Pb(2). Let $(A,*)$ be a semigroup such that $a,b\in A$. If $a\neq b$ the $a*b\neq b*a$ then show that

(i) $a*a=a \quad , \forall\, a\in A$

(ii) $a*b*a=a \quad , \forall\, a,b\in A$

iii) $a*b*c=a*c \quad , \forall\, a,b,c\in A$

Ans: If $a\neq b$ then $a*b\neq b*a$, which is equivalent to, if $a*b=b*a$ the $a=b$ — (1)

$x \;\; * \;\; y \;\; = \;\; y \;\; * \;\; x$ then $x=y$

(i) $(a*a)*a = a*(a*a)$ (associative) To prove. $a*a=a$

$\Rightarrow a*a=a \quad , \forall\, a\in A$ by (1)

ii) $(a*b*a)*a = a*(b*a*a)$ (associative)

$(a*a)*(b*a)$ $\because a*a=a$, $a=a*a$

$=a*(a*b*a)$ (associative)

By (1) $\underline{\underline{a*b*a=a}}$

(iii) $(a*b*c)*(a*c) = (a*b)*(c*a*c)$ (by associative)

$=[(a*c*a)*b]*c$ (from ii)

$=(a*c)*(a*b*c)$ (associative)

By (1), $\underline{\underline{a*b*c=a*c}}$

Pb③: In a group $(G, *)$, if $(a*b)^2 = a^2 * b^2$, $\forall\, a, b \in G$, Show that G is abelian.

Ans: $(a*b)^2 = a^2 * b^2$

$$(a*b)*(a*b) = (a*a)*(b*b)$$

$$a*(b*a)*b = a*(a*b)*b$$

Using left & right cancellation law,

$b*a = a*a$, commutative.

$\Rightarrow G$ is abelian.

Pb④: Let G be group in which every element is inverse of itself. Then show that G is abelian.

$(a^{-1} = a)$

Ans: Let $a, b \in G$, we have $a*a = e$, $b*b = e$

and $\underset{d}{a*b} \in G$ (by closure).

$d*d = e$

$$(a*b)*(a*b) = e = e*e$$
$$= (a*a)*(b*b)$$

Using left & right cancellation law,

$b*a = a*b$, commutative.

Hence G is abelian.

Note: The converse of the above statement is not true.

Eg: $(\mathbb{Z}, +)$ is abelian in which all the elements are not self inverse.

Problem: In a group $(G, *)$, if $a^2 = e$, $\forall a \in G$, S.T G is abelian.

Pb ⑥: If a group $(G, *)$ has even number of elements then show that atleast one element must be its own inverse.

Pf: Let G consists of even number of elements.

$$G = \{a_1, a_2, a_3, \dots, a_{2n-1}, a_{2n} = e\}$$

We know that $e^{-1} = e$

Suppose that a_1 and a_2, a_3 and a_4, ..., a_{2n-3} and a_{2n-2} be inverses of each other.

Then $a_{2n-1}^{-1} = a_{2n-1}$

Hence proved

Subgroup:

Let $(G, *)$ be a group and H be a non-empty subset of G.

H is said to be a subgroup of G, if H itself forms a group under '$*$'.

Eg ① $(\mathbb{Z}, +)$ is a subgroup of $(\mathbb{Q}, +)$ $\qquad \mathbb{Z}, + \downarrow \mathbb{Q}, +$

② $(\mathbb{Q} \setminus \{0\}, \cdot)$ is a subgroup of $(\mathbb{R} \setminus \{0\}, \cdot)$

Eg ③: Let $G = \{1, -1, i, -i\}$??

$\cdot$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

identity

From table,
all the element in G
→ closure satisfies
→ associative satisfies
→ 1 is identity elt

$1^{-1} = 1$, $\quad -1^{-1} = -1$.

$i^{-1} = -i$, $\quad (-i)^{-1} = i$

Clearly G is a group.

Let $H_1 = \{1, -1\}$ is a subset of G & it forms a group w.r.to multiplication.

$\therefore$ H_1 is a subgroup.

Let $H_2 = \{i, -i\}$ is not a subgroup of G.

because $i \cdot i = -1 \notin H_2$

<u>Note</u>: $\{e\}$ & G are always subgroups of G & are called trivial subgroups.