Q: Show that group of prime order is abelian.

If we prove that prime order group is cyclic, then it is abelian.

Let $O(G) = p$, p is a prime number.

then, there is an element $a \in G$ and $a \neq e$.

Let $H = (a)$ $\{$ i.e., $H = \{a^n / n \in Z\}\}$

The H is a subgroup of G.

By Lagrange's theorem, $O(H) \mid O(G)$

then, either $O(H) = 1$ or $O(H) = p$

Since $a \neq e$, $O(H) \neq 1$

$$O(H) = p = O(G)$$

$$\Rightarrow H = G$$

i.e., $(a) = G$

Hence G is cyclic.

$\therefore$ G is abelian

The converse is not true.

i.e., a cyclic group need not be a prime order group.

Eg: $G = \{1, -1, i, -i\}$

$$O(G) = 4$$

Q: Show that any group with atmost 5 elements is abelian.

Groups with order 2, 3 or 5 (prime order) are abelian.

If $O(G) = 1$ then $G = \{e\}$ which is abelian.

Let $\underline{O(G)} = 4$

then, there is an element $a \neq e \in G$

Let $H = (a)$

$O(H) = 1$ or 2 or 4 $\quad$ ($\because$ By Lagrange's thm $O(H) \mid O(G)$, $O(H) \mid 4$)

But $a \neq e$, $\therefore O(H) \neq 1$

If $O(H) = 4 = O(G) \Rightarrow G = H$.

then $G = (a)$, Hence G is cyclic & it is abelian.

If $O(H) = 2$, for every $a \neq e \in G$

then every element is its own inverse,

hence G is abelian

Q: Show that order of an element divides order of the group.

Let G be a finite group.

$a^n = e$

$O(a) = n$

Let $a \in G$

Let 'n' be the order of a

i.e., $O(a) = n$

Let $H = (a)$

Since H is cyclic group with generator (a).

$O(H) = n.$ ($\because$ If H is cyclic group with generator (a), then $O(a) = O(H)$)

By Lagrange's theorem,

$$O(H) \mid O(G)$$

$$n \mid O(G) \Rightarrow \underline{O(a) \mid O(G)}$$

Q: If G is a finite group and $a \in G$, the prove that $a^{O(G)} = e$.

Since order of an element divides the order of group.

i.e., $O(a) \mid O(G)$

$$\frac{O(G)}{O(a)} = m$$

$$O(G) = m\,O(a)$$

$$a^{O(G)} = a^{mO(a)} = \left[a^{O(a)}\right]^m = e^m = e$$

$a^{O(a)} = a^n = e$ if $O(a) = n$