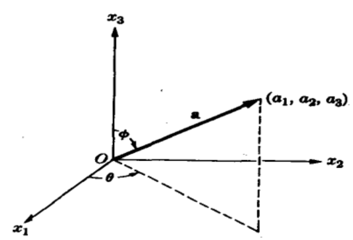
Linear Algebra

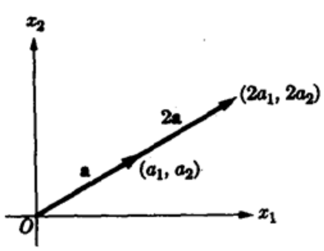
Introduction to generalization of vector concept:

Vectors are generally used in many areas of physical and engineering sciences. The need for a vector concept arose very naturally in machines. The force on a body cannot in general be completely described by a single number. Force has two properties, magnitude and direction, and so requires more than a single number of its description. We learn a concept of vector and generalization of vector in higher dimensions. Vector is a physical quantity that has both direction and magnitude (For example: Force). Mathematically vector is referred as directed line segment.

The following diagram shows the vector in three dimensions.

Consider OA as in figure. It has magnitude





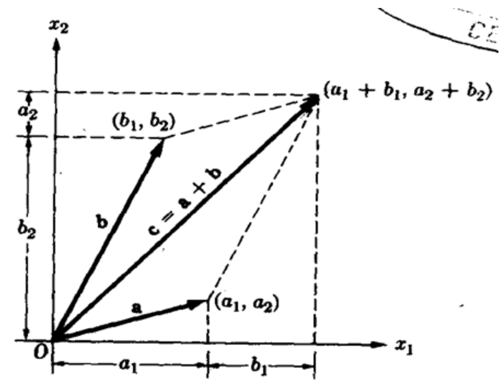
Scalar multiplication is one of the basic operations defining a vector space in linear algebra (or more generally, a module in abstract algebra). Note that scalar multiplication is different from scalar product which is an inner product between two vectors. The scalar multiplication is defined as $\mathbf{a} = (a_1, a_2)$ be any vector and α be any scalar, then $\alpha\mathbf{a} = (\alpha a_1, \alpha a_2)$.

Example: If $\mathbf{a} = (2, 3)$, $6\mathbf{a} = (12, 18)$. If $\mathbf{a} = (1, -1)$, then $-4\mathbf{a} = (-4, 4)$.

Graphical Illustration:

Consider the mechanics for our intuitive foundations. It is well known that if two forces act on a particle (a proton, for example) to produce some resultant motion, the same motion can be produced by applying a single force. This single force can, in a real sense, be considered to be the stun of the original two forces. The rule which we use in obtaining magnitude and direction of a single force which replaces the original two forces is rather interesting:

If **a**, **b** are the original forces, then the single force **c**, which we shall call the sum of **a**, **b**, is the diagonal of the parallelogram with sides **a**, **b**. This is illustrated in the following diagram.



a = (a_1, a_2), **b** = (b_1, b_2),

c = **a** + **b** = $(a_1 + b_1, a_2 + b_2) = (c_1, c_2)$.

To add three vectors, the first two are added to obtain the resultant, and the third is then added to the resultant of the first two.

We recollect that for a non-empty set G, a binary operation on G is mapping from $G \times G$ into G.

Definition: A non-empty set G together with a binary operation * is called a **group** if the algebraic system (G, *) satisfies the following four axioms:

(i) <u>Associative axiom:</u> (a*b)*c = a*(b*c) for all elements a, b, c $\in$ G.

(ii) <u>Identity axiom:</u> There exists an element e $\in$ G such that a*e = e*a = a for all elements a $\in$ G.

(iii) <u>Inverse axiom:</u> To each element a $\in$ G there corresponds an element b $\in$ G such that a*b = e = b*a.

Moreover G is said to be **abelian** if a * b = b * a for all a, b $\in$ G (similarly, G is commutative if the multiplication a $\cdot$ b = b $\cdot$ a for all a, b $\in$ G).

Examples: (i) The set of all integers with respect to addition is an abelian group.

(ii) The set of all 2×2 matrices with real or complex entries is a group with respect to matrix addition.

Definition: A non-empty set F is said to be a **field** if there exists two binary operations + and "." on F such that

(i) (F, +) is an abelian group

(ii) $(F \setminus \{0\}, .)$ is a multiplicative group and

(iii) for any a, b, c $\in$ F, we have $a(b + c) = ab + ac$, $(a + b)c = ac + bc$.

Examples: (i) Set of real numbers with usual addition and multiplication is a field.

(ii) The set of all complex numbers is also a field with respect to the addition and multiplication of complex numbers.

Vector Spaces

Introduction (Elementary Notations)

A **set** is a well-defined collection of objects in which we can say whether a given object is in the collection. The fact that a is a member of a set A is denoted by $a \in A$ and we call it as 'a belongs to A'. The members of a set are called **elements**.

A set is usually specified either by listing all of its elements inside a pair of braces or by stating the property that determines whether or not an object x belongs to the set. We might write $S = \{x_1, x_2, \ldots, x_n\}$.

Example: If E is the set of even positive integers, we describe E by writing either $E = \{2, 4, 6, \ldots\}$

Or $E = \{x \mid x \text{ is an even integer and } x > 0\}$.

We write $2 \in E$ when we want to say that 2 is in the set E, and $-3 \notin E$ to say that -3 is not in the set E.

Notations: Some of the more important set notations are given below:

$\mathbb{N}$: The set of all natural numbers = $\{n \mid n$ is a natural number$\}$ = $\{1, 2, 3, \ldots\}$;
$\mathbb{Z}$: The set of all integers = $\{x \mid x$ is an integer$\}$ = $\{\ldots, -1, 0, 1, 2, \ldots\}$;
$\mathbb{Q}$: The set of all rational numbers = $\{p/q \mid p, q \in \mathbb{Z}$ where $q \neq 0\}$;
$\mathbb{R}$: The set of all real numbers = $\{x \mid x$ is a real number$\}$;
$\mathbb{C}$: The set of all complex numbers = $\{z \mid z$ is a complex number$\}$.

Definition: If x is not an element of A then we write $x \notin A$. Suppose A and B are two sets. We say that A is a **subset** of B (written as $A \subseteq B$) if every element of A is also an element of B. Two sets A and B are said to be **equal** (denoted by $A = B$) if A is a subset of B, and B is a subset of A. A set B is a proper subset of A if $B \subset A$ (that is, B is a subset of A, but not equal to A). Trivially, every set is a subset of it self. A set which contains no elements at all is called the **Null set** (denoted by ϕ).

For example, $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

Example: Consider the sets $A = \{x \mid x$ is an even positive integer$\}$ and $B = \{x \mid x$ is a positive integer$\}$. Then $A \subseteq B$.

Operations on Sets:

(i) If A and B are two sets, then the set $\{x \,/\, x \in A$ or $x \in B\}$ is denoted by $A \cup B$ and we call it as the **union** of A and B.

(ii) The set $\{x \,/\, x \in A$ and $x \in B\}$ is denoted by $A \cap B$ and we call it as the **intersection** of A and B.

(iii) If A and B are two sets, then the set $\{x \in B \,/\, x \notin A\}$ is denoted by $B - A$ (or $B \setminus A$) and it is called as the **complement** A in B.

(iv) The set that contains no members is called the **empty set** and it is denoted by ϕ. Empty set is a subset of every set.

Example: Let $\mathbb{R}$ be the universal set and suppose that $A = \{x \in \mathbb{R} \mid 0 < x \leq 3\}$ and $B = \{x \in \mathbb{R} \mid 2 \leq x < 4\}$. Then
$A \cap B = \{x \in \mathbb{R} \mid 2 \leq x \leq 3\}$;
$A \cup B = \{x \in \mathbb{R} \mid 0 < x < 4\}$;
$A \setminus B = \{x \in \mathbb{R} \mid 0 < x < 2\}$;

$A^1 = \{x \in \mathbb{R} \mid x \le 0 \text{ or } x > 3\}$.

Note: The operations of union and intersection can be defined for three or more sets in the similar way.
$A \cup B \cup C = \{x / x \in A \text{ or } x \in B \text{ or } x \in C\}$ and
$A \cap B \cap C = \{x / x \in A, x \in B, x \in C\}$

In general, let A_i be a collection of sets – one for each element i belongs to I, where I is some set (I may be the set of all positive integers). We define
$\bigcap_{i \in I} A_i = \{a / a \in A_i \text{ for all } i \in I\}$, and
$\bigcup_{i \in I} A_i = \{a / a \in A_i \text{ for some } i \in I\}$.
A collection $\{A_i\}_{i \in I}$ of sets is said to be *mutually disjoint* if $A_i \cap A_j = \phi$ for all $i \in I$, $j \in I$ such that $i \neq j$.

Example:
(i) Write $A_i = \{i, i+1, i+2, \ldots\}$ for each $i \in N$, the set of natural numbers. Then it is easy to observe that $\bigcup_{i \in N} A_i = N$ and $\bigcap_{i \in I} A_i = \Phi$.
(ii) If $B_i = \{2i, 2i+1\}$ for all $i \in N$, then $\{B_i\}_{i \in N}$ is a collection of mutually disjoint sets.

Definition: Let A and B are two sets. We define their symmetric difference as the set $A \Delta B = (A-B) \cup (B-A)$. Sometimes it is denoted by $A \oplus B$.

Example: If $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 5, 7\}$, then $A \oplus B = \{3, 4, 5, 7\}$.

The set operations satisfy the following properties.
1. $A \cup B = B \cup A$; $A \cap B = B \cap A$ (commutative properties)
2. $A \cup (B \cup C) = (A \cup B) \cup C$; $A \cap (B \cap C) = (A \cap B) \cap C$ (Associative)
3. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive)
4. $A \cup A = A$; $A \cap A = A$ (Idempotent)
5. $(A')' = A$
6. $A \cup A' = U$(universal set)
7. $A \cap A' = \phi$

8. $\phi' = U$
9. $U' = \phi$
10. $(A \cup B)' = A' \cap B'$; $(A \cap B)' = A' \cup B'$ (D' Morgan laws)
11. $A \cup \Phi = A$; $A \cap \Phi = \Phi$; $A \cup U = U$; $A \cap U = A$ (Universal)

Note: Let $|A|$ denote the number of elements in the set A.
For any two sets P and Q, we have
(i). $|P \cup Q| \leq |P| + |Q|$ (ii). $|P \cap Q| \leq \min(|P|, |Q|)$
(iii). $|P \oplus Q| = |P| + |Q| - 2|P \cap Q|$ where $\oplus$ is the symmetric difference.

Definition: (i) If S and T are two sets, then the set $\{(s, t) / s \in S \text{ and } t \in T\}$ is called the **Cartesian product** of S and T
(here $(a, b) = (s, t) \Leftrightarrow a = s$ and $b = t$). The Cartesian product of S and T is denoted by $S \times T$.
Thus $S \times T = \{(s, t) / s \in S \text{ and } t \in T\}$.
Note that if S and T are two sets, then $S \times T$ and $T \times S$ may not be equal.
(ii) If $S_1, S_2, .., S_n$ are n sets, then the **Cartesian product** is defined as
$S_1 \times S_2 \times \ldots \times S_n = \{(s_1, s_2, \ldots, s_n) / s_i \in S_i \text{ for } 1 \leq i \leq n\}$.
Here the elements of $S_1 \times S_2 \times \ldots \times S_n$ are called **ordered** n-**tuples**. For any two n-tuples, we have $(s_1, s_2, \ldots, s_n) = (t_1, t_2, \ldots, t_n) \Leftrightarrow s_i = t_i$, $1 \leq i \leq n$.

Example: If $X = \{a, b\}$ and $Y = \{x, y\}$, then
$X \times Y = \{(a, x), (a, y), (b, x), (b, y)\}$ and $Y \times X = \{(x, a) (x, b), (y, a), (y, b)\}$.
Note that $X \times Y \neq Y \times X$.

Definition: Let S and T be sets. A **function** f from S to T is a subset f of $S \times T$ such that

(i) for $s \in S$, there exists $t \in T$ with $(s, t) \in f$;

(ii) $(s, u) \in f$ and $(s, t) \in f \Rightarrow t = u$.

If $(s, t) \in f$, then we write $(s, f(s))$ or $f(s) = t$.

Here t is called the **image** of s; and s is called the **preimage** of t.

The set S is called the **domain** of f and T is called the **codomain**.

The set $\{f(s) / s \in S\}$ is a subset of T and it is called the **image** of S under f (or image of f). We denote the fact: 'f is a function from S to T' by "$f : S \to T$".

Example: Let R be the set of real numbers. Define $f(x) = x^2$ for every $x \in R$. This represents a function $f = \{(x, x^2) \mid x \in R\}$.

Example: Let $f: N \to N$ be a function such that $f(x) = x \pmod 3$. That is $f(x)$ is the remainder obtained when x is divided by 3. Then the domain of f is N and the range of f is $\{0, 1, 2\}$.

Definition: $f: S \to T$ is said to be

(i) **one-one function** (or **injective function**) if it satisfies the following condition: $f(s_1) = f(s_2) \Rightarrow s_1 = s_2$.

(ii) onto function (or **surjective function**) if it satisfies the following condition: $t \in T \Rightarrow$ there corresponds an element s in S such that $f(s) = t$.

(iii) a **bijection** if it is both one-one and onto.

Definition: Let $g: S \to T$ and $f : T \to U$. The **composition** of f and g is a function $fog : S \to U$ defined by $(fog)(s) = f(g(s))$ for all s in S.

That is, $fog = \{(s, u) \mid s \in S, u \in U \text{ and } \exists\, t \in T \text{ and } t = g(s) \text{ and } u = f(t)\}$.

Example: Let $f: R \to R$ and $g: R \to R$ where R is the set of real numbers. If $f(x) = x^2-2$ and $g(x) = x + 4$. Find gof and fog

Solution: $(gof)(x) = g(f(x)) = g(x^2-2) = (x^2-2) + 4 = x^2 + 2$; and

$(fog)(x) = f(g(x)) = f(x + 4) = (x + 4)^2 – 2 = x^2 + 8x +14$.

Definition: A non-empty set V is said to be a *vector space* over a field F (generally, we take the field as the set of real numbers or the set of complex numbers) if it satisfying the following:

(i) $(V, +)$ is an abelian group,

(ii) V is closed under scalar multiplication (that is, for every $\alpha \in F$, $v \in V$ we have $\alpha v \in V$) and also this scalar multiplication satisfies the following conditions:

(a) $\alpha(v + w) = \alpha v + \alpha w$,

(b) $(\alpha + \beta)\, v = \alpha v + \beta v$,

(c) $\alpha(\beta v) = (\alpha\beta)v$ and

(d) $1.v = v$ for all $\alpha, \beta \in F$ and $v, w \in V$ (here 1 is the identity of F with respect to multiplication).

Here + is addition either in the field or in the vector space, as appropriate; and 0 is the additive identity in either. Juxtaposition indicates either scalar multiplication or the multiplication operation in the field.

Note: We use F for field, also the elements of F are called scalars and the elements of V are called vectors.

Definition: (i) An n-component vector **a** is an ordered n-tuple of numbers written as a row $(a_1,...,a_n)$ or as a column $\begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{pmatrix}$ where a_i, i = 1, 2, …, n are assumed to be real numbers and are called the **components** of the vector.

(ii) A **unit vector** denoted by e_i and defined with unity as value of its i^{th} component and all other components zero.

(iii) **Null vector** is a vector all of whose components are zero.

(iii) A **sum vector** is a vector having unity as a value for each component; it will be written as 1.

(iv) Let **a** and **b** be two n-component vectors. Then **a** and **b** are **equal** if and only if $a_i = b_i$ for each i.

(v) The **scalar product** of two n-component vectors **a, b** is defined to be the scalar

$a_1b_1 + a_2b_2 + \ldots + a_nb_n = \sum_{i=1}^{n} a_i b_i$

Definition: The **norm** (or Euclidean norm) of an n-component vector **a** = $(a_1, a_2, \ldots, a_n)$ is denoted and defined by $\| \mathbf{a} \| = \sqrt{<\mathbf{a}, \mathbf{a}>} = \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$.

Definition: An **n-dimensional Euclidean space** is defined as the collection of all vectors (also called as, points) **a** = $(a_1, a_2, \ldots, a_n)$ where $a_1, a_2, \ldots, a_n$ are real or complex number. The addition of vectors and multiplication of vector by a scalar are respectively defined as follows.

$E^n = \{(a_1, a_2, \ldots, a_n) \mid$ each a_i is a real or complex number, $1 \leq i \leq n\}$.

Define addition and multiplication by scalar in E^n as, for any $(a_1, a_2, \ldots, a_n)$, $(b_1, b_2, \ldots, b_n) \in E^n$,

$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) \in E^n$, and

for any scalar λ, $\lambda(a_1, a_2, \ldots, a_n) = (\lambda a_1, \lambda a_2, \ldots, \lambda a_n) \in E^n$.

Therefore E^n is a vector space (over R).

Example: Let $M_{2\times 2} = \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} / a_i \text{ is a real number} \right\}$. Then $M_{2\times 2}$ is a vector space over the set of real numbers, with respect to the addition of matrices and the multiplications of a matrix by a scalar.

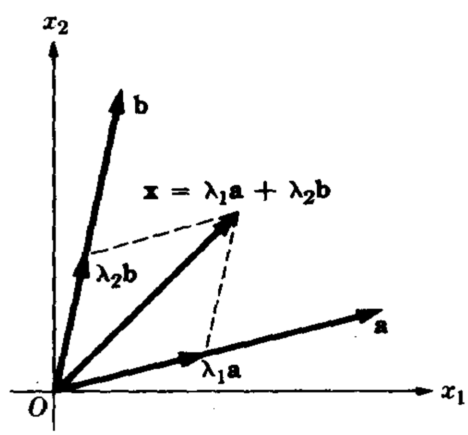
Example: R is a set of real number. Then R over R is a vector space.

Definition: Let V be a vector space over F and $W \subseteq V$. Then W is called a ***subspace*** of V if W is a vector space over F under the same operation.

Definition: Suppose V is a vector space over F, $v_i \in V$ and $\alpha_i \in F$ for $1 \leq i \leq n$. Then

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$$

is called the **linear combination** (over F) of $\{v_1, v_2, \ldots, v_n\}$.



Definition: Let V be a vector space and $S \subseteq V$. We write

$L(S) = \{\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n \,/\, n \in N, v_i \in S \text{ and } \alpha_i \in F \text{ for } 1 \leq i \leq n\}$, which is the set of all linear combinations of finite number of elements of S. This $L(S)$ is called the **linear span of S.**

Example: The set of vectors $\{(1, 0), (0, 1)\}$ spans R^2. Take $(2, 3) \in R^2$. Then $(2, 3) = 2\cdot(1, 0) + 3\cdot(0, 1)$.

Example: Let the field K be the set **R** of real numbers, and let the vector space V be the Euclidean space $\mathbf{R}^3$. Consider the three unit vectors $e_1 := (1,0,0)$, $e_2 := (0,1,0)$ and $e_3 = (0,0,1)$. Then any vector in $\mathbf{R}^3$ is a linear combination of e_1, e_2 and e_3.

Example: Let a = (2, 3, 4, 7), b = (0, 0, 0, 1), c = (1, 0, 1, 0). Then the vector d = (5, 3, 7, 9) is a linear combination of the vectors a, b, c as follows.

$$(5, 3, 7, 9) = (2, 3, 4, 7) + 2(0, 0, 0, 1) + 3(1, 0, 1, 0).$$

Note: Let $v \in S$. Then$1.v$ is a linear combination and hence $v \in L(S)$. Therefore $S \subseteq L(S)$.

Problem: Let V be a vector space over F and $\phi \neq W \subseteq V$. Then the following two conditions are equivalent. (i) W is a subspace of V

(ii) $\alpha, \beta \in F$ and $w_1, w_2 \in W \Rightarrow \alpha w_1 + \beta w_2 \in W$.

Examples: The following are the subspaces of R^2.

1. The lines passing through origin in the complex plane.
2. The set $\{(x, 0) \mid x \in R\}$ is a subspace of R^2.
3. The set $\{(x, y) \mid x \geq 0, y \geq 0\}$ is not a subspace of R^2.

Example: The set $S_{2\times 2}=\left\{\begin{pmatrix} a & a \\ a & a \end{pmatrix} / a \text{ is a real number}\right\}$ is a subspace of $M_{2\times 2}$.

Example: Verify that set $W = \{\alpha(1, 1, 0) \mid \alpha \text{ is a real number}\}$ is a subspace of R^3.

Verification: Clearly $(0, 0, 0) \in W$

Take x, y $\in$ W. Then x = α(1, 1, 0) and y = β(1, 1, 0) for some $\alpha, \beta \in$ R.

Now for any $\gamma, \delta \in$ R, consider the linear combination

$$\gamma x + \delta y = \gamma\, \alpha(1, 1, 0) + \delta\, \beta(1, 1, 0)$$

$$= \gamma\, (\alpha, \alpha, 0) + \delta\, (\beta, \beta, 0) \qquad \text{(by scalar multiplication)}$$

$$= (\gamma\alpha, \gamma\alpha, 0) + (\delta\beta, \delta\beta, 0) \qquad \text{(by scalar multiplication)}$$

$$= (\gamma\alpha + \delta\beta, \gamma\alpha + \delta\beta, 0) \qquad \text{(by addition of vectors)}$$

$$= (\gamma\alpha + \delta\beta)\, (1, 1, 0) \in W \text{ (here } \gamma\alpha + \delta\beta \in R).$$

Problem: If S is any subset of a vector space V, then show that L(S) is a subspace of V.

Solution: Let $v, w \in$ L(S) and $\alpha, \beta \in F$.

Since $v, w \in$ L(S), we have that

$$v = \alpha_1 v_1 + ... + \alpha_n v_n \text{ and } w = \beta_1 w_1 + ... + \beta_m w_m$$

for some $v_i \in S$, $\alpha_i \in F$ for $1 \le i \le n$ and $w_j \in S$, $\beta_j \in F$ for $1 \le j \le m$.

Now

$$\alpha v + \beta w = \alpha(\alpha_1 v_1 + ... + \alpha_n v_n) + \beta(\beta_1 w_1 + ... + \beta_m w_m)$$

$$= \alpha\alpha_1 v_1 + \alpha\alpha_2 v_2 + ... + \alpha\alpha_n v_n + \beta\beta_1 w_1 + \beta\beta_2 w_2 + ... + \beta\beta_m w_m, \text{ which}$$

is a linear combination of elements from S.

Hence $\alpha v + \beta w \in$ L(S). This shows that L(S) is a subspace of V.

Properties of linear span: If S and T are subsets of a vector space V then

(i) $S \subseteq T \Rightarrow L(S) \subseteq L(T)$,

(ii) $L(S \cup T) = L(S) + L(T)$,

(iii) $L(L(S)) = L(S)$.

Definitions: (i) The vector space V is said to be **finite-dimensional** (over F) if there is a finite subset S in V such that $L(S) = V$.

(ii) The vectors $v_i \in V$ for $1 \leq i \leq n$, are **linearly dependent** over F if there exists elements $a_i \in F$, $1 \leq i \leq n$, <u>not all of them equal to zero</u>, such that $a_1v_1 + a_2v_2 + ... + a_nv_n = 0$. If the vectors v_i, $1 \leq i \leq n$ are not linearly dependent over F then they are said to be **linearly independent** over F.

Example: Consider R, the set of real numbers and take V = R. Then R is a vector space over R.

Examples: Consider R, the set of real numbers and take $V = R^2$, then

(i) V is vector space over the field R.

(ii) Consider $S = \{(1, 0), (1, 1), (0, 1)\} \subseteq R^2$. Then the linear span of S is

$$L(S) = \{\alpha_1(1, 0) + \alpha_2(1, 1) + \alpha_3(0, 1) \ / \ \alpha_i \in R, 1 \leq i \leq 3\}$$

$$= \{(\alpha_1 + \alpha_2, \alpha_2 + \alpha_3) \ / \ \alpha_i \in R, 1 \leq i \leq 3\} \subseteq R^2.$$

If $(x, y) \in R^2$ then write $\alpha_1 = x$, $\alpha_2 = 0$, $\alpha_3 = y$ then

$(x, y) = (\alpha_1 + \alpha_2, \alpha_2 + \alpha_3) \in L(S)$. Hence $L(S) = R^2 = V$.

This shows that R^2 is finite dimensional.

(iii) Write $v_1 = (1, 0)$, $v_2 = (2, 2)$, $v_3 = (0, 1)$, $v_4 = (3, 3)$. Then

$$\alpha_1v_1 + \alpha_2 v_2 + \alpha_3v_3 + \alpha_4v_4 = \mathbf{0} \text{ where } \alpha_1 = 2, \alpha_2 = -1, \alpha_3 = 2, \alpha_4 = 0.$$

Thus there exists scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ not all of them equal to zero such that

$$\alpha_1v_1 + \alpha_2 v_2 + \alpha_3v_3 + \alpha_4v_4 = \mathbf{0}.$$

Hence $\{v_i \ / \ 1 \leq i \leq 4\}$ is a linearly dependent set.

(iv) Suppose $v_1 = (1, 0)$, $v_2 = (0, 1)$. Suppose $\alpha_1v_1 + \alpha_2v_2 = 0$ for some $\alpha_1, \alpha_2 \in R$. Then

$$\alpha_1(1, 0) + \alpha_2(0, 1) = \mathbf{0} = (0, 0)$$

$$\Rightarrow (\alpha_1, \alpha_2) = (0, 0)$$

$$\Rightarrow \alpha_1 = 0 = \alpha_2.$$

Hence the vectors v_1, v_2 are linearly independent.

Lemma: Let V be a vector space over F. If $v_1, v_2, ..., v_n \in V$ are linearly independent, then every element in their linear span has a unique representation in the form $\lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_n v_n$ with $\lambda_i \in F$, $1 \leq i \leq n$.

Proof: Let $S = \{v_i / 1 \leq i \leq n\}$. Consider $L(S)$. Let $v \in L(S)$. Then

$$v = \alpha_1 v_1 + ... + \alpha_n v_n$$

for some $\alpha_i \in F$, $v_i \in S$, $1 \leq i \leq n$ (by the definition of $L(S)$).

<u>Uniqueness</u>: Suppose $v = \alpha_1 v_1 + ... + \alpha_n v_n = \beta_1 v_1 + ... + \beta_n v_n$ for some $\alpha_i, \beta_i \in F$, $1 \leq i \leq n$

$\Rightarrow (\alpha_1 v_1 + ... + \alpha_n v_n) - (\beta_1 v_1 + ... + \beta_n v_n) = 0$

$\Rightarrow (\alpha_1 - \beta_1)v_1 + ... + (\alpha_n - \beta_n)v_n = 0$

$\Rightarrow \alpha_1 - \beta_1 = 0\ \alpha_2 - \beta_2 = 0, ..., \alpha_n - \beta_n = 0$ (since v_i, $1 \leq i \leq n$ are linearly independent)

$\Rightarrow \alpha_1 = \beta_1, \alpha_2 = \beta_2, ..., \alpha_n = \beta_n$.

Hence every element in the linear span can be expressed as in a unique way as a linear combination of v_i, $1 \leq i \leq n$

Theorem: If $v_i \in V$, $1 \leq i \leq n$, then either they are linearly independent or some v_k is a linear combination of the preceding ones $v_1, v_2, ..., v_{k-1}$.

Proof: If v_i, $1 \leq i \leq n$ are linearly independent then there is nothing to prove. Suppose that v_i, $1 \leq i \leq n$ are <u>not</u> linearly independent (that is, linearly dependent). Then there exist scalars α_i, $1 \leq i \leq n$ <u>not all zero</u>, such that

$$\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n = \mathbf{0}.$$

Since all the α_i are not zero, there exists largest k such that $\alpha_k \neq 0$. Then

$$\alpha_{k+1} = 0, \ldots, \alpha_n = 0 \text{ and so } \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k = \mathbf{0}$$

$$\Rightarrow \alpha_k v_k = -\alpha_1 v_1 - \alpha_2 v_2 - \ldots - \alpha_{k-1} v_{k-1}$$

$$\Rightarrow v_k = \alpha_k^{-1}(-\alpha_1 v_1 - \alpha_2 v_2 - \ldots - \alpha_{k-1} v_{k-1}) = -\alpha_k^{-1}\alpha_1 v_1 - \ldots - \alpha_k^{-1}\alpha_{k-1} v_{k-1} \text{ and}$$

$$\alpha_k^{-1}\alpha_i \in F \text{ for each } 1 \le i \le k\text{-}1.$$

Hence v_k is a linear combination of v_i, $1 \le i \le k$-1.

Definition: *A* subset *S* of a vector space *V* is called a ***basis*** of *V* if

(i) *S* consists of linearly independent elements (that is, any finite number of elements in *S* is a linearly independent), and

(ii) *S* spans *V (that is, V* = *L*(*S*)).

Note: A subset B of a vector space is a basis if and only if any of the following equivalent conditions are met:

- B is a minimal generating set of *V*, i.e., it is a generating set but no proper subset of B is.
- B is a maximal set of linearly independent vectors, i.e., it is a linearly independent set but no other linearly independent set contains it as a proper subset.
- Every vector in *V* can be expressed as a linear combination of vectors in *B* in a unique way.

Examples:

(i) Consider $\mathbf{R}^2$, the vector space of all coordinates (a, b) where both a and b are real

numbers. Then a very natural and simple basis is simply the vectors $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$: suppose that $v = (a, b)$ is a vector in $\mathbf{R}^2$, then $v = a\,(1,0) + b\,(0,1)$.

(ii) The set of vectors $\{(1, 1), (-1, 2)\}$ form a basis of $\mathbf{R}^2$.

Verification: Part (i): First we prove that $\{(1, 1), (-1, 2)\}$ is a linearly independent set.

Suppose that there are numbers a, b such that: $a(1, 1) + b(-1, 2) = (0, 0)$.

$\Rightarrow (a - b, a + 2b) = (0, 0) \Rightarrow a - b = 0$ and $a + 2b = 0$. Subtracting we get $3b = 0$ and so $b = 0$. From the first equation we get that $a = 0$.

Part (ii): In this part we show that $\{(1, 1), (-1, 2)\}$ generates R^2. Let $(a, b) \in R^2$. Now we show that there exists two numbers x and y such that

$x(1, 1) + y(-1, 2) = (a, b)$. Then we have to solve the equations:

$x - y = a$ and $x + 2y = b$

Subtracting the first equation from the second, we get $3y = b - a \Rightarrow y = (b - a)/3$ and $x = (b + 2a)/3$.

Therefore $\{(1, 1), (-1, 2)\}$ generates (or spans) $\mathbf{R}^2$.

Observation:

(i) Since (-1, 2) is clearly not a multiple of (1, 1) and since (1, 1) is not the zero vector, these two vectors are linearly independent. Since the dimension of $\mathbf{R}^2$ is 2, the two vectors already form a basis of $\mathbf{R}^2$ without needing any extension.

(ii) The **standard basis** (also called **natural basis** or **canonical basis**) of the *n*-dimensional Euclidean space $\mathbf{R}^n$ is the basis obtained by taking the *n* basis vectors,

($e_1, e_2, \ldots, e_n$) where e_i is the vector with a 1 in the i^{th} coordinate and 0 elsewhere. In many ways, it is the "obvious" basis.

Example: The set of all n unit vectors $e_1, e_2, \ldots, e_n$ form a basis for R^n.

To show that the set $\{e_1, e_2, \ldots, e_n\}$ is linearly independent:

Suppose $\alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n = \mathbf{0}$.

This means: $\alpha_1(1, 0, \ldots, 0) + \alpha_2(0, 1, 0, \ldots, 0) + \ldots + \alpha_n(0, 0, \ldots, 1) = (0, 0, \ldots 0)$.

$\Rightarrow (\alpha_1, 0, \ldots, 0) + (0, \alpha_2, \ldots, 0) + \ldots + (\alpha_n, \ldots, 1) = (0, 0, \ldots, 0)$.

$\Rightarrow (\alpha_1, \alpha_2, \ldots, \alpha_n) = (0, 0, \ldots, 0)$.

This means that $\alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_n = 0$.

Therefore $\{e_1, e_2, \ldots, e_n\}$ is linearly independent.

Next to show that $\{e_1, e_2, \ldots, e_n\}$ spans R^n. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in R^n$. Then $\mathbf{x}$ can be represented as a linear combination of $\{e_1, e_2, \ldots, e_n\}$ in the following way.

$$\mathbf{x} = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n$$

Therefore the set $\{e_1, e_2, \ldots, e_n\}$ spans R^n and hence a basis.

Example: Prove that the set B = $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ is a basis for $M_{2\times 2}$.

Solution: The set B is linearly Independent:

Suppose $\alpha_1\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \alpha_2\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + \alpha_3\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} + \alpha_4\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$. This implies that $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, $\alpha_4 = 0$.

We show that the set B spans $M_{2\times2}$:

Let $\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in M_{2\times2}$. Then

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} = a\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + b\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + c\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} + d\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}.$$

Therefore the set B spans $M_{2\times2}$, and hence B is a basis.

Note: Any n linearly independent vectors in R^n forms a basis for R^n

Example: Test whether the set of vectors {(1,1,0), (1,0,-2), (1,1,1)} form basis for R^3. If so express (1, 2, 3) in terms of basis vectors.

Solution: Consider the matrix determinant of the coefficients.

$$D = \begin{vmatrix}1 & 1 & 0\\ 1 & 0 & -2\\ 0 & 1 & 1\end{vmatrix} = -1 \neq 0.$$

Therefore the given set of vectors is a maximal linearly independent set in R^3 (as these are three linearly independent vectors in R^3). Hence they form a basis for R^3. Now (1, 2, 3) = 1 · (1, 1, 0) + 1 · (1, 1, 1) + (-1) · (1, 0, -2).

Example: Test whether the set B = {(2, 1, 0), (3, 0, 1), (5, 2, 2)} forms a basis for R^3.

Solution: Consider the matrix determinant of the coefficients.

$$D = \begin{vmatrix} 2 & 3 & 5 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{vmatrix} = 2(-2) – 3(2) + 5(1) = -5 \neq 0.$$

Therefore B is linearly independent in R^3. Hence B forms a basis for R^3.

Problem: Any vector in $\mathbf{R}^n$ can be expressed as a linear combination of a set of vectors in only one way.

Proof: Let $b \in \mathbf{R}^n$ and $\{a_1, a_2, \ldots, a_r\}$ a set of basis vectors.

Suppose that b is expressed in two ways as follows:

$b = \alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_n a_r$ and $b = \beta_1 a_1 + \beta_2 a_2 + \ldots + \beta_n a_r$ where α's and β's are scalars.

Now $(\alpha_1 - \beta_1)a_1 + (\alpha_2 - \beta_2)a_2 + \ldots + (\alpha_r - \beta_r)\, a_r = 0$.

Since $\{a_1, a_2, \ldots, a_r\}$ is linearly independent, we have that

$$\alpha_1 - \beta_1 = \alpha_2 - \beta_2 = \ldots \alpha_r - \beta_r = 0.$$

This means $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, …, $\alpha_r = \beta_r = 0$.

Remark: If V is a finite dimensional vector space over F, then any two bases of V have that same number of elements.

Problem: Define minimal spanning set of vectors. Prove that a minimal spanning set of vectors forms a basis.

Solution:

<u>Minimal spanning set</u>: A subset S of a vector space V is said to be a *minimal spanning set* if (i) S is a spanning set for V, and (ii) $S \setminus \{v\}$ do not span V for any $v \in S$.

Let $S = \{v_1, v_2, ..., v_n\}$ be a minimal spanning set. This means $L(S) = V$.

In order to prove S is a basis, it suffices to prove S is linearly independent.

In a contrary way, suppose that S is not linearly independent.

Then there exists v_j (for some j, $1 \leq j \leq n$) is a linear combination of its preceding ones. That is.,

$$v_j = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_{j-1} v_{j-1} \text{ for some } \alpha_i \in F, 1 \leq i \leq (j-1).$$

Clearly $L(\{v_1, v_2, ..., v_{j-1}, v_{j+1}, ..., v_n\}) \subseteq L(\{v_i / 1 \leq i \leq n\}) = L(S)$.

On the other hand, take $x \in L(S)$.

Then $x = \beta_1 v_1 + \beta_2 v_2 + ... + \beta_n v_n$ for some $\beta_i \in F$, $1 \leq i \leq n$

$$\Rightarrow x = \beta_1 v_1 + ... + \beta_{j-1} v_{j-1} + \beta_j(\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_{j-1} v_{j-1}) + \beta_{j+1} v_{j+1} + ... + \beta_n v_n$$

$$\Rightarrow x = (\beta_1 + \beta_j \alpha_1) v_1 + ... + (\beta_{j-1} + \beta_j \alpha_{j-1}) v_{j-1} + \beta_{j+1} v_{j+1} + ... + \beta_n v_n$$

$$\in L(\{v_1, v_2, ..., v_{j-1}, v_{j+1}, ..., v_n\}).$$

Therefore, $L(\{v_1, v_2, ..., v_{j-1}, v_{j+1}, ..., v_n\}) = L(S) = V$, which is a contradiction to the fact that n is minimum with S spans V. Therefore S is linearly independent.

Problem: Define a maximal linearly independent set. Prove that a maximal linearly independent set is a basis.

Solution:

Maximal linearly independent set **:** A subset S of a vector space V is said to be a *maximal linearly independent set* if (i) S is a linearly independent set, and (ii) $S \cup \{v\}$ is linearly dependent for any $v \in V \setminus S$.

Let $S = \{v_1, v_2, ..., v_n\}$ be a maximal linearly independent set. In order to prove S is a basis, it suffices to prove S spans V.

Take $v \in V$. Suppose $\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n + \alpha v = 0$.

If $\alpha = 0$, then since $v_1, v_2, ..., v_n$ are linearly independent, we get $\alpha_i = 0$ for all $1 \leq i \leq n$.

This means $v, v_1, v_2, ..., v_n$ are (which are n + 1, in number) linearly independent, a contradiction to the maximality of n.

Therefore, $\alpha \neq 0$. Now $\alpha v = (-\alpha_1 v_1) + (-\alpha_2 v_2) + ... + (-\alpha_n v_n)$.

This implies $v = (-\alpha_1\alpha^{-1})v_1 + (-\alpha_2\alpha^{-1})v_2 + ... + (\alpha_n\alpha^{-1})v_n$.

Therefore *S* spans *V*. Hence *S* is a basis.

Problem 2: Whether the vectors $\{a_1 = (4, 2, 1),\ a_2 = (2, -6, -5),\ a_3 = (1, -2, 3)\}$ are linearly independent?

Solution: The set $\{a_1, a_2, a_3\}$ is linearly independent if $\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 = 0$ implies each of the $\lambda_i = 0$ for i = 1, 2, 3.

That is, $\lambda_1(4, 2, 1) + \lambda_2(2, -6, -5) + \lambda_3(1, -2, 3) = 0$

This yields the following system of homogeneous equations

$$4\lambda_1 + 2\lambda_2 + \lambda_3 = 0$$

$$2\lambda_1 - 6\lambda_2 - 2\lambda_3 = 0$$

$$\lambda_1 - 5\lambda_2 + 3\lambda_3 = 0$$

The above system will have trivial solution $\lambda_1 = \lambda_2 = \lambda_3 = 0$ if and only if $|A| \neq 0$, where A is the coefficient matrix $\begin{bmatrix} 4 & 2 & 1 \\ 2 & -6 & -2 \\ 1 & -5 & 3 \end{bmatrix}$

We observe that $|A| = \begin{vmatrix} 4 & 2 & 1 \\ 2 & -6 & -2 \\ 1 & -5 & 3 \end{vmatrix} = -132 \neq 0$. Hence $\lambda_1 = \lambda_2 = \lambda_3 = 0$.
Thus the given vectors are linearly independent.

Note:

1. A set of vectors are linearly independent if $|A| \neq 0$ where A is the matrix formed from the given vectors.
2. Maximum number of linearly independent vectors in E^n is n. Hence any set of n+1 vectors are always linearly dependent.

Problem

Verify whether or not, the following set of vectors is linearly independent?

{(4, 2, -1), (3, -6, -5)}

Solution: The vectors are linearly independent.

Problem Verify whether or not, the following set of vectors is linearly independent?

{(1, 1, 1), (0, 2, 3), (1, -2, 3)}

Solution: The set of vectors is linearly independent.

Problem 5 Verify whether or not, the set of vectors {(1, 2, 3), (1, 1, 1) , (1, 0, 1)} is linearly independent?

Ans: The set of vectors {(1, 2, 3), (1, 1, 1) , (1, 0, 1)} is linearly independent.

Problem 6 Verify whether or not, the following set of vectors is linearly independent?

{(1, 3, -2), (2, -1, 4) , (1, -11, 14)}

Ans: The set is not linearly independent

Problem: Show that if a set of vectors are linearly independent then every subset is also linearly independent.

Solution: Suppose that set of vectors $\{a_1, a_2, \ldots, a_r\}$ are linearly independent.

Then $\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_r a_r = 0$ implies that all λ_i are zero.

Consider the subset $\{a_1, a_2, a_3\}$. If these vectors are linearly dependent then $\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 = 0$ implies there exists say $\lambda_1 \neq 0$. But this contradicts that $\{a_1, a_2, \ldots, a_r\}$ are linearly independent. The proof is complete.

Exercises:

1. Test whether or not the following set of vectors is linearly independent?

i) $\{(2, 2, 3), (-1, -2, 1), (0, 1, 0)\}$
ii) $\{(4, 2, 1), (2, -6, -5)\}$
iii) $\{(4, 2, 1), (2, -6, -5), (1, -2, 3), (1,-1,2)\}$

2. Show that if a set of vectors are linearly dependent then every superset is also linearly dependent.

3. Which of the following set of vectors forms a basis for E^3? Express (3,1,2) as linear combination of basis vectors.

i) $\{(2, 2, 3), (-1, -2, 1), (0, 1, 0)\}$

ii) $\{(4, 2, 1), (2, -6, -5)\}$

iii) $\{(4, 2, 1), (2, -6, -5), (1, -2, 3), (1,-1,2)\}$

Exercises

1. Prove that the set $A=\{(2,-1,0),(3,5,1),(1,1,2)\}$ forms a basis for $\mathbb{R}^3$ and express $(2,4,5)$ in terms of elements of A.
2. Check whether the following set of vectors form a basis for $\mathbb{R}^3$.

   $(i)\ B=\{(2,1,4),(1,-1,2),(3,1,-2)\}$
   $(ii)\ C=\{(1,2,1),(2,1,0),(1,-1,2)\}$
3. Let $S = \{v_i \,/\, 1 \leq i \leq n\}$ is a subset of vector space V. If v_j linear combination of its preceding ones, then prove that
   $L(\{v_1, v_2, ..., v_{j-1}, v_{j+1}, ..., v_n\}) = L(S)$.
4. Prove that for a vector space V over a field K, if $v_i \in V$, $1 \leq i \leq n$, then either they are linearly independent or some v_k is a linear combination of the preceding ones $v_1, v_2, ..., v_{k-1}$.

Definition: Two vectors $\vec{a}$ and $\vec{b}\left(\vec{a},\vec{b}\neq 0\right)$ are said to be **orthogonal** if their scalar product is zero. That is $\vec{a}.\vec{b}=0$

Definition: A set $\{\vec{a}_1,...,\vec{a}_r\}$ of vectors is said to be an **orthogonal set** if the vectors are pairwise orthogonal.

An orthogonal set is said to be **orthonormal** if magnitude of each vector is unity.

Theorem: Any orthogonal set of nonzero vectors is linearly independent.

Proof: Let $\{\vec{a}_1,...,\vec{a}_r\}$ be set of nonzero orthogonal vectors.

Let $\lambda_1\overrightarrow{a_1}+...+\lambda_r\overrightarrow{a_r}=0$................................(1)

Taking scalar product of (1) with $\overrightarrow{a_1}$, we get

$\lambda_1\left|\overrightarrow{a_1}\right|^2+\lambda_2\overrightarrow{a_2}.\overrightarrow{a_1}...+\lambda_r\overrightarrow{a_r}.\overrightarrow{a_1}=0\Rightarrow\lambda_1=0$

Similarly we can prove that $\lambda_2=\lambda_3=...=\lambda_r=0\Rightarrow$ Given set is linearly independent.

Exercises

1. Prove that the set of vectors $B=\left\{\left(3/5,0,4/5\right),\left(-4/5,0,3/5\right),(0,1,0)\right\}$ is an orthonormal set in $\mathbb{R}^3$.

1. Prove that the set of vectors $B=\left\{\frac{1}{\sqrt{3}}(1,1,1),\sqrt{\frac{3}{2}}\left(\frac{-1}{3},\frac{2}{3},\frac{1}{3}\right),\frac{1}{\sqrt{8}}(-2,0,2)\right\}$ is an orthonormal set in $\mathbb{R}^3$.

2. Show that an orthonormal set of nonzero vectors is linearly independent.
3. Verify the set of vectors form an orthonormal set for $\mathbb{R}^3$
   $(i)\,B=\left\{(1,5,7),(4,0,6),(1,0,0)\right\}$
   $(ii)\,C=\left\{(3,0,2),(7,0,9),(4,1,2)\right\}$
   $(iii)\,D=\left\{(1,1,0),(3,0,1),(5,2,1)\right\}$

Orthogonal bases

A basis $\{\vec{a}_1,\ldots,\vec{a}_n\}$ from E^{n} is said to be an orthogonal basis if its vectors are pairwise orthogonal.

In addition, if magnitude of each vector is unity then the basis is said to be orthonormal.

e.g. If $\boldsymbol{x_1} = (1,0,2), \boldsymbol{x_2} = (-2,0,1) \& \boldsymbol{x_3} = (0,1,0)$ form an orthogonal set in R^3 then $\boldsymbol{u_1} = \left(\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right)$, $\boldsymbol{u_2} = \left(-\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right)$ are unit vectors in the direction of $\boldsymbol{x_1}$and$\boldsymbol{x_2}$respectively. $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{x_3}\}$ is an orthonormal set as $\boldsymbol{x_3}$is a unit vector.

ILLUSTRATION:

1. Let $S = \{u_1, u_2, u_3\}$be an orthonormal basis for R^3 where $u_1 = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$, $u_2 = \left(\frac{2}{3}, \frac{1}{3}, -\frac{1}{3}\right)$, $u_3 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$. Write vector $v = (3,4,5)$ as a linear combination of vectors in S.

Sol. $v = c_1 u_1 + c_2 u_2 + c_3 u_3$

$v \cdot u_1 = c_1, v \cdot u_2 = c_2, \& v \cdot u_3 = c_3$ lead to $c_1 = 1, c_2 = 0, c_3 = 7$

$Therefore\ \ v = u_1 + 7u_3$

GRAM-SCHMIDT PROCESS:

This process operates in finite dimensional Euclidean space and produces an orthonormal basis modifying a given non-orthogonal basis.

The **Gram Schmidt process** for computing an orthonormal basis is as follows:

Let $\{\vec{a}_1,\ldots,\vec{a}_n\}$ be linearly independent in E^{n}.

Let $\overrightarrow{u_1} = \frac{\overrightarrow{a_1}}{\left|\overrightarrow{a_1}\right|}$

Let $\overrightarrow{v_2} = \overrightarrow{a_2} - \left(\overrightarrow{a_2}.\overrightarrow{u_1}\right)\overrightarrow{u_1}$

$\therefore \overrightarrow{u_2} = \frac{\overrightarrow{v_2}}{\left|\overrightarrow{v_2}\right|}$ In general, $\overrightarrow{v_r} = \overrightarrow{a_r} - \sum_{i=1}^{r-1}\left(\overrightarrow{a_r}.\overrightarrow{u_i}\right)\overrightarrow{u_i}$ and $\overrightarrow{u_r} = \frac{\overrightarrow{v_r}}{\left|\overrightarrow{v_r}\right|}$

$\overrightarrow{u_i}.\overrightarrow{u_j} = 0,\ i \neq j;\ i, j = 1, \dots, n$

Therefore $\left\{\vec{u}_1, \dots, \vec{u}_n\right\}$ forms an orthonormal basis.

Example: Let $S = \{a_1, a_2, a_3\}$ be a basis for R^3 where $a_1 = (1,1,1)$,$a_2 = (-1,0,-1)$, $a_3 = (-1,2,3)$. Use Gram Schmidt process to transform S to an orthonormal basis of R^3 .

Sol. Let $\overrightarrow{a_1} = (1,1,1)$, $\overrightarrow{u_1} = \frac{(1,1,1)}{\sqrt{3}}$

$$\overrightarrow{v_2} = \overrightarrow{a_2} - \left(\overrightarrow{a_2}.\overrightarrow{u_1}\right)\overrightarrow{u_1} = (-1, 0, -1) - \left(-\frac{2}{3}\right)(1,1,1) = \left(-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right)$$

$$\overrightarrow{u_2} = \frac{(-1,2,-1)}{\sqrt{6}}$$

$$\overrightarrow{v_3} = \overrightarrow{a_3} - \left(\overrightarrow{a_3}.\overrightarrow{u_1}\right)\overrightarrow{u_1} - \left(\overrightarrow{a_3}.\overrightarrow{u_2}\right)\overrightarrow{u_2} = (-1,2,3) - \frac{4}{3}(1,1,1) - \frac{2}{6}(-1,2,-1)$$

$= (-2,0,2)$

$$\overrightarrow{u_3} = \frac{(-1,0,1)}{\sqrt{2}}$$

Therefore, $\{u_1, u_2, u_3\}$ is an orthonormal basis for R^3.

Example: Using Gram-Schmidt process construct an orthonormal basis from the set of vectors $\{a_1 = (1, 1, 1), a_2 = (2, -1, 2), a_3 = (1, 2, 3)\}$ in E^3.

Solution: Let $\overrightarrow{a_1} = (1,1,1)$, $\overrightarrow{u_1} = \frac{(1,1,1)}{\sqrt{3}}$

$$\overrightarrow{v_2} = \overrightarrow{a_2} - \left(\overrightarrow{a_2}.\overrightarrow{u_1}\right)\overrightarrow{u_1} = (2,-1,2) - (1,1,1) = (1,-2,1)$$

$$\overrightarrow{u_2} = \frac{(1,-2,1)}{\sqrt{6}}$$

$$\overrightarrow{v_3} = \overrightarrow{a_3} - \left(\overrightarrow{a_3}.\overrightarrow{u_1}\right)\overrightarrow{u_1} - \left(\overrightarrow{a_3}.\overrightarrow{u_2}\right)\overrightarrow{u_2} = (-1,0,1)$$

$$\overrightarrow{u_3} = \frac{(-1,0,1)}{\sqrt{2}}$$

The set $\{u_1, u_2, u_3\}$ is an orthonormal basis of R^3.

Example: Using Gram-Schmidt process construct an orthonormal basis from the set of vectors $\{a_1 = (3, 0, 4), a_2 = (-1, 0, 7), a_3 = (2, 9, 11)\}$ in E^3.

Solution: Let $\overrightarrow{a_1} = (3, 0\ ,4)$, $\overrightarrow{u_1} = \frac{(3,0,4)}{5}$

$$\overrightarrow{v_2} = \overrightarrow{a_2} - \left(\overrightarrow{a_2}.\overrightarrow{u_1}\right)\overrightarrow{u_1} = (-4, 0, 3).$$

$$\overrightarrow{u_2} = \frac{(-4,0,3)}{5}$$

$$\overrightarrow{v_3} = \overrightarrow{a_3} - \left(\overrightarrow{a_3}.\overrightarrow{u_1}\right)\overrightarrow{u_1} - \left(\overrightarrow{a_3}.\overrightarrow{u_2}\right)\overrightarrow{u_2} = (0,9,0)$$

$\overrightarrow{u_3} = (0,1,0)$

The set $\{u_1, u_2, u_3\}$ is an orthonormal basis of R^3 .

Problems

1. Apply **Gram Schmidt process** to the following vectors:

i. $\boldsymbol{v_1} = (1,0,3)$, $\boldsymbol{v_2} = (2,2,0)$, $\boldsymbol{v_3} = (3,1,2)$

ii. $\boldsymbol{v_1} = (12,1)$, $\boldsymbol{v_2} = (1,0,1)$, $\boldsymbol{v_3} = (3,1,0)$

iii. $\boldsymbol{v_1} = (0,0,1,0)$, $\boldsymbol{v_2} = (1,0,1,0)$, $\boldsymbol{v_3} = (1,1,1,1)$

iv. $\boldsymbol{v_1} = (1,-2,0,1)$, $\boldsymbol{v_2} = (-1,0,0,-1)$, $\boldsymbol{v_3} = (1,1,0,0)$

2. State **true** or **false**

i. If the components of the vector $\boldsymbol{v}$ are not integers then the length of $\boldsymbol{v}$ is not an integer.

ii. An orthogonal set of vectors is linearly independent.

iii. If $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{x}$ be three vectors in R^n such that $\boldsymbol{x} \perp \boldsymbol{u} + \boldsymbol{v}$ and $\boldsymbol{x} \perp \boldsymbol{u} - \boldsymbol{v}$ then $\boldsymbol{x} \perp \boldsymbol{u}$ and $\boldsymbol{x} \perp \boldsymbol{v}$.

3. Let $\boldsymbol{u} = (1,1,-2)$ and $\boldsymbol{v} = (a,-1,2)$. For what values of a, b is $\{\boldsymbol{u}, \boldsymbol{v}\}$ an orthogonal set?

4. Let $u = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, $v = \left(a, \frac{1}{\sqrt{2}}, -b\right)$. For what values of a, b is $\{\boldsymbol{u}, \boldsymbol{v}\}$ an orthonormal set?

5. Which of the following are orthonormal sets?

a) $\left(\frac{1}{3},\frac{2}{3},\frac{2}{3}\right),\left(\frac{2}{3},\frac{1}{3},-\frac{2}{3}\right),\left(\frac{2}{3},-\frac{2}{3},\frac{1}{3}\right)$ b) $(0,2,2,1),(1,1,-2,2),(0,-2,1,2)$