Higher order Differential Equation

$\frac{dy}{dx}+P(x)y=X$

The differential equation of the form

$$b_0(x)\frac{d^ny}{dx^n}+b_1(x)\frac{d^{n-1}y}{dx^{n-1}}+b_2(x)\frac{d^{n-2}y}{dx^{n-2}}+\cdots+b_n(x)y=X \longrightarrow (1).$$

is called the n^{th} order linear differential equation of variable co-efficients, where $b_0, b_1, b_2 \cdots b_n$ & X are functions of x only.

Note:- If $X=0$ in (1), then eqⁿ (1) is called n^{th} order linear homogeneous differential equation.

If $X\neq 0$ in (1), then eq (1) is called n^{th} order linear non-homogeneous differential equation.

If in eqⁿ (1), $b_0, b_1, b_2 \cdots b_n$ are constants, then eqⁿ (1) is called n^{th} order linear differential equation with constant co-efficient.

i.e $$\frac{d^ny}{dx^n}+b_1\frac{d^{n-1}y}{dx^{n-1}}+b_2\frac{d^{n-2}y}{dx^{n-2}}+\cdots b_ny=X. \longrightarrow (2)$$

Note:- (1) If y_1 & y_2 are only the solution of

$$\frac{d^ny}{dx^n}+b_1\frac{d^{n-1}y}{dx^{n-1}}+b_2\frac{d^{n-2}y}{dx^{n-2}}+\cdots+b_ny=0 \longrightarrow (3).$$

then $c_1y_1+c_2y_2$ is also the solution of (3).

Proof:- Given y_1 & y_2 are the only solution of (3).

i.e $$\frac{d^ny_1}{dx^n}+b_1\frac{d^{n-1}y_1}{dx^{n-1}}+b_2\frac{d^{n-2}y_1}{dx^{n-2}}+\cdots+b_ny_1=0 \rightarrow (i)$$

$$\frac{d^ny_2}{dx^n}+b_1\frac{d^{n-1}y_2}{dx^{n-1}}+b_2\frac{d^{n-2}y_2}{dx^{n-2}}+\cdots+b_ny_2=0 \rightarrow (ii)$$

Consider $u=c_1y_1+c_2y_2$

$$\frac{d^nu}{dx^n}+b_1\frac{d^{n-1}u}{dx^{n-1}}+b_2\frac{d^{n-2}u}{dx^{n-2}}+\cdots+b_nu \boxed{=0 \ ?}$$

$$\frac{d^n u}{dx^n} + b_1\frac{d^{n-1} u}{dx^{n-1}} + b_2\frac{d^{n-2} u}{dx^{n-2}} + \cdots + b_n u \boxed{= 0 \;?}$$

$$\frac{d^n}{dx^n}(c_1y_1+c_2y_2) + b_1\frac{d^{n-1}}{dx^{n-1}}(c_1y_1+c_2y_2) + b_2\frac{d^{n-2}}{dx^{n-2}}(c_1y_1+c_2y_2) + \cdots + b_n(c_1y_1+c_2y_2).$$

$$\Rightarrow \quad c_1\left[\frac{d^n y_1}{dx^n} + b_1\frac{d^{n-1} y_1}{dx^{n-1}} + b_2\frac{d^{n-2} y_1}{dx^{n-2}} + \cdots + b_n y_1\right] + c_2\left[\frac{d^n y_2}{dx^n} + b_1\frac{d^{n-1} y_2}{dx^{n-1}} + b_2\frac{d^{n-2} y_2}{dx^{n-2}} + \cdots + b_n y_2\right]$$

$$c_1\{0\} + c_2\{0\}$$

?

$\Rightarrow$ "u" is the solution of (3).

i.e $u = c_1y_1 + c_2y_2$ is also the solⁿ of (3).

Since the general solution of differential equation of n^{th} order contains 'n' arbitrary constant. It follows from the above note if,

$y_1, y_2, \ldots\ldots y_n$ are 'n' independent solution of (3), then

$c_1y_1 + c_2y_2 + c_3y_3 + \cdots + c_ny_n$ is the complete solution of (3).

Note-2 :- If 'V' be a particular solution of

$$\frac{d^n y}{dx^n} + b_1\frac{d^{n-1} y}{dx^{n-1}} + b_2\frac{d^{n-2} y}{dx^{n-2}} + \cdots + b_n y = X \neq 0 \quad \longrightarrow (4).$$

then

$$\frac{d^n V}{dx^n} + b_1\frac{d^{n-1} V}{dx^{n-1}} + b_2\frac{d^{n-2} V}{dx^{n-2}} + \cdots + b_n V = X. \quad \longrightarrow (5).$$

W.K.T $u = c_1y_1 + c_2y_2 + \cdots + c_ny_n$ is the solution of

$$\frac{d^n y}{dx^n} + b_1\frac{d^{n-1} y}{dx^{n-1}} + \cdots + b_n y = 0 \quad \longrightarrow (*)$$

$$\frac{d^n u}{dx^n} + b_1\frac{d^{n-1} u}{dx^{n-1}} + b_2\frac{d^{n-2} u}{dx^{n-2}} + \cdots + b_n u = 0. \quad \longrightarrow (6).$$

Adding (5) & (6)

$$\frac{d^n}{dx^n}(u+v) + b_1 \frac{d^{n-1}}{dx^{n-1}}(u+v) + b_2 \frac{d^{n-2}}{dx^{n-2}}(u+v) + \cdots + b_n(u+v) = X \rightarrow (7)$$

$\Rightarrow \quad y = u + v$

u: Solution of homogeneous differential equation.
v: Solution of non-homogeneous differential equation

'u' is called Complementary function (CF) & 'v' is called the particular integral (P I).

We will study two methods:-

1) Inverse differential operator method
2) Method of variation of parameters

Inverse differential operator method

Denoting $D, D^2, D^3, \cdots$ by $\frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3} \cdots$

$$Dy = \frac{dy}{dx}, \quad D^2y = \frac{d^2y}{dx^2}, \quad \cdots$$

D: Differential operator.

Solution procedure of finding the CF is as follows

$$\frac{d^ny}{dx^n} + b_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + b_ny = 0 \longrightarrow (1).$$

write (1) in Symbolic form

$$D^ny + b_1 D^{n-1}y + \cdots + b_ny = 0.$$

$$(D^n + b_1 D^{n-1} + b_2 D^{n-2} + \cdots + b_n)y = 0 \longrightarrow (2)$$

$$f(D)y = 0.$$

$\Rightarrow$ where $f(D) = D^n + b_1 D^{n-1} + b_2 D^{n-2} + \cdots + b_n$

To find CF, equate the Co-efficient in (2) to zero

i.e $D^n + b_1 D^{n-1} + b_2 D^{n-2} + \cdots + b_n = 0 \longrightarrow (3)$.

Eqn (3) is called "Auxillary Equation" (A E).

Case-I :- If the roots of eqn (3) are real & different, say

$m_1, m_2, m_3, \cdots m_n$ are roots of (3).

then eqn (2) can be written as

$$(D-m_1)(D-m_2)(D-m_3) \cdots (D-m_n) y = 0 \longrightarrow (4)$$

Eqn (4) has to satisfy

$y(IF) = \int IF X + c$

$$(D-m_n) y = 0$$

$$\Rightarrow \frac{dy}{dx} - m_n y = 0 \rightarrow (K)$$

$$IF \text{ of } (K) = e^{\int -m_n dx} = e^{-m_n x}$$

Its solution is

$$y e^{-m_n x} = c$$

$$\underline{y = c e^{m_n x}}$$

$$\Rightarrow y = c_1 e^{m_1 x}, \; y = c_2 e^{m_2 x}, \; y = c_3 e^{m_3 x} \cdots y = c_n e^{m_n x}$$

$$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}} \longrightarrow (5)$$

Case-II :- If the roots of the equation are real & equal.

Say $\underline{m_1 = m_2}, m_3, \cdots m_n$.

Eqn (2) can be written as

$$(D-m_1)(D-m_1)(D-m_3) \cdots (D-m_n) y = 0$$

From (5)

$$y = c_1 e^{m_1 x} + c_2 e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

$$y = c e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

This can not be a general soln of the differential equation.

$$\boxed{(D-m_1)(D-m_1)y=0}$$

$(D-m_1)y=0$

$\frac{dy}{dx}-m_1y=0$

$y=c_1e^{m_1x}$

Consider

$$z=(D-m_1)y.$$

$$z=c_1e^{m_1x}$$

$$(D-m_1)y=c_1e^{m_1x}$$

$$\frac{dy}{dx}-m_1y=c_1e^{m_1x}\longrightarrow(*)$$

$$\Rightarrow\quad I.F=e^{-\int m_1dx}=e^{-m_1x}$$

Sol^n $(*)$ is

$$ye^{-m_1x}=\int c_1e^{m_1x}\cdot e^{-m_1x}dx+c_2$$

$$\Rightarrow\quad ye^{-m_1x}=c_1x+c_2$$

$$\Rightarrow\quad y=(c_1x+c_2)e^{m_1x}$$

$\frac{dy}{dx}+P(x)y=X$

$\Rightarrow\quad IF=e^{\int P(x)dx}$

$y\,IF=\int X\,IF+C$

$\therefore\quad (5)\Rightarrow$

$$y=(c_1x+c_2)e^{m_1x}+c_3e^{m_3x}+c_4e^{m_4x}+\cdots+c_ne^{m_nx}\quad(**)$$

Note :- If $m_1=m_2=m_3,\ m_4,m_5\cdots m_n$ are roots

$$y=(c_1x^2+c_2x+c_3)e^{m_1x}+c_4e^{m_4x}+\cdots+c_ne^{m_nx}$$

Case (III) :- If the roots of A.E having a pair of imaginary roots, i.e $m_1=\alpha+i\beta,\ m_2=\alpha-i\beta,\ m_3,m_4\cdots m_n$, then

W.K.T

$$y=c_1e^{m_1x}+c_2e^{m_2x}+c_3e^{m_3x}+\cdots+c_ne^{m_nx}$$

$$y=c_1e^{(\alpha+i\beta)x}+c_2e^{(\alpha-i\beta)x}+c_3e^{m_3x}+\cdots+c_ne^{m_nx}$$

$$y=c_1\{e^{\alpha x}e^{i\beta x}\}+c_2\{e^{\alpha x}e^{-i\beta x}\}+c_3e^{m_3x}+\cdots+c_ne^{m_nx}$$

$$=e^{\alpha x}[c_1(\cos\beta x+i\sin\beta x)+c_2(\cos\beta x-i\sin\beta x)]+c_3e^{m_3x}+\cdots+c_ne^{m_nx}$$

$$y=e^{\alpha x}\{c_1\cos\beta x+c_2\sin\beta x\}+c_3e^{m_3x}+\cdots+c_ne^{m_nx}\longrightarrow(*)$$

(where $c_1=c_1+c_2$
$c_2=c_1i-c_2i$

Note :- For repeated complex root

$m_1 = m_2 = \alpha + i\beta$

$m_3 = m_4 = \alpha - i\beta.$

$$y = e^{\alpha x}\left\{(c_1 x + c_2)\cos\beta x + (c_3 x + c_4)\sin\beta x\right\} + c_5 e^{m_5 x} + \cdots + c_n e^{m_6 x}.$$

Roots	CF
1) Real & different (m_1, m_2, m_3)	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$ ✓
2) Real & equal $(m_1 = m_2, m_3)$.	$y = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_3 x}$ ✓
3) Roots are imaginary $(m_1 = \alpha + i\beta, m_2 = \alpha - i\beta), m_3$	$y = e^{\alpha x}\{c_1 \cos\beta x + c_2 \sin\beta x\} + c_3 e^{m_3 x}$ ✓
4) Repeated imaginary roots	$y = e^{\alpha x}\{(c_1 x + c_2)\cos\beta x + (c_3 x + c_4)\sin\beta x\}$

Q) Solve $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$, given $x(0) = 0$ & $\frac{dx}{dt} = 15$ when $t = 0$.

Sol^n:- The given differential equation in symbolic form is

$D^2 x + 5Dx + 6x = 0$ ✓ , where $D = \frac{d}{dt}$

$\Rightarrow (D^2 + 5D + 6)x = 0$ ✓ $\qquad f(D)y = 0$

A.E is $D^2 + 5D + 6 = 0$ ✓

Roots of A.E is

$D^2 + 3D + 2D + 6 = 0$

$D(D+3) + 2(D+3) = 0$

$\Rightarrow (D+2)(D+3) = 0$

$D = -2, -3.$ ✓

Roots are real & different

$\Rightarrow x(t) = c_1 e^{-2t} + c_2 e^{-3t} \rightarrow (1)$

Given $x(0) = 0$

Substitute $t = 0$ in (1)

$x(0) = c_1 \times 1 + c_2$

$\Rightarrow c_1 + c_2 = 0 \rightarrow (i).$

Given $\frac{dx}{dt} = 15$, when $t = 0$.

$$\frac{dx}{dt} = -2c_1 e^{-2t} - 3c_2 e^{-3t} \rightarrow (2)$$

Put $t=0$ in (2)

$$15 = -2c_1 - 3c_2 \longrightarrow (ii)$$

$\boxed{c_1 + c_2 = 0 \rightarrow (i)}$

Solving (i) & (2)

$$15 = 2c_2 - 3c_2$$

$$\Rightarrow -c_2 = 15 \Rightarrow c_2 = -15$$

$$\Rightarrow c_1 = 15.$$

$$\therefore \boxed{x(t) = 15e^{-2t} - 15e^{-3t}}$$

Q) Solve $D^4x + 4x = 0$.

Sol^n:- Given $(D^4+4)x = 0$

A.E

$$D^4 + 4 = 0$$

$$(D^2+2)^2 - 4D^2 = 0$$

$$(D^2+2)^2 - (2D)^2 = 0$$

$[a^2-b^2 = (a+b)(a-b)]$

$$\Rightarrow \underbrace{(D^2+2+2D)}_{(i)}\underbrace{(D^2+2-2D)}_{(ii)} = 0$$

$$D = \frac{-2 \pm \sqrt{4-8}}{2}, \quad D = \frac{2 \pm \sqrt{4-8}}{2}$$

$$= \frac{-2 \pm \sqrt{-4}}{2} \qquad = \frac{2 \pm \sqrt{-4}}{2}$$

$$= -1 \pm i \qquad = 1 \pm i$$

$$D = (-1+i), (-1-i), (1+i), (1-i).$$

$$D = (-1+i), (-1-i), (1+i), (1-i).$$

$$\therefore \quad x = e^{-t}\left[c_1\cos t + c_2\sin t\right] + e^{t}\left[c_3\cos t + c_4\sin t\right]$$

Q) Solve $D^3 y + y = 0$

Sol^n:- Given $(D^3+1)y = 0$

A.E $D^3+1 = 0$

$(D+1)(D^2-D+1) = 0$, $D = -1, \frac{1\pm\sqrt{3}i}{2}$

$D = \frac{1\pm\sqrt{1-4}}{2}$ $\quad \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$D = \frac{1\pm\sqrt{3}i}{2}$

$$y = c_1 e^{-x} + e^{\frac{1}{2}x}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right]$$

Q) Solve $(4D^4-8D^3-7D^2+11D+6)y = 0$

Sol^n:- AE is $4D^4-8D^3-7D^2+11D+6 = 0$. $\rightarrow$ Synthetic division method.

	D^4	D^3	D^2	D	D^0
-1	4	-8	-7	11	6
		-4	12	-5	-6
2	4	-12	5	6	0
		8	-8	-6	
	4	-4	-3	0	

$4D^2-4D-3 = 0$

$D = \frac{3}{2}, -\frac{1}{2}$

$D = -1, 2, \frac{3}{2}, -\frac{1}{2}$

$$y = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{\frac{3}{2}x} + c_4 e^{-\frac{1}{2}x}.$$

The solution of linear Non-homogeneous differential equation with Constant Coefficients

We know that the solution of differential equation of the form

$$\frac{d^n y}{dx^n} + b_1 \frac{d^{n-1} y}{dx^{n-1}} + b_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + b_n y = X \rightarrow (1)$$

is $\quad y = y_c + y_p \qquad [y = CF + PI]$

where

y_c = General solution of linear homogeneous differential equation

y_p = Particular integral of eq^n (1)

To find y_p we use the following methods

1) Inverse differential operator

2) Method of variation of parameter.

Inverse differential operator :-

To find the sol^n of D.E of the form

$$f(D) y = \phi(x)$$

we introduce the inverse differential operator as $\frac{1}{f(D)}$,

i.e $\quad y = \frac{1}{f(D)} \phi(x).$

$$\left\{ \begin{array}{l} D(x^2) = 2x \\ \frac{1}{D}(x^2) = \int x^2 dx = \frac{x^3}{3} \end{array} \right.$$

Case (i) :- when $\phi(x) = e^{ax}$ $\qquad y = \frac{1}{f(D)}\phi(x)$

i.e particular integral

$$= \frac{1}{f(D)} e^{ax} = \begin{cases} \frac{e^{ax}}{f(a)} & \text{if } f(a) \neq 0. \\ \frac{x e^{ax}}{f'(a)} & \text{if } f(a) = 0 \text{ provided } f'(a) \neq 0. \end{cases}$$

Proof :- W.K.T

$$D e^{ax} = a e^{ax}$$

$$D^2 e^{ax} = a^2 e^{ax}$$

$$\vdots$$

$$D^n e^{ax} = a^n e^{ax}$$

$$f(D) e^{ax} = f(a) e^{ax}$$

$$e^{ax} = \frac{1}{f(D)} f(a) e^{ax}$$

$$\Rightarrow \boxed{\frac{e^{ax}}{f(a)} = \frac{1}{f(D)} e^{ax}} \qquad f(a) \neq 0.$$

$$\boxed{f(D) \neq 0}$$

when $f(a) = 0$.

$$f(D) = (D-a)\phi(D), \qquad \phi(a) \neq 0.$$

$$\Rightarrow \frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)\phi(D)} e^{ax} = \frac{1}{(D-a)\phi(a)} e^{ax} \rightarrow (*).$$

Note :- W.K.T $\frac{1}{D-a} X = e^{ax} \int X e^{-ax} dx$.

$$\frac{1}{D-a} X = y$$

$$X = Dy - ay$$

$$X = \frac{dy}{dx} - ay$$

$$IF \quad e^{-ax}$$

$$y e^{-ax} = \int X e^{-ax} dx$$

$$\frac{1}{D-a} e^{ax} = e^{ax} \int e^{ax} e^{-ax} dx = x e^{ax}$$

where $f(D) = \frac{d^n}{dx^n} + b_1 \frac{d^{n-1}}{dx^{n-1}} + b_2 \frac{d^{n-2}}{dx^{n-2}} + \cdots + b_n$

$$\Rightarrow \frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)\,\phi(D)} e^{ax} = \frac{1}{(D-a)\,\phi(a)} e^{ax} \rightarrow (*)$$

$$= \frac{1}{\phi(a)} x e^{ax}$$

$$\therefore \frac{1}{f(D)} e^{ax} = \frac{x e^{ax}}{\phi(a)} \text{ , when } f(a) = 0.$$

$\boxed{f(D) = 0}$

W.K.T

$$f(D) = (D-a)\,\phi(D)$$

$$f'(D) = (D-a)\,\phi'(D) + \phi(D)$$

$$f'(a) = 0 + \phi(a) \Rightarrow \phi(a) = f'(a).$$

Thus $\frac{1}{f(D)} e^{ax} = \frac{x e^{ax}}{f'(a)}$ when $f(a) = 0$. Provided $f'(a) \neq 0$.

Illy

$\frac{1}{f(D)} e^{ax} = x^2 \frac{e^{ax}}{f''(a)}$ If $f'(a) = 0$ Provided $f''(a) \neq 0$

$\vdots$

Case 2:- $\phi(x) = \sin(ax+b)$ or $\cos(ax+b)$.

i.e $f(D)y = \sin(ax+b)$ or $\cos(ax+b)$.

Particular integral is

$$y = \frac{1}{f(D)}\{\sin(ax+b) \text{ or } \cos(ax+b)\}.$$

$$D\sin(ax+b) = a\cos(ax+b)$$
$$D^2\sin(ax+b) = -a^2\sin(ax+b)$$
$$D^3\sin(ax+b) = -a^3\cos(ax+b)$$
$$D^4\sin(ax+b) = a^4\sin(ax+b)$$
$$(D^2)^2\sin(ax+b) = (-a^2)^2\sin(ax+b)$$
$$(D^2)^n\sin(ax+b) = (-a^2)^n\sin(ax+b)$$

$$f(D^2)\sin(ax+b) = f(-a^2)\sin(ax+b)$$

$$\Rightarrow \sin(ax+b) = \frac{1}{f(D^2)} f(-a^2)\sin(ax+b)$$

$$\frac{\sin(ax+b)}{f(-a^2)} = \frac{1}{f(D^2)}\sin(ax+b). \quad | \; f(-a^2) \neq 0.$$

If $f(-a^2) = 0$, above case fails.

$$\boxed{\frac{1}{f(D^2)}\sin(ax+b) = x\frac{1}{f'(-a^2)}\sin(ax+b)}$$

when $f(-a^2) = 0$
Provided $f'(-a^2) \neq 0$.

W.K.T

$$e^{i(ax+b)} = \cos(ax+b) + i\sin(ax+b)$$

$$I.P \frac{1}{f(D^2)} e^{i(ax+b)} = I.P \; x \frac{1}{f'(-a^2)} e^{i(ax+b)}$$ when $f(-a^2)=0$ & $f'(-a^2) \neq 0$

i.e

$$\frac{1}{f(D^2)} \sin(ax+b) = x \frac{1}{f'(-a^2)} \sin(ax+b)$$ when $f(-a^2)=0$ & $f'(-a^2) \neq 0$.

If $f'(-a^2)=0$

$$\frac{1}{f(D^2)} \sin(ax+b) = x^2 \frac{1}{f''(-a^2)} \sin(ax+b)$$ when $f''(-a^2) \neq 0$ & $f'(-a^2) = 0$.

Case 3:- when $\phi(x) = x^m$

$$P.I = \frac{1}{f(D)} x^m = [f(D)]^{-1} x^m$$

Expand $[f(D)]^{-1}$ in ascending powers of D as far as the term in D^m & operate on x^m term by term. Since $(m+1)$th & higher derivatives of x^m are zero, we need not consider terms beyond D^m.

For Ex:- $\boxed{\frac{1}{1-D} x^2}$

$$= (1-D)^{-1} x^2$$

$$= (1+D+D^2+\cdots)x^2$$

$$= (x^2+2x+2+0\cdots)$$

Case 4: $\phi(x) = e^{ax} V$, where 'V' is function of 'x'

$$P.I = \frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$$

Proof :-

$$D\{e^{ax} u\} = a e^{ax} u + e^{ax} Du$$

$$= e^{ax} (D+a) u$$

$$D^2[e^{ax} u] = e^{ax} D^2 u + Du e^{ax}.a + a e^{ax} Du + u a^2 e^{ax}$$

$$= e^{ax} [D^2 + 2Da + a^2] u.$$

$$= e^{ax} (D+a)^2 u$$

$$D^3[e^{ax} u] = e^{ax} (D+a)^3 u.$$

$$\vdots$$

$$D^n\{e^{ax} u\} = e^{ax} [D+a]^n u.$$

$$f(D)(e^{ax} u) = e^{ax} f(D+a) u.$$

Operating B.S by $\frac{1}{f(D)}$

$$e^{ax} u = \frac{1}{f(D)} [e^{ax} f(D+a) u] \longrightarrow (1).$$

Now put $f(D+a) u = V$, i.e $u = \frac{V}{f(D+a)}$

So that

$$e^{ax} \times \frac{V}{f(D+a)} = \frac{1}{f(D)} (e^{ax} V)$$

$$\therefore \frac{1}{f(D)} (e^{ax} V) = e^{ax} \frac{1}{f(D+a)} V.$$