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<u>Another derivation</u>

Let $F = I - 2\,(vv^T/v^Tv)$. Find v s.t. $Fx \in \text{span}\{e_1\}$.

$$
\begin{aligned}
Fx &= x - 2\,(v^Tx/v^Tv)\,v \\
&\in \text{span}\{e_1\} \quad \Leftrightarrow \quad v \in \text{span}\{x, e_1\}
\end{aligned}
$$

Let

$$
\begin{aligned}
v &= x + \alpha\, e_1 \\
v^T x &= x^Tx + \alpha\, e_1^T x = x^Tx + \alpha\, x_1 \\
v^T v &= (x + \alpha\, e_1)^T (x + \alpha\, e_1) \\
&= x^Tx + 2\,\alpha\, x_1 + \alpha^2
\end{aligned}
$$

$\therefore$

$$
\begin{aligned}
Fx &= x - 2\,\frac{v^Tx}{v^Tv}(x + \alpha e_1) \\
&= \left(1 - 2\,\frac{v^Tx}{v^Tv}\right)x - 2\alpha\,\frac{v^Tx}{v^Tv}\,e_1 \\
&= \left(1 - 2\,\frac{x^Tx + \alpha x_1}{x^Tx + 2\alpha x_1 + \alpha^2}\right)x - 2\alpha\,\frac{v^Tx}{v^Tv}\,e_1 \\
&= \underbrace{\frac{x^Tx + 2\alpha x_1 + \alpha^2 - 2x^Tx - 2\alpha x_1}{x^Tx + 2\alpha x_1 + \alpha^2}}_{=0 \quad \text{if} \quad \alpha^2 - x^Tx = 0}\,x - 2\alpha\,\frac{v^Tx}{v^Tv}\,e_1
\end{aligned}
$$

$$\alpha = \pm\|x\|$$

Hence $\quad v = x \pm \|x\| e_1$

and $\quad Fx = \mp \|x\| e_1$

Example

$$x = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$v = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + (1)\sqrt{1+2^2+2^2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$$

$$F = I - 2\tfrac{vv^T}{v^T v}$$

$$= \tfrac{1}{3} \begin{bmatrix} -1 & -2 & -2 \\ -2 & 2 & -1 \\ -2 & -1 & 2 \end{bmatrix}$$

$$Fx = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$$

- Note: no need to actually perform F x to compute the product.

- For convenience, v is sometimes normalized; i.e. $||v|| = 1$.
  In this case,

$$F\,x = x - 2\,v\,(v^T x / v^T v)$$

$$= x - 2\,v\,(v^T x)$$

Householder QR factorization algorithm (full version)

```
for  k = 1, 2, . . . , n
   x  =  A(k:m, k)
   v_k  =  x + sign(x_1) ||x|| e_1
   v_k  =  v_k / ||v_k||
   for  j = k, k+1, . . . , n
      A(k:m, j) = A(k:m, j) - 2 v_k (v_k^T A(k:m, j))
   end
end
```

(Notation: $A(k{:}m, j)$ = j-th column of A from row k to row m.)

Notes

1) At the end of the algorithm, A is reduced to R.
2) Q is not constructed. In fact, only v_k's are kept.

Note:
$$Q^T = Q_n \ldots Q_2 Q_1$$
$$Q = Q_1 Q_2 \ldots Q_n \qquad (Q_k = Q_k^T)$$

To compute $Q^T b$:

```
for  k = 1, 2, . . . , n
   b(k:m) = b(k:m) - 2 v_k (v_k^T b(k:m))
end
```

To compute $Q x$:

```
for  k = n, n-1, . . . , 1
   x(k:m) = x(k:m) - 2 v_k (v_k^T x(k:m))
end
```

3) To compute the reduced QR, i.e. $A = \hat{Q}\hat{R} = \begin{bmatrix} & \\ & \end{bmatrix}_{m\times n} \begin{bmatrix} & \end{bmatrix}_{n\times n}$

Then

$$\hat{Q} = \begin{bmatrix} Qe_1 & Qe_2 & \cdots & Qe_n \end{bmatrix} \qquad e_j = \text{j-th col of I}$$

<u>Complexity</u>

- The work is dominated by the inner-most loop:

$$A(k:m, j) = A(k:m, j) - 2 v_k (v_k^T A(k:m, j))$$

flops(v_k^T A(k:m, j)) ~ 2(m-k+1)

flops(2 v_k (v_k^T A(k:m, j))) ~ m-k+1

flops(subtraction) ~ m-k+1

∴ subtotal ~ 4(m-k+1)

This operation is done n-k+1 times (j-loop)

i.e. flops = 4(m-k+1)(n-k+1)

∴ Total flops = sum_{k=1:n} 4(m-k+1)(n-k+1)

~ $2mn^2 - 2/3\, n^3$

When m = n, flops(QR) ~ $4/3\, n^3$ = 2 × flops (LU)

Note: It does not include the computation of Q

Example: Find the QR factorization of A = $\begin{bmatrix} 1 & -4 \\ 2 & 3 \\ 2 & 2 \end{bmatrix}$

$$x = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \qquad v_- = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + \sqrt{1+2^2+2^2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$$

$$\text{or} \quad v_+ = -\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + \sqrt{1+2^2+2^2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} - \tfrac{2}{v_+^T v_+}\begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}\begin{bmatrix} 2 & -2 & -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix} = Q_1$$

$$Q_1 A = Q_1\begin{bmatrix} 1 & -4 \\ 2 & 3 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 0 & -3 \\ 0 & -4 \end{bmatrix}$$

$$x = \begin{bmatrix} -3 \\ -4 \end{bmatrix} \qquad v = -\begin{bmatrix} -3 \\ -4 \end{bmatrix} + \sqrt{3^2+4^2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} - \tfrac{2}{v^T v}\begin{bmatrix} 8 \\ 4 \end{bmatrix}\begin{bmatrix} 8 & 4 \end{bmatrix} = \begin{bmatrix} \frac{-3}{5} & \frac{-4}{5} \\ \frac{-4}{5} & \frac{2}{5} \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{F_2}} \\ 0 & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-3}{5} & \frac{-4}{5} \\ 0 & \frac{-4}{5} & \frac{2}{5} \end{bmatrix}$$

$$Q_2(Q_1 A) = Q_2\begin{bmatrix} 3 & 2 \\ 0 & -3 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} = R$$

$$A = Q_1 Q_2 R$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-3}{5} & \frac{-4}{5} \\ 0 & \frac{-4}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 0 & 5 \\ 0 & 0 \end{bmatrix}$$

$$= \underbrace{\frac{1}{15} \begin{bmatrix} 5 & -14 & -2 \\ 10 & 5 & -10 \\ 10 & 2 & 11 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 3 & 2 \\ 0 & 5 \\ 0 & 0 \end{bmatrix}}_{R}$$