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EXACT DIFFERENTIAL EQUATIONS

Partial derivatives.

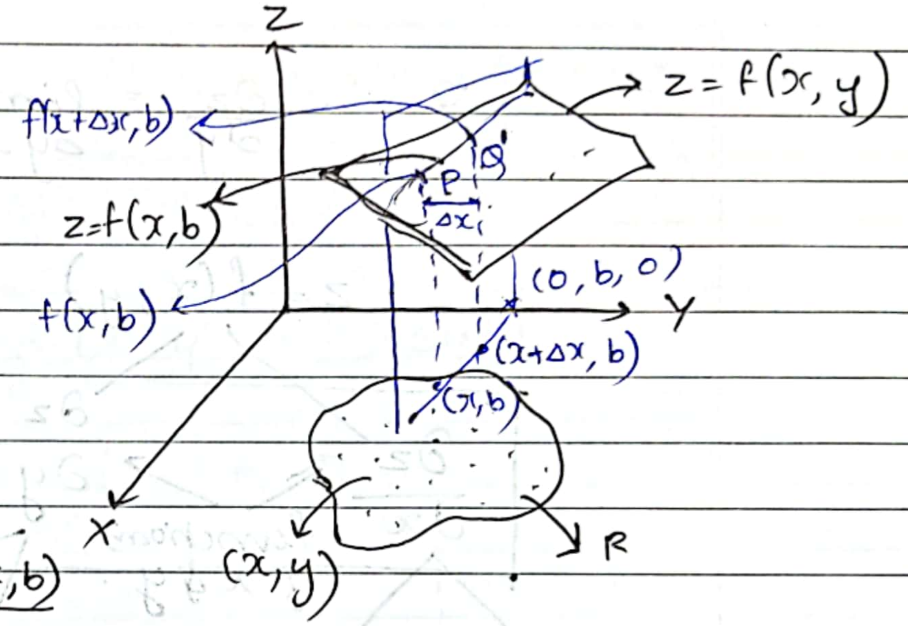
Let $z = f(x,y)$ be a function of two independent ~~random~~ variables.

dependent

independent variables.

If y is constant

$y = b$



Slope of chord PQ

$$= \frac{f(x+\Delta x, b) - f(x,b)}{(x+\Delta x) - x}$$

$$= \frac{f(x+\Delta x, b) - f(x,b)}{\Delta x}$$

$$\left(\frac{\partial z}{\partial x}\right)_P = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, b) - f(x,b)}{\Delta x}$$ exists.

Let $z = f(x,y)$ be a function of two indpt variables x and y. If we keep y as constant then z is a function of x alone. Then the partial derivative of z w.r.t x is denoted by $\frac{\partial z}{\partial x}$ or z_x or f_x or $\frac{\partial f}{\partial x}$.

is defined as total derivative of z w.r.t x by keeping y as constant

ie $$\frac{\partial z}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}.$$

→ If we keep x as constant then z is a function of y alone. Then the partial derivative of z w.r.t y is denoted by $\frac{\partial z}{\partial y}$ or z_y or $\frac{\partial f}{\partial y}$ or f_y, is defined as the total derivative of z w.r.t y by keeping x constant.

$$\therefore \frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

$$z = f(x,y)$$

$$\boxed{\frac{\partial z}{\partial x}, \quad \frac{\partial z}{\partial y}}$$

functions of x & y.

first order partial derivative of z w.r.t x & y.

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} \quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$$

second order partial derivatives of z w.r.t x & y.

If $u = f(x,y)$ & $v = f(x,y)$

(1) $$\frac{\partial}{\partial x}(uv) = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}$$

$$\frac{\partial}{\partial y}(uv) = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}$$

(2) $\frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$

$\frac{\partial}{\partial y}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}}{v^2}$

Q. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = x^3 + 3x^2y^2 + 5y^4$

Ans $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(x^3 + 3x^2y^2 + 5y^4\right)$

$= \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial x}(3x^2y^2) + \frac{\partial}{\partial x}(5y^4)$

$= 3x^2 + 3y^2\frac{\partial}{\partial x}(x^2) + 0$

$= 3x^2 + 6xy^2$

$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^3 + 3x^2y^2 + 5y^4)$

$= 0 + 3x^2\frac{\partial}{\partial y}(y^2) + 5\frac{\partial}{\partial y}(y^4)$

$= 6yx^2 + 20y^3$

TOTAL DIFFERENTIAL:

Let $z = f(x,y)$ be a function of two independent variables x and y

Then the total differential of z is defined as,

$$\boxed{dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy}$$

If $z = f(u, v, w)$ then

$$\boxed{dz = \frac{\partial z}{\partial u} \cdot du + \frac{\partial z}{\partial v} dv + \frac{\partial z}{\partial w} dw.}$$

Eg i) Suppose $z = xy$ then.

$$dz = \frac{\partial}{\partial x}(xy)dx + \frac{\partial}{\partial y}(xy dy)$$

$$d(xy) = ydx + xdy$$

ii) Suppose $z = y/x$ then.

$$dz = \frac{\partial}{\partial x}\left(\frac{y}{x}\right)dx + \frac{\partial}{\partial y}(y/x)\,dy.$$

$$= y\left(\frac{-1}{x^2}\right)dx + \frac{1}{x}(1)\,dy$$

$$d(y/x) = \frac{xdy - ydx}{x^2}$$

EXACT DIFFERENTIAL EQUATION.

A d.e of the form $\underbrace{M(x,y)}_{\text{fn of } x \,\&\, y} dx + \underbrace{N(x,y)}_{\text{fn of } x \,\&\, y} dy = 0 \rightarrow (*)$

This eqn is exact if LHS of $(*)$ is a total differential or exact differential of some function of x and y.

Suppose LHS of ⊛ is the ~~oldest~~ total differential of $u(x,y)$ then ⊛ becomes $d(u)=0$.

Integrating both side we get

$\boxed{u(x,y) = U = C}$, is the soln of ⊛

Eg: ① Consider $x\,dy + y\,dx = 0$

$(N\,dy + M\,dx = 0)$ form.

This can be written as $d(xy) = 0$

$xy = c$ is the soln

- Consider the d.e $M(x,y)dx + N(x,y)dy = 0$ — ⊛

Assume that $M(x,y)$, $N(x,y)$, $\frac{\partial N}{\partial y}$, $\frac{\partial M}{\partial x}$ are continuous functions then eqn ⊛ is exact if and only if

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

Working rule to get the solution of an exact diff eq

Soln is

$$\underbrace{\int M(x,y)\,dx}_{\text{treating 'y' as constant}} + \int \left(\text{Terms in } N \text{ not containing } x\right) dy = C.$$

Q. Verify if the given differential equation is exact or not. If so then solve it

$$y\sin 2x\, dx + (1+y^2+\cos^2 x)\, dy = 0 \quad \text{---} \quad (*)$$

Ans Here $M = y\sin 2x$ | $N = -(1+y^2+\cos^2 x)$

$\frac{\partial M}{\partial y} = \sin 2x \cdot (1) = \sin 2x$ | $\frac{\partial N}{\partial x} = 0 + 0 + 2\cos x \sin x = \sin 2x$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ the equation $(*)$ is exact.

$\therefore$ Sol^n,

$$\underset{\text{treating 'y' terms as constant}}{\int M\,dx} + \int \left(\text{Terms in N not containing } x\right) dy = C$$

$$\Rightarrow \int y\sin 2x\, dx + \int -(1+y^2)\, dy = c$$

$$\Rightarrow y\int \sin 2x\, dx - \int (1+y^2)\, dy = c$$

$$\Rightarrow y\left(\frac{-\cos 2x}{2}\right) - \left(y + \frac{y^3}{3}\right) = c$$

$$\therefore -\frac{y\cos 2x}{2} - y - \frac{y^3}{3} = c$$

Q. Verify if the given d.e is exact or not. And if so, solve.

$$3x(xy-2)\,dx + (x^3+2y)\,dy = 0 \longrightarrow (*)$$

Ans. $M = 3x^2y - 6x$ $\qquad N = x^3 + 2y$.

$\frac{\partial M}{\partial y} = 3x^2 - 0 = 3x^2$ $\qquad \frac{\partial N}{\partial x} = 3x^2 + 0 = 3x^2$.

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ (*) is exact

$\therefore$ Sol^n is,

$\int (3x^2y - 6x)\,dx + \int 2y\,dy = c$

Treating 'y' as constant

$\Rightarrow 3y\frac{x^3}{3} - \frac{6x^2}{2} + \frac{\cancel{2}y^2}{\cancel{2}} = c$

$x^3y - 3x^2 + y^2 = c$.

Q Verify the d.e and solve.

$(\cos 2y - 3x^2y^2)\,dx + (\cos 2y - 2x\sin 2y - 2x^3y)\,dy = 0$. $\longrightarrow$ (*)

Ans. $\frac{\partial M}{\partial y} = -2\sin 2y - 3x^2(2y) = -2\sin 2y - 6x^2y$ $\qquad \frac{\partial N}{\partial x} = 0 - 2\sin 2y(1) - 2y(3x^2) = -2\sin 2y - 6x^2y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ (*) is exact.

$\therefore$ Soln is :

$\int_{y \text{ is constant}} (\cos 2y - 3x^2y^2)\,dx + \int \cos 2y\,dy = C$

$\cos 2y \int dx - 3y^2\int x^2\,dx + \int \cos 2y\,dy = C$

$x\cos 2y - x^3y^2 + \frac{\sin 2y}{2} = C$

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EQUATIONS REDUCIBLE TO EXACT DIFF EQN.

Equations that are not exact can be made exact by suitable multiplication of a function of x and y. Such multiplier is called integrating factor (I.F). of the differential eqn

- Inspection Method.
  - $dx \pm dy = d(x \pm y)$
  - $x dx \pm y dy = d\left(\frac{x^2 \pm y^2}{2}\right)$
  - $x dy + y dx = d(xy)$
  - $\frac{x dx + y dy}{x^2+y^2} = d\left(\frac{\log(x^2+y^2)}{2}\right)$
  - $\frac{x dx + y dy}{\sqrt{x^2+y^2}} =$
  - $\frac{x dy + y dx}{xy} =$
  - $\frac{x dy - y dx}{x^2} = d\left(\frac{y}{x}\right)$
  - $\frac{x dy - y dx}{xy}$
  - $\frac{x dy - y dx}{x^2+y^2}$

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Q. Solve $ydx - xdy + 3x^2y^2e^{x^3}dx = 0$.

Ans. Divide throughout by y^2 we get,

$$\frac{ydx - xdy}{y^2} + 3x^2e^{x^3}dx = 0.$$

$$d\left(\frac{x}{y}\right) + d\left(e^{x^3}\right) = 0$$

$$d\left(\frac{x}{y} + e^{x^3}\right) = 0.$$

Integration both sides.

$$\frac{x}{y} + e^{x^3} = c$$

Q. Solve $xdy - ydx = x\sqrt{x^2 - y^2}\,dx$.

Ans. Given $xdy - ydx = x\sqrt{x^2\left(1 - \frac{y^2}{x^2}\right)}\,dx$

$$xdy - ydx = x^2\sqrt{1 - y^2/x^2}\,dx$$

$$\frac{\frac{xdy - ydx}{x^2}}{\sqrt{1 - \frac{y^2}{x^2}}} = dx.$$

$$\frac{1}{\sqrt{1 - (y/x)^2}}\left(\frac{xdy - ydx}{x^2}\right) = dx$$

$$d\left(\sin^{-1}(y/x)\right) = dx.$$

$$\sin^{-1}(y/x) = x + c.$$

Q. Solve $ydx - xdy + 3x^2y^2e^{x^3}dx = 0$.

Ans. Divide throughout by y^2 we get,

$$\frac{ydx - xdy}{y^2} + 3x^2e^{x^3}dx = 0.$$

$$d\left(\frac{x}{y}\right) + d\left(e^{x^3}\right) = 0$$

$$d\left(\frac{x}{y} + e^{x^3}\right) = 0.$$

Integration both sides.

$$\frac{x}{y} + e^{x^3} = C$$

Q. Solve $xdy - ydx = x\sqrt{x^2 - y^2}\, dx$.

Ans. Given $xdy - ydx = x\sqrt{x^2\left(1 - \frac{y^2}{x^2}\right)}\, dx$

$$xdy - ydx = x^2\sqrt{1 - y^2/x^2}\, dx$$

$$\frac{\frac{xdy - ydx}{x^2}}{\sqrt{1 - \frac{y^2}{x^2}}} = dx.$$

$$\frac{1}{\sqrt{1 - (y/x)^2}}\left(\frac{xdy - ydx}{x^2}\right) = dx$$

$$d\left(\sin^{-1}(y/x)\right) = dx.$$

$$\sin^{-1}(y/x) = x + c.$$

From eq^n ② we've,

$M = \frac{1}{y} - \frac{2}{x}$. $N = -\frac{x}{y^2} + \frac{3}{y}$

Sol^n is $\int\left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y}\,dy = c \Rightarrow \frac{1}{y}x - 2\log x + 3\log y = C$