EXERCISES 1-4

1 Show the validity of the following arguments, for which the premises are given on the left and the conclusion on the right.

(*a*)	$\neg(P \wedge \neg Q),\ \neg Q \vee R,\ \neg R$	$\neg P$
(*b*)	$(A \rightarrow B) \wedge (A \rightarrow C),\ \neg(B \wedge C),\ D \vee A$	D
(*c*)	$\neg J \rightarrow (M \vee N),\ (H \vee G) \rightarrow \neg J,\ H \vee G$	$M \vee N$
(*d*)	$P \rightarrow Q,\ (\neg Q \vee R) \wedge \neg R,\ \neg(\neg P \wedge S)$	$\neg S$
(*e*)	$(P \wedge Q) \rightarrow R,\ \neg R \vee S,\ \neg S$	$\neg P \vee \neg Q$
(*f*)	$P \rightarrow Q,\ Q \rightarrow \neg R,\ R,\ P \vee (J \wedge S)$	$J \wedge S$
(*g*)	$B \wedge C,\ (B \rightleftarrows C) \rightarrow (H \vee G)$	$G \vee H$
(*h*)	$(P \rightarrow Q) \rightarrow R,\ P \wedge S,\ Q \wedge T$	R

2 Derive the following, using rule **CP** if necessary.

(*a*) $\neg P \vee Q,\ \neg Q \vee R,\ R \rightarrow S \Rightarrow P \rightarrow S$

(*b*) $P,\ P \rightarrow (Q \rightarrow (R \wedge S)) \Rightarrow Q \rightarrow S$

(*c*) $P \rightarrow Q \Rightarrow P \rightarrow (P \wedge Q)$

(*d*) $(P \vee Q) \rightarrow R \Rightarrow (P \wedge Q) \rightarrow R$

(*e*) $P \rightarrow (Q \rightarrow R),\ Q \rightarrow (R \rightarrow S) \Rightarrow P \rightarrow (Q \rightarrow S)$

3 Prove $\neg P \wedge (P \vee Q) \Rightarrow Q$, using E_{12}, E_6, E_3, and I_2 only.

4 Show that the following sets of premises are inconsistent.

(*a*) $P \rightarrow Q,\ P \rightarrow R,\ Q \rightarrow \neg R,\ P$

(*b*) $A \rightarrow (B \rightarrow C),\ D \rightarrow (B \wedge \neg C),\ A \wedge D$

Hence show that $P \rightarrow Q,\ P \rightarrow R,\ Q \rightarrow \neg R,\ P \Rightarrow M$, and

$$A \rightarrow (B \rightarrow C),\ D \rightarrow (B \wedge \neg C),\ A \wedge D \Rightarrow P$$

5 Show the following (use indirect method if needed).

(*a*) $(R \rightarrow \neg Q),\ R \vee S,\ S \rightarrow \neg Q,\ P \rightarrow Q \Rightarrow \neg P$

(*b*) $S \rightarrow \neg Q,\ S \vee R,\ \neg R,\ \neg R \rightleftarrows Q \Rightarrow \neg P$

(*c*) $\neg(P \rightarrow Q) \rightarrow \neg(R \vee S),\ ((Q \rightarrow P) \vee \neg R),\ R \Rightarrow P \rightleftarrows Q$

6 Show the following, using the system given in Sec. 1-4.4.

(*a*) $P \Rightarrow (\neg P \rightarrow Q)$

(*b*) $P \wedge \neg P \wedge Q \Rightarrow R$

(*c*) $R \Rightarrow P \vee \neg P \vee Q$

(*d*) $P,\ \neg P \vee (P \wedge Q) \Rightarrow Q$

(*e*) $\neg(P \wedge Q) \Rightarrow \neg P \vee \neg Q$

1-5 THE PREDICATE CALCULUS

So far our discussion of symbolic logic has been limited to the consideration of statements and statement formulas. The inference theory was also restricted in the sense that the premises and conclusions were all statements. The symbols $P, Q, R, \ldots, P_1, Q_1, \ldots$ were used for statements or statement variables. The statements were taken as basic units of statement calculus, and no analysis of any atomic statement was admitted. Only compound formulas were analyzed, and this analysis was done by studying the forms of the compound formulas, i.e., the connections between the constituent atomic statements. It was not possible to express the fact that any two atomic statements have some features in common. In order to investigate questions of this nature, we introduce the concept of a predicate in an atomic statement. The logic based upon the analysis of predicates in any statement is called *predicate logic.*

1-5.1 Predicates

Let us first consider the two statements

John is a bachelor.

Smith is a bachelor.

Obviously, if we express these statements by symbols, we require two different symbols to denote them. Such symbols do not reveal the common features of these two statements; viz., both are statements about two different individuals who are bachelors. If we introduce some symbol to denote "is a bachelor" and a method to join it with symbols denoting the names of individuals, then we will have a symbolism to denote statements about any individual's being a bachelor. The part "is a bachelor" is called a *predicate*. Another consideration which leads to some similar device for the representation of statements is suggested by the following argument.

All human beings are mortal.

John is a human being.

Therefore, John is a mortal.

Such a conclusion seems intuitively true. However, it does not follow from the inference theory of the statement calculus developed earlier. The reason for this deficiency is the fact that the statement "All human beings are mortal" cannot be analyzed to say anything about an individual. If we could separate the part "are mortal" from the part "All human beings," then it might be possible to consider any particular human being.

We shall symbolize a predicate by a capital letter and the names of individuals or objects in general by small letters. We shall soon see that using capital letters to symbolize statements as well as predicates will not lead to any confusion. Every predicate describes something about one or more objects (the word "object" is being used in a very general sense to include individuals as well). Therefore, a statement could be written symbolically in terms of the predicate letter followed by the name or names of the objects to which the predicate is applied.

We again consider the statements

1 John is a bachelor.
2 Smith is a bachelor.

Denote the predicate "is a bachelor" symbolically by the predicate letter B, "John" by j, and "Smith" by s. Then Statements (1) and (2) can be written as $B(j)$ and $B(s)$ respectively. In general, any statement of the type "p is Q" where Q is a predicate and p is the subject can be denoted by $Q(p)$.

A statement which is expressed by using a predicate letter must have at least one name of an object associated with the predicate. When an appropriate number of names are associated with a predicate, then we get a statement. Using a capital letter to denote a predicate may not indicate the appropriate number of names associated with it. Normally, this number is clear from the context or from the notation being used. This numbering can also be accomplished by attaching a superscript to a predicate letter, indicating the number of names that are to be appended to the letter. A predicate requiring m ($m > 0$) names is called an m-place predicate. For example, B in (1) and (2) is a 1-place predicate. Another example is that "L: is less than" is a 2-place predicate. In order to extend our definition to $m = 0$, we shall call a statement a 0-place predicate because no names are associated with a statement.

Let R denote the predicate "is red" and let p denote "This painting." Then the statement

3 This painting is red.

can be symbolized by $R(p)$. Further, the connectives described earlier can now be used to form compound statements such as "John is a bachelor, and this painting is red," which can be written as $B(j) \wedge R(p)$. Other connectives can also be used to form statements such as

$$B(j) \rightarrow R(p) \qquad \neg R(p) \qquad B(j) \vee R(p) \qquad \text{etc.}$$

Consider, now, statements involving the names of two objects, such as

4 Jack is taller than Jill.
5 Canada is to the north of the United States.

The predicates "is taller than" and "is to the north of" are 2-place predicates because names of two objects are needed to complete a statement involving these predicates. If the letter G symbolizes "is taller than," j_1 denotes "Jack," and j_2 denotes "Jill," then Statement (4) can be translated as $G(j_1, j_2)$. Note that the order in which the names appear in the statement as well as in the predicate is important. Similarly, if N denotes the predicate "is to the north of," c: Canada, and s: United States, then (5) is symbolized as $N(c, s)$. Obviously, $N(s, c)$ is the statement "The United States is to the north of Canada."

Examples of 3-place predicates and 4-place predicates are:

6 Susan sits between Ralph and Bill.
7 Green and Miller played bridge against Johnston and Smith.

In general, an n-place predicate requires n names of objects to be inserted in fixed positions in order to obtain a statement. The position of these names is important. If S is an n-place predicate letter and $a_1, a_2, \ldots, a_n$ are the names of objects, then $S(a_1, a_2, \ldots, a_n)$ is a statement. If we use this convention, every predicate symbol is followed by an appropriate number of letters, which are the names of objects, enclosed in parentheses and separated by commas. Occasionally, the parentheses and the commas are dropped. The definition does not require that the names be chosen from any fixed set. For example, if B denotes the predicate "is a bachelor" and t denotes "This table," then $B(t)$ symbolizes "This table is a bachelor." In our everyday language, the only admissible name in this context would be that of an individual. However, such restrictions are not necessary according to the rules given above. We show a method of imposing such restrictions in Sec. 1-5.5.

1-5.2 The Statement Function, Variables, and Quantifiers

Let H be the predicate "is a mortal," b the name "Mr. Brown," c "Canada," and s "A shirt." Then $H(b)$, $H(c)$, and $H(s)$ all denote statements. In fact, these statements have a common form. If we write $H(x)$ for "x is mortal," then $H(b)$, $H(c)$, $H(s)$, and others having the same form can be obtained from $H(x)$ by replacing x by an appropriate name. Note that $H(x)$ is not a statement, but it results in a statement when x is replaced by the name of an object. The letter x used here is a placeholder. From now on we shall use small letters as individual or object variables as well as names of objects.

A *simple statement function* of one variable is defined to be an expression consisting of a predicate symbol and an individual variable. Such a statement function becomes a statement when the variable is replaced by the name of any object. The statement resulting from a replacement is called a *substitution instance* of the statement function and is a formula of statement calculus.

The word "simple" in the above definition distinguishes the simple statement function from those statement functions which are obtained from combining one or more simple statement functions and the logical connectives. For example, if we let $M(x)$ be "x is a man" and $H(x)$ be "x is a mortal," then we can form *compound statement functions* such as

$$M(x) \wedge H(x) \qquad M(x) \rightarrow H(x) \qquad \neg H(x) \qquad M(x) \vee \neg H(x) \qquad \text{etc.}$$

An extension of this idea to the statement functions of two or more variables is straightforward. Consider, for example, the statement function of two variables:

1 $G(x, y)$: x is taller than y.

If both x and y are replaced by the names of objects, we get a statement. If m represents Mr. Miller and f Mr. Fox, then we have

$G(m, f)$: Mr. Miller is taller than Mr. Fox.

and

$G(f, m)$: Mr. Fox is taller than Mr. Miller.

It is possible to form statement functions of two variables by using statement functions of one variable. For example, given

$M(x)$: x is a man.

$H(y)$: y is a mortal.

then we may write

$M(x) \wedge H(y)$: x is a man and y is a mortal.

It is not possible, however, to write every statement function of two variables using statement functions of one variable.

One way of obtaining statements from any statement function is to replace the variables by the names of objects. There is another way in which statements can be obtained. In order to understand this alternative method, we first consider some familiar equations in elementary algebra.

2 $x + 2 = 5$
3 $x^2 + 1 = 0$
4 $(x - 1) * (x - \frac{1}{2}) = 0$
5 $x^2 - 1 = (x - 1) * (x + 1)$

In algebra, it is conventional to assume that the variable x is to be replaced by numbers (real, complex, rational, integer, etc.). In the above equations, we would not normally consider substituting for x the name of a person or object instead of numbers. We may state this idea by saying that the *universe* of the variable x is the set of real numbers or complex numbers or integers, etc. The restriction depends upon the problem under consideration. For example, we may be interested in only the real solution or the positive solution in a particular case. In Statement (2), if x is replaced by a real number, we get a statement. The resulting statement is true when 3 is substituted for x, while, for every other substitution, the resulting statement is false. In (3) there is no real number which, when substituted for x, gives a true statement. If, however, the universe of x includes complex numbers as well, then we find that there are two substitution instances which give true statements. In (4), if the universe of x is assumed to be integers, then there is only one number which produces a true statement when substituted. The situation is slightly different in (5) in the sense that if any number is substituted for x, then the resulting statement is true. Therefore, we may say that

6 For any number x, $x^2 - 1 = (x - 1) * (x + 1)$.

Note that (6) is a statement and not a statement function even though a variable x appears in it. In fact, the addition of the phrase "For any number x," has changed the situation. The letter x, as used in (6), is different from the variable x used in Statements (2) to (5). In (6) the variable x need not be replaced by any name to obtain a statement. In mathematics this distinction is often not made. Occasionally when a statement involves an equality, a distinction is made by using the symbol $\equiv$ instead of the equality sign to show that it is a statement. In this case, (6) would be written as $x^2 - 1 \equiv (x - 1) * (x + 1)$. A similar situation occurs when a statement function does not involve an equality, and a distinction is necessary in logic between these two different uses of the variables.

Let us first consider the following statements. Each one is a statement about all individuals or objects belonging to a certain set.

7 All men are mortal.
8 Every apple is red.
9 Any integer is either positive or negative.

Let us paraphrase these in the following manner.

7a For all x, if x is a man, then x is a mortal.
8a For all x, if x is an apple, then x is red.
9a For all x, if x is a integer, then x is either positive or negative.

We have already shown how statement functions such as "x is a man," "x is an apple," or "x is red" can be written by using predicate symbols. If we introduce a symbol to denote the phrase "For all x," then it would be possible to symbolize Statements (7a), (8a), and (9a).

We symbolize "For all x" by the symbol "$(\forall x)$" or by "(x)" with an understanding that this symbol be placed before the statement function to which this phrase is applied. Using

$M(x)$: x is man. $\quad H(x)$: x is a mortal.

$A(x)$: x is an apple. $\quad R(x)$: x is red.

$N(x)$: x is an integer. $\quad P(x)$: x is either positive or negative.

we write (7a), (8a), and (9a) as

7b $\quad (x)(M(x) \rightarrow H(x))$
8b $\quad (x)(A(x) \rightarrow R(x))$
9b $\quad (x)(N(x) \rightarrow P(x))$

Sometimes $(x)(M(x) \rightarrow H(x))$ is also written as $(\forall x)(M(x) \rightarrow H(x))$. The symbols (x) or $(\forall x)$ are called *universal quantifiers*. Strictly speaking, the quantification symbol is "( )" or "($\forall$)," and it contains the variable which is to be quantified. It is now possible for us to quantify any statement function of one variable to obtain a statement. Thus $(x)M(x)$ is a statement which can be translated as

10 $\quad$ For all x, x is a man.
10a $\quad$ For every x, x is a man.
10b $\quad$ Everything is a man.

In order to determine the truth values of any one of these statements involving a universal quantifier, one may be tempted to consider the truth values of the statement function which is quantified. This method is not possible for two reasons. First, statement functions do not have truth values. When the variables are replaced by the names of objects, we get statements which have a truth value. Second, in most cases there is an infinite number of statements that can be produced by such substitutions.

Note that the particular variable appearing in the statements involving a quantifier is not important because the statements remain unchanged if x is replaced by y throughout. Thus the statements $(x)(M(x) \rightarrow H(x))$ and $(y)(M(y) \rightarrow H(y))$ are equivalent.

Sometimes it is necessary to use more than one universal quantifier in a statement. For example consider

$$G(x, y) : x \text{ is taller than } y.$$

We can state that "For any x and any y, if x is taller than y, then y is not taller than x" or "For any x and y, if x is taller than y, then it is not true that y is taller than x." This statement can now be symbolized as

$$(x)(y)(G(x, y) \rightarrow \neg G(y, x))$$

The universal quantifier was used to translate expressions such as "for all," "every," and "for any." Another quantifier will now be introduced to symbolize expressions such as "for some," "there is at least one," or "there exists some" (note that "some" is used in the sense of "at least one").

Consider the following statements:

11 There exists a man.
12 Some men are clever.
13 Some real numbers are rational.

The first statement can be expressed in various ways, two such ways being

11a There exists an x such that x is a man.
11b There is at least one x such that x is a man.

Similarly, (12) can be written as

12a There exists an x such that x is a man and x is clever.
12b There exists at least one x such that x is a man and x is clever.

Such a rephrasing allows us to introduce the symbol "$(\exists x)$," called the *existential quantifier*, which symbolizes expressions such as "there is at least one x such that" or "there exists an x such that" or "for some x." Writing

$M(x)$: x is a man.
$C(x)$: x is clever.
$R_1(x)$: x is a real number.
$R_2(x)$: x is rational.

and using the existential quantifier, we can write the Statements (11) to (13) as

11c $(\exists x)(M(x))$
12c $(\exists x)(M(x) \wedge C(x))$
13c $(\exists x)(R_1(x) \wedge R_2(x))$

It may be noted that a conjunction has been used in writing the statements of the form "Some A are B," while a conditional was used in writing statements of the form "All A are B." To a beginner this usage may appear confusing. We show in Sec. 1-5.5 why these connectives are the right ones to be used in these cases.

1-5.3 Predicate Formulas

Recall that capital letters were first used to denote some definite statements. Subsequently they were used as placeholders for the statements, and, in this sense, they were called statement variables. These statement variables were also considered as special cases of statement formulas.

In Secs. 1-5.1 and 1-5.2 the capital letters were introduced as definite predicates. It was suggested that a superscript n be used along with the capital letters in order to indicate that the capital letter is used as an n-place predicate. However, this notation was not necessary because an n-place predicate symbol must be followed by n object variables. Such variables are called *object* or *individual variables* and are denoted by lowercase letters. When used as an n-place predicate, the capital letter is followed by n individual variables which are enclosed in parentheses and separated by commas. For example, $P(x_1, x_2, \ldots, x_n)$ denotes an n-place predicate formula in which the letter P is an n-place predicate and

$x_1, x_2, \ldots, x_n$ are individual variables. In general, $P(x_1, x_2, \ldots, x_n)$ will be called an *atomic formula* of predicate calculus. It may be noted that our symbolism includes the atomic formulas of the statement calculus as special cases ($n = 0$). The following are some examples of atomic formulas.

$$R \qquad Q(x) \qquad P(x, y) \qquad A(x, y, z) \qquad P(a, y) \qquad \text{and} \qquad A(x, a, z)$$

A well-formed formula of predicate calculus is obtained by using the following rules.

1 An atomic formula is a well-formed formula.
2 If A is a well-formed formula, then $\neg A$ is a well-formed formula.
3 If A and B are well-formed formulas, then $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, and $(A \rightleftarrows B)$ are also well-formed formulas.
4 If A is a well-formed formula and x is any variable, then $(x)A$ and $(\exists x)A$ are well-formed formulas.
5 Only those formulas obtained by using rules (1) to (4) are well-formed formulas.

Since we will be concerned with only well-formed formulas, we shall use the term "formula" for "well-formed formula." We shall follow the same conventions regarding the use of parentheses as was done in the case of statement formulas.

1-5.4 Free and Bound Variables

Given a formula containing a part of the form $(x)P(x)$ or $(\exists x)P(x)$, such a part is called an x-bound part of the formula. Any occurrence of x in an x-bound part of a formula is called a *bound occurrence* of x, while any occurrence of x or of any variable that is not a bound occurrence is called a *free occurrence*. Further, the formula $P(x)$ either in $(x)P(x)$ or in $(\exists x)P(x)$ is described as the *scope* of the quantifier. In other words, the scope of a quantifier is the formula immediately following the quantifier. If the scope is an atomic formula, then no parentheses are used to enclose the formula; otherwise parentheses are needed. As illustrations, consider the following formulas:

$$(x)P(x, y) \tag{1}$$
$$(x)(P(x) \rightarrow Q(x)) \tag{2}$$
$$(x)(P(x) \rightarrow (\exists y)R(x, y)) \tag{3}$$
$$(x)(P(x) \rightarrow R(x)) \vee (x)(P(x) \rightarrow Q(x)) \tag{4}$$
$$(\exists x)(P(x) \wedge Q(x)) \tag{5}$$
$$(\exists x)P(x) \wedge Q(x) \tag{6}$$

In (1), $P(x, y)$ is the scope of the quantifier, and both occurrences of x are bound occurrences, while the occurrence of y is a free occurrence. In (2), the scope of the universal quantifier is $P(x) \rightarrow Q(x)$, and all occurrences of x are bound. In (3), the scope of (x) is $P(x) \rightarrow (\exists y)R(x, y)$, while the scope of $(\exists y)$ is $R(x, y)$. All occurrences of both x and y are bound occurrences. In (4), the scope of the first quantifier is $P(x) \rightarrow R(x)$, and the scope of the second is

$P(x) \rightarrow Q(x)$. All occurrences of x are bound occurrences. In (5), the scope of $(\exists x)$ is $P(x) \wedge Q(x)$. However, in (6) the scope of $(\exists x)$ is $P(x)$, and the last occurrence of x in $Q(x)$ is free.

It is useful to note that in the bound occurrence of a variable, the letter which is used to represent the variable is not important. In fact, any other letter can be used as a variable without affecting the formula, provided that the new letter is not used elsewhere in the formula. Thus the formulas

$$(x)P(x, y) \qquad \text{and} \qquad (z)P(z, y)$$

are the same. Further, the bound occurrence of a variable cannot be substituted by a constant; only a free occurrence of a variable can be. For example, $(x)P(x) \wedge Q(a)$ is a substitution instance of $(x)P(x) \wedge Q(y)$. In fact, $(x)P(x) \wedge Q(a)$ can be expressed in English as "Every x has the property P, and a has the property Q." A change of variables in the bound occurrence is not a substitution instance. Sometimes it is useful to change the variables in order to avoid confusion. In (6), it is better to write $(y)P(y) \wedge Q(x)$ instead of $(x)P(x) \wedge Q(x)$, so as to separate the free and bound occurrences of variables. Occasionally, one may come across a formula of the type $(x)P(y)$ in which the occurrence of y is free and the scope of (x) does not contain an x; in such a case, we have a vacuous use of (x). Finally, it may be mentioned that in a statement every occurrence of a variable must be bound, and no variable should have a free occurrence. In the case where a free variable occurs in a formula, then we have a statement function.

EXAMPLE 1 Let

$$P(x): x \text{ is a person.}$$
$$F(x, y): x \text{ is the father of } y.$$
$$M(x, y): x \text{ is the mother of } y.$$

Write the predicate "x is the father of the mother of y."

SOLUTION In order to symbolize the predicate, we name a person called z as the mother of y. Obviously we want to say that x is the father of z and z the mother of y. It is assumed that such a person z exists. We symbolize the predicate as

$$(\exists z)(P(z) \wedge F(x, z) \wedge M(z, y)) \qquad ////$$

EXAMPLE 2 Symbolize the expression "All the world loves a lover."

SOLUTION First note that the quotation really means that everybody loves a lover. Now let

$$P(x): x \text{ is a person.}$$
$$L(x): x \text{ is a lover.}$$
$$R(x, y): x \text{ loves } y.$$

The required expression is

$$(x)(P(x) \rightarrow (y)(P(y) \wedge L(y) \rightarrow R(x, y))) \qquad ////$$

1-6.4 Theory of Inference for The Predicate Calculus

The method of derivation involving predicate formulas uses the rules of inference given for the statement calculus and also certain additional rules which are required to deal with the formulas involving quantifiers. The rules **P** and **T**, regarding the introduction of a premise at any stage of derivation and the introduction of any formula which follows logically from the formulas already introduced, remain the same. If the conclusion is given in the form of a conditional, we shall also use the rule of conditional proof called **CP**. Occasionally, we may use the indirect method of proof in introducing the negation of the conclusion as an additional premise in order to arrive at a contradiction.

The equivalences and implications of the statement calculus can be used in the process of derivation as before, except that the formulas involved are generalized to predicates. But these formulas do not have any quantifiers in them, while some of the premises or the conclusion may be quantified. In order to use the equivalences and implications, we need some rules on how to eliminate quantifiers during the course of derivation. This elimination is done by *rules of specification* called rules **US** and **ES**. Once the quantifiers are eliminated, the derivation proceeds as in the case of the statement calculus, and the conclusion is reached. It may happen that the desired conclusion is quantified. In this case, we need *rules of generalization* called rules **UG** and **EG**, which can be used to attach a quantifier.

The rules of generalization and specification follow. Here $A(x)$ is used to denote a formula with a free occurrence of x. $A(y)$ denotes a formula obtained by the substitution of y for x in $A(x)$. Recall that for such a substitution $A(x)$ must be free for y.

Rule **US** (Universal Specification) From $(x)A(x)$ one can conclude $A(y)$.

Rule **ES** (Existential Specification) From $(\exists x)A(x)$ one can conclude $A(y)$ provided that y is not free in any given premise and also not free in any prior step of the derivation. These requirements can easily be met by choosing a new variable each time **ES** is used. (The conditions of **ES** are more restrictive than ordinarily required, but they do not affect the possibility of deriving any conclusion.)

Rule **EG** (Existential Generalization) From $A(x)$ one can conclude $(\exists y)A(y)$.

Rule **UG** (Universal Generalization) From $A(x)$ one can conclude $(y)A(y)$ provided that x is not free in any of the given premises and provided that if x is free in a prior step which resulted from use of **ES**, then no variables introduced by that use of **ES** appear free in $A(x)$.

We shall now show, by means of an example, how an invalid conclusion could be arrived at if the second restriction on rule **UG** were not imposed. The other restrictions on **ES** and **UG** are easy to understand.

Let $D(u, v)$: u is divisible by v. Assume that the universe of discourse is $\{5, 7, 10, 11\}$, so that the statement $(\exists u)D(u, 5)$ is true because both $D(5, 5)$

and $D(10, 5)$ are true. On the other hand, $(y)D(y, 5)$ is false because $D(7, 5)$ and $D(11, 5)$ are false. Consider now the following derivation.

{1}	(1)	$(\exists u)D(u, 5)$	P
{1}	(2)	$D(x, 5)$	ES, (1)
{1}	(3)	$(y)D(y, 5)$	UG, (2) (neglecting second restriction)

In step 3 we have obtained from $D(x, 5)$ the conclusion $(y)D(y, 5)$. Obviously x is not free in the premise, and so the first restriction is satisfied. But x is free in step 2 which resulted by use of ES, and that x has been introduced by use of ES and appears free in $D(x, 5)$; hence it cannot be generalized. This is the reason why we obtained a false conclusion from a true premise.

We now give several examples with comments to explain the method of derivation. In the first two examples we use the principles **UG** and **US**, but not **EG** and **ES**.

EXAMPLE 1 Show that $(x)(H(x) \to M(x)) \land H(s) \Rightarrow M(s)$. Note that this problem is a symbolic translation of a well-known argument known as the "Socrates argument" which is given by:

All men are mortal.

Socrates is a man.

Therefore Socrates is a mortal.

If we denote $H(x)$: x is a man, $M(x)$: x is a mortal, and s: Socrates, we can put the argument in the above form.

SOLUTION

{1}	(1)	$(x)(H(x) \to M(x))$	P
{1}	(2)	$H(s) \to M(s)$	US, (1)
{3}	(3)	$H(s)$	P
{1, 3}	(4)	$M(s)$	T, (2), (3), I_{11}

Note that in step 2 first we remove the universal quantifier. ////

EXAMPLE 2 Show that

$$(x)(P(x) \to Q(x)) \land (x)(Q(x) \to R(x)) \Rightarrow (x)(P(x) \to R(x))$$

SOLUTION

{1}	(1)	$(x)(P(x) \to Q(x))$	P
{1}	(2)	$P(y) \to Q(y)$	US, (1)
{3}	(3)	$(x)(Q(x) \to R(x))$	P
{3}	(4)	$Q(y) \to R(y)$	US, (3)
{1, 3}	(5)	$P(y) \to R(y)$	T, (2), (4), I_{13}
{1, 3}	(6)	$(x)(P(x) \to R(x))$	UG, (5)

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EXAMPLE 3 Show that $(\exists x)M(x)$ follows logically from the premises

$$(x)(H(x) \to M(x)) \quad \text{and} \quad (\exists x)H(x)$$

SOLUTION

{1}	(1)	$(\exists x)H(x)$	**P**
{1}	(2)	$H(y)$	**ES**, (1)
{3}	(3)	$(x)(H(x) \to M(x))$	**P**
{3}	(4)	$H(y) \to M(y)$	**US**, (3)
{1, 3}	(5)	$M(y)$	**T**, (2), (4), I_{11}
{1, 3}	(6)	$(\exists x)M(x)$	**EG**, (5)

Note that in step 2 the variable y is introduced by **ES**. Therefore a conclusion such as $(x)M(x)$ could not follow from step 5 because it would violate the rules given for **UG**.

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EXAMPLE 4 Prove that

$$(\exists x)(P(x) \land Q(x)) \Rightarrow (\exists x)P(x) \land (\exists x)Q(x)$$

SOLUTION

{1}	(1)	$(\exists x)(P(x) \land Q(x))$	**P**
{1}	(2)	$P(y) \land Q(y)$	**ES**, (1), y fixed
{1}	(3)	$P(y)$	**T**, (2), I_1
{1}	(4)	$Q(y)$	**T**, (2), I_2
{1}	(5)	$(\exists x)P(x)$	**EG**, (3)
{1}	(6)	$(\exists x)Q(x)$	**EG**, (4)
{1}	(7)	$(\exists x)P(x) \land (\exists x)Q(x)$	**T**, (4), (5), I_9

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It is instructive to try to prove the converse which does not hold. The derivation is

(1)	$(\exists x)P(x) \land (\exists x)Q(x)$	**P**
(2)	$(\exists x)P(x)$	**T**, (1), I_1
(3)	$(\exists x)Q(x)$	**T**, (1), I_2
(4)	$P(y)$	**ES**, (2)
(5)	$Q(z)$	**ES**, (3)

Note that in step 4, y is fixed, and it is no longer possible to use that variable again in step 5.

EXAMPLE 5 Show that from

(a) $(\exists x)(F(x) \land S(x)) \to (y)(M(y) \to W(y))$
(b) $(\exists y)(M(y) \land \neg W(y))$

the conclusion $(x)(F(x) \to \neg S(x))$ follows.

SOLUTION

{1}	(1)	$(\exists y)(M(y) \land \neg W(y))$	P
{1}	(2)	$M(z) \land \neg W(z)$	ES, (1)
{1}	(3)	$\neg(M(z) \to W(z))$	T, (2), E_{17}
{1}	(4)	$(\exists y)\neg(M(y) \to W(y))$	EG, (3)
{1}	(5)	$\neg(y)(M(y) \to W(y))$	E_{26}, (4)
{6}	(6)	$(\exists x)(F(x) \land S(x)) \to (y)(M(y) \to W(y))$	P
{1, 6}	(7)	$\neg(\exists x)(F(x) \land S(x))$	T, (5), (6), I_{12}
{1, 6}	(8)	$(x)\neg(F(x) \land S(x))$	T, (7), E_{25}
{1, 6}	(9)	$\neg(F(x) \land S(x))$	US, (8)
{1, 6}	(10)	$F(x) \to \neg S(x)$	T, (9), E_9, E_{16}, E_{17}
{1, 6}	(11)	$(x)(F(x) \to \neg S(x))$	UG, (10)

EXAMPLE 6 Show that

$$(x)(P(x) \lor Q(x)) \Rightarrow (x)P(x) \lor (\exists x)Q(x)$$

SOLUTION We shall use the indirect method of proof by assuming $\neg((x)P(x) \lor (\exists x)Q(x))$ as an additional premise.

{1}	(1)	$\neg((x)P(x) \lor (\exists x)Q(x))$	P (assumed)
{1}	(2)	$\neg(x)P(x) \land \neg(\exists x)Q(x)$	T, (1), E_9
{1}	(3)	$\neg(x)P(x)$	T, (2), I_1
{1}	(4)	$(\exists x)\neg P(x)$	T, (3), E_{26}
{1}	(5)	$\neg(\exists x)Q(x)$	T, (2), I_2
{1}	(6)	$(x)\neg Q(x)$	T, (5), E_{25}
{1}	(7)	$\neg P(y)$	ES, (4)
{1}	(8)	$\neg Q(y)$	US, (6)
{1}	(9)	$\neg P(y) \land \neg Q(y)$	T, (7), (8), I_9
{1}	(10)	$\neg(P(y) \lor Q(y))$	T, (9), E_9
{11}	(11)	$(x)(P(x) \lor Q(x))$	P
{11}	(12)	$P(y) \lor Q(y)$	US, (11)
{1, 11}	(13)	$\neg(P(y) \lor Q(y)) \land (P(y) \lor Q(y))$	T, (10), (12), I_9 contradiction

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1-6.5 Formulas Involving More Than One Quantifier

So far we have considered only those formulas in which the universal and existential quantifiers appear singly. We shall now consider cases in which the quanti-