Generating Function

A function $f(x)$ is said to be a generating function for the sequence $\{a_r\}_{r=1}^{\infty}$, if

$$f(x) = \sum_{r=1}^{\infty} a_r \, x^r$$

i.e., a_r can be obtained as coefficient of x^r in the expansion of $f(x)$.

Example: $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = \sum_{r=1}^{\infty} \frac{1}{r!} \, x^r$; Here $a_r = \frac{1}{r!}$

$$(1+x)^n = 1 + {}^nC_1 \, x + {}^nC_2 \, x^2 + {}^nC_3 \, x^3 + \cdots + {}^nC_r \, x^r + \cdots + {}^nC_n \, x^n \quad ; a_r = C(n,r)$$

$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{r=1}^{\infty} r\, x^{r-1}$;

1, 2, 3, … are coefficient of $x^0, x^1, x^2, \ldots$ in the expansion of $(1-x)^{-2}$.

If the terms of the sequence can be obtained as coefficient of $\frac{x^r}{r!}$ in the expansion of $f(x)$, then $f(x)$ is said to be an exponential generating function.

Generating Function for Combinations:

Consider the three distinct objects a, b and c, and form the polynomial

$$(1 + ax)(1 + bx)(1 + cx) = 1 + (a + b + c)x + (ab + bc + ac)x^2 + abc\, x^3 \text{ ----- (1)}$$

On the RHS,

If we consider, three ways of selecting one object $(a \text{ or } b \text{ or } c)$ and represent it by $(a + b + c)$, then it is co-efficient of x^1 in equation (1)

Similarly, the three ways of selecting two objects $(ab \text{ or } bc \text{ or } ca)$ may be represented $(ab + bc + ca)$ which is the coefficient of x^2 in equation (1)

There is only one way of selecting all three objects it is represented by abc, which is the coefficient of x^3 in equation (1)

Thus, coefficient of x is all possible combinations of 3 objects taken 1 at a time

coefficient of x^2 is all possible combinations of 3 objects taken 2 at a time and so on.

In general, the coefficient of x^r gives the number of r- combination of n objects.

This result may be interpreted as follows:

The factor $(1 + ax)$ can be representing symbolically two ways:

(i) $1\ or\ x^0$ represent non-selection of object 'a'

(ii) ax represent selection of the object 'a'

The factors $(1 + bx)$ and $(1 + cx)$ can be interpreted in a similar manner.

The product of the three factors $(1+ax)(1+bx)(1+cx)$ indicates the selection or non-selection of all three objects a, b and c, and the powers of x in the product indicate the number of objects selected.

If $a = b = c = 1$, we get the polynomial,

$$(1+x)^3 = 1 + 3x + 3x^2 + x^3$$

i.e., there are three ways of selecting one object, three ways of selecting two objects and one way of selecting all three objects.

Case 1: If n distinct objects, say $a_1, a_2, \dots, a_n$ are given, then corresponding to each object we have,

$$(1 + a_1 x)(1 + a_2 x) \dots (1 + a_n x) = 1 + (a_1 + a_2 + \cdots + a_n)x + \cdots + (a_1 a_2 \dots a_n)x^n$$

Where coefficient of x^r gives all possible r-combination of n objects

If $a_1 = a_2 = \cdots = a_n = 1$, then

$$(1 + x)^n = 1 + {}^nC_1\, x + {}^nC_2\, x^2 + {}^nC_3\, x^3 + \cdots + {}^nC_r\, x^r + \cdots + {}^nC_n\, x^n$$

∴ number r-combination of n-objects without repetition is the coefficient of x^r.

Thus $(1+x)^n$ is a generating function for r $-$combinations of n distinct objects without repetition.

A generating function used in this way is called an ‘ENUMERATOR’

Case 2: If an object is allowed an unlimited repetition, the corresponding factor in the enumerator must have every power of x present in it.

Hence, the factor will be $\mathbf{1+x+x^2+x^3+\cdots=\frac{1}{1-x}=(1-x)^{-1}}$

Thus, the enumerator for $r-$combination of n objects with unlimited repetition is $(1-x)^{-n}$

$$(1-x)^{-n} = 1 + {}^{-n}C_1\ (-x) + {}^{-n}C_2\,(-x)^2 + {}^{-n}C_3\,(-x)^3 + \cdots + {}^{-n}C_r\,(-x)^r + \cdots$$

Number of $r-$combination of n objects with repetition $=$ coefficient of x^r

$$= {}^{-n}C_r(-1)^r$$

Number of r −combination of n objects with repetition

$$= \frac{-n(-n-1)(-n-2)\ldots\big(-n-(r-1)\big)(-n-r)!}{r!\quad(-n-r)!}\;(-1)^r$$

$$= (-1)^r\;\frac{n(n+1)(n+2)\ldots\big(n+(r-1)\big)}{r!}\;(-1)^r$$

$$= \frac{n(n+1)(n+2)\ldots\big(n+(r-1)\big)}{r!}$$

$$= {}^{n+r-1}C_r$$

Note: If an object is allowed unlimited repetition, the corresponding factor in the enumerator must have every power of x present in it and hence the factor will be $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$.

Problems:

1. Write the generation function for combination of 3 objects can be formed if 1st object can be selected at most once, 2nd object at most twice and 3rd object at most 3 times.

2. Obtain the generation function for the number of ways to select r objects with repetition from 5 distinct objects with at least 2 of each type.

3. Use a generating function to count all selections of 6 objects from 3 types of objects with repetitions up to 4-times of each type. Also model the function with unlimited repetition.

Generating Function for permutations:

In ordinary algebraic multiplication, it is not possible to distinguish between ab and ba.

Thus, the method discussed earlier cannot be used to obtain generating function for permutation.

To overcome this difficulty, we proceed as follows,

$$(1+x)^n = 1 + {}^nC_1\, x + {}^nC_2\, x^2 + {}^nC_3\, x^3 + \cdots + {}^nC_r\, x^r + \cdots +$$

$$= 1 + \frac{{}^nP_1}{1!} x + \frac{{}^nP_2}{2!} x^2 + \frac{{}^nP_3}{3!} x^3 + \cdots + \frac{{}^nP_r}{r!} x^r + \cdots$$

i.e, ${}^nP_r$ = coefficient of $\frac{x^r}{r!}$ in the expansion of $(1+x)^n$

Thus $(1+x)^n$ is an exponential generating function for r −permutations of n distinct objects, without repetition.

If repetition is allowed, then factor for each object must represent the fact that the object may not appear, may appear once, may appear twice and so on in the permutation.

Hence factor for each object is: $1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$

$$\therefore \text{Enumerator} = \left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)^n = (e^x)^n = e^{nx} = \sum_{r=0}^{\infty}\frac{(nx)^r}{r!} = \sum_{r=0}^{\infty} n^r \, \frac{x^r}{r!}$$

Number of permutations with repetition = = coefficient of $\frac{x^r}{r!} = n^r$

1. In how many ways can 4 letters of the word ‘ENGINE’ be arranged using generating function?

2. A ship carries 48 flags, 12 each of the colors white, red, blue and black. Twelve of these flags are placed on a vertical pole in order to communicate a signal to other ships.

a) How many of these signals use an even number of blue flags and an odd number of black flags?

b) How many of these signals use at least 3 white flags or no white flags at all?

3. Find the number of ways to place 25 people into three rooms with at least one person in each room.

Partitions and Compositions of Integers:

A positive integer n can be represented as sum of one or more positive integers.

i.e, $n = a_1 + a_2 + \cdots + a_m$, where each $a_i > 0$, is an integer.

Divisions of a positive integer n are of two types:

i) Ordered divisions- are called **Compositions**

ii) Unordered divisions- are called **Partitions**

Example: The partitions and compositions of the integer $n = 5$

The various compositions of an integer 5 are:

5, 4+1, 1+4, 3+2, 2+3, 3+1+1, 1+3+1, 1 + 1 + 3, 2+2+1, 2+1+2, 1 + 2+2,

2+1 + 1 +1, 1 + 2+1+1, 1+1+2+1, 1 + 1 + 1+2, 1+1 + 1+1+1

The various partitions of an integer 5 are:

5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1 + 1 + 1, 1+1+1 + 1 + 1

Note: Partitions or compositions say $2 + 1 + 1 + 1$ will be written 2111, or $2\ 1^3$, and for partitions the largest parts will be written first.

Enumerators for Compositions:

Consider a positive integer n, and n ones in a row.

If there is no restriction on the number of parts, we may or may not put a marker in any of the $(n-1)$ spaces between the ones in order to form groups; this may be done in 2^{n-1} ways.

Hence, the number of compositions of n with no restrictions on the number of parts is 2^{n-1}.

If we restrict the compositions to have exactly m parts, then just $(m-1)$ markers are required to form m groups.

We can place $(m-1)$ markers in the $(n-1)$ available spaces in $C(n-1, m-1)$ ways.

Therefore, number of compositions of n with exactly m parts $= {}^{n-1}C_{m-1}$

Problems:

1. How many compositions of n with m parts are there when zero parts are allowed?

2. How many ways can an examiner assign 30 marks to 8 questions so that no question receives less than 2 marks?

Generating functions for Partitions:

Let p_n be the number of unrestricted partition of n so that the generating function is,

$$p(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n + \cdots$$

Consider the polynomial,

$$1 + x + x^2 + x^3 + \cdots + x^k + \cdots + x^n$$

- The appearance of x^k can be interpreted as the existence of just k number of one's (1's) in a partition of the integer n.

Similarly the polynomial

$$1 + x^2 + (x^2)^2 + (x^2)^3 + \cdots + (x^2)^k + \cdots = 1 + x^2 + x^4 + x^6 + \cdots + x^{2k} + \cdots$$

- is concerned with the twos (2's) in the partition.
- The coefficient $x^{2k} = (x^2)^k$ represents the case of just k number of 2's in the partition.

In general the polynomial,

$$1 + x^r + x^{2r} + \cdots + x^{kr} + \cdots$$

can represent the k number of r's in the partition.

The generating function for partition should contain one factor for ones, another factor for twos, and so on.

Hence, generating function for partition of n is

$$p(x) = \left(1 + x + x^2 + x^3 + \cdots + x^k + \cdots\right)\left(1 + x^2 + x^4 + x^6 + \cdots + x^{2k} + \cdots\right) \ldots$$

$$\ldots (1 + x^r + x^{2r} + \cdots + x^{kr} + \cdots)$$

$$= (1 - x)^{-1}\,(1 - x^2)^{-1} \ldots (1 - x^r)^{-1} \ldots.$$

The number of unrestricted partitions of n is therefore coefficient of x^n

1. Find the number of ways that can be obtained, change for Rs.10, in terms of Rs.5, Rs.2 and Rs.1.

2. Prove that the number of partitions of n in which no integer occurs more than twice as a part is equal to the number of partitions of n in to parts not divisible by 3.

3. Show that the number of partitions of n in which every part is odd is equal to the number of partitions of n with unequal(or distinct) parts.

Ferrers Graph:

It is a graph to represent a partition by an array of dots.

A Ferrers graph has the following property:

(i) There is one row for each part.

(ii) The number of dots in a row is the size of that part.

(iii) An upper row always contains at least as many dots as a lower row.

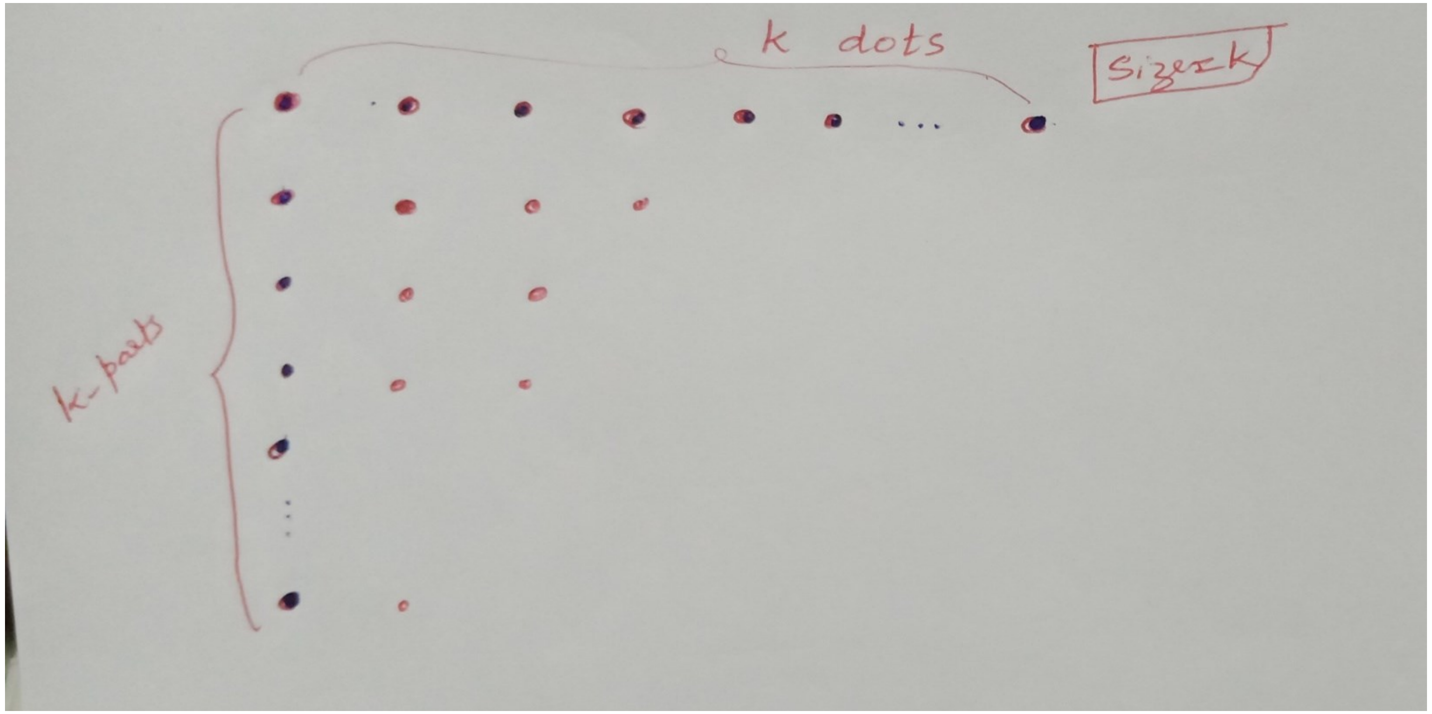
(iv) The rows are aligned on the left

The partition obtained by reading the Ferrers graph by **columns** is called the conjugate partition.

A partition whose Ferrers graph reads the same by rows and by columns is called self-conjugate.

1). Show that the number of partitions of an integer n with no part greater than k is equal to the number of partition of n with at most k parts.

Consider, a partitions of an integer n with no part greater than k and consider the Ferrers graph representation



If we read this partition by column, then the number of parts is $\leq k$

Thus, for a given a partitions of an integer n with no part greater than k, there corresponds a partition of n with at most k parts.

Conversely,

Consider a partition of n with at most k parts, then number of rows is $\leq k$ in Ferrers graph.

And hence in the conjugate partition, size of the partition is $\leq k$.

Thus, for every partition with at most k parts, there corresponds a partition with no part greater than k.

Hence, the number of partitions of an integer n with no part greater than k is equal to the number of partition of n with at most k parts.

2. Obtain the generating function for the partition of n with exactly k parts.

3. Show that the number of partitions of n is equal to number of partitions of $2n$ with exactly n parts.

4. Show that the number of partitions of n in which odd parts are not repeated but even parts can occur any number times is equal to the number of partitions of n in which every part is either odd or a multiple of 4.

5. Prove that the number of partitions of n with exactly k parts is equal to the number of partition of '$n-k$' with no part greater than k.

Show what these partitions are for $n = 11, k = 3$.