A DISCRETE TIME AND FREQUENCY WIGNER-VILLE DISTRIBUTION: PROPERTIES AND IMPLEMENTATION

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ABSTRACT

Time-frequency distributions are used in the analysis and processing of nonstationary signals. The Wigner-Ville distribution (WVD) is a fundamental time-frequency distribution uniquely satisfying many desirable mathematical properties. The realisation of this distribution for hardware or software platforms requires a discrete version. Historically the majority of the work on deriving discrete versions of the WVD has focused on creating alias-free distributions, often resulting in a loss of some desirable properties. Here a new discrete time and frequency WVD will be presented for nonperiodic signals and will be examined both in terms of its properties and aliasing. In particular unitarity, an assumed property for optimum time-frequency detection and signal estimation, and invertibility, a useful property especially for time-frequency filtering, will be examined. An efficient implementation of the distribution using standard real-valued fast Fourier transforms will also be presented.

1. INTRODUCTION

Time-frequency (TF) signal analysis and processing is concerned with the study of nonstationary signals or signals with time-varying frequency content. Most real life signals, e.g. audio, video, biomedical etc., are nonstationary in nature and require nonstationary analysis for proper evaluation. Time-frequency distributions (TFDs) jointly describe the TF energy content of the signal. A particular class of TFDs, namely quadratic TFDs, have proven very useful in the analysis of nonstationary signals [1]. The Wigner-Ville distribution (WVD) uniquely satisfies most of the desirable properties of quadratic TFDs and is the basis for all the members of this class. The WVD is defined for a signal $x(t)$ as [1]

$$W_z(t,f) = \int_{-\infty}^{\infty} z(t+\tfrac{\tau}{2}) z^*(t-\tfrac{\tau}{2}) \mathrm{e}^{-j2\pi f\tau} \mathrm{d}\tau. \quad (1)$$

where $z(t)$ is the analytic associative of $x(t)$. The Wigner distribution is simply defined when the real signal $x(t)$ is used instead of the analytic one $z(t)$.

An important part of TF analysis and processing is signal detection and/or estimation. A basic property a quadratic TFD needs to satisfy to be optimal in the context of TF detection is unitarity [2]. This is mathematically expressed for the case of the WVD by Moyal's formula:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_x(t,f) W_y(t,f) \mathrm{d}t\mathrm{d}f = \left| \int_{-\infty}^{\infty} x(t) y^*(t) \mathrm{d}t \right|^2 \quad (2)$$

where $x(t)$ and $y(t)$ are analytical signals. Another useful property of quadratic TFDs is invertibility, i.e. the ability to obtain the original signal (up to a constant phase factor) from the TFD. This is expressed for the case of the WVD as

$$\int_{-\infty}^{\infty} W_z(t/2, f) \mathrm{e}^{j2\pi f t} \mathrm{d}f = z(t) z^*(0). \quad (3)$$

The advantage of such a property should be obvious, particularly if time-frequency filtering is required.

This communication is concerned with the formation of the WVD in discrete form required for implementation on arbitrary hardware/software platforms. An ideal discrete WVD (DWVD) should closely resemble the continuous WVD and satisfy all its properties. A DWVD is proposed which aims to approximate this ideal DWVD and is examined in terms of aliasing and its ability to satisfy desirable quadratic TFDs properties, in particular unitarity and invertibility as described in Eq. (1) and Eq. (2). The proposed distribution will be defined for nonperiodic bandlimited signals with finite time support. To the authors knowledge it is the first DWVD to satisfy unitarity whilst remaining alias free for this particular class of signals.

Finally an efficient implementation of the distribution will be given using standard fast Fourier transform (FFT) routines resulting in $\mathcal{O}(N^2 \log_2 N)$ real multiplications for the $2N \times N$ distribution.

2. DISCRETE TIME AND FREQUENCY WIGNER-VILLE DISTRIBUTIONS

Claasen and Mecklenbräuker introduced the discrete-time Wigner distribution as [3]

$$W_x^{(cm)}[\tfrac{n}{f_s}, f) = \frac{2}{f_s} \sum_{m=-N/2+1}^{N/2-1} x[n+m]x^*[n-m] \cdot \mathrm{e}^{-j4\pi f(n/f_s)} \quad (4)$$

where finite signal $x(t)$ is sampled at sample rate f_s. Aliasing is avoided when $x[n]$ is analytic or when f_s is twice the normal Nyquist rate. The discrete-time discrete-frequency WD, henceforth referred to as the DWD (or DWVD if the analytic signal is used), is obtained by sampling in the frequency domain. The vast majority of the work in this area has been concerned with creating alias free distributions. A good review of various DWDs and their relationship to aliasing is described in [4].

A problem with the DWD represented in Eq. (4) is that the distribution does not have a proper discrete quadratic signal representation, i.e. some form including the terms $x[a]x^*[b]$, where $0 \leq a, b \leq N-1$. In the above representation, $x[a]x^*[b]$ exists only for a, b both even or a, b both odd and not a even b odd (and visa-versa). This leads to an immediate problem with the distribution satisfying the unitarity and invertibility property expressed for the continuous case in Eq. (2) and Eq. (3). As unitarity is an assumed property for optimum time-frequency detection, it has been empirically shown, as expected, that detectors using DWDs not satisfying this property exhibit serious performance degradation [5]. Therefore distributions satisfying this property will now be examined.

Chassande-Mottin and Pai [6] recently introduced a DWVD with the explicit reason of satisfying the unitarity property. This distribution for a signal $z(t)$ is described as

$$W_z^{(C)}[\tfrac{n}{f_s}, \tfrac{kf_s}{2N}] = \sum_{m=-N+1}^{N-1} \hat{z}[2n+m]\hat{z}^*[2n-m] \cdot \mathrm{e}^{-j(2\pi/2N)mk}$$

where $\hat{z}[2n] = \hat{z}[2n+1] = z[n], \forall n \in \{0, 1, \ldots, N-1\}$, thus resulting in a $N \times 2N$ grid. This distribution has reduced aliasing though is not an alias-free distribution. The DWD for periodic discrete time-domain signals were considered by O'Neill *et al.* [7] and Richman *et al.* [8]. Independently they defined a DWD

$$W_x^{(R)}[\tfrac{n}{f_s}, \tfrac{kf_s}{N}] = \sum_{m=0}^{N-1} x[(n+cm)_N]x^*[(n-cm)_N] \cdot \mathrm{e}^{-j(2\pi/N)mk}$$

where $(a)_N \equiv a \mod N$. The distribution only exists when N is odd and where $c = \frac{N+1}{2}$. It is described on a $N \times N$ sample grid. Although this is an alias-free distribution, due the quadratic nature and periodic assumptions [7] of the DWD artifacts are present in the distribution. Therefore this distribution does not closely represent the continuous version. An alias free distribution will be proposed that closely resembles its continuous counterpart whilst satisfying both unitarity and invertibility properties.

2.1. Proposed DWVD

Peyrin and Prost [9] introduced a DWVD assuming that the discrete signal is periodic, represented as

$$W_x^{(P)}[\tfrac{n}{2f_s}, \tfrac{kf}{2N}] = \mathrm{e}^{j(\pi/N)nk} \sum_{m=0}^{N-1} x[m]x^*[(n-m)_N] \cdot \mathrm{e}^{-j(2\pi/N)mk}. \quad (5)$$

To avoid aliasing either the analytic associative of $x[n]$ is used or $x[n]$ is oversampled by at least a factor of two. The distribution contains an interpolation in both the discrete time and frequency domains and is represented on a $2N \times 2N$ sample grid.

The proposed distribution is based on the Peyrin and Prost distribution with two simple modifications. One is that the analytic signal is always used, thus defining a DWVD, and the other is that the signal is assumed to be nonperiodic. This distribution can be expressed as a function of a time-lag domain kernel as

$$W_z^{(M)}[\tfrac{n}{2f_s}, \tfrac{kf_s}{2N}] = \mathrm{e}^{j(\pi/N)kn} \sum_{m=l_1}^{l_2} z[m]z^*[n-m] \cdot \mathrm{e}^{-j(2\pi/N)km} \quad (6)$$

for $l_1 = \max\{0, n-(N-1)\}$ and $l_2 = \min\{n, N-1\}$. It can also be defined as a function of a frequency-doppler domain kernel as

$$W_z^{(M)}[\tfrac{n}{2f_s}, \tfrac{kf_s}{2N}] = \frac{\mathrm{e}^{-j(\pi/N)kn}}{N} \sum_{u=0}^{2N-1} \hat{Z}[u] \cdot \hat{Z}^*[(2k-u)_N]\mathrm{e}^{j(2\pi/2N)un} \quad (7)$$

where $\hat{Z}[k]$ is the discrete frequency domain representation of $z[n]$ with $f = kf_s/2N$, for $0 \leq k \leq 2N-1$, i.e. twice the usual frequency domain sampling rate. The distribution is represented on a $2N \times N$ sample grid. Henceforth the sampling information displayed in the argument of the DWVD function will be dropped for the simpler notation of $W_z^{(M)}[n, k]$.

An important feature of the proposed distribution is that it can be expressed in quadratic form, as

$$W_z^{(M)}[n, k] = \sum_{p=l_1}^{l_2} \sum_{q=l_1}^{l_2} z[p]z^*[q]H[n, k; p, q]$$

with

$$H[n, k; p, q] = \mathrm{e}^{j(\pi/N)kn}\mathrm{e}^{-j(2\pi/N)kp}\delta[q-n+p]$$

where the Kronecker delta function $\delta[n]$ is defined as $\delta[n] = 1$ for $n = 0$ and $\delta[n] = 0$ for $n \neq 0$.

To illustrate the differences between different distributions a simple example can provide some insight. The four previously discussed distributions, namely the Chassande-Mottin and Pai ($W_z^{(C)}$) , Richman *et al.* and O'Neill ($W_z^{(R)}$), Peyrin and Prost ($W_z^{(P)}$) and modified Peyrin and Prost ($W_z^{(M)}$) are all tested with an analytic linear frequency modulated signal of length $N = 128$ samples (except for $W_z^{(R)}$ where $N = 127$) and displayed in Fig. 1. Both $W_z^{(R)}$ and $W_z^{(P)}$ suffer from artifacts (also known as cross terms [1]) due to the assumed periodicity of the signal and quadratic nature of the distribution. $W_z^{(R)}$ also contains similar artifacts due to the assumed periodicity in the discrete frequency domain. $W_z^{(C)}$ more closely represents the expected distribution, however an artifact (aliasing [6]) is present in the negative spectrum. $W_z^{(M)}$ however is free from artifacts arising from both aliasing and periodic effects. Implementations of $W_z^{(P)}$ and $W_z^{(M)}$ from their respective definitions in Eq. (5) and Eq. (6) contain only positive spectral components as illustrated in Fig. 1.

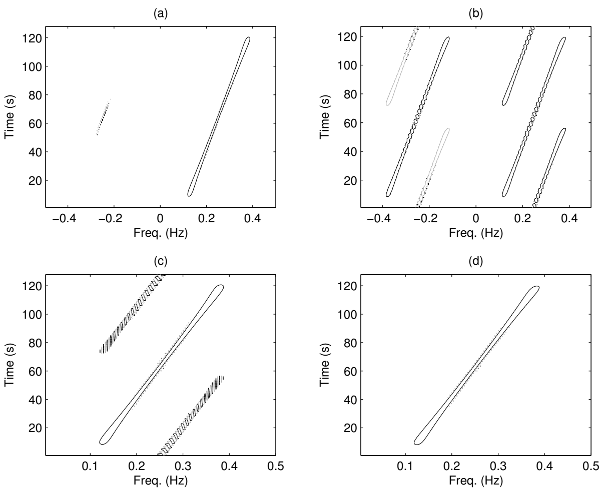

Fig. 1. DWVDs for (a) $W_z^{(C)}$, (b) $W_z^{(R)}$, (c) $W_z^{(P)}$ and (d) $W_z^{(M)}$ distributions of a linear FM signal where the frequency law linearly increases from 0.1Hz to 0.4Hz. Two contour levels are displayed with level $E_z/4$ for black and $-E_z/4$ for grey, where E_z represents the signal's energy.

3. PROPERTIES OF PROPOSED DWVD

The proposed distribution will now be examined in terms of some of its properties, namely the time and frequency marginals, unitarity and invertibility. The inclusion of the time and frequency marginal properties relates the distribution to the notion of a time-frequency energy distribution, which its continuous counterpart adheres to. Unitarity and invertibility are included here due to their importance in signal detection and/or estimation and time-frequency filtering, as previously discussed.

1) Time Marginal: The summation of the distribution over k yields the instantaneous power of the signal at even time samples in the distribution:

$$\frac{1}{N}\sum_{k=0}^{N-1} W_z^{(M)}[2n,k] = |z[n]|^2.$$

Proof:

$$\begin{aligned}
&\frac{1}{N}\sum_{k=0}^{N-1} W_z^{(M)}[n,k] \\
&= \frac{1}{N}\sum_{k=0}^{N-1} \mathrm{e}^{j(\pi/N)kn} \sum_{m=l_1}^{l_2} z[m]z^*[n-m]\mathrm{e}^{-j(2\pi/N)mk} \\
&= \sum_{m=l_1}^{l_2} \frac{1}{N}\sum_{k=0}^{N-1} \mathrm{e}^{-j(2\pi/N)k(m-n/2)} z[m]z^*[n-m] \\
&= \sum_{m=l_1}^{l_2} \delta[m-n/2] z[m]z^*[n-m] \\
&= |z[\tfrac{n}{2}]|^2.
\end{aligned}$$

where $\delta[n] = 1/N \sum_{k=0}^{N-1} \mathrm{e}^{-j(2\pi/N)kn}$.

2) Frequency Marginal: The summation of the distribution over n yields the energy spectrum of the signal at even frequency samples in the distribution:

$$\sum_{n=0}^{2N-1} W_z^{(M)}[n,2k] = |Z[k]|^2$$

Proof:

$$\begin{aligned}
&\sum_{n=0}^{2N-1} W_z^{(M)}[n,k] \\
&= \frac{1}{N}\sum_{n=0}^{2N-1} \mathrm{e}^{-j(\pi/N)kn} \sum_{u=0}^{2N-1} \hat{Z}[u]\hat{Z}^*[(2k-u)_N] \\
&\quad \cdot \mathrm{e}^{j(2\pi/2N)un} \\
&= \sum_{u=0}^{2N-1} \frac{1}{N}\sum_{n=0}^{2N-1} \mathrm{e}^{j(\pi/N)n(u-k)} \hat{Z}[u]\hat{Z}^*[(2k-u)_N] \\
&= \sum_{u=0}^{2N-1} \delta[u-k]\hat{Z}[u]\hat{Z}^*[(2k-u)_N] \\
&= |\hat{Z}[k]|^2.
\end{aligned}$$

3) Unitarity or Moyal's Formula: Moyal's formula, described in Eq. (2), is preserved by the proposed discrete distribution for two analytic signals $x[n]$ and $y[n]$ as

$$\frac{1}{N}\sum_{n}^{2N-1}\sum_{k}^{N-1} W_x^{(M)}[n,k]W_y^{(M)}[n,k] = \left|\sum_{m}^{N-1} x[m]y^*[m]\right|^2.$$

Proof:

$$\frac{1}{N}\sum_{n=0}^{2N-1}\sum_{k=0}^{N-1} W_x^{(M)}[n,k]W_y^{(M)}[n,k]$$

$$= \frac{1}{N}\sum_{n=0}^{2N-1}\sum_{k=0}^{N-1} \mathrm{e}^{j(\pi/N)kn} \sum_{m_1=l_1}^{l_2} x[m_1]x^*[n-m_1]$$

$$\cdot\, \mathrm{e}^{-j(2\pi/N)km_1}\mathrm{e}^{j(\pi/N)kn}$$

$$\cdot \sum_{m_2=l_1}^{l_2} y[m_2]y^*[n-m_2]\mathrm{e}^{-j(2\pi/N)km_2}$$

$$= \sum_{n=0}^{2N-1}\sum_{m_1=l_1}^{l_2}\sum_{m_2=l_1}^{l_2} \frac{1}{N}\sum_{k=0}^{N-1} \mathrm{e}^{-j(2\pi/N)k(m_1+m_2-n)}$$

$$\cdot\, x[m_1]x^*[n-m_1]y[m_2]y^*[n-m_2]$$

$$= \sum_{n=0}^{2N-1}\sum_{m_1=l_1}^{l_2}\sum_{m_2=l_1}^{l_2} \delta[m_1+m_2-n]$$

$$\cdot\, x[m_1]x^*[n-m_1]y[m_2]y^*[n-m_2]$$

$$= \sum_{m_1=0}^{N-1} x[m_1]y^*[m_1] \sum_{m_2=0}^{N-1} x^*[m_2]y[m_2]$$

$$= \left|\sum_{m_1=0}^{N-1} x[m_1]y^*[m_1]\right|^2.$$

4) Invertibility: The ability to extract the time domain signal from the distribution, up to a constant phase factor, is called the invertibility property and the proposed distribution satisfies this property as

$$\frac{1}{N}\sum_{k=0}^{N-1} W_z^{(M)}[n,k]\mathrm{e}^{j(\pi/N)kn} = z[n]z^*[0].$$

Proof:

$$\frac{1}{N}\sum_{k=0}^{N-1} W_z^{(M)}[n,k]\mathrm{e}^{j(\pi/N)kn}$$

$$= \sum_{k=0}^{N-1} \mathrm{e}^{j(\pi/N)kn} \sum_{m=l_1}^{l_2} z[m]z^*[n-m]$$

$$\cdot\, \mathrm{e}^{-j(2\pi/N)mk}\mathrm{e}^{j(\pi/N)nk}$$

$$= \sum_{m=l_1}^{l_2} \frac{1}{N}\sum_{k=0}^{N-1} \mathrm{e}^{j(2\pi/N)k(n-m)} z[m]z^*[n-m]$$

$$= \sum_{m=l_1}^{l_2} \delta[n-m]z[m]z^*[n-m]$$

$$= z[n]z^*[0].$$

4. EFFICIENT IMPLEMENTATION OF DWVD

Peyrin and Prost provided an implementation of their distribution based on independent computation of odd and even time slices [9] of Eq. (5) using FFTs. This representation can be simply rewritten for the proposed distribution as

$$W_z^{(M)}[2n,k] = \sum_{m=0}^{N-1} K_1[n,m]\mathrm{e}^{-j(2\pi/N)mk} \quad (8)$$

$$W_z^{(M)}[2n+1,k] = \mathrm{e}^{j(\pi/N)k}\sum_{m=0}^{N-1} K_2[n,m] \cdot \mathrm{e}^{-j(2\pi/N)mk} \quad (9)$$

with the kernels defined as $K_1[n,m] \equiv z[n+m]z^*[n-m]$ and $K_2[n,m] \equiv z[n+m]z^*[n-m+1]$. The order of computational complexity of the DWVD is dominated by the number and type of FFTs required. An efficient implementation will be presented here to reduce this computational burden.

4.1. DWVD for Even Time Samples

The half of the DWVD containing the even time samples is equivalent to the DWVD version of Eq. (4) spectrally sampled at $f = kf_s/2N$. Boashash and Black [10] provided an efficient implementation using a complex FFT algorithm by taking advantage of the Hermitian nature of the time-lag kernel as $K_1[n,-m] = K_1^*[n,m]$. Therefore the discrete Fourier transform (DFT) produces a purely real output and two time slices can be implemented in one complex DFT. This implementation therefore requires $N/2\times$ complex DFTs of length N.

A more efficient method would be to use a FFT routine specifically designed to take advantage of the conjugate symmetry of the kernel, as the 'packing' of two time-slices into one FFT requires some computational overhead. Chan and Ho suggest [11] using an inverse real-valued FFT algorithm, as the output of a real DFT transform results in a conjugate symmetrical sequence. This appears to the most efficient method as a real FFT requires half the multiplications and less than half the additions of its complex counterpart [12].

Although complex and (forward) real-valued FFT routines are readily available in signal processing tools such MATLAB™ and OCTAVE, the same is not true for the inverse real-valued FFT routines. With a small overhead ($2N$ real additions using a DFT of length N), a real-valued FFT routine can be utilised by using some properties of the DFT [13]; namely that an even real input produces a purely real output and an odd real input produces a purely imaginary output. A new real input can be formed for the conjugate symmetrical sequence $k[n]$ by expressing

$$u[n] = \Re\{k[n]\} + \Im\{k[n]\} \quad (10)$$

where $\Re\{k[n]\}$ is even and $\Im\{k[n]\}$ is odd ($\Re$ and $\Im$ represent the real and imaginary parts respectively). Therefore the DFT of $k[n]$, labelled $K[k]$, can be calculated using

$$K[k] = \Re\{U[k]\} - \Im\{U[k]\} \quad (11)$$

with $U[k]$ representing the DFT of the real valued function $u[n]$. This method results in $N\times$ real DFTs of length N and is preferable to Boashash-Black method as the overall computational load is smaller.

4.2. DWVD for Odd Time Samples

The conjugate symmetry of the kernel in Eq. (8) is unfortunately not present in Eq. (9). However a method is presented that again halves the number of required FFTs and also reduces the total number of multiplications. Letting $\mathrm{e}^{j(\pi/N)k} = A_k + jB_k$ Eq. (9) can be rewritten, for $A_k \neq 0$, as

$$W_z^{(M)}[2n+1,k] = (A_k + \frac{B_k^2}{A_k}) \cdot \Re\left\{\sum_{m=0}^{N-1} K_2[n,m]\mathrm{e}^{-j(2\pi/N)mk}\right\} \quad (12)$$

as $W_z^{(M)}[2n+1,k] \in \mathbb{R}$. This expression replaces the N^2 complex multiplications by N^2 real multiplications. Also as only the real part of the right hand side of the above expression is needed the kernel can be replaced by $\hat{K}_2[n,m]$ as

$$\sum_{m=0}^{N-1} \hat{K}_2[n,m]\mathrm{e}^{-j(2\pi/N)mk} = \Re\left\{\sum_{m=0}^{N-1} K_2[n,m]\mathrm{e}^{-j(2\pi/N)mk}\right\}$$

where the new kernel is formed by forcing conjugate symmetry on $K_2[n,m]$, i.e.

$$\hat{K}_2[n,m] = \frac{1}{2}\left(K_2[n,m] + K_2^*[n,N-m]\right) \quad (13)$$

for $1 \leq m \leq N/2-1$ and $\hat{K}_2[n,-m] = \hat{K}_2^*[n,m]$. Thus Eq. (9) can be rewritten as

$$W_z^{(M)}[2n+1,k] = (A_k + \frac{B_k^2}{A_k}) \sum_{m=0}^{N-1} \hat{K}_2[n,m] \cdot \mathrm{e}^{-j(2\pi/N)mk}$$

for $A_k \neq 0$ which can be implemented with real FFTs using the method described for Eq. (8).

4.3. Special Case for Odd Time Samples

The special case when $A_k = 0$ arises for $k = N/2$ and $W_z^{(M)}[2n+1, N/2]$ must be calculated as Eq. (12) does not exist. Using a decimation-in-frequency approach [13] to the odd values of n and substituting $k = N/2$ into Eq. (7) results in

$$W_z^{(M)}[2n+1,k]\Big|_{k=N/2} = \frac{\mathrm{e}^{-j\frac{\pi}{2}n}}{N}\sum_{u=0}^{N-1}\mathrm{e}^{j(\pi/N)u} \cdot (K_{\hat{Z}}[u] - K_{\hat{Z}}[u+N])\mathrm{e}^{j(2\pi/N)un} \quad (14)$$

with $K_{\hat{Z}}[u] \equiv \hat{Z}[u]\hat{Z}^*[(N-u)_N]$ for $0 \leq u \leq 2N-1$. As certain symmetries exist in the kernel $K_{\hat{Z}}[u]$, in particular $K_{\hat{Z}}[N+u] = K_{\hat{Z}}^*[2N-u]$ and $K_{\hat{Z}}[u+N/2] = K_{\hat{Z}}^*[N/2-u]$ for $0 \leq u \leq N/2-1$, the total kernel $\hat{K}_{\hat{Z}} \equiv \mathrm{e}^{j(\pi/N)u}\left(K_{\hat{Z}}[u] - K_{\hat{Z}}[u+N]\right)$ is conjugate anti-symmetrical. The inverse DFT (IDFT) of this sequence results in a purely imaginary output [13] and by summing the real and imaginary parts of $\hat{K}_{\hat{Z}}[u]$ to create a real input, as in Eq. (10), the IDFT output can be obtained by summing the output, as opposed to the subtraction in Eq. (11), coupled with a simple multiplication by j (with $j \equiv \sqrt{-1}$). This results in a real IDFT, however a real DFT is desired to take advantage of the real FFT algorithms. The IDFT can be represented as a DFT by using the duality property of DFTs [13] which requires a multiplication by N and a reverse of the indices of the output sequence.

4.4. Computational Cost

The computational cost in terms of real multiplications and real additions, denoted M_r and A_r respectively, for the proposed DWVD will be accessed. It is assumed that one complex multiplication requires $M_r = 4$ and $A_r = 2$. It is also assumed that the real FFT routine used is a split-radix type [12] requiring $M_r = N/2\log_2 N - 3N/2 + 2$ and $A_r = 3N/2\log_2 N - 5N/2 + 2$ for real signal of length N.

For even time samples of n Eq. (8) requires the computation of $K_1[n,m]$ from $z[n]$ with $0 \leq n \leq N-1$ and $-N/2+1 \leq m \leq N/2-1$. This requires $M_r = N^2$ and $A_r = N^2/2$ by taking advantage of the symmetry $K_1[n,-m] = K_1^*[n,m]$ and using the identity $K[n,0] = |z[n]|^2$. $N\times$ real FFTs of length N are required with the overhead involved equal to $A_r = (3/2)N^2 - N$.

For the odd n case the kernel $\hat{K}_1[n,m]$, defined within $0 \leq n \leq N-2$ and $-N/2+1 \leq m \leq N/2-1$, represented in Eq. (13) can be formed from $z[n]$ with $M_r = N^2 + 4N - 2$ and $A_r = N^2 + 2N - 3$. This is achieved by exploiting the symmetry in the kernel $K_1[n,m]$, as $K_1[n,-m] = K_1^*[n,m+1]$ for $1 \leq m \leq N/2-1$. The formation of $(A_k + B_k^2/A_k)$ for $0 \leq k \leq N-1$ and multiplication by the output of the DFT requires $M_r = N^2 + N - 2$ and $A_r = N - 1$. $(N-1)\times$ real FFTs of length N are required with the overhead involved equal to $A_r = (3/2)N^2 - N - 2$.

Also for the odd n case $W_z^{(M)}[2n+1,k]\,|_{k=N/2}$, expressed in Eq. (14), for $0 \leq n \leq N-2$ needs to calculated. The kernel $\hat{K}_{\hat{Z}}[u]$ can be formed from $\hat{Z}[u]$ with $M_r = 8N + 4$ and $A_r = 6N + 2$, exploiting the symmetries of kernel $K_{\hat{Z}}$ discussed in Section 4.3. $1\times$ real FFT of length N is used requiring overhead of $A_r = 2N$. Also $M_r = N$ is required for the multiplication of the complex exponential in front of the right hand side of Eq. (14).

For completeness the formation of $\hat{Z}[k]$ from $z[n]$ can be considered, which exists for $0 \leq k \leq 2N -$

1. Assuming that $Z[k]$ is known (as this is required to form the analytic signal $z[n]$ [1]) this represents $\hat{Z}[2k]$. Therefore only the frequency domain interpreted samples $\hat{Z}[2k+1]$ need be computed. Applying a decimation-in-frequency approach [13] this can be calculated as

$$Z[2k+1] = \sum_{n=0}^{N-1} \mathrm{e}^{-j(\pi/N)n} z[n] \mathrm{e}^{-j(2\pi/N)nk}.$$

Thus this requires $M_r = 4N$ and $A_r = 2N$ plus $1\times$ complex FFT routine.

The total computations can be added for both even and odd n which results in the following computational load:

$$M_r = N^2 \log_2 N + N(19 + \log_2 N) + 4$$
$$A_r = N^2(3\log_2 N - \frac{1}{2}) + N(12 + 3\log_2 N).$$

Thus the order of complexity for $W_z^{(M)}[n,k]$ resulting in a $2N \times N$ grid is $M_r = \mathcal{O}(N^2 \log_2 N)$ and $A_r = \mathcal{O}(3N^2 \log_2 N)$. This is a significant reduction on the original DWVD $W_z^{(P)}[n,k]$ method by Peyrin and Prost as they suggested a method resulting in $M_r = \mathcal{O}(8N^2 \log_2 2N)$ and $A_r = \mathcal{O}(10N^2 \log_2 2N)$ [9].

5. CONCLUSION

An alias-free discrete time and frequency Wigner-Ville distribution for nonperiodic signals has been proposed. This distribution, a modified version of the Peyrin and Prost distribution, has many advantages over other D-WVDs. Firstly it satisfies the unitarity property (or Moyal's formula) making it a suitable distribution for use in time-frequency detection. To-date no other alias-free DWVD satisfies this property for nonperiodic signals. Secondly it satisfies the invertibility property, thus the signal can be easily completely recovered (up to a constant phase factor) from the distribution. This is a particularly useful property for time-frequency filtering.

An efficient implementation of the proposed distribution is also presented using only standard real FFT routines. The implementation requires $\mathcal{O}(N^2 \log_2 N)$ and $\mathcal{O}(3N^2 \log_2 N)$ real multiplications and real additions respectively for signal length N. This can be compared with the implementation suggested by Peyrin and Prost which results in $\mathcal{O}(8N^2 \log_2 2N)$ real multiplications and $\mathcal{O}(10N^2 \log_2 2N)$ real additions.

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