## 1. Eigen values and eigen vectors

**Properties 1.1.**

- *For any square matrix $A$, the sum of the eigen values of $A$ is equal to the sum of the diagonal elements of $A$. The sum of the eigen values of $A$ is called the* ***trace of A****.*
- *For any square matrix $A$, the product of the eigen values of $A$ is equal to the* ***determinant A****.*
- *If $X$ is an eigen vector of a matrix $A$ corresponding to an eigen value $\lambda$ then $kX$ is also an eigen vector of $A$ for $\lambda$ where $k$ is any non-zero number.*
- *If $X_1$ & $X_2$ are non zero-eigen vectors of a matrix $A$ corresponding to an eigen value $\lambda$ then $k_1X_1 + k_2X_2$ is also an eigen vector of $A$ for $\lambda$ where $k_1, k_2$ are non-zero numbers.*
- *The eigen vectors corresponding to distinct eigen values of a matrix $A$ are linearly independent.*

**Problem 1.2.** *Find the eigenvalues and eigenvectors of* $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$. (Given)

**Solution:**

Ans:- characteristic eqⁿ ; $|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(3-\lambda) - 8 = 0$

$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$

$\Rightarrow \lambda = 5, -1$, eigen values

When $\lambda = 5$

Let $X = \begin{pmatrix} x \\ y \end{pmatrix}$ be the non zero vector such that $AX = 5X$. $\Rightarrow (A - 5I)X = 0$

$\Rightarrow \begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow -4x + 2y = 0$

Let $y = k$ be any real no. then $x = k/2$

$\therefore X = \begin{pmatrix} k/2 \\ k \end{pmatrix} = k \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$

coeff. matrix $= \begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix}$

$\overset{R_2 \to R_2 + R_1}{\sim} \begin{pmatrix} -4 & 2 \\ 0 & 0 \end{pmatrix}$ ✓

$\begin{pmatrix} -4 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow -4x + 2y = 0$

$\therefore$ The eigen vector corresponding to $\lambda = 5$ is

$X_1 = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$

When $\lambda = -1$;

Contd....

Let $X = \begin{pmatrix} x \\ y \end{pmatrix}$ be the nonzero vector

such that $AX = -1 \cdot X \Rightarrow (A+I)X = 0$

$$\Rightarrow \begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Echelon form of $\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} \sim \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$ $R_2 \to R_2 - 2R_1$

$$\Rightarrow \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 2x + 2y = 0$$

$$\Rightarrow x + y = 0 \checkmark$$

Let $x = k$ be any real no.

then $y = -k$

$$\therefore X = \begin{pmatrix} k \\ -k \end{pmatrix} = k \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\therefore$ eigen vector corresponding to $\lambda = -1$

is $X_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

When $\lambda = -3$ :-

**Contd...**

Let $X = \begin{pmatrix} x \\ y \end{pmatrix}$ be non-zero vector

s.t $AX = -3X \Rightarrow (A + 3I)X = 0$

$$\Rightarrow \begin{pmatrix} 5 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Echelon form

$$\Rightarrow \begin{pmatrix} 5 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 5x + y = 0$$

Let $x = k$ be any real no then

$$y = -5k$$

$$\therefore \quad X = \begin{pmatrix} k \\ -5k \end{pmatrix} = k \begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

$\therefore$ Eigen vector corresponding to $\lambda = -3$ is $X_2 = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$

Q. Find the eigen values and eigen vectors of the matrix $A = \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix}$. (Given)



Contd...

Ans:- Ch. eq$^n$. $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 \\ 5 & -2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda = \pm i,$$

are the eigen values

When $\lambda = i$

Let $X = \begin{pmatrix} x \\ y \end{pmatrix}$ be a nonzero vector

Such that $AX = iX \Rightarrow (A - iI)X = 0$

$$\Rightarrow \begin{pmatrix} 2-i & -1 \\ 5 & -2-i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ ——— } (*)$$

Coeff. matrix is $\begin{pmatrix} 2-i & -1 \\ 5 & -2-i \end{pmatrix}$

$R_1 \to \frac{R_1}{2-i}$

$$\frac{-1}{2-i} = \frac{-2-i}{5}$$

$$\sim \begin{pmatrix} 1 & -1/2-i \\ 5 & -2-i \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{-2-i}{5} \\ 5 & -2-i \end{pmatrix}$$

$R_2 \to R_2 - 5R_1$

$$\sim \begin{pmatrix} 1 & \frac{-2-i}{5} \\ 0 & 0 \end{pmatrix}$$ Echelon form

$$(*) \Rightarrow \begin{pmatrix} 1 & \frac{-2-i}{5} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x + \left(\frac{-2-i}{5}\right) y = 0$$

Contd...

Let $y = k$ be any real no then $x = \left(\frac{2+i}{5}\right)k$

$$\therefore X = \begin{pmatrix} \left(\frac{2+i}{5}\right)k \\ k \end{pmatrix} = k \begin{pmatrix} \frac{2+i}{5} \\ 1 \end{pmatrix}$$

$\therefore$ eigen vector corresponding to $\lambda = i$ is $X_1 = \begin{pmatrix} \frac{2+i}{5} \\ 1 \end{pmatrix}$

(Ex:) when $\lambda = -i$

Hint:- $x + \left(\frac{-2+i}{5}\right)y = 0$

If $y = k$ then $x = \left(\frac{2-i}{5}\right)k$

$\therefore$ eigen vector corresponding to $\lambda = -i$ is, $X_2 = \begin{pmatrix} \frac{2-i}{5} \\ 1 \end{pmatrix}$

**Problem 1.4.** *Find the eigen values and eigen vectors of* $A = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ (Given)

**Solution:**

Ch. eq$^n$ is, $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & -2 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$$

$$\Rightarrow \lambda = \underline{\underline{-1, 1, 2}} \text{ eigen values of } A.$$

<u>when $\lambda = -1$:</u>

Let $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the non-zero vector

such that $AX = -X \Rightarrow (A+I)X = 0$

$$\Rightarrow \begin{pmatrix} 3 & 1 & -2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ ——— } (*)$$

$$\text{coeff matrix} = \begin{pmatrix} 3 & 1 & -2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} R_1 \leftrightarrow R_2$$

$$\sim \begin{pmatrix} 1 & 1 & 0 \\ 3 & 1 & -2 \\ 0 & 1 & 1 \end{pmatrix} R_2 \to R_2 - 3R_1 \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{pmatrix} R_3 \to R_3 + \tfrac{1}{2}R_2$$

$$\sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \text{ echelon form}$$



**Contd....**

$$\therefore (*) \Rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x+y \\ -2y-2z \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{matrix} x+y=0 \\ y+z=0 \end{matrix}\right\}$$

Let $z = k$ be any real no then $y = -k$

$\Rightarrow x = k$.

$$\therefore \quad X = \begin{pmatrix} k \\ -k \\ k \end{pmatrix} = k\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$\therefore$ eigen vector corresponding to $\lambda = 1$ is

$$X_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

When $\lambda = 1$ :-

Let $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the non-zero vector

Such that $AX = 1 \cdot X \Rightarrow (A - I)X = 0$

$$\Rightarrow \begin{pmatrix} 1 & 1 & -2 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Echelon form

$$\Rightarrow \begin{pmatrix} 1 & 1 & -2 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Contd....

$$\Rightarrow \left.\begin{array}{r} x+y-2z=0 \\ y-z=0 \end{array}\right\}$$

Let $z=k$ be any real no then

$y=k$

$\Rightarrow x=k$

$\therefore X = \begin{pmatrix} k \\ k \\ k \end{pmatrix} = k \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\therefore$ eigen vector corresponding to $\lambda=1$ is $X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ //

When $\lambda = 2$.

Hint

$AX = 2X \Rightarrow (A-2I)X = 0$

$$\Rightarrow \begin{pmatrix} 0 & 1 & -2 \\ 1 & -2 & 0 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Echelon form

$$\Rightarrow \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{array}{r} x-2y=0 \\ y-2z=0 \end{array}\right\} \checkmark$$

Let $z=k$ be any real no. then $y=2k$

$\Rightarrow x=4k$

**Contd....**

$$\therefore \quad X = \begin{pmatrix} 4K \\ 2K \\ K \end{pmatrix} = K \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

$\therefore$ eigen vector corresponding to $\lambda = 2$

is $\quad X_3 = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$

**Problem 1.5.** *Find the eigen values and eigen vectors of* $A = \begin{pmatrix} 3 & -1 & 2 \\ 3 & -1 & 6 \\ -2 & 2 & -2 \end{pmatrix}$

Ans:- ch. eqⁿ; $|A - \lambda I| = 0$

$\Rightarrow \lambda = \underbrace{2, 2}_{\text{repeated}}, -4$ ✓ are the eigen values of $A$.

<u>when $\lambda = 2$ :-</u>

Let $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be a non-zero vector

such that $(A - 2I)X = 0$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 2 \\ 3 & -3 & 6 \\ -2 & 2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Echelon form $\Rightarrow$

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x - y + 2z = 0$$

Let $y = k_1$ and $z = k_2$ be any real no's then $x = k_1 - 2k_2$

then $X = \begin{pmatrix} k_1 - 2k_2 \\ k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} k_1 \\ k_1 \\ 0 \end{pmatrix} + \begin{pmatrix} -2k_2 \\ 0 \\ k_2 \end{pmatrix}$

$$X = k_1\begin{pmatrix}1\\1\\0\end{pmatrix} + k_2\begin{pmatrix}-2\\0\\1\end{pmatrix}$$



**Contd...**

$\therefore$ the two independent eigenvectors for $\lambda = 2$ are $X_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}$ and $X_2 = \begin{pmatrix}-2\\0\\1\end{pmatrix}$

(Ex)

When $\lambda = -4$ :-

Hint:- $(A + 4I)X = 0$

$$\Rightarrow \begin{pmatrix}7 & -1 & 2\\3 & 3 & 6\\-2 & 2 & 2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Echelon form

$$\Rightarrow \begin{pmatrix}7 & -1 & 2\\0 & 2 & 3\\0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow \quad 7x - y + 2z = 0$$
$$2y + 3z = 0$$

Let $z = k$ be any real no. then

$$y = -\tfrac{3}{2}k$$

$$7x = -\tfrac{3}{2}k - 2k = -\frac{7}{2}k \Rightarrow x = -\frac{k}{2}$$

**Contd...**

$$\therefore\ X = \begin{pmatrix} -K/2 \\ -3/2\,K \\ K \end{pmatrix} = \frac{K}{2}\begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix}$$

$\therefore$ eigen vector corresponding to $\lambda = 4$ is $X_3 = \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix}$

HW

Q. Find the eigen values and the corresponding eigen vectors of

(i) $A = \begin{pmatrix} -2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ Here : $\lambda = 1, -2, -2$

(ii) $B = \begin{pmatrix} 2 & 1 \\ 2 & -2 \end{pmatrix}$

(iii) $C = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$

**Problem 1.6.** *Find the eigen values and eigen vectors of* $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 4 \end{pmatrix}$

**2. Numerically largest eigen value using RAYLEIGH POWER METHOD**

Let $A$ be a square matrix

$$\underline{AX = \lambda X} \qquad X^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Let $X^{(0)}$ be the initial approx vector

then <u>Iteration I</u>:- $A_{n\times n} X^{(0)}_{n\times 1} = \begin{pmatrix} \ \end{pmatrix}_{n\times 1}$

$$= \lambda^{(1)} \left( X^{(1)} \right)$$

<u>Iteration II</u>: $AX^{(1)} = \begin{pmatrix} \ \end{pmatrix}_{n\times 1} = \lambda^{(2)} X^{(2)}$

<u>Iteration III</u> $AX^{(2)} = \begin{pmatrix} \ \end{pmatrix}_{n\times 1} = \lambda^{(3)} X^{(3)}$

<u>Iteration IV</u> $AX^{(4)} = \begin{pmatrix} \ \end{pmatrix}_{n\times 1} = \lambda^{4} X^{(4)}$

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Q. Determine the largest eigen value and the corresponding eigen vector for the matrix

$$A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$$ (correct to two decimal places) by taking $X^{(0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Ans:- Iteration I :- $AX^{(0)} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 0.2 \end{pmatrix} = \lambda^{(1)} X^{(1)}$

Iteration II :- $AX^{(1)} = \begin{pmatrix} 5.8 \\ 1.4 \end{pmatrix} = 5.8\begin{pmatrix} 1 \\ 0.241 \end{pmatrix} = \lambda^{(2)} X^{(2)}$

Iteration III :- $AX^{(2)} = \begin{pmatrix} 5.964 \\ 1.482 \end{pmatrix} = 5.964\begin{pmatrix} 1 \\ 0.24844 \end{pmatrix} = \lambda^{(3)} X^{(3)}$

Iteration IV :- $AX^{(3)} = \begin{pmatrix} 5.99376 \\ 1.49688 \end{pmatrix} = 5.99376\begin{pmatrix} 1 \\ 0.24973 \end{pmatrix} = \lambda^{(4)} X^{(4)}$

Iteration V :- $AX^{(4)} = \begin{pmatrix} 5.99892 \\ 1.49946 \end{pmatrix} = 5.99892\begin{pmatrix} 1 \\ 0.24995 \end{pmatrix} = \lambda^{(5)} X^{(5)}$

Here $\left.\begin{matrix} \lambda^{(5)} \text{ and } \lambda^{(4)} \\ X^{(5)} \text{ and } X^{(4)} \end{matrix}\right\}$ are equal upto two decimal decimal places

$\therefore$ Largest eigen value is $5.9989 \approx 6$

and the corresponding eigen vector is

$$X = \underline{\underline{\begin{pmatrix} 1 \\ 0.25 \end{pmatrix}}}.$$