* Poset : (A, R) s.t R is i) reflexive
ii) antisym
iii) transive

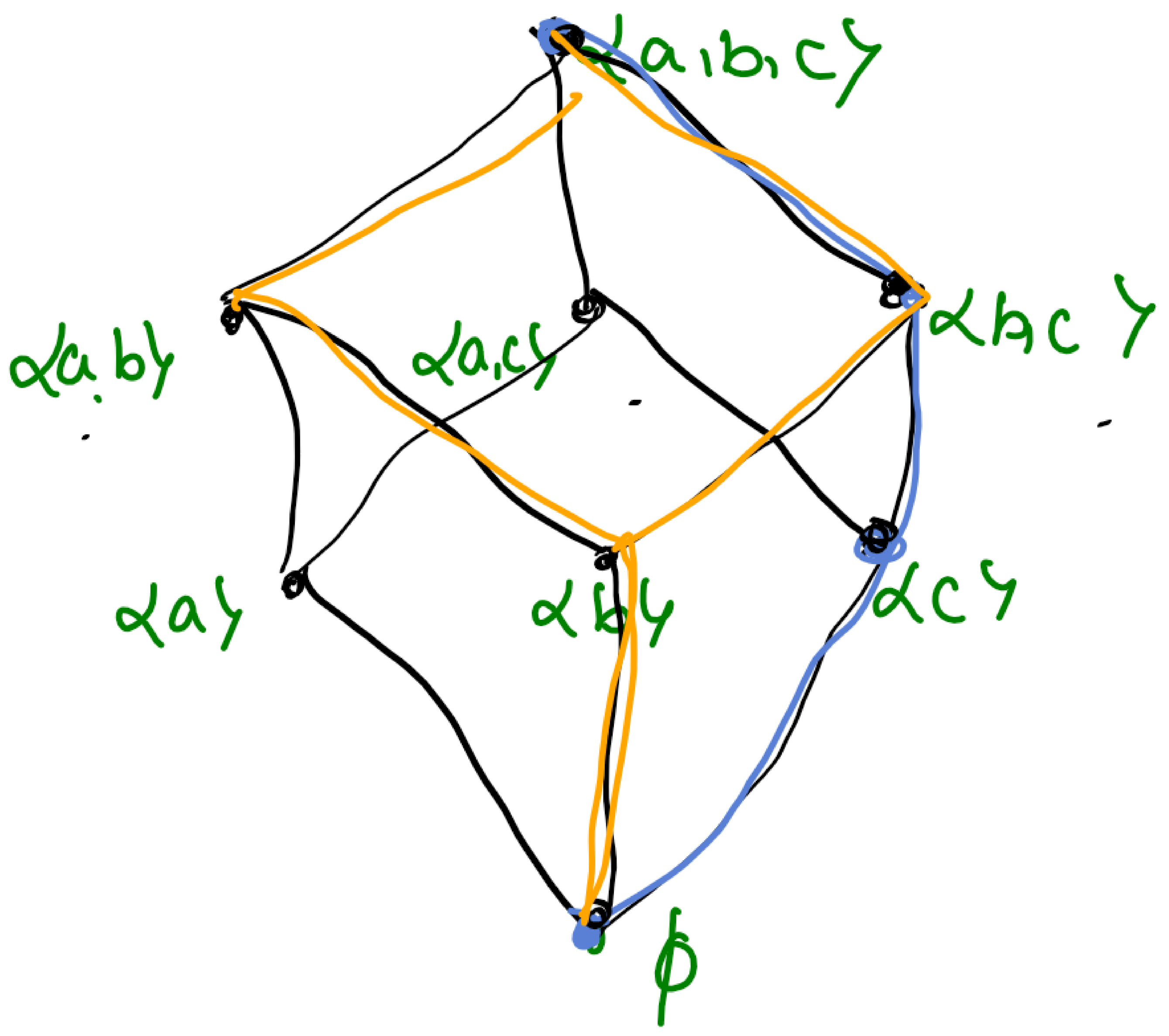
* Totally ordered set / chain :- For any two elts a & b, either aRb or bRa

* Antichain :- For any 2 elts a & b, neither aRb nor bRa

* maximal elts
minimal elts

* Hasse Diagram :

* $(P(S), \subseteq)$ is a poset, where $S = \{a, b, c\}$
$P(S)$ = powerset



ϕ, $\{c\}$, $\{b, c\}$ $\{a, b, c\}$

Length of Chain :- No of elts in the chain

Length of the Longest chain = 4

$\{\phi\}$, $\{a\}, \{b\}, \{c\}$, $\{a, b\} \{b, c\} \{a, c\}$, $\{a, b, c\}$

Theorem: Let $(P, \leq)$ be a poset. Suppose length of the longest chain in P is 'n'. Then elements in P can be partitioned into n disjoint antichains

Proof:- Induction on the length of the longest chain 'n'.

When $n = 1$; No 2 elts are related, clearly they form an antichain

Assume that the result is true when the length of the longest chain in the poset is $(n-1)$

Let P be a poset in which length of the longest chain is 'n',

Let M be the set of all maximal elts. Clearly M is a nonempty & an antichain (no 2 elts of M are related)

consider $(P-M, \leq)$

Since there is no chain of length 'n' in $(P-M)$, the length of the longest chain would be atmost $(n-1)$.

Further, since no 2 elts of M are related, obviously the length of the longest chain in $(P-M)$ is exactly $(n-1)$

By induction hypothesis, $(P-M)$ can be partitioned into $(n-1)$ disjoint antichains. Thus, P can be partitioned into n disjoint antichains

* $a \leq a \vee b$ & $a \wedge b \leq a$

* If $a \leq b$ & $c \leq d$, then $a \vee c \leq b \vee d$
  $a \wedge c \leq b \wedge d$

Lattice :- unique lub & glb

Basic prop^n of algebraic sys defined by lattices

Let $(A, \leq, \vee, \wedge)$ be the algebraic sys defined by the lattice $(A, \leq)$

i) Commutative law :- For any 2 elts $a, b \in A$

i) $a \vee b = b \vee a$

ii) $a \wedge b = b \wedge a$

ii) associative law :- For any 3 elts $a, b, c \in A$

i) $a \vee (b \vee c) = (a \vee b) \vee c$

ii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

Proof :-

Let $a \vee (b \vee c) = g$ $\qquad (a \vee b) \vee c = h$

To show that $g = h$, Let me show $g \leq h$ & $h \leq g$

$\Rightarrow g = h$ (antisym)

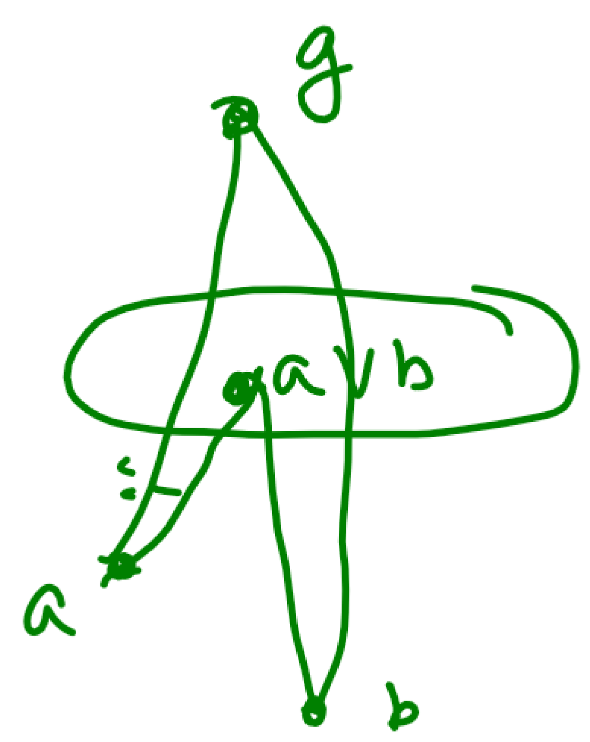
$a \leq a \vee (b \vee c)$ $\qquad$ $a \leq a \vee b$

$a \leq g$ ——— (1)

$(b \vee c) \leq g$

$b \leq b \vee c \leq g \Rightarrow b \leq g$ ——— (2)

$c \leq b \vee c \leq g \Rightarrow c \leq g$ ——— (3)



Since $(a \vee b)$ is the lub of a & b, g is some ub of both a & b $\Rightarrow$ $a \vee b \leq g$

$a \vee b \leq g$ and $c \leq g$

$(a \vee b) \vee c \leq g$

$h \leq g$

$(a \vee b) \vee c$ is the least upper bound of $(a \vee b)$ & c.
But g is some upper bound.

Similarly one show $g \leq h$ (Try)

$\therefore g = h$

$a \vee (b \vee c) = (a \vee b) \vee c$

On taking dual ;

$a \wedge (b \wedge c) = (a \wedge b) \wedge c$

③ Idempotent law: $a \vee a = a$, $a \wedge a = a$ for all $a \in A$

④ Absorption Law:

i) $a \vee (a \wedge b) = a$ for all $a, b \in A$

ii) $a \wedge (a \vee b) = a$

Proof:-

$a \leq a$ (reflexive) —— ①

$a \wedge b \leq a$ ($\because$ know) —— ②

If $a \leq b$ & $c \leq d$ then $a \vee c \leq b \vee d$

$a \vee (a \wedge b) \leq a \vee a$

$a \vee (a \wedge b) \leq a$ ($\because$ Idempotent)

Distributive Lattice:

A lattice is said to be a distributive lattice if the meet the operation is distributive over the join operation & the join operation is distributive over the meet operation

For any elements $a, b, c \in A$

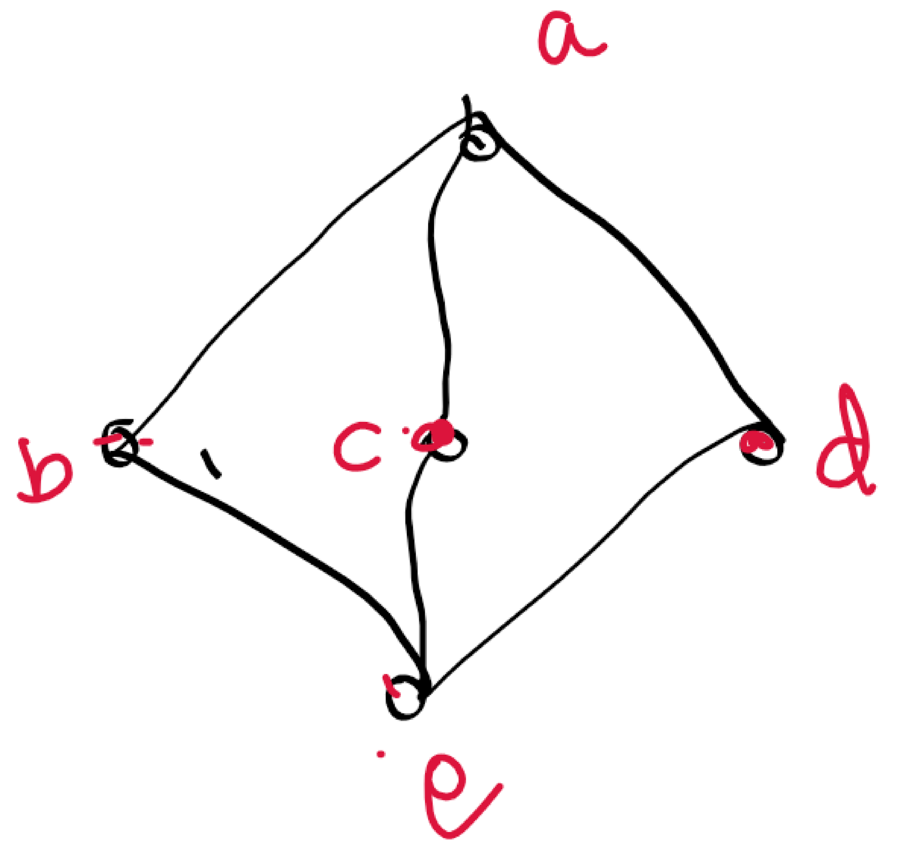
i) $a \wedge (b \vee c) = (a \wedge b) \vee (a \vee c)$

ii) $a \vee (b \wedge c) = a \vee b) \wedge (a \vee c)$

} Distributive Laws

Ex:- ① $(P(S), \subseteq)$:

②



Not distributive

$b \wedge (c \vee d) = b \wedge a$
$= b$

$(b \wedge c) \vee (b \wedge d) = e \wedge e$
$= e$

LHS $\neq$ RHS

Theorem :-

If the meet operation is distributive over the join operatn in a lattice, then join is distributive over the meet operation. Similarly, if join is distributive over meet, then meet is distributive over join.

Proof

Consider the lattice $(A, \leq)$

Let $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, I've to show that

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

consider RHS $= (a \vee b) \wedge (a \vee c)$

$= [(a \vee b) \wedge a] \vee [(a \vee b) \wedge c]$ ($\because$ meet is dist over join)

$= a \vee [(a \vee b) \wedge c]$ ($\because$ absorption)

$= a \vee [c \wedge (a \vee b)]$ ($\because$ comm prop)

$= a \vee [(c \wedge a) \vee (c \wedge b)]$ ($\because$ meet is dist over join)

$A \vee (B \vee C) = (A \vee B) \vee C$

$= [a \vee (c \wedge a)] \vee (c \wedge b)$ ($\because$ associative)

$= a \vee (c \wedge b)$ ($\because$ absorption)

$= a \vee (b \wedge c)$ ($\because$ commutative)

$=$ LHS

Another statement follows from the duality principle

* ① Let a & b be two elements in the lattice $(A, \leq)$ s.t $a \wedge b = b$ iff $a \vee b = a$

Proof:-

Let $a \wedge b = b$, prove that $a \vee b = a$

consider $a \vee (a \wedge b) = a$ ($\because$ absorption)

$a \vee b = a$ ($\because a \vee b = b$)

conversely, let $a \vee b = a$, I've prove $a \wedge b = b$

consider $b \wedge (a \vee b) = b$ ($\because$ absorption)

$$b \wedge a = b$$

$$a \wedge b = b$$

② Let a, b, c be elements of lattice $(A, \leq)$. Show that if $a \leq b$, then $a \vee (b \wedge c) \leq b \wedge (a \vee c)$

Proof:-

Let $a \leq b$ —— ①

$a \leq a \vee c$ —— ②

① $\wedge$ ② $\Rightarrow$ $a \leq b \wedge (a \vee c)$ —— ③

w.k.t $b \leq b$ and $c \leq a \vee c$

$b \wedge c \leq b \wedge (a \vee c)$ —— ④

From ③ & ④

$$a \vee (b \wedge c) \leq b \wedge (a \vee c)$$

Prob

① Let a, b, c be elements in the lattice $(A, \leq)$. Show that

i) $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

ii) $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$

(2) Let $(A, \vee, \wedge)$ be an algebraic sys. where $\vee$ & $\wedge$ satisfy commutative, associative & absorption laws.

a) Define a binary relation $\leq$ as follows:

For all $a, b \in A$, $a \leq b$ iff $a \wedge b = a$

S.T $\leq$ is a partial ordering relation

b) S.T $(a \vee b)$ is the lub of a & b in $(A, \leq)$

$(a \wedge b)$ is the glb of a & b in $(A, \leq)$