Lattice Theory

Set: A set is a collection of distinct objects.

Subset: Every element in a set A is also an element of a set B, then A is called a subset of B

The union of two sets A and B, denoted by $A \cup B$, is the set of all elements which belong to A or B. i.e., $A \cup B = \{x\colon \ x \in A \ \ or \ \ x \in B\}$

The intersection of two sets A and B, denoted by $A \cap B$, is the set of all elements which belong to both A and B; i.e., $\mathrm{A} \cap B = \{x\colon \ x \in A \ \ and \ \ x \in B\}$

A' or A^c or $\bar{A}$, is the set of elements which belong to U but which do not belong to A;

i.e., $\mathrm{A}^{\mathrm{c}} = \{x\colon \ x \in U, \ x \notin A\}$

The difference of A and B, denoted by $A\backslash\mathrm{B}$, is the set of elements which belong to A but do not belong to B

i.e., $A\backslash\mathrm{B} = \{x : x \in A,\ \ x \notin B\}$

If A and B are two sets, the Cartesian product (or cross product or direct product) of A and B is the set, $A \times B = \{(a, b)/a \in A \ and\ b \in B\}$

The **power set** is a set which includes all the subsets including the empty set and the original set itself.

Example:

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Relations:

A binary relation R from a set A to B is a subset of $A \times B$.

i.e., $R = \{(a, b);\ a \in A, b \in B\} \subseteq A \times B$

If $(a, b) \in R$, then we say that the element 'a is related to b' and write aRb.

Example: $A = \{1, 2, 3\}\ and\ B = \{a, b)$

$R_1 = \{(1, a), (2, a), (3, b)\}$

$R_2 = \{(1, b), (2, b)\}$ are relations from A to B

$R_3 = \{(a, 1), (a, 2)\}$ is a relation from B to A.

Identity relation: A binary relation from one set to itself is known as identity relation.

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Types of Relations:

1. Reflexive relation

A binary relation R on a set A is said to be a reflexive relation if $(a, a) \in R,$ for every $a \in A$.

Example 1: Let A be a set of +ve integers

And let us define a relation R on A such that (a, b) is in R iff $a\ divides\ b$.

Since an integer always divides itself, R is a reflexive relation

Example 2: Consider the set all straight lines in a plane and let us define a relation ‘ is parallel to’ . This relation is reflexive.

Example 3: Consider the set all straight lines in a plane and let us define a relation ‘ is perpendicular to’ . Since no straight line is $\perp r$ itself, this relation is not reflexive.

2. Symmetric Relation:

A relation R is said to be a symmetric relation if $(a, b) \in R$ implies that (b, a) is also in R, where $a, b \in A$

Example: Consider the set all straight lines in a plane

The relation ' is parallel to' and 'is perpendicular to' are symmetric relation.

3. **Transitive Relation:**

A relation R is said to be a transitive relation on A if (a, c) is in R whenever both (a, b) and (b, c) are in R, where $a, b, c \in A$

Example: The relation 'is equal to' on the set of straight lines is transitive.

4. Equivalence relation:

Let A be a nonempty set, R be a relation on A. R is said to be an equivalence relation if it is reflexive, symmetric and transitive.

5. Anti Symmetric Relation:

Let R be a relation on A. If both (a, b) and (b, a) are in R, then $a = b$.

[Or, R is said to be an antisymmetric relation if $(\mathrm{a}, \mathrm{b}) \in R \Longrightarrow (b, a) \notin R$ unless $a = b$.

Example: Let $A = \{a, b, c\}$

Let $S = \{(a, a), (b, b)\}$ and $N = \{(a, b), (a, c), (c, a)\}$ be relations on A

Here S is both symmetric and Antisymmetric

N is Neither symmetric nor Antisymmetric.

Irreflexive Relation:

If for all $a \in A,$ at least one a exists such that $(a, a) \notin R$ then R is irreflexive.

Example: The relation 'is $\perp r$ to' is an Irreflexive Relation.

Partial Ordering relations:

A relation is said to be a Partial Ordering relation if it is reflexive, antisymmetric, and transitive.

Example: Let A be a set of positive integers, and let R be a relation on A such that $(a, b) \in R$ if $a\ divides\ b$.

- Since any integer divides itself, R is a reflexive relation.
- Since if '$a\ divides\ b$' means '$b\ does\ not\ divide\ a$' unless $a = b$, R is an antisymmetric relation.
- Since if '$a\ divides\ b$' and '$b\ divides\ c$', then '$a\ divides\ c$' , R is a transitive relation

Consequently, R is a partial ordering relation.

Partially ordered set (poset):

A nonempty set A with a partial ordering relation on A is a partially ordered set (poset) and is denoted by $\langle A, R\rangle$ or (A, R).

Example: Let A be the set of positive integers and R be the binary relation on A defined by $a \leq b$ if and only if $a\ divides\ b$. Then $(A, \leq)$ is a poset.

Comparable Elements:

Let $\langle A,\ R\rangle$ be a partially ordered set. Two element $a, b \in A$ are said to be comparable if either aRb or bRa.

Example:

Two elements are said to be non-comparable if they are not comparable.

Total Ordering:

Let $\langle A, \leq \rangle$ be a poset. If Every two elements of a set are comparable, then the relation is called total ordering or total ordered relation.

A totally ordered set is also called **Chain**.

Example:

Antichain: No two elements are comparable in a set

Example:

Cover of an element:

Let $\langle A, \leq \rangle$ be a poset. An element $b \in A$ is said to cover an element $a \in A$ if $a \leq b$ **and there is no** $c \in A$ such that $a \leq c \leq b$.