9/12/24 # Higher order Differential eqn's

$\Rightarrow P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_n y = X$ — ①

n^{th} order D.eqn. $P_0, P_1, P_2, \dots, P_n, X$ are constants (or) functions of x.

X: e^x, Polynomial, $\sin x, \dots$

⊛ Soln $y = \underbrace{C.F}_{\text{Complementary function}} + \underbrace{P.I}_{\text{Particular Integral}}$

$\Rightarrow$ In eqn ① C.F is the soln of : [: = 0]

$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + P_n y = 0$ — ②

Suppose y_1 is a soln of ②

$P_0 \frac{d^n y_1}{dx^n} + P_1 \frac{d^{n-1} y_1}{dx^{n-1}} + \dots + P_n y_1 = 0$ — ③

Suppose y_2 is a soln of ③

$P_0 \frac{d^n y_2}{dx^n} + P_1 \frac{d^{n-1} y_2}{dx^{n-1}} + P_2 \frac{d^{n-2} y_2}{dx^{n-2}} + \dots + P_n y_2 = 0$ — ④

As we already know y_1 & y_2 are solns then

Let us check $C_1 y_1 + C_2 y_2$ be a soln? [C_1, C_2 are consts]

$P_0 \frac{d^n (C_1 y_1 + C_2 y_2)}{dx^n} + P_1 \frac{d^{n-1}}{dx^{n-1}} (C_1 y_1 + C_2 y_2) + \dots + P_n (C_1 y_1 + C_2 y_2)$

$= C_1 \left\{ P_0 \frac{d^n y_1}{dx^n} + P_1 \frac{d^{n-1} y_1}{dx^{n-1}} + \dots + P_n y_1 \right\} + C_2 \left\{ P_0 \frac{d^n y_2}{dx^n} + \dots + P_n y_2 \right\}$

$= C_1 \cdot 0 + C_2 \cdot 0 = 0.$

(*) $P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \text{---} + P_n y = 0$ — (1)

then $y_1, y_2, \text{---}, y_n$ are solⁿ of (1) then

$$C.F = \underbrace{C_1 y_1 + C_2 y_2 + C_3 y_3}_{\text{Soln}} + \text{---} + C_n y_n$$

⇒ Particular Integral ≈ Particular Solⁿ.

Solⁿ of $P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \text{--} + P_n y = \underset{\text{function of } x}{X}$ — (1)

If V is P.I solⁿ then:-

$P_0 \frac{d^n V}{dx^n} + P_1 \frac{d^{n-1} V}{dx^{n-1}} + \text{---} + P_n V = \underset{\text{we get fnx.}}{X}$

Finding Complementary Function;

$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \text{---} + P_n y = 0$ — (1)

Let $\frac{d}{dx} = D$ (Differential operator)

then (1) ⇒ $P_0 D^n y + P_1 D^{n-1} y + \text{---} + P_n y = 0$

Auxillary eqⁿ:- $K^n y + K^{n-1} y + \text{---} + K_1 y = 0$

$K^n + K^{n-1} + \text{--} + K_1 = 0$ of degree n is A.eqⁿ.

ex:- $\frac{d^2 y}{dx^2} + 3\frac{dy}{dx} + 2y = 0 \Rightarrow D^2 y + 3Dy + 2y = 0$

$\Rightarrow K^2 + 3K + 2 = 0 \rightarrow$ Aux. eqⁿ

Let the roots of eqn (1) be $m_1, m_2, m_3, ——, m_n$

case (1) → Roots are real & distinct.

case (2) → Roots are real & repeated.

case (3) → Roots are complex & distinct.

case (4) → Roots are complex & repeated roots

Case (1): Roots are real & Distinct

$m_1, m_2, ——, m_n$

$D^n y + D^{n-1} y + \cdots + Dy + y = 0$

$(D - m_1)(D - m_2) —— (D - m_n) y = 0$

if m_n is a root $(D - m_n) y = 0$

$\Rightarrow \frac{dy}{dx} - m_n y = 0$

$\Rightarrow y = C_n e^{m_n x}$

if m_1 is a root $\Rightarrow y = C_1 e^{m_1 x}$

if m_2 is a root $\Rightarrow y = C_2 e^{m_2 x}$

C.F → $C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x} + —— + C_n e^{m_n x}$

(*) $D^n y + D^{n-1} y + —— + Dy + y = 0$

$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \ldots —— + C_n e^{m_n x}$

$m_1, m_2, ——, m_n$ are roots of Aux-eqn
real & distinct

Case-② : If 2 roots are same, $m = m_1$

$$y = C_1e^{m_1x} + C_2e^{m_1x} + C_3e^{m_3x} + \cdots + C_ne^{m_nx}$$

$$= e^{m_1x}(C_1+C_2) + C_3e^{m_3x} + \cdots + C_ne^{m_nx}$$

$$= Ce^{m_1x} + C_3e^{m_3x} + \cdots + C_ne^{m_nx}$$

There are only $(n-1)$ arbitrary const but it should be n.

$$(D-m_1)(D-m_1)y = 0 \quad \text{--- ①}$$

let $(D-m_1)y = z$ --- ②

① $\Rightarrow (D-m_1)z = 0 \Rightarrow \frac{dz}{dx} - m_1z = 0$

$$\Rightarrow z = C_1e^{m_1x}$$

$$(D-m_1)y = C_1e^{m_1x}.$$

$$\frac{dy}{dx} - m_1y = C_1e^{m_1x}$$

$$\Rightarrow y = (C_1x + C_2)e^{m_1x}$$

★ : real & repeated.

2 repeated $\Rightarrow y = (C_1x + C_2)e^{m_1x} + C_3e^{m_3x} + \cdots + C_ne^{m_nx}$.

3 repeated $\Rightarrow$

$$y = (C_1x^2 + C_2x + C_3)e^{m_1x} + C_4e^{m_4x} + \cdots + C_ne^{m_nx}$$

Case-(3): If one pair of root is imaginary

$m_1 = \alpha + i\beta,\ m_2 = \alpha - i\beta$

$y = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x} + C_3 e^{m_3 x} + \cdots + C_n e^{m_n x}$

$= e^{\alpha x}(C_1 e^{i\beta x} + C_2 e^{-i\beta x}) + \cdots$

$= e^{\alpha x}(C_1 \cos\beta x + C_1 i \sin\beta x + C_2 \cos\beta x - i C_2 \sin\beta x) + \cdots$

$\{\because e^{i\theta} = \cos\theta + i\sin\theta\}$

$= e^{\alpha x}(A\cos\beta x + B\sin\beta x) + \cdots$

(A) One pair complex.

$y = e^{\alpha x}(C_1\cos\beta x + C_2 \sin\beta x) + C_3 e^{m_3 x} + \cdots + C_n e^{m_n x}$

Case-(4): If two pairs of roots are imaginary & repeated

$y = e^{\alpha x}((C_1 + C_2 x)\cos\beta x + (C_3 + C_4 x)\sin\beta x) + C_5 e^{m_5 x} + \cdots$

Conclusion:

(1) Real & distinct $\Rightarrow y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \cdots + C_n e^{m_n x}$

(2) Real & Repeated $\Rightarrow y = (C_1 + C_2 x)e^{m_1 x} + C_3 e^{m_3 x} + \cdots + C_n e^{m_n x}$

(3) $\Rightarrow y = (C_1 + C_2 x + C_3 x^2)e^{m_1 x} + C_4 e^{m_4 x} + \cdots + C_n e^{m_n x}$.

(3) One pair imaginary $\Rightarrow y = e^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x) + C_3 e^{m_3 x} + \cdots$

$(\alpha \pm i\beta)$

(4) 2 pairs (of) repeated imaginary roots: rep: $\alpha \pm i\beta, \alpha \pm i\beta$

rep $\Rightarrow y = e^{\alpha x}((C_1 + C_2 x)\cos\beta x + (C_3 + C_4 x)\sin\beta x) + C_5 e^{m_5 x} + \cdots$

2 pairs $\Rightarrow y = e^{\alpha x}(C_1\cos\beta x + C_2 \sin\beta x) + e^{\gamma x}(C_3\cos\theta x + C_4\sin\theta x) + C_5 e^{m_5 x} + \cdots$

$(\alpha \pm i\beta), (\gamma \pm i\theta)$.

Examples:

(1) $2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 12y = 0$

Sol: $(2D^2 + 5D - 12)y = 0$

Aux eqn $\rightarrow$ $2m^2 + 5m - 12 = 0$

$m_1 = \frac{3}{2}, m_2 = -4.$

$\therefore y = C_1 e^{\frac{3}{2}x} + C_2 e^{-4x}$

(2) $\frac{d^2y}{dx^2} + 4y = 0$

Sol: $(D^2 + 4)y = 0$

Aux eqn $\rightarrow$ $m^2 + 4 = 0$

$m = \pm 2i = 0 \pm 2i$

$\therefore y = C_1 \cos 2x + C_2 \sin 2x$

(3) $\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = 0$

Sol: A.E: $m^3 + 4m^2 + 4m = 0$

$m_1 = 0, m_2 = -2, m_3 = -2$

$y = \{C_1 + C_2 x\}e^{-2x} + C_3 e^{0x} = (C_1 + C_2 x)e^{-2x} + C_3$

(4) $\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

Sol: $m_1 = 1, m_2 = 1, 0 \pm i$

$y = (C_1 + C_2 x)e^{x} + e^{0x}(C_3 \cos x + C_4 \sin x)$

(8) $\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0$

Sol: $m = \pm 2i, \pm 2i$

$y = e^{0x}[(C_1 + C_2 x)\cos 2x + (C_3 + C_4 x)\sin 2x]$

13/12/24 **Finding Particular Integral:**

$\Rightarrow P_n \frac{d^n y}{dx^n} + P_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_1 y = X$

$D^n y + D^{n-1}y + D^{n-2}y + \cdots + Dy = x \quad \left[D \to \frac{d}{dx}\right]$

$(D^n + D^{n-1} + \cdots + D)y = X$

$f(D)y = x \quad —(1)$

$\left\{\frac{1}{f(D)} \to \text{Inverse operator}\right]$

$\frac{1}{f(D)} \cdot f(D)y = \frac{1}{f(D)} X \Rightarrow \boxed{y = \frac{1}{f(D)} x}$

$\Rightarrow$ **Rules for finding particular Integral:**

Inverse operator: $\frac{1}{f(D)} X$ is that function of x, not containing any arbitrary constants which when operated upon by $f(D)$ gives x.

Note: • $f(D)$ & $\frac{1}{f(D)}$ are inverse operators.

• D - differential operator $\frac{1}{D}$ - Integral operator.

$D = \frac{d}{dx} \qquad \frac{1}{D} = \int dx$

Result 1:- If x is a function of x.

$$\frac{1}{D}x = \int x\,dx \quad \{\because D \text{ is differential op}$$

Pf:- $\frac{1}{D}x = y$ — (1)

$$D \cdot \frac{1}{D}x = Dy = \frac{dy}{dx}$$

$$x = \frac{dy}{dx} \Rightarrow dy = x\,dx$$

Variable separable

$$y = \int x\,dx \quad \text{— (2)}$$

From (1) $\therefore$ $\boxed{\frac{1}{D}x = \int x\,dx}$

Result-2:- $\frac{1}{D-a}x = e^{ax}\int x e^{-ax}\,dx$

Pf:- $\frac{1}{D-a}x = y$ — (1)

$$(D-a)\frac{1}{(D-a)}x = (D-a)y$$

$$x = (D-a)y = \frac{dy}{dx} - ay$$

$$\Rightarrow \frac{dy}{dx} - ay = x$$

$$I.F \rightarrow e^{-ax}$$

$$ye^{-ax} = \int x e^{-ax}\,dx \Rightarrow y = e^{ax}\int x e^{-ax}\,dx \text{ — (2)}$$

From (1) $\therefore$ $\boxed{\frac{1}{D-a}x = e^{ax}\int x e^{-ax}\,dx}$

Case-①: $X = e^{ax}$

$$X = e^{ax}$$

$$De^{ax} = \frac{d}{dx}(e^{ax}) = a e^{ax}$$

$$D^2 e^{ax} = a^2 e^{ax}$$

$$\vdots$$

(+) $D^n e^{ax} = a^n e^{ax}$

$$(D^n + D^{n-1} + \cdots + D + k_n)e^{ax} = (a^n + a^{n-1} + \cdots + a + k_n)e^{ax}$$

$$f(D)e^{ax} = f(a)e^{ax} \quad \text{—①}$$

$$\frac{1}{f(D)} \cdot f(D)\, e^{ax} = \frac{1}{f(D)} f(a) e^{ax}$$

$$e^{ax} = \frac{1}{f(D)} f(a) e^{ax}$$

$$\Rightarrow \boxed{P.I = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}}$$

If $f(a) = 0$, $(D-a)$ is a factor of $f(D)$

$$f(D) = (D-a)\,\phi(D)$$

$$\therefore P.I = \frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)\phi D} e^{ax} = \frac{1}{(D-a)} \frac{1}{\phi(D)} e^{ax}$$

$$= \frac{1}{\phi(a)} \frac{1}{D-a} e^{ax} = \frac{1}{\phi(a)} e^{ax} \int e^{ax} \cdot e^{-ax} dx$$

$$= \frac{1}{\phi(a)} e^{ax} \int dx = \frac{1}{\phi(a)} e^{ax} \cdot x = \frac{1}{f'(a)} x e^{ax}$$

$(\because f'(a) \neq 0)$

$$\left[\because f(D) = (D-a)\,\phi(D) \\ f'(D) = (D-a)\,\phi'(D) + \phi(D) \\ f'(a) = \phi(a)\right]$$

Q: $f(D)y = e^{ax}$

Sol: P.I $\Rightarrow y = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \quad [\because f(a) \neq 0]$

if $f(a) = 0$, $y = x \cdot \frac{1}{f'(a)} e^{ax} \quad [\because f'(a) \neq 0]$

if $f'(a) = 0$, $y = x^2 \frac{1}{f''(a)} e^{ax} \quad [\because f''(a) \neq 0]$

Examples

① Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^x$

Sol:

$m^2 - 6m + 9 = 0$

$\Rightarrow m = 3, 3.$

$C.F = (C_1x + C_2)e^{3x}$

$P.I = \frac{1}{f(D)} x = \frac{1}{D^2 - 6D + 9} e^{1x} = \frac{1}{f(a)} e^{ax}$

$= \frac{1}{1^2 - 6 \cdot 1 + 9} e^x = \frac{1}{4} e^x.$

$\therefore y = C.F + P.I$

$= (C_1x + C_2)e^{3x} + \frac{1}{4}e^x.$

(2) $(D^2-3D-4)y = e^{-x}+e^{-2x}$

Sol: $m^2-3m-4=0$

$\Rightarrow m = 4, -1$

$\Rightarrow C.F = c_1 e^{4x} + c_2 e^{-x}$

$$P.I = \underbrace{\frac{1}{D^2-3D-4}e^{-x}}_{P.I_1} + \underbrace{\frac{1}{D^2-3D-4}e^{-2x}}_{P.I_2}$$

$$P.I_1 = \frac{1}{(-1)^2-3(-1)-4} = X = x \cdot \frac{1}{f'(D)} e^{-x}$$

$$= x \cdot \frac{1}{2D-3} e^{-x} = x \cdot \frac{1}{2(-1)-3} e^{-x}$$

$$= x \cdot \frac{1}{(-5)} e^{-x} = \frac{-x}{5} e^{-x}$$

$$P.I_2 = \frac{1}{D^2-3D-4} e^{-2x} = \frac{1}{(-2)^2-3(-2)-4} e^{-2x} = \frac{1}{6} e^{-2x}$$

$$\therefore y = c_1 e^{4x} + c_2 e^{-x} + \frac{-x}{5} e^{-x} + \frac{1}{6} e^{-2x}$$

(★) $\cosh x = \frac{1}{2}(e^x + e^{-x})$

$\sinh x = \frac{1}{2}(e^x - e^{-x})$

$\cosh 2x = \frac{1}{2}(e^{2x} + e^{-2x})$

③ $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=\cosh 2x$

Sol:
$$m=\frac{-1\pm\sqrt{3}i}{2}$$

$$\Rightarrow C.F=e^{-\frac{1}{2}x}\left(c_1\cos\left(\frac{\sqrt{3}}{2}\right)x+c_2\sin\left(\frac{\sqrt{3}}{2}\right)x\right)$$

$$P.I=\frac{1}{D^2+D+1}\left[\frac{1}{2}e^{2x}+\frac{1}{2}e^{-2x}\right]$$

$$=\frac{1}{D^2+D+1}\left(\frac{1}{2}e^{2x}\right)+\frac{1}{D^2+D+1}\left(\frac{1}{2}e^{-2x}\right)$$

$$=\frac{1}{2}\left\{\frac{1}{2^2+2+1}e^{2x}\right\}+\frac{1}{2}\left[\frac{1}{(-2)^2+(-2)+1}e^{-2x}\right]$$

$$=\frac{1}{14}e^{2x}+\frac{1}{6}e^{-2x}$$

$$\therefore y=C.F+P.I.$$

④ $(D^2-4)y=\sin^2 hx$

Sol: $\sin^2 hx=\left\{\frac{1}{2}(e^x-e^{-x})\right\}^2=\frac{1}{4}(e^{2x}-2e^xe^{-x}+e^{-2x})$

$$=\frac{1}{4}(e^{2x})-\frac{1}{2}+\frac{1}{4}e^{-2x}$$

$$\Rightarrow C.F=c_1e^{2x}+c_2e^{-2x}$$

$$P.I_1=\frac{1}{D^2-4}\left(\frac{1}{4}e^{2x}\right)=\frac{1}{4}x\frac{1}{2D}e^{2x}=\frac{1}{4}x\frac{1}{4}e^{2x}=\frac{1}{16}xe^{2x}$$

$$P.I_2=\frac{1}{D^2-4}\left(-\frac{1}{2}e^{0x}\right)=-\frac{1}{2}\frac{1}{0-4}e^{0x}=-\frac{1}{2}\cdot\frac{-1}{4}=\frac{1}{8}$$

$$P.I_3=\frac{1}{D^2-4}\left(\frac{1}{4}e^{-2x}\right)=x\cdot\frac{1}{4}\frac{1}{2D}e^{-2x}=x\cdot\frac{1}{4}\cdot\frac{-1}{4}e^{-2x}=\frac{-1}{16}xe^{-2x}$$

$$\therefore y=c_1e^{2x}+c_2e^{-2x}+\frac{1}{16}xe^{2x}+\frac{-1}{16}xe^{-2x}+\frac{1}{8}$$

Case ②: $x = \sin(ax+b)$ (or) $\cos(ax+b)$

Suppose $\boxed{x = \sin(ax+b)}$

$D(\sin(ax+b)) = a\cos(ax+b)$

$D^2(\sin(ax+b)) = -a^2\sin(ax+b)$

$D^3(\sin(ax+b)) = -a^3\cos(ax+b)$

$D^4(\sin(ax+b)) = a^4\sin(ax+b)$

→ we want only sine function its comming only in even powers. so we take only them.

$(D^2)^r(\sin(ax+b)) = (-a^2)^r(\sin(ax+b))$

$f(D^2)(\sin(ax+b)) = f(-a^2)(\sin(ax+b))$

$\frac{1}{f(D^2)} f(D^2)(\sin(ax+b)) = \frac{1}{f(D^2)} f(-a^2)(\sin(ax+b))$

$\Rightarrow \sin(ax+b) = \frac{1}{f(D^2)} f(-a^2)(\sin(ax+b))$

$\Rightarrow \boxed{\frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{f(-a^2)}(\sin(ax+b))}$, $\{f(-a^2) \neq 0\}$.

Suppose $f(-a^2) = 0$, $\{\because e^{i\theta} = \cos\theta + i\sin\theta\}$

$\sin(ax+b) =$ Imaginary Part of $e^{i(ax+b)}$

$P.I = \text{Im.Pa}\ \frac{1}{f(-a^2)} e^{i(ax+b)}$ $\{f(-a^2) \neq 0\}$

$f(-a^2) = 0 \Rightarrow P.I = \text{Im.Pa}\ x \cdot \frac{1}{f'(-a^2)} e^{i(ax+b)}$ $\{f'(-a^2) \neq 0\}$

$= x \cdot \frac{1}{f'(-a^2)} \sin(ax+b)$

Q: $f(D)y = \sin(ax+b)$ (or) $\cos(ax+b)$

Sol: P.I $\Rightarrow y = \frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{f(-a^2)}\sin(ax+b)$

$f(-a^2)=0 \Rightarrow$ P.I $=$ Im.Pa $x.\frac{1}{f'(-a^2)}e^{i(ax+b)}$

$= x\frac{1}{f'(-a^2)}\sin(ax+b)$

$f'(-a^2)=0 \Rightarrow$ P.I $=$ Im.Pa $x^2\frac{1}{f''(-a^2)}e^{i(ax+b)}$

$= x^2\frac{1}{f''(-a^2)}\sin(ax+b)$

If $\cos(ax+b)$

P.I $\Rightarrow y = \frac{1}{f(D^2)}\cos(ax+b) = \frac{1}{f(-a^2)}\cos(ax+b)$

$f(-a^2)=0 \Rightarrow$ P.I $=$ Real Pa $x.\frac{1}{f'(-a^2)}e^{i(ax+b)}$

$= x.\frac{1}{f'(-a^2)}\cos(ax+b)$

$f'(-a^2)=0 \Rightarrow$ P.I $= x^2\frac{1}{f''(-a^2)}\cos(ax+b)$

Examples:

① Solve $(D^2-3D+2)y = \sin 3x$

Sol: C.F $= c_1e^x + c_2e^{2x}$

P.I $= \frac{1}{D^2-3D+2}\sin 3x$

$= \frac{1}{-3^2-3D+2}\sin 3x = \frac{1}{-7-3D}\sin 3x$

$= \frac{-1}{3D+7}\sin 3x = \frac{-(3D-7)}{(3D+7)(3D-7)}\sin 3x$

$$= \frac{-(3D-7)}{(9D^2-49)} \sin 3x$$

$$= \frac{-(3D-7)}{9(-3^2)-49} \sin 3x = \frac{-(3D-7)}{9(-9)-49} \sin 3x$$

$$= \frac{(-3D+7)}{-130} \sin 3x = \frac{-1}{130}(-3D+7)\sin 3x$$

$$= \frac{-1}{130}(-3D\sin 3x + 7\sin 3x)$$

$$= \frac{-1}{130}(-3\times 3\cos 3x + 7\sin 3x)$$

$$= \frac{-1}{130}[-9\cos 3x + 7\sin 3x]$$

$\therefore y = C.F + P.I$

(2) $(D^2+D+1)y = \sin 2x$

$\Rightarrow C.F = e^{-x/2}\left(c_1\cos\left(\frac{\sqrt{3}}{2}x\right) + c_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right)$

$$P.I = \frac{1}{D^2+D+1}\sin 2x = \frac{1}{-2^2+D+1}\sin 2x.$$

$$= \frac{1}{-4+D+1}\sin 2x = \frac{1}{D-3}\sin 2x$$

$$= \frac{(D+3)}{(D-3)(D+3)}\sin 2x = \frac{(D+3)}{D^2-9}\sin 2x$$

$$= \frac{D+3}{-2^2-9}\sin 2x = \frac{(D+3)}{-4-9}\sin 2x$$

$$= \frac{-1}{13}(D\sin 2x + 3\sin 2x)$$

$$= \frac{-1}{13}(2\cos 2x + 3\sin 2x)$$ $\therefore y = CF + PI$

③ Solve $\frac{d^2y}{dx^2} + y = 2\cos x$

<u>Sol:</u> C.F $y = C_1 \cos x + C_2 \sin x$

$$P.I = \frac{1}{D^2+1} 2\cos x = 2 \frac{1}{-1^2+1} \cos x$$

$$\frac{1}{0}$$

$$= \cancel{2} \cdot x \cdot \frac{1}{\cancel{2}D} \cos x = x \cdot \frac{1}{D} \cos x$$

$$= x \int \cos x \, dx = x \sin x$$

$$\therefore y = C.F + P.I$$

16/12/24

④ Solve $\frac{d^3y}{dx^3} + 4 \frac{dy}{dx} = \sin 2x$

<u>Sol:</u> $m^3 + 4m = 0 \Rightarrow m(m^2+4) = 0 \Rightarrow m = 0, \pm 2i$

$$C.F = C_1 e^{0x} + e^{0x}(C_2 \cos 2x + C_3 \sin 2x)$$

$$= C_1 + C_2 \cos 2x + C_3 \sin 2x$$

$$P.I = \frac{1}{D^3+4D} \sin 2x = \frac{1}{D(D^2+4)} \sin 2x$$

$$= x \frac{1}{3D^2+4} \sin 2x = x \cdot \frac{1}{3(-2^2)+4} \sin 2x$$

$$= x \cdot \frac{1}{-12+4} \sin 2x$$

$$= -\frac{x}{8} \sin 2x$$

<u>G. Sol^n:</u> $y = C.F + P.I$

(5) $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} = 3e^x + \sin x$

Sol: $C.F = (c_1 + c_2 x) + c_3 e^x$ — (1)

$P.I_1 = \frac{1}{D^3 - D^2} 3e^x = 3x \frac{1}{3D^2 - 2D} e^x = 3xe^x$ — (2)

$P.I_2 = \frac{1}{D^3 - D^2} \sin x = \frac{1}{D^2(D-1)} \sin x = \frac{1}{-1^2(D-1)} \sin x$

$= \frac{-1(D+1)}{D^2 - 1} \sin x = \frac{-1}{(-1^2)-1} (D+1) \sin x$

$= \frac{1}{2} (D+1) \sin x = \frac{1}{2} (D \sin x + \sin x)$

$= \frac{1}{2} \left(\frac{d}{dx} (\sin x) + \sin x \right)$

$= \frac{1}{2} (\cos x + \sin x)$

$y = C.F + P.I_1 + P.I_2$

Case (3): when X = a polynomial in "x^m", m is +ve integer

$f(D) y = x^m$

$\Rightarrow P.I = \frac{1}{f(D)} x^m$

$\Rightarrow \{1+a\}^n = 1 + na + \frac{n(n-1)}{1 \cdot 2} a^2 + \text{---}$

$\{1+a\}^{-1} = 1 - a + a^2 - a^3 + a^4 \text{---}$

$[1-a]^{-1} = 1 + a + a^2 + a^3 + \text{---}$

$\{1+a\}^{-2} = 1 - 2a + 3a^2 - 4a^3 + \text{---}$

$\{1-a\}^{-2} = 1 + 2a + 3a^2 + 4a^3 + \text{--}$

Examples:

(1) Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2 + 2x$

Sol: $C.F = e^{-x}(C_1 + C_2 x)$ — (1)

$P.I = \frac{1}{D^2+2D+1}(x^2+2x) = \frac{1}{(D+1)^2}(x^2+2x)$

$= (1+D)^{-2}(x^2+2x)$

$= \{1 - 2D + 3D^2 - 4D^3 + \cdots\}(x^2+2x)$

$= 1(x^2+2x) - 2D(x^2+2x) + 3D^2(x^2+2x) - 0$ —

$= x^2 + 2x - 2(2x+2) + 3(2)$

$= x^2 - 2x + 2$

$y = C.F + P.I$

(2) Solve $(D^2 - 6D + 9)y = x^2 + x + 1$

Sol: $C.F = (C_1 + C_2 x)e^{3x}$ — (1)

$P.I = \frac{1}{D^2 - 6D + 9}(x^2+x+1) = \frac{1}{(D-3)^2}(x^2+x+1)$

$= \frac{1}{(3-D)^2}(x^2+x+1) = \frac{1}{9\left(1-\frac{D}{3}\right)^2}(x^2+x+1)$

$= \frac{1}{9}\left(1 - \frac{D}{3}\right)^{-2}(x^2+x+1)$

$= \frac{1}{9}\left(1 + \frac{2D}{3} + 3\left(\frac{D}{3}\right)^2 + 4\left(\frac{D}{3}\right)^3 + \cdots\right)(x^2+x+1)$

$= \frac{1}{9}\left(1 + \frac{2D}{3} + \frac{D^2}{3} + \cdots\right)(x^2+x+1)$

$= \frac{1}{9}\{x^2+x+1+\frac{2}{3}(2x+1)+\frac{2}{3}\}$

$= \frac{1}{27}(3x^2+7x+7)$

<u>G.Sol:</u> $y = C.F + P.I.$

(3) Solve $(D^2+4D+5)y = \cos 4x + x^3$

Sol: $C.F = e^{-2x}(c_1 \cos x + c_2 \sin x)$ — (1)

$P.I_1 = \frac{1}{D^2+4D+5}\cos 4x = \frac{1}{-4^2+4D+5}\cos 4x = \frac{1}{4D-11}\cos 4x$

$= \frac{4D+11}{16D^2-121}\cos 4x = \frac{(4D+11)}{16(-16)-121}\cos 4x$

$= \frac{-1}{377}(4(-4\sin 4x)+11\cos 4x)$

$= \frac{1}{377}(16\sin 4x - 11\cos 4x)$

$P.I_2 = \frac{1}{D^2+4D+5}x^3 = \frac{1}{5(1+\frac{4D+D^2}{5})}x^3$

$= \frac{1}{5}\{1+(\frac{4D}{5}+\frac{D^2}{5})\}^{-1}x^3$

$= \frac{1}{5}\{1-(\frac{4}{5}D+\frac{D^2}{5})+(\frac{4}{5}D+\frac{D^2}{5})^2-(\frac{4}{5}D+\frac{D^2}{5})^3+\}(x^3)$

$= \frac{1}{5}\{1-\frac{4}{5}D-\frac{D^2}{5}+\frac{16}{25}D^2+\frac{8}{25}D^3-(\frac{4}{5})^3D^3 - - \}(x^3)$

$= \frac{1}{5}(x^3-\frac{12}{5}x^2+\frac{66}{25}x-\frac{144}{125})$

$y = C.F + P.I_1 + P.I_2$

Case (4): $X = e^{ax}.V$

$$f(D)y = e^{ax}.V$$

$$\Rightarrow \boxed{P.I = e^{ax}\frac{1}{f(D+a)}V}$$

Let U be a function of x

Now $D(e^{ax}.U) = e^{ax}DU + ae^{ax}U = e^{ax}(D+a)U$

$$D^2(e^{ax}.U) = D(e^{ax}DU) + D(ae^{ax}U)$$

$$= e^{ax}D^2U + ae^{ax}DU + a^2e^{ax}U + ae^{ax}DU$$

$$= e^{ax}D^2U + 2ae^{ax}DU + a^2e^{ax}U$$

$$= e^{ax}(D+a)^2U$$

In general $D^n(e^{ax}U) = e^{ax}(D+a)^nU$

$$\therefore f(D)(e^{ax}.U) = e^{ax}f(D+a)U$$

$$\frac{1}{f(D)}f(D)(e^{ax}.U) = \frac{1}{f(D)}e^{ax}f(D+a)U$$

$$\Rightarrow (e^{ax}.U) = \frac{1}{f(D)}e^{ax}f(D+a)U \quad \text{--- (1)}$$

Let $f(D+a)U = V$, i.e $U = \frac{1}{f(D+a)}V$

i.e, (1) $\Rightarrow e^{ax}.\frac{1}{f(D+a)}V = \frac{1}{f(D)}e^{ax}.V$

$\therefore$ P.I when $x = e^{ax}.V$ is $e^{ax}\frac{1}{f(D+a)}V$

Examples:

(1) Solve $(D^2-3D+2)y = e^x \cdot x$

Sol: $C.F = C_1 e^x + C_2 e^{2x}$

$$P.I = \frac{1}{f(D)} e^x \cdot x = \frac{1}{D^2-3D+2} e^x \cdot x$$

$$= e^x \frac{1}{(D+1)^2-3(D+1)+2} x$$

$$= e^x \frac{1}{D^2-D} x = e^x \frac{1}{-D\{-D+1\}} x$$

$$= -e^x \frac{1}{D} \{1-D\}^{-1} x$$

$$= -e^x \frac{1}{D} \{1+D+D^2+ \text{—} \} x$$

$$= -e^x \cdot \frac{1}{D} \{x+1\} = -e^x \left(\frac{x^2}{2}+x\right)$$

$$\rightarrow y = C_1 e^x + C_2 e^{2x} + e^x \frac{x^2}{2} + x(-e^x)$$

(2) Solve $(D^2-2D+4)y = e^x \cos x$

Sol: $C.F = e^x (C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x)$

$$P.I = \frac{1}{D^2-2D+4} (e^x \cos x) = e^x \left[\frac{1}{\{(D+1)^2-2(D+1)+4\}} \cos x\right]$$

$$= e^x \left[\frac{1}{D^2+3} \cos x\right] = \frac{1}{2} e^x \cos x$$

$y = C.F + P.I$

(3) Solve $(D^3 - 7D - 6)y = e^{2x}(1+x)$

Sol: $C.F = c_1 e^{-x} + c_2 e^{3x} + c_3 e^{-2x}$

$$P.I = e^{2x} \cdot \frac{1}{D^3 + 6D^2 + 5D - 12}(1+x)$$

$$= \frac{-1}{12} e^{2x} \left(1 - \left(\frac{5D + 6D^2 + D^3}{12}\right)\right)^{-1}$$

$$= \frac{-1}{12} e^{2x} \left(\frac{17}{12} + x\right)$$

20/12/24

(4) Solve $(D^2 - 1)y = x \sin 3x$

Sol: $C.F \rightarrow c_1 e^{x} + c_2 e^{-x}$

$$P.I = \frac{1}{D^2 - 1}(x \sin 3x)$$

$\sin 3x \downarrow e^{3ix}$

$\sin 3x$ = Im part of e^{3ix}

$e^{3ix} = \cos 3x + i \sin 3x$

$$= \text{Im. Part} \left\{ \frac{1}{D^2 - 1}(x e^{3ix}) \right\}$$

$$= \text{Im. Part} \left\{ e^{3ix} \left\{ \frac{1}{(D + 3i)^2 - 1} x \right\} \right\}$$

$$= \text{Im part of} \left\{ e^{3ix} \left\{ \frac{1}{D^2 + 6Di - 9 - 1} x \right\} \right\}$$

$$= \text{Im part of} \left\{ e^{3ix} \left\{ \frac{1}{D^2 + 6Di - 10} x \right\} \right\}$$

$$= \text{Im part of} \left\{ e^{3ix} \left\{ \frac{1}{-10\left(1 - \left(\frac{6Di + D^2}{10}\right)\right)} x \right\} \right\}$$

$= \text{ImPart of} \left[e^{3ix} \left(\frac{-1}{10}\right) \left\{ 1 - \left(\frac{6Di + D^2}{10}\right) \right\} x \right]$

$= \text{ImPart of} \left[e^{3ix} \left(\frac{-1}{10}\right) \left\{ 1 + \frac{6Di + D^2}{10} + \ldots \right\} (x) \right]$

$= \text{ImPart of} \left\{ e^{3ix} \left(\frac{-1}{10}\right) \left(x + \frac{6}{10} i\right) \right.$

$= \text{ImPart of} \left\{ \frac{-1}{10} (\cos 3x + i \sin 3x) \left(x + \frac{6}{10} i\right) \right\}$

$= \frac{-1}{10} \left(\frac{6}{10} \cos 3x + x \sin 3x \right)$

$= \frac{-6}{100} \cos 3x - \frac{x}{10} \sin 3x$

$y = C.F + P.I$

(5) Solve $(D^4 + 2D^2 + 1) y = x^2 \cos x$

Sol: $CF \rightarrow (C_1 + C_2 x) \cos x + (C_3 + C_4 x) \sin x$

$P.I = \frac{1}{D^4 + 2D^2 + 1} x^2 \cos x = \text{Re.Part} \left\{ \frac{1}{(D^2+1)^2} x^2 e^{ix} \right\}$

$= \text{RePart} \left\{ e^{ix} \cdot \frac{1}{\{(D+i)^2 + 1\}^2} x^2 \right\}$

$= \text{RePart} \left\{ e^{ix} \cdot \frac{1}{(D^2 + 2iD)^2} x^2 \right\}$

$= \text{RePart} \left[e^{ix} \frac{1}{-4D^2} \left\{ 1 + \frac{D}{2i} \right\}^{-2} (x^2) \right]$

$= \text{RePart} \left\{ e^{ix} \frac{-1}{4} \frac{1}{D^2} \left[1 - \frac{2D}{2i} - \frac{3D^2}{4} \right\} (x^2) \right]$

$= \text{Repart}\left\{-\frac{e^{ix}}{4}\frac{1}{D^2}\left(x^2+2xi-\frac{3}{2}\right)\right\}$

$= \text{Repart}\left\{-\frac{e^{ix}}{4}\frac{1}{D}\frac{1}{D}\left(x^2+2xi-\frac{3}{2}\right)\right\}$

$= \text{Repart}\left(-\frac{e^{ix}}{4}\cdot\frac{1}{D}\left\{\frac{x^3}{3}+x^2i-\frac{3}{2}x\right\}\right)$

$= \text{Repart}\left(-\frac{e^{ix}}{4}\left(\frac{x^4}{12}+\frac{x^3}{3}i-\frac{3}{4}x^2\right)\right)$

$= \frac{-1}{48}\cos x\cdot x^4+\frac{3}{16}x^2\cos x+\frac{1}{12}\sin x\cdot x^3$

$\Rightarrow y = C.F + P.I$

(6) Solve $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y = xe^x\sin x$

Sol: $C.F \rightarrow (c_1+c_2x)e^x$

$P.I = \frac{1}{D^2-2D+1}(e^x\sin x\cdot x)$

$= e^x\frac{1}{(D+1)^2-2(D+1)+1}(x\sin x)$

$= e^x\cdot\frac{1}{D^2}(x\sin x)$

$= -xe^x\sin x-2e^x\cos x$

$\Rightarrow y = C.F + P.I$

Method of Variation of Parameters:

→ This method applies to the eqⁿ of the form

$$y'' + Py' + Qy = X \quad \text{--- (1)}$$

where P, Q are constants & X is a fn of x.

In this case $P.I. = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$

where y_1 & y_2 are soln of $y'' + Py' + Qy = 0$

and $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ which is called the

Wronskian of y_1 & y_2.

Examples:

(1) Solve $\frac{d^2y}{dx^2} + 4y = \tan 2x$

Sol: $C.F. = C_1 \underbrace{\cos 2x}_{y_1} + C_2 \underbrace{\sin 2x}_{y_2}$

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

$$P.I. = -\cos 2x \int \frac{\sin 2x \cdot \tan 2x}{2} dx +$$

$$\sin 2x \int \frac{\cos 2x \tan 2x}{2} dx$$

$$= -\frac{1}{2} \cos 2x \int (\sec 2x - \cos 2x) dx + \frac{1}{2} \sin 2x \int \sin 2x \, dx$$

$$= \frac{-1}{2}\cos 2x\left(\log(\sec 2x+\tan 2x)\frac{1}{2} - \frac{\sin 2x}{2}\right) +$$

$$\frac{1}{2}\sin 2x\left(-\frac{\cos 2x}{2}\right)$$

$$= -\frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$$

General Solⁿ: $y = C.F + P.I$

② Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^{3x}}{x^2}$

Sol: $CF = (C_1 + C_2 x)e^{3x}$

$y_1 = e^{3x}, \; y_2 = xe^{3x}$

$W = e^{6x}$

$$P.I = -e^{3x}\int \frac{xe^{3x}\,\frac{e^{3x}}{x^2}}{e^{6x}}dx + xe^{3x}\int \frac{e^{3x}\cdot e^{3x}}{x^2 e^{6x}}dx$$

$$= -e^{3x}\int \frac{1}{x}dx + xe^{3x}\int \frac{1}{x^2}dx$$

$$= -e^{3x}(\log x + 1)$$

$\Rightarrow y = C.F + P.I$

③ $\frac{d^2y}{dx^2} - y = \frac{2}{(1+e^x)}$

Sol: $CF = C_1 e^x + C_2 e^{-x}$

$W = -2$

$$P.I = e^{x}\int \frac{e^{-x}}{(1+e^{x})}dx - e^{-x}\int \frac{e^{x}}{1+e^{x}}dx$$

$$= e^{x}\left(-e^{-x}+\log(1+e^{-x})\right) - e^{-x}\left(\log(1+e^{x})\right)$$

$1+e^{x}=U$

$e^{x}dx=dU$

$$\int \frac{e^{-x}}{1+e^{x}}dx = \int \frac{1}{e^{x}(1+e^{x})}dx$$

$$= \int \left(\frac{1}{e^{x}} - \frac{1}{1+e^{x}}\right)dx$$

$$= \int e^{-x}dx - \int \frac{1}{1+e^{x}}dx$$

$$= -e^{-x} - \int \frac{1}{e^{x}\left(\frac{1}{e^{x}}+1\right)}dx$$

$$= -e^{-x} - \int \frac{e^{-x}}{(e^{-x}+1)}dx$$

$1+e^{-x}=v$

$-e^{-x}dx=dv$

$$= -e^{-x}+\log(1+e^{-x})$$

$$P.I = -1+e^{x}\log(e^{-x}+1) - e^{-x}\log(1+e^{x})$$

$$y = C.F + P.I$$

(4) Solve $y''-2y'+y=e^{x}\log x$

Sol: $(c_1+c_2x)e^{x}+\frac{1}{4}x^{2}e^{x}(2\log x-3)$

Legendre's Linear eqⁿ:

→ An eqⁿ of the form:

$$(ax+b)^n \frac{d^ny}{dx^n} + K_1 (ax+b)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \dots + K_n y = X \quad (1)$$

where K_i's, a, b are constants and

X is a function of x.

(S-I) $\boxed{ax+b=e^t}$ $\Rightarrow \log(ax+b) = t$

$$\Rightarrow \frac{dt}{dx} = \frac{a}{ax+b}$$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{a}{ax+b}\frac{dy}{dt} = \frac{a}{ax+b} Dy$$

where $D = \frac{d}{dt}$.

$$\Rightarrow (ax+b)\frac{dy}{dx} = aDy.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{a}{ax+b}\cdot\frac{dy}{dt}\right)$$

$$= \frac{-a^2}{(ax+b)^2}\frac{dy}{dt} + \frac{a}{ax+b}\frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$= \frac{-a^2}{(ax+b)^2}\frac{dy}{dt} + \frac{a}{ax+b}\frac{d}{dt}\left(\frac{dy}{dx}\right)$$

$$= \frac{-a^2}{(ax+b)^2}\frac{dy}{dt} + \frac{a}{ax+b}\frac{d}{dt}\left(\frac{dy}{dx}\right)$$

$$= \frac{-a^2}{(ax+b)^2}\frac{dy}{dt} + \frac{a}{ax+b}\frac{d}{dt}\left(\frac{a}{ax+b}\frac{dy}{dt}\right)$$

$$= \frac{-a^2}{(ax+b)^2}\frac{dy}{dt} + \frac{a}{ax+b}\cdot\frac{a}{ax+b}\frac{d^2y}{dt^2}$$

$$= \frac{-a^2}{(ax+b)^2}\frac{dy}{dt} + \frac{a^2}{(ax+b)^2}\frac{d^2y}{dt^2}$$

$$\frac{d^2y}{dx^2} = \frac{a^2}{(ax+b)^2}(-Dy + D^2y)$$

$$= \frac{a^2}{(ax+b)^2} D(D-1)y$$

$$(ax+b)^2\frac{d^2y}{dx^2} = a^2 D(D-1)y$$

$$(ax+b)\frac{dy}{dx} = aDy$$

$$(ax+b)^2\frac{d^2y}{dx^2} = a^2 D(D-1)y \qquad D = \frac{d}{dt}$$

$$(ax+b)^3\frac{d^3y}{dx^3} = a^3 D(D-1)(D-2)y$$

23/12/24

$$K_n(ax+b)^n\frac{d^ny}{dx^n} + K_{n-1}(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + K_1 y = X$$

$K_1, \ldots, K_n, a, b$

$$(ax+b) = e^t \Rightarrow t = \log(ax+b)$$

$$(ax+b)\frac{dy}{dx} = aDy$$

$$(ax+b)^2\frac{d^2y}{dx^2} = a^2 D(D-1)y \qquad D = \frac{d}{dt}$$

$$(ax+b)^3\frac{d^3y}{dx^3} = a^3 D(D-1)(D-2)y$$

Cauchy's differential equation:

If $a=1, b=0$ in Legendre's D.Eq^n then it's Cauchy.

$$K_n x^n \frac{d^n y}{dx^n} + K_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \text{---} + K_1 y = X \quad (1)$$

$\rightarrow K_n, K_{n-1}, - K_n$ are const

X is a fn of x

$\boxed{x = e^t} \Rightarrow t = \log x.$

$$x \frac{dy}{dx} = Dy$$

$$x^2 \frac{d^2y}{dx^2} = D(D-1)y \qquad D = \frac{d}{dt}$$

$$x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

Examples:

(1) $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin(\log(1+x))$

Sol^n: $ax+b$, $a=1, b=1$

$\boxed{x+1 = e^t} \Rightarrow t = \log(x+1)$

$(1+x)\frac{dy}{dx} = 1.Dy = Dy$, $D = \frac{d}{dt}$

$(1+x)^2 \frac{d^2y}{dx^2} = 1^2 D(D-1)y = D(D-1)y$

$ax+b = e^t$

$(ax+b)\frac{dy}{dx} = aDy$

$(ax+b)^2 \frac{d^2y}{dx^2} = a^2D(D-1)$

$\Rightarrow D(D-1)y + Dy + y = 2\sin(t)$

$D^2y - \cancel{Dy} + \cancel{Dy} + y = 2\sin t$

$D^2y + y = 2\sin t \Rightarrow \frac{d^2y}{dt^2} + y = 2\sin t$ (1)

$C.F = c_1 \cos t + c_2 \sin t$

$= c_1 \cos[\log(1+x)] + c_2 \sin[\log(1+x)]$

$P.I = \frac{1}{(D^2+1)} 2 \sin t = 2/t \frac{1}{2D} \sin t = -t \cos t$

$= -\log(1+x) \cos(\log(1+x))$

$y = C.F + P.I = c_1 \cos t + c_2 \sin t - t \cos t$

where $t = \log(1+x)$

(2) Solve $(2x-1)^2 \frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} - 2y = 8x^2 - 2x + 3$

Sol: $a=2$, $2x-1 = e^t$, $x = \frac{1+e^t}{2}$

$(2x-1)^2 \frac{d^2y}{dx^2} = 2^2 D(D-1)y = 4D(D-1)y$

$(2x-1)\frac{dy}{dx} = 2Dy$

$4D(D-1)y + 2Dy - 2y = 8\left(\frac{1+e^t}{2}\right)^2 - 2\left(\frac{1+e^t}{2}\right) + 3$

$\Rightarrow (2D^2 - D - 1)y = e^{2t} + \frac{3}{2}e^t + 2$

$C.F = c_1 e^t + c_2 e^{-t/2}$ where, $t = \log(2x-1)$

$P.I = \frac{1}{(2D^2-D-1)}(e^{2t} + \frac{3}{2}e^t + 2)$ $\cancel{= \frac{1}{(2D^2-D-1)} e}$

$= \frac{1}{2D^2-D-1} e^{2t} + \frac{3}{2} \cdot \frac{1}{2D^2-D-1} e^t + 2 \cdot \frac{1}{2D^2-D-1} e^{0t}$

$= \frac{1}{5} e^{2t} + \frac{1}{2} t e^t - 2$

$y = C.F + P.I$

③ Solve $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=\log x\sin(\log x)$

Sol: $\boxed{x=e^t} \Rightarrow t=\log x$

$D(D-1)y+Dy+y=t\sin(t)$

$(D^2+1)y=t\sin t$ $\qquad D\to\frac{d}{dt}$

$C.F=c_1\cos t+c_2\sin t$

$P.I=\frac{1}{D^2+1}t\sin t$

$=\text{Im pa}\left\{\frac{1}{D^2+1}.te^{it}\right\}$

$=\text{Im pa}\left\{e^{it}.\frac{1}{(D+i)^2+1}t\right\}$

$=\text{Im pa}\left\{e^{it}\frac{1}{D^2+2iD}t\right\}$

$=\text{Im pa}\left\{e^{it}\frac{1}{2i}\frac{1}{D}\frac{1}{1+\frac{D}{2i}}t\right\}$

$=\text{Im pa}\left\{e^{it}\frac{1}{2i}\frac{1}{D}\left\{1+\frac{D}{2i}\right\}^{-1}t\right\}$

$=\text{Im pa}\left\{\frac{e^{it}}{2i}\frac{1}{D}\left\{1-\frac{D}{2i}+\text{---}\right\}t\right\}$

$=\text{Im pa}\left\{\frac{e^{it}}{2i}\frac{1}{D}\left[t-\frac{1}{2i}\right]\right\}$

$=\text{Im pa}\left\{\frac{e^{it}}{2i}\left(\frac{t^2}{2}-\frac{t}{2i}\right)\right\}$

$= \text{Im Pa}\left\{(\cos t + i\sin t)\left(\frac{-it^2}{4} + \frac{t}{4}\right)\right\}$

$= \frac{-t^2}{4}\cos t + \frac{t\sin t}{4}$

$y = C.F + P.I$

$= C_1\cos t + C_2\sin t - \frac{t^2}{4}\cos t + \frac{t\sin t}{4},$

where $t = \log x$

(4) Solve $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^5$

Sol: $(D^2 - 5D + 6)y = e^{5t}$

$C.F \rightarrow C_1e^{2t} + C_2e^{3t}$

$P.I \rightarrow \frac{e^{5t}}{6}$

Solution of simultaneous differential eqn:

$x \rightarrow x(t),\ y \rightarrow y(t),\ x, y.$

Examples:

(1) Solve $\frac{dx}{dt} + 5x - 2y = t$, $\frac{dy}{dt} + 2x + y = 0$
—(1) —(2)

Given $x = y = 0$ when $t = 0$

Sol: $Dx + 5x - 2y = t \Rightarrow (D+5)x - 2y = t$ —(1)

$2x + Dy + y = 0 \Rightarrow 2x + (D+1)y = 0$ —(2)

Eliminate x, $[(1) \times 2] - [(2) \times (D+5)]$

$\Rightarrow -4y - (D+1)(D+5)y = 2t$

$\boxed{(D^2 + 6D + 9)y = -2t}$

$D \rightarrow \frac{d}{dt}$

$$C.F = (C_1 + C_2 t)e^{-3t}$$

$$P.I = \frac{1}{(D+3)^2}(-2t) = -2\frac{1}{9\left(1+\frac{D}{3}\right)^2}(t)$$

$$= \frac{-2}{9}\left\{1+\frac{D}{3}\right\}^{-2}(t).$$

$$= \frac{-2}{9}\left(1-\frac{2D}{3}+\cdots\right)(t) = \frac{-2}{9}t+\frac{4}{27}$$

$$\boxed{y = (C_1 + C_2 t)e^{-3t} - \frac{2}{9}t + \frac{4}{27}} \longrightarrow (A)$$

$$x = -\frac{1}{2}(Dy+y) = \left[\left(C_1 - \frac{1}{2}C_2\right)+C_2 t\right]e^{-3t} + \frac{t}{9} + \frac{1}{27} \longrightarrow (B)$$

$$C_2 e^{-3t} + (C_1 + C_2 t)(-3)e^{-3t} - 2/9$$

$$(A) \Rightarrow 0 = C_1 + \frac{4}{27} \Rightarrow C_1 = \frac{-4}{27}$$

$$(B) \Rightarrow 0 = \left(C_1 - \frac{1}{2}C_2\right) + \frac{1}{27} \Rightarrow C_2 = \frac{-2}{9}$$

(2) $\frac{dx}{dt} + 2y + \sin t = 0, \frac{dy}{dt} - 2x - \cos t = 0$

Given that $x=0$ & $y=1$ where $t=0$.

Sol: $Dx + 2y = -\sin t$ — (1) $\quad D \to \frac{d}{dt}$

$Dy - 2x = \cos t$ — (2)

(1) $\times 2$ + (2) $\times D$ $\Rightarrow (D^2+4)y = -3\sin t$

$C.F \to C_1 \cos 2t + C_2 \sin 2t$

$P.I = -\sin t$

$$\boxed{y = C_1 \cos 2t + C_2 \sin 2t - \sin t}$$

$$x = \frac{1}{2}(Dy - \cos t) = -c_1 \sin 2t + c_2 \cos 2t - \cos t$$

$$\Downarrow$$

$$-2c_1 \sin 2t + 2c_2 \cos 2t - \cos t.$$

$t=0,\ x=0 \Rightarrow \boxed{c_1 = 1}$

$t=0,\ y=1 \Rightarrow \boxed{c_2 = 1}$