MA-105 Tutorial-1

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1. Define $a_1 = \sqrt{2}$, $a_2 = \sqrt{2 + \sqrt{2}}$, and generally $a_n = \sqrt{2 + a_{n-1}}$ for $n \geq 1$. Prove that the sequence is monotonically increasing and bounded above. Also find its limit.

2. Let
$$x_n = \frac{1}{n+1} + \ldots + \frac{1}{n+n}$$
Prove that (x_n) is monotonically increasing and bounded above. Prove that the limit of (x_n) lies between $\frac{1}{2}$ and 1.

3. Suppose (x_n) is a monotonically increasing sequence. Prove that the sequence of averages $y_n = \frac{1}{n}(x_1 + \ldots + x_n)$ is also monotonically increasing.

4. Discuss whether the given sequence:
$$x_n = \frac{n}{n^2+1} + \frac{n}{n^2+2} + \ldots + \frac{n}{n^2+n}$$
is convergent.

5. Let
$$x_n = \frac{1 \times \ldots \times (2n-1)}{2 \times \ldots \times (2n)}$$

Show that nx_n^2 is monotonically increasing and $(n+\frac{1}{2})x_n^2$ is monotonically decreasing.

Check the convergence of x_n and nx_n.

6. Let (a_n) be a sequence such that $a_j \in \{0, 1, \ldots, 9\}\ \forall j$.

   Now for $n = 1, 2, 3, \ldots$, construct

$$x_n = \frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_n}{10^n}$$

   Show that the sequence (x_n) is monotonically increasing and bounded above.

   Call $\lim_{n\to\infty} x_n = a$, $a \in [0,\ 1]$. Is it true that given any a, there exists a corresponding sequence (a_n) as above such that

$$\lim_{n\to\infty}\left(\frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_n}{10^n}\right) = a$$