Formation of a D.E.

→ By eliminating arbitrary constants

Eg. Eliminate the arbitrary constants and form the d.e. from the eqn

$$y = e^x(A\cos x + B\sin x)$$

$$\frac{dy}{dx} = y + e^x(-A\sin x + B\cos x) \quad (1)$$

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x(-A\sin x + B\cos x) + e^x(-A\cos x$$

→ $y = e^x(A\cos x + B\sin x)$

$\Rightarrow \frac{dy}{dx} = e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x)$

$\Rightarrow \frac{dy}{dx} = y + e^x(-A\sin x + B\cos x)$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x(-A\sin x + B\cos x) + e^x(-A\cos x - B\sin x)$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x(-A\sin x$

Q. Form the diff. eq^n from $y = ax^3 + bx^2$ by eliminating the arbitrary constants

$y = ax^3 + bx^2$ —— (*)

$y' = 3ax^2 + 2xb$ —— (1)

$y'' = 6ax + 2b$ —— (2)

~~$2\times(1) - (2)\times x \Rightarrow 2y' - xy'' = 3ax^2 + 2xb -$~~

$2\times(1) - x\times(2) \Rightarrow 2y' - xy'' = 2xb$

$\Rightarrow \boxed{b = \frac{2y' - xy''}{2x}}$

$\Rightarrow \boxed{2xb = 2y' - xy''}$

$(1) \Rightarrow y' = 3ax^2 + 2y' - xy''$

$\boxed{a = \frac{y + xy''}{3x^2}}$

(*) $y = ax^3 + bx^2$

$= x^3\left(\frac{xy'' - y'}{3x^2}\right) + b$

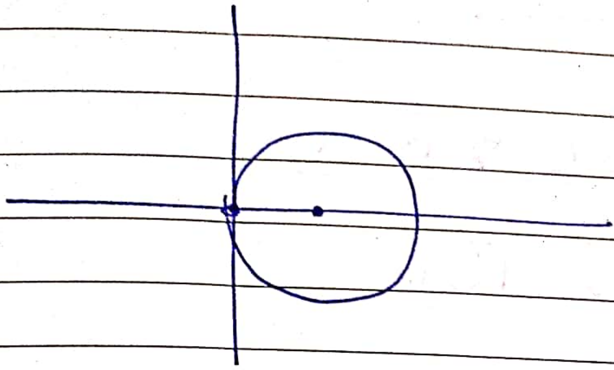
$y = \frac{x}{3}(xy'' - y') + \frac{x}{2}(2y' - xy'')$

$6y = 2xy'' - 2xy' + 6xy' - 3x^2y''$

$\boxed{0 = -3x^2y'' + 2xy'' + 4xy'}$

Page :
Date :

Q. Form the diff. eq^n of all circles touching the ~~axes~~ axis of y at the origin and centre on the x axis



$$(x^2 + y^2 + 2ax + 2by + c) = 0$$

or, $$\boxed{(x-a)^2 + y^2 = a^2}$$

$$x^2 + a^2 - 2ax + y^2 = a^2$$

$$\boxed{y^2 + x^2 - 2ax = 0} \quad — (1)$$

diff w.r.t. x

$$\Rightarrow 2yy' + 2x - 2a = 0$$

$$\boxed{a = yy' + x} \quad — (2)$$

replacing a in (1) using (2)

$$x^2 + y^2 - 2(yy' + x)x = 0$$

$$x^2 + y^2 - 2yy'x - 2x^2 = 0$$

$$\boxed{y^2 - x^2 - 2yy'x = 0}$$

Solution of differential Eq^n

1st order 1st degree

(1.1) Variable Seperable Form

if $\frac{dy}{dx} = \phi(x)\,\psi(y)$

$$\frac{dy}{\psi(y)} = \phi(x)\,dx$$

Integrate both side,

$$\int \frac{dy}{\psi(y)} = \int \phi(x)\,dx + C$$

$$y = h(x) + c$$

Q. $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

$$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

$$\tan^{-1}(y) = \tan^{-1}(x) + C$$

Q. $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

$$\int \frac{\tan y\, dy}{\sec^2 y} + \int \cancel{\frac{\tan x\, dx}{\sec^2 d(x)}} \; \frac{\sec^2 x\, dx}{\tan x}$$

$\cancel{\sin y \cos y\, dy}$

$\log \tan x + \log \tan y = k$

$\log(\tan x \tan y) = k$

$\boxed{\tan x \tan y = e^k = k_1}$

Q. $\dfrac{dy}{dx} = \dfrac{x(2\log x + 1)}{\sin y + y\cos y}$

$(\sin y + \underset{\text{I}}{y}\,\underset{\text{II}}{\cos y})dy = (2x\log x + x)\,dx$

$-\cos y + (y\sin y + \cos y) = 2\log x \cdot \left(\frac{x^2}{2}\right) - 2\int \left(\frac{1}{x}\left(\frac{x^2}{2}\right)\right)dx + \frac{x^2}{2} + c$

$y\sin y = \cancel{2}\log x\left(\frac{x^2}{\cancel{2}}\right) - \cancel{\frac{x^2}{2}} + \cancel{\frac{x^2}{2}} + c$

$\boxed{y\sin y = x^2\log x + c}$

Q $e^x(y-1)\,dx + 2(e^x + 4)\,dy = 0$

ans: $(e^x + 4)(y-1)^2 = c$

Initial Value Problems

Q. solve $\frac{dy}{dx} = \cancel{y}\, x e^{y-x^2}$, given $y(0)=0$

$\rightarrow$ $\frac{dy}{e^y} = \frac{x\,dx}{e^{x^2}}$ $\quad (-x^2 = t)$ $\quad (\cancel{2}x\,dx = \frac{dt}{2})$

$\frac{dy}{e^y} = \frac{dt}{\cancel{2e^t}} \quad \frac{-1}{2} e^t dt$

$\frac{-1}{e^y} = \frac{-1}{2e^{x^2}} + C$.

$+\frac{1}{e^y} = +\frac{1}{2e^{x^2}} + C$

when $x=0, y=0$:

$\frac{1}{1} = +\frac{1}{2} + C$

$\boxed{\frac{+1}{2} = C}$

$\therefore$ Ans $= \frac{1}{e^y} + \frac{1}{2e^{x^2}} + \frac{1}{2}$

Page :
Date :

(1.2) Reducable to variable seperable form

Type I : $\frac{dy}{dx} = f(ax+by+c)$

$$ax+by+c = t$$

$$\frac{dt}{dx} = a + b\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{b}\left(\frac{dt}{dx} - a\right)$$

$$\frac{1}{b}\left(\frac{dt}{dx} - a\right) = f(t)$$

$$\frac{dt}{dx} - a = bf(t)$$

$$\frac{dt}{dx} = bf(t) + a$$

$$\boxed{\frac{dt}{(bf(t)+a)} = dx}$$

variable seperable form

Type2: $\frac{dy}{dx} = \frac{(ax+by)+c}{k(ax+by)+c_1}$ — (1)

$$ax+by = t$$

$$\frac{dt}{dx} = a + b\frac{dy}{dx}$$

$$\frac{dy}{dx} = \left(\frac{dt}{dx} - a\right)\frac{1}{b} \quad -(2)$$

(1) = (2)

$$\frac{t+c}{kt+c_1} = \left(\frac{dt}{dx} - a\right)\frac{1}{b}$$

Page :
Date :

$\Rightarrow b\left(\frac{t+c}{kt+c_1}\right) = \frac{dt}{dx} - a$

$\Rightarrow \frac{dt}{dx} = \left(\frac{t+c}{kt+c_1}\right)b + a$

(8)

Q. $\frac{dy}{dx} = 9x + y + 1$

Sol. $9x + y + 1 = t$, then $\frac{dt}{dx} = 9 + \frac{dy}{dx}$

$\frac{dy}{dx} = 9 - \frac{dt}{dx}$

$\boxed{\frac{dy}{dx} = \frac{dt}{dx} - 9}$

(1) $\Rightarrow \frac{dt}{dx} - 9 = t^2$

$\Rightarrow \frac{dt}{dx} = t^2 + 9 \Rightarrow \frac{dt}{t^2+9} = dx$ (var. sep. form)

$\int \frac{dt}{t^2+9} = \int dx + c$

$\frac{1}{3} \tan^{-1}\left(\frac{t}{3}\right) = x + c$

$\boxed{\tan^{-1}\left(\frac{9x+y+1}{3}\right) = 3x + C'}$

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Q. $\frac{dy}{dx} = \cos(x+y+1)$

put $x+y+1 = t \Rightarrow \frac{dt}{dx} = 1 + \frac{dy}{dx} \Rightarrow \boxed{\frac{dy}{dx} = \frac{dt}{dx} - 1}$

$$\frac{dt}{dx} - 1 = \cos t$$

$$\int \frac{dt}{1+\cos t} = \int dx$$

$$\int \frac{dt}{2\cos^2(x/2)} = x + c$$

$$\frac{1}{2}\int \sec^2\left(\frac{t}{2}\right) dt = x + c$$

$$\frac{t}{2} = u$$

$$\cancel{dt}\; du = \frac{dt}{2}$$

~~tanox~~ $\cancel{2}\int \sec^2 u \, du = x + c$

~~tan~~ $\cancel{2}\cdot \tan(u) = x + c$

$$\boxed{\tan(t/2) = x + c}$$

$$\tan\left(\frac{x+y+1}{2}\right) = x + c$$

2 Homogenous diff. Eq^n

2.1 Homogenous f^n

Let $u = f(x,y)$ be a f^n of x and y then u is said to be a homogenous f^n in x and y of degree n if

$$u = x^n \phi(y/x) \quad \text{or} \quad u = y^n \psi\left(\frac{x}{y}\right)$$

Eg. $f(x,y) = x^{1/2} + y^{1/2}$

Here
$$f(x,y) = x^{1/2}\left(1 + \frac{y^{1/2}}{x^{1/2}}\right)$$
$$= x^{1/2}\underbrace{\left(1 + \sqrt{\frac{y}{x}}\right)}_{\phi(y/x)}$$

$f(x,y)$ is a homog. funct. of deg $1/2$

$$f(x,y) = y^{1/2}\left(1 + \frac{x^{1/2}}{y^{1/2}}\right)$$
$$= y^{1/2}\underbrace{\left(1 + \sqrt{\frac{x}{y}}\right)}$$
$$y^{1/2}\left(\phi\left(\frac{x}{y}\right)\right)$$

Note: replace x by λx and y by λy in given f^n then if $f(\lambda x, \lambda y) = \lambda^n f(x,y)$

2.2 Homogenous diff^n eq^n

A. D.E. of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$

where $g(x,y) \neq 0$, $f(x,y)$ and $g(x,y)$ are homogeneous f^n of same degree

Sol^n of a homogeneous eq^n:

put $y = vx$

$v = \frac{y}{x}$

Q. $(x^2 - y^2)\, dx = xy\, dy$

$$\frac{dy}{dx} = \frac{x}{y} - \frac{y}{x}$$

$y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{1-v^2}{v}$$

$$x\frac{dv}{dx} = \frac{1}{v} - v - v$$

$$x\frac{dv}{dx} = \frac{1}{v} - 2v$$

$$\int \frac{v}{1-2v^2}\, dv = \int \frac{dx}{x} + \log K$$

$$-\frac{1}{4}\log(1-2v^2) = \log(xK)$$

$$-\log\left(1-\frac{2y^2}{x^2}\right) = 4\log(xk)$$

$$-\log\left(\frac{x^2-2y^2}{x^2}\right) = \log(xk)^4$$

$$\frac{x^2}{x^2-2y^2} = (xk)^4$$

$$\frac{1}{x^2-2y^2} = cx^2 \quad (c = k^4)$$

Q. $\dfrac{dy}{dx} = \dfrac{x}{x-\sqrt{xy}}$ ← degree = 1

degree = 1 ↳ degree = 1

→ $Sol^n = 2\sqrt{\dfrac{x}{y}} + \log y = K$

(3) Reducible homogenous D.E.

Consider the D.E. $\dfrac{dy}{dx} = \dfrac{ax+by+c}{a_1x+b_1y+c_1}$ —— (*)

case I : when $\dfrac{a}{a_1} \neq \dfrac{b}{b_1}$

put $x = X+h$ and $y = Y+K$

$dx = dX$ and $dy = dY$

then (*) $\Rightarrow \dfrac{dy}{dx} = \dfrac{a(X+h)+b(Y+K)+c}{a_1(X+h)+b_1(Y+K)+c_1}$

$$\Rightarrow \frac{dy}{dx} = \frac{aX + bY + (ah + bK + c)}{a_1X + b_1Y + (a_1h + b_1K + c_1)}$$

Take $ah + bK + c = 0$

and $a_1h + b_1K + c_1 = 0$

$\therefore$ (I) becomes $\frac{dY}{dX} = \frac{aX + bY}{a_1X + b_1Y}$, a homo. DE

Put $Y = vX$

Q. $\frac{dy}{dx} = \frac{y + x - 2}{y - x - 2}$ —— (*)

put $x = X + h$ and $y = Y + K$ $\Rightarrow dx = dX$, $dy = dY$

$\therefore$ (*) $\Rightarrow$ $\frac{dY}{dX} = \frac{Y + X + (h + K - 2)}{Y - X + (-h + K - 2)}$

$$h + K - 2 = 0$$
$$-h + K + 2 = 0$$
$$2K = 4$$

$\boxed{K = 1}$

$\boxed{h = 0}$

$$\frac{dY}{dX} = \frac{Y + X}{Y - X}$$

$$Y = vX$$

$$\boxed{\frac{dY}{dx} = x\frac{dv}{dx} + v}$$

$$x\frac{dv}{dx} + v = \frac{Y+X}{Y-X}$$

$$x\frac{dv}{dx} = \frac{v+1-v(v-1)}{v-1}$$

$$x\frac{dv}{dx} = \frac{v+1-v^2+v}{v-1}$$

$$x\frac{dv}{dx} = \frac{2v-v^2+1}{v-1}$$

$$\frac{v-1}{2v-v^2+1}\,dv = \frac{dx}{x}$$

$$2v-v^2+1 = t$$

$$(2-2v)\,dv = \frac{dt}{2}$$

$$\frac{-dt}{2t} = \frac{dx}{x}$$

$$-\frac{1}{2}\log(2v-v^2+1) = \log x$$

$$-\frac{1}{2}\log\left(2\frac{y}{x}-\left(\frac{y}{x}\right)^2+1\right) = \log xc$$

$$2\frac{y}{x}-\left(\frac{y}{x}\right)^2+1 = \left(\frac{1}{xc}\right)^2$$