Boolean ALGEBRA

Department of Mathematics
Manipal Institute of Technology
Manipal Academy of Higher Education Manipal
Manipal

Boolean lattice: A complemented and distributive lattice is called a Boolean lattice.

Example 1.1.

$(P(S), \subseteq)$ is a Boolean lattice.

Let $(A, \leq)$ be a Boolean lattice. Since every element a has a unique complement $\bar{a}$, we have another unary operation known as complementation and denoted by $^{-}$. Thus we can say that the lattice $(A, \leq)$ defines an algebraic system $(A, \leq, \vee, \wedge)$ where $\vee$ and $\wedge$ are the join and meet operations respectively. The algebraic system defined by a Boolean lattice is known as **Boolean algebra.**

Theorem 1.2.

DeMorgan's laws: For any elements a, b in a Boolean algebra $(A, \leq, \vee, \wedge)$,

- $\overline{a \vee b} = \bar{a} \wedge \bar{b}$
- $\overline{a \wedge b} = \bar{a} \vee \bar{b}$

Proof.

We have to prove that $(a \vee b) \vee (\bar{a} \wedge \bar{b}) = 1$ and $(a \vee b) \wedge (\bar{a} \wedge \bar{b}) = 0$.

$$
\begin{aligned}
\text{Consider } (a \vee b) \vee (\bar{a} \wedge \bar{b}) &= [(a \vee b) \vee \bar{a}] \wedge [(a \vee b) \vee \bar{b}] \text{ (distributive law)} \\
&= [\bar{a} \vee (a \vee b)] \wedge [a \vee (b \vee \bar{b})] \text{ (associative law)} \\
&= [(\bar{a} \vee a) \vee b] \wedge [a \vee 1] \text{ (associative law)} \\
&= [1 \vee b] \wedge [a \vee 1] \\
&= 1 \wedge 1 = 1 \\
\text{Similarly, } (a \vee b) \wedge (\bar{a} \wedge \bar{b}) &= (\bar{a} \wedge \bar{b}) \wedge (a \vee b) \text{ (commutative law)} \\
&= [(\bar{a} \wedge \bar{b}) \wedge a] \vee [(\bar{a} \wedge \bar{b}) \wedge b] \text{ (distributive law)} \\
&= [a \wedge (\bar{a} \wedge \bar{b})] \vee [(\bar{a} \vee \bar{b}) \wedge b] \text{ (commutative law)} \\
&= [(a \wedge \bar{a}) \wedge \bar{b}] \vee [\bar{a} \vee (\bar{b} \wedge b)] \text{ (associative law)} \\
&= [0 \wedge \bar{b}] \vee [(\bar{a} \wedge 0] \\
&= 0 \vee 0 = 0
\end{aligned}
$$

The second part follows from principle of duality. □

Uniqueness of finite Boolean algebras:

Definition 1.3.

An element of a Boolean algebra is called an atom if it covers 0.

Note: Finite Boolean algebra has 2^n elements for some n.

Lemma 1.4.

In a distributive lattice, if $b \wedge \bar{c} = 0$, then $b \leq c$.

Proof.

We know that $0 \vee c = c$

$(b \wedge \bar{c}) \vee c = c$ (given)

$c \vee (b \wedge \bar{c}) = c$ (commutative law)

$(c \vee b) \wedge (c \vee \bar{c}) = c$ (distributive law)

$(c \vee b) \wedge 1 = c$

$(c \vee b) = c$

$(b \vee c) = c$

Thus $b \leq b \vee c \implies b \leq c$. $\square$

Lemma 1.5.

Let $(A, \leq)$ be a finite Boolean lattice with universal lower bound 0. Then for any element b (which is not universal lower bound 0), there exists at least one atom $'a`$ such that $a \leq b$.

Proof.

If b is an atom, then there is nothing to prove as $b \leq b$.

Suppose b is not an atom, since $(A, \leq)$ is a finite lattice, there must be chain in $(A, \leq)$ such that $(0, b_i, \ldots, b_2, b_1, b)$ where b_i is an atom. □

Lemma 1.6.

Let $(A, \leq, \vee, \wedge, ^{-})$ *be a finite Boolean algebra. Let* b *be any nonzero element in* A *and* $a_1, a_2, \ldots, a_k$ *be all the atoms of* A *such that* $a_i \leq b$. *Then* $b = a_1 \vee a_2 \vee \ldots a_k$.

Proof.

For notational convenience, let $c = a_1 \vee a_2 \vee \ldots a_k$.
We should Prove $b = c$.
First let us prove $b \leq c$, then $c \leq b$.
Since $a_1 \leq b, a_2 \leq b, \ldots, a_k \leq b$, it follows that $a_1 \vee a_2 \vee \ldots a_k \leq b$
$\implies c \leq b$ $-----------------------(1)$
Suppose $b \wedge \bar{c} \neq 0$, then there exists an atom a_i such that $a_i \leq b \wedge \bar{c}$.
$a_i \leq b \wedge \bar{c}$ and $b \wedge \bar{c} \leq \bar{c} \implies a_i \leq \bar{c}$ $--------(2)$
As $a_1 \vee a_2 \vee \ldots a_k = c$, we have $a_i \leq c$ $----------(3)$.
From (2) and (3), we get $a_i \leq c \wedge \bar{c}$.
$\implies a_i \leq 0$.
which is contradiction to definition of an atom. □

Proof continues....

Thus $b \wedge \bar{c} = 0 \implies b \leq c$.
That is $b \leq a_1 \vee a_2 \vee \ldots a_k - - - - - - - - - - - - - - - - - -(3)$
Form (1) and (3) and antisymmetric property, $a_1 \vee a_2 \vee \ldots a_k = b$.

Lemma 1.7.

Let $(A, \leq, \vee, \wedge, ^{-})$ be a finite Boolean algebra. Let b be any nonzero element in A and $a_1, a_2, \ldots, a_k$ be all the atoms of A such that $a_i \leq b$. Then $b = a_1 \vee a_2 \vee \ldots a_k$ is the unique way to represent b as a join of atoms.

Proof.

Suppose that we have alternative representation $b = a_{j1} \vee a_{j2} \vee \ldots a_{jt}$.
Since b is the lub of $a_{j1}, a_{j2}, \ldots a_{jt}$, it is true that
$a_{j1} \leq b, a_{j2} \leq b, \ldots, a_{jt} \leq b$.
Consider an atom a_{ju} $(1 \leq u \leq t)$.
Since $a_{ju} \leq b$, we have $a_{ju} \wedge b = a_{ju}$
$a_{ju} \wedge (a_1 \vee a_2 \vee \ldots a_k) = a_{ju}$
$(a_{ju} \wedge a_1) \vee (a_{ju} \wedge a_2) \vee \ldots (a_{ju} \wedge a_k) = a_{ju}$
Then for some a_i $(1 \leq i \leq k)$, $a_{ju} \wedge a_i \neq 0$.
Since a_{ju} and a_i are atoms, we must have $a_{ju} = a_i$.
Thus each atom in the alternative representation is an atom in the original one, and the lemma follows. $\square$

From the above lemmas, it is clear that there is one to one correspondence between the elements of a Boolean lattice and subset of atoms. As a matter of fact, there is one to one correspondence from $(A, \leq)$ to $(P(S), \subseteq)$, where S is the set of all atoms.

Theorem 1.8.

Let $(A, \vee, \wedge, ^-)$ be a finite Boolean algebra. Let S be the set of all atoms. Then $(A, \vee, \wedge, ^-)$ is isomorphic to the algebraic system defined by the lattice $(P(S), \subseteq)$.

It follows from the above lemmas that **there exists a finite Boolean algebra of 2^n elements for any $n > 0$.**

Example 1.9.

Let P be the set of all positive factors of 60 and let '|' denote the 'divides' relation. Then poset $(P, |)$ a Boolean Lattice?

Solution: Positive factors of 60 are

$\{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$

Atoms: 2,3,5

Number of elements in the lattice=12

Not a boolean lattice.

Boolean expression: Let Let $(A, \vee, \wedge, ^{-})$ be a finite Boolean algebra. A Boolean expression over $(A, \vee, \wedge, ^{-})$ is defined as follows:

- An element of A is a Boolean expression.
- Any variable name is a Boolean expression.
- If e_1 and e_2 are Boolean expressions, then $\overline{e_1}, e_1 \vee e_2$ and $e_1 \wedge e_2$ are Boolean expressions.

Example 2.1.

$0 \vee x$, $(x_1 \vee x_2) \wedge \overline{(2 \vee 3)}$

Let $E(x_1, x_2, \ldots, x_n)$ be a Boolean expression of n variables over a Boolean algebra $(A, \vee, \wedge, ^{-})$. By assignment of values to the variables $x_1, x_2, \ldots, x_n$, we mean an assignment of elements of A to be the values of the variables. For an assignment of values to the variables, we can evaluate $E(x_1, x_2, \ldots, x_n)$ by substituting the variables in the expression by their values.

Two Boolean expressions of n variables are said to be equivalent, if they assume same values for every assignment of values to the n variables. If $E_1(x_1, x_2, \dots, x_n)$ and $E_2(x_1, x_2, \dots, x_n)$ are equivalent, then we write $E_1(x_1, x_2, \dots, x_n) = E_2(x_1, x_2, \dots, x_n)$.

Example 2.2.

$(x_1 \wedge x_2) \vee (x_1 \wedge \bar{x_3})$ is equivalent to $x_1 \wedge (x_2 \vee \bar{x_3})$.

Boolean function: A function $f : A^n \to A$ is said to be a Boolean function if it can be specified by a Boolean expression of n variables.

Minterm: A booleam expression of n variables $x_1, x_2, \ldots, x_n$ is said to be a minterm if it is of the form $\tilde{x_1} \wedge \tilde{x_2} \wedge \cdots \wedge \tilde{x_n}$ where $\tilde{x_i}$ is either x_i or $\bar{x_i}$.
Disjunctive normal form (DNF): A Boolean expression over $(\{0,1\}, \wedge, \vee, ^{-})$ is said to be in disjunctive normal form if it is join of minterms.

Example 2.3.

$(x_1 \wedge \bar{x_2} \wedge x_3) \vee (\bar{x_1} \wedge \bar{x_2} \wedge x_3) \vee (x_1 \wedge x_2 \wedge x_3)$

Maxterm: A booleam expression of n variables $x_1, x_2, \ldots, x_n$ is said to be a maxterm if it is of the form $\tilde{x_1} \vee \tilde{x_2} \vee \cdots \vee \tilde{x_n}$ where $\tilde{x_i}$ is either x_i or $\bar{x_i}$.
Conjunctive normal form (CNF): A Boolean expression over $(\{0,1\}, \wedge, \vee, ^{-})$ is said to be in conjunctive normal form if it is meet of maxterms.

Example 2.4.

$(x_1 \vee \bar{x_2} \vee x_3) \wedge (\bar{x_1} \vee \bar{x_2} \vee x_3)$

DNF: Given a function $\{0,1\}^n \to \{0,1\}$, we can obtain a Boolean expression in DNF corresponding to this function by having a minterm corresponding to each ordered n tuple of $0's$ and $1's$ for which the value of the function is 1. For each n tuple with the functional value is 1, we have the minterm $\tilde{x_1} \vee \tilde{x_2} \vee \cdots \vee \tilde{x_n}$ where $\tilde{x_i} = \begin{cases} x_i & if\ \ i^{th}\ componet\ is\ 1 \\ \bar{x}_i & if\ \ i^{th}\ componet\ is\ 0 \end{cases}$.

CNF: We can obtain a Boolean expression in CNF corresponding to this function by having a maxterm corresponding to each ordered n tuple of $0's$ and $1's$ for which the value of the function is 1. For each n tuple with the functional value is 0, we have the minterm $\tilde{x_1} \vee \tilde{x_2} \vee \cdots \vee \tilde{x_n}$ where

$$\tilde{x_i} = \begin{cases} x_i & if\ \ i^{th}\ componet\ is\ 0 \\ \bar{x}_i & if\ \ i^{th}\ componet\ is\ 1 \end{cases}$$

Q1. Let $E(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee (x_1 \wedge x_3) \vee (\bar{x_2} \wedge x_3)$ be a Boolean expression over $(\{0,1\}, \wedge, \vee, ^-)$. Write the Boolean expression in both DNF and CNF.

Sol:

x_1	x_2	x_3	$(x_1 \wedge x_2)$	$(x_1 \wedge x_3)$	$(\bar{x_2} \wedge x_3)$	f
0	0	0	0	0	0	0
0	0	1	0	0	1	1
0	1	0	0	0	0	0
0	1	1	0	0	0	0
1	0	0	0	0	0	0
1	0	1	0	1	1	1
1	1	0	1	0	0	1
1	1	1	1	1	0	1

DNF : $(\bar{x_1} \wedge \bar{x_2} \wedge x_3) \vee (x_1 \wedge \bar{x_2} \wedge x_3) \vee (x_1 \wedge x_2 \wedge \bar{x_3}) \vee (x_1 \wedge x_2 \wedge x_3)$

CNF :$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x_2} \vee x_3) \wedge (x_1 \vee \bar{x_2} \vee \bar{x_3}) \wedge (\bar{x_1} \vee x_2 \vee x_3)$ □

References

[1] Liu C L, Elements of discrete mathematics,2^{nd} *edition*, *McGraw Hill Book Company, New Dehli*, (2007).

THANK YOU