Functions of two dimensional random variables

Let (X, Y) be a continuous two dimensional random variable with pdf $f(x, y)$. If $Z = H_1(X, Y)$ is a continuous function of (X, Y), then Z will be a continuous (one-dimensional) random variable. In order to find the pdf of Z, we shall follow the procedure discussed below.

To find the pdf of $Z = H_1(X, Y)$, we first introduce a second random variable, say $W = H_2(X, Y)$, and obtain the joint pdf of Z and W, say $k(z, w)$. From the knowledge of $k(z, w)$, we can then obtain the desired pdf of Z, say $g(z)$, by simply integrating $k(z, w)$ with respect to w. That is, $g(z) = \int_{-\infty}^{\infty} k(z, w)dw$.

Two problems which arise here are

i. how to find the joint pdf $k(z, w)$of $Z$and W

ii. how to choose the appropriate random variable $W = H_2(X, Y)$

To resolve these problems, let us simply state that we usually make the simplest possible choice for W as it plays only an intermediate role. In order to find the joint pdf $k(z, w)$, we need the following theorem.

Theorem:

Suppose that (X, Y) is a two-dimensional continuous random variable with joint pdf $f(x, y)$. Let $Z = H_1(X, Y)$ and $W = H_2(X, Y)$ and assume that the functions H_1 and H_2 satisfy the following conditions:

i. The equations $z = H_1(x, y)$ and $w = H_2(x, y)$ may be uniquely solved for x and y in terms of z and w, say $x = G_1(z, w)$ and $y = G_2(z, w)$.

ii. The partial derivatives $\frac{\partial x}{\partial z}, \frac{\partial x}{\partial w}, \frac{\partial y}{\partial z}$ and $\frac{\delta y}{\delta w}$ exist and are continuous.

Then the joint pdf (Z, W), say $k(z, w)$, is given by the following expression:

$k(z, w) = f[G_1(z, w), G_2(z, w)]|J(z, w)|$,

where $J(z, w)$ is the following 2×2 determinant:

$$J(z, w) = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\delta y}{\delta w} \end{vmatrix}$$

This determinant is called the 'Jacobian' of the transformation $(x, y) \rightarrow (z, w)$ and is sometimes denoted by $\frac{\delta(x,y)}{\delta(z,w)}$. We note that $k(z, w)$ will be nonzero for those values of (z, w) corresponding to values of (x, y) for which $f(x, y)$ is nonzero.

Problems

1. Suppose that X and Y are two independent random variables having pdf $f(x) = e^{-x}, 0 \leq x \leq \infty$ and $g(y) = 2e^{-2y}, 0 \leq y \leq \infty$. Find the pdf of X+Y

   Solution:

   Since X and Y are independent, the joint pdf of (X, Y) is given by,

   $$f(x, y) = f(x)g(y) = 2e^{-(x+2y)}, 0 \leq x, y \leq \infty$$

   Let $Z = X + Y$ and $W = Z$, that is $Y = W$ and $X = Z - W$.

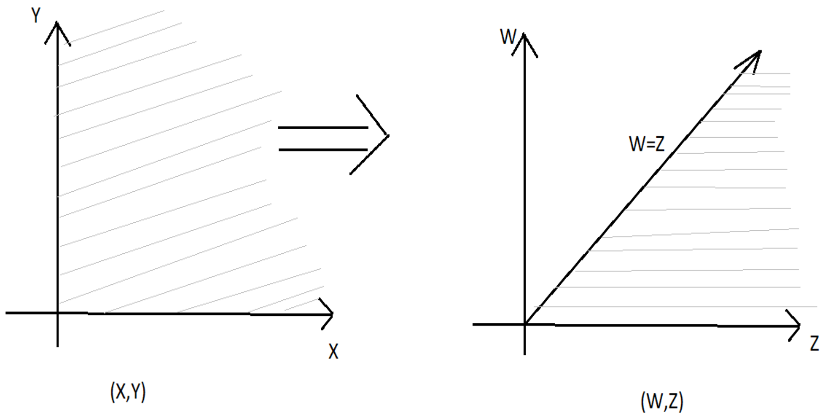
The Jacobian $J = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\delta y}{\delta w} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$

Thus joint pdf of (W, Z) is,

$$k(z, w) = f(x, y)|J| = 2e^{-(x+2y)} = 2e^{-(z+w)}$$

$$0 \leq y \leq \infty \Rightarrow 0 \leq w \leq \infty$$

$$0 \leq x \leq \infty \Rightarrow 0 \leq z - w \leq \infty \Rightarrow w \leq z \leq \infty$$



Thus $k(w, z) = 2e^{-(z+w)}, 0 \leq w \leq z \leq \infty$

The required pdf of z, $h(z) = \int_{w=0}^{z} 2e^{-(z+w)} dw$

$$= 2(e^{-z} - e^{-2z}), 0 \leq z \leq \infty.$$

2. If $X \sim N(0, \sigma^2)$, $Y \sim N(0, \sigma^2)$ and X, Y are independent. Find the pdf of $R = \sqrt{X^2 + Y^2}$

Solution:

Pdf of X: $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}, -\infty \leq x \leq \infty$

Pdf of Y: $g(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2}, -\infty \leq y \leq \infty$

Since X and Y are independent, the joint pdf of (X, Y) is given by,

$$f(x, y) = f(x)g(y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}, -\infty \leq x, y \leq \infty$$

Let $R = \sqrt{X^2 + Y^2}$ and $\theta = \tan^{-1}\left(\frac{X}{Y}\right)$, that is $X = R\cos\theta$ and $Y = R\sin\theta$ and the Jacobian $J = R$.

Thus joint pdf of (R, θ) is,

$k(r, \theta) = f(x, y)|J| = \frac{R}{2\pi\sigma^2} e^{-R^2/2\sigma^2}, R \geq 0, 0 \leq \theta \leq 2\pi$

The required pdf of z, $h(z) = \int_{\theta=0}^{2\pi} \frac{R}{2\pi\sigma^2} e^{-R^2/2\sigma^2} d\theta$

$$= \frac{R}{\sigma^2} e^{-R^2/2\sigma^2}, R \geq 0.$$

3. If X_1, X_2 are independent and have standard normal distribution $X_1, X_2 \sim N(0,1)$. Find the pdf of $\frac{X_1}{X_2}$.

Solution:

Pdf of X_1: $f(x_1) = \frac{1}{\sqrt{2\pi}} e^{-x_1^2/2}, -\infty \leq x_1 \leq \infty$

Pdf of X_2: $g(x_2) = \frac{1}{\sqrt{2\pi}} e^{-y_1^2/2}, -\infty \leq y_1 \leq \infty$

SinceX_1, X_2 are independent, the joint pdf of $(X_1, X_2)$is given by,

$$f(x_1, x_2) = \frac{1}{2\pi} e^{-(x_1^2+x_2^2)/2}, -\infty \leq x_1, y_1 \leq \infty$$

Let $Z = \frac{X_1}{X_2}$and $W = X_2$, that is $X_2 = W$ and $X_1 = ZW$.

The Jacobian $J = \begin{vmatrix} w & z \\ 0 & 1 \end{vmatrix} = w$

Thus joint pdf of (W, Z) is,

$k(z, w) = \frac{|w|}{2\,\pi} e^{-w^2(1+z^2)/2}, \quad -\infty \leq w, z \leq \infty.$

The required pdf of Z, $h(z) = \int_{-\infty}^{\infty} \frac{|w|}{2\pi} e^{-w^2(1+z^2)/2}\, dw$

$$= \frac{2}{2\,\pi} \int_0^{\infty} |w| e^{-w^2\left(1+z^2\right)/2}\, dw$$

On substitution: $-w^2(1+z^2)/2 = t$

$$-w(1+z^2)dw = dt$$

We get, $h(z) = \frac{1}{\pi} \int_0^{\infty} \frac{e^{-t}}{1+z^2} dt = \frac{1}{\pi(1+z^2)}, -\infty \leq z \leq \infty.$

4. The joint pdf of the random variable(X, Y) is given by

$$f(x, y) = \frac{x}{2} e^{-y}, 0 < x < 2, y > 0$$

Find the pdf of $X + Y$

Solution:

Let $Z = X + Y$ and $W = Z$, that is $Y = W$ and $X = Z - W$.

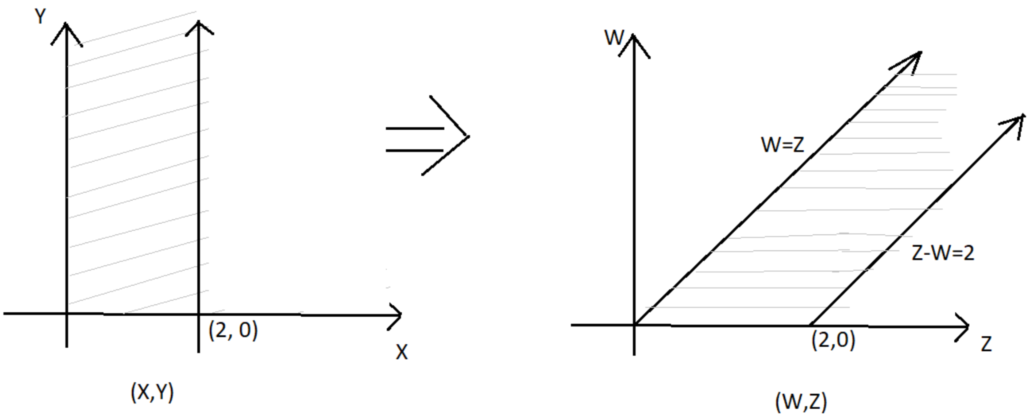
The Jacobian $J = 1$

Thus joint pdf of (W, Z) is, $k(z, w) = f(x, y)|J| = \frac{z-w}{2} e^{-w}$

$$0 \leq y \leq \infty \Rightarrow 0 \leq w \leq \infty$$

$$0 \leq x \leq 2 \Rightarrow 0 \leq z - w \leq 2 \Rightarrow w \leq z \leq 2 + w$$

$k(z, w) = \frac{z-w}{2} e^{-w}, 0 \leq w \leq z \leq 2 + w$



The required pdf of z, $h(z) = \begin{cases} \int_0^z \left(\frac{z-w}{2}\right) e^{-w}\, dw, & when\ 0 < z < 2 \\ \int_{z-2}^z \left(\frac{z-w}{2}\right) e^{-w}\, dw, & when\ 2 < z < \infty \end{cases}$

$$h(z) = \begin{cases} \frac{1}{2}(z + e^{-z} - 1), & when\ 0 < z < 2 \\ \frac{1}{2}(e^{z} + e^{2-z}), & when\ 2 < z < \infty \end{cases}$$

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