Problems for Wednesday Tut Aug 13, 2025

(1) $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

prove that $f$ is diff. at origin. Do higher derivatives exist at the origin? If so find them.

(2) $f(x) = x^3 - 3x$

(i) Sketch the graph of $f$

(ii) Determine the values of $a \in \mathbb{R}$ for which the equation $f(x) = a$ has

(a) 3 distinct real roots

(b) Only one real root

(c) two distinct real roots.

The points $p$ at which $f'(p) = 0$ (for any $f: I \to \mathbb{R}$ diff) are called critical points

Codomain is $\mathbb{R}$

For future reference: In case the Codomain is $\mathbb{R}^k$ the definition of a critical point would be different: $I$ would be a domain in $\mathbb{R}^n$. We shall return to this point later. (much later)

The values $\{ f(p) \,/\, p \text{ is a critical pt} \}$ are called Critical values.

Can you describe what happens to the # of roots of $f(x) = a$ when $a$ crosses a critical value?

Can you experiment with other polynomials? Rational functions??

(3) Let $I$ be a closed bounded interval $f: I \to \mathbb{R}$ a bounded function

$M_I = \text{l.u.b} \{ f(x) \,/\, x \in I \}$

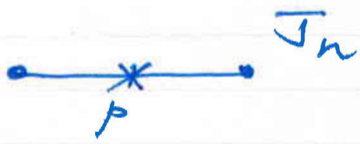
$m_I = \text{glb} \{ f(x) \,/\, x \in I$

$M_I - m_I = \sigma(I)$ is called the
Spread of $f$ across $I$

(Standard terminology is oscillation of $f$ over $I$. But the word osc. can create a false image of $f$ being some sort of sinusoidal graph. So we prefer to use the word ~~A~~spread of $f$ *)

(a) If $I \subset J$ what can you say about $\sigma(I)$ and $\sigma(J)$ ?

(b) Fix $p \in I$
Consider a seq of closed intervals $J_n$ all of them enclosing $p$




length $J_n \to 0$.

Is it necessary $\sigma(J_n) \to 0$ as $n \to \infty$ ?

If not provide a counter examples
(offer quite a few)
What if $f$ is continuous at $p$. ?
Can you say $\sigma(J_n) \to 0$ as $n \to \infty$ ?
Here: All the intervals contain $p$ (so the drama is localized in a nbd of $p$)

(c) Take an interval $[a, b]$
and do repeated bisections
$[a, b]$ : $\left[a, \frac{a+b}{2}\right]$ : $\left[\frac{a+b}{2}, b\right]$ etc.,

For simplicity we take $a = 0$, $b = 1$
(Gen. Case: Do it yourself)

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*) This terminology is certainly not mine.
It appears in
F. Bowman and Gerard: Higher Calculus
Camb. Univ. Press.

Let $f: [0,1] \longrightarrow \mathbb{R}$ continuous

$I_2 = [0,1]$

$I_3 = [0, \frac{1}{2}]$. $I_4 = [\frac{1}{2}, 1]$

$I_5 = [0, \frac{1}{4}]$, $I_6 = [\frac{1}{4}, \frac{1}{2}]$, $I_7 = [\frac{1}{2}, \frac{3}{4}]$,
$I_8 = [\frac{3}{4}, 1]$

etc;

The block $I_{2^{n-1}+1}, \ldots, I_{2^{n-1}+2^{n-1}}$
of $2^{n-1}$ intervals are all equal in length, non overlapping and length $= \frac{1}{2^{n-1}}$

~~[illegible]~~

Prove that $\lim_{n\to\infty} \sigma(I_n) = 0$
as $n \to \infty$

prove that if ~~[illegible]~~

$$A_n = \operatorname{Max}\{\sigma(I_v) / v = 2^{n-1}+1, \ldots, 2^{n-1}+2^{n-1}\}$$

then $A_n \to 0$ as $n \to \infty$.
Here the intervals go all across $[0,1]$ so drama is <u>NOT</u> localised in a nbd of a specific pt

(4) Prove the Basic Slope lemma for convex functions (see the notes).