LECTURE 11 - DATE : **08 JUNE 2021**

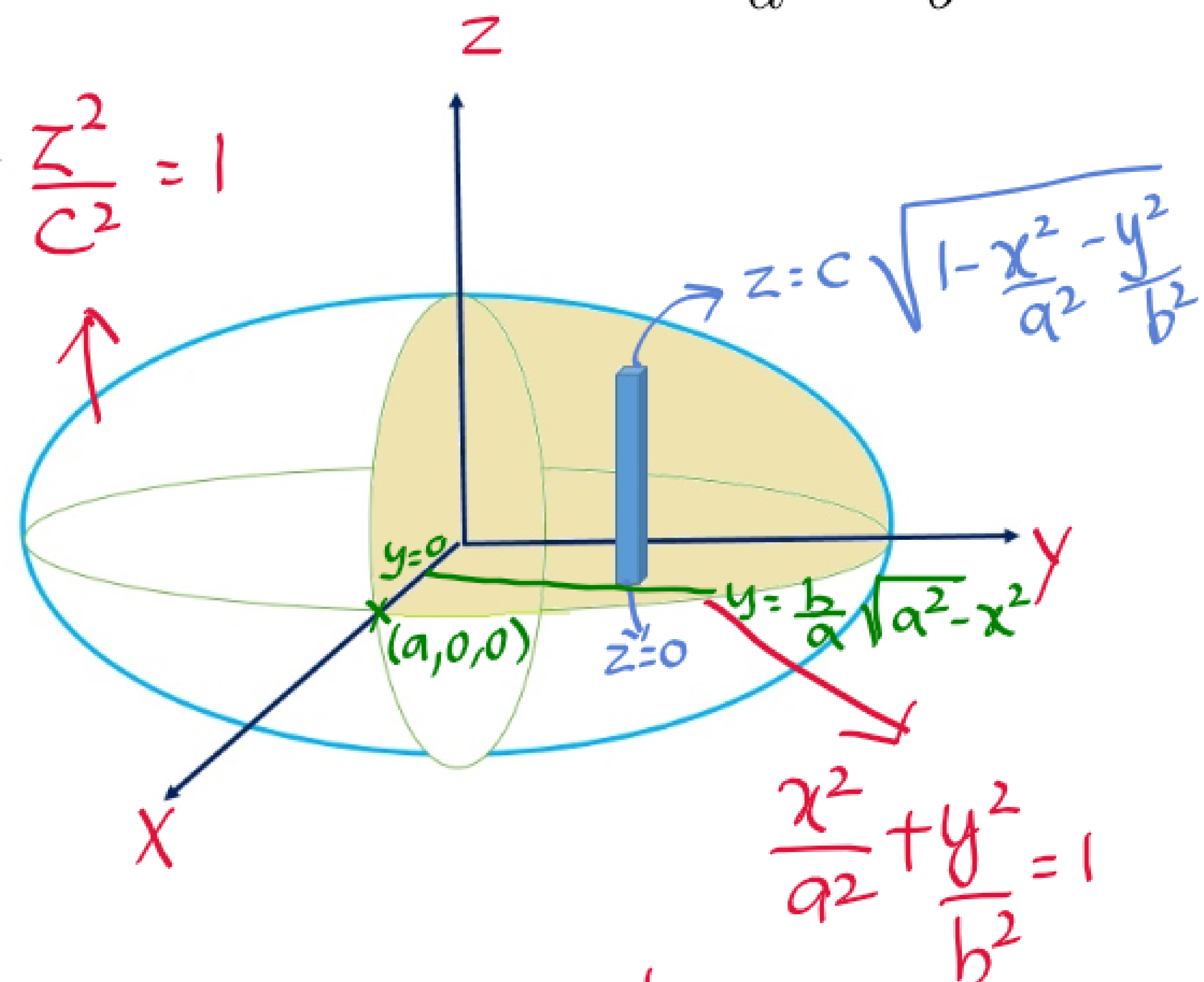
1. PROBLEMS ON TRIPLE INTEGRALS

Problem 1.1. *Using triple integrals, find the volume of the ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$\Rightarrow \frac{z^2}{c^2} = 1-\frac{x^2}{a^2}-\frac{y^2}{b^2}$$

$$\Rightarrow z = \pm c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$



Req'd volume, V = 8x Volume of the solid in the first octant.

$$= 8\times \iiint_{1^{st}\text{ octant}} dx\,dy\,dz$$

$$= 8\times \int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}}\int_{z=0}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx$$

$$= 8\times \int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dx$$

$$= 8c \int_{x=0}^{a} \int_{y=0}^{b/a\sqrt{a^2-x^2}} \frac{1}{b} \sqrt{b^2\left(1-\frac{x^2}{a^2}\right) - y^2}\; dy\, dx$$

$$= \frac{8c}{b} \int_{x=0}^{a} \int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} \sqrt{\left(b\sqrt{1-\frac{x^2}{a^2}}\right)^2 - y^2}\;\; dy\, dx$$

$$= \frac{8c}{b} \int_{x=0}^{a} \left[\frac{y}{2}\sqrt{\left(b\sqrt{1-\frac{x^2}{a^2}}\right)^2 - y^2} + \frac{b^2\left(1-x^2/a^2\right)}{2} \sin^{-1}\left(\frac{y}{b\sqrt{1-\frac{x^2}{a^2}}}\right)\right]_{y=0}^{y=\frac{b}{a}\sqrt{a^2-x^2}} dx$$

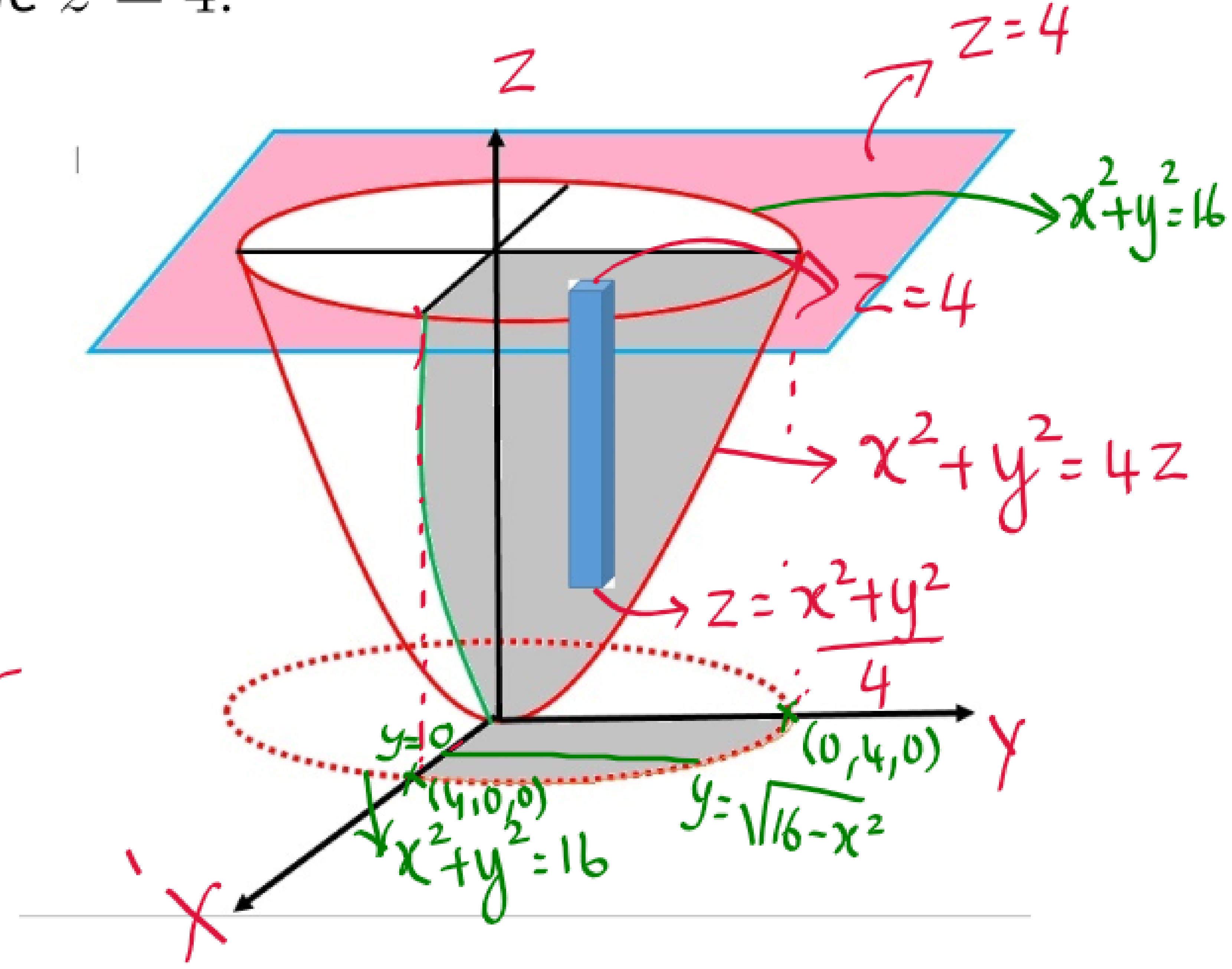
$$= \frac{8c}{b} \int_{x=0}^{a} \left(0 + \frac{b^2}{2a^2}(a^2-x^2)\sin^{-1}(1) - 0\right) dx$$

$$= \frac{4cb}{a^2}\frac{\pi}{2} \int_{x=0}^{a} (a^2-x^2)\, dx = \frac{2cb}{a^2}\pi\left(a^2x - \frac{x^3}{3}\right)_0^a$$

$$= \frac{2cb\pi}{a^2}\,\frac{2a^3}{3} = \underline{\underline{\frac{4\pi abc}{3}}} \text{ c.units}$$

$x^2+y^2=az \rightarrow$ paraboloid

Problem 1.2. *Using triple integrals, find the volume of the paraboloid of revolution* $x^2+y^2=4z$ *cut off by the plane* $z=4$.



Ans:-

Req'd volume

V = 4 x Volume of the portion in the 1^{st} Octant

$$= 4\times\int_{x=0}^{4}\int_{y=0}^{\sqrt{16-x^2}}\int_{z=\frac{x^2+y^2}{4}}^{z=4} dz\,dy\,dx$$

$$= 4\times\int_{x=0}^{4}\int_{y=0}^{\sqrt{16-x^2}} (z)_{\frac{x^2+y^2}{4}}^{4}\,dy\,dx$$

$$= 4\times\int_{x=0}^{4}\int_{y=0}^{\sqrt{16-x^2}} \left(4-\frac{x^2+y^2}{4}\right)dy\,dx$$

$$= \int_{x=0}^{4}\int_{y=0}^{\sqrt{16-x^2}} \left[(16-x^2)-y^2\right]dy\,dx$$

$$= \int_{x=0}^{4} \left[(16-x^2)y - \frac{y^3}{3} \right]_{y=0}^{y=\sqrt{16-x^2}} dx$$

$$= \int_{x=0}^{4} \left[(16-x^2)^{3/2} - \frac{(16-x^2)^{3/2}}{3} \right] dx$$

$$= \frac{2}{3} \int_{x=0}^{4} (16-x^2)^{3/2} \, dx$$

put $x = 4\sin\theta$

$dx = 4\cos\theta \, d\theta$

$x=0 \Rightarrow \theta = 0$

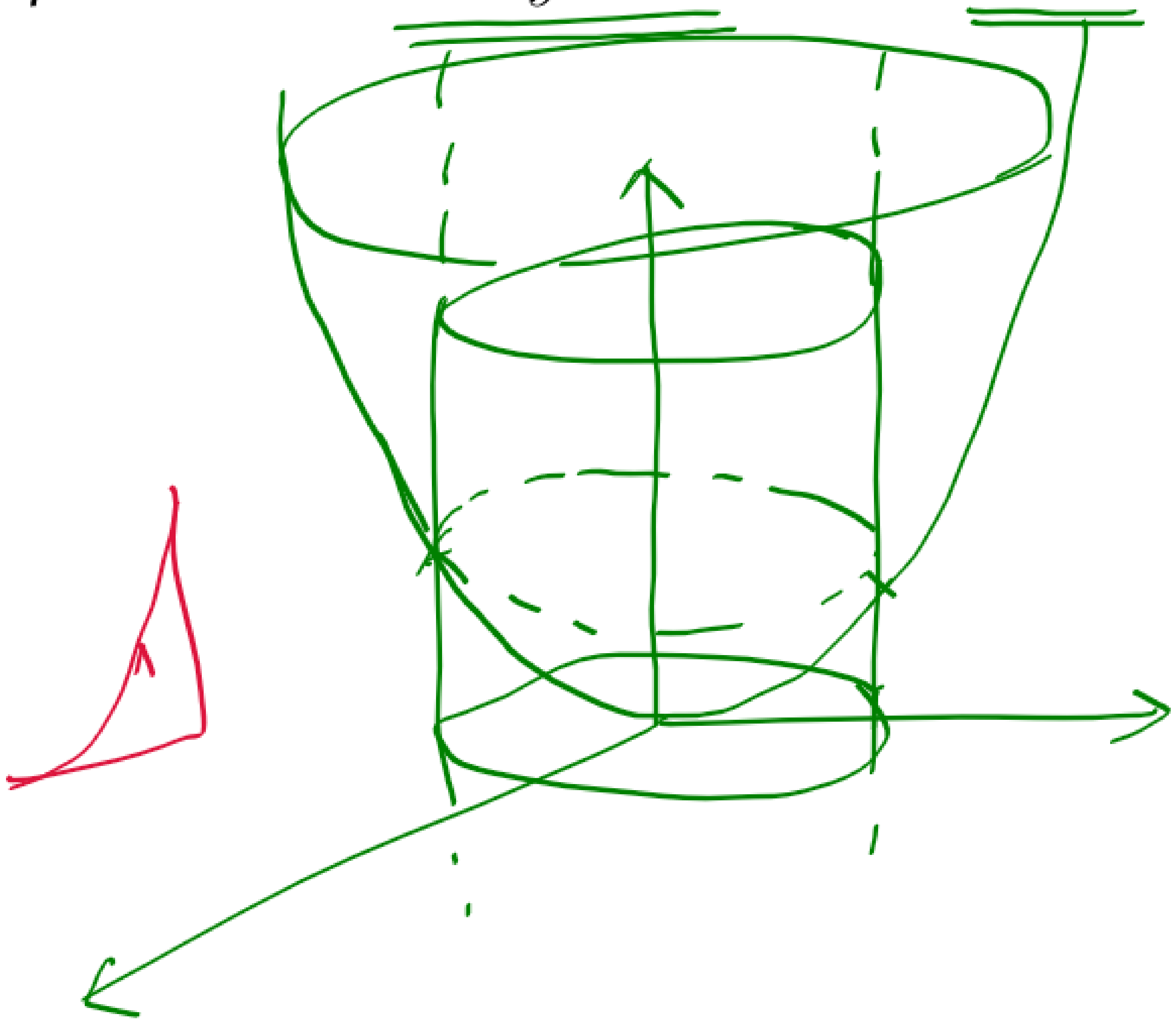
$x=4 \Rightarrow \theta = \pi/2$

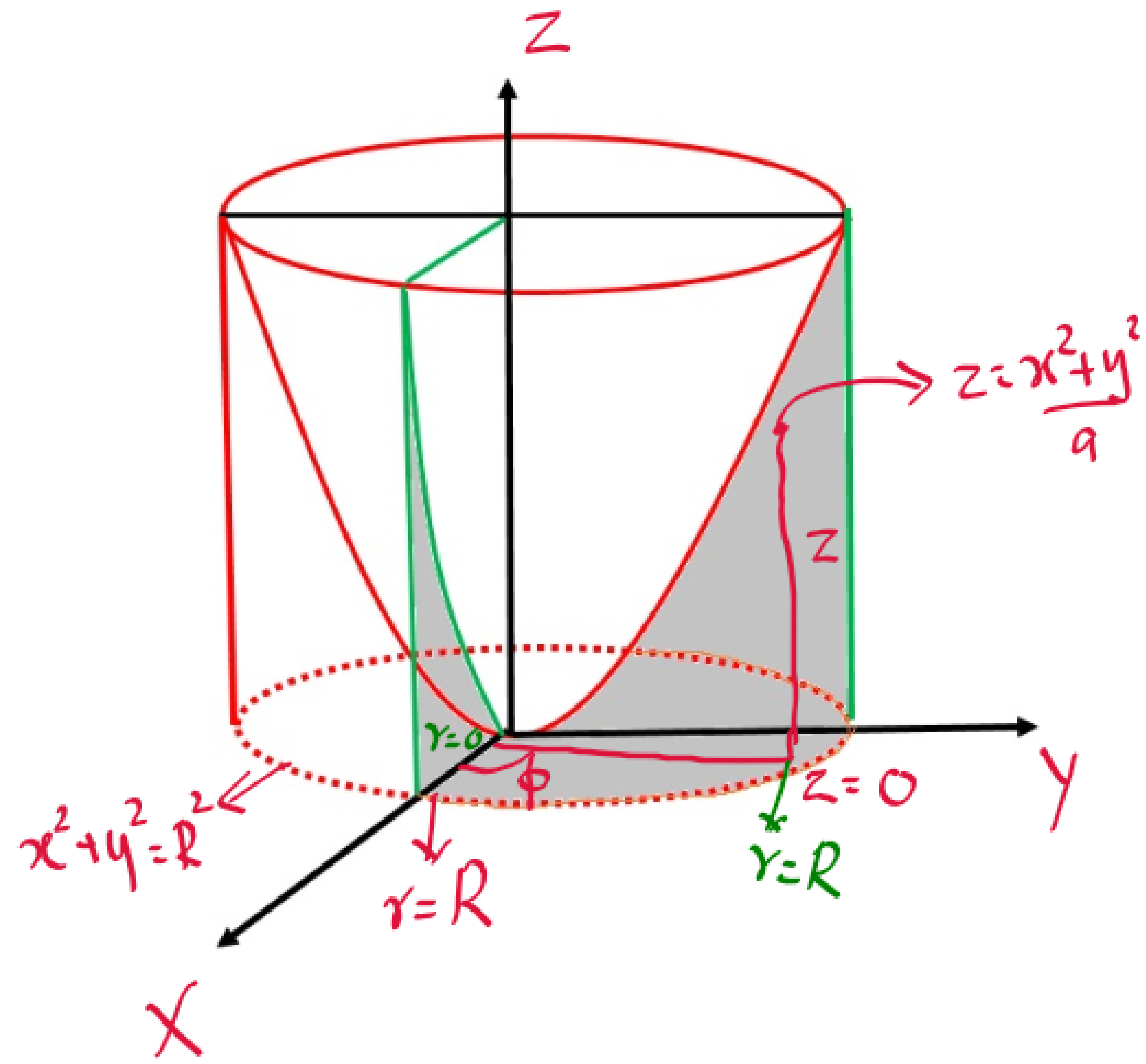
$$= \frac{2}{3} \int_{\theta=0}^{\pi/2} 4^3 \cos^3\theta \; 4\cos\theta \, d\theta$$

$$= \frac{2}{3} \times 4^4 \int_{\theta=0}^{\pi/2} \cos^4\theta \, d\theta = \frac{2}{3} \times 4^4 \times \frac{3 \times 1}{4 \times 2} \times \frac{\pi}{2}$$

$$= \underline{\underline{32\pi \text{ cubic units}}}$$

Problem 1.3. *Using triple integrals, find the volume of the region bounded by the paraboloid $x^2 + y^2 = az$ where $a > 0$ and the cylinder $x^2 + y^2 = R^2$.*





Req'd volume $V = 4 \times \underset{1^{st}\text{ Octant}}{\iiint} dx\,dy\,dz$.

Changing to cylindrical polar coordinates,

$x = r\cos\phi, \quad y = r\sin\phi, \quad z = z, \quad dx\,dy\,dz = r\,dz\,dr\,d\phi$

$\therefore$ Req'd volume, $V = 4 \times \int_{\phi=0}^{\pi/2} \int_{r=0}^{R} \int_{z=0}^{\frac{x^2+y^2}{a}} r\,dz\,dr\,d\phi$

$= 4 \times \int_{\phi=0}^{\pi/2} \int_{r=0}^{R} \int_{z=0}^{r^2/a} r\,dz\,dr\,d\phi$

$= 4 \times \int_{\phi=0}^{\pi/2} \int_{r=0}^{R} r\,(z)_0^{r^2/a}\,dr\,d\phi$

$$= 4 \int_{\theta=0}^{\pi/2} \int_{r=0}^{R} \frac{r^3}{a} \, dr \, d\phi$$

$$= \frac{4}{a} \int_{\theta=0}^{\pi/2} \left(\frac{r^4}{4} \right)_0^R d\phi = \frac{R^4}{a} \frac{\pi}{2} \text{ cubic units}$$

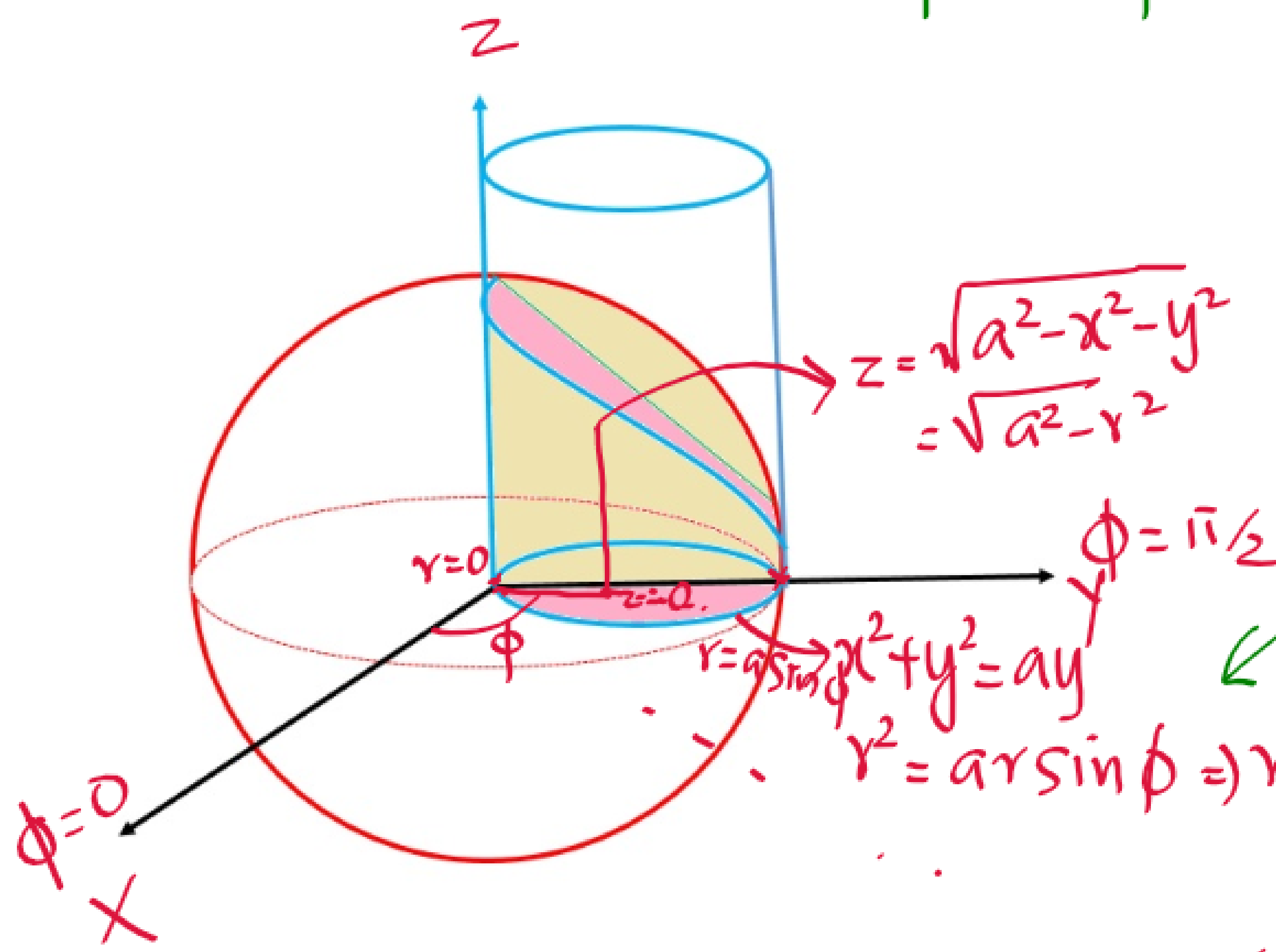
Problem 1.4. *Using triple integrals, find the volume of the portion of the sphere $x^2+y^2+z^2=a^2$ lying inside the cylinder $x^2+y^2=ay$ where $a>0$ above xy plane.*

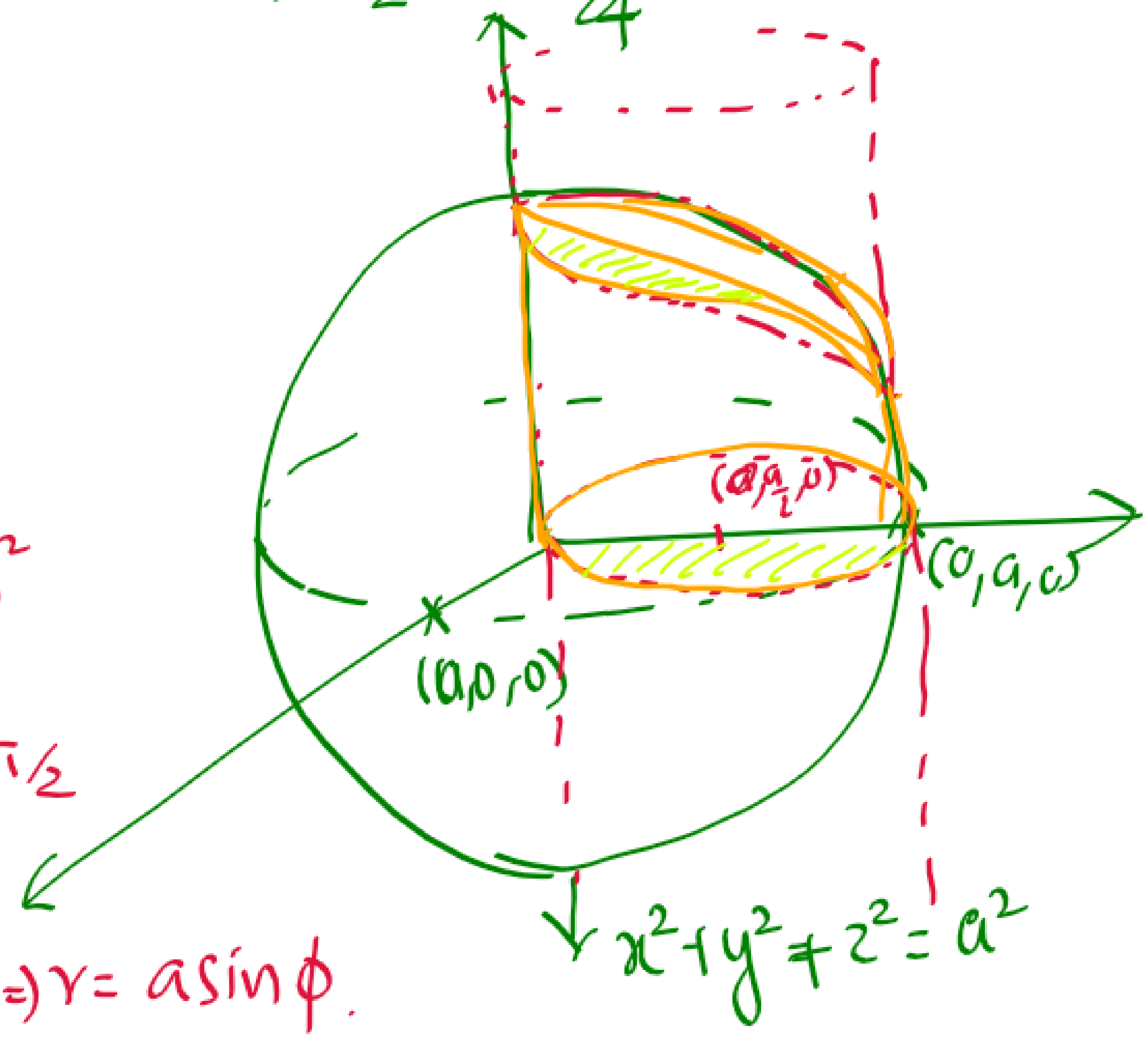
$$x^2+y^2-ay=0$$

$$\Rightarrow x^2+y^2-ay+\frac{a^2}{4}=\frac{a^2}{4}$$

$$\Downarrow$$

$$x^2+\left(y-\frac{a}{2}\right)^2=\frac{a^2}{4}$$





Req'd Volume $= 2\times\iiint dx\,dy\,dz$

$$=2\times\int_{\phi=0}^{\pi/2}\int_{r=0}^{a\sin\phi}\int_{z=0}^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\phi$$

$= \quad ?$

$=$

By changing to cylindrical polar coordinate,

$x=r\cos\phi$

$y=r\sin\phi$

$z=z$

$dx\,dy\,dz=r\,dz\,dr\,d\phi$

Ans:- $\frac{2a^3}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right)$ cubic units

PROPER INTEGRAL

(i) $[a,b]$, finite interval

(ii) $f(x)$ is bounded in $[a,b]$

ie; $f(x)$ doesn't take '∞' for any point x in $[a,b]$.

(iii) $\frac{d}{dx}[\phi(x)] = f(x)$ then $\int_a^b f(x)\,dx = \phi(b) - \phi(a)$

then such integrals are called ✓ **PROPER INTEGRALS**.

If (i) doesn't hold (ie, either a or b or both a & b are infinite) $\int_a^b f(x)\,dx$ is called an ✓ **IMPROPER INTEGRAL OF FIRST KIND**.

$(m,n) \longrightarrow \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

$(m_1,n_1) \longrightarrow \mathbb{R}$

Beta and Gamma function

Dfn (Beta Function)

Let $m,n>0$ then the definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is a function of m and n is called beta function of m and n.

It is denoted by $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Properties of Beta function

Beta function is symmetric. i.e. $\beta(m,n) = \beta(n,m)$**..**

By def^n, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

put $(1-x)=y \Rightarrow dx = -dy$

when $x=0$, $y=1$

$x=1$, $y=0$

$\therefore \beta(m,n) = \int_1^0 (1-y)^{m-1}(y)^{n-1}\,-dy = \int_0^1 y^{n-1}(1-y)^{m-1}\,dy$

$= \underline{\underline{\beta(n,m)}}$

$\circledast\ \beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$..

For, By defn, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

put $x = \sin^2\theta \Rightarrow \theta = \sin^{-1}(\sqrt{x})$

$dx = 2\sin\theta\cos\theta\,d\theta$

$x=0 \Rightarrow \theta = 0$

$x=1 \Rightarrow \theta = \pi/2$

$\therefore \beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} (1-\sin^2\theta)^{n-1} 2\sin\theta\cos\theta\,d\theta$

$= 2\int_0^{\pi/2} \sin^{2m-2}\theta \cos^{2n-2}\theta \sin\theta\cos\theta\,d\theta$

$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta$

$\Rightarrow p+1>0$ $\quad$ $\Rightarrow q+1>0$

⊛ **For** $p > -1$ **and** $q > -1$, $\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$.

for, we've,
$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

put $2m-1=p$ $\quad$ and $2n-1=q$

$\Rightarrow m = \frac{p+1}{2}$ $\quad\quad$ $\Rightarrow n = \frac{q+1}{2}$

$$\therefore \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$$

$$\Rightarrow \int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

for $\frac{p+1}{2} > 0$, $\frac{q+1}{2} > 0$

ie, for $p>-1$, $q>-1$

$\circledast$ $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$..

For, By def^n,

$$\beta(m+1,n) + \beta(m,n+1) = \int_0^1 x^{(m+1)-1}(1-x)^{n-1}\,dx + \int_0^1 x^{m-1}(1-x)^{(n+1)-1}\,dx$$

$$= \int_0^1 x^{m}(1-x)^{n-1}\,dx + \int_0^1 x^{m-1}(1-x)^{n}\,dx$$

$$= \int_0^1 \left[x^m (1-x)^{n-1} + x^{m-1}(1-x)^n\right] dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}\left[\, x + (1-x)\,\right] dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \beta(m,n)$$

Gamma Function: Let $n>0$. Then the definite integral $\int_0^\infty e^{-x} x^{n-1}\,dx$, is a function of n, is called the Gamma function of n. It is denoted by $\Gamma n = \int_0^\infty e^{-x} x^{n-1}\,dx$

Note:- $\Gamma 1 = \int_0^\infty e^{-x} x^0\,dx = \left(\frac{e^{-x}}{-1}\right)_0^\infty$

$$\boxed{\Gamma 1 = 1}$$

$= 0 + 1 = 1$

Reduction formula for Γn.

$$\Gamma n = (n-1)\,\Gamma n-1 \quad \text{for all } n>0$$

For, By def^n, $\Gamma n = \int_0^\infty e^{-x} x^{n-1}\,dx$

$$\Rightarrow \Gamma n = \left(x^{n-1}\frac{e^{-x}}{-1}\right)_0^\infty - \int_0^\infty \left((n-1)x^{n-2}\frac{e^{-x}}{-1}\right)dx$$

$$= 0 + (n-1)\int_0^\infty e^{-x} \cdot x^{(n-1)-1}\,dx$$

$$= (n-1)\,\Gamma n-1$$

Result:- $\Gamma n = (n-1)! \quad \forall\, n \in Z^+$

For, By the reduction formula,

$$\Gamma n = (n-1)\,\Gamma(n-1)$$
$$= (n-1)(n-2)\,\Gamma(n-2)$$
$$= (n-1)(n-2)(n-3)\,\Gamma(n-3)$$
$$\vdots$$
$$= (n-1)(n-2)(n-3)\cdots\cdots 3.2.1\,\Gamma 1$$
$$= (n-1)(n-2)(n-3)\cdots 3.2.1$$

$$\underline{\underline{\Gamma n = (n-1)!}}$$

Result:- $\Gamma \tfrac{1}{2} = \sqrt{\pi}$

For, By def^n, $\Gamma n = \int_0^\infty e^{-t}.\, t^{n-1} dt$

put $t = x^2 \Rightarrow dt = 2x\,dx$

$t = 0 \Rightarrow x = 0$; $t = \infty \Rightarrow x = \infty$

$$\therefore \Gamma n = \int_0^\infty e^{-x^2}.\, x^{2n-2}.\, 2x\,dx$$

$$\Rightarrow \Gamma n = 2\int_0^\infty e^{-x^2}.\, x^{2n-1}\,dx$$

$$\Rightarrow \Gamma \tfrac{1}{2} = 2\int_0^\infty e^{-x^2}\,dx \quad \text{--- (1)}$$

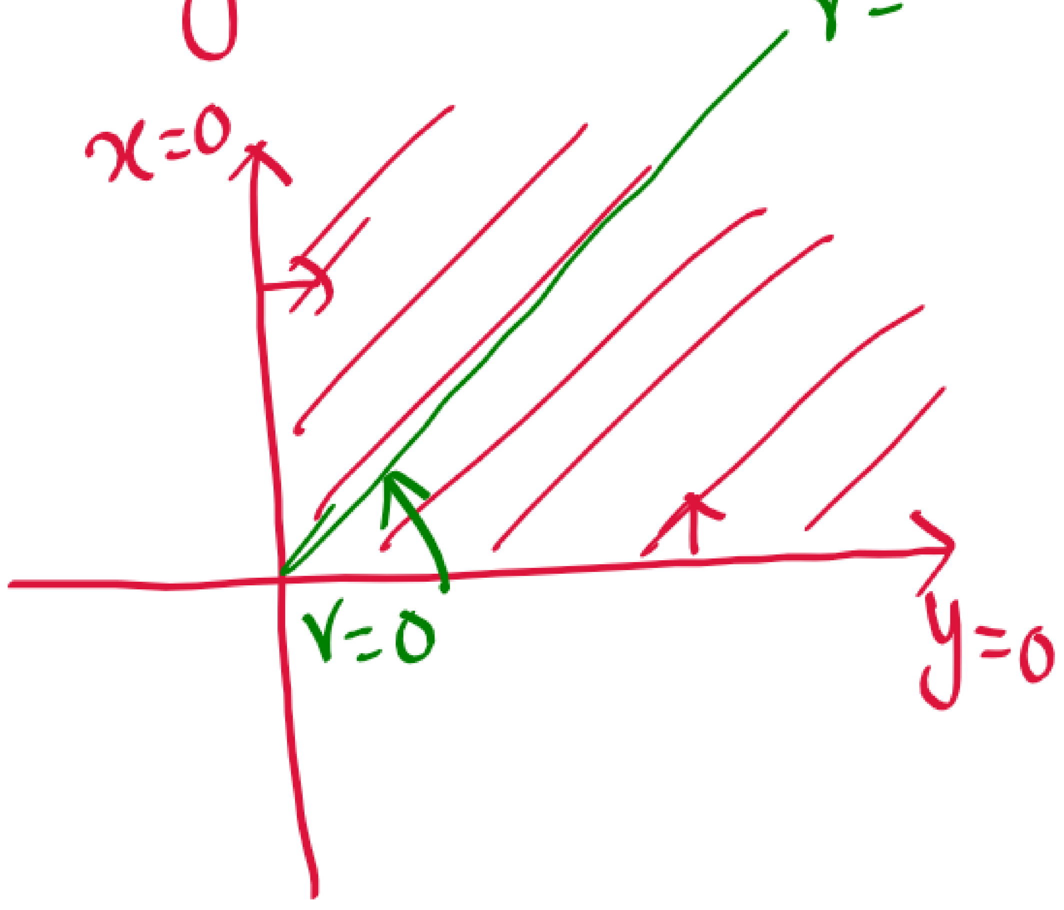
$|||^{ly}$ if we put $t=y^2$ we get

$$\Gamma\tfrac{1}{2} = 2\int_0^\infty e^{-y^2}\,dy \quad \text{—— ②}$$

From ① & ②

$$\therefore \left(\Gamma\tfrac{1}{2}\right)^2 = 2\int_0^\infty e^{-x^2}\,dx \times 2\times\int_0^\infty e^{-y^2}\,dy$$

$$= 4\int_{x=0}^{\infty}\int_{y=0}^{\infty} e^{-x^2-y^2}\,dy\,dx$$



Changing to polar coordinates,

$x = r\cos\theta \quad y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$$\therefore \left(\Gamma\tfrac{1}{2}\right)^2 = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\,r\,dr\,d\theta$$

put $r^2 = u$

$\Rightarrow du = 2r\,dr$

$\Rightarrow r\,dr = \frac{du}{2}$

$r=0 \Rightarrow u=0$

$r=\infty \Rightarrow u=\infty$

$$= 4\times\int_{\theta=0}^{\pi/2}\int_{u=0}^{\infty} e^{-u}\cdot\frac{du}{2}\,d\theta$$

$$= 2\int_{\theta=0}^{\pi/2}\left(\frac{e^{-u}}{-1}\right)_0^\infty d\theta = 2\times\int_0^{\pi/2} d\theta = \pi$$

ie; $\left(\Gamma\frac{1}{2}\right)^2 = \pi \Rightarrow \underline{\underline{\Gamma\frac{1}{2} = \sqrt{\pi}}}$

Eg:- Find $\int_0^\infty x^4 e^{-x}\, dx$

Ans:- $\int_0^\infty x^4 e^{-x}\, dx = \int_0^\infty e^{-x} x^{5-1}\, dx = \Gamma 5$

$= (5-1)!$

$= 4!$

$= \underline{\underline{24}}$

Q. Find $\int_0^\infty x^2 e^{-2x^2}\, dx$

Ans:- Let $I = \int_0^\infty x^2 e^{-2x^2}\, dx = \int_0^\infty x e^{-2x^2} \cdot x\,dx$

Put $2x^2 = t \Rightarrow \frac{dt}{dx} = 4x \Rightarrow \underline{\underline{x\,dx = \frac{dt}{4}}}$

$\Rightarrow x^2 = t/2$

$\Rightarrow x = \sqrt{t/2}$

when $x = 0 \Rightarrow t = 0$

$x = \infty \Rightarrow t = \infty$

$\therefore I = \frac{1}{\sqrt{2}} \int_0^\infty t^{1/2} e^{-t} \frac{dt}{4}$ $\left(\Gamma n = (n-1)\Gamma n-1 \quad \forall n > 0\right)$

$= \frac{1}{4\sqrt{2}} \int_0^\infty e^{-t} \cdot t^{3/2 - 1}\, dt = \frac{1}{4\sqrt{2}} \Gamma 3/2$

$= \frac{1}{4\sqrt{2}} \frac{1}{2} \Gamma\frac{1}{2} = \frac{\sqrt{\pi}}{8\sqrt{2}}$ //

(Since $\Gamma\frac{1}{2} = \sqrt{\pi}$)

Q. Find $\int_0^1 (\log x)^4 \, dx$

Ans:- Let $I = \int_0^1 (\log x)^4 \, dx$

Put $t = \log x$

$\Rightarrow -\frac{dt}{dx} = \frac{1}{x}$

$\Rightarrow dx = -x\,dt$

$\Rightarrow dx = -e^{-t} dt$

when $x = 0 \Rightarrow t = \infty$

when $x = 1 \Rightarrow t = 0$

$\therefore I = \int_\infty^0 (-t)^4 \cdot -e^{-t} \, dt$

$= \int_0^\infty e^{-t} \cdot t^4 \, dt$

$= \int_0^\infty e^{-t} \cdot t^{5-1} \, dt$

$= \Gamma 5 = 4! = \underline{\underline{24}}$

Q. Evaluate $\int_0^\infty \sqrt{x} \, e^{-x^3} dx$

Ans: put $x^3 = t$

Let $I = \int_0^\infty \sqrt{x} \, e^{-x^3} dx$

Ex?

$= \frac{1}{3} \int_0^\infty e^{-t} \, t^{-1/2} dt = \frac{1}{3} \int_0^\infty e^{-t} \cdot t^{1/2 - 1} dt$

$= \frac{1}{3} \Gamma \frac{1}{2} = \underline{\underline{\frac{\sqrt{\pi}}{3}}}$