GENERATING FUNCTIONS

$\{a_r\}_{r=0}^{\infty}$

Definition:

Let $\{a\}_{r=0}^{\infty}$ be a sequence. A function $f(x)$ is said to be a generating function for the sequence Let $\{a\}_{r=0}^{\infty}$ if

$$f(x) = \sum_{r=0}^{\infty} a_r x^r$$

seq: $a_0, a_1, a_2, a_3, \ldots$

$f(x) = \sum_{r=0}^{\infty} a_r x^r$

coeff of x^r in the expansn $\rightarrow$ a_r (r^{th} term of the series)

ex:- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots$

The function $f(x) = e^x$ is the generating fn for $\{1/n!\}_{n=0}^{\infty}$

ie coeff of x^r in $f(x) = e^x$ is $1/r!$

Ex 2: $(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$

The function $(1+x)^n$ is the g.f for the seq $\{{}^nC_r\}_{r=0}^{n}$

Ex 3 :- $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots$

The function $\frac{1}{(1-x)^2}$ is the g.f for $\{1, 2, 3, \ldots\}$

ie $\{r+1\}_{r=0}^{\infty}$

{ coeff of x^r $\longrightarrow$ r^{th} term of the seq

$\longrightarrow a_r$ }

Exponential generating function:

If the terms of the sequence can be obtained as the coefficient of $\frac{x^r}{r!}$ In the expansion of $f(x)$, then $f(x)$ is said to be exponential generating function.

If a_r = coeff of $\frac{x^r}{r!}$, then "Exponential gf"

Ex:- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

The function e^x is the exponential gf for $\{1, 1, 1, 1, \ldots\}$

Generating function for combination

consider 3 objects, a, b and c and consider a polynomial

$(1+ax)(1+bx)(1+cx) = 1 + (a+b+c)x + (ab+bc+ca)x^2 + (abc)x^3$

RHS

coeff of x → ways of selecting one object out of 3 (ie a or b or c)
$a+b+c$

coeff x^2 → ways of selecting 2 objects out of 3
$ab+bc+ca$

coeff of x^3 → ways of selecting 3 objects out of 3
(abc)

LHS

$(1+ax)$ $\xrightarrow{\text{stands for}}$ symbollically represents either selecting a or not selecting a

x^0 → Not selecting a

ax → select a

$(1+bx)$ ⟶ selectn & nonselectn of 'b'

$(1+cx)$ ⟶ " " 'c'

$(1+ax)(1+bx)(1+cx)$ ⟶ selectn & nonselectn of the 3 objects a & b & c

Since we are concentrating on the no of ways, rather than the objs // ie since we want to enumerate, we just take $a = b = c = 1$

The polynomial turns into

$$\boxed{(1+x)^3 = 1 + 3x + 3x^2 + x^3}$$ → coeff of x^3 → no of ways of selecting 3

x^2 → no of ways of selecting 2 objs // out of 3

coeff $x \to {}^3C_1$

coeff of $x^2 \to {}^3C_2$

coeff of $x^3 \to {}^3C_3$

x → no of ways of selecting 1 obj out of 3

In general, if there are n objs // : $a_1\ a_2\ a_3 \cdots a_n$

$$(1+a_1x)(1+a_2x)(1+a_3x)\cdots(1+a_nx)$$
$$= 1 + (a_1 + a_2 + \cdots + a_n)x + (a_1a_2 + a_2a_3 + \cdots)x^2 + \cdots$$

For enumeration, $a_1 = a_2 = \cdots = a_n = 1$

$$(1+x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \cdots + {}^nC_nx^n$$

$$\boxed{(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r}$$

∴ The fn $(1+x)^n$ is the gf for $\{{}^nC_r\}_{r=0}^{n}$

The gf used in this way is called 'enumerator'

The function $(1+x)^n$ is the gf for r-combination of n objs // without reptn

* If there are 3 obj's. Ist obj can be selected atmost once
IInd atmost twice
IIIrd atmost thrice

No of ways of selecting 4 obj's out of these satisfying the above condⁿs :- ($\underbrace{abbc}_{\checkmark}$) ($\underbrace{aabc}_{\times}$)

writing the gf for this :-

$$\underbrace{(\overset{no}{1}+\overset{once}{x})}_{\text{Ist obj}}\underbrace{(\overset{no}{1}+\overset{once}{x}+\overset{twice}{x^2})}_{\text{second}}\underbrace{(1+x+x^2+x^3)}_{\text{third obj}}$$

To count the no of ways of selecting 4 obj's :-

coeff of x^4

* obtain the gf to count the no of ways to select 'r' obj's with the given condⁿ ; There are 5 distinct obj
Each obj can be selected atleast twice
more that 3 times
5th obj cannot be selected

Sol^n

$$(x^2+x^3+x^4+\dots)^4(x^2+x^3)$$

coeff of x^r →

G.f for combⁿs with unlimited repetition

If there are n obj's
For the first obj :- $(1+x+x^2+x^3+\dots\dots) = \frac{1}{1-x}$

∴ For the first obj, the corresp term $(1-x)^{-1}$

∴ For n objects :-

$$[(1-x)^{-1}]^n = (1-x)^{-n} = \sum {}^{n+r-1}C_r x^r \quad (?)$$

Note :-

The function $(1-x)^{-n}$ is the gf for combⁿ with unlimited reptⁿ

ie coeff of x^r is ${}^{n-r+1}C_r$

* Combinations :- (There are n objs)

of no reptⁿ $\longrightarrow (1+x)^n$

of unlimited repⁿ $\longrightarrow (1-x)^{-n} = \begin{cases} 1+x+x^2+x^3+\dots \\ = (1-x)^{-1} \end{cases}$

$(1+x)^n = \sum {}^nC_r x^r \longrightarrow {}^nC_r \rightarrow$ no of r-combⁿ with no rept

$(1-x)^{-n} = \sum {}^{n+r-1}C_r \, x^r \rightarrow {}^{n+r-1}C_r \rightarrow$ no of r-combⁿˢ with reptⁿ.

* No of ways of distributing 30M to 8 questⁿˢ s.t each quest gets atleast two marks

(Ans $= {}^{21}C_{14}$ (done already using logic))

solⁿ

$q_1 \, q_2 \cdots q_8$

$(x^2+x^3+x^4+\dots\dots)^8$

coeff of $x^{30} \Rightarrow$

$(x^2)^8 (1+x+\dots\dots)^8$

$x^{16}(1-x)^{-8}$

$$x^{16} \underbrace{\sum_{r=0}^{\infty} {}^{8+r-1}C_r \, x^r}_{\text{coeff of } x^{14}}$$

Ans $= {}^{8+14-1}C_{14} = {}^{21}C_{14}$

→ There are 3 obj, Each obj can be selected atmost once

$$(1+x)^3$$

→ There are 3 obj's, gf for selectn with no repitition

$$(1+x)^3$$

→ A bag has balls of 6 colours. No of ways of selecting 8 balls with the condn that the blue balls are selected atleast once and red balls are selected atmost twice.

coeff of x^8 from the gf:

$$gf :- (1+x+x^2+\dots\dots)^4 (x+x^2+x^3+\dots\dots)(1+x+x^2)$$

* No of selectn of 6 obj's out of 3 types of obj's with reptn upto 4 times of each type.

soln

coeff of x^6 from the gf

$$gf \rightarrow (1+x+x^2+x^3+x^4)^3$$

of unlimited reptn is allowed for all the 3 types :-

$$(1+x+x^2+\dots\dots\dots)^3 = \left[(1-x)^{-1}\right]^3$$

$$= (1-x)^{-3}$$

1. $1+x+x^2+x^3+\cdots+x^{r-1} = \dfrac{1-x^r}{1-x}$

2. $1+x+x^2+\cdots\cdots = \dfrac{1}{1-x}$

3. $(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$ (gf which generates $\{{}^nC_r\}$
gf for r-comb^n with no rept^n)

4. $(1-x)^{-n} = \sum_{r=0}^{\infty} {}^{n+r-1}C_r x^r$ (gf which generates $\{{}^{n+r-1}C_r\}$
gf for r-comb^n with unlimited rept^n)

5. $(1+x)^{-n} = \sum {}^{n+r-1}C_r (-1)^r x^r$

6. $(1-x^m)^n = \sum {}^nC_r (-x^m)^r$

7. If $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$g(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots$

coeff of x^r in the product $f(x)g(x)$ is

$$a_0 b_r + a_1 b_{r-1} + a_2 b_{r-2} + \cdots + a_r b_0$$

f

7

$\cancel{b_5}$

$+ \cancel{a_6 b_4} + \cancel{a_7\ 3}$

$a_9 b_1$

1 $a_{10} b_0$

① How many ways are there to select 25 toys from 7 types of toys with betwn 2 to 6 of each type

Soln

coeff of x^{25} from the g f $f(x)$

$$f(x) = (x^2+x^3+\dots+x^6)^7$$

$$= (x^2)^7(1+x+x^2+\dots+x^4)^7$$

$$= x^{14}\left(\frac{1-x^5}{1-x}\right)^7$$

$$f(x) = x^{14}\underbrace{(1-x^5)^7(1-x)^{-7}}_{\text{I want coeff of } x^{11}}$$

$$= x^{14}\left\{\sum_{r=0}^{7} {}^7C_r(-x^5)^r \sum_{r=0}^{\infty} {}^{7+r-1}C_r\, x^r\right\}$$

$\rightarrow a_0 b_{11} + a_1 b_{10} + a_2 b_9 + \dots + a_{11} b_0$ } only a_0, a_5, a_{10}

$\rightarrow$

$\therefore$ coeff is $\Rightarrow$ $a_0 b_{11} + a_5 b_6 + a_{10} b_1$

$r=0$ $r=11$ $r=1$ $r=6$ $r=2$ $r=1$

$${}^7C_0\ {}^{7+11-1}C_{11} + {}^7C_1(-1)\ {}^{7+6-1}C_6 + {}^7C_2\ {}^{7+1-1}C_1$$

Ans $\Rightarrow$ $\boxed{{}^7C_0\,{}^{17}C_{11} - {}^7C_1\,{}^{12}C_6 + {}^7C_2\,{}^7C_1}$