Bertrand's Postulate

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Bertrand's Postulate

For any integer $n > 1$, there always exists at least one prime number p such that $n < p < 2n$.

Beweis eines Satzes von Tschebyschef.

Von P. Erdős in Budapest.

Für den zuerst von Tschebyschef bewiesenen Satz, laut dessen es zwischen einer natürlichen Zahl und ihrer zweifachen stets wenigstens eine Primzahl gibt, liegen in der Literatur mehrere Beweise vor. Als einfachsten kann man ohne Zweifel den Beweis von Ramanujan[1]) bezeichnen. In seinem Werk *Vorlesungen über Zahlentheorie* (Leipzig, 1927), Band I, S. 66—68 gibt Herr Landau einen besonders einfachen Beweis für einen Satz über die Anzahl der Primzahlen unter einer gegebenen Grenze, aus welchem unmittelbar folgt, daß für ein geeignetes q zwischen einer natürlichen Zahl und ihrer q-fachen stets eine Primzahl liegt. Für die augenblicklichen Zwecken des Herrn Landau kommt es nicht auf die numerische Bestimmung der im Beweis auftretenden Konstanten an; man überzeugt sich aber durch eine numerische Verfolgung des Beweises leicht, daß q jedenfalls größer als 2 ausfällt.

In den folgenden Zeilen werde ich zeigen, daß man durch eine Verschärfung der dem Landauschen Beweis zugrunde liegenden Ideen zu einem Beweis des oben erwähnten Tschebyschefschen Satzes gelangen kann, der — wie mir scheint — an Einfachkeit nicht hinter dem Ramanujanschen Beweis steht. Griechische Buchstaben sollen im Folgenden durchwegs positive, lateinische Buchstaben natürliche Zahlen bezeichnen; die Bezeichnung p ist für Primzahlen vorbehalten.

1. Der Binomialkoeffizient

$$\binom{2a}{a} = \frac{(2a)!}{(a!)^2}$$

[1]) Sr. Ramanujan, A Proof of Bertrand's Postulate, *Journal of the Indian Mathematical Society*, **11** (1919), S. 181—182 = *Collected Papers of* Srinivasa Ramanujan (Cambridge, 1927), S. 208—209.

Satz von Tschebyschef. 197

also, da das erste Produkt rechts höchstens $\sqrt{2n}$ Faktoren besitzt und da allgemein

$$2n \cdot \binom{2n}{n} = \frac{2}{1} \cdot \frac{3}{1} \cdot \frac{4}{2} \cdot \frac{5}{2} \cdots \frac{2n-2}{n-1} \cdot \frac{2n-1}{n-1} \cdot \frac{2n}{n} \cdot \frac{2n}{n} > 2^{2n}$$

gilt, die Ungleichung

$$(6) \qquad 2^{2n} < (2n)^{1+\sqrt{2n}} \prod_{\sqrt{2n} < p \leqq \frac{2}{3} n} p \prod_{n < p \leqq 2n} p.$$

4. Es sei nun $n \geqq 50$ und setzen wir voraus, daß es zwischen n und $2n$ keine Primzahl gibt. Dann ist das zweite Produkt in (6) leer; für den ersten gilt wegen $\sqrt{2n} \geqq 10$ und (3)

$$\prod_{\sqrt{2n} < p \leqq \frac{2}{3} n} p \leqq \prod_{10 < p \leqq \frac{2}{3} n} p < 2^{2\left[\frac{2}{3} n\right]} \leqq 2^{\frac{4}{3} n};$$

also geht (6) in

$$(7) \qquad 2^{2n} < (2n)^{1+\sqrt{2n}}\, 2^{\frac{4}{3} n}$$

über, was für hinreichend große n unmöglich ist.

Um eine nicht allzu hohe Schranke zu gewinnen, von welcher ab (7), d. h.

$$(8) \qquad 2^{\frac{2}{3} n} < (2n)^{1+\sqrt{2n}}$$

nicht mehr gültig ist, schätzen wir wie folgt ab. Wegen der Ungleichung $a \leqq 2^{a-1}$ (was man etwa durch vollständige Induktion leicht zeigen kann) ist

$$2n = \left(\sqrt[6]{2n}\right)^6 < \left(\left[\sqrt[6]{2n}\right] + 1\right)^6 \leqq 2^{6\left[\sqrt[6]{2n}\right]} \leqq 2^{6\sqrt[6]{2n}}$$

also folgt aus (8) (falls immer $n \geqq 50$ vorausgesetzt wird)

Lemma (1)

By the binomial theorem,

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Consider the $(2n+1)$-term sum

$$\sum_{i=0}^{2n} \binom{2n}{i}.$$

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Consider the $(2n+1)$-term sum

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The binomial theorem states that

$$(x+y)^{2n} = \sum_{i=0}^{2n} \binom{2n}{i} x^{2n-i} y^i$$

Let $x = y = 1$

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

Goal

We know the previous lower bound to be true. We want to show that if there is no prime p with $n < p < 2n$ where $n > 1$, then there is some upper bound on $\binom{2n}{n}$ that is smaller than $\frac{4^n}{2n+1}$(unless n is small but then computer verifies). This would be a contradiction, so it must be true that there is always prime p with $n < p < 2n$.

$O_p(n)$

Definition

Define $O_p(n)$ to be the largest exponent of p that divides n.

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The following properties hold

- $O_p(ab) = O_p(a) + O_p(b)$
- $O_p\left(\frac{a}{b}\right) = O_p(a) - O_p(b)$

Legendre's Formula

Corollary (1)

Legendre's Formula states that for any prime p, and any positive integer n, then

$$O_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor.$$

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You could think of is saying that since $n!$ is the product of integers 1 through n, that $O_p(n!)$ is the amount of multiples of p less than n (which is $\lfloor \frac{n}{p} \rfloor$). But you actually need to double count for p^2 and triple count for p^3 and so on.

Example

Let $n = 4$. Then $n! = 4! = 24 = 2^3 \cdot 3^1$. It follows that

$$O_2(24) = \sum_{i=1}^{\infty} \left\lfloor \frac{4}{2^i} \right\rfloor = \left\lfloor \frac{4}{2} \right\rfloor + \left\lfloor \frac{4}{2^2} \right\rfloor + \cdots = 2 + 1 + 0 + \cdots = 3.$$

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and

$$O_3(24) = \sum_{i=1}^{\infty} \left\lfloor \frac{4}{3^i} \right\rfloor = \left\lfloor \frac{4}{3} \right\rfloor + \cdots = 1 + 0 + \cdots = 1.$$

Important Fact(s)

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$$n \geq 3, \text{ and } \frac{2}{3}n < p \leq n.$$

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This actually guarantees that there is one factor of p in $n!$, and two factors of p in $(2n)!$. Consider $1 < p \leq n < n+1 < 2p \leq 2n$ and

$$\binom{2n}{n} = \frac{(2n)!}{(n!)(2n-n)!} = \frac{(2n)!}{(n!)^2} = \frac{2n(2n-1)(2n-2)\cdots(n+1)}{n(n-1)(n-2)\cdots 1}$$

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which shows that there is no such p such that $p|\binom{2n}{n}$ given the inequality. It also shows that there is no prime factor greater than $2n$.

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Let $r(p)$ be such that $p^{r(p)} \leq 2n < p^{r(p)+1}$. It follows that

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Let $r(p)$ be such that $p^{r(p)} \leq 2n < p^{r(p)+1}$. It follows that

$$\begin{aligned} O_p\left(\binom{2n}{n}\right) &= O_p((2n)!) - 2O_p(n!) = \sum_{i=1}^{r(p)} \left\lfloor \frac{2n}{p^i} \right\rfloor - 2\sum_{i=1}^{r(p)} \left\lfloor \frac{n}{p^i} \right\rfloor \\ &= \sum_{i=1}^{r(p)} \left(\left\lfloor \frac{2n}{p^i} \right\rfloor - 2\left\lfloor \frac{n}{p^i} \right\rfloor \right) \\ &\leq r(p). \end{aligned}$$

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The last line is justified because for all real x, $0 \leq \lfloor 2x \rfloor - 2\lfloor x \rfloor \leq 1$ which is either 0 or 1. Therefore $p^{O_p\left(\binom{2n}{n}\right)} \leq p^{r(p)} \leq 2n$. □

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Let $p_1, ..., p_\ell$ be the distinct primes dividing $\binom{2n}{n}$. It follows that

$$\binom{2n}{n} = \prod_{i=1}^{\ell} p_i^{O_{p_i}(\binom{2n}{n})} \leq (2n)^\ell$$

because of Lemma (1). $\square$

Product of Primes

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For all $n \geq 2$, $\prod_{p \leq n} p \leq 4^n$, where the product is over primes.

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When n is even, it follows that

$$\prod_{p \leq n-1} p = \prod_{p \leq n} p \leq 4^{n-1} \leq 4^n.$$

When n is odd, we work with $n = 2m + 1$...

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By the induction hypothesis and $n - 1 = 2m$, $\prod_{p \le m+1} p \le 4^{m+1}$. Also

because

$$\binom{2m+1}{m} = \frac{(2m+1)!}{(m+1)!} = (2m+1)(2m) \cdots (m+2),$$

this shows that that all primes between $m + 2$ and $2m + 1$ divide $\binom{2m+1}{m}$ it must be true that...

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Suppose $\binom{2m+1}{m} > 2^{2m}$. Then since $2m+1$ is odd

$$\binom{2m+1}{m} + \binom{2m+1}{m+1} = 2\binom{2m+1}{m} > 2(2^{2m}) = 2^{2m+1}$$

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But,

$$\sum_{i=1}^{2m+1} \binom{2m+1}{i} = 2^{2m+1}$$

and $n \geq 2$, so $2m+1 \geq 3$, and so this is a 3-term sum so there is a contradiction. Therefore $\binom{2m+1}{m} \leq 2^{2m}$.

Product of Primes

Proof.

I have that
$$\prod_{p\leq n} p \leq 4^{m+1}\binom{2m+1}{m}$$
and $\binom{2m+1}{m} \leq 2^{2m}$. Therefore I conclude that
$$\prod_{p\leq n} p \leq 4^{m+1}2^{2m} = 4^{2m+1} = 4^n$$
This concluses the proof. $\square$

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- If $p|\binom{2n}{n}$ then $p^{O_p(\binom{2n}{n})} \leq 2n$.
- If $n \geq 3$, and $\frac{2}{3}n < p \leq n$ there is no p such that $p|\binom{2n}{n}$.

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- If $p|\binom{2n}{n}$ then $p^{O_p(\binom{2n}{n})} \leq 2n$.
- If $n \geq 3$, and $\frac{2}{3}n < p \leq n$ there is no p such that $p|\binom{2n}{n}$.
- If $n \geq 3$, and $\frac{2}{3}n < p \leq n$ there is no prime factor greater than $2n$.

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Note that $\binom{2n}{n}$ has at most $\sqrt{2n}$ prime factors that do not exceed $\sqrt{2n}$. By Lemma 1, the most a prime factor can contribute is $2n$. Now, our hypothesis says that there is no p such that $n < p \leq 2n$. Finally, remember now, the important fact!, if $n \geq 3$, and $\frac{2}{3}n < p \leq n$, then there is no prime factors of $\binom{2n}{n}$ between $\frac{2}{3}n$ and n or greater than $2n$.

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and using Corollary 3, it follows that

$$\binom{2n}{n} \leq (2n)^{\sqrt{2n}} 4^{\frac{2}{3}n}.$$

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So,

$$\frac{4^n}{2n+1} \leq (2n)^{\sqrt{2n}} 4^{\frac{2}{3}n}.$$

Proof.

$$\frac{4^n}{2n+1} \leq (2n)^{\sqrt{2n}} 4^{\frac{2}{3}n}$$

BUT this inequality fails! Specifically for all $n \geq 468$. Therefore there is a contradiction, and so it is concluded that there exists a p such that $n < p < 2n$ given $n \geq 468$. For all $n < 468$ this is verified by manually checking the following primes satisfy the theorem for all $n < 468$.

$$2, 3, 5, 7, 13, 23, 43, 83, 163, 317, 631.$$

$\square$

Applications

Theorem (1.1)

The set of integers $\{1, 2, 3, \cdots, 2n\}, n \geq 1$ can be partitioned into pairs $\{a_i, b_i\}$ such that $a_i + b_i$ is prime for all $i = 1, 2, \cdots, n$.

Applications

Theorem (1.1)

The set of integers $\{1, 2, 3, \cdots, 2n\}$, $n \geq 1$ can be partitioned into pairs $\{a_i, b_i\}$ such that $a_i + b_i$ is prime for all $i = 1, 2, \cdots, n$.

Theorem (1.2)

There are infinitely many G_{2n}'s that have a Hamiltonian cycle. (Not neccesarily all G_{2n})

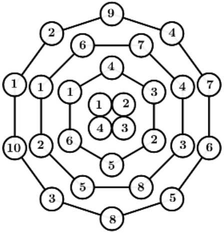

Figure 2: Examples for small G_{2n}'s with a Hamilton cycle.

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Theorem (1.3)

$G_{2n},\ n \geq 2$ contains a Hamiltonian cycle if there exists two primes $p_1 < p_2$ in $[1, 2n]$ such that $2n + p_1$ and $2n + p_2$ are primes and $\gcd\left(\frac{p_2 - p_1}{2}, n\right) = 1$.

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Theorem (2)

n'th Harmonic Number is not an integer, $n \geq 2$.

Questions?