Hasse Diagram :

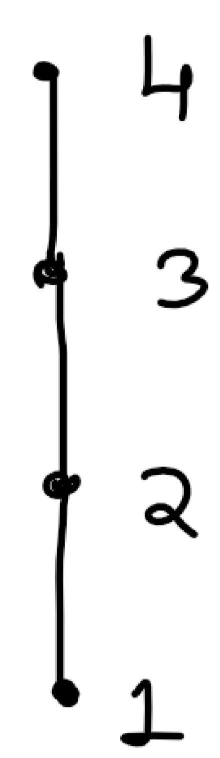
A poset $(A, \leq)$ can be represented by a Hasse Diagram.

Rules :

1) Each element of A is represented by small circle or dot.
2) The circle or dot for $x \in A$ is drawn below the dot for $y \in A$, if $x \leq y$. A line is drawn between x and y if y covers x.
3) If $x \leq y$ but y does not cover x, then x and y are not connected directly by a single line.

Example

1) Let $X = \{1, 2, 3, 4\}$ & $(X, \leq)$ be a POSET.



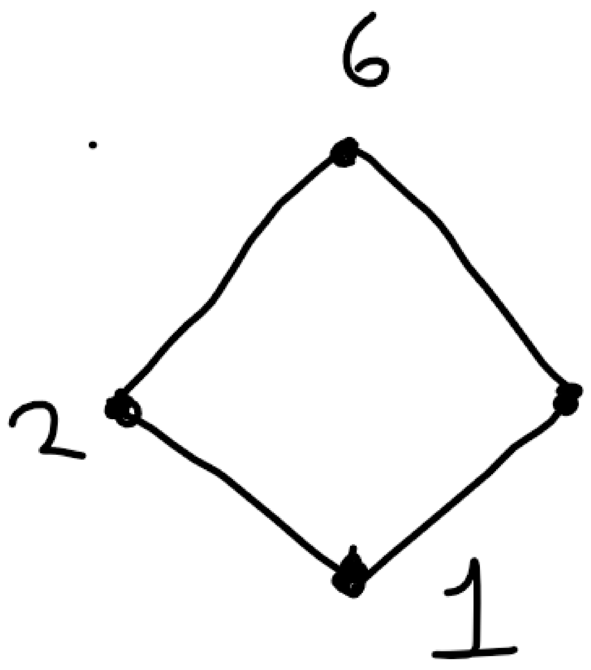
$1 \leq 2$ $\quad$ $1 \leq 3$

$2 \leq 3$ $\quad$ $1 \leq 2 \leq 3$

$3 \leq 4$

2) Let $A = \{1, 2, 3, 6\}$ & $(A, /)$ be a Poset.

Draw the Hasse diagram.



$1 | 2$ ✓

$1 | 3$ ✓ $\quad 2 \nmid 3 \rightarrow$ same level No Line

$1 | 6 \quad 2 | 6 \quad 3 | 6$

↑ No line $\because$ $1 | 2$ & $2 | 6$

Chain : $\{1, 2, 6\}$

Chain : $\{1, 3, 6\}$

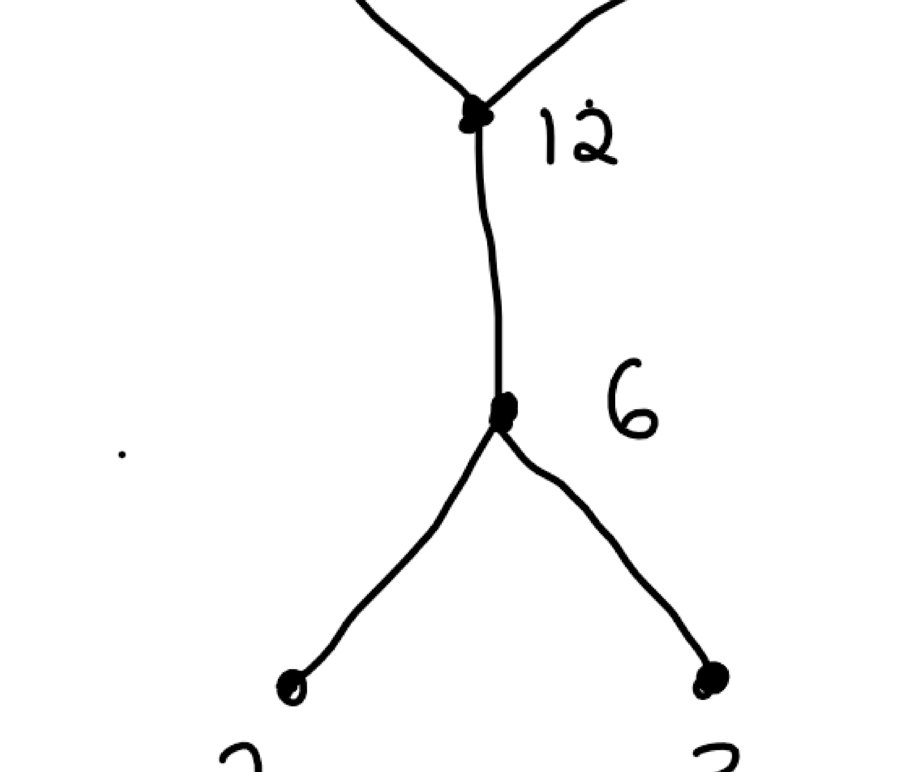
Antichain : $\{1\}$

$\{2, 3\}$

$\{6\}$

3) Let $X = \{2, 3, 6, 12, 24, 36\}$ & $(X, /)$ be a poset.

Draw a Hasse diagram.



$2 \nmid 3$

$2 | 6 \quad 6 | 36$ No Line

$3 | 6$

$6 | 12 \quad 6 | 12$ &

$12 | 36$

$\{3, 6, 12, 36\} \rightarrow$ Chain

$\{2, 6, 12, 24\} \rightarrow$ Chain

$\{2, 6, 12, 36\} \rightarrow$ Chain

Antichain : $\{2, 3\}$

$\{24, 36\}, \{6\}, \{12\}$

4) Let $A = \{a, b, c\}$ & $P(A)$ be its power set.
Draw a Hasse diagram.

$P(A) = \{ \phi, \{a\}, \{b\}, \{c\}, \{a\ b\} \{a\ c\} \{b\ c\} \{a\ b\ c\} \}$

$(P(A), \subseteq) \rightarrow$ is a Poset.

$\{\phi\} \subseteq \{a\}$

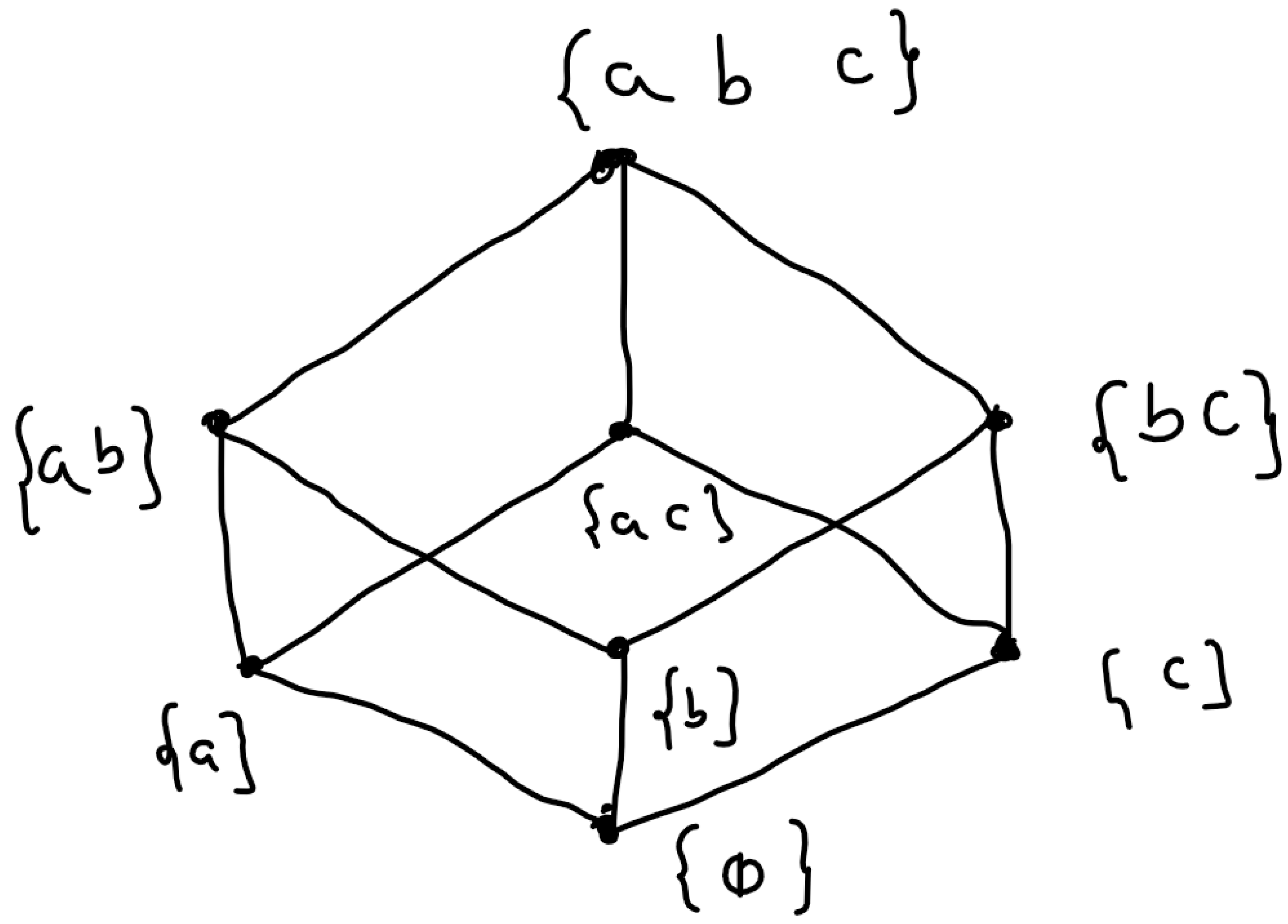
$\{a\} \subseteq \{a, b\}$

$\{a\} \nsubseteq \{b\}$

$\{b\} \nsubseteq \{c\}$

$\{a\} \subseteq \{a\ b\ c\}$

$\{ab\} \nsubseteq \{ac\}$



$\{\{\phi\}, \{a\}, \{ac\}, \{a\ b\ c\}\} \rightarrow$ Chain

<u>Longest</u> chain : $\{\{\Phi\}\ \{a\}\ \{ac\}\ \{a\ bc\}\}$

Chain: $\{\{b\}, \{abc\}\}$

Antichain: $\{\{\phi\}\}$

$\{\{a\}\ \{b\}\ \{c\}\}$

$\{\{ab\}\ \{ac\}\ \{b\ c\}\}$

Maximal and Minimal Element :

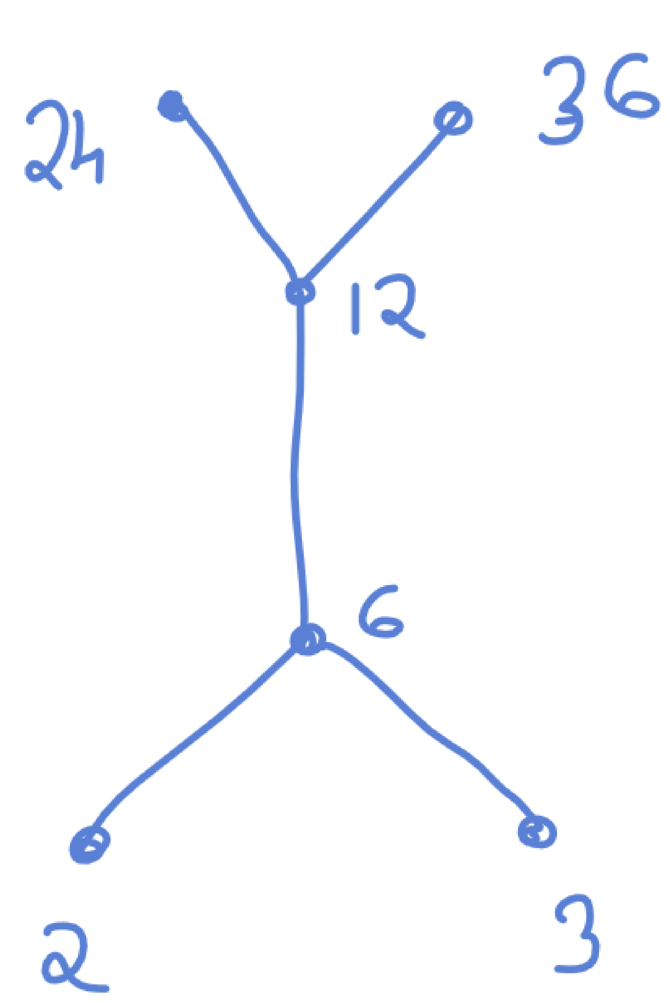
Let $(A, \leq)$ be a Poset.

An element $a \in A$ is said to be a maximal element of A if there is no $b \in A$, such that $a \neq b$, $a \leq b$.

An element $a \in A$ is said to be a minimal element of A, if there is no $b \in A$, such that $a \neq b$; $b \leq a$.

Example :

Find the maximal and minimal element of $A = \{2, 3, 6, 12, 24, 36\}$ & $(A, \backslash)$ be a Poset.



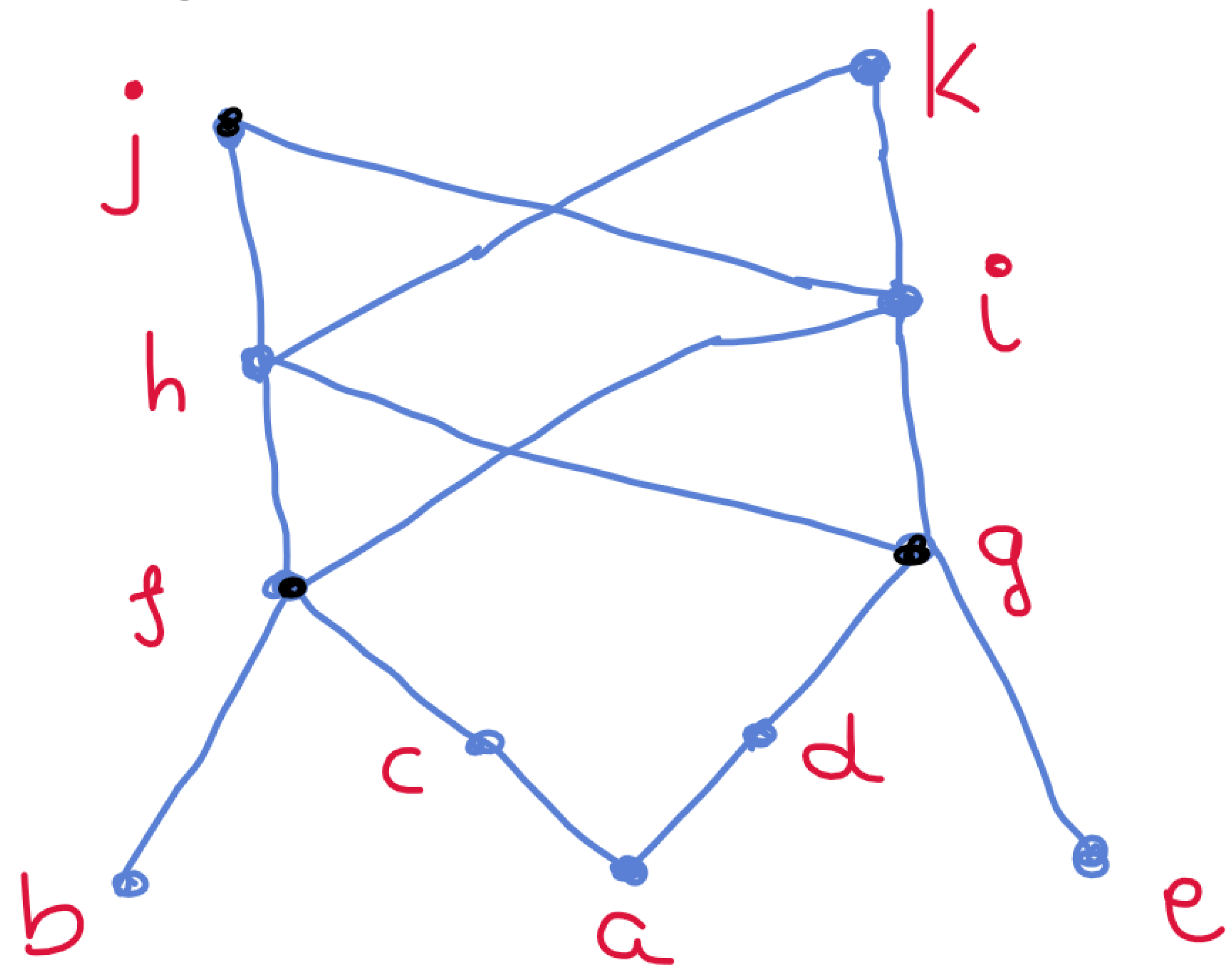
maximal elements : 24, 36

Minimal elements : 2, 3

Definition: Let $(A, \leq)$ be a Poset, and let $a, b \in A$.
An element 'c' is said to be an <u>upper</u> bound of 'a' and 'b' if $a \leq c$ and $b \leq c$.

An element 'c' is said to be a <u>least upper bound</u> (or supremum) of a and b, if c is an upper bound of a and b and if there is no other upper bound d of a and b such that $d \leq c$.

Consider the following Hasse diagram.



$f \leq h$
$g \leq h$

$f \leq i$
$g \leq i$

$f \leq j$
$g \leq j$

<u>h, i, j, k</u> ~~is~~ are ~~an~~ the upper bounds of f and g.

<u>h, i</u> are the <u>least upper</u> bound of f & g.

<u>j, k, i</u> ~~is~~ are the upper bound of f and i.

<u>i</u> is the least upper bound of f and i.

h is not the upper bound of f and i.

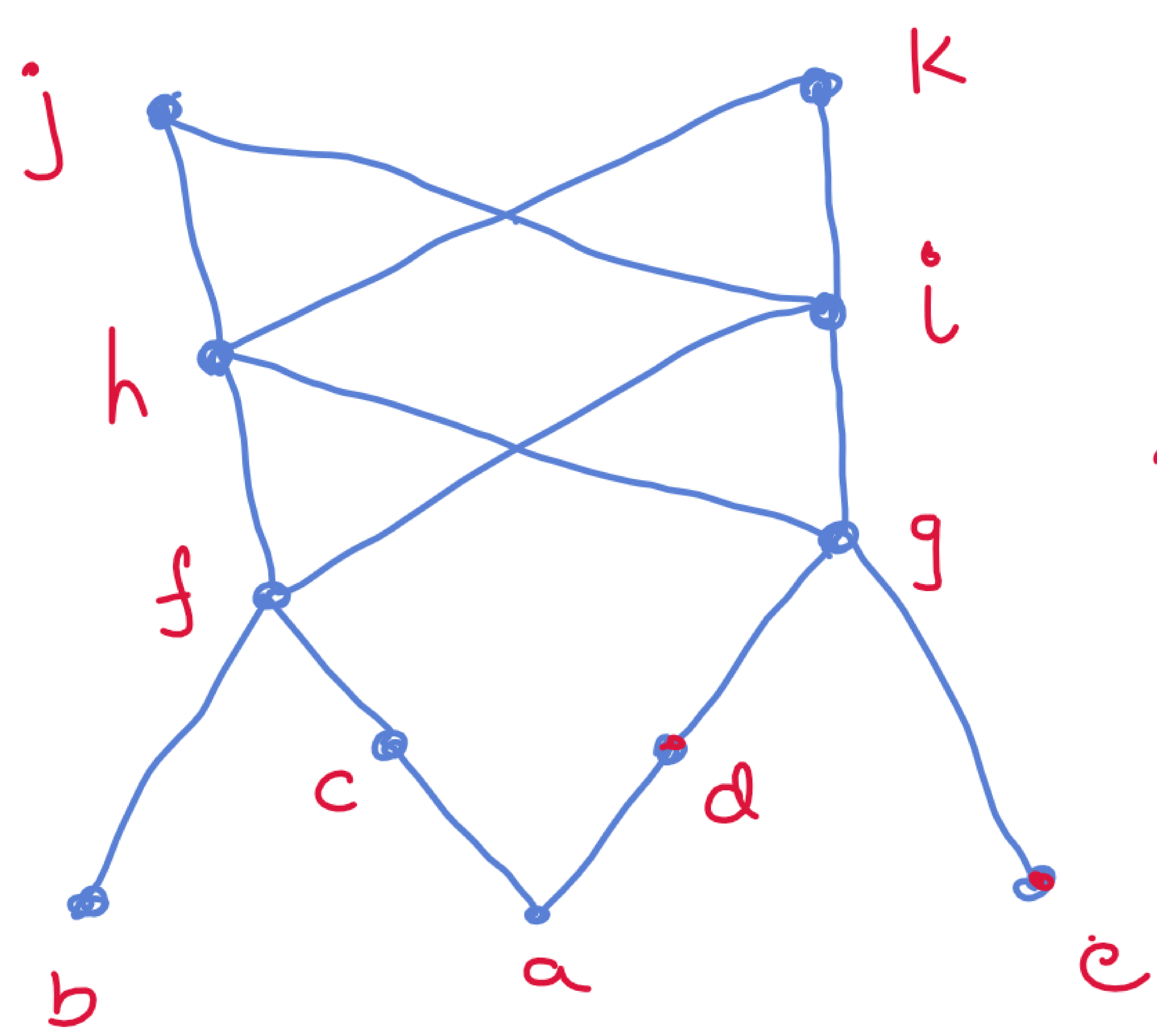
<u>h, i, j, k</u> are the upper bound of b and g

<u>h, e</u> ~~is~~ are. the least upper bound of b and g.

An element c is said to be a lower bound of a and b if $c \leq a$ and $c \leq b$.

An element c is said to be a greatest lower bound (or Infimum) of a and b if c is a lower bound of a and b and there is no other lower bound d of a and b such that $c \leq d$.

Consider the following Hasse diagram :



Longest chain: $\{a, c, f, i, j\}$

f, g, b, c, a, d, e are the lower bounds of h and i.

f and g are the greatest lower bounds of h & i

d a g e are the lower bound of h and g

g is the greatest lower bound of h & g.

Longest chain :

Length of the longest chain is 5

Antichains : $\{a\ b\ e\}$, $\{c, d\}$ $\{f\ g\}$ $\{h\ i\}$ $\{j k\}$

Let A be the set of integers and $(A, \backslash)$ be a Poset. For two integers a and b, common multiple of a and b is an upper bound of a and b. And the least common multiple of a and b is a least upper bound (only one) of a and b.

Similarly, a common divisor of a and b is a lower bound of a and b. The greatest common divisor of a and b is a greatest lower bound (only one) of a and b.

4 and 6: u.b: 12, 24, 36, ... –

l.u.b: 12

12 and 24: l.b: 1, 2, 3, 4 6, 12

g.l.b: 12

Extra Problems

1. Draw a Hasse diagram for $A = \{2, 4, 8, 12, 16, 20, 24, 32\}$ and $(A, \backslash)$ be a Poset. Find Maximal element, Minimal element, longest chain. lub of 8, 12 and glb of 12, 20.

Definition : The number of elements in a chain is the length of the chain.

Theorem : Let $(P, \leq)$ be a Poset. Suppose the length of the longest chain in P is n, then the elements in P can be partitioned into n disjoint antichains.

Proof : Proof is by induction on n.
For $n=1$, no 2 elements are related. $\Rightarrow$ They constitute an antichain.
For $n=2$, let $a, b \in A$. Let $a \leq b$
$\Rightarrow$ Length of longest chain in 2 and can be partitioned into 2 disjoint sets.
Induction step : we assume that the theorem holds when the length of the longest chain is $n-1$.
Let P be a Poset with length of the longest chain is n.
Let M denote the set of maximal element in P.
Clearly, M is a non empty set in P which is an antichain. (i.e. elements which are not comparable)
Consider the Poset $(P-M, \leq)$.
Since there is no chain of length n in P-M, length of longest chain is atmost $n-1$. Suppose if length of longest chain in P-M is less than $(n-1)$, then M contains 2 or more elements that are members of same chain, which is not possible.
$\Rightarrow$ length of the longest chain in P-M is $n-1$. &
from induction, P-M can be partitioned into $(n-1)$ disjoint antichains. Thus P can be partitioned into n disjoint antichains.

Theorem: Let $(P, \leq)$ be a Poset consisting of $mn+1$ elements. Either there is an antichain consisting of $m+1$ elements or there is a chain of length $(n+1)$ in P.

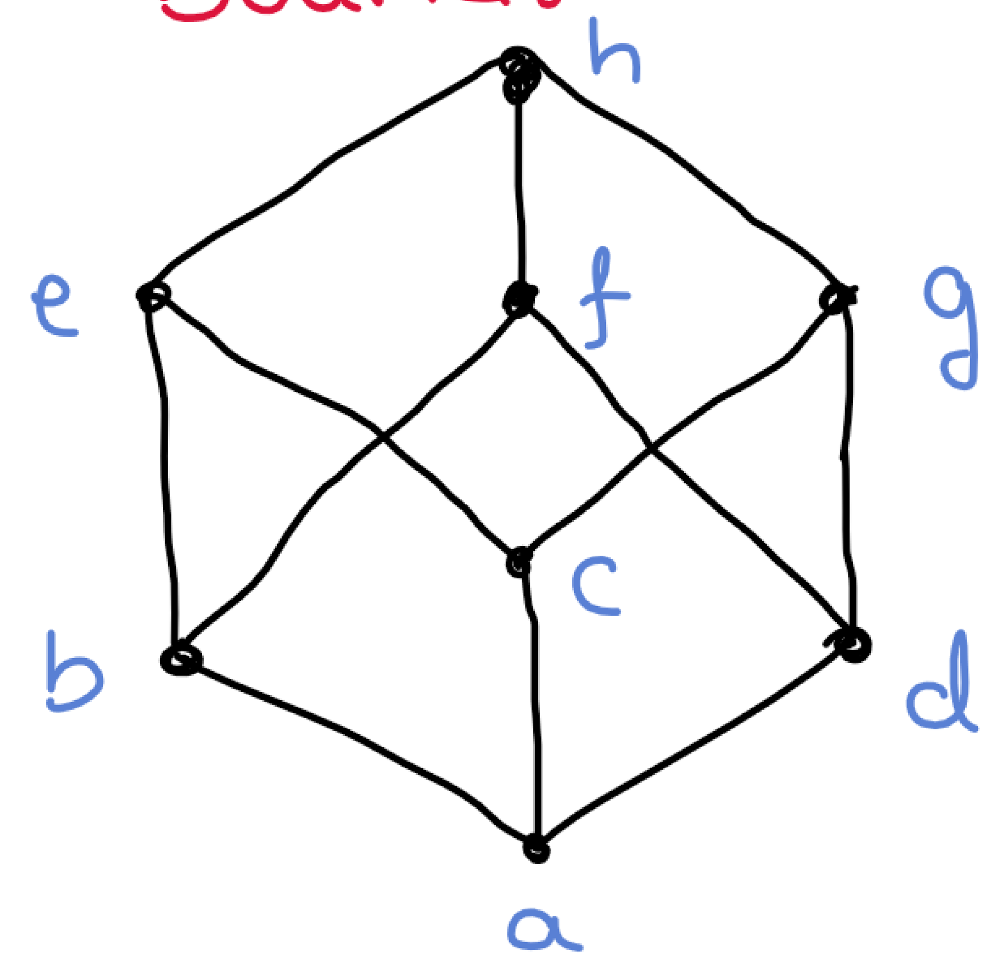
Proof: Suppose the length of the longest chain in P is n. Then P can be partitioned into n disjoint antichains.

If each of these antichains consists of $\leq m$ elements, then the total number of elements in P is atmost mn which is a contradiction.

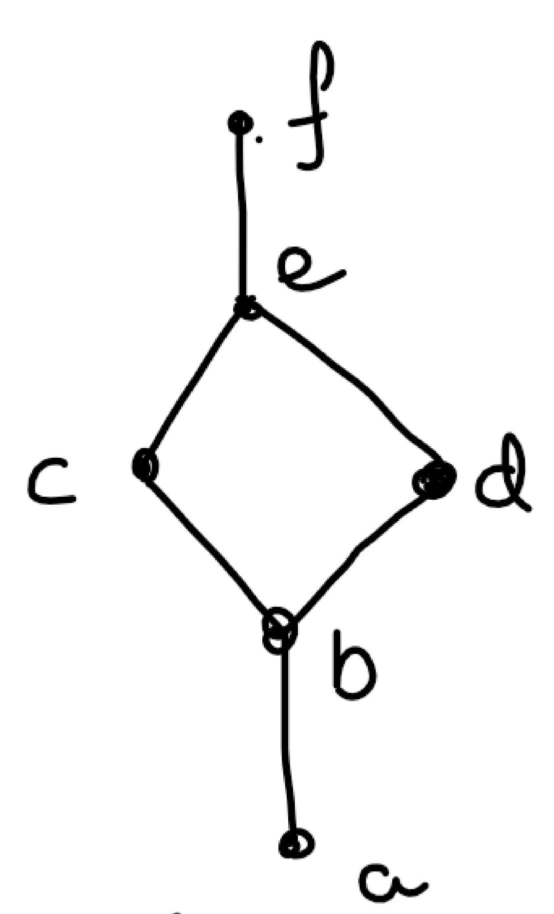
$\Longrightarrow$ There is an antichain consisting of $m+1$ elements.

Definition: A poset is said to be a lattice if every two elements in the set have a unique least upper bound and a unique greatest lower bound.

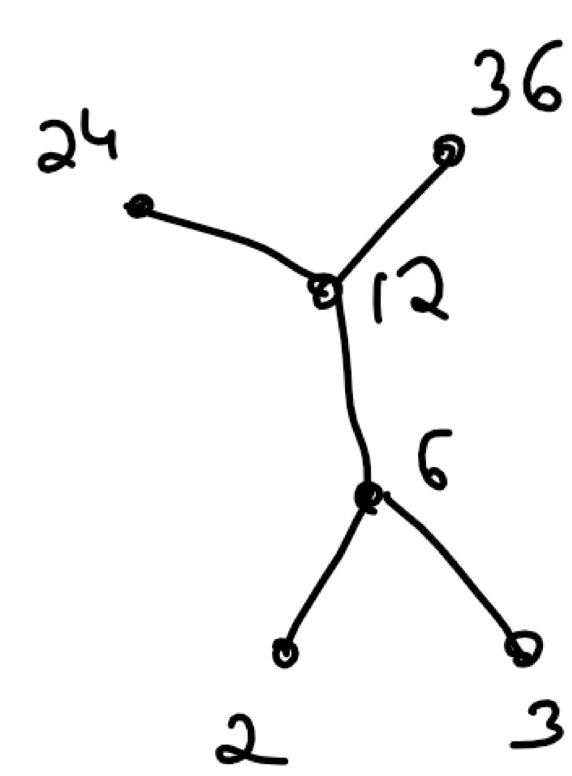
Example:



e, g → lub = h
e, g → glb = c
e, d → lub = h
e, d → glb = a



Lattice

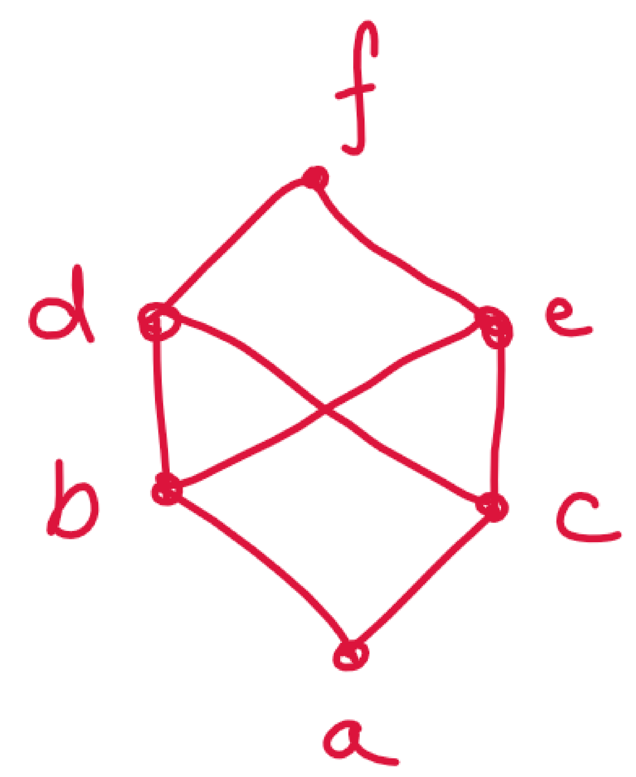


← {24, 36} l.u.b: NO

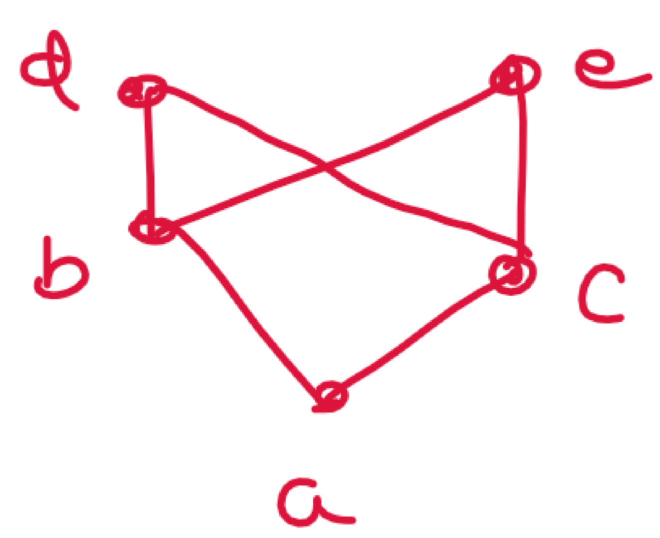
Not a lattice

Note: $(Z^+, \backslash)$ is an infinite lattice.

Is it a lattice ?



No. lub(b, c) = ?



No. lub(b, c) = ?