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PERFORMANCE EVALUATION OF THE B-DISTRIBUTION

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ABSTRACT

In this paper, the properties of the recently proposed B-distribution are investigated. We show that this distribution satisfies most of the desirable properties sought for a time-frequency distribution. That is, we show that the B-distribution is real, time and frequency shift invariant and its first moment with respect to frequency yields the instantaneous frequency of the signal. We also show, using real and synthetic data, that the B-distribution outperforms other quadratic time-frequency distributions in resolving signals' components in the time-frequency domain. It achieves a better time-frequency resolution and a better energy concentration around the instantaneous frequency of the signal, and still significantly suppresses the cross-terms.

1. INTRODUCTION

Both natural and man-made signals may exhibit some degree of non-stationarity. Classical signal analysis tools, however, do not take this into account, assuming that the signal characteristics are stationary.

A solution to the problem of representing non-stationary signals is found in their joint time and frequency representations. The introduction of time-frequency signal analysis (TFSA) has led to define new tools to represent and characterise the time-varying contents of non-stationary signals using time-frequency distributions (TFDs) [3, 4, 10].

Among the most studied time-frequency distributions are the quadratic distributions. In this paper, we propose a new member of the quadratic class of TFDs, referred to as the B-distribution, which can resolve close signals in the time-frequency domain that other members fail to do so. In addition to that, the B-distribution is shown to outperform existing reduced interference distributions in suppressing the cross-terms of a multicomponent signal, while keeping a high time-frequency resolution.

The paper is organised as follows. In section 2, we discuss the relationships between the quadratic class TFDs and the ambiguity function. The concept of two-dimensional filtering in the ambiguity domain is presented in section 3. The time-lag and the Doppler-lag B-distribution kernel, as well as its properties, are defined in section 4. Examples of comparison of the B-distribution with other commonly used TFDs in analysis of both synthetic and real-life multicomponent signals are given in section 5. Section 6 is devoted to conclusions.

2. QUADRATIC TIME-FREQUENCY DISTRIBUTIONS AND AMBIGUITY FUNCTION

Quadratic (also known as Cohen's) class of time-frequency distributions is defined by the following expression [3]:

$$\rho_z(t,f) = \iiint g(\nu,\tau) z(u+\frac{\tau}{2}) z^*(u-\frac{\tau}{2}) e^{j2\pi(\nu u - \nu t - f\tau)} d\nu du d\tau \tag{1}$$

where $g(\nu,\tau)$ is the kernel function and $z(t)$ is the analytic signal associated with the real one[1].

The key to understanding time-frequency relationships is through understanding of the ambiguity domain [3].

If we define $R_z(t,\tau)$ to be the instantaneous autocorrelation function of $z(t)$ as:

$$R_z(t,\tau) = z(t+\frac{\tau}{2}) z^*(t-\frac{\tau}{2}) \tag{2}$$

then the Wigner-Ville distribution (WVD) of the same signal $z(t)$ is defined as the Fourier transform of $R_z(t,\tau)$ with respect to the lag variable τ:

$$W_z(t,f) = \int z(t+\frac{\tau}{2}) z^*(t-\frac{\tau}{2}) e^{-j2\pi f\tau} d\tau \tag{3}$$

The symmetrical ("Sussman's") ambiguity function (AF), on the other hand, is defined as the Fourier transform of $R_z(t,\tau)$ with respect to time t [3, 5, 9]:

$$A_z(\nu,\tau) = \int z(t+\frac{\tau}{2}) z^*(t-\frac{\tau}{2}) e^{-j2\pi\nu t} dt \tag{4}$$

From equations (3) and (4) we can see that the WVD and the AF are related by a two-dimensional Fourier transform [9, 12]:

$$W_z(t,f) = \iint A_z(\nu,\tau) e^{j2\pi(\nu t - f\tau)} d\nu d\tau \tag{5}$$

Using equations (1) and (5), the following expression can also be derived [9]:

$$\rho_z(t,f) = \iint g(\nu,\tau) A_z(\nu,\tau) e^{j2\pi(\nu t - f\tau)} d\nu d\tau \tag{6}$$

Therefore, the quadratic class of distributions may be found by first multiplying the kernel by the symmetrical ambiguity function and then carrying out the two-dimensional Fourier transform.

[1] All integrals have limits from $-\infty$ to $+\infty$ unless specified otherwise.

3. CROSS-TERMS FILTERING IN THE AMBIGUITY DOMAIN

Various methods have been proposed for analysing multicomponent signals [1, 3, 5, 11]. A technique which is of a particular interest here, is the filtering of cross-terms in the ambiguity domain.

It was shown that a signal mapped by the AF into the Doppler-lag domain always traverses the origin of that plane, while the cross-terms, having oscillating amplitude in the time-frequency domain, are located away from the origin in the Doppler-lag plane, the distance being directly proportional to the time and frequency distance of the signal components [7].

Since the WVD is related to the ambiguity function by a two-dimensional Fourier transform, the simplest way to reduce cross-terms of the WVD would be their filtering out in the ambiguity domain, followed by a two-dimensional Fourier transform.

Let us illustrate this by the following example (the corresponding plots are given in Figure 1).

Consider a multicomponent signal, $z(t)$, consisting of two components, $x(t)$ and $y(t)$, where:

$$x(t) = e^{j2\pi 0.1t}, 16 < t < 32;$$

$$y(t) = e^{j2\pi 0.25t}, 64 < t < 80; \text{ and}$$

$$z(t) = x(t) + y(t)$$

We can easily show, using equation (3), that the WVD of $z(t)$ is given by:

$$W_z(t,f) = W_x(t,f) + W_y(t,f) + 2\Re\{W_{xy}(t,f)\} \quad (7)$$

The cross WVD, $W_{xy}(t,f)$, is defined as [9]:

$$W_{xy}(t,f) = \int x(t+\frac{\tau}{2})y^*(t-\frac{\tau}{2})e^{-j2\pi f\tau}d\tau \quad (8)$$

The signal symmetrical ambiguity function is found to be [3, 9]:

$$A_z(\nu,\tau) = A_x(\nu,\tau) + A_y(\nu,\tau) + A_{xy}(\nu,\tau) + A_{yx}(\nu,\tau) \quad (9)$$

where $A_x(\nu,\tau)$ and $A_y(\nu,\tau)$ are the auto ambiguity functions (defined by equation (4)) concentrated around the origin, and $A_{xy}(\nu,\tau)$ and $A_{yx}(\nu,\tau)$ are the cross ambiguity functions concentrated away from the origin and defined by the following expression:

$$A_{xy}(\nu,\tau) = \int x(t+\frac{\tau}{2})y^*(t-\frac{\tau}{2})e^{-j2\pi\nu t}dt \quad (10)$$

4. THE B-DISTRIBUTION KERNEL

The property of the AF to have the auto-terms around the origin in the ambiguity plane, and the cross-terms away from it, has inspired searches for a kernel $g(\nu,\tau)$ such that the generalised ambiguity function, $g(\nu,\tau)A_z(\nu,\tau)$, is enhanced in the vicinity of the origin and suppressed everywhere else.

Barkat and Boashash [2] recently proposed a kernel for a new quadratic TFD, known as the B-distribution. They applied the technique of cross-terms filtering in the ambiguity domain to the design of a low-pass filter-type kernel with a sharp cut-off, essential to ensure the clear separation of the auto-terms from the cross-terms.

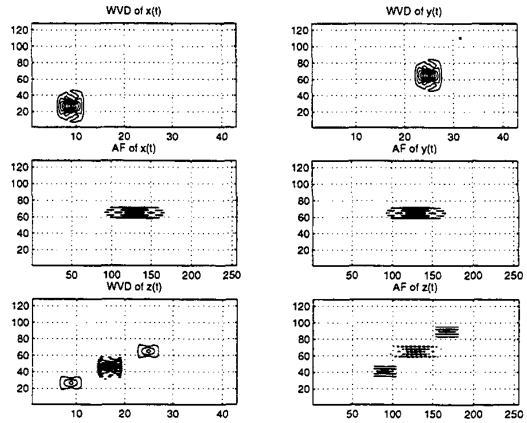

Figure 1: *Location of the auto-terms and the cross-terms of a multicomponent signal in the time-frequency and the Doppler-lag domain*

The B-distribution has its time-lag kernel defined in the following manner [2]:

$$G(t,\tau) = \left(\frac{|\tau|}{\cosh^2(t)}\right)^{\sigma} \quad (11)$$

Application dependent parameter σ, ($0 \le \sigma \le 1$), controls the sharpness of the cut-off of the filter and hence the trade-off between the cross-terms elimination and the time-frequency resolution.

In the sequel, we will define the Doppler-lag domain equivalent of the kernel and outline some of its important properties.

The Doppler-lag kernel is defined as the Fourier transform of the time-lag kernel with respect to time t:

$$\begin{aligned} g(\nu,\tau) &= \int \left(\frac{|\tau|}{\cosh^2(t)}\right)^{\sigma} e^{-j2\pi\nu t}dt \\ &= |\tau|^{\sigma} \int \frac{e^{-j2\pi\nu t}}{\cosh^{2\sigma}(t)}dt \end{aligned} \quad (12)$$

Since the kernel is an even and a real-valued function, equation (12) can be further simplified to:

$$g(\nu,\tau) = 2|\tau|^{\sigma} \int_0^{\infty} \frac{\cos(2\pi\nu t)}{\cosh^{2\sigma}(t)}dt \quad (13)$$

Using [8](3.985.1), the following formula is a possible solution to the integral which must be calculated in (13):

$$\int_0^{\infty} \frac{\cos(\alpha t)}{\cosh^{\gamma}(\beta t)}dt = \frac{2^{\gamma-2}}{\beta\Gamma(\gamma)}\Gamma\left(\frac{\gamma}{2}+\frac{j\alpha}{2\beta}\right)\Gamma\left(\frac{\gamma}{2}-\frac{j\alpha}{2\beta}\right) \quad (14)$$

for $\Re\{\beta\} > 0, \Re\{\gamma\} > 0$, and $\alpha > 0$.

Hence, the equation for the B-distribution kernel in the Doppler–lag domain may be written in the following form:

$$g(\nu,\tau) = |\tau|^{\sigma}\frac{2^{2\sigma-1}}{\Gamma(2\sigma)}\Gamma(\sigma + \jmath\pi\nu)\Gamma(\sigma - \jmath\pi\nu) \qquad (15)$$

4.1. The B-distribution Properties

Ideally, there is a set of desirable properties that a TFD should satisfy [3, 6, 9, 11].

The B-distribution satisfies the following properties:

- *Realness*
  To show this property we must have [3]:
  $$g(\nu,\tau) = g^*(-\nu,-\tau)$$
  In our case:
  $$g^*(-\nu,-\tau) = \{|-\tau|^{\sigma}\frac{2^{2\sigma-1}}{\Gamma(2\sigma)}\Gamma(\sigma - \jmath\pi\nu)\Gamma(\sigma + \jmath\pi\nu)\}^*$$
  $$g^*(-\nu,-\tau) = |\tau|^{\sigma}\frac{2^{2\sigma-1}}{\Gamma(2\sigma)}\Gamma(\sigma + \jmath\pi\nu)\Gamma(\sigma - \jmath\pi\nu)$$
  Hence, we have $g(\nu,\tau) = g^*(-\nu,-\tau)$.
- *Time Shift Invariance*
  To prove that a distribution is time shift invariant, it is sufficient to note that $g(\nu,\tau)$ does not depend on t [3].
  From equation (15), the B-distribution Doppler-lag kernel is not a function of time t.
- *Frequency Shift Invariance*
  To prove this property, it is sufficient to note that $g(\nu,\tau)$ does not depend on f [3].
  Similarly, from equation (15), we see that the B-distribution Doppler-lag kernel does not depend on frequency f.
- *The Instantaneous Frequency*
  The first moment of the B-distribution yields the IF of the signal. That is,
  $$f_i(t) = \frac{\int f\rho_z(t,f)df}{\int \rho_z(t,f)df}$$
  This property is satisfied since [9]:
  $$\frac{\partial g(\nu,\tau)}{\partial \nu}|_{(0,0)} = \frac{\partial g(\nu,\tau)}{\partial \tau}|_{(0,0)} = 0$$
  and
  $$g(\nu,\tau) = constant\ \forall\tau.$$

5. PERFORMANCE EVALUATION OF THE B-DISTRIBUTION

The performance of the B-distribution is compared to several commonly-chosen time-frequency distributions used to represent multicomponent signals. Two examples are presented in this paper: a synthetic and a real-life signal.

5.1. Example 1

Here, we analyse a multicomponent signal consisting of the following noiseless signals: a linear FM signal of frequency range f = 0.1 – 0.2 Hz, a sinusoidal FM signal of centre frequency f = 0.25 Hz and a hyperbolic FM signals of frequency range f = 0.3 – 0.1 Hz. All three signals have the same length N = 128 and the same sampling frequency $f_s = 1$ Hz.

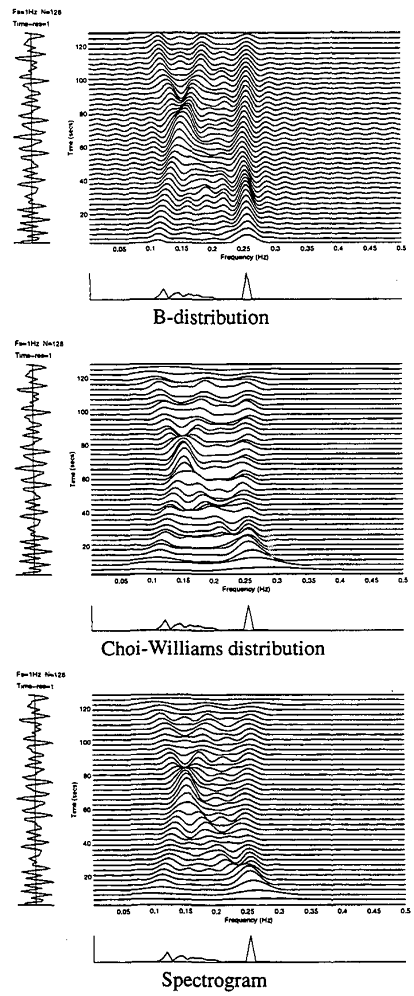

Figure 2: *Multicomponent signal consisting of a linear FM, a sinusoidal FM and a hyperbolic FM signal*

Figure 2 shows the superiority of the B-distribution ($\sigma = 0.01$) over the Choi-Williams distribution ($\sigma = 3$) and the spectrogram in resolving the three closely spaced components, as well as in reducing the cross-terms.

5.2. Example 2

In this example (Figure 3), a bat signal of length N = 400 and the sampling frequency $f_s = 1$ Hz is analysed using the B-distribution ($\sigma = 0.031$), Choi-Williams distribution ($\sigma = 3$) and the spectrogram.

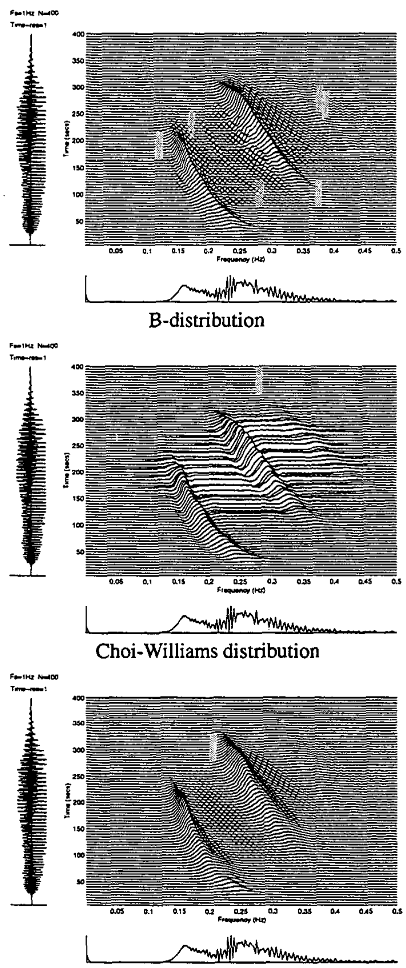

Figure 3: *Bat signal*

The B-distribution again outperforms the other two distributions by achieving a better time-frequency resolution than the spectrogram does, and by significantly suppressing the interference that affects the Choi-Williams distribution.

6. CONCLUSIONS

In this paper, we have reviewed the fundamental concepts of the cross-terms elimination using the ambiguity domain filtering, based on which the B-distribution kernel was proposed. The kernel has been defined in both the time-lag and the Doppler-lag domain, and important properties it satisfies have been presented.

Using synthetic and real-life multicomponent signals, it has been shown that the B-distribution achieves a better time-frequency resolution and energy concentration around the instantaneous frequency of a signal, while still significantly suppressing the cross-terms, than other commonly-chosen distributions for multicomponent signals analysis.

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