ESTIMATION OF PARAMETERS:

In this chapter, we discuss the parameters which are associated with the probability distribution of random variable X.

Definition: Let X be a random variable with some probability distribution depending on an unknown parameter θ. Let $X_1, X_2, X_3, \dots X_n$ be sample of size n taken from distribution of X. If $g(X_1, X_2, X_3, \dots X_n)$ is a function of sample to be used for estimating θ. We refer g as an estimator of θ. The value of g assumes will be refer as an estimate of θ. We write $\hat{\theta} = g(X_1, X_2, X_3, \dots X_n)$.

Definition: Let $\hat{\theta}$ be an estimate for the unknown parameter θ associated with the distribution of random variable X. Then $\hat{\theta}$ is an unbiased estimator for θ if $E(\hat{\theta}) = \theta$, $\forall\ \theta$.

Note: Any good estimate should be close to the value it is estimating, unbiasedness refers the average value of the estimate will be close to the true parameter value.

Definition: Let $\hat{\theta}$ be an estimate of the parameter θ we say that $\hat{\theta}$ is a consistent estimate of θ if $\lim\limits_{n\to\infty} P\{|\hat{\theta} - \theta| > \epsilon\} = 0\ \ \forall \epsilon > 0\ \ or\ \lim\limits_{n\to\infty} P\{|\hat{\theta} - \theta| \le \epsilon\} = 1$.

Note: As sample size increases the estimate becomes 'better' is indicated in above definition.
We shall find unbiasedness and consistent of estimate using the following theorem.

Theorem: Let $\hat{\theta}$ be an estimate of the parameter θ based on a sample size n. If $E(\hat{\theta}) = \theta$, $\lim\limits_{n\to\infty} V(\hat{\theta}) = 0$ then $\hat{\theta}$ is a consistent estimate of θ.

Proof: We shall prove by using Chebyshev's inequality.

$\therefore\ \lim\limits_{n\to\infty} P\{|\hat{\theta} - \theta| > \epsilon\} \le \frac{1}{\epsilon^2} E(\hat{\theta} - \theta)^2 = \frac{1}{\epsilon^2} E\{(\hat{\theta} - \mathrm{E}(\hat{\theta})) + (\mathrm{E}(\hat{\theta}) - \theta)\}^2$ (Add &subtract $\mathrm{E}(\hat{\theta})$)

$= \frac{1}{\epsilon^2}\left\{E[\hat{\theta} - \mathrm{E}(\hat{\theta})]^2 + 2E\{[\hat{\theta} - \mathrm{E}(\hat{\theta})\,](\mathrm{E}(\hat{\theta}) - \theta)\} + \mathrm{E}(\mathrm{E}(\hat{\theta}) - \theta)^2\right\}$

$= \frac{1}{\epsilon^2}\left\{V(\hat{\theta}) + 2E(\hat{\theta} - \mathrm{E}(\hat{\theta})(\mathrm{E}(\hat{\theta}) - \theta) + (\mathrm{E}(\hat{\theta}) - \theta)^2\right\}$

$= \frac{1}{\epsilon^2}\left\{V(\hat{\theta}) + (\mathrm{E}(\hat{\theta}) - \theta)^2\right\} \to 0\ as\ n \to \infty$ using given condition.

$\therefore\ \hat{\theta}$ is a consistent estimate of θ.

Examples:

1. Show that sample mean is an unbiased and consistent estimate of population mean.

Solution: Let $X_1, X_2, \dots X_n$ are the samples taken from the distribution of X having mean value μ.

$\therefore\ \bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$ is the sample mean.

To prove: $E\ (\bar{X}\ ) = \mu \quad and\ V(\bar{X}\ ) = 0.$

Consider $E\ (\bar{X}\ ) = E\left(\frac{\sum_{i=1}^{n} X_i}{n}\right) = \frac{1}{n}.\sum_{i=1}^{n} X_i = \frac{\sum_{i=1}^{n} X_i}{n} = \mu.$

That is, the sample mean is an unbiased estimate of population mean.

$$V(\bar{X}) = V\left(\frac{\sum_{i=1}^{n} X_i}{n}\right) = \frac{1}{n^2}\, V\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} V(X_i) = \frac{nV(X)}{n^2} = \frac{\sigma^2}{n}.$$

$$\therefore \lim_{n\to\infty} V(\bar{X}) = \lim_{n\to\infty} \frac{\sigma^2}{n} = 0$$

That is, the sample mean is consistent estimate of population mean.

2. Show that sample variance S^2 is not an unbiased estimate of population variance.

Solution: Let σ^2 be the variance for the distribution of X. That is, population variance. Let S^2 be the sample variance.

To prove: $E(S^2) \neq \sigma^2$

By definition, $S^2 = \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{n} = \frac{\sum_{i=1}^{n}\left({X_i}^2+(\bar{X})^2-2\bar{X}\,X_i\right)}{n} = \frac{[\sum_{i=1}^{n}({X_i}^2)] + n(\bar{X})^2 - 2n(\bar{X}^2)}{n}$

$$= \frac{\sum_{i=1}^{n}({X_i}^2)}{n} - (\bar{X})^2 \quad \{ since\ \sum_{i=1}^{n}(\bar{X})^2 = n(\bar{X})^2\ and\ \bar{X}\,n = \sum_{i=1}^{n} X_i\ \}$$

Consider, E(S^2) = $E\left\{\frac{\sum_{i=1}^{n}({X_i}^2)}{n} - \overline{(X)}^2\right\} = E\left\{\frac{\sum_{i=1}^{n}({X_i}^2)}{n}\right\} - E\{\overline{(X)}^2\}$

$= \frac{1}{n}\ E\left(\sum_{i=1}^{n}\left({X_i}^2\right)\right) - \{E\overline{(X)}^2\} \quad [\ since\ E(X^2) = V(X) + \mu^2\ , E(\bar{X}^2) = V(\bar{X}) + \mu^2\ ]$

$= \frac{1}{n}\ [n\ (\sigma^2 + \mu^2)] - \{\frac{\sigma^2}{n} + \mu^2\}$ (sample variance is $\frac{\sigma^2}{n}$)

$= (\sigma^2 + \mu^2) - \{\frac{\sigma^2}{n} + \mu^2\} = (\sigma^2) - \left\{\frac{\sigma^2}{n}\right\} = \sigma^2\left(\frac{n-1}{n}\right)$

$\therefore E(S^2) \neq \sigma^2$

3. Show that $\bar{X}$ is a random sample of size n, $f(x,\theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}} & 0 < x < \infty,\ 0 < \theta < \infty \\ 0 & elsewhere \end{cases}$

is an unbiased estimate of θ and has variance $\frac{\theta^2}{n}$.

Solution: First we shall find the mean and variance of X.

$E(X) = \int_{-\infty}^{\infty} x\ f(x,\theta)\, dx\ = \int_0^{\infty} x\ \frac{1}{\theta} e^{-\frac{x}{\theta}}\ dx$

$$= \frac{1}{\theta}\int_0^{\infty} x\, e^{-\frac{x}{\theta}}\ dx\ = \frac{1}{\theta}\ \{\ x\ \frac{e^{-\frac{x}{\theta}}}{\frac{-1}{\theta}}\ - \frac{e^{-\frac{x}{\theta}}}{\frac{1}{\theta^2}}\}_0^{\infty}\ = \frac{\theta^2}{\theta} = \theta$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2\ f(x,\theta)\, dx\ = \int_0^{\infty} x^2\ \frac{1}{\theta} e^{-\frac{x}{\theta}}\ dx$$

$$= \frac{1}{\theta}\int_0^{\infty} x^2\, e^{-\frac{x}{\theta}}\ dx\ = \frac{1}{\theta}\ \{\ x^2\ \frac{e^{-\frac{x}{\theta}}}{\frac{-1}{\theta}}\ - 2x\frac{e^{-\frac{x}{\theta}}}{\frac{1}{\theta^2}}\ +\ 2\frac{e^{-\frac{x}{\theta}}}{\frac{-1}{\theta^3}}\ \}_0^{\infty}\ = \frac{2\theta^3}{\theta} = 2\theta^2$$

$\therefore\ V(X) = 2\theta^2 - \theta^2 = \theta^2$

The random variable X has mean θ and variance θ^2. Hence, $(\bar{X}) = \theta$, $V(\bar{X}) = \frac{\theta^2}{n}$

4. Let $X_1, X_2, \dots X_n$ are the samples taken from a normal distribution with $\mu = 0$ and variance $\sigma^2 = \theta$, $0 < \theta < \infty$. Show that $Y = \frac{\sum x_i^2}{n}$ is an unbiased and consistent estimate of θ.

Solution: Let X has $N(\,0,\ \theta)$ and sample mean $(\bar{X})$ has $N(\,0,\ \frac{\theta}{n})$

Therefore, $(\bar{X}) = 0$, $V(\bar{X}) = \frac{\theta}{n}$

Let $Y = \frac{\sum x_i^2}{n}$.

To prove $E(Y) = 0$ and $V(Y) = 0\ as\ n \to \infty$

Consider $E(Y) = E\left(\frac{\sum X_i^2}{n}\right) = \frac{1}{n} E\left(\sum X_i^2\right) = \frac{n\,E(X^2)}{n} = E(X^2)$

$E(Y) = V(X) + [\,E(X)]^2 = \theta$ which implies Y has unbiased estimate of θ.

We know that, X has (μ, σ^2), $Z = \frac{X-\mu}{\sigma}$ has $N(0, 1)$ and $Z^2 = \left(\frac{X-\mu}{\sigma}\right)^2 = \left(\frac{X-0}{\sqrt{\theta}}\right)^2 = \frac{X^2}{\theta}$ has $\chi^2(1)$

$\therefore E\left(\frac{X^2}{\theta}\right) = 1 \; and \; V\left(\frac{X^2}{\theta}\right) = 2$

This implies $E(X^2) = \theta$, $V(X^2) = 2\theta^2$

Consider $V(Y) = V\left(\frac{\sum X_i^2}{n}\right) = \frac{1}{n^2}, V\left(\sum X_i^2\right) = \frac{n\,V(X^2)}{n^2} = \frac{V(X^2)}{n} = \frac{2\theta^2}{n}$

$\therefore \lim_{n\to\infty} V(Y) = \lim_{n\to\infty} \frac{2\theta^2}{n} = 0$

5. Let Y_1, Y_2 be two independent unbiased statistics for θ. The variance of Y_1 is twice the variance of Y_2. Find the constants k_1 *and* k_2 such that $Y = k_1Y_1 + k_2Y_2$ is an unbiased statistics for θ with smallest possible variance for such a linear combination.

Solution: Given that $E(Y_1) = E(Y_2) = \theta, V(Y_1) = 2V(Y_2) = 2\sigma^2$.

To find k_1 *and* k_2 such that $E(Y) = E(k_1Y_1 + k_2Y_2) = \theta$.

That is, $k_1E(Y_1) + k_2E(Y_2) = \theta \Rightarrow k_1\theta + k_2\theta = \theta \Rightarrow k_1 + k_2 = 1 \Rightarrow k_2 = 1 - k_1$.

Consider $V(Y) = V(k_1Y_1 + k_2Y_2) = {k_1}^2\, V(Y_1) + {k_2}^2\, V(Y_2) = \sigma^2\,(2{k_1}^2 + {k_2}^2)$

$V(Y) = \sigma^2\left(2{k_1}^2 + {k_2}^2\right) = \sigma^2\left(2{k_1}^2 + (1-k_1)^2\right)$

V(Y) has minima if $\frac{dV(Y)}{dk_1} = 0$

$\Rightarrow \frac{dV(Y)}{dk_1} = \frac{d[\sigma^2\left(2{k_1}^2 + (1-k_1)^2\right)]}{dk_1} = 0 \Rightarrow 4k_1 - 2(1-k_1) = 0 \Rightarrow k_1 = \frac{1}{3}$ and $k_2 = \frac{2}{3}$

6. Let $X_1, X_2 \,..X_{25}, Y_1, Y_2 \,...Y_{25}$ be two independent random samples from the distribution $N(3,16)$, $N(4,9)$ respectively. Evaluate P $\left(\frac{\bar{X}}{\bar{Y}} > 1\right)$

Solution: Let $X \sim N(3,16), Y \sim N(4,9)$. Then, $\bar{X} \sim N\left(3, \frac{16}{25}\right)$, $\bar{Y} \sim N(4, \frac{9}{25})$

Now $\frac{\bar{X}}{\bar{Y}} > 1 \Rightarrow \bar{X} > \bar{Y} \Rightarrow \bar{X} - \bar{Y} > 0$. Since, $\bar{X} - \bar{Y} \sim N[\,3\times 1 + 4\times(-1),\; 1^2 \times \frac{16}{25} + (-1)^2 \times \frac{9}{25}] \sim N(-1,1)$ and Z= $\bar{X} - \bar{Y} + 1 \sim N(0,1)$

Consider P $\left(\frac{\bar{X}}{\bar{Y}} > 1\right) = P(\bar{X} - \bar{Y} > 0) = P(\bar{X} - \bar{Y} + 1 > 1) = P(Z > 1)$

$= 1 - \Phi(1)$=1- 0.841=0.159

Interval estimation:

Let 'X' be a random variable with some probability distribution, depending on an unknown parameter θ. An estimate of θ given by two magnitudes within which θ can lie is called an interval estimate of the parameter θ. The process of obtaining an interval estimate for θ is called interval estimation.

Let θ be an unknown parameter to be determined by a random sample $X_1, X_2, X_3, \dots X_n$ of size n. The confidence interval for the parameter θ is a random interval containing the parameter with high probability say $1-\alpha$; $1-\alpha$ is called confidence coefficient.

Suppose that $P\{(H(X_1, X_2, \dots X_n) < \theta < G(X_1, X_2, \dots X_n)\} = 1-\alpha$ then $\{(H(X_1, X_2, \dots X_n), G(X_1, X_2, \dots X_n)\}$ is a $[(1-\alpha)\times 100]\%$ confidence interval.

Note: Let $X_1,\ X_2 \dots X_n$ be a random sample of size n from a normal distribution $N(\mu,\ \sigma^2)$.

1. $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} \sim N(\mu, \frac{\sigma^2}{n})$ for μ, σ^2 is known.
2. $T = \frac{\bar{X}-\mu}{\frac{S}{\sqrt{n-1}}} \sim T(n-1)$ for μ, σ^2 is unknown.
3. $Y = \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\sigma^2} \sim \chi^2$(n) for σ, μ is known
4. $Y = \frac{nS^2}{\sigma^2} \sim \chi^2$(n-1) for σ, μ is unknown

Confidence Interval for mean:

1) σ^2 is known: Consider $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} \sim N(\mu, \frac{\sigma^2}{n})$

$$\therefore Z = \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$

To find a such that $P(-a < Z < a) = 1-\alpha$

$$\Rightarrow P\left(-a < \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} < a\right) = 1-\alpha$$

$$\Rightarrow P\left(\bar{X} - \frac{a\sigma}{\sqrt{n}} < \mu < \bar{X} + \frac{a\sigma}{\sqrt{n}}\right) = 1-\alpha$$

$$\Rightarrow \mu \in \left(\bar{X} - \frac{a\sigma}{\sqrt{n}}, \quad \bar{X} + \frac{a\sigma}{\sqrt{n}}\right)$$

Examples:

1. Let the observed value of $\bar{X}$ of size 20 from a normal distribution with μ and σ^2=80 be 81.2. Obtain 95% confidence interval for the mean μ.

Solution: Let $X \sim N(\mu,\ 80)$, $\bar{X} \sim N\left(\mu, \frac{80}{20}\right) = N(\mu, 4)$

$$\therefore Z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\bar{X} - \mu}{2} \sim N(0,1)$$

$P(-a < Z < a) = 0.95 \Rightarrow 2\,\Phi(a)\text{-}1\text{=}0.95 \Rightarrow \Phi(a)\text{=}\frac{1.95}{2} = 0.975 \Rightarrow a = 1.96$

$$\Rightarrow \mu \in \left(\bar{X} - \frac{a\sigma}{\sqrt{n}}, \quad \bar{X} + \frac{a\sigma}{\sqrt{n}}\right) \in (81.2 - 1.96 \times 2, \quad 81.2 + 1.96 \times 2)$$

$\Rightarrow \mu \in (77.28, 85.12)$

2. Let the observed value of $\bar{X}$ of size n from a normal distribution with μ and σ^2= 9. Find n such that $P(\bar{X} - 1 < \mu < \bar{X} + 1) = 0.9$ approximately.

Solution: Let $X \sim N(\mu,\ 9)$, $\bar{X} \sim N\left(\mu,\ \frac{9}{n}\right)$

$$\therefore Z = \frac{\bar{X} - \mu}{\frac{3}{\sqrt{n}}} \sim N(0,1)$$

$\therefore P(-a < Z < a) = 0.9 \Rightarrow 2\,\Phi(a)\text{-}1\text{=}0.9 \Rightarrow \Phi(a)\text{=}\frac{1.9}{2} = 0.95 \Rightarrow a = 1.65$

Given that, $P(\bar{X} - 1 < \mu < \bar{X} + 1) = 0.9 \Rightarrow P(1 - \bar{X} > -\mu > -\bar{X} - 1) = 0.9$

$\Rightarrow P(1 > \bar{X} - \mu > -1) = 0.9$

$\Rightarrow P(-1 < \bar{X} - \mu < 1) = 0.9$

$$\Rightarrow P\left(\frac{-1}{3/\sqrt{n}} < \frac{\bar{X} - \mu}{3/\sqrt{n}} < \frac{1}{3/\sqrt{n}}\right) = 0.9$$

$$\Rightarrow P\left(\frac{-1}{3/\sqrt{n}} < Z < \frac{1}{3/\sqrt{n}}\right) = 0.9$$

$$\Rightarrow 2\Phi\left(\frac{\sqrt{n}}{3}\right) - 1 = 0.9$$

$$\Rightarrow \Phi\left(\frac{\sqrt{n}}{3}\right) = {}^{1.9}/_{2} = 0.95$$

$$\Rightarrow \left(\frac{\sqrt{n}}{3}\right) = 1.65 \;\Rightarrow \sqrt{n} = 4.95 \;\Rightarrow n = 24.5025 \cong 25$$

2) σ^2 is unknown:

Consider $T = \frac{\bar{X}-\mu}{\frac{S}{\sqrt{n-1}}} \sim T(n-1)$ is in t- distribution with (n-1) degrees of freedom.

To find a such that $P(-a < T < a) = 1 - \alpha$

$$\Rightarrow P\left(-a < \frac{\bar{X}-\mu}{\frac{S}{\sqrt{n-1}}} < a\right) = 1 - \alpha$$

$$\Rightarrow P\left(\bar{X} - \frac{aS}{\sqrt{n-1}} < \mu < \bar{X} + \frac{aS}{\sqrt{n-1}}\right) = 1 - \alpha$$

$$\Rightarrow \mu \in \left(\bar{X} - \frac{aS}{\sqrt{n-1}}, \qquad \bar{X} + \frac{aS}{\sqrt{n-1}}\right)$$

Examples:

1. Let a random sample of size 17 from $N(\mu,\, \sigma^2)$ yields $\bar{X}$=4.7 and S^2=5.76. Determine 90% confidence interval for μ.

Solution: Given that $n = 17, \bar{X}$=4.7 and S^2=5.76

Let $T = \frac{\bar{X}-\mu}{\frac{S}{\sqrt{n-1}}} = \frac{4(\bar{X}-\mu)}{\sqrt{5.76}} \sim T(17-1) \sim T(16)$

To find a such that $P(-a < T < a) = 0.90 \Rightarrow 2\Phi(a) - 1 = 0.90 \Rightarrow a = 1.75$

$$\therefore \mu \in \left(4.7 - \frac{1.75 \times \sqrt{5.76}}{\sqrt{16}}, \qquad 4.7 + \frac{1.75 \times \sqrt{5.76}}{\sqrt{16}}\right) \Rightarrow \mu \in (3.65, 5.75)$$

Confidence interval for variance:

1) μ is known:

Let $Y = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{\sigma^2} \sim \chi^2$(n)

To find a and b such that $P(a < Y < b) = 1 - \alpha$

$\Rightarrow P(Y < a) = \frac{\alpha}{2}$, $P(Y > b) = \frac{\alpha}{2}$

$\therefore P\left(a < \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{\sigma^2} < b\right) = 1 - \alpha$

$$\Rightarrow P\left(\frac{1}{b} < \frac{\sigma^2}{\sum_{i=1}^{n}(X_i - \bar{X})^2} < \frac{1}{a}\right) = 1 - \alpha$$

$$\Rightarrow P\left(\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{b} < \sigma^2 < \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{a}\right) = 1 - \alpha$$

$\Rightarrow \sigma^2 \in \left(\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{b}, \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{a}\right)$ where $\mu = \frac{\sum X_i}{n} = \bar{X}$

Examples: 1. If 8.6, 7.9, 8.3, 6.4, 8.4, 9.8, 7.2, 7.8, 7.5 are the observed values of a random sample of size 9 from a distribution $N(8, \sigma^2)$, construct 90% confidence interval for σ^2.

Solution: Let $\mu = 8, n = 9$ and $Y = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{\sigma^2} = \frac{1}{\sigma^2}\{(0.6)^2 + (0.1)^2 + (0.3)^2 + (1.6)^2 + (0.4)^2 + (1.8)^2 + (0.8)^2 + (0.2)^2 + (0.5)^2\} = \frac{7.35}{\sigma^2} \sim \chi^2(9)$

To find a and b such that $P(a < Y < b) = 1 - \alpha = 0.90 \Rightarrow \alpha = 0.10$

$\Rightarrow P(Y < a) = \frac{\alpha}{2} = \frac{0.10}{2} = 0.05 \Rightarrow a = 3.33$, $P(Y > b) = \frac{\alpha}{2} = \frac{0.10}{2} = 0.05$

$\Rightarrow (Y < b) = 1 - 0.05 = 0.95 \Rightarrow b = 16.9$ using chi square table for 9 degrees of freedom.

$\therefore \sigma^2 \in \left(\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{b}, \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{a}\right) = \left(\frac{7.35}{16.9}, \frac{7.35}{3.33}\right) = (0.43, 2.21)$

2) μ is unknown:

Let $Y = \frac{nS^2}{\sigma^2} \sim \chi^2(n-1)$

To find a and b such that $P(a < Y < b) = 1 - \alpha$

$\Rightarrow P(Y < a) = \frac{\alpha}{2}$, $P(Y > b) = \frac{\alpha}{2}$

$\therefore P\left(a < \frac{nS^2}{\sigma^2} < b\right) = 1 - \alpha$

$$\Rightarrow P\left(\frac{1}{b} < \frac{\sigma^2}{nS^2} < \frac{1}{a}\right) = 1 - \alpha$$

$$\Rightarrow P\left(\frac{nS^2}{b} < \sigma^2 < \frac{nS^2}{a}\right) = 1 - \alpha$$

$$\Rightarrow \ \sigma^2 \in \left(\frac{nS^2}{b}, \frac{nS^2}{a}\right)$$

Examples:

1. A random sample of size 15 from a normal distribution $N(\mu,\ \sigma^2)$ yields $\bar{X} = 3.2$, $S^2 = 4.24$. Determine a 90% confidence interval for σ^2.

Solution: Given that, $1 - \alpha = 0.9 \Rightarrow \alpha = 0.1$ & $\frac{\alpha}{2} = 0.05$, $Y = \frac{nS^2}{\sigma^2} \sim \chi^2(15-1) = \chi^2(14)$

$$\therefore P(Y < a) = \frac{\alpha}{2} = \frac{0.10}{2} = 0.05 \Rightarrow a = 6.57 \text{ , } P(Y > b) = \frac{\alpha}{2} = \frac{0.10}{2} = 0.05$$

$\Rightarrow \ (Y < b) = 1 - 0.05 = 0.95 \Rightarrow b = 23.7$ using chi square table for 14 degrees of freedom.

$$\therefore \sigma^2 \in \left(\frac{15 \times 4.24}{23.7}, \frac{15 \times 4.24}{6.57}\right) = (2.68, 9.68)$$

Extra Problems: 1. A random sample of size 9 from a normal distribution $N(\mu,\ \sigma^2)$ yields $S^2 = 7.63$. Determine a 95% confidence interval for σ^2.

ANS: $\sigma^2 \in (3.924, 31.5)$

2. A random sample of size 15 from a normal distribution $N(\mu,\ \sigma^2)$ yields $\bar{X} = 3.2$, $S^2 = 4.24$. Determine a 95% confidence interval for μ.

   ANS: $\mu \in (2.02, 4.38)$

3. A random sample of size 25 from a normal distribution $N(\mu, 4)$ yields $\bar{X} = 78.3$, $S^2 = 4.24$. Determine a 99% confidence interval for μ.

   ANS: $\mu \in (77.268, 79.332)$