Analytic functions:

* Defined & differentiable

Theorem: (Cauchy - Reimann's theorem)

The necessary and the sufficient conditions for a function $f(z) = u(x,y) + iv(x,y)$ to be analytic for every 'z' in the region 'R' are given by

i) The partial derivatives u_x, u_y, v_x, v_y are continous in R

ii) The real and imaginary parts satisfy a set of eqns known as CR eqn $\longrightarrow$

$$u_x = v_y$$
$$u_y = -v_x$$

ex:- $f(z) = z^2 = u + iv$

To check if f(z) is analytic

I've to check if CR eqns are satisfied

$$f(z) = (x+iy)^2 = \underbrace{x^2 - y^2}_{u} + \underbrace{2ixy}_{v}$$

$u = x^2 - y^2 \qquad v = 2xy$

$u_x = 2x \qquad v_x = 2y$

$u_y = -2y \qquad v_y = 2x.$

$u_x = v_y$ & $u_y = -v_x$

$\therefore f(z) = z^2$ is analytic

Proof:-

Let $f(z) = u(x,y) + iv(x,y)$ is analytic at all points $z \in R$

I've to prove the CR eqns.

Given $f(z) = u + iv$ is analytic, it is defined & differentiable

Differentiable $\Rightarrow \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = f'(z)$ exists

Evaluating the limit, (consider $\Delta z \to 0$ from real axis
$\Delta z \to 0$ from imag axis)

If $\Delta z \to 0$ from the real axis :- $\Delta x \to 0$, $\Delta y = 0$

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - [u(x,y) + iv(x,y)]}{\Delta x}$$

$$\frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$f(z) = u(x,y) + iv(x,y)$

$\Delta z = \Delta x + i\Delta y$

$f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)$

$$= \lim_{\Delta x \to 0} \left\{ \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \right\}$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$f'(z) = u_x + iv_x$ ——— (1) (when $\Delta z \to 0$ from real axis)

If $\Delta z \to 0$ from imaginary axis, i.e. $\Delta x = 0$
$\Delta y \to 0$

Then, we get

$f'(z) = -i u_y + v_y$ ——— (2)

since the fn is analytic $\Rightarrow$ differentiable $\Rightarrow$ that lim exists

∃ the real axis ka lim = the imaginary axis ka limit

$\therefore\ u_x + i v_x = -i u_y + v_y$

On equating : $\boxed{\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}}$ CR eqns

<u>converse part</u> :-

Let the condns given are satisfied by the function

$f(z) = u(x,y) + i v(x,y)$

I've to prove that $f(z)$ is analytic

consider $f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + i v(x+\Delta x, y+\Delta y)$

Use Taylor series exp

$= u(x,y) + \Delta x\, u_x + u_y\, \Delta y + \cdots$

$+ i\left[v(x,y) + v_x \Delta x + v_y \Delta y + \cdots\right]$

$u(x,y) + i v(x,y) = f(z)$

$= f(z) + \Delta x\{u_x + i v_x\} + \Delta y\{u_y + i v_y\}$

CR eqns ($-v_x$ u_x

$f(z+\Delta z) - f(z) = \Delta x\{u_x + i v_x\} + \Delta y\{-v_x + i u_x\}$

$= \Delta x\{u_x + i v_x\} + i\Delta y\left\{\frac{-1}{i} v_x + u_x\right\}$

$z = x + iy$

$\left(\frac{-1}{i} \times \frac{i}{i} = \frac{-i}{-1} = i\right.$

$= \Delta x\{u_x + i v_x\} + i\Delta y\{i v_x + u_x\}$

$= \underbrace{(\Delta x + i\Delta y)}_{\Delta z}(u_x + i v_x)$

$$f(z+\Delta z) - f(z) = (\Delta z)(u_x + i v_x)$$

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = u_x + i v_x$$

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} u_x + i v_x$$

$$f'(z) = u_x + i v_x$$

Summary

Analytic $\Longleftrightarrow$ CR eqns

CR eqn :- $f(z) = u + iv$

$$u_x = v_y$$

$$u_y = -v_x$$

$$f'(z) = u_x + i v_x$$

$$f'(z) = v_y - i u_y$$

① $f(z) = \bar{z}$ is analytic nowhere

soln

$f(z) = x - iy$

$u = \text{Real part} = x$

$v = \text{Im part} = -y$

$u_x = 1 \qquad v_x = 0$

$u_y = 0 \qquad v_y = -1$

$u_x \neq v_y \quad \therefore$ CR eqn fails, $\therefore$ Not analytic anywhere

② $f(z) = |z|^2$. Check if it is analytic.

soln

$f(z) = x^2 + y^2$

$u = x^2 + y^2 \qquad v = 0$

$u_x = 2x \qquad v_x = 0$

$u_y = 2y \qquad v_y = 0$

$\therefore f(z) = |z|^2$ is analytic only at the origin

only at $z = 0$

$x + iy = 0$

③ $f(z) = \log z$

soln

$z = re^{i\theta}$

$r = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1}(y/x)$

$f(z) = \log(x + iy)$

$f(z) = \log(re^{i\theta}) = \log r + \log e^{i\theta}$

$= \underline{\log r} + i\theta$

$= \underbrace{\log\left(\sqrt{x^2+y^2}\right)}_{u} + i\underbrace{\tan^{-1}(y/x)}_{v}$

$$u = \log(\sqrt{x^2+y^2}) \qquad v = \tan^{-1}(y/x)$$

$$= \log(x^2+y^2)^{1/2}$$

$$= 1/2 \log(x^2+y^2)$$

$$u_x = \frac{1}{2} \cdot \frac{1}{x^2+y^2} 2x = \frac{x}{x^2+y^2}$$

$$u_y = \frac{y}{x^2+y^2}$$

$$v_x = \frac{1}{1+(y/x)^2}\left(-y/x^2\right) = \frac{-y}{x^2+y^2}$$

$$v_y = \frac{1}{1+(y/x)^2}(1/x) = \frac{x}{x^2+y^2}$$

$$u_x = v_y \qquad u_y = -v_x$$

$\therefore f(z) = \log z$ is analytic everywhere expt at $z=0$

$$f'(z) = u_x + i v_x$$

$$= \underbrace{\frac{x}{x^2+y^2} + i\frac{(-y)}{x^2+y^2}}$$

$$= \frac{x-iy}{x^2+y^2} = \frac{\cancel{x-iy}}{(\cancel{x-iy})(x+iy)}$$

$$= \frac{1}{(x+iy)} = \frac{1}{z}$$

$x = z$
$y = 0$

④ $f(z) = \sin z$

Soln

$f(z) = \sin(x+iy)$

$= \sin x \cos iy + \cos x \sin iy$

$\cos i\theta = \cosh\theta$

$\sin i\theta = i\sinh\theta$

$$\begin{cases} \cosh\theta = \dfrac{e^{\theta}+e^{-\theta}}{2} \\ \sinh\theta = \dfrac{e^{\theta}-e^{-\theta}}{2} \end{cases} \qquad \begin{cases} \therefore \cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2} \\ \sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i} \end{cases}$$

$f(z) = \sin x \cosh y + \cos x (i\sinh y)$

$= \sin x \cosh y + i(\cos x \sinh y)$

$u = \sin x \cosh y$ $\qquad$ $v = \cos x \sinh y$

$u_x = \cosh y \cos x$ $\qquad$ $v_x = -\sin x \sinh y$

$u_y = \sin x \sinh y$ $\qquad$ $v_y = \cos x \cosh y$

$\therefore u_x = v_y$ $\qquad$ $u_y = -v_x$ $\qquad$ $\therefore f(z)$ is analytic

$f'(z) = u_x + i v_x$

$= \cosh y \cos x + i(-\sin x \sinh y)$

$= \cos iy \cos x - i(\sin x \div \sin(iy))$

$= \cos iy \cos x - \sin x \sin iy$

$= \cos(x+iy)$

$= \cos z$

$\therefore \frac{d}{dz} \sin z = \cos z$

Check if the follow fns are analytic. If so, find the derivatives

(i) $\cos z$

(2) e^z

(3) $\cosh z$

(4) $\sinh z$

(5) z^n