Let $(L, \leq)$ be a lattice.

Define 2 binary operations '$\vee$' (join or sum) or or
or and
and '$\wedge$' (meet or product) as follows.

For any 2 elements a and b in L,

$a \vee b$ is the least upper bound of a and b.

$a \wedge b$ is the greatest lower bound of a and b

We can get an algebraic system $(L, \wedge, \vee)$ from a lattice $(L, \leq)$.

Examples for Lattice:

1) $(N, \backslash)$ is a Lattice. We can define an algebraic system $(N, \wedge, \vee)$ as follows.

For any 2 elements $m, n \in N$,

$$m \vee n = \text{lcm}(m, n)$$
$$m \wedge n = \gcd(m, n)$$

$1 \wedge 2 = 1$

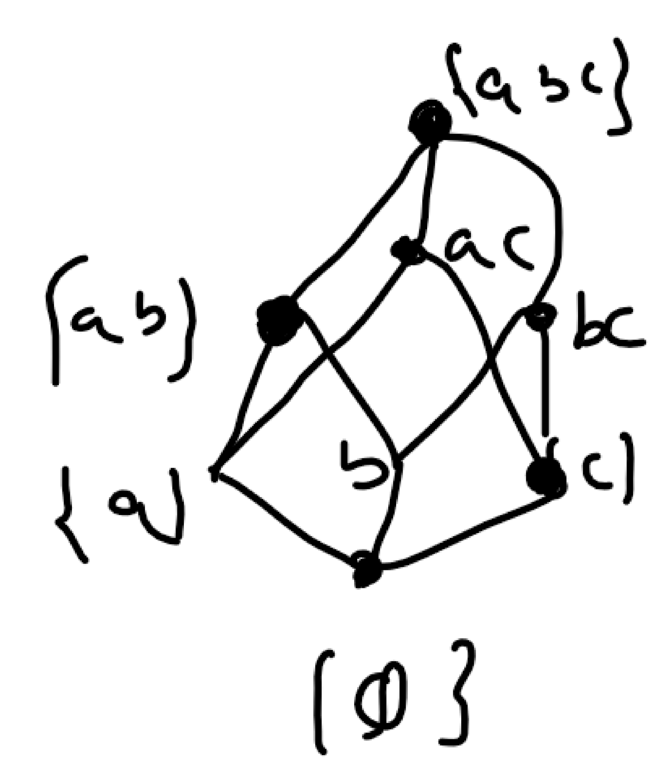
$1 \vee 2 = 2$

$1 \vee 4 = 4$

$2 \vee 4 = 4$

$12 \wedge 24 = 12$

$4 \vee 6 = 12$



2) Let A be a nonempty set. $(P(A), \subseteq)$ is a Lattice. For $B_1, B_2 \in P(A)$, we can define

$$B_1 \vee B_2 = B_1 \cup B_2$$
$$B_1 \wedge B_2 = B_1 \cap B_2$$

Hence $(P(A), \cup, \cap)$ is an algebraic system.

3) $(N, \leq)$ is a lattice

Define $(N, \vee, \wedge)$ as follows.

$a \vee b = \max(a, b)$

$a \wedge b = \min(a, b)$

4) Let n be +ve integer.

Let S_n denote set of +ve divisors of n.

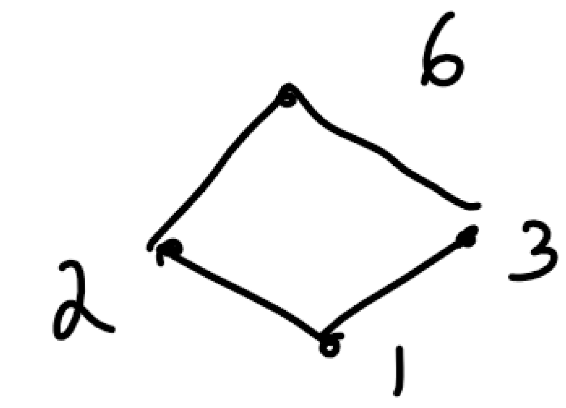
Then $(S_n, \backslash)$ is a Lattice.

$a, b \in S_n$, $\quad a \vee b = \text{lcm}(a, b)$

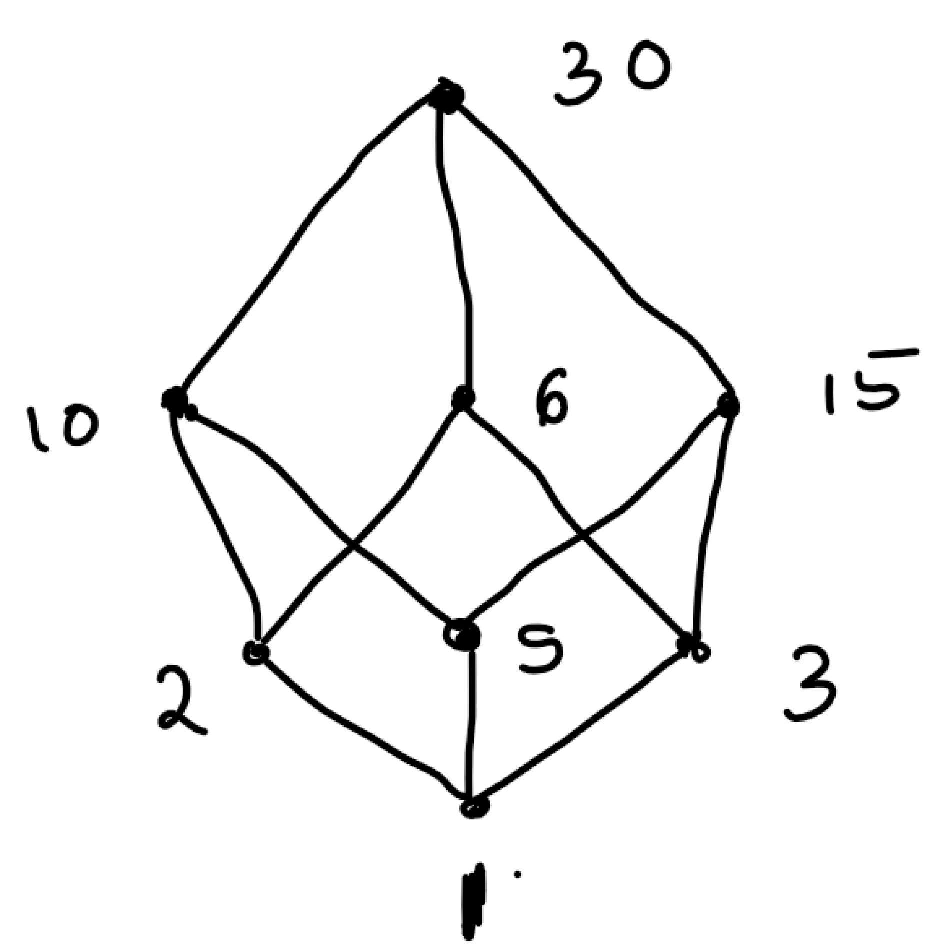
$a \wedge b = \gcd(a, b)$

So $(S_n, \vee, \wedge)$ is an algebraic system.

$S_6 = \{1, 2, 3, 6\}$



$S_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$



$2 \wedge 15 = 1$

$3 \vee 10 = 30$

$10 \vee 6 = 30$

Theorem 1: For any 'a' and 'b' in a lattice $(A, \leq)$, $a \leq a \vee b$ and $a \wedge b \leq a$.

Proof: The join of a and b is an upper bound of a. Therefore $a \leq a \vee b$

The meet of a and b is a lower bound of a $\Rightarrow$ $a \wedge b \leq a$.

Theorem 2: For any a, b, c, d in a lattice $(A, \leq)$, if $a \leq b$ and $c \leq d$, then

(i) $a \vee c \leq b \vee d$

(ii) $a \wedge c \leq b \wedge d$

Proof: (i) Given $a \leq b$ and we know $b \leq b \vee d$ (from Th 1)

From transitive law, $a \leq b \vee d$ — (1)

Given $c \leq d$ and we know $d \leq b \vee d$ (Th 1)

From transitive law, $c \leq b \vee d$ — (2)

From (1) and (2), we observe $(b \vee d)$ is a upper bound of a and c.

But $(a \vee c)$ is least upper bound of a and c, we get $a \vee c \leq b \vee d$.

(ii) $c \leq d$ (given) and we know $a \wedge c \leq c$ (Th 1)

using transitive, $a \wedge c \leq d$ — (3)

$a \leq b$ (given) & we know $a \wedge c \leq a$

$\Longrightarrow$ $a \wedge c \leq b$ — (4)

From (3),(4) $\Rightarrow$ $(a \wedge c)$ is a lower bound of b and d. As $(b \wedge d)$ is glb of b and d $\Rightarrow$ $a \wedge c \leq b \wedge d$

Duality: Let $(A, \leq)$ be a poset. Define a relation R on A as follows:

aRb if and only if $b \leq a$

R is Reflexive : $a \leq a$ for all $a \in A$
$\Rightarrow aRa$

R is Antisymmetric : Suppose aRb and bRa
i.e., $b \leq a$ and $a \leq b$
$\Rightarrow a = b$

R is Transitive : Suppose aRb and bRc
$\Rightarrow b \leq a$ and $c \leq b$
$\Rightarrow c \leq a$
$\Rightarrow aRc$

This relation R is Partial ordering Relation.
we denote this relation by $\geq$.

Hence, if $(A, \leq)$ is a lattice, then $(A, \geq)$ is also a lattice and the lattice is called Dual of $(A, \leq)$.

The symbols $\leq$ and $\geq$ are called duals of each other.

Principle of Duality:

If ϕ is a statement about a lattice, then the statement ϕ^* obtained from ϕ by interchanging the operational symbols $\vee$ and $\wedge$ and also the operations $\leq$ and $\geq$ is called dual of ϕ.

Eg: ϕ: $a \vee b \geq c \wedge d$

Dual ϕ^*: $a \wedge b \leq c \vee d$

If the statement remains the same after dualization, then such a statement is called a Self Dual.

Principle of duality states that any statement about lattices involving $\vee$ and $\wedge$ and $\leq$ and $\geq$ remains true if $\wedge$ is replaced with $\vee$ and $\leq$ is replaced with $\geq$.

Properties of Algebraic systems defined by Lattices

Let $(A, \vee, \wedge)$ be an algebraic system defined by a lattice $(A, \leq)$.

1) Commutative Property :

Let $a, b \in A$.

(i) $a \vee b = b \vee a$

(ii) $a \wedge b = b \wedge a$

2) Associative Law :

Let $a, b, c \in A$.

(i) $(a \vee b) \vee c = a \vee (b \vee c)$

(ii) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$

Proof: (i) Let $a \vee (b \vee c) = g$ and $(a \vee b) \vee c = h$

Since g is the join of a and $b \vee c$ we have

$a \leq g$ and $b \vee c \leq g$

Further, $b \vee c \leq g$

$\Rightarrow b \leq g$ and $c \leq g$

We have, $a \leq g$ and $b \leq g$

$\Rightarrow$ g is an upper bound of a and b.

But $a \vee b$ is the least upper bound of a and b.

$\Rightarrow a \vee b \leq g$

Further $a \vee b \le g$ and $c \le g$

$\Longrightarrow (a \vee b) \vee c \le g$

[$\because$ g is an ub of $(a \vee b)$ and c.

But as $(a \vee b) \vee c$ is lub of $(a \vee b)$ and c]

$\Longrightarrow h \le g$ — (1)

Similarly, we can show $g \le h$ — (2)

By Antisymmetric property, (from (1) & (2))

$g = h$

According to principle of duality, the meet operation $\wedge$ is also Associative.

3. Idempotent Property:

Let $(A, \le)$ be a lattice. For every $a \in A$,

$a \vee a = a$ and $a \wedge a = a$

4. Absorption Property :

Let $a, b \in A$. Then $a \vee (a \wedge b) = a$

$a \wedge (a \vee b) = a$

<u>Proof:</u> Since $a \vee (a \wedge b)$ is the join of a and $(a \wedge b)$, we have

$$a \leq a \vee (a \wedge b) \quad - (1)$$

Since $a \leq a$, and $a \wedge b \leq a$

From Theorem (2), $a \vee (a \wedge b) \leq a \vee a$

$$a \vee (a \wedge b) \leq a \quad - (2)$$

from (1) & (2) $\Longrightarrow$ $a \vee (a \wedge b) = a$ (antisymmetry)

And,

$a \wedge (a \vee b) = a$ from Duality Principle

Example : Consider lattice $(N, \leq)$.

Since the max (min) of 2 elements a & b is same as max (min) of b and a

$\Rightarrow$ Join (Meet) operation is commutative

We know,

$$\max(\max(a,b), c) = \max(a, \max(b,c))$$

$\Rightarrow$ Associative.

As,

$$\left.\begin{array}{l} \max(a,a) = a \\ \min(a,a) = a \end{array}\right\} \Rightarrow \text{Idempotent}$$

$$\text{As, } \left.\begin{array}{l} \max(a, \min(a,b)) = a \\ \min(a, \max(a,b)) = a \end{array}\right\} \Rightarrow \text{Absorption.}$$