IN ORDER FOR THE MATRIX TO BE INVERTIBLE, WE MUST HAVE

$$\det [\vec{y}_1 \ \vec{y}_2 \ \cdots \ \vec{y}_n] \neq 0$$; THIS DETERMINANT IS THE

nth-DIMENSIONAL <u>WRONSKIAN</u>.

EX. DOES THIS DEFINITION CORRESPOND TO OUR PREVIOUS DEFINITION IN THE CONTEXT OF 2nd ORDER EQUATIONS $y'' + P(x)y' + Q(x)y = 0$? HOW WOULD $\vec{y}_1$ & $\vec{y}_2$ BE DEFINED ABOVE?

IN ANOLOGY WITH ABEL'S IDENTITY FOR 2nd-ORDER SYSTEMS, LIOUVILLE'S FORMULA PROVIDES A CONNECTION BETWEEN THE COEFFICIENTS $\mathbb{A}(x)$ AND THE WRONSKIAN:

$$W(x) = W_0 \exp\left[\int_{x_0}^{x} \mathrm{Tr}[\mathbb{A}(x')] \, dx' \right]$$

WHERE $\mathrm{Tr}[\mathbb{A}(x)] = a_{11}(x) + a_{22}(x) + \cdots + a_{nn}(x)$ IS THE SUM OF THE DIAGONAL ELEMENTS OF $\mathbb{A}(x)$.

THE FUNDAMENTAL MATRIX

THE HOMOGENEOUS & INHOMOGENEOUS SOLUTIONS TO A SYSTEM OF 1st ORDER DIFFERENTIAL EQUATIONS IS CONVENIENTLY EXPRESSED IN TERMS OF THE LINEARLY-INDEPENDENT SOLUTIONS $\vec{Z}_i$ THAT SOLVE THE HOMOGENEOUS INITIAL-VALUE PROBLEM SUBJECT TO:

$$\vec{Z}_i(x_0) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$ ONE IN THE ith ROW.

THINK OF THE HOMOGENEOUS SOLUTIONS $\vec{y}_n$ AS LINEARLY-INDEPENDENT BASIS FUNCTIONS; $\vec{Z}_n$ PLAY THE ROLE OF AN <u>ORTHONORMAL</u> BASIS.

WITH THIS CANONICAL SET $\vec{Z}_n(x)$ WE CAN CONSTRUCT THE MATRIX

$$\Phi(x, x_0) = \begin{bmatrix} | & | & & | \\ \vec{Z}_1(x) & \vec{Z}_2(x) & \cdots & \vec{Z}_n(x) \\ | & | & & | \end{bmatrix}$$

WHICH IS CALLED THE <u>FUNDAMENTAL MATRIX</u> (OR PROPAGATOR) FOR THE SYSTEM.

EXAMPLE, CALCULATE THE FUNDAMENTAL MATRIX $\Phi(x,2)$ FOR THE SYSTEM: $\frac{dy_1}{dx} = \frac{1}{x} y_1 + y_2$; $\frac{dy_2}{dx} = \frac{2}{x} y_2$ $x > 0$.

THE COEFFICIENT MATRIX $A(x) = \begin{pmatrix} \frac{1}{x} & 1 \\ 0 & \frac{2}{x} \end{pmatrix}$. TRIANGULAR MATRICES ARE EASY TO DEAL WITH.

IT MEANS AT LEAST ONE OF THE FUNCTION CAN BE DETERMINED IN ISOLATION VIA SEPARABILITY, AND THE REST, UPON SUBSTITUTION, ARE SOLVED AS LINEAR - FIRST ORDER DES.

IN THIS EXAMPLE, $\frac{dy_2}{dx} = \frac{2}{x} y_2$ HAS NO DEPENDENCE ON y_1.

THERE ARE TWO SOLUTIONS: $y_2 = 0$ OR $y_2 = x^2$.

IF $y_2 = 0$, THEN THE EQ. FOR y_1 IS: $\frac{dy_1}{dx} = \frac{1}{x} y_1$ SO $\cancel{y_1 = 0}$ OR $y_1 = x$.

$y_1 = 0, y_2 = 0$ IS TRIVIAL

IF $y_2 = x^2$, THEN $\frac{dy_1}{dx} = \frac{1}{x} y_1 + x^2$ ← LINEAR FIRST-ORDER $y_1 = \frac{1}{2}x^3 - \frac{1}{2}x$.

WE HAVE TWO SETS OF SOLUTIONS $\vec{Y}_1 = (\underset{y_1}{x}, \underset{y_2}{0})$ AND $\vec{Y}_2 = (\underset{y_1}{\tfrac{1}{2}x^3 - \tfrac{1}{2}x}, \underset{y_2}{x^2})$

FOR THE FUNDAMENTAL MATRIX,
WE NEED TO FIND CONSTANTS TO COMBINE THESE SOLUTIONS SUCH THAT WE PRODUCE ORTHONORMAL SOLUTIONS $\vec{Z}_1(2) = (1,0)$ & $\vec{Z}_2(2) = (0,1)$.

ie. $\begin{bmatrix} \vec{y}_1 & \vec{y}_2 \end{bmatrix} = \begin{bmatrix} x & \frac{1}{2}x^3 - \frac{1}{2}x \\ 0 & x^2 \end{bmatrix} \xrightarrow{\text{AT } x=2:} \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$.

FIND $[c_1\ c_2]^T$ SO THAT:

$$\begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$$

FIND $[d_1\ d_2]^T$ SO THAT

$$\begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} -3/8 \\ 1/4 \end{bmatrix}$$

$$\vec{Z}_1 = \begin{bmatrix} x & \frac{1}{2}x^3 - \frac{1}{2}x \\ 0 & x^2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{x}{2} \\ 0 \end{bmatrix}.$$

$$\vec{Z}_2 = \begin{bmatrix} x & \frac{1}{2}x^3 - \frac{1}{2}x \\ 0 & x^2 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} -\frac{x}{2} + \frac{x^3}{8} \\ \frac{x^2}{4} \end{bmatrix}$$

SO $$\Phi(x,2) = \begin{bmatrix} \vec{Z}_1 & \vec{Z}_2 \end{bmatrix} = \begin{bmatrix} x/2 & -x/2 + x^3/8 \\ 0 & x^2/4 \end{bmatrix}.$$

EXERCISE: SHOW THAT IN GENERAL, $\underline{\Phi}(x,x_0) = \begin{bmatrix} x/x_0 & -x/2 + x^3/2x_0^2 \\ 0 & x^2/x_0^2 \end{bmatrix}$ $x_0 > 0$

—//—

PROPERTIES OF THE FUNDAMENTAL MATRIX

① BY THE DEFINITION OF $\vec{Z}_n(x)$:

$$\underline{\Phi}(x,x_0) = \begin{bmatrix} | & | & & | \\ \vec{Z}_1(x_0) & \vec{Z}_2(x_0) & \cdots & \vec{Z}_n(x_0) \\ | & | & & | \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = \mathbb{I}_n$$ ← $n \times n$ IDENTITY MATRIX.

② BY LINEARITY,

$$\frac{d}{dx}\underline{\Phi}(x,x_0) = \mathbb{A}(x)\cdot\underline{\Phi}(x,x_0)$$ ← SHOW THIS!

③ FROM ②, FOR ANY INITIAL VECTOR $\vec{y}^{\,0}$, THE SOLUTION OF THE INITIAL VALUE PROBLEM:

$$\frac{d}{dx}\vec{y} = \mathbb{A}(x)\cdot\vec{y} \quad ; \quad \vec{y}(x_0) = \vec{y}^{\,0}$$

IS GIVEN BY: $\vec{y}(x) = \underline{\Phi}(x,x_0)\cdot\vec{y}^{\,0}$

HENCE THE FUNDAMENTAL MATRIX $\underline{\Phi}(x,x_0)$ PROPAGATES (OR MOVES) THE SOLUTION $\vec{y}(x)$ FROM 'x_0' TO 'x'.

PROOF: INITIAL CONDITIONS: $\vec{y}(x_0) = \underline{\Phi}(x_0,x_0)\vec{y}^{\,0} = \mathbb{I}_n\cdot\vec{y}^{\,0} = \vec{y}^{\,0}$. ✓

DYNAMICS: $\frac{d}{dx}\vec{y} = \frac{d}{dx}\underline{\Phi}(x,x_0)\vec{y}^{\,0} = \mathbb{A}(x)\underline{\Phi}(x,x_0)\vec{y}^{\,0} = \mathbb{A}(x)\cdot\vec{y}$ ✓ ∎

—//—

SOME NOTATION: $\underline{\Phi}(x, x_0)$ — x_0: WHERE WE START; x: WHERE WE FINISH.

④ SEMI-GROUP PROPERTY. FOR ANY $x_0 < x_1 < x_2$,

$$\underline{\Phi}(x_2,x_1)\,\underline{\Phi}(x_1,x_0) = \underline{\Phi}(x_2,x_0) \qquad (*).$$

PROOF: IF $\vec{y}(x)$ SATISFIES THE INITIAL CONDITIONS $\vec{y}(x_0) = y^0$, THEN

$\vec{y}(x_2) = \underline{\Phi}(x_2,x_0)\vec{y}^{\,0}$ AND $\vec{y}(x_1) = \underline{\Phi}(x_1,x_0)\vec{y}^{\,0}$

BUT WE COULD GET FROM $x_0 \to x_2$ VIA x_1: i.e. $x_0 \to x_1 \to x_2$

THEN,

$$\vec{y}(x_2) = \underline{\Phi}(x_2, x_1)\, \vec{y}(x_1) = \underline{\Phi}(x_2, x_1)\, \underline{\Phi}(x_1, x_0)\, \vec{y}^{\,0}$$

BY UNIQUENESS, $\vec{y}(x_2)$ IS THE SAME WHETHER WE GO $x_0 \to x_2$ OR $x_0 \to x_1 \to x_2$,

SO:

$$\boxed{\underline{\Phi}(x_2, x_0) = \underline{\Phi}(x_2, x_1)\, \underline{\Phi}(x_1, x_0)}$$

R

5) THE BACKWARD PROPAGATOR: FOR $x_{-1} < x_0$, THE PROPAGATOR $\underline{\Phi}(x_0, x_{-1})$ WILL MOVE THE INITIAL VALUE $\vec{y}(x_{-1})$ FORWARD TO $\vec{y}(x_0)$. THE BACKWARD PROPAGATOR $\underline{\Phi}(x_{-1}, x_0)$ STARTS AT ~~$x=x_0$~~ $\vec{y}(x_0)$ AND MOVES THE SOLUTION BACKWARD TO $\vec{y}(x_{-1})$

NOTICE:

$$\underbrace{\underline{\Phi}(x_0, x_{-1})}_{\text{FORWARD}}\, \underbrace{\underline{\Phi}(x_{-1}, x_0)}_{\text{BACKWARD}} \cdot \vec{y}^{\,0} = \vec{y}^{\,0}$$

SO,

$$\underbrace{\underline{\Phi}(x_{-1}, x_0)}_{\text{BACKWARD}} = \underbrace{\left[\underline{\Phi}(x_0, x_{-1})\right]^{-1}}_{\text{INVERSE-FORWARD.}}$$

6.) THE FUNDAMENTAL MATRIX SIMPLIFIES CONSIDERABLY FOR AUTONOMOUS SYSTEMS $\mathbb{A}(x) = \mathbb{A}$. THEN $\underline{\Phi}(x, x_0) = \underline{\Phi}(x - x_0, 0)$ i.e. DEPENDS ONLY ON RELATIVE DIFFERENCE $x - x_0$. IN THIS CASE, WE WRITE THE FUNDAMENTAL MATRIX AS A FUNCTION OF A SINGLE-VARIABLE

$$\underline{\Phi}(x, 0) \equiv \underline{\Phi}(x).$$

AND FROM THE PREVIOUS PROPERTIES:

$$\left.\begin{aligned} \underline{\Phi}(0) &= \mathbb{I}_n \\ \frac{d}{dx}\underline{\Phi}(x) &= \mathbb{A}\,\underline{\Phi}(x) \\ \underline{\Phi}(x_1 + x_2) &= \underline{\Phi}(x_1)\,\underline{\Phi}(x_2) \\ \underline{\Phi}(-x) &= \left[\underline{\Phi}(x)\right]^{-1} \end{aligned}\right\}$$

THESE ARE THE DEFINING PROPERTIES OF THE EXPONENTIAL FUNCTION...

WE'LL RETURN TO THIS.

NHOMOGENEOUS EQUATIONS - VARIATION-OF-PARAMETERS

TO DERIVE THE PARTICULAR SOLUTION SYMBOLICALLY IS MUCH SIMPLER IN THE VECTOR-MATRIX FORMULATION. AS WE DID IN THE 2^{nd} ORDER CASE, ASSUME THE PARTICULAR SOLUTION $\vec{y}_P$ IS A WEIGHTED COMBINATION OF HOMOGENEOUS SOLUTIONS:

$$\vec{y}_P = \Phi(x, x_0) \cdot \vec{v}(x). \quad (*)$$

TAKING THE DERIVATIVE -

$$\frac{d}{dx}\vec{y}_P = \left[\left[\frac{d}{dx}\Phi(x,x_0)\right] \cdot \vec{v}(x) + \Phi(x,x_0)\left[\frac{d\vec{v}}{dx}\right] \right] \quad \leftarrow \text{PRODUCT RULE}$$

$$= \left[\mathbb{A}(x)\,\Phi(x,x_0)\right] \cdot \vec{v}(x) + \Phi(x,x_0) \cdot \left[\frac{d\vec{v}}{dx}\right] \quad \leftarrow \text{VIA PROPERTY (2).}$$

$$= \mathbb{A}(x)\,\vec{y}_P + \Phi(x,x_0) \cdot \frac{d\vec{v}}{dx} \quad \leftarrow \text{FROM } (*) \text{ ABOVE.}$$

FROM THE INHOMOGENEOUS EQ: $\frac{d\vec{y}_P}{dx} = \mathbb{A}(x) \cdot \vec{y}_P + \vec{b}(x)$,

$$\frac{d\vec{y}_P}{dx} = \cancel{\mathbb{A}(\vec{y}_P)}\ \cancel{\mathbb{A}(x) \cdot \vec{y}_P} + \Phi(x,x_0) \cdot \frac{d\vec{v}}{dx} = \cancel{\mathbb{A}(x)\vec{y}_P} + \vec{b}(x)$$

SO,

$$\frac{d\vec{v}}{dx} = \left[\Phi(x, x_0)\right]^{-1} \cdot \vec{b}(x) \quad \text{OR,} \quad \vec{v} = \int_{x_0}^{x} \left[\Phi(x', x_0)\right]^{-1} \cdot \vec{b}(x')\,dx'$$

THEN,

$$\vec{y}_P = \Phi(x,x_0) \cdot \vec{v}(x) = \Phi(x,x_0) \int_{x_0}^{x} \Phi(x_0, x') \cdot \vec{b}(x')\,dx'$$

$$\boxed{\vec{y}_P = \int_{x_0}^{x} \Phi(x, x') \cdot \vec{b}(x')\,dx'}$$

CONVOLUTION INTEGRAL.

$\Phi(x,x')$ IS THE 'GREEN'S FUNCTION'

FOR AUTONOMOUS SYSTEMS,

$$\vec{y}_P = \int_{x_0}^{x} \Phi(x-x')\,\vec{b}(x')\,dx'$$

ALSO CALLED 'THE IMPULSE RESPONSE.' ...

HOW TO ACTUALLY SOLVE THESE SYSTEMS

1) REDUCTION-OF-ORDER: FOR A SYSTEM OF N FIRST-ORDER EQUATIONS, IF ONE HOMOGENEOUS SOLUTION IS KNOWN, THEN VIA THE SUBSTITUTION $\vec{y} = \vec{y}_1 \cdot \vec{u}$, THE SYSTEM CAN BE REDUCED TO AN (N-1) SYSTEM. THAT IS NOT GENERALLY USEFUL UNLESS N=2.

2.) AUTONOMOUS SYSTEMS IF $\mathbb{A}(x) = \mathbb{A}$, THEN THE FUNDAMENTAL MATRIX HAS A SIMPLE FORM. FOR A DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS, THE ~~F~~ CHARACTERISTIC SOLUTION IS $e^{\lambda x}$. LOOKING FOR EXPONENTIAL SOLUTIONS: $\vec{y}(x) = \vec{v} \cdot e^{\lambda x}$ (WHERE $\vec{v}$ IS CONSTANT).

UPON SUBSTITUTION INTO THE DIFFERENTIAL EQUATION $\frac{d}{dx}\vec{y} = \mathbb{A}\vec{y}$,

$$\frac{d}{dx}\vec{y} = \lambda \vec{v} e^{\lambda x} = \mathbb{A}\vec{v}e^{\lambda x}$$

OR,
$$(\lambda \mathbb{I} - \mathbb{A}) \cdot \vec{v} = 0$$

SO EITHER $\vec{v} = 0$ (TRIVIAL; UNINTERESTING) OR $\det[\lambda \mathbb{I} - \mathbb{A}] = 0$

THAT IS, λ IS AN EIGENVALUE OF $\mathbb{A}$ & $\vec{v}$ IS THE ASSOCIATED EIGENVECTOR: $\quad \mathbb{A}\vec{v} = \lambda \vec{v}$.

EX. FIND THE GENERAL SOLUTION OF $\frac{d\vec{y}}{dx} = \mathbb{A}\vec{y}$ WITH $\mathbb{A} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$

TO FIND THE EIGENVALUES:

$$\det[\lambda \mathbb{I} - \mathbb{A}] = \det \begin{bmatrix} \lambda-2 & 3 \\ -1 & \lambda+2 \end{bmatrix} = (\lambda-2)(\lambda+2)+3 = \lambda^2 - 1.$$

SO $\lambda = \pm 1$.

FOR $\lambda = 1$,

$$\underbrace{\begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix}}_{[\lambda \mathbb{I} - \mathbb{A}]} \underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{\vec{v}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

FOR $\lambda = -1$

$$\begin{bmatrix} -3 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

THE GENERAL SOLUTION IS:

$$\vec{y}_H(x) = C_1 e^{x} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + C_2 e^{-x} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

i.e., $$\vec{y}_H = \begin{bmatrix} 3e^{x} & e^{-x} \\ e^{x} & e^{-x} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

MATRIX EXPONENTIAL. RECALL THAT FOR AN AUTONOMOUS SYSTEM,

$$\frac{d\vec{y}}{dx} = \mathbb{A}\cdot\vec{y}$$

THE FUNDAMENTAL MATRIX $\Phi(x,0) \equiv \Phi(x)$ HAS THE FOLLOWING PROPERTIES:

i) $\Phi(0) = \mathbb{I}_n$ iii) $\Phi(x_1 + x_2) = \Phi(x_1)\Phi(x_2)$

ii) $\frac{d\Phi}{dx} = \mathbb{A}\cdot\Phi$ iv) $\Phi(-x) = [\Phi(x)]^{-1}$.

THESE ARE ALL PROPERTIES OF THE EXPONENTIAL. SUGGEST WE DEFINE A NEW FUNCTION CALLED THE MATRIX EXPONENTIAL

$$e^{\mathbb{A}} = \sum_{n=0}^{\infty} \frac{\mathbb{A}^n}{n!} = \mathbb{I} + \mathbb{A} + \frac{\mathbb{A}^2}{2!} + \frac{\mathbb{A}^3}{3!} + \cdots + \frac{\mathbb{A}^n}{n!} + \cdots$$

CONVERGES FOR ALL $\mathbb{A}$.

NOTICE:

i) $e^0 = \mathbb{I}$

ii) $$\frac{d}{dx}e^{\mathbb{A}x} = \frac{d}{dx}\sum_{n=0}^{\infty}\frac{\mathbb{A}^n x^n}{n!} = \sum_{n=1}^{\infty}\frac{\mathbb{A}^n x^{n-1}}{(n-1)!} = \mathbb{A}\sum_{n=1}^{\infty}\frac{\mathbb{A}^{n-1}x^{n-1}}{(n-1)!}$$

$$= \mathbb{A}\cdot\sum_{n=0}^{\infty}\frac{\mathbb{A}^n x^n}{n!} = \mathbb{A}\,e^{\mathbb{A}x}$$

HAVE TO PAY ATTENTION TO COMMUTIVITY

iii) $$e^{\mathbb{A}+\mathbb{B}} = \mathbb{I} + (\mathbb{A}+\mathbb{B}) + \frac{(\mathbb{A}+\mathbb{B})^2}{2} + \cdots = \mathbb{I} + (\mathbb{A}+\mathbb{B}) + \frac{1}{2}\left(\mathbb{A}^2 + \underline{\mathbb{A}\mathbb{B} + \mathbb{B}\mathbb{A}} + \mathbb{B}^2\right) + \cdots$$

FOR THE PRODUCT:

$$e^{\mathbb{A}}e^{\mathbb{B}} = \left(\sum_{n=0}^{\infty}\frac{\mathbb{A}^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{\mathbb{B}^n}{n!}\right) = \left(\mathbb{I} + \mathbb{A} + \frac{\mathbb{A}^2}{2} + \cdots\right)\left(\mathbb{I} + \mathbb{B} + \frac{\mathbb{B}^2}{2} + \cdots\right)$$

$$= \mathbb{I} + (\mathbb{A}+\mathbb{B}) + \left(\frac{\mathbb{A}^2}{2} + \mathbb{A}\mathbb{B} + \frac{\mathbb{B}^2}{2}\right) + \cdots$$

IN GENERAL,

$$e^{\mathbb{A}+\mathbb{B}} = e^{\mathbb{A}}e^{\mathbb{B}}$$ IF $\mathbb{A}$ & $\mathbb{B}$ COMMUTE i.e. $\mathbb{A}\mathbb{B} = \mathbb{B}\mathbb{A}$.

iv) $\mathbb{A}$ COMMUTES WITH $-\mathbb{A}$ SO, $e^{\mathbb{A}}e^{-\mathbb{A}} = e^{-\mathbb{A}}e^{\mathbb{A}} = e^{(\mathbb{A}-\mathbb{A})} = e^0 = \mathbb{I}$

AND $e^{-\mathbb{A}} = [e^{\mathbb{A}}]^{-1}$

HOW DOES THIS MATRIX EXPONENTIAL CONNECT TO THE SOLUTION OF OUR DIFFERENTIAL EQ. $\frac{d\vec{y}}{dx} = \mathbb{A}\cdot\vec{y}$ & $\vec{y}(0) = \vec{y}^{\,o}$?

WE CAN INTEGRATE THIS FIRST-ORDER EQUATION:

$$\vec{y} = \vec{y}^{\,o} + \int_0^x \mathbb{A}\cdot\vec{y}(x')\,dx'$$

THIS IS AN EQUIVALENT <u>INTEGRAL</u> EQUATION FOR $\vec{y}(x)$. ALTHOUGH NO EASIER TO SOLVE THAN THE ORIGINAL DE., IT DOES SUGGEST AN APPROXIMATION SCHEME: MAKE A GUESS FOR $\vec{y}(x)$ AND SUBSTITUTE INTO THE RIGHT-HAND SIDE. USE THE RESULTING EXPRESSION AS AN UPDATED-GUESS, THEN ITERATE...

i.e.
$$\vec{y}^{(n)}(x) = \vec{y}^{\,o} + \int_0^x \mathbb{A}\cdot\vec{y}^{(n-1)}(x')\,dx'$$

START WITH $\vec{y}^{(0)} = \vec{y}^{\,o}$:

$$\vec{y}^{(1)} = \vec{y}^{\,o} + \int_0^x \mathbb{A}\cdot\vec{y}^{\,o}\,dx'$$
$$= \vec{y}^{\,o} + (\mathbb{A}x)\cdot\vec{y}^{\,o} = (\mathbb{I} + \mathbb{A}x)\cdot\vec{y}^{\,o}$$

AGAIN:

$$y^{(2)} = \vec{y}^{\,o} + \int_0^x \mathbb{A}\cdot\vec{y}^{(1)}(x')\,dx' = \vec{y}^{\,o} + \int_0^x \mathbb{A}\cdot(\mathbb{I} + \mathbb{A}x')\vec{y}^{\,o}\,dx'$$
$$= \vec{y}^{\,o} + (\mathbb{A}x)\vec{y}^{\,o} + \left(\frac{\mathbb{A}^2x^2}{2}\right)\vec{y}^{\,o} = \left(\mathbb{I} + \mathbb{A}x + \tfrac{1}{2}(\mathbb{A}x)^2\right)\cdot\vec{y}^{\,o}$$

KEEP GOING: $\lim_{n\to\infty}\vec{y}^{(n)} = \left(\sum_{n=0}^{\infty}\frac{\mathbb{A}^n x^n}{n!}\right)\cdot\vec{y}^{\,o} = e^{\mathbb{A}x}\cdot\vec{y}^{\,o}$

AND SO, $\Phi(x) = e^{\mathbb{A}x}$.

WE CAN USE THIS SAME IDEA (AT LEAST FORMALLY) EVEN IF $\mathbb{A}(x)$ IS <u>NOT</u> CONSTANT, WHICH WE'LL CONSIDER SHORTLY.

FIRST, HOW DO WE ACTUALLY CALCULATE $e^{\mathbb{A}x}$?