Q4: Mary buys 12 oranges for her children A, B and C. In how many ways can she distribute the oranges such that A gets at least 4, B and C gets at least 2 but C gets not more than 5.

Soln: $(x^4 + x^5 + x^6 + x^7 + x^8)(x^2 + x^3 + \cdots + x^6)(x^2 + x^3 + x^4 + x^5)$

A B C

$$x^8 (1 + x + x^2 + x^3 + x^4)(1 + x + \cdots + x^4)(1 + x + x^2 + x^3)$$

$$x^8 (1 - x^5)(1-x)^{-1}(1 - x^5)(1 - x)^{-1}(1 - x^4)(1 - x)^{-1}$$

$$x^8 (1 - x^5)^2 (1 - x^4)(1 - x)^{-3} \quad \text{— (1)}$$

coeff of x^{12} in (1) is same as

coeff of x^4 in $(1 - x^5)^2 (1 - x^4)(1 - x)^{-3}$

$1 \cdot 1 \cdot$ select coeff of x^4 in $(1-x)^{-3}$ $\quad x^0 \cdot x^0 \cdot x^4$

or $1 \cdot -1 \cdot 1$ $\quad x^0 \cdot x^4 \cdot x^0$

Ans: $1 \cdot 1 \cdot {}^6C_4 + 1 \cdot (-1)(1) = {}^6C_4 - 1 = \underline{\underline{14}}$

A	B	C
4	3	5
4	4	4
4	5	3
4	6	2
5	..	

$\underline{\underline{14}}$

Q 5. Use generating to find the number of ways to collect \$15 from 20 distinct people of each of the 19 people can give a dollar or nothing and 20th person can give either \$1 or \$5 or nothing.

Soln : GF: $(1+x)^{19}(1+x+x^5)$

Coeff of x^{15}: $(1+x+x^5)(1+x)^{19}$

$x^0(1)$ & coeff of x^{15} in $(1+x)^{19}$

or coeff x & " " x^{14} in $(1+x)^{19}$

or x^5 & " " x^{10} in $(1+x)^{19}$

Note 3: $(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_n x^n$

$19C_{15} + 19C_{14} + 19C_{10} = \underline{\underline{107882}}$

Q6: How many ways are there to distribute 25 identical balls into 7 distinct boxes if the first box can have no more than 10 balls but any number of balls can go into each of the other 6 boxes.

Soln: G.F: $(1+x+\cdots+x^{10})(1+x+x^2+\cdots+x^{25})^6$ from Note 1

1^{st} box $\quad 2^{nd} \ldots 7^{th}$ box

$$= (1-x^{11})(1-x)^{-1}(1-x^{26})^6(1-x)^{-6} = (1-x^{11})(1-x^{26})^6(1-x)^{-7}$$

coeff of x^{25} in $(1-x^{11})(1-x^{26})^6(1-x)^{-7}$

coeff x^0 × coeff of x^0 × coeff of x^{25} in $(1-x)^{-7}$

or coeff x^{11} × " " × " " x^{14} in $(1-x)^{-7}$

$$1 \cdot 1 \cdot {}^{7+25-1}C_{25} + (-1)(1)\,{}^{7+14-1}C_{14}$$

$$31C_{25} - 20C_{14} = \underline{\underline{697521}}$$

Q7: How many ways are there to select 25 toys from 7 types of toys with between 2 and 6 of each type?

Soln: GF : $(x^2+x^3+x^4+x^5+x^6)^7$

$= x^{14}(1+x+x^2+x^3+x^4)^7$ — ①

coeff of x^{25} in ① is same as

coeff of x^{11} in $(1+x+x^2+x^3+x^4)^7$

$= (1-x^5)^7(1-x)^{-7}$ by note 4

$= (1-{}^7C_1 x^5 + {}^7C_2 x^{10} + \cdots)(1-x)^{-7}$

coeff of x^0 × coeff of x^{11} in $(1-x)^{-7}$

or " x^5 × " x^6 " "

or x^{10} × " x^1 " "

${}^{7+11-1}C_{11} + (-{}^7C_1) \times {}^{7+6-1}C_6 + {}^7C_2 \times {}^{7+1-1}C_1$

${}^{17}C_{11} - 7 \times {}^{12}C_6 + 21 \times {}^7C_1 = \underline{\underline{6055}}$

Generating function for Permutations

Exponential generating function:

If the terms of the sequence can be obtained as coefficient of $\frac{x^r}{r!}$ in the expansion of $f(x)$ is said to be an exponential generating function.

Consider $(1+x)^n = 1 + nc_1 x + nc_2 x^2 + \cdots + nc_r x^r + \cdots + nc_n x^n$

$$= 1 + np_1 \cdot x + \frac{np_2}{2!} x^2 + \cdots + \frac{np_r}{r!} x^r + \cdots + \frac{np_n}{n!} x^n$$

$$= 1 + np_1 \frac{x}{1!} + np_2 \frac{x^2}{2!} + \cdots + np_r \frac{x^r}{r!} + \cdots$$

Thus $(1+x)^n$ is an exponential generating function for r- permutations of n objects without repetition.

If repetition is allowed, then the factor for each object must represent the fact that the object may not appear, may appear once, may appear twice & so on. Hence factor for each object is

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$$

$\therefore$ Enumerator is $\left(1 + x + \frac{x^2}{2!} + \cdots\right)^n = e^{nx}$

$$e^{nx} = 1 + nx + n^2\frac{x^2}{2!} + \cdots + \frac{n^r x^r}{r!} + \cdots = \sum_{r=0}^{\infty} \frac{(nx)^r}{r!} = \sum \frac{n^r x^r}{r!}$$

$\therefore$ Number of r- permutations of n objects with repetition is coeff of $x^r/r!$ i.e. n^r.

Thus e^{nx} is an exponential generating function for r- permutation of n objects with repetition.

Problems

Q1. In how many ways can 4 letters of the word ENGINE be arranged using generating function?

Soln:
$$\underset{E}{\left(1+x+\frac{x^2}{2!}\right)}\underset{N}{\left(1+x+\frac{x^2}{2!}\right)}\underset{I+G}{(1+x)^2}$$

$$\left(1+x+\frac{x^2}{2!}\right)^2(1+x)^2$$

E N E I
N I N G

If we want 6 letter arrangement

coeff of $\frac{x^6}{6!}$ is $\frac{6!}{4}$ ✓

— — — — — —

$\frac{6!}{2!2!}$ ✓

If we want 4 letter arrangement — — — —

coeff of $\frac{x^4}{4!}$ is $\left(1+2x^2+2x+\frac{x^4}{4}+x^3\right)\left(1+x^2+2x\right)$

$$4!\left(2\cdot1+\frac{1}{4}\cdot1+1\cdot2\right)=\underline{\underline{102}}$$

Q2. A ship carries 48 flags, 12 each of the colors white, red, blue and black. Twelve of these flags are placed on a vertical pole in order to communicate a signal to other ships.

(i) How many of these signals use an even number of blue flags and an odd number of black flags?

(ii) How many of these signals use at least 3 white flags or no white flags at all?

Soln: (i)

$$\underbrace{\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)}_{\text{Blue}}\underbrace{\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)}_{\text{black}}$$

$$\underbrace{\left(1+x+\frac{x^2}{2!}+\cdots\right)^2}_{\text{white \& red}}$$

$$=\left(\frac{e^{x}+e^{-x}}{2}\right)\left(\frac{e^{x}-e^{-x}}{2}\right)\left(e^{2x}\right)$$

$$=\frac{1}{4}\left(e^{4x}-1\right)$$

$$=\frac{1}{4}\left(\sum_{r=0}^{\infty}\frac{(4x)^r}{r!}-1\right)$$

coeff of $\frac{x^{12}}{12!}$ is $\frac{1}{4}\left(4^{12}\right)=4^{11}$