Q: Let $(G,*)$ be a group. Let H_1 and H_2 be subgroups of G. Test whether $H_1 \cap H_2$ and $H_1 \cup H_2$ are subgroups of G.

Ans: Let H_1 and H_2 be subgroups of G.

So, by definition of subgroup, $e \in H_1$ and $e \in H_2$

Hence $e \in H_1 \cap H_2$

$\therefore H_1 \cap H_2 \neq \phi$.

Let $a, b \in H_1 \cap H_2$.

$\therefore a, b \in H_1$ and $a, b \in H_2$

$\Rightarrow a * b^{-1} \in H_1$ and $a * b^{-1} \in H_2$ ($\because H_1$ & H_2 are subgroups of G)

$\Rightarrow a * b^{-1} \in H_1 \cap H_2$

$\Rightarrow \underline{H_1 \cap H_2}$ is a subgroup of G.

If $H_1 \cup H_2 = H_1$ OR $H_1 \cup H_2 = H_2$ then clearly $H_1 \cup H_2$ is a subgroup of G.

Suppose that neither $H_1 \cup H_2 = H_1$ nor $H_1 \cup H_2 = H_2$

then there is an element $x \in H_1$ with $x \notin H_2$ and $y \in H_2$ & $y \notin H_1$

If $x * y \in H_1$ then $x^{-1} * (x * y) \in H_1$ $\quad (\because x^{-1} \in H_1)$

$\Rightarrow y \in H_1$

which is a contradiction.

If $x * y \in H_2$ then $(x * y) * y^{-1} \in H_2$ $\quad (\because y^{-1} \in H_2)$

$\Rightarrow x \in H_2$, a contradiction.

$\Rightarrow x * y \notin H_1 \cup H_2$

$\Rightarrow H_1 \cup H_2$ is not a subgroup of G.

Example: $H_1 \cup H_2$ is not necessarily a subgroup

Consider a group $(\mathbb{Z}, +)$

Let $H_1 = \{2n \mid n \in \mathbb{Z}\} = \{\cdots -4, -2, 0, 2, 4, 6, \cdots\}$

$H_2 = \{3n \mid n \in \mathbb{Z}\} = \{\cdots -6, -3, 0, 3, 6, \cdots\}$

H_1 and H_2 are subgroups of G.

But $H_1 \cup H_2$ is not a subgroup of G.

because, $2, 3 \in H_1 \cup H_2$

$2 + 3 \notin H_1 \cup H_2$

closure does not satisfy.

H and K

Q: Let $(H, \cdot)$ and $(K, \cdot)$ be two subgroups of a group $(G, \cdot)$.

Define $HK = \{hk \mid h \in H, k \in K\}$.

Prove that $(HK, \cdot)$ is a subgroup of $(G, \cdot)$ iff $HK = KH$.

($\Rightarrow$) Suppose HK is a subgroup of G

To prove $HK = KH$.

$\cancel{(a*b)^{-1} = b^{-1}*a^{-1}}$

Let $x \in KH$

$x = kh$ for some $k \in K$, $h \in H$

$x^{-1} = (kh)^{-1} = h^{-1}k^{-1} \in HK$

& HK is a subgroup.

H is subgroup
Since $h \in H$
$h^{-1} \in H$

$\therefore (x^{-1})^{-1} \in HK$

K is subgroup
$k \in K$
$k^{-1} \in K$

i.e $x \in HK$

Since $x \in KH \Rightarrow x \in HK$.

Hence $KH \subseteq HK$.

Similarly, $HK \subseteq KH$

Hence $HK = KH$ //

Conversely, Let $HK = KH$.

We shall prove that HK is a subgroup.

$e \in H$, $e \in K$

$\therefore e \cdot e = e \in HK$, Hence $H \neq \phi$

Let $a, b \in HK$

$a = h_1 k_1$, $b = h_2 k_2$ for some $h_1, h_2 \in H$
$k_1, k_2 \in K$

$$ab^{-1} = (h_1 k_1)(h_2 k_2)^{-1}$$

$$= h_1 \underbrace{k_1 k_2^{-1}}_{\in K} h_2^{-1} \quad \text{———} \; 1$$

Now, $k_1 k_2^{-1} h_2^{-1} = k_3 h_2^{-1} \in KH = HK$

$\therefore k_1 k_2^{-1} h_2^{-1} = h_3 k_3$ for some $h_3 \in H$
$k_3 \in K$

$1 \Rightarrow$

$$ab^{-1} = h_1 h_3 k_3 = hk_3 \in HK$$

$\Rightarrow HK$ is a subgroup of G.

Right coset:

Let G be a group and H be a subgroup of G. For any $a \in G$, the set Ha is called a right coset of H in G and is denoted as,

*

$$Ha = \{ha \mid h \in H, a \in G\}.$$

Left coset:- of H in G is denoted by aH and is defined as, $aH = \{ah \mid a \in G, h \in H\}$.

Example: Consider the group $(G, \cdot)$ where $G = \{1, -1, i, -i\}$ and $H = \{1, -1\}$ is a subgroup of G.

$H_1 = \{1, -1\}$ $\qquad$ $-iH = \{-i, i\}$

$H_i = \{i, -i\}$

Note: A coset is a nonempty subset of G but it need not be a subgroup of G.

Binary operation '$*$' is defined, then the right coset we can also define as follows:

$$Ha = \{h * a \mid h \in H, a \in G\}$$

Note: Any two left (right) cosets of H in G have the same (finite or infinite) number of elements

Theorem: Let G be a group. Let H be a subgroup. Then any two right (Left) cosets of H in G are either identical or disjoint.

Proof! Let Ha and Hb be two right cosets of H in G.

If they are disjoint, there is nothing to prove.

disjoint
↓
Intersection is empty

Suppose the are not disjoint, we must prove that they are identical.

If Ha and H_b are NOT disjoint $\Rightarrow$ $H_a \cap H_b \neq \phi$

Let $x \in Ha \cap H_b$

$x \in Ha$, $x \in Hb$

$x = h_1 a$, $x = h_2 b$ for some $h_1, h_2 \in H$
$a, b \in G$

$h_2^{-1} x = b$

$h_2^{-1} h_1 a = b$ $\quad (\because x = h_1 a)$

Let $y \in Hb \Rightarrow y = hb$ for smae $h \in H$

$= h(h_2^{-1} h_1 a)$

$= h h_2^{-1} h_1 a$

$= h_3 a$ where $h_3 = h h_2^{-1} h_1 \in H$

$\Rightarrow y \in Ha$

$\Rightarrow Hb \subseteq Ha$ — (i)

By similar argument, $Ha \subseteq Hb$ — (ii)

From (i) & (ii) $Ha = Hb$