# III - Mean Value Theorems of differential Calculus.

Title refers to a group of theorems that are of paramount importance.
All these follow from Rolle's theorem which in turn is a simple consequence of the fundamental result:
Thm B on p32 and its supplement Thm D on p34. This is precisely the reason for discussing in detail the proofs on p 32 ff.

**Thm (Rolle's thm):** Suppose $f: [a,b] \to \mathbb{R}$ is continuous, and differentiable on $(a,b)$. Further, assume $f(a) = f(b)$
Then there exists a $c \in (a,b)$ (open interval!) such that $f'(c) = 0$.

**proof.** $M = \text{l.u.b}\{f(x) / x \in [a,b]\}$
$m = \text{g.l.b}\{f(x) / x \in [a,b]\}$
$M = f(p)$ : $m = f(q)$ for $p, q \in [a,b]$
if $p, q$ are both endpoints of $[a,b]$ ($p = a$, $q = b$ or $q = a$, $p = b$) then by virtue of $f(a) = f(b)$ we conclude
$M = m$ and $f$ is const on $[a,b]$
and at every point in $(a,b)$, $f'$ vanishes ✓

Assume $p$ or $q$ is in the open interval $(a,b)$ then the derivative at that point is 0

Assume $p \in (a,b)$: $f(p) = M = \sup\{f(x) / a \le x \le b\}$
$p$ is a point of local Max and so $f'(p) = 0$
(See p 41). Rollé's thm is proved

Cor 1: Suppose $f, g, h : [a,b] \to \mathbb{R}$ cont. and differentiable on $(a,b)$ then there exists $c \in (a,b)$ such that

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0. \quad \text{——} (*)$$

proof:
Let $\phi(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$

$\phi(a) = \phi(b) = 0$
$\phi$ is Cont. on $[a,b]$ and diff. on $(a,b)$
By Rollé's thm there is a $c \in (a,b)$ s.t.
$\phi'(c) = 0$.

Exercise: Check that $\phi'(c) = 0$ is precisely $(*)$

Cor 2: Cauchy's MVT (CMVT) Suppose $f, g : [a,b] \to \mathbb{R}$ Cont. diff on $(a,b)$. Then there exists $c \in (a,b)$ such that
$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a))$$
pf: Take $h(x) = 1$ in Cor 1 (Exercise)

Cor 3 Lagrangé's MVT (LMVT): If $f : [a,b] \to \mathbb{R}$ Continuous and diff on $(a,b)$ then there exists $c \in (a,b)$ s.t.

$$f(b) - f(a) = f'(c)(b-a)$$

Pf: Exercise. Take $g(x) = x$ in CMVT

Cor 4: If $f : (a,b) \to \mathbb{R}$ is differentiable and $f'(x) > 0$ throughout $(a,b)$ then $f$ is strictly increasing.

That is $x, y \in (a,b)$ with $x < y$

$$\Rightarrow f(x) < f(y)$$

Pf: Apply LMVT to $f$ on $[x,y]$

Formulate a parallel result when $f'(x) < 0$ throughout $(a,b)$

Cor 5: (L'Hospital's Rule)

Suppose $f, g : (a,b) \to \mathbb{R}$ differentiable and $p \in (a,b)$ such that $g(x) \neq 0$

(i) $f(p) = g(p) = 0$ : $g'(x) \neq 0$ on $(a,b)$

(ii) $\lim\limits_{x \to p} \frac{f'(x)}{g'(x)}$ exists $= \ell$

Then

$$\lim_{x \to p} \frac{f(x)}{g(x)} \text{ exists and } = \ell.$$

Note: We need (ii) ~~and (iii)~~ and (i) to hold only on some interval $(p-c, p+c)$

That $g'(x)$ could vanish at points in $(a,b)$ as long as we are guarenteed $g(x)$ and $g'(x) \neq 0$ in an interval $(p-c, p+c)$

For computing limits and derivatives of functions at a point $p$ all that matters is behaviour in some interval $(p-c, p+c)$

Proof: We have to prove that if $(x_n)$ is a seq of points such that $x_n \to a$ and $x_n \neq a$ the corresp. sequence $\frac{f(x_n)}{g(x_n)} \to l$.

Well, $\frac{f(x_n)}{g(x_n)} = \frac{f(x_n) - f(p)}{g(x_n) - g(p)} = \frac{f'(c_n)}{g'(c_n)}$

by CMVT:

$c_n$ lies between $p$ and $x_n$

So, $|c_n - p| \leq |p - x_n|$ but $|p - x_n| \to 0$

$\therefore \quad c_n \to p$.

Since $\lim_{x \to p} \frac{f'(x)}{g'(x)}$ is given to exist and $= l$.

We conclude $\frac{f'(c_n)}{g'(c_n)} \to l$

$\therefore \quad \frac{f(x_n)}{g(x_n)} \to l$. Thm is proved!

Exercises: If $P(x)$ is a polynomial a root $c$ is called a ~~double~~ simple root if $P(x) = (x-c)\,Q(x)$: $Q(c) \neq 0$

$c$ is called a double root if

$P(x) = (x-c)^2\,Q(x)$ : $Q(c) \neq 0$.

$c$ is called a triple root if

$P(x) = (x-c)^3\,Q(x)$: $Q(c) \neq 0$ etc:

Prove that if $c$ is a $k$-fold root of $p(x)$ (Polynomial)
namely

$$p(x) = (x-c)^k Q(x) : \quad Q(c) \neq 0$$

then

$$\left.\begin{array}{l} p(c) = 0, \ p'(c) = 0, \ldots, \ p^{(k-1)}(c) = 0 \\ \text{and} \ \ p^{(k)}(c) \neq 0 \end{array}\right\} (*)$$

Conversely if $(*)$ holds then $c$ is a $k$-fold root of $p(x)$

Exercise. Suppose $f$ is a polynomial such that $a, b$ are both double roots of $f$ $(a<b)$ then $f'(x)$ vanishes at least at two distinct points in $(a, b)$.

Exercise: Prove that the polynomial

$$\left(\frac{d}{dx}\right)^n (x^2-1)^n \qquad (**)$$

has $n$ distinct roots in $(-1, 1)$

Hint: $f(a) = f(b) = 0$
Apply Rolle's thm to get $f'(c) = 0$; $a<c<b$
Apply Rolle's thm again on $(a, c)$ and $(c, b)$ (to what function?)

The polynomial $(**)$ has degree $n$ and so cannot have more than $n$ roots. With the factor $\frac{1}{2^n n!}$ it is denoted by $P_n(x)$:

$$P_n(x) = \frac{1}{2^n n!} D^n (x^2-1)^n \quad (\text{Legendre Poly}).$$

Sufficient Cond. for a point to be a local Max or a local Min: $I$ is an open interval

$f : I \to \mathbb{R}$ differentiable

$f'(p) = 0$ : $p \in I$.

$c > 0$ is such that $(p-c, p+c) \subset I$

We shall put further cond. to ensure $p$ is a point of local Max/Min

Thm:

Suppose $\left.\begin{array}{l} f'(x) > 0 \text{ on } (p-c, p) \\ f'(x) < 0 \text{ on } (p, p+c) \end{array}\right\} \Rightarrow p$ is a pt of local Max (i)

Suppose $\left.\begin{array}{l} f'(x) < 0 \text{ on } (p-c, p) \\ f'(x) > 0 \text{ on } (p, p+c) \end{array}\right\} \Rightarrow p$ is a pt of local Min (ii)

We shall prove (i)

proof of (ii) Exercise for the Student.

By Cor 4 on p. 44

$f$ is strictly inc on $(p-c, p)$

In particular, let $x \in (p-c, p)$ ; $x < p$

$\therefore$ $x < p - \frac{1}{n}$ for all $n$ sufficiently large

$\therefore$ $f(x) < f(p - 1/n)$ ...

But $p - 1/n$ monotone inc. to $p$

Continuity of $f$ now implies the seq

$f(p - 1/n)$ monotone inc & $\to f(p)$

$\therefore$ $f(x) < f(p)$

Thus $f(x) < f(p)$ For all $x \in (p-c, p)$

Next $f$ is strictly dec. on $(p, p+c)$

Let $x \in (p, p+c)$

$p + \frac{1}{n} < x$ for all $n$ sufficiently large

$p + 1/n$ monotone $\downarrow$ and conv. to $p$.

$\therefore$ $f(p+1/n)$ is monotone increasing and converges to $f(p)$

$f(x) < f(p+\frac{1}{n}) \to f(p)$

also

$\therefore$ $f(x) < f(p)$ for $x \in (p, p+c)$

$\therefore$ $f(x) < f(p)$ $\forall$ $x \in (p-c, p+c)$, $x \neq p$

$p$ is a point of local Max

<u>Cor</u>: If $f : I \to \mathbb{R}$ is twice differentiable and $f'(p) = 0$.

$f''(p) > 0 \Rightarrow p$ is a point of local Min

$f''(p) < 0 \Rightarrow p$ is a point of local Max

Well, $f''(p) < 0 \Rightarrow f'$ is strictly dec. at $p$ so there is a $c > 0$ s.t.

$(p-c, p+c) \subset I$ and

$f'(x) < f'(p) = 0$ on $(p, p+c)$

$f'(x) > f'(p) = 0$ on $(p-c, p)$

By the Main thm (p 47) we get that $p$ is a point of local Max.

The Other Case $f''(p) > 0$ left for the Student to work out . . .

(Reading Ex for Student ↓)

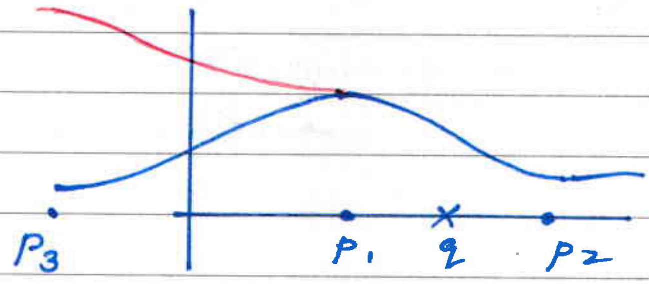
<u>Remark</u>: In many problems one computes the roots of $f'(x) = 0$

They are usually finite in number say $p_1$, $p_2$, $p_3$ for example.

$f'(p_1) = 0$; $f'(p_2) = 0$, $f'(p_3) = 0$

There is NO need to compute $f''(p)$

BECAUSE, $f'$ maintains the same sign between $p_1$ and $p_2$
One can simply evaluate $f'(q)$ at some convenient point $q \in (p_1, p_2)$
and figure out the sign of $f'$ there
if $f' < 0$ on $(p_1, p_2)$ then
$f$ must decrease from $p_1, p_2$




Now check the sign of $f'(q)$ for some convenient $q$ in $(p_3, p_1)$
If this sign is +ve then we know $f$ increases on $(p_3, p_1)$ (blue curve)
However if this sign is −ve then $f$ decreases from $p_3$ to $p_1$ (Red ~~curve~~ Arc)

$p_1$ is a maxima for the completely blue curve
$p_1$ is neither a Max nor a Min for the Red-blue curve

This is useful because computing the second derivative can be quite a tedious task.

Try $f(x) = e^{-x^2}\left(\dfrac{1-x^2}{1+x^2}\right)$. Is origin a Max/Min ?

Polynomial
Compute this Expression

$f'(x) = 2x\,Q(x)\,(\cdots)$ $(1+x^2)^2$ clubbed with $Q(x)$

$Q(x)$ is never vanishing and always positive

$f'(q) = 2q$ (positive)

$\text{Sign } f'(x) = \text{Sign } x(\cdots)$

$2\,Q(x)$ has been ignored !

Note that Computing $f''(x)$ would be quite troublesome

(End of Reading Ex)

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## Points of Inflection

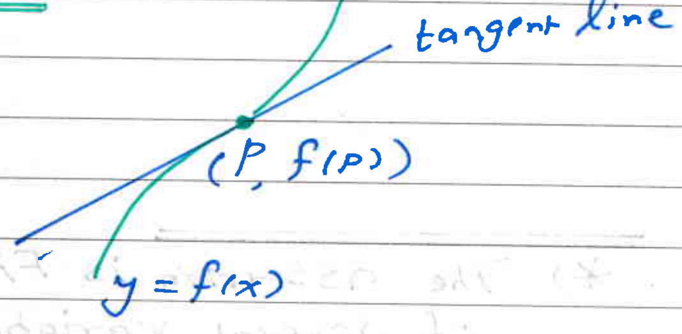
(In older books the spelling is inflexion : G. A. Gibson, elementary treatise on the Calculus)

For a twice differentiable function $f : I \to \mathbb{R}$, a point $p \in I$ is said to be a point of inflection if $f''(p) = 0$

(Thomas-Finney, 9th Ed; page 211)

Geometrically at the point $(p, f(p))$ on the graph, the graph crosses its tangent and bends away from it in opposite directions on opposite sides

(p35 of G. A. Gibson)



Thomas-Finney (9th Edition p 219)

Problem 76: $f'(x) = (x-1)^2(x-2)(x-4)$

locate the points of local Max / Min

Does the function have points of inflection?

Problem 79: For what value of $b$ will $x^3 + bx^2 + cx + d$ have an inflection point at $x = 1$ ?

Exercise: Let $I$ be an open interval and $f : I \to \mathbb{R}$ have a unique $p$ at which $f'(p) = 0$.

Suppose that $f''(p) > 0$

we know $p$ is a point of local Min

Can we assert that

$$f(p) = \inf \{ f(x) / x \in I \}$$

Would Rolle's thm help?

The case with functions of several variables is more interesting* and we shall return to this in a later chapter

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*) The assertion is FALSE for functions of several variables.

## Convex functions:

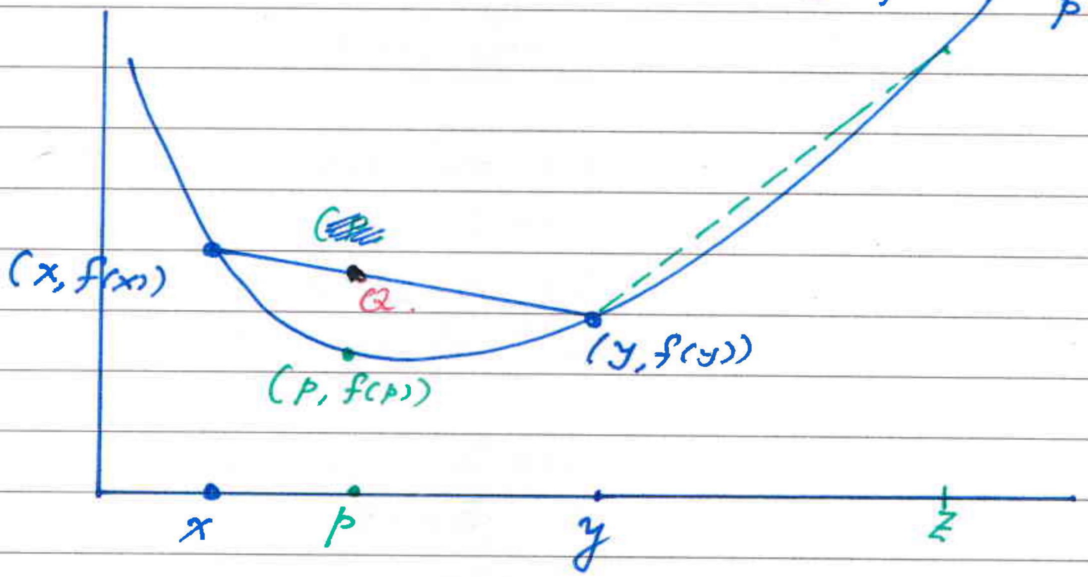
$I$ is an open interval.

$f : I \to \mathbb{R}$ is said to be convex if given $x, y \in I$

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y) \quad (*)$$

In otherwords, the chord joining $(x, f(x))$ and $(y, f(y))$ lies above the graph of $f$

The nicest example is the parabola (slightly skewed in this pic)




$$p = tx + (1-t)y$$

Using say section formula:

$$Q = (p, \; tf(x) + (1-t)f(y))$$

Condition $(*)$ simply says the point

$Q$ lies above the point $(p, f(p))$

## Basic slope lemma:

Let $f: I \to \mathbb{R}$ be convex

Assume $x < y < z$ ; $x, y, z \in I$

Slope chord joining $(x, f(x))$ and $(y, f(y))$

$\leq$ slope chord joining $(y, f(y))$ and $(z, f(z))$

( See the picture on p 52)

**Exercise (Tutorial)** Prove the basic slope lemma

You have nothing but (*) and section formula to play with!

How do these two slopes compare with slope of chord joining $(x, f(x))$, $(z, f(z))$

Can you make a guess? Prove that your guess is correct.

Use the basic slope lemma to prove that for $f: I \to \mathbb{R}$ differentiable,

$f$ is convex iff $f'$ is increasing.

(In one direction LMVT will help)

Deduce that if $f: I \to \mathbb{R}$ is twice differentiable

$f$ is convex iff $f'' \geq 0$.

**Example:** $e^x$ is convex

$-\log x$ is convex on $(0, \infty)$

$e^{-x}$ is convex

- $\sin x$ is convex on $(0, \pi/2)$

  $\sin x$ is Concave on $(0, \pi/2)$ The chord joining two points lies below the graph

Ex: prove that $\sin x \geq \frac{2}{\pi} x$ for $0 < x < \pi/2$

(Draw the picture. Use Concavity)

What is $\lim_{R \to \infty} \int_0^{\pi/2} e^{-R\sin\theta}\, d\theta$ ?

Exercises:

Which of the following functions are Convex ?

(i) $\log \sec\theta$ on $(-\pi/2, \pi/2)$

(ii) $\log(e^x + e^{-x})$ on $\mathbb{R}$

(iii) $\tan^{-1} e^x$ on $(0, \infty)$ ?

(iv) $\tan^{-1} e^x$ on $(-\infty, 0)$

(v) $\tan^{-1} e^x$ on $\mathbb{R}$

What are the points of inflection on the graph of $\tan^{-1} e^x$ ?

$e^x$ and $x^2$ are both Convex

If $f : \mathbb{R}^+ \to \mathbb{R}$ is Concave then $f(e^x)$, $f(x^2)$ may not be Convex. That is "Convexity can be lost by applying a Concave function to it"

Discuss this through several examples.

$\sec\theta$ is Convex and it retains its Convexity even after applying log (!)

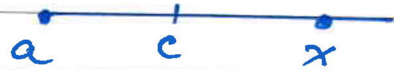
## Polynomial Approximation

Taylor's Theorem: Before we state and prove this let us make a simple observation:
Assume $f:[a,b]\to\mathbb{R}$ cont and twice differentiable* on $[a,b)$. For $a<x<b$

$$f(x) = f(a) + (x-a)f'(c)$$

$c$ between $a$ and $x$

We can write (LMVT again)




$$f'(c) = f'(a) + (c-a)f''(\xi)$$

for some $\xi \in (a,c)$

So

$$f(x) = f(a) + (x-a)f'(a) + (c-a)(x-a)f''(\xi)$$

Note that $|(c-a)(x-a)| \leq |x-a|^2$

So if $x$ is close to $x$ then,

$f(x) \doteq f(a) + (x-a)f'(a)$ is a good approximation since the "Error"

$$|(c-a)(x-a)f''(\xi)| \leq M|x-a|^2$$

$$M = \underset{[a,x]}{\text{l.u.b}}|f''|$$

is small due to the quadric $|x-a|^2$

In fact we can do much better than this

Instead of the two $c,\xi$ that ~~are not~~ arise out of successive applications of LMVT

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* differentiable twice on an interval $(a-c, b)$. This is purely Motivational

We will obtain a formula with "just one single c" - namely one single application of Rolle's thm

Let's begin: Assume $f: I \to \mathbb{R}$ is twice differentiable on an open interval and $a, b \in I$ with $a < b$


Consider

$$\phi(x) = f(b) - f(x) - (b-x) f'(x)$$

$$\phi(b) = 0$$

But $\phi(a) \neq 0$. Let us Modify $\phi(x)$ as

$$\Phi(x) = \phi(x) - \left(\frac{b-x}{b-a}\right)^2 \phi(a)$$

So $\Phi(a) = 0 = \Phi(b)$

Rolle's thm can be applied. Let us Compute $\Phi'(x)$:

$$\Phi'(x) = \phi'(x) + \frac{2(b-x)}{(b-a)^2} \phi(a)$$

$$= -(b-x) f''(x) + \frac{2(b-x)}{(b-a)^2} \phi(a)$$

Now $\Phi'(c) = 0$ for some $c \in (a,b)$

$$\therefore \quad \phi(a) = \frac{1}{2}(b-a)^2 f''(c)$$

$$\therefore f(b) = f(a) + (b-a) f'(a) + \frac{1}{2}(b-a)^2 f''(c)$$

Now suppose $f: I \to \mathbb{R}$ is thrice differentiable. Define

$$\phi(x) = f(b) - f(x) - (b-x)f'(x) - \frac{(b-x)^2}{2} f''(x)$$

$\phi(b) = 0$ but $\phi(a) \neq 0$.

So $\Phi(x) = \phi(x) - \left(\frac{b-x}{b-a}\right)^3 \phi(a)$

$\Phi(b) = \Phi(a) = 0$.

Exercise: Prove that

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!} f''(a) + \frac{(b-a)^3}{3!} f'''(c)$$

Thm: (Taylor's Thm) If $f: I \to \mathbb{R}$ is $n$-times differentiable and $a, b \in I$ Then, $\exists$ $c$ between $a$ and $b$ such that

$$f(b) = f(a) + (b-a)f'(a) + \dots + \frac{(b-a)^{n-1} f^{(n-1)}(a)}{(n-1)!} + R_n(a, b, c) \qquad (*)$$

$R_n(a,b,c) = \frac{(b-a)^n}{n!} f^{(n)}(c)$. Called the Remainder in Taylor's Theorem - called Young's form of remainder

Remark: There are several different versions but this one suffices for our purpose. A great deal is known about the dependence of $c$ on $a, b$ .... We shall not discuss.

Take $f(x) = \sin x$

$$f^{(k)}(x) = \sin\left(x + k\frac{\pi}{2}\right)$$

Take $a = 0$ : $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1$

So (*) reads:

$$\sin b = b - \frac{b^3}{3!} + \dots + (-1)^{n-1}\frac{b^{2n-1}}{(2n-1)!} + \frac{b^{2n}}{(2n)!}\sin\left(c + n\pi\right)$$

$$R_n = \frac{b^{2n}}{(2n)!}\sin(c + n\pi)$$

Exercise: Prove that if $x \in \mathbb{R}$

$$\left|\frac{x^n}{n!}\right| \to 0 \quad \text{as } n \to \infty$$

So

$$\lim_{n\to\infty}\left(\sin b - \left(b - \frac{b^3}{3!} + \dots + (-1)^{n-1}\frac{b^{2n-1}}{(2n-1)!}\right)\right)$$

$$= \lim_n R_n = 0.$$

$$\therefore \sin b = \lim_n\left(b - \frac{b^3}{3!} + \dots + (-1)^{n-1}\frac{b^{2n-1}}{(2n-1)!}\right)$$

$$= b - \frac{b^3}{3!} + \frac{b^5}{5!} - \dots$$

Prove: $\cos b = 1 - \frac{b^2}{2!} + \frac{b^4}{4!} - \dots$

Prove: $e^b = 1 + b + \frac{b^2}{2!} + \frac{b^3}{3!} + \dots$

Ex:

$f(x) = \log(1+x)$ ; ~~xxx~~ $x \in (-1, \infty)$

Calculate the $n$th derivative of $f$ at the origin.

Write down the Taylor series for $f$ with remainder $R_n(b)$:

$$f(b) = f(0) + b f'(0) + \ldots + \frac{b^{n-1} f^{(n-1)}(0)}{(n-1)!} + R_n(b)$$

Suppose $0 < b < 1$ Can you say $R_n(b) \to 0$ ?

Is it Correct to say (Strict Ineq)

$$\log(1+b) = b - \frac{b^2}{2} + \frac{b^3}{3} - \ldots ; \quad 0 < b < 1$$

Is it Correct to say

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \ldots$$

} Dangerous! Result is Correct but needs Serious argument

Ex: Calculate the Successive derivatives of $(1+x)^{-1/2}$ at the Origin and Write out the Taylor Series, $x \in (-1, \infty)$ Showing that $R_n(b) \to 0$ is going to involve Some work!

Take $0 < b < 1$. Then $0 < c < b$

$0 < \dfrac{b}{c+1} < b$. Write out $R_n(b)$.

Would Stirling's formula help in proving $R_n(b) \to 0$? Try it.

Que: What about things like $(1+x)^{-1/3}$ or $(1+x)^{\alpha}$? More troublesome. Shall return to this later.