First-Order Logic Part2

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[with material from “Mathematical Logic for Computer Science”, by Zhongwan, published by World Scientific]

Objectives

- Substitution Theorem
- First-Order Logic Proofs
- Soundness and Completeness
- Maximal Consistency

Satisfiability and Validity

- **A modal formula φ is**

- *valid* iff $I, \theta \models \varphi$ for all interpretations I and all valuations θ (i.e., true in all models),
- *satisfiable* iff $I, \theta \models \varphi$ for some interpretation I and some valuation θ (i.e., has a model), and
- *unsatisfiable* otherwise.

- **Some Lemmas of Satisfiability:**
  - If Σ is satisfiable in D then Σ is satisfiable
  - If φ is valid then φ is valid in D
  - φ is satisfiable in D iff ¬φ is not valid in D
  - φ is valid in D iff ¬φ is unsatisfiable in D
  - $\exists x_1, \ldots x_n . \varphi(x_1, \ldots x_n)$ is satisfiable in D iff $\varphi(u_1, \ldots u_n)$ is satisfiable in D
  - $\forall x_1, \ldots x_n . \varphi(x_1, \ldots x_n)$ is valid in D iff $\varphi(u_1, \ldots u_n)$ is valid in D

Substitution

- A **syntactic substitution** of a term *t* for a variable *x*, written $(.)\,^{x}_{t} : \mathsf{WFF} \to \mathsf{WFF}$, is a mapping of terms to terms and formulæ to formulæ

1. For a term t_1, $(t_1)\,^{x}_{t}$ is t_1 with each occurrence of the variable x replaced by the term t.
2. For $\varphi = P(t_1, \ldots, t_{\mathsf{ar}(P)})$, $(\varphi)\,^{x}_{t} = P\big((t_1)\,^{x}_{t}, \ldots, (t_{\mathsf{ar}(P)})\,^{x}_{t}\big)$.
3. For $\varphi = (\neg\psi)$, $(\varphi)\,^{x}_{t} = (\neg(\psi)\,^{x}_{t})$;
4. For $\varphi = (\psi \to \eta)$, $(\varphi)\,^{x}_{t} = ((\psi)\,^{x}_{t} \to (\eta)\,^{x}_{t})$, and
5. for $\varphi = (\forall y.\psi)$, there are two cases:
   - if x is y, then $(\varphi)\,^{x}_{t} = \varphi = (\forall y.\psi)$, and
   - otherwise, then $(\varphi)\,^{x}_{t} = (\forall z.(\psi\,^{y}_{z})\,^{x}_{t})$, where z is any variable that is not free in t or in φ.

- $\varphi\,^{x}_{t}$ stands for applying the substitution $(.)\,^{x}_{t}$ to φ

Substitution Lemma

- Let *I* be an interpretation, θ a valuation. *t* a term, and *x* a variable. Then, $I, \theta \models \varphi^x_t$ iff $I, \theta[x/(t)^{I,\theta}] \models \varphi$ for all $\varphi \in$ WFF.

First-Order Hilbert System /1

- **The First-Order Hilbert System** is a deduction system for first-order logic defined by the tuples generated by the following schemes:

Ax1	$\langle \forall^*(\varphi \to (\psi \to \varphi)) \rangle$;	
Ax2	$\langle \forall^*((\varphi \to (\psi \to \eta)) \to ((\varphi \to \psi) \to (\varphi \to \eta))) \rangle$;	
Ax3	$\langle \forall^*(((\neg\varphi) \to (\neg\psi)) \to (\psi \to \varphi)) \rangle$;	
Ax4	$\langle \forall^*(\forall x.(\varphi \to \psi)) \to ((\forall x.\varphi) \to (\forall x.\psi)) \rangle$;	
Ax5	$\langle \forall^*(\forall x.\varphi) \to \varphi\,{}^{x}_{t} \rangle$	for $t \in \mathsf{TS}$ a term;
Ax6	$\langle \forall^*(\varphi \to \forall x.\varphi) \rangle$	for $x \notin \mathsf{FV}(\varphi)$; and
MP	$\langle \varphi, (\varphi \to \psi), \psi \rangle$.	

- $\forall^*$ is a finite sequence of universal quantifiers (e.g., $\forall x_1.\forall y.\forall x$)

First-Order Hilbert System /2

- **The First-Order Hilbert System is Sound:**

  $\Sigma \vdash \varphi$ then $\Sigma \models \varphi$

- **The First-Order Hilbert System is Complete:**

  $\Sigma \models \varphi$ then $\Sigma \vdash \varphi$

- **Generalization Lemma:**

  Let $\Sigma \vdash \varphi$ and $x \notin \mathsf{FV}(\Sigma)$. Then $\Sigma \vdash \forall x.\varphi$.

- **Deduction Theorem:**
  - For φ, $\omega \in$ WFF and $\Sigma \subseteq$ WFF,
    $\Sigma \vdash \varphi \rightarrow \omega$ iff $\Sigma \cup \{\varphi\} \vdash \omega$

First-Order Hilbert System /3

- **Lewis Carrol's Logic**
  - Taken from *"Symbolic Logic"* by Lewis Carrol
  - **Assumptions:**

    Babies are illogical $\forall x(B(x) \rightarrow \neg L(x))$

    Nobody is despised who can manage a crocodile

    $$\forall x(C(x) \rightarrow \neg D(x))$$

    Illogical persons are despised (hmmm…)

    $$\forall x(\neg L(x) \rightarrow D(x))$$

  - **Prove:**

    Therefore, babies cannot manage crocodiles

    $$\forall x(B(x) \rightarrow \neg C(x))$$

First-Order Hilbert System /4

- **Recall the following theorems proved for H:**

  $\vdash_H (\neg\neg A \rightarrow A)$

  $\vdash_H (A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$

  $\{(A \rightarrow B), (B \rightarrow C)\} \vdash_H (A \rightarrow C)$ (Transitivity Theorem)

- **Formally prove:** $\forall x(B(x) \rightarrow \neg L(x)), \forall x(C(x) \rightarrow \neg D(x)), \forall x(\neg L(x) \rightarrow D(x)) \vdash \forall x(B(x) \rightarrow \neg C(x))$

  1. $\forall x(B(x) \rightarrow \neg L(x))$ (by Assumptions)
  2. $\forall x(B(x) \rightarrow \neg L(x)) \rightarrow (B(x) \rightarrow \neg L(x))$ (by Ax5)
  3. $B(x) \rightarrow \neg L(x)$ (by MP, (2), (1))
  4. $\forall x(C(x) \rightarrow \neg D(x))$ (by Assumptions)
  5. $\forall x(C(x) \rightarrow \neg D(x)) \rightarrow (C(x) \rightarrow \neg D(x))$ (by Ax5)
  6. $C(x) \rightarrow \neg D(x)$ (by MP, (5), (4))

First-Order Hilbert System /5

- **Formally prove:** $\forall x(B(x) \rightarrow \neg L(x)), \forall x(C(x) \rightarrow \neg D(x)), \forall x(\neg L(x) \rightarrow D(x)) \vdash \forall x(B(x) \rightarrow \neg C(x))$

7. $\forall x(\neg L(x) \rightarrow D(x))$ (by Assumptions)
8. $\forall x(\neg L(x) \rightarrow D(x)) \rightarrow (\neg L(x) \rightarrow D(x))$ (by Ax5)
9. $\neg L(x) \rightarrow D(x)$ (by MP, (8), (7))
10. $B(x) \rightarrow D(x)$ (by Transitivity Theorem, (3), (9))
11. $\neg\neg C(x) \rightarrow C(x)$ (by $\neg\neg A \rightarrow A$ Theorem)
12. $\neg\neg C(x) \rightarrow \neg D(x)$ (by Transitivity Theorem, (11), (6))
13. $(\neg\neg C(x) \rightarrow \neg D(x)) \rightarrow (D(x) \rightarrow \neg C(x))$ (by Ax3)
14. $D(x) \rightarrow \neg C(x)$ (by MP, (13), (12))
15. $B(x) \rightarrow \neg C(x)$ (by Transitivity Theorem, (10), (14))
16. $\forall x(B(x) \rightarrow \neg C(x))$ (by Generalization Lemma, (15))

First-Order Hilbert System /6

- **Formally prove $\vdash$ A(a) $\rightarrow \exists$x A(x)**
  1. $\forall$**x** $\neg$**A(x)** $\rightarrow$ $\neg$**A(a)** (by Ax5)
  2. $\neg\neg$**(**$\forall$**x** $\neg$**A(x))** $\rightarrow$ $\forall$**x** $\neg$**A(x)** (by $\neg\neg$A $\rightarrow$ A Theorem)
  3. $\neg\neg$**(**$\forall$**x** $\neg$**A(x))** $\rightarrow$ $\neg$**A(a)** (by Transitivity Theorem, (2), (1))
  4. **(**$\neg\neg$**(**$\forall$**x** $\neg$**A(x))** $\rightarrow$ $\neg$**A(a))** $\rightarrow$ **(A(a)** $\rightarrow\neg\forall$**x** $\neg$**A(x))** (by Ax3)
  5. **A(a)** $\rightarrow\neg\forall$**x** $\neg$**A(x)** (by MP, (4), (3))
  6. **A(a)** $\rightarrow \exists$**x A(x)** (by definition of $\exists$, (5))

First-Order Hilbert System /7

- **Formally prove**
  $\vdash \forall x(A(x) \rightarrow B(x)) \rightarrow (\forall x\, A(x) \rightarrow \forall x\, B(x))$

1. $\forall x\, A(x) \vdash \forall x\, A(x)$ (by Deduction System Definition)
2. $\forall x(A(x) \rightarrow B(x)), \forall x\, A(x) \vdash \forall x\, A(x)$ (by Weakening, (1))
3. $\forall x(A(x) \rightarrow B(x)), \forall x\, A(x) \vdash \forall x\, A(x) \rightarrow A(a)$ (by Ax5)
4. $\forall x(A(x) \rightarrow B(x)), \forall x\, A(x) \vdash A(a)$ (by MP, (3), (2))
5. $\forall x(A(x) \rightarrow B(x)), \forall x\, A(x) \vdash \forall x(A(x) \rightarrow B(x))$
   (by Assumptions)
6. $\forall x(A(x) \rightarrow B(x)), \forall x\, A(x) \vdash$
   $\forall x(A(x) \rightarrow B(x)) \rightarrow (A(a) \rightarrow B(a))$ (by Ax5)
7. $\forall x(A(x) \rightarrow B(x)), \forall x\, A(x) \vdash A(a) \rightarrow B(a)$ (by MP, (6), (5))

First-Order Hilbert System /8

- **Formally prove**
  $\vdash \forall x(A(x) \to B(x)) \to (\forall x\, A(x) \to \forall x\, B(x))$

8. **$\forall x(A(x) \to B(x)), \forall x\, A(x) \vdash B(a)$** (by MP, (7), (4))
9. **$\forall x(A(x) \to B(x)), \forall x\, A(x) \vdash \forall x\, B(x)$** (by Gen. Lemma)
10. **$\forall x(A(x) \to B(x)) \vdash \forall x\, A(x) \to \forall x\, B(x)$**
    (by Deduction Theorem)
11. **$\vdash \forall x(A(x) \to B(x)) \to (\forall x\, A(x) \to \forall x\, B(x))$**
    (by Deduction Theorem)

Axioms of Equality

- **Axioms of Equality:**
  - Let $\approx$ be a binary predicate symbol written in infix
  - We define the First-Order Axioms of Equality as

$$
\begin{array}{ll}
\text{EqId} & \langle \forall x.(x \approx x) \rangle; \\
\text{EqCong} & \langle \forall x.\forall y.(x \approx y) \rightarrow (\varphi\,{}^{z}_{x} \rightarrow \varphi\,{}^{z}_{y}) \rangle;
\end{array}
$$

- **Godel's Theorem [1930]:**
  - Hilbert System with axiomatized equality is sound and complete with respect to first-order logic with equality

Consistency and Compactness

- **Formula Consistency:** (Definition 5.2.2)
  - A set $\Sigma \subseteq$ WFF is consistent iff there is no $\varphi \in$ WFF such that $\Sigma \vdash \varphi$ and $\Sigma \vdash (\neg\varphi)$
- **Compactness Theorem:**
  - A set Σ is consistent iff every finite $\Sigma_0 \subseteq \Sigma$ is consistent
- **Maximal Consistency:** (Definition 5.3.1)
  - A set $\Sigma \subseteq$ WFF is maximal consistent iff

1. Σ is consistent, and
2. For any $\varphi \in$ WFF such that $\varphi \notin \Sigma$, $\Sigma \cup \{\varphi\}$ is inconsistent

Maximal Consistency /1

- **Maximal Consistency Lemma1:** (Lemma 5.3.2)
  - Let $\Sigma \subseteq$ WFF be maximal consistent
  - Then $\varphi \in \Sigma$ iff $\Sigma \vdash \varphi$
- Proof:
  - If $\varphi \in \Sigma$ then by $\Sigma \vdash \varphi$ by definition
  - Assume that $\Sigma \vdash \varphi$ and $\varphi \notin \Sigma$
  - Since Σ is maximal consistent then $\Sigma \cup \{\varphi\}$ is inconsistent; $\Sigma \vdash \neg\varphi$ since φ cannot infer $\neg\varphi$; as a result, Σ is inconsistent but that is a contradiction
  - Hence, $\varphi \in \Sigma$ as required

Maximal Consistency /2

- **Maximal Consistency Lemma2:** (Lemma 5.3.3)
  - Let $\sum \subseteq$ WFF is maximal consistent; then it holds
  1. $\neg\varphi \in \sum$ iff $\varphi \notin \sum$
  2. $\varphi \wedge \omega \in \sum$ iff $\varphi \in \sum$ and $\omega \in \sum$
  3. $\varphi \vee \omega \in \sum$ iff $\varphi \in \sum$ or $\omega \in \sum$
  4. $\varphi \rightarrow \omega \in \sum$ iff $\varphi \in \sum$ implies $\omega \in \sum$
  5. $\varphi \leftrightarrow \omega \in \sum$ iff ($\varphi \in \sum$ iff $\omega \in \sum$)
- **Maximal Consistency Lemma3:** (Lemma 5.3.4)
  - Let $\sum \subseteq$ WFF be maximal consistent
  - Then $\sum \vdash \neg\varphi$ iff $\sum \vdash \varphi$ does not hold
- **Maximal Consistency Lemma4:** (Lemma 5.3.5)
  - Any consistent set of formulas $\sum$ can be extended to some maximal consistent set

Definability

- **Definability in an Interpretation:**
  - Let $I = (D, (.)^I)$ be a first-order interpretation and $\varphi \in$ WFF
  - A set S of k-tuples over D is defined by the formula φ if
    $S = \{(\theta(x_1), \ldots, \theta(x_k)) \mid I, \theta \models \varphi\}$
  - A set S is **definable in first-order logic** if it is defined by some $\varphi \in$ WFF
- **Definability of a Set of Interpretations:**
  - Let $\sum$ be a set of first-order sentences and K a set of interpretations
  - We say that $\sum$ defines K if $I \in \mathcal{K}$ if and only if $I \models \Sigma$
  - A set K is definable if it is defined by a set of first-order formulas $\sum$; K is strongly definable if $\sum$ is finite

Food for Thought

- **Read:**
  - Chapter 3, Section 3.4, and Chapter 4 from Zhongwan
  - Chapter 5, Sections 5.1 – 5.4 from Zhongwan
    - Read the material discussed in class in more detail
    - Follow the notation conventions discussed in class
    - Cursory reading of the material not emphasized in class
  - Handout on “First-Order Logic”
    - Available from the course schedule web page or through LEARN
  - Chapter 6, Sections 6.1 – 6.2 from Zhongwan
    - Read the material on compactness and definability
- **Answer Assignment #4 questions**
  - Assignment #4 includes several practice exercises related to First-Order Logic