Partitions and compositions of integers

Any positive integer n can be represented as a sum of one or more positive integers (a_i)

i.e., $n = a_1 + a_2 + \cdots + a_n$.

This division of an integer into parts are of two types depending on whether or not the ordering of a_i's are important or not. Unordered divisions are called partitions while ordered divisions are called compositions.
(This is same as permutation & combinations).
Another distinction that can be made in divisions such as partitions is whether the numbers of parts is stated or not.

Ex: Partitions and composition of $n=5$.

Seven unrestricted partitions:

$5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1$

and 15 unrestricted compositions are

$5,\ 4+1,\ 1+4,\ 3+2,\ 2+3,\ 3+1+1,\ 1+3+1,\ 1+1+3.$
$2+2+1,\ 1+2+2,\ 2+1+2;\ 2+1+1+1,\ 1+2+1+1,\ 1+1+2+1,$
$1+1+1+2,\ 1+1+1+1+1$

Notation: The partition or composition $2+1+1+1$ can be written as, 2111 or 21^3. (as short notation)

Enumerator for composition

Enumerating the unrestricted compositions of n:
~~if~~ we consider n ones in a row. Since there is no restriction, on the number of parts, we may or may not put a marker in any of the $(n-1)$ spaces between the ones inorder to form groups; this may be done in 2^{n-1} ways.

$\therefore$ No. of compositions of 'n' with no restriction on no. of parts $= 2^{n-1}$.

Similarly, if we restrict the composition to have exactly m parts, then $(m-1)$ markers are needed to form m groups ~~from~~ and the no. of ways of placing $m-1$ ~~m~~ markers in the $n-1$ spaces between ones is, $^{n-1}C_{m-1}$ Thus, no. of compositions of 'n' with m parts is $^{n-1}C_{m-1}$.

Enumerator for composition - Generating function

Let $C_m(x)$ denote the enumerator for composition of n with exactly m parts, where

$$C_m(x) = \sum_{n=} C_{mn} x^n$$

and C_{mn}, the coef. of x^n is the number of composition of n into exactly 'm' parts.

Each part of any composition can be one, two three or any greater number so that the factor in the enumerator must contain each of these powers of x, and so is

$$x + x^2 + x^3 + \dots + x^k + \dots = x(1-x)^{-1}$$

Since there are exactly m parts, the generating functions is the product of m such factors:

$$C_m(x) = (x + x^2 + \dots + x^k + \dots)^m$$

Which can be rewritten,

$$C_m(x) = x^m (1-x)^{-m} = x^m \sum_{i=0}^{\infty} \binom{m+i-1}{i} x^i$$

Replacing $m+i$ by r in the summation

$$C_m(x) = \sum_{r=m}^{\infty} \binom{r-1}{r-m} x^r = \sum_{r=m}^{\infty} \binom{r-1}{m-1} x^r$$

So that the coef. of x^n in this enumerator is $\binom{n-1}{m-1}$, as before.

The enumerating generating function for compositions with no restriction on the number of parts $C(x)$, can be obtained from $C_m(x)$ by summing

$$C(x) = \sum_{m=1}^{\infty} C_m(x) = \sum_{m=1}^{\infty} x^m (1-x)^{-m}$$

Substituting $t = x/(1-x)$ in the series we get,

$$C(x) = (t + t^2 + t^3 + \dots)$$

$$C(x) = \frac{t}{1-t} = \frac{x}{1-2x} = \sum_{n=1}^{\infty} 2^{n-1} x^n.$$

Since the coef. of x^n in the enumerator is 2^{n-1} this yields again the number of unrestricted composition of n.

Generating function for partitions

We have a simple relationship between r-combinations and r-permutations. But no such simple relation exists between the number of partitions and no. of compositions because each partition will in general give rise to a dif. no. of compositions. For example the two partitions of 10, 811 and 4321, give respectively three and 24 compositions. Thus, it is impossible to get any conclusion from results obtained for compositions.

(*) Let p_n be the number of partitions of n so that the generating function is,

$$p(x) = p_0 + p_1 x + p_2 x^2 + \dots + p_n x^n + \dots$$

Consider the polynomial,

$$1+x+x^2+\dots +x^k+\dots x^n$$

the appearance of x^k can be interpreted as the existance of just k ones in a partition of the integer n.

Similarly, polynomial $1+x^2+\dots+x^{2k}+\dots$

i.e. considered with the twos in the partition, and in particular the coefficient $x^{2k}=(x^2)^k$ represents the case of just k twos in the partition. In general the polynomial $1+x^r+x^{2r}+\dots+x^{kr}+\dots$

can represent the r's in the partition.

The generating function will need one factor for the ones, one for twos, and so on. Collecting together these polynomials, the generating functions ~~of n~~ for the partitions of n is obtained as,

$$p(x)=(1+x+x^2+x^3+\dots+x^k+\dots)(1+x^2+x^4+\dots+x^{2k}+\dots)$$

$$*\dots(1+x^r+x^{2r}+\dots x^{kr}+\dots)\dots$$

$$=(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\dots(1-x^r)^{-1}\dots \longrightarrow (1)$$

Thus, the no. of unrestricted partition of n is the coef. of the term x^n in eqn (1).