Two Dimensional Random Variables

Definition: Let S be the sample space associated with a random experiment Let $X = X(s)$ and $Y = Y(s)$ be two functions each assigning a real number to each outcome $s \in S$. Then (X, Y) is called a Two Dimensional Random Variables.

If the possible values of (X, Y) are finite or countably infinite, (X, Y) is called a 2D discrete random variable.

i.e., when (X, Y) is a 2D discrete random variable, the possible values of (X, Y) may be represented as (x_i, y_j), $i = 1, 2, \dots, n;\quad j = 1, 2, \dots, m$

If (X, Y) can assume all values in a specified region R in the XY-plane, then (X, Y) is called a 2D continuous random variables.

Probability function of (X, Y) [Joint pmf]:

If (X, Y) is a 2D discrete random variables such that $P(X = x_i, Y = y_j) = p_{ij}$, then p_{ij} is called the probability mass function of (X, Y) provided the following conditions are satisfied:

(i). $p_{ij} \geq 0, for\ all\ i\ and\ j$

(ii). $\sum_j \sum_i p_{ij} = 1$

The set of triples $\{x_i, y_j, p_{ij}\}$, $i = 1, 2, \ldots, n; j = 1, 2, \ldots, m$ is called the joint probability distribution of (X, Y). This is expressed in the form of rectangular table.

X \ Y	y_1	y_2	…	y_m
x_1	p_{11}	p_{12}	…	p_{1m}
x_2	p_{21}	p_{22}	…	p_{2m}
⋮	⋮	⋮	⋮	⋮
x_n	p_{n1}	p_{n2}	…	p_{nm}

Joint Probability density function [Joint pdf]:

If (X, Y) is a 2D continuous random variables and if there exist a function $f(x, y)$ called the joint pdf of (X, Y), provided $f(x, y)$ satisfies the following conditions:

(i). $f(x, y) \geq 0$, for all $(x, y) \in R$, R is the range space

(ii). $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy$=1

(iii). $P(a < X < b,\ c < Y < d) = \int_a^b \int_c^d f(x, y)\, dx\, dy$

Cumulative distribution function (cdf):

If (X, Y) is a 2D random variables (discrete or continuous), then $F(x, y) = P(X \leq x, Y \leq y)$ is called the cdf of (X, Y).

If (X, Y) is a 2D discrete random variables; $F(x, y) = \sum_{y_j \leq y} \sum_{x_i \leq x} p_{ij}$

If (X, Y) is a 2D continuous random variables; $F(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(x, y)\, dx\, dy$

Properties:

1. $F(-\infty, y) = 0 = F(x, -\infty)$ and $F(\infty, \infty) = 1$

2. $P(a < X < b,\ Y \leq y) = F(b, y) - F(a, y)$

3. $P(X \leq x,\ c < Y < d) = F(x, d) - F(x, c)$

4. $P(a < X < b,\ c < Y < d) = F(b, d) - F(a, d) - F(b, c) + F(a, c)$

5. At point of continuity of $F(x, y)$,

$$\frac{\partial^2 F}{\partial x \partial y} = f(x, y)$$

Marginal probability function:

Let (X, Y) is a 2D discrete random variables with joint pmf $\mathrm{P}(x_i, y_j)$.Then

$f(x_i) = \sum_{j=1}^{m} \mathrm{P}(x_i, y_j)$ is called the Marginal probability function of X. And

$g(y_j) = \sum_{i=1}^{n} \mathrm{P}(x_i, y_j)$ is called the Marginal probability function of Y.

$f(x_i) = \sum_{j=1}^{m} \mathrm{P}(x_i, y_j)$ - Marginal distribution of X.

$g(y_j) = \sum_{i=1}^{n} \mathrm{P}(x_i, y_j)$ - Marginal distribution function of Y.

Y / X	y_1	y_2	…	y_m	Row Sum
x_1	p_{11}	p_{12}	…	p_{1m}	$f(x_1)$
x_2	p_{21}	p_{22}	…	p_{2m}	$f(x_2)$
⋮	⋮	⋮	⋮	⋮	⋮
x_n	p_{n1}	p_{n2}	…	p_{nm}	$f(x_n)$
Column Sum	$g(y_1)$	$g(y_2)$	…	$g(y_m)$	**1**

Marginal Distribution or Marginal pdf:

Let (X, Y) is a 2D continuous random variables with joint pdf $f(x, y)$. We define $g\ and\ h$, the marginal pdf of X and Y respectively as follows:

$g(x) = \int_{-\infty}^{\infty} f(x, y)dy$ - called marginal pdf of X

and $h(y) = \int_{-\infty}^{\infty} f(x, y)dx$ - called marginal pdf of Y

Note: $P(a \leq X \leq b) = P(a \leq X \leq b, -\infty < Y < \infty)$

$$= \int_{-\infty}^{\infty} \int_{a}^{b} f(x, y)dx\, dy = \int_{a}^{b} \left[\int_{-\infty}^{\infty} f(x, y)dy\right] dx = \int_{a}^{b} g(x)dx$$

Similarly,

$$P(c \le Y \le d) = P(-\infty < X < \infty, c \le Y \le d)$$

$$= \int_{-\infty}^{\infty} \int_{c}^{d} f(x, y) dx\, dy = \int_{c}^{d} \left[\int_{-\infty}^{\infty} f(x, y) dx \right] dy = \int_{c}^{d} h(y) dy$$

Conditional Probability Distribution:

Let (X, Y) is a 2D discrete random variables

$P(X = x_i | Y = y_j) = \frac{P(X=x_i, Y=y_j)}{P(Y=y_j)}$ is called conditional probability finction of X, given that $Y = y_j$.

Conditional PDF:

Let (X, Y) is a 2D continuous random variables with joint pdf $f(x, y)$. Let $g(x)$ and $h(y)$ be the marginal pdf of x and Y respectively. Then conditional pdf of X for given y is

$$g(x|y) = \frac{f(x, y)}{h(y)}$$

And conditional pdf of Y for given X is

$$h(y|x) = \frac{f(x, y)}{g(x)}$$

Independent Random Variables:

Let (X, Y) is a 2D discrete random variables with joint pmf $P(x_i, y_j)$ and marginal pmf $f(x_i)\ and\ g(y_j)$. We say that X and Y are independent random variables if

$$P(x_i, y_j) = f(x_i)\, g(y_j) \qquad \forall\ i\ and\ j$$

Let (X, Y) is a 2D continuous random variables with joint pdf $f(x, y)$. Let $g(x)$ and $h(y)$ be the marginal pdf of X and Y respectively. We say that X and Y are independent random variables if

$$f(x, y) = g(x)\, h(y) \quad for\ all\ (x, y)$$

Mean and Variance:

Let (X, Y) is a 2D discrete random variables with joint pmf $P(x_i, y_j)$ and marginal pmf $f(x_i)\ and\ g(y_j)$.

$$E(X) = \sum_i x_i\, f(x_i)$$

$$E(Y) = \sum_i y_j\, g(y_j)$$

$$E(XY) = \sum_{i,j} x_i\, y_j\ P(x_i, y_j)$$

Let (X, Y) is a 2D continuous random variables with joint pdf $f(x, y)$. Let $g(x)$ and $h(y)$ be the marginal pdf of x and Y respectively.

$$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\ f(x, y)\, dx\, dy = \int_{-\infty}^{\infty} x\, g(x) dx \qquad [\because \int_{-\infty}^{\infty} f(x, y) dy = g(x)]$$

$$E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\ f(x, y)\, dx\, dy = \int_{-\infty}^{\infty} y\, h(y) dy \qquad [\because \int_{-\infty}^{\infty} f(x, y) dx = h(y)]$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\ f(x, y)\, dx\, dy$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$V(Y) = E(Y^2) - [E(Y)]^2$$

Properties:

$$E(X+Y) = E(X) + E(Y)$$

$$E(XY) = E(X)E(Y) \textit{ if and only if } X \textit{ and } Y \textit{ are independent}$$

$$V(X+Y) = V(X) + V(Y) \textit{ if and only if } X \textit{ and } Y \textit{ are independent}$$

Covariance of X and Y is defined as,

$$Cov\,(X,Y) = E(XY) - E(X)E(Y)$$

Let (X,Y) be a 2D random variable. We define ρ_{xy} , the coefficient of correlation between X and Y as follows:

$$\rho = \rho_{xy} = \frac{E(XY) - E(X)E(Y)}{\sqrt{V(X)\,V(Y)}} = \frac{Cov\,(X,Y)}{\sqrt{V(X)\,V(Y)}}$$

Note:

- If X and Y are independent, then $\rho = 0$ (Because $E(XY) = E(X)E(Y)$)
- If $\rho = 0$, we say that X and Y are uncorrelated. (but X and Y not necessarily independent)