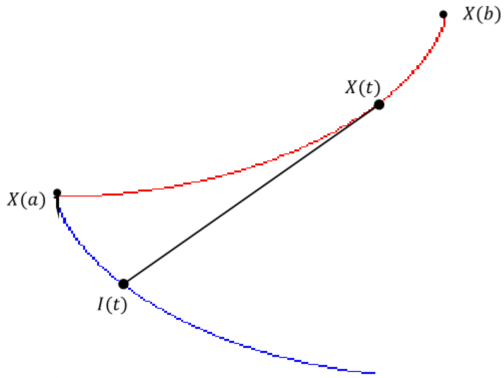
Problem 1. Show that

$$\vec{I}(t) = \vec{X} - s\vec{T}$$



Solution:

It is important to see in the diagram that the line created between $X(t)$ and $I(t)$ is the same length as the arc length between $X(a)$ and $X(t)$ which is equal to $s(t)$. Notice that the line is also in the opposite direction of the tangent vector at $X(t)$ since the function is increasing from $a \leq t \leq b$. This line can be written as $-s\vec{T}(t)$ because $\vec{T}(t)$ is the tangent unit vector and we multiply by -1 because the line is in the opposite direction. Then we multiply by $s(t)$ because that is its magnitude or length. Now, since $\vec{X}(t)$ has its tail where $-s\vec{T}(t)$ has its head the sum is a vector from $\vec{X}(a)$ to $\vec{I}(t)$ which draws the blue path. In other words, this shows that $\vec{I}(t) = \vec{X}(t) + (-s\vec{T}(t)) = \vec{X}(t) - s\vec{T}(t)$.

Problem 2. Show that the involute of the section of cycloid given by

$$x = a(\theta - \sin(\theta))$$

$$y = a(1 - \cos(\theta))$$

from the point $X(0) = (0,0)$ is given by

$$I(t) = \begin{bmatrix} a(\theta + \sin(\theta)) \\ a(3 + \cos(\theta)) \end{bmatrix}, -\pi \leq \theta \leq 0$$

Solution:

Since I have shown that $\vec{I}(t) = \vec{X}(t) - s(t)\vec{T}(t)$, I will use this fact. I am given that

$$x = a(\theta - \sin(\theta))$$

$$y = a(1 - \cos(\theta))$$

$$-\pi \leq \theta \leq 0$$

Using substitution I get

$$\vec{I}(\theta) = \vec{X}(\theta) - s(\theta)\vec{T}(\theta) = \begin{bmatrix} a(\theta - \sin(\theta)) \\ a(1 - \cos(\theta)) \end{bmatrix} - s(\theta)\vec{T}(\theta)$$

Then, the arclength is defined as

$$s(\theta) = \int_{\theta=-\pi}^{\theta} \sqrt{\left(\frac{dx}{d\theta}^2\right) + \left(\frac{dy}{d\theta}^2\right)}\, d\theta$$

$$= \int_{\theta=-\pi}^{\theta} \sqrt{(a - a\cos(\theta))^2 + (a\sin(\theta))^2}$$

$$= \int_{\theta=-\pi}^{\theta} \sqrt{a^2 - 2a^2\cos(\theta) + (a\cos(\theta))^2 + (a\sin(\theta))^2}\, d\theta$$

$$= \int_{\theta=-\pi}^{\theta} \sqrt{2a^2 - 2a^2\cos(\theta)}\, d\theta$$

$$= \int_{\theta=-\pi}^{\theta} \sqrt{2a^2(1 - \cos(\theta))}\, d\theta$$

$$= \sqrt{2a} \int_{\theta=-\pi}^{\theta} \sqrt{1 - \cos(\theta)}\, d\theta$$

$$= 2a \int_{\theta=-\pi}^{\theta} \sqrt{2\sin^2(\frac{\theta}{2})}\, d\theta$$

$$= 2a \int_{\theta=-\pi}^{\theta} \left|\sin(\frac{\theta}{2})\right|\, d\theta = 2a \int_{\theta=-\pi}^{\theta} \sin(\frac{\theta}{2})\, d\theta = 4a\cos(\frac{\theta}{2})$$

Plugging this function into the equation I get

$$\vec{I}(t) = \begin{bmatrix} a(\theta - \sin(\theta)) \\ a(1 - \cos(\theta)) \end{bmatrix} - 4a\cos(\frac{\theta}{2})\vec{T}(t)$$

By defintion, $\vec{T}(t) = \frac{\frac{d\vec{X}}{d\theta}}{\left|\frac{d\vec{X}}{d\theta}\right|}$. So, differentiating and plugging in values I get

$$\vec{T}(t) = \begin{bmatrix} \frac{a(1-\cos(\theta))}{-2a\sin(\frac{\theta}{2})} \\ \frac{a\sin(\theta)}{-2a\sin(\frac{\theta}{2})} \end{bmatrix}$$

Finally, substituting all values I get

$$\vec{I}(t) = \begin{bmatrix} a(\theta - \sin(\theta)) \\ a(1 - \cos(\theta)) \end{bmatrix} - 4a\cos(\frac{\theta}{2}) \begin{bmatrix} \frac{a(1-\cos(\theta))}{-2a\sin(\frac{\theta}{2})} \\ \frac{a\sin(\theta)}{-2a\sin(\frac{\theta}{2})} \end{bmatrix} = \begin{bmatrix} a(\theta + \sin(\theta)) \\ a(3 + \cos(\theta)) \end{bmatrix}.$$

Problem 3. Use substitution $\alpha = \theta + \pi$ to show that this involute is a section of cycloid shifted by the vector

$$\begin{bmatrix} -\pi a \\ 2a \end{bmatrix}$$

Solution:

$$
\begin{aligned}
\vec{I}(t) &= \begin{bmatrix} a(\alpha-\pi)+\sin(\alpha-\pi) \\ a(3+\cos(\alpha-\pi)) \end{bmatrix} \\
&= \begin{bmatrix} a(\alpha-\pi)+\sin(\alpha)\cos(\pi)-\cos(\alpha)\sin(\pi) \\ a(3+(\cos(\alpha)\cos(\pi)+\sin(\alpha)\sin(\pi))) \end{bmatrix} \\
&= \begin{bmatrix} a(\alpha-\pi)-\sin(\alpha) \\ a(3-\cos(\alpha)) \end{bmatrix} \\
&= \begin{bmatrix} \alpha^2-\alpha\pi-a\sin(\alpha) \\ 3a-a\cos(\alpha) \end{bmatrix} \\
&= \begin{bmatrix} a(\alpha-\sin(\alpha)) \\ a(1-\cos(\alpha)) \end{bmatrix} + \begin{bmatrix} -\pi a \\ 2a \end{bmatrix}, 0 \leq \alpha \leq \pi
\end{aligned}
$$