Tutorial 8 Sept 24, 2025

(1) Compute the arc length of the cycloid (notes. p125)

$\vec{r}(t) = (a(t-\sin t),\ a(1-\cos t))$; $0 \le t \le 2\pi$

(2) Parametrize $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and set up the integral for the perimeter in terms of eccentricity.

(integral cannot be evaluated in elementary terms)

(3) $\hat{a}, \hat{b}$ two unit vectors in $\mathbb{R}^2$ s.t $\hat{a} \times \hat{b} \ne 0$.

Show that $\{\hat{a}\cos t + \hat{b}\sin t \,/\, 0 \le t \le 2\pi\}$ is an ellipse

(4) Identify the parametrized surface (p135 notes).

$\vec{r}(u,v) = (\sqrt{1+v^2}\cos u)i + (\sqrt{1+v^2}\sin u)j + vk$

(5) Find a bijective continuous map from the cylinder $\{(x,y,z) / x^2+y^2 \le 1\}$ onto

$\{(x,y,z) / x^2+y^2-z^2 \le 1\}$.

Further arrange it so that inverse is also continuous.

Which of these sets is convex?

(6) Find the unit normal $N(u,v) = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$

for the Möbius band (See p 142-143 of the notes).

Fix $v=0$ and let u vary from 0 to 2π.

What happens to ~~....~~ $N(u,0)$? Trace the curve $\vec{r}(0,v)$ on the surface

Discuss with pictures.

(7) For a scalar valued $f: \mathbb{R}^3 \to \mathbb{R}$, $\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$

$= \text{grad } f$

For a vect. valued $F: \mathbb{R}^3 \to \mathbb{R}^3$, $\underline{\nabla \cdot F} = \text{Div} F = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

$F = f_1 i + f_2 j + f_3 k$ only a notation and $\quad \text{Curl } F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$

Only a notation ~~for~~ as a mnemonic aid. not to be taken literally.

$$= i\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - j\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + k\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

<u>Def:</u> $\text{Div } F = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

$$\text{Curl } F = i\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + j\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + k\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right).$$

<u>Prove:</u> $\text{Curl}(\text{grad } f) = 0.$

$\text{Div}(\text{Curl } F) = 0$

Everything is twice cont. diff..

(8) What is wrong with the following derivation

$\text{Div}(F \times G) = (\text{Curl } F) \cdot G \qquad (?)$

$\nabla \cdot (F \times G) = (\nabla \times (F)) \cdot G = (\text{Curl } F) \cdot G$

point out the mistake. Correct it and prove the correct result