Exam 2 Problem

1a.

$$A = \left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) \quad \text{and} \quad L = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)$$

$\hat{A}$ is the original contents of $A$ so, $A$ at the start of the Algo is equal to $\hat{A}$.

According to the Cholesky factorization $A = LL^T$ if $A$ is SPD and $L$ is a lower triangular matrix. If the diagonal elements of $L$ are positive then, $L$ is unique.

$$\left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) = \left(\begin{array}{c|c} L_{00} & \emptyset \\ \hline l_{10}^T & \lambda_{11} \end{array}\right) \left(\begin{array}{c|c} L_{00} & \emptyset \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)^T$$

$$= \left(\begin{array}{c|c} L_{00} & \emptyset \\ \hline l_{11}^T & \lambda_{11} \end{array}\right) \left(\begin{array}{c|c} L_{00}^T & l_{10} \\ \hline \emptyset & \lambda_{11} \end{array}\right)$$

$$= \left(\begin{array}{c|c} L_{00} L_{00}^T & L_{00} l_{10} \\ \hline l_{10}^T L_{00}^T & l_{10}^T l_{10} + (\lambda_{11})^2 \end{array}\right)$$

$A_{00} = L_{00} L_{00}^T$

$a_{10}^T = l_{10}^T L_{00}^T$

$a_{01} = L_{00} l_{10}$

$\alpha_{11} = l_{10}^T l_{10} + (\lambda_{11})^2$

to justify this $\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c|c} L_{TL} L_{TL} & \hat{A}_{TR} \\ \hline \hat{A}_{BL} & \hat{A}_{BR} \end{array}\right) \wedge$

$L_{TL} L_{TL}^T = \hat{A}_{TL}$ after updating $A$ it must contain.

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$$\left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right) = \left(\begin{array}{c|c|c} L_{00} & \cancel{*} & \hat{A}_{02} \\ l_{10}^T & \lambda_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right) \wedge$$

$$\left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)\left(\begin{array}{c|c} L_{00}^T & l_{10} \\ \hline 0 & \lambda_{11} \end{array}\right) = \left(\begin{array}{c|c} \hat{A}_{00} & \hat{a}_{01} \\ \hline \hat{a}_{10}^T & \hat{\alpha}_{11} \end{array}\right)$$

so All of $\hat{A}$ entries are know due them being the entries in the unaltered $A$ matrix. $A_{33}$ is $L_{00}$ due to this proposed algo being bordered.

unknowns are $l_{10}, l_{10}^T, \lambda_{11}$

$L_{00} L_{00}^T = \hat{A}_{00}$

$L_{00} l_{10} = \hat{a}_{01}$

$l_{10}^T L_{00}^T = \hat{a}_{10}^T$ where $(\hat{a}_{10}^T)^T = (l_{10}^T L_{00}^T)^T = L_{00} l_{10}$

$l_{10}^T l_{10} + \lambda^2 = \hat{\alpha}_{11}$

Solve $a_{10} := L_{00} l_{10}$ while overwriting $a_{10}$ with the result
<triangular solve>

· $a_{10}^T := l_{10}^T = (l_{10})^T = (a_{10})^T$ <replace $a_{10}^T$ with $(a_{01})^T$
<transposition of terms>

· $\alpha_{11} := \lambda = \sqrt{\alpha_{11} - a_{10}^T a_{01}}$ <overwrite $\alpha$ with this result>

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The proposed Alogrithm is

$$A = LL^T\text{-var1 Bordered}(A)$$

$$A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$$

$A_{TL} \rightarrow$ is $0 \times 0$

while $n(A_{TL}) < n(A)$

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|cc} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ A_{20} & a_{21} & A_{22} \end{array}\right)$$

Solve $a_{10} := A_{00} l_{10}$ overwriting $a_{10}^T$ with the result transposed

$\alpha_{11} := \sqrt{\alpha_{11} - a_{10}^T a_{10}}$ overwriting $\alpha_{11}$ with the result

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{cc|c} A_{00} & a_{01} & A_{02} \\ a_{10}^T & \alpha_{11} & a_{12} \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$$

endwhile

Test

$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ is SPD. Proof $\begin{pmatrix} 1 \\ 0 \end{pmatrix}^T \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} =$

$(1\ 0)\begin{pmatrix} 2 \\ -1 \end{pmatrix} = 2$

loop 1 $\left(\begin{array}{c|cc} 0 \times 0 & & \\ \hline & 2 & -1 \\ & -1 & 2 \end{array}\right)$

$a_{10}^T = (0 \times 0)(l_{10}) = X$

$\alpha_{11} = \sqrt{2 - (0)(0)}$

$\left(\begin{array}{c|c} \sqrt{2} & -1 \\ -1 & 2 \end{array}\right)$ New matrix

$\alpha_{11} = 2$

$A_{00} = 0 \times 0$

$a_{10}^T = (0 \times 0)$

loop 2 $\left(\begin{array}{c|c} \sqrt{2} & -1 \\ \hline -1 & 2 \end{array}\right)$

$a_{10} = \sqrt{2}(l_{10}) = -1 = \sqrt{2}\, l_1 = l_1 = \frac{-1}{\sqrt{2}} =: a_{10}$

$\alpha_{11} = \sqrt{2 - \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right)} = \sqrt{2 - \frac{1}{2}} = \sqrt{\frac{3}{2}}$

$A_{00} = \sqrt{2}$

$a_{10} = -1$

$a_{10} = -1$

$\alpha_{11} = 2$

$\begin{pmatrix} \sqrt{2} & 0 \\ -1/\sqrt{2} & \sqrt{\frac{3}{2}} \end{pmatrix}$ Finish

$\begin{pmatrix} \sqrt{2} & \emptyset \\ -\frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & -1/\sqrt{2} \\ \emptyset & \sqrt{\frac{3}{2}} \end{pmatrix} = \begin{pmatrix} (\sqrt{2})(\sqrt{2}) & (\sqrt{2})(-\frac{1}{\sqrt{2}}) \\ (\sqrt{2})(-\frac{1}{\sqrt{2}}) & (-\frac{1}{\sqrt{2}})(-\frac{1}{\sqrt{2}}) + (\sqrt{\frac{3}{2}})(\sqrt{\frac{3}{2}} \end{pmatrix}$

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$$\begin{pmatrix} (\sqrt{2})(\sqrt{2}) & (\sqrt{2})(-1/\sqrt{2}) \\ (\sqrt{2})(-1/\sqrt{2}) & \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right) + \left(\sqrt{\frac{3}{2}}\right)\left(\sqrt{\frac{3}{2}}\right) \end{pmatrix} =$$

$$\frac{1}{2} + \frac{3}{2} = 4$$

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

algo works

1.b

Theorem: Given a SPD matrix $A$, there exists a lower triangular matrix $L$ such that $A = LL^T$. If the diagonal elements of $L$ are restricted to be positive, $L$ is unique.

Proof by induction

1. Base case: $n=1$

- $A = \alpha = \alpha_{11}$
- For $\alpha_{11}$ to be SPD, then $x^T \alpha x > 0$ for all $x$ that are not the zero vector.

Let $x = e_j$, then

$$e_j^T \alpha e_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}^T \alpha_{11} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \alpha(0) \\ \alpha(0) \\ \vdots \\ \alpha(1) \\ \vdots \\ \alpha(0) \\ \alpha(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ (\alpha) \\ \vdots \\ 0 \\ 0 \end{pmatrix} =$$

$j$th element

$$(000\cdots1\cdots000) \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \alpha \\ \vdots \\ 0 \\ 0 \end{pmatrix} = \begin{matrix} (0)(0)+(0)(0)+(0)(0)+\ldots \\ (1)(\alpha)+(0)(0)+(0)(0)+\ldots = \alpha \\ (0)(0) \end{matrix}$$

If $\alpha$ is positive and real, the $x^T \alpha x > 0$ holds, then the 1 property of SPD holds. Since $\alpha = \alpha^T$, all properties of SPD hold and $A$ is SPD. Apply the Bordered Cholesky factorization on $A$ yields

$$\alpha \rightarrow A \rightarrow \begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix} = \begin{array}{c|c} 0\times0 & \\ \hline 1 & \alpha \end{array}$$

$A_{TL} = 0\times0$

$A_{TL} = 0\times0$
$A_{BL} = 0\times0$
$A_{TR} = 0\times0$
$A_{BR} = 1\times1$

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \begin{array}{c|cc} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ A_{20} & a_{21} & A_{22} \end{array}$$

$a_{10} := A_{00} l_{10} = (0\times0)(0\times0)$

$\alpha_{11} = \sqrt{\alpha - (0\times0)(0\times0)} = \sqrt{\alpha}$

end

$L = \sqrt{\alpha}$ $\qquad LL^T = (\sqrt{\alpha})(\sqrt{\alpha}) = \alpha = A$

Theorem holds for case $n=1$.

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2. Inductive Step: Assume the results are true for $n = k$. we will show that it holds for $n = k+1$.

Let $A^{k+1 \times k+1}$ be SPD partitioned

$$A = \left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) \text{ and } L = \left(\begin{array}{c|c} L_{00} & \emptyset \\ \hline l_{10}^T & \lambda \end{array}\right)$$

$$\left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) = \left(\begin{array}{c|c} L_{00} & \emptyset \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)\left(\begin{array}{c|c} L_{00}^T & l_{10} \\ \hline \emptyset & \lambda_{11} \end{array}\right) =$$

$A_{00} = L_{00} L_{00}^T$

$a_{01} = L_{00} l_{10}$

$a_{10}^T = l_{10}^T L_{00}^T$

$\alpha_{11} = l_{10}^T l_{10} + \lambda^2$ where $\lambda = \sqrt{\alpha_{11} - l_{10}^T l_{10}}$

We have shown for the base case that for a $1 \times 1$ SPD matrix that $L_{00}$ is solved and known. So by adding $n = k+1$

$a_{01} = L_{00} l_{10}$ which is a triangular solve.
The solution to a triangular solve is well defined and unique.
$a_{10}^T = a_{01}^T$ since $A$ = SPD so by solving for $l_{10}$ and replacing it with $a_{01}$ we have solved for $a_{10}^T$

$\alpha_{11}$ must be greater than $l_{10}^T l_{10}$ for the alg. to complete
since $\lambda = \sqrt{\alpha_{11} - l_{10}^T l_{10}}$.

So
$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{cc|c} L_{00} & \emptyset & A_{02} \\ l_{10}^T & \lambda & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$$

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Thus by the principle of induction, the hypothesis holds.