Theorem: If $A \subset B$, then $P(A) \leq P(B)$

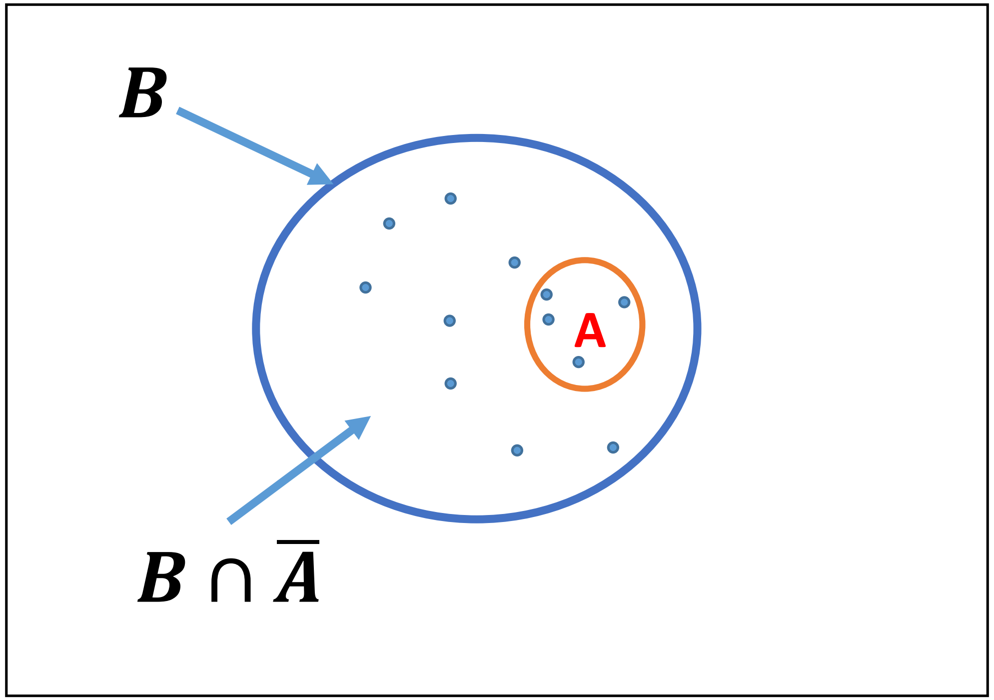
Proof:

$$\boldsymbol{B = A \cup [B \cap \bar{A}]}$$

$$\boldsymbol{P(B) = P(A) + P[B \cap \bar{A}]} \;\; \geq \boldsymbol{P(A)}$$

(Since $\boldsymbol{0 \leq P(B \cap \bar{A}) \leq 1}$)

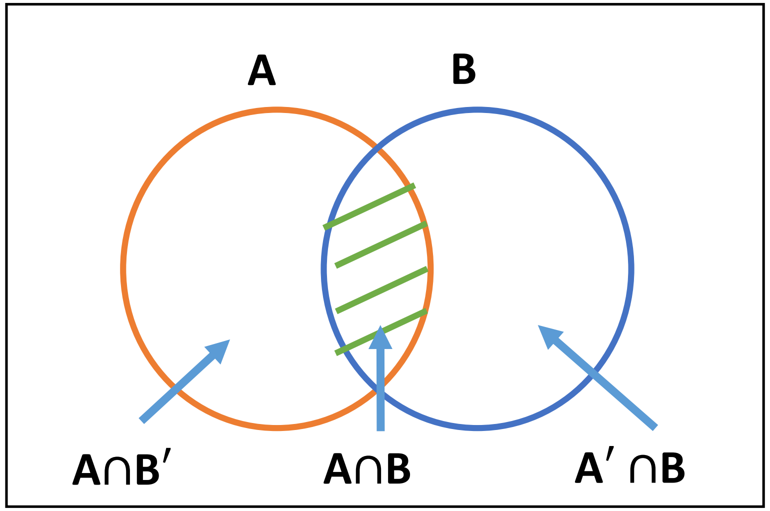
Therefore $\boldsymbol{P(B) \geq P(A)}$



Note:

$$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

$$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$



Show that the probability that exactly one of the event A or B occurs is equal to $P(A) + P(B) - 2P(A \cap B)$

Proof: $P(A) + P(B) = P(A \cup B) - P(A \cap B)$

$$= P(A) + P(B) - P(A \cap B) - P(A \cap B)$$

$$= P(A) + P(B) - 2P(A \cap B)$$

Methods of Enumeration:

Multiplication Principle: Suppose that a procedure, say **procedure A** can be done in $\boldsymbol{n}$ different ways and another **procedure say B** can be done in $\boldsymbol{m}$ different ways. Also suppose that any way of doing procedure A can be followed by any way of doing procedure B. Then, the procedure consisting of 'A followed by B' can be performed in $\boldsymbol{mn}$ ways.

Addition Principle: The number ways in which either A or B, but not both, can be performed is $\boldsymbol{m + n}$.

Permutation: (Arrangement of given objects; Order is important)

The number of permutations of $\boldsymbol{n}$ distinct objects taken $\boldsymbol{r}$ at a time is

- $${}^{n}P_{r} = \frac{n!}{(n-r)!}, \; r \le n$$ if repetition is not allowed (r permutation of n objects).
- n^r if repetition is allowed.
- $$\frac{n!}{k_1!\, k_2! \ldots k_m!}$$, if there are $\boldsymbol{n}$ objects, k_1 are of one kind, k_2 are of a second kind, …, k_m are of m^{th} kind
- Number of circular permutation of $\boldsymbol{n}$ distinct objects is $(\boldsymbol{n}-\mathbf{1})!$
- If clockwise and anticlockwise permutations are indistinguishable, then number of circular permutations would be $\frac{(\boldsymbol{n}-\mathbf{1})!}{\mathbf{2}}$

Combination: (Selection of objects; Order is not important)

An r **–combination** of n **objects** is an unordered selection of r – objects from given n – objects.

$$^{n}C_{r} = \frac{n!}{(n-r)!\ r!}$$ without repetition

$$^{n+r-1}C_{r} = \frac{n!}{(n-r)!\ r!}$$ with repetition

Questions:

Conditional Probability:

The conditional probability of an event B, given that an event A already happened, is denoted by $P(B|A)$ and is defined as,

$$\boldsymbol{P(B|A) = \frac{P(A \cap B)}{P(A)}}, \qquad \text{provided } P(A) \neq 0$$

Properties:

- If $A \subset B$, $\boldsymbol{P(B|A) = 1}$ $\quad (\because A \cap B = A)$
- If $B \subset A$, $\boldsymbol{P(B|A) \geq P(B)}$ $\quad \left(\because A \cap B = \boldsymbol{B}\right.$
  $\left. P(B|A) = \frac{P(B)}{P(A)} \geq P(B), as\ P(A) \leq P(S) = 1 \right)$
- If A and B are mutually exclusive, $\boldsymbol{P(B|A) = 0}$ $\quad (\because P(A \cap B) = 0)$
- If $\boldsymbol{P(A) > P(B)}$, $\boldsymbol{P(A|B) > P(B|A)}$
- **If** $\boldsymbol{A_1 \subset \mathrm{A}_2}$, $\boldsymbol{\mathrm{P}(\mathrm{A}_1|\mathrm{B}) \leq P(A_2|B)}$
- $\boldsymbol{P(S|A) = 1}$
- $\boldsymbol{P[(B_1 \cup B_2)|A] = P(B_1|A) + P(B_2|A) \quad if \quad B_1 \cap B_2 = \emptyset}$

Independent Event:

A **set** of **events** are said to be **independent,** if the occurrence of any one of them does not depend on the occurrence or non occurrence of the others.

Multiplication law of probability for independent events:

- A and B are independent iff $P(A \cap B) = P(A)P(B)$
- A and B are independent iff $P(A|B) = P(A)$ and $P(B|A) = P(B)$ provided $P(A) \neq 0, P(B) \neq 0$

Theorem: If A and B are independent, then prove that

$(i)\ \ A$ and $\overline{B}$

$(ii)\,\overline{A}\ \ and\ B$

$(iii)\,\overline{A}\ \ and\ \overline{B}$ are also independent

Proof:

(i). $(A \cap B') \cup (A \cap B) = A$

$$\boldsymbol{P}(A \cap B') + \boldsymbol{P}(A \cap B) = \boldsymbol{P}(A)$$

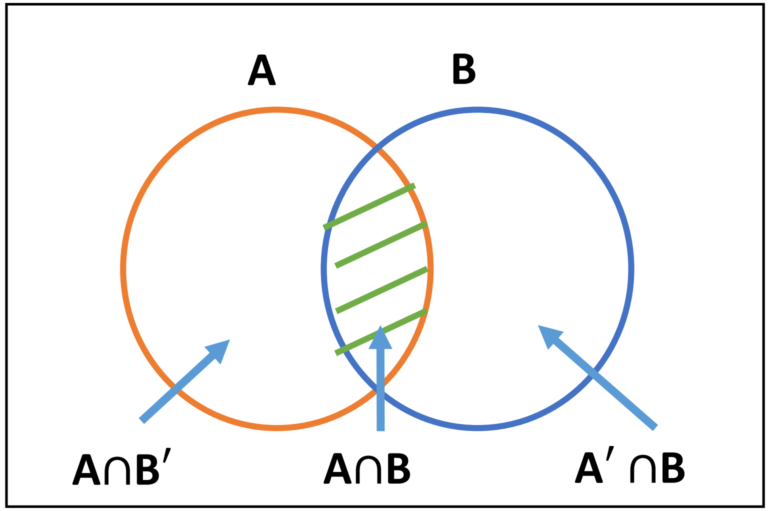
$$\boldsymbol{P}(A \cap B') = \boldsymbol{P}(A) - \boldsymbol{P}(A \cap B)$$

$$= \boldsymbol{P}(A) - \boldsymbol{P}(A)\boldsymbol{P}(B)$$

$$= \boldsymbol{P}(A)[\mathbf{1} - \boldsymbol{P}(B)]$$

$$= \boldsymbol{P}(A)\boldsymbol{P}(B')$$

$\therefore$ A and B' are independent.



(ii). $(A \cap B) \cup (A' \cap B) = B$

$$\boldsymbol{P}(A \cap B) + \boldsymbol{P}(A' \cap B) = \boldsymbol{P}(B)$$

$$\boldsymbol{P}(A' \cap B) = \boldsymbol{P}(B) - \boldsymbol{P}(A \cap B)$$

$$= \boldsymbol{P}(\boldsymbol{B}) - \boldsymbol{P}(A)\boldsymbol{P}(B)$$

$$= \boldsymbol{P}(\boldsymbol{B})[\mathbf{1} - \boldsymbol{P}(\boldsymbol{A})]$$

$$= \boldsymbol{P}(A')\boldsymbol{P}(B)$$

$\therefore$ A' and B are independent.

(iii). $\bar{A} \cap \bar{B} = \overline{A \cup B}$

$$\boldsymbol{P}(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$$

$$= \mathbf{1} - \boldsymbol{P}(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - P(A) - P(B) + P(A)P(B)$$

$$= [1 - P(A)] - P(B)[1 - P(A)] = [1 - P(A)][1 - P(B)]$$

$$= P(\bar{A})P(\bar{B})$$

$\therefore A'$ and B' are independent.

Probability Questions:

1. Find the probability that a leap year selected at random will contains 53 Sundays.

2. The probability that a contractor will get a plumbing contract is $\frac{2}{3}$ and probability that he will not get an electrical contract is $\frac{5}{9}$. If the probability of his getting at least one contract is $\frac{4}{5}$. What is the probability that he will be getting both contracts?

3. Three groups of children contain respectively (3 girls, 1boy), (2 girls, 2 boys), (1 girl, 3 boys). One child is selected at random from each group. Show that the chance that the 3 selected consists 1 girl and 2 boys is $\frac{13}{32}$.

4. Six persons toss a coin turn by turn. The game is won by the player who first throws head. Find the probability of success of the 4^{th} player.

5. Two defective tubes get mixed with 2 good ones. The tubes are tested one by one, until both defective are found. What is the probability that the last defective tube is obtained on

(i). the second test

(ii). the third test

(iii). the fourth test.

6. If 4 squares are chosen at random on a chess board, find the chance that they should be in a diagonal line.

7. If A and B are two independent events of S, such that $P(\bar{A} \cap B) = \frac{2}{15}$, $P(A \cap \bar{B}) = \frac{1}{6}$, then find $P(B)$.

8. A bag contains 10 gold and 8 silver coins. Two successive drawing of 4 coins are made such that,

(i) The coins are replaced before the second trial

(ii) The coins are not replaced before the second trial.

Find the probability that the first drawing will give 4 gold and second drawing will give 4 silver coins.

9. The probability that a student passes a certain examination is 0.8 (given that he has studied). The probability that he passes the exam without studying is 0.2 Assuming that the probability of the student studying for the exam is 0.6. Given that the student passes the examination, what is the probability that he has studied?

10. The odds that a book will be reviewed favourably by 3 independent critics are 5 to 2, 4 to 3, 3 to 4. What is the probability that of the 3 reviews, a majority will be favourable?

11. Three students A, B, C are in a swimming race. A and B have the same probability of winning and each is twice as likely to win as C. Find the probability that B or C wins the race.

12. A and B throw alternatively a pair of dice. A wins if he throws 6 before B throws 7 and B wins if he throws 7 before A throws 6. If A begins, find his chances of winning the game.

13. The coefficients a, b, c of quadratic equation $ax^2 + bx + c = 0$ are determined by throwing a die 3 times. Find the probability that

(i) Roots are real

(ii) Roots are complex