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<u>Notes</u>

1) This is upper bound only. CG convergence is usually better.

2) CG convergence depends only all $\{\lambda_j\}$, not just λ_{min}, λ_{max}.

   E.g. A has 3 distinct eigenvalues: $\lambda_1 < \lambda_2 < \lambda_3$

   Define Lagrange poly. $P_3(x)$ of deg ≤ 3 such that

   $$P_3(0) = 1, \quad P_3(\lambda_j) = 0 \qquad j = 1, 2, 3.$$

   Then $$\left\|e^3\right\|_A^2 \le \left\|P_3(A)e^0\right\|_A^2 = \sum_{j=1}^{3} \xi_j^2 P_3^2(\lambda_j)\lambda_j = 0$$

   $\Rightarrow$ CG converges in 3 iterations, independent of $\kappa(A)$.

3) For Poisson equation, convergence rates for SOR and CG are the same. However, no optimal parameter needed for CG.

Preconditioning

Idea: to construct a preconditioning matrix $M \approx A$ s.t.

$$\kappa(M^{-1} A) \ll \kappa(A)$$

M is called a preconditioner.

Instead of solving $A x = b$, we solve the preconditioned system:

$$M^{-1} A x = M^{-1} b$$

Since $\kappa(M^{-1} A) \ll \kappa(A)$, CG should converge much faster on the preconditioned system.

If A is SPD, we also want M to be SPD.

Note: A, M SPD does not imply $M^{-1} A$ is SPD.

Symmetric preconditioning

- Since M is SPD, we can write

$$M = L L^T$$

where L = lower Δ.

- "Split" the preconditioner between left and right:

$$\underbrace{(L^{-1} A L^{-T})}_{\tilde{A}}\underbrace{(L^T x)}_{\tilde{x}} = \underbrace{L^{-1} b}_{\tilde{b}}$$

- $\tilde{A}$ = $L^{-1} A L^{-T}$ is SPD.

- Apply CG to $\tilde{A}\tilde{x} = \tilde{b}$

$$\tilde{x}^0 = \text{initial guess; } \tilde{r}^0 = \tilde{b} - \tilde{A}\tilde{x}^0$$
$$\textit{for } \ k = 0, 1, 2, \ldots, n-1$$
$$\text{compute } \tilde{\beta}_k, \tilde{p}_k, \tilde{\alpha}_k, \tilde{x}^{k+1}, \tilde{r}^{k+1}$$
$$\textit{end}$$

- It turns out no need to form $\tilde{A}$ explicitly.

<u>Preconditioned Conjugate Gradient</u> (PCG)

x^0 = initial guess; $r^0 = b - A x^0$
for k = 0, 1, 2, . . . , n-1
  $z^k = M^{-1} r^k$
  $\beta^k = (z^k, r^k) / (z^{k-1}, r^{k-1})$
  $p^k = z^k + \beta^k p^{k-1}$
  $\alpha^k = (z^k, r^k) / (p^k, Ap^k)$
  $x^{k+1} = x^k + \alpha^k p^k$
  $r^{k+1} = r^k - \alpha^k A p^k$
end

- Basically, add one line: $z^k = M^{-1} r^k$

- If M = I, then the same as CG.

Modern linear solvers = CG methods + good preconditioners

Overview

GE for structured matrices
- symmetric, band, tridiagonal, SPD

Sparse matrices
- PDEs, 1D + 2D Laplacian matrix

GE for sparse matrices
- ordering methods: CM, RCM, min deg

Iterative methods
- Richardson, Jacobi, GS, SOR
- Derivation of CG

Least squares problem
- What is LS problem? What is the geometric meaning?
- QR factorization
- (modified) Gram-Schmidt, Householder transform, Givens rotation

Eigenvalue problem
- Eigenvalue and eigenvectors of a matrix
- Gershgorin's theorem
- Computational methods: power iteration, inverse iteration, idea of shift, QR iteration, shifted QR iteration

SVD
- What is SVD?
- Properties of SVD, rank-k approximation
- Applications

Convergence of iterative methods
- Results for SPD A. Results for 2D Laplacian matrix
- Convergence analysis for Richardson, Jacobi, CG