Problem 1. Let G be an abelian group of order 1200.

(a) List its isomorphism classes in the form $\bigoplus_{i=1}^{k} \mathbb{Z}_{m_i}$ where each m_i is a prime-power

(b) List its isomorphism classes in the form $\bigoplus_{i=1}^{k} \mathbb{Z}_{m_i}$ in which $m_{i+1} \mid m_i$ for $1 \leq i \leq k-1$

(c) If G contains an element a of order at least 300, what could it be isomporphic to? Explain! Express the answers in the form specified in (b).

Solution:

(a) Since $|G| = 1200 = 2^4 \cdot 3^1 \cdot 5^2$, the isomorphism classes are:

- $\mathbb{Z}_{16} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{25}$
- $\mathbb{Z}_{16} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$
- $\mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{25}$
- $\mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$
- $\mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{25}$
- $\mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$
- $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$
- $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{25}$
- $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{25}$
- $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$

(b)
- $\mathbb{Z}_{1200}$
- $\mathbb{Z}_{600} \oplus \mathbb{Z}_2$
- $\mathbb{Z}_{300} \oplus \mathbb{Z}_4$
- $\mathbb{Z}_{300} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$
- $\mathbb{Z}_{240} \oplus \mathbb{Z}_5$

(c) If G contains an element of order at least 300, that means that the order of each group in the isomorphism classes must be at least 300. This is when $G \cong \mathbb{Z}_{16} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{25}$, and $G \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{25}$.

Problem 2. How many isomorphism classes are there for an abelian group of order p^6, where p is a prime? List them explicitly.

Solution:

The isomorphism classes for an abelian group of order p^6 are, $\mathbb{Z}_{p^6}, \mathbb{Z}_{p^5} \oplus \mathbb{Z}_p, \mathbb{Z}_{p^4} \oplus \mathbb{Z}_{p^2}$, and $\mathbb{Z}_{p^3} \oplus \mathbb{Z}_{p^3}$.

Problem 3. Consider the subgroup $H = \left\langle \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \right\rangle$ of the group.

$$G = \left\{ \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \;\middle|\; k \in \mathbb{Z}_{24} \right\}.$$

Describe its left cosets by naming them with proper notation, and listing their elements explicitly.

Solution:

It can be shown that $G \cong \mathbb{Z}_{24}$ so $|G| = 24$ and $\text{ord}_{\mathbb{Z}_{24}}(\langle 4\rangle) = 6 = \text{ord}_G(\langle \left[\begin{smallmatrix}1&4\\0&1\end{smallmatrix}\right]\rangle)$. Using this, I know that the index, $(G:H) = \frac{|G|}{|H|} = \frac{24}{6} = 4$. Those 4 elements are:

$$\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]H = \{\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1&4\\0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1&8\\0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1&12\\0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1&16\\0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1&20\\0&1\end{smallmatrix}\right]\},$$

$$\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]H = \{\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1&5\\0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1&9\\0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1&13\\0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1&17\\0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1&21\\0&1\end{smallmatrix}\right]\},$$

$$\left[\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right]H = \{\left[\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1&6\\0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1&10\\0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1&14\\0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1&18\\0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1&22\\0&1\end{smallmatrix}\right]\},$$

$$\left[\begin{smallmatrix}1&3\\0&1\end{smallmatrix}\right]H = \{\left[\begin{smallmatrix}1&3\\0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1&7\\0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1&11\\0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1&15\\0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1&19\\0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1&23\\0&1\end{smallmatrix}\right]\}.$$

Problem 4. List the cosets of $H = \langle (4,2),(2,3)\rangle$ in $\mathbb{Z}_6 \oplus \mathbb{Z}_4$. Display the elements in the lexicographic or dictionary order. Be sure to use proper notation.

Solution:

$$H = \{(4,2),(2,0),(0,2),(4,0),(2,2),(0,0),(2,3),(0,1),(4,3),(2,1),(0,3),(4,1)\}$$

and there is one other coset which is:

$$(1,0)H = \{(5,2),(3,0),(1,2),(5,0),(3,2),(1,0),(3,3),(1,1),(5,3),(3,1),(1,3),(5,1)\}.$$

Problem 5. Find the left cosets of the subgroup $K = \{R_0, F_1\}$ in D_4. Write each coset with R_i first if it can be found in the coset. Arrange the elements in the Cayley table of D_4 according to the order in which they appear in the cosets, and shade the table as it was done in the textbook.

Solution:

The order of a dihedral group D_n is $2n$. So the order of D_4 is 8. Also, F_1 generates K with order 2. So, $(D_4 : K) = \frac{8}{2} = 4$. So there are 4 left cosets. They are represented as so.

$\circ$	R_0	F_1	R_{90}	E_2	R_{180}	F_2	R_{270}	E_1
R_0	R_0	F_1	R_{90}	E_2	R_{180}	F_2	R_{270}	E_1
F_1	F_1	R_0	E_1	R_{270}	F_2	R_{180}	E_2	R_{90}
R_{90}	R_{90}	E_2	R_{180}	F_2	R_{270}	E_1	R_0	F_1
E_2	E_2	R_{90}	F_1	R_0	E_1	R_{270}	F_2	R_{180}
R_{180}	R_{180}	F_2	R_{270}	E_1	R_0	F_1	R_{90}	E_2
F_2	F_2	R_{180}	E_2	R_{90}	F_1	R_0	E_1	R_{270}
R_{270}	R_{270}	E_1	R_0	F_1	R_{90}	E_2	R_{180}	F_2
E_1	E_1	R_{270}	F_2	R_{180}	E_2	R_{90}	F_1	R_0

Problem 6. Find the right cosets of the subgroup $K = \{R_0, F_1\}$ in D_3. Write each coset with R_i first if it can be found in the coset. Arrange the elements in the Cayley table of D_3 according to the order in which they appear in the cosets, and shade the table as it was done in the textbook.

Solution:

The order of a dihedral group D_n is $2n$. So the order of D_3 is 6. Also, F_1 generates K with order 2. So, $(D_3 : K) = \frac{6}{2} = 3$. So there are 3 left cosets. They are represented as so.

$\circ$	R_0	F_1	R_{120}	F_3	R_{240}	F_2
R_0	R_0	F_1	R_{120}	F_3	R_{240}	F_2
F_1	F_1	R_0	F_3	R_{120}	F_2	R_{240}
R_{120}	R_{120}	F_2	R_{240}	F_1	R_0	F_3
F_3	F_3	R_{240}	F_2	R_0	F_1	R_{120}
R_{240}	R_{240}	F_3	R_0	F_2	R_{120}	F_1
F_2	F_2	R_{120}	F_1	R_{240}	F_3	R_0

Problem 7. Find the index of $\langle 3 \rangle$ in the group $\mathbb{Z}_{24}$. How many different cosets does $\langle 3 \rangle$ have?

Solution:

The order of $\mathbb{Z}_{24}$ is 24 and $\langle 3 \rangle = \{3, 6, 9, 12, 15, 18, 21, 0\}$, so $|\langle 3 \rangle| = 8$. Therefore the index of $\langle 3 \rangle$ is $(\mathbb{Z}_{24} : \langle 3 \rangle) = \frac{24}{8} = 3$. This means that there are 3 left cosets. Since $\mathbb{Z}_{24}$ is abelian, $a\langle 3 \rangle = \langle 3 \rangle a,\ \forall a \in \mathbb{Z}_{24}$. So, $\langle 3 \rangle$ only has 3 different cosets.

Problem 8. Let $\sigma = (1, 2, 5)(2, 3)$ in S_5. Find the index of $\langle \sigma \rangle$ in S_5.

Solution:

$|\langle (1,2,5)(2,3) \rangle| = \left|\langle \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{smallmatrix}\right) \rangle\right| = \left|\left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{smallmatrix}\right)\right| = \text{LCM}(4, 1) = 4$ and $|S_5| = 5!$, so the index of $|\langle (1,2,5)(2,3) \rangle|$ is $(S_5 : \langle (1,2,5)(2,3) \rangle) = \frac{120}{4} = 30$.

Problem 9. Let H be a normal subgroup of G, and let $m = (G : H)$. Show that $\forall a \in G,\ a^m \in H$.

Solution:

Since H is a normal subgroup it forms the factor group G/H. The index is its order which is m. So, $(aH)^m = a^m H = H$. Using theorem 10.1.4, it is true that $aH = H$ iff $a \in H$. Since I have shown that $a^m H = H$, this shows that $a^m \in H$.

Problem 10. Evaluate the order of the element $26 + \langle 12 \rangle$ in the factor group $\mathbb{Z}_{60}/\langle 12 \rangle$.

Solution:

$\langle 12\rangle = \{0, 12, 24, 36, 48\}$ and $26 + \langle 12\rangle = \{26, 38, 50, 2, 14\}$. I need to find an n such that $n26 = H$. I can eliminate sets easily if they do not share any elements with H because if so, they cannot possibly be H. So, $2 \cdot 26 (\text{mod } 60) \equiv 52$ which is not in H, $3 \cdot 26 (\text{mod } 60) \equiv 18$ which is not in H, $4 \cdot 26 (\text{mod } 60) \equiv 44$ which is not in H, and $5 \cdot 26 (\text{mod } 60) \equiv 10$ which is not in H, but $6 \cdot 26 (\text{mod } 60) \equiv 36$ which is in H, so checking to see if all elements are the same I get $6 \cdot 26 = \{36, 48, 0, 12, 24\}$. Therefore 6 is the order of $26 + \langle 12\rangle$.

Problem 11. Find the order of the factor group $(\mathbb{Z}_{12} \oplus \mathbb{Z}_{18})/\langle(4, 3)\rangle$.

Solution:

The order of a factor group is its index. The order of $\mathbb{Z}_{12} \oplus \mathbb{Z}_{18}$ is equal to $|\mathbb{Z}_{12}| \cdot |\mathbb{Z}_{18}| = 216$. $\langle(4, 3)\rangle = \{(4, 3), (8, 6), (0, 9), (4, 12), (8, 15), (0, 0)\}$, so the order of $\langle(4, 3)\rangle$ is 6. Hence, the index is $(\mathbb{Z}_{12} \oplus \mathbb{Z}_{18} : \langle(4, 3)\rangle) = \frac{216}{6} = 36$ so, $|(\mathbb{Z}_{12} \oplus \mathbb{Z}_{18})/\langle(4, 3)\rangle| = 36$.

Problem 12. Let H be a normal subgroup of G, and let $a \in G$. If the coset aH has order 3 in the factor group G/H, and $|H| = 10$, what are the possible orders of a in G?

Solution:

Since aH has order 3, $a^3H = H$. So, $a^3 \in H$. Since the order of H is 10, $h^{10} = e$. By corollary 10.2.4, it follows that $(a^3)^{10} = a^{30} = e$. So, $|a| \mid 30$ is true and 1 and 2 are not orders because $|aH| = 3$. So, $|a| \in \{3, 5, 6, 10, 15, 30\}$.