Exact differential Equations

1. Reducible to exact differential equation - Continued

Type 1.1. *Consider the non exact equation $Mdx + Ndy = 0$. If*

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x), \text{ a function of } x \text{ alone.}$$

Then, the integrating factor is,

$$I.F. = e^{\int f(x)}dx.$$

→ Consider $Mdx + Ndy = 0$ ——— (1)

If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ (1) is not exact

(*) $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$, a fnct of 'x' alone

Then $I.f = e^{\int f(x) dx}$

(*) $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(y)$, a fnct of 'y' alone

Then $I.f = e^{-\int g(y).dy}$

Type 1.2. *Consider the non exact equation $Mdx + Ndy = 0$. If*

$$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(y), \text{ a function of } y \text{ alone.}$$

Then, the integrating factor is,

$$I.F. = e^{-\int g(y)} dy.$$

Problem 1.3. *Solve* $\left(xy^2 - e^{\frac{1}{x^3}}\right)dx - x^2ydy = 0$ —— ①

Ans:- In ①, $M = xy^2 - e^{1/x^3}$, $N = -x^2y$

$$\frac{\partial M}{\partial y} = x(2y) - 0 \qquad \frac{\partial N}{\partial x} = -2xy$$
$$= 2xy$$

ie; $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ ① is not exact.

$$\therefore \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4xy$$

$$\therefore \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{-x^2y} \times 4xy = \frac{-4}{x} = f(x)$$

$$\therefore I.F = e^{\int -\frac{4}{x}dx} = e^{-4\log x} = \frac{1}{x^4}$$

$$\therefore ① \Rightarrow \left(\frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}\right)dx - \frac{y}{x^2}dy = 0 \text{ —— ②}$$

In ②, $M = \frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}$, $N = -y/x^2$

$\therefore$ Sol^n is,

$$\int\left(\frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}\right)dx + \int 0\,dy = C$$

y as constant

Contd..

$$\Rightarrow y^2 \int x^{-3} dx - \int \frac{e^{1/x^3}}{x^4} dx = C$$

put $1/x^3 = t$

$$\Rightarrow y^2 \left(\frac{-1}{2x^2}\right) + \frac{1}{3}\int e^t dt = C$$

$$\Rightarrow \frac{dt}{dx} = \frac{-3}{x^4}$$

$$\Rightarrow -\frac{y^2}{2x^2} + \frac{e^{1/x^3}}{3} = C$$

$$\Rightarrow \frac{dx}{x^4} = -\frac{dt}{3}$$

Problem 1.4. *Solve* $(xy^3 + y)\,dx + 2\,(x^2y^2 + x + y^4)\,dy = 0.$ —— ①

Ans:- In ①, $M = xy^3 + y$; $N = 2x^2y^2 + 2x + 2y^4$

$\frac{\partial M}{\partial y} = 3xy^2 + 1$; $\frac{\partial N}{\partial x} = 4xy^2 + 2$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ ① is not exact.

$$\therefore \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3xy^2 + 1 - 4xy^2 - 2$$
$$= -1 - xy^2 = -(1 + xy^2)$$

$$\therefore \frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{y(xy^2+1)} - (1+xy^2) = \frac{-1}{y} = g(y)$$

$$\therefore I.F = e^{-\int g(y)dy} = e^{\int \frac{1}{y}dy} = e^{\log y} = y$$

$$\therefore ① \Rightarrow (xy^4 + y^2)\,dx + 2(x^2y^3 + xy + y^5)\,dy = 0 \quad —— ②$$

In ②, $M = xy^4 + y^2$; $N = 2x^2y^3 + 2xy + 2y^5$

$$\therefore \text{Sol}^n \text{ is, } \int\limits_{y \text{ as constant}} (xy^4 + y^2)\,dx + \int 2y^5\,dy = C$$

Contd..

Ans:- $$\frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = C$$

Problem 1.5. *Solve* $(y^4+2y)\,dx+(xy^3+2y^4-4x)\,dy=0.$ —— ②

Ans:: Here $M = y^4+2y$; $N = xy^3+2y^4-4x$

$$\Rightarrow \frac{\partial M}{\partial y} = 4y^3+2 \quad ; \quad \frac{\partial N}{\partial x} = y^3-4$$

ie; $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ ① is not exact

$$\therefore \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3(y^3+2)$$

$$\therefore \frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{y(y^3+2)}3(y^3+2)$$

$$= 3/y = g(y)$$

$$\therefore I.F = e^{-\int 3/y\,dy} = \frac{1}{y^3}$$

$$\therefore ① \Rightarrow \left(y+\frac{2}{y^2}\right)dx + \left(x+2y-\frac{4x}{y^3}\right)dy = 0 \quad \text{—— ②}$$

In ②, $M = y+\frac{2}{y^2}$; $N = x+2y-\frac{4x}{y^3}$

$\therefore$ Sol^n is,

$$\underset{y \text{ as constant}}{\int}\left(y+2/y^2\right)dx + \int(2y)\,dy = C$$

Contd..

$$\Rightarrow \left(y + \frac{2}{y^2}\right)x + y^2 = C$$

Problem 1.6. *Solve* $(6x^2 + 4y^3 + 12y)\,dx + 3x\,(1 + y^2)\,dy = 0.$ —— ①

Ans:- $\dfrac{\partial M}{\partial y} = 12y^2 + 12$; $\dfrac{\partial N}{\partial x} = 3 + 3y^2$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{3}{x} = f(x)$$

$$\therefore \text{I.F} = e^{\int f(x)\,dx} = x^3$$

$$① \Rightarrow (6x^5 + 4x^3y^3 + 12x^3y)\,dx + 3x^4(1+y^2)\,dy = 0 \quad \text{—— ②}$$

Sol^n,

$$\int (6x^5 + 4x^3y^3 + 12x^3y)\,dx + \int 0\,dy = C$$

y as constant

$$\Rightarrow x^6 + x^4y^3 + 3x^4y = C$$

Contd..

Consider $M\,dx + N\,dy = 0$ —— (1), is not exact

(1) can be written as

$$\underbrace{y\,f(xy)}_{M}\,dx + \underbrace{x\,g(xy)}_{N}\,dy = 0 \quad \text{—— (2)}$$

If $Mx - Ny \neq 0$ then $I.F = \dfrac{1}{Mx - Ny}$

Type 1.7. *Suppose the equation $Mdx + Ndy = 0$ is of the form*
$$yf(xy)dx + xg(xy)dy = 0.$$
If $Mx - Ny \neq 0$ then, the integrating factor is, $I.F. = \frac{1}{Mx - Ny}$.

Problem 1.8. *Solve $(xy^2 + y)dx - (x^2y - x)dy = 0.$* —— ①

Ans:- In ①, $M = xy^2 + y \quad N = -x^2y + x$

$$\frac{\partial M}{\partial y} = 2xy + 1 \qquad \frac{\partial N}{\partial x} = -2xy + 1$$

ie; $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ ① is not exact.

$$① \Rightarrow y(xy+1)dx - x(xy-1)dy = 0 \quad —— ②$$

In ②, $M = y(xy+1) \quad N = -x(xy-1)$

$\therefore Mx - Ny = 2x^2y^2 \neq 0$

$$\therefore I.F = \frac{1}{Mx - Ny} = \frac{1}{2x^2y^2}$$

$$\therefore ② \Rightarrow \underbrace{\left(\frac{xy+1}{2x^2y}\right)}_{M} dx + \underbrace{\frac{(1-xy)}{2xy^2}}_{N} dy = 0 \quad —— ③$$

$$= \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)$$

Solⁿ, $\int_{y \text{ as constant}} \left(\frac{1}{2x} + \frac{1}{2x^2y}\right) dx + \int \frac{-1}{2y} dy = C$

Contd..

$$\Rightarrow \int \frac{1}{2x}\,dx + \frac{1}{2y}\int \frac{1}{x^2}\,dx - \frac{1}{2}\int \frac{1}{y}\,dy = C$$

$$\Rightarrow \log\left(\frac{x}{y}\right) - \frac{1}{xy} = C'$$

Problem 1.9. *Solve*

$$\underbrace{\left(xy^2\sin(xy)+y\cos(xy)\right)}_{=M}dx+\underbrace{\left(x^2y\sin(xy)-x\cos(xy)\right)}_{=N}dy=0. \quad \text{—— (1)}$$

Ans:-

Ex

$$\frac{\partial M}{\partial y}=x\cdot\frac{\partial}{\partial y}\left(y^2\sin(xy)\right)+\frac{\partial}{\partial y}\left(y\cos(xy)\right)$$

$$\frac{\partial N}{\partial x}=y\,\frac{\partial}{\partial x}\left(x^2\sin(xy)\right)-\frac{\partial}{\partial x}\left(x\cos(xy)\right)$$

$$\therefore \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}\Rightarrow (1) \text{ is not exact}$$

$$(1)\Rightarrow y\left(\underbrace{xy\sin(xy)+\cos(xy)}\right)dx+x\left(\underbrace{xy\sin(xy)-\cos(xy)}\right)dy=0$$

$$\therefore Mx-Ny=2xy\cos(xy)$$

$$\therefore \text{I.F}=\frac{1}{Mx-Ny}=\frac{1}{2xy\cos(xy)}$$

$$(1)\Rightarrow\underbrace{\left(\frac{y}{2}\tan(xy)+\frac{1}{2x}\right)}_{M}dx+\underbrace{\left(\frac{x}{2}\tan(xy)-\frac{1}{2y}\right)}_{N}dy=0 \quad \text{—— (2)}$$

Contd..

$\therefore$ Sol^n is,

$$\int\left(\frac{y}{2}\tan(xy)+\frac{1}{2x}\right)dx+\int\frac{-1}{2y}dy=C$$

y as constant

$$\Rightarrow \frac{y}{2}\int\tan(xy)\,dx+\frac{1}{2}\int\frac{1}{x}dx-\frac{1}{2}\int\frac{dy}{y}=C$$

$$\Rightarrow \frac{y}{2}\,\frac{1}{y}\log\sec(xy)+\frac{1}{2}\log x-\frac{1}{2}\log y=C$$

$$\Rightarrow \log(\sec(xy))+\log(x/y)=\log K$$

$$\Rightarrow \log\left[\sec(xy)\cdot\frac{x}{y}\right]=\log K$$

$$\Rightarrow \left(\frac{x}{y}\right)\sec(xy)=K$$

Problem 1.10. *Solve*

$$\left(xy^2 + 2x^2y^3\right)dx + \left(x^2y - x^3y^2\right)dy = 0$$

<u>Hint:-</u> $M x - N y = 3x^3y^3$

$$I.F = \frac{1}{3x^3y^3}$$

$\therefore$ Sol^n is, $\log\left(\frac{x^2}{y}\right) - \frac{1}{xy} = K$

Contd..

Consider $M\,dx + N\,dy = 0$ —— (1)

$$x^{k_1} y^{k_2} (c_1 y\,dx + c_2 x\,dy) + x^{k_3} y^{k_4} (c_3 y\,dx + c_4 x\,dy) = 0$$

then I.F $= \underline{\underline{x^{a} y^{b}}}$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Type 1.11. *Suppose the equation $Mdx + Ndy = 0$ is of the form*

$$x^{k_1}y^{k_2}(c_1ydx + c_2xdy) + x^{k_3}y^{k_4}(c_3ydx + c_4xdy) = 0,$$

where k_1, k_2, k_3, k_4 and c_1, c_2, c_3, c_4 are constants, then, the integrating factor is, x^ay^b. The constants a and b are determined such that the condition for exact equation is satisfied.

Problem 1.12. *Solve*

$$x(3ydx + 2xdy) + 8y^4(ydx + 3xdy) = 0 \quad \text{--- (1)}$$

(1) becomes,

$$(3xy + 8y^5)dx + (2x^2 + 24xy^4)dy = 0 \quad \text{--- (2)}$$

Multiply (2) by $x^a y^b$ we get,

$$(\underbrace{3x^{a+1}y^{b+1} + 8x^a y^{b+5}}_{M})dx + (\underbrace{2x^{a+2}y^b + 24x^{a+1}y^{b+4}}_{N})dy = 0 \quad \text{--- (3)}$$

Here, $M = 3x^{a+1}y^{b+1} + 8x^a y^{b+5}$

$N = 2x^{a+2}y^b + 24x^{a+1}y^{b+4}$

Contd..

Since (3) is exact, $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$$\Rightarrow 3x^{a+1}(b+1)y^{b} + 8x^{a}(b+5)y^{b+4} = 2(a+2)x^{a+1}y^{b} + 24(a+1)x^{a}y^{b+4}$$

Comparing both sides we get,

$$3(b+1) = 2(a+2) \text{ and}$$
$$8(b+5) = 24(a+1)$$

$$\Rightarrow \begin{array}{l} 2a-3b=-1 \\ 3a-b=2 \end{array} \Rightarrow a=1 \,;\, b=1$$

$$\therefore \; M = 3x^{2}y^{2} + 8xy^{6}$$
$$N = 2x^{3}y + 24x^{2}y^{5}$$

$$\therefore \text{ Sol}^{n} \text{ is, } \int_{y \text{ as constant}} (3x^{2}y^{2} + 8xy^{6})\,dx + \int 0\,dy = C$$

$$\Rightarrow x^{3}y^{2} \underline{+4x^{2}}y^{6} = C$$

Problem 1.13. *Solve*

$$x\,(4ydx + 2xdy) + y^3\,(3ydx + 5xdy) = 0$$

Ans:. $(4xy + 3y^4)dx + (2x^2 + 5xy^3)dy = 0$ —— (1)

Multiply both sides of (1) by $x^a y^b$

we get,

$$(\underbrace{4x^{a+1}y^{b+1} + 3x^a y^{b+4}}_{=M})dx + (\underbrace{2x^{a+2}y^b + 5x^{a+1}y^{b+3}}_{=N})dy = 0$$ —— (2)

Since (2) is exact, $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$a = 2$ and $b = 1$

$\therefore$ $M = 4x^3y^2 + 3x^2y^5$

$N = 2x^4y + 5x^3y^4$

Sol^n, $x^4y^2 + x^3y^5 = C$

Definition 2.1. (**Matrix representation of an n-dimensional vector**) The matrix representation of an n-dimensional vector with components $a_1, a_2, ..., a_n$ is $\vec{X} = \begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ . \\ a_n \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & . & . & . & a_n \end{pmatrix}^T$

Example 2.2. Let $\vec{a} = 2\vec{i} + 3\vec{j} - 5\vec{k}$ and $\vec{b} = 2\vec{i} + 6\vec{k}$ then we write, $\vec{a} = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix}$.

→ 'm' equations (linear)

→ 'n' unknowns

$x_1, x_2 \cdots x_n$

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n &= b_3 \\ \vdots \qquad\qquad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} (*)$$

The matrix eq^n of $(*)$ is,

$$AX = B$$

where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m\times n}$ $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1}$ $B = {}_{m\times 1}\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

Matrix equation of system of linear equations: Consider the system of m linear equations in $n-$unknowns $x_1, x_2, ..., x_n$ as below:

consistent.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_1 \\ a_{31}x_1 + a_{12}x_2 + ... + a_{3n}x_n &= b_1 \\ &..... \\ &..... \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &= b_1 \end{aligned}\right\} (*)$$

Then the matrix equation of the above system is $AX = B$ where

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ a_{31} & a_{32} & ... & a_{3n} \\ . & . & . & . \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix} \quad B = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_m \end{pmatrix}$$

If $B = 0$ then the system of equations is said to be *homogeneous*, otherwise *non-homogeneous.*

Any $n-$tuple $x = (x_1, x_2, ..., x_n)$ which satisfies the above system of equations is called the *solution* of the system.

→ Consider $AX = B$.

Augmented matrix:- $\left(A \,\middle|\, B \right)$

Eg:- $2x_1 + 3x_2 = 6$

$3x_1 + x_2 = 5$

$AX = B$ where $A = \begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}$ $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\therefore$ Aug. matrix $[A:B]$ or $[A|B]$ or $(A|B)$ $\quad B = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$

Definition 2.5. (**Augmented Matrix**) Consider the system of linear equations $AX = B$ then the augmented matrix is obtained by placing the column matrix B to the right of the matrix A. It is denoted by $[A : B]$. $[A|B]$