# MA-105 Tutorial-8 Solutions

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1. **Compute the arc-length of the cycloid**

$$\vec{r}(t) = \big(a(t - \sin t),\ a(1 - \cos t)\big) \qquad 0 \le t \le 2\pi$$

**Sol.**

$$\begin{aligned} \vec{r}(t) &= \big(a(t - \sin t),\ a(1 - \cos t)\big) \\ \Rightarrow \quad \vec{r}'(t) &= \big(a(1 - \cos t),\ a \sin t\big) \end{aligned}$$

Now:

$$\begin{aligned} \|\vec{r}'(t)\|^2 &= \big(a(1 - \cos t)\big)^2 + \big(a \sin t\big)^2 \\ &= a^2(1 - \cos t)^2 + a^2 \sin^2 t \\ &= a^2\Big((1 - \cos t)^2 + \sin^2 t\Big) \end{aligned}$$

$$\begin{aligned} (1 - \cos t)^2 + \sin^2 t &= (1 - 2\cos t + \cos^2 t) + \sin^2 t \\ &= 1 - 2\cos t + (\cos^2 t + \sin^2 t) \\ &= 1 - 2\cos t + 1 \\ &= 2 - 2\cos t \\ &= 2(1 - \cos t) \end{aligned}$$

Hence

$$\|\vec{r}'(t)\|^2 = a^2 \cdot 2(1 - \cos t) = 2a^2(1 - \cos t)$$

Use the half-angle identity $1 - \cos t = 2\sin^2\frac{t}{2}$ to rewrite:

$$\|\vec{r}'(t)\|^2 = 2a^2 \cdot 2\sin^2\frac{t}{2} = 4a^2\sin^2\frac{t}{2}$$

$$\|\vec{r}'(t)\| = \sqrt{4a^2\sin^2\frac{t}{2}} = 2a\left|\sin\frac{t}{2}\right|$$

The arc-length is

$$L = \int_0^{2\pi} \|\vec{r}'(t)\|\,dt = \int_0^{2\pi} 2a\left|\sin\frac{t}{2}\right|\,dt$$

$$L = \int_{u=0}^{\pi} 2a\,|\sin u| \cdot 2\,du = 4a\int_0^{\pi} |\sin u|\,du = 8a$$

**Final answer:** $L = 8a$

2. **Parametrize the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and set up the perimeter integral in terms of the eccentricity $e=\sqrt{1-\frac{b^2}{a^2}}$ assuming $b<a$**

**Sol.**

A standard parametrization of the ellipse is

$$x = a\cos\theta \qquad y = b\sin\theta \qquad \theta \in [0, 2\pi]$$

Differentiate component-wise:

$$\vec{r}(\theta) = (a\cos\theta,\ b\sin\theta) \qquad \vec{r}'(\theta) = (-a\sin\theta,\ b\cos\theta)$$
$$\|\vec{r}'(\theta)\|^2 = a^2\sin^2\theta + b^2\cos^2\theta$$

Therefore the perimeter (circumference) is

$$L = \int_0^{2\pi} \|\vec{r}'(\theta)\|\, d\theta = \int_0^{2\pi} \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta$$

**Symmetry reduction** The integrand has period $\pi$ and symmetry in the four quadrants hence

$$L = 4\int_0^{\pi/2} \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta$$

Write $b^2 = a^2(1-e^2)$ where $0 \le e < 1$ since $b < a$ Then

$$\begin{aligned} a^2\sin^2\theta + b^2\cos^2\theta &= a^2\sin^2\theta + a^2(1-e^2)\cos^2\theta \\ &= a^2\big(\sin^2\theta + (1-e^2)\cos^2\theta\big) \\ &= a^2\big(\sin^2\theta + \cos^2\theta - e^2\cos^2\theta\big) \\ &= a^2\big(1 - e^2\cos^2\theta\big) \end{aligned}$$

Therefore the integrand simplifies to

$$\|\vec{r}'(\theta)\| = a\sqrt{1 - e^2 \cos^2 \theta}$$

Thus the perimeter becomes

$$\boxed{L = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \cos^2 \theta}\, d\theta}$$

(One often sees the alternative form with $\sin^2$ inside by the substitution $\theta \mapsto \frac{\pi}{2} - \theta$

$$\boxed{L = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta}\, d\theta}$$

which is the standard complete elliptic integral of the second kind)

**Remark:** This integral cannot be expressed in elementary functions for arbitrary $e$ it is denoted

$$L = 4a\, E(e)$$

where $E(e) = \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta}\, d\theta$ is the complete elliptic integral of the second kind

3. **If $\hat{a}$ and $\hat{b}$ are two unit vectors in $\mathbb{R}^2$ such that $\hat{a} \times \hat{b} \neq \mathbf{0}$ show that the set**

$$\{\hat{a}\cos t + \hat{b}\sin t \mid t \in [0, 2\pi]\}$$

**is an ellipse**

**Sol.**

Write the unit vectors in coordinates:

$$\hat{a} = (a_1, a_2) \qquad \hat{b} = (b_1, b_2)$$

The condition $\hat{a} \times \hat{b} \neq \mathbf{0}$ in $\mathbb{R}^2$ means the scalar (2D) cross product (or determinant)

$$D := a_1 b_2 - a_2 b_1 \neq 0$$

so $\hat{a}$ and $\hat{b}$ are linearly independent

Form the parametric coordinates $(x, y)$ for $t \in [0, 2\pi]$:

$$x(t) = a_1 \cos t + b_1 \sin t \qquad y(t) = a_2 \cos t + b_2 \sin t$$

We solve these two linear equations for $\cos t$ and $\sin t$ Treat it as a $2 \times 2$ linear system in the unknowns $\cos t, \sin t$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

Since the coefficient matrix has determinant $D \neq 0$ it is invertible Using Cramer's rule the solutions are

$$\cos t = \frac{1}{D} \det \begin{pmatrix} x & b_1 \\ y & b_2 \end{pmatrix} = \frac{b_2 x - b_1 y}{D}$$

$$\sin t = \frac{1}{D} \det \begin{pmatrix} a_1 & x \\ a_2 & y \end{pmatrix} = \frac{a_1 y - a_2 x}{D}$$

Now impose the trigonometric identity $\cos^2 t + \sin^2 t = 1$ Substitute the expressions above:

$$\left(\frac{b_2 x - b_1 y}{D}\right)^2 + \left(\frac{a_1 y - a_2 x}{D}\right)^2 = 1$$

Multiply by $D^2$ and expand to obtain a quadratic equation in $(x, y)$:

$$(b_2 x - b_1 y)^2 + (a_1 y - a_2 x)^2 = D^2$$

Expand and collect like terms:

$$\begin{aligned}(b_2 x - b_1 y)^2 &= b_2^2 x^2 - 2b_1 b_2 xy + b_1^2 y^2 \\ (a_1 y - a_2 x)^2 &= a_1^2 y^2 - 2a_1 a_2 xy + a_2^2 x^2\end{aligned}$$

Adding yields

$$(b_2^2 + a_2^2)x^2 - 2(b_1 b_2 + a_1 a_2)xy + (b_1^2 + a_1^2)y^2 = D^2$$

Thus the curve satisfies a quadratic form

$$Ax^2 + Bxy + Cy^2 = D^2$$

with

$$A = a_2^2 + b_2^2 \qquad B = -2(a_1 a_2 + b_1 b_2) \qquad C = a_1^2 + b_1^2$$

To determine the type of conic compute the discriminant $\Delta = B^2 - 4AC$

$$
\begin{aligned}
\Delta &= \big(-2(a_1a_2 + b_1b_2)\big)^2 - 4(a_2^2 + b_2^2)(a_1^2 + b_1^2) \\
&= 4(a_1a_2 + b_1b_2)^2 - 4\big(a_1^2 + b_1^2\big)\big(a_2^2 + b_2^2\big) \\
&= 4\Big((a_1a_2 + b_1b_2)^2 - (a_1^2 + b_1^2)(a_2^2 + b_2^2)\Big)
\end{aligned}
$$

The expression inside the parentheses simplifies to $-(a_1b_2 - a_2b_1)^2 = -D^2$ Hence

$$\Delta = 4(-D^2) = -4D^2 < 0$$

Since $\Delta < 0$ the curve is an ellipse

4. **Identify the parametrized surface**

$$\vec{r}(u,v) = \left(\sqrt{1+v^2}\cos u,\ \sqrt{1+v^2}\sin u,\ v\right)$$

**Sol.**

Write the components as $x = \sqrt{1+v^2}\cos u$ $y = \sqrt{1+v^2}\sin u$ $z = v$ Compute $x^2 + y^2$

$$\begin{aligned} x^2 + y^2 &= (\sqrt{1+v^2}\cos u)^2 + (\sqrt{1+v^2}\sin u)^2 \\ &= (1+v^2)(\cos^2 u + \sin^2 u) \\ &= 1 + v^2 \end{aligned}$$

But $z = v$ so $v^2 = z^2$ Substitute:

$$x^2 + y^2 = 1 + z^2$$

Rearrange:

$$x^2 + y^2 - z^2 = 1$$

This is the equation of a one-sheeted hyperboloid

5. **Find a bijective continuous map from the cylinder**

$$\mathbb{S}_1 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1\}$$

**onto**

$$\mathbb{S}_2 := \{(X, Y, Z) \in \mathbb{R}^3 \mid X^2 + Y^2 - Z^2 \leq 1\}$$

**Sol.**

Define the map $f : \mathbb{S}_1 \to \mathbb{R}^3$ by

$$f(x, y, z) = \big(X, Y, Z\big) = \big(x\sqrt{1+z^2},\ y\sqrt{1+z^2},\ z\big)$$

**(i) $f$ maps $\mathbb{S}_1$ into $\mathbb{S}_2$**

Take $(x, y, z) \in \mathbb{S}_1$ so $x^2 + y^2 \leq 1$ Compute

$$\begin{aligned} X^2 + Y^2 - Z^2 &= (1 + z^2)(x^2 + y^2) - z^2 \\ &= x^2 + y^2 + z^2(x^2 + y^2 - 1) \end{aligned}$$

Because $x^2 + y^2 - 1 \leq 0$ we have

$$X^2 + Y^2 - Z^2 \leq x^2 + y^2 \leq 1$$

Hence $f(\mathbb{S}_1) \subseteq \mathbb{S}_2$

**(ii) Injectivity**

Suppose $f(x_1, y_1, z_1) = f(x_2, y_2, z_2)$ Then $z_1 = z_2 = z$ and

$$x_1\sqrt{1+z^2} = x_2\sqrt{1+z^2} \qquad y_1\sqrt{1+z^2} = y_2\sqrt{1+z^2}$$

Thus $x_1 = x_2$ and $y_1 = y_2$ so injective

**(iii) Surjectivity**

Let $(X, Y, Z) \in \mathbb{S}_2$ Define

$$(x, y, z) := \left(\frac{X}{\sqrt{1+Z^2}},\ \frac{Y}{\sqrt{1+Z^2}},\ Z\right)$$

Then

$$x^2 + y^2 = \frac{X^2 + Y^2}{1 + Z^2} \leq 1$$

and $f(x, y, z) = (X, Y, Z)$ Hence surjective

**(iv) Continuity and inverse**

Both $f$ and

$$f^{-1}(X, Y, Z) = \left(\frac{X}{\sqrt{1+Z^2}},\ \frac{Y}{\sqrt{1+Z^2}},\ Z\right)$$

are continuous Hence a homeomorphism

6. **Vector calculus identities** For a scalar $f : \mathbb{R}^3 \to \mathbb{R}$ and a vector field $F = (F_1, F_2, F_3) : \mathbb{R}^3 \to \mathbb{R}^3$ (assume all components are $C^2$) prove

$$\nabla \times (\nabla f) = \mathbf{0} \qquad \nabla \cdot (\nabla \times F) = 0$$

**Sol.**

**(A)** $\nabla \times (\nabla f) = \mathbf{0}$

$$\nabla f = (f_x, f_y, f_z)$$

By definition

$$\nabla \times (\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix} = \big(f_{zy} - f_{yz},\ f_{xz} - f_{zx},\ f_{yx} - f_{xy}\big)$$

By equality of mixed partial derivatives (Clairaut's theorem) all terms vanish so

$$\nabla \times (\nabla f) = (0, 0, 0) = \mathbf{0}$$

**(B)** $\nabla \cdot (\nabla \times F) = 0$

$$\nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix} = (F_{3y} - F_{2z},\ F_{1z} - F_{3x},\ F_{2x} - F_{1y})$$

Now take divergence:

$$\nabla \cdot (\nabla \times F) = (F_{3y} - F_{2z})_x + (F_{1z} - F_{3x})_y + (F_{2x} - F_{1y})_z$$

Expand:

$$= F_{3yx} - F_{2zx} + F_{1zy} - F_{3xy} + F_{2xz} - F_{1yz}$$

By equality of mixed partials each pair cancels hence

$$\nabla \cdot (\nabla \times F) = 0$$