MA 105

Calculus One and Several Variables

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Text Books: G. B. Thomas and R. L. Finney
Calculus and analytic Geo, 9th Ed.

Suppl:
For those who are more inclined towards a rigrous development, the book J. C. Burkill: First Course in Math. Analysis Camb. Univ. press 1991 would be an excellent place to begin.

However Thomas - Finney cited above suffices (with its huge collection of solved and unsolved exercises)

Standard Notations

$\mathbb{R}$ = Set of real numbers with the usual operations and relations
$+$, $\cdot$, $\leq$ etc;

$\mathbb{Q}$ = Set of all rational numbers
$\mathbb{Z}$ = set of all integers.
$\mathbb{N} = \{0, 1, 2, \ldots\}$.

$|x|$ denotes the absolute value of x
$\sqrt{p}$ will always denote the Unique positive Square root if $p \geq 0$.
Generally $p^{1/n}$ will always denote the Unique positive nth root of p when $p \geq 0$

The least Upper bound property (l.u.b)

If A is a non empty subset of $\mathbb{R}$, a real no. c is said to be an upper bound for A if $x \leq c$ for all $x \in A$
If such a c exists we say A is bounded above and c is called an upper bound of A

l.u.b ppty: Every non empty subset $A \subseteq \mathbb{R}$ which is bounded above has a least upper bound namely there exists a c_0 such that

(i) c_0 is an upper bound for A:

$x \leq c_0$ for all $x \in A$

(ii) If c is ANY upper bound then

$c_0 \leq c$.

That is, an ~~an~~ upper bound exists and there is a least one among all upper bounds. Notation: $c_0 = \text{lub} A$

A little thought would convince you that $\text{lub} A$ is unique when A is a non empty subset of $\mathbb{R}$ bounded above.

Remark: It is possible to give a precise mathematical def of a real number and l.u.b is then a theorem. We shall not get into these subtle issues. Interested students should look up to MA 403 course or notes on the website.

We shall take the l.u.b property for granted.

We begin with the basic concept of a sequence of real numbers and the notion of limit of a sequence.

Sequences of real Numbers:

A seq of reals is simply a function

$$x : \mathbb{N} \to \mathbb{R}$$

Instead of writing $x(1)$ (the value of the function x at 1) it is customary to write x_1.

The sequence is usually displayed as a list: $x_1, x_2, x_3, \ldots$

or compactly as $(x_n)_{n \in \mathbb{N}}$ or even more briefly as (x_n)

Ex: $x_n = (-1)^n$. The seq is

$-1, 1, -1, 1, \ldots$ $(n = 1, 2, \ldots)$

Ex: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$; $(x_n = \frac{1}{n})$

Ex: $1, 4, 9, 16, 25, 36, \ldots$ are all examples of sequences; $(x_n = n^2)$

Here is a more interesting example:

$0, 1, \frac{1}{2}, 1, \frac{1}{3}, \frac{2}{3}, 1, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \ldots$

Is it possible to find a formula for the nth term?

Notion of Convergence of a Seq:

Informally, a sequence (x_n) of reals

is said to Converge to $l \in \mathbb{R}$
(denoted $x_n \to l$ as $n \to \infty$)
if the terms ultimately approach l as closely as we please.

To make this into a rigorous def.
Assume given an error bound E (or a tolerance bound)
All terms beyond a certain stage must lie within E of l.
That is to say $x_N, x_{N+1}, \ldots$ must all lie in $(l-E, l+E)$. So we formulate a rigorous def of Convergence

Def. A seq. of reals (x_n) Converges to $l \in \mathbb{R}$ if
given any $E > 0$ (error bound)
there exists $N \in \mathbb{N}$ such that
$|x_n - l| < E$ for $n = N, N+1, N+2, \ldots$

Purpose of this def is of course to prove theorems on Convergence.
First a notation: $x_n \to l$ as $n \to \infty$ is often written as
$$\lim_{n \to \infty} x_n = l.$$

Then for real sequences $(x_n), (y_n)$ Converging to l, m respectively,
$x_n + y_n \to l + m$ as $n \to \infty$
$x_n y_n \to l m$

Or, $\lim_{n\to\infty}(x_n+y_n) = (\lim_n x_n) + (\lim_n y_n)$

$$\lim_n (x_n y_n) = (\lim_n x_n)(\lim_n y_n)$$

provided $\lim_n x_n$ and $\lim_n y_n$ exist

What about quotients?
Well, if $y_n \neq 0$ for all $n \in \mathbb{N}$
$\left(\frac{x_n}{y_n}\right)$ is again a seq of reals.

$$\lim_n \left(\frac{x_n}{y_n}\right) = \frac{\lim_n x_n}{\lim_n y_n} = \frac{l}{m}$$

provided $m \neq 0$.

These rules for manipulating limits can all be proved using the def given on p4.
We shall take these for granted.

Sandwich Thm: Suppose (x_n), (y_n), (z_n) are three real sequences such that
(i) $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$
(ii) $\lim_n x_n$ and $\lim_n z_n$ both exist and are equal. $= l$.
Then $\lim_n y_n$ exists and $= l$.

This is one result that will be used frequently.

Thm: Suppose (x_n) is a Seq. of real numbers Such that

$$x_1 \leq x_2 \leq x_3 \leq \dots$$

and the set $\{x_1, x_2, x_3, \dots\}$ is bounded above then

$\lim_n x_n$ exists and equals the Supremum of the above set

In Short a monotone increasing seq bounded above Converges to its l.u.b.

Examples: $1, \frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \dots$

is a monotone increasing seq bdd. above by 2.

Is 2 the l.u.b ? Justify.

What about a monotone decreasing Seq. bounded below?

Formulate a theorem.

What is the meaning of a Subset of $\mathbb{R}$ Which is bounded below?

What is the analogue of l.u.b ppty?

g.l.b = greatest lower bound.

A monotone decreasing Seq bdd below Converges to its g.l.b.

Ex: $1, \frac{1}{2}, \frac{1}{3}, \ldots$ monotone dec.
bounded below by 0

Is 0 the g.l.b ? Discuss.

$\lim_n \frac{1}{n} = 0$. More generally $\lim_n \frac{c}{n} = 0$.

Example: If $c > 0$, let
$$x_n = c^{1/n} \quad (n = 1, 2, 3, \ldots)$$
Then $\lim_n x_n = 1$

Example: $\lim_n n^{1/n}$ exists $= 1$

Example: $\lim_{n \to \infty} (n^3 + 7n^2 - 3n - 36)^{\frac{1}{n}} = ?$
Does it exist ??

The first two are basic models.

Recall: Your favourite tool: AM-GM ineq!

If $a_1, \ldots, a_n > 0$
then
$$(a_1 \ldots a_n)^{\frac{1}{n}} \leq \frac{1}{n}(a_1 + \ldots + a_n)$$
For a delightful account
See Courant and Robbins: What is Mathematics

We shall assume known this important inequality GM $\leq$ AM. for n pos. reals.

Let us apply this to the n real numbers

$c, 1, 1, 1, \ldots, 1$

$$AM = \frac{n-1+c}{n}$$

$$GM = c^{1/n}.$$

So if $c \geq 1$ to begin with we have

$$1 \leq \underset{(GM)}{c^{\frac{1}{n}}} \leq \underset{(AM)}{\frac{n-1+c}{n}}$$

Now note that $\frac{n-1+c}{n} \to 1$ as $n \to \infty$

So by Sandwich thm, we conclude

$$\lim_n c^{1/n} = 1$$

Case (ii) $0 < c < 1$. Put $d = 1/c > 1$

$$\lim_n c^{1/n} = \lim_n \frac{1}{d^{1/n}} = \frac{1}{1} = 1$$

<u>Ex:</u> Let us prove $\lim_n n^{1/n} = 1$

Take the n real #s as

$\sqrt{n}, \sqrt{n}, 1, 1, \ldots, 1$

$$AM = \frac{2\sqrt{n} + n-2}{n} = \frac{2}{\sqrt{n}} + \frac{n-2}{n}$$

$$\lim_n \left(\frac{2}{\sqrt{n}} + \frac{n-2}{n}\right) = 1$$

$$GM = n^{1/n}$$

So $1 \leq n^{1/n} \leq \frac{2}{\sqrt{n}} + \left(\frac{n-2}{n}\right)$

Sandwich thm at once gives $\lim_n n^{1/n} = 1$

<u>Ex:</u> $\left(1+\frac{1}{n}\right)^n = x_n$ is a monotone inc. seq.

That is $x_n \leq x_{n+1}$ for all n

Apply AM-GM ineq to the $n+1$ numbers

$$\underbrace{1+\frac{1}{n}, \dots\dots, 1+\frac{1}{n}}_{n} : 1$$

$$AM = \frac{1}{n+1}\left(n\left(1+\frac{1}{n}\right)+1\right)$$

$$\therefore AM = 1 + \frac{1}{n+1}.$$

$$GM = \left\{ \left(1+\frac{1}{n}\right)^n \right\}^{\frac{1}{n+1}}$$

$GM \leq AM$ at once gives

$$\left(1+\frac{1}{n}\right)^n \leq \left(1+\frac{1}{n+1}\right)^{n+1}$$ Done.

More generally for any $x > 0$

$$\left(1+\frac{x}{n}\right) \leq \left(1+\frac{x}{n+1}\right)^{n+1}.$$

<u>Exercise:</u> Let $y_n = \left(1-\frac{1}{n}\right)^{-n}$; $n \geq 2$

Show that (y_n) is monotone dec.

$$y_2 \geq y_3 \geq y_4 \geq \dots$$

Check whether

$$x_n \leq y_n \text{ for } n \geq 2$$

So, $x_n \leq y_2$ for all $n \geq 2$

$x_2 \leq y_n$ for all $n \geq 2$

So (x_n) is monotone inc and bdd above $(n \geq 2)$

(y_n) is monotone dec. and bdd. below.

So $\lim_n x_n = l$

$\lim_n y_n = m$ both exist !

useful !

Now

$$\frac{x_n}{y_n} = \left(1 - \frac{1}{n^2}\right)^n \boxed{= \left(y_{n^2}\right)^{-\frac{1}{n}}}$$

What happens to this ?

Well, $\left(1 - \frac{1}{n^2}\right)^{n^2} = \left(y_{n^2}\right)^{-1}$

We know $x_2 \leq y_{n^2} \leq y_2 \quad \forall n \geq 2$

We also know

$x_2^{1/n}$ and $y_2^{1/n}$ both converge to the same value 1.

So

$$\lim_n \left(y_{n^2}\right)^{\frac{1}{n}} = 1$$

We get that $\lim \left(\frac{x_n}{y_n}\right) = 1$

$\therefore l = m$

$$\therefore \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = \lim_n \left(1 - \frac{1}{n}\right)^{-n}$$

Common value $= e$

(Universally accepted Notation)

Note that in the Sandwich thm the Condition

$$x_n \leq y_n \leq z_n \quad \text{for } \underline{\text{all}} \quad n \in \mathbb{N}$$

Can be slightly weakened

It is enough if

$$x_n \leq y_n \leq z_n \quad \text{holds for all} \quad n \geq n_0$$

for some $\underline{\text{fixed}}$ $n_0 \in \mathbb{N}$

The result goes through

Similarly: It is enough if
(x_n) is eventually monotone increasing and bounded above for
$\lim_{n \to \infty} x_n$ to exists.

That is, if $x_n \leq x_{n+1}$ for all $n \geq n_0$
and $\{x_1, x_2, x_3, \ldots\}$ bounded above
then (x_n) Converges to l.u.b of the above set.

Why this is relevant ?

Look at $x_n = \left(1 - \frac{x}{n}\right)^n \; : \; x > 0$
x is a fixed real no.
The first few terms may actually Oscillate in Sign.

Example: $x = 1000$

$x_1 < 0$, $x_2 > 0$, $x_3 < 0$, ... but eventually all terms are positive !

Use AM-GM ineq to prove that

$\left(1-\frac{x}{n}\right)^n$ is eventually monotone increasing $(x>0)$

However (for $x>0$)

$\left(1+\frac{x}{n}\right)^n$ is monotone inc right from the start.

prove that $\lim\limits_{n\to\infty}\left(1 \underline{+} \frac{x}{n}\right)^n$ exists ; $x \geq 0$

So $\lim\limits_{n\to\infty}\left(1+\frac{x}{n}\right)^n$ exists <u>For all</u> $x \in \mathbb{R}$

<u>This limit</u> will be <u>denoted by $\exp x$</u>

<u>Def.</u> The function $\exp : \mathbb{R} \to \mathbb{R}$

given by

$$\exp x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

is called the exponential function

$$\exp 1 = e$$
$$\exp 0 = 1.$$

For $x \geq 0$,

Since $\left(1+\frac{x}{n}\right)^n$ is monotone increasing and bounded above

$$\exp x \geq \left(1+\frac{x}{1}\right)^1 = 1+x \qquad (x \geq 0)$$

So we get $\boxed{1+x \leq \exp x}$ $(x \geq 0)$

The exponential function is one of the most important functions in all of Science.

Exponential Addition Thm:

$$\exp(x+y) = (\exp x)(\exp y) : \quad x, y \in \mathbb{R}$$

Begin with

$$\left(1+\frac{x}{n}\right)\left(1+\frac{y}{n}\right) = \left(1+\frac{x+y}{n}+\frac{xy}{n^2}\right)$$

$$= \left(1+\frac{x+y}{n}\right)(1+A_n)$$

Assume $x, y > 0$.

$$1 \leq 1+A_n \leq 1+\frac{xy}{n^2}$$

$$\therefore 1 \leq 1+A_n \leq \left\{\left(1+\frac{xy}{n^2}\right)^{n^2}\right\}^{\frac{1}{n^2}}$$

$$\therefore 1 \leq (1+A_n)^n \leq \left(\left(1+\frac{xy}{n^2}\right)^{n^2}\right)^{\frac{1}{n}}$$

Can you use the Sandwich Thm to prove $\left(\left(1+\frac{xy}{n^2}\right)^{n^2}\right)^{\frac{1}{n}} \to 1$ as $n \to \infty$

$$\left(1+\frac{x}{n}\right)^n\left(1+\frac{y}{n}\right)^n = \left(1+\frac{x+y}{n}\right)^n(1+A_n)^n$$

letting $n \to \infty$

$$(\exp x)(\exp y) = \exp(x+y)$$

A minor modification leads to the truth of this for all x, y.

So $\exp 2 = \exp(1+1) = e^2$

$\exp 3 = e^3$ etc;

In general for pos. rational x

$$\exp x = e^x$$

We shall now use the notation e^x for all $x \in \mathbb{R}$ and this means $\exp x$.

$$(\exp x)(\exp(-x)) = \exp 0 = 1$$

$$\text{so} \quad \exp(-x) = (\exp x)^{-1}$$

That is

$$(e^x)^{-1} = e^{-x}$$

It follows:

e^x takes only positive values.

Well,

Since $(\exp x)(\exp(-x)) = 1$

$\exp x \neq 0$ for all $x \in \mathbb{R}$

So

$e^x = \exp x$ has its range in $(0, \infty)$

Further properties of e^x:

Since $e^x \geq 1+x$ we see $e^x \underset{\neq}{>} 1$ if $x \underset{\neq}{>} 0$

Now,

if $x < y$ then by exp. addition thm

$$e^y = e^x \cdot e^{y-x} \underset{\neq}{>} e^x$$

So the function e^x is strictly increasing

Ex: If $0<c<1$ prove that the sequence (x_n) given by

$$x_n = 1 + c + \dots + c^{n-1}$$

is monotone inc and bounded above

$\lim_n x_n$ is $\dfrac{1}{1-c}$

We write $1 + c + c^2 + \dots = \dfrac{1}{1-c}$.

Exercise: Expand $\left(1+\frac{x}{n}\right)^n$ by the binomial thm.

For $x>0$ prove that

$$\left(1+\frac{x}{n}\right)^n < 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$$

Now $\dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dots + \dfrac{x^n}{n!}$

$$\leq \frac{x^2}{2} + \frac{x^3}{4} + \dots + \frac{x^n}{2^n}$$

Deduce that if $0<x<2$,

$$1+x < e^x \leq 1 + x + \frac{x^2/2}{1-x/2}$$

VERY USEFUL!

$\lim\limits_{x\to 0+} e^x = 1$ (We shall formally discuss $\lim\limits_{x\to a} f(x)$ later but for now you can use your JEE concepts)

Prove: $\lim\limits_{x\to 0-} e^x = 1$. Use $e^{-x} = (e^x)^{-1}$

Prove that e^x is a continuous function

i.e $\lim_{x \to a} e^x = e^a \quad \forall\, a \in \mathbb{R}$

$\lim_{x \to -\infty} e^x = 0 \;;\; \lim_{x \to +\infty} e^x = +\infty$

So by intermediate value theorem e^x takes on all values in $(0, \infty)$

The derivative of e^x: Recall notion of derivative (JEE concept)

We shall discuss this formally later.

Using the exercise on p 15

$$\lim_{h \to 0+} \frac{e^h - 1}{h} = 1$$

Further $\frac{e^h - 1}{h} = e^h\left(\frac{e^{-h} - 1}{-h}\right)$ for $h < 0$

$$\lim_{h \to 0-} \frac{e^h - 1}{h} = 1$$

Using exponential addition thm

$$\lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = e^x \lim_{h \to 0} \frac{e^h - 1}{h} = e^x$$

So exponential fun. is diff. and

$$\frac{d}{dx}(e^x) = e^x.$$

The Logarithm Function

We have seen that the function

$$\exp : \mathbb{R} \to (0,\infty)$$

$$\exp x = e^x \qquad \left(= \lim_n \left(1+\frac{x}{n}\right)^n\right)$$

is a strictly increasing continuous surjective function.

Strictly increasing $\Rightarrow$ injective (why?)

So $\exp : \mathbb{R} \to (0,\infty)$ is bijective

The inverse function?

Def: The inverse function of $\exp : \mathbb{R} \to (0,\infty)$ is called the (natural) Log function: $\log : (0,\infty) \to \mathbb{R}$

Thus,

$$e^{\log x} = x \quad \text{for all } x \in (0,\infty)$$

$$\log e^x = x \quad \text{for all } x \in \mathbb{R}$$

We shall see later that if for open int. I, J

$f : I \to J$ is differentiable, surjective

$f'(x) \neq 0$ for all $x \in I$

then $f^{-1} : J \to I$ is also differentiable

In particular $\log x$ is differentiable.

Let us now apply the chain rule to the equation

$$e^{\log x} = x \quad \text{and we get}$$

$$e^{\log x} \frac{d}{dx}(\log x) = 1 \quad \text{or} \quad x \frac{d}{dx} \log x = 1$$

$$x > 0.$$

We have : $\frac{d}{dx} \log x = \frac{1}{x}$; $x > 0$.

From the exponential addition thm we get at once the following:

$$\log(xy) = \log x + \log y : \quad x, y > 0$$

Next $e^{\log 1} = 1 = e^0$

$\therefore \log 1 = 0$.

Ex. Prove that $\log \sqrt{a} = \frac{1}{2} \log a$; $a > 0$

More generally $\log a^x = x \log a$ for all rational x

Que. Why is it at this stage we are not writing $\log(a^x) = x \log a$, $a > 0$ for all real numbers x ?

Ex: Prove that $(1/\log n)$ is monotone decreasing ; $n = 2, 3, 4, \ldots$

What is the limit ?

prove that $\frac{(\log n)^{1000}}{n^{1/7}} \to 0$ as $n \to \infty$

Some more exercises on sequences :

Define $a_1 = \sqrt{2}$; $a_2 = \sqrt{2 + \sqrt{2}}$

generally $a_{n+1} = \sqrt{2 + a_n}$; $n \geq 1$

Is the seq (a_n) monotone increasing ?

Is it bounded above.

Problem will be done in Tutorial 1.

Prove by induction $a_n \leq 2$ for all n

Since the limit $\lim_n a_n = l$ exists

$$\lim_n a_{n+1} = l \quad (\text{why ?})$$

$$\therefore \lim_n a_{n+1}^2 = l^2 \;;\; \text{but } a_{n+1}^2 = 2 + a_n$$

$$\therefore l^2 = 2 + l$$

$$\therefore l = 2$$

Ex: Let $x_n = \frac{1}{n+1} + \cdots + \frac{1}{n+n}$

Prove that (x_n) is monotone increasing and bounded above. $\lim_n x_n$ exists $= l$.
Show that l lies between $\frac{1}{2}$ and 1.
We shall see later that $l = \ln 2$.

To get started, Calculate $x_{n+1} - x_n$ and see.

Tutorial Problem:

Suppose (x_n) is monotone increasing
Is the sequence of averages

$$\frac{1}{n}(x_1 + \cdots + x_n) \text{ also monotone increasing ?}$$

Ex:

Discuss whether the seq (x_n) where

$$x_n = \frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n} \text{ is}$$

Convergent. (Easy!)

Exercise: Let $x_n = \dfrac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}$

Show that $n x_n^2$ is monotone increasing
$(n+\frac{1}{2}) x_n^2$ is monotone decreasing

Is (x_n) Convergent? NO
Is $(n x_n)$ Convergent? Yes.
It is not easy to "Guess" the limit !!

J.C. Burkill, p 46.

Cauchy's first limit theorem:

If (x_n) is a seq of real numbers such that $\lim_n x_n = l$ then the seq of arith. Means also $\to l$.

i.e $\lim_n \frac{1}{n}(x_1 + \ldots + x_n) = l$.

Proof is not difficult but we shall not discuss it here.

Discuss (without attempting a rigorous proof) why is this plausible.

Cor: Suppose (x_n) is a seq of positive reals and $x_n \to l$ as $n \to \infty$

Then,
$$\lim_n (x_1 \ldots x_n)^{\frac{1}{n}} = l.$$

Proof: Case 1: $l = 0$.
Use AM-GM ineq.

$$0 < (x_1 \ldots x_n)^{\frac{1}{n}} \le \frac{1}{n}(x_1 + \ldots + x_n)$$

Now Cauchy's first lim. thm gives the result.

Case 2: $l > 0$. Note that $\frac{1}{x_n} \to \frac{1}{l}$

So
$$\frac{1}{n}\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right) \to l^{-1}$$

$$\frac{1}{n}(x_1 + \cdots + x_n) \to l.$$

But $HM \leq GM \leq AM$ (how?)

HM = harmonic mean
$$= \left(\frac{1}{n}\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right)\right)^{-1}$$

~~Use~~. $HM \to l$ as $n \to \infty$

Use Sandwich thm.

Cor. Cauchy's 2nd limit theorem

Suppose (x_n) is a seq. of positive reals. Such that
$$\lim_n \left(\frac{x_{n+1}}{x_n}\right) = l$$

Then $\lim_n \sqrt[n]{x_n} = l$.

(i.e $\lim_n \sqrt[n]{x_n}$ exists and equals l).

Proof: Define $y_1 = x_1$

$y_2 = x_2/x_1$, $\ldots$, $y_n = \dfrac{x_n}{x_{n-1}}$

Then $\lim_n y_n = l$

So by the previous Corollary

$\lim_n (y_1 \cdots y_n)^{\frac{1}{n}} = l \quad \therefore \lim_n (x_n)^{\frac{1}{n}} = l$

<u>Ex:</u> $\lim_n \left(\frac{n!}{n^n}\right)^{\frac{1}{n}} = \frac{1}{e}$

<u>proof:</u> $x_n = \frac{n!}{n^n}$; $\frac{x_{n+1}}{x_n} = \frac{1}{(1+\frac{1}{n})^n}$

By Cauchy's 2nd limit thm we get

$$\frac{\sqrt[n]{n!}}{n} \to \frac{1}{e}$$

So, loosely speaking this result can be written as

$$\sqrt[n]{n!} \sim n/e$$

Or for $n >> 1$

$$\log n! \sim n \log n - n \qquad (*)$$

This approx is extremely useful in modern phy. See A. Beiser, Perspectives in Mod. phy. p 354 McGraw Hill 1969.

One MUST RESIST the temptation of exponentiating (*) and writing

$n! \sim n^n e^{-n}$ ⟵ This is WRONG

The correct statement is given by

Thm (<u>Stirling's Approx Formula</u>)

$$n! \sim n^n e^{-n}\sqrt{2n\pi} \quad \text{or}$$

$$\lim_{n\to\infty} \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} = 1.$$

James Stirling Methodus Differentialis (1730)

We cannot prove this here. See W. Feller. Intro to Prob. th. vol I.

Problems:

(1) Let $x_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$ (Tutorial)
(x_n) is monotone inc but not bounded above and so cannot converge.
Show: $x_4 > 2$, $x_8 > \cancel{3} 5/2$, $x_{16} >$?

(2) Let $a > 0$ and
$$x_n = 1 + a + \frac{a^2}{2!} + \dots + \frac{a^n}{n!}$$

Show (x_n) is monotone increasing and bounded above. Let us call $\lim_{n\to\infty} x_n = f(a)$
This $f(a) = e^a$

$\cancel{(3)}$ Thm: $e^a = 1 + a + \frac{a^2}{2!} + \dots + \dots$ all the way to infinity

This thm is Extremely imp.
You must remember the statement of this thm - will be used often.
Proof of the thm? That's altogether a different matter.

(3) Optional Exercise: Prove the theorem for $a > 0$
Well $x_n < f(a)$
Now expand $(1 + \frac{a}{n})^n$ using binomial thm and check:
$(1 + a/n)^n \leq f(a)$. Letting $n \to \infty$
we get $e^a \leq f(a)$. Now the reverse ineq
Expand $(1 + a/n)^n$ using binomial and truncate it to m terms with $m < n$

Keep m fixed and let $n \to \infty$

$(1 + \frac{a}{n})^n \geq$ (Truncation to m terms)

letting $n \to \infty$ we get

$$e^a \geq 1 + a + \frac{a^2}{2!} + \dots + \frac{a^m}{m!} \quad \forall m \in \mathbb{N}$$

(4) Compute : $\lim_{n \to \infty} n(\sqrt[n]{5} - 1)$

Does $\lim_{n \to \infty} n(\sqrt[n]{n} - 1)$ exist ?

(5) Let (a_n) be a sequence such that each $a_j \in \{0, 1, 2, \dots 9\}$

Now we construct for $n = 1, 2, 3, \dots$

$$x_n = \frac{a_1}{10} + \frac{a_2}{10^2} + \dots + \frac{a_n}{10^n}$$

Show that this new seq (x_n) is monotone increasing and bounded above.

Call $\lim_{n \to \infty} x_n = \underline{\underline{a}} \in [0, 1]$.

Is it true that given any $\underline{\underline{a}} \in [0, 1]$ there exists a seq (a_n) as above such that

$$\lim_{n \to \infty} \left(\frac{a_1}{10} + \dots + \frac{a_n}{10^n} \right) = \underline{\underline{a}} \quad ?$$

Discuss this result in a more informal and elementary (non rigorous) language that was familar to you

Important:

Prove that given any real number x, there is a sequence of rational numbers (q_n) converging to x.

End of the Chapter. Concluding remarks

Here in this Chapter we have discussed

- The notion of Convergence of a Seq.
- A few theorems that are frequently used.
- Introduced two functions

$$\exp x \quad \text{or } e^x \ ; \quad \text{and } \log x$$

proved some basic results on these

$$\exp(x+y) = (\exp x)(\exp y)$$

$$\log(xy) = \log x + \log y$$

Both $\exp : \mathbb{R} \to (0,\infty)$ and $\log : (0,\infty) \to \mathbb{R}$ are bijective Cont. functions and inverses of each other.

$$\frac{d}{dx}(\exp x) = \exp x : \quad \frac{d}{dx}(\log x) = \frac{1}{x}$$

$$\exp 0 = 1, \ \log 1 = 0 : \quad \exp 1 = e$$

Transcendental Functions: $\exp x$ and $\log x$

These are two fundamental examples of non algebraic functions.

The function $f : [-1,1] \to \mathbb{R}$ given by

$$f(x) = \sqrt{1-x^2}$$

is an example of an algebraic function and so are $x + \frac{1}{x}$, $1 - \frac{1}{x^2}$ etc;

Simplest functions are polynomials, next come rational functions $\frac{P(x)}{Q(x)}$

$P(x)$, $Q(x)$ polynomials

$Q(x) \neq$ Zero polynomial and $\frac{P(x)}{Q(x)}$ has its domain $\mathbb{R}$ minus roots of $Q(x)$

For example $\left(1+\frac{x}{n}\right)^n$ is an example of a rational fn. (infact poly) and $\left(1-\frac{x}{n}\right)^{-n}$ is also a rational fn.

Next in ~~the~~ hierarchy come algebraic functions such as

$\sqrt{x}$ and $x^{1/3}$ defined on $[0,\infty)$

and $\sqrt{1-x^2}$ defined on $[-1,1]$

$x^{1/3}$ satisfies the algebraic equation

$$y^3 - x = 0$$

$\sqrt{1-x^2}$ satisfies the alg. eq^n $y^2 = 1-x^2$

It is a fact and we shall not prove it here that $\exp x$ and $\log x$ do not

For a plausibility, consult G.H. Hardy, Course in Pure Math

satisfy such algebraic equations namely

$$y^n + R_1(x)\, y^{n-1} + \dots + R_n(x) = 0$$

$R_1(x), \dots, R_n(x)$ rational functions.

There are other such functions that you are familiar with: $\cos x$ and $\sin x$

We seem to have developed e^x and $\log x$ from scratch giving precise, arguments for definitions/ support of the basic thm identities such as exp. addition thm.

Is it possible to develop properties of $\cos x$, $\sin x$ rigorously starting from precise definitions?

Yes! It can be done. See J.C. Burkill

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