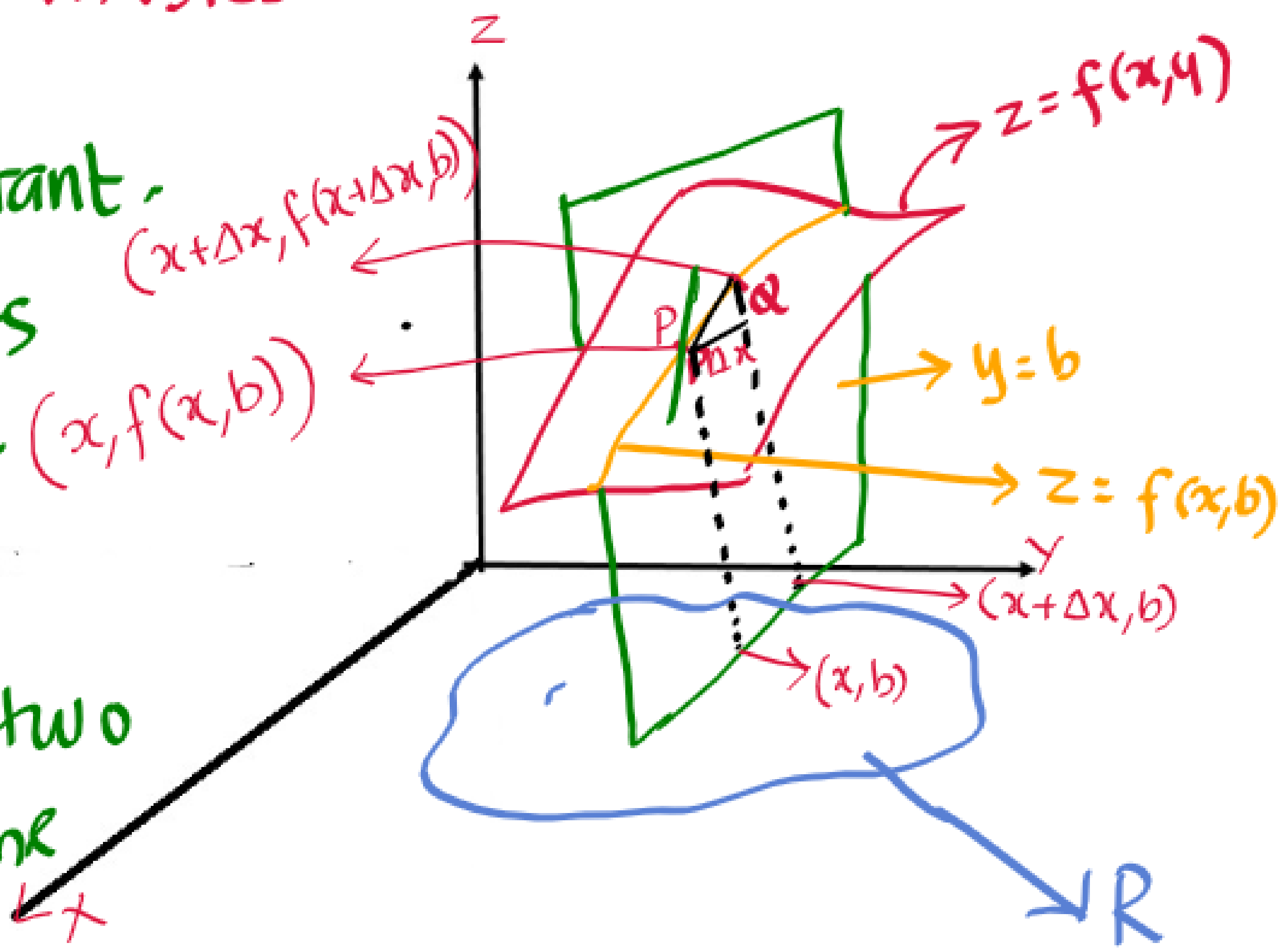
Exact differential Equations

1. Partial derivatives

dpt var.

Let $z = f(x, y)$ be a function of two independent ~~random~~ variables.

indpt. variables



If we. y as constant.
Take $y = b$, represents a plane $\perp^r$ to xy plane.

let $P(x, f(x,b))$ and $Q(x+\Delta x, f(x+\Delta x, b))$ be two neighbouring points on the curve $z = f(x,b)$.

Then the slope of the chord PQ is,

$$\frac{f(x+\Delta x, b) - f(x,b)}{(x+\Delta x) - x} = \frac{f(x+\Delta x, b) - f(x,b)}{\Delta x}$$

as $\Delta x \to 0$, the chord PQ becomes a tangent at P.

$\therefore$ Slope of the tangent at $P(x, f(x,b))$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x, b) - f(x,b)}{\Delta x} \text{ if exists } = \left(\frac{\partial z}{\partial x}\right)_{P(x, f(x,b))}$$

* Let $z = f(x,y)$ be a function of two indpt variables x and y. If we keep y as constant then z is a function of x alone. Then the partial derivative of z w.r.t. x is denoted by $\frac{\partial z}{\partial x}$ or z_x or f_x or $\frac{\partial f}{\partial x}$, is defined as total derivative of z w.r.t. x by keeping y as constant.

ie;
$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}, \text{ if exists}$$

* If we keep x as constant then z is a function of y alone. Then the partial derivative of z w.r.t y is denoted by $\frac{\partial z}{\partial y}$ or z_y or $\frac{\partial f}{\partial y}$ or f_y, is defined as the total derivative of z w.r.t y by keeping x as constant

$$\therefore \frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$ if exists.

Eg: $z = x^2 y$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2 y) = y \cdot \frac{d}{dx}(x^2) = 2xy$$

$z = f(x,y)$

$\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ — functions of x & y

first order partial derivati. of z w.r.t x & y

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \qquad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} \qquad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} \qquad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$$

second order partial derivatives of z w.r.t x & y

* If $u = f(x,y)$ & $v = f(x,y)$. then

① $$\frac{\partial}{\partial x}(uv) = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} \qquad \frac{\partial}{\partial y}(uv) = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}$$

② $$\frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$$

$$\frac{\partial}{\partial y}\left(u/v\right) = \frac{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}}{v^2}$$

2. Total differential

Let $z = f(x, y)$ be a function of two independent random variables.

Q. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$z = x^3 + 3x^2y^2 + 5y^4$.

Ans:- $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(x^3 + 3x^2y^2 + 5y^4\right)$

$= \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial x}(3x^2y^2) + \frac{\partial}{\partial x}(5y^4)$

$= 3x^2 + 3y^2\frac{\partial}{\partial x}(x^2) + 0$

$= 3x^2 + 3y^2 \times (2x) = 3x^2 + 6xy^2$

$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^3) + \frac{\partial}{\partial y}(3x^2y^2) + \frac{\partial}{\partial y}(5y^4)$

$= 0 + 3x^2(2y) + 5 \times 4y^3$

$= 6x^2y + 20y^3$ //

Total differential:- Let $z=f(x,y)$ be a function of two independent variables x and y.

Then the total differential of z is defined as,

$$\boxed{dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy} \checkmark$$

$\rightarrow$ If $z = f(u,v,w)$ then

$$dz = \frac{\partial z}{\partial u}\cdot du + \frac{\partial z}{\partial v}dv + \frac{\partial z}{\partial w}dw$$

Eg:- (i) Suppose $z = xy$ then

$$dz = \frac{\partial}{\partial x}(xy)\cdot dx + \frac{\partial}{\partial y}(xy)\,dy$$

$$\Rightarrow d(xy) = y\,dx + x\,dy$$

(ii) Suppose $z = (y/x)$ then.

$$\frac{x\,dy - y\,dx}{x^2}$$

$$dz = \frac{\partial}{\partial x}(y/x)\,dx + \frac{\partial}{\partial y}(y/x)\cdot dy = y\cdot\left(\frac{-1}{x^2}\right)dx + \frac{1}{x}(1)\,dy$$

3. Exact differential equation

Definition 3.1. A d.e. of the form

$\underbrace{M(x,y)}_{\text{function of } x \,\&\, y} dx + \underbrace{N(x,y)}_{\text{function of } x \,\&\, y} dy = 0$ — (*) is said

to be exact if LHS of (*) is a total differential or exact differential of some function of x and y.

Suppose LHS of (*) is the total differential of $u(x,y)$ then (*) becomes, $d(u) = 0$

Integrating both sides we get,

$\boxed{u(x,y) = u = C}$, is the sol^n of (*).

Eg: (1) Consider $\overset{N\,dy \;+\; M\,dx \;=\; 0}{x\,dy + y\,dx = 0}$

This can be written as $d(xy) = 0$

$\Rightarrow$ $xy = C$ is the sol^n

The following theorem gives a necessary and sufficient condition for a first order first degree differential equation to be an exact differential equation.

Theorem 3.2. Consider the d.e.

$M(x,y)\,dx + N(x,y)\,dy = 0$ ——— ⊛

Assume that $M(x,y)$, $N(x,y)$, $\frac{\partial N}{\partial x}$, $\frac{\partial M}{\partial y}$ are continuous functions then eq^n ⊛ is exact if and only if ⟺

$$\boxed{\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}}.$$

⊛ **Working rule to get the solution of an exact differential equation**

Sol^n is,

$$\underset{\text{treating 'y' as constant}}{\int M(x,y)\,dx} + \int \left(\begin{array}{c}\text{Terms in 'N' not}\\ \text{containing } x\end{array}\right) dy = C$$

Problem 3.3. *Verify the given differential equation is exact or not. If so, then solve it.*

$$\underbrace{y\sin 2x}_{M(x,y)}\, dx - \underbrace{(1+y^2+\cos^2 x)}_{=N(x,y)}dy = 0. \quad \text{———} \quad (*)$$

Solution:

Here $M = y\sin 2x$ $\qquad$ $N = -(1+y^2+\cos^2 x)$

$$\frac{\partial M}{\partial y} = \sin 2x \cdot (1) = \sin 2x \qquad \frac{\partial N}{\partial x} = 0+0+2\cos x\sin x = \sin 2x$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow (*) \text{ is } \underline{\underline{\text{exact}}}.$$

$\therefore$ Sol^n,

$$\underbrace{\int M\,dx}_{\text{treating 'y' terms as constant}} + \int \begin{pmatrix}\text{Terms in } N \text{ not}\\ \text{containing } x\end{pmatrix} dy = C$$

$$\Rightarrow \int y\sin 2x\,dx + \int -(1+y^2)\,dy = C$$

$$\Rightarrow y\int \sin 2x\,dx - \int (1+y^2)\,dy = C$$

$$\Rightarrow y\left(\frac{-\cos 2x}{2}\right) - \left(y+\frac{y^3}{3}\right) = C$$

$$\Rightarrow -\frac{y\cos 2x}{2} - y - y^3/3 = C \quad //$$

Problem 3.4. *Verify the given differential equation is exact or not. If so, then solve it.*

$$\underbrace{3x\,(xy-2)}_{M}\,dx+\underbrace{\left(x^3+2y\right)}_{N}\,dy=0. \quad \text{—— } (*)$$

Solution:

Here $M = 3x^2y - 6x$ $\quad N = x^3+2y$

$\Rightarrow \dfrac{\partial M}{\partial y} = 3x^2 - 0 = 3x^2$ $\quad \dfrac{\partial N}{\partial x} = 3x^2+0 = 3x^2$

$\therefore \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \Rightarrow (*)$ is exact.

$\therefore$ Sol^n is,

$$\int (3x^2y-6x)\,dx + \int 2y\,dy = C$$

Treating 'y' as constant

$$\Rightarrow 3y\left(x^3/3\right) - 6\left(\frac{x^2}{2}\right) + 2\left(y^2/2\right) = C$$

$$\Rightarrow x^3y - 3x^2 + y^2 = C$$

Problem 3.5. *Verify the given differential equation is exact or not. If so, then solve it.*

$$\underbrace{\left(\cos 2y - 3x^2y^2\right)}_{M} dx + \left(\underbrace{\cos 2y - 2x\sin 2y - 2x^3y}_{N}\right) dy = 0. \longrightarrow (*)$$

Solution:

$$\frac{\partial M}{\partial y} = -2\sin 2y - 3x^2(2y) = -2\sin 2y - 6x^2y$$

$$\frac{\partial N}{\partial x} = 0 - 2\sin 2y(1) - 2y(3x^2) = -2\sin 2y - 6x^2y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow (*) \text{ is exact}$$

$\therefore$ Sol^n is,

$$\int (\cos 2y - 3x^2y^2)\,dx + \int \cos 2y\,dy = C$$

treating y as constant

$$\Rightarrow \cos 2y \int dx - 3y^2 \int x^2 dx + \int \cos 2y\,dy = C$$

$$\Rightarrow x\cos 2y - x^3y^2 + \frac{\sin 2y}{2} = C$$

Recall:- If $z=f(x,y)$ then $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$

4. Equations reducible to exact differential equation

Equations that are not exact, can be made exact, by suitable multiplication of a function of x and y. Such multiplier is called an **integrating factor (I.F.)** of the differential equation.

Type 4.1. ***Inspection Method:*** ✓

Some of the frequently occurring exact differentials are

- $dx \pm dy =$ $d(x \pm y)$
- $xdx \pm ydy =$ $d\left(\frac{x^2+y^2}{2}\right)$
- $xdy + ydx =$ $d(xy)$
- $\dfrac{xdx + ydy}{x^2 + y^2} =$ $d\left(\frac{\log(x^2+y^2)}{2}\right)$
- $\dfrac{xdx + ydy}{\sqrt{x^2 + y^2}} =$ $d\left(\sqrt{x^2+y^2}\right)$
- $\dfrac{xdy + ydx}{xy} =$ $d(\log(xy))$
- $\dfrac{xdy - ydx}{x^2} =$ $d(y/x)$
- $\dfrac{xdy - ydx}{xy} =$ $d(\log(y/x))$
- $\dfrac{xdy - ydx}{x^2 + y^2} =$ $d(\tan^{-1}(y/x))$

verify it

Problem 4.2. *Solve* $ydx - xdy + 3x^2y^2e^{x^3}dx = 0.$

Ans:- Divide both sides by 'y^2' we get,

$$\frac{ydx - xdy}{y^2} + 3x^2e^{x^3}dx = 0$$

$$\Rightarrow d\left(\frac{x}{y}\right) + d\left(e^{x^3}\right) = 0$$

$$\Rightarrow d\left(\frac{x}{y} + e^{x^3}\right) = 0$$, exact d.e

Integrating both sides we get,

$$\frac{x}{y} + e^{x^3} = C$$

$$d(x/y) = \frac{\partial}{\partial x}(x/y)dx + \frac{\partial}{\partial y}(x/y)dy$$

$$\rightarrow d(f(x)) + d(g(x)) = d\left(f(x) + g(x)\right)$$

Problem 4.3. *Solve* $xdy - ydx = x\sqrt{x^2 - y^2}dx$.

<u>Ans:-</u> Given $xdy - ydx = x\sqrt{x^2\left(1 - y^2/x^2\right)}dx$

$$\Rightarrow xdy - ydx = x^2\sqrt{1 - y^2/x^2}\ dx$$

$$\Rightarrow \left(\frac{xdy - ydx}{x^2}\right) = \sqrt{1 - y^2/x^2}\ dx$$

$$\Rightarrow \frac{\left(\frac{xdy - ydx}{x^2}\right)}{\sqrt{1 - y^2/x^2}} = dx$$

$$\Rightarrow \frac{1}{\sqrt{1 - (y/x)^2}}\left(\frac{xdy - ydx}{x^2}\right) = dx$$

$$\Rightarrow d\left(\sin^{-1}(y/x)\right) = dx$$, exact

Integrating both sides

$$\Rightarrow \sin^{-1}(y/x) = x + C$$

Problem 4.4. *Solve* $y\left(2xy+e^{x}\right)dx = e^{x}dy$.

Ans:- $2xy^2dx + ye^x dx = e^x dy$

$$\Rightarrow 2x\,dx + \frac{ye^x dx - e^x dy}{y^2} = 0$$

$$\Rightarrow d(x^2) + d\left(\frac{e^x}{y}\right) = 0$$

$$\Rightarrow d\left(x^2 + \frac{e^x}{y}\right) = 0,$$ exact d.e.

Integrating

$$\Rightarrow x^2 + \frac{e^x}{y} = C$$

Type 4.5. *Consider the non exact equation* $Mdx + Ndy = 0$. *If the given differential equation is homogenous and* $Mx + Ny \neq 0$, *then the I.F. is* $\frac{1}{Mx + Ny}$.

Problem 4.6. *Solve* $\underbrace{(x^2y - 2xy^2)}_{M}\,dx - \underbrace{(x^3 - 3x^2y)}_{N}\,dy = 0$ —— (1)

Here $M = (x^2y - 2xy^2)$ & $N = -x^3 + 3x^2y$

(1) is a homo. eq^n of deg 3.

$$Mx + Ny = x(x^2y - 2xy^2) + y(-x^3 + 3x^2y)$$
$$= x^3y - 2x^2y^2 - x^3y + 3x^2y^2$$
$$= x^2y^2 \neq 0$$

$$\therefore \text{I.F} = \frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$$

Multiply eq^n (1) by $\frac{1}{x^2y^2}$ we get,

$$\underbrace{\left(\frac{1}{y} - \frac{2}{x}\right)}_{M} dx + \underbrace{\left(\frac{-x}{y^2} + \frac{3}{y}\right)}_{N} dy = 0,$$ will be an exact d.e —— (2)

from eq^n (2), we've,

$$M = \frac{1}{y} - \frac{2}{x} \qquad N = \frac{-x}{y^2} + \frac{3}{y}$$

$\therefore$ Sol^n is $\int\left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y}\,dy = C \Rightarrow \frac{1}{y}x - 2\log x + 3\log y = C$

Treating y as constant

Problem 4.7. *Solve* $(x^2 - 3xy + 2y^2)\,dx + x\,(3x - 2y)\,dy = 0.$

Ans:- $\log x + \dfrac{3y}{x} - \dfrac{y^2}{x^2} = C$

$\left(\text{Hint } \quad I.F = \dfrac{1}{x^3}\right)$

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

$$d\left(\sin^{-1}(y/x)\right) = \frac{\partial}{\partial x}\left(\sin^{-1}(y/x)\right)dx + \frac{\partial}{\partial y}\left(\sin^{-1}(y/x)\right)dy \quad \text{---} (*)$$

$$\frac{\partial}{\partial x}\left(\sin^{-1}(y/x)\right) = \frac{1}{\sqrt{1-(y/x)^2}} \cdot \frac{\partial}{\partial x}(y/x)$$

$$= \frac{1}{\sqrt{1-(y/x)^2}} \cdot y\left(\frac{-1}{x^2}\right)$$

$$|||^{ly}\ \frac{\partial}{\partial y}\left(\sin^{-1}(y/x)\right) = \frac{1}{\sqrt{1-(y/x)^2}} \cdot \frac{\partial}{\partial y}(y/x) = \frac{1}{\sqrt{1-(y/x)^2}} \cdot \frac{1}{x} \times 1$$

Type 4.8. *Consider the non exact equation* $Mdx + Ndy = 0$. *If*

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x), \text{ a function of } x \text{ alone.}$$

Then, the integrating factor is,

$$I.F. = e^{\int f(x)}dx.$$

$$\therefore (*) \Rightarrow d\left(\sin^{-1}(y/x)\right)$$

$$= \left(\frac{1}{\sqrt{1-(y/x)^2}} \cdot \frac{-y}{x^2}\right)dx + \left(\frac{1}{\sqrt{1-(y/x)^2}} \cdot \frac{1}{x}\right)dy$$

$$= \frac{1}{\sqrt{1-(y/x)^2}}\left[\frac{xdy - ydx}{x^2}\right]$$