**Lecture 7 - Date : 24 May 2021**

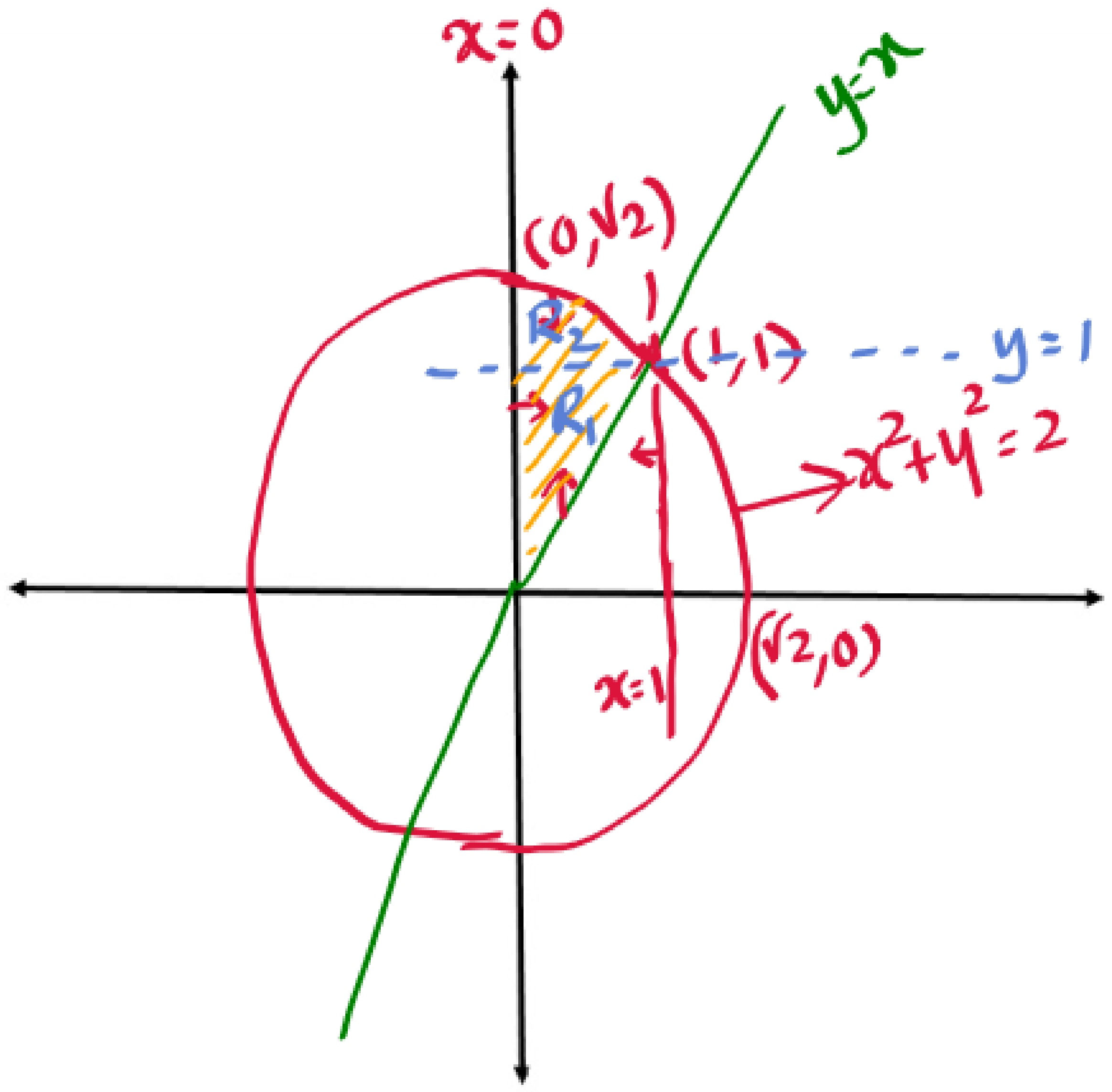
## 1. Problems on Change of order of integration

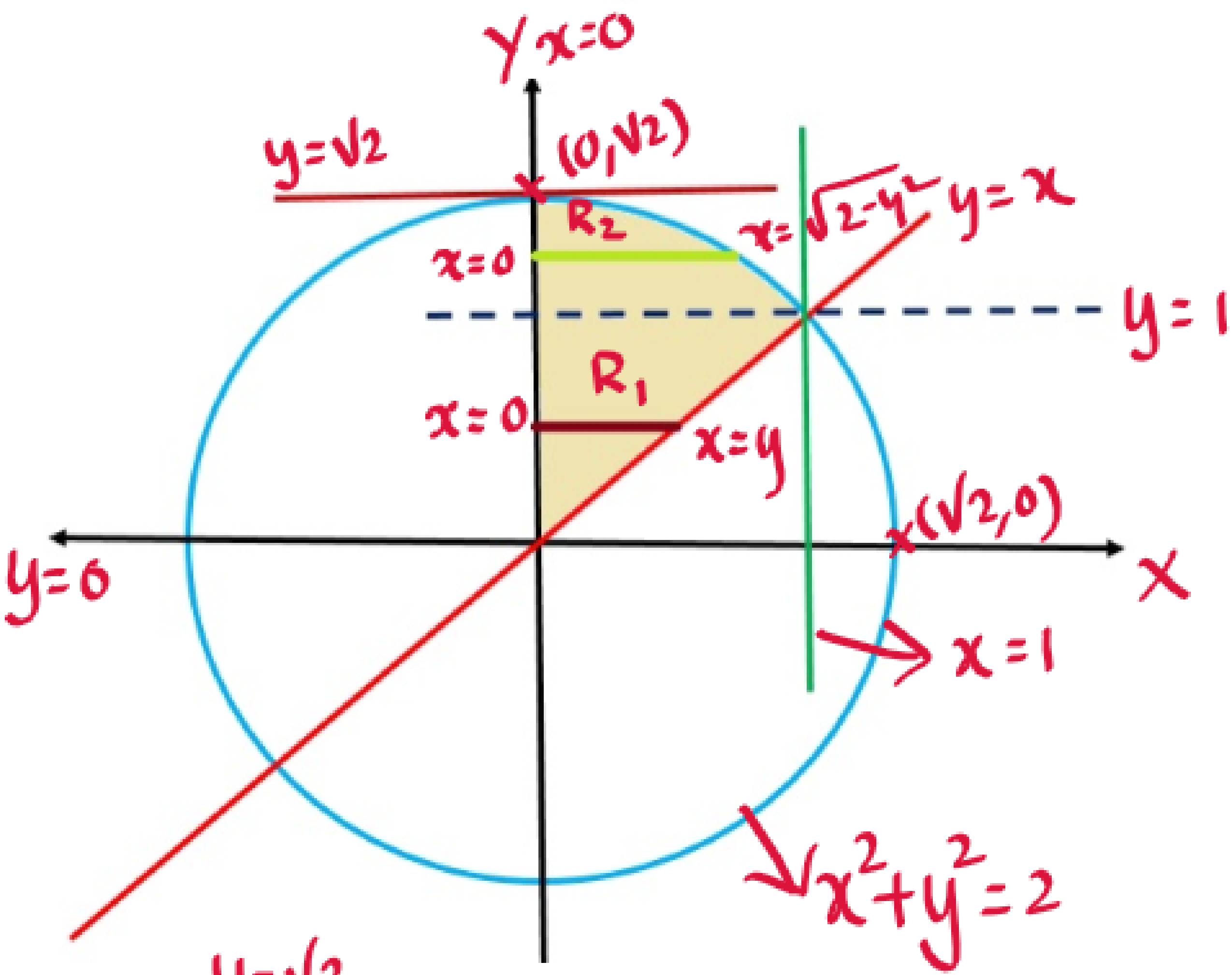
**Problem 1.1.** *Change the order of integration and evaluate*

**Ans:-** Let $$I = \int_{x=0}^{1}\int_{y=x}^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$$

Here $y$ varies from the line $y=x$ to the curve $y=\sqrt{2-x^2}$ i.e; $x^2+y^2=2$

$x$ varies from the line $x=0$ to the line $x=1$.




$$\therefore I = \iint_{R_1} \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy + \iint_{R_2} \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy$$

$$= \int_{y=0}^{y=1}\left(\int_{x=0}^{x=y} \frac{x}{\sqrt{x^2+y^2}}\,dx\right)dy + \int_{y=1}^{y=\sqrt{2}}\left(\int_{x=0}^{x=\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}}\,dx\right)dy$$

$$= \int_{y=0}^{1} \left( \int_{t=y^2}^{2y^2} \frac{dt/2}{\sqrt{t}} \right) dy$$

$$+ \int_{y=1}^{\sqrt{2}} \left( \int_{t=y^2}^{2} \frac{dt/2}{\sqrt{t}} \right) dy$$

put $x^2+y^2 = t$

$\Rightarrow \frac{dt}{dx} = 2x$

$\Rightarrow x\,dx = \frac{dt}{2}$

when $x=0 \Rightarrow t = y^2$

when $x=y \Rightarrow t = 2y^2$

when $x=0 \Rightarrow t = y^2$

when $x=\sqrt{2-y^2} \Rightarrow t = 2$

$$= \int_{y=0}^{1} \frac{1}{2} \left(2\sqrt{t}\right)_{y^2}^{2y^2} dy + \int_{y=1}^{\sqrt{2}} \frac{1}{2} \left(2\sqrt{t}\right)_{y^2}^{2} dy$$

$$= \int_{y=0}^{1} (\sqrt{2}-1)\, y\, dy + \int_{y=1}^{\sqrt{2}} (\sqrt{2} - y)\, dy$$

$$= \left( (\sqrt{2}-1) \frac{y^2}{2} \right)_0^1 + \left( \sqrt{2}y - \frac{y^2}{2} \right)_1^2$$

$$= \frac{1}{2}(\sqrt{2}-1) + (2\sqrt{2} - 2) - \left( \sqrt{2} - \frac{1}{2} \right)$$

$$= \underline{\underline{1 - \frac{1}{\sqrt{2}}}}$$

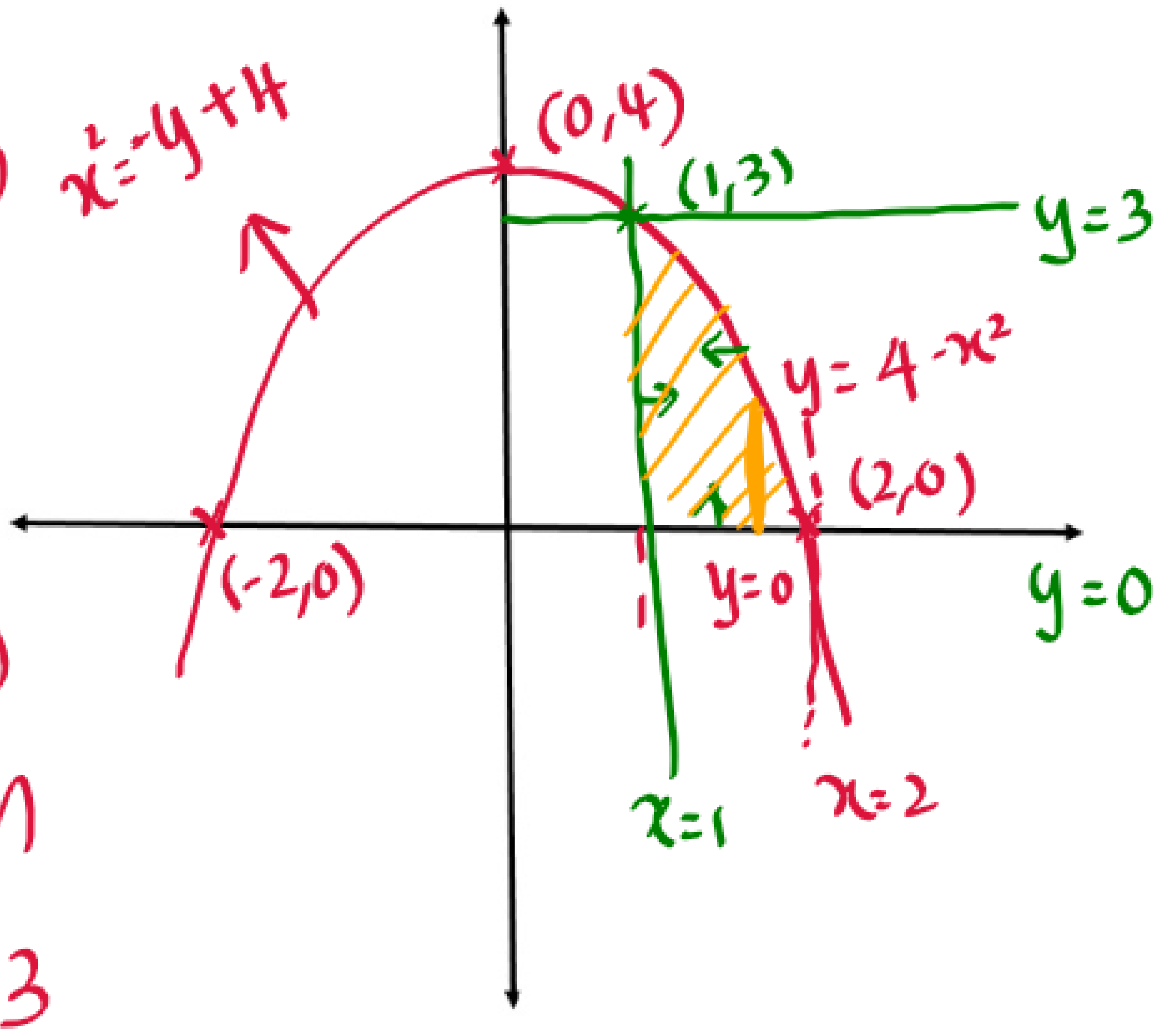
**Problem 1.2.** *Change the order of integration and evaluate*

**Ans:** Let $I = \int_{y=0}^{3} \int_{x=1}^{\sqrt{4-y}} (x+y)\, dx\, dy.$

Here x varies from the line $x=1$ to the curve $x=\sqrt{4-y}$

i.e; $x^2 = -(y-4)$

and y varies from $y=0$ to $y=3$




$$\therefore I = \int_{x=1}^{2} \left( \int_{y=0}^{y=4-x^2} (x+y)\, dy \right) dx$$

$$= (\text{Ex})$$

$$= \frac{241}{60} \text{ (Ans)}$$

→ Evaluation of Double integrals

Cartesian

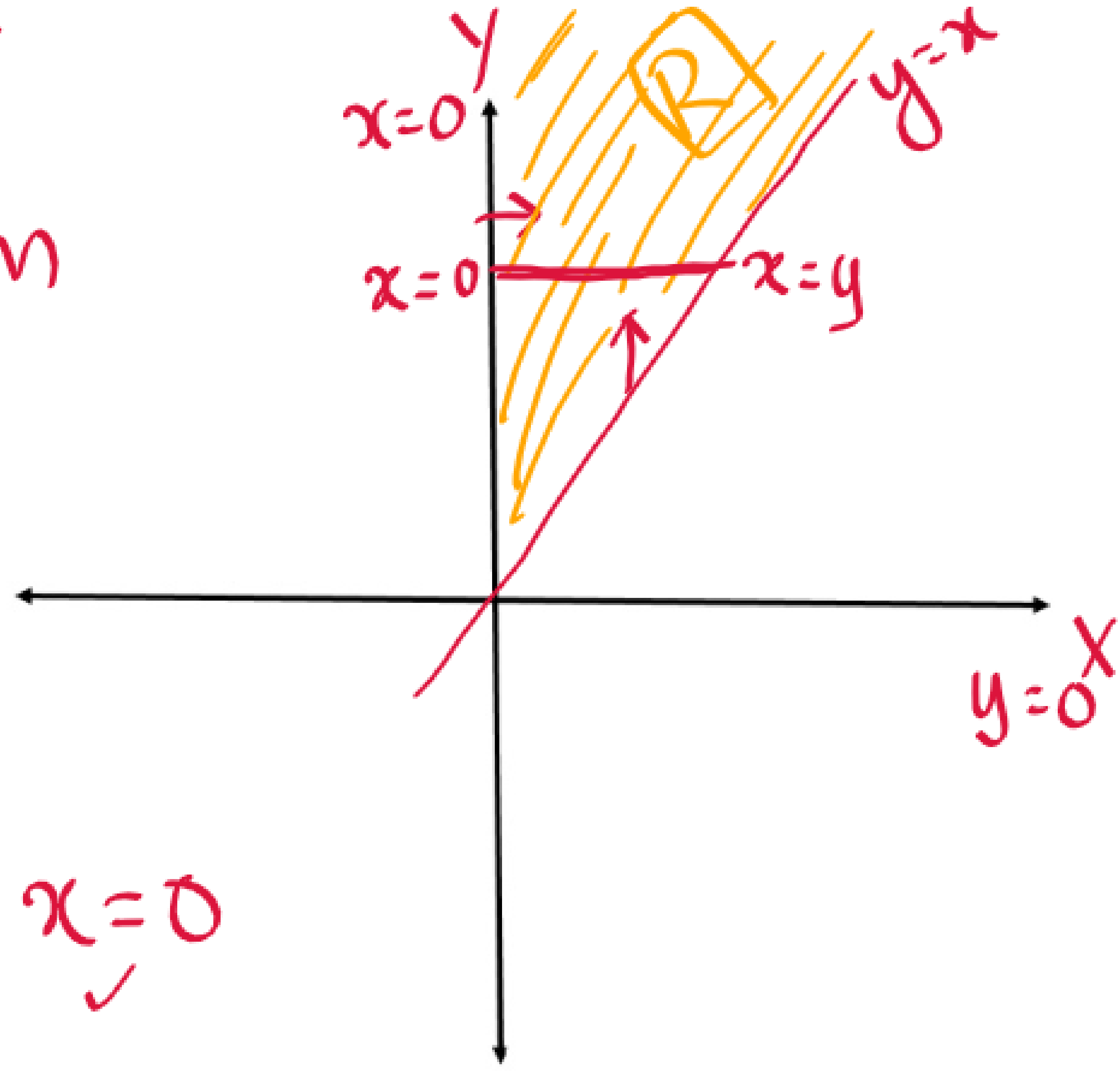
Polar

→ Change of order of integration

→ Change of variables

**Problem 1.3.** *Change the order of integration and evaluate*

Ans: Let $I = \int_{x=0}^{\infty}\int_{y=x}^{\infty} \frac{e^{-y}}{y}\, d\cancel{x}\,y \; d\cancel{y}\,x.$




Here y varies from the line $y=x$ to $y=\infty$ and x varies from the line $x=0$ to $x=\infty$

$$\therefore I = \int_{y=0}^{y=\infty}\left(\int_{x=0}^{x=y} \frac{e^{-y}}{y}\, dx\right) dy$$

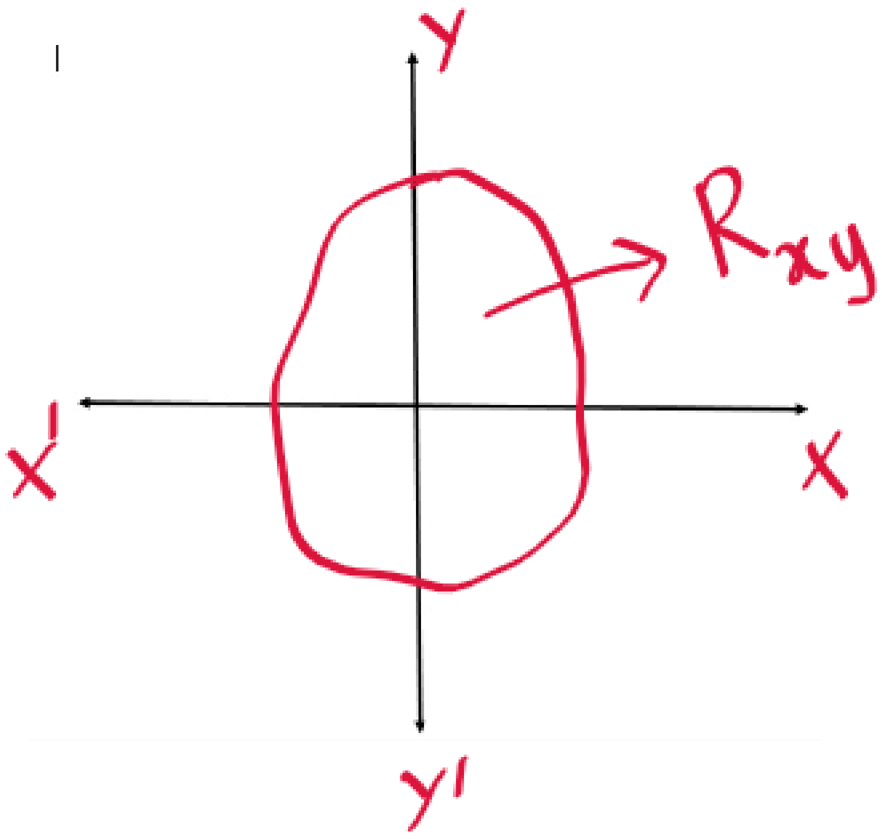
$$= \int_{y=0}^{\infty} \frac{e^{-y}}{y}\,(x)_0^y\, dy \qquad \Big|\ \lim_{y\to\infty} e^{-y} = 0$$

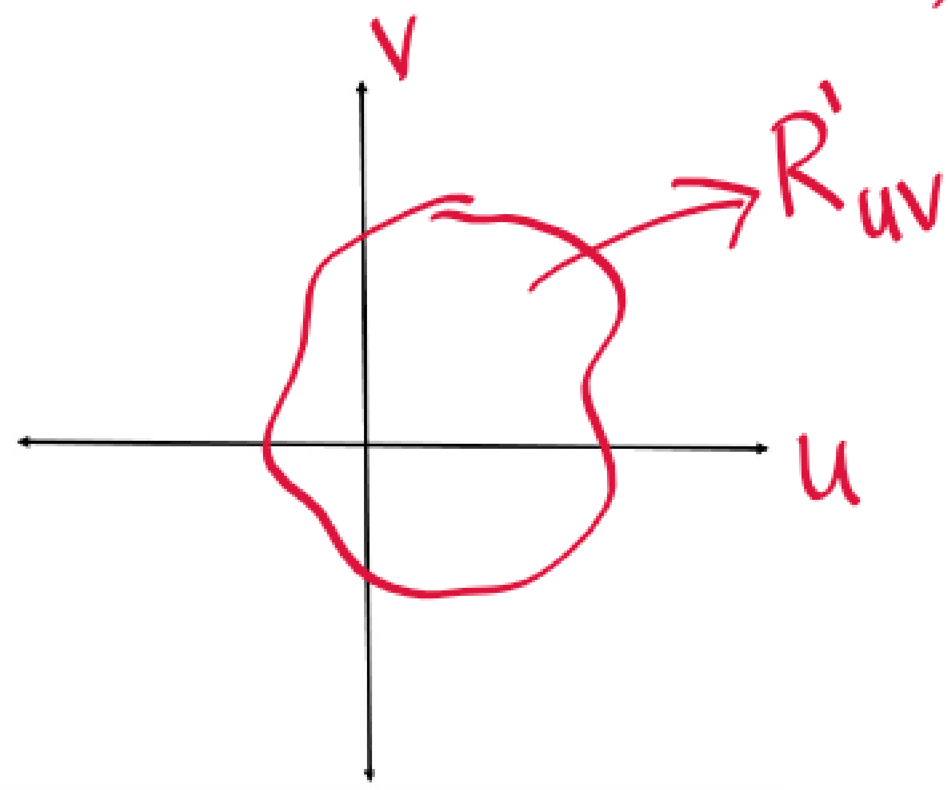
$$= \int_{y=0}^{\infty} e^{-y}\, dy = \left(\frac{e^{-y}}{-1}\right)_0^{\infty} = 0+1 = \underline{\underline{1}}$$

$$\iint\limits_{R_{xy}} f(x,y)\,dx\,dy = \iint\limits_{R_{uv}} f(\phi(u,v),\psi(u,v))\,|J|\,du\,dv \quad \text{where } J = \frac{\partial(x,y)}{\partial(u,v)}$$

## 2. CHANGE OF VARIABLES

### ✓ 2.1. Jacobian Transformation




Let $x = \phi(u,v)$ and $y = \psi(u,v)$ are continuous and have continuous first order partial derivatives in some region $R'_{uv}$ in the uv-plane corresponding to the region $R_{xy}$ in xy plane then

$$\iint\limits_{R_{xy}} f(x,y)\,dx\,dy = \iint\limits_{R'_{uv}} f\left[\phi(u,v),\psi(u,v)\right]\,|J|\,du\,dv$$

where $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$, is called the Jacobian Transformation from xy plane to uv plane.

**2.2. To change the cartesian coordinates $(x, y)$ to the polar coordinates $(r, \theta)$.**

In this case, the respective transformation,

$$x = r\cos\theta \ , \ y = r\sin\theta.$$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\sin^2\theta + \cos^2\theta) = r$$

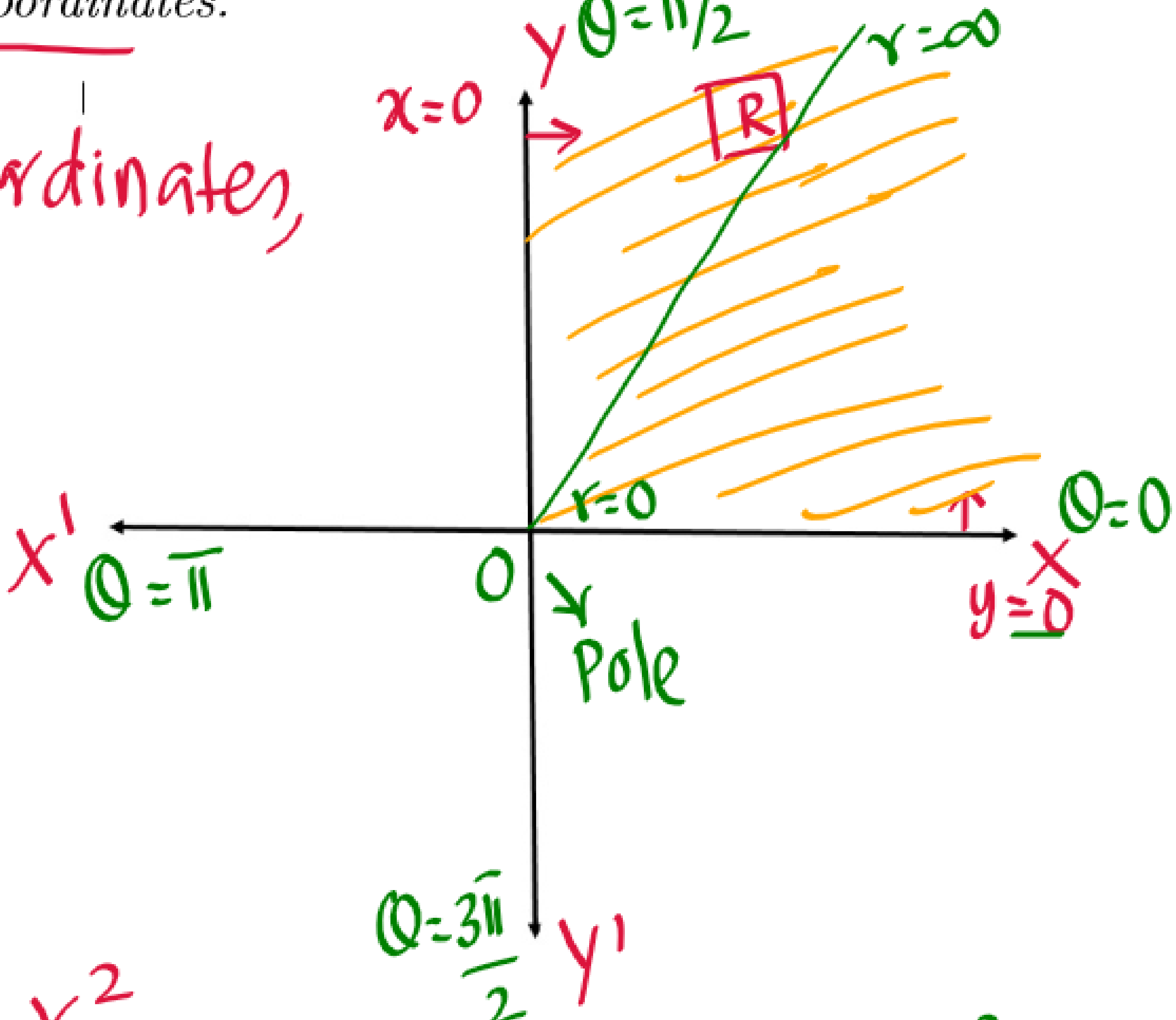
$$\therefore |J| = r$$

$$\therefore \iint_{R_{xy}} f(x,y)\,dx\,dy = \iint_{R'_{r\theta}} f[r\cos\theta, r\sin\theta]\, r\,dr\,d\theta$$

**Problem 2.1.** *Evaluate*

$$\text{Let } I = \int_{y=0}^{\infty}\int_{x=0}^{\infty} e^{-(x^2+y^2)}\,dx\,dy.$$

*by changing to polar coordinates.*




Ans:- In polar coordinates,

$x = r\cos\theta$

$y = r\sin\theta$

& $J = r$

$\therefore x^2+y^2 = r^2$

$$\therefore I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\, r\,dr\,d\theta$$

put $r^2 = t$

$\Rightarrow r\,dr = \frac{dt}{2}$

when $r=0$

$\Rightarrow t=0$

when $r=\infty$

$\Rightarrow t=\infty$

$$= \int_{\theta=0}^{\pi/2}\left(\int_{t=0}^{\infty} e^{-t}\,\frac{dt}{2}\right) d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/2}\left(\frac{e^{-t}}{-1}\right)_0^{\infty} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4}$$ //