Problem 1. Consider the differential one form $\omega = f\,dx + g\,dy$. Show that the following are equivalent

a. $\oint\limits_L \omega = 0$ for any closed loop L

b. $\int\limits_C \omega$ is independent of path C from X_0 to X_1

c. $\omega = d\phi$ for some function $\phi(x,y)$

Proof. To show this equivalency, I will show that a $\implies$ b $\implies$ c $\implies$ a. First, assume that $\oint_L \omega = 0$ for any closed loop L. Also notice that in $\mathbb{R}^n$, $n \geq 2$ there exists at least two paths, C_1 and C_2, from a point X_0 to X_1. Which means there is one path, C_1, from X_0 to X_1 and another path, $-C_2$, from X_1 to X_0. This forms a closed loop $L = C_1 - C_2$. So it must be true that,

$$0 = \oint\limits_L \omega = \int\limits_{C_1} \omega + \int\limits_{-C_2} \omega = \int\limits_{C_1} \omega - \int\limits_{C_2} \omega = 0$$

Which implies that

$$\int\limits_{C_1} \omega = \int\limits_{C_2} \omega$$

This shows that a $\implies$ b.

Now I show that b $\implies$ c. To do this I assume that $\int_C \omega$ is independent of path where C is some curve. Now I define a special curve, C_1 from X_0 to X_3. Let C_2 be any path from X_0 to X_1 and let C_3 be a path from X_1 to X_3 such that the x component of C_3 does not change. Note that $C_1 = C_2 + C_3$. The work done by a point on this curve, can be described as $\omega = f(x,y)dx + g(x,y)dy$. Now, let $\phi = \int_C \omega$. It follows that

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x} \int\limits_{C_1} \omega = \frac{\partial}{\partial x}\left(\int\limits_{C_2} f\,dx + g\,dy + \int\limits_{X_1}^{X_3} f\,dx + g\,dy \right)$$

and since C_3 is not changing in the x direction this implies that

$$\int\limits_{X_1}^{X_3} f\,dx + g\,dy = 0$$

which means that

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left(\int\limits_{C_2} f\,dx + g\,dy + 0 \right) \overset{F.T.C}{\implies} \frac{\partial \phi}{\partial x} = f(x,y).$$

Now consider another special, but different curve, C_4, from X_0 to X_3. Let C_5 be any path from X_0 to X_2. Now, let C_6 be a path from X_2 to X_3 such that the y direction does not change. Note that $C_4 = C_5 + C_6$. The work done by a point on this curve can also be described as ω. Now, I look at $\frac{\partial \phi}{\partial y}$ from this new curve. That is,

$$\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\left(\int\limits_{C_5} f\,dx + g\,dy + \int\limits_{X_2}^{X_3} f\,dx + g\,dy \right)$$

and since the y component on C_6 is constant, this means that

$$\frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}\left(\int\limits_{C_5} f dx + g dy + 0\right) \stackrel{F.T.C}{\Longrightarrow} \frac{\partial\phi}{\partial y} = g$$

Finally, I have shown that $\frac{\partial\phi}{\partial x} = f$ and $\frac{\partial\phi}{\partial y} = g$. Since this is true, and by the definition of $d\phi(x,y)$, that, $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy$, by substitiution I show that,

$$d\phi = f dx + g dy = \omega \implies \int\limits_C \omega = \int\limits_C d\phi.$$

Since ω is differentiable, $\int_C d\phi = \phi$ exists. This shows that b $\implies$ c.

Finally, to show that c $\implies$ a, I assume that $\omega = d\phi$ for some $\phi(x,y)$. Since,

$$\int\limits_C \omega = \int\limits_C d\phi = \phi(X_1) - \phi(X_0)$$

where C is a path from X_0 to X_1. When C is a closed loop, L from X_0 to X_0, then

$$\oint\limits_L \omega = \oint\limits_{X_0}^{X_0} d\phi = \phi(X_0) - \phi(X_0) = 0.$$

This shows that c $\implies$ a.
Since a $\implies$ b $\implies$ c $\implies$ a, this proves that all three statements are equivalent. $\square$

Problem 2. Suppose that $\vec{F}$ is a conservative vector field and that $\vec{F} = -\nabla\phi$. Consider a particle of mass m moving along a path C from point X_0 to X_1. Show that

$$\phi(X_0) + \frac{1}{2}mv^2\Big|_{X_0} = \phi(X_1) + \frac{1}{2}mv^2\Big|_{X_1}$$

Solution:

Using the true statement that $-\int_C \vec{F}\cdot d\vec{X} = -\int_C \vec{F}\cdot d\vec{X}$ and that

$$-\int\limits_C \vec{F}\cdot d\vec{X} = -\int\limits_C m\vec{a}\cdot d\vec{X}$$

$$= -m\int\limits_C \frac{d\vec{v}}{dt}\cdot d\vec{X} = \int\limits_C d\vec{v}\cdot\frac{d\vec{X}}{dt}$$

$$= -m\int\limits_C \vec{v}\cdot d\vec{v} = -m\left(\frac{1}{2}v^2\Big|_{X_1} - \frac{1}{2}v^2\Big|_{X_0}\right)$$

and

$$-\int_C \vec{F} \cdot d\vec{X} = -\int_C \nabla\phi \cdot d\vec{X}$$

$$= -\int_{X_0}^{X_1} d\phi = -(\phi(X_1) - \phi(X_0))$$

then

$$-(\phi(X_1) - \phi(X_0)) = -m\left(\frac{1}{2}v^2\bigg|_{X_1} - \frac{1}{2}v^2\bigg|_{X_0}\right)$$

$$\implies \phi(X_0) + \frac{1}{2}mv^2\bigg|_{X_0} = \phi(X_1) + \frac{1}{2}mv^2\bigg|_{X_1}$$

where $\vec{F} = \begin{bmatrix} \frac{\partial\phi}{\partial x_1} \\ \vdots \\ \frac{\partial\phi}{\partial x_n} \end{bmatrix}$.

Problem 3. In a PTQ, we showed that the electric field $\vec{E}$ located at $P = (x, y, z)$ generated by two point charges, one of charge -1 coulombs located at $(0,0,0)$ and the other of charge +1 coulombs located at the point $(1,2,3)$ is given by

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{((x-1)^2+(y-2)^2+(z-3)^2)^{\frac{3}{2}}}\begin{bmatrix} x-1 \\ y-2 \\ z-3 \end{bmatrix} - \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$$

Show that the work done by $\vec{E}$ to move the particle from point A to point B is independent of path.

Solution:

Since the work done by the particle moving along the path in the field is

$$\int_C \vec{E} \cdot d\vec{X}$$

it is clear that since $\vec{E}$ is only a function of x, y and z, that the computation of $\vec{E}$ has nothing to do with the path. For example, if there are two curves C_1 and C_2 that go from $(0,0,0)$ to (x, y, z) then it is true that

$$\int_{C_1} \vec{E} \cdot d\vec{X} = \int_{(0,0,0)}^{(x,y,z)} \vec{E} \cdot d\vec{X}$$

and

$$\int_{C_2} \vec{E} \cdot d\vec{X} = \int_{(0,0,0)}^{(x,y,z)} \vec{E} \cdot d\vec{X}$$

So, clearly $\int_{C_1} \vec{E} \cdot d\vec{X} = \int_{C_2} \vec{E} \cdot d\vec{X}$ for any and all paths C_1 and C_2. Therefore it is path independent.

Problem 4. Suppose $\vec{T}$ is a unit tangent vector to the curve C, $\vec{r} = \vec{r}(u)$. Show that the work done in moving a particle in a force field $\vec{F}$ along C is given by $\int_C \vec{F} \cdot \vec{T} ds$ where s is the arc length.

Solution:

By using the fact that

$$\vec{T} = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|} = \frac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}}$$

and the work done on a particle in $\vec{F}$ along C is

$$\int\limits_C \vec{F} \cdot d\vec{r}$$

So, by substitution I get

$$\int\limits_C \vec{F} \cdot \vec{T} ds = \int\limits_C \vec{F} \frac{\frac{d\vec{r}}{dt}}{\left|\frac{ds}{dt}\right|} ds = \int\limits_C \vec{F} \cdot d\vec{r}.$$

This shows that the work done by moving a particle in a force field $\vec{F}$ along C is given by $\int_C \vec{F} \cdot d\vec{r}$.