

MA-105 Tutorial-1 Solutions

Daksh Maahor

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1. **Define** $a_1 = \sqrt{2}$, $a_2 = \sqrt{2 + \sqrt{2}}$, **and generally** $a_n = \sqrt{2 + a_{n-1}}$ **for** $n \geq 2$. **Prove by induction that the sequence is monotonically increasing and bounded above. Find its limit**

Proof. **Monotonicity by induction**

Base case:

$$a_2 = \sqrt{2 + \sqrt{2}} > \sqrt{2} = a_1$$

Inductive step: assume for some $n \geq 2$ that

$$a_n > a_{n-1}$$

We show $a_{n+1} > a_n$ Consider

$$a_{n+1} - a_n = \sqrt{2 + a_n} - \sqrt{2 + a_{n-1}}$$

Rationalize the difference:

$$a_{n+1} - a_n = \frac{(2 + a_n) - (2 + a_{n-1})}{\sqrt{2 + a_n} + \sqrt{2 + a_{n-1}}} = \frac{a_n - a_{n-1}}{\sqrt{2 + a_n} + \sqrt{2 + a_{n-1}}}$$

By the inductive hypothesis $a_n - a_{n-1} > 0$ and the denominator is positive, hence

$$a_{n+1} - a_n > 0$$

Thus $a_{n+1} > a_n$ and the induction closes Therefore (a_n) is strictly increasing

Boundedness above by induction

Base case: $a_1 = \sqrt{2} < 2$

Inductive step: assume $a_n < 2$ for some $n \geq 1$. Then

$$a_{n+1} = \sqrt{2 + a_n} < \sqrt{2 + 2} = \sqrt{4} = 2$$

So $a_{n+1} < 2$ and by induction $a_n < 2$ for all n Hence (a_n) is bounded above by 2

Limit

Since (a_n) is increasing and bounded above it converges Let

$$L = \lim_{n \rightarrow \infty} a_n$$

Passing to the limit in the recursion yields

$$L = \sqrt{2 + L}$$

Square both sides

$$L^2 = 2 + L \iff L^2 - L - 2 = 0 \iff (L - 2)(L + 1) = 0$$

Thus $L = 2$ or $L = -1$ Since every $a_n > 0$ we must have $L = 2$
Therefore

$$\boxed{\lim_{n \rightarrow \infty} a_n = 2}$$

□

2. Let $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$. Prove that (x_n) is monotonically increasing and bounded above. Prove that the limit of (x_n) lies between $\frac{1}{2}$ and 1

Proof. We can write

$$x_n = \sum_{k=n+1}^{2n} \frac{1}{k} = H_{2n} - H_n$$

where $H_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}$ is the m -th harmonic number.

Monotonicity: Compute $x_{n+1} - x_n$:

$$\begin{aligned} x_{n+1} - x_n &= (H_{2n+2} - H_{n+1}) - (H_{2n} - H_n) \\ &= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} \end{aligned}$$

Simplify:

$$\begin{aligned} &= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} \\ &= \frac{1}{2n+1} - \frac{1}{2n+2} \\ &= \frac{1}{(2n+1)(2n+2)} \end{aligned}$$

After simplification, the difference becomes positive, hence $x_{n+1} > x_n$. Therefore (x_n) is monotonically increasing.

Boundedness above: For each $k \in \{n+1, \dots, 2n\}$ we have $\frac{1}{k} < \frac{1}{n}$. Since there are n terms,

$$x_n < \frac{n}{n} = 1$$

so the sequence is bounded above by 1.

Limit bounds: Note that for $k = n + 1, \dots, 2n$ we also have $k \leq 2n$, hence $\frac{1}{k} \geq \frac{1}{2n}$. Thus

$$x_n \geq n \cdot \frac{1}{2n} = \frac{1}{2}$$

So $\frac{1}{2} \leq x_n < 1$ for all n . Therefore the limit $L = \lim_{n \rightarrow \infty} x_n$ exists and lies between $\frac{1}{2}$ and 1.

$$\boxed{\frac{1}{2} \leq \lim_{n \rightarrow \infty} x_n \leq 1}$$

□

3. **Suppose (x_n) is a monotonically increasing sequence. Prove that the sequence of averages $y_n = \frac{1}{n}(x_1 + \cdots + x_n)$ is also monotonically increasing**

Proof. We want to prove $y_{n+1} \geq y_n$ for all n .

$$y_n = \frac{1}{n} \sum_{k=1}^n x_k, \quad y_{n+1} = \frac{1}{n+1} \sum_{k=1}^{n+1} x_k$$

Multiply to compare:

$$(n+1)y_{n+1} = x_{n+1} + \sum_{k=1}^n x_k, \quad ny_n = \sum_{k=1}^n x_k$$

So

$$(n+1)y_{n+1} - ny_n = x_{n+1} \geq x_n$$

Now

$$(n+1)(y_{n+1} - y_n) = x_{n+1} - y_n$$

But since $x_{n+1} \geq x_k$ for each $k \leq n$, the average y_n cannot exceed x_{n+1} . Thus $x_{n+1} \geq y_n$ and hence $y_{n+1} \geq y_n$. Therefore (y_n) is monotonically increasing. \square

4. **Discuss whether the sequence** $x_n = \frac{n}{n^2 + 1} + \frac{n}{n^2 + 2} + \cdots + \frac{n}{n^2 + n}$ **is convergent**

Proof. Write the sum as

$$x_n = \sum_{k=1}^n \frac{n}{n^2 + k}$$

For each $k \in \{1, \dots, n\}$ we have the inequalities

$$\frac{n}{n^2 + n} \leq \frac{n}{n^2 + k} \leq \frac{n}{n^2 + 1}$$

Summing these inequalities over $k = 1, \dots, n$ gives

$$\sum_{k=1}^n \frac{n}{n^2 + n} \leq x_n \leq \sum_{k=1}^n \frac{n}{n^2 + 1}$$

The left and right sums simplify to

$$\frac{n^2}{n^2 + n} \leq x_n \leq \frac{n^2}{n^2 + 1}$$

Divide numerator and denominator on both bounds by n^2 to obtain

$$\frac{1}{1 + \frac{1}{n}} \leq x_n \leq \frac{1}{1 + \frac{1}{n^2}}$$

Take limits as $n \rightarrow \infty$ The left bound tends to 1 and the right bound tends to 1 Therefore by the sandwich theorem

$$\boxed{\lim_{n \rightarrow \infty} x_n = 1}$$

□

5. Let $x_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$. Show that nx_n^2 is monotonically increasing and $(n+\frac{1}{2})x_n^2$ is monotonically decreasing.

Proof. Notice that

$$x_{n+1} = \frac{1 \cdot 3 \cdots (2n-1) \cdot (2n+1)}{2 \cdot 4 \cdots 2n \cdot 2(n+1)} = x_n \cdot \frac{2n+1}{2n+2}$$

Consider the ratio

$$\begin{aligned} \frac{(n+1)x_{n+1}^2}{nx_n^2} &= \frac{n+1}{n} \left(\frac{2n+1}{2n+2} \right)^2 \\ &= \frac{n+1}{n} \cdot \frac{(2n+1)^2}{(2n+2)^2} \\ &= \frac{(n+1)(2n+1)^2}{n(2n+2)^2} \end{aligned}$$

Simplify the denominator: $2n+2 = 2(n+1)$, so

$$\begin{aligned} \frac{(n+1)(2n+1)^2}{n \cdot 4(n+1)^2} &= \frac{(2n+1)^2}{4n(n+1)} \\ &= \frac{4n^2 + 4n + 1}{4n(n+1)} \\ &= 1 + \frac{1}{4n(n+1)} > 1 \end{aligned}$$

Hence

$$(n+1)x_{n+1}^2 > nx_n^2$$

so nx_n^2 is monotonically increasing

Consider the ratio

$$\frac{(n + 3/2)x_{n+1}^2}{(n + 1/2)x_n^2} = \frac{n + 3/2}{n + 1/2} \left(\frac{2n + 1}{2n + 2} \right)^2 = \frac{n + 3/2}{n + 1/2} \cdot \frac{(2n + 1)^2}{(2n + 2)^2}$$

Write $2n + 2 = 2(n + 1)$:

$$\begin{aligned} \frac{n + 3/2}{n + 1/2} \cdot \frac{(2n + 1)^2}{4(n + 1)^2} &= \frac{2n + 3}{2n + 1} \cdot \frac{(2n + 1)^2}{4(n + 1)^2} \\ &= \frac{2n + 3}{4(n + 1)^2} (2n + 1) \\ &= \frac{(2n + 1)(2n + 3)}{4(n + 1)^2} \end{aligned}$$

Observe

$$(2n + 1)(2n + 3) = 4n^2 + 8n + 3 < 4n^2 + 8n + 4 = 4(n + 1)^2$$

Thus the ratio is less than 1, so

$$\left(n + \frac{1}{2}\right)x_n^2$$

is monotonically decreasing

□

6. Let (a_n) be a sequence such that $a_j \in \{0, 1, \dots, 9\}$ for all j . For $n = 1, 2, 3, \dots$, construct

$$x_n = \frac{a_1}{10} + \frac{a_2}{10^2} + \dots + \frac{a_n}{10^n}$$

Show that (x_n) is monotonically increasing and bounded above. Call $\lim_{n \rightarrow \infty} x_n = a$, with $a \in [0, 1]$. Is it true that given any a there exists a corresponding sequence (a_n) as above such that $\lim_{n \rightarrow \infty} x_n = a$?

Proof. Monotonicity: Clearly $x_{n+1} = x_n + \frac{a_{n+1}}{10^{n+1}} \geq x_n$, so (x_n) is monotonically increasing.

Boundedness: Since each $a_j \leq 9$,

$$x_n \leq \frac{9}{10} + \frac{9}{10^2} + \dots + \frac{9}{10^n} < \frac{9}{10} \cdot \frac{1}{1 - \frac{1}{10}} = 1$$

Thus $x_n < 1$, and bounded above by 1.

Limit existence: An increasing bounded sequence converges, so $\lim_{n \rightarrow \infty} x_n = a$ exists, with $a \in [0, 1]$.

Representation of any $a \in [0, 1]$: This construction is precisely the decimal expansion of real numbers in $[0, 1]$. Every real number in $[0, 1]$ has such a decimal representation. The only subtlety is that some numbers (like $0.4999\dots = 0.5000\dots$) have two representations, but at least one sequence (a_n) always exists.

Every $a \in [0, 1]$ can be obtained as $\lim_{n \rightarrow \infty} x_n$

□