Problem 1. Does there exist any cycle σ such that σ^2 is not a cycle?

Yes. $(1,2,3,4)^2 = (1,3)(2,4)$ which is not a cycle.

Problem 2. Write the following product of cycles as a cycle:

$$(2,3,6,8)(4,9,1,3)(1,5,7).$$

The product (4,9,1,3)(1,5,7) is equal to (1,5,7,3,4,9) and the product of (2,3,6,8)(1,5,7,3,4,9) is equal to (1,5,7,6,8,2,3,4,9). Therefore, (2,3,6,8)(4,9,1,3)(1,5,7) = (1,5,7,6,8,2,3,4,9)

Problem 3. Determine whether

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 5 & 1 & 6 & 8 & 9 & 2 & 7 & 4 \end{pmatrix}$$

is even or odd. How about (2,3)α and (3,6,1,7)α?

α can be broken into 6 transpositions. Those are (1,3)(2,7)(2,8)(2,5)(4,9)(4,6). Since α requires an even number of transpositions in its decomposition α is even. Since α is even, composing another transposition would make it odd. Therefore (2,3)α is odd. (3,6,1,7)α is also odd because (3,6,1,7) can be composed into 3 transpositions, (3,7)(3,1)(3,6) and an even number of transpositions plus an odd number of transpositions is always odd so (3,6,1,7)α is odd.

Problem 4. Determine whether the following statement is true or false, and explain:

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 6 & 7 & 5 & 8 & 1 & 3 & 4 \end{pmatrix} \in A_8.$$

$\gamma \notin A_8$ because γ is represented by an odd number of transpositions. Those are (1,6)(1,2)(3,7)(4,8)(4,5). Since A_8 contains only even permutations, γ cannot be in A_8. So, the statement is false.

Problem 5. Consider

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 8 & 5 & 6 & 4 & 7 & 3 & 1 \end{pmatrix} \in S_8.$$

(a) Evaluate β^{2014}.

(b) Find $\alpha \in S_8$ such that $\alpha^4 = \beta$.

(a) Since β can be written as β = (1,2,8)(3,5,4,6,7), I know that the order of β is equal to the LCM of the lengths of these cycles. That is, ord(β) = LCM(3,5) = 15. Since the order is 15, that means that $\beta^{2010} = \iota$ because 2010 is divisible by 15. $\beta^{2014} = \beta^{2010}\beta^4$, so $\beta^{2014} = \iota\beta^4 = \beta^4 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 8 & 7 & 5 & 3 & 4 & 6 & 1 \end{smallmatrix}\right)$.

(b) Since β is written as (1,2,8)(3,5,4,6,7), it is written as two cycles with different lengths. This means that α must be the product of at least two cycles of different lengths. Assume $\alpha = \gamma\sigma$, so that $\alpha^4 = \gamma^4\sigma^4 = (1,2,8)(3,5,4,6,7)$. γ is a 3-cycle because by theorem 6.4.1 if γ is an m-cycle, then γ^4 is a product of $\gcd(m,4)$ cycles and has length $\frac{m}{\gcd(m,k)}$. Since the length of γ^4 is 3, $\frac{m}{\gcd(m,4)} = 3$. Since γ^4 is a product of 1 cycle, $\gcd(m,4) = 1$. Both of these equations are solved when $m = 3$. Therefore γ is a 3-cycle. Using this fact, $\gamma^4 = \gamma^3\gamma = \iota\gamma = \gamma$. σ is a 5-cycle because by theorem 6.4.1, if σ is an m-cycle, then σ^4 has length $\frac{m}{\gcd(m,4)}$ and is a product of $\gcd(m,4)$ cycles. Since the length of $\sigma^4 = 5$, $\frac{m}{\gcd(m,4)} = 5$. Since σ^4 is a product of 1 cycle, $\gcd(m,4) = 1$. Both equations are satasfied when $m = 5$. Therefore σ is a 5-cycle. Knowing that σ is a 5-cycle, it is easy to see that $\sigma = (3,7,6,4,5)$. So, $\alpha = \gamma\sigma = (1,2,8)(3,7,6,4,5)$.

Problem 6. For any $H \leq S_n$ and any fixed permutation $\sigma \in S_n$, define

$$K = \left\{\sigma\tau\sigma^{-1} \mid \tau \in H\right\}.$$

In other words, K is the set consisting of all permutations of the form $\sigma\tau\sigma^{-1}$ for some element $\tau \in H$. Show that $K \leq S_n$.

Since H is a subgroup of S_n it contains ι. If I let $\sigma = \iota$ and $\tau = \iota$, this shows that $\iota \in K$ because $\iota\iota\iota = \iota$. So, K is a non-empty subset of S_n. K is closed under S_n's binary operation. To show this I will show the product of two elements $a, b \in K$ is also in K. Since $a, b \in K$ $a = \sigma\tau_1\sigma^{-1}, b = \sigma\tau_2\sigma^{-1}$. So the product is $ab = \sigma\tau_1\sigma^{-1}\sigma\tau_2\sigma^{-1} = \sigma\tau_1\iota\tau_2\sigma^{-1} = \sigma\tau_1\tau_2\sigma^{-1}$. Since $\tau \in H$ and $H \leq S_n$, H is closed so, let the product $\tau_1\tau_2 = \tau_3$. Then, $ab = \sigma\tau_3\sigma^{-1}$. This takes the form of K so $ab \in K$ therefore K is closed. For all $a \in K$, the inverse also exists. That is, there is an element $a^{-1} \in K$ such that $aa^{-1} = \iota$. Rewritten, this is $\sigma\tau_1\sigma^{-1}\sigma\tau_2\sigma = \sigma\tau_1\tau_2\sigma^{-1} = \iota$. Since I want to show that $aa^{-1} = \iota$, it must be that $\tau_2 = \tau_1^{-1}$ which exists because $\tau_1 \in H$ and $H \leq S_n$. So, $aa^{-1} = \sigma\tau_1\tau_1^{-1}\sigma^{-1} = \sigma\sigma^{-1} = \iota$. This shows that the inverse exists for all $a \in K$. Since I have shown K is non-empty, the binary operation of S_n is closed in K, and for all $a \in K$ a^{-1} exists, by corollary 4.2.2 $K \leq S_n$.

Problem 7. Let α be a 10-cycle. Find the integers k, where $2 \leq k \leq 10$, such that α^k also a 10-cycle? Explain.

To find the values of k where α^k is a 10-cycle, I will use theorem 6.4.1. According to the theorem, α^k is a product of $\gcd(10, k)$ cycles and has length $\frac{10}{\gcd(10,k)}$. Since I want to find the values of k such that α^k is a 10-cycle, then $\frac{10}{\gcd(10,k)} = 10$. The values in which this happens is when $k = 1, 3, 7, 9$. But, I also want the number of cycle products to be 1 and the k value to not be greater than one and less than 11. The k values that satisfy this are $k = 3, 7, 9$.

Problem 8. Let $\beta \in S_n$ such that $\text{ord}_{S_n}(\beta) = 36$.

(a) How many elements are there in $\langle\beta\rangle$?

(b) What are the generators of $\langle\beta\rangle$?

(c) Which subgroup H of $\langle\beta\rangle$ has order 12?

(d) Find the generators of the subgroup H of $\langle\beta\rangle$ of order 12.

(e) How many elements in $\langle\beta\rangle$ have order 12? Does your answer agree with part (d)? Explain.

(a) Since the order of β is 36, this means that there are 36 elements in $\langle\beta\rangle$.

(b) Using theorem 5.1.6 $|\beta^k| = \frac{36}{\gcd(36,k)}$. Since I want to find the generators of β, I need the orders to be the same. That is, $|\beta^k| = \frac{36}{\gcd(36,k)} = 36$. This is when $\gcd(36, k) = 1$. Solving for k I find that they are all that are coprime to 36. $k = 1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35$. But the generators of β must be $\{\beta, \beta^5, \beta^7, \beta^{11}, \beta^{13}, \beta^{17}, \beta^{19}, \beta^{23}, \beta^{25}, \beta^{29}, \beta^{31}, \beta^{35}\}$.

(c) $H = \{\beta^{3n} \mid n \in \mathbb{Z}_{12}\}$

(d) Again using theorem 5.1.6 I want to find the values of k where $|\beta^k| = \frac{36}{\gcd(36,k)} = 12$. This is when $k = 3, 15, 21, 33$. So, the generators of H are $\{\beta^3, \beta^{15}, \beta^{21}, \beta^{33}\}$.

(e) There are 4 elements in $\langle\beta\rangle$ which I computed in (d). Those are β^k when $k = 3, 15, 21, 33$. By using the theorem 5.2.2 and the euler totient function, I should get $\phi(12) = 4$ values. Since I got 4 values and was expecting 4 values my answer agrees with part (d).

Problem 9. It can be shown that (but you do not have to)

$$H = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \right\}$$

is a group.

(a) Compute the operation table for H.

(b) For brevity, name the four permutations as $\iota, \alpha, \beta, \gamma$. Rewrite the operation table in terms of $\iota, \alpha, \beta, \gamma$.

(c) This operation table is identical to the operation table of which familiar group?

(d) Express β and γ in terms of α.

(e) What is the order of α? Does this conform with your answer to part (c)?

(f) What are the generators of H?

(a)

$\cdot$	$\begin{pmatrix} 1 2 3 4 \\ 1 2 3 4 \end{pmatrix}$	$\begin{pmatrix} 1 2 3 4 \\ 2 3 4 1 \end{pmatrix}$	$\begin{pmatrix} 1 2 3 4 \\ 3 4 1 2 \end{pmatrix}$	$\begin{pmatrix} 1 2 3 4 \\ 4 1 2 3 \end{pmatrix}$
$\begin{pmatrix} 1 2 3 4 \\ 1 2 3 4 \end{pmatrix}$	$\begin{pmatrix} 1 2 3 4 \\ 1 2 3 4 \end{pmatrix}$	$\begin{pmatrix} 1 2 3 4 \\ 2 3 4 1 \end{pmatrix}$	$\begin{pmatrix} 1 2 3 4 \\ 3 4 1 2 \end{pmatrix}$	$\begin{pmatrix} 1 2 3 4 \\ 4 1 2 3 \end{pmatrix}$
$\begin{pmatrix} 1 2 3 4 \\ 2 3 4 1 \end{pmatrix}$	$\begin{pmatrix} 1 2 3 4 \\ 2 3 4 1 \end{pmatrix}$	$\begin{pmatrix} 1 2 3 4 \\ 3 4 1 2 \end{pmatrix}$	$\begin{pmatrix} 1 2 3 4 \\ 4 1 2 3 \end{pmatrix}$	$\begin{pmatrix} 1 2 3 4 \\ 1 2 3 4 \end{pmatrix}$
$\begin{pmatrix} 1 2 3 4 \\ 3 4 1 2 \end{pmatrix}$	$\begin{pmatrix} 1 2 3 4 \\ 3 4 1 2 \end{pmatrix}$	$\begin{pmatrix} 1 2 3 4 \\ 4 1 2 3 \end{pmatrix}$	$\begin{pmatrix} 1 2 3 4 \\ 1 2 3 4 \end{pmatrix}$	$\begin{pmatrix} 1 2 3 4 \\ 2 3 4 1 \end{pmatrix}$
$\begin{pmatrix} 1 2 3 4 \\ 4 1 2 3 \end{pmatrix}$	$\begin{pmatrix} 1 2 3 4 \\ 4 1 2 3 \end{pmatrix}$	$\begin{pmatrix} 1 2 3 4 \\ 1 2 3 4 \end{pmatrix}$	$\begin{pmatrix} 1 2 3 4 \\ 2 3 4 1 \end{pmatrix}$	$\begin{pmatrix} 1 2 3 4 \\ 3 4 1 2 \end{pmatrix}$

(b)

$\cdot$	ι	α	β	γ
ι	ι	α	β	γ
α	α	β	γ	ι
β	β	γ	ι	α
γ	γ	ι	α	β

(c) The subgroup of rotations in D_4

(d) $\beta = \alpha^2$ and $\gamma = \alpha \cdot \beta = \alpha^3$

(e) $\text{ord}(\alpha) = 4$. This does conform with my answer to part (c).

(f) The generators of H are α and γ because if I multiply them by themselves I get all elements in H and reach ι when the power is 4.

Problem 10. Determine whether the function $\phi : \langle \mathbb{R}, + \rangle \to \langle \mathbb{R}^+, \cdot \rangle$ defined by $\phi(x) = 3^{x/2}$ is an isomorphism between the two groups $\langle \mathbb{R}, + \rangle$ and $\langle \mathbb{R}^+, \cdot \rangle$

To determine that ϕ is an isomorphism between $\langle \mathbb{R}, + \rangle$ and $\langle \mathbb{R}^+, \cdot \rangle$, I will need to show that there is a bijection between these two and that ϕ is operation preserving. ϕ forms a bijection between $\langle \mathbb{R}, + \rangle$ and $\langle \mathbb{R}^+, \cdot \rangle$. To show this I show that ϕ is onto. That is, $\phi(x_1) = \phi(x_2) \implies x_1 = x_2$. Assume $\phi(x_1) = \phi(x_2)$. Then, I can rewrite this as $3^{x_1/2} = 3^{x_2/2}$. If I take the natural log of both sides I get $\frac{x_1}{2}\ln(3) = \frac{x_2}{2}\ln(3)$. Multiplying by two and then dividing by $\ln(3)$ on both sides gives me $x_1 = x_2$. Next it is easy to see that ϕ is onto because for every element $y \in \mathbb{R}^+$ there is an element $x \in \mathbb{R}$ such that $\phi(x) = y$. Since ϕ is one to one and onto, ϕ is a bijection. ϕ is also operation preserving. This is true because $\phi(x_1 + x_2) = \phi(x_1) \cdot \phi(x_2)$. Looking at the left hand side, $\phi(x_1 + x_2) = 3^{\frac{x_1+x_2}{2}}$. Now looking at the right hand side, $\phi(x_1) \cdot \phi(x_2) = 3^{\frac{x_1}{2}} \cdot 3^{\frac{x_2}{2}} = 3^{\frac{x_1}{2} + \frac{x_2}{2}} = 3^{\frac{x_1+x_2}{2}}$. Since $\phi(x_1 + x_2) = \phi(x_1) \cdot \phi(x_2) = 3^{\frac{x_1+x_2}{2}}$, ϕ is operation preserving. Finally, since ϕ forms a bijection between $\langle \mathbb{R}, + \rangle$ and $\langle \mathbb{R}^+, \cdot \rangle$ and is operating preserving, ϕ is an isomorphism between $\langle \mathbb{R}, + \rangle$ and $\langle \mathbb{R}^+, \cdot \rangle$.

Problem 11. Show that $\langle 5\mathbb{Z}, + \rangle \cong \langle 8\mathbb{Z}, + \rangle$

To show that $\langle 5\mathbb{Z}, + \rangle \cong \langle 8\mathbb{Z}, + \rangle$, I need to show that there is a function $\phi : \langle 5\mathbb{Z}, + \rangle \to \langle 8\mathbb{Z}, + \rangle$ that is a bijection and is operation preserving. Noticing a common pattern I guess the function $\phi(x) = x + \frac{3x}{5}$ forms an isomorphism. To show that it does, I first show a bijection. Assume $\phi(x_1) = \phi(x_2)$, this implies $x_1 = x_2$. I show this by finding that $\phi(x_1) = x_1 + \frac{3x_1}{5} = x_1(1+\frac{3}{5})$ and $\phi(x_2) = x_2 + \frac{3x_2}{5} = (x_2)(1+\frac{3}{5})$. Since I assume $\phi(x_1) = \phi(x_2)$, then $x_1(1+\frac{3}{5}) = x_2(1+\frac{3}{5})$. Since $(1+\frac{3}{5})$ is constant it is clear that $x_1 = x_2$. This shows that ϕ is one to one. To show that $\phi(x)$ is onto, I must show that for every $y \in 8\mathbb{Z}$, there is an $x \in 5\mathbb{Z}$ such that $\phi(x) = y$. I do this by rewriting ϕ as $\phi(x) = x(1+\frac{3}{5}) = \frac{8x}{5} = 8(\frac{x}{5})$. Clearly, since $x \in 5\mathbb{Z}$, $\frac{x}{5}$ is an integer and $8(\frac{x}{5}) \in 8\mathbb{Z}$ when $\frac{x}{5}$ is an integer. This shows that ϕ is onto. ϕ is also operation preserving. To show this I must show that $\phi(x_1+x_2) = \phi(x_1)+\phi(x_2)$. Expanding the left hand side I get $\phi(x_1+x_2) = (x_1+x_2)(1+\frac{3}{5})$. Expanding the right hand side I get $\phi(x_1) + \phi(x_2) = (x_1)(1+\frac{3}{5}) + (x_2)(1+\frac{3}{5}) = (1+\frac{3}{5})(x_1+x_2)$. Since $\phi(x_1+x_2) = (x_1+x_2)(1+\frac{3}{5}) = \phi(x_1)+\phi(x_2)$, this shows that ϕ is operation preserving. Finally, since ϕ is a bijection that is also operation preserving, ϕ is an isomorphism between $\langle 5\mathbb{Z}, +\rangle$ and $\langle 8\mathbb{Z}\rangle$. Since an isomorphism exists between these two groups, $\langle 5\mathbb{Z}, +\rangle \cong \langle 8\mathbb{Z}, +\rangle$.

Problem 12. Let

$$G = \left\{ a + b\sqrt{2} \mid a, b \in \mathbb{Q} \right\}$$

and

$$H = \left\{ \begin{bmatrix} x & 2y \\ y & x \end{bmatrix} \;\middle|\; x, y \in \mathbb{Q} \right\}$$

It is known that $\langle G, +\rangle$ and $\langle H, +\rangle$ are groups. Show that $\langle G, +\rangle \cong \langle H, +\rangle$.

To show that $\langle G, +\rangle \cong \langle H, +\rangle$ I will find a function, ϕ, that is an isomorphism between these two groups. The function $\phi : \langle G, +\rangle \to \langle H, +\rangle$ I define as $\phi(c) = \left[\begin{smallmatrix} a & 2b \\ b & a \end{smallmatrix}\right]$ is an isomorphism. To show this I will show that there is a bijection and that ϕ is operation preserving. Assume $\phi(c_1) = \phi(c_2)$. Expanding the left side of this equation I get, $\phi(c_1) = \left[\begin{smallmatrix} a_1 & 2b_1 \\ b_1 & a_1 \end{smallmatrix}\right]$. Expanding the right side I get $\phi(c_2) = \left[\begin{smallmatrix} a_2 & 2b_2 \\ b_2 & a_2 \end{smallmatrix}\right]$. This shows that $\left[\begin{smallmatrix} a_1 & 2b_1 \\ b_1 & a_1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} a_2 & 2b_2 \\ b_2 & a_2 \end{smallmatrix}\right]$. Since the matricies are equal, the entries must be equal such that $a_1 = a_2, 2b_1 = 2b_2$, and $b_1 = b_2$. Therefore, $c_1 = a_1 + b_1\sqrt{2} = a_2 + b_2\sqrt{2} = c_2$. This shows that ϕ is one to one. ϕ is onto because for every $d \in H$, there is clearly an $e \in G$ such that $\phi(e) = d$. ϕ is operation preserving. This is because $\phi(c_1+c_2) = \phi(c_1)+\phi(c_2)$. I will show this by expanding the left hand side to be $\phi(c_1+c_2) = \phi(a_1+b_1\sqrt{2}+a_2+b_2\sqrt{2}) = \left[\begin{smallmatrix} a_1+a_2 & 2(b_1+b_2) \\ b_1+b_2 & a_1+a_2 \end{smallmatrix}\right]$. Now I will expand the right hand side of the equation to get $\phi(c_1)+\phi(c_2) = \left[\begin{smallmatrix} a_1 & 2b_1 \\ b_1 & a_1 \end{smallmatrix}\right] + \left[\begin{smallmatrix} a_2 & 2b_2 \\ b_2 & a_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} a_1+a_2 & 2(b_1+b_2) \\ b_1+b_2 & a_1+a_2 \end{smallmatrix}\right]$. This shows that $\phi(c_1+c_2) = \left[\begin{smallmatrix} a_1+a_2 & 2(b_1+b_2) \\ b_1+b_2 & a_1+a_2 \end{smallmatrix}\right] = \phi(c_1)+\phi(c_2)$ and therefore ϕ is operation preserving. Since ϕ is a bijection and operation preserving function between $\langle G, +\rangle$ and $\langle H, +\rangle$, ϕ is an isomorphism between $\langle G, +\rangle$ and $\langle H, +\rangle$. Since an isomorphism exists between these two groups, $\langle G, +\rangle \cong \langle H, +\rangle$.