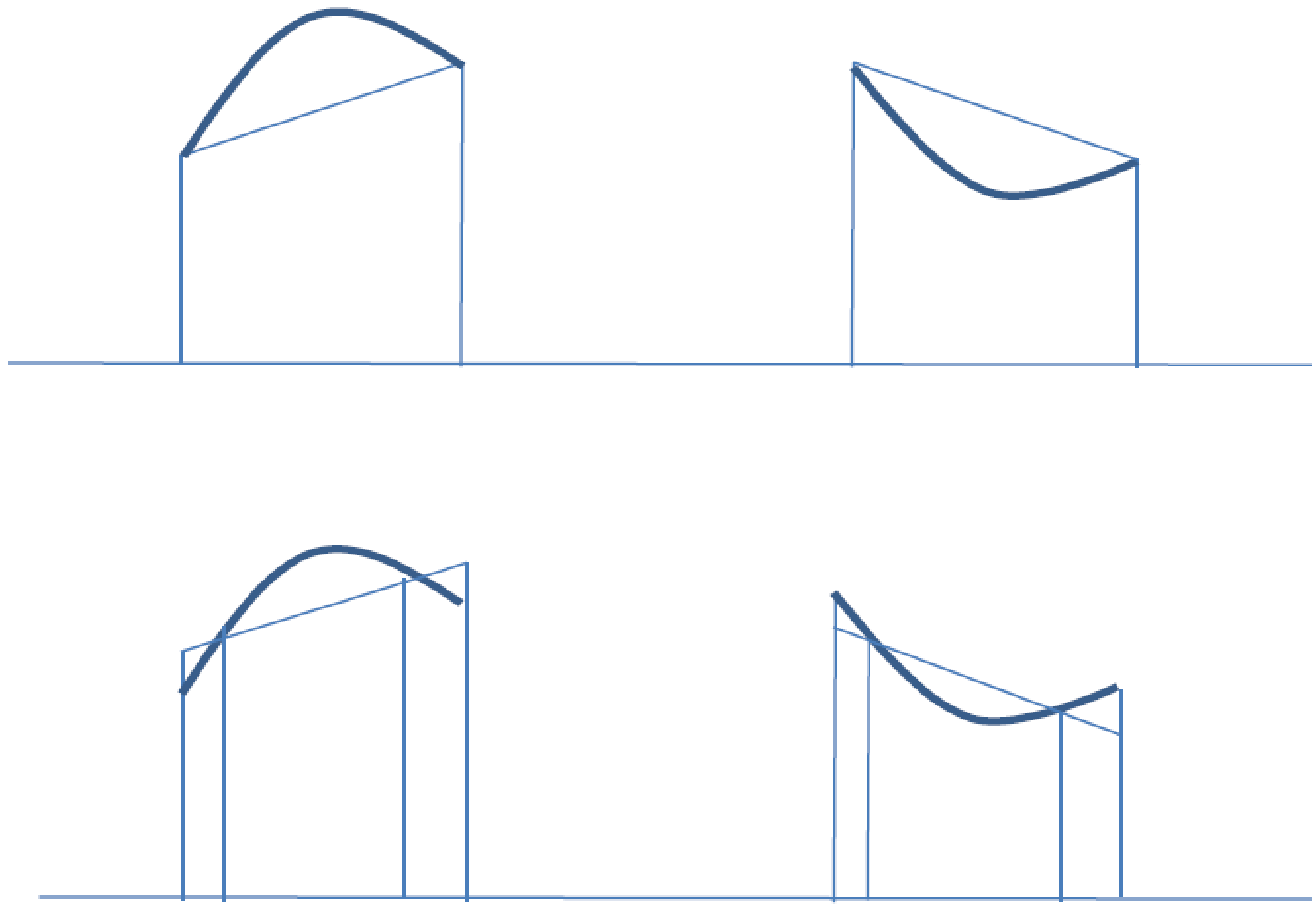
Newton-Cotes Methods for $\int_a^b f(x)dx$

- Approximate f(x) by a polynomial of degree n
- Integrate the interpolating polynomial
- Evaluate f(x) at n+1 evenly-spaced points
- Can we reduce the error further by consider other choices of points and approaches?

Can we do better if we allow different points?



Gaussian Quadrature Rules

- Choose values
  - $x_0, x_1, \ldots, x_{n-1}$ in [a,b]
  - coefficients $c_0, c_1, \ldots, c_{n-1}$
- to minimize expected error when approximating $\int_a^b f(x)dx$ with $\sum_{i=0}^{n-1} c_i f(x_i)$
- We want to be exact for as many degrees of polynomials as possible

Gaussian Quadrature Rules (con't)

- We have 2n unknowns
- We can impose 2n conditions: being exact for polynomials of degree 2n-1
  – $1, x, x^2, x^3, \ldots, x^{2n-1}$
  – And any linear combination of them.

Consider a special case: restrict the interval to [-1,1] and choose n=3

- $\int_{-1}^{1} f(x)dx = c_0 f(x_0) + c_1 f(x_1) + c_2 f(x_2)$
- Our approximating polynomial can then be of degree at most 5:
- $p(x) = a_0 + + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$
- There are now more unknowns (a_k). How does this help?
- $\int_{-1}^{1} f(x)dx =$
  $\int_{-1}^{1} (a_0 + + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5) dx$

Which leads to

- Approximate $\int_{-1}^{1} f(x)dx$ by

$$\frac{5}{9}f\left(-\sqrt{\frac{3}{5}}\right)+\frac{8}{9}f(0)+\frac{5}{9}f\left(\sqrt{\frac{3}{5}}\right)$$

Example: $\int_{-1}^{1} x \sin x \, dx$

- Actual solution: $(-x \cos(x) + \sin(x))_{-1}^{1} = 0.6023$
- Simpson's Rule: 2/6(f(-1)+4f(0)+f(1)) = 0.5610
- Gaussian Quadrature:

$$\frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right) = 0.6020$$

How useful is this?

- Only works for interval [-1,1]
- Need to solve a system of nonlinear equations to define unknowns c_k and x_k.

First: the interval

- Any interval x Є[a,b] can be mapped onto
  tЄ [-1,1] by noting that:

$$x = \frac{a}{2}(1-t) + \frac{b}{2}(1+t)$$

- Rewrite the interval and update the variable

Example: Integrate $\int_1^{1.5} e^{-x^2}\, dx$

- Exact (to 7 decimals) = 0.1093643
- Convert the interval to [-1,1] and change variable
- Use our 3-point Gaussian Quadrature rule

What about solving the system of unknowns?

- The x_k values are actually the roots of the Legendre polynomials, defined recursively as:
  - $P_0(x) = 1$
  - $P_1(x) = x$
  - $P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) + \frac{n}{n+1} \; P_{n-1}(x)$
- For example:
  - $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$ and $P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$

Legendre Polynomials

- Roots (our x_k points) are well-established
- The coefficients c_k can be calculated as:

$$c_k = \int_{-1}^{1} \prod_{\substack{j=0 \\ j\neq k}}^{n-1} \frac{x - x_j}{x_k - x_j} dx$$

- These values are also well-established and are accessible.
- These polynomials have many very interesting properties (that we won't get into)

What about the error using the Gaussian quadrature approach?

- The error is proportional to $|f^{(2n)}(x)|$
- So, when using n=3 points, it will be proportional to $|f^{(6)}(x)|$
- Recall:
  - Simpson's Rules error was proportional to $|f^{(4)}(x)|$
  - Midpoint/Trapezoid's error proportional to $|f^{(2)}(x)|$
- Will be exact for more polynomials than the other approaches, using the same number of points
  - Does not necessarily mean it will always be more accurate for higher order or non-polynomial functions