L19 Applications Time Evolution

5.8 Time evolution of free Hamiltonian

5.8.1 Free Hamiltonian:

the Hamiltonian of a system without outer influence

$$H_{free} = \sum_{j=1}^{n} E_j |E_j\rangle\langle E_j|$$

Free Hamiltonian operators are usually easy to construct by looking at the classical energies associated with different constellations of a system.

Example: if an atom can be placed at two different heights in a gravitational field, then the associated Hamilton operator describing the atom is given as

$$H = mgh_1 |h_1\rangle\langle h_1| + mgh_2 |h_2\rangle\langle h_2|$$

5.8.2 Invariance under constant energy shift

Constant Energy Shift

Hamilton Operators:

shift of eigen-energies by Δ

$$H = \sum_n E_n |E_n\rangle\langle E_n| \qquad H' = \sum_n (E_n + \Delta) |E_n\rangle\langle E_n|$$

initial state:

$$|\Psi(0)\rangle = \sum_n c_n |E_n\rangle \qquad |\Psi'(0)\rangle = \sum_n c_n |E_n\rangle$$

at time t:

$$|\Psi(t)\rangle = \sum_n c_n\, e^{-i\frac{E_n t}{\hbar}} |E_n\rangle \qquad |\Psi'(t)\rangle = \sum_n c_n\, e^{-i\frac{(E_n+\Delta)t}{\hbar}} |E_n\rangle$$

$$= e^{-i\frac{\Delta t}{\hbar}} \sum_n c_n\, e^{-i\frac{E_n t}{\hbar}} |E_n\rangle$$

$$= e^{-i\frac{\Delta t}{\hbar}} |\Psi(t)\rangle$$

Energy Shift leads to (time-dependent) global phase

➔ physically irrelevant!

In classical mechanics, we are use to the idea that Energy should be defined only up to a constant (think potential energy), so everything should be only about energy differences!

5.9 Applications

5.9.1 Neutrino Oscillations

(See McIntyre 3.3 for more details!)

5.9.1.1. Background

Neutrinos: relativistic particles (Leptons), interact only via weak interaction

electron neutrino:

$$p \to n + e^{+} + \nu_e$$

positron decays into neutron, positron and electron neutrino

muon neutrino:

$$\pi^{+} \to \mu^{+} + \nu_\mu$$

pion decays into muon and muon neutrino

(third neutrino: *tau neutrino* ν_τ)

Actually:

the three neutrino types correspond to states in a three-dimensional vector state!

So we have

$$\nu_e \leftrightarrow |\nu_e\rangle$$

$$\nu_\mu \leftrightarrow |\nu_\mu\rangle$$

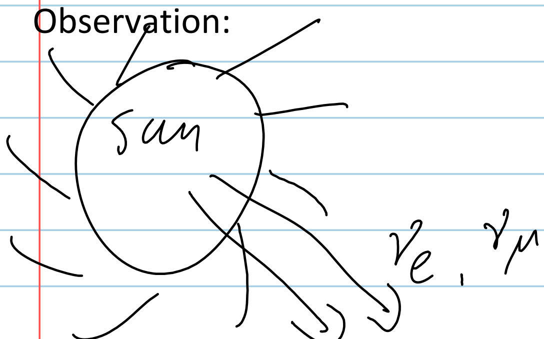
$$\nu_\tau \leftrightarrow |\nu_\tau\rangle$$

so a general state of a neutrino is given as

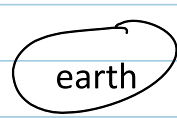
$$|\psi\rangle = \alpha\,|\nu_e\rangle + \beta\,|\nu_\mu\rangle + \gamma\,|\nu_\tau\rangle$$

How come?

Observation:



can neglect tau neutrinos here



sun produces lots of electron and muon neutrinos!

==> good model for all nuclear processes at the sun

BUT: the predicted ration of muon and electron neutrinos does not correspond to observation!

How come? Took a while to find explanation!

5.9.1.2 Quantum Mechanical explanation

1) Neutrino generation:

decay produces states $|\nu_e\rangle, \; |\nu_\mu\rangle$

with some probability distribution: Cause is weak interaction

2) Neutrinos travel through free space to earth

==> weak interaction can be neglected

==> neutrinos behave like free particles

==> need to consider free Hamiltonian for neutrinos!

3) Free Hamiltonian for neutrinos

neutrinos have mass, mass corresponds to energy,

neutrinos have momentum, momentum corresponds to kinetic energy

free Hamiltonian: relativistic expression ...

eigenstates of free Hamiltonian are not $|\nu_e\rangle, |\nu_\mu\rangle$

but some states (mass eigenstates) $|\nu_1\rangle, \; |\nu_2\rangle$

$$H = \sum_j E_j |\nu_j\rangle\langle\nu_j|$$

$$E_j = \sqrt{(pc)^2 + (m_j c^2)^2}$$

4) Mixing angle (choose mass eigenstates as standard basis)

$$|e\rangle = \cos\frac{\theta}{2}|\nu_1\rangle + \sin\frac{\theta}{2}|\nu_2\rangle \rightarrow \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

$$|\nu_\mu\rangle = \sin\frac{\theta}{2}|\nu_1\rangle - \cos\frac{\theta}{2}|\nu_2\rangle \leftrightarrow \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix}$$

the angle θ is determined by the decay process that generates the neutrinos

5) Then time evolution of electron neutrino in two-dimensional subspace:

$$|\psi(0)\rangle = |\nu_e\rangle \leftrightarrow \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

$$|\psi(t)\rangle = \begin{pmatrix} e^{-i\frac{E_1 t}{\hbar}}\cos\frac{\theta}{2} \\ e^{-i\frac{E_2 t}{\hbar}}\sin\frac{\theta}{2} \end{pmatrix}$$

6) prediction for observation at later time:

$$P_r(\nu_e \to \nu_\mu) = |\langle \nu_\mu | \psi(t) \rangle|^2$$

$$= \left| e^{-i\frac{E_1 t}{\hbar}} \cos\frac{\theta}{2} \sin\frac{\theta}{2} - e^{-i\frac{E_2 t}{\hbar}} \cos\frac{\theta}{2} \sin\frac{\theta}{2} \right|^2$$

$$\ldots = \sin^2\theta \sin^2\left(\frac{(E_1 - E_2)t}{2\hbar}\right)$$

7) Experimental matching of parameters

non-relativistic approximation: $E \approx m_i c^2$

flight time Earth sun: $t = \frac{L}{c}$ L: distance sun-earth

$$P_r(\nu_e \to \mu_e) \approx \sin^2\frac{2\theta}{2} \sin^2\left(\frac{(m_1^2 - m_2^2) L c^3}{4 E \hbar}\right)$$

experimental parameters

$$\Rightarrow m_1^2 - m_2^2 = 8 \cdot 10^{-5} eV$$

Theoretical models predict some ratio of electron to muon neutrinos, and the a different ratio is observed on earth.

Using the above model (with transition probabilities from electron to muon neutrino, and also for the reverse process), researchers concluded that

$$\theta = 69^\circ$$

5.9.2 Quantum Systems as Clocks

5.9.2.1 Prepraration: Measuring time with a mechanical pendulum

can be used to measure time:

long times: number of oscillation periods (trivial)

short times: difference in position corresponding to short times

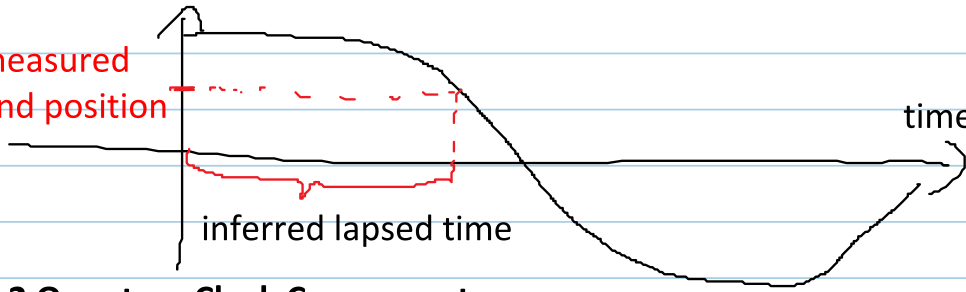
Step 1: Preparation of Pendulum, e.g. place it in right turning point

Step 2: Start signal: let pendulum go

Step 3: Stop signal, stop signal

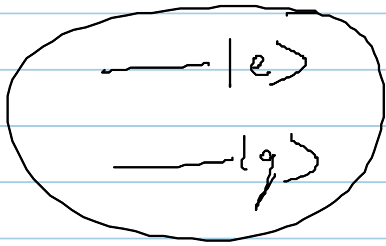
Step 4: determine position of pendulum at stop position

Relate position to elapsed time



5.9.2.2 Quantum Clock Components

System description

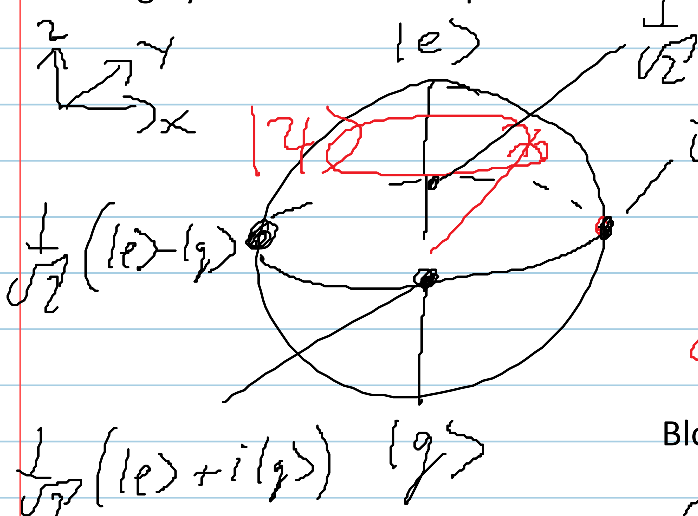


two internal states of an atom,
- ground state $|g\rangle$
- exited state $|e\rangle$

Hamilton operator:

$$H = E_g |g\rangle\langle g| + E_e |e\rangle\langle e|$$

resulting dynamics in Bloch Sphere:



state

$\Leftrightarrow \rho = |\psi\rangle\langle\psi|$ density matrix

$\Leftrightarrow \vec{r}$ Bloch vector

Bloch vector precesses around z-axis with frequency

$$\Omega = \frac{(E_e - E_g)}{\hbar}$$

Example:

Initial state $\frac{1}{\sqrt{2}}\left(|e\rangle + |g\rangle\right)$

at time t $\frac{1}{\sqrt{2}}\left(e^{i\frac{E_e}{\hbar}t}|e\rangle + e^{i\frac{E_g}{\hbar}t}|g\rangle\right)$

$$= e^{i\frac{E_g}{\hbar}t}\frac{1}{\sqrt{2}}\left(e^{i\frac{(E_e-E_g)}{\hbar}t}|e\rangle + |g\rangle\right)$$

$\longrightarrow$ Bloch vector rotating in the x-y plane

Tools:

1) free time evolution with Hamilton operator H as above
2) ability to prepare system in the ground state $|g\rangle$
3) Ability to perform a unitary

Unitary rotates Bloch vectors around y-axis by 90 degrees:

Method varies according to system:

spin 1/2 system:

switch on additional strong magnetic field along y-axis

dominates over free evolution, so effective B-field along y-axis

atom: short laser pulse at transition frequency corresponding to energy difference between levels ... (time dependent Hamiltonian)

$$U_{\frac{\pi}{2}} = \frac{1}{\sqrt{2}}\left(|e\rangle + |g\rangle\right)\langle g| + \frac{1}{\sqrt{2}}\left(|e\rangle - |g\rangle\right)\langle e|$$

$$\doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

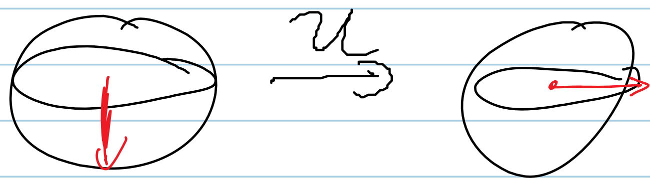
Inverse operation:

$$U_{-\frac{\pi}{2}} = U_{\frac{3}{2}\pi} = |g\rangle\left[\frac{1}{\sqrt{2}}\left(\langle e| + \langle g|\right)\right] + |e\rangle\left[\frac{1}{\sqrt{2}}\left(\langle e| - \langle g|\right)\right]$$

$$\doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Example:

$$U_{\frac{\pi}{2}}|g\rangle = \frac{1}{\sqrt{2}}\left(|e\rangle + |g\rangle\right)$$



4) Ability to perform measurement in the basis

$$\{|e\rangle, |g\rangle\}$$

For atoms, this can be done using

- resonance fluorescence, or
- ionisation measurement

5.9.2.3 Clock Protocol:

1) initialize system in state

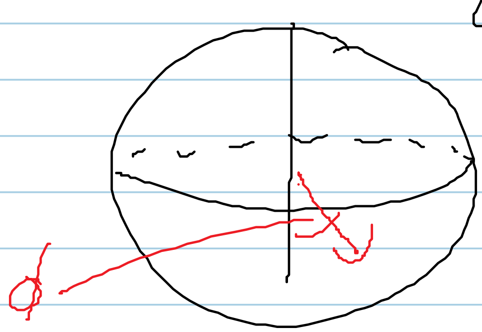
2) Apply unitary $U_{\pi/2}$

3) Let system evolve under Hamiltonian H for time Δt

(Bloch vector rotates in x-y plane)

Bloch vector rotates by angle $\phi = \Omega \Delta t$

4) Apply inverse operation $U_{-\frac{\pi}{2}}$

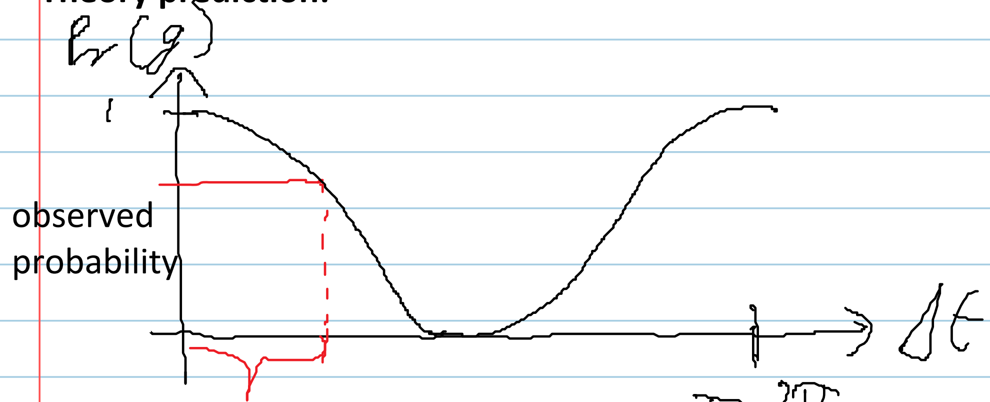


Bloch vector end up in y-z plane!

5) Perform measurement in the standard basis $\{|e\rangle, |g\rangle\}$

6) Repeat steps 1-5 many times (with the same time Δt)
to obtain the probabilities to find the system in the exicted state $|e\rangle$
or the ground state $|g\rangle$

Theory prediction:



$$Pr(g) = \cos^2 \frac{\Omega}{2} t$$