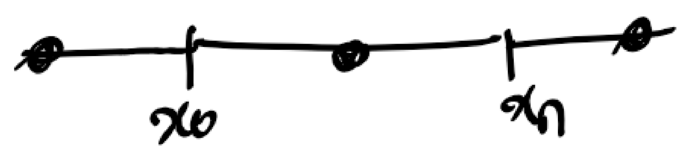
Numerical Methods

Introduction:-

Let $y = f(x)$ in $x_0 \le x \le x_n$.

The above expression means corresponding to every value of x in the range x_0 to x_n, there exists one or more values of "y". Assuming that $f(x)$ is single valued, continuous and that it is known explicitly then the values of y corresponding to certain given values of x say $x_0, x_1, x_2 \cdots x_n$ can easily be computed & tabulated.

The central problem of numerical analysis is the converse of this given the set of tabular values $(x_0, y_0), (x_1, y_1), \cdots (x_n, y_n)$ satisfying the relation $y = f(x)$ where the explicit nature of $f(x)$ is not known. It is required to find a simpler function say $\phi(x)$ which approximates $f(x)$ such that $f(x)$ & $\phi(x)$ agrees at the set of tabulated points.



Interpolation:- It is the process of finding the value of "y" for any x b/w x_0 & x_n.

Extrapolation:- It is the process of finding the value of "y" outside the given range.

Weierstrass Theorem:-

If $f(x)$ is continuous function in $x_0 \le x \le x_n$, then $\forall\ \epsilon > 0, \exists$ a polynomial $p(x)$ $\ni' |f(x) - p(x)| < \epsilon\ \forall\ x$ in $x_0 \le x \le x_n$.

Remark:-

Through two distinct points we can construct a unique polynomial of degree one.

Through three distinct points we can construct a unique polynomial degree two. eg: Parabola.

In general through $(n+1)$ distinct points we can construct a unique polynomial of degree n.

Interpolation with Equally Spaced Points

To construct the interpolating polynomial we use the "finite difference" concept.

Suppose that $\boxed{x_i = x_0 + ih}$; $h > 0$, $i = 0, 1, 2, \cdots n$.

i.e The values of x, all equally spaced. The following are three types of finite differences.

1) Forward difference ✓
2) Backward difference ✓
3) Central difference. ✓

x_0	y_0	
		y_1-y_0
x_1	y_1	
		y_2-y_1
x_2	y_2	
$\vdots$		$\vdots$
		y_n-y_{n-1}
x_n	y_n	

Forward difference:-

The differences $y_1-y_0,\ y_2-y_1,\ y_3-y_2\ ---\ y_n-y_{n-1}$ are called first order forward difference of y's are respectively denoted by

$\Delta y_0, \Delta y_1, \Delta y_2, --- \Delta y_n$. i.e $\Delta y_n = y_{n+1}-y_n$.

Δ:- Forward difference operator.

The differences of the first order forward differences are called second order forward differences and are called second order forward differences and are denoted by $\Delta^2 y_0, \Delta^2 y_1, \Delta^2 y_2, --- \Delta^2 y_{n-1}$

Δy_0	
	$\Delta y_1-\Delta y_0$
Δy_1	
	$\Delta y_2-\Delta y_1$
Δy_2	
$\vdots$	$\vdots$
Δy_n	

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$
$$= (y_2-y_1)-(y_1-y_0)$$
$$\Delta^2 y_0 = y_2+y_0-2y_1$$

$$\therefore\ \Delta^2 y_1 = \Delta y_2 - \Delta y_1$$
$$\vdots$$
$$\Delta^2 y_n = \Delta y_{n+1} - \Delta y_n$$
$$= y_{n+2}-y_{n+1}-(y_{n+1}-y_n)$$
$$= y_{n+2}-2y_{n+1}+y_n.$$

In general

$$\Delta^k y_n = \Delta^{k-1} y_{n+1} - \Delta^{k-1} y_n.$$

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_0	y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$	$\Delta^5 y_0$	$\Delta^6 y_0$
x_1	y_1	Δy_1	$\Delta^2 y_1$	$\Delta^3 y_1$	$\Delta^4 y_1$	$\Delta^5 y_1$	
x_2	y_2	Δy_2	$\Delta^2 y_2$	$\Delta^3 y_2$	$\Delta^4 y_2$		
x_3	y_3	Δy_3	$\Delta^2 y_3$	$\Delta^3 y_3$			
x_4	y_4	Δy_4	$\Delta^2 y_4$				
x_5	y_5	Δy_5					
x_6	y_6						

These differences can be tabulated and table so obtained is called forward difference table.

Backward differences:-

The differences $y_1 - y_0, y_2 - y_1, y_3 - y_2, \ldots y_n - y_{n-1}$ are called first order backward differences of 'y' and are respectively denoted by

$$\nabla y_1, \nabla y_2, \ldots \nabla y_n.$$

The operator (∇) is called backward difference operator.

The differences of the first order backward differences are called second order backward differences and are called second order backward differences and are denoted by $\nabla^2 y_1, \nabla^2 y_2, \nabla^2 y_3 \ldots$

One can define the n^{th} order backward difference i.e

$$\nabla^n y_k = \nabla^{n-1} y_k - \nabla^{n-1} y_{k-1}$$

$n = 1, 2, \ldots \infty.$

Relation b/w Δ & ∇ is given by

$$\nabla y_n = \Delta y_{n-1}$$

x	y	∇	∇^2	∇^3	∇^4
x_0	y_0				
		∇y_1			
x_1	y_1		$\nabla^2 y_1$		
		∇y_2		$\nabla^3 y_1$	
x_2	y_2		$\nabla^2 y_2$		$\nabla^4 y_1$
		∇y_3		$\nabla^3 y_2$	
x_3	y_3		$\nabla^2 y_3$		
		∇y_4			
x_4	y_4				

Alternate notations for $y = f(x)$

$$\left.\begin{aligned} \Delta y &= f(x+h) - f(x) \\ \nabla y &= f(x) - f(x-h). \end{aligned}\right\}$$

Shift operator:-

The shift operator is denoted by "E" & defined as

$$E y_n = y_{n+1} \quad , \quad E(f(x)) = f(x+h)$$

Note :-

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Delta f(x) = E(f(x)) - f(x)$$

$$\Delta f(x) = (E-1) f(x).$$

$$\Rightarrow \Delta = E - 1$$

$$\boxed{E = 1 + \Delta}$$

$f(x)$
$f(x+h)$
$f(x+2h)$
$f(x+3h)$
$\vdots$

Also $\quad E^2 f(x) = f(x+2h).$

In general $\quad E^n f(x) = f(x+nh)$

$$\boxed{E^n y_m = y_{m+n}.}$$

Inverse shift operator :-

The inverse shift operator is denoted by E^{-1} & defined as

$$E^{-1} y_n = y_{n-1}.$$

$$E^{-1}(f(x)) = f(x-h).$$

y_0
y_1
y_2
y_3
$\vdots$
y_{n-2}
y_{n-1}
y_n

$$\nabla(f(x)) = f(x) - f(x-h)$$
$$= f(x) - E^{-1}(f(x))$$

$$\nabla f(x) = (1 - E^{-1}) f(x)$$

$$\Rightarrow \nabla = 1 - E^{-1} \Rightarrow \boxed{E^{-1} = 1 - \nabla.} \quad \& \quad \boxed{E = 1 + \Delta}$$

Also

$$E^{-1}(f(x)) = f(x-h)$$

$$E^{-2}(f(x)) = f(x-2h).$$

In general $\quad E^{-n} f(x) = f(x-nh)$ or $E^{-n} y_m = y_{m-n}.$ //

Properties of finite difference

1) Linearity property: All finite difference are linear i.e.

For any two constants a, b & for any two functions $f(x)$, $g(x)$ we have

$$\Delta\left(af(x) \pm b\,g(x)\right) = a\,\Delta f(x) \pm b\,\Delta g(x).$$

2) Index law:-

If m, n are +ve integers then $\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x)$.

3) The first order difference of a polynomial of degree 'n' is a polynomial of degree '$n-1$', the n^{th} order difference is a constant & $(n+1)^{th}$ order difference of the polynomial of n^{th} degree is zero.

Proof:-

Let $y = f(x) = ax^n + bx^{n-1} + cx^{n-2} + \text{----} + kx + l$ be a polynomial of degree 'n' $\rightarrow (1)$

Giving an increment h to x we get

$$y + \Delta y = a(x+h)^n + b(x+h)^{n-1} + \text{------} + k(x+h) + l \rightarrow (2)$$

where $h = \Delta x$

$(2) - (1)$

$$\Delta y = y + \Delta y - y = a\left[(x+h)^n - x^n\right] + b\left[(x+h)^{n-1} - x^{n-1}\right] + \text{----} + kh.$$

$$\Delta y = a\left\{\cancel{x^n} + nhx^{n-1} + \frac{n(n-1)}{2!}h^2 x^{n-2} + \text{----} - \cancel{x^n}\right\} + b\left[\cancel{x^{n-1}} + (n-1)x^{n-2}h + \frac{(n-1)(n-2)}{2!}x^{n-3}h^2 + \text{---} - \cancel{x^{n-1}}\right] + \text{----} + kh$$

$$\Delta y = anh\,x^{n-1} + \left\{ah^2\frac{n(n-1)}{2!} + hb(n-1)\right\}x^{n-2} + \text{------}$$

$$\Delta y = anh\,x^{n-1} + b'x^{n-2} + c'x^{n-3} + \text{---} + k'x + l \rightarrow (3)$$

$\therefore$ The first difference of a polynomial of n^{th} degree is a polynomial of degree $(n-1)$.

To find the second difference, we give x an increment $\Delta x = h$,

$$\Delta(y+\Delta y) = \Delta y + \Delta(\Delta y) = anh(x+h)^{n-1} + b'(x+h)^{n-2} + c'(x+h)^{n-3} + \cdots + k'(x+h) + l' \quad \text{(4)}$$

(4) − (3)

$$\cancel{\Delta y} + \Delta^2 y - \cancel{\Delta y} = anh\left[(x+h)^{n-1} - x^{n-1}\right] + b'\left[(x+h)^{n-2} - x^{n-2}\right] + \cdots + k'h.$$

$$\Delta^2 y = anh\left[\left\{\cancel{x^{n-1}} + (n-1)h(x)^{n-2} + \frac{(n-1)(n-2)}{2!}h^2 x^{n-3} \cdots - \cancel{x^{n-1}}\right\}\right] +$$

$$b'\left\{\cancel{x^{n-2}} + (n-2)h\,x^{n-3} + \frac{(n-2)(n-3)h^2}{2}x^{n-4} + \cdots - \cancel{x^{n-2}}\right\} + \cdots + k'h$$

$$\Delta^2 y = an(n-1)h^2 x^{n-2} + b'' x^{n-3} + c'' x^{n-4} + \cdots + k'' x + l'' \quad (*)$$

⇒ The second order difference of a polynomial of degree 'n' is a polynomial of degree $(n-2)$.

In general

$$\Delta^n y = \{a\,n(n-1)(n-2)\cdots 2\cdot 1\}h^n x^0$$

$$\Delta^n y = a\,n!\,h^n \quad \text{which is a constant}$$

⇒ Since n^{th} difference is therefore constant & all higher differences are zero.