21-10-2021

Q. Form the differential eq^n from $y = ax^3 + bx^2$ by eliminating arbitrary constants.

Ans:- Given $y = ax^3 + bx^2$ —— (*)

$y' = 3ax^2 + 2xb$ —— (1)

$y'' = 6ax + 2b$ —— (2)

$2 \times (1) - x \times (2) \Rightarrow 2y' - xy'' = 2xb$

$\Rightarrow b = \dfrac{2y' - xy''}{2x}$

$\therefore (1) \Rightarrow y' = 3ax^2 + 2y' - xy''$

$\Rightarrow a = \dfrac{-y' + xy''}{3x^2}$

(*) $\Rightarrow y = ax^3 + bx^2$

$= x^3 \left(\dfrac{xy'' - y'}{3x^2} \right) + \left(\dfrac{2y' - xy''}{2x} \right) x^2$

$\Rightarrow y = \dfrac{x(xy'' - y')}{3} + \dfrac{x}{2}(2y' - xy'')$

$\Rightarrow x^2y'' - 4xy' + 6y = 0$ //

Solution of differential Equations

1. Solution of differential equations

1.1. Variable separable form

Consider $\frac{dy}{dx} = f(x,y)$ — ⊛

then ⊛ is seperable if ⊛ can be expressed

as $\frac{dy}{dx} = \phi(x)\psi(y)$ or $\frac{1}{\psi(y)}\,dy = \phi(x)\,dx$.

Integrate both sides we get,

$$\int \frac{1}{\psi(y)}\,dy = \int \phi(x)\,dx + C$$

after simplification we get, $\boxed{y = h(x) + c.}$

is the required sol^n.

Problem 1.1. *Solve the differential equation* $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$ —— (*)

Ans:- (*) $\Rightarrow \frac{1}{1+y^2}\,dy = \frac{1}{1+x^2}\,dx$,

variable sep form

$$\Rightarrow \int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2} + K$$

$$\Rightarrow \tan^{-1}(y) = \tan^{-1}(x) + K$$

Problem 1.2. *Solve the differential equation* $\sec^2 x \tan y\, d\overset{x}{\cancel{y}} + \sec^2 y \tan x\, d\overset{y}{\cancel{x}} = 0$

Ans:- Given, $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

$\div$ by $\tan x \tan y$ we get

$$\int \frac{\sec^2 x}{\tan x} dx + \int \frac{\sec^2 y}{\tan y} dy = K \text{ or } \log K_1$$

$$\Rightarrow \log(\tan x) + \log(\tan y) = K \text{ or } \log K_1$$

$$\Rightarrow \log(\tan x \tan y) = K \text{ or } \log K_1$$

$$\Rightarrow \tan x \tan y = e^K = K_1$$

Problem 1.3. *Solve the differential equation* $\dfrac{dy}{dx} = \dfrac{x(2\log x + 1)}{\sin y + y\cos y}$ —— (*)

(*) $\Rightarrow$

<u>Ans:</u> $(\sin y + y\cos y)\,dy = (2x\log x + x)\,dx$,

Var. sep. form

$$\Rightarrow \int \sin y\,dy + \int y\cos y\,dy = \int 2x\log x\,dx + \int x\,dx + C$$

$$\Rightarrow -\cos y + (y\sin y + \cos y) = 2\log x\left(\frac{x^2}{2}\right) - 2\int \frac{1}{\cancel{x}}\left(\frac{x^{\cancel{2}}}{2}\right)dx + \frac{x^2}{2} + C$$

$$\Rightarrow y\sin y = x^2\log x + C$$

HW

Problem 1.4. *Solve the differential equation* $e^x(y-1)dx + 2(e^x+4)dy = 0$

Solution:

Ans:- $(e^x+4)(y-1)^2 = C$

Problem 1.5. *Solve* $\dfrac{dy}{dx} = xe^{y-x^2}$ *given that* $y(0) = 0.$

intial value problem.

Solution:

Given $\dfrac{dy}{dx} = xe^{y-x^2}$

$$\Rightarrow \frac{dy}{dx} = x.e^{y}.e^{-x^2}$$

$$\Rightarrow \frac{dy}{e^y} = xe^{-x^2}dx, \text{ var. sep form}$$

$$\Rightarrow e^{-y}dy = e^{-x^2}.xdx$$

$$\Rightarrow \int e^{-y}dy = \frac{-1}{2}\int e^{-x^2}.(-2x)dx + C$$

$$\boxed{\int e^{f(x)}.f'(x)\,dx = e^{f(x)} + C}$$

put $-x^2 = t$

$\Rightarrow -2xdx = dt$

$$\int e^{-x^2}.xdx = \int e^t dt = e^t = e^{f(x)}$$

$$\Rightarrow -e^{-y} = \frac{-1}{2}e^{-x^2} + C$$

Given, when $x=0$, $y=0$ $\Rightarrow -e^{0} = \frac{-1}{2}e^{0} + C$

$\Rightarrow C = \frac{-1}{2}$

$\therefore$ Sol^n is, $-e^{-y} = \frac{-1}{2}e^{-x^2} - \frac{1}{2} \Rightarrow 2e^{-y} = e^{-x^2} + 1$

1.2. Reducible to variable separable form

Type 1: If $\frac{dy}{dx} = f(ax + by + c)$ then, —— (1)

put $ax+by+c=t$ ✓

$\Rightarrow \frac{dt}{dx} = a + b\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{b}\left(\frac{dt}{dx} - a\right)$

$\therefore (1) \Rightarrow \frac{1}{b}\left(\frac{dt}{dx} - a\right) = f(t)$

$\Rightarrow \frac{dt}{dx} - a = bf(t)$

$\Rightarrow \frac{dt}{dx} = bf(t) + a$

$\Rightarrow \frac{dt}{bf(t)+a} = dx$, var. sep form

Type : If $\frac{dy}{dx} = \frac{(ax + by) + c}{k(ax + by) + c_1}$ then, —— (2)

put $ax+by=t$

$\Rightarrow \frac{dt}{dx} = a + b\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{b}\left(\frac{dt}{dx} - a\right)$

$\Rightarrow \frac{1}{b}\left(\frac{dt}{dx} - a\right) = \frac{t+c}{kt+c_1}$

$\Rightarrow \frac{dt}{dx} - a = \left(\frac{t+c}{kt+c_1}\right)b$ $\Rightarrow \frac{dt}{dx} = \left(\frac{t+c}{kt+c_1}\right)b + a$

Convert it into var. sep. form and proceed for the sol^n.

Problem 1.6. *Solve the differential equation* $\frac{dy}{dx} = (9x + y + 1)^2$

Solution: Given $\frac{dy}{dx} = (9x+y+1)^2$ —— (1)

put $9x+y+1 = t$ then $\frac{dt}{dx} = 9 + \frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 9$$

$$(1) \Rightarrow \frac{dt}{dx} - 9 = t^2$$

$$\Rightarrow \frac{dt}{dx} = t^2 + 9 \quad \Rightarrow \frac{dt}{t^2+9} = dx$$

, var. sep. form

$$\Rightarrow \int \frac{dt}{t^2+9} = \int dx + C$$

$$\Rightarrow \frac{1}{3} \tan^{-1}\left(t/3\right) = x + C$$

$$\Rightarrow \tan^{-1}\left(\frac{9x+y+1}{3}\right) = 3x + C'$$

Problem 1.7. *Solve the differential equation* $\frac{dy}{dx} = \cos(x + y + 1)$ —— (1)

Solution: put $x+y+1 = t \Rightarrow \frac{dt}{dx} = 1 + \frac{dy}{dx}$

(1) $\Rightarrow \frac{dt}{dx} - 1 = \cos t \Rightarrow \frac{dt}{1+\cos t} = dx$, var. sep. form

$$\Rightarrow \int \frac{dt}{1+\cos t} = \int dx + C$$

$$\Rightarrow \int \frac{dt}{2\cos^2(t/2)} = x + C$$

$$\Rightarrow \int \sec^2(t/2)\,\frac{dt}{2} = x + C$$

put $t/2 = u \Rightarrow du = \frac{dt}{2}$

$$\Rightarrow \int \sec^2(u)\,du = x + C$$

$$\Rightarrow \tan(u) = x + C$$

$$\Rightarrow \tan(t/2) = x + C \Rightarrow \underline{\underline{\tan\left(\frac{x+y+1}{2}\right) = x + C}}$$

1.3. Practice problems

Solve the following differential equation

1. $(xy + x)dx + (x^2y^2 + x^2 + y^2 + 1)dy = 0$
2. $\dfrac{y}{x}\dfrac{dy}{dx} = \sqrt{1 + x^2 + y^2 + x^2y^2}$
3. $y - x\dfrac{dy}{dx} = y^2 + \dfrac{dy}{dx}$
4. $\tan y\dfrac{dy}{dx} = \cos(x + y) + \cos(x - y)$
5. $\dfrac{dy}{dx} = \sin(x + y) + \cos(x + y)$
6. $(x + y + 1)^2\dfrac{dy}{dx} = 1$

2. Homogeneous differential equation

2.1. Homogeneous function

Let $u = f(x,y)$ be a function of x and y then u is said to be a homogenous function in x & y of degree n if

$$u = x^n \phi(y/x) \quad \text{or} \quad u = y^n \psi(x/y)$$

Eg:- $f(x,y) = x^{1/2} + y^{1/2}$

Here $f(x,y) = x^{1/2}\left(1 + \frac{y^{1/2}}{x^{1/2}}\right) = x^{1/2}\underbrace{\left(1 + \sqrt{y/x}\right)}_{\phi(y/x)}$

$\therefore f(x,y)$ is a homog. funct of deg $1/2$.

$$f(x,y) = y^{1/2}\left(\frac{x^{1/2}}{y^{1/2}} + 1\right) = y^{1/2}\underbrace{\left(\sqrt{x/y} + 1\right)}$$

$$= y^{1/2}\,\psi(x/y)$$

Note:-
Replace x by λx and y by λy in $f(x,y)$ then; if $f(\lambda x, \lambda y) = \lambda^n f(x,y)$ we say

that $f(x,y)$ is a homo. funct in x & y of deg. n.

2.2. Homogeneous differential equation

A d.e. of the form $\boxed{\dfrac{dy}{dx} = \dfrac{f(x,y)}{g(x,y)}}$

where $g(x,y) \neq 0$, $f(x,y)$ & $g(x,y)$ are homogenous functions of same degree, such d.e's are called homogenous d.e.

Sol^n of homo. d.e.

put $y = vx \Rightarrow v = y/x$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substitute it in the given eq^n and reduce the diff. eq^n to var. sep. form and integrate it for the req'd sol^n.

Problem 2.1. *Solve the differential equation* $(x^2 - y^2)\ dx = xy\ dy$

Ans:- $\dfrac{dy}{dx} = \dfrac{x^2-y^2}{xy}$, homo. d.e. ——— ①

put $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$① \Rightarrow v + x\frac{dv}{dx} = \frac{1-v^2}{v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-v^2}{v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-2v^2}{v}$$

$$\Rightarrow \left(\frac{v}{1-2v^2}\right)dv = \frac{dx}{x}$$, var. sep form

$$\Rightarrow \int \frac{v}{1-2v^2}dv = \int \frac{dx}{x} + \log k$$

$$\Rightarrow \frac{-1}{4}\int \frac{-4v}{1-2v^2}dv = \log x + \log k$$

$$\Rightarrow \frac{-1}{4}\log(1-2v^2) = \log(xk)$$

$y = vx$
$\Rightarrow v = y/x.$

Solution:

$$\Rightarrow -\log\left(1 - 2\frac{y^2}{x^2}\right) = 4\log(xK)$$

$$\Rightarrow -\log\left(\frac{x^2 - 2y^2}{x^2}\right) = \log(xK)^4$$

$$\Rightarrow \log\left(\frac{x^2}{x^2 - 2y^2}\right) = \log(xK)^4$$

$$\Rightarrow \frac{x^2}{x^2 - 2y^2} = (xK)^4$$

$$\Rightarrow \frac{x^2}{x^2 - 2y^2} = Cx^4 \quad (\text{where } C = K^4)$$

$$\Rightarrow \underline{\underline{x^2(x^2 - 2y^2) = \frac{1}{C} = C'}}, \text{ req'd sol}^n$$

Problem 2.2. *Solve the differential equation* $x^2y\ dx - (x^3+y^3)\ dy = 0$

Ans:- $\frac{dy}{dx} = \frac{x^2y}{x^3+y^3}$. hom. d. e

Sol^n:- $-\frac{x^3}{3y^3} + \log y = C$

Problem 2.3. *Solve the differential equation* $\dfrac{dy}{dx} = \dfrac{y}{x - \sqrt{xy}}$

Ans:-

Sol^n:- $2\sqrt{x/y} + \log y = K$

2.3. Practice problems

Solve the following differential equation

1. $(y^3 - 3x^2y)dx - (x^3 - 3xy^2)dy = 0$
2. $y - x\dfrac{dy}{dx} = x + y\dfrac{dy}{dx}$
3. $xdy - ydx = \sqrt{x^2 + y^2}dx$
4. $y\ dy + \sin^2\left(\frac{x}{y}\right)[x\ dy - y\ dx] = 0$

3. Reducible to homogeneous differential equation (Non-homogeneous differential equations)

Consider the d.e. $\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1}$ —— (*)

<u>Case(i)</u> :- when $\frac{a}{a_1} \neq \frac{b}{b_1}$

put $x = X+h$ and $y = Y+k$

$dx = dX$ & $dy = dY$

then (*) $\Rightarrow$ $\frac{dY}{dX} = \frac{a(X+h)+b(Y+k)+c}{a_1(X+h)+b_1(Y+k)+c_1}$

$\Rightarrow \frac{dY}{dX} = \frac{aX+bY+(ah+bk+c)}{a_1X+b_1Y+(a_1h+b_1k+c_1)}$ —— (1)

Take $\left.\begin{aligned} ah+bk+c&=0 \\ \text{\&}\ a_1h+b_1k+c_1&=0 \end{aligned}\right\}$ $h = ?$ $k = ?$

$\therefore$ (1) becomes $\boxed{\frac{dY}{dX} = \frac{aX+bY}{a_1X+b_1Y}}$, a homo. d.e.

put $Y = vX$ ✓ Y interms of X

Problem 3.1. *Solve the differential equation* $\frac{dy}{dx} = \frac{y+x-2}{y-x-2}$, non-homo. d.e. — (*)

Solution: put $x = X+h$ and $y = Y+k$ $\Rightarrow dx = dX$, $dy = dY$

$$\therefore (*) \Rightarrow \frac{dY}{dX} = \frac{Y+X+(h+k-2)}{Y-X+(-h+k-2)} \quad \text{——} (1)$$

Take $h+k-2=0$, $-h+k-2=0$ $\Rightarrow 2k=4 \Rightarrow k=2$ $\therefore h=0$

$$\therefore (1) \Rightarrow \frac{dY}{dX} = \frac{Y+X}{Y-X}, \text{ is a homog. d.e.} \quad \text{——} (2)$$

put $Y = vX \Rightarrow \frac{dY}{dX} = v + X\frac{dv}{dX}$

$$\therefore (2) \Rightarrow v + X\frac{dv}{dX} = \frac{v+1}{v-1}$$

$$\Rightarrow X\frac{dv}{dX} = \frac{v+1}{v-1} - v$$

$$\Rightarrow X\frac{dv}{dX} = \frac{v+1-v^2+v}{v-1}$$

$$\Rightarrow \frac{v-1}{1+2v-v^2}dv = \frac{dX}{X}, \text{ var. sep form}$$

$$\Rightarrow \int \frac{v-1}{1+2v-v^2}dv = \int \frac{dX}{X} + \log C$$

Solution:

$$\Rightarrow \frac{-1}{2}\int \frac{2-2v}{1+2v-v^2}\,dv = \int \frac{dX}{X} + \log C$$

$$\Rightarrow \log(1+2v-v^2) = -2\log X + \log C'$$

$$\Rightarrow \log\left(1+2\frac{Y}{X} - \frac{Y^2}{X^2}\right) + \log X^2 = \log C'$$

$$\Rightarrow \log\left(\left(\frac{X^2+2YX-Y^2}{X^2}\right)\cdot X^2\right) = \log C'$$

$$\Rightarrow X^2+2XY-Y^2 = C'$$

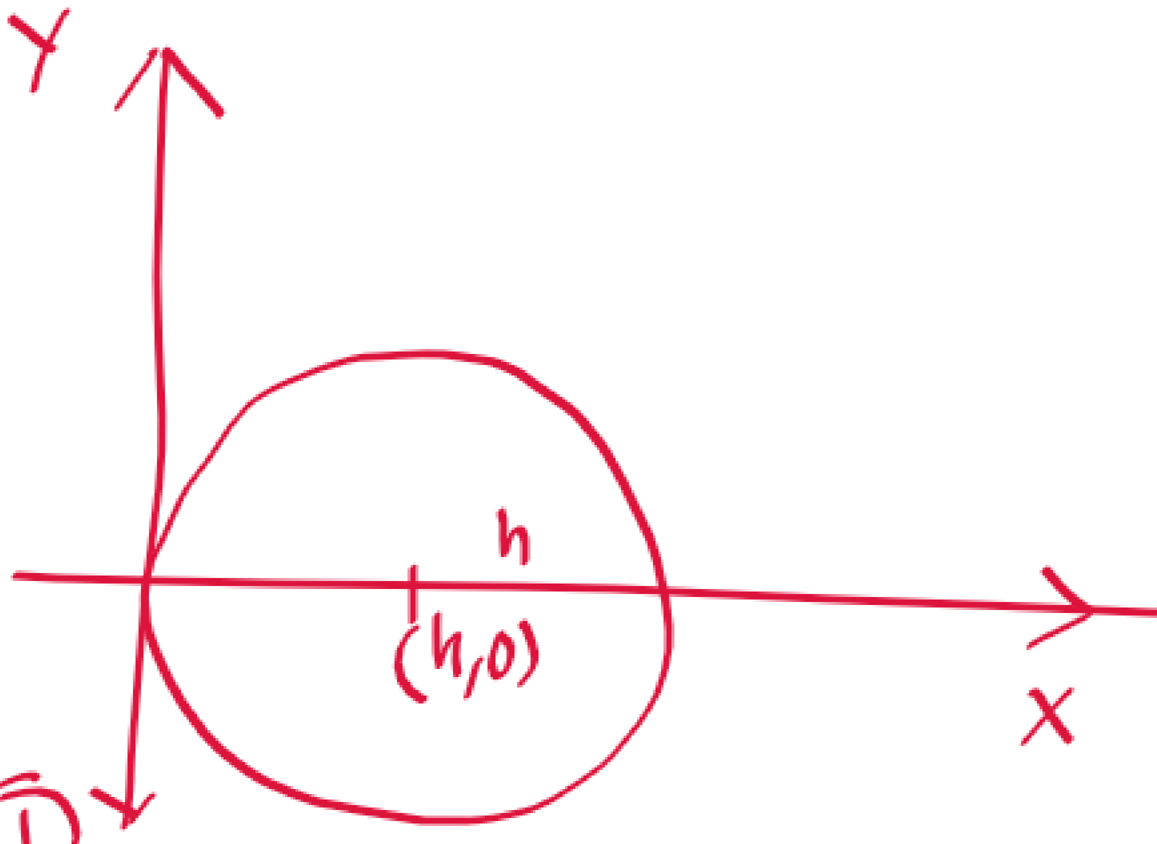
put $X = x-h = x$ & $Y = y-k = y-2$

$\therefore$ Req'd solⁿ is

$$x^2+2x(y-2)-(y-2)^2 = C'$$

Q. Form the d.e. of all circles touching the axis of y at the origin and centre on the x-axis.

Ans:- Eq^n of the given circle is



$(x-h)^2 + y^2 = h^2$

$\Rightarrow x^2 - 2hx + y^2 = 0$ —— ①

Differentiate both sides w.r.t. x we get,

$$2x - 2h + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad h = x + y\frac{dy}{dx}$$

$$① \Rightarrow \quad x^2 - 2\left(x + y\frac{dy}{dx}\right)x + y^2 = 0$$

$$\Rightarrow -x^2 - 2yx\frac{dy}{dx} + y^2 = 0$$

$$\Rightarrow x^2 + 2yxy' - y^2 = 0$$