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MATHS

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Unit 1 : Mean Value Theorems
Indeterminate
Solid Geometry
} (10 Hrs)

Unit 2 : Partial Differentiation
Errors and Approximation } (6 Hrs)

Unit 3 : Taylor Series for two variable functions
Maxima and minima
} (6)

Unit 4 : Multiple integral
Beta and Gamma function
} (12 hrs)

Unit 5 : Laplace Transform (10 hrs)

Unit 6 : Infinite Series (6 hrs).

* UNIT 1

Cauchy's Mean Value Theorem

function → $A = \pi r^2$

$\boxed{A = f(r)}$ ↳ single variable func.

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Rolle's Theorem : Let $f(x)$ be a single variable function.

(i) $f(x)$ is continuous in $[a,b]$

(ii) $f'(x)$ exists for every value (a,b)

(iii) $f(a) = f(b)$, then there exists atleast one value c in (a,b) such that $\boxed{f'(c) = 0}$ //

Lagrange's Mean Value Theorem :

(i) $f(x)$ is continuous in $[a,b]$

(ii) $f'(x)$ exists in (a,b), then $\exists$ atleast one 'c' in (a,b) Such that (s)

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}} //$$

Cauchy's Mean Value Theorem :-

(i) $f(x)$ and $g(x)$ be continuous in $[a,b]$

(ii) $f'(x)$ and $g'(x)$ exist in (a,b)

(iii) $g'(x) \neq 0$, for $x \in (a,b)$, then $\exists$ atleast one value 'c' in (a,b) 's' $\boxed{\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}}$

Proof : Consider $\phi(x) = f(x) - \frac{f(b)-f(a)}{g(b)-g(a)} \cdot g(x)$

(i) $\phi(x)$ is continuous in $[a,b]$

(ii) $\phi'(x)$ exists in (a,b)

(iii) $\phi(a) = f(a) - \frac{f(b)-f(a)}{g(b)-g(a)} \cdot g(a)$

$\phi(b) = f(b) - \frac{f(b)-f(a)}{g(b)-g(a)} \cdot g(b)$

$\phi(a) = \phi(b)$

$\therefore$ $\phi(x)$ satisfies all the three conditions of Rolle's Theorem.

Then $\exists$ one value c in (a,b)

$\Rightarrow \phi'(c) = 0$

$\phi'(c) = f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)} \cdot g'(c) = 0$

$$\boxed{\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}}$$ → important

when $g(x) = x$; $f(x) \rightarrow$ generalisation of lagrange's mean value theorem.

$$\frac{f'(c)}{1} = \frac{f(b)-f(a)}{b-a}$$

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$

(1) Verify cauchy's mean value theorem for the functions

$$f(x) = e^x \text{ and } g(x) = e^{-x} \text{ in } [a,b]$$

Sol. $f(x) = e^x$; $g(x) = e^{-x}$

(i) exponential functions it is always continuous in $[a,b]$

(ii) $f'(x) = e^x$ and $g'(x) = -e^{-x}$

$f'(x)$ and $g'(x)$ exist, in (a,b)

(iii) $g'(x) = -e^{-x} \neq 0$

i.e $f(x)$ and $g(x)$ satisfy all the conditions of CMVT.

Hence $\exists$ one c in (a,b) s

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{e^c}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-a}}$$

$$\Rightarrow -e^{2c} = \frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}}$$

$$-e^{2c} = \frac{e^b - e^a}{\frac{e^a - e^b}{e^{a+b}}}$$

$$-e^{2c} = -e^{a+b}$$

$$e^{a+b} = e^{2c}$$

$$2c = a+b$$

or $\boxed{c = \frac{a+b}{2}}$

(2) Verify CMVT. $f(x) = x^3$; $g(x) = x^2$ in $[1, 2]$ ($a = 1$, $b = 2$)

(i) $f(x) = x^3$, $g(x) = x^2$ are continuous

(ii) $f'(x) = 3x^2$, $g'(x) = 2x$

(iii) $g'(x) \neq 0$ $\forall$ x in $[1, 2]$

Hence $g(x)$ and $f'(x)$ satisfy CMVT

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{3c^2}{2c} = \frac{\cancel{b^3}-\cancel{a^3}}{\cancel{b^2}-\cancel{a^2}} \quad \frac{2^3-1^3}{2^2-1^2}$$

$$\frac{3c}{2} = \frac{7}{3}$$

$$\boxed{c = \frac{14}{9}}$$

HW

(1) $f(x) = \frac{1}{x^2}$ $g(x) = \frac{1}{x}$ in $[a,b] \Rightarrow b>a>0$

(2) $f(x) = \sqrt{x}$ $g(x) = \frac{1}{\sqrt{x}}$ in $[3,5]$

(3) $f(x) = \sin x$ $g(x) = \cos x$ in $[a,b]$, $0 \le a < b \le \pi/2$

Taylor's theorem and Taylor's Series

Theorem : Let $f(x)$ satisfy the following conditions

(i) $f(x)$ and its $(n-1)$ derivatives are continuous in $[a,b]$.

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(ii) $f(x)^{n-1}$ exists in (a,b), then $\exists$ one c in (a,b) 's'

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) +$$

$$\frac{(b-a)^3}{3!}f'''(a) + \cdots \frac{(b-a)^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{(b-a)^n}{n!}f^n(c)$$

for $n=1$, it reduces to LMVT

for $n = 2, 3, 4$; we get second, third, ... mean value theorems.

Alternate form

Take $(b-a) = h$ and $c = a + \theta h$; $0 < \theta < 1$

where $\theta = \frac{c-a}{h} = \frac{c-a}{b-a}$

then, the above eqn. reduces to

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots$$

$$\frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \underbrace{\frac{h^n}{n!}f^n(a+\theta h)}_{\text{Remainder}} \rightarrow (2)$$

Remainder in Lagrange form

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Taylor's Series

Taking $(a+h) = x$ in eqn. (2)

$$f(x) = f(a) + (x-a)\cdot f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$$

$$\frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(x-a)^n}{n!}\cdot f^n[a+\theta(x-a)]$$

$$= S_n(x) + R_n(x)$$

when $S_n(x) = f(a) + (x-a)\cdot f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$

$$\frac{(x-a)^{n-1}}{(n-1)!}\cdot f^{n-1}(a)$$

$$R_n(x) = \frac{(x-a)^n}{n!} f^n[a+\theta(x-a)]$$

Suppose, $R_n(x) \to 0$ as $n \to \infty$ then

$$\boxed{f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(x-a)^n}{n!} f^n(a) + \dots \infty}$$

~~in powers of $(x-a)$~~

when you are given a function $f(x)$ then the Taylor's series expansion of $f(x)$ about $x = a$ is

$$f(x) = f(a) + (x-a) \cdot f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots$$

same { Or in powers of $(x-a)$ "
Or in the neighbourhood of $x = a$ "

(1) Expand $f(x) = \log x$ in powers of $(x-1)$ and hence evaluate $\log(1.1)$ correct to four decimals.

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a)$$

in powers of $x-1$

$\boxed{a=1}$

$f(x) = \log x \quad f(a) = f(1) = \log 1 = 0$

$f'(x) = \frac{1}{x} = 1$

$f''(x) = \frac{-1}{x^2} \quad f''(1) = -1$

$f'''(x) = \frac{2}{x^3} = \quad f'''(1) = 2$

$$f(x) = \log x = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \ldots$$

$$0 + (x-1)\cdot 1 + \frac{(x-1)^2}{2!}\cdot(-1) + \frac{(x-1)^3}{3!} \times 2 + \ldots$$

$$\Rightarrow \boxed{(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \ldots}$$

taking $x = 1.1$

$\log(1.1) = (1.1-1) - \frac{(1.1-1)^2}{2!} + \frac{(1.1-1)^3}{3}$

≈ 0.0953

(2) Obtain the power series exp. of $\cos x$ about $x = \frac{\pi}{3}$. Hence find approx. value of $\cos 61°$.

$a = \pi/3$

$f(x) = \cos x \qquad f(a) = \cos(\pi/3) = 1/2$

$f'(x) = -\sin x \qquad f'(a) = -\sin(\pi/3) = -\sqrt{3}/2$

$f''(x) = -\cos x \qquad f''(a) = -\cos(\pi/3) = -1/2$

$f'''(x) = \sin x \qquad f'''(a) = \sin(\pi/3) = \sqrt{3}/2$

$f(x) = \cos x = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots$

$\Rightarrow \frac{1}{2} + \left(x - \frac{\pi}{3}\right)\left(-\frac{\sqrt{3}}{2}\right) + \frac{\left(x - \frac{\pi}{3}\right)^2\left(-\frac{1}{2}\right)}{2!} + \frac{\left(x - \frac{\pi}{3}\right)^3\left(\frac{\sqrt{3}}{2}\right)}{3!}$

about $x = \pi/3$.

$$\Rightarrow \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{3}\right)^3 + \cdots$$

To determine $\cos 61^\circ$

$x = 61^\circ \Rightarrow x - \frac{\pi}{3} = 1^\circ = \frac{\pi}{180}$ rad.

$$\cos 61^\circ = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{\pi}{180}\right) - \frac{1}{4}\left(\frac{\pi}{180}\right)^2 + \frac{\sqrt{3}}{12}\left(\frac{\pi}{180}\right)^3 + \cdots$$

$\cos 61^\circ \approx 0.4848$.

Mclaurin's Expansion

$$f(x) = f(a) + (x-a)f'(a) + \cdots \frac{(x-a)^n}{n!} f^n[x + \theta(x-a)]$$

↳ Taylor's theorem

for $a = 0$, above exp. becomes

$* \; f(x) = f(0) + (x).f'(0) + \frac{x^n}{n!} f^n[x + \theta(x)]$ which is called Maclaurin's exp.

Maclaurin's Series exp. is

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^n(0) + \cdots$$

Examples

(1) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$f(x) = e^x$; $a = 0$

(2) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$f^n(n) = e^x$

$f^n(0) = 1$

(3) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

(4) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

Note: in powers of x; about origin, in ascending powers of x

(1) Exp. $e^{\sin x}$ in powers of x upto first four non-zero terms.

$f(x) = e^{\sin x} \rightarrow f(0) = e^{\sin 0} = 1$

$f'(1) = e^{\sin x} \cdot \cos x \rightarrow f'(0) = 1$

$f''(0) = e^{\sin x}(-\sin x) + \cos^2 x\, e^{\sin x} \rightarrow f''(0) = 1$

$f'''(0) = -(e^{\sin x}\cos x + \underbrace{\sin x \cos x}_{\frac{\sin 2x}{2}} \cdot e^{\sin x}) +$
$\cos^3 x \cdot e^{\sin x} + \underbrace{2\sin x \cos x}_{\sin 2x} \cdot e^{\sin x} \Rightarrow f'''(0) = 0$

$f'''(x) = f^{iv}(0) = 3$

$$e^{\sin x} = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \ldots$$

$$1 + x(1) + \frac{x^2}{2}(1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(-3) + \ldots$$

$$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8}$$

(2) Expand $e^{a\sin^{-1}x}$ in ascending powers of x.

→ Let $y = e^{a\sin^{-1}x}$

Differentiate both sides

$$y_1 = e^{a\sin^{-1}x} \cdot a \cdot \frac{1}{\sqrt{1-x^2}}$$

i.e $\sqrt{1-x^2}\, y_1 = a e^{a\sin^{-1}x}$

$$\sqrt{1-x^2}\, y_1 = ay$$

S.O.B.S

$$(1-x^2) y_1^2 = a^2 y^2$$

Differentiate again

~~$(1-x^2)y_2$~~ product rule

$$(1-x^2)\cancel{2}\cancel{y_1}\cdot y_2 + y_1^{\cancel{2}}(-\cancel{2}x) = a^2\cdot \cancel{2}y\cdot \cancel{y_1}$$

$$\boxed{(1-x^2)y_2 - xy_1 - a^2y = 0} \rightarrow (1)$$

$$\frac{d}{dx}(uv) = uv_1 + vu_1$$

$$\frac{d^2}{dx^2}(uv) = \frac{d}{dx}\left(\frac{d}{dx}(uv)\right) = \frac{d}{dx}\left(uv_1 + vu_1\right)$$

$$= uv_2 + u_1v_1 + vu_2 + v_1u_1$$

$$= uv_2 + 2u_1v_1 + vu_2.$$

$$\boxed{\frac{d^n}{dx^n}(uv) = uv_n + {}^nC_1\, u_1v_{n-1} + {}^nC_2\, u_2v_{n-2} + \cdots u_nv}$$

differentiate n times, we get

$$\frac{d^n}{dx^n}\Big[\overset{u}{(1-x^2)}\overset{v}{y_2}\Big] - \frac{d^n}{dx^n}\left(\overset{u}{x}\overset{v}{y_1}\right) - \frac{d^n}{dx^n}\left(a^2y\right) = 0$$

$$\left[(1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2!}(-2)y_n\right]$$

$$- \left[x\,y_{n+1} + n(1)y_n\right] - a^2 y_n = 0$$

(2) $\rightarrow$ $(1-x^2)y_{n+2} - \underline{\underline{(2n+1)x}}\cdot y_{n+1} - (n^2+a^2)y_n = 0$

$y(0) = 1$

$y'(0) = y_1(0) = a$

(2) $\rightarrow$ $n=0 \Rightarrow y_2(0) = y''(0) = a^2$

(2) $\rightarrow$ $n=1 \Rightarrow y_3(0) = a(1+a^2)$

(2) $\rightarrow$ $n=2 \Rightarrow y_4(0) = a^2(2^2+a^2)$

$\therefore$ exp. of $e^{a\sin^{-1}x}$ is

$$1 + x(a) + \frac{x^2}{2!}(a^2) + \frac{x^3}{3!}\left(a(1+a^2)\right) + \frac{x^4}{4!}\left[a^2(2+a^2)\right]$$

INDETERMINATE FORMS

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$\lim\limits_{x \to a} f(x)$ exists and $\lim\limits_{x \to a} g(x)$ exists

Then $\lim\limits_{x \to a} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$

But $\lim\limits_{x \to a} f(x) = 0$ and $\lim\limits_{x \to a} g(x) = 0$. Then

the above $\Rightarrow \lim\limits_{x \to a} \left(\dfrac{f(x)}{g(x)}\right) = \dfrac{0}{0}$ which doesn't mean that value is meaningless. In many cases it has a finite value.

Such expression is called ~~indet~~ indeterminate form.

Form 1 $\longrightarrow \dfrac{0}{0}$

$\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{f(a)}{g(a)} = \dfrac{0}{0}$

in such case, we have to apply L-hospital rule

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$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$ provided $g'(a)\neq 0$

then $$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)}$$ provided $g'(a)\neq 0$

in general, $$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f^n(x)}{g^n(x)}.$$

Std. results

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \; ; \; \lim_{x\to 0}\frac{\tan x}{x} = 1$$

Q1) Evaluate $\lim_{x\to 0}\frac{xe^x - \log(1+x)}{x^2}$

Soln. $$\lim_{x\to 0}\frac{xe^x - \log(1+x)}{x^2} = \frac{f(x)}{g(x)} = \frac{0}{0} \text{ form}$$

Apply L'H.P rule

$$\lim_{x\to 0}\frac{xe^x + e^x - 1/1+x}{2x} \quad (0/0)$$

$$\lim_{x\to 0}\frac{\left(xe^x + e^x + e^x + 1/(1+x)^2\right)}{2}$$

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Q2. Evaluate $\underset{x\to 0}{Lt} \frac{\sin x}{(e^x-1)^2}$ $\frac{0}{0}$ form

L.H.P.

$$\underset{x\to 0}{Lt} \frac{\cos x}{2(e^x-1)\cdot e^x} = \frac{\cos x}{2(e^{2x}-e^x)}$$

$$\underset{x\to 0}{Lt} \frac{-\sin x}{2(2e^{2x}-e^x)} = \frac{0}{2(2-1)} = \underline{\underline{0}}$$

Q3. Find a and b such that

$$\underset{x\to 0}{Lt} \frac{x(1+a\cos x)-b\sin x}{x^3} = 1$$

Consider $\underset{x\to 0}{Lt} \frac{x(1+a\cos x)-b\sin x}{x^3}$ $(0/0)$

$$\underset{x\to 0}{Lt} \frac{(1+a\cos x)-a\sin x-b\cos x}{3x^2} \rightarrow (1)$$

As the denominator is zero as $x=0$ eqn. (1) will tend to a finite limit iff. numerator also becomes zero for $x=0$

i.e $1+a-b=0 \rightarrow (2)$

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(1) $\displaystyle \lim_{x\to 0} \frac{-a\sin x - a(x\cos x + \sin x) + b\sin x}{6x}$

diff. $\Rightarrow \displaystyle \lim_{x\to 0} \frac{-a\cos x - a(x\sin x + \cos x + \cos x) + b\cos x}{6}$

$\frac{-a-a-a+b}{6} = 1$

$\frac{b-3a}{6} = 1 \Rightarrow b - 3a = 6 \rightarrow (3)$

Solve, (a) and (b) $\Rightarrow a = -\frac{5}{2}$; $b = \frac{-3}{2}$.

Form 2 $\frac{\infty}{\infty} \rightarrow$ L.H.P rule.

(1) $\displaystyle \lim_{x\to 0} \frac{\log x}{\cot x} \rightarrow \frac{\infty}{\infty}$

$\displaystyle \lim_{x\to 0} \frac{1/x}{-\operatorname{cosec}^2 x} = -\lim_{x\to 0} \frac{\sin^2 x}{x} = -\lim_{x\to 0}\left(\frac{\sin x}{x} \cdot \sin x\right)$

$\displaystyle = -\left[\lim_{x\to 0}\left(\frac{\sin x}{x}\right) \cdot \lim_{x\to 0}(\sin x)\right] = -[1 \cdot 0]$

$= 0$

Q2. $\displaystyle \operatorname*{Lt}_{x\to\pi/2} \frac{\log(\cos x)}{\tan x} = \frac{\infty}{\infty}$

$$\operatorname*{Lt}_{x\to\pi/2} \frac{\frac{1}{\cos x}(-\sin x)}{\sec^2 x}$$

$$\operatorname*{Lt}_{x\to\pi/2} \frac{-\tan x}{\sec^2 x} \quad (\infty/\infty)$$

$$= -\operatorname*{Lt}_{x\to\pi/2} \frac{\sec^2 x}{2\sec x\cdot\sec x\cdot\tan x}$$

$$= -\operatorname*{Lt}_{x\to\pi/2} \frac{1}{2\tan x} = 0$$

3. $\displaystyle \operatorname*{Lt}_{x\to 0} \frac{e^x\sin x - x - x^2}{x^2 + x\log(1-x)}$

$$\operatorname*{Lt}_{x\to 0} \frac{\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\dots\right)\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}\dots\right) - x - x^2}{x^2 + x\left[-x-\frac{x^2}{2}-\frac{x^3}{3}-\dots\right]}$$

$$\lim_{x\to 0} \frac{\left(\cancel{x}+\cancel{x^2}+\frac{x^3}{3}+\cancel{x^4}\right)-\cancel{x}-\cancel{x^2}}{\cancel{x^2}-\cancel{x^2}-\frac{x^3}{2}-\frac{x^4}{3}-\dots} = -\frac{2}{3}$$

$$\frac{\frac{x^3}{3}+0x^4+0x^5}{-\frac{x^3}{2}-\frac{x^4}{3}+\dots} = -\frac{2}{3}$$

Evaluate $\lim_{x\to 0} \frac{(1+x)^{1/x}-e}{x}$ $(0/0)$

Let $y=(1+x)^{1/x} \Rightarrow \log y = \frac{1}{x}\log(1+x)$

$$\log y = \frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots\right)$$

$$= 1-\frac{x}{2}+\frac{x^2}{3}+\dots$$

$$y = e^{\left(1-x/2+\frac{x^2}{3}\dots\right)}$$

e^t where $t=\left(\frac{-x}{2}+\frac{x^2}{3}\right)$

$$= e\cdot e^{-x/2+x^2/3}$$

$$= e\left(1+t+\frac{t^2}{2!}+\dots\right) = e$$

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$$e\left(1+\left(-\frac{x}{2}+\frac{x^2}{3}+\cdots\right)+\frac{\left(-\frac{x}{2}+\frac{x^2}{3}+\cdots\right)^2}{2!}+\cdots\right)$$

$$e\left[1-\frac{x}{2}+\frac{11}{24}x^2+\cdots\right]$$

$$\lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x}=\lim_{x\to 0}\frac{e\left[1-\frac{x}{2}+\frac{11}{24}x^2+\cdots\right]-e}{x}$$

III Forms reducible to $0/0$ / ∞/∞

(a) Form : $0 \times \infty$.

$$\underbrace{\lim_{x \to a} f(x)}_{0} \cdot \underbrace{\lim_{x \to a} g(x)}_{\infty}$$

To evaluate this, we write

$$f(x) \cdot g(x) = \frac{f(x)}{1/g(x)} \quad (0/0 \text{ form})$$

OR

$$f(x) \cdot g(x) = \frac{g(x)}{1/f(x)} \quad (\infty/\infty \text{ form})$$

(1) $\lim_{x \to 0} x^n \log x$, $x > 0$

$$\lim_{x \to 0} \frac{\log x}{x^{-n}} \quad (\infty/\infty)$$

L'Hospital rule

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$$\lim_{x\to 0} \frac{1/x}{-n\cdot x^{-n-1}} = \lim_{x\to 0} \frac{-x^n}{n} = 0$$

2. $\lim_{x\to\infty} \left(a^{1/x}-1\right)x$ form $0\cdot\infty$

$$\lim_{x\to\infty} \frac{a^{1/x}-1}{1/x} \quad \left(\frac{0}{0} \text{ form}\right)$$ put $\frac{1}{x}=t$

$x\to\infty$ then $t=0$

$$\lim_{t\to 0} \frac{a^t-1}{t} \quad \left(\frac{0}{0} \text{ form}\right)$$

L'Hospital

$$\lim_{t\to 0} \frac{a^t \log a}{1} = \log a$$

(b) Form : $\infty-\infty$ $\quad \lim_{x\to a}\left[f(x)-g(x)\right]$

Method: Rewrite $f(x)-g(x) =$

$$\frac{\left(\frac{1}{g(x)}-\frac{1}{f(x)}\right)}{\frac{1}{f(x)\cdot g(x)}} \quad (0/0)$$

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Ex: $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$ $(\infty-\infty)$

Sol: $\lim_{x\to 0}\frac{x-\sin x}{x\cdot\sin x}$ (0/0 form)

$\lim_{x\to 0}\frac{1-\cos x}{x\cos x+\sin x}$ (0/0 form)

$\lim_{x\to 0}\frac{\sin x}{-x\sin x+\cos x+\cos x} = 0$

(c) Forms $0^0, 1^\infty, \infty^0$ $\lim_{x\to a}[f(x)]^{g(x)}$

Method: Take $y=\lim_{x\to a}[f(x)]^{g(x)}$ taking log,

$\log y = \lim_{x\to a} g(x)\log f(x)$ which takes $0\cdot\infty$ form and can be evaluated by the method explained above.

Let $\log y = l$ Hence $y=e^{l}$

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(1) $\underset{x\to 0}{Lt}\ x^{\sin x}$ (form 0^0)

Sol: Let $y = \underset{x\to 0}{Lt}\ x^{\sin x} \Rightarrow \log y = \underset{x\to 0}{Lt}\ \sin x \cdot \log x$

(0 . ∞ form)

$\log y = \dfrac{\log x}{\operatorname{Cosec} x}$ ($\dfrac{\infty}{\infty}$ form) - Apply L' rule

$\log y = \underset{x\to 0}{Lt}\ \dfrac{1/x}{-\operatorname{Cosec} x \cot x}$

$= \underset{x\to 0}{Lt}\ -\dfrac{\sin x}{x} \cdot \tan x = -1.0 = 0$

$\therefore y = e^0 = 1$ i.e. $\underset{x\to 0}{Lt}\ x^{\sin x} = 1$

(2) $\underset{x\to a}{Lt}\left(2 - \dfrac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$ (form 1^∞)

Sol. Let $y = \underset{x\to a}{Lt}\left(2 - \dfrac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$

$\log y = \underset{x\to a}{Lt}\ \tan\left(\dfrac{\pi x}{a}\right) \cdot \log\left(2 - \dfrac{x}{a}\right)$ (form $0.\infty$)

$$\lim_{x\to a} \frac{\log\left(2-\frac{x}{a}\right)}{\cot\left(\frac{\pi x}{2a}\right)} \qquad (0/0 \text{ form})$$

Apply L' rule.

$$\log y = \lim_{x\to a} \frac{\left(\frac{1}{2-x/a}\right)\cdot\left(\frac{-1}{a}\right)}{-\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)\cdot\frac{\pi}{2a}}$$

$$\lim_{x\to a} \frac{\frac{2}{\pi}\cdot\sin^2\left(\frac{\pi x}{2a}\right)}{2-\frac{x}{a}} = 2/\pi.$$

i.e $\log y = 2/\pi \Rightarrow y = e^{(2/\pi)}$

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Sphere

Defn.: a locus of a pt. which remains at a const. distance from a fixed point. The fixed pt. is called the centre and the const. the r of the sphere.

Eqn. of a sphere with centre (a,b,c) with radius r is

$$(x-a)^2+(y-b)^2+(z-c)^2 = r^2.$$

If centre is at origin, $x^2+y^2+z^2 = r^2$

General form:

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

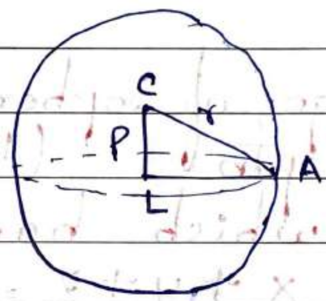
↳ represents a sphere whose centre is $(-u,-v,-w)$ and radius is

$$r = \sqrt{u^2+v^2+w^2-d}$$

(ii) Section of a sphere by a plane is a circle and the section of a sphere through its centre is called a great circle.

Let $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ (sphere)
$Ax+By+Cz+D=0$ (plane)

taken together, considered as a circle with centre L and radius $LA = \sqrt{r^2-p^2}$



(iii) Eqn. of any sphere through the circle of intersection of the sphere $S=0$ and the plane $U=0$ is

$$\boxed{S+kU=0}$$

Q1. Find the eqn. of sphere whose centre is $(3,-1,4)$ and passes through the pt. $(1,-2,0)$

$(a,b,c)=(3,-1,4)$

$(x-a)^2+(y-b)^2+(z-c)^2=r^2$

$\Rightarrow (x-3)^2+(y+1)^2+(z-4)^2=r^2$

Since sphere passes through $(1,-2,0)$

$(1-3)^2 + (-2+1)^2 + (0-4)^2 = r^2$

$4 + 1 + 16 \Rightarrow r = \sqrt{21}$ (or) $r^2 = 21$.

$(x-3)^2 + (y+2)^2 + (z-4)^2 = 21$.

Q2. Obtain the eqn. of sphere which passes through the pts (1,0,0); (0,1,0); (0,0,1) and which has its centre on the plane $x+y+z=6$

Let $x^2+y^2+z^2+2ux+2vy+2wz+d=0$.

Since it passes through the given points, we have

$\Rightarrow 1+2u+d=0 \rightarrow$ (i)

$1+2v+d=0 \rightarrow$ (ii)

$1+2w+d=0 \rightarrow$ (iii)

$u = -\frac{(d+1)}{2}$; $w = -\frac{(d+1)}{2}$; $v = -\left(\frac{d+1}{2}\right)$

its centre lies on the plane $x+y+z=6$

Centre $\rightarrow (-u, -v, -w)$

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i.e. $-u-v-w=6$

$$\Rightarrow \frac{d+1}{2}+\frac{d+1}{2}+\frac{d+1}{2}=6$$

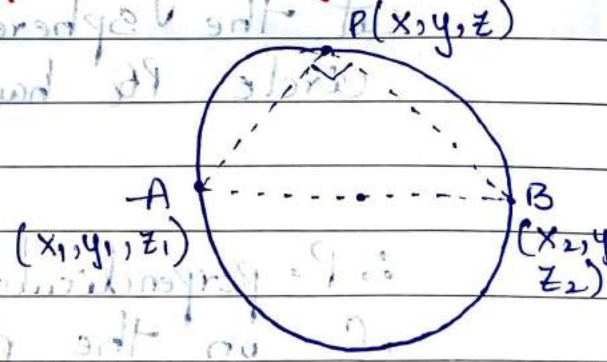
$$d=3$$

$\therefore \quad u=-2;\ v=-2,\ w=-2.$

$\therefore \quad x^2+y^2+z^2-4x-4y-4z+3=0$

8. (a) Find eqn. of sphere which has (x, y, z) and (x_2, y_2, z_2) as the extremities of a diameter.

(b) Deduce the eqn. of sphere described on the line joining the points $(2,-1,4)$ and $(-2,2,-2)$ as diameter. Find the area of the circle in which the sphere is intersected by the plane $2x+y-z=3$



Let $P(x,y,z)$ be any pt. on the sphere

Then AP and BP are @ right angles.

Now DR's of AP : $x-x_1,\ y-y_1,\ z-z_1$

DR's of BP : $x-x_2,\ y-y_2,\ z-z_2$

$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)$
which is the required eqn.

(b) We are given, $(x_1, y_1, z_1) = (2,-1,4)$
$(x_2, y_2, z_2) = (-2,2,-2)$

The eqn. is $(x-2)(x+2)+(y+1)(y-2)+(z-4)(z+2)$

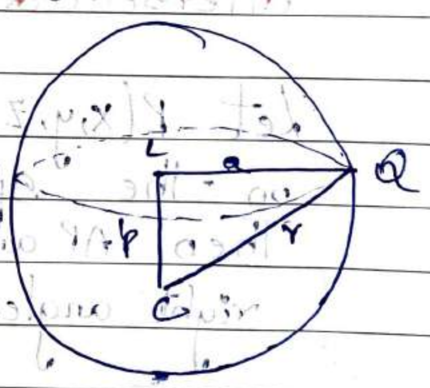
$x^2+y^2+z^2-y-2z-14=0 \rightarrow (i)$

$(2x+y-z=3)$

$(-u,-v,-w) \Rightarrow$ centre is $C(0,\frac{1}{2},1)$

radius $r = \sqrt{0+1/4+1+14} = \sqrt{\frac{61}{4}}$

Let the given plane $2x+y-z-3=0$ cut the sphere (1) in the circle PQ having centre L



$\therefore$ P = perpendicular CL from C on the plane.

$\frac{\frac{1}{2}-1-3}{\sqrt{4+1+1}} = \frac{7}{2\sqrt{6}}$

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Now radius of the circle be 'a' then

$$a^2 = r^2 - p^2$$

$$= \frac{61}{4} - \frac{49}{24} = \frac{317}{24}.$$

$\therefore$ The area of the circle is πa^2.

$$\Rightarrow \pi\left(\frac{317}{24}\right)$$

(or)

$\frac{317}{24}\pi$ sq. units

Q. Find eqn. of the sphere having the circle $x^2+y^2+z^2+10y-4z-8=0$; $x+y+z=3$, as a great circle.

Eqn. of sphere passing through the given circle is

$$x^2+y^2+z^2+10y-4z-8+k(x+y+z-3)=0$$

Since the given circle is a great circle of this sphere, then centre of the sphere and centre of the circle coincide.

which is possible if centre of the sphere lies on the plane $x+y+z-3=0$ of given circle.

Centre of the sphere is $\cancel{C\left(-\frac{K}{2}, -5, 2\right)}$

$$C\left(-\frac{K}{2}, -5-\frac{K}{2}, 2-\frac{K}{2}\right)$$

This centre lies on the plane

$$-\frac{K}{2} - 5 - \frac{K}{2} + 2 - \frac{K}{2} = 3$$

$$\boxed{K = -4}$$

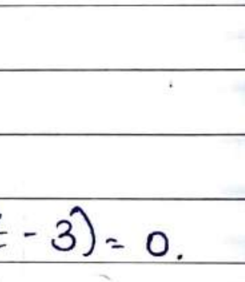
Hence, the req. eqn. is

$$x^2+y^2+z^2+10y-4z-8-4(x+y+z-3)=0.$$

$$x^2+y^2+z^2-4x+6y-8z+4=0$$

Eqn. of the tangent plane

(i) The eqn. of the tangent plane at any point (x,y,z) of sphere $x^2+y^2+z^2=r^2$ is



$$\boxed{xx_1+yy_1+zz_1=r^2}$$

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(ii) The tangent plane at (x_1, y_1, z_1) to the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ is

$$\boxed{xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0.}$$

NOTE: The condition of a plane (or a line) to touch a sphere is that ⊥ cular distance of the centre from the plane (line) = radius.

Q1. Find the eqn. of the sphere passing through the circle $x^2+y^2+z^2-6x-2z+5=0$; $y=0$ touching the plane $3y+4z+5=0$

Sol. The eqn. of sphere passing through the given circle is

$$x^2+y^2+z^2-6x-2z+5+ky=0 \rightarrow (i)$$

Centre $= \left(3, -\frac{k}{2}, 1\right)$

Radius $= \sqrt{9+\frac{k^2}{4}+1} = \sqrt{\frac{k^2+20}{4}}$

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The sphere (i) will touch the plane $3y+4z+5=0$ of $\perp^r$ distance of the centre $\left(3, -\frac{k}{2}, 1\right)$ from the plane = radius.

i.e $$\frac{3\left(-\frac{k}{2}\right)+4(1)+5}{\sqrt{9+16}} = \sqrt{\frac{k^2+20}{4}}$$

i.e $4k^2+27k+44=0 \Rightarrow k = \frac{-11}{4}$ (or) -4.

Substituting k in eqn. (i)

$$x^2+y^2+z^2-6x-\frac{11}{4}y-2z+5=0$$

and

$x^2+y^2+z^2-6x-4y-2z+5=0$ as the two required spheres.

Q. Find the tangent plane to the sphere $x^2+y^2+z^2-4x+2y-6z+5=0$ which are $\parallel$el to the plane $2x+2y-z=0$.

Sol. Any plane parallel to the given sphere is

$2x+2y-z+k=0$

The plane is a tangent plane to the given sphere if $\perp^r$ dist. of $C = r$.

$C = (2, -1, 3)$

$r = \sqrt{4+1+9-5}$

$$\frac{4-2-3+k}{\sqrt{4+4+1}} = 3$$

$k - 1 = 9$

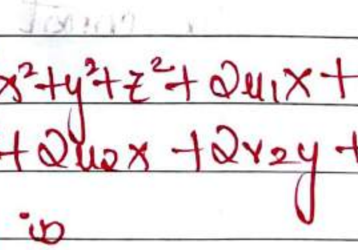
$\boxed{k = 10}$

Hence, the eqn. of tangent plane = $\underline{\underline{2x + 2y - z + 10 = 0}}$.

Orthogonal spheres

Defn. Two spheres are said to be orthogonal if the tangent planes at a P.O.I are at right angles.

$\boxed{(C_1C_2)^2 = r_1^2 + r_2^2}$



Q. Show the condition for spheres $x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$ and $x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$ to cut orthogonally is $\underline{\underline{2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1+d_2}}$!

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Sol. The centres of the spheres are

$$C_1(-u_1, -v_1, -w_1) \quad r_1 = \sqrt{u_1^2+v_1^2+w_1^2-d_1}$$
$$C_2(-u_2, -v_2, -w_2) \quad r_2 = \sqrt{u_2^2+v_2^2+w_2^2-d_2}$$

Now these spheres cut orthogonally if

$$(C_1C_2)^2 = r_1^2 + r_2^2$$

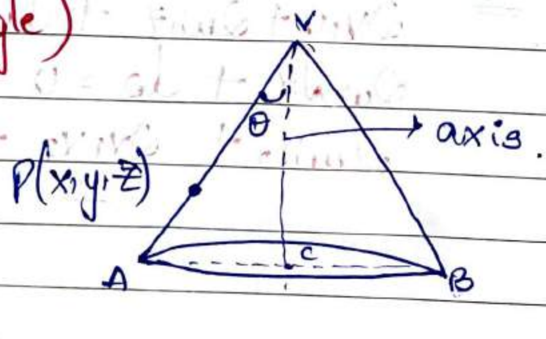
i.e $(u_1-u_2)^2 + (v_1-v_2)^2 + (w_1-w_2)^2 = u_1^2+v_1^2+w_1^2-d_1 + u_2^2+v_2^2+w_2^2-d_2$

$$-2u_1u_2 - 2v_1v_2 - 2w_1w_2 = -d_1 - d_2$$

i.e $2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2$

Right Circular Cone

Defn. A right circular cone is a surface generated by a st. line which passes through a fixed pt. and makes a const. angle with a fixed line.
↳ (vertex)
↳ (axis)
↳ (semi-vertical angle)



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Note: The section of a right circular cone by a plane $\perp^r$ to its axis is called a circle.

Eqn. of Right Circular Cone

Let (x_0, y_0, z_0) be the coordinates of the vertex V and (a, b, c) be the direction ratios of the axis. Consider any pt. $P(x, y, z)$ on the cone. Then the DR's of the generator VP are

$$x - x_0,\ y - y_0,\ z - z_0 \text{ and}$$

$$\cos\theta = \frac{a(x-x_0) + b(y-y_0) + c(z-z_0)}{\sqrt{a^2+b^2+c^2} \cdot \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}}$$ which is the req. eq. of right circular cone

Q. Find the eqn. of right circular cone whose vertex is $(1, -1, 0)$, semi-vertical angle is $\pi/3$, radius and axis is the line $\frac{x+1}{1} = \frac{y+1}{3} = \frac{z}{2}$.

Sol. $V(1, -1, 0)$; Let $P(x, y, z)$ be any pt. on the surface

DR's of VP $= x-1,\ y+1,\ z-0 = 0 \Rightarrow x-1,\ y+1,\ z.$

DR's of axis $= (a, b, c) = (1, 3, 2)$

$$\cos\theta = \frac{a(x-1)+b(y+1)+c(z)}{\sqrt{a^2+b^2+c^2}\sqrt{(x-1)^2+(y+1)^2+z^2}}$$

$$\cos\frac{\pi}{3} = \frac{(x-1)+3(y+1)+2z}{\sqrt{1+9+4}\sqrt{(x-1)^2+(y+1)^2+z^2}}$$

$$\Rightarrow 7(x^2+y^2-2x+2y+z^2+2) = 2(x+3y+2z+2)^2$$

Q. Find eqn. of right circular cone having semi-vertical angle 60° and the line $\frac{x-1}{3} = \frac{y+2}{-4} = \frac{z+1}{5}$ as its axis

Since the axis passes through the vertex, we can take vertex $(1,-2,-1)$

dr's of axis $(3,-4,5)$

$$\cos 60° = \frac{3(x-1)-4(y+2)+5(z+1)}{\sqrt{9+16+25}\sqrt{(x-1)^2+(y+2)^2+(z+1)^2}}$$

$$25[(x-1)^2+(y+2)^2+(z+1)^2] = 2(3x-4y+5z-6)^2$$

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Q. Find the eqn. of right circular cone when passes through the point $(2,1,3)$ has its vertex at $(1,1,2)$ and axis parallel to

$$\frac{x-2}{2} = \frac{y-1}{-4} = \frac{z+2}{3}.$$

Sol. Dr's of axis: $(2,-4,3)$, Vertex $(1,1,2)$ passes through $A(2,1,3)$.
Then VA is the generator and θ be the angle b/w VA and axis.

DR's of $VA = (2-1, 1-1, 3-2) = (1,0,1)$

$$\cos\theta = \frac{2(1) + (-4\times 0) + (3\times 1)}{\sqrt{4+16+9}\cdot\sqrt{1+0+1}}.$$

$$\cos\theta = \frac{5}{\sqrt{58}}$$

on the other hand, if $P(x,y,z)$ be any point on the cone, the dr's of VP are

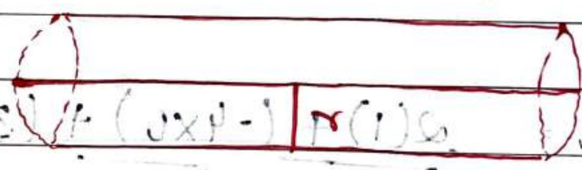
$(x-1, y-1, z-2)$. Then

$$\cos\theta = \frac{5}{\sqrt{58}} = \frac{2(x-1) - 4(y-1) + 3(z-2)}{\sqrt{4+16+9}\cdot\sqrt{(x-1)^2+(y-1)^2+(z-2)^2}}$$

$$25[(x-1)^2+(y-1)^2+(z-2)^2] = 2(2x-4y+3z-4)^2.$$

RIGHT CIRCULAR CYLINDER

Defn. A right circular cylinder is a surface generated by a st. line which is parallel to a fixed line and is at a const. dist. from it. The const. distance is called the radius of the cylinder.



Eqn. of right circular cylinder having radius r units and the line $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$ as its axis.

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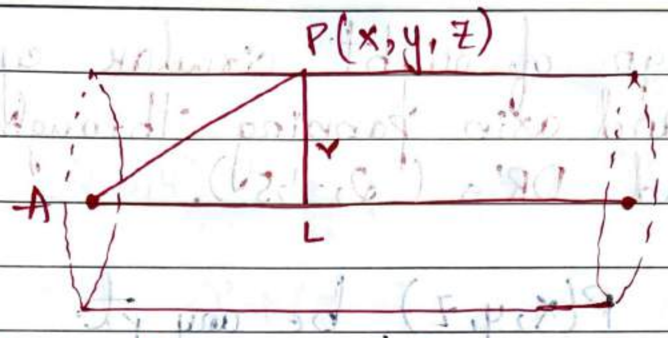

Let $P(x, y, z)$ be any point on the surface of the cylinder and

let $A(x_0, y_0, z_0)$ be a pt. on the axis.

DR's of axis are a, b, c and hence

DC's are, $l = \frac{a}{\sqrt{a^2+b^2+c^2}}$, $m = \frac{b}{\sqrt{a^2+b^2+c^2}}$, $n = \frac{c}{\sqrt{a^2+b^2+c^2}}$

Join AP and draw PL $\perp$ cular to the axis

$PL = r$

Also AL = projection of AP on the axis

$= l(x-x_0) + m(y-y_0) + n(z-z_0)$ → (by projection formula)

From the figure ⇒ $AP^2 = AL^2 + PL^2$

i.e

$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = \left[l(x-x_0) + m(y-y_0) + n(z-z_0)\right]^2 + r^2$$

which is the req. eqn. of right circular cone

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Q. Find the eqn. of right circular cylinder of $r = 2$ units and axis passing through the pt. $(1, -3, 2)$ and DR's $(2, -1, 5)$.

Sol. $A(1, -3, 2)$ $P(x, y, z)$ be any pt.

dc's of the axis $\Rightarrow l = \frac{2}{\sqrt{4+1+25}} = \frac{2}{\sqrt{30}}$

$m = \frac{-1}{\sqrt{30}}$

$n = \frac{5}{\sqrt{30}}$

$$(x-1)^2 + (y+3)^2 + (z-2)^2 = \left[\frac{2(x-1) - 1(y+3) + 5(z-2)}{\sqrt{30}}\right]^2 + (2)^2.$$

Q2. Find the eqn of right circular cylinder having radius 3 units, and the line passing through the points $(2, 3, 4)$ and $(4, 4, 2)$ as the axis.

Sol. Eqn. of line passing through $(2, 3, 4)$ and $(4, 4, 2)$

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = \frac{x-2}{2} = \frac{y-3}{1} = \frac{z-4}{-2}.$$

Dr's of axis $\Rightarrow (2, 1, -2)$.

DC's are $l = \frac{2}{3}$, $m = \frac{1}{3}$, $n = -\frac{2}{3}$

$A = (2,3,4)$; $P = p(x,y,z)$. Then the eqn. is

$$(x-2)^2+(y-3)^2+(z-4)^2 = \left[\frac{2}{3}(x-2)+\frac{1}{3}(y-3)-\frac{2}{3}(z-4)\right]^2+3^2$$

$$9\left[(x-2)^2+(y-3)^2+(z-4)\right]^2 = (2x+y-2z+1)^2+81$$

Q. Find the eqn. of right circular cylinder whose axis is $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ and a generator is $\frac{x-4}{3} = \frac{y-3}{4} = \frac{z-2}{5}$.

Sl. Given, $A = (2,3,4)$ $P = p(x,y,z)$

dr's of axis $= (3,4,5)$, dc's $= \left(\frac{3}{\sqrt{50}}, \frac{4}{\sqrt{50}}, \frac{5}{\sqrt{50}}\right)$

$$AL = \frac{3}{\sqrt{50}}(x-2)+\frac{4}{\sqrt{50}}(y-3)+\frac{5}{\sqrt{50}}(z-4)$$

$$AP^2 = AL^2+PL^2$$

$$(x-2)^2+(y-3)^2+(z-4)^2 = \frac{\left[3(x-2)+4(y-3)+5(z-4)\right]^2}{50}+r^2$$

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Now, the cylinder passes through (4, 3, 2)
(from generator)

$$r^2 = \frac{384}{50}$$

Hence the req. eqn. is

$$50\left[(x-2)^2 + (y-3)^2 + (z-4)^2\right] = (3x+4y+5z-38)^2 + 384.$$

PARTIAL DIFFRENTIATION

Function of several variables : The func. depends on more than one independent variables.

$u = f(x,y)$ is a func. of 2 variables

$u = f(x,y,z)$ is a func. of 3 variables.

* Partial derivates

Let u be a func. of two independent variables x and y i.e $u = f(x,y)$, the partial derivative of u w.r.t one of the independent variable say x, treat the other variable y as const. and is denoted by $\frac{\partial u}{\partial x}$ (or) u_x.

mathematically $\frac{\partial u}{\partial x}$ is defined as

$$\frac{\partial u}{\partial x} = \underset{\delta x \to 0}{Lt} \frac{f(x+\delta x, y) - f(x,y)}{\delta x} = u_x$$

$$\frac{\partial y}{\partial x} = \underset{\delta y \to 0}{Lt} \frac{f(x, y+\delta y) - f(x,y)}{\delta y} = u_x$$

Here u_x, u_y are called 1st order partial derivatives of u. w.r.t x and y

Partial derivatives of higher order

The second order partial derivatives of $u(x, y)$ defined as

Let $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are first order derivati

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} \text{ (or) } u_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \cdot \partial x} = u_{yx}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \cdot \partial y} = u_{x,y}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{d^2 u}{\partial y^2} = u_{y,y}$$

Here u_{xx}, u_{xy}, u_{yx} & u_{yy} are called second order partial derivatives.

u_{xy} and u_{yx} are called mixed partial derivatives.

for most of the funcs, the two mixed derivative are equal.

$$\boxed{\frac{\partial^2 u}{\partial x \cdot \partial y} = \frac{\partial^2 u}{\partial y \cdot \partial x}}$$ (is not true always.)

Rules of P.d.

Let u and v be two functions of x and y.
i.e $u = u(x,y)$ and $v = v(x,y)$; then

(i) $\frac{\partial}{\partial x}(u+v) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}$

(or)

$\frac{\partial}{\partial y}(u+v) = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

(ii) $\frac{\partial}{\partial x}(uv) = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}$

(or)

$\frac{\partial}{\partial y}(uv) = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}$

(iii) $\frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$

(or)

$\frac{\partial}{\partial y}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}}{}$

NOTE: In case of ordinary derivatives,

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \quad \text{(or) vice versa}$$

$$\boxed{\frac{\partial y}{\partial x} \neq \frac{1}{\left(\frac{\partial x}{\partial y}\right)}}$$

INDIRECT P.D

Given a relation of the form $f(x, y, z) = c$ we need to identify dependent and independent variables based on required partial derivatives.

Ex: if $\frac{\partial z}{\partial x}$, $\frac{\partial^2 z}{\partial y^2}$, $\frac{\partial^2 z}{\partial x \cdot \partial y}$ we have identified z as dependent, x and y as independent.

(1) $u = x^3 - 3xy^2 + x + e^x \cos y + 1$.

There $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\frac{\partial u}{\partial x} =$

$$\rightarrow \frac{\partial u}{\partial x} = 3x^2 - 3y^2(1) + 1 + \cos y\,(e^x) + 0$$

$$\rightarrow \frac{\partial u}{\partial y} = \cancel{3}0 - 3x(2y) + 0 + e^x(-\sin y) + 0$$

$$\rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = 6x - 0 + 0 + \cos y\,(e^x)$$

$$\rightarrow \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = -3x(2) + (\overset{-\cos y}{\cancel{\cos y}})e^x$$

$$= -6x - e^x \cos y$$

$$\rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (6x + e^x \cos y) - (6x + e^x \cos y)$$

$$= 0$$

2. $u = \tan^{-1}\left(\frac{y}{x}\right)$. Then show that

$$\frac{\partial^2 u}{\partial y \cdot \partial x} = \frac{\partial^2 u}{\partial x \cdot \partial y}$$

$$\rightarrow u = \tan^{-1}\left(\frac{y}{x}\right)$$

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$$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{1}{1+\frac{y^2}{x^2}} \left\{ y\left(-\frac{1}{x^2}\right) \right\}$$

$$\frac{\partial u}{\partial x} = \frac{\cancel{x^2}}{x^2+y^2}\left(\frac{-y}{\cancel{x^2}}\right) = \frac{-y}{x^2+y^2} = \frac{\partial u}{\partial x} \rightarrow (i)$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$\frac{1}{1+\frac{y^2}{x^2}} \cdot \left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} \rightarrow (ii)$$

Differentiating (i) w.r.t y.

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x \cdot \partial y}$$

$$\frac{\partial}{\partial y}\left\{\frac{-y}{(x^2+y^2)}\right\}$$

$$\Rightarrow -\left\{\frac{(x^2+y^2)\cdot\frac{\partial}{\partial y}\cdot y - y\cdot\frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)}\right\}$$

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$$\frac{\partial^2 u}{\partial y\,\partial x} = -\left\{\frac{(x^2+y^2)(1) - y(\cancel{x}^{0} + 2y)}{(x^2+y^2)}\right\}$$

$$\frac{\partial^2 u}{\partial x\cdot\partial y} = -\frac{y^2-x^2}{x^2+y^2}.$$

(3) $u = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}$ Then S.T

$x u_x + y u_y + z u_{\cancel{x}} = 0.$

→ $u_x \Rightarrow \frac{\partial u}{\partial x} = \frac{1}{y+z} + y\left\{\frac{-1}{(z+x)^2}\right\} + z\left\{\frac{-1}{(x+y)^2}\right\}$

$$x u_x = \frac{x}{y+z} - \frac{xy}{(z+x)^2} - \frac{xz}{(x+y)^2}.$$

→ $u_y \Rightarrow \frac{\partial u}{\partial y} = x\left(\frac{-1}{(y+z)^2}\right) + \left(\frac{1}{z+x}\right) + z\left(\frac{-1}{(x+y)^2}\right)$

$$y u_y = \frac{-xy}{(y+z)^2} + \frac{y}{(z+x)} - \frac{zy}{(x+y)^2}$$

→ $u_z = \frac{\partial u}{\partial z} = x\left(\frac{-1}{(y+z)^2}\right) + y\left(\frac{-1}{(z+x)^2}\right) + \frac{1}{(x+y)}$

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$$z u_z = \frac{-zx}{(y+z)^2} - \frac{zy}{(z+x)^2} + \frac{z}{(x+y)}$$

$$x u_x + y u_y + z u_z = \frac{x}{y+z} - \frac{xy}{(x+z)^2} - \frac{xz}{(x+z)^2}$$

$$\frac{-xy}{(y+z)^2} + \frac{y}{(z+x)} - \frac{yz}{(x+y)^2} - \frac{xz}{(y+z)^2} - \frac{yz}{(x+z)^2} + \frac{z}{(x+y)}$$

$$= 0$$

(4) $u = \log(x^3+y^3+z^3-3xyz)$ P.T

(i) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$

$$\rightarrow \frac{\partial u}{\partial x} \Rightarrow \frac{1}{x^3+y^3+z^3-3xyz} \frac{\partial}{\partial x}\left(x^3+y^3+z^3-3xyz\right)$$

$$\Rightarrow \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

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$$\frac{\partial u}{\partial y} = \frac{1}{x^3+y^3+z^3-3xyz} \cdot \frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz)$$

$$\frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz} \cdot \frac{\partial}{\partial z}(x^3+y^3+z^3-3xyz)$$

$$= \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2)-3(yz+xz+xy)}{x^3+y^3+z^3-3xyz}$$

using $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

$$\frac{3\cancel{[x^2+y^2+z^2-xy-yz-zx]}}{(x+y+z)\cancel{[x^2+y^2+z^2-xy-yz-zx]}} = \frac{3}{x+y+z}.$$

(iii) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u.$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$3\left(\frac{-1}{(x+y+z)^2}\right) + 3\left(\frac{-1}{(x+y+z)^2}\right) + 3\left(\frac{-1}{(x+y+z)^2}\right)$$

$$= \frac{-9}{(x+y+z)^2}$$

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6) $r^2 = x^2+y^2+z^2$ and $v = r^m$. Then P.T

$v_{xx} + v_{yy} + v_{zz} = m(m+1)r^{m-2}$.

$\Rightarrow$ Given $v = r^m$ and $r^2 = x^2+y^2+z^2$.

Diff. v w.r.t x

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(r^m) = \frac{\partial}{\partial r}(r^m)\frac{\partial r}{\partial x}$$

$$r^2 = x^2+y^2+z^2$$

$$\cancel{2}r\frac{\partial r}{\partial x} = \cancel{2}x \quad = \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial r}(r^m)\left\{\frac{x}{r}\right\} = mr^{m-1}\left(\frac{x}{r}\right)$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(mr^{m-1}\left(\frac{x}{r}\right)\right)$$

$$= m\frac{\partial}{\partial x}\left[r^{m-1}\left(\frac{x}{r}\right)\right]$$
(1) (2)

$$\frac{\partial^2 r}{\partial x^2} = m\left\{ r^{m-1}\cdot\frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{x}{r}\frac{\partial}{\partial x}\left(r^{m-1}\right)\right\}$$

$$m\left\{ r^{m-1}\left[\frac{r\frac{\partial}{\partial x}x - x\frac{\partial}{\partial x}r}{r^2}\right] + \frac{x}{r}\left[\frac{\partial}{\partial r}\left(r^{m-1}\right)\cdot\frac{\partial r}{\partial x}\right]\right\}$$

$$= m\left\{ r^{m-1}\left[\frac{r - x\left(x/r\right)}{r^2}\right] + \frac{x}{r}\left[(m-1)\, r^{m-2}\cdot\frac{x}{r}\right]\right\}$$

$$= m\left\{ r^{m-1}\left[\frac{r^2-x^2}{r^3}\right] + \frac{x^2}{r^2}(m-1)r^{m-2}\right\}$$

$$m\left\{ r^{m-4}\left(r^2-x^2\right) + (m-1)x^2 r^{m-4}\right\}$$

$$m\left\{ r^{m-4}\left(r^2-x^2+mx^2-x^2\right)\right\}$$

$$m\left\{ r^{m-4}\left(r^2+mx^2-2x^2\right)\right\}$$

$$= m r^{m-1}\left[r^2+(m-2)x^2\right]$$

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$V_{xx} = m r^{m-2} + m(m-2) x^2 r^{m-4}$

$V_{yy} = m r^{m-2} + m(m-2) y^2 r^{m-4}$

$V_{zz} = m r^{m-2} + m(m-2) z^2 r^{m-4}$

$V_{xx} + V_{yy} + V_{zz} = 3 m r^{m-2} + (x^2 + y^2 + z^2) m(m-2) r^{m-4}$

$= 3 m r^{m-2} + m(m-2) r^{m-4} r^2$

$= r^{m-2} [3m + m^2 - 2m]$

$= m(m+1) r^{m-2}$.

7) If $\theta = t^n e^{r^2/4t}$. Then find the value of n such that $\frac{1}{r^2} \frac{\partial}{\partial r} \left\{ r^2 \frac{\partial \theta}{\partial r} \right\} = \frac{\partial \theta}{\partial t}$

Given θ is a func. of t and r $\theta = f(t, r)$

$\theta = t^n e^{r^2/4t}$ diff. w.r.t.

$$\frac{\partial \theta}{\partial t} = t^n \times \frac{\partial}{\partial t}\left(e^{r^2/4t}\right) + e^{r^2/4t} \cdot \frac{\partial}{\partial t}\left(t^n\right)$$

$$\frac{d\theta}{dt} = t^n e^{r^2/4t} \frac{\partial}{\partial t}\left(\frac{r^2}{4t}\right) + e^{r^2/4t} \cdot n t^{n-1}$$

$$= t^n e^{r^2/4t} \left\{ \frac{r^2}{4}\left(\frac{-1}{t^2}\right) \right\} + e^{r^2/4t} n t^{n-1}$$

$$\frac{\partial \theta}{\partial t} = \frac{-t^{n-2}}{4} r^2 e^{r^2/4t} + e^{r^2/4t} n t^{n-1} \rightarrow \text{(i)}$$

Diff. w.r.t r

$$\frac{\partial \theta}{\partial r} = t^n \frac{\partial}{\partial r} e^{r^2/4t} = t^n e^{r^2/4t} \frac{\partial}{\partial r}\left(\frac{r^2}{4t}\right)$$

$$\frac{\partial \theta}{\partial r} = t^n e^{r^2/4t} \frac{2r}{4t} \rightarrow (\times \text{ by } r^2)$$

$$r^2 \frac{\partial \theta}{\partial r} = t^n e^{r^2/4t} \cdot \frac{r^3}{2t} \rightarrow \text{(2)}$$

Using (i) and (ii)

$$\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$$

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$$\frac{1}{r^2}\left\{\frac{\partial}{\partial r}\left(t^n e^{r^2/4t}\frac{r^3}{2t}\right)\right\}$$

$$= \frac{-t^{n-2}}{4} r^2 e^{r^2/4t} + e^{r^2/4t} n t^{n-1}$$

$$\frac{1}{r^2}\left\{\frac{t^n}{2t}\left[e^{r^2/4t}\, 3r^2 + r^3 e^{r^2/4t}\frac{2r}{4t}\right]\right.$$

$$e^{r^2/4t}\left\{n t^{n-1} - r^2\frac{t^{n-2}}{4}\right\}$$

$$\frac{1}{r^2}\left\{\frac{t^n}{2t}\left[3r^2 + \frac{r^4}{2t}\right]\right\} = n t^{n-1} - r^2\frac{t^{n-2}}{4}$$

$$\Rightarrow n = \frac{3}{2} + \frac{r^2}{2t}$$ //

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Q. If $u = f(r)$ where $r = \sqrt{x^2+y^2+z^2}$ then S.T $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r} f'(r)$ and hence reduce the relation for $u = r^n$

Sol. Let $u = f(r)$ diff. w.r.t x

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial r} f(r) \frac{\partial r}{\partial x}$$

$$r = \sqrt{x^2+y^2+z^2}$$

$$\frac{\partial u}{\partial x} = f'(r)\frac{x}{r} \qquad \frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2+z^2}}(2x) = \frac{x}{r}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left\{f'(r)\cdot\frac{x}{r}\right\} = f'(r)\frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{x}{r}\frac{\partial}{\partial x}f'(r)$$

$$\frac{\partial^2 u}{\partial x^2} = f'(r)\left\{\frac{r(1) - x\frac{\partial r}{\partial x}}{r^2}\right\} + \frac{x}{r} f''(r)\frac{\partial r}{\partial x} =$$

$$f'(r)\left\{\frac{r - x\,(x/r)}{r^2}\right\} + \frac{x}{r} f''(r)\left(\frac{x}{r}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = f'(r)\left\{\frac{r^2-x^2}{r^3}\right\} + \frac{x^2}{r^2} f''(r) \rightarrow (i)$$

$$\frac{\partial^2 u}{\partial y^2} = f'(r)\left\{\frac{r^2-y^2}{r^3}\right\} + \frac{y^2}{r^2} f''(r) \rightarrow (ii)$$

$$\frac{\partial^2 u}{\partial z^2} = f'(r)\left\{\frac{r^2 - z^2}{r^3}\right\} + \frac{z^2}{r^2} f''(r) \rightarrow (iii)$$

$(1) + (2) + (3)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{f'(r)}{r^3}\left\{3r^2 - (x^2+y^2+z^2)\right\} + \frac{f''(r)}{r^2}(x^2+y^2+z^2)$$

$$\frac{f''(r)}{r^3}\left\{3r^2 - r^2\right\} + \frac{f''(r)}{\cancel{r^2}}(\cancel{r^2})$$

$$\frac{f''(r)}{r^3}(2r^2) + f''(r) \Rightarrow f''(r) + \frac{2}{r} f'(r)$$

Q. If $z = u + v$ where $u = x^m f\left(\frac{y}{x}\right)$ and $v = y^n g$

Sol. Given $z = u + v = x^m f\left(\frac{y}{x}\right) + y^n g\left(\frac{x}{y}\right)$

$$\frac{\partial z}{\partial x} = x^m f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) + f\left(\frac{y}{m}\right) m x^{m-1} + y^n g'\left(\frac{x}{y}\right)\frac{1}{y}$$

XIe by x

$$x\frac{\partial z}{\partial x} = -x^{m-1} y f'\left(\frac{y}{x}\right) + m x^m f\left(\frac{y}{x}\right) + x y^{n-1} g'\left(\frac{x}{y}\right)$$

Diff. w.r.t y

$$\frac{\partial z}{\partial y} = x^m f'\left(\frac{y}{x}\right)\frac{1}{x} + y^n g'\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right) + g\left(\frac{x}{y}\right) n y^{n-1}$$

x'^{c} by y O.B.S

$$y\frac{\partial z}{\partial y} = x^{m-1} y f'\left(\frac{y}{x}\right) - y^{n-1} x g'\left(\frac{x}{y}\right) + n y^n g\left(\frac{x}{y}\right) \rightarrow (2)$$

adding (1) and (2)

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = mu + nv$$

HOMOGENOUS FUNCTIONS

A func. $u = f(x,y)$ is said to be a homogenous func. of degree n, it can be expressed ~~as~~ in the form of

$x^n \phi\left(\frac{y}{x}\right)$ (or) $y^n \phi\left(\frac{x}{y}\right)$ where ϕ is arbitrary func.

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Similarly a func. of three variable is said to be a Homogenous func. of degree n if it can be expressed as

$$u = x^n \phi\left(\frac{y}{x}, \frac{z}{x}\right)$$

(or)

$$u = y^n \phi\left(\frac{x}{y}, \frac{z}{y}\right)$$

(or)

$$u = z^n \phi\left(\frac{x}{z}, \frac{y}{z}\right)$$

Ex: $u = x^3 + 3xy^2 = x^3\left\{1 + 3\left(\frac{y}{x}\right)^2\right\} = x^3 \phi\left(\frac{y}{x}\right)$

u is a highest degree of order $\underline{\underline{3}}$.

$$u = \frac{x^2+y^2}{\sqrt{x}+\sqrt{y}} = \frac{x^2\left[1+\left(\frac{y}{x}\right)^2\right]}{x^{1/2}\left[1+\left(\frac{y}{x}\right)^{1/2}\right]}$$

$$= x^{2-1/2}\,\phi\left(\frac{y}{x}\right) = x^{3/2}\,\phi\left(\frac{y}{x}\right)$$

$\underline{\underline{n = 3/2}}$.

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$\Rightarrow$ $u = x^2 \sin\left(\frac{y}{x}\right) + y^2 \cos\left(\frac{y}{x}\right) + xy$

$$u = x^2 \left\{ \sin\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 \cos\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right) \right\}$$

$\therefore$ u is a H.F of degree 2.

$\Rightarrow$ $u = \dfrac{x^3 y^4}{x^2 + y^2}$ $\quad u = \dfrac{x^{\cancel{7}\,5}(y/x)^4}{\cancel{x^2}\left[1 + \left(\frac{y}{x}\right)^2\right]}$

$\boxed{n = 5}$ //.

$$\frac{x^7 \left(\frac{y}{x}\right)^4}{x^2 \left[1 + \left(\frac{y}{x}\right)^2\right]} \Rightarrow x^5 \phi(y/x)$$

$\boxed{n = 5}$ //.

Euler's Theorem on Homogenous function

If $u = f(x, y)$ is a homogenous func. of degree n then P.T

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(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$

Proof: Given u is a H.F of degree n.

$u = x^n \phi\left(\frac{y}{x}\right) \rightarrow (i)$ [diff. w.r.t. x]

$\frac{\partial u}{\partial x} = x^n \phi'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + \phi\left(\frac{y}{x}\right) n x^{n-1}$ x'e by x

$x\frac{\partial u}{\partial x} = -x^{n-1} y\,\phi'\left(\frac{y}{x}\right) + n x^n \phi\left(\frac{y}{x}\right) \rightarrow (2)$

diff. (i) w.r.t y

$\frac{\partial u}{\partial y} = x^n \phi'\left(\frac{y}{x}\right)\frac{1}{x}(1)$

x'e by y

$y\frac{\partial u}{\partial y} = x^{n-1} y\,\phi'\left(\frac{y}{x}\right) \rightarrow (3)$

Adding (2) and (3) $\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

$$= -x^{n-1} y \cancel{\phi'}\left(\frac{y}{x}\right) + n x^n \phi\left(\frac{y}{x}\right) + x^{n-1} y \cancel{\phi'}\left(\frac{y}{x}\right)$$

$$\boxed{\begin{gathered} x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu \\ \text{(or)} \\ x u_x + y u_y = nu \end{gathered}} \rightarrow (4)$$

diff. (4) w.r.t (x) partially.

$$x u_{xx} + u_x(1) + y u_{xy} = n u_x$$

($\times$'le by x O.B.S)

$$x^2 u_{xx} + x u_x + xy u_{xy} = nx u_x \rightarrow (5)$$

diff. (4) w.r.t (y) partially.

$$x u_{xy} + y u_{yy} + u_y(1) = n u_y$$

($\times$'le by y O.B.S)

$$xy u_{xy} + y^2 u_{yy} + y u_y = ny u_y \rightarrow (6)$$

Adding (5) and (6)

$$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} + (x u_x + y u_y) = n(x u_x + y u_y)$$

$$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = n(nu) - nu$$

$$\Rightarrow \quad x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = n(n-1)u.$$

Q. Verify Euler's theorem for $u = \cos^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

$$u = \cos^{-1}\left(\frac{x}{y}\right) + \cot^{-1}\left(\frac{x}{y}\right)$$

$u = y^0 \left\{ \phi\left(\frac{x}{y}\right) \right\}$ Here, u is a H.F of degree 0 $(n = 0)$

By Euler's theorem,

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0u = 0 \quad \rightarrow (i)$$

Verify (1) by diff. u w.r.t x and y partially

$$\cancel{\frac{\partial u}{\partial x} = \frac{-1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\left(\frac{1}{y}\right)} = \cancel{\frac{-y}{\sqrt{x^2+y^2}}\left(\frac{1}{y}\right)}$$

x^{le} ~~by~~ x O.B.S

$$\frac{\partial u}{\partial x} = \frac{-1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\left(\frac{1}{y}\right) - \frac{1}{1+(x/y)^2}\left(\frac{1}{y}\right)$$

$$\frac{-y(1/y)}{\sqrt{y^2-x^2}} - \frac{y^{\cancel{2}}}{x^2+y^2}\left(\frac{1}{\cancel{y}}\right)$$

x'le by x O.B.S

$$x\frac{\partial u}{\partial x} = \frac{-x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} \rightarrow (2)$$

$$\frac{\partial u}{\partial y} = \frac{-1}{\sqrt{1-(x/y)^2}}\left(\frac{-x}{y^2}\right) - \frac{1}{1+(x/y)^2}\left(\frac{-x}{y^2}\right)$$

$$\frac{\partial u}{\partial y} = \frac{y}{\sqrt{y^2-x^2}}\left(\frac{x}{y^2}\right) + \frac{\cancel{y^2}}{x^2+y^2}\left(\frac{x}{\cancel{y^2}}\right)$$

x'le by y O.B.S

$$y\frac{\partial u}{\partial y} = \frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} \rightarrow (3)$$

Adding (2) and (3)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{-\cancel{xy}}{\sqrt{\cancel{y^2-x^2}}} - \frac{\cancel{xy}}{\cancel{x^2+y^2}} + \frac{\cancel{x}}{\sqrt{\cancel{y^2-x^2}}} + \frac{\cancel{xy}}{\cancel{x^2+y^2}}$$

$$= 0$$

$\therefore$ Euler's theorem is verified.