Chapter 2 : Lattice Theory

Definition: Let A and B be 2 sets. The Cartesian product or cross product or direct product of A and B is the set

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

In other words, Cartesian product of A and B is denoted by $A \times B$, is set of all ordered pairs of the form (a, b) where $a \in A$ and $b \in B$.

Eg: $A = \{a, b\}$ $\quad B = \{1, 2, 3\}$

$A \times B = \{(a\ 1)(a\ 2)(a\ 3)(b\ 1)(b\ 2)(b\ 3)\}$

$B \times A = \{(1\ a)(1\ b)(2\ a)(2\ b)(3\ a)(3\ b)\}$

$A \times A = \{(a\ a)(a\ b)(b\ a)(b\ b)\}$

In general $A \times B \neq B \times A$.

Relation on a set :

A binary relation 'R' from a set A into a set B is a subset of $A \times B$ i.e., $R \subseteq A \times B$.

$$R = \{(a, b) \mid a \in A \text{ and } b \in B\} \subseteq A \times B$$

Eg: $A = \{a, b\}$ $\quad B = \{1, 2, 3\}$

$R_1 = \{(a\ 1)(b\ 1)\}$

$R_2 = \{(a\ 1)(a\ 2)(a\ 3)\}$

Relations from A to B

Note: If order of A i.e., $n(A) = m$ and $n(B) = n$, then $n(A \times B) = mn$.

We can define define 2^{mn} number of relations.

A relation R on a set A is a subset of $A \times A$.

Types of Relation:

1. Reflexive Relation: A relation R is said to be a reflexive relation, if $(a,a) \in R$ or aRa for every $a \in A$.

2. Irreflexive Relation: If for all $a \in A$, atleast one 'a' exists, such that $(a,a) \notin R$, then R is irreflexive. We denote it by $a \not R a$ for $a \in A$.

Examples

1. Let A be a set of positive integers. Let us define a relation R on A such that (a,b) is in R if and only if a divides b.

Since an integer always divides itself, R is a Reflexive relation.

$A \times A = \{ \cdot \cdot \quad \}$

$A = \{1, 2, 3, \cdot \cdot\}$

a divides b

$a | b \Rightarrow b = ka$

$2|4$

$4 = 2 \times 2$

$R = \{(1,1)(2,2),(2,4), \cdots\}$

2) Consider the set of all straight lines in a plane and define a relation R "is parallel to."
This relation is Reflexive. $(A \parallel A)$

Consider the relation "is perpendicular to".
As no straight line is $\perp^r$ to itself, the relation "is $\perp^r$ to" is irreflexive. $(A \perp A \times)$

3) The relation "$\leq$" is reflexive in the set of real numbers.

$2 \leq 2$ ✓
$0.123 \leq 0.123$ ✓

3. Symmetric Relation: A relation R is said to be a symmetric relation if $(a, b) \in R$ implies that (b, a) is also in R, where $a, b \in A$, i.e., if $(a, b) \in R \Rightarrow (b, a) \in R$
$aRb \Rightarrow bRa$

$$(a, b) \in R \Longrightarrow (b, a) \in R$$

$$\text{or} \quad aRb \Longrightarrow bRa$$

$A \parallel B$ $\quad A \perp^r B$
$B \parallel A$ $\quad B \perp^r A$

Example: The relation is parallel to and is perpendicular to are symmetric relation.

4. Transitive Relation: A relation R is said to be Transitive Relation on A if $(a, c) \in R$ when both (a, b) and (b, c) are in R, where $a, b, c \in A$ i.e., aRb and $bRc \Longrightarrow aRc$.

Example :

The relation "is parallel to" on the set of st. lines is Transitive.

$A \parallel B$ & $B \parallel C \implies A \parallel C$.

The relation "is $\perp^r$ to" is not transitive.

$A \perp B$ & $B \perp C$

$\not\Rightarrow A \perp C$

5. Equivalence Relation :

Let A be a nonempty set. Let R be a relation on A. The relation R is said to be an Equivalence relation if it is reflexive, symmetric and transitive.

aRb & bRa

aRa $\quad 2|2$

$2|4$ & $4|2$

$\implies$

6. Anti Symmetric Relation :

Let R be a relation on A. If both (a,b) & (b,a) are in R, then $a=b$.

i.e., aRb, $bRa \implies a=b$.

Example : The relation "is a subset of" is antisymmetric.

$A \subseteq B$ & $B \subseteq A \implies A = B$.

The relation "a divides b" is antisymmetric.

i.e if $a|b$ and $b|a \implies a=b$.

(b,b)

$R_1 : \{ (a\ a)\ (a,\ b)\ (b\ a) \}$

symmetric

is reflexive $\quad (b, b)$

Partial Ordering Relation

A relation is said to be a partial ordering relation if it is reflexive, antisymmetric and transitive.

Eg: 1) Let A be a set of $+^{ve}$ integers and let R be a relation on A such that $(a,b) \in R$ if a divides b.

Since any integer divides itself, R is reflexive.
If a divides b means b doesnot divides a unless $a=b$. Hence R is antisymmetric.
If a divides b and b divides c, then a divides c. Therefore R is transitive.
Hence, R is a partial ordered relation.

2) On the set of $+^{ve}$ integers, the relation "is a multiple of" is a partial ordering relation.

Definition: A nonempty set 'A' with a partial ordering relation R on A is a partial ordered set (<u>POSET</u>) and is denoted by $\langle A, R \rangle$.

Example: $\leq$, $\geq$, inclusion.

<u>Note:</u> If $(a,b) \in R$, we write $a \leq b$ and we call $\langle A, \leq \rangle$ is a POSET.

$2 \leq 3$ $\quad$ $2 \leq 3$

<u>Examples:</u>

1) Consider the relation "is parallel to" on a set of straight lines in a plane.

This relation is reflexive, symmetric, transitive.

But this relation is not antisymmetric.

2) The relation "is $\perp^r$ to" is only symmetric.

3) The relation "is less than or equal to" on a set of real numbers.

This relation is reflexive, antisymmetric and transitive. Not symmetric. Hence it is a Partial ordering relation.

4) Consider the relation " is a subset of " on P(A) (Power set of a nonempty set A).

Power set - Set of all subsets

$A = \{a, b, c\}$

$\{\phi\}$ $\{a\}$ $\{b\}$ $\{c\}$ $\{a b\}$ $\{a c\}$ $\{b c\}$ $\{a b c\}$

This relation is reflexive, Transitive, antisymmetric.

~~$A \subseteq B$ $B \subseteq A$~~

$$A_1 \subseteq B_1 \ \& \ B_1 \subseteq A_1 \Rightarrow A_1 = B_1$$

5) Consider the relation " divides ".

Let $a, b \in N = \{1, 2, \cdots\}$

we say a divides b if there is a $k \in N$, such that $b = ka$ & we denote by $a|b$. Show that the " divides " relation is partial ordering.

(i) To show reflexive $\Rightarrow$ $a \in N$, $a = 1 . a \Rightarrow a|a$

(ii) To show antisymmetric $\Rightarrow$ suppose $a, b \in N$

$$a|b \ \& \ b|a$$

$$\Rightarrow b = k_1 a \ \& \ a = k_2 b$$

$$b = k_1 (k_2 b) = (k_1 k_2) b$$

$$\Rightarrow k_1 k_2 = 1 \Rightarrow k_1 = k_2 = 1$$

$$\Rightarrow b = a$$

(iii) Transitive $\Rightarrow$ $a|b$ & $b|c \Rightarrow b = k_1 a$ and $c = k_2 b$

$c = k_2 k_1 a \Rightarrow$ $c = ka \Rightarrow a|c$ Hence Partial ordering

Prove that $\leq$ is a partial ordering relation on R.

<u>Soln</u>: Let $a \in R$.

(i) $a \leq a \Rightarrow aRa \Rightarrow$ Reflexive

(ii) Suppose $a \leq b$ & $b \leq a$

Together it implies if $a = b$

$\Rightarrow$ antisymmetric

(iii) Suppose $a \leq b$ & $b \leq c$

$a \leq b \leq c \Longrightarrow a \leq c$

$\longrightarrow$ Transitive.

$\Longrightarrow$ This relation is partial ordering.

<u>Note</u>:

$\langle A, \leq \rangle$ POSET

<u>Note</u>: A set 'A' with a partial ordered relation such as $\leq, \geq, \subseteq, /$ are represented by $\langle A, \leq \rangle$.

Let Z be a set of integers and n be a fixed integer. Define a relation "congruent modulo n" denoted by $\cong$ as follows.

For all $(a, b) \in Z$, $a \cong b \pmod{n}$ if and only if n divides $a-b$ i.e., $n \mid a-b$.

Show that the above relation is an equivalence relation.

Soln: To show reflexive. We know $n \mid 0$

$$\Rightarrow n \mid a-a$$

$$\Rightarrow a \cong a \pmod{n}$$

To show symmetric: If $a \cong b \pmod{n}$ $\Rightarrow n \mid a-b \Rightarrow (a-b) = kn$

$$b-a = (-k)n$$

$$\Rightarrow b \cong a \pmod{n}$$

To show transitive..

$a \cong b \pmod{n}$ and $b \cong c \pmod{n}$

$b-a = k_1 n$ and $c-b = k_2 n$

$$c-a = (k_1 + k_2)n = kn$$

$$a \cong c \pmod{n}$$

$\Rightarrow$ Transitive

$\Rightarrow$ Equivalence.

Comparable Elements :

Let $<A, \leq>$ be a POSET. Two elements a and b are said to be comparable if either $a \leq b$ or $b \leq a$.

Two elements are said to be noncomparable, if they are not comparable.

Eg : $A : \{2, 3, 6, 9, 10\}$ R : is a multiple of.

$\{(2,10)(2,6) \dots\}$

2 and 3 are non comparable.

3 and 6 are comparable.

$A : \{2, 5, 7\} \rightarrow$ Anti Chain

$A : \{2, 4, 8, 16\} \rightarrow$ Chain

Total ordering :

If every 2 elements of a set are comparable, then the relation is called total ordering or total ordered relation.

A totally ordered set is called Chain.

If no two distinct elements are comparable in a set then the set is called Antichain.

Eg : 1) $(\mathbb{Z}, \leq) \rightarrow$ is totally ordered set or Chain.

$a, b \in \mathbb{Z}$ $\quad a \leq b$ or $b \leq a$

2) $(\mathbb{Z}, \mid)$ is not totally ordered relation. Not chain. Not Antichain

$2 \nmid 3$ $\quad a \mid b$ $\quad 2 \nmid 7$

$2 \mid 6$ ✓

$3 \mid 9$ ✓

Cover of an element: Let $\langle A, \leq \rangle$ be a POSET

An element $b \in A$ is said to cover an element $a \in A$, if $a \leq b$ and there is no element $c \in A$, such that

$$a \leq c \leq b$$

Eg: $(N, /)$

4 and 2: Here $2|4$. $\therefore$ 4 covers 2. There is no $c \in N$ s.t $2|c$ and $c|4$.

8 and 2. Here $2|8$. But we have $2|4$ and $4|8$ $\therefore$ 8 doesnot cover 2.