Exam 2

2.

For the Bordered LU factorization algorithm, the backward error analysis can be written as

$$A + \Delta A = \check{L}\check{U} \text{ with } |\Delta A| \le \gamma_n |\check{L}||\check{U}| \text{ where}$$

$\check{L}\check{U}$ equals the computed LU factorization for matrix A

$$A = \left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) \quad L = \left(\begin{array}{c|c} L_{00} & \emptyset \\ \hline l_{10}^T & 1 \end{array}\right) \text{ and } U = \left(\begin{array}{c|c} U_{00} & u_{01} \\ \hline \emptyset & v_{11} \end{array}\right)$$

(a) Proof by induction Base case: $n = 1$

$A = \alpha$

$$A = \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$$

where $A_{00} = 0 \times 0$, and $\alpha_{11} = 1 \times 1$

$$\text{So } \left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) = \left(\begin{array}{c|c} \check{L}_{00} & \emptyset \\ \hline \check{l}_{10}^T & 1 \end{array}\right)\left(\begin{array}{c|c} \check{U}_{00} & \check{u}_{01} \\ \hline \emptyset & \check{v}_{11} \end{array}\right) =$$

$A_{00} = L_{00} U_{00}$

$a_{10}^T = \check{l}_{10}^T \check{U}_{00}$ and $(a_{10}^T)^T = (l_{10}^T U_{00})^T = a_{10} = U_{00}^T l_{10}$

$a_{01} = \check{L}_{00} \check{v}_{01}$

$\alpha_{11} = \check{l}_{10}^T \check{v}_{01} + \check{v}_{11}$ and $v_{11} = \alpha_{11} - l_{10}^T v_{01}$

$A_{00} = \check{L}_{00} \check{U}_{00} = (0 \times 0)(0 \times 0) = 0$

$a_{01} = \check{L}_{00} \check{U}_{01} = (0 \times 0)(0 \times 0) = 0$

$a_{10}^T = \check{U}_{00}^T \check{l}_{10} = (0 \times 0)(0 \times 0) = 0$

$\check{v}_{11} = \alpha_{11} - \check{a}_{10}^T \check{a}_{01} = \alpha_{11} - (0 \times 0)(0 \times 0) = \alpha$

$\check{v}_{11} = \alpha_{11}$

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$a_{10}^T = l_{10}^T$

$a_{01} = v_{01}$

$\check{v}_{11} = \alpha_{11} - \check{l}_{10}^T \check{v}_{01}$

$\check{v}_{11} = \alpha(1+\epsilon_s) - \check{l}_{10}^T \check{v}_{01}$ (stored as a floating point)

$= \alpha + \epsilon_s \alpha - l_{10}^T v_{01}$ $\quad \epsilon_s \leq \epsilon_{mach}$

$\check{v}_{11} = \alpha + d\alpha - l_{10}^T v_{01}$

So $\check{v}_{11} = \check{V}$ $\quad \check{L} = (1)$ $\quad \alpha = A$ $\quad d\alpha = \Delta A$

$A + \Delta A = \check{L}\check{V}$ with $|d\alpha| \leq |\theta_1||1||\check{v}_{11}| = |\theta_1||L||\check{V}| \leq \gamma_1 |L||V|$ for $n=1$

Base Case $n=1$ holds.

So Assume that for $A_{00} \in R^{n \times n}$ the bordered LU factorization computes (in floating point arithmetic) $\check{L}_{00}$ $\check{L}_{00}$ where

$$A_{00} + \Delta A_{00} = \check{L}_{00} \check{U}_{00} \text{ with } |\Delta A| \leq \gamma_n |\check{L}_{00}||\check{U}_{00}|$$

Must show that the bordered LU factorization applied to

$$\left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) \text{ computes } \left(\begin{array}{c|c} \check{L}_{00} & 0 \\ \hline \check{l}_{10}^T & 1 \end{array}\right) \text{ and } \left(\begin{array}{c|c} \check{U}_{00} & \check{v}_{11} \\ \hline \emptyset & \check{v}_{11} \end{array}\right)$$

where

$$\left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) + \left(\begin{array}{c|c} \Delta A_{00} & da_{01} \\ \hline da_{10}^T & d\alpha_{11} \end{array}\right) = \left(\begin{array}{c|c} \check{L}_{00} & \emptyset \\ \hline \check{l}_{10}^T & 1 \end{array}\right)\left(\begin{array}{c|c} \check{U}_{00} & \check{v}_{01} \\ \hline \emptyset & \check{v}_{11} \end{array}\right)$$

with

$$\left|\left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right)\right| \leq \gamma_{n+1} \left|\left(\begin{array}{c|c} \check{L}_{00} & \emptyset \\ \hline \check{l}_{10}^T & 1 \end{array}\right)\right|\left|\left(\begin{array}{c|c} \check{U}_{00} & \check{v}_{01} \\ \hline \emptyset & \check{v}_{11} \end{array}\right)\right|$$

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$$\left(\begin{array}{c|c} |\Delta A_{00}| & |da_{01}| \\ \hline |da_{10}^T| & |d\alpha_{11}| \end{array}\right) \le \gamma_{n+1} \left(\begin{array}{c|c} |\check{L}_{00}| & \emptyset \\ \hline |l_{10}^T| & |1| \end{array}\right)\left(\begin{array}{c|c} |\check{V}_{00}| & |\check{v}_{01}| \\ \hline \emptyset & |\check{v}_{11}| \end{array}\right)$$

I $|\Delta A_{00}| \le \gamma_{n+1} |\check{L}_{00}||\check{V}_{00}|$

II $|da_{01}| \le \gamma_{n+1} |\check{L}_{00}||\check{v}_{01}|$

III $|da_{10}^T| \le \gamma_{n+1} |l_{10}^T||\check{V}_{00}|$

IV $|d\alpha_{11}| \le \gamma_{n+1} \left(|l_{10}^T||\check{v}_{01}| + |\check{v}_{11}|\right)$

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✓ I. $\underbrace{|\Delta A_{00}| \le \gamma_n |L_{00}||V_{00}|} \le \gamma_{n+1} |L_{00}||V_{00}|$

Our inductive step must be true for the loop to continue.

✓ II $a_{01} = \check{L}_{00} \check{v}_{01}$ $\wedge$

$|da_{01}| \le \max(\gamma_2, \gamma_{n-1}) |\check{L}_{00}||\check{v}_{01}|$
$\le \gamma_{n+1} |\check{L}_{00}||\check{v}_{01}|$

From Piazza post 909

III $a_{10}^T = l_{10}^T V_{00}$ $\quad (a_{10}^T)^T = (l_{10}^T \check{V}_{00})^T =$

$(a_{10} + da_{10}) = ($ $\quad a_{10}^T = \check{V}_{00}^T l_{10}$ $\wedge$

$|da_{10}| \le \max(\gamma_2, \gamma_{n-1}) |\check{V}_{00}^T||l_{10}| \le$

as above piazza post 909 $\quad \gamma_{n+1} |\check{V}_{00}^T||l_{10}|$

IV $\check{\nu}_{11} = \alpha_{11} - \check{l}_{10}^T \check{v}_{01}$

$$(\alpha_{11} - l_{10}^T(I+\Sigma^{n+1})v_{01})(1+\theta_2)$$

$$(\alpha_{11}(I+\theta_1) - l_{10}^T(I+\Sigma^{n-1})(1+\theta_2)\check{v}_{01})$$

$$\alpha_{11} + \theta_1\alpha_{11}$$

$$\alpha_{11} + \delta\alpha_{11} - l_{10}^T\left(\begin{pmatrix}(1+\theta_n) & & & \\ & (1+\theta_n) & & \\ & & (1+\theta_{n-1}) & \\ & & & \ddots \; (1+\theta_2)\end{pmatrix}(1+\theta_2)v_{01}\right)$$

$$l_{10}^T\begin{pmatrix}(1+\theta_n)(1+\theta_2) & & & \\ & (1+\theta_n)(1+\theta_2) & & \\ & & (1+\theta_n)(1+\theta_{n-1}) & \\ & & & \ddots \; (1+\theta_2)(1+\theta_2)\end{pmatrix}v_{01}$$

$$l_{10}^T\begin{pmatrix}(1+\theta_{n+1}) & & & \\ & (1+\theta_{n+1}) & & \\ & & (1+\theta_n) & \\ & & & \ddots \; (1+\theta_3)\end{pmatrix}v_{01}$$

$$\check{\nu} = (\alpha_{11} + \delta\alpha_{11}) - l_{10}^T(I+\Sigma^{n+1})v_{01}$$

$$|\delta\alpha_{11}| \le \max(\gamma_1, \gamma_{n+1})\,|\alpha_{11} - l_{10}^T v_{01}| \le$$

$$\gamma_{n+1}|\alpha_{11} - l_{10}^T v_{01}| \le \gamma_{n+1}|\alpha_{11}| + |\check{l}_{10}^T||\check{v}_{01}|$$

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} + \Delta A_{00} & a_{01} + \delta a_{01} & A_{02} \\ \hline a_{10}^T + \delta a_{10}^T & \alpha_{11} + \delta\alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$$

By the principle of mathematic induction our solution holds.