MA-105 Tutorial-1 Solutions

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1. **Define** $a_1 = \sqrt{2}$, $a_2 = \sqrt{2+\sqrt{2}}$**, and generally** $a_n = \sqrt{2 + a_{n-1}}$ **for** $n \geq 2$**. Prove by induction that the sequence is monotonically increasing and bounded above. Find its limit**

   Proof. **Monotonicity by induction**

   Base case:
$$a_2 = \sqrt{2+\sqrt{2}} > \sqrt{2} = a_1$$

   Inductive step: assume for some $n \geq 2$ that
$$a_n > a_{n-1}$$

   We show $a_{n+1} > a_n$ Consider
$$a_{n+1} - a_n = \sqrt{2 + a_n} - \sqrt{2 + a_{n-1}}$$

   Rationalize the difference:
$$a_{n+1} - a_n = \frac{(2 + a_n) - (2 + a_{n-1})}{\sqrt{2 + a_n} + \sqrt{2 + a_{n-1}}} = \frac{a_n - a_{n-1}}{\sqrt{2 + a_n} + \sqrt{2 + a_{n-1}}}$$

   By the inductive hypothesis $a_n - a_{n-1} > 0$ and the denominator is positive, hence
$$a_{n+1} - a_n > 0$$

Thus $a_{n+1} > a_n$ and the induction closes Therefore (a_n) is strictly increasing

Boundedness above by induction

Base case: $a_1 = \sqrt{2} < 2$

Inductive step: assume $a_n < 2$ for some $n \geq 1$. Then

$$a_{n+1} = \sqrt{2 + a_n} < \sqrt{2 + 2} = \sqrt{4} = 2$$

So $a_{n+1} < 2$ and by induction $a_n < 2$ for all n Hence (a_n) is bounded above by 2

Limit

Since (a_n) is increasing and bounded above it converges Let

$$L = \lim_{n \to \infty} a_n$$

Passing to the limit in the recursion yields

$$L = \sqrt{2 + L}$$

Square both sides

$$L^2 = 2 + L \iff L^2 - L - 2 = 0 \iff (L - 2)(L + 1) = 0$$

Thus $L = 2$ or $L = -1$ Since every $a_n > 0$ we must have $L = 2$ Therefore

$$\boxed{\lim_{n \to \infty} a_n = 2}$$

$\square$

2. **Let** $x_n = \dfrac{1}{n+1} + \dfrac{1}{n+2} + \cdots + \dfrac{1}{2n}$**. Prove that** (x_n) **is monotonically increasing and bounded above. Prove that the limit of** (x_n) **lies between** $\frac{1}{2}$ **and** 1

Proof. We can write

$$x_n = \sum_{k=n+1}^{2n} \frac{1}{k} = H_{2n} - H_n$$

where $H_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}$ is the m-th harmonic number.

Monotonicity: Compute $x_{n+1} - x_n$:

$$\begin{aligned} x_{n+1} - x_n &= \Big(H_{2n+2} - H_{n+1}\Big) - \Big(H_{2n} - H_n\Big) \\ &= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} \end{aligned}$$

Simplify:

$$\begin{aligned} &= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} \\ &= \frac{1}{2n+1} - \frac{1}{2n+2} \\ &= \frac{1}{(2n+1)(2n+2)} \end{aligned}$$

After simplification, the difference becomes positive, hence $x_{n+1} > x_n$. Therefore (x_n) is monotonically increasing.

Boundedness above: For each $k \in \{n+1, \ldots, 2n\}$ we have $\frac{1}{k} < \frac{1}{n}$. Since there are n terms,

$$x_n < \frac{n}{n} = 1$$

so the sequence is bounded above by 1.

Limit bounds: Note that for $k = n+1, \ldots, 2n$ we also have $k \leq 2n$, hence $\frac{1}{k} \geq \frac{1}{2n}$. Thus

$$x_n \geq n \cdot \frac{1}{2n} = \frac{1}{2}$$

So $\frac{1}{2} \leq x_n < 1$ for all n. Therefore the limit $L = \lim_{n\to\infty} x_n$ exists and lies between $\frac{1}{2}$ and 1.

$$\boxed{\tfrac{1}{2} \leq \lim_{n\to\infty} x_n \leq 1}$$

$\square$

3. **Suppose (x_n) is a monotonically increasing sequence. Prove that the sequence of averages $y_n = \frac{1}{n}(x_1 + \cdots + x_n)$ is also monotonically increasing**

Proof. We want to prove $y_{n+1} \geq y_n$ for all n.

$$y_n = \frac{1}{n}\sum_{k=1}^{n} x_k, \qquad y_{n+1} = \frac{1}{n+1}\sum_{k=1}^{n+1} x_k$$

Multiply to compare:

$$(n+1)y_{n+1} = x_{n+1} + \sum_{k=1}^{n} x_k, \quad ny_n = \sum_{k=1}^{n} x_k$$

So

$$(n+1)y_{n+1} - ny_n = x_{n+1} \geq x_n$$

Now

$$(n+1)(y_{n+1} - y_n) = x_{n+1} - y_n$$

But since $x_{n+1} \geq x_k$ for each $k \leq n$, the average y_n cannot exceed x_{n+1}. Thus $x_{n+1} \geq y_n$ and hence $y_{n+1} \geq y_n$. Therefore (y_n) is monotonically increasing. $\square$

4. **Discuss whether the sequence** $x_n = \frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n}$ **is convergent**

Proof. Write the sum as

$$x_n = \sum_{k=1}^{n} \frac{n}{n^2+k}$$

For each $k \in \{1, \ldots, n\}$ we have the inequalities

$$\frac{n}{n^2+n} \leq \frac{n}{n^2+k} \leq \frac{n}{n^2+1}$$

Summing these inequalities over $k = 1, \ldots, n$ gives

$$\sum_{k=1}^{n} \frac{n}{n^2+n} \leq x_n \leq \sum_{k=1}^{n} \frac{n}{n^2+1}$$

The left and right sums simplify to

$$\frac{n^2}{n^2+n} \leq x_n \leq \frac{n^2}{n^2+1}$$

Divide numerator and denominator on both bounds by n^2 to obtain

$$\frac{1}{1+\frac{1}{n}} \leq x_n \leq \frac{1}{1+\frac{1}{n^2}}$$

Take limits as $n \to \infty$ The left bound tends to 1 and the right bound tends to 1 Therefore by the sandwich theorem

$$\boxed{\lim_{n\to\infty} x_n = 1}$$

$\square$

5. **Let** $x_n = \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$**. Show that** nx_n^2 **is monotonically increasing and** $\left(n+\frac{1}{2}\right)x_n^2$ **is monotonically decreasing.**

Proof. Notice that

$$x_{n+1} = \frac{1 \cdot 3 \cdots (2n-1) \cdot (2n+1)}{2 \cdot 4 \cdots 2n \cdot 2(n+1)} = x_n \cdot \frac{2n+1}{2n+2}$$

Consider the ratio

$$\begin{aligned}
\frac{(n+1)x_{n+1}^2}{nx_n^2} &= \frac{n+1}{n}\left(\frac{2n+1}{2n+2}\right)^2 \\
&= \frac{n+1}{n} \cdot \frac{(2n+1)^2}{(2n+2)^2} \\
&= \frac{(n+1)(2n+1)^2}{n(2n+2)^2}
\end{aligned}$$

Simplify the denominator: $2n+2 = 2(n+1)$, so

$$\begin{aligned}
\frac{(n+1)(2n+1)^2}{n \cdot 4(n+1)^2} &= \frac{(2n+1)^2}{4n(n+1)} \\
&= \frac{4n^2+4n+1}{4n(n+1)} \\
&= 1 + \frac{1}{4n(n+1)} > 1
\end{aligned}$$

Hence

$$(n+1)x_{n+1}^2 > nx_n^2$$

so nx_n^2 is monotonically increasing

Consider the ratio

$$\frac{(n+3/2)x_{n+1}^2}{(n+1/2)x_n^2} = \frac{n+3/2}{n+1/2}\left(\frac{2n+1}{2n+2}\right)^2 = \frac{n+3/2}{n+1/2}\cdot\frac{(2n+1)^2}{(2n+2)^2}$$

Write $2n+2 = 2(n+1)$:

$$\begin{aligned}\frac{n+3/2}{n+1/2}\cdot\frac{(2n+1)^2}{4(n+1)^2} &= \frac{2n+3}{2n+1}\cdot\frac{(2n+1)^2}{4(n+1)^2}\\ &= \frac{2n+3}{4(n+1)^2}(2n+1)\\ &= \frac{(2n+1)(2n+3)}{4(n+1)^2}\end{aligned}$$

Observe

$$(2n+1)(2n+3) = 4n^2+8n+3 < 4n^2+8n+4 = 4(n+1)^2$$

Thus the ratio is less than 1, so

$$\left(n+\frac{1}{2}\right)x_n^2$$

is monotonically decreasing

$\square$

6. **Let** (a_n) **be a sequence such that** $a_j \in \{0, 1, \dots, 9\}$ **for all** j**. For** $n = 1, 2, 3, \dots$**, construct**

$$x_n = \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n}$$

Show that (x_n) **is monotonically increasing and bounded above. Call** $\lim_{n\to\infty} x_n = a$**, with** $a \in [0, 1]$**. Is it true that given any** a **there exists a corresponding sequence** (a_n) **as above such that** $\lim_{n\to\infty} x_n = a$**?**

Proof. **Monotonicity:** Clearly $x_{n+1} = x_n + \frac{a_{n+1}}{10^{n+1}} \geq x_n$, so (x_n) is monotonically increasing.

Boundedness: Since each $a_j \leq 9$,

$$x_n \leq \frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} < \frac{9}{10} \cdot \frac{1}{1 - \frac{1}{10}} = 1$$

Thus $x_n < 1$, and bounded above by 1.

Limit existence: An increasing bounded sequence converges, so $\lim_{n\to\infty} x_n = a$ exists, with $a \in [0, 1]$.

Representation of any $a \in [0, 1]$**:** This construction is precisely the decimal expansion of real numbers in $[0, 1]$. Every real number in $[0, 1]$ has such a decimal representation. The only subtlety is that some numbers (like $0.4999\cdots = 0.5000\dots$) have two representations, but at least one sequence (a_n) always exists.

$$\boxed{\text{Every } a \in [0,1] \text{ can be obtained as } \lim_{n\to\infty} x_n}$$

$\square$