Compositions

composit^n of '4' :- 13, 31, 112, 212, 21

Partit^n of '4' :- 13, 112

* The no of compositions of an integer 'n' :- 2^{n-1}

* The no of compositions of an integer 'n' into 'm' parts : $^{n-1}C_{m-1}$

The no of compositns of 10 into 4 parts :-

$$^9C_3 = 84$$

6122
2162

The total no of compositns :-

$$2^9 = 512$$

10
1+9, 2+8, . . . +5+5
9+1, 8+2, . . .

N o of compositns of '10' into 10 parts :

11111. . . . 1 $\Rightarrow {}^{n-1}C_{m-1} \Rightarrow {}^9C_9$

Counting the no of compositions of an integer 'n' into 'm' parts using generating function

Let $f_m(x)$ be the required g.f (enumerator)

Then coeff of x^n = no of compositn of n into m parts $= {}^{n-1}C_{m-1}$

Defn of gf

Each part :-

$$(\underset{1}{x} + \underset{2}{x^2} + \underset{3}{x^3} + \cdots)^m$$

$$f_m(x) = (x+x^2+x^3+\dots)^m = x^m(1+x+x^2+\dots)^m$$

$$= \frac{x^m}{(1-x)^m}$$

$$\boxed{f_m(x) = x^m(1-x)^{-m}}$$

To prove $f_m(x)$ is the right required gf :-

I've to S.T coeff of x^n in $f_m(x)$ is ${}^{n-1}C_{m-1}$

$$f_m(x) = x^m \sum_{r=0}^{\infty} {}^{m+r-1}C_r \, x^r$$

$m+r=k$

$$= \sum_{k=m}^{\infty} {}^{k-1}C_{k-m} \, x^k$$

$\left({}^nC_r = {}^nC_{n-r}\right.$

${}^{k-1}C_{k-m} = {}^{k-1}C_{k-1-(k-m)}$

$$\frac{x^m}{(1-x)^m} = \sum_{k=m}^{\infty} {}^{k-1}C_{m-1} \, x^k$$

coeff of $x^n = {}^{n-1}C_{m-1}$

$\therefore$ $f_m(x) = \left(\frac{x}{1-x}\right)^m$ is the required gf which generates the sequence ${}^{n-1}C_{m-1}$

$f_m(x) = \left(\frac{x}{1-x}\right)^m$ is the required gf for no of compositⁿs o the integer 'n' into 'm' parts

$$f_m(x) = \left(\frac{x}{1-x}\right)^m$$

No of compositⁿ of the integer 'n'

Let $f(x)$ be the gf

$$f(x) = \underbrace{f_1(x)}_{\substack{\text{gf for}\\ \text{comp of}\\ \text{'n' into}\\ \text{1 part}}} + \underbrace{f_2(x)}_{\substack{\text{into 2}\\ \text{parts}}} + \underbrace{f_3(x)}_{\substack{\text{into 3}\\ \text{parts}}} + \cdots$$

$$f(x) = \sum_{m=1}^{\infty} f_m(x) = \sum_{m=1}^{\infty} \left(\frac{x}{1-x}\right)^m$$

$$= \left(\frac{x}{1-x}\right) + \left(\frac{x}{1-x}\right)^2 + \left(\frac{x}{1-x}\right)^3 + \left(\frac{x}{1-x}\right)^4 + \cdots$$

$$= t + t^2 + t^3 + t^4 + \cdots \quad \text{where } t = \frac{x}{1-x}$$

$$= t\left[1 + t + t^2 + \cdots\right] = t \cdot \frac{1}{1-t}$$

$$f(x) = \frac{\frac{x}{1-x}}{1 - \left(\frac{x}{1-x}\right)} = \frac{x}{1-2x}$$

$\therefore$ $f(x) = \dfrac{x}{1-2x}$ is the gf :

To prove this is the right gf, I've sit the coeff of x^n gives 2^{n-1}

$$f(x) = x \cdot \left(1 + 2x + 2^2x^2 + 2^3x^3 + \cdots\right)$$

$$= x + 2x^2 + 2^2x^3 + 2^3x^4 + \cdots$$

coeff of $x^n \rightarrow 2^{n-1}$

* The no of compositⁿ of an integer 'n' into m parts $\Rightarrow \left(\frac{x}{1-x}\right)^m$

* The total no of compositⁿ : $\frac{x}{1-2x}$

PARTITIONS.

Generating function for partition of an integer 'n'

* No of partⁿ of '5' in which no part > 2
* in which every part is distinct

No of partitions of 5:

5
32, 41
311, 221
2111
11111

For a value as a part, the corresp term is

For integer 1 :- $(1+x+x^2+x^3+\dots\dots)$

integer 2 :- $(1+x^2+(x^2)^2+(x^2)^3+\dots\dots)$

3 :- $(1+x^3+(x^3)^2+\dots\dots)$

$$g(x)=(1+x+x^2+\dots)(1+x^2+(x^2)^2+(x^2)^3+\dots)(1+x^3+(x^3)^2+(x^3)^2+\dots)\dots\dots$$

$$\boxed{g(x)=(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\dots\dots\dots\dots}$$

* Gf for partition of integer 'n' as the sum of +ve integers where no integer appears more than thrice

$$\left(1+x+x^2+x^3\right)\left(1+x^2+(x^2)^2+(x^2)^3\right)$$

$$\left(1+x^3+(x^3)^2+(x^3)^3\right)\cdots\cdots$$

14: 222242 X

14: 22253 ✓

coeff of x^n is needed

* Gf for partition in which no part is > 5

(1) 2, 3, 4, 5

14: 4442 ✓

14: 11 $\cancel{3}$6 2 X

$$\left(1+x+x^2+x^3+\cdots\right)\left(1+x^2+(x^2)^2+\cdots\right)\cdots\left(1+x^5+(x^5)^2+\cdots\right)$$

$$(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}(1-x^4)^{-1}(1-x^5)^{-1}$$

* Gf for counting the no of partitions of 100, in which every part is distinct

soln

coeff of x^{100} from the gf

The gf :-

$$(1+x)(1+x^2)(1+x^3)\cdots\cdots$$

① P.T the no of partitions of 'n' in which no integer occurs more than twice as part is equal to the no of partitions of 'n' into parts which are not divisible by 3.

Soln

C_1 : No of partitions in which no integer occurs more than twice

C_2 : No of partitions in which no part is divisible by 3

Let $f(x)$ be the gf for C1 :
Let $g(x)$ be the gf for C2.
To P.T $f(x) = g(x)$

* $f(x) = (1+x+x^2)(1+x^2+x^4)(1+x^3+x^6)\ldots\ldots$

$g(x) = (1+x+x^2+\cdots)(1+x^2+x^4+\cdots)(1+x^4+x^8+\cdots)$

* $g(x) = (1-x)^{-1}(1-x^2)^{-1}(1-x^4)^{-1}(1-x^5)^{-1}(1-x^7)^{-1}\ldots\ldots$

To proving:

$$f(x) = \left(\frac{1-x^3}{1-x}\right)\cdot\frac{1-(x^2)^3}{1-x^2}\cdot\frac{1-(x^3)^3}{1-x^3}\cdot\frac{1-(x^4)^3}{1-x^4}\cdots\cdots$$

$$= \frac{\cancel{1-x^3}}{1-x}\cdot\frac{\cancel{1-x^6}}{1-x^2}\cdot\frac{\cancel{1-x^9}}{\cancel{1-x^3}}\cdot\frac{\cancel{1-x^{12}}}{1-x^4}\cdots\cdots$$

Only Xles of 3 are getting cancelled.

$$= \frac{1}{(1-x)(1-x^2)(1-x^4)(1-x^5)(1-x^7)\cdots\cdots}$$

$$= (1-x)^{-1}(1-x^2)^{-1}(1-x^4)^{-1}(1-x^5)^{-1}\cdots\cdots$$

$$= g(x)$$

5:

C1: no part occurs more than twice

$$\begin{cases} 5 \\ 41, 32 \\ 221, 311 \\ \cancel{2111} \\ \cancel{11111} \end{cases}$$

C2: No part is divisible by 3

$$\begin{cases} 5 \\ 41, \cancel{32} \\ 221, \cancel{311} \\ 2111 \\ 11111 \end{cases}$$

* S.T the no of partitions of 'n' in which every part is odd is equal to the no of partitions of 'n' with unequal (distinct) parts

Soln

$$f(x) = (1-x)^{-1}(1-x^3)^{-1}(1-x^5)^{-1} \cdots\cdots$$

$$g(x) = (1+x)(1+x^2)(1+x^3) \cdots\cdots$$