II - Limits, Continuity and the Derivative

We shall consider functions

$f : I \to \mathbb{R}$ where I is an

interval which ~~could~~ could be open / closed bounded or unbounded. Here is a list:

$I = \mathbb{R} = (-\infty, \infty)$

$I = [a, \infty)$ or $(-\infty, a]$

$I = (a, \infty)$ or $(-\infty, a)$

$I = (a, b)$ or $[a, b]$ or $(a, b]$ or $[a, b)$

a, b are finite and $a \leq b$.

$\left.\begin{array}{l} [a, b] = \{a\} \text{ if } a = b \\ (a, b) \text{ is empty if } a = b. \end{array}\right\}$ Conventions.

We now define $\lim_{x \to p} f(x)$

When $p \in I$ or p is an end point of I

in case $I = (-\infty, a)$ p could be a

in case $I = (a, b)$ then p could be a, b

etc;

Def: $\lim_{x \to p} f(x) = l$ (a real number)

if the following condition holds.

Whenever (x_n) is a sequence in I converging to p (with $x_n \neq p$ $\forall n$) the corresponding seq $(f(x_n))$ converges to l

Note that l is supposed to be a fixed real no.

That is to say for ~~ANY~~ EVERY choice of the Seq (x_n) converging to p (as described above) the corresponding sequences $(f(x_n))$ converges to the SAME limit l

Example: $f(x) = x^2+1$

$\lim_{x \to p} f(x)$ exists $= p^2+1$

let $x_n \to p$. Examine $\lim_n f(x_n)$

$= \lim_n ((x_n)^2 + 1) = p^2+1$

So For EVERY Seq (x_n) Conv. to p. the Corresp. Seq $(f(x_n))$ converges to p^2+1

So $\lim_{x \to p} f(x)$ exists and $= p^2+1$

Exercise: Show that if $f(x)$ is any polynomial, $\lim_{x \to p} f(x)$ exists and $= f(p)$

Show that if $f(x) = \dfrac{P(x)}{Q(x)}$ is a rational function, then at any point p s.t $Q(p) \neq 0$

$\lim_{x \to p} f(x)$ exists and $= \dfrac{P(p)}{Q(p)}$.

Example: Let $f: \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \sin \frac{\pi}{2} \frac{[x]}{|x|} \quad \text{if } x \neq 0$$

$$= 0 \quad \text{if } x = 0.$$

Let us take $x_n = 1/n \to 0$

$$f(x_n) = \sin \frac{\pi}{2}\left(\frac{[1/n]}{1/n}\right) = 0$$

So $\lim_n f(x_n) = 0$.

However, taking $x_n = \frac{-1}{n}$

we get $f(x_n) = \sin\left(\frac{\pi}{2} \cdot n\,[-1/n]\right)(-1)$

$= -\sin(-n\pi/2)$ and this

Sequence does not converge to 0.

So $\lim_{x \to 0} f(x)$ doesn't exist.

<u>Ex:</u>

Formulate a Sandwich-thm for $\lim_{x \to p} f(x)$.

Given 3 functions $f, g, h : I \to \mathbb{R}$

assume $f(x) \le g(x) \le h(x)$ $\forall x \in I$

$p \in I$ or p is an end point of I.

$\lim_{x \to p} f(x)$, $\lim_{x \to p} h(x)$ both exist and are equal to ℓ

Then,

$\lim_{x \to p} g(x)$ also exists and $= \ell$.

<u>Ex:</u> Prove this using the Sandwich-thm for sequences.

<u>Thm:</u> Suppose $f, g : I \to \mathbb{R}$ and $p \in I$ (or p is an endpoint of I)

$\lim_{x \to p} f(x)$, $\lim_{x \to p} g(x)$ exists $= \ell, m$ respectively

Then $\lim_{x \to p} f(x) + g(x))$ exists $= l+m$

$\lim_{x \to p} (f(x)g(x))$ exists $= lm$

The usual rules for limits that is familiar to you can be proved using the corresponding results for sequences.

Def: Let $f : I \to \mathbb{R}$ be a function and $p \in I$ (In case $I = (a,b)$ then p cannot be a or b), we say f is continuous at p if $\lim_{x \to p} f(x)$ exists and $= f(p)$.

Again, if $f, g : I \to \mathbb{R}$ are cont. at $p \in I$ then $f+g$, fg are also continuous at p. Discuss the result for quotients

Need to add the further cond. that $g(x) \neq 0$ throughout I.

So polynomials are continuous and rational functions are continuous on their domains ($= \mathbb{R}$ minus zeros of the denominator)

e^x and $\log x$ are continuous on $\mathbb{R}$ and $(0, \infty)$ respectively.

Note: $0 \notin$ domain of $\log x$

It is incorrect to say $\log x$ is discontinuous at the origin. question of continuity at p arises only if $p \in$ domain

Ex: Discuss the points at which the function $f(x) = 10x - [10x]$ is not continuous (Tutorial problem)

(A) **Thm** Suppose $f : I \to \mathbb{R}$ is continuous at the point $p \in I$; $f(p) > 0$

then there is a $c > 0$ such that

f ~~is~~ (x) is positive for all $x \in (p-c, p+c)$

Note: c is small enough that

$(p-\cancel{p}^{c}, c+p) \subset I$

In case p is the left endpoint of I then the conclusion should be modified as $f(x)$ is positive for all $x \in [p, p+c)$ etc;

proof is EASY:

Suppose no such c exists.

Then $c = 1/n$ doesn't work. That is, for some $x_n \in (p - 1/n, p + 1/n)$

$f(x_n) \leq 0$. This is so for all $n \gg 1$

$x_n \to p$. So continuity of f implies

$f(x_n) \to f(p)$

$\lim_n f(x_n) = f(p)$

But LHS ≤ 0 (since $f(x_n) \leq 0$ for all n)

RHS > 0

Contradiction

We now discuss two more theorems that we call theorem (B) and theorem (C)

Theorem B: Let $f : [a,b] \to \mathbb{R}$ be cont.
Then f is bounded on $[a,b]$
That is the set $\{ |f(x)| \,/\, a \le x \le b \}$ is bounded (note: $a, b \in \mathbb{R}$, $a \le b$)

Theorem C: If $f : [a,b] \to \mathbb{R}$ is cont and is such that ~~f(a)~~ $f(a)$, $f(b)$ have opposite signs
namely $f(a) \cdot f(b) < 0$
Then there is a point $c \in (a,b)$ such that $f(c) = 0$.

Remarks: From thm C the intermediate value theorem follows at once. Discuss.

Call $I_0 = [a,b]$

Intervals in this discussion will be closed and bounded.

Thm B says a cont. function on a closed bdd interval is bounded.

Thm C says if f assumes (values with) opp. signs at the end points then f must vanish somewhere in the interval.

Although the two theorems appear dissimilar it is remarkable that the two can be proved by the Same Method

Known as the Method of bisection.

Bisect I_0 at its mid point $\frac{1}{2}(a+b)$

Look at the two halves

$$I_0' = \left[a, \tfrac{a+b}{2}\right] : \quad I_0'' = \left[\tfrac{a+b}{2}, b\right]$$

proof by Contradiction.

Suppose Thm B / Thm C is false.

Then,

for Thm B, f must fail to be bounded in I_0' or I_0'' (or both).

For Thm C, f does not vanish in I_0' or I_0''

In particular $f\left(\frac{a+b}{2}\right) \neq 0$

So f must take (values with) opp signs at endpoints of I_0' or I_0''

Let I_1 be the one (I_0' or I_0'') on which f is unbounded (Thm B)

I_1 be the one for which f takes opp. sign at the end points (Thm C)

We can now repeat the process with I_1

Bisect it at the mid point and select that half I_2 such that

f is unbdd on I_2 (Thm B)

f takes (values of) opp. signs at endpts of I_2

...

$I_0 \supset I_1 \supset I_2 \supset \ldots$

The left endpoints a_j of I_j form a
right b_j

monotone increasing / dec. seq.

Both Converge but their limits must be Equal (Why?)

Say p.

Thus $a_1 \leq a_2 \leq \dots \leq p \leq \dots \leq b_2 \leq b_1 \leq b_0$

Now, for Thm B: f is unbounded on I_n

So n is not an upper bound for $|f(x)|$

So $\exists$ a point $c_n \in I_n$ such that

$$|f(c_n)| \geq n$$

For Thm C: Simply take $c_n = \frac{1}{2}(a_n + b_n)$

or any point in $[a_n, b_n]$ for that matter

Then $a_n \leq c_n \leq b_n \quad \forall n$

$\Rightarrow \lim_n c_n = p$ (Sandwich Thm)

By Continuity $(f(c_n))$ converges to $f(p)$

For Thm B: $|f(c_n)| \geq n$ and this denies the convergence of $(f(c_n))$ to $f(p)$. (Contradiction)

For Thm C: $f(a_n) \cdot f(b_n) < 0 \quad \forall n$

$a_n \to p$, $b_n \to p$, f continuous

$\Rightarrow f(p)^2 \leq 0$

$\Rightarrow f(p) = 0$. Contradiction

Since we have assumed f does not vanish in $[a,b]$

Both theorems have been established.

Thm D: Let $f : [a,b] \to \mathbb{R}$ be cont.

$M = \sup \{ f(x) / a \leq x \leq b \}$ and

Sup = l.u.b

$m = \inf \{ f(x) / a \leq x \leq b \}$

(M, m exist thanks to Thm B)

M and m are both attained namely $\exists$ $c, d \in [a,b]$ such that

$$f(c) = m, \quad f(d) = M$$

<u>proof:</u> Suppose M is NOT attained

i.e $f(x) < M \quad \forall \; x \in [a,b]$

Then

$$g(x) = \frac{1}{M - f(x)} \text{ is cont. on } [a,b] \text{ and positive!}$$

So $g(x)$ is bounded above by thm B.

let μ be ~~the~~ an upper bound.

$\therefore \; g(x) \leq \mu$ for all $x \in [a,b]$

$$\therefore \; 0 < \frac{1}{M - f(x)} \leqslant \mu \qquad \forall \, x \in [a,b]$$

$$\therefore \; M - f(x) \geqslant 1/\mu > 0.$$

$$f(x) \leqslant M - 1/\mu \not\leq M$$

This is a contradiction since M is the <u>LEAST</u> upper bound!

<u>Examples:</u> $f(x) = 1 - x \; : \; 0 \leqslant x \leqslant 1$

$M = \sup f = 1 \qquad (\sup = \ell.u.b)$

$m = \inf f = 0 \qquad (\inf = g.\ell.b)$

M attained at $x=0$; m attained at $x=1$

Both $\ell.u.b$ and $g.\ell.b$ attained at the end points.

<u>Example:</u> $f(x) = x(1-x) \; : \; 0 \leqslant x \leqslant 1$

$g.\ell.b \; f = 0$ attained at both end points.

l.u.b f attained at $x = \frac{1}{2}$

Note: $f'(\frac{1}{2}) = 0$: $f'(0) \neq 0$
$f'(1) \neq 0$.

In the first example also $f'(0) \neq 0$: $f'(1) \neq 0$
So if the l.u.b or g.l.b is attained in the Open interval (a,b) at point p then $f'(p) = 0$
If l.u.b / g.l.b attained at an endpoint the derivative need not be zero at that point.

Ex: $f(x) = \sin \frac{1}{x}$; $\frac{1}{3} \leq x \leq 1$
Find all points where f attains it l.u.b and g.l.b.

The Derivative: Here we shall assume f is defined on an Open interval (a,b) ($a = -\infty$ and/or $b = +\infty$) is allowed.

We say $f : (a,b) \to \mathbb{R}$ is differentiable at $p \in (a,b)$ if

$$\lim_{x \to p} \frac{f(x) - f(p)}{x - p}$$ exists in

which case the limit is denoted by $f'(p)$ or $Df(p)$ called the derivative of f at the point p.
If the derivative exists at each $p \in (a,b)$ we say f is differentiable

throughout (a,b) we get then the new function $f' : (a,b) \to \mathbb{R}$ assigning to each $x \in (a,b)$ the derivative $f'(x)$.

If f' is differentiable at $p \in (a,b)$ its derivative will be denoted by $f''(p)$ the second derivative at p.

Successive derivatives can be defined inductively.

Rules of Calculus for derivatives.

If $f, g : (a,b) \to \mathbb{R}$ are differentiable at p then $f+g$, fg are also diff. at p and further

$$(f+g)'(p) = f'(p) + g'(p)$$

$$(fg)'(p) = f(p)\, g'(p) + f'(p)\, g(p)$$

Leibnitz' rule.

In case $g(x) \neq 0$ in an interval $(p-c, p+c) \subset (a,b)$ then $\frac{f}{g}$ which is defined on $(p-c, p+c)$ is also differentiable at p and the quotient rule holds:

$$\left(\frac{f}{g}\right)'(p) = \frac{g(p) f'(p) - f(p) g'(p)}{(g(p))^2}$$

<u>The Chain Rule</u>:

If $f : (a,b) \to \mathbb{R}$ is differentiable at $p \in (a,b)$ and g is defined on an open interval J containing image of f and g is differentiable at $f(p)$

Then the Composite $(g \circ f)$ is differentiable at p and $(g \circ f)'(p) = g'(f(p))\, f'(p)$
(Chain Rule).

<u>Note:</u> f diff. at $p \Rightarrow f$ is cont. at p.
The nth derivative of f at p is denoted by $D^n f(p)$.

<u>Exercise:</u> Is the function $\dfrac{\log(x+1)}{\sin x} = f(x)$ defined on $\left(-\frac{1}{2}, \frac{1}{2}\right)$
differentiable at the origin? $f(0) = 1$
Calculate the derivative.

<u>Compute</u> $D^n f(1)$ when $f(x) = \dfrac{(x^2-1)^n}{2^n n!}$

Prove that if f, g are both n-times diff. at p then

$$(D^n fg)(p) = {}^nC_0\, D^n f(p)\, g(p) + {}^nC_1\, D^{n-1} f(p)\, g'(p) + \dots + {}^nC_n\, f(p)\, D^n g(p)$$

This will be called generalized Leibnitz' rule.

<u>Prove that</u> $D^n \sin x = \sin(x + n\pi/2)$

<u>Ex.</u> Suppose $y = \exp(\tan^{-1} x) = f(x)$

$$(1+x^2)\, y'' + (2x-1)\, y' = 0$$

Find a formula connecting
$D^{n+2} f(0)$, $D^n f(0)$ and $D^{n+1} f(0)$

<u>Ans:</u> $D^{n+2} f(0) + n(n+1)\, D^n f(0) = D^{n+1} f(0)$

(I hope I have done the algebra correctly!)
J. Edwards. Diff Calc. p 90 (1886)