Lecture 12 - Date : 11 June 2021

Beta and Gamma functions

Recall:

* $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, for $m,n>0$

* $\beta(m,n) = \beta(n,m)$

* $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$

* If $p>-1$, $q>-1$, $\int_0^{\pi/2} \sin^p x \cos^q x\, dx = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

* $\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$ for $n>0$

* $\Gamma 0 = 1$

* $\Gamma n = (n-1)\Gamma n-1$ for $n>0$

* $\Gamma n = (n-1)!$ for $n \in \mathbb{Z}^+$

Relation between Beta and Gamma Functions:

$$\beta(m.n) = \frac{\Gamma n \Gamma m}{\Gamma(m+n)}$$

i.e., $\beta(m,n) = \dfrac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$

Proof:- By def^n, $\Gamma m = \int_0^\infty e^{-t} t^{m-1} \, dt$

put $t = x^2 \Rightarrow dt = 2x\,dx$ when $t=0 \Rightarrow x=0$
$t=\infty \Rightarrow x=\infty$

$\therefore \Gamma m = \int_0^\infty e^{-x^2} \cdot x^{2m-2} \cdot 2x\,dx = 2\int_0^\infty e^{-x^2} x^{2m-1} \, dx$ —— (1)

$|||^{ly}$ $\Gamma n = \int_0^\infty e^{-t} \cdot t^{n-1} \, dt$ Here put $t = y^2$

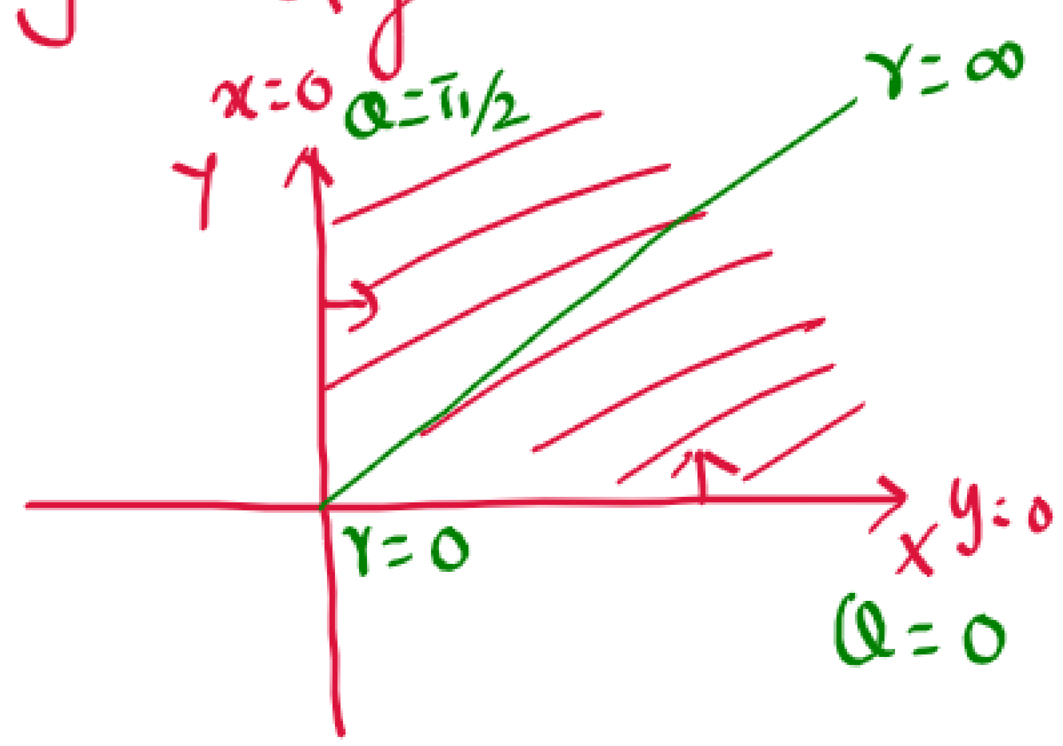
As above, $\Gamma n = 2\int_0^\infty e^{-y^2} y^{2n-1} \, dy$ —— (2)

$\therefore$ From (1) & (2) we get,

$$\Gamma m \, \Gamma n = 2\int_0^\infty e^{-x^2} x^{2m-1} \, dx \times 2\int_0^\infty e^{-y^2} y^{2n-1} \, dy$$

$$= 4 \int_{x=0}^{\infty} \int_{y=0}^{\infty} e^{-(x^2+y^2)} \cdot x^{2m-1} y^{2n-1} \, dy \, dx$$

By changing to polar coordinates



$x = r\cos\theta \quad y = r\sin\theta \quad dx\,dy = r\,dr\,d\theta$

r: 0 to ∞ $\quad\theta$: 0 to $\pi/2$

$$\Gamma m \, \Gamma n = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2} (r\cos\theta)^{2m-1} (r\sin\theta)^{2n-1} \cdot r\,dr\,d\theta$$

$$= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2} \cdot r^{2(m+n)-1} \cos^{2m-1}\theta \sin^{2n-1}\theta \, dr\,d\theta$$

$$= 2\int_{\theta=0}^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta \, d\theta \times 2 \times \int_{r=0}^{\infty} e^{-r^2} r^{2(m+n)-1} dr$$

$$= \beta(n,m) \times \Gamma m+n$$

$$\Gamma m \, \Gamma n = \beta(m,n) \times \Gamma m+n$$

$$\Rightarrow \beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n} \quad //$$

<u>Note</u>:- we've, $\Gamma n = (n-1)\, \Gamma n-1$

$$\Rightarrow \Gamma n-1 = \frac{\Gamma n}{n-1} \qquad \forall \; n>0$$

put $n=\frac{1}{2}$, we get,

$$\Gamma -\tfrac{1}{2} = \frac{\Gamma \tfrac{1}{2}}{(-\tfrac{1}{2})} = \underline{\underline{-2\sqrt{\pi}}}$$

Problem 0.1. *Prove that* $\beta\left(m, \frac{1}{2}\right) = 2^{2m-1}\beta(m,m)$. *Hence prove that* ✓

$$\Gamma m \, \Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma 2m \quad \text{(Duplication Formula)}$$

<u>Ans:</u> we've, $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$ —— (*)

Then $\beta\left(m, \frac{1}{2}\right) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta$ —— (1)

we've, $\beta(m,m) = 2\int_0^{\pi/2} (\sin\theta \cos\theta)^{2m-1} d\theta$

$$= 2\int_0^{\pi/2} \frac{(\sin 2\theta)^{2m-1}}{2^{2m-1}} d\theta.$$

$$= \frac{2}{2^{2m-1}} \int_0^{\pi/2} \sin^{2m-1} 2\theta \, d\theta$$

put $2\theta = \phi$
$\Rightarrow d\theta = \frac{d\phi}{2}$
$\theta = 0 \Rightarrow \phi = 0$
$\theta = \pi/2 \Rightarrow \phi = \pi$

$$= \frac{\cancel{2}}{2^{2m-1}} \int_0^{\pi} \sin^{2m-1}\phi \, \frac{d\phi}{\cancel{2}}$$

$$= \frac{1}{2^{2m-1}} \int_0^{\pi} \sin^{2m-1}\phi \, d\phi$$

$$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$$

$$\text{ie; } \beta(m,m) = \frac{1}{2^{2m-1}} \times 2 \int_0^{\pi/2} \sin^{2m-1}\phi \, d\phi$$

$$\Rightarrow \beta(m,m) = \frac{1}{2^{2m-1}} \beta\left(m, \tfrac{1}{2}\right)$$

$$\Rightarrow \beta\left(m, \tfrac{1}{2}\right) = 2^{2m-1} \beta(m,m) \quad \text{—— (a)}$$

From eqn (a) we get,

$$\beta\left(m, \tfrac{1}{2}\right) = 2^{2m-1} \beta(m,m)$$

$$\Rightarrow \frac{\cancel{\Gamma m}\, \Gamma \tfrac{1}{2}}{\Gamma\left(m+\tfrac{1}{2}\right)} = 2^{2m-1} \frac{\Gamma m \, \cancel{\Gamma m}}{\Gamma 2m}$$

$$\Rightarrow \Gamma m \, \Gamma\left(m+\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma 2m$$

Problem 0.2. *Find* $\dfrac{\Gamma(8/3)}{\Gamma(2/3)}$.

Ans: We've, $\Gamma n = (n-1)\Gamma n-1$

$$\therefore \frac{\Gamma 8/3}{\Gamma 2/3} = \frac{5/3\,\Gamma 5/3}{\Gamma 2/3} = \frac{(5/3)(2/3)\,\Gamma 2/3}{\Gamma 2/3} = \frac{10}{9}$$

Problem 0.3. *Evaluate*

$$\text{Let } I = \int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$$

Ans:- Then $I = \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta \, d\theta$

We've, $\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$; $p>-1$, $q>-1$

$\therefore$ Here $I = \frac{1}{2}\beta\left(\frac{1/2+1}{2}, \frac{-1/2+1}{2}\right)$

By Duplication formula,

$\Gamma m\, \Gamma m+1/2 = \frac{\sqrt{\pi}\,\Gamma 2m}{2^{2m-1}}$

put $m = 1/4$ we get,

$\Gamma 1/4\, \Gamma 3/4 = \frac{\sqrt{\pi}\,\Gamma 1/2}{2^{-1/2}}$

$= \sqrt{2}\pi$

$$= \frac{1}{2}\beta(3/4, 1/4) = \frac{1}{2}\frac{\Gamma 3/4\, \Gamma 1/4}{\Gamma 1}$$

$$= \frac{1}{2}\Gamma 3/4\, \Gamma 1/4$$

$$= \frac{1}{2}(\sqrt{2}\pi) = \frac{\pi}{\sqrt{2}}$$

Problem 0.4. *Evaluate*

let $I = \int_0^1 x^4(1-x)^3\, dx$

Ans:-

then $I = \int_0^1 x^{5-1}(1-x)^{4-1}\,dx$

$= \beta(5,4)$

$= \dfrac{\Gamma 5\,\Gamma 4}{\Gamma 9} = \dfrac{4!\,3!}{8!}$

$= \dfrac{1}{280}$

Problem 0.5. *Express*

$$\text{Let } I = \int_0^1 \frac{dx}{\sqrt{1-x^4}}$$

in terms of gamma functions

Ans:- put $x^2 = \sin\theta \Rightarrow 2x\,dx = \cos\theta\,d\theta$

$$\Rightarrow dx = \frac{\cos\theta\,d\theta}{2x}$$

$$\Rightarrow dx = \frac{\cos d\theta}{2\sqrt{\sin\theta}}$$

when $x = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \theta = \pi/2$

$$\therefore I = \int_0^{\pi/2} \frac{1}{\cos\theta} \cdot \frac{\cos\theta\,d\theta}{2\sqrt{\sin\theta}} = \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta\,\cos^0\theta\,d\theta$$

$$= \frac{1}{2} \times \frac{1}{2}\,\beta\left(\frac{-1/2+1}{2}, \frac{0+1/2}{2}\right)$$

$$= \frac{1}{4}\beta(1/4, 1/2) = \frac{1}{4}\,\frac{\Gamma(1/4)\,\Gamma(1/2)}{\Gamma(3/4)}$$

$$I = \frac{\sqrt{\pi}}{4}\,\frac{\Gamma(1/4)}{\Gamma(3/4)}$$

$$= \frac{\sqrt{\pi}}{4}\,\frac{(\Gamma(1/4))^2}{\sqrt{2}\,\pi} = \frac{1}{4\sqrt{2\pi}}(\Gamma(1/4))^2$$

we've, $\Gamma(1/4)\,\Gamma(3/4) = \sqrt{2}\pi$

$$\Rightarrow \Gamma(3/4) = \frac{\sqrt{2}\pi}{\Gamma(1/4)}$$

Problem 0.6. *Evaluate*

$$\int_a^b (x-a)^p (b-x)^q \, dx$$

Ans: Let $I = \int_a^b (x-a)^p (b-x)^q \, dx$

put $x = a + (b-a)t$ $dx = (b-a)\,dt$

when $x = a \Rightarrow t = 0$; when $x = b \Rightarrow t = 1$

$$\therefore I = \int_0^1 [(b-a)t]^p \, [b-a-(b-a)t]^q (b-a)\,dt$$

$$= \int_0^1 (b-a)^p t^p \, (b-a)^q (1-t)^q (b-a)\,dt$$

$$= (b-a)^{p+q+1} \int_0^1 t^{(p+1)-1} (1-t)^{(q+1)-1} \, dt$$

$$= (b-a)^{p+q+1} \beta(p+1, q+1)$$

Problem 0.7. *Evaluate*

Let $I = \int_0^2 x^4\sqrt{4-x^2}\,dx$

using beta and gamma functions.

Ans:- put $x^2 = 4t$ then $2x\,dx = 4\,dt$

$\Rightarrow x = 2\sqrt{t}$ $\Rightarrow x\,dx = 2\,dt$.

when $x = 0 \Rightarrow t = 0$

$x = 2 \Rightarrow t = 1$

$$\therefore I = \int_0^2 x^3\sqrt{4-x^2}\;x\,dx$$

$$= \int_0^1 8\,t^{3/2}(4-4t)^{1/2}\cdot 2\,dt$$

$$= 32\int_0^1 t^{3/2}(1-t)^{1/2}\,dt$$

$$= 32\int_0^1 t^{5/2-1}(1-t)^{3/2-1}\,dt$$

$$= 32\;\beta(5/2, 3/2) = 32\cdot\frac{\Gamma 5/2\,\Gamma 3/2}{\Gamma 4}$$

$$= 32\,\frac{\left(\frac{3}{2}\right)\Gamma 3/2\,\left(1/2\right)\Gamma 1/2}{3!}$$

$$= 32\,\frac{(3/2)(1/2)\Gamma 1/2\,(1/2)\Gamma 1/2}{6}$$

$$= \underline{\underline{2\pi}}$$

Problem 0.8. *Prove that*

$$\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \pi$$

Ans:- we've, $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

$$LHS = \int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta \, d\theta \times \int_0^{\pi/2} \sin^{-1/2}\theta \cos^0\theta \, d\theta$$

$$= \frac{1}{2}\beta\left(\frac{1/2+1}{2}, \frac{0+1}{2}\right) \times \frac{1}{2}\beta\left(\frac{-1/2+1}{2}, \frac{0+1}{2}\right)$$

$$= \frac{1}{2}\beta(3/4, 1/2) \times \frac{1}{2}\beta(1/4, \tfrac{1}{2})$$

$$= \frac{1}{4} \times \frac{\cancel{\Gamma 3/4}\, \Gamma 1/2}{\Gamma 5/4} \times \frac{\Gamma 1/4 \, \Gamma 1/2}{\cancel{\Gamma 3/4}} = \frac{\pi}{4} \cdot \frac{\Gamma 1/4}{\frac{1}{4}\Gamma 1/4}$$

$$= \underline{\underline{\pi}}$$

Problem 0.9. *Evaluate*

$$\int_0^2 \frac{x^2}{\sqrt{2-x}}\,dx$$

Hint : put $x = 2t$ $\longrightarrow$ $I = 4\sqrt{2}\,\beta(3, 1/2)$

$$= 4\sqrt{2}\,\frac{\Gamma 3\,\Gamma 1/2}{\Gamma 7/2}$$

Ans:-

$$= \frac{4\sqrt{2}\cdot 2!\,\sqrt{\pi}}{5/2\,\Gamma 5/2}$$

$$= \frac{4\sqrt{2}\times 2\cancel{\sqrt{\pi}}}{(5/2)(3/2)(1/2)\cancel{\Gamma 1/2}}$$

$$= \frac{8\sqrt{2}\times 8}{15} = \frac{64\sqrt{2}}{15}$$

Problem 0.10. *Evaluate*

Let $I = \displaystyle\int_0^\infty \frac{dx}{1+x^4}$ using beta and gamma functions.

Ans:- put $x^2 = \tan\theta$

$$\Rightarrow 2x\,dx = \sec^2\theta\, d\theta \Rightarrow dx = \frac{\sec^2\theta}{2\sqrt{\tan\theta}}\, d\theta$$

$$\theta = \tan^{-1}(x^2) \Rightarrow x = 0 \Rightarrow \theta = 0$$
$$x = \infty \Rightarrow \theta = \pi/2$$

$$\therefore I = \int_0^{\pi/2} \frac{1}{1+\tan^2\theta}\, \frac{\sec^2\theta}{2\sqrt{\tan\theta}}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta\, \cos^{1/2}\theta\, d\theta$$

$$= \frac{1}{2} \times \frac{1}{2}\, \beta\left(\frac{-1/2+1}{2}, \frac{1/2+1}{2}\right) = \frac{1}{4}\, \beta\left(1/4, 3/4\right)$$

$$= \frac{1}{4}\, \frac{\Gamma(1/4)\,\Gamma(3/4)}{\Gamma(1)}$$

$$= \frac{1}{4}\sqrt{2}\,\pi = \frac{\pi}{2\sqrt{2}}$$

Problem 0.11. *Evaluate*

Let $I = \int_0^\infty \frac{x^a}{a^x} dx$ for $a > 1$

Ans:-

put $a^x = e^t \Rightarrow x \log a = t$

$\Rightarrow x = \frac{1}{\log a} t$

$\Rightarrow dx = \frac{dt}{\log a}$

when $x = 0 \Rightarrow t = 0$

when $x = \infty \Rightarrow t = \infty$

$$\therefore I = \int_0^\infty \left(\frac{t}{\log a}\right)^a \cdot \frac{1}{e^t} \cdot \frac{dt}{\log a}$$

$$= \frac{1}{(\log a)^{a+1}} \int_0^\infty e^{-t} \cdot t^a \, dt$$

$$= \frac{1}{(\log a)^{a+1}} \int_0^\infty e^{-t} \cdot t^{(a+1)-1} dt$$

$$= \frac{\Gamma(a+1)}{(\log a)^{a+1}}$$

Problem 0.12. *Evaluate*

$$\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}dx \times \int_0^\infty x^2 e^{-x^4} dx$$

Ans: Let $I_1 = \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}} dx$ and

$I_2 = \int_0^\infty x^2 e^{-x^4} dx$

For I_1

put $x^2 = t$

$\Rightarrow 2x\,dx = dt$

$\Rightarrow dx = \frac{dt}{2x} = \frac{dt}{2t^{1/2}}$

when $x = 0 \Rightarrow t = 0$

$x = \infty \Rightarrow t = \infty$

$\therefore I_1 = \int_0^\infty \frac{e^{-t}}{t^{1/4}} \cdot \frac{dt}{2t^{1/2}}$

$= \frac{1}{2}\int_0^\infty e^{-t} \cdot t^{-3/4} dt$

$= \frac{1}{2}\int_0^\infty e^{-t} \cdot t^{1/4 - 1} dt$

$= \frac{1}{2}\Gamma\frac{1}{4}$

For I_2

put $x^4 = u$

$\Rightarrow 4x^3 dx = du$

$\Rightarrow dx = \frac{du}{4x^3} = \frac{du}{4u^{3/4}}$

when $x = 0 \Rightarrow u = 0$

$x = \infty \Rightarrow u = \infty$

$I_2 = \int_0^\infty e^{-u} \frac{du}{4u^{1/4}}$

$= \frac{1}{4}\int_0^\infty e^{-u} u^{-1/4} du$

$= \frac{1}{4}\int_0^\infty e^{-u} \cdot u^{3/4 - 1} du$

$= \frac{1}{4}\Gamma\frac{3}{4}$

$\therefore I_1 \times I_2 = \frac{1}{8}\Gamma\frac{1}{4}\,\Gamma\frac{3}{4} = \frac{1}{8}\sqrt{2}\pi = \frac{\pi}{4\sqrt{2}}$

PRACTICE PROBLEMS

Evaluate.

① $\int_0^\infty x^7 e^{-2x^3}\,dx$ Ans:- $3/16$

② $\int_0^1 (x\log x)^4\,dx$ Ans:- $\frac{24}{5^5}$

③ $\int_0^1 x^3 (1-\sqrt{x})^5\,dx$ Ans:- $2\,\beta(8,6)$ (Simplify)

④ $\int_0^\infty \frac{x\,dx}{1+x^6}$

Q. Express $\int_0^1 x^m(1-x^n)^p\,dx$ in terms of gamma function and hence find $\int_0^1 x^5(1-x^3)^{10}\,dx$.

Ans:- Let $I = \int_0^1 x^m(1-x^n)^p\,dx$

put $x^n = t \Rightarrow x = t^{1/n}$

$\Rightarrow dx = \frac{1}{n} t^{1/n - 1}\,dt$

$x=0 \Rightarrow t=0$
$x=1 \Rightarrow t=1$

$$\therefore I = \int_0^1 t^{m/n}(1-t)^p \cdot \frac{1}{n} t^{1/n-1}\,dt$$

$$= \frac{1}{n}\int_0^1 t^{\frac{m+1}{n}-1}(1-t)^{(p+1)-1}\,dt$$

$$= \frac{1}{n}\beta\left(\frac{m+1}{n}, p+1\right)$$

$$\int_0^1 x^m(1-x^n)^p\,dx = \frac{1}{n}\frac{\Gamma\frac{m+1}{n}\;\Gamma p+1}{\Gamma\frac{m+1}{n}+p+1}$$

Put $m=5$
$n=3$
$p=10$

$$\therefore \int_0^1 x^5(1-x^3)^{10}\,dx = \frac{1}{3}\frac{\Gamma 2\;\Gamma 11}{\Gamma 13} = \frac{1}{3}\frac{1!\,10!}{12!} = 1/396$$

Q. Express $\int_0^\infty x^n e^{-a^2x^2}\,dx$ in terms of gamma functions.

Ans: Let $I = \int_0^\infty x^n e^{-a^2x^2}\,dx$

put $a^2x^2 = t \Rightarrow x = \sqrt{t}/a \Rightarrow \frac{dt}{dx} = 2a^2x$

$\Rightarrow dx = \frac{dt}{2a\sqrt{t}}$

when $x=0 \Rightarrow t=0$

$x=\infty \Rightarrow t=\infty$

$$\therefore I = \int_0^\infty \frac{t^{n/2}}{a^n} \cdot e^{-t} \cdot \frac{dt}{2a\sqrt{t}}$$

$$= \frac{1}{2a^{n+1}} \int_0^\infty e^{-t} \cdot t^{\left(\frac{n-1}{2}\right)}\,dt$$

$$= \frac{1}{2a^{n+1}} \int_0^\infty e^{-t} \cdot t^{\left(\frac{n+1}{2}\right)-1}\,dt$$

$$= \frac{1}{2a^{n+1}} \Gamma\left(\frac{n+1}{2}\right)$$