Double Integrals

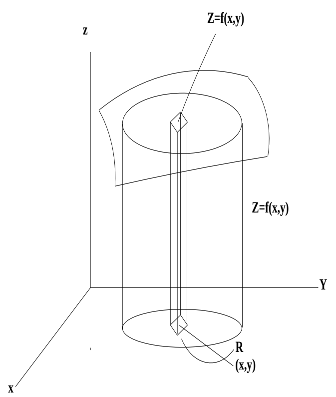
Consider a function of two independent variables $z = f(x, y)$. Geometrically, for each (x, y) in the xy plane, $z = f(x, y)$ is defined, and represents a surface in the space.

Let R be a region in the xy plane. To define the double integral of $z = f(x, y)$ over the region R, divide R into finite number of sub-regions, say n.

Let i^{th} sub-region has area ΔS_i. Let (x_i, y_i) be a point on the i^{th} sub-region. Let $z_i = f(x_i, y_i)$. Consider $\sum_{i=1}^{n} f(x_i, y_i)\Delta S_i$. As $n \to \infty$ such that maximum value of $\Delta S_i \to 0$, the above sum converges to a value called the double integral of $z = f(x, y)$ over the region R denoted by $\iint_R f(x, y)dS$.

$$\therefore \iint_R f(x, y)dS = \lim_{\substack{n\to\infty \\ Max.\Delta S_i \to 0}} \sum_{i=1}^{n} f(x_i, y_i)\Delta S_i$$

Note that $f(x_i, y_i)\Delta S_i$ is the volume of a cylindrical solid region with the base ΔS_i and height $f(x_i, y_i)$.



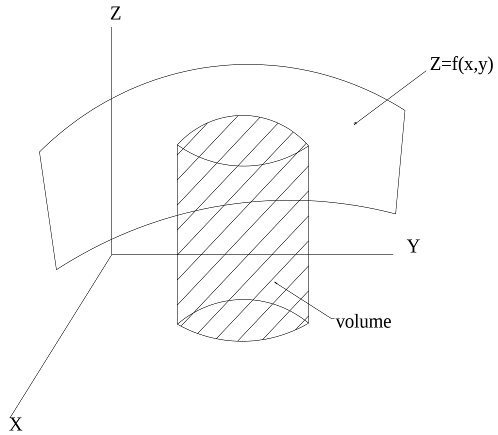
$\therefore$

Geometrically $\iint_R f(x, y)dS$ is the volume of the cylindrical region with the base R in the xy plane and the top surface is cut off by the surface $z = f(x, y)$.

If $f(x,y)=1$, then $\iint_R f(x,y)dS = \iint_R dS$

$$= \lim_{\substack{n\to\infty \\ \Delta S_i \to 0}} \sum_{i=1}^{n} \Delta S_i$$

$= $ Area of the region R in the xy plane



Evaluation of double integrals

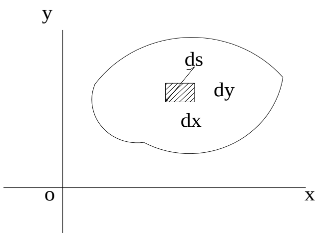
Let the elementary area dS be the area of a rectangle with sides dx and dy parallel to the co-ordinate axes.

$\therefore dS = dxdy$

$\therefore \iint_R f(x,y)dS = \iint_R f(x,y)dxdy$

A double integral can be considered as an iterated integral.

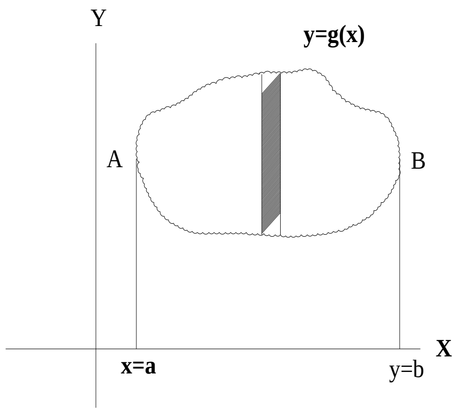
i.e. , $\iint_R f(x,y)dxdy = \underbrace{\int \underbrace{\left[\int f(x,y)dy \right]}_{\text{inner integral}} dx}_{\text{outer integral}}$



The dependent variable among x and y is considered as the variable of the inner integral and the independent variable is the variable of the outer integral. While evaluating the inner integral, the variable of the outer integral is treated as a constant (i.e. the inner integration is the partial integration w.r.t the corresponding variable), then the outer integral is evaluated.

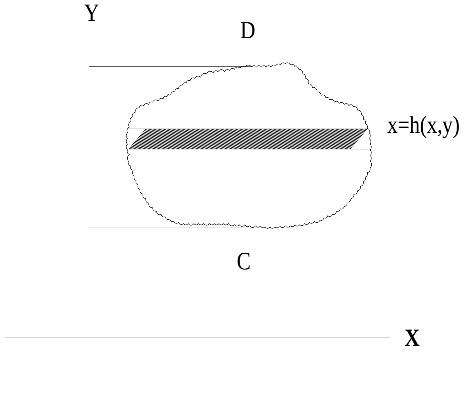
If the region R is bounded by the curves $y = g_1(x)$, $y = g_2(x)$ for $a \le x \le b$, then

$$\iint_R f(x,y)dxdy = \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} f(x,y)dydx .$$



If R is bounded by the curves $x = h_1(y)$, $x = h_2(y)$ for $c \le y \le d$, then

$$\iint_R f(x,y)\,dydx = \int_{y=c}^{y=d} \int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\,dxdy$$



<u>Properties of the double integrals</u>

1) If $f(x,y)$ and $g(x,y)$ are functions of x and y defined over R, c_1 and c_2 are constants, then
$$\iint_R (c_1 f(x,y) + c_2 g(x,y))\,dS = c_1 \iint_R f(x,y)\,dS + c_2 \iint_R g(x,y)\,dS$$
2) If the region R is the union of two non-overlapping regions R_1 and R_2 i.e. $R = R_1 \cup R_2$, then
$$\iint_R f(x,y)\,dS = \iint_{R_1} f(x,y)\,dS + \iint_{R_2} f(x,y)\,dS$$
3) If $f(x,y) = g(x).h(y)$ where $g(x)$ is a function of x alone and $h(y)$ is a function of y alone and if $c\,\&\,d$ are independent of x and $a\,\&\,b$ are independent of y, then
$$\int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x,y)\,dS = \int_{x=a}^{x=b} g(x)\,dx \times \int_{y=c}^{y=d} h(y)\,dy$$
4) $\iint_R f(x,y)\,dS =$ Volume of the cylindrical region with base R in the xy plane and top surface is cut off by the surface $z = f(x,y)$.

5) $\iint\limits_R dS =$ Area of R

Double integral in the polar co-ordinates

If the region R in the plane is bounded by the polar curves $r = f_1(\theta)$ and $r = f_2(\theta)$ for $c_1 \le \theta \le c_2$, then

$$\iint\limits_R f(r,\theta)dS = \int\limits_{\theta=c_1}^{\theta=c_2} \int\limits_{r=f_1(\theta)}^{r=f_2(\theta)} f(r,\theta)drd\theta$$

Problems:

1. Evaluate $\int\limits_0^{\frac{\pi}{2}} \int\limits_0^1 (y\sin x)dydx$.

   Solution:

$$\int\limits_0^{\frac{\pi}{2}} \int\limits_0^1 (y\sin x)dydx = \int\limits_{x=0}^{x=\frac{\pi}{2}} \left.\frac{y^2}{2}\sin x\right|_0^1 dx$$

$$= \int\limits_{x=0}^{x=\frac{\pi}{2}} \frac{1}{2}\sin xdx$$

$$= \left.\frac{-\cos x}{2}\right|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}$$ or

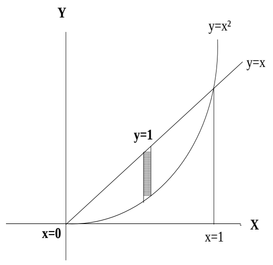
$$\int\limits_0^{\frac{\pi}{2}} \int\limits_0^1 (y\sin x)dydx = \int\limits_{x=0}^{x=\frac{\pi}{2}} \sin xdx \times \int\limits_{y=0}^{y=1} ydy$$

$$= \int\limits_{x=0}^{x=\frac{\pi}{2}} \frac{1}{2}\sin xdx$$

$$= -\cos x\Big|_0^{\frac{\pi}{2}} \times \left.\frac{y^2}{2}\right|_0^1$$

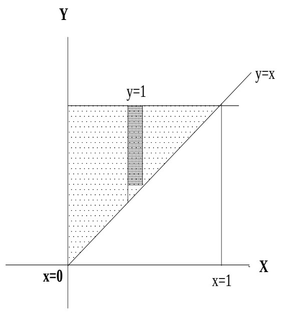
$$= \frac{1}{2}$$

2. Evaluate $\int_{x=0}^{2}\int_{y=x^2}^{x}(y^2x)dydx$



$$\int_{x=0}^{2}\int_{y=x^2}^{x}(y^2x)dydx=\int_{x=0}^{2}\left.\frac{y^3x}{3}\right|_{y=x^2}^{x}dx$$
$$=\int_{x=0}^{2}\frac{x}{3}(x^3-x^6)dx$$
$$=\int_{x=0}^{2}\left(\frac{x^4}{3}-\frac{x^7}{3}\right)dx$$
$$=\left.\frac{x^5}{15}-\frac{x^8}{24}\right|_0^2=\frac{32}{15}-\frac{256}{24}=\frac{128}{15}$$

3. Evaluate $\iint_R (xy-y^3)dS$ where $R=\{(x,y)/0<\mathrm{x}<\mathrm{y}<1\}$



$$\iint_R (xy-y^3)dS = \int_{x=0}^{1}\int_{y=x}^{1}(xy-y^3)dydx$$

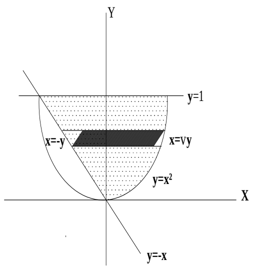
$$= \int_{x=0}^{1}\left(\frac{xy^2}{2}-\frac{y^4}{4}\right)\Bigg|_x^1 dx$$

$$= \int_{x=0}^{1}\left(\frac{x}{2}(1-x^2)-\frac{1}{4}(1-x^4)\right)dx$$

$$= \int_{x=0}^{1}\left(\frac{x}{2}-\frac{x^3}{2}-\frac{1}{4}+\frac{x^4}{4}\right)dx$$

$$= \frac{1}{4}-\frac{1}{8}-\frac{1}{4}+\frac{1}{20}=\frac{2-5}{40}=\frac{-3}{40}$$

4. Evaluate $\iint_R xydS$ where R is bounded by $y=-x$, $x=\sqrt{y}$ and $y=1$.



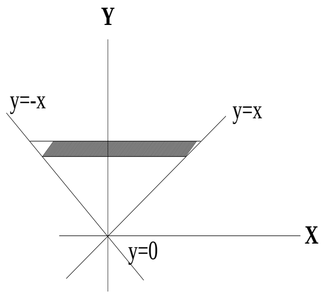
$$\iint_R xydS = \int_{y=0}^{1}\int_{x=-y}^{\sqrt{y}}(xy)dxdy$$

$$= \int_{y=0}^{1}\left(\frac{x^2y}{2}\right)\Bigg|_{-y}^{\sqrt{y}} dy$$

$$= \int_{y=0}^{1}\left[\frac{y}{2}\left(y-\frac{y^2}{2}\right)\right]dy$$

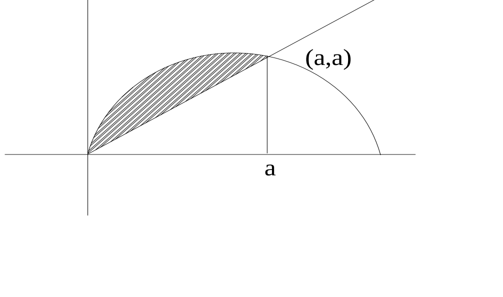
$$= \int_{y=0}^{1}\left(\frac{y^2}{2}-\frac{y^3}{4}\right)dy = \frac{1}{6}-\frac{1}{16}=\frac{5}{48}$$

5. Evaluate $\iint_D (x+e^{-y})ds$ when $D=\{(x,y)/y>|x|\}$



$$\iint_D (x+e^{-y})ds = \int_{y=0}^{\infty}\int_{x=-y}^{y}(x+e^{-y})dxdy$$
$$= \int_{y=0}^{\infty}\left(\frac{x^2}{2}+xe^{-y}\right)\Bigg|_{-y}^{y} dy$$
$$= \int_{y=0}^{\infty}\frac{1}{2}(y^2-y^2+2ye^{-y})dy$$
$$= 2\int_{y=0}^{\infty} ye^{-y}dy$$
$$= 2[y(-e^{-y})-(1)(e^{-y})]_0^{\infty}$$
$$= 2[0-(0-1)] = 2$$

6. Evaluate $\iint_D r dr d\theta$ over the region D bounded by $x^2+y^2=2ax$ and $y=x$ in the first quadrant.



$$\iint_R r\,dr\,d\theta = \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{r=0}^{2a\cos\theta} r\,dr\,d\theta$$

$$= \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}} \left.\frac{r^2}{2}\right|_0^{2a\cos\theta} d\theta$$

$$= \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{4a^2\cos^2\theta}{2}\,d\theta$$

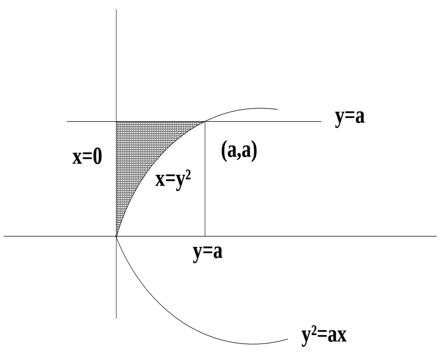
$$= a^2 \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}} (1+\cos 2\theta)\,d\theta$$

$$= a^2\left[\theta + \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= a^2\left(\frac{\pi}{4} - \frac{\pi}{2} + \frac{1}{2}\left(\sin\pi - \sin\frac{\pi}{2}\right)\right)$$

$$= a^2\left(\frac{\pi}{4} - \frac{1}{2}\right)$$

7. Change the order of the integration and evaluate $\int_0^a \int_{\sqrt{ax}}^a \left(\frac{y^2}{\sqrt{y^4 - a^2x^2}}\right) dydx$.



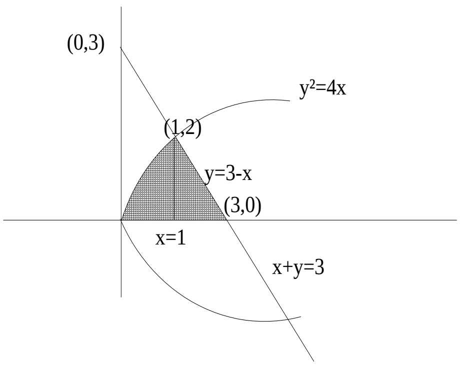
Changing the order of integration we get $\int_{y=0}^{a}\int_{0}^{\frac{y^2}{2}} \frac{y^2}{\sqrt{y^4-a^2x^2}}dxdy$

$$= \int_{y=0}^{a} \frac{y^2}{a}\sin^{-1}\left(\frac{ax}{y^2}\right)\Bigg|_0^{y^2/2} dy$$

$$= \int_{y=0}^{a} \frac{y^2}{a}\left(\sin^{-1}1-\sin^{-1}0\right)dy$$

$$= \int_{y=0}^{a} \frac{\pi}{2}\frac{y^2}{a}dy = \frac{\pi}{2a}\frac{y^3}{3}\Bigg|_0^{a} = \frac{\pi a^2}{6}$$

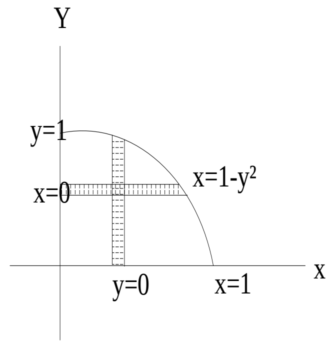
8. Evaluate $\int_{0}^{2}\int_{y^2/4}^{3-y}\left(x^2+y^2\right)dxdy$ by changing the order of integration.



$$\int_{x=0}^{1}\int_{y=0}^{2\sqrt{x}}(x^2+y^2)dydx + \int_{x=1}^{3}\int_{y=0}^{3-x}(x^2+y^2)dydx$$

$$= \frac{172}{105}+\frac{22}{3} = \frac{942}{105} = \frac{314}{35}$$

9. Evaluate $\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}} \frac{dydx}{(1+e^y)(\sqrt{1-x^2-y^2})}$ by changing the order of integration.



$$\int_0^1 \int_0^{\sqrt{1-y^2}} \frac{dydx}{(1+e^y)(\sqrt{1-x^2-y^2})}$$

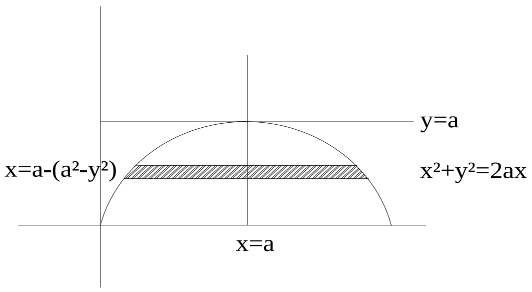
$$= \int_{x=0}^{1} \frac{1}{1+e^y} . \sin^{-1}\left(\frac{x}{\sqrt{1-y^2}}\right)\Bigg|_0^{\sqrt{1-y^2}} dy$$

$$= \int_{x=0}^{1} \frac{1}{1+e^y}\left(\frac{\pi}{2}-0\right)dy$$

$$= \frac{\pi}{2}\int_{x=0}^{1} \frac{e^{-y}}{1+e^{-y}}dy$$

$$= \frac{\pi}{2}\left[-\log(1+e^{-y})\right]_0^1 = \frac{\pi}{2}\left(\log 2 - \log(1+\mathrm{e}^{-1})\right)$$

10. Evaluate $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} dydx$ by reversing the order of integration.



$$\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} dydx = \int_{y=0}^{a}\int_{x=a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dydx$$

$$= \int_0^a \left(a+\sqrt{a^2-y^2}\right)-\left(a-\sqrt{a^2-y^2}\right)dy$$

$$= 2\int_0^a \sqrt{a^2-y^2}\,dy = 2\sin^{-1}\left(\frac{y}{a}\right)\Bigg|_0^a$$

$$= \pi$$

<u>Jacobians</u>

Let u = u(x, y) and v = v(x, y) be continuous functions of x and y having continuous partial derivatives. Then Jacobian of u, v w.r.t. x & y denoted by J or $\frac{\partial(u,v)}{\partial(x,y)}$ is defined as

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial x} \end{vmatrix}$$

The definition can be extended to any number of variables.

If $u_i = u_i(x_1, x_2, ...x_n)$ is a function of $x_1, x_2, ...x_n$ having continuous partial derivatives , I = 1,2,…n then Jacobian of $u_1, u_2, ...u_n$ w.r.t. $x_1, x_2, ...x_n$ denoted by $\frac{\partial(u_1,u_2,...u_n)}{\partial(x_1,x_2,...x_n)}$ is defined as

$$J = \frac{\partial(u_1,u_2,...u_n)}{\partial(x_1,x_2,...x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_n} \\ & & \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

Properties of Jacobians:

1. If $J_1 = \frac{\partial(u,v)}{\partial(x,y)}$, $J_2 = \frac{\partial(x,y)}{\partial(u,v)}$ then $J_1 J_2 = 1$

Proof: Let $u = u(x, y)$ and $v = v(x, y)$ be one-to-one and onto functions of x and y. Then the inverses of these functions are defined and $x = x(u, v)$ and $y = y(u, v)$. Hence $\frac{\partial(x, y)}{\partial(u, v)}$ is defined.

Consider

$$J_1 J_2 = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} . \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} . \frac{\partial x}{\partial u} + \frac{\partial u}{\partial y} . \frac{\partial y}{\partial u} & \frac{\partial u}{\partial x} . \frac{\partial x}{\partial v} + \frac{\partial u}{\partial y} . \frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x} . \frac{\partial x}{\partial u} + \frac{\partial v}{\partial y} . \frac{\partial y}{\partial u} & \frac{\partial v}{\partial x} . \frac{\partial x}{\partial v} + \frac{\partial v}{\partial y} . \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial u} & \frac{\partial u}{\partial v} \\ \frac{\partial v}{\partial u} & \frac{\partial v}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Using the total differential formula.

Note that if $J_1 \neq 0$, then only J_2 is defined and $J_2 = \frac{1}{J_1}$.

2. If $J_1 = \frac{\partial(u, v)}{\partial(z, w)}, J_2 = \frac{\partial(z, w)}{\partial(x, y)}$ and $J_3 = \frac{\partial(u, v)}{\partial(x, y)}$ then $J_1 J_2 = J_3$.

Proof: Consider $J_1 J_2 = \frac{\partial(u, v)}{\partial(z, w)} . \frac{\partial(z, w)}{\partial(x, y)}$

$$= \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial w} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial w} \end{vmatrix} . \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial z} . \frac{\partial z}{\partial x} + \frac{\partial u}{\partial w} . \frac{\partial w}{\partial x} & \frac{\partial u}{\partial z} . \frac{\partial z}{\partial y} + \frac{\partial u}{\partial w} . \frac{\partial w}{\partial y} \\ \frac{\partial v}{\partial z} . \frac{\partial z}{\partial x} + \frac{\partial v}{\partial w} . \frac{\partial w}{\partial x} & \frac{\partial v}{\partial z} . \frac{\partial z}{\partial y} + \frac{\partial v}{\partial w} . \frac{\partial w}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)} = J_3$$

3. If $x = x(u,v)$ and $y = y(u,v)$ are one-one and onto functions of u and v with Jacobian $J = \dfrac{\partial(x,y)}{\partial(u,v)} \neq 0$, then $\iint\limits_S f(x,y)dxdy = \iint\limits_S f(x(u,v), y(u,v))|J|dudv$.

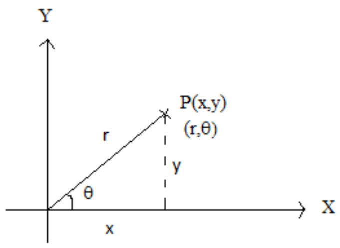
Examples:

1) Let $u = x^2 y$ and $v = 5x + \sin y$.

Then $J = \dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial u}{\partial y} \end{vmatrix} = \begin{vmatrix} 2xy & x^2 \\ 5 & \cos y \end{vmatrix} = 2xy\cos y - 5x^2$.

2) Consider the transformation from the Cartesian co-ordinates (x,y) to polar co-ordinates (r,θ), i.e $x = r\cos\theta,\ 0 \le r \le \infty$

$y = r\sin\theta,\ 0 \le \theta \le 2\pi$



$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r > 0$$

$$\therefore \iint\limits_R f(x,y)dxdy = \iint\limits_R f(r\cos\theta, r\sin\theta)rdrd\theta$$

3) Consider the transformation from Cartesian co-ordinate system to cylindrical co-ordinate system, $x = \rho\cos\phi,\ 0 \le \rho \le \infty$

$y = \rho\sin\phi,\ 0 \le \phi \le 2\pi$

$z = z,\ -\infty < z < \infty$

$\rho = c_1$, a constant represents a cylinder.

$\phi = c_2$, a constant represents a half-plane.

$z = c_3$, a constant represents a plane parallel to xy-plane.

$\rho = c_1$ and $\phi = c_2$ represents a line.

$\rho = c_1$ and $z = c_3$ represents a circle.

$\phi = c_2$ and $z = c_3$ represents a half line (ray).

$$J=\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)}=\begin{vmatrix}\frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\phi} & \frac{\partial x}{\partial z}\\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\phi} & \frac{\partial y}{\partial z}\\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\phi} & \frac{\partial z}{\partial z}\end{vmatrix}=\begin{vmatrix}\cos\phi & -\rho\sin\phi & 0\\ \sin\phi & \rho\cos\phi & 0\\ 0 & 0 & 1\end{vmatrix}=\rho$$

4) Consider the transformation from Cartesian co-ordinate system to spherical co-ordinate system, i.e $x=r\sin\theta\cos\phi,\ 0\le r\le\infty$

$y=r\sin\theta\sin\phi,\ 0\le\phi\le 2\pi$

$z=r\cos\theta,\ 0\le\theta\le\pi$

$r=c_1$, a constant represents a sphere with centre at the origin and radius c_1.

$\phi=c_2$ represents a half plane.

$\theta=c_3$ represents a cone with axis as z-axis and semi-vertical angle c_3.

$r=c_1$ and $\phi=c_2$ represents a semi-circle.

$r=c_1$ and $\theta=c_3$ represents a circle.

$\theta=c_1$ and $\phi=c_2$ represents a half line.

$$J=\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi}\end{vmatrix}=\begin{vmatrix}\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi\\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi\\ \cos\theta & -r\sin\theta & 0\end{vmatrix}=r^2\sin\theta.$$

Problems:

1) Evaluate the following by changing to polar co-ordinates : $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy$

Solution: Changing over to polar co-ordinates $x=r\cos\theta$, $y=r\sin\theta$, $x^2+y^2=r^2$ for $0\le r<\infty, 0\le\theta\le\frac{\pi}{2}$, we get

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy=\int_0^{\pi/2}\int_0^\infty e^{-r^2}rdrd\theta=\int_0^{\pi/2}-\frac{e^{-r^2}}{2}\Bigg|_0^\infty d\theta$$

$$=\int_0^{\pi/2}\frac{-1}{2}(0-1)d\theta=\frac{1}{2}.\frac{\pi}{2}$$

$$=\frac{\pi}{4}$$

2) Evaluate $\iint_R \sqrt{x^2+y^2}\,dxdy$ where R is the region bounded by circles $x^2+y^2=a^2,\ x^2+y^2=b^2,\ a<b$.

Solution: Changing over to polar co-ordinates $x=r\cos\theta,\ y=r\sin\theta,\ a\le r\le b,\ 0\le\theta\le 2\pi,$ we get

$$\iint_R \sqrt{x^2+y^2}\,dxdy = \int_{\theta=0}^{2\pi}\int_{r=a}^{b} r.\,rdrd\theta = \int_{\theta=0}^{2\pi} \left.\frac{r^3}{3}\right|_a^b d\theta$$

$$= \frac{b^3-a^3}{3}\int_0^{2\pi} d\theta$$

$$= \frac{b^3-a^3}{3}.2\pi$$

3) Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dxdy$ by changing to polar co-ordinates.

Solution: $y=\sqrt{2x-x^2} \iff x^2+y^2=2x \iff r=2\cos\theta$

Changing over to polar co-ordinates $x=r\cos\theta,\ y=r\sin\theta,\ 0\le r\le 2\cos\theta,\ 0\le\theta\le\pi/2,$

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dxdy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{2\cos\theta} \frac{r\cos\theta}{r}.\,rdrd\theta = \int_{\theta=0}^{\pi/2} \left.\frac{r^2}{2}\right|_0^{2\cos\theta} \cos\theta d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/2} 4\cos^2\theta.\cos\theta d\theta = 2\int_{\theta=0}^{\pi/2}\cos^3\theta d\theta$$

$$= 2.\frac{2}{3}.1 = \frac{4}{3}$$

4) Evaluate $\int_{x=0}^{1}\int_{y=0}^{1-x} e^{\frac{y}{x+y}}\,dydx$ using the transformations $x+y=u,\ y=uv$.

Solution: $x=u-y=u-uv=u(1-v)$ and $y=uv$

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}1-v & -u\\ v & u\end{vmatrix}=u$$

Now $x=0 \Rightarrow u=0$ or $v=1$

$y=1-x \Rightarrow x+y=1$, i.e $u=1$

$y=0 \Rightarrow u=0$ or $v=0$

$\therefore 0\le u,v\le 1$

Changing the variables, we get

$$\int_{x=0}^{1}\int_{y=0}^{1-x} e^{\frac{y}{x+y}}dydx = \int_{u=0}^{1}\int_{v=0}^{1} e^{\frac{uv}{u}}udvdu$$

$$= \int_{u=0}^{1}\int_{v=0}^{1} e^{u}udvdu$$

$$= \int_{u=0}^{1} e^{v}\Big|_0^1 udu = (e-1)\int_{u=0}^{1} udu$$

$$= (e-1)\frac{u^2}{2}\Big|_0^1 = \frac{(e-1)}{2}$$

5) Evaluate $\int_0^a\int_y^a \frac{x^2}{\sqrt{x^2+y^2}}dxdy$ by changing over to polar co-ordinates.

Solution: Changing over to polar co-ordinates, $x = r\cos\theta,\ \ y = r\sin\theta$,

$x = a \Rightarrow r\cos\theta = a$ or $r = a\sec\theta$

$\therefore 0 \le r \le a\sec\theta,\ \ 0 \le \theta \le \frac{\pi}{4}$

$$\int_0^a\int_y^a \frac{x^2}{\sqrt{x^2+y^2}}dxdy = \int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sec\theta} \frac{r^2\cos^2\theta}{r}.rdrd\theta$$

$$= \int_{\theta=0}^{\pi/4} \frac{r^3}{3}\cos^2\theta\Big|_0^{a\sec\theta} d\theta$$

$$= \frac{1}{3}\int_{\theta=0}^{\pi/4} a^3\sec^3\theta\cos^2\theta d\theta$$

$$= \frac{a^3}{3}\int_{\theta=0}^{\pi/4} \sec\theta d\theta$$

$$= \frac{a^3}{3}\log(\sec\theta+\tan\theta)\Big|_{\theta=0}^{\pi/4}$$

$$= \frac{a^3}{3}\log(\sqrt{2}+1)$$

Exercises:

1) Evaluate $\int_0^a\int_0^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}dydx$ by changing over to polar co-ordinates.

2) Evaluate where R is bounded by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(Hint: Use the transformation $x = ar\cos\theta, y = ar\sin\theta$, $0 \le r \le 1,\ 0 \le \theta \le 2\pi$)

3) Evaluate $\iint_R (x+y)^2\,dxdy$ where R is the parallelogram in the xy-plane with vertices $(1,0),(3,1),(2,2)$ and $(0,1)$ using the transformations $u = x+y, u = x-2y$.

4) Evaluate $\iint_R \frac{dxdy}{\sqrt{1+x^2+y^2}}$ where R is the region bounded by one loop of $r^2 = \cos 2\theta,\ -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$.

5) Evaluate $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dydx$ by transforming into polar co-ordinates.

Area enclosed by plane curves

1. Cartesian co-ordinates

Consider the area enclosed by the curves $y = f_1(x)$ and $y = f_2(x)$ and the ordinates $x = x_1$ and $x = x_2$ as shown in figure 1. Divide this area into vertical strips of width δx. If $P(x,y)$ and $Q(x+\delta x, y+\delta y)$ be two neighboring points, then the area of the small rectangle PQ is $\delta x \delta y$.

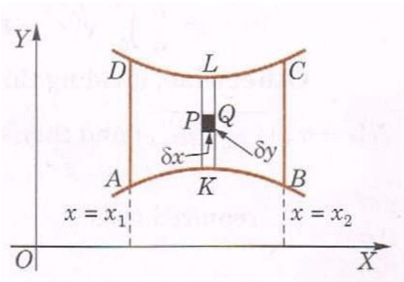


Figure 1:

$\therefore$ The area of the strip $KL = \underset{\delta y \to 0}{Lt} \sum \delta x \delta y$ Since for all rectangles in the strip, δx is the same and y varies from $y = f_1(x)$ to $y = f_2(x)$, the area of the strip $KL = \delta x \underset{\delta y \to 0}{Lt} \sum_{f_1(x)}^{f_2(x)} dy = \delta x \int_{f_1(x)}^{f_2(x)} dy$. Adding up all such strips from $x = x_1$ to $x = x_2$, we get

$$\text{The area } ABCD = \underset{\delta x \to 0}{Lt} \sum_{x_1}^{x_2} \delta x \int_{f_1(x)}^{f_2(x)} dy = \int_{x_1}^{x_2} dx \int_{f_1(x)}^{f_2(x)} dy = \int_{x=x_1}^{x=x_2} \int_{y=f_1(x)}^{y=f_2(x)} dxdy.$$

Similarly, dividing the area $A'B'C'D'$ (figure 2) into horizontal strips of width δy, we get the area $A'B'C'D' = \int_{y_1}^{y_2} \int_{f_1(y)}^{f_2(y)} dxdy$.

<u>Figure 2:</u>

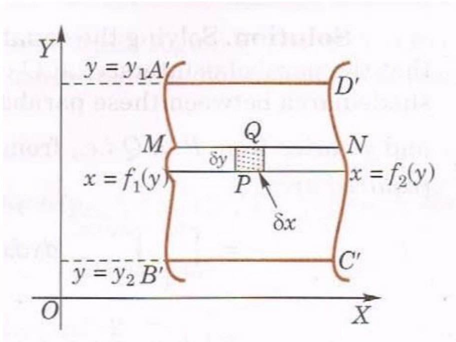
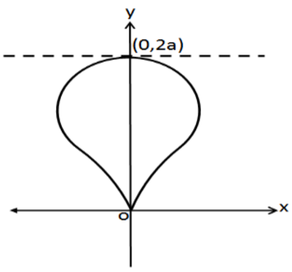


Illustration:

1. By double integration, find the whole area of the curve $a^2x^2 = y^3\left(2a-y\right)$.

   Solution:



The required area is $A = 2\int\limits_{y=0}^{2a}\int\limits_{x=0}^{f(y)} dxdy = 2\int\limits_{y=0}^{2a}\int\limits_{x=0}^{\frac{y}{a}\sqrt{y(2a-y)}} dxdy = 2\int\limits_{y=0}^{2a}\frac{y}{a}\sqrt{y\left(2a-y\right)}dy$

Put $y = 2a\sin^2\theta$. Then $dy = 2a.2\sin\theta\cos\theta d\theta$

$$\therefore A = 2\int\limits_{\theta=0}^{\frac{\pi}{2}} 2\sin^2\theta\sqrt{2a\sin^2\theta 2a\cos^2\theta}.4a\sin\theta\cos\theta d\theta$$

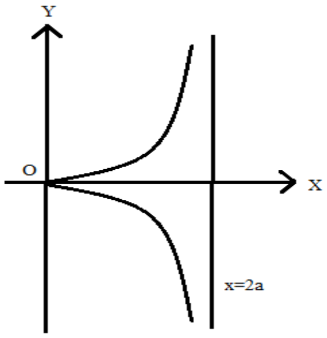
$$= 32a^2\int\limits_{\theta=0}^{\frac{\pi}{2}} \sin^4\theta\cos^2\theta d\theta$$

$$= 32a^2.\frac{3}{6}.\frac{1}{4}.\frac{1}{2}.\frac{\pi}{2}$$

$$= \pi a^2 sq.units$$

2. Find by double integration, the area included between the curve $y^2(2a-x)=x^3$ and its asymptote.

Solution: The required area is



$$A=2\int_{x=0}^{2a}\int_{y=0}^{f(x)=\frac{x\sqrt{x}}{\sqrt{2a-x}}} dydx$$

$$=2\int_0^{2a}\sqrt{\frac{x^3}{2a-x}}dx$$

$$=2\int_0^{\pi/2}\sqrt{\frac{\left(2a\sin^2\theta\right)^3}{2a\cos^2\theta}}4a\sin\theta\cos\theta d\theta$$

by putting $x=2a\sin^2\theta$

$$=16a^2\int_0^{\pi/2}\sin^4\theta$$

$$=16a^2.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$$

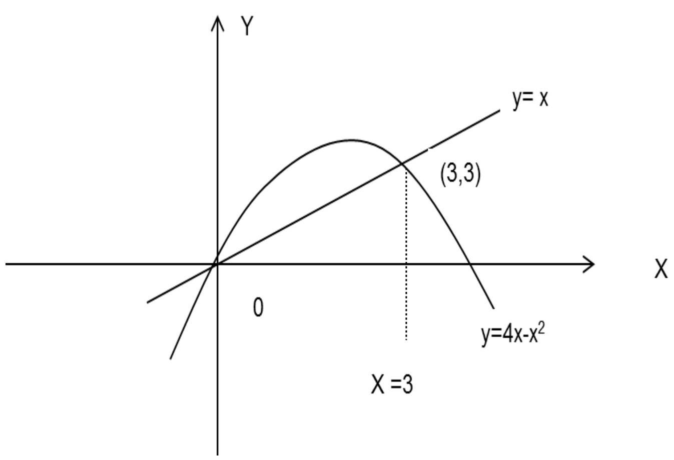
$$=3\pi a^2 sq.units$$

3. Find by double integration, the area included between the curve $y=4x-x^2$ and the line $y=x$.

Solution: The required area is

$$A=\int_{x=0}^{3}\int_{y=x}^{4x-x^2} dydx$$

$$=\int_0^3\left(4x-x^2-x\right)dx=4.5sq.units$$



Self-learning exercise:

1. Find using double integrals, the area of a plate in the form of a quadrant of the ellipse
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$
2. Using double integrals, find the area between the parabolas

$y^2 = 4ax$
and
$x^2 = 4ay$

Solutions:

1. $\frac{\pi ab}{4}$
2. (16/3)πa²

Polar Co-ordinates

Consider an area A enclosed by a curve whose equation is in polar co-ordinates. Let $P(r,\theta)$, $Q(r+\delta r,\theta+\delta\theta)$ be two neighbouring points. Mark circular areas of radii r and $r+\delta r$ meeting OQ in R and OP (produced) in S. Since arc PR=$r\delta\theta$ and PS=δr, area of the curvilinear rectangle PRQS is approximately=PR. PS=$r\delta\theta\,.\,\delta r$.

If the whole area is divided into such curvilinear rectangles, the sum $\sum\sum r\delta\theta\delta r$ taken for all these rectangles gives in the limit the area A. Hence A= $\underset{\delta r,\delta\theta\to 0}{Lt}\sum\sum r\delta\theta\delta r=\iint rd\theta dr$ where the limits are to be so chosen as to cover the entire area.

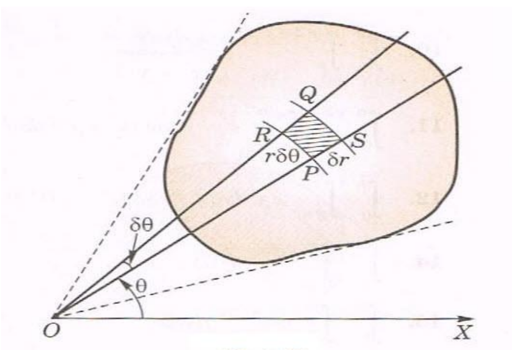
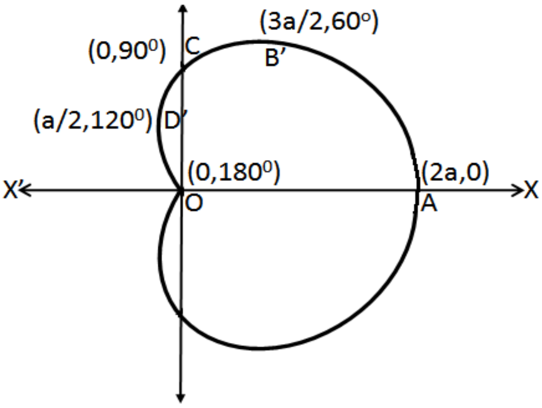


Illustration:

1. Find the area of the cardioid $r = a(1+\cos\theta)$, by double integration.

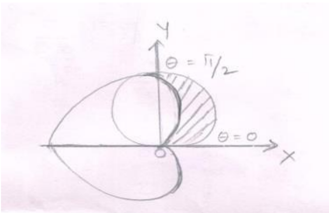
   Solution:



:

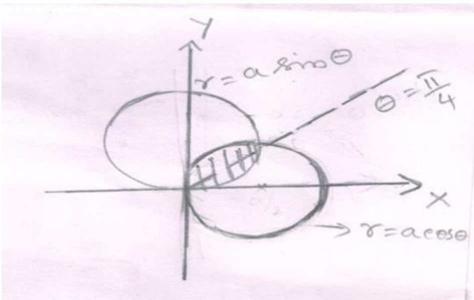
Area $A = 2\int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r\,dr\,d\theta = \int_{\theta=0}^{\pi} a^2(1+\cos\theta)^2\,d\theta = 3\pi a^2/2$ sq.units

2. Find by double integration, the area lying inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1-\cos\theta)$.



Solution: Area $A = \int_{\theta=0}^{\pi/2}\int_{r=a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta = a^2(4-\pi)/4$ sq.units.

3. Find the area common to the circles $r = a\sin\theta$ and $r = a\cos\theta$ by double integration.



Solution: Area $A = \int_{\theta=0}^{\pi/4}\int_{r=a\sin\theta}^{a\cos\theta} r\,dr\,d\theta = a^2/4$ sq.units.

Exercise:

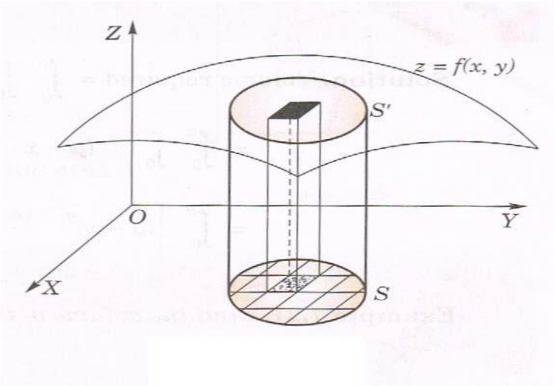
1. Calculate the area included between the curve $r = a(\sec\theta + \cos\theta)$ and its asymptote, using double integration.
2. Find by double integration, the area lying inside the cardioid $r = a(1+\cos\theta)$ and the circle $r = a$

   Solutions:

   1. $\frac{5\pi a^2}{4}$
   2. $\frac{a^2}{4}(\pi + 8)$

Volumes of solids

1. Volumes as double integrals:



Consider a surface $z = f(x, y)$. Let the orthogonal projection on XY-plane of its portion S' be the area S. Divide S into elementary rectangles of area $\delta x \delta y$ by drawing lines parallel to X and Y axes. With each of these rectangles as base, erect a prism having its length parallel to OZ.

$\therefore$ volume of this prism between S and the given surface $z = f(x, y)$ is $\delta x \delta y$

Hence the volume of the solid cylinder on S as base, bounded by the given surface with generators parallel to Z-axis is = $\underset{\delta x, \delta y \to 0}{Lt} \sum\sum z\delta x\delta y = \iint z dxdy = \iint f(x, y) dxdy$

where the integration is carried over the area S.

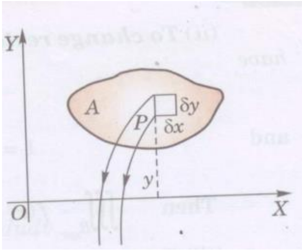
Note: While using polar co-ordinates, divide S into elements of area $r\delta\theta . \delta r$. Therefore replacing $dxdy$ by $r\delta\theta . \delta r$, we get the required volume as $\iint zrd\theta dr$

2. **Volumes as solids of revolution**

Consider an elementary area $\delta x \delta y$ at the point $P(x, y)$ of a plane area A. As this elementary area revolves about x-axis, we get a ring of volume = $2\pi y \delta x \delta y$. Hence the total volume of the solid formed by the revolution of the area A about the x-axis = $\iint_A 2\pi y dxdy$.

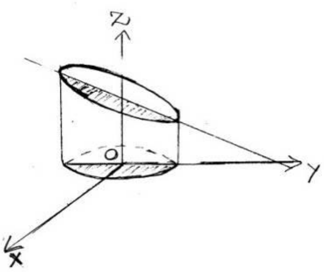
In polar co-ordinates, the above formula for the volume becomes $\iint_A 2\pi r \sin\theta . r d\theta dr$.

Similarly, the volume of the solid formed by the revolution of the area A about the y-axis is $\iint_A 2\pi x dx dy$



1. Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$ using a double integral.

Solution



$$V = 2\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} z dx dy$$

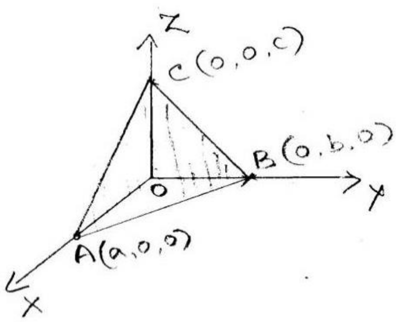
$$= 2\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} (4-y) dx dy$$

$$= 16\pi \text{ cubic units}$$

Using a triple integral, $V = 2\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}\int_{0}^{4-y} dz dx dy$

2. Using a double integral, find the volume of the tetrahedron bounded by the co-ordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Solution:



$$V = \int_{0}^{a}\int_{0}^{b\left(1-\frac{x}{a}\right)} z dy dx$$

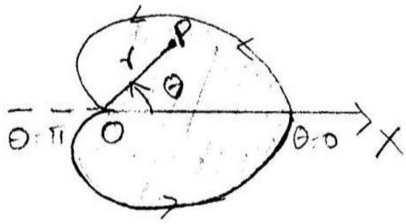
$$= \int_{0}^{a}\int_{0}^{b\left(1-\frac{x}{a}\right)} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy dx$$

$$= \frac{1}{6} abc \text{ cubic units}$$

Using a triple integral, V

$$V = \int_{0}^{a}\int_{0}^{b\left(1-\frac{x}{a}\right)}\int_{0}^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz dy dx$$

3. Using a double integral, find the volume generated by the revolution of the cardioid $r = a(1+\cos\theta)$ about its axis.



Solution:

$$V = \int_0^{\pi} \int_0^{a(1+\cos\theta)} 2\pi r^2 \sin\theta \, dr \, d\theta$$

$$= \frac{8}{3}\pi a^3 \text{ cubic units}$$