Distributions

Distribution	PMF/PDF	Mean	Variance
Binomial distribution $X \sim B(n,p)$	$P(x) = {}^{n}C_{k}\, p^{k}(1-p)^{n-k} \quad ,k = 0,1,2,\dots,n$	$E(x) = np$	$V(x) = np(1-p)$
Poisson's Distribution $X \sim P(\alpha)$	$P(x) = \dfrac{e^{-\alpha}\alpha^{k}}{k!}, \quad k = 0,1,2,\dots, \alpha > 0$	$E(x) = \alpha = np$	$V(x) = \alpha = np$
Uniform Distribution $X \sim U(a,b)$	$f(x) = \begin{cases} \dfrac{1}{b-a}, & a < x < b \\ 0, & otherwise \end{cases}$	$E(x) = \dfrac{b+a}{2}$	$V(x) = \dfrac{(b-a)^{2}}{12}$
Normal Distribution $X \sim N(\mu,\sigma^{2})$	$f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\frac{(x-\mu)^{2}}{\sigma^{2}}}, \quad -\infty < x,\mu < \infty,\ \sigma > 0$	$E(x) = \mu$	$V(x) = \sigma^{2}$
Exponential Distribution $X \sim E(\lambda)$	$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & otherwise \end{cases}$	$E(x) = \dfrac{1}{\lambda}$	$V(X) = \dfrac{1}{\lambda^{2}}$
Gamma Distribution $X \sim G(r,\alpha)$	$f(x) = \begin{cases} \dfrac{x^{r-1}e^{-\alpha x}\alpha^{r}}{\Gamma(r)}, & x > 0,\ \alpha, r > 0 \\ 0, & elsewhere \end{cases}$	$E(x) = \dfrac{r}{\alpha}$	$V(x) = \dfrac{r}{\alpha^{2}}$
Chi-square Distribution $X \sim \chi^{2}(n)$	$f(x) = \begin{cases} \dfrac{x^{\frac{n}{2}-1} e^{-\frac{x}{2}}}{\Gamma(n/2)2^{\frac{n}{2}}}, & x > 0 \\ 0, & elsewhere \end{cases}$	$E(x) = n$	$V(x) = 2n$

Binomial Distribution: BIPARAMETRIC

Based on the above explanation, the properties of a Binomial Distribution are:

1. Each trial is independent.
2. There are only two possible outcomes in a trial – success or failure.
3. A total number of n identical trials are conducted.
4. The probability of success and failure is the same for all trials. (Trials are identical.)

The mathematical representation of binomial distribution is given by:

$$P(x) = {}^{n}C_{k} p^{k} (1-p)^{n-k} \quad ,k = 0,1,2,\ldots,n$$

Mean and Variance:

E(X)= $\sum_{x=1}^{n} \boldsymbol{x P(x)}$

$= \sum_{x=1}^{n} \boldsymbol{x} \ {}^{n}C_x \ p^x q^{n-x}$

$= \sum_{x=1}^{n} \boldsymbol{x} \quad {}^{n}C_x \ p^x q^{n-x}$

$= \sum_{x=1}^{n} \boldsymbol{x} \ \frac{n!}{x!\,(n-x)!} \ p^x q^{n-x}$

$=$ np $\sum_{x=1}^{n} \ \frac{(n-1)!}{(x-1)!\,(n-x)!} \ p^{x-1} q^{n-x}$

Substitute, x-1=s

We get, E(X)= np $\sum_{s=0}^{n-1} \ \frac{(n-1)!}{s!\,(n-s-1)!} \ p^s q^{n-s-1}$

$=$ np $\sum_{s=0}^{n-1} \ \frac{(n-1)!}{s!\,(n-s-1)!} \ p^s q^{n-1-s}$

$=$ np $(p+q)^{n-1}$

$=$ np

V(X)= $E(x^2) - [E(x)]^2 = E(x^2) - (np)^2$

Where, $E(x^2) = \sum_{x=1}^{n} (\boldsymbol{x^2 - x + x}) \ {}^{n}C_x \ p^x q^{n-x}$

$= \sum_{x=1}^{n} (\boldsymbol{x^2 - x}) \ {}^{n}C_x \ p^x q^{n-x} + \sum_{x=1}^{n} \boldsymbol{x} \ {}^{n}C_x \ p^x q^{n-x}$

$= \sum_{x=2}^{n} \boldsymbol{x\,(x-1)} \ \frac{n!}{x!\,(n-x)!} \ p^x q^{n-x} +$ np

$= n(n-1)p^2 \sum_{x=2}^{n} \ \frac{(n-2)!}{(x-2)!\,(n-x)!} \ p^{x-2} q^{n-x} +$ np

Substitute, x-2=s

$= n(n-1)p^2 \sum_{s=0}^{n-2} \ \frac{(n-2)!}{s!\,(n-s-2)!} \ p^s q^{n-s-2} +$ np

$= n(n-1)p^2 \ (p+q)^{n-2} +$ np

$= n(n-1)p^2 +$ np

V(X)= $E(x^2) - [E(x)]^2 = n(n-1)p^2 +$ np $- (np)^2$

V(X)= npq

Problems:

1. Six coins are tossed. Find the probability of getting:
   a) Exactly 3 heads --------P(X=3)--------5/16
   b) At least 3 heads -------P($X \geq 3$)-----21/32
   c) At most 3 heads -------P($X \leq 3$)-- --21/32
   d) At least 1 head ---------P($X \geq 1$) = 1-P(zero heads)

   Ans: n=6 X: number of heads
   p: getting head = ½
   q: getting tail = ½
2. What is the probability of getting a 6 at least once in 2 throws of a fair die.
   Ans: n=2 X: number of times 6 obtained
   P($X \geq 1$) ------------11/36

p: getting six = 1/6
q: not getting six = 5/6

3. A fair die is thrown 180 times. What is the expected number of sixes.
   Ans: n=180 X: number of times 6 obtained
   E(x) =np=30
4. A die is thrown 8 times. Find the probability that 3 appears.
   i. Two times……….P(X=2)= 0.2604
   ii. At least 7 times ……….P(X$\geq$ 7)= 0.0000244
   iii. Exactly one time ……..P(X=1)= 0.372
5. Two percent fuses manufactured by company are defective. Find the probability that a box containing 200 fuses contains.
   i. NO defective …………P(X=0)= 0.01758
   ii. 3 or more defective……….P(3 or more)=1- P(0,1,2)= 0.7649
6. Find the probability in a family of 4 children there will be at least one boy by assuming that probability of male birth is ½……….P(At least 1)= 1- P(NO boy)
7. A family has 6 children. Find the probability that there are fewer boys than girls.
   Ans: n=6 and X- number of boys can take the values 0, 1, or 2
8. The sum and product of mean and variance of binomial distribution are 24 and 128. Find the distribution.
   Solution: np+npq=24
   np.npq=128
   q= 2 and 0.5
   2 not possible therefore q=0.5 hence p=0.5
   We have np.npq=128 therefore n=32
   The binomial distribution = $^{32}C_x\ (0.5)^x\ (0.5)^{32-x}$
9. Numbers are selected at random one at a time from the two digit numbers 00,01,…, 99 with replacement. An event occurs iff the product of the two digits of selected number is 18. If 4 numbers are selected find the probability that the event occurs at least 3 times.
   Solution: n= 4
   p= 4/100 and q= 24/25
   P($X \geq 3$)= 97/254
10. A perfect die is tossed 100 times in sets of 8 the occurrence of 5 and 6 is called a success. How many times do you expect to get 3 success.
    Solution:
    p=1/3 and q=2/3 and n=8
    P (3 success)= $^8C_3\,(1/3)^3\,(2/3)^5$= 0.2731
    E (3 success)=100 x 0.2731= 27.31
11. Suppose that the probability for A to win a game of tennis against B is 0.4. A has an option of playing either a best of 3 games or a best of 5 games. Which option A should choose so that his probability of winning is greater.
    Solution:
    X: number of games A wins against B
    P= 0.4 and q= o.6
    When n= 3
    P($X \geq 2$)= 0.352
    When n= 5
    P($X \geq 3$)= 0.31744
12. An aircraft knows that 5% of the people making reservation on a certain flight will not show up. Consequently, their policy is to sell 52 tickets for the flight that can only hold 50 passengers. What is the probability that there will be a seat available for every passenger who turns up?
    Solution: n=52
    X: number of passengers who won't turn up
    p= passenger will not turn up= 0.05
    q= 0.95
    P($X \geq 2$)= 1- [P(X=0)+P(X=1)]= 0.7405

Poisson's Distribution: UNIPARAMETRIC

$$P(x) = \frac{e^{-\alpha}\alpha^k}{k!}, \quad k = 0,1,2,\ldots, \alpha > 0$$

$$\boldsymbol{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots}$$

Theorem: Let X be a binomial random variable with parameters n, p and pdf P(X=x)= $^nC_x\, p^x q^{n-x}$. Suppose n→∞; np=α then $\lim_{n\to\infty} P(X = x) = \frac{e^{-\alpha}\alpha^x}{x!}$ is a Poisson's distribution with parameter α.

Proof: General expression for binomial distribution is P(X)= $^nC_x\, p^x q^{n-x}$

$$= {}^nC_x\, p^x (1-p)^{n-x}$$

$$= \frac{n(n-1)(n-2)\ldots(n-x+1)}{x!} p^x (1-p)^{n-x}$$

Let $np = \alpha$

$$= \frac{n(n-1)(n-2)\ldots(n-x+1)}{x!} \left(\frac{\alpha}{n}\right)^x \left(1-\frac{\alpha}{n}\right)^{n-x}$$

$$= \frac{\alpha^x}{x!} [1.\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots(1-\frac{\{x+1\}}{n})] \frac{\left(1-\frac{\alpha}{n}\right)^n}{\left(1-\frac{\alpha}{n}\right)^x}$$

Let n→∞ and $\alpha = np$

$$\lim_{n\to\infty} P(X = x) = \lim_{n\to\infty} \left(\frac{\alpha^x}{x!}[1.\left(1-\frac{1}{n}\right)\ldots(1-\frac{\{x+1\}}{n})] \frac{\left(1-\frac{\alpha}{n}\right)^n}{\left(1-\frac{\alpha}{n}\right)^x}\right)$$

$$\lim_{n\to\infty} P(X = x) = \frac{\alpha^x\, e^{-\alpha}}{x!}$$

Mean and Variance:

E(X)= $\sum_{x=0}^{\infty} \boldsymbol{xP(x)}$

= $\sum_{x=1}^{\infty} \boldsymbol{x} \frac{e^{-\alpha}\alpha^x}{x!}$

= $\sum_{x=1}^{\infty} \frac{e^{-\alpha}\alpha^x}{(x-1)!}$

= $\alpha e^{-\alpha} \sum_{x=1}^{\infty} \frac{\alpha^{(x-1)}}{(x-1)!}$

Substitute, x-1=s

= $\alpha e^{-\alpha} \sum_{s=0}^{\infty} \frac{\alpha^s}{s!}$

$= \alpha e^{-\alpha} e^{\alpha} = \alpha$

Therefore,

E(X)= α

V(X)= $E(x^2) - [E(x)]^2 = E(x^2) - [\alpha]^2$

$E(x^2) = \sum_{x=1}^{\infty} \boldsymbol{x^2} \frac{e^{-\alpha}\alpha^x}{x!}$

= $\alpha \sum_{x=1}^{\infty} \boldsymbol{x} \frac{e^{-\alpha}\alpha^{x-1}}{(x-1)!}$

Substitute, x-1=s

$$E(x^2) = \alpha \sum_{s=0}^{\infty} (\boldsymbol{s}+\boldsymbol{1}) \frac{e^{-\alpha}\alpha^s}{s!}$$

= $\alpha \sum_{s=1}^{\infty} s \frac{e^{-\alpha}\alpha^s}{s!} + \alpha e^{-\alpha} \sum_{s=0}^{\infty} \frac{\alpha^s}{s!}$

= $\alpha^2 e^{-\alpha} \sum_{s=1}^{\infty} \frac{\alpha^{s-1}}{(s-1)!} + \alpha e^{-\alpha} e^{\alpha}$

= $\alpha^2 + \alpha$

Therefore,

V(X)= $E(x^2) - [E(x)]^2 = \alpha$

Problems:

a) 2% of fuses manufactured by a company are defective. Find the probability that a box having 200 fuses contains.

i. No defective fuse.

ii. 3 or more defective fuses.

Solution:

$$\alpha = np$$

n=200, p=0.02, q= 0.98. Therefore, $\alpha = 4$.

i. P(X=0) = 0.0183

ii. $P(X \geq 3) = 0.769$

b) A pot has 10% of defective items. What should be the number of items such that the probability of finding at least 1 defective item is at least 0.95.

Solution:

n=? p= 0.1 and q= 0.9

P $(x \geq 1) \geq 0.95$

[1-P(x<1)] ≥ 0.95

P(x<1) ≤ 0.05

P(x=0) ≤ 0.05

$$e^{-\alpha}\alpha^0 \leq 0.05$$

$$e^{-np} \leq 0.05$$

$$n \geq 29.95$$

c) Suppose that a container contains 10,000 particles. The probability that such a particle escapes from the container equals 0.0004. What is the probability that more than 5 such escape occurs.

Solution:

X: number of particle escapes

n= 10,000 p= 0.0004 and $\alpha = 4$

P $(X > 5)$= $1 - P(X \leq 5) = 1 - \sum_{x=0}^{5} \frac{e^{-4}4^x}{x!}$

d) X is a Poisson's variable and it's found that the probability that X=2 is two third of the probability that X=1. Find the probability that X=0 and X=3. What is the probability that X exceeds 3?

Solution: P(X=2) = 2/3 P(X=1)

$$\alpha = \frac{4}{3}$$

P(X=0) =0.2635

P(X=3) = ……….

P(X>3) =………..

e) An insurance company has discovered that only about 0.1% of the population is limited in a certain type of accidents each year. If its 10000 policy holders were randomly selected from the population. What is the probability that not more than 5 of the clients are involved in such accidents each year.

Solution:

X: number of clients involved in accidents
p=0.001 n=1000 $\alpha = 10$
P($X \geq 5$) = 0.8686

f) Suppose that a book of 585 pages contains 43 typographical errors. If these errors are randomly distributed throughout the book, what is the probability that 10 pages selected at random will be errorfree.
Solution:
X: number of errors
n=10
p= 43/585

$$\alpha = 0.735$$

P(X=0)=0.4795

g) Probability that an individual suffers a bad reaction from an injection is 0.001. Find the probability that out of 2000 individuals (i) exactly 3 (ii) not more than 2 suffer from bad reaction.

Continuous Distributions

Exponential Distribution: UNIPARAMETRIC

This family of distributions is characterized by a single parameter λ, which is called the rate. Intuitively, λ can be thought of as the instantaneous "failure rate" of a "device" at any time t, given that the device has survived up to t. The exponential distribution is typically used to model time intervals between "random events".

Examples:

a. The length of time between telephone calls
b. The length of time between arrivals at a service station
c. The lifetime of electronic components, i.e., an inter failure time.

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & otherwise \end{cases}$$

Mean and Variance:

$$E(X) = \int_{-\infty}^{\infty} x\, f(x) dx$$

$$E(X) = \int_{0}^{\infty} x\, \lambda e^{-\lambda x} dx$$

$$= \lambda \left\{ x \frac{e^{-\lambda x}}{-\lambda} - \frac{e^{-\lambda x}}{\lambda^2} \right\} = \frac{1}{\lambda}$$

$$Similarly, V(X) = \frac{1}{\lambda^2}$$

Gamma Distribution

A continuous random variable X is said to have a gamma distribution with parameters $\alpha > 0$ and $r > 0$, shown as $X \sim G(\alpha, r)$, if its PDF is given by

$$f(x) = \begin{cases} \dfrac{x^{r-1} e^{-\alpha x} \alpha^r}{\Gamma(r)}, & x > 0,\ \alpha, r > 0 \\ 0, & elsewhere \end{cases}$$

Note: When we sub r=1 in gamma distribution we get exponential distribution.

Mean and Variance:

$$E(X) = \int_{-\infty}^{\infty} x\, f(x) dx$$

$$E(X) = \int_{0}^{\infty} x \left\{\frac{x^{r-1}\, e^{-\alpha x}\, \alpha^r}{\Gamma(r)}\right\} dx$$

Multiply and divide by "$\alpha\, r$"

$E(X) = \frac{r}{\alpha} \int_0^\infty \left\{\frac{x^r\, e^{-\alpha x}\, \alpha^{r+1}}{\Gamma(r+1)}\right\} dx$ where $\int_0^\infty \left\{\frac{x^r\, e^{-\alpha x}\, \alpha^{r+1}}{\Gamma(r+1)}\right\} dx = 1$ $being\ a\ pdf\ of\ Gamma\ function$

$$E(X) = \frac{r}{\alpha}$$

V(X)= $E(x^2) - [E(x)]^2$

$$E(x^2) = \int_{-\infty}^{\infty} x^2\, f(x) dx$$

$$E(x^2) = \int_{0}^{\infty} x^2 \left\{\frac{x^{r-1}\, e^{-\alpha x}\, \alpha^r}{\Gamma(r)}\right\} dx$$

Multiply and divide by "$\alpha^2\, r(r+1)$"

$E(x^2) = \frac{r(r+1)}{\alpha^2} \int_0^\infty \left\{\frac{x^{r+1}\, e^{-\alpha x}\, \alpha^{r+2}}{\Gamma(r+2)}\right\} dx = \frac{r(r+1)}{\alpha^2}$

where $\int_0^\infty \left\{\frac{x^{r+1}\, e^{-\alpha x}\, \alpha^{r+2}}{\Gamma(r+2)}\right\} dx = 1$ $being\ a\ pdf\ of\ Gamma\ function$

Therefore,

V(X)= $\frac{r}{\alpha^2}$

To check the pdf defined is valid:

$$f(x) \geq 0$$

$$\int_{-\infty}^{\infty} f(x) dx = \int_0^\infty f(x) dx = \int_0^\infty \left\{\frac{x^{r-1}\, e^{-\alpha x}\, \alpha^r}{\Gamma(r)}\right\} dx$$

$$substitute, \alpha x = v$$

$$= \frac{\alpha^r}{\Gamma(r)} \int_0^\infty \left\{\frac{v}{\alpha}\right\}^{r-1} e^{-v}\, (1/\alpha) dv$$

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$$

$$= \frac{1}{\Gamma(r)} \int_0^\infty \{v\}^{r-1}\, e^{-v}\, dv$$

$$= 1$$

Problems:

1. The number of road accidents per day in a city is following a gamma distribution with an average of 6 and variance of 18. Find the probability that there will be (i) more than 8 accidents (ii) between 5 to 8 in a day.

   Solution: $\frac{r}{\alpha}$ =6 and $\frac{r}{\alpha^2} = 18 \rightarrow$ r=2 and $\alpha = \frac{1}{3}$

   P(X>8) = $\int_0^8 \left\{\frac{(1/3)\, e^{-\left(\frac{1}{3}\right)x}\left(\frac{1}{3}x\right)}{\Gamma(1)}\right\} dx$ = 1/9 $[11e^{-1/3} - 1]$

2. The daily consumption of electric power is in million Kw in a certain city is a random variable x having the pdf f(x)= $\begin{cases} \frac{1}{9} x\, e^{-\frac{x}{3}} & 0 < x \\ 0 & otherwise \end{cases}$. Find the probability that the power supply is inadequate on any given date if the daily capacity of the power plant is 12 million Kw.

   Solution: P(x>12) = $\int_{12}^{\infty} \left\{\frac{1}{9} x\, e^{-\frac{x}{3}}\right\} dx$

   Note: V(X)= $\frac{r}{\alpha^2} = \frac{2}{\left(\frac{1}{3}\right)^2} = 18$

3. If X has the pdf f(x)= $\begin{cases} \frac{1}{4} x\, e^{-\frac{x}{2}} & 0 < x < \infty \\ 0 & otherwise \end{cases}$. Find the mean and variance.

4. Solution:

   $E(X) = \frac{r}{\alpha}$ and V(X)= $\frac{r}{\alpha^2}$

   We have, r-1=1 i.e r=2 and $-\alpha x = -\frac{x}{2}$ $i.e\ \alpha = \frac{1}{2}$

   $E(X) = \frac{r}{\alpha} = 4$ and V(X)= $\frac{r}{\alpha^2}$ = 8

5. The amount of time required to repair a TV is exponentially distributed with mean 2. Find (i) the probability that the required time exceeds 2 hours. (ii) The conational probability that the required time taken at least 10 hours given that already 9 hours have been spent on repairing the TV.

   Solution: E(X)=$\frac{1}{\lambda} = 2$ $\qquad \lambda = 0.5$

   f(x)= $\begin{cases} \frac{1}{2}\, e^{-\frac{x}{2}} & 0 < x \\ 0 & \text{otherwise} \end{cases}$

   (i) P(X>2)=1/e

   (ii) $P(X \geq 10 \quad |x > 9) = \frac{P(X\geq 10 \cap |x>9)}{P(x>9)} = e^{-\frac{1}{2}}$

6. If $X \sim E(\lambda)$ with P($X \leq 1$) = P(x>1) then find V(X).

   Solution: $\alpha = \ln 2$ therefore $V(X) = \frac{1}{\lambda^2} = \frac{1}{(\ln 2)^2}$

Chi-square Distribution:

Special case of Gamma distribution: $\boldsymbol{r = \frac{n}{2}}$ and $\boldsymbol{\alpha = \frac{1}{2}}$ in $\boldsymbol{\Gamma}$ function we get $\boldsymbol{\chi^2}$distribution.

A continuous random variable X is said to have a chi-square distribution if its PDF is given by.

$$f(x) = \begin{cases} \dfrac{x^{\frac{n}{2}-1} e^{-\frac{x}{2}}}{\Gamma(n/2)2^{\frac{n}{2}}}, & x > 0 \\ 0, & elsewhere \end{cases}$$

Mean and Variance:

$E(X) = \frac{r}{\alpha}$ = n and V(X)= $\frac{r}{\alpha^2}$ = 2n

Normal Distribution: (Gaussian Distribution)

1. The mean, median, and mode of the distribution coincide.
2. The curve of the distribution is bell-shaped and symmetrical about the line x=μ.
3. The total area under the curve is 1.
4. Exactly half of the values are to the left of the center, and the other half to the right.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\frac{(x-\mu)^2}{\sigma^2}} , \qquad -\infty < x, \mu < \infty, \ \sigma > 0$$

To check the pdf defined is valid:

$$f(x) \geq 0$$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

Sub, $\frac{x-\mu}{\sigma} = z$

$$\int_{-\infty}^{\infty} f(x)dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-(z)^2}{2}} dz$$

$$\int_{-\infty}^{\infty} f(x)dx = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{\frac{-(z)^2}{2}} dz$$

$$\int_{-\infty}^{\infty} f(x)dx = \frac{\sqrt{2}}{\sqrt{\pi}} \int_{0}^{\infty} e^{\frac{-(z)^2}{2}} dz$$

$$Sub, \frac{z^2}{2} = t$$

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} \, dt$$

$$\int_{-\infty}^{\infty} f(x)dx = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{\infty} \frac{e^{-t}}{\sqrt{2t}} \ dt = 1$$

Mean and Variance: Prove the following.

E(X)=μ and V(X)= σ^2

Problem on Chebyshev's Inequality:

The number of patients requiring ICU in a hospital is a random variable with mean 18 and S.D 2.5. Determine minimum probability- that number of patients are between 8 and 28.

Solution: $P(8 \leq X \leq 28) > 1 - \frac{1}{k^2}$

$$P(\mu + k\sigma \leq X \leq \mu - k\sigma) > 1 - \frac{1}{k^2}$$

$\mu + k\sigma = 8$ therefore k = 4

$$P(8 \leq X \leq 28) > \frac{15}{16}$$

Standard normal distribution

The **standard normal distribution**, also called the z-**distribution**, is a special normal distribution where the mean(μ) is 0 and the standard deviation(σ) is 1. The curve is symmetric about x=0. Denoted by $Z \sim N(0,1)$.

Its PDF is,

$$\phi(z) = \frac{e^{\frac{-z^2}{2}}}{\sqrt{2\pi}}, \quad -\infty<z<\infty$$

where, $z = \frac{x-\mu}{\sigma}$ when μ is 0 and and σ is 1 z=x.

Properties:

1. Area under the curve is one.
2. P(a<x<b) is area under the curve from a to b.
3. Cdf of $\boldsymbol{\phi(a) = P(Z \leq a)}$
4. P($\boldsymbol{a}$ ≤Z≤ $\boldsymbol{b}$)= $\boldsymbol{\phi(b) - \phi(a)}$
5. $\boldsymbol{\phi(-a) = P(Z \leq -a) = P(Z \geq a)}$= 1-P(Z<a)= 1- $\boldsymbol{\phi(a)}$
6. $P(a \leq x \leq b)$= $P(\frac{a-\mu}{\sigma} \leq z \leq \frac{b-\mu}{\sigma})$

Problems:

1. Suppose $\boldsymbol{X \sim N(75, 100). Find}$

(i) P(X<60)

(ii) P (70< X<100)

(iii) P(X<65)

Solution: $\mu = 75$ and $\sigma^2 = 100$.

Z= $\frac{X-75}{10}$ and $Z \sim N(0,1)$

(i) P(X<60) = P (Z<$\frac{X-75}{10}$)

= P(Z<-1.5)

= $\phi(-1.5)$

= $1 - \phi(1.5)$ = $1 - 0.9332$= 0.0668

(ii) P(70<X<100) = P(-0.5< z <2.5)

= $\phi(2.5) - \phi(-0.5) = \phi(2.5) - [1 - \phi(0.5)]$

= 0.9938-1+0.6915

= 0.6853

(iii) P(X<65)= 0.1587

2. Suppose $X \sim N(2,\ 0.16). Find.$

(i) $P(X \geq 2.3)$→ 0.2266

(ii) $P(1.8 \leq X \leq 2.1)$ → 0.2902

3. Diameter of an electric cable is normally distributed with mean 0.8 and variance 0.0004. What is the probability that the diameter exceeds 0.81 inches.

Solution:

$$\mu = 0.8 \ and\ \sigma = 0.02$$

$$P(X > 0.81) = P\ (Z > \frac{0.81-\mu}{\sigma})= 0.3085$$

4. $X \sim N(1, 4), Find\ P(|x| > 4). Ans = 0.073$

5. $X \sim N(75, 25). Find\ P(X > 80|X > 77). Ans: 0.4605$

6. The height of 500 soldiers is found to have normal distribution. Of them 258 are found to be within 2c.m of the mean height of 170c.m. Find the standard deviation of X.

   Solution: $\mu = 170$
   X: height of soldiers

$$X \sim N(170, \quad \sigma^2)$$
$$P(168 < X < 172) = \frac{258}{500} = 0.516$$
$$P\left(\frac{168-\mu}{\sigma} < Z < \frac{172-\mu}{\sigma}\right) = 0.516$$
$$2\phi\left(\frac{2}{\sigma},\right) = 1.516$$
$$2/\sigma \ = \phi^{-1}(0.758)$$
$$\frac{2}{\sigma} = 0.7$$
$$\sigma = 2.857$$

7. In normal distribution 31% of item are <45 and 8% are over 64. Find mean and Standard distribution.
   Solution: P(X<45)=0.31
   P(X>64) = 0.08
   Simplifying, 45- μ= -0.5σ
   $64 - \mu = 1.41\sigma$
   Solving, $\mu = \ 49.97\ and\ \sigma = 9.94$

8. Suppose X has $N(3, 4)$. Find "c" such that P$(X > c)$= $2P(X \le c)$.
   Ans: c= 2.14

9. Suppose that the life span of two electronic device A and B have distribution N(40, 36) and N(45, 9). If the electronic device is to be used for 45 hours period which device is to be preferred. If it is used for 48 hours which device is to be preferred.
   Solution:

Device A	Device B
N (40 , 36) Mean = 40 SD = 6 device A is to be used for a 45h period and above. $P(X \ge 45) = 0.2025$ device A is to be used for a 48h period. $P(X \ge 48) = 0.0915$	N (45 , 9) Mean = 45 SD = 3 device B is to be used for a 45 –h period. $P(X \ge 45) = 0.5$ 0.5 > 0.2025 Hence Device B is better for 45 hours. Device B is to be used for a 48 –h period. $P(X \ge 48) = 0.1587$ 0.1587 > 0.0915 Hence Device B is better for 48 hours. in Both cases Device B is better

10. In a normal distribution, 7% of the item are under 35 and 89% of the item are under 63. Find the mean and variance of the distribution.
    Solution: P(X<35) = 0.07 and P(X<65)= 0.89 gives $X \sim N(50, 100)$

11. Obtain the percentage of students who are graded A, B, C, D, E and F.

(10 points) An examination is often regarded as good (i.e., has a valid grade spread) if the test scores of those taking it can be approximated by a normal distribution. The instructor uses the test scores to estimate parameters μ and σ^2. Then she assigns grades according to the following chart:

Grade	Score Range
A	Students who score greater than $\mu + \sigma$
B	Students who score between μ and $\mu + \sigma$
C	Students who score between $\mu - \sigma$ and μ
D	Students who score between $\mu - 2\sigma$ and $\mu - \sigma$
F	Students who score below $\mu - 2\sigma$

Solution:

$P(X > \mu + \sigma) = 0.1587$

$P(\mu < X < \mu + \sigma) = 0.3413$

$P(\mu - \sigma < X < \mu) = 0.3413$

$$P(\mu - 2\sigma < X < \mu - \sigma) = 0.1359$$

$$P(X < \mu - 2\sigma) = 0.0228$$

12. The monthly income of a group of 10,000 person were found to be normally distributed with mean 750 rupees and SD rupees 50. Show that of this group about 95% had income exceeding rupees 668 and only 5% had income exceeding rupees 832. What was the lowest income among the richest 100?

Solution:

X: Monthly income of a group

To Show P(X>668) =0.95 and P(X>832) =0.05

Consider, P(X>668) and P(X>832) and solve.

To find "C" such that P(X>μ+C) = 100/10000 =0.01

C= 116.5

Therefore, lowest income among the richest 100 = $\mu + C$= 866.5

13. The annual rainfall at a certain locality is known to be normally distributed random variable with mean 29.5inches and SD 2.5inches. How many inches of rain annually exceeds about 5% of the time?

Solution:

X: annual rainfall at certain locality

$$\mu = 29.5 \text{ } and \text{ } \sigma = 2.5$$

$$P(X > \mu + c) = 0.05$$

$$1 - P\left(Z \leq \frac{c}{\sigma}\right) = 0.95$$

$$C = 4.125$$

Rain exceeds about 5% of the time is $\mu + c$= 33.625 inches.