Introduction to time-frequency analysis

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Lecture 6
Stationary and non-stationary spectral analysis

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The spectrogram

The **spectrogram** for a signal $x(t)$ is defined as

$$S_x(t,f) = |X(t,f)|^2 = |\int_{-\infty}^{\infty} x(t_1)h^*(t_1 - t)e^{-i2\pi f t_1}dt_1|^2,$$

where the unit energy window function $h(t)$ centered at time t is multiplied with the signal $x(t)$ before the Fourier transform.

- The window length should match the component length.
- The window shape should be chosen for the best trade off between sidelobe suppression and mainlobe width.

The Wigner distribution

The **Wigner distribution**, or sometimes the Wigner-Ville distribution, is defined as

$$W_x(t,f) = \int_{-\infty}^{\infty} x(t+\frac{\tau}{2})x^*(t-\frac{\tau}{2})e^{-i2\pi f\tau}d\tau.$$

- Usually, a real-valued signal $x(t)$ is replaced by the **analytic signal** correspondence $z(t)$ in the calculations.
- The Wigner distribution of a Gaussian function has twice the concentration of a spectrogram using the optimal matching Gaussian window.
- The Wigner distribution suffers from cross-terms.

The discrete Wigner distribution

The discrete-time and discrete-frequency Wigner distribution is defined as

$$W_x[n,l] = 2 \sum_{m=-\min(n,N-1-n)}^{\min(n,N-1-n)} x_{n+m} x^*_{n-m} e^{-i2\pi m \frac{l}{L}}$$

- To avoid **aliasing**, the highest normalized frequency of a **real-valued signal** could not be higher than $f_{max} = 0.25$.
- For the corresponding **analytic signal**, $f_{max} = 0.5$.

The ambiguity function

The word "ambiguity" usually specifies something that is not clearly defined. However, there is nothing ambiguous about the **ambiguity function** (AF), defined as

$$A_z(\nu, \tau) = \int_{-\infty}^{\infty} z(t + \frac{\tau}{2}) z^*(t - \frac{\tau}{2}) e^{-i2\pi\nu t} dt.$$

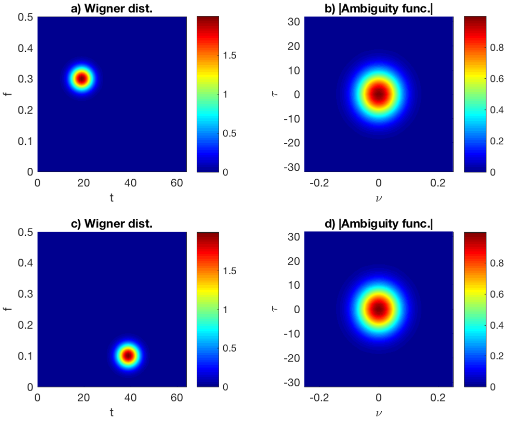
The **ambiguity spectrum** is defined as

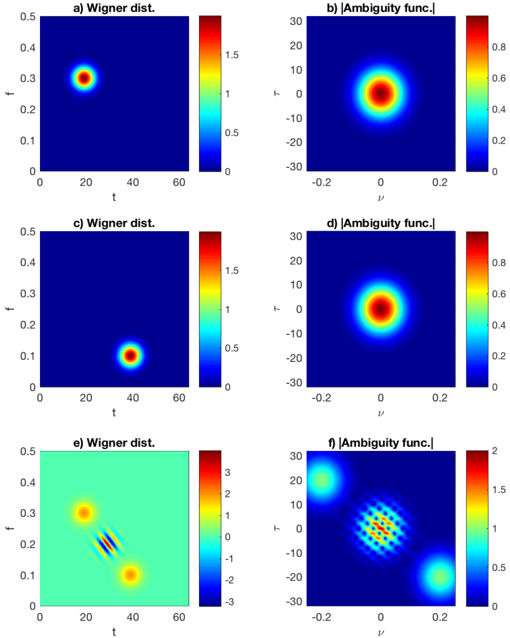
$$A_x^s(\nu, \tau) = \int_{-\infty}^{\infty} r_x(t, \tau) e^{-i2\pi\nu t} dt,$$

where the instantaneous autocorrelation function, (IAF), $r_x(t, \tau) = E[x(t + \frac{\tau}{2}) x^*(t - \frac{\tau}{2})]$.

Ambiguity function example

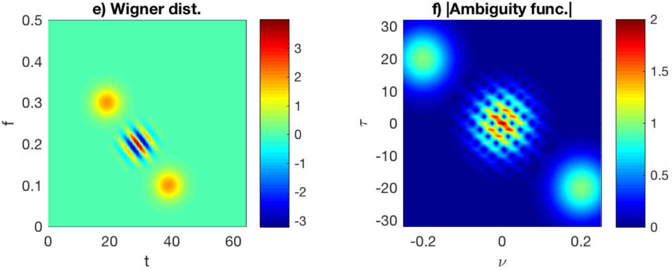
- The ambiguity function is complex-valued.
- The absolute value of the AF is not affected by time- and frequency shifts of a single component.



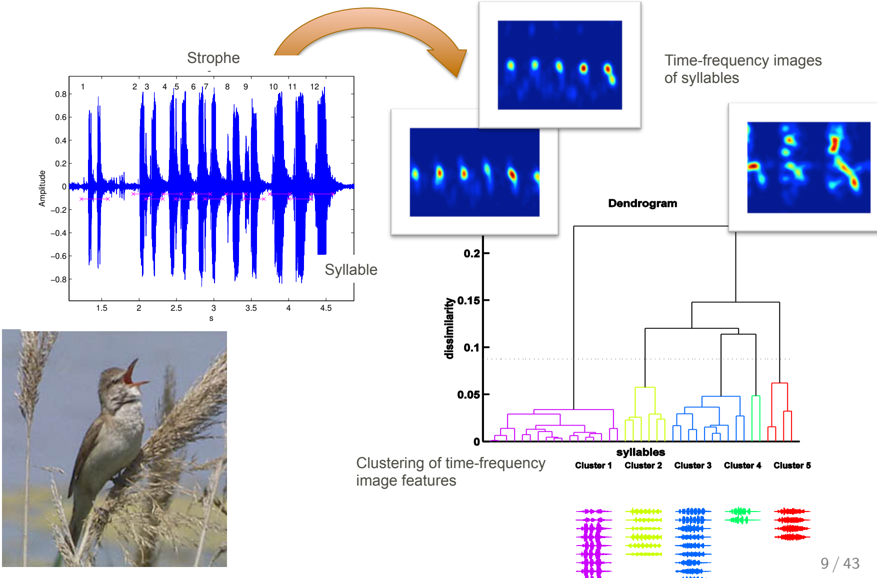

a) Wigner dist.
b) |Ambiguity func.|
c) Wigner dist.
d) |Ambiguity func.|
e) Wigner dist.
f) |Ambiguity func.|
f
t
τ
ν

The cross-terms of the ambiguity function

- The time- and frequency locations of the Wigner distribution cross-terms are $(t_1 + t_2)/2$ and $(f_1 + f_2)/2$ respectively.
- In the ambiguity function, the autoterms are summed at the centre, i.e $\tau = \nu = 0$, and the cross-terms show up located at doppler frequencies $\pm(f_2 - f_1)$ and at lags $\pm(t_2 - t_1)$.

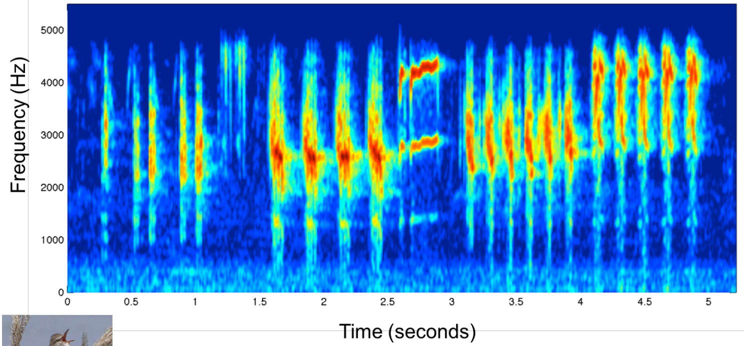


Bird song syllable similarity



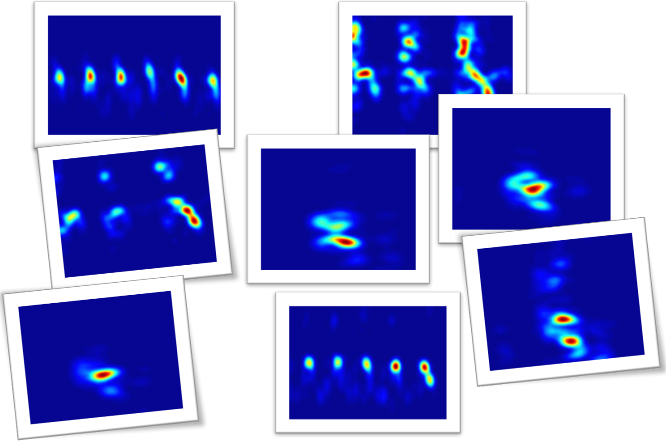
Bird song syllable similarity

The spectrogram of a strophe from the song of the Great Reed Warbler.

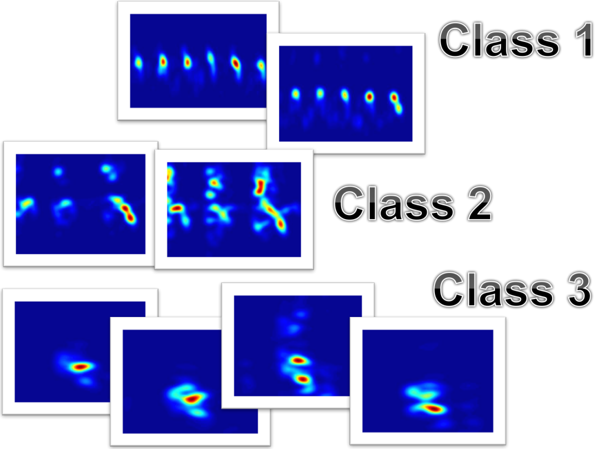


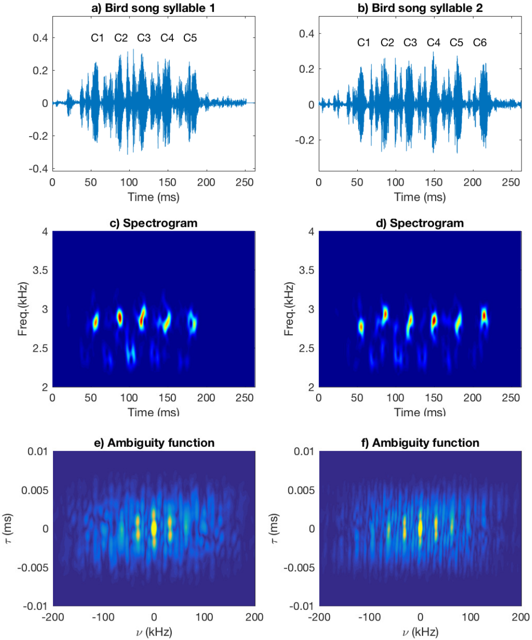
Mareile Große Ruse, Dennis Hasselquist, Bengt Hansson, Maja Tarka and Maria Sandsten, "Automated Analysis of Song Structure in Complex Bird Songs", Animal Behaviour, Vol. 112, pp. 39-51, 2016.

Bird song syllable similarity

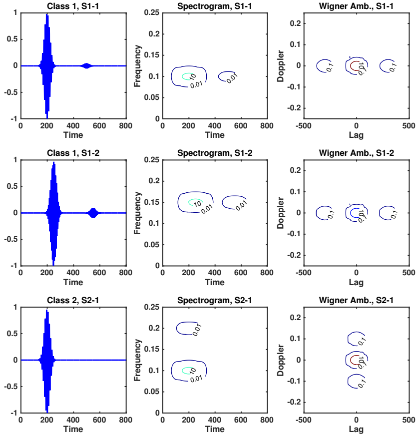


Bird song syllable similarity




a) Bird song syllable 1
C1 C2 C3 C4 C5
0.4
0.2
0
-0.2
-0.4
0 50 100 150 200 250
Time (ms)
b) Bird song syllable 2
C1 C2 C3 C4 C5 C6
0.4
0.2
0
-0.2
0 50 100 150 200 250
Time (ms)
c) Spectrogram
Freq.(kHz)
4
3.5
3
2.5
2
0 50 100 150 200 250
Time (ms)
d) Spectrogram
Freq.(kHz)
4
3.5
3
2.5
2
0 50 100 150 200 250
Time (ms)
e) Ambiguity function
τ (ms)
0.01
0.005
0
-0.005
-0.01
-200 -100 0 100 200
ν (kHz)
f) Ambiguity function
τ (ms)
0.01
0.005
0
-0.005
-0.01
-200 -100 0 100 200
ν (kHz)

Classification based on cross-terms



Maria Sandsten and Johan Brynolfsson, 'Classification of Bird Song Syllables Using Wigner-Ville Ambiguity Function Cross-Terms', EUSIPCO, Kos Island, Greece, 2017.

Cross-term enhancement

To illustrate, we study the following simple example with a two-component signal,

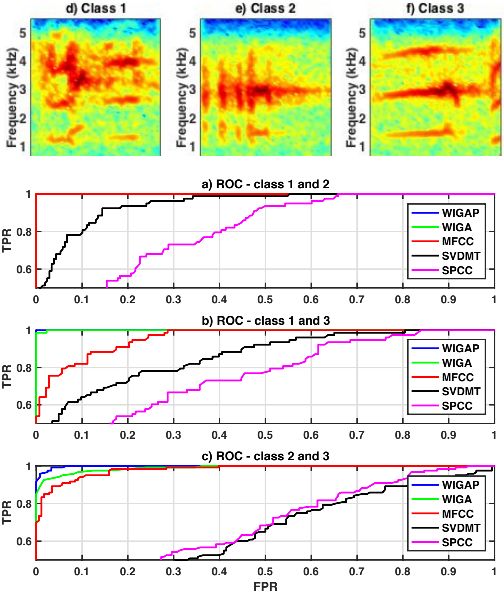
$$z(t) = z_1(t) + z_2(t) = c_1 e^{-i2\pi f_1 t} + c_2 e^{-i2\pi f_2 t},$$

where the absolute value of the resulting ambiguity function is

$$|A_z(\nu,\tau)| = (c_1^2 + c_2^2)\delta(\nu) + c_1 c_2(\delta(\nu + (f_1 - f_2)) + \delta(\nu - (f_1 - f_2)).$$

We consider the case when one component is much weaker than the other, $c_2 << c_1$.

Classification based on cross-terms



Ambiguity kernels

A **filtered ambiguity function** can be formulated as the multiplication of the ambiguity function and an **ambiguity kernel**,

$$A_z^F(\nu, \tau) = A_z(\nu, \tau) \cdot \phi(\nu, \tau).$$

The ambiguity kernel is most often defined to preserve the auto-terms at the centre and to suppress the cross-terms away from the centre.

Time-frequency kernels

The corresponding **time-frequency kernel** is given as,

$$\Phi(t,f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(\nu,\tau) e^{-i2\pi(f\tau - \nu t)} d\tau d\nu,$$

where the corresponding **smoothed Wigner distribution** is found as the 2-dimensional convolution

$$W_z^F(t,f) = W_z(t,f) * *\Phi(t,f).$$

We note that the original Wigner distribution has the simple ambiguity kernel $\phi(\nu,\tau) = 1$ for all ν and τ, and the corresponding time-frequency (non-)smoothing kernel is $\Phi(t,f) = \delta(t)\delta(f)$.

The quadratic class - Cohen's class

In the 1940s to 60s, a lot of distributions and time-frequency methods were invented. Many of these satisfied the marginals. A formulation was then made by Leon Cohen, **the Cohen's class**, which included these and an infinite number of other methods with different kernel functions. Later the **quadratic class** was defined, also including kernels which do not satisfy the marginals, the restriction of the original Cohen's class.

The quadratic class

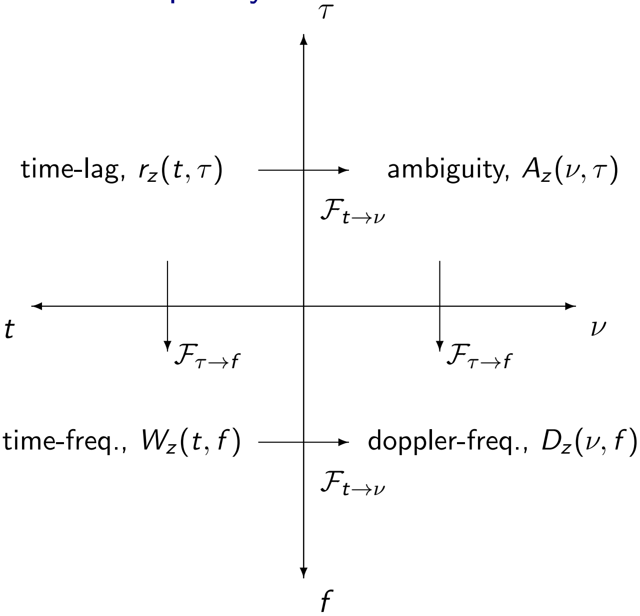
The quadratic class is defined as

$$W_z^Q(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_z(\nu,\tau)\phi(\nu,\tau)e^{-i2\pi(\tau f - t\nu)}d\tau d\nu,$$

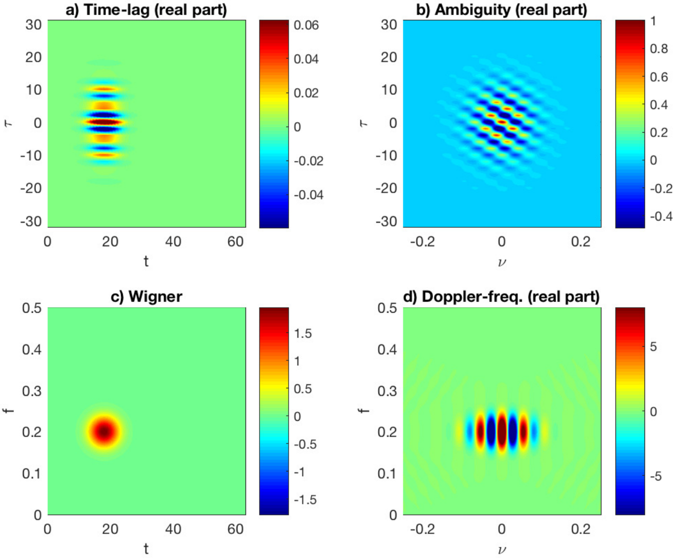
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z(u+\frac{\tau}{2})z^*(u-\frac{\tau}{2})\phi(\nu,\tau)e^{i2\pi(\nu t - f\tau - \nu u)}du\, d\tau\, d\nu,$$

where the last formulation is the one that usually appears in literature.

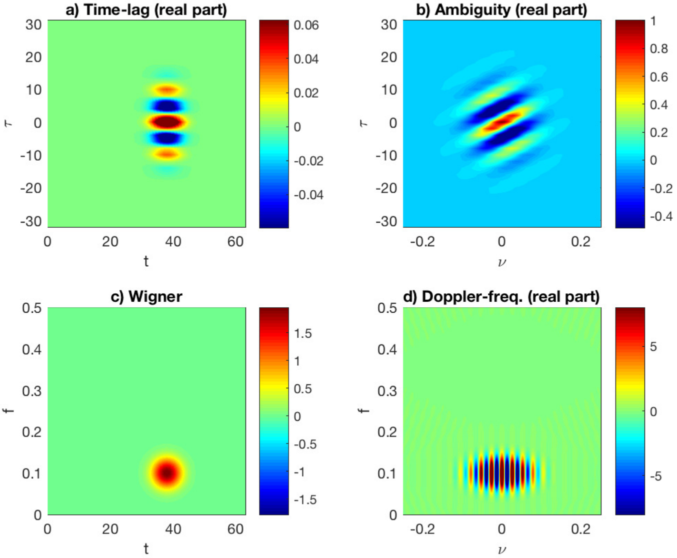
The four time-frequency domains



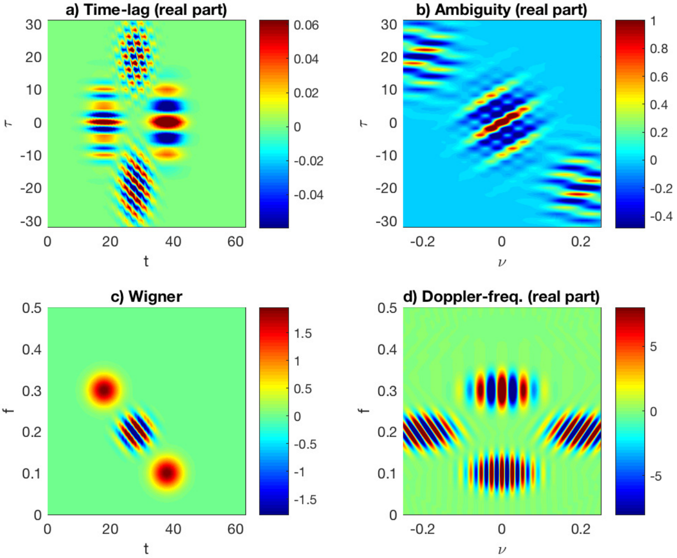
Four-domain representation examples



Four-domain representation examples



Four-domain representation examples



The doppler-frequency distribution

The **doppler-frequency distribution** can be expressed as

$$D_z(\nu, f) = Z(f + \frac{\nu}{2})Z^*(f - \frac{\nu}{2}),$$

where $Z(f)$ is $\mathcal{F}\{z(t)\}$, and $D_z(\nu, f)$ is the corresponding **spectral autocorrelation function**, i.e. the frequency dual of the IAF function $r_z(t, \tau) = z(t + \frac{\tau}{2})z^*(t - \frac{\tau}{2})$.

Time-frequency marginals

We recall that the Wigner distribution satisfies the time- and frequency marginals,

$$\int_{-\infty}^{\infty} W_z(t,f)df = |z(t)|^2,$$

$$\int_{-\infty}^{\infty} W_z(t,f)dt = |Z(f)|^2,$$

and the total energy condition,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_z(t,f)dtdf = \int_{-\infty}^{\infty} |z(t)|^2 dt = \int_{-\infty}^{\infty} |Z(f)|^2 df = E_z,$$

where E_z is the energy of the signal.

Marginals in the ambiguity domain

Using the definition of the ambiguity function

$$A_z(\nu,\tau) = \int_{-\infty}^{\infty} z(t+\frac{\tau}{2}) z^*(t-\frac{\tau}{2}) e^{-i2\pi\nu t} dt,$$

with $\tau = 0$ we get

$$A_z(\nu,0) = \int_{-\infty}^{\infty} z(t) z^*(t) e^{-i2\pi\nu t} dt = \int_{-\infty}^{\infty} |z(t)|^2 e^{-i2\pi\nu t} dt,$$

i.e. the Fourier transform of the time marginal.

Marginals in the ambiguity domain

Similarly we find

$$A_z(0,\tau) = \int_{-\infty}^{\infty} Z(f)Z^*(f)e^{i2\pi\tau f}\,df = \int_{-\infty}^{\infty} |Z(f)|^2 e^{i2\pi\tau f}\,df,$$

which is the inverse Fourier transform of the frequency marginal.

The axis $A_z(0,\tau)$ can be interpreted as the usual covariance function as the frequency marginal is the usual spectral density.

Properties of the ambiguity kernel

We can conclude that in order to preserve the marginals of the smoothed time-frequency distribution, the ambiguity kernel must fulfill

$$\phi(0, \tau) = \phi(\nu, 0) = 1.$$

We also find that

$$\phi(0, 0) = 1,$$

will preserve the total energy of the signal.

Properties of the ambiguity kernel

The Wigner distribution is real-valued and for the smoothed Wigner distribution to also become real-valued, the kernel must fulfill

$$\phi(\nu, \tau) = \phi^*(-\nu, -\tau). \text{ Assignment!}$$

The Choi-Williams distribution

The Choi-Williams distribution or the Exponential distribution is defined by the ambiguity kernel

$$\phi_{ED}(\nu, \tau) = e^{-\frac{\nu^2\tau^2}{\sigma}},$$

where σ is a design parameter. The Choi-Williams distribution falls into the subclass of **product kernels**, where the dimensionality is reduced as $x = \nu\tau$. Product kernels are advantageous in parameter optimization.

Properties of the Choi-Willams kernel

The Choi-Williams distribution satisfies the marginals,

$$\phi_{ED}(0,\tau)=\phi_{ED}(\nu,0)=1,$$

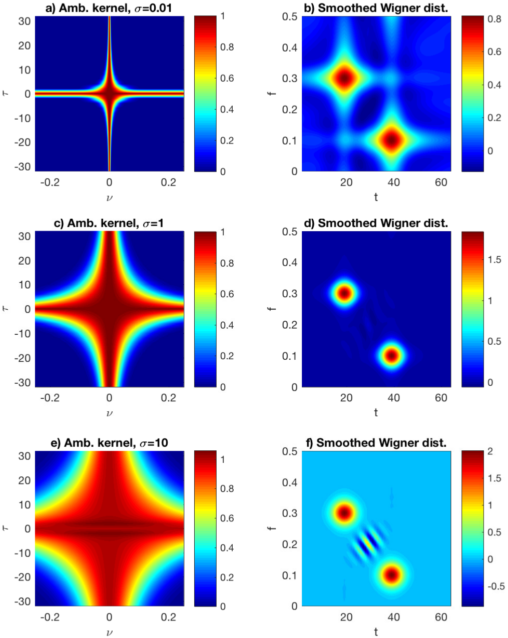
and the total energy condition,

$$\phi_{ED}(0,0)=1.$$

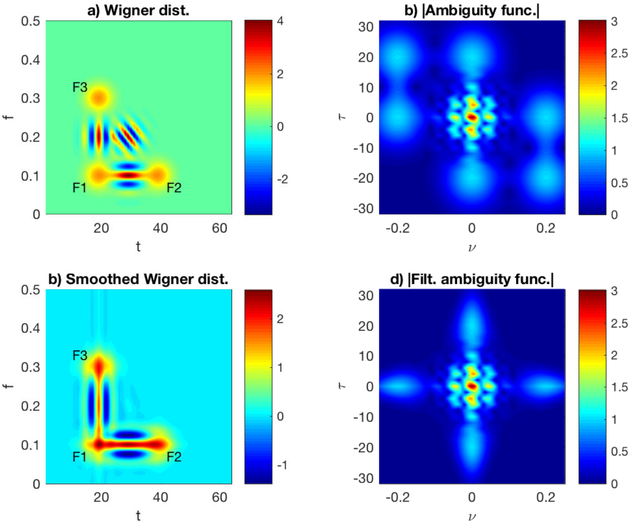
The kernel also satsifies

$$\phi^*_{ED}(-\nu,-\tau)=e^{-\frac{\nu^2\tau^2}{\sigma}}=\phi_{ED}(\nu,\tau),$$

which preserves the Choi-Williams distribution to be real-valued.


a) Amb. kernel, σ=0.01
b) Smoothed Wigner dist.
c) Amb. kernel, σ=1
d) Smoothed Wigner dist.
e) Amb. kernel, σ=10
f) Smoothed Wigner dist.
τ
ν
f
t

Another example



Separable kernels

A **separable** kernel is defined by

$$\phi(\nu, \tau) = G_1(\nu) \cdot g_2(\tau),$$

and the resulting filtered ambiguity function is

$$A_z^F(\nu, \tau) = G_1(\nu) \cdot A_z(\nu, \tau) \cdot g_2(\tau).$$

Separable kernels

This formulation transfers easily to the time-frequency domain as

$$\Phi(t, f) = g_1(t) \cdot G_2(f),$$

with $g_1(t) = \mathcal{F}^{-1}\{G_1(\nu)\}$ and $G_2(f) = \mathcal{F}\{g_2(\tau)\}$. The smoothed Wigner distribution becomes

$$W_z^F(t, f) = W_z(t, f) * *\Phi(t, f) = g_1(t) * W_z(t, f) * G_2(f).$$

The separable kernel replaces the 2-D convolution of the quadratic time-frequency representation with two 1-D convolutions, which might be beneficial for some signals.

Doppler-independent kernel

If

$$G_1(\nu) = 1,$$

a **doppler-independent** kernel is given as $\phi(\nu, \tau) = g_2(\tau)$, and the quadratic time-frequency distribution reduces to

$$W_z^F(t, f) = W_z(t, f) * G_2(f),$$

which is a smoothing in the frequency domain. The doppler-independent kernel is also given the name **Pseudo-Wigner** or **windowed Wigner distribution**.

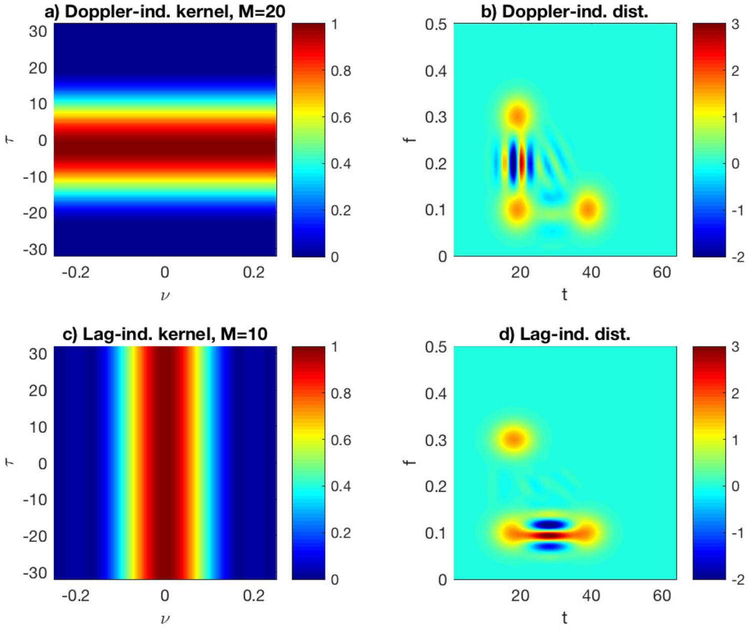
Lag-independent kernel

The other case is when

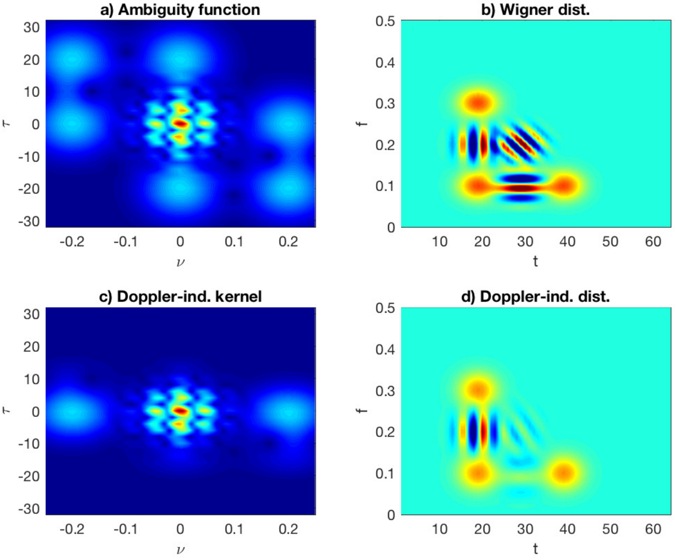
$$g_2(\tau) = 1,$$

giving the **lag-independent** kernel, $\phi(\nu, \tau) = G_1(\nu)$ where the time-frequency formulation gives only a smoothing in t,

$$W_z^F(t, f) = g_1(t) * W_z(t, f).$$


a) Doppler-ind. kernel, M=20
b) Doppler-ind. dist.
c) Lag-ind. kernel, M=10
d) Lag-ind. dist.
τ
ν
f
t
30
20
10
0
-10
-20
-30
-0.2
0
0.2
0.5
0.4
0.3
0.2
0.1
0
20
40
60
1
0.8
0.6
0.4
0.2
0
3
2
1
0
-1
-2

Example


a) Ambiguity function
b) Wigner dist.
c) Doppler-ind. kernel
d) Doppler-ind. dist.
τ
ν
f
t
30
20
10
0
-10
-20
-30
-0.2
-0.1
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0.4
0.3
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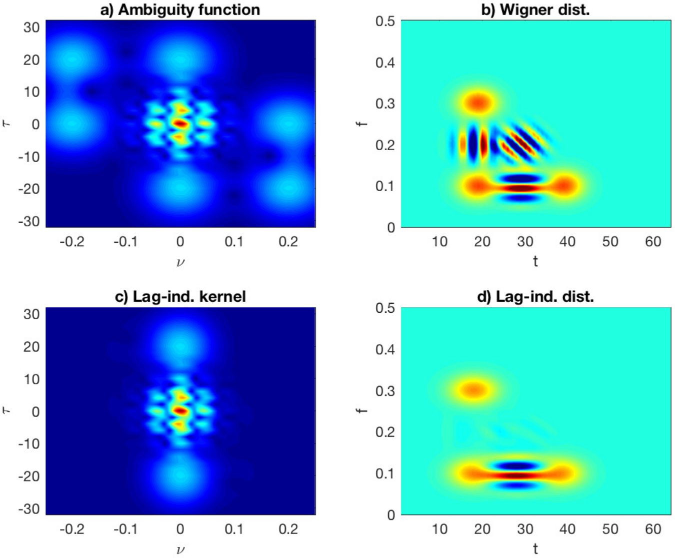
50
60

Example


a) Ambiguity function
b) Wigner dist.
c) Lag-ind. kernel
d) Lag-ind. dist.
τ
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0
0.1
0.2
0.5
0.4
0.3
0.2
0.1
0
10
20
30
40
50
60

Homework 3
Introduction to time-frequency analysis

1) A signal $x(t) = \delta(t - t_0)$ is windowed with a Gaussian function $h(t) = (a/\pi)^{1/4}e^{-at^2/2}$, $-\infty < t < \infty$. Calculate the STFT and the spectrogram. (1p)

2) Show that the Wigner distribution must be real-valued even if the signal is complex-valued. Hint: Consider the complex conjugate of the Wigner distribution. (1p)

3) Show that the Wigner distribution is time-shift and frequency-shift invariant by replacing the input signal $x(t)$ with $y(t) = x(t - t_0)e^{i2\pi f_0 t}$, $-\infty < t < \infty$. (1p)

4) Calculate the Wigner distribution of

$$x(t) = e^{i2\pi f_1 t} + e^{i2\pi f_2 t}, \quad -\infty < t < \infty.$$

Based on this expression, what is the Wigner distribution of $x(t) = A\cos(2\pi f_0 t)$. (1p)

5) Create a signal X_c using the Matlab-function `gaussdata` with two Gaussian windowed *complex-valued* sinusoids of equal amplitudes and individual component length 64 samples. The time-frequency centers are $(80, 0.1)$, $(180, 0.3)$ and the total data length of the signal should be 256 samples with sampling frequency 1 Hz. Compute the Wigner distribution using `quadtf`. Also construct a corresponding *real-valued* sinusoid X_r (using `real(`X_c`)`) and compute the corresponding spectrogram using `mtspectrogram` and the Wigner distribution of X_r using `quadtf`. (The Matlab-functions are found in the folder `code/TimeFrequency`).

   a) Compare the three resulting figures, the Wigner distribution of X_c, the Wigner distribution of X_r and the spectrogram of X_r regarding the occurrence of **cross-terms**. Explain why they show up at their positions. (1p)

   b) Does **aliasing** occur and in such case, where? (1p)

6) Calculate the ambiguity function of

$$x(t) = e^{i2\pi f_1 t} + e^{i2\pi f_2 t}, \quad -\infty < t < \infty.$$

Using this expression, what is the ambiguity function of $x(t) = A\cos(2\pi f_0 t)$. (1p)

7) Show that

$$\phi(\nu, \tau) = \phi^*(-\nu, -\tau),$$

is a necessary condition for the real-valued property of the resulting smoothed Wigner distribution. (1p)

8) Load the complex-valued data of the file `hw2exdata.mat` into Matlab, and compute the Wigner distribution of the variable `Xtotal` using `quadtf`. Also compute the absolute value of the ambiguity function using `quadamb`. The signal `Xtotal` contains five components, each of them found in the variables `Xc1` to `Xc5`. Visualize the Wigner distribution and ambiguity function of each of the separate components. Don't hand in these figures!

   a) Check that the ambiguity function figures of `Xc2` and `Xc3` are identical. Show that this is generally true for mono-component signals that just differ in their time- and frequency locations. Calculate (by hand) the ambiguity functions, of two example signals, e.g. $x_1(t) = e^{i2\pi f_1 t}$ and $x_2(t) = e^{i2\pi f_2 t}$ and show that the absolute value of the ambiguity functions are equal. (1p)

   b) Based on your separate views of the ambiguity functions of the five components included in the signal `Xtotal`, sketch by hand, an area in the ambiguity function figure of `Xtotal`, representing an ambiguity kernel that should be efficient in reducing the cross-terms. Hand in this sketch. (1p)

The following useful Matlab-functions are found in the folder `code/TimeFrequency`: `gaussdata`, `quadtf`, `quadamb`, `mtspectrogram`

This assignment is due on Monday, February 17th, 2020.