Q: Show that $r \wedge (p \vee q)$ is a valid conclusion from the premises $\underset{3}{p \vee q}$, $\underset{4}{q \to r}$, $\underset{(1)}{p \to m}$ and $\underset{2}{\neg m}$

1) $p \to m$ (Rule P)

2) $\neg m$ (Rule P) $\quad \because \neg q \wedge (p \to q) \Rightarrow \neg p$

3) $\neg p$ (Rule T) by (1),(2)

4) $p \vee q$ (Rule P) $\quad \because \neg p \wedge (p \vee q) \Rightarrow q$

5) q (Rule T) by (3),(4)

6) $q \to r$ (Rule P) $\quad \because p \wedge (p \to q) \Rightarrow q$.

7) r (Rule T) by (5),(6)

8) $r \wedge (p \vee q)$ (Rule T) by (4),(7).

Q: If A works hard, then either B or C will enjoy themselves. If B enjoys himself, then A will not work hard. If D enjoys himself, then C will not. Therefore, prove that if A works hard, D will not enjoy himself.

Let A : A works hard
B : B will enjoy himself.
C : C will enjoy himself
D : D will enjoy himself

To prove $A \to \neg D$ which follows from,

$\underset{(1)}{A \to (B \vee C)}$, $\underset{2}{B \to \neg A}$ and $D \to \neg C$

(1) $A \to (B \vee C)$ (Rule P)

$\because (p \to q) \Leftrightarrow (\neg q \to \neg p)$

2) $\neg(B \vee C) \to \neg A$ (Rule T) by (1).

3) $(\neg B \wedge \neg C) \to \neg A$ (Rule T) by(2).

$\because \neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$

$\because (p \wedge q) \to r \Leftrightarrow p \to (q \to r)$

4) $\neg B \to (\neg C \to \neg A)$ (Rule T) by(3)

✓ 5) $\neg B \to (A \to C)$ (Rule T)

6) $B \to \neg A$ (Rule P)

$\because (p \to q) \Leftrightarrow (\neg q \to \neg p)$

✓ 7) $A \to \neg B$ (Rule T)

$\because (p \to q) \wedge (q \to r) \Leftrightarrow (p \to r)$

8) $A \to (A \to C)$ (Rule T) by (7), (5)

9) $(A \wedge A) \to C$ (Rule T)

$\because (p \wedge q) \to r \Leftrightarrow p \to (q \to r)$

✓ 10) $A \to C$ (Rule T) by (9)

11) $D \to \neg C$ (Rule P)

$\because (p \to q) \Leftrightarrow (\neg q \to \neg p)$

✓ 12) $C \to \neg D$ (Rule T)

$\because (p \to q) \wedge (q \to r) \Leftrightarrow (p \to r)$

(13) $A \to \neg D$ (Rule T) by (10) & (12).

Predicate Calculus

A part of a declarative sentence describing the properties of an object or relation among objects is called a predicate.

Eg: Consider two propositions

Ram is a bachelor
Shyam is a bachelor

Both Ram and Shyam have the same property of being bachelor.

The part "is a bachelor" is called a predicate.

The two propositions can be replaced by a single proposition "x is a bachelor".

By replacing x by Ram, Shyam or by any other name we get many propositions.

In logic predicates can be obtained by removing any ~~nouns~~ nouns from a statement.

The predicate is denoted by capital letters and names of individuals or objects by small letters.

Let B denote the predicate "is a bachelor" then the sentence "x is a bachelor" can be written as $B(x)$ where 'x' is a predicate variable.

$B(x)$ is also called a propositional function, which becomes a statement when concrete values are submitted in place of 'x'.

A predicate requiring $m\ (>0)$ names is called an m-place predicate.

Eg: x is taller that y : $T(x, y)$ – the two place predicate

Universal and Existential Quantifiers:

Quantifiers are words that refer to quantifies such as "some" or "all" and indicate how frequently a certain statement is true.

The phrase "for all" ($\forall$) is called the universal quantifier.

Eg: All human beings are ~~mortat~~ mortal.

Note: $(\forall x)\, P(x)$ is true iff $P(x)$ is true $\forall x$ in U

$(\forall x)\, P(x)$ is false iff $P(x)$ is false for atleast one x in U

The phrase "there exists" ($\exists$) is called the Existential quantifier.

Eg: There exists x such that $x^2 = 5$

~~Thi~~

This can be written as $(\exists x)\, P(x)$, where $P(x): x^2 = 5$

$\exists x$ represents the following:

- there exists an x
- there is an x
- for some x
- there is at least one x

Note: $(\exists x) P(x)$ is true if $P(x)$ is true for at least one x in U

$(\exists x) P(x)$ is false if $P(x)$ is false for every x in U.

Example:

Let $D = \{1, 2, 3, 4, \ldots, 9\}$. Determine the truth values of each of the following statements.

(1) $(\forall x)\ x+4 < 15$ T

(2) $(\exists x)\ x+4 = 10$ T

(3) $(\forall x)\ x+4 \leq 10$ F

(4) $(\exists x)\ x+4 > 15$ F

Symbolize the Statement:

1) All men are moral

Let, $M(x)$: x is a man
$H(x)$: x is a moral

$$(\forall x)\left(M(x) \rightarrow H(x)\right)$$

2) An integer is either +ve or −ve

Let, $M(x)$: x is an integer
$N(x)$: x is eithe +ve or −ve

$$(\forall x)\left[M(x) \rightarrow N(x)\right]$$

(3) There exists a man

We can rewrite, there exists an x such that x is a man OR there is atleast one x such that x is a man.

$M(x)$: x is a man

$$(\exists x)M(x)$$

Nagation of quantified statements –

$$\neg(\forall x)P(x) \Leftrightarrow (\exists x)\neg P(x)$$

$$\neg(\exists x)P(x) \Leftrightarrow (\forall x)\neg P(x)$$

Example:

Nagate the statement

For all real numbers x, if $x > 3$ then $x^2 > 9$.

Ans: Let $P(x)$: $x > 3$

$Q(x)$: $x^2 > 9$

$\therefore (\forall x)(P(x) \rightarrow Q(x))$

Nagation is:

$$\neg(\forall x)(P(x) \rightarrow Q(x))$$

$$\Leftrightarrow (\exists x)\neg(P(x) \rightarrow Q(x))$$

$$\Leftrightarrow (\exists x)(P(x) \wedge \neg Q(x))$$

$\because \neg(p \rightarrow q) \Leftrightarrow (p \wedge \neg q)$

i.e, there exist a real number x such that $x > 3$ and $x^2 \leq 9$

Rules of Inference

The method of derivation involving predicate formulas uses the rules of inference given for the statement calculus and also certain additional rules which are required to deal with the formulas involving quantifiers.

(In addition to Rule P and Rule T)

Rule US (Universal Specification)

From $(\forall x)\, A(x)$, we can conclude $A(y)$.

i.e., $(\forall x)\, A(x) \Rightarrow A(y)$

Rule ES (Existential Specification)

From $(\exists x)\, A(x)$, we can conclude $A(y)$ provided that y is not free in any ~~premise~~ of the given premise and also not free in any prior step of the derivation.

$$(\exists x)\, A(x) \Rightarrow A(y)$$

Rule UG (Universal Generalization)

From $A(x)$, one can conclude $(\forall y) A(y)$ provided that x is not free in any of the given premises and provided that if x is free in a prior step which resulted from use of ES, then no variables introduced by that use of ES appear free in $A(x)$.

$$A(x) \Rightarrow (\forall y) A(y)$$

Rule EG (Existential Generalization)

From $A(x)$, one can conclude $(\exists y) A(y)$

i.e., $A(x) \Rightarrow (\exists y) A(y)$.

Rule US : $(\forall x) A(x) \Rightarrow A(y)$

Rule ES : $(\exists x) A(x) \Rightarrow A(y)$

Rule UG : $A(x) \Rightarrow (\forall y) A(y)$

Rule EG : $A(y) \Rightarrow (\exists y) A(y)$

Q: Show that $(\forall x)[H(x) \to M(x)] \wedge H(s) \Rightarrow M(s)$

1) $(\forall x)[H(x) \to M(x)]$ (Rule P)

2) $H(s) \to M(s)$ (Rule US) by (1)

3) $H(s)$ (Rule P)

4) $M(s)$ (Rule T) by (2)(3)

$\because p \wedge (p \to q) \Rightarrow q$

Q: Show that $(\forall x)[P(x) \to Q(x)] \wedge (\forall x)[Q(x) \to R(x)] \Rightarrow (\forall x)[P(x) \to R(x)]$

1) $(\forall x)[P(x) \to Q(x)]$ (P)

2) $P(y) \to Q(y)$ (US) by (1)

3) $(\forall x)[Q(x) \to R(x)]$ (P)

4) $Q(y) \to R(y)$ (US) by (3)

5) $P(y) \to R(y)$ (T) by (2),(4)

$(p \to q) \wedge (q \to r) \Rightarrow (p \to r)$

6) $(\forall x)[P(x) \to R(x)]$ (UG) by (5)

Q: S.T $(\exists x) M(x)$ follows logically from the premises $(\forall x)[H(x) \to M(x)]$ and $(\exists x) H(x)$
(1)

1) $(\exists x) H(x)$ (P)

2) $H(y)$ (ES) by (1)

3) $(\forall x)[H(x) \to M(x)]$ (P)

4) $H(y) \to M(y)$ (US) by (3)

$\because p \wedge (p \to q) \Rightarrow q$

5) $M(y)$ (T) by (2),(4)

6) $(\exists x) M(x)$ (EG) by (5)

Q: Show that from

a). $(\exists x)[F(x) \wedge S(x)] \rightarrow (\forall y)[M(y) \rightarrow W(y)]$ (2)

b). $(\exists y)[M(y) \wedge \neg W(y)]$ (1) the conclusion

$(\forall x)[F(x) \rightarrow \neg S(x)]$ follows.

1) $(\exists y)[M(y) \wedge \neg W(y)]$ (P)

2) $M(z) \wedge \neg W(z)$ (ES) by (1)

3) $\neg[M(z) \rightarrow W(z)]$ (T) by (2). $\because (p \wedge \neg q) \Leftrightarrow \neg(p \rightarrow q)$

4) $(\exists y)[\neg(M(y) \rightarrow W(y))]$ (EG) by (3)

5) $\neg(\forall y)[M(y) \rightarrow W(y)]$ (T) by (4) $(\exists y)\neg A(x) \Leftrightarrow \neg(\forall x) A(x)$

6) $(\exists x)[F(x) \wedge S(x)] \rightarrow (\forall y)[M(y) \rightarrow W(y)]$ (P)

7) $\neg(\exists x)[F(x) \wedge S(x)]$ (T) by (5),(6). $\neg q \wedge (p \rightarrow q) \Rightarrow \neg p$

8) $(\forall x)\neg[F(x) \wedge S(x)]$ (T) by (7)

9) $\neg[F(y) \wedge S(y)]$ (US) by (8)

10) $\underbrace{\neg F(y)}_{\neg p} \vee \underbrace{\neg S(y)}_{q}$ (T) by (9) $\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$

11) $F(y) \rightarrow \neg S(y)$ (T) by (10) $(p \rightarrow q) \Leftrightarrow \neg p \vee q$

12) $(\forall x)[F(x) \rightarrow \neg S(x)]$ (UG) by (11)

End Sem Q.P.

1A 3M
1B 3M
1C 4M

$\left.\begin{array}{l}3M\\3M\\4M\end{array}\right\}$ 10M

2A ⋮
2B
2C

⋮

5A ⋮
5B ⋮
5C

Dijkstra's Algorithm → 4M

Lexico | Reverse Lexico | Fike → 4M

CNF & DNF → 3M

Propositional & Predicate calculus → 3+3 = 6M

Chapter 1 → 14M (P & C)
Chapter 2 → 12M (Lattice)
Chapter 3 → 10M (Graph)
Chapter 4 → 8M (Group)
Chapter 5 → 6M (P&P)

$4 < \begin{array}{l}2M\\2M\end{array}$