Q2. Show that number of partitions of an integer n with no part greater than k is equal to the number of partition of n with atmost k parts.

Soln : Consider the ferrers graph representation of a partition of n with no part greater than k.

Then number of dots in each row must be $\leq k$. (size of any row $\leq k$)

If we read columnwise then the no. of parts is $\leq k$.

Thus, for a given partition of n with no part greater than k, there correspond a partition of n with atmost k parts.

Conversely, consider a F.G representation of a partition of n with at most k parts. Then no. of rows is $\leq k$.

In the conjugate partition, size of any col is $\leq k$.

Thus, for every partition of n with atmost k parts there corresponds a partition of n with no part greater than k.

Hence, the no. of partition of n with no part greater than k must be equal to the no of partition of n with at most k parts.

Q3. Obtain a generating function for the partition of n with exactly k parts.
Solution:

G.F Partition of n with exactly k parts = Enumerator with atmost k parts − Enumerator with atmost (k−1) parts.

= Enumerator with no part greater than k − Enumerator with no part greater than k−1

$$= \left[(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\cdots(1-x^k)^{-1}\right] - \left[(1-x)^{-1}(1-x^2)^{-1}\cdots\cdots(1-x^{k-1})^{-1}\right]$$

$$= (1-x)^{-1}(1-x^2)^{-1}\cdots(1-x^{k-1})^{-1}\left\{(1-x^k)^{-1}-1\right\}$$

$$= (1-x)^{-1}(1-x^2)^{-1}\cdots(1-x^{k-1})^{-1}\frac{x^k}{(1-x^k)}$$

$$= x^k(1-x)^{-1}(1-x^2)^{-1}\cdots(1-x^k)^{-1} \quad \text{—} (*)$$

is the G.F

coeff of x^n in $(*)$ is the no. of partition of n with exactly k parts.

Q4. Prove that the number of partition of n with exactly k parts is equal to the number of partition of n – k with no part greater than k.

Solution:

GF Number of partition of n with exactly k parts

$$G_1(x) = x^k \underbrace{(1-x)^{-1}(1-x^2)^{-1} \cdots (1-x^k)^{-1}} \quad -(1)$$

coeff of x^n

Let

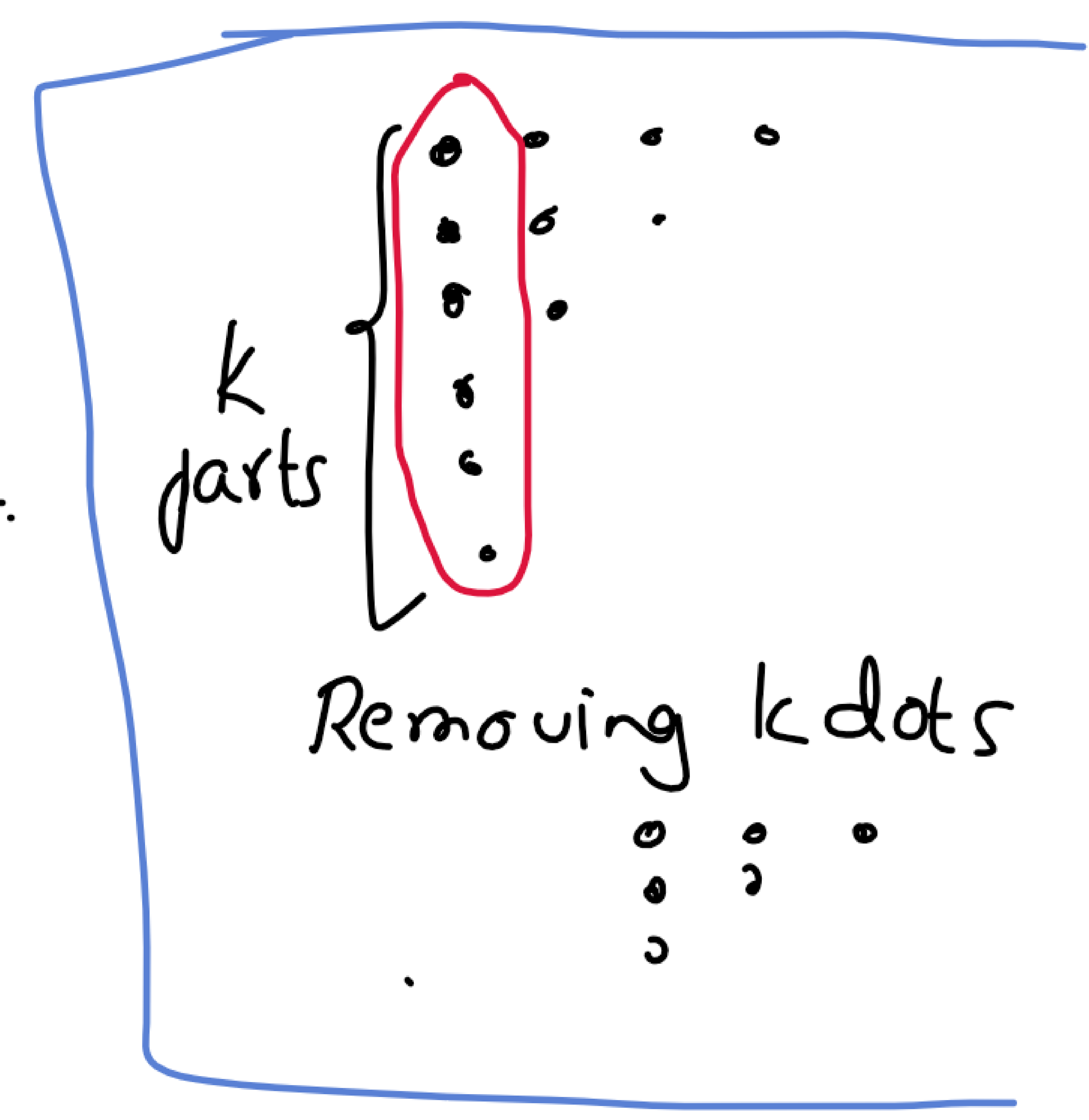
$$G_2(x) = (1-x)^{-1}(1-x^2)^{-1} \cdots (1-x^k)^{-1}$$

No. of partition of $(n-k)$ with no part greater than k.

coeff of x^n in (1) = coeff of x^{n-k} in $G_2(x)$

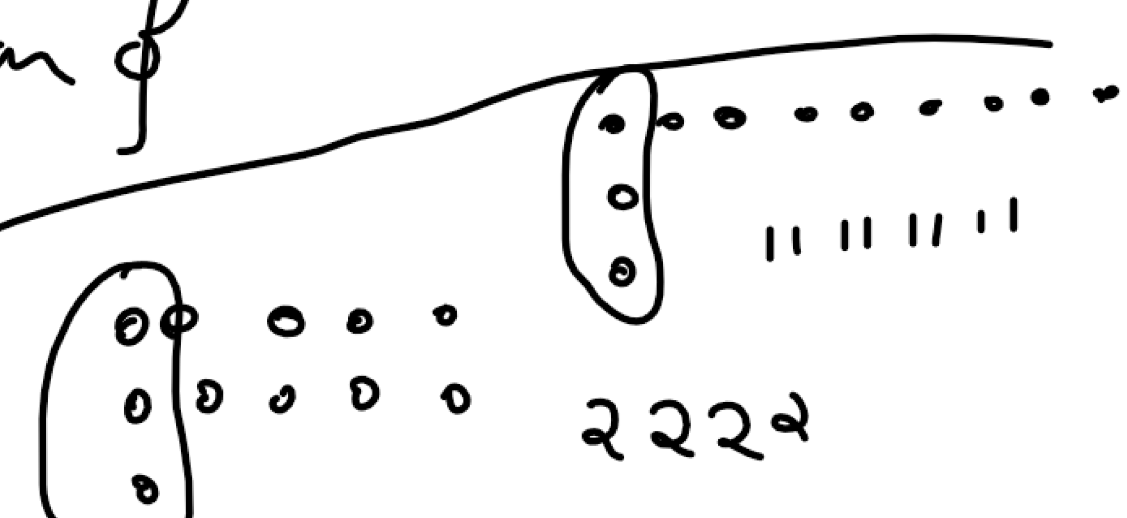
Or Consider F.G representation of a partition of n with k parts.

Delete the first colⁿ then reading col wise it represents a partition of $(n-k)$ with no part greater than k.



$\Rightarrow \ldots$

Conversely consider a partition of $(n-k)$ with no part greater than k. We add k dots as first col. This represents a partition of n with exactly k parts.



$k=3$

$n=11$ 911 ✓ 821 · 731 · 722 641 632 551 ·

$n-k = 8$ with no part greater than (3) $\Rightarrow$ 2111111, 2222, 332, 11111111 ✓

Q5. Show that the number of partitions of n in which the largest part is not repeated is equal to the number of unrestricted partitions of $n-1$, where $n > 1$.

Hint: Add one dot to the first row in the ferrers graph of $n-1$.

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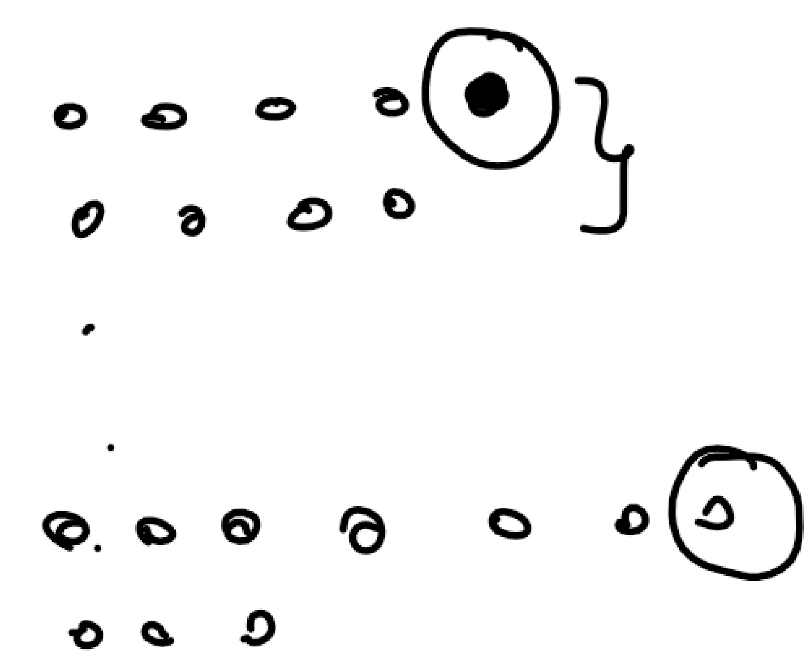
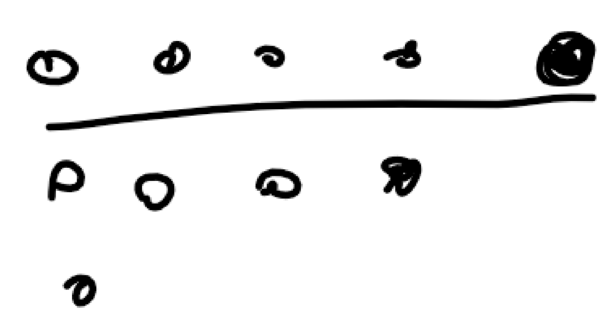


$n-1 = 9$



Ordering

(i) Given a set of items, produce a systematic ordering

(ii) Given a new item, insert it into its proper place in the ordering

(iii) Given an item in the ordering, find its position.

(iv) Given a position, determine the item in the ordering which occupies that position.

(v) Given an element in the ordering, generate the next element

(vi) Generate a random number of item in the list.

1. Lexicographic Ordering (Dictionary Ordering)

Consider the permutation of n objects say 1, 2, . . . , n. In the Lexicographic ordering of permutation, we keep 1 in the first position as far as possible, 2 in the second position as far as possible & so on.

Example : All permutations of 4 objects 1, 2, 3, 4 with no repetition in lexicographical ordering are given by,

1234	2134	3124	4123
1243	2143	3142	4132
1324	2314	3214	4213
1342	2341	3241	4231
1423	2413	3412	4312
1432	2431	3421	4321

17^{th} permutation of $n = 4$ is 3412

Immediate next permutation of 2431 is 3124

$n=5$:

12345 13 14 15 . .
12354
12435
12
12
12 13 14 15432 (24^{th})

21345 23145 (31)
23154
23415
23451
23514 (35^{th})
21 23541 (48)

31245
31254 (50)

(72)

41

(96)

51234
51243
51324
51342 (100)

54321 (120)

Q1. Given the permutation 1, 2, 3, 4, 5. Find 50th, 100th, 35th, 79th permutation in lexicographical ordering.

Soln:

50th : 31254 ✓

100th : 51342

35th : 23514

79th : 42135

Q2. Immediate next permutation of 1432 in lexicographic ordering is 2134

Q3. Immediate next permutation of 43215 in lexicographic ordering is 43251

Q4. Immediate next permutation of 431250 in lexicographic ordering is 431502

Q5. Immediate next permutation of cabdfe in lexicographic ordering is cabedf

Algorithm to find k^{th} permutation of n lexicographic ordering.

Step 1: $K-1 = c_{n-1}(n-1)! + c_{n-2}(n-2)! + \cdots + c_1 1!$

where $0 \le c_i \le i$

Step 2: Consider

$c_{n-1} \quad c_{n-2} \quad \cdots \quad c_2 \; c_1 \longleftrightarrow 1 \; 2 \; 3 \cdots n$

Step 3: Pick the element in the position $c_{n-1}+1$ as 1st element of the permutation.

Step 4: Remove c_{n-1} and the selected 1st element

Step 5: Repeat step 2 onwards.

Example:

To get 35^{th} permutation when $n=5$.

$0 \le c_i \le i$

$34 = c_4 \times 4! + c_3 \times 3! + c_2 \times 2! + c_1 \times 1$

$= 1 \times 4! + 1 \times 3! + 2 \times 2! + 0 \times 1$

$\Rightarrow c_4 = 1, \; c_3 = 1, \; c_2 = 2$ and $c_1 = 0$

$1120 \longleftrightarrow 12345 \longrightarrow 2$

$(c_4+1 = 1+1 = 2$
Pick 2nd element)

$120 \longleftrightarrow 1345 \longrightarrow 3$

$\boxed{23514}$

$c_3+1 = 1+1 = 2$

$20 \longleftrightarrow 145 \longrightarrow 5$

$0 \longleftrightarrow 14 \longrightarrow 1$

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