Graph Theory

References:

- Graph Theory by Frank Harary
- Graph theory with Application to computer science by Narasingh Deo

Definition: A graph G = (V, E) consists of a nonempty set $V = V(G)$ whose elements are called vertices (or points, or nodes) of G and a set $E(G)$ of unordered pairs of distinct elements of V(G), whose elements are called edges (or lines, or arc) of G.

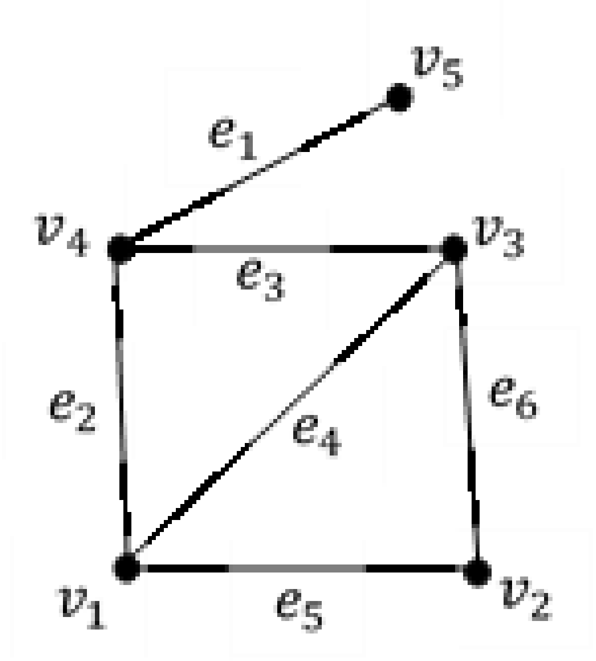

Fig.1 Graph G

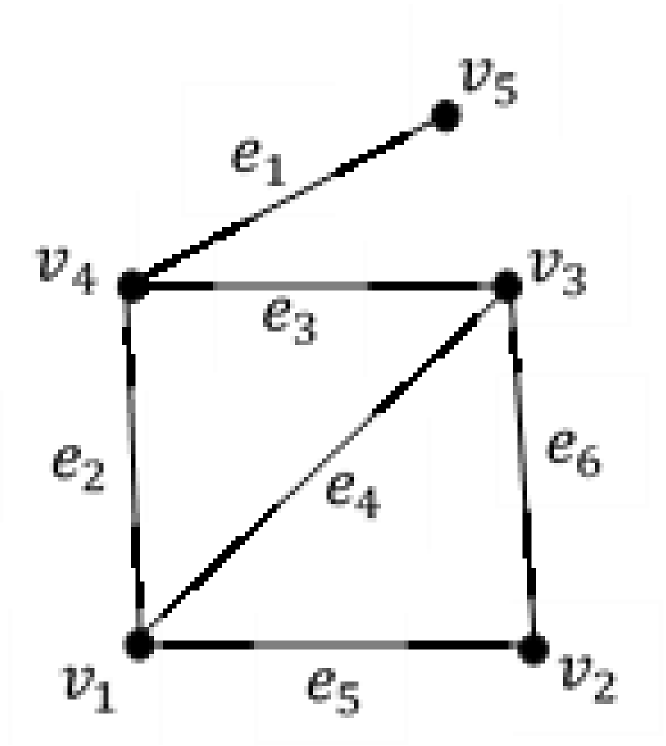

Fig.1 Graph G

$$V(G) = \{v_1, v_2, v_3, v_4, v_5\}$$

$$E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

$$|V(G)| = 5 \; and \; |E(G)| = 6$$

Two **vertices** in a graph G are said to be ***adjacent*** if there is an edge between them.

Example: In fig.1, v_1 is adjacent with v_2, i. e., $v_1 \sim v_2$

v_1 is adjacent with v_3, i. e., $v_1 \sim v_3$ etc.

Two **edges** are said to be ***adjacent*** if they have a vertex in common.

Example: In fig.1, e_1 and e_3 are adjacent, e_1 and e_2 are adjacent, etc.

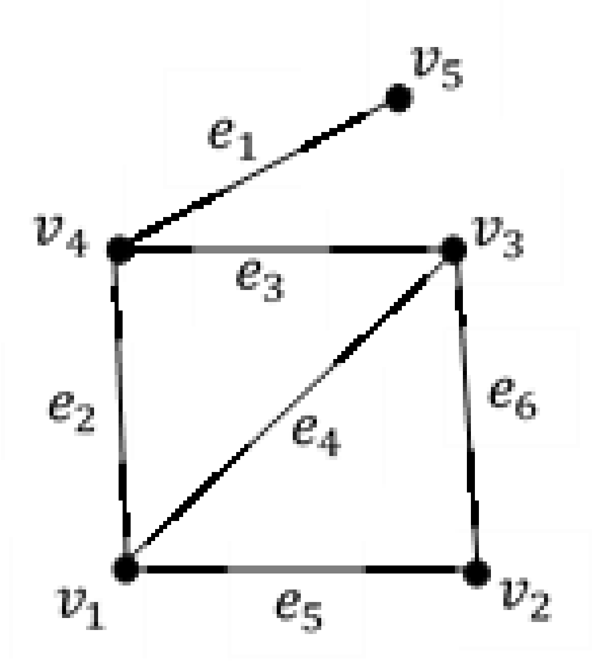

Fig.1 Graph G

In a definition of a graph G = (V, E), it is possible for the edge set E to be empty.

Such a graph without any edges, is called a ***null graph*.**

If a and b are two vertices, and e is the edge between a and b in a graph G,

then we say that the edge e is incident with the vertices a and b.

A graph with 'p' vertices and 'q' edges is called a (p, q) graph.

Sub graph: A sub graph H of G is a graph having all of its vertices and edges in G.

If G_1 is a sub graph of G, then G is a **super graph** of G_1.

A ***spanning sub graph*** is a sub graph containing all the vertices of G. For any set S of vertices of G, the ***induced sub graph*** $\langle S \rangle$ is the maximal subgraph of G with vertex set S. Thus two vertices of S are adjacent in $\langle S \rangle$ if and only if they are adjacent in G.

Example: In Fig.2. G_1 is a induced sub graph of G but G_2 is not; G_2 is a spanning sub graph of G but G_1 is not.

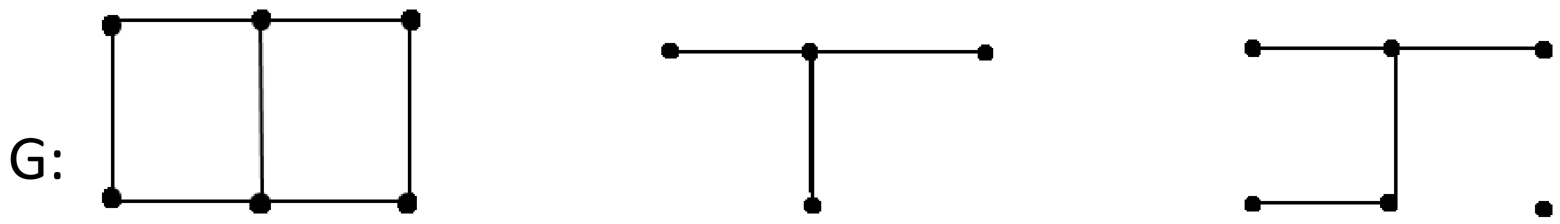

Fig.2. A graph and two sub graphs

The **removal of a vertex** $\boldsymbol{v}$ from a graph G results in that sub graph $G - v$ of G consisting of all vertices of G except v and all edges not incident with v. Thus $G - v$ is the maximal sub graph of G not containing v.

Removal of an edge $\boldsymbol{e}$ from a graph G results in that sub graph $G - e$ of G containing all edges of G except e. Thus $G - e$ is the maximal sub graph of G not containing e.

If two vertices u and v are not adjacent in G, the addition of edge uv results in the minimal super graph of G containing the edge uv.

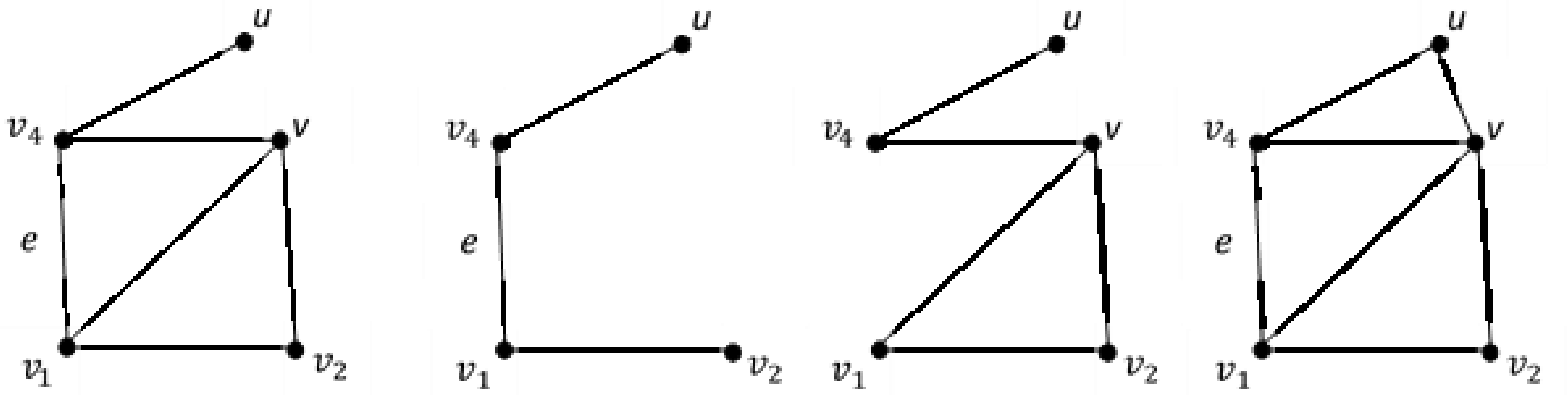

Fig. 3 Graphs G, $G - v$, $G - e$ and $G + uv$

Isomorphic graph: Two graphs G and H are isomorphic if there exists a one-to-one correspondence between their vertex sets which preserves adjacency

Example:

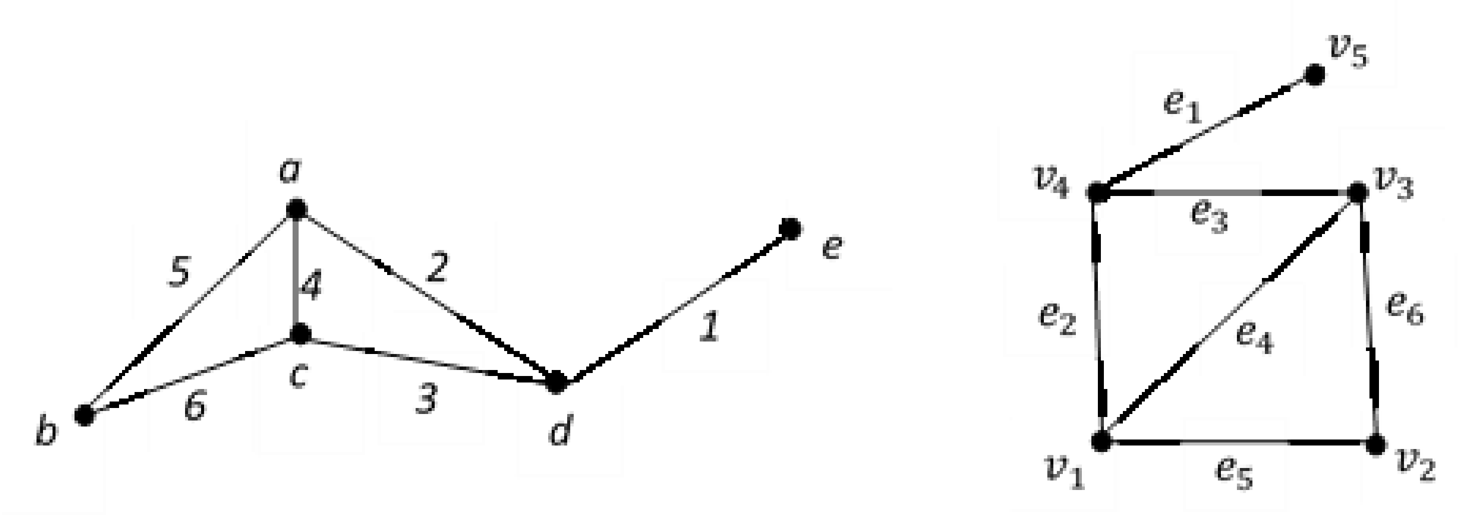

Fig.4. Isomorphic graphs

The correspondence between the two graphs in Fig.4 is as follows:

The vertices *a, b, c, d,* and *e* correspond to v_1, v_2, v_3, v_4, and v_5, respectively. The edges 1, 2, 3, 4. 5 and 6 correspond to e_1, e_2, e_3, e_4, e_5, and e_6, respectively.

Walk: A walk of a graph G is an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n$ beginning and ending with vertices, in which each edge is incident with the two vertices immediately preceding and following it.

A walk is ***closed*** if $v_0 = v_n$ and is ***open*** otherwise.

It is a ***trail*** if all the edges are distinct and a ***path*** if all the vertices and edges are distinct.

If the walk is closed, then it is a ***cycle*** provided its *n* vertices are distinct and $n \geq 3$. A cycle with *n* vertices denoted by C_n.

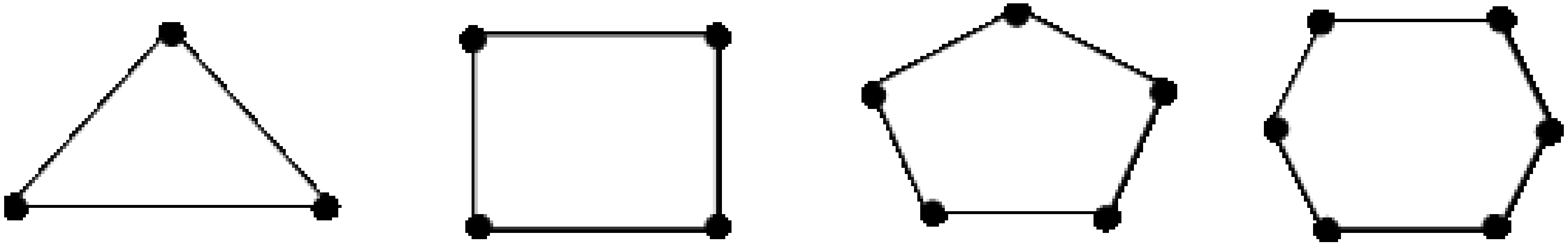

Fig. 5 Cycles C_3, C_4, C_5 and C_6

The length of a walk $v_0, v_1, v_2, \dots, v_n$ is n, the number of occurrence of edges in it.

Distance between two vertices: The distance *d*(*u, v*) between two vertices *u* and *v* in G is the length of the shortest path joining them, if any; otherwise d(*u, v*) = ∞.

In a connected graph G,

$d(u, v) \geq 0$ with $d(u, v) = 0$ if and only if $u = v$.

$$d(u, v) = d(v, u)$$

$$d(u, v) + d(v, w) \geq d(u, w)$$

A shortest $u - v$ path is called a **geodesic.**

Eccentricities: The eccentricity $e(v)$ of a vertex v in a connected graph G is maximum of $d(u, v)$ for all u in G.

The radius r(G) is the minimum eccentricity of the vertices of G.

The maximum eccentricity is the **diameter.** A vertex v is a central vertex if $e(v) = r(G)$, and the center of G is the set of all central vertices.

The ***girth*** of a graph G, denoted g(G), is the length of a shortest cycle in G; the ***circumference*** c(G) the length of any longest cycle.

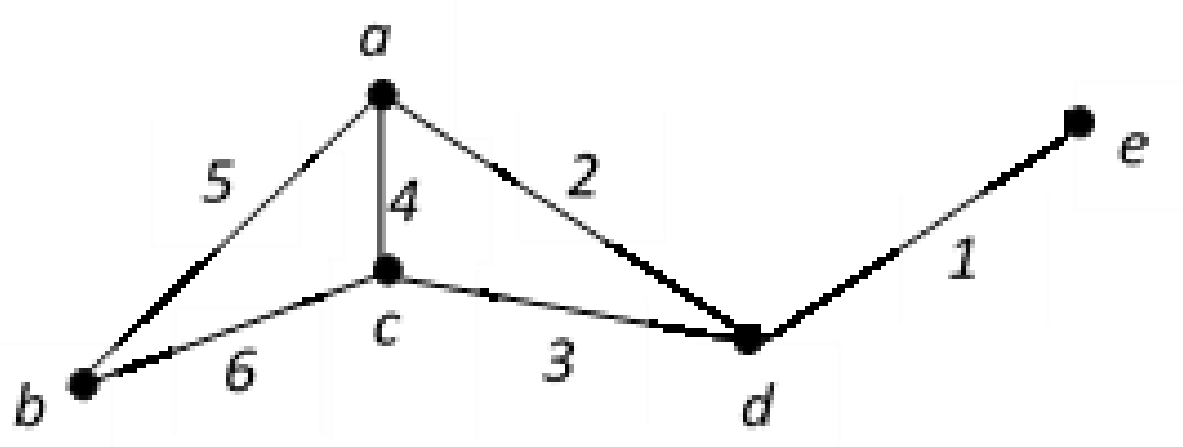

Figure 6. Graph G

$d(b, d) = 2,$

diameter of G = 3, radius of G =2

girth $g(G) = 3$, *circumference* $c(G) = 4$