Maximum Likelihood Estimate for θ (MLE)

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample from random variable X and let $x_1, x_2, x_3, \ldots, x_n$ be sample values. We define likelihood function L as following function

$L(X_1, X_2, X_3, \ldots, X_n; \theta) = f(x_1, \theta) f(x_2, \theta) \ldots f(x_n, \theta)$.

MLE of θ say $\hat{\theta}$ based on random sample $X_1, X_2, X_3, \ldots, X_n$ is that value of θ that maximizes $L(X_1, X_2, X_3, \ldots, X_n; \theta)$.

Example 1. Let $X_1, X_2, X_3, \ldots, X_n$ denote a random sample of size n from a distribution having pdf $f(x,\theta) = \begin{cases} \theta^x (1-\theta^{1-x}), & 0 \le \theta \le 1 \\ 0, & elsewhere \end{cases}$. Find a MLE for θ.

Solution: $L(X_1, X_2, X_3, \ldots, X_n; \theta) = f(x_1, \theta) f(x_2, \theta) \ldots f(x_n, \theta)$

$$= \theta^{x_1}(1-\theta^{1-x_1}).\theta^{x_2}(1-\theta^{1-x_2})\ldots\theta^{x_n}(1-\theta^{1-x_n}),$$

$$= \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{n-\sum_{i=1}^{n} x_i}$$

Taking logarithm on both sides, and then partially differentiating with respect to θ,

$$\frac{\partial(\log L)}{\partial\theta} = \sum_{i=1}^{n} x_i \cdot \frac{1}{\theta} + \left(n - \sum_{i=1}^{n} x_i\right)\left(\frac{-1}{1-\theta}\right)$$

For maximum, $\dfrac{\partial(\log L)}{\partial\theta} = 0$.

On simplifying , MLE of θ, $\hat{\theta} = \bar{X}$.

Example 2. Let $X_1, X_2, X_3, \ldots, X_n$ denote a random sample of size n from a distribution having pdf $f(x,\theta) = \begin{cases} \dfrac{\theta^x e^{-\theta}}{x!}, & 0 \le \theta \le 1 \\ 0, & elsewhere \end{cases}$. Find MLE for θ.

Solution. $L(X_1, X_2, X_3, \ldots, X_n; \theta) = f(x_1, \theta) f(x_2, \theta) \ldots f(x_n, \theta)$

$$= \frac{\theta^{x_1} e^{-\theta}}{x_1!} \cdot \frac{\theta^{x_2} e^{-\theta}}{x_2!} \ldots \frac{\theta^{x_n} e^{-\theta}}{x_n!}$$

$$= \frac{\theta^{\sum_{i=1}^{n} x_i} n e^{-\theta}}{\prod_{i=1}^{n} x_i !}.$$

Taking log and then putting $\frac{\partial(\log L)}{\partial \theta} = 0$, we get $\hat{\theta} = \bar{X}$.

Example 3. Find MLE for normal distribution $N(\theta_1, \theta_2)$ where $-\infty < \theta_1 < \infty$, $0 < \theta_2 < \infty$.

Solution:

Pdf: $f(x, \theta_1, \theta_2) = \frac{1}{\sqrt{2\pi\theta_2}} e^{\frac{-(x-\theta_1)^2}{2\theta_2^2}}$

$$L(X_1, X_2, X_3, \ldots, X_n; \theta_1, \theta_2) = f(x_1, \theta_1, \theta_2) f(x_2, \theta_1, \theta_2) \ldots f(x_n, \theta_1, \theta_2)$$

$$= \frac{1}{\sqrt{2\pi\theta_2}} e^{\frac{-(x_1-\theta_1)^2}{2\theta_2^2}} . \frac{1}{\sqrt{2\pi\theta_2}} e^{\frac{-(x_2-\theta_1)^2}{2\theta_2^2}} \ldots \frac{1}{\sqrt{2\pi\theta_2}} e^{\frac{-(x_n-\theta_1)^2}{2\theta_2^2}}$$

Taking logarithm on both sides and then differentiating partially with respect to θ_1 ,

$$\frac{\partial}{\partial \theta_1}(\log L) = \frac{1}{2\theta_2}\left[2\sum_{i=1}^{n}(x_i - \theta_1)\right]$$

For maximum, $\frac{\partial}{\partial \theta_1}(\log L)$ =0. Simplifying, $\hat{\theta}_1 = \bar{X}$.

Differentiating partially with respect to θ_2 ,

$$\frac{\partial}{\partial \theta_2}(\log L) = -\frac{n}{2\theta_2} + \frac{1}{2\theta_2^2}\left[\sum_{i=1}^{n}(x_i - \theta_1)^2\right]$$

For maximum, $\frac{\partial}{\partial \theta_2}(\log L) = 0$, we get

$$\theta_2 = \frac{\sum_{i=1}^{n}(x_i - \bar{X})^2}{n} = s^2 .$$