Ordering of Permutations

1 Lexicographical Order

To find the k^{th}permutation of n marks (letter, symbols), say $a_1, a_2, \ldots, a_n$, when the permutations are sorted lexicographically, proceed as given below:

1. Write $k - 1$ in the form

$$k - 1 = c_{n-1}(n-1)! + c_{n-2}(n-2)! + \cdots + c_1 1!$$

   where each integer c_i has the maximum possible value, $0 \leq c_i \leq i$. In other words, we divide $k-1$ by $(n-1)!$, and take c_{n-1} as the quotient; then divide the remainder by $(n-2)!$ and take c_{n-2} as the quotient; and so on. This gives us a sequence $c_{n-1}c_{n-2}\cdots c_1$.

   Example 1. To compute the 35^{th}permutation of the five marks $1, 2, 3, 4, 5$, we note that $k = 35$ and $n = 5$. Now

$$35 - 1 = 34 = \underline{1} \times 4! + \underline{1} \times 3! + \underline{2} \times 2! + \underline{0} \times 1!$$

   so that the sequence is $\boxed{1120}$.

2. Next, the sequence $c_{n-1}\cdots c_1$ is treated as a sequence of array indices (the range being 0 to $n-1$). Then the k^{th}permutation is constructed in the following manner. Start with the array of marks $1, 2, \ldots, n$, and pick the element indexed by c_{n-1} as the first element of the permutation. Remove this element from the array to get a new array, and also remove c_{n-1} from the sequence of indices to get the new sequence $c_{n-2}\cdots c_1$. Now continue until the sequence of indices is exhausted. At this point, exactly one mark will remain in the array, and write this down as the last element of the permutation.

   Example 2. Continuing from the previous example, to compute the 35^{th} permutation of the five marks $1, 2, 3, 4, 5$, we have already obtained the sequence

1120. Now, consider the array of marks 12345.
The first index is 1 (the first element of 1120), and the element of the array indexed by this is 2. Thus, the permutation is 2____.
The new array is 1345, and the new sequence of indices is 120. Now, the element indexed by the first index 1 is 3. Thus the permutation is 23___.
The new array is 145, and the indices are 20. The element indexed by 2 is 5, so the permutation is 235__.
The array is now 14, and the only index remaining is 0. The corresponding element is 1, and the permutation is 2351_.
The only remaining element 4 is the last element of the permutation, so the complete permutation is $\boxed{23514}$.

Solved Problems

1. Find the 23^{rd} permutation of the four marks $1, 2, 3, 4$ in lexicographical order.

$$23 - 1 = 22 = \underline{3} \times 3! + \underline{2} \times 2! + \underline{0} \times 1! \longrightarrow 320$$

Index	Marks	Mark
$\underline{3}20$	$123\underline{4}$	$\longrightarrow 4$
$\underline{2}0$	$12\underline{3}$	$\longrightarrow 3$
$\underline{0}$	$\underline{1}2$	$\longrightarrow 1$
	$\underline{2}$	$\longrightarrow 2$

Thus, the 23^{rd} permutation of $1, 2, 3, 4$ in lexicographical order is $\boxed{4312}$.

2. Find the 18^{th} permutation of the marks a, b, c, d in lexicographical order.

$$18 - 1 = 17 = \underline{2} \times 3! + \underline{2} \times 2! + \underline{1} \times 1! \longrightarrow 221.$$

$\underline{2}21$	$ab\underline{c}d$	$\longrightarrow c$
$\underline{2}1$	$ab\underline{d}$	$\longrightarrow d$
$\underline{1}$	$a\underline{b}$	$\longrightarrow b$
	$\underline{a}$	$\longrightarrow a$

Thus, the 18^{th} permutation of the marks a, b, c, d in lexicographical order is $\boxed{cdba}$.

3. Find the 50^{th} permutation of the marks $0, 1, 2, 3, 4$ in lexicographical order.

$$50 - 1 = 49 = \underline{2} \times 4! + \underline{0} \times 3! + \underline{0} \times 3! + \underline{1} \times 1! \longrightarrow 2001$$

$\underline{2}001$	$01\underline{2}34$	$\to 2$
$\underline{0}01$	$\underline{0}134$	$\to 0$
$\underline{0}1$	$\underline{1}34$	$\to 1$
$\underline{1}$	$3\underline{4}$	$\to 4$
	$\underline{3}$	$\to 3$

Thus, the 50^{th}permutation of $0, 1, 2, 3, 4$ in lexicographical order is $\boxed{20143}$.

4. Find the 268^{th}permutation of `LISTEN` in lexicographical order.

$$268 - 1 = 267 = \underline{2} \times 5! + \underline{1} \times 4! + \underline{0} \times 3! + \underline{1} \times 2! + \underline{1} \times 1! \longrightarrow 21011$$

$\underline{2}1011$	$\mathtt{LI\underline{S}TEN}$	$\to \mathtt{S}$
$\underline{1}011$	$\mathtt{L\underline{I}TEN}$	$\to \mathtt{I}$
$\underline{0}11$	$\mathtt{\underline{L}TEN}$	$\to \mathtt{L}$
$\underline{1}1$	$\mathtt{T\underline{E}N}$	$\to \mathtt{E}$
$\underline{1}$	$\mathtt{T\underline{N}}$	$\to \mathtt{N}$
	$\mathtt{\underline{T}}$	$\to \mathtt{T}$

Thus, the 268^{th}permutation of `LISTEN` in lexicographical order is $\boxed{\texttt{SILENT}}$.

2 Reverse Lexicographical Order

To obtain the k^{th}permutation of n marks $a_1, a_2, \ldots, a_n$ in reverse lexicographical order, first reverse the order of marks to get $a_n, a_{n-1}, \ldots, a_1$, compute the k^{th}permutation of these marks in lexicographical order, and then reverse the resulting permutation.

Solved Problems

1. Find the 50^{th}permutation of the five marks $0, 1, 2, 3, 4$ in reverse lexicographical order.

$$50 - 1 = 49 = \underline{2} \times 4! + \underline{0} \times 3! + \underline{0} \times 3! + \underline{1} \times 1! \longrightarrow 2001$$

$\underline{2}001$	$43\underline{2}10$	$\to 2$
$\underline{0}01$	$\underline{4}310$	$\to 4$
$\underline{0}1$	$\underline{3}10$	$\to 3$
$\underline{1}$	$1\underline{0}$	$\to 0$
	$\underline{1}$	$\to 1$

Thus, the 50^{th}permutation of $0, 1, 2, 3, 4$ in reverse lexicographical order is $\boxed{10342}$.

2. Find the 100^{th} permutation of the marks $1,2,3,4,5$ in reverse lexicographical order.

$$100-1=99=\underline{4}\times 4!+\underline{0}\times 3!+\underline{2}\times 1!+\underline{1}\times 1!\longrightarrow 4011$$

$\underline{4}011$	$5432\underline{1}$	$\longrightarrow 1$
$\underline{0}11$	$\underline{5}432$	$\longrightarrow 5$
$\underline{1}1$	$4\underline{3}2$	$\longrightarrow 3$
$\underline{1}$	$4\underline{2}$	$\longrightarrow 2$
	4	$\longrightarrow 4$

Thus, the 100^{th} permutation of $1,2,3,4,5$ in reverse lexicographical order is $\boxed{42351}$.

3 Fike's Order

To obtain the k^{th} permutation of n marks $a_1,a_2,\dots,a_n$ in Fike's order, proceed as follows.

1. First, we must generate Fike's sequence, using which the permutation is to be computed. To find the sequence, first write $k-1$ in the form

$$k-1=c_1\times n(n-1)\cdots 3+c_2\times n(n-1)\cdots 4+\cdots+c_{n-2}\times n+c_{n-1}\times 1.$$

That is, the place values are $\frac{n!}{2!},\ \frac{n!}{3!},\ \dots,\ \frac{n!}{n!}=1$. Now, **subtract this sequence** $c_1c_2\cdots c_{n-1}$ from the sequence $12\cdots(n-1)$ to get the sequence $d_1d_2\cdots d_{n-1}$. That is, $d_i=i-c_i$, $i=1,\dots,n-1$. This is Fike's sequence.

Example 3. To compute the 65^{th} permutation of the five marks $1,2,3,4,5$ in Fike's order, we note that $k=65$ and $n=5$. First, compute the place values $\dfrac{n!}{2!},\dots,\dfrac{n!}{n!}$. For $n=5$, these are 60, 20, 5, 1. Then,

$$65-1=64=\underline{1}\times 60+\underline{0}\times 20+\underline{0}\times 5+\underline{4}\times 1\longrightarrow 1004.$$

Now, Fike's sequence is

$$\begin{array}{l} 1234\ - \\ 1004\ = \\ \hline 0230. \end{array}$$

2. Using the Fike's sequence, the permutation is generated from the initial permutation $12\cdots n$ by a sequence of interchanges, in the following manner. For the sequence $d_1d_2\cdots d_{n-1}$, first the element of the permutation index 1 is interchanged

with the element at index d_1. Similarly, at each stage, the element at index i is interchanged with the element at index d_i, until the sequence is exhausted. The resulting permutation is the k^{th}permutation in Fike's order.

Example 4. For the sequence 0230 obtained in the previous example, we start with the original arrangement of the marks: 12345. Now, the element at index 1 is 2, and the element at index $d_1 = 0$ is 1. Therefore, interchanging 2 and 1, we get 21345. Next, the element at index 2 is 3, and the element at index $d_2 = 2$ is 3 (the same). "Interchanging" these, we get 21345 (i.e., the permutation remains the same). The element at index 3 is 4, and that at index $d_3 = 3$ is again the same, so once more, the permutation is 21345. Lastly, the element at index 4 is 5, and that at index $d_4 = 0$ is 2. Interchanging these, we get 51342. Thus, the 65^{th}permutation of $1, 2, 3, 4, 5$ in Fike's order is $\boxed{51342}$. We can write this succinctly as given below. First, Fike's sequence is written as a column. Then we write the original permutation in the first row, and underline the element at index 1, which is to be interchanged with the element at index d_1.

$$\begin{array}{ll}
0 & 1\underline{2}345 \\
2 & \\
3 & \\
0 &
\end{array}$$

The interchange is performed and the result is written in the next row, and this process is repeated until the sequence is exhausted.

$$\begin{array}{lll}
0 & 1\underline{2}345 & \longrightarrow \\
2 & 21\underline{3}45 & \longrightarrow \\
3 & 213\underline{4}5 & \longrightarrow \\
0 & 2134\underline{5} & \longrightarrow \\
 & \boxed{51342} &
\end{array}$$

Solved Problems

1. Obtain the 40^{th}permutation of the five marks $0, 1, 2, 3, 4$ in Fike's order. Since $n = 5$, the place values are 60, 20, 5, 1.

$$40 - 1 = 39 = \underline{0} \times 60 + \underline{1} \times 20 + \underline{3} \times 5 + \underline{4} \times 1 \longrightarrow 0134$$

Then Fike's sequence is

$$\begin{array}{l}
1234\ - \\
0134\ = \\
\hline
1100.
\end{array}$$

The permutation is then obtained as follows.

$$\begin{array}{ccc} 1 & 0\underline{1}234 & \longrightarrow \\ 1 & 01\underline{2}34 & \longrightarrow \\ 0 & 021\underline{3}4 & \longrightarrow \\ 0 & 3210\underline{4} & \longrightarrow \\ & \boxed{42103} & \end{array}$$

2. Obtain the 50^{th} permutation of the five marks $1, 2, 3, 4$ in Fike's order. Since $n = 5$, the place values are 60, 20, 5, 1.

$$50 - 1 = 49 = \underline{0} \times 60 + \underline{2} \times 20 + \underline{1} \times 5 + \underline{4} \times 1 \longrightarrow 0214$$

Then Fike's sequence is

$$\begin{array}{l} 1234\, - \\ 0214\, = \\ \hline 1020. \end{array}$$

The permutation is then obtained as follows.

$$\begin{array}{ccc} 1 & 1\underline{2}345 & \longrightarrow \\ 0 & 12\underline{3}45 & \longrightarrow \\ 2 & 321\underline{4}5 & \longrightarrow \\ 0 & 3241\underline{5} & \longrightarrow \\ & \boxed{52413} & \end{array}$$

3. Obtain the 111^{th} permutation of the five marks $1, 2, 3, 4, 5$ in Fike's order.

$$111 - 1 = 110 = \underline{1} \times 60 + \underline{2} \times 20 + \underline{2} \times 5 + \underline{0} \times 1 \longrightarrow 1220$$

Then Fike's sequence is

$$\begin{array}{l} 1234\, - \\ 1220\, = \\ \hline 0014. \end{array}$$

The permutation is then obtained as follows.

$$\begin{array}{ccc} 0 & 1\underline{2}345 & \longrightarrow \\ 0 & 21\underline{3}45 & \longrightarrow \\ 1 & 312\underline{4}5 & \longrightarrow \\ 4 & 3421\underline{5} & \longrightarrow \\ & \boxed{34215} & \end{array}$$

Remarks

1. The sequence used in the case of each of these three orderings is completely determined by the values of n, the total number of marks, and k, the number of the permutation to be found. It does not depend on the values of the marks at all. Indeed, the marks have no value, even when they are $1, 2, 3, 4, 5$, for example. The marks could also be a, b, c, d, e. They merely represent objects being permuted. In particular, the marks being $1, 2, 3, 4, 5$, or $0, 1, 2, 3, 4$ makes no difference to the sequence (of indices) found.

2. In the case of lexicographical (or reverse lexicographical) order, the sequence used is of the form $c_{n-1}c_{n-2}\cdots c_1$, where no c_i exceeds i. Thus, for example if $n = 5$, 1322 is **not** a valid sequence, since $c_1 = 2 > 1$. Also note that the number of terms in the sequence is always $n - 1$. If the first term of the sequence is 0, then this cannot be dropped, as the sequence is not a simple number, but rather a collection of indices in a particular order.

3. In the case of Fike's order, the first sequence obtained (the one that is used for computing Fike's sequence) must have i^{th}term not exceeding i, for any i. For instance, when $n = 5$, 1232 is a valid sequence, but not 1322, since the second term is $3 > 2$. Note that we subtract the sequence from 1234 term-wise. Once again, this is a sequence and not an actual number, so that there is no carrying involved in the subtraction. This is always true because any valid sequence will be term-wise smaller than or equal to the sequence $1234\cdots(n-1)$.

Exercises

1. Find the 119^{th}permutation of $1, 2, 3, 4, 5$ in lexicographical, reverse lexicographical, and Fike's orders.

2. Find the 123_{rd} permutation of $1, 2, 3, 4, 5, 6$ in lexicographical, reverse lexicographical, and Fike's orders.

3. Find the k^{th}permutation of $1, 2, \ldots, n$ in lexicographical and reverse lexicographical orders, where $k = m! + 1$ for some $m < n$.