22/11/24

Linear Algebra

→ $\underset{\vec{v_1}}{(1,2,3)}$ $\underset{\vec{v_2}}{(4,5,6)}$ $\underset{\vec{v_3}}{(0,1,4)}$ ← E3

$x_1(1,2,3)+x_2(4,5,6)+x_3(0,1,4)=(0,0,0)$ → L·C

$$\left.\begin{aligned} x_1+4x_2+0x_3&=0 \\ 2x_1+5x_2+x_3&=0 \\ 3x_1+6x_2+4x_3&=0 \end{aligned}\right\} \rightarrow x_1=0, x_2=0, x_3=0$$

↓

Linearly Independent.

→ (1,0,0) (0,1,0) (0,0,1) ← E3

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \neq 0$$

$\rho(A)=3$

⇒ $x(1,0,0)+y(0,1,0)+z(0,0,1) = (0,0,0)$ → L.I

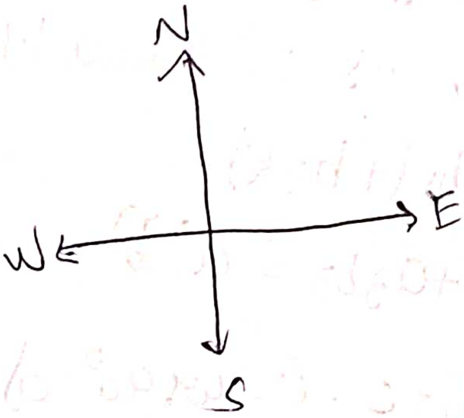
⇒ $(x,y,z)=(0,0,0)$

↓

Linearly Independent.

→ If infinite sol^{n}s then linearly dependent.

→



From East 1m → Indep.

From North 1m → Indep

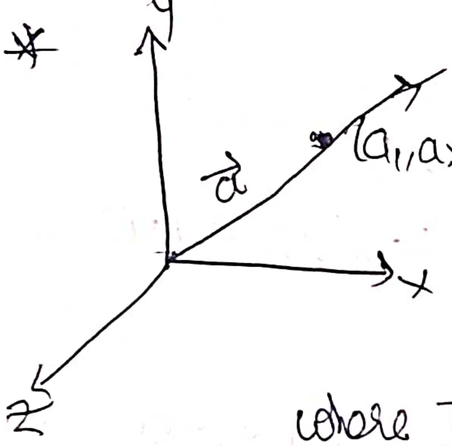
From Northeast 5m

↳ L.C dependent

25/11/24 ⇒ Vector & Scalar:

* Vector is physical quantity that has both Magnitude & direction.
* Scalar has only magnitude no direction.

*



$\vec{a} = (a_1, a_2, a_3)$

$= a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1)$

$\vec{a} = a_1\vec{e_1} + a_2\vec{e_2} + a_3\vec{e_3}$

where $\vec{e_1}, \vec{e_2}, \vec{e_3}$ are identity vector.

$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

(*) $\vec{a} = (a_1, a_2, a_3)$ $\vec{b} = (b_1, b_2, b_3)$

Equality: $\vec{a} = \vec{b}$; $a_1 = b_1, a_2 = b_2, a_3 = b_3$.

Addition: $\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

Scalar Multiplication: $\lambda\vec{a} = (\lambda a_1, \lambda a_2, \lambda a_3)$

Scalar product: $\vec{a} \cdot \vec{b} = |a||b| \cos\theta$.

$\rightarrow$ $i \cdot i = 1$, $j \cdot j = 1$, $k \cdot k = 1$, $i \cdot j = 0$, $j \cdot k = 0$, $i \cdot k = 0$

$(a_1 i + a_2 j + a_3 k) \cdot (b_1 i + b_2 j + b_3 k)$

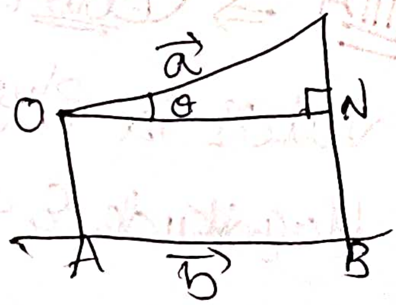
$= a_1 b_1 + a_2 b_2 + a_3 b_3 = \vec{a} \cdot \vec{b}$

$\Leftrightarrow$ $\vec{a} \perp \vec{b}$

$\vec{a} \cdot \vec{b} = 0$. ($\because \cos 90° = 0$)

Projection: $ON = |\vec{a}| \cos\theta$

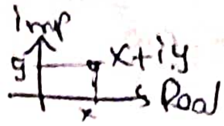
Proj of $\vec{a}$ on $\vec{b}$



$\vec{a} \cdot \vec{b} = |a||b| \cos\theta$.

$\vec{a} \cdot \vec{b} = |b|$ Proj $\vec{a}$ on $\vec{b}$

$\therefore$ $\boxed{\text{Proj } \vec{a} \text{ on } \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|b|}}$

⇒ C – Set of Complex no $\{x+iy\}$. 

Q – Set of Rational number. $\{p/q \mid p, q \in Z, q \neq 0\}$

R – Set of Real no.

Q – Set of Rational numbers.

N – Set of Natural number $\{1, 2, 3, --\}$

Z or I – Set of Integers $\{--, -1, 0, 1, ---\}$

⇒ Binary operation (*)

* in a set G is a mapping.

$* : G \times G \rightarrow G$ (i.e).

Binary operation "*" in a set G associates a unique element of G to every pair of element of G.

→ $\boxed{(a,b) \in G * G \text{ then } a*b \in G}$

ex: ① The usual addition in Z are binary oper

$(2,3)$ $a+b \in Z$ $2+3 = 5 \in Z$.

② Subtraction is not binary operation in N

→ If * is a binary operation defined in a set G then we say G is closed under the binary composition *, simply G is close under *.

→ N is closed under "addition" but it is not closed under "subtraction" & "division"

Group:

A non empty set G together with binary operation $*$ is called group if the algebraic system $(G, *)$ satisfies the following axioms:

① Associative Axiom: $a, b, c \in G$
$(a*b)*c = a*(b*c)$

② Identity Axiom: $e \in G, a \in G$ $a*e = e*a = a$
e is identity element

③ Inverse Axiom: $a \in G, b \in G$ $a*b = b*a = e$
$a*a^{-1} = a^{-1}*a = e$

→ A group $(G, *)$ is said to be an abelian group if the binary operation $*$ satisfies all 4 Axioms Associative, Identity, Inverse, Commutative.

④ Commutative Axiom: $a \in G, b \in G$ $a*b = b*a$.

⇒ Closure, Associative → Semigroup
Closure, Associative, Identity → Monoid.
Closure, Ass., Iden, Inverse → group
Closure, Ass, Iden, Inv, Commutative → Abelian group

eg: ① $\{Z, +\}$

① Clos: $2, 3 \in Z$; $2+3 = 5 \in Z$ ⑤ Comm: $2+3 = 3+2 = 5$.

② Ass: $(2+3)+4 = 2+(3+4)$

③ Iden: $2+0 = 0+2 = 2$. ∴ Abelian group

④ Inverse: $2+(-2) = 0$.

(2) $\{N, +\} \rightarrow$ Semigroup.

(3) $\{N, \times\} \rightarrow$ Monoid

(4) $\{Z, +\} \rightarrow$ abelian

(5) $\{R, +\} \rightarrow$ abelian

(6) $\{1, \omega, \omega^2\}$ is a group under multiplication & also abelian.

<u>Subgroup</u>:

A Subset H of a group $(G, *)$ is said to be a Subgroup if H itself is a group under the Same binary operation defined on G.

Eg: $\{Z, +\} \rightarrow$ group
$\{Z^+, +\} \rightarrow$ group

As we know $Z^+ \subset Z$ & $\{Z^+, +\}$ is subgroup of $\{Z, +\}$

<u>Field</u>:

A non empty set F is said to be a field if there exists two binary operations "$+$ & $\cdot$" on F such that:

(1) $(F, +) \rightarrow$ abelian group

(2) $(F / \{0\}, \cdot) \rightarrow$ group

(3) $a, b, c \in F \rightarrow$ $a(b+c) = ab + ac$, $(a+b)c = ac + bc$ } Distributive law.

<u>Ex</u>: Set of real number with usual addition & multiplication is a field.

$\Rightarrow$ we use F for field, also the elements of F are called Scalars & the elements of V are called vectors.

Vector Space:

A non empty set V is said to be a vector space over a field F if it satisfying the followings:

1. $(V,+)$ is an abelian group.
2. V is closed under scalar multiplication.
   $\alpha \in F, v \in V \Rightarrow \alpha v \in V$
3. Distributivity of scalar multiplication w.r.t vector addition.
   $\alpha \in F; v, w \in V \Rightarrow \alpha(v+w) = \alpha v + \alpha w$
4. Distributivity of scalar multiplication w.r.t to field addition.
   $\alpha, \beta \in F; v \in V \Rightarrow (\alpha+\beta)v = \alpha v + \beta v$
5. Scalar multiplication with field multiplication
   $\alpha(\beta v) = (\alpha\beta)v$
6. Identity element of scalar multiplication.
   $1 \cdot v = v$

(1 is identity of F w.r.t multiplication)

$\Rightarrow$ A n-component vector $\vec{a}$ is an ordered n-tuple of numbers written as row $(a_1, a_2, \ldots, a_n)$ or as a column $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

$a_1, a_2, \ldots, a_n$ are real numbers & are called components of the vector.

$\vec{a} = (a_1, a_2, a_3) \Rightarrow \vec{a} = (a_1, a_2, \ldots, a_n)$

Unit vector:

A Unit vector denoted by e_i is a vector with Unity $(i = 1, 2, \ldots, n)$.

$$e_1 = (1, 0, 0, \ldots, 0)$$

$$e_2 = (0, 1, 0, \ldots, 0)$$

$$e_n = (0, 0, 0, \ldots, 1)$$

Null vector:

It is a vector all of whose components are zero.

$$\vec{0} = (0, 0, 0, \ldots, 0)$$

Equality:

Let a & b be two n-component vectors then a & b are equal if & only if $a_i = b_i$ (for each i)

$$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$$

$$\Rightarrow a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n$$

→

* Scalar product of two n-component vectors a, b is defined to be the scalar.

* The norm of an n-component vector $a = (a_1, a_2, \ldots, a_n)$ is denoted & defined by:

$$\|a\| = |a| = \sqrt{a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2}$$

Euclidean Space:

→ An n-dimensional Euclidean space is defined as collection of all vector $a = (a_1, a_2, \ldots, a_n)$ where $a_1, a_2, \ldots, a_n$ are real (or) complex number.

→ The addition of vectors & multiplication of vector by a scalar are respectively defined as follows:

$E^n = \{(a_1, a_2, \ldots, a_n) \mid$ each a_i is a real (or) complex, $1 \le i \le n\}$

→ $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in E^n$

$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n)$

$= (a_1+b_1, a_2+b_2, \ldots, a_n+b_n) \in E^n$

→ $\lambda \in F$

$\lambda(a_1, a_2, \ldots, a_n) = (\lambda a_1, \lambda a_2, \ldots, \lambda a_n) \in E^n$

→ Therefore E^n is vector space (over $\mathbb{R}$).

⇒ Note:

Let V be a vector space over F and $W \subseteq V$. Then W is called a subspace of V if W is a vector space over F under same operation.

Linear Combination:

Suppose V is a vector space over F, $v_i \in V$ and $\alpha_i \in F$ for $1 \le i \le n$ then $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$ is called linear combination (over F) of $\{v_1, v_2, \ldots, v_n\}$.

Ex: $(2,3)\ (3,2) \in E^2$

$\alpha_1(2,3) + \alpha_2(3,2)$

Linear Span:

Let V be a vector space & $S \subseteq V$.

we write:

$$L(S) = \{ \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \mid n \in N, v_i \in S \text{ \& } \alpha_i \in F \}$$
$$1 \leq i \leq n$$

which is the set of all linear combination of finite number of elements of S. This $L(S)$ is called linear span of S.

ex. $(2,3) \in E^2$ $\quad \{(1,0)(0,1)\} \in E^2$

$$(2,3) = 2(1,0) + 3(0,1)$$

Theorm - 1

St: If S is any subset of a vector space V, then S.T $L(S)$ is a subspace of V.

Pf: Let $v, w \in L(S)$ $\quad \alpha, \beta \in F$

Since $v, w \in L(S)$

we have that,

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$

$$w = \beta_1 w_1 + \beta_2 w_2 + \dots + \beta_m w_m$$

$$\left| \begin{array}{l} v_i \in S, \alpha_i \in F \text{ for } 1 \leq i \leq n \\ w_j \in S, \beta_j \in F \text{ for } 1 \leq j \leq m \end{array} \right.$$

$$\alpha v + \beta w = \alpha \{\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n\} + \beta \{\beta_1 w_1 + \beta_2 w_2 + \dots + \beta_m w_m\}$$

$$= \alpha \alpha_1 v_1 + \alpha \alpha_2 v_2 + \dots + \alpha \alpha_n v_n + \beta \beta_1 w_1 + \beta \beta_2 w_2 + \dots + \beta \beta_m w_m$$

which is linear combination of elements from

Hence $\alpha v + \beta w \in L(S)$

This shows that $L(S)$ is a subspace of V.

$\Rightarrow$ The above is proved by using this axiom.

$\rightarrow$ V be a vector space over F & $W \subset V$, than the following conditions are equivalent.

1. W is subspace of V
2. $\cdot \alpha, \beta \in F$
   $w_1, w_2 \in W$
   $\alpha w_1 + \beta w_2 \in W$

(*) So, we observe in proof of above theorm that if (2) in above axion is true then we hold that (1) is true.

2/12/24 **Linearly Independent & Dependent**

* The vectors $v_i \in V$ for $1 \le i \le n$ are linearly independent over F if there exist elements $a_i \in F$, $1 \le i \le n$ not all of them equal to zero, such that $a_1 v_1 + a_2 v_2 + \dots + a_n v_n = 0$.

* If the vectors v_i, $1 \le i \le n$ are not linearly dependent over F then they are said to be linearly independent over F.

(*) $v_1, v_2, \dots, v_n \rightarrow$ L.I $\alpha_i \in F$

then $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$ & also

$\alpha_i = 0 \ \forall i = 1, 2, \dots, n$

ex: (1) $\{(1,1,0), (3,0,1), (5,2,2)\}$

m(1)

$\alpha_1(1,1,0) + \alpha_2(3,0,1) + \alpha_3(5,2,2) = 0$

$$\left.\begin{aligned} \alpha_1 + 3\alpha_2 + 5\alpha_3 &= 0 \\ \alpha_1 + 0\alpha_2 + 2\alpha_3 &= 0 \\ 0\alpha_1 + \alpha_2 + 2\alpha_3 &= 0 \end{aligned}\right\} \begin{bmatrix} 1 & 3 & 5 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 3 & 5 \\ 0 & -3 & -3 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2/-3} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \ \rho(A) = n = 3$

$$\left.\begin{aligned} \alpha_1 + 3\alpha_2 + 5\alpha_3 &= 0 \\ \alpha_2 + \alpha_3 &= 0 \\ \alpha_3 &= 0 \end{aligned}\right\} \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$$

$\Rightarrow$ linearly independent

m(2). Minors method.

Trivial sol^n: $\rho(A) = n = 3$.

$$\begin{vmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 5 & 2 & 2 \end{vmatrix}_{3\times3} = 1\{-2\} - 1\{6-5\} + 0() = -2-1 = -3 \neq 0$$

$\rho(A) = 3 = n \ \therefore \alpha_1 = \alpha_2 = \alpha_3 = 0.$

(2) $\{(1,1,0)\ (3,0,1)\ (5,2,2)\ (1,2,3)\} \in F$?

$\alpha_1(1,1,0) + \alpha_2(3,0,1) + \alpha_3(5,2,2) + \alpha_4(1,2,3)$

$\alpha_1 + 3\alpha_2 + 5\alpha_3 + \alpha_4 = 0$

$\alpha_1 + 0\alpha_2 + 2\alpha_3 + 2\alpha_4 = 0$

$0\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 = 0$

$$\begin{bmatrix} 1 & 3 & 5 & 1 \\ 1 & 0 & 2 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 3 & 5 & 1 \\ 0 & -3 & -3 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 3 & 5 & 1 \\ 0 & 1 & 1 & -1/3 \\ 0 & 0 & 1 & 10/3 \end{bmatrix} \xleftarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 3 & 5 & 1 \\ 0 & 1 & 1 & -1/3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$\downarrow$$

$$\left.\begin{aligned} \alpha_1 + 3\alpha_2 + 5\alpha_3 + \alpha_4 &= 0 \\ \alpha_2 + \alpha_3 - \tfrac{1}{3}\alpha_4 &= 0 \\ \alpha_3 + \tfrac{10}{3}\alpha_4 &= 0 \end{aligned}\right\}$$

$\alpha_4 = K$ then we can write $\alpha_1, \alpha_2, \alpha_3$ in terms of α_4 (i.e) K.

$\therefore$ linearly dependent.

⊛ Also, if 4 vectors are linearly dependent then one vector can be expressed as linear combination of other 3 vectors which are linearly independent if only they are considered.

eg: $(1,2,3) = -\frac{14}{3}(1,1,0) + -\frac{11}{3}(3,0,1) + \frac{10}{3}(5,2,2)$. $\boxed{\text{detail at back}}$

⇒ To Check L·depen (or) L·Indep:

① Minor method → $\rho(A) = n = 3$: L·Ind., $= 0$. Trivial

→ $\rho(A) < n$: L·Dep. Infinite sol.

② Sol^n's method

(*) If the vectors $v_i, 1 \le i \le n$ are not linearly independent over F then they are said to be linearly dependent over F.

→ if there exists elements $a_i \in F$ $1 \le i \le n$ not all of them are equal to zero.

Theorem-(22):

St: Let V be vector space over F. If $v_1, v_2, \ldots, v_n \in V$ are linearly independent, then every element in their linear span has a unique representation in the form $\lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n$ with $\lambda_i \in F, 1 \le i \le n$.

P.f: Let $S = \{ v_i \mid 1 \le i \le n \}$

Consider $L(S)$, Let $v \in L(S)$ then

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n \quad \alpha_i \in F, v_i \in S.$$

Suppose: $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$ $\quad \alpha_i, \beta_i \in F$

$(-)$ $\quad v = \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n$ $\quad 1 \le i \le n$

$0 = (\alpha_1 - \beta_1) v_1 + \ldots + (\alpha_n - \beta_n) v_n$

⇒ we know by def of linearly indep scalars = 0.

So, $\alpha_i - \beta_i = 0$ $\quad 1 \le i \le n$

$$\Rightarrow \alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_n = \beta_n$$

∴ Hence proved.

Theorm - (3) :

St: If $v_i \in V, 1 \le i \le n$, are linearly dependent if and only if one of the vectors is a linear combination of the other.

P.f: if & only if means we have to prove both cases.

Suppose that one of the vector is linear combination of other.

$$V_n = \alpha_1 V_1 + \text{---} + \alpha_{n-1} V_{n-1} \quad (\because V_1, V_2, \text{---}, V_n \in V)$$

$$\Rightarrow \alpha_1 V_1 + \alpha_2 V_2 + \text{--} + \alpha_{n-1} V_{n-1} + (-1) V_n = 0$$

if $\alpha_1 = \alpha_2 = \text{---} = \alpha_{n-1} = 0$ but $-1 \neq 0$ so all the set of vectors $V_1, V_2, \text{---}, V_n$ are L. Dep.

($\because$ If any one scalar is not equal to zero then given set $V_1, \text{---}, V_n$ are L.D).

$\leftarrow$

Suppose $V_1, V_2, \text{---}, V_n \rightarrow L.D$

then there exist α_i not all zero such that:

$$\alpha_1 V_1 + \alpha_2 V_2 + \text{---} + \alpha_n V_n = 0$$

$\rightarrow$ suppose $\alpha_n \neq 0$

$$\alpha_n V_n = -\alpha_1 V_1 - \alpha_2 V_2 \text{---} - \alpha_{n-1} V_{n-1}$$

$$V_n = -\frac{\alpha_1}{\alpha_n} V_1 - \frac{\alpha_2}{\alpha_n} V_2 \text{---} - \frac{\alpha_{n-1}}{\alpha_n} V_{n-1}$$

$$V_n = \beta_1 V_1 + \beta_2 V_2 + \text{---} + \beta_{n-1} V_{n-1} \rightarrow L.C$$

$\therefore$ Hence proved.

(*) A set containing single vector "a" is linearly dependent, if there exist scalar $\lambda \neq 0$ such that $\lambda a = 0$ true if & only if $a = 0$.

(*) A set containing single vector a is linearly independent, if there exist scalar $\lambda = 0$ such that $\lambda a = 0$ iff $a \neq 0$

Theorem-④:-

St:- If a set of vectors is linearly independent then any subset of these vectors is also linearly independent.

Pf:- Let $S = \{v_1, v_2, ---, v_n\}$ are L.I set.

⊙ If possible $v_1, v_2, ---, v_k$, $k<n$ be L.Dep. there exist scalar λ_i not all zero such that:

$$\lambda_1 v_1 + \lambda_2 v_2 + --- + \lambda_k v_k = 0.$$

$$\Rightarrow \lambda_1 v_1 + \lambda_2 v_2 + --- + \lambda_k v_k + 0 v_{k+1} + 0 v_{k+2} + --- + 0 v_n = 0.$$

Now we have $(v_1, ---, v_n)$ atleast one $\lambda_i \neq 0$

Thus $(v_1, v_2, -, v_n)$ are Linearly Dependent.

This is contradiction to what we had taken.

So, as our assumption is wrong,

every subset L.I set is L.I.

every superset L.D set is L.D.

∴ Hence Proved

Maximal linearly independent set:

A Subset S of a vector space V is said to be a Maximal linearly independent set if:

1. S is a linearly independent set
2. $S \cup \{v\}$ is linearly dependent for any $v \in V/S$

eg: $S = \{(1,1,0)(3,0,1)(5,2,1)\} \to$ L. Indep

$S \cup \{v\} = \{(1,1,0)(3,0,1)(5,2,1)(1,2,3)\} \to$ L. dep.

Ex: Check wheather linearly Ind (or) Dep.

1. $\{(4,2,1)(2,-6,-5)(1,-2,3)\} \to E^3$

Sol: (m-1)

$$\begin{vmatrix} 4 & 2 & 1 \\ 2 & -6 & -5 \\ 1 & -2 & 3 \end{vmatrix} \neq 0 \quad \text{L.I}$$

(m-2) $\alpha_1(4,2,1) + \alpha_2(2,-6,-5) + \alpha_3(1,-2,3) = 0$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$

$\therefore$ L.I

Note:

- Maximum no of linearly independent vectors in E^n is n.
- Hence any set of $(n+1)$ vectors are always linearly dependent.

Basis:

A Subset S of a vector space V is called a basis of V if:

(ii) S consists of linearly independent elements ((i.e) any finite no of elements in S is a linearly independent).

(iii) S spans V (i.e $V = L(S)$)

eg:- $S = \{(1,1,0)(3,0,1)(5,2,2)\} \to$ L.I

S subset of V.

$(1,2,3) \in V$.

$\Rightarrow (1,2,3) = \alpha_1(1,1,0) + \alpha_2(3,0,1) + \alpha_3(5,2,2)$

$\alpha_1 + 3\alpha_2 + 5\alpha_3 = 1$

$\alpha_1 + 0\alpha_2 + 2\alpha_3 = 2$

$0\alpha_1 + \alpha_2 + 2\alpha_3 = 3$

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 1 \\ 1 & 0 & 2 & 2 \\ 0 & 1 & 2 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 5 & 1 \\ 0 & -3 & -3 & 1 \\ 0 & 1 & 2 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 5 & 1 \\ 0 & 1 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{10}{3} \end{array}\right]$$

$\therefore \alpha_1 = -\frac{14}{3}, \alpha_2 = \frac{-11}{3}, \alpha_3 = \frac{10}{3}$

$(1,2,3) = -\frac{14}{3}(1,1,0) + -\frac{11}{3}(3,0,1) + \frac{10}{3}(5,2,2)$ is L-comb

6/12/17

(*) Standard basis of the n-dimensional euclidean space R^n is the basis obtained by taking the n basis vectors, $(e_1, e_2, \text{—}, e_n)$ where e_i is the vector with 1 in the i^{th} coordinate & 0 elsewhere.

Theorem (5):
(*) The set of all n unit vectors $e_1, e_2, \ldots, e_n$ form a basis for R^n.

Pf: $\{e_1, e_2, \ldots, e_n\}$ are L.I

$$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \ldots + \alpha_n e_n = 0$$

$$\alpha_1(1,0,0,\ldots,0) + \alpha_2(0,1,0,\ldots,0) + \ldots + \alpha_n(0,0,\ldots,1) = (0,0,\ldots,0)$$

$$(\alpha_1,0,0,\ldots,0) + (0,\alpha_2,0,\ldots,0) + \ldots + (0,0,\ldots,\alpha_n) = (0,0,\ldots,0)$$

$$(\alpha_1,\alpha_2,\ldots,\alpha_n) = (0,0,\ldots,0)$$

$\alpha_i = 0$ for $i = 1, \ldots, n$.

$\therefore e_1, \ldots, e_n$ are L.I.

$x \in R^n$. $\quad x = (x_1, \ldots, x_n)$

$$= x_1(1,0,\ldots,0) + x_2(0,1,0,\ldots,0) + \ldots + x_n(0,0,\ldots,1)$$

$$x = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n$$

$\Rightarrow$ Subset B of vector space is a basis iff:

1. Maximal set of linearly Independent
2. Minimal generating set (means in E^3 space min 3 vector should be)
3. Every vector v can be expressed as a linear combination of vectors in a unique way.

Ex: (1) Test wheather the set of vectors $\{(1,1,0)\ (1,0,-2)\ (1,1,1)\}$ forms basis for R^3. If so express $(1,2,3)$ interms of basis vectors.

Sol: For basis check wheather it is L.I (a) Dep.

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{vmatrix} = 1(+2) - 1(1+2) \neq 0 \quad \therefore \text{L.I}$$

$\therefore$ Basis.

$(x, y, z) \in R^3$

$(x, y, z) = \alpha_1(1,1,0) + \alpha_2(1,0,-2) + \alpha_3(1,1,1)$

$$\left.\begin{array}{l} \alpha_1 + \alpha_2 + \alpha_3 = x \\ \alpha_1 + 0\alpha_2 + \alpha_3 = y \\ 0\alpha_1 - 2\alpha_2 + \alpha_3 = z \end{array}\right\} \left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 1 & 0 & 1 & y \\ 0 & -2 & 1 & z \end{array}\right]$$

$\downarrow R_2 \to R_2 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & 1 & 0 & x-y \\ 0 & 0 & 1 & z+2x-2y \end{array}\right] \xleftarrow{R_3 \to R_3 + 2R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & 1 & 0 & \cancel{y}+x \\ 0 & -2 & 1 & z \end{array}\right]$$

$\therefore \alpha_3 = z + 2x - 2y, \ \alpha_2 = x - y,$

$\alpha_1 = x - (x-y) - (z + 2x - 2y)$

$\therefore$ It is Basis

$(x, y, z) = (1, 2, 3)$

$\alpha_3 = 3+2-4 = 1, \ \alpha_2 = 1-2 = -1, \ \alpha_1 = 1-(1-2)-(3+2-4)$
$= 1.$

$(1,2,3) = 1(1,1,0) + -1(1,0,-2) + 1(1,1,1)$
$= (1,2,3)$ ✓ $\therefore \boxed{\alpha_1 = 1, \alpha_2 = -1, \alpha_3 = 1}$

Theorem-⑥:

(*) Any vector in R^n can be expressed as a linear combination of a set of vectors in only one way.

P-f:- $b \in R^n$ $\{a_1, a_2, \ldots, a_n\}$ basis vectors.

$$b = \alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_n a_n \quad (1)$$

$$b = \beta_1 a_1 + \beta_2 a_2 + \ldots + \beta_n a_n - (2)$$

(1) − (2)

$$(\alpha_1 - \beta_1)a_1 + (\alpha_2 - \beta_2)a_2 + \ldots + (\alpha_n - \beta_n)a_n = 0$$

As we know, $\{a_1, \ldots, a_n\} \to L.I$

$\alpha_i - \beta_i = 0 \quad \forall i = 1 \to n$

$\Rightarrow \alpha_i = \beta_i$

Minimal spanning set:-

* A subset S of a vector space V is said to be minimal spanning set iff

① S is spanning set for V.

② $S \setminus v$ do not span V for any $v \in S$.

(can't if we remove one element)

eg:- $\{(1,0), (0,1)\} \in S \to L.I$
$\hookrightarrow$ Spans V.

$(2,3) \in V$

$(2,3) = \alpha_1(1,0) + \alpha_2(0,1) = 2(1,0) + 3(0,1)$

If we add $(1,1)$ to S $\{(1,0)(0,1)(1,1)\} \to L.D$ ✗

If we remove $(0,1)$ from S : $\{(1,0)\} \to L.D$ ✗

Theorem-(2):

St: Prove that minimal spanning set of vectors forms a basis.

P.f: let $S = \{V_1, V_2, \ldots, V_n\}$ be a minimal spanning set. $L(S) = V$.

Inorde to prove S is a basis $\Rightarrow S \to L.I$

So, in a contrary way suppose that S is not L.I.

then there exist V_j $(1 \leq j \leq n)$ is a linear combination of its preceding ones.

$$V_j = \alpha_1 V_1 + \alpha_2 V_2 + \cdots + \alpha_{j-1} V_{j-1} + \alpha_{j+1} V_{j+1} + \cdots + \alpha_n V_n.$$

$x \in L(S)$

$x = \beta_1 V_1 + \beta_2 V_2 + \cdots + \beta_n V_n \quad \beta_i \in F, 1 \leq i \leq n.$

$$x = \beta_1 V_1 + \beta_2 V_2 + \beta_j (\alpha_1 V_1 + \cdots + \alpha_{j-1} V_{j-1} + \alpha_{j+1} V_{j+1} + \cdots + \alpha_n V_n) + \cdots + \beta_n V_n.$$

$$x = (\beta_1 + \alpha_1 \beta_j) V_1 + \cdots + \beta_n V_n.$$

$L(V_1, V_2, \ldots, V_{j-1}, V_{j+1}, \ldots, V_n) = L(S) = V$

which is a contradiction to the fact that n is minimum with S spans V.

$\therefore S \to L.I$

$\therefore$ Hence proved.

Proof with example for understanding:-

$\{(1,0)\ (0,1)\} \in S \rightarrow$ basis then it should be L.I

So, To prove L.I.

Contradiction: $\{(1,0)(0,1)\} \rightarrow$ L.D

then $(1,0) = \alpha(0,1)$ — ①

$\vec{\alpha} \in L(S)$

$(2,3) \in L(S)$

$(2,3) = \beta_1(1,0) + \beta_2(0,1)$

$(2,3) = \beta_1 \alpha(0,1) + \beta_2(0,1)$ [∵ ①]

$(2,3) = \gamma_1(0,1)$

No value satisfies γ_1.

∴ It is contradiction

∴ S → L. Indep.

Theorem-⑧:

St: P.T a maximal linearly independent set is basis

P.f:

Orthogonal:

* $a, b \in V$

$\boxed{\vec{a} \cdot \vec{b} = 0}$ → than a, b are orthogonal vectors

Orthogonal set:

* A set $\{a_1, a_2, \ldots a_n\}$ of vectors is said to be an orthogonal set. if the vectors are pairwise orthogonal.

$(a_1 \cdot a_2 = a_1 \cdot a_n = a_3 \cdot a_5 = \cdots = 0)$.

Orthonormal set:

* An orthogonal set is said to be orthonormal if magnitude of each vector is unity.

eg: $\{e_1, e_2, \ldots, e_n\}$ are orthonormal set.

$e_1 = (1, 0, 0)$ $e_2 = (0, 1, 0)$

$e_1 \cdot e_2 = 0$ $|e_1| = \sqrt{1+0+1} = 1$ ∴ ✓

Theorem (9):

St: Any orthogonal set of nonzero vectors is linearly independent.

Pf: Let $\{a_1, \ldots, a_n\}$ set of non zero orthogonal vectors.

$\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n = 0$ — (1)

$\lambda_1 a_1 \cdot a_1 + \lambda_2 a_2 \cdot a_1 + \cdots + \lambda_n a_n \cdot a_1 = 0$

$\lambda_1 1 + \lambda_2 0 + \cdots + \lambda_n 0 = 0.$

$\Rightarrow \lambda_1 = 0$

Similarly, if we do with $a_2, \ldots, a_n$ dot Product

$\lambda_2 = 0$

$\ldots \quad \lambda_n = 0$. $\quad \therefore$ L-I

Orthogonal Basis:-

* $\{a_1, a_2, \ldots, a_n\}$ basis for E^n, is said to be an orthogonal basis if its vector are Pairwise orthogonal.

* In addition if magnitude of each vector is unity then the basis is said to be "Orthonormal".

Gram - Schmidt orthogonalization Process.

* To construct orthonormal basis:- vectors should be

① L.I

② Pair wise orthogonal

③ magnitude each vector must be unity.

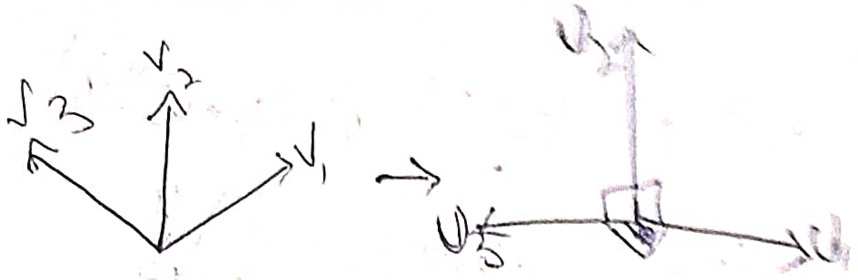
$\Rightarrow \{v_1, v_2, v_3\} \in E^3 \rightarrow$ L-I

$\{U_1, U_2, U_3\} \rightarrow$ orthonormal basis E^3

$U_1 \cdot U_2 = 0 \qquad |U_1| = 1$

$U_1 \cdot U_3 = 0 \qquad |U_2| = 1$

$U_3 \cdot U_2 = 0 \qquad |U_3| = 1$



$\{V_1, V_2, V_3\} \rightarrow \{U_1, U_2, U_3\}$

Given normal vectors — Final ans orthonormal vectors of V_1, V_2, V_3.

$U_1 = \frac{\vec{V_1}}{|\vec{V_1}|} = \frac{V_1}{|V_1|}$ $\quad |\hat{U_1}| = 1$ $\quad$ for clarity see next page

$U_2 = \frac{V_2}{|V_2|}$ $\quad |U_2| = 1 \quad V_2 = U_2 - (U_2 \cdot V_1)V_1$

Projection

$U_3 = \frac{V_3}{|V_3|}$ $\quad |U_3| = 1 \quad V_3 = U_3 - (U_3 \cdot V_1)V_1 - (U_3 \cdot V_2)V_2$

Ex:- ① Construct the orthonormal basis for:-

$\{(1,1,1)\ (-1,0,-1)\ (-1,2,3)\}$

$\quad \vec{a_1} \quad \vec{a_2} \quad \vec{a_3}$

Sol:-

$\vec{U_1} = \frac{(1,1,1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ $\quad |\vec{a_1}| = \sqrt{3}$, $\quad |\vec{U_1}| = \sqrt{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 1$

$\vec{U_2} = \frac{\vec{V_2}}{|\vec{V_2}|}$ $\quad \vec{V_2} = \vec{a_2} - (\vec{a_2} \cdot \vec{U_1})\vec{U_1}$

$= (-1,0,-1) - \left(\frac{-1}{\sqrt{3}} + 0 - \frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$= (-1,0,-1) + \frac{2}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$= \left(\frac{-1}{3}, \frac{2}{3}, \frac{-1}{3}\right)$

$\vec{U_2} = \frac{\left(\frac{-1}{3}, \frac{2}{3}, \frac{-1}{3}\right)}{\sqrt{\frac{1+4+1}{9}}} = \left(\frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)$ $\quad |U_2| = \sqrt{\frac{1+4+1}{6}} = 1.$

$U_1 \cdot U_2 = \frac{-1}{\sqrt{18}} + \frac{2}{\sqrt{18}} - \frac{1}{\sqrt{18}} = 0.$

$$\vec{U_3} = \frac{\vec{V_3}}{|\vec{V_3}|} \qquad \vec{V_3} = \vec{a_3} - (\vec{a_3}\cdot\vec{U_1})\vec{U_1} - (a_3 \cdot \vec{U_2})\vec{U_2}$$

$$= (-1,2,3) - \left(\cancel{\ \ }\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right)\frac{4}{\sqrt3}$$

$$- \frac{2}{\sqrt6}\left(-\frac{1}{\sqrt6}, \frac{2}{\sqrt6}, -\frac{1}{\sqrt6}\right)$$

$$= \left(-1-\frac{4}{3}+\frac{2}{6},\ 2-\frac{4}{3}-\frac{4}{6},\ 3-\frac{4}{3}+\frac{2}{6}\right)$$

$$= \left(-\frac{12}{6}, 0, \frac{12}{6}\right) = (-2,0,2)$$

$$\vec{U_3} = \frac{(-2,0,2)}{\sqrt{4+4}} = \left(-\frac{1}{\sqrt2}, 0, \frac{1}{\sqrt2}\right)$$

$$\|\vec{U_3}\| = \sqrt{\tfrac12 + \tfrac12} = 1$$
$$U_3 \cdot U_1 = 0$$
$$U_3 \cdot U_2 = 0$$

$\therefore$ Hence $\left\{\left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right)\left(-\frac{1}{\sqrt6}, \frac{2}{\sqrt6}, -\frac{1}{\sqrt6}\right)\left(-\frac{1}{\sqrt2}, 0, \frac{1}{\sqrt2}\right)\right\}$ is

a ~~orthogonal~~ basis.
orthonormal.

$\Longrightarrow$ Gram–Schmidt method:- What we did above detaily

$$\{a_1, a_2, a_3\} \longrightarrow \{U_1, U_2, U_3\}$$

Given ones $\qquad$ L.I $\qquad\qquad$ orthonormal finalans

$$U_1 = \frac{\vec{a_1}}{|\vec{a_1}|} \qquad U_1 = V_1 = \frac{\vec{a_1}}{|\vec{a_1}|}$$

$$U_2 = \frac{\vec{V_2}}{|\vec{V_2}|} \qquad V_2 = \vec{a_2} - (a_2 \cdot U_1)U_1$$

$$U_3 = \frac{\vec{V_3}}{|\vec{V_3}|} \qquad V_3 = \vec{a_3} - (\vec{a_3}\cdot\vec{U_1})\vec{U_1} - (\vec{a_3}\cdot\vec{U_2})\vec{U_2}$$