Q. Solve the system, By Gauss-Jacobi's method and Gauss-Seidel method.

$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$
$$20x + y - 2z = 17$$

We can rewrite

$$20x + y - 2z = 17 \quad - (1)$$
$$3x + 20y - z = -18 \quad - (2)$$
$$2x - 3y + 20z = 25 \quad - (3)$$

$\therefore$ $|20| > |1| + |-2|$

$|20| > |3| + |-1|$

$|20| > |2| + |-3|$

$\therefore$ This satisfies diagonal dominance.

Master Eqns. (*)

$$\begin{cases} x = \frac{1}{20}(17 - y + 2z) \\ y = \frac{1}{20}(-18 - 3x + z) \\ z = \frac{1}{20}(25 - 2x + 3y) \end{cases}$$

Gauss – Jacobi's Method :-

Let $x^0 = y^0 = z^0 = 0$.

Iteration I :-

$$x^{(1)} = \frac{1}{20}\left(17 - y^{(0)} + 2z_0\right) = 0.85$$

$$y^{(1)} = \frac{1}{20}\left(-18 - 3x^{(0)} + z^{(0)}\right) = -0.9$$

$$z^{(2)} = \frac{1}{20}\left(25 - 2x^{(0)} + 3y^{(0)}\right) = 1.25$$

Iteration II :-

$$x^{(2)} = \frac{1}{20}\left(17 - y^{(1)} - 2z^{(1)}\right) = 1.020.$$

$$y^{(2)} = \frac{1}{20}\left(-18 - 3x^{(1)} + z^{(1)}\right) = -0.965$$

$$z^{(2)} = \frac{1}{20}\left(25 - 2x^{(1)} + 3y^{(1)}\right) = 1.03$$

Iteration 3 :-

$$x^{(3)} = \frac{1}{20}\left(17 - y^{(2)} - 2z^{(2)}\right) = 1.00125$$

$$y^{(3)} = \frac{1}{20}\left(-18 - 3x^{(1)} + z^{(2)}\right) = -1.0015$$

$$z^{(3)} = \frac{1}{20}\left(+25 + 3y^{(2)} - 2x^{(2)}\right) = 1.00325$$

Iteration IV :-

$$x^{(4)} = \frac{1}{20}\left(17 - y^{(3)} + 2z^{(3)}\right) = 1.0004$$

$$y^{4} = \frac{1}{20}\left(-18 - 3x^{(3)} + z^{(3)}\right) = -1.000025$$

$$z^{4} = \frac{1}{20}\left(25 - 2x^{(3)} + 3y^{(3)}\right) = 0.99934$$

Iteration V :-

$$x^{(5)} = \frac{1}{20}\left(17 - y^{(4)} + 2z^{(4)}\right) = 0.99996625$$

$$y^{(5)} = \frac{1}{20}\left(-18 - 3x^{(4)} + z^{(4)}\right) = -1.0000775$$

$$z^{(5)} = \frac{1}{20}\left(25 - 2x^{4} + 3y^{4}\right) = 0.9995625$$

Iteration VI :-

$$x^{(6)} = \frac{1}{20}\left(17 - y^{(5)} + 2z^{(5)}\right) = 0.9999995$$

$$y^{(6)} = \frac{1}{20}\left(-18 - 3x^{(5)} + z^{(5)}\right) = 0.99999725$$

$$z^{(6)} = \frac{1}{20}\left(+25 - 2x^{(5)} + 3y^{(5)}\right) = 1.000022125$$

$\therefore$ From Iterations (5) & (6)

the values of x, y and z are same upto 3 decimal places.

for upto 3 decimal places.

$\therefore$ Approx. soln is

$$\boxed{\begin{aligned} x &= \overset{0.999}{0.9999995} \approx 1 \\ y &= -\overset{0.999}{0.9999997} \approx -1 \\ z &= \overset{1.000}{1.000022} \approx 1. \end{aligned}}$$

- Gauss-Seidal Method:-

Let $y^{(0)} = z^{(0)} = 0$.

Iteration (1)

$$x^{(1)} = \frac{1}{20}\left(17 - y^{(0)} + 2z^{(0)}\right) = 0.85$$

$$y^{(1)} = \frac{1}{20}\left(-18 - 3x^{(1)} + z^{(0)}\right) = -1.0275$$

$$z^{(1)} = \frac{1}{20}\left(25 - 2x^{(1)} + 3y^{(1)}\right) = 1.010875.$$

Iteration (2)

$$x^{(2)} = \frac{1}{20}\left(17 - y^{(1)} + 2z^{(1)}\right) = 1.0024625$$

$$y^{(2)} = \frac{1}{20}\left(-18 - 3x^{(2)} + z^{(1)}\right) = -0.9982563$$

$$z^{(3)} = \frac{1}{20}\left(25 - 2x^{(2)} + 3y^{(2)}\right) = 0.9997799$$

Iteration ③

$$x^{(3)} = \frac{1}{20}\left(17 - y^{(2)} + 2z^{(2)}\right) = 0.999969271$$

$$y^{(3)} = \frac{1}{20}\left(-18 - 3x^{(3)} + z^{(2)}\right) = -1.000006395$$

$$z^{(3)} = \frac{1}{20}\left(25 - 2(x)^{(3)} + 3y^{(3)}\right) = 0.99991007$$

Iteration ④

$$x^{(4)} = \frac{1}{20}\left(17 - y^{(3)} + 2z^{(3)}\right) = 0.9999913$$

$$y^{(4)} = \frac{1}{20}\left(-18 - 3x^{(4)} + z^{(3)}\right) = -1.0000032$$

$$z^{(4)} = \frac{1}{20}\left(+25 - 2x^{(4)} + 3y^{(4)}\right) = 1.00000039.$$

$\therefore$ The values in Iteration ③ & ④ are same upto 3 decimal places.

$\therefore$

$$\boxed{\begin{aligned} x &= 0.999 \\ y &= -1.000 \\ z &= 0.99991007 \approx 1.000 \end{aligned}}$$

Q. Solve the ~~of~~ system of equations (stop after 4 Iterations)

$$5x - y + z = 10$$
$$2x + 4y = 12$$
$$x + y + 5z = -1.$$

a) Gauss - Jacobi's method (correct to two decimal places)
b) Gauss- Seidal method (correct to three decimal place).

∵ The given system of ~~eq~~ eqn satisfies diagonal dominance condition.

$$\therefore \quad x = \frac{1}{5}(10 + y - z)$$

$$y = \frac{1}{4}(12 - 2x)$$

$$z = +\frac{1}{5}(-1 - x - y)$$

Gauss - Jacobi method

Let $x^{(0)} = y^{(0)} = z^{(0)} = 0.$ be the initial approximation

$$x^{(1)} = \frac{1}{5}(10 + y^{(0)} - z^{(0)}) = 2.$$

$$y^{(1)} = \frac{1}{4}(12 - 2x^{(0)}) = 3$$

$$z^{(1)} = \frac{1}{5}(-1 - x^{(0)} - y^{(0)}) = -0.2.$$

Iteration 2

$$x^{(2)} = \frac{1}{5}\left(10 + y^{(1)} - z^{(1)}\right) = 2.64$$

$$y^{(2)} = \frac{1}{4}\left(12 - 2x^{(1)}\right) = 2.$$

$$z^{(2)} = \frac{1}{5}\left(-1 - x^{(1)} - y^{(1)}\right) = -1.2.$$

Iteration 3

$$x^{(3)} = \frac{1}{5}\left(10 + y^{(2)} - z^{(2)}\right) = 2.64$$

$$y^{(3)} = \frac{1}{4}\left(12 - 2x^{(2)}\right) = 1.68$$

$$z^{(3)} = \frac{1}{5}\left(-1 - x^{(2)} - y^{(2)}\right) = -1.128.$$

Iteration 4 :-

$$x^{(4)} = \frac{1}{5}\left(10 + y^{(3)} - z^{(3)}\right) = 2.5616$$

$$y^{(4)} = \frac{1}{4}\left(12 - 2x^{(3)}\right) = 1.68.$$

$$z^{(4)} = \frac{1}{5}\left(-1 - x^{(3)} - y^{(3)}\right) = -1.064$$

- After 4 Iterations the approx soln correct upto 2 decimal places are :-

$$\rightarrow \boxed{x = 2.56 \quad , \quad y = 1.68 \quad , \quad z = -1.06}$$

Gauss - Seidal Method :-

Let $y^{(0)} = z^{(0)} = 0$ be the initial approximation.

Iteration 1 :-

$$x^{(1)} = \frac{1}{5}\left(10 + y^{(0)} - z^{(0)}\right) = 2.$$

$$y^{(1)} = \frac{1}{4}\left(12 - 2x^{(1)}\right) = 2$$

$$z^{(1)} = \frac{1}{5}\left(-1 - x^{(1)} - y^{(1)}\right) = -1$$

Iteration 2:-

$$x^{(2)} = \frac{1}{5}\left(10 + y^{(1)} - z^{(1)}\right) = 2.6$$

$$y^{(2)} = \frac{1}{4}\left(12 - 2x^{(2)}\right) = 1.7$$

$$z^{(2)} = \frac{1}{5}\left(-1 - x^{(2)} - y^{(2)}\right) = -1.06.$$

Iteration ③

$$x^{(3)} = \frac{1}{5}\left(10 + y^{(2)} - z^{(2)}\right) = 2.552$$

$$y^{(3)} = \frac{1}{4}\left(12 - 2x^{(3)}\right) = 1.724$$

$$z^{(3)} = \frac{1}{5}\left(-1 - y^{(3)} - x^{(3)}\right) = -1.0552$$

Iteration ④ :-

$$x^{(4)} = \frac{1}{5}\left(10 + y^{(3)} - z^{(3)}\right) = \cancel{2.552}\ 2.55584$$

$$y^{(4)} = \frac{1}{4}\left(12 - 2x^{(4)}\right) = 1.72208$$

$$z^{(4)} = \frac{1}{5}\left(-1 - y^{(4)} - x^{(4)}\right) = -1.055584$$

→ After 4 iteration the approx soln upto 3 decimal places is

$$\boxed{\begin{aligned} x &\approx 2.556 \\ y &\approx 1.722 \\ z &\approx -1.056 \end{aligned}}$$

2. <u>Eigen Values and eigen Vectors</u> :-

→ ~~term~~ Consider a matrix $A = \begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix}$. Let $e_1 = (1 \ 0)$ and $e_2 = (0 \ 1)$ be two 2-dimensional vectors.

$$\therefore \ \underline{e_1} = (1 \ 0) = \hat{i} + 0\hat{j}$$
$$\boxed{= \hat{i}}$$

$$\underline{e_2} = (0 \ 1) = 0\hat{i} + \hat{j}$$
$$\boxed{= \hat{j}}$$

$$Ae_1 = \begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = (1, 2).$$

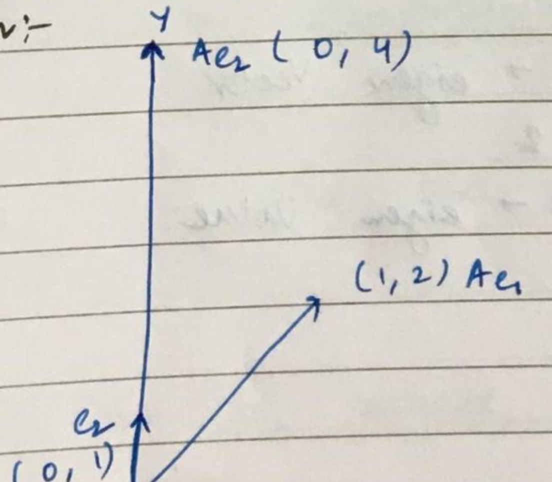
$\neq$ any multiple of e_1

$$Ae_2 = \begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} = (0, 4)$$
$$= 4e_2$$

e_2 → eigen Vector ($\because Ae_2 \parallel e_2$).

4 → eigen Value

→ <u>Graphical Interpretation</u>:-



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• Eigen Vector → $\boxed{\text{must be non-zero.}}$ ✓✓
& → $\boxed{AX = \lambda X}$ ✓✓

→ Let A be a square matrix then eigen vectors of A are the non-zero vectors X, that after being multiplied by the matrix A, the two vectors X and AX remain parallel.

ie. X and AX are parallel then we can write

$$AX = \lambda X \quad \text{for some } \lambda.$$

(like $Ae_2 = 4e_2$ from prev. example)

Q. Consider the matrix $A = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$

For $X = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

then $AX = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$= 2X.$

$\therefore \boxed{AX = \lambda X}$

∴ ie. X and AX are parallel.

∴ X → eigen vector
&
2 → eigen value

Q. Consider the matrices $B = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$ for $V = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Then $BV = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ +4 \end{pmatrix} = -2 \begin{pmatrix} +1 \\ -2 \end{pmatrix}$

$= -2V.$

$\therefore \boxed{BV = \lambda V}$ $(\lambda = -2)$

eigen vector → eigen value.

$\therefore$ BV and V are parallel.

→ **Working Rules:-**

→ Let A be a square matrix of order n.

Then corresponding to the given matrix A if we calculate;

→ Let λ be a scalar

→ Then → $\boxed{(A - \lambda I)_{n \times n}}$ ✓ is called the characteristic matrix.

$\therefore \boxed{|A - \lambda I| = 0}$ is called the characteristic eqn of A.

→ $\boxed{|A - \lambda I| = 0.}$ is eqn of degree n in λ.

→ Roots of the equation $\boxed{|A - \lambda I| = 0}$ is called the eigen values of A.

→ Let $\lambda_1, \lambda_2, \ldots \lambda_n$ be the eigen value

∴ For each λ find a non-zero vector X such that

→ $\boxed{AX = \lambda_i X.}$ (where $i = 1, 2 \ldots$

∴

$AX = \lambda_i (IX)$ for $i = 1, 2, \ldots n$.

identity matrix.

∴ $AX - (\lambda_i I) X = 0$.

$\boxed{(A - \lambda_i I) X = 0.}$ → system of non homogeneous equations.

∴ Solving this eqn. and putting the values of λ_i (i.e. $\lambda_1, \lambda_2 \ldots$ we get n Xs i.e.

i.e. X_1 → eigen vector for λ_1 value (eigen)

X_2 → eigen vector for λ_2 eigen value and so on.

- **Properties of eigen Values:-**

Let A be an $n \times n$ matrix. Assume that A has n distinct eigen values say $\lambda_1, \lambda_2, \ldots \lambda_n$ then :-

- The eigen values of A^T are $\lambda_1, \lambda_2 \ldots \lambda_n$.

- The eigen values of A^{-1} (if it exists) are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots \frac{1}{\lambda_n}$

- The eigen values of the matrix $A - \alpha I$ are $\lambda_1 - a, \lambda_2 - a, \ldots \lambda_n - a$.

- for any non negative integer k, the eigen values of A^k are $\cancel{\lambda_1^{k_1}}, \cancel{\lambda_2^{k_2}}, \cancel{\lambda_3^{k_3}} \ldots \cancel{\lambda_n^{k_n}}$

$\lambda_1^k, \lambda_2^k, \lambda_3^k \ldots \lambda_n^k$

i.e. If $AX = \lambda X$

then $A^2 X = \lambda^2 X$.