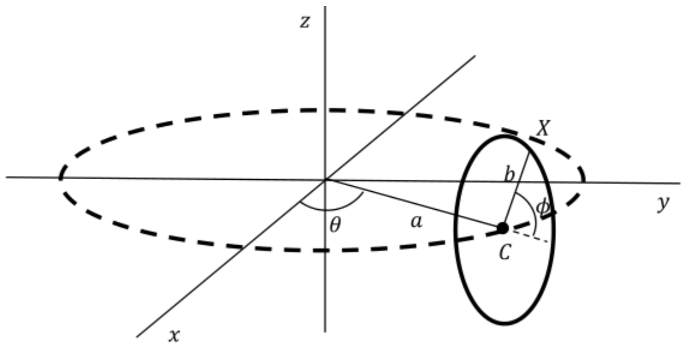
Problem 1. Consider the following torus obtained by revolving a circle of radius b around a circle of radius a, where $a \leq b$.



a. Use the fact that $\vec{X} = \vec{C} + \vec{CX}$ to derive the following paramatric equations of the torus.

$$\begin{gathered} x = a\cos(\theta) + b\cos(\phi)\cos(\theta) \\ y = a\sin(\theta) + b\cos(\phi)\sin(\theta) \\ z = b\sin(\phi) \\ 0 \leq \phi \leq 2\pi, \quad 0 \leq \theta \leq 2\pi \end{gathered}$$

b. Use this parameterization to find the surface area of the torus.

Solution:

a. Using basic trigonometry properties and the diagram above, I get that the x component of $\vec{C}$ is just $a\cos(\theta)$, the y component is just $a\sin(\theta)$ and the z component is zero. This is from the triangle that $\vec{C}$ makes with the x axis. Next I get that the z component of $\vec{CX}$ is $b\sin(\phi)$ and by using the diagram by projecting $\vec{X}$ onto the x,y plane, I get that the y component of $\vec{CX}$ is $b\cos(\phi)\sin(\theta)$ and finally, the x component to be $b\cos(\phi)\cos(\theta)$. So, it follows that

$$\vec{X} = \vec{C} + \vec{CX} = \begin{bmatrix} a\cos(\theta) \\ a\sin(\theta) \\ 0 \end{bmatrix} + \begin{bmatrix} b\cos(\phi)\cos(\theta) \\ b\cos(\phi)\sin(\theta) \\ b\sin(\phi) \end{bmatrix} = \begin{bmatrix} a\cos(\theta) + b\cos(\phi)\cos(\theta) \\ a\sin(\theta) + b\cos(\phi)\sin(\theta) \\ b\sin(\phi) \end{bmatrix}$$
$$\implies x = a\cos(\theta) + b\cos(\phi)\cos(\theta),\ y = a\sin(\theta) + b\cos(\phi)\sin(\theta),\ z = b\sin(\phi).$$

b. It is known that the surface area over a surface Σ is

$$\iint_{\Sigma} dS = \iint_{R} \sqrt{\left(\frac{\partial(y,z)}{\partial(\phi,\theta)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(\phi,\theta)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(\phi,\theta)}\right)^2}\, d\phi\, d\theta$$

given a transformation $(x,y,z) \to (\phi,\theta)$ which maps Σ to R. So,

$$\left.\begin{aligned} \frac{\partial(y,z)}{\partial(\phi,\theta)} &= \left|\begin{bmatrix} -b\sin(\phi)\cos(\theta) & a\cos(\theta) + b\cos(\phi)\cos(\theta) \\ b\cos(\phi) & 0 \end{bmatrix}\right| \\ \frac{\partial(z,x)}{\partial(\phi,\theta)} &= \left|\begin{bmatrix} b\cos(\phi) & 0 \\ -b\sin(\phi)\cos(\theta) & -a\sin(\theta) - b\cos(\phi)\sin(\theta) \end{bmatrix}\right| \\ \frac{\partial(x,y)}{\partial(\phi,\theta)} &= \left|\begin{bmatrix} -b\sin(\phi)\cos(\theta) & -a\sin(\theta) - b\cos(\phi)\sin(\theta) \\ -b\sin(\phi)\sin(\theta) & a\cos(\theta) + b\cos(\phi)\cos(\theta) \end{bmatrix}\right| \end{aligned}\right. \implies \iint_{\Sigma} dS = \iint_{R} ab + b^2\cos(\phi)\, d\phi\, d\theta.$$

Since $0 \le \phi \le 2\pi,\ 0 \le \theta \le 2\pi$, the surface area of the torus is

$$\int_0^{2\pi}\left[\int_0^{2\pi} ab + b^2\cos(\phi)\,d\phi\right]d\theta = 4\pi^2 ab.$$

Problem 2.

a. Show that if a surface is described by $\Sigma = \{(x,y,z) \mid z = z(x,y),\ (x,y) \in R\}$, then the surface area is given by

$$S = \iint_\Sigma dS = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dxdy$$

b. Find the area of the surface

$$z = x^{\frac{3}{2}} + y^{\frac{3}{2}}$$

located above the square $0 \le x,\ y \le 1$.

Solution:

a.

It is known that $S = \iint_\Sigma dS = \iint_R \sqrt{\left(\frac{\partial(y,z)}{\partial(x,y)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(x,y)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(x,y)}\right)^2}\,dx\,dy.$

Given that $\Sigma = \{(x,y,z) \mid z = z(x,y),\ (x,y) \in R\}$ I can parameterize the surface by this transformation.

$$\begin{aligned} x &= x \\ y &= y \\ z &= z(x,y) \end{aligned}$$

This allows me to compute the corresponding jacobians.

$$\begin{aligned} \frac{\partial(y,z)}{\partial(x,y)} &= \left|\begin{bmatrix} 0 & 1 \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix}\right| = -\frac{\partial z}{\partial x} \\ \frac{\partial(z,x)}{\partial(x,y)} &= \left|\begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ 1 & 0 \end{bmatrix}\right| = -\frac{\partial z}{\partial y} \\ \frac{\partial(x,y)}{\partial(x,y)} &= \left|\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = 1 \end{aligned}$$

So by substitution, the surface area is

$$S = \iint_\Sigma dS = \iint_R \sqrt{\left(-\frac{\partial z}{\partial x}\right)^2 + \left(-\frac{\partial z}{\partial y}\right)^2 + 1^2}\,dx\,dy = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

b.

Since this surface is described by $\Sigma = \{(x,y,z) \mid z(x,y),\ (x,y) \in R\}$, where $R = \{(x,y) \mid 0 \le x,y \le 1\}$ the surface area can be described as

$$\iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy. = \int_0^1\left[\int_0^1 \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dx\right]dy$$

and it is true that

$$\frac{\partial z}{\partial x} = \frac{3}{2}x^{\frac{1}{2}}$$
$$\frac{\partial z}{\partial y} = \frac{3}{2}y^{\frac{1}{2}}$$

So by substitution the surface area is equal to

$$\int_0^1 \left[\int_0^1 \sqrt{\left(\frac{3}{2}x^{\frac{1}{2}}\right)^2 + \left(\frac{3}{2}y^{\frac{1}{2}}\right)^2 + 1}\, dx \right] dy = \int_0^1 \left[\int_0^1 \sqrt{\frac{9x}{4} + \frac{9y}{4} + 1}\, dx \right] dy = \frac{968\sqrt{22} - 676\sqrt{13} + 64}{1215}.$$

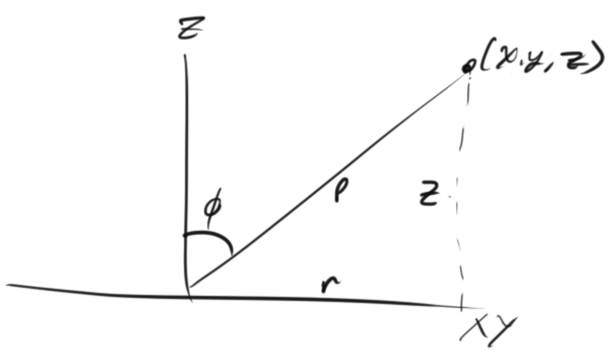
Problem 3.

a. Show that the unit sphere can be parameterized by

$$x = \sin(\phi)\cos(\theta), \quad y = \sin(\phi)\cos(\theta), \quad z = \cos(\phi)$$
$$0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi$$

b. Find the flux of the vector field $\vec{A} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ through the unit sphere oriented so that its orientation matches that of the ϕ, θ plane.

Solution:



a. The diagram above shows a point on the sphere in x, y, z space with radius ρ. Using the diagram and trigonometry properties I see that the following is true

$$z = \rho\cos(\phi)$$
$$r = \rho\sin(\phi)$$

I can also describe the (x, y, z) coordinate by using cylindrical coordinates. That is,

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$
$$z = z.$$

Now, by substitution it is true that the parameterization of the (unit $\rho = 1$) sphere is

$$\begin{matrix} x = \rho\sin(\phi)\cos(\theta) & & x = \sin(\phi)\cos(\theta) \\ y = \rho\sin(\phi)\sin(\theta) & \overset{\rho=1}{\Longrightarrow} & y = \sin(\phi)\sin(\theta) \\ z = \rho\cos(\phi) & & z = \cos(\phi). \end{matrix}$$

b. Given some transformation from (x,y,z) to (ϕ,θ) I know that flux is described by

$$\iint_R A_1 \frac{\partial(y,z)}{\partial(\phi,\theta)} + A_2 \frac{\partial(z,x)}{\partial(\phi,\theta)} + A_3 \frac{\partial(x,y)}{\partial(\phi,\theta)}\, d\phi\, d\theta$$

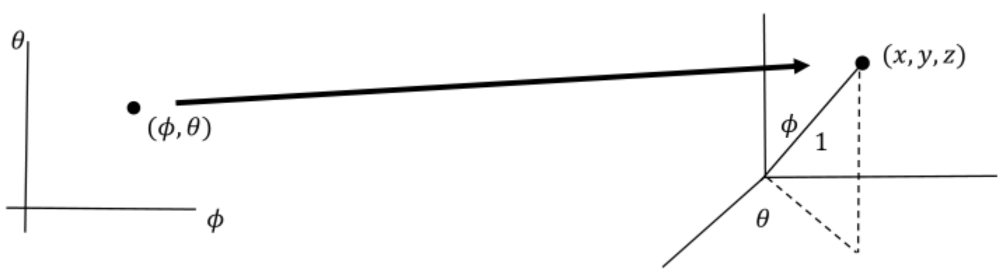
where $\vec{A} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$, R is the region that describes the unit sphere, and that

$$\frac{\partial(y,z)}{\partial(\phi,\theta)} = \left| \begin{bmatrix} \cos(\phi)\sin(\theta) & \sin(\phi)\cos(\theta) \\ -\sin(\phi) & 0 \end{bmatrix} \right| = \sin^2(\phi)\cos(\theta)$$
$$\frac{\partial(z,x)}{\partial(\phi,\theta)} = \left| \begin{bmatrix} -\sin(\phi) & 0 \\ \cos(\phi)\cos(\theta) & -\sin(\phi)\sin(\theta) \end{bmatrix} \right| = \sin^2(\phi)\sin(\theta)$$
$$\frac{\partial(x,y)}{\partial(\phi,\theta)} = \left| \begin{bmatrix} \cos(\phi)\cos(\theta) & -\sin(\phi)\cos(\theta) \\ \cos(\phi)\sin(\theta) & \sin(\phi)\cos(\theta) \end{bmatrix} \right| = \sin(\phi)\cos(\phi)$$

By subtitution the flux become as simple as

$$\iint_R \sin(\phi)\, d\phi\, d\theta$$

But I first need to describe R. I look at the unit sphere in (x,y,z) space, $\Sigma = \{(x,y,z) \mid x^2+y^2+z^2 = 1,\ x,y,z \in \mathbb{R}^+\}$, what I want to do is to transform this into (ϕ,θ) space. To do this I insert a diagram below



and I notice that in (x,y,z) space ϕ and θ are constricted by $x^2+y^2+z^2 = 1$ and only have the freedom of $0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi$. So, in the (ϕ,θ) space the coordinate (ϕ,θ) can only move around the rectangle bounded by $\theta = 2\pi,\ \theta = 0$ and $\phi = \pi,\ \phi = 0$. So, it follows that

$$R = \{(\phi,\theta) \mid 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi\}$$

So, the flux of the field A through the field through the unit sphere is

$$\int_0^{2\pi} \left[\int_0^{\pi} \sin(\phi)\, d\phi \right] d\theta = 4\pi.$$

Problem 4. Suppose we have a fluid flowing with a velocity vector $\vec{A} = \begin{bmatrix} y^2 \\ 2z \\ -1 \end{bmatrix}$. Find the flux of this fluid through the paraboloid $z = x^2 + y^2,\ 0 \le x^2 + y^2 \le 1$, where the orientation matches our usual orientation in the x, y plane.

Solution:

Consider the transformation

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \\ z &= z. \end{aligned}$$

It follows that $z = x^2 + y^2 = r^2$, $\vec{A} = \begin{bmatrix} r^2\sin^2(\theta) \\ 2r^2 \\ -1 \end{bmatrix}$, and, $0 \le r^2 \le 1$. Using this information it follows that the surface $\Sigma = \{(x,y,z) \mid z = x^2 + y^2,\ 0 \le z \le 1,\ x, y \in \mathbb{R}\}$ becomes $R = \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$. So the flux of the vector field $\vec{A}$ through this surface is

$$\iint_R A_1 \frac{\partial(y,z)}{\partial(r,\theta)} + A_2 \frac{\partial(z,x)}{\partial(r,\theta)} + A_3 \frac{\partial(x,y)}{\partial(r,\theta)}\, dr\, d\theta$$

and since $R = \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$ and

$$\begin{aligned} \frac{\partial(y,z)}{\partial(\phi,\theta)} &= \left| \begin{bmatrix} \sin(\theta) & r\cos(\theta) \\ 2r & 0 \end{bmatrix} \right| = -2r^2\cos(\theta) \\ \frac{\partial(z,x)}{\partial(\phi,\theta)} &= \left| \begin{bmatrix} 2r & 0 \\ \cos(\theta) & -r\sin(\theta) \end{bmatrix} \right| = -2r^2\sin(\theta) \\ \frac{\partial(x,y)}{\partial(\phi,\theta)} &= \left| \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix} \right| = r \end{aligned}$$

then it must be true that the flux of $\vec{A}$ through the surface is

$$\iint_R -2r^4\cos(\theta)\sin^2(\theta) - 4r^4\sin(\theta) - r\, dr\, d\theta = \int_0^{2\pi} \left[\int_0^1 -2r^4\cos(\theta)\sin^2(\theta) - 4r^4\sin(\theta) - r dr \right] d\theta = -\pi.$$