MA-105 Tutorial-5 – Solutions

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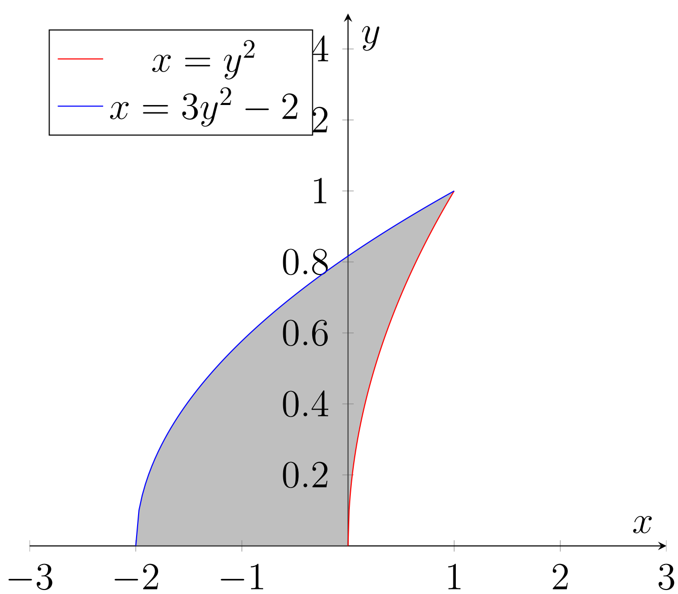
1. **Find the volume of the solid obtained by revolving the given shaded region about the x-axis.**



Solution :

The given region is bounded by the curves:

$$x = y^2, \quad x = 3y^2 - 2$$

Their intersection points occur when:

$$y^2 = 3y^2 - 2 \implies 2y^2 = 2 \implies y = \pm 1$$

The limits for y are from 0 to 1. Revolving around the x-axis means each horizontal strip at height y generates a shell of radius y and thickness dy. Since our equations are in terms of y, we use the shell method:

$$\begin{aligned} V &= 2\pi \int_0^1 y(x_{right} - x_{left}) dy \\ &= 2\pi \int_0^1 y[(y^2) - (3y^2 - 2)] dy \\ &= 2\pi \int_0^1 y(2 - 2y^2) dy \\ &= 2\pi \left(y^2 - \frac{y^4}{2} \right) \Bigg|_0^1 \\ &= \pi \end{aligned}$$

$$\boxed{V = \pi}$$

2. **Prove that if $f : [a,b] \to \mathbb{R}$ is continuous, then $\exists c \in [a,b]$ such that**

$$\int_a^b f(x)dx = (b-a)f(c)$$

and deduce that $\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_a^x f(t)dt = f(x)$.

Solution :

Since f is continuous on $[a,b]$, it attains its maximum M and minimum m. By the properties of integrals:

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

Dividing both sides by $(b-a)$ gives:

$$m \leq \frac{1}{b-a}\int_a^b f(x)dx \leq M$$

By the Intermediate Value Theorem, there exists $c \in [a,b]$ such that:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

Multiplying both sides by $(b-a)$, we obtain:

$$\boxed{\int_a^b f(x)dx = (b-a)f(c)}$$

To deduce $\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_a^x f(t)dt = f(x)$, define $F(x) = \int_a^x f(t)dt$. For any $h > 0$,

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)dt$$

By the Mean Value Theorem for integrals, there exists $c_h \in [x, x+h]$ such that:

$$\frac{F(x+h) - F(x)}{h} = f(c_h)$$

Taking $h \to 0$, continuity of f implies $c_h \to x$ and thus:

$$\boxed{F'(x) = f(x)}$$

3. **Prove that the set** $\left\{(x,y) \mid \frac{x^2}{44} - \frac{y^2}{37} < 1\right\}$ **is open.**

Solution :

Let

$$g(x,y) = \frac{x^2}{44} - \frac{y^2}{37}, \qquad (x,y) \in \mathbb{R}^2,$$

and let

$$S = \{(x,y) \in \mathbb{R}^2 \mid g(x,y) < 1\}.$$

Take an arbitrary point $p = (x_0, y_0) \in S$. Then $g(p) < 1$; set

$$\varepsilon := 1 - g(p) > 0.$$

We claim there exists $\delta > 0$ such that for every point $q = (x,y)$ with $\|(x,y) - (x_0,y_0)\| < \delta$ we have $g(q) < 1$. This will show the open ball $B_\delta(p) \subset S$ and hence that p is an interior point; since p was arbitrary, S is open.

To prove the claim use the limit (continuity) of g at p. Note that g is a polynomial in x, y, so

$$\lim_{(x,y)\to(x_0,y_0)} g(x,y) = g(x_0,y_0).$$

By the ε-δ definition of a limit, since $|g(x_0,y_0) - 1| = \varepsilon > 0$, there exists $\delta > 0$ such that

$$|g(x,y) - g(x_0,y_0)| < \varepsilon \quad \text{whenever} \quad \|(x,y) - (x_0,y_0)\| < \delta.$$

For such (x,y) we get

$$g(x,y) < g(x_0,y_0) + \varepsilon = g(x_0,y_0) + (1 - g(x_0,y_0)) = 1.$$

Hence every point q in the ball $B_\delta(p)$ satisfies $g(q) < 1$, so $B_\delta(p) \subset S$. Therefore p is an interior point of S. As this holds for every $p \in S$, the set S is open.

4. **Is the set** $\left\{(x,y) \mid \frac{x^2}{4}+\frac{y^2}{9}<1\right\}$ **convex?**

Solution :

Let S be the given set. For any two points $(x_1,y_1),(x_2,y_2)\in S$, we must show that every point on the line segment joining them is also in S.

Consider $(x,y)=t(x_1,y_1)+(1-t)(x_2,y_2)$, where $0\le t\le 1$. Then:

$$\frac{x^2}{4}+\frac{y^2}{9}\le t\left(\frac{x_1^2}{4}+\frac{y_1^2}{9}\right)+(1-t)\left(\frac{x_2^2}{4}+\frac{y_2^2}{9}\right)<t+(1-t)=1$$

using convexity of the quadratic function. Hence the set is convex.

$$\boxed{S \text{ is convex.}}$$

5. **Prove that a polynomial in two variables is a continuous function.**

Solution :

Let $p(x,y)=\sum_{i,j}a_{ij}x^iy^j$. Each monomial x^iy^j is continuous as it is a product of continuous functions. A finite linear combination of continuous functions is continuous, hence $p(x,y)$ is continuous on $\mathbb{R}^2$.

$$\boxed{p \text{ is continuous everywhere.}}$$

6. **Prove that if $f(x, y)$ is continuous, then the level set**

$$L_c = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = c\}$$

is closed.

Solution :

Let (x_n, y_n) be a sequence in L_c that converges to some point $(x_0, y_0) \in \mathbb{R}^2$. We want to show that $(x_0, y_0) \in L_c$, that is, $f(x_0, y_0) = c$.

Since f is continuous at (x_0, y_0), we have

$$\lim_{n\to\infty} f(x_n, y_n) = f\big(\lim_{n\to\infty} (x_n, y_n)\big) = f(x_0, y_0).$$

But each $(x_n, y_n) \in L_c$, so $f(x_n, y_n) = c$ for all n. Therefore,

$$\lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to\infty} c = c.$$

Combining the two limits, we get

$$f(x_0, y_0) = c.$$

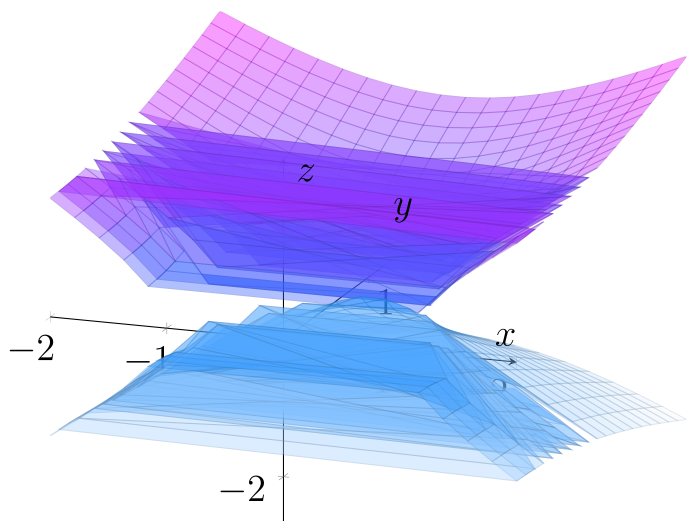
Hence, $(x_0, y_0) \in L_c$, proving that L_c contains all its limit points. By definition, this means L_c is closed.

$$\boxed{L_c \text{ is closed.}}$$

7. **Sketch the level sets of $f(x, y, z) = x^2 + y^2 - z^2$.**

We have $f(x, y, z) = k \implies x^2 + y^2 - z^2 = k$.

- For $k > 0$, we get a **two-sheeted hyperboloid**. - For $k = 0$, we get a **double cone**. - For $k < 0$, we get a **one-sheeted hyperboloid**.



These represent the level surfaces for $f(x, y, z) = 1$ and $f(x, y, z) = -1$, respectively.