Generating functions

combinations (identical objects)

① With no reptⁿ $\xrightarrow{{}^nC_r}$ $(1+x)^n = {}^nC_r x^r$

② With reptⁿ $\xrightarrow{{}^{n+r-1}C_r}$ $(1-x)^{-n} = \sum {}^{n+r-1}C_r x^r$

Enumerators → we pick the coeff of x^r

ex:- Ist obj can be repeated twice
2nd → no reptⁿ
3rd → no restrictⁿ

$$f(x) = x^2(1+x)(1+x+x^2+x^3+\dots)$$

No of ways of selecting 6 objects out of them, I've to pick the coeff of x^6 from the gf $f(x)$

Permutations (Distinct obj / ppl / letter formatⁿ / No formatⁿ)

① With no reptⁿ $\xrightarrow{{}^nP_r}$ $(1+x)^n = \sum {}^nP_r \frac{x^r}{r!}$

② With reptⁿ $\xrightarrow{n^r}$ $e^{nx} = \sum n^r \frac{x^r}{r!}$

Exponential gf → we pick the coeff of $\frac{x^r}{r!}$

select 6 ppl out of 3 groups :- CS, CC, IT

CS → atmost 2 ppl
CC → atleast one person, but not more than 3
IT → no restⁿ

$$\left(1+\frac{x}{1!}+\frac{x^2}{2!}\right)\left(\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}\right)\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\right)$$

we have to pick coeff of $\frac{x^6}{6!}$

* Find the no of r - dig quaternary sequences (whose dig are 0, 1, 2, 3) with even no of 0's and odd no of 1's

Sol^n

$$\underbrace{\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots\right)}_{\text{dig } 0}\underbrace{\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)}_{\text{dig } 1}\underbrace{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)^2}_{\text{dig } 2 \,\&\, 3}$$

$$f(x)=\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^x-e^{-x}}{2}\right)\left(e^x\right)^2$$

The coeff of $\frac{x^r}{r!}$ is to be to be picked from $f(x)$

On simplifying,

$$f(x)=\frac{1}{4}\left[e^{2x}-e^{-2x}\right]e^{2x}$$

$$=\frac{1}{4}\left[e^{4x}-1\right]$$

$$=\frac{1}{4}\left[\sum\frac{(4x)^r}{r!}-1\right]$$

coeff is $\longrightarrow \frac{1}{4}4^r = \underline{\underline{4^{r-1}}}$

② How many 10 letter words are there with each of e, r, n, s occur i) atmost once ii) atleast once

Soln

i) atmost once :

enabcdgfhk ✓

enebs srkfh → ✗

enaaaaaaaa → ✓

aaaaaaaaaa ✓

$$f(x) = \left(1+\frac{x}{1!}\right)^4 \left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)^{22}$$

$$f(x) = (1+x)^4 (e^x)^{22}$$

$$= e^{22x}\left[(1+x^2+2x)(1+x^2+2x)\right]$$

$$= e^{22x}\left[1+x^2+2x+x^2+x^4+2x^3+2x+2x^3+4x^2\right]$$

$$= e^{22x}\left[x^4+4x^3+6x^2+4x+1\right] \qquad \left(e^x = \sum \frac{x^n}{n!}\right)$$

coeff of $\frac{x^{10}}{10!}$

$$f(x) = x^4 \sum \frac{(22x)^n}{x!} + 4x^3 \sum \frac{(22x)^n}{n!} + 6x^2 \sum \frac{(22x)^n}{n!} + 4x \sum \frac{(22x)^n}{n!} + \sum \frac{(22x)^n}{n!}$$

$$= x^4 \left[\frac{22^6 x^6}{6!} \frac{10!}{10!}\right] \quad \frac{(22x)^7}{7!} \frac{10!}{10!}$$

∴ coeff is ⇒

$$\frac{10!}{6!}\cdot(22)^6 + 4\cdot 22^7 \cdot \frac{10!}{7!} + 6\cdot 22^8 \cdot \frac{10!}{8!} + 4(22)^9 \frac{10!}{9!} + (22)^{10}$$

ii) e, n, r, s atleast once

$$\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)^{22}\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)^4$$

$$f(x) = e^{22x}(e^x-1)^4$$

Coeff of $\frac{x^{10}}{10!}$

Principle of inclusion and exclusion
(Sieve's method)

consider N objects and 2 properties a & b (prop a & prob b)

$N(a) \to$ No of objects having prop a

$N(b) \to$ " " prop b

$N(a'b') \to$ No of objects having none of the props a & b

$$\boxed{N(a'b') = N - [N(a) + N(b)] + N(ab)}$$

in general, $N \to$ objects

r properties, say, $a_1\, a_2\, a_3 \cdots a_r$

$N(a_i) \to$ No of objects satisfying i^{th} prop a_i, $1 \le i \le r$

$N(a_1' a_2' \cdots a_r') \to$ No of objects satisfying none of the 'r' properties

$$\boxed{N(a_1' a_2' \cdots a_r') = N - \sum N(a_i) + \sum N(a_i a_j) - \sum N(a_i a_j a_k) + \cdots + (-1)^r N(a_1 a_2 \cdots a_r)}$$

Ex:- How many integers b/wn 1 to 6300 are neither divisible by 3, nor $\div$ble by 5

soln

$a_1 \to$ prop that the no is $\div$ble 3

$a_2 \to$ prop that the no is $\div$ble 5

$N(a_1' a_2') = ?$

$N(a_1' a_2') = N - (N(a_1) + N(a_2)) + N(a_1 a_2)$

$N(a_1) \rightarrow$ No of integers satisfying the prop a_1

$\rightarrow$ No of integers $\div$ble by 3

$1, 2, 3, \ldots 6300$

$N(a_1) = \left\lfloor \frac{6300}{3} \right\rfloor = 2100$

$N(a_2) = \left\lfloor \frac{6300}{5} \right\rfloor = 1260$

$N(a_1 a_2)$ = Nos of integers $\div$ble by both 3 & 5

$= \left\lfloor \frac{6300}{3 \cdot 5} \right\rfloor = 420$

$N(a_1' a_2') = 6300 - [2100 + 1260] + 420$

$= \underline{\underline{3360}}$

$n(\underline{\underline{A \cup B}}) = n(A) + n(B) - n(A \cap B)$

$n(A' \cap B') = T - n(A \cup B)$

Derrangements

Permutation / arrangement of objects s.t no obj is in its proper position

123
arrangement of the nos 1,2,3 s.t
1 → not 1st positn
2 → not in 2nd positn
3 → not in 3rd posit

213 ✗
231 ✓
312, 231,

No of derrangements of 3 objects is → 2

No of ways of arranging the digits 1,2,3 s.t no dig is in its proper positn
→ 2

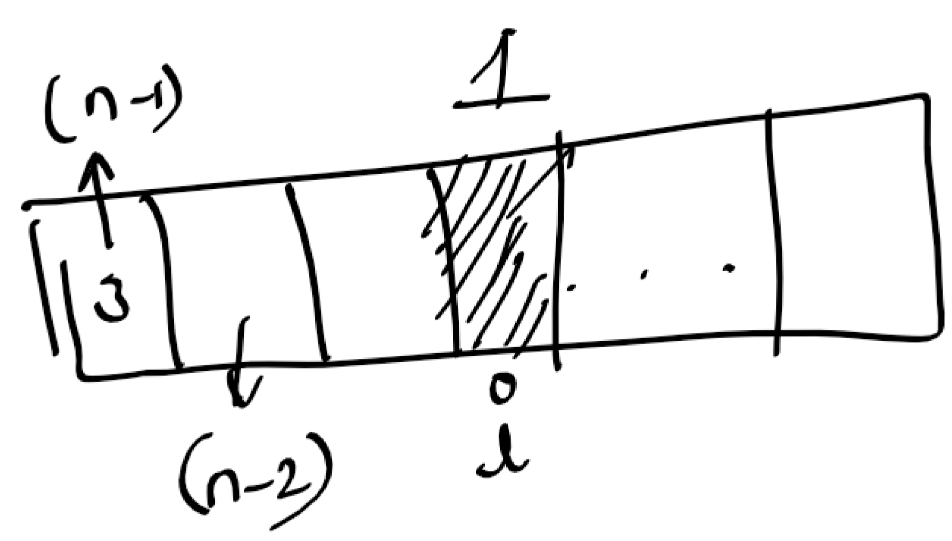
* How many permutations of distinct elements 1,2,3,...n are there s.t kth elt is not in kth positio for every k, $1 \le k \le n$

Proof:-
Let a_i be the prop that the i^{th} elt is in i^{th} positn
$1 \le i \le n$

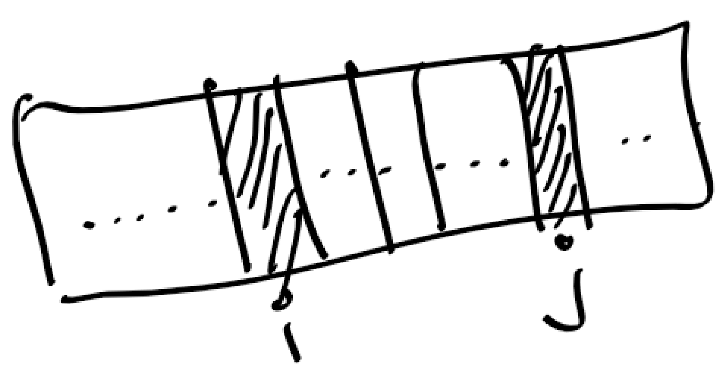
$N(a_1' a_2' a_3' \cdots a_n') = ?$

$N(a_i)$ → No of elts satisfying prop a_i
→ No of arrangements where i^{th} elt is in i^{th} positn

$N(a_i) = (n-1)!$



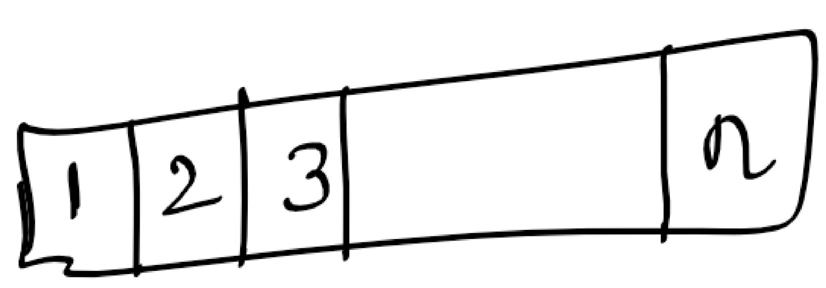
$N(a_i a_j) = (n-2)!$



$N(a_i a_j a_k) = (n-3)!$

⋮

$N(a_1 a_2 a_3 \cdots a_n) = 1$



$$N(a_1'a_2'\ldots a_n') = N - \sum N(a_i) + \sum_{i\neq j} N(a_i a_j) - \sum N(a_i a_j a_k)$$

$1, 2 \ldots n$

$$+ \cdots + (-1)^n N(a_1 a_2 \ldots a_n)$$

$$= N - n(n-1)! + {}^nC_2 (n-2)! - {}^nC_3 (n-3)!$$

$$+ \cdots + (-1)^n 1$$

$$N(a_1'a_2'a_3'\ldots a_n') = N - n! + \left(\frac{n(n-1)}{2!}\right)(n-2)! - \left(\frac{n(n-1)(n-2)}{3!}\right)(n-3)!$$

$$+ \cdots + (-1)^n$$

$$= N - n! + \frac{n!}{2!} - \frac{n!}{3!} + \frac{n!}{4!} + \cdots + (-1)^n \frac{n!}{n!}$$

$N \to$ Total no

$\to$ Total of arrangements of $1, 2, 3 \ldots n$ with no restrictn

$\to n!$

$$= \cancel{n!} - \cancel{n!} + \frac{n!}{2!} - \frac{n!}{3!} + \frac{n!}{4!} - \cdots + (-1)^n \frac{n!}{n!}$$

$$= n!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots + \frac{(-1)^n}{n!}\right].$$

$$= n!\left[\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \cdots + \frac{(-1)^n}{n!}\right]$$

$$\boxed{N(a_1' a_2' \cdots a_n') = n! \sum_{r=0}^{n} \frac{(-1)^r}{r!}} = D_n$$

But if n is large, $N(a_1' a_2' \cdots a_n') = n! \sum \frac{(-1)^r}{r!}$

$$= n! \, e^{-1}$$

$$= \frac{n!}{e}$$

<u>1, 2, 3 :-</u>

$$D_3 = 3! \sum_{r=0}^{3} \frac{(-1)^r}{r!} = 3! \left[\frac{(-1)^0}{0!} - \frac{(-1)}{1!} + \frac{(-1)^2}{2!} - \frac{(-1)^3}{3!} \right]$$

$$= 3! \left[1 - 1 + \frac{1}{2} - \frac{1}{6} \right]$$

$$= 2$$

For the digits 1, 2, 3, 4

$$D_4 = 4! \sum_{r=0}^{4} \frac{(-1)^r}{r!} = 9$$

2143	3421
2341	4321
2413	4312
3412	4123
3142	