MA-105 Tutorial-2 Solutions

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1. **Let $x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Show that $x_4 > 2$, $x_8 > \frac{5}{2}$ and $x_{16} > 3$**

   Proof. We know the sequence $x_n = \sum_{k=1}^{n} \frac{1}{k}$ is monotonically increasing

$$x_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = 1 + 0.5 + 0.333\ldots + 0.25 \approx 2.083 > 2$$

$$x_8 = x_4 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 2 + 4 \cdot \frac{1}{8} = 2 + 0.5 = 2.5 = \frac{5}{2}$$

$$x_{16} = x_8 + \sum_{k=9}^{16} \frac{1}{k} > \frac{5}{2} + 8 \cdot \frac{1}{16} = \frac{5}{2} + \frac{1}{2} = 3$$

   Hence the inequalities hold

$$x_4 > 2, \quad x_8 > \frac{5}{2}, \quad x_{16} > 3$$

$\square$

2. **Let $a > 0$ and $x_n = 1 + a + \frac{a^2}{2} + \cdots + \frac{a^n}{n!}$. Show that (x_n) is monotonically increasing and bounded above**

Proof. The sequence satisfies

$$x_{n+1} = x_n + \frac{a^{n+1}}{(n+1)!} \geq x_n$$

Hence (x_n) is monotonically increasing

Notice that for $a > 0$, for $n \geq 2$

$$x_n = \sum_{k=0}^{n} \frac{a^k}{k!} < 1+a+\frac{a^2}{2}+\frac{a^3}{4}+\cdots = 1+a+\frac{\frac{a^2}{2}}{1-\frac{a}{2}} = 1+a+\frac{a^2}{2-a}$$

Hence (x_n) is bounded above by $\frac{2+a}{2-a}$.

Conclusion: (x_n) is monotonically increasing and bounded above

$\square$

3. **Is the function** $f(x) = \frac{\log(x+1)}{\sin x}$ **defined on** $\left(-\frac{1}{2}, \frac{1}{2}\right)$ **differentiable at** 0**, given** $f(0) = 1$**? Find** $f'(0)$

Proof. Compute the derivative at 0 using the definition

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{\frac{\log(1+x)}{\sin x} - 1}{x} = \lim_{x\to 0} \frac{\log(1+x) - \sin x}{x \sin x}$$

Use the Taylor expansions around $x = 0$:

$$\log(1+x) = x - \frac{x^2}{2} + O(x^3), \qquad \sin x = x - \frac{x^3}{6} + O(x^5)$$

Thus

$$\log(1+x) - \sin x = -\frac{x^2}{2} + O(x^3), \quad x \sin x = x^2 + O(x^4)$$

Hence

$$f'(0) = \lim_{x\to 0} \frac{-\frac{x^2}{2} + O(x^3)}{x^2 + O(x^4)} = -\frac{1}{2}$$

Conclusion: f is differentiable at 0 and

$$\boxed{f'(0) = -\frac{1}{2}}$$

$\square$

4. **Compute** $D^n f(1)$ **when** $f(x) = \frac{(x^2-1)^n}{2^n n!}$

Proof. Notice $f(x) = \frac{(x-1)^n(x+1)^n}{2^n n!}$

Apply Leibniz formula:

$$D^n[(x-1)^n(x+1)^n] = \sum_{k=0}^{n} \binom{n}{k} D^k(x-1)^n D^{n-k}(x+1)^n$$

Only $D^n(x-1)^n = n!$, $D^n(x+1)^n = n!$, and $D^k(x-1)^n = 0$ for $k > n$

Evaluate at $x = 1$: $(x-1)^n = 0$ so all terms vanish except $k = n$

$$D^n f(1) = \frac{1}{2^n n!} \cdot \binom{n}{n} n! \cdot D^0(x+1)^n|_{x=1} = \frac{(1+1)^n}{2^n} = 1$$

Conclusion: $D^n f(1) = 1$ $\square$

5. **Prove that if f, g are two n-times differentiable functions at p then** $(D^n(fg))(p) = \sum_{k=0}^{n} \binom{n}{k} D^k f(p) D^{n-k} g(p)$

Proof. This is the Leibniz rule.

Induction on n:

Base case $n = 1$: $(fg)' = f'g + fg'$ holds.

Induction step: Assume formula holds for n, then

$$\begin{aligned} D^{n+1}(fg) &= D(D^n(fg)) \\ &= D\left(\sum_{k=0}^{n} \binom{n}{k} D^k f \cdot D^{n-k} g\right) \\ &= \sum_{k=0}^{n} \binom{n}{k} \left(D^{k+1} f \cdot D^{n-k} g + D^k f \cdot D^{n-k+1} g\right) \end{aligned}$$

Reindex sums and use $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ to obtain

$$D^{n+1}(fg) = \sum_{k=0}^{n+1} \binom{n+1}{k} D^k f \cdot D^{n+1-k} g$$

Hence the formula holds for all $n \in \mathbb{N}$ $\square$

6. **Prove that** $D^n \sin x = \sin(x + \frac{n\pi}{2})$

Proof. Use induction on n:

Base case $n = 0$**:** $D^0 \sin x = \sin x = \sin(x + 0)$

Induction step: Assume $D^n \sin x = \sin(x + n\pi/2)$. Then

$$\begin{aligned} D^{n+1} \sin x &= D(\sin(x + n\pi/2)) \\ &= \cos(x + n\pi/2) \\ &= \sin(x + n\pi/2 + \pi/2) \\ &= \sin(x + (n+1)\pi/2) \end{aligned}$$

Hence formula holds for all $n \in \mathbb{N}$ $\square$

7. **Find the n^{th} derivative of $\sin^{-1} x$ at $x = 0$**

Proof. Let $y = \sin^{-1} x$, so $x = \sin y$ and $\frac{dx}{dy} = \cos y$

Step 1: First derivative:

$$y' = \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$$

Step 2: Higher derivatives: By induction, only odd derivatives are nonzero at $x = 0$ due to symmetry, and we have

$$D^{2n}(\sin^{-1} x)(0) = 0$$

Step 3: Odd derivatives:

$$D^{2n+1}(\sin^{-1} x)(0) = \frac{((2n)!)^2}{4^n (n!)^2}$$

Hence the n^{th} derivative at 0 is

$$D^n(\sin^{-1} x)(0) = \begin{cases} 0 & \text{if } n \text{ even} \\ \dfrac{((2m)!)^2}{4^m (m!)^2} & \text{if } n = 2m + 1 \text{ odd} \end{cases}$$

$\square$

8. **Suppose $y(x)$ is infinitely differentiable and satisfies $(1-x^2)y'' - 2xy' + p(p+1)y = 0$. Prove that if $y(1) = 0$, then $D^n y(1) = 0$ for all $n \in \mathbb{N}$**

Proof. **Step 1: Factor $(1-x^2)$**

Rewrite the equation:

$$(1-x)(1+x)y'' - 2xy' + p(p+1)y = 0$$

Step 2: Evaluate at $x = 1$

At $x = 1$, $(1-x^2) = 0$, so the first term vanishes. Using $y(1) = 0$, the equation gives

$$-2y'(1) + p(p+1)y(1) = -2y'(1) + 0 = 0 \implies y'(1) = 0$$

Step 3: Higher derivatives by induction

Assume $D^k y(1) = 0$ for $k \leq n$, differentiate the ODE and evaluate at $x = 1$. Each differentiation yields a factor $(1-x)$ multiplying the highest derivative, which vanishes at $x = 1$, and the lower-order terms vanish by induction hypothesis. Hence $D^{n+1}y(1) = 0$

Conclusion: $D^n y(1) = 0$ for all $n \in \mathbb{N}$ □