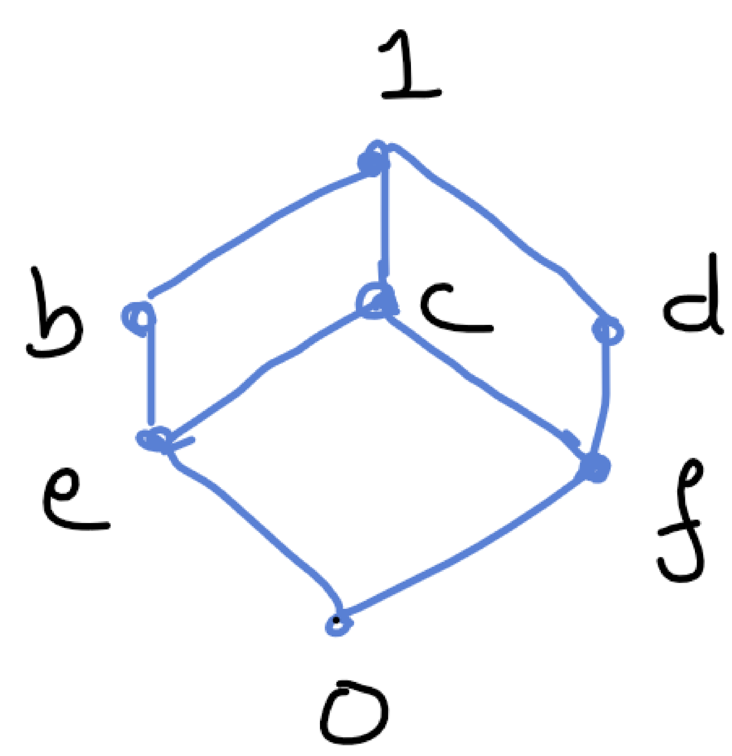
Uniqueness of Finite Boolean Algebra.

Definition: Let $(A, \leq)$ be a finite Boolean lattice with universal lower bound 0. An element is called an atom if it covers 0.



Atoms are e and f.

Lemma: In a distributive lattice, if $b \wedge \bar{c} = 0$, then $b \leq c$.

Proof: Consider $b \wedge \bar{c} = 0$

$$(b \wedge \bar{c}) \vee c = 0 \vee c$$

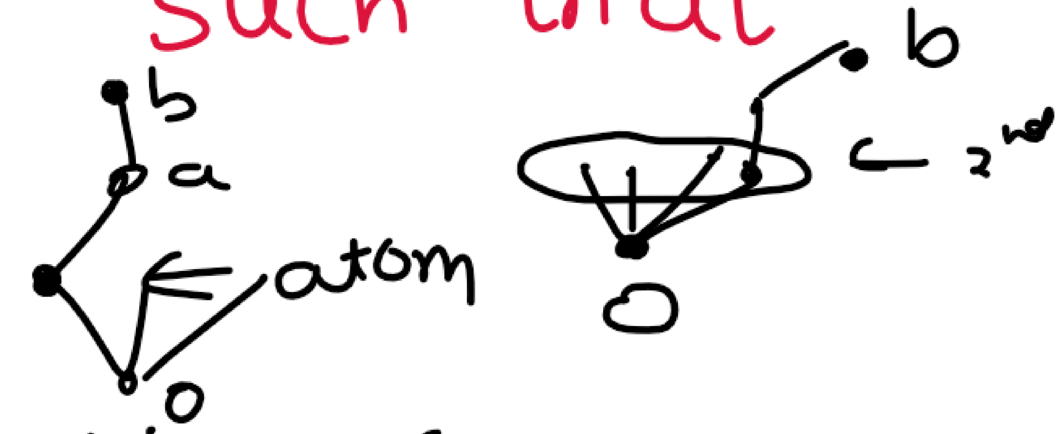
$$(c \vee b) \wedge (c \vee \bar{c}) = c \qquad \text{Distributive}$$

$$(c \vee b) \wedge 1 = c$$

$$c \vee b = c$$

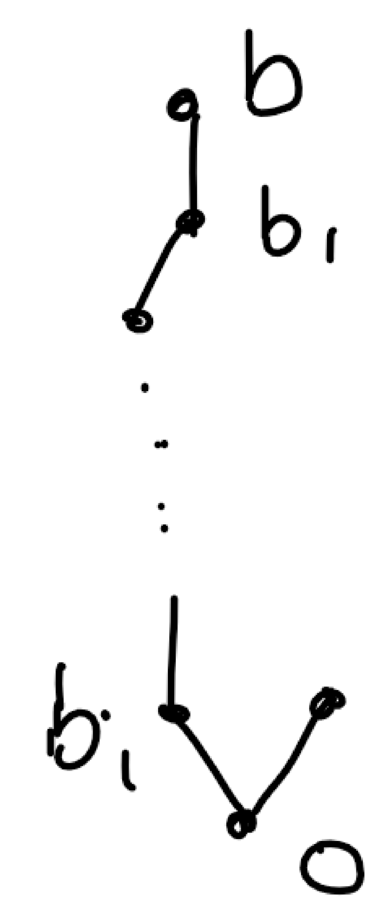
$$\Rightarrow b \leq c$$

Lemma 2: Let $(A, \leq)$ be a finite lattice with a universal lower bound 0. Then for any non zero element b (which is not universal lower bound 0), there exists at least one atom 'a' such that $a \leq b$.



Proof: If $\underline{b}$ is an atom, then there is nothing to prove as $b \leq b$.

Suppose b is not an atom, since $(A, \leq)$ is a finite lattice, there must be a chain in $(A, \leq)$ such that $(0, \underline{b_i}, \dots, b_2, b_1, b)$ where b_i is an atom.



$\implies b_i \leq b$

Lemma 3 : Let $(A, \vee, \wedge, -)$ be a finite boolean algebra. Let b be any nonzero element in A, and $a_1, a_2, \ldots a_k$ be all the atoms of A such that $a_i \leq b$. Then,

$$b = a_1 \vee a_2 \vee \cdots \vee a_k$$

Proof: Let $a_1 \vee a_2 \vee \cdots \vee a_k = c$.

To show that $b = c$, we first show $b \leq c$ and then we show $c \leq b$. By antisymmetry we will get the result.

As $a_i \leq b$, $i = 1, 2, \ldots k$

$\Rightarrow a_1 \leq b, \; a_2 \leq b, \ldots \; a_k \leq b$

$a_1 \vee a_2 \vee \cdots \vee a_k \leq b \Rightarrow c \leq b$ — (i)

To prove $b \leq c$.

suppose $b \wedge \bar{c} \neq 0$, then from Lemma 2, there exists an atom

In a D.L	Lemma 1
if $b \wedge \bar{c} = 0 \Rightarrow b \leq c$	

a_i such that $a_i \leq b \wedge \bar{c}$

We know $b \wedge \bar{c} \leq \bar{c}$. From transitive property

$a_i \leq \bar{c}$ — (ii)

As $a_1 \vee a_2 \vee \cdots \vee a_k = c \Rightarrow a_i \leq c$ — (iii)

From (ii) & (iii), $a_i \leq c \wedge \bar{c}$ (Th 2)

$a_i \leq 0$

which is a contradiction to the defination of atom

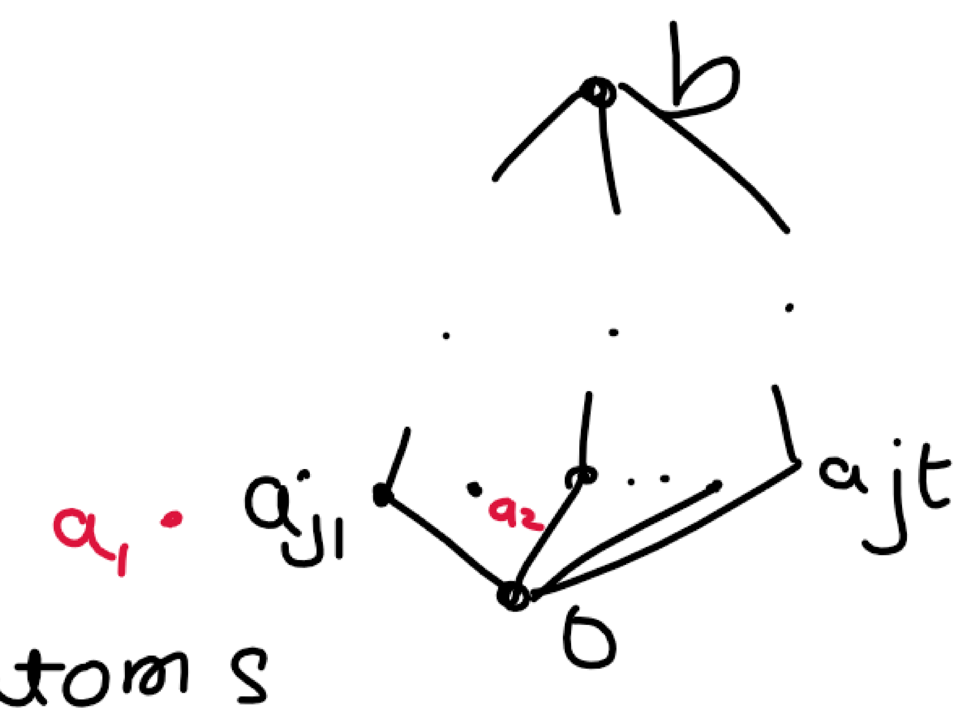
$\Rightarrow b \wedge \bar{c} = 0 \Rightarrow b \leq c$ — (iv)

From (i) & (iv) $\Rightarrow b = c$ (Antisym.)

$b = a_1 \vee a_2 \vee \cdots \vee a_k$.

Lemma 4: Let $(A, \vee, \wedge, -)$ be a finite boolean algebra. Let b be any non zero element in A, and $a_1, a_2, \ldots, a_k$ be all the atoms of A such that $a_i \leq b$. Then $b = a_1 \vee a_2 \vee \cdots \vee a_k$ is the unique way to represent b as join of atoms.

Proof: Suppose that we have an alternate representation for b, i.e.,



$$b = a_{j_1} \vee a_{j_2} \vee \cdots \vee a_{j_t}$$

Since $(a_1, a_2 \cdots a_k)$ are all the atoms of A, and $(a_{j_1}, a_{j_2} \cdots a_{j_t})$ are some atoms among $(a_1 \cdots a_k)$, an atom $a_{j_u} = a_r$.

Now, to show an atom a_i is equal to some atom among the alternate representation. i.e., to prove $a_i = a_{j_s}$

Since $a_i \leq b$

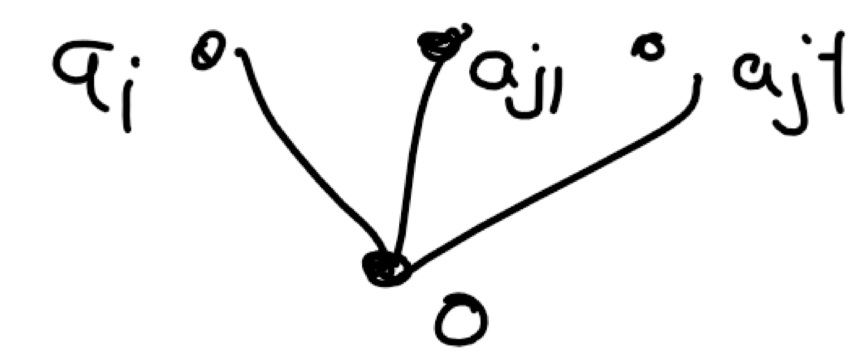
$$\Rightarrow \quad a_i \wedge b = a_i$$

$$a_i \wedge (a_{j_1} \vee a_{j_2} \vee \cdots \vee a_{j_s} \vee \cdots \vee a_{j_t}) = a_i$$

$a_i \wedge (a_{j_1} \vee a_{j_2} \vee \cdots \vee a_{j_s} \vee \cdots \vee a_{j_t}) = a_i$

$(a_i \wedge a_{j_1}) \vee (a_i \wedge a_{j_2}) \vee \cdots \vee (a_i \wedge a_{j_t}) = a_i$

$\Rightarrow$ Then for some a_{j_s}, $1 \leq s \leq t$



$a_{j_s} \wedge a_i \neq 0$

As both are atoms we must have

$a_i = a_{j_s}$

Thus each atom in the original representation is equal to one atom in the alternate representation.

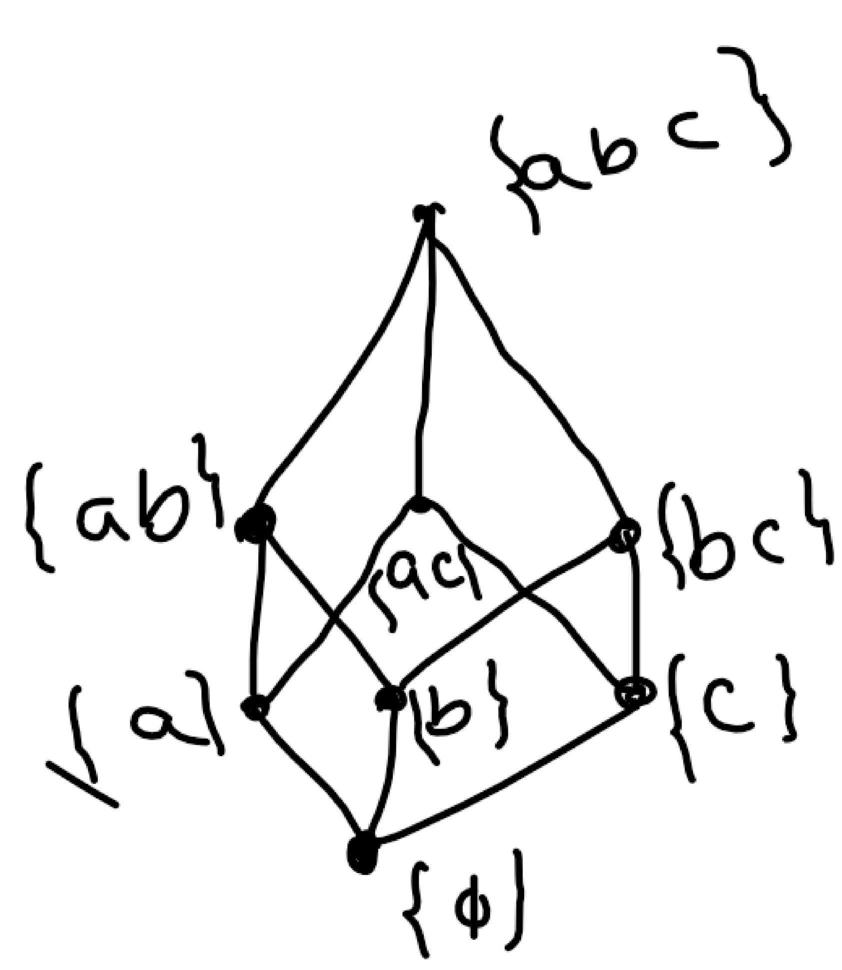
Theorem: Let $(A, \vee, \wedge, -)$ be a finite boolean algebra. Let S be the set of all atoms. Then $(A, \vee, \wedge, -)$ is isomorphic to the algebraic system defined by the lattice $(P(S), \subseteq)$.

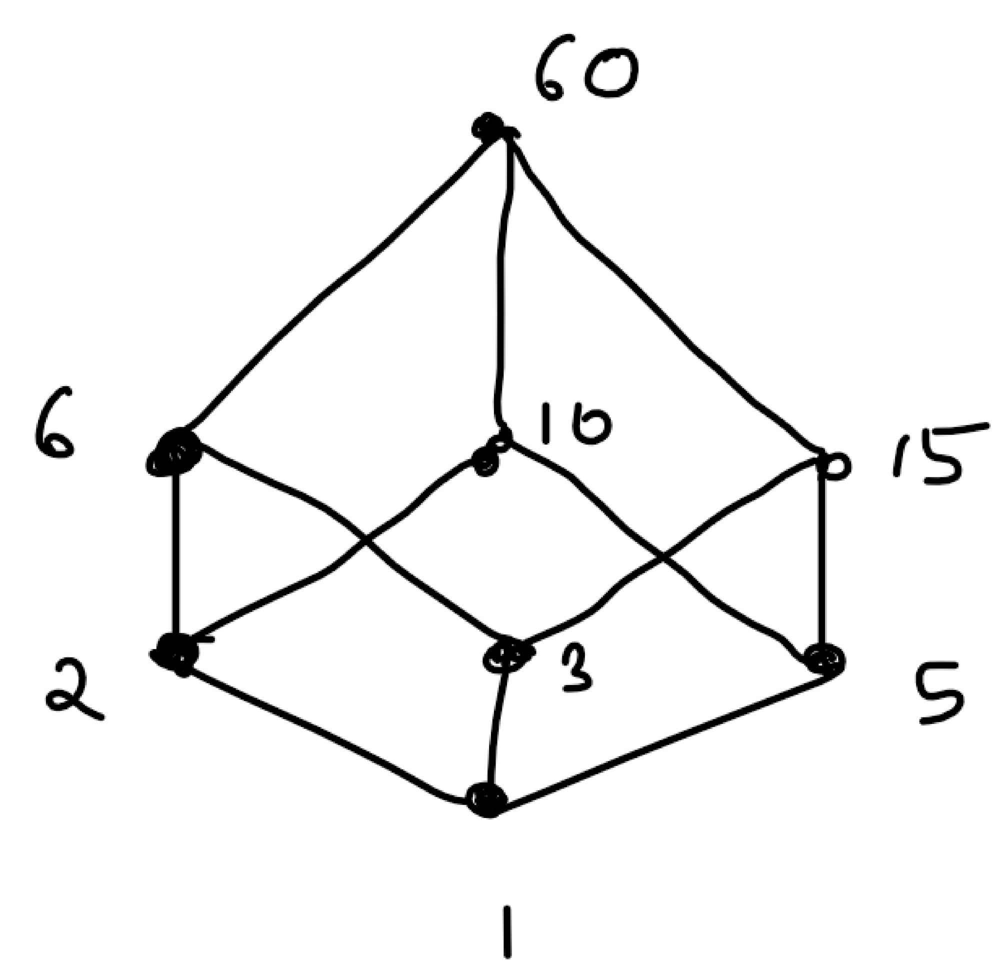
There is a one to one correspondance between the elements of a Boolean Lattice and subsets of the atoms.

Note: There exists a unique finite boolean algebra of $\underline{2^n}$ elements for any $n>0$.

Further, there are no other finite boolean algebras.

$\underline{2^n \text{ elements}}$





$\{1, 2, 3, 5, 6, 10, 15, 60\}$

No. of atoms $= \underline{\underline{3}}$

No. of elements in this Boolean Lattice is 2^3.

Q1. Let P be the set of all positive factors of 60, and let / denote the 'divides' relation. Then the poset (P, /) a Boolean lattice? Justify.

Soln: Positive factors of 60 are

$\{1, 2, 3, 4, 5, 10, 12, 15, 20, 30, 60\}$

Atoms: 2, 3, 5

Number of atoms = 3

No. of elements in the lattice = 11

Not a boolean lattice.

Boolean Expressions and Boolean functions

Let $(A, \vee, \wedge, -)$ be a boolean algebra. A Boolean expression over $(A, \vee, \wedge, -)$ is defined as follows:

1) Any element of A is a boolean expression
2) Any variable name is a boolean expression.
3) If e_1 and e_2 are boolean expressions, then $\bar{e}_1$, $e_1 \vee e_2$, $e_1 \wedge e_2$ are also boolean expressions.

Example: $0 \vee x$, $[(\overline{2 \wedge 3}) \vee (x_1 \wedge \bar{x}_2)] \wedge [x_1 \wedge \bar{x}_3]$ are boolean expressions over the boolean algebra $(\{0,1,2,3\} \vee, \wedge, -)$.

Let $E(x_1, x_2 \cdots x_n)$ be a boolean expression of n variables over a boolean algebra $(A, \vee, \wedge, -)$.

For an assignment of values to the variables we can evaluate the expression $E(x_1, x_2 \cdots x_n)$.

Example: For the boolean expression,

$E(x_1\ x_2\ x_3) = (x_1 \vee x_2) \wedge (\bar{x}_1 \vee \bar{x}_2) \wedge (\overline{x_2 \vee x_3})$ over the boolean algebra $(\{0\ 1\}, \vee, \wedge, -)$, the assignment of values $x_1 = 0$, $x_2 = 1$, $x_3 = 0$ gives

$E(0\ 1\ 0) = (0 \vee 1) \wedge (\bar{0} \vee \bar{1}) \wedge (\overline{1 \vee 0}) =$
$1 \wedge 1 \wedge 0 = 0$

Two Boolean expressions of n variables are said to be equivalent if they assume same value for every assignment of values to the n variables.

Eg: $x_1 \wedge (x_2 \vee \bar{x}_3) \approx (x_1 \wedge x_2) \vee (x_1 \wedge \bar{x}_3)$

Let $(A, \vee, \wedge, -)$ be a boolean algebra. Let f be a mapping from A^n to A.

A function $f: A^n \to A$ is called a Boolean function if it can be specified by a boolean expression of n variables.

Example: $(0, 1)^3 \to (0\ 1)$ is a boolean function.

Any 2-valued Boolean algebra is a Boolean function. $(0\ 1)^n \to (0\ 1)$ is a boolean function.

$f: \{0, a, b, 1\}^2 \to \{0, a, b, 1\}$ is not a boolean function.

Example : Let $f: A^n \to A$, where $A = \{0,1\}$.

The boolean expression $E(x_1, x_2) = (x_1 \wedge \bar{x}_1) \vee x_2$ over the boolean algebra $(\{0,1\}, \wedge, \vee, -)$ defines a boolean function. f.

x_1	x_2	$\bar{x}_1$	$x_1 \wedge \bar{x}_1$	$(x_1 \wedge \bar{x}_1) \vee x_2$	f
0	0	1	0	0	0
0	1	1	0	1	1
1	0	0	0	0	0
1	1	0	0	1	1

② Let $f: A^n \to A$ where $A = \{0,1\}$. The boolean expression $E(x_1, x_2, x_3) = (\bar{x}_1 \wedge x_2 \wedge \bar{x}_3) \vee (x_1 \wedge \bar{x}_2) \vee (x_1 \wedge x_3)$ over the boolean algebra $(\{0,1\}, \vee, \wedge, -)$ is a boolean function.

x_1	x_2	x_3	$x_1 \wedge \bar{x}_2$	$x_1 \wedge x_3$	$\bar{x}_1 \wedge x_2 \wedge \bar{x}_3$	f
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	1	1
1	0	0	0	0	0	0
0	1	1	1	0	0	1
1	0	1	1	1	0	1
1	1	0	0	0	0	0
1	1	1	0	1	0	1

A boolean expression of n variables $x_1, x_2, \cdots x_n$ is said to be a minterm, if it is of the form $\tilde{x}_1 \wedge \tilde{x}_2 \wedge \cdots \wedge \tilde{x}_n$, where $\tilde{x}_i$ is either x_i or $\overline{x_i}$.

A boolean expression over $(\{0,1\}, \vee, \wedge, -)$ is said to be in Disjunctive normal forms (DNF) if it is join of minterms.

Example: $(x_1 \wedge \overline{x_2} \wedge x_3) \vee (\overline{x_1} \wedge \overline{x_2} \wedge x_3) \vee (x_1 \wedge x_2 \wedge \overline{x_3})$ is a boolean expression in DNF.

A boolean expression of n variables $x_1, x_2, \cdots, x_n$ is said to be maxterm if it is of the form $\tilde{x}_1 \vee \tilde{x}_2 \vee \cdots \vee \tilde{x}_n$ where $\tilde{x}_i$ is either x_i or $\overline{x_i}$.

A boolean expression over $(\{0,1\}, \vee, \wedge, -)$ is said to be in Conjunctive Normal forms (CNF) if it is meet of maxterms.

Example: $(x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_2 \vee \overline{x_3})$ is in CNF.