Problem 1.

a. Given the transformation for polar coordinates

$$x = r\cos(\theta),\ y = r\sin(\theta)$$

compute the Jacobian $\frac{\partial(x,y)}{\partial(r,\theta)}$.

b. Let R be the quarter annulus given by

$$R = \left\{(x,y)|1 \le x^2+y^2 \le 2,\ y \ge |x|\right\}$$

Find the center of mass (centroid) of R

Solution:

a. $J = \left| \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} \right| = \left| \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix} \right| = r$

b. I know that the center of mass is $(\bar{x},\bar{y})$ where

$$\bar{x} = \frac{\iint\limits_R x\,dx\,dy}{\iint\limits_R dx\,dy},\ \bar{y} = \frac{\iint\limits_R y\,dx\,dy}{\iint\limits_R dx\,dy}$$

and

$$R = \left\{(x,y)|1 \le x^2+y^2 \le 2,\ y \ge |x|\right\}$$

but this integral is calculated much easier if I do integration by a pullback. To do this I will need to rewrite my region and coordinates the correct way. That is,

$$\bar{x} = \frac{\iint\limits_P r\cos(\theta)J\,dr\,d\theta}{\iint\limits_P J\,dr\,d\theta},\ \bar{y} = \frac{\iint\limits_P r\sin(\theta)J\,dr\,d\theta}{\iint\limits_P J\,dr\,d\theta}$$

where

$$\begin{aligned} P &= \left\{(r,\theta)|1 \le r^2 \le 2,\ r\sin(\theta) \ge |r\cos(\theta)|\right\} \\ &= \left\{(r,\theta)|\sqrt{1} \le r \le \sqrt{2},\ \sin(\theta) \ge |\cos(\theta)|\right\} \\ &= \left\{(r,\theta)|1 \le r \le \sqrt{2},\ \frac{\pi}{4} \le \theta \le \frac{3\pi}{4}\right\}. \end{aligned}$$

Substituing the region and integrating, I get

$$\bar{x} = \frac{\int\limits_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left[\int\limits_1^{\sqrt{2}} r\cos(\theta)J\,dr\right] d\theta}{\int\limits_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left[\int\limits_1^{\sqrt{2}} J\,dr\right] d\theta} = 0,\ \bar{y} = \frac{\int\limits_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left[\int\limits_1^{\sqrt{2}} r\sin(\theta)J\,dr\right] d\theta}{\int\limits_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left[\int\limits_1^{\sqrt{2}} J\,dr\right] d\theta} = \frac{\frac{4-\sqrt{2}}{3}}{\frac{\pi}{4}} \implies (\bar{x},\bar{y}) = (0, \frac{4(4-\sqrt{2})}{3\pi})$$

Problem 2. Find the center of mass (centroid) of the following region S bounded by

$$y = \frac{1}{2}x,\ y = 3x,\ xy = 1,\ \text{and } xy = 4.$$

Solution:

To find the centroid I will do integration by a pullback. The coordinates of the centroid are described as $(\overline{x}, \overline{y})$ where

$$\overline{x} = \frac{\iint\limits_S x\,dx\,dy}{\iint\limits_S dx\,dy},\ \overline{y} = \frac{\iint\limits_S y\,dx\,dy}{\iint\limits_S dx\,dy}.$$

Since S is not a type I or type II region, let $u = \frac{y}{x},\ v = xy$. Then, the new region is

$$P = \left\{(u,v)\middle|\frac{1}{2} \leq u \leq 3,\ 1 \leq v \leq 4\right\}$$

and since

$$u = \frac{y}{x},\ v = xy \implies x = \sqrt{\frac{v}{u}},\ y = u\sqrt{\frac{v}{u}} \implies J = \left|\begin{bmatrix} \frac{-v}{2u^2\sqrt{\frac{v}{u}}} & \frac{1}{2u\sqrt{\frac{v}{u}}} \\ \frac{v}{2u\sqrt{\frac{v}{u}}} & \frac{1}{2\sqrt{\frac{v}{u}}} \end{bmatrix}\right| = -\frac{1}{2u} \overset{\text{ignore orientation}}{\Longrightarrow} J = \frac{1}{2u}$$

the coordinates of the centroid become

$$\overline{x} = \frac{\iint\limits_P x\,dx\,dy}{\iint\limits_P dx\,dy} = \frac{\int_1^4 \left[\int_{\frac{1}{2}}^3 \sqrt{\frac{v}{u}}\frac{1}{2u}\,du\right]dv}{\int_1^4 \left[\int_{\frac{1}{2}}^3 \frac{1}{2u}\,du\right]dv} = \frac{\frac{14}{3}\frac{\sqrt{6}-1}{\sqrt{3}}}{-\frac{3}{2}(\ln(3)+\ln(2))},$$

$$\overline{y} = \frac{\iint\limits_P y\,dx\,dy}{\iint\limits_P dx\,dy} = \frac{\int_1^4 \left[\int_{\frac{1}{2}}^3 u\sqrt{\frac{v}{u}}\frac{1}{2u}\,du\right]dv}{\int_1^4 \left[\int_{\frac{1}{2}}^3 \frac{1}{2u}\,du\right]dv} = \frac{\frac{7(\sqrt{2}-2\sqrt{3})}{3}}{-\frac{3}{2}(\ln(3)+\ln(2))}.$$

So, for a visual, the approximation of the centroid is around $(1.45, 1.78)$.

Problem 3.

a. Suppose $x = x(u,v),\ y = y(u,v)$ and $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$. Show that

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}}.$$

b. Evaluate

$$\iint\limits_S (x^2+y^2)\,dx\,dy$$

where S is the region bounded by $y = 0,\ y = x,\ xy = 1$ and $x^2 - y^2 = 1$.

Solution:

a. Let J_x be the Jacobian matrix that is $\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$ and J_u be the Jacobian matrix that is $\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$. It is true that

$$d\vec{X} = J_x\, d\vec{U}$$

where $d\vec{X} = \begin{bmatrix} dx \\ dy \end{bmatrix}$ and $d\vec{U} = \begin{bmatrix} du \\ dv \end{bmatrix}$. A square matrix is invertible iff its determinant is nonzero. J_x is square and in this problem we assume $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$ therefore J_x is invertible. Since J_x is invertible, it must also be true that by left multiplication,

$$d\vec{U} = {J_x}^{-1}\, d\vec{X}.$$

and since it is also true that

$$d\vec{U} = J_u\, d\vec{X}$$

this implies that

$$J_x^{-1} = J_u.$$

Using the property that $|A^{-1}| = \frac{1}{|A|}$, for any nonzero invertible matrix A, it follows that

$$|J_u| = |J_x^{-1}| = \frac{1}{|J_x|}.$$

In other notation,

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}}.$$

b. Consider the transformation $u = xy,\ v = x^2 - y^2$. The jacobian of this transformation is $\frac{\partial(u,v)}{\partial(x,y)} = \left| \begin{bmatrix} y & x \\ 2x & 2y \end{bmatrix} \right| = -2(x^2+y^2)$. From problem a. I know that $\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}}$ so,

$$\iint\limits_S (x^2+y^2)\, dx\, dy = \iint\limits_P x^2+y^2 \frac{1}{-2(x^2+y^2)}\, du\, dv = -\frac{1}{2}\iint\limits_P du\, dv.$$

To find this region I use the facts

$$y = 0 \implies u = xy = 0,\ y = x \implies y^2 = x^2 \implies x^2 - y^2 = 0 = v,\ u = xy = 1,\ v = x^2 - y^2 = 1$$

to set my bounds correctly. That is,

$$P = \{(u,v) | 0 \le u \le 1,\ 0 \le v \le 1\}.$$

So,

$$\iint\limits_S (x^2+y^2)\, dx\, dy = \frac{1}{2}\iint\limits_P du\, dv = -\frac{1}{2}\int_0^1 \left[\int_0^1 du\right] dv = -\frac{1}{2}$$

and ignoring orientation

$$\iint\limits_S (x^2+y^2)\, dx\, dy = \frac{1}{2}.$$

Problem 4. To derive the normal (Gaussian) distribution, Gauss needed to compute the value of

$$I = \int_{x=-\infty}^{\infty} e^{-x^2} dx$$

a. Show that

$$I^2 = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)} \, dx \, dy$$

b. Change variables to polar coordinates to show that

$$\int_{x=-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}$$

Solution:

a.

$$I = \int_{x=-\infty}^{\infty} e^{-x^2} \, dx \implies I^2 = \int_{x=-\infty}^{\infty} e^{-x^2} \, dx \int_{y=-\infty}^{\infty} e^{-y^2} \, dy = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx \right] dy = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)} \, dx \, dy$$

where y acts as a dummy variable.

b. Since it is true that

$$I^2 = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx \right] dy$$

converting to polar coordinates with the Jacobian being $\left| \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix} \right| = r$ it is true that

$$I^2 = \int_0^{2\pi} \left[\int_0^{\infty} re^{-r^2} dr \right] d\theta = \frac{1}{2} \int_0^{2\pi} d\theta = \pi \implies I = \sqrt{\pi}$$

Problem 5. Show that under suitable conditions on F and G,

$$\int_0^{\infty} \left[\int_0^{\infty} e^{-s(x+y)} F(x) \, G(y) \, dx \right] dy = \int_0^{\infty} e^{-st} \left[\int_0^{t} F(u) \, G(t-u) \, du \right] dt$$

Solution:

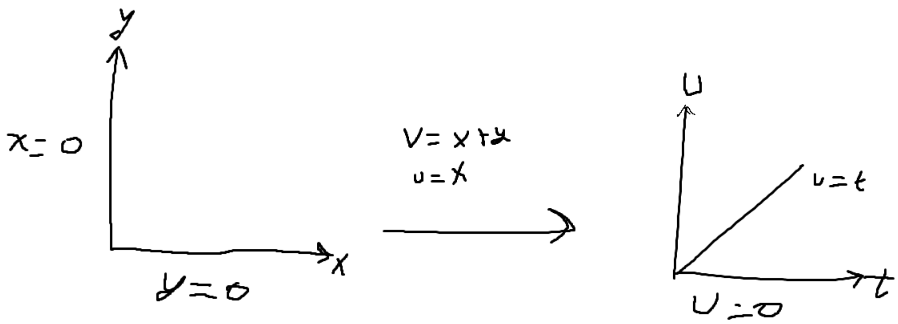
Consider the transformation $u = x, t = x + y$. The region in the (x, y) plane can be described as

$$S = \{(x,y) | 0 \leq x < \infty, 0 \leq y < \infty\}$$

This transformation brings the region to be

$$P = \{(u,v) | 0 \leq t < \inf, \; 0 \leq u \leq t\}$$

in the (t,u) plane. This is because the transformation takes $x = 0$ in the (x,y) to $t = y$ and $u = 0$ in the (t,u) plane. Then the transformation takes $y = 0$ in the (x,y) plane to $t = x = u$ in the (t,u) plane. This can be described by the following



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Furthermore, the jacobian of this transformation is $\left|\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\right| = 1$ so by substitution,

$$\int_0^\infty \left[\int_0^\infty e^{-s(x+y)}\, F(x)\, G(y)\, dx\right] dy = \int_0^\infty e^{-st} \left[\int_0^t F(u)\, G(t-u)\, du\right] dt.$$