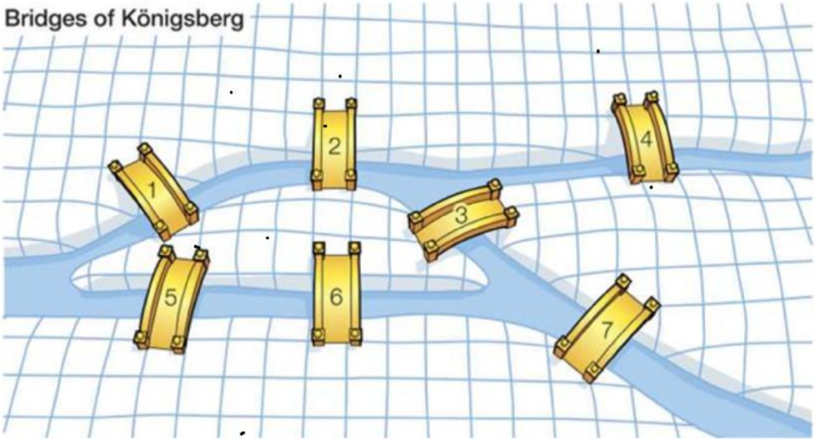
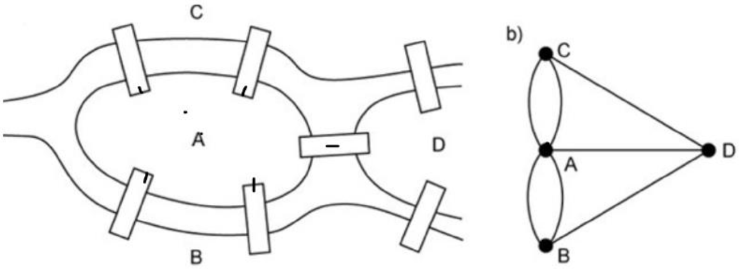
Graph Theory

Konigsberg Bridge Problem

Graph theory was originated from the Konigsberg Bridge Problem, were two islands linked to each other and to the banks of the Pregel River by seven bridges. The problem was to begin at any of the four land areas, walk across each bridge exactly once and return to the initial point. This problem was solved in **1736 by Euler. In 1847**.





Definition: A **graph G** consists of a finite nonempty set **V = V(G)** of points called vertices together with a prescribed set **E(G)** unordered pair of distinct points of **V** whose elements are called edges of **G**.

Terms and Terminologies:

1. A graph with p points and q lines is called a **(p, q) graph**.
2. The (1, 0) graph is **trivial graph**.
3. For a graph G, the number of elements in V(G) is called order of G denoted by |V(G)| and the number of elements in E(G) is called the size of G i.e |E(G)|.

4. If two distinct lines x and y are incident with a common point, then they are **adjacent lines**.
5. We write **x = uv** and say that u and v are adjacent points, denoted **u adj v or u ~ v**.

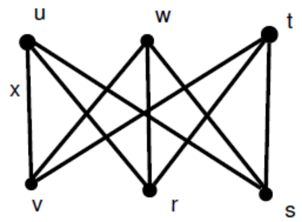


Figure: G: A Graph to illustrate adjacency

6. An edge with identical ends is called a **loop** and two edges with same end vertices are called **parallel edges**.
7. A graph is finite if both its vertex set, and edge set are **finite**.
8. A graph is **simple** if it has no loops or parallel edges. A graph is finite if both its vertex set, and edge set are finite.

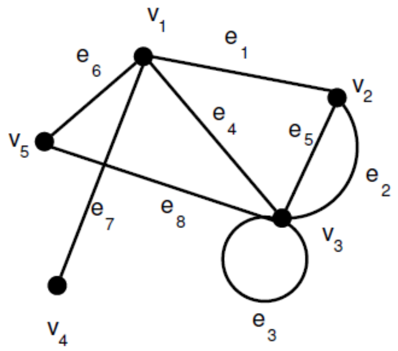


Figure: A graph G with loops and multiple edges

9. In **multigraph**, no loops are allowed but more than one line can join two points; these are called multiple lines. If both loops and multiple lines are permitted, we have a **pseudograph**.

10. A **directed graph or digraph** D consists of a finite nonempty set V of points together with a prescribed collection X of ordered pairs of distinct points. The elements of X are directed lines or arcs. By definition, a digraph has no loops or multiple arcs **(simple graph).**

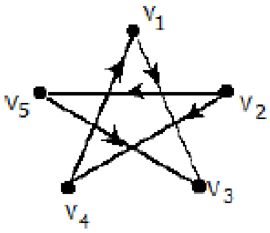


Figure: A directed graph

11. A **labelled graph** is a graph in which every vertex and every edge is labelled, most of the times, a graph means a labelled graph.
12. Two graphs G and H are **isomorphic** (written G ≅ H or sometimes G = H) if there exists a one-to-one correspondence between their point sets which preserves adjacency.
13. An invariant of a graph G is a number associated with G which has the same value for any graph isomorphic to G. Thus, the numbers p and q are certainly **invariants.**

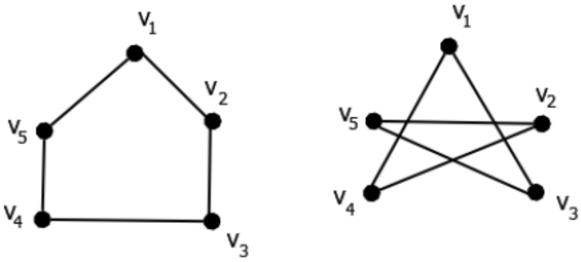


Figure: Isomorphic Graphs

14. A graph H is called a **subgraph of a graph** G if V(H) ⊆ V(G) and E(H) ⊆E(G).
15. A subgraph H of G is called a **spanning subgraph** if V(H) = V(G).
    Let S be a subset of the vertex set V(G) of G. Then, the subgraph induced by S, denoted by $\langle S \rangle$ is the maximal subgraph of G with S as the vertex set. Thus, two points of S are adjacent in $\langle S \rangle$ if and only if they are adjacent in G.

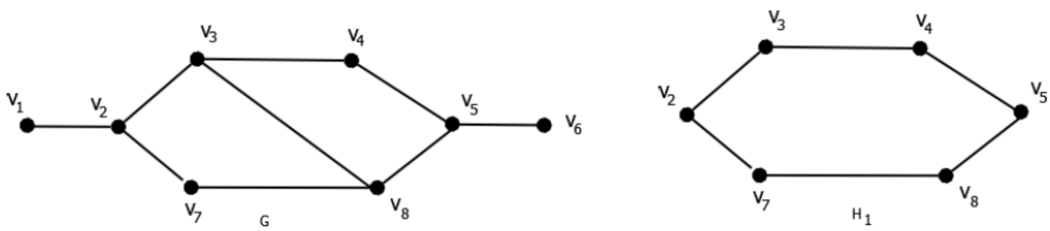


Figure: Graph of G and H_1 subgraph of G

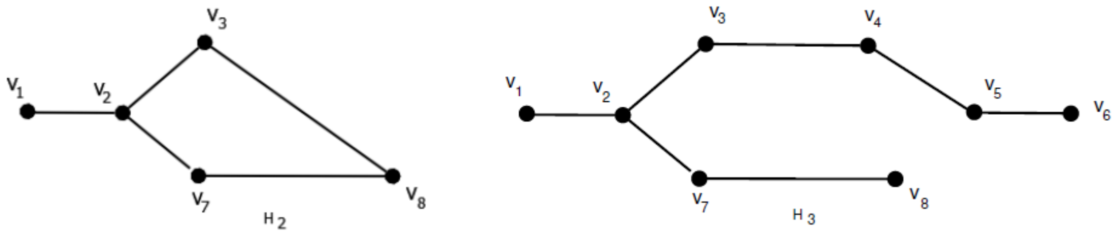


Figure: H_2 induced subgraph of G and H_3 Spanning subgraph of G

16. The removal of a point v_i from a graph G results in that subgraph $G - v_i$ of G consisting of all points of G except v_i and all lines not incident with v_i. Thus, G −vi is the maximal subgraph of G not containing v_i.
17. On the other hand, the removal of a line x_j from G yields the spanning subgraph G – xj containing all lines of G except x_j . Thus, $G - x_j$ is the maximal subgraph of G not containing x_j . The removal of a set of points or lines from G is defined by the removal of single elements in succession. On the other hand, if v_i and v_j are not adjacent in G, the addition of line v_iv_j results in the smallest supergraph of G containing the line v_iv_j .
18. A **walk** in G is a finite non-null sequence $W = v_0e_1v_1e_2v_2e_3 \ldots e_kv_k$ whose terms are alternately vertices and edges such that, for $1 \leq i \leq k$, the ends of e_i are v_{i-1} and v_i. We say that- W is a walk from v_0 to v_k or a (v_0, v_k)–walk.

The vertices v_0 and v_k are called the origin and terminus of W respectively. Other vertices in a walk are called internal vertices. The integer k, which is the number of edges in W, is the **length of W**. In a simple graph, a walk $v_0e_1v_1e_2 \ldots e_kv_k$ is determined by the sequence $v_0v_1 \ldots v_k$ of vertices. If the edges $e_1, e_2, \ldots, e_k$ of a walk W are distinct, then W is called a **trail**; in addition, if the vertices $v_0v_1 \ldots v_k$ are distinct, W is called a **path**. The following figure illustrates a walk, a trail, and a path in a graph.

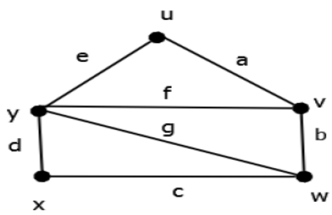


Figure: Walk, trial,path in Graph G

Walk: $u\ a\ v\ f\ y\ g\ w\ b\ v\ f\ y$.
Trail: $u\ e\ y\ f\ b\ w\ g\ y\ d\ x$.
Path: $u\ a\ v\ f\ y\ g\ w$.

19. Two vertices u and v of G are said to be **connected** if there is a (u, v)-path in G. This connection is an equivalence relation on the vertex set V(G).
20. Thus, there is a partition of V(G) into nonempty subsets V_1, V_2, . . ., Vk such that two vertices u and v are connected if and only if both u and v belongs to the same set $V_{i.}$ The induced subgraphs $\langle V_1 \rangle$, $\langle V_2 \rangle$, . . ., $\langle V_k \rangle$ are called the **components of G**. If G has exactly one component, G is connected; otherwise, G is disconnected. We denote the number of components of G by ω(G). In the following figure, G is disconnected with three components whereas H is connected.

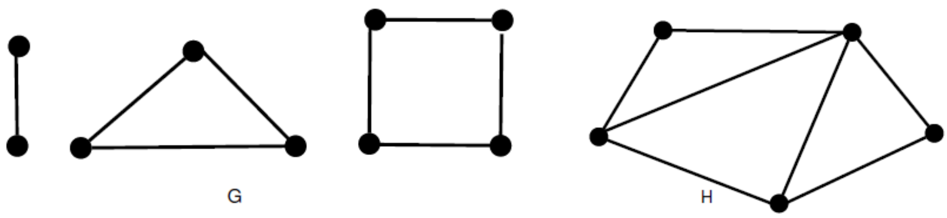


Figure: A graph G with 3 components and Connected Graph H

21. A trail whose origin is same as terminus, is called a **circuit** and such a path is called a **cycle.**
22. Let G be a graph with vertex set V(G) and edge set E(G). The **complement of** G, denoted by $\bar{G}$ is a graph with V(G) = V(G) and two vertices are adjacent in $\bar{G}$ if and only if they are not adjacent in G.

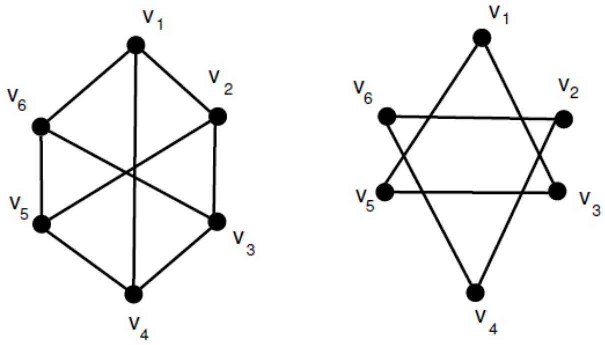


Figure: A graph G and its complement $\overline{G}$

23. A graph G is said to be **self-complementary** if G is isomorphic to its complements.

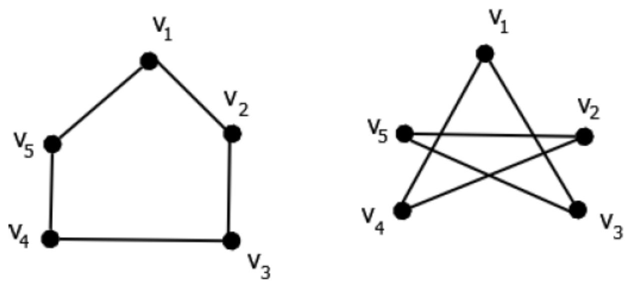


Figure: Self complementary graphs

24. Let G be a connected graph and let u, v be two vertices in G. Shortest path between u and v in G is a (u, v) − path with minimum number of edges in it.
25. The **distance between u and v** in G is the length of a shortest path between them. The **distance** between u and v in G is denoted by $d_G(u, v)$.
26. The $min_{u,v\in G}\{d(u, v)\}$ is called the **radius of G** and $max_{u,v\in G}\{d(u, v)\}$ is called the **diameter of G**. They are usually denoted by rad(G) and dia(G), respectively.
27. **Eccentricity** of a vertex v, in a connected graph G, denoted by e(G) is defined as follows. $e_G(v) = max_{u,v\in G}\{d(u, v)\}$. Obviously, the minimum and maximum of the eccentricities of vertices of G are radius and diameter of the graph G.
28. A vertex v in G with minimum eccentricity is called a **central vertex** and set of all central vertices in G is called the **centre of G**. In the following figure, diameter = 2, radius = 1, V_1 is the central vertex.

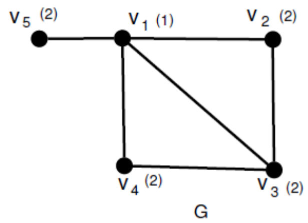


Figure: Center, Radius, Diameter of G

29. **Degree** of a vertex v in G, denoted by **$deg_G v$** is the number of vertices adjacent to v in G. A vertex in a graph G is said to be isolated when its degree is zero. A vertex is said to be a **pendant vertex** if its degree is 1. The minimum degree among the vertices of G is denoted by $\delta(G)$. The maximum degree among the vertices of G is denoted by $\Delta(G)$. In the following figure, degGv1 = 0, degGv2 = degGv6 = 2, degGv3 = degGv4 = 3, degGv5 = 1. Here, v1 is the isolated vertex and v5 is the pendant vertex.

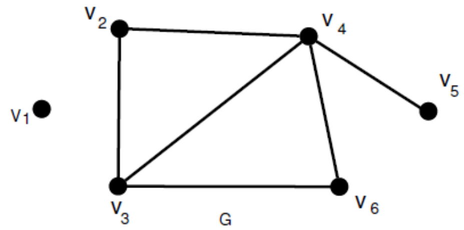


Figure: Degree of vertices of G

Theorem 1:

Let G be a (p, q) graph. The sum of the degrees of vertices of a graph G is twice the number of edges, $\sum deg(v) = 2q,\ v \in V$

Theorem 2:

In any graph, the number of vertices of odd degree is even.

30. A graph on n vertices, in which every two vertices are adjacent, is called a **complete graph** and is denoted by K_n. A graph G in which every vertex is of same degree is called a **regular graph**. When G is regular, $\delta(G) = \Delta(G)$ and the common value is called regularity of G. A connected regular graph with regularity two is called a cycle. A **cycle** on n vertices is denoted by C_n.

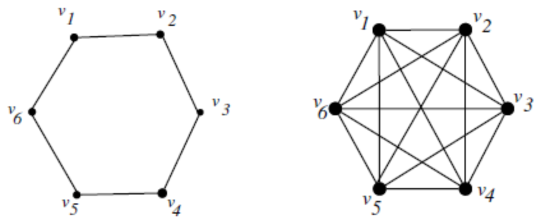


Figure: Cycle graph and Complete graph on six vertices

31. A **bipartite graph** is one whose vertex set can be partitioned into 2 subsets X and Y so that each edge has one end vertex in X and one end vertex in Y. Such a partition (X, Y) is called a bipartition of the graph G. A **complete bipartite graph** is a bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y; if |X| = m and |Y | = n, such a graph is denoted by $K_{m, n}$.
The graphs (a) and (b) below are complete bipartite and bipartite graphs respectively.

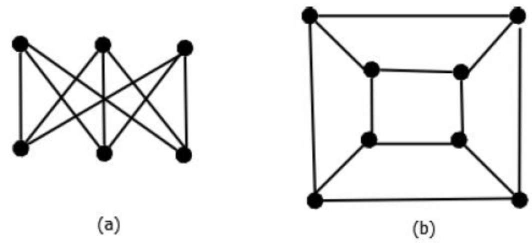


Figure: Complete bipartite and bipartite graphs

Theorem 3:

A graph is bipartite if and only if all its cycles are even.

32. Complement of a complete graph on n vertices is called the totally disconnected graph. since it has no edges at all.
33. A graph G is said to be self-cantered if every vertex of G has the same eccentricity. In such a graph, radius is equal to the diameter.
    Example: C_n is a self-centred graph and is the complete graph K_n.
34. Let G be a graph, v be a vertex in G and e be an edge in G. Then $G - \{v\}$ or $G - v$ is the subgraph of G obtained by removing the vertex v and all the edges in G which are incident with v, from the graph G. But $G - \{e\}$ is a subgraph of G obtained by removing only the edge e from G.

Theorem 4:

For any Graph G with six vertices, G or $\overline{G}$ contains a triangle.

Theorem 5:

Let G be a self complementary graph. Show that the number of vertices in G is of the form $4n$ or $4n+1$.

Theorem 6:

If G has p vertices and minimum degree of a graph $\delta(G) \geqslant (p-1)/2$, then G is connected.

Theorem 7:

If $diam(G) \geq 3$, then $diam(\overline{G}) \leq 3$.

Theorem 8:

Every nontrivial self complementary graph has diameter 2 or 3.

Theorem 9:

For any graph G, show that either G or $\overline{G}$ is connected.

35. The **cartesian product** $G_1 \times G_2$ is defined as follows. Vertex set is $V(G_1) \times V(G_2)$. The vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent in $G_1 \times G_2$ whenever [$u_1 = v_1$ and u_2 is adjacent to v_2] or [u_1 is adjacent to v_1

and $u_2 = v_2$]. Let G_1 be the path graph P_2 and G_2 be P_3, then $G_1 \times G_2$ is as shown in the figure below.

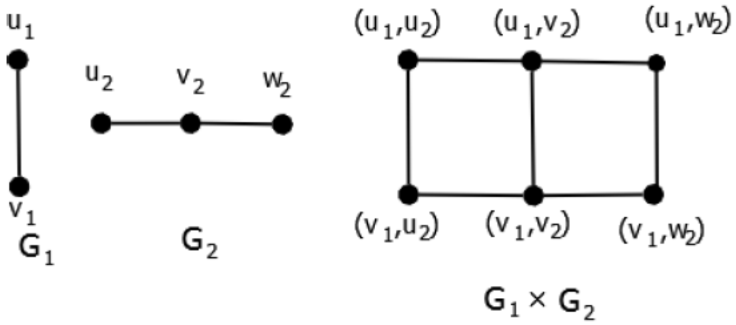
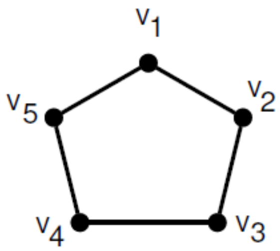


Figure: Cartesian product of two graphs

36. A **cut vertex** of a graph is one whose removal increases the number of components and bridge are such an edge. A **non-separable graph** is connected, nontrivial, and has no cut vertices. A **block** of a graph is a maximal non separable sub graph. We note that every nontrivial connected graph has at least two vertices which are not cut vertices.

37. For a graph G with $V(G) = \{v_1, v_2, . . ., v_n\}$, the **adjacency matrix** of G, denoted by A(G) is the $n \times n$ matrix defined as follows. The rows and columns of A(G) are indexed by V(G). If $i \neq j$ then the $(i, j)^{th}$– entry of A(G) is 0 for vertices v_i and v_j non adjacent, and $(i, j)^{th}$– entry of A(G) is 1 for vertices v_i and v_j adjacent. The $(i, i)^{th}$– entry of A(G) is 0 for $i = 1, 2, . . ., n$. We often denoted by A(G) or simply A.

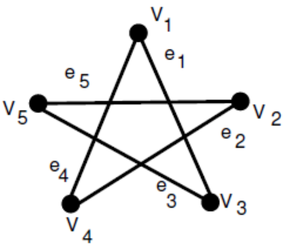


$$A(G)=\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Figure: Graph G and its adjacency matrix $A(G)$

38. For a graph G with $V(G) = \{v_1, v_2, . . ., v_n\}$ and $E(G) = \{e_1, e_2, . . ., e_m\}$, the (vertex- edge) **incidence matrix** of G, which we denote by B(G) is

the $n \times m$ matrix defined as follows. The $(i, j)^{th}$– entry of $B(G)$ is 0 if vertex v_i and edge e_j are not incident, and otherwise $(i, j)^{th}$– entry of $B(G)$ is 1. This is often referred to as the $(0, 1)$ − incidence matrix.



$$B(G) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Figure: Graph G and its incidence matrix $B(G)$

TREES:

39. An acyclic is one that contains no cycles. It is also called a forest. A **tree** is a connected acyclic graph. In a tree, any two vertices are connected by a unique path. All the trees on six vertices are given below.

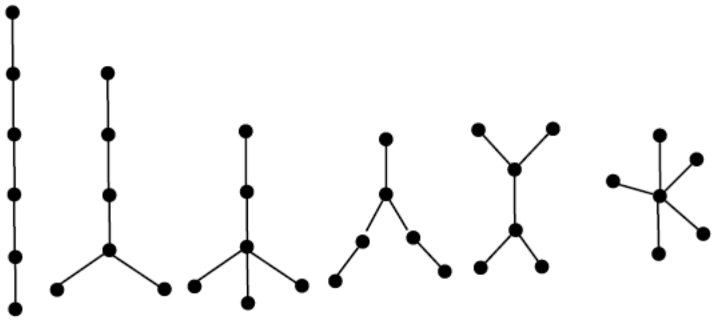


Figure: Trees

40. If G is a (n,m) tree then $m = n - 1$. Every nontrivial tree has at least two vertices of degree one (pendant vertices). A tree with exactly two vertices of degree one is a path. A tree in which all the vertices except one is of degree one is called a **star**. If G is a tree with $\Delta(G) \geq k$, then G has at least k vertices of degree 1. Centre of a tree contains either a single vertex or two adjacent vertices. Accordingly, a tree is called uni-central or bi-central.
41. Every simple graph G with $\Delta(G) \geq k$ has a subgraph isomorphic to a tree on $(k + 1)$ vertices. A connected graph is a tree if and only if every edge of the graph is a cut edge or bridge.
42. A **spanning tree** of G is a spanning subgraph of G that is a tree. We note that every connected subgraph has a spanning tree. Hence, if G is a connected (n,m) graph then m _ $n - 1$.

Theorem 10:

A graph G is a tree if and only if between every pair of vertices there exist a unique path.

Theorem 11:

A tree with p vertices has $p-1$ edges.

Theorem 12:

Every tree has a center consisting of either one vertex or two adjacent vertices.

43. A walk that traverses every edge of G exactly once, goes through all vertices and ends at the starting vertex is called **Eulerian circuit or Eulerian cycle**. A graph G is said to be Eulerian if it has an Eulerian cycle.

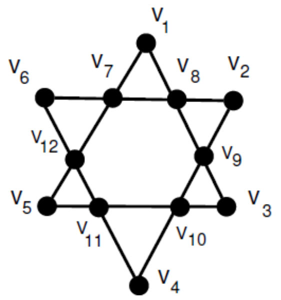


Figure: *An Eulerian graph*

Theorem 13:

A non empty connected graph is Eulerian if and only if all of its vertices of even degree.

44. A path that contains every vertex of G is called a Hamilton path of G. Similarly, a Hamilton cycle of G is a cycle that contains every vertex of G. A graph is **Hamiltonian** if it contains a Hamilton cycle.

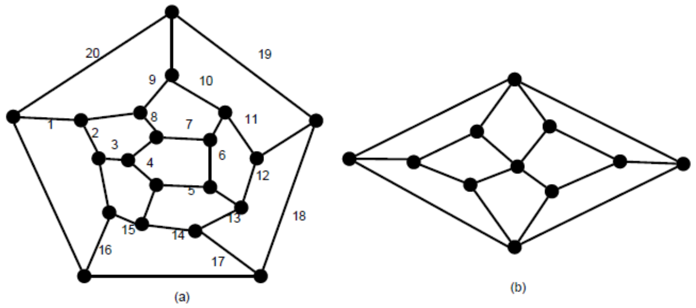


Figure: Hamiltonian and Non hamiltonian graphs

The dodecahedron in figure (a) is Hamiltonian. The Herschel graph in figure (b) is non Hamiltonian.

Algorithm:

Shortest paths in graphs: The graph G has n vertices and a distance associated with each edge of the graph G (such a graph is often called a network). The representation of the network will be as a distance matrix D.

The **distance matrix** D = (d_{ij})

where, d_{ij} = 0, if i = j.

d_{ij} = ∞, if i is not joined to j by an edge.

d_{ij} = distance associated with an edge from i to j, if i is joined to j by an edge.

We shall find the shortest distance between the vertices of a graph G using **Dijkstra's algorithm.**

Let us define two sets K and U, where K consists of those vertices which have been fully investigated and between which the best path is known, and U of those vertices which have not yet been processed. Clearly, every vertex belongs to either K or U but not both. Let a vertex **r** be selected from which we shall find the shortest paths to all the other vertices of the network. Let the array bestd(i) hold the length of the shortest path so far formed from r to vertex i, and another array tree(i) the next vertex to i on the current shortest path.

Dijkstra's algorithm:

Step 1: Intialise K = {r}, U=f all other vertices of G except rg. Set bestd(i)= dri and tree(i)= r.

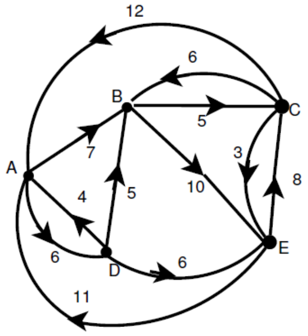
Step 2: Find the vertex s in U which has the minimum value of bestd. Remove s from U and put it in K.

Step 3: For each vertex u in U, find bestd(s) + dsu and if it is less than bestd(u) replace bestd(u) by this new value and let tree(u)= s.(a shorter path to u has been found by going via vertex s.)

Step 4: If U contains only one vertex then stop the process or else go to step 2. The array bestd(i) contains the length of shortest path from r to i.

Examples 1:

Implement Dijkstra’s algorithm to find shortest path from the vertex B to all other vertices of following graph G.



$$D(G) = \begin{bmatrix} 0 & 7 & \infty & 6 & \infty \\ \infty & 0 & 5 & \infty & 10 \\ 12 & 6 & 0 & \infty & 3 \\ 4 & 5 & \infty & 0 & 6 \\ 11 & \infty & 8 & \infty & 0 \end{bmatrix}$$

Figure: Graph G and its distance matrix $D(G)$

Step 1: Intialise $K = \{B\}$, U={ A,C,D,E}.

	A	**C**	**D**	**E**
best d	∞	**5**	∞	**10**
tree	**B**	**B**	**B**	**B**

Therefore, bestd(C)= 5 (minimum distance) and tree(C)= B.

Step 2: Remove C from U and put it in K. Now, U={ A,D,E}. K={ B,C}. Find minimum distance from B to A, D, E via C. Therefore, bestd(A) = $17 < \infty$, tree(A)=C and bestd(E)= $8 < 10$, tree(E)=C.

	A	**D**	**E**
best d	**17**	∞	**8**
tree	**C**	**B**	**C**

Therefore, bestd(E)= 8 and tree(E)= C

Step 3: Remove E from U having minimum distance and put it in K. Now, U={ A,D}. K={ B, C, E}. Find minimum distance from B to A, D via C and E.The distances from B to A are 17 (B *to* C *to* A), 30 (B *to* E *to* C *to* A), 21 (B *to* E *to* A). Therefore, bestd(A) = 17 $<$ (30 , 21), tree(A)=C and bestd(D)= 8 $<$ 10, tree(E)=C.

	A	**D**
best d	**17**	∞
tree	**C**	**B**

Therefore, bestd(A)= 17 and tree(A)= C

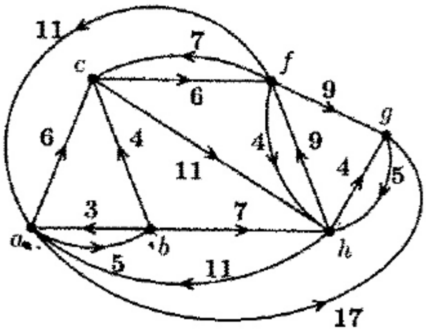
Step 4: Remove A from U having minimum distance and put it in K. Now, U = { D}, K={ B, C, E, A}. Find minimum distance from B to D via A, E, C. The distances from B to D are 23 (B *to* C *to* A *to* D), 36 (B *to* E *to* C *to* A *to* D), 27 (B *to* E *to* A D). Therefore, bestd(D) = 23 $<$ (36 *and* 27), tree(D)=A. Now, U contains only one vertex and stop the process.Therefore, bestd(D)= 23 and tree(D)= A.

The array bestd contains the length of shortest path from B to all other vertices of G. The shortest path from the vertex B to all other vertices of a graph G is given by

B	**A**	**C**	**D**	**E**
best d	**17**	**5**	**23**	**8**
tree	**C**	**B**	**A**	**C**

Examples 2:

Implement Dijkstra0s algorithm to find shortest path from c to all other vertices of the following network.



$$D(G) = \begin{bmatrix} 0 & 5 & 6 & \infty & 17 & \infty \\ 3 & 0 & 4 & \infty & \infty & 7 \\ \infty & \infty & 0 & 6 & \infty & 11 \\ 11 & \infty & 7 & 0 & 9 & 4 \\ \infty & \infty & \infty & \infty & 0 & 5 \\ 11 & \infty & \infty & 9 & 4 & 0 \end{bmatrix}$$

Figure: Graph G and its distance matrix $D(G)$

Step 1: Intialise $K = \{c\}$, $U = \{a, b, f, g, h\}$.

	a	**b**	**f**	**g**	**h**
best d	∞	∞	**6**	∞	**11**
tree	**c**	**c**	**c**	**c**	**c**

Therefore, bestd(f)= 6 (minimum distance) and tree(f)= c.

Step 2: Remove f from U and put it in K.

Now, U={ a, b, g, h}. K={ c, f}.

Find minimum distance from c to a, b, g, h via f.

Therefore, bestd(a) $= 17 < \infty$, tree(a)=f, bestd(g)= 15 $<\infty$, tree(g)=f and bestd(h)= 10 $<$ 11 , tree(h)=f

	a	**b**	**g**	**h**
best d	**17**	∞	**15**	**10**
tree	**f**	**c**	**f**	**f**

Therefore, bestd(h)= 10 and tree(h)= f.

Step 3: Remove h from U having minimum distance and put it in K. Now, U={ a, b, g}. K={ c, f, h}. Find minimum distance from c to a, b, g via h and f.The distances from c to a are 17 , 21 , 22. Therefore, bestd(a) = 17, tree(a)=f and bestd(g)= 14 $<$ 15, tree(g)=h.

	a	**b**	**g**
best d	**17**	∞	**14**
tree	**f**	**c**	**h**

Therefore, bestd(g)= 14 and tree(g)= h

Step 4: Remove g from U having minimum distance and put it in K. Now, U={ a, b}, K={c, f, g, h}. Find minimum distance from c to a and b via c, f, g, h. The distances from c to a and b are 17 and ∞. Therefore, bestd(a) = 17, tree(a)=f.

	a	**b**
best d	**17**	∞
tree	**f**	**c**

Therefore, bestd(a)= 17 and tree(a)= f

Step 5: Remove a from U having minimum distance and put it in K. Now, U={ b}, K={c, f, g, h, a}. Find minimum distance from c to b via c, f, g, h and a. The distances from c to b are 22, 26,27... Therefore, bestd(b) = 22, tree(b)=a. Now, U contains only one vertex and stop the process.

The array bestd contains the length of shortest path from c to all other vertices of G. The shortest path from the vertex c to all other vertices of a graph G is given by

	a	**b**	**f**	**g**	**h**
best d	**17**	**22**	**6**	**14**	**10**
tree	**f**	**a**	**c**	**h**	**f**

Proofs of the theorems:

Theorem 1:

Let G be a (p, q) graph. The sum of the degrees of vertices of a graph G is twice the number of edges, $\sum deg(v) = 2q$, $v \in V$

Theorem 2:

In any graph, the number of vertices of odd degree is even.

Proof:

Let Se= Sum of all degree of all even degree vertices. Let So= Sum of all degree of all odd degree vertices. So + $Se = 2q$. i.e, $So = 2q - Se = even$.
Each term in the sum So is odd. Therefore, So can be even, only if even number of terms in So. Hence, the theorem.

Theorem 3:

A graph is bipartite if and only if all its cycles are even.

Proof:

Let G be a connected bipartite graph.
Then its vertex set V can be partitioned into two sets V_1 and V_2 such that every edge of G joins a vertex of V_1 with a vertex of V_2.
Thus, every cycle $v_1, v_2, ...v_n, v_1$ in G necessarily has its oddly subscripted vertices in V_1(say). i.e., $v_1, v_3, ... \in V_1$ and other vertices $v_2, v_4, ... \in V_2$.
In a cycle $v_1, v_2, ...v_n, v_1$: $v_n\, v_1$ is an edge in G.
Since, $v_1 \in V_1$ we must have $v_n \in V_2$. This implies n is even. Hence, the length of the cycle is even.
Conversely, suppose that G is a connected graph with no odd cycles.
Let $u \in G$ be any vertex. Let $V_1 = \{v \in V/d(u, v) = even\}$,
$V_2 = \{v \in V/d(u, v) = odd\}$.
Then, $V = V_1 \cup V_2$, $V_1 \cap V_2 = \phi$.
We must prove that no two vertices in V_1 and V_2 are adjacent.
Suppose that $x,w \in V_1$ be adjacent. $w \in V_1 \Rightarrow d(u,w) = 2k$ and
$x \in V_1 \Rightarrow d(u, x) = 2l$. Thus, the path $u - w - x - u$ forms a cycle of length $2k + 2l + 1$, odd a contradiction. Therefore, x and w cannot be adjacent. That is no two vertices in V_1 are adjacent. Similarly, we can prove no two vertices in V_2 are adjacent. Hence, the graph is bipartite.

Theorem 4:

For any Graph G with six vertices, G or $\overline{G}$ contains a triangle.

Proof:

Let G be a graph with six vertices. Let v be any vertex in G. Since v is adjacent to other five vertices either in G or in $\bar{G}$. We assume that v is adjacent with v_1, v_2, v_3 in G. If any 2 of these vertices say v_1, v_2 are adjacent then v_1, v_2, v forms a triangle in G. If no two of them are adjacent in G then v_1, v_2, v_3 are the vertices of a triangle in $\bar{G}$.

Theorem 5:

Let G be a self complementary graph. Show that the number of vertices in G is of the form $4n$ or $4n+1$.

Proof:

Let G be a (p, q) graph. Number of edges in $Kp = p(p-1)/2 = {}^pC_2$
Since G is self-complementary, number of edges in G = number of edges in $\bar{G}$ = q
Number of edges in Kp = number of edges in G + number of edges in $\bar{G}$.
Number of edges in $G = p(p-1)/2 - q$
$q = p(p-1)/2 - q$,
$4q = p(p-1)$
Therefore, q = p(p-1)/4
$\Rightarrow 4/p$ *or* $4/(p-1)$
$p = 4n$ *or* $p-1 = 4n$
$p = 4n$ *or* $p = 4n+1$

Theorem 6:

If G has p vertices and minimum degree of a graph $\delta(G) \geqslant (p-1)/2$, then G is connected.

Proof:

Suppose that the graph G is disconnected. Let us assume that G has two (or more) components say C_1 and C_2.
Suppose that a component C_1 has a vertex of minimum degree $(p-1)/2$.
Then, C_1 must contain at least $[(p-1)/2+1]$ vertices.
Similarly, suppose that a component C_2 has a vertex of minimum degree $(p-1)/2$. Then, C_2 must contain at least $[(p-1)/2+1]$ vertices.
Now, total number of vertices in G is equal to
$[(p-1)/2+1+(p-1)/2+1] = p-1+2 = p+1$ which is a contradiction to the fact that G has p vertices. Hence, G is connected.

Theorem 7:

If $diam(G) \geq 3$, then $diam(\overline{G}) \leq 3$.

Proof:

Let x and y be any two vertices in $\bar{G}$. Since $diam(G) \geq 3$, there exist vertices u and v at distance 3 in G. Hence, uv is an edge in $\bar{G}$. Since u and v have no common neighbour in G, both x and y are each adjacent to u or v in $\bar{G}$. It follows that $d(x, y) \leq 3$ in $\bar{G}$ and hence $d(\bar{G}) \leq 3$).

Theorem 8:

Every nontrivial self complementary graph has diameter 2 or 3.

Proof:

Let G be a self-complementary graph. Clearly, G cannot have diameter 1.Since $G \cong Kn$ which is not self-complementary graph. Hence, self complementary graphs have diameter at least 2. Suppose that $diam(G) \geq 3$. By the above theorem, diam($\bar{G}$) ≤ 3. Hence, diameter of every self-complementary graph is either 2 or 3.

Theorem 9:

For any graph G, show that either G or $\overline{G}$ is connected.

Proof:

If G itself is connected, there is nothing to prove.

Suppose that the graph G is disconnected and has two components C_1 and C_2.

Let u and v be any two vertices, we have the following cases.

a) If u and v are in different components and are not adjacent in G. Then u and v are adjacent in $\bar{G}$. We have, uv path, hence $\bar{G}$ is connected.
b) If u and v belong to the same component but they are not adjacent in G. Hence, they are adjacent in $\bar{G}$. Hence, we have uv path.
c) Suppose that u and v are adjacent in G (Obviously, they belong to the same component). Then we can find w in another component (which does not contain u and v). We have a uv path via w in $\bar{G}$. That is, $u \sim w$ and $v \sim w$.

Theorem 10:

A graph G is a tree if and only if between every pair of vertices there exist a unique path.

Proof:

Let G be a tree then G is connected. Hence, there exist at least one path between every pair of vertices. Suppose that between two vertices say u and v, there are two distinct paths then union of these two paths will contain a cycle; a contradiction. Thus, if G is a tree, there is at most one path joining any two vertices.

Conversely, suppose that there is a unique path between every pair of vertices in G. Then G is connected. A cycle in the graph implies that there is at least one pair of vertices u and v such that there are two distinct paths between u and v. Which is not possible because of our hypothesis. Hence, G is acyclic and therefore it is a tree.

Theorem 11:

A tree with p vertices has $p - 1$ edges.

Proof:

The theorem will be proved by induction on the number of vertices.

If $p = 1$, we get a tree with one vertex and no edge. If $p = 2$, we get a tree with two vertices and one edge. If $p = 3$, we get a tree with three vertices and two edges. Assume that the statement is true with all tree with k vertices ($k < p$). Let G be a tree with p vertices. Since G is a tree there exist a unique path between every pair of vertices in G. Thus, removal of an edge e from G will disconnect the graph G. Further, $G-e$ consists of exactly two components with number of vertices say m and n with $m+n = p$. Each component is again a tree. By induction, the component with m vertices has $m-1$ edges and the component with n vertices has $n-1$ edges. Thus, the number of edges in $G = m - 1 + n - 1 + 1 = m + n - 1 = p - 1$.

Theorem 12:

Every tree has a center consisting of either one vertex or two adjacent vertices.

Proof:

The result is obvious for the trees K_1 and K_2. We show that any other tree T has the same central vertices as the tree $T1$ obtained by removing all end vertices of T. Clearly, the maximum of the distances from a given vertex u of T to any other vertex v of T will occur only when v is an end vertex. Thus, the eccentricity of each vertex in T_1 will be exactly one less than the eccentricity of the same vertex in T. Hence, the vertices of T which possess minimum eccentricity in T are the same vertices having minimum eccentricity in T_1. That is, T and T_1 have the same centre. If the process of removing end vertices is repeated, we obtain successive trees having the same centre as T. Since T is finite, we eventually obtain a tree which is either K_1 or K_2. In either case all vertices of this ultimate trees constitute the centre of T which consists of just a single vertex or of two adjacent vertices.

Theorem 13:

A non empty connected graph is Eulerian if and only if all of its vertices of even degree.

Proof:

Suppose that G is connected and Eulerian. Since G has a Eulerian circuit which passes through each edge exactly once, goes through all vertices and all its vertices are of even degree.

Conversely, Let G be a connected graph such that every vertex of G is of even degree. Since, G is connected, no vertex can be of degree zero. Thus, every vertex of degree ≥ 2, so G contains a cycle. Let C be a cycle in a graph G. Remove edges of the cycle C from the graph G. The resulting graph (Say G_1) may not be connected, but every vertex of the resulting graph is of even degree. Suppose G consists only of this cycle C, then G is obviously Eulerian. Otherwise, there is another cycle C_1 with a vertex v in common with C. The walk beginning at v and consisting of the cycles C and C_1 in succession is a closed trial containing the edges of these two cycles. By continuing this process, we can construct a closed trial containing all edges of G, hence G is Eulerian.