$\rightarrow \dfrac{dy}{dx} = \dfrac{ax+by+c}{a_1x+b_1y+c_1} \xrightarrow[y=Y+k]{\checkmark x=X+h \checkmark} \dfrac{dY}{dX} = \dfrac{aX+bY}{a_1X+b_1Y}$, H.D $\checkmark$

if $\dfrac{a}{a_1} \neq \dfrac{b}{b_1}$ where $ah+bk+c=0$

$a_1h+b_1k+c_1=0$

$\text{Sol}^n \leftarrow$ var. sep form $\checkmark \xleftarrow{Y=vX}$

$\dfrac{a}{a_1} = \dfrac{b}{b_1}$

Ordinary differential Equations

$\dfrac{dy}{dx} = \dfrac{ax+by+c}{k(ax+by)+c}$ put $ax+by=t \rightarrow$ var. sep. form.

1. Problems on reducible to homogenous differential equations

Problem 1.1. *Solve* $(x-4y-9)dx + (4x+y-2)dy = 0$

Ans:- $\dfrac{dy}{dx} = \dfrac{-x+4y+9}{4x+y-2}$, non. homo. d.e. —— ①

put $x = X+h$ & $y = Y+k \Rightarrow dy = dY$ & $dx = dX$

$\therefore$ ① $\Rightarrow \dfrac{dY}{dX} = \dfrac{(-X+4Y)+(-h+4k+9)}{(4X+Y)+(4h+k-2)}$ —— ②

Take $\left.\begin{aligned} -h+4k+9=0 \\ 4h+k-2=0 \end{aligned}\right\} \Rightarrow h=1,\ k=-2$

$\therefore$ ② $\Rightarrow \dfrac{dY}{dX} = \dfrac{-X+4Y}{4X+Y}$, homo. d.e. —— ③

put $Y = vX \Rightarrow \dfrac{dY}{dX} = v + X\dfrac{dv}{dX}$

$\therefore$ ③ $\Rightarrow v + X\dfrac{dv}{dX} = \dfrac{-1+4v}{4+v}$

$$\Rightarrow \frac{dv}{dx} = \frac{-1+4V}{4+V} - V \Rightarrow x\frac{dv}{dx} = \frac{-1+4V-V(4+V)}{4+V}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-1-V^2}{4+V}$$

$$\Rightarrow \left(\frac{4+V}{1+V^2}\right)dV = -\frac{dx}{x}$$, var. sep. form.

$$\Rightarrow \frac{4}{1+V^2}dv + \frac{V}{1+V^2}dv = -\frac{dx}{x}$$

$$\Rightarrow 4\int\frac{1}{1+V^2}dv + \frac{1}{2}\int\frac{2V}{1+V^2}dv = -\int\frac{dx}{x} + C$$

$$\Rightarrow 4\tan^{-1}(V) + \frac{1}{2}\log(1+V^2) = -\log X + C$$

$$\Rightarrow 4\tan^{-1}\left(\frac{Y}{X}\right) + \frac{1}{2}\log\left(1+\frac{Y^2}{X^2}\right) + \log X = C$$

$$\Rightarrow 4\tan^{-1}\left(\frac{Y}{X}\right) + \log\left(\frac{\sqrt{X^2+Y^2}}{X}\right) + \log X = C$$

$$\Rightarrow 4\tan^{-1}\left(\frac{Y}{X}\right) + \log\sqrt{X^2+Y^2} = C$$

Replace $X = x-1$ and $Y = y+2$ we get

$$4\tan^{-1}\left(\frac{y+2}{x-1}\right) + \log\sqrt{(x-1)^2+(y+2)^2} = C$$, is the required $\underline{\text{Sol}^n}$

Problem 1.2. *Solve* $(x + y + 1)dx - (2x + 2y + 3)dy = 0$

Ans:- $\frac{dy}{dx} = \frac{x+y+1}{2x+2y+3} = \frac{(x+y)+1}{2(x+y)+3}$ —— (1)

put $x+y=t \Rightarrow \frac{dt}{dx} = 1+\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{dt}{dx}-1$

$\therefore (1) \Rightarrow \frac{dt}{dx} - 1 = \frac{t+1}{2t+3}$

$\Rightarrow \frac{dt}{dx} = \frac{t+1}{2t+3} + 1 = \frac{3t+4}{2t+3}$

$\Rightarrow \left(\frac{2t+3}{3t+4}\right)dt = dx$, Var. sep. form.

Dividend = (quotient)(divisor) + Remainder

$$3t+4 \,\overline{\big)\, 2t+3 \,}^{\,2/3}$$
$$\frac{\quad 2t + 8/3}{\quad\quad 1/3}$$

$\therefore \frac{2t+3}{3t+4} = \frac{2}{3} + \frac{1/3}{3t+4}$ ✓

$\therefore \left(\frac{2}{3} + \frac{1}{3}\cdot\frac{1}{3t+4}\right)dt = dx$

$\Rightarrow \frac{2}{3}\int dt + \frac{1}{3}\int \frac{dt}{3t+4} = \int dx + C$

$$\Rightarrow \frac{2}{3}t + \frac{1}{3}\cdot\frac{1}{3}\log(3t+4) = x + C$$

$$\Rightarrow \frac{2}{3}(x+y) + \frac{1}{9}\log(3x+3y+4) = x + C$$

$$\Rightarrow -\frac{1}{3}x + \frac{2}{3}y + \frac{1}{9}\log(3x+3y+4) = C$$

$$\Rightarrow -3x + 6y + \log(3x+3y+4) = K$$

Problem 1.3. *Solve* $(3y + 2x + 4)dx - (4x + 6y + 5)dy = 0$

Ans:- Given $\frac{dy}{dx} = \frac{2x+3y+4}{4x+6y+5} = \frac{(2x+3y)+4}{2(2x+3y)+5}$ —— (1)

put $2x+3y=t \Rightarrow \frac{dt}{dx} = 2+3\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{3}\left(\frac{dt}{dx} - 2\right)$

$\therefore$ (1) $\Rightarrow \frac{1}{3}\left(\frac{dt}{dx} - 2\right) = \frac{t+4}{2t+5}$

$\Rightarrow \left(\frac{2t+5}{7t+22}\right)dt = dx$, var. sep. form

$$\begin{array}{r|l} & \ \ 2/7 \\ \hline 7t+22 & 2t+5 \\ & 2t+44/7 \\ \hline & \ \ -9/7 \end{array}$$

$\therefore \frac{2t+5}{7t+22} = \frac{2}{7} - \frac{9}{7} \times \frac{1}{7t+22}$

$\Rightarrow \int\left(\frac{2}{7} - \frac{9}{7} \cdot \frac{1}{7t+22}\right)dt = \int dx + C$

$$\Rightarrow \frac{2}{7}t - \frac{9}{7}\cdot\frac{1}{7}\log(7t+22) = x + C$$

$$\Rightarrow \frac{2}{7}(2x+3y) - \frac{9}{49}\log(14x+21y+22) = x + C$$

$$\Rightarrow 14(2x+3y) - 9\log(14x+21y+22) = 49x + K$$

$$\Rightarrow -21x + 42y - 9\log(14x+21y+22) = K$$

2. Practice problems

Problem 2.1. *Solve* $(x-4y-9)\,dx+(4x+y-2)\,dy=0$
Ans: $8\tan^{-1}\left(\dfrac{y+2}{x-1}\right)+\log\left[(x-1)^2+(y+2)^2\right]=C$

Problem 2.2. *Solve* $(3y-7x-7)\,dx+(7y-3x+3)\,dy=0$
Ans: $(x+y-1)^5(x-y-1)^2=C$

Problem 2.3. *Solve* $(x+2y-3)\,dx+(2x+y-3)\,dy=0$
Ans: $(x+y-2)=C(x-y)^3$

Problem 2.4. *Solve* $(x+y+1)\,dx+(2x+2y+3)\,dy=0$
Ans: $x=2y+\dfrac{1}{3}\log\left(3x+3y+4\right)+C$

Problem 2.5. *Solve* $\dfrac{dy}{dx}=\dfrac{4x-6y-1}{2x-3y+2}$
Ans: $x=\dfrac{-1}{4}\left[(2x-3y)+\dfrac{15}{4}\log\left(2x-3y-\dfrac{7}{4}\right)\right]+C$

3. Linear differential equation

Definition 3.1. A differential equation is said to be **linear** if the dependent variable and its differential coefficient occur only in the first degree and not multiplied together.

The linear differential equation of the first order, also known as **Leibnitz's linear equation.**

The general form of a linear differential equation is,

$$\frac{dy}{dx} + P(x)y = Q(x) \text{ (linear in } y\text{)} \quad (1)$$

or

$$\frac{dx}{dy} + P(y)x = Q(y) \text{ (linear in } x\text{)}$$

Let $e^{\int P(x)dx}$ be the I.F.

Multiply both sides of (1) by $e^{\int P(x)dx}$

we get,

$$e^{\int P(x)dx} \cdot \frac{dy}{dx} + \Big(\underbrace{e^{\int P(x)dx}}_{= \frac{d}{dx}\left(e^{\int P(x)dx}\right)} \cdot P(x)\Big) y = e^{\int P(x)dx} Q(x)$$

$$\frac{d}{dx}\left(e^{\int P(x)dx} \cdot y\right) = e^{\int P(x)dx} \cdot Q(x)$$

Integrating both sides w.r.t x,

$$\Rightarrow e^{\int P(x)dx} \cdot y = \int \left[e^{\int P(x)dx} \cdot Q(x)\right] dx + C$$

$$\Rightarrow y \cdot I.F = \int (I.F \cdot Q(x))\, dx + C$$

Problem 3.2. *Solve* $\frac{dy}{dx} + y\sec x = \tan x.$ —— ①

Solution: Here $P(x) = \sec x$ $\quad Q(x) = \tan x$

$\therefore$ $I.F = e^{\int P(x)dx} = e^{\int \sec x\, dx}$

$= e^{\log(\sec x + \tan x)}$

$= \sec x + \tan x$

$\therefore$ Solⁿ of ① is,

$$y \cdot (I.F) = \int (I.F)\, Q(x)\, dx + C$$

$$\Rightarrow y(\sec x + \tan x) = \int \tan x(\sec x + \tan x)dx + C$$

$$\Rightarrow y(\sec x + \tan x) = \int \left((\sec x \tan x) + \tan^2 x\right)dx + C$$

$$\Rightarrow y(\sec x + \tan x) = \sec x + \int (\sec^2 x - 1)dx + C$$

$$\Rightarrow y(\sec x + \tan x) = \sec x + \tan x - x + C$$

Problem 3.3. *Solve* $(1+x^2)\dfrac{dy}{dx}+y=e^{\tan^{-1}x}$.

Solution:

$$\frac{dy}{dx}+\left(\frac{1}{1+x^2}\right)y=\frac{e^{\tan^{-1}x}}{1+x^2}, \text{ linear d.e.}$$

Here $P(x)=\dfrac{1}{1+x^2}$

$$\therefore \text{I.F} = e^{\int P(x)\,dx} = e^{\tan^{-1}x}$$

$$\therefore \text{Sol}^n \text{ is, } y e^{\tan^{-1}x} = \int \frac{e^{\tan^{-1}x}}{1+x^2}\, e^{\tan^{-1}x}\cdot dx + C$$

put $e^{\tan^{-1}x}=t$

$$\Rightarrow \frac{dt}{dx} = e^{\tan^{-1}x}\cdot\frac{1}{1+x^2}$$

$$\Rightarrow \frac{e^{\tan^{-1}x}}{1+x^2}\,dx = dt$$

$$\Rightarrow y e^{\tan^{-1}x} = \int t\cdot dt + C$$

$$\Rightarrow y e^{\tan^{-1}x} = \frac{t^2}{2}+C$$

$$\Rightarrow y e^{\tan^{-1}x} = \frac{\left(e^{\tan^{-1}x}\right)^2}{2}+C$$

Problem 3.4. *Solve* $(1+y^2)\,dx = (\tan^{-1} y - x)\,dy$.

Solution: we can write,

$$\frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2}$$

$$\Rightarrow \frac{dx}{dy} + \left(\frac{1}{1+y^2}\right)x = \frac{\tan^{-1}y}{1+y^2}$$, linear d.e in x

Here $P(y) = \frac{1}{1+y^2}$

$$\therefore \text{I.F} = e^{\int P(y)dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

$\therefore$ Sol^n, $$x \cdot e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2} \cdot e^{\tan^{-1}y}\,dy + C$$

$$\Rightarrow x \cdot e^{\tan^{-1}y} = \int t e^{+t}\,dt + C$$

put $\tan^{-1}y = t$

$$\frac{dt}{dy} = \frac{1}{1+y^2}$$

$$\Rightarrow dt = \frac{dy}{1+y^2}$$

$$\Rightarrow xe^{\tan^{-1}y} = te^t - \int e^t\,dt + C$$

$$\Rightarrow xe^{\tan^{-1}y} = e^t(t-1) + C$$

$$\Rightarrow xe^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + C$$

$$\Rightarrow x = (\tan^{-1}y - 1) + Ce^{-\tan^{-1}y}$$

$\frac{dx}{dy} + P(y)x = Q(y)x^n$ ✓

4. Bernoulli's differential equation

A differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \quad \text{--- (1)}$$

is reducible to the Leibnitz's linear equation, is called a **Bernoulli's differential equation.**

Divide bothsides of (1) by 'y^n'

we get,

$$\frac{1}{y^n}\frac{dy}{dx} + P(x)\cdot\frac{1}{y^{n-1}} = Q(x) \quad \text{--- } (*)$$

put $t = \frac{1}{y^{n-1}} = y^{-(n-1)} \Rightarrow \frac{dt}{dx} = -(n-1)y^{-n}\cdot\frac{dy}{dx}$

$$\Rightarrow \frac{dt}{dx} = \frac{-(n-1)}{y^n}\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{y^n}\frac{dy}{dx} = \frac{-1}{n-1}\frac{dt}{dx}$$

$$\left(\frac{-1}{n-1}\right)\frac{dt}{dx} + t\,P(x) = Q(x)$$

$$\Rightarrow \frac{dt}{dx} - (n-1)t\,P(x) = -(n-1)Q(x),$$

linear d.eq^n in 't'

$$\frac{dy}{dx}+P(x)y=Q(x)y^n$$

Problem 4.1. *Solve* $x\frac{dy}{dx}+y=x^3y^6$. —— (1)

(1) $\Rightarrow \frac{dy}{dx}+\frac{y}{x}=x^2y^6$ —— (2)

Divide by y^6 we get,

$$\frac{1}{y^6}\frac{dy}{dx}+\frac{1}{x}\cdot\frac{1}{y^5}=x^2 \quad \text{—— (3)}$$

put $\frac{1}{y^5}=t \Rightarrow \frac{dt}{dx}=\frac{-5}{y^6}\frac{dy}{dx}$

$$\Rightarrow \frac{1}{y^6}\frac{dy}{dx}=\frac{-1}{5}\frac{dt}{dx}$$

(3) $\Rightarrow \frac{-1}{5}\frac{dt}{dx}+t\cdot\frac{1}{x}=x^2$

$\Rightarrow \frac{dt}{dx}+\left(\frac{-5}{x}\right)t=-5x^2$, linear d.e in 't'

Here $P(x)=-5/x$

$\Rightarrow$ I.F $= e^{\int -5/x\,dx}=e^{\log(1/x^5)}=\frac{1}{x^5}$

$\therefore\ t\left(\frac{1}{x^5}\right)=\int -5x^2\cdot\frac{1}{x^5}dx+C$

$\Rightarrow \frac{1}{x^5y^5}=-5\cdot\frac{1}{-2x^2}+C \Rightarrow \frac{1}{x^5y^5}=\frac{5}{2x^2}+C$

Contd..

Problem 4.2. *Solve* $\frac{dz}{dx} + \frac{z}{x}\log z = \frac{z}{x}(\log z)^2$ —— (1)

Ans:- $\frac{dz}{dx} + (z\log z)\cdot\frac{1}{x} = \frac{1}{x}\underbrace{z(\log z)^2}$

Divide both sides by $z(\log z)^2$ we get

$$\frac{1}{z(\log z)^2}\frac{dz}{dx} + \frac{1}{\log z}\cdot\frac{1}{x} = \frac{1}{x}. \quad \text{—— (2)}$$

put $\frac{1}{\log z} = t \Rightarrow t = (\log z)^{-1}$

$$\Rightarrow \frac{dt}{dx} = -1(\log z)^{-2}\cdot\frac{1}{z}\cdot\frac{dz}{dx}$$

$$\Rightarrow -\frac{dt}{dx} = \frac{1}{z(\log z)^2}\cdot\frac{dz}{dx}$$

$$(2) \Rightarrow -\frac{dt}{dx} + t\cdot\frac{1}{x} = \frac{1}{x}$$

linear d.e in 't'

$$\Rightarrow \frac{dt}{dx} + \left(\frac{-1}{x}\right)t = \frac{-1}{x}$$

$$I.F = e^{\int\frac{-1}{x}dx} = e^{\log(1/x)} = \frac{1}{x}$$

$$\therefore \quad t\cdot\left(\frac{1}{x}\right) = \int\left(\frac{-1}{x}\cdot\frac{1}{x}\right)dx + C$$

Contd..

$$\Rightarrow \frac{1}{x\log z} = -\int x^{-2}\,dx + c.$$

$$\Rightarrow \frac{1}{x\log z} = \frac{1}{x} + c, \text{ req'd sol}^n.$$

Problem 4.3. *Solve* $xy(1+xy^2)dy = dx$

Ans:- $\frac{dx}{dy} = xy(1+xy^2) = xy + x^2y^3$

$\Rightarrow \frac{dx}{dy} - xy = y^3 \cdot x^2$ —— (1)

Divide both sides of (1) by x^2,

$\frac{1}{x^2}\frac{dx}{dy} - \frac{y}{x} = y^3$ —— (2)

put $\frac{-1}{x} = t \Rightarrow \frac{dt}{dy} = \frac{+1}{x^2} \cdot \frac{dx}{dy}$

$\therefore$ (2) $\Rightarrow \frac{dt}{dy} + yt = y^3$, linear d.e. in t

I.F $= e^{\int y\,dy} = e^{y^2/2}$

$\therefore\ t \cdot e^{y^2/2} = \int e^{y^2/2} \cdot y^3\,dy + C$

$\Rightarrow -\frac{e^{y^2/2}}{x} = \int e^{y^2/2}\, y^2 \cdot y\,dy + C$

$\Rightarrow -\frac{e^{y^2/2}}{x} = \int e^{z} \cdot 2(z)\,dz + C$

put $\frac{y^2}{2} = z$

$\Rightarrow \frac{dz}{dy} = y \Rightarrow y\,dy = dz$

Contd..

$$\Rightarrow -\frac{e^{y^2/2}}{x} = 2\int z e^{z}\,dz + c$$

$$= 2\left(z e^{z} - e^{z}\right) + c$$

$$\Rightarrow -\frac{e^{y^2/2}}{x} = 2 e^{y^2/2}\left(\frac{y^2}{2} - 1\right) + c$$

$$\Rightarrow \frac{1}{x} = \underline{\underline{(2 - y^2) + c_1 e^{-y^2/2}}}$$

Problem 4.4. *Solve* $\frac{dy}{dx} + x\sin 2y = x^3 \cos^2 y$ —— (1)

Ans:- Divide both sides by $\cos^2 y$ in (1)

$$\frac{1}{\cos^2 y}\frac{dy}{dx} + 2x\tan y = x^3 \quad \text{——} \ (2)$$

put $2\tan y = t \Rightarrow \frac{dt}{dx} = 2\sec^2 y \cdot \frac{dy}{dx}$

$$\Rightarrow \frac{1}{\cos^2 y}\frac{dy}{dx} = \frac{1}{2}\frac{dt}{dx}$$

$$(2) \Rightarrow \frac{1}{2}\frac{dt}{dx} + x \cdot t = x^3$$

$$\Rightarrow \frac{dt}{dx} + 2x \cdot t = 2x^3 \quad \text{——} \ (3)$$

linear d.e. in 't'

$$I.F = e^{\int 2x\,dx} = e^{x^2}$$

$$\therefore \ t \cdot e^{x^2} = \int 2x^3 \cdot e^{x^2}\,dx + C$$

$$\Rightarrow 2\tan y\,(e^{x^2}) = \int x^2 e^{x^2} \cdot 2x\,dx + C$$

$$\Rightarrow 2e^{x^2}\tan y = \int z e^z\,dz + C$$

put $x^2 = z$

$2x\,dx = dz$

Contd..

$$\Rightarrow 2e^{x^2}\tan y = e^{z}(z-1) + C$$

$$\Rightarrow 2e^{x^2}\tan y = e^{x^2}(x^2-1) + C$$

$$\Rightarrow 2\tan y = (x^2-1) + Ce^{-x^2}$$

5. Practice problems

Problem 5.1. *Solve* $(x+1)\dfrac{dy}{dx} - y = e^{3x}(x+1)^2$

Ans: $y = \dfrac{1}{3}e^{3x}(x+1) + C(x+1)$

Problem 5.2. *Solve* $\left(\dfrac{e^{-2\sqrt{x}}}{\sqrt{x}} - \dfrac{y}{\sqrt{x}}\right)\dfrac{dx}{dy} = 1$

Ans: $ye^{2\sqrt{x}} = 2\sqrt{x} + C$

Problem 5.3. *Solve* $\tan y\dfrac{dy}{dx} + \tan x = \cos y\cos^2 x$

Ans: $\sec y = (C + \sin x)\cos x$

Problem 5.4. *Solve* $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1+x)e^x \sec y$

Ans: $\sin y = (1+x)(e^x + C)$

Problem 5.5. *Solve* $y(2xy + e^x)dx = e^x dy$

Ans: $y = \dfrac{e^x}{C - x^2}$