Partially ordered set :-

$(A, R) \rightarrow$ A : nonempty set
R : Relation satisfying i) Reflexive, ii) Antisym, iii) Transive } Partial ordering relation

ex:-

① $(\mathbb{Z}^+, \leq)$ where $\mathbb{Z}^+ \rightarrow$ set of all +ve integers
$\leq \rightarrow$ $a \leq b$ iff a is less than & equal to b

② $(\mathbb{Z}^+, \geq)$ where $\mathbb{Z}^+ \rightarrow$ set of all +ve integers
$\geq \rightarrow$ $a \geq b$ iff a is greater than & equal to b

③ $(\mathbb{Z}^+, |)$ where $\mathbb{Z}^+ \rightarrow$ set of all +ve integers
$| \rightarrow$ $a|b$ iff a decides b

④ $(\mathbb{Z}^+, R)$ where $R \rightarrow$ aRb iff a is multiple of b

⑤ $(P(A), \subseteq)$ where $P(A) \rightarrow$ Powerset of a nonempty set A
$\subseteq \rightarrow$ $S_1 \subseteq S_2$ iff S_1 is contained in S_2

$A = \{a, b, c\}$

$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{c,a\}, \{a,b,c\}\}$

$(P(A), \subseteq)$ forms a poset

comparable elements

Let $(A, \leq)$ be a poset. Two elements a and b in A are said to be comparable if either $a \leq b$ (a is related to b)

& $b \leq a$
(b is related to a)

① Ex:- $(P(A), \subseteq)$, where $A = \{a, b, c\}$

$P(A) = \{ \phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{c,a\}, \{a,b,c\} \}$

* The elements $\{a\}$ and $\{a,b\}$ are comparable ($\because \{a\} \subseteq \{a,b\}$)

* The elements $\{a\}$ and $\{b\}$ are not comparable ($\because$ Neither $\{a\} \subseteq \{b\}$ nor $\{b\} \subseteq \{a\}$

* ex2 :- $(Z^+, |)$ is a poset

The elts 2, 4 are comparable since $2|4$

The elts 3, 4 are not combarable, neither $3|4$ nor $4|3$.

* $(Z^+, \leq)$ where $\leq$: less than & equal to

In the above poset, every two elements are comparable

Totol ordering:

If every two elements of the set are comparable, then the relation is called the total ordering & totally ordered relation.

Chain:

A totally ordered set

OR

Let $(A, \leq)$ be a poset. A subset B of A is called a chain if every 2 elts of B are comparable.

ex:- $(P(A), \subseteq)$ is a poset

$A = \{a, b, c\}$

$P(A) = \left\{ \begin{array}{l} \phi, \{a\}, \{b\}, \{c\} \\ \{a,b\}, \{b,c\}, \{c,a\} \\ \{a,b,c\} \end{array} \right\}$

$B = \{ \phi, \{a\}, \{a,b\}, \{a,b,c\} \}$ is a chain

$B_2 = \{ \phi, \{a\}, \{b\}, \{a,b\} \}$ is not a chain

Bcz $\{a\}$ & $\{b\}$ are not comparable.

Antichain

A set in which no two elements are comparable.

$C = \{ \{a\}, \{b\}, \{c\} \}$ is an antichain

$C_2 = \{ \phi, \{a\}, \{b\}, \{c\} \}$ is neither a chain nor an antichain

cover of an element

Let $(A, \leq)$ be a poset. An element $b \in A$ is said to cover the element $a \in A$ if $a \leq b$ and there is no other element $c \in A$ s.t $a \leq c \leq b$

ex:- In $(Z^+, |)$ where $a|b$ if a divides b

i) 4 covers 2, $2|4$, there no other elt 'c' s.t $2|c$ and $c|4$

ii) 8 does not cover 2, $2|8$, but there is an elt 4 s.t $2|4$ and $4|8$

Hasse Diagram

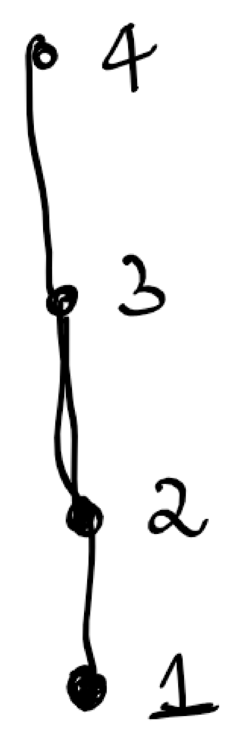
A poset $(A, \leq)$ can be represented by a diagram called Hasse diagram.

i) Each elt of A is represented as either a dot or as a small circle

ii) Consider any two elements $x, y \in A$, the elt x is drawn below y if $x \leq y$. A line is drawn b/wn x and y if y covers x. If y doesnt cover x, then donot draw the line connecting x & y

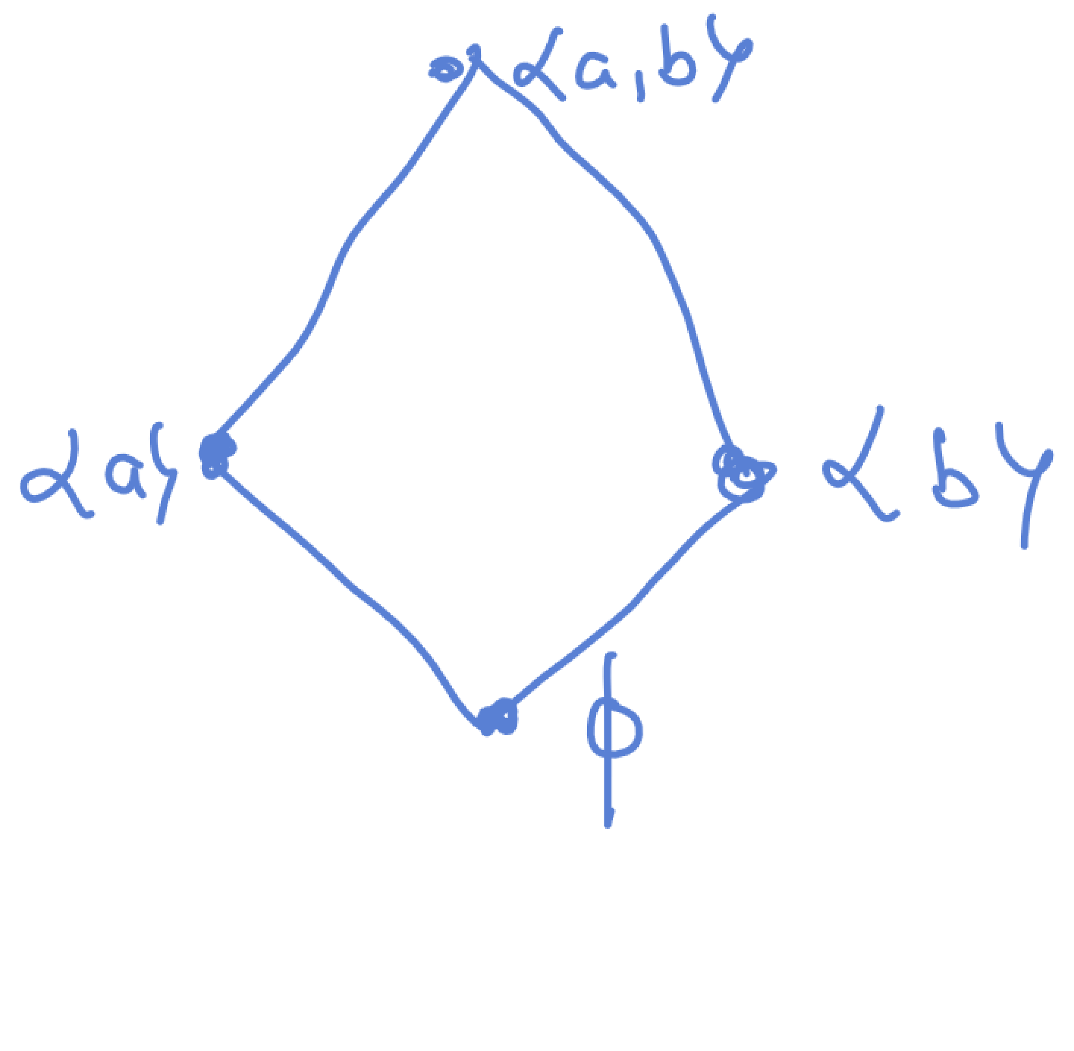
* $X = \{1, 2, 3, 4\}$ and $\leq$: less than & equal to

$(X, \leq)$ is a poset.

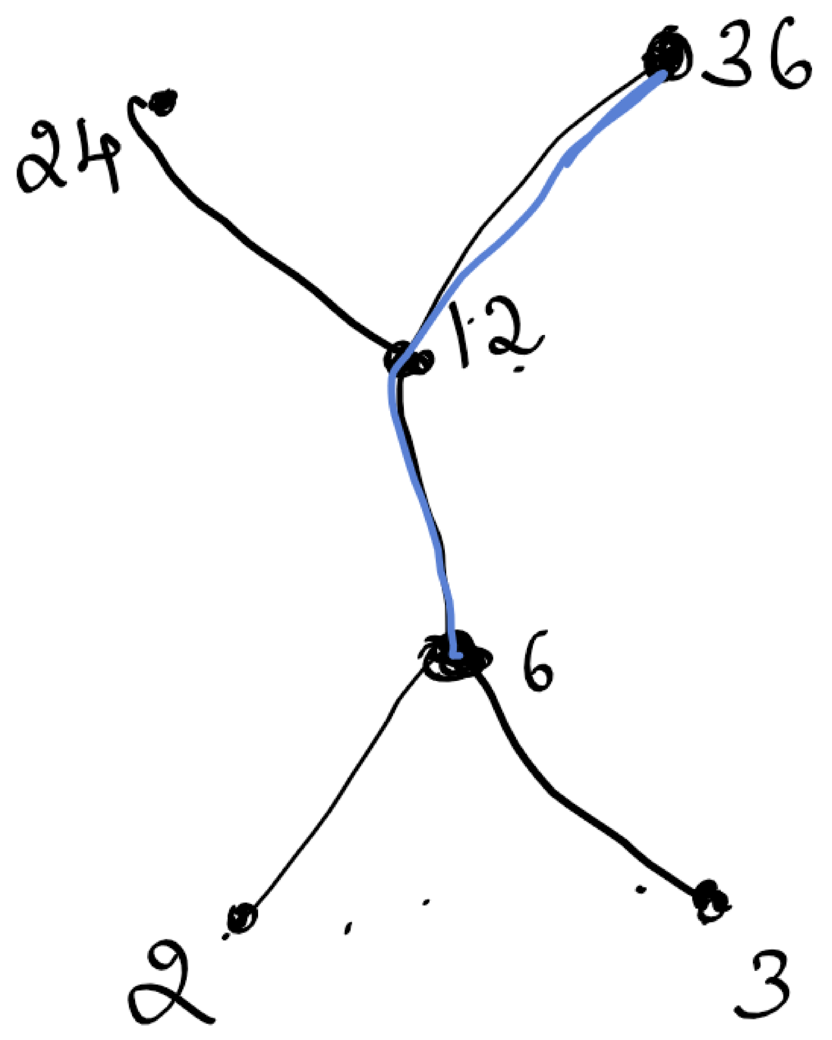


* $(P(A), \subseteq)$ where $A = \{a, b\}$

$P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$



(3) $A = \{2, 3, 6, 12, 24, 36\}$ and $(A, |)$



maximal : 24, 36
minimal : 2, 3

* 6 covers 3

$3|6$ & there is no intermediate elt 'c' s.t $3|c$ & $c|6$

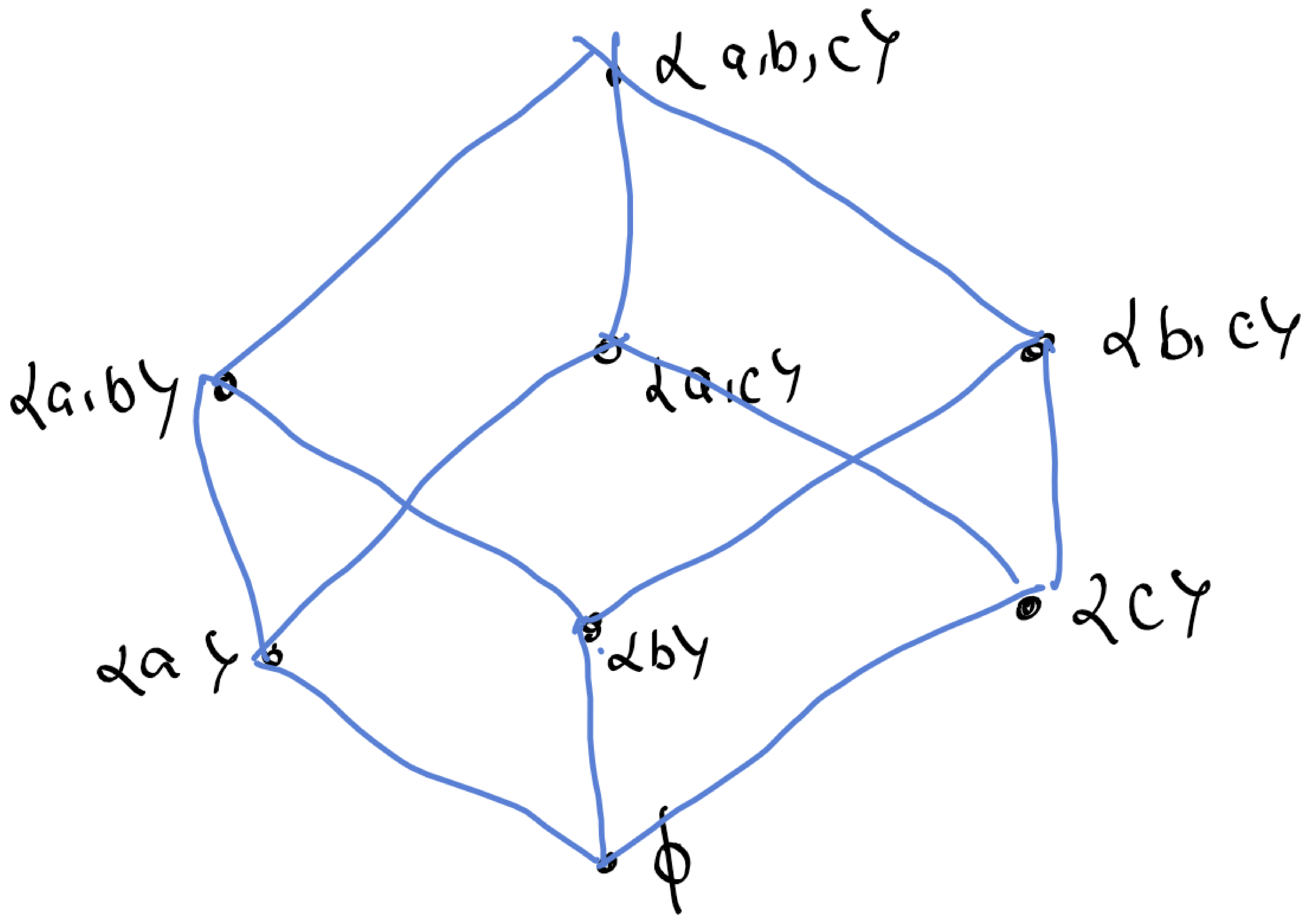
* 3 doesnot cover 12

$3|12$, but there is an intermediate elt 6 s.t $3|6$ & $6|12$

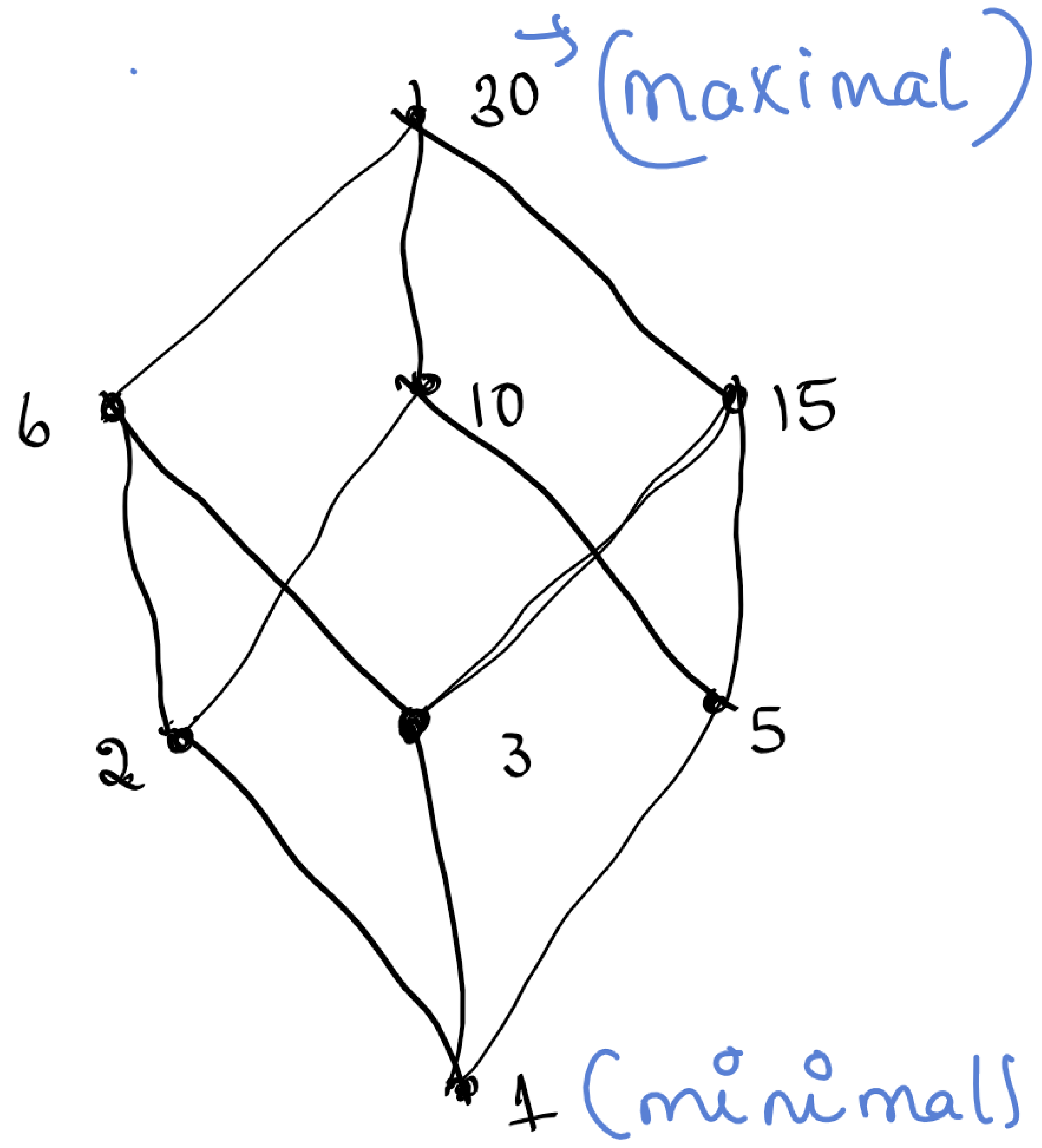
* 6 & 36 are related, but 36 doesnot cover 6

(4) $(P(A), \subseteq)$ where $A = \{a, b, c\}$

$$P(A) = \left\{ \phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{c,a\}, \{a,b,c\} \right\}$$



* $S = \{1, 2, 3, 5, 6, 15, 10, 30\}$ $(S, |)$



maximal element:

Let $(A, \leq)$ be a poset. An elt $a \in A$ is said to be a maximal element of A if there is no other element $b \in A$ s.t $a \neq b$ & $a \leq b$

An elt $a \in A$ is said to be a minimal elt of A if there is no other elt $b \in A$ s.t $a \neq b$ & $b \leq a$

* Let A be a nonempty set, $(P(A), \subseteq)$ be a poset

Then the maximal elt $\rightarrow$ A

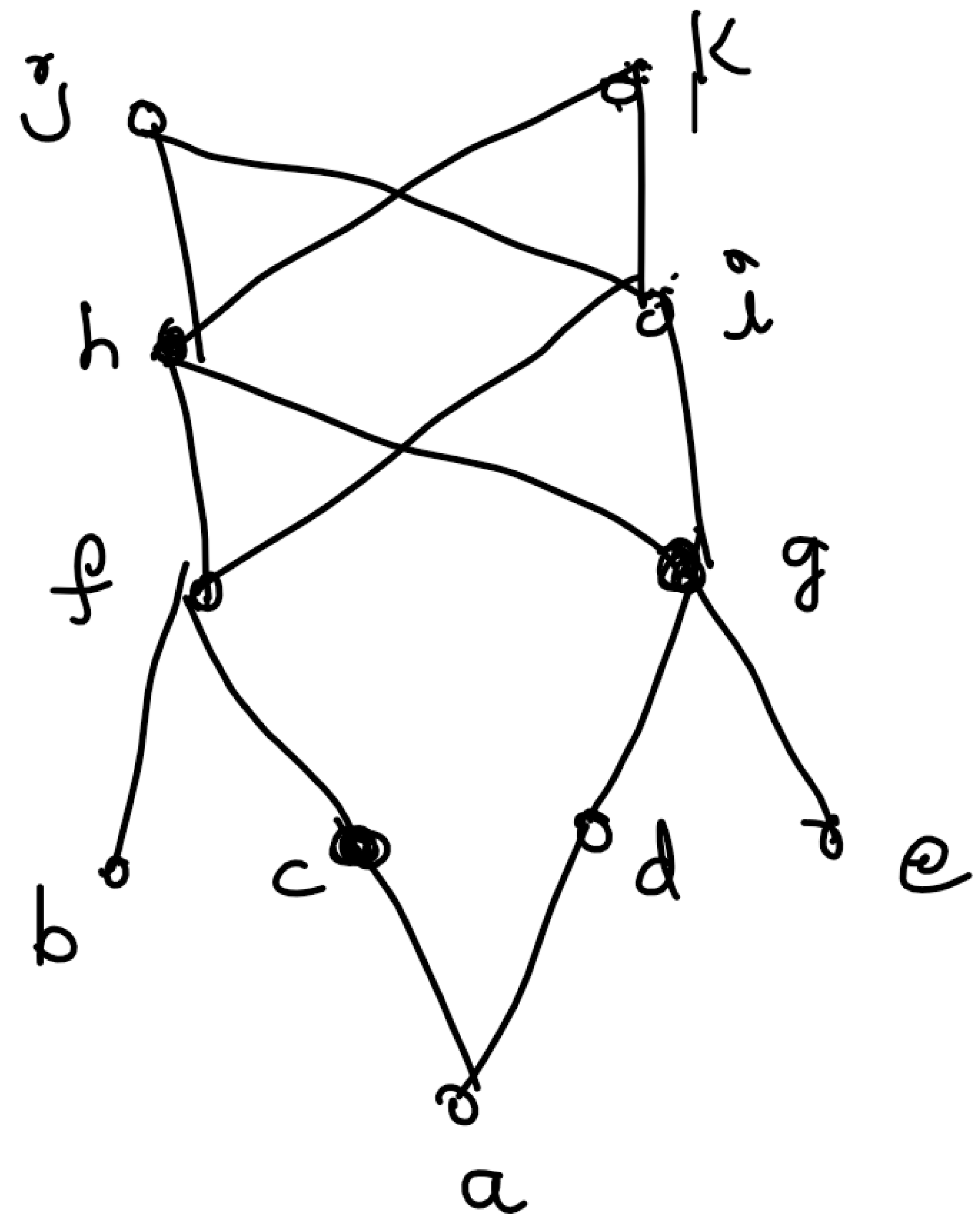
minimal elt $\rightarrow$ ϕ

* $(N, |)$

minimal elt $\longrightarrow$ 1

Upper bound of two elts

Let $(A, \leq)$ be a poset. and let $a, b \in A$. An elt c is said to be an upper bound of a & b if $a \leq c$ & $b \leq c$



* K is an upper bound of h & g

$h \leq k, \quad g \leq k$

* K is an upper bound of c & g

$c \leq K \quad g \leq K$

* f is an upper bound of f & b

$f \leq f \quad b \leq f$

Lower bound

An elt c is said to be a lower bound of a & b if $c \leq a, \quad c \leq b$