat equation this corresponds to no-flux at the boundary. So instead of losing the stuff at the boundary it would get reflected, or possibly accumulate at the boundary.

- 3) A mixed or Robin boundary condition is of the form,

$$a \frac{\partial u}{\partial x}(0, t) + bu(0, t) = 0.$$

- 4) Periodic boundary conditions. That is to say the solution is equal at the left and right ends of the domain.

Each of these boundary conditions gives rise to very different motion and therefore we should expect that the solution depends inherently on what condition is imposed.



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