Row-reduced Echelon Form: Let A be the given matrix. Apply row elementary transformations to the matrix A such that

- First row, first nonzero element should be 1 and all the elements below in that column should be zeros.
- In the reduced matrix, convert the first nonzero element in the second row as 1 and all the elements below in that column should be zeros.
- Continue the same procedure till all the rows are exhausted.
- The final reduced matrix is the **Row-reduced Echelon Form** of A.

Note 0.1. *Let A be the given matrix and C be the row reduced Echelon form of A, then the number of nonzero rows in C is the rank of A.*

Hence we can define, **the rank of a matrix A is the maximum number of linearly independent rows (or columns) of A.**

Note :- Rank of a matrix A is denoted by $\rho(A)$.

1. Rank of a matrix

Definition 1.1. Let A be a given matrix then **rank** of the matrix A is the maximal number of linearly independent rows or maximal number of linearly independent columns of A. It is denoted by $\rho(A)$.

Example 1.2. Consider a matrix $A = \begin{pmatrix} 9 & 9 & 9 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 7 & 7 & 7 \end{pmatrix}$ $\rho(A) = \underline{\underline{2}}$.

Ans:-

$$A = \begin{pmatrix} 9 & 9 & 9 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 7 & 7 & 7 \end{pmatrix} \quad R_1 \to \frac{R_1}{9}, \; R_4 \to \frac{R_4}{7}, \; R_3 \to \frac{R_3}{2}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \end{array}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad R_3 \to R_3 - R_2$$

$$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ Echelon form of } A.$$

no. of non zero rows $= \underline{\underline{2 = \rho(A)}}$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ✓

1.1. Properties of rank of a matrix

1. If A is a square matrix of order n then $\rho(A) \leq n$. ✓
2. If A is an $m \times n$ matrix then $\rho(A) \leq \min\{m, n\}$..
3. If I_n is an identity matrix of order n then $\rho(I_n) = n$.
4. A matrix and its transpose have the same rank. ✓
5. If $\rho(A) = 0$ then A is a null matrix.

6. If A is not a null matrix, then $\rho(A) \geq 1$.

Problem 1.3. *Using row elementary transformations, find the rank of the matrix* $A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ -1 & 3 & 0 & -4 \\ 2 & 1 & 3 & -2 \\ 1 & 1 & 1 & -1 \end{pmatrix}$ (Given)

<u>Ans:-</u> $R_2 \to R_2 + R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - R_1$

$$\therefore A \sim \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 5 & -1 & -4 \\ 0 & -3 & 5 & -2 \\ 0 & -1 & 2 & -1 \end{pmatrix} \quad R_2 \leftrightarrow R_4$$

$$\sim \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & -3 & 5 & -2 \\ 0 & 5 & -1 & -4 \end{pmatrix} \quad R_2 \to -R_2$$

$$\sim \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & -3 & 5 & -2 \\ 0 & 5 & -1 & -4 \end{pmatrix} \quad \begin{matrix} R_3 \to R_3 + 3R_2 \\ R_4 \to R_4 - 5R_2 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 9 & -9 \end{pmatrix} \quad R_3 \to -R_3 \ \& \ R_4 \to \frac{R_4}{9}$$

Contd...

$$\sim \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$R_4 \to R_4 - R_3$$

$$\sim \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ Echelon form}$$

$\therefore$ no. of nonzero rows $= 3 = \underline{\rho(A)}$

Problem 1.4. *Using row elementary transformations, find the rank of the matrix* $A = \begin{pmatrix} 2 & -2 & 5 & 3 \\ 4 & -1 & 1 & 1 \\ 3 & -2 & 3 & 4 \\ 1 & -3 & 7 & 6 \end{pmatrix}$ (Given) $R_1 \leftrightarrow R_4$

Ans:-

$$A \sim \begin{pmatrix} 1 & -3 & 7 & 6 \\ 4 & -1 & 1 & 1 \\ 3 & -2 & 3 & 4 \\ 2 & -2 & 5 & 3 \end{pmatrix} \begin{matrix} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 2R_1 \end{matrix}$$

$$\therefore A \sim \begin{pmatrix} 1 & -3 & 7 & 6 \\ 0 & 11 & -27 & -23 \\ 0 & 7 & -18 & -14 \\ 0 & 4 & -9 & -9 \end{pmatrix} R_2 \to \frac{R_2}{11}$$

$-9 + \frac{(23 \times 4)}{11}$

$-9 + \frac{(27 \times 4)}{11}$

$$\sim \begin{pmatrix} 1 & -3 & 7 & 6 \\ 0 & 1 & -27/11 & -23/11 \\ 0 & 7 & -18 & -14 \\ 0 & 4 & -9 & -9 \end{pmatrix}$$

$-14 + \frac{(23 \times 7)}{11}$

$-18 + \frac{(27 \times 7)}{11}$

$R_3 \to R_3 - 7R_2$, $R_4 \to R_4 - 4R_2$

$$\sim \begin{pmatrix} 1 & -3 & 7 & 6 \\ 0 & 1 & -27/11 & -23/11 \\ 0 & 0 & -9/11 \checkmark & 7/11 \\ 0 & 0 & 9/11 & -7/11 \end{pmatrix}$$

Contd...

$$R_4 \to R_4 + R_3$$

$$\sim \begin{pmatrix} 1 & -3 & 7 & 6 \\ 0 & 1 & -27/11 & -23/11 \\ 0 & 0 & -9/11 & 7/11 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

Echelon form of A.

$\therefore$ Here no. of non zero rows $= 3$

$= \rho(A)$

Problem 1.5. *Using row elementary transformations, find the rank of the matrix* $A = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 2 & 6 \\ 0 & 3 & 5 & 1 \end{pmatrix}$ (Given)

$R_2 \to R_2 - 2R_1$

$$A \sim \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 5 & 1 \end{pmatrix} \quad R_2 \leftrightarrow R_3$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 3 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ Echelon form}$$

no. of nonzero rows $= 2 = \rho(A)$

2. Linear Equations

Consider the system of m linear equations in $n-$unknowns $x_1, x_2, ..., x_n$ as below:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{12}x_2 + ... + a_{3n}x_n &= b_3 \\ &..... \\ &..... \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &= b_m \end{aligned}\right\} (*)$$

Then the matrix equation of the above system is $AX = B$ where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ . & . & . & . \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \; X = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix} \; B = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_m \end{pmatrix}$$

If $B = 0$ then the system of equations is said to be *homogeneous*, otherwise *non-homogeneous.*

Any $n-$tuple $x = (x_1, x_2, ..., x_n)$ which satisfies the above system of equations is called the *solution* of the system.

Note 2.1. *It is clear that* $0 = (0, 0, ..., 0)$ *is a solution of the system* $AX = 0$, *called the trivial solution.*

Note 2.2. *The set of all solutions of the homogeneous system of linear equations are closed under addition and scalar multiplication.*

i.e. If y_1 *and* y_2 *are the solutions of the homogeneous system* $AX = 0$ *then* $y = c_1y_1 + c_2y_2$ *is again a solution of* $AX = 0$.

Definition 2.3. (**Augmented Matrix**) Consider the system of linear equations $AX = B$ then the augmented matrix is obtained by placing the column matrix B to the right of the matrix A. It is denoted by $[A : B]$ or $[A \mid B]$.

Gauss Elimination Method ✓

Definition 2.4. (**Consistency**) A system of linear equations $AX = B$ tis said to be, **consistent** if the system has a solution, otherwise it is **inconsistent**.

Consider $\boxed{AX = B}$ —— (*)

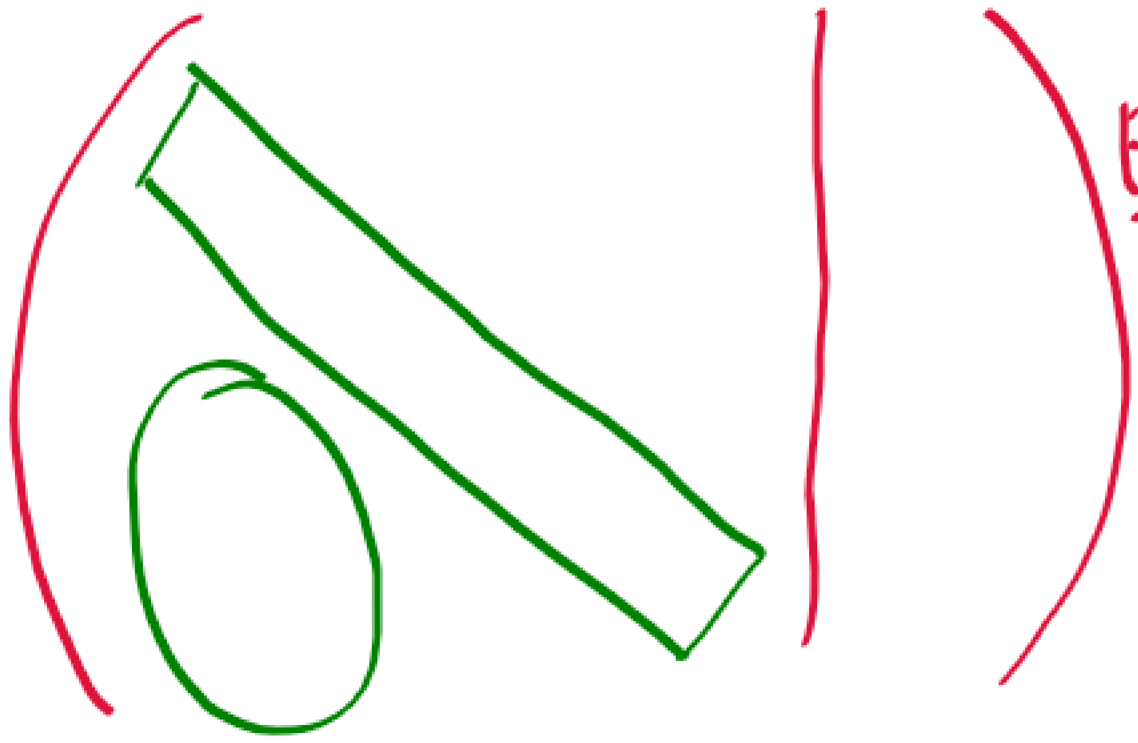
no. of
$n \to$ unknowns

Augmented matrix $[A : B]$

$[A:B] = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 3 & 2 & 1 & 8 \\ 5 & 9 & 7 & 9 \end{array}\right)$

$\to \rho([A:B])$?

$[A : B] \sim$



Echelon form

$\rho(A) = ?$

$\to$ whether $\rho([A:B]) = \rho(A)$ or not ?

Yes — Then the System is consistent

- $\underbrace{\rho(A) = \rho[A:B]}= n$ — Unique Soln
- $\rho(A) = \rho[A:B] < n$ — $\boxed{\text{Infinitely many}}$

No — Then the system is consistent

Problem 2.5. *Test the consistency and solve the system of equations by Gauss elimination method.*

Given
$$\left.\begin{aligned} x+y+z&=6\\ x-y+2z&=5\\ 3x+y+z&=8 \end{aligned}\right\} (*)$$

Ans:- The matrix eq^n of $(*)$ is

$AX=B$ where $A=\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \end{pmatrix}$ $X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$B=\begin{pmatrix} 6 \\ 5 \\ 8 \end{pmatrix}$

Augmented matrix $[A:B]=\begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & -1 & 2 & : & 5 \\ 3 & 1 & 1 & : & 8 \end{bmatrix}$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 3R_1$

$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 1 & : & -1 \\ 0 & -2 & -2 & : & -10 \end{bmatrix}$ $R_3 \to R_3 - R_2$

$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 1 & : & -1 \\ 0 & 0 & -3 & : & -9 \end{bmatrix}$, Echelon form

$\therefore \rho[A:B]=3$

Contd...

Echelon form of A is $\begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{pmatrix}$

$\therefore \rho(A) = 3$

ie, $\rho(A) = \rho(A:B) = 3 =$ no. of unknowns

$\therefore$ System is consistent and has a unique sol^n.

$\therefore$ The equivalent matrix eq^n is,

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ -9 \end{pmatrix}$$

$$\Rightarrow \quad x + y + z = 6 \quad \text{——— (1)}$$
$$-2y + z = -1 \quad \text{——— (2)}$$
$$-3z = -9 \quad \text{——— (3)}$$

(3) $\Rightarrow z = 3$

(2) $\Rightarrow y = 2$

(1) $\Rightarrow x = 1$

$\therefore$ Req'd sol^n is $X = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ //

Problem 2.6. *Test the consistency and solve the system of equations by Gauss elimination method.*

Given,
$$\left.\begin{aligned} x + 2y + 3z &= 14 \\ 4x + 5y + 7z &= 35 \\ 3x + 3y + 4z &= 21 \end{aligned}\right\} (*)$$

Ans:- The matrix eqⁿ of (*) is $AX=B$

where $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \\ 3 & 3 & 4 \end{pmatrix}$ $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$B = \begin{pmatrix} 14 \\ 35 \\ 21 \end{pmatrix}$

Augmented matrix, $[A:B] = \left[\begin{array}{ccc:c} 1 & 2 & 3 & 14 \\ 4 & 5 & 7 & 35 \\ 3 & 3 & 4 & 21 \end{array}\right]$

$R_2 \to R_2 - 4R_1$, $R_3 \to R_3 - 3R_1$

$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 2 & 3 & 14 \\ 0 & -3 & -5 & -21 \\ 0 & -3 & -5 & -21 \end{array}\right]$ $R_3 \to R_3 - R_2$

$\sim \left[\begin{array}{ccc:c} 1 & 2 & 3 & 14 \\ 0 & -3 & -5 & -21 \\ 0 & 0 & 0 & 0 \end{array}\right]$, Echelon form

$\therefore \rho[A:B] = 2$ Echelon form of A is $\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -5 \\ 0 & 0 & 0 \end{pmatrix}$

Contd... $\therefore \rho(A) = 2$

i.e; $\rho[A:B] = \rho(A) = 2 < 3 =$ no. of unknowns

$\therefore$ System is consistent and have infinitely many solutions.

The equivalent matrix eq^n is,

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -5 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 14 \\ -21 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad x + 2y + 3z = 14 \quad \text{--- (1)}$$
$$-3y - 5z = -21 \quad \text{--- (2)}$$

Let $x = k$ then $2y + 3z = 14 - k$
$-3y - 5z = -21$

Try

Try it.

~~or~~

let $z = k$ be any real no.

from (2) we get $-3y = -21 + 5k$
$\Rightarrow y = 7 - \frac{5}{3}k$

(1) $\Rightarrow x = 14 - 2y - 3z$
$= 14 - 14 + \frac{10}{3}k - 3k = \frac{k}{3}$

Contd...

$\therefore$ Req'd solⁿ is

$$X = \begin{pmatrix} k/3 \\ 7 - \frac{5}{3}k \\ k \end{pmatrix}$$

where k is any real no.

Problem 2.7. *For what values of λ and μ such that the system of equations*

Given
$$\left.\begin{aligned} x + y + z &= 6 \\ x + 2y + 3z &= 10 \\ x + 2y + \lambda z &= \mu \end{aligned}\right\} (*)$$

may have,

1. *unique solution*
2. *infinite number of solutions*
3. *No solution*

Ans:- The matrix eq^n of (*) is $AX = B$

where $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{pmatrix}$ $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$B = \begin{pmatrix} 6 \\ 10 \\ \mu \end{pmatrix}$

Augmented matrix $[A:B] = \begin{pmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{pmatrix}$

$R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$[A:B] \sim \begin{pmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda-1 & : & \mu-6 \end{pmatrix}$

$R_3 \to R_3 - R_2$

$\sim \begin{pmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda-3 & : & \mu-10 \end{pmatrix}$, Echelon form

Contd...

(i) System has unique sol^n if

$\rho[A:B] = \rho(A) = 3$.

i.e; possible only if $\lambda - 3 \neq 0$ &

μ can be any real no.

i.e; if $\lambda \neq 3$ and μ be any real no.

(ii) System have infinitely many sol^n

if $\rho[A:B] = \rho(A) < 3$

i.e; if $\rho[A:B] = \rho(A) = 2$

i.e; if $\lambda - 3 = 0$ and $\mu - 10 = 0$

i.e if $\lambda = 3$ and $\mu = 10$

(iii) System has no sol^n if

$\rho[A:B] \neq \rho(A)$

i.e; if $\lambda = 3$ and $\mu \neq 10$.