PERMUTATION AND COMBINATION

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COMBINATIONS AND PERMUTATIONS

Combinations, permutations, partitions and compositions are the simplest and the most widely used mathematical objects in combinatorics.

Rule of product: If one experiment has m possible outcomes and another experiment has n possible outcomes, then there are $m \times n$ possible outcomes when both of these experiments take place.

Rule of sum: If one experiment has m possible outcomes and another experiment has n possible outcomes, then there are m + n possible outcomes when exactly one of these experiments takes place.

Definition 1.1

A combination of n objects taken r at a time (called an r-combination of n elements) is a selection of r of the objects where the order of the objects in the selection is immaterial.

Definition 1.2

A permutation of n objects taken r at a time (called an r- permutation of n elements) is an ordered selection of r of the objects.

Suppose there are four objects a, b, c and d and selections of them are made two at a time.
The combinations without repetition are six in number, namely
ab, ac, ad, bc, bd, cd
and there are ten with repetition, which are
$aa, ab, ac, ad, bb, bc, bd, cc, cd, dd$.

The twelve permutations without repetition are
$ab, ac, ad, ba, bc, bd, ca, cb, cd, da, db, dc$
and the sixteen with repetition are
$aa, ab, ac, ad, ba, bb, bc, bd, ca, cb, cc, cd, da, db, dc, dd$.

The Enumerators for Permutations and Combinations

1. The number of r-permutations of n objects, $P(n,r) = {}^n P_r$, is given by
$$P(n,r) = n(n-1)...(n-r+1) = \frac{n!}{(n-r)!}$$
2. When unlimited repetition in an r-permutation of n objects is allowed then after choosing the first object in n ways, the next object can also be chosen in n ways and so on. This can be done in n^r ways.
3. Consider n objects of which m_1 are of the first kind, m_2 are of the second kind, ..., m_k are of the k^{th} kind, so that $\sum_{i=1}^{k} m_i = n$.
The number of permutations of all the objects in this case is
$$\frac{n!}{m_1!m_2!\dots m_k!}$$

4. The number of r-combinations of n objects without repetition is denoted by $C(n,r) = {}^nC_r = \dfrac{n!}{r!(n-r)!}$

5. The general formula for r-combinations of n objects when unlimited repetition is allowed is more difficult to obtain - we cannot simply divide the permutation result for unlimited repetition, n^r, by an appropriate factor since different combinations with repetition will not in general give rise to the same number of permutations.
For example (aab) gives rise to three different permutations while (abc) gives six permutations.

Let one r-combination of n objects (which are considered to be the digits $1, 2, 3, ..., n$) in which repetition is allowed be $(c_1, c_2, \ldots, c_r)$, and suppose $c_1, c_2, \ldots, c_r$ are in rising order, i. e. $c_1 \leq c_2 \leq \ldots \leq c_r$. Form the set of d's ,$d_1, \ldots, d_r$, by the rule
$d_1 = c_1 + 0, d_2 = c_2 + 1, , \ldots d_i = c_i + i - 1 \ldots, d_r = c_r + r - 1$. This transformation ensures that the d's are unalike whatever the c's were. It is clear that the sets of c's and d's are equinumerous since every distinct r-combination of the c's produces a distinct set of d's and vice versa. The number of sets of d's is the number of r-combinations without repetition of the objects $1, 2, \ldots, n + r - 1$ since the largest d is d_r when c_r has its maximum value n. Thus the number of sets of d's is ${}^{n+r-1}C_r$ and this is equal to the number of r-combinations of n objects with unlimited repetition.

Definition 1.3

*A **distribution** is defined as a separation of a set of objects into a number of classes - for example, the assignment of the objects to cells (or boxes); problems about distributions are very closely related to problems of permutations and combinations.*

Consider first the case of assigning r different objects to n distinct cells such that each cell has at most one object. If $n >$r then there are $P(n, r)$ ways, since the first object may be assigned to any of the n cells, the second object to one of the $n - 1$ remaining cells etc. Alternatively if $r > n$ then there are $P(r, n)$ ways, since the object assigned to the first cell may be done in r ways, the object assigned to the second cell in $r - 1$ ways etc.

Continuing with r different objects and n distinct cells but now allowing each cell to hold any number of objects, we obtain n ways of distributing the objects. This is true whether n is larger or smaller than r since the first object can be assigned to any one of the n cells and so can the second and the other objects.

When the r objects to be distributed are not all different suppose that m_1, of them are of the first kind, m_2 of the second kind, ..., m_k of them of the k^{th} kind, so that $r = \sum_{i=1}^{k} m_i$. First suppose that each of the n distinct cells may hold at most one object $(n \geq r)$.

The r cells are selected from the n cells (in $C(n, r)$ ways) and then the r objects are distributed into these r cells which is equivalent to forming a permutation with repetition of the objects. Therefore there are $\frac{r!}{m_1!m_2!\ldots m_k!}$ such permutations.

Therefore the number of these distributions is

$$C(n, r)\frac{r!}{m_1!m_2!\ldots m_k!} = \frac{n!}{(n-r)!m_1!m_2!\ldots m_k!}.$$

The r like objects are placed in n distinct cells without any restriction on the number going into each cell. The number of ways of doing this is equivalent to selecting r cells from n with repetition of cells allowed and the number of such distributions is $C(n+r-1,r)$.

Example: There are five different Spanish books, six different French books, and eight different Transylvanian books. How many ways are there to pick an (unordered) pair of two books not both in the same language? Ans: $5.6+5.8+6.8 = 30+40+48 = 118$.

Problems

1. Find the number of different letter arrangement can be formed using the word "SYSTEMS"?
Ans: $\dfrac{7!}{3!}$

2. Find the number of ways in which 3 exams can be scheduled in a 5 day period such that (i) No two exams are scheduled on the same day?
(ii) There are no restrictions on number of exams conducted on a day?
Ans: (i) Considering the three examinations as distinctly colored balls and the five days as distinctly numbered boxes, we obtain the result 5 x 4 x 3 $= 60$ i.e., ${}^{5}P_{3}$ (ii) 5^{3}.

3. If 5 men A,B,C,D,E intend to speak at a meeting, in how many orders can they do so without B speaking before A? How many orders are there in which A speaks immediately before B?
Ans: $4! + 3.3! + 6.2! + 3!$.

4. How many ways may one right and one left shoe be selected from six pairs of shoes without obtaining a pair?
Ans: 30.

5. How many ways can twelve white pawns and twelve black pawns be placed on the black squares of an 8×8 chess board?
Ans: $\frac{32.31.....9}{12!12!}$.

6. Find the sum of all the four digit numbers that can be obtained by using the digits 1, 2, 3, 4 once in each.
Ans: Each digit occupies each place 6 times. Threfore, sum is
$6([1+2+3+4]1000+[1+2+3+4]100+[1+2+3+4]10+[1+2+3+4]) = 66660$

7. How many points of intersection are formed by n lines drawn in a plane if no two are parallel and no three concurrent? Into how many regions is the plane divided?

Ans:Two lines 1 intersecting point

3 lines 1+2 intersecting points

So n lines 1+2+...+(n-1) intersecting points.

Number of intersecting points is ${}^nC_2$.

Similarly one line 2 regions

2 lines 4 regions

So, n lines $\frac{(n+1)n}{2} + 1$ regions.

8. How many ways can three integers be selected from 3n consecutive integers so that the sum is a multiple of 3 ?

Ans: ${}^nC_3 + {}^nC_3 + {}^nC_3 + {}^nC_1\,{}^nC_1\,{}^nC_1$

9. How many ways can five different messages be delivered by three messengers if no messenger is left unemployed? The order in which a messenger delivers his messages is immaterial.
Ans: $3\dfrac{5!}{3!1!1!} + 3\dfrac{5!}{2!2!1!} = 150.$

10. In how many ways can a lady wear five rings on the fingers (not the thumb) of her right hand?
Ans: Consider 5 rings as identical. Then distribution of 5 rings in to 4 fingers such that finger can hold any number of rings is ${}^{4+5-1}C_5 = {}^{8}C_5$.
As 5 rings can be arranged in 5! ways the answer is $5!{}^{8}C_5 = 6720$.

11. Six distinct symbols are transmitted through a communication channel. A total of twelve blanks are to be inserted between the symbols with at least two blanks between every pair of symbols. In how many ways can we arrange the symbols and blanks ?
Ans: ${}^{5+2-1}C_2 = 15$

12. A new national flag is to be designed with six vertical stripes in yellow, green, blue and red. In how many ways can this be done so that no two adjacent stripes have the same colour?
Ans: 4.3.3.3.3.3

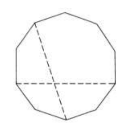
13. In how many ways can an examiner assign 30 marks to 8 questions so that no question receives less than 2 marks?
Ans: ${}^{8+14-1}C_4 = {}^{21}C_4$ ways.

14. Suppose we print all FIVE-digit numbers on slips of paper with one number on each slip. However, since the digits 0, 1, 6, 8, and 9 become 0, 1, 9, 8, and 6 when they are read upside down, there are pairs of numbers that can share the same slip if the slips are read right side up or upside down. For example, we can make up one slip for the numbers 89166 and 99168. The question is then how many distinct slips will we have to make up for all five-digit numbers?

We note first that there are 10^5 distinct five-digit numbers. Among these numbers, 5^5 of them can be read either right side up or upside down. (They are made up of the digits 0, 1, 6, 8, and 9.) However, there are numbers that read the same either right side up or upside down, for example, 16091, and there are $3(5^2)$ such numbers. (The center digit of these numbers must be either 1, 0, or 8; further, the fifth digit must be the first digit turned upside down, and the fourth digit must be the second digit turned upside down.) Consequently, there are $5^5 - 3(5^2)$ numbers that can be read either right side up or upside down but will read differently. These numbers can be divided into pairs so that every pair of numbers can share one slip. It follows that the total number of distinct slips we need is $10^5 - \dfrac{5^5 - 3(5^2)}{2}$.

15. If no three diagonals of a convex decagon meet at the same point inside the decagon, into how many line segments are the diagonals divided by their intersections?

Ans:



First of all, the number of diagonals is equal to ${}^{10}C_2 - 10 = 45 - 10 = 35$ as there are ${}^{10}C_2$ straight lines joining the ${}^{10}C_2$ pairs of vertices, but 10 of these 45 lines are the sides of the decagon. Since for every four vertices we can count exactly one intersection between the diagonals, as Figure shows (the decagon is convex), there are a total of ${}^{10}C_4 = 210$ intersections between the diagonals. Since a diagonal is divided into k + 1 straight-line segments when there are k intersecting points lying on it, and since each intersecting point lies on two diagonals, the total number of straight-line segments into which the diagonals are divided is $35 + 2 \times 210 = 455$.