Partition:

The family of sets $A_1, A_2, \dots, A_n$ is said to be a partition of S if-

(i) $\bigcup_{i=1}^{n} A_i = S$, collectively exhaustive

(ii) $A_i \cap A_j = \emptyset, \ \forall\, i, j, \ i \neq j$, mutually exclusive.

Theorem of Total probability:

If $B_1, B_2, \dots, B_k$ are partitions of S with $P(B_i) \neq 0$ and A be any event of S, then

$$P(A) = \sum_{i=1}^{k} P(A|B_i)P(B_i)$$

Proof:

Baye's Theorem:

Let $B_1, B_2, \dots, B_k$ are partitions of S with $P(B_i) \neq 0$ and A be any arbitrary event of S, then

$$P(B_i|A) = \frac{P(B_i)\ P(A|B_i)}{\sum_{i=1}^{k} P(B_i)\ P(A|B_i)}$$

Proof: By total probability theorem, $P(A) = \sum_{i=1}^{k} P(A|B_i)P(B_i)$ ------ (i)

By definition of conditional probability,

$$P(A \cap B_i) = P(B_i)P(A|B_i) = P(A)P(B_i|A)$$

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{P(A)} \text{ ------- (ii)}$$

Subdtituting (i) in (ii), we get $P(B_i|A) = \frac{P(B_i)\ P(A|B_i)}{\sum_{i=1}^{k} P(B_i)\ P(A|B_i)}$

Problems on Total probability and Baye's Theorem:

1. Suppose 3 companies X, Y and Z produce televisions. X produce twice as many as Y, while Y and Z produce the same number. It is known that 2% of X, 2% of Y and 4% of Z are defective. All the televisions produced are put into one shop and then one TV is selected at random.

(i) What is the probability that the television is defective?

(ii) Suppose a TV chosen is defective, what is the probability that this TV is produced by the company X?

2. Box 1 contains 10 white and 3 black balls. Box 2 contains 3 white and 5 black balls. 2 balls are drawn at random from box 1 and placed in box 2. Then one ball is drawn at random from the box 2. Then one ball is drawn at random from box 2. What is the probability that it is a white ball?

3. It is suspected that a patient has one of the diseases A_1, A_2, A_3. Suppose that the population suffering from these illness are in the ratio 2:1:1. The patient is given a test, which turns out to be positive in 25% of the case of A_1 , 50% of A_2 and 90% of A_3. Given that out of 3 tests taken by the patient, 2 are positive, then find the probability for each of these illness.

Let A_i: Event that the patient has an illness $A_i,\ i = 1, 2, 3$

B: Event that the result of two tests are positive

4. A box contains 10 items in which 4 are defective. The items are chosen one after another and tested for defectiveness and kept out of the box. What is the probability that the last defective item is selected in

(i) 5^{th} test?

(ii) 9^{th} test?

A: Selection of 3 defective items in first 4 test

B: Selecting a last defective item in the 5^{th} test

Probability that the last defective item is selected in 5^{th} test $= P(A \cap B) = P(A).P(B|A)$

5. For a certain binary communication channel, the probability that a transmitted '0' is received as '0' is 0.95 and the probability that a transmitted '1' is received as '1' is 0.90. If the probability that a '0' is transmitted is 0.4, find the probability that

(i) a '1' is received

(ii) a '1' was transmitted given that '1' was received.

A: Event of transmitting 1 ;

$\bar{A}$: *Event of transmitting 0 ;*

B: Event of receiving 1 ;

$\bar{B}$: *Event of receiving 0 ;*

6. An archer with an accuracy of 75% fires three arrows at one target. The probability of the target falling is 0.6 if it is hit once, 0.7 if it is hit twice, and 0.8 if it is hit three times. Given that the target has fallen, find the probability that it was hit twice.

Let H_1, H_2, H_3 be the events of target being hit once, twice, thrice respectively.

A be the event of target falling

7. N letters are placed at random in N envelops. Show that the probability of each letter will be placed in a wrong envelope is $\sum_{k=2}^{N}(-1)^k \frac{1}{k!}$

Let A_k: k^{th} letter is place in k^{th}envelope

P[each letter will be placed in a wrong envelope]

$$= 1 - P[at\ least\ one\ letter\ placed\ in\ correct\ envelope]$$

$$= 1 - P\left[\bigcup_{k=1}^{N} A_k\right] \text{\_\_\_\_\_\_} (1)$$

We know that,

$$P(A_1 \cup A_2 \cup \cdots \cup A_k) = \sum_{i=1}^{k} P(A_i) - \sum_{i<j=2}^{k} P\left(A_i \cap A_j\right) + \sum_{i<j<r=3}^{k} P\left(A_i \cap A_j \cap A_r\right) - \ldots + (-1)^{k-1} P(A_1 \cap A_2 \cap \cdots \cap A_k)$$

$$P(A_i) = \frac{(N-1)!}{N!} \text{ and } \sum P(A_i) = {}^{N}C_1\, P(A_i) = N\, \frac{(N-1)!}{N!} = 1$$

$$P\left(A_i A_j\right) = \frac{(N-2)!}{N!} \text{ and } \sum P(A_i A_j) = {}^{N}C_2\, P\left(A_i A_j\right) = \frac{N(N-1)}{2!} . \frac{(N-2)!}{N!} = \frac{1}{2!}$$

$$P\left(A_i A_j A_k\right) = \frac{(N-3)!}{N!} \quad \text{and}$$

$$\sum P(A_i A_j A_k) = {}^{N}C_3\, P\left(A_i A_j A_k\right) = \frac{N(N-1)(N-2)}{3!} \cdot \frac{(N-3)!}{N!} = \frac{1}{3!}$$

$$P(A_1 A_2 \ldots . A_N) = \frac{1}{N!}$$

$Eqn\ (1) \Longrightarrow P$[each letter will be placed in a wrong envelope]

$$= 1 - P\left[\bigcup_{k=1}^{N} A_k\right]$$

$$= 1 - \left\{1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{N-1} \frac{1}{N!}\right\}$$

$$= \sum_{k=2}^{N} (-1)^k \frac{1}{k!}$$

8. Two absent minded roommates forget their umbrellas in some way or another. A always takes his umbrella when he goes out, while B forgets to take his umbrella with probability ½. Each of them forgets his umbrella at a shop with probability ¼. After visiting 3 shops they return home. Find the probability that

(a) They have both the umbrellas

(b) They have only one umbrella

(c) B has lost his umbrella given that there is only one umbrella after their return.

$Let,\ A_i$ be the event that A forget his umbrella in i^{th}shop

B_i be the event that B forget his umbrella in i^{th}shop

B_0 be the event that B has left his umbrella at home

$$P(A_i) = \frac{1}{4} = P(B_i), \qquad i = 1, 2, 3$$

$P(\overline{A_i}) = P(\overline{B}_i) = \frac{3}{4}$ and $P(B_0) = \frac{1}{2}$

(a). P(They have both the umbrellas)= $P\,(\overline{A_1}\,\overline{A_2}\,\overline{A_3})\,P(B_0) +$

$$P\,(\overline{A_1}\,\overline{A_2}\,\overline{A_3})\,P(\overline{B_0}\,\overline{B_1}\,\overline{B_2}\,\overline{B_3})$$

(b).

P(They have only one umbrella)= $P(B_0)[\mathrm{P}(\mathrm{A}_1) + \mathrm{P}(\overline{A_1}\,\boldsymbol{A_2}) + P\,(\overline{A_1}\,\overline{A_2}\,A_3)]$

$$+\ P(\overline{B_0}\,\overline{B_1}\,\overline{B_2}\,\overline{B_3})\ [\mathrm{P}(\mathrm{A}_1) + \mathrm{P}(\overline{A_1}\,\boldsymbol{A_2}) +\ P\,(\overline{A_1}\,\overline{A_2}\,A_3)]$$

$$+ P\,(\overline{A_1}\,\overline{A_2}\,\overline{A_3})\,[P(\overline{B_0}\,B_1) + P(\overline{B_0}\,\overline{B_1}\,B_2) + P(\overline{B_0}\,\overline{B_1}\,\overline{B_2}\,B_3)]$$

(c)

P(B has lost his umbrella | there is only one umbrella after their return)

$$= \frac{P(\text{B has lost his umbrella and there is only one umbrella})}{P(\textit{They have only one umbrella})}$$

$$= \frac{P\,(\overline{A_1}\,\overline{A_2}\,\overline{A_3})\,[P(\overline{B_0}\,B_1) + P(\overline{B_0}\,\overline{B_1}\,B_2) + P(\overline{B_0}\,\overline{B_1}\,\overline{B_2}\,B_3)]}{P(\textit{They have only one umbrella})}$$

Practice Questions:

1. A businessman goes to the hotels X, Y, Z respectively $20\%, 50\%$ and 30% of the time. It is known that $5\%, 4\%, 8\%$ of the rooms in X, Y, Z have faulty plumbing.

Determine the probability that businessman goes to the hotel room with faulty plumbing.

What is the probability that his room having faulty plumbing is assigned to hotel Z.

2. There are two bags one of which contains 4 white and 3 black balls and the other has 4 black and 3 white balls. A fair die is rolled. If a 1 or a 3 turns up, a ball is drawn from the first bag. Otherwise a ball is drawn from the second bag. What is the probability that the ball is black?

3.

. In an exam each candidate is admitted or rejected according whether he has passed or failed the test. Of the students who are really capable, 80% pass the test and of the incapable students, 25% pass the test. Given that 40% of the candidates are really capable. Find the proportion of capable college students.