29/11/2021

1. Vector Space

Definition 1.1. (**Basis**)A set of all vectors $S = \{v_1, v_2, ..., v_n\}$ in V is said to be a basis for V if,

- . S is linearly independent.
- . S spans V.

→ generates

Definition 1.2. (**Dimension of a vector space**) The number elements in the basis of a vector space V is called the dimension of a vector space. It is denoted by $dim(V)$.

Problem 1.3. *Prove that* $S = \{(1,1,1), (1,1,0), (1,0,0)\}$ $\subseteq \mathbb{R}^3$ *form a basis for* $\mathbb{R}^3$. *Express* $(2,-3,5)$ *in terms of basis elements in* S.

Ans:- Let $v_1 = (1,1,1)$, $v_2 = (1,1,0)$, $v_3 = (1,0,0)$

we've, $\mathbb{R}^3$ is a vector space over $\mathbb{R}$

Let

$$c_1v_1 + c_2v_2 + c_3v_3 = \vec{0}$$

$$\Rightarrow c_1(1,1,1) + c_2(1,1,0) + c_3(1,0,0) = (0,0,0)$$

$$\Rightarrow (c_1+c_2+c_3, c_1+c_2, c_1) = (0,0,0)$$

$$\Rightarrow c_1+c_2+c_3 = 0 \Rightarrow c_3 = 0$$

$$c_1 + c_2 = 0 \Rightarrow c_2 = 0$$

$$c_1 = 0$$

$\therefore c_1 = c_2 = c_3 = 0$ $\therefore$ S is linearly independent

Let $(x,y,z) \in \mathbb{R}^3 = \mathbb{R}\times\mathbb{R}\times\mathbb{R}$, $x, y, z \in \mathbb{R}$

Let $(x,y,z) = c_1v_1 + c_2v_2 + c_3v_3$

$\Rightarrow (x,y,z) = c_1(1,1,1) + c_2(1,1,0) + c_3(1,0,0)$

$\Rightarrow (x,y,z) = (c_1+c_2+c_3,\ c_1+c_2,\ c_1)$

$\Rightarrow \quad c_1+c_2+c_3 = x \Rightarrow c_3 = x-y \in \mathbb{R}$

$\quad c_1+c_2 = y \Rightarrow c_2 = y-z \in \mathbb{R}$

$\quad c_1 = z \in \mathbb{R}$

$\therefore \underline{\underline{(x,y,z)}} = z(1,1,1) + (y-z)(1,1,0) + (x-y)(1,0,0)$

$= zV_1 + (y-z)V_2 + (x-y)V_3$

$\therefore$ S spans $\mathbb{R}^3$.

$\therefore$ S form a basis for $\mathbb{R}^3$.

$\therefore \underset{=x}{(2},\underset{=y}{-3},\underset{=z}{5)} = 5V_1 + {}^{-}8V_2 + 5V_3$

Q. We know, $\mathbb{R}^n$ is a vector space over $\mathbb{R}$. Let $S = \{e_1 = (1,0,0 \cdots 0),\ e_2 = (0,1,0 \cdots 0), \cdots\ e_n = (0,0, \cdots 0,1)\} \subseteq \mathbb{R}^n$ whether S form a basis for $\mathbb{R}^n$ or not?

Ans:

Let $c_1e_1 + c_2e_2 + \cdots + c_ne_n = \vec{0}$ ✓

then $c_1(1,0,0 \ldots 0) + c_2(0,1,0 \ldots 0) + \cdots + c_n(0,0, \ldots 0,1) = (0,0, \ldots, 0)$

$\Rightarrow (c_1, c_2, \ldots c_n) = (0,0, \ldots, 0)$

$\Rightarrow c_1 = 0, \; c_2 = 0 \; \ldots \; c_n = 0$

$\Rightarrow S$ is linearly independent.

Let $(\underbrace{x_1, x_2, \ldots x_n}_{\in \mathbb{R}}) \in \mathbb{R}^n$ then

$$(x_1, x_2, \ldots x_n) = x_1(1,0, \ldots 0) + x_2(0,1,0, \ldots 0) + x_3(0,0,1, \ldots 0) + \cdots + x_n(0,0, \ldots 0,1)$$

$$= x_1e_1 + x_2e_2 + \cdots + x_ne_n$$

$\dim(\mathbb{R}^n) = n$.

$\therefore S$ spans $\mathbb{R}^n$.

$\Rightarrow S$ is a basis for $\mathbb{R}^n$. This is called the STANDARD BASIS for $\mathbb{R}^n$

Problem 1.4. *Test whether the set of vectors*
✓$S = \{(1,1,2),(1,2,3),(0,-1,1)\}$ *form a basis for* $\mathbb{R}^3$ *or not. If so, express the vector* $(1,1,1)$ *in terms of basis elements.*

Ans:- Let $v_1 = (1,1,2)$, $v_2 = (1,2,3)$, $v_3 = (0,-1,1)$

We know, $\mathbb{R}^3$ is a vector space over $\mathbb{R}$. ✓

Let $c_1v_1 + c_2v_2 + c_3v_3 = \vec{0} = (0,0,0)$

then $(c_1, c_1, 2c_1) + (c_2, 2c_2, 3c_2) + (0, -c_3, c_3) = (0,0,0)$

$\Rightarrow (c_1+c_2,\ c_1+2c_2-c_3,\ 2c_1+3c_2+c_3) = (0,0,0)$

$$\Rightarrow \left.\begin{aligned} c_1+c_2 &= 0 \\ c_1+2c_2-c_3 &= 0 \\ 2c_1+3c_2+c_3 &= 0 \end{aligned}\right\} (*)$$

$\therefore$ |coeff. matrix of $(*)$| $= \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 2 & 3 & 1 \end{vmatrix}$

$= 2 \neq 0$

$\therefore c_1 = c_2 = c_3 = 0$ $\therefore$ S is linearly independent.

Let $\overbrace{(x,y,z)}^{\in\mathbb{R}} \in \mathbb{R}^3$ and

$$(x,y,z) = c_1 v_1 + c_2 v_2 + c_3 v_3$$

then $(x,y,z) = (c_1 + c_2,\ c_1 + 2c_2 - c_3,\ 2c_1 + 3c_2 + c_3)$

$$\Rightarrow \quad \begin{aligned} c_1 + c_2 &= x \\ c_1 + 2c_2 - c_3 &= y \\ 2c_1 + 3c_2 + c_3 &= z \end{aligned}$$

$\therefore$ By Cramer's rule,

$$c_1 = \frac{1}{2}\begin{vmatrix} x & 1 & 0 \\ y & 2 & -1 \\ z & 3 & 1 \end{vmatrix} = \frac{5x - y - z}{2} \in \mathbb{R}$$

$$c_2 = \frac{1}{2}\begin{vmatrix} 1 & x & 0 \\ 1 & y & -1 \\ 2 & z & 1 \end{vmatrix} = \frac{-3x + y + z}{2} \in \mathbb{R}$$

$$c_3 = \frac{1}{2}\begin{vmatrix} 1 & 1 & x \\ 1 & 2 & y \\ 2 & 3 & z \end{vmatrix} = \frac{-x - y + z}{2} \in \mathbb{R}$$

$$\therefore (x,y,z) = \left(\frac{5x - y - z}{2}\right) v_1 + \left(\frac{-3x + y + z}{2}\right) v_2 + \left(\frac{-x - y + z}{2}\right) v_3$$

$\therefore$ S spans $\mathbb{R}^3$.

$\Rightarrow$ S forms a basis for $\underline{\underline{\mathbb{R}^3}}$.

$$\therefore (\overset{=x}{1}, \overset{=y}{1}, \overset{=z}{1}) = \frac{3}{2} V_1 - \frac{1}{2} V_2 - \frac{1}{2} V_3$$

Problem 1.5. *Test whether the set of vectors* $S = \{(1,1,1),(1,0,1),(1,1,0)\}$ *form a basis for* $\mathbb{R}^3$ *or not. If so, express the vector* $(1,2,3)$ *in terms of basis elements.*

Dfn:- Let $S = \{v_1, v_2, \cdots v_n\}$ be the set of n-dimensional vectors then S is said to be

(i) orthogonal if $\langle v_i, v_j \rangle = 0 \; \forall i \neq j$

innerproduct

Here, for n-dimensional vectors

$$\langle v_i, v_j \rangle = v_i \cdot v_j = 0 \quad \forall \; i \neq j$$

(ii) orthonormal if

$\|v_i\| = \sqrt{\langle v_i, v_i \rangle} = \sqrt{v_i \cdot v_i}$

(a) S is orthogonal

(b) $\langle v_i, v_j \rangle = 1$ if $i = j$

or

Also, $\langle v_i, v_i \rangle = \underbrace{\|v_i\|}_{\text{norm of } v_i}{}^2 = 1$

OR

S is orthonormal if

$$\langle v_i, v_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Definition 1.6. • **Orthogonal vectors:** The set of n-dimensional vectors $\{v_1, v_2, ..., v_n\}$ is said to be orthogonal if $\langle v_i, v_j\rangle = v_i.v_j = 0$ if $i \neq j$..

- **Orthonormal vectors:** The set of n-dimensional vectors $\{v_1, v_2, ..., v_n\}$ is said to be orthonormal if $\langle v_i, v_j\rangle = v_i.v_j = 0$ if $i \neq j$. and $\langle v_i, v_i\rangle = \|v_i\|^2 = 1$.

Note 1.7. *For any n-dimensional vector v_1, we have, the norm of v_1, is defined as* $\|v_1\| = \sqrt{\langle v_1, v_1\rangle} = \sqrt{v_1.v_1}$

Problem 1.8. *Show that the set of vectors $B = \{v_1 = (3, 0, 4), v_2 = (-4, 0, 3), v_3 = (0, 1, 0)\}$ is an orthogonal set.*

Ans:- Here $\langle V_1, V_2\rangle = V_1 \cdot V_2 = -12+0+12 = 0 = \langle V_2, V_1\rangle$

$\langle V_1, V_3\rangle = V_1 \cdot V_3 = 0+0+0 = 0 = \langle V_3, V_1\rangle$

$\langle V_2, V_3\rangle = V_2 \cdot V_3 = 0+0+0 = 0 = \langle V_3, V_2\rangle$

$\therefore$ B is an orthogonal set in R^3

Problem 1.9. *Show that the set of vectors* $B = \left\{ v_1 = \left(\frac{3}{5}, 0, \frac{4}{5}\right), v_2 = \left(\frac{-4}{5}, 0, \frac{3}{5}\right), v_3 = (0, 1, 0) \right\}$ *is an orthonormal set.*

Ans:- Here, $\langle V_1, V_2 \rangle = V_1 \cdot V_2 = \frac{-12}{25} + 0 + \frac{12}{25} = 0 = \langle V_2, V_1 \rangle$

$\langle V_1, V_3 \rangle = V_1 \cdot V_3 = 0 + 0 + 0 = 0 = \langle V_3, V_1 \rangle$

$\langle V_2, V_3 \rangle = V_2 \cdot V_3 = 0 + 0 + 0 = 0 = \langle V_3, V_2 \rangle$

$\therefore$ B is an orthogonal set in $\mathbb{R}^3$.

Also,

$\langle V_1, V_1 \rangle = V_1 \cdot V_1 = \frac{9}{25} + 0 + \frac{16}{25} = 1 = \|V_1\|^2$

$\langle V_2, V_2 \rangle = V_2 \cdot V_2 = \frac{16}{25} + 0 + \frac{9}{25} = 1 = \|V_2\|^2$

$\langle V_3, V_3 \rangle = V_3 \cdot V_3 = 0 + 1 + 0 = 1 = \|V_3\|^2$

$\therefore$ B is an orthonormal set in $\mathbb{R}^3$.

2. Gram-Schmidt Orthogonalization Process

Construction of orthonormal set from a linearly independent set of vectors

Consider a linearly independent set of n-dimensional vectors $S = \{a_1, a_2, ..., a_n\}$

1. Take $v_1 = a_1$ then $u_1 = \dfrac{v_1}{\|v_1\|}$ where $\|v_1\| = \sqrt{\langle v_1, v_1\rangle} = v_1.v_1$
2. $v_2 = a_2 - \langle a_2, u_1\rangle u_1$ then $u_2 = \dfrac{v_2}{\|v_2\|}$
3. $v_3 = a_3 - \langle a_3, u_1\rangle u_1 - \langle a_3, u_2\rangle u_2$ then $u_3 = \dfrac{v_3}{\|v_3\|}$ and so on

then, the orthonormal set of vectors are $U = \{u_1, u_2, u_3, ...\}$

Let $S = \{a_1, a_2, \cdots a_n\}$ be the set of linearly independent n-dimensional vectors.

Step1: Let $V_1 = a_1$ then $u_1 = \dfrac{V_1}{\|V_1\|}$ where

$$\|V_1\| = \sqrt{\langle V_1, V_1\rangle} = \sqrt{V_1 \cdot V_1}$$

Step2: Let $V_2 = a_2 - \langle a_2, u_1\rangle u_1$

then $u_2 = \dfrac{V_2}{\|V_2\|}$ where

$$\|V_2\| = \sqrt{\langle V_2, V_2\rangle}$$

$u_1 \quad u_2, u_3$

Step 3:- Let $V_3 = a_3 - \langle a_3, u_1 \rangle u_1$
$\qquad - \langle a_3, u_2 \rangle u_2$

then $u_3 = \dfrac{V_3}{\|V_3\|}$

Step 4:- Let $V_4 = a_4 - \langle a_4, u_1 \rangle u_1$
$\qquad - \langle a_4, u_2 \rangle u_2$
$\qquad - \langle a_4, u_3 \rangle u_3$

then $u_4 = \dfrac{V_4}{\|V_4\|}$

$\vdots$

Continue like this,

Let $V_n = a_n - \langle a_n, u_1 \rangle u_1 - \langle a_n, u_2 \rangle u_2$
$\qquad - - - - \; - \langle a_n, u_{n-1} \rangle u_{n-1}$

then $u_n = \dfrac{V_n}{\|V_n\|}$

$\therefore U = \{u_1, u_2, \ldots u_n\}$ is the req'd orthonormal set.

Problem 2.1. *Construct an orthonormal set of vectors from the given set of linearly independent vectors* $S = \{(1,2),(2,3)\}$. $\subseteq \mathbb{R}^2$

Ans:- Let $a_1 = (1,2)$ & $a_2 = (2,3)$

Step 1:. Let $v_1 = a_1 = (1,2)$ then $u_1 = \dfrac{v_1}{\|v_1\|}$

Here, $\|v_1\| = \sqrt{\langle v_1, v_1\rangle} = \sqrt{v_1 \cdot v_1} = \sqrt{1+2^2} = \sqrt{5}$

$$\therefore u_1 = \frac{1}{\sqrt{5}}(1,2) = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right).$$

Step 2:. Let $v_2 = a_2 - \langle a_2, u_1\rangle u_1$

Here, $\langle a_2, u_1\rangle = a_2 \cdot u_1 = \dfrac{2}{\sqrt{5}} + \dfrac{6}{\sqrt{5}} = \dfrac{8}{\sqrt{5}}$

$$\therefore v_2 = (2,3) - \frac{8}{\sqrt{5}}\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

$$= (2,3) - \left(\frac{8}{5}, \frac{16}{5}\right) = \left(\frac{2}{5}, \frac{-1}{5}\right)$$

$\therefore u_2 = \dfrac{v_2}{\|v_2\|}$ where $\|v_2\| = \sqrt{v_2 \cdot v_2} = \sqrt{\dfrac{4}{25} + \dfrac{1}{25}}$

$$= \frac{1}{\sqrt{5}}$$

$\therefore\ u_2 = \sqrt{5}\left(\frac{2}{5}, \frac{-1}{5}\right) = \left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$

$\therefore$ The required orthonormal set is

$$\left\{ u_1 = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right),\ u_2 = \left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right) \right\}$$

Using Gram- Schmidt Process,

Problem 2.2. *Construct an orthonormal set of vectors from the given set of linearly independent vectors*

$$B = \{(1,1,1), (-1,0,-1), (-1,2,3)\} \subseteq \mathbb{R}^3$$

<u>Ans</u>:- Let $a_1 = (1,1,1)$, $a_2 = (-1,0,-1)$, $a_3 = (-1,2,3)$

<u>Step1</u>:- Let $v_1 = a_1 = (1,1,1)$ then $u_1 = \frac{v_1}{\|v_1\|}$

Here, $\|v_1\| = \sqrt{v_1 \cdot v_1} = \sqrt{1+1+1} = \sqrt{3}$

$\therefore u_1 = \frac{1}{\sqrt{3}}(1,1,1) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

<u>Step2</u>:- Let $v_2 = a_2 - \langle a_2, u_1 \rangle u_1$

Here, $\langle a_2, u_1 \rangle = (-1,0,-1) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{-2}{\sqrt{3}}$

$\therefore \langle a_2, u_1 \rangle u_1 = \frac{-2}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \left(\frac{-2}{3}, \frac{-2}{3}, \frac{-2}{3}\right)$

$\therefore v_2 = (-1,0,-1) - \left(\frac{-2}{3}, \frac{-2}{3}, \frac{-2}{3}\right) = \left(\frac{-1}{3}, \frac{2}{3}, \frac{-1}{3}\right)$

$\therefore u_2 = \frac{v_2}{\|v_2\|}$ where $\|v_2\| = \sqrt{v_2 \cdot v_2}$

$= \sqrt{2/3}$

$\therefore u_2 = \sqrt{3/2}\left(\frac{-1}{3}, \frac{2}{3}, \frac{-1}{3}\right) = \left(\frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)$

Step 3 :.

Let $v_3 = a_3 - \langle a_3, u_1 \rangle u_1 - \langle a_3, u_2 \rangle u_2$

Here,

$\langle a_3, u_1 \rangle = a_3 \cdot u_1 = (-1, 2, 3) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$

$= 4/\sqrt{3}$

$\therefore \langle a_3, u_1 \rangle u_1 = \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3} \right).$

Also,

$\langle a_3, u_2 \rangle = a_3 \cdot u_2 = (-1, 2, 3) \cdot \left(\frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \right)$

$= 2/\sqrt{6}$

$\therefore \langle a_3, u_2 \rangle u_2 = \left(\frac{-2}{6}, \frac{4}{6}, \frac{-2}{6} \right)$

$\therefore v_3 = (-1, 2, 3) - \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3} \right) - \left(\frac{-2}{6}, \frac{4}{6}, \frac{-2}{6} \right)$

$= (-2, 0, 2)$

then $u_3 = \frac{v_3}{\|v_3\|}$ where $\|v_3\| = \sqrt{v_3 \cdot v_3}$

$= \sqrt{8}$

$\therefore u_3 = \frac{1}{\sqrt{8}} (-2, 0, 2) = \left(\frac{-2}{\sqrt{8}}, 0, \frac{2}{\sqrt{8}} \right)$

$\therefore$ The required orthonormal set is,

$$U = \left\{ u_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), u_2 = \left(\frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right), u_3 = \left(\frac{-2}{\sqrt{8}}, 0, \frac{2}{\sqrt{8}}\right) \right\}$$

Q. Show that an orthonormal set of non-zero vectors is linearly independent.

Proof:- Let $A = \{v_1, v_2, \cdots v_n\}$ be an orthonormal set of nonzero vectors.

To prove A is linearly independent.

Let $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \vec{0}$ ✓

then $\langle \vec{0}, v_k \rangle = 0 \quad \forall \ v_k \in A$ ✓

$\Rightarrow \langle \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n, v_k \rangle = 0$ ✓

$$\Rightarrow \langle \alpha_1 v_1, v_k\rangle + \langle \alpha_2 v_2, v_k\rangle + \cdots + \langle \alpha_k v_k, v_k\rangle + \cdots$$
$$+ \langle \alpha_n v_n, v_k\rangle = 0 \checkmark$$

$$\Rightarrow \alpha_1 \underbrace{\langle v_1, v_k\rangle}_{=0} + \alpha_2 \underbrace{\langle v_2, v_k\rangle}_{=0} + \cdots + \alpha_k \underbrace{\langle v_k, v_k\rangle}_{=1} + \cdots$$
$$+ \alpha_n \underbrace{\langle v_n, v_k\rangle}_{=0} = 0 \quad \text{———} (*) \checkmark$$

Since A is orthonormal, $\langle v_i, v_j\rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

$\therefore (*) \Rightarrow \quad \alpha_k \cdot 1 = 0 \quad \forall \; k = 1, 2 \ldots n$

$\Rightarrow \quad \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0 \Rightarrow A$ is linearly independent.

Q. Show that an orthogonal set of non-zero vectors is linearly independent.

Proof: Let $A = \{v_1, v_2, \ldots v_n\}$ be the orthogonal set of non-zero vectors.

To prove A is linearly independent.

Let $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \vec{0}$ then

$$\langle \vec{0}, v_k\rangle = 0 \quad \forall \; v_k \in A$$

$$\Rightarrow \langle \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n, v_k\rangle = 0 \quad \forall \; k = 1, 2 \ldots n$$

$$\Rightarrow \langle \alpha_1 v_1, v_k\rangle + \langle \alpha_2 v_2, v_k\rangle + \cdots + \langle \alpha_k v_k, v_k\rangle + \cdots + \langle \alpha_n v_n, v_k\rangle = 0$$
$$\forall \; k = 1, 2 \ldots n$$

$$\Rightarrow \alpha_1 \underbrace{\langle v_1, v_k\rangle}_{=0} + \alpha_2 \underbrace{\langle v_2, v_k\rangle}_{=0} + \cdots + \alpha_k \underbrace{\langle v_k, v_k\rangle}_{=\|v_k\|^2} + \cdots + \alpha_n \underbrace{\langle v_n, v_k\rangle}_{0=} = 0 \quad \forall k = 1, 2 \ldots n \quad (*)$$

Since A is orthogonal, $\langle v_i, v_j \rangle = 0 \ \forall \ i \neq j$.

Also, $\langle v_i, v_i \rangle = \|v_i\|^2$

$\therefore \circledast \Rightarrow \alpha_k \|v_k\|^2 = 0 \quad \forall \ k = 1, 2, \ldots n$

$\Rightarrow$ either $\alpha_k = 0$ or $\|v_k\|^2 = 0 \quad \forall \ k = 1, 2 \ldots n$

Since v_k's are non-zero vectors, $\|v_k\|^2 = 0$ is impossible.

$\therefore$ The only possibility is $\alpha_k = 0 \ \forall \ k = 1, 2 \ldots n$

$\therefore$ A is linearly independent.