1. Rayleigh Power method

Problem 1.1. *Using Rayleigh power method, ~~to~~ find the numerically largest eigen value and the corresponding eigen vector of the matrix* $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$, *Carry out 5 iterations, correct to two decimal places.*

Take the initial approximation of the eigenvector as $X^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ✓

Ans:- Iteration 1:- $AX^{(0)} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ -0.5 \\ 0 \end{pmatrix}$

$= \lambda^{(1)} X^{(1)}$

Iteration 2:- $AX^{(1)} = \begin{pmatrix} 2.5 \\ -2 \\ 0.5 \end{pmatrix} = 2.5 \begin{pmatrix} 1 \\ -0.8 \\ 0.2 \end{pmatrix}$

$= \lambda^{(2)} X^{(2)}$

Iteration 3:- $AX^{(2)} = \begin{pmatrix} 2.8 \\ -2.8 \\ 1.2 \end{pmatrix} = 2.8 \begin{pmatrix} 1 \\ -1 \\ 0.4285 \end{pmatrix}$

$= \lambda^{(3)} X^{(3)}$

Iteration 4:-

$AX^{(3)} = \begin{pmatrix} 3 \\ -3.4285 \\ 1.8571 \end{pmatrix} = 3.4285 \begin{pmatrix} 0.87500 \\ -1 \\ 0.54166 \end{pmatrix}$

$= \lambda^{(4)} X^{(4)}$

Iteration 5:-

$$AX^{(4)} = \begin{pmatrix} 2.75 \\ -3.41666 \\ 2.08332 \end{pmatrix} = 3.41666 \begin{pmatrix} 0.8048 \\ -1 \\ 0.60976 \end{pmatrix}$$

$$= \lambda^{(5)} X^{(5)}$$

$\therefore$ After 5 iterations, the largest eigen value is 3.42 and the corresponding eigen vector is $\begin{pmatrix} 0.81 \\ -1 \\ 0.61 \end{pmatrix}$

HW

Q. Find the numerically largest eigen value and the corresponding eigen vector of the matrix

$$A = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}$$ with intial approximation $X^{(0)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, correct to two decimal places.

$\mathbb{R} \rightarrow$ set of all real numbers

$\mathbb{C} \rightarrow$ set of all complex numbers.

F or $\mathbb{F} \rightarrow$ field of scalars (either $\mathbb{R}$ or $\mathbb{C}$)

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$ ✓

Binary operation :- Let S be a non-empty set then a binary operation $*$ on S is a mapping, $*: S \times S \to S$ defined by $*(a,b) = a*b \in S$.

2. Vector Space

Notation 2.1. . $\mathbb{R}$ - The set of all real numbers.

Eg:- '+' is a binary operation on $\mathbb{R}$.
→ "÷" is not a binary operation on $\mathbb{R}$.

. $\mathbb{C}$ - The set of all complex numbers.

. $\mathbb{F}$ - called the **Field** of scalars. Here we choose $\mathbb{F}$ as either $\mathbb{R}$ or $\mathbb{C}$.

. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x_1, x_2)\,; x_1, x_2 \in \mathbb{R}\}$ — ordered pair

. $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x_1, x_2, x_3)\,; x_1, x_2, x_3 \in \mathbb{R}\}$. — ordered triplet

. $\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times ... \times \mathbb{R}}_{n \text{ times}} = \{(x_1, x_2, x_3, .., x_n)\,; x_1, x_2, x_3..., x_n \in \mathbb{R}\}$. — ordered n-tuple

⊛ The set of all n- dimensional vectors is called the **Euclidean Space,** denoted by $\mathbb{E}^n$ or $\mathbb{R}^n$.

⊛ **How to define the operation '+' (called addition) on $\mathbb{R}^n$?**

Let $x \in \mathbb{R}^n$ then $x = (x_1, x_2, \dots x_n)$ $x_i \in \mathbb{R}$, $i = 1, 2 \dots n$

and $y \in \mathbb{R}^n$ then $y = (y_1, y_2 \dots y_n)$ $y_i \in \mathbb{R}$

$+ : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$

Then, define $x + y = (x_1, x_2 \dots x_n) + (y_1, y_2 \dots y_n)$

$= (x_1 + y_1, x_2 + y_2 \dots x_n + y_n) \in \mathbb{R}^n$

The operation addition is a binary operation in $\mathbb{R}^n$ or '+' is closed in $\mathbb{R}^n$.

$\cdot : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by $\cdot(\alpha, x) = \alpha \cdot x$

⊛ **How to define the operation '.' (called scalar multiplication) on $\mathbb{R}^n$?**

→ Take $\mathbb{F} = \mathbb{R}$, the collection of scalars.

$$\mathbb{R} \times \mathbb{R}^n = \{(\alpha, x) \mid \alpha \in \mathbb{R},\ x \in \mathbb{R}^n\}$$

Let $\alpha \in \mathbb{F} = \mathbb{R}$ and $x \in \mathbb{R}^n$

then $x = (x_1, x_2, \cdots x_n)$ where $x_i \in \mathbb{R}$ for $i = 1, 2 \cdots n$

Define
$$\alpha \cdot x = \alpha \cdot (x_1, x_2, \cdots x_n)$$
$$= (\underbrace{\alpha \cdot x_1}_{\in \mathbb{R}}, \underbrace{\alpha \cdot x_2}_{\in \mathbb{R}}, \cdots \underbrace{\alpha \cdot x_n}_{\in \mathbb{R}})$$
$$\in \mathbb{R}^n$$

$\therefore$ Scalar multiplication is closed in $\mathbb{R}^n$.

Definition 2.2. Let $\mathbb{R}^n$ be the set of all n- dimensional vectors and $\mathbb{F} = \mathbb{R}$ be the set of all scalars then **$\mathbb{R}^n$ is said to be a vector space over $\mathbb{R}$** if the two operations '+' (called addition) and '.' (called scalar multiplication) on $\mathbb{R}^n$ are closed in $\mathbb{R}^n$.

Here '+' (addition) is a function from $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and

'.' (scalar multiplication) is a function from $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$

$\mathbb{R}\setminus\{0\}$ × → multiplication

Examples:. $\mathbb{R}$, + is a binary operation on $\mathbb{R}$.

(i) + is associative on $\mathbb{R}$

ie; $a+(b+c) = (a+b)+c \quad \forall\, a,b,c \in \mathbb{R}$

(ii) There is a unique element '0' in $\mathbb{R}$ such that

$$a + \underline{0} = a = \underline{0} + a \quad \forall\, a \in \mathbb{R}$$

Then '0' is called additive identity in $\mathbb{R}$

(iii) For all $a \in \mathbb{R}$ there is an element $\underline{-a}$ in $\mathbb{R}$ such that

$$a + \underline{-a} = 0 = \underline{-a} + a$$

$\therefore$ '$-a$' is the additive inverse of a in $\mathbb{R}$.

Then $(\mathbb{R}, +)$ is called a group.

* Consider $\mathbb{R}$, the set of all real numbers with the binary operation multiplication "x".

Then we know that,

(i) 'x' is associative in $\mathbb{R}$

ie; $a \times (b \times c) = (a \times b) \times c \quad \forall\, a, b, c \in \mathbb{R}$

(ii) There is a unique element $1 \in \mathbb{R}$ such that $a \times 1 = a = 1 \times a \quad \forall\, a \in \mathbb{R}$.

(iii) For $0 \in \mathbb{R}$, the multiplicative inverse doesn't exist in $\mathbb{R}$.

$\therefore$ $(\mathbb{R}, \times)$ is <u>not</u> a group.

→ From the above, we can construct a group as below,

Consider $\mathbb{R} \setminus \{0\}$ together with the binary operation multiplication then $(\mathbb{R} \setminus \{0\}, \times)$ forms a group.

From the above examples we formally define the concept of GROUP as below.

Definition 3.1. Let G be a non-empty set and $*$ be a binary operation defined on G. Then $(G, *)$ is said to be a **group** if the following conditions holds:

1. $*$ **is associative in** G.
   i.e., $a * (b * c) = (a * b) * c$ for all $a, b, c \in G$..
2. **Existence of identity element in** G.
   i.e., There is a unique element $e \in G$ such that $a * e = a = e * a$ for all $a \in G$.
3. **Existence of inverse element in** G.
   i.e., For all $a \in G$ there exists an element $a' \in G$ such that $a * a' = e = a' * a$.

A group $(G, *)$ is said to be **abelian** if $*$ is **commutative** in G.
i.e., $a * b = b * a$ for all $a, b \in G$.

"abelian group"

Eg:- $G = M_{2\times 2}(\mathbb{R})$ = set of all 2×2 matrices with real entries.

Define '+' on G as "addition of matrices".
then "addition of matrices" is a binary operation on G.

(i) $A + (B + C) = (A + B) + C \quad \forall\, A, B, C \in G$

(ii) There is a unique elt $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in G$ such that $A_{2\times 2} + O_{2\times 2} = A_{2\times 2} = O_{2\times 2} + A_{22}$

(iii) For any $A_{2\times 2}$ in G then $-A_{2\times 2} \in G$ such that $A_{2\times 2} + (-A_{2\times 2}) = O_{2\times 2} = -A_{2\times 2} + A_{2\times 2}$

$\therefore (M_{2\times 2}(\mathbb{R}), +)$ is a group.

4. ~~Vector Space~~ Also "addition of matrices is commutative"

$\therefore (M_{2\times 2}(\mathbb{R}), +)$ is an abelian group.

Vector space :- Let V be a non-empty set. Let $\mathbb{F}$ be the field of scalars. Define two operations on V;

addition: $+ : V \times V \to V$ by $+(u,v) \to u+v \in V$

Scalar multiplication: $\cdot : \mathbb{F} \times V \to V$ by $\cdot(\alpha, u) \to \alpha \cdot u \in V$

Then V is said to be a Vector space over $\mathbb{F}$ if;

(1) $(V, +)$ is an abelian group ✓

(2) For all $\alpha, \beta \in \mathbb{F}$ and For all $u, v \in V$

(i) $\alpha \cdot (u+v) = \alpha \cdot u + \alpha \cdot v$

(ii) $\alpha \cdot (\beta u) = (\alpha\beta) \cdot u$

(iii) $(\alpha+\beta) \cdot u = \alpha \cdot u + \beta \cdot u$

(iv) $1 \cdot u = u$ where '1' is the multiplicative identity in F

Eg: Let $V = \mathbb{R}$ and $\mathbb{F} = \mathbb{R}$

$V = \mathbb{R}$ $\mathbb{F} = \mathbb{C}$

Define addition '+' on V as

$$+(a,b) = \underset{\in \mathbb{R} = V}{a+b} \quad \forall\ a,b \in \mathbb{R}$$

Scalar multiplication '$\cdot$' on V as

$$\cdot : \mathbb{F} \times V \to V \text{ ie; } \cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

by $\cdot(\alpha, a) = \underset{\in \mathbb{R} = V}{\alpha \cdot a} \quad \forall\ \alpha \in \mathbb{F} = \mathbb{R}$ & $a \in V = \mathbb{R}$

(i) $(\mathbb{R}, +)$ is ✓ an abelian group.

(ii) For all $\alpha, \beta \in \mathbb{F} = \mathbb{R}$ and $a, b \in V = \mathbb{R}$

(i) $\alpha \cdot (a+b) = \alpha \cdot a + \alpha \cdot b$ ✓

(ii) $\alpha \cdot (\beta a) = (\alpha\beta) a$ ✓

(iii) $(\alpha+\beta) \cdot a = \alpha \cdot a + \beta \cdot a$

(iv) $1 \cdot a = a \qquad 1 \in \mathbb{F} = \mathbb{R}$

$\therefore$ $V = \mathbb{R}$ is a vector space over $\mathbb{F} = \mathbb{R}$

Definition 4.1. Let V be a non empty set and F be the field of Scalars. Define two operations on Vas below;

$$\textbf{addition} - + : V \times V \to V \text{ by } +(u,v) \mapsto u + v \in V$$
$$\textbf{scalar multiplication} - \cdot : F \times V \to V \text{ by } +(\alpha,u) \mapsto \alpha \cdot u \in V$$

Then, V is said to be a **vector space** over F if the following conditions holds;

1. $(V,+)$ is an abelian group.
2. For all $u, v \in V$ and for all $\alpha, \beta \in F$.
   (a) $\alpha \cdot (u+v) = \alpha \cdot u + \alpha \cdot v$ for all $u, v \in V$ and for all $\alpha \in F$.
   (b) $(\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot u$
   (c) $\alpha \cdot (\beta \cdot u) = (\alpha\beta) \cdot u$
   (d) $1 \cdot v = v$, where 1 denotes the multiplicative identity in F.

let $a = 2$ $\alpha = 1+i$ the $\alpha \cdot a = (1+i)2 \notin \mathbb{R}$.

→ $\mathbb{R}$ is not a vector space over $\mathbb{C}$. (Reason:- Scalar multiplication is not closed)

→ Q. whether $\mathbb{C}$ is a vector space over $\mathbb{R}$?