Prob on combinations using generating fns :-

① How many ways are there to place an order of 12 scoops of icecreams if there are 5 flavours & atmost 4 scoops of each flavour is allowed.

Soln

selectn of 12 scoops s.t each flavours can be selected atmost 4 times

coeff of x^{12} from the gf $f(x)$

$$f(x) = (1+x+x^2+x^3+x^4)^5$$

$$= \left\{\frac{(1-x^5)}{1-x}\right\}^5$$

$$= (1-x^5)^5(1-x)^{-5}$$

$$= \sum {}^5C_r(-x^5)^r \sum {}^{5+r-1}C_r\, x^r$$

$1+x+x^2+\dots = \frac{1}{1-x}$

$1+x+\dots+x^{r-1} = \frac{1-x^r}{1-x}$

$(1+x)^n = \sum {}^nC_r\, x^n$

$(1-x)^{-n} = \sum {}^{n+r-1}C_r\, x^r$

$$f(x) = \Big(\underbrace{\sum (-1)^r\, {}^5C_r\, x^{5r}}_{\text{xles of } 5}\Big)\Big(\sum {}^{5+r-1}C_r\, x^r\Big)$$

coeff of x^{12} : $a_0b_{12} + a_1b_{11} + a_2b_{10} + \dots + a_{12}b_0$

$a_i \to$ coeff x^i

$b_i \to$ coeff of x^i

$\therefore$ The coeff is $\to$ $a_0b_{12} + a_5b_7 + a_{10}b_2$

$a_0 \to$ coeff of x^0 : $r=0$, $r=12$

$a_5 \to$ coeff of x^5 : $r=1$, $r=7$

$r=2$, $r=2$

$$\text{Ans : } \left({}^5C_0\, {}^{5+12-1}C_{12}\right) + \left((-1)\,{}^5C_1\, {}^{5+7-1}C_7\right) + \left({}^5C_2\, {}^{5+2-1}C_2\right)$$

$$\text{Ans} \Rightarrow {}^5C_0\, {}^{16}C_{12} - {}^5C_1\, {}^{11}C_7 + {}^5C_2\, {}^6C_2$$

(2) A man buys 12 oranges for his kids Grace, Mary & Frank. In how many ways can he distribute the oranges s.t Grace gets atleast 4
Mary & Frank get atleast 2, but Frank gets not more than 5?

GMF

Soln

Dist 12 oranges to 3 kids

Pick the coeff of x^{12} from the gf $f(x)$,

$$f(x) = (x^4+x^5+\ldots\ldots)(x^2+x^3+\ldots\ldots)(x^2+x^3+x^4+x^5)$$

$$= x^4(1+x+x^2+\ldots)\,x^2(1+x+x^2+\ldots)\,x^2(1+x+x^2+x^3)$$

$$= x^8(1+x+x^2+x^3)\left((1-x)^{-1}\right)^2$$

$$= x^8\underbrace{(1+x+x^2+x^3)(1-x)^{-2}}_{\text{coeff of } x^4}$$

$$= x^8\left[1(1-x)^{-2} + x(1-x)^{-2} + x^2(1-x)^{-2} + x^3(1-x)^{-2}\right]$$

$$= x^8\left[\sum {}^{2+r-1}C_r x^r + x\sum {}^{2+r-1}C_r x^r + x^2\sum {}^{2+r-1}C_r x^r + x^3\sum {}^{2+r-1}C_r x^r\right]$$

coeff of x^4

$$\text{Ans} = {}^{2+4-1}C_4 + {}^{2+3-1}C_3 + {}^{2+2-1}C_2 + {}^{2+1-1}C_1$$

(3) Find the no of ways to collect \$15 from 20 ppl if each of the first 19 ppl can give one dollar & nothing. And the 20th person can give one dollar & 5 dollars & nothing

solⁿ

coeff of x^{15} from the gf $f(x)$

$$f(x) = (1+x)^{19}(1+x+x^5)$$

$$= (1+x)^{19} + x(1+x)^{19} + x^5(1+x)^{19}$$

$$= \sum {}^{19}C_r x^r + x\sum {}^{19}C_r x^r + x^5 \sum {}^{19}C_r x^r$$

coeff of x^{15} is :-

$$\underline{{}^{19}C_{15} + {}^{19}C_{14} + {}^{19}C_{10}}$$

Generating functions for permutations

w.k.t $(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \dots + {}^nC_n x^n$

$= \sum {}^nC_r x^r$ (ie $\{{}^nC_r\}$ is generated by the fn $(1+x)^n$

Permutations with no repetⁿ

I want to generate $\{{}^nP_r\}$

w.k.t $(1+x)^n = \sum {}^nC_r x^r$, ${}^nC_r = \frac{{}^nP_r}{r!}$

$\therefore (1+x)^n = \sum \frac{{}^nP_r}{r!} x^r$

$= \sum {}^nP_r \left(\frac{x^r}{r!}\right)$

coeff $\frac{x^r}{r!} = {}^nP_r$ ie

$\therefore (1+x)^n$ is the exponential generating function for ${}^nP_r$.

Gf for permutations with repetⁿ

For one obj :- $1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = e^x$

If there are 'n' objs : $(e^x)^n$

$\therefore e^{nx}$ is the exponential gf for permutatⁿˢ with unlimited repetⁿ combⁿ

(enumerators)

↓

coeff of x^r is picked

With no repⁿ $({}^nC_r)$ ⟶ $(1+x)^n$ is the g f

With unlimited rep $({}^{n+r-1}C_r)$ ⟶ $(1-x)^{-n}$ is the g f

permutations

coeff $\frac{x^r}{r!}$ is picked

With no repⁿ $({}^nP_r)$ ⟶ $(1+x)^n$ is exponential g f

With unlimited repⁿ (n^r) ⟶ e^{nx} is the exponential g f

① In how many ways can 4 letters of the word "ENGINE" be arranged? Use gf & solve.

Soln

E N G I

E, N: atmost twice

G, I: atmost once

coeff of $\frac{x^4}{4!}$

$$\left(1+x+\frac{x^2}{2!}\right)^2 (1+x)^2$$

$\boxed{\text{Ans} = 102}$ (manually)

② A ship carries 48 flags, 12 each of the colors white, red, blue & black. 12 of these flags are placed on a vertical pole in order to communicate a signal with other ships,

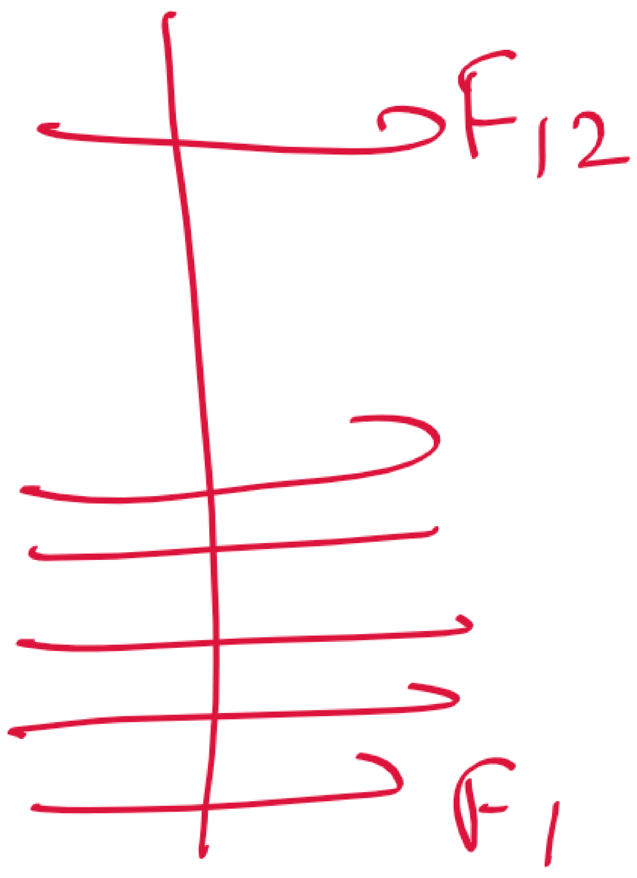
a) How many of these signals use an even no of blue flags & odd no of black flags

b) How many of them use atleast 3 white flags & no white flags at all

Soln

Arrange 12 flags out of 48 flags

W, Blue, Black, Red



a) Even no of Blue & odd no of Black
no restⁿ on White & Red

For white, Red $\longrightarrow \left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\cdots\right)$

For blue $\longrightarrow \left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots\right)$

Black $\longrightarrow \left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}\cdots\cdots\right)$

The gf is

$$f(x)=\left(1+x+\frac{x^2}{2!}+\cdots\right)^2\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)\left(1+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)$$

$$f(x)=\left(e^x\right)^2\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^x-e^{-x}}{2}\right)$$

$$1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots=\frac{e^x+e^{-x}}{2}$$

$$x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots=\frac{e^x+e^{-x}}{2}$$

coeff of $\frac{x^{12}}{12!}$

$$f(x)=\frac{e^{2x}}{4}\left(e^{2x}-e^{-2x}\right)$$

$$f(x)=\frac{1}{4}\left[e^{4x}-e^{0}\right] \quad = \frac{1}{4}\left[e^{4x}-1\right]$$

$$f(x)=\frac{1}{4}\left[\sum\frac{(4x)^r}{r!}-1\right]$$

$\therefore$ coeff of $\frac{x^{12}}{12!} \longrightarrow \frac{1}{4}\left(4^{12}\right) = \underline{\underline{4^{11}}}$

ii) Either atleast 3 white flags & no white flags.

$$\underbrace{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)^3}_{\text{Blue, Black, Red}}\underbrace{\left(1+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots\right)}_{\text{White}}$$

$$\therefore f(x) = e^{3x}\left(e^x - x - \frac{x^2}{2!}\right)$$

$$= e^{4x} - xe^{3x} - \frac{x^2}{2}e^{3x}$$

$$= \sum\frac{(4x)^r}{r!} - x\sum\frac{(3x)^r}{r!} - \frac{x^2}{2}\sum\frac{(3x)^r}{r!}$$

$$x\;\frac{(3x)^{11}}{11!}\times\frac{12!}{12!} \qquad \frac{(3x)^{10}}{10!}\times\frac{12!}{12!}$$

coeff of $\frac{x^{12}}{12!}$

$$\text{Ans} = 4^{12} - \frac{12!}{11!}\,3^{11} - \frac{1}{2}\cdot 3^{10}\cdot\frac{12!}{10!}$$