Lecture 10 - Date : **04 June 2021**

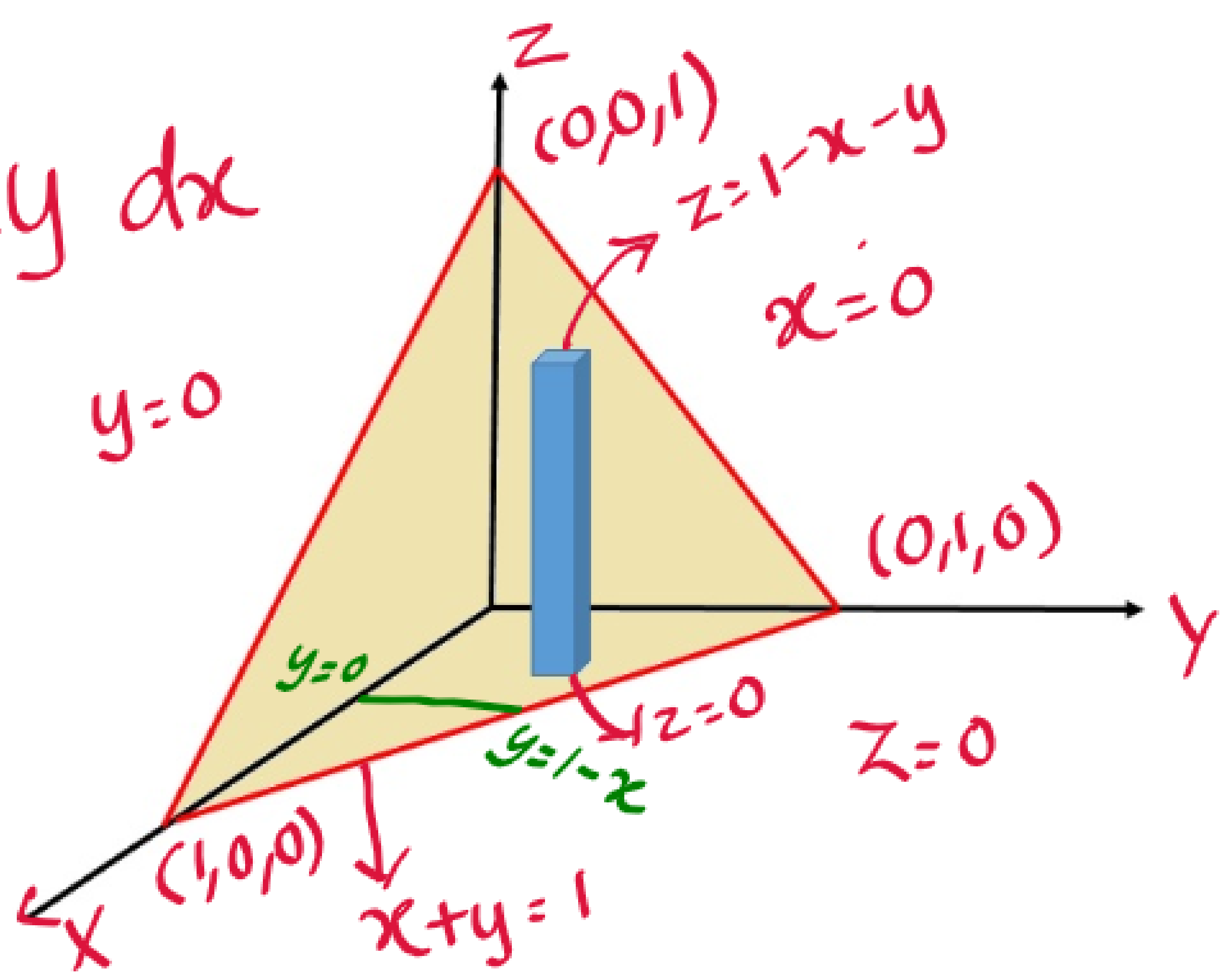
1. Problems on Triple Integrals

Problem 1.1. *Evaluate*

$$\text{Let } I = \iiint_V (x+y+z)\,dx\,dy\,dz$$

where V is the volume of the solid formed by the tetrahedran $x+y+z=1$ with the coordinate planes.

Ans:- $I = \displaystyle\int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{z=1-x-y}(x+y+z)\,dz\,dy\,dx$



$$\Rightarrow I = \int_{x=0}^{1}\int_{y=0}^{1-x}\left[(x+y)z+\frac{z^2}{2}\right]_{z=0}^{z=1-x-y} dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}\left[(x+y)(\overbrace{1-x-y}^{=1-(x+y)}) + \frac{(\overbrace{1-x-y}^{=1-(x+y)})^2}{2}\right] dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}\left[\cancel{(x+y)} - (x+y)^2 + \frac{1}{2} - \cancel{(x+y)} + \frac{(x+y)^2}{2}\right] dy\,dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} \left[\frac{1}{2} - \frac{1}{2}(x+y)^2 \right] dy\, dx$$

$$= \int_{x=0}^{1} \left[\frac{y}{2} - \frac{1}{2} \frac{(x+y)^3}{3} \right]_{y=0}^{1-x} dx$$

$$= \int_{x=0}^{1} \left[\frac{(1-x)}{2} - \frac{1}{6} + \frac{1}{6} x^3 \right] dx$$

$$= \left(\frac{1}{2}x - \frac{x^2}{4} - \frac{1}{6}x + \frac{x^4}{24} \right)_0^1 = \frac{1}{2} - \frac{1}{4} - \frac{1}{6} + \frac{1}{24}$$

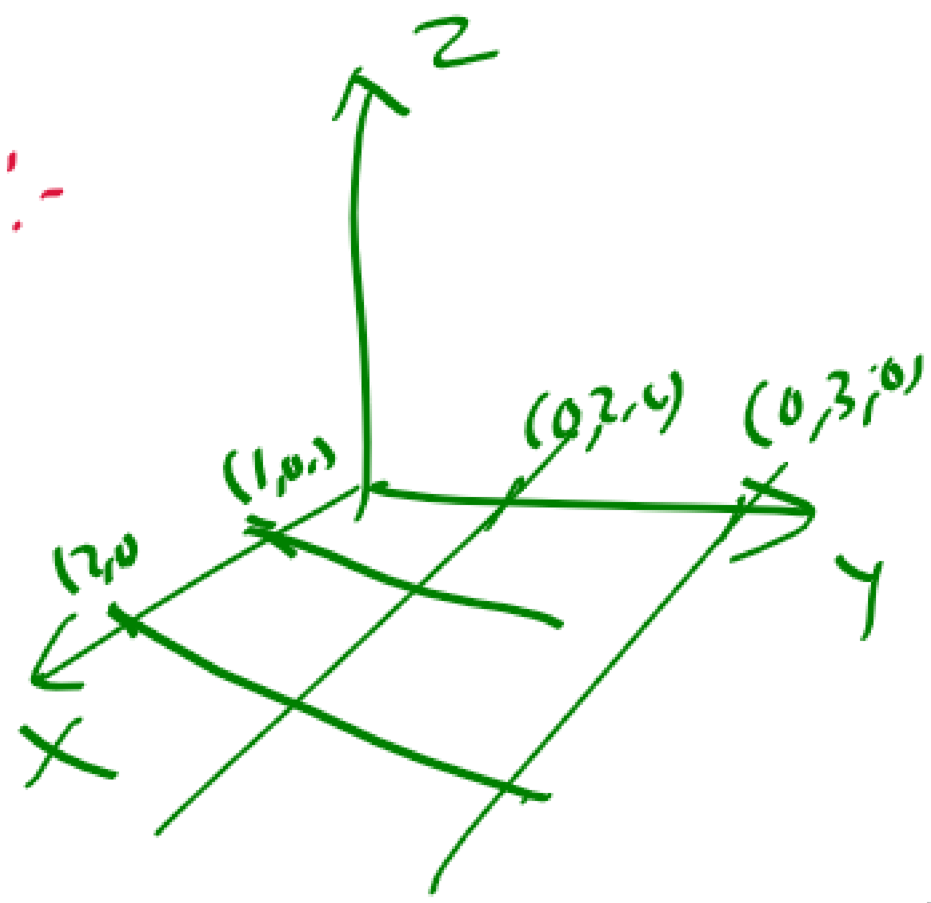
$$= \frac{1}{8}$$

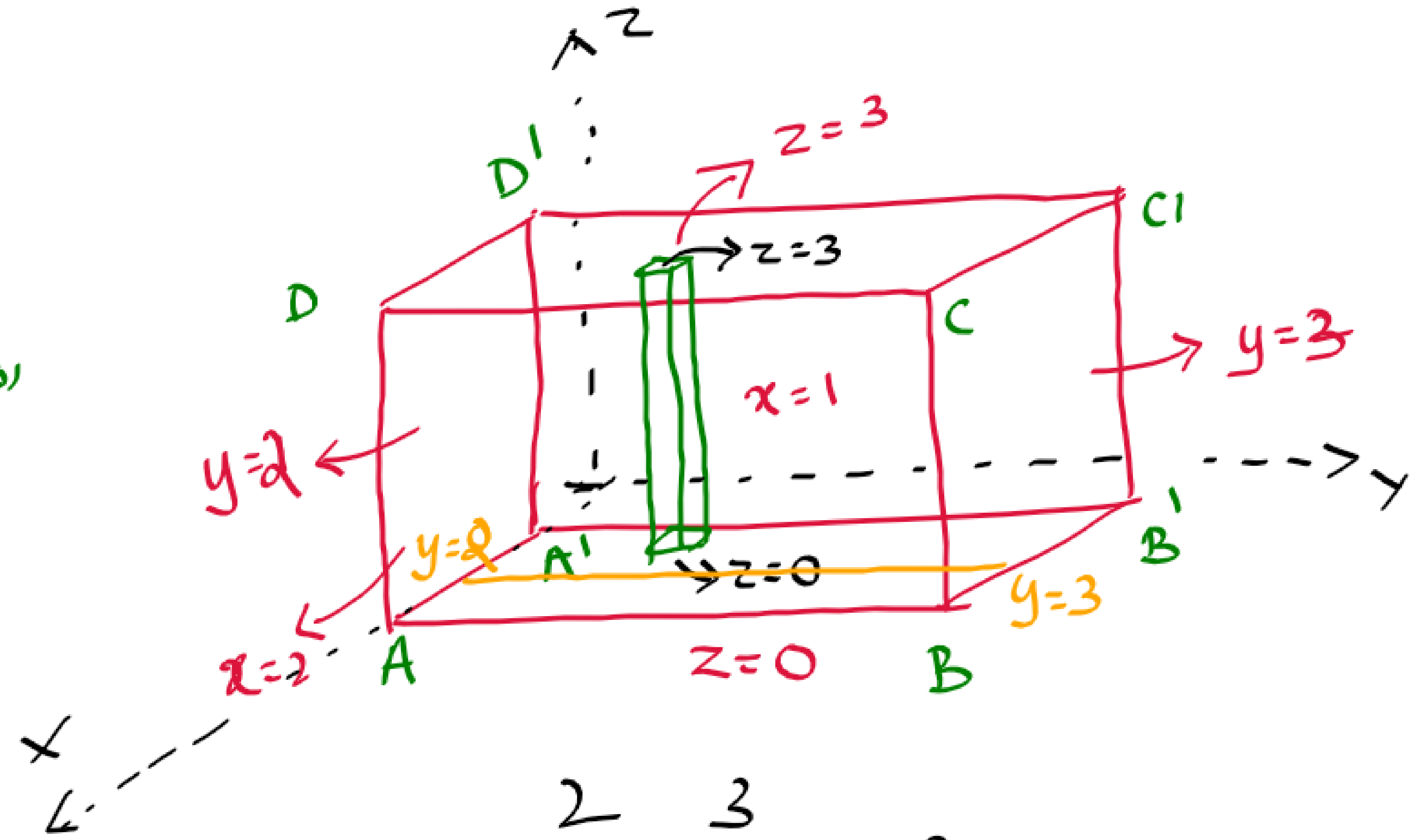
Problem 1.2. *Evaluate*

$$\iiint_V \left(x^2 + y^3 + z\right) dx\, dy\, dz$$

where V *is the volume bounded by the planes* $x = 1, x = 2, y = 2,$ $y = 3, z = 0, z = 3.$

Ans:-

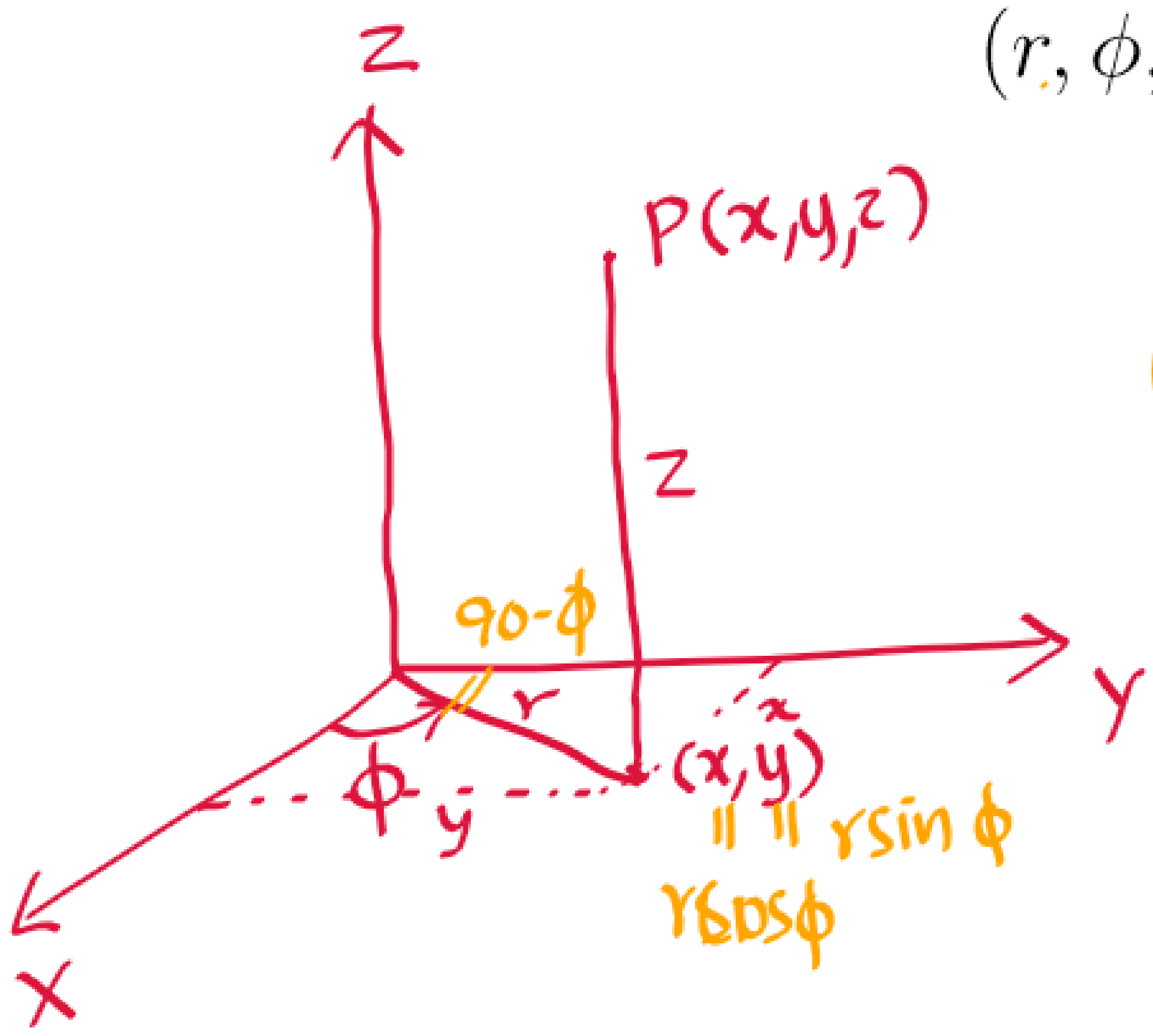




$$\therefore \iiint_V (x^2+y^3+z)\,dx\,dy\,dz = \int_{x=1}^{2}\int_{y=2}^{3}\left(\int_{z=0}^{3}(x^2+y^3+z)\,dz\right)dy\,dx$$

$= (\text{Ex})$. Ans: $\dfrac{241}{4}$

TO CHANGE CARTESIAN COORDINATES (x, y, z) TO CYLINDRICAL polar COORDINATES (r, ϕ, z)



$$(x,y,z) \longmapsto (r\cos\phi, r\sin\phi, z)$$

By changing from cartesian coordinates to cylindrical polar cordinates

$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$z = z$$

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint f(r\cos\phi, r\sin\phi, z)\,|J|\,dr\,d\phi\,dz$$

where $|J| = \dfrac{\partial(x,y,z)}{\partial(r,\phi,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix}$

$$\therefore |J| = \begin{vmatrix} \cos\phi & -r\sin\phi & 0 \\ \sin\phi & r\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

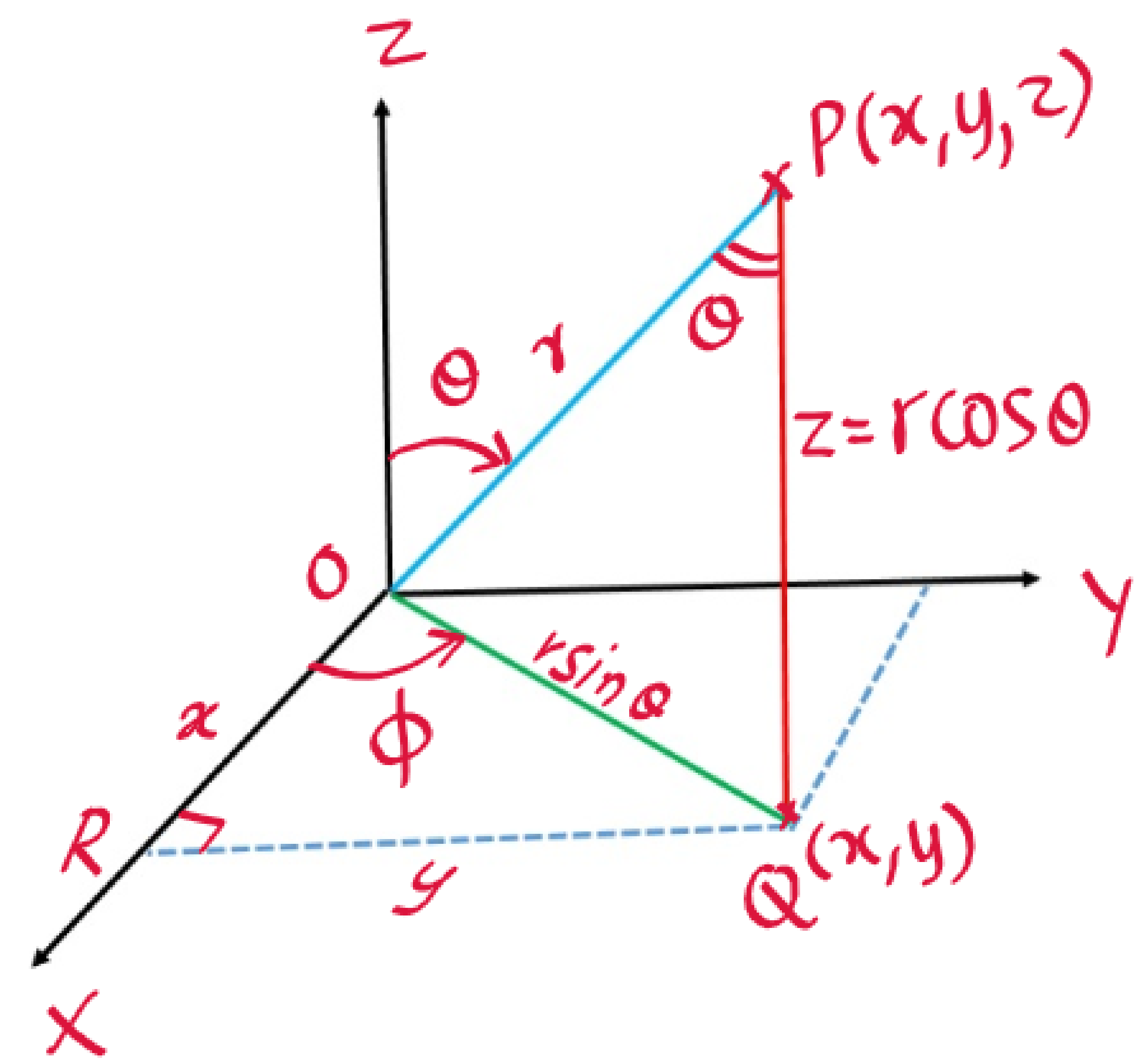
$$\therefore \iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint f(r\cos\phi, r\sin\phi, z)\, r\,dr\,d\phi\,dz$$

To change cartesian coordinates (x, y, z) to spherical polar coordinates (r, θ, ϕ)

From r'gt Δ OQP,

$$\cos\theta = \frac{PQ}{OP} \Rightarrow z = r\cos\theta$$

$$\sin\theta = \frac{OQ}{OP} \Rightarrow OQ = r\sin\theta$$



From the r'gt Δ^{le}, ORQ, $\cos\phi = \dfrac{x}{r\sin\theta} \Rightarrow x = r\sin\theta\cos\phi$

$$\sin\phi = \frac{y}{r\sin\theta} \Rightarrow y = r\sin\theta\sin\phi$$

$\therefore$
$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$\therefore \iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\,|J|\,dr\,d\theta\,d\phi$$

where
$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = ? = r^2\sin\theta$$

(Ex)

$$\therefore \iiint f(x,y,z)\,dx\,dy\,dz = \iiint f[r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta]\, r^2\sin\theta\,dr\,d\theta\,d\phi$$

Volume of a solid using triple integrals

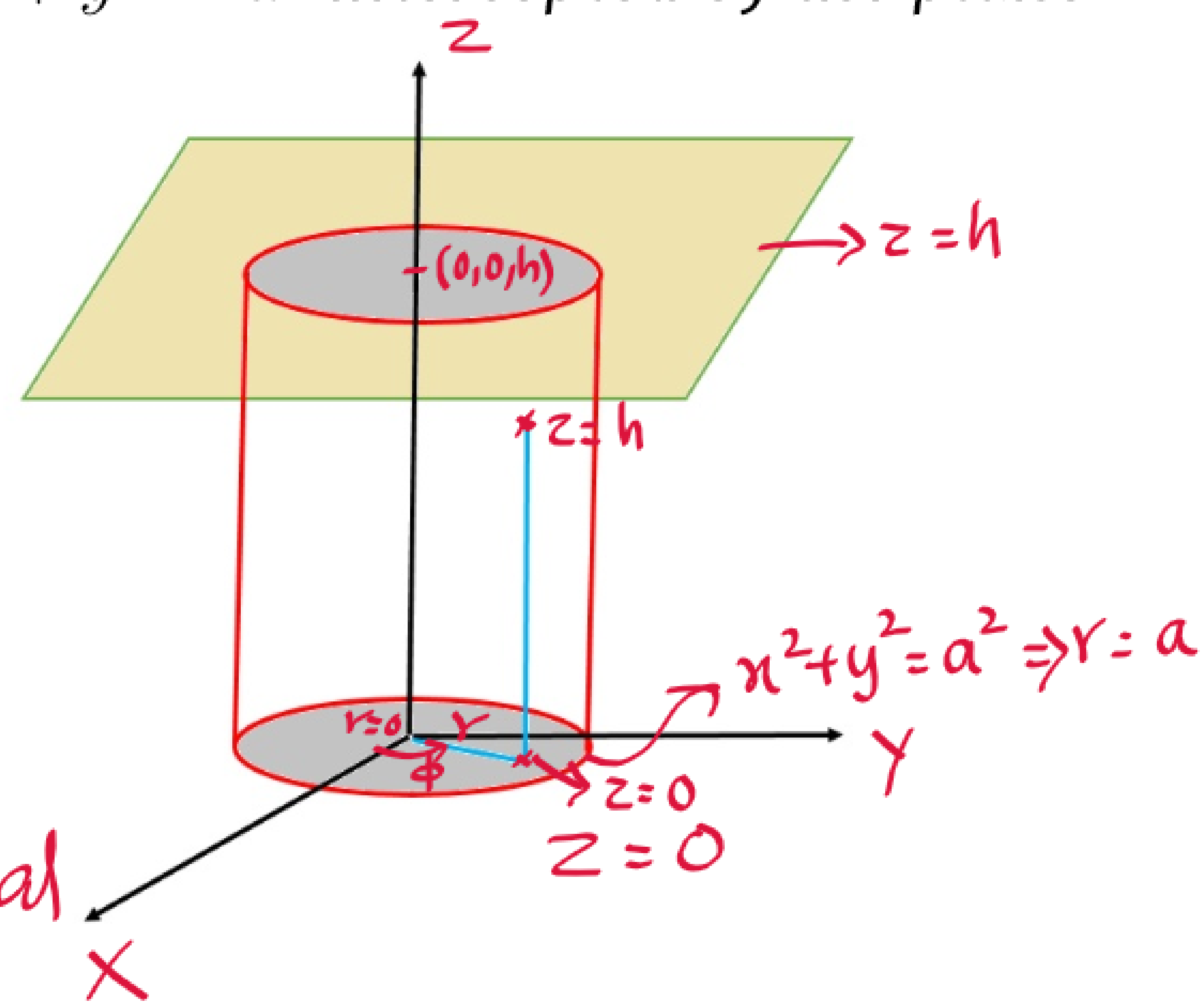
Volume of a solid in 3'dimensional region is given by,

$$\iiint_V dx\,dy\,dz$$

Problem 1.3. *Evaluate*

$$\iiint_V z\left(x^2+y^2+z^2\right)dx\,dy\,dz$$

where V is the volume of the cylinder $x^2+y^2=a^2$ intercepted by the plane $z=0$ and $z=h$.



Ans:-

Let $I = \iiint_V z(x^2+y^2+z^2)\,dx\,dy\,dz$

Changing (x,y,z) to cylindrical polar coordinates,

put $x = r\cos\phi$, $y = r\sin\phi$ $z=z$

and $dx\,dy\,dz = r\,dr\,d\phi\,dz$

$$\therefore \iiint_V z(x^2+y^2+z^2)\,dx\,dy\,dz = \int_{\phi=0}^{2\pi}\int_{r=0}^{a}\int_{z=0}^{h} z(r^2+z^2)\,r\,dz\,dr\,d\phi$$

$$= \int_{\phi=0}^{2\pi}\int_{r=0}^{a} r\left(\frac{r^2z^2}{2}+\frac{z^4}{4}\right)_{z=0}^{h} dr\,d\phi$$

$$= \int_{\phi=0}^{2\pi}\int_{r=0}^{a} r\left(\frac{h^2}{2}r^2+\frac{h^4}{4}\right) dr\,d\phi$$

$$= \int_{\phi=0}^{2\pi} \left(h^2 \frac{r^4}{8} + h^4 \frac{r^2}{8} \right)_{r=0}^{a} d\phi$$

$$= \frac{h^2 a^4 + h^4 a^2}{8} \int_{\phi=0}^{2\pi} d\phi = \underline{\underline{\frac{\pi h^2 a^2}{4} \left(a^2 + h^2 \right)}}.$$

Problem 1.4. *Evaluate*

Let $I = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{z=\sqrt{1-x^2-y^2}} \frac{dz\, dy\, dx}{\sqrt{1-x^2-y^2-z^2}}$

by changing to spherical polar coordinates.

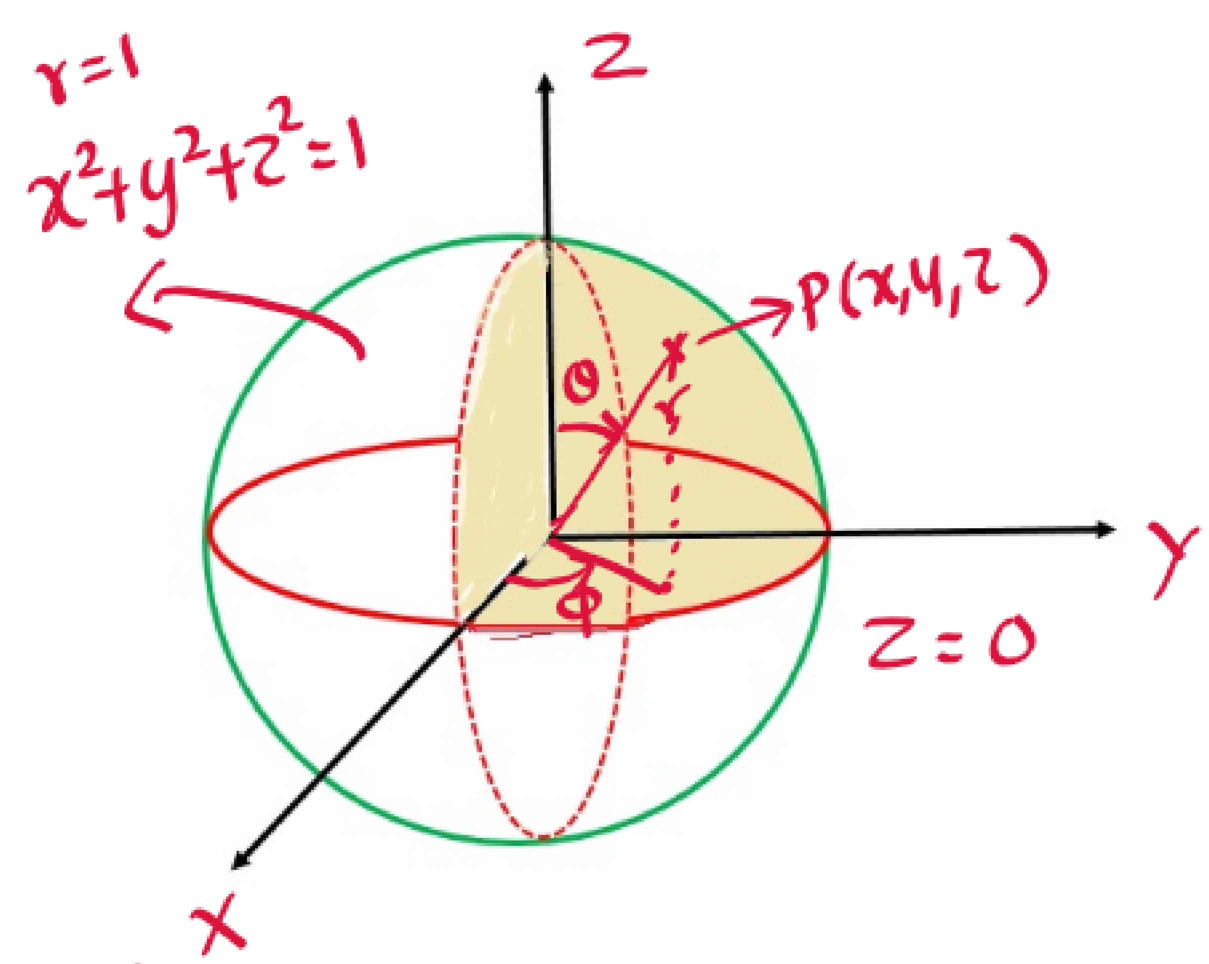
Ans: Region bdd by,

plane $z=0$ to the surface $z=\sqrt{1-x^2-y^2}$
$\Rightarrow x^2+y^2+z^2=1$

line $y=0$ to the curve $y=\sqrt{1-x^2}$
ie; $x^2+y^2=1$

line $x=0$ to the line $x=1$

$\therefore$ the enclosed region is the volume of the sphere in the first octant.



By changing to Spherical coordinates

$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$ $\quad$ & $dx\,dy\,dz = r^2\sin\theta\, dr\, d\theta\, d\phi$

Here: r varies from 0 to 1; θ: 0 to $\pi/2$
ϕ: 0 to $\pi/2$

$$\therefore\ I = \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} \frac{1}{\sqrt{1-r^2}} \cdot r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \frac{\left[1-(1-r^2)\right]}{\sqrt{1-r^2}} \sin\theta \, dr\, d\theta\, d\phi$$

$$= \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \sin\theta \left[\frac{1}{\sqrt{1-r^2}} - \sqrt{1-r^2}\right] dr\, d\theta\, d\phi$$

$$= \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \sin\theta \left[\sin^{-1}(r) - \frac{r}{2}\sqrt{1-r^2} - \frac{1}{2}\sin^{-1}(r)\right]_{r=0}^{1} d\theta\, d\phi$$

$$= \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \frac{\pi}{4} \sin\theta \, d\theta\, d\phi$$

$$= \frac{\pi}{4}\int_{\phi=0}^{\pi/2} \left(-\cos\theta\right)_{\theta=0}^{\pi/2} d\phi \quad = \quad \frac{\pi}{4}\int_{\phi=0}^{\pi/2} d\phi \; = \underline{\underline{\frac{\pi^2}{8}}}$$

Problem 1.5. *Evaluate*

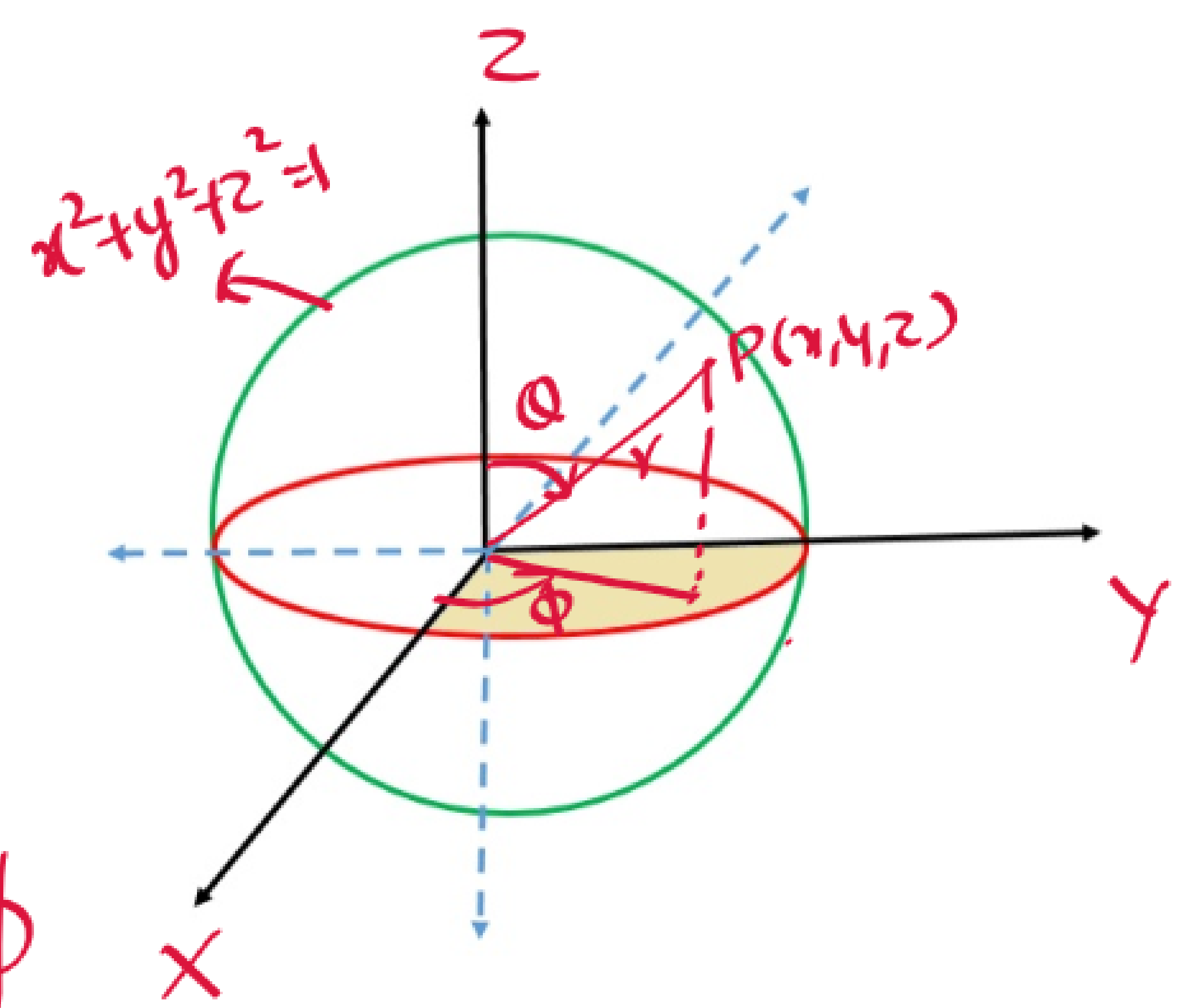
$$\iiint_V \left(x^2 + y^2 + z^2\right)\, dx\, dy\, dz$$

by changing to spherical polar coordinates where V *is the volume of the spherical region* $x^2 + y^2 + z^2 \leq 1$.

Ans:- By changing to spherical polar coordinates,

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

and $dx\,dy\,dz = r^2\sin\theta\, dr\, d\theta\, d\phi$



$$\therefore I = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{1} r^2\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

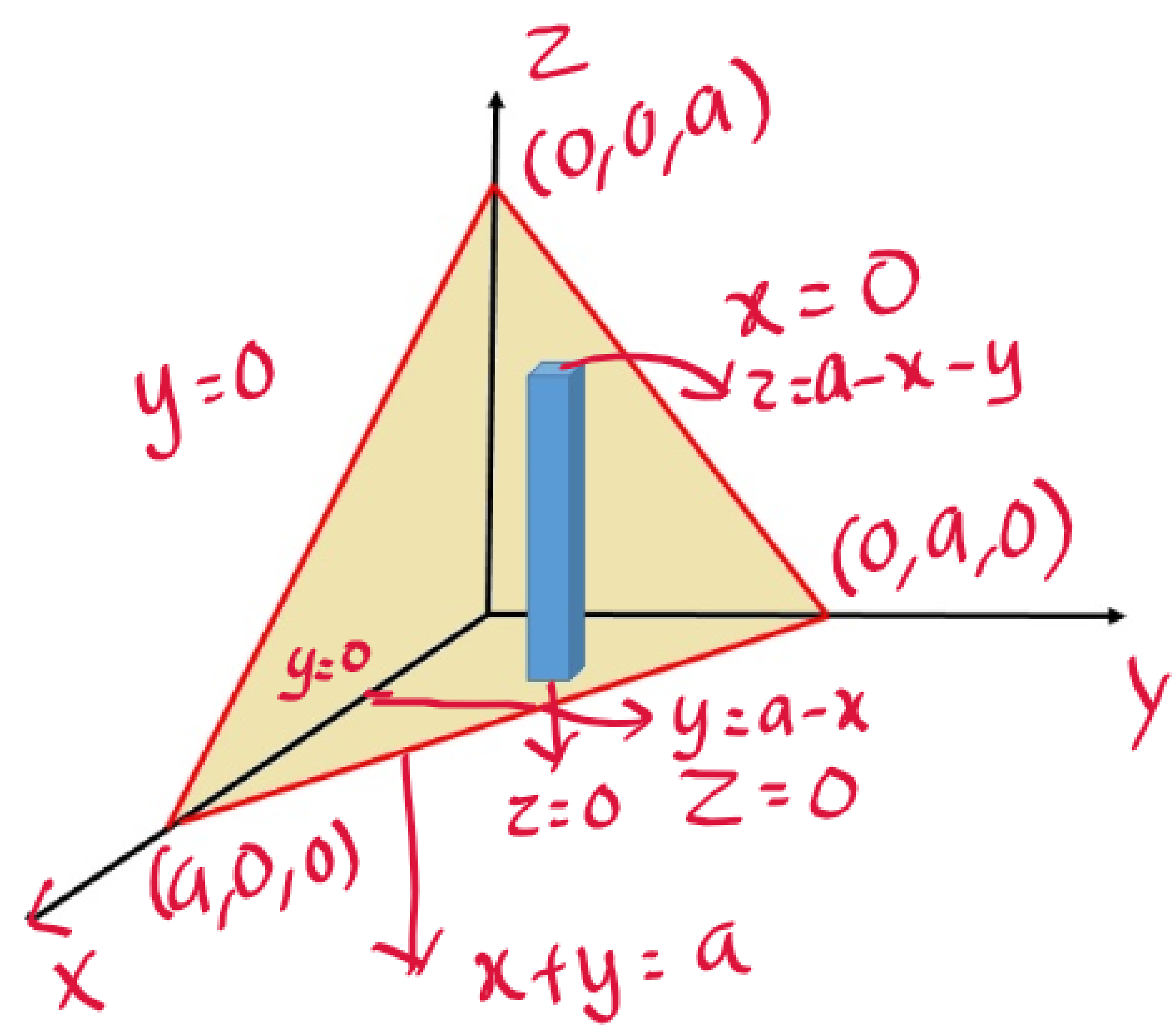
$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{1} r^4 \sin\theta\, dr\, d\theta\, d\phi$$

$$= (Ex) \qquad = \frac{4\pi}{5}$$

Problem 1.6. *Using triple integrals, find the volume of the solid bounded by the planes* $x = 0, y = 0, x + y + z = a$ *and* $z = 0$.

Ans:- we've,

$$\text{Volume} = \iiint_V dx\,dy\,dz$$



$$= \int_{x=0}^{a}\int_{y=0}^{a-x}\int_{z=0}^{a-x-y} dz\,dy\,dx$$

$$= \int_{x=0}^{a}\int_{y=0}^{a-x} (z)_{z=0}^{a-x-y}\,dy\,dx = \int_{x=0}^{a}\int_{y=0}^{a-x} (a-x-y)\,dy\,dx$$

$$= \int_{x=0}^{a}\left[(a-x)y - \frac{y^2}{2}\right]_{y=0}^{a-x} dx$$

$$= \int_{x=0}^{a}\left[(a-x)^2 - \frac{(a-x)^2}{2}\right] dx$$

$$= \frac{1}{2}\int_{x=0}^{a}(a-x)^2\,dx = \left(\frac{-1}{6}(a-x)^3\right)_0^a$$

Ans: $\frac{a^3}{6}$ (cubic units)