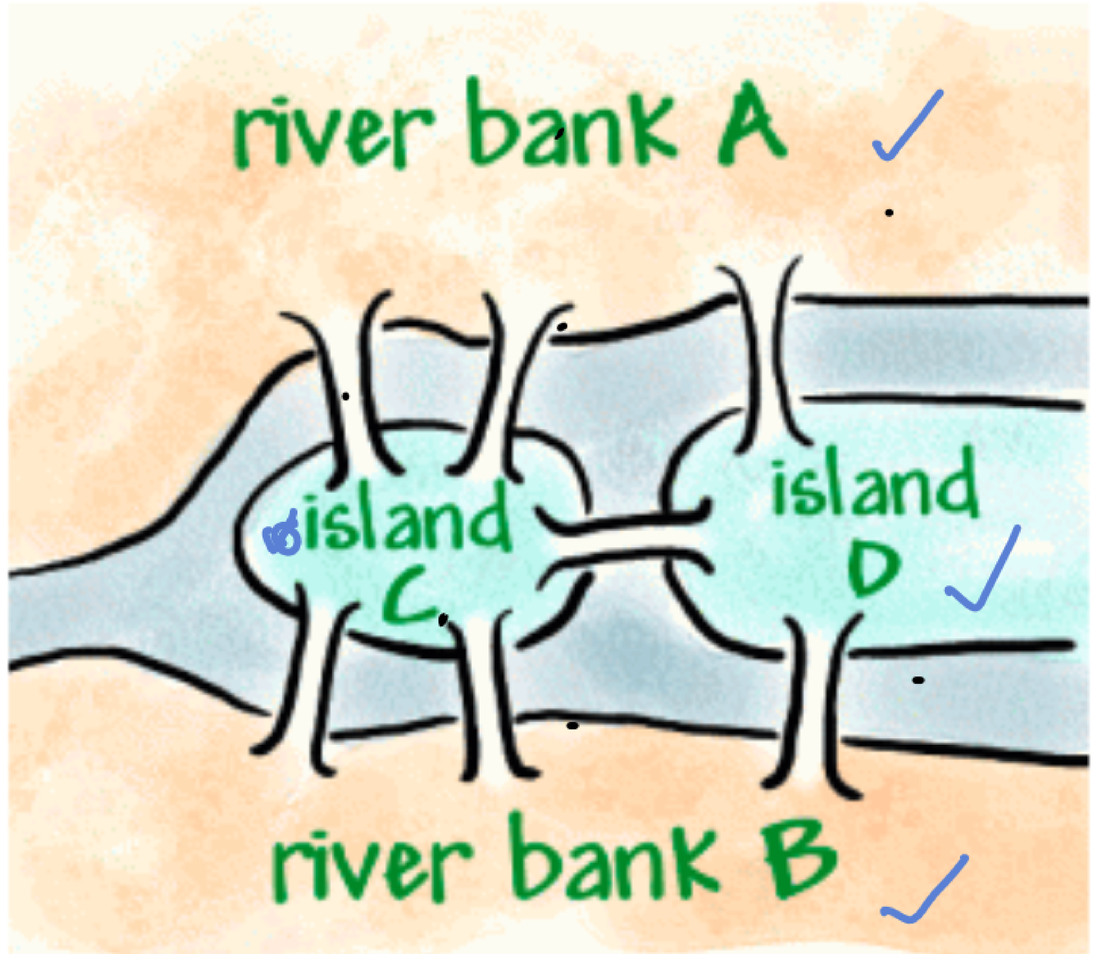
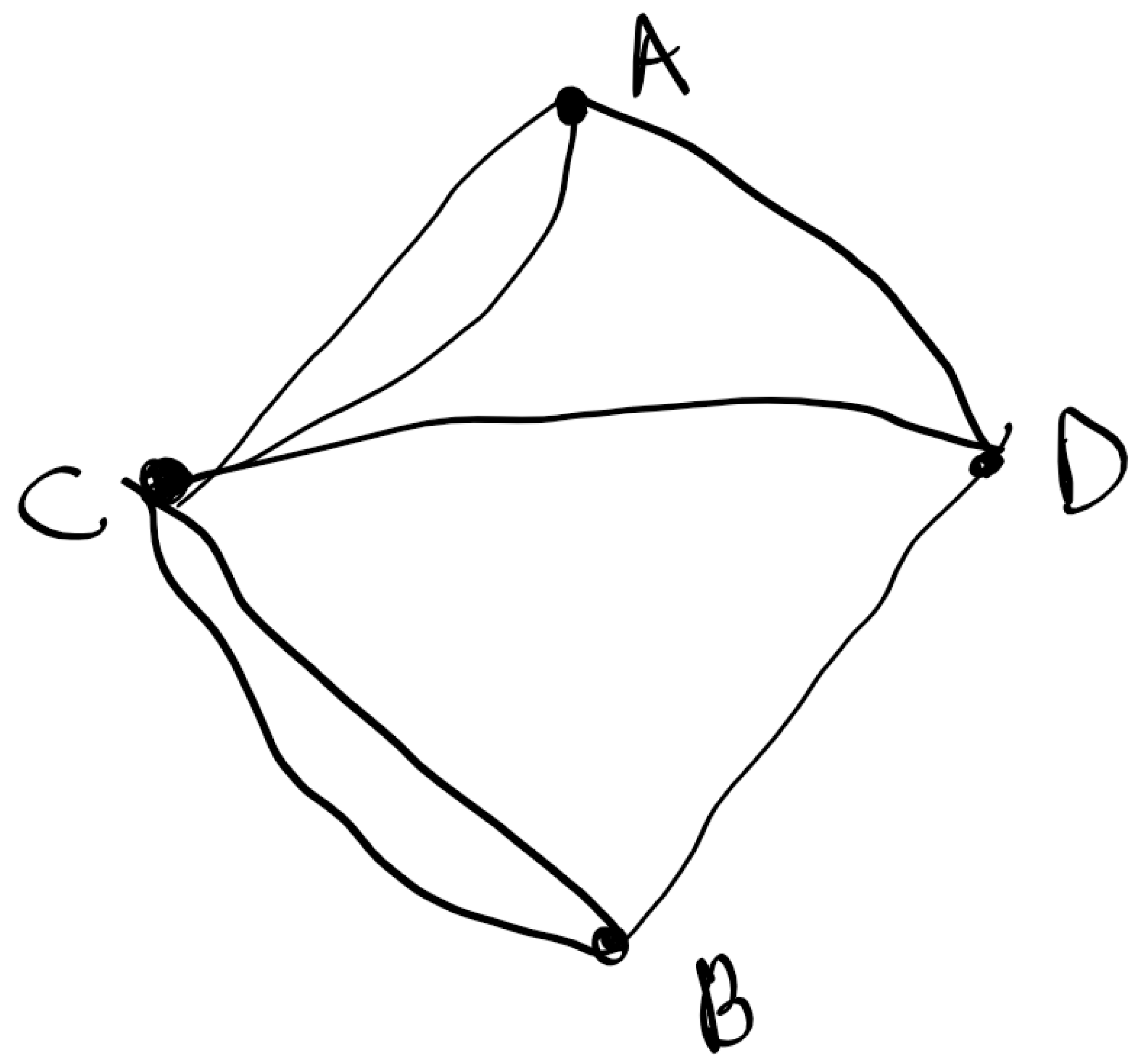
Graph Theory

Konigsberg Bridge problem:



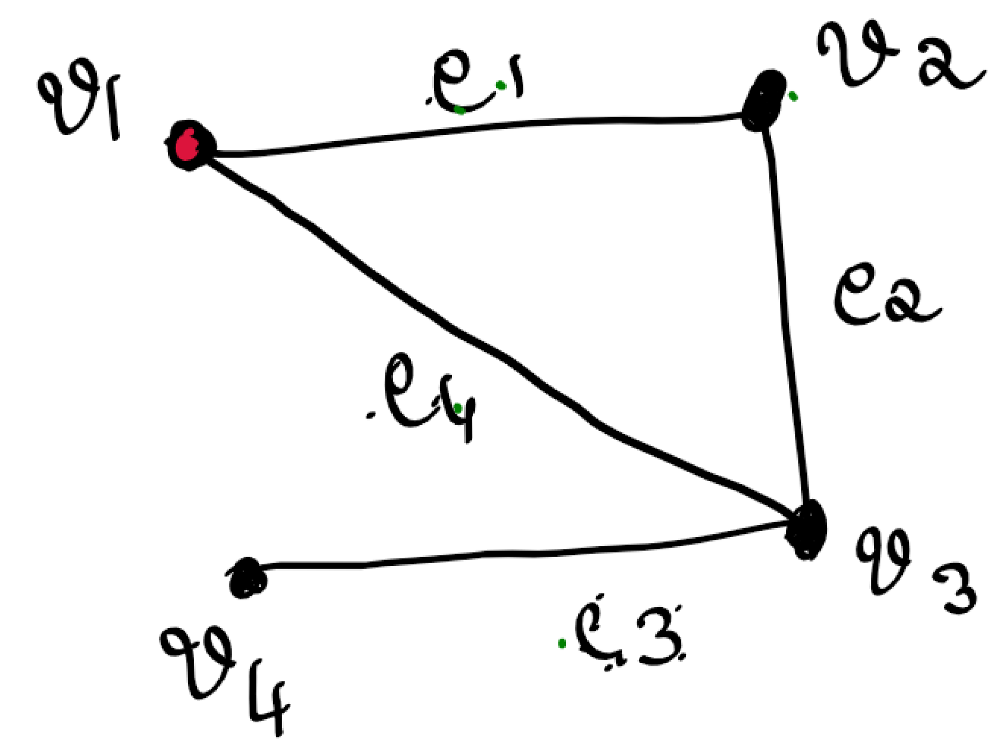
Euler
↓
Father of Graph Theory

Unsolved problem



Graph

A graph $G=(V,E)$ consists of a finite nonempty set $V=V(G)$ whose elts are called 'vertices' of G and a set $E=E(G)$ which contains the unordered pairs of elements of $V(G)$. The elements of the set E are called 'edges'



$V = \{\underbrace{v_1, v_2, v_3, v_4}_{\text{vertices}}\}$

$E = \{\underbrace{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_1, v_3)}_{\text{edges}}\}$

$G = (V, E)$

$(4, 4)$ - graph

$\deg(v_1) = 2$

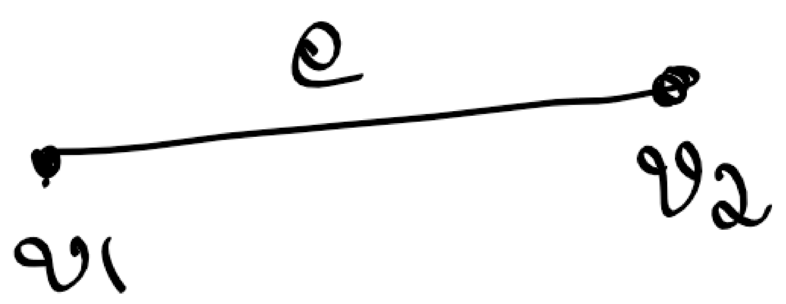
$\deg(v_3) = 3$

* Two vertices in a graph G are said to be 'adjacent' if there is an edge b/wn them

  ex: v_1 and v_2 are adjacent

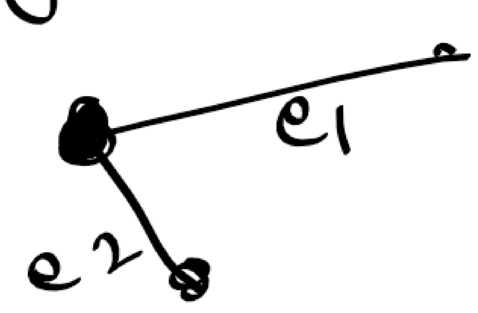
  v_1 & v_4 are not adjacent

* If v_1 & v_2 are the 2 vertices and e is the edge b/wn v_1 & v_2, then we say that the edge 'e' is incident with the vertices v_1 & v_2



* Two edges are said to be adjacent if they have a vertex in common

  ex:-



* A graph with p vertices and q edges is called a (p,q) graph

* **Degree of a vertex :** the no of edges incident with that vertex OR the no of vertices adjacent to that vertex

* Degree of a vertex v is denoted by $d(v)$ & $deg(v)$

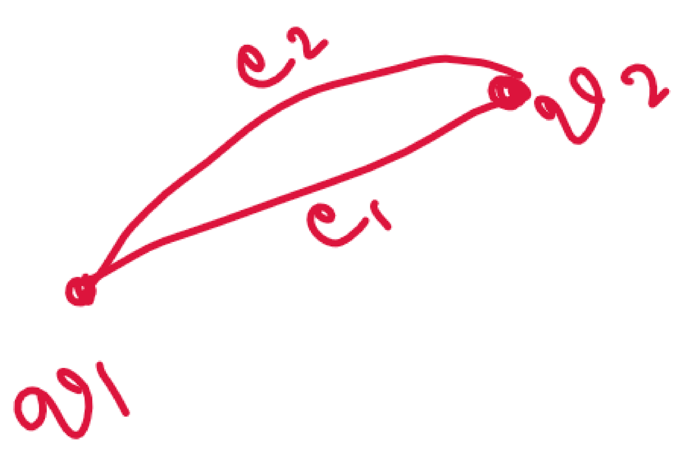
* In a (p,q) graph, $0 \leq deg(v) \leq p-1$

* A vertex v with $deg(v)=0$ is called an isolated vertex

* If G has an isolated vertex, there can't be a vertex of degree $(p-1)$

* **Loop :** If a vertex is joined to itself by an edge, then that edge is called a 'loop'

* **multiple edges :** If 2 & more edges join same pair of vertices, such edges are called multiple edges (parallel edges)

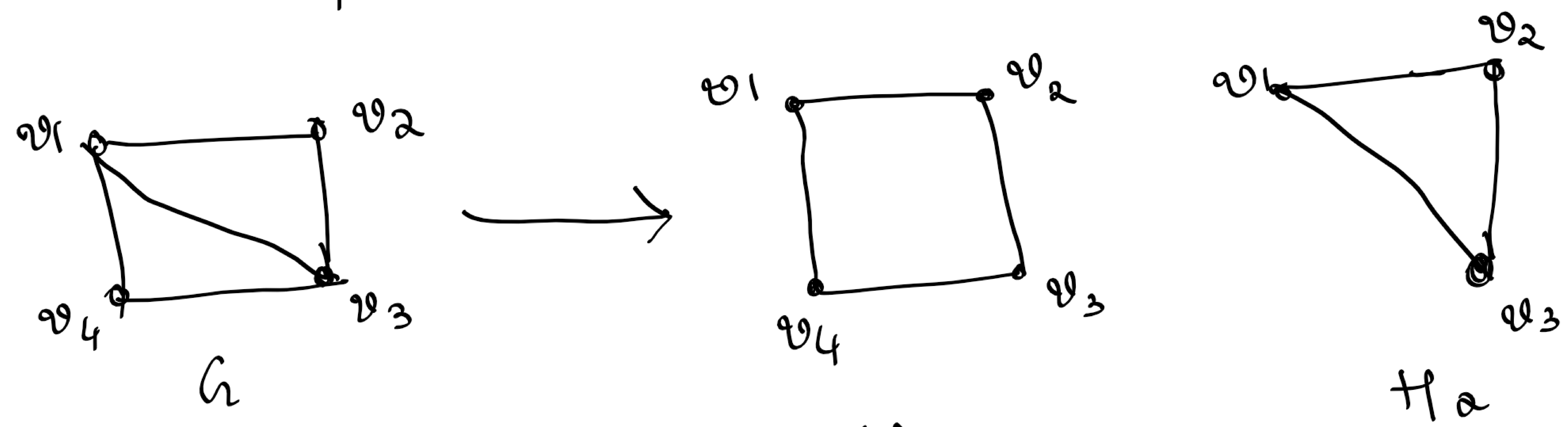


e_1 & e_2 are parallel edges

* **multigraph :** with multiple edges, but no loop
* **pseudograph :** with multiple edges & loops
* **Simple graph :** with no loops & no multiple edges.

Subgraph : Let $G \equiv (V, E)$ be a (p,q) graph. A graph H is said to be a subgraph of G if all the vertices & all the edges of H are in G.

ie $H \equiv (V_1, E_1)$ is a subgraph of $G \equiv (V, E)$ if $V_1 \subseteq V$ and $E_1 \subseteq E$



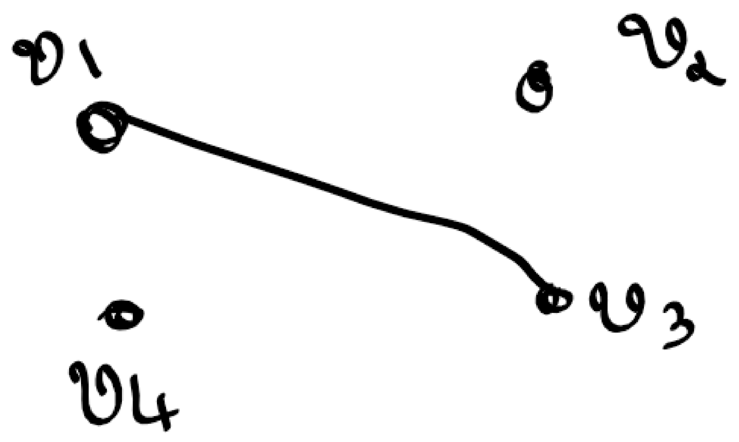
G H₁ H₂

Types of subgphs :-

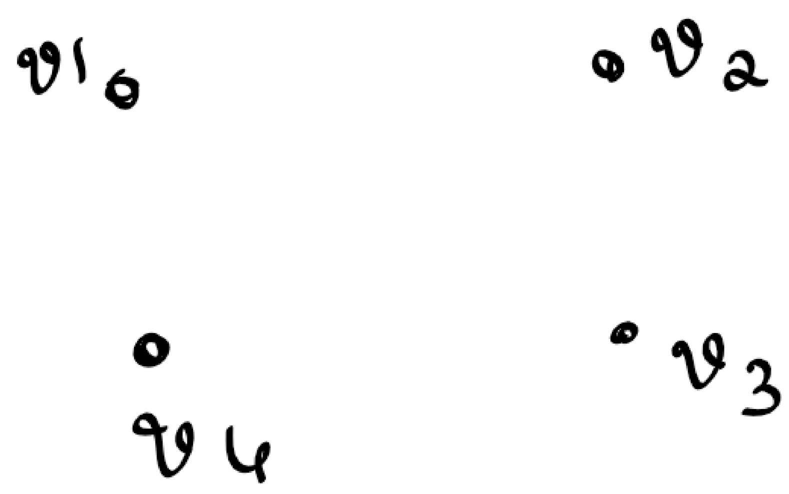
① Spanning subgraph : A spanning subgph of a graph G is a subgraph which contains all the vertices of G.

Ex :- H_1 is a spanning subgraph

H_2 :

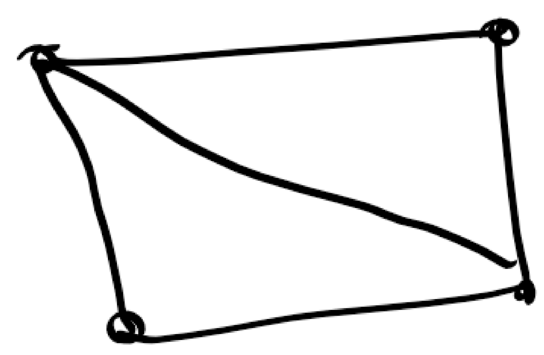


H_3 :



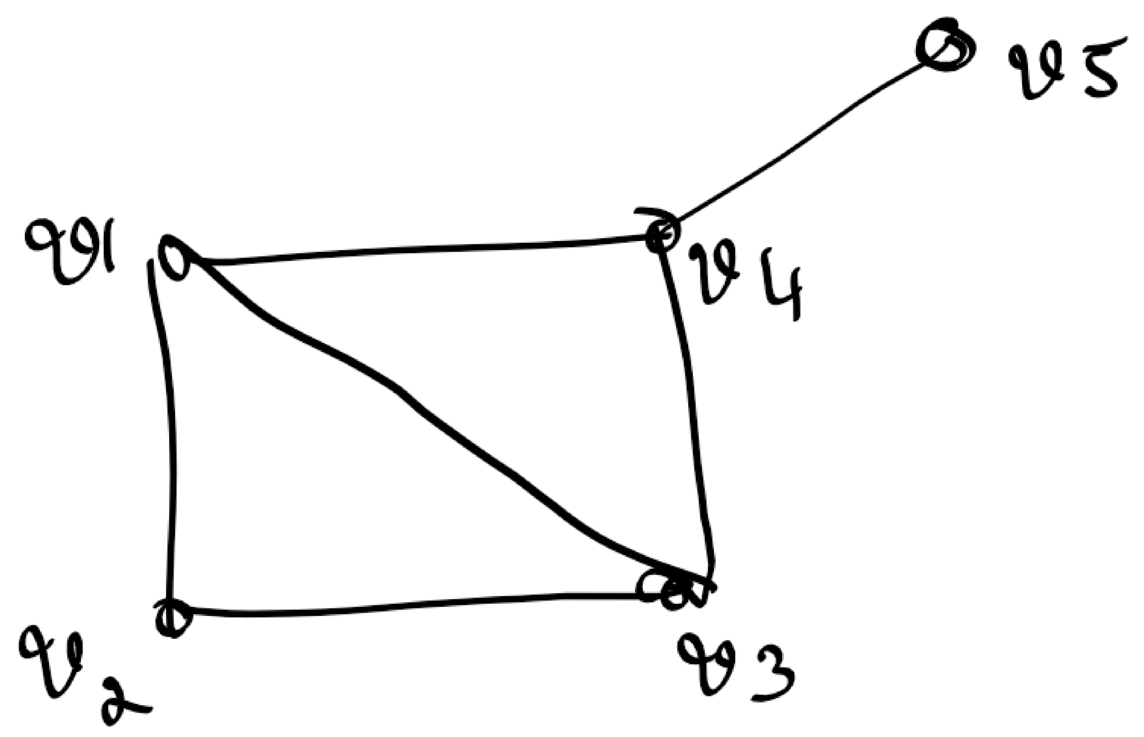
→ Trivial

H_4 :



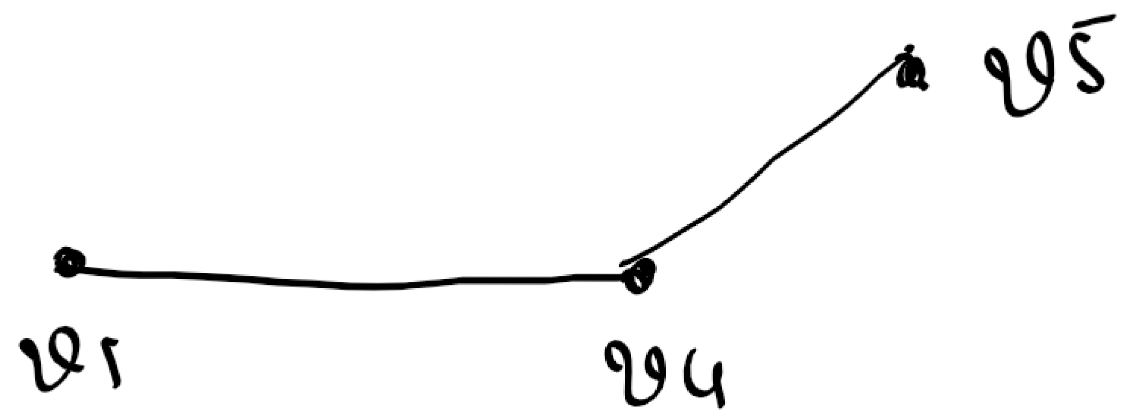
→ Trivial

② **Induced subgraph:** for any set S of vertices of G, the induced subgraph, denoted by $\langle S \rangle$ is the maximal subgraph of G with the vertex set 'S'



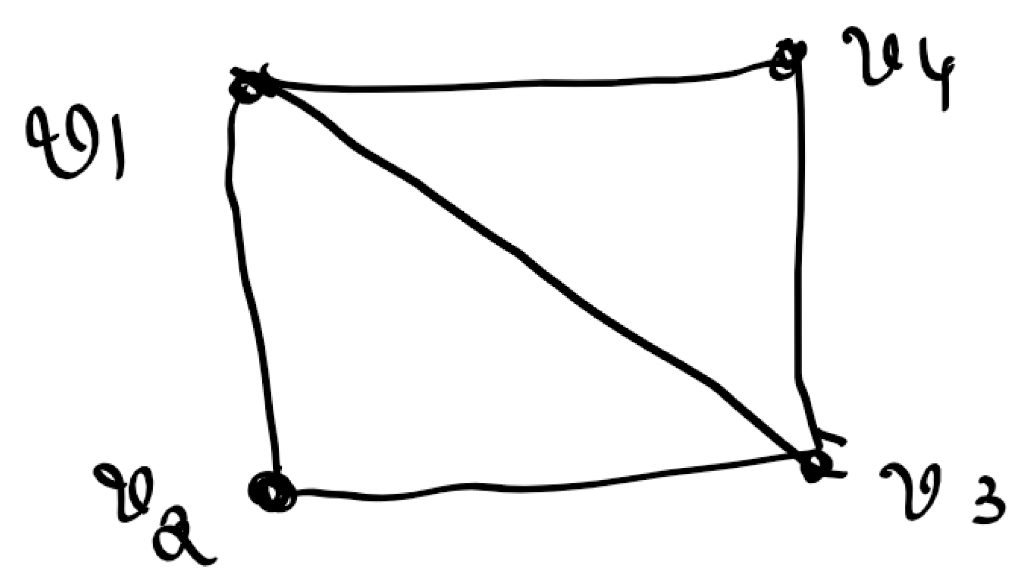
$S_1 = \{v_1, v_5, v_4\}$

$\langle S_1 \rangle$

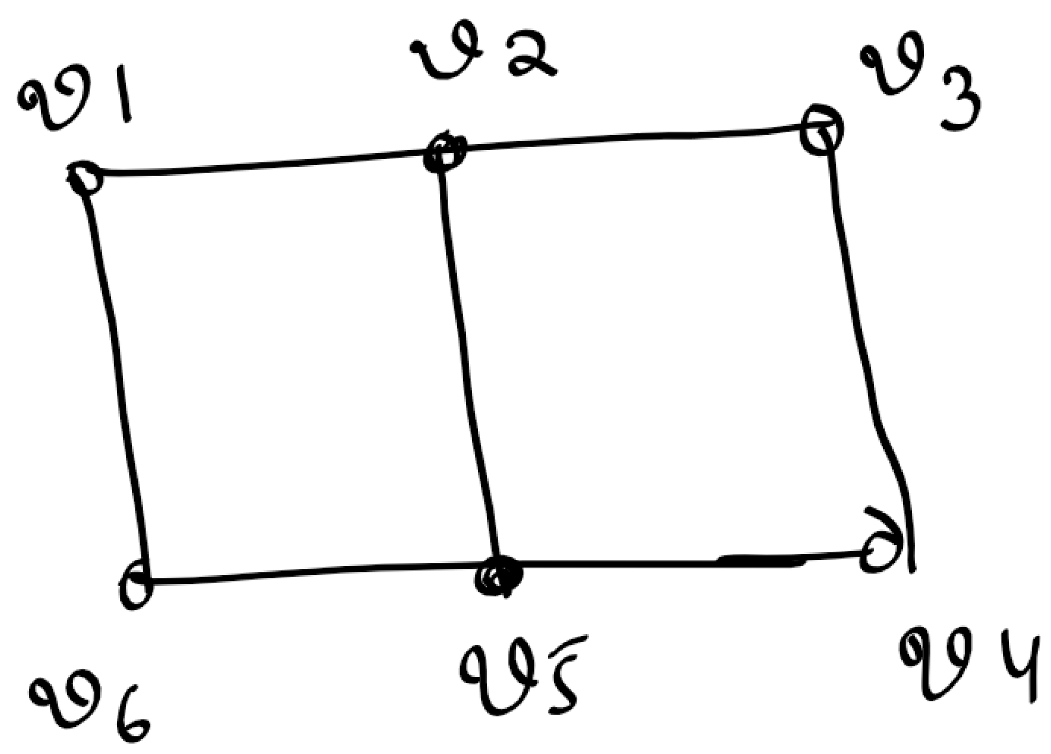


② $S_2 = \{v_1\ v_2\ v_3\ v_4\}$

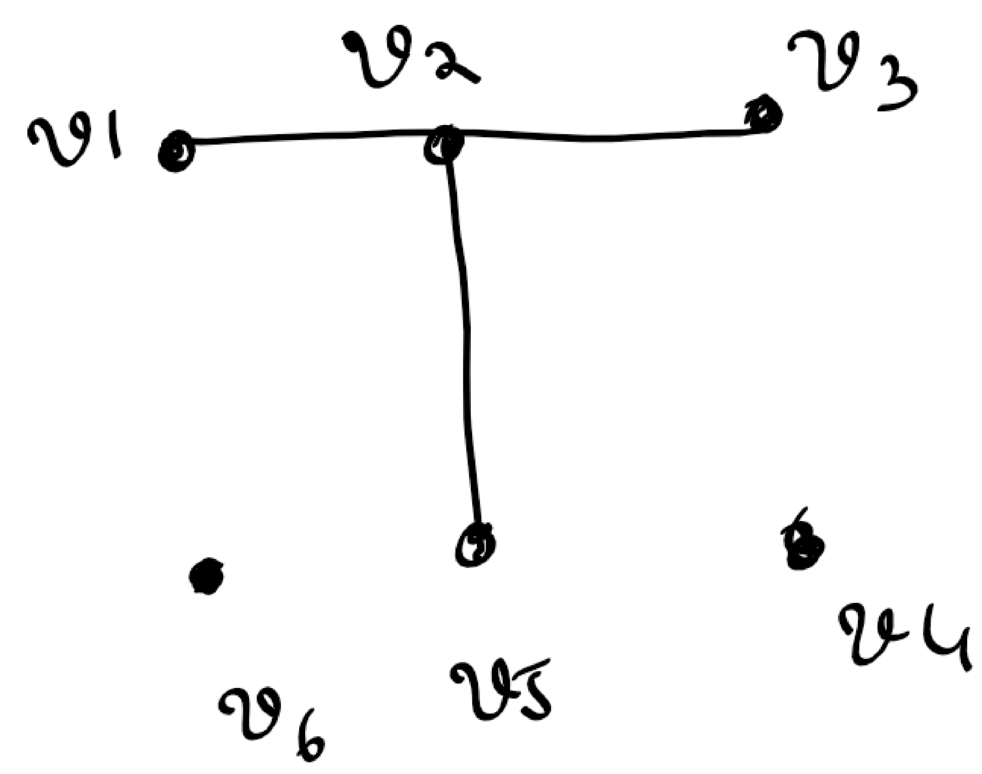
$\langle S_2 \rangle$



ex:-



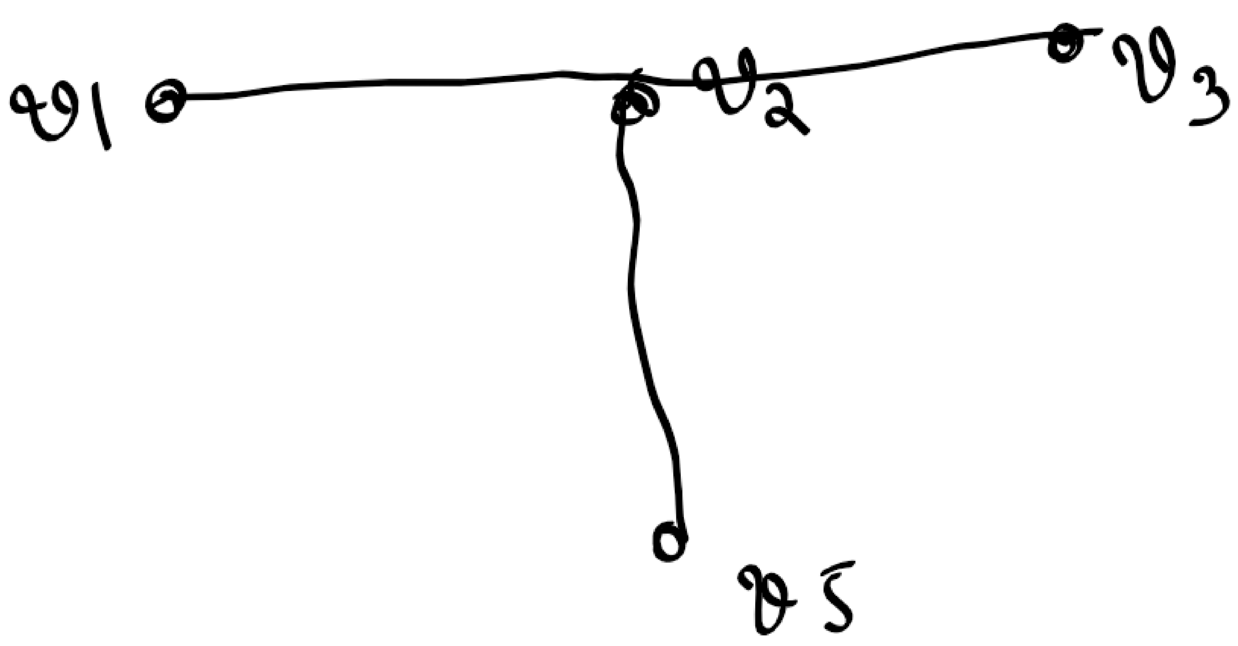
$\underline{\underline{G}}$

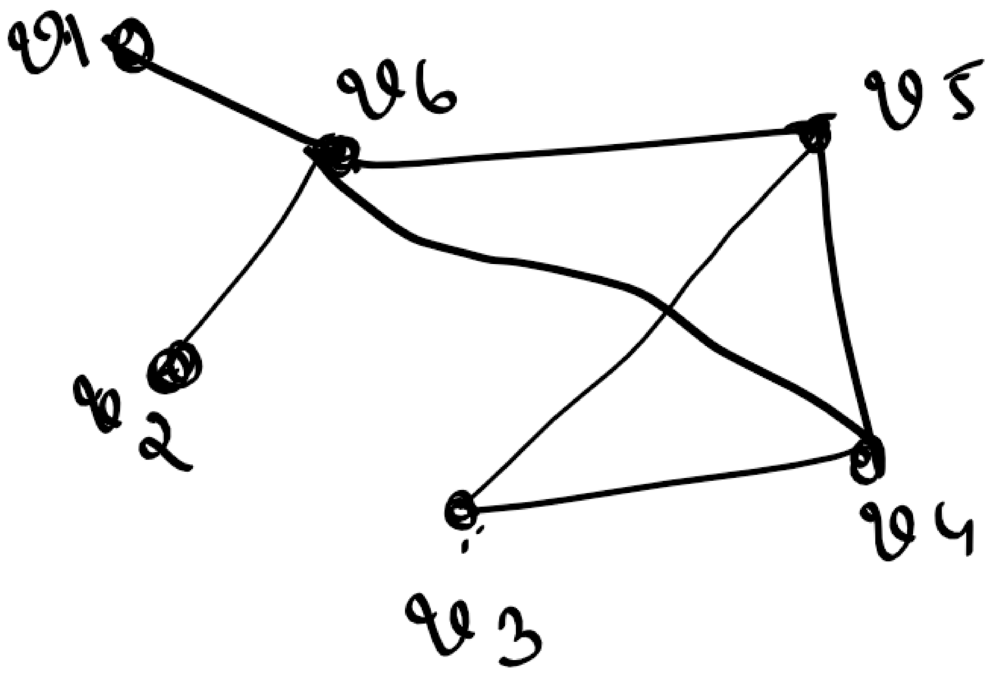


spanning subgraph

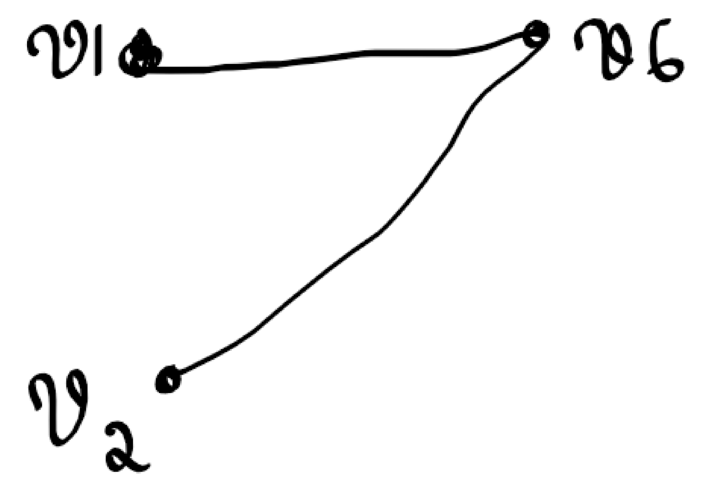
$S = \{v_1, v_2, v_3, v_5\}$

$\langle S \rangle \Rightarrow$



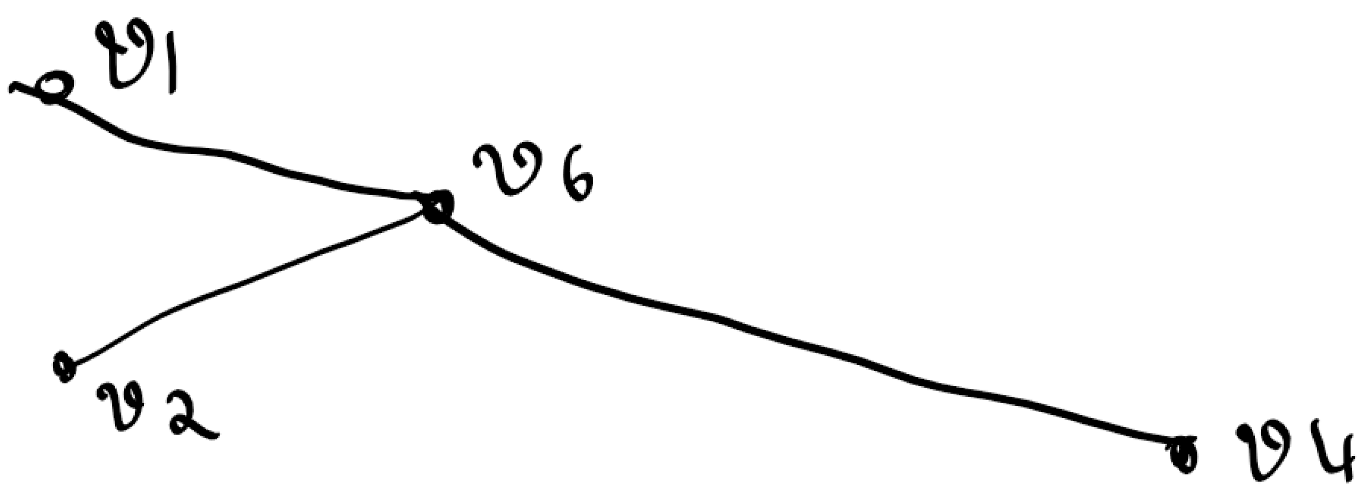


$S = \{ v_1, v_2, v_6 \}$



$S_2 = \{ v_1, v_2, v_6, v_4 \}$

$\langle S_2 \rangle$

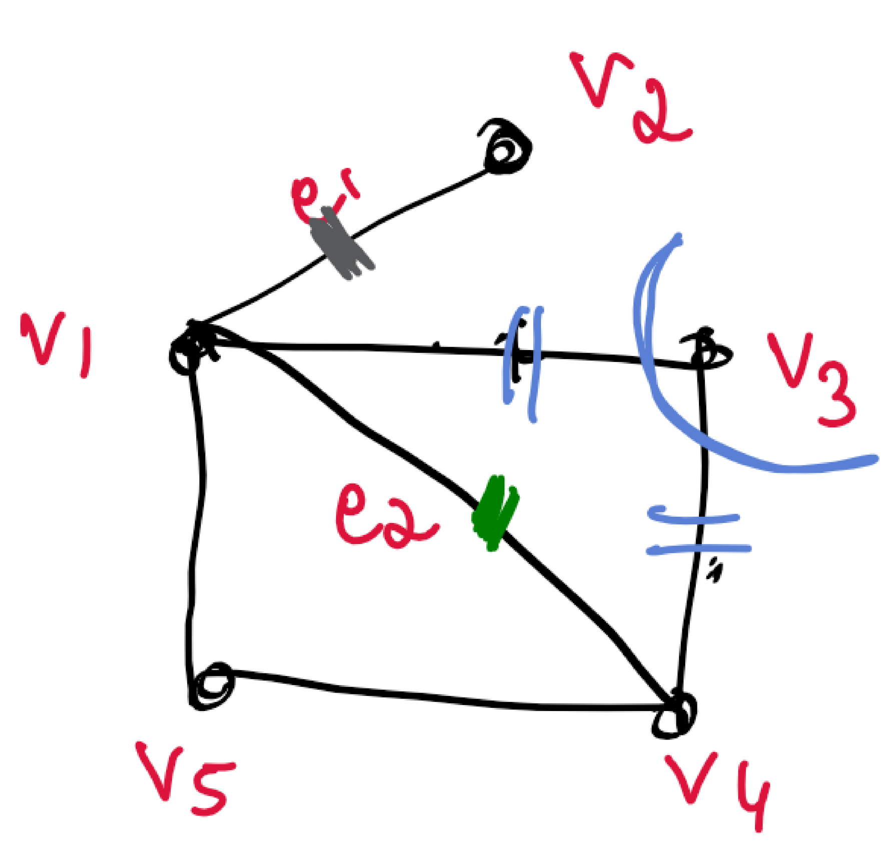


① Removing a vertex from a graph.

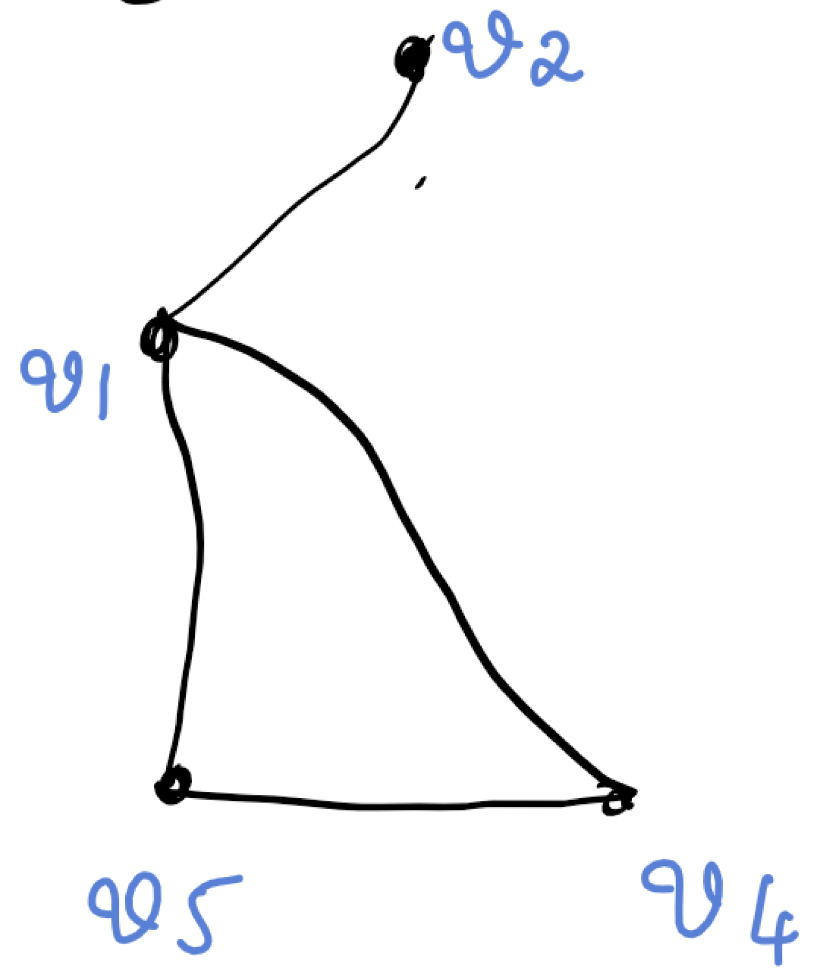
Removing a vertex v from G, results in $G - v$, which contains all the vertices of G expt v & expt the edges incident on v

Removing an edge :

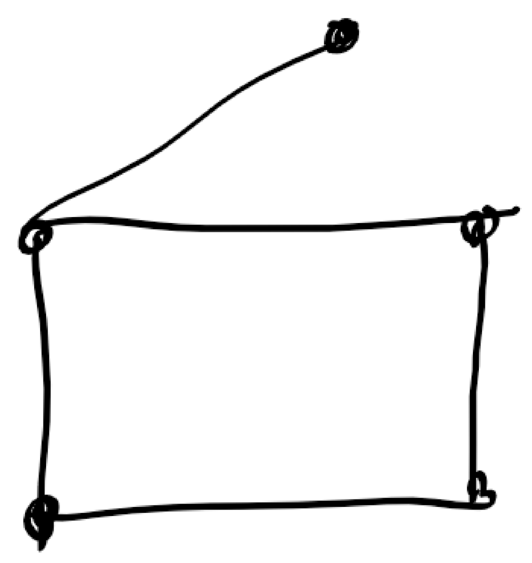
$G - e$ consists of all the vertices of G and all the edges of G expt e



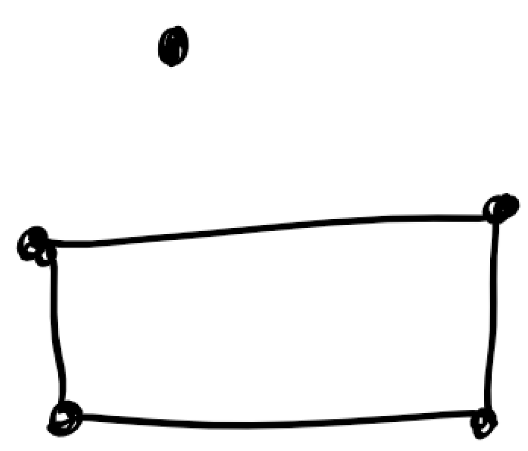
$G - v_3$:



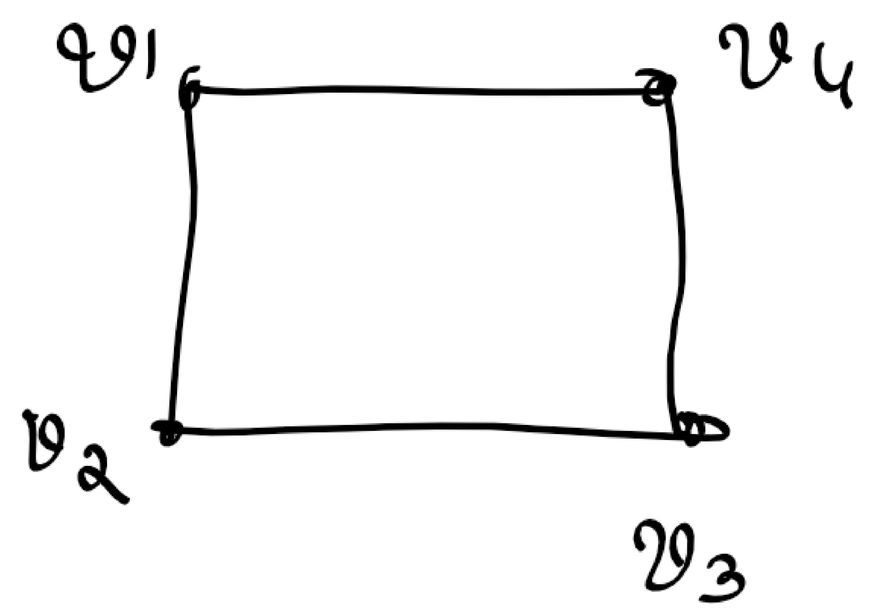
$G - e_2$:



$G - e_1$:

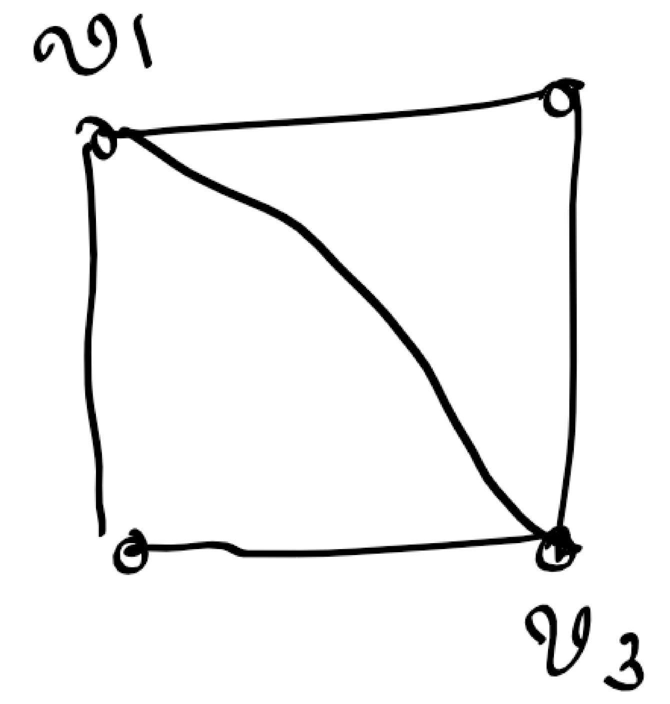


Add^n of an edge :



G

$G + (v_1 v_3)$:



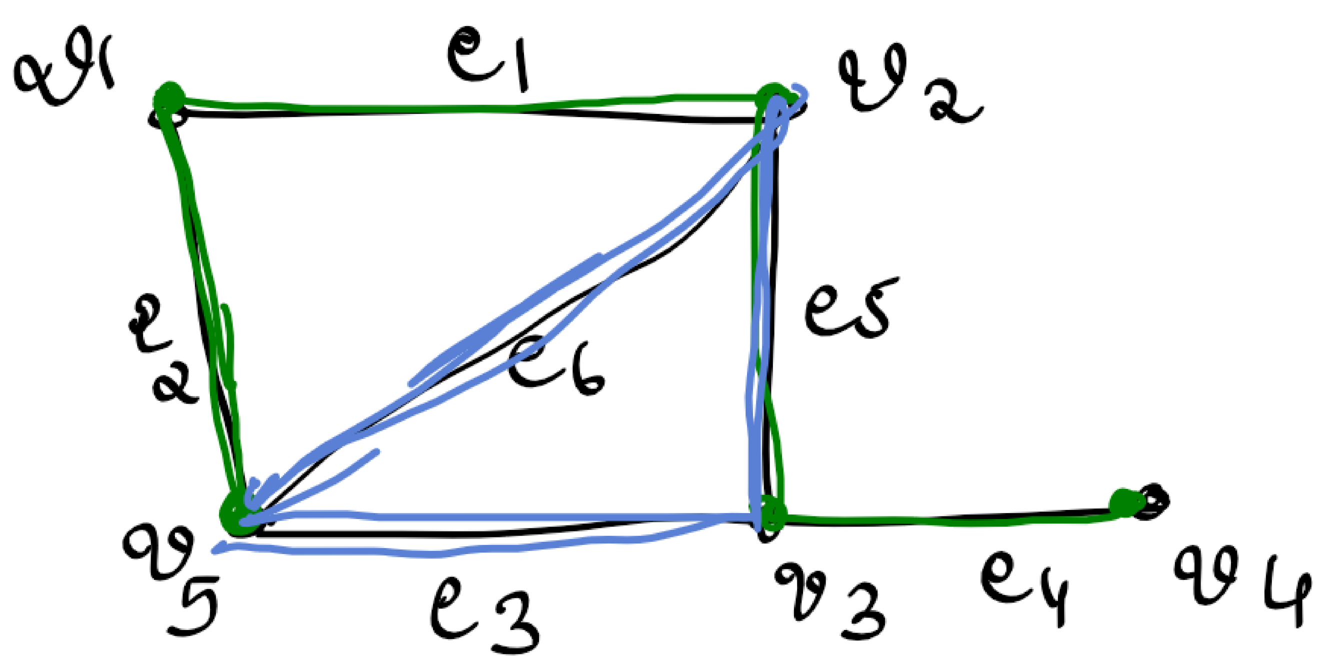
Walk :

A walk in a graph G is an alternating sequence of vertices & edges, beginning & ending with vertices,

$$v_0 e_1 v_1 e_2 v_2 e_3 \ldots \ldots v_{n-1} e_n v_n$$

in which each edge is incident with the two vertices immediately preceding and following it ie

e_i is adj with v_{i-1} & v_i



$v_5 e_2 v_1 e_1 v_2 e_5 v_3 e_4 v_4 \rightarrow$ open

$v_5 e_6 v_2 e_5 v_3 e_3 v_5 \rightarrow$ closed

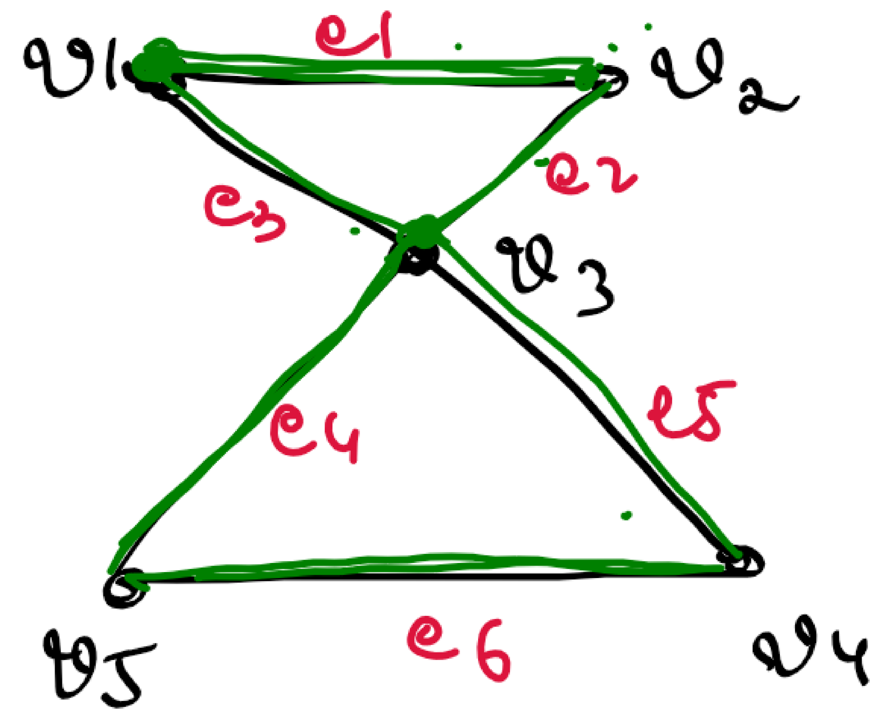
$v_3 e_4 v_4 \rightarrow$ open

closed walk : $v_0 = v_n$

ex:- $v_5 e_6 v_2 e_5 v_3 e_3 v_5$

Trail : A walk in which all the edges are distinct

path : A walk in which all the vertices & all the edges are distinct

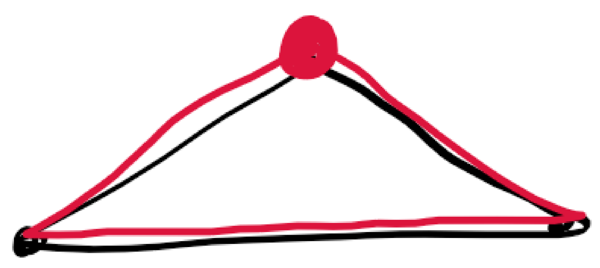


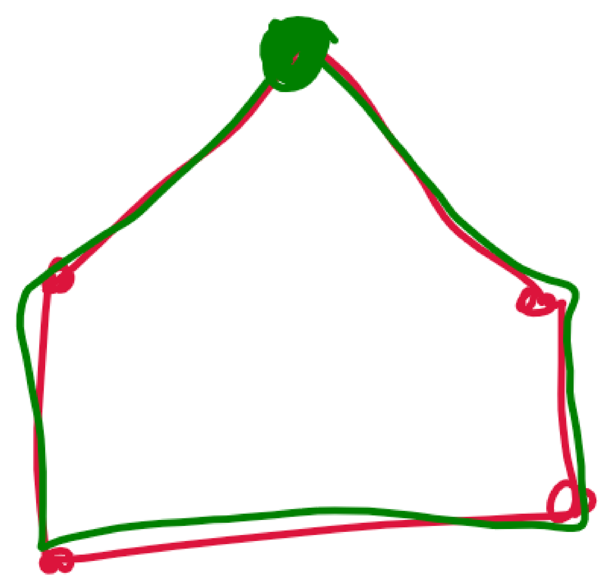
$v_1 e_1 v_2 e_2 v_3 e_4 v_5 e_6 v_4 e_5 v_3 e_3 v_1$

Trail $\longrightarrow$ No rept^n of edges

Path $\longrightarrow$ Not a path
v_3 is repeated

Cycle : A closed path





Length of a walk : No of edges in the walk