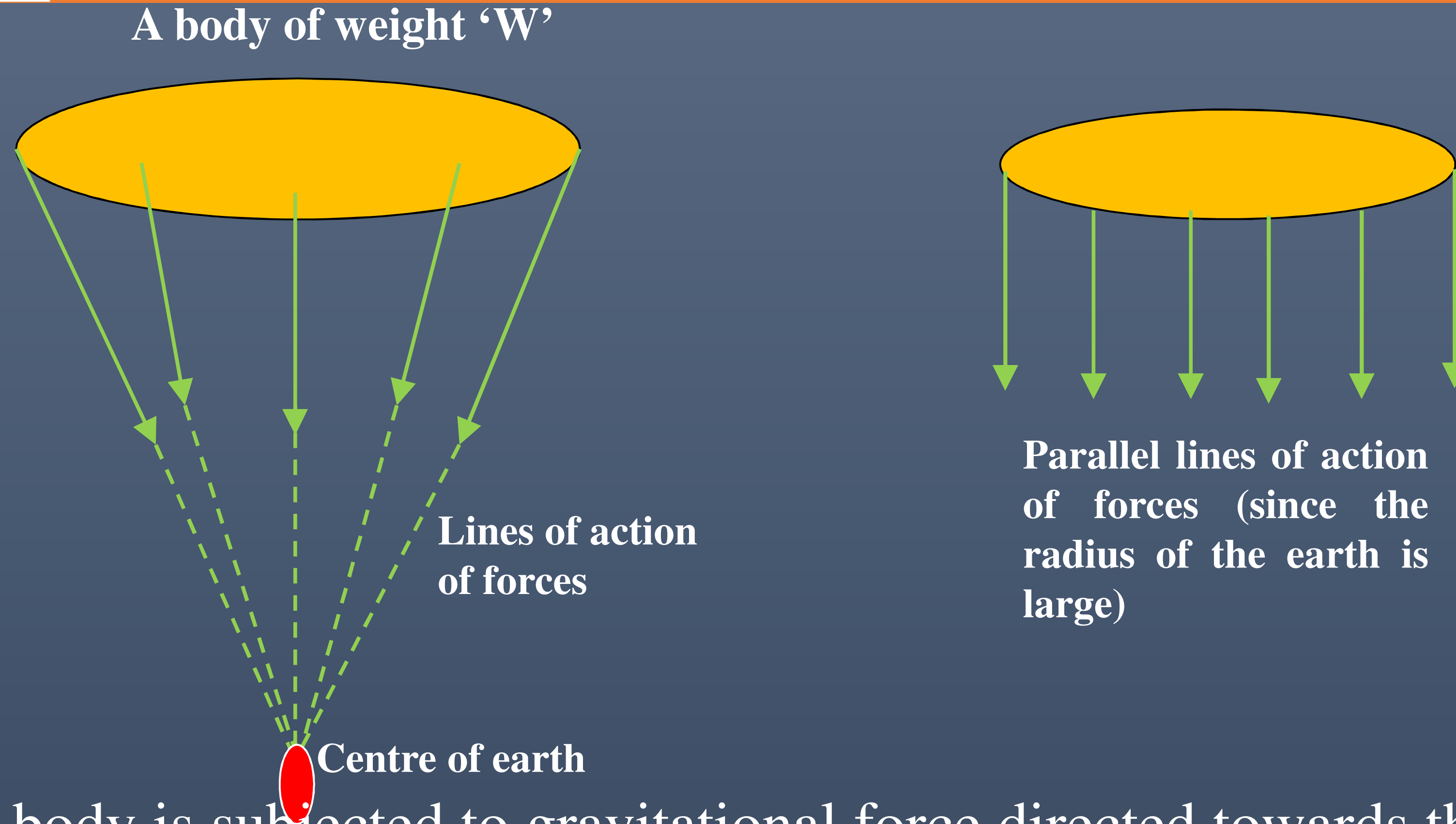
CENTROID
AND
MOMENT OF INERTIA

LECTURE 10

- Introduction
- Centre of Gravity, Centroid
- Centroid of Simple figures: using method of moment (First moment of area)
- Axis of Symmetry
- To locate centroid of plane areas – Rectangle and triangle

INTRODUCTION



A body is subjected to gravitational force directed towards the center of the earth.

The magnitude of this force depends on the mass of the body.

Since the size of the bodies are small when compared to the size of the earth, gravitational forces can be assumed to be parallel.

CENTRE OF GRAVITY

The point at which the whole weight of the body may be assumed to be concentrated.

The point through which the resultant gravitational force (weight) of the body acts for any orientation of the body.

A body has only one center of gravity for all positions of the body and is represented by **CG** or simply **G** or **C**

CENTROID

The term *"center of gravity"* applies to the bodies with mass and weight, while the term *"centroid"* applies to plane areas (when the calculation concerns a geometrical shape only).

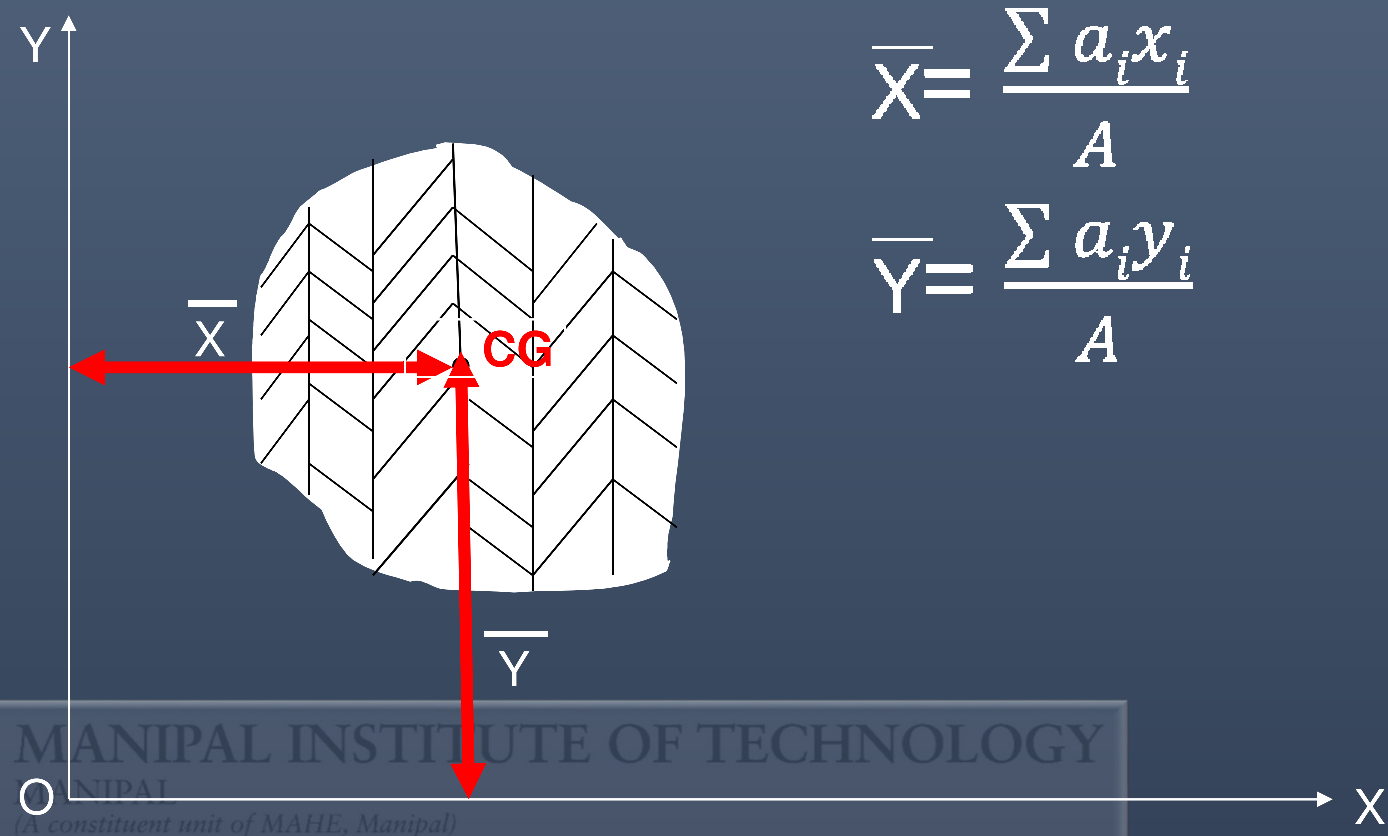
In case of plane figures such as rectangle, triangle, circle, semicircle, etc if the total area is concentrated at one and only one point it is defined as centroid.

Centroid is the point in a plane area such that the moment of the area, about any axis, through that point is zero.

CENTROID OF SIMPLE FIGURES

Using method of moment (First moment of area)

Moment of Total area 'A' about y-axis = Algebraic Sum of moment of elemental '*d*A' about the same axis



$$\bar{X} = \frac{\sum a_i x_i}{A}$$

$$\bar{Y} = \frac{\sum a_i y_i}{A}$$

AXIS OF SYMMETRY

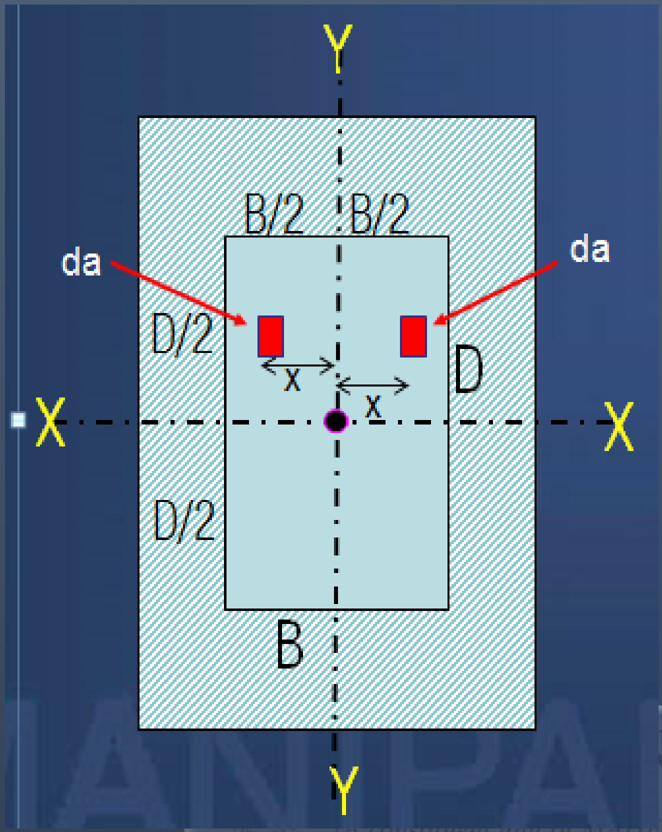
It is an axis w.r.t. which for an elementary area on one side of the axis , there is a corresponding elementary area on the other side of the axis (the first moment of these elementary areas about the axis balance each other)

- If an area has an axis of symmetry, then the centroid must lie on that axis.

- If an area has two axes of symmetry, then the centroid must lie at the point of intersection of these axes.

For example:

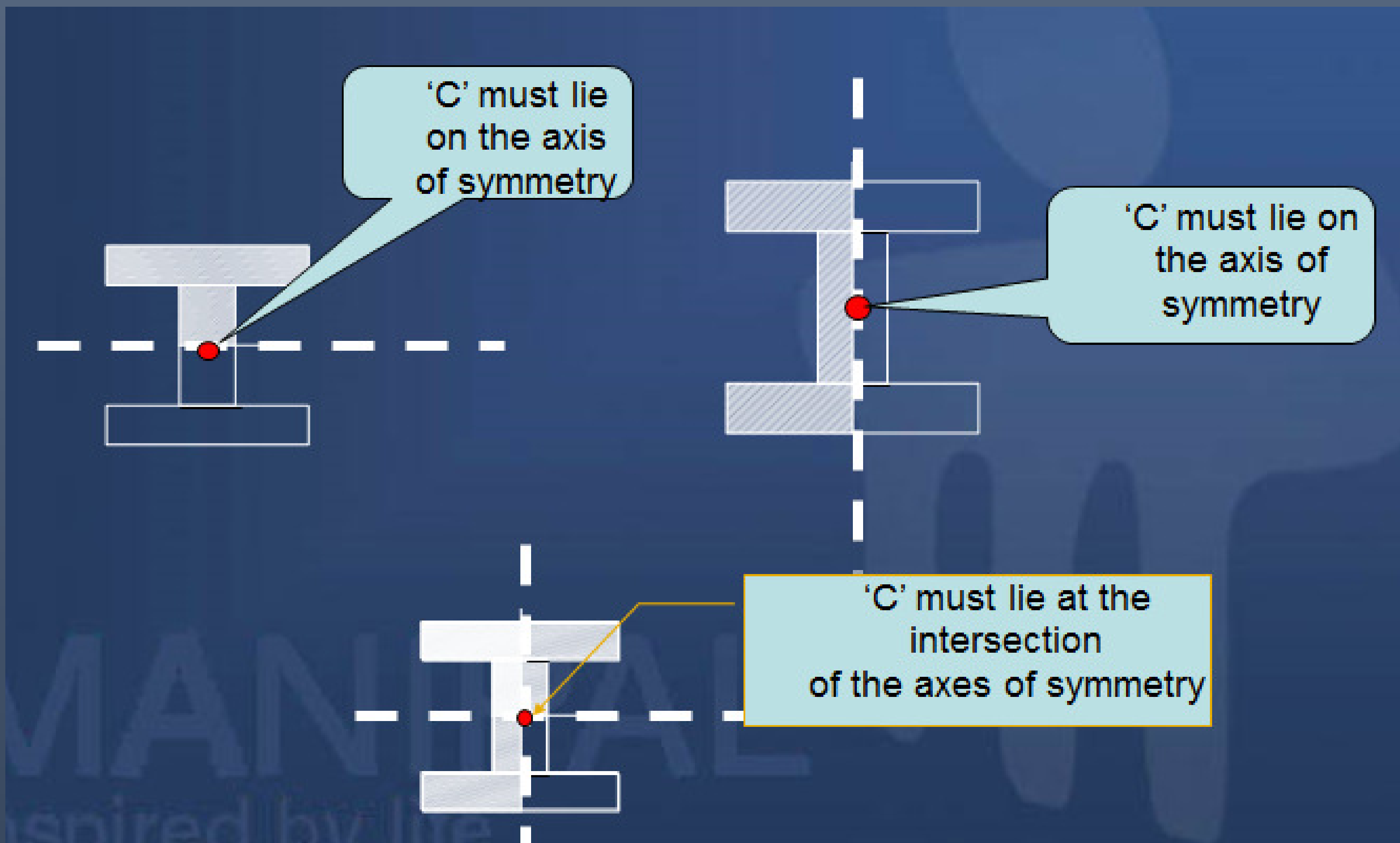
The rectangular shown in the figure has two axis of symmetry, X-X and Y-Y. Therefore intersection of these two axes gives the centroid of the rectangle.



Moment of areas, ‘da’ about y-axis cancel each other

$$da \times (-x) + da \times x = 0$$

$$da \times x = da \times x$$

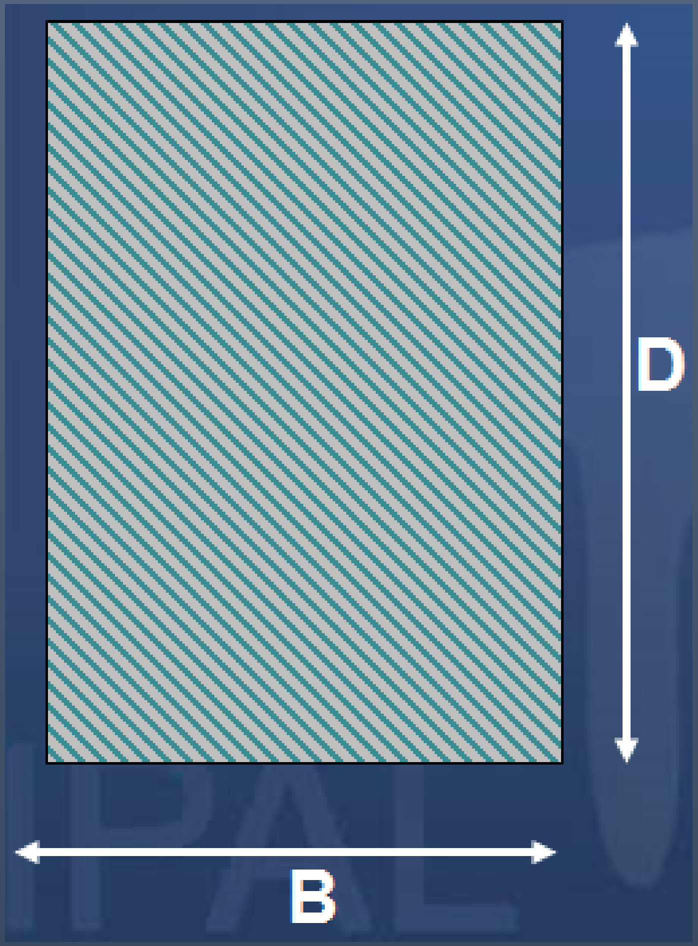

'C' must lie on the axis of symmetry
'C' must lie on the axis of symmetry
'C' must lie at the intersection of the axes of symmetry

CENTROID OF SIMPLE FIGURES

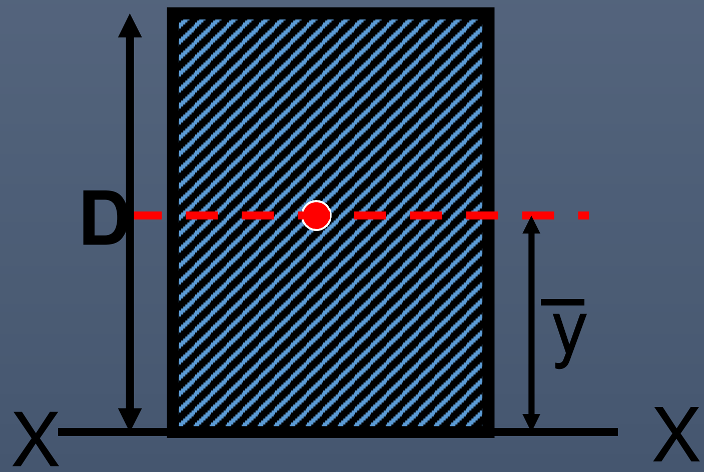
Using method of moment (First moment of area)

- Rectangle
- Triangle

To locate the centroid of rectangular area from first principles



To locate the centroid w.r.t. the base line x-x



Let the distance of centroid from the base line x-x be $\bar{y}$

Then from the the ***Principle of Moments***

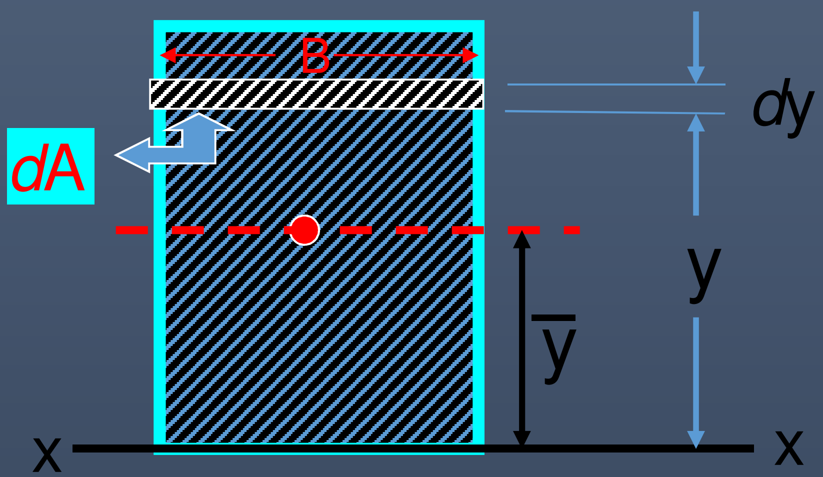
$$A \cdot \bar{y} = \int y \cdot da$$

Moment of Total area ***A*** about *x*-axis	=	Sum of moment of elemental area ***dA*** about the same axis

Contd.

Consider a elemental area ***dA*** at a distance ***y*** from the base line (x-x)

Let the thickness of the element be '***dy***'



Area of small element

= ***dA* = B .*dy***

Moment of this elemental area about x-x axis

= (area) x (distance)

= (B.dy) . (y)

Contd.

Sum of Moment of all such elemental areas comprising the total area =

$$= \int y \cdot da$$

$$= \int B \cdot dy \cdot y$$

$$= \left[\frac{By^2}{2} \right]_0^D \qquad = \left[\frac{BD^2}{2} \right]$$

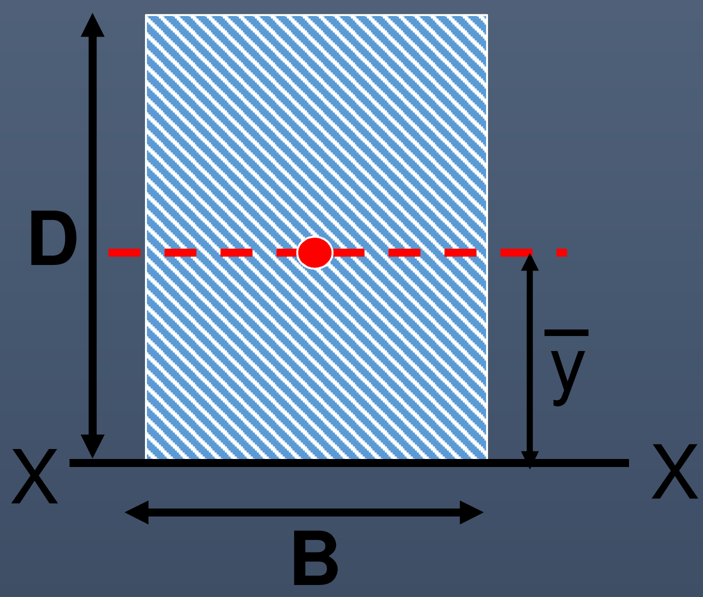
Then from the ***Principle of Moments***

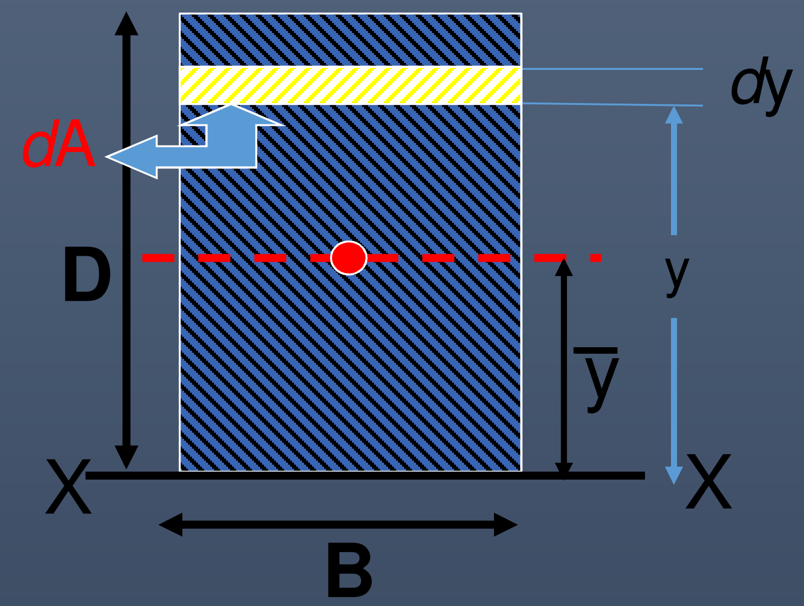
$$A\bar{y} = \frac{BD^2}{2}$$

$$\bar{y} = \frac{BD^2}{2A}, \qquad \bar{y} = \frac{BD^2}{2BD}, \qquad \boxed{\bar{y} = \frac{D}{2}}$$

Contd.

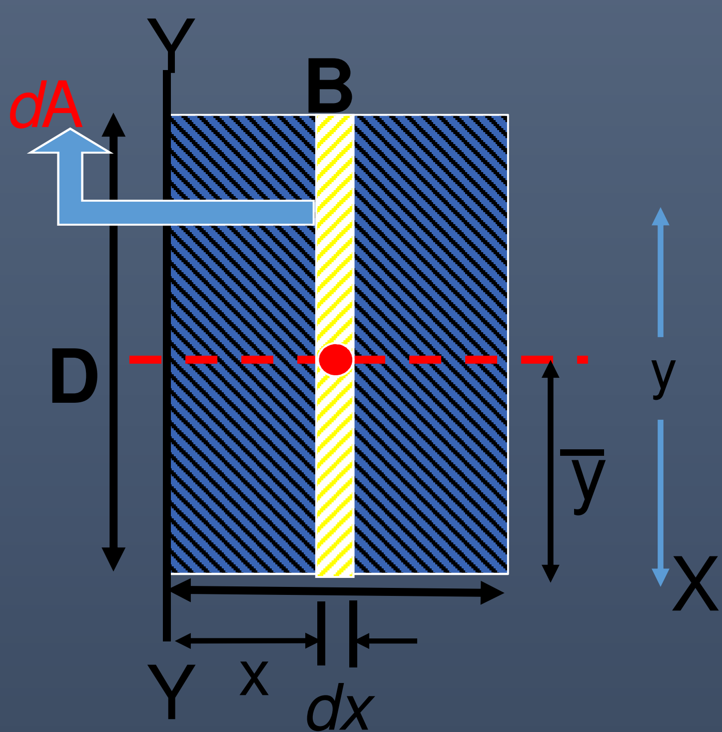
To locate centroid w.r.t the base line x-x

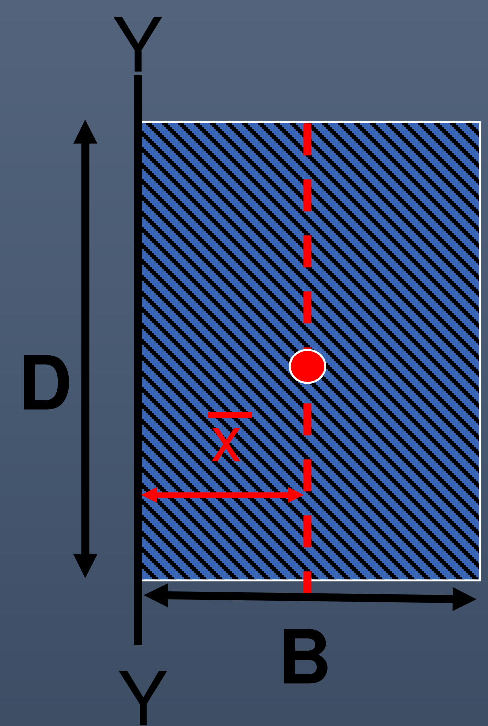




$\bar{y} = D/2$

Similarly, we can show

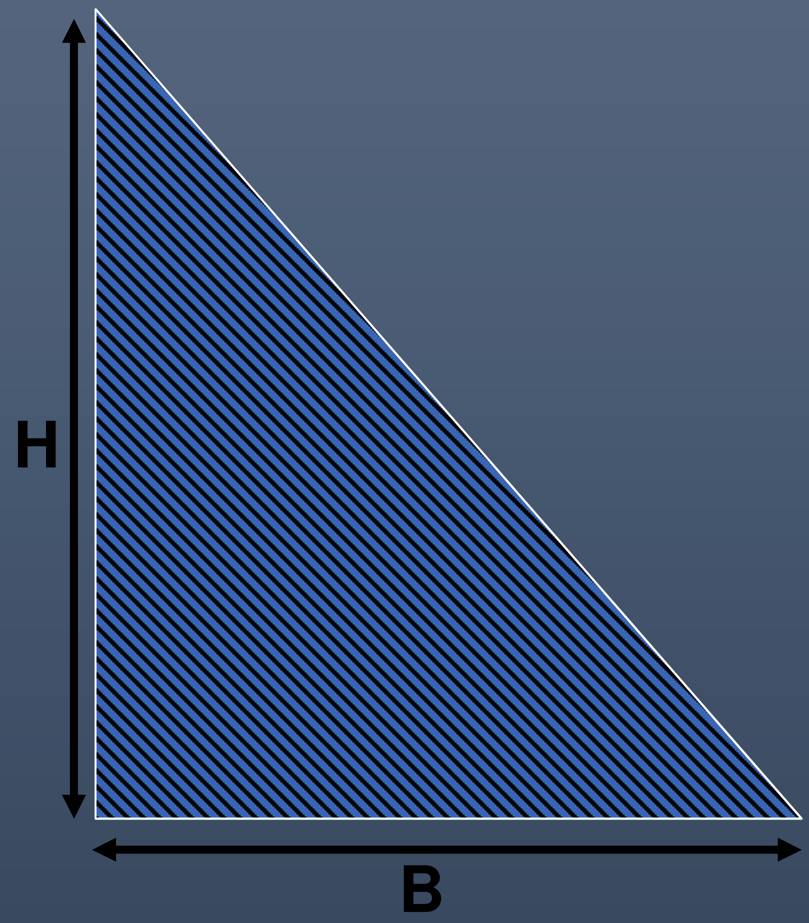




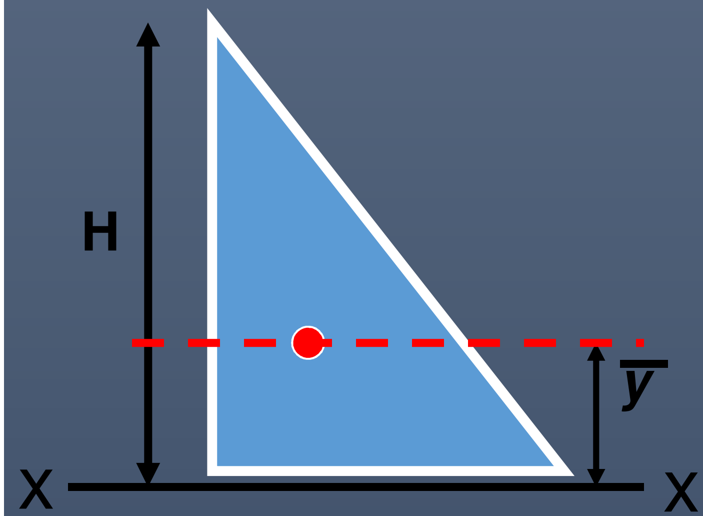
$\bar{x} = B/2$

LECTURE 11

To locate the centroid of right angled triangular area from first principles



To locate the centroid w.r.t. the base line x-x.



Let the distance of centroid from the base line x-x be $\bar{y}$

Then from the ***Principle of Moments***

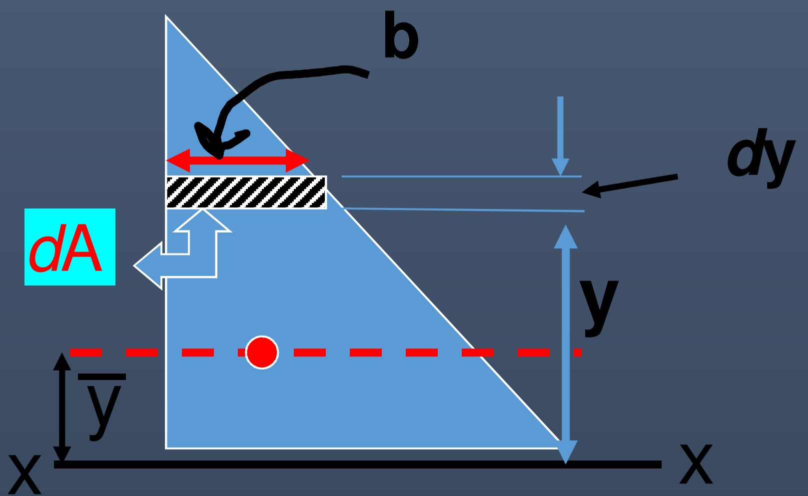
$$A \cdot \bar{y} = \int y \cdot da$$

Moment of Total area A about x-axis = *Algebraic Sum of moment of elementary area 'dA' about the sam*

Contd.

Consider a small elemental area '***d*A**' at a distance '***y***' from the base line (x-x)

Let the thickness of the element be '***d*y**'



Area of small element
= ***d*A = b .*d*y**

Moment of this small elemental area about x-x axis

= (area) x (distance)

= (b.dy) . (y)

Contd.

Then from the ***Principle of Moments***

$$A \cdot \bar{y} = \int y \cdot da$$

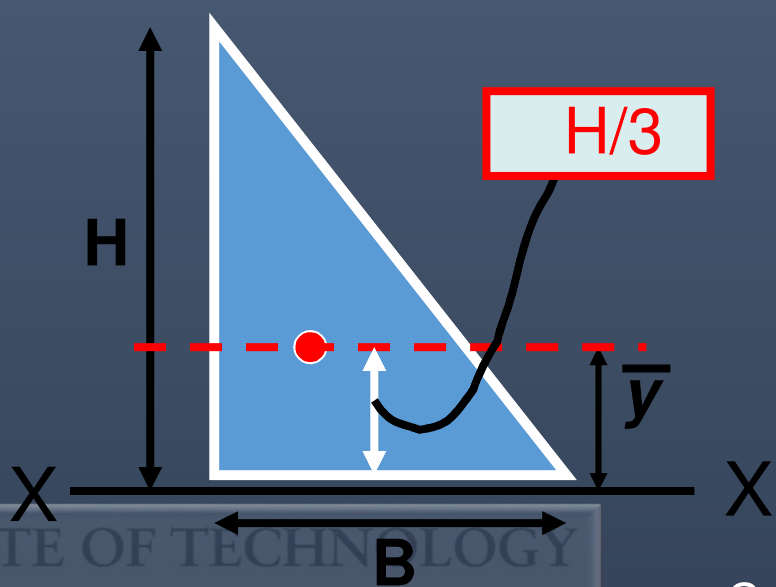
$$da = b \times dy = \frac{B(H-y)}{H} \times dy$$

$$A \cdot \bar{y} = \int b \cdot dy \cdot y$$

$$b = \frac{B(H-y)}{H}$$

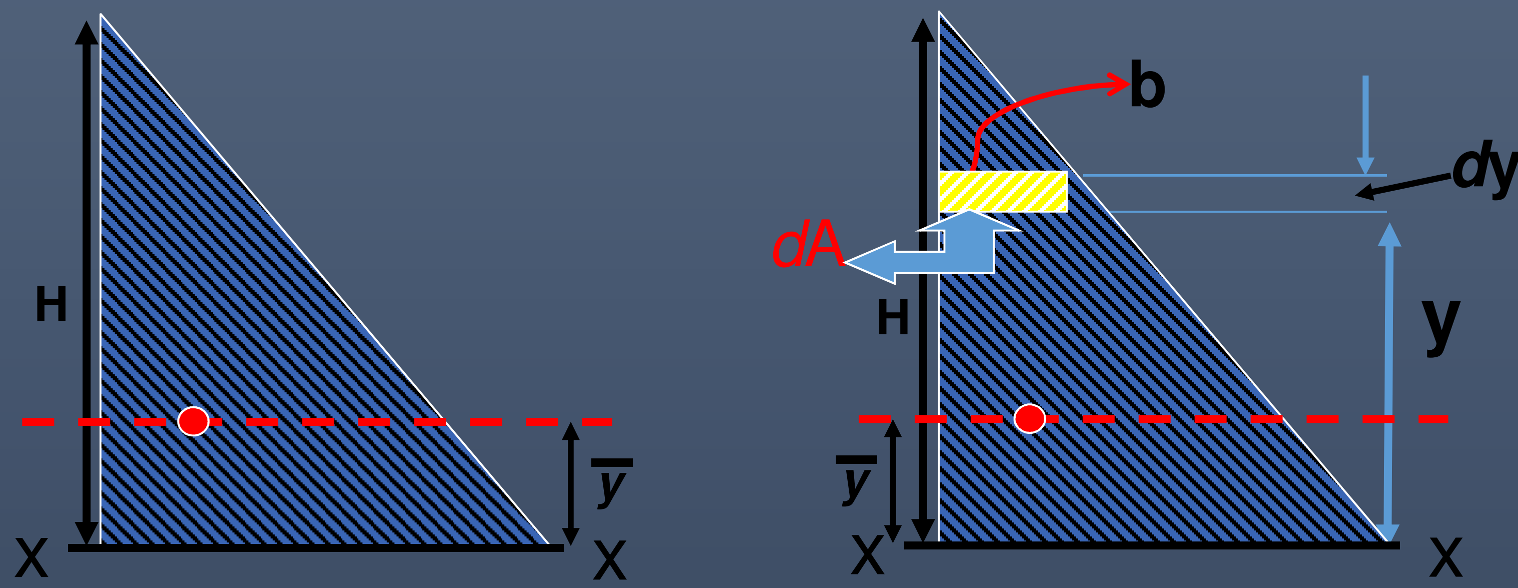
$$A \cdot \bar{y} = \int \frac{B(H-y)}{H} \cdot dy \cdot y$$

$\bar{y}$ = H/3



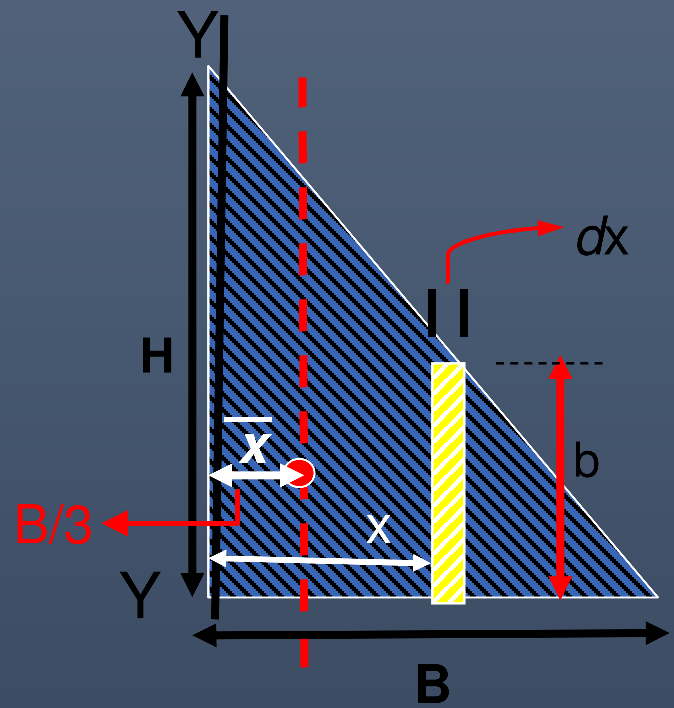
Contd.

To locate centroid w.r.t the base line x-x



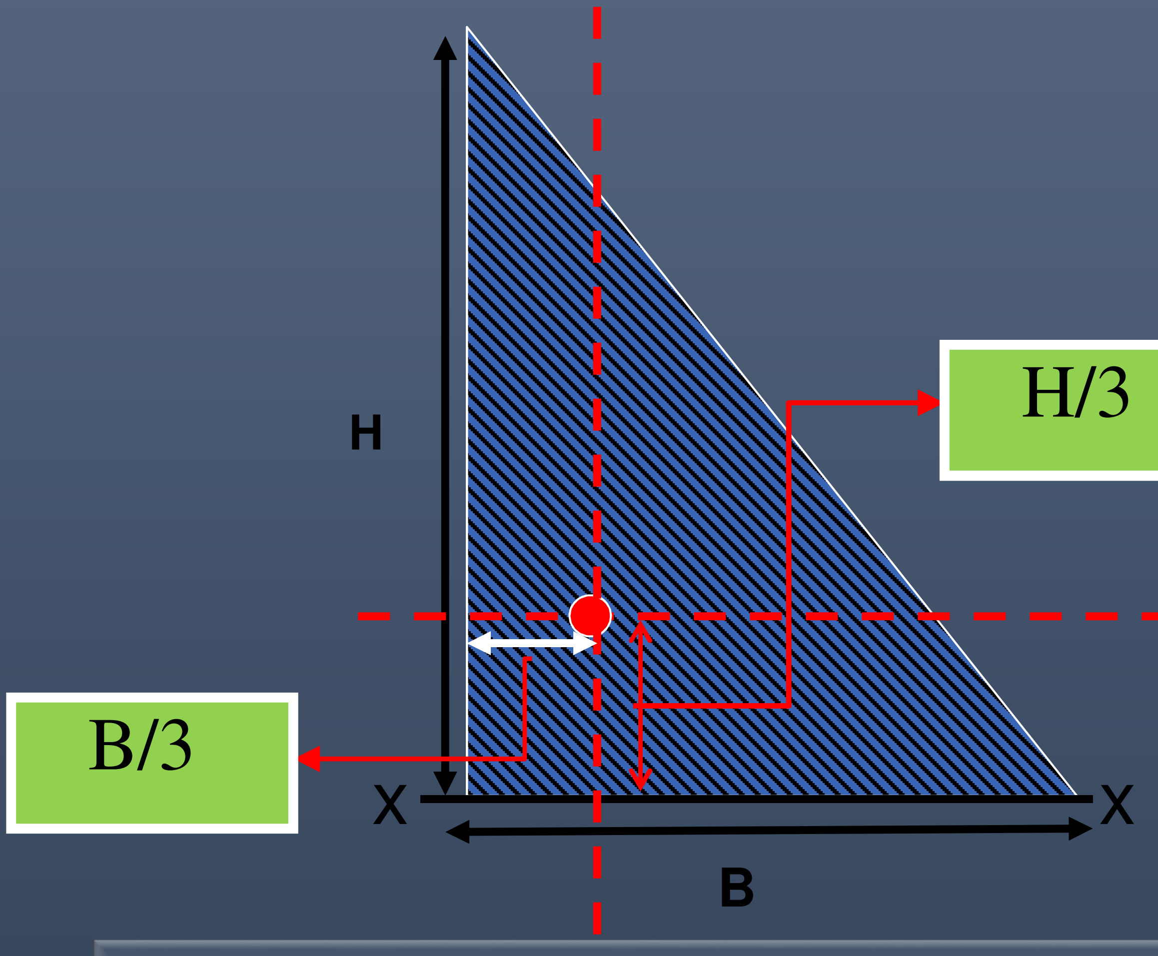
$\bar{y} = H/3$

Similarly, we can show

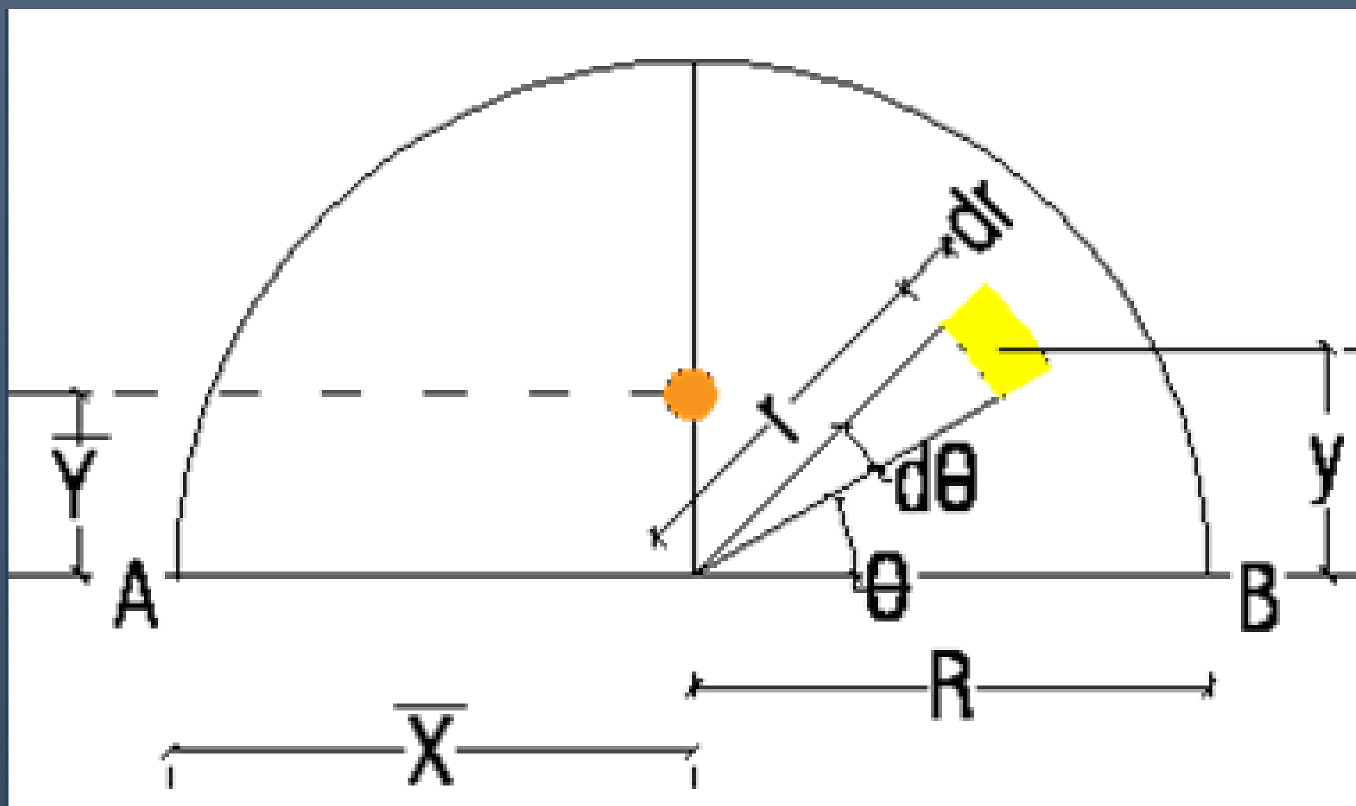


$$\bar{x} = B/3$$

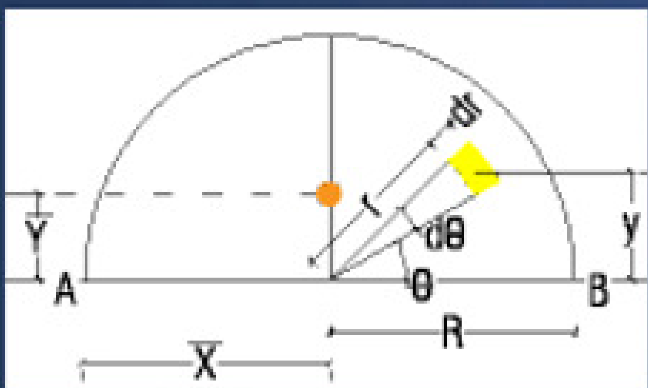
The centroid of right angled triangular area from the base



To locate the centroid of Semi Circular area w.r.t diameter from first principles



To locate the centroid of Semi Circular Area w.r.t. the diameter AB from first principles



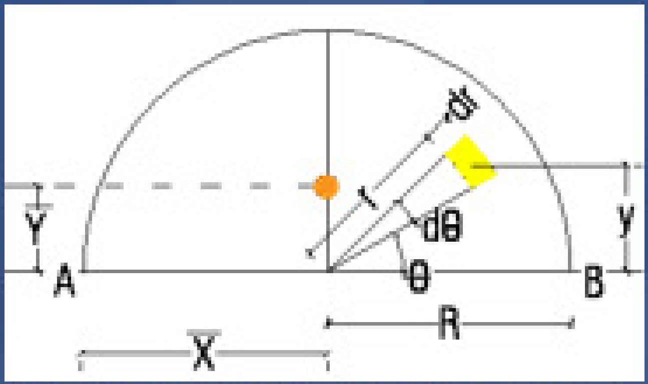
Consider a semicircle of radius R,

A = area = $\frac{\pi D^2}{8}$ D = 2R

Let 'G' be the centroid of the Semicircle, and $\bar{y}$ is its distance from the diameter AB.

Contd.

Consider a small elemental area ***da***, located at distance **y** from the diameter **AB**,



Let ‘**r**’ =radial distance of area ‘*d*a’ from centre of the semi circle.

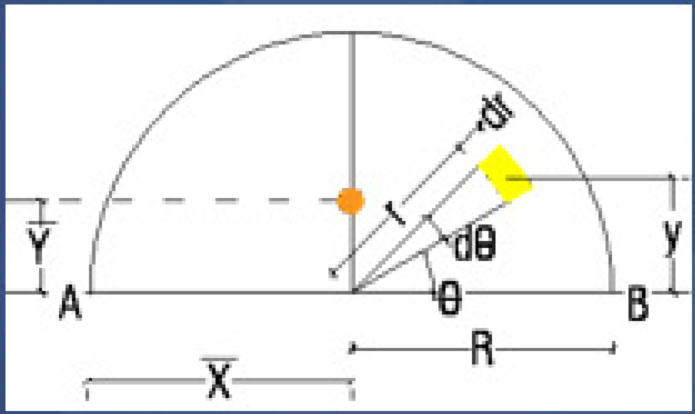
$$da = r \times d\theta \times dr$$

$$y = r \times \sin\,\theta$$

Moment of this elemental area about the diameter AB =

$$= r^2 \sin\,\theta \cdot dr \cdot d\theta$$

Moment of all such elemental area '*da*' about the diameter AB



$$= \int_0^{\pi}\int_0^{R} r^2 \cdot \sin\theta \cdot dr.d\theta$$

$$= \int_0^{\pi}\left[\frac{r^3}{3}\right]_0^R \cdot \sin\theta \cdot .d\theta$$

$$= \frac{R^3}{3}\left(-\left[\cos\theta\right]_0^{\pi}\right)$$

$$= \frac{2.R^3}{3}$$

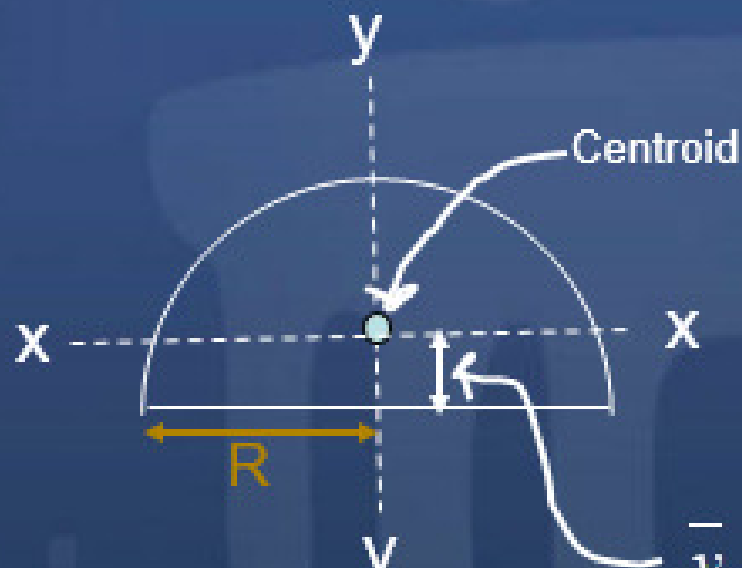
Then from the ***Principle of Moments***

$$A \cdot \bar{y} = \int y \cdot da$$

$$A \cdot \bar{y} = \int_0^{\pi}\int_0^{R} r^2 \cdot \sin\theta \cdot dr.d\theta$$

$$A \cdot \bar{y} = \frac{2.R^3}{3}$$

$$\boxed{\bar{y} = \frac{4R}{3\pi}}$$



$$\bar{y} = \frac{4R}{3\pi}$$

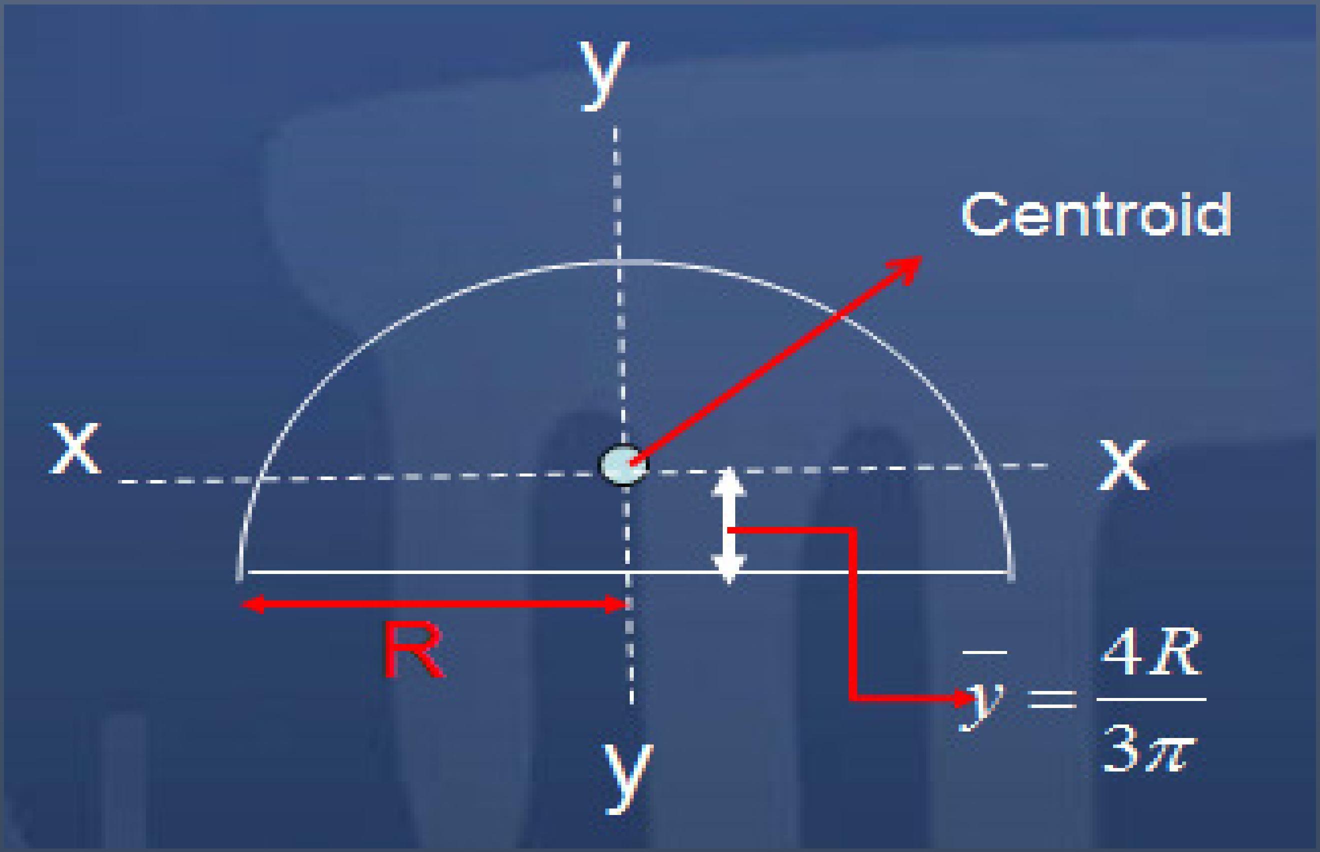
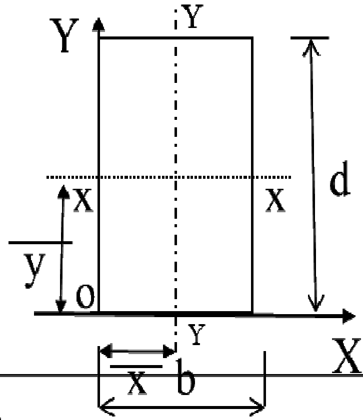
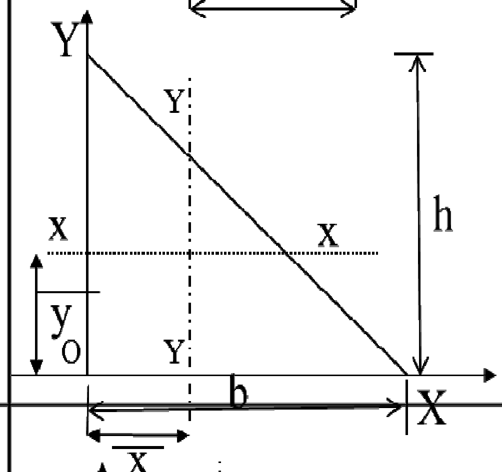
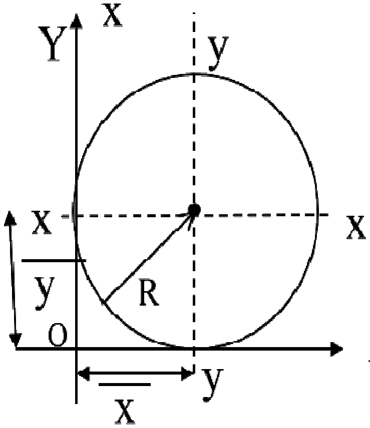
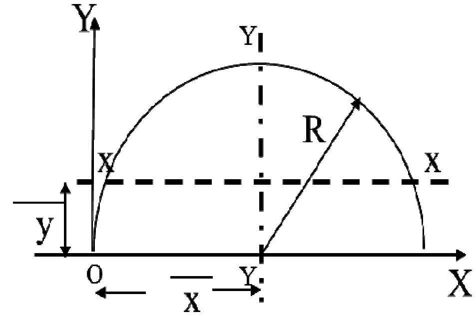
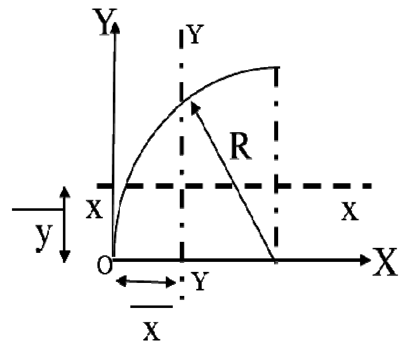
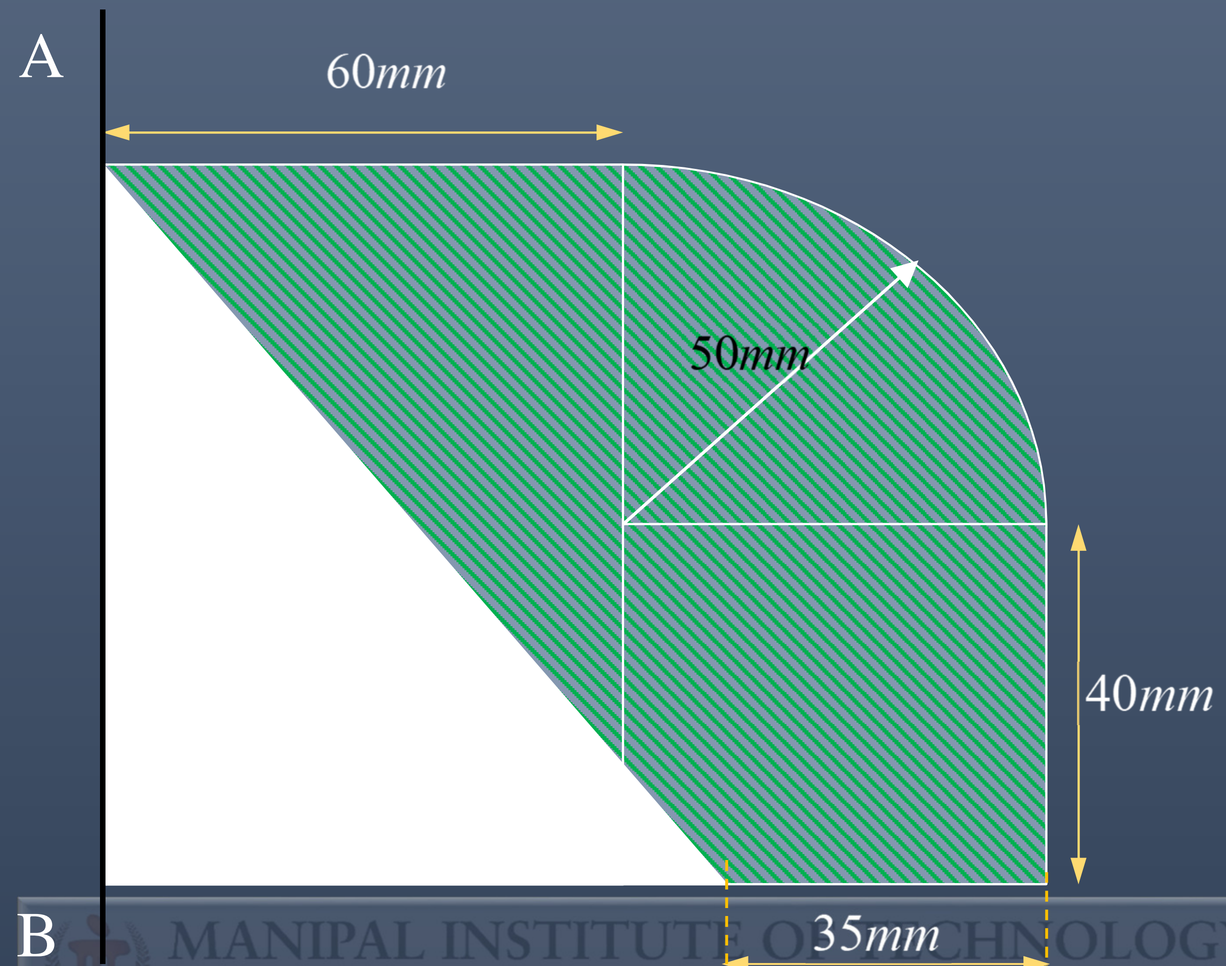
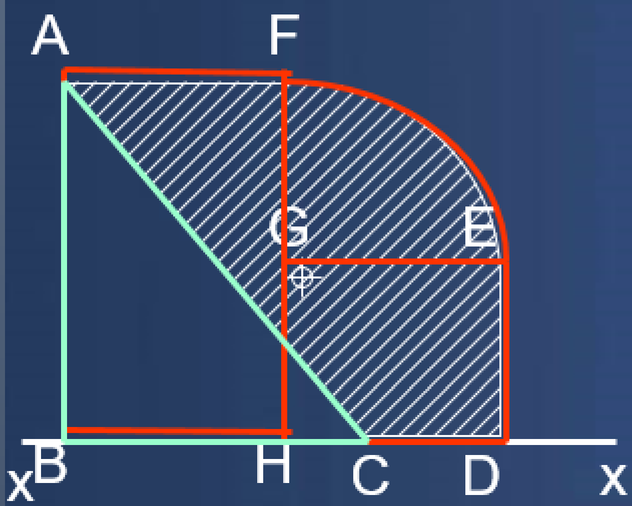

y
Centroid
X
X
R
$\bar{y} = \frac{4R}{3\pi}$
y

	FIGURE	$\bar{x}$	$\bar{y}$
1		b/2	d/2
2		b/3	h/3
3		R	R
4		R	$4R/3\pi$
5		R- 4R/3 π	$4R/3\pi$

TUTORIAL 4

T1. Locate the centroid of the shaded area shown in figure.





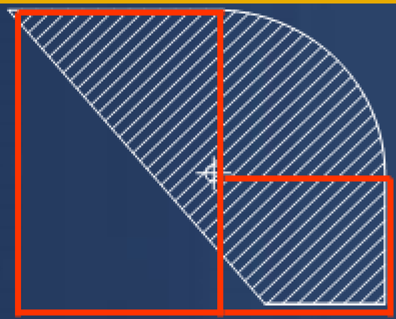
a_1= Rectangle- ABHF

a_2= Rectangle- GHDE

a_3= Quarter of circle- GEF

$-a_4$= Triangle- ABC

Now the total shaded area = $a_1+a_2+a_3+(-a_4)$

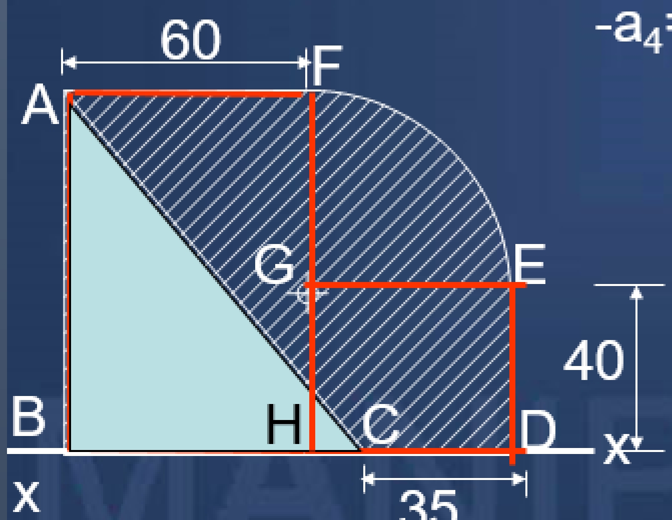


a_1= Rectangle- 60x90=5400

a_2= Rectangle- 40x50=2000

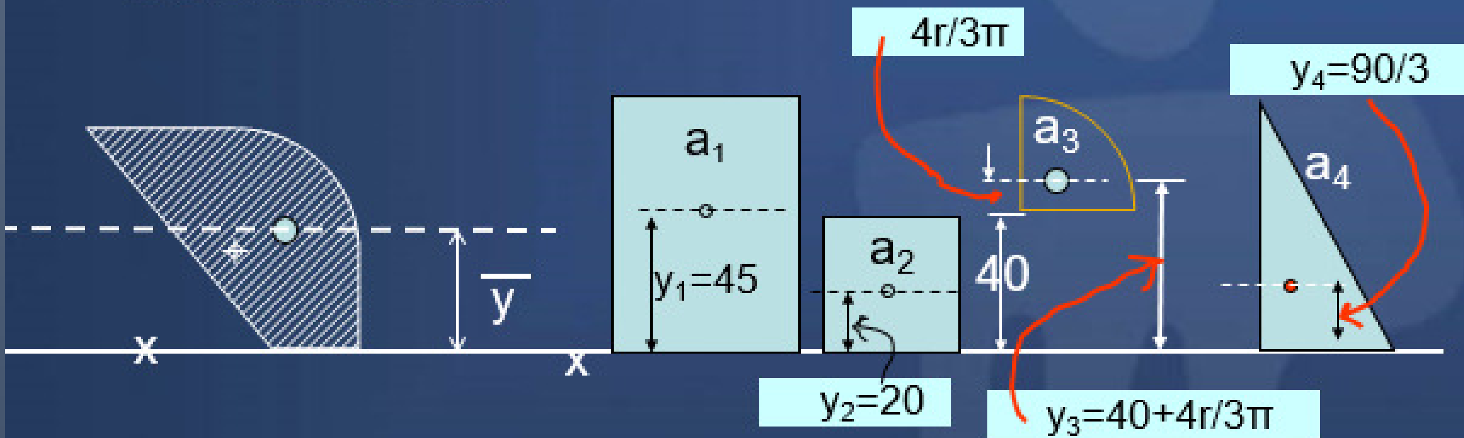
a_3= Quarter of circle- (¼)πr^2=1963.5

$-a_4$= Triangle- ½ (75x90)= (-3375)



Total shaded area=5988.5

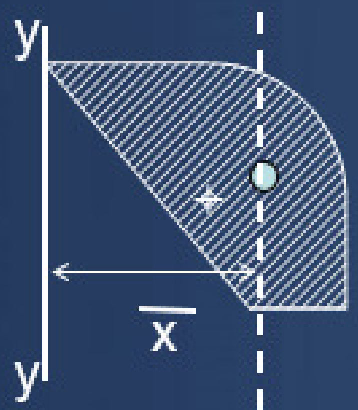
To find $\bar{y}$ apply the principle of moments about the reference line **'xx'**

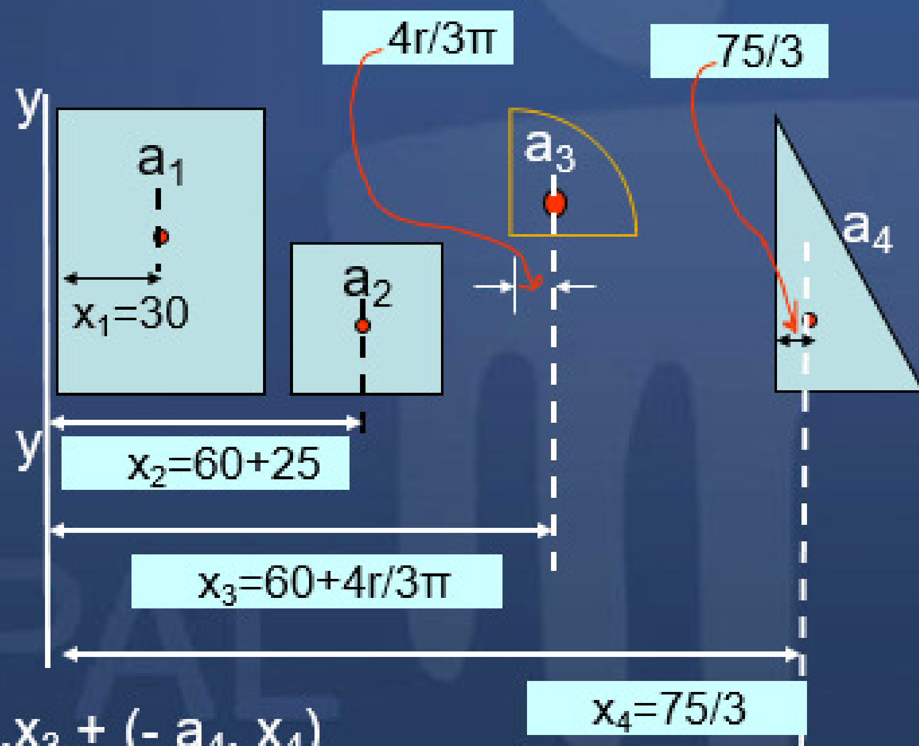


$$A\,\bar{y} = a_1.y_1 + a_2.\,y_2 + a_3.y_3 + (-\,a_4.\,y_4)$$

$\bar{y}$ **= 50.42**

To find $\overline{x}$ apply the principle of moments about the reference line **'yy'** ***shown***



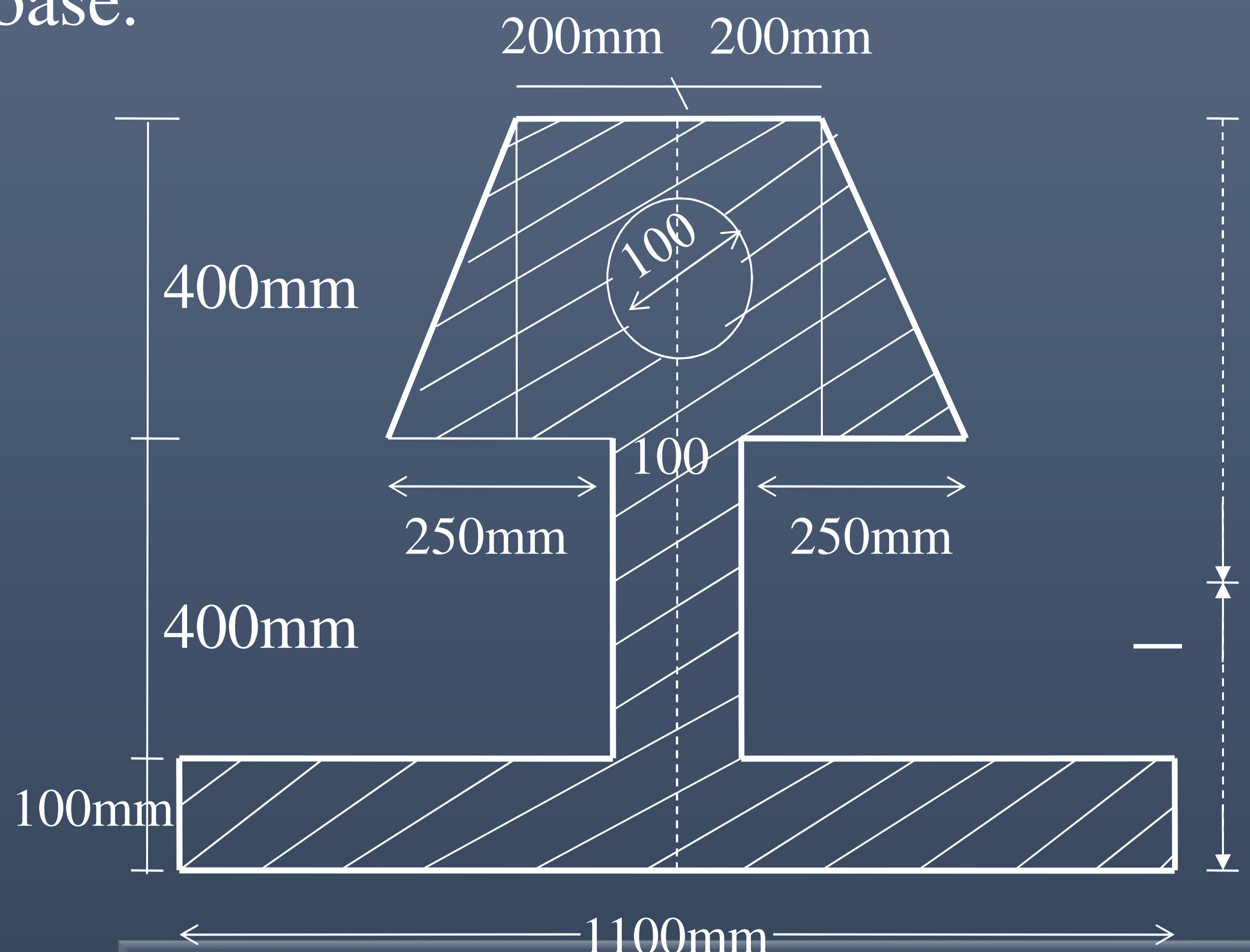


$$A\,\overline{y} = a_1.x_1 + a_2.\ x_2 + a_3.x_3 + (-\ a_4.\ x_4)$$

$\overline{\mathbf{x}}$ **= 67.98**

Contd.

T2. Determine the centroid of shaded area with respect to the base.



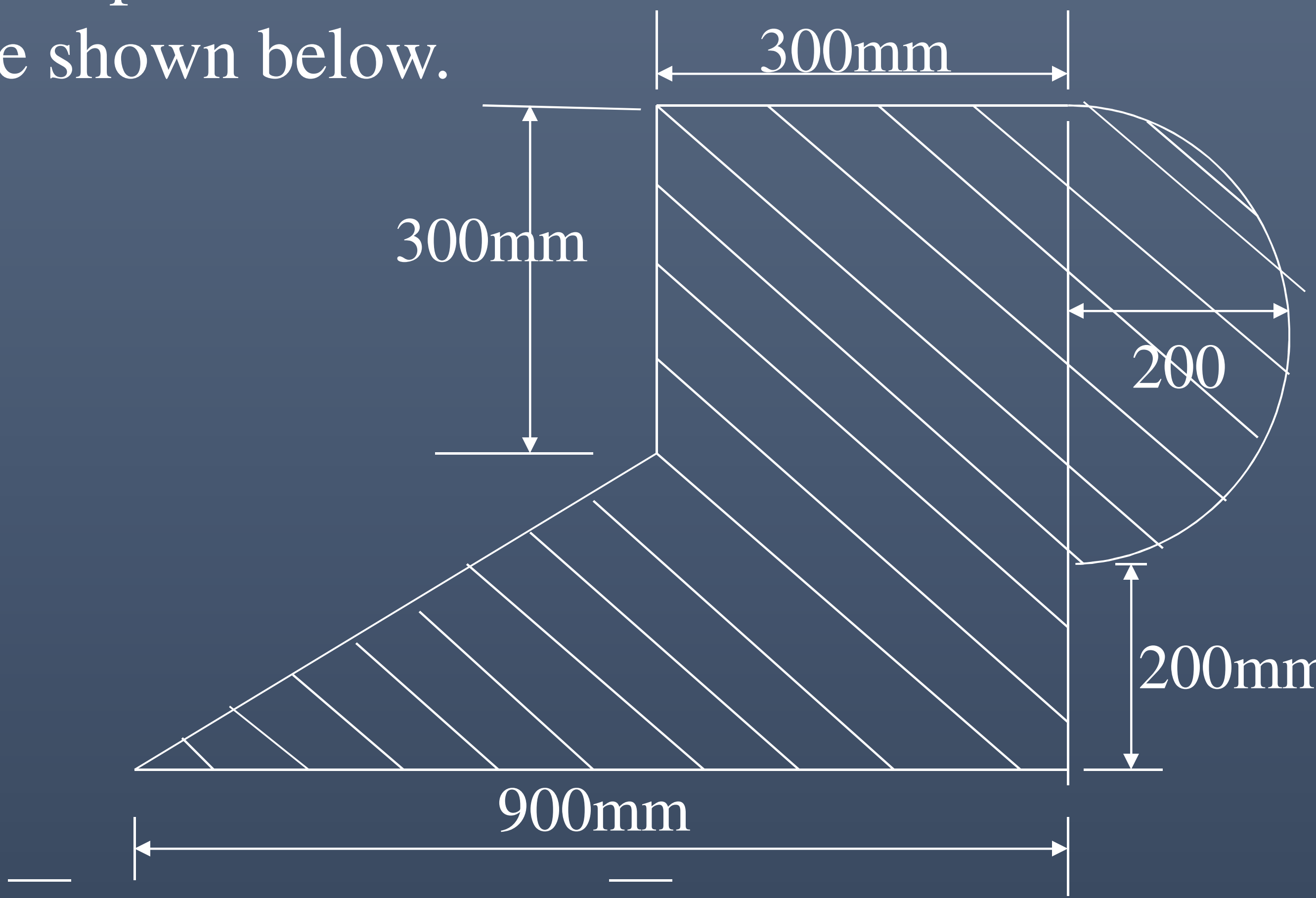
$$\sum A = 1100*100+400*100+400*400+2(1/2*100*400)-\pi*50^2$$

$$=3,42,150\ mm^2$$

$$\sum AY = 1100*100*50+100*400*300+400*400*700+[(1/2)*100*400*633.3]*2 - \pi*50^2*700 \quad =14,93,38,200 mm^3$$

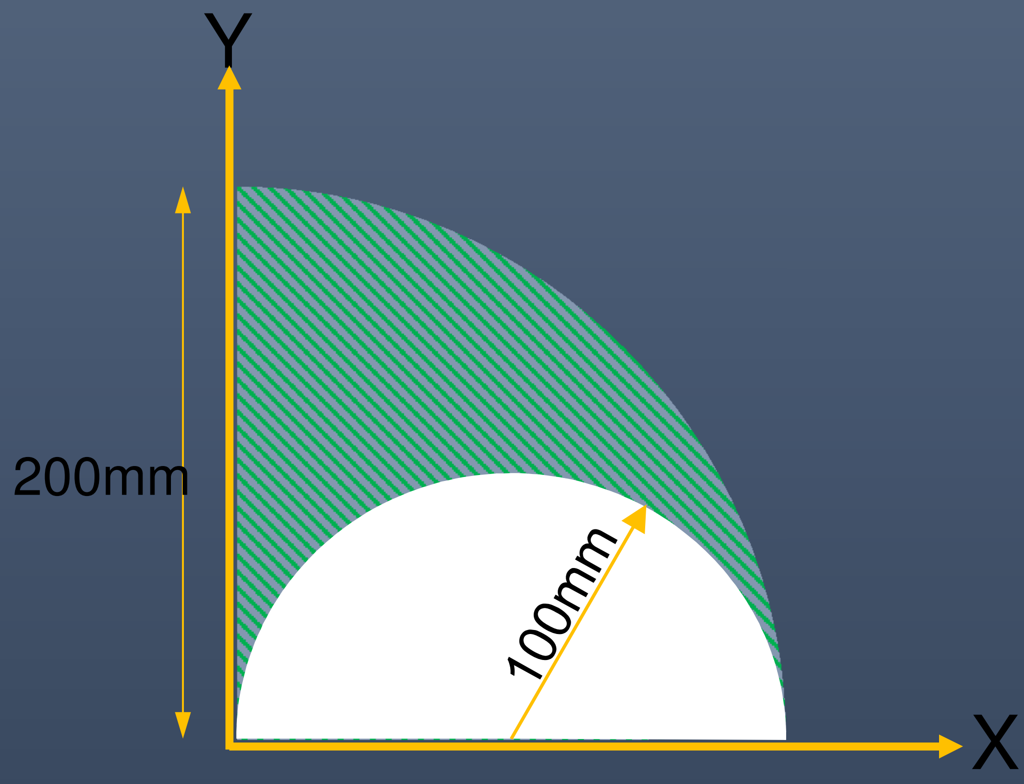
$$\overline{Y} = \sum AY / \sum A = 436.5mm$$

T3. Locate the position of horizontal and vertical centroid of the figure shown below.



[Ans: X = 699.7mm , Y = 265 mm

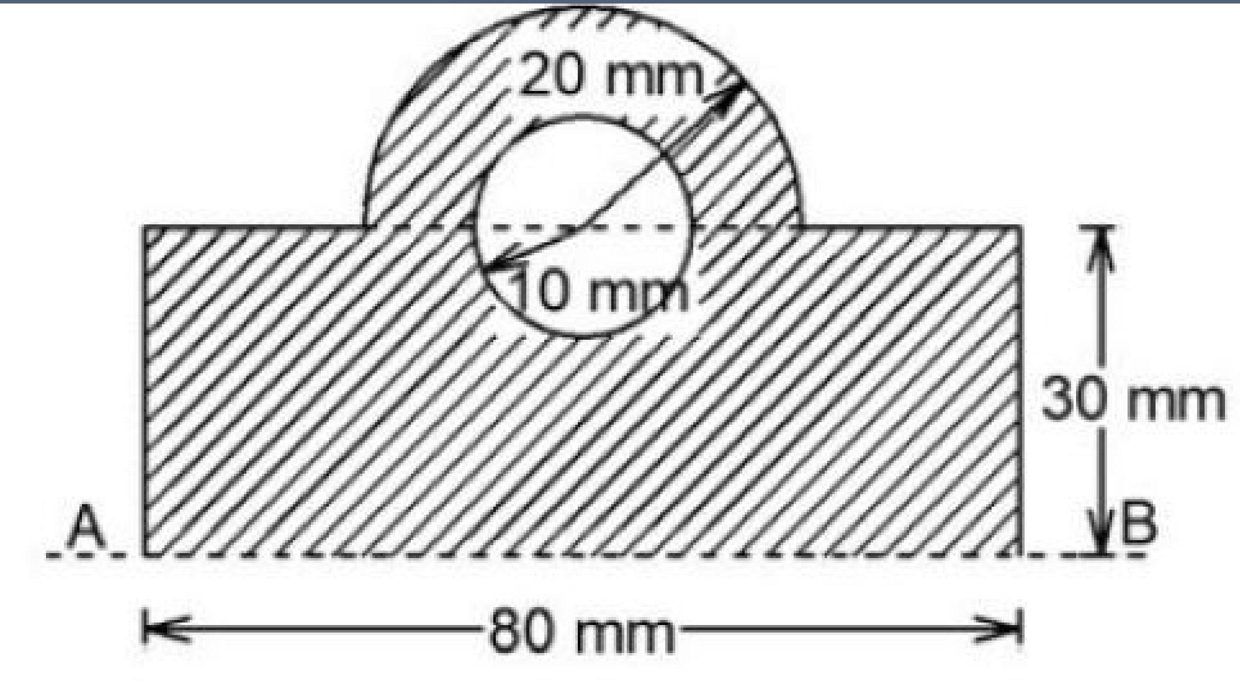
T4 . Locate the position of centroid of hatched portion with respect to reference axes shown.



[Ans: X = 19.99mm , Y = 42.44 mm

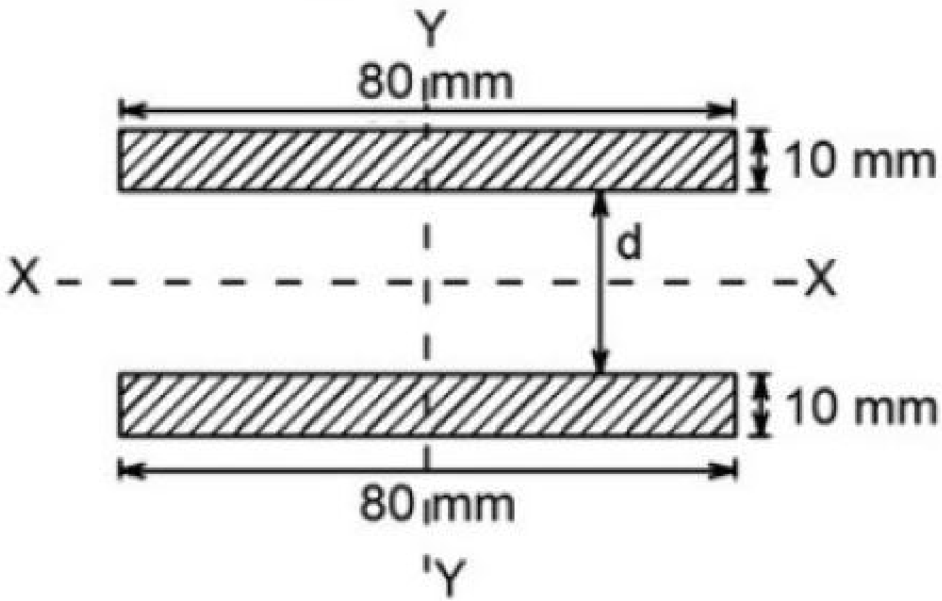
TUTORIAL – ADDITIONAL PROBLEMS

1. Determine the centroid of shaded area w.r.t given reference axis AB.



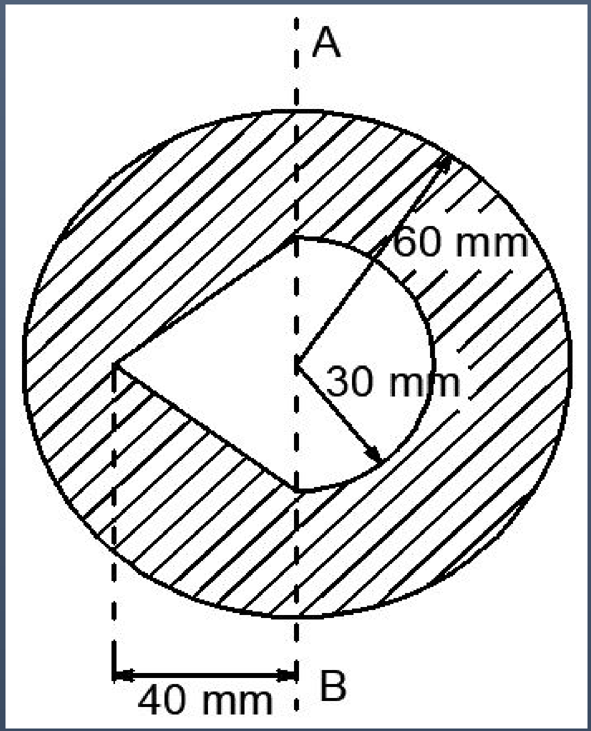
Ans: 18.7mm

2. The shaded area shown in the figure is symmetrical about X-X and Y-Y axes marked. Determine the distance 'd', such that, for the total shaded area shown, $I_{XX}=I_{YY}$.



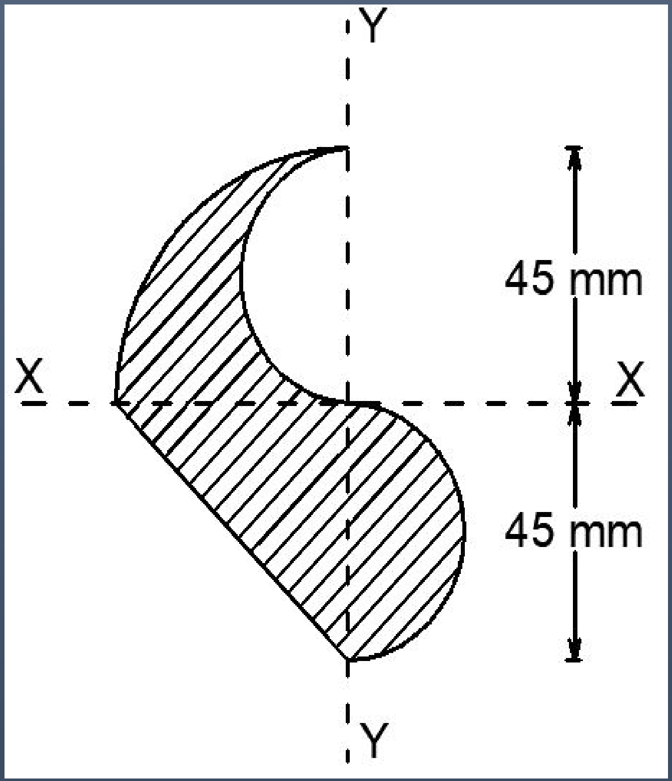
Ans: 35.82mm

3. Calculate the centroid of shaded area shown in figure with respect to given reference axis AB.



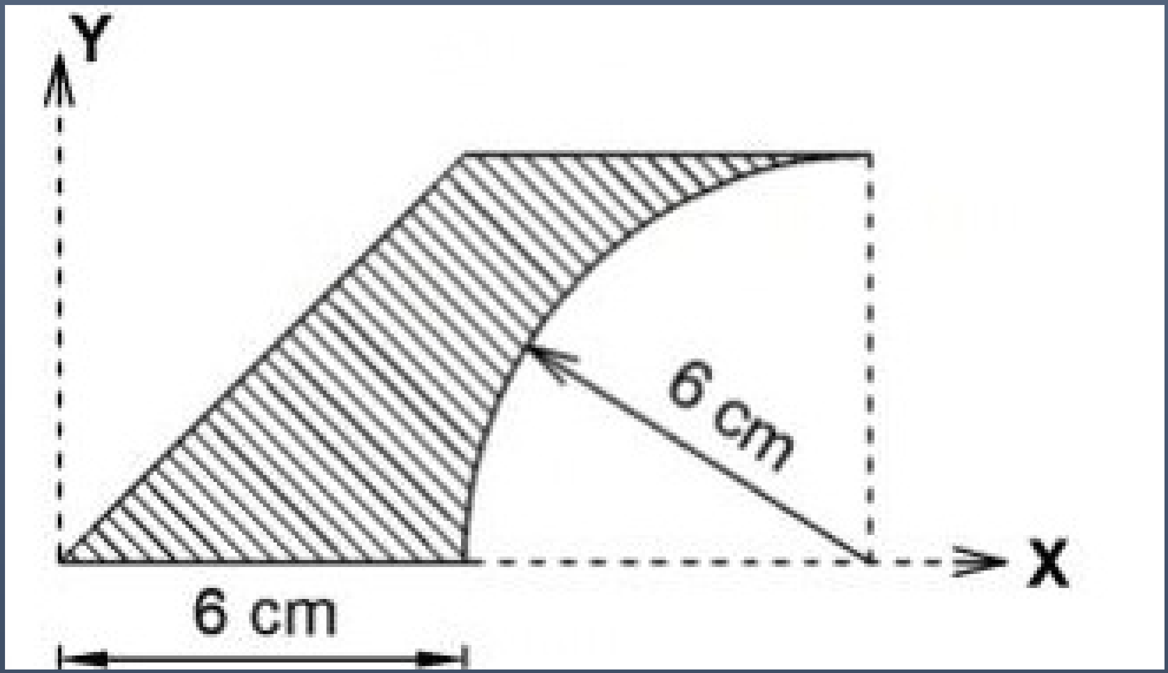
Ans: -0.23mm

4. Locate the centroid of shaded area with respect to the axes shown in the figure.(5 marks)



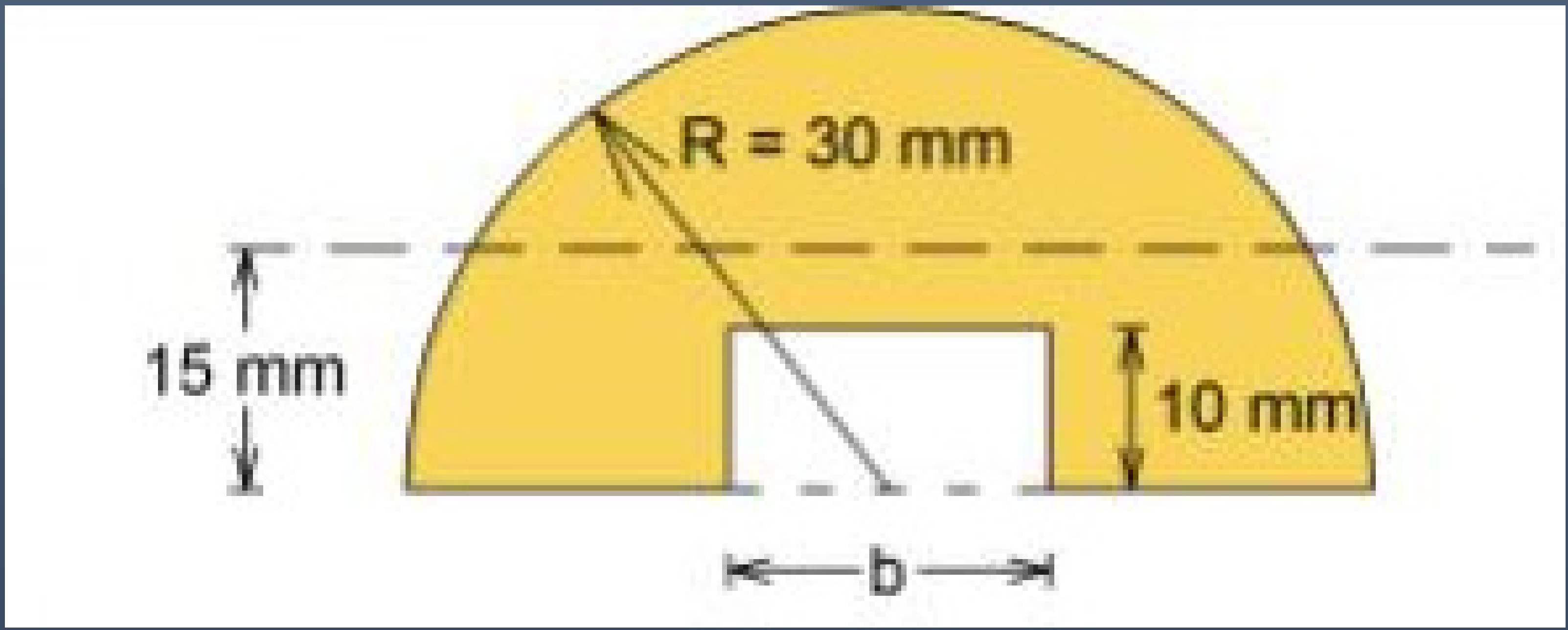
Ans: -11.67, 3.757mm

5. Locate the centroid with respect to the given axes shown in the figure for the shaded region.



Ans: x = 5cm , y = 2.8cm

6. In a semi-circular lamina a rectangular cut is made as shown in the figure. Determine the dimension 'b' of the rectangle, such that centroid of lamina is at a height of 15 mm from the base.



Ans : b = 32.06mm

LECTURE 12

MOMENT OF INERTIA

- Second Moment of Area.
- Radius of gyration.
- Perpendicular Axis Theorem.
- Parallel Axis Theorem.

SECOND MOMENT OF AREA

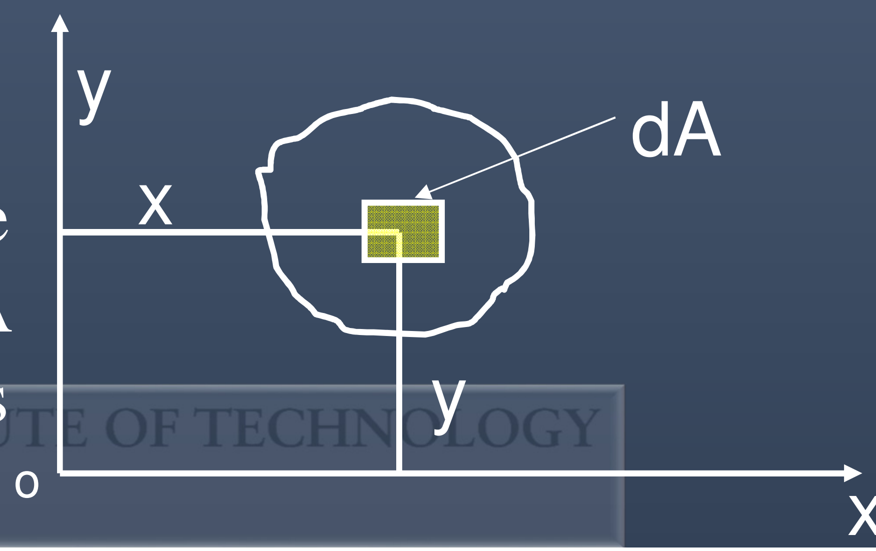
- The product of area and square of the distance from reference axis is called second moment of area about the reference axis.
- Also known as Moment of Inertia represented by 'I'.
- The axis about which the second moment of area of a plane figure is considered, is denoted by the subscripts.

$I_{ox} = \int y^2 dA$

$I_{oy} = \int x^2 dA$, respectively,

where:

'y' and 'x' are the distance of the elemental area dA from XX and YY axes respectively.

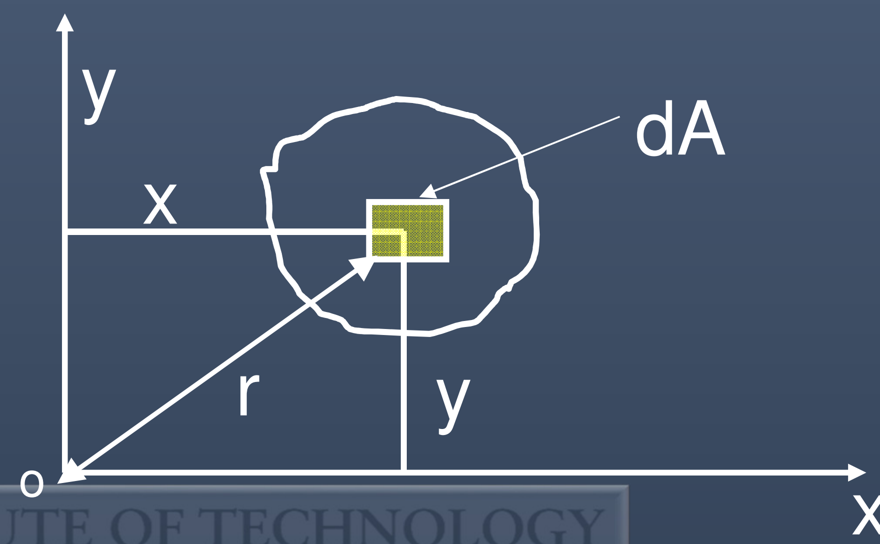


The moment of inertia of the entire area A about the pole 'O' (Z-axis) is, by similar definition

$I_{oz} = \int r^2 dA$, known as polar moment of inertia.

$I_{oz} = I_{ox} + I_{oy}$, i.e., $(r^2 = x^2 + y^2)$

The moment of inertia of an area is purely mathematical property of the area and in itself has no physical significance.



Second moment of area can be considered as the sum of a number of elements each consisting of an area multiplied by distance squared

$$I_{ox} = da_1\, y_1^2 + da_2\, y_2^2 + da_3\, y_3^2 + \text{--}$$

$$= \sum da\, y^2$$

$$I_{oy} = da_1\, x_1^2 + da_2\, x_2^2 + da_3\, x_3^2 + \text{----}$$

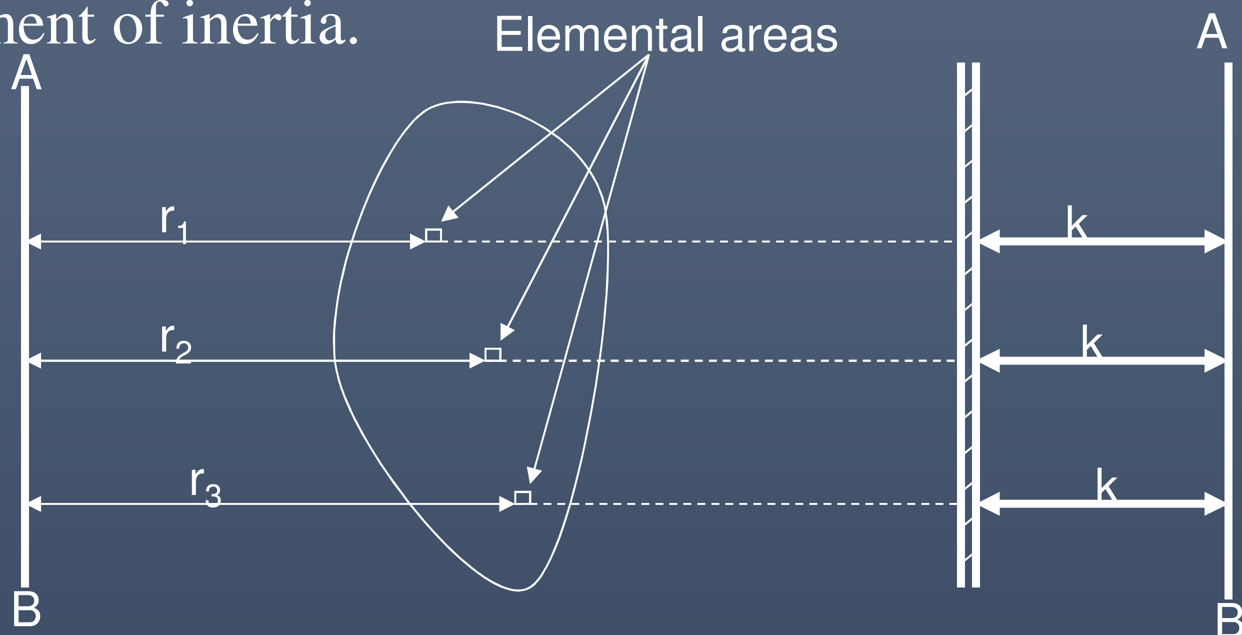
$$= \sum da\, x^2$$

- Second moment of area can be considered as the sum of a number of elements each consisting of an area multiplied by distance squared.
- The dimension of the second moment of area is given by L^4 and its unit is m^4.
- Sign of each term is +ve since the distance is squared.
- The first moment of area about the centroidal axis is zero where as the second moment of area about the centroidal axis is non zero.
- Moment of inertia plays a major role in design of beams, columns, machine and also helps in selection of members in structural design.

APPLICATION
OF
MOMENT OF INERTIA

RADIUS OF GYRATION

Radius of gyration is defined as a constant distance of all elemental areas which have been rearranged with out altering the total moment of inertia.



It is the distance from the axis to a point where the concentrated area of the same size could be placed to have the same second moment of area with respect to the given axis.

$I_{AB} = da\ k^2 + da\ k^2 + \text{----------}$ $\qquad$ $I_{AB} = A\ k^2$

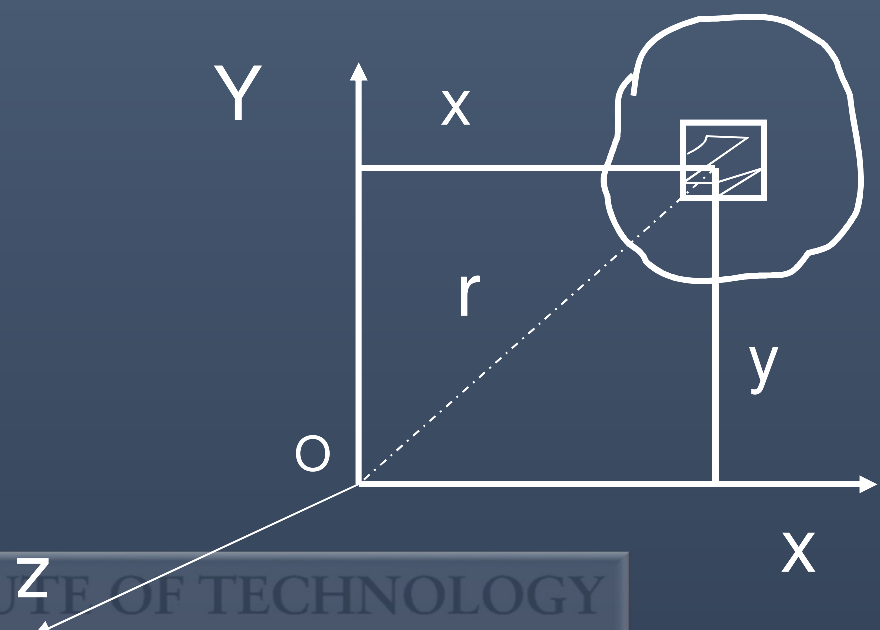
$I_{AB} = \sum da\ k^2$ $\qquad$ $k = \sqrt{I_{AB}/A}$

POLAR MOMENT OF INERTIA

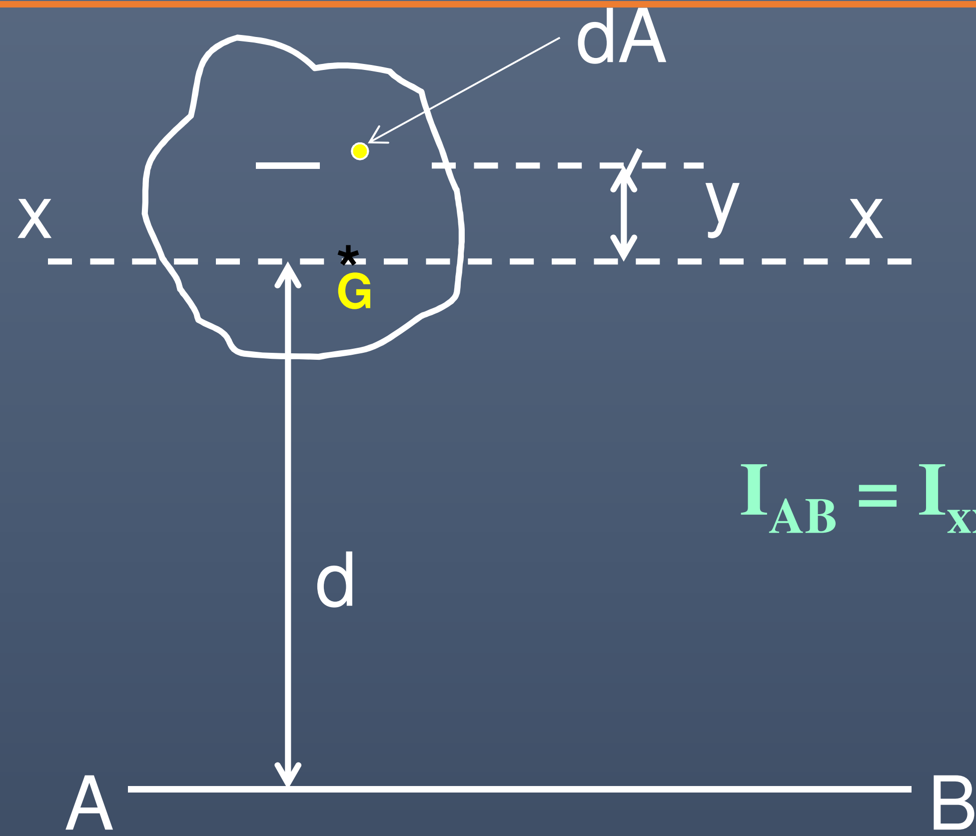
(Perpendicular axis theorem)

The polar M.I. for an area w.r.t. an axis perpendicular to its plane of area is equal to the sum of the M.I. about any two mutually perpendicular axes in its plane, passing through the point of intersection of the polar axis and the area.

$$
\begin{aligned}
I_{oz} = I_p = J = \int r^2 dA \\
&= \int (x^2 + y^2)\, dA \\
&= \int x^2 dA + \int y^2 dA \\
&= I_{oy} + I_{ox}
\end{aligned}
$$



Parallel Axes Theorem



$$I_{AB} = I_{xx} + Ad^2$$

Moment of inertia of any area about an axis AB is equal to the M.I. about parallel centroidal axis plus the product of the total area and square of the distance between the two axes.

‘xx’ ……… Centroidal axis passing through centroid G

‘AB’ …….. reference axis parallel to centroidal axis ‘xx’

$I_{AB} = \int H^2 dA$

$I_{AB} = \int (y+d)^2 dA$

$I_{AB} = \int y^2 dA + d^2 \int dA + 2d \int y dA$

Since $\int ydA$ about centroidal axis is zero,

Therefore

$I_{AB} = I_{xx} + Ad^2$

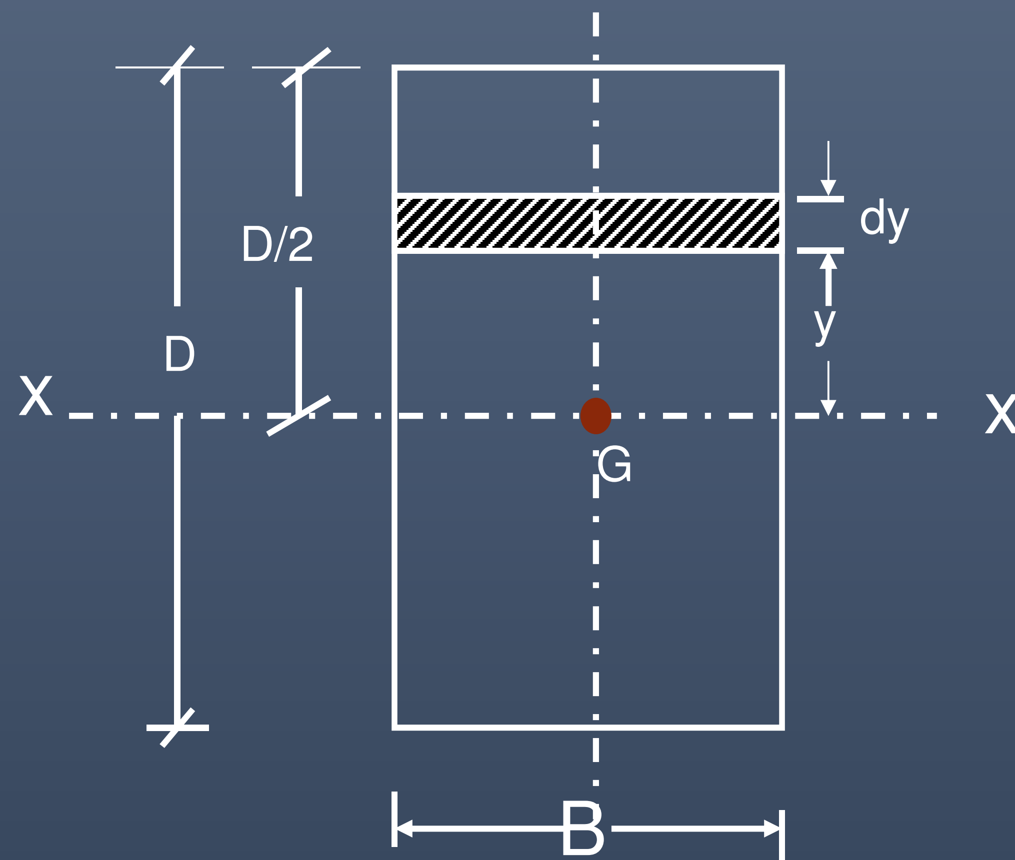
MOMENT OF INERTIA
BY
DIRECT INTEGRATION

MI of rectangular area about centroidal horizontal axis by direct integration

$$I_{\bar{x}\bar{x}} = \int_{-D/2}^{+D/2} (dA)\, y^2$$

$$I_{\bar{x}\bar{x}} = \int_{-D/2}^{+D/2} (B.\, dy)\, y^2$$

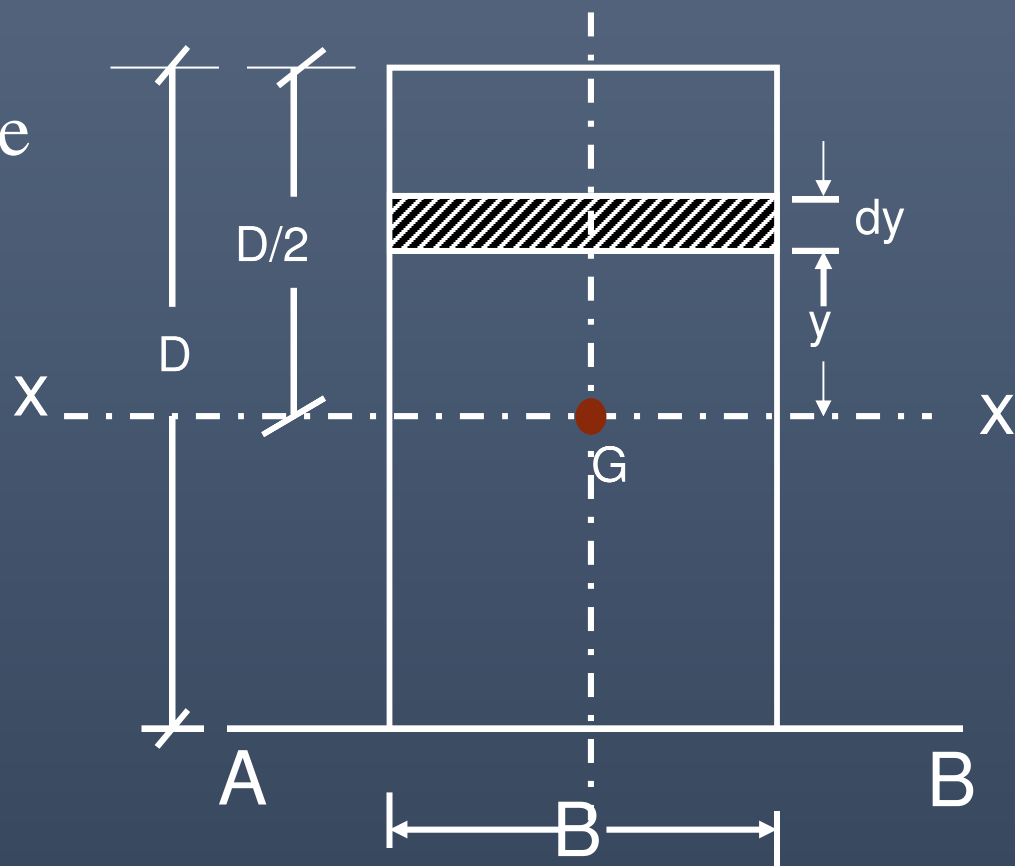
$$= BD^3 \;/\; 12$$



MI of rectangular area about the base by parallel axis theorem

$I_{AB} = I_{XX} + A(d)^2$

Where d = D/2, the distance between axes xx and AB



$$I_{AB} = BD^3/12+(BD)(D/2)^2$$

$$= BD^3/12+BD^3/4$$

$$= BD^3/3$$

LECTURE 13

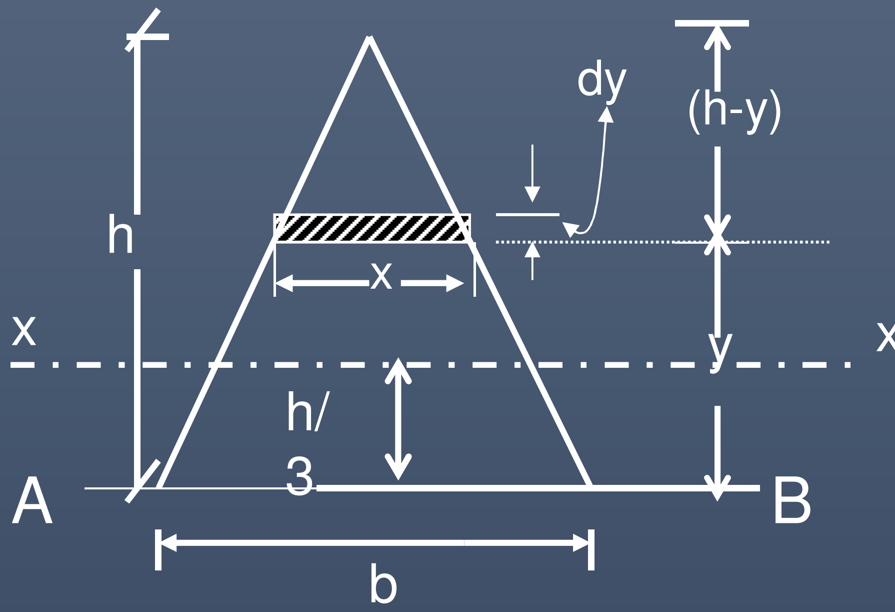
MI of triangular area about the base by direct integration

$I_{AB} = \int dA.y^2 = \int (x.dy)\ y^2$

$$I_{AB} = \int_0^h (b(h-y)\ y^2.dy)\ /h$$

$$= b[\ h\ (y^3/3) - y^4/4\ ]/h$$

$$= bh^3/12$$



From similar triangles

$b/h = x/(h-y)$

$\therefore\ x = b\ .\ (h-y)/h$

Using Parallel axis theorem

$I_{AB} = I_{xx} + Ad^2$

$I_{xx} = I_{AB} - Ad^2$

$= bh^3/12 - bh/2\ .\ (h/3)^2$

$= bh^3/36$

MI of circular area about the centroidal axis

$$I_{xx} = \int dA \, . \, y^2$$

$$= \int_0^R \int_0^{2\pi} (r.d\theta.dr)\, r^2 Sin^2\theta$$

$$= \int_0^R \int_0^{2\pi} r^3.dr\ Sin^2\theta\ d\theta$$

$$= \int_0^R r^3\ dr \int_0^{2\pi} \{(1 - Cos2\theta)/2\} d\theta$$

$$= [r^4/4]_0^R \quad [\theta/2 - Sin2\theta/4]_0^{2\pi}$$

$$= R^4/4[\pi - 0] = \pi R^4/4$$

$$= \pi\, R^4/4 \ = \pi D^4/64$$

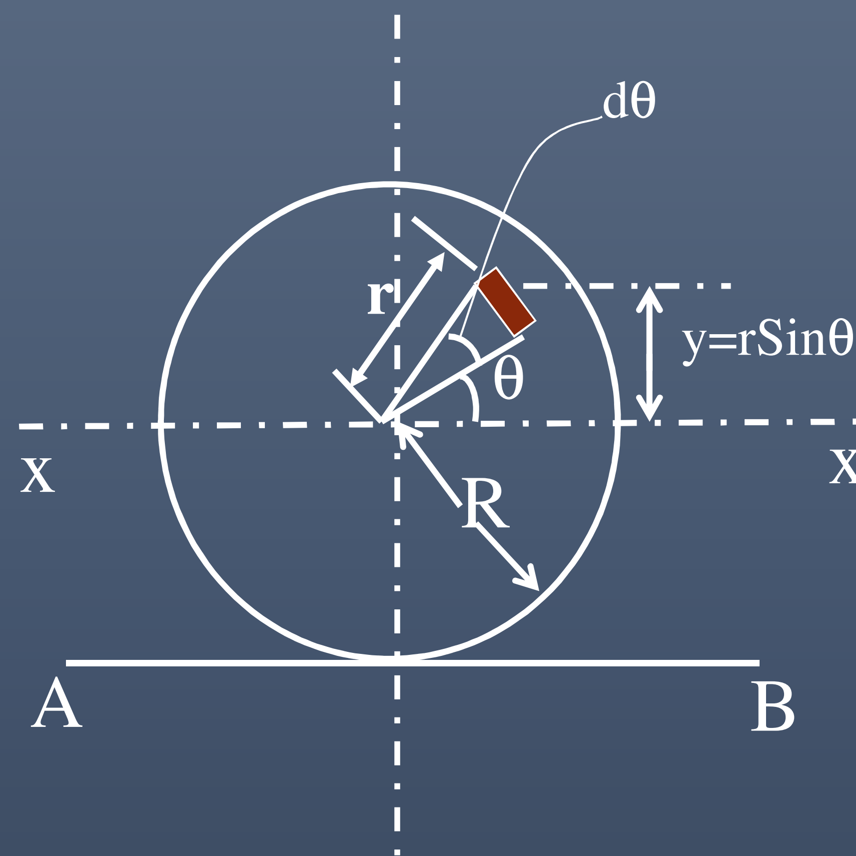


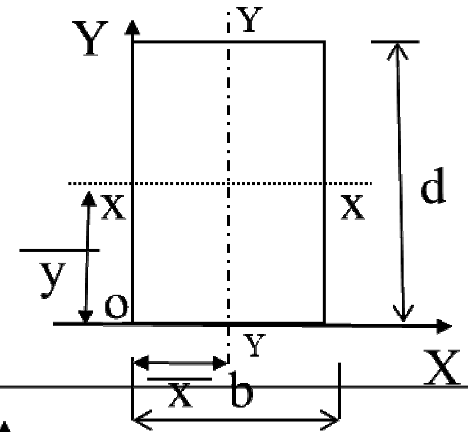
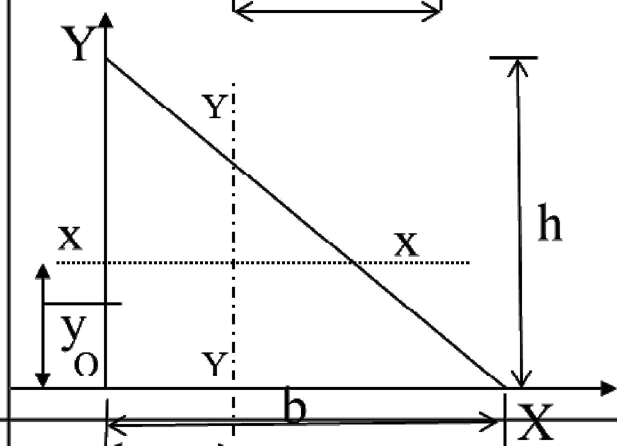
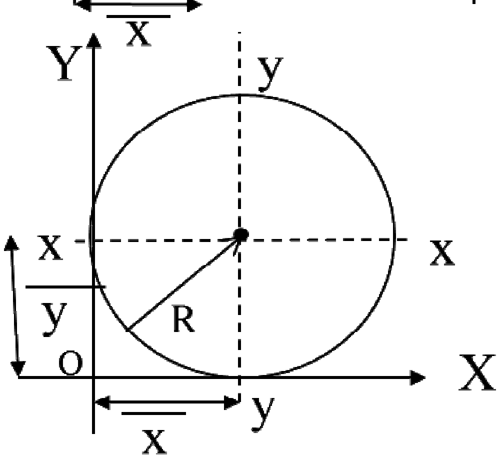
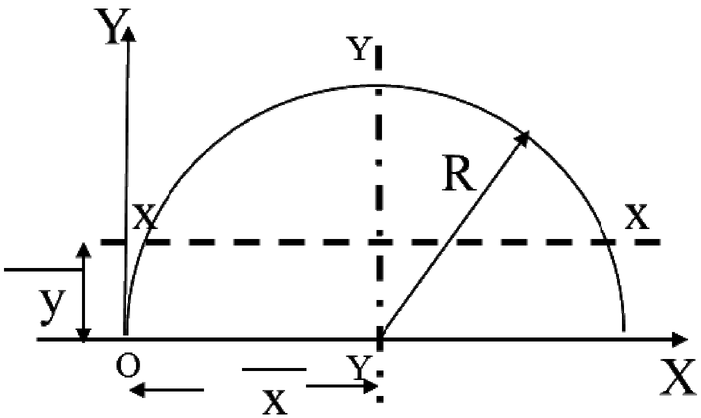
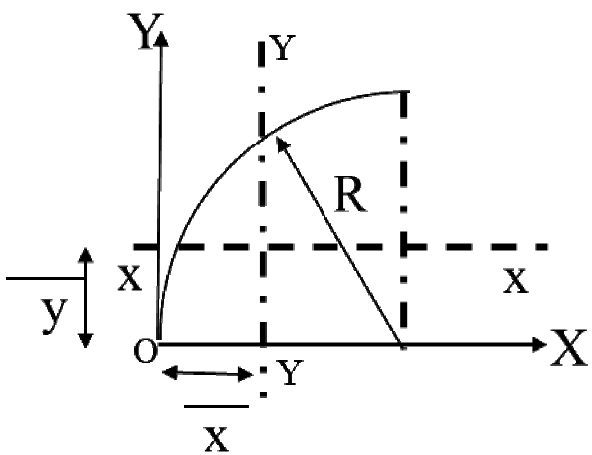
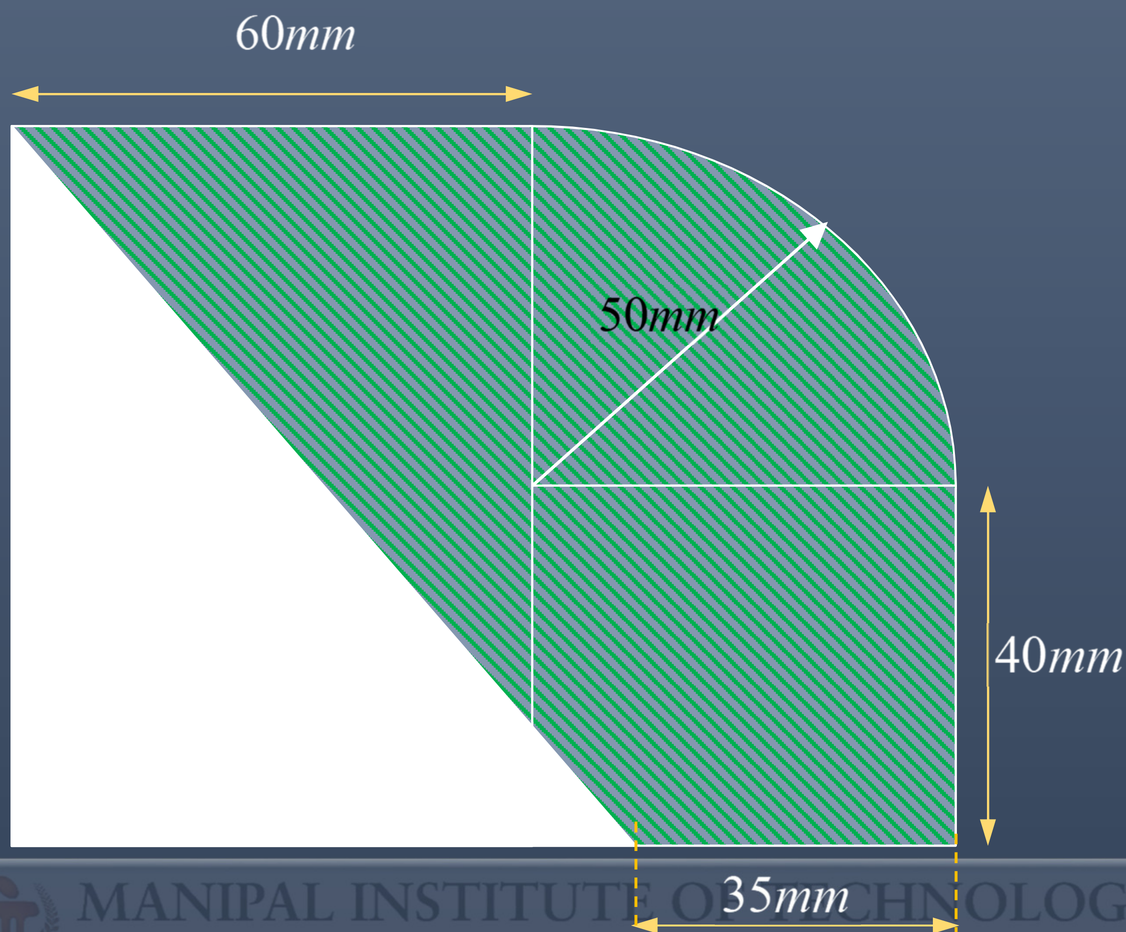
	FIGURE	$\bar{x}$	$\bar{y}$	I_{x-x}	I_{y-y}
1		b/2	d/2	$bd^3/12$	$db^3/12$
2		b/3	h/3	$bh^3/36$	$hb^3/36$
3		R	R	$\pi R^4/4$	$\pi R^4/4$

	FIGURE	$\bar{x}$	$\bar{y}$	I_{x-x}	I_{y-y}
4		R	$4R/3\pi$	$0.11R^4$	$\pi R^4/8$
5		R-$4R/3\pi$	$4R/3\pi$	$0.055R^4$	$0.055R^4$

TUTORIAL 5

T4. Compute MI about horizontal centroidal axis.



Sol:- Constructing triangle 'A', then, consider AB as reference axis to find $\bar{y}$.

Sl. No.	Shape	Area	$\bar{y}$	$A\bar{y}$
1.	Rectangle	$60 \times 90 = 5400$	$\frac{90}{2} = 45$	243000
2.	Rectangle	$50 \times 40 = 2000$	$\frac{40}{2} = 20$	40000
3.	Quarter circle	$\frac{\pi \times 50^2}{4} = 1963.5$	$\frac{4 \times 50}{3\pi} + 40 = 61.22$	120205.47
~~4.~~	~~Triangle~~			
4.	Triangle	$\frac{1}{2} \times 75 \times 90 = -3375$	$\frac{90}{3} = 30$	-101250

$\Sigma A = 5988.5\,mm^2$ $\qquad \Sigma a\bar{y} = 301955.47$

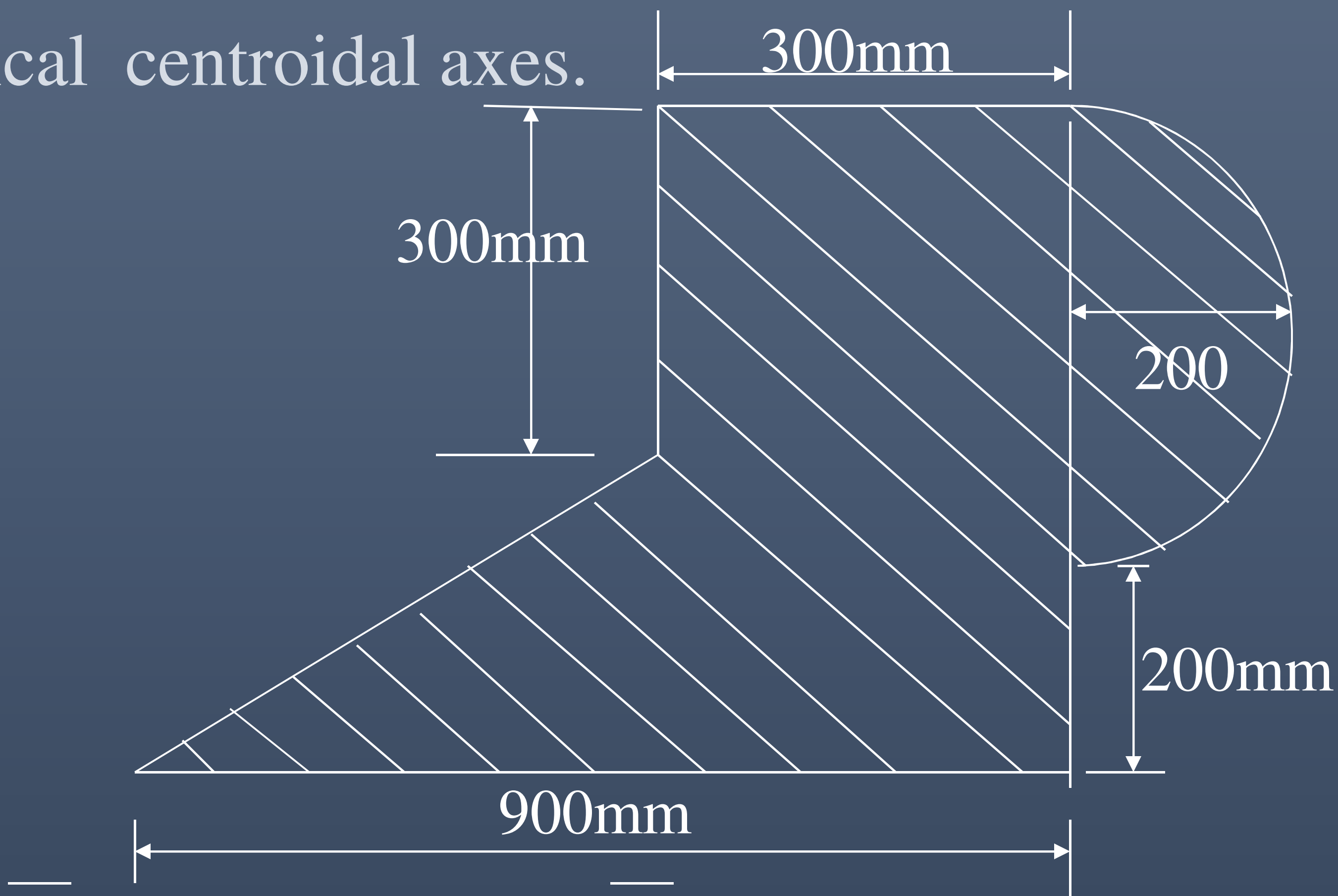
$$\bar{y} = \frac{\Sigma A\bar{y}}{\Sigma A} = \frac{301955.47}{5988.5} = 50.42\,mm$$

To find $I_{x_0x_0}$.

Sl. No.	Shape	$I_{x_0x_0}$
1.	Rectangle	$\frac{90^3 \times 60}{12} + 60 \times 90 \times (50.42 - 45)^2 = 3803632.56$
2.	Rectangle	$\frac{50 \times 40^3}{12} + 50 \times 40 \times (50.42 - 20)^2 = 2117419.47$
3.	Quarter circle	$0.055 \times 50^4 + \frac{\pi \times 100^2}{16} \times (61.22 - 50.42)^2 = 572772.10$
4.	Triangle	$\frac{75 \times 90^3}{36} + \frac{1}{2} \times 75 \times 90 \times (50.42 - 30)^2 = -2926045.35$

$$\therefore I_{x_0x_0} = 3.56 \times 10^6\,mm^4$$

T5. Determine second moment of area about horizontal and vertical centroidal axes.

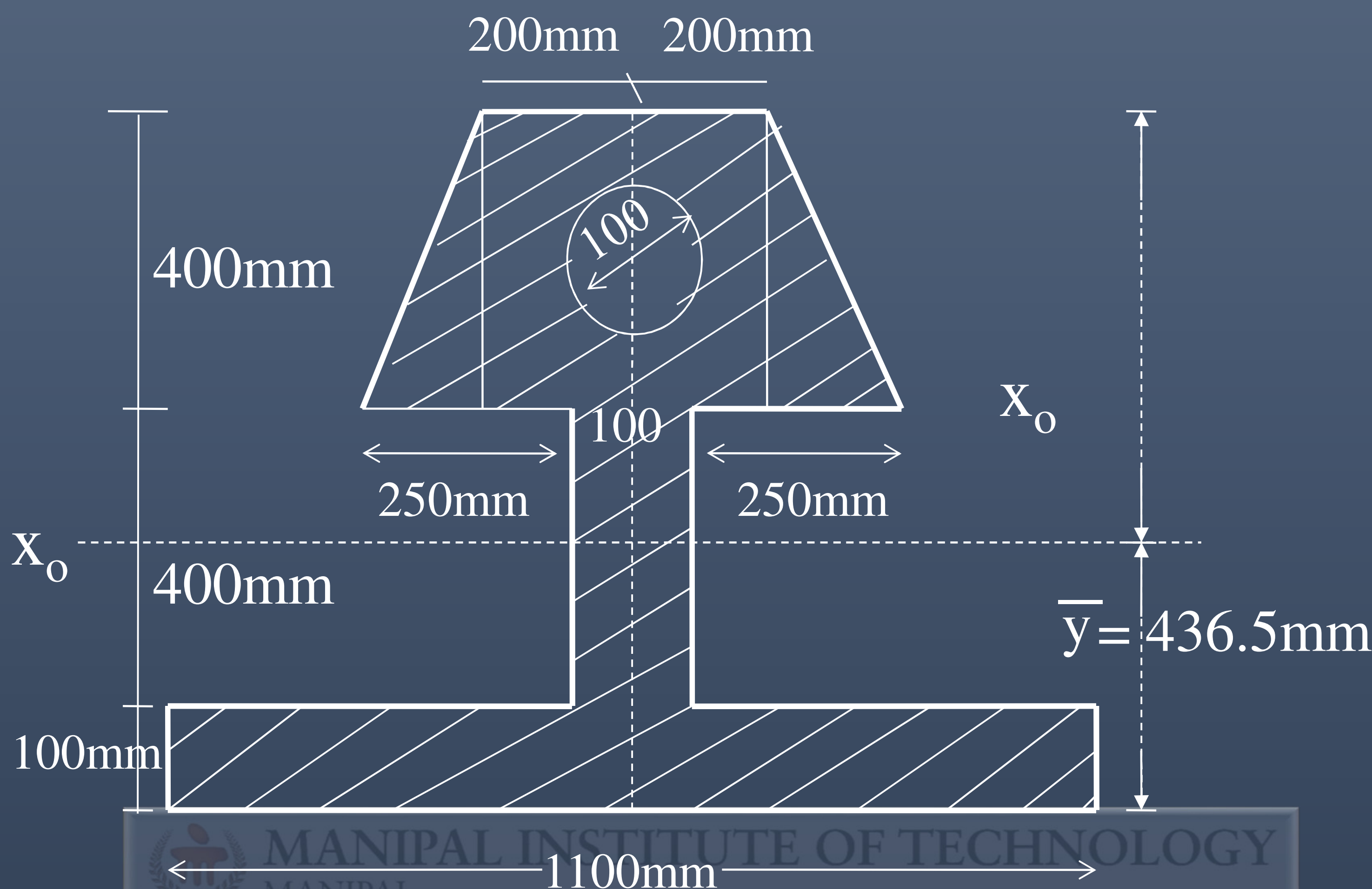


[Ans: X = 699.7mm from A, Y = 265 mm

I_{xx} = 10.29 x 10^9mm^4, I_{yy} = 16.97 x 10^9mm^4]

TUTORIAL 6

T6. Find M.I about horizontal centroidal axis for the area shown. Also calculate radius of gyration.



$\sum A = 1100*100+400*100+400*400+2(1/2*100*400)-\pi*50^2$

$= 3,42,150\ mm^2$

$\sum AY = 1100*100*50+100*400*300+400*400*700+[(1/2)*100*400*633.3]*2 - \pi*50^2*700$

$= 14,93,38,200 mm^3$

$\overline{Y} = \sum AY / \sum A = 436.5 mm$

Moment of Inertia about horizontal centroidal

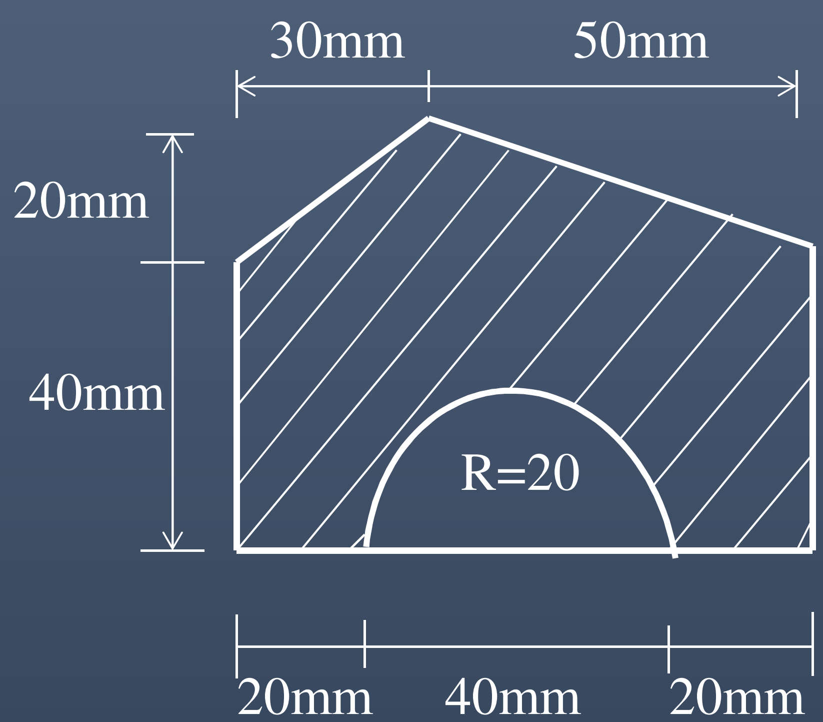
Axis:-

$$I_{XoXo} = [1100*100^3/12 + 1100*100(386.5)^2] + [100*400^3/12 + (100*400)*(136.5)^2] + [400*400^3/12 + 400*400*(263.5)^2] + 2[100*400^3/36 + (1/2*400*100)*(196.8)^2] - [\pi*(50)^4/4 + \pi*50^2*(263.5)^2]$$

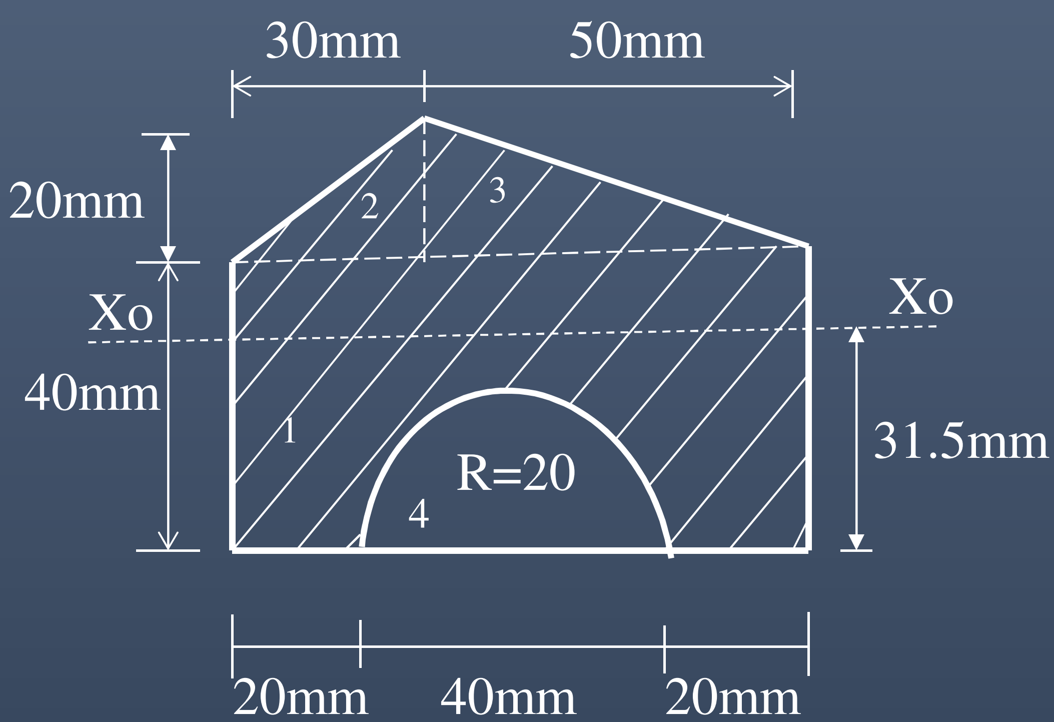
$$I_{XoXo} = 32.36*10^9 mm^4$$

(radiu of gyration) $r_{XoXo} = \sqrt{(I_{XoXo}/A)} = 307.536mm.$

T7. Find the second moment of the shaded area shown in fig.about its centroidal x-axis.



solution:-



$$\sum A = 40*80 + 1/2*30*30 + 1/2*50*30 - 1/2*\pi*(20)^2 = 3772 mm^2$$

$$\sum A_i X_i = 3200*40 + 450*2/3*30 + 750*(30+50/3)$$

$$-1/2*\pi*20^2*40 = 146880 mm^3$$

$$\sum A_i Y_i = 3200*20 + 450*50 + 750*50 - 628*4*20/3\pi = 118666.67 mm^3$$

$$\bar{Y} = 118666.67/3772 = 31.5 mm$$

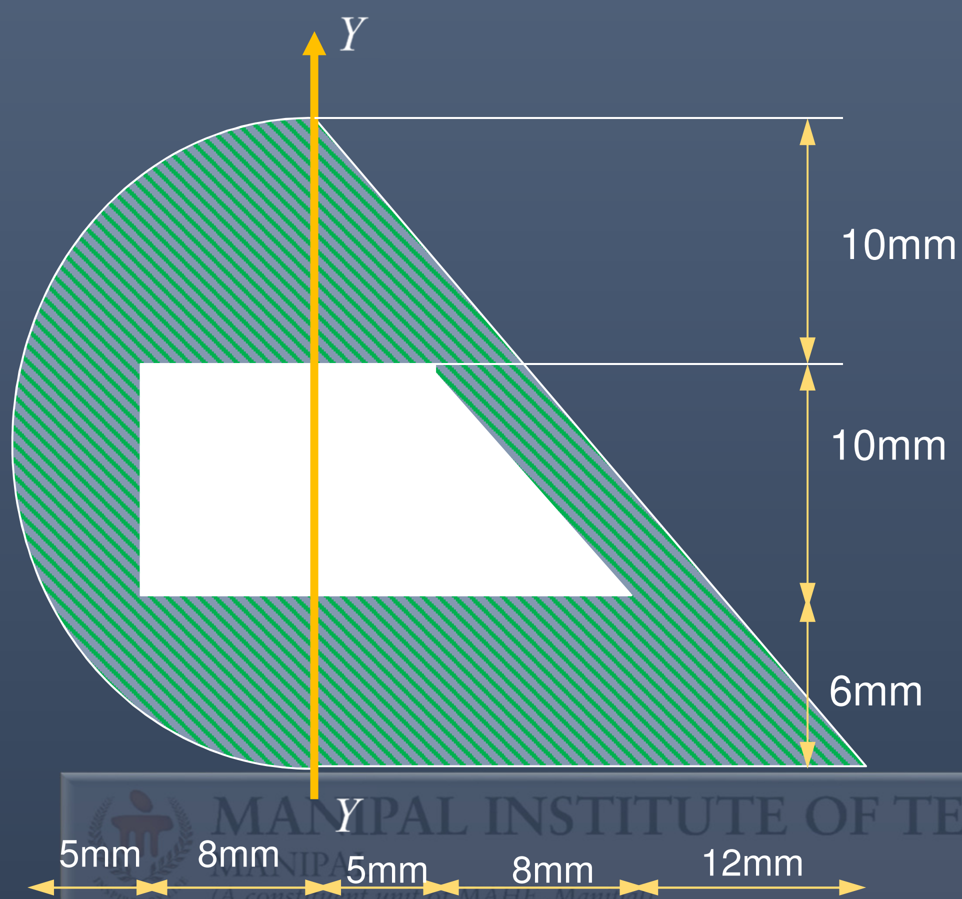
$$I_{XoXo} = [80*40^3/12 + (80*40)(11.5)^2] + [30*30^3/36 +$$

$$1/2*30*30(18.5)^2] + [50*30^3/36 + 1/2+50*30*(18.50)^2] -$$

$$[0.11*20^4) + \pi/2*(20)^2(31.5 - 0.424*20)]$$

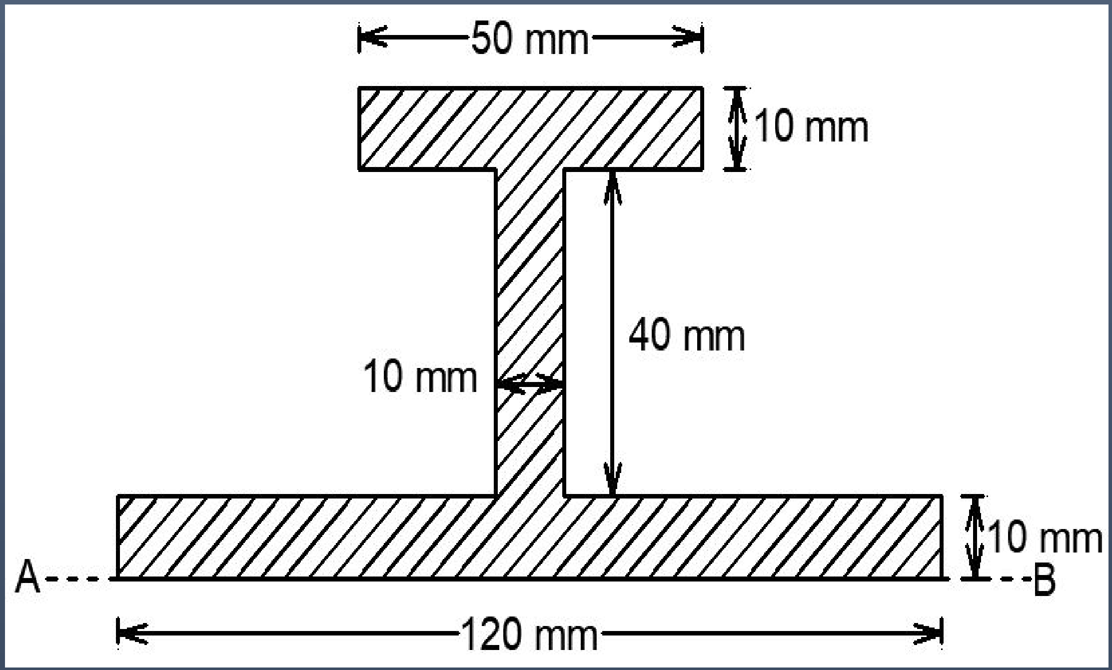
$$= 970.3*10^3 mm^4$$

T8. Determine M.I with respect to reference axes Y-Y shown.

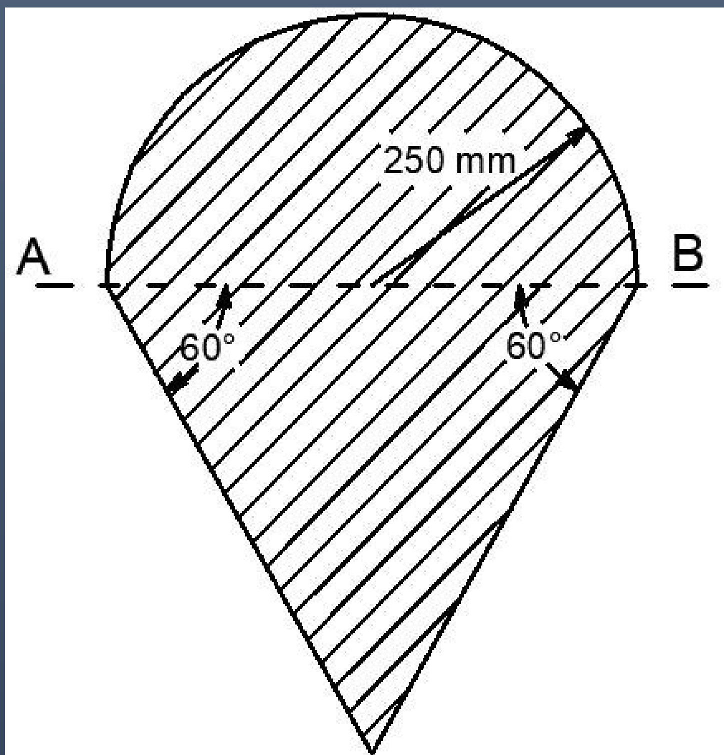


TUTORIAL – 7
(Additional problems)

1. Determine the second moment of area for the hatched portion shown in figure with respect to given reference axis AB. (3 marks)

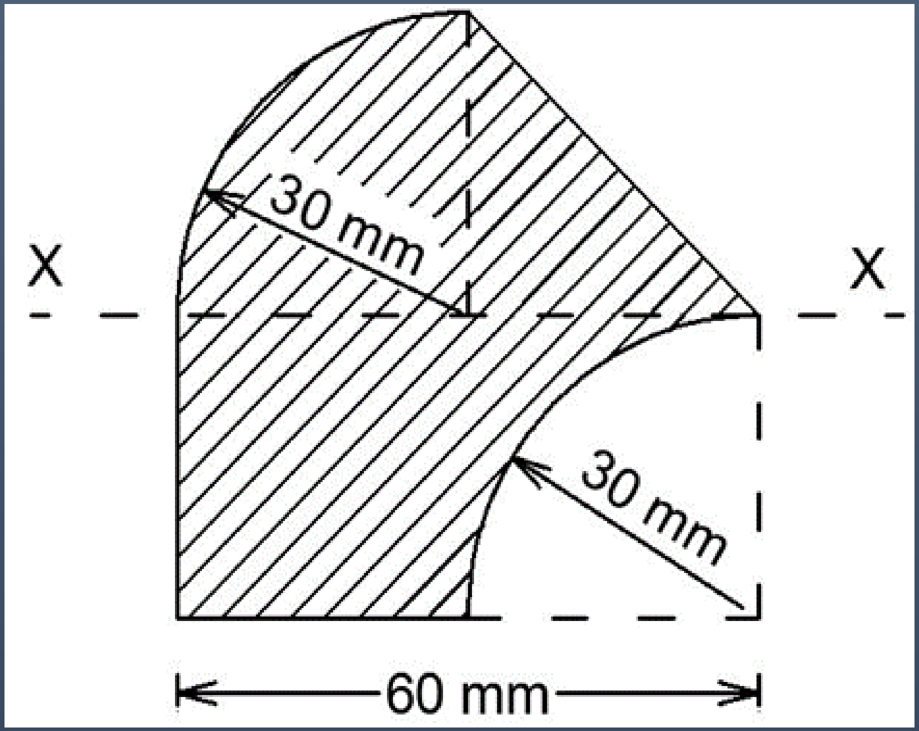


2. Determine moment of inertia of the shaded area shown in figure with respect to the given reference axis AB.5 marks)



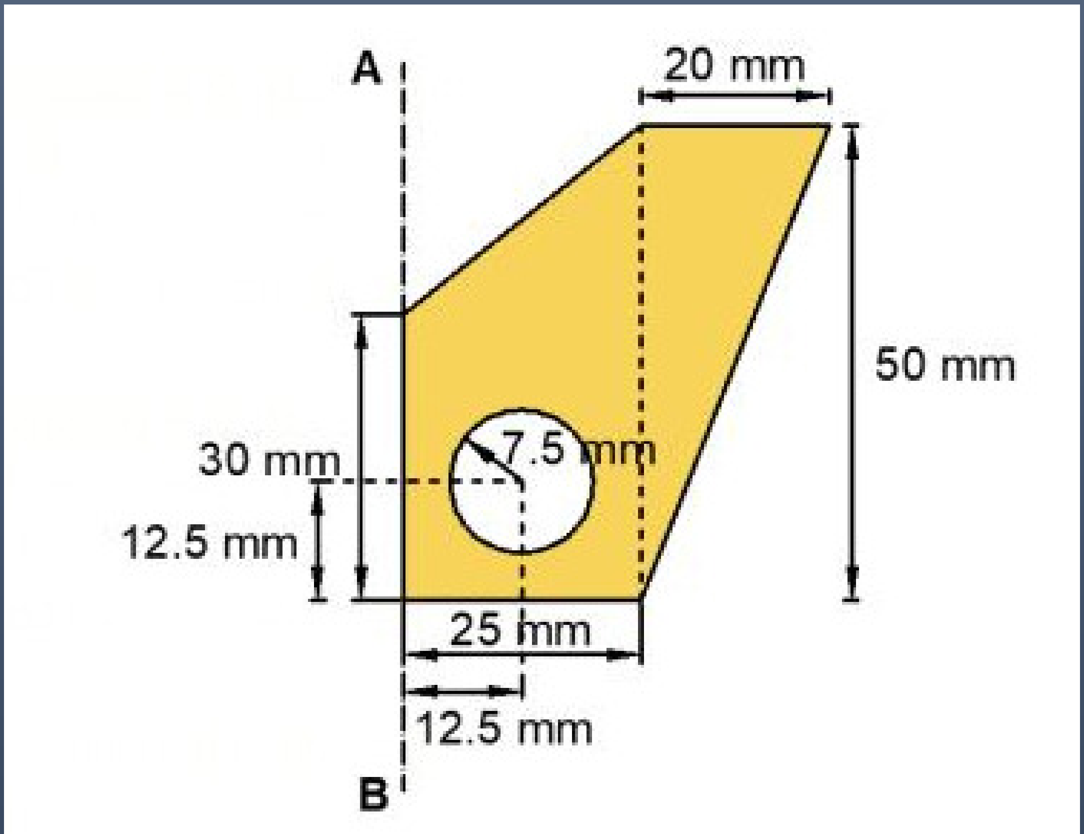
Ans: $4.995 \times 10^{6} mm^{4}$

3. Determine the second moment of area for the shaded area shown in the figure w.r.t given axis X-X.
.5 marks)



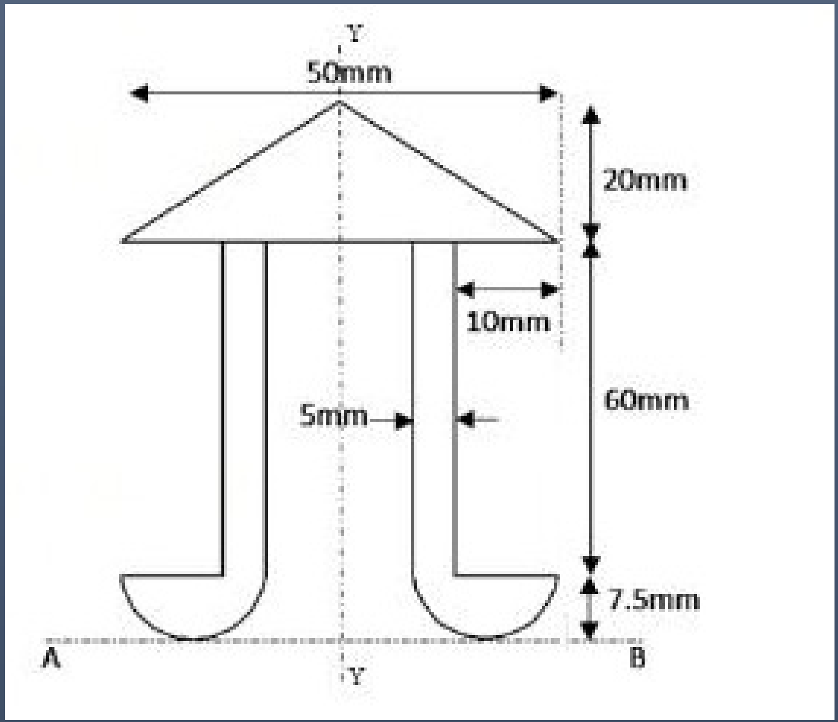
Ans: $5.113 X 10^5 mm^4$

4. Determine the second moment of area with respect to given reference axis AB for the shaded region.



Ans: $I_{AB} = 716778.3\ mm^4$

5. Determine the second moment of area for the figure shown below w.r.t given axis AB.



Ans: $I_{AB} = 3.7892 \times 10^6 \ mm^4$