18/10/21 # Differential eqn's & application's

⟹ Definition: An eqn involving one or more derivative of an unknown function is called a D.E.

→ D.E
- → Ordinary D.E : Single independent variable (ODE) eg: $\frac{dy}{dx} = 3x^2+y$
- → Partial D.E : Two or more independent variables. (PDE) eg: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

⟹ Ordinary Differential eqn's {ODE}:

* Order: highest derivative occurring in D.E.
* Degree: degree of highest derivative, after the eqn is free from fractional powers

eg: ① $y' = 3x^2+y$ O: 1, D: 1

② $3(y'')^5 + (y')^2 = \sec x$ O: 2, D: 5

③ $\frac{d^2y}{dx^2} = c\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}$ O: 2, D: 2.

* A 'Solution' of D.E is a relation b/w the variable which satisfies the given D.E.
* A 'General Solution' of an nth order eqn is a soln containing n arbitary independent constants.
* A soln obtained from the general soln by assigning specific values to the arbitrary constants is called a 'Particular soln'.

eg: $\frac{dy}{dx} = xy$, $y(1)=1$

Sol: $\int \frac{dy}{y} = \int x\,dx$

$\therefore \boxed{\ln y = \frac{x^2}{2} + C} \to$ Solⁿ of DE

General Solⁿ

$y(1)=1 \Rightarrow \ln 1 = \frac{1}{2} + C$

$\Rightarrow C = -\frac{1}{2}$

$\therefore \boxed{\ln y = \frac{x^2}{2} - \frac{1}{2}} \to$ Particular Solⁿ.

→ Initial Value problem: which specifies the value of unknown function at a point in the domain

eg: ① $y' = 3x^3 + y$; $y(0)=1$ at one point "0"

② $y'' + 5y' + 6y = 3x^3 + \cos x$; $y(0)=1$, $y'(0)=0$ at one pt "0"

→ Boundary Value Problem: Conditions specified at more than one point.

eg: ① $y'' = x + y$, $y(0) = y(1) = 0$ at 2 pts 0, 1

② $y'' + 5y + 6y = 3x^3 + \cos x$, $y(0)=1$, $y'(2)=0$ at 2 pts 0, 2.

⇒ Given an IVP / BVP, it is not necessary that a solution exists. Even if a solⁿ exists it may be valid only in an interval. Solⁿ need not to be unique.

eg: ① $x\frac{dy}{dx} = y$, $y(0)=1$

It has no solution because every solⁿ satisfies $y(0)=0$.

→ In a D.eq^n, if the order of eq^n is "n" then the eq^n. has "n" arbitary constants.

→ **Homogeneous D.E**: There should be no constant

eg: $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$.

NonHomogenous D.E: There should be constant.

eg: $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin x$

21/10/21

⇒ **Variable seperable method**:

* Seperate x variables on one side & y variables on other side and integrate both sides you will get sol^n of D.Eq^n.

Examples:

① $x^2dx + y^2dy = 0$

Sol: $\int y^2dy = \int -x^2dx$

$$\boxed{\frac{y^3}{3} = -\frac{x^3}{3} + C}$$

② The sol^n of D.E $\frac{1}{x^2}\left(\frac{dy}{dx}\right)^2 + 6 = \frac{5}{x}\frac{dy}{dx}$ is $y = \lambda x^2 + C$ (where, C is arbitary const). The sum of all values of λ is —

Sol: $\left(\frac{dy}{dx}\right)^2 + 6x^2 = 5x\frac{dy}{dx} \Rightarrow \left(\frac{dy}{dx}\right)^2 - 5x\left(\frac{dy}{dx}\right) + 6x^2 = 0$

$\boxed{\frac{dy}{dx} = m} \Rightarrow m^2 - 5xm + 6x^2 = 0$

$\Rightarrow m^2 - 2xm - 3xm + 6x^2 = 0$

$\Rightarrow m(m-2x) - 3x(m-2x) = 0$

$\Rightarrow (m-2x)(m-3x) = 0 \Rightarrow \boxed{\left(\frac{dy}{dx} - 2x\right)\left(\frac{dy}{dx} - 3x\right) = 0}$

$\frac{dy}{dx} - 3x = 0 \Rightarrow \int dy = \int 3x\,dx \Rightarrow \boxed{y = 3\frac{x^2}{2} + C}$

$\frac{dy}{dx} - 2x = 0 \Rightarrow \int dy = \int 2x\,dx \Rightarrow \boxed{y = x^2 + C}$

$\boxed{\lambda = \frac{3}{2}, 1}$

Sum $= \frac{5}{2}$

③ The Solⁿ of D.E $\frac{dy}{dx} = \frac{x^3+xy^2}{y^3-yx^2}$ is $y^K - x^K = 2x^2y^2 + \lambda$; K = ___

Sol: This can be solved by ~~Homogeneous~~ method also but hard. Easy by variable seperable

④ $y' = \sin^2(x-y+1)$

Sol: $x-y+1 = z$

$$1 - \frac{dy}{dx} = \frac{dz}{dx} \Rightarrow \frac{dy}{dx} = 1 - \frac{dz}{dx}$$

$$\Rightarrow 1 - \frac{dz}{dx} = \sin^2(z) \Rightarrow \frac{dz}{dx} = (1-\sin^2 z)$$

$$\Rightarrow \int dx = \int \frac{dz}{\cos^2 z}$$

$$\Rightarrow x = \tan z$$

$$\Rightarrow \boxed{\tan(x-y+1) = x + C}$$

③ Sol: $\frac{dy}{dx} = \frac{x(x^2+y^2)}{y(y^2-x^2)} \Rightarrow \frac{y\,dy}{x\,dx} = \frac{x^2+y^2}{y^2-x^2}$

$y^2 = Y$, $x^2 = X$

$2y\,dy = dY$, $2x\,dx = dX$ $\Rightarrow \frac{dY/2}{dX/2} = \frac{X+Y}{Y-X}$

$$\Rightarrow Y\,dY - X\,dY = X\,dX + Y\,dX$$

$$\Rightarrow X\,dY + Y\,dX = Y\,dY - X\,dX$$

☆☆ $$\Rightarrow d(XY) = Y\,dY - X\,dX$$

$$\Rightarrow XY = \frac{Y^2}{2} - \frac{X^2}{2} + \lambda$$

Put $X = x^2$, $Y = y^2$ $\Rightarrow \boxed{y^4 - x^4 = 2x^2y^2 + \lambda}$

$\therefore$ K = 4

⇒ Homogeneous Method:

* For function $f(x,y)$; $f(\lambda x, \lambda y) = \lambda^n f(x,y)$
  Then it is homogeneous eqn of degree "n".
* Another method: $f(x,y)$; $f(x,y) = x^n f\left(\frac{y}{x}\right)$ (or) $y^n f\left(\frac{x}{y}\right)$
  then it is homogenous eqn of degree 'n'.

eg: (i) $f(x,y) = x^2 + xy + y^2$

M-1 $f(\lambda x, \lambda y) = \lambda^2 x^2 + \lambda x \lambda y + \lambda^2 y^2 = \lambda^2 \{x^2 + xy + y^2\}$

$f(\lambda x, \lambda y) = \lambda^2 f(x,y)$ → Homogenous eqn of degree 2.

M-2 $f(x,y) = x^2 + xy + y^2 = x^2\left\{1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2\right\}$

$= x^2 f\left(\frac{y}{x}\right)$ → H eqn of degree 2

$f(x,y) = y^2\left\{\left(\frac{x}{y}\right)^2 + \frac{x}{y} + 1\right\} = y^2 f\left(\frac{x}{y}\right)$ → "

→ To find soln of DE, if we see the eqn is Homogenous then substitute: $\boxed{y = vx}$ (or) $\boxed{x = vy}$

$dy = v\,dx + x\,dv$ | $dx = v\,dy + y\,dv$

$\boxed{\frac{dy}{dx} = v + x\frac{dv}{dx}}$, $\boxed{\frac{dx}{dy} = v + y\frac{dv}{dy}}$

☆ If u see $\frac{dy}{dx}$ in eqn use $y = vx$, $\frac{dx}{dy}$ than $x = vy$.

Examples:

(1) $(y + \sqrt{x^2+y^2})dx - x\,dy = 0$; $y(1) = 0$

Sol: $\frac{dy}{dx} = \frac{y + \sqrt{x^2+y^2}}{x}$ → Homogeneous of degree 1.

$y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} \Rightarrow v + x\frac{dv}{dx} = v + \sqrt{1+v^2}$

$\Rightarrow x\frac{dv}{dx} = \sqrt{1+v^2} \Rightarrow \int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x}$

$\Rightarrow \ln|v + \sqrt{1+v^2}| = \ln x + \ln c$

$\Rightarrow \ln(xc) = \ln|v + \sqrt{1+v^2}|$

$\Rightarrow \boxed{xc = \frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}}$

$y(1) = 0$

$\Rightarrow \boxed{c = 1}$

$\therefore \boxed{x = \frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}}$

(2) $\frac{y}{x}\frac{dy}{dx}+\frac{x^2+y^2-1}{2(x^2+y^2)+1}=0$

Sol: $x^2+y^2=t$

$2x\,dx+2y\,dy=dt$

$2x+2y\frac{dy}{dx}=\frac{dt}{dx} \Rightarrow x+y\frac{dy}{dx}=\frac{dt}{2dx}$

$\Rightarrow 1+\frac{y}{x}\frac{dy}{dx}=\frac{dt}{2x\,dx}$

$\Rightarrow \boxed{\frac{y}{x}\frac{dy}{dx}=\frac{dt}{2x\,dx}-1}$

$\Rightarrow \left(\frac{1}{2x}\frac{dt}{dx}-1\right)+\frac{t-1}{2t+1}=0$

$\Rightarrow \frac{1}{2x}\frac{dt}{dx}=-\frac{t-1}{2t+1}+1=\frac{-t+1+2t+1}{2t+1}=\frac{t+2}{2t+1}$

$\Rightarrow \int\frac{2t+1}{t+2}dt=\int 2x\,dx$

$\int\frac{2t+4}{t+2}-\frac{3}{t+2}=x^2$

$\Rightarrow \int 2-\frac{3}{t+2}=x^2 \Rightarrow \int 2-\int\frac{3}{t+2}=x^2$

$\Rightarrow x^2=2t-3\ln(t+2)+C$

$\Rightarrow x^2=2(x^2+y^2)-3\ln(x^2+y^2+2)+C$

$\therefore \boxed{x^2-2(x^2+y^2)+3\ln(x^2+y^2+2)+C=0}$

$\Rightarrow$ Non Homogeneous method (or) Reducible to Homogeneous:-

$\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$

case-(1): $\frac{a_1}{a_2}=\frac{b_1}{b_2}$ (Parallel) then, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = m$ (or) $\frac{1}{m}$ then $a_1 = ma_2$, $b_1 = mb_2$ (or) $a_2 = ma_1$, $b_2 = mb_1$

$$\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{m(a_1x+b_1y)+c_2} \quad \text{let } \boxed{a_1x+b_1y = z}$$

$$= \frac{z+c_1}{mz+c_2}$$ & solved by Homogeneous (or) variable separable.

eg: ① $\frac{dy}{dx} = \frac{2x+3y+1}{4x+6y+4}$

Sol: $\frac{dy}{dx} = \frac{2x+3y+1}{2(2x+3y)+4} = \frac{z+1}{2z+4}$

Case-II: $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ (h,k) intersecting

$\boxed{x = X+h, \ y = Y+k.}$

$$\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$$

$dx = dX$
$dy = dY$

$a_1x+b_1y+c_1 = 0$
$a_2x+b_2y+c_2 = 0$
as (h,k) is also on these lines

$$\frac{dY}{dX} = \frac{a_1X+b_1Y+a_1h+b_1k+c_1}{a_2X+b_2Y+a_2h+b_2k+c_2}$$

$\star \begin{cases} a_1h+b_1k+c_1 = 0 \\ a_2h+b_2k+c_2 = 0 \end{cases}$

$$\frac{dY}{dX} = \frac{a_1X+b_1Y}{a_2X+b_2Y}$$ & this is Homogeneous & solved by $Y = vX$

examples:

① $(x-2y+1)dx + (4x-3y-6)dy = 0$

Sol: $\frac{dy}{dx} = \frac{-x+2y-1}{4x-3y-6}$ $\quad \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

$x = X+3, \ y = Y+2$

$dx = dX$
$dy = dY$

$4x-3y-6 = 0$
$-4x+8y-4 = 0$
$5y = 10$
$y = 2$
$x = 3$
$\therefore (h,k) = (3,2)$

$\frac{dY}{dX} = \frac{-X+2Y}{4X-3Y} \rightarrow$ Homogeneous

$Y = vX$
$\frac{dY}{dX} = v + X\frac{dv}{dX}$

$$v + X\frac{dv}{dX} = \frac{-X+2(vX)}{4X-3(vX)}$$

$$\therefore v + x\frac{dv}{dx} = \frac{2v-1}{4-3v}$$

$$x\frac{dv}{dx} = \frac{2v-1}{4-3v} - v = \frac{2v-1-4v+3v^2}{4-3v}$$

$$\int \frac{4-3v}{3v^2-2v-1}\,dv = \int \frac{dx}{x}$$

$$\int \frac{3v-4}{3v^2-2v-1}\,dv = -\int \frac{dx}{x}$$

$$\int \frac{3v-1}{3v^2-2v-1}\,dv - \int \frac{3}{3v^2-2v-1}\,dv = -\int \frac{dx}{x}$$

$$\therefore \frac{1}{2}\ln(3v^2-2v-1) - \frac{3}{4}\ln\left(\frac{3v-3}{3v+1}\right) = -\ln x + \ln c$$

Put $v = \frac{y}{x} = \frac{y-2}{x-3}$ & it is final answer.

(2) $\frac{dy}{dx} = \frac{y-x}{y-x+2}$.

Sol: $\frac{a_1}{a_2} = \frac{b_1}{b_2}$. $\Rightarrow \frac{dy}{dx} = \frac{z}{z+2}$ $z = y-x$

$$\frac{dz}{dx} = \frac{dy}{dx} - 1$$

$$\Rightarrow \frac{dz}{dx} = \frac{z}{z+2} - 1 = \frac{z-z-2}{z+2} = \frac{-2}{z+2}$$

$$\Rightarrow (z+2)\,dz = -2\,dx$$

$$\Rightarrow \int z\,dz + \int 2\,dz = \int -2\,dx$$

$$\Rightarrow \frac{z^2}{2} + 2z = -2x + c$$

$$z = y-x \Rightarrow \frac{(y-x)^2}{2} + 2(y-x) = -2x + c$$

$$\Rightarrow \boxed{(y-x)^2 + 4y - 2x + c = 0}$$

③ $(2x^2+3y^2-7)x\,dx = (3x^2+2y^2-8)y\,dy$

Sol: $\frac{y\,dy}{x\,dx} = \frac{2x^2+3y^2-7}{3x^2+2y^2-8}$

$\left.\begin{array}{l} x^2 = X,\ y^2 = Y \\ 2x\,dx = dX,\ 2y\,dy = dY \end{array}\right\} \Rightarrow \frac{dY}{dX} = \frac{2X+3Y-7}{3X+2Y-8}$

$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$. $X = X+2,\ Y = Y+1$

$\frac{dY}{dX} = \frac{2X+3Y}{3X+2Y} \rightarrow$ Homogeneity

$6X+4Y-16=0$
$(-)6X\pm 9Y\mp 21=0$
$-5Y+5=0$
$-5Y=-5$
$\Rightarrow Y=1$
$X=2$
$\therefore (h,k)=(2,1)$

$Y = vX$

$\frac{dY}{dX} = v + X\frac{dv}{dX} \Rightarrow v + X\frac{dv}{dX} = \frac{2X+3(vX)}{3X+2(vX)}$

$\Rightarrow v + X\frac{dv}{dX} = \frac{\cancel{X}(2+3v)}{\cancel{X}(3+2v)}$

$\Rightarrow X\frac{dv}{dX} = \frac{2+3v}{3+2v} - v = \frac{2+\cancel{3v}-\cancel{3v}-2v^2}{3+2v} = \frac{2-2v^2}{3+2v}$

$\Rightarrow \int \frac{3+2v}{2-2v^2}\,dv = \int \frac{1}{X}\,dX = \ln X + \ln c.$ — answer.

25/10/21

⟹ Partial Derivatives:

* $\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$z = f(x,y) \Rightarrow \frac{\partial z}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} = z_x$

$\frac{\partial z}{\partial y} = \lim_{k\to 0} \frac{f(x,y+k)-f(x,y)}{k} = z_y$

eg: ① $z = x^2+y^2$

$\frac{\partial z}{\partial x} = 2x,\ \frac{\partial z}{\partial y} = 2y,\ \underbrace{\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)}_{=z_{xy}} = \frac{\partial}{\partial y}(2x) = 0,\ \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)}_{=z_{yx}} = 0$

$\boxed{z_{xy} = z_{yx}}$

* $\frac{\partial^2 z}{\partial x^2} = \lim_{h \to 0} \frac{\frac{\partial}{\partial x} f((x+h), y) - \frac{\partial}{\partial x} f(x, y)}{h}$

* $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = z_{yx}$

* $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = z_{xy}$

eg: ① $u(x,y) = x^3 + y^3 + 3x^2y + \sin x + \cos y$

Sol: $\frac{\partial u}{\partial x} = 3x^2 + 0 + 6xy + \cos x + 0$

$\frac{\partial u}{\partial y} = 0 + 3y^2 + 3x^2 + 0 - \sin y$

⟹ Total derivative: $u(x,y)$.

* $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$

$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{dy}{dx}$

⟹ Exact method:

* A Differential eqn of the form $M(x,y)dx + N(x,y)dy = 0$ is said to be exact if its Left hand member is exact differential of some function $u(x,y) = c$ (i.e) $du = 0$

* Consider the eqn $M(x,y)dx + N(x,y)dy = 0$. Suppose that there exists a function $F(x,y)$ such that $dF = Mdx + Ndy$, then the D.eqn is said to be an exact D.eqn & its soln is given by $F(x,y) = c$.

$M(x,y)dx + N(x,y)dy = 0$: exact D.E

$f(x,y)$ అనేది soln

Checking exact & finding sol^n if exact:

Imp Given D.E: $M(x,y)\,dx + N(x,y)\,dy = 0$

Checking exact: $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Sol^n: $\int_{y\,const} M\,dx + \int N$ terms not containing x terms dy $= c$

*. Proof of checking exact & sol^n of exact:

$M(x,y)\,dx + N(x,y)\,dy = 0$ is exact. There exists

$f(x,y)$ such that:

$$M(x,y)\,dx + N(x,y)\,dy = df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$M = \frac{\partial f}{\partial x},\ N = \frac{\partial f}{\partial y}$$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x},\ \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y} \Rightarrow f_{xy} = f_{yx} \Rightarrow \boxed{M_y = N_x}$$

$$\therefore \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

→ Now, we will show that the above condition is sufficient for $M\,dx + N\,dy = 0$.

$f(x,y)$ such that $\dfrac{\partial f}{\partial x} = M,\ \dfrac{\partial f}{\partial y} = N$

$\dfrac{\partial f}{\partial x} = M$

Integrate partially w.r.t x

$$f(x,y) = \int_{y\,const} M(x,y)\,dx + g(y) \quad \text{—(1)}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\int_{y\,const} M\,dx\right) + g'(y)$$

$$N = \frac{\partial}{\partial y}\left(\int_{y\,const} M\,dx\right) + g'(y)$$

$$\int g'(y) = \int N - \frac{\partial}{\partial y}\left(\int_{y\,const} M\,dx\right) \quad \text{—(2)}$$

$= g(y)$

$$f(x,y) = \int_{y\,const} M(x,y)\,dx + \int N - \frac{\partial}{\partial y}\left(\int_{y\,const} M\,dx\right)$$

{∵ Here x terms will get cancelled see in next example}

$$\therefore \boxed{f(x,y) = \int_{y\,const} M\,dx + \int N \text{ terms not containing } x\ dy}$$

eg ① $\left(x+\frac{2}{y}\right)dy + y\,dx = 0$

Sol: $M(x,y) = y,\ N(x,y) = x + \frac{2}{y}$

$$\frac{\partial M}{\partial y} = 1,\quad \frac{\partial N}{\partial x} = 1 \Rightarrow \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

$$f(x,y) = \int_{y\ \text{constant}} y\,dx + g(y) = yx + g(y) \quad \text{— ①}$$

$$g'(y) = N - \frac{\partial}{\partial y}\left(\int_{y\ \text{const}} y\,dx\right) = N - \frac{\partial}{\partial y}(xy)$$

$$= \cancel{x} + \frac{2}{y} - \cancel{x} = \frac{2}{y}$$

$$\cancel{f(x,y) = yx + \frac{2}{y}} \quad \therefore \cancel{\boxed{xy + \frac{2}{y} = c}}$$

$$\int g'(y)\,dy = \int \frac{2}{y} \Rightarrow \boxed{g(y) = 2\log y} \quad \text{— ②}$$

$$f(x,y) = yx + 2\log y \quad \therefore \boxed{xy + 2\log y = C}$$

Shortcut:
$$\int_{y\ \text{const}} M\ \text{terms}\ dx + \int N\ \text{terms not containing}\ x\ dy = C$$

$$\int_{y\ \text{const}} y\,dx + \int \frac{2}{y}\,dy = c$$

$$\therefore \boxed{xy + 2\log y = C}$$

Examples:

① $\underbrace{e^y}_{M}dx + (\underbrace{xe^y + 2y}_{N})dy = 0$

Sol: $\frac{\partial M}{\partial y} = e^y \quad \frac{\partial N}{\partial x} = e^y \Rightarrow \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

$$\Rightarrow \int e^y\,dx + \int 2y\,dy = C$$

$$\Rightarrow \boxed{xe^y + y^2 = C}$$

② $(\underbrace{\cos x - x\cos y}_{N})dy - (\underbrace{\sin y + y\sin x}_{M})dx = 0$

$\frac{\partial M}{\partial y} = -\cos y - \sin x \quad \frac{\partial N}{\partial x} = -\sin x - \cos y \Rightarrow \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

$\boxed{y\cos x - x\sin y + C} \Leftarrow \int -\sin y - y\sin x + 0 = C$

28/10/21

(3) $(3x^2y + \frac{y}{x})dx + (x^3 + \log x)dy = 0$

Sol: $\frac{\partial M}{\partial y} = 3x^2 + \frac{1}{x}$ $\quad \frac{\partial N}{\partial x} = 3x^2 + \frac{1}{x}$ $\Rightarrow \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

$\int (3x^2y + \frac{y}{x})dx + 0 = c \Rightarrow \boxed{yx^3 + y\log x = c}$

(4) $(r + \sin\theta - \cos\theta)dr + r(\sin\theta + \cos\theta)d\theta = 0$

Sol: $M(r,\theta)\,dr + N(r,\theta)\,d\theta = 0$

$\frac{\partial M}{\partial \theta} = \cos\theta + \sin\theta$ $\quad \frac{\partial N}{\partial r} = \sin\theta + \cos\theta$

$\Rightarrow \boxed{\frac{\partial M}{\partial \theta} = \frac{\partial N}{\partial r}}$

$\Rightarrow \int (r + \sin\theta - \cos\theta)dr + 0 = \frac{r^2}{2} + r\sin\theta - r\cos\theta + c$

(5) $(\sin\theta - 2r\cos^2\theta)\,dr + r\cos\theta(2r\sin\theta + 1)d\theta = 0$

Sol: $M(r,\theta)\,dr + N(r,\theta)\,d\theta = 0$

$\Rightarrow M = \sin\theta - 2r\cos^2\theta$

$N = 2r^2\cos\theta\sin\theta + r\cos\theta.$

$\frac{\partial M}{\partial \theta} = \cos\theta + 4r\cos\theta\sin\theta.$

$\frac{\partial N}{\partial r} = 4r\cos\theta\sin\theta + \cos\theta$

$\boxed{\frac{\partial M}{\partial \theta} = \frac{\partial N}{\partial r}}$

$\Rightarrow \int \sin\theta - 2r\cos^2\theta \, dr + 0 = c$

$\Rightarrow -\cos\theta - \theta - \frac{\sin 2\theta}{2} = c$

$\Rightarrow \theta + \cos\theta + \frac{\sin 2\theta}{2} = c.$

→ Reduction of Non-exact D.Eqⁿ:

* $M(x,y)\,dx + N(x,y)\,dy = 0$

If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then the D.Eqⁿ is Non-exact.

For making it exact we should multiply by Integrating Factor (I.F) for which $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Case (1): $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \underset{\sim}{f(x)} \quad \text{then } I.F = e^{\int f(x)dx}.$$

function of x alone

Case (2): $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = g(y) \quad \text{then } I.F = e^{-\int g(y)dy}.$$

function of y alone

eg: (1) $\underbrace{y(2x^2 - xy + 1)}_{M}dx + \underbrace{(x-y)}_{N}dy = 0$

Sol: $\frac{\partial M}{\partial y} = 2x^2 - 2xy + 1 \neq \frac{\partial N}{\partial x} = 1$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2x^2 - 2xy \Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2x^2 - 2xy}{x-y} = \underset{f(x)}{2x}$$

$$I.F = e^{\int f(x)dx} = e^{\int 2x\,dx} = e^{x^2}$$

$$\Rightarrow e^{x^2}(2x^2y - xy^2 + y)dx + e^{x^2}(x-y)dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\to \int e^{x^2}(2x^2y - xy^2 + y)dx + \int 0 + c.$$

Case ③: $m(x,y)dx + N(x,y)dy = 0 \rightarrow$ Homogenous of any degree.

$$I.F = \frac{1}{mx+Ny} \quad (mx+Ny \neq 0)$$

eg: ① $(\underbrace{y^3 - 2x^2y}_{M})dx + (\underbrace{2y^2x - x^3}_{N})dy = 0$

Sol: $\frac{\partial m}{\partial y} \neq \frac{\partial N}{\partial x}$. $\quad I.F = \frac{1}{mx+Ny}$

$$I.F = \frac{1}{xy^3 - 2x^3y + 2y^3x - x^3y} = \frac{1}{3xy^3 - 3x^3y} = \frac{1}{3xy(y^2-x^2)}$$

Case ④: $m(x,y)dx + N(x,y)dy = 0$ is of the form

$y\, g(xy)dx + x\, h(xy)dy = 0$ then

$$I.F = \frac{1}{mx - Ny} \quad (mx - Ny \neq 0)$$

eg: ① $(x^2y^2 + xy + 1)y\,dx + (x^2y^2 - xy + 1)x\,dy = 0$.

④ **Proofs:**

Case ①: $\because m(x,y)dx + N(x,y)dy = 0$ ① ; $\frac{\partial m}{\partial y} \neq \frac{\partial N}{\partial x}$

Suppose $I.F = U(x,y)$ the we multiply I.F to eq^n ①

$$mUdx + NUdy = 0$$

$$m_1dx + N_1dy = 0 \Rightarrow \boxed{\frac{\partial m_1}{\partial y} = \frac{\partial N_1}{\partial x}} \text{ - exact.}$$

$$\Rightarrow \frac{\partial}{\partial y} mU = \frac{\partial}{\partial x} NU$$

$$\Rightarrow m\frac{\partial U}{\partial y} + U\frac{\partial m}{\partial y} = N\frac{\partial U}{\partial x} + U\frac{\partial N}{\partial x}$$

• Suppose $U(x,y)$ is function of x alone

$$m \cdot 0 + U\frac{\partial m}{\partial y} = N\frac{\partial U}{\partial x} + U\frac{\partial N}{\partial x}$$

$$U\left[\frac{\partial m}{\partial y} - \frac{\partial N}{\partial x}\right] = N\frac{\partial U}{\partial x} \qquad \left[\because \frac{\partial U}{\partial x} = \frac{dU}{dx}\right]$$

as $\frac{\partial U}{\partial y} = 0$ as U is only x functi

$$\Rightarrow \frac{1}{U}\frac{dU}{dx} = \frac{\frac{\partial m}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

$$\Rightarrow \int \frac{dU}{U} = \int \frac{m_y - N_x}{N}\, dx$$ {variable separable}

$$\Rightarrow \ln U = \int \frac{m_y - N_x}{N}\, dx$$

$$\Rightarrow U = I.F = e^{\int \frac{m_y - N_x}{N} dx} = e^{\int f(x) dx}$$

Case ②: $\frac{\partial}{\partial y} mU = \frac{\partial N U}{\partial x}$

Suppose U = function y alone

$$m\frac{\partial U}{\partial y} + U\frac{\partial m}{\partial y} = N\frac{\partial U}{\partial x} + U\frac{\partial N}{\partial x}$$

$$U\{-m_y + N_x\} = m\frac{dU}{dy}$$

$$\int \frac{dU}{U} = \int \frac{N_x - m_y}{m}\, dy$$

$$\log U = \int \frac{N_x - m_y}{m}\, dy$$

$$\Rightarrow U = e^{\int \frac{N_x - m_y}{m} dy} = e^{-\int \frac{m_y - N_x}{m} dy}$$

$$\therefore I.F = e^{-\int g(y) dy}.$$

Problems:

① $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$

Sol: $\frac{\partial m}{\partial y} = 12x^2y^3 + 2x \quad \neq \quad \frac{\partial N}{\partial x} = 6x^2y^3 - 2x$

$$\frac{\frac{\partial m}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{6x^2y^3 + 4x}{3x^2y^4 + 2xy} = \frac{2}{y}$$

$$\Rightarrow I.F = e^{-\int \frac{2}{y} dy} = e^{-2\log y} = \frac{1}{y^2}$$

$\times$ I.F $\Rightarrow (3x^2y^2 + \frac{2x}{y})\,dx + (2x^3y - \frac{x^2}{y^2})\,dy = 0$

$$\frac{\partial m}{\partial y} = \frac{\partial N}{\partial x}.$$

$$\Rightarrow \int (3x^2y^2 + \frac{2x}{y})\,dx = c$$

$$\Rightarrow 3\frac{x^3}{3}y^2 + \frac{2}{y}\frac{x^2}{2} + c \Rightarrow \boxed{x^3y^2 + \frac{x^2}{y} = c}$$

② $\underbrace{(x-y)}_{m}dx - \underbrace{dy}_{N} = 0,\ y(0) = 2.$

Sol: $\frac{\partial m}{\partial y} = -1 \quad \frac{\partial N}{\partial x} = 0$

$$\frac{\partial m}{\partial y} - \frac{\partial N}{\partial x} = -1 \Rightarrow \frac{\frac{\partial m}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-1}{-1} = 1$$

$$I.F = e^{\int f(x)dx} = e^{\int 1 dx} = e^x.$$

$\times$ I.F $\Rightarrow e^x(x-y)dx - e^x dy = 0$

$\Rightarrow \int e^x(x-y)dx = c.$ $\quad [\because \int uv\,dx = uv_1 - u'v_2 + u''v_3 + \ldots]$

$$\Rightarrow xe^x - e^x.1 - ye^x + c.$$

③ $\underbrace{(x^2+y^2+2x)}_{m}dx + \underbrace{2y}_{N}\,dy = 0$

Sol: $\frac{\partial m}{\partial y} = 2y. \quad \frac{\partial N}{\partial x} = 0 \Rightarrow \frac{\partial m}{\partial y} \neq \frac{\partial N}{\partial x}$

$$\frac{\frac{\partial m}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y}{2y} = \underbrace{1}_{f(x)} \Rightarrow e^{\int 1 dx} = e^x.$$

$$\Rightarrow e^x\{x^2+y^2+2x\}dx + e^x 2y\,dy = 0$$

$$\Rightarrow \int x^2e^x + y^2e^x + 2xe^x + 0 + c$$

$$\Rightarrow x^2e^x - e^x 2x + e^x 2 + y^2e^x + 2(xe^x - e^x) + c$$

$$\Rightarrow x^2e^x + y^2e^x + c$$

Case-③: $M(x,y)dx + N(x,y)dy = 0$

M, N are homogeneous then $I.F = \frac{1}{Mx+Ny}$

Examples:

① $y(\underbrace{y^2-2x^2}_{M})dx + x(\underbrace{2y^2-x^2}_{N})dy = 0$

Sol: $\frac{\partial M}{\partial y} = 3y^2 - 2x^2$ & $\frac{\partial N}{\partial x} = 2y^2 - 3x^2$

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = y^2 + x^2$ If we ÷ M (or) N no use it is fn of both x & y

As it is Homogeneous [both M & N]

M-①: $y = vx$.

M-②: $I.F = \frac{1}{Mx+Ny} = \frac{1}{xy^3 - 2x^3y + 2xy^3 - yx^3}$

$= \frac{1}{3xy^3 - 3yx^3} = \frac{1}{3xy\{y^2-x^2\}}$

$\Rightarrow \frac{\cancel{y}(y^2-2x^2)}{3x\cancel{y}(y^2-x^2)}dx + \frac{\cancel{x}(2y^2-x^2)}{3\cancel{x}y(y^2-x^2)}dy = 0$

$\Rightarrow \frac{y^2-x^2-x^2}{x(y^2-x^2)}dx + \frac{y^2+y^2-x^2}{y(y^2-x^2)}dy = 0$

$\Rightarrow \frac{dx}{x} - \frac{x}{y^2-x^2}dx + \frac{y}{(y^2-x^2)}dy + \frac{dy}{y} = 0$

$\Rightarrow \int\frac{dx}{x} + \int\frac{dy}{y} + \int\frac{ydy - xdx}{y^2-x^2} = 0$

$\Rightarrow \log x + \log y + \frac{1}{2}\int\frac{2(ydy - xdy)}{y^2-x^2} = c$

$\Rightarrow \ln x + \ln y + \frac{1}{2}\int d\log(y^2-x^2) = c$

$\Rightarrow \ln x + \ln y + \frac{1}{2}\ln(y^2-x^2) = c$

Case (4) $M(x,y)dx + N(x,y)dy = 0$

$y\, f(xy)\, dx + x\, g(xy)\, dy = 0$

$I.F = \frac{1}{Mx - Ny}$ $(Mx - Ny \neq 0)$

Examples:

(1) $(x^2y^2 + xy + 1)\, y\, dx + (x^3y^2 - x^2y + x)\, dy = 0$

M N

Sol: $\frac{\partial M}{\partial y} = 3y^2x^2 + 2yx + 1$ $\neq$ $\frac{\partial N}{\partial x} = 3x^2y^2 - 2xy + 1$

$\Rightarrow y(x^2y^2 + xy + 1)dx + x(x^2y^2 - xy + 1)dy = 0$

$I.F = \frac{1}{Mx - Ny} = \frac{1}{x^3y^3 + x^2y^2 + xy - x^3y^3 + x^2y^2 - xy}$

$= \frac{1}{2x^2y^2}$

$\Rightarrow \left[\frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y}\right]dx + \left[\frac{x}{2} - \frac{1}{2y} + \frac{1}{2xy^2}\right]dy = 0$

$\Rightarrow \int \left[\frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y}\right]dx + \int -\frac{1}{2y}\, dy = c$

$\Rightarrow \frac{y}{2}x + \frac{1}{2}\ln x + \frac{1}{2y}\frac{x^{-1}}{-1} - \frac{1}{2}\ln y = c$

Case (5) $M(x,y)dx + N(x,y)dy = 0$

$\downarrow$ form

$\boxed{x^a y^b (my\,dx + nx\,dy) + x^c y^d (py\,dx + qx\,dy) = 0}$

a, b, c, d, m, n, p, q are all constants.

$\boxed{I.F = x^h y^k}$

To find ~~xy~~ h, k; If I.F multipled eq^n becoms exact

$\Rightarrow (m x^{a+h} y^{b+k+1} + p x^{c+h} y^{d+k+1})dx + (n x^{a+h+1} y^{b+k} + q x^{c+h+1} y^{d+k})dy = 0$

$\rightarrow$ exact

$$\frac{\partial M_1}{\partial y} = m x^{a+h}(b+K+1)y^{b+K} + P x^{c+h}(d+K+1)y^{d+K}$$

$$\frac{\partial N_1}{\partial x} = n x^{a+h}(a+h+1)y^{b+K} + Q y^{d+K}(c+h+1)x^{c+h}$$

$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \Rightarrow$ equating the coeff x^{a+h} & y^{b+K} & x^{c+h} & y^{d+K}.

$\Rightarrow m(b+K+1) = n(a+h+1)$ | $P(d+K+1) = Q(c+h+1)$

$$\boxed{\frac{a+h+1}{m} = \frac{b+K+1}{n}} \text{ — (1)} \qquad \boxed{\frac{c+h+1}{P} = \frac{d+K+1}{Q}} \text{ — (2)}$$

Solving these 2 eq^n's we get h & K & hence I.F $= x^h y^K$

Examples →

1. $(2y^2 + 4x^2y)dx + (4xy + 3x^3)dy = 0$

Sol^n $\Rightarrow 2y^2 dx + 4x^2 y dx + 4xy dy + 3x^3 dy = 0$

$x^2(4y\,dx + 3x\,dy) + y(2y\,dx + 4x\,dy) = 0$

$$\boxed{x^a y^b(my\,dx + nx\,dy) + x^c y^d(Py\,dx + Qx\,dy) = 0}$$

$a = 2, b = 0, m = 4, n = 3$ | $c = 0, d = 1, P = 2, Q = 4$

$\frac{a+h+1}{m} = \frac{b+K+1}{n} \Rightarrow \frac{2+h+1}{4} = \frac{0+K+1}{3}$ — (1)

$\frac{c+h+1}{P} = \frac{d+K+1}{Q} \Rightarrow \frac{0+h+1}{2} = \frac{1+K+1}{4}$ — (2)

From (1) & (2) $h = 1, K = 2$. $\therefore \boxed{I.F = x^h y^K} = xy^2$

$\Rightarrow (2xy^4 + 4x^3y^3)dx + (4x^2y^3 + 3x^4y^2)dy = 0$

$\Rightarrow \int d(x^2y^4) + \int d(x^4y^3) = 0$

$\Rightarrow \boxed{x^2y^4 + x^4y^3 = c}$

$\Rightarrow$ Inspection Method [Grouping of Terms]

Case 6

Group of Terms	I.F	exact differential
① $xdy+ydx$	1	$d(xy)$
② $xdy+ydx$	$\frac{1}{xy}$	$\frac{xdy+ydx}{xy}=d(\ln xy)$
③ $xdy+ydx$	$\frac{1}{(xy)^n}(n\neq 1)$	$\frac{xdy+ydx}{(xy)^n}=d\left\{\frac{(xy)^{1-n}}{1-n}\right\}$
④ $xdx+ydy$	$\frac{1}{x^2+y^2}$	$\frac{xdx+ydy}{x^2+y^2}=\frac{1}{2}d(\ln(x^2+y^2))$
⑤ $xdx+ydy$	2	$2(xdx+ydy)=d(x^2+y^2)$
⑥ $xdy-ydx$	$\frac{1}{x^2}$	$\frac{xdy-ydx}{x^2}=d\left(\frac{y}{x}\right)$
⑦ $xdy-ydx$	$\frac{1}{y^2}$	$\frac{-(ydx-xdy)}{y^2}=-d\left(\frac{x}{y}\right)$
⑧ $xdy-ydx$	$\frac{1}{xy}$	$\frac{dy}{y}-\frac{dx}{x}=d\left(\ln\left(\frac{y}{x}\right)\right)$
⑨ $xdy-ydx$	$\frac{1}{x^2+y^2}$	$\frac{xdy-ydx}{x^2+y^2}=d\left(\tan^{-1}\frac{y}{x}\right)$

Example:

① $xdy+ydx+xy\,dx=0$

Sol: $\frac{xdy+ydx}{xy}+dx=0 \Rightarrow \int d(\ln xy)+\int dx=0$

$\Rightarrow \ln(xy)+x=C$

② $xdy-ydx+(x^2+y^2)dx=0$

Sol: $\frac{xdy-ydx}{x^2+y^2}+dx=0$

$\int d\left(\tan^{-1}\left(\frac{y}{x}\right)\right)+\int dx=0$

$\Rightarrow \tan^{-1}\frac{y}{x}+x=C$

(3) $y(y^3-x)dx + x(y^3+x)dy = 0$

Sol: $y^4 dx + xy^3 dy + x^2 dy - xy dx = 0$

$y^3(ydx + xdy) + x(xdy - ydx) = 0$

$y^3(ydx+xdy) + x \cdot x^2 \frac{(xdy-ydx)}{x^2} = 0$

$y^3 d(xy) + x^3 d\left(\frac{y}{x}\right) = 0$

$d(xy) + \frac{x^3}{y^3} d\left(\frac{y}{x}\right) = 0$

$d(xy) + \frac{y^{-3}}{x^{-3}} d\left(\frac{y}{x}\right) = 0$

$\Rightarrow d(xy) + d\left[-\frac{1}{2}\left(\frac{y}{x}\right)^{-2}\right] = 0$

$\Rightarrow xy - \frac{1}{2}\left(\frac{y}{x}\right)^{-2} = c$

$d\left(\frac{y}{x}\right)^{-2} = -2\left(\frac{y}{x}\right)^{-3} \frac{xdy-ydx}{x^2}$

$= -2\left(\frac{y}{x}\right)^{-3} d\left(\frac{y}{x}\right)$

$d\left(-\frac{1}{2}\left(\frac{y}{x}\right)^{-2}\right) = \left(\frac{y}{x}\right)^{-3} d\left(\frac{y}{x}\right)$

(4) $(x^3y^3+1)dx + x^4y^2 dy = 0$

Sol: $x^3y^3 dx + x^4y^2 dy + dx = 0$

$x^3y^2(ydx + xdy) + dx = 0$

$x^2y^2 d(xy) + \frac{dx}{x} = 0$

$\int (xy)^2 d(xy) + \int \frac{dx}{x} = 0$

$\Rightarrow \frac{(xy)^3}{3} + \ln x = c$

$\Rightarrow x^3y^3 + 3\ln x = c$

(5) $y(x^3e^{xy} - y)dx + x(xy + x^3e^{xy})dy = 0$

Some Standard differentials:

(1) $x\,dy + y\,dx = d(xy)$

(2) $dx + dy = d(x+y)$

(3) $dx - dy = d(x-y)$

(4) $\dfrac{y\,dx - x\,dy}{y^2} = d\left(\dfrac{x}{y}\right)$

(5) $\dfrac{x\,dy - y\,dx}{x^2} = d\left(\dfrac{y}{x}\right)$

(6) $\dfrac{2xy\,dy - y^2dx}{x^2} = d\left(\dfrac{y^2}{x}\right)$

(7) $\dfrac{2xy\,dx - x^2dy}{y^2} = d\left(\dfrac{x^2}{y}\right)$

(8) $2xy\,dy + y^2dx = d(xy^2)$

(9) $2xy\,dy + x^2dy = d(x^2y)$

(10) $\dfrac{x\,dx + y\,dy}{\sqrt{x^2+y^2}} = d\left(\sqrt{x^2+y^2}\right)$

(11) $\dfrac{y\,dx - x\,dy}{x^2+y^2} = d\left(\operatorname{Tan}^{-1}\left(\dfrac{x}{y}\right)\right)$

(12) $\dfrac{x\,dx + y\,dy}{x^2+y^2} = d\left\{\dfrac{1}{2}\log(x^2+y^2)\right\}$ (or) $d\left\{\log\left(\sqrt{x^2+y^2}\right)\right\}$

(13) $\dfrac{x\,dy - y\,dx}{x\sqrt{x^2-y^2}} = d\left\{\operatorname{Sin}^{-1}\left(\dfrac{y}{x}\right)\right\}$

(14) $\dfrac{y\,dx - x\,dy}{y\sqrt{y^2-x^2}} = d\left\{\operatorname{Sin}^{-1}\left(\dfrac{x}{y}\right)\right\}$

(15) $\dfrac{x\,dy + y\,dx}{1+(xy)^2} = d\left\{\operatorname{Tan}^{-1}(xy)\right\}$

(16) $x^3dy + 3x^2y\,dx = d(x^3y)$

(17) $x\cos y\,dy + \sin y\,dx = d(x\sin y)$

(18) $-x\sin y\,dy + \cos y\,dx = d(x\cos y)$

(19) $x^2\sec^2 y\,dy + 2x\tan y\,dx = d(x^2\tan y)$

(20) $\cos(xy)(x\,dy + y\,dx) = d[\sin(xy)]$

(21) $e^{x/y}\left[\dfrac{y\,dx - x\,dy}{y^2}\right] = d[e^{x/y}]$

(22) $$\frac{e^{y/x}\{x\,dy - y\,dx\}}{x^2(1+e^{2y/x})} = d\{\tan^{-1}(e^{y/x})\}$$

(23) $x\sin 2y\,dy + \sin^2 y\,dx = d(x\sin^2 y)$

(24) $$\frac{y\,dx - x\log x\,dy}{xy^2} = d\left(\frac{\log x}{y}\right)$$

(25) $$\frac{(1-y^2)dx + xy\,dy}{(1-y^2)^{3/2}} = d\left(\frac{x}{\sqrt{1-y^2}}\right)$$

(26) $$\frac{dx+dy}{x+y} = d\{\log(x+y)\}$$

(27) $$\frac{dx-dy}{x-y} = d\{\log(x-y)\}$$

(28) $$\frac{e^x(dx-dy)}{e^y} = d(e^{x-y}) \text{ (or) } d\left(\frac{e^x}{e^y}\right)$$

(29) $$\frac{x(y\,dx - x\,dy)}{y^3} = \frac{x}{y}\left(\frac{y\,dx - x\,dy}{y^2}\right) = \frac{x}{y}\,d\left(\frac{x}{y}\right)$$

(30) $$\frac{xy\,dy - y^2\,dx}{x^3} = \frac{y}{x}\left(\frac{x\,dy - y\,dx}{x^2}\right) = \frac{y}{x}\,d\left(\frac{y}{x}\right)$$

⇒ Leibnitz's Linear eqⁿ:

* $\frac{dy}{dx} + P(x)\,y = Q(x)$ → P, Q function of x.

$\uparrow$ linear eqⁿ.

$\Rightarrow dy + [P(x)y - Q(x)]dx = 0$

$M = P(x)y - Q(x)$, $N = 1$

$\frac{\partial M}{\partial y} = P(x)$, $\frac{\partial N}{\partial x} = 0$ $\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{P(x)}{1} = \underbrace{P(x)}_{\text{function of } x}$

$$\boxed{I.F = e^{\int P(x)dx}}$$

$\Rightarrow \frac{dy}{dx} e^{\int P(x)dx} + P(x)\,y\,e^{\int P(x)dx} = Q(x)\,e^{\int P(x)dx}$

$\int d(y \cdot e^{\int Pdx}) = \int Q e^{\int Pdx}$

$$\boxed{y \cdot e^{\int Pdx} = \int Q \cdot e^{\int Pdx} dx + C}$$

$\left[\because \frac{d}{dx}(y e^{\int Pdx}) = y \frac{d}{dx} e^{\int Pdx} + e^{\int Pdx}\frac{dy}{dx} = y \cdot P(x) e^{\int Pdx} + e^{\int Pdx}\frac{dy}{dx}\right]$

Shortcut:

⊛ $\frac{dy}{dx} + P(x)\,y = Q(x)$ then,

① $I.F = e^{\int P(x)dx}$

② Solⁿ: $y \cdot e^{\int Pdx} = \int Q\, e^{\int Pdx} dx + C$

$$\boxed{y\, I.F = \int Q \cdot I.F\, dx + C}$$

$\frac{dx}{dy} + P(y)x = Q(y)$ then,

① $I.F = e^{\int P(y)dy}$.

② Sol^n: $x e^{\int P(y)dy} = \int Q \cdot e^{\int P(y)dy} dy + C$

$$\boxed{x \cdot I.F = \int Q \cdot I.F \, dy + C}$$

Examples:

① $y' = 4y + 2x - 4x^2$

Sol: $\frac{dy}{dx} = 4y + 2x - 4x^2$

$\frac{dy}{dx} - 4y = 2x - 4x^2$

$I.F = e^{\int P(x)dx} = e^{\int -4dx} = e^{-4x}$

$\Rightarrow y \cdot e^{-4x} = \int (2x - 4x^2) e^{-4x} dx + C$

$\Rightarrow y e^{-4x} = 2\left\{x \frac{e^{-4x}}{-4} - 1 \frac{e^{-4x}}{(-4)^2}\right\}$

$-4\left\{x^2 \frac{e^{-4x}}{-4} - 2x \frac{e^{-4x}}{(-4)^2} + 2 \frac{e^{-4x}}{(-4)^3}\right\} + C$

② $\frac{dy}{dx} = \frac{1}{1+x^2}(e^{\tan^{-1}x} - y)$

Sol: $\frac{dy}{dx} + \frac{y}{1+x^2} = \frac{e^{\tan^{-1}x}}{1+x^2}$

$I.F = e^{\int \frac{1}{1+x^2}dx} = e^{\tan^{-1}x}$

$\Rightarrow y e^{\tan^{-1}x} = \int \frac{e^{\tan^{-1}x}}{1+x^2} e^{\tan^{-1}x} dx + C$

$\Rightarrow y e^{\tan^{-1}x} = \frac{(e^{\tan^{-1}x})^2}{2} + C$

③ $y^2 dx - (3xy-1)dy = 0$

Sol: $\frac{dy}{dx} = \frac{y^2}{3xy-1}$, $\underbrace{\frac{dx}{dy}}_{easy} = \frac{3xy-1}{y^2} = \frac{3x}{y} - \frac{1}{y^2}$

$\Rightarrow \frac{dx}{dy} - \frac{3}{y}x = -\frac{1}{y^2}$

$I.F = e^{\int -\frac{3}{y}dy} = e^{-3\ln y} = \frac{1}{y^3}$

$x \frac{1}{y^3} = \int -\frac{1}{y^2} \frac{1}{y^3} dy + c$

$\frac{x}{y^3} = -\frac{y^{-5+1}}{-5+1} + c = \frac{y^{-4}}{4} + c$

$\therefore \frac{x}{y^3} = \frac{1}{4y^4} + c$

④ $\frac{dy}{dx} + \frac{y\log y}{x - \log y} = 0$

Sol: $\frac{dy}{dx} = -\frac{y\log y}{x-\log y} \Rightarrow \frac{dx}{dy} = \frac{\log y - x}{y\log y}$

$\Rightarrow \frac{dx}{dy} + \frac{1}{y\log y}x = \frac{1}{y}$

$I.F = e^{\int \frac{1}{y\log y}} = e^{\log|\log y| + c} = \log y + c$

$x \log y = \int \frac{1}{y}\log y \, dy + c$

$$\boxed{x \log y = \frac{1}{2}(\ln y)^2 + c}$$

(5) $(1-4y)dx - (x^2+1)\,dy = 0,\ y(2)=1$

Sol: $(1-4y)dx = (x^2+1)dy$

$$\frac{dy}{dx} = \frac{1-4y}{x^2+1} = \frac{1}{x^2+1} - \frac{4y}{x^2+1}$$

$$\frac{dy}{dx} + \frac{4}{x^2+1}y = \frac{1}{x^2+1}$$

$$I.F = e^{\int \frac{4}{x^2+1}} = e^{4\tan^{-1}x}$$

$$\Rightarrow y e^{4\tan^{-1}x} = \int \frac{1}{x^2+1} e^{4\tan^{-1}x} + C$$

$\Longrightarrow$ Bernoulli eqn:

* $\frac{dy}{dx} + P(x)y = Q(x)y^a \quad \Big| \quad \frac{dx}{dy} + P(y)x = Q(y)x^a$

which is non linear for any value of real no. 'a' (except for $a=0$ & 1)

For $a=0$ Reduces to linear first order D.E

$a=1$ Reduces to linear seperable D.E

(i.e) $\frac{dy}{dx} + P(x)y = Q(x)y$

$$\Rightarrow \frac{dy}{dx} + \{P(x)-Q(x)\}y = 0$$

Other than 0 or 1 the non linear first order D.E can be reduced to linear D.E by the Substitution: $\boxed{z = y^{1-a}}$

$$\Rightarrow \frac{dy}{dx} + P(x)y = Q(x)y^a$$

$$\frac{1}{y^a}\frac{dy}{dx} + P(x)\frac{1}{y^{a-1}} = Q(x) \quad \div \text{by } y^a$$

Put $\boxed{t = y^{1-a}}$ (i.e) $t = \frac{1}{y^{a-1}}$

$$\frac{dt}{dx} = (1-a)y^{1-a-1}\frac{dy}{dx}$$

$$\boxed{\frac{1}{1-a}\frac{dt}{dx} = y^{-a}\frac{dy}{dx}}$$

$$\rightarrow \frac{dt}{dx}\frac{1}{1-a} + P(x)t = Q(x)$$

$$\boxed{\frac{dt}{dx} + (1-a)P(x)t = (1-a)Q(x)}$$

Linear

· Examples:

(1) · $3y' + xy = xy^{-2}$

Sol: $\frac{dy}{dx} + \frac{x}{3}y = \frac{x}{3}y^{-2}$

$$\Rightarrow \frac{1}{y^{-2}}\frac{dy}{dx} + \frac{x}{3}\frac{1}{y^{-3}} = \frac{x}{3} \quad \div \text{by } y^{-2}$$

Put $t = \frac{1}{y^{-3}}$

$$\frac{dt}{dx} = 3\frac{1}{y^{-2}}\frac{dy}{dx}$$

$$\frac{1}{3}\frac{dt}{dx} + \frac{x}{3}t = \frac{x}{3}$$

$$\boxed{\frac{dt}{dx} + xt = x}$$

Linear

$$I.F = e^{\int x\,dx} = e^{x^2/2}$$

$$te^{x^2/2} = \int xe^{x^2/2}dx + C$$

Put $u = x^2/2$

$du = x\,dx$

$\int xe^{x^2/2} = \int e^u du$

$$te^{x^2/2} = e^u + C$$

$$\Rightarrow \boxed{y^3e^{x^2/2} = e^{x^2/2} + C}$$

(2) $\cos x\, dy = y(\sin x - y)\, dx$

Sol: $\frac{dy}{dx} = \frac{y(\sin x - y)}{\cos x}$

$\frac{dy}{dx} = \frac{y \sin x}{\cos x} - \frac{y^2}{\cos x}$

$\frac{dy}{dx} - \tan x\, y = -\frac{1}{\cos x} y^2$

$\div y^2$ $\quad \frac{1}{y^2}\frac{dy}{dx} - \frac{\tan x}{y} = -\frac{1}{\cos x}$

Put $t = \frac{1}{y}$

$\frac{dt}{dx} = -1 \cdot y^{-2}\frac{dy}{dx}$

$\frac{1}{y^2}\frac{dy}{dx} = -\frac{dt}{dx}$

$$\boxed{\frac{dt}{dx} + \tan x\, t = \frac{1}{\cos x}}$$

Linear

$I.F = e^{\int \tan x\, dx} = e^{-\log|\cos x|} = \frac{1}{\cos x}$

$\Rightarrow \frac{1}{\cos x} t = \int \frac{1}{\cos x} \frac{1}{\cos x} + C$

$t \sec x = \int \sec^2 x + C$

$\therefore \boxed{t \sec x = \tan x + C}$ (or) $\boxed{t \sec x = -\tan x + C}$

4/01/21

(3) $2xyy' = y^2 - 2x^3,\ y(1) = 2$

Sol: $2xy\frac{dy}{dx} = y^2 - 2x^3$

$\frac{dy}{dx} = \frac{y^2 - 2x^3}{2xy} = \frac{y}{2x} - \frac{x^2}{y}$

$\frac{dy}{dx} - \frac{1}{2x} y = -\frac{x^2}{y}$

$\times y$ $\quad y\frac{dy}{dx} - \frac{y^2}{2x} = -x^2$

$$\frac{1}{2}\frac{dt}{dx} - \frac{t}{2x} = -x^2$$

$$\Rightarrow \frac{dt}{dx} - \frac{t}{x} = -2x^2$$

Put $t = y^2$

$\frac{dt}{dx} = 2y\frac{dy}{dx}$

$y\frac{dy}{dx} = \frac{1}{2}\frac{dt}{dx}$

$$I.F = e^{-\int \frac{1}{x}} = e^{-\log x} = \frac{1}{x}$$

$$\Rightarrow \frac{1}{x}t = \int -2x^2 \cdot \frac{1}{x}\, dx + C$$

$$\Rightarrow \boxed{\frac{y^2}{x} = -2\frac{x^2}{2} + C}$$

$y(1) = 2 \Rightarrow x = 1, y = 2 \Rightarrow 4 = -1 + C \Rightarrow \boxed{C = 5}$

$$\therefore \boxed{\frac{y^2}{x} = -2\frac{x^2}{2} + 5}$$

(4) $(xy^5 + y)dx - dy = 0$

Sol: $\frac{dy}{dx} = xy^5 + y \Rightarrow \frac{dy}{dx} - y = xy^5$

$$\div y^5 \Rightarrow \frac{1}{y^5}\frac{dy}{dx} - \frac{1}{y^4} = x$$

Put $t = \frac{1}{y^4}$

$\frac{dt}{dx} = -4\frac{1}{y^5}\frac{dy}{dx}$

$$\Rightarrow -\frac{1}{4}\frac{dt}{dx} - t = x$$

$$\Rightarrow \frac{dt}{dx} + 4t = 4x$$

$$I.F = e^{\int 4 dx} = e^{4x}$$

$$\Rightarrow e^{4x}t = \int e^{4x} \cdot 4x \cdot dx + C$$

(5) $y' - 2\cos x \cot y + \sin^2 x \csc y \cos x = 0$

Sol: $\frac{dy}{dx} - 2\cos x \cot y = -\sin^2 x \csc y \cos x$

$\frac{dy}{dx} + Py = Qy^n$

$\div \csc y$ $\frac{1}{\csc y}\frac{dy}{dx} - 2\cos x \frac{\cos y}{\sin y} \sin y = -\sin^2 x \cos x$

$\Rightarrow \frac{dt}{dx} + 2\cos x . t = -\sin^2 x \cos x$

$t = -\cos y$

$\frac{dt}{dx} = \sin y \frac{dy}{dx}$

I.F $= e^{\int 2\cos x dx} = e^{2\sin x}$

$\Rightarrow t . e^{2\sin x} = \int -\sin^2 x \cos x \, e^{2\sin x} dx + C$

$\int e^{2\sin x} \sin^2 x \cos x dx$

$\int e^z \frac{z^2}{4} \frac{dz}{2}$ $z = 2\sin x$, $dz = 2\cos x dx$

$\frac{1}{8}[e^z z^2 - e^z 2z + e^z . 2]$

$\Rightarrow t . e^{2\sin x} = -\frac{1}{8}\{e^{2\sin x}(2\sin x)^2 - e^{2\sin x} 4\sin x + e^{2\sin x} . 2\} + C$

(6) $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$

Sol: $\div \cos y$ $\sec y \tan y \frac{dy}{dx} + \sec y \tan x = \cos^2 x$

$t = \sec y$

$\frac{dt}{dx} = \sec y \tan y \frac{dy}{dx}$

$\frac{dt}{dx} + \cancel{\sec y}\, t \tan x = \cos^2 x$

I.F $= e^{\int \tan x} = e^{\log|\sec x|} = \sec x$

$\sec x \, t = \int \cos^2 x \sec x \, dx + C$

$\boxed{\sec x \sec y = \sin x + C}$

Final all methods problems:

(1) $y\,dx + (x + x^3y^2)\,dy = 0$

Sol: $\frac{dy}{dx} = \frac{y}{-(x+x^3y^2)} \Rightarrow \frac{dx}{dy} = \frac{-x-x^3y^3}{y} = \frac{-x}{y} - x^3y^2$

$\Rightarrow \frac{dx}{dy} + \frac{1}{y}x = -x^3y^2$

$\div x^3 \Rightarrow \frac{1}{x^3}\frac{dx}{dy} + \frac{1}{yx^2} = -y^2$

$t = \frac{1}{x^2}$

$\frac{dt}{dy} = -2 \cdot \frac{1}{x^3}\frac{dx}{dy}$

$\Rightarrow -\frac{1}{2}\frac{dt}{dy} + t\frac{1}{y} = -y^2$

$\Rightarrow \frac{dt}{dy} - \frac{2t}{y} = 2y^2$

$I.F = e^{-\int \frac{2}{y}} = e^{-2\log|y|} = \frac{1}{y^2}$

$\Rightarrow \frac{1}{y^2}t = \int 2y^2 \frac{1}{\cancel{y^2}} + c$

$\Rightarrow \boxed{\frac{1}{x^2y^2} = \frac{2}{y} + c}$

(2) $(1 + 3x\sin y)\,dx - x^2\cos y\,dy = 0$

Sol: $\frac{dy}{dx} = \frac{1+3x\sin y}{x^2\cos y} = \frac{1}{x^2\cos y} + \frac{3x\sin y}{x^2\cos y}$

$\div \cos y$ $\quad \cos y\frac{dy}{dx} = \frac{1}{x^2} + \frac{3\sin y}{x}$

$t = \sin y$

$\frac{dt}{dx} = \cos y\frac{dy}{dx}$

$\frac{dt}{dx} = \frac{1}{x^2} + \frac{3t}{x}$

$I.F = e^{\int \frac{-3}{x}} = e^{-3\log|x|} = \frac{1}{x^3}$

$\frac{1}{x^3}t = \int \frac{1}{x^2}\frac{1}{x^3} + c$

$\therefore \boxed{\frac{1}{x^3}\sin y = -\frac{1}{4x^4} + c}$

③ $\frac{1}{y}\, dy \tan x + (x\tan x - \log y)dx = 0$

Sol: $\dfrac{dy}{dx} = \dfrac{\log y - x\tan x}{\frac{\tan x}{y}} = \dfrac{y\log y}{\tan x} - \dfrac{xy\cancel{\tan x}}{\cancel{\tan x}}$

$\left(\div y\right) \Rightarrow \dfrac{1}{y}\dfrac{dy}{dx} - \dfrac{\log y}{\tan x} = -x$

$\dfrac{dt}{dx} - \dfrac{1}{\tan x}t = -x$ | $t = \log y$, $\dfrac{dt}{dx} = \dfrac{1}{y}\dfrac{dy}{dx}$

$I.F = e^{-\int \cot x} = e^{-\log \sin x} = \csc x$

$\csc x \log y = \int -x\csc x + C$

$$\boxed{\csc x \log y = -\left\{\frac{x^2}{2}\csc x + x\log|\csc x - \cot x|\right\}}$$

④ $\dfrac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

Sol: $\left(\div \cos^2 y\right)$ $\dfrac{1}{\cos^2 y}\dfrac{dy}{dx} + \dfrac{2x\sin y}{\cos y} = x^3$

$\dfrac{dt}{dx} + 2xt = x^3$ | $t = \tan y$, $\dfrac{dt}{dx} = \sec^2 x\dfrac{dy}{dx}$

$I.F = e^{2\int x} = e^{x^2}$

$e^{x^2}\tan y = \int x^3 e^{x^2} + c$

$e^{x^2}\tan y = x^3\dfrac{e^{x^2}}{2x} + e^{x^2}\dfrac{x^4}{4} + C$

(5) $\frac{dz}{dx} + \frac{z}{x}\log z = \frac{z}{x}(\log z)^2$.

Sol:

$$\frac{1}{z(\log z)^2}\frac{dz}{dx} + \frac{1}{x\log z} = \frac{1}{x}$$

$$t = \frac{1}{\log z}$$

$$\frac{dt}{dx} = \frac{-1}{(\log z)^2}\frac{1}{z}\frac{dz}{dx}$$

$$\frac{dt}{dx} - \frac{1}{x}t = -\frac{1}{x}.$$

$$I.F = e^{-\int \frac{1}{x}} = e^{-\log x} = \frac{1}{x}.$$

$$\frac{1}{x}\frac{1}{\log z} = \int \frac{1}{x}\frac{1}{x} + c.$$

$$\boxed{\frac{1}{x}\frac{1}{\log z} = -\frac{1}{x} + C}$$

(6) $2(x^2y^2 + x + y^4)dy + (xy^3 + y)dx = 0$

Sol: $\frac{\partial M}{\partial y} = 3xy^2 + 1$, $\frac{\partial N}{\partial x} = 4xy^2 + 2 \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -xy^2 - 1 \Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{-xy^2 - 1}{y(xy^2+1)} = \frac{-1}{y}$$

$$I.F = e^{-\int g(y)dy} = e^{+\int \frac{1}{y}dy} = y.$$

$\times I.F \Rightarrow 2(x^2y^3 + xy + y^5)dy + (xy^4 + y^2)dx = 0$

$$\Rightarrow \int (xy^4 + y^2)dx + \int 2y^5\, dy = C$$

$$\Rightarrow \boxed{\frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = C}$$

(7) $y(x^3e^{xy}-y)dx + x(xy+x^3e^{xy})dy = 0$

Sol: $x^3e^{xy}(ydx+xdy) + y(xdy-ydx)=0$

$x^3 d(e^{xy}) + y\cdot x^2 \frac{(xdy-ydx)}{x^2} = 0$

$\Rightarrow d(e^{xy}) + \frac{y}{x} d\left(\frac{y}{x}\right)=0$

$\Rightarrow \boxed{e^{xy} + \frac{1}{2}\left(\frac{y}{x}\right)^2 = C}$

(8) If a substance cools from 370K to 330K in 10 minutes, when the temperature of the surrounding air is 290K, then the temperature of the substance after 40 minutes is → (or)

Solve DE $\frac{dT}{dt} = \alpha(T_0-T)$ & hence find $T(40)$.

where $T(0)=370$, $T_0=290$ & $T(10)=330$.

Sol: Newton's law cooling, $\frac{dT}{dt} \propto (T_0-T)$

$\Rightarrow \frac{dT}{dt} = \alpha(T_0-T)$

$T(0)=370$, $T(10)=330$, T_0 = surrounding temp = 290

$\Rightarrow \frac{dT}{dt} = \alpha T_0 - \alpha T$ → linear.

$\Rightarrow \frac{dT}{dt} + \alpha T = \alpha T_0$

$I.F = e^{\int \alpha dt} = e^{\alpha t}$

$Te^{\alpha t} = \int e^{\alpha t} \alpha T_0 + C$

$$T(t)e^{\lambda t} = k T_0 \frac{e^{\lambda t}}{\lambda} + c$$

$$\cancel{T(t)e^{\lambda t} = T_0 + c}$$

$$\boxed{T(t) = T_0 + ce^{-\lambda t}}$$

$\Rightarrow T(0) = 370 = 290 + ce^{-\lambda 0} = 290 + c$

$\therefore \boxed{c = 80}$

$\Rightarrow T(10) = 330 = 290 + 80e^{-\lambda 10}$

$\therefore \boxed{\lambda = 0.069314718}$

$\Rightarrow T(40) = 290 + 80e^{-0.069314718 \times 40}$

$\therefore \boxed{T(40) = 295K}$

$\Rightarrow \boxed{T(t) = 290 + 80e^{-0.069314718t}}$

(9) In a certain house, a police were called about 3:45 AM where a murder victim was found. Police took the temp of body which was found to be 34.5°C. After 1hr, police again took the temp of the body which was found to be 33.9°C. The Temp of room was 15°C. So, what's murder time?

Sol: $\frac{d\phi}{dt} \propto (\phi - 15) \Rightarrow \frac{d\phi}{dt} = K(\phi - 15)$

$\Rightarrow \int \frac{d\phi}{(\phi - 15)} = \int K dt$ [variable separable form]

$\boxed{\ln(\phi - 15) = Kt + C}$

$T(0) = 34.8°C$

$T(1) = 33.9°C$

T_0 - Surrounding $= 15°C$

$\Rightarrow \ln(34.8-15) = K(0) + C$

$\Rightarrow \therefore \boxed{C = \ln 19.5}$

$\Rightarrow \ln(33.9-15) = K(1) + \ln 19.5$

$\therefore \boxed{K = -0.032}$

$\Rightarrow \therefore \boxed{\ln(\phi - 15) = -0.032t + \ln 19.5}$

when the person is alive, person will have normal body temperature (i.e) $\phi = 37°C$

$\Rightarrow \ln(37-15) = -0.032t + \ln 19.5$

$\therefore \boxed{t = -3.86 \text{ hrs} = -3 \text{ hrs } 51 \text{ min}}$

That means at 3:45 Am, before 3hrs 51min person died

$\Rightarrow$ 3:45Am − 3hrs 51min = $\boxed{11:54 \text{ PM}}$ before day

Practice problems of Homogeneous method:

① $(y^3 - 3x^2y)dx - (x^3 - 3xy^2)dy = 0$

② $y - x\frac{dy}{dx} = x + y\frac{dy}{dx}$

③ $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$

④ $y\,dy + \sin^2\left(\frac{x}{y}\right)\{x\,dy - y\,dx\} = 0$

⑤ $\frac{dy}{dx} = \frac{y}{x - \sqrt{xy}}$