Recall :- E^n or $\mathbb{R}^n$ → set of n-dimensional vector
+ , · closed in $\mathbb{R}^n$

1. Vector Space

Definition 1.1. Let V be a non empty set and F be the field of Scalars. Define two operations on Vas below; ($\mathbb{R}$ or $\mathbb{C}$)

addition - $+ : V \times V \to V$ by $+(u, v) \mapsto u + v \in V$
scalar multiplication - $\cdot : F \times V \to V$ by $\cdot(\alpha, u) \mapsto \alpha \cdot u \in V$

Then, V is said to be a **vector space** over F if the following conditions holds;

1. $(V, +)$ is an abelian group.
2. For all $u, v \in V$ and for all $\alpha, \beta \in F$.
   (a) $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$ for all $u, v \in V$ and for all $\alpha \in F$.
   (b) $(\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot u$
   (c) $\alpha \cdot (\beta \cdot u) = (\alpha\beta) \cdot u$
   (d) $1 \cdot v = v$, where 1 denotes the multiplicative identity in F.

Examples:- (i) $\mathbb{R}$ is a vectorspace over $\mathbb{R}$.
(ii) $V = \mathbb{R}$ is not a vector space over $F = \mathbb{C}$.
(iii) $V = \mathbb{R}^n$ is a vector space over $F = \mathbb{R}$.

Q. (iv) whether $V = \mathbb{C}$ is a vector space over $F = \mathbb{R}$?

Here:- $+ : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by $+(z_1, z_2) = z_1 + z_2 \in \mathbb{C}$

$\cdot : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ by $\cdot(\alpha, z_1) = \alpha \cdot z_1 \in \mathbb{C}$

① $(\mathbb{C}, +)$ is a abelian group.

$0 = 0 + i0 \in \mathbb{C}$

② For $\alpha, \beta \in \mathbb{R}$ and $u, v \in \mathbb{C}$

$z_1 + ? = -z_1$

(i) $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$

(ii) $\underbrace{(\alpha+\beta)}_{\in \mathbb{R}} \cdot u = \alpha \cdot u + \beta \cdot u$

(iii) $(\alpha\beta)\cdot u = \alpha(\beta u)$ ✓

(iv) we've $1 \in \mathbb{R}$ $\quad$ 'α' $= a+i0$

Such that $1 \cdot u = u \quad \forall\, u \in \mathbb{C}$

$\therefore$ $\mathbb{C}$ is a vector space over $\mathbb{R}$.

→ Note: If V is a vector space over the field $\mathbb{F}$ then elements of V are called vectors. (Note that it is not the usual vector)

Eg: let $V = M_{n\times n}(\mathbb{R})$ – set of all $n\times n$ matrices with real entries

and $F = \mathbb{R}$

'+' : matrix addition

'$\cdot$' : multiply a matrix by scalar

(i) $(\mathcal{M}_{n\times n}, +)$ is an abelian group.

+ : is associate.
+ : is commutative
additive identity :- $O_{n\times n}$
additive inverse :-
For all $A_{n\times n} \in \mathcal{M}_{n\times n}$

$$A_{n\times n} + \underline{-A_{n\times n}} = O_{n\times n}$$

② For $\alpha, \beta \in \mathbb{R}$, $A, B \in V$

(i) $\alpha\cdot(A+B) = \alpha\cdot A + \alpha\cdot B$

(ii) $(\alpha+\beta)\cdot A = \alpha\cdot A + \beta\cdot A$

(iii) $(\alpha\beta)A = \alpha(\beta A)$

(iv) $1 \in \mathbb{R}$ Such that $1\cdot A = A$

$\therefore$ $(\mathcal{M}_{n\times n}(\mathbb{R}), +, \cdot)$ is a vector space over $\underline{\underline{\mathbb{R}}}$.

Definition 1.2. (**Subspace**) Let $S \subseteq V$ then S is said to be a **Subspace** of V, if S itself is a vector space over the field F under the same operations '+' and '.' defined on V.

Dfn:- Let V be a vector space over F.
Let $S \subseteq V$ then S is said to be a SUBSPACE of V if S itself is a vector space over F under the same operations defined on V.

Eg:. $V = \mathbb{C}$ is a vector space over $F = \mathbb{R}$.
Let $S = \mathbb{R} \subset \mathbb{C} = V$
Also, we know that $S = \mathbb{R}$ is a vector space over $F = \mathbb{R}$
$\therefore$ $\mathbb{R}$ is a subspace of $\mathbb{C}$.

Result: Let V be a vector space over the field F and S be a non-empty subset of V then, S is a subspace of V if and only if $\alpha \cdot u + \beta \cdot v \in S$ for all $u, v \in S$ and for all $\alpha, \beta \in F$.

S is a subspace of $V \Longleftrightarrow$

$\boxed{\alpha \cdot u + \beta \cdot v \in S}$ $\forall\, \alpha, \beta \in F$, $\forall\, u, v \in S$

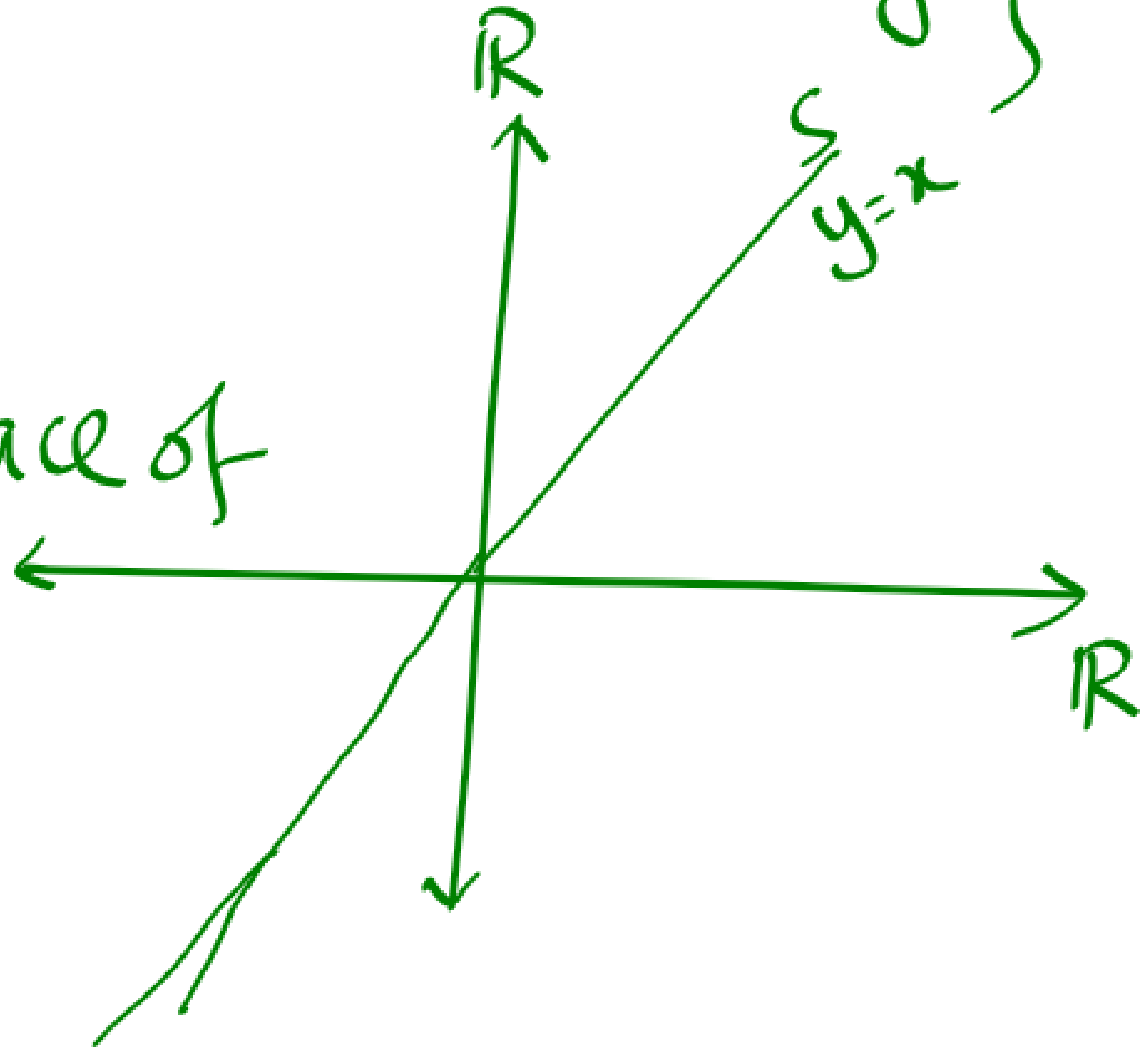
F S

Eg.: Let $V = \mathbb{R}^2$ and $F = \mathbb{R}$ then $\mathbb{R}^2$ is a vector space over $\mathbb{R}$.

Consider $S = \{(x, y) \mid x, y \in \mathbb{R} \text{ and } x = y\}$

then $S \subset \mathbb{R}^2$

Q. S is a subspace of $\mathbb{R}^2$ or not?



Let $\alpha, \beta \in \mathbb{R} = F$

and $u, v \in S$

then $u = (x, x)$, $v = (y, y)$

$$\alpha \cdot u = \alpha \cdot (x, x) = (\alpha \cdot x, \alpha \cdot x) \qquad \Bigg| \qquad \beta \cdot v = \beta \cdot (y, y) = (\beta \cdot y, \beta \cdot y)$$

$\in S$

$$\alpha \cdot u + \beta \cdot v = (\alpha \cdot x, \alpha \cdot x) + (\beta \cdot y, \beta \cdot y)$$

$\in F$

$$= (\underbrace{\alpha \cdot x + \beta \cdot y}_{\in \mathbb{R}}, \underbrace{\alpha x + \beta \cdot y}_{\in \mathbb{R}})$$

$\in S$

$\therefore$ S is a subspace of $\mathbb{R}^2$.

Linearly dependent and independent set of vectors

Dfn:- Let V be a vector space over a field $\mathbb{F}$.

Let $S = \{v_1, v_2, \dots v_n\}$ be a subset of V. Then

(i) S is said to be linearly independent if the linear combination

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$$

then $c_1 = c_2 = \dots = c_n = 0$

(ii) S is said to be linearly dependent if $\exists$ scalars $c_1, c_2 \dots c_n$ in F (not all zeros)

Such that $c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$

Consider a vector space V over the field F.

Definition 1.3. (**Linearly independent vectors**) Let $S = \{v_1, v_2, ..., v_n\}$ be the set of all vectors in V. Then $v_1, v_2, ..., v_n$ are said to be linearly independent if $c_1v_1 + c_2v_2 + ... + c_nv_n = 0$ implies $c_1 = c_2 = ... = c_n = 0$.

Definition 1.4. (**Linearly dependent vectors**) Let $S = \{v_1, v_2, ..., v_n\}$ be the set of all vectors in V. Then $v_1, v_2, ..., v_n$ are said to be linearly dependent if there exists scalars $c_1, c_2, ..., c_n$, not all zeros in F such that $c_1v_1 + c_2v_2 + ... + c_nv_n = 0$.

Q. Consider $S = \{\overset{V_1}{(1,0)}, \overset{V_2}{(1,1)}, \overset{V_3}{(0,1)}\}$

* is S a linearly independent or dependent set of vectors in $\mathbb{R}^2$?

<u>Ans:-</u> S is linearly dependent.

Reason

Since $V_2 = V_1 + V_3$. $\left((1,1) = (1,0) + (0,1) \right.$

$\Rightarrow V_1 + (-V_2) + V_3 = 0$

$\Rightarrow \underset{c_1}{1} \cdot V_1 + \underset{c_2}{(-1)} V_2 + \underset{c_3}{1} (V_3) = 0$

$\to \mathbb{R}^2$ is a vector space over $\mathbb{R}$

Problem 1.5. *Test whether the set of vectors* $S = \{(1,2), (2,5)\}$ *is linearly independent in* $\mathbb{R}^2$ *or not.*

Ans:- Let $S = \{v_1 = (1,2),\ v_2 = (2,5)\} \subset \mathbb{R}^2$

Let $c_1 v_1 + c_2 v_2 = 0$ then

$$c_1(1,2) + c_2(2,5) = (0,0)$$

$$\Rightarrow (c_1, 2c_1) + (2c_2, 5c_2) = (0,0)$$

$$\Rightarrow (c_1 + 2c_2,\ 2c_1 + 5c_2) = (0,0)$$

$$\Rightarrow \left.\begin{array}{l} c_1 + 2c_2 = 0 \\ 2c_1 + 5c_2 = 0 \end{array}\right\} (*)$$

$$|\text{coeff. matrix of } (*)| = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 5 - 4 = 1 \neq 0$$

$\Rightarrow (*)$ has a trivial sol^n

$\therefore c_1 = c_2 = 0$

$\Rightarrow S$ is linearly independent.

$\rightarrow \mathbb{R}^3$ is a vector space over $\mathbb{R}$.

Problem 1.6. *Test whether the set of vectors* $\{(1,0,1)\},(1,2,5),(1,-1,1)\}$ *is linearly independent in* $\mathbb{R}^3$ *or not.*

Ans: Let $S = \{v_1 = (1,0,1),\ v_2 = (1,2,5),\ v_3 = (1,-1,1)\}$

Let $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$ then

$$c_1(1,0,1) + c_2(1,2,5) + c_3(1,-1,1) = (0,0,0)$$

$$\Rightarrow (c_1,0,c_1) + (c_2,2c_2,5c_2) + (c_3,-c_3,c_3) = (0,0,0)$$

$$\Rightarrow (c_1+c_2+c_3,\ 2c_2-c_3,\ c_1+5c_2+c_3) = (0,0,0)$$

$$\Rightarrow \left.\begin{aligned} c_1+c_2+c_3&=0 \\ 2c_2-c_3&=0 \\ c_1+5c_2+c_3&=0 \end{aligned}\right\} (*)$$

$$\therefore \left|\text{coeff matrix of } (*)\right| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 1 & 5 & 1 \end{vmatrix} = 4 \neq 0$$

$\therefore c_1 = c_2 = c_3 = 0$ $\therefore$ S is linearly independent

Definition 1.7. (**Spanning set**) A set of vectors $S = \{v_1, v_2, ..., v_n\}$ in V is said to be a spanning set of V if any vector in V can be expressed as a linear combination of elements of S.

The spanning set of S is denoted by $L(S)$.

Theorem 1.8. *Let V be a vector space over the field F and $S \subset V$ then, $L(S)$ is a subspace of V.*

Proof:- Let $S = \{v_1, v_2, \ldots v_n\}$ be the subset of V

To prove $L(S)$ is a subspace of V.

we've, $L(S) = \{ c_1v_1 + c_2v_2 + \cdots + c_nv_n \mid c_i \in F, 1 \leq i \leq n \}$

Let $\alpha, \beta \in F$ and $u, v \in L(S)$

then $u = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$
$v = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$ where $\alpha_i, \beta_i \in F$, $1 \leq i \leq n$

$$\begin{aligned}
\therefore \alpha u + \beta v &= \alpha(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) + \beta(\beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n) \\
&= [(\alpha\alpha_1)v_1 + (\alpha\alpha_2)v_2 + \cdots + (\alpha\alpha_n)v_n] + [(\beta\beta_1)v_1 + (\beta\beta_2)v_2 + \cdots + (\beta\beta_n)v_n] \\
&= \underbrace{(\alpha\alpha_1 + \beta\beta_1)}_{\in F}v_1 + \underbrace{(\alpha\alpha_2 + \beta\beta_2)}_{\in F}v_2 + \cdots + \underbrace{(\alpha\alpha_n + \beta\beta_n)}_{\in F}v_n
\end{aligned}$$

RHS is a linear combination of $v_1, v_2, \ldots v_n$

$\therefore \alpha u + \beta v \in L(S)$

$\therefore L(S)$ is a subspace of V //

<u>Dfn</u>: Let V be a vector space over $\mathbb{F}$.

Let $S = \{v_1, v_2, \cdots v_n\} \subset V$

then <u>linear span of S</u> is the collection of all linear combinations of $v_1, v_2, \cdots v_n$.

It is denoted by $L(S)$.

ie; $L(S) = \left\{ c_1v_1 + c_2v_2 + \cdots + c_nv_n \;\middle|\; c_i \in F \text{ for } 1 \le i \le n \right\}$

→ S is a <u>spanning set</u> (or generating set) of V if

$$L(S) = V$$

ie; any element in V can be expressed as the linear combination of elements in S.

$V \to$ vector space

Definition 1.9. (**Basis**)A set of all vectors $S = \{v_1, v_2, ..., v_n\}$ in V is said to be a basis for V if,

- S is linearly independent.
- S spans V. $($ie; $L(S) = V)$

Definition 1.10. (**Dimension of a vector space**) The number elements in the basis of a vector space V is called the dimension of a vector space. It is denoted by $dim(V)$.

Problem 1.11. *Prove that* $S = \{(1,1), (2,3)\}$ *form a basis for* $\mathbb{R}^2$. $= V$

Ans:- we know, $\mathbb{R}^2$ is a vector space over $\mathbb{R}$.

$\to$ S is linearly independent

For Let $S = \{V_1 = (1,1), V_2 = (2,3)\}$

Let $c_1 V_1 + c_2 V_2 = 0$

$$\Rightarrow c_1(1,1) + c_2(2,3) = (0,0)$$

$$\Rightarrow (c_1 + 2c_2, c_1 + 3c_2) = (0,0)$$

$$\Rightarrow \left.\begin{array}{l} c_1 + 2c_2 = 0 \\ c_1 + 3c_2 = 0 \end{array}\right\} (*)$$

$$|\text{coeff matrix of } (*)| = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1 \neq 0$$

$\therefore$ S is linearly independent //

14 Verify that S spans $\mathbb{R}^2$ or not.

Let $(x, y) \in \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ $\in \mathbb{R}$

Let $(x, y) = c_1 v_1 + c_2 v_2$

then $(x, y) = c_1(1,1) + c_2(2,3)$

$\Rightarrow (c_1 + 2c_2,\ c_1 + 3c_2) = (x, y)$

$\Rightarrow c_1 + 2c_2 = x$ —— ①

$c_1 + 3c_2 = y$ —— ②

① − ② $\Rightarrow -c_2 = x - y$

$\Rightarrow c_2 = y - x \in \mathbb{R}$

① $\Rightarrow c_1 = x - 2c_2 = x \cancel{-2y} \cancel{+2x}$

$c_1 = 3x - 2y \in \mathbb{R}$

ie; $(x, y) = 3x - 2y \overset{v_1}{(1,1)} + y - x \overset{v_2}{(2,3)}$

$\therefore$ S spans $\mathbb{R}^2$

$\therefore$ S is a basis for $\mathbb{R}^2$