* A Boolean lattice has $2^n$ elts, $n \to$ no of atoms

## Boolean functions

A function $f: A^n \longrightarrow A$ said to be a function if it can be specified by a boolean expression of $n$ variables.

Ex :- $f: A^3 \longrightarrow A$ where $f(x_1, x_2, x_3) = (x_1 \vee x_2 \wedge x_3)$

$$f(1,0,0) = 1 \vee 0 \wedge 0 = 1$$

Ex :- $f = (x_1 \wedge \bar{x}_1) \vee x_2$ $f: A^2 \longrightarrow A$ where $A = \{0,1\}$

This is a boolean fn over the boolean algebra $(\{0,1\}, \vee, \wedge, -)$

| $x_1$ | $x_2$ | $(x_1 \wedge \bar{x}_1)$ | $(x_1 \wedge \bar{x}_1) \vee x_2 = f$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 |

$\bar{1} = 0$ $\quad 0 \vee 0 = 0$ $\quad 0 \wedge 0 = 0$

$\bar{0} = 1$ $\quad 0 \vee 1 = 1$ $\quad 0 \wedge 1 = 0$

$1 \vee 0 = 1$ $\quad 1 \wedge 0 = 0$

$1 \vee 1 = 1$ $\quad 1 \wedge 1 = 1$

(2) Let $f : A^3 \to A$ where $A = \{0, 1\}$

$$f(x_1\, x_2\, x_3) = (\bar{x}_1 \wedge x_2 \wedge \bar{x}_3) \vee (x_1 \wedge \bar{x}_2) \vee (x_1 \wedge x_3)$$

| $x_1$ | $x_2$ | $x_3$ | $(\bar{x}_1 \wedge x_2 \wedge \bar{x}_3)$ | $(x_1 \wedge \bar{x}_2)$ | $(x_1 \wedge x_3)$ | $f'$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 | 0 | 0 | 1 |
| 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 1 | 0 | 1 |
| 1 | 0 | 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 | 1 | 1 |

## Minterm :

A boolean expression of n variables $x_1\, x_2 \ldots x_n$ is said to be a minterm if its of the form

$$\tilde{x}_1 \wedge \tilde{x}_2 \wedge \tilde{x}_3 \wedge \ldots \wedge \tilde{x}_n \quad \text{where } \tilde{x}_i = x_i \;\&\; \bar{x}_i$$

## Disjunctive Normal form (DNF)

A boolean exp over $(\{0,1\}, \wedge, \vee, -)$ is said to be in DNF if its join of minterms ex:- $\underbrace{(x_1 \wedge x_2 \wedge x_3)}_{\text{minterm}} \vee \underbrace{(x_1 \wedge \bar{x}_2 \wedge \bar{x}_3)}_{\text{minterm}} \vee \underbrace{(\bar{x}_1 \wedge \bar{x}_2 \wedge \bar{x}_3)}_{\text{minterm}}$

DNF: Join of minterms

## maxterm

$\tilde{x}_1 \vee \tilde{x}_2 \vee \cdots \vee \tilde{x}_n$ where $\tilde{x}_i = x_i$ & $\overline{x_i}$

## Conjunctive Normal form (CNF)

Meet of Maxterms

ex:- $(x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$

## How to get a boolean expresn in DNF

Given a fn $\{0,1\}^n \to \{0,1\}$, we can obtain the boolean exp in DNF corresp to this fn by having a minterm corresp to each ordered n tuple of 0's & 1's for which final value is 1.

For each n tuple with functional value 1, we write the corresp minterm $\tilde{x}_1 \wedge \tilde{x}_2 \wedge \cdots \wedge \tilde{x}_n$

where $\tilde{x}_i = \begin{cases} x_i & \text{if } i\text{th component is } 1 \\ \overline{x_i} & \text{if } i\text{th component is } 0 \end{cases}$

| | f |
|---|---|
| 1 0 1 | 1 |

$x_1 \wedge \overline{x_2} \wedge x_3$

## How to get CNF (meet of maxterms)

Pick the ordered n tuple for which the functional value is 0.

For each n tuple with functional value zero, we write a maxterm $\tilde{x}_1 \vee \tilde{x}_2 \vee \cdots \vee \tilde{x}_n$

where $\tilde{x}_i = \begin{cases} x_i & i^{th} \text{ comp is } 0 \\ \overline{x_i} & \text{"} \quad \text{"} \quad 1 \end{cases}$

(DNF : join of minterms
CNF : meet of maxterms)

(1) Let $E(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee (x_1 \wedge x_3) \vee (\bar{x}_2 \wedge x_3)$ be a boolean expn in a 2 valued boolean algebra. Write the given boolean exp in DNF & CNF

Soln

| $x_1$ | $x_2$ | $x_3$ | $(x_1 \wedge x_2)$ | $(x_1 \wedge x_3)$ | $(\bar{x}_2 \wedge x_3)$ | $f$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 | | 0 | 0 |
| 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 | 0 | 1 |

$$\left.\begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{matrix}\right\} \rightarrow 1$$

DNF : join of minterms

write the minterms

$1 \rightarrow x_i$

$0 \rightarrow \bar{x}_i$

**DNF :**

$$(\bar{x}_1 \wedge \bar{x}_2 \wedge x_3) \vee (x_1 \wedge \bar{x}_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge \bar{x}_3) \vee (x_1 \wedge x_2 \wedge x_3)$$

**CNF** (meet of maxterms)

$$\left.\begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{matrix}\right\} \rightarrow 0$$

$0 \longrightarrow x_i$

$1 \rightarrow \bar{x}_i$

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_2 \vee x_3)$$

② $E(x_1, x_2, x_3) = \overline{(\overline{x_1 \vee x_2}) \vee (\overline{x}_1 \vee x_3)}$

Soln

| $x_1$ | $x_2$ | $x_3$ | $(\overline{x_1 \vee x_2})$ | $(\overline{x}_1 \vee x_3)$ | $(\overline{x_1 \vee x_2}) \vee (\overline{x}_1 \vee x_3)$ | $f$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 0 |
| 0 | 0 | 1 | 1 | 1 | 1 | 0 |
| 0 | 1 | 0 | 0 | 1 | 1 | 0 |
| 0 | 1 | 1 | 0 | 1 | 1 | 0 |
| ✓ 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 | 1 | 1 | 0 |
| ✓ 1 | 1 | 0 | 0 | 0 | 0 | 1 |
| 1 | 1 | 1 | 0 | 1 | 1 | 0 |

DNF : $(x_1 \wedge \overline{x}_2 \wedge \overline{x}_3) \vee (x_1 \wedge x_2 \wedge \overline{x}_3)$

CNF : $(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee \overline{x}_3) \wedge (x_1 \vee \overline{x}_2 \vee x_3)$

$\wedge (x_1 \vee \overline{x}_2 \vee \overline{x}_3) \wedge (\overline{x}_1 \vee x_2 \vee \overline{x}_3) \wedge$

$(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3)$

## POSET :

(Partially ordered set)

Relation satisfying ① ref
② antisy
③ Transitive

ex:- $(P(S), \subseteq)$

$(Z^+, |)$

$(N, \leq)$

$(S_n, |)$ $S_n \rightarrow$ +ve divisors of n

**Chain** : Set where every 2 elements are comparable (Totally ordered set)

① $(Z^+, |)$ is not a totally ordered set

Take 2, 3 $\Rightarrow$ Neither $2|3$, nor $3|2$

$\therefore$ 2 & 3 are not comparable

② $(N^+, \leq)$ is a totally ordered set

**Antichain** : No 2 elts are comparable.

**Lattice** : Poset where every 2 elts have unique lub & unique glb

①* $(P(S), \subseteq) \longrightarrow$ $\vee \rightarrow \cup$ $\wedge \rightarrow \cap$

② $(Z^+, |) \longrightarrow$ $\vee \rightarrow$ lcm $\wedge \rightarrow$ gcd

③ $(N, \leq) \longrightarrow$ $\vee \rightarrow$ max $\wedge \rightarrow$ min

Properties of lattice : ① commutative

② Associative

③ Absorbtion $\rightarrow$ $a \vee (a \wedge b) = a$

$a \wedge (a \vee b) = a$

④ Idempotent

$\hookrightarrow$ $a \vee a = a$

$a \wedge a = a$

Distributive lattice :

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

ex:- $(P(S), \subseteq)$

$(Z^{+}, |)$

## Complemented lattice

* Every elt of the lattice has a complement

complement of an elt → $a \vee b = 1$

$a \wedge b = 0$

a is comp of b

* An elt may have more than 1 complement
* In a distributive lattice, complement of an elt is unique

## Boolean Lattice

Distribute + complemented

Boolean algebra $(L, \leq, \vee, \wedge, \bar{\ })$

* Has $2^n$ elts, $n \to$ no of atoms
* 

$$\overline{(a \vee b)} = \bar{a} \wedge \bar{b}$$

$$\overline{(a \wedge b)} = \bar{a} \vee \bar{b}$$