Lecture 1

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Differential Equation

1) Differential equation of first order :

Definition :- It is an equation which involves differential ~~of~~ coefficients or differentials

Example :-

① $\frac{dy}{dx} = 1$

② $e^x dx + e^y dy = 0$

③ $\frac{d^2y}{dx^2} = 0 - x^2$

$\therefore \frac{d^2y}{dx^2} + x^2 = 0$

④ $\frac{dy}{dx} = \frac{\sqrt{x}}{\sqrt{1+y^2}}$

⑤ $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = c\left(\frac{d^2y}{dx^2}\right)$

⑥ $\frac{dy}{dx} = y$

⑦ $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$

⑧ $\frac{\partial^2 y}{\partial t^2} = 16\frac{\partial^2 y}{\partial x^2}$

-> An ordinary D.E is that in which all the diff coefficients have reference to a single independent variables

Eg : 1, 2, 3, 4, 5, 6

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→ A partial D.E is that in which there are two or more independent variable and partial differential coefficients wrt any of them.

Eg: 7 and 8

2) Order :

The order of a D.E is the order of the highest derivative appearing in it.

Order 1 → 1, 2, 4, 6
Order 2 → 3, 5

3) Degree :

It is the degree of the highest ordered derivative when the derivatives are cleared of radicals and fractional powers.

Degree 1 → 1, 2, 3, 4, 6
Degree 2 → 5

4) Solutions :

i) General solution,

General solution is that in which the no. of arbitrary constants = order of the D.E.

Eg: $y = A\cos x + B\sin x$

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ii) Particular Solution,

Particular solution is that which can be obtained from general solution by giving particular values to the arbitrary constants.

5) Solutions of D.E. :

i) Variable separable form
ii) Homogeneous equations
iii) Non-homogeneous equations OR equations reducible to homogeneous form

i) Variable separable form;

General form :- $f(y)\,dy = \phi(x)\,dx$

Example 1 :-

$$\frac{dy}{dx} = e^{2x-y} + x^3 e^{-y}$$

$$\frac{dy}{dx} = e^{2x} \cdot e^{-y} + x^3 e^{-y}$$

$$\frac{dy}{dx} = e^{-y}\left(e^{2x} + x^3\right)$$

dividing by e^{-y} on both sides

$$\therefore \frac{dy}{dx}\frac{1}{e^{-y}} = e^{2x} + x^3$$

multiplying both sides with dx

$$\frac{dy}{e^{-y}} = (e^{2x} + x^3)\,dx$$

Integrating

$$\int e^y\,dy = \int (e^{2x} + x^3)\,dx$$

$$e^y = \frac{e^{2x}}{2} + \frac{x^4}{4} + C$$

Example 2 :-

$$\frac{dy}{dx} = (4x + y + 1)^2, \quad y(0) = 1$$

Taking $4x + y + 1 = t$

$$\therefore\ 4 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\therefore\ \frac{dy}{dx} = \frac{dt}{dx} - 4$$

$$\therefore\ \frac{dy}{dx} = (4x + y + 1)^2$$

$$\frac{dt}{dx} - 4 = t^2$$

$$\therefore\ \frac{dt}{dx} = t^2 + 4$$

dividing by (t^2+4) on both sides

$\therefore \frac{dt}{(t^2+4)dx} = 1$

multiply by dx on both sides

$\therefore \frac{dt}{t^2+4} = dx$

Integrating

$\int \frac{dt}{t^2+4} = \int dx$

$\therefore \frac{1}{2} \tan^{-1}\left(\frac{t}{2}\right) = x + C$

$\therefore \frac{1}{2} \tan^{-1}\left(\frac{4x+y+1}{2}\right) = x + C$

$\therefore \tan^{-1}\left(\frac{4x+y+1}{2}\right) = 2x + 2C \rightarrow (1)$

$\therefore x = 0, y = 1$

$\tan^{-1}\left(\frac{0+1+1}{2}\right) = 2(0) + 2C$

$\tan^{-1}(1) = 2C$

$\therefore \pi/4 = 2C$

$\therefore \pi/8 = C$

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substituting in equation (1)

$$\tan^{-1}\left(\frac{4x+y+1}{2}\right) = 2x + 2\left(\frac{\pi}{8}\right)$$

$$\therefore \tan^{-1}\left(\frac{4x+y+1}{2}\right) = 2x + \frac{\pi}{4}$$

//

ii) Homogeneous Equation;

H.E are of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$

where f and g are homogeneous functions of same degree in x and y.

Eg $\rightarrow$ $a_0 x^n + a_1 x^{n+1} y + \cdots a_n y^n$

$$x^n\left(a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + \cdots a_n\left(\frac{y}{x}\right)^n\right)$$

$$\therefore x^n \phi\left(\frac{y}{x}\right) \quad \text{OR} \quad y^n f\left(\frac{x}{y}\right)$$

1) $\frac{x^3+y^3}{x+y}$

$$\therefore \frac{x^3(1+(y/x)^3)}{x(1+y/x)} = x^2 \frac{(1+(y/x)^3)}{(1+y/x)}$$

$\therefore$ degree is 2

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How to solve a H.E.,
put $y = vx$ or $x = vy$

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Separate the variables V and x and integrate sin (x/y)

Example 1 :

$$(x+2y)dx + (2x+y)\,dy = 0$$

$$(2x+y)dy = -(x+2y)\,dx$$

$$\therefore \frac{dy}{dx} = \frac{-(x+2y)}{(2x+y)}$$

$$y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore v + x\frac{dv}{dx} = \frac{-(x+2y)}{(2x+y)}$$

$$= \frac{-(x+2vx)}{(2x+vx)}$$

$$v + x\frac{dv}{dx} = \frac{-(1+2v)}{(2+v)}$$

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$\therefore \quad x\frac{dv}{dx} = -\left(\frac{1+2v}{2+v}\right) - v$

$= -\left(\frac{1+2v+2v+v^2}{2+v}\right)$

$x\frac{dv}{dx} = \frac{-(v^2+4v+1)}{2+v}$

Integrating

$\int \frac{v+2}{v^2+4v+1}\, dv = -\int \frac{dx}{x}$

$\therefore$ taking $t = v^2+4v+1$

$dt = (2v+4)\, dv$

$= 2(v+2)\, dv$

$\frac{1}{2} dt = (v+2)\, dv$

$\therefore \quad \frac{1}{2}\int \frac{dt}{t} = -\log x$

$\frac{1}{2}\log t + \log x = c$

$\therefore \log(v^2+4v+1) + \log x^2 = 2c$

$\therefore \log[(v^2+4v+1)(x^2)] = \log 2c$

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$$\therefore [(v^2+4v+1)x^2] = 2c$$

$$\therefore \left(\frac{y^2}{x^2} + \frac{4y}{x} + 1\right)x^2 = 2c$$

$$\therefore y^2 + 4xy + x^2 = 2c$$

By completing the square:-

$$(x+2y)dx + (2x+y)dy = 0$$

$$x\,dx + y\,dy + 2(y\,dx + x\,dy) = 0$$

$$\therefore x\,dx + y\,dy + 2\,d(xy)$$

$$\therefore \int x\,dx + \int y\,dy + 2\int d(xy) = 0$$

$$\therefore \frac{x^2}{2} + \frac{y^2}{2} + 2xy = 0$$

$$\therefore x^2 + y^2 + 4xy = 0$$

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iii) Non-homogeneous Equation;

1) Equation of the type $\dfrac{dy}{dx} = \dfrac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$

where $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$.

This is not homogeneous, but can be reducible to homogeneous, by substitution

$$x = X + h \qquad \frac{dy}{dx} = \frac{dY}{dX}$$
$$y = Y + k$$

Substituting in the equation

$$\frac{dY}{dX} = \frac{a_1(X+h) + b_1(Y+k) + c_1}{a_2(X+h) + b_2(Y+k) + c_2}$$

$$= \frac{a_1X + b_1Y + (a_1h + b_1k + c_1)}{a_2X + b_2Y + (a_2h + b_2k + c_2)}$$

Choose

$$\left.\begin{array}{l} a_1h + b_1k + c_1 = 0 \\ a_2h + b_2k + c_2 = 0 \end{array}\right\}$$

and — Solving this two will give h and k

Then,
$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$$

2) $\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$

where, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = m$ (say)

then substitute $a_1 x + b_1 y = t$, then the given equation can be reduced to variable separable form.

1) Question solving :

i) Non-homogeneous equation reducible to homogeneous;

a) $\frac{dy}{dx} = \frac{y+x-2}{y-x-4}$

Solution,

$$\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

$$\therefore \frac{a_1}{a_2} = \frac{1}{-1} = -1 \qquad \frac{b_1}{b_2} = \frac{1}{1} = 1$$

$$\therefore -1 \neq 1$$

$\therefore x = X + h$
$y = Y + k$

$$\frac{dY}{dX} = \frac{Y+k+X+h-2}{Y+k-X-h-4}$$

$$= \frac{X+Y+(h+k-2)}{Y-X+(k-h-4)}$$

solving,

$$\left.\begin{aligned} h+k-2 &= 0 \\ k-h-4 &= 0 \end{aligned}\right\} \text{simultaneous equation}$$

$\therefore h = -1$
$k = 3$

$$\frac{dy}{dx} = \frac{x+y}{y-x}$$, which is homogeneous

substituting,
$y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore \quad v + x\frac{dv}{dx} = \frac{x+vx}{vx-x}$$

$$= \frac{1+v}{v-1}$$

$$\therefore \quad x\frac{dv}{dx} = \frac{1+v}{v-1} - v$$

$$= \frac{1+v-v^2+v}{v-1}$$

$$= \frac{1+2v-v^2}{v-1}$$

$$\therefore \quad \frac{v-1}{1+2v-v^2}\, dv = \frac{dx}{x}$$

Integrating,

$$\int \frac{v-1}{-v^2+2v+1}\,dv = \int \frac{dx}{x}$$

$$\therefore \cancel{\frac{-1}{2} \ln|1+2v-v^2|}$$

Taking,

$$1+2v-v^2 = t$$
$$(2-2v)\,dv = dt$$
$$(1-v)\,dv = \frac{1}{2}\,dt$$

$$\therefore \frac{-1}{2}\int \frac{dt}{t}$$

$$\therefore -\frac{1}{2} \ln|1+2v-v^2| = \ln x + C$$

$$\ln|1+2v-v^2| + \log x^2 = -2C$$

$$\therefore \ln\left|1+\frac{2y}{x}-\frac{y^2}{x^2}\right| + \log x^2 = -2C$$

$$\therefore \ln\left|\frac{x^2+2xy-y^2}{x^2}\right| + \log x^2 = -2C$$

$$\therefore \ln|x^2+2xy-y^2| - \log x^2 + \log x^2 = -2C$$

$$\therefore x^2+2xy-y^2 = e^{-2C} = C'$$

$x = X + h$ $\qquad\qquad$ $y = Y + k$

$= X - 1$ $\qquad\qquad$ $= Y + 3$

$\therefore X = x + 1$ $\qquad\qquad$ $\therefore Y = y \cancel{+} - 3$

substituting the values

$$\therefore (x+1)^2 + 2[(x+1)(y-3)] - (y-3)^2 = c'$$

6) Solve $(3x - 2y + 1)dy + (4y - 6x - 3)dx = 0$

Solution,

$$\frac{dy}{dx} = \frac{-(4y - 6x - 3)}{3x - 2y + 1}$$

$$= \frac{6x - 4y + 3}{3x - 2y + 1}$$

Checking,

$$\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

$\frac{a_1}{a_2} = \frac{6}{3} = 2$ $\qquad\qquad$ $\frac{b_1}{b_2} = \frac{-4}{-2} = 2$

$$\therefore 2 = 2$$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2}$$

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$$\frac{dy}{dx} = \frac{2(3x-2y)+3}{3x-2y+1}$$

Putting $3x - 2y = t$

$$3 - 2\frac{dy}{dx} = \frac{dt}{dx}$$

$$\therefore -2\frac{dy}{dx} = \frac{dt}{dx} - 3$$

$$\frac{dy}{dx} = \frac{\left(\frac{dt}{dx} - 3\right)}{-2}$$

$$\therefore \frac{-1}{2}\left(\frac{dt}{dx} - 3\right) = \frac{2t+3}{t+1}$$

$$\therefore \frac{dt}{dx} - 3 = \frac{-(4t+6)}{t+1}$$

$$\therefore \frac{dt}{dx} = \frac{-(4t+6)}{t+1} + 3$$

$$= \frac{-4t-6+3t+3}{t+1}$$

$$= \frac{-(t+3)}{t+1}$$

$$\therefore -\frac{(t+1)}{t+3}\,dt = dx$$

Integrating,

$$\therefore \frac{t+3-2}{t+3}\,dt = dx$$

$$\therefore \frac{t+3-2}{t+3}\,dt + dx = 0$$

$$\int\left(\frac{t+3}{t+3} - \frac{2}{t+3}\right)dt + \int dx = 0$$

$$\therefore \int dt - 2\int\frac{dt}{t+3} + x = 0$$

$$\therefore t - 2\ln|t+3| + x = 0$$

$$\therefore (3x-2y) - 2\ln|3x-2y+3| + x = 0 \;//$$

—— X —— X —— X —— X ——

H.W

1) Solve $\dfrac{dy}{dx} = \dfrac{2x-y+1}{2y-x-1}$

2) $\dfrac{dy}{dx} = \dfrac{y-x}{y-x+2}$

3) $(x+y-10)\,dx + (x-y-2)\,dy = 0$

—— X —— X —— X —— X ——

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2) Exact Differential Equation :

A D.E. of the form $M(x,y)dx + N(x,y)dy = 0$ is said to be exact if its left hand member is the exact differential of some function $u(x,y)$

$$du = Mdx + Ndy = 0 \quad \} \text{ exact differential}$$

$\downarrow$

Solution $\Rightarrow$ $u(x,y) = c$

Theorem : The necessary and sufficient condition for the D.E., $Mdx + Ndy = 0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solution of exact D.E. :

i) First evaluate $\int M dx$ w.r.t. x, treating y as a constant.

ii) Then evaluate $\int N dy$ w.r.t. y, omitting terms in x

iii) Add ① and ② and equate to a constant.

iv) Solution is $\int M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = 0$

$\downarrow$

Keeping 'y' as constant

Example :-

$$(3x + 4y + 9)\,dx + (3xy + 9y^2 + 4x)\,dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\int (3x + 4y + 9)\,dx + \int 9y^2\,dy = c$$

$$\therefore \frac{3x^2}{2} + 4y(x) + 9x + 9\frac{y^3}{3} = c$$

$$\therefore \frac{3}{2}x^2 + 4y(x) + 9x + 3y^3 = c$$

3) Solving Questions:

i) $(x^2 - y^2)dx = 2xy\, dy$

Solution,

$$(x^2-y^2)dx - 2xy\, dy = 0$$

$\therefore M = x^2 - y^2$; $N = -2xy$

$\frac{\partial M}{\partial y} = -2y$; $\frac{\partial N}{\partial x} = -2y$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \text{ exact D.E}$$

$$\therefore \int_{y\,const} (x^2-y^2)\, dx + \int 0\, dy = c$$

$$\therefore \frac{x^3}{3} - xy^2 = c$$

//

ii) $ye^{xy}dx + (xe^{xy} + 2y)dy = 0$

Solution,

$M = ye^{xy}$; $N = xe^{xy} + 2y$

$$\frac{\partial M}{\partial y} = y(e^{xy}(x)) + e^{xy}(1)$$

$$= xye^{xy} + e^{xy}$$

$$\frac{\partial N}{\partial x} = x\left(e^{xy}(y)\right) + e^{xy}(1)$$

$$= x\left(e^{xy}\right)y + e^{xy}$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \text{ exact D.E.}$$

$$\therefore \int y e^{xy}\,dx + \int 2y\,dy = 0$$

$$\therefore y\int e^{xy}\,dx + \int 2y\,dy = 0$$

$$\therefore \quad y\frac{e^{xy}}{y} + \frac{2y^2}{2} = 0$$

$$\therefore \quad e^{xy} + y^2 + C = 0$$

—— X —— X —— X —— X ——

H.W.

1) $\left(3x^2y + \frac{y}{x}\right)dx + (x^3 + \ln x)\,dy = 0$

2) $(\sec x \tan x \tan y - e^x)\,dx + (\sec x \sec^2 y)\,dy = 0$

—— X —— X —— X —— X ——

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4) Equations Reducible to exact equations:

Some linear D.E. is not exact, can be made exact on multiplication by a suitable factor called integrating factor (I.F.)

i) Rules for finding I.F. of the equation $Mdx + Ndy = 0$:

a) IF found by inspection

Example:

a) $x\,dy + y\,dx = d(xy)$

b) $\dfrac{x\,dy - y\,dx}{x^2} = d\left(\dfrac{y}{x}\right)$

c) $\dfrac{x\,dy - y\,dx}{y^2} = -d\left(\dfrac{x}{y}\right)$

d) $\dfrac{x\,dy - y\,dx}{xy} = d\left(\log\left(\dfrac{y}{x}\right)\right)$

e) $\dfrac{x\,dy - y\,dx}{x^2+y^2} = d\left(\tan^{-1}\left(\dfrac{y}{x}\right)\right)$ proof

$\therefore$ dividing by x^2

$$\frac{\left(\dfrac{x\,dy - y\,dx}{x^2}\right)}{\left(\dfrac{x^2+y^2}{x^2}\right)} = 1 + \left(\frac{y}{x}\right)^2$$

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Question 1 →

$$y\,dx = x\,dy + xy^3\,dy$$

Solution, Method 1

$$y\,dx - x\,dy = xy^3\,dy$$

Dividing by xy

$$\therefore \frac{y\,dx - x\,dy}{xy} = y^2\,dy$$

$$\int \frac{y\,dx - x\,dy}{xy} = \int y^2\,dy + C$$

$$-\int d\log\left(\frac{y}{x}\right) = \frac{y^3}{3} + C$$

$$-\log\left(\frac{y}{x}\right) = \frac{y^3}{3} + C \quad //$$

Method 2

$$y\,dx = x\,dy + xy^3\,dy$$

$$\therefore y\,dx = x(1+y^3)\,dy$$

$$\int \frac{1}{x}\,dx = \int\left(\frac{1+y^3}{y}\right)dy$$

$$\therefore \log x = \log y + \frac{y^3}{3} + C$$

$$\therefore \log\left(\frac{x}{y}\right) = \frac{y^3}{3} + C \quad //$$

Ans $\rightarrow \frac{e^x}{y} + x^2 = C$

Question 2 $\rightarrow$

$y(2xy + e^x)dx = e^x dy$

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b) I.F. of a homogeneous equation

If $Mdx + Ndy = 0$ be a homogeneous equation in x and y, then $\frac{1}{Mx+Ny}$ is an I.F. $(Mx + Ny \neq 0)$

Example:

$$(x+2y)dx + (2x+y)dy = 0 \quad \rightarrow (1)$$

$$I.F. = \frac{1}{Mx+Ny} = \frac{1}{(x+2y)x + (2x+y)y}$$

$$= \frac{1}{x^2+4xy+y^2}$$

Multiplying (1) with I.F

$$\therefore \left[(x+2y)dx + (2x+y)dy\right]\frac{1}{x^2+4xy+y^2} = 0$$

$$\therefore \frac{x+2y}{x^2+4xy+y^2}dx + \frac{2x+y}{x^2+4xy+y^2}dy = 0$$

Integrating

$$\int \frac{x+2y}{x^2+4xy+y^2}dx + \int \frac{2x+y}{x^2+4xy+y^2}dy = 0$$

$$\therefore \int M\,dx + \int 0\,dy = 0$$

$$\therefore \int \frac{x+2y}{x^2+4xy+y^2}\,dx = 0$$

Substituting,

$$x^2+4xy+y^2 = t$$

$$2x + 4y = \frac{dt}{dx}$$

$$\therefore 2(x+2y) = \frac{dt}{dx}$$

$$\therefore (x+2y)\,dx = \frac{dt}{2}$$

$$\therefore \frac{1}{2}\int \frac{dt}{t} = 0$$

$$\frac{1}{2}\log(t) = 0$$

$$\therefore \frac{1}{2}\log|x^2+4xy+y^2| = 0$$

$$\therefore \log|x^2+4xy+y^2| = 2c$$

c) I.F. for an equation of the type

$$f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$$

If the equation $M dx + N dy = 0$ be of the above form, then $\frac{1}{Mx - Ny}$ is an I.F. (Denominator $\neq 0$)

Example:

$$(1+xy)y\,dx + (1-xy)x\,dy = 0 \rightarrow ①$$

$$I.F. = \frac{1}{Mx \mp Ny} = \frac{1}{(1+xy)xy - (1-xy)xy}$$

$$= \frac{1}{2x^2y^2}$$

Multiplying I.F with eq ①

$$\left[(1+xy)y\,dx + (1-xy)x\,dy\right]\frac{1}{2x^2y^2} = 0$$

$$\therefore \frac{y+xy^2}{2x^2y^2}dx + \frac{x-x^2y}{2x^2y^2}dy = 0$$

Integrating

$$\int \frac{y+xy^2}{2x^2y^2}dx + \int \frac{x-x^2y}{2x^2y^2}dy = 0$$

$$\therefore \int\left(\frac{y}{2x^2y^2} + \frac{xy^2}{2x^2y^2}\right)dx - \int \frac{x^2y}{2x^2y^2} = 0$$

$\therefore \int\left(\frac{y}{2x^2y^2} + \frac{xy^2}{2x^2y^2}\right)dx - \frac{1}{2}\int\frac{1}{y}\,dy = 0$

$\therefore \frac{1}{2y}\int\frac{1}{x^2}\,dx + \frac{1}{2}\int\frac{1}{x}\,dx - \frac{1}{2}\int\frac{1}{y}\,dy = 0$

$\therefore \frac{1}{2y}\left(\frac{-1}{x}\right) + \frac{1}{2}\ln x - \frac{1}{2}\ln y = 0$

d) In the equation $Mdx + Ndy = 0$

→ If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ (function of x only)

then the I.F. is $e^{\int f(x)\,dx}$

→ If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = g(y)$ (function of y alone)

then the I.F. is $e^{\int g(y)\,dy}$

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Example 1:

$$3(x^2+y^2)dx + x(x^2+3y^2+6y)dy = 0 \rightarrow ①$$

$$\frac{\partial M}{\partial y} = 3x^2+3y^2 \qquad \frac{\partial N}{\partial x} = x^3+3xy^2+6xy$$

$$\therefore \frac{\partial M}{\partial y} = 6y \qquad \frac{\partial N}{\partial x} = 3x^2+3y^2+6y$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{3x^2+3y^2+6y-6y}{3x^2+3y^2}$$

$$= 1 \quad \text{function of } y$$

$$I.F. = e^{\int 1 dy} = e^y$$

multiplying I.F. with eq ①

$$(3x^2+3y^2)e^y dx + (x^3+3xy^2+6yx)e^y = 0$$

Integrating

$$\int_{y \text{ constant}} (3x^2+3y^2)e^y dx = 0$$

$$e^{y}(x^3 + 3y^2 x) = 0$$

---X---X---X---X---

H.W.

1) $2xy\,dy - (x^2 + y^2 + 1)\,dx = 0$

2) $x^4 \dfrac{dy}{dx} + x^3 y + \operatorname{cosec}(xy) = 0$

3) $2y\,dx + x(2\log x - y)\,dy = 0$

---X---X---X---X---

5) Exact Differential Equation Theory:

The necessary and sufficient condition for D.E. $M\,dx + N\,dy = 0$ to be exact is $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

i) Necessary Condition:

The equation $M\,dx + N\,dy = 0$ will be exact if $M\,dx + N\,dy = du$, $\rightarrow (1)$

where U is a function of x and y.

But $du = \dfrac{\partial u}{\partial x}\,dx + \dfrac{\partial u}{\partial y}\,dy \rightarrow (2)$

Comparing (1) and (2)

$M = \frac{\partial u}{\partial x}$ $\qquad$ $N = \frac{\partial u}{\partial y}$

Differentiating with x and y respectively

$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}$ $\qquad$ $\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$

we know that,

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

ii) Sufficient Condition ;

If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then $Mdx + Ndy = 0$

is exact.

Let $\int Mdx = u$, where y is supposed to be a constant while doing integration

Then,

$$\frac{\partial}{\partial x} \int Mdx = \frac{\partial u}{\partial x}$$

$$\Rightarrow M = \frac{\partial u}{\partial x} \quad \rightarrow (1)$$

Differentiating ① partially wrt y

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

$$\therefore \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) \qquad \left(\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\right)$$

$$\therefore \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) \rightarrow ②$$

Integrating ② wrt x

$$\int \frac{\partial N}{\partial x} dx = \int \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) dx$$

$$\therefore N = \frac{\partial u}{\partial y} + f(y)$$ → constant function

$$\therefore M dx + N dy = \frac{\partial u}{\partial x} dx + \left(\frac{\partial u}{\partial y} + f(y)\right) dy$$

$$= \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + f(y)\, dy$$

$$= dU + f(y)\,dy$$

$$= d\left(U + \int f(y)\,dy\right)$$

$$\therefore\ M\,dx + N\,dy = d\left(U + \int f(y)\,dy\right)$$

which shows that,

$$M\,dx + N\,dy = 0 \text{ is exact}$$

Intrgrating,

$$U + \int f(y)\,dy = 0$$

$$\therefore \int M\,dx + \int f(y)\,dy = c$$

term of N not containing x

1) Solution for D.E : (continued)

iv) Linear Differential equation;

A D.E. is said to be linear if the dependent variable and its differential coefficients occurs in the first degree and not multiplied together.

Standard form also known as Leibnitz's linear equation is of the form

$$\frac{dy}{dx} + Py = Q \rightarrow ①$$

where,
P and Q are functions of x

$$\frac{dx}{dy} + Px = Q$$

where
P and Q are functions of y

a) To solve equation ①,

Multiply both sides of ① by integrating factor which is $e^{\int P dx}$

$$\Rightarrow \frac{dy}{dx} e^{\int P dx} + y e^{\int P dx} (P) = Q \cdot e^{\int P dx}$$

$$\therefore \frac{d}{dx}\left(y e^{\int P dx}\right) = Q \cdot e^{\int P dx}$$

* when we diff the above equation we get the 2^{nd} term in LHS of eq ① *

Integrating,

$$\int \frac{d}{dx}\left(y \cdot e^{\int P dx}\right) = \int Q \cdot e^{\int P dx} \, dx$$

$$\therefore \; y e^{\int P dx} = \int Q \cdot e^{\int P dx} \, dx + C$$

The factor $e^{\int P dx}$ on multiplying by which the LHS of eq ① becomes the differential coefficient of a single function is called the integrating factor (I.F.) of the linear equation ①

The solution is,

$$y(I.F) = \int Q \, (I.F) \, dx + C$$

Question 2:

$$(1+x^2)dy + 2xy\,dx = \cot x \, dx$$

$$(1+x^2)dy = (-2xy + \cot x)dx$$

* The more no. of terms with dx or dy will come in the numerator and the lesser term will come in the denominator *

$$\therefore \; \frac{dy}{dx} = \frac{-2xy + \cot x}{1+x^2}$$

$$\therefore \; \frac{dy}{dx} + \frac{2x}{1+x^2}(y) = \frac{\cot x}{1+x^2}$$

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$$P = \frac{2x}{1+x^2} \quad , \quad Q = \frac{\cot x}{1+x^2}$$

$$\therefore I.F = e^{\int \frac{2x}{1+x^2} dx}$$

substituting,
$1+x^2 = t$
$2x\,dx = dt$

$$= e^{\int \frac{1}{t} dt}$$

$$= \cancel{e^{\int \log t}} = e^{\log t}$$

$$= t$$
$$= 1+x^2$$

$$\therefore y(I.F) = \int Q(I.F.)\,dx + C$$

$$\therefore y(1+x^2) = \int \frac{\cot x}{1+x^2}(1+x^2)\,dx + C$$

$$\therefore y(1+x^2) = \int \cot x\,dx + C$$

$$\therefore y(1+x^2) = \ln|\sin x| + C$$

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Question 2:

$$(1+y^2)dx + (x - e^{-\tan^{-1}y})dy = 0$$

* Here the coefficient of dy has more terms so it comes in the numerator *

$$(1+y^2)dx = -(x - e^{-\tan^{-1}y})dy$$

$$\therefore \frac{dx}{dx} = \frac{-(x - e^{-\tan^{-1}y})}{1+y^2}$$

$$= \frac{-x}{1+y^2} + \frac{e^{-\tan^{-1}y}}{1+y^2}$$

$$\therefore \frac{dx}{dy} + \frac{1}{1+y^2}(x) = \frac{e^{-\tan^{-1}y}}{1+y^2}$$, linear in x

$$I.F = e^{\int \frac{1}{1+y^2} dy}$$

$$= e^{\tan^{-1}y}$$

$$\therefore x(I.F) = \int Q\,(I.F)\,dy + C$$

$$\therefore x(e^{\tan^{-1}y}) = \int \frac{e^{-\tan^{-1}y}}{1+y^2}(e^{\tan^{-1}y})dy + C$$

$$\therefore x\left(e^{\tan^{-1}y}\right) = \int \frac{1}{1+y^2}\,dy + C$$

$$\therefore x\left(e^{\tan^{-1}y}\right) = \tan^{-1}y + C$$

Question 3:

$$y(\log y)\,dx + (x - \log y)\,dy = 0$$

$$y(\log y)\,dx = -(x - \log y)\,dy$$

$$\therefore \frac{dx}{dxy} = \frac{-(x - \log y)}{y\log y}$$

$$= \frac{-x}{y\log y} + \frac{\log y}{y\log y}$$

$$\therefore \frac{dx}{dy} + \frac{x}{y\log y} = \frac{1}{y}$$

$$P = \frac{x}{y\log y}, \quad Q = \frac{1}{y}$$

$$\text{I.F.} = e^{\int P\,dy} = e^{\int \frac{1}{y\log y}}$$

substituting,
$\log y = t$
$\frac{1}{y} = \frac{dt}{dy}$
$dt = \frac{dy}{y}$

$$I.F. = e^{\int \frac{1}{t} dt}$$

$$= e^{\log t}$$

$$= t$$

$$= \log y$$

$$\therefore I.F. = \log y$$

$$x(IF) = \int Q(I.F)\, dy + C$$

$$\therefore x(\log y) = \int \frac{1}{y} (\log y)\, dy + C$$

substituting,
$\log y = t$
$\frac{dy}{y} = dt$

$$\therefore x(t) = \int t\, dt + C$$

$$= \frac{t^2}{2} + C$$

$$\therefore x(\log y) = \frac{(\log y)^2}{2} + C$$

$$\therefore x = \frac{1}{2} \log y + \frac{C}{\log y}$$

//

v) Bernoulli's equation ;

The equation $\frac{dy}{dx} + Py = Qy^{m}$ $\rightarrow$ ①

where n is any real number except for 0 and 1.

P and Q are functions of x, is reducible to linear and is usually called the Bernoulli's equation.

a) To solve eqn ①

Divide by y^n

$$\therefore \frac{1}{y^n}\frac{dy}{dx} + \frac{Py}{y^n} = Q.$$

$$\therefore y^{-n}\frac{dy}{dx} + P\cdot y^{1-n} = Q \quad \rightarrow ②$$

Put $z = y^{1-n}$

$$\frac{dz}{dx} = (1-n)\, y^{1-n-1} \frac{dy}{dx}$$

$$= (1-n)\, y^{-n} \frac{dy}{dx}$$

$$\therefore \left(\frac{1}{1-n}\right)\frac{dz}{dx} = y^{-n}\frac{dy}{dx}$$

Equation ② becomes,

$$\left(\frac{1}{1-n}\right)\frac{dz}{dx} + Pz = Q$$

multiplying by $(1-n)$

$$\frac{dz}{dx} + P(1-n)z = Q(1-n),$$ linear in z and can be solved.

Question 1:

$$2xy' = 10x^3y^5 + y$$

Dividing by $2x$

$$\frac{dy}{dx} = \frac{10x^3y^5}{2x} + \frac{y}{2x}$$

$$= 5x^2y^5 + \frac{y}{2x}$$ Taking this on LHS as there are less y terms

$$\therefore \frac{dy}{dx} - \frac{1}{2x}(y) = 5x^2y^5$$

Dividing y^5.

$$\therefore \frac{1}{y^5}\frac{dy}{dx} - \frac{1}{2x}\left(\frac{1}{y^4}\right) = 5x^2 \rightarrow (1)$$

substituting,

$$t = \frac{1}{y^4}$$

$$\frac{dt}{dx} = \frac{-4}{y^5}\frac{dy}{dx}$$

Dividing by -4

$$\therefore \frac{-1}{4}\frac{dt}{dx} = \frac{1}{y^5}\frac{dy}{dx}$$

substituting in eq^n (1)

$$\frac{-1}{4}\frac{dt}{dx} - \frac{1}{2x}(t) = 5x^2$$

multiplying by -4

$$\frac{dt}{dx} + \frac{4}{2x}(t) = -4(5x^2)$$

$$\frac{dt}{dx} + \frac{2t}{x} = -20x^2$$, linear in t

$P = \frac{2}{x}$, $Q = -20x^2$

$\therefore IF = e^{\int P dx} = e^{\int \frac{2}{x} dx}$

$= e^{2\int \frac{1}{x} dx}$

$\therefore IF = e^{\log x^2}$

$= x^2$

$\therefore t(x^2) = \int -20x^2(x^2)dx + C$

$= -20\int x^4 dx + C$

$\therefore \frac{1}{y^4}(x^2) = -20\left(\frac{x^5}{5}\right) + C$

$\therefore \frac{1}{y^4}(x^2) = -4x^5 + C$

$\therefore x^2 = -4x^5y^4 + y^4C$ //

Example 2:

$$\frac{dy}{dx} = \frac{y}{x - \sqrt{xy}}$$

$$y\,dx = (x - \sqrt{xy})\,dy$$

$$\therefore \frac{dx}{dy} = \frac{x - \sqrt{xy}}{y}$$

$$= \frac{x}{y} - \frac{\sqrt{xy}}{y}$$

$$\therefore \frac{dx}{dx} - \frac{x}{y} = \frac{-\sqrt{xy}}{y}$$

$$= -\sqrt{x} \cdot y^{-1/2} \quad \left(\because \frac{\sqrt{y}}{y} = y^{-1/2}\right)$$

$$\left[\because \frac{dx}{dy} + P(y)\,x = Q(y)\right]$$

Dividing by $\sqrt{x}$

$$\therefore \frac{1}{\sqrt{x}}\frac{dx}{dx} - \frac{x}{\sqrt{x}}\frac{1}{y} = -y^{-1/2}$$

$$\frac{1}{\sqrt{x}}\frac{dx}{dx} - \sqrt{x}\,\frac{1}{y} = -y^{-1/2}$$

substitute

$t = \sqrt{x}$

$$\frac{dt}{dy} = \frac{1}{2\sqrt{x}}\frac{dx}{dy}, \quad 2\frac{dt}{dy} = \frac{1}{\sqrt{x}}\frac{dx}{dx}$$

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$$2\frac{dt}{dy} - \frac{1}{y}t = -y^{-1/2}$$

$$\frac{dt}{dt} - \frac{1}{2y}t = \frac{-1}{2}y^{-1/2}$$, linear in t

$$\therefore I.F = e^{-\int \frac{1}{2y} dy}$$

$$= e^{-\frac{1}{2}\int \frac{1}{y} dy} = e^{\frac{-1}{2}\log y}$$

$$= e^{\log y^{-1/2}}$$

$$\therefore I.F = y^{-1/2}$$

$$\therefore t(y^{-1/2}) = \frac{-1}{2}\int y^{-1/2} y^{-1/2} dy + C$$

$$\therefore \frac{t}{\sqrt{y}} = \frac{-1}{2}\int \frac{1}{y} dy + C$$

$$\therefore \frac{t}{\sqrt{y}} = \frac{-1}{2}\log y + C$$

$$\therefore \frac{\sqrt{x}}{\sqrt{y}} = \log y^{-1/2} + C$$

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Example 3:

$$r\sin\theta - \cos\theta \frac{dr}{d\theta} = r^2$$

$$\cos\theta \frac{dr}{d\theta} = r\sin\theta - r^2$$

$$\therefore \frac{dr}{d\theta} = \frac{r\sin\theta - r^2}{\cos\theta}$$

$$\therefore \frac{dr}{d\theta} - \frac{r\sin\theta}{\cos\theta} = \frac{-r^2}{\cos\theta}$$

Dividing by $-r^2$

$$\therefore \frac{-1}{r^2}\frac{dr}{d\theta} + \frac{1}{r}\frac{\sin\theta}{\cos\theta} = +\frac{1}{\cos\theta}$$

substituting $\frac{1}{r} = t$

$$-\frac{1}{r^2}\frac{dr}{d\theta} = \frac{dt}{d\theta}$$

$$\therefore \frac{dt}{d\theta} + \tan\theta\,(t) = \sec\theta$$

$$I.F = e^{\int \tan\theta\, d\theta}$$

$$= e^{\log\sec\theta}$$

$$= \sec\theta$$

$t(\sec\theta) = \int \sec\theta(\sec\theta)\, d\theta + c$

$\therefore\ t\sec\theta = \tan\theta + c$

$\therefore\ \frac{1}{r}\sec\theta = \tan\theta + c$ //

Example 4:

$$\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$$

$\tan y(e^{x^2}) = \frac{(x^2-1)}{2}e^{x^2}$

$$\frac{dy}{dx} + 2x\sin y\cos y = x^3\cos^2 y$$

Dividing by $\cos^2 y$

$$\frac{1}{\cos^2 y}\frac{dy}{dx} + \frac{2x\sin y\cos y}{\cos^2 y} = x^3$$

$$\sec^2 y\frac{dy}{dx} + 2x\tan y = x^3$$

$t = \tan y$

$\frac{dt}{dx} = \sec^2 y\frac{dy}{dx}$

$\therefore\ \frac{dt}{dx} + 2xy = x^3$, linear in t

$$I.F = e^{\int 2x\,dx} = e^{x^2}$$

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$$\therefore t(e^{x^2}) = \int x^3 e^{x^2} dx + C$$

$$= \int x^2 \cdot x \cdot e^{x^2} dx$$

$u = x^2$
$du = 2x\,dx$
$x\,dx = \frac{1}{2} du$

$$\therefore t(e^{x^2}) = \frac{1}{2}\int u \cdot du \cdot e^u$$

$$= \frac{1}{2}\int e^u u \, du$$

using by parts

$$\therefore t(e^{x^2}) = \frac{1}{2}\left(ue^u - e^u\right)$$

$$= \frac{1}{2} e^u (u-1)$$

$$\tan y(e^{x^2}) = \frac{1}{2} e^{x^2}(x^2-1)$$ //

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2) Practive Problems :

i) $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$

$M = y^4 + 2y$, $N = xy^3 - 2y^4 - 4x$

$\frac{\partial M}{\partial y} = 4y^3 + 2$ $\qquad \frac{\partial N}{\partial x} = y^3 - 4$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{y^3 - 4 - 4y^3 - 2}{y^4 + 2y}$$

$$= \frac{-3y^3 - 6}{y(y^3 + 2)}$$

$$= \frac{-3\cancel{(y^3+2)}}{y\cancel{(y^3+2)}} = \frac{-3}{y}$$

$$\therefore I.F = e^{-\int \frac{3}{y} dy} = e^{-3\log y}$$

$$= y^{-3}$$

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Given Equation,

$$\frac{(y^4+2y)}{y^3}dx + \frac{(xy^3+2y^4-4x)}{y^3}dy = 0$$

$$\left(y+\frac{2}{y^2}\right)dx + \left(x+2y-\frac{4x}{y^3}\right)dy = 0$$

Integrating,

$$\int\left(y+\frac{2}{y^2}\right)dx + \int 2y\,dy = C$$

$$\therefore\ xy + \frac{2x}{y^2} + y^2 = C$$

$$\therefore\ xy^3 + 2x + y^4 = Cy^2$$

ii) $xy(1+xy^2)\frac{dy}{dx} = 1$

iii) $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

iv) $3x^2y\,dx + (y^4 - x^3)\,dy = 0$

v) $6y^2dx - x(2x^3+y)\,dy = 0$

vi) $(x+\tan y)\,dy = (\sin 2y)\,dx$