PREDICATE CALCULUS

January 7, 2022

A part of declarative sentence describing the properties of an object or relation among objects is called a "**PREDICATE**"

Example

Consider two propositions, Ram is a Bachelor, Shyam is a Bachelor. Both Ram and Shyam have the same property of having bachelor. The part "is a bachelor" is called a predicate.

Notations

The predicate is denoted by capital letters and names of individuals or objects by small letters. Let "B" denote the predicate "is bachelor", then the sentence "x is a bachelor" can be written as $B(x)$, where x is a predicate variable $B(x)$ is also called a propositional function, which becomes a statement when values are submitted in place of x. A predicate requiring $m(>0)$ names is called $m-$place predicate

Example

x is taller than y; T(x,y)- the two place predicate

Quantifiers are words that refer to quantifiers such as "some" or "all" and indicate now frequently certain statement is true.
The phrase "for all" ($\forall$) is called the **UNIVERSAL QUANTIFIERS**

Example

All human beings are mortal. For all natural numbers "n", "$2n$" is an even number.

The phrase "there exists" ($\exists$) is called the **EXISTENTIAL QUANTIFIER**

Example

There exists x such that $x^2 = 5$. This can be written as $\exists x P(x)$ where $P(x) : x^2 = 5$.

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- for some x
- there is atleast one x

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- $(\forall x)P(x)$ is false if $P(x)$ is false for every x in U

Question 1

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- $(\exists x)x + 4 = 10 \qquad T$

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- $(\exists x)x + 4 = 10 \qquad T$
- $(\forall x)x + 4 > 15 \qquad F$
- $(\forall x)x + 4 \leq 10 \qquad F$

Question 2

Example

Symbolize the statement: All men are mortal

Solution: Let $M(x)$: x is an integer

$N(x)$: x is either positive or negative

$(\forall x)(M(x) \rightarrow N(x))$

Question 3

Example

Symbolize the statement: An integer is either positive or negative.
Solution: Let $M(x)$: x is a man
$H(x)$: x is a mortal
$(\forall x)(M(x) \rightarrow H(x))$

- $\neg(\forall x)P(x) \Leftrightarrow (\exists x)\neg P(x)$

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Question 4

Negate the following statements: For all real numbers x, if $x > 3$ then $x^2 > 9$

Solution: Given : Let $P(x) : x > 3$

$Q(x) : x^2 > 9$,

$\therefore (\forall x)(P(x) \rightarrow Q(x))$

$$\begin{aligned} \text{Negation is} \quad : \quad & \neg\forall x(P(x) \rightarrow Q(x)) \\ & \Leftrightarrow (\exists x)\neg(P(x) \rightarrow Q(x)) \\ & \Leftrightarrow (\exists x)\neg(P(x) \wedge \neg Q(x)) \end{aligned}$$

that is there exists a real number x such that $x > 3$ and $x^2 \leq 9$

Exercise Q5 : Negate the following statement " Every city in Canada is clean"

Rules of Inference

The method of derivation involving predicate formulas uses the rules of inference given for the statement calculus and also certain additional rules which are required to deal with the formulas involving quantifiers (In addition to Rule P and T)

(1) Rule US(Universal Specification)
From $(\forall x)A(x)$, we can conclude $A(y)$.

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(1) Rule US(Universal Specification)
From $(\forall x)A(x)$, we can conclude $A(y)$.

$$(\forall x)A(x) \implies A(y)$$

(2) Rule ES (Existential Specification)
From $(\exists x)A(x)$ one can conclude $A(y)$ provided that y is not free in any given premise and also not free in any prior step of the derivation.

$$(\exists x)A(x) \implies A(y)$$

(3) Rule EG(Extential Generalization)
From $A(x)$ one can conclude $(\exists y)A(y)$.

$$A(x) \implies (\exists y)A(y)$$

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From $A(x)$ one can conclude $(\exists y)A(y)$.

$$A(x) \implies (\exists y)A(y)$$

(4) Rule UG (Universal Generalization)
From $A(x)$ one can conclude $(\forall y)A(y)$ provided that x is not free in any of the given premises and provided that if x is free in a prior step which resulted from use of ES, then no variables introduced by that use of ES appear free in $A(x)$

Question 5

$$\text{Show that } (\forall x)[H(x) \to M(x)] \wedge H(s) \implies M(s)$$

[Note that this problem is a symbolic representation or translation of a well known arguement known as "Socrates arguement" which is given by
"All men are mortal
Socrates is a man
Therefore Socrates is a mortal"]

Solution : Denote $H(s)$: x is a man
$M(s)$: Socrates is mortal

$$\begin{array}{llr} (1) & (\forall x)[H(x) \to M(x)] & \textit{Rule P} \\ (2) & H(s) \to M(s) & \textit{Rule US} \\ (3) & H(s) & \textit{Rule P} \\ (4) & M(s) & \textit{Rule T} \end{array}$$

Question 6

Show that $(\exists x)M(x)$ *follows logically from the premises*

$$(\forall x)[H(x) \rightarrow M(x)] \text{ and } (\exists x)H(x)$$

Solution

(1)	$(\exists x)$	*Rule P*
(2)	$H(y)$	*Rule ES*
(3)	$(\forall x)[H(x) \rightarrow M(x)]$	*Rule P*
(4)	$[H(y) \rightarrow M(y)]$	*Rule US by* (3)
(5)	$M(y)$	*Rule T by* (2) *and* (4)
(6)	$(\exists x)M(x)$	*Rule EG*

Exercise 7

Show that

$$(\forall x)[P(x) \rightarrow Q(x)] \wedge \forall x[Q(x) \rightarrow R(x)] \implies (\forall x[P(x) \rightarrow R(x)]$$

Exercise 8

Show that from

$$(a)(\forall x)[F(x) \wedge S(x)] \rightarrow \forall y[M(y) \rightarrow W(y)]$$

$$(b)(\exists y)[M(y) \wedge \neg W(y)]$$

the conclusion $(\forall x)[F(x) \rightarrow \neg S(x)]$ follows