A graph G in which every vertex is of same degree is called a regular graph.

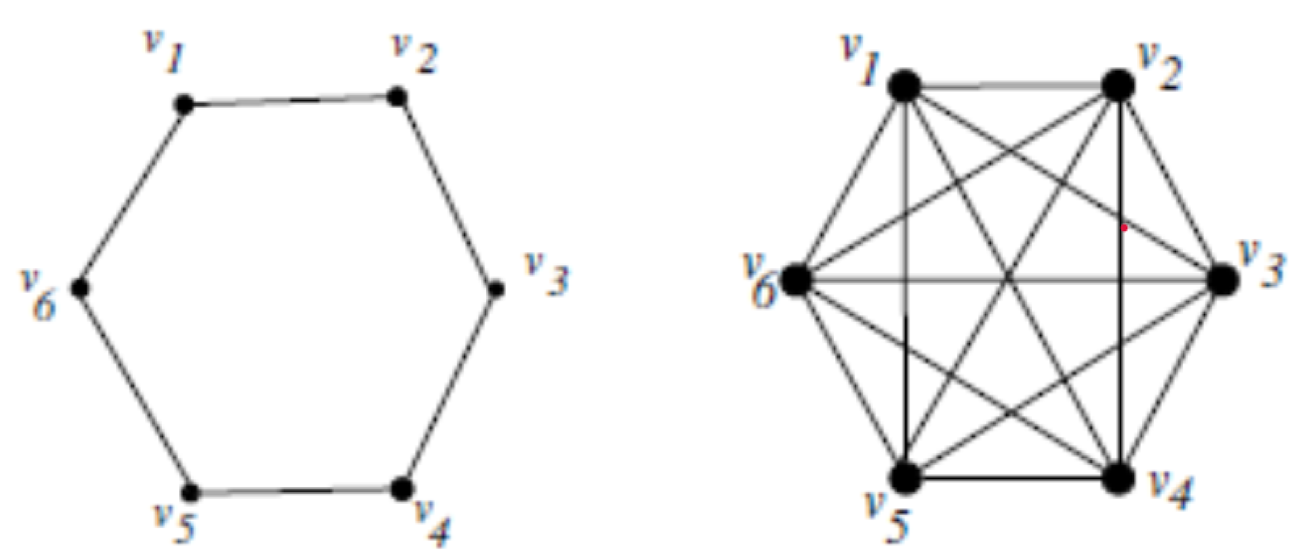
A regular graph with degree 3 is called a cubic graph.
A cubic graph has always even number of vertices.

A graph on n vertices, in which every two vertices are adjacent, is called a complete graph and is denoted by Kn.

A connected regular graph with regularity two is called a cycle.
A cycle on n vertices is denoted by Cn.

radius = diam = 1
In K_n
$e(v_1) = 1$

In C_6,
$e(v_1) = 3$
Radius = 3 = Diameter



Cycle graph and Complete graph on six vertices.

A graph G is said to be self centered if every vertex of G has the same eccentricity. In such a graph, radius is equal to the diameter.
The cycle graph C_n is a self-centered graph and is the complete graph K_n.

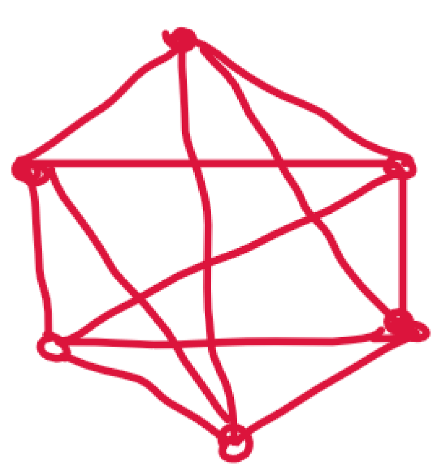
Question 1: Draw a regular graph on regularity 4 and number of vertices 6.
Question 2: Draw a complete graph on 6 vertices.
Question 3: Draw a cycle graph on 8 vertices.
Question 4: Draw the complement of cycle graph C_8.
Question 5: The complete graph Kp has pC_2 edges.

Question 6: The cycle graph Cn has n edges.

Question 7: The complete graph Kp has diameter= 1

Question 8: Draw a regular graph on 6 vertices with regularity 1.

1



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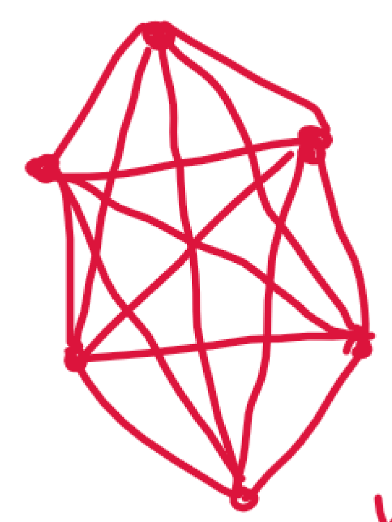
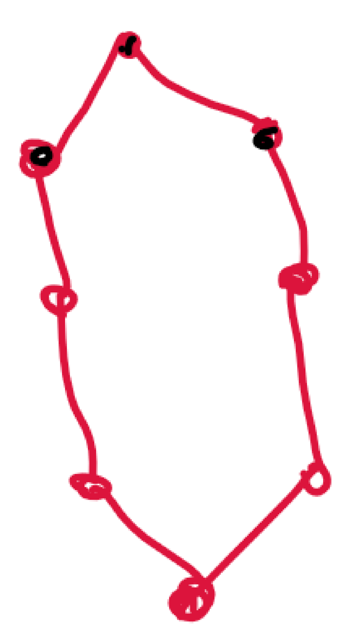


K_6

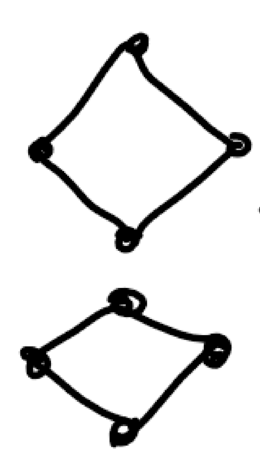
3



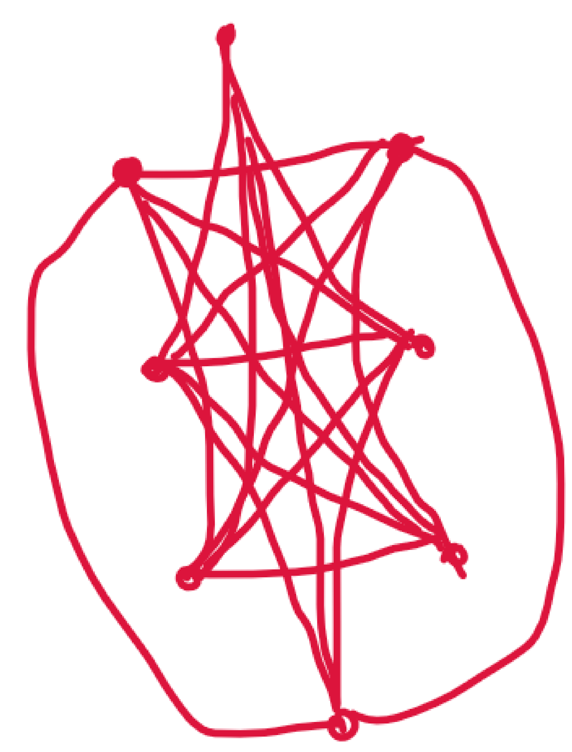
C_8

✓

4

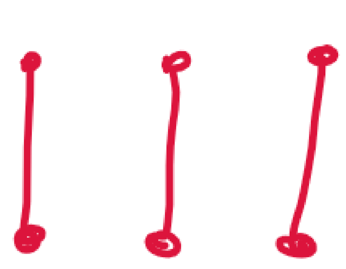


X C_8



$\overline{C_8}$

8



Theorem 3: If diam $(G) \geq 3$, then diam $(\overline{G}) \leq 3$.

<u>Proof:</u> Let G be a graph with diameter ≥ 3.

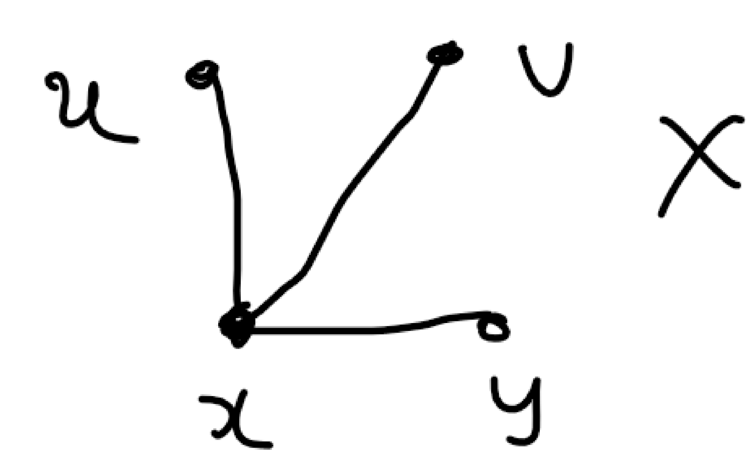
$\Rightarrow$ There are 2 vertices u and v in G, such that $d(u, v) \geq 3$.

$\Rightarrow$ u and v are not adj in G.

(bw u and v are adj in $\overline{G}$)

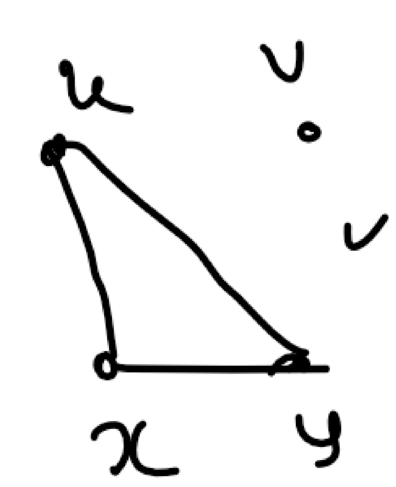
Consider any 2 vertices x and y which are adj to each other in G. (x and y are not adj in $\overline{G}$). To prove, $d(x, y)$ in $\overline{G}$ is ≤ 3.

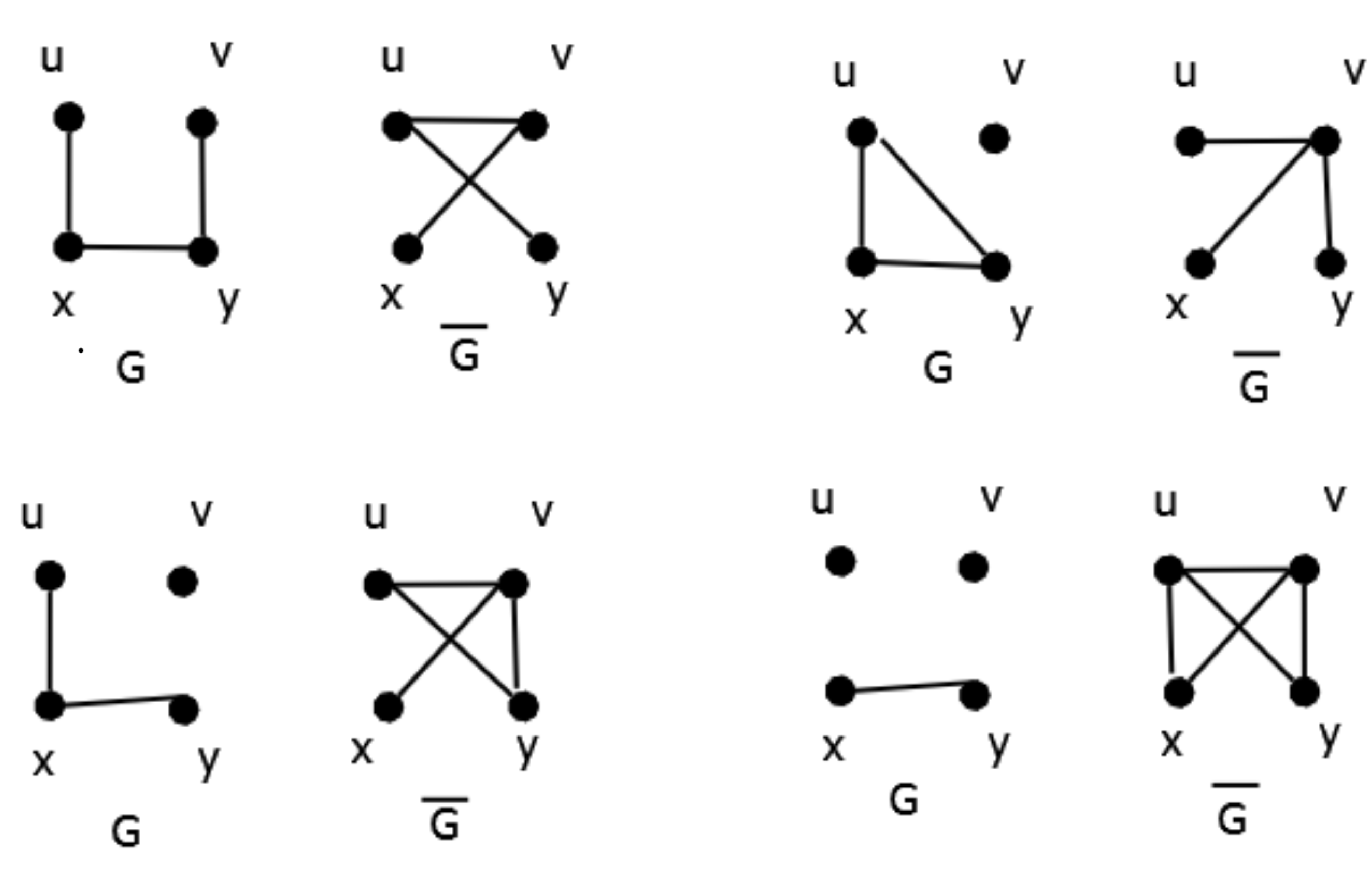
We note x cannot be adj to both u and v. ($\because d(u,v) \geq 3$).
G



x is adj to atmost one of u or v.

1) x is adj to u, y is adj to v
2) x is adj to u, y is not adj to v
3) x is adj to u, y is adj to u
4) x and y not adj to u or v.



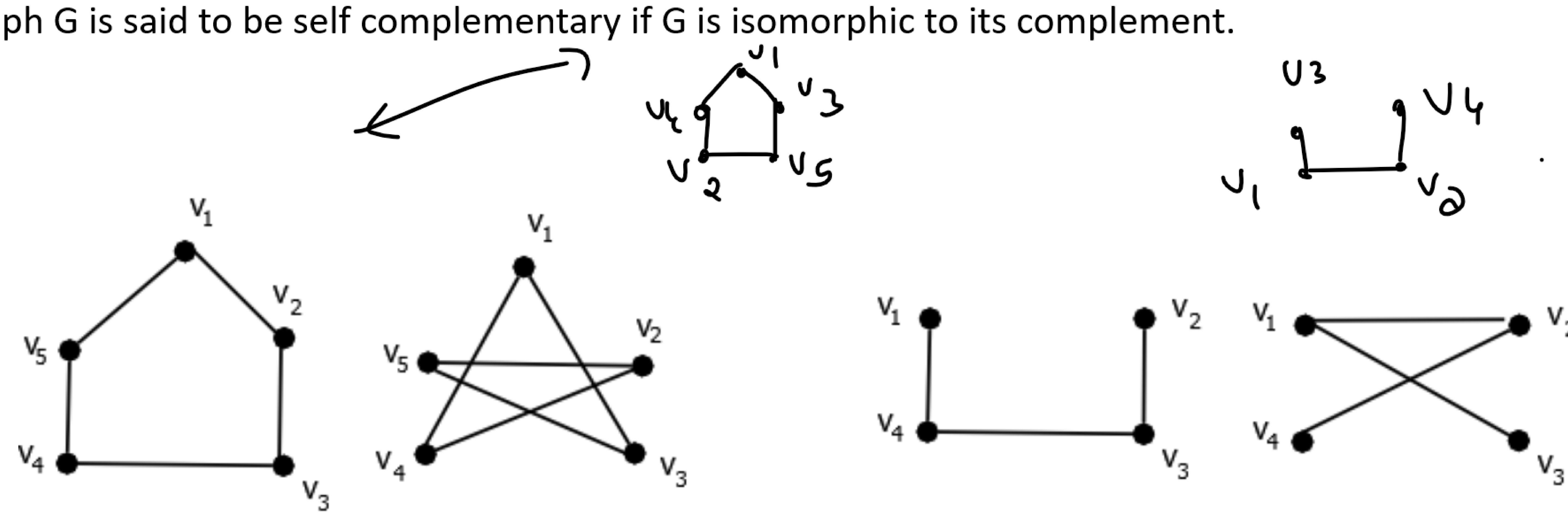


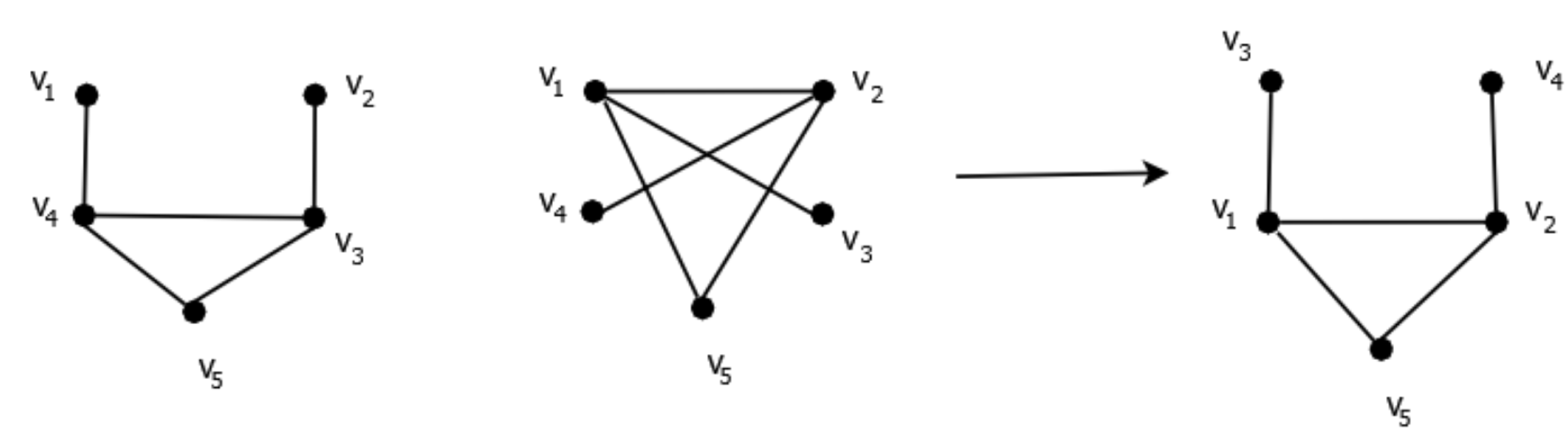
Since u and v have common neighbor in G,
both x and y are adj to u or v in $\bar{G}$.

$\Rightarrow d(x, y) \leq 3$ in $\bar{G}$.

$\Rightarrow \operatorname{diam}(\bar{G}) \leq 3$.

A graph G is said to be self complementary if G is isomorphic to its complement.





Theorem 4: Let G be a self complementary graph. Show that the number of vertices in G is of the form 4n or 4n + 1.

Proof : Let G be (p, q) graph. Number of edges in $K_p = {}^pC_2 = \frac{p(p-1)}{2}$.

Since G is self complementary, no. of edges in G = no. of edges in $\bar{G} = q$

No. of edges in K_p = no. of edges in G + no. of edges in $\bar{G}$.

$$\frac{p(p-1)}{2} = q+q \Rightarrow q = \frac{p(p-1)}{4} \Rightarrow 4 \mid p \text{ or } 4 \mid p-1$$

$\Rightarrow p = 4n$ or $4n+1$.

Theorem 5: Every nontrivial self complementary graph has diameter 2 or 3.

Proof : Let G be a self complementary graph

Then G cannot have diameter 1.

Since if G is of diam = 1, then $G \cong K_n$ which is not a self complementary graph.

$\Rightarrow$ self complementary graphs have diameter atleast 2.

Suppose diam (G) ≥ 3, Then we know diam $(\bar{G}) \leq 3$.

$\Rightarrow$ Diameter of every self complementary graph is either 2 or 3.

Extra questions

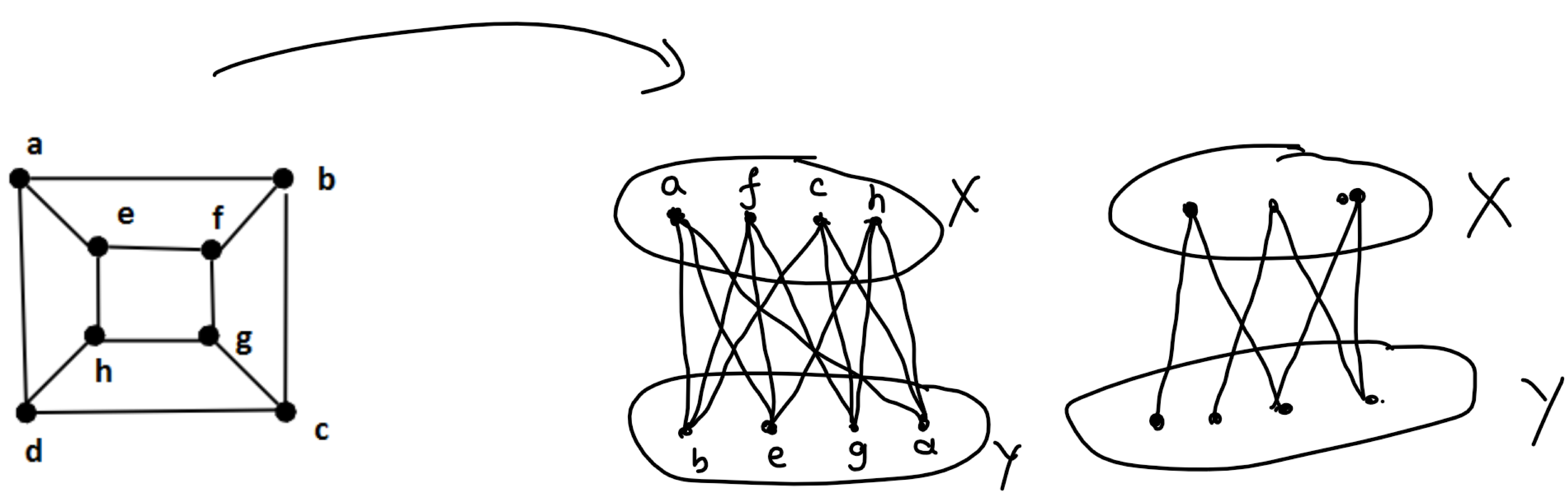
1. There exists a self complementary graph on —— vertices.
(i) 3 (ii) 8 (iii) 11 (iv) 14

2. Let G be a simple graph with 6 vertices. The degrees of 5 vertices are (2,3,3,3,5). Then the degree of 6th vertex is ——-.
(i) 0 (ii) 1 (iii) 2 (iv) 4

3. Let G be a simple graph with 6 vertices. The degrees of 5 vertices are (2,3,3,3,5). Then the number of edges is equal to ——-.
(i) 7 (ii) 8 (iii) 9 (iv) 10.

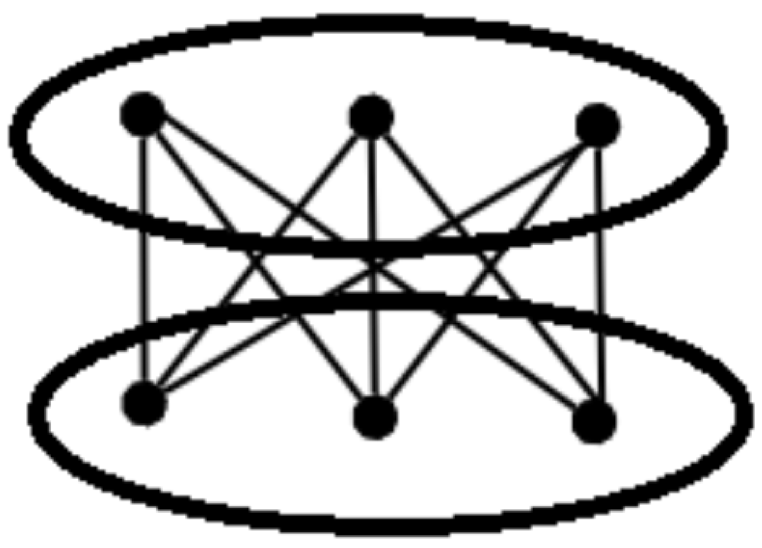
Bipartite Graph:

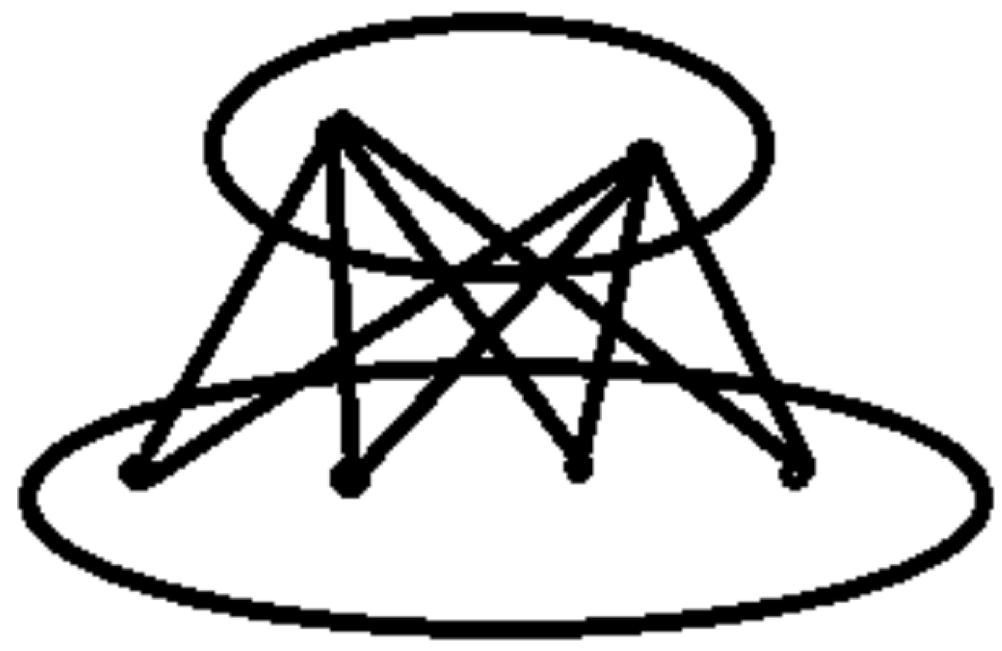
A bipartite graph is one whose vertex set can be partitioned into 2 subsets X and Y so that each edge has one end vertex in X and one end vertex in Y . Such a partition (X, Y) is called a bipartition of the graph G.

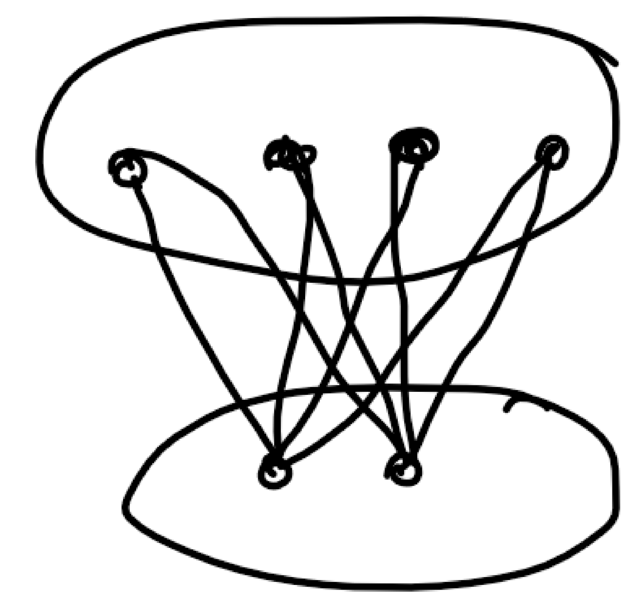


Bipartite graph on 8 vertices with partition X = {a, f , c, h} and Y = {b, e, g, d}.

A complete bipartite graph is a bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y ; if |X| = m and |Y | = n, such a graph is denoted by Km,n.

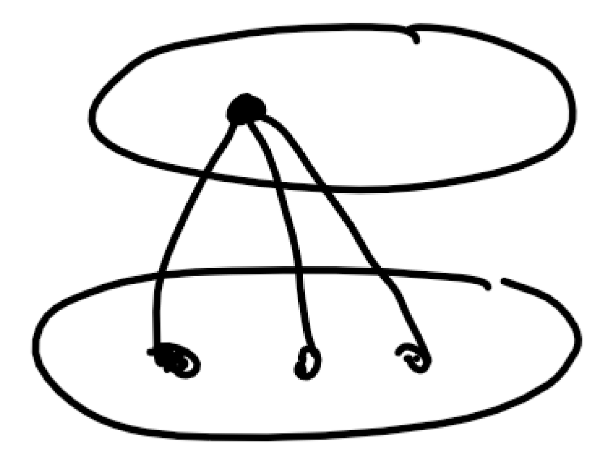






Complete bipartite graph $K_{3,3}$ $K_{2,4}$ K_{42}

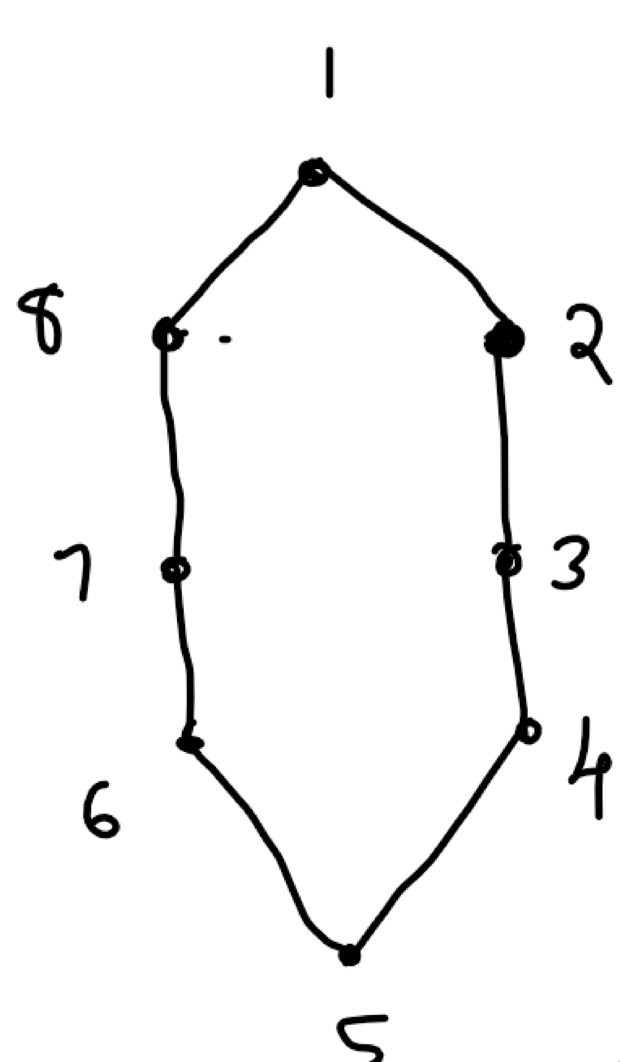
A complete bipartite graph Km,n has mn edges.



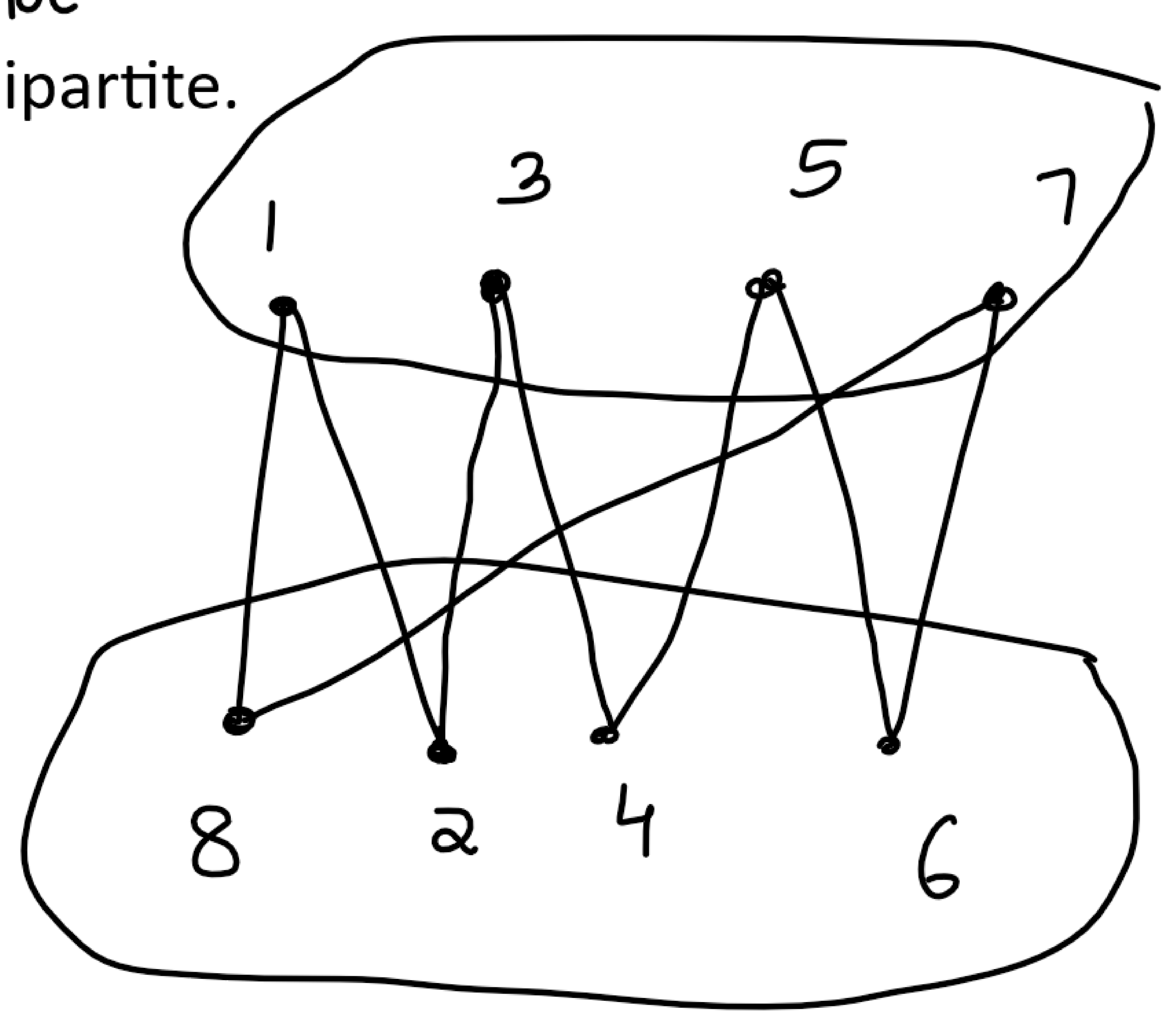
K_{13}

Question: Check whether C_8 and C_7 are bipartite graphs?

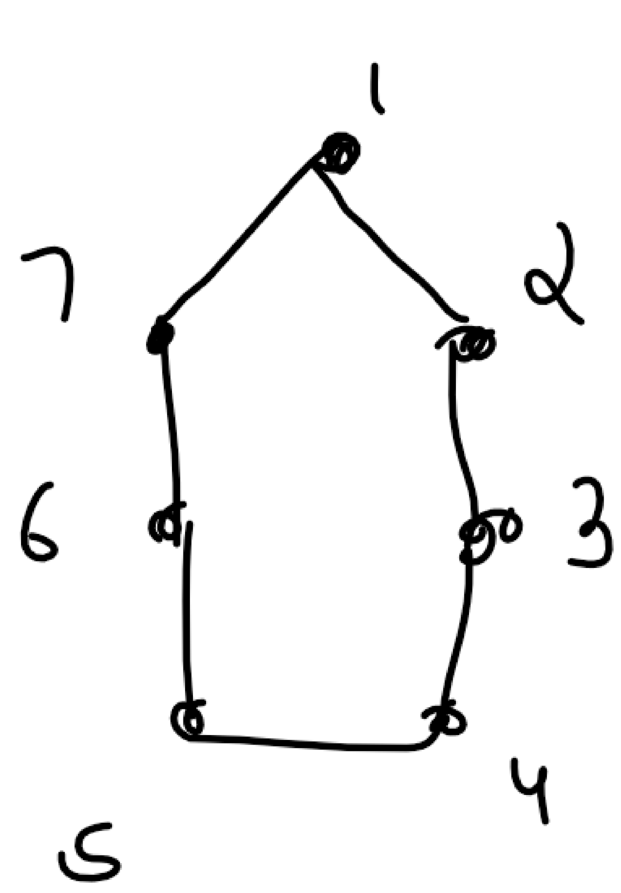
Soln: The graph C_8 is bipartite but C_7 is not bipartite.

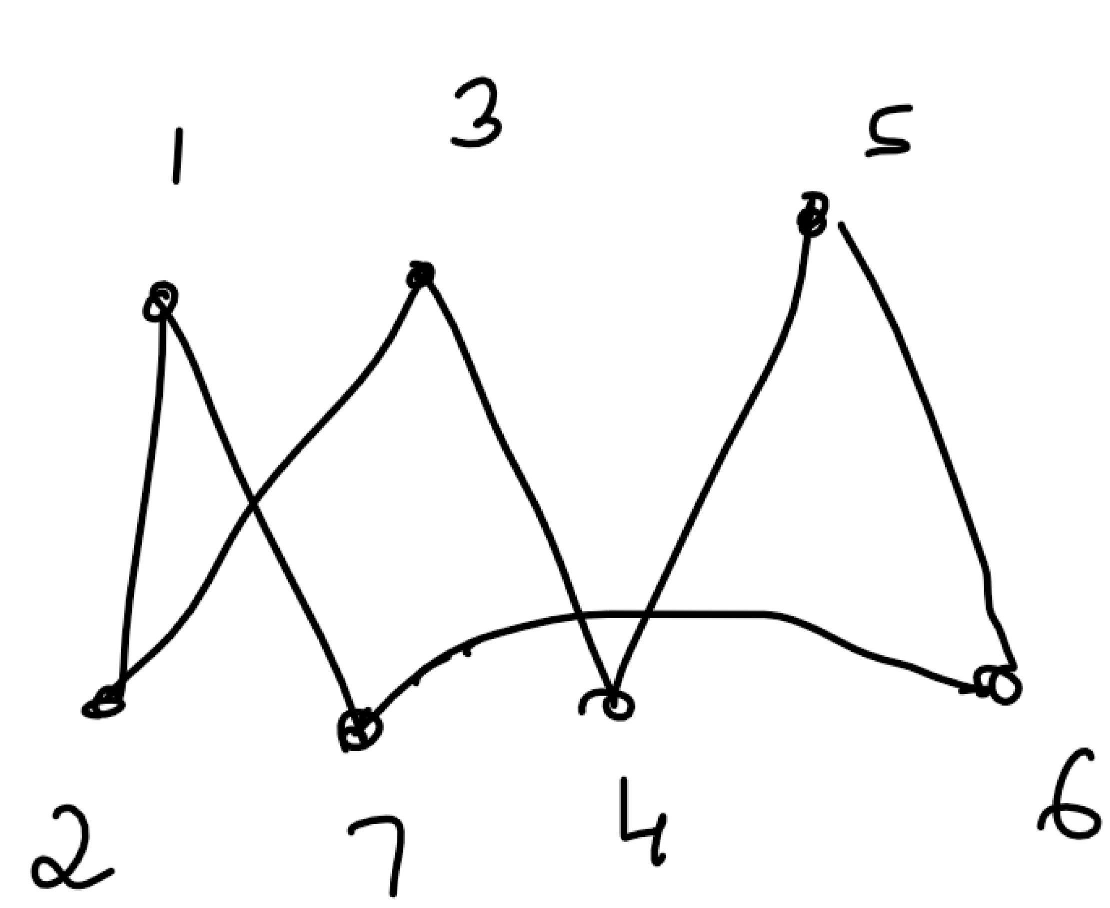


C_8:



C_8





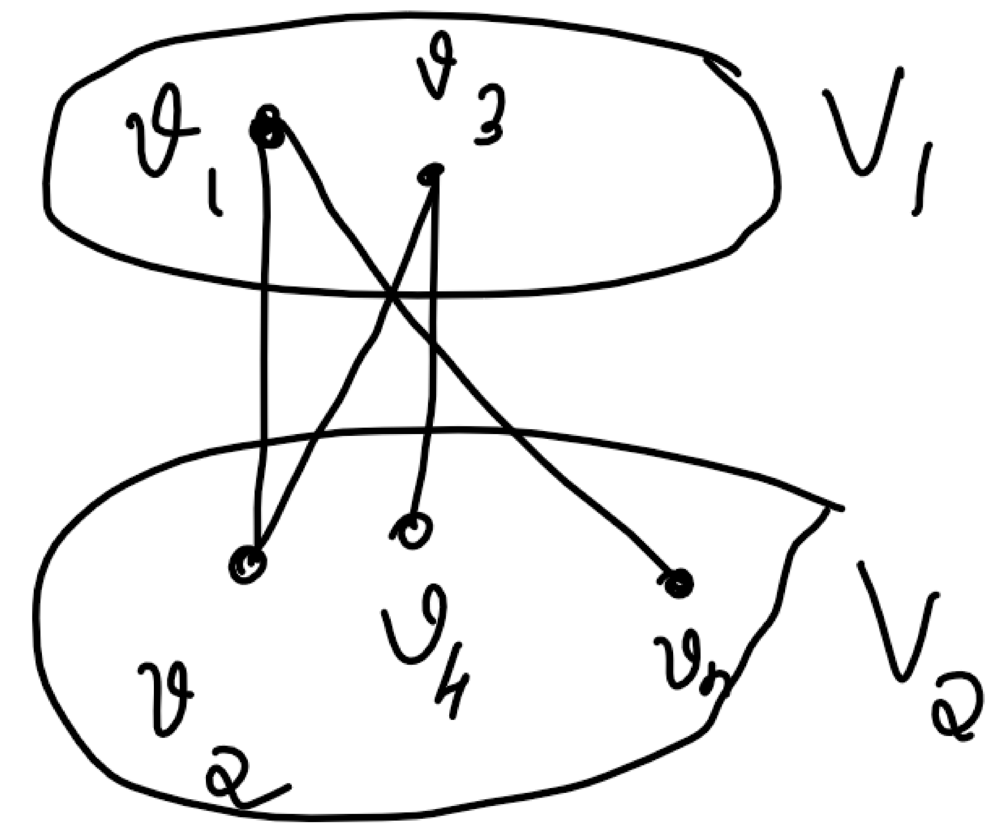
Not

Theorem 6: A graph is bipartite if and only if all its cycles are even.

Proof: Let G be a connected bipartite graph. Then its vertex set can be partitioned into 2 sets V_1 and V_2 s.t each edge of G joins a vertex of V_1 with a vertex of V_2. Thus every cycle $v_1 v_2 \cdots v_n v_1$ in G necessarily has its oddly subscripted vertices in V_1

i.e, $v_1 v_3 \cdots \in V_1$ and

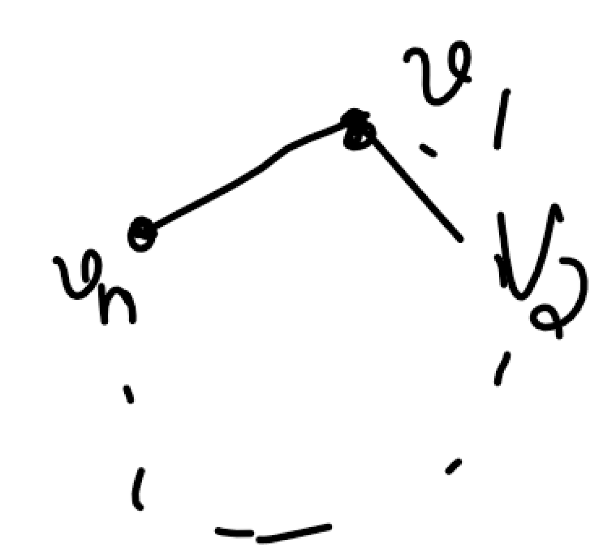
$v_2 v_4 \cdots \in V_2$.



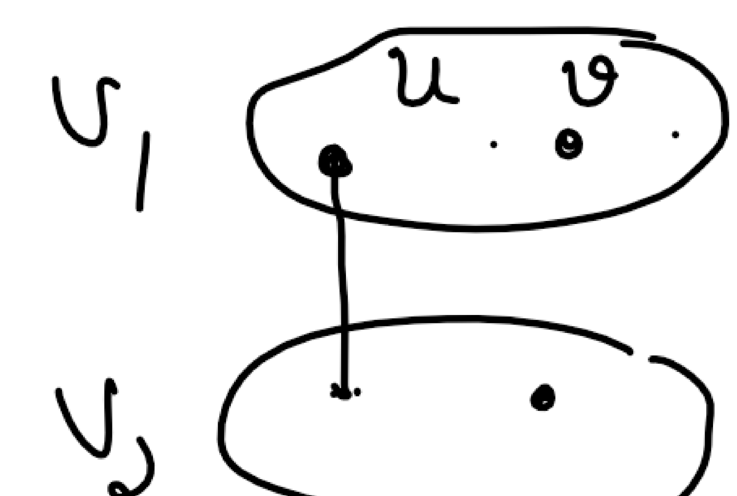
Suppose in a cycle $v_1 v_2 \cdots v_n v_1$, as ~~if~~ $v_1 v_n$ is an edge $\Rightarrow$

$v_n \in V_2 \Rightarrow n$ is even.

$\Rightarrow$ length of the cycle is even.



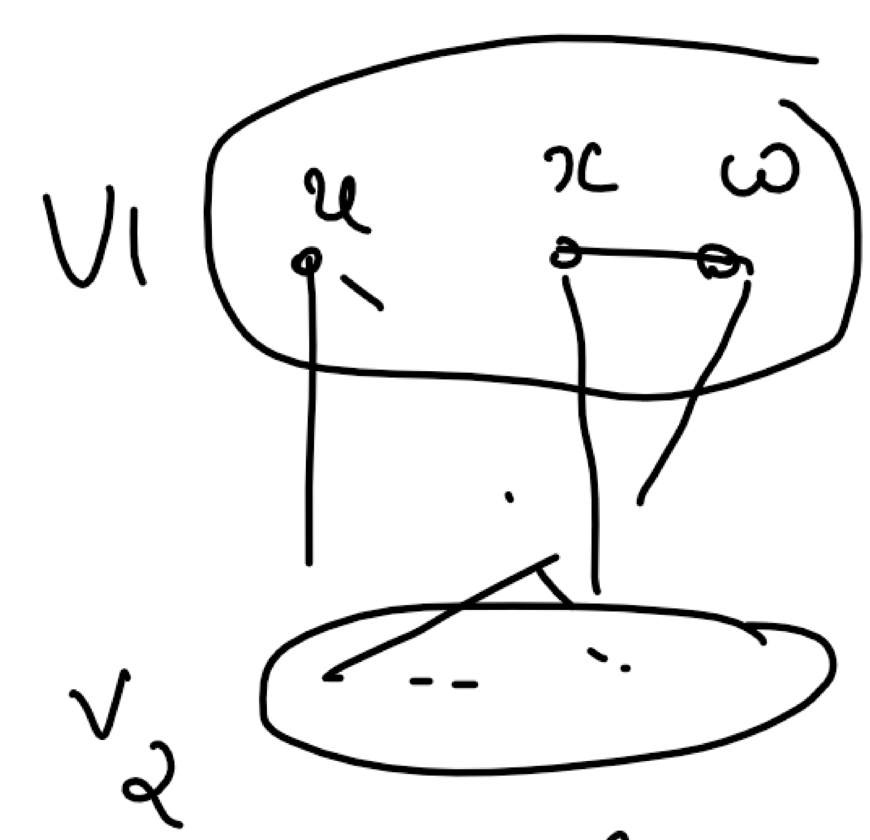
Conversely, suppose G is a connected graph with no odd cycles. Let $u \in V(G)$.



Let $V_1 = \{v \in V(G) \mid d(u,v) = \text{even}\}$

$V_2 = \{v \in V(G) \mid d(u,v) = \text{odd}\}$

$V_1 \cup V_2 = V(G)$, $\quad V_1 \cap V_2 = \phi$



Suppose, $x, w \in V_1$ are adjacent.

as $w \in V_1 \Rightarrow d(u,w) = 2k$

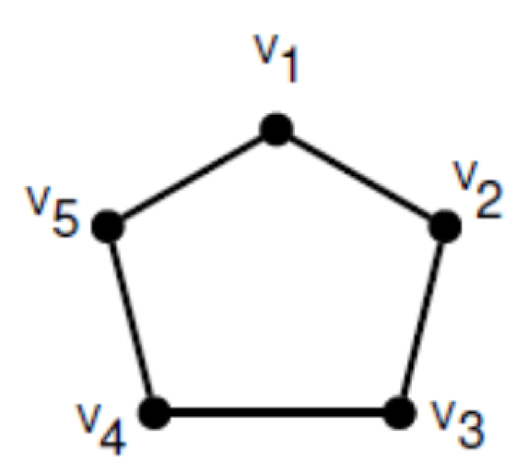
& $x \in V_1 \Rightarrow d(u,x) = 2l$

Thus the path $u - w - x - u$ forms a cycle of length $2k + 1 + 2l$ an odd number. $\Rightarrow x$ & w are not adjacent $\Rightarrow$ No 2 vertices in V_1 are adj.

Similarly no two vertices in V_2 are adj.

$\Longrightarrow G$ is bipartite.

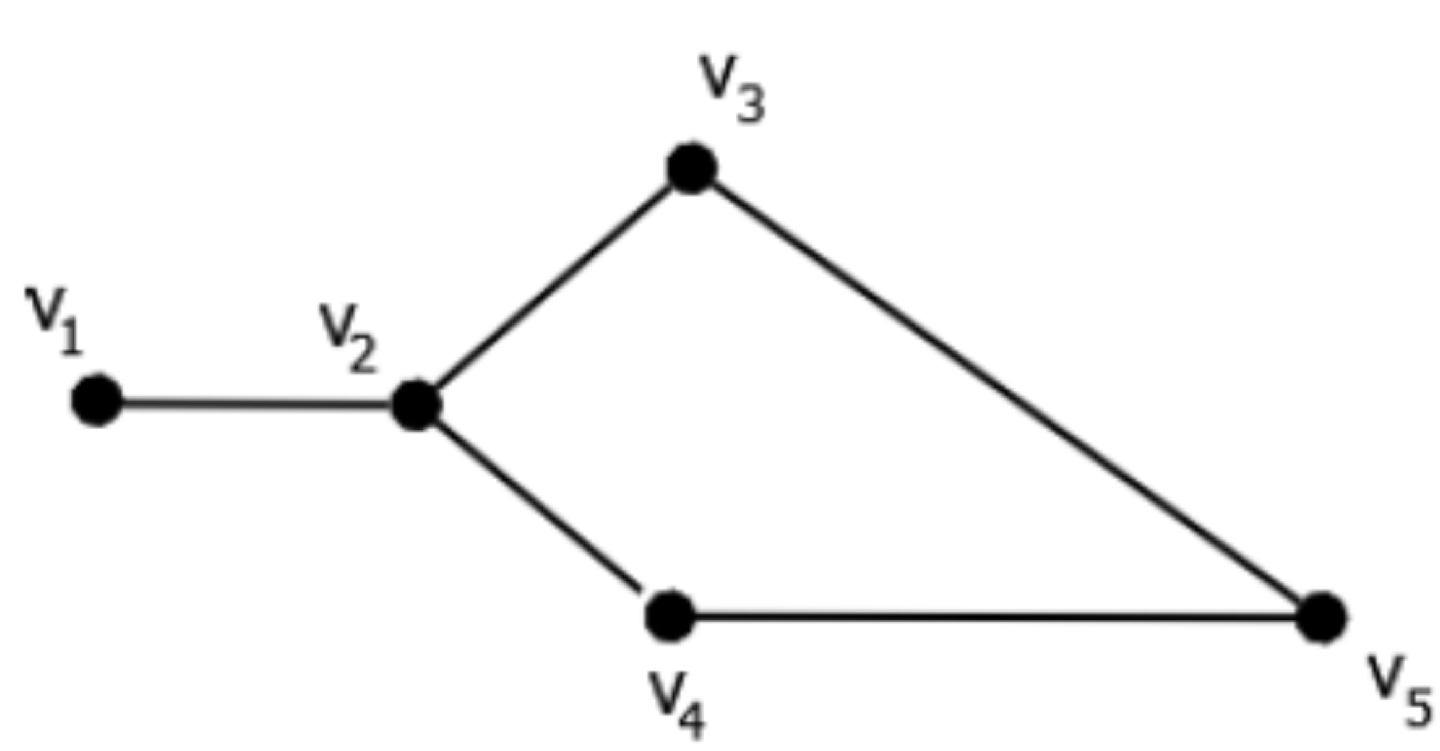
The adjacency matrix of G, denoted by A(G) with $V(G) = \{v_1\ v_2 \cdots v_n\}$. is The $n \times n$ matrix defined as follows. The rows & colns are indexed by $V(G)$. If $i \neq j$, then $(i,j)^{th}$ entry is 0 if v_i & v_j are not adjacent. and $(i,j)^{th}$ entry 1 if v_i and v_j are adjacent. The $(i,i)^{th}$ entry is 0 for $i = 1, 2, \cdots n$.



$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Graph G and its adjacency matrix $A(G)$

Question 17: Write the adjacency matrix of the graph.

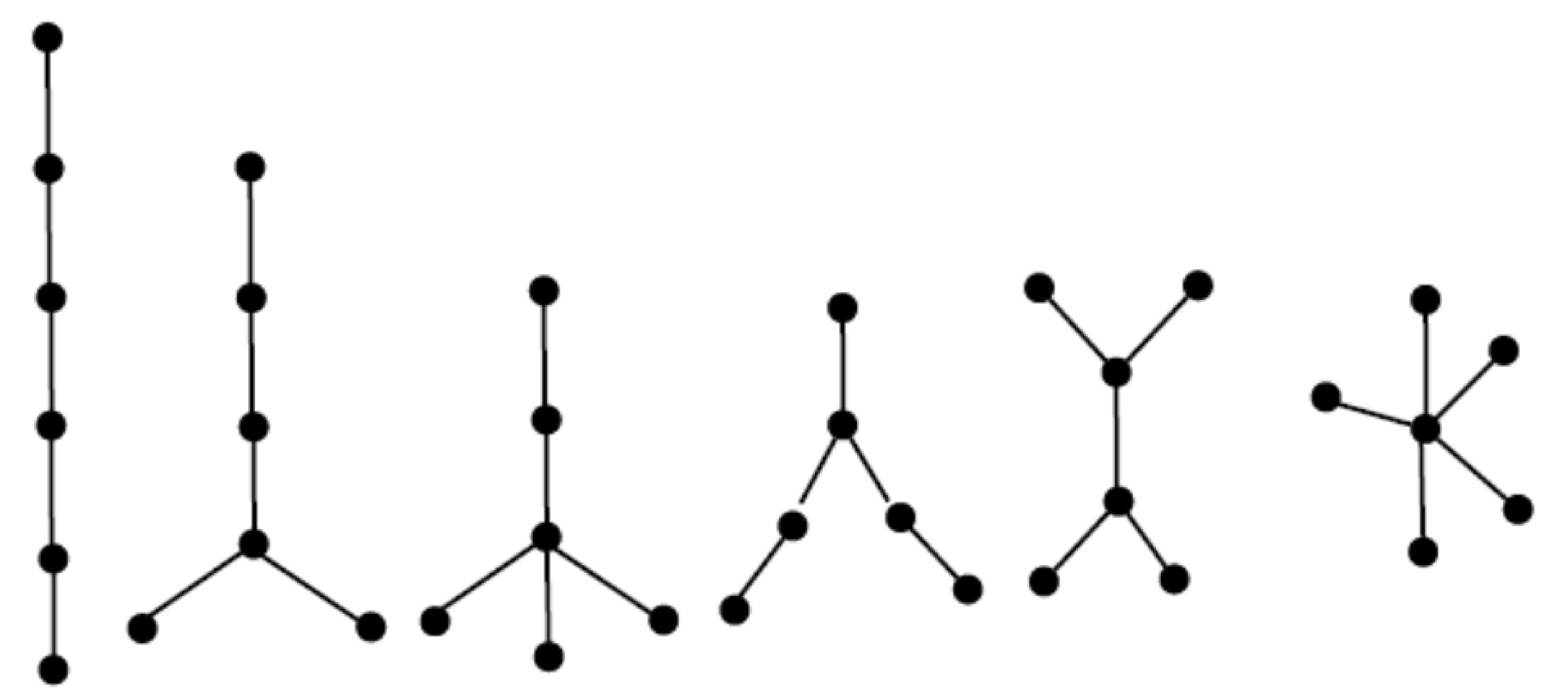


$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Trees

A graph G is acyclic if it has no cycles.
A tree is a connected acyclic graph.
In a tree, any two vertices are connected by a unique path.

All trees on 6 vertices:



Theorem : A graph G is a tree if and only if between every pair of vertices there exist a unique path.

Proof: Let G be a tree. Then G is connected
There exist at least one path between every pair of vertices. Suppose that between 2 vertices say u and v, there are 2 distinct paths then, union of these 2 paths creates a cycle. A contradiction

⇒ If G is a tree then there is a unique path joining any 2 vertices
Conversely, suppose that there is a unique path between every pair of vertices in G, then G is connected.
A cycle in G implies that there is at least one pair of vertices u and v such that there are 2 distinct paths between u and v.
which is not possible because of our hypothesis
Hence G is acyclic & therefore it is a tree.