PROPOSITIONAL AND PREDICATE CALCULUS

October 21, 2020

Proposition Calculus

A declarative sentence that is either true or false is called "**PROPOSITION**"

Example

It rained yesterday.

"True" or "False" are called the truth vales of the proposition and are denoted by T and F respectively.

A proposition that is true under all circumstances is called "**Tautology**"

Example

15 is divisible by 3.

A proposition that is false under all circumstances is called "**Contradiction**"

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Example

-3 is a natural number.

Definition

Two or more propositions can be combined using words like "and, "or", "iff", "if, then" etc. These are called **Logical Connectivities**.

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A proposition having one or more logical connectivities is called a **Compound Proposition**. Otherwise is called **Simple**/ **Atom**

Definition

Two propositions p and q are said to be **Equivalent** if when p is T, q is also T and when p is F, q is also F and conversely.

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Example

p: He was born in 1934

q: He'll be 60 years old in 1994

p and q are equivalent

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Example

p: He was born in 1934
q: He'll be 60 years old in 1994
p and q are equivalent

Example

p: x is a prime number
q: x is not divisible by 2
p and q are not equivalent, as x not divisible by 2 doesn't mean its prime

Definition

Let p be a proposition, we define **Negation** of p denoted by $\neg$**p** to be a proposition which is true when p is false and is false when p is true

$\neg$	p	$\neg p$
	T	F
	F	T

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Example

If p is " monthly volume of sales is less than 20K", then negation p is " monthly volume of sales exceeds or equal to 20K"

Definition

Let p and q be two propositions. The **Disjunction** of two propositions is denoted by **p** $\vee$ **q(read as p or q)**

$\vee$	p	q	$p \vee q$
	T	T	T
	T	F	T
	F	T	T
	F	F	F

Definition

Let p and q be two propositions. The **Conjunction** of two propositions is denoted by **p** $\wedge$ **q(read as p and q)**

$\wedge$	p	q	$p \wedge q$
	T	T	T
	T	F	F
	F	T	F
	F	F	F

Definition

The **Conditional** statement is denoted by $\mathbf{p} \rightarrow \mathbf{q}$(**read as if p then q**)

$\rightarrow$	p	q	$p \rightarrow q$
	T	T	T
	T	F	F
	F	T	T
	F	F	T

Note: 1) p is called the " first component" or "ANTECEDENT" and q is called the "second component" or "CONSEQUENT"

Note: 2) For the conditional $p \to q$,

(i) $q \to p$ is called "converse"

p	q	$p \to q$	$q \to p$	$\neg p \to \neg q$	$\neg q \to \neg p$	$\neg p \vee q$
T	T	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	F	F	T	T
F	F	T	T	T	T	T

Table: 1

Note: 2) For the conditional $p \to q$,

(i) $q \to p$ is called "converse"

(ii) $\neg p \to \neg q$ is called " inverse"

p	q	$p \to q$	$q \to p$	$\neg p \to \neg q$	$\neg q \to \neg p$	$\neg p \vee q$
T	T	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	F	F	T	T
F	F	T	T	T	T	T

Table: 1

Note: 2) For the conditional $p \to q$,

(i) $q \to p$ is called "converse"

(ii) $\neg p \to \neg q$ is called " inverse"

(iii) $\neg q \to \neg p$ is called "contrapositive"

p	q	$p \to q$	$q \to p$	$\neg p \to \neg q$	$\neg q \to \neg p$	$\neg p \vee q$
T	T	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	F	F	T	T
F	F	T	T	T	T	T

Table: 1

Observations

Note: 3) From Table 1 we make the following observations:

(i) $p \to q$ and $\neg q \to \neg p$ are logically equivalent

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(ii) Inverse and Converse are logically equivalent. i.e., $q \to p$ and $\neg p \to \neg q$ are logically equivalent

Observations

Note: 3) From Table 1 we make the following observations:

(i) $p \to q$ and $\neg q \to \neg p$ are logically equivalent

(ii) Inverse and Converse are logically equivalent. i.e., $q \to p$ and $\neg p \to \neg q$ are logically equivalent

(iii) $p \to q$ and $\neg p \vee q$ are logically equivalent

Problem 1

Question: There are two restaurants next to each other. One has a sign that says " Good food is not cheap". The other has a sign that says " Cheap food is not good". Are the signs saying the same thing?

Solution : Let A: Food is good

B: Food is cheap

We have to examine $A \to \neg B$ and $B \to \neg A$

A	B	$A \to \neg B$	$B \to \neg A$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

Inference: Both are saying the same thing.

Exercise 1

Question: John made two statements:

- **I love Lucy**

Given that John either told the truth or lied in both the cases, determine whether John really loves Lucy?

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Question: John made two statements:

- **I love Lucy**
- **If I love lucy, then I also love Vivian.**

Given that John either told the truth or lied in both the cases, determine whether John really loves Lucy?

Definition

Let p and q be two propositions. The **Biconditional** is denoted by $p \leftrightarrow q$ read as "**p iff q**

$\leftrightarrow$	p	q	$p \leftrightarrow q$
	T	T	T
	T	F	T
	F	T	F
	F	F	T

Problem 2

Question: An island has 2 tribes of natives. Any native from the first tribe always tells the truth, while any native from the other tribe always lies. You arrive at island and ask a native if there is gold at the island. He answers "there is gold on the island iff I always tell the truth". Which tribe is he from? Is there gold on the island?

Solution : Let p: There is gold on the island

q: I always tell the truth

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Consider the following cases:

Case 1: If the person belongs to first tribe. Then q is true and the statement $p \leftrightarrow q$ is true. From the truth table above, p is also true. Therefore, "there is gold"

Thus there is gold on the island and the native could have been from either tribe.

Consider the following cases:

Case 1: If the person belongs to first tribe. Then q is true and the statement $p \leftrightarrow q$ is true. From the truth table above, p is also true. Therefore, "there is gold"

Case 2: If the person belongs to second tribe. Then q is false and the statement $p \leftrightarrow q$ must be false. From the truth table above, p is true. Therefore, "there is gold"

Thus there is gold on the island and the native could have been from either tribe.

Well formed formulas

A WFF is a formula generated using the following groups:

i) A statement variable is a WFF

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i) A statement variable is a WFF

ii) If A is WFF, then $\neg A$ is also a WFF

iii) If A and B are WFF's, $A \vee B$, $A \wedge B$, $A \rightarrow B$, $A \leftrightarrow B$ are also WFFs.

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i) A statement variable is a WFF

ii) If A is WFF, then $\neg A$ is also a WFF

iii) If A and B are WFF's, $A \vee B$, $A \wedge B$, $A \rightarrow B$, $A \leftrightarrow B$ are also WFFs.

iv) A string of symbols consisting of statement variables, connectivities and parenthesis is a WFF iff it can be obtained by finite many applications of the rules (i), (ii) and (iii)

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i) A statement variable is a WFF

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iii) If A and B are WFF's, $A \vee B$, $A \wedge B$, $A \rightarrow B$, $A \leftrightarrow B$ are also WFFs.

iv) A string of symbols consisting of statement variables, connectivities and parenthesis is a WFF iff it can be obtained by finite many applications of the rules (i), (ii) and (iii)

Example

(1) $p \wedge q$, $\neg(p \wedge q)$, $(\neg(p \rightarrow q)) \vee r$, $((p \rightarrow q) \rightarrow r$ are WFFs.
(2) $p \wedge q \rightarrow r$ is not a WFF as it can be $(p \wedge q) \rightarrow r$ or $p \wedge (q \rightarrow r)$

Equivalence of formulas

Definition

Let A and B be two statement formulas and $P_1, P_2, \cdots P_n$ denote all the variables occurring in A and B. If the truth value of A is same as that of B for each of 2^n possible set of assignments to the variables $P_1, P_2, \cdots P_n$, then A and B are said to be equivalent. We write as $A \Leftrightarrow B$.
Two statement formulas A and B are equivalent iff $A \leftrightarrow B$ is a Tautology.

Table of equivalence

(1) $\neg\neg p \Leftrightarrow q$

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(2) Commutative: (a) $p \vee q \Leftrightarrow q \vee p$
(b) $p \wedge q \Leftrightarrow q \wedge p$

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(2) Commutative: (a) $p \vee q \Leftrightarrow q \vee p$
(b) $p \wedge q \Leftrightarrow q \wedge p$

(3) Associative: (a) $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$
(b) $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

Table of equivalence

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(2) Commutative: (a) $p \vee q \Leftrightarrow q \vee p$
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(3) Associative: (a) $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$
(b) $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

(4) Distributive: (a) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
(b) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

Table of equivalence

(1) $\neg\neg p \Leftrightarrow q$

(2) Commutative: (a) $p \vee q \Leftrightarrow q \vee p$
(b) $p \wedge q \Leftrightarrow q \wedge p$

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(4) Distributive: (a) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
(b) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

(5) Absorption: (a) $p \vee (p \wedge q) \Leftrightarrow p$
(b) $p \wedge (p \vee q) \Leftrightarrow p$

Table of equivalence

(1) $\neg\neg p \Leftrightarrow q$

(2) Commutative: (a) $p \vee q \Leftrightarrow q \vee p$
(b) $p \wedge q \Leftrightarrow q \wedge p$

(3) Associative: (a) $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$
(b) $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

(4) Distributive: (a) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
(b) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

(5) Absorption: (a) $p \vee (p \wedge q) \Leftrightarrow p$
(b) $p \wedge (p \vee q) \Leftrightarrow p$

(6) Idempotent: (a) $(p \wedge p) \Leftrightarrow p$
(b) $(p \vee p) \Leftrightarrow p$

Table of equivalence

(7) (a) $p \wedge (\neg p) \Leftrightarrow F$
(b) $p \vee (\neg p) \Leftrightarrow T$

Table of equivalence

(7) (a) $p \wedge (\neg p) \Leftrightarrow F$
    (b) $p \vee (\neg p) \Leftrightarrow T$

(8) (a) $p \vee F \Leftrightarrow p$
    (b) $p \wedge F \Leftrightarrow F$

Table of equivalence

(7) (a) $p \wedge (\neg p) \Leftrightarrow F$
(b) $p \vee (\neg p) \Leftrightarrow T$

(8) (a) $p \vee F \Leftrightarrow p$
(b) $p \wedge F \Leftrightarrow F$

(9) (a) $p \vee T \Leftrightarrow T$
(b) $p \wedge T \Leftrightarrow p$

Table of equivalence

(7) (a) $p \wedge (\neg p) \Leftrightarrow F$
(b) $p \vee (\neg p) \Leftrightarrow T$

(8) (a) $p \vee F \Leftrightarrow p$
(b) $p \wedge F \Leftrightarrow F$

(9) (a) $p \vee T \Leftrightarrow T$
(b) $p \wedge T \Leftrightarrow p$

(10) (a) $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$
(b) $\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$

Table of equivalence

(7) (a) $p \wedge (\neg p) \Leftrightarrow F$
(b) $p \vee (\neg p) \Leftrightarrow T$

(8) (a) $p \vee F \Leftrightarrow p$
(b) $p \wedge F \Leftrightarrow F$

(9) (a) $p \vee T \Leftrightarrow T$
(b) $p \wedge T \Leftrightarrow p$

(10) (a) $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$
(b) $\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$

(11) (a) $p \rightarrow q \Leftrightarrow \neg p \vee q$
(b) $\neg(p \rightarrow q) \Leftrightarrow p \wedge \neg q$

Table of equivalence

(7) (a) $p \wedge (\neg p) \Leftrightarrow F$
(b) $p \vee (\neg p) \Leftrightarrow T$

(8) (a) $p \vee F \Leftrightarrow p$
(b) $p \wedge F \Leftrightarrow F$

(9) (a) $p \vee T \Leftrightarrow T$
(b) $p \wedge T \Leftrightarrow p$

(10) (a) $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$
(b) $\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$

(11) (a) $p \rightarrow q \Leftrightarrow \neg p \vee q$
(b) $\neg(p \rightarrow q) \Leftrightarrow p \wedge \neg q$

(12) (a) $p \rightarrow q \Leftrightarrow (\neg q \rightarrow \neg p)$
(b) $q \rightarrow p \Leftrightarrow (\neg p \rightarrow \neg q)$

Problem 4

Question: Show that $(\neg p \wedge (\neg q \wedge r)) \vee (q \wedge r) \vee (p \wedge r) \Leftrightarrow r$

$$\begin{aligned}
(\neg p \wedge (\neg q \wedge r)) \vee (q \wedge r) \vee (p \wedge r) &\Leftrightarrow (\neg p \wedge (\neg q \wedge r)) \vee (r \wedge (q \vee p)) \\
&\Leftrightarrow ((\neg p \wedge \neg q) \wedge r) \vee (r \wedge (p \vee q)) \\
&\Leftrightarrow (\neg (p \vee q) \wedge r) \vee (r \wedge (p \vee q)) \\
&\Leftrightarrow r \wedge [(p \vee q) \vee \neg (p \vee q)] \\
&\Leftrightarrow r \wedge T \\
&\Leftrightarrow r
\end{aligned}$$

Excerise 5

Question: Show that $p \to (q \to r) \Leftrightarrow p \to (\neg q \vee r) \Leftrightarrow (p \wedge q) \to r$

$$\begin{aligned} p \to (q \to r) &\Leftrightarrow p \to (\neg q \vee r) \\ &\Leftrightarrow \neg p \vee (\neg q \vee r) \\ &\Leftrightarrow (\neg p \vee \neg q) \vee r \\ &\Leftrightarrow \neg(p \wedge q) \vee r \\ &\Leftrightarrow (p \wedge q) \to r \end{aligned}$$

Problem 4

Question: Show that
$((p \vee q) \wedge \neg[(\neg p) \wedge (\neg q \vee \neg r)]) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$ **is a tautology**.
Solution: $[(p \vee q) \wedge \neg((\neg p) \wedge (\neg q \vee \neg r))] \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$
$= [(p \vee q) \wedge \neg(\neg p \wedge (\neg q \vee \neg r))] \vee (\neg p \wedge (\neg q \vee \neg r))$
$= [(p \vee q) \wedge \neg\neg p \vee (q \wedge r)] \vee (\neg p \wedge \neg(q \wedge r))$
$= [(p \vee q) \wedge (p \vee (q \wedge r))] \vee \neg(p \vee (q \wedge r))$
$= [(p \vee q) \wedge (p \vee q) \wedge (p \vee r)] \vee \neg(p \vee (q \wedge r))$
$= [(p \vee q) \wedge (p \vee r)] \vee \neg(p \vee (q \wedge r))$
$= [(p \vee (q \wedge r)] \vee \neg(p \vee (q \wedge r))$
$= T$

Excerise 6

Question: Show that $q \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$ **is a tautology**

Tautological Implications

Definition

A is said to tautologically imply to statement B if $A \to B$ is a tautology. In this case, we write $A \Rightarrow B$ (read as A implies B)

Table of Tautological Implications

$$
\begin{aligned}
(1)\quad p \wedge q &\implies p \\
p \wedge q &\implies q
\end{aligned}
$$

Table of Tautological Implications

$$
\begin{aligned}
(1)\quad & p \wedge q \implies p \\
& p \wedge q \implies q \\
(2)\quad & p \implies p \vee q \\
& q \implies p \vee q
\end{aligned}
$$

Table of Tautological Implications

(1) $p \wedge q \implies p$

$p \wedge q \implies q$

(2) $p \implies p \vee q$

$q \implies p \vee q$

(3) $\neg p \implies p \rightarrow q$

Table of Tautological Implications

(1) $p \wedge q \implies p$

$p \wedge q \implies q$

(2) $p \implies p \vee q$

$q \implies p \vee q$

(3) $\neg p \implies p \to q$

(4) $q \implies p \to q$

Table of Tautological Implications

$$
\begin{aligned}
&(1) \quad p \wedge q \implies p \\
& \quad p \wedge q \implies q \\
&(2) \quad p \implies p \vee q \\
& \quad q \implies p \vee q \\
&(3) \quad \neg p \implies p \to q \\
&(4) \quad q \implies p \to q \\
&(5) \quad \neg(p \to q) \implies p
\end{aligned}
$$

Table of Tautological Implications

(1) $p \wedge q \implies p$

$p \wedge q \implies q$

(2) $p \implies p \vee q$

$q \implies p \vee q$

(3) $\neg p \implies p \to q$

(4) $q \implies p \to q$

(5) $\neg(p \to q) \implies p$

(6) $\neg(p \to q) \implies \neg q$

Table of Tautological Implications

(1) $p \wedge q \implies p$

$p \wedge q \implies q$

(2) $p \implies p \vee q$

$q \implies p \vee q$

(3) $\neg p \implies p \to q$

(4) $q \implies p \to q$

(5) $\neg(p \to q) \implies p$

(6) $\neg(p \to q) \implies \neg q$

(7) $p \wedge (p \to q) \implies q$

Table of Tautological Implications

(8) $\neg q \wedge (p \to q) \implies \neg p$

Table of Tautological Implications

(8) $\neg q \wedge (p \to q) \implies \neg p$

(9) $\neg p \wedge (p \vee q) \implies q$

Table of Tautological Implications

(8) $\neg q \wedge (p \to q) \implies \neg p$

(9) $\neg p \wedge (p \vee q) \implies q$

(10) $(p \to q) \wedge (q \to r) \implies p \to r$

Table of Tautological Implications

$$
\begin{aligned}
&(8) && \neg q \wedge (p \to q) \implies \neg p \\
&(9) && \neg p \wedge (p \vee q) \implies q \\
&(10) && (p \to q) \wedge (q \to r) \implies p \to r \\
&(11) && (p \vee q) \wedge (p \to r) \wedge (q \to r) \implies r
\end{aligned}
$$

Problem 7

Question: Show that $\neg q \wedge (p \rightarrow q) \implies \neg p$

Solution: Suppose $\neg q \wedge (p \rightarrow q)$ is true.

$\neg q$ is true and $p \rightarrow q$ is true

q is false and $p \rightarrow q$ is true

$\implies p$ is false

$\implies \neg p$ is true

$\therefore \neg q \wedge (p \rightarrow q) \implies \neg p$

Remark

To show that $A \implies B$*, we can assume* B *is false and show that* A *is false. So the above problem can also be analyzed as follows: Consider again* $\neg p$ *is false,* $\implies p$ *is true. If* q *is true,* $\neg q$ *is false and its understood that* $\neg q \wedge (p \rightarrow q)$ *is false. If* q *is false,* $\neg q$ *is true and* $p \rightarrow q$ *is false. Again* $\neg q \wedge (p \rightarrow q)$ *is false.*

Problem 8

Question: Show that $\neg(p \to q) \implies \neg q$

Solution: We say that $A \implies b$ if $A \to B$ is true in all conditions

p	q	$p \to q$	$\neg(p \to q)$	$\neg q$	$\neg(p \to q) \to \neg q$
T	T	T	F	F	T
T	F	F	T	T	T
F	T	T	F	F	T
F	F	T	F	T	T

Exercise 9

Question: Show that $(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r) \implies r$

Exercise 10

Question: Prove that

(i) $\neg p \implies p \to q$

Exercise 10

Question: Prove that

(i) $\neg p \implies p \to q$

(ii) $p \wedge (p \to q) \implies q$

Exercise 10

Question: Prove that

(i) $\neg p \implies p \to q$

(ii) $p \wedge (p \to q) \implies q$

(iii) $p \wedge q \implies p$

Rules of Inference

Let A and B be two statement formula. We say that " B logically follows from A" or "B is a valid conclusion of the premise A" iff $A \to B$ is a Tautology i.e., $A \implies B$.
To demonstrate that a particular formula is valid consequence of a given set of premises, we use the follow rules of inference.

Rule P	:	*A premise may be introduced at any point in the derivation*
Rule T	:	*A formula S may be introduced in a derivation if S is tautologically implied by any one or more of the preceding formulas in the derivation*

Question : Demonstrate that r is a valid inference from the premises $p \to q,\ \ q \to r\ \ and\ \ p$

Solution :

$$\begin{array}{rl}
p \to q & (Rule\ P) \\
p & (Rule\ P) \\
q & (Rule\ T,\ p \wedge (p \to q) \implies q) \\
q \to r & (Rule\ P) \\
r & (Rule\ T, q \wedge (q \to r) \implies r)
\end{array}$$

Question : RVS follows logically from the premises $C \wedge D$, $C \vee D \to \neg H$, $\neg H \to (A \wedge \neg B)$, $(A \wedge \neg B) \to (R \vee S)$.
Solution :

$$\begin{aligned} C \vee D \to \neg H &\qquad (Rule\ P) \\ \neg H \to A \wedge \neg B &\qquad (Rule\ P) \\ C \vee D \to A \wedge \neg B &\qquad (Rule\ T) \\ A \wedge \neg B \to R \vee S &\qquad (Rule\ P) \\ C \vee D \to R \vee S &\qquad (Rule\ T) \\ C \vee D &\qquad (Rule\ P) \\ R \vee S &\qquad (Rule\ T) \end{aligned}$$

Question : Show that $S \vee r$ is tautologically implied by $(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow s)$

Solution :

$$\begin{array}{rl}
p \vee q & (Rule\ P) \\
\neg p \rightarrow q & (Rule\ T\ i.e.,\ p \rightarrow q \Leftrightarrow \neg p \vee q) \\
q \rightarrow S & (Rule\ P) \\
\neg p \rightarrow S & (Rule\ T) \\
\neg S \rightarrow p & (Rule\ T, p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p) \\
p \rightarrow r & (Rule\ P) \\
\neg S \rightarrow r & (Rule\ T) \\
S \vee r & (Rule\ T, p \rightarrow q \Leftrightarrow \neg(\neg p \vee q))
\end{array}$$

Exercise Q1 : $R \wedge (p \vee q)$ is a valid conclusion from the premises $p \vee q$, $q \to r$, $p \to M$ and $\neg M$.

Exercise Q2 : If A works hard, then either B or C will enjoy themselves. If B enjoys himself, then A will not work hard. If D enjoys himself, then C will not, therefore prove that if A works hard, D will not enjoy himself.

Solution hint :

$$\begin{aligned} A &: \textit{A works hard} \\ B &: \textit{B will enjoy himself} \\ C &: \textit{C will enjoy himself} \\ D &: \textit{D will enjoy himself} \end{aligned}$$

To prove $A \to \neg D$ follows from $A \to B \vee C$, $B \to \neg A$ and $D \to \neg C$

A part of declarative sentence describing the properties of an object or relation among objects is called a "**PREDICATE**"

Example

Consider two propositions, Ram is a Bachelor, Shyam is a Bachelor. Both Ram and Shyam have the same property of having bachelor. The part "is a bachelor" is called a predicate.

Notations

The predicate is denoted by capital letters and names of individuals or objects by small letters. Let "B" denote the predicate "is bachelor", then the sentence "x is a bachelor" can be written as $B(x)$, where x is a predicate variable $B(x)$ is also called a propositional function, which becomes a statement when values are submitted in place of x. A predicate requiring $m(>0)$ names is called $m-$place predicate

Example

x is taller than y; T(x,y)- the two place predicate

Universal and existential quantifiers

Quantifiers are words that refer to quantifiers such as "some" or "all" and indicate now frequently certain statement is true.
The phrase "for all" ($\forall$) is called the **UNIVERSAL QUANTIFIERS**

Example

All human beings are mortal. For all natural numbers "n", "$2n$" is an even number.

The phrase "there exists" ($\exists$) is called the
EXISTENTIAL QUANTIFIER

Example

There exists x such that $x^2 = 5$. This can be written as $\exists x P(x)$ where $P(x) : x^2 = 5$.

NOTE 1

"there exists" ($\exists$) represents the following:

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"there exists" ($\exists$) represents the following:

- there exists an x
- there is an x
- for some x
- there is atleast one x

NOTE 2

- $(\forall x)P(x)$ is true iff $P(x)$ is true $\forall x$ in U

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- $(\forall x)P(x)$ is true if $P(x)$ is true for atleast one x in U
- $(\forall x)P(x)$ is false if $P(x)$ is false for every x in U

Question 1

Example

Let $D = \{1, 2, 3, ...9\}$ Determine the truth value of each of the follow statements.

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- $(\forall x)x + 4 > 15$ $\quad F$

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Let $D = \{1, 2, 3, ...9\}$ Determine the truth value of each of the follow statements.

- $(\forall x)x + 4 < 15 \qquad T$
- $(\exists x)x + 4 = 10 \qquad T$
- $(\forall x)x + 4 > 15 \qquad F$
- $(\forall x)x + 4 \leq 10 \qquad F$

Question 2

Example

Symbolize the statement: All men are mortal

Solution: Let $M(x)$: x is an integer

$N(x)$: x is either positive or negative

$(\forall x)(M(x) \rightarrow N(x))$

Question 3

Example

Symbolize the statement: An integer is either positive or negative.

Solution: Let $M(x)$: x is a man

$H(x)$: x is a mortal

$(\forall x)(M(x) \rightarrow H(x))$

Negation of quantified statements

- $\neg(\forall x)P(x) \Leftrightarrow (\exists x)\neg P(x)$

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- $\neg(\exists x)P(x) :\Leftrightarrow (\forall x)\neg P(x)$

Question 4

Negate the following statements: For all real numbers x, if $x > 3$ then $x^2 > 9$

Solution: Given : Let $P(x) : x > 3$

$Q(x) : x^2 > 9$,

$\therefore (\forall x)(P(x) \rightarrow Q(x))$

$$\begin{aligned} \text{Negation is} \quad : \quad & \neg \forall x (P(x) \rightarrow Q(x)) \\ & \Leftrightarrow (\exists x) \neg (P(x) \rightarrow Q(x)) \\ & \Leftrightarrow (\exists x) \neg (P(x) \wedge \neg Q(x)) \end{aligned}$$

that is there exists a real number x such that $x > 3$ and $x^2 \leq 9$

Exercise Q5 : Negate the following statement " Every city in Canada is clean"

Rules of Inference

The method of derivation involving predicate formulas uses the rules of inference given for the statement calculus and also certain additional rules which are required to deal with the formulas involving quantifiers (In addition to Rule P and T)

(1) Rule US(Universal Specification)
From $(\forall x)A(x)$, we can conclude $A(y)$.

$$(\forall x)A(x) \implies A(y)$$

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(1) Rule US(Universal Specification)
From $(\forall x)A(x)$, we can conclude $A(y)$.

$$(\forall x)A(x) \implies A(y)$$

(2) Rule ES (Existential Specification)
From $(\exists x)A(x)$ one can conclude $A(y)$ provided that y is not free in any given premise and also not free in any prior step of the derivation.

$$(\exists x)A(x) \implies A(y)$$

(3) Rule EG(Extential Generalization)
From $A(x)$ one can conclude $(\exists y)A(y)$.

$$A(x) \implies (\exists y)A(y)$$

(3) Rule EG(Extential Generalization)
From $A(x)$ one can conclude $(\exists y)A(y)$.

$$A(x) \implies (\exists y)A(y)$$

(4) Rule UG (Universal Generalization)
From $A(x)$ one can conclude $(\forall y)A(y)$ provided that x is not free in any of the given premises and provided that if x is free in a prior step which resulted from use of ES, then no variables introduced by that use of ES appear free in $A(x)$

Question 5

$$\text{Show that } (\forall x)[H(x) \to M(x)] \wedge H(s) \implies M(s)$$

[Note that this problem is a symbolic representation or translation of a well known argument known as "Socrates argument" which is given by "All men are mortal
Socrates is a man
Therefore Socrates is a mortal"]
Solution : Denote $H(s)$: x is a man
$M(s)$: Socrates is mortal

$$\begin{array}{lrl}
(1) & (\forall x)[H(x) \to M(x)] & \textit{Rule P} \\
(2) & H(s) \to M(s) & \textit{Rule US} \\
(3) & H(s) & \textit{Rule P} \\
(4) & M(s) & \textit{Rule T}
\end{array}$$

Question 6

Show that $(\exists x)M(x)$ *follows logically from the premises*

$$(\forall x)[H(x) \to M(x)] \textit{ and } (\exists x)H(x)$$

Solution

(1)	$(\exists x)$	*Rule P*
(2)	$H(y)$	*Rule ES*
(3)	$(\forall x)[H(x) \to M(x)]$	*Rule P*
(4)	$[H(y) \to M(y)]$	*Rule US by* (3)
(5)	$M(y)$	*Rule T by* (2) *and* (4)
(6)	$(\exists x)M(x)$	*Rule EG*

Exercise 7

Show that

$$(\forall x)[P(x) \to Q(x)] \wedge \forall x[Q(x) \to R(x)] \implies (\forall x[P(x) \to R(x)]$$

Solution

(1)	$(\forall x)[P(x) \to Q(x)]$	*Rule P*
(2)	$[P(y) \to Q(y)]$	*Rule US by* (1)
(3)	$(\forall x)[Q(x) \to R(x)]$	*Rule P*
(4)	$[Q(y) \to R(y)]$	*Rule US by* (3)
(5)	$[Q(y) \to R(y)]$	*Rule T by* (2) *and* (4)
(6)	$(\forall x[P(x) \to R(x)]$	*Rule UG*

Exercise 8

Show that from

$$(a)(\forall x)[F(x) \wedge S(x)] \rightarrow \forall y[M(y) \rightarrow W(y)]$$

$$(b)(\exists y)[M(y) \wedge \neg W(y)]$$

the conclusion $(\forall x)[F(x) \rightarrow \neg S(x)]$ follows