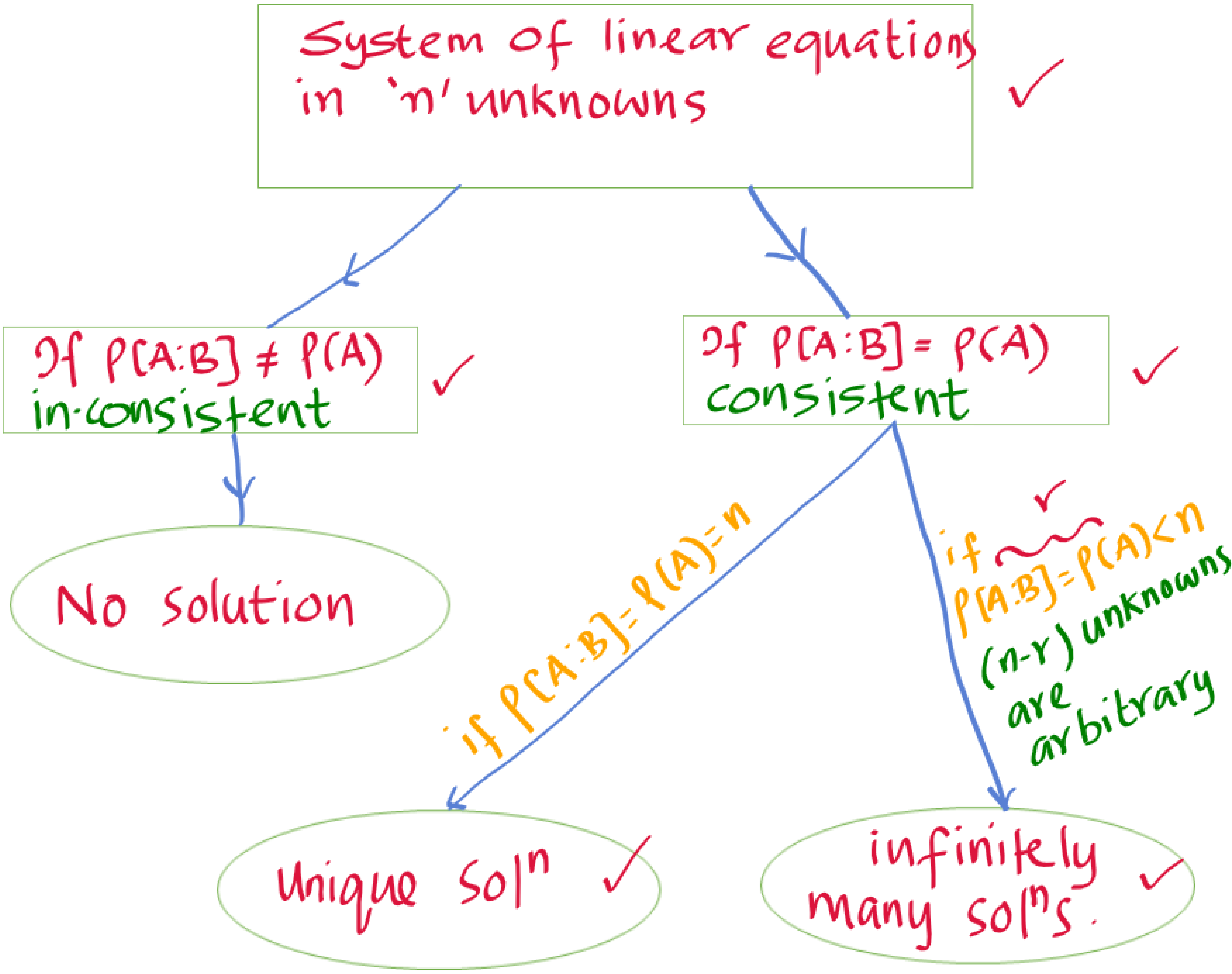
**1. Nature of the solution of system of linear equations**




Q. Solve by Gauss elimination method,

$$x_1 + 2x_2 - 3x_3 - 4x_4 = 6$$
$$x_1 + 3x_2 + x_3 - 2x_4 = 4 \quad \text{(Given)}$$
$$2x_1 + 5x_2 - 2x_3 - 5x_4 = 10.$$

Ans:-

Augmented matrix $[A:B] = \left[\begin{array}{cccc:c} 1 & 2 & -3 & -4 & 6 \\ 1 & 3 & 1 & -2 & 4 \\ 2 & 5 & -2 & -5 & 10 \end{array}\right]$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 2R_1$

2 $\sim \begin{bmatrix} 1 & 2 & -3 & -4 & : & 6 \\ 0 & 1 & 4 & 2 & : & -2 \\ 0 & 1 & 4 & 3 & : & -2 \end{bmatrix}$ $R_3 \to R_3 - R_2$

$\sim \begin{bmatrix} 1 & 2 & -3 & -4 & : & 6 \\ 0 & 1 & 4 & 2 & : & -2 \\ 0 & 0 & 0 & 1 & : & 0 \end{bmatrix}$ Echelon form

$\rho[A:B] = 3$ Also, Echelonform of A is

$A \sim \begin{bmatrix} 1 & 2 & -3 & -4 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\therefore \rho(A) = 3$

ie; $\rho[A:B] = \rho(A) = 3 < 4 =$ no. of unknowns

$\therefore$ System is consistent and have infinitely many $sol^n$.

The equivalent matrix $eq^n$ is

$$\begin{bmatrix} 1 & 2 & -3 & -4 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} x_1 + 2x_2 - 3x_3 - 4x_4 &= 6 \\ x_2 + 4x_3 + 2x_4 &= -2 \end{aligned}\right\}$$

$X = \begin{pmatrix} 11k+10 \\ -2-4k \\ k \\ 0 \end{pmatrix}$

$x_4 = 0$ ✓

$\Rightarrow x_1 + 2x_2 - 3x_3 = 6$ and $x_2 + 4x_3 = -2$

Let $x_3 = k$ be any real no Then $x_2 = -2-4k$

$\therefore x_1 = 11k + 10$. $\therefore Sol^n$ is $X = \begin{pmatrix} 11k+10 \\ -2-4k \\ k \\ 0 \end{pmatrix}$ where k is any real no.

## 2. Inverse of a matrix using row reduced elementary transformation

Let $A$ be a square matrix of order $n$ then the **inverse** of $A$ is a square matrix $B$ of order $n$ such that $AB = BA = I$ where $I$ is an identity matrix of order $n$.

**Working rule:** Let $B$ be the inverse of the given matrix $A$. Then,

$$A\ B = I$$
$$(A)\ \ B = I$$

Apply same row elementary transformations to the matrices $A$ and $I$ simultaneously to make $A$ an identity matrix.

## 3. Guass Jordan Method (To find the inverse of a matrix)

Let A be a square matrix.

$$\left( A \mid I \right)$$

Apply row elementary transformations to make A as an identity matrix, simultaneously apply the same transformations to I.

Gauss Jordan method.

**Problem 3.1.** *Using ~~row reduced Echelon form~~ find the inverse of the matrix* $A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$

Ans:- Here

$$(A|I) = \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right)$$

$$R_1 \to R_1/2$$

$$\sim \left(\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right)$$

$$R_2 \to R_2 - 3R_1$$

$$\sim \left(\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & -3/2 & 1 \end{array}\right)$$

$$R_1 \to R_1 - R_2$$

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1/2 & -3/2 & 1 \end{array}\right)$$

$$R_2 \to 2R_2$$

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -3 & 2 \end{array}\right)$$

$$\therefore A^{-1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

✓ Gauss Jordan method.

**Problem 3.2.** *Using ~~row reduced Echelon form~~ find the inverse of the matrix* $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & 2 \end{pmatrix}$

HW **Problem 3.3.** *Using row elementary transformation, (GAUSS - JORDAN METHOD) find the inverse of the matrix* $A = \begin{pmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{pmatrix}$ ✓

Ans (Prob 3.2) :- $(A|I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right)$

$R_2 \to R_2 - R_1$ , $R_3 \to R_3 - R_1$

$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 0 & -2 & 2 & -1 & 0 & 1 \end{array}\right)$

$R_3 \to R_3 - R_2$

$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right)$

$R_2 \to R_2 - R_3$

$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right)$

$R_2 \to R_2/-2$

**Contd...**

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & -1 & 1/2 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right)$$

$$R_1 \to R_1 - R_2$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1 & -1/2 \\ 0 & 1 & 0 & 1/2 & -1 & 1/2 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right)$$

$$\therefore \quad A^{-1} = \underline{\underline{\begin{pmatrix} 1/2 & 1 & -1/2 \\ 1/2 & -1 & 1/2 \\ 0 & -1 & 1 \end{pmatrix}}}$$

$\check{\underline{A}} X = \underline{B}$



## 4. GAUSS-JORDAN METHOD (To solve system of linear equations)

**Problem 4.1.** *Using Gauss-Jordan method, solve the system of equations .*

$$\begin{aligned} x + y + z &= 9 \\ 2x - 3y + 4z &= 13 \\ 3x + 4y + 5z &= 40 \end{aligned}$$ (Given)

<u>Ans:-</u> The matrix eq$^n$ is

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, \; R_3 \to R_3 - 3R_1$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ 13 \end{bmatrix}$$

$R_3 \to R_3 + \frac{1}{5} R_2$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & 0 & 12/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ 12 \end{bmatrix}$$

$R_3 \to R_3 \times \frac{5}{12}$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ 5 \end{bmatrix}$$

$R_2 \to R_2 - 2R_3$

**Contd...**

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ -15 \\ 5 \end{pmatrix}$$

$$R_2 \to \frac{R_2}{-5}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \\ 5 \end{pmatrix}$$

$$R_1 \to R_1 - R_2$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix}$$

$$R_1 \to R_1 - R_3$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

**Problem 4.2.** *Using Gauss-Jordan method, solve the system of equations .*

HW

$$\begin{aligned} x + 2y - 3z &= 2 \\ x + 3y - 9z &= 6 \\ 7x + 14y - 21z &= 13 \end{aligned}$$

**Problem 4.3.** *Using Gauss-Jordan method, solve the system of equations .*

$$
\begin{aligned}
4y + z &= 2 \\
2x + 6y - 2z &= 3 \\
4x + 8y - 5z &= 4
\end{aligned}
$$

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## 5. Iterative method to solve the system of linear equation

### 5.1. Gauss - Jacobi's Method and Gauss -Seidel Method

Consider the system of linear equations in three unknowns $x, y, z$ as below.

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} (*)$$

Diagonal dominance condition

$$|a_1| \geq |b_1| + |c_1|$$

$$|b_2| \geq |a_2| + |c_2|$$

$$|c_3| \geq |a_3| + |b_3|$$

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1y - c_1z) \\ y &= \frac{1}{b_2}(d_2 - a_2x - c_2z) \\ z &= \frac{1}{c_3}(d_3 - a_3x - b_3y) \end{aligned}\right\} (1)$$

Gauss Jacobi Method

Let $x_0 = y_0 = z_0 = 0$ be the intial approximate value.

**contd..**

<u>Iteration 1</u>:- put $x = x_0$ $y = y_0$ $z = z_0$ in (1)

$$x^{(1)} = x_1 = \frac{1}{a_1}(d_1 - b_1 y_0 - c_1 z_0) = \frac{d_1}{a_1}$$

$$y^{(1)} = y_1 = \frac{1}{b_2}(d_2 - a_2 x_0 - c_2 z_0) = \frac{d_2}{b_2}$$

$$z^{(1)} = z_1 = \frac{1}{c_3}(d_3 - a_3 x_0 - b_3 y_0) = \frac{d_3}{c_3}$$

<u>Iteration 2</u>:-

$$x^{(2)} = \frac{1}{a_1}\left(d_1 - b_1 y^{(1)} - c_1 z^{(1)}\right) = ?$$

$$y^{(2)} = \frac{1}{b_2}\left(d_2 - a_2 x^{(1)} - c_2 z^{(1)}\right) = ?$$

$$z^{(2)} = \frac{1}{c3}\left(d_3 - a_3 x^{(1)} - b_3 y^{(1)}\right. = ?$$

III$^{ly}$ <u>Iteration 3</u>:- $x^{(3)} = \frac{1}{a_1}(d_1 - b_1 y^{(2)} - c_1 z^{(2)})$

$$y^{(3)} = \frac{1}{b_2}(d_2 - a_2 x^{(2)} - c_2 z^{(2)})$$

and so on. $z^{(3)} = \frac{1}{c_3}(d_3 - a_3 x^{(2)} - b_3 y^{(2)})$

**contd..**

# Gauss Seidal Method

Let $y_0 = z_0 = 0$.

**Iteration 1:-** $x^{(1)} = \frac{1}{a_1}(d_1 - b_1 y_0 - c_1 z_0) = ?$

$y^{(1)} = \frac{1}{b_2}(d_2 - a_2 x^{(1)} - c_2 z_0) = ?$

$z^{(1)} = \frac{1}{c}(d_3 - a_3 x^{(1)} - b_3 y^{(1)}) = ?$

**Iteration 2:-** $x^{(2)} = \frac{1}{a_1}(d_1 - b_1 y^{(1)} - c_1 z^{(1)})$

$y^{(2)} = \frac{1}{b_2}(d_2 - a_2 x^{(2)} - c_2 z^{(1)})$

$z^{(3)} = \frac{1}{c_3}(d_3 - a_3 x^{(2)} - b_3 y^{(2)})$

|||ly **Iteration 3:-** $x^{(3)} = \frac{1}{a_1}(d_1 - b_1 y^{(2)} - c_1 z^{(2)})$

$y^{(3)} = \frac{1}{b_2}(d_2 - a_3 x^{(3)} - c_3 z^{(2)})$

$z^{(3)} = \frac{1}{c_3}(d_3 - a_3 x^{(3)} - b_3 y^{(3)})$

and so on..