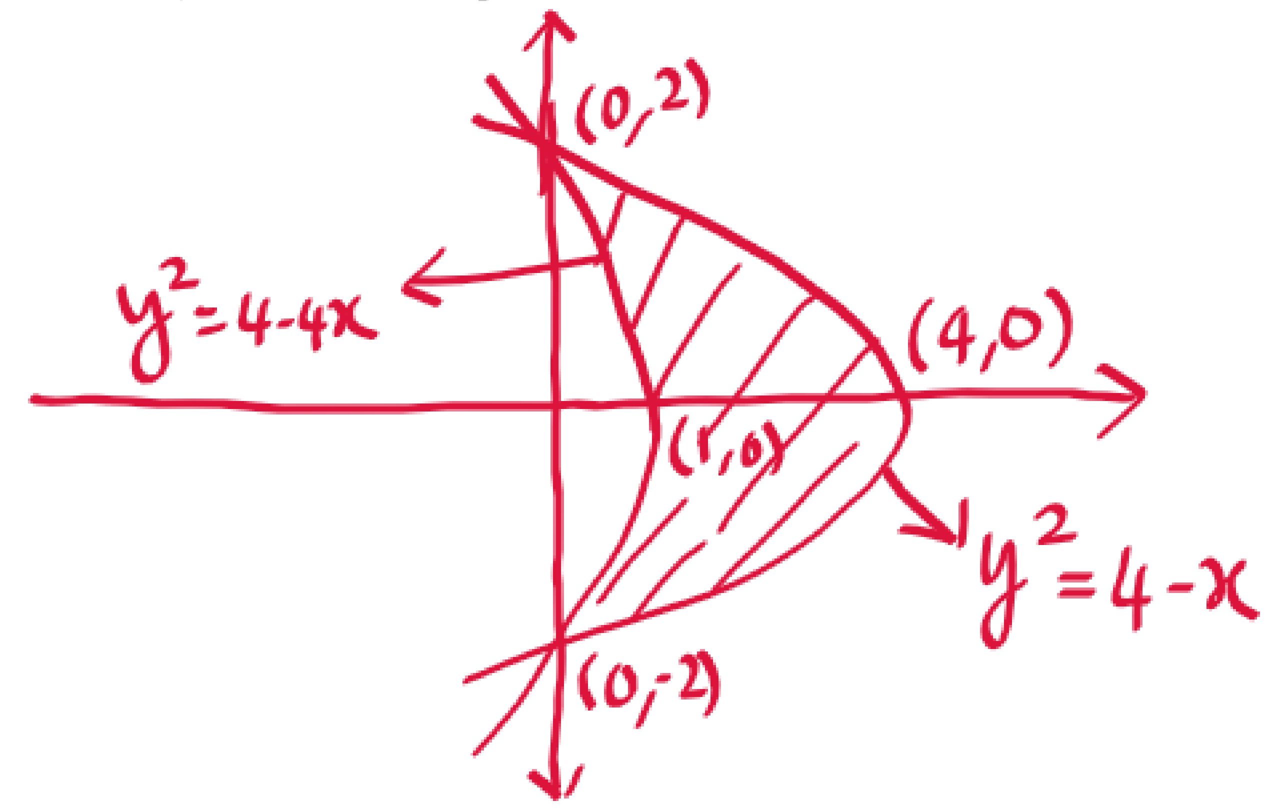
Lecture 9 - Date : 01 June 2021

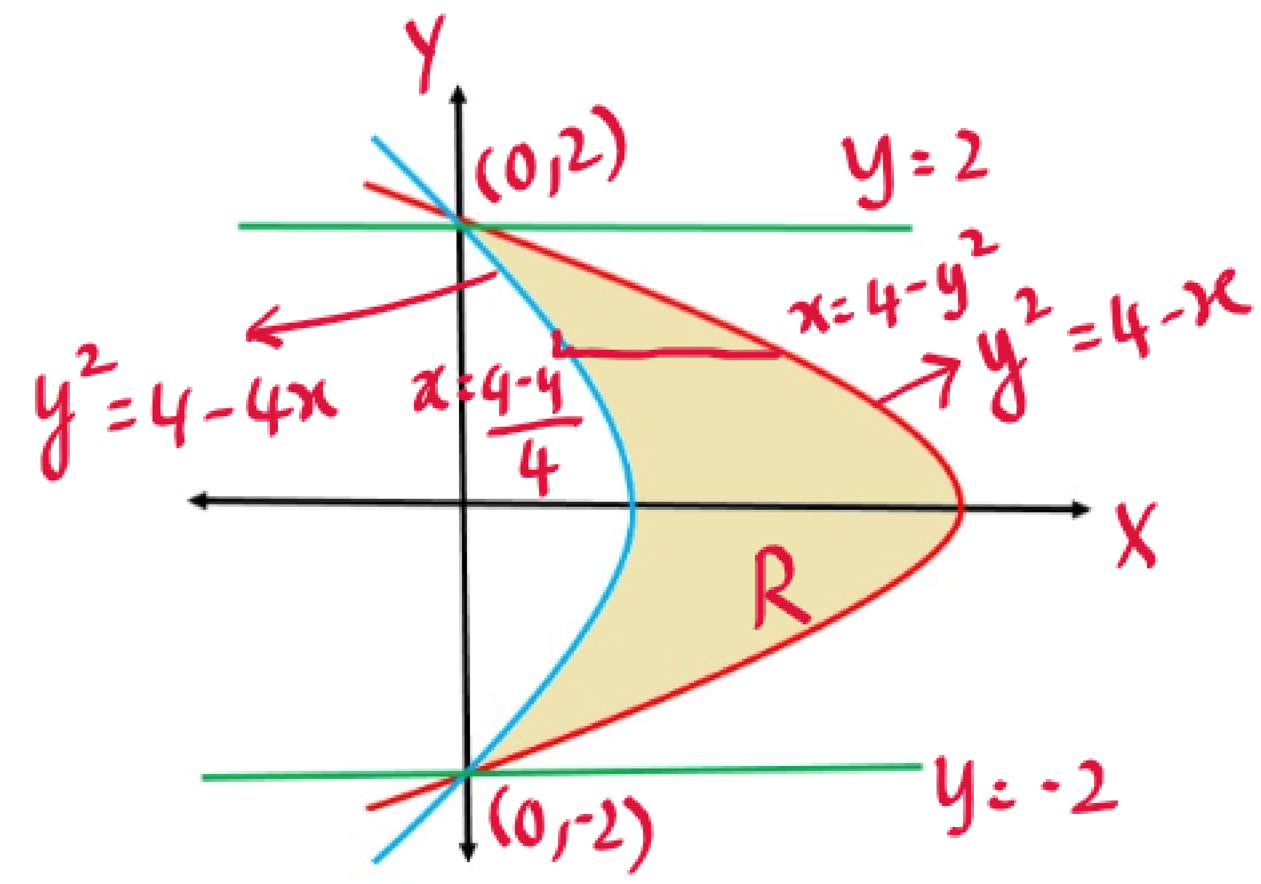
Problem 0.1. *Using double integration, find the area of the region between the curves* $y^2 = 4 - x$ *and* $y^2 = 4 - 4x$.



Ans: $y^2 = 4 - x \Rightarrow y^2 = -(x-4)$

$y^2 = -4(x-1)$

By double integrals,



$$\text{Area}(R) = \iint_R dx\,dy$$

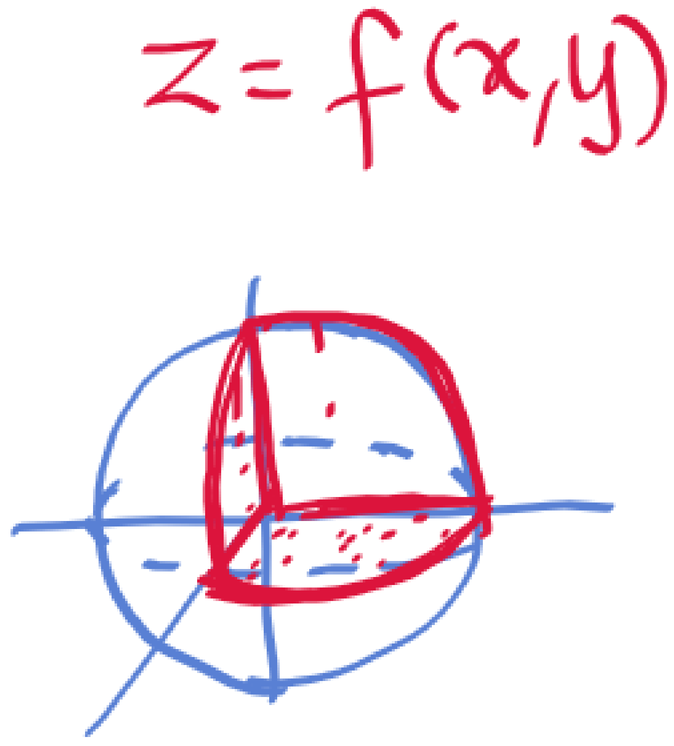
$$= \int_{y=-2}^{2} \int_{x=\frac{4-y^2}{4}}^{4-y^2} dx\,dy$$

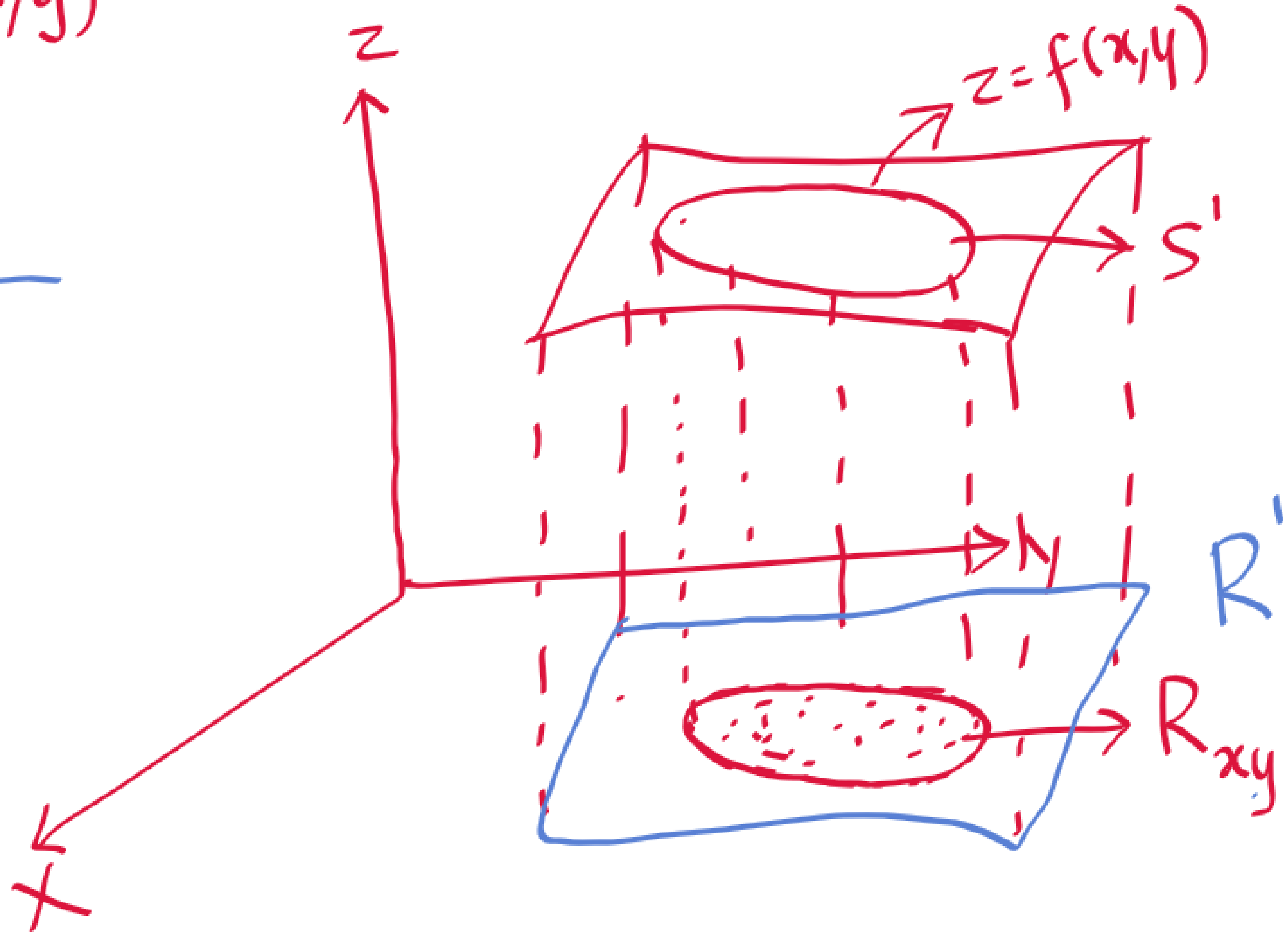
$$= \int_{y=-2}^{2} (x)_{\frac{4-y^2}{4}}^{4-y^2} dy = \int_{y=-2}^{2} \left[(4-y^2) - \frac{(4-y^2)}{4}\right] dy$$

$$= \frac{3}{4}\int_{y=-2}^{2} (4-y^2)\,dy = \frac{3}{2}\int_0^2 (4-y^2)\,dy$$

$= ?$

$= 8$ sq. units



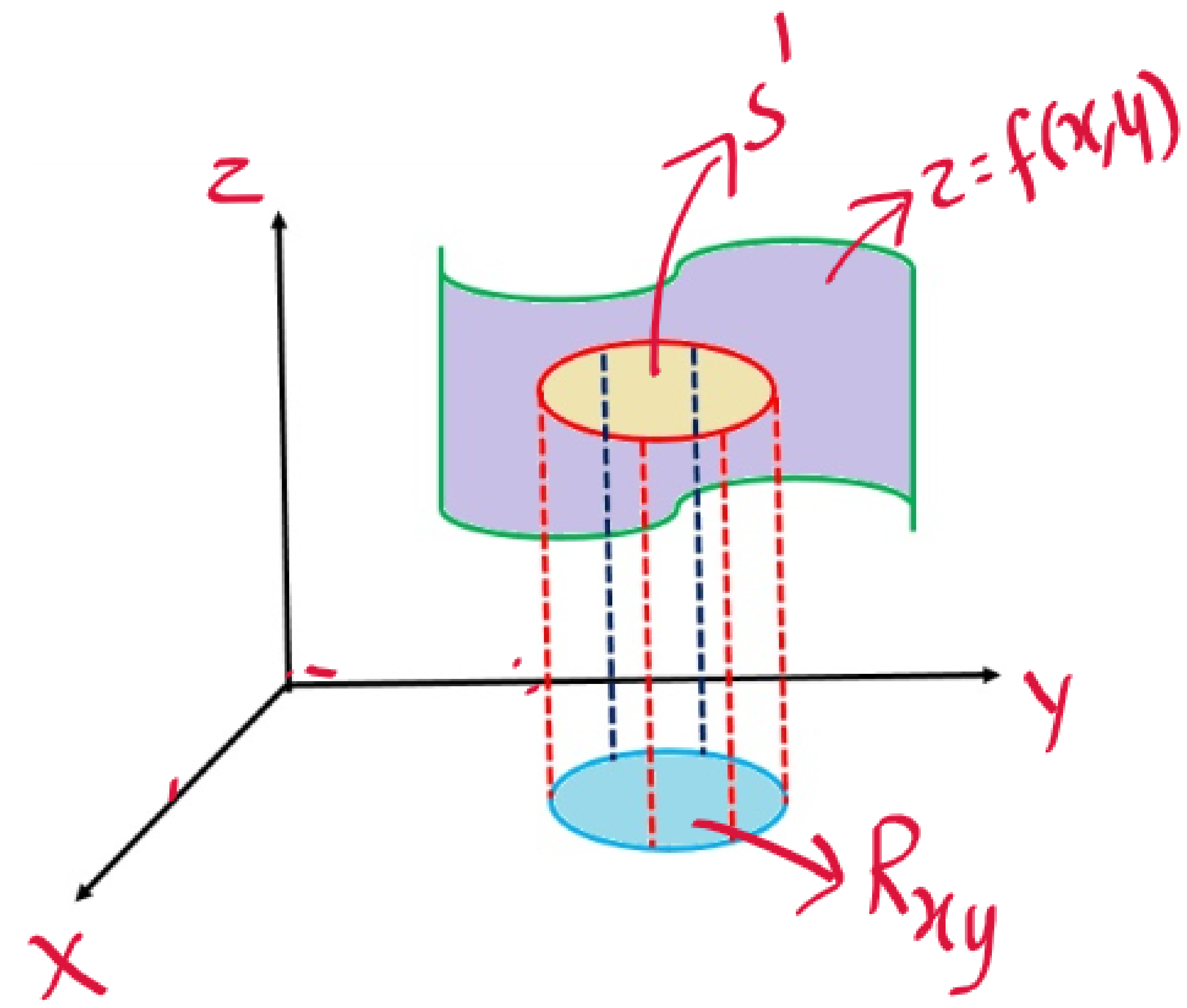


$$\text{Volume } (V) = \iint_{R_{xy}} z \, dx\,dy = \iint_{R_{xy}} f(x,y) \, dx\,dy$$

1. Volume of a solid using double integrals

Consider a surface $z = f(x, y)$.
Let S' be a portion on the surface $z = f(x, y)$.
Let R_{xy} be the orthogonal projection of S' in XY plane.

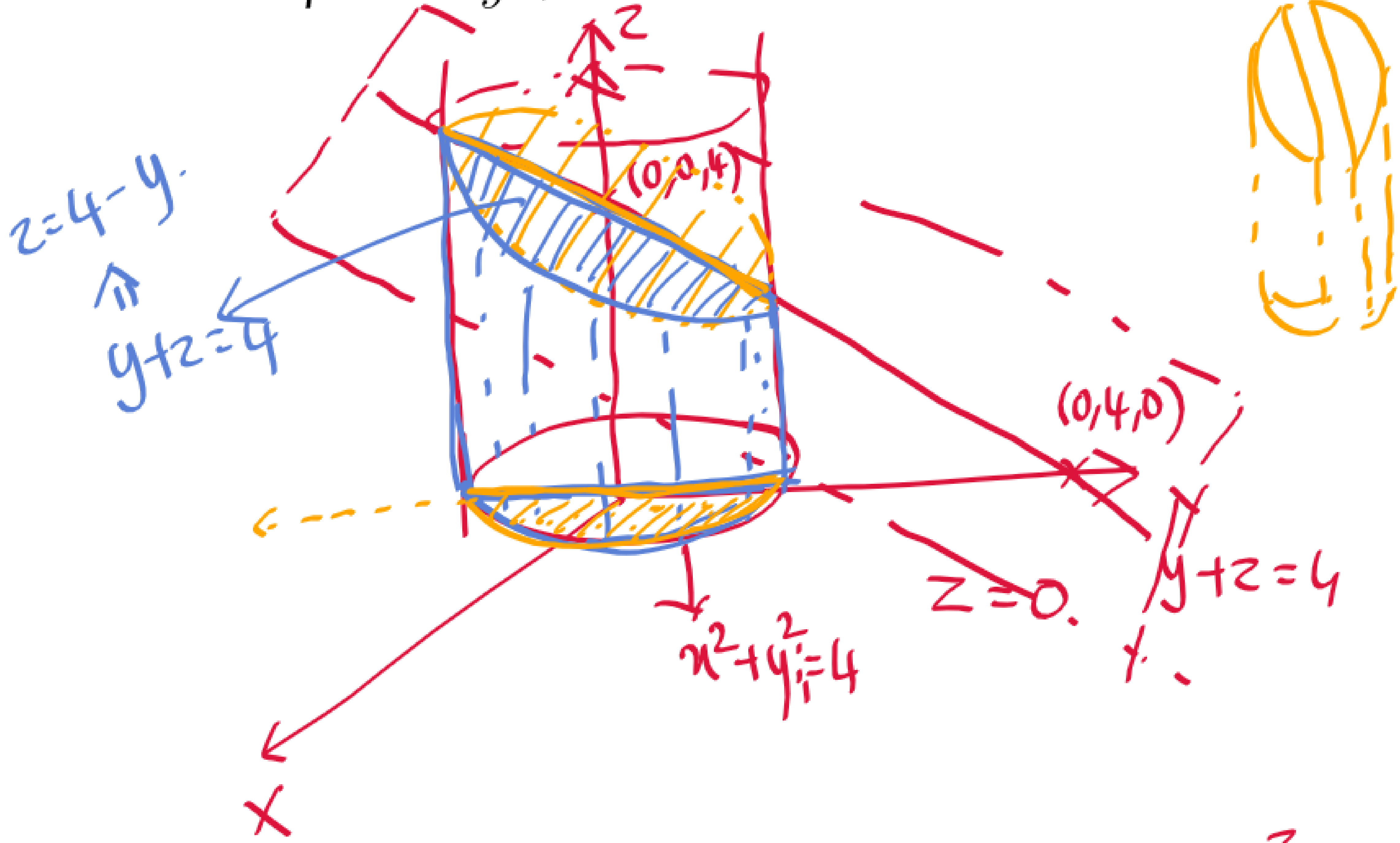
$\therefore$ The volume of the solid having R_{xy} as the base and S' as the top surface is given by

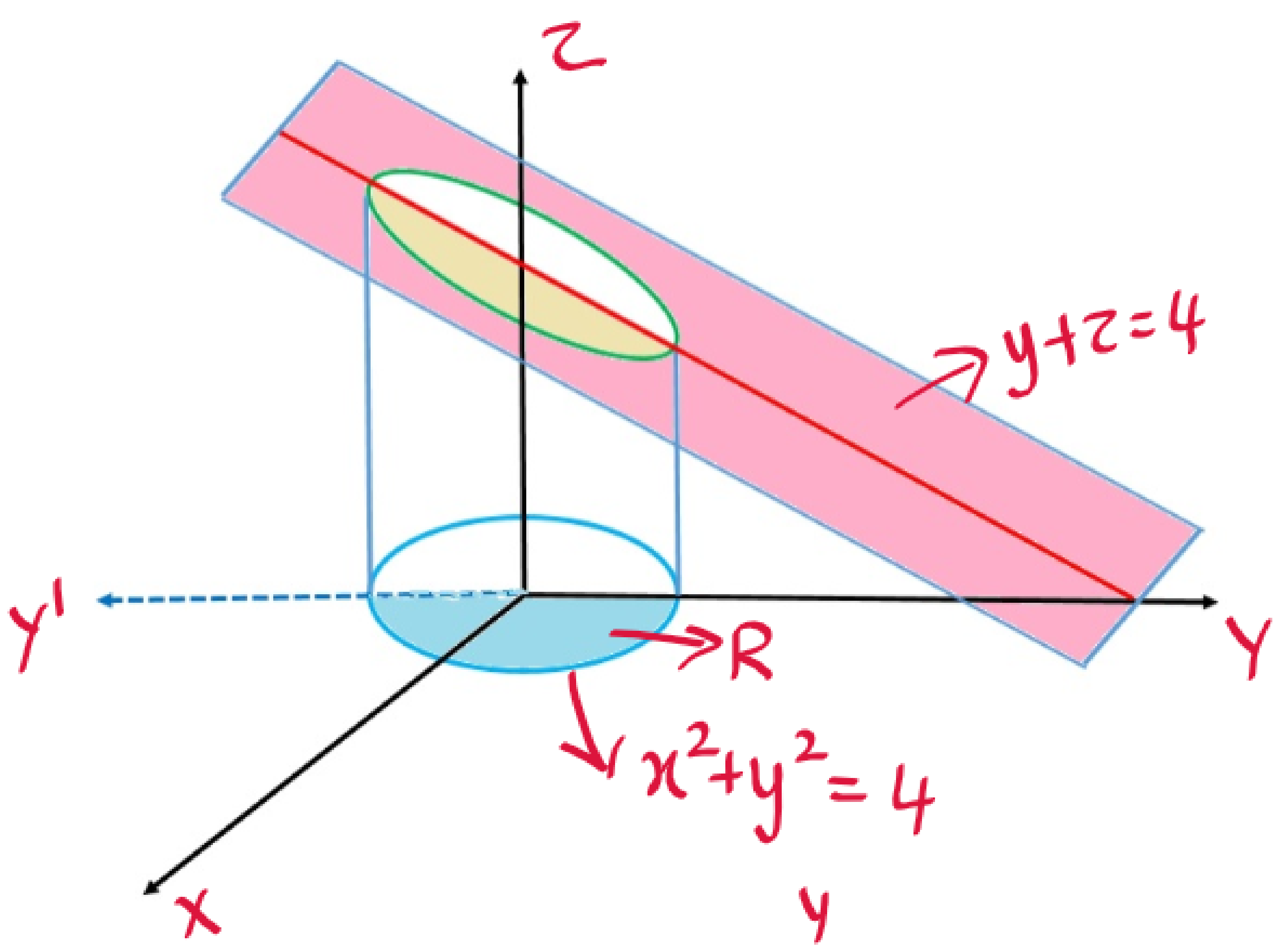


$$\text{Volume, } V = \iint_{R_{xy}} z \, dx\,dy$$

$$= \iint_{R_{xy}} f(x,y) \, dx\,dy$$

Problem 1.1. *Using double integration, find the volume of the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.*

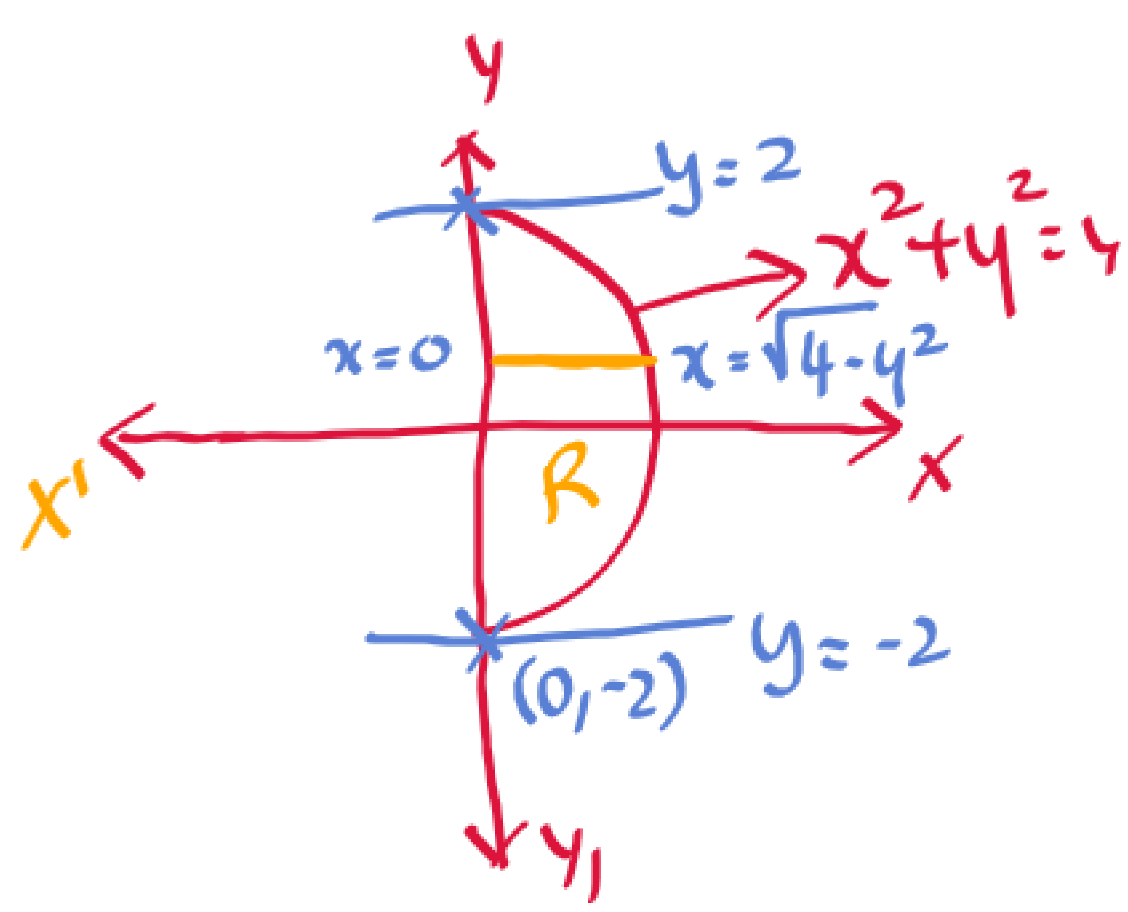




∴ Req'd volume,

$$V = 2 \times \iint_R z \, dx\, dy$$

$$= 2 \times \int_{y=-2}^{2} \int_{x=0}^{\sqrt{4-y^2}} (4-y) \, dx \, dy$$



$$= 2 \times \int_{y=-2}^{2} (4-y) \, (x)_0^{\sqrt{4-y^2}} \, dy$$

$$= 2 \times \int_{y=-2}^{2} (4-y) \sqrt{4-y^2} \, dy$$

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$$= 2 \times \int_{y=-2}^{2} 4\sqrt{4-y^2}\,dy - 2 \underbrace{\int_{y=-2}^{2} y\sqrt{4-y^2}\,dy}_{\text{odd fnct.}}$$

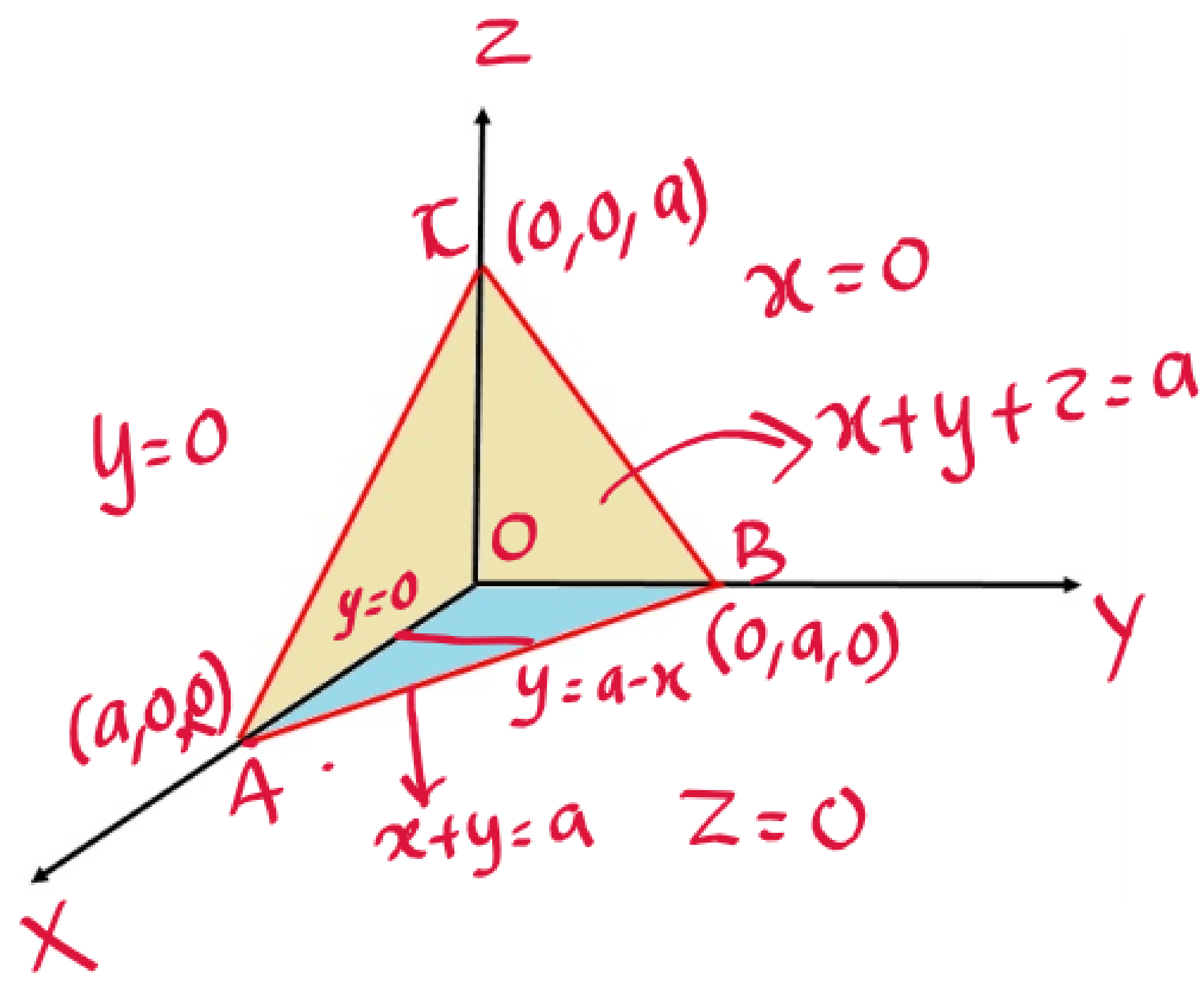
$$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| \; 0$$

$$= 16 \int_{y=0}^{2} \sqrt{4-y^2}\,dy$$

$$= 16 \left[\frac{y}{2}\sqrt{4-y^2} + \frac{4}{2}\sin^{-1}(y/2) \right]_0^2$$

$$= 16\left[0 + \pi\right] = 16\pi \; \underline{\underline{\text{c. units.}}}$$

Problem 1.2. *Using double integration, find the volume of the solid bounded by the planes* $x = 0, y = 0, x + y + z = a$ *and* $z = 0$.



Req'd volume, surface

$$V = \iint_{OAB} z \, dx\, dy$$

$$= \int_{x=0}^{a} \int_{y=0}^{a-x} (a-x-y)\, dy\, dx$$

$$= \int_{x=0}^{a} \left[(a-x)y - \frac{y^2}{2} \right]_{y=0}^{a-x} dx$$

$$= \int_{x=0}^{a} \left[(a-x)^2 - \frac{(a-x)^2}{2} \right] dx \quad = \frac{1}{2} \int_{x=0}^{a} (a-x)^2 dx$$

$$= \left[\frac{-1}{2} \frac{(a-x)^3}{3} \right]_0^a$$

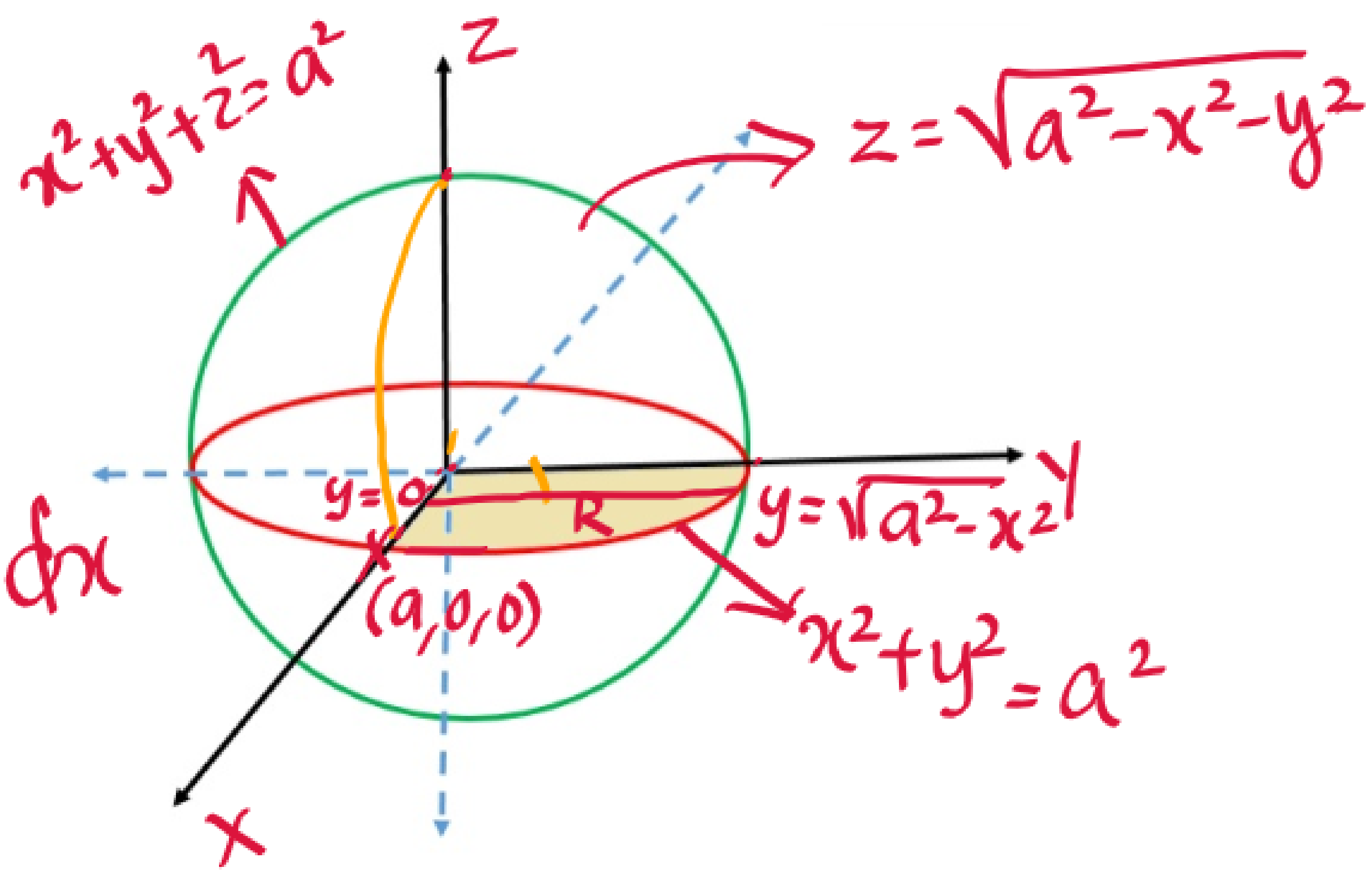
$$= \frac{a^3}{6} \text{ cubic units}$$

Problem 1.3. *Using double integration, find the volume of the sphere of radius* a., centre origin.

Ans: Eq^n of the sphere $x^2+y^2+z^2=a^2$

Req'd volume, V= 8x Volume of the sphere in the Ist octant

$$V = 8 \times \iint_R \sqrt{a^2-x^2-y^2}\, dx\, dy$$



$$= 8 \times \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\, dx$$

= (Ex) ?

$$= \frac{4}{3}\pi a^3 \text{ cubic units.}$$

2. Triple Interals

Let $u = f(x, y, z)$ be function defined in a three dimensional region V in space. Divide the region V into n cuboids $V_1, V_2, \dots V_n$ each of volume $\delta V_1, \delta V_2, \dots \delta V_n$

Let (x_i, y_i, z_i) be a point on the ith cuboid, having volume δV_i

If
$$\lim_{\substack{n \to \infty \\ \delta V_i \to 0}} \sum_{i=1}^{n} f(x_i, y_i, z_i)\, \delta V_i \text{ exists}$$

then it is called the triple integral of $f(x, y, z)$ w.r.t $x, y,$ and z over V. It is denoted by $\iiint_V f(x, y, z)\, dx\, dy\, dz$.

Problem 2.1. *Evaluate*

Let $I = \displaystyle\int_{x=1}^{2}\int_{y=2}^{3}\int_{z=1}^{3} (x^2y+z)\,dz\,dy\,dx$

Ans:-

$$= \int_{x=1}^{2}\int_{y=2}^{3}\left(x^2yz+\frac{z^2}{2}\right)_{z=1}^{3} dy\,dx$$

$$= \int_{x=1}^{2}\int_{y=2}^{3}\left[\left(3x^2y+\frac{9}{2}\right)-\left(x^2y+\frac{1}{2}\right)\right] dy\,dx$$

$$= \int_{x=1}^{2}\int_{y=2}^{3}\left(2x^2y+4\right) dy\,dx$$

$$= \int_{x=1}^{2}\left(x^2y^2+4y\right)_{y=2}^{3} dx$$

$$= \int_{x=1}^{2}\left(5x^2+4\right) dx = \left(\frac{5x^3}{3}+4x\right)_{x=1}^{2}$$

$$= \frac{47}{3}$$

$$e^{x+y+z} = e^{x+y} \cdot e^{z}$$

Problem 2.2. *Evaluate*

$$\text{Let } I = \int_{x=0}^{2} \int_{y=0}^{x} \int_{z=0}^{z=x+y} e^{x+y+z} dz\, dy\, dx$$

$$= \int_{x=0}^{2} \int_{y=0}^{x} \int_{z=0}^{x+y} e^{x+y} \cdot e^{z}\, dz\, dy\, dx$$

$$= \int_{x=0}^{2} \int_{y=0}^{x} e^{x+y} \left(e^{z}\right)_{z=0}^{x+y} dy\, dx$$

$$= \int_{x=0}^{2} \int_{y=0}^{x} e^{x+y} \left[e^{x+y} - 1\right] dy\, dx$$

$$= \int_{x=0}^{2} \int_{y=0}^{x} \left[e^{2x} \cdot e^{2y} - e^{x} \cdot e^{y}\right] dy\, dx$$

$$= \int_{x=0}^{2} \left(e^{2x} \cdot \frac{e^{2y}}{2} - e^{x} e^{y}\right)_{y=0}^{x} dx$$

$$= \int_{x=0}^{2} \left(\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^{x}\right) dx$$

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$$= \int_{x=0}^{2} \left(\frac{e^{4x}}{2} - \frac{3e^{2x}}{2} + e^{x} \right) dx$$

$$= \left(\frac{e^{4x}}{8} - \frac{3e^{2x}}{4} + e^{x} \right)_0^2$$

$$= \frac{e^8}{8} - \frac{3e^4}{4} + e^2 - \left(\frac{1}{8} - \frac{3}{4} + 1 \right)$$

$$= \frac{e^8}{8} - \frac{3e^4}{4} + e^2 - \frac{3}{8}$$

Problem 2.3. *Evaluate*

$$\int_{-1}^{1} \int_{0}^{z} \int_{x-z}^{x+z} (x + y + z)\, dx\, dy\, dz$$

Practice Problems

Problem 2.5. *Using double integration, find the volume common to the cylinders* $x^2 + y^2 = a^2$ *and* $x^2 + z^2 = a^2$. *Ans:* $\boxed{\dfrac{16a^3}{3}}$

Problem 2.6. *Using double integration, find the volume of the ellipsoid* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. *Ans:* $\boxed{\dfrac{4\pi abc}{3}}$

Problem 2.7. *Using double integration, find the volume of the tetrahedran* $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ *and the coordinate planes. Ans:* $\boxed{\dfrac{abc}{6}}$

Problem 2.8. *Evaluate*

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1 - x^2 - y^2 - z^2}} dz\, dy\, dx \; Ans : \boxed{\frac{\pi^2}{8}}$$

Problem 2.9. *Evaluate*

$$\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz\, dx\, dy \; Ans : \boxed{\frac{e^2}{4} - 2e + \frac{13}{4}}$$