Distributive Lattice :

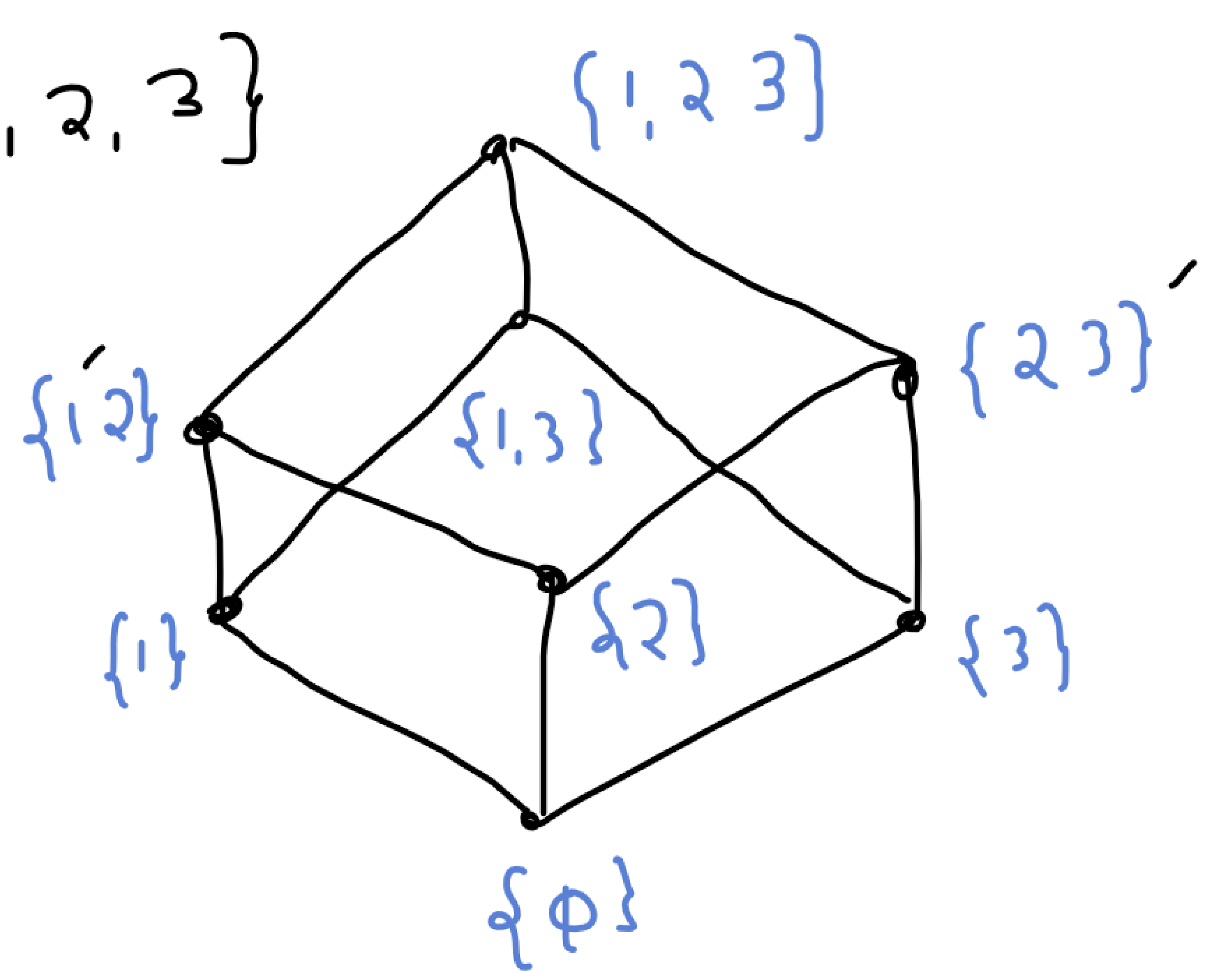
A Lattice is said to be distributive lattice, if the meet operation distributes over the join operation and the join operation distributes over the meet operation.

i.e., for any a, b, c,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

Example: i) $(P(S), \subseteq)$ is a distributive lattice.

$S = \{1, 2, 3\}$



$a = \{1\ 2\}$, $b = \{2, 3\}$, $c = \{1\ 2\ 3\}$

Check whether

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

$a \wedge (b \vee c) = \{1\ 2\} \wedge \{1, 2, 3\} = \{1\ 2\}$ ✓

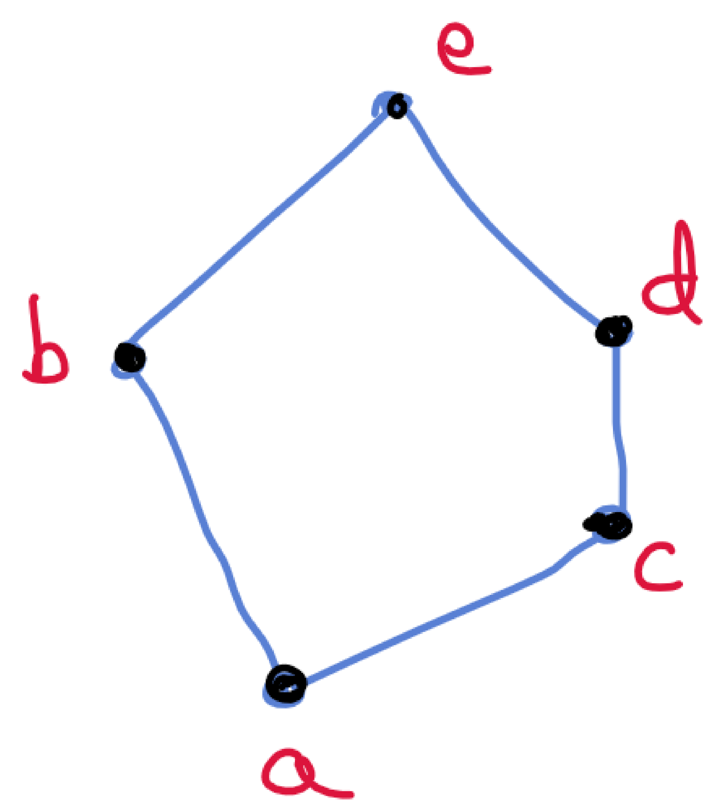
$(a \wedge b) \vee (a \wedge c) = \{2\} \vee \{1, 2\} = \{1\ 2\}$ ✓

To Check

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$a \vee (b \wedge c) = \{1\ 2\ 3\}$, $(a \vee b) \wedge (a \vee c) = \{1, 2, 3\}$

Is it a Distributive Lattice?

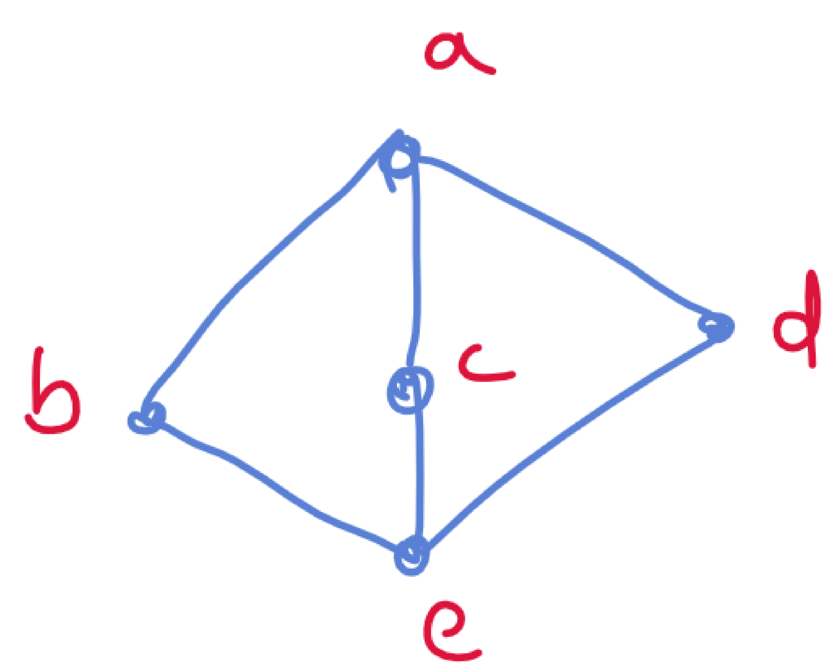


$$d \wedge (b \vee c) = \underset{\cdots}{d \wedge e} = d$$

$$(\underset{glb}{d \wedge b}) \vee (d \wedge c) = a \vee c = c$$

$$c \neq d$$

Not a distributive Lattice



$$b \wedge (c \vee d) = b \wedge a = b$$

$$(b \wedge c) \vee (b \wedge d) = e \vee e = e$$

$$b \neq e$$

Not a Distributive Lattice

Check whether $S_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$ is a Distributive Lattice?

Theorem: If the meet operation is distributive over join operation in a lattice, then the join operation is also distributive over meet operation (vice versa).

i.e., if the join operation is distributive over the meet operation, then meet operation is also distributive over the join operation.

Proof: Given $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ — (1)

To prove $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ — (2)

Consider $(a \vee b) \wedge (a \vee c) = [(a \vee b) \wedge a] \vee [(a \vee b) \wedge c]$ (Applying (1))

$= a \vee [(a \vee b) \wedge c]$ (Absorption law)

$= a \vee [c \wedge (a \vee b)]$ Commutativity

$= a \vee [(c \wedge a) \vee (c \wedge b)]$ Distributive

$= [a \vee (c \wedge a)] \vee (c \wedge b)$ Associative

$= a \vee (c \wedge b)$ Absorption

$= a \vee (b \wedge c)$ Commutative.

By duality, we obtain that if join is distributive over meet then meet operation also distributed over join operation.

Problems

1) Let a and b be two elements in a lattice $(A, \leq)$. Show that $a \wedge b = b$ if and only if $a \vee b = a$.

Soln : Let $a \wedge b = b$ (i)

To prove $a \vee b = a$

$$a \vee (a \wedge b) = a \qquad \leftarrow \text{Absorption}$$

$$a \vee (b) = a \qquad \text{by (i)}$$

$$a \vee b = a$$

Let $a \vee b = a$, (ii) to prove $a \wedge b = b$.

$$b \wedge (a \vee b) = b \qquad \leftarrow \text{Absorption}$$

$$b \wedge (a) = b \qquad \text{by (ii)}$$

$$a \wedge b = b \qquad \text{Commutative}$$

2. Let a, b, c be elements in a lattice $(A, \leq)$. Show that, if $a \leq b$, then

$$a \vee (b \wedge c) \leq b \wedge (a \vee c).$$

Th 2

$a \leq b$ &

$c \leq d$

$a \vee c \leq b \vee d$

$a \wedge c \leq b \wedge d$

Soln :

First we show $a \leq b \wedge (a \vee c) \checkmark$ — (1)

Then we show, $b \wedge c \leq b \wedge (a \vee c)$ — (2)

Given $a \leq b$, and by Theorem (1), $a \leq a \vee c$

By Theorem (2), $a \wedge a \leq b \wedge (a \vee c)$

$$a \leq b \wedge (a \vee c) \quad \text{— (1)}$$

We know $b \leq b$ and $c \leq a \vee c$

By Theorem (2),

$$b \wedge c \leq b \wedge (a \vee c) \quad \text{— (2)}$$

By Theorem (2), To eqn (1) and (2)

$$a \vee (b \wedge c) \leq [b \wedge (a \vee c)] \vee [b \wedge (a \vee c)]$$

$$\Rightarrow \quad a \vee (b \wedge c) \leq b \wedge (a \vee c)$$

Idempotent

$\because a \vee a = a$

3. Let a, b, c be elements in a lattice $(A, \leq)$. Show that,

(i) $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

(ii) $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$

Soln: (i) We know $a \leq a \vee b$ (Th 1)

and $a \leq a \vee c$

From Theorem ②,

$$a \wedge a \leq (a \vee b) \wedge (a \vee c)$$

$$a \leq (a \vee b) \wedge (a \vee c) \quad \text{— ①}$$

We know $b \leq a \vee b$ & $c \leq a \vee c$

From Theorem ②, $b \wedge c \leq (a \vee b) \wedge (a \vee c)$ — ②

Using eqn ① & ② and Theorem 2,

$$a \vee (b \wedge c) \leq [(a \vee b) \wedge (a \vee c)] \vee [(a \vee b) \wedge (a \vee c)]$$

$$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c) \quad \text{by idempotent}$$

Since (i) is true, by <u>duality</u> (ii) is also true.

4. Let $(A, \vee, \wedge)$ be an algebraic system, where $\vee$ and $\wedge$ are binary operations satisfying absorption property. Show that $\wedge$ and $\vee$ also satisfies idempotent law.

Soln: Given for all $a, b \in A$.

$$\left.\begin{aligned} a \vee (a \wedge b) &= a \quad \text{and} \\ a \wedge (a \vee b) &= a \end{aligned}\right\} \text{Absorption law.}$$

Then, to prove $a \vee a = a$ and $a \wedge a = a$

Consider $a \vee a = a \vee (a \wedge (a \vee b))$

$= a$ (by absorption)

Consider $a \wedge \underline{a} = a \wedge (a \vee (a \wedge b))$

$= a$ by absorption.

5. Let $(A, \leq)$ be a distributive lattice. Show that if $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ for some $a \in A$, then $x = y$.

Soln : Consider $x \vee (a \wedge x) = x$ → Absorption

$x \vee (a \wedge y) = x$ → $a \wedge x = a \wedge y$

$(x \vee a) \wedge (x \vee y) = x$ → Distributive

$(y \vee a) \wedge (x \vee y) = x$ → $x \vee a = y \vee a$

$y \vee (a \wedge x) = x$ Distributive

$y \vee (a \wedge y) = x$ $a \wedge x = a \wedge y$

$y = x$ Absorption

6. Show that a lattice is distributive if and only if for any elements a, b, c in the lattice,

$$(a \vee b) \wedge c \leq a \vee (b \wedge c)$$

Soln : Assume the lattice is distributive.

Then $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ —— ①

we know $c \leq a \vee c$ and $a \vee b \leq a \vee b$

$(a \vee b) \wedge c \leq (a \vee b) \wedge (a \vee c)$ Theorem ②

$(a \vee b) \wedge c \leq a \vee (b \wedge c)$ From eqn ①

Conversely, suppose $(a \vee b) \wedge c \leq a \vee (b \wedge c)$, — ✱

to prove $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$(a \vee b) \wedge (a \vee c) \leq a \vee (b \wedge (a \vee c))$

Applying ✱

$\leq a \vee ((a \vee c) \wedge b)$ → Applying ✱

$\leq a \vee (a \vee (c \wedge b))$

$\leq (a \vee a) \vee (c \wedge b)$ Associative

$(a \vee b) \wedge (a \vee c) \leq a \vee (c \wedge b)$ —— (i)

From Problem ③, we have

$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$ —— (ii)

from (i) & (ii) $\Longrightarrow$ $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$