# DIFFERENTIAL EQUATIONS

## Introduction:

An equation involving one or more derivatives of an unknown function is called a differential equation. Differential equations arise in many physical phenomena and mathematical analysis of any engineering problems.

A mathematician is interested in exploring whether a given differential equation possesses a solution; if so, he is keen on obtaining the solution and deduce a few properties of that solution. A physicist or an engineer on the other hand is usually interested in the specific expression of the solution. The usual compromise is to find the solution.

## Fundamental Definitions:

An ordinary differential equation is an equation which involves ordinary derivatives of an unknown function $y$ of a single variable $x$.

Ordinary differential equations (ODEs) arise in many different contexts throughout mathematics and science (social and natural) one way or another, because when describing changes mathematically, the most accurate way uses differentials and derivatives (related, though not quite the same). Since various differentials, derivatives, and functions become inevitably related to each other via equations, a differential equation is the result, describing dynamical phenomena, evolution, and variation. Often, quantities are defined as the rate of change of other quantities (time derivatives), or gradients of quantities, which is how they enter differential equations.

Specific mathematical fields include geometry and analytical mechanics. Scientific fields include much of physics and astronomy (celestial mechanics), geology (weather modeling), chemistry (reaction rates), biology (infectious diseases, genetic variation), ecology and population modelling (population competition), economics (stock trends, interest rates and the market equilibrium price changes).

Eminent mathematicians who have studied differential equations and contributed to the field, include Newton, Leibniz, the Bernoulli family, Riccati, Clairaut, d'Alembert and Euler.

**Examples:**

$i)\ y' = 3x^3 + y \quad ii)\ 3(y'')^5 + (y')^2 = \sec x$

A partial differential equation is an equation which involves partial derivatives of unknown functions of two or more independent variables.

**Examples:**

$$i)\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad ii)\ \frac{\partial u}{\partial t} = c\ \frac{\partial^2 u}{\partial x^2}$$

In this chapter, we consider only ordinary differential equations.

The **order** of a differential equation is the order of the highest derivative occurring in it.

The **degree** of a differential equation is the degree of the highest order derivative occurring in the differential equation, after the equation is made free from fractional powers and radicals.

The order and degree of the differential equations in the above examples are respectively

i) 1, 1 ii) 2, 5

A differential equation is said to be **linear** if it is a linear function of the dependent variable and its derivatives, i.e., it is of degree one in the dependent variable *y* and its derivatives, and the dependent variable and the derivatives are not multiplied.

General linear differential equation of order n is of the form

$$b_0(x)\frac{d^n y}{dx^n} + b_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots .. + b_{n-1}(x)\frac{dy}{dx} + b_n(x)y = R(x)$$

where $b_0, b_1, \ldots, b_n,$ and R are functions of x alone.

A solution of a differential equation is a relation between the variables which satisfies the given differential equation.

The general solution of a differential equation is a linear combination of all linearly independent solutions of the given equation. An $n^{th}$ order differential equation has n linearly independent solutions and hence its general solution has precisely n arbitrary constants.

A particular solution is a solution obtained from the general solution by giving specific values to the arbitrary constants.

Examples:

1) Consider the differential equation $\frac{dy}{dx} - 2y = 0.$

For this equation, general solution is $y = Ae^{2x}$ , where $A$ is an arbitrary constant.

Also, $y = e^{2x}$, $y = -½ e^{2x}$, $y = 3e^{2x}$ are some particular solutions.

2) Consider the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$.

For this equation, general solution is $y = Ae^{2x} + Be^{x}$ , where $A$ and $B$ are arbitrary constants.

Also, $y = e^{2x}$, $y = -e^{x}$, $y = 2e^{2x} + 3e^{x}$, $y = -e^{2x} + 5e^{x}$ are some particular solutions.

**Note:** A differential equation together with an initial condition is called an Initial Value problem. The initial condition is used to determine the value of the arbitrary constants in the general solution.

**Formulation of differential equations by eliminating arbitrary constants**:

In this section we start with a relation involving arbitrary constants, and by elimination of those arbitrary constants obtain a differential equation which is consistent with the original relation. In other words we will obtain a differential equation for which the given relation is the general solution.

Methods for elimination of arbitrary constants vary. Since each differentiation yields a new relation, the number of derivatives that needs to be used is same as that of the number of arbitrary constants to be eliminated. Thus in eliminating arbitrary constants from a relation we obtain a differential equation that is

(i) Of order equal to the number of arbitrary constants in the equation.

(ii) Consistent with relation.

(iii) Free from arbitrary constants.

**Example (1):** Eliminate the arbitrary constants $c_1$ and $c_2$ from the relation

$$y = c_1 e^{2x} + c_2 e^{-3x}. ------(1)$$

**Solution:** Since two constants are to be eliminated, we need to differentiate twice.

$$y' = 2c_1e^{2x} - 3c_2e^{-3x}, --------- (2)$$
$$y'' = 4c_1e^{2x} + 9c_2e^{-3x}. \quad --------- (3)$$

Equations (1), (2) and (3) considered as equations in two unknowns $c_1$ and $c_2$ can have solutions only if

$$\begin{vmatrix} 1 & 1 & y \\ 2 & -3 & y' \\ 4 & 9 & y'' \end{vmatrix} = 0$$
$$\Rightarrow \; y'' + y' - 6y = 0,$$

which is the required differential equation.

**Example (2):** Eliminate the constant $a$ from the equation $(x-a)^2 + y^2 = a^2.$

**Solution:** Direct differentiation of the relation yields $2(x-a) + 2yy' = 0,$ from which $a = x + yy'.$

Therefore, using the original equation, we find that $y^2 = x^2 + 2xyy',$ which may be written in the form $(x^2 - y^2)dx + 2xydy = 0.$

**Note:** In above case we can also isolate the arbitrary constant and then differentiate.

The equation $(x-a)^2 + y^2 = a^2$ may be put in the form $\dfrac{x^2+y^2}{x} = 2a.$

Then differentiation of both sides leads to

$$\frac{x(2xdx + 2ydy) - (x^2+y^2)dx}{x^2} = 0, \text{ i.e., } (x^2 - y^2)dx + 2xydy = 0, \text{ as desired.}$$

**Example (3):** Eliminate $c$ from the equation $cxy + c^2x - 7 = 0.$

**Solution:** By differentiating we get, $c(y + xy') + c^2 = 0.$

Since $c \neq 0, c = -(y + xy')$ and substitution into the original gives the differential equation

$$x^3(y')^2 + x^2yy' + 4 = 0.$$

**Example (4):** Eliminate $B$ and $\alpha$ from the relation $x = B\sin(\omega t + \alpha),$ in which ω is a parameter (not to be eliminated).

**Solution:** Since there are two arbitrary constants, we need to differentiate twice.

$$\frac{dx}{dt} = \omega B \cos(\omega t + \alpha),$$
$$\frac{d^2x}{dt^2} = -\omega^2 B \sin(\omega t + \alpha).$$

From the above we get , $\dfrac{d^2x}{dt^2} + \omega^2 x = 0.$

**Exercises:**

In each of the following, eliminate the arbitrary constants.

1. $x = c_1 \cos \omega t + c_2 \sin \omega t; \omega$ a parameter.
2. $x^2 = 4ay.$
3. $y = x^2 + c_1 e^{-x} + c_2 e^{3x}.$
4. $y = Ae^{3x} + Bxe^{3x}.$
5. $y = c_1 e^{ax} \cos bx + c_2 e^{ax} \sin bx;$ a and b are parameters.

**Families of curves:**

A relation involving a parameter, and one or both the coordinates of a point in a plane, represents a family of curves. Each value of the parameter gives rise to a member of the family.

For instance, the equation $(x-c)^2 + (y-c)^2 = 2c^2$ represents the family of circles, each having its centre on the line y = x and each passing through the origin.

If the constant *c* is treated as an arbitrary constant and eliminated, then the resulting differential equation is called the *differential equation of the family represented by the equation.* In this case, the elimination of *c* is easily performed by isolating *c*, and then differentiating.

Thus, from $\dfrac{x^2 + y^2}{x + y} = 2c$, we find that $x^2 + 2xy - y^2 + (2xy + y^2 - x^2)y' = 0$.

Note that for a two parameter family of curves, the differential equation will be of order 2.

**Example (1):** Find the differential equation of the family of parabolas, having their vertices at the origin and their foci on the x-axis.

**Solution:** The equation of this family of parabolas with vertex at origin and foci on x-axis is given by

$$y^2 = 4ax.$$

Then from

$$\frac{y^2}{x} = 4a,$$

we get

$$2xy' - y = 0$$

on differentiation.

**Example (2):** Find the differential equation of the family of circles having their centres on y-axis.

**Solution:** Since a member of the family of circles of this example may have its centre anywhere on y-axis and radius of any magnitude, we are dealing with the two-parameter family

$$x^2 + (y-b)^2 = r^2.$$

We shall eliminate both b and r and arrive, of course, at a second-order differential equation for the family.

At once $x + (y-b)y' = 0,$ from which $\dfrac{x + yy'}{y'} = b.$

Then

$$\frac{y'[1 + yy'' + (y')^2] - y''(x + yy')}{(y')^2} = 0,$$

so the desired differential equation is $xy'' - (y')^3 - y' = 0.$

**<u>Exercises:</u>**

In each exercise, obtain the differential equation of the family of plane curves described and sketch several representative members of the family.

1. Straight lines with slope and x-intercept equal.
2. Straight lines passing through the origin
3. Circles with fixed radius r and touching x-axis.
4. Parabolas with axis parallel to the y-axis.

## DIFFERENTIAL EQUATIONS OF ORDER ONE AND DEGREE ONE

In our study we initially consider only first order and first degree differential equations. Such equations can be written in the form

$$\frac{dy}{dx} = f(x,y) \quad or \quad M(x,y)dx + N(x,y)dy = 0.$$

Before discussing some of the analytic techniques for finding solutions we shall state an important theorem concerning to the uniqueness and existence of solution. Consider

$$\frac{dy}{dx} = f(x,y) \qquad (1)$$

Let T denote the rectangular region defined by $|x - x_0| \le a$ and $|y - y_0| \le b$, a region with the point $(x_0, y_0)$ at its centre. Suppose that $f$ and $\frac{\partial f}{\partial y}$ are continuous functions of $x$ and $y$ in $T$.

Under the conditions imposed on $f(x,y)$ above, an interval exists about $x_0$ , $|x - x_0| \le h$ and $y(x)$ satisfies the properties.

a) A solution of equation (1) in $|x - x_0| \le h$

b) On the interval $|x - x_0| \le h$, $y(x)$ satisfies $|y(x) - y_0| \le b$

c) At $x = x_0$, $y = y(x_0) = y_0$

d) $y(x)$ is unique in $|x - x_0| \le h$ satisfying the above conditions.

In other words, the theorem states that if $f(x,y)$ is sufficiently well behaved near the point $(x_0, y_0)$, then the differential equation ,

$$\frac{dy}{dx} = f(x,y)$$

has a solution that passes through the point $(x_0, y_0)$ and that solution is unique near $(x_0, y_0)$.

We consider some first order differential equation in the following forms.

### <u>1. Variable Separable Equations:</u>

A differential equation M dx + N dy = 0 is said to be variable separable if it can be put in the form $f(x)\,dx + g(y)\,dy = 0$ , where $f$ is a function of $x$ alone and $g$ is a function of $y$ alone. In this case the solution is given by

$\int f(x)dx + \int g(y)dy = c$, where $c$ is an arbitrary constant.

**Example (1):** Solve $2x\ dx + e^y\ dy = 0$.

**Solution:** The equation is variable separable. Hence the solution is

$$\int 2x\ dx + \int e^y\ dy = c$$

i.e. $x^2 + e^y = c$, where $c$ is some arbitrary constanst.

**Example (2):** Solve $\sec^2 x \tan y\ dx + \sec^2 y \tan x\ dy = 0$.

**Solution:** Here the variables are not separated but it can be reduced to that form by dividing the equation by $\tan x \tan y$. Then

$$\frac{\sec^2 x}{\tan x} dx + \frac{\sec^2 y}{\tan y} dy = 0.$$

The solution is

$$\int \frac{\sec^2 x}{\tan x} dx + \int \frac{\sec^2 y}{\tan y} dy = c$$

$$logtanx + logtany = c \quad \text{i.e., } \tan x \tan y = c'$$

**Example (3):** $Solve\ \frac{dy}{dx} = e^{2x-3y} + 4x^2 e^{-3y}$

**Solution:** $\frac{dy}{dx} = e^{-3y}(e^{2x} + 4x^2)$

$$e^{3y} dy = (e^{2x} + 4x^2)dx$$

Integrating we get,

$$\frac{e^{3y}}{3} = \frac{e^{2x}}{2} + \frac{4}{3}x^3 + c.$$

**Example (4):** $Solve\ (xy + x)dx + (x^2y^2 + x^2 + y^2 + 1)dy = 0$

**Solution:** We have

$$xy + x = x(y + 1)$$

$$x^2y^2 + x^2 + y^2 + 1 = x^2(y^2 + 1) + 1(y^2 + 1) = (x^2 + 1)(y^2 + 1)$$

Hence the given equation is

$$x(y + 1)dx + (x^2 + 1)(y^2 + 1)dy = 0$$

$$\int \frac{x}{x^2+1}\ dx + \int \frac{y^2+1}{y+1}\ dy = c$$

$$\frac{1}{2}\log(x^2 + 1) + \int \frac{(y^2-1)+1}{y+1}\ dy = c$$

$$\frac{1}{2}\log(x^2 + 1) + \int \left(y - 1 + \frac{1}{y+1}\right) dy = c$$

$$\frac{1}{2}\log(x^2 + 1) + \frac{y^2}{2} - y + \log(y + 1) = c.$$

**Note:** Some differential equations which are not variable separable, can be reduced to variable separable by suitable substitutions.

**Example (5):** $Solve\ \frac{dy}{dx} = \sin(x+y)$

**Solution:** $Put\ z = x + y$. Then

$$\frac{dz}{dx} = 1 + \frac{dy}{dx}.$$

Hence

$$\frac{dz}{dx} = 1 + \sin z$$

$$\frac{dz}{1+\sin z} = dx$$

$$\frac{1-\sin z}{\cos^2 z}\ dz = dx$$

Integrating,

$$\tan z - \sec z = x + c$$

$$\tan(x+y) - \sec(x+y) = x + c$$

**Example (6):** $Solve\ \frac{dy}{dx} - xtan(y-x) = 1$

**Solution:** Put $y - x = z$. Then

$$\frac{dy}{dx} = \frac{dz}{dx} + 1.$$

$$\frac{dz}{dx} + 1 - xtan(z) = 1.$$

$$\frac{dz}{dx} = xtan(z).$$

$$\frac{dz}{tanz} = xdx.$$

$$\log sinz = \frac{x^2}{2} + c \ \text{ i.e., } \log\sin(y-x) = \frac{x^2}{2} + c.$$

**Exercises: Solve the following**

1. $(xy - x)dx + (xy + y)dy = 0$
2. $\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$
3. $(x + y + 1)^2\ \frac{dy}{dx} = 1$
4. $\frac{dy}{dx} = xe^{y-x^2}$
5. $\frac{dy}{dx} = (4x + y + 1)^2, y(0) = 1$

6. $y' = 2(3x + y)^2 - 1; when\ x = 0, y = 1$

7. $(1 + x^3)\ dy - x^2y\ dx = 0; when\ x = 1, y = 2$

8. $xyy' = 1 + y^2$

9. $(x + y)^2 \frac{dy}{dx} = a^2$

10. $\frac{dy}{dx} = \frac{\sin x + y\cos x}{y(2\log y + 1)}$

**2. Differential Equations with Homogenous Coefficients:**

Consider the differential equation of the form

$$\frac{dy}{dx} = \frac{f(x,y)}{\varphi(x,y)},$$

where $f(x, y)$ and $\varphi(x, y)$ are homogeneous functions of the same degree in $x$ and $y$.

To solve this equation, we note that $\frac{f(x,y)}{\varphi(x,y)}$ being a homogenous equation of degree zero, is a function of (y/x) only. Let

$$\frac{f(x,y)}{\varphi(x,y)} = g\left(\frac{y}{x}\right).$$

This suggests the substitution

$$y = vx, \text{ then } \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Substituting in the given equation we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx} = g(v)$$

$$\text{or } x\frac{dv}{dx} = g(v) - v,$$

which can be solved by separating the variables and integrating.

**Example (1):** $Solve\ (2x + y)^2\ dx = xy\ dy$

**Solution:**

Given equation can be put in the form

$$\frac{dy}{dx} = \frac{(2x+y)^2}{xy}.$$

Put $y = vx$ then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{(2x+vx)^2}{xvx}$$

Simplifying we get

$$\frac{4}{x}\, dx = \frac{v}{1+v}\, dv$$

$$4\log x = v - \log(v+1) + c$$

$$4\log x = \frac{y}{x} - \log\left(\frac{y+x}{x}\right) + c$$

**Example (2):** $Solve\ 3(3x^2 + y^2)dx - 2xydy = 0$

**Solution:** The equation can be put in the form

$$\frac{dy}{dx} = \frac{3(3x^2+y^2)}{2xy}$$

Put $y = vx$ then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Hence

$$v + x\frac{dv}{dx} = \frac{3(3+v^2)}{2v}$$

$$\frac{2vdv}{v^2+9} = \frac{dx}{x}$$

Integrating we get,

$$\log(v^2 + 9) = logx + logc$$

$$y^2 + 3x^2 = cx^3.$$

**Remark:**

It is quite immaterial whether one uses $y = vx$ or $x = vy$. However, it is sometimes easier to solve by substituting for the variable whose differential has the simpler coefficient.

**Example (3):** $Solve\ \ \frac{dy}{dx} = \frac{y}{x+\sqrt{xy}}$

**Solution:** $Put\ x = vy\ \ then\ dx = v\ dy + y\ dv$

$$\left(vy + y\sqrt{v}\right) dy = y(v\ dy + y\ dv)$$

$$\left(v + \sqrt{v}\right)dy = v\ dy + y\ dv$$

$$\sqrt{v}dy = y\ dv$$

Integrating,

$$2\sqrt{v} = \log y + c \ \text{ i.e., } \ 2\sqrt{\frac{x}{y}} = \log y + c$$

**Exercises: Solve the following differential equations**

1. $2xy\frac{dy}{dx} = 3y^2 + x^2$
2. $(y^2 - 2xy)dx = (x^2 - 2xy)dy$
3. $xdy - ydx = \sqrt{x^2 + y^2}\, dx$
4. $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$
5. $\frac{dy}{dx} = \frac{y}{x} + \sin\frac{y}{x}$
6. $y\, e^{\frac{x}{y}}\, dx = \left(xe^{\frac{x}{y}} + y^2\right) dy$
7. $\left(1 + e^{\frac{x}{y}}\right) dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right) dy = 0$
8. $\left(x\tan\left(\frac{y}{x}\right) - y\sec^2\frac{y}{x}\right) dx + x\sec^2\frac{y}{x}\, dy = 0$

## 3. Differential Equations with linear coefficients

The equations of the form $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$ can be reduced to either variable separable form or to the homogenous form and hence can be solved.

Consider the equations $ax + by + c = 0\ and\ a'x + b'y + c' = 0.$ They represent straight lines in plane. The lines may be parallel or intersecting.

**Case 1:** Suppose that the lines are parallel. Then there exists a number m such that

$$a'x + b'y = m(ax + by).$$

Substitute $ax + by = z$, the equation reduces to variable separable equation and hence can be solved.

**Case 2:** Suppose that the lines are intersecting. Let $(h, k)$ be the point of intersection.

Putting $x = X + h$ and $y = Y + k$, the equation reduces to $\frac{dY}{dX} = \frac{aX+bY}{a'X+b'Y}$ which is of homogenous coefficients in $X, Y$ and hence can be solved.

**Example (1):** Solve $\frac{dy}{dx} = \frac{2x-6y+7}{x-3y+4}$

**Solution:** We observe that $2x - 6y = 2(x - 3y)$.

Put $z = x - 3y \Rightarrow \frac{dz}{dx} = 1 - 3\frac{dy}{dx}$

i.e $\frac{dz}{dx} = 1 - 3\frac{2z+7}{z+4}$

i.e $(z+4)\frac{dz}{dx} = z + 4 - 6z - 21$

Separating, $dx + \frac{z+4}{5z+17}\ dz = 0$

i.e Solution is $5x + z + \frac{3}{5}\log(5z + 17) = c$

i.e., $6x - 3y + \frac{3}{5}\log(5x - 15y + 17) = c$.

**Example (2):** Solve $\frac{dy}{dx} = \frac{y+x-2}{y-x-4}$

**Solution:** Consider $y + x - 2 = 0\ and\ y - x - 4 = 0$.

The lines are intersecting. Point of intersection is (-1, 3).

Put x = X -1, y = Y + 3

$$\frac{dY}{dX} = \frac{Y+X}{Y-X} \quad \Rightarrow \quad \text{put } Y = VX$$

$$\therefore \frac{dY}{dX} = V + X\frac{dV}{dX}$$

Substituting, $V + X\frac{dV}{dX} = \frac{V+1}{V-1}$

$$\Rightarrow \frac{V-1}{1 + 2V - V^2}\ dV = \frac{dX}{X}$$

Integrating both sides, $-\frac{1}{2}\log(1 + 2V - V^2) = \log X + c$

$\log(X^2 + 2XY - Y^2) = -2c$

Finally, $x^2 + 2xy - y^2 - 4x + 8y - 14 = c'$.

**Exercises: Solve the following equations.**

1. $(2x - y)dx + (4x + y - 6)dy = 0$
2. $(x + 3y - 4)dx + (x + 4y - 5)dy = 0$
3. $(x + 2y - 1)dx - (2x + y - 5)dy = 0$
4. $(2x + 3y + 4)dx - (4x + 6y + 5)dy = 0$
5. $(x - 2y + 1)dx + (2x - 4y + 3)dy = 0$
6. $\frac{dy}{dx} + \frac{10x+8y-12}{7x+5y-9} = 0$
7. $\frac{dy}{dx} + \frac{x-2y+1}{2x-4y+3} = 0$

8. $(x-1)dx-(3x-2y-5)dy=0$
9. $(2x+4y-1)dx-(x+2y-3)dy=0$
10. $(4x-6y-1)dx+(3y-2x-2)dy=0$

## 4. Exact Equations

Consider the equation $M(x,y)dx+N(x,y)dy=0$. Suppose that there exists a function $F(x,y)$ such that $dF=Mdx+Ndy$, then the differential equation is said to be an exact differential equation and its solution is given by $F(x,y)=c$.

**Theorem:** If $M, N, \frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$ are continuous functions of *x* and *y*, then a necessary and sufficient condition that

$$Mdx+Ndy=0 \text{ ------(1)}$$

is exact, is that $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

**Proof:** Suppose that (1) is exact.

Then by definition $there\ exists\ a\ F$ such that

$$dF=Mdx+Ndy$$

But from total derivate formula,

$$dF=\frac{\partial F}{\partial x}\,dx+\frac{\partial F}{\partial y}\,dy$$

$comparing\ the\ above\ equations\ we\ get\ \ M=\frac{\partial F}{\partial x},\quad N=\frac{\partial F}{\partial y}.$

These two equations lead to $\frac{\partial M}{\partial y}=\frac{\partial^2 F}{\partial y\partial x}$ and $\frac{\partial N}{\partial x}=\frac{\partial^2 F}{\partial x\partial y}$

Since $\frac{\partial^2 F}{\partial y\partial x}=\frac{\partial^2 F}{\partial x\partial y}$ we get $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

Conversely suppose that $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

Let $\phi(x,y)$ be a function for which $\frac{\partial \phi}{\partial x}=M(x,y)$.

Then $\frac{\partial^2\phi}{\partial y\partial x}=\frac{\partial M}{\partial y}; \quad \Rightarrow \quad \frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}=\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

On integrating both sides of this equation w.r.t. x holding y fixed we get,

$\frac{\partial\phi}{\partial y}=N+B(y),$ where B(y) is an arbitrary function of y.

Now define a function F as $F(x,y)=\phi(x,y)-\int B(y)dy.$

Then $dF=\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy=\frac{\partial \phi}{\partial x}dx+\left(\frac{\partial \phi}{\partial y}-B(y)\right)dy=Mdx+Ndy.$

Hence the given equation is exact.

**Note:**

If the differential equation $Mdx + Ndy = 0$ is exact then its solution can be obtained as follows.

Let $F$ be a function of $x$ and $y$ such that $dF = Mdx+Ndy$.

Then comparing with the equation $dF=\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy$ we get $\frac{\partial F}{\partial x}=M(x,y)$.

On partially integrating w.r.t. $x$ we get $F(x,y) = \int M(x,y)dx+B(y)$, where integration is done partially w.r.t. $x$ holding $y$ as a constant and $B(y)$ is an arbitrary function of $y$ alone. Now $\frac{\partial F}{\partial y}=\frac{\partial}{\partial y}\left\{\int M(x,y)dx\right\}+B'(y)=N(x,y)$ implies that $B'(y)$ consists of terms in $N(x,y)$ which does not contain $x$.

Thus the solution of the equation is given by

$$\int M(x,y)dx+\int (Terms\, in\, N(x,y)\, not\, containg\; x)\,dy = c.$$

The solution can also be obtained from

$$\int N(x,y)dx+\int (Terms\, in\, M(x,y)\, not\, containg\; y)\,dx = c.$$

**Example (1):** Solve $3x(xy-2)dx+(x^2+2y)dy=0$

**Solution:** M = 3x(xy - 2), N= $x^2$ +2y

First $\frac{\partial M}{\partial y}=3x^2$ and $\frac{\partial N}{\partial x}=3x^2$.

i.e., $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. Also the functions and derivatives are continuous. Hen e the equation is exact.

Therefore the solution is given by

$$\int M(x,y)dx+\int (Terms\, in\, N(x,y)\, not\, containg\; x)\,dy = \mathrm{C}$$

$$\int(3x^2y-6x)dx+\int 2ydy=c$$

Or, $x^3y-3x^2+y^2=C.$

**Example (2):** Solve $(2xy - tany)dx + (x^2 - xsec^2y)dy = 0$

**Solution:** $M = 2xy - tany$, $N = x^2 - xsec^2y$

$$\frac{\partial M}{\partial y} = 2x - sec^2y, \quad \frac{\partial N}{\partial x} = 2x - sec^2y.$$

i.e., $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Also the functions and derivatives are continuous. Hence the equation is exact.

Therefore the solution is given by

$$\int M(x,y)dx + \int (Terms\, in\, N(x,y)\, not\, containg\, x)\, dy = \text{C}$$

$$\int (2xy - tany)dx = c$$

i.e., $x^2y - xtany = c$

**Exercises: Solve the following differential equations**

1. $(x + y)dx + (x - y)dy = 0$
2. $(6x + y^2)dx + y(2x - 3y)dy = 0$
3. $(2xy - 3x^2)dx + (x^2 + y)dy = 0$
4. $(2xy + y)dx + (x^2 - x)dy = 0$
5. $(1 - xy)^{-2}dx + [y^2 + x^2(1 - xy)^{-2}]dy = 0$
6. $(x^2sin^3y - y^2cosx)dx + (x^3cosysin^2y - 2ysinx)dy = 0$
7. $(x^2 - 2xy - y^2)d - (x + y)^2dy = 0$
8. $\frac{dy}{dx} + \frac{ycosx + siny + y}{sinx + xcosy + x} = 0$

## 5. Equations Reducible to Exact Equations

Equations that are not exact, can be made exact, by suitable multiplication of a function of x and y. Such multiplier is called an integrating factor (I.F.) of the differential equation.

**(I)Integrating factors found by inspection**.

Here we are concerned with the equations that are simple enough to enable us to find the integrating factors by inspection. The ability to do this depends largely upon recognition of certain common exact differential and upon experience. Below are four exact differential that occur frequently:

(i) $d(xy) = xdy + ydx$

(ii) $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$

(iii) $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$

(iv) $d\left(\tan^{-1}\frac{y}{x}\right) = \frac{xdy - ydx}{x^2 + y^2}$

Using these exact differentials it is possible to group the terms in given differential equation and obtain the integrating factors.

**Example (1):** Solve $ydx + (x + x^3y^2)dy = 0.$

Grouping the terms we get, $(ydx + xdy) + x^3y^2dy = 0$

Dividing by $(xy)^3$, $\frac{d(xy)}{(xy)^3} + \frac{dy}{y} = 0$

Solution on integration is $-\frac{1}{2x^2y^2} + \log y = c.$

**Example (2):** Solve $x\frac{dy}{dx} = y + \cos^2\frac{y}{x}$

$$xdy - ydx = \cos^2\frac{y}{x}\,dx$$

$$d\left(\frac{y}{x}\right).x^2 = \cos^2\frac{y}{x}dx$$

$$\sec^2\frac{y}{x}\,d\left(\frac{y}{x}\right) = \frac{dx}{x^2}$$

$$\therefore \tan\frac{y}{x} = -\frac{1}{x} + c.$$

**Exercises: Solve the following**

1. $[2x + y\cos(xy)]dx + x\cos(xy)\,dy = 0$
2. $(x^2 + y^2)dx - xydy = 0$
3. $(3x^2y^2 + x^2)dx + (2x^3y + y^2)dy = 0$
4. $y(2xy + 1)dx - xdy = 0$
5. $y(y^3 - x)dx + x(y^3 + x)dy = 0$
6. $(x^3y^3 + 1)dx + x^4y^2dy = 0$
7. $y(y^2 + 1)dx + x(y^2 - 1)dy = 0$
8. $3x^2ydx + (y^4 - x^3)dy = 0$
9. $y(x^3 - y)dx - x(x^3 + y)dy = 0$
10. $y(2xy + e^x)dx = e^xdy$

**(II). Another Method of finding Integrating Factor**

Consider a non-exact equation $Mdx + Ndy = 0$…………………(1)

Suppose that $u$, possibly a function of both $x$ and $y$, is to be an integrating factor of (1). Then $u\,Mdx + u\,Ndy = 0$ must be exact.

$\therefore \frac{\partial}{\partial y}(uM) = \frac{\partial}{\partial x}(uN)$

Hence, $u$ must satisfy $u\frac{\partial M}{\partial y} + M\frac{\partial u}{\partial y} = u\frac{\partial N}{\partial x} + N\frac{\partial u}{\partial x}$.

$\therefore u\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{\partial u}{\partial x} - M\frac{\partial u}{\partial y}$.

First let $u$ be a function of $x$ alone. Then $\frac{\partial u}{\partial y} = 0$ and $\frac{\partial u}{\partial x}$ becomes $\frac{du}{dx}$.

Then we have $u\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{du}{dx}$ or $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx = \frac{du}{u}$

If the left member of the above equation as function of $x$ alone, we have $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$

Then the desired I.F is $u = e^{\int f(x)dx}$

By a similar argument, assuming $u$ is a function of $y$ alone, we get $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(y)$

Then an integrated factor is $u = e^{-\int g(y)dy}$.

Using the above integrating factors, one can convert the equation to exact form and solve.

**Example (1):** Solve $(4xy + 3y^2 - x)dx + x(x + 2y)dy = 0$

$M = 4xy + 3y^2 - x$ and $N = x^2 + 2xy$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4x + 6y - (2x + 2y) = 2x + 4y = 2(x + 2y)$$

Hence, $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{2x+4y}{x(x+2y)} = \frac{2}{x}$.

$\therefore I.F. = e^{2\int\frac{dx}{x}} = x^2$.

Multiply to the above equation by $x^2$, we get

$$(4x^3ydx + x^4dy) + (3x^2y^2dx + 2x^3ydy) - x^3dx = 0$$

which is exact and hence the solution is

$$x^4y + x^3y^2 - \frac{x^4}{4} = \frac{1}{4}c = c'.$$

**Example (2):** Solve $y(x+y+1)dx + x(x+3y+2)dy = 0$.

$$\frac{\partial M}{\partial y} = x + 2y + 1,\ \frac{\partial N}{\partial x} = 2x + 3y + 2$$

$$\therefore\ \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -x - y - 1$$

So $\quad \frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{x+y+1}{y(x+y+1)} = -\frac{1}{y}$

I.F. = $e^{\int \frac{1}{y} dy} = y$

Multiplying $y$ to the equation we get,

$$(xy^2 + y^3 + y^2)dx + (x^2y + 3xy^2 + 2xy)dy = 0$$

which is exact. Hence the solution is

$$xy^2(x + 2y + 2) = c.$$

**Exercises: Solve the following equations**

1. $(x^2 + y^2 + 1)dx + x(x - 2y)dy = 0$
2. $2y(x^2 - y + x)dx + (x^2 - 2y)dy = 0$
3. $y(2x - y + 1)dx + x(3x - 4y + 3)dy = 0$
4. $y(4x + y)dx - 2(x^2 - y)dy = 0$
5. $(xy + 1)dx + x(x + 4y - 2)dy = 0$
6. $(2y^2 + 3xy - 2y + 6x)dx + x(x + 2y - 1)dy = 0$
7. $y(y + 2x - 2)dx - 2(x + y)dy = 0$
8. $y^2dx + (3xy + y^2 - 1)dy = 0$
9. $cosysin2xdx + (cos^2y - cos^2x)dy = 0$
10. $y(8x - 9y)\mathrm{dx} + 2\mathrm{x}(\mathrm{x} - 3\mathrm{y})\mathrm{dy} = 0$

## 6. Linear Equations:

A differential equation is said to be linear if dependent variable and its differential coefficient occur only in the first degree and not multiplied together.

Linear differential equation of first order can be put in the form

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \text{(Legendre's equation).}$$

To solve, we multiply both sides by $e^{\int P dx}$ so that

$$\frac{dy}{dx}\, e^{\int P dx} + y\, e^{\int P dx}\, P = Q\, e^{\int P dx}$$

The above equation is equivalent to

$$\frac{d}{dx}\left(y\, e^{\int P dx}\right) = Q\, e^{\int P dx}$$

Integrating, we get $y\, e^{\int P dx} = \int Q\, e^{\int P dx}\ dx + c$ as required solution. If $I.F. = e^{\int P dx}$, then solution can be rewritten as

$$y\,(I.F.) = \int Q(x)(I.F.)\ dx + c.$$

Note: Equation linear in x is of the form

$$\frac{dx}{dy} + P(y)x = Q(y).$$

For this equation

$$I.F. = e^{\int P dy}$$

and solution is given by

$$x\,(I.F.) = \int Q(y)(I.F.)\ dy + c.$$

**Example (1):** Solve $2(y - 4x^2)dx + xdy = 0$

**Solution:** The equation can be rewritten as

$$x\frac{dy}{dx} + 2y = 8x^2$$

Dividing by x, we get

$$\frac{dy}{dx} + \frac{2}{x}y = 8x$$

The equation is linear in y.

$$I.F. = e^{\int \frac{2dx}{x}} = e^{2logx} = x^2$$

Solution is

$$y(x^2) = \int 8x(x^2)dx + c \quad i.e.,\ x^2y = 2x^4 + c$$

**Example (2):** Solve $(x+1)\ \frac{dy}{dx} - y = e^{3x}(x+1)^2$

**Solution:** Dividing throughout by $(x+1)$,

$$\frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(x+1).$$

$$P = -\frac{1}{x+1}, \quad I.F = \frac{1}{x+1}$$

Thus the solution is $\frac{y}{x+1} = \int e^{3x}(x+1).\frac{1}{x+1}dx + c = \frac{1}{3}e^{3x} + c.$

**Example (3):** Solve $(1+y^2)dx = (\tan^{-1} y - x)dy$

**Solution:** Clearly, equation is linear in $x$.

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}$$

$$P(y) = \frac{1}{1+y^2}, \quad Q(y) = \frac{\tan^{-1} y}{1+y^2}$$

$$I.F = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

Thus the solution is

$$x\, e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} . e^{\tan^{-1} y}\, dy + c$$

$$x\, e^{\tan^{-1} y} = \int t\, e^t\, dt + c \quad \text{where} \quad t = \tan^{-1} y$$

$$= t\, e^t - e^t + c$$

$$= (\tan^{-1} y - 1)e^{\tan^{-1} y} + c$$

$\therefore x = \tan^{-1} y - 1 + c\, e^{-\tan^{-1} y}$

**Note:** Some equations which are not linear, can be reduced to linear by suitable substitutions. General equation reducible to Leibnitz's linear equation is of the form

$$f'(y)\frac{dy}{dx} + P(x)f(y) = Q(x).$$

To solve it put $f(y) = z$.

**Example (4):** $Solve \quad \frac{dy}{dx} + xsin2y = x^3cos^2y.$

**Solution:** Dividing by $cos^2y$

$$sec^2y\frac{dy}{dx} + xtany = x^3 \qquad \text{( use sin2y = 2 siny cosy)}$$

Put tany = z. Then $sec^2y\frac{dy}{dx} = \frac{dz}{dx}$

Hence the equation reduces to

$$\frac{dz}{dx} + xz = x^3$$

which is linear in z and hence can be solved.

**Exercises: Solve the following**

1. $\cos^2 x \frac{dy}{dx} + y = \tan x$

2. $x\log x\frac{dy}{dx}+y=\log x^2$

3. $\frac{dy}{dx}=x^3-2xy,\;\; y(1)=2$

4. $\frac{dy}{dx}+y\cot x=4x\operatorname{cosec} x\,,\quad y\left(\frac{\pi}{2}\right)=0$

5. $\sqrt{1-y^2}\,dx=(\sin^{-1}y-x)dy$

6. $ydx+(3x-xy+2)dy=0$

7. $(2xy+x^2+x^4)dx-(1+x^2)dy=0$

8. $3x(1-x^2)y^2\frac{dy}{dx}+(2x^2-1)y^3=x^3$

9. $\frac{dy}{dx}-\frac{tany}{1+x}=(1+x)e^x secy$

10. $\frac{dy}{dx}=\frac{x^2+y^2+1}{2xy}$

## 7. Bernoulli Equation

A differential equation of the form

$$\frac{dy}{dx}+P(x)y=Q(x)y^n$$

is called Bernoulli's equation. This equation can be solved by reducing it to a linear equation.

To solve, divide both sides by $y^n$, so that

$$y^{-n}\frac{dy}{dx}+P(x)y^{1-n}=Q(x).$$

Put $y^{1-n}=z$. Equation reduces to

$$\frac{dz}{dx}+P(x)(1-n)z=Q(x)(1-n)$$

which is linear in $z$ and can be solved.

**Example (1):** Solve $x\frac{dy}{dx}+y=x^3y^6$.

**Solution:** Dividing by $xy^6$, $y^{-6}\frac{dy}{dx}+\frac{y^{-5}}{x}=x^2$.

Put $y^{-5}=z$, so $-5y^{-6}\frac{dy}{dx}=\frac{dz}{dx}$

i.e $-\frac{1}{5}\frac{dz}{dx}+\frac{z}{x}=x^2$, i.e $\frac{dz}{dx}-\frac{5}{x}z=-5x^2$ which is linear in $z$.

I.F = $e^{-\int\frac{5}{x}dx}$ = $e^{-5\log x}$ = $x^{-5}$

Solution is $z(I.F)=\int -5x^2(I.F)dx+c$

i.e $\quad zx^{-5} = \int -5x^2 . x^{-5} dx + c$

i.e $\quad y^{-5}x^{-5} = -5\frac{x^{-2}}{(-2)} + c \quad i.e., (xy)^{-5} = \frac{5}{2}x^{-2} + c$

**Example (2):** Solve $\frac{dz}{dx} + \left(\frac{z}{x}\right)\log z = \frac{z}{x}(\log z)^2$

Dividing by $z$, $\frac{1}{z}\frac{dz}{dx} + \frac{1}{x}\log z = \frac{1}{x}(\log z)^2$

$y = \log z \qquad \frac{dy}{dx} = \frac{1}{z}\frac{dz}{dx}$

$\therefore \frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x}$ , which is variable separable.

$$x\frac{dy}{dx} = y^2 - y$$

$$\frac{dy}{y(y-1)} = \frac{dx}{x}$$

$$\left(\frac{1}{y-1} - \frac{1}{y}\right) dy = \frac{dx}{x}$$

Integrating we get

$$y - 1 = cxy$$

**Exercises: Solve the following**

1. $x\frac{dy}{dx} + y = x^4y^6$
2. $\frac{dy}{dx}(x^2y^3 + xy) = 1$
3. $\frac{dy}{dx} + y\tan x = y^3 \sec x$
4. $3y^2\frac{dy}{dx} + 2xy^3 = 4xe^{-x^2}$
5. $y(2xy + e^x)dx - xe^x dy = 0$
6. $\frac{dy}{dx} - \frac{1}{2}\left(x + \frac{1}{x}\right)y + \frac{3}{2}y^3 = 0$
7. $xy(1 + xy^2)\frac{dy}{dx} = 1$
8. $x^2y - x^3\frac{dy}{dx} = y^4\cos x, \qquad y(\pi) = \pi$

## Linear Differential Equations

The general linear differential equation of order n is of the form

$$b_0(x)\frac{d^ny}{dx^n}+b_1(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots..+b_{n-1}(x)\frac{dy}{dx}+b_n(x)y=R(x)\ \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

If $R(x)=0,\ \forall x$, then the equation is called a homogenous linear differential equation; otherwise it is called non-homogeneous differential equation. If the coefficients are constants we get

$$b_0\frac{d^ny}{dx^n}+b_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots..+b_{n-1}\frac{dy}{dx}+b_ny=R(x)\ldots\ldots\ldots\ldots\ldots\ldots\ (2)$$

which is a linear differential equation with constant coefficients. We will study two methods

(i) Inverse differential operator method (ii) Method of variation of parameters,

to solve equations of type (2).

For a homogeneous equation,

$$b_0(x)\frac{d^ny}{dx^n}+b_1(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots..+b_{n-1}(x)\frac{dy}{dx}+b_n(x)y=0$$

the general solution is of the form

$$y=c_1y_1+c_2y_2+\cdots+c_ny_n$$

where $y_1,y_2,..y_n$ are the linearly independent solutions of the given equation.

For the nonhomogeneous equation

$$b_0(x)\frac{d^ny}{dx^n}+b_1(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots..+b_{n-1}(x)\frac{dy}{dx}+b_n(x)y=R(x)$$

the general solution is of the form $y=y_c+y_p$ , where $y_c$ is the general solution of the corresponding homogenous equation

$$b_0(x)\frac{d^ny}{dx^n}+b_1(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots..+b_{n-1}(x)\frac{dy}{dx}+b_n(x)y=0$$

and $y_p$ is a particular solution of the given equation , which does not contain any arbitrary constants.

We call $y_c$ the complementary function (CF) and $y_p$ the particular integral(PI).

Linear Independence of Solutions:

Given the functions $f_1,f_2,\ldots f_n$ if constants $c_1,c_2,\ldots c_n$, not all zero, exist such that $c_1f_1(x)+c_2f_2(x)+\cdots+c_nf_n(x)=0$ for all $x$ in $a\le x\le b$, then the functions $f_1,f_2,\ldots,f_n$ are said to be linearly dependent on that interval. If no such relation exists, the functions are said to be linearly independent.

**The Wronskian of Solution:**

To test whether $n$ functions are linearly independent on an interval $a \leq x \leq b$, let us assume that each of the functions $f_1, f_2, \dots f_n$ is differentiable atleast $(n-1)$ times in $a \leq x \leq b$.

Then from the equation

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0,$$

it follows by successive differentiation that

$$c_1 f_1' + c_2 f_2' + \cdots + c_n f_n' = 0$$

$$c_1 f_1'' + c_2 f_2'' + \cdots + c_n f_n'' = 0$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$c_1 f_1^{(n-1)} + c_2 f_2^{(n-1)} + \cdots + c_n f_n^{(n-1)} = 0$$

For any fixed value of $x$ in $a \leq x \leq b$, the nature of solutions of these will be determined by the determinant $W(x) = \begin{vmatrix} f_1(x) & f_2(x) \dots\dots\dots\dots\dots & f_n(x) \\ f_1'(x) & f_2'(x) \dots\dots\dots\dots\dots & f_n'(x) \\ & \dots\dots\dots & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) \dots\dots\dots\dots & f_n^{(n-1)}(x) \end{vmatrix}$.

If $W(x_0) \neq 0$ for some $x_0$ on $a \leq x \leq b$, then $c_1 = c_2 = \cdots . c_n = 0$. Hence, functions are linearly independent on $a \leq x \leq b$. The function $W(x)$ is called Wronskian of the functions $f_1, f_2, \dots f_n$.

**Example1.** Let $y_1 = e^{ax}$ *and* $y_2 = e^{bx}$ , *then* $y_1' = ae^{ax}, y_2' = be^{bx}$ .

*Then wronskian is given by* $W = \begin{vmatrix} e^{ax} & e^{bx} \\ ae^{ax} & be^{bx} \end{vmatrix} = (b-a)e^{(a+b)x} = 0$ only if a=b.

T herefore for a≠b, $y_1 = e^{ax}$ *and* $y_2 = e^{bx}$ are linearly independent.

**Example2**. Let $y_1 = 1$ *and* $y_2 = x$, *then* $y_1' = 0, y_2' = 1$

*Then wronskian is given by* $W = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix} = 1 \neq 0$. Therefore $y_1 = 1$ *and* $y_2 = x$ are linearly independent.

**The Differential operator:**

Let $D = \frac{d}{dx}$ *then* $D^k = \frac{d^k}{dx^k}$ *for* $k = 1, 2, .....$

Then the equation (1) can be expressed as

$$b_0 D^n y + b_1 D^{n-1} y + ........... + b_n y = R(x)$$
$$i.e., (b_0 D^n + b_1 D^{n-1} + ........... + b_n) y = R(x)$$

i.e., f(D) y = R(x) where f(D) = $b_0 D^n + b_1 D^{n-1} + ........... + b_n$. D is called the differential operator.

**Properties of Differential Operator:**

**1. f(D)$e^{ax}$ = $e^{ax}$f(a)**

**Proof:** Let f(D) = $b_0 D^n + b_1 D^{n-1} + ........... + b_n$ .

Since $D^k e^{ax} = a^k e^{ax}$, for k = 1, 2, 3……n. We have,

$$f(D)e^{ax} = b_0 D^n e^{ax} + b_1 D^{n-1} e^{ax} + ........... + b_n e^{ax}$$
$$= b_0 a^n e^{ax} + b_1 a^{n-1} e^{ax} + ........ + b_n e^{ax}$$
$$= (b_0 a^n + b_1 a^{n-1} + ........ + b_n) e^{ax} = f(a) e^{ax}$$

2. f(D)$e^{ax}$ y = $e^{ax}$f(D+a)y

**Proof:** Let f(D) = $b_0 D^n + b_1 D^{n-1} + ........... + b_n$

We have D $e^{ax}$ y = y D$e^{ax}$ + $e^{ax}$ Dy = y $e^{ax}$ a + $e^{ax}$ Dy = $e^{ax}$ (D+a)y,

$\Rightarrow$ $D^2(e^{ax} y)$ = D(D($e^{ax}$ y)) = D($e^{ax}$ (D+a)y) = $e^{ax}$ (D+a)(D+a)y= $e^{ax}$ $(D+a)^2$y

Similarly we can show that $D^k(e^{ax} y) = e^{ax} (D+a)^k y$, k= 1,2,…..n.

Substituting in the formula we get the required result.

3. $$(D-a)^k e^{ax} x^j = \begin{cases} 0, & j = 0,1,2,....k-1 \\ e^{ax} k!, & j = k \end{cases}$$

**Proof:** We know that $D^k x^j = \begin{cases} 0, & j = 0,1,2,....k-1 \\ k!, & j = k \end{cases}$

From the property (2), the result follows.

**The solution of linear homogeneous differential equation with constant coefficients**

Consider the linear homogeneous differential equation with constant coefficients

$b_0\frac{d^ny}{dx^n}+b_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots..+b_{n-1}\frac{dy}{dx}+b_ny=R(x)$ where $b_0, b_1, \ldots b_n$ are all constants.

This equation can be rewritten as f(D)y = R(x) where f(D) = $b_0D^n+b_1D^{n-1}+...........+b_n$.

If y = $e^{ax}$ , then by property (1), f(D)y = f(D)$e^{ax}$ = $e^{ax}$f(a) = 0 $\Rightarrow$ f(a) = 0.

This equation is called the auxiliary or characteristic equation associated with the given differential equation. For an $n^{th}$ order differential equation, the auxiliary equation has n roots say, $a_1, a_2, ......a_n$.

Then $y_1=e^{a_1x}, y_2=e^{a_2x}, ..........., y_n=e^{a_nx}$ are all solutions of the given differential equation.

**Case 1**: If the roots of the axillary equations are all real and distinct then

$y_1=e^{a_1x}, y_2=e^{a_2x}, ..........., y_n=e^{a_nx}$ are linearly independent solutions of the given equation.

Hence the general solution is given by $y=C_1e^{a_1x}+C_2e^{a_2x}+..........+C_ne^{a_nx}$.

**Case 2**: Suppose that roots are real and not all roots are distinct. Let $a_1=a_2=......=a_k=a.$

Then the solution $y_1=y_2=.........=y_k=e^{ax}$. Then the solution is given by $y=e^{a_1x}+e^{a_2x}+..........+e^{a_nx}$ does not contain n arbitrary constants and hence cannot be the general solution. Since first k roots of the auxiliary equations are equal the given differential equation can rewritten as $g(D)(D-a)^ky=0$. Then by property (3) we observe that $y_j=e^{ax}x^j,\ j=0,1,......,k-1$ are all solutions of the given solution. Hence the general solution is

$y=(C_1+C_2x+C_3x^2+......+C_kx^{k-1})e^{ax}+C_{k+1}e^{a_{k+1}x}+......+C_ne^{a_nx}$.

**Case 3**. If not all roots of the auxiliary equation are real. Since for equation with real coefficients the roots exists in conjugate pairs, let $\alpha_1=a+ib$ $and$ $\alpha_2=a-ib$ be two roots. Then $y_1=e^{\alpha_1x}=e^{(a+ib)x}=e^{ax}(\cos bx+i\sin bx)$ $and$ $y_2=e^{\alpha_2x}=e^{(a-ib)x}=e^{ax}(\cos bx-i\sin bx)$ are the two distinct solutions. Therefore the corresponding solution is $y=C_1e^{\alpha_1x}+C_2e^{\alpha_2x}$, which can be expressed as $y_1=e^{ax}(A_1\cos bx+A_2\sin bx)$ where $A_1$ $and$ $A_2$ are arbitrary constants.

***Example 1***: Solve $(2D^2 + 5D - 12)y = 0$, where $D = \frac{d}{dx}$.

***Solution***: Auxiliary equation is $2m^2 + 5m - 12 = 0$.

i.e., $(2m - 3)(m + 4) = 0$ and it has the roots $m_1 = 3/2$ and $m_2 = -4$ which are real and distinct.

Therefore the general solution is $y = C_1 e^{3x/2} + C_2 e^{-4x}$

***Example 2***: Solve $\frac{d^2y}{dx^2} + 4y = 0$

***Solution***: Auxiliary equation is $m^2 + 4 = 0$ or $m^2 = -4$

Therefore $m = \pm 2i = 0 \pm 2i$

Therefore $y = e^{0x}(C_1 \cos 2x + C_2 \sin 2x)$

or $y = C_1 \cos 2x + C_2 \sin 2x$.

***Example 3***: Solve $\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = 0$.

***Solution***: Auxiliary equation is $m^3 + 4m^2 + 4m = 0$.

That is, $m(m + 2)^2 = 0$

Therefore $m_1 = 0$, $m_2 = -2$, $m_3 = -2$ are its roots.

Here we find that two roots are equal. Hence the general solution is given by

$y = C_1 e^{0x} + (C_2 + C_3x) e^{-2x}$

Or $y = C_1 + (C_2 + C_3x) e^{-2x}$

***Example 4*** : Solve $\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0$

***Solution***: Here the auxiliary equation is $m^4 + 8m^2 + 16 = 0$, which is simply $(m^2 + 4)^2 = 0$.

Therefore we have $m^2 + 4 = 0$ repeated. It gives $m^2 = -4$

Therefore $m = \pm 2i$.

Thus the roots are $m = \pm 2i, \pm 2i$ (imaginary, repeated).

Hence the general solution is

$y = e^{0x} [(C_1 + C_2x) \cos 2x + (C_3 + C_4x) \sin 2x]$

That is, $y = (C_1 + C_2x) \cos 2x + (C_3 + C_4x) \sin 2x$.

***Example 5*** : Solve $\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

***Solution***: Hence the A.E. is $m^4 - 2m^3 + 2m^2 - 2m + 1 = 0$.

The roots are m = 1, 1, $\pm i$

Therefore the general solution is

$y = (C_1 + C_2x)\, e^x + e^{0x}\,(C_3 \cos x + C_4 \sin x)$

That is, $y = (C_1 + C_2x)e^x + (C_3 \cos x + C_4 \sin x)$.

**The solution of linear non-homogeneous differential equation with constant coefficients**

We know that the general solution of nonhomogeneous linear differential equation

$$b_0 \frac{d^n y}{dx^n} + b_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots .. + b_{n-1} \frac{dy}{dx} + b_n y = R(x)$$

is of the form $y = y_c + y_p$ , where $y_c$ is the general solution of the corresponding homogenous equation $b_0 \frac{d^n y}{dx^n} + b_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots .. + b_{n-1} \frac{dy}{dx} + b_n y = 0$ , called the complementary function and $y_p$ is a particular solution of the given equation , which does not contain any arbitrary constants. The complementary solution can be determined using the method described above. To determine the particular solution $y_p$ , we use the following methods.

**Inverse Differential Operator Method**

If $f(D)y = \phi(x)$ then we define the inverse differential operator denoted by $\frac{1}{f(D)}$ as

$$\frac{1}{f(D)}[\phi(x)] = y \text{ , where 'D' is the differential operator } \frac{d}{dx}.$$

Thus f(D) is also a differential operator and $\frac{1}{f(D)}$ can be treated as its inverse.

**Example 1 :** $\frac{1}{D} y = \int y\,dx \;\because\; DR(x) = y \Rightarrow R(x) = \int y\,dx.$

**Properties of inverse differential operator:**

1. $\frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)} \;\; if \;\; f(a) \neq 0.$

**Proof:** We know that $f(D)e^{ax} = e^{ax}f(a)$.

If $f(a) \neq 0$, then dividing by f(a) we get

$$\frac{1}{f(a)} f(D)e^{ax} = f(D)\frac{e^{ax}}{f(a)}$$

$$\therefore \frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}$$

2. $\frac{1}{f(D-a)} e^{ax} y = e^{ax} \frac{1}{f(D)} y$

**Proof:** From the property $f(D)e^{ax}\, y = e^{ax} f(D+a)y$ the result follows.

3. $\frac{1}{D-a} y = e^{ax} \int e^{-ax} y\, dx$

**Proof:** The result follows directly from the result (2) and example (1) .

To determine the particular solution of a linear non-homogenous differential equation, we use inverse differential operators. We have , if $f(D)y = \phi(x)$ ,then $y_p = \frac{1}{f(D)} [\phi(x)]$

**Case(i):** If $\phi(x) = e^{ax}$ , then $y_p = \frac{e^{ax}}{f(a)}$ if $f(a) \neq 0$.

If $f(a) = 0$, then $f(D)$ can expressed as $f(D) = (D-a)^k \psi(D)$, where $\psi(a) \neq 0$, for k = 1,2,3,….

Then $$\frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)^k \psi(D)} e^{ax} = \frac{1}{(D-a)^k}\left\{\frac{1}{\psi(D)} e^{ax}\right\} = \frac{1}{(D-a)^k}\left\{\frac{e^{ax}}{\psi(a)}\right\}$$

$$= \frac{1}{\psi(a)}\left\{\frac{1}{(D-a)^k} e^{ax}\right\} = \frac{e^{ax}}{\psi(a)} \frac{1}{D^k} 1 = \frac{e^{ax}}{\psi(a)} \frac{x^k}{k!}$$

**Case(ii):** If $\phi(x)$= cosax or sinax then, since cosax = Re($e^{iax}$) and sinax = Im($e^{iax}$) this case reduces to the case(i) and hence can be solved.

**Case(iii)** If $\phi(x) = x^m$, for some positive integer m, then $\frac{1}{f(D)}$ can expanded as a series in positive powers of D and hence $\frac{1}{f(D)} x^m$ can be determined.

***Working Rule***:

1. If $\phi(x) = e^{ax}$, then

$$\frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(a)} \text{ if } f(a) \neq 0.$$

$$\text{If } f(a) = 0, \text{then } \frac{1}{f(D)}e^{ax} = x\frac{1}{f'(D)}e^{ax} = x\frac{e^{ax}}{f'(a)}, \text{if } f'(a) \neq 0,$$

$$= x^2\frac{1}{f''(D)}e^{ax}, \text{if } f'(a) = 0.$$

2. If $\phi(x)$ = cos ax or sin ax and if the differential operator can be written as $f(D^2)$ then

$$\frac{1}{f(D^2)}[\phi(x)] = \frac{1}{f(-a^2)}\phi(x), \text{ provided } f(-a^2) \neq 0.$$

If $f(-a^2) = 0$ and $f'(-a^2) \neq 0$, then $\frac{1}{f(D^2)}(\phi(x)) = x.\ \frac{1}{f'(-a^2)}(\phi(x))$ and so on.

**Example 1:** Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 10y = \cos 2x + e^{-3x}$

***Solution***: Auxiliary equation is $m^2 - 6m + 10 = 0$.

Therefore $m = \frac{6 \pm \sqrt{36-40}}{2.1} = \frac{6 \pm \sqrt{-4}}{2} = \frac{6 \pm 2i}{2} = 3 \pm i$

Or $m = \alpha \pm i\beta$, where $\alpha = 3$ and $\beta = 1$.

Therefore C.F. $= e^{3x}(C_1 \cos x + C_2 \sin x)$

$$\text{P.I.} = \frac{1}{D^2 - 6D + 10}(\cos 2x) + \frac{1}{D^2 - 6D + 10}(e^{-3x})$$

$$= \frac{1}{-2^2 - 6D + 10}(\cos 2x) + \frac{1}{(-3)^2 - 6(-3) + 10}e^{-3x}$$

$$= \frac{1}{6 - 6D}(\cos 2x) + \frac{1}{9 + 18 + 10}.e^{-3x}$$

$= \frac{1}{6}\frac{1+D}{1-D^2}(\cos 2x) + \frac{1}{37}e^{-3x}$, multiplying numerator and denominator by 1 + D in the first expression.

That is, P.I. $= \frac{1}{6}\frac{1+D}{1-(-2^2)}(\cos 2x) + \frac{1}{37}e^{-3x}$, by P.I. 2(a),

$\frac{1}{30}$ {1 . cos 2x + D(cos 2x)} + $\frac{1}{37}$ $e^{-3x}$.

That is, P.I = $\frac{1}{30}$(cos 2x – 2 sin 2x) + $\frac{1}{37}$ $e^{-3x}$, since D = $\frac{d}{dx}$

General solution is y = C.F. + P.I.

Therefore y = $e^{3x}$ ($C_1$ cos x + $C_2$ sin x) + $\frac{1}{30}$ (cos 2x – 2 sin 2x) + $\frac{1}{37}$ $e^{-3x}$

**Example 2:** Solve ($D^4$ + 18 $D^2$ + 81) y = $\cos^2$ x.

***Solution***: Auxiliary equation is $m^4$ + 18 $m^2$ + 81 = 0

That is $(m^2 + 9)^2 = 0$. Therefore m = ±3i, ± 3i.

Thus C.F. = $e^{0.x}$ {($C_1$ + $C_2$ x) cos 3x + ($C_3$ + $C_4$x) sin 3x}

C.F. = ($C_1$ + $C_2$x) cos 3x + ($C_3$ + $C_4$x) sin 3x.

$$\text{P.I} = \frac{1}{D^4+18D^2+81}(\cos^3 x)$$

$$= \frac{1}{D^4+18D^2+81}\left[\frac{3}{4}\cos x+\frac{1}{4}\cos 3x\right]. \qquad \{\cos^3 A = \frac{3}{4}\cos A + \frac{1}{4}\cos 3A.\}$$

$$\text{P.I} = \frac{3}{4}\,\frac{1}{D^4+18D^2+81}(\cos x) + \frac{1}{4}\,\frac{1}{D^4+18D^2+8}(\cos 3x)$$

$$= \frac{3}{4}\,\frac{1}{(-1^2)^2+18(-1)^2+81}\ (\cos x) + \frac{1}{4}\,\frac{1}{D^4+18D^2+81}(\cos 3x), \text{ by P.I. rule 2.}$$

In the second term, we observe that $f(-a^2) = f(-9) = 81 – 162 + 81 = 0$

Also $f'(D) = 4D^3 + 36 D + 0 = 4 D(D^2 + 9)$. Hence $f'(-9) = 4 D(-9 + 9) = 0$.

$$\text{Hence } \frac{1}{D^4+18D^2+81}(\cos 3x) = x^2\ \frac{1}{f''(D^2)}(\cos 3x),$$

$f''(D) = 12 D^2 + 36$

$$\text{Hence } \frac{1}{D^4+18D^2+81}(\cos 3x) = x^2\frac{1}{f''(-3^2)}\cos 3x = x^2\ \frac{1}{12(-9)+36}\ \cos 3x = -\ \frac{1}{72}\ x^2\cos 3x$$

$$\text{Thus P.I.} = \frac{3}{4}\,.\frac{1}{1-18+81}\ \cos x + \frac{1}{4}\left(-\frac{1}{72}x^2\cos 3x\right)$$

$$\text{Therefore P.I.} = \frac{3}{256}.\cos x - \frac{1}{288}.x^2\cos 3x.$$

Hence the complete solution is

$$y = (C_1 + C_2x) \cos 3x + (C_3 + C_4x) \sin 3x + \frac{3}{256} \cos x - \frac{1}{288} x^2 \cos 3x.$$

**Example 3**: Solve $(D^2 - 6D + 9)y = x^2 + x + 1$

***Solution***: Auxiliary equation $m^2 - 6m + 9 = 0$.

That is $(m - 3)^2 = 0$. Therefore $m = 3, 3$

Thus C.F. = $(C_1 + C_2\, x)e^{3x}$.

$$\text{P.I} = \frac{1}{D^2 - 6D + 9}(x^2 + x + 1)$$

$$= \frac{1}{9\left(1 - \frac{2}{3}D + \frac{1}{9}D^2\right)}(x^2 + x + 1)$$

$$= \frac{1}{9}.\left[1 - \left(\frac{2}{3}D - \frac{1}{9}D^2\right)\right]^{-1}(x^2 + x + 1) \qquad \text{(Binomial expansion } (1 - a)^{-1} = 1 + a + a^2 + \ldots)$$

$$= \frac{1}{9}\left\{1 + \left(\frac{2}{3}D - \frac{1}{9}D^2\right) + \left(\frac{2}{3}D - \frac{1}{9}D^2\right)^2 + \ldots\right\}(x^2 + x + 1)$$

$$= \frac{1}{9}\{1 + \frac{2}{3} D - \frac{1}{9}D^2 + \frac{4}{9}D^2 + \text{higher powers}\}\ (x^2 + x + 1)$$

$= \frac{1}{9}(1 + \frac{2}{3}D + \frac{1}{3}D^2)(x^2 + x + 1)$, neglecting $D^3$ and higher powers since we have only 2nd degree polynomial $x^2 + x + 1$.

$$\text{Therefore P.I.} = \frac{1}{9}\{1(x^2 + x + 1) + \frac{2}{3}\frac{d}{dx}(x^2 + x + 1) + \frac{1}{3}\frac{d^2}{dx^2}(x^2 + x + 1)\}$$

$$= \frac{1}{9}\{(x^2 + x + 1) + \frac{2}{3}(2x + 1) + \frac{1}{3}(2)\}$$

$$= \frac{1}{9}x^2 + \frac{7}{27}x + \frac{7}{27} = \frac{1}{27}(3x^2 + 7x + 7)$$

General solution $y$ = C.F. + P. I. = $(C_1 + C_2x)\, e^{3x} + \frac{1}{27}(3x^2 + 7x + 7)$.

**Example 4** : Solve $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} = 1 + x + e^{-3x}$

***Solution***: Auxiliary equation is $m^3 + 3m^2 = 0$

Thus C.F. = $(C_1 + C_2x)\, e^{0x} + C_3\, e^{-3x} = C_1 + C_2\, x + C_3\, e^{-3x}$

P.I. = $\frac{1}{D^3+3D^2}(1 + x + e^{-3x})$

$= \frac{1}{3D^2\left(1+\frac{D}{3}\right)}(1 + x) + \frac{1}{D^3+3D^2}\, e^{-3x}$

$= \frac{1}{3D^2}\left(1+\frac{D}{3}\right)^{-1}(1 + x) + x\,.\frac{1}{3D^2+6D}\, e^{-3x},$

$= \frac{1}{3D^2}\left(1-\frac{D}{3}+\frac{D^2}{9}-\ldots\right)(1 + x) + x.\ \frac{1}{3D^2+6D} e^{-3x}$

$= \frac{1}{3D^2}\left(1+x-\frac{1}{3}\right)+ x.\ \frac{1}{3(-3)^2+6(-3)}\, e^{-3x}$

$= \frac{1}{3D^2}\left(x+\frac{2}{3}\right)+\frac{x}{9}e^{-3x}.$

$= \frac{x^3}{18}+\frac{x^2}{9}+\frac{x}{9}e^{-3x}.$

General Solution is y = C.F. + P.I. = $C_1 + C_2x + C_3\, e^{-3x} + \frac{1}{9}x^2 + \frac{1}{18}x^3 + \frac{x}{9}\, e^{-3x}$

**Example 5**: Solve $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{2x} + x^3$

***Solution***: Auxiliary equation is $m^3 + 2m^2 + m = 0$

$m(m^2 + 2m + 1) = 0$ or $m\,(m + 1)^2 = 0$

Therefore m = 0, -1, -1

Thus C.F. = $C_1\, e^{0x} + (C_2 + C_3\, x)\, e^{-x} = C_1 + (C_2 + C_3\, x)\, e^{-x}$

P.I. = $\frac{1}{D^3+2D^2+D}(e^{2x}) + \frac{1}{D^3+2D^2+D}(x^3)$

Now, $\frac{1}{D^3+2D^2+D}(e^{2x}) = \frac{1}{2^3+2.2^2+2}.e^{2x} = \frac{e^{2x}}{18}$

Consider

$$\frac{1}{D^3 + 2D^2 + D}(x^3)$$

$$= \frac{1}{D(1+D)^2} x^3.$$

$$= \frac{1}{D}\left(1 - 2D + 3D^2 - 4D^3\right)x^3 \qquad \text{(Using } (1 + a)^{-2} = 1 - 2a + 3a^2 - + \ldots).$$

$$= \frac{x^4}{4} - 2x^3 + 9x^2 - 24x$$

Thus P.I. = $\frac{1}{18} e^{2x} + \frac{x^4}{4} - 2x^3 + 9x^2 - 24x$.

Hence the general solution is y = C.F. + P.I.

That is, $y = C_1 + (C_2 + C_3x) e^{-x} + \frac{1}{18}e^{2x} + \frac{x^4}{4} - 2x^3 + 9x^2 - 24x$.

**Further rules**

**Rule of P.I 4**: (i) If $\phi(x) = e^{ax}\psi(x)$ where $\psi(x)$ is a function of x then

$$\frac{1}{f(D)}\phi(x) = e^{ax}\frac{1}{f(D+a)}[\psi(x)]$$

(ii) If $\phi(x) = x.\,\psi(x)$ where $\psi(x)$ is a function of x, then

$$\frac{1}{f(D)}\phi(x) = x.\frac{1}{f(D)}[\psi(x)] - \frac{f^1(D)}{[f(D)]^2}[\psi(x)].$$

**Exercises:**

1. Solve $4\frac{d^2y}{dx^2} + 16\frac{dy}{dx} - 9y = 4e^{x/2} + 3\sin(x/4)$
2. Solve $(D^2 + 1)y = e^x + x^4 + \sin x$.

**<u>Method of Variation of Parameters:</u>**

Consider the second order linear differential equation

$$y'' + p(x)y' + q(x)y = R(x). ------(1)$$

Suppose that we know the general solution of the homogeneous equation

$y'' + p(x)y' + q(x)y = 0.$ Suppose $y_c = c_1y_1 + c_2y_2$ is the general solution.

Let us see what happens if we replace both the constants $c_1$ *and* $c_2$ with functions of x.

That is, we consider $y = A(x)y_1 + B(x)y_2$ ---------(2) and try to determine

$A(x)$ *and* $B(x)$ so that $A(x)y_1 + B(x)y_2$ is a solution of equation (1).

Note that we are involved with two unknown functions $A(x)$ *and* $B(x)$ and that we have only insisted that these functions satisfy one condition: the function in (2) is to be a solution of equation (1). We may therefore expect to impose a second condition on $A(x)$ *and* $B(x)$ in some way which would be to our advantage.

From (2) it follows that $y' = Ay_1' + By_2' + A'y_1 + B'y_2.$ -------(3)

Rather than involved with the derivatives of $A(x)$ *and* $B(x)$ of higher order than the first, we now choose some particular function for the expression $A'y_1 + B'y_2$.

Technically, we could let this function be $\sin x, e^x$, or any other suitable function. For simplicity we choose $A'y_1 + B'y_2 = 0.$ --------(4)

It then follows from (3) that

$$y'' = Ay_1'' + By_2'' + A'y_1' + B'y_2'. \text{-------}(5)$$

Since y was to be a solution of (1) we substitute from (2), (3), and (5) into equation (1) to obtain

$$A(y_1'' + py_1' + qy_1) + B(y_2'' + py_2' + qy_2) + A'y_1' + B'y_2' = R(x).$$

But $y_1$ and $y_2$ are solutions of the homogeneous equation, so that finally

$$A'y_1' + B'y_2' = R(x). \text{-------}(6)$$

Equations (4) and (6) now give us two equations that we wish to solve for $A'(x)$ *and* $B'(x)$.

This solution exists providing the determinant $W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ does not vanish. But this determinant is precisely the Wronskian of the functions $y_1$ and $y_2$, which are presumed to be linearly independent. Therefore, the Wronskian does not vanish on that interval and we can find $A'$ *and* $B'$. By integration we get

$$A(x) = -\int \frac{y_2 R(x)}{W(x)} dx \quad and \quad B(x) = \int \frac{y_1 R(x)}{W(x)} dx.$$

This argument can easily be extended to equations of order higher than two, but no essentially new ideas appear. Moreover, there is nothing in the method that prohibits the linear differential equation involved from having variable coefficients.

**Example 1 :** Solve the equation $(D^2+1)y = \sec x \tan x.$

**Solution:** Then,

$$y_c = c_1 \cos x + c_2 \sin x.$$

Let us seek a particular solution by variation of parameters. Put $y_p = A\cos x + B\sin x.$

Then $$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$$

$$A(x) = -\int \frac{y_2 R(x)}{W(x)} dx = -\int \frac{\sin x \sec x \tan x}{1} dx = -\int \tan^2 x dx = -\int \left(\sec^2 x - 1\right) dx = x - \tan x.$$

(Constant of integration has been disregarded because we are seeking only a particular solution.)

and

$$B(x) = \int \frac{y_1 R(x)}{W(x)} dx = \int \frac{\cos x \sec x \tan x}{1} dx = \int \tan x dx = \ln|\sec x|.$$

Therefore, the complete solution is

$$y(x) = c_1 \cos x + c_2 \sin x + \cos x(x - \tan x) + \sin x \ln|\sec x|$$
$$= c_1 \cos x + c_3 \sin x + x\cos x + \sin x \ln|\sec x|,$$

where the term $(-\sin x)$ in $y_p$ has been absorbed in the complementary function term $c_3 \sin x$, since $c_3$ is an arbitrary constant.

**Example 2:** Solve the equation $(D^2 - 3D + 2)y = \dfrac{1}{1+e^{-x}}.$

**Solution:** Here

$$y_c = c_1 e^x + c_2 e^{2x},$$

so we put

$$y_p = Ae^x + Be^{2x}.$$

Then $$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x}.$$

$$A(x) = -\int \frac{y_2 R(x)}{W(x)} dx = -\int \frac{e^{2x}}{(1+\mathrm{e}^{-x})\mathrm{e}^{3x}} dx = -\int \frac{e^{-x}}{1+e^{-x}} dx = \ln(1+e^{-x}).$$

$$B(x) = \int \frac{y_1 R(x)}{W(x)} dx = \int \frac{e^{-2x}}{(1+e^{-x})} dx = \int \left[ e^{-x} - \frac{e^{-x}}{1+e^{-x}} \right] dx = -e^{-x} + \ln(1+e^{-x}).$$

Therefore, the complete solution is $y(x) = c_3 e^x + c_2 e^{2x} + (e^x + e^{2x}) \ln(1+e^{-x}).$

**Exercises:**

Solve using variation of parameters:

1. $(D^2+1)y = \csc x \cot x.$
2. $(D^2+1)y = \sec^4 x.$
3. $(D^2+1)y = \tan^2 x.$
4. $(D^2+1)y = \sec^2 x \csc x.$
5. $(D^2-2D+1)y = e^{2x}(e^x+1)^{-2}.$
6. $(D^2-3D+2)y = \cos(e^{-x}).$
7. $(D^2-1)y = 2(1-e^{-2x})^{-1/2}.$

**Cauchy's homogeneous linear equation:**

An equation of the form

$$x^n \frac{d^n y}{dx^n} + k_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + k_{n-1} x \frac{dy}{dx} + k_n y = R(x) \dots\dots\dots\dots\dots\dots (1),$$

where R(x) is a function of $x$, and $k_i'$ s, $i = 1, 2..., n,$ are constants is called Cauchy's homogeneous linear equation.

Equations of this type can be reduced to linear differential equations with constant coefficients by letting $x=e^t$. Thus $t=\log x$.

If $D=\frac{d}{dt}$, then

$$\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx}=\frac{dy}{dt}.\frac{1}{x};\ \text{i.e.,}\ x\frac{dy}{dx}=Dy$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{x}.\frac{dy}{dt}\right)=\frac{1}{x^2}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right);\ \text{i.e.,}\ x^2\frac{d^2y}{dx^2}=D\left(D-1\right)y.$$

Similarly, $x^3\frac{d^3y}{dx^3}=D\left(D-1\right)\left(D-2\right)y$ and so on.

After making these substitutions in equation (1), we get a linear equation with constant coefficients which can be solved as before.

**Example 1:** Solve $x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+6y=x^2$

**Solution**: Put $x=e^t$. Then $t=\log x$. Let $D=\frac{d}{dt}$, then $x\frac{dy}{dx}=Dy$, $x^2\frac{d^2y}{dx^2}=D\left(D-1\right)y$.

The given equation becomes

$$\left(D\left(D-1\right)-4D+6\right)y=e^{2t};\quad \text{i.e.,}\ \left(D^2-5D+6\right)y=e^{2t}$$

The auxiliary equation is $m^2-5m+6=0$; i.e., $\left(m-2\right)\left(m-3\right)=0$.

$\therefore$ The roots are $m=2,3$. The complementary function is $y_c=c_1e^{2t}+c_2e^{3t}$

The particular integral is

$$y_p=\frac{1}{D^2-5D+6}e^{2t}=t\frac{1}{2D-5}e^{2t}=t\frac{e^{2t}}{2\times 2-5}=-te^{2t}$$

The complete solution is $y=y_c+y_p=c_1x^2+c_2x^3-x^2\log x.$

**Example 2**: $x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-12y=x^3\log x$

**Solution**: Put $x=e^t$. Then $t=\log x$. Let $D=\frac{d}{dt}$, then $x\frac{dy}{dx}=Dy$, $x^2\frac{d^2y}{dx^2}=D\left(D-1\right)y$.

The given equation becomes $\left(D^2+D-12\right)y=te^{3t}$

The roots of the auxiliary equation are $m=3,-4$

$\therefore$ The complementary function is $y_c = c_1 e^{3t} + c_2 e^{-4t}$

The particular integral is $y_p = \dfrac{1}{D^2 + D - 12} te^{3t} = \dfrac{e^{3t}}{7}\left(\dfrac{t^2}{2} - \dfrac{1}{7}t + \dfrac{1}{49}\right)$

The complete solution is $y = y_c + y_p = c_1 x^3 + c_2 x^{-4} + \dfrac{x^3}{7}\left(\dfrac{(\log x)^2}{2} - \dfrac{1}{7}\log x + \dfrac{1}{49}\right)$

**Example 3:** Solve $x^3 \dfrac{d^3 y}{dx^3} + 3x^2 \dfrac{d^2 y}{dx^2} + x\dfrac{dy}{dx} + y = x + \log x$

**Solution**: Put $x = e^t$. Then $t = \log x$.

Let $D = \dfrac{d}{dt}$, then $x\dfrac{dy}{dx} = Dy$, $x^2 \dfrac{d^2 y}{dx^2} = D(D-1)y$, $x^3 \dfrac{d^3 y}{dx^3} = D(D-1)(D-2)y$

The given equation becomes $\left(D^3 + 1\right) y = e^t + t$

The roots of the auxiliary equation are $m = -1, \dfrac{1 \pm \sqrt{3}i}{2}$;

$\therefore$ The complementary function is $y_c = c_1 e^{-t} + e^{t/2}\left(c_2 \cos\dfrac{\sqrt{3}}{2}t + c_3 \sin\dfrac{\sqrt{3}}{2}t\right)$

The particular integral is $y_p = \dfrac{1}{D^3 + 1}\left(e^t + t\right) = \dfrac{e^t}{2} + t$

The complete solution is

$$y = y_c + y_p = c_1 x^{-1} + \sqrt{x}\left(c_2 \cos\left(\frac{\sqrt{3}}{2}\log x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}\log x\right)\right) + \frac{x}{2} + \log x$$

**Exercise**

**Solve the following differential equations.**

**1**. $x^2 \dfrac{d^2 y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = x^4$

**2.** $x\dfrac{d^2 y}{dx^2} - \dfrac{2}{x} y = x + \dfrac{1}{x^2}$

**3.** $x^2 \dfrac{d^2 y}{dx^2} + x\dfrac{dy}{dx} + y = \sin(\log x)\log x$

**Legendre's Differentila Equation:**

This equation is of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + k_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + ... + k_{n-1}(ax+b)\frac{dy}{dx} + k_n y = R(x)...............(1),$$

where R(x) is a function of $x$, and $k_i^{'}$s, $i = 1, 2..., n,$ are constants.

Equations of this type can be reduced to linear differential equations with constant coefficients by letting $ax+b = e^t$. Thus $t = \log(ax+b)$.

If $D = \frac{d}{dt}$, then

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{dy}{dt}.\frac{1}{ax+b}.a \text{; i.e.,} (ax+b)\frac{dy}{dx} = aDy$$

$$\frac{d^2 y}{dx^2} = \frac{a^2}{(ax+b)^2} D(D-1)y\text{; i.e., } (ax+b)^2 \frac{d^2 y}{dx^2} = a^2 D(D-1)y.$$

Similarly, $(ax+b)^3 \frac{d^3 y}{dx^3} = a^3 D(D-1)(D-2)y$ and so on.

After making these substitutions in equation (1), we get a linear equation with constant coefficients which can be solved as before.

**Example 1:** Solve $(2x+3)^2 \frac{d^2 y}{dx^2} - (2x+3)\frac{dy}{dx} - 12y = 6x$

**Solution**: Put $(2x+3) = e^t$. Then $t = \log(2x+3)$.

Let $D = \frac{d}{dt}$, then $(2x+3)\frac{dy}{dx} = 2Dy$, $(2x+3)^2 \frac{d^2 y}{dx^2} = 2^2 D(D-1)y$.

The given equation becomes $(4D(D-1) - 2D - 12)y = 6\left(\frac{e^t - 3}{2}\right)$;

i.e., $(4D^2 - 6D - 12)y = 3e^t - 9$

The roots of the auxiliary equation are $m = \frac{3 \pm \sqrt{57}}{4}$;

$\therefore$ The complementary function is $y_c = c_1 e^{\frac{3+\sqrt{57}}{4}t} + c_2 e^{\frac{3-\sqrt{57}}{4}t}$

The particular integral is $y_p = \dfrac{1}{4D^2 - 6D - 12}\left(3e^t - 9\right) = \dfrac{3e^t}{-14} + \dfrac{3}{4}$

The complete solution is $y = y_c + y_p = c_1\left(2x+3\right)^{\frac{3+\sqrt{57}}{4}} + c_2\left(2x+3\right)^{\frac{3-\sqrt{57}}{4}} - \dfrac{3}{14}\left(2x+3\right) + \dfrac{3}{4}$

**Example 2**. Solve $\left(x-1\right)^3 \dfrac{d^3y}{dx^3} + 2\left(x-1\right)^2 \dfrac{d^2y}{dx^2} - 4\left(x-1\right)\dfrac{dy}{dx} + 4y = 4\log\left(x-1\right)$

**Solution**: Put $\left(x-1\right) = e^t$. Then $t = \log\left(x-1\right)$.

Let $D = \dfrac{d}{dt}$, then $\left(x-1\right)\dfrac{dy}{dx} = Dy$, $\left(x-1\right)^2 \dfrac{d^2y}{dx^2} = D\left(D-1\right)y$, $\left(x-1\right)^3 \dfrac{d^3y}{dx^3} = D\left(D-1\right)\left(D-2\right)y$

The given equation becomes $\left(D^3 - D^2 - 4D + 4\right)y = 4t$

The roots of the auxiliary equation are $m = 1, 2, -2$;

$\therefore$ The complementary function is $y_c = c_1 e^t + c_2 e^{2t} + c_3 e^{-2t}$

The particular integral is

$$y_p = \frac{1}{D^3 - D^2 - 4D + 4}4t = \frac{1}{4}\left(1 - \frac{D^3 - D^2 - 4D}{4}\right)4t = t + 1$$

The complete solution is $y = y_c + y_p = c_1\left(x-1\right) + c_2\left(x-1\right)^2 + c_3\left(x-1\right)^{-2} + \log\left(x-1\right) + 1$

**Exercise**

**Solve the following differential equations.**

**1**. $\left(3x+2\right)^2 \dfrac{d^2y}{dx^2} + 5\left(3x+2\right)\dfrac{dy}{dx} - 3y = x^2 + x + 1$

**2.** $\left(x+1\right)^2 \dfrac{d^2y}{dx^2} + \left(x+1\right)\dfrac{dy}{dx} + y = 2\sin\left(\log\left(x+1\right)\right)$

**3.** $\left(2x-1\right)^2 \dfrac{d^2y}{dx^2} + \left(2x-1\right)\dfrac{dy}{dx} - 2y = 8x^2 - 2x + 3$

**System of linear differential equations with constant coefficients**

Quite often we come across a system of linear differential equations with constant coefficients in which there are two or more dependent variables and a single independent variable exists. Such a system of equation can be solved by eliminating all but one of the dependent variables, and solving the resulting equation. Then using he given equations, the other dependent variables can be expressed in terms of the dependent variable which is obtained earlier and can be determined.

**Example 1:**

Solve the simultaneous equations:

$$\frac{dx}{dt}+5x-2y=t;\frac{dy}{dt}+2x+y=0, \text{ given } x=y=0 \text{ when } t=0.$$

**Solution:** Taking $\frac{d}{dt}\equiv D,$ the given equations become

$$(D+5)x-2y=t-----------(i)$$
$$2x+(D+1)y=0-----------(ii)$$

Eliminate x as if D were an ordinary algebraic multiplier. Multiplying (i) by 2 and operating on (ii) by (D+5) and then subtracting, we get

$$(D^2+6D+9)y=-2t.$$

Its complementary function is $y_c(t)=(c_1+c_2t)e^{-3t}$ and a particular integral is

$$y_p(t)=\frac{1}{(D+3)^2}(-2t)=-\frac{2t}{9}+\frac{4}{27}.$$

Thus $y(t)=y_c+y_p.$

when t=0, $0=y\Rightarrow c_1=-\frac{4}{27}.$

Substituting the value of y in (ii), we obtain

$$x(t)=\left[\left(-\frac{4}{27}-\frac{1}{2}c_2\right)+c_2t\right]e^{-3t}+\frac{t}{9}+\frac{1}{27}.$$

when t = 0, $0 = x \Rightarrow c_2 = -\frac{2}{9}.$

Hence the desired solutions are

$$x(t) = -\frac{1}{27}(1+6t)e^{-3t} + \frac{1}{27}(1+3t);$$
$$y(t) = -\frac{2}{27}(2+3t)e^{-3t} + \frac{2}{27}(2-3t).$$

**Example 2:**

Solve the simultaneous equations

$\frac{dx}{dt} + 2y + \sin t = 0; \frac{dy}{dt} - 2x - \cos t = 0$ given that x = 0 and y = 1 when t = 0.

**Solution:** Taking $\frac{d}{dt} \equiv D,$ the given equations become

$$Dx + 2y = -\sin t ---------- (i)$$
$$-2x + Dy = 0\cos t ---------(ii)$$

Eliminating x by multiplying (i) by 2 and operating on (ii) by D and then adding, we get

$$(D^2 + 4)y = -3\sin t.$$

Its complementary function is $y_c(t) = c_1 \cos 2t + c_2 \sin 2t$ and a particular integral is

$$y_p(t) = -3\frac{1}{D^2+4}\sin t = -\sin t.$$

Thus $y(t) = y_c + y_p.$

When t=0, $1 = y \Rightarrow c_1 = 1.$

Substituting the value of y in (ii), we obtain

$$x(t) = -\sin 2t + c_2 \cos 2t - \cos t.$$

When t = 0, $0 = x \Rightarrow c_2 = 1.$

Hence the desired solutions are

$$x(t) = \cos 2t - \sin 2t - \cos t;$$
$$y(t) = \cos 2t + \sin 2t - \sin t.$$

**Example 3:**

Solve the simultaneous equations

$$\frac{dx}{dt}+\frac{dy}{dt}-2y=2\cos t-7\sin t;\ \frac{dx}{dt}-\frac{dy}{dt}+2x=4\cos t-3\sin t.$$

**Solution:** Taking $\frac{d}{dt}\equiv D,$ the given equations become

$$Dx+(D-2)y=2\cos t-7\sin t----------(i)$$
$$(D+2)x-Dy=4\cos t-3\sin t---------(ii)$$

Eliminating y by operating on (i) by D and operating on (ii) by (D-2) and then adding, we get

$$(D^2-2)x=-9\cos t.$$

Its complementary function is $x_c(t)=c_1e^{\sqrt{2}t}+c_2e^{-\sqrt{2}t}$ and a particular integral is

$$x_p(t)=3\cos t.$$

Thus $x(t)=x_c+x_p.$

Substituting the value of x in (ii), we obtain

$$x(t)=(\sqrt{2}+1)c_1e^{\sqrt{2}t}-(\sqrt{2}-1)c_2e^{-\sqrt{2}t}+2\sin t+c_3.$$

where $c_1,c_2$ and $c_3$ are arbitrary constants.

**Exercise:**

*Solve the following simultaneous equations:*

i. $\frac{dx}{dt}+y-\sin t=0;\frac{dy}{dt}+x-\cos t=0$ *given that x = 2 and y = 0 when t = 0.*

ii. $(D-1)x+Dy=2t+1;(2D+1)x+2Dy=t$ .

iii. $(D+1)x+(2D+1)y=e^t;(D-1)x+(D+1)y=1.$

iv. $(D^2+1)y4(D-1)v=4e^x;(D-1)y+(D+9)v=0$ *given y = 5, dy/dx = 0, v=1/2 at x = 0.*