MA-105 Tutorial-8 Solutions

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1. Compute the arc-length of the cycloid $\vec{r}(t) = (a(t-\sin t),\ a(1-\cos t));\ 0 \le t \le 2\pi$.

   Sol:

$$
\begin{aligned}
l &= \int_0^{2\pi} ||\vec{r}'(t)||dt \\
\text{Now,} & \\
\vec{r}(t) &= (a(t-\sin t),\ a(1-\cos t)) \\
\vec{r}'(t) &= (a(1-\cos t),\ a\sin t) \\
||\vec{r}'(t)|| &= \sqrt{(a(1-\cos t))^2 + (a\sin t)^2} \\
&= \sqrt{a^2(1+\cos^2 t + \sin^2 t - 2\cos t)} \\
&= a\sqrt{2-2\cos t} \\
&= a\sqrt{4\sin^2\frac{t}{2}} \\
||\vec{r}'(t)|| &= 2a\left|\sin\frac{t}{2}\right| \\
\text{So,} & \\
l &= \int_0^{2\pi} 2a\left|\sin\frac{t}{2}\right| dt \\
&= \int_0^{\pi} 4a|\sin x|dx \\
&= 8a
\end{aligned}
$$

2. Parametrize the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and set-up the integral for the perimeter of the ellipse in terms of its eccentricity $e=\sqrt{1-\frac{b^2}{a^2}}$. Assume $b<a$.

Sol:

The parametrization of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is given by:

$$\begin{aligned} x &= a\cos\theta \\ y &= b\sin\theta \\ \theta &\in [0,2\pi] \end{aligned}$$

Hence, an ellipse can be described as the following function:

$$\vec{r}(\theta) = (a\cos\theta,\ b\sin\theta)$$

Now,

$$\begin{aligned} \vec{r}'(\theta) &= (-a\sin\theta, b\cos\theta) \\ ||\vec{r}'(\theta)|| &= \sqrt{(-a\sin\theta)^2+(b\cos\theta)^2} \\ &= \sqrt{a^2\sin^2\theta+b^2\cos^2\theta} \end{aligned}$$

So,

$$\begin{aligned} l &= \int_0^{2\pi} ||\vec{r}'(\theta)||d\theta \\ &= \int_0^{2\pi} \sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\ d\theta \\ &= 4\int_0^{\frac{\pi}{2}} \sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\ d\theta \\ &= 4a\int_0^{\frac{\pi}{2}} \sqrt{\sin^2\theta+\frac{b^2}{a^2}\cos^2\theta\ d\theta} \end{aligned}$$

$$
\begin{aligned}
&= 4a \int_0^{\frac{\pi}{2}} \sqrt{\sin^2\theta + \left(\frac{b^2}{a^2} - 1\right)\cos^2\theta + \cos^2\theta}\; d\theta \\
&= 4a \int_0^{\frac{\pi}{2}} \sqrt{1 - e^2\cos^2\theta}\; d\theta \\
&= 4a \int_0^{\frac{\pi}{2}} \sqrt{1 - e^2\sin^2\theta}\; d\theta
\end{aligned}
$$

The above integral, known as elliptical integral of second kind, is non-elementary and hence, cannot be calculated by our commonly used methods. To calculate the integral, we use numerical approximation.

3. If $\hat{a}$ and $\hat{b}$ are two unit vectors in $\mathbb{R}^2$ such that $\hat{a} \times \hat{b} \neq \vec{0}$. Show that $\{\hat{a}\cos t + \hat{b}\sin t \mid t \in [0,\ 2\pi]\}$ is an ellipse.

Sol: Let $\hat{a} = a_1\hat{\imath} + a_2\hat{\jmath}$ and $\hat{b} = b_1\hat{\imath} + b_2\hat{\jmath}$

$$\begin{aligned}\hat{a}\cos t + \hat{b}\sin t &= (a_1\hat{\imath} + a_2\hat{\jmath})\cos t + (b_1\hat{\imath} + b_2\hat{\jmath})\sin t \\ &= (a_1\cos t + b_1\sin t)\hat{\imath} + (a_2\cos t + b_2\sin t)\hat{\jmath}\end{aligned}$$

Thus,

$$\begin{aligned}x &= a_1\cos t + b_1\sin t \\ y &= a_2\cos t + b_2\sin t\end{aligned}$$

Solving the above equations for $\sin t$ and $\cos t$ we'll get:

$$\begin{aligned}\sin t &= \frac{a_1y - a_2x}{a_1b_2 - a_2b_1} \\ \cos t &= \frac{b_2x - b_1y}{a_1b_2 - a_2b_1}\end{aligned}$$

Then,

$$\begin{aligned}\sin^2 t + \cos^2 t &= 1 \\ \left(\frac{a_1y - a_2x}{a_1b_2 - a_2b_1}\right)^2 + \left(\frac{b_2x - b_1y}{a_1b_2 - a_2b_1}\right)^2 &= 1\end{aligned}$$

$$\frac{1}{(a_1b_2 - a_2b_1)^2}[(a_1y - a_2x)^2 + (b_2x - b_1y)^2] = 1$$

$$\frac{1}{(a_1b_2 - a_2b_1)^2}[(a_1^2 + b_1^2)y^2 + (a_2^2 + b_2^2)x^2 - 2(a_1a_2 + b_1b_2)xy] = 1$$

The discriminant Δ of the above equation will be

$$\begin{aligned}\Delta &= (2(a_1a_2 + b_1b_2))^2 - 4(a_1^2 + b_1^2)(a_2^2 + b_2^2) \\ &= 4(2a_1a_2b_1b_2 - a_1^2b_2^2 - a_2^2b_1^2) \\ &= -4(a_1b_2 - a_2b_1)^2 \leq 0\end{aligned}$$

Thus, the above equation describes an ellipse.

4.