Graph Theory

References:

- Graph Theory by Frank Harary
- Graph theory with Application to computer science by Narasingh Deo

Definition: A graph G = (V, E) consists of a nonempty set $V = V(G)$ whose elements are called vertices (or points, or nodes) of G and a set $E(G)$ of unordered pairs of distinct elements of V(G), whose elements are called edges (or lines, or arc) of G.

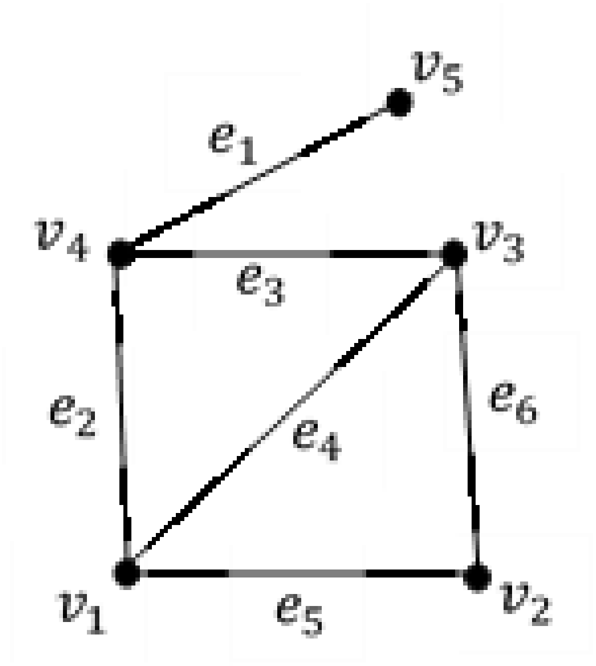


Fig.1 Graph G

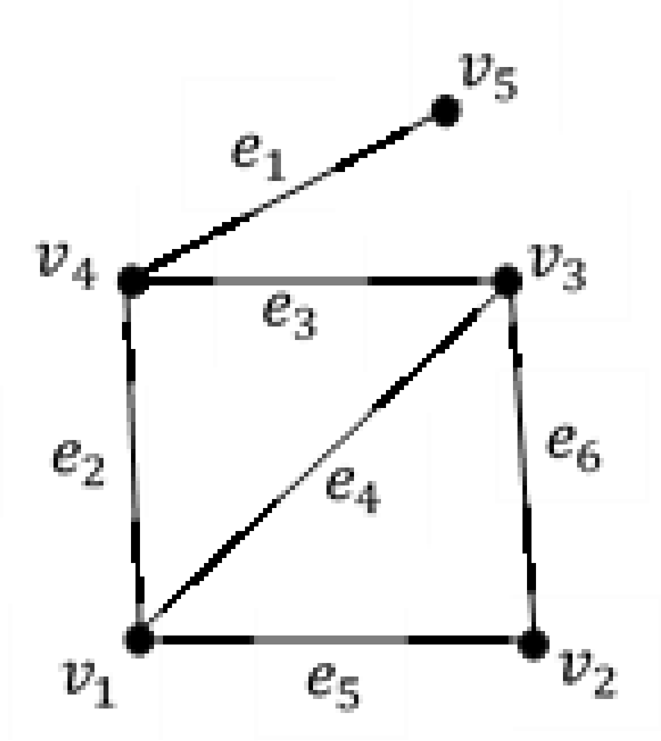


Fig.1 Graph G

$$V(G) = \{v_1, v_2, v_3, v_4, v_5\}$$

$$E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

$$|V(G)| = 5 \; and \; |E(G)| = 6$$

Two **vertices** in a graph G are said to be ***adjacent*** if there is an edge between them.

Example: In fig.1, v_1 is adjacent with v_2, i. e., $v_1 \sim v_2$

v_1 is adjacent with v_3, i. e., $v_1 \sim v_3$ etc.

Two **edges** are said to be ***adjacent*** if they have a vertex in common.

Example: In fig.1, e_1 and e_3 are adjacent, e_1 and e_2 are adjacent, etc.

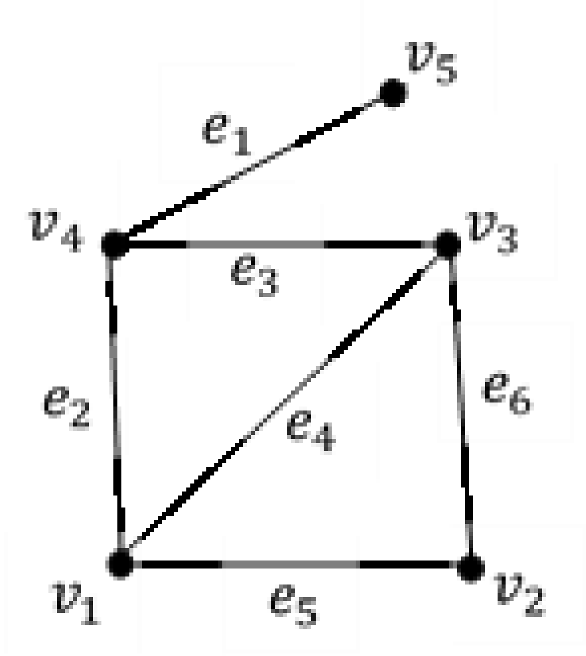


Fig.1 Graph G

In a definition of a graph G = (V, E), it is possible for the edge set E to be empty.

Such a graph without any edges, is called a ***null graph*.**

If a and b are two vertices, and e is the edge between a and b in a graph G,

then we say that the edge e is incident with the vertices a and b.

A graph with 'p' vertices and 'q' edges is called a (p, q) graph.

Sub graph: A sub graph H of G is a graph having all of its vertices and edges in G.

If G_1 is a sub graph of G, then G is a **super graph** of G_1.

A ***spanning sub graph*** is a sub graph containing all the vertices of G. For any set S of vertices of G, the ***induced sub graph*** $\langle S \rangle$ is the maximal subgraph of G with vertex set S. Thus two vertices of S are adjacent in $\langle S \rangle$ if and only if they are adjacent in G.

Example: In Fig.2. G_1 is a induced sub graph of G but G_2 is not; G_2 is a spanning sub graph of G but G_1 is not.

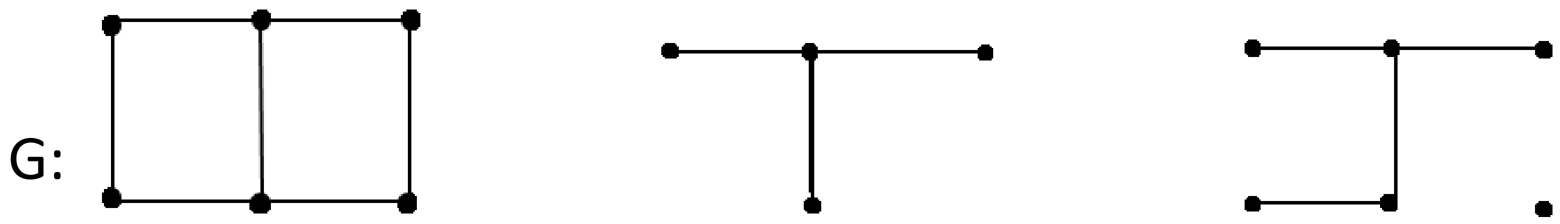


Fig.2. A graph and two sub graphs

The **removal of a vertex v** from a graph G results in that sub graph $G - v$ of G consisting of all vertices of G except v and all edges not incident with v. Thus $G - v$ is the maximal sub graph of G not containing v.

Removal of an edge $\boldsymbol{e}$ from a graph G results in that sub graph $G - e$ of G containing all edges of G except e. Thus $G - e$ is the maximal sub graph of G not containing e.

If two vertices u and v are not adjacent in G, the addition of edge uv results in the minimal super graph of G containing the edge uv.

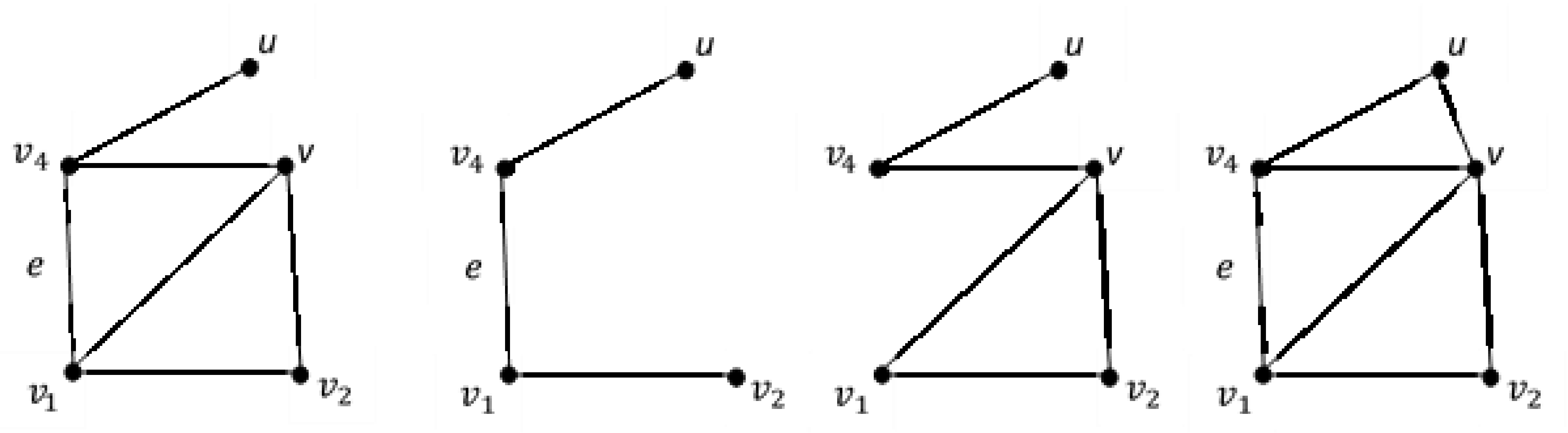


Fig. 3 Graphs G, $G - v$, $G - e$ and $G + uv$

Isomorphic graph: Two graphs G and H are isomorphic if there exists a one-to-one correspondence between their vertex sets which preserves adjacency

Example:

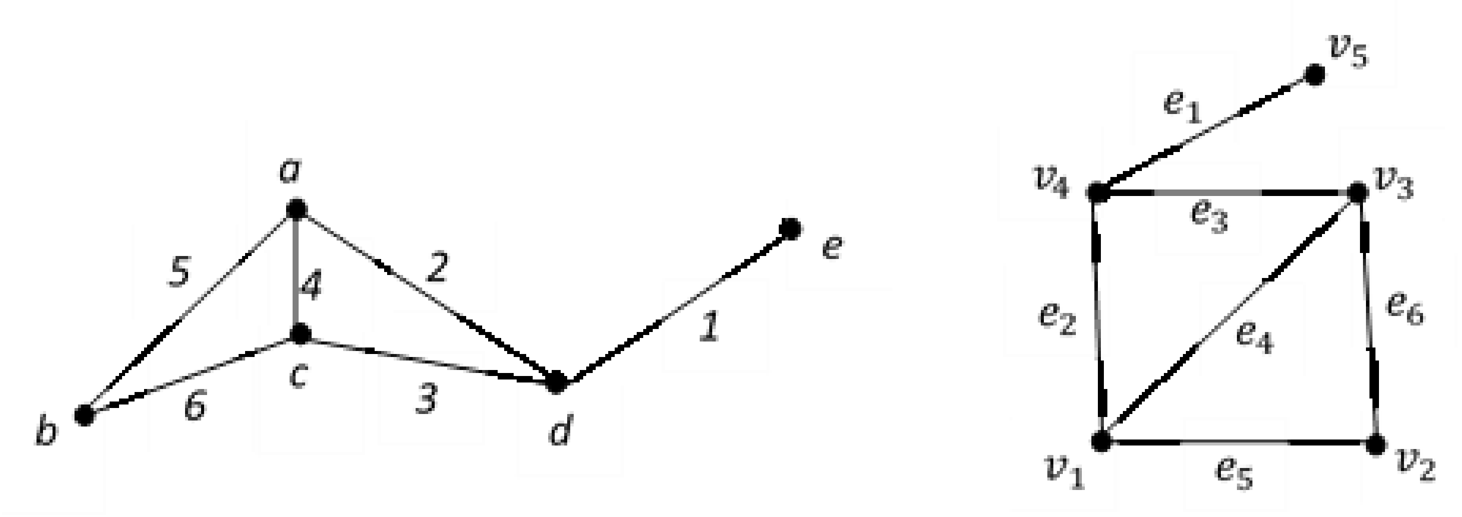


Fig.4. Isomorphic graphs

The correspondence between the two graphs in Fig.4 is as follows:

The vertices *a, b, c, d,* and *e* correspond to v_1, v_2, v_3, v_4, and v_5, respectively. The edges 1, 2, 3, 4. 5 and 6 correspond to e_1, e_2, e_3, e_4, e_5, and e_6, respectively.

Walk: A walk of a graph G is an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n$ beginning and ending with vertices, in which each edge is incident with the two vertices immediately preceding and following it.

A walk is ***closed*** if $v_0 = v_n$ and is ***open*** otherwise.

It is a ***trail*** if all the edges are distinct and a ***path*** if all the vertices and edges are distinct.

If the walk is closed, then it is a ***cycle*** provided its *n* vertices are distinct and $n \geq 3$. A cycle with *n* vertices denoted by C_n.

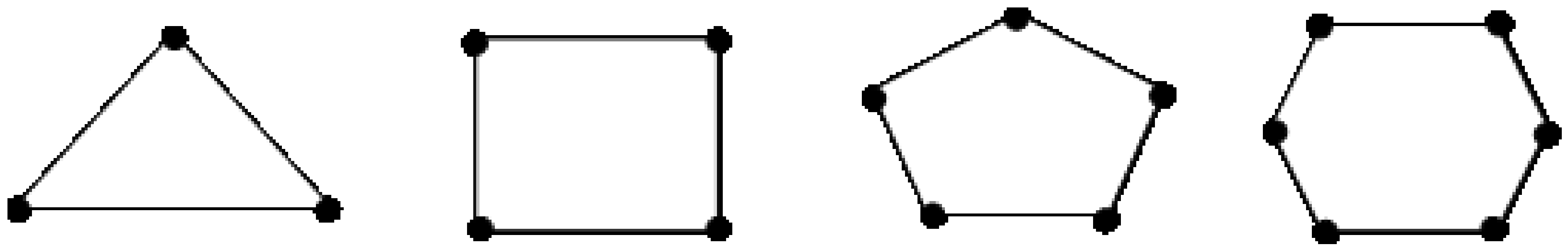


Fig. 5 Cycles C_3, C_4, C_5 and C_6

The length of a walk $v_0, v_1, v_2, \ldots, v_n$ is n, the number of occurrence of edges in it.

An edge with identical ends is called a loop and two edges with same end vertices are called parallel edges.

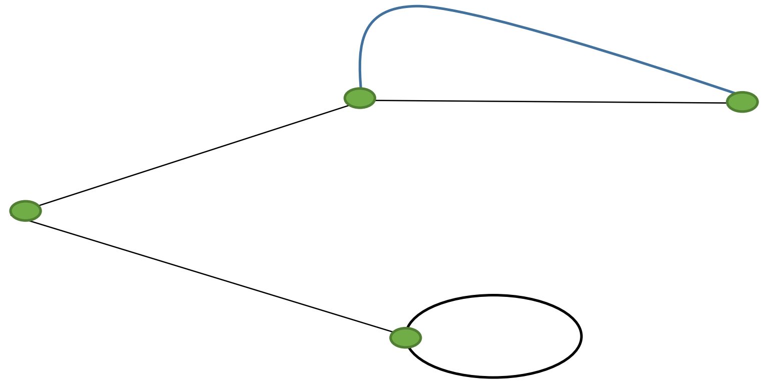
A graph is finite if both its vertex set and edge set are finite.

A graph is simple if it has no loops or parallel edges

In a **multigraph**, no loops are allowed but more than one line can join two points; these are called multiple lines.

If both loops and multiple lines are permitted, we have a **pseudograph**.

Ex: A graph *G* with loops and multiple edges



Distance between two vertices: The distance $d(u, v)$ between two vertices u and v in G is the length of the shortest path joining them, if any; otherwise $d(u, v) = \infty$.

In a connected graph G,

$d(u,v) \geq 0$ with $d(u,v) = 0$ if and only if $u = v$.

$$d(u,v) = d(v,u)$$

$$d(u,v) + d(v,w) \geq d(u,w)$$

A shortest $u - v$ path is called a **geodesic.**

Eccentricities: The eccentricity $e(v)$ of a vertex v in a connected graph G is maximum of $d(u, v)$ for all u in G.

The radius $r(G)$ is the minimum eccentricity of the vertices of G.

The maximum eccentricity is the **diameter.** A vertex v is a central vertex if $e(v) = r(G)$, and the center of G is the set of all central vertices.

The ***girth*** of a graph G, denoted $g(G)$, is the length of a shortest cycle in G; the ***circumference*** $c(G)$ the length of any longest cycle.

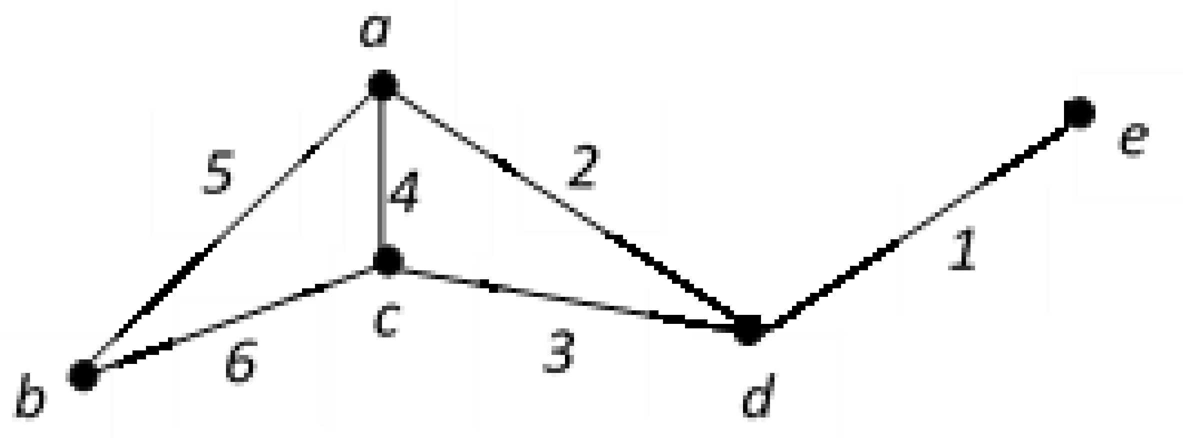


Figure 6. Graph G

$d(b, d) = 2,$

diameter of G = 3, radius of G =2

girth g(G) = 3, *circumference* c(G) = 4

Degree: The degree of a vertex v in a graph G, denoted $\deg(v)$, is the number of edges incident with v.

A vertex in a graph G is said to be **isolated** when its degree is '0'.

A vertex in a graph G is said to be an **end vertex** or **pendent vertex** if its degree is 1.

The minimum degree among the vertices of G is denoted by δG, the maximum degree among the vertices of G is denoted by ΔG.

Example: In a graph G shown in fig.6, $\delta G = 1$ and $\Delta G = 3$.

Regular graph: A graph in which all vertices are of equal degree is called a *regular graph*.

A regular graph of degree 3 is called **cubic** graph.

A cubic graph has always even number of vertices

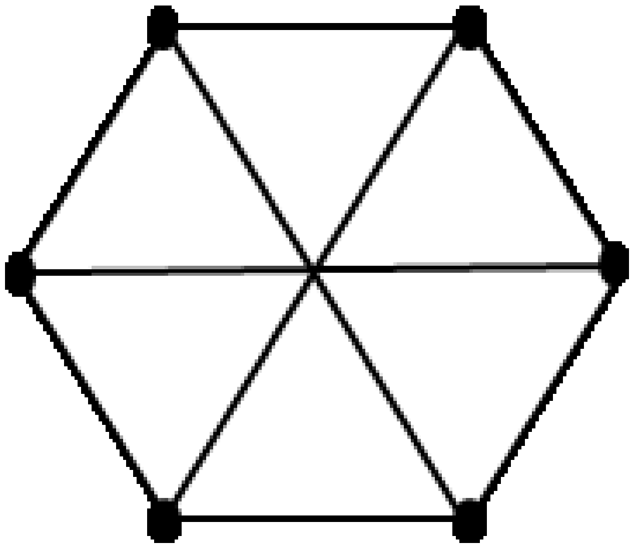


Figure 7. Regular graph

Hand shaking lemma: The sum of the degree of all vertices in a graph G is an even number, and this number is equal to twice the number of edges in the graph.

Proof: Let us consider a graph G with q edges and *n* vertices v_1, v_2, v_3, …, v_n. Since each edge contributes two degrees, the sum of the degrees of all vertices in G is twice the number of edges in G. i.e., $\sum_{i=1}^{n} \deg v_i$ = 2*q*.

Theorem : In any graph, the number of vertices of odd degree is even.

Proof: Let *Se*= Sum of all degree of all even degree vertices.

Let *So*= Sum of all degree of all odd degree vertices.

By Hand shaking lemma, $So + Se = 2q$.

i.e, $So = 2q - Se = even$.

Each term in the sum *So* is odd.

Therefore, *So* can be even, only if even number of terms in *So*. Hence, the theorem.

Complete graph: A simple graph in which there exists an edge between every pair of vertices is called a ***complete graph***.

A complete graph with p vertices is denoted by K_p. The graph K_p has $\binom{p}{2} = \frac{p(p-1)}{2}$ edges and K_p is a regular graph of degree $p - 1$.

Figure 8. Complete graph K_5, K_2

Definition: A graph is said to be **perfect** if no two vertices are of same degree.

Question: Show that no graph is perfect.

Ans: Let G be a (p, q) graph.

For any vertex v in G, $0 \leq \deg(v) \leq p - 1$.

If we have a vertex with degree 0, then we cannot have a vertex with degree $p - 1$.

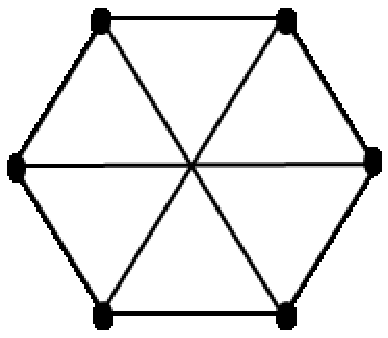
Similarly, if we have a vertex with degree $p - 1$, then we cannot have a vertex with degree 0. Hence degree of a vertex has $p - 1$ choices.

The $p - 1$ integers are to be associated as degrees to p vertices.

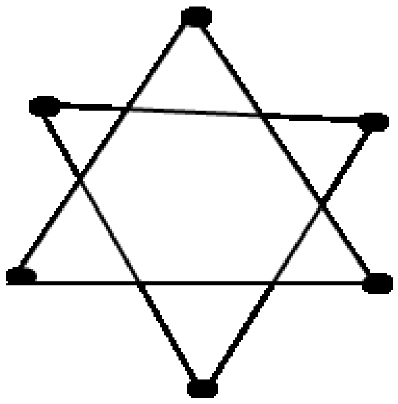
From the Pigeonhole principle, there are at least two vertices which are of same degree. Hence no graph is perfect.

Complement of a graph: The complement $\bar{G}$ of a graph G also has V(G) as its vertex set, but two vertices are adjacent in $\bar{G}$ if and only if they are not adjacent in G.

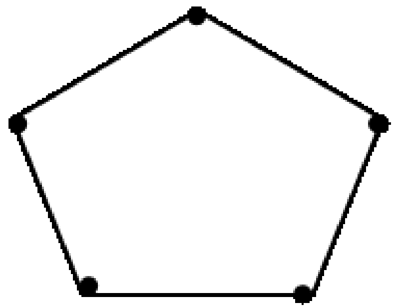
G_1:



$\overline{G_1}$:



G_2:



$\overline{G_2}$:

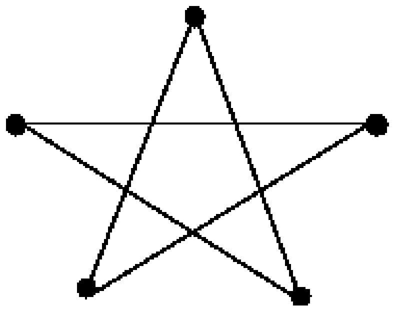
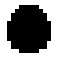


Fig.9. A graph and its complement

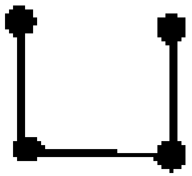
A ***self-complementary*** graph is isomorphic with its complement.

Example (1): In fig.9, Graph G_2 and $\overline{G}\ _2$

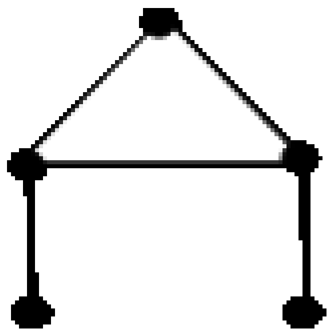
Other examples:



K_1



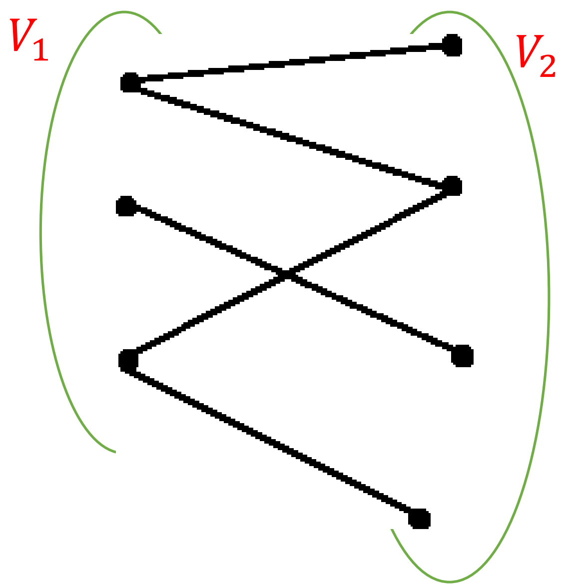
P_4



Bipartite graph: A graph G is called bipartite whose vertex set V can be partitioned into two subsets V_1 and V_2, such that every edge of G joins a vertex of V_1 and a vertex of V_2 (i.e., each edge has one end vertex in V_1 and one end vertex in V_2).

If G contains every edge joining V_1 and V_2 [i.e., each vertex of V_1 is joined to each vertex of V_2] then G is a complete bipartite graph.

If V_1 and V_2 have m and n vertices, we write $G = K_{m,\,n}$.



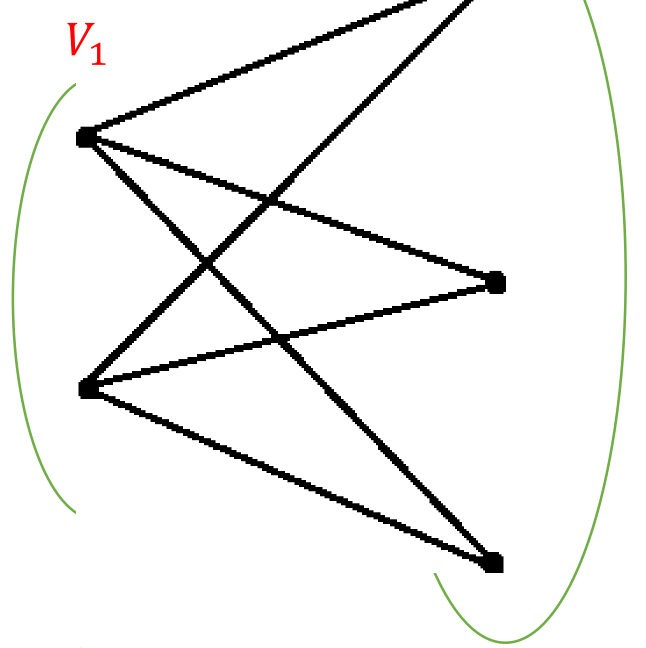


Fig.10. Bipartite and Complete bipartite graph

Theorem: A graph is bipartite if and only if all its cycles are even.

Proof: Let G be a connected bipartite graph, then its vertex set V can be partitioned into two sets V_1 and V_2, such that every edge of G joins a vertex of V_1 with a vertex of V_2.

Thus every cycle $v_1 v_2 \ldots . v_k v_1$ in G necessarily has its oddly subscripted vertices in V_1 (say).

i.e., $v_1, v_3, v_5, \ldots \in V_1$ and the other vertices $v_2, v_4, v_6, \ldots \in V_2$.

In a cycle $v_1 v_2 \ldots . v_k v_1$, $v_k v_1$ is an edge in G. Since $v_1 \in V_1$, the vertex v_k must be V_2. This implies that, k is even. Hence length of the cycle is even.

Conversely,

Suppose that G is a connected graph with no odd cycles.

Let u be any vertex in G and let $V_1 = \{v \in V(G)/d(u,v) = even\}$ and

$$V_2 = \{v \in V(G)/d(u,v) = odd\}$$

Then $V(G) = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$.

We must prove that no two vertices in V_1 and no two vertices in V_2 are adjacent.

Suppose that, the vertices $x, w \in V_1$ be adjacent. Then $d(u,w) = 2k$ and $d(u,x) = 2l$.

Thus a path $u - w - x - u$ forms a cycles of length $2k + 2l + 1$ $(= odd)$, a contradiction.

Therefore, x and w cannot be adjacent, i.e., no two vertices in V_1 are adjacent.

Similarly, we can prove that no two vertices in V_2 are adjacent.

Hence the graph is bipartite.

Theorem: **Let G be a self-complementary graph. Show that the number of vertices in G is of the form $4n$ or $4n + 1$.**

Proof: Let G be a (p, q) graph.

Number of edges in a complete graph $K_p = \frac{p(p-1)}{2}$.

Since G is self –complementary, Number of edges in $G =$ Number of edges in $\overline{G} = q$

Number of edges in $K_p =$ Number of edges in G + Number of edges in $\overline{G}$.

$$\Longrightarrow \quad \frac{p(p-1)}{2} = q + q$$

$$\Longrightarrow 2q = \frac{p(p-1)}{2} \Longrightarrow 4q = p(p-1)$$

$$\Longrightarrow q = \frac{p(p-1)}{4}$$

$\Longrightarrow$ either $4|p$ or $4|(p-1)$

$\Longrightarrow p = 4n$ or $p - 1 = 4n$

$\Longrightarrow$ $p = 4n$ or $p = 4n + 1.$

Theorem: For any graph G with six vertices, G or $\overline{G}$ contains a triangle.

Proof: Let G be a graph with 6 vertices $v, v_1, v_2, v_3, v_4, v_5$.

Since v is adjacent to other 5 vertices either in G or $\overline{G}$

We assume that, v is adjacent with v_1, v_2, v_3 in G.

If any two of these vertices say v_1 and v_2 are adjacent then v, v_1 and v_2 forms a triangle.

If no two of them are adjacent in G, then v_1, v_2 and v_3 are the vertices of a triangle in $\overline{G}$.

Theorem: If diameter of $G \geq 3$, then diam($\overline{G}$) ≤ 3.

Proof: As G is a graph with diameter ≥ 3, there are two vertices u and v in G such that $d(u, v) = 3$.

For any vertex x in G can be adjacent to at most one of u and v, also u and v are nonadjacent in G.

Hence vertices u and v are adjacent in $\overline{G}$.

Consider two vertices x and y in $\overline{G}$.

Since u and v have no common neighbor in G, both x and y are each adjacent u or v in $\overline{G}$.

it follows that $d(x, y) \leq 3$ in $\overline{G}$, and hence diameter($\overline{G}$) ≤ 3.

Theorem: Prove that every self-complementary graph has diameter either 2 or 3.

Ans. Let G be a self-complementary graph.

Clearly, G cannot have diameter 1, since then $G \cong K_n$ which is not self-complementary.

Hence self-complementary graph have diameter at least 2.

Suppose that diam$(G) \geq 3$, by previous theorem, diam $(\overline{G}\ ) \leq 3$.

Hence diameter of every self-complementary graph is either 2 or 3.

Theorem: For any graph G, show that either G or $\overline{G}$ is connected.

Ans: If G itself is connected, then there is nothing to prove.

Suppose that G is disconnected.

Let C_1 and C_2 be two components of G, u and v be any two vertices in G.

If u and v are in different components, and they are not adjacent in G.

Then u and v are adjacent in $\overline{G}$. Hence there is a $u - v$ path and $\overline{G}$ is connected.

If u and v belong to the same component but they are not adjacent in G. Then u and v are adjacent in $\overline{G}$. Hence there is an $u - v$ path.

If u and v are adjacent in G (they belong to the same component). Then we can find w in another component such that we have a $u - v$ path via w in $\overline{G}$.

i.e., $u{\sim}w$ and $w{\sim}v$.

Theorem: If G has p vertices and minimum degree of a graph $\delta(G) \geq \frac{p-1}{2}$, then G is connected.

Suppose that the graph G is disconnected.

Let us assume that G has two(or more) components say C_1 and C_2.

Suppose that a component C_1 has a vertex of minimum degree $\frac{p-1}{2}$. Then, C_1 must contain at least $[\frac{p-1}{2} + 1]$ vertices.

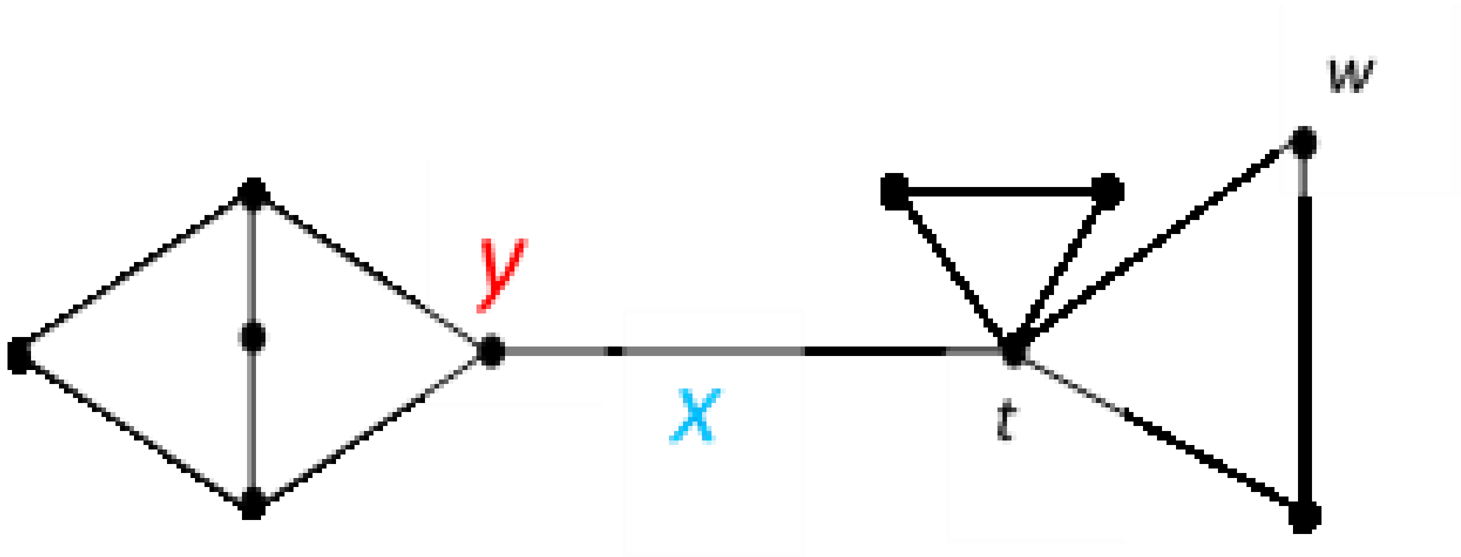
Similarly, suppose that a component C_2 has a vertex of minimum degree $\frac{p-1}{2}$. Then, C_2 must contain atleast $[\frac{p-1}{2} + 1]$ vertices.

Now, total number of vertices in G is equal to $[\frac{p-1}{2} + 1]$+$[\frac{p-1}{2} + 1] = p + 1$, which is a contradiction to the fact that G has p vertices.

Hence, G is connected.

Cut vertex: A ***cutvertex*** of a graph is one whose removal increases the number of components, and a ***bridge*** is such an edge. Thus if *v* is a cutvertex of a connected graph G, then G – *v* is disconnected.

A ***nonseparable*** graph is connected nontrivial, and has no cutvertices.

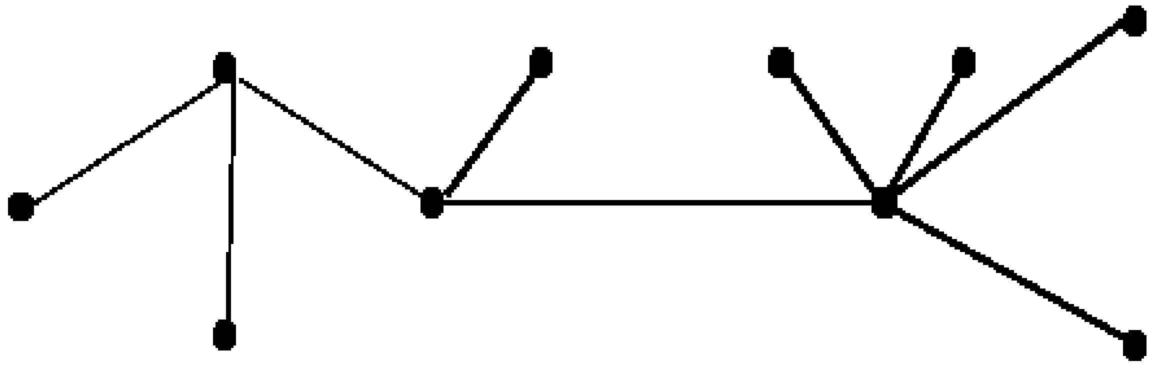


Cutvertex is y and cutedge *is* x

TREES

A graph is ***acyclic*** if it has no cycles. It is also called a **forest.**

A ***tree*** is a connected acyclic graph.



A tree in which all the vertices except one is of degree one is called a star.

A **spanning tree** of G is a spanning subgraph of G that is a tree.

We note that every connected subgraph has a spanning tree.

Hence, if G is a connected (n,m) graph then $m \geq n - 1$.

Theorem: A graph G is a tree if and only if between every pair of vertices there exist a unique path.

Proof: : Let G be a tree, then is connected graph.

Hence there must exist at least one path between every pair of vertices in G.

Now suppose that between two vertices a and b of G there are two distinct paths.

Then union of these two paths will contain a **cycle** and G cannot be a tree.

Thus if G is a tree, there is at most one path joining any two vertices.

Conversely,

Suppose that there is a unique path between every pair of vertices in G. Then G is connected.

A cycle in the graph implies that there is at least one pair of vertices u, v such that there are two distinct paths between u and v.

Since G has one path between every pair of vertices, G cannot have cycle.

Hence G is a tree.

Show that a tree with p vertices has $p-1$ edges.

Proof: This result can be proved by induction on number of vertices.

If $p=1$, we get a tree with one vertex and no edge.

If $p=2$, we get a tree with two vertices and one edge.

If $p=3$, we get a tree with three vertices and two edges.

Assume that, the statement is true for all tree with k vertices, where $k<p$.

Let G be a tree with p vertices.

Since G is a tree, there exist a unique path between every pair of vertices in G.

Thus removal of an edge 'e' from G results disconnect graph G.

Furthermore, $G - e$ consists of exactly two components with number of vertices say 'm' and 'n' with $m + n = p$.

Each component is again a tree.

Hence by induction, the component with m vertices has $m - 1$ edges and the component with n vertices has $n - 1$ edges.

Thus number of edges in $G = (m-1) + (n-1) + 1$

$$= m + n - 1$$

$$= p - 1$$

A connected graph G is said to be **minimally connected** if removal of an edge disconnects the graph.

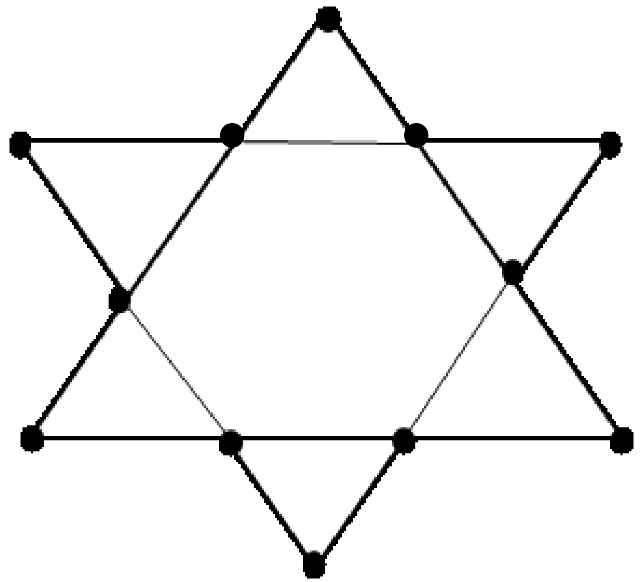
Questions:

1. Show that a graph G is a tree if and only if it is minimally connected.

2. In any tree (with two or more vertices), prove that there are at least two pendent vertices.

3. Prove that every tree has a center consisting of either one vertex or two adjacent vertices.

Eulerian graph: In a graph G, a walk that traverses each edge exactly once, goes through all vertices, and ends at the starting vertex, then the graph is called **Eulerian circuit or Eulerian cycle**. A graph *G* is said to be Eulerian if it has an Eulerian cycle.



Theorem: A connected graph G is Eulerian if and only if all of its vertices are of even degree.

Proof: Suppose that *G* is connected and Eulerian.

Since *G* has an Eulerian circuit which passes through each edge exactly once, goes through all vertices and all of its vertices are of even degree.

Conversely,

Let G be a connected graph such that every vertex of G is of even degree.

Since, *G* is connected, no vertex can be of degree zero.

Thus, every vertex of degree$\geq$ 2, so *G* contains a cycle.

Let C be a cycle in a graph G.

Remove edges of the cycle C from the graph G. The resulting graph (say G_1) may not be connected, but every vertex of the resulting graph is of even degree.

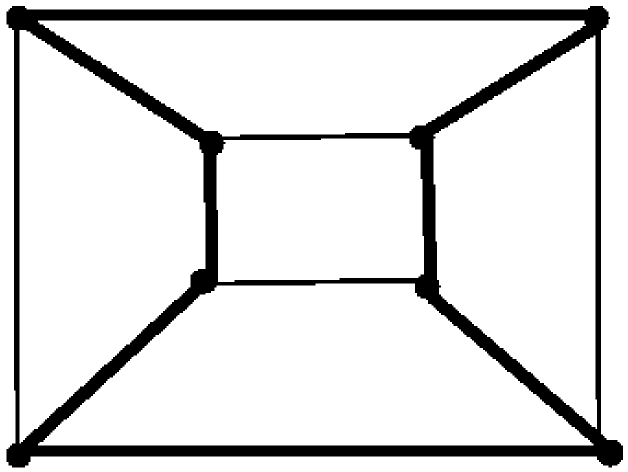
Suppose G consists only of this cycle C, then G is obviously Eulerian.

Otherwise, there is another cycle C_1 with a vertex v in common with C.

The walk beginning at v and consisting of the cycles C and C_1 in succession is a closed trial containing the edges of these two cycles.

By continuing this process, we can construct a closed trial containing all edges of G, hence G is Eulerian.

Hamiltonian graph: If there is a cycle in connected graph G that contains all the vertices of G, then that cycle is called a Hamilton cycle in G. A graph that contains a Hamilton cycle is called a Hamilton graph.



Question: Prove that a simple graph with n vertices and k components can have at most $(n-k)\left(\frac{n-k+1}{2}\right)$ edges.

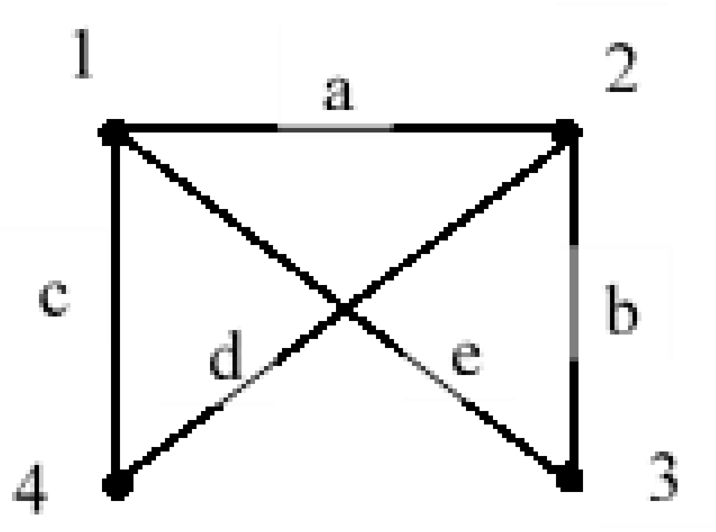
Ans: Let $n_1, n_2, \ldots, n_k$ be the number of vertices in each of the k components of a graph G. Thus $n_1 + n_2 + \ldots + n_k = n$, where $n_i \geq 1,\ 1 \leq i \leq k$.

A component of a graph with n vertices and k components, say $n_i,\ 1 \leq i \leq k$, may have maximum of $[n-(k-1)]$ vertices, and this component has maximum number of edges $\binom{n-k+1}{2} = \frac{(n-k+1)(n-k+1-1)}{2} = \frac{(n-k+1)(n-k)}{2}$ when it is complete.

Matrix Representation of a graphs

Incidence Matrix: Let G be a graph with n vertices, e edges and no self-loops. Define an $n \times e$ matrix I = $[i_{ij}]$, whose n rows correspond to the n vertices and the e columns correspond to the e edges, as follows:

$$i_{ij} = \begin{cases} 1, & \text{if } j^{th} \text{ edge } e_j \text{ is incident on } i^{th} \text{ vertex } v_i \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

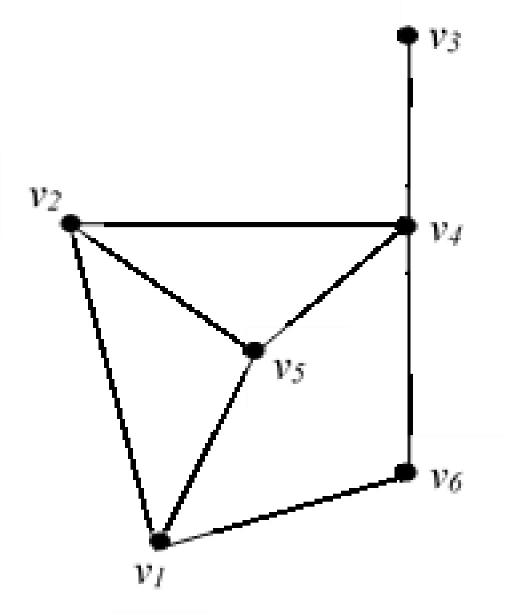


$$
I=\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4
\end{array}
\begin{array}{c}
\begin{array}{ccccc} a & b & c & d & e \end{array} \\
\begin{bmatrix}
1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0
\end{bmatrix}
\end{array}
$$

Graph and its Indence matrix

Adjacency matrix: An adjacency matrix of a graph G with n vertices and no parallel edges is an $n \times n$ symmetric binary matrix A = [a_{ij}] defined by,

$$a_{ij} = \begin{cases} 1, \text{ if there is an edge between ith and jth vertices, and} \\ \quad\quad 0, \quad \text{if there is no edge between them.} \end{cases}$$



$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Graph and its adjacency matrix

Distance Matrix: The distance matrix D = (d_{ij}) is defined as follows:

$$d_{ij} = \begin{cases} 0, & \text{if } i = j \\ \infty, & \text{if } i \text{ and } j \text{ are not adjacent} \\ \text{distance of the edge from } i \text{ to } j & \end{cases}$$

Shortest Paths in graphs: Dijkstra's algorithm

This algorithm is used to find the shortest path between the vertices when each edge is associated with a distance.

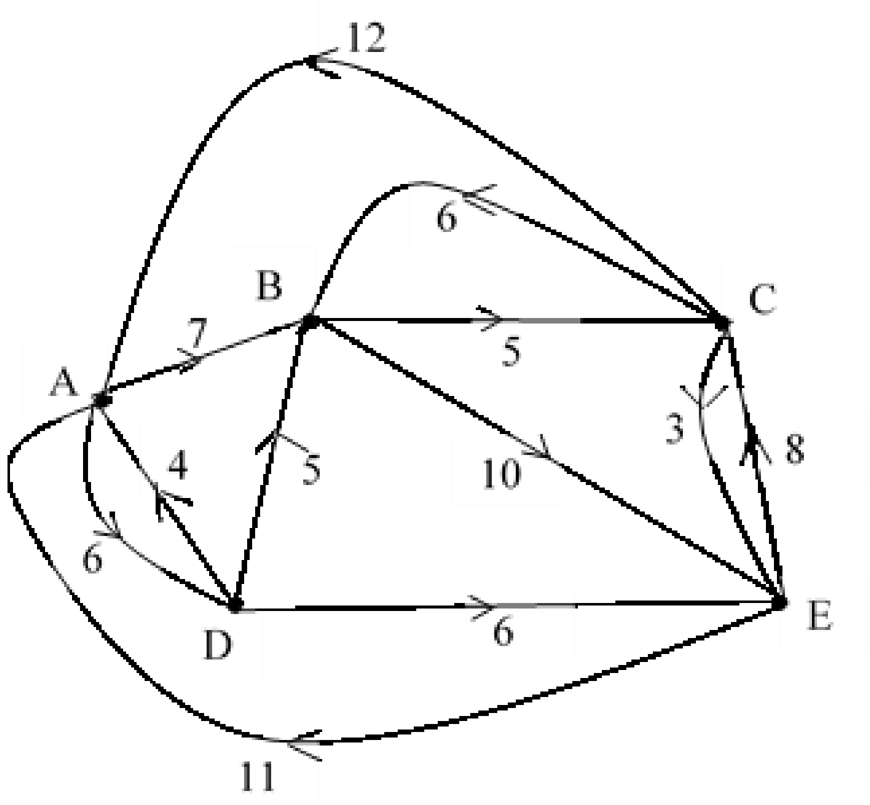
Step 1: We consider **two sets** of vertices K {vertex r}, U {all the other vertices except r}. For all vertices except r, set best $d(i) = d_{ri}$ and tree(i) = r.

Step 2: Find the vertex s in U which has the minimum value of best d. Remove s from U and put it in K.

Step 3: For each vertex u in U find best $d(s) + d_{su}$ and if it is less than best $d(u)$ replace best $d(u)$ by this new value and let tree(u) = s. (In other words a shortest path to u has been found by going via vertex s.)

Step 4: If U contains only one vertex stop else go back to step 2.

Using Dijkstra's algorithm, find the shortest weighted path from B to all other vertices in the following network



Network with distances on the lines

$$\text{Distance matrix} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 7 & \infty & 6 & \infty \\ \infty & 0 & 5 & \infty & 10 \\ 12 & 6 & 0 & \infty & 3 \\ 4 & 5 & \infty & 0 & 6 \\ 11 & \infty & 8 & \infty & 0 \end{bmatrix} \end{array}$$

In step 1 we have K={B} and U={A, C, D, E}, with the arrays

	A	C	D	E
best d	∞	5	∞	10
tree	B	B	B	B

1st iteration: Minimum best d is 5. Remove C from U, Put it in K.

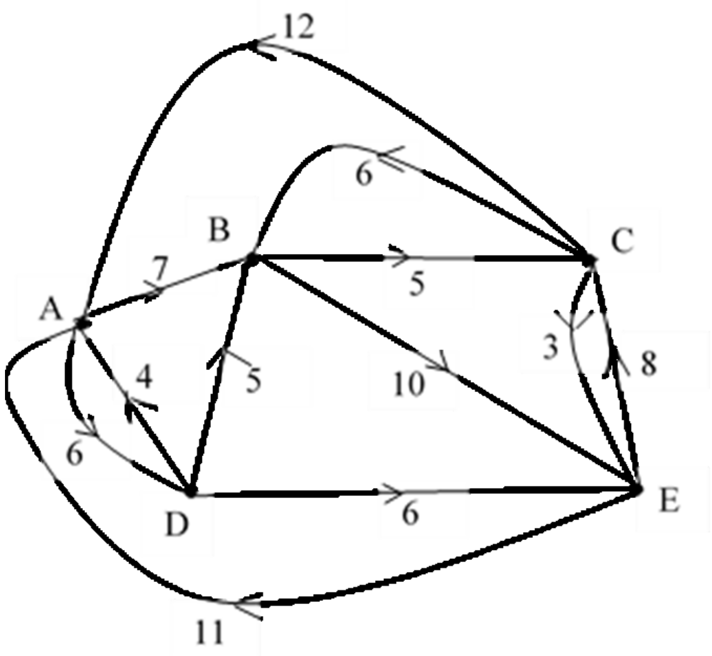
Therefore, K={B, C} and U= {A, D, E}

Distance from B to A via C = 5+12=17 $< \infty$

Distance from B to D via C = $5+\infty = \infty$

Distance from B to E via C = 5+3 = 8 < 10

	A	D	E
best d	17	∞	8
tree	C	B	C



Network with distances on the lines

$$
\begin{array}{c}
\\ A \\ B \\ C \\ D \\ E
\end{array}
\begin{array}{c}
\begin{array}{ccccc} A & B & C & D & E \end{array} \\
\begin{bmatrix}
0 & 7 & \infty & 6 & \infty \\
\infty & 0 & 5 & \infty & 10 \\
12 & 6 & 0 & \infty & 3 \\
4 & 5 & \infty & 0 & 6 \\
11 & \infty & 8 & \infty & 0
\end{bmatrix}
\end{array}
$$

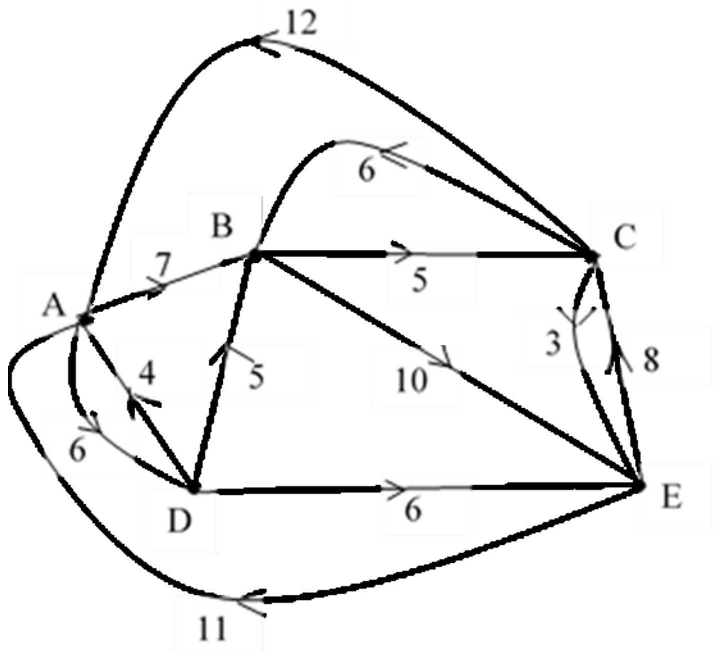
2nd iteration: Minimum best d is 8. Remove E from U, Put it in K.

Therefore, K={B, C, E} and U= {A, D}

Distance from B to A via E = 8+11=19 > 17

Distance from B to D via E = $8+\infty = \infty$

	A	D
best d	17	∞
tree	C	B



Network with distances on the lines

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 7 & \infty & 6 & \infty \\ \infty & 0 & 5 & \infty & 10 \\ 12 & 6 & 0 & \infty & 3 \\ 4 & 5 & \infty & 0 & 6 \\ 11 & \infty & 8 & \infty & 0 \end{bmatrix} \end{array}$$

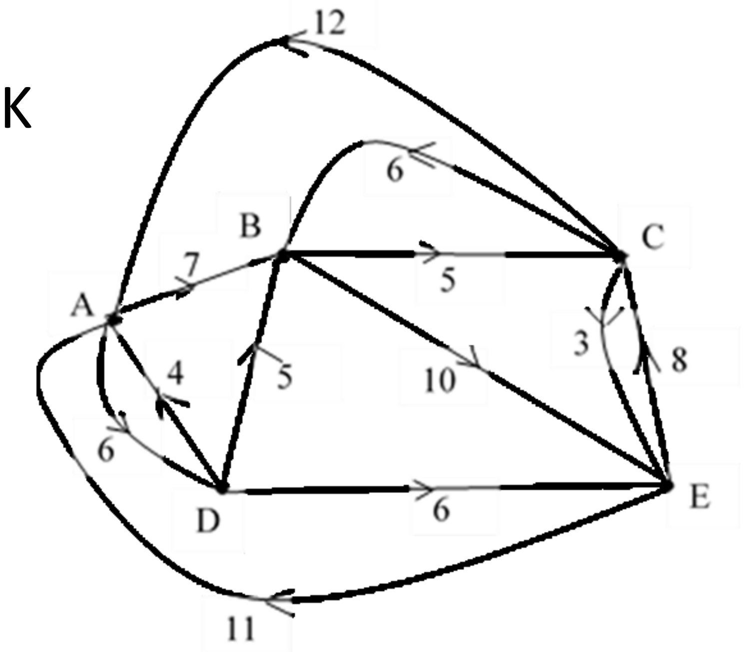
3rd iteration: Minimum best d is 17. Remove A from U, Put it in K

Therefore, K={A, B, C, E} and U= {D}

Distance from B to D via A = 17 + 6 = 23 < ∞

Hence,

	A	C	D	E
best d	17	5	23	8
tree	C	B	A	C



Network with distances on the lines

$$\begin{array}{c|ccccc} & A & B & C & D & E \\ \hline A & 0 & 7 & \infty & 6 & \infty \\ B & \infty & 0 & 5 & \infty & 10 \\ C & 12 & 6 & 0 & \infty & 3 \\ D & 4 & 5 & \infty & 0 & 6 \\ E & 11 & \infty & 8 & \infty & 0 \end{array}$$