Higher order Differential Equation

$\frac{dy}{dx} + P(x)y = X$

The differential equation of the form

$$b_0(x)\frac{d^n y}{dx^n} + b_1(x)\frac{d^{n-1}y}{dx^{n-1}} + b_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \text{-----} + b_n(x)y = X \longrightarrow (1).$$

is called the n^{th} order linear differential equation of variable coefficients, where $b_0, b_1, b_2 \text{--} b_n$ & X are functions of x only.

Note:- If $X=0$ in (1), then eqn (1) is called n^{th} order linear homogeneous differential equation.

If $X \neq 0$ in (1), then eq (1) is called n^{th} order linear non-homogeneous differential equation.

If in eqn (1), $b_0, b_1, b_2 \text{---} b_n$ are constants, then eqn (1) is called n^{th} order linear differential equation with constant co-efficient.

i.e
$$\frac{d^n y}{dx^n} + b_1\frac{d^{n-1}y}{dx^{n-1}} + b_2\frac{d^{n-2}y}{dx^{n-2}} + \text{---} b_n y = X. \longrightarrow (2)$$

Note:- (1) If y_1 & y_2 are only the solution of

$$\frac{d^n y}{dx^n} + b_1\frac{d^{n-1}y}{dx^{n-1}} + b_2\frac{d^{n-2}y}{dx^{n-2}} + \text{---} + b_n y = 0 \longrightarrow (3).$$

then $C_1y_1 + C_2y_2$ is also the solution of (3).

Proof:- Given y_1 & y_2 are the only solution of (3).

i.e
$$\frac{d^n y_1}{dx^n} + b_1\frac{d^{n-1}y_1}{dx^{n-1}} + b_2\frac{d^{n-2}y_1}{dx^{n-2}} + \text{--} + b_n y_1 = 0 \rightarrow (i)$$

$$\frac{d^n y_2}{dx^n} + b_1\frac{d^{n-1}y_2}{dx^{n-1}} + b_2\frac{d^{n-2}y_2}{dx^{n-2}} + \text{--} + b_n y_2 = 0 \rightarrow (ii)$$

Consider $u = C_1y_1 + C_2y_2$

$$\frac{d^n u}{dx^n} + b_1\frac{d^{n-1}u}{dx^{n-1}} + b_2\frac{d^{n-2}u}{dx^{n-2}} + \text{--} + b_n u \boxed{= 0 \; ?}$$

$$\frac{d^n u}{dx^n} + b_1 \frac{d^{n-1} u}{dx^{n-1}} + b_2 \frac{d^{n-2} u}{dx^{n-2}} + \cdots + b_n u \boxed{= 0 \ ?}$$

$$\frac{d^n}{dx^n}(c_1 y_1 + c_2 y_2) + b_1 \frac{d^{n-1}}{dx^{n-1}}(c_1 y_1 + c_2 y_2) + b_2 \frac{d^{n-2}}{dx^{n-2}}(c_1 y_1 + c_2 y_2) + \cdots + b_n (c_1 y_1 + c_2 y_2).$$

$$\Rightarrow \quad c_1 \left[\frac{d^n y_1}{dx^n} + b_1 \frac{d^{n-1} y_1}{dx^{n-1}} + b_2 \frac{d^{n-2} y_1}{dx^{n-2}} + \cdots + b_n y_1 \right] + c_2 \left[\frac{d^n y_2}{dx^n} + b_1 \frac{d^{n-1} y_2}{dx^{n-1}} + b_2 \frac{d^{n-2} y_2}{dx^{n-2}} + \cdots + b_n y_2 \right]$$

$$c_1 \{0\} + c_2 \{0\}$$

☺ 0

$\Rightarrow$ "u" is the solution of (3).

i.e $u = c_1 y_1 + c_2 y_2$ is also the sol^n of (3).

Since the general solution of differential equation of n^{th} order contains 'n' arbitrary constant. It follows from the above note if,

$y_1, y_2, \cdots\cdots y_n$ are 'n' independent solution of (3), then

$c_1 y_1 + c_2 y_2 + c_3 y_3 + \cdots + c_n y_n$ is the complete solution of (3).

Note-2 :- If 'v' be a particular solution of

$$\frac{d^n y}{dx^n} + b_1 \frac{d^{n-1} y}{dx^{n-1}} + b_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + b_n y = X \neq 0 \rightarrow (4).$$

then $$\frac{d^n v}{dx^n} + b_1 \frac{d^{n-1} v}{dx^{n-1}} + b_2 \frac{d^{n-2} v}{dx^{n-2}} + \cdots + b_n v = X . \rightarrow (5).$$

W.K.T $u = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ is the solution of

$$\frac{d^n y}{dx^n} + b_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + b_n y = 0 \longrightarrow (*)$$

$$\frac{d^n u}{dx^n} + b_1 \frac{d^{n-1} u}{dx^{n-1}} + b_2 \frac{d^{n-2} u}{dx^{n-2}} + \cdots + b_n u = 0. \rightarrow (6).$$

Adding (5) & (6)

$$\frac{d^n}{dx^n}(u+v) + b_1 \frac{d^{n-1}}{dx^{n-1}}(u+v) + b_2 \frac{d^{n-2}}{dx^{n-2}}(u+v) + \cdots + b_n(u+v) = X \quad \to (7)$$

$$\Rightarrow \quad y = u + v$$

u: Solution of homogeneous differential equation.

v: Solution of non-homogeneous differential equation

'u' is called Complementary function (CF) & 'v' is called the particular integral (P I).

We will study two methods:-

1) Inverse differential operator method

2) Method of variation of parameters

<u>Inverse differential operator method</u>

Denoting $D, D^2, D^3, \cdots$ by $\frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3} \cdots$

$$Dy = \frac{dy}{dx}, \quad D^2 y = \frac{d^2 y}{dx^2}, \cdots$$

D: Differential operator.

<u>Solution procedure of finding the CF is as follows</u>

$$\frac{d^n y}{dx^n} + b_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + b_n y = 0 \quad \to (1)$$

write (1) in Symbolic form

$$D^n y + b_1 D^{n-1} y + \cdots + b_n y = 0.$$

$$(D^n + b_1 D^{n-1} + b_2 D^{n-2} + \cdots + b_n) y = 0 \quad \to (2)$$

$$f(D) y = 0.$$

$\Rightarrow$ where $f(D) = D^n + b_1 D^{n-1} + b_2 D^{n-2} + \cdots + b_n$

To find CF, equate the co-efficient in (2) to zero

i.e $D^n + b_1 D^{n-1} + b_2 D^{n-2} + \cdots + b_n = 0 \longrightarrow (3)$.

Eqn (3) is called "Auxillary Equation" (AE).

Case-I :- If the roots of eqn (3) are real & different, say

$m_1, m_2, m_3, \cdots m_n$ are roots of (3).

then eqn (2) can be written as

$$(D-m_1)(D-m_2)(D-m_3) \cdots (D-m_n) y = 0 \longrightarrow (4)$$

Eqn (4) has to satisfy

$y(IF) = \int IF X + c$

$$(D-m_n) y = 0$$

$$\Rightarrow \frac{dy}{dx} - m_n y = 0 \rightarrow (K)$$

$$IF \text{ of } (K) = e^{\int -m_n dx} = e^{-m_n x}$$

Its solution is

$$y e^{-m_n x} = c$$

$$y = c e^{m_n x}$$

$$\Rightarrow y = c_1 e^{m_1 x}, \; y = c_2 e^{m_2 x}, \; y = c_3 e^{m_3 x} \cdots y = c_n e^{m_n x}$$

$$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}} \longrightarrow (5)$$

Case-II :- If the roots of the equation are real & equal.

Say $m_1 = m_2, m_3, \cdots m_n$.

Eqn (2) can be written as

$$(D-m_1)(D-m_1)(D-m_3) \cdots (D-m_n) y = 0$$

From (5)

$$y = c_1 e^{m_1 x} + c_2 e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

$$y = c e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

This can not be a general soln of the differential equation.

$$\boxed{(D-m_1)(D-m_1)y=0}$$

$(D-m_1)y=0$

$\frac{dy}{dx}-m_1y=0$

$y=c_1e^{m_1x}$

Consider

$$z=(D-m_1)y.$$

$$z=c_1e^{m_1x}$$

$$(D-m_1)y=c_1e^{m_1x}$$

$$\frac{dy}{dx}-m_1y=c_1e^{m_1x}\rightarrow(*)$$

$\frac{dy}{dx}+P(x)y=X$

$\Rightarrow IF=e^{\int P(x)dx}$

$y\,IF=\int X\,IF+c$

$$\Rightarrow I.F=e^{-\int m_1dx}=e^{-m_1x}$$

$$\text{Sol}^n\ (*)\text{ is}\quad ye^{-m_1x}=\int c_1e^{m_1x}\cdot e^{-m_1x}dx+c_2$$

$$\Rightarrow ye^{-m_1x}=c_1x+c_2$$

$$\Rightarrow y=(c_1x+c_2)e^{m_1x}$$

$\therefore (S)\Rightarrow$

$$y=\underline{(c_1x+c_2)}e^{m_1x}+c_3e^{m_3x}+c_4e^{m_4x}+\dots+c_ne^{m_nx}\rightarrow(**)$$

Note :- If $m_1=m_2=m_3,\ m_4, m_5 \dots m_n$ are roots

$$y=(c_1x^2+c_2x+c_3)e^{m_1x}+c_4e^{m_4x}+\dots+c_ne^{m_nx}$$

Case (III) :- If the roots of A.E having a pair of imaginary roots, i.e $m_1=\alpha+i\beta,\ m_2=\alpha-i\beta,\ m_3, m_4 \dots m_n$, then

W.K.T

$$y=c_1e^{m_1x}+c_2e^{m_2x}+c_3e^{m_3x}+\dots+c_ne^{m_nx}$$

$$y=c_1e^{(\alpha+i\beta)x}+c_2e^{(\alpha-i\beta)x}+c_3e^{m_3x}+\dots+c_ne^{m_nx}$$

$$y=c_1\{e^{\alpha x}e^{i\beta x}\}+c_2\{e^{\alpha x}e^{-i\beta x}\}+c_3e^{m_3x}+\dots+c_ne^{m_nx}$$

$$=e^{\alpha x}[c_1(\cos\beta x+i\sin\beta x)+c_2(\cos\beta x-i\sin\beta x)]+c_3e^{m_3x}+\dots+c_ne^{m_nx}$$

$$y=e^{\alpha x}\{c_1\cos\beta x+c_2\sin\beta x\}+c_3e^{m_3x}+\dots+c_ne^{m_nx}\longrightarrow(*)$$

(where $c_1=c_1+c_2$, $c_2=c_1i-c_2i$)

Note :- For repeated Complex root

$$m_1 = m_2 = \alpha + i\beta$$
$$m_3 = m_4 = \alpha - i\beta.$$

$$y = e^{\alpha x}\left\{(c_1x+c_2)\cos\beta x + (c_3x+c_4)\sin\beta x\right\} + c_5 e^{m_5 x} + \dots + c_n e^{m_6 x}.$$

Roots	CF
1) Real & different (m_1, m_2, m_3)	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$ ✓
2) Real & equal $(m_1 = m_2, m_3)$.	$y = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_3 x}$ ✓
3) Roots are imaginary $(m_1 = \alpha + i\beta, m_2 = \alpha - i\beta), m_3$	$y = e^{\alpha x}\{c_1\cos\beta x + c_2 \sin\beta x\} + c_3 e^{m_3 x}$ ✓
4) Repeated imaginary roots	$y = e^{\alpha x}\{(c_1x+c_2)\cos\beta x + (c_3x+c_4)\sin\beta x\}$

Q) Solve $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$, given $x(0) = 0$ & $\frac{dx}{dt} = 15$ when $t = 0$.

Sol^n:- The given differential equation in symbolic form is

$$D^2x + 5Dx + 6x = 0 \checkmark \quad , \text{ where } D = \frac{d}{dt}$$

$$\Rightarrow (D^2 + 5D + 6)x = 0 \checkmark \qquad f(D)y = 0$$

A.E is $\quad D^2 + 5D + 6 = 0$ ✓

Roots of A.E is

$$D^2 + 3D + 2D + 6 = 0$$
$$D(D+3) + 2(D+3) = 0$$
$$\Rightarrow (D+2)(D+3) = 0$$

$$D = -2, -3. \checkmark$$

Roots are real & different

$$\Rightarrow x(t) = c_1 e^{-2t} + c_2 e^{-3t} \rightarrow (1)$$

Given $x(0) = 0$

Substitute $t = 0$ in (1)

$$x(0) = c_1 \times 1 + c_2$$
$$\Rightarrow \quad c_1 + c_2 = 0 \rightarrow (i).$$

Given $\frac{dx}{dt} = 15$, when $t=0$.

$$\frac{dx}{dt} = -2c_1 e^{-2t} - 3c_2 e^{-3t} \rightarrow (2)$$

Put $t=0$ in (2)

$$15 = -2c_1 - 3c_2 \longrightarrow (ii) \qquad \boxed{c_1 + c_2 = 0 \rightarrow (i)}$$

Solving (i) & (2)

$$15 = 2c_2 - 3c_2$$

$$\Rightarrow -c_2 = 15 \Rightarrow c_2 = -15$$

$$\Rightarrow c_1 = 15.$$

$$\therefore \boxed{x(t) = 15e^{-2t} - 15e^{-3t}}$$

Q) Solve $D^4x + 4x = 0$.

Sol^n:- Given $(D^4+4)x = 0$

A.E $\quad D^4 + 4 = 0$

$$(D^2+2)^2 - 4D^2 = 0$$

$$(D^2+2)^2 - (2D)^2 = 0 \qquad [a^2-b^2 = (a+b)(a-b)]$$

$$\Rightarrow \underbrace{(D^2+2+2D)}_{(i)}\underbrace{(D^2+2-2D)}_{(ii)} = 0$$

$$D = \frac{-2 \pm \sqrt{4-8}}{2}, \qquad D = \frac{2 \pm \sqrt{4-8}}{2}$$

$$= \frac{-2 \pm \sqrt{-4}}{2} \qquad = \frac{2 \pm \sqrt{-4}}{2}$$

$$= -1 \pm i \qquad = 1 \pm i$$

$$D = (-1+i), (-1-i), (1+i), (1-i).$$

$D = (-1+i), (-1-i), (1+i), (1-i).$

$$\therefore \quad x = e^{-t}\left[c_1\cos t + c_2\sin t\right] + e^{t}\left[c_3\cos t + c_4\sin t\right]$$

Q) Solve $D^3y + y = 0$

Sol^n:- Given $(D^3+1)y = 0$

A.E $D^3+1 = 0$

$(D+1)(D^2-D+1) = 0$, $\quad D = -1, \dfrac{1\pm\sqrt{3}i}{2}$

$D = \dfrac{1\pm\sqrt{1-4}}{2}$ $\quad \dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2}$

$D = \dfrac{1\pm\sqrt{3}i}{2}$

$$y = c_1 e^{-x} + e^{\frac{1}{2}x}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right]$$

Q) Solve $(4D^4-8D^3-7D^2+11D+6)y = 0$

Sol^n:- AE is $4D^4-8D^3-7D^2+11D+6 = 0.$

$\rightarrow$ Synthetic division method.

	D^4	D^3	D^2	D	D^0
-1	4	-8	-7	11	6
		-4	12	-5	-6
2	4	-12	5	6	0
		8	-8	-6	
	4	-4	-3	0	

$4D^2-4D-3 = 0.$

$D = \frac{3}{2}, -\frac{1}{2}$

$D = -1, 2, \frac{3}{2}, -\frac{1}{2}$

$$y = c_1e^{-x} + c_2e^{2x} + c_3e^{\frac{3}{2}x} + c_4e^{-\frac{1}{2}x}.$$

The solution of linear Non-homogeneous differential equation with Constant Coefficients

We know that the solution of differential equation of the form

$$\frac{d^n y}{dx^n} + b_1 \frac{d^{n-1} y}{dx^{n-1}} + b_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + b_n y = X \rightarrow (1)$$

is $\quad y = y_c + y_p \qquad [y = CF + PI]$

where

y_c = General solution of linear homogeneous differential equation

y_p = Particular integral of eqⁿ (1)

To find y_p we use the following methods

1) Inverse differential operator

2) Method of variation of parameter.

Inverse differential operator :-

To find the solⁿ of D.E of the form

$$f(D) y = \phi(x)$$

we introduce the inverse differential operator as $\frac{1}{f(D)}$,

i.e $\quad y = \frac{1}{f(D)} \phi(x).$

$$\left\{ \begin{array}{l} D(x^2) = 2x \\ \frac{1}{D}(x^2) = \int x^2 dx = \frac{x^3}{3} \end{array} \right.$$

Case (i) :- when $\phi(x) = e^{ax}$ $\qquad$ $y = \frac{1}{f(D)}\phi(x)$

i.e Particular integral

$$= \frac{1}{f(D)} e^{ax} = \begin{cases} \frac{e^{ax}}{f(a)} & \text{if } f(a) \neq 0. \\ \frac{x e^{ax}}{f'(a)} & \text{if } f(a) = 0 \text{ provided } f'(a) \neq 0. \end{cases}$$

Proof :- w.k.T

$$D e^{ax} = a e^{ax}$$

$$D^2 e^{ax} = a^2 e^{ax}$$

$$\vdots$$

$$D^n e^{ax} = a^n e^{ax}$$

$$f(D) e^{ax} = f(a) e^{ax}$$

$$e^{ax} = \frac{1}{f(D)} f(a) e^{ax}$$

$$\Rightarrow \boxed{\frac{e^{ax}}{f(a)} = \frac{1}{f(D)} e^{ax}} \qquad f(a) \neq 0.$$

$\boxed{f(D) \neq 0}$

when $f(a) = 0$.

$f(D) = (D-a)\phi(D)$, $\qquad \phi(a) \neq 0$.

$$\Rightarrow \frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)\phi(D)} e^{ax} = \frac{1}{(D-a)\phi(a)} e^{ax} \rightarrow (*)$$

Note :- w.k.T $\frac{1}{D-a} X = e^{ax} \int X e^{-ax} dx$.

$\frac{1}{D-a} X = y$

$X = Dy - ay$

$X = \frac{dy}{dx} - ay$

IF e^{-ax}

$y e^{-ax} = \int X e^{-ax} dx$

$$\frac{1}{D-a} e^{ax} = e^{ax} \int e^{ax} e^{-ax} dx = x e^{ax}$$

where $f(D) = \frac{d^n}{dx^n} + b_1 \frac{d^{n-1}}{dx^{n-1}} + b_2 \frac{d^{n-2}}{dx^{n-2}} + \cdots + b_n$

$$\Rightarrow \frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)\phi(D)} e^{ax} = \frac{1}{(D-a)\phi(a)} e^{ax} \rightarrow (*)$$

$$= \frac{1}{\phi(a)} x e^{ax}$$

$$\therefore \frac{1}{f(D)} e^{ax} = \frac{x e^{ax}}{\phi(a)} ? \quad , \text{ when } f(a) = 0.$$

$\boxed{f(D) = 0}$

W.K.T

$$f(D) = (D-a)\phi(D)$$

$$f'(D) = (D-a)\phi'(D) + \phi(D)$$

$$f'(a) = 0 + \phi(a) \Rightarrow \phi(a) = f'(a).$$

Thus $\frac{1}{f(D)} e^{ax} = \frac{x e^{ax}}{f'(a)}$ when $f(a) = 0$. Provided $f'(a) \neq 0$.

Illy $\frac{1}{f(D)} e^{ax} = \frac{x^2 e^{ax}}{f''(a)}$ If $f'(a) = 0$ Provided $f''(a) \neq 0$

⋮

Case 2 :- $\phi(x) = \sin(ax+b)$ or $\cos(ax+b)$.

i.e $f(D)y = \sin(ax+b)$ or $\cos(ax+b)$.

Particular integral is

$$y = \frac{1}{f(D)}\{\sin(ax+b) \text{ or } \cos(ax+b)\}.$$

$$D\sin(ax+b) = a\cos(ax+b)$$
$$D^2\sin(ax+b) = -a^2\sin(ax+b)$$
$$D^3\sin(ax+b) = -a^3\cos(ax+b)$$
$$D^4\sin(ax+b) = a^4\sin(ax+b)$$
$$(D^2)^2\sin(ax+b) = (-a^2)^2\sin(ax+b)$$
$$(D^2)^n\sin(ax+b) = (-a^2)^n\sin(ax+b)$$

$$f(D^2)\sin(ax+b) = f(-a^2)\sin(ax+b)$$

$$\Rightarrow \quad \sin(ax+b) = \frac{1}{f(D^2)} f(-a^2)\sin(ax+b)$$

$$\frac{\sin(ax+b)}{f(-a^2)} = \frac{1}{f(D^2)}\sin(ax+b). \quad | \; f(-a^2) \neq 0.$$

If $f(-a^2) = 0$, above case fails.

$$\boxed{\frac{1}{f(D^2)}\sin(ax+b) = x\frac{1}{f'(-a^2)}\sin(ax+b)}$$

when $f(-a^2) = 0$
Provided $f'(-a^2) \neq 0$.

W.K.T $e^{i(ax+b)} = \cos(ax+b) + i\sin(ax+b)$

$$I.P \frac{1}{f(D^2)} e^{i(ax+b)} = I.P \; x \frac{1}{f'(-a^2)} e^{i(ax+b)}$$ when $f(-a^2)=0$ & $f'(-a^2) \neq 0$

i.e $$\frac{1}{f(D^2)} \sin(ax+b) = x \frac{1}{f'(-a^2)} \sin(ax+b)$$ when $f(-a^2)=0$ & $f'(-a^2) \neq 0$.

If $f'(-a^2)=0$

$$\frac{1}{f(D^2)} \sin(ax+b) = x^2 \frac{1}{f''(-a^2)} \sin(ax+b)$$ when $f''(-a^2) \neq 0$ & $f'(-a^2)=0$.

Case 3:- when $\phi(x) = x^m$

$$P.I = \frac{1}{f(D)} x^m = [f(D)]^{-1} x^m$$

Expand $[f(D)]^{-1}$ in ascending powers of D as far as the term in D^m & operate on x^m term by term. Since $(m+1)$th & higher derivatives of x^m are zero, we need not consider terms beyond D^m.

For EX:- $\boxed{\frac{1}{1-D} x^2}$

$$= (1-D)^{-1} x^2$$
$$= (1+D+D^2+\dots)x^2$$
$$= (x^2+2x+2+0\ldots)$$

Case 4: $\phi(x) = e^{ax} \underline{V}$, where 'V' is function of 'x'

$$P.I = \frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$$

Proof :-

$$D\{e^{ax} u\} = \underset{\sim}{a e^{ax} u} + \underset{\sim}{e^{ax} Du}$$

$$= e^{ax} (D+a) u$$

$$D^2 [e^{ax} u] = e^{ax} D^2 u + Du e^{ax} \cdot a + a e^{ax} Du + u a^2 e^{ax}$$

$$= e^{ax} [D^2 + 2Da + a^2] u.$$

$$= e^{ax} (D+a)^2 u$$

$$D^3 [e^{ax} u] = e^{ax} (D+a)^3 u.$$

$$\vdots$$

$$D^n \{e^{ax} u\} = e^{ax} [D+a]^n u.$$

$$f(D)(e^{ax} u) = e^{ax} f(D+a) u.$$

Operating B.S by $\frac{1}{f(D)}$

$$e^{ax} u = \frac{1}{f(D)} [e^{ax} \underset{\sim}{f(D+a) u}] \longrightarrow (1).$$

Now put $f(D+a) u = V$, i.e $u = \frac{V}{f(D+a)}$

So that

$$e^{ax} \times \frac{V}{f(D+a)} = \frac{1}{f(D)} (e^{ax} V)$$

$$\therefore \frac{1}{f(D)} (e^{ax} V) = e^{ax} \frac{1}{f(D+a)} V.$$

Particular integral of equations

$f(D) = D^n + b_1 D^{n-1} + b_2 D^{n-2} + \dots + b_n.$

X	P.I.
1) e^{ax}	$\frac{e^{ax}}{f(a)}$, $f(a) \neq 0$ $x\frac{e^{ax}}{f'(a)}$, $f(a) = 0,\ f'(a) \neq 0.$ $x^2\frac{e^{ax}}{f''(a)}$, $f'(a) = 0,\ f''(a) \neq 0.$
2) $\sin(ax+b)$ or $\cos(ax+b)$	$\frac{\sin(ax+b)}{f(-a^2)}$, $f(-a^2) \neq 0.$ $x\frac{\sin(ax+b)}{f'(-a^2)}$, $f(-a^2) = 0$ & $f'(-a^2) \neq 0.$
3) x^m	$\{f(D)\}^{-1} x^m.$
4) $e^{ax} V$	$e^{ax}\frac{1}{f(D+a)} V.$

Q) Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = (1-e^x)^2$

Sol:- Write the given D.E in symbolic form

$$(D^2y + Dy + y) = (1-e^x)^2.$$

A.E is $D^2 + D + 1 = 0$

$D = \alpha \pm i\beta$

$$D = \frac{-1 \pm \sqrt{-3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$e^{\alpha x}\{c_1\cos\beta x + c_2\sin\beta x\}$

$$CF = e^{-\frac{x}{2}}\left\{c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right\}$$

$\boxed{ae^{0x} = a}$

W.K.T

$$PI = \frac{1}{f(D)}(1-e^x)^2 = \frac{1}{D^2+D+1}\left[1 + e^{2x} - 2e^x\right]$$

$$= \frac{1}{D^2+D+1}e^{0x} + \frac{1}{D^2+D+1}e^{2x} - 2\frac{1}{D^2+D+1}e^x$$

$$PI = 1 + \frac{1}{7}e^{2x} - \frac{2e^x}{3}$$

$\therefore \quad y = CF + PI$

$$y = e^{-\frac{x}{2}}\left\{c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right\} + 1 + \frac{e^{2x}}{7} - \frac{2e^x}{3}$$

Q) Solve $(D-2)^2 y = 8\{e^{2x} + \sin(2x) + x^2\}$.

Sol^n:- A.E

$(D-2)^2 = 0. \quad \Rightarrow (D-2)(D-2) = 0.$

$\Rightarrow D = 2, 2.$

$|m_1 = m_2 \neq m_3 |\; CF = (c_1x + c_2)e^{m_1x} + c_3e^{m_3x}$

$CF = (c_1x + c_2)e^{2x}$

W.K.T $\quad PI = \underbrace{\frac{1}{f(D)}8e^{2x}}_{(1)} + \underbrace{\frac{1}{f(D)}8\sin(2x)}_{(2)} + \underbrace{\frac{1}{f(D)}8x^2}_{(3)}.$

① $\frac{1}{f(D)}8e^{2x} = 8\frac{1}{(D-2)^2}e^{2x} = 8\times x\frac{1}{2(D-2)}e^{2x} = 8\times x^2\frac{1}{2}e^{2x}$

$= \frac{\cancel{8}^{4}x^2e^{2x}}{\cancel{2}} = 4x^2e^{2x}.$

② $\frac{1}{f(D)}8\sin(2x) = 8\frac{1}{(D-2)^2}\sin(2x) = 8\frac{1}{D^2+4-4D}\sin 2x = \frac{\cancel{8}^{2}\times 1}{\cancel{-4}\cancel{+4}-4D}\sin 2x$

$= -2\frac{1}{D}\sin 2x = -2\int \sin 2x\, dx$

$= -\cancel{2}\left[-\frac{\cos 2x}{\cancel{2}}\right] = \cos 2x$

③ $\frac{1}{f(D)}8x^2 = 8\frac{1}{(D-2)^2}x^2 = \cancel{8}^{2}\frac{1}{\cancel{4}(1-\frac{D}{2})^2}x^2$

$= 2\left[1-\frac{D}{2}\right]^{-2}x^2$

$(1-x)^{-1} = 1+x+x^2+\cdots$

$(1+x)^{-1} = 1-x+x^2-x^3+\cdots$

$(1-x)^{-2} = 1+2x+3x^2+\cdots$

$(1+x)^{-2} = 1-2x+3x^2-\cdots$

$= 2\left\{1+\cancel{2}\frac{D}{\cancel{2}}+3\left(\frac{D}{2}\right)^2\right\}x^2$

$= 2\left\{x^2+2x+\frac{3\times 2}{4}\right\} = 2x^2+4x+3.$

$\therefore PI = 4x^2e^{2x} + \cos 2x + 2x^2+4x+3.$

$\therefore y = CF + PI$

$$y = (c_1x + c_2)e^{2x} + 4x^2e^{2x} + \cos 2x + 2x + 4x + 3.$$

Qs Solve $\frac{d^2y}{dx^2} + 2y = x^2e^{3x} + e^x\cos 2x.$

Sol^n:- The given D.E in symbolic form.

$$(D^2+2)y = x^2e^{3x} + e^x\cos 2x. \qquad \text{where } D = \frac{d}{dx}$$

A.E is $D^2+2 = 0$

$D = \pm i\sqrt{2}.$

$$CF = C_1\cos\sqrt{2}x + C_2\sin\sqrt{2}x.$$

$$\left| \begin{array}{l} e^{ax}V = \\ PI = e^{ax}\frac{1}{f(D+a)}V. \end{array}\right.$$

W.K.T $PI = \underbrace{\frac{1}{f(D)}x^2e^{3x}}_{①} + \underbrace{\frac{1}{f(D)}e^x\cos 2x}_{②}$

①

$$\frac{1}{f(D)}x^2e^{3x} = e^{3x}\frac{1}{(D+3)^2+2}x^2 = e^{3x}\frac{1}{D^2+9+6D+2}x^2 = e^{3x}\frac{1}{D^2+6D+11}x^2.$$

$$= e^{3x}\frac{1}{11\left(1+\frac{D^2+6D}{11}\right)}x^2 = \frac{e^{3x}}{11}\left[1+\left(\frac{D^2+6D}{11}\right)\right]^{-1}x^2$$

$$= \frac{e^{3x}}{11}\left[1-\left(\frac{D^2+6D}{11}\right)+\left(\frac{D^2+6D}{11}\right)^2+\cdots\right]x^2$$

$$= \frac{e^{3x}}{11}\left[x^2 - \frac{1}{11}(2+12x) + \frac{1}{11^2}(D^4+36D^2+12D^3)x^2\right]$$

$$= \frac{e^{3x}}{11}\left\{x^2 - \frac{2}{11} - \frac{12x}{11} + \frac{1}{11^2}(0+36\times 2+0)\right\}$$

$$\frac{1}{f(D)}x^2e^{3x} = \frac{e^{3x}}{11}\left[x^2 - \frac{2}{11} - \frac{12x}{11} + \frac{72}{121}\right]$$

② $\frac{1}{f(D)} e^{x}\cos 2x = \frac{1}{(D^2+2)} e^{x}\cos 2x = e^{x}\frac{1}{(D+1)^2+2}\cos 2x$

$$= e^{x}\frac{1}{D^2+1+2D+2}\cos 2x$$

$$= e^{x}\frac{1}{-4+2D+3}\cos 2x$$

$$= e^{x}\frac{1}{2D-1}\cos 2x$$

$$= e^{x}\frac{2D+1}{(2D-1)(2D+1)}\cos 2x$$

$$= e^{x}\frac{(2D+1)}{4D^2-1}\cos 2x$$

$$= e^{x}\frac{(2D+1)\cos 2x}{-17}$$

$$\frac{1}{f(D)} e^{x}\cos 2x = \frac{e^{x}}{-17}[-4\sin 2x+\cos 2x].$$

$$\therefore PI = \frac{e^{3x}}{11}\left[x^2 - \frac{2}{11} - \frac{12}{11}x + \frac{72}{121}\right] - \frac{e^{x}}{17}(-4\sin 2x+\cos 2x)$$

$$= \frac{e^{3x}}{11}\left[x^2 - \frac{12}{11}x + \frac{50}{121}\right] - \frac{e^{x}}{17}(-4\sin 2x+\cos 2x)$$

$\therefore$ $y = CF + PI$

$$= c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x + \frac{e^{3x}}{11}\left[x^2 - \frac{12}{11}x + \frac{50}{121}\right] - \frac{e^{x}}{17}(-4\sin 2x+\cos 2x)$$

8) Solve $(D^2-1)y = x\sin x + (1+x^2)e^{x}$.

Soln:- A.E $D^2-1=0$

$D = \pm 1$ or $1, -1$

$CF = c_1 e^{x} + c_2 e^{-x}$

W.K.T $PI = \frac{1}{f(D)} x\sin x + \frac{1}{f(D)}(1+x^2)e^x$
(1) (2)

$e^{i\theta} = \cos\theta + i\sin\theta$
$R.P\, e^{i\theta} = \cos\theta$
$I.P\, e^{i\theta} = \sin\theta$

(1)

$$\frac{1}{f(D)} x\sin x = \frac{1}{D^2-1} x\{I.P\, e^{ix}\}$$

$$= I.P\, e^{ix} \frac{1}{(D+i)^2-1} x = I.P\, e^{ix} \frac{1}{D^2-1+2Di-1} x$$

$$= I.P\, e^{ix} \frac{1}{D^2+2Di-2} x = I.P\, e^{ix} \frac{1}{-2\left\{1-\left(\frac{D^2+2Di}{2}\right)\right\}} x$$

$$= I.P\, \frac{e^{ix}}{-2} \left\{1-\left(\frac{D^2+2Di}{2}\right)\right\}^{-1} x$$

$$= I.P\, \frac{e^{ix}}{-2} \left[1 + \frac{D^2+2Di}{2} + \left(\frac{D^2+2Di}{2}\right)^2 + 0 + \cdots\right] x$$

$$= I.P\, \frac{e^{ix}}{-2} \left\{x + \frac{2i}{2} + \frac{1}{4}(0) + \cdots\right\} = I.P\, \frac{e^{ix}}{-2}(x+i)$$

$$= I.P \left\{ \frac{\cos x + i\sin x}{-2} (x+i) \right\}$$

$$\frac{1}{f(D)} x\sin x = -\frac{\cos x}{2} - \frac{x\sin x}{2}$$

(2) $$\frac{1}{f(D)}(1+x^2)e^x = \frac{1}{(D^2-1)} e^x + \frac{1}{(D^2-1)} e^x x^2$$

$$= \frac{x}{2D} e^x + e^x \frac{1}{(D+1)^2-1} x^2$$

$$= x/2\, e^x + e^x \frac{1}{D^2+\cancel{1}+2D-\cancel{1}} x^2$$

$$= \frac{x e^x}{2} + e^x \frac{1}{D^2+2D} x^2$$

$$= \frac{x}{2} e^x + e^x \frac{1}{2D\left[1+\frac{D}{2}\right]} x^2$$

$$= \frac{x}{2} e^x + \frac{e^x}{2D}\left[1+\frac{D}{2}\right]^{-1} x^2$$

$$= \frac{x}{2} e^x + \frac{e^x}{2D}\left\{1-\frac{D}{2}+\frac{D^2}{4}+0+0\cdots\right\} x^2$$

$$= \frac{x}{2} e^x + \frac{e^x}{2D}\left\{x^2 - \frac{\cancel{2}x}{\cancel{2}} + \frac{\cancel{2}}{\cancel{4}2}\right\}$$

$$= \frac{x}{2} e^x + \frac{e^x}{2}\left[\int x^2 dx - \int x dx + \int \frac{1}{2} dx\right]$$

$$\frac{1}{f(D)}(1+x^2)e^x = \frac{xe^x}{2} + \frac{e^x}{2}\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{2}\right]$$

$$\therefore PI = \frac{-\cos x}{2} - \frac{x\sin x}{2} + \frac{xe^x}{2} + \frac{e^x}{2}\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{2}\right]$$

$$\therefore y = c_1 e^x + c_2 e^{-x} - \frac{\cos x}{2} - \frac{x\sin x}{2} + \frac{xe^x}{2} + \frac{e^x}{2}\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{2}\right]$$

Q5 Solve $(D^3-6D^2+11D-6)y = 2^x + \cos 2x$.

Sol^n:- A.E is $D^3-6D^2+11D-6=0$

$$D = 1,2,3$$

$$\left[\; e^{\log 2^x} = e^{x\log 2}\right.$$

$$\therefore CF = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}.$$

W.K.T

$$PI = \underbrace{\frac{1}{f(D)} 2^x}_{①} + \underbrace{\frac{1}{f(D)} \cos 2x}_{②}$$

(1) $\frac{1}{f(D)} 2^x = \frac{1}{D^3-6D^2+11D-6} e^{\log 2^x} = \frac{1}{D^3-6D^2+11D-6} e^{x\log 2}.$

$$= \frac{e^{x\log 2}}{(\log 2)^3 - 6(\log 2)^2 + 11\log 2 - 6}$$

(2) $\frac{1}{f(D)} \cos 2x = \frac{1}{D^3-6D^2+11D-6} \cos 2x$

$\because D^3 = D^2 D$

$$= \frac{1}{-4D - 6X - 4 + 11D - 6} \cos 2x$$

$$= \frac{1}{7D+18} \cos 2x = \frac{7D-18}{(7D+18)(7D-18)} \cos 2x.$$

$$= \frac{(7D-18)}{49D^2 - 324} \cos 2x = \frac{(7D-18)}{-520} \cos 2x$$

$$= \frac{7}{\cancel{+520}_{260}} \times \cancel{+2}\sin 2x + \frac{\cancel{18}^{9}}{-\cancel{520}_{260}} \cos 2x$$

$$\frac{1}{f(D)} \cos 2x = \frac{7\sin 2x}{260} + \frac{9\cos 2x}{260}$$

$\therefore PI = \frac{1}{260}(7\sin 2x + 9\cos 2x) +$

$$\frac{e^{x\log 2}}{(\log 2)^3 - 6(\log 2)^2 + 11\log 2 - 6}$$

$y = CF + PI$

$= (c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}) +$

$\frac{1}{260}(7\sin 2x + 9\cos 2x) +$

$$\frac{e^{x\log 2}}{(\log 2)^3 - 6(\log 2)^2 + 11\log 2 - 6}$$

Q5 Solve $(D^2+2D+1)y = x\cos x$.

A.E is $(D+1)^2 = 0$.

$D = -1, -1$

$CF = (c_1 x + c_2)e^{-x}$

$PI = \frac{\cos x}{2} + \frac{x\sin x}{2} - \frac{\sin x}{2}$

Method of variation of parameters

This method is used to solve the differential equation of the form

$$y'' + Py' + qy = X \longrightarrow (*)$$

It gives PI as: $\boxed{y_2 \int \frac{X y_1}{W} dx - y_1 \int \frac{y_2 X}{W} dx} \longrightarrow (**)$

where W is called Woronskain.

i.e $\quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

where y_1 & y_2 are the solution of $y'' + Py' + qy = 0$

Q) Solve $(D^2 - 3D + 2)y = \sin(e^{-x})$.

Sol^n:- A.E is $D^2 - 3D + 2 = 0$

$$D = 1, 2$$

$$\therefore \quad CF = c_1 e^{x} + c_2 e^{2x}$$

Let $y_1 = e^{x}$, $y_2 = e^{2x}$

W.K.T $\quad PI = y_2 \int \frac{y_1 X}{w} dx - y_1 \int \frac{y_2 X}{w} dx \longrightarrow (1)$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{x} & e^{2x} \\ e^{x} & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x}.$$

Given $X = \sin(e^{-x})$,

Substitute y_1, y_2, w & X in (1).

$$\therefore PI = e^{2x} \int \underbrace{\frac{e^{x}\sin(e^{-x})}{e^{3x}} dx}_{(1)} - e^{x} \int \underbrace{\frac{e^{2x}\sin(e^{-x})}{e^{3x}} dx}_{(2)} \longrightarrow (2)$$

(1) $\quad e^{2x} \int \frac{e^{x}\sin(e^{-x})}{e^{3x}} dx$, put $e^{-x} = t$

$$-e^{-x} dx = dt$$

$$= e^{2x} \int \frac{e^{-2x}\sin(e^{-x})}{e^{-x} \cdot e^{-x}} dx.$$

$$= e^{2x} \int -t \sin(t)\, dt$$

$$= -e^{2x}\left[-t\cos t + \sin t\right]$$

$$= -e^{2x}\left[-e^{-x}\cos(e^{-x}) + \sin(e^{-x})\right]$$

(2)

$$e^{x}\int \underbrace{e^{-x}\sin(e^{-x})\,dx}$$

Put $e^{-x}=t$

$-e^{-x}dx=dt$

$$e^{x}\int -\sin t\,dt$$

$$= -e^{x}\{-\cos t\} = e^{x}\cos(e^{-x})$$

$$\therefore PI = -e^{2x}\left[-e^{-x}\cos(e^{-x})+\sin(e^{-x})\right] - e^{x}\cos(e^{-x})$$

$$= e^{x}\cancel{\cos(e^{-x})} - e^{2x}\sin(e^{-x}) - \cancel{e^{x}\cos(e^{-x})}$$

$$\therefore PI = -e^{2x}\sin(e^{-x}).$$

$$\therefore \quad y = CF + PI$$

$$y = c_1e^{x}+c_2e^{2x}-e^{2x}\sin(e^{-x})$$

Q) Solve $(D^2+3D+2)y = e^{e^{x}}$

Sol:- A.E is $D^2+3D+2=0$

$D=-1,-2.$

$CF = c_1e^{-x}+c_2e^{-2x}$

$$PI = y_2\int \frac{y_1X}{w}dx - y_1\int \frac{Xy_2}{w}dx \longrightarrow (1).$$

Given $X = e^{e^{x}}$, $y_1=e^{-x}$, $y_2=e^{-2x}$ $\Rightarrow W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x}\end{vmatrix}$

$$= -2e^{-3x}+e^{-3x}$$

$$= -e^{-3x}.$$

$\therefore (1) \Rightarrow$

$$P.I = \underbrace{e^{-2x}\int \frac{e^{-x}e^{e^{x}}}{-e^{-3x}}dx}_{(1)} - \underbrace{e^{-x}\int \frac{e^{-2x}e^{e^{x}}}{-e^{-3x}}dx.}_{(2)}$$

(1) =>

$$e^{-2x}\int \frac{e^{-x}e^{e^{x}}}{-e^{-3x}}dx = -e^{-2x}\int e^{2x}e^{e^{x}}dx$$

$$e^{x}\cdot e^{x}\, e^{e^{x}}dx$$

put $e^{x}=t$ $\quad e^{x}dx = dt$

$$= -e^{-2x}\int te^{t}dt = -e^{-2x}\{te^{t}-e^{t}\}$$

$$= -e^{-2x}\{e^{x}e^{e^{x}}-e^{e^{x}}\}.$$

(2)

$$e^{-x}\int \frac{e^{-2x}e^{e^{x}}}{-e^{-3x}}dx. = -e^{-x}\int e^{x}e^{e^{x}}dx.$$

put $e^{x}=t$, $\quad e^{x}dx=dt$

$$= -e^{-x}\int e^{t}dt = -e^{-x}\{e^{t}\}$$

$$= -e^{-x}e^{e^{x}}$$

$$PI = -e^{-2x}\{e^{x}e^{e^{x}}-e^{e^{x}}\}+e^{-x}e^{e^{x}}$$

$$= -\cancel{e^{-x}e^{e^{x}}}+e^{-2x}e^{e^{x}}+\cancel{e^{-x}e^{e^{x}}}$$

$$\therefore\ PI = e^{-2x}e^{e^{x}}$$

$$\therefore\ y = CF + PI$$

$$y = c_1e^{-x}+c_2e^{-2x}+e^{-2x}e^{e^{x}}$$

Q) Solve $(D^2+4)y = \tan 2x.$

Sol:- A.E is $D^2+4=0$ $\qquad \alpha \pm i\beta$

$$D = \pm 2i$$

$\Rightarrow CF = c_1\cos 2x + c_2\sin 2x$

$\Rightarrow y_1 = \cos 2x$, $y_2 = \sin 2x$

$$\Rightarrow W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2\cos^2 2x + 2\sin^2 2x$$

$$= 2\{1\} = 2.$$

$$\therefore PI = y_2\int \frac{y_1 X}{W}dx - y_1\int \frac{y_2 X}{W}dx$$

$$= \sin 2x\int \frac{\cos 2x \cdot \tan 2x}{2}dx - \cos 2x\int \frac{\sin 2x \tan 2x}{2}dx$$

$$= \frac{\sin 2x}{2}\int \frac{\cancel{\cos 2x}\times \sin 2x}{\cancel{\cos 2x}}dx - \frac{\cos 2x}{2}\int \frac{\sin 2x \times \sin 2x}{\cos 2x}dx$$

$$= \frac{\sin 2x}{2}\left[-\frac{\cos 2x}{2}\right] - \frac{\cos 2x}{2}\int \frac{\sin^2 2x}{\cos 2x}dx$$

$$= \frac{-\sin 2x \cos 2x}{4} - \frac{\cos 2x}{2}\int \frac{1-\cos^2 2x}{\cos 2x}dx$$

$$= -\frac{\sin 2x\cos 2x}{4} - \frac{\cos 2x}{2}\left[\int \sec 2x\,dx - \int \cos 2x\,dx\right]$$

$$= -\frac{\sin 2x\cos 2x}{4} - \frac{\cos 2x}{2}\left[\frac{\log(\sec 2x+\tan 2x)}{2} - \frac{\sin 2x}{2}\right]$$

$$= \cancel{-\frac{\sin 2x\cos 2x}{4}} - \frac{\cos 2x\log(\sec 2x+\tan 2x)}{4} + \cancel{\frac{\sin 2x\cos 2x}{4}}$$

$$PI = \frac{-\cos 2x\log(\sec 2x+\tan 2x)}{4}.$$

HW
$(D^2+a^2)y=\tan ax$
$(D^2+a^2)y=\sec ax$

$$\therefore y = CF + PI$$

$$= c_1\cos 2x + c_2\sin 2x - \frac{\cos 2x\log(\sec 2x+\tan 2x)}{4}.$$

8) Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \frac{1}{1+e^{-x}}$.

Sol^n:- Given $(D^2 - 3D + 2)y = \frac{1}{1+e^{-x}}$.

A.E is $D^2 - 3D + 2 = 0$

$D = 1, 2$.

$\therefore$ CF $= c_1 e^{x} + c_2 e^{2x}$.

$\Rightarrow y_1 = e^{x}, \quad y_2 = e^{2x}$

$$w = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x}.$$

$$\therefore \ PI = \underbrace{y_2 \int \frac{y_1 X}{w} dx}_{①} - \underbrace{y_1 \int \frac{y_2 X}{w} dx}_{②}.$$

$$y_2 \int \frac{y_1 X}{w} dx = e^{2x} \int \frac{e^{x}}{e^{3x}} \cdot \frac{1}{1+e^{-x}} dx = e^{2x} \int \frac{e^{-2x}}{1+e^{-x}} dx$$

$$= e^{2x} \int \frac{e^{-2x}}{e^{-x}\{1+\frac{1}{e^{-x}}\}} dx = e^{2x} \int \frac{e^{-x}}{(1+e^{x})} dx$$

$$= e^{2x} \int \frac{e^{-x}}{e^{x}(1+\frac{1}{e^{x}})} dx = e^{2x} \int \frac{1}{e^{2x}(1+\frac{1}{e^{x}})} dx$$

$\frac{1}{e^{x}.e^{x}}$

Put $\frac{1}{e^{x}} + 1 = t$

$-e^{-x} dx = dt \Rightarrow -\frac{1}{e^{x}} dx = dt$

$$= e^{2x} \int \frac{-dt\,(t-1)}{t} = -e^{2x} \int \frac{t-1}{t} dt$$

$$= -e^{2x} \left[\int dt - \int \frac{1}{t} dt \right].$$

$$= -e^{2x} [t - \log t]$$

$$= -e^{2x} \left[\frac{1}{e^{x}} + 1 - \log\left(1 + \frac{1}{e^{x}}\right) \right].$$

② $y_1 \int \frac{y_2 X}{W} dx = e^x \int \frac{e^{2x}}{e^{3x}} \times \frac{1}{1+e^{-x}} dx$ ✓

$= e^x \int \frac{e^{-x}}{1+e^{-x}} dx$

$= e^x \int \frac{1}{e^x(1+\frac{1}{e^x})} dx$, Put $1+\frac{1}{e^x} = t$

$-\frac{1}{e^x} dx = dt$

$= e^x \int -\frac{dt}{t}$

$= -e^x \log t = -e^x \log(1+\frac{1}{e^x})$.

$\therefore PI = -e^{2x}\left[\frac{1}{e^x}+1 - \log(1+\frac{1}{e^x})\right] + e^x \log(1+\frac{1}{e^x})$

$PI = -e^x - e^{2x} + e^{2x}\log(1+\frac{1}{e^x}) + e^x \log(1+1/e^x)$ //.

$\therefore y = CF + PI$

$= c_1 e^x + c_2 e^{2x} - e^x - e^{2x} + e^{2x}\log(1+\frac{1}{e^x}) + e^x\log(1+1/e^x)$ //.

Q) Solve $\boxed{y'' + 2y - 2y' = e^x \tan x.}$

Sol:- Given $(D^2+2-2D)y = e^x \tan x$

$\therefore A.E = D^2 - 2D + 2 = 0.$

$D = 1 \pm i$

$CF = e^x(c_1\cos x + c_2 \sin x)$

$= c_1 e^x \cos x + c_2 e^x \sin x$

$\Rightarrow y_1 = e^x \cos x$, $y_2 = e^x \sin x$.

$$\Rightarrow W = \begin{vmatrix} e^x\cos x & e^x \sin x \\ -e^x\sin x + e^x\cos x & e^x\cos x + e^x \sin x \end{vmatrix}$$

$e^{2x}\cos^2 x + \cancel{e^{2x}\sin x\cos x} + e^{2x}\sin^2 x - \cancel{e^{2x}\sin x\cos x} = e^{2x}$ //

$$PI = y_2\int \frac{y_1 X}{w}dx - y_1\int \frac{y_2 X}{w}dx.$$

$$= e^x\sin x\int \frac{\cancel{e^x}\cos x \times \cancel{e^x}\tan x}{\cancel{e^{2x}}}dx - e^x\cos x\int \frac{\cancel{e^x}\sin x \times \cancel{e^x}\tan x}{\cancel{e^{2x}}}dx$$

$$= e^x\sin x\int \cos x\tan x\,dx - e^x\cos x\int \sin x\tan x\,dx$$

$$= e^x\sin x\int \cancel{\cos x}\times\frac{\sin x}{\cancel{\cos x}}dx - e^x\cos x\int \sin x\times\frac{\sin x}{\cos x}dx$$

$$= -e^x\sin x\cos x - e^x\cos x\int \frac{1-\cos^2 x}{\cos x}dx$$

$$PI = -e^x\sin x\cos x - e^x\cos x\{\log(\sec x+\tan x) - \sin x\}$$

$$PI = -\cancel{e^x\sin x\cos x} - e^x\cos x\log(\sec x+\tan x) + \cancel{e^x\sin x\cos x}$$

$$\therefore\ y = CF + PI$$

$$= e^x\{c_1\cos x + c_2\sin x\} - e^x\cos x\log(\sec x+\tan x).$$

Equations Reducible to Linear Equations with Constant Coefficients.

I) Cauchy's differential equations :-

The differential equation of the form

$$x^n\frac{d^n y}{dx^n} + k_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + k_2 x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \dots + k_n y = R(x) \longrightarrow (*)$$

where $R(x)$ is a function of x & k_i's, where $i = 1,2,\dots n$ are constant.

This type of differential equation can be reduced to linear differential equation with constant coefficients by putting $x = e^t \Rightarrow t = \log x \Rightarrow \frac{dt}{dx} = \frac{1}{x}$

$$\left.\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\ &= \frac{dy}{dt}\cdot\frac{1}{x}\end{aligned}\right\} \Rightarrow x\frac{dy}{dx} = \frac{dy}{dt}$$

i.e

$x\frac{dy}{dx} = Dy$, where $D = \frac{d}{dt}$

Now
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dt}\frac{1}{x}\right)$$
$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx}$$
$$= -\frac{1}{x^2}Dy + \frac{1}{x}\frac{d^2y}{dt^2} \times \frac{1}{x}$$
$$\frac{d^2y}{dx^2} = \frac{-1}{x^2}Dy + \frac{1}{x^2}D^2y$$
$$x^2\frac{d^2y}{dx^2} = D^2y - Dy$$
$$= D(D-1)y.$$

$x\frac{dy}{dx} = Dy.$

$x^2\frac{d^2y}{dx^2} = D(D-1)y$

$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y.$

Illy
$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y.$$

Substituting $x^n\frac{d^ny}{dx^{n-1}}$, $x^{n-1}\frac{d^{n-1}y}{dx}$, ------ in (*) we will get

n^{th} order linear differential equation with constant coefficient.

Q) Solve $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^2.$

Sol^n:- W.K.T $x^2\frac{d^2y}{dx^2} = D(D-1)y$ & $x\frac{dy}{dx} = Dy$, where $D = \frac{d}{dt}$

& $t = \log x$ or $x = e^t$.

The given D.E becomes.
$$D(D-1)y - 4Dy + 6y = e^{2t}$$
$$(D^2 - D - 4D + 6)y = e^{2t}$$
$$(D^2 - 5D + 6)y = e^{2t}$$

$\therefore$ A.E is $D^2 - 5D + 6 = 0$

$D = 3, 2$

$\therefore$ CF $= c_1e^{3t} + c_2e^{2t}$

$$PI = \frac{1}{f(D)}e^{2t} = \frac{1}{D^2-5D+6}e^{2t}$$
$$= t\frac{1}{2D-5}e^{2t}$$
$$PI = -te^{2t}$$

$\therefore$ $y = CF + PI$
$$= c_1e^{3\log x} + c_2e^{2\log x} - \log x\, e^{2\log x}$$
$$= c_1x^3 + c_2x^2 - x^2\log x$$

Q5 Solve $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 12y = x^3 \log x$.

Sol^n:- W.K.T $x^2\frac{d^2y}{dx^2} = D(D-1)y$, $x\frac{dy}{dx} = Dy$, & $x = e^t$, $t = \log x$

$\therefore$ $D(D-1)y + 2Dy - 12y = e^{3t}t.$

$D^2 - Dy + 2Dy - 12y = te^{3t}$

$(D^2 + D - 12)y = te^{3t}$

$\therefore$ A.E $D^2 + D - 12 = 0$

$D = 3, -4$

$\therefore$ CF $= c_1 e^{3t} + c_2 e^{-4t}$

$$PI = \frac{1}{D^2+D-12} te^{3t} = e^{3t}\frac{1}{(D+3)^2+(D+3)-12} t$$

$$= e^{3t}\frac{1}{D^2+\cancel{9}+6D+D+\cancel{3}-\cancel{12}} t$$

$$= e^{3t}\frac{1}{D^2+7D} t$$

$$= e^{3t}\frac{1}{7D(1+\frac{D}{7})} t = e^{3t}\frac{1}{7D}\left(1+\frac{D}{7}\right)^{-1} t$$

$$= e^{3t}\frac{1}{7D}\left(1-\frac{D}{7}\right)t$$

$$= e^{3t}\frac{1}{7D}\left(t-\frac{1}{7}\right)$$

$$PI = \frac{e^{3t}}{7}\left(\frac{t^2}{2}-\frac{t}{7}\right) //$$

$y =$ CF + PI

$=$