Generating Function

A function $f(x)$ is said to be a generating function for the sequence $\{a_r\}_{r=1}^{\infty}$ if

$$f(x) = \sum_{r=1}^{\infty} a_r x^r.$$

i.e., a_r can be obtained as coefficients of x^r in the expansion of $f(x)$.

Eg: 1) $$e^x = 1 + x + \frac{x^2}{2!} + \cdots \qquad = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$\left\{ 1, \frac{1}{2!}, \frac{1}{3!}, \cdots \right\}$$

e^x is the generating function for the sequence $\{\frac{1}{r!}$

2) $$1 + nc_1 x + nc_2 x^2 + \cdots \qquad = \sum_{r=0}^{\infty} nc_r x^r$$

$a_r = nC_r$ $\qquad \{ nc_0, nc_1, nc_2 \cdots \}$

3) $$1 + 2x + 3x^2 + 4x^3 + \cdots \qquad = \frac{1}{(1-x)^2} = (1-x)^{-2} = \sum_{r=1}^{\infty} r \cdot x^{r-1}$$

$\{1, 2, 3, \cdots\}$ are the coefficients of $(1-x)^{-2}$.

$(1-x)^{-2}$ is a G.F for the sequence $\{1, 2, 3 \cdots\}$

Generating function for Combination

Consider the polynomial

$$(1+ax).(1+bx).(1+cx) = 1+(a+b+c)x^1+(ab+bc+ca)x^2 + abc\,x^3 \quad - (1)$$

If we consider, 3 ways of selecting one object (a or b or c) and represent it by $(a+b+c)$ then it is coeff of x in (1).

Similarly the 3 ways of selecting 2 objects (ab or ac or bc) may be represented by $(ab+ac+bc)$ which is coeff of x^2 in (1).

There is only one way of selecting all 3 objects & it is represented by abc, which is coeff of x^3 in (1).

Coeff of x is all possible combination of 3 objects taken 1 at a time & so on

consider a, b, c (3 objects)

$(1+ax^1) \longrightarrow$ can be represented as

(i) 1 or x^0 $\rightarrow$ non - selection of object a

(ii) ax $\rightarrow$ selection of " "

Similarly $(1+bx)$, $(1+cx)$

Thus the product $(1+ax)(1+bx)(1+cx)$ indicates selection or non selection of 3 objects and powers of x in the product indicate the number of objects being selected.

Put $a=b=c=1$

$$(1+x)(1+x)(1+x) = 1+3x^1+3x^2+x^3$$

Generalization

Case i : If n objects say $a_1, a_2, \ldots, a_n$ given, then $(1+a_1x)(1+a_2x)\ldots(1+a_nx) =$

$1 + (a_1+a_2+\cdots+a_n)x + \cdots + (a_1a_2\cdots a_n)x^n$

where coefficient of x^r gives all possible r-combination of n objects.

If $a_1 = a_2 = \cdots = a_n = 1$ we get

$$(1+x)^n = 1 + nc_1 x + nc_2 x^2 + \cdots + nc_r x^r + \cdots + nc_n x^n$$

Number of r- combination of n objects without repetition is coeff of x^r i.e. nc_r

Thus $(1+x)^n$ is the G.F for r- combination of n objects without repetition.

Note : A generating function used in this way is called as Enumerator.

Case ii : If an object is allowed an unlimited repetition, the corresponding factor in the enumerator must have every power of x present in it.

So,

$$(1+x+x^2+\cdots)(1+x+x^2+\cdots)\cdots(1+x+x^2+\cdots)$$

$$= (1-x)^{-1}(1-x)^{-1}\cdots(1-x)^{-1}$$

$$= (1-x)^{-n}$$

$$= 1+{}^{n}C_1 x + {}^{n+1}C_2 x^2 + {}^{n+2}C_3 x^3 + \cdots + {}^{n+r-1}C_r x^r + \cdots$$

$\Rightarrow$ ${}^{n+r-1}C_r$ coeff of x^r is the r- combination of n objects with repetition.

$(1-x)^{-n}$ is the G.F for r- combination of n objects with repetition.

Note : 1) $1+x+x^2+\dots+x^m = \dfrac{1-x^{m+1}}{1-x}$

$\Rightarrow \quad 1+x+x^2 = \dfrac{1-x^3}{1-x}$ ✓

2) ✓ $(1-x)^{-1} = 1+x+x^2+\dots$

3) ✓ $(1+x)^n = 1+{}^nC_1\,x + {}^nC_2\,x^2 + \dots + {}^nC_n\,x^n$

4) ✓ $(1-x^m)^n = 1-{}^nC_1\,x^m + {}^nC_2\,x^{2m} + \dots + (-1)^K\,{}^nC_K\,x^{Km} + \dots + (-1)^n\,{}^nC_n\,x^{mn}$

5) ✓ $(1-x)^{-n} = 1+{}^nC_1\,x + {}^{n+1}C_2\,x^2 + \dots + {}^{n+r-1}C_r\,x^r + \dots$

6) $(1+x)^{-n} = 1-{}^nC_1 x + {}^{n+1}C_2\,x^2 - {}^{n+2}C_3\,x^3 + \dots + (-1)^r\,{}^{n+r-1}C_r\,x^r + \dots$

7) $(1-x)^n = 1-{}^nC_1\,x + {}^nC_2\,x^2 - {}^nC_3\,x^3 + \dots + (-1)^n\,x^n$

Example

Find the coefficient of x^{16} in $(x^2+x^3+\dots)^5$.

Find coefficient of x^r.

Soln : $x^{10}(1+x+x^2+\dots)^5 \quad -(1)$

coeff of x^{16} in (1) is same as coeff of x^6 in $(1+x+x^2+\dots)^5 = (1-x)^{-5}$

$(1-x)^{-5} = 1+{}^5C_1\,x + {}^6C_2\,x^2 + \dots + {}^{5+6-1}C_6\,x^6 + \dots \quad (2)$

coeff of x^{16} is ${}^{10}C_6$ $\qquad {}^{n+r-1}C_r$

coeff of x^r is ${}^{r-6}C_{r-10}$ ✓ $\qquad$ coeff of x^{r-10} in (2)

Problems

Q1. How many combination of 3 objects can be formed if I^{st} object can be selected at most once, 2^{nd} object at most twice and 3^{rd} object at most 3 times.

Soln : $(1+x).(1+x+x^2)(1+x+x^2+x^3)$

$\quad\ I^{st} \qquad\qquad 2^{nd} \qquad\qquad 3^{rd}$

$$= 1+3x+5x^2+6x^3+5x^4+3x^5+x^6$$ ✓

1 combination of 3 objects is → 3 (a, b, c)

2 " " " " → 5

{ab, bb, bc, ac, cc}

3 " " 3 " is → 6

{abc, abb, acc, bbc, bcc, ccc}

{a bb ccc}

Q2. How many ways can an examiner assign 30 marks to 8 questions so that no question receives less than 2 marks?

Soln: 14 marks to 8 questions

$$\left(1+x+x^2+\cdots+x^{14}\right)^8$$

coeff of x^{14} in $\left(1+x+x^2+\cdots+x^{14}\right)^8$

$$= \left(1-x^{15}\right)^8 (1-x)^{-8}$$

coeff of x^{14} in $(1-x)^{-8}$ is $^{8+14-1}C_{14} = {}^{21}C_{14}$

Q3. Use generating function to count all selections of 6 objects from 3 types of objects with repetition up to 4 times of each type.

Soln:

$$\left(1+x+x^2+x^3+x^4\right)^3 \quad — \text{(1)}$$

coeff of x^6 in (1)

$$\left(1-x^5\right)^3 (1-x)^{-3}$$

$$\left(1-3x^5+3x^{10}-x^{15}\right)(1-x)^{-3}$$

coeff of x^6 is

coeff of x^0 in $\left(1-3x^5+3x^{10}-x^{15}\right)$ & x^6 from $(1-x)^{-3}$

or coeff of x^5 in $\left(1-3x^5+3x^{10}-x^{10}\right)$ & x from $(1-x)^{-3}$

$$1 \cdot {}^{3+6-1}C_6 + (-3)\cdot {}^3C_1 = {}^8C_6 - 3\cdot 3 = \underline{\underline{19}}$$

aaaa
bbbb
cccc

a	b	c
4	1	1
1	4	1
2	2	2
0	4	2
4	0	2
0	2	4

19 options