Q2. A ship carries 48 flags, 12 each of the colors white, red, blue and black. Twelve of these flags are placed on a vertical pole in order to communicate a signal to other ships:

(i) How many of these signals use an even number of blue flags and an odd number of black flags?

(ii) How many of these signals use at least 3 white flags or no white flags at all?

<u>soln</u> (ii) GF: $\left(1+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)\left(1+x+\frac{x^2}{2!}+\cdots\right)^3$

white

$$= \left(e^x - x - \frac{x^2}{2!}\right)e^{3x} = e^{4x} - xe^{3x} - \frac{x^2}{2!}e^{3x}$$

coeff of $\frac{x^{12}}{12!}$ is $= \frac{\sum (4x)^r}{r!} - x\sum\frac{(3x)^r}{r!} - \frac{x^2}{2!}\cdot\sum\frac{(3x)^r}{r!}$

$$4^{12} - 3^{11}\times 12 - \frac{3^{10}\times 11\times 12}{2} =$$

Q3. Find the number of ways to place 25 people into 3 rooms with at least one person in each room?

Soln: GF: $\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)^3=\left(e^x-1\right)^3$

$$= e^{3x}-3\cdot e^{2x}+3e^{x}-1=\sum\frac{(3x)^r}{r!}-3\sum\frac{(2x)^r}{r!}+3\sum\frac{x^r}{r!}-1$$

coeff of $\frac{x^{25}}{25!}$ is $3^{25}-3\times 2^{25}+3=$ $\underline{\underline{\qquad}}$

Q4. Find the number of r-digit quaternary sequence (whose digits are 0, 1, 2 and 3) with an even number of 0's and odd number of 1's?

Soln: GF: $\underbrace{\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)}_{0's}\underbrace{\left(x+\frac{x^3}{3!}+\cdots\right)}_{1's}\underbrace{\left(1+x+\frac{x^2}{2!}+\cdots\right)^2}_{2's\ \&\ 3's}$

$$\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^x-e^{-x}}{2}\right)e^{2x}$$

$$=\frac{1}{4}\left(e^{4x}-1\right)=\frac{1}{4}\left(\sum\frac{(4x)^r}{r!}-1\right)$$

coeff of $\frac{x^r}{r!}$ is $\frac{1}{4}4^r=\underline{\underline{4^{r-1}}}$

Q5. How many n-digit ternary $(0, 1, 2)$ sequences are there with at least one 0, atleast one 1, and at least one 2?

Q6. How many r-digit quaternary sequences are there that have even number of 0's and an even number of 1's.

Q5. <u>Ans</u>: $3^n - 3 \times 2^n + 3$

Q6: Ans: $\frac{4^r}{4} + \frac{2^r}{2}$

Q7. How many ways are there to split 6 copies of one book, 7 copies of a second book and 11 copies of a third book between 2 teachers if each teacher gets 12 books and each teacher gets at least 2 copies of each book?

Hint: G.F $\underbrace{(x^2+x^3+x^4)}_{1^{st}\text{ book}}\underbrace{(x^2+x^3+x^4+x^5)}_{2^{nd}}\underbrace{(x^2+x^3+\cdots+x^9)}_{3^{rd}}$

one teacher:

Coeff of x^{12} is $8C_6 - 5C_3 - 4C_2 = \underline{\underline{12}}$

Q8. How many ways are there to select 300 chocolates from 7 types if each type comes in boxes of 20 and if at least one but not more than 5 boxes of each type are chosen?

Hint: G.F $\left(x^{20} + (x^{20})^2 + \cdots + (x^{20})^5\right)^7$

$x^{140}\left(1 + x^{20} + \cdots + (x^{20})^4\right)^7$

coeff of x^{300} is $14C_8 - 7 \times 9C_3 = \underline{2415}$

Principle of Inclusion and Exclusion

Consider N objects and 2 properties say 'a' and 'b'. Let N(a) and N(b) denote the number of objects having the properties 'a' and 'b' respectively.

Let N(ab) denote the no. of objects having both the properties a and b.

Then, the number of objects without the properties 'a' and 'b' is given by

$$N(a'b') = N - N(a) - N(b) + N(ab)$$

Generalization: Consider N objects and r properties $a_1, a_2, \ldots, a_r$. Let $N(a_i)$ denote the number of objects having property a_i.

Let $N(a_i a_j)$ denote the number of objects having properties $a_i a_j$ and so on.

Let $N(a_1 a_2 \cdots a_r)$ denote the number of objects having properties $a_1 a_2 \cdots a_r$.

Then the principle of inclusion and exclusion states that the number of objects having none of the properties is

$$N(a_1' a_2' \cdots a_r') = N - N(a_1) - N(a_2) - \cdots - N(a_r) + N(a_1 a_2) + N(a_1 a_3) + \cdots + (-1)^r N(a_1 a_2 \cdots a_r)$$

$$N(a_1'a_2'\cdots a_r') = N - \sum_{i=1}^{r} N(a_i) + \sum_{\substack{i\neq j \\ i<j}} N(a_i a_j) - \sum_{i<j<k} N(a_i a_j a_k) + \cdots + (-1)^r N(a_1 a_2 \cdots a_r)$$

Problems :

Q1. How many integers between 1 and 6300 are not divisible by 3, 5 or 7?

Soln : $N = 6300$

a_1 : property that it is divisible by 3

a_2 : " 5

a_3 : " 7

$N(a_1) = 2100$

$N(a_3) = 900$

$N(a_2) = 1260$

$N(a_1 a_2) = 420$

$N(a_2 a_3) = 180$

$N(a_1 a_3) = 300$

$N(a_1 a_2 a_3) = 60$

$$N(a_1' a_2' a_3') = N - N(a_1) - N(a_2) - N(a_3) + N(a_1 a_2) + N(a_1 a_3) + N(a_2 a_3) - N(a_1 a_2 a_3)$$

$= \underline{2880}$ ✓

Q2. How many arrangements of the digits 0, 1, 2, ..., 9 are there in which the first digit is greater than 1 and the last digit is less than 8?

Solution:

Let F: Set of all arrangements with 0 or 1 in the first digit.

L: Set of all arrangements with 8 or 9 in the last digit.

0 2 3 _ _ _

$N = 10!$

$N(F) = 2 \times 9!$

$\underset{0 \text{ or } 1}{\_}\ \underbrace{\_\ \_\ \_\ \_}_{9!}$

$N(L) = 2 \times 9!$

$\_\ \_\ \ldots\ \underset{8 \text{ or } 9}{\_}$

$N(F \cdot L) = 2 \times 2 \times 8!$

$\underset{0 \text{ or } 1}{\_}\ \_\ \ldots\ \underset{8 \text{ or } 9}{\_}$

Req Ans: $N(F'L') = N - N(F) - N(L) + N(FL)$

$=$ ______

Q3. How many positive integers less than or equal to 70 are relatively prime to 70?

$\{3, 9, 11, \cdots 69\}$

Solution:
Relatively prime to 70 means the numbers that have no common divisors with 70.
The prime divisors of 70 are 2,5,7.

We want to count the number of integers less than or equal to 70 that do not have 2 or 5 or 7 as divisors.

a_1 : set of +ve integers that are divisible by 2

a_2 : " 5

a_3 : " 7

$N = 70$

$N(a_1) =$

$N(a_2) =$

$N(a_1 a_2) =$

$N(a_1 a_3) =$

$N(a_2 a_3) =$

$N(a_1 a_2 a_3) =$

$$N(a_1' a_2' a_3') = N - (N(a_1) + N(a_2) + N(a_3)) + (N(a_1 a_2) + N(a_2 a_3) + N(a_1 a_3)) - N(a_1 a_2 a_3)$$

$$= \underline{\underline{24}}$$

Q4. How many n-digit ternary (0, 1, 2) sequences are there with at least one '0', at least one '1', and at least one 2?

Soln: A_0 : number of n-digit ternary sequences with no 0's

A_1 : " " " " 1's

A_2 : " " " " 2s

$N = 3^n$

$N(A_0) = N(A_1) = N(A_2) = 2^n$

1 or 2

$N(A_0 A_1) = N(A_0 A_2) = N(A_1 A_2) = 1$

$N(A_0 A_1 A_2) = 0$

Total number of n-digit sequences with at least one of each digit $= N(A_0' A_1' A_2')$

$= N - N(A_0) - N(A_1) - N(A_2) + N(A_0 A_1) + N(A_0 A_2) + N(A_1 A_2) - N(A_0 A_1 A_2)$

$= 3^n - 3 \times 2^n + 3 \times 1$

OR

GF : $\left(x + \frac{x^2}{2!} + \cdots\right)^3 = (e^x - 1)^3$

coeff of $\frac{x^n}{n!}$: $3^n - 3 \times 2^n + 3$

Q5. Find number of ways in which 25 distinct objects can be placed in 3 distinct boxes such that no box is empty?

Soln: CF: $\left(x+\frac{x^2}{2!}+\cdots\right)^3=(e^x-1)^3$

$= e^{3x}-3e^{2x}+3e^{x}-1$

$= \sum\frac{(3x)^r}{r!}-3\sum\frac{(2x)^r}{r!}+3\sum\frac{x^r}{r!}-1$

Coeff of $\frac{x^{25}}{25!}$ is $3^{25}-3\times 2^{25}+3$

OR $N(a_1)$ = number of ways the first box is empty $=2^{25}$

$\_\ \_\ \_$ $\uparrow$ empty, 25

$N(a_2)=N(a_3)=2^{25}$

$N(a_1a_2)$ = No. of ways such that I^{st} and 2^{nd} box are empty $=1$

$N(a_1a_3)=N(a_2a_3)=1$

$N(a_1a_2a_3)=0$

No. of ways such that no box is empty is

$N(a_1'a_2'a_3')=N-N(a_1)-N(a_2)-N(a_3)$

$+N(a_1a_2)+N(a_1a_3)+N(a_2a_3)-N(a_1a_2a_3)$

$=3^{25}-3\times 2^{25}+3$

Q6. If a number 'n' has only 2 distinct prime factors p_1 and p_2, Show that $f(n)$ the number of +ve integers $< n$ and relatively prime to n is

$$f(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right).$$

Hence find $f(135)$.

Q7. How many of the first 1000 integers are not divisible by 2,3, 5 or 7?
ANS:228
Q8. Seven people enter a lift. The lift stops only at 3 floor(unspecified). At each of the 3 floors no one enters the lift but at least one person leaves the lift. After 3 stops, the lift is empty. In how many ways this can happen?

GF:(x+x^2/2+⋯)^3=(e^x-1)^3
Coeff of x^7/7! Is 3^7-3(2^7)+3=1806

or
a_i: no body leave the lift at the ith stop, $i = 1, 2, 3$.

$N(a_i) = 2^7$ $\quad$ $N(a_i a_j) = 1$ $\quad$ $N(a_1 a_2 a_3) = 0$

$$N(a_1' a_2' a_3') = 3^7 - 3 \times 2^7 + 3 = \underline{1806}$$

Derangements

It is a permutation of objects that leaves no object in its position.

Eg: 231, 312 are the derangements of 123.

For $n=4$, the total number of derangements are 9

2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, 4321.

1234	2134	3124	4123
1243	2143 ✓	3142	4132
1324	2314 ✗	3214	4213
1342	2341 ✓	3241	4231
1423	2413	3412	4312
1432	2431	3421	4321

$N = 4!$

a_i: digit i in the i^{th} position

$$N(a_1) = 3! = N(a_2) = N(a_3) = N(a_4)$$

$$N(a_1 a_2) = 2! = N(a_1 a_3) = N(a_2 a_3) = \dots$$

$$N(a_1 a_2 a_3) = 1 = \dots \qquad N(a_1 a_2 a_3 a_4) = 1$$

$$N(a_1' a_2' a_3' a_4') = N - \Sigma N(a_i) + \Sigma N(a_i a_j) - \Sigma N(a_i a_j a_k) + N(a_1 a_2 a_3 a_4)$$

$$= 4! - 4 \times 3! + 4C_2 \times 2! - 4C_3 \times 1! + 1$$

$$= \underline{\underline{9}}$$

<u>Question</u>: How many permutation of n distinct elements 1, 2, ... n are there in which the element k is not in the k^{th} position?

<u>Soln</u>: Let a_k be the property that the k^{th} element is in the k^{th} position, $1 \leq k \leq n$.

$$N = n!$$

$$N(a_1) = N(a_2) = \dots = N(a_n) = (n-1)!$$

$$N(a_i a_j) = (n-2)!$$

$$N(a_i a_j a_k) = (n-3)! \quad \dots \quad N(a_1 a_2 \dots a_n) = 1$$

$$N(a_1' a_2' \dots a_n') = N - \sum N(a_i) + \sum N(a_i a_j) - \sum N(a_i a_j a_k) + \dots + (-1)^n N(a_1 a_2 \dots a_n)$$

$$= n! - n \times (n-1)! + nC_2 (n-2)! - nC_3 (n-3)! + \dots + (-1)^n \cdot 1$$

$$= n! - n! + \frac{n(n-1)}{2}(n-2)! - \frac{n(n-1)(n-2)}{3!}(n-3)! + \cdots + (-1)^n$$

$$= \frac{n!}{2!} - \frac{n!}{3!} + \frac{n!}{4!} - \cdots + (-1)^n \frac{n!}{n!}$$

$$= n! \left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + (-1)^n \frac{1}{n!} \right)$$

Eg: For $n = 6$, Ans is $6! \left(\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{6!} \right) = \underline{265}$

<u>Note</u>: $n! \left(\frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right)$

$$= n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!} \right)$$

If n is large, $= \frac{n!}{e}$