7. Let $(A, \vee, \wedge)$ be an algebraic system where $\vee$ and $\wedge$ are binary operations satisfying the commutative, associative and absorption laws.

(a) Define a binary operation $\leq$ on A as follows: for all $a, b \in A$, $a \leq b$ if and only if $a \wedge b = a$. Show that $\leq$ is a partial ordering relation.

(b) Show that $a \vee b$ is least upper bound of a and b in $(A, \leq)$

(c) Show that $a \wedge b$ is greatest lower bound and a and b in $(A \leq)$.

Soln: (a) From Q No 4, we know that if $\vee$ and $\wedge$ satisfies absorption law, than $\vee$ and $\wedge$ also satisfies idempotent law.

By idempotent law, we have

$$a \wedge a = a \implies a \leq a \implies \text{'}\leq\text{' is reflexive.}$$

To prove antisymmetry,

If $a \leq b$ and if $b \leq a$, $a \wedge b = a$ and $b \wedge a = b$.

we have
$$a \wedge b = a \quad \text{(i)}$$
$$b \wedge a = b \quad \text{(ii)}$$
by commutativity,
$$a \wedge b = b \quad \text{(iii)}$$

From (i) & (iii) $\implies \underline{\underline{a = b}} \implies$ Antisymmetric

To prove Transitive law, if $a \leq b$ and $b \leq c$ then

$a \wedge b = a$ and $b \wedge c = b$.

$a = a \wedge b = a \wedge (b \wedge c) = (a \wedge b) \wedge c = a \wedge c$

As $a \wedge c = a \Rightarrow a \leq c \Rightarrow$ Transitive

$\Rightarrow$ '$\leq$' is partial ordering relation

(b) To prove $a \vee b$ is lub of a and b.

(we will show first $a \vee b$ is an ub of a and b)

From absorption law $a \wedge (a \vee b) = a$

$a \leq a \vee b$ — ①

Similarly $b \wedge (a \vee b) = b$

$b \leq a \vee b$ — ②. ① & ② $\Rightarrow$ $a \vee b$ is

an ub of a and b.

suppose 'd' is any other ub of a and b i.e. $a \leq d$, $b \leq d$, then to prove $a \vee b \leq d$.

Given, $a \wedge d = a$, $b \wedge d = b$. To prove

$\boxed{(a \vee b) \wedge d = a \vee b}$

$a \vee b = (a \vee b) \wedge ((a \vee \underline{b}) \vee d)$ → absorption

$x = x \wedge (x \vee y)$

$= (a \vee b) \wedge ((a \vee (b \wedge d)) \vee d)$ → $b = b \wedge d$

$= (a \vee b) \wedge (a \vee [(b \wedge d) \vee d])$ Associativity

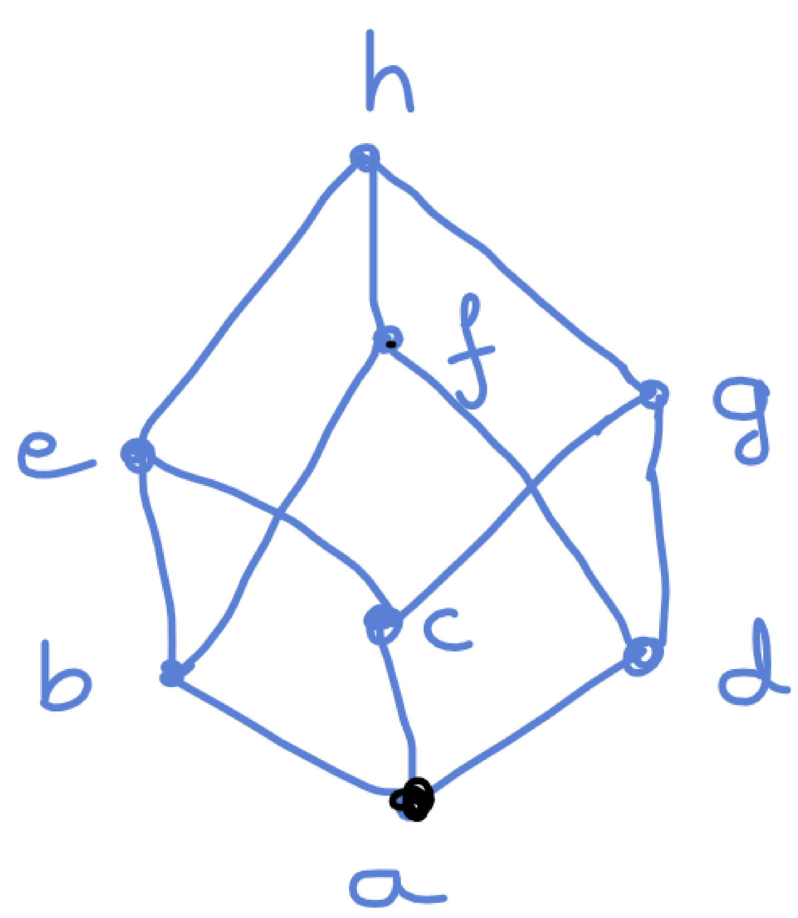
$= (a \vee b) \wedge (a \vee d)$ Absorption

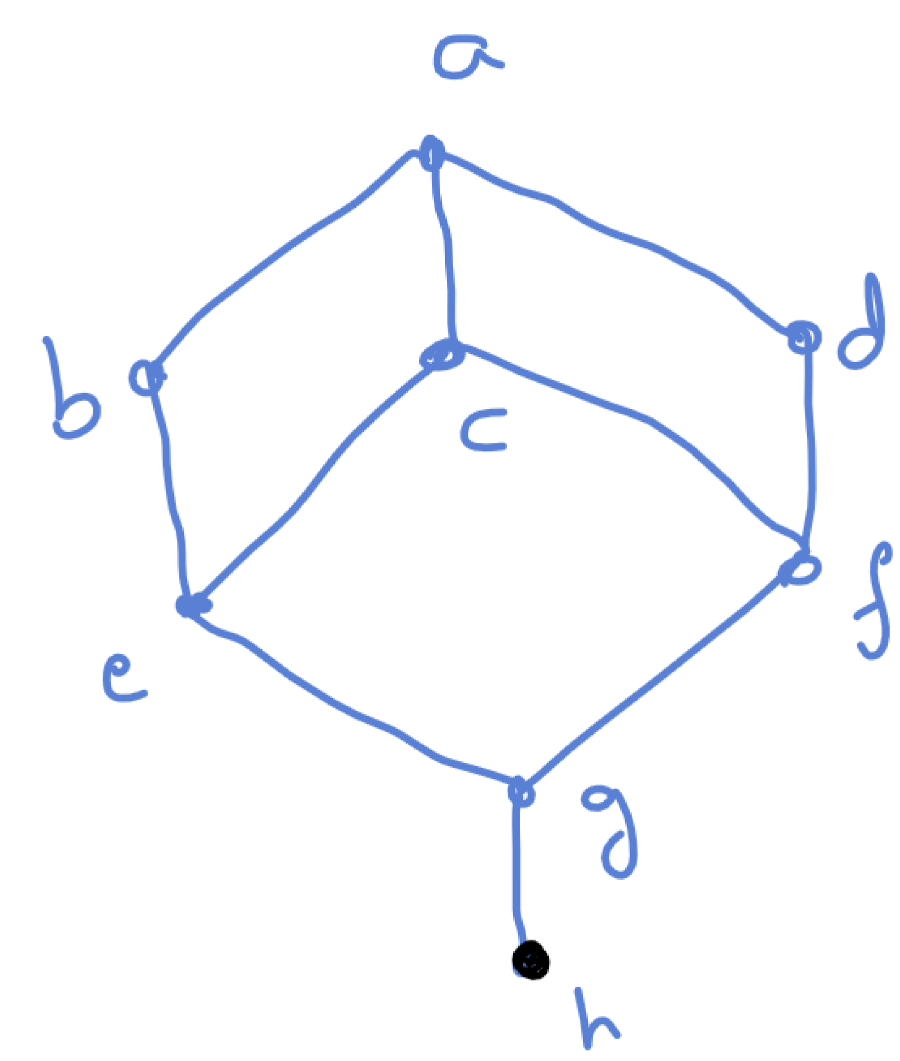
$= (a \vee b) \wedge ((a \wedge d) \vee d)$ ∵ $a \wedge d = a$ given

$= (a \vee b) \wedge d$ absorption.

Universal Lower Bound :

An element 'a' in a lattice $(A, \leq)$ is called a universal lower bound if for every element $b \in A$, $a \leq b$.





universal lower bound = a

h universal lower bound =

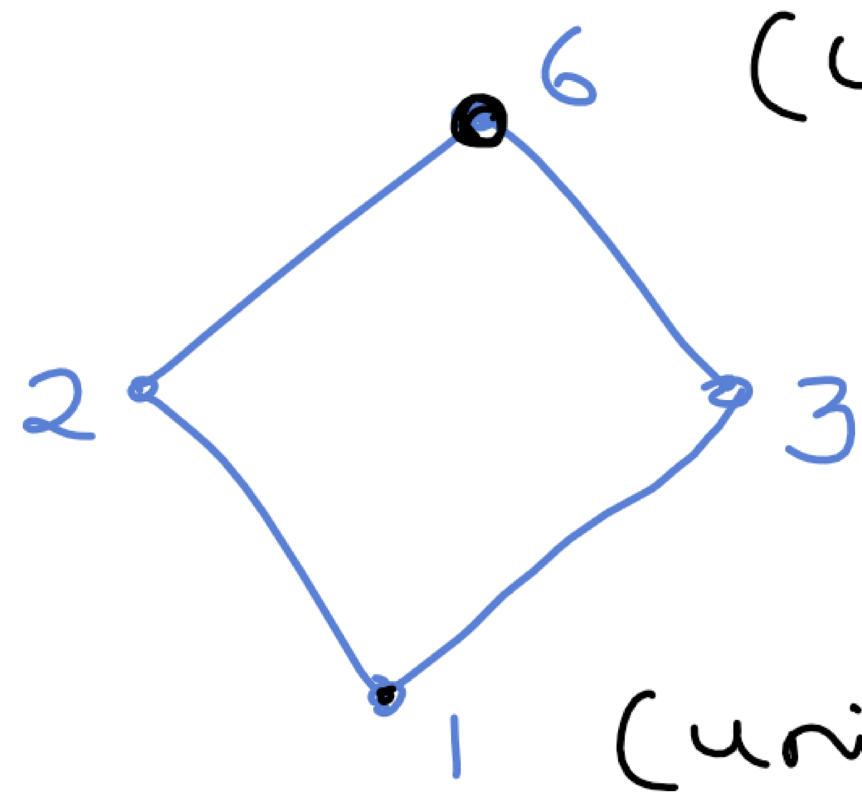
Note : From the definition of lattice, it is clear that if a lattice has a universal lower bound, then it is unique.

For if we assume there are 2 universal lower bounds a and b, then $a \leq b$ and $b \leq a$.

$\Rightarrow a = b$.

Universal upper bound

An element 'a' in a Lattice $(A, \leq)$ is called a universal upper bound if for every element $b \in A$, $b \leq a$.



(universal ub.

(universal lower bound)

Note: If a Lattice has a universal upper bound, then it is unique.

We use 0 to denote the universal lower bound and 1 to denote the universal upper bound of a Lattice (if such bounds exist)

Example: In a Lattice $(P(A), \subseteq)$, the empty set is the universal lower bound and set A is the universal upper bound.

Theorem: Let $(A, \leq)$ be a lattice with universal upper and lower bounds 1 and 0. For any element a in A,

$$a \vee 1 = 1 \quad \text{— (i)}$$
$$a \wedge 1 = a \quad \text{— (ii)}$$
$$a \wedge 0 = 0 \quad \text{— (iii)}$$
$$a \vee 0 = a \quad \text{— (iv)}$$

Proof: (i) We know $1 \leq a \vee 1$ — ①

Since '1' is the universal upper bound, we have

$a \vee 1 \leq 1$ — ② $\Rightarrow$ $a \vee 1 = 1$

(ii) From theorem ①, $a \wedge 1 \leq a$ — ③

$a \leq a$ and $a \leq 1 \Rightarrow$ From Theorem ②

$$a \wedge a \leq a \wedge 1$$

$$a \leq a \wedge 1 \quad \text{— ④}$$

③ ④ $\Rightarrow$ $a \wedge 1 = a$

(iii)

We know, $a \wedge 0 \leq 0$ — ⑤

As '0' is universal l.b we have, $0 \leq a \wedge 0$ — ⑥

$$\Rightarrow a \wedge 0 = 0$$

(iv) we know, $a \leq a \vee 0$ — (7)

$a \leq a$, $0 \leq a$

$\Rightarrow$ $a \vee 0 \leq a$ — (8)

$\Rightarrow$ $a \vee 0 = a$

Note: The element 0 is an identity element of join operation and 1 is an identity element of meet operation. i.e;

$a \vee 0 = a$

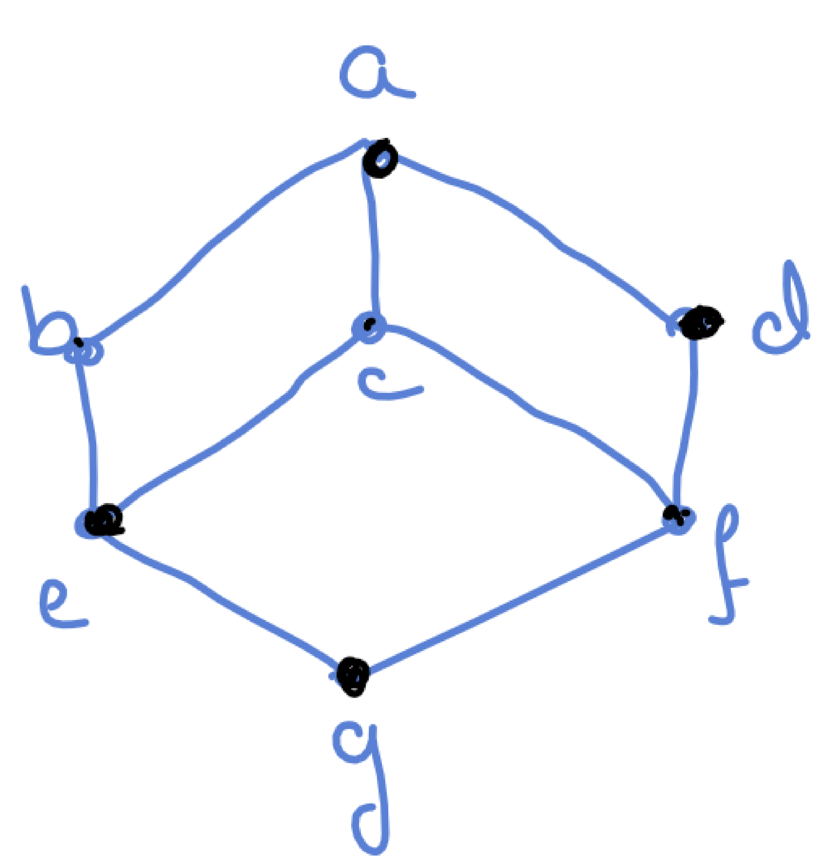
$a \wedge 1 = a$

Complemented Lattice

Let $(A, \leq)$ be a lattice with universal lower and upper bounds 0 and 1.

For an element $a \in A$, an element b is said to be a complement of a if $a \vee b = 1$ & $a \wedge b = 0$.

Note: Because of commutativity, if a is a complement of b, then b is also a complement of a.



$\because d \vee (e) = a$, $d \wedge (e) = g$ $\quad d \vee b = a$, $d \wedge b = g$

$(\because a = 1, \; g = 0)$

complement of $d = e, b$

complement of $f = b$

complement of $g = a$

complement of c = Not there

$f \vee b = a \quad f \wedge b = g$

Note: An element may have more than one complement.

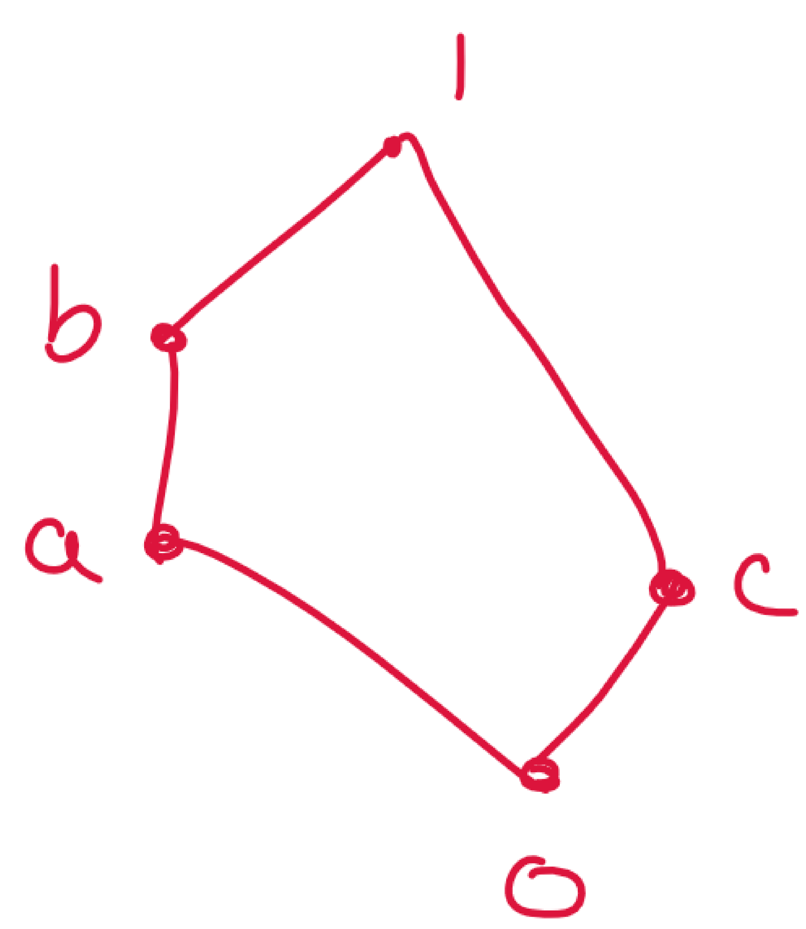
Note: 0 is the unique complement of 1 and 1 is the unique complement of 0.

In a lattice, there may exist an element without a complement.

Definition: A lattice is said to be a **Complemented lattice** if every element in the lattice has a complement.

Clearly, a complemented lattice must have universal lower and upper bound.

Example:



← Complemented Lattice.

$a \vee c = 1$

$a \wedge c = 0$

Complement of a is c

Complement of b is c

complement of 0 is 1

complement of c is a, b.

Theorem: In a distributive lattice, if an element has a complement, then this complement is unique.

Proof: Suppose that an element a has 2 complements say b and c, then

$$a \vee b = 1, \quad a \wedge b = 0$$
$$a \vee c = 1, \quad a \wedge c = 0$$

$$\begin{aligned} b &= b \wedge 1 \\ &= b \wedge (a \vee c) \\ &= (b \wedge a) \vee (b \wedge c) \\ &= 0 \vee (b \wedge c) \\ &= (a \wedge c) \vee (b \wedge c) \\ &= (a \vee b) \wedge c \\ b &= 1 \wedge c = c \end{aligned}$$

$a \vee c = 1$

distributive

$b \wedge a = 0$

$0 = a \wedge c$

Boolean Lattice and Boolean Algebra

Boolean Lattice: A lattice is said to be a boolean lattice, if it is distributive and complemented lattice.

Example: $(P(A), \subseteq)$ is a Boolean Lattice.

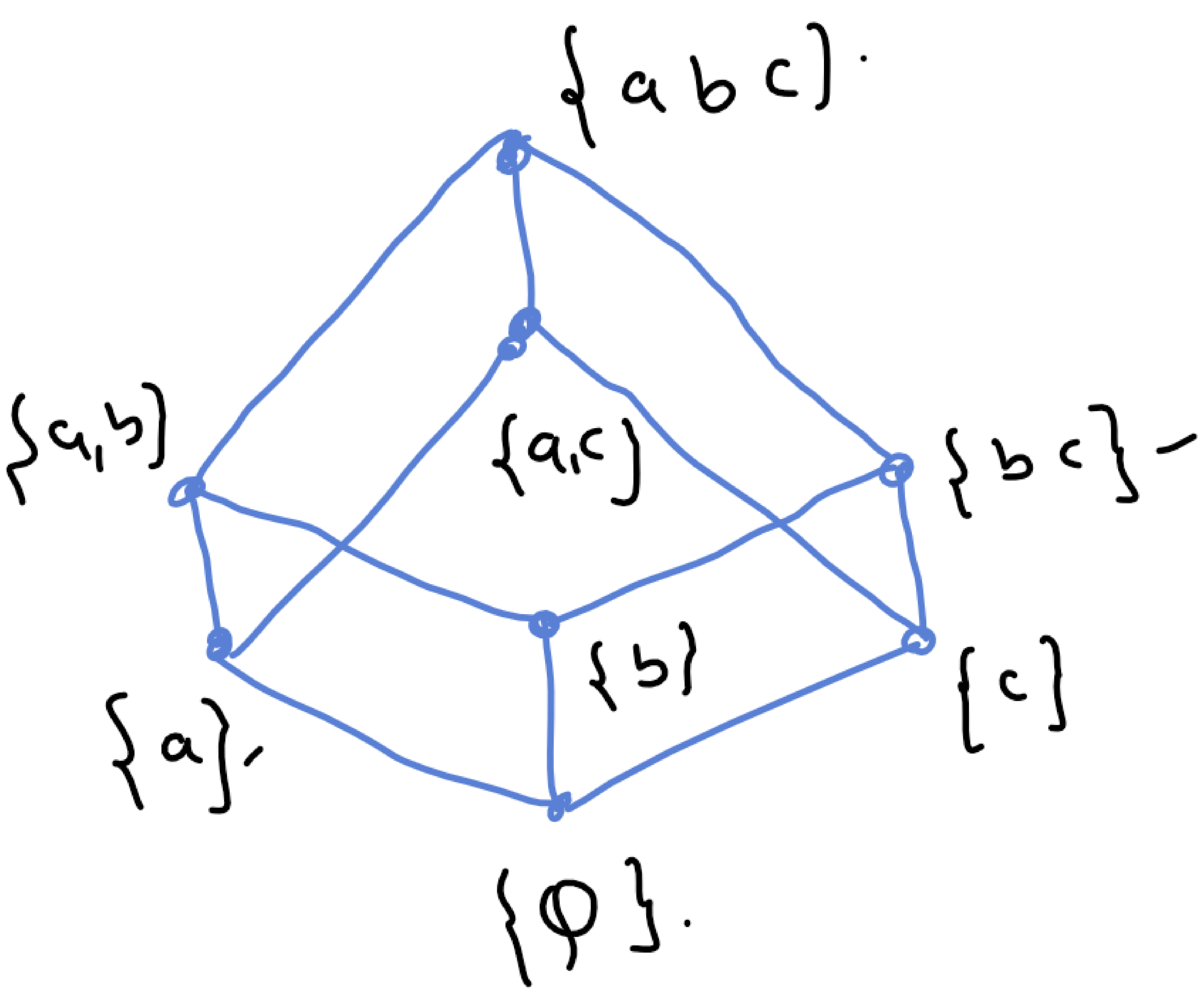
Let $(A, \leq)$ be a boolean lattice. Every $a \in A$, has a unique complement denoted by $\bar{a}$.
(since in a distributive lattice if an element has a complement, then the complement is unique)

Thus, we have a unary operation known as complementation '$-$'.

$\therefore$ a boolean lattice $(A, \leq)$ defines an algebraic system $(A, \vee, \wedge, -)$ where $\vee, \wedge$ and $-$ are join, meet and complementation operation, respectively, which is known as Boolean Algebra.

Example: Let S be a finite set, and $(P(S), \subseteq)$ is a Boolean lattice.

In this universal upper bound is $\underline{S}$ and universal lower bound is $\underline{\{\phi\}}$ and the complement of any set T is $\underline{S - T}$.



Complement of $\{a\}$ is $\underline{\{b\ c\}}$

Complement of $\{b\}$ is $\underline{\{a c\}}$

Complement of $\{c\}$ is $\underline{\{a, b\}}$

Complement of $\{\phi\}$ is $\underline{\{a b c\}}$

De' Morgan's law:

Theorem: For any a and b in a boolean algebra $(A, \vee, \wedge, -)$,

(i) $\overline{a \vee b} = \bar{a} \wedge \bar{b}$

(ii) $\overline{a \wedge b} = \bar{a} \vee \bar{b}$

Proof: (i) To prove

$$\left.\begin{aligned}(a \vee b) \vee (\bar{a} \wedge \bar{b}) &= 1 \\ (a \vee b) \wedge (\bar{a} \wedge \bar{b}) &= 0\end{aligned}\right\}$$

$$(a \vee b) \vee (\bar{a} \wedge \bar{b}) = \left[(a \vee b) \vee \bar{a}\right] \wedge \left[(a \vee b) \vee \bar{b}\right]$$

$$= ((a \vee \bar{a}) \vee b) \wedge (a \vee (b \vee \bar{b}))$$

$$= (1 \vee b) \wedge (a \vee 1)$$

$$= 1 \wedge 1 = \underline{\underline{1}}$$

$$(a \vee b) \wedge (\bar{a} \wedge \bar{b}) = \left[a \wedge (\bar{a} \wedge \bar{b})\right] \vee \left[b \wedge (\bar{a} \wedge \bar{b})\right]$$

$$= ((a \wedge \bar{a}) \wedge \bar{b}) \vee ((b \wedge \bar{b}) \wedge \bar{a})$$

$$= (0 \wedge \bar{b}) \vee (0 \wedge \bar{a})$$

$$= 0 \vee 0 = 0$$

Thus $\bar{a} \wedge \bar{b}$ is the complement of $a \vee b$.

i.e., $\overline{a \vee b} = \bar{a} \wedge \bar{b}$.

From principle of duality, $\overline{a \wedge b} = \bar{a} \vee \bar{b}$.