Problem 1. Suppose that $\vec{F}$ is an n-dimensional vector field defined on $\mathbb{R}^n$ and let $\vec{X} = \vec{X}(t)$, $t_0 \leq t \leq t_1$ represent some curve in $\mathbb{R}^n$. Suppose further that $\phi : \mathbb{R}^n \to \mathbb{R}$ is a scalar function such that $\nabla\phi = \vec{F}$. Show that the work done by $\vec{F}$ in moving a particle along the curve is given by

$$\phi(X(t_1)) - \phi(X(t_0))$$

Solution:

The work of the particle moving along the path can be described as

$$\int_{X(t_0)}^{X(t_1)} \vec{F} \cdot d\vec{X} = \int_{X(t_0)}^{X(t_1)} \nabla\phi \cdot d\vec{X}$$

and since

$$\nabla\phi \cdot d\vec{X} = \begin{bmatrix} \frac{\partial\phi}{\partial x_1} \\ \vdots \\ \frac{\partial\phi}{\partial x_n} \end{bmatrix} \cdot \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix} = (\frac{\partial\phi}{\partial x_1}dx_1 + \cdots + \frac{\partial\phi}{\partial x_n}dx_n) = d\phi$$

by substitution I get

$$\int_{X(t_0)}^{X(t_1)} d\phi = \phi\Big|_{X=X(t_0)}^{X(t_1)} = \phi(X(t_1)) - \phi(X(t_0))$$

So it is clear that the work done by the particle moving along the path is

$$\phi(X(t_1)) - \phi(X(t_0))$$

Problem 2. Suppose $\nabla\psi = y^2 - 2xyz^3\mathbf{i} + 3 + 2xy - x^2z^3\mathbf{j} + 6z^3 - 3x^2yz^2\mathbf{k}$. Find ψ.

Solution:

Since $\nabla\psi$ is written as

$$\nabla\psi = \frac{\partial\psi}{\partial x}\mathbf{i} + \frac{\partial\psi}{\partial y}\mathbf{j} + \frac{\partial\psi}{\partial z}\mathbf{k},$$

if I integrate the first component I get

$$\psi = \int \frac{\partial\psi}{\partial x}dx = \int y^2 - 2xyz^3 dx = y^2x - x^2yz^3 + f(y,z).$$

Now, if I differentiate ψ with respect to y, I get

$$\frac{\partial\psi}{\partial y} = 2xy - x^2z^3 + \frac{\partial f(y,z)}{\partial y}$$

It is also given that

$$\frac{\partial\psi}{\partial y} = 3 + 2xy - x^2z^3$$

Setting these terms equal I get,

$$\frac{\partial\psi}{\partial y} = 2xy - x^2z^3 + \frac{\partial f(y,z)}{\partial y} = 3 + 2xy - x^2z^3 \implies \frac{\partial f(y,z)}{\partial y} = 3.$$

Now, integrating this derivative with respect to y I get

$$f(y,z)=\int \frac{\partial f(y,z)}{\partial y}dy = 3y+f(z)$$

and substituting $f(y,z)$ back into ψ I get

$$\psi = y^2x - x^2yz^3 + 3y + f(z).$$

Now, differentiating ψ with respect to z I get

$$\frac{d\psi}{dz} = 3x^2yz^2 + \frac{df(z)}{dz}.$$

It is also given that

$$\frac{d\psi}{dz} = 6z^3 - 3x^2yz^2.$$

Setting these two equal I get

$$\frac{d\psi}{dz} = 3x^2yz^2 + \frac{df(z)}{dz} = 6z^3 - 3x^2yz^2 \implies \frac{df(z)}{dz} = 6z^3.$$

Now, integrating $\frac{df(z)}{dz}$ with respect to dz I get

$$f(z) = \int \frac{df(z)}{dz}dz = \frac{3z^4}{2} + c.$$

Finally, subsituting $f(z)$ into ψ I get

$$\psi = y^2x - x^2yz^3 + 3y + \frac{3z^4}{2} + c.$$

Problem 3a. Let $X = (x_1, x_2, \cdots, x_n)$ represent a generic point in $\mathbb{R}^n$ and let $P = (p_1, p_2, \cdots, p_n)$ be a fixed point. Let $\vec{U} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ be a unit vector. Finally, let $\phi = \phi(X)$ be a real valued function of n variables. If we parameterize the line through P with direction $\vec{U}$ by $\vec{X}(t) = P + t\vec{U}$, then ultimately, $\phi(X(t))$ is a function of t. Show that

$$\left.\frac{d\phi}{dt}\right|_{t=0} = \nabla\phi(P)\cdot\vec{U}$$

Solution:

I can rewrite $d\phi$ as $\nabla\phi \cdot d\vec{X}$ so,

$$\frac{d\phi}{dt} = \nabla\phi\cdot\frac{d\vec{X}}{dt}$$

and $\frac{d\vec{X}}{dt} = \vec{U}$, so when $\frac{d\phi}{dt}$ is evaluated at $t=0$ I get

$$\left.\frac{d\phi}{dt}\right|_{t=0} = \nabla\phi(\vec{X}(0))\cdot\left.\frac{d\vec{X}}{dt}\right|_{t=0} = \nabla\phi(P)\cdot\vec{U}.$$

Problem 3b. Use the Cauchy Schwarz Inequality to show that

$$|D_{\vec{U}}(\phi)| \leq |\nabla\phi|$$

and that this maximum rate of change occurs in the direction $\frac{\nabla\phi}{|\nabla\phi|}$

Solution:

The Cauchy Schwarz Inequalty states that given vectors $\vec{V}, \vec{W}$, that $|\vec{V} \bullet \vec{W}| \leq |V| \cdot |W|$. Using this inequality and the definition of the directional derivitive, it must be true that

$$\begin{aligned} &|\nabla\phi \bullet \vec{U}| \leq |\nabla\phi| \cdot |\vec{U}| \\ &\implies |\nabla\phi \bullet \vec{U}| \leq |\nabla\phi| \cdot 1 \\ &\implies |D_{\vec{U}}(\phi)| \leq |\nabla\phi| \end{aligned}$$

It must also be true that in the direction $\vec{U} = \frac{\nabla\phi}{|\nabla\phi|}$ is when the maximum rate of change occurs because it is the upper bound to the inequality $|D_{\vec{U}}(\phi)| \leq |\nabla\phi|$. To show this I subsitute $\vec{U}$ for $\frac{\nabla\phi}{|\nabla\phi|}$

$$|D_{\frac{\nabla\phi}{|\nabla\phi|}}(\phi)| = \left|\nabla\phi \bullet \frac{\nabla\phi}{|\nabla\phi|}\right| = |\nabla\phi| \cdot \left|\frac{\nabla\phi}{|\nabla\phi|}\right| \cdot \cos(0) = |\nabla\phi|$$

Problem 4. Consider the $n-1$ dimensional (hyper) surface in $\mathbb{R}^n$ given by $\phi(x_1, x_2, \cdots, x_n) = c$ where c is a constant. Show that at any point on the surface, $\nabla\phi$ is orthogonal to the surface.

Solution:

Let $\vec{X}(t)$ represent any curve lying on the surface $\phi = c$. Then,

$$\phi(\vec{X}(t)) = c$$

and now $\phi(\vec{X}(t))$ is a function of one variable. So, differentiating both sides of this equation I get,

$$\frac{d\phi}{dt} = \frac{d\phi(\vec{X}(t))}{dt} = \nabla\phi \bullet \frac{d\vec{X}}{dt} = \frac{dc}{dt} = 0$$

Since I have shown that $\nabla\phi \bullet \frac{d\vec{X}}{dt} = 0$, I have shown that $\nabla\phi \perp \frac{d\vec{X}}{dt}$ which implies that $\nabla\phi \perp \phi(\vec{X}(t))$

Problem 5. Find an equation for the tangent plane to the surface $xz^2 + x^2y = z - 1$ at the point $(1, -3, 2)$

Solution:

An equation for a plane tangent to the surface, $w = f(x,y,z)$, is $w = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)$. Using the function and points I am given it is true that

$$\begin{aligned} &f_x = z^2 + 2xy, f_x(1,-3,2) = -2 \\ &f_y = x^2, f_y(1,-3,2) = 1 \\ &f_z = 2xz - 1, f_z(1,-3,2) = 3 \\ &f(1,-3,2) = -1 \end{aligned}$$

and by substitution I get an equation for a plane perpendicular to the point (1,-3,2) on the surface to be

$$w = -1 + -2(x-1) + 1(y+3) + 3(z-2) = -2x + y + 3z - 2$$