Problem 1. Utilizing the parameterization of x,y,z in terms of u,v, prove the following rules of the product of differential forms.

1. $d(x+y)dz = dxdz + dydz$
2. $dxdy = -dydx$. In particular, $dxdx = 0$

Solution:

1. *Proof.* From the definitions of $dxdz$ and $dydz$ I expand out the R.H.S of the equation to be

$$dx\,dz + dy\,dz = \frac{\partial(x,z)}{\partial(u,v)}dudv + \frac{\partial(y,z)}{\partial(u,v)}dudv.$$

It is also true that

$$\frac{\partial(x,z)}{\partial(u,v)}dudv = det\left(\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix}\right) dudv,$$

$$\frac{\partial(y,z)}{\partial(u,v)}dudv = det\left(\begin{bmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix}\right) dudv.$$

Using the fact that $det[A+B|C] = det[A|C] + det[B|C]$ I rewrite the R.H.S of the original equation to be

$$\begin{aligned} dx\,dz + dy\,dz &= \left(det\left(\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix}\right) + det\left(\begin{bmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix}\right)\right) dudv \\ &= det\left(\begin{bmatrix} \frac{\partial x}{\partial u} + \frac{\partial y}{\partial u} & \frac{\partial x}{\partial v} + \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix}\right) dudv \\ &= \frac{\partial(x+y,z)}{\partial(u,v)}dudv. \end{aligned}$$

Now, notice that

$$d(x+y)dz = \frac{\partial(x+y,z)}{\partial(u,v)}du\,dv$$

By substitution,

$$d(x+y)dz = \frac{\partial(x+y,z)}{\partial(u,v)}du\,dv = dx\,dz + dy\,dz$$

Therefore, $d(x+y)dz = dxdz + dydz$. $\square$

2. *Proof.* I will rewrite the L.H.S of the original equation to be

$$dxdy = \frac{\partial(x,y)}{\partial(u,v)}dudv = det\left(\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}\right) dudv$$

and I can also write

$$dydx = \frac{\partial(y,x)}{\partial(u,v)}dudv = det\left(\begin{bmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{bmatrix}\right) dudv$$

which implies that

$$-dydx = -det\left(\begin{bmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{bmatrix}\right) dudv.$$

Using the property that $det([A|B]) = -det([B|A])$, it follows that

$$det\left(\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}\right) dudv = -det\left(\begin{bmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{bmatrix}\right) dudv$$

which then by substitution on the previous equation means that

$$dxdy = -dydx.$$

□

Problem 2.

a. Use the rules for multiplying differential forms on cylindrical coordinates

$$x = r\cos(\theta),\ y = r\sin(\theta),\ z = z$$

to show that

$$dxdydz = r\,drd\theta dz.$$

b. Use the rules for multiplying differential forms on spherical coordinates

$$x = \rho\sin(\phi)\cos(\theta),\ y = \rho\sin(\phi)\sin(\theta),\ z = \rho\cos(\phi)$$

to show that

$$dxdydz = \rho^2\sin(\phi)d\rho d\phi d\theta.$$

Solution:

a. It is true that $dxdydz = (dxdy)dz$. Expanding this using the multiplication rules for differential forms I get

$$\begin{aligned} dxdydz = (dxdy)dz = \left(\frac{\partial(x,y)}{\partial(r,\theta)} drd\theta\right) dz = \left(det\left(\begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix}\right) drd\theta\right) dz \\ = r(\cos^2(\theta) + \sin^2(\theta))drd\theta dz = r\,drd\theta dz \end{aligned}$$

This shows that $dxdydz = r\,drd\theta dz$.

b. By expanding out the L.H.S of the equation I get

$$dxdydz = \left(\frac{\partial x}{\partial \rho}d\rho + \frac{\partial x}{\partial \phi}d\phi + \frac{\partial x}{\partial \theta}d\theta\right)\left(\frac{\partial y}{\partial \rho}d\rho + \frac{\partial y}{\partial \phi}d\phi + \frac{\partial y}{\partial \theta}d\theta\right)\left(\frac{\partial z}{\partial \rho}d\rho + \frac{\partial z}{\partial \phi}d\phi + \frac{\partial z}{\partial \theta}d\theta\right).$$

Multiplying the first two terms out I get,

$$\begin{aligned} dxdy =& \sin^2(\phi)\cos(\theta)\sin(\theta)d\rho d\rho + \rho\sin(\phi)\cos(\phi)\cos(\theta)\sin(\theta)d\rho d\phi \\ &+ \rho\sin^2(\phi)\cos^2(\theta)d\rho d\theta + \rho\cos(\phi)\sin(\phi)\cos(\theta)\sin(\theta)d\phi d\rho \\ &+ \rho^2\cos^2(\phi)\cos(\theta)\sin(\theta)d\phi d\phi + \rho^2\cos(\phi)\sin(\phi)\cos^2(\theta)d\phi d\theta \\ &- \rho\sin^2(\phi)\sin^2(\theta)d\theta d\rho - \rho^2\sin(\phi)\cos(\phi)\sin^2(\theta)d\theta d\phi \\ &- \rho^2\sin^2(\phi)\sin(\theta)\cos(\theta)d\theta d\theta. \end{aligned}$$

I can now simplify this because of the property that a differential form multiplied by itself is zero. This gives me

$$\begin{aligned} dxdy =& \rho\sin(\phi)\cos(\phi)\cos(\theta)\sin(\theta)d\rho d\phi \\ &+ \rho\sin^2(\phi)\cos^2(\theta)d\rho d\theta + \rho\cos(\phi)\sin(\phi)\cos(\theta)\sin(\theta)d\phi d\rho \end{aligned}$$

$$+\rho^2\cos(\phi)\sin(\phi)\cos^2(\theta)d\phi d\theta$$
$$-\rho\sin^2(\phi)\sin^2(\theta)d\theta d\rho - \rho^2\sin(\phi)\cos(\phi)\sin^2(\theta)d\theta d\phi.$$

Using property 2 of problem 1 and rewriting I get that

$$dxdy = \rho\sin^2(\phi)(\cos^2(\theta)+\sin^2(\theta))d\rho d\theta + \rho^2\cos(\phi)\sin(\phi)(\cos^2(\theta)+\sin^2(\theta))d\phi d\theta$$

and rewriting this I get

$$dxdy = \rho\sin^2(\phi)d\rho d\theta + \rho^2\cos(\phi)\sin(\phi)d\phi d\theta.$$

Now, multiplying this by dz I get

$$dxdydz = (\rho\sin^2(\phi)d\rho d\theta + \rho^2\cos(\phi)\sin(\phi)d\phi d\theta)(\cos(\phi)d\rho - \rho\sin(\phi)d\phi)$$

which then factors out to be

$$dxdydz = (\rho\sin^2(\phi)\cos(\phi)d\rho d\theta d\rho - \rho^2\sin^3(\phi)d\rho d\theta d\phi + \rho^2\cos^2(\phi)\sin(\phi)d\phi d\theta d\rho - \rho^3\cos(\phi)\sin^2(\phi)d\phi d\theta d\phi)$$

and by using the fact that a differential multiplied by itself is zero and property 2 of problem 1, I get that

$$dxdydz = -\rho^2\sin^3(\phi)d\rho d\theta d\phi + \rho^2\cos^2(\phi)\sin(\phi)d\phi d\theta d\rho = \rho^2\sin(\phi)d\rho d\phi d\theta.$$

Problem 3. For each of the following differential forms ω, compute $d\omega$. Use the rules of multiplication to combine terms as much as possible in your answer.

a. $\omega = (x^2-z)dx + (yz)dy + (e^x+y^3)dz$

b. $\omega = (\sin(xz))dydz + (x+yz)dzdx + (xyz)dxdy$

c. $\omega = f(r) = f\left(\sqrt{x^2+y^2}\right)$

Solution:

a. By using the rules for differential forms I get

$$d\omega = (2dxdx - dz)dx + (zdy+ydz)dy + (e^xdx+3y^2dy)dz = -dzdx + ydzdy + (e^xdx+3y^2dy)dz$$
$$= -(e^x+1)dzdx + (3y^2-y)dydz$$

b. By using the rules for differential forms I get

$$d\omega = (z\cos(x)dx + x\cos(z)dz)dydz + (dx+zdy+ydz)dzdx + (yzdx+xzdy+xydz)dxdy$$

which becomes

$$d\omega = z\cos(x)dxdydz + zdydzdx + xydzdxdy = (z\cos(x)+z+xy)dxdydz$$

c. By using the rules for differential forms I get that

$$d\omega = d(f(r)) = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x}dx + \frac{\partial f}{\partial r}\frac{\partial r}{\partial y}dy = \frac{\partial f}{\partial r}\frac{1}{\sqrt{x^2+y^2}}(xdx+ydy).$$

Problem 4. Suppose that $\omega = f(x,y,z)dx + g(x,y,z)dy + h(x,y,z)dz$ is a 1-form defined on a 2-dimensional surface $S \subset \mathbb{R}^3$ and its (1-dimensional) boundary ∂S. Show that

$$\oint_{\partial S} \omega = \iint_S d\omega$$

which becomes

$$\oint_{\partial S} f\,dx + g\,dy + h\,dz = \iint_S \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dy\,dz + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) dz\,dx + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

which is (the original) Stokes' Theorem.

Solution:

Computing $d\omega$ I get

$$d\omega = df\,dx + dg\,dy + dh\,dz$$

I rewrite this as

$$d\omega = \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right) dx + \left(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz\right) dy + \left(\frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy + \frac{\partial h}{\partial z}dz\right) dz.$$

Using properties about differential forms shown in previous problems I simplify to get

$$d\omega = \frac{\partial f}{\partial y}dy\,dx + \frac{\partial f}{\partial z}dz\,dx + \frac{\partial g}{\partial x}dx\,dy + \frac{\partial g}{\partial z}dz\,dy + \frac{\partial h}{\partial x}dx\,dz + \frac{\partial h}{\partial y}dy\,dz$$

which combines to

$$d\omega = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dy\,dz + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) dz\,dx + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy.$$

Given that

$$\oint_{\partial S} \omega = \iint_S d\omega$$

By substitution and since ω is a 1-form defined on a surface with a 1-dimensional boundary ∂S and $d\omega$ is a 2-form on S, it follows that

$$\oint_{\partial S} f\,dx + g\,dy + h\,dz = \iint_S \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dy\,dz + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) dz\,dx + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

which is Stokes' Theorem.

Problem 5. Suppose that $\omega = f(x,y,z)dy\,dz + g(x,y,z)dz\,dx + h(x,y,z)dx\,dy$ is a 2-form defined on a 3-dimensional region $S \subset \mathbb{R}^3$ and its 2-dimensional boundary ∂S. Show that

$$\oint_{\partial S} \omega = \iint_S d\omega$$

becomes

$$\oiint_{\partial S} f\,dy\,dz + g\,dz\,dx + h\,dx\,dy = \iiint_S \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) dx\,dy\,dz$$

which is the Divergence Theorem.

Solution:

Computing $d\omega$ I get

$$d\omega = \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right)dydz + \left(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz\right)dzdx + \left(\frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy + \frac{\partial h}{\partial z}dz\right)dxdy.$$

Using the properties about differential forms shown in previous problems I simplify to get

$$d\omega = \frac{\partial f}{\partial x}dxdydz + \frac{\partial g}{\partial y}dydzdx + \frac{\partial h}{\partial z}dzdxdy$$

which then simplifies by rules of differential forms to become

$$d\omega = \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right)dxdydz.$$

Given that

$$\oint_{\partial S} \omega = \iint_S d\omega$$

By substitution and since ω is a 2-form defined on a surface with a 2-dimensional boundary ∂S and $d\omega$ is a 3-form on S, it follows that

$$\oiint_{\partial S} f\,dydz + g\,dzdx + h\,dxdy = \iiint_S \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right)dxdydz$$

which is the Divergence Theorem.