WE CAN APPLY THIS SAME APPROACH TO SOLVE OUR 2ND-ORDER EQUATION $y'' + P(x)y' + Q(x)y = 0$. FURTHERMORE, THE SERIES SOLUTION MOTIVATES A USEFUL CLASSIFICATION OF SINGULAR POINTS.

ORDINARY POINTS

IF $P(x)$ & $Q(x)$ ARE ANALYTIC AT $x = x_0$, THEN WE CALL x_0 AN 'ORDINARY' POINT. THE POINT x_0 IS CALLED 'SINGULAR' OTHERWISE. AT ORDINARY POINTS, THE SOLUTION $y(x)$ WILL LIKEWISE BE ANALYTIC, AND WE ARE JUSTIFIED IN ASSUMING A POWER-SERIES FORM. MORE FORMALLY:

> FOR AN ORDINARY POINT x_0 & CONSTANTS α & β, THERE EXISTS A UNIQUE FUNCTION $y(x)$ THAT IS ANALYTIC AT x_0 SOLVING THE HOMOGENEOUS DIFFERENTIAL EQ IN THE NEIGHBOURHOOD OF x_0, AND SATISFYING THE INITIAL CONDITIONS $y(x_0) = \alpha$ & $y'(x_0) = \beta$.
>
> IF $P(x)$ & $Q(x)$ ARE ANALYTIC ON $|x - x_0| < \rho$, THEN THE POWER SERIES FOR $y(x)$ IS VALID ON THE SAME INTERVAL.

<u>PROOF</u>: SEE SIMMONS, p. 180.

EX. $y'' + \omega^2 y = 0$. FOR THIS EQUATION, ALL POINTS x ARE ORDINARY.

WE ASSUME $y = \sum_{n=0}^{\infty} a_n x^n$ AND $y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$

$$= a_0 + a_1 x + a_2 x^2 + \ldots \qquad = 2a_2 + 3\cdot 2\cdot a_3 x + 4\cdot 3 x^2 + \ldots$$

TO SATISFY THE DIFFERENTIAL EQUATION, WE MUST CHOOSE a_n SO THAT

$$2a_2 + 3\cdot 2\cdot a_3 x + \ldots + (n+2)(n+1) a_{n+2} x^n + \ldots = -\omega^2 \left[a_0 + a_1 x + \ldots + a_n x^n + \ldots \right]$$

OR, $$\boxed{a_{n+2} = -\frac{\omega^2 a_n}{(n+2)(n+1)} \qquad n = 0, 1, 2, \ldots}$$

THERE IS A SURPRISE HIDDEN IN THIS RECURSION - IF WE START FROM AN <u>EVEN</u> INDEX, THE RECURSION MAPS BACK TO $\underline{a_0}$; WHEREAS IF WE START FROM AN <u>ODD</u> INDEX, WE MAP BACK TO $\underline{a_1}$.

$$a_2 = \frac{-\omega^2 a_0}{2!} \qquad a_3 = \frac{-\omega^2 a_1}{3!}$$

$$a_4 = \frac{\omega^4 a_0}{4!} \qquad a_5 = \frac{\omega^4 a_1}{5!}$$

$$\vdots \qquad\qquad \vdots$$

$$a_{2n} = \frac{(-1)^n \omega^{2n} a_0}{(2n)!} \qquad a_{2n+1} = \frac{(-1)^n \omega^{2n} a_1}{(2n+1)!} = \frac{(-1)^n \omega^{2n+1}}{(2n+1)!} \cdot \frac{a_1}{\omega}$$

THESE INDEPENDENT SEQUENCES CORRESPOND TO THE TWO LINEARLY-INDEPENDENT SOLUTIONS!

PUTTING THEM TOGETHER, $y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n (\omega x)^{2n}}{(2n)!} + \frac{a_1}{\omega} \sum_{n=0}^{\infty} \frac{(-1)^n (\omega x)^{2n+1}}{(2n+1)!}$

WE RECOGNIZE THESE -

$$= A \cos(\omega x) + B \sin(\omega x).$$

THIS FRACTURING OF THE SERIES INTO LINEARLY-INDEPENDENT PARTS IS A UNIVERSAL FEATURE OF NONSINGULAR ORDINARY DIFFERENTIAL EQS. WE CAN, OF COURSE, APPLY THE SAME APPROACH TO DETERMINE SOLUTIONS IN THE MORE GENERAL CASE OF NON-CONSTANT COEFFICIENTS $P(x)$ & $Q(x)$

EX. LEGENDRE'S EQUATION: $(1-x^2)y'' - 2xy' + p(p+1)y = 0$ ['p' A POSITIVE INTEGER, $p \in \mathbb{Z}^+$]

IN THIS EXAMPLE, FOR A GIVEN 'p', THE LINEARLY-INDEPENDENT SOLUTIONS ARE 1.) A POLYNOMIAL OF ORDER 'p' [LEGENDRE POLYNOMIALS] AND 2). AN INFINITE SERIES THAT DEFINES A 'SPECIAL FUNCTION' THAT CANNOT BE EXPRESSED IN TERMS OF ELEMENTARY FUNCTIONS.

IN STANDARD FORM, $y'' - \frac{2x}{(1-x^2)} y' + \frac{p(p+1)}{(1-x^2)} y = 0$

BOTH $P(x) = -2x/(1-x^2)$ AND $Q(x) = p(p+1)/(1-x^2)$ ARE ANALYTIC AT $x=0$, BUT SINGULAR AT $x = \pm 1$. LET'S TRY A POWER SERIES ABOUT $x=0$:

$$(1-x^2)y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1) a_n x^n$$

$$-2xy' = -2 \sum_{n=0}^{\infty} n a_n x^n \quad \text{AND} \quad p(p+1)y = \sum_{n=0}^{\infty} p(p+1) a_n x^n$$

SUBSTITUTING INTO THE DIFFERENTIAL EQ, WE HAVE,

$$(n+2)(n+1)a_{n+2} + [p(p+1) - n(n+1)]a_n = 0$$

OR,

$$a_{n+2} = -\frac{[p(p+1) - n(n+1)]}{(n+2)(n+1)} a_n$$

$$= -\frac{(p-n)(p+n+1)}{(n+2)(n+1)} a_n$$

AS IN THE HARMONIC OSCILLATOR EXAMPLE, EVEN INDICES RECURSE BACK TO $\underline{a_0}$ AND ODD INDICES RECURSE BACK TO $\underline{a_1}$:

$$a_2 = -\frac{p(p+1)}{2!} a_0 \qquad a_3 = -\frac{(p-1)(p+2)}{3!} a_1$$

$$a_4 = \frac{p(p+1)(p-2)(p+3)}{4!} a_0 \qquad a_5 = \frac{(p-1)(p+2)(p-3)(p+4)}{5!} a_1$$

$$\vdots$$

$$a_{2n} = (-1)^n \frac{\prod_{i=1}^{n}(p-2i+2)(p+2i-1)}{(2n)!} a_0 \qquad a_{2n+1} = (-1)^n \frac{\prod_{i=1}^{n}(p-2i+1)(p+2i)}{(2n+1)!} a_1$$

WE DENOTE THESE SOLUTIONS BY EVEN AND ODD LEGENDRE POLYNOMIALS $L^0_p(x)$ & $L^1_p(x)$, RESPECTIVELY:

$$y(x) = \sum_{n=0}^{\infty} a_{2n}x^{2n} + \sum_{n=0}^{\infty} a_{2n+1}x^{2n+1} = a_0 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n \prod_{i=1}^{n}(p-2i+2)(p+2i-1)}{(2n)!} x^{2n}}_{L^0_p(x)} + a_1 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n \prod_{i=1}^{n}(p-2i+1)(p+2i)}{(2n+1)!} x^{2n+1}}_{L^1_p(x)}$$

$$= a_0 \, L^0_p(x) + a_1 \, L^1_p(x)$$

FOR EVEN 'p', $L^0_{EVEN}(x)$ TERMINATES INTO A 'p' ORDER POLYNOMIAL; FOR ODD 'p', $L^1_{ODD}(x)$ TERMINATES, eg.

$$L^0_0 = 1 \qquad L^1_1 = x$$

$$L^0_2 = 1 - 3x^2 \qquad L^1_3 = x - \tfrac{5}{3}x^3$$

$$L^0_4 = 1 - 10x^2 + \frac{35}{3}x^4 \qquad L^1_5 = x - \frac{14}{3}x^3 + \frac{21}{5}x^5$$

THESE LEGENDRE POLYNOMIALS HAVE A STRANGE, BUT USEFUL, PROPERTY. THE PRODUCT OF ANY TWO POLYNOMIALS WILL VANISH WHEN INTEGRATED FROM $x=-1$ TO $x=1$ IF, AND ONLY IF, THE POLYNOMIALS ARE DISTINCT. (ie. DIFFENT 'p').

$$\int_{-1}^{1} \underset{L_0^0}{(1)} \cdot \underset{L_1^0}{(x)}\, dx = 0 \quad , \quad \int_{-1}^{1} \underset{L_2^0}{(1-3x^2)} \underset{L_0^0}{(1)}\, dx = 0 \quad , \text{ ETC..}$$

BUT,

$$\int_{-1}^{1} \underset{L_2^0}{(1-3x^2)} \underset{L_2^0}{(1-3x^2)}\, dx = 8/5 \neq 0 .$$

LIKE A FOURIER SERIES IN $\sin(nx)$ & $\cos(mx)$, LEGENDRE POLYNOMIALS CAN BE USED AS ORTHOGONAL BASIS FUNCTIONS ON THE INTERVAL $x \in [-1,1]$.

SINGULAR POINTS

IF $x = x_0$ IS NOT AN ORDINARY POINT, THEN IT IS SINGULAR. SOME SINGULARITIES ARE EASIER TO WORK WITH THAN OTHERS.

IF $x = x_0$ IS A SINGULAR POINT OF $y'' + P(x)y' + Q(x)y = 0$, BUT

$$(x - x_0)P(x) \quad \text{AND} \quad (x - x_0)^2 Q(x)$$

ARE ANALYTIC AT $x = x_0$, THEN x_0 IS A <u>REGULAR</u> SINGULAR POINT. IT IS AN IR<u>REGULAR</u> SINGULAR POINT OTHERWISE.

<u>EX</u>. LEGENDRE'S EQUATION IN STANDARD FORM IS,

$$y'' - \frac{2x}{1-x^2} y' + \frac{p(p+1)}{1-x^2} y = 0$$

THE POINTS $x = \pm 1$ ARE REGULAR SINGULAR POINTS. AT $x = 1$,

$$(x-1)P(x) = \frac{2x}{1+x} \quad \text{AND} \quad (x-1)^2 Q(x) = -\frac{(x-1)p \cdot (p+1)}{(1+x)}$$

BOTH OF WHICH ARE ANALYTIC AT $x = 1$. SAME GOES FOR $x = -1$.

REGULAR SINGULAR POINTS ARE TRACTABLE INSOFAR AS A SERIES SOLUTION IS STILL POSSIBLE IN THEIR NEIGHBOURHOOD.

HERE IS A CLASSIC EXAMPLE FROM ASTROPHYSICS,

EMDEN'S EQUATION: STUDYING THE DENSITY & INTERNAL TEMPERATURE OF STARS, EMDEN ARRIVED AT THE FOLLOWING PROBLEM -

DETERMINE THE FIRST POINT ON THE POSITIVE X-AXIS WHERE THE SOLUTION $y(x)$ TO THE DIFFERENTIAL EQUATION

$$xy'' + 2y' + xy = 0; \quad y(0) = 1 \ \& \ y'(0) = 0$$

IS ZERO.

IN STANDARD FORM - $y'' + \frac{2}{x}y' + y = 0$; SO $x = 0$ IS A REGULAR SINGULAR POINT. ASSUMING A POWER SERIES $y = \sum_{n=0}^{\infty} a_n x^n$,

$$xy'' = x\left[2a_2 + 3\cdot 2a_3 x + \ldots + (n+1)n\, a_{n+1}\, x^{n-1} + \ldots\right] = \sum_{n=0}^{\infty} n(1+n)\, a_{n+1} x^n$$

$$2y' = 2\left[a_1 + 2a_2 x + \ldots + (n+1)a_{n+1} x^n + \ldots\right] = \sum_{n=0}^{\infty} n(1+n)\, a_{n+1} x^n$$

$$xy = x\left[a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} + \ldots\right] = \sum_{n=1}^{\infty} a_{n-1} x^n$$

AFTER SUBSTITUTION INTO THE DIFFERENTIAL EQ,

$$[2a_1] + [6a_2 + a_0]x + [12a_3 + a_1]x^2 + \ldots + \left[\left((n+1)n + 2(n+1)\right)a_{n+1} + a_{n-1}\right]x^n + \ldots = 0$$

OR,

$$2a_1 = 0 \implies a_1 = 0$$

$$6a_2 + a_0 = 0 \implies a_2 = -\tfrac{1}{6}a_0$$

$$12a_3 + a_1 = 0 \implies a_3 = -\tfrac{1}{12}a_1 = 0$$

$$\vdots$$

BECAUSE $a_1 = 0$, ALL ODD COEFFICIENTS VANISH.

FOR EVEN INDICES

$$a_{n+1} = -\frac{a_{n-1}}{(n+2)(n+1)} \implies a_{2k} = -\frac{a_{2k-2}}{(2k+1)(2k)} = \ldots = \frac{(-1)^k a_0}{(2k+1)!}$$

ALTOGETHER,

$$y(x) = a_0\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots\right]$$

FROM THE BOUNDARY CONDITIONS $y(0) = 1$, $y'(0) = 0$ WE HAVE $a_0 = 1$.

Q: WHAT HAPPENED TO THE OTHER LINEARLY-INDEPENDENT SOLUTION?

$$= \left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots\right] = \frac{\sin x}{x}$$

THE SMALLEST POSITIVE ROOT OF $y(x)$ IS $x = \pi$. //

WE WERE LUCKY WITH THIS PROBLEM BECAUSE $\sin x / x$ HAS A TAYLOR SERIES THAT BEGINS WITH A NON-NEGATIVE POWER OF X. FOR REGULAR SINGULAR PROBLEMS, THIS WILL NOT ALWAYS BE THE CASE. NEVERTHELESS, WE CAN STILL GENERATE POWER SERIES SOLUTIONS! THE APPROACH IS DUE TO FROBENIUS.

FROBENIUS SERIES & REGULAR SINGULAR PROBLEMS

TO MOTIVATE THE IDEA, CONSIDER THE PROTOTYPICAL EQUATION WITH REGULAR SINGULARITIES AT $x=0$:

$$\underbrace{x^2 y'' + p_0 x y' + q_0 y = 0}_{\text{EULER'S EQUATION}}$$

THE GENERAL SOLUTION IS x^r & r IS A CONSTANT. WITH SUBSTITUTION INTO THE DIFFERENTIAL EQUATION, WE OBTAIN AN 'INDICIAL EQUATION' FOR 'r':

$$r(r-1) + p_0 r + q_0 = 0$$

ASIDE: IF THIS QUADRATIC HAS: i) DISTINCT ROOTS r_1 & r_2, THEN $y_H(x) = C_1 x^{r_1} + C_2 x^{r_2}$.

ii) REPEATED ROOTS $r = r_1$, THEN $y_H(x) = C_1 x^r + C_2 x^r \ln(x)$.

iii) COMPLEX ROOTS $r_1, r_2 = \alpha \pm j\beta$, THEN $y_H(x) = x^\alpha [C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)]$ $[x>0]$.

FOR A GENERAL DIFFERENTIAL EQUATION WITH REGULAR SINGULAR POINT AT $x=0$, WE CAN WRITE $P(x) = \frac{p(x)}{x}$ AND $Q(x) = \frac{q(x)}{x^2}$ AND EXPAND THE ANALYTIC FUNCTIONS $p(x)$ AND $q(x)$ AS POWER SERIES.

$$y'' + \left(\frac{p_0 + p_1 x + p_2 x^2 + \cdots}{x}\right) y' + \left(\frac{q_0 + q_1 x + \cdots}{x^2}\right) y = 0$$

THE SINGULAR PART OF THIS EQUATION CORRESPONDS TO THE EULER EQUATION ABOVE & SUGGESTS WE LOOK FOR SOLUTIONS OF THE FORM:

$$y(x) = \underbrace{x^r}_{\text{SOLVES SINGULAR PART}} \cdot \sum_{n=0}^{\infty} a_n x^n = \underbrace{\boxed{\sum_{n=0}^{\infty} a_n x^{n+r} \quad (a_0 \neq 0)}}_{\text{FROBENIUS SERIES}}$$

EXERCISE. RETURN TO EMDEN'S EQUATION $y'' + \frac{2}{x}y' + y = 0$.
HERE, $p_0 = 2$ & $q_0 = 0$. THE INDICIAL EQ. IS:

$$r(r-1) + 2r = 0 \quad \text{WITH} \quad r = 0, -1.$$

PREVIOUSLY, WE USED $r=0$ TO DETERMINE ONE SOLUTION $y_1 = a_0\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} \dots\right]$
USE $r=-1$ TO FIND THE OTHER: $y_2 = \frac{1}{x}\sum_{n=0}^{\infty} a_n x^n$.

EXAMPLE - $2x^2y'' + x(2x+1)y' - y = 0$ OR, $y'' + \frac{\left(\frac{1}{2}+x\right)}{x}y' + \frac{\left(-\frac{1}{2}\right)}{x^2}y = 0$

$$\left.\begin{array}{l} xP(x) = \frac{1}{2} + x \\ x^2Q(x) \quad -\frac{1}{2} \end{array}\right\}$$ BOTH ARE ANALYTIC ~~AT x=0~~
SO $x=0$ IS A 'REGULAR SINGULAR POINT'.

THE ASSOCIATED INDICIAL EQUATION IS: $r(r-1) + \frac{1}{2}(r-1) = 0$ OR $r = 1, -\frac{1}{2}$.
TRY A FROBENIUS SERIES - $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ WITH $a_0 \neq 0$.

$$xy' = x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r}$$

$$2x^2y' = 2x^2\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} = \sum_{n=1}^{\infty}2(n+r-1)a_{n-1}x^{n+r}$$

$$2x^2y'' = 2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} = \sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r}$$

SUBSTITUTING INTO THE DIFFERENTIAL EQUATION,

$$a_0[(2r+1)(r-1)] + \sum_{n=1}^{\infty}(n+r-1)[(2(n+r)+1)a_n + 2a_{n-1}]x^{n+r} = 0.$$

WITH THE RESTRICTION THAT $a_0 \neq 0$, THE LEADING TERM PROVIDES A CONSTRAINT ON 'r': $r = 1, -\frac{1}{2}$. THIS IS EXACTLY WHAT WE FOUND FROM THE INDICIAL EQ., AND IS A GENERAL FEATURE OF THE FROBENIUS METHOD - YOU DON'T NEED TO INVESTIGATE THE INDICIAL EQUATION SEPARATELY; IT IS INCLUDED AUTOMATICALLY.

FOR THE REMAINING COEFFICIENTS,

$$a_n = -\frac{a_{n-1}}{\left(n+r+\frac{1}{2}\right)}$$

FOR $r=1$,

$$a_1 = -\frac{2}{5}a_0$$

$$a_2 = -\frac{2}{7}a_1 = \frac{4}{35}a_0$$

$$\vdots$$

FOR $r=-1/2$,

$$a_1 = -a_0$$

$$a_2 = -\frac{1}{2}a_1 = \frac{1}{2}a_0$$

$$\vdots$$

ALTOGETHER, THE TWO LINEARLY-INDEPENDENT SOLUTIONS ARE:

$$y_1 = \underbrace{x}_{r=1}\left(1 - \frac{2}{5}x + \frac{4}{35}x^2 + \cdots\right) \quad \text{AND} \quad y_2 = \underbrace{\frac{1}{\sqrt{x}}}_{r=-1/2}\left(1 - x + \frac{1}{2}x^2 + \cdots\right)$$

CAUTION: FROM THE FORM OF THE FROBENIUS SERIES: $\sum_{n=0}^{\infty} a_n x^{n+r}$, IF r_1 & r_2 (THE ROOTS OF THE INDICIAL EQ) DIFFER BY AN INTEGER, IT MAY NOT BE POSSIBLE TO FIND THE SECOND LINEARLY-INDEPENDENT SOLUTION. SAME GOES FOR REPEATED ROOTS $r_1 = r_2$; IN THOSE CASES, YOU MUST APPLY A 'REDUCTION-OF-ORDER'.