Problem 1. Compute

$$\begin{bmatrix} 3 & 5 \\ -5 & 3 \end{bmatrix}^2 \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} - 3 \begin{bmatrix} 6 & 7 \\ -7 & 6 \end{bmatrix}$$

Use the result to determine the value of $(3-5i)^2(1+2i) - 3(6-7i)$.
Since any matrix in the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

is isomorphic to the complex number $a + bi$, and all of these matricies take this form, if I do arithmetic on these matricies I will get the same result as if I were to perform the arithmetic on the complex numbers. So, doing the computations I get the following.

$$\begin{aligned} \begin{bmatrix} 3 & 5 \\ -5 & 3 \end{bmatrix}^2 \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} - 3 \begin{bmatrix} 6 & 7 \\ -7 & 6 \end{bmatrix} &= \begin{bmatrix} -16 & 30 \\ -30 & -16 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} - 3 \begin{bmatrix} 6 & 7 \\ -7 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 44 & 62 \\ -62 & 44 \end{bmatrix} - 3 \begin{bmatrix} 6 & 7 \\ -7 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 44 & 62 \\ -62 & 44 \end{bmatrix} - \begin{bmatrix} 18 & 21 \\ -21 & 18 \end{bmatrix} \\ &= \begin{bmatrix} 26 & 41 \\ -41 & 26 \end{bmatrix} \end{aligned}$$

So, I determine that the answer is $26 - 41i$ because it is equivalent to my computation of $\begin{bmatrix} 26 & 41 \\ -41 & 26 \end{bmatrix}$.

Problem 3. Define a binary operator $*$ on $\mathbb{Q}$ by $a * b = ab + 1$. Evaluate $3 * 5$ and $2 * -\frac{1}{2}$.
Determine whether $*$ is commutative and associative.

First I evaluate $3 * 5 = 16$ and $2 * -\frac{1}{2} = 0$. Then, to show that $*$ is not associative I will show that $(a * b) * c \neq a * (b * c)$ where $a, b, c \in \mathbb{Q}$.

$$(a * b) * c = (ab + 1) * c = (ab + 1)c + 1 = abc + c + 1$$

$$a * (b * c) = a * (bc + 1) = a(bc + 1) + 1 = abc + a + 1$$

Since $abc + c + 1 \neq abc + a + 1$, $(a * b) * c \neq a * (b * c)$. Therefore, $*$ is not associative. Now to show that $*$ is commutative I show that $a * b = b * a$.

$$a * b = ab + 1$$

$$b * a = ba + 1$$

Since $a * b = ab + 1 = ba + 1 = b * a$, $a * b = b * a$ therefore $*$ is commutative.

Problem 4. Define a binary operator $*$ on $\mathbb{Z}^+$ as $a * b = (2^a)^b$. Evaluate $3 * 5$ and $2 * 3$.
Determine whether $*$ is commutative and associative.

First I compute $3 * 5 = (2^3)^5 = 2^{15} = 32768$ and $2 * 3 = (2^2)^3 = 2^6 = 64$. To show that $*$ is associative I will show that $(a * b) * c = a * (b * c)$ where $a, b, c \in \mathbb{Z}^+$.

$$(a * b) * c = (2^a)^b * c = \left(2^{(2^a)^b}\right)^c = 2^{2abc}$$

$$a * (b * c) = a * ((2^b)^c) = (2^a)^{((2^b)^c)} = 2^{a^{(2bc)}} = 2^{a2bc}$$

$$2^{a2bc} = 2^{2abc}$$

Since $(a * b) * c = 2^{a2bc} = 2^{2abc} = a * (b * c)$, $*$ is associative.

To show that $*$ is commutative, I must show that $a * b = b * a$.

$$a * b = (2^a)^b = 2^{ab}$$

$$b * a = (2^b)^a = 2^{ba}$$

$$2^{ba} = 2^{ab}$$

Since $a * b = 2^{ab} = 2^{ba} = b * a$, $*$ is commutative.

Problem 5. Define a binary operator $*$ on $\mathbb{Z}^+$ as $a * b = 2^{a^b}$. Evaluate $3 * 5$ and $2 * 3$. Determine whether $*$ is commutative and associative.

First I evaluate $3 * 5 = 2^{3^5} = 2^{243}$ and $2 * 3 = 2^{2^3} = 2^8$ Now, to show that $*$ is not associative I show that $(a * b) * c \neq a * (b * c)$ where $a, b, c \in \mathbb{Z}^+$.

$$(a * b) * c = (2^a)^b * c = \left((2^{(2^a)})^b\right)^c = \left(2^{(2^{ab})}\right)^c = 2^{c(2^{ab})}$$

$$a * (b * c) = a * (2^b)^c = (2^a)^{(2^b)^c} = (2^a)^{2^{bc}} = 2^{a2^{bc}}$$

Since, $2^{c(2^{ab})} \neq 2^{a2^{bc}}$, $(a * b) * c \neq a * (b * c)$. Therefore $*$ is not associative. To show that $*$ is commutative I show that $a * b = b * a$.

$$a * b = (2^a)^b = 2^{ab}$$

$$b * a = (2^b)^a = 2^{ba}$$

Since $2^{ab} = 2^{ba}$, it follows that $a * b = b * a$. Therefore $*$ is commutative.

Problem 6. The binary operation $*$ on $\mathbb{R}^+ \cup \{0\}$ is defined as

$$a * b = \max(a, a - b),$$

where $\max(x, y)$ denotes the maximum value of the real numbers x and y.

(a) Explain why $a * 0 = a$ for all real numbers a.

$$a * 0 = a,\ \forall a \in \mathbb{R} \text{ because } a * 0 = \max(a, a - 0) = \max(a, a) = a$$

(b) Does this mean that 0 is the identity element? Explain.

The identity element is defined as the element which is the left identity and the right identity. Although 0 is the right identity, $a * 0 = \max(a, a) = a$, it is not the left identity because $0 * a = \max(0, 0 - a)$. Here, if a is positive, then $\max(0, 0 - a) = 0 \neq a$. Therefore 0 is not the identity element.

Problem 7. Define a binary operation $*$ on Z according to

$$a * b = 3a + 5b$$

(a) Is $*$ commutative? Explain.

(b) Is $*$ associative? Explain.

(c) Is a^3 well-defined? Explain.

(d) Does the identity element e exist? Explain.

(a) $*$ is not commutative because $a * b \neq b * a$.

$$a * b = 3a + 5b$$

$$b * a = 3b + 5a$$

Since $3a + 5b \neq 3b + 5a$, $a * b \neq b * a$. Therefore, $*$ is not commutative.

(b) $*$ is not associative because $(a * b) * c \neq a * (b * c)$.

$$(a * b) * c = (3a + 5b) * c = 3(3a + 5b) + 5c$$

$$a * (b * c) = a * (3b + 5c) = 3a + 5(3b + 5c)$$

Since $(a * b) * c = 3(3a + 5b) + 5c \neq 3a + 5(3b + 5c) = a * (b * c)$, $*$ is not associative.

(c) a^3 is not well defined because it is not associative. This means that a^3 is ambiguous.

(d) The identity element does not exist. I will show there is no such identity element e. Suppose there was an element $e \in \mathbb{Z}$, then

$$a * e = a \text{ and } e * a = a \text{ so } a * e = e * a$$

It also follows that

$$a * e = a = 3a + 5e$$

solving for e I get

$$\frac{-2a}{5} = e$$

It also follows that

$$e * a = a = 3e + 5a$$

solving for e I get

$$\frac{-4a}{3} = e$$

Since e is equal to two separate numbers, this is a contradiction with the assumption that e is the identity element because it is not unique. Therefore the identity element does not exist.

Problem 8. Let $m \geq 2$ be an integer, and define

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \middle| k \in \mathbb{Z}_m \right\}$$

Show that S is a group under multiplication, as follows.

(a) First of all, describe, using proper notation, the meaning of the underlying binary operation $*$.

(b) Show that S is closed under $*$.

(c) Do we need to prove that $*$ is associative? Explain.

(d) What is the identity element? Is it an element in S? Be sure to explain why this is an element of S.

(e) Given an element $z \in S$, does its inverse always exist? If yes, what is it? Is it an element of S?

(f) What is your conclusion about S?

(a) $*$ is normal matrix multiplication but with elements of $\mathbb{Z}_m$

(b)
$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ k_1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ k+k_1 & 1 \end{bmatrix} \text{ where } k, k_1 \in \mathbb{Z}_m$$

Since $k, k_1 \in \mathbb{Z}_m,\ k + k_1 \in \mathbb{Z}_m$ then it follows that the prodcut $\begin{bmatrix} 1 & 0 \\ k+k_1 & 1 \end{bmatrix} \in S$

(c) We don't need to prove associativity on $*$ because we know that matrix multiplication is associative.

(d) The identity element is
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
This is an element of S because it takes the form of $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix},\ k = 0 \in \mathbb{Z}_m$

(e) Yes, there is always an inverse because the determinant is non-zero. Using the fact that the inverse of a 2×2 matrix is $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, I can show that
$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}^{-1} = \frac{1}{1-0}\begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix}$$

(f) S is a group because the set is closed, and it has associativity, inverse, and identity on its operator.

Problem 9. Show that the set $T = \{nd \mid n \in \mathbb{Z}\}$ forms an additive group for any fixed nonzero integer d. What if $d = 0$? Is $\langle T, + \rangle$ still a group? Explain.

To show that T is an additive group I need to show that T is closed under $+$, associative, has an identity and has an inverse. First, T is closed under $+$ because $a + b = n_1 d_1 + n_2 d_2,\ a, b \in T$ and any two integers multiplied or added to each other is another integer which can always be represented by the product nd. Next I show then T is associative. To do this I show that $(a + b) + c = a + (b + c)$. Let $n_1, n_2, n_3 \in \mathbb{Z}$ and $d_1, d_2, d_3 \in \mathbb{Z} - \{0\}$

$$(a + b) + c = (n_1 d_1 + n_2 d_2) + c = (n_1 d_1 + n_2 d_2) + n_3 d_3 = n_1 d_1 + n_2 d_2 + n_3 d_3$$
$$a + (b + c) = a + (n_2 d_2 + n_3 d_3) = n_1 d_1 + (n_2 d_2 + n_3 d_3) = n_1 d_1 + n_2 d_2 + n_3 d_3$$

So, clearly it is shown that $(a + b) + c = a + (b + c)$. Next I state that the identity is $e = 0 \cdot 1$. This is the identity because any element, a, in T added with this element, e, in T yeilds the same element a. Finally I show that the additive inverse exists. The additive inverse is $-a,\ \forall a \in T$ because any element, $a = nb$, added with $-a = -nb$ is equal to the additive identity $0 \cdot 1$. All of the properties of T are shown to satisfy the requirements of a group so T is a group.

Problem 10. Prove that the set
$$R = \{3^k \mid k \in \mathbb{Z}\}$$
forms a group under multiplication.

Proof. To prove that R forms a group under multiplication, I will show that R is closed under $+$, R is associative, has an identity element and has an inverse. R is closed because any integer raised to another integer power is another integer. To prove associativity, I must show that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. Let $k_1, k_2 \in \mathbb{Z}$. Then,
$$(a \cdot b) \cdot c = (3^k \cdot 3^{k_1}) \cdot c = (3^{k+k_1}) \cdot c = 3^{k+k_1} \cdot k^{k_2} = 3^{k+k_1+k_2}$$

and

$$a \cdot (b \cdot c) = a \cdot (3^{k_1} \cdot 3^{k_2}) = a \cdot (3^{k_1+k_2}) = 3^k \cdot 3^{k_1+k_2} = 3^{k+k_1+k_2}$$

Therefore $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. Now to I show that the identity of R is $e = 3^0$.That is, for any element, $a = 3^k$, in R , $3^k \cdot 3^0 = 3^k \cdot 1 = 3^k$. Finally I show that the inverse of any element, $a = 3^k$, in R is 3^-k. This is because $3^k \cdot 3^{-k} = 3^{k-k} = 3^0 = e$. Since all the requirements for a group is satisfied for multiplication, R must be a group under multiplication. □

Problem 11. Let S be the set of 3×3 real matricies of the form

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Show that S form a group under matrix multiplication. Is it abelian? What is the inverse of the matrix given above?

To show that S is a group under matrix multiplication, I must show that S is closed under matrix multiplication, it is associative, has an indentity element, and has an inverse.
S is closed because matrix multiplication is known to be closed. S is also associative because matrix multiplication is associative. This group can be shown to be abelian. In otherwords $AB = BA,\ A, B \in S$ is true, which I show now

$$AB = \begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a_1 & b_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a_1 + a & b_1 + b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad a_1, b_1 \in \mathbb{R}$$

$$BA = \begin{bmatrix} 1 & a_1 & b_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a_1 & b_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a + a_1 & b + b_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since AB=BA, matrix multiplication under S is commutative and therefore S is abelian
The identity element of S can be shown with $a = b = 0$.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is because for any matrix $A \in S$, $AE = A$ and $EA = A$.
Next, the inverse can be shown to exist like so,

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = rref\left(\left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \right) = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -a & -b \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

So,

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -a & -b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

S satisfies all properties of an abelian group under matrix multiplication. Therefore S is an abelian group under matrix multiplication.