Convergence

Suppose $\{x_k\}_{k=0}^{\infty}$ converges to x^*. Let $e_k = x_i$-x^*.

If $\exists q > 0, \lambda > 0$ such that

$$\lim_{n\to\infty} \frac{|x_{k+1} - x^*|}{|x_k - x^*|^q} = \frac{|e_{k+1}|}{|e_k|^q} = \lambda$$

then, $\{x_k\}_{k=0}^{\infty}$ is said to converge to x^* with order q and asymptotic error constant λ.

(Note: terminology in deSterck is equivalent, but different)

Note that:

- Higher order q will generally converge faster
- Smaller λ is generally better (but not as important as value of q).

Linear vs Quadratic Convergence

Consider two sequences that converge to 0:

- *Linear*
- $\{p_n\}_{n=0}^{\infty} \rightarrow 0$ with $\lambda_p = ½$ and $|p_{n+1}| \sim ½\,|p_n|$

  So, $|p_{n+1}| \sim (½)^n\,|p_0|$

- *Quadratic*
- $\{q_n\}_{n=0}^{\infty} \rightarrow 0$ with $\lambda_q = ½$ and $|q_{n+1}| \sim ½\,|q_n|^2$

  So, $|q_{n+1}| \sim (½)^{2^n-1}\,|q_0|^{2^n}$

Linear vs Quadratic: Does it matter?

n	Linear $\{p_n\}$	Quadratic $\{q_n\}$
1	5.0000 e -1	5.0000 e -1
2	2.5000 e -1	1.2500 e -1
3	1.2500 e -1	7.8125 e -3
4	6.2500 e -2	3.0518 e -5
5	3.1250 e -2	4.6566 e -10
6	1.5625 e -2	1.0842 e -19
7	7.8125 e -3	5.8775 e -39

Some terminology

- Double root: x^* is a double root of $f(x^*)=0$ iff $f(x^*)=0$ and $f'(x^*) = 0$.
- Root of multiplicity m: $f(x^*)=0$, $f'(x)=0$, ..., $f^{(m-1)}(x^*) = 0$, for some $m>0$
- Simple root: multiplicity 1
- Multiple roots may affect convergence rates.

Convergence of Newton-Raphson Method

Thm: If $f(x^*)=0$, $f'(x^*)\neq 0$, and f, f', f'' are all continuous in an interval about x^* (e.g. over $[x^*-\delta, x^*+\delta]$) with x_0 sufficiently close to x^*, then the sequence $\{x_k\}$ converges quadratically to x^*.

Suppose x* is a simple root of f. By Taylor's expansion, there exists ϑ_k between x^* and x_k such that

$$f(x^*) = f(x_k) + f'(x_k)(x^* - x_k) + \frac{1}{2} f''(\vartheta_k)(x^* - x_k)^2$$

Additional notes on Newton-Raphson

- if $f'(x^*)=0$ (i.e. x^* has multiplicity ≥ 2), convergence becomes linear
- If conditions not met, sequence may not converge.
- If conditions not met, sequence might still converge or may diverge or even cycle

Secant:

- Similar analysis to Newton
- Not as fast as Newton (due to additional approximations)
- $q \sim 1.6$
- Like Newton, not guaranteed to converge if conditions not met
- Note: like Newton, not guaranteed to diverge if conditions not met either.

Bisection Method:

- Nature of algorithm does not easily apply to definition
- At each stage the interval decrease by a half
- Error is no more than length of interval
- Length of interval converges to 0
- Convergence is (roughly/informally) linear
- Recall: convergence is guaranteed

Regula falsi:

- Bracketing – again hard to apply convergence definition directly
- Generally will be at least as fast as bisection, though could be particularly slow
- Recall: convergence is guaranteed

Convergence of Fixed Point Methods

- Let g be continuous over interval $[a, b]$,
- g(x) ∈ $[a, b]$ for all $x \in [a, b]$,
- x* is a fixed point of g ∈ $[a, b]$,
- $\exists \delta \; s.t. \, g'(x)$ is continuous on $[x^* - \delta, x^* + \delta]$
- Define $x_k = g(x_{k+1})$

Then

- If $|g'(x^*)| < 1$, ∃ε s.t. $\{x_k\}$ converges to x* for $|x_0 - x^*| < \varepsilon$
- If $|g'(x^*)| > 1$ then $\{x_k\}$ diverges for any x_0.