Welcome to Physics 242

Electricity and Magnetism 1

Instructor: Hartwig Peemoeller (room 366)
peemoell@uwaterloo.ca)

Office hours: Tuesdays, 10:30-11:30
Wednesday, 9:30-10:30

Some administrative stuff (see LEARN for details):

- Lectures: T, Th in RCH 309, 8:30 - 9:50
  -- attending lectures is a course requirement

- Tutorials: will be arranged as the course progresses

- Assignments: 6 to 8 over the term
  – hand in by depositing in drop-off box (opposite room 204)
  -- marked assignments will be placed outside of my office (see note re assignment returns under administrative details on LEARN)

- Texts: “Elements of Electromagnetism” by Sadiku, 5th edition and

“Physics for Scientists and Engineers 7th edition” by Serway and Jewett, or equivalent text

(e.g., “Essential University Physics” by Wolfson

“Fundamentals of Physics” by Halliday, Resnick and Walker

“University Physics 13th edition” by Young and Freedman)

- Course web site: https://learn.uwaterloo.ca

- Midterm test: 90 minutes, time and place will be arranged shortly
- Final exam: 150 minutes, scheduled by registrar
- Course mark:

  Assignment 20%

  Midterm test 20%

  Final exam 60%

- Academic regulations: carefully read the academic regulations detailed on the PHYS 242 LEARN site

- Note on PHYS 242 lecture format
  - good deal of material on blackboard
  - most power point slides or overheads I present in class will be placed on LEARN before the material is presented in class

  - the problems/examples done in class will not be available on LEARN (only the example/problem statements will be placed there).

  - none of the problems/examples considered in the tutorials will be placed on LEARN (only the example/problem statements will be placed there).

- Course Summary (details are on LEARN):

-- from the prescribed texts -

- Electrostatic E, F_E , Coulomb's Law
- Electric Flux density, Gauss's law
- Electric potential V
- Maxwell's Eq
- Energy density
- E fields in materials
- Convection and Conduction Currents
- Polarization in dielectrics
- Continuity equation, boundary conditions
- Faraday's Law, Induced emf and Electric Fields
- Inductance, RLC circuits
- AC circuits, RLC circuits, resonance, power
- augmented by lecture material

- Note re course content

-- ALL material covered in the following:

1) lectures
2) lecture notes on LEARN
   (updates will be added throughout term)
3) textbook material (as detailed on LEARN)
4) assignments

(At first glance, the course outline for PHYS 241 or for the E&M part of PHYS 122 looks rather similar to that for PHYS 242)

- In this course we will

i) Develop a more rigorous mathematical framework for the physics of electricity (some magnetism) than was done in your introductory course on E&M
(for example, the divergence and curl of the E field, as well as Maxwell's equations will be explored in some depth)

ii) Electric fields in material space will be looked at in depth

ii) AC circuits will be looked at in depth

I Electrostatic Fields

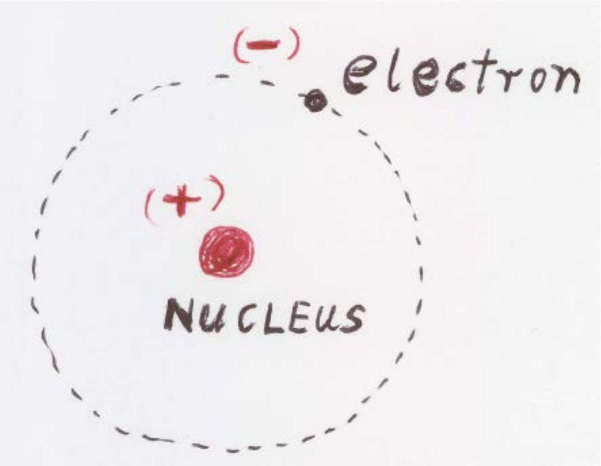
- First a few thoughts re electricity and magnetism:

- Electromagnetism is one of the fundamental forces

(strong interaction, electromagnetic force, weak force, gravitational force)

- Electric and magnetic phenomena are all around us and touch many aspects of everything!

- atoms
- (their very existence) :



Spectacular example of electric phenomenon (charge flow):

- Lightning -

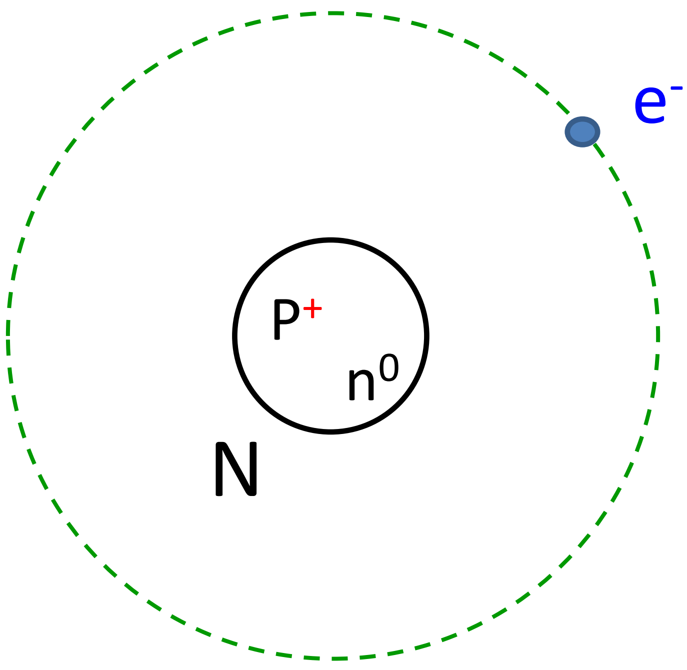
And of electromagnetic phenomenon:

- Northern Lights -

Examples of electromagnetic effects and applications:

- Lightning, northern lights
- cellular membrane charge/potential
- ECG, EEG
- Kitchen toaster
- Defibrillator
- Radio, TV, IPOD, cellphone, computers
- Electric machines, motors, generators
- Lasers, superconducting rail transit systems, etc., etc.

A starting point for our discussion:
Charges and forces between charges



e^- : $q = -1.60\times10^{-19}$ Coulomb

$m_e = (1/1830)\ m_p$

P^+ : $q = +1.60\times10^{-19}$ Coulomb

$m_p = 1.67\times10^{-27}$ kg

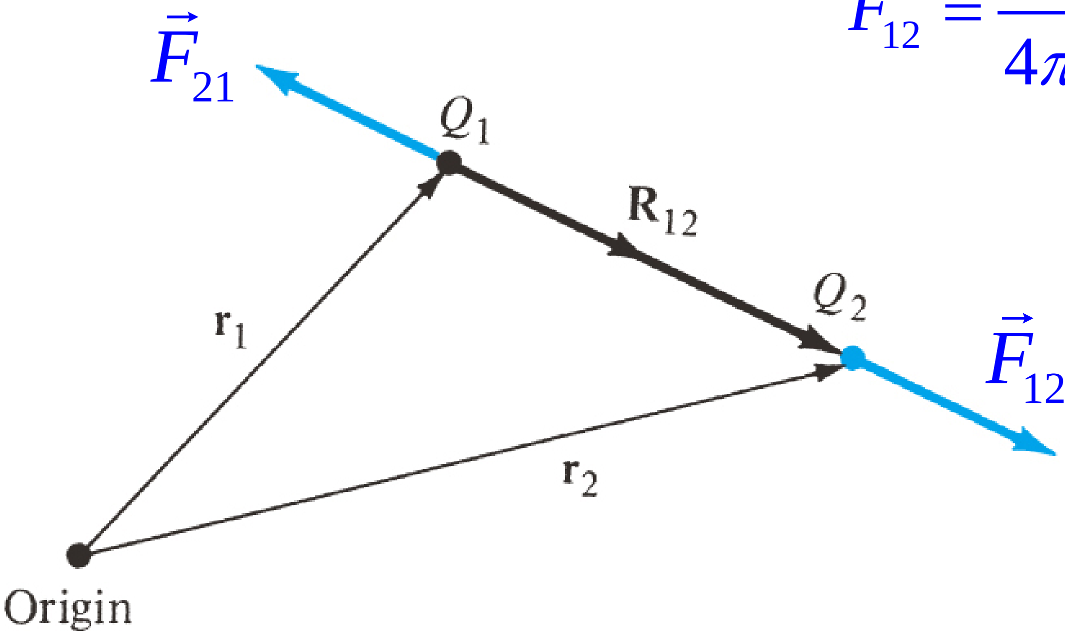
n: $q = 0$; $m_n = m_p$

$r_{nucleus} \sim 10^{-14}$ m

$r_{atom} \sim 10^{-10}$ m or about 1 Angstrom

- +ve ion – if one or more electrons have been removed
- -ve ion – if one or more electrons has been added

1 Coulomb's Law and Field intensity



$$\vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\hat{a}_{12} \qquad Coulomb's\ Law$$

$$\varepsilon_o \equiv permittivity\ of\ free\ space$$

$$= 8.854\text{x}10^{-12}\ F/m$$

$$also\ k \equiv Coulomb\ constant$$

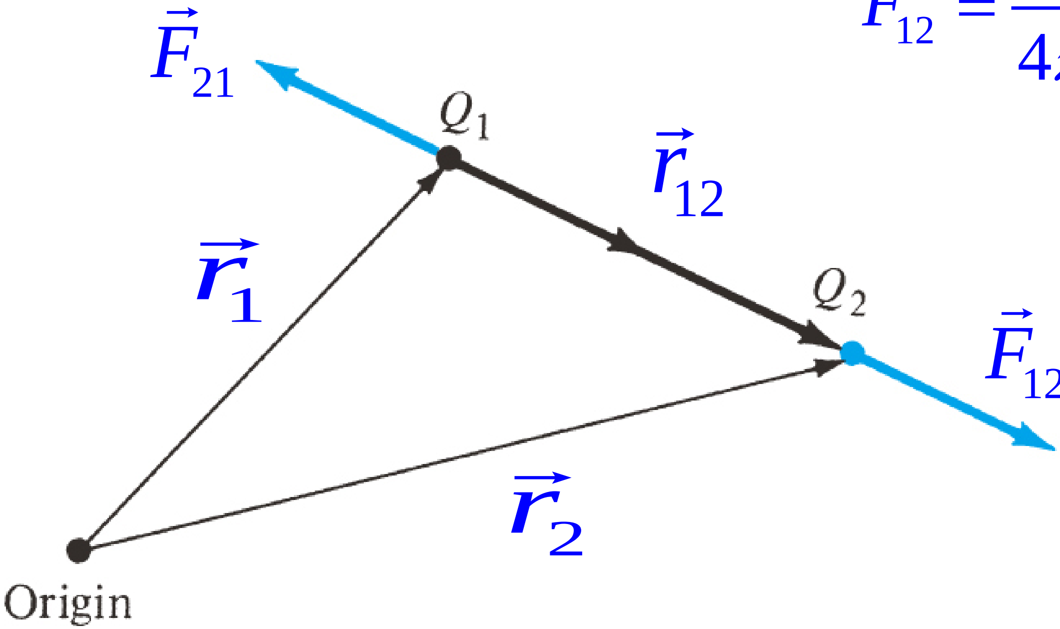
$$= \frac{1}{4\pi\varepsilon_o} \simeq 9\text{x}10^{9}\ m/F$$

$$also\ \vec{R}_{12} = \vec{r}_2 - \vec{r}_1\ ;\ R = \left|\vec{R}_{12}\right| \quad ;\ \hat{a}_{R_{12}} = \frac{\vec{R}_{12}}{R}$$

$$\therefore\ \vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^3}\vec{R}_{12} \qquad or\ \vec{F}_{12} = \frac{Q_1 Q_2 (\vec{r}_2 - \vec{r}_1)}{4\pi\varepsilon_0 \left|\vec{r}_2 - \vec{r}_1\right|^3}$$

$$also\ \vec{F}_{21} = \left|\vec{F}_{12}\right|\hat{a}_{R_{21}} = \left|\vec{F}_{12}\right|(-\hat{a}_{R_{12}}) = -\vec{F}_{12}$$

Alternate notation:



$$\vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r_{12}}\hat{r}_{12}$$ *Coulomb's Law*

$$\varepsilon_o \equiv permittivity\ of\ free\ space$$

$$= 8.854\text{x}10^{-12}\ F/m$$

$$also\ k \equiv Coulomb\ constant$$

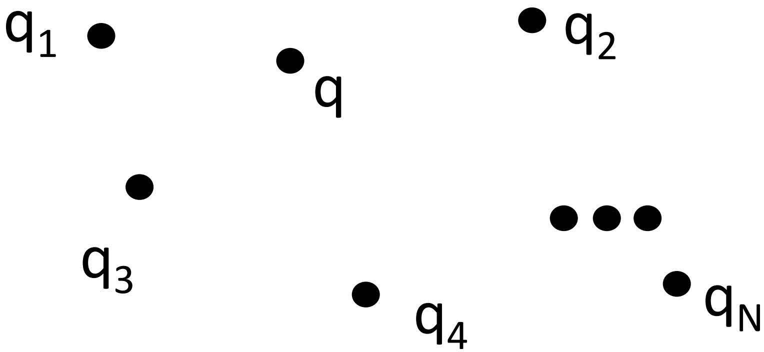
$$= \frac{1}{4\pi\varepsilon_o} \simeq 9\text{x}10^9\ m/F$$

$$also\ \vec{r}_{12} = \vec{r}_2 - \vec{r}_1\ ;\ r_{12} = |\vec{r}_{12}|\quad ;\ \hat{r}_{12} = \frac{\vec{r}_{12}}{r_{12}}$$

$$\therefore\ \vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r_{12}^3}\vec{r}_{12} \quad or\ \vec{F}_{12} = \frac{Q_1 Q_2 (\vec{r}_2 - \vec{r}_1)}{4\pi\varepsilon_0 |\vec{r}_2 - \vec{r}_1|^3}$$

$$also\ \vec{F}_{21} = |\vec{F}_{12}|\hat{r}_{21} = |\vec{F}_{12}|(-\hat{r}_{12}) = -\vec{F}_{12} \quad since\ \hat{r}_{21} = -\hat{r}_{12}$$

Principle of Superposition for more than two charges; N q's:



now have position vector $\vec{r}$ for q and N position vectors for the N charges

$$\vec{F}_q = \frac{qq_1(\vec{r}-\vec{r}_1)}{4\pi\varepsilon_0\left|\vec{r}-\vec{r}_1\right|^3} + \frac{qq_2(\vec{r}-\vec{r}_2)}{4\pi\varepsilon_0\left|\vec{r}-\vec{r}_2\right|^3} + \cdots + \frac{qq_N(\vec{r}-\vec{r}_N)}{4\pi\varepsilon_0\left|\vec{r}-\vec{r}_N\right|^3}$$

$$\vec{F}_q = \frac{q}{4\pi\varepsilon_0}\sum_{k=1}^{N}\frac{q_k(\vec{r}-\vec{r}_k)}{\left|\vec{r}-\vec{r}_k\right|^3}$$

Note re point charges:

In principle mathematical points are involved. We at least require:

$$r_{12} >> \textit{size of the charged objects}$$

$\vec{E} \equiv electric\ field\ intensity\ or\ electric\ field\ strength$

$= force\ that\ a\ unit\ charge\ experiences\ when\ placed\ in\ this\ field$

$$\vec{E} = \frac{\vec{F}}{q} \quad (N / C\ or\ V / m)$$

$i.e.\ to\ find\ the\ field\ \vec{E}\ at\ some\ point\ in\ space$

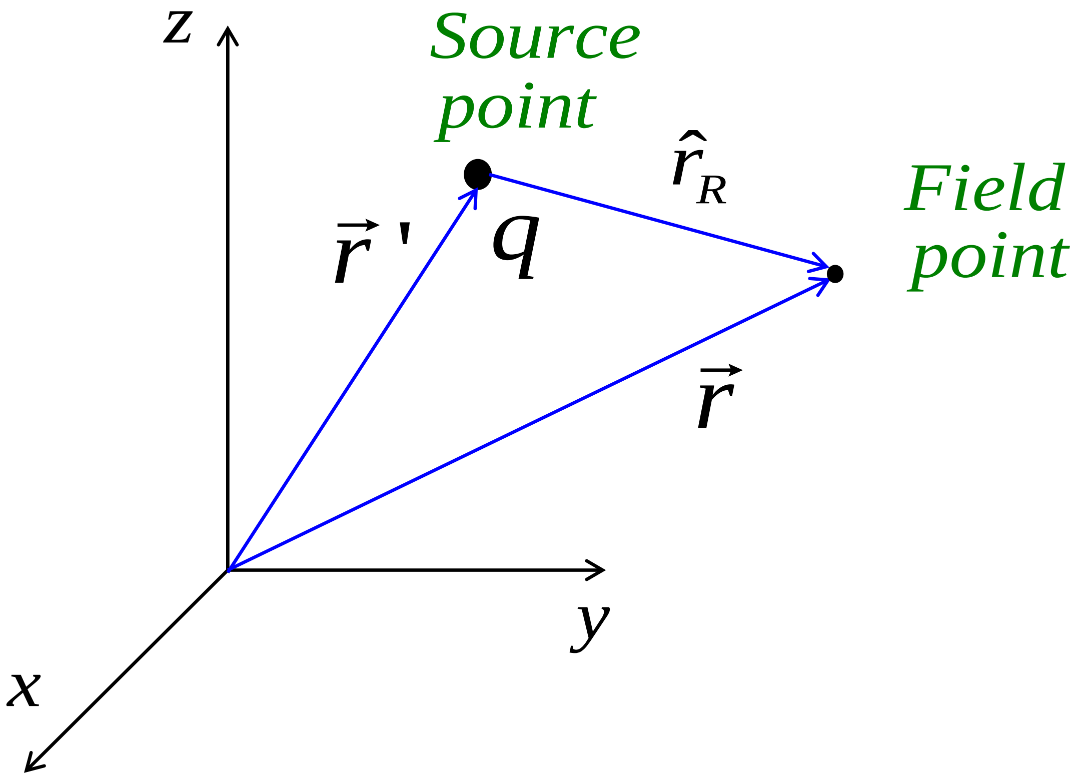
$--\ place\ a\ charge\ at\ the\ point\ of\ interest\ and\ measure\ \vec{F}$

- $\vec{F}\ also\ gives\ direction\ of\ \vec{E}$

- in various practical situations placing a q in some field can change the charge distribution producing E you are trying to measure (e.g., $\vec{\mathrm{E}}$ produced by q's on a conductor)

$$\vec{E} = \lim_{q \to 0} \frac{\vec{F}}{q} \quad ;\ q \to q_0,\ or\ \delta q,\ a\ small\ "test\ charge"$$

For a point charge:



$$\vec{E} = \frac{q}{4\pi\varepsilon_0 R^2}\hat{r}_R$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{(\vec{r}-\vec{r}\,')}{\left|\vec{r}-\vec{r}\,'\right|^3}$$

For N point charges: $\vec{E} = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\sum_{k=1}^{N}\frac{q_k(\vec{r}-\vec{r}_k)}{\left|\vec{r}-\vec{r}_k\right|^3}$

$$In\ rectangular\ coordinates: \vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{N}\frac{q_k\left[(x-x_k)\hat{i}+(y-y_k)\hat{j}+(z-z_k)\hat{k}\right]}{\left[(x-x_k)^2+(y-y_k)^2+(z-z_k)^2\right]^{\frac{3}{2}}}$$

Practice Excercise 4.1

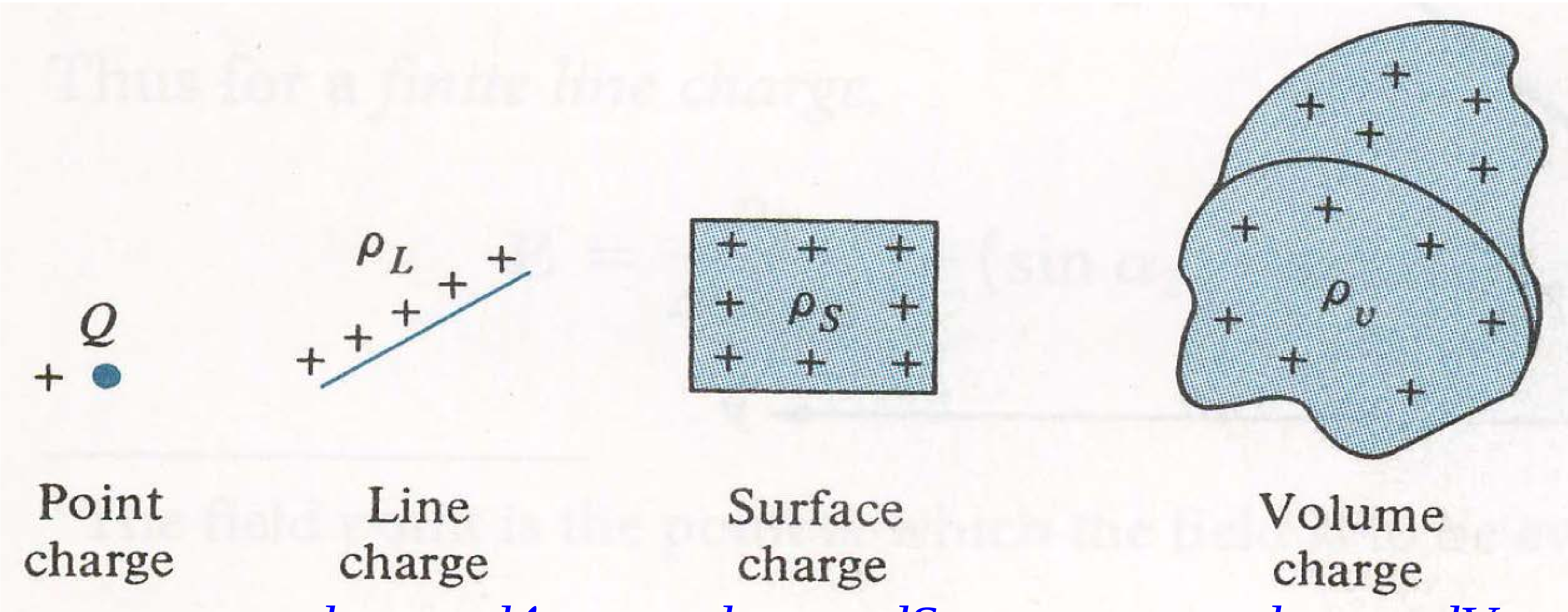
Point charges $5\ nC$ *and* $-2\ nC$ *are located at* $(2,0,4)$ *and* $(-3,0,5)$, *respectively.*

$a)$ *Determine the force on a* $1\ nC$ *point charge located at* $(1,-3,7)$.

$b)$ *Find the electric field* $(\vec{E})$ *at* $(1,-3,7)$.

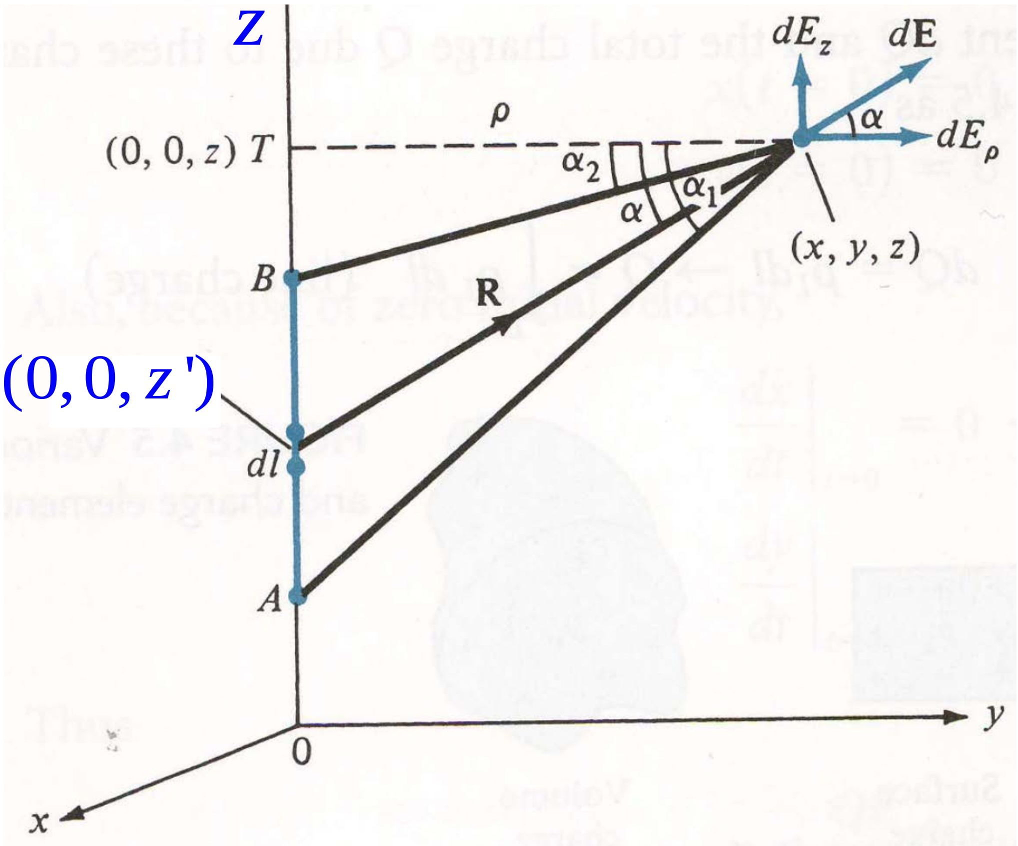
- *convenient to use* $\varepsilon_o \simeq \dfrac{10^{-9}}{36\pi}\ F/m$
- *Also look at examples* 4.2 *and* 4.3.

2 E due to Continuous Charge Distributions



Line	Surface	Volume
$dq = \rho_L d\ell$	$dq = \rho_S dS$	$dq = \rho_V dV$
$dq = \lambda d\ell$	$dq = \sigma dS$	$dq = \rho dV$
$Q = \int_L \lambda d\ell$	$Q = \int_S \sigma dS$	$Q = \int_V \rho dV$
$\vec{E} = \int_L \frac{\lambda d\ell}{4\pi\varepsilon_0 R^2}\hat{r}_R$	$\vec{E} = \int_S \frac{\sigma dS}{4\pi\varepsilon_0 R^2}\hat{r}_R$	$\vec{E} = \int_V \frac{\rho dV}{4\pi\varepsilon_0 R^2}\hat{r}_R$

Line charge (find $\vec{E}$ due to a line of charge):



$$dq = \lambda d\ell = \lambda dz$$

$$Q = \int_{Z_A}^{Z_B} \lambda dz$$

field point (x, y, z)

source point (x', y', z')

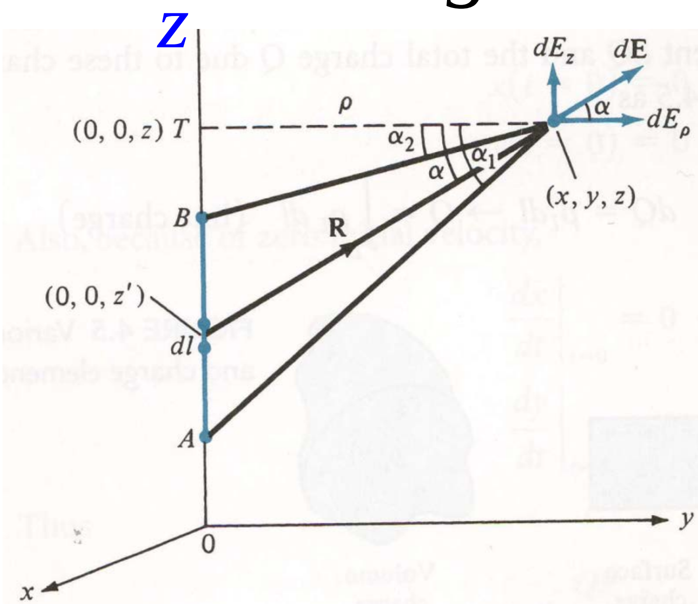
Then $d\ell = dz'$

$$\vec{R} = (x, y, z) - (0, 0, z') \quad = x\hat{a}_x + y\hat{a}_y + (z - z')\hat{a}_z$$

$$= x\hat{i} + y\hat{j} + (z - z')\hat{k}$$

$$\text{or} \quad \vec{R} = \rho\hat{\rho} + (z - z')\hat{k}$$

Line charge:



$$dq = \lambda d\ell = \lambda dz'$$

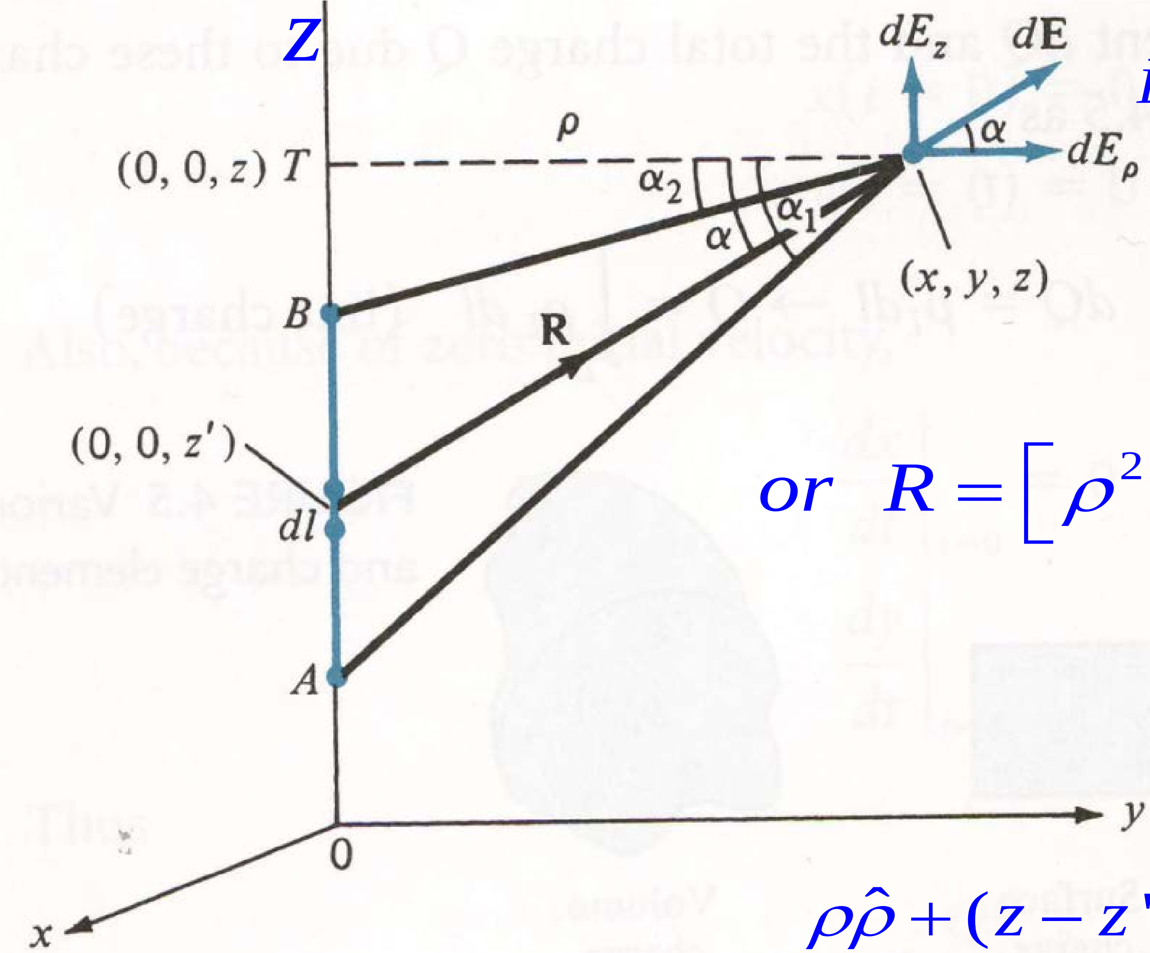
$$\vec{R} = \rho\hat{\rho} + (z - z')\hat{k}$$

$$\vec{E} = \int_L \frac{\lambda d\ell}{4\pi\varepsilon_0 R^2}\hat{r}_R \quad ; \; need \;\; \frac{\hat{r}_R}{R^2}$$

$$R^2 = \left|\vec{R}\right|^2 = \rho^2 + (z - z')^2$$

$$\frac{\hat{r}_R}{R^2} = \frac{\vec{R}}{\left|\vec{R}\right|^3} = \frac{\rho\hat{\rho} + (z - z')\hat{k}}{\left[\rho^2 + (z - z')^2\right]^{\frac{3}{2}}}$$

$$or \;\; \vec{E} = \frac{\lambda}{4\pi\varepsilon_0}\int \frac{\rho\hat{\rho} + (z - z')\hat{k}}{\left[\rho^2 + (z - z')^2\right]^{\frac{3}{2}}} dz'$$



$$\vec{E} = \frac{\lambda}{4\pi\varepsilon_0}\int \frac{\rho\hat{\rho} + (z - z')\hat{k}}{\left[\rho^2 + (z - z')^2\right]^{\frac{3}{2}}}\, dz'$$

$$\frac{R}{\rho} = \sec\alpha$$

$$or \quad R = \left[\rho^2 + (z - z')^2\right]^{\frac{1}{2}} = \rho\sec\alpha$$

$$z' = OT - \rho\tan\alpha$$

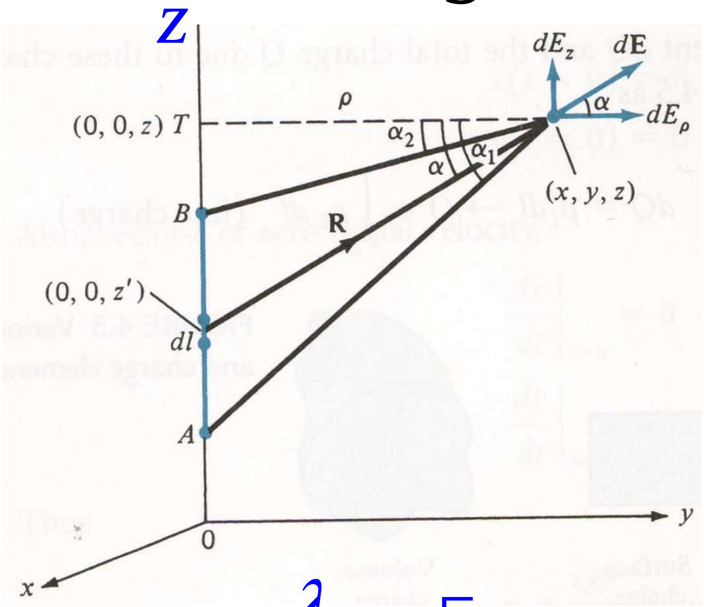
$$dz' = -\rho\sec^2\alpha\ d\alpha$$

$$\rho\hat{\rho} + (z - z')\hat{k} = R\cos\alpha\hat{\rho} + R\sin\alpha\hat{k}$$

$$\vec{E} = \frac{\lambda}{4\pi\varepsilon_0}\int \frac{\rho\sec\alpha\cos\alpha\hat{\rho} + \rho\sec\alpha\sin\alpha\hat{k}}{\rho^3\sec^3\alpha}\,(-\rho\sec^2\alpha d\alpha)$$

$$\vec{E} = -\frac{\lambda}{4\pi\varepsilon_0\rho}\int_{\alpha_1}^{\alpha_2}(\cos\alpha\hat{\rho} + \sin\alpha\hat{k})\, d\alpha$$

Line charge:



$$\vec{E} = -\frac{\lambda}{4\pi\varepsilon_0\rho}\int_{\alpha_1}^{\alpha_2} (\cos\alpha\hat{\rho} + \sin\alpha\hat{k})\, d\alpha$$

$$\vec{E} = \frac{\lambda}{4\pi\varepsilon_0\rho}\left[-(\sin\alpha_2 - \sin\alpha_1)\hat{\rho} + (\cos\alpha_2 - \cos\alpha_1)\hat{k}\right]$$

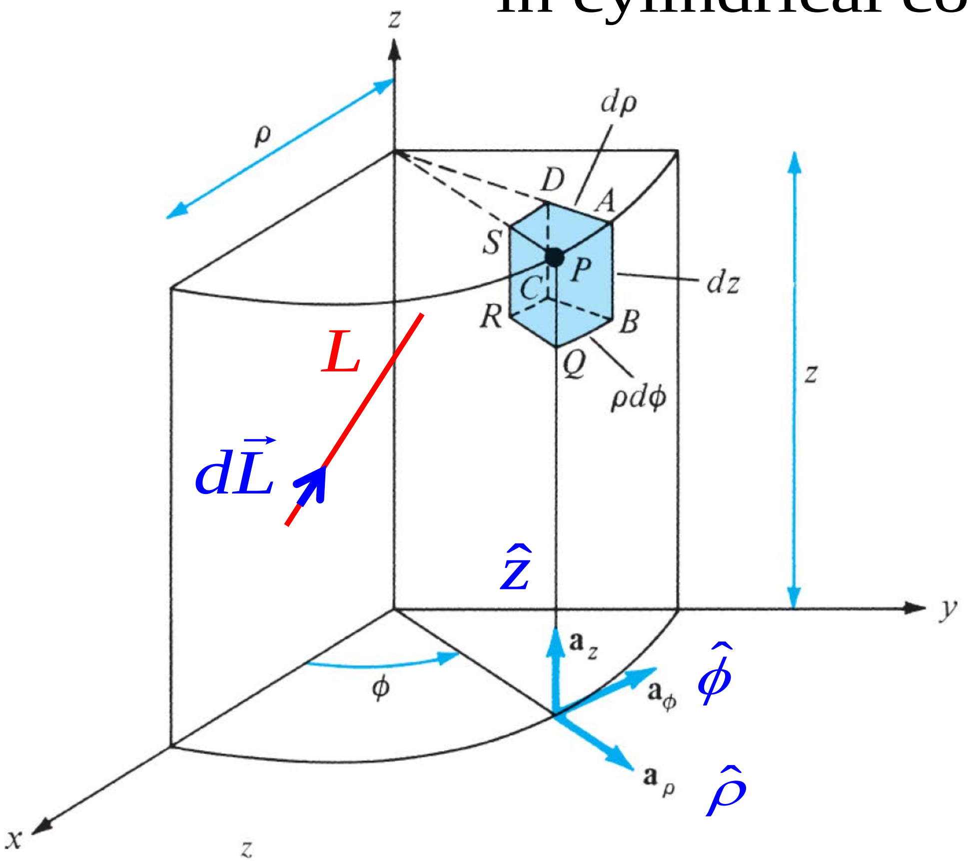
- *for infinite line charge* $B = (0,0,\infty)$ *and* $A = (0,0,-\infty)$

$$\alpha_1 = \pi/2,\ \ \alpha_2 = -\pi/2$$

$$\therefore\ \vec{E} = \frac{\lambda}{2\pi\varepsilon_0\rho}\hat{\rho}$$

– what if the line of charge is not along z ?

– in cylindrical coordinates -



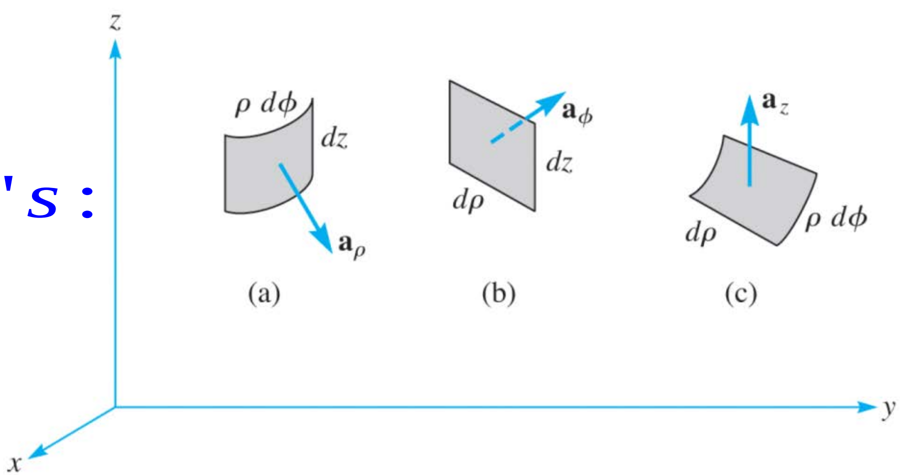
$$d\vec{L} = d\rho\ \hat{\rho} + \rho d\phi\ \hat{\phi} + dz\ \hat{z}$$

$$d\vec{A} = \rho d\phi\ dz\,\hat{\rho}$$

$$d\rho dz\ \hat{\phi}$$

$$\rho d\phi\ d\rho\ \hat{z}$$

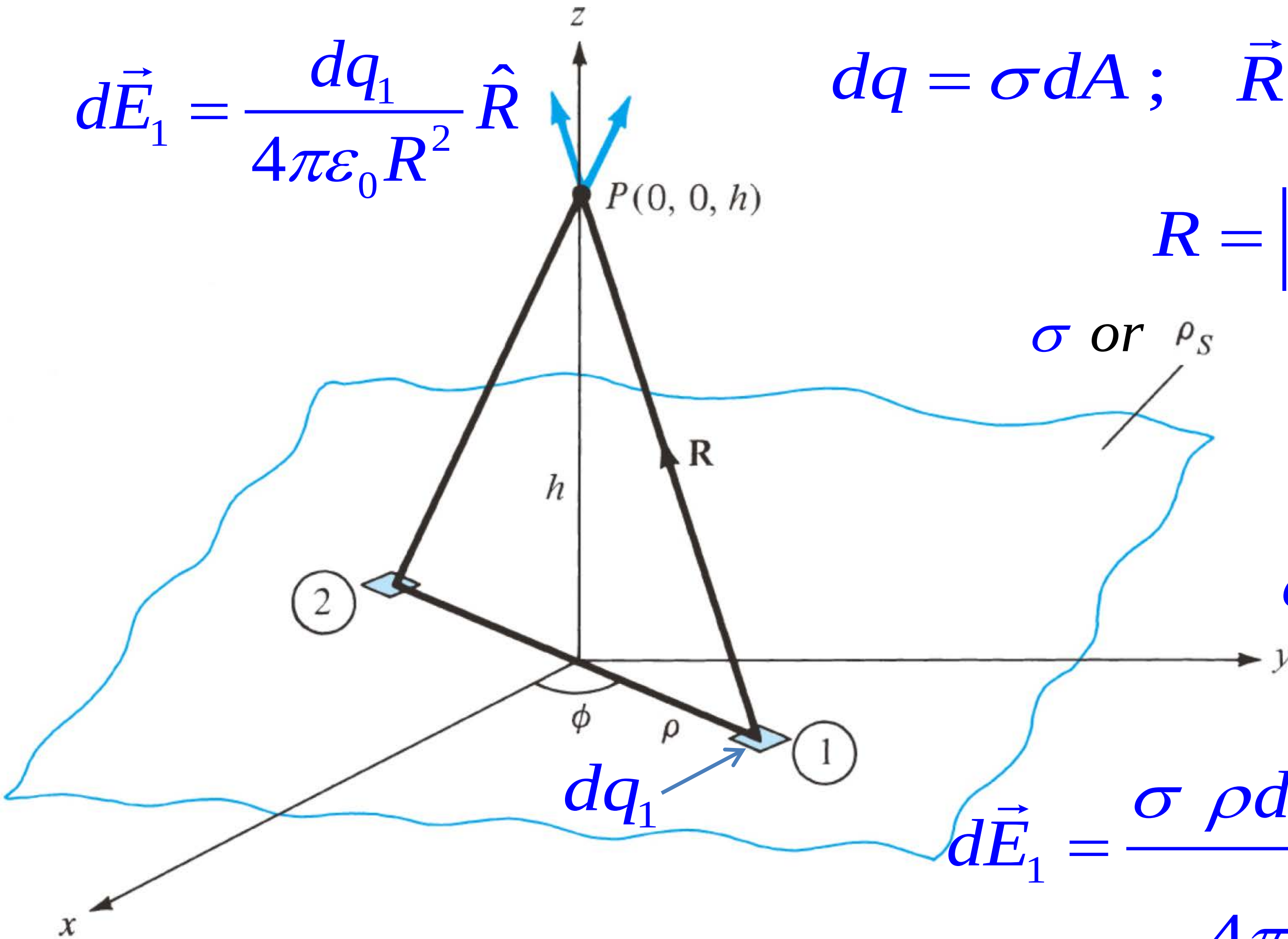
$d\vec{A}$'s :



$$dV = \rho d\phi\ d\rho\ \ dz$$

Surface charge (*find* $\vec{E}$ *due to infinite sheet of charge*):

$$d\vec{E}_1 = \frac{dq_1}{4\pi\varepsilon_0 R^2}\hat{R}$$



$$dq = \sigma dA\,; \quad \vec{R} = \rho(-\hat{\rho}) + h\hat{z}$$

$$R = \left|\vec{R}\right| = \left[\rho^2 + h^2\right]^{\frac{1}{2}}$$

$$\hat{R} = \frac{\vec{R}}{R}$$

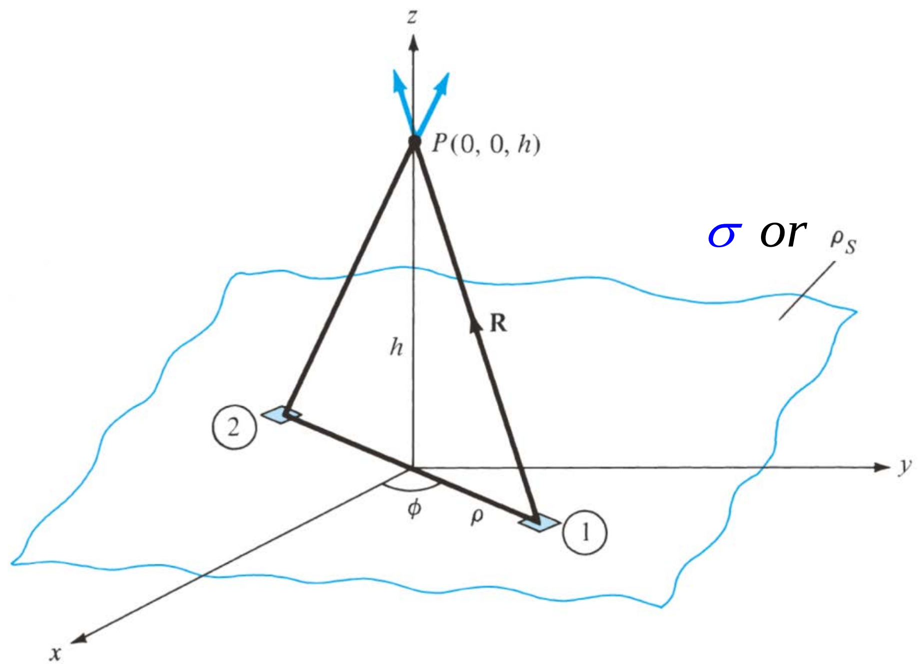
$$dq = \sigma\ \rho d\phi d\rho$$

$$d\vec{E}_1 = \frac{\sigma\ \rho d\phi d\rho\left[-\rho\hat{\rho} + h\hat{z}\right]}{4\pi\varepsilon_0\left[\rho^2 + h^2\right]^{\frac{3}{2}}}$$

- *from symmetry*:

for every element 1 *have corresponding element* 2:

$$dE_{1\rho} = dE_{2\rho} \Rightarrow dE_{1\rho} - dE_{2\rho} = 0$$



–only elements

dE_z remain

$$d\vec{E}_z = \frac{\sigma\ \rho d\phi d\rho\left[h\hat{z}\right]}{4\pi\varepsilon_0\left[\rho^2+h^2\right]^{\frac{3}{2}}}$$

$$\vec{E} = \int_A d\vec{E}_z$$

$$\vec{E} = \frac{\sigma}{4\pi\varepsilon_0}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{\infty}\frac{h\rho d\phi d\rho}{\left[\rho^2+h^2\right]^{\frac{3}{2}}}\hat{z}$$ (could let $u=\rho^2;\ du=2\rho d\rho$)

$$\vec{E} = \frac{\sigma h}{4\pi\varepsilon_0}2\pi\int_0^{\infty}\left[\rho^2+h^2\right]^{-\frac{3}{2}}\frac{1}{2}d(\rho^2)\hat{z}$$

$$\vec{E} = \frac{\sigma h}{2\varepsilon_0}\left\{-\left[\rho^2+h^2\right]^{-\frac{1}{2}}\right\}_0^{\infty}\hat{z}$$ or $$\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{z}$$

PRACTICE EXERCISE 4.5

A square plate described by $-2 \leq x \leq 2, -2 \leq y \leq 2, z = 0$ carries a charge $12\ |y|\ \text{mC/m}^2$. Find the total charge on the plate and the electric field intensity at $(0, 0, 10)$.

Answer: 192 mC, $16.6\ \mathbf{a}_z$ MV/m.

$$-\,plane\ lies\ in\ xy - plane$$

$$Q_A = \int \sigma dA = \int_{-2}^{2}\int_{-2}^{2} 12|y|dxdy \qquad = 192\ mC$$

$$\vec{E}(0,0,10):$$

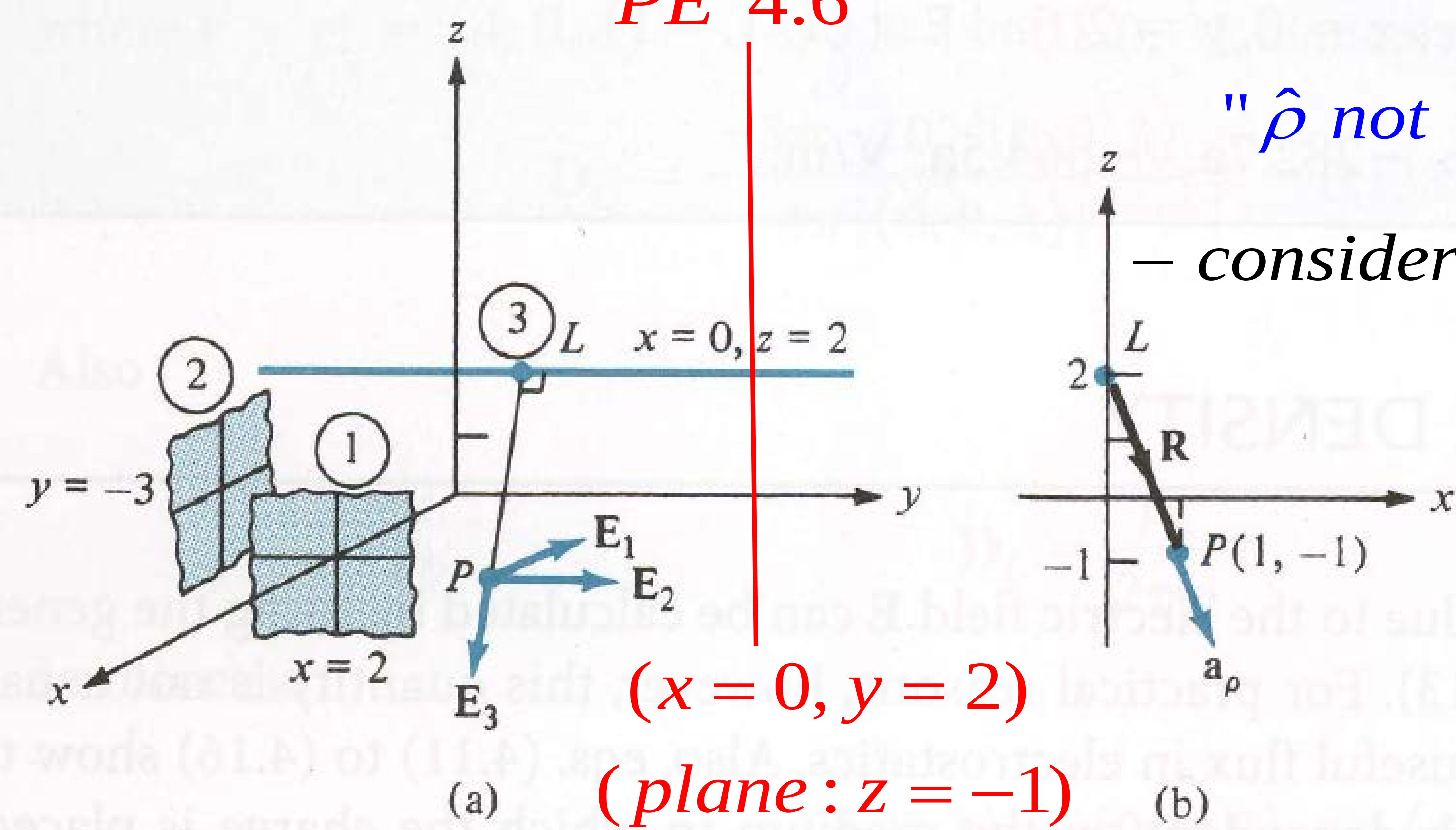
Comment on Example 4.6

Planes $x = 2$ and $y = -3$, respectively, carry charges 10 nC/m^2 and 15 nC/m^2. If the line $x = 0$, $z = 2$ carries charge 10π nC/m, calculate $\mathbf{E}$ at $(1, 1, -1)$ due to the three charge distributions.

Text gives $\vec{E}_3 = \dfrac{\lambda}{2\pi\varepsilon_0\rho}\hat{\rho}$;

"$\hat{\rho}$ not regular $\hat{\rho}$"

PE 4.6



$(x = 0, y = 2)$

$(plane: z = -1)$

– consider plane $y = 1$:

$\vec{R}$ is vector from L to P

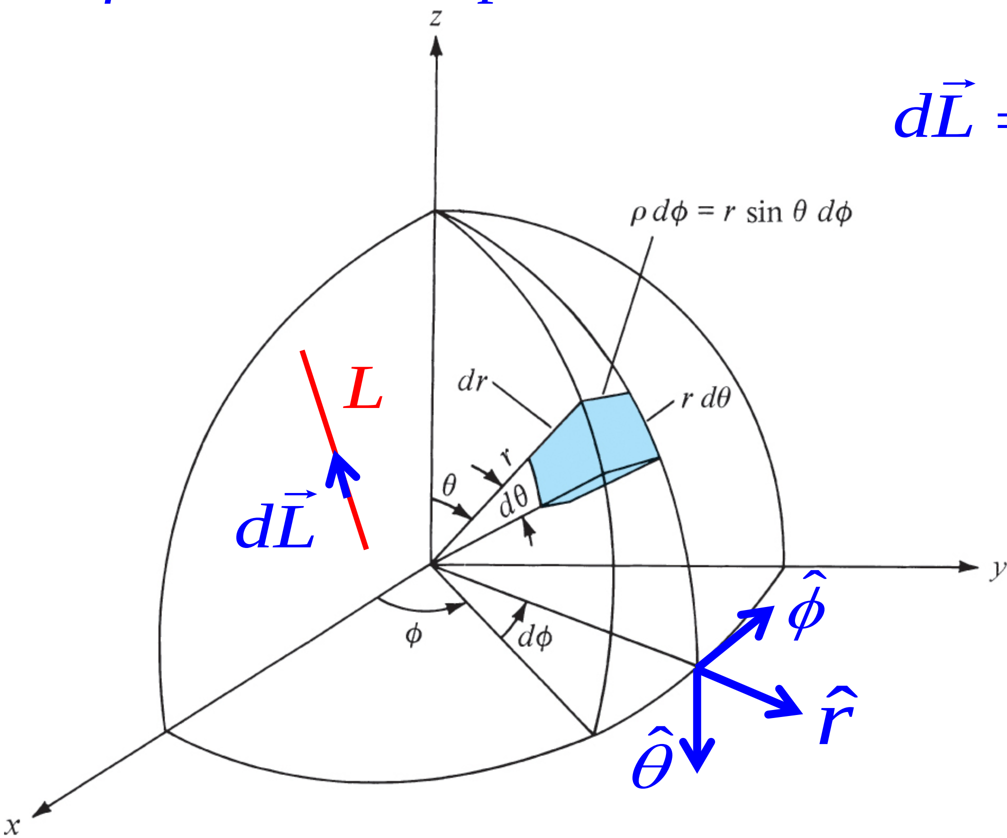
$\vec{R} = -3\hat{z} + \hat{x}$

$\rho = \left|\vec{R}\right|$

$\hat{\rho} = \dfrac{\vec{R}}{\left|\vec{R}\right|}$

FIGURE 4.10 For Example 4.6: (**a**) three charge distributions, (**b**) finding ρ and $\mathbf{a}_\rho$ on plane $y = 1$.

- *next want to consider volume charge distributions for which a spherical coordinate system is useful*



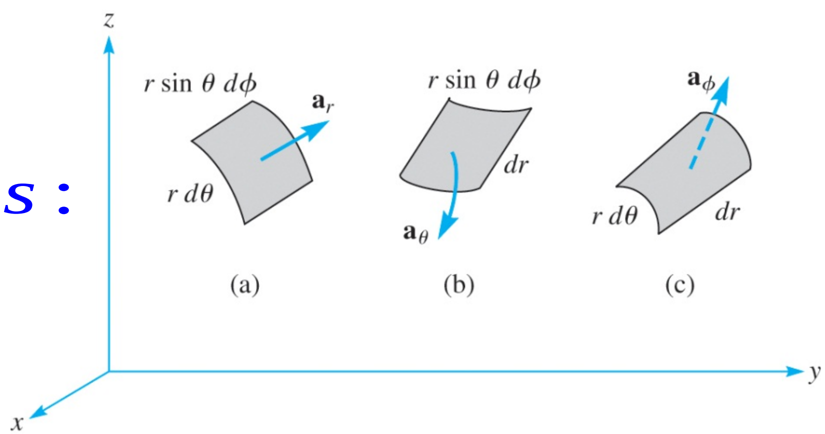
$$d\vec{L} = dr\ \hat{r} + rdr\ \hat{\theta} + r\sin\theta d\phi\ \hat{\phi}$$

$$d\vec{A} = r^2 \sin\theta d\theta d\phi\ \hat{r}$$

$$r\sin\theta dr d\phi \hat{\theta}$$

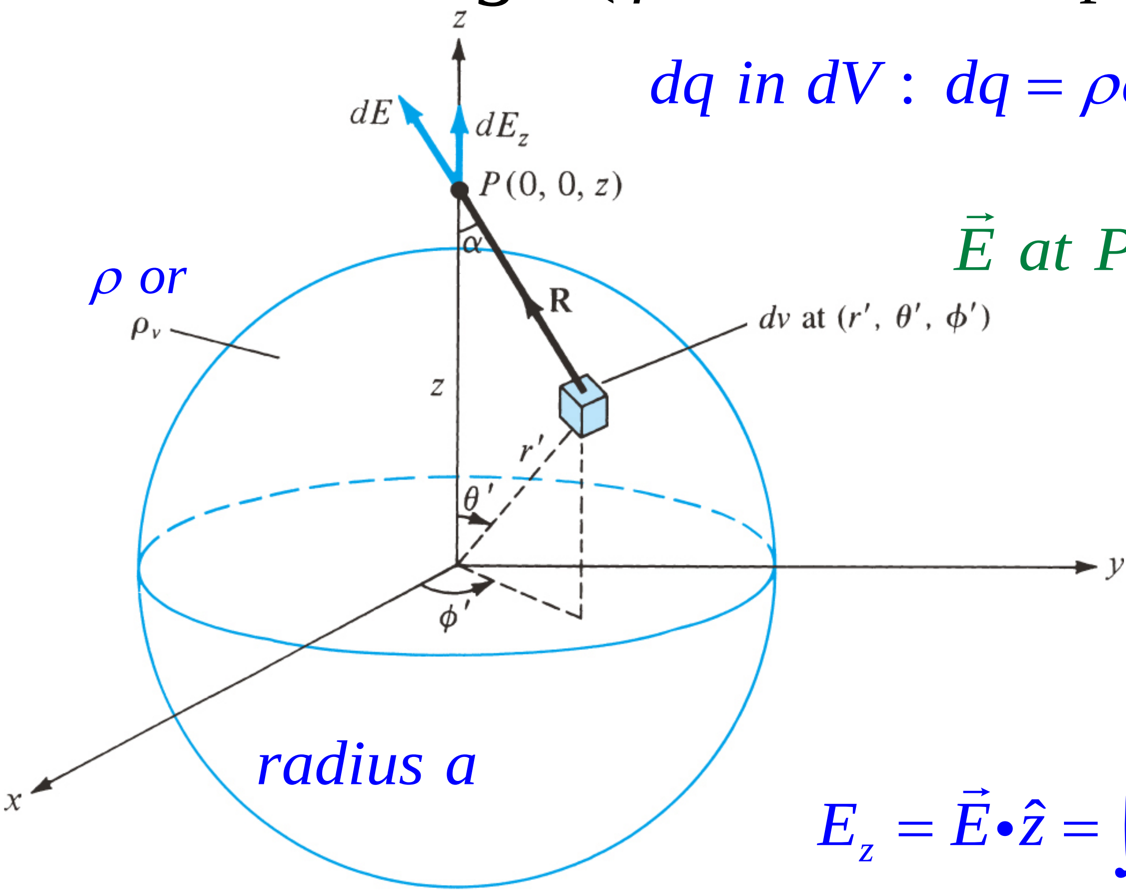
$$r dr d\theta \hat{\phi}$$

$d\vec{A}$'s :



$$dV = r^2 \sin\theta dr d\theta d\phi$$

Volume charge (*find* $\vec{E}$ *due to sphere of uniform charge*) :



dq *in* dV : $dq = \rho dV$; $\quad Q = \int_V dq = \rho \int_V dV = \rho \dfrac{4\pi a^3}{3}$

$\vec{E}$ *at* $P(0,0,z)$? $\quad d\vec{E} = \dfrac{\rho dV}{4\pi\varepsilon_0 R^2} \hat{R}$

$$\hat{R} = \cos\alpha \ \hat{z} + \sin\alpha \ \hat{\rho}$$

– *symmetry of the charge distribution*

$\Rightarrow$ *only* E_z *survives*

$$E_z = \vec{E} \bullet \hat{z} = \int dE \cos\alpha = \frac{\rho}{4\pi\varepsilon_0} \int \frac{dV \cos\alpha}{R^2}$$

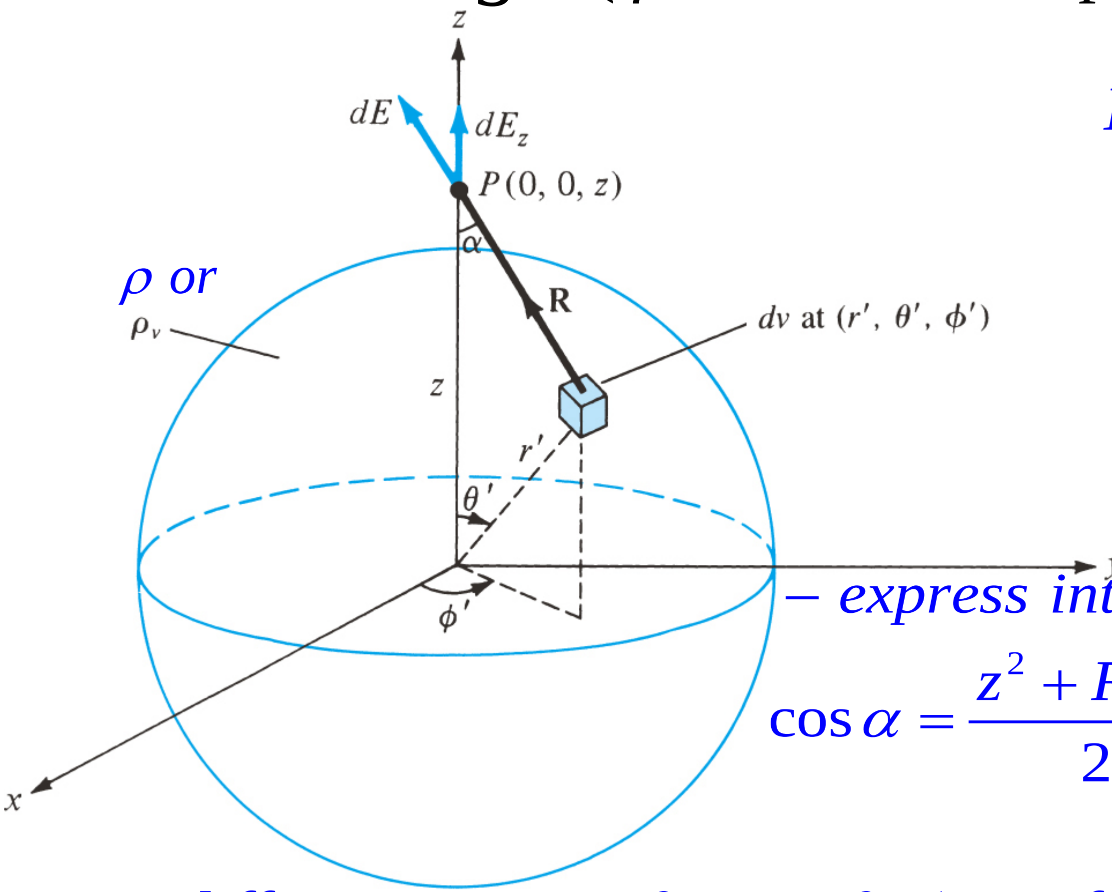
–*need expressions for* dV, R^2, $\cos\alpha$

$$dV = r'^2 \sin\theta' d\theta' dr' d\phi'$$

Apply cosine rule :

$$R^2 = z^2 + r'^2 - 2zr'\cos\theta' \qquad r'^2 = z^2 + R^2 - 2zR\cos\alpha$$

Volume charge (*find* $\vec{E}$ *due to sphere of uniform charge*):



$$E_z = \frac{\rho}{4\pi\varepsilon_0}\int \frac{dV\cos\alpha}{R^2}$$

$$dV = r'^2 \sin\theta' d\theta' dr' d\phi'$$

$$R^2 = z^2 + r'^2 - 2zr'\cos\theta'$$

$$r'^2 = z^2 + R^2 - 2zR\cos\alpha$$

– *express integral in terms of R and r'*:

$$\cos\alpha = \frac{z^2 + R^2 - r'^2}{2zR} \;;\; \cos\theta' = \frac{z^2 + r'^2 - R^2}{2zr'}$$

– *differentiate* $\cos\theta'$ *wrt* θ' $(z, r'$ *fixed*$)$: $\sin\theta' d\theta' = \dfrac{RdR}{zr'}$

$$E_z = \frac{\rho}{4\pi\varepsilon_0}\int_{\phi'=0}^{2\pi} d\phi' \int_{r'=0}^{a} \int_{R=z-r'}^{z+r'} r'^2 \frac{RdR}{zr'} dr' \frac{z^2 + R^2 - r'^2}{2zR}\frac{1}{R^2} = \frac{1}{4\pi\varepsilon_0}\frac{1}{z^2}\left(\frac{4}{3}\pi a^3\rho\right)$$

(*as* θ' *varies from* 0 *to* π)

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 z^2}\hat{z} \quad \text{or here } \vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

Example

4.9 Given that $\rho_v = 4\rho^2 z \cos\phi$ nC/m^3, find the total charge contained in a wedge defined by $0 < \rho < 2, 0 < \phi < \pi/4, 0 < z < 1$.

-- will deal with volume charge distributions much more a little later in connection with Gauss's Law

3 Electric Flux Density, Gauss's Law, Divergence of Electrostatic Fields, Maxwell's Eq.

$Define\ \vec{D} \equiv electric\ flux\ density$ $or\ the\ electric\ displacement\ [C/m^2]$

$$\vec{\mathrm{D}} = \varepsilon_0 \vec{E}$$ $-\ this\ is\ a\ vector\ field$

$-define\ the\ electric\ flux\ \phi_E = \int_S \vec{D} \bullet d\vec{A} \quad [C]$

- $in\ SI\ units:$

$$+1C \xrightarrow[\text{one line of electric flux}]{} -1C$$

$Can\ use\ the\ previous\ formulas\ for\ E\ to\ get\ D:$

$$e.g.\quad for\ infinite\ sheet:\ \vec{D} = \frac{\sigma}{2}\hat{n}$$

$$for\ volume\ distribution:\ \vec{D} = \int_V \frac{\rho dV}{4\pi R^2}\hat{R}$$

PRACTICE EXERCISE 4.7

A point charge of 30 nC is located at the origin, while plane $y = 3$ carries charge 10 nC/m^2. Find $\mathbf{D}$ at $(0, 4, 3)$.

Answer: $5.076\mathbf{a}_y + 0.0573\mathbf{a}_z$ nC/m^2.

–also look at Example 4.6

Gauss's Law:

–total electric flux ϕ_E *passing through a closed surface*

= total charge enclosed by the closed surface

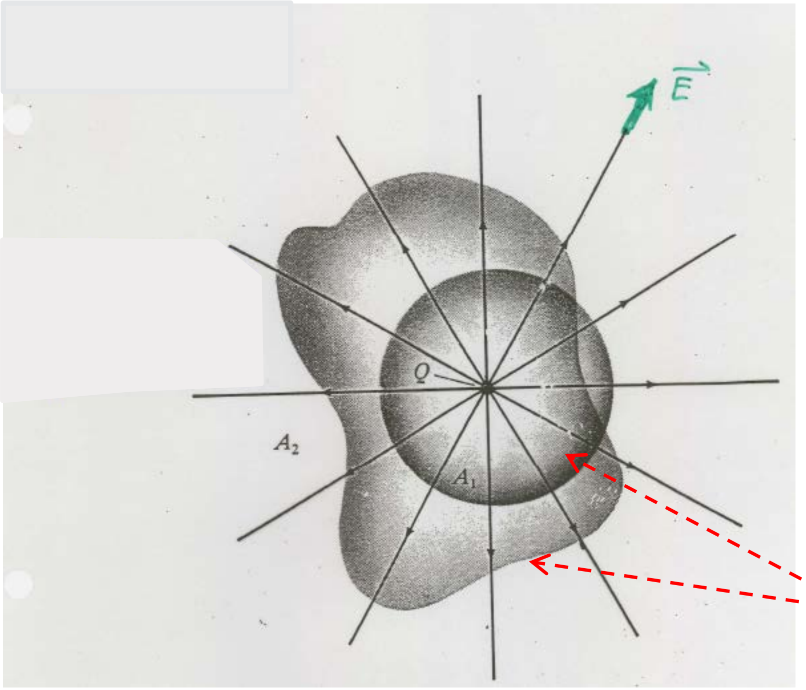
$$\phi_E = q_{enc}$$

$$\phi_E = \oint_S d\phi_E = \oint_S \vec{D} \bullet d\vec{A} = q_{enc} = \int_V \rho dV$$

- *an amazing feature:*

for give q_{enc} , ϕ_E *independent of shape of the closed surface*

– consistent with our understanding of electric flux :



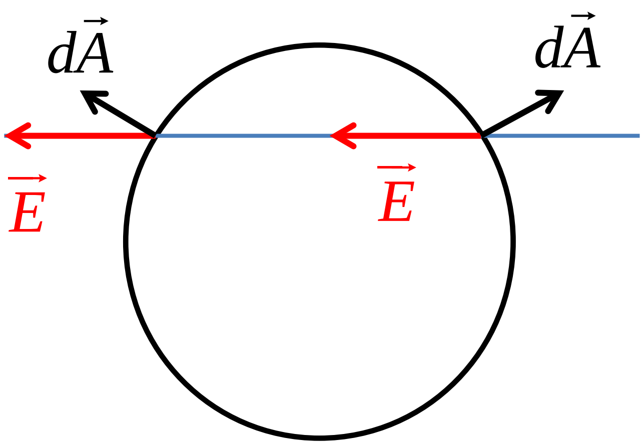
Gaussian surfaces

In problems involving Gauss' Law we are interested in finding Φ_E through a <u>closed surface</u> ; i.e.

$$\Phi_E = \underset{\substack{\text{Surface}\\ \text{integral}\\ \text{over C.S.}}}{\oint} \vec{D}\bullet d\vec{A} = \oint_{C.S.} D_n dA$$

component of D $\perp$ to surface

Note that $d\vec{A}$ is always taken to <u>point outward:</u>



- for D lines entering: $\theta > 90^0$ and Φ is -ve

- for D lines leaving: $\theta < 90^0$ and Φ is +ve

Gauss' Law for a number of charges:

$$\Phi_E = \oint_{G.S.} \vec{D} \bullet d\vec{A} = q_{encl} \quad \text{with } q_{encl} = \sum_i q_i(inside)$$

- *In favorable cases (high degree of symmetry for charge distribution) it is possible to find E through Gauss' Law.*
  - *a suitable* Gaussian *surface would have*

$1)\ \left|\vec{D}\right| = constant$

$2)\ \cos\theta = constant$ - ensure every part of G.S. is either $\perp$ or tangent to E-field

$$\oint_{G.S.} \vec{D} \bullet d\vec{A} = \left|\vec{D}\right| \oint_{G.S.} \left|d\vec{A}\right| \cos\theta = q_{enc}$$

- spherical symmetry
- cylindrical symmetry
- planar symmetry

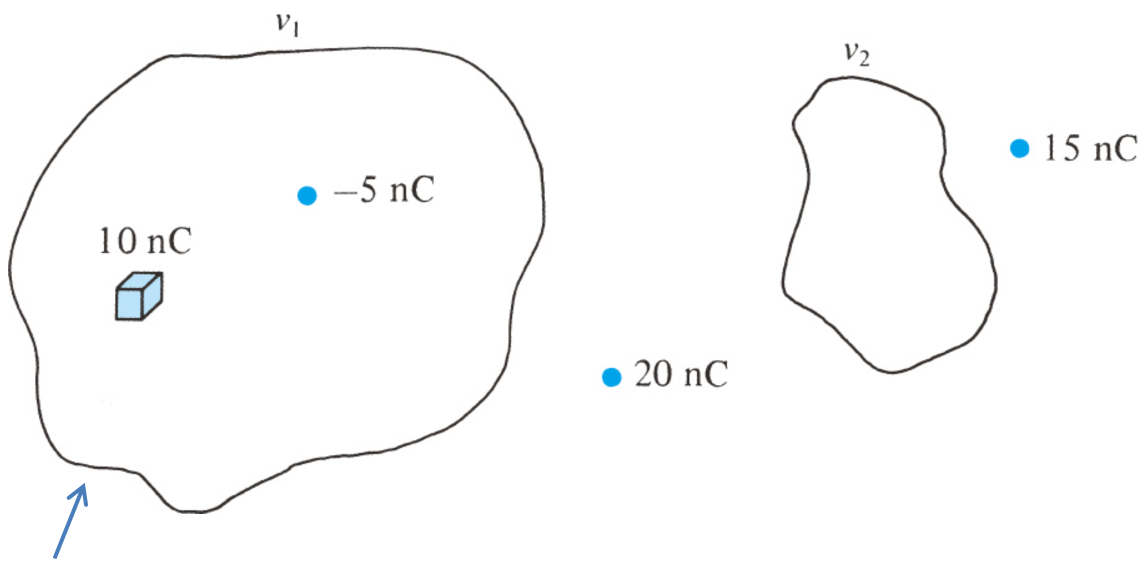
$If\ \ charge\ distribution\ has:$

- $spherical\ \ symmetry: Gaussian\ \ surface \equiv concentric\ \ sphere$
- $cylindrical\ \ symmetry: Gaussian\ \ surface \equiv coaxial\ \ cylinder$
- $plane\ symmetry: Gaussian\ \ surface \equiv\ pill\ \ box\ \ crossing\ \ surface$

- *Although Gauss's Law always holds no matter what the shape of the charge distribution,*

it may not always be useful for finding $\vec{E}$

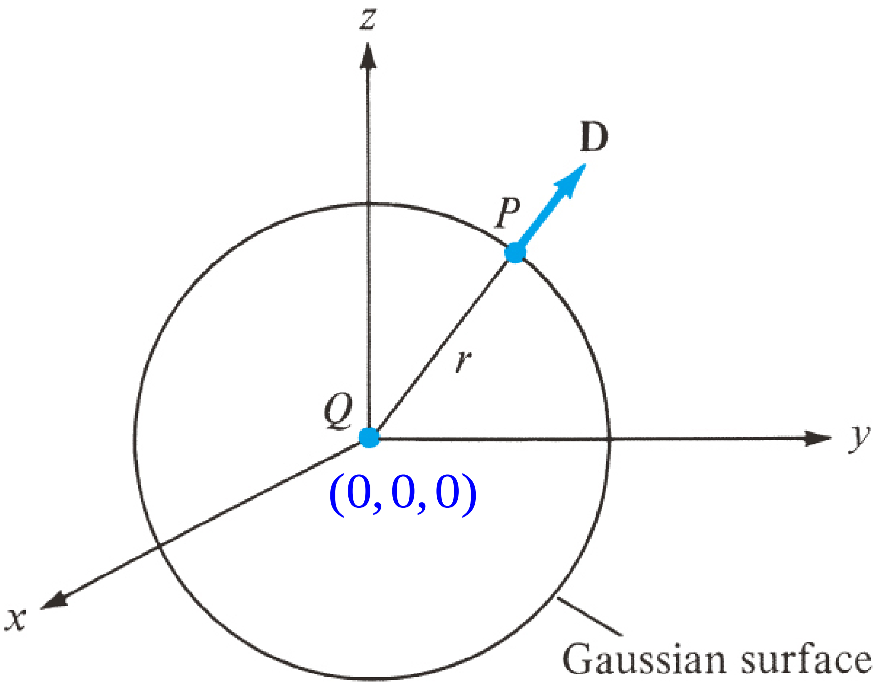
Ex.



$flux = 10 - 5 = 5nC$

Note: the 20 nC and 15 nC charges contribute to flux crossing V_1*, however, the net flux through* V_1 *is only due to the two charges inside* V_1*.*

Point charge:



Spherical symmetry suggested:

- *to find* $\vec{D}_P$ *use a spherical surface centered on* $(0,0,0)$

$-$*then* $\vec{D} \perp$ *surface*

$$\vec{D} \parallel \hat{r} \quad ; \quad \vec{D} = D_r \hat{r}$$

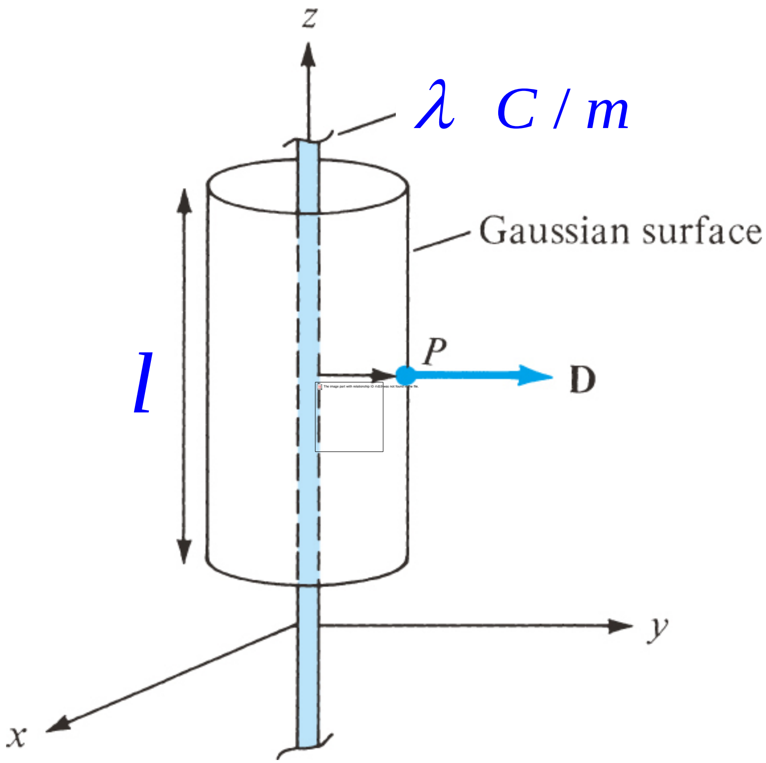
$$\vec{D} \parallel d\vec{A}$$

$$Q = \oint_{\text{G.S.}} \vec{D} \bullet d\vec{A} = D_r \oint_{\text{G.S.}} dA$$

$$= D_r \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi = D_r 4\pi r^2$$

$$\vec{D} = \frac{Q}{4\pi r^2} \hat{r}$$

Infinite line charge:



Cylindrical symmmetry suggested:

- *to find* $\vec{D}_P$ *use a* coaxial *cylinder* *(centered on line of charge)*

- *then* $\vec{D} \perp$ *Gaussian surface*

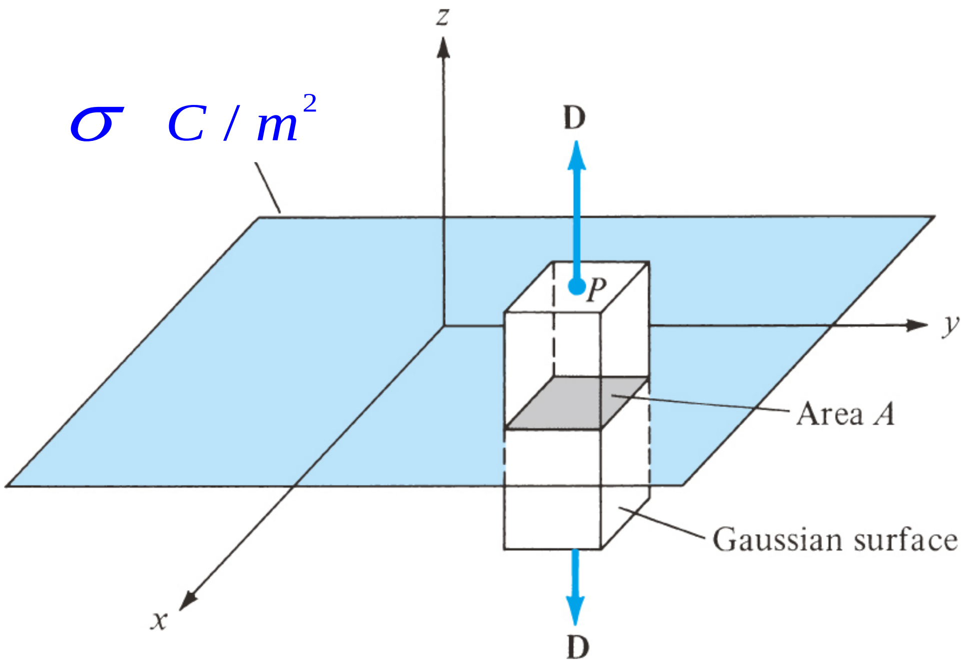
- $|D|$ *constant on Gaussian surface*

Then for some length l*:* $\quad q_{enc} = \lambda l = \oint_{G.S.} \vec{D} \bullet d\vec{A} = D_\rho \oint_{G.S.} dA$

$-$ *over the end caps:* $\vec{D} \bullet d\vec{A} = 0$

$$\therefore \; D_\rho \oint_{G.S.} dA = D_\rho 2\pi\rho l \qquad \text{and} \quad \vec{D} = \frac{\lambda}{2\pi\rho}\hat{\rho}$$

Infinite sheet of charge:



Plane symmmetry:

Gaussian surface

$\equiv$ *pill box crossing surface*

$\vec{D} \perp$ *sheet* $\qquad \vec{D} = D_z \hat{z}$

$\Rightarrow \vec{D} \bullet d\vec{A} = 0$ *everywhere on the sides of pill box*

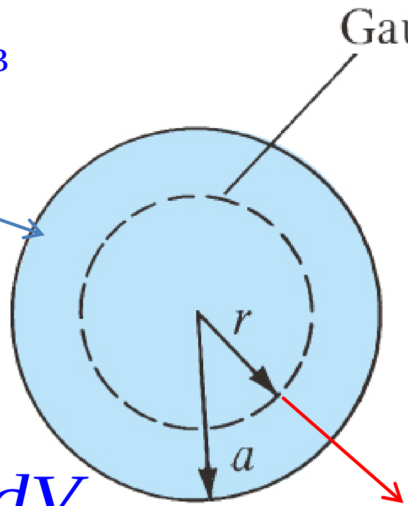
$$\therefore q_{enc} = \sigma \int_S dA = \sigma A = \oint_{G.S.} \vec{D} \bullet d\vec{A} = D_z \oint_{G.S.} dA = D_z (2A)$$

cross sectional area in plane of the sheet

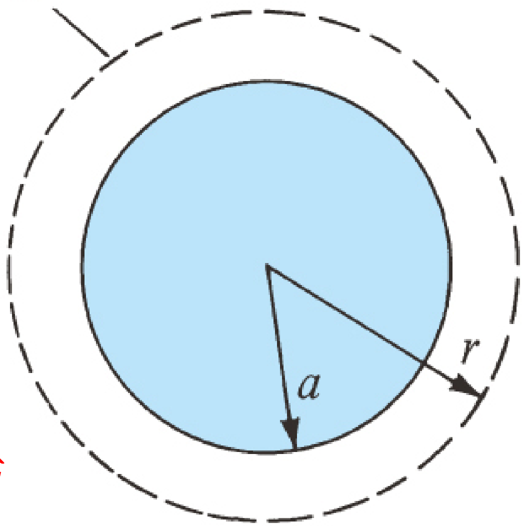
$$\text{or} \quad \vec{D} = \frac{\sigma}{2} \hat{z}$$

Uniformly charged sphere:

$\rho_0 \quad C/m^3$



$\vec{D} = D\hat{r}$



$0 < r \leq a:$

$$q_{enc} = \int_V \rho dV$$

$$= \rho_0 \int_{\phi=0}^{2\pi} \int_{\theta}^{\pi} \int_{r=0}^{r} r^2 \sin\theta dr d\theta d\phi$$

$$= \rho_0 \frac{4}{3} \pi r^3$$

$-$ *from Gauss's Law show*:

$$D_r 4\pi r^2 = \rho_0 \frac{4}{3} \pi r^3$$

get $\vec{D} = \dfrac{\rho_0}{3} r \ \hat{r} \quad (0 < r \leq a)$

$r \geq a:$

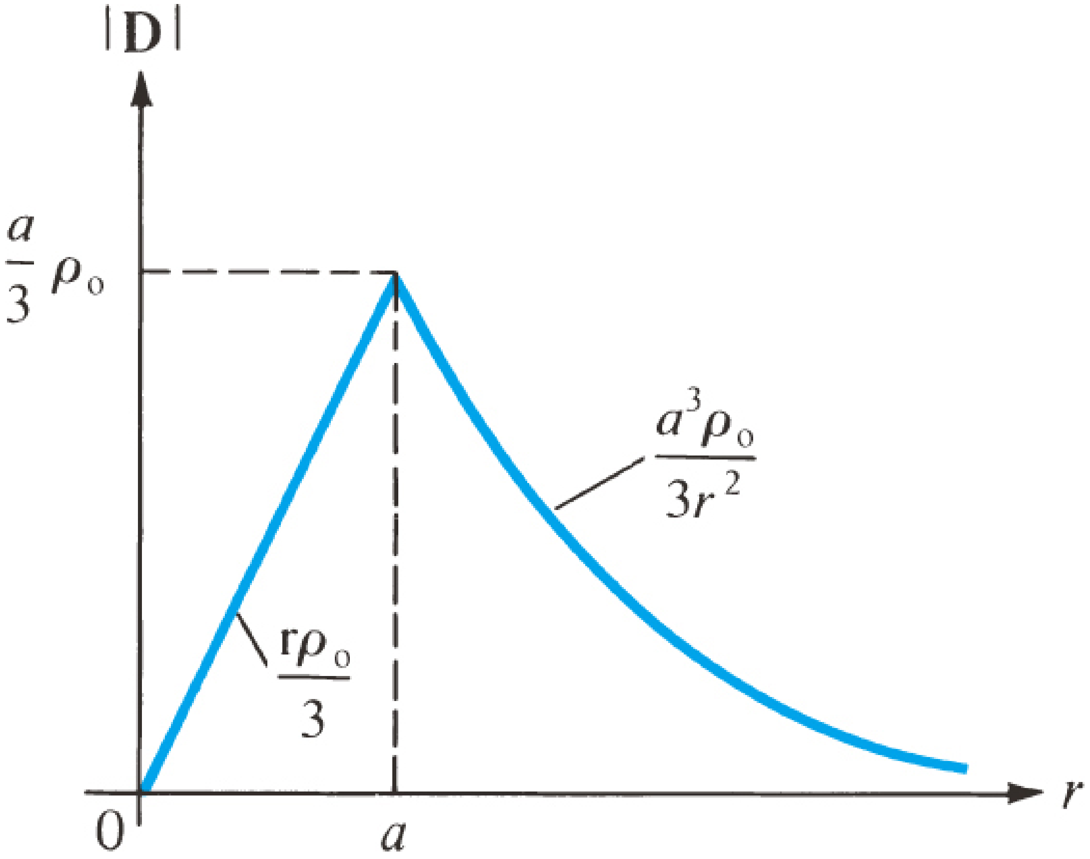
$$q_{enc} = \rho_0 \frac{4}{3} \pi a^3$$

$-$ *from Gauss's Law*:

$$D_r 4\pi r^2 = \rho_0 \frac{4}{3} \pi a^3$$

get

$$\vec{D} = \frac{\rho_0 a^3}{3r^2} \ \hat{r} \quad (r \geq a)$$



–also look at Examples 4.8 and 4.9

PRACTICE EXERCISE 4.9

A charge distribution in free space has $\rho_v = 2r\ \text{nC/m}^3$ for $0 \leq r \leq 10$ m and zero otherwise. Determine **E** at $r = 2$ m and $r = 12$ m.

Answer: $226\mathbf{a}_r$ V/m, $3.927\mathbf{a}_r$ kV/m.

Divergence of $\vec{D}$: $-$had $\phi_E = \oint_S \vec{D} \bullet d\vec{A}$

i.e. outward flow of flux from the surface

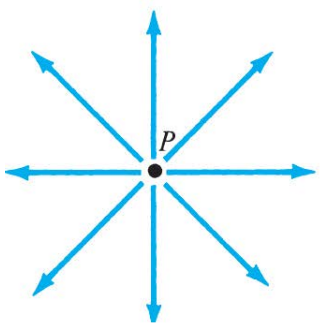
- *define* $div\ \vec{D} = \vec{\nabla} \bullet \vec{D} = \lim\limits_{\Delta V \to 0} \dfrac{\oint_S \vec{D} \bullet d\vec{A}}{\Delta V}$

i.e. outward flow of flux (per unit volume) as $\Delta V \to 0$; i.e. a point

- *how much the field diverges or emanates from a point*
- *a limit of the field source strength per unit volume*

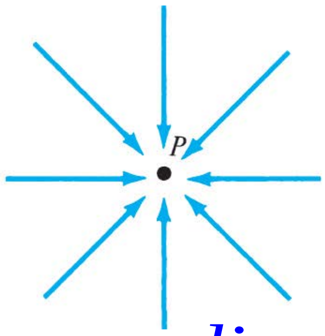
(source density)

Ex.



+ve div.

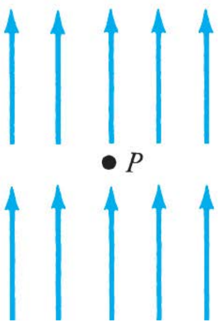
(source pt.)



$-$ve div.

(convergence)

(sink pt.)



zero div.

- *in Cartesian coordinates:*

$$\vec{\nabla}\bullet\vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

- *in cylindrical coordinates:*

$$\vec{\nabla}\bullet\vec{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

- *in spherical coordinates:*

$$\vec{\nabla}\bullet\vec{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(D_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}$$

- *some properties of divergence for vectors* $\vec{A}$ *and* $\vec{B}$:

1) *It produces a scalar field (a scalar product).*

2) $\vec{\nabla}\bullet(\vec{A}+\vec{B}) = \vec{\nabla}\bullet\vec{A} + \vec{\nabla}\bullet\vec{B}$

3) $\vec{\nabla}\bullet(V\vec{A}) = V\vec{\nabla}\bullet\vec{A} + \vec{A}\bullet\vec{\nabla}V$

- *Divergence Theorem*

$$\underbrace{\oint_S \vec{D}\bullet d\vec{A}}_{\substack{\text{outward flux} \\ \text{of vector field} \\ \vec{D}\text{ through } S}} = \underbrace{\int_V \vec{\nabla}\bullet\vec{D}\,dV}_{\substack{\text{volume integral} \\ \text{of divergence} \\ \text{of } \vec{D}}}$$

$Maxwell's\ Equation:$

$-$ *reconsider Gauss's Law:*

$$q_{enc} = \int_V \rho dV = \oint_S \vec{D}\bullet d\vec{A} \xrightarrow[\text{theorem}]{\text{by divergence}} = \int_V \vec{\nabla}\bullet\vec{D}dV$$

$-$ *comparing volume integrals:*

$$\vec{\nabla}\bullet\vec{D} = \rho \qquad 1^{st} \text{ of four Maxwell's Equations}$$

or $$\vec{\nabla}\bullet\vec{E} = \frac{\rho}{\varepsilon_0}$$

Note: this is Gauss's Law in differential form!

PRACTICE EXERCISE 4.8

If $\mathbf{D} = (2y^2 + z)\mathbf{a}_x + 4xy\mathbf{a}_y + x\mathbf{a}_z$ C/m^2, find

(a) The volume charge density at $(-1, 0, 3)$

(b) The flux through the cube defined by $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$

(c) The total charge enclosed by the cube

Answer: (a) -4 C/m^3, (b) 2 C, (c) 2 C.

***4.16** State Gauss's law. Deduce Coulomb's law from Gauss's law, thereby affirming that Gauss's law is an alternative statement of Coulomb's law and that Coulomb's law is implicit in Maxwell's equation $\nabla \cdot \mathbf{D} = \rho_v$.

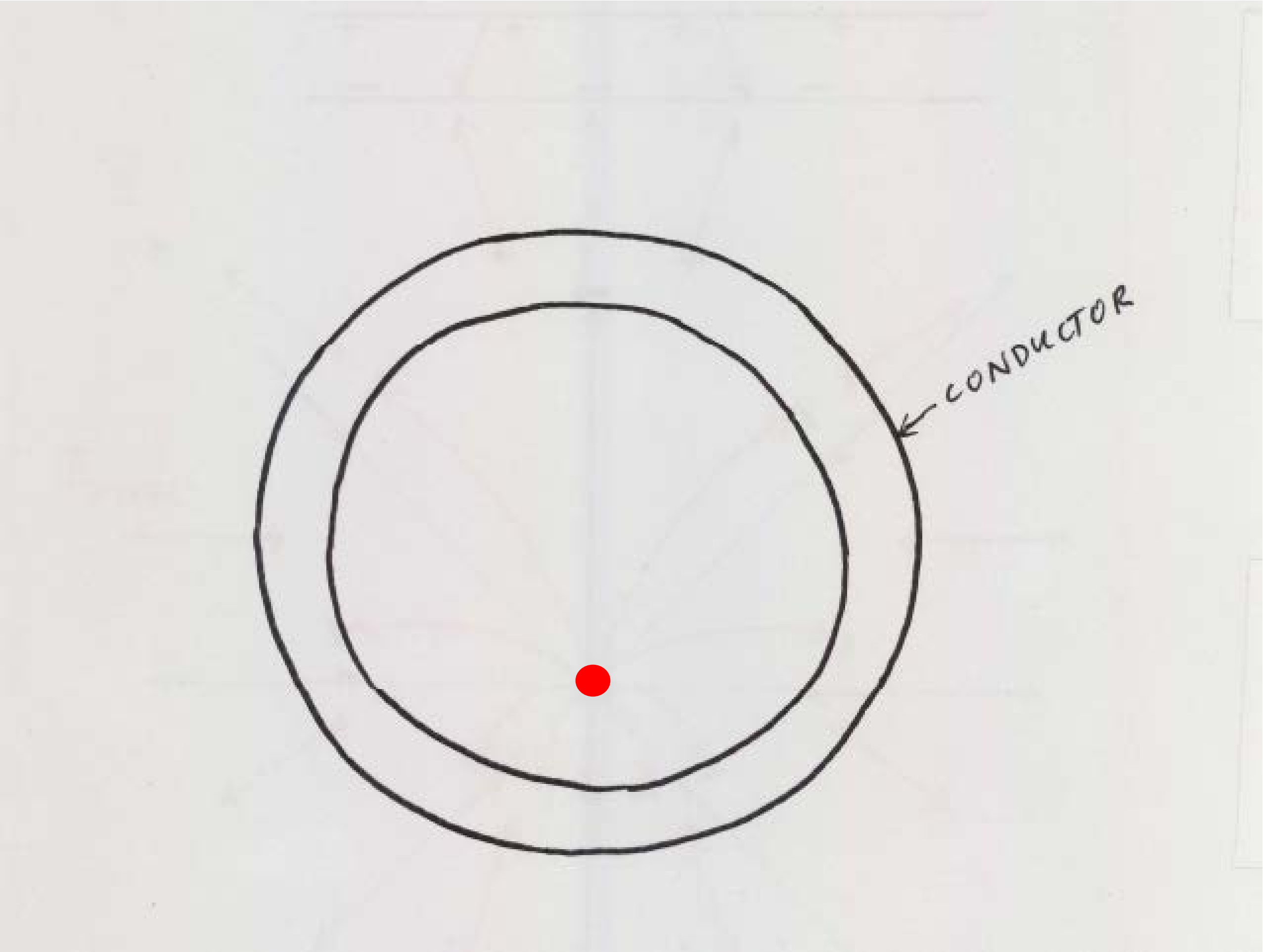
4.24 Let $\rho_v = \rho_o/r$ nC/m^3, $0 < r < a$, where ρ_o is a constant. (a) Find **E** inside and outside $r = a$. (b) Calculate the total charge.

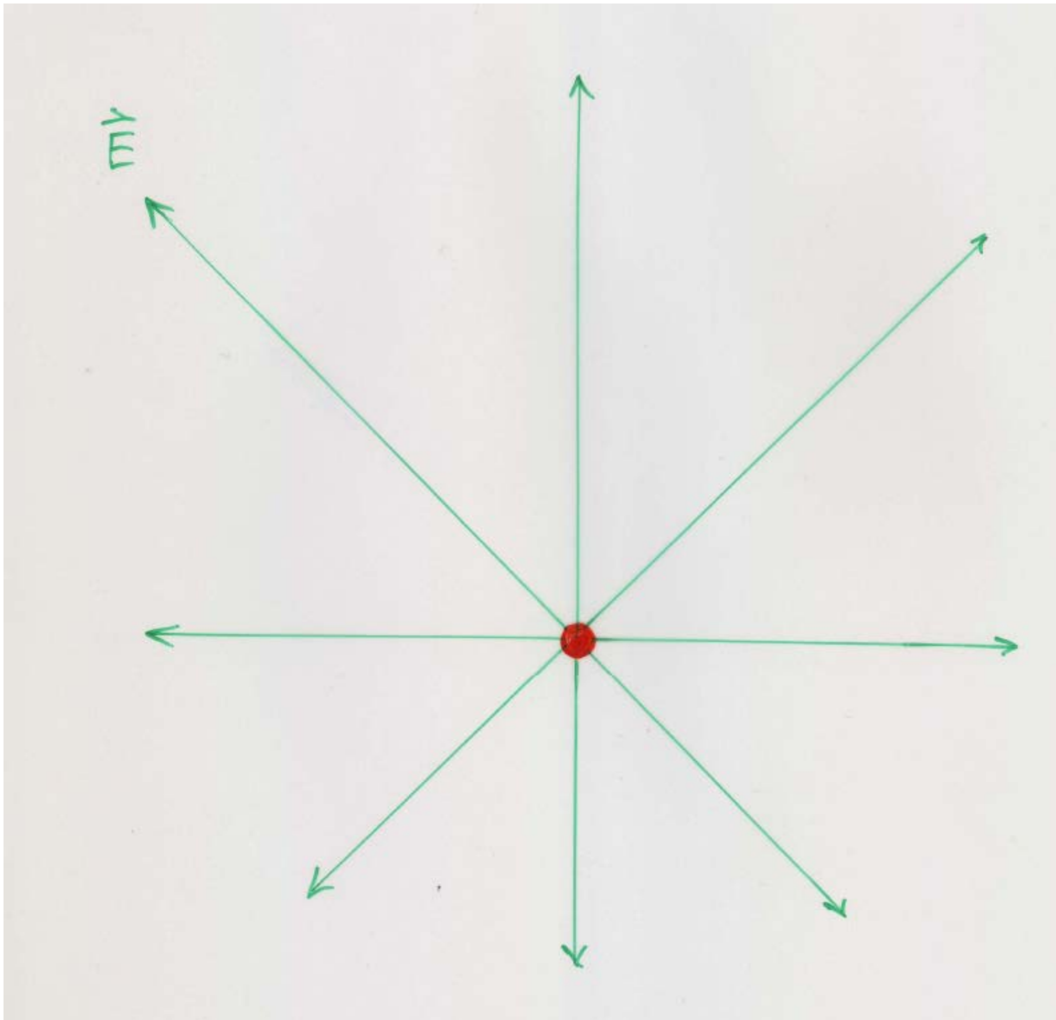
c) Now place a conducting, spherical shell with net charge Q, inner radius b and outer radius c, over the sphere, centered on the origin.

i) What is $\vec{E}$ everywhere?

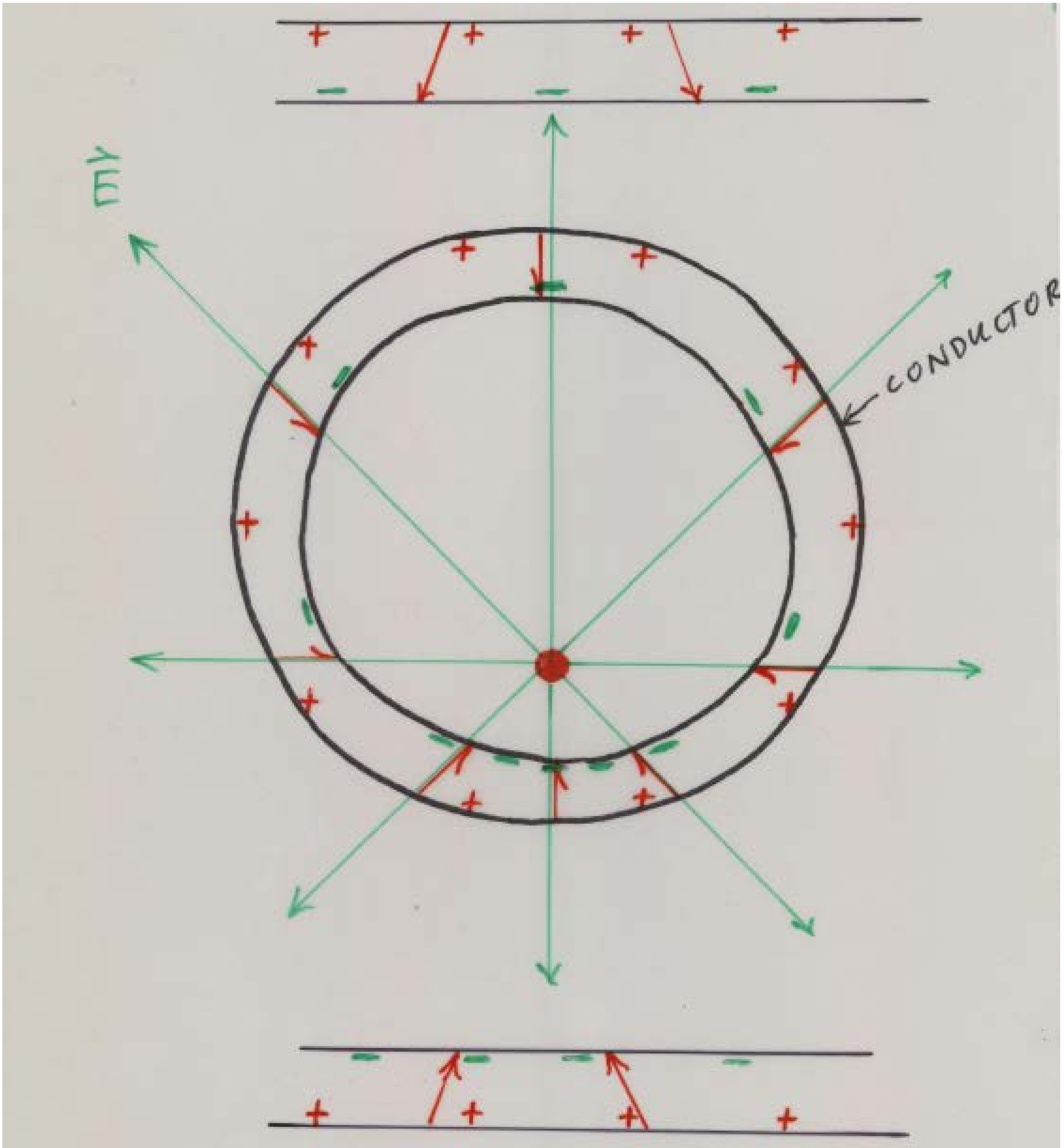
ii) Find the induced charge / unit area on innner and outer surfaces of the shell.

Additional commentary about E-fields in conductor
- charge +q inside cavity within an uncharged conductor:

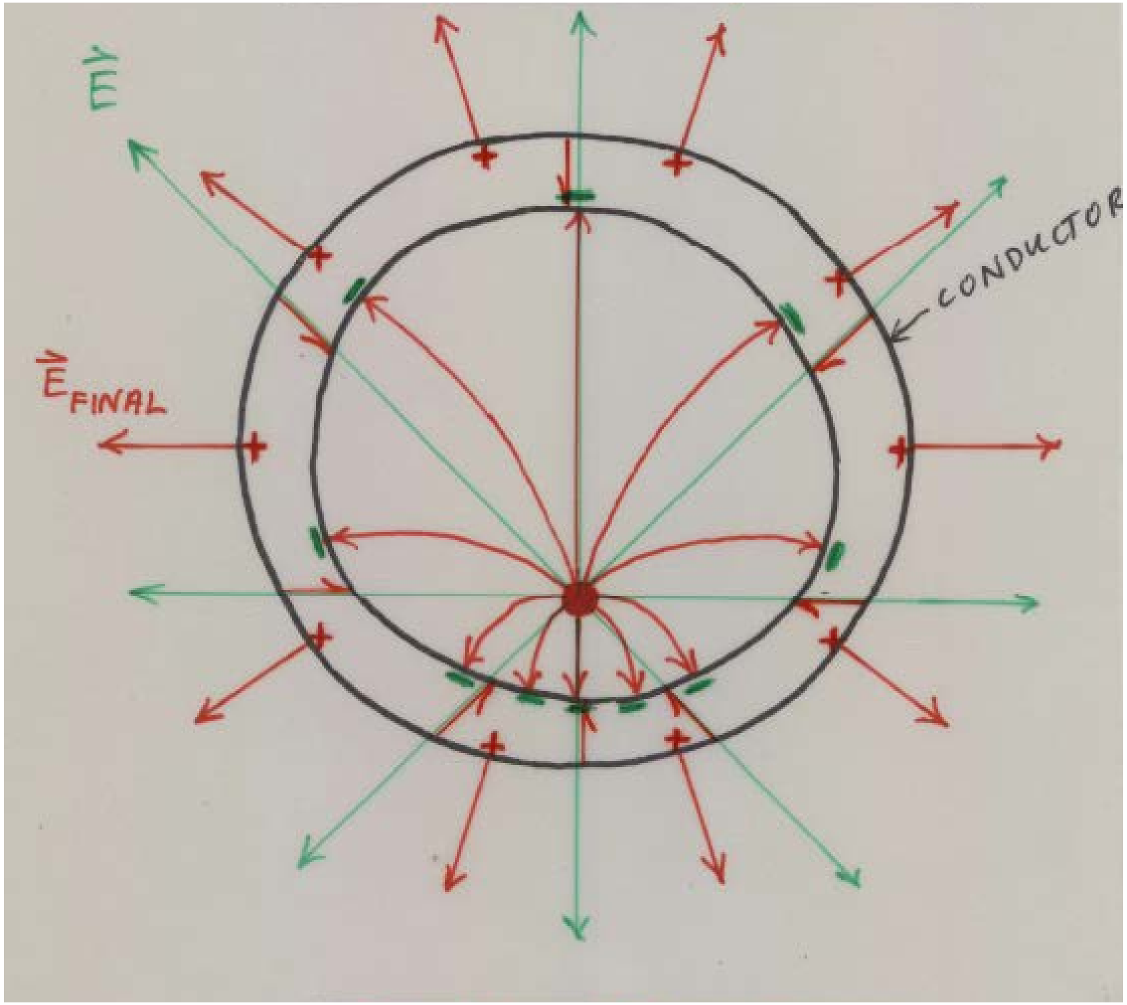




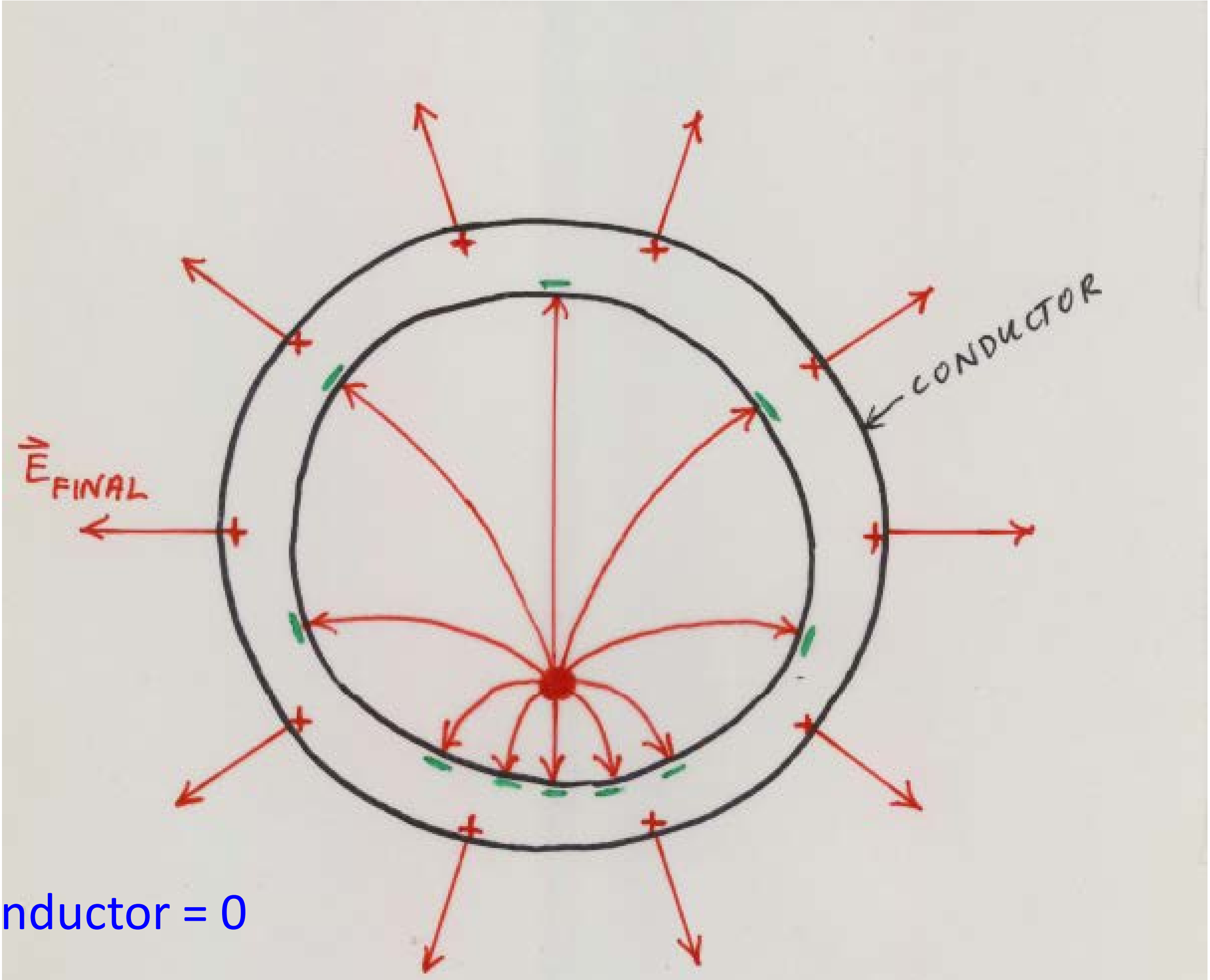
$\vec{E}$



$\vec{E}$
CONDUCTOR

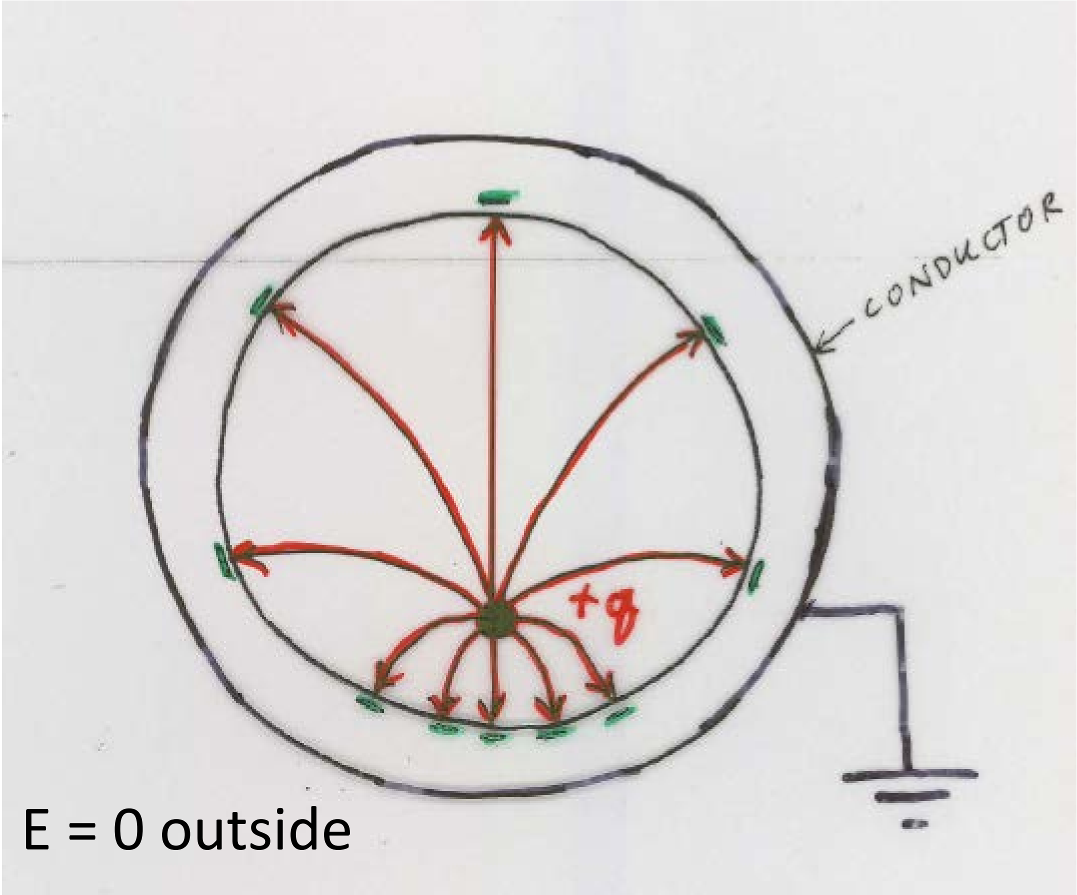


$\vec{E}$
$\vec{E}_{FINAL}$
CONDUCTOR



- E inside conductor = 0
- E inside cavity is modified as discussed
- E outside is that of a conductor with outer surface carrying a charge equal to net charge inside cavity – distributed in keeping with shape of outer surface of conductor

If outer surface of conductor is grounded:



Faraday's cage

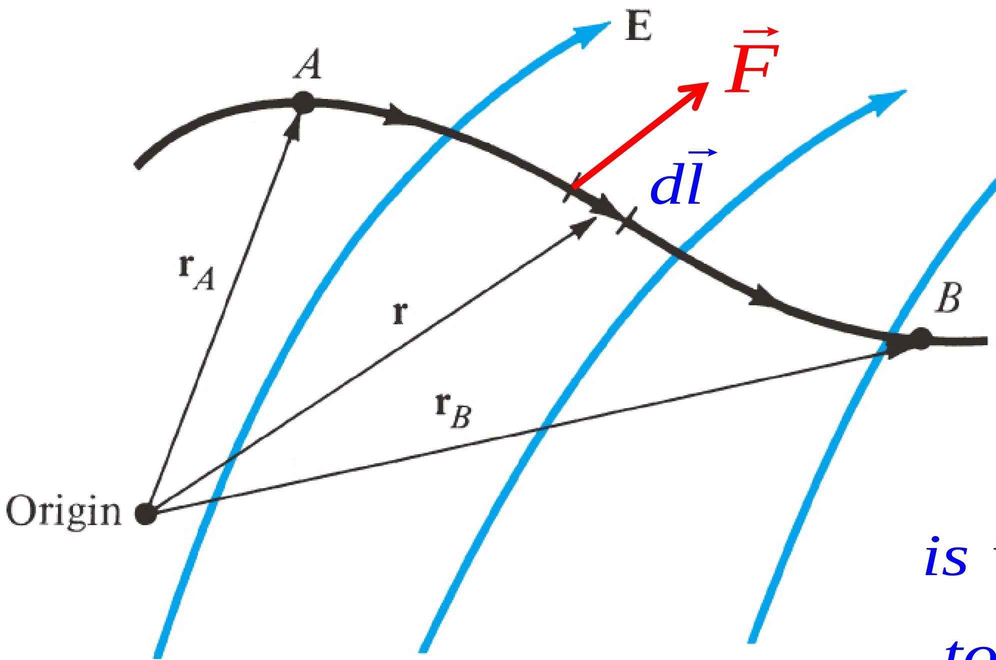
- No communication about static charge distribution across the enclosing conductor

- Also, electromagnetic shielding - important in communications (shielding of electronic equipment, military, etc.)

- Ground acts as infinite "sea" of charge

- Any field line due to excess charges will terminate or originate there - as needed to satisfy the rest of the problem (e.g., E = 0 inside conductor)

4 Electric Potential



Moving Q from A to B in E – field requires *work.*

$$\vec{F} = Q\vec{E}$$

$$dW = -\vec{F} \bullet d\vec{l} = -Q\vec{E} \bullet d\vec{l}$$

is work done by external agent to move Q over $d\vec{l}$

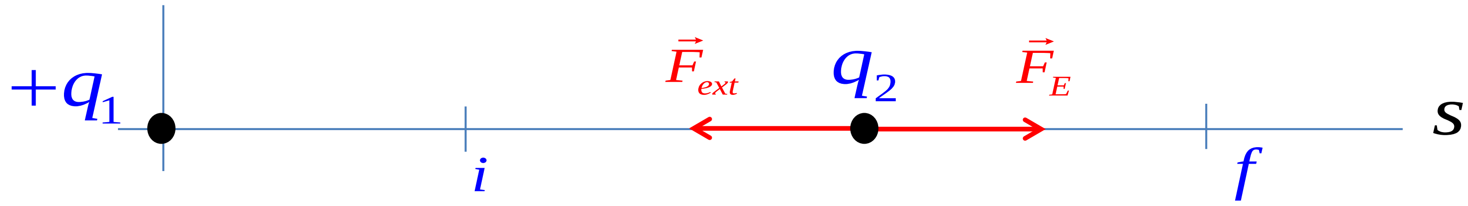
$\therefore$ total *work* required: $W = -Q\int_A^B \vec{E} \bullet d\vec{l}$

Potential difference $V_B - V_A = V_{AB} = \dfrac{W}{Q} = -\int_A^B \vec{E} \bullet d\vec{l}$ $[J / C]$ *or in volts* $[V]$

Note $V_{AB} < 0 \Rightarrow$ *loss in* electric potential energy *U* *(E did the work)*

$V_{AB} > 0 \Rightarrow$ *gain in U* *(external agent did the work)*

$$Reconsider\ U\ for\ a\ moment - of\ point\ charges:$$



$$Find\ the\ change\ in\ electric\ potential\ energy$$
$$as\ you\ move\ q_2\ from\ i\ to\ f,\ without\ acceleration.$$

$$\Delta U = U_f - U_i = W_{ext} = W_{if} = -q_2 \int_i^f \vec{E} \bullet d\vec{s}$$

$$here\ \ \vec{E} = \frac{q_1}{4\pi\varepsilon_0 r^2}\hat{r} \quad --note\ that\ \hat{r}\ is\ a\ unit\ vector\ in\ direction\ of\ increasing\ r \qquad d\vec{s} = d\vec{r} = |dr|\hat{r}$$

$$\therefore \Delta U = U_f - U_i = -q_2 \int_i^f \frac{q_1}{4\pi\varepsilon_0 r^2}\hat{r} \bullet dr\hat{r} = -q_2 \int_i^f \frac{q_1}{4\pi\varepsilon_0 r^2} dr$$

$$= -\frac{q_1 q_2}{4\pi\varepsilon_0}\int_i^f \frac{dr}{r^2} = -\frac{q_1 q_2}{4\pi\varepsilon_0}\left[-\frac{1}{r}\right]_{r_i}^{r_f} = \frac{q_1 q_2}{4\pi\varepsilon_0}\left(\frac{1}{r_f} - \frac{1}{r_i}\right) \quad (-ve)$$

$What\ about\ U\ rather\ than\ \Delta U\ ?$

$Only\ \Delta U\ has\ physical\ meaning!$

$Could\ take\ any\ reference\ point\ in\ the\ field!$

$In\ previous\ example\ take\ r_{initial} = \infty\ and\ move\ to\ r_f\ :$

$(\text{at}\ \infty\ \ E\ due\ to\ q_1 = 0\ and\ F_E\ on\ q_2\ is\ 0)$

$$\Delta U = U_f - U_i = U_f - U_\infty$$

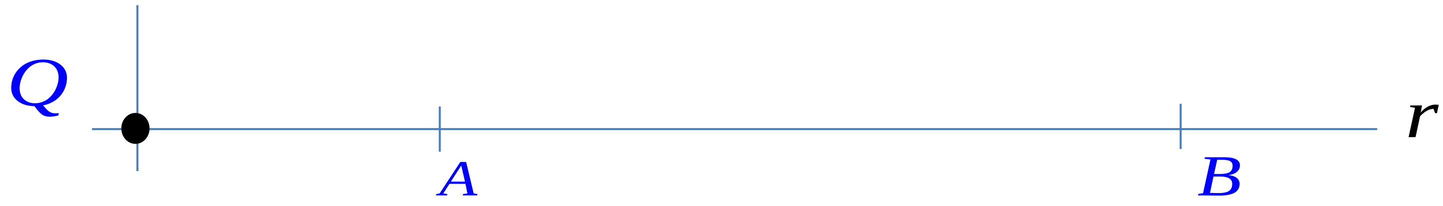
$$\text{then}\ \Delta U = \frac{q_1\ q_2}{4\pi\varepsilon_0}\left(\frac{1}{r = r_f} - \frac{1}{\infty}\right) = U_f$$

$$\mathrm{U} = \frac{1}{4\pi\varepsilon_0}\frac{q_1\ q_2}{r}\ (= U_{elec})$$

we say
"energy of q_2
in the field of q_1"

-- but remember that this is the energy of the system!

Electric Potential in E of point charge:



Find the change in electric potential as you move from A to B

$$\text{Potential difference } V_B - V_A = V_{AB} = -\int_{r_A}^{r_B} \vec{E} \bullet d\vec{l}$$

$$-\text{ here } \vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

$--$ *note that $\hat{r}$ is a unit vector in direction of increasing r*

$$d\vec{l} = d\vec{r} = |dr|\hat{r}$$

$$V_{AB} = V_B - V_A = -\int_{r_A}^{r_B} \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} \bullet dr\hat{r} = -\frac{Q}{4\pi\varepsilon_0}\int_{r_A}^{r_B} \frac{1}{r^2} dr$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_B} - \frac{1}{r_A}\right] \quad (-ve)$$

- *potential at B relative to that at A*

What about V rather than ΔV ?

For point charges one uses infinity as a reference point:

i.e. let $V_A = 0$ as $r_A \to \infty$

$V(r) \equiv$ potential due to Q at the origin

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{(r_B = r)} - \frac{1}{(r_A = \infty)}\right] = \frac{Q}{4\pi\varepsilon_0 r}$$

i.e. $\quad V(r) = -\int_{\infty}^{r} \vec{E}\bullet d\vec{l}$

- if Q not at origin, but at r': $V(x,y,z) = V(\vec{r}) = \dfrac{Q}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}\,'\right|}$

- for n point charges: $V(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\sum_{k=1}^{n}\frac{Q_k}{\left|\vec{r} - \vec{r}_k\right|}$

- *for continuous charge distributions:*

  – *line charge:* $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_L \frac{\lambda(\vec{r}\,')dl\,'}{|\vec{r}-\vec{r}\,'|}$$

  – *surface charge:* $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma(\vec{r}\,')dA'}{|\vec{r}-\vec{r}\,'|}$$

  – *volume charge:* $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\vec{r}\,')dV\,'}{|\vec{r}-\vec{r}\,'|}$$

$Notes:$

1) *In the last several expressions for* $V(\vec{r})$ *a reference point of* $V=0$ *at infinity was used.*

$$-- \text{ in general } V(\vec{r}) = [above\ V(\vec{r})] + C$$

$$where\ C \equiv potential\ at\ reference\ point$$

2) *If the charge distribution is not known, but* $\vec{E}$ *is:*

$$V = -\int \vec{E} \bullet d\vec{l} + C$$

$$and\ V_{AB} = V_B - V_A = -\int_A^B \vec{E} \bullet d\vec{l} = \frac{W}{Q}$$

Example 4.10 Two point charges $-4\ \mu C$ and 5 μC are located at (2,-1,3) and (0,4,-2), respectively. Find the potential at (1,0,1), assuming $V(\infty) = 0$.

Example

A line charge carrying uniform charge density λ *lies along the* x*-axis*. Showing all steps, *find* $V(\rho)$.

- to answer this the reference point needs to be specified!

Example 4.11 A point charge of 5 nC is located at (-3,4,0), while line $y = 1$, $z = 1$ carries uniform charge 2 nC/m.

a) If $V = 0$ at O(0,0,0), find V at A(5,0,1).

- also look at parts b and c.

Relation between E and V, Maxwell's Equation:

For this discussion we need the

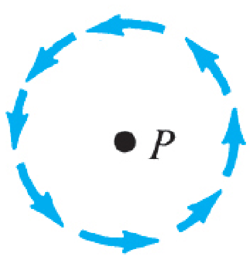
Curl and Stokes's Theorem:

$$curl\ \vec{A} = \vec{\nabla} \mathrm{x} \vec{A} = \left(\lim_{\Delta S \to 0} \frac{\oint_L \vec{A} \bullet d\vec{l}}{\Delta S} \right)_{\max} \hat{n}$$

$\equiv$ *axial (or rotational) vector whose mag. is the* max. *circulation of* $\vec{A}$ / *unit area* (circulation density), *as the area* ΔS *tends to zero and whose* direction *is the normal direction of* ΔS *when* ΔS *is oriented* *to make the circulation* max.

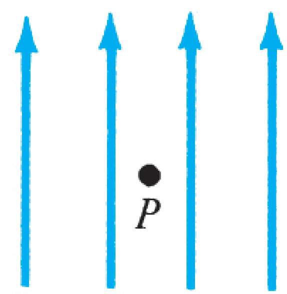
• *a measure of the circulation or how much the field "curls" around a point P*

e.g.



$\vec{E}$

curl points out of page



curl at P is 0

$$\vec{\nabla}\text{x}\vec{A}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}$$

$$\vec{\nabla}\text{x}\vec{A}=\frac{1}{\rho}\begin{vmatrix}\hat{\rho} & \rho\hat{\phi} & \hat{z}\\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z}\\ A_\rho & \rho A_\phi & A_z\end{vmatrix}$$

$$\vec{\nabla}\text{x}\vec{A}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi}\\ A_r & rA_\theta & r\sin\theta A_\phi\end{vmatrix}$$

Some properties of the curl :

1) *Curl of a vector field is another vector field*

2) $\vec{\nabla}\text{x}(\vec{A}+\vec{B})=\vec{\nabla}\text{x}\vec{A}+\vec{\nabla}\text{x}\vec{B}$

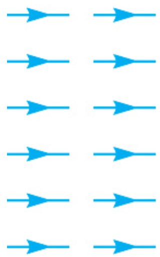
3) $\vec{\nabla}\text{x}(\vec{A}\text{x}\vec{B})=\vec{A}(\vec{\nabla}\bullet\vec{B})-\vec{B}(\vec{\nabla}\bullet\vec{A})+(\vec{B}\bullet\vec{\nabla})\vec{A}-(\vec{A}\bullet\vec{\nabla})\vec{B}$

4) $\vec{\nabla}\text{x}(V\vec{A})=V\vec{\nabla}\text{x}\vec{A}+\vec{\nabla}V\text{x}\vec{A}$

5) *Div of the curl of a vector field vanishes;* $\vec{\nabla}\bullet(\vec{\nabla}\text{x}\vec{A})=0$

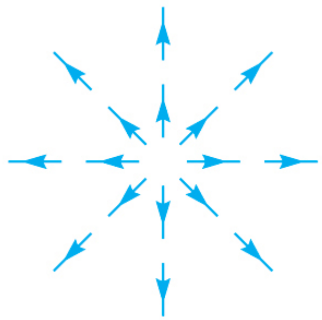
6) *Curl of the gradient of a scalar field vanishes;*

$$\vec{\nabla}\text{x}\vec{\nabla}V=0 \text{ or } \vec{\nabla}\text{x}\vec{\nabla}=0$$



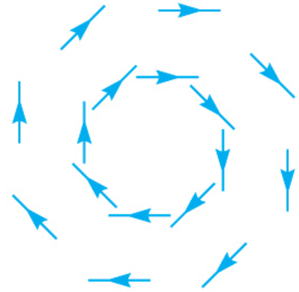
$$\vec{\nabla} \bullet \vec{A} = 0$$

$$\vec{\nabla} \mathrm{x} \vec{A} = 0$$



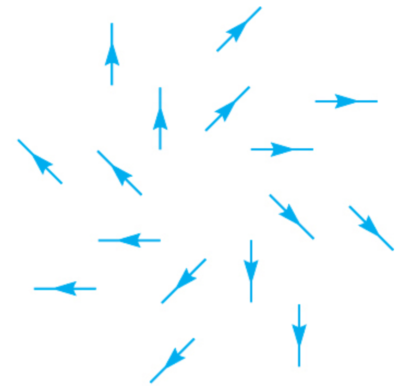
$$\vec{\nabla} \bullet \vec{A} \neq 0$$

$$\vec{\nabla} \mathrm{x} \vec{A} = 0$$



$$\vec{\nabla} \bullet \vec{A} = 0$$

$$\vec{\nabla} \mathrm{x} \vec{A} \neq 0$$



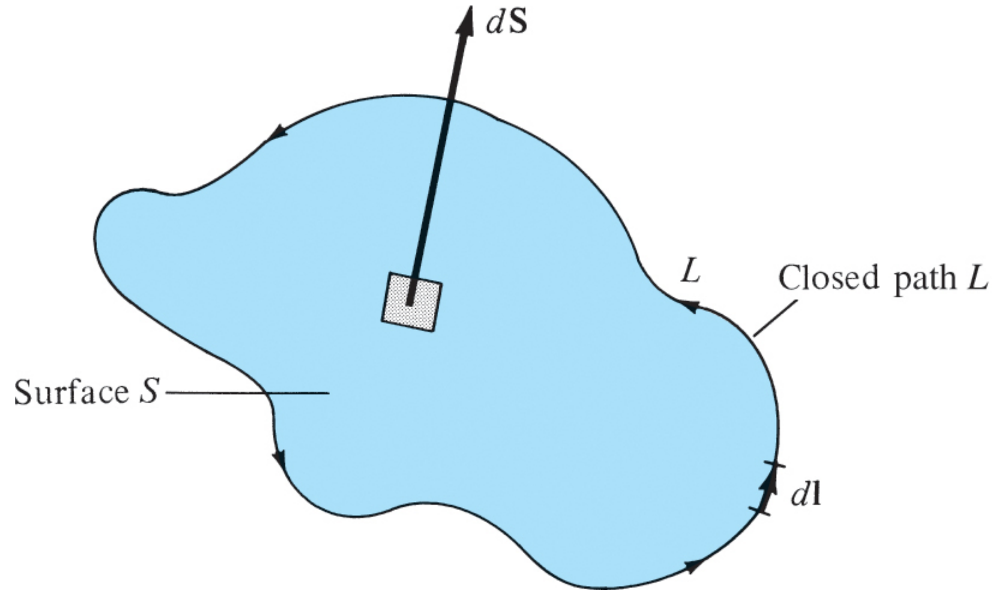
$$\vec{\nabla} \bullet \vec{A} \neq 0$$

$$\vec{\nabla} \mathrm{x} \vec{A} \neq 0$$

Stokes's Theorem:

$$\oint_L \vec{A} \bullet d\vec{l} = \int_S (\vec{\nabla} \mathrm{x} \vec{A}) \bullet d\vec{S}$$

circulation around closed path — *surface integral of curl over open surface bounded by L*



(compare with divergence theorem:

$$\oint_S \vec{D} \bullet d\vec{A} = \int_V \vec{\nabla} \bullet \vec{D} dV \quad)$$

Back to our electricity discussion:

We had for a point charge:

$$V_{AB} = V_B - V_A = -\int_{r_A}^{r_B} \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} \bullet dr\hat{r}$$

Since E points in the radial direction, the dot product $\vec{E} \bullet d\vec{l} = E\cos\alpha dl = Edr$ *excludes any contributions from displacements in* $\hat{\theta}$ *or* $\hat{\phi}$ *directions* *i.e. only* Δr *matters*

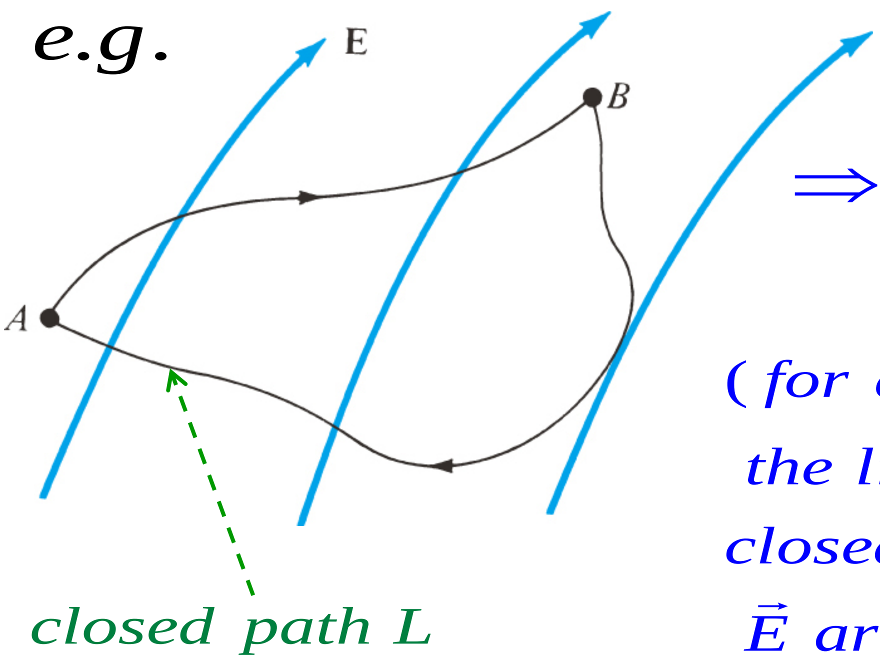
$\Rightarrow$ *the line integral / the potential difference* V_{AB} *is independent of path - this is true for any electrostatic E field.*

- *such a vector field is called a conservative vector field (vector is conservative)*

$$\Rightarrow V_{AB} = -V_{BA} \Rightarrow V_{AB} + V_{BA} = \oint \vec{E} \bullet d\vec{l} = 0$$

$$\oint \vec{E} \bullet d\vec{l} = 0$$

e.g.



$$\oint_L \vec{E} \bullet d\vec{l} = 0$$

$\Rightarrow W = 0$ *to move a charge over the (any) closed path*

(for a conservative vector field $\vec{E}$ *the line integral of* $\vec{E}$ *around a closed path L - the circulation of* $\vec{E}$ *around L - equals 0)*

Using Stokes's theorem:

$$\oint_L \vec{E} \bullet d\vec{l} = \int_S (\vec{\nabla} \mathrm{x} \vec{E}) \bullet d\vec{S} = 0$$

$\Rightarrow \vec{\nabla} \mathrm{x} \vec{E} = 0$ $\quad 2^{nd}$ *Maxwell's Equation for static E – fields (differential form)*

$\left(\oint_L \vec{E} \bullet d\vec{l} = 0 \right.$ *is the integral form*$\left. \right)$

Note that a conservative vector field (curl-free) is also called an irrotational vector field.

Gradient of V: $\quad -$ *had defined* $V = -\int \vec{E} \bullet d\vec{l}$

$$\therefore \; dV = -\vec{E} \bullet d\vec{l} = -E_x dx - E_y dy - E_z dz$$

$$\text{but } dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$$

$$\Rightarrow \; E_x = -\frac{\partial V}{\partial x}, \; E_y = -\frac{\partial V}{\partial y}, \; E_z = -\frac{\partial V}{\partial z}$$

$$\text{or} \quad \vec{E} = -\vec{\nabla} V$$

- 3 *orthogonal vector components from a scalar!!*
- 3 *components of* $\vec{E}$ *not independent of each other:* *interrelated by* $\vec{\nabla} \mathrm{x} \vec{E} = 0$

Example 4.12

Given the potential $V = \dfrac{10}{r^2}\sin\theta\cos\phi$,

a) *Find the electric flux density D at* $(2,\ \pi/2,\ 0)$.

b) *Calculate the work done in moving a* $10\ \mu C$ *charge from point* $A(1,\ 30^o,\ 120^o)$ *to* $B(4,\ 90^o,\ 60^o)$.

Problem 4.32

$V = x^2 y(z+3)$ V. *Find*

a) $\vec{E}$ *at* $(3, 4, -6)$

b) *the charge within the cube* $0 < x < 1,\ 0 < y < 1,\ 0 < z < 1.$

Problem 4.34 - *some thoughts on this only (assignment)*

A spherically symmetric charge distribution is given by

$$\rho = \begin{cases} \rho_0\left(1-\dfrac{r}{a}\right)^2, & r \le a \\ 0, & r \ge a \end{cases}$$

$a)$ *Find* $\vec{E}$ *and* V *for* $r \le a$.

Can use Gauss's Law to find $\vec{E}$. *Then find* $V = (\ldots) + C_1$

Take $V(\infty) = 0$ *to find* C_1.

$b)$ *Find* $\vec{E}$ *and* V *for* $r \ge a$.

Can use Gauss's Law to find $\vec{E}$. *Then find* $V = (\ldots) + C_2$

To find C_2 *take* $V(r = a) =$ *to the value*

of V *from the* $a-$ *part for* $r = a$.

$a)$ *Find the total charge. (usual methods)*

Problem 4.47

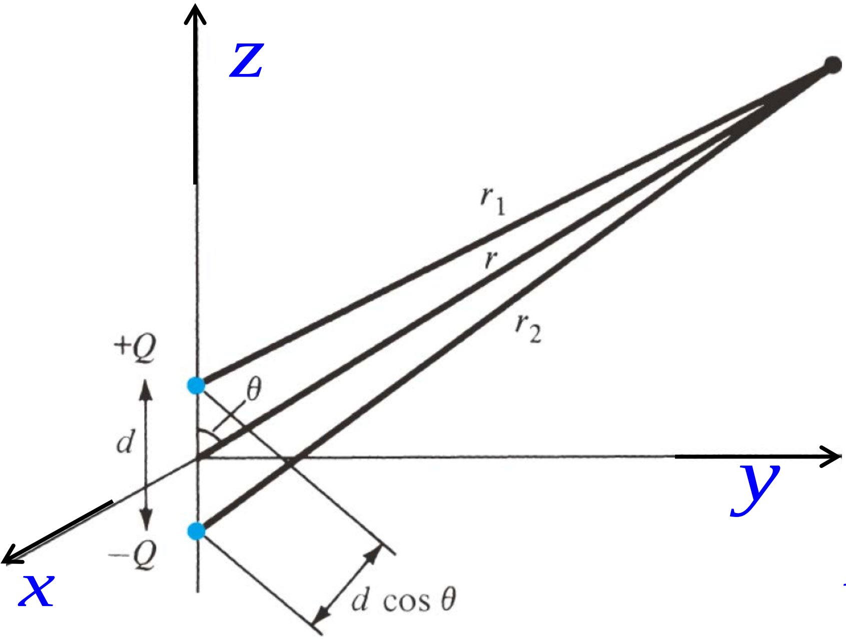
If $\vec{D} = 2\rho\sin\phi\hat{\rho} - \dfrac{\cos\phi}{2\rho}\hat{\phi}\ C/m_2$, *determine whether or not* $\vec{D}$ *is a genuine electric flux density.*

Determine the flux crossing $\rho = 1,\ 0 \le \phi \le \pi/4,\ 0 < z < 1.$

for cylindrical coordinates

$$\vec{\nabla}\mathrm{x}\vec{D} = \left[\frac{1}{\rho}\frac{\partial D_z}{\partial\phi} - \frac{\partial D_\phi}{\partial \mathrm{z}}\right]\hat{\rho} + \left[\frac{\partial D_\rho}{\partial \mathrm{z}} - \frac{\partial D_z}{\partial\rho}\right]\hat{\phi} + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho D_\phi) - \frac{\partial D_\rho}{\partial\phi}\right]\hat{z}$$

Electric Dipole and Flux Lines:

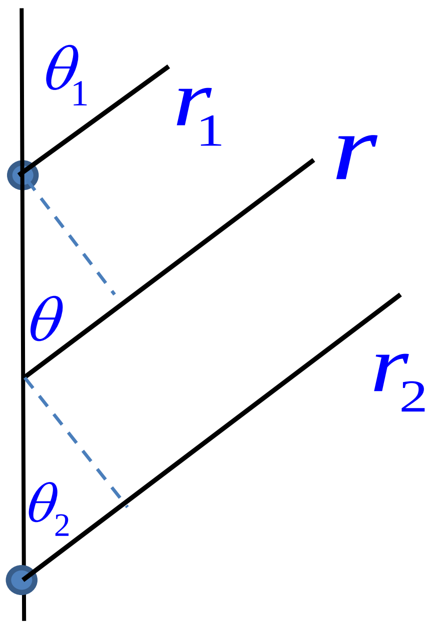


Find V, and then $\vec{E}$ from V

$$V = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right] = \frac{Q}{4\pi\varepsilon_0}\left[\frac{r_2 - r_1}{r_1 r_2}\right]$$

Now assume $r \gg d$:

$$\theta_1 \simeq \theta \simeq \theta_2 \Rightarrow r_2 \simeq r + \frac{d}{2}\cos\theta\ ,\ r_1 \simeq r - \frac{d}{2}\cos\theta$$



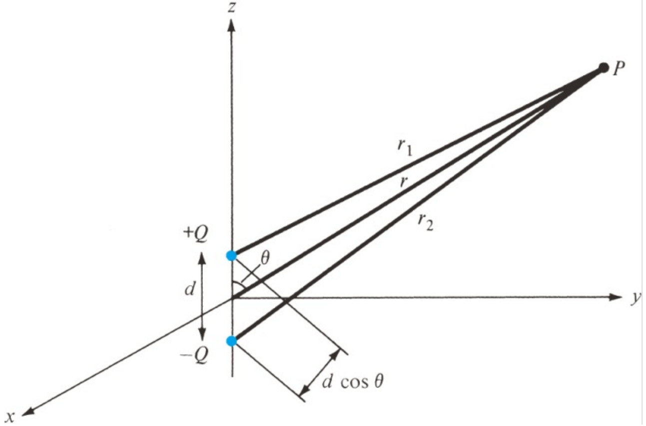
$$\left.\begin{aligned}\therefore\ r_2 - r_1 &\simeq d\cos\theta \\ r_2 r_1 &\simeq r^2\end{aligned}\right\} V = \frac{Q}{4\pi\varepsilon_0}\frac{d\cos\theta}{r^2}$$

but $d\cos\theta = \vec{d}\bullet\hat{r}$ *where* $\vec{d} = d\hat{z}$

dipole moment:

$$\vec{p} = Q\vec{d}$$

$$\therefore\ V = \frac{\vec{p}\bullet\hat{r}}{4\pi\varepsilon_0 r^2}$$



$$V = \frac{\vec{p}\bullet\hat{r}}{4\pi\varepsilon_0 r^2}$$

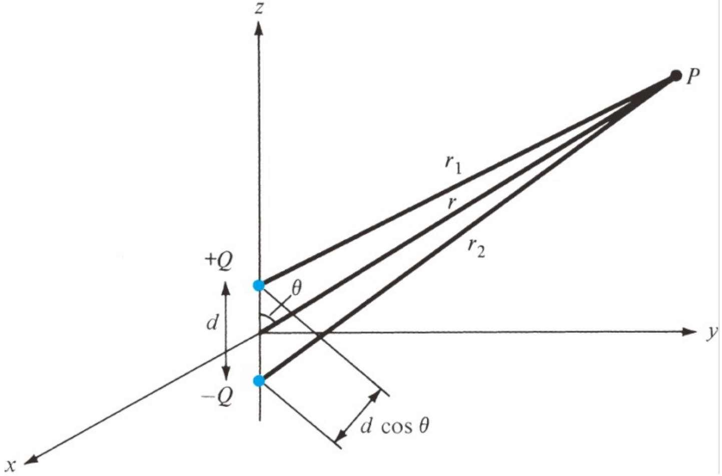
$\vec{E}$ can be obtained directly from the negative gradient of V :

$$\vec{E} = -\vec{\nabla}V = -\left[\frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta}\right]$$

$$= -\frac{Qd}{4\pi\varepsilon_0}\left[\frac{\partial}{\partial r}\left(\frac{\cos\theta}{r^2}\right)\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{r^2}\right)\hat{\theta}\right]$$

$$\vec{E} = \frac{Qd\cos\theta}{2\pi\varepsilon_0 r^3}\hat{r} \quad + \frac{Qd\sin\theta}{4\pi\varepsilon_0 r^3}\hat{\theta}$$

$$\vec{E} = \frac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta\ \hat{r} + \sin\theta\ \hat{\theta})$$



$$V = \frac{\vec{p} \bullet \hat{r}}{4\pi\varepsilon_0 r^2}$$

- *if $\vec{p}$ not centered on origin (still along $\hat{z}$) but at $\vec{r}\,'$:*

$$V(\vec{r}) = \frac{\vec{p} \bullet (\hat{r} - \vec{r}\,')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}\,'\right|^3}$$

Dependence of E and V of multipoles on r:

	E	V
monopole	$\propto \dfrac{1}{r^2}$	$\propto \dfrac{1}{r}$
dipole (2 *monopoles*)	$\propto \dfrac{1}{r^3}$	$\propto \dfrac{1}{r^2}$
quadrupoles (2 *dipoles*)	$\propto \dfrac{1}{r^4}$	$\propto \dfrac{1}{r^3}$
octupoles (2 *quadrupoles*)	$\propto \dfrac{1}{r^5}$	$\propto \dfrac{1}{r^4}$

PRACTICE EXERCISE 4.13

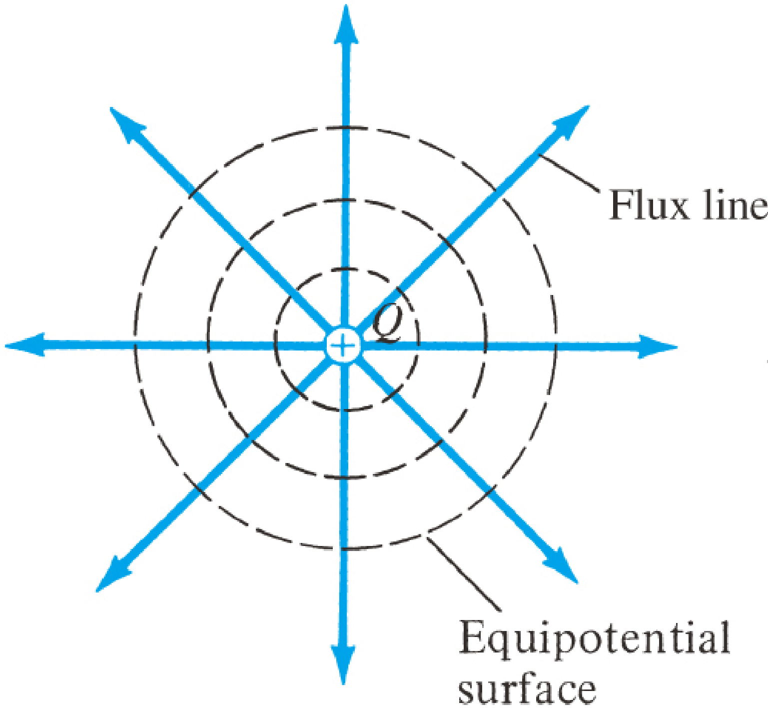
An electric dipole of 100 $\mathbf{a}_z$ pC $\cdot$ m is located at the origin. Find V and $\mathbf{E}$ at points

(a) $(\overset{10}{\cancel{0}}, 0, \overset{0}{\cancel{10}})$

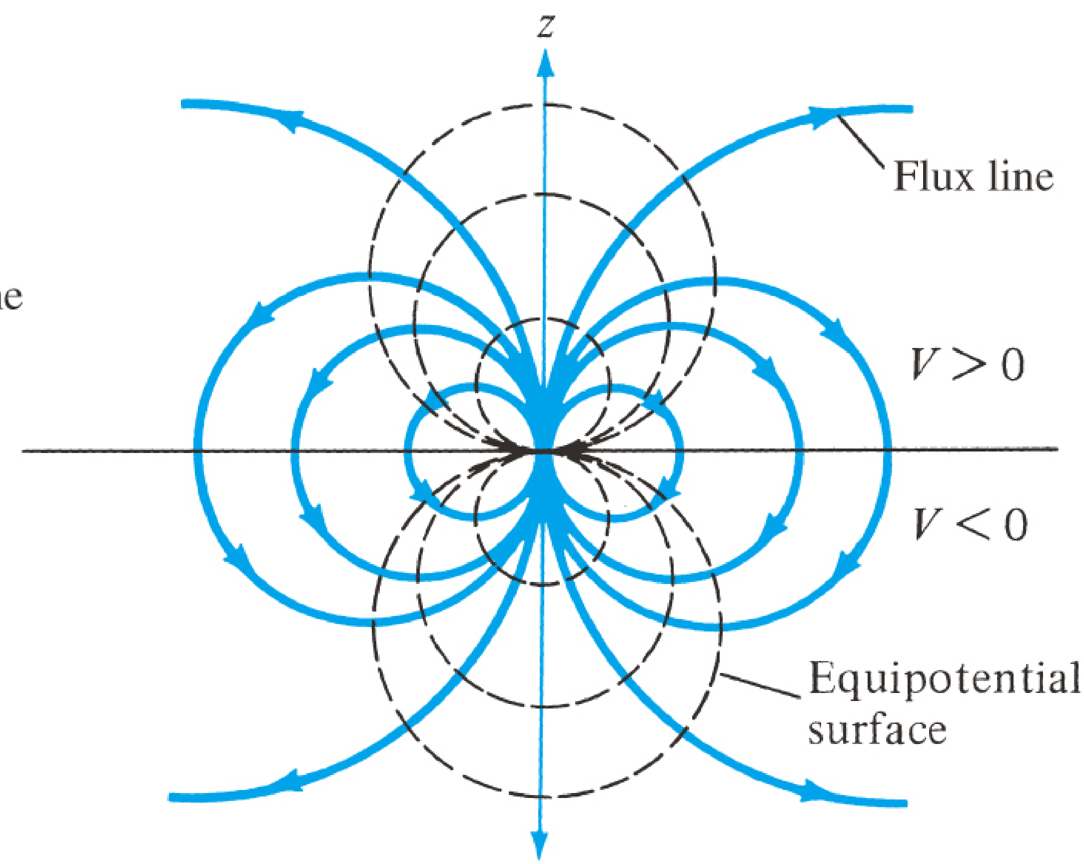
(b) $(1, \pi/3, \pi/2)$

Answer: (a) 9 mV, $1.8\mathbf{a}_r$ mV/m, (b) 0.45 V, $0.9\mathbf{a}_r + 0.7794\mathbf{a}_\theta$ V/m.

Electric flux lines (electric lines of force) introduced by Michael Faraday:



(a)

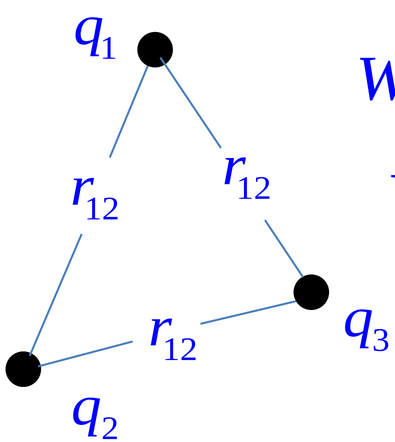


(b)

$$\int_L \vec{E} \bullet dl = 0$$ *along an equipotential line or surface*

Energy Density in Electrostatic fields :

– consider energy needed to assemble a point charge configuration



$$W_T = U_T = W_1 + W_2 + W_3$$

$$W_T = U_T = 0 + q_2 V_{21} + q_3 (V_{31} + V_{32})$$

where $V_{ij} \equiv V$ at i due to q_j

We can postion the charges in reverse order - same energy / work:

$$W_T = U = W_3 + W_2 + W_1 = 0 + q_2 V_{23} + q_1 (V_{12} + V_{13})$$

– adding : $W = \dfrac{1}{2}(q_1 V_1 + q_2 V_2 + q_3 V_3)$ where V_i = potential at i due to charges j and k

- for n point charges: $W = \dfrac{1}{2}\sum_{k=1}^{n} q_k V_k$
- to assemble continuous distributions of charge:

$$W = \frac{1}{2}\int \lambda V dl \qquad W = \frac{1}{2}\int \sigma V dA \qquad W = \frac{1}{2}\int \rho V dv$$

(Note that these W's (work done by external agent) are equal to U's of system.)

$V \equiv$ electric potential

$dv \equiv$ volume element

PRACTICE EXERCISE 4.14

Point charges $Q_1 = 1\text{ nC}$, $Q_2 = -2\text{ nC}$, $Q_3 = 3\text{ nC}$, ~~and $Q_4 = -4\text{ nC}$~~ are positioned one at a time and in that order at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 0, -1)$, ~~and $(0, 0, 1)$~~, respectively. Calculate the energy in the system after each charge is positioned.

Answer: 0, -18 nJ, -29.18 nJ, -68.27 nJ.

$$W = \frac{1}{2}\sum_{k=1}^{n} q_k V_k \qquad \textit{where } V_i = \textit{potential at } i \textit{ due to} \text{ all but } \textit{the } i^{th} \textit{ charge}$$

Energy Density:

Reconsider $W = U = \frac{1}{2}\int \rho V dv \quad = \frac{1}{2}\int (\vec{\nabla}\bullet\vec{D})V dv$

In addition, we had $\vec{\nabla}\bullet(V\vec{A}) = V(\vec{\nabla}\bullet\vec{A}) + \vec{A}\bullet\vec{\nabla}V$

or $(\vec{\nabla}\bullet\vec{A})V = \vec{\nabla}\bullet V\vec{A} - \vec{A}\bullet\vec{\nabla}V$

– substitution into the above integral:

$$W = \frac{1}{2}\int (\vec{\nabla}\bullet V\vec{D})dv - \frac{1}{2}\int (\vec{D}\bullet\vec{\nabla}V)dv$$

Divergence Theorem

$$W = \frac{1}{2}\oint_S (V\vec{D})\bullet d\vec{A} - \frac{1}{2}\int (\vec{D}\bullet\vec{\nabla}V)dv$$

or $W = \frac{\varepsilon_0}{2}\oint_S V\vec{E}\bullet d\vec{A} + \frac{1}{2}\int (\vec{D}\bullet\vec{E})dv$

Started with $W = \dfrac{1}{2}\int_V \rho V dv \quad (a)$

now have $W = \dfrac{\varepsilon_0}{2}\oint_S V\vec{E}\bullet d\vec{A} + \dfrac{\varepsilon_0}{2}\int_V E^2 dv \quad (b)$

What volume is involved? $\Big\}\ \displaystyle\int_{volume\ containing\ \rho} (\quad)dv\ \Big\{$ volume with non-zero ρ

$-$ *if additional volume is included, outside of that containing* ρ *, the initial volume integral (a) is unchanged*

- *for the volume term of (b):*

$\uparrow$ *as volume* $\uparrow$ *since* E^2 *always* $+ve$

-then, since (a)=(b) $\Rightarrow$ the area term of (b) must correspondingly decrease

e.g. for point charges $VE \propto \dfrac{1}{r}\dfrac{1}{r^2}$ *while area* $\uparrow$ *as* $r^2 \Rightarrow$ *this term* $\downarrow$ *as* $\dfrac{1}{r}$

If we let $r \to \infty$, 1^{st} *term in (b)* $\to 0$

i.e. integrate over all space $\Rightarrow\ W = U = \dfrac{\varepsilon_0}{2}\displaystyle\int_{all\ space} E^2 dv$

$$W = U = \frac{1}{2}\int_V \vec{D}\bullet\vec{E}dv = \frac{\varepsilon_0}{2}\int_V \vec{E}\bullet\vec{E}dv = \frac{\varepsilon_0}{2}\int_V E^2 dv$$

$$\frac{W}{unit\ volume} = w = u = \frac{dW}{dv} = \frac{1}{2}\vec{D}\bullet\vec{E} = \frac{1}{2}\varepsilon_0 E^2 = \frac{D^2}{2\varepsilon_0}$$

PRACTICE EXERCISE 4.15

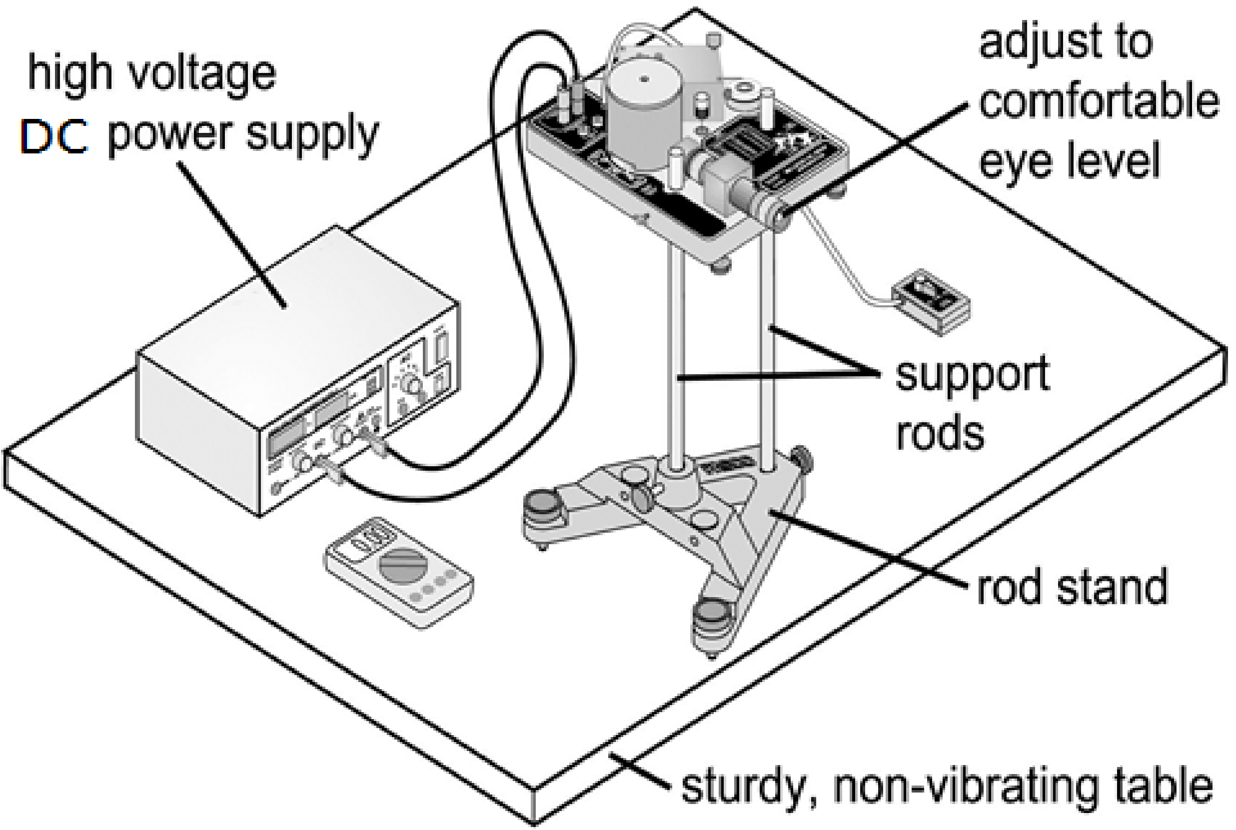
If $V = x - y + xy + 2z$ V, find **E** at $(1, 2, 3)$ and the electrostatic energy stored in a cube of side 2 m centered at the origin.

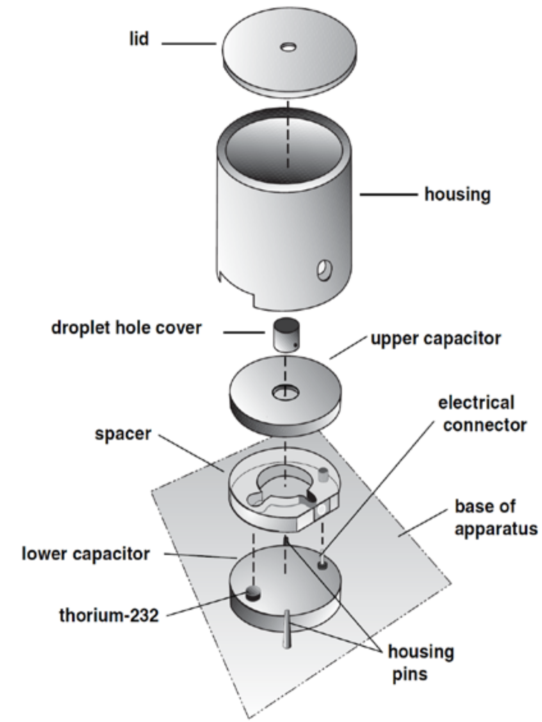
Answer: $-3\mathbf{a}_x - 2\mathbf{a}_z$ V/m, 0.2358 nJ.

$$W = U = \frac{1}{2}\int_V \vec{D} \bullet \vec{E} dv = \frac{\varepsilon_0}{2}\int_V \vec{E} \bullet \vec{E} dv = \frac{\varepsilon_0}{2}\int_V E^2 dv$$

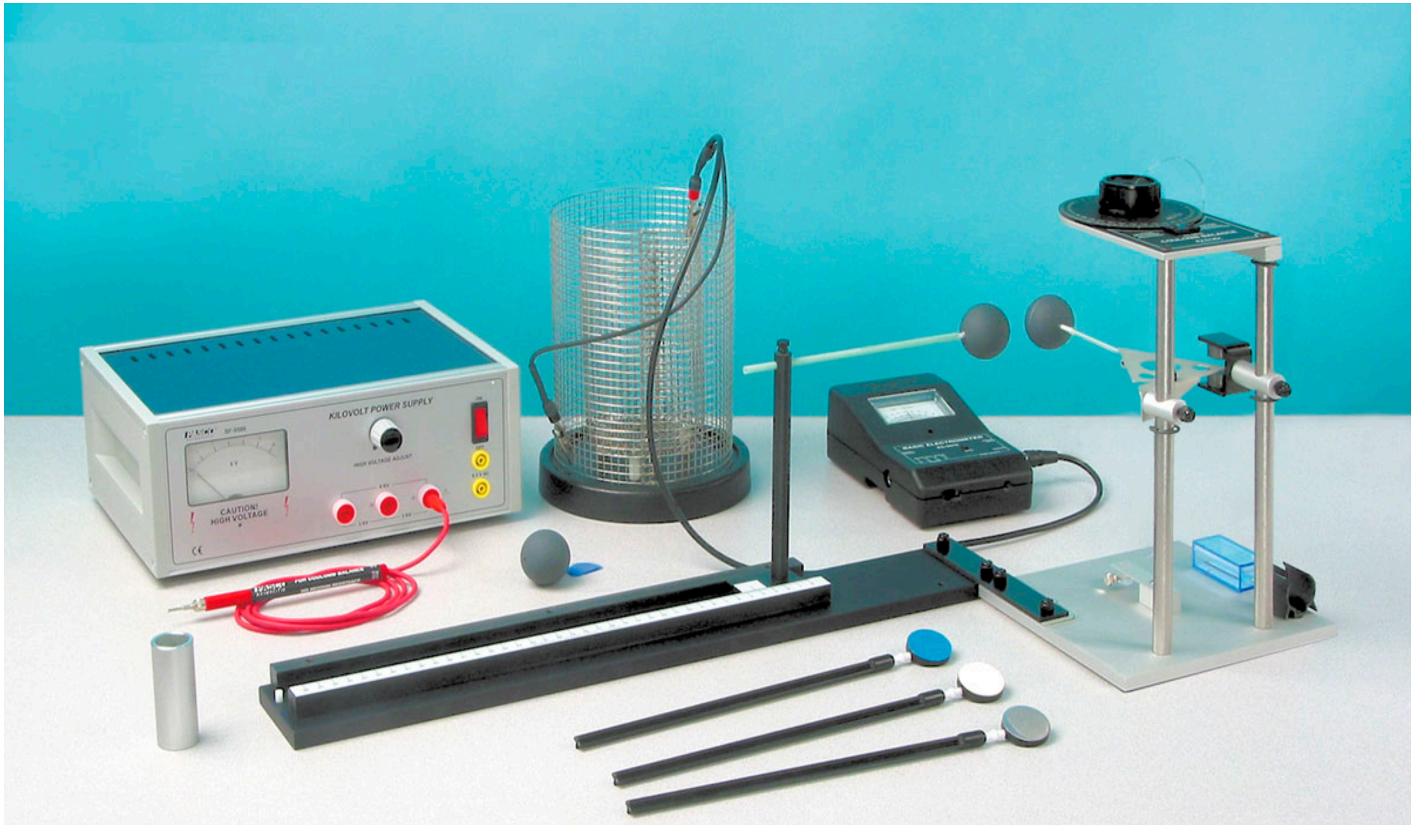
$You\ have\ looked\ at\ \vec{F}_E,\ \vec{E}\ in\ connection\ with\ PHYS242L:$

$Millikan\ Oil\ Drop\ experiment\ -$





Coulomb's Law experiment, Gauss's Law verification –



Electric Field mapping –

–next experiment : map the field with Overbeck apparatus

applying idea of $\vec{\nabla}V$