Problem 1. We have learned that

$$H = \left\{ \begin{bmatrix} 1 & 0 \\ 6n & 1 \end{bmatrix} \middle| \, n \in \mathbb{Z} \right\}$$

is a subgroup of $SL(2,\mathbb{R})$. Is H cyclic? What are its generators?

H is cyclic because there is an element a that generates all elements in H. That is, when $n = 1$, or, $a = \left[\begin{smallmatrix} 1 & 0 \\ 6 & 1 \end{smallmatrix}\right]$. To show that this is a generator, I use the fact that

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}^x = \begin{bmatrix} 1 & 0 \\ kx & 1 \end{bmatrix}, \; k, x \in \mathbb{Z}.$$

If I let $6n = k$ then the resulting matrix becomes

$$\begin{bmatrix} 1 & 0 \\ 6n & 1 \end{bmatrix}^x = \begin{bmatrix} 1 & 0 \\ (6n)x & 1 \end{bmatrix}$$

and when $n = 1$,

$$\begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}^x = \begin{bmatrix} 1 & 0 \\ 6x & 1 \end{bmatrix}.$$

Since this takes the form of any element in H, it is clear that $\left[\begin{smallmatrix} 1 & 0 \\ 6 & 1 \end{smallmatrix}\right]$ is the only generator for H.

Problem 2. Find the generator of $\langle S, \cdot \rangle$, where

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \middle| \, k \in \mathbb{Z}_{12} \right\}.$$

The generator of S is when $k = 1$. I know this because

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 0 \\ kn & 1 \end{bmatrix} \pmod{12}, n \in \mathbb{Z}_{12}$$

and when $k = 1$ this is equivalent to

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix} \pmod{12}$$

So, it is clear that when $k = 1$, the resulting matrix generates S.

Problem 3. Let a be an element in a group with $\text{ord}(a) = 18$. Use an appropriate formula to compute the values of $\text{ord}_{\langle a \rangle}(a^k)$, for k = 2, 3, 4, 5.

I use theorem 5.1.6 to get that, $|a^2| = \frac{18}{\gcd(18,2)} = \frac{18}{2} = 9$, $|a^3| = \frac{18}{\gcd(18,3)} = \frac{18}{3} = 6$, $|a^4| = \frac{18}{\gcd(18,4)} = \frac{18}{2} = 9$, and $|a^5| = \frac{18}{\gcd(18,5)} = \frac{18}{1} = 18$.

Problem 4. Let G be a group with an element a such that $\text{ord}_G(a) = 72$. Let $H = \langle a \rangle$.

(a) What is the value of $|H|$?

(b) Compute $\text{ord}_H(a^6)$?

(c) Let $K = \langle a^6 \rangle$. Determine $\text{ord}_K(a^{48})$

(a) Since H is generated by a, the order of a is equal to the order of the group. So, $|H| = |\langle a \rangle| = 72$

(b) Using theorem 5.1.6, I get that $|a^6| = \frac{72}{\gcd(72,6)} = \frac{72}{6} = 12$.

(c) If a^6 generates K, then $|K| = |\langle a^6\rangle| = 12$. Using theorem 5.1.6, I get $|\langle a^{48}\rangle| = \frac{12}{\gcd(12,48)} = \frac{12}{4} = 3$.

Problem 5. Find the elements of $\mathbb{Z}_{96}$ that have order 12.

Since $\mathbb{Z}_{96}$ is cyclic because $\mathbb{Z}_{96} = \langle 1\rangle$, I can use theorem 5.1.6 to show the elements of order 12. To find these elements, using theorem 5.1.6 I need to find the values of k where $|1^k| = \frac{96}{\gcd(96,k)} = 12$. I find that $\gcd(96,8) = \gcd(96,40) = \gcd(96,56) = \gcd(96,88) = 8$. Since theorem 5.2.2 says that if a group has order n, that given a divisor d of n, the number of elements in that group having order d is $\phi(d)$, I can say that this group, $\mathbb{Z}_{96}$, with order 96 and 12 being the order which is a divisor of 96, that the amount of elements which has order 12 is $\phi(12) = 4$. So, these four elements are $\{8, 40, 56, 88\}$.

Problem 6. Let a be an element of a group G such that $\mathrm{ord}(a) = 36$. Find all the elements of $H = \langle a\rangle$ with order 9.

Since $H = \langle a\rangle$, $|H| = |\langle a\rangle| = 36$. With this fact and theorem 5.1.6, I can find the elements of H with order 9 to be a^k where $a^k = \frac{36}{\gcd(36,k)} = 9$. These k values are $4, 8, 16, 20, 28, 32$. So, the elements of order 9 are $a^4, a^8, a^{16}, a^{20}, a^{28}$, and a^{32}.

Problem 7. Let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 4 & 6 & 7 & 5 & 3 \end{pmatrix}, \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 3 & 2 & 1 & 4 \end{pmatrix}$$

Evaluate α^2, β^3 and $\alpha^2\beta^3$. Express the answers in the 2-line form.

$$\alpha^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 6 & 5 & 3 & 7 & 4 \end{pmatrix}$$
$$\beta^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 7 & 6 & 2 & 1 & 4 & 5 \end{pmatrix}$$
$$\alpha^2\beta^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 4 & 7 & 2 & 1 & 5 & 3 \end{pmatrix}$$

Problem 8. Let the 2-line form of $\sigma, \gamma \in S_5$ be

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 5 & 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 2 & 3 \end{pmatrix}.$$

(a) Find $\sigma\gamma, \gamma\sigma, \sigma^2\gamma^3\sigma$, and γ^{-1}

(b) What are the orders of σ and γ?

(c) Find σ^{23}

(a) $\sigma\gamma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 2 & 4 & 3 \end{smallmatrix}\right)$, $\gamma\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 1 & 3 & 5 \end{smallmatrix}\right)$, $\sigma^2\gamma^3\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 5 & 1 \end{smallmatrix}\right)^2 \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 2 & 3 \end{smallmatrix}\right)^3 \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 5 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 2 & 4 \end{smallmatrix}\right)$,
$\gamma^{-1} = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{smallmatrix}\right)$, $\sigma^{-1} = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 3 & 2 & 4 \end{smallmatrix}\right)$

(b) The order of σ is 4 because 4 is the smallest positive integer greater than zero that gives ι when σ is raised to the 4th power. The order of γ is 6 because 6 is the smallest positive integer greater than zero that gives ι when γ is raised to the 6th power.

(c) Since σ has an order of 4, $\sigma^{20} = \iota$. It follows that $\sigma^{23} = \sigma^{20}\sigma^3 = \iota\sigma^3 = \sigma^3 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 3 & 2 & 4 \end{smallmatrix}\right)$