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1. Vector Space

Let V be a vector space over F.

Theorem 1.1. *Let $S = \{v_1, v_2, \ldots, v_n\}$. Then S is a linearly dependent set in V if and only if some one of the vectors in S can be expressed as a linear combination of the remaining vectors.*

Proof:

Assume that S is linearly dependent.

Then $\exists$ scalars $\alpha_1, \alpha_2, \ldots \alpha_n$ in F (not all zeros) such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0 \quad \text{——— (1)}$$

Let k be the largest +ve integer such that $\alpha_k \neq 0$.

$\therefore$ (1) can be written as,

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_{k-1} v_{k-1} + \alpha_k v_k + \alpha_{k+1} v_{k+1} + \cdots + \alpha_n v_n = 0$$

$$\Rightarrow \alpha_k v_k = -\alpha_1 v_1 + -\alpha_2 v_2 + -\alpha_3 v_3 + \cdots - \alpha_{k-1} v_{k-1} + -\alpha_{k+1} v_{k+1} + \cdots + -\alpha_n v_n$$

$$\Rightarrow v_k = \left(\frac{-\alpha_1}{\alpha_k}\right) v_1 + \left(\frac{-\alpha_2}{\alpha_k}\right) v_2 + \cdots + \left(\frac{-\alpha_{k-1}}{\alpha_k}\right) v_{k-1} + \left(-\frac{\alpha_{k+1}}{\alpha_k}\right) v_{k+1} + \cdots + \left(\frac{\alpha_n}{\alpha_k}\right) v_n$$

ie; v_k is a linear combination of $v_1, v_2, \ldots, v_{k-1}, v_{k+1}, \ldots v_n$

Conversely, assume that some one of the vector in S is a linear combination of remaining vectors in S

To prove S is linearly dependent.

Let $v_j = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_{j-1} v_{j-1} + \alpha_{j+1} v_{j+1} + \cdots + \alpha_n v_n$

where $\alpha_1, \alpha_2 \cdots \alpha_{j-1}, \alpha_{j+1} \cdots \alpha_n \in F$

Then $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_{j-1} v_{j-1} + (-1) v_j + \alpha_{j+1} v_{j+1} + \cdots + \alpha_n v_n = 0$

$\therefore$ $S = \{v_1, v_2, \cdots v_n\}$ is L.D.

Let V be a vector space over the field F.

Theorem 1.2. *Let S be a linearly independent subset of V and $T \subset S$ then T is linearly independent.*
In other words, ***Every subset of a linearly independent set in V is linearly independent.***

Proof:

Let $S = \{v_1, v_2, \dots v_n\}$ be a linearly independent subset of V.

Let $T = \{v_1, v_2, \dots v_k\}$ where $1 \leq k < n$

Then $T \subset S$.

To prove T is linearly independent.

Assume that T is linearly dependent.

$\exists$ scalars $\alpha_1, \alpha_2, \dots \alpha_k$ in F (not all zeros) such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k = 0 \checkmark$$

$$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k + 0 v_{k+1} + 0 v_{k+2} + \dots + 0 v_n = 0 \checkmark$$

Where some of the α_i's are non zero.

which is a contradiction to S is linearly independent.

$\therefore$ Our assumption is wrong.

$\therefore$ T is linearly independent.

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* If $S = \{v_1, v_2 \cdots v_n\} \subseteq V$

then, the linear span of S,

$$L(S) = \{k_1v_1 + k_2v_2 + \cdots + k_nv_n \mid k_1, \cdots k_2 \in F\}$$

Eg: We know that $V = \mathbb{R}^2$ is a vector space over $F = \mathbb{R}$.

Consider $S = \{\overset{v_1 =}{(1,1)}, \overset{v_2 =}{(1,2)}\} \subset \mathbb{R}^2$

Then the linear span of S is

$$L(S) = \{k_1v_1 + k_2v_2 \mid k_1, k_2 \in \mathbb{R}\}$$

$$= \{k_1(1,1) + k_2(1,2) \mid k_1, k_2 \in \mathbb{R}\}$$

$$= \{(k_1 + k_2, k_1 + 2k_2) \mid k_1, k_2 \in \mathbb{R}\} \subseteq \mathbb{R}^2$$

Theorem 1.3. *If $S = \{v_1, v_2, ..., v_n\}$ is a linearly independent* Set *in V then every element in the linear span of S has a unique representation of the form $\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n$ where $\alpha_i \in F$, $1 \leq i \leq n$.*

Proof:

Given $S = \{v_1, v_2 \cdots v_n\}$ then

$$L(S) = \left\{ \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \;\middle|\; \alpha_i \in \mathbb{F},\ 1 \leq i \leq n \right\}$$

Let $v \in L(S)$ then

$$v = \underbrace{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n}_{} \quad \text{--- } (1)$$

Assume that $v = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$ --- (2)

where $\beta_1, \beta_2 \cdots \beta_n \in F$

$(1) - (2) \Rightarrow 0 = (\alpha_1 - \beta_1) v_1 + (\alpha_2 - \beta_2) v_2 + \cdots + (\alpha_n - \beta_n) v_n$

$$\Rightarrow \begin{array}{l} \alpha_1 - \beta_1 = 0 \\ \alpha_2 - \beta_2 = 0 \\ \vdots \\ \alpha_n - \beta_n = 0 \end{array} \quad (\because S \text{ is linearly independent})$$

$\Rightarrow \alpha_1 = \beta_1, \alpha_2 = \beta_2 \cdots \alpha_n = \beta_n$

$\therefore$ The representation in $L(S)$ is unique.

$A = B$ if $A \subseteq B$ and $B \subseteq A$

Theorem 1.4. *If $S = \{v_1, v_2, ..., v_n\}$ is a subset of the vector space V. If v_j is a linear combination of its preceeding ones, then*

$$L\{v_1, v_2, ..., v_{j-1}, v_{j+1}, ...v_n\} = L(S).$$

Proof:

Since v_j is the linear combination of $v_1, v_2, \dots v_{j-1}$, we can write

$$v_j = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_{j-1} v_{j-1}$$ where $\alpha_1, \alpha_2 \dots \alpha_{j-1} \in F$

To prove,

$$L\{v_1, v_2, \dots v_{j-1}, v_{j+1} \dots v_n\} \subseteq L(S)$$

Let $v \in L\{v_1, v_2, \dots v_{j-1}, v_{j+1} \dots v_n\}$

then $v = \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_{j-1} v_{j-1} + \beta_{j+1} v_{j+1} + \dots + \beta_n v_n$

$$\Rightarrow v = \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_{j-1} v_{j-1} + 0 v_j + \beta_{j+1} v_{j+1} + \dots + \beta_n v_n$$

$$\Rightarrow v \in L(S)$$

$$\therefore L\{v_1, v_2, \dots v_{j-1}, v_{j+1}, \dots v_n\} \subseteq L(S) \quad \text{(1)}$$

To prove,

$$L(S) \subseteq L\{v_1, v_2, \ldots v_{j-1}, v_{j+1} \ldots v_n\}$$

Let $u \in L(S)$ then

$$u = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_{j-1} v_{j-1} + \beta_j v_j + \beta_{j+1} v_{j+1} + \cdots + \beta_n v_n$$ where

$\beta_1, \beta_2 - \cdots \beta_n \in F$

$$\Rightarrow u = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_{j-1} v_{j-1} + \beta_j (\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_{j-1} v_{j-1}) + \beta_{j+1} v_{j+1} + \cdots + \beta_n v_n$$

$$\Rightarrow u = \underbrace{(\beta_1 + \beta_j \alpha_1)}_{\in F} v_1 + \overbrace{(\beta_2 + \beta_j \alpha_2)}^{\in F} v_2 + \cdots + \overbrace{(\beta_{j-1} + \beta_j \alpha_{j-1})}^{\in F} v_{j-1} + \beta_{j+1} v_{j+1} + \cdots + \beta_n v_n$$

i.e; u is a linear combination of

$v_1, v_2, \ldots v_{j-1}, v_{j+1} \ldots v_n$

$\Rightarrow u \in L\{v_1, v_2, \ldots v_{j-1}, v_{j+1} \ldots v_n\}$

$\Rightarrow L(S) \subseteq L\{v_1, v_2 \ldots v_{j-1}, v_{j+1} \ldots v_n\}$ —— 2

$\therefore$ 1 & 2 $\Rightarrow L\{v_1 \ldots v_{j-1}, v_{j+1}, \ldots v_n\} = L(S)$

Theorem 1.5. *If $S = \{v_1, v_2, ..., v_n\}$ is a basis for the vector space V then the representation of any element in V in terms of basis elemnts is unique.*

Proof: If S is a basis for V

then (i) S is linearly independent

(ii) S spans V

i.e; $L(S) = V$.

Since the representation of elements in $L(S)$ is unique; we' ue,

The representation of elements in V in terms of basis elements is unique

Definition 1.7. (**Minimal Spanning set**)

Let V be a vector space over F. Let $S \subseteq V$ then S is said to be a minimal spanning set if

(i) S is a spanning set for V.

(ii) $S \setminus \{v\}$ donot span V for any $v \in S$

Theorem 1.8. *In a vector spave V, a minimal spanning set of vectors forms a basis.*

Proof: $= \{v_1, v_2, \dots v_n\}$

Let S be a minimal spanning set for V. Then S spans V. ie; $L(S) = V$

In order to prove that S is a basis for V, it is enough to show that S is linearly independent.

Assume that S is linearly dependent. then there exists v_j (for some $j, 1 \le j \le n$) is a linear combination of its preceeding ones.

ie; $v_j = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_{j-1} v_{j-1}$

where $\alpha_1, \alpha_2 \dots \alpha_{j-1} \in F$

Then

$$L\{v_1, \dots v_{j-1}, v_{j+1} \dots v_n\} = L(S) = V$$

$\Rightarrow \{v_1, v_2 \ldots v_{j-1}, v_{j+1} \ldots v_n\}$ spans V

$\Rightarrow$ $S \setminus \{v_j\}$ spans V, contradiction to S is a minimal - spanning set.

$\therefore$ Our assumption is wrong.

$\therefore$ Only possibility is S is linearly independent.

$\therefore$ S forms a basis for V ✓.

Definition 1.9. (**Maximal linearly independent set**)

Let V be a vector space over F. Let $S \subseteq V$ then S is said to be a <u>maximal linearly independent set</u> if

(i) S is linearly independent

(ii) $S \cup \{v\}$ is linearly dependent for any $v \in V \setminus S$

Theorem 1.10. *In a vector spave V, a maximal linearly independent set forms a basis.*

Proof:- Let $S = \{v_1, v_2, \dots v_n\}$ be a maximal linearly independent set for V.

To prove S forms a basis for V.

It is enough to prove that S spans V.

Take $v \in V$.

Suppose $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n + \alpha v = 0$ —— (*)
where $\alpha_i \in F$, $\alpha \in F$ for $1 \le i \le n$.

when $\alpha = 0$

(*) $\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$

$\Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$ ($\because S$ is linearly independent)

$\Rightarrow S \cup \{v\}$ is linearly independent, a contradiction to the maximality of S.

$\therefore$ The only possibility is $\alpha \neq 0$

$\therefore \circledast \Rightarrow \alpha v = -\alpha_1 v_1 + (-\alpha_2)v_2 + \cdots + (-\alpha_n)v_n$

$$\Rightarrow v = \underbrace{\left(\frac{-\alpha_1}{\alpha}\right)}_{\in F} v_1 + \underbrace{\left(\frac{-\alpha_2}{\alpha}\right)}_{\in F} v_2 + \cdots + \underbrace{\left(\frac{-\alpha_n}{\alpha}\right)}_{\in F} v_n$$

$\Rightarrow$ v is a linear combination of elements from S

$\Rightarrow$ S spans V

$\therefore$ S forms a basis for V.

* We know that $\mathbb{R}^n$ is a vector space over $\mathbb{R}$.

Also, we have, $\dim(\mathbb{R}^n) = n$. (since, no. of elements in the basis of $\mathbb{R}^n$ is 'n')

So, the maximum number of linearly independent vectors in $\mathbb{R}^n$ is 'n'.

$\therefore$ By the previous theorem, <u>a linearly independent set with 'n' vectors in $\mathbb{R}^n$</u> forms <u>a basis for $\mathbb{R}^n$</u>.

<u>Eg:</u> Let $S = \{\underset{v_1 =}{(1,-1)}, \underset{v_2 =}{(-1,2)}\} \subseteq \mathbb{R}^2$

Here,

let $\alpha_1 v_1 + \alpha_2 v_2 = 0$ then,

$$\alpha_1(1,-1) + \alpha_2(-1,2) = 0 = (0,0)$$

$$\Rightarrow (\alpha_1 - \alpha_2, -\alpha_1 - 2\alpha_2) = (0,0)$$

$$\Rightarrow \left.\begin{array}{l} \alpha_1 - \alpha_2 = 0 \\ -\alpha_1 - 2\alpha_2 = 0 \end{array}\right\} \Rightarrow \alpha_1 = \alpha_2 = 0$$

$\therefore$ S is a linearly independent set in $\mathbb{R}^2$ with '2' vectors.

$\therefore$ S forms <u>a basis for</u> $\mathbb{R}^2$