1. Gauss - Jacobi's Method and Gauss -Seidel Method

Problem 1.1. *Solve the system of equations by Gauss -Jacobi's method and Gauss- Seidel method,*

Given
$$\left.\begin{aligned} 3x + 20y - z &= -18 \\ 2x - 3y + 20z &= 25 \\ 20x + y - 2z &= 17 \end{aligned}\right\} (*)$$

Ans: We can rewrite (*) as

$$\left.\begin{aligned} 20x+y-2z &= 17 \quad \text{--- (1)} \\ 3x+20y-z &= -18 \quad \text{--- (2)} \\ 2x-3y+20z &= 25 \quad \text{--- (3)} \end{aligned}\right\} (A)$$

In (1); $|20| > |1| + |-2|$

In (2); $|20| > |3| + |-1|$

In (3); $|20| > |2| + |-3|$

$\therefore$ (A) satisfies the diagonal dominance condition.

$$\left.\begin{aligned} (1) &\Rightarrow x = \frac{1}{20}(17 - y + 2z) \\ (2) &\Rightarrow y = \frac{1}{20}(-18 - 3x + z) \\ (3) &\Rightarrow z = \frac{1}{20}(25 - 2x + 3y) \end{aligned}\right\}$$

Gauss – Jacobi's Method.

Let $x^{(0)} = y^{(0)} = z^{(0)} = 0$. be the initial approx. soln.

contd..

Iteration I

$$x^{(1)} = \frac{1}{20}\left(17 - y^{(0)} + 2z^{(0)}\right) = 0.85$$

$$y^{(1)} = \frac{1}{20}\left(-18 - 3x^{(0)} + z^{(0)}\right) = -0.9$$

$$z^{(1)} = \frac{1}{20}(25 - 2x^{(0)} + 3y^{(0)} = 1.25$$

Iteration II :

$$x^{(2)} = \frac{1}{20}\left(17 - y^{(1)} + 2z^{(1)}\right) = 1.020$$

$$y^{(2)} = \frac{1}{20}\left(-18 - 3x^{(1)} + z^{(1)}\right) = -0.965$$

$$z^{(2)} = \frac{1}{20}(25 - 2x^{(1)} + 3y^{(1)} = 1.03$$

Iteration III :

$$x^{(3)} = \frac{1}{20}\left(17 - y^{(2)} + 2z^{(2)}\right) = 1.00125$$

$$y^{(3)} = \frac{1}{20}\left(-18 - 3x^{(2)} + z^{(2)}\right) = -1.0015$$

$$z^{(3)} = \frac{1}{20}\left(25 - 2x^{(2)} + 3y^{(2)}\right) = 1.00325$$

Iteration IV

$$x^{(4)} = \frac{1}{20}\left(17 - y^{(3)} + 2z^{(3)}\right) = 1.0004$$

$$y^{(4)} = \frac{1}{20}\left(-18 - 3x^{(3)} + z^{(3)}\right) = -1.000025$$

$$z^{(4)} = \frac{1}{20}\left(25 - 2x^{(3)} + 3y^{(3)}\right) = 0.99934$$

contd..

Iteration V :- $x^{(5)} = \frac{1}{20}(17 - y^{(4)} + 2z^{(4)}) = 0.99996625$

$$y^{(5)} = \frac{1}{20}(-18 - 3x^{(4)} + z^{(4)}) = -1.0000775$$

$$z^{(5)} = \frac{1}{20}(25 - 2x^{(4)} + 3y^{(4)}) = 0.99995625$$

Iteration VI :-

$$x^{(6)} = \frac{1}{20}(17 - y^{(5)} + 2z^{(5)}) = 0.9999995$$

$$y^{(6)} = \frac{1}{20}(-18 - 3x^{(5)} + z^{(5)}) = -0.9999997125$$

$$z^{(6)} = \frac{1}{20}(25 - 2x^{(5)} + 3y^{(5)}) = 1.00002212$$

From Iterations 5 and 6 the values of x, y and z are same upto three decimal places.

$\therefore$ approx. sol^n is $x = \underbrace{0.9999995}_{=0.999} \approx 1$

$$y = \underbrace{-0.999999}_{=-0.999} \approx -1$$

$$z = \underbrace{1.000022}_{=1.000} \approx \underline{\underline{1}}$$

Gauss-Seidal Method.

Let $y^{(0)} = z^{(0)} = 0$. Then

Iteration I:

contd..

$$x^{(1)} = \frac{1}{20}\left(17 - y^{(0)} + 2z^{(0)}\right) = 0.85$$

$$\checkmark\; y^{(1)} = \frac{1}{20}\left(-18 - 3x^{(1)} + z^{(0)}\right) = -1.0275$$

$$z^{(1)} = \frac{1}{20}\left(25 - 2x^{(1)} + 3y^{(1)}\right) = 1.010875$$

Iteration II

$$x^{(2)} = \frac{1}{20}\left(17 - y^{(1)} + 2z^{(1)}\right) = 1.0024625$$

$$y^{(2)} = \frac{1}{20}\left(-18 - 3x^{(2)} + z^{(1)}\right) = -0.99982563$$

$$z^{(2)} = \frac{1}{20}\left(25 - 2x^{(2)} + 3y^{(2)}\right) = 0.9997799$$

Iteration III

$$x^{(3)} = \frac{1}{20}\left(17 - y^{(2)} + 2z^{(2)}\right) = 0.99996927$$

$$y^{(3)} = \frac{1}{20}\left(-18 - 3x^{(3)} + z^{(2)}\right) = -1.0000062$$

$$z^{(3)} = \frac{1}{20}\left(25 - 2x^{(3)} + 3y^{(3)}\right) = 0.99991007$$

Iteration IV:

$x^{(4)} = 0.9999913$; $y^{(4)} = -1.0000032$

$z^{(4)} = 1.00000039$

$\therefore$ From Iterations 3 and 4 the values of x, y, z are same upto three decimal places. $\therefore$ approx. solⁿ is $x = 0.9999 \approx 1$

Problem 1.2. *Solve the system of equations*

$y = -1.0000 \approx -1$

$z = 1.0000 \approx 1$

Given, $(*) \begin{cases} 5x - y + z = 10 \\ 2x + 4y = 12 \\ x + y + 5z = -1 \end{cases}$

by (a) Gauss -Jacobi's method (Correct to two decimal places)
(b) Gauss- Seidal method (Correct to three decimal places)

→ Carryout 4 iterations.

Ans:- System $(*)$ satisfies diagonal dominance condition.

$$(*) \Rightarrow \left.\begin{aligned} x &= \frac{1}{5}(10 + y - z) \\ y &= \frac{1}{4}(12 - 2x) \\ z &= \frac{1}{5}(-1 - x - y) \end{aligned}\right\} (1)$$

Gauss Jacobi method.

Let $x^{(0)} = y^{(0)} = z^{(0)} = 0$ be the initial approximation.

Iteration I :- $x^{(1)} = \frac{1}{5}(10 + y^{(0)} - z^{(0)}) = 2$

$y^{(1)} = \frac{1}{4}(12 - 2x^{(0)}) = 3$

$z^{(1)} = \frac{1}{5}(-1 - x^{(0)} - y^{(0)}) \doteq 0.2$

Iteration II : $x^{(2)} = \frac{1}{5}(10 + y^{(1)} - z^{(1)}) = 2.64$

$y^{(2)} = \frac{1}{4}(12 - 2x^{(1)}) = 2$

$$z^{(2)} = \frac{1}{5}(-1 - x^{(1)} - y^{(1)}) = -1.2$$

Iteration III :

contd..

$$x^{(3)} = \frac{1}{5}(10 + y^{(2)} - z^{(2)}) = 2.64$$

$$y^{(3)} = \frac{1}{4}(12 - 2x^{(2)}) = 1.68$$

$$z^{(3)} = \frac{1}{5}(-1 - x^{(2)} - y^{(2)}) = -1.128$$

Iteration IV

$$x^{(4)} = \frac{1}{5}(10 + y^{(3)} - z^{(3)}) = 2.5616$$

$$y^{(4)} = \frac{1}{4}(12 - 2x^{(3)}) = 1.68$$

$$z^{(4)} = \frac{1}{5}(-1 - x^{(3)} - y^{(3)}) = -1.064$$

After 4 iterations the approx. sol^n correct to two decimal places are, $x = 2.56$, $y = 1.68$, $z = -1.06$

(b) Gauss - Seidal Method.

Let $y^{(0)} = z^{(0)} = 0$ be the initial approx.

Iteration I :.

contd..

$$x^{(1)} = \frac{1}{5}\left(10 + y^{(0)} - z^{(0)}\right) = 2$$

$$y^{(1)} = \frac{1}{4}\left(12 - 2x^{(1)}\right) = 2$$

$$z^{(1)} = \frac{1}{5}\left(-1 - y^{(1)} - x^{(1)}\right) = {}^{-}1$$

Iteration II

$$x^{(2)} = \frac{1}{5}\left(10 + y^{(1)} - z^{(1)}\right) = 2.6$$

$$y^{(2)} = \frac{1}{4}\left(12 - 2x^{(2)}\right) = 1.7$$

$$z^{(2)} = \frac{1}{5}\left(-1 - y^{(2)} - x^{(2)}\right) = {}^{-}1.06$$

Iteration III

$$x^{(3)} = 1/5\left(10 + y^{(2)} - z^{(2)}\right) = 2.552$$

$$y^{(3)} = 1/4\left(12 - 2x^{(3)}\right) = 1.724$$

$$z^{(3)} = 1/5\left(-1 - y^{(3)} - x^{(3)}\right) = {}^{-}1.0552$$

Iteration IV

$$x^{(4)} = 1/5\left(10 + y^{(3)} - z^{(3)}\right) = 2.55584$$

$$y^{(4)} = 1/4\left(12 - 2x^{(4)}\right) = +1.72208$$

$$z^{(4)} = 1/5\left(-1 - y^{(4)} - x^{(4)}\right) = {}^{-}1.055584$$

After 4 iterations the approx sol^n correct to 3 decimal places are, $x \approx 2.556$

$y \approx 1.722$; $z \approx \underline{{}^{-}1.056}$

Problem 1.3. *Solve the system of equations*

$$\begin{aligned} 2x + y + 6z &= 9 \\ 8x + 3y + 2z &= 13 \\ x + 5y + z &= 7 \end{aligned}$$

HW

by Gauss- Seidal method (Correct to 3 decimal places)

Ans:- $x = 1.625$

$y = 1.075$

$z = 0.7792$

Problem 1.4. *Solve the system of equations*

HW

$$\begin{aligned} 10x - 2y - z - w &= 5 \\ -2x + 10y - z - w &= 15 \\ -x - y + 10z - 2w &= 27 \\ -x - y - 2z + 10w &= -9 \end{aligned}$$

by Gauss- Seidal method

Ans:- $x = 1.2167$

$y = 2.05$

$z = 3.0333$

$w = 0.0333$

Eg: $\vec{a} = 2\hat{i} + 3\hat{j} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = (2,3)$

2. Eigen values and eigen vectors

$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hat{i} \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hat{j}$

Consider a matrix $A = \begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix}$. Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$ be two 2-dimensional vectors

$Ae_1 = \begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = (1,2) \neq$ any multiple of e_1

$Ae_2 = \begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} = (0,4) = 4e_2$

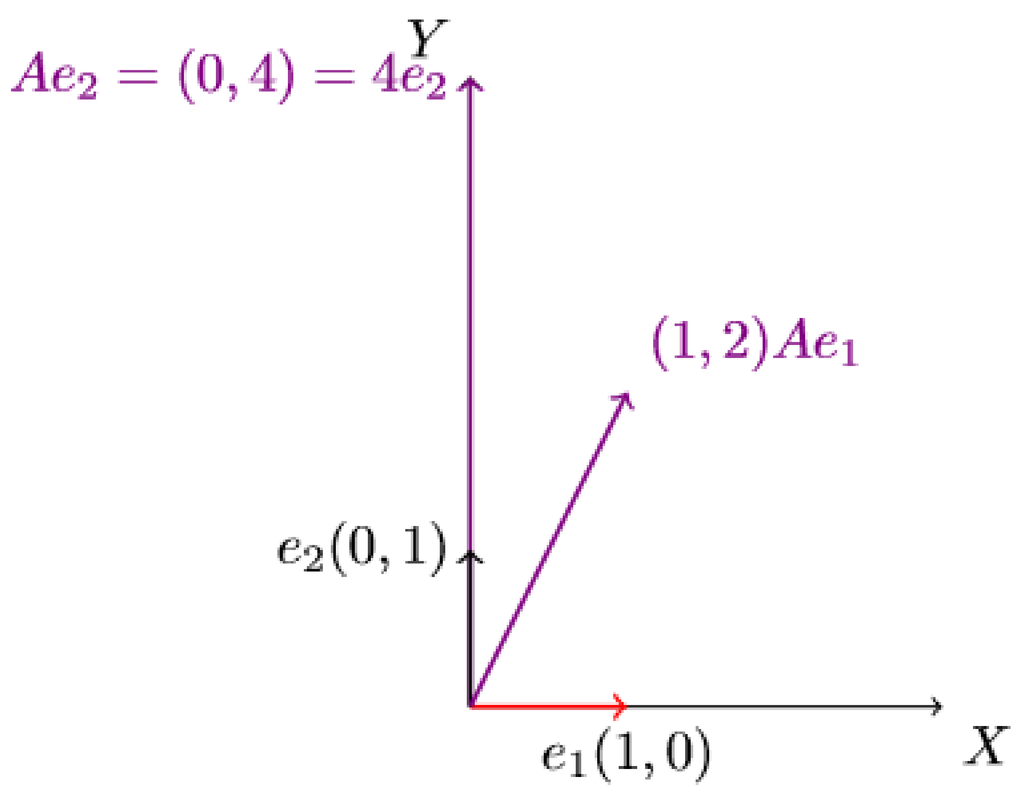
eigen vector

eigen value.

$\vec{x} = 10e_2$

$A\vec{x} \parallel \vec{x}$

Graphical interpretation:



$AX = \lambda X$ real or complex scalar

Let A be a square matrix then the *eigen vectors* of A are the non zero vectors X, that after being multiplied by the matrix A, the two vectors X and AX remain parallel.

i.e., X and AX are parallel then we can write $AX = \lambda X$ for some scalar λ.

Example 2.1. Consider the matrices, $A = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$

For $X = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

then $AX = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

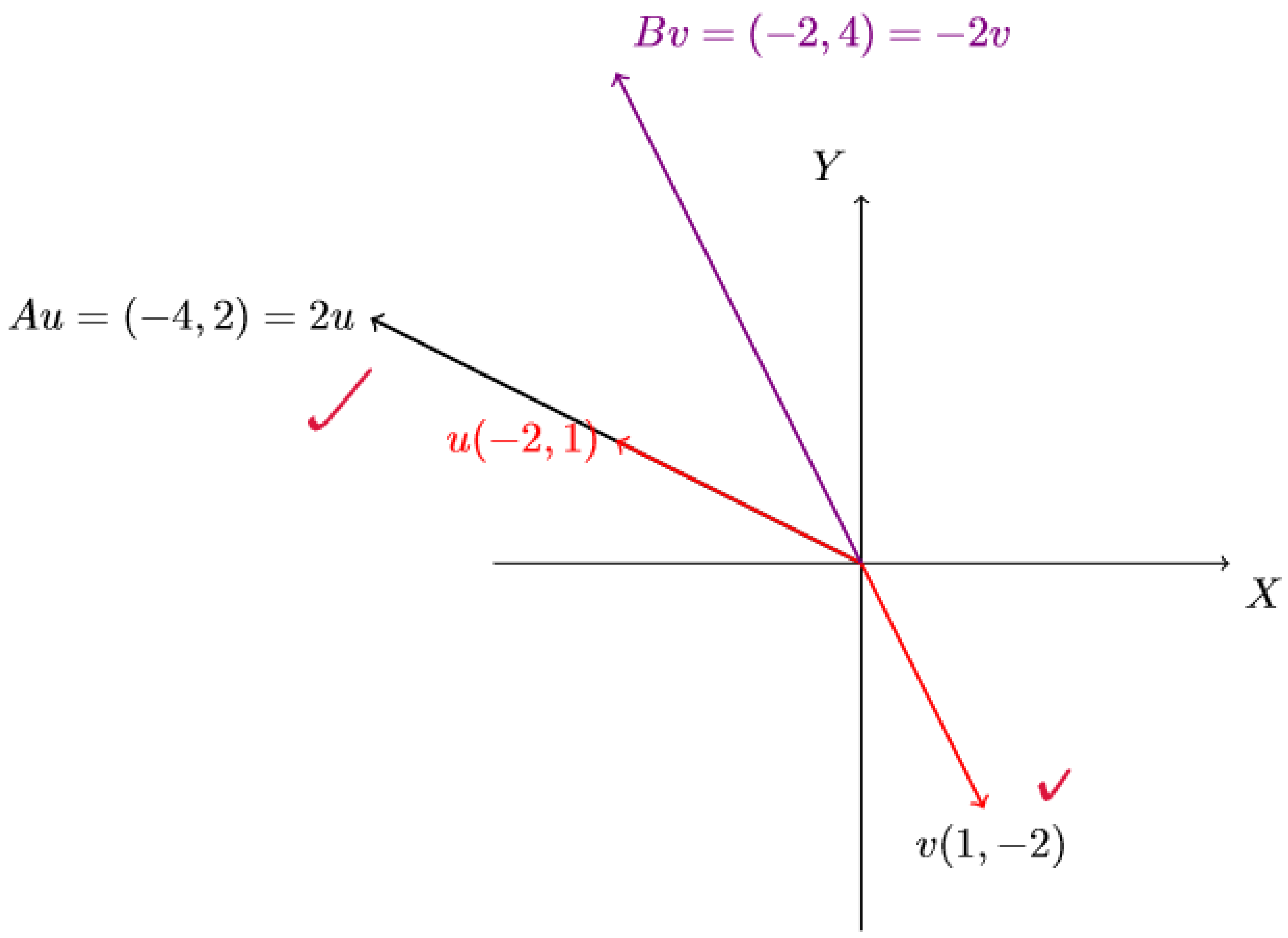
$= \begin{pmatrix} -4 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} = 2X$

ie; X and AX are parellel

$\therefore$ $X = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is the eigen vector of A

and the respective eigen value is 2.

Graphical interpretation:



Example 2.2. Consider the matrix $B = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$.

For $v = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$$Bv = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ +4 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$
$$= -2v$$

ie; v and Bv are $\|^{el}$

$\therefore$ v is the eigen vector for B and the respective eigen value is -2.

$X \longrightarrow$ non-zero vectors ✓
$AX = \lambda X$ ✓

Now, we can give a formal mathematical description of this idea,

Definition 2.3. Given a square matrix A, let us consider the problem of finding numbers λ (real or complex) and non zero vectors (column matrix) X such that $AX = \lambda X$. This problem is called **eigen value problem.** The numbers λ are called **eigen values** of the matrix A, and the non zero vectors X are called the **eigen vectors** corresponding to the eigen value λ.

→ Let A be a square matrix of order n. Let λ be a scalar. then $(A - \lambda I)_{n \times n}$, is called the Characteristic matrix.

→ $|A - \lambda I| = 0$, is called the Characteristic eqn of A.

→ $|A - \lambda I| = 0$, is a eqn of degree 'n' in 'λ'.

→ Roots of the equation $|A - \lambda I| = 0$ is called the eigen values of A.

→ Let $\lambda_1, \lambda_2, \ldots \lambda_n$ be the eigen values of A. ✓

→ Find a non zero vector X ∋: $AX = \lambda_i X$
where $i = 1, 2, \ldots n$

Finding eigen values

Let A be a square matrix then the eiqgen vector of A is a **non-zero** vector X such that $AX = \lambda X$ for some scalar λ.

i.e.,

$$AX = \lambda X$$
$$AX - \lambda X = 0$$
$$(A - \lambda I)X = 0$$

The matrix $A - \lambda I$ is called the **characteristic matrix** and the equation $|A - \lambda I| = 0$ is called the **characteristic equation** of A.

The values of λ which satisfies the characteristic equation $|A - \lambda I| = 0$ of A are called the **eigen values or characteristic roots or latent roots** of A.

Corresponding to each eigen value λ the non zero vector X satisfies $AX = \lambda X$ is called the eigen vector of λ.

ie; $AX = \lambda_i (IX)$ for $i = 1, 2, \ldots n$

$\Rightarrow AX - (\lambda_i I)X = 0$

$\Rightarrow \boxed{(A - \lambda_i I)X = 0}$

$A_{n \times n}$

$X_{n \times 1} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$I_{n \times n}$.

Properties of eigen values::

Let A be an $n \times n$ matrix. Assume that A has n distinct eigen values say, $\lambda_1, \lambda_2, ..., \lambda_n$ then

- the eigen values of A^T are $\lambda_1, \lambda_2, ..., \lambda_n$
- the eigen values of A^{-1} (if it exists) are $\dfrac{1}{\lambda_1}, \dfrac{1}{\lambda_2}, ..., \dfrac{1}{\lambda_n}$
- the eigen values of the matrix $A - \alpha I$ are $\lambda_1 - \alpha, \lambda_2 - \alpha, ..., \lambda_n - \alpha$
- for any non negative integer k, the eigen values of A^k are $\lambda_1^k, \lambda_2^k, ..., \lambda_n^k$

$\left(A_{n\times n} - \alpha I_{n\times n}\right)_{n\times n}$

α - Scalar

A^k