Problem 1. Let

$$S = \{A \in M_n(\mathbb{R}) | \det(A) = 7^k \text{ for some integer } k\}.$$

(a) Use Corollary 4.2.2 to prove that $S \leq GL(n, \mathbb{R})$.

(b) Use Corollary 4.2.3 to prove that $S \leq GL(n, \mathbb{R})$.

(a) *Proof.* S is a nonempty set because when $k = 0$, the identity matrix, $I \in S$ because $\det(I) = 1 = 7^0$ and $I^{-1} = I$. S is closed under multiplication in $GL(n, \mathbb{R})$ because matrix multiplication is known to be closed and since $\det(AB) = \det(A)\det(B) = \det(7^a)\det(7^b) = \det(7^{a+b}),\ a, b \in \mathbb{Z},\ B \in S$ and $7^{a+b} \neq 0$, it follows that AB is invertible and has a determinant in the form 7^k which means $AB \in S$. To show that an inverse exists $\forall A \in S$, I use the fact that $A \in GL(n, \mathbb{R})$, so A is invertible and $\det(A^{-1}) = \frac{1}{\det(A)}$. It follows that since $A \in S, \det(A) = 7^n$, and that $\det(A^{-1}) = \frac{1}{7^n} = 7^{-n}$. Since A^{-1} exists and has a determinant in the form of 7^k, $A^{-1} \in S$. Since S is closed under matrix multiplication and $\forall a \in S$, $a^{-1} \in S$, by Corollary 4.2.2, S is a subset of $GL(n, R)$. □

(b) *Proof.* S is a nonempty set because when $k = 0$, the identity matrix, $I \in S$ because $\det(I) = 1 = 7^0$ and $I^{-1} = I$. Since $\det(A) = 7^k$ and $\det(B^{-1}) = \frac{1}{7^n} = 7^{-n}$ where $B \in S$ and $\det(B) = 7^n$, then $\det(AB^{-1}) = \det(A)\det(B^{-1}) = 7^k 7^{-n} = 7^{k-n}$. Since $\det(AB^{-1}) = 7^{k-n}$, AB^{-1} is invertible and takes the form 7^k. Therefore $AB^{-1} \in S$. Since $AB^{-1} \in S$, $\forall A, B \in S$, this satasfies all of the conditions for Corollary 4.2.3 and therefore S is a subgroup of $GH(n, \mathbb{R})$. □

Problem 2. Recall that

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \middle| k \in \mathbb{Z}_{120} \right\}$$

forms a group under multiplication. Prove, in the easiest way, that

$$T = \left\{ \begin{bmatrix} 1 & 0 \\ 3k & 1 \end{bmatrix} \middle| k \in \mathbb{Z}_{120} \right\}$$

is a subgroup of S.

Since $\forall a, b \in T$, $ab \in T$ because $\begin{bmatrix} 1 & 0 \\ 3k & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3n & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3(k+n) & 1 \end{bmatrix}$ and since ab takes the form $\begin{bmatrix} 1 & 0 \\ 3k & 1 \end{bmatrix}, k \in \mathbb{Z}_{120}$, this means $ab \in T$. It is also true that $\forall a \in T$, a^{-1} exists. To show that a^{-1} exists, I show that there is an element $a^{-}1$ such that $aa^{-1} = I$. If I let $n = -k$, $\begin{bmatrix} 1 & 0 \\ 3k & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3n & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3(k+-k) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3(0) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ therefore the inverse is $\begin{bmatrix} 1 & 0 \\ -3k & 1 \end{bmatrix}$. Since T is closed under S's binary operation and $\forall a \in T, a^{-1}$ exists, by Corollary 4.2.2, T is a subgroup of S.

Problem 3. Consider D_{12}, the dihedral group of degree 12.

(a) List its elements using the notation we used in class.

(b) Use the geometric interpretation of the rigid transformations to explain why

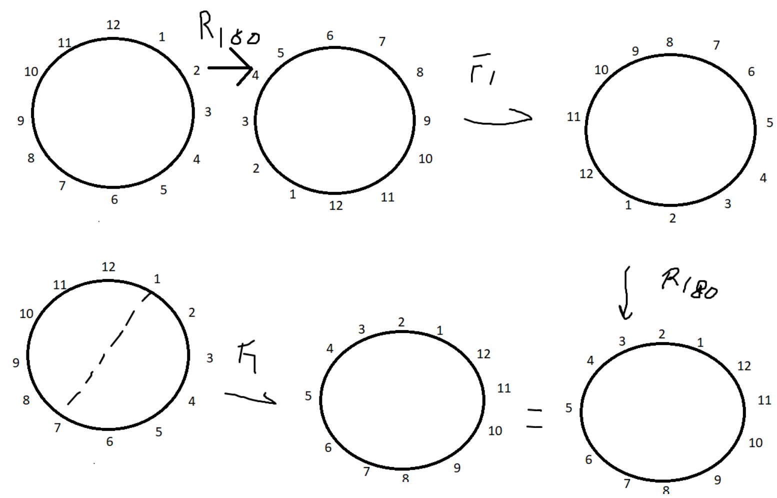
$$R_{180} F_1 R_{180} = F_1.$$

(c) Is it true that $R_{180} F_i F_{180} = F_i$for any i? Explain.

(d) Is it true that $R_{180} E_i F_{180} = E_i$ for any i? Explain.

(a)
$D_{12} = R_0, R_{30}, R_{60}, R_{90}, R_{120}, R_{150}, R_{180}, R_{210}, R_{240}, R_{270}, R_{300}, R_{330}, R_{360}, F_1, F_2, F_3, F_4, F_5, F_6, E_1, E_2, E_3, E_4, E_5, E_6$

(b)



(c) Yes because rotating 180 degrees clockwise and flipping just makes the next rotation be counterclockwise. This cancels undoes the other rotation just leaving F_i

(d) Same reasoning as (c), because rotating 180 degrees clockwise and flipping just makes the next rotation be counterclockwise. This cancels undoes the other rotation just leaving E_i

Problem 4. Prove that an abelian group G with two elements a and b of order 2 must have a subgroup H of order 4.

If G is an abelian group with 2 elements a, b with order 2, then $a^2, b^2 = e$. Since $a * a = e$ and $b * b = e$, $a^{-1} = a$ and $b^{-1} = b$. It follows that $a * b * a * b = a * a * b * b = e * e = e$ so $a * b^{-1} = a * b$ and $e^{-1} = e$. If I let $H = \langle \{e, a, b, a * b\}, * \rangle$, every element in H has an inverse and H is closed. I can show H is closed by showing all operations by left multiplication because it is an abelian group, commutativity holds. They are $e*e = e, e*a = a, e*b = b, e*(a*b) = a*a = e, a*b = a*b, a*(a*b) = e*b = b, b*b = e, b*(a*b) = b*b*a = a$. This shows H is closed. Since H is closed under the binary operation of G and for all $a \in H$, a^{-1} exists, by Corollary 4.2.2 H is a subgroup of G.

Problem 5. Consider U_{18}, the group of all the 18th roots of unity.

(a) List the elements of U_{18} in the form of ω^k for some fixed complex number ω. What should be your choice for ω? In other words, which complex number should ω be?

(b) Let $S = \{1, \omega^3, \omega^6, \omega^9, \omega^{12}, \omega^{15}\}$. Show that $S \leq U_{18}$.

(c) Let $T = \{1, \omega^6, \omega^{12}\}$. Show that $T \leq S$.

(a) $U_{18} = \{1, \omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6, \omega^7, \omega^8, \omega^9, \omega^{10}, \omega^{11}, \omega^{12}, \omega^{13}, \omega^{14}, \omega^{15}, \omega^{16}, \omega^{17}\}$, and $\omega = \text{cis}(\frac{360}{18}) = \text{cis}(20)$

(b)

$*$	1	ω^3	ω^6	ω^9	ω^{12}	ω^{15}
1	1	ω^3	ω^6	ω^9	ω^{12}	ω^{15}
ω^3	ω^3	ω^6	ω^9	ω^{12}	ω^{15}	1
ω^6	ω^6	ω^9	ω^{12}	ω^{15}	1	ω^3
ω^9	ω^9	ω^{12}	ω^{15}	1	ω^3	ω^6
ω^{12}	ω^{12}	ω^{15}	1	ω^3	ω^6	ω^9
ω^{15}	ω^{15}	1	ω^3	ω^6	ω^9	ω^{12}

Since $\forall a, b \in S$, $a * b \in S$ as shown above, is closed. Now to show $\forall a \in S, a^{-1} \in S$, I show there is an a^{-1} such that $a * a^{-1} = e$. This is when $a^-1 = \omega^{18-k}$ where $a = \omega^k$. Using Corollary 4.2.2, this proves $S \leq U_{18}$

(c)

$*$	1	ω^6	ω^{12}
1	1	ω^6	ω^{12}
ω^6	ω^6	ω^{12}	1
ω^{12}	ω^{12}	1	ω^6

Since $\forall a, b \in T$, $a * b \in T$ is closed. Now to show $\forall a \in T, a^{-1} \in T$, I show there is an a^{-1} such that $a * a^{-1} = e$. This is when $a^-1 = \omega^{18-k}$ where $a = \omega^k$. Using Corollary 4.2.2, this proves $T \leq U_{18}$

Problem 6. Let $H = \{a + bi \mid a, b \in \mathbb{R} \text{ and } a^2 + b^2 = 1\}$. Describe the elements of H geometrically. Prove or disprove that H is a subgroup of $\mathbb{C}^*$ under multiplication.

H represents the unit complex circle. H is a subgroup of $\mathbb{C}^*$ under multiplication because it satasfies all requirements of Corollary 4.2.3. First, $\forall a, b \in \mathbb{C}^*$, b^{-1} exists because I can find an element b^{-1} such that $bb^{-1} = 1 + 1i$. If $b = c + di$, $b^{-1} = \frac{1}{c} + \frac{1}{d}i$ and if I let $e = \frac{1}{c}$ and $f = \frac{1}{d}$, then $b^{-1} = e + fi$. To show that $ab^{-1} \in H$, I will multiply them out and show that the real part squared plus the coeficcient of the imaginary part squared is equal to one. That is, if $a = (x + yi)$ and $b^{-1} = e + fi$ then,

$$ab^{-1} = (x + yi)(e + fi) = -xe - xfi - yei + yf = -xe + yf + (-xf - ye)i$$

and to show

$$(-xe + yf)^2 + (-xfi - yei)^2 = 1$$

I simplify further to find

$$(-xe + yf)^2 = (xe)^2 - 2xeyf + (yf)^2$$

and

$$(-xfi - yei)^2 = (xf)^2 + 2xfye + (ye)^2$$

adding these two together I get

$$(xe)^2 + (xf)^2 + (yf)^2 + (ye)^2 = x^2(e^2 + f^2) + y^2(e^2 + f^2) = x^2(1) + y^2(1) = x^2 + y^2 = 1$$

This shows that $ab^{-1} \in H$. Since $ab^{-1} \in H$ and H is nonempty because ab^{-1} exists, by Corollary 4.2.3, H is a subgroup of $\mathbb{C}^*$ under multiplication.

Problem 7. Prove that if G is an abelian group with identity e, then all elemennts x of G satisfying the equation $x^2 = e$ form a subgroup of G. Be sure to show all the steps in your argument.

I want to show that $H = \{x \in G \mid x^2 = e\} \leq G$. To do this I will use Corollary 4.2.3. H is nonempty because it contains e. $b^{-1} = b$ because $b^2 = b*b = e$. Then, $a*b^{-1} \in H$ because $(a*b^{-1})^2 = (a*b)^2 = a*a*b*b = e*e = e$ which also means $(a * b)^{-1} = a * b$. Since H is nonempty and ab^{-1} exists for all $a, b \in H$, then by Corollary 4.2.3, H is a subgroup of G.

Problem 8. Let p and q be distinct primes. Suppose $H < \mathbb{Z}$ and H contains *exactly* three of the five elements $p, p + q, pq, p^q$, and q^p. Determine which of the following are these three elements.

The answer is (iii) $p, p + q, pq$

Problem 9. List the elements of $\langle \frac{1}{2} \rangle$ in $\langle \mathbb{Q}, + \rangle$ and in $\langle \mathbb{Q}^*, \cdot \rangle$.

The elements that $\langle Q, + \rangle$ and $\langle Q^*, \cdot \rangle$ share is equal to $\{2^n \mid n \geq -1 \ , n \in \mathbb{Z}\}$

Problem 10. Recall that $U(20) = \mathbb{Z}_{20}^*$. What are its elements? How can you check whether $U(20)$ is cyclic?

The elements are

$$U(20) = \{7, 9, 11, 13, 17, 19\}$$

I could check to see whether $U(20)$ is cyclic by checking if there is a generator in $U(20)$ that generates $U(20)$. If there is, it is cyclic, but if there isn't it is not.

Problem 11. The group $U(15)$ has six cyclic subgroups. List them.

The cyclic subgroups of $U(15)$ are

$$\begin{aligned}
\langle 1 \rangle &= \{1\} \\
\langle 2 \rangle &= \{2, 4, 8, 1\} \\
\langle 4 \rangle &= \{4, 1\} \\
\langle 7 \rangle &= \{7, 4, 13, 1\} \\
\langle 11 \rangle &= \{11, 1\} \\
\langle 14 \rangle &= \{14, 1\}
\end{aligned}$$