Problem 1. Express the complex number $\frac{4+7i}{2-5i}$ in polar form. First write the answer in exact form, *then* approximate to 2 decimal places. You may leave the angle in degrees.

First, I start with by getting the number into $a+bi$ form. To do this I multiply by the conjugate

$$\frac{4+7i}{2-5i}\cdot\frac{2+5i}{2+5i}=\frac{-27+34i}{29}=\frac{-27}{29}+\frac{34i}{29}$$

Now I need to get this in the exact form form $z=r(\cos\theta+i\sin\theta)$.

$$\theta=180^\circ-\arg\left(\frac{-27}{29}+\frac{34}{29}i\right)=180^\circ-\tan^{-1}\left(\frac{34}{27}\right)=180^\circ-51.55^\circ=128.45^\circ$$

$$r=\sqrt{\left(\frac{-27}{29}\right)^2+\left(\frac{34}{29}\right)^2}=\sqrt{\frac{65}{29}}$$

Therefore $z=\sqrt{\frac{65}{29}}\cdot\text{cis}(180^\circ-\tan^{-1}\left(\frac{34}{27}\right))=1.50\cdot\text{cis}\big(128.45^\circ\big)=-1.40+5.21i$

Problem 2. Let $z=\sqrt{2}(\cos 32^\circ+i\sin 32^\circ)$. Find the *exact value* of z^{15}, which means you should leave your answer in terms of radical, sine and cosine functions. However, simplifiy your answer as much as possible.

By using De Moivre's Theorem:

$$z^n=(r\text{cis}(\theta))^n=r^n\text{cis}(n\theta)$$

I can find z^{15} by substituting n,θ and r and expanding. It follows that

$$\begin{aligned}z^{15}&=\sqrt{2}^{15}\cdot(\cos(15\cdot 32)+i\sin(15\cdot 32))\\&=\sqrt{2}^{15}\cdot(\cos(480)+i\sin(480))\end{aligned}$$

Problem 3. Approximate, in 4 decimal places, all of the 5th roots of $-17-26i$.

By using De Moivre's Theorem I can use the fact

$$z^{\frac{1}{n}}=r^{\frac{1}{n}}\cdot\text{cis}\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)$$

I can find $(-17-26i)^{\frac{1}{5}}$ by substituting $r,n,k,$ and θ with the corresponding values:

$$\theta=\tan^{-1}(26/17)+\pi$$

$$r=\sqrt{(-17)^2+(-26)^2}=\sqrt{965}$$

$$n=5$$

$$k=0,1,2,...,n-1$$

Substituting, I get the roots to be

$$\begin{aligned}z^{\frac{1}{5}}&=\sqrt{965}^{\frac{1}{5}}\text{cis}\left(\frac{\tan^{-1}(\frac{26}{17})}{5}+\frac{0\cdot\pi}{5}\right)=1.3467+1.4626i\\z^{\frac{1}{5}}&=\sqrt{965}^{\frac{1}{5}}\text{cis}\left(\frac{\tan^{-1}(\frac{26}{17})}{5}+\frac{2\cdot\pi}{5}\right)=-0.9749+1.7327i\\z^{\frac{1}{5}}&=\sqrt{965}^{\frac{1}{5}}\text{cis}\left(\frac{\tan^{-1}(\frac{26}{17})}{5}+\frac{4\cdot\pi}{5}\right)=-1.9492-0.3918i\\z^{\frac{1}{5}}&=\sqrt{965}^{\frac{1}{5}}\text{cis}\left(\frac{\tan^{-1}(\frac{26}{17})}{5}+\frac{6\cdot\pi}{5}\right)=-0.2297-1.9748i\\z^{\frac{1}{5}}&=\sqrt{965}^{\frac{1}{5}}\text{cis}\left(\frac{\tan^{-1}(\frac{26}{17})}{5}+\frac{8\cdot\pi}{5}\right)=1.8072-0.8288i\end{aligned}$$

Problem 4. Use the binomial theorem to expand $(\cos\theta + i\sin\theta)^5$. Use the result to express $\cos 5\theta$ and $\sin 5\theta$ in terms of $\cos\theta$ and $\sin\theta$.

Expanding $(\cos\theta + i\sin\theta)^5$ I get

$$(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5i\sin\theta\cos^4\theta - 10\sin^2\theta\cos^3\theta - 10i\sin^3\theta\cos^2\theta + 5\sin^4\theta\cos\theta + i\sin^5\theta$$

Using De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

I show that

$$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$$

so I can separate the real parts and imaginary parts like so

$$\cos 5\theta = \cos^5\theta - 10\sin^2\theta\cos^3\theta + 5\sin^4\theta\cos\theta$$

and

$$\sin 5\theta = 5\sin\theta\cos^4\theta - 10\sin^3\theta\cos^2\theta + \sin^5\theta$$

Problem 5. Let a be a fixed number. Use induction to show that

$$\begin{bmatrix}1 & a\\0 & 1\end{bmatrix}^n = \begin{bmatrix}1 & an\\0 & 1\end{bmatrix}$$

for all integers $n \geq 0$.

Proof. Using the steps of induction I show that this holds for the base case $n = 1$

$$\begin{bmatrix}1 & a\\0 & 1\end{bmatrix}^1 = \begin{bmatrix}1 & a\cdot 1\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & a\\0 & 1\end{bmatrix}$$

Now, I show that for any $n = k, k \geq 0$ that if this holds for $n = k$, then this also holds for $n = k+1$. To do this, I assume the induction hypothesis is true for some $n = k, k \geq 0$, then show the induction hypothesis is true for $n = k+1$. Using the induction hypothesis, I get

$$\begin{bmatrix}1 & a\\0 & 1\end{bmatrix}^{k+1} = \begin{bmatrix}1 & a\cdot(k+1)\\0 & 1\end{bmatrix}.$$

Simplifying this matrix, I get

$$\begin{bmatrix}1 & a\\0 & 1\end{bmatrix}^{k+1} = \begin{bmatrix}1 & a\\0 & 1\end{bmatrix}^k \begin{bmatrix}1 & a\\0 & 1\end{bmatrix}.$$

So, I want to show that

$$\begin{bmatrix}1 & a\\0 & 1\end{bmatrix}^k \begin{bmatrix}1 & a\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & a\cdot(k+1)\\0 & 1\end{bmatrix}.$$

Multiplying out the L.H.S I get

$$\begin{bmatrix}1 & ak\\0 & 1\end{bmatrix} \begin{bmatrix}1 & a\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & a+ak\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & a\cdot(k+1)\\0 & 1\end{bmatrix}$$

Therefore, this shows that induction holds for $n = k+1$. Since the base case holds and the inductive step holds, by mathematical induction

$$\begin{bmatrix}1 & a\\0 & 1\end{bmatrix}^n = \begin{bmatrix}1 & an\\0 & 1\end{bmatrix}$$

holds for all positive integers n. $\square$

Problem 6. Evaluate

$$\text{(a)} \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}^{-1} \pmod{11} \qquad \text{(b)} \begin{bmatrix} 1+i & 2-3i \\ 3-i & 1+2i \end{bmatrix}^{-1}$$

$$\begin{aligned} \text{(a)} \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}^{-1} &\equiv \frac{1}{-7} \begin{bmatrix} 4 & -3 \\ -5 & 4 \end{bmatrix} \equiv (-7)^{-1} \begin{bmatrix} 4 & -3 \\ -5 & 4 \end{bmatrix} \\ &\equiv 3 \begin{bmatrix} 4 & -3 \\ -5 & 4 \end{bmatrix} \equiv \begin{bmatrix} 1 & 2 \\ 7 & 6 \end{bmatrix} \pmod{11} \end{aligned}$$

$$\text{(b)} \begin{bmatrix} 1+i & 2-3i \\ 3-i & 1+2i \end{bmatrix}^{-1} = \frac{1}{-4+14i} \begin{bmatrix} 1+2i & -2+3i \\ -3+i & 1+i \end{bmatrix}$$

Problem 7. Find the cube roots of the matrix $\begin{bmatrix} -1 & -3 \\ 3 & -1 \end{bmatrix}$ in 3 decimal places.

Since $\begin{bmatrix} -1 & -3 \\ 3 & -1 \end{bmatrix}$ is isomorphic to $-1+3i$, I can find the cube roots of $-1+3i$ and show the matrix representation of those roots so the problem becomes a lot easier. To solve the for the cube roots of this complex number I will use the property

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} \operatorname{cis}\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right).$$

So if I let $z = -1+3i$, then $\theta = arg(z) = \pi - \tan^{-1}(3)$ and $r = \|z\| = \sqrt{10}$. Then, substituting the values I get

$$z^{\frac{1}{3}} = \sqrt[3]{\sqrt{10}} \operatorname{cis}\left(\frac{\pi - \tan^{-1}(3)}{3} + \frac{2k\pi}{3}\right), \; k = 0,1,2$$

These roots are

$$\begin{aligned} z^{\frac{1}{3}} &= 1.185 + 0.866i \\ z^{\frac{1}{3}} &= -1.342 + .0594i \\ z^{\frac{1}{3}} &= 0.157 + 1.459i \end{aligned}$$

or in matrix form

$$z^{\frac{1}{3}} = \begin{bmatrix} 1.185 & -0.866 \\ 0.866 & 1.185 \end{bmatrix}$$

$$z^{\frac{1}{3}} = \begin{bmatrix} -1.342 & -.0594 \\ .0594 & -1.342 \end{bmatrix}$$

$$z^{\frac{1}{3}} = \begin{bmatrix} 0.157 & -1.459 \\ 1.459 & 0.157 \end{bmatrix}$$

Problem 9. Complete the following transformation and operation tables for D_4:

	1	2	3	4
R_0	1	2	3	4
R_{90}	4	1	2	3
R_{180}	3	4	1	2
R_{270}	2	3	4	1
F_1	1	4	3	2
F_2	3	2	1	4
E_1	2	1	4	3
E_2	4	3	2	1

$\cdot$	R_0	R_{90}	R_{180}	R_{270}	F_1	F_2	E_1	E_2
R_0	R_0	R_{90}	R_{180}	R_{270}	F_1	F_2	E_1	E_2
R_{90}	R_{90}	R_{180}	R_{270}	R_0	E_2	E_1	F_1	F_2
R_{180}	R_{180}	R_{270}	R_0	R_{90}	F_2	F_1	E_2	E_1
R_{270}	R_{270}	R_0	R_{90}	R_{180}	E_1	E_2	F_2	F_1
F_1	F_1	E_1	F_2	E_2	R_0	R_{180}	R_{90}	R_{270}
F_2	F_2	E_2	F_1	E_1	R_{180}	R_0	R_{270}	R_{90}
E_1	E_1	F_2	E_2	F_1	R_{270}	R_{90}	R_0	R_{180}
E_2	E_2	F_1	E_1	F_2	R_{90}	R_{270}	R_{180}	R_0

Problem 10. Using the table from the last problem, compute the value of

$$R_{90}F_2R_{270}, \quad F_2R_{90}E_1, \quad E_2F_1F_2E_1 \quad \text{and} \quad R_{180}R_{90}F_1E_1R_{270}$$

$$R_{90}F_2R_{270} = R_{90}E_1 = F_1$$
$$F_2R_{90}E_1 = F_2F_1 = R_{180}$$
$$E_2F_1F_2E_1 = E_2F_1R_{270} = E_2E_2 = R_0$$
$$R_{180}R_{90}F_1E_1R_{270} = R_{180}R_{90}F_1F_1 = R_{180}R_{90}R_0 = R_{180}R_{90} = R_{270}$$