Boolean Lattice

A $\underbrace{\text{complemented}}_{\text{Every elt must have complement}}$ & $\underbrace{\text{distributive}}_{\text{Distributive laws satisfied}}$ lattice.

* Since a boolean lattice is distributive, every elt in boolean lattice has unique complement

* Let $(A, \leq)$ be a boolean lattice, since every elt 'a' has a unique complement, say $\bar{a}$, we have a unary operation defined on the elts of A, $(\bar{\ })$

Thus a boolean lattice defines an algebraic sys

$$(\underset{\text{relat}^n}{A, \leq}, \underset{\text{join}}{\vee}, \underset{\text{meet}}{\wedge}, \underset{\text{complement}}{\bar{\ }})$$

Boolean algebra : An algebraic sys defined by the boleean lattice. ie $(A, \leq, \vee, \wedge, \bar{\ })$

ex:- $(P(S), \subseteq)$ is a boolean lattice.

Boolean algebra : $(P(S), \subseteq, \cup, \cap, \bar{\ })$

$S = \{a, b, c\}$ $\quad P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{c,a\}, \{a,b,c\}\}$

$0 = \phi$ $\quad 1 = \{a,b,c\}$

complement of $\{a,b\} \equiv \{c\}$

" $\{b\} = \{a,c\}$

$\therefore$ complement of 'A' $\in P(S)$, $S \setminus A$

we say a is comp of b if $a \vee b = 1$, $a \wedge b = 0$

Demorgan's law

For any elts a, b in a boolean algebra $(A, \leq, \vee, \wedge, ^{-})$

i) $\overline{a \vee b} = \bar{a} \wedge \bar{b}$

ii) $\overline{a \wedge b} = \bar{a} \vee \bar{b}$

Proof:-

To prove (i) : we've to prove that complement of $(a \vee b)$ is $\bar{a} \wedge \bar{b}$

ie we've to prove $(a \vee b) \vee (\bar{a} \wedge \bar{b}) = 1$

$(a \wedge b) \wedge (\bar{a} \wedge \bar{b}) = 0$

consider $(a \vee b) \vee (\bar{a} \wedge \bar{b}) = [(a \vee b) \vee \bar{a}] \wedge [(a \vee b) \vee \bar{b}]$ (dist law)

$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

$= [\bar{a} \vee (a \vee b)] \wedge [a \vee (b \vee \bar{b})]$ (com law) (Assoi)

$= [(\bar{a} \vee a) \vee b] \wedge [a \vee 1]$ $(b \vee \bar{b} = 1)$

$= [1 \vee b] \wedge 1$ $(a \vee 1 = 1)$

$= 1 \wedge 1$

$= \underline{\underline{1}}$

$\therefore (a \vee b) \vee (\bar{a} \wedge \bar{b}) = 1$

Now we prove $(a \vee b) \wedge (\bar{a} \wedge \bar{b}) = 0$

consider $(a \vee b) \wedge (\bar{a} \wedge \bar{b}) = (\bar{a} \wedge \bar{b}) \wedge (a \vee b)$

$$= [(\bar{a} \wedge \bar{b}) \wedge a] \vee [(\bar{a} \wedge \bar{b}) \wedge b] \quad \text{(dist)}$$

(comm)

$$= [a \wedge (\bar{a} \wedge \bar{b})] \vee [a \wedge (\bar{b} \wedge b)]$$

(associativity)

$$= [(a \wedge \bar{a}) \wedge \bar{b}] \vee [a \wedge 0]$$

$(\because b \wedge \bar{b} = 0)$

$$= (0 \wedge \bar{b}) \vee (a \wedge 0)$$

$$= 0 \vee 0$$

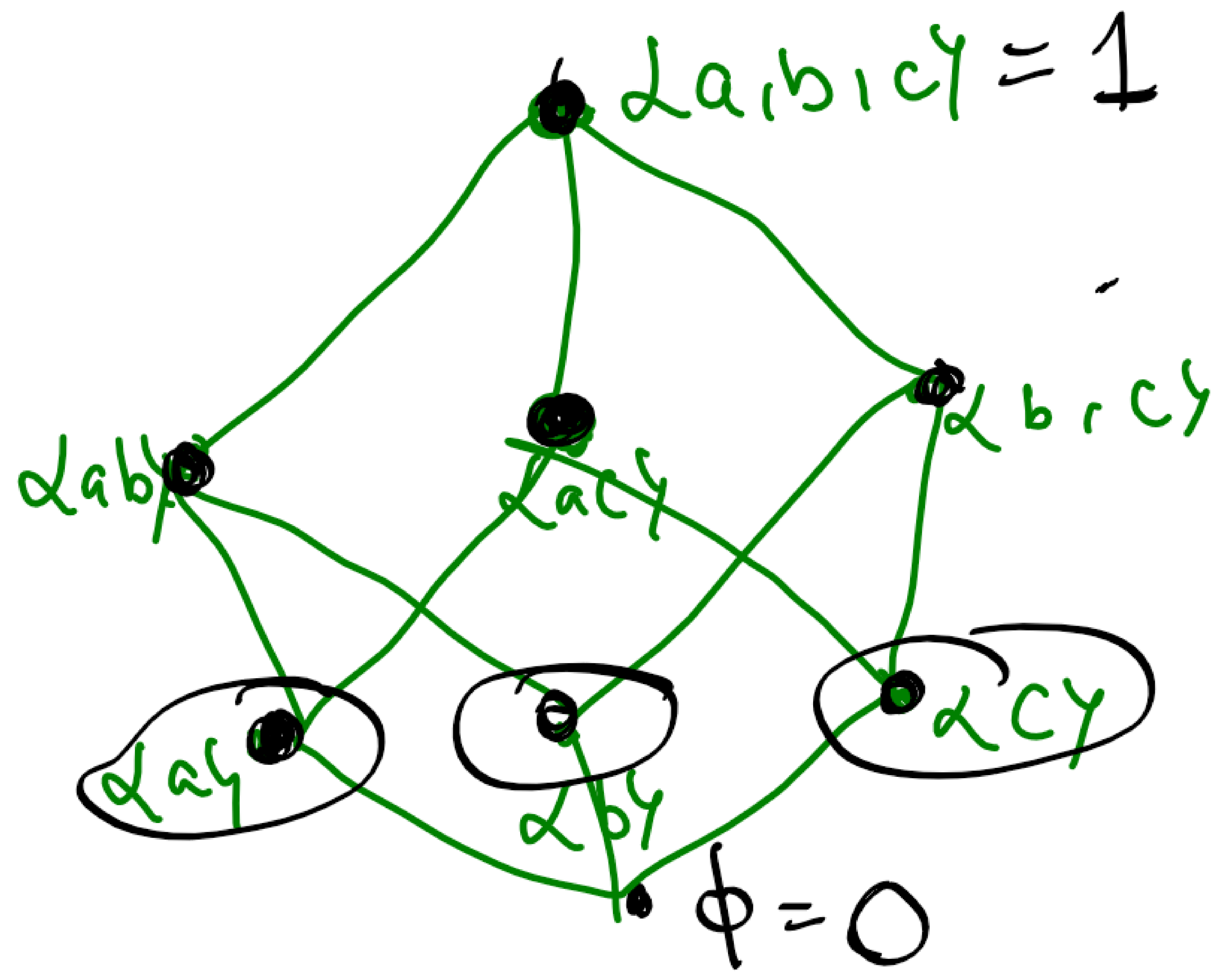
$$= \underline{0}$$

$$\therefore \overline{(a \vee b)} = \bar{a} \wedge \bar{b}$$

* atom : Let $(A, \leq)$ be a boolean lattice with ulb '0' and uub 1. An element is called an atom if it covers '0'

In the case of $(P(S), \subseteq)$, the atoms are the singleton sets :

$S = \{a, b, c\}$ $\qquad P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{ab\}, \{bc\}, \{ca\}, \{a, b, c\}\}$



Note: Let $(A, \leq)$ be a finite lattice with universal lower bound 0. Then for any nonzero element 'b', there exits atleast one atom 'a' s.t $a \leq b$

Lemma 1

In a distributive lattice, if $b \wedge \bar{c} = 0$, then $b \leq c$

Proof

Let $b \wedge \bar{c} = 0$, then we have to prove that $b \leq c$

w.k.t $\quad 0 \vee c = c$

$(b \wedge \bar{c}) \vee c = c \quad$ ($\because$ given $b \wedge \bar{c} = 0$)

$c \vee (b \wedge \bar{c}) = c \quad$ (commutative)

$(c \vee b) \wedge (c \vee \bar{c}) = c \quad$ (Distri law)

$(c \vee b) \wedge 1 = c \quad$ ($\because c \vee \bar{c} = 1$)

$(c \vee b) = c \quad$ ($1 \wedge c = c$)

$b \vee c = c$

$\therefore b \leq b \vee c$

$$\boxed{b \leq c}$$

Lemma 2:

Let $(A, \vee, \wedge, -)$ be a boolean algebra. Let 'b' be any nonzero element in A, and $a_1, a_2 \ldots a_k$ be all the atoms of b s.t $a_i \leq b$. Then $b = a_1 \vee a_2 \vee a_3 \vee \ldots \vee a_k$

$\alpha\{a, b\} = \alpha\{a\} \vee \alpha\{b\}$. $\qquad \alpha\{c, b\} = \alpha\{c\} \vee \{b\}$

Proof:- We've P.T $b = \underbrace{a_1 \vee a_2 \vee \dots \vee a_k}_{c}$. i.e. P.T $b = c$

Since $a_1, a_2 \dots a_k$ are all atoms s.t $a_i \leq b$ (for $i = 1, 2, \dots k$)

ie $a_1 \leq b$ $a_2 \leq b$ $\dots$ $a_k \leq b$

$$a_1 \vee a_2 \vee \dots \vee a_k \leq b$$

$$c \leq b \text{ ——— (1)}$$

Next, P.T $b \leq c$. "Lemma 1: If $b \wedge \bar{c} = 0 \Rightarrow b \leq c$"

To prove that $b \leq c$, it enough if we prove $b \wedge \bar{c} = 0$

$\therefore$ To prove $b \wedge \bar{c} = 0$

Suppose $b \wedge \bar{c} \neq 0$, then there exists atleast one atom, say 'a' s.t $a \leq (b \wedge \bar{c})$

Now $a \leq b \wedge \bar{c}$ and $b \wedge \bar{c} \leq b \Rightarrow a \leq b$ ——— (2)

$a \leq b \wedge \bar{c}$ and $b \wedge \bar{c} = \bar{c} \Rightarrow a \leq \bar{c}$ ——— (3)

By (2), $a \leq b$ and a is an atom, implies a is one among $a_1, a_2 \dots a_k$

$$a \leq a_1 \vee a_2 \vee \dots \vee a_k$$

$$a \leq c \text{ ——— (4)}$$

Now, from (3) and (4)

$$a \wedge a \leq c \wedge \bar{c}$$

$$a \leq 0, \text{ impossible}$$

Our assumption is wrong. $\therefore$ $b \wedge \bar{c} = 0$

$\therefore$ By lemma 1, $b \wedge \bar{c} = 0 \Rightarrow b \leq c$ ——— (5)

$\therefore$ From (1) & (5) $\Rightarrow$ $\boxed{b = c}$

Lemma 3 : Let $(A, \vee, \wedge, ^-)$ be a boolean algebra. Let b be a nonzero element in A. and $a_1, a_2 \ldots a_k$ be the atoms s.t $a_i \leq b$ $(i=1,2\ldots k)$. Then

$b = a_1 \vee a_2 \vee \ldots \vee a_k$ is the unique way to represent b as join of atoms

$\langle b, c \rangle = \langle b \rangle \cup \langle c \rangle$
$\langle a, b \rangle = \langle a \rangle \cup \langle b \rangle$ } unique

Proof :- Let $b = a_1 \vee a_2 \vee \ldots \vee a_k$ —(1)

Let $b = a_1' \vee a_2' \vee \ldots \vee a_t'$ (2) be an alternate rep of 'b' as join of atoms $a_1', a_2', \ldots a_t'$

(We prove that, $\forall$ a_i' in (2), there exists one a_i in (1)
$\forall$ a_i in (1), there exists one a_j' in (2)

Since b is the lub of $a_1' a_2' a_3' \ldots a_t'$, its true that

$a_1' \leq b \quad a_2' \leq b \quad \ldots \quad a_t' \leq b$

consider an arbitrary elt a_i' in (2) $(1 \leq i \leq t)$

Since $a_i' \leq b$ and a_i' is an atom, obviously

$a_i' \wedge b = a_i'$

$a_i' \wedge (a_1 \vee a_2 \vee \ldots a_k) = a_i'$

$(a_i' \wedge a_1) \vee (a_i' \wedge a_2) \vee \ldots \vee (a_i' \wedge a_k) = a_i'$

→ nonzero means, there is atleast one nonzero term

For some j, $a_i' \wedge a_j \neq 0$ $(1 \leq j \leq k)$

But since both a_i' & a_j are atoms, $a_i' = a_j$

$\therefore$ a_i' is equal to some a_j

Thus for each atom in the alternate rep, there is one atom in the original rep

similarly, one can prove that some arbitary a_j is equal to some a_i' $(1 \le i \le t)$

$\therefore$ The representation $b = a_1 \vee a_2 \cdot \vee a_k$ is unique

Theorem

Let $(A, \vee, \wedge, -)$ finite boolean algebra. Let S be the set of all atoms. Then $(A, \vee, \wedge, -)$ is isomorphic to the algebraic sys defined by $(P(S), \subseteq)$

Imp

Any finite boolean algebra has 2^n elts for some $n > 0$

$n \to$ the no of atoms

* If a boolean lattice has n atoms, then the corresp boolean algebra has 2^n elts

Boolean functions

Let $(A, \vee, \wedge, -)$ be a boolean algebra. A boolean expression over $(A, \vee, \wedge, -)$ is defined as follows

i) Any elt of A is a boolean expression

ii) Any variable name is "

iii) If e_1 & e_2 are boolean expressions, then $e_1 \vee e_2$, $e_1 \wedge e_2$, $\bar{e_1}$, $\bar{e_2}$ are also boolean expressions

$$E(x_1, x_2, x_3) = x_1 \vee (x_2 \wedge x_3) \vee (x_3 \wedge x_1)$$

$E(1,0,1) \Rightarrow \{x_1 = 1, x_2 = 0, x_3 = 1$ Assignment of vals"

$E(x_1, x_2 \cdots x_n) \rightarrow$ Boolean expresn of n variables

Equivalent boolean expression

Two boolean expressions $E_1(x_1 \cdot x_2 \cdots x_n)$ & $E_2(x_1, x_2 \cdots x_n)$ are equivalent if they assume same value for every assignments of values.

$$E_1(x_1 \cdots x_n) = E_2(x_1\ x_2 \cdots x_n)$$