Module 06: Fourier Analysis Working with periodic data and some cool algorithms

Starting Wednesday, March 12, 2014

Fourier Analysis

- The process of approximating periodic functional data (from observations or a known function) with a (possibly infinite) combination of sine and cosine waves.
- Involves the conversion of time or spatial information info frequency information
- The following all involve Fourier transforms in some way: cell phones, disc drives, DVDs, JPEGs.

Touch-tone dialing

- Each row and column of a keypad has an associated frequency
- When you press a button, both frequencies play
- Two frequencies in different ranges, to try to ensure that speech would not interfere with it.

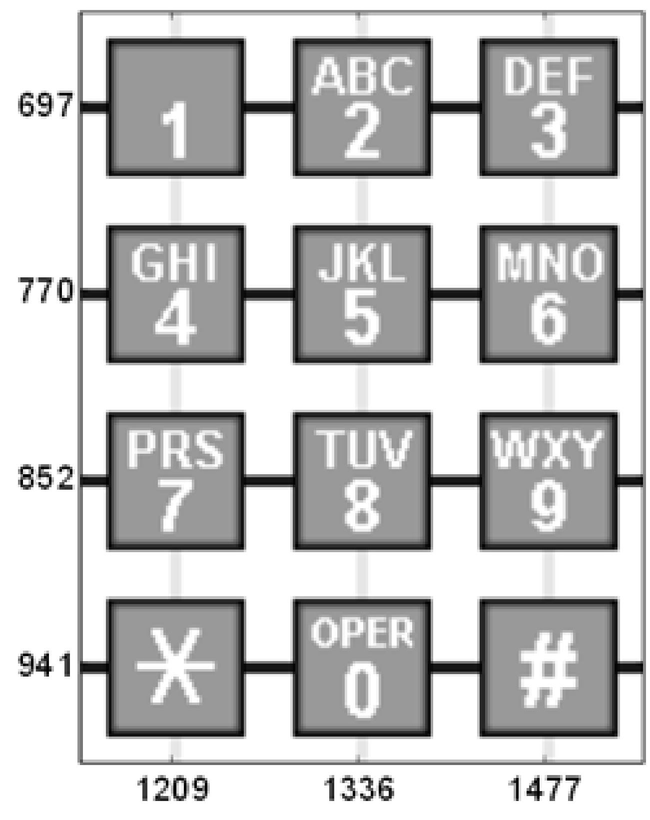


Figure 8.1. *Telephone keypad.*

- The following corresponds to an 11-digit phone number being called:

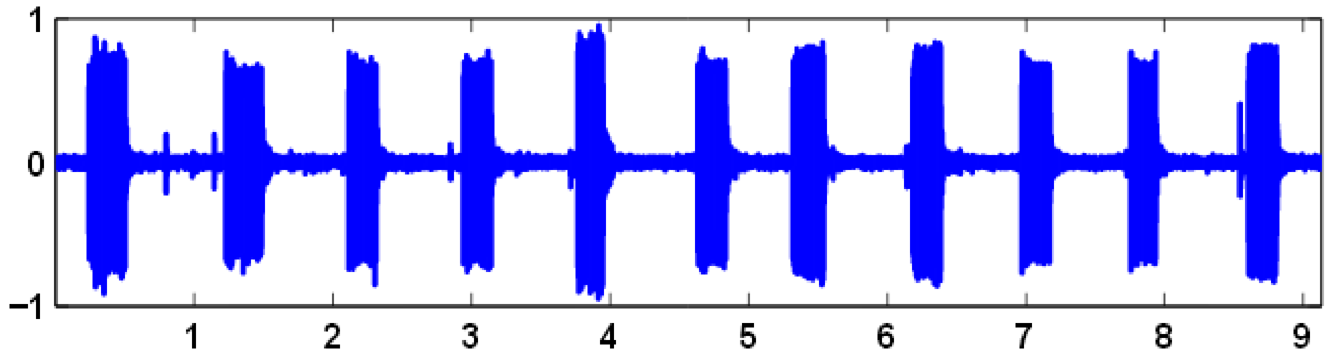


Figure 8.3. *Recording of an 11-digit telephone number.*

- It is "noisy" - more noises than just the numbers
- We can see 11 spikes – but cannot determine which buttons were pressed from this.
- Further analysis is needed to identify them.

- Applying the FFT process to this data isolates the primary frequencies in the call:

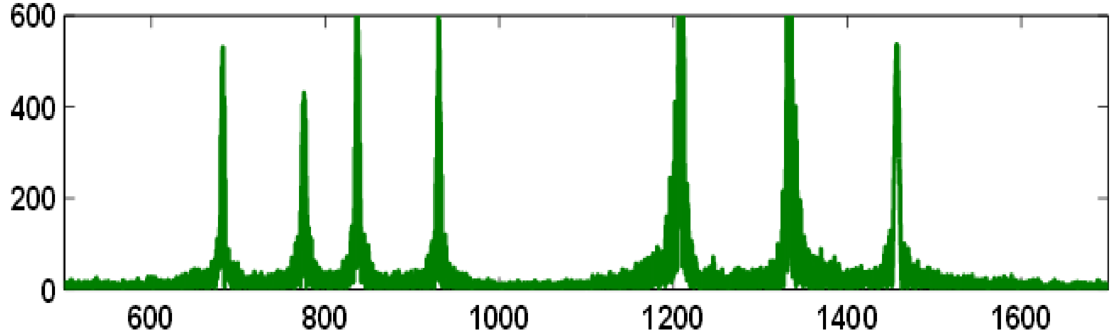
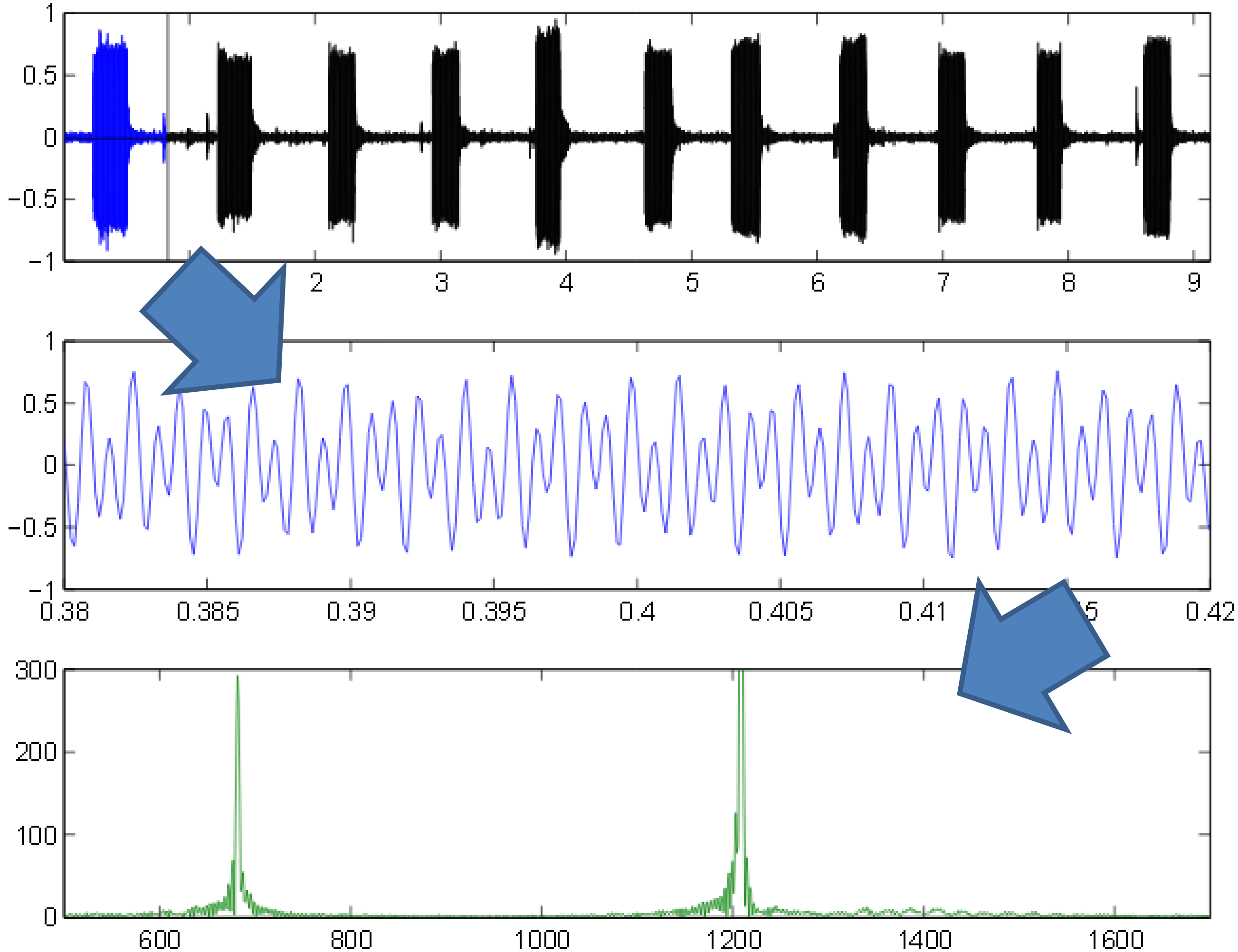


Figure 8.4. *FFT of the recorded signal.*

- The seven peaks correspond to the seven frequencies of the tones.

Expand this region – first number dialed



Apply FFT to find frequencies

Figure 8.5. *The first segment and its FFT.*

A few comments as we start …

- Fast Fourier Transform (FFT) – "a Top 10 Algorithm" – editors of SIAM
- We will study FFT later – we will start with Fourier series first.
- Note: Previous images are from http://www.mathworks.com/moler/fourier.pdf

What we will study

- Consider the continuous function f(t).
- Any periodic function can be written as a combination of trig functions

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2\pi k t}{T}\right) + \sum_{k=1}^{\infty} b_k \sin\left(\frac{2\pi k t}{T}\right)$$

- How to determine a_0, a_k, b_k from a function f(t)?
- How to determine a_0, a_k, b_k from discrete data?
- How to do this computation quickly?

Fourier Series

- Continuous function f(t)
- Periodic with period T: *f(t±T) = f(t)*
- e.g. $\mathrm{g}(\mathrm{t}) = \cos\left(\frac{2\pi k t}{T}\right), \mathrm{h}(\mathrm{t}) = \sin\left(\frac{2\pi k t}{T}\right)$ are periodic with period T
- Every periodic function f(t) can be written as a combination of sine and cosine functions:

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2\pi k t}{T}\right) + \sum_{k=1}^{\infty} b_k \sin\left(\frac{2\pi k t}{T}\right)$$

Useful information

- Assume (for simplicity) that t$\in[0, 2\pi]$ and $\mathrm{T} = 2\pi$
- $\int_0^{2\pi} \cos kt \sin kt \; dt = 0$
- $\int_0^{2\pi} \cos kt \cos jt \; dt = 0 \,, k \neq j$
- $\int_0^{2\pi} \sin kt \sin jt \; dt = 0 \,, k \neq j$
- $\int_0^{2\pi} \sin kt \; dt = 0 \,,$
- $\int_0^{2\pi} \cos kt \; dt = 0,$
- i.e. the functions {1, cos kt, sin kt} are orthogonal on $[0, 2\pi]$

Rewrite using Euler's Formula

Recall, for $i = \sqrt{-1}$*, we can write:*

- $e^{i\theta} = \cos\theta + \mathrm{i}\sin\theta$
- $e^{-i\theta} = \cos\theta - \mathrm{i}\sin\theta$