

# BOOLEAN ALGEBRA

Department of Mathematics  
Manipal Institute of Technology  
Manipal Academy of Higher Education Manipal  
Manipal



**MANIPAL**  
ACADEMY of HIGHER EDUCATION

(Deemed to be University under Section 3 of the UGC Act, 1956)

# Contents

- 1 Binary relations
- 2 Lattice
- 3 Distributive and complemented lattice
- 4 Boolean lattice and boolean algebra
- 5 Boolean functions and boolean expressions
- 6 Reference

- A binary relation  $R$  from a set  $A$  to  $B$  is a subset of  $A \times B$ . That is,  $R = \{(a, b) | a \in A, b \in B\} \subseteq A \times B$ . If  $(a, b) \in R$ , then we say that the element ' $a$  is related to  $b$ ' and write  $aRb$ .
- A binary relation a set  $A$  to  $A$  is said to be a binary relation on  $A$ .

## Types of relations:

- ① **Reflexive relation:** A binary relation  $R$  on a  $A$  is said to be a reflexive relation if  $(a, a) \in R$  for all  $a \in A$ .  
Ex: Let  $A$  be the set of positive integers and  $R$  be the binary relation on  $A$  defined by  $(a, b) \in R$  if and only if  $a$  divides  $b$ . Then  $R$  is reflexive as every integer divides itself.
- ② **Symmetric relation:** A binary relation  $R$  on a  $A$  is said to be symmetric if  $(a, b) \in R \implies (b, a) \in R$  for all  $a, b \in A$ .  
Ex: The relations “is parallel to ” and “is perpendicular to ” are symmetric relations on the set of all straight lines.

- ③ **Antisymmetric relation:** A binary relation  $R$  on a set  $A$  is said to be antisymmetric if  $(a, b) \in R \implies (a, b) \notin R$  unless  $a = b$ .  
Ex: The binary relation  $R$  defined by  $(a, b) \in R$  if and only if  $a \geq b$  is antisymmetric on the set of positive integers.
- ④ **Transitive relation:** A binary relation  $R$  on a  $A$  is said to be transitive if  $(a, c) \in R$  whenever both  $(a, b) \in R$  and  $(b, c) \in R$ .  
Ex: The relation “is parallel to ” is transitive, but the relation “is perpendicular to ” is not transitive on the set of straight lines.
- ⑤ **Equivalence relation:** A binary relation on a set is said to be an equivalence relation if it is reflexive, symmetric and transitive.

- ③ **Partial ordering relation:** A binary relation on a set is said to be an equivalence relation if it is reflexive, antisymmetric and transitive. A nonempty set  $A$  with a partial ordering relation  $R$  is a partially ordered set (abbreviated as poset). **For each ordered pair  $(a, b) \in R$ , we write  $a \leq b$  instead of  $aRb$  where  $\leq$  is a generic symbol and commonly read as “less than or equal to ”.** It is often denoted as  $(A, R)$  or  $\langle A, R \rangle$  or  $(A, \leq)$ .  
 Ex: Let  $A$  be the set of positive integers and  $R$  be the binary relation on  $A$  defined by  $a \leq b$  if and only if  $a$  divides  $b$ . Then  $(A, \leq)$  is a poset.

- **Comparable elements:** Let  $(A, \leq)$  is a poset. Two elements  $a, b \in A$  are said to be comparable if either  $a \leq b$  or  $b \leq a$ .
- **Chain:** Let  $(A, \leq)$  is a poset. A subset of  $A$  is called a chain if every two elements in the subset are comparable. The number of elements in a chain is known as the length of the chain.
- **Antichain:** Let  $(A, \leq)$  is a poset. A subset of  $A$  is called an antichain if no two distinct elements in the subset are comparable.
- **Totally ordered set:** A poset  $(A, \leq)$  is called a totally ordered set if  $A$  is a chain. In this case, the binary relation  $\leq$  is called a total ordering relation.
- **Cover of an element:** Let  $(A, \leq)$  is a poset. An element  $b \in A$  is said to cover an element  $a \in A$  if  $a \leq b$  and there is no element  $c \in A$  such that  $a \leq c \leq b$ .

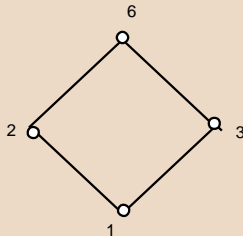
## Hasse diagram

A poset  $(A, \leq)$  is graphically represented by Hasse diagram. The following steps are to be followed to draw Hasse diagram corresponding to a given poset  $(A, \leq)$ .

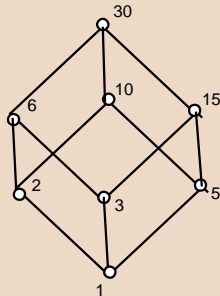
- Each element of  $A$  is represented by a small circle or a dot.
- The circle for  $x \in A$  is drawn below the circle for  $y \in A$  if  $x \leq y$ . A line is drawn if  $y$  covers  $x$ .
- If  $x \leq y$  but  $y$  doesn't cover  $x$ , then  $x$  and  $y$  are not connected directly by a single line.



## Example 1.1.



The poset  $(A, |)$  where  
 $A = \{1, 2, 3, 6\}$



The poset  $(D, |)$  where  
 $D = \{1, 2, 3, 5, 6, 10, 15, 30\}$



The poset  $(C, \leq)$   
 where  $C = \{1, 2, 3, 4\}$

Here  $|$  is the relation “divides ” and  $\leq$  is the relation “less than or equal to ”.

We note the following terminologies for a given poset  $(A, \leq)$ .

- Maximal element:** An element  $a \in A$  is said to be a maximal element of  $A$  if there is no  $b \in A$  such that  $a \neq b$  and  $a \leq b$ . We note that 6, 30 and 4 are the maximal elements of  $(A, |)$ ,  $(D, |)$  and  $(C, \leq)$  respectively.
- Minimal element:** An element  $a \in A$  is said to be a minimal element of  $A$  if there is no  $b \in A$  such that  $a \neq b$  and  $b \leq a$ . 1 is the minimal element of  $(A, |)$ ,  $(D, |)$  and  $(C, \leq)$ .
- Upper bound:** Let  $a, b \in A$ . An element  $c \in A$  is said to be an upper bound of  $a$  and  $b$  if  $a \leq c$  and  $b \leq c$ .
- Lower bound:** An element  $c \in A$  is said to be a lower bound of  $a$  and  $b$  if  $c \leq a$  and  $c \leq b$ .

- **Least upper bound (lub):** An element  $c \in A$  is said to be a least upper bound of  $a$  and  $b$  if  $c$  is an upper bound for  $a$  and  $b$ , and there is no upper bound  $d$  of  $a$  and  $b$  such that  $d \leq c$ .  
In  $(D, |)$  of example 1.1, the element 30 is an upper bound of 2 and 3, but it is not the least upper bound. The lub for 2 and 3 is 6.
- **Greatest lower bound (glb):** An element  $c \in A$  is said to be an greatest lower bound of  $a$  and  $b$  if  $c$  is a lower bound for  $a$  and  $b$ , and there is no lower bound  $d$  of  $a$  and  $b$  such that  $c \leq d$ .

## Lattice:

A partially ordered set is said to be a lattice if every two elements in the set have a unique glb and unique lub. Let  $(L, \leq)$  be a lattice. For any two elements  $a, b$ , let

$a \vee b$  : **lub of  $a$  and  $b$**  and  $a \wedge b$  : **glb of  $a$  and  $b$** .

Then  $(L, \leq, \vee, \wedge)$  is an algebraic system defined by the lattice  $(L, \leq)$ .

### Example 2.1.

Let  $P(S)$  be the power set of a nonempty set  $S$ . Then  $(P(S), \subseteq)$  is a lattice where  $A \vee B = A \cup B$  and  $A \wedge B = A \cap B$ . This defines the algebraic system  $(P(S), \subseteq, \cup, \cap)$ .

### Example 2.2.

Let  $N^+$  be the set of all positive integers. Then  $(N^+, |)$  ( $a|b$  if  $a$  divides  $b$ ) is a lattice where  $a \vee b = lcm(a, b)$  and  $a \wedge b = gcd(a, b)$ .

### Theorem 2.3.

*For any elements  $a, b$  in a lattice  $(A, \leq)$ ,*

- $a \leq a \vee b$  and  $b \leq a \vee b$
- $a \wedge b \leq a$  and  $a \wedge b \leq b$

### Theorem 2.4.

*For any elements  $a, b, c, d$  in a lattice  $(A, \leq)$ , if  $a \leq b$  and  $c \leq d$*

- $a \vee c \leq b \vee d$
- $a \wedge c \leq b \wedge d$

## Duality Principle

Let  $(A, \leq)$  be a poset. Let  $\geq$  be a binary relation on  $A$  such that for any  $a, b$  in  $A$ ,  $a \geq b$  if and only if  $b \leq a$ . We note that  $(A, \geq)$  is a poset.

- If  $(A, \leq)$  is a lattice, then so is  $(A, \geq)$
- The join operation of the algebraic system defined by the lattice  $(A, \leq)$  is the meet operation of the algebraic system defined by  $(A, \geq)$  and vice versa.
- Consequently, given any valid statement concerning the general properties of the lattices, we can obtain another valid statement by replacing the relation  $\leq$  with  $\geq$ , the meet operation with the join operation and the join operation with the meet operation. This is known as principle of duality for lattices.
- If the statement remains the same after dualism, then such a statement is called self dual.

## Properties of algebraic systems defined by lattices:

Let  $(A, \leq, \vee, \wedge)$  be the algebraic system defined by the lattice  $(A, \leq)$ . For any elements  $a, b, c \in A$ ,

① Commutative property:

- $a \vee b = b \vee a$
- $a \wedge b = b \wedge a$

② Associative property:

- $(a \vee b) \vee c = a \vee (b \vee c)$
- $(a \wedge b) \wedge c = a \wedge (b \wedge c)$

③ Idempotent property:

- $a \vee a = a$
- $a \wedge a = a$

④ Absorption property:

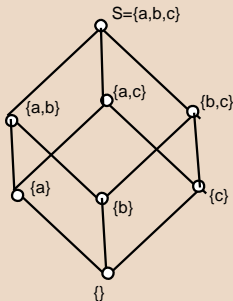
- $a \wedge (a \vee b) = a$
- $a \vee (a \wedge b) = a$

**Distributive lattice:** A lattice is said to be a distributive lattice if the meet operation distributes over the join operation and the join operation distributes over the meet operation. For any  $a, b, c$

- $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
- $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

### Example 3.1.

Let  $S = \{a, b, c\}$ . Then  $(P(S), \subseteq)$  is a distributive lattice.





## Theorem 3.2.

*If the meet operation is distributive over the join operation in a lattice, then the join operation is also distributive over the meet operation. If the join operation is distributive over the meet operation in a lattice, then the meet operation is also distributive over the join operation.*

### Proof.

Given that  $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$  — — — — — (1)

To prove  $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ ,

$$\begin{aligned}
 \text{Consider } (a \vee b) \wedge (a \vee c) &= [(a \vee b) \wedge a] \vee [(a \vee b) \wedge c] && \text{from (1)} \\
 &= a \vee [(a \vee b) \wedge c] && \text{(absorption law)} \\
 &= a \vee [c \wedge (a \vee b)] && \text{(commutative law)} \\
 &= a \vee [(c \wedge a) \vee (c \wedge b)] && \text{from (1)} \\
 &= [a \vee (c \wedge a)] \vee (c \wedge b) && \text{(associative law)} \\
 &= a \vee (c \wedge b) && \text{(absorption law)} \\
 &= a \vee (b \wedge c) && \text{(commutative law)}
 \end{aligned}$$

Second part follows from the principle of duality. □

**Problems:**

**Q1.** Let  $a$  and  $b$  be two elements in a lattice  $(A, \leq)$ . Show that  $a \wedge b = b$  if and only if  $a \vee b = a$ .

**Sol.**

Let

$$a \wedge b = b \text{ ----- (2)}$$

$$\text{Consider } a \vee (a \wedge b) = a \quad \text{(absorption law)}$$

$$a \vee b = a \quad \text{from (2)}$$

$$\text{Conversely, let } a \vee b = a \text{ ----- (3)}$$

$$\text{Consider } b \wedge (a \vee b) = b \quad \text{(absorption law)}$$

$$a \wedge b = b$$



**Q2.** Let  $a, b, c$  be elements in a lattice  $(A, \leq)$ . Show that

- i.  $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$
- ii.  $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$

Sol.

$$\text{i. } a \leq a \vee b \text{ and } a \leq a \vee c \implies a \leq (a \vee b) \wedge (a \vee c) \text{ --- (4)}$$

$$b \leq a \vee b \text{ and } c \leq a \vee c \implies b \wedge c \leq (a \vee b) \wedge (a \vee c) \text{ --- (5)}$$

From (4) and (5),  $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$ . (By Theorem 2.4)

$$\text{ii. } (a \wedge b) \leq a \text{ and } (a \wedge c) \leq a \implies (a \wedge b) \vee (a \wedge c) \leq a \text{ --- (6)}$$

$$(a \wedge b) \leq b \text{ and } (a \wedge c) \leq c \implies (a \wedge b) \vee (a \wedge c) \leq (b \vee c) \text{ --- (7)}$$

From (6) and (7),  $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$ . (By Theorem 2.4)



**Q3.** Let  $a, b, c$  be elements in a lattice  $(A, \leq)$ . Show that if  $a \leq b$ , then  $a \vee (b \wedge c) \leq b \wedge (a \vee c)$ .

**Q4.** Let  $(A, \leq, \vee, \wedge)$  be an algebraic system where  $\vee, \wedge$  are binary operations satisfying absorption law. Show that  $\vee$  and  $\wedge$  also satisfy idempotent law.

**Q5.** Let  $(A, \vee, \wedge)$  be an algebraic system where  $\vee, \wedge$  are binary operations satisfying commutative, associative and absorption laws. Define a binary operation  $\leq$  as follows: for all  $a, b \in A$ ,  $a \leq b$  if and only if  $a \wedge b = a$ . Show that  $\leq$  is a poset. Also show that  $a \vee b$  is lub of  $a$  and  $b$  and  $a \wedge b$  is glb of  $a$  and  $b$  in  $(A, \leq)$ .

**Q6.(Cancellation laws)** Let  $(A, \leq)$  be a distributive lattice. Show that if  $a \wedge x = a \wedge y$  and  $a \vee x = a \vee y$  for some  $a$ , then  $x = y$ .

**Q7.** Show that a lattice is distributive if and only if for any elements  $a, b, c$  in lattice  $(a \vee b) \wedge c \leq a \vee (b \wedge c)$ .

**Universal lower and upper bounds:** An element  $a$  in a lattice  $(A, \leq)$  is called a universal lower bound if for every element  $b \in A$ ,  $a \leq b$ . We use '0' to denote universal lower bound. An element  $a$  in a lattice  $(A, \leq)$  is called a universal upper bound if for every element  $b \in A$ ,  $b \leq a$ . We use '1' to denote universal upper bound. If a lattice has a universal lower (upper) bound, then it is unique. In the lattice  $(P(S), \subseteq)$ , the null set  $\phi$  and the set  $S$  are the universal lower and upper bounds respectively.

### Theorem 3.3.

*Let  $(A, \leq)$  be a lattice with universal upper and lower bounds 1 and 0. For any elements  $a \in A$*

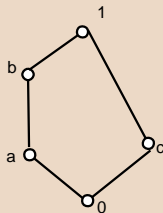
- $a \vee 1 = 1$                                       •  $a \wedge 1 = a$
- $a \vee 0 = a$                                         •  $a \wedge 0 = 0$

**Complement of an element:** Let  $(A, \leq)$  be a lattice with universal upper and lower bounds 1 and 0. For any element  $a \in A$ , an element  $b$  is said to be a complement of  $a$  if  $a \vee b = 1$  and  $a \wedge b = 0$

An element in a lattice may have more than one complement. Not all the elements in a lattice have complements. It's evident that '0' is the unique complement of '1' and vice versa.

**Complemented lattice:** A lattice is said to be a complemented lattice if every element in the lattice has a complement. Clearly, a complemented lattice has a universal lower and upper bounds.

### Example 3.4.



Complement of  $a$  and  $b$  is  $c$ . Complement of  $c$  are  $a, b$ .

### Theorem 3.5.

*In a distributive lattice, if an element has a complement, it is unique.*

#### Proof.

Suppose an element  $a$  has two complements  $b$  and  $c$ . i.e.

$$a \vee b = a \vee c = 1 \text{ and } a \wedge b = a \wedge c = 0.$$

Consider  $b = b \wedge 1$

$$\begin{aligned} &= b \wedge (a \vee c) \\ &= (b \wedge a) \vee (b \vee c) \\ &= 0 \vee (b \vee c) \\ &= (a \wedge c) \vee (b \vee c) \\ &= c \wedge (a \vee b) \\ &= c \wedge 1 \\ &= c \end{aligned}$$

Thus  $b = c$



**Boolean lattice:** A complemented and distributive lattice is called a boolean lattice.

### Example 4.1.

$(P(S), \subseteq)$  is a boolean lattice.

Let  $(A, \leq)$  be a boolean lattice. Since every element  $a$  has a unique complement  $\bar{a}$ , we have another unary operation known as complementation and denoted by  $\bar{\phantom{x}}$ . Thus we can say that the lattice  $(A, \leq)$  defines an algebraic system  $(A, \leq, \vee, \wedge)$  where  $\vee$  and  $\wedge$  are the join and meet operations respectively. The algebraic system defined by a boolean lattice is known as **boolean algebra**.

### Theorem 4.2.

*DeMorgan's laws: For any elements  $a, b$  in a boolean algebra  $(A, \leq, \vee, \wedge)$ ,*

- $\overline{a \vee b} = \bar{a} \wedge \bar{b}$
- $\overline{a \wedge b} = \bar{a} \vee \bar{b}$



## Proof.

We have to prove that  $(a \vee b) \vee (\bar{a} \wedge \bar{b}) = 1$  and  $(a \vee b) \wedge (\bar{a} \wedge \bar{b}) = 0$ .

$$\begin{aligned}
 \text{Consider } (a \vee b) \vee (\bar{a} \wedge \bar{b}) &= [(a \vee b) \vee \bar{a}] \wedge [(a \vee b) \vee \bar{b}] \text{ (distributive law)} \\
 &= [\bar{a} \vee (a \vee b)] \wedge [a \vee (b \vee \bar{b})] \text{ (associative law)} \\
 &= [(\bar{a} \vee a) \vee b] \wedge [a \vee 1] \text{ (associative law)} \\
 &= [1 \vee b] \wedge [a \vee 1] \\
 &= 1 \wedge 1 = 1
 \end{aligned}$$

$$\begin{aligned}
 \text{Similarly, } (a \vee b) \wedge (\bar{a} \wedge \bar{b}) &= (\bar{a} \wedge \bar{b}) \wedge (a \vee b) \text{ (commutative law)} \\
 &= [(\bar{a} \wedge \bar{b}) \wedge a] \vee [(\bar{a} \wedge \bar{b}) \wedge b] \text{ (distributive law)} \\
 &= [a \wedge (\bar{a} \wedge \bar{b})] \vee [(\bar{a} \wedge \bar{b}) \wedge b] \text{ (commutative law)} \\
 &= [(a \wedge \bar{a}) \wedge \bar{b}] \vee [\bar{a} \vee (\bar{b} \wedge b)] \text{ (associative law)} \\
 &= [0 \wedge \bar{b}] \vee [(\bar{a} \wedge 0)] \\
 &= 0 \vee 0 = 0
 \end{aligned}$$

The second part follows from principle of duality. □

**Uniqueness of finite boolean algebras:** We show that a finite boolean algebra has  $2^n$  elements for some  $n$ . An element of a boolean algebra is called an atom if it covers 0.

### Lemma 4.3.

*In a distributive lattice, if  $b \wedge \bar{c} = 0$ , then  $b \leq c$ .*

#### Proof.

We know that  $0 \vee c = c$

$$(b \wedge \bar{c}) \vee c = c \quad (\text{given})$$

$$c \vee (b \wedge \bar{c}) = c \quad (\text{commutative law})$$

$$(c \vee b) \wedge (c \vee \bar{c}) = c \quad (\text{distributive law})$$

$$(c \vee b) \wedge 1 = c$$

$$(c \vee b) = c$$

$$(b \vee c) = c$$

$$\text{Thus } b \leq b \vee c \implies b \leq c.$$



**Lemma 4.4.**

Let  $(A, \leq, \vee, \wedge, \neg)$  be a finite boolean algebra. Let  $b$  be any nonzero element in  $A$  and  $a_1, a_2, \dots, a_k$  be all the atoms of  $A$  such that  $a_i \leq b$ . Then  $b = a_1 \vee a_2 \vee \dots a_k$ .

**Proof.**

Since  $a_1 \leq b, a_2 \leq b, \dots, a_k \leq b$ , it follows that

$$a_1 \vee a_2 \vee \dots a_k \leq b \text{ ----- (8)}$$

For notational convenience, let  $c = a_1 \vee a_2 \vee \dots a_k$ . Suppose  $b \wedge \bar{c} \neq 0$ , then there exists an atom  $a$  such that  $a \leq b \wedge \bar{c}$ .

$$\text{Now } a \leq b \wedge \bar{c} \text{ and } b \wedge \bar{c} \leq b \implies a \leq b \text{ ----- (9)}$$

$$a \leq b \wedge \bar{c} \text{ and } b \wedge \bar{c} \leq \bar{c} \implies a \leq \bar{c}$$

From (9),  $a$  is equal to one of the atoms  $a_1, a_2, \dots, a_k$ . Also  $a \leq c$ .

Combining  $a \leq c$  and  $a \leq \bar{c}$ , we get  $a \leq c \wedge \bar{c} \implies a \leq 0$ , which is impossible. Thus  $b \wedge c = 0 \implies b \leq c$ . That is

$$b \leq a_1 \vee a_2 \vee \dots a_k. \text{ ----- (10)}$$

Form (8) and (10) and antisymmetric property,  $a_1 \vee a_2 \vee \dots a_k = b$ .  $\square$

### Lemma 4.5.

Let  $(A, \leq, \vee, \wedge, \neg)$  be a finite boolean algebra. Let  $b$  be any nonzero element in  $A$  and  $a_1, a_2, \dots, a_k$  be all the atoms of  $A$  such that  $a_i \leq b$ . Then  $b = a_1 \vee a_2 \vee \dots \vee a_k$  is the unique way to represent  $b$  as a join of atoms.

### Proof.

Suppose that we have alternative representation  $b = a_{j1} \vee a_{j2} \vee \dots \vee a_{jt}$ . Since  $b$  is the lub of  $a_{j1}, a_{j2}, \dots, a_{jt}$ , it is true that  $a_{j1} \leq b, a_{j2} \leq b, \dots, a_{jt} \leq b$ . Consider an atom  $a_{ju}$  ( $1 \leq u \leq t$ ). Since  $a_{ju} \leq b$ ,

we have  $a_j \wedge b = a_{ju}$

$$a_{ju} \wedge (a_1 \vee a_2 \vee \dots \vee a_k) = a_{ju}$$

$$(a_{ju} \wedge a_1) \vee (a_{ju} \wedge a_2) \vee \dots \vee (a_{ju} \wedge a_k) = a_{ju}$$

Then for some  $a_i$  ( $1 \leq i \leq k$ ),  $a_{ju} \wedge a_i \neq 0$ .

Since  $a_{ju}$  and  $a_i$  are atoms, we must have  $a_{ju} = a_i$ . Thus each atom in the alternative representation is an atom in the original one, and the lemma follows. □

From the above lemmas, it is clear that there is one to one correspondence between the elements of a boolean lattice and subset of atoms. As a matter of fact, there is one to one correspondence from  $(A, \leq)$  to  $(P(S), \subseteq)$ , where  $S$  is the set of all atoms.

### Theorem 4.6.

*Let  $(A, \vee, \wedge, -)$  be a finite boolean algebra. Let  $S$  be the set of all atoms. Then  $(A, \vee, \wedge, -)$  is isomorphic to the algebraic system defined by the lattice  $(P(S), \subseteq)$ .*

It follows from the above lemmas that **there exists a finite boolean algebra of  $2^n$  elements for any  $n > 0$ .**

**Boolean expression:** Let  $(A, \vee, \wedge, \neg)$  be a finite boolean algebra. A boolean expression over  $(A, \vee, \wedge, \neg)$  is defined as follows:

- An element of  $A$  is a boolean expression.
- Any variable name is a boolean expression.
- If  $e_1$  and  $e_2$  are boolean expressions, then  $\neg e_1$ ,  $e_1 \vee e_2$  and  $e_1 \wedge e_2$  are boolean expressions.

### Example 5.1.

$$0 \vee x, (x_1 \vee x_2) \wedge \overline{(2 \vee 3)}$$

Let  $E(x_1, x_2, \dots, x_n)$  be a boolean expression of  $n$  variables over a boolean algebra  $(A, \vee, \wedge, \neg)$ . By assignment of values to the variables  $x_1, x_2, \dots, x_n$ , we mean an assignment of elements of  $A$  to be the values of the variables. For an assignment of values to the variables, we can evaluate  $E(x_1, x_2, \dots, x_n)$  by substituting the variables in the expression by their values.

Two boolean expressions of  $n$  variables are said to be equivalent if they assume the same values for every assignment of values to the  $n$  variables. If  $E_1(x_1, x_2, \dots, x_n)$  and  $E_2(x_1, x_2, \dots, x_n)$  are equivalent, then we write  $E_1(x_1, x_2, \dots, x_n) = E_2(x_1, x_2, \dots, x_n)$ .

### Example 5.2.

$(x_1 \wedge x_2) \vee (x_1 \wedge \bar{x}_3)$  is equivalent to  $x_1 \wedge (x_2 \vee \bar{x}_3)$ .

**Boolean function:** A function  $f : A^n \rightarrow A$  is said to be a boolean function if it can be specified by a boolean expression of  $n$  variables.

**Minterm:** A boolean expression of  $n$  variables  $x_1, x_2, \dots, x_n$  is said to be a minterm if it is of the form  $\tilde{x}_1 \wedge \tilde{x}_2 \wedge \dots \wedge \tilde{x}_n$  where  $\tilde{x}_i$  is either  $x_i$  or  $\bar{x}_i$ .

**Disjunctive normal form (DNF):** A boolean expression over  $(\{0, 1\}, \wedge, \vee, -)$  is said to be in disjunctive normal form if it is join of minterms.

### Example 5.3.

$$(x_1 \wedge \bar{x}_2 \wedge x_3) \vee (\bar{x}_1 \wedge \bar{x}_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge x_3)$$

**Maxterm:** A boolean expression of  $n$  variables  $x_1, x_2, \dots, x_n$  is said to be a maxterm if it is of the form  $\tilde{x}_1 \vee \tilde{x}_2 \vee \dots \vee \tilde{x}_n$  where  $\tilde{x}_i$  is either  $x_i$  or  $\bar{x}_i$ .

**Conjunctive normal form (CNF):** A boolean expression over  $(\{0, 1\}, \wedge, \vee, -)$  is said to be in conjunctive normal form if it is meet of maxterms.

### Example 5.4.

$$(x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3)$$



**DNF:** Given a function  $\{0, 1\}^n \rightarrow \{0, 1\}$ , we can obtain a boolean expression in DNF corresponding to this function by having a minterm corresponding to each ordered  $n$  tuple of 0's and 1's for which the value of the function is 1. For each  $n$  tuple with the functional value is 1, we have the minterm  $\tilde{x}_1 \vee \tilde{x}_2 \vee \cdots \vee \tilde{x}_n$  where  $\tilde{x}_i = \begin{cases} x_i & \text{if } i^{\text{th}} \text{ componet is 1} \\ \bar{x}_i & \text{if } i^{\text{th}} \text{ componet is 0} \end{cases}$ .

**CNF:** We can obtain a boolean expression in CNF corresponding to this function by having a maxterm corresponding to each ordered  $n$  tuple of 0's and 1's for which the value of the function is 1. For each  $n$  tuple with the functional value is 0, we have the minterm  $\tilde{x}_1 \vee \tilde{x}_2 \vee \cdots \vee \tilde{x}_n$  where  $\tilde{x}_i = \begin{cases} x_i & \text{if } i^{\text{th}} \text{ componet is 0} \\ \bar{x}_i & \text{if } i^{\text{th}} \text{ componet is 1} \end{cases}$ .

**Q8.** Let  $E(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee (x_1 \wedge x_3) \vee (\bar{x}_2 \wedge x_3)$  be a boolean expression over  $(\{0, 1\}, \wedge, \vee, -)$ . Write the boolean expression in both DNF and CNF.

Sol:

	$(x_1 \wedge x_2)$	$(x_1 \wedge x_3)$	$(\bar{x}_2 \wedge x_3)$	$f$
000	0	0	0	0
001	0	0	1	1
010	0	0	0	0
011	0	0	0	0
100	0	0	0	0
101	0	1	1	1
110	1	0	0	1
111	1	1	0	1

DNF :  $(\bar{x}_1 \wedge \bar{x}_2 \wedge x_3) \vee (x_1 \wedge \bar{x}_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge \bar{x}_3) \vee (x_1 \wedge x_2 \wedge x_3)$

CNF :  $(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_2 \vee x_3)$



# References

- [1] Liu C L, Elements of discrete mathematics, 2<sup>nd</sup> edition, *McGraw Hill Book Company, New Dehli*, (2007).

**THANK YOU**