MA-105 Tutorial-5 – Solutions

Daksh Maahor

August 2025

1. **Find the volume of the solid obtained by revolving the given shaded region about the x-axis.**

   The given region is bounded by the curves:

$$x = y^2, \quad x = 3y^2 - 2$$

   Their intersection points occur when:

$$y^2 = 3y^2 - 2 \implies 2y^2 = 2 \implies y = \pm 1$$

   The limits for y are from -1 to 1. Revolving around the x-axis means each horizontal strip at height y generates a disk of radius y and thickness dx. However, since our equations are in terms of y, we use the shell method:

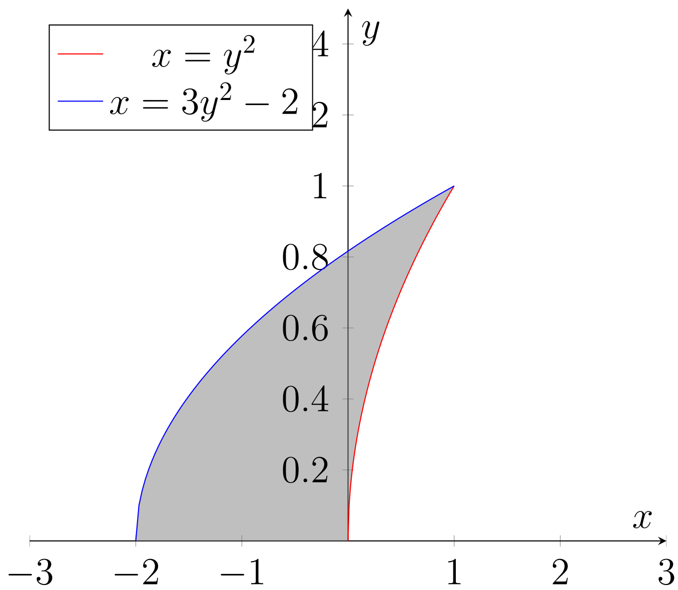
$$V = 2\pi \int_{y=-1}^{1} y(x_{right} - x_{left})dy = 2\pi \int_{-1}^{1} y[(3y^2 - 2) - (y^2)]dy = 2\pi \int_{-1}^{1} y(2y^2 - 2)dy$$

   Because the integrand is odd, the volume is twice the integral from 0 to 1:

$$V = 4\pi \int_{0}^{1} y(2y^2 - 2)dy = 4\pi \left[\frac{y^4}{2} - y^2\right]_0^1 = 4\pi \left(\frac{1}{2} - 1\right) = -2\pi$$

Since volume is positive:

$$\boxed{V = 2\pi}$$



2. **Prove that if $f : [a, b] \to \mathbb{R}$ is continuous, then $\exists c \in [a, b]$ such that**

$$\int_a^b f(x)dx = (b - a)f(c)$$

and deduce that $\dfrac{\mathrm{d}}{\mathrm{d}x} \displaystyle\int_a^x f(t)dt = f(x)$.

Since f is continuous on $[a, b]$, it attains its maximum M and minimum m. By the properties of integrals:

$$m(b - a) \leq \int_a^b f(x)dx \leq M(b - a)$$

Dividing both sides by $(b - a)$ gives:

$$m \leq \frac{1}{b - a} \int_a^b f(x)dx \leq M$$

By the Intermediate Value Theorem, there exists $c \in [a, b]$ such that:

$$f(c) = \frac{1}{b-a} \int_a^b f(x)dx$$

Multiplying both sides by $(b - a)$, we obtain:

$$\boxed{\int_a^b f(x)dx = (b-a)f(c)}$$

To deduce $\dfrac{\mathrm{d}}{\mathrm{d}x} \displaystyle\int_a^x f(t)dt = f(x)$, define $F(x) = \int_a^x f(t)dt$. For any $h > 0$,

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t)dt$$

By the Mean Value Theorem for integrals, there exists $c_h \in [x, x+h]$ such that:

$$\frac{F(x+h) - F(x)}{h} = f(c_h)$$

Taking $h \to 0$, continuity of f implies $c_h \to x$ and thus:

$$\boxed{F'(x) = f(x)}$$

3. **Prove that the set $\left\{(x, y) \mid \frac{x^2}{44} - \frac{y^2}{37} < 1\right\}$ is open.**

   Let $g(x, y) = \frac{x^2}{44} - \frac{y^2}{37}$. This function is continuous everywhere. The set can be written as:

   $$S = g^{-1}((-\infty, 1))$$

   Since $(-\infty, 1)$ is open in $\mathbb{R}$ and g is continuous, the preimage S is open in $\mathbb{R}^2$.

   $$\boxed{S \text{ is open.}}$$

4. **Is the set $\left\{(x,y) \mid \frac{x^2}{4}+\frac{y^2}{9}<1\right\}$ convex?**

   Let S be the given set. For any two points $(x_1,y_1),(x_2,y_2) \in S$, we must show that every point on the line segment joining them is also in S.

   Consider $(x,y) = t(x_1,y_1)+(1-t)(x_2,y_2)$, where $0 \le t \le 1$. Then:

$$\frac{x^2}{4}+\frac{y^2}{9} \le t\left(\frac{x_1^2}{4}+\frac{y_1^2}{9}\right)+(1-t)\left(\frac{x_2^2}{4}+\frac{y_2^2}{9}\right) < t+(1-t)=1$$

   using convexity of the quadratic function. Hence the set is convex.

$$\boxed{S \text{ is convex.}}$$

5. **Prove that a polynomial in two variables is a continuous function.**

   Let $p(x,y)=\sum_{i,j} a_{ij}x^iy^j$. Each monomial x^iy^j is continuous as it is a product of continuous functions. A finite linear combination of continuous functions is continuous, hence $p(x,y)$ is continuous on $\mathbb{R}^2$.

$$\boxed{p \text{ is continuous everywhere.}}$$

6. **Prove that if $f(x,y)$ is continuous then the level set $\{(x,y) \mid f(x,y)=c\}$ is closed.**

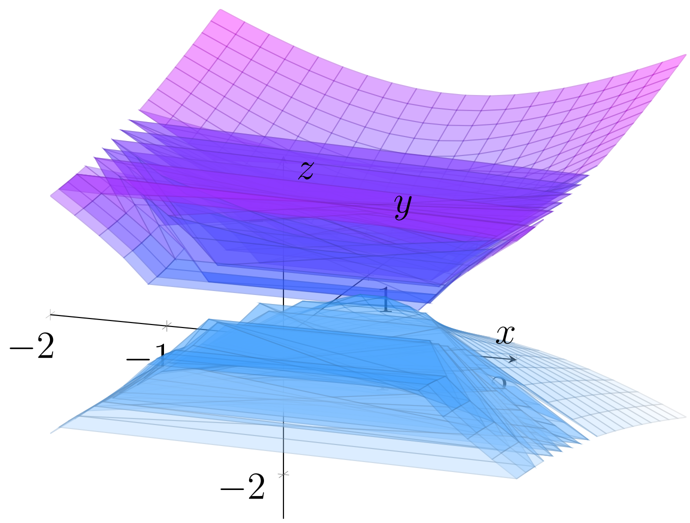
   Let $A=\{(x,y) \mid f(x,y)=c\}$. Then $A=f^{-1}(\{c\})$. The singleton set $\{c\}$ is closed in R and f is continuous, so the preimage of a closed set under a continuous map is closed. Hence:

$$\boxed{A \text{ is closed.}}$$

7. **Sketch the level sets of $f(x,y,z)=x^2+y^2-z^2$.**

   We have $f(x,y,z)=k \implies x^2+y^2-z^2=k$.

- For $k > 0$, we get a **two-sheeted hyperboloid**. - For $k = 0$, we get a **double cone**. - For $k < 0$, we get a **one-sheeted hyperboloid**.



These represent the level surfaces for $f(x, y, z) = 1$ and $f(x, y, z) = -1$, respectively.