MA-105 Tutorial-3 Solutions

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1. **Let a function $f(x)$ be defined as:**

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Prove that f is differentiable at origin. Do higher order derivatives exist at origin? If so, find them.

Solution:

We first show differentiability at 0. For $x \neq 0$ the function is smooth; the only nontrivial point is $x = 0$. Consider the difference quotient:

$$\frac{f(x) - f(0)}{x - 0} = \frac{e^{-1/x^2}}{x} = \frac{1}{x}e^{-1/x^2}.$$

Let $t = 1/|x|$. Then

$$\left|\frac{e^{-1/x^2}}{x}\right| = te^{-t^2}.$$

As $x \to 0$, $t \to \infty$ and $te^{-t^2} \to 0$ because exponential decay dominates any polynomial growth. Therefore the difference quotient tends to 0, so $f'(0) = 0$. Thus f is differentiable at 0.

Next we show that *all* higher derivatives at 0 exist and are 0. We prove by induction that for every integer $n \geq 0$ there exists a polynomial $P_n(1/x)$ (in $1/x$) such that for $x \neq 0$:

$$f^{(n)}(x) = P_n\big(1/x\big)e^{-1/x^2}.$$

This is standard: differentiation of e^{-1/x^2} yields factors of powers of $1/x$ times e^{-1/x^2}, so the representation holds for each n. In particular each $f^{(n)}(x)$ (for $x \neq 0$) decays faster than any power of x as $x \to 0$, because e^{-1/x^2} dominates.

To compute the n-th derivative at 0 we use the limit definition:

$$f^{(n)}(0) = \lim_{x \to 0} \frac{f^{(n-1)}(x) - f^{(n-1)}(0)}{x - 0}.$$

By the inductive representation, for $x \neq 0$ we can write

$$f^{(n-1)}(x) = Q(1/x)e^{-1/x^2}$$

for some polynomial Q. Hence

$$\left|\frac{f^{(n-1)}(x) - f^{(n-1)}(0)}{x}\right| = \frac{|Q(1/x)|}{|x|}e^{-1/x^2} \to 0$$

as $x \to 0$. Therefore $f^{(n)}(0) = 0$ for every $n \geq 1$. The base case $f(0) = 0$ already holds, so all derivatives of all orders exist at 0 and equal 0.

Remark: This shows f is a C^∞ function on $\mathbb{R}$ whose Taylor series about 0 is identically 0, yet f is not the zero function (so the Taylor series does not converge to f away from 0).

2. **Let $f(x) = x^3 - 3x$. Sketch the graph of f(x). Determine the values of $a \in \mathbb{R}$ for which the equation $f(x) = a$ has 3, 1, and no real roots.**

   The points p at which $f'(p) = 0$ are called critical points, and the value of $f(x)$ at that point, $f(p)$ are called critical values. Describe what happens to the roots of $f(x) = a$ when a crosses a critical value of $f(x)$.

   Solution:

   Compute derivative:

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1).$$

   Critical points satisfy $f'(x) = 0$, so $x = \pm 1$. Evaluate f at these points:

$$f(1) = 1^3 - 3 \cdot 1 = -2, \qquad f(-1) = (-1)^3 - 3(-1) = 2.$$

   Thus $x = -1$ is a local maximum with value 2, and $x = 1$ is a local minimum with value -2. The cubic has the usual shape: rising to $+\infty$ as $x \to +\infty$ and falling to $-\infty$ as $x \to -\infty$, with a hill at $(-1, 2)$ and a valley at $(1, -2)$.

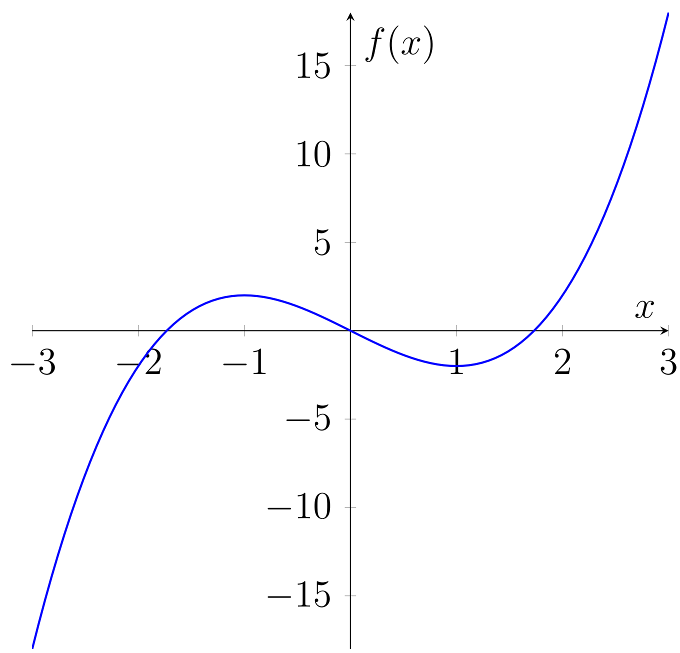
   Because a cubic polynomial is odd-degree with real coefficients, it always has at least one real root. The number of *distinct* real roots of $f(x) - a = 0$ depends on a relative to the critical values 2 and -2:

   - If $-2 < a < 2$, the horizontal line $y = a$ intersects the graph in three distinct points $\Rightarrow$ three distinct real roots.
   - If $a = 2$ or $a = -2$, the horizontal line is tangent to the curve at one of the critical points, producing a double root and a simple root (so two real roots counting multiplicity, but only two distinct roots if one is double).

- If $a > 2$ or $a < -2$, the horizontal line meets the cubic only once $\Rightarrow$ exactly one real root.

In particular there are never *no* real roots for a real cubic; at least one real root always exists.

When the parameter a crosses a critical value of f (for instance goes from slightly below 2 to slightly above 2), two of the real roots collide at the critical point and then become complex (conjugate) for parameter values beyond the critical value. Thus crossing a critical value is precisely the event where two real roots merge into a double root and then cease to be two distinct real roots.



3. **Let $\mathbb{I}$ be a closed bounded interval, and $f : \mathbb{I} \to \mathbb{R}$ be a bounded function.**

$$M_{\mathbb{I}} = \text{lub}\{f(x) \mid x \in \mathbb{I}\}$$
$$m_{\mathbb{I}} = \text{glb}\{f(x) \mid x \in \mathbb{I}\}$$

The value $M_{\mathbb{I}} - m_{\mathbb{I}} = \sigma(\mathbb{I})$ is called the spread of f across $\mathbb{I}$.

(a) If $\mathbb{I} \subset \mathbb{J}$ what can you say about $\sigma(\mathbb{I})$ and $\sigma(\mathbb{J})$?

(b) Fix $p \in \mathbb{I}$, consider a set of closed intervals $\mathbb{J}_n$ that all enclose p, such that length of $\mathbb{J}_n \to 0$ as $n \to \infty$. Is it necessary that for any function $f : \mathbb{I} \to \mathbb{R}$, $\sigma(\mathbb{J}_n) \to 0$ as $n \to \infty$ for the function f. If not provide a counter-example. What if f is continuous at p? Can you then say $\sigma(\mathbb{J}_n) \to 0$ as $n \to \infty$?

(c) Take an interval $[a,\ b]$ and do repeated bisections on it $[a,\ b]; [a,\ \frac{a+b}{2}]\ [\frac{a+b}{2},\ b]; [a,\ \frac{3a+b}{4}]\ [\frac{3a+b}{4},\ \frac{a+b}{2}]\ [\frac{a+b}{2},\ \frac{a+3b}{4}]\ [\frac{a+3b}{4}, b]$ etc. For simplicity, take $a = 0$ and $b = 1$.
Let $f : [0,\ 1] \to \mathbb{R}$ be a continuous function.
Let $\mathbb{I}_2 = [0,\ 1]$
Let $\mathbb{I}_3 = [0,\ \frac{1}{2}]$, $\mathbb{I}_4 = [\frac{1}{2},\ 1]$
Let $\mathbb{I}_5 = [0,\ \frac{1}{4}]$, $\mathbb{I}_6 = [\frac{1}{4},\ \frac{1}{2}]$, $\mathbb{I}_7 = [\frac{1}{2},\ \frac{3}{4}]$, $\mathbb{I}_8 = [\frac{3}{4},\ 1]$ etc.
The 2^{n-1} intervals $\mathbb{I}_{2^{n-1}+1},\ \ldots,\ \mathbb{I}_{2^n}$ are all non overlapping and equal in length $l = \frac{1}{2^{n-1}}$.
Prove that $\lim_{n\to\infty} \sigma(\mathbb{I}_n) = 0$
If $A_n = \max\{\sigma(\mathbb{I}_j) \mid j = 2^{n-1}+1, \ldots, 2^n\}$, prove that $\lim_{n\to\infty} \mathbb{A}_n = 0$

Solution:

(a) If $\mathbb{I} \subset \mathbb{J}$ then the set of values $f(\mathbb{I})$ is a subset of $f(\mathbb{J})$. Hence

$$m_{\mathbb{J}} \leq m_{\mathbb{I}} \leq M_{\mathbb{I}} \leq M_{\mathbb{J}},$$

so

$$\sigma(\mathbb{I}) = M_{\mathbb{I}} - m_{\mathbb{I}} \leq M_{\mathbb{J}} - m_{\mathbb{J}} = \sigma(\mathbb{J}).$$

Thus $\sigma(\mathbb{I}) \leq \sigma(\mathbb{J})$.

(b) It is *not* necessary in general. Counterexample: let f be the Dirichlet-type function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q}, \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

This function is bounded on any interval but is discontinuous at every point. For any sequence of closed intervals $\mathbb{J}_n$ containing a point p with lengths shrinking to 0, each interval contains both rationals and irrationals, so

$$\sup\{f(x) \mid x \in \mathbb{J}_n\} = 1, \qquad \inf\{f(x) \mid x \in \mathbb{J}_n\} = 0$$

for every n. Hence $\sigma(\mathbb{J}_n) = 1$ for all n and does *not* tend to 0.

If, however, f is continuous at p, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - p| < \delta$ then $|f(x) - f(p)| < \varepsilon/2$. For sufficiently large n, the interval $\mathbb{J}_n$ lies inside $(p - \delta, p + \delta)$, so for such n we have

$$M_{\mathbb{J}_n} - f(p) < \varepsilon/2, \qquad f(p) - m_{\mathbb{J}_n} < \varepsilon/2,$$

and therefore

$$\sigma(\mathbb{J}_n) = M_{\mathbb{J}_n} - m_{\mathbb{J}_n} < \varepsilon.$$

Thus $\sigma(\mathbb{J}_n) \to 0$ as $n \to \infty$ when f is continuous at p.

(c) Now suppose f is continuous on the compact interval $[0,1]$. Then f is uniformly continuous. Fix $\varepsilon > 0$. By uniform continuity there exists $\delta > 0$ such that whenever $|x-y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$. Choose N large enough so that the length $\ell_n = 2^{-(n-1)} < \delta$ for all $n \geq N$. Every subinterval $\mathbb{I}_j$ at the n-th stage has length $\ell_n < \delta$, so the oscillation (spread) on each such interval is less than ε. Hence for all $n \geq N$ and all j in the range $2^{n-1}+1, \ldots, 2^n$ we have $\sigma(\mathbb{I}_j) < \varepsilon$. Therefore

$$A_n = \max \sigma(\mathbb{I}_j) \leq \varepsilon \qquad \text{for all } n \geq N,$$

which implies $A_n \to 0$ as $n \to \infty$. In particular for any sequence of nested intervals whose lengths go to 0 we also get $\sigma(\mathbb{I}_n) \to 0$.

4. **Let $f : \mathbb{I} \to \mathbb{R}$ be a convex function on a closed interval $\mathbb{I}$. Assume $x, y, z \in \mathbb{I}$, $x < y < z$. Prove that the slope of chord joining $(x, f(x))$ and $(y, f(y)) \leq$ slope of chord joining $(y, f(y))$ and $(z, f(z))$.**

   Solution:

   Since $x < y < z$, the point y lies between x and z. Hence we can express y as a convex combination of x and z. Concretely, set

$$t = \frac{y - x}{z - x}.$$

   Notice that since $x < y < z$, the fraction satisfies $0 < t < 1$. Then we can write

$$y = (1 - t)x + tz.$$

   Now recall the definition of convexity: a function f is convex on $\mathbb{I}$ if for every $u, v \in \mathbb{I}$ and $\lambda \in [0, 1]$,

$$f\big((1 - \lambda)u + \lambda v\big) \leq (1 - \lambda)f(u) + \lambda f(v).$$

   Applying this definition with $u = x$, $v = z$, and $\lambda = t$, we obtain

$$f(y) = f\big((1 - t)x + tz\big) \leq (1 - t)f(x) + tf(z).$$

   This inequality is the starting point. Subtract $f(x)$ from both sides:

$$f(y) - f(x) \leq (1 - t)\big(f(x) - f(x)\big) + t\big(f(z) - f(x)\big),$$

   which simplifies to

$$f(y) - f(x) \leq t\big(f(z) - f(x)\big).$$

Now divide both sides by $y - x$. Since $y > x$, this denominator is positive, so the inequality direction is preserved:

$$\frac{f(y) - f(x)}{y - x} \leq \frac{t}{y - x}\big(f(z) - f(x)\big).$$

But recall that $t = \dfrac{y - x}{z - x}$. Substituting this expression for t, the right-hand side becomes

$$\frac{t}{y - x}(f(z) - f(x)) = \frac{1}{z - x}(f(z) - f(x)).$$

So we now have

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(x)}{z - x}.$$

Next, we want to compare $\dfrac{f(y) - f(x)}{y - x}$ with $\dfrac{f(z) - f(y)}{z - y}$. To do this, observe that

$$\frac{f(z) - f(x)}{z - x} = \frac{(f(z) - f(y)) + (f(y) - f(x))}{(z - y) + (y - x)}.$$

This is a weighted average of the two slopes:

$$\frac{f(z) - f(x)}{z - x} = \frac{z - y}{z - x} \cdot \frac{f(z) - f(y)}{z - y} + \frac{y - x}{z - x} \cdot \frac{f(y) - f(x)}{y - x}.$$

Since the coefficients $\dfrac{z - y}{z - x}$ and $\dfrac{y - x}{z - x}$ are positive and sum to 1, this shows that the slope $\dfrac{f(z) - f(x)}{z - x}$ is a convex combination of the two smaller slopes.

From the inequality we derived earlier,

$$\frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(x)}{z-x},$$

it follows that $\dfrac{f(y)-f(x)}{y-x}$ is less than or equal to this weighted average of the two slopes. The only way this can hold is if

$$\frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(y)}{z-y}.$$

Thus, the slope of the secant line through $(x, f(x))$ and $(y, f(y))$ is less than or equal to the slope of the secant line through $(y, f(y))$ and $(z, f(z))$.

$\square$