PERMUTATION AND COMBINATION

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COMBINATIONS AND PERMUTATIONS

Combinations, permutations, partitions and compositions are the simplest and the most widely used mathematical objects in combinatorics.

Rule of product: If one experiment has m possible outcomes and another experiment has n possible outcomes, then there are $m \times n$ possible outcomes when both of these experiments take place.

Rule of sum: If one experiment has m possible outcomes and another experiment has n possible outcomes, then there are m + n possible outcomes when exactly one of these experiments takes place.

Definition 1.1

A combination of n objects taken r at a time (called an r-combination of n elements) is a selection of r of the objects where the order of the objects in the selection is immaterial.

Definition 1.2

A permutation of n objects taken r at a time (called an r- permutation of n elements) is an ordered selection of r of the objects.

Suppose there are four objects a, b, c and d and selections of them are made two at a time.
The combinations without repetition are six in number, namely
ab, ac, ad, bc, bd, cd
and there are ten with repetition, which are
$aa, ab, ac, ad, bb, bc, bd, cc, cd, dd$.

The twelve permutations without repetition are
$ab, ac, ad, ba, bc, bd, ca, cb, cd, da, db, dc$
and the sixteen with repetition are
$aa, ab, ac, ad, ba, bb, bc, bd, ca, cb, cc, cd, da, db, dc, dd$.

The Enumerators for Permutations and Combinations

1. The number of r-permutations of n objects, $P(n,r) =^n P_r$, is given by
$$P(n,r) = n(n-1)...(n-r+1) = \frac{n!}{(n-r)!}$$
2. When unlimited repetition in an r-permutation of n objects is allowed then after choosing the first object in n ways, the next object can also be chosen in n ways and so on. This can be done in n^r ways.
3. Consider n objects of which m_1 are of the first kind, m_2 are of the second kind, ..., m_k are of the k^{th} kind, so that $\sum_{i=1}^{k} m_i = n$.
The number of permutations of all the objects in this case is
$$\frac{n!}{m_1!m_2!\dots m_k!}$$

4. The number of r-combinations of n objects without repetition is denoted by $C(n, r) =^n C_r = \dfrac{n!}{r!(n-r)!}$

5. The general formula for r-combinations of n objects when unlimited repetition is allowed is more difficult to obtain - we cannot simply divide the permutation result for unlimited repetition, n^r, by an appropriate factor since different combinations with repetition will not in general give rise to the same number of permutations.

For example (aab) gives rise to three different permutations while (abc) gives six permutations.

Let one r-combination of n objects (which are considered to be the digits $1, 2, 3, ..., n$) in which repetition is allowed be $(c_1, c_2, \ldots, c_r)$, and suppose $c_1, c_2, \ldots, c_r$ are in rising order, i. e. $c_1 \leq c_2 \leq \ldots \leq c_r$. Form the set of d's ,$d_1, \ldots, d_r$, by the rule $d_1 = c_1 + 0, d_2 = c_2 + 1, , \ldots d_i = c_i + i - 1 \ldots, d_r = c_r + r - 1$. This transformation ensures that the d's are unalike whatever the c's were. It is clear that the sets of c's and d's are equinumerous since every distinct r-combination of the c's produces a distinct set of d's and vice versa. The number of sets of d's is the number of r-combinations without repetition of the objects $1, 2, \ldots, n + r - 1$ since the largest d is d_r when c_r has its maximum value n. Thus the number of sets of d's is ${}^{n+r-1}C_r$ and this is equal to the number of r-combinations of n objects with unlimited repetition.

Definition 1.3

*A **distribution** is defined as a separation of a set of objects into a number of classes - for example, the assignment of the objects to cells (or boxes); problems about distributions are very closely related to problems of permutations and combinations.*

Consider first the case of assigning r different objects to n distinct cells such that each cell has at most one object. If $n >$r then there are $P(n, r)$ ways, since the first object may be assigned to any of the n cells, the second object to one of the $n-1$ remaining cells etc. Alternatively if $r > n$ then there are $P(r, n)$ ways, since the object assigned to the first cell may be done in r ways, the object assigned to the second cell in $r-1$ ways etc.

Continuing with r different objects and n distinct cells but now allowing each cell to hold any number of objects, we obtain n ways of distributing the objects. This is true whether n is larger or smaller than r since the first object can be assigned to any one of the n cells and so can the second and the other objects.

When the r objects to be distributed are not all different suppose that m_1, of them are of the first kind, m_2 of the second kind, ..., m_k of them of the k^{th} kind, so that $r = \sum_{i=1}^{k} m_i$. First suppose that each of the n distinct cells may hold at most one object $(n \geq r)$.

The r cells are selected from the n cells (in $C(n, r)$ ways) and then the r objects are distributed into these r cells which is equivalent to forming a permutation with repetition of the objects. Therefore there are

$\dfrac{r!}{m_1!m_2!\dots m_k!}$ such permutations.

Therefore the number of these distributions is

$$C(n, r)\frac{r!}{m_1!m_2!\dots m_k!} = \frac{n!}{(n-r)!m_1!m_2!\dots m_k!}.$$

The r like objects are placed in n distinct cells without any restriction on the number going into each cell. The number of ways of doing this is equivalent to selecting r cells from n with repetition of cells allowed and the number of such distributions is $C(n+r-1, r)$.

Example: There are five different Spanish books, six different French books, and eight different Transylvanian books. How many ways are there to pick an (unordered) pair of two books not both in the same language? Ans: $5.6+5.8+6.8 = 30+40+48 = 118$.

Problems

1. Find the number of different letter arrangement can be formed using the word "SYSTEMS"?
Ans: $\dfrac{7!}{3!}$

2. Find the number of ways in which 3 exams can be scheduled in a 5 day period such that (i) No two exams are scheduled on the same day?
(ii) There are no restrictions on number of exams conducted on a day?
Ans: (i) Considering the three examinations as distinctly colored balls and the five days as distinctly numbered boxes, we obtain the result 5 x 4 x 3 $= 60$ i.e., ${}^5P_3$ (ii) 5^3.

3. If 5 men A,B,C,D,E intend to speak at a meeting, in how many orders can they do so without B speaking before A? How many orders are there in which A speaks immediately before B?
Ans: $4! + 3.3! + 6.2! + 3!$.

4. How many ways may one right and one left shoe be selected from six pairs of shoes without obtaining a pair?
Ans: 30.

5. How many ways can twelve white pawns and twelve black pawns be placed on the black squares of an 8×8 chess board?
Ans: $\dfrac{32.31.....9}{12!12!}$.

6. Find the sum of all the four digit numbers that can be obtained by using the digits 1, 2, 3, 4 once in each.
Ans: Each digit occupies each place 6 times. Threfore, sum is
$6([1+2+3+4]1000+[1+2+3+4]100+[1+2+3+4]10+[1+2+3+4]) = 66660$

7. How many points of intersection are formed by n lines drawn in a plane if no two are parallel and no three concurrent? Into how many regions is the plane divided?

Ans:Two lines 1 intersecting point

3 lines 1+2 intersecting points

So n lines 1+2+...+(n-1) intersecting points.

Number of intersecting points is ${}^nC_2$.

Similarly one line 2 regions

2 lines 4 regions

So, n lines $\dfrac{(n+1)n}{2} + 1$ regions.

8. How many ways can three integers be selected from 3n consecutive integers so that the sum is a multiple of 3 ?

Ans: ${}^nC_3 + {}^nC_3 + {}^nC_3 + {}^nC_1\ {}^nC_1\ {}^nC_1$

9. How many ways can five different messages be delivered by three messengers if no messenger is left unemployed? The order in which a messenger delivers his messages is immaterial.

Ans: $3\dfrac{5!}{3!1!1!}+3\dfrac{5!}{2!2!1!}=150.$

10. In how many ways can a lady wear five rings on the fingers (not the thumb) of her right hand?

Ans: Consider 5 rings as identical. Then distribution of 5 rings in to 4 fingers such that finger can hold any number of rings is ${}^{4+5-1}C_5 = {}^{8}C_5$. As 5 rings can be arranged in 5! ways the answer is $5!\,{}^{8}C_5 = 6720$.

11. Six distinct symbols are transmitted through a communication channel. A total of twelve blanks are to be inserted between the symbols with at least two blanks between every pair of symbols. In how many ways can we arrange the symbols and blanks ?

Ans: ${}^{5+2-1}C_2 = 15$

12. A new national flag is to be designed with six vertical stripes in yellow, green, blue and red. In how many ways can this be done so that no two adjacent stripes have the same colour?
Ans: 4.3.3.3.3.3

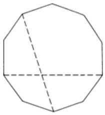
13. In how many ways can an examiner assign 30 marks to 8 questions so that no question receives less than 2 marks?
Ans: ${}^{8+14-1}C_4 = {}^{21}C_4$ ways.

14. Suppose we print all FIVE-digit numbers on slips of paper with one number on each slip. However, since the digits 0, 1, 6, 8, and 9 become 0, 1, 9, 8, and 6 when they are read upside down, there are pairs of numbers that can share the same slip if the slips are read right side up or upside down. For example, we can make up one slip for the numbers 89166 and 99168. The question is then how many distinct slips will we have to make up for all five-digit numbers?

We note first that there are 10^5 distinct five-digit numbers. Among these numbers, 5^5 of them can be read either right side up or upside down. (They are made up of the digits 0, 1, 6, 8, and 9.) However, there are numbers that read the same either right side up or upside down, for example, 16091, and there are $3(5^2)$ such numbers. (The center digit of these numbers must be either 1, 0, or 8; further, the fifth digit must be the first digit turned upside down, and the fourth digit must be the second digit turned upside down.) Consequently, there are $5^5 - 3(5^2)$ numbers that can be read either right side up or upside down but will read differently. These numbers can be divided into pairs so that every pair of numbers can share one slip. It follows that the total number of distinct slips we need is $10^5 - \dfrac{5^5 - 3(5^2)}{2}$.

15. If no three diagonals of a convex decagon meet at the same point inside the decagon, into how many line segments are the diagonals divided by their intersections?

Ans:



First of all, the number of diagonals is equal to ${}^{10}C_2 - 10 = 45 - 10 = 35$ as there are ${}^{10}C_2$ straight lines joining the ${}^{10}C_2$ pairs of vertices, but 10 of these 45 lines are the sides of the decagon. Since for every four vertices we can count exactly one intersection between the diagonals, as Figure shows (the decagon is convex), there are a total of ${}^{10}C_4 = 210$ intersections between the diagonals. Since a diagonal is divided into k + 1 straight-line segments when there are k intersecting points lying on it, and since each intersecting point lies on two diagonals, the total number of straight-line segments into which the diagonals are divided is $35 + 2 \times 210 = 455$.

Generating function

We see how the Generating function concepts is applied to the enumeration of permutations and combinations.
Consider the three distinct objects a, b and c, and form the polynomial $(1+ax)(1+bx)(1+cx) = 1+(a+b+c)x+(ab+bc+ca)x^2+abcx^3$.
The coefficients of x on the right hand side have some interesting properties; if we consider the three ways of selecting one object (a or b or c) and represent it a+b+c then we have the coefficient of the first power of x , $x^1 = x$. Similarly the three ways of selecting two objects (ab or be 2 or ca) may be represented ab+bc+ca which is the coefficient of x^2. Finally the single way of selecting all three objects, namely abc, is the coefficient of x^3.
The factor 1+ax can be considered as representing symbolically the two ways of selecting a or not, the 1 or x representing the 'non-selection of a' and the ax representing the 'selection of a'. The factors (1+bx) and (1+cx) can be interpreted in a similar manner.

Combinations

The product of the three factors (1+ax)(1+bx)(1+cx) indicates the selection or non-selection of all three objects a, b and c , and the powers of x in the product indicate the number of objects selected. Thus the coefficient of x is an enumeration of the ways in which we can select two objects.
The polynomial (1+ax)(1+bx)(1+cx) represent the different ways of selecting the three objects a, b and c. This can be generalised to n objects, say $a_1, a_2, ..., a_n$, by expanding in powers of x the polynomial
$(1 + a_1x)(1 + a_2x)(1 + a_3x)...(1 + a_nx) = 1 + (a_1 + a_2 + ... + a_n)x + (a_1a_2 + a_1a_3 + ...)x^2 + ...$
The coefficient of x on the right hand side represents the r-combination chosen from the objects $a_1, a_2, ..., a_n$. However, we often require the number of r-combinations of n objects rather than the actual combinations. In this case we can use the above formula with $a_1 = a_2 = ... = a_n = 1$. In the simple case of three objects we obtain $(1 + x)^3 = 1 + 3x + 3x^2 + x^3$.

For n objects $(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + ... + .. + x^n$
$= C(n,0) + C(n,1)x^1 + C(n,2)x^2 + ... + C(n,r)x^r + ... + C(n,n)x^n$.
A generating function used in this way is usually called an enumerator.

Example:

How many combinations of three objects can be formed if one object can be selected at most once, the second object at most twice, and the third object at most three times?

Since the first object can appear either once or not at all, the factor in the enumerator must contain x^0 and x^1 but no other terms; similarly the factor for the second object will have x^2 appearing as well. The enumerator can thus be written down one factor at a time corresponding to each of the objects. The enumerator is
$(1+x)(1+x+x^2)(1+x+x^2+x^3) = 1+3x+5x^2+6x^3+5x^4+3x^5+x^6$.

If an object is allowed unlimited repetition the corresponding factor in the enumerator must have every power of x present and so is
$(1+x+x^2+...+x^i+...) = (1-x)^{-1}$.

Thus the enumerator of r-combinations of n objects with unlimited repetition is

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + ... + \frac{n(n+1)...(n+r-1)}{r!}x^r + ...$$
$$= \sum_{r=0}^{\infty} {}^{n+r-1}C_r x^r.$$

Permutations

We develop a generating function for permutation

In the case of three objects a, b and c, the generating function desired has the form

$1 + (a+b+c)x + (ab+ba+bc+cb+ca+ac)x + (abc+acb+bac+bca+cab+cba)x = 1 + (a+b+c)x + 2(ab+bc+ca)x^2 + 6abcx^3$.

Consider $(1+x)^n = 1 + C(n,1)x + C(n,2)x^2 + ... + C(n,r)x^r + ... + x^n$

and replace C(n, r) by $\dfrac{P(n,r)}{r!}$

$$(1+x)^n = 1 + \frac{P(n,1)}{1!}x + \ldots + \frac{P(n,r)}{r!}x^r + ... + \frac{p(n,n)}{n!}x^n$$

is an Exponential generating function for $r-$ permutations of n distinct objects, without repetition.

If repetition is allowed, then factor for each object must represent the fact that the object may not appear, may appear once, and so on in the permutation. Hence, factor for each object is $1 + \frac{x}{1!} + ... = e^x$. And Enumerator is $(1 + x + \frac{x^2}{2!} + ...)^n = \sum \frac{x^r n^r}{r!}$.

Note:

1. $1 + x + x^2 + ... + x^m = \dfrac{1 - x^{m+1}}{1 - x}$.
2. $(1 - x)^{-1} = 1 + x + x^2 + ...$
3. $(1 + x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + ... + {}^nC_n x^n$.
4. $(1 - x^m)^n = 1 - {}^nC_1 x^m + {}^nC_2 x^{2m} + ... + (-1)^n\, {}^nC_n x^{mn}$.
5. $(1 - x)^{-n} = 1 + {}^nC_1 x + {}^{n+1}C_2 x^2 + ... + {}^{n+r-1}C_r x^r$.

Problems

1. Use generating functions to find the number of ways to collect \$15 from 20 distinct people if each of the first 19 people can give a dollar (or nothing) and the twentieth person can give either \$1 or \$5 (or nothing).
Ans: Generating function is $(1+x)^{19}(1+x+x^5)$. We want the coefficient of x^{15}.
Hece answer is ${}^{19}C_{15} +^{19} C_{14} +^{19} C_{10}$.

2. How many ways are there to distribute 25 identical balls into seven distinct boxes if the first box can have no more than 10 balls but any number can go into each of the other six boxes?
Ans: The generating function for the number of ways to distribute r balls into seven boxes with at most 10 balls in the first box is
$(1+x+x^2+\ldots+x^{10})(1+x+x^2+\ldots)^6 = (1-x^{11})(1-x)^{-7}$.
Coeff of x^{25} in $(1-x)^{-7}$ or (-1) Coeff of x^{14} in $-(1-x)^{-7}$.
Ans is ${}^{7+25-1}C_{25} + (-1).{}^{7+14-1}C_{14} =^{31} C_{25} -^{20} C_{14}$.

3. How many ways are there to select 25 toys from seven types of toys with between two and six of each type?
Ans: GF is $(x^2 + x^3 + x^4 + x^5 + x^6)^7$.
We want the coefficient of x^{25} in
$x^{14}(1 + x + x^2 + x^3 + x^4)^7 = x^{14}(1 - x^5)^7(1 - x)^{-7}$.
Ans is ${}^{17}C_{11} - 7.{}^{12}C_6 + 21.{}^{7}C_1$.

4. How many ways are there to get a sum of 25 when 10 distinct dice are rolled?
Ans: GF$=(x + x^2 + ... + x^6)^{10} = x^{10}(1 - x^6)^{10}(1 - x)^{-10}$.
Coeff of x^{25} is
${}^{24}C_{15} - 10.{}^{18}C_9 + 45.{}^{12}C_3$.

5. Find the number of ways to place 25 people into three rooms with at least one person in each room.
Ans: The exponential generating function for this problem is
$(x + \frac{x^2}{2!} + ...)^3 = (e^x - 1)^3 = e^{3x} - 3.e^{2x} + 3e^x - 1$
$= \sum \frac{(3x)^r}{r!} - 3\sum \frac{(2x)^r}{r!} + 3\sum \frac{x^r}{r!} - 1.$
Coeff of $\frac{x^25}{25!}$ is $3^{25} - 3.2^{25} + 3$.

6. Find the number of r-digit quaternary sequences (whose digits are 0, 1, 2, and 3) with an even number of 0s and an odd number of 1s.
Ans: $(1 + \frac{x^2}{2!} + ...)(x + \frac{x^3}{3!} + ...)(1 + x + \frac{x^2}{2!} + ...)^2$
$= (\frac{e^x + e^{-x}}{2})(\frac{e^x - e^{-x}}{2})e^x e^x = \frac{1}{4}(e^{2x} - e^{-2x})e^{2x} = \frac{1}{4}(e^{4x} - 1)$
$= \frac{1}{4}(\frac{(4x)^r}{r!} - 1),$

Coeff of $\frac{x^r}{r!}$ is 4^{r-1}.

THE PRINCIPLE OF INCLUSION AND EXCLUSION

The principle of inclusion and exclusion (sometimes known as the sieve method) is an important combinatorial counting theorem. Consider first a simple illustrative example: A golf club has 125 members, 89 are male and 18 are left-handed players, 11 of the men also play left-handed. How many right-handed lady golfers does the club have? The answer can be written $125\text{-}89\text{-}18+11 = 29$.

Generalising this example, consider N objects and the two properties a and b ; then the number without both these properties is given by $N(a'b') = N - N(a) - N(b) + N(ab)$ for in subtracting from N those with property a and those with property b we have subtracted those with both properties N(ab) twice and they must be replaced.

Theorem 2.1

*If, of N objects, $N(a_1)$ have the property a_1, $N(a_2)$ the property a_2,...,
$N(a_r)$ the property a_r, $N(a_1 a_2)$ both the properties a_1 and a_2, ...,
$N(a_1 a_2..a_r)$ all the properties $a_1, a_2, ..., a_r$, then the number $N(a'_1 a'_2...a'_r)$
with none of these properties is given by*

$$N(a'_1 a'_2...a'_r) =$$
$$N - N(a_1) - N(a_2) - ... - N(a_r) + N(a_1 a_2) + N(a_1 a_3) + ... + (-1)^r N(a_1...a_r).$$

Problems

1 How many of the first 1000 integers are not divisible by 2, 3, 5 or 7 ?
Ans: Let a_1, a_2, a_3 and a_4 be the properties of divisibility by 2, 3, 5 and 7 respectively.
$N(a_1) = 500, N(a_2) = 333, N(a_3) = 200, N(a_4) = 142, N(a_1a_2) = 166, N(a_1a_3) = 100, N(a_1a_4) = 71, N(a_2a_3) = 66 N(a_2a_4) = 47, N(a_3a_4) = 28 N(a_1a_2a_3) = 33, N(a_1a_2a_4) = 23, N(a_1a_3a_4) = 14, N(a_2a_3a_4) = 9 N(a_1a_2a_3a_4) = 4.$
$N(a_1'a_2'a_3'a_4') = 1000 - 500 - 333 - 200 - 142 + 166 + 100 + 71 + 66 + 47 + 28 - 33 - 23 - 14 - 9 + 4 = 228.$

2. Find the number of ways in which 25 distinct objects can be placed in 3 distinct boxes such that no box is empty.
Ans: $N = 3^{25}, N(a_1) = N(a_2) = N(a_3) = 2^{25}, N(a_1a_2) = N(a_1a_3) = N(a_2a_3) = 1, N(a_1a_2a_3) = 0$
$N(a_1'a_2'a_3'a_4') = 3^{25} - 3.2^{25} + 3.$

How many permutations of the n distinct elements (1,2,3,...,n) are there in which the element k is not in the kth position?
Ans: Let $b_1, b_2 ... b_n$ be a permutation of 1, 2,...,n. Let the property that $b_i = i$ be a_i.
Then for any s properties $a_{i_1} ... a_{i_s}$
$N(a_{i_1} ... a_{i_s}) = (n-s)!$
because if s positions are fixed the remaining positions give $(n-s)!$ permutations. Then the principle of inclusion and exclusion gives
$N(a'_1 a'_2 ..., a'_n) = n! - n(n-1)! - {}^nC_2(n-2)! + ... + (-1)^i \, {}^nC_i(n-i)! + ... + (-1)^n$
$= n!(1 - \frac{1}{1!} + \frac{1}{2!} + ... + \frac{(-1)^n}{n!}) \cong \frac{n!}{e}$, if n large.

For n = 4 the derangements number 9 = 12-4+1 , and they are the permutations 2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, 4321.

PARTITIONS AND COMPOSITIONS

It is clearly possible to represent any positive integer n as a sum of one or more positive integers (a_i).

$n = a_1 + a_2 + ... + a_m$

Divisions of a positive integer n are of two types depending on whether the ordering of the parts $a_1, a_2, ..., a_m$ is regarded as important or not. Ordered divisions are called compositions while unordered divisions are called partitions.

Consider, for example, the partitions and compositions of the integer 5 . There are seven unrestricted partitions, namely 5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1 + 1 + 1, 1+1+1 + 1 + 1 and two of these, namely 4+1 and 3+2 , have exactly two parts.
There are sixteen unrestricted compositions of n=5 5, 4+1, 1+4, 3+2, 2+3, 3+1+1, 1+3+1, 1 + 1 + 3, 2+2+1, 2+1+2, 1 + 2+2, 2+1 + 1 + 1, 1 + 2+1+1, 1+1+2+1, 1 + 1 + 1+2, 1+1 + 1+1+1 and four of these have exactly two parts.

When we write partitions or compositions we
will omit the + signs, thus 2+1+1+1 will be written 2111, or 21^3, and for partitions the largest parts will be written first.

Compositions

One of the easiest ways of enumerating the unrestricted compositions of n is to consider n ones in a row. Since there is no restriction on the number of parts, we may or may not put a marker in any of the (n-1) spaces between the ones in order to form groups; this may be done in 2 ways. The same type of argument can be applied when we restrict the compositions to have exactly m parts. Just (m-1) markers are needed to form m groups and the number of ways of placing (m-1) markers in the (n-1) spaces between the ones is ${}^{n-1}C_{m-1}$.

These results can also be obtained as follows using generating functions. Let $C_m(x)$ be the enumerator for compositions of n with exactly m parts, where $C_m(x) = \sum_n C_{mn}x^n$ and C_{mn}, the coefficient of x^n in this series, is the number of compositions of n into exactly m parts. Each part of any composition can be one, two, three or any greater number so that the factor in the enumerator must contain each of these powers of x, and so is $x + x^2 + x^3 + ... + x^k + ... = x(1-x)^{-1}$.

Since there are exactly m parts, the GF is the product of m such factors.
$C_m(x) = (x + x^2 + ...)^m = x^m(1-x)^{-m}$
$= x^m(1 +^m C_1 x +^{m+1} C_2 x^2 + ...) = \sum_{n=m}^{\infty} x^n$.
The coefficient of x^n in this enumeratoris ${}^{n-1}C_{m-1}$. So number of compositions of n with exactly m parts $=^{n-1} C_{m-1}$.

The enumerating GF for composition with no restriction on the number of parts $C(x)$ can be obtained from $C_m(x)$.
$C(x) = \sum_{m=1}^{\infty} C_m(x) = \sum_{m=1}^{\infty} x^m(1-x)^{-m} = x(1-2x)^{-1}$
$= x(1 + 2x + (2x)^2 + ...) = \sum_{n=1}^{\infty} 2^{n-1}x^n$.
Number of compositions with no restriction$=2^{n-1}$.

Example: How many compositions of n with m parts are there when zero as a part are allowed.
Ans: Consider a compositions of n with m parts are there when zero as a part are allowed. Add one to each part, then it will represent a composition of $(n+m)$ with exactly m parts.
Therefore, number of compositions$=^{n+m-1}C_{m-1}$.

Generating function for unrestricted partitions

The polynomial $1 + x + x^2 + ... + x^k + ... + x^n$ is concerned with one's in the partition. The polynomial $1 + x^2 + (x^2)^2 + ...x^{2k} + ...$ is concerned with two's in the partition and in particular, the coefficient of $x^{2k} = (x^2)^k$ represents the case of just k number of 2's in the partition.
So the GF for partition should contain one factor for 1's, one for 2's and so on.
GF$=(1 + x + x^2 + ...)(1 + x^2 + x^4 + ...).... = (1 - x)^{-1}(1 - x^2)^{-1}....$

Problems

Prove that number of partitions of n in which no integer occurs more than twice as apart is equal to the number of partitions of n into parts not divisible by 3.

SOLUTION: Enumerator for partitions of n into parts not divisible by 3 is $G_0(x) = (1-x)^{-1}(1-x^2)^{-1}(1-x^4)^{-1}....$

Enumerator for partitions of n in which no integer occurs more than twice as apart is equal $G_1(x) = (1+x+x^2)(1+x^2+x^4)(1+x^3+x^6)....$

$G_1(x) = \dfrac{(1-x)(1+x+x^2)}{(1-x)}.\dfrac{(1-x^22)(1+x^2x^4)}{(1-x^2)}... =$

$\dfrac{(1-x^3)(1-x^6)(1-x^9)....}{(1-x)(1-x^2)(1-x^3)...}$

$= (1-x^{-1})(1-x^2)^{-1}(1-x^4)^{-1}... = G_0(x).$

Ferrors graph

It is a graph to represent a partiotion by an array of dots.
It has the following property.
(i) There is one row for each part.
(ii) The number of dots in arow is the size of that part.
(iii) An upper row always contains at least as many dots as a lower row.
(iv) The rows are alligned to the left.
Example:
Consider the partition 5 3 2 2.

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. . .

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. .

The partition obtained by reading the Ferror's graph by column is called *conjugate partition*.

The conjugate partition of 5 3 2 2 is 4 4 2 1 1.

A partition whose Ferror's graph reads the same by rows and by columns is called self-conjugate.

Example: 5 4 2 2 1, 4 3 2 1.

. . . . .

. . . .

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Problems

1. Show that the number of partitions of n is equal to number of partitions of $2n$ with exactly n parts.

Soln: Consider a Ferror's graph of a partition of n. Add a column of n dots on the left of the graph. Then the graph represents a partition of 2n with exactly n parts.
Consider a partition of 2n with exactly n parts. Then the left most column contains n dots. Eliminating the first column, results in a partition of n.
Thus, for every partition of n dots there corresponds a partition of 2n with exactly n parts and vice versa. Hence, the number of partitions of n is equal to number of partitions of $2n$ with exactly n parts.

Lexicographical Order

To find the k^{th}permutation of n marks (letter, symbols), say $a_1, a_2, \ldots, a_n$, when the permutations are sorted lexicographically, proceed as given below: Write $k-1$ in the form

$$k - 1 = c_{n-1}(n-1)! + c_{n-2}(n-2)! + \cdots + c_1 1!$$

where each integer c_i has the maximum possible value, $0 \leq c_i \leq i$. In other words, we divide $k-1$ by $(n-1)!$, and take c_{n-1} as the quotient; then divide the remainder by $(n-2)!$ and take c_{n-2} as the quotient; and so on. This gives us a sequence $c_{n-1}c_{n-2}\cdots c_1$.

Example 4.1

To compute the 35^{th}*permutation of the five marks* $1, 2, 3, 4, 5$*, we note that* $k = 35$ *and* $n = 5$*. Now*

$$35 - 1 = 34 = \underline{1} \times 4! + \underline{1} \times 3! + \underline{2} \times 2! + \underline{0} \times 1!$$

so that the sequence is $\boxed{1120}$.

Next, the sequence $c_{n-1} \cdots c_1$ is treated as a sequence of array indices (the range being 0 to $n-1$). Then the k^{th}permutation is constructed in the following manner. Start with the array of marks $1, 2, \ldots, n$, and pick the element indexed by c_{n-1} as the first element of the permutation. Remove this element from the array to get a new array, and also remove c_{n-1} from the sequence of indices to get the new sequence $c_{n-2} \cdots c_1$. Now continue until the sequence of indices is exhausted. At this point, exactly one mark will remain in the array, and write this down as the last element of the permutation.

Example 4.2

Continuing from the previous example, to compute the 35^{th} *permutation of the five marks* $1, 2, 3, 4, 5$*, we have already obtained the sequence* 1120*. Now, consider the array of marks* 12345*.*
The first index is 1 *(the first element of* 1120*), and the element of the array indexed by this is* 2*. Thus, the permutation is* $2\_\_\_\_$*.*
The new array is 1345*, and the new sequence of indices is* 120*. Now, the element indexed by the first index* 1 *is* 3*. Thus the permutation is* $23\_\_\_$*.*
The new array is 145*, and the indices are* 20*. The element indexed by* 2 *is* 5*, so the permutation is* $235\_\_$*.*
The array is now 14*, and the only index remaining is* 0*. The corresponding element is* 1*, and the permutation is* $2351\_$*.*
The only remaining element 4 *is the last element of the permutation, so the complete permutation is* $\boxed{23514}$*.*

1. Find the 23rd permutation of the four marks $1, 2, 3, 4$ in lexicographical order.

$$23 - 1 = 22 = \underline{3} \times 3! + \underline{2} \times 2! + \underline{0} \times 1! \to 320$$

Index	Marks	Mark
$\underline{3}20$	$123\underline{4}$	$\to 4$
$\underline{2}0$	$12\underline{3}$	$\to 3$
$\underline{0}$	$\underline{1}2$	$\to 1$
	$\underline{2}$	$\to 2$

Thus, the 23rd permutation of $1, 2, 3, 4$ in lexicographical order is $\boxed{4312}$.

2. Find the 18^{th} permutation of the marks a, b, c, d in lexicographical order.

$$18 - 1 = 17 = \underline{2} \times 3! + \underline{2} \times 2! + \underline{1} \times 1! \to 221.$$

$\underline{2}21$	$ab\underline{c}d$	$\to c$
$\underline{2}1$	$ab\underline{d}$	$\to d$
$\underline{1}$	$a\underline{b}$	$\to b$
	$\underline{a}$	$\to a$

Thus, the 18^{th} permutation of the marks a, b, c, d in lexicographical order is $\boxed{cdba}$.

3. Find the 268$^{\text{th}}$permutation of `LISTEN` in lexicographical order.

$$268 - 1 = 267 = \underline{2} \times 5! + \underline{1} \times 4! + \underline{0} \times 3! + \underline{1} \times 2! + \underline{1} \times 1! \to 21011$$

$\underline{2}1011$	LI<u>S</u>TEN	$\to$ S
$\underline{1}011$	L<u>I</u>TEN	$\to$ I
$\underline{0}11$	<u>L</u>TEN	$\to$ L
$\underline{1}1$	T<u>E</u>N	$\to$ E
$\underline{1}$	T<u>N</u>	$\to$ N
	<u>T</u>	$\to$ T

Thus, the 268$^{\text{th}}$permutation of `LISTEN` in lexicographical order is $\boxed{\texttt{SILENT}}$.

Reverse Lexicographical Order

To obtain the k^{th}permutation of n marks $a_1, a_2, \ldots, a_n$ in reverse lexicographical order, first reverse the order of marks to get $a_n, a_{n-1}, \ldots, a_1$, compute the k^{th}permutation of these marks in *lexicographical order*, and then reverse the resulting permutation.

1. Find the 50^{th}permutation of the five marks $0, 1, 2, 3, 4$ in reverse lexicographical order.

$$50 - 1 = 49 = \underline{2} \times 4! + \underline{0} \times 3! + \underline{0} \times 3! + \underline{1} \times 1! \rightarrow 2001$$

$\underline{2}001$	$43\underline{2}10$	$\rightarrow 2$
$\underline{0}01$	$\underline{4}310$	$\rightarrow 4$
$\underline{0}1$	$\underline{3}10$	$\rightarrow 3$
$\underline{1}$	$1\underline{0}$	$\rightarrow 0$
	$\underline{1}$	$\rightarrow 1$

Thus, the 50^{th}permutation of $0, 1, 2, 3, 4$ in reverse lexicographical order is $\boxed{10342}$.

2. Find the 100^{th} permutation of the marks $1, 2, 3, 4, 5$ in reverse lexicographical order.

$$100 - 1 = 99 = \underline{4} \times 4! + \underline{0} \times 3! + \underline{2} \times 1! + \underline{1} \times 1! \rightarrow 4011$$

$\underline{4}011$	$5432\underline{1}$	$\rightarrow 1$
$\underline{0}11$	$\underline{5}432$	$\rightarrow 5$
$\underline{1}1$	$4\underline{3}2$	$\rightarrow 3$
$\underline{1}$	$4\underline{2}$	$\rightarrow 2$
	4	$\rightarrow 4$

Thus, the 100^{th} permutation of $1, 2, 3, 4, 5$ in reverse lexicographical order is $\boxed{42351}$.

Fike's order

To obtain the k^{th}permutation of n marks $a_1, a_2, \ldots, a_n$ in Fike's order, proceed as follows.
First, we must generate *Fike's sequence*, using which the permutation is to be computed. To find the sequence, first write $k-1$ in the form

$$k-1 = c_1 \times n(n-1)\cdots 3 + c_2 \times n(n-1)\cdots 4 + \cdots + c_{n-2} \times n + c_{n-1} \times 1.$$

That is, the place values are $\frac{n!}{2!}$, $\frac{n!}{3!}$, $\ldots$, $\frac{n!}{n!} = 1$. Now, **subtract this sequence** $c_1 c_2 \cdots c_{n-1}$ from the sequence $1\ 2 \cdots (n-1)$ to get the sequence $d_1 d_2 \cdots d_{n-1}$. That is, $d_i = i - c_i$, $i = 1, \ldots, n-1$. This is Fike's sequence.

Example 4.3

To compute the 65^{th}*permutation of the five marks* $1, 2, 3, 4, 5$ *in Fike's order, we note that* $k = 65$ *and* $n = 5$*. First, compute the place values* $\frac{n!}{2!}$*, ...,* $\frac{n!}{n!}$*. For* $n = 5$*, these are 60, 20, 5, 1. Then,*

$$65 - 1 = 64 = \underline{1} \times 60 + \underline{0} \times 20 + \underline{0} \times 5 + \underline{4} \times 1 \to 1004.$$

Now, Fike's sequence is

$$\begin{array}{l} 1234 - \\ 1004 = \\ \hline 0230. \end{array}$$

Using the Fike's sequence, the permutation is generated from the initial permutation $12\cdots n$ by a sequence of interchanges, in the following manner. For the sequence $d_1 d_2 \cdots d_{n-1}$, first the element of the permutation index 1 is interchanged with the element at index d_1. Similarly, at each stage, the element at index i is interchanged with the element at index d_i, until the sequence is exhausted. The resulting permutation is the k^{th}permutation in Fike's order.

For the sequence 0230 obtained in the previous example, we start with the original arrangement of the marks: 12345. Now, the element at index 1 is 2, and the element at index $d_1 = 0$ is 1. Therefore, interchanging 2 and 1, we get 21345. Next, the element at index 2 is 3, and the element at index $d_2 = 2$ is 3 (the same). "Interchanging" these, we get 21345 (i.e., the permutation remains the same). The element at index 3 is 4, and that at index $d_3 = 3$ is again the same, so once more, the permutation is 21345. Lastly, the element at index 4 is 5, and that at index $d_4 = 0$ is 2. Interchanging these, we get 51342. Thus, the 65$^{\text{th}}$permutation of $1, 2, 3, 4, 5$ in Fike's order is $\boxed{51342}$. We can write this succinctly as given below. First, Fike's sequence is written as a column. Then we write the original permutation in the first row, and underline the element at index 1, which is to be interchanged with the element at index d_1.

0	1$\underline{2}$345
2	
3	
0	

The interchange is performed and the result is written in the next row, and this process is repeated until the sequence is exhausted.

0	1$\underline{2}$345	$\rightarrow$
2	21$\underline{3}$45	$\rightarrow$
3	213$\underline{4}$5	$\rightarrow$
0	2134$\underline{5}$	$\rightarrow$
	$\boxed{51342}$	

Problems

1. Obtain the 40^{th}permutation of the five marks $0, 1, 2, 3, 4$ in Fike's order. Since $n = 5$, the place values are 60, 20, 5, 1.

$$40 - 1 = 39 = \underline{0} \times 60 + \underline{1} \times 20 + \underline{3} \times 5 + \underline{4} \times 1 \rightarrow 0134$$

Then Fike's sequence is

$$\begin{array}{l} 1234 - \\ 0134 = \\ \hline 1100. \end{array}$$

The permutation is then obtained as follows.

$$\begin{array}{ccc} 1 & 0\underline{1}234 & \rightarrow \\ 1 & 01\underline{2}34 & \rightarrow \\ 0 & 021\underline{3}4 & \rightarrow \\ 0 & 3210\underline{4} & \rightarrow \\ & \boxed{42103} & \end{array}$$

2. Obtain the 50$^{\text{th}}$permutation of the five marks $1, 2, 3, 4$ in Fike's order. Since $n = 5$, the place values are 60, 20, 5, 1.

$$50 - 1 = 49 = \underline{0} \times 60 + \underline{2} \times 20 + \underline{1} \times 5 + \underline{4} \times 1 \rightarrow 0214$$

Then Fike's sequence is

$$\begin{array}{l} 1234\ - \\ 0214\ = \\ \hline 1020. \end{array}$$

The permutation is then obtained as follows.

$$\begin{array}{lcl} 1 & 1\underline{2}345 & \rightarrow \\ 0 & 12\underline{3}45 & \rightarrow \\ 2 & 321\underline{4}5 & \rightarrow \\ 0 & 3241\underline{5} & \rightarrow \\ & \boxed{52413} & \end{array}$$

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THANK YOU