Problem 1. Consider the formula for the work done moving a mass m along the path $\vec{X}(t)$ from X_0 to X which is

$$\int_{t=t_0}^{t_1} \vec{F}(t) \bullet d\vec{X}$$

Show that

$$\int_{t=t_0}^{t_1} \vec{F}(t) \bullet d\vec{X} = \frac{1}{2}mv^2(t_1) - \frac{1}{2}mv^2(t_0)$$

Solution:

First I make a substitution

$$\int_{t=t_0}^{t_1} m\vec{a} \bullet d\vec{X}$$

Then I rewrite this as

$$m\int_{t=t_0}^{t_1} \frac{d\vec{v}}{dt} \bullet d\vec{X} = m\int_{t=t_0}^{t_1} d\vec{v} \bullet \frac{d\vec{X}}{dt} = m\int_{t=t_0}^{t_1} \vec{v} \bullet d\vec{v}.$$

Note that $\vec{v} \bullet d\vec{v} = v_x dv_x + v_y dv_y$ by definition of dot product. Now, by substitution I get

$$m\int_{t=t_0}^{t_1} \vec{v} \bullet d\vec{v} = m\int_{t=t_0}^{t_1} v_x dv_x + v_y dv_y$$

which means that

$$\begin{aligned}
m\int_{t=t_0}^{t_1} \vec{v} \bullet d\vec{v} &= m\left(\int_{t=t_0}^{t_1} v_x dv_x + \int_{t=t_0}^{t_1} v_y dv_y\right) \\
&= m\left(\frac{1}{2}v_x^2(t_1) - \frac{1}{2}v_x^2(t_0)\right) + \left(\frac{1}{2}v_y^2(t_1) - \frac{1}{2}v_y^2(t_0)\right) \\
&= \frac{1}{2}m\left(\left(v_x^2(t_1) - v_x^2(t_0)\right) + \left(v_y^2(t_1) - v_y^2(t_0)\right)\right) \\
&= \frac{1}{2}m\left(v_x^2(t_1) - v_x^2(t_0) + v_y^2(t_1) - v_y^2(t_0)\right) \\
&= \frac{1}{2}m\left(v_x^2(t_1) - v_y^2(t_1) + v_x^2(t_0) - v_y^2(t_0)\right) \\
&= \frac{1}{2}m(-v^2(t_1) + -v^2(t_0)) = \frac{1}{2}m(v^2(t_1) - v^2(t_0)) \\
&= \frac{1}{2}mv^2(t_1) - \frac{1}{2}mv^2(t_0))
\end{aligned}$$

Problem 2a. Show that the work done by gravity moving the mass along the path $\vec{X}(t)$ from X_0 to X is also given by

$$\int_{t=t_0}^{t_1} \vec{F}(t) \bullet d\vec{X} = mg(y(t_1) - y(t_0))$$

Solution:

First I make a substitution using the fact that $\vec{F} = m\vec{a}$. That is,

$$\int_{t=t_0}^{t_1} \vec{F} \cdot d\vec{X} = \int_{t=t_0}^{t_1} m\vec{a} \cdot d\vec{X}.$$

Then using the fact that $\vec{a} = \begin{bmatrix} 0 \\ g \end{bmatrix}$, I make the substitution,

$$\int_{t=t_0}^{t_1} m\vec{a} \cdot d\vec{X} = \int_{t=t_0}^{t_1} m \begin{bmatrix} 0 \\ g \end{bmatrix} \cdot d\vec{X}.$$

Now by rewriting what I have I get

$$\begin{aligned} &\int_{t=t_0}^{t_1} m \begin{bmatrix} 0 \\ g \end{bmatrix} \cdot d\vec{X} \\ &= m \int_{t=t_0}^{t_1} \begin{bmatrix} 0 \\ g \end{bmatrix} \cdot \begin{bmatrix} dx(t) \\ dy(t) \end{bmatrix} \\ &= m \int_{t=t_0}^{t_1} g dy(t) \\ &= mg \int_{t=t_0}^{t_1} dy(t) = mg(y(t_1) - y(t_0)) \end{aligned}$$

This shows that

$$\int_{t=t_0}^{t_1} \vec{F}(t) \cdot d\vec{X} = mg(y(t_1) - y(t_0))$$

is the work done by gravity.

Problem 2b. Use our two formulas for work to show that if the mass starts from rest then

$$v(t) = \sqrt{2g(y(t_1) - y(t_0))}$$

Solution:

Since the initial state is at rest, this means velocity is equal to zero at t_0. That is, $v(t_0) = 0$. Plugging this into the equation I got in the first problem I get,

$$\int_{t=t_0}^{t_1} \vec{F}(t) \cdot d\vec{X} = \frac{1}{2} m v^2(t_1) - 0$$

Plugging this into the equation I got in the second problem I get,

$$\int_{t=t_0}^{t_1} \vec{F}(t) \cdot d\vec{X} = mg(y(t_1) - y(t_0)).$$

This means that
$$\int_{t=t_0}^{t_1} \vec{F}(t) \cdot d\vec{X} = \frac{1}{2}mv^2(t_1) = mg(y(t_1) - y(t_0)).$$
Using algebraic manipulation I get
$$\begin{aligned} &\frac{1}{2}mv^2(t_1) = mg(y(t_1) - y(t_0)) \\ &\implies \frac{1}{2}v^2(t_1) = g(y(t_1) - y(t_0)) \\ &\implies v^2(t_1) = 2g(y(t_1) - y(t_0)) \\ &\implies v(t) = \sqrt{2g(y(t_1) - y(t_0))} \end{aligned}$$
This shows that $v(t) = \sqrt{2g(y(t_1) - y(t_0))}$ when the mass is initially at rest.

Problem 3a. Use the above information to show that the time it takes for the mass to travel from X_0 to X_1 under the influence of gravity (starting from rest) is given by
$$\int_{X=X_0}^{X_1} dt = \int_{X=X_0}^{X_1} \frac{ds}{\sqrt{2g(y(t_1) - y(t_0))}}$$

Solution:

It is a fact that $\frac{ds}{dt} = v(t)$. Using this fact it is clear that $dt = \frac{ds}{v(t)}$. Using the equation I got from problem 2a (because we are accounting for gravity), I can rewrite $v(t) = \sqrt{2g(y(t_1) - y(t_0))}$. So, rewriting the given integral I get,
$$\int_{X=X_0}^{X_1} dt = \int_{X=X_0}^{X_1} \frac{ds}{v(t)} = \int_{X=X_0}^{X_1} \frac{ds}{\sqrt{2g(y(t_1) - y(t_0))}}$$
This shows the time it takes for the mass to travel from X_0 to X_1 under the influence of gravity.

Problem 3b. Use the formula from 3a to show that the time it takes for the mass to move along the cycloid
$$\begin{aligned} &x = a(\theta - \sin(\theta)) \\ &y = a(1 - \cos(\theta)) \\ &\theta_0 \leq \theta \leq \pi \end{aligned}$$
is $\pi\sqrt{\frac{a}{g}}$ which is independent of θ_0.

Solution:

When I use the formula
$$\int_{X=X_0}^{X_1} \frac{ds}{v(t)} = \int_{X=X_0}^{X_1} \frac{ds}{\sqrt{2g(y(t_1) - y(t_0))}}$$
I can solve for ds knowing that it is the change in arc length. That is,
$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{a^2(\cos(\theta) - 1)^2 + (a\sin(\theta))^2}.$$

When I plug this into the integral I get

$$\int_{X=X_0}^{X_1} \frac{\sqrt{a^2(\cos(\theta)-1)^2+(a\sin(\theta))^2}}{\sqrt{2g(y(t_1)-y(t_0))}}$$

(idk notation is very weird I will ask questions)