

MECHANICS OF MATERIALS 1

*An Introduction to the Mechanics of Elastic and
Plastic Deformation of Solids and Structural Materials*

THIRD EDITION

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Butterworth-Heinemann
Linacre House, Jordan Hill, Oxford OX2 8DP
225 Wildwood Avenue, Woburn, MA 01801-2041
A division of Reed Educational and Professional Publishing Ltd



First published 1977
Reprinted with corrections 1980, 1981, 1982
Second edition 1985
Reprinted with corrections 1988
Reprinted 1989, 1991, 1993, 1995, 1996
Third edition 1997
Reprinted 1998, 1999, 2000

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British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

ISBN 0 7506 3265 8

Library of Congress Cataloguing in Publication Data

Hearn, E. J. (Edwin John)

Mechanics of materials 1: an introduction to the mechanics of elastic and plastic deformation of solids and structural components/E. J. Hearn. - 3rd ed.

p. cm.

Includes bibliographical references and index.

ISBN 0 7506 3265 8

1. Strength of materials. I. Title

TA405.H3

620.1'23-dc21

96-49967

CIP

Printed and bound in Great Britain by Scotprint, Musselburgh



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INTRODUCTION

This text is the suitably revised and extended third edition of the highly successful text initially published in 1977 and intended to cover the material normally contained in degree and honours degree courses in mechanics of materials and in courses leading to exemption from the academic requirements of the Engineering Council. It should also serve as a valuable reference medium for industry and for post-graduate courses. Published in two volumes, the text should also prove valuable for students studying mechanical science, stress analysis, solid mechanics or similar modules on Higher Certificate and Higher Diploma courses in the UK or overseas and for appropriate NVQ* programmes.

The study of mechanics of materials is the study of the behaviour of solid bodies under load. The way in which they react to applied forces, the deflections resulting and the stresses and strains set up within the bodies, are all considered in an attempt to provide sufficient knowledge to enable any component to be designed such that it will not fail within its service life. Typical components considered in detail in this volume include beams, shafts, cylinders, struts, diaphragms and springs and, in most simple loading cases, theoretical expressions are derived to cover the mechanical behaviour of these components. Because of the reliance of such expressions on certain basic assumptions, the text also includes a chapter devoted to the important experimental stress and strain measurement techniques in use today with recommendations for further reading.

Each chapter of the text contains a summary of essential formulae which are developed within the chapter and a large number of worked examples. The examples have been selected to provide progression in terms of complexity of problem and to illustrate the logical way in which the solution to a difficult problem can be developed. Graphical solutions have been introduced where appropriate. In order to provide clarity of working in the worked examples there is inevitably more detailed explanation of individual steps than would be expected in the model answer to an examination problem.

All chapters (with the exception of Chapter 16) conclude with an extensive list of problems for solution of students together with answers. These have been collected from various sources and include questions from past examination papers in imperial units which have been converted to the equivalent SI values. Each problem is graded according to its degree of difficulty as follows:

- A Relatively easy problem of an introductory nature.
- A/B Generally suitable for first-year studies.
- B Generally suitable for second or third-year studies.
- C More difficult problems generally suitable for third year studies.

*National Vocational Qualifications

Gratitude is expressed to the following examination boards, universities and colleges who have kindly given permission for questions to be reproduced:

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Union of Educational Institutions	U.E.I.
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Both volumes of the text together contain 150 worked examples and more than 500 problems for solution, and whilst it is hoped that no errors are present it is perhaps inevitable that some errors will be detected. In this event any comment, criticism or correction will be gratefully acknowledged.

The symbols and abbreviations throughout the text are in accordance with the latest recommendations of BS 1991 and PD 5686†.

As mentioned above, graphical methods of solution have been introduced where appropriate since it is the author's experience that these are more readily accepted and understood by students than some of the more involved analytical procedures; substantial time saving can also result. Extensive use has also been made of diagrams throughout the text since in the words of the old adage "a single diagram is worth 1000 words".

Finally, the author is indebted to all those who have assisted in the production of this volume; to Professor H. G. Hopkins, Mr R. Brettell, Mr R. J. Phelps for their work associated with the first edition and to Dr A. S. Tooth¹, Dr N. Walker², Mr R. Winters² for their contributions to the second edition and to Dr M. Daniels for the extended treatment of the Finite Element Method which is the major change in this third edition. Thanks also go to the publishers for their advice and assistance, especially in the preparation of the diagrams and editing, to Dr. C. C. Perry (USA) for his most valuable critique of the first edition, and to Mrs J. Beard and Miss S. Benzing for typing the manuscript.

E. J. HEARN

† Relevant Standards for use in Great Britain: BS 1991; PD 5686: Other useful SI Guides: *The International System of Units*, N.P.L. Ministry of Technology, H.M.S.O. (Britain), Mechty, *The International System of Units (Physical Constants and Conversion Factors)*, NASA, No SP-7012, 3rd edn. 1973 (U.S.A.) *Metric Practice Guide*, A.S.T.M. Standard E380-72 (U.S.A.).

1. §23.27. Dr. A. S. Tooth, University of Strathclyde, Glasgow.
 2. §26. D. N. Walker and Mr. R. Winters, City of Birmingham Polytechnic.
 3. §24.4 Dr M. M. Daniels, University of Central England.

NOTATION

<i>Quantity</i>	<i>Symbol</i>	<i>SI Unit</i>
Angle	$\alpha, \beta, \theta, \gamma, \phi$	rad (radian)
Length	L, s	m (metre) mm (millimetre)
Area	A	m^2
Volume	V	m^3
Time	t	s (second)
Angular velocity	ω	rad/s
Velocity	v	m/s
Weight	W	N (newton)
Mass	m	kg (kilogram)
Density	ρ	kg/m^3
Force	F or P or W	N
Moment	M	N m
Pressure	P	Pa (Pascal) N/m^2 bar (= 10^5 N/m^2) N/m^2
Stress	σ	—
Strain	ϵ	—
Shear stress	τ	N/m^2
Shear strain	γ	—
Young's modulus	E	N/m^2
Shear modulus	G	N/m^2
Bulk modulus	K	N/m^2
Poisson's ratio	ν	—
Modular ratio	m	—
Power	—	W (watt)
Coefficient of linear expansion	α	$\text{m/m } ^\circ\text{C}$
Coefficient of friction	μ	—
Second moment of area	I	m^4
Polar moment of area	J	m^4
Product moment of area	I_{xy}	m^4
Temperature	T	$^\circ\text{C}$
Direction cosines	l, m, n	—
Principal stresses	$\sigma_1, \sigma_2, \sigma_3$	N/m^2
Principal strains	$\epsilon_1, \epsilon_2, \epsilon_3$	—
Maximum shear stress	τ_{\max}	N/m^2
Octahedral stress	σ_{oct}	N/m^2

Quantity	Symbol	SI Unit
Deviatoric stress	σ'	N/m ²
Deviatoric strain	ε'	—
Hydrostatic or mean stress	$\bar{\sigma}$	N/m ²
Volumetric strain	Δ	—
Stress concentration factor	K	—
Strain energy	U	J
Displacement	δ	m
Deflection	δ or y	m
Radius of curvature	ρ	m
Photoelastic material fringe value	f	N/m ² /fringe/m
Number of fringes	n	—
Body force stress	X, Y, Z F_R, F_θ, F_Z	N/m ³
Radius of gyration	k	m
Slenderness ratio	L/k	—
Gravitational acceleration	g	m/s ²
Cartesian coordinates	x, y, z	—
Cylindrical coordinates	r, θ, z	—
Eccentricity	e	m
Number of coils or leaves of spring	n	—
Equivalent J or effective polar moment of area	J_{eq} or J_E	m ⁴
Autofrettage pressure	P_A	N/m ² or bar
Radius of elastic-plastic interface	R_p	m
Thick cylinder radius ratio R_2/R_1	K	—
Ratio elastic-plastic interface radius to internal radius of thick cylinder R_p/R_1	m	—
Resultant stress on oblique plane	p_n	N/m ²
Normal stress on oblique plane	σ_n	N/m ²
Shear stress on oblique plane	τ_n	N/m ²
Direction cosines of plane	l, m, n	—
Direction cosines of line of action of resultant stress	l', m', n'	—
Direction cosines of line of action of shear stress	l_s, m_s, n_s	—
Components of resultant stress on oblique plane	p_{xn}, p_{yn}, p_{zn}	N/m ²
Shear stress in any direction ϕ on oblique plane	τ_ϕ	N/m ²
Invariants of stress	$\begin{cases} I_1 \\ I_2 \\ I_3 \end{cases}$	N/m ² (N/m ²) ² (N/m ²) ³
Invariants of reduced stresses	J_1, J_2, J_3	—
Airy stress function	ϕ	—

Quantity	Symbol	SI Unit
'Operator' for Airy stress function biharmonic equation	∇	—
Strain rate	$\dot{\epsilon}$	s^{-1}
Coefficient of viscosity	η	
Retardation time (creep strain recovery)	t'	s
Relaxation time (creep stress relaxation)	t''	s
Creep contraction or lateral strain ratio	$J(t)$	—
Maximum contact pressure (Hertz)	p_0	N/m^2
Contact formulae constant	Δ	$(N/m^2)^{-1}$
Contact area semi-axes	a, b	m
Maximum contact stress	$\sigma_c = -p_0$	N/m^2
Spur gear contact formula constant	K	N/m^2
Helical gear profile contact ratio	m_p	—
Elastic stress concentration factor	K_t	—
Fatigue stress concentration factor	K_f	—
Plastic flow stress concentration factor	K_p	—
Shear stress concentration factor	K_{ts}	—
Endurance limit for n cycles of load	S_n	N/m^2
Notch sensitivity factor	q	—
Fatigue notch factor	K_f	—
Strain concentration factor	K_ϵ	—
Griffith's critical strain energy release	G_c	
Surface energy of crack face	γ	N m
Plate thickness	B	m
Strain energy	U	N m
Compliance	C	$m N^{-1}$
Fracture stress	σ_f	N/m^2
Stress Intensity Factor	K or K_I	$N/m^{3/2}$
Compliance function	Y	—
Plastic zone dimension	r_p	m
Critical stress intensity factor	K_{IC}	$N/m^{3/2}$
"J" Integral	J	
Fatigue crack dimension	a	m
Coefficients of Paris Erdogan law	C, m	—
Fatigue stress range	σ_r	N/m^2
Fatigue mean stress	σ_m	N/m^2
Fatigue stress amplitude	σ_a	N/m^2
Fatigue stress ratio	R_s	—
Cycles to failure	N_f	—
Fatigue strength for N cycles	σ_N	N/m^2
Tensile strength	σ_{TS}	N/m^2
Factor of safety	F	—

<i>Quantity</i>	<i>Symbol</i>	<i>SI Unit</i>
Elastic strain range	$\Delta\varepsilon_e$	—
Plastic strain range	$\Delta\varepsilon_p$	—
Total strain range	$\Delta\varepsilon_t$	—
Ductility	D	—
Secondary creep rate	ε_s^0	s^{-1}
Activation energy	H	N m
Universal Gas Constant	R	J/kg K
Absolute temperature	T	$^{\circ}\text{K}$
Arrhenius equation constant	A	—
Larson–Miller creep parameter	P_1	—
Sherby–Dorn creep parameter	P_2	—
Manson–Haford creep parameter	P_3	—
Initial stress	σ_i	N/m ²
Time to rupture	t_r	s
Constants of power law equation	β, n	—

CONTENTS

Introduction	xv
Notation	xvii
1 Simple Stress and Strain	1
1.1 <i>Load</i>	1
1.2 <i>Direct or normal stress (σ)</i>	2
1.3 <i>Direct strain (ϵ)</i>	2
1.4 <i>Sign convention for direct stress and strain</i>	2
1.5 <i>Elastic materials – Hooke's law</i>	3
1.6 <i>Modulus of elasticity – Young's modulus</i>	3
1.7 <i>Tensile test</i>	4
1.8 <i>Ductile materials</i>	8
1.9 <i>Brittle materials</i>	8
1.10 <i>Poisson's ratio</i>	9
1.11 <i>Application of Poisson's ratio to a two-dimensional stress system</i>	10
1.12 <i>Shear stress</i>	11
1.13 <i>Shear strain</i>	11
1.14 <i>Modulus of rigidity</i>	12
1.15 <i>Double shear</i>	12
1.16 <i>Allowable working stress – factor of safety</i>	12
1.17 <i>Load factor</i>	13
1.18 <i>Temperature stresses</i>	13
1.19 <i>Stress concentrations – stress concentration factor</i>	14
1.20 <i>Toughness</i>	14
1.21 <i>Creep and fatigue</i>	15
<i>Examples</i>	17
<i>Problems</i>	25
<i>Bibliography</i>	26
2 Compound Bars	27
<i>Summary</i>	27
2.1 <i>Compound bars subjected to external load</i>	28

2.2	<i>Compound bars – “equivalent” or “combined” modulus</i>	29
2.3	<i>Compound bars subjected to temperature change</i>	30
2.4	<i>Compound bar (tube and rod)</i>	32
2.5	<i>Compound bars subjected to external load and temperature effects</i>	34
2.6	<i>Compound thick cylinders subjected to temperature changes</i>	34
	<i>Examples</i>	34
	<i>Problems</i>	39
3	Shearing Force and Bending Moment Diagrams	41
	<i>Summary</i>	41
3.1	<i>Shearing force and bending moment</i>	41
3.1.1	<i>Shearing force (S.F.) sign convention</i>	42
3.1.2	<i>Bending moment (B.M.) sign convention</i>	42
3.2	<i>S.F. and B.M. diagrams for beams carrying concentrated loads only</i>	43
3.3	<i>S.F. and B.M. diagrams for uniformly distributed loads</i>	46
3.4	<i>S.F. and B.M. diagrams for combined concentrated and uniformly distributed loads</i>	47
3.5	<i>Points of contraflexure</i>	48
3.6	<i>Relationship between S.F. Q, B.M. M, and intensity of loading w</i>	49
3.7	<i>S.F. and B.M. diagrams for an applied couple or moment</i>	50
3.8	<i>S.F. and B.M. diagrams for inclined loads</i>	52
3.9	<i>Graphical construction of S.F. and B.M. diagrams</i>	54
3.10	<i>S.F. and B.M. diagrams for beams carrying distributed loads of increasing value</i>	55
3.11	<i>S.F. at points of application of concentrated loads</i>	55
	<i>Examples</i>	56
	<i>Problems</i>	59
4	Bending	62
	<i>Summary</i>	62
	<i>Introduction</i>	63
4.1	<i>Simple bending theory</i>	64
4.2	<i>Neutral axis</i>	66
4.3	<i>Section modulus</i>	68
4.4	<i>Second moment of area</i>	68
4.5	<i>Bending of composite or flitched beams</i>	70
4.6	<i>Reinforced concrete beams – simple tension reinforcement</i>	71
4.7	<i>Skew loading</i>	73
4.8	<i>Combined bending and direct stress – eccentric loading</i>	74

4.9 "Middle-quarter" and "middle-third" rules	76
4.10 Shear stresses owing to bending	77
4.11 Strain energy in bending	78
4.12 Limitations of the simple bending theory	78
<i>Examples</i>	79
<i>Problems</i>	88
5 Slope and Deflection of Beams	92
<i>Summary</i>	92
<i>Introduction</i>	94
5.1 Relationship between loading, S.F., B.M., slope and deflection	94
5.2 Direct integration method	97
5.3 Macaulay's method	102
5.4 Macaulay's method for u.d.l's	105
5.5 Macaulay's method for beams with u.d.l. applied over part of the beam	106
5.6 Macaulay's method for couple applied at a point	106
5.7 Mohr's "area-moment" method	108
5.8 Principle of superposition	112
5.9 Energy method	112
5.10 Maxwell's theorem of reciprocal displacements	112
5.11 Continuous beams – Clapeyron's "three-moment" equation	115
5.12 Finite difference method	118
5.13 Deflections due to temperature effects	119
<i>Examples</i>	123
<i>Problems</i>	138
6 Built-in Beams	140
<i>Summary</i>	140
<i>Introduction</i>	141
6.1 Built-in beam carrying central concentrated load	141
6.2 Built-in beam carrying uniformly distributed load across the span	142
6.3 Built-in beam carrying concentrated load offset from the centre	143
6.4 Built-in beam carrying a non-uniform distributed load	145
6.5 Advantages and disadvantages of built-in beams	146
6.6 Effect of movement of supports	146
<i>Examples</i>	147
<i>Problems</i>	152

7 Shear Stress Distribution	154
<i>Summary</i>	154
<i>Introduction</i>	155
7.1 <i>Distribution of shear stress due to bending</i>	156
7.2 <i>Application to rectangular sections</i>	157
7.3 <i>Application to I-section beams</i>	158
7.3.1 <i>Vertical shear in the web</i>	159
7.3.2 <i>Vertical shear in the flanges</i>	159
7.3.3 <i>Horizontal shear in the flanges</i>	160
7.4 <i>Application to circular sections</i>	162
7.5 <i>Limitation of shear stress distribution theory</i>	164
7.6 <i>Shear centre</i>	165
<i>Examples</i>	166
<i>Problems</i>	173
8 Torsion	176
<i>Summary</i>	176
8.1 <i>Simple torsion theory</i>	177
8.2 <i>Polar second moment of area</i>	179
8.3 <i>Shear stress and shear strain in shafts</i>	180
8.4 <i>Section modulus</i>	181
8.5 <i>Torsional rigidity</i>	182
8.6 <i>Torsion of hollow shafts</i>	182
8.7 <i>Torsion of thin-walled tubes</i>	182
8.8 <i>Composite shafts – series connection</i>	182
8.9 <i>Composite shafts – parallel connection</i>	183
8.10 <i>Principal stresses</i>	184
8.11 <i>Strain energy in torsion</i>	184
8.12 <i>Variation of data along shaft length – torsion of tapered shafts</i>	186
8.13 <i>Power transmitted by shafts</i>	186
8.14 <i>Combined stress systems – combined bending and torsion</i>	187
8.15 <i>Combined bending and torsion – equivalent bending moment</i>	187
8.16 <i>Combined bending and torsion – equivalent torque</i>	188
8.17 <i>Combined bending, torsion and direct thrust</i>	189
8.18 <i>Combined bending, torque and internal pressure</i>	189
<i>Examples</i>	190
<i>Problems</i>	195

9 Thin Cylinders and Shells	198
<i>Summary</i>	198
9.1 <i>Thin cylinders under internal pressure</i>	198
9.1.1 <i>Hoop or circumferential stress</i>	199
9.1.2 <i>Longitudinal stress</i>	199
9.1.3 <i>Changes in dimensions</i>	200
9.2 <i>Thin rotating ring or cylinder</i>	201
9.3 <i>Thin spherical shell under internal pressure</i>	202
9.3.1 <i>Change in internal volume</i>	203
9.4 <i>Vessels subjected to fluid pressure</i>	203
9.5 <i>Cylindrical vessel with hemispherical ends</i>	204
9.6 <i>Effects of end plates and joints</i>	205
9.7 <i>Wire-wound thin cylinders</i>	206
<i>Examples</i>	208
<i>Problems</i>	213
10 Thick cylinders	215
<i>Summary</i>	215
10.1 <i>Difference in treatment between thin and thick cylinders – basic assumptions</i>	216
10.2 <i>Development of the Lamé theory</i>	217
10.3 <i>Thick cylinder – internal pressure only</i>	219
10.4 <i>Longitudinal stress</i>	220
10.5 <i>Maximum shear stress</i>	221
10.6 <i>Change of cylinder dimensions</i>	221
10.7 <i>Comparison with thin cylinder theory</i>	222
10.8 <i>Graphical treatment – Lamé line</i>	223
10.9 <i>Compound cylinders</i>	224
10.10 <i>Compound cylinders – graphical treatment</i>	226
10.11 <i>Shrinkage or interference allowance</i>	226
10.12 <i>Hub on solid shaft</i>	229
10.13 <i>Force fits</i>	229
10.14 <i>Compound cylinder – different materials</i>	230
10.15 <i>Uniform heating of compound cylinders of different materials</i>	231
10.16 <i>Failure theories – yield criteria</i>	233
10.17 <i>Plastic yielding – “auto-frettagge”</i>	233
10.18 <i>Wire-wound thick cylinders</i>	234
<i>Examples</i>	236
<i>Problems</i>	251

11 Strain Energy	254
<i>Summary</i>	254
<i>Introduction</i>	256
11.1 <i>Strain energy – tension or compression</i>	257
11.2 <i>Strain energy – shear</i>	259
11.3 <i>Strain energy – bending</i>	260
11.4 <i>Strain energy – torsion</i>	261
11.5 <i>Strain energy of a three-dimensional principal stress system</i>	262
11.6 <i>Volumetric or dilatational strain energy</i>	262
11.7 <i>Shear or distortional strain energy</i>	263
11.8 <i>Suddenly applied loads</i>	263
11.9 <i>Impact loads – axial load application</i>	264
11.10 <i>Impact loads – bending applications</i>	265
11.11 <i>Castigliano's first theorem for deflection</i>	266
11.12 <i>"Unit-load" method</i>	268
11.13 <i>Application of Castigliano's theorem to angular movements</i>	269
11.14 <i>Shear deflection</i>	269
<i>Examples</i>	274
<i>Problems</i>	292
12 Springs	297
<i>Summary</i>	297
<i>Introduction</i>	299
12.1 <i>Close-coiled helical spring subjected to axial load W</i>	299
12.2 <i>Close-coiled helical spring subjected to axial torque T</i>	300
12.3 <i>Open-coiled helical spring subjected to axial load W</i>	301
12.4 <i>Open-coiled helical spring subjected to axial torque T</i>	304
12.5 <i>Springs in series</i>	305
12.6 <i>Springs in parallel</i>	306
12.7 <i>Limitations of the simple theory</i>	306
12.8 <i>Extension springs – initial tension</i>	307
12.9 <i>Allowable stresses</i>	308
12.10 <i>Leaf or carriage spring: semi-elliptic</i>	309
12.11 <i>Leaf or carriage spring: quarter-elliptic</i>	312
12.12 <i>Spiral spring</i>	314
<i>Examples</i>	316
<i>Problems</i>	324

13 Complex Stresses	326
<i>Summary</i>	326
13.1 <i>Stresses on oblique planes</i>	326
13.2 <i>Material subjected to pure shear</i>	327
13.3 <i>Material subjected to two mutually perpendicular direct stresses</i>	329
13.4 <i>Material subjected to combined direct and shear stresses</i>	329
13.5 <i>Principal plane inclination in terms of the associated principal stress</i>	331
13.6 <i>Graphical solution – Mohr's stress circle</i>	332
13.7 <i>Alternative representations of stress distributions at a point</i>	334
13.8 <i>Three-dimensional stresses – graphical representation</i>	338
<i>Examples</i>	342
<i>Problems</i>	358
14 Complex Strain and the Elastic Constants	361
<i>Summary</i>	361
14.1 <i>Linear strain for tri-axial stress state</i>	361
14.2 <i>Principal strains in terms of stresses</i>	362
14.3 <i>Principal stresses in terms of strains – two-dimensional stress system</i>	363
14.4 <i>Bulk modulus K</i>	363
14.5 <i>Volumetric strain</i>	363
14.6 <i>Volumetric strain for unequal stresses</i>	364
14.7 <i>Change in volume of circular bar</i>	365
14.8 <i>Effect of lateral restraint</i>	366
14.9 <i>Relationship between the elastic constants E, G, K and v</i>	367
14.10 <i>Strains on an oblique plane</i>	370
14.11 <i>Principal strain – Mohr's strain circle</i>	372
14.12 <i>Mohr's strain circle – alternative derivation from the general stress equations</i>	374
14.13 <i>Relationship between Mohr's stress and strain circles</i>	375
14.14 <i>Construction of strain circle from three known strains (McClintock method) – rosette analysis</i>	378
14.15 <i>Analytical determination of principal strains from rosette readings</i>	381
14.16 <i>Alternative representations of strain distributions at a point</i>	383
14.17 <i>Strain energy of three-dimensional stress system</i>	385
<i>Examples</i>	387
<i>Problems</i>	397
15 Theories of Elastic Failure	401
<i>Summary</i>	401
<i>Introduction</i>	401

15.1	<i>Maximum principal stress theory</i>	402
15.2	<i>Maximum shear stress theory</i>	403
15.3	<i>Maximum principal strain theory</i>	403
15.4	<i>Maximum total strain energy per unit volume theory</i>	403
15.5	<i>Maximum shear strain energy per unit volume (or distortion energy) theory</i>	403
15.6	<i>Mohr's modified shear stress theory for brittle materials</i>	404
15.7	<i>Graphical representation of failure theories for two-dimensional stress systems (one principal stress zero)</i>	406
15.8	<i>Graphical solution of two-dimensional theory of failure problems</i>	410
15.9	<i>Graphical representation of the failure theories for three-dimensional stress systems</i>	411
15.9.1	<i>Ductile materials</i>	411
15.9.2	<i>Brittle materials</i>	412
15.10	<i>Limitations of the failure theories</i>	413
15.11	<i>Effect of stress concentrations</i>	414
15.12	<i>Safety factors</i>	414
15.13	<i>Modes of failure</i>	416
	<i>Examples</i>	417
	<i>Problems</i>	427
16	Experimental Stress Analysis	430
	<i>Introduction</i>	430
16.1	<i>Brittle lacquers</i>	431
16.2	<i>Strain gauges</i>	435
16.3	<i>Unbalanced bridge circuit</i>	437
16.4	<i>Null balance or balanced bridge circuit</i>	437
16.5	<i>Gauge construction</i>	437
16.6	<i>Gauge selection</i>	438
16.7	<i>Temperature compensation</i>	439
16.8	<i>Installation procedure</i>	440
16.9	<i>Basic measurement systems</i>	441
16.10	<i>D.C. and A.C. systems</i>	443
16.11	<i>Other types of strain gauge</i>	444
16.12	<i>Photoelasticity</i>	445
16.13	<i>Plane-polarised light – basic polariscope arrangements</i>	446
16.14	<i>Temporary birefringence</i>	446
16.15	<i>Production of fringe patterns</i>	448
16.16	<i>Interpretation of fringe patterns</i>	449
16.17	<i>Calibration</i>	450

16.18 <i>Fractional fringe order determination – compensation techniques</i>	451
16.19 <i>Isoclinics – circular polarisation</i>	452
16.20 <i>Stress separation procedures</i>	454
16.21 <i>Three-dimensional photoelasticity</i>	454
16.22 <i>Reflective coating technique</i>	454
16.23 <i>Other methods of strain measurement</i>	456
<i>Bibliography</i>	456

Appendix 1. Typical mechanical and physical properties for engineering materials	
	xxi

Appendix 2. Typical mechanical properties of non-metals	
	xxii

Appendix 3. Other properties of non-metals	
	xxiii

Index	xxv
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CHAPTER 1

SIMPLE STRESS AND STRAIN

1.1. Load

In any engineering structure or mechanism the individual components will be subjected to external forces arising from the service conditions or environment in which the component works. If the component or member is in equilibrium, the resultant of the external forces will be zero but, nevertheless, they together place a load on the member which tends to deform that member and which must be reacted by internal forces which are set up within the material.

If a cylindrical bar is subjected to a direct pull or push along its axis as shown in Fig. 1.1, then it is said to be subjected to *tension* or *compression*. Typical examples of tension are the forces present in towing ropes or lifting hoists, whilst compression occurs in the legs of your chair as you sit on it or in the support pillars of buildings.

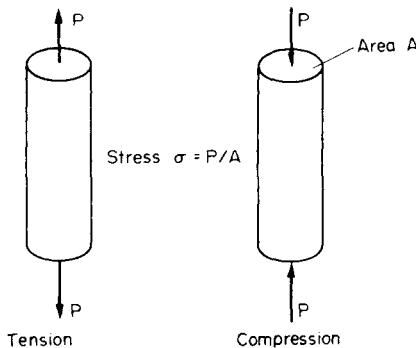


Fig. 1.1. Types of direct stress.

In the SI system of units load is measured in *newtons*, although a single newton, in engineering terms, is a very small load. In most engineering applications, therefore, loads appear in SI multiples, i.e. kilonewtons (kN) or meganewtons (MN).

There are a number of different ways in which load can be applied to a member. Typical loading types are:

- Static* or *dead* loads, i.e. non-fluctuating loads, generally caused by gravity effects.
- Live* loads, as produced by, for example, lorries crossing a bridge.
- Impact* or *shock* loads caused by sudden blows.
- Fatigue*, *fluctuating* or *alternating* loads, the magnitude and sign of the load changing with time.

1.2. Direct or normal stress (σ)

It has been noted above that external force applied to a body in equilibrium is reacted by internal forces set up within the material. If, therefore, a bar is subjected to a uniform tension or compression, i.e. a direct force, which is uniformly or equally applied across the cross-section, then the internal forces set up are also distributed uniformly and the bar is said to be subjected to a uniform *direct or normal stress*, the stress being defined as

$$\text{stress } (\sigma) = \frac{\text{load}}{\text{area}} = \frac{P}{A}$$

Stress σ may thus be compressive or tensile depending on the nature of the load and will be measured in units of newtons per square metre (N/m^2) or multiples of this.

In some cases the loading situation is such that the stress will vary across any given section, and in such cases the stress at any point is given by the limiting value of $\delta P/\delta A$ as δA tends to zero.

1.3. Direct strain (ϵ)

If a bar is subjected to a direct load, and hence a stress, the bar will change in length. If the bar has an original length L and changes in length by an amount δL , the *strain* produced is defined as follows:

$$\text{strain } (\epsilon) = \frac{\text{change in length}}{\text{original length}} = \frac{\delta L}{L}$$

Strain is thus a measure of the deformation of the material and is non-dimensional, i.e. it has no units; it is simply a ratio of two quantities with the same unit (Fig. 1.2).

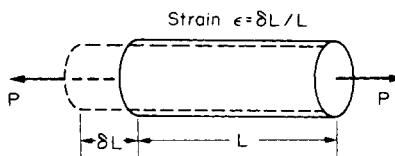


Fig. 1.2.

Since, in practice, the extensions of materials under load are very small, it is often convenient to measure the strains in the form of $\text{strain} \times 10^{-6}$, i.e. *microstrain*, when the symbol used becomes $\mu\epsilon$.

Alternatively, strain can be expressed as a *percentage strain*

i.e.

$$\text{strain } (\epsilon) = \frac{\delta L}{L} \times 100\%$$

1.4. Sign convention for direct stress and strain

Tensile stresses and strains are considered **POSITIVE** in sense producing an *increase* in length. Compressive stresses and strains are considered **NEGATIVE** in sense producing a *decrease* in length.

1.5. Elastic materials – Hooke's law

A material is said to be *elastic* if it returns to its original, unloaded dimensions when load is removed. A particular form of elasticity which applies to a large range of engineering materials, at least over part of their load range, produces deformations which are proportional to the loads producing them. Since loads are proportional to the stresses they produce and deformations are proportional to the strains, this also implies that, whilst materials are elastic, stress is proportional to strain. *Hooke's law*, in its simplest form*, therefore states that

$$\text{stress } (\sigma) \propto \text{strain } (\varepsilon)$$

i.e.

$$\frac{\text{stress}}{\text{strain}} = \text{constant}^*$$

It will be seen in later sections that this law is obeyed within certain limits by most ferrous alloys and it can even be assumed to apply to other engineering materials such as concrete, timber and non-ferrous alloys with reasonable accuracy. Whilst a material is elastic the deformation produced by any load will be *completely* recovered when the load is removed; there is no permanent deformation.

Other classifications of materials with which the reader should be acquainted are as follows:

A material which has a uniform structure throughout without any flaws or discontinuities is termed a *homogeneous* material. *Non-homogeneous* or *inhomogeneous* materials such as concrete and poor-quality cast iron will thus have a structure which varies from point to point depending on its constituents and the presence of casting flaws or impurities.

If a material exhibits uniform properties throughout in all directions it is said to be *isotropic*; conversely one which does not exhibit this uniform behaviour is said to be *non-isotropic* or *anisotropic*.

An *orthotropic* material is one which has different properties in different planes. A typical example of such a material is wood, although some composites which contain systematically orientated "inhomogeneities" may also be considered to fall into this category.

1.6. Modulus of elasticity – Young's modulus

Within the elastic limits of materials, i.e. within the limits in which Hooke's law applies, it has been shown that

$$\frac{\text{stress}}{\text{strain}} = \text{constant}$$

This constant is given the symbol E and termed the *modulus of elasticity* or *Young's modulus*.

Thus

$$E = \frac{\text{stress}}{\text{strain}} = \frac{\sigma}{\varepsilon} \quad (1.1)$$

$$= \frac{P}{A} \div \frac{\delta L}{L} = \frac{PL}{A \delta L} \quad (1.2)$$

* Readers should be warned that in more complex stress cases this simple form of Hooke's law will not apply and mis-application could prove dangerous; see §14.1, page 361.

Young's modulus E is generally assumed to be the same in tension or compression and for most engineering materials has a high numerical value. Typically, $E = 200 \times 10^9 \text{ N/m}^2$ for steel, so that it will be observed from (1.1) that strains are normally very small since

$$\epsilon = \frac{\sigma}{E} \quad (1.3)$$

In most common engineering applications strains do not often exceed 0.003 or 0.3% so that the assumption used later in the text that deformations are small in relation to original dimensions is generally well founded.

The actual value of Young's modulus for any material is normally determined by carrying out a standard tensile test on a specimen of the material as described below.

1.7. Tensile test

In order to compare the strengths of various materials it is necessary to carry out some standard form of test to establish their relative properties. One such test is the standard tensile test in which a circular bar of uniform cross-section is subjected to a gradually increasing tensile load until failure occurs. Measurements of the change in length of a selected *gauge length* of the bar are recorded throughout the loading operation by means of extensometers and a graph of load against extension or stress against strain is produced as shown in Fig. 1.3; this shows a typical result for a test on a mild (low carbon) steel bar; other materials will exhibit different graphs but of a similar general form see Figs 1.5 to 1.7.

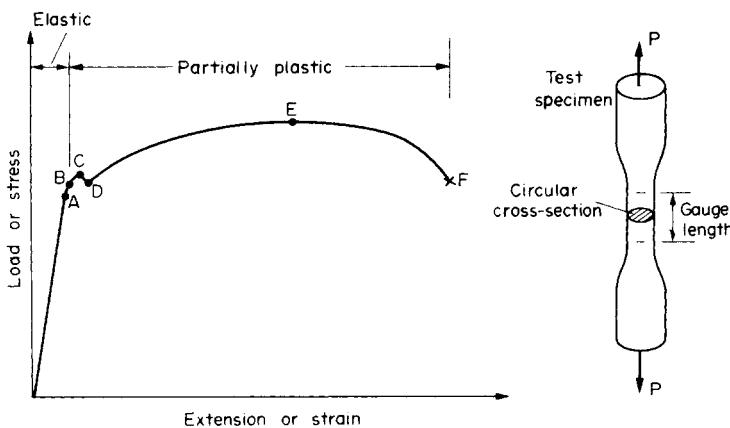


Fig. 1.3. Typical tensile test curve for mild steel.

For the first part of the test it will be observed that Hooke's law is obeyed, i.e. the material behaves elastically and stress is proportional to strain, giving the straight-line graph indicated. Some point A is eventually reached, however, when the linear nature of the graph ceases and this point is termed the *limit of proportionality*.

For a short period beyond this point the material may still be elastic in the sense that deformations are completely recovered when load is removed (i.e. strain returns to zero) but

Hooke's law does not apply. The limiting point *B* for this condition is termed the *elastic limit*. For most practical purposes it can often be assumed that points *A* and *B* are coincident.

Beyond the elastic limit *plastic deformation* occurs and strains are not totally recoverable. There will thus be some permanent deformation or *permanent set* when load is removed. After the points *C*, termed the *upper yield point*, and *D*, the *lower yield point*, relatively rapid increases in strain occur without correspondingly high increases in load or stress. The graph thus becomes much more shallow and covers a much greater portion of the strain axis than does the elastic range of the material. The capacity of a material to allow these large plastic deformations is a measure of the so-called *ductility* of the material, and this will be discussed in greater detail below.

For certain materials, for example, high carbon steels and non-ferrous metals, it is not possible to detect any difference between the upper and lower yield points and in some cases no yield point exists at all. In such cases a *proof stress* is used to indicate the onset of plastic strain or as a comparison of the relative properties with another similar material. This involves a measure of the permanent deformation produced by a loading cycle; the 0.1% proof stress, for example, is that stress which, when removed, produces a permanent strain or "set" of 0.1% of the original gauge length – see Fig. 1.4(a).

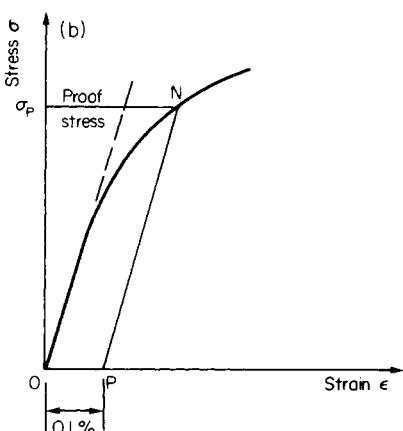


Fig. 1.4. (a) Determination of 0.1% proof stress.

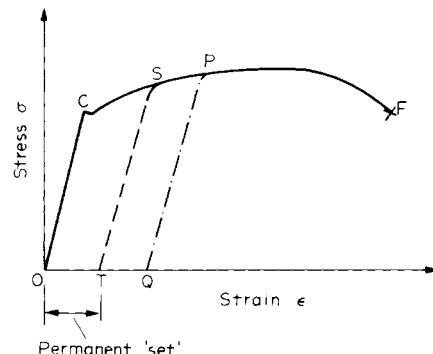


Fig. 1.4. (b) Permanent deformation or "set" after straining beyond the yield point.

The 0.1% proof stress value may be determined from the tensile test curve for the material in question as follows:

Mark the point *P* on the strain axis which is equivalent to 0.1% strain. From *P* draw a line parallel with the initial straight line portion of the tensile test curve to cut the curve in *N*. The stress corresponding to *N* is then the 0.1% *proof stress*. A material is considered to satisfy its specification if the permanent set is no more than 0.1% after the proof stress has been applied for 15 seconds and removed.

Beyond the yield point some increase in load is required to take the strain to point *E* on the graph. Between *D* and *E* the material is said to be in the *elastic-plastic* state, some of the section remaining elastic and hence contributing to recovery of the original dimensions if load is removed, the remainder being plastic. Beyond *E* the cross-sectional area of the bar

begins to reduce rapidly over a relatively small length of the bar and the bar is said to neck. This necking takes place whilst the load reduces, and fracture of the bar finally occurs at point *F*.

The nominal stress at failure, termed the *maximum* or *ultimate tensile stress*, is given by the load at *E* divided by the original cross-sectional area of the bar. (This is also known as the *tensile strength* of the material of the bar.) Owing to the large reduction in area produced by the necking process the actual stress at fracture is often greater than the above value. Since, however, designers are interested in maximum loads which can be carried by the complete cross-section, the stress at fracture is seldom of any practical value.

If load is removed from the test specimen after the yield point *C* has been passed, e.g. to some position *S*, Fig. 1.4(b), the unloading line *ST* can, for most practical purposes, be taken to be linear. Thus, despite the fact that loading to *S* comprises both elastic (*OC*) and partially plastic (*CS*) portions, the unloading procedure is totally elastic. A second load cycle, commencing with the permanent elongation associated with the strain *OT*, would then follow the line *TS* and continue along the previous curve to failure at *F*. It will be observed, however, that the repeated load cycle has the effect of increasing the elastic range of the material, i.e. raising the effective yield point from *C* to *S*, while the tensile strength is unaltered. The procedure could be repeated along the line *PQ*, etc., and the material is said to have been *work hardened*.

In fact, careful observation shows that the material will no longer exhibit true elasticity since the unloading and reloading lines will form a small *hysteresis loop*, neither being precisely linear. Repeated loading and unloading will produce a yield point approaching the ultimate stress value but the elongation or strain to failure will be much reduced.

Typical stress-strain curves resulting from tensile tests on other engineering materials are shown in Figs 1.5 to 1.7.

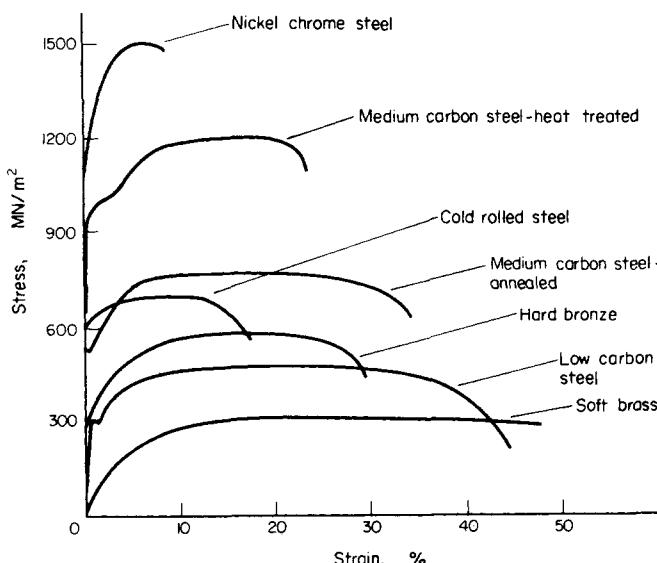


Fig. 1.5. Tensile test curves for various metals.

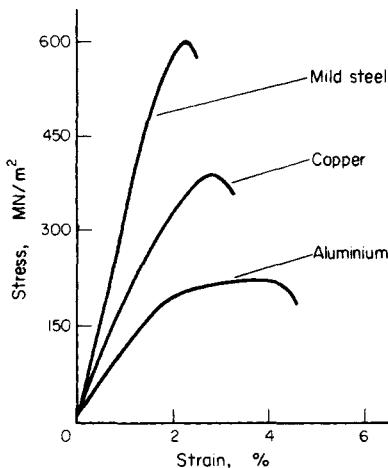


Fig. 1.6. Typical stress-strain curves for hard drawn wire materials—note large reduction in strain values from those of Fig. 1.5.

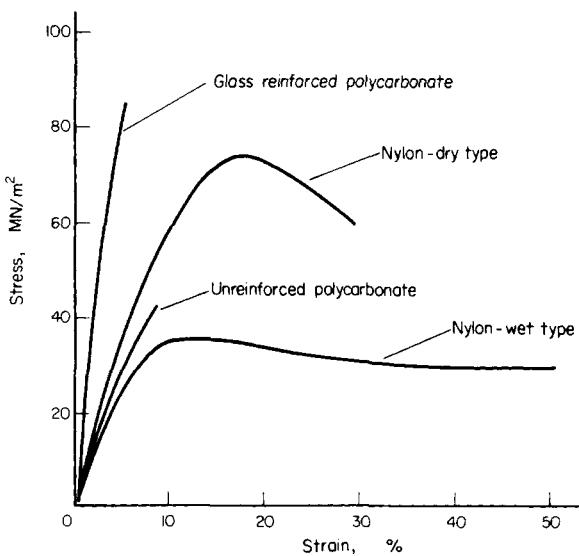


Fig. 1.7. Typical tension test results for various types of nylon and polycarbonate.

After completing the standard tensile test it is usually necessary to refer to some “British Standard Specification” or “Code of Practice” to ensure that the material tested satisfies the requirements, for example:

- BS 4360 British Standard Specification for Weldable Structural Steels.
- BS 970 British Standard Specification for Wrought Steels.
- BS 153 British Standard Specification for Steel Girder Bridges.
- BS 449 British Standard Specification for the use of Structural Steel in Building, etc.

1.8. Ductile materials

It has been observed above that the partially plastic range of the graph of Fig. 1.3 covers a much wider part of the strain axis than does the elastic range. Thus the extension of the material over this range is considerably in excess of that associated with elastic loading. The capacity of a material to allow these large extensions, i.e. the ability to be drawn out plastically, is termed its *ductility*. Materials with high ductility are termed *ductile* materials, members with low ductility are termed *brittle* materials. A quantitative value of the ductility is obtained by measurements of the *percentage elongation* or *percentage reduction in area*, both being defined below.

$$\text{Percentage elongation} = \frac{\text{increase in gauge length to fracture}}{\text{original gauge length}} \times 100$$

$$\text{Percentage reduction in area} = \frac{\text{reduction in cross-sectional area of necked portion}}{\text{original area}} \times 100$$

The latter value, being independent of any selected gauge length, is generally taken to be the more useful measure of ductility for reference purposes.

A property closely related to ductility is *malleability*, which defines a material's ability to be hammered out into thin sheets. A typical example of a malleable material is lead. This is used extensively in the plumbing trade where it is hammered or beaten into corners or joints to provide a weatherproof seal. Malleability thus represents the ability of a material to allow permanent extensions in all lateral directions under compressive loadings.

1.9. Brittle materials

A brittle material is one which exhibits relatively small extensions to fracture so that the partially plastic region of the tensile test graph is much reduced (Fig. 1.8). Whilst Fig. 1.3 referred to a low carbon steel, Fig. 1.8 could well refer to a much higher strength steel with a higher carbon content. There is little or no necking at fracture for brittle materials.

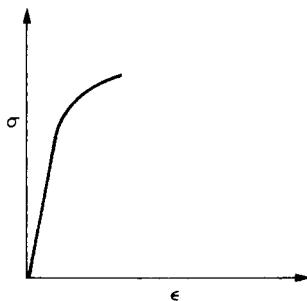


Fig. 1.8. Typical tensile test curve for a brittle material.

Typical variations of mechanical properties of steel with carbon content are shown in Fig. 1.9.

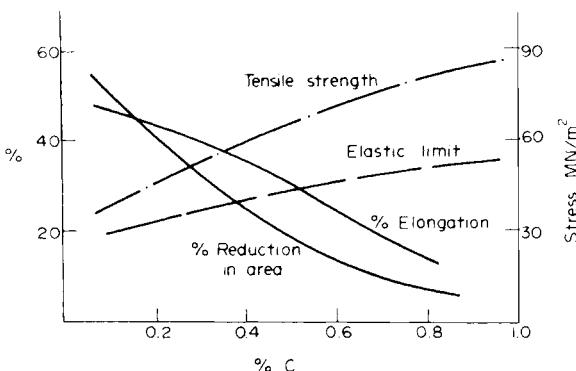


Fig. 1.9. Variation of mechanical properties of steel with carbon content.

1.10. Poisson's ratio

Consider the rectangular bar of Fig. 1.10 subjected to a tensile load. Under the action of this load the bar will increase in length by an amount δL giving a longitudinal strain in the bar of

$$\varepsilon_L = \frac{\delta L}{L}$$

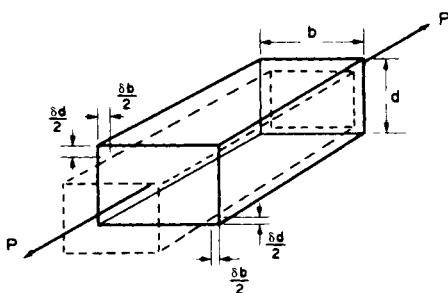


Fig. 1.10.

The bar will also exhibit, however, a *reduction* in dimensions laterally, i.e. its breadth and depth will both reduce. The associated lateral strains will both be equal, will be of opposite sense to the longitudinal strain, and will be given by

$$\varepsilon_{\text{lat}} = -\frac{\delta b}{b} = -\frac{\delta d}{d}$$

Provided the load on the material is retained within the elastic range the ratio of the lateral and longitudinal strains will always be constant. This ratio is termed *Poisson's ratio*.

i.e. $\text{Poisson's ratio } (\nu) = \frac{\text{lateral strain}}{\text{longitudinal strain}} = \frac{(-\delta d/d)}{\delta L/L}$ (1.4)

The negative sign of the lateral strain is normally ignored to leave Poisson's ratio simply as

a ratio of strain magnitudes. It must be remembered, however, that the longitudinal strain induces a lateral strain of opposite sign, e.g. tensile longitudinal strain induces compressive lateral strain.

For most engineering materials the value of ν lies between 0.25 and 0.33. Since

$$\text{longitudinal strain} = \frac{\text{longitudinal stress}}{\text{Young's modulus}} = \frac{\sigma}{E} \quad (1.4a)$$

Hence

$$\text{lateral strain} = \nu \frac{\sigma}{E} \quad (1.4b)$$

1.11. Application of Poisson's ratio to a two-dimensional stress system

A two-dimensional stress system is one in which all the stresses lie within one plane such as the $X-Y$ plane. From the work of §1.10 it will be seen that if a material is subjected to a tensile stress σ on one axis producing a strain σ/E and hence an extension on that axis, it will be subjected simultaneously to a lateral strain of ν times σ/E on any axis at right angles. This lateral strain will be compressive and will result in a compression or reduction of length on this axis.

Consider, therefore, an element of material subjected to two stresses at right angles to each other and let both stresses, σ_x and σ_y , be considered tensile, see Fig. 1.11.

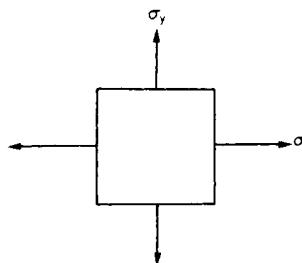


Fig. 1.11. Simple two-dimensional system of direct stresses.

The following strains will be produced:

- (a) in the X direction resulting from $\epsilon_x = \sigma_x/E$,
- (b) in the Y direction resulting from $\epsilon_y = \sigma_y/E$.
- (c) in the X direction resulting from $\epsilon_x = -\nu(\sigma_y/E)$,
- (d) in the Y direction resulting from $\epsilon_y = -\nu(\sigma_x/E)$.

strains (c) and (d) being the so-called *Poisson's ratio strain*, opposite in sign to the applied strains, i.e. compressive.

The total strain in the X direction will therefore be given by:

$$\epsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} = \frac{1}{E} (\sigma_x - \nu \sigma_y) \quad (1.5)$$

and the total strain in the Y direction will be:

$$\varepsilon_y = \frac{\sigma_y}{E} - v \frac{\sigma_x}{E} = \frac{1}{E} (\sigma_y - v \sigma_x) \quad (1.6)$$

If any stress is, in fact, compressive its value must be substituted in the above equations together with a negative sign following the normal sign convention.

1.12. Shear stress

Consider a block or portion of material as shown in Fig. 1.12a subjected to a set of equal and opposite forces Q . (Such a system could be realised in a bicycle brake block when contacted with the wheel.) There is then a tendency for one layer of the material to slide over another to produce the form of failure shown in Fig. 1.12b. If this failure is restricted, then a *shear stress* τ is set up, defined as follows:

$$\text{shear stress } (\tau) = \frac{\text{shear load}}{\text{area resisting shear}} = \frac{Q}{A}$$

This shear stress will always be *tangential* to the area on which it acts; direct stresses, however, are always *normal* to the area on which they act.

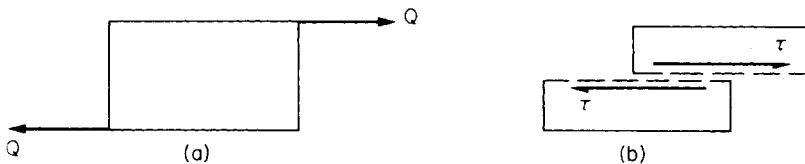


Fig. 1.12. Shear force and resulting shear stress system showing typical form of failure by relative sliding of planes.

1.13. Shear strain

If one again considers the block of Fig. 1.12a to be a bicycle brake block it is clear that the rectangular shape of the block will not be retained as the brake is applied and the shear forces introduced. The block will in fact change shape or "strain" into the form shown in Fig. 1.13. The angle of deformation γ is then termed the *shear strain*.

Shear strain is measured in radians and hence is non-dimensional, i.e. it has no units.

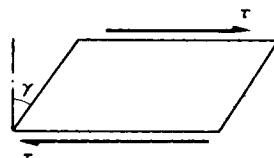


Fig. 1.13. Deformation (shear strain) produced by shear stresses.

1.14. Modulus of rigidity

For materials within the elastic range the shear strain is proportional to the shear stress producing it,
i.e.

$$\frac{\text{shear stress}}{\text{shear strain}} = \frac{\tau}{\gamma} = \text{constant} = G \quad (1.7)$$

The constant G is termed the *modulus of rigidity* or *shear modulus* and is directly comparable to the modulus of elasticity used in the direct stress application. The term *modulus* thus implies a ratio of stress to strain in each case.

1.15. Double shear

Consider the simple riveted lap joint shown in Fig. 1.14a. When load is applied to the plates the rivet is subjected to shear forces tending to shear it on one plane as indicated. In the butt joint with two cover plates of Fig. 1.14b, however, each rivet is subjected to possible shearing on two faces, i.e. *double shear*. In such cases twice the area of metal is resisting the applied forces so that the shear stress set up is given by

$$\text{shear stress } \tau \text{ (in double shear)} = \frac{P}{2A} \quad (1.8)$$

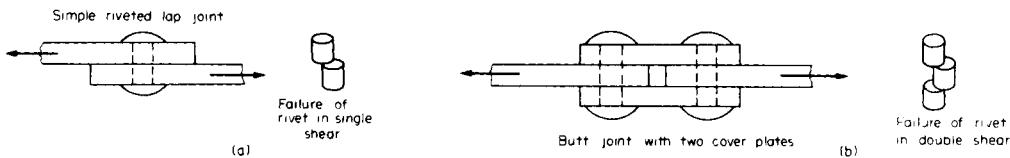


Fig. 1.14. (a) Single shear. (b) Double shear.

1.16. Allowable working stress – factor of safety

The most suitable strength or stiffness criterion for any structural element or component is normally some maximum stress or deformation which must not be exceeded. In the case of stresses the value is generally known as the *maximum allowable working stress*.

Because of uncertainties of loading conditions, design procedures, production methods, etc., designers generally introduce a *factor of safety* into their designs, defined as follows:

$$\text{factor of safety} = \frac{\text{maximum stress}}{\text{allowable working stress}} \quad (1.9)$$

However, in view of the fact that plastic deformations are seldom accepted this definition is sometimes modified to

$$\text{factor of safety} = \frac{\text{yield stress (or proof stress)}}{\text{allowable working stress}}$$

In the absence of any information as to which definition has been used for any quoted value of safety factor the former definition must be assumed. In this case a factor of safety of 3 implies that the design is capable of carrying three times the maximum stress to which it is expected the structure will be subjected in any normal loading condition. There is seldom any realistic basis for the selection of a particular safety factor and values vary significantly from one branch of engineering to another. Values are normally selected on the basis of a consideration of the social, human safety and economic consequences of failure. Typical values range from 2.5 (for relatively low consequence, static load cases) to 10 (for shock load and high safety risk applications)—see §15.12.

1.17. Load factor

In some loading cases, e.g. buckling of struts, neither the yield stress nor the ultimate strength is a realistic criterion for failure of components. In such cases it is convenient to replace the safety factor, based on stresses, with a different factor based on loads. The *load factor* is therefore defined as:

$$\text{load factor} = \frac{\text{load at failure}}{\text{allowable working load}} \quad (1.10)$$

This is particularly useful in applications of the so-called plastic limit design procedures.

1.18. Temperature stresses

When the temperature of a component is increased or decreased the material respectively expands or contracts. If this expansion or contraction is not resisted in any way then the processes take place free of stress. If, however, the changes in dimensions are restricted then stresses termed *temperature stresses* will be set up within the material.

Consider a bar of material with a linear coefficient of expansion α . Let the original length of the bar be L and let the temperature increase be t . If the bar is free to expand the change in length would be given by

$$\Delta L = Lat \quad (1.11)$$

and the new length

$$L' = L + Lat = L(1 + \alpha t)$$

If this extension were totally prevented, then a compressive stress would be set up equal to that produced when a bar of length $L(1 + \alpha t)$ is compressed through a distance of Lat . In this case the bar experiences a compressive strain

$$\epsilon = \frac{\Delta L}{L} = \frac{Lat}{L(1 + \alpha t)}$$

In most cases αt is very small compared with unity so that

$$\epsilon = \frac{Lat}{L} = \alpha t$$

But

$$\frac{\sigma}{\varepsilon} = E$$

$$\therefore \text{stress } \sigma = E\varepsilon = Eat \quad (1.12)$$

This is the stress set up owing to total restraint on expansions or contractions caused by a temperature rise, or fall, t . In the former case the stress is compressive, in the latter case the stress is tensile.

If the expansion or contraction of the bar is *partially* prevented then the stress set up will be less than that given by eqn. (1.10). Its value will be found in a similar way to that described above except that instead of being compressed through the total free expansion distance of $L\alpha t$ it will be compressed through some proportion of this distance depending on the amount of restraint.

Assuming some fraction n of $L\alpha t$ is *allowed*, then the extension which is prevented is $(1 - n)L\alpha t$. This will produce a compressive strain, as described previously, of magnitude

$$\varepsilon = \frac{(1 - n)L\alpha t}{L(1 + \alpha t)}$$

or, approximately,

$$\varepsilon = (1 - n)L\alpha t/L = (1 - n)\alpha t.$$

The stress set up will then be E times ε .

i.e.

$$\sigma = (1 - n)Eat \quad (1.13)$$

Thus, for example, if one-third of the free expansion is prevented the stress set up will be two-thirds of that given by eqn. (1.12).

1.19. Stress concentrations – stress concentration factor

If a bar of uniform cross-section is subjected to an axial tensile or compressive load the stress is assumed to be uniform across the section. However, in the presence of any sudden change of section, hole, sharp corner, notch, keyway, material flaw, etc., the local stress will rise significantly. The ratio of this stress to the nominal stress at the section in the absence of any of these so-called *stress concentrations* is termed the *stress concentration factor*.

1.20. Toughness

Toughness is defined as the ability of a material to withstand cracks, i.e. to prevent the transfer or propagation of cracks across its section hence causing failure. Two distinct types of toughness mechanism exist and in each case it is appropriate to consider the crack as a very high local stress concentration.

The first type of mechanism relates particularly to ductile materials which are generally regarded as tough. This arises because the very high stresses at the end of the crack produce local yielding of the material and local plastic flow at the crack tip. This has the action of blunting the sharp tip of the crack and hence reduces its stress-concentration effect considerably (Fig. 1.15).

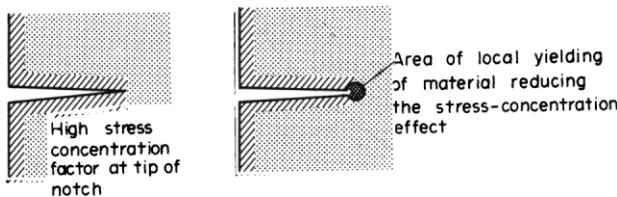


Fig. 1.15. Toughness mechanism - type

The second mechanism refers to fibrous, reinforced or resin-based materials which have weak interfaces. Typical examples are glass-fibre reinforced materials and wood. It can be shown that a region of local tensile stress always exists at the front of a propagating crack and provided that the adhesive strength of the fibre/resin interface is relatively low (one-fifth the cohesive strength of the complete material) this tensile stress opens up the interface and produces a crack sink, i.e. it blunts the crack by effectively increasing the radius at the crack tip, thereby reducing the stress-concentration effect (Fig. 1.16).

This principle is used on occasions to stop, or at least delay, crack propagation in engineering components when a temporary "repair" is carried out by drilling a hole at the end of a crack, again reducing its stress-concentration effect.

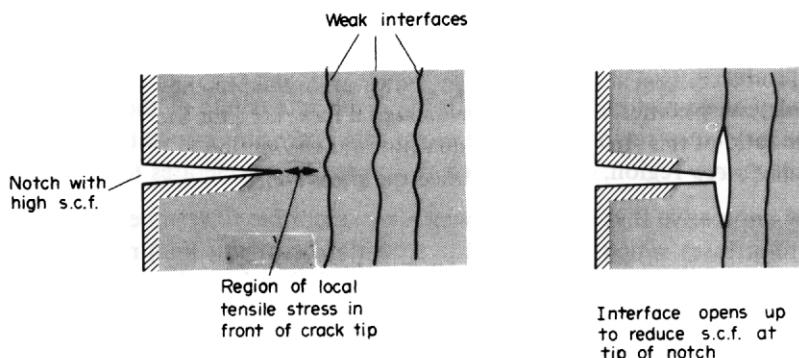


Fig. 1.16. Toughness mechanism - type 2.

1.21. Creep and fatigue

In the preceding paragraphs it has been suggested that failure of materials occurs when the ultimate strengths have been exceeded. Reference has also been made in §1.15 to cases where excessive deformation, as caused by plastic deformation beyond the yield point, can be considered as a criterion for effective failure of components. This chapter would not be complete, therefore, without reference to certain loading conditions under which materials can fail at stresses much less than the yield stress, namely *creep* and *fatigue*.

Creep is the gradual increase of plastic strain in a material with time at constant load. Particularly at elevated temperatures some materials are susceptible to this phenomenon and even under the constant load mentioned strains can increase continually until fracture. This form of fracture is particularly relevant to turbine blades, nuclear reactors, furnaces, rocket motors, etc.

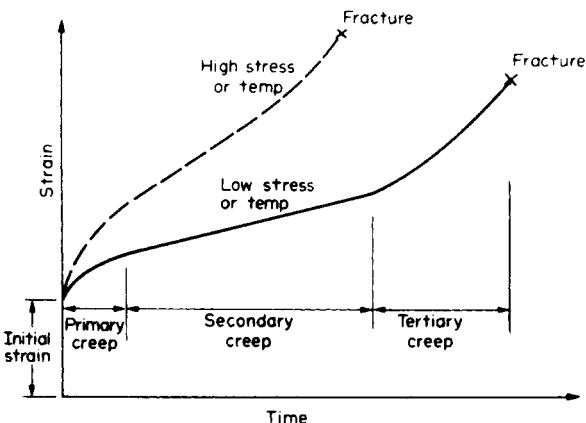


Fig. 1.17. Typical creep curve.

The general form of the strain versus time graph or *creep curve* is shown in Fig. 1.17 for two typical operating conditions. In each case the curve can be considered to exhibit four principal features.

- An *initial strain*, due to the initial application of load. In most cases this would be an elastic strain.
- A *primary creep* region, during which the creep rate (slope of the graph) diminishes.
- A *secondary creep* region, when the creep rate is sensibly constant.
- A *tertiary creep* region, during which the creep rate accelerates to final fracture.

It is clearly imperative that a material which is susceptible to creep effects should only be subjected to stresses which keep it in the secondary (straight line) region throughout its service life. This enables the amount of creep extension to be estimated and allowed for in design.

Fatigue is the failure of a material under fluctuating stresses each of which is believed to produce minute amounts of plastic strain. Fatigue is particularly important in components subjected to repeated and often rapid load fluctuations, e.g. aircraft components, turbine blades, vehicle suspensions, etc. Fatigue behaviour of materials is usually described by a *fatigue life* or *S-N curve* in which the number of stress cycles N to produce failure with a stress peak of S is plotted against S . A typical *S-N curve* for mild steel is shown in Fig. 1.18. The particularly relevant feature of this curve is the limiting stress S_n since it is assumed that stresses below this value will not produce fatigue failure however many cycles are applied, i.e. there is *infinite life*. In the simplest design cases, therefore, there is an aim to keep all stresses below this limiting level. However, this often implies an over-design in terms of physical size and material usage, particularly in cases where the stress may only occasionally exceed the limiting value noted above. This is, of course, particularly important in applications such as aerospace structures where component weight is a premium. Additionally the situation is complicated by the many materials which do not show a defined limit, and modern design procedures therefore rationalise the situation by aiming at a prescribed, long, but *finite life*, and accept that service stresses will occasionally exceed the value S_n . It is clear that the number of occasions on which the stress exceeds S_n , and by how

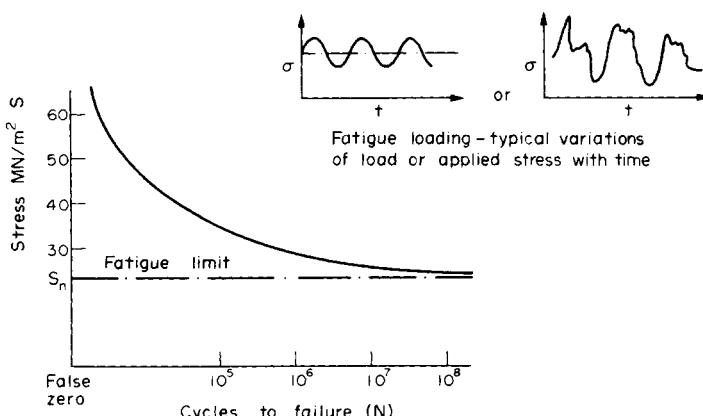


Fig. 1.18. Typical S-N fatigue curve for mild steel.

much, will have an important bearing on the prescribed life and considerable specimen, and often full-scale, testing is required before sufficient statistics are available to allow realistic life assessment.

The importance of the creep and fatigue phenomena cannot be over-emphasised and the comments above are only an introduction to the concepts and design philosophies involved. For detailed consideration of these topics and of the other materials testing topics introduced earlier the reader is referred to the texts listed at the end of this chapter.

Examples

Example 1.1

Determine the stress in each section of the bar shown in Fig. 1.19 when subjected to an axial tensile load of 20 kN. The central section is 30 mm square cross-section; the other portions are of circular section, their diameters being indicated. What will be the total extension of the bar? For the bar material $E = 210 \text{ GN/m}^2$.

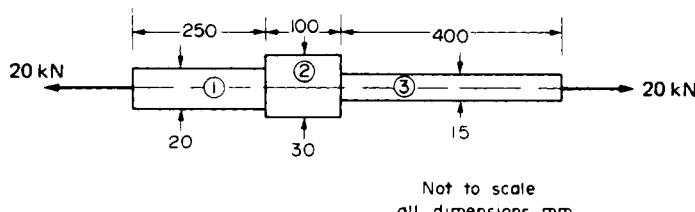


Fig. 1.19.

Solution

$$\text{Stress} = \frac{\text{force}}{\text{area}} = \frac{P}{A}$$

$$\text{Stress in section (1)} = \frac{20 \times 10^3}{\frac{\pi(20 \times 10^{-3})^2}{4}} = \frac{80 \times 10^3}{\pi \times 400 \times 10^{-6}} = 63.66 \text{ MN/m}^2$$

$$\text{Stress in section (2)} = \frac{20 \times 10^3}{30 \times 30 \times 10^{-6}} = 22.2 \text{ MN/m}^2$$

$$\text{Stress in section (3)} = \frac{20 \times 10^3}{\frac{\pi(15 \times 10^{-3})^2}{4}} = \frac{80 \times 10^3}{\pi \times 225 \times 10^{-6}} = 113.2 \text{ MN/m}^2$$

Now the extension of a bar can always be written in terms of the stress in the bar since

$$E = \frac{\text{stress}}{\text{strain}} = \frac{\sigma}{\delta/L}$$

i.e.

$$\delta = \frac{\sigma L}{E}$$

$$\therefore \text{extension of section (1)} = 63.66 \times 10^6 \times \frac{250 \times 10^{-3}}{210 \times 10^9} = 75.8 \times 10^{-6} \text{ m}$$

$$\text{extension of section (2)} = 22.2 \times 10^6 \times \frac{100 \times 10^{-3}}{210 \times 10^9} = 10.6 \times 10^{-6} \text{ m}$$

$$\text{extension of section (3)} = 113.2 \times 10^6 \times \frac{400 \times 10^{-3}}{210 \times 10^9} = 215.6 \times 10^{-6} \text{ m}$$

$$\begin{aligned}\therefore \text{total extension} &= (75.8 + 10.6 + 215.6)10^{-6} \\ &= 302 \times 10^{-6} \text{ m} \\ &= \mathbf{0.302 \text{ mm}}\end{aligned}$$

Example 1.2

(a) A 25 mm diameter bar is subjected to an axial tensile load of 100 kN. Under the action of this load a 200 mm gauge length is found to extend 0.19×10^{-3} mm. Determine the modulus of elasticity for the bar material.

(b) If, in order to reduce weight whilst keeping the external diameter constant, the bar is bored axially to produce a cylinder of uniform thickness, what is the maximum diameter of bore possible given that the maximum allowable stress is 240 MN/m²? The load can be assumed to remain constant at 100 kN.

(c) What will be the change in the outside diameter of the bar under the limiting stress quoted in (b)? ($E = 210 \text{ GN/m}^2$ and $v = 0.3$).

Solution

(a) From eqn. (1.2),

$$\begin{aligned}\text{Young's modulus } E &= \frac{PL}{A \delta L} \\ &= \frac{100 \times 10^3 \times 200 \times 10^{-3}}{\frac{\pi(25 \times 10^{-3})^2}{4} \times 0.19 \times 10^{-3}} \\ &= 214 \text{ GN/m}^2\end{aligned}$$

(b) Let the required bore diameter be d mm; the cross-sectional area of the bar will then be reduced to

$$A = \left[\frac{\pi \times 25^2}{4} - \frac{\pi d^2}{4} \right] 10^{-6} = \frac{\pi}{4} (25^2 - d^2) 10^{-6} \text{ m}^2$$

$$\therefore \text{stress in bar} = \frac{P}{A} = \frac{4 \times 100 \times 10^3}{\pi(25^2 - d^2) 10^{-6}}$$

But this stress is restricted to a maximum allowable value of 240 MN/m².

$$\therefore 240 \times 10^6 = \frac{4 \times 100 \times 10^3}{\pi(25^2 - d^2) 10^{-6}}$$

$$\therefore 25^2 - d^2 = \frac{4 \times 100 \times 10^3}{240 \times 10^6 \times \pi \times 10^{-6}} = 530.5$$

$$\therefore d^2 = 94.48 \quad \text{and} \quad d = 9.72 \text{ mm}$$

The maximum bore possible is thus **9.72 mm**.

(c) The change in the outside diameter of the bar will be obtained from the lateral strain,

$$\text{i.e. lateral strain} = \frac{\delta d}{d}$$

$$\text{But Poisson's ratio } v = \frac{\text{lateral strain}}{\text{longitudinal strain}}$$

$$\text{and longitudinal strain} = \frac{\sigma}{E} = \frac{240 \times 10^6}{210 \times 10^9}$$

$$\therefore \frac{\delta d}{d} = -v \frac{\sigma}{E} = -\frac{0.3 \times 240 \times 10^6}{210 \times 10^9}$$

$$\begin{aligned}\therefore \text{change in outside diameter} &= -\frac{0.3 \times 240 \times 10^6}{210 \times 10^9} \times 25 \times 10^{-3} \\ &= -8.57 \times 10^{-6} \text{ m (a reduction)}\end{aligned}$$

Example 1.3

The coupling shown in Fig. 1.20 is constructed from steel of rectangular cross-section and is designed to transmit a tensile force of 50 kN. If the bolt is of 15 mm diameter calculate:

- the shear stress in the bolt;
- the direct stress in the plate;
- the direct stress in the forked end of the coupling.

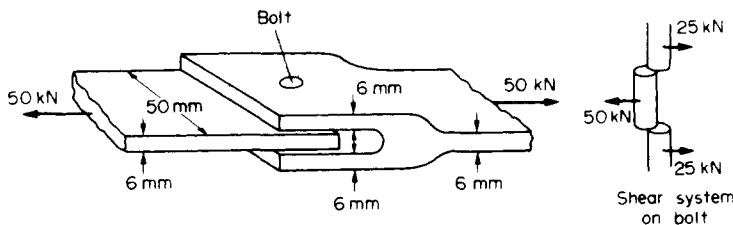


Fig. 1.20.

Solution

- (a) The bolt is subjected to double shear, tending to shear it as shown in Fig. 1.14b. There is thus twice the area of the bolt resisting the shear and from eqn. (1.8)

$$\begin{aligned} \text{shear stress in bolt} &= \frac{P}{2A} = \frac{50 \times 10^3 \times 4}{2 \times \pi(15 \times 10^{-3})^2} \\ &= \frac{100 \times 10^3}{\pi(15 \times 10^{-3})^2} = 141.5 \text{ MN/m}^2 \end{aligned}$$

- (b) The plate will be subjected to a direct tensile stress given by

$$\sigma = \frac{P}{A} = \frac{50 \times 10^3}{50 \times 6 \times 10^{-6}} = 166.7 \text{ MN/m}^2$$

- (c) The force in the coupling is shared by the forked end pieces, each being subjected to a direct stress

$$\sigma = \frac{P}{A} = \frac{25 \times 10^3}{50 \times 6 \times 10^{-6}} = 83.3 \text{ MN/m}^2$$

Example 1.4

Derive an expression for the total extension of the tapered bar of circular cross-section shown in Fig. 1.21 when it is subjected to an axial tensile load W .

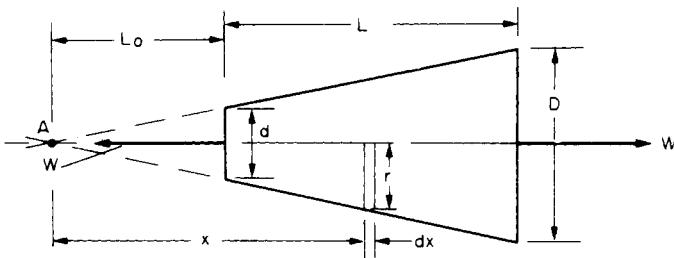


Fig. 1.21.

Solution

From the proportions of Fig. 1.21,

$$\frac{d/2}{L_0} = \frac{(D-d)/2}{L}$$

$$\therefore L_0 = \frac{d}{(D-d)} L$$

Consider an element of thickness dx and radius r , distance x from the point of taper A .

$$\text{Stress on the element} = \frac{W}{\pi r^2}$$

But

$$\frac{r}{x} = \frac{d}{2L_0}$$

$$\therefore r = d \left(\frac{D-d}{2dL} \right) x = \frac{x(D-d)}{2L}$$

$$\therefore \text{stress on the element} = \frac{4WL^2}{\pi(D-d)^2x^2}$$

$$\therefore \text{strain on the element} = \frac{\sigma}{E}$$

$$\text{and extension of the element} = \frac{\sigma dx}{E}$$

$$= \frac{4WL^2}{\pi(D-d)^2x^2E} dx$$

$$\therefore \text{total extension of bar} = \int_{L_0}^{L_0+L} \frac{4WL^2}{\pi(D-d)^2E} \frac{dx}{x^2}$$

$$= \frac{4WL^2}{\pi(D-d)^2E} \left[-\frac{1}{x} \right]_{L_0}^{L_0+L}$$

$$= \frac{4WL^2}{\pi(D-d)^2E} \left[-\frac{1}{(L_0+L)} - \left(-\frac{1}{L_0} \right) \right]$$

But

$$L_0 = \frac{d}{(D-d)} L$$

$$\therefore L_0 + L = \frac{d}{(D-d)} L + L = \frac{(d+D-d)}{D-d} L = \frac{DL}{(D-d)}$$

\therefore total extension

$$\begin{aligned} &= \frac{4WL^2}{\pi(D-d)^2 E} \left[-\frac{(D-d)}{DL} + \frac{(D-d)}{dL} \right] = \frac{4WL}{\pi(D-d)E} \left[\frac{(-d+D)}{Dd} \right] \\ &= \frac{4WL}{\pi D d E} \end{aligned}$$

Example 1.5

The following figures were obtained in a standard tensile test on a specimen of low carbon steel:

diameter of specimen, 11.28 mm;
gauge length, 56 mm;
minimum diameter after fracture, 6.45 mm.

Using the above information and the table of results below, produce:

- (1) a load/extension graph over the complete test range;
- (2) a load/extension graph to an enlarged scale over the elastic range of the specimen.

Load (kN)	2.47	4.97	7.4	9.86	12.33	14.8	17.27	19.74	22.2	24.7
Extension ($m \times 10^{-6}$)	5.6	11.9	18.2	24.5	31.5	38.5	45.5	52.5	59.5	66.5
Load (kN)	27.13	29.6	32.1	33.3	31.2	32	31.5	32	32.2	34.5
Extension ($m \times 10^{-6}$)	73.5	81.2	89.6	112	224	448	672	840	1120	1680
Load (kN)	35.8	37	38.7	39.5	40	39.6	35.7	28		
Extension ($m \times 10^{-6}$)	1960	2520	3640	5600	7840	11200	13440	14560		

Using the two graphs and other information supplied, determine the values of:

- (a) Young's modulus of elasticity;
- (b) the ultimate tensile stress;
- (c) the stress at the upper and lower yield points;
- (d) the percentage reduction of area;
- (e) the percentage elongation;
- (f) the nominal and actual stress at fracture.

Solution

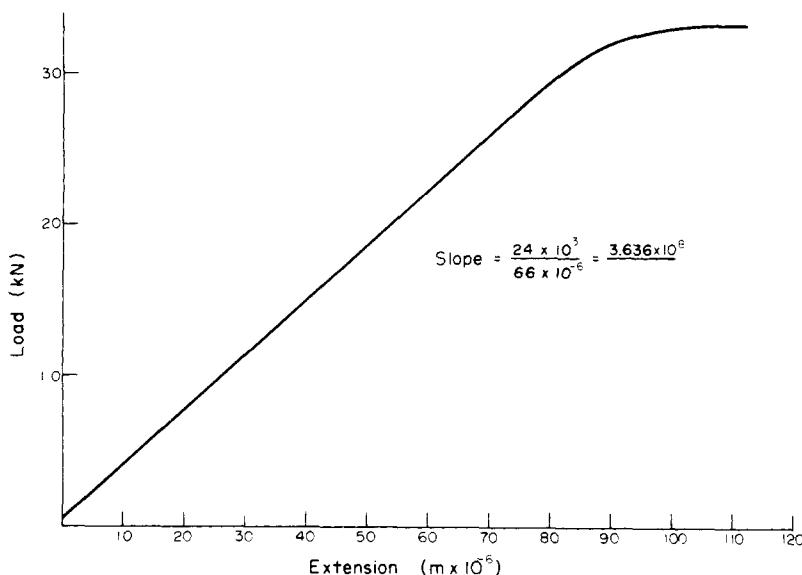


Fig. 1.22. Load-extension graph for elastic range.

$$(a) \quad \text{Young's modulus } E = \frac{\sigma}{\epsilon} = \frac{\text{load}}{\text{area}} \times \frac{\text{gauge length}}{\text{extension}}$$

$$= \frac{\text{load}}{\text{extension}} \times \frac{\text{gauge length}}{\text{area}}$$

$$\text{i.e. } E = \text{slope of graph} \times \frac{L}{A} = 3.636 \times 10^8 \times \frac{56 \times 10^{-3}}{100 \times 10^{-6}}$$

$$= 203.6 \times 10^9 \text{ N/m}^2$$

$$\therefore E = 203.6 \text{ GN/m}^2$$

$$(b) \quad \text{Ultimate tensile stress} = \frac{\text{maximum load}}{\text{cross-section area}} = \frac{40.2 \times 10^3}{100 \times 10^{-6}} = 402 \text{ MN/m}^2$$

(see Fig. 1.23).

$$(c) \quad \text{Upper yield stress} = \frac{33.3 \times 10^3}{100 \times 10^{-6}} = 333 \text{ MN/m}^2$$

$$\text{Lower yield stress} = \frac{31.2 \times 10^3}{100 \times 10^{-6}} = 312 \text{ MN/m}^2$$

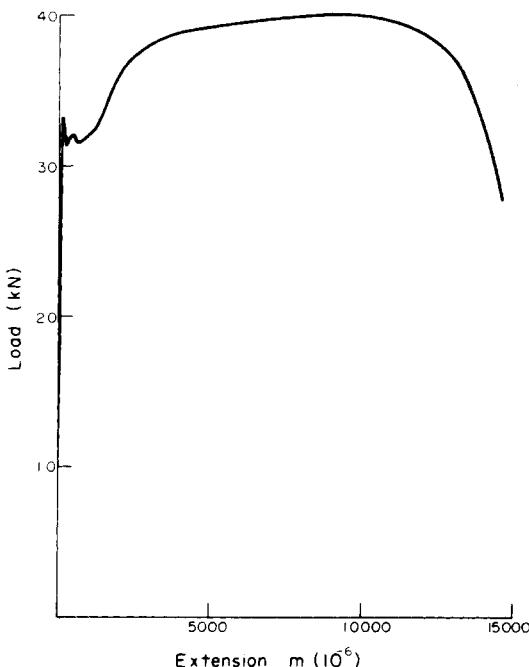


Fig. 1.23. Load-extension graph for complete load range.

$$(d) \quad \text{Percentage reduction of area} = \frac{\left(\frac{\pi}{4} D^2 - \frac{\pi}{4} d^2 \right)}{\frac{\pi}{4} D^2} \times 100 \\ = \frac{(D^2 - d^2)}{D^2} \times 100 \\ = \frac{(11.28^2 - 6.45^2)}{11.28^2} = 67.3\%$$

$$(e) \quad \text{Percentage elongation} = \frac{(70.56 - 56)}{56} \times 100 \\ = 26\%$$

$$(f) \quad \text{Nominal stress at fracture} = \frac{28 \times 10^3}{100 \times 10^{-6}} = 280 \text{ MN/m}^2$$

$$\text{Actual stress at fracture} = \frac{28 \times 10^3}{\frac{\pi}{4} (6.45)^2 \times 10^{-6}} = 856.9 \text{ MN/m}^2$$

Problems

1.1 (A). A 25 mm square-cross-section bar of length 300 mm carries an axial compressive load of 50 kN. Determine the stress set up in the bar and its change of length when the load is applied. For the bar material $E = 200 \text{ GN/m}^2$. [80 MN/m²; 0.12 mm.]

1.2 (A). A steel tube, 25 mm outside diameter and 12 mm inside diameter, carries an axial tensile load of 40 kN. What will be the stress in the bar? What further increase in load is possible if the stress in the bar is limited to 225 MN/m²? [106 MN/m²; 45 kN.]

1.3 (A). Define the terms *shear stress* and *shear strain*, illustrating your answer by means of a simple sketch.

Two circular bars, one of brass and the other of steel, are to be loaded by a shear load of 30 kN. Determine the necessary diameter of the bars (a) in single shear, (b) in double shear, if the shear stress in the two materials must not exceed 50 MN/m² and 100 MN/m² respectively. [27.6, 19.5, 19.5, 13.8 mm.]

1.4 (A). Two fork-end pieces are to be joined together by a single steel pin of 25 mm diameter and they are required to transmit 50 kN. Determine the minimum cross-sectional area of material required in one branch of either fork if the stress in the fork material is not to exceed 180 MN/m². What will be the maximum shear stress in the pin? [$1.39 \times 10^{-4} \text{ m}^2$; 50.9 MN/m².]

1.5 (A). A simple turnbuckle arrangement is constructed from a 40 mm outside diameter tube threaded internally at each end to take two rods of 25 mm outside diameter with threaded ends. What will be the nominal stresses set up in the tube and the rods, ignoring thread depth, when the turnbuckle carries an axial load of 30 kN? Assuming a sufficient strength of thread, what maximum load can be transmitted by the turnbuckle if the maximum stress is limited to 180 MN/m²? [39.2, 61.1 MN/m², 88.4 kN.]

1.6 (A). An I-section girder is constructed from two 80 mm \times 12 mm flanges joined by an 80 mm \times 12 mm web. Four such girders are mounted vertically one at each corner of a horizontal platform which the girders support. The platform is 4 m above ground level and weighs 10 kN. Assuming that each girder supports an equal share of the load, determine the maximum compressive stress set up in the material of each girder when the platform supports an additional load of 15 kN. The weight of the girders may *not* be neglected. The density of the cast iron from which the girders are constructed is 7470 kg/m³. [2.46 MN/m².]

1.7 (A). A bar *ABCD* consists of three sections: *AB* is 25 mm square and 50 mm long, *BC* is of 20 mm diameter and 40 mm long and *CD* is of 12 mm diameter and 50 mm long. Determine the stress set up in each section of the bar when it is subjected to an axial tensile load of 20 kN. What will be the total extension of the bar under this load? For the bar material, $E = 210 \text{ GN/m}^2$. [32, 63.7, 176.8 MN/m², 0.062 mm.]

1.8 (A). A steel bar *ABCD* consists of three sections: *AB* is of 20 mm diameter and 200 mm long, *BC* is 25 mm square and 400 mm long, and *CD* is of 12 mm diameter and 200 mm long. The bar is subjected to an axial compressive load which induces a stress of 30 MN/m² on the largest cross-section. Determine the total decrease in the length of the bar when the load is applied. For steel $E = 210 \text{ GN/m}^2$. [0.272 mm.]

1.9 (A). During a tensile test on a specimen the following results were obtained:

Load (kN)	15	30	40	50	55	60	65
Extension (mm)	0.05	0.094	0.127	0.157	1.778	2.79	3.81
Load (kN)	70	75	80	82	80	70	
Extension (mm)	5.08	7.62	12.7	16.0	19.05	22.9	

Diameter of gauge length = 19 mm Gauge length = 100 mm

Diameter at fracture = 16.49 mm Gauge length at fracture = 121 mm

Plot the complete load extension graph and the straight line portion to an enlarged scale. Hence determine:

- (a) the modulus of elasticity;
- (b) the percentage elongation;
- (c) the percentage reduction in area;
- (d) the nominal stress at fracture;
- (e) the actual stress at fracture;
- (f) the tensile strength.

[116 GN/m²; 21%; 24.7%; 247 MN/m²; 328 MN/m²; 289 MN/m².]

1.10 Figure 1.24 shows a special spanner used to tighten screwed components. A torque is applied at the tommy-bar and is transmitted to the pins which engage into holes located into the end of a screwed component.

- (a) Using the data given in Fig. 1.24 calculate:

- (i) the diameter *D* of the shank if the shear stress is not to exceed 50 N/mm²,
- (ii) the stress due to bending in the tommy-bar,
- (iii) the shear stress in the pins.

- (b) Why is the tommy-bar a preferred method of applying the torque?

[C.G.] [9.14 mm; 254.6 MN/m²; 39.8 MN/m².]

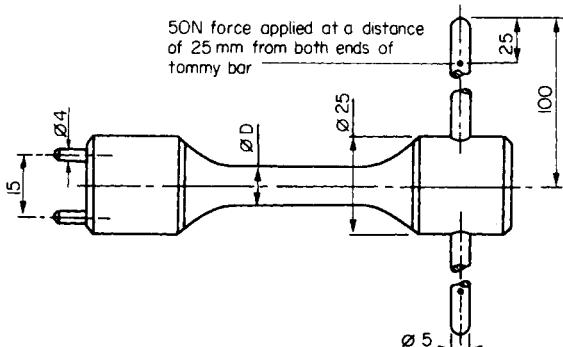


Fig. 1.24.

1.11 (a) A test piece is cut from a brass bar and subjected to a tensile test. With a load of 6.4 kN the test piece, of diameter 11.28 mm , extends by 0.04 mm over a gauge length of 50 mm . Determine:

(i) the stress, (ii) the strain, (iii) the modulus of elasticity.

(b) A spacer is turned from the same bar. The spacer has a diameter of 28 mm and a length of 250 mm , both measurements being made at 20°C . The temperature of the spacer is then increased to 100°C , the natural expansion being entirely prevented. Taking the coefficient of linear expansion to be $18 \times 10^{-6}/^\circ\text{C}$ determine:

(i) the stress in the spacer, (ii) the compressive load on the spacer.

[C.G.] [64 MN/m^2 , 0.0008 , 80 GN/m^2 , 115.2 MN/m^2 , 71 kN .]

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CHAPTER 2

COMPOUND BARS

Summary

When a compound bar is constructed from members of different materials, lengths and areas and is subjected to an external tensile or compressive load W the load carried by any single member is given by

$$F_1 = \frac{\frac{E_1 A_1}{L_1}}{\sum \frac{EA}{L}} W$$

where suffix 1 refers to the single member and $\sum \frac{EA}{L}$ is the sum of all such quantities for all the members.

Where the bars have a common length the compound bar can be reduced to a single equivalent bar with an equivalent Young's modulus, termed a *combined E*.

$$\text{Combined } E = \frac{\sum EA}{\sum A}$$

The free expansion of a bar under a temperature change from T_1 to T_2 is

$$\alpha(T_2 - T_1)L$$

where α is the coefficient of linear expansion and L is the length of the bar.

If this expansion is prevented a stress will be induced in the bar given by

$$\alpha(T_2 - T_1)E$$

To determine the stresses in a compound bar composed of two members of different free lengths two principles are used:

- (1) The tensile force applied to the short member by the long member is equal in magnitude to the compressive force applied to the long member by the short member.
- (2) The extension of the short member plus the contraction of the long member equals the difference in free lengths.

This difference in free lengths may result from the tightening of a nut or from a temperature change in two members of different material (i.e. different coefficients of expansion) but of equal length initially.

If such a bar is then subjected to an additional external load the resultant stresses may be obtained by using the *principle of superposition*. With this method the stresses in the members

arising from the separate effects are obtained and the results added, taking account of sign, to give the resultant stresses.

N.B.: Discussion in this chapter is concerned with compound bars which are symmetrically proportioned such that no bending results.

2.1. Compound bars subjected to external load

In certain applications it is necessary to use a combination of elements or bars made from different materials, each material performing a different function. In overhead electric cables, for example, it is often convenient to carry the current in a set of copper wires surrounding steel wires, the latter being designed to support the weight of the cable over large spans. Such combinations of materials are generally termed *compound bars*. Discussion in this chapter is concerned with compound bars which are symmetrically proportioned such that no bending results.

When an external load is applied to such a compound bar it is shared between the individual component materials in proportions depending on their respective lengths, areas and Young's moduli.

Consider, therefore, a compound bar consisting of n members, each having a different length and cross-sectional area and each being of a different material; this is shown diagrammatically in Fig. 2.1. Let all members have a common extension x , i.e. the load is positioned to produce the same extension in each member.

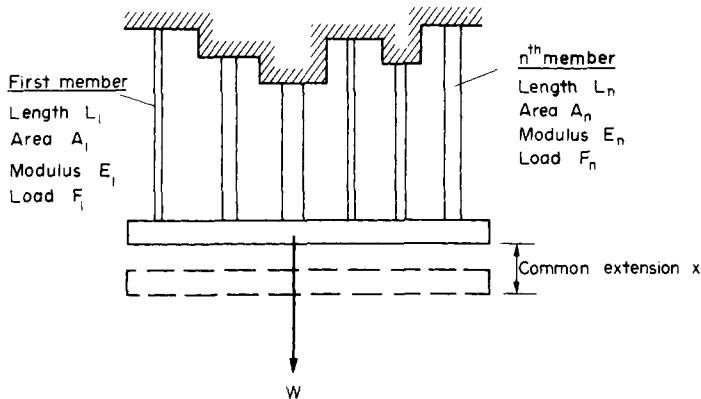


Fig. 2.1. Diagrammatic representation of a compound bar formed of different materials with different lengths, cross-sectional areas and Young's moduli.

For the n^{th} member,

$$\frac{\text{stress}}{\text{strain}} = E_n = \frac{F_n L_n}{A_n x_n}$$

$$\therefore F_n = \frac{E_n A_n x}{L_n} \quad (2.1)$$

where F_n is the force in the n^{th} member and A_n and L_n are its cross-sectional area and length.

The total load carried will be the sum of all such loads for all the members

$$\text{i.e. } W = \sum \frac{E_n A_n x}{L_n} = x \sum \frac{E_n A_n}{L_n} \quad (2.2)$$

Now from eqn. (2.1) the force in member 1 is given by

$$F_1 = \frac{E_1 A_1 x}{L_1}$$

But, from eqn. (2.2),

$$x = \frac{W}{\sum \frac{E_n A_n}{L_n}}$$

$$\therefore F_1 = \frac{\frac{E_1 A_1}{L_1}}{\frac{\Sigma E A}{L}} W \quad (2.3)$$

i.e. each member carries a portion of the total load W proportional to its EA/L value.

If the wires are all of equal length the above equation reduces to

$$F_1 = \frac{E_1 A_1}{\Sigma E A} W \quad (2.4)$$

The stress in member 1 is then given by

$$\sigma_1 = \frac{F_1}{A_1} \quad (2.5)$$

2.2. Compound bars—“equivalent” or “combined” modulus

In order to determine the common extension of a compound bar it is convenient to consider it as a single bar of an imaginary material with an *equivalent* or *combined* modulus E_c . Here it is necessary to assume that both the extension and the original lengths of the individual members of the compound bar are the same; the strains in all members will then be equal.

Now total load on compound bar = $F_1 + F_2 + F_3 + \dots + F_n$ where F_1, F_2 , etc., are the loads in members 1, 2, etc.

But $\text{force} = \text{stress} \times \text{area}$

$$\therefore \sigma(A_1 + A_2 + \dots + A_n) = \sigma_1 A_1 + \sigma_2 A_2 + \dots + \sigma_n A_n$$

where σ is the stress in the equivalent single bar.

Dividing through by the common strain ε ,

$$\frac{\sigma}{\varepsilon} (A_1 + A_2 + \dots + A_n) = \frac{\sigma_1}{\varepsilon} A_1 + \frac{\sigma_2}{\varepsilon} A_2 + \dots + \frac{\sigma_n}{\varepsilon} A_n$$

$$\text{i.e. } E_c (A_1 + A_2 + \dots + A_n) = E_1 A_1 + E_2 A_2 + \dots + E_n A_n$$

where E_c is the *equivalent* or *combined* E of the single bar.

$$\therefore \text{combined } E = \frac{E_1 A_1 + E_2 A_2 + \dots + E_n A_n}{A_1 + A_2 + \dots + A_n}$$

i.e.

$$E_c = \frac{\Sigma EA}{\Sigma A} \quad (2.6)$$

With an external load W applied,

$$\text{stress in the equivalent bar} = \frac{W}{\Sigma A}$$

and

$$\text{strain in the equivalent bar} = \frac{W}{E_c \Sigma A} = \frac{x}{L}$$

\therefore since

$$\frac{\text{stress}}{\text{strain}} = E$$

$$\begin{aligned} \text{common extension } x &= \frac{W L}{E_c \Sigma A} \\ &= \text{extension of single bar} \end{aligned} \quad (2.7)$$

2.3. Compound bars subjected to temperature change

When a material is subjected to a change in temperature its length will change by an amount

$$\alpha Lt$$

where α is the coefficient of linear expansion for the material, L is the original length and t the temperature change. (An increase in temperature produces an increase in length and a decrease in temperature a decrease in length except in very special cases of materials with zero or negative coefficients of expansion which need not be considered here.)

If, however, the free expansion of the material is prevented by some external force, then a stress is set up in the material. This stress is equal in magnitude to that which would be produced in the bar by initially allowing the free change of length and then applying sufficient force to return the bar to its original length.

Now

$$\text{change in length} = \alpha Lt$$

$$\therefore \text{strain} = \frac{\alpha Lt}{L} = \alpha t$$

Therefore, the stress created in the material by the application of sufficient force to remove this strain

$$= \text{strain} \times E$$

$$= Eat$$

Consider now a compound bar constructed from two different materials rigidly joined together as shown in Fig. 2.2 and Fig. 2.3(a). For simplicity of description consider that the materials in this case are steel and brass.

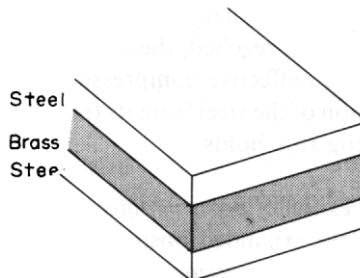


Fig. 2.2.

In general, the coefficients of expansion of the two materials forming the compound bar will be different so that as the temperature rises each material will attempt to expand by different amounts. Figure 2.3b shows the positions to which the individual materials will extend if they are completely free to expand (i.e. not joined rigidly together as a compound bar). The extension of any length L is given by

$$\alpha Lt$$

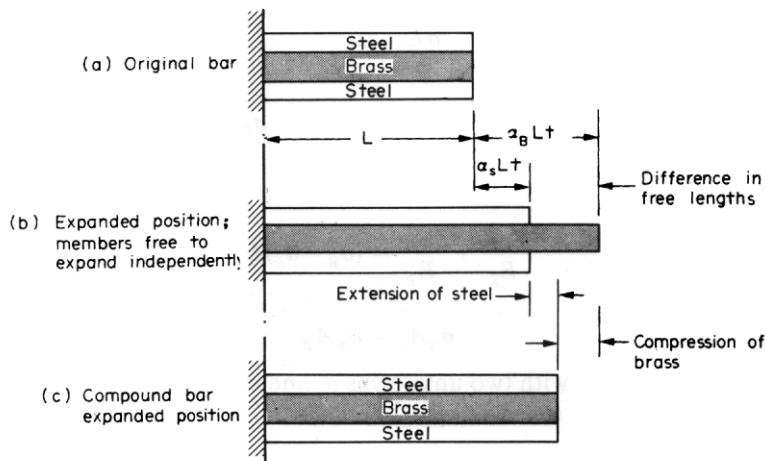


Fig. 2.3. Thermal expansion of compound bar.

Thus the difference of "free" expansion lengths or so-called *free lengths*

$$= \alpha_b L t - \alpha_s L t = (\alpha_b - \alpha_s) L t$$

since in this case the coefficient of expansion of the brass α_b is greater than that for the steel α_s . The initial lengths L of the two materials are assumed equal.

If the two materials are now rigidly joined as a compound bar and subjected to the same temperature rise, each material will attempt to expand to its free length position but each will be affected by the movement of the other. The higher coefficient of expansion material (brass) will therefore seek to pull the steel up to its free length position and conversely the lower

coefficient of expansion material (steel) will try to hold the brass back to the steel "free length" position. In practice a compromise is reached, the compound bar extending to the position shown in Fig. 2.3c, resulting in an effective compression of the brass from its free length position and an effective extension of the steel from its free length position. From the diagram it will be seen that the following rule holds.

Rule 1.

Extension of steel + compression of brass = difference in "free" lengths.

Referring to the bars in their free expanded positions the rule may be written as

Extension of "short" member + compression of "long" member = difference in free lengths.

Applying Newton's law of equal action and reaction the following second rule also applies.

Rule 2.

The tensile force applied to the short member by the long member is equal in magnitude to the compressive force applied to the long member by the short member.

Thus, in this case,

$$\text{tensile force in steel} = \text{compressive force in brass}$$

Now, from the definition of Young's modulus

$$E = \frac{\text{stress}}{\text{strain}} = \frac{\sigma}{\delta/L}$$

where δ is the change in length.

$$\therefore \delta = \frac{\sigma L}{E}$$

Also

$$\text{force} = \text{stress} \times \text{area} = \sigma A$$

where A is the cross-sectional area.

Therefore Rule 1 becomes

$$\frac{\sigma_s L}{E_s} + \frac{\sigma_b L}{E_b} = (a_b - a_s)Lt \quad (2.8)$$

and Rule 2 becomes

$$\sigma_s A_s = \sigma_b A_b \quad (2.9)$$

We thus have two equations with two unknowns σ_s and σ_b and it is possible to evaluate the magnitudes of these stresses (see Example 2.2).

2.4. Compound bar (tube and rod)

Consider now the case of a hollow tube with washers or endplates at each end and a central threaded rod as shown in Fig. 2.4. At first sight there would seem to be no connection with the work of the previous section, yet, in fact, the method of solution to determine the stresses set up in the tube and rod when one nut is tightened is identical to that described in §2.3.

The compound bar which is formed after assembly of the tube and rod, i.e. with the nuts tightened, is shown in Fig. 2.4c, the rod being in a state of tension and the tube in compression. Once again Rule 2 applies, i.e.

$$\text{compressive force in tube} = \text{tensile force in rod}$$

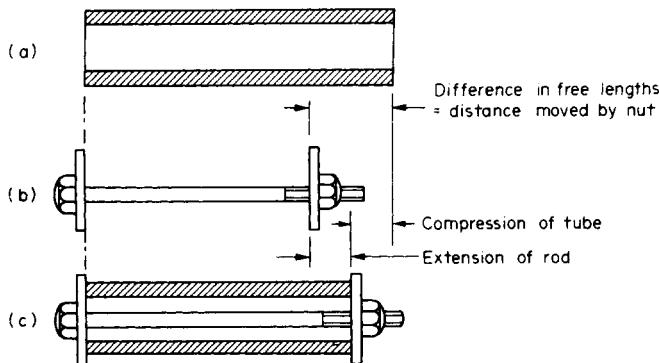


Fig. 2.4. Equivalent "mechanical" system to that of Fig. 2.3.

Figure 2.4a and b show, *diagrammatically*, the effective positions of the tube and rod before the nut is tightened and the two components are combined. As the nut is turned there is a simultaneous compression of the tube and tension of the rod leading to the final state shown in Fig. 2.4c. As before, however, the diagram shows that Rule 1 applies:

compression of tube + extension of rod = difference in free lengths = axial advance of nut
i.e. the axial movement of the nut (= number of turns $n \times$ threads per metre) is taken up by combined compression of the tube and extension of the rod.

Thus, with suffix t for tube and R for rod,

$$\frac{\sigma_t L}{E_t} + \frac{\sigma_R L}{E_R} = n \times \text{threads/metre} \quad (2.10)$$

also

$$\sigma_R A_R = \sigma_t A_t \quad (2.11)$$

If the tube and rod are now subjected to a change of temperature they may be treated as a normal compound bar of §2.3 and Rules 1 and 2 again apply (Fig. 2.5),

i.e.

$$\frac{\sigma'_t L}{E_t} + \frac{\sigma'_R L}{E_R} = (\alpha_t - \alpha_R)Lt \quad (2.12)$$

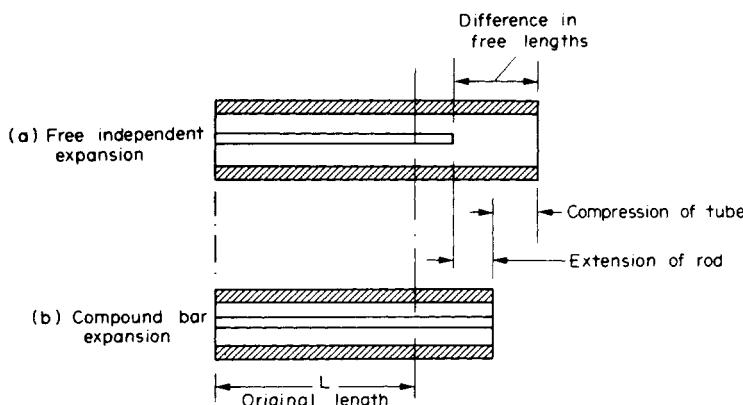


Fig. 2.5.

where σ'_t and σ'_R are the stresses in the tube and rod due to temperature change only and α_t is assumed greater than α_R . If the latter is not the case the two terms inside the final bracket should be interchanged.

Also

$$\sigma'_R A_R = \sigma'_t A_t$$

2.5. Compound bars subjected to external load and temperature effects

In this case the *principle of superposition* must be applied, i.e. provided that stresses remain within the elastic limit the effects of external load and temperature change may be assessed separately as described in the previous sections and the results added, taking account of sign, to determine the resultant total effect;

i.e. *total strain = sum of strain due to external loads and temperature strain*

2.6. Compound thick cylinders subjected to temperature changes

The procedure described in § 2.3 has been applied to compound cylinders constructed from tubes of different materials on page 230.

Examples

Example 2.1

(a) A compound bar consists of four brass wires of 2.5 mm diameter and one steel wire of 1.5 mm diameter. Determine the stresses in each of the wires when the bar supports a load of 500 N. Assume all of the wires are of equal lengths.

(b) Calculate the “equivalent” or “combined” modulus for the compound bar and determine its total extension if it is initially 0.75 m long. Hence check the values of the stresses obtained in part (a).

For brass $E = 100 \text{ GN/m}^2$ and for steel $E = 200 \text{ GN/m}^2$.

Solution

(a) From eqn. (2.3) the force in the steel wire is given by

$$\begin{aligned} F_s &= \frac{E_s A_s}{\Sigma E A} W \\ &= \left[\frac{200 \times 10^9 \times \frac{\pi}{4} \times 1.5^2 \times 10^{-6}}{200 \times 10^9 \times \frac{\pi}{4} \times 1.5^2 \times 10^{-6} + 4(100 \times 10^9 \times \frac{\pi}{4} \times 2.5^2 \times 10^{-6})} \right] 500 \\ &= \left[\frac{2 \times 1.5^2}{(2 \times 1.5^2) + (4 \times 2.5^2)} \right] 500 = 76.27 \text{ N} \end{aligned}$$

∴ total force in brass wires = $500 - 76.27 = 423.73 \text{ N}$

$$\therefore \text{stress in steel} = \frac{\text{load}}{\text{area}} = \frac{76.27}{\frac{\pi}{4} \times 1.5^2 \times 10^{-6}} = 43.2 \text{ MN/m}^2$$

and $\text{stress in brass} = \frac{\text{load}}{\text{area}} = \frac{423.73}{4 \times \frac{\pi}{4} \times 2.5^2 \times 10^{-6}} = 21.6 \text{ MN/m}^2$

(b) From eqn. (2.6)

$$\begin{aligned} \text{combined } E &= \frac{\Sigma EA}{\Sigma A} = \frac{200 \times 10^9 \times \frac{\pi}{4} \times 1.5^2 \times 10^{-6} + 4(100 \times 10^9 \times \frac{\pi}{4} \times 2.5^2 \times 10^{-6})}{\frac{\pi}{4}(1.5^2 + 4 \times 2.5^2)^{10-6}} \\ &= \frac{(200 \times 1.5^2 + 400 \times 2.5^2)}{(1.5^2 + 4 \times 2.5^2)} 10^9 = 108.26 \text{ GN/m}^2 \end{aligned}$$

Now $E = \frac{\text{stress}}{\text{strain}}$

and the stress in the equivalent bar

$$= \frac{500}{\Sigma A} = \frac{500}{\frac{\pi}{4}(1.5^2 + 4 \times 2.5^2)10^{-6}} = 23.36 \text{ MN/m}^2$$

$$\therefore \text{strain in the equivalent bar} = \frac{\text{stress}}{E} = \frac{23.36 \times 10^6}{108.26 \times 10^9} = 0.216 \times 10^{-3}$$

$$\begin{aligned} \therefore \text{common extension} &= \text{strain} \times \text{original length} \\ &= 0.216 \times 10^{-3} \times 0.75 = 0.162 \times 10^{-3} \\ &= 0.162 \text{ mm} \end{aligned}$$

This is also the extension of any single bar, giving a strain in any bar

$$= \frac{0.162 \times 10^{-3}}{0.75} = 0.216 \times 10^{-3} \text{ as above}$$

$$\therefore \text{stress in steel} = \text{strain} \times E_s = 0.216 \times 10^{-3} \times 200 \times 10^9 \\ = 43.2 \text{ MN/m}^2$$

and $\text{stress in brass} = \text{strain} \times E_B = 0.216 \times 10^{-3} \times 100 \times 10^9 \\ = 21.6 \text{ MN/m}^2$

These are the same values as obtained in part (a).

Example 2.2

(a) A compound bar is constructed from three bars 50 mm wide by 12 mm thick fastened together to form a bar 50 mm wide by 36 mm thick. The middle bar is of aluminium alloy for which $E = 70 \text{ GN/m}^2$ and the outside bars are of brass with $E = 100 \text{ GN/m}^2$. If the bars are initially fastened at 18°C and the temperature of the whole assembly is then raised to 50°C , determine the stresses set up in the brass and the aluminium.

$$\alpha_B = 18 \times 10^{-6} \text{ per } ^\circ\text{C} \quad \text{and} \quad \alpha_A = 22 \times 10^{-6} \text{ per } ^\circ\text{C}$$

- (b) What will be the changes in these stresses if an external compressive load of 15 kN is applied to the compound bar at the higher temperature?

Solution

With any problem of this type it is convenient to let the stress in one of the component members or materials, e.g. the brass, be x .

Then, since

$$\text{force in brass} = \text{force in aluminium}$$

and

$$\text{force} = \text{stress} \times \text{area}$$

$$x \times 2 \times 50 \times 12 \times 10^{-6} = \sigma_A \times 50 \times 12 \times 10^{-6}$$

i.e. stress in aluminium $\sigma_A = 2x$

Now, from eqn. (2.8),

$$\text{extension of brass} + \text{compression of aluminium} = \text{difference in free lengths}$$

$$= (\alpha_A - \alpha_B)(T_2 - T_1)L$$

$$\frac{xL}{100 \times 10^9} + \frac{2xL}{70 \times 10^9} = (22 - 18)10^{-6}(50 - 18)L$$

$$\frac{(7x + 20x)}{700 \times 10^9} = 4 \times 10^{-6} \times 32$$

$$27x = 4 \times 10^{-6} \times 32 \times 700 \times 10^9$$

$$x = 3.32 \text{ MN/m}^2$$

The stress in the brass is thus **3.32 MN/m² (tensile)** and the stress in the aluminium is $2 \times 3.32 = 6.64 \text{ MN/m}^2$ (**compressive**).

(b) With an external load of 15 kN applied each member will take a proportion of the total load given by eqn. (2.3).

$$\begin{aligned} \text{Force in aluminium} &= \frac{E_A A_A}{\Sigma EA} W \\ &= \left[\frac{70 \times 10^9 \times 50 \times 12 \times 10^{-6}}{(70 \times 50 \times 12 + 2 \times 100 \times 50 \times 12)10^9 \times 10^{-6}} \right] 15 \times 10^3 \\ &= \left[\frac{70}{(70 + 200)} \right] 15 \times 10^3 \\ &= 3.89 \text{ kN} \end{aligned}$$

$$\therefore \text{force in brass} = 15 - 3.89 = 11.11 \text{ kN}$$

$$\begin{aligned} \therefore \text{stress in brass} &= \frac{\text{load}}{\text{area}} = \frac{11.11 \times 10^3}{2 \times 50 \times 12 \times 10^{-6}} \\ &= 9.26 \text{ MN/m}^2 \text{ (compressive)} \end{aligned}$$

$$\text{Stress in aluminium} = \frac{\text{load}}{\text{area}} = \frac{3.89 \times 10^3}{50 \times 12 \times 10^{-6}}$$

$$= 6.5 \text{ MN/m}^2 \text{ (compressive)}$$

These stresses represent the *changes* in the stresses owing to the applied load. The total or resultant stresses owing to combined applied loading plus temperature effects are, therefore,

$$\begin{aligned}\text{stress in aluminium} &= -6.64 - 6.5 = -13.14 \text{ MN/m}^2 \\ &= 13.14 \text{ MN/m}^2 \text{ (compressive)} \\ \text{stress in brass} &= +3.32 - 9.26 = -5.94 \text{ MN/m}^2 \\ &= 5.94 \text{ MN/m}^2 \text{ (compressive)}\end{aligned}$$

Example 2.3

A 25 mm diameter steel rod passes concentrically through a bronze tube 400 mm long, 50 mm external diameter and 40 mm internal diameter. The ends of the steel rod are threaded and provided with nuts and washers which are adjusted initially so that there is no end play at 20°C.

- (a) Assuming that there is no change in the thickness of the washers, find the stress produced in the steel and bronze when one of the nuts is tightened by giving it one-tenth of a turn, the pitch of the thread being 2.5 mm.
- (b) If the temperature of the steel and bronze is then raised to 50°C find the changes that will occur in the stresses in both materials.

The coefficient of linear expansion per °C is 11×10^{-6} for steel and 18×10^{-6} for bronze. E for steel = 200 GN/m². E for bronze = 100 GN/m².

Solution

- (a) Let x be the stress in the tube resulting from the tightening of the nut and σ_R the stress in the rod.

Then, from eqn. (2.11),

$$\begin{aligned}\text{force (stress} \times \text{area) in tube} &= \text{force (stress} \times \text{area) in rod} \\ x \times \frac{\pi}{4} (50^2 - 40^2) 10^{-6} &= \sigma_R \times \frac{\pi}{4} \times 25^2 \times 10^{-6} \\ \sigma_R &= \frac{(50^2 - 40^2)}{25^2} x = 1.44x\end{aligned}$$

And since compression of tube + extension of rod = axial advance of nut, from eqn. (2.10),

$$\begin{aligned}\frac{x \times 400 \times 10^{-3}}{100 \times 10^9} + \frac{1.44x \times 400 \times 10^{-3}}{200 \times 10^9} &= \frac{1}{10} \times 2.5 \times 10^{-3} \\ 400 \frac{(2x + 1.44x)}{200 \times 10^9} 10^{-3} &= 2.5 \times 10^{-4} \\ \therefore 6.88x &= 2.5 \times 10^8 \\ x &= 36.3 \text{ MN/m}^2\end{aligned}$$

The stress in the tube is thus 36.3 MN/m^2 (compressive) and the stress in the rod is $1.44 \times 36.3 = 52.3 \text{ MN/m}^2$ (tensile).

(b) Let p be the stress in the tube resulting from temperature change. The relationship between the stresses in the tube and the rod will remain as in part (a) so that the stress in the rod is then $1.44p$. In this case, if free expansion were allowed in the independent members, the bronze tube would expand more than the steel rod and from eqn. (2.8)

$$\text{compression of tube} + \text{extension of rod} = \text{difference in free length}$$

$$\therefore \frac{pL}{100 \times 10^9} + \frac{1.44pL}{200 \times 10^9} = (\alpha_B - \alpha_S)(T_2 - T_1)L$$

$$\frac{(2p + 1.44p)}{200 \times 10^9} = (18 - 11)10^{-6}(50 - 20)$$

$$3.44p = 7 \times 10^{-6} \times 30 \times 200 \times 10^9$$

$$p = 12.21 \text{ MN/m}^2$$

and

$$1.44p = 17.6 \text{ MN/m}^2$$

The changes in the stresses resulting from the temperature effects are thus 12.2 MN/m^2 (compressive) in the tube and 17.6 MN/m^2 (tensile) in the rod.

The final, resultant, stresses are thus:

$$\text{stress in tube} = -36.3 - 12.2 = 48.5 \text{ MN/m}^2 \text{ (compressive)}$$

$$\text{stress in rod} = 52.3 + 17.6 = 69.9 \text{ MN/m}^2 \text{ (tensile)}$$

Example 2.4

A composite bar is constructed from a steel rod of 25 mm diameter surrounded by a copper tube of 50 mm outside diameter and 25 mm inside diameter. The rod and tube are joined by two 20 mm diameter pins as shown in Fig. 2.6. Find the shear stress set up in the pins if, after pinning, the temperature is raised by 50°C .

For steel $E = 210 \text{ GN/m}^2$ and $\alpha = 11 \times 10^{-6}$ per $^\circ\text{C}$.

For copper $E = 105 \text{ GN/m}^2$ and $\alpha = 17 \times 10^{-6}$ per $^\circ\text{C}$.

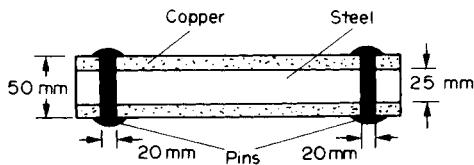


Fig. 2.6.

Solution

In this case the copper attempts to expand more than the steel, thus tending to shear the pins joining the two.

Let the stress set up in the steel be x , then, since

$$\text{force in steel} = \text{force in copper}$$

$$x \times \frac{\pi}{4} \times 25^2 \times 10^{-6} = \sigma_c \times \frac{\pi}{4} (50^2 - 25^2) 10^{-6}$$

i.e.

$$\text{stress in copper } \sigma_c = \frac{x \times 25^2}{(50^2 - 25^2)} = 0.333x = \frac{x}{3}$$

Now the extension of the steel from its freely expanded length to its forced length in the compound bar is given by

$$\frac{\sigma L}{E} = \frac{xL}{210 \times 10^9}$$

where L is the original length.

Similarly, the compression of the copper from its freely expanded position to its position in the compound bar is given by

$$\frac{\sigma L}{E} = \frac{x}{3} \times \frac{L}{105 \times 10^9}$$

Now the extension of steel + compression of copper

$$= \text{difference in "free" lengths}$$

$$= (\alpha_2 - \alpha_1)(T_2 - T_1)L$$

$$\therefore \frac{xL}{210 \times 10^9} + \frac{xL}{3 \times 105 \times 10^9} = (17 - 11)10^{-6} \times 50 \times L$$

$$\frac{3x + 2x}{6 \times 105 \times 10^9} = 6 \times 10^{-6} \times 50$$

$$5x = 6 \times 10^{-6} \times 50 \times 6 \times 105 \times 10^9$$

$$x = 37.8 \times 10^6 = 37.8 \text{ MN/m}^2$$

\therefore load carried by the steel = stress \times area

$$= 37.8 \times 10^6 \times \frac{\pi}{4} \times 25^2 \times 10^{-6}$$

$$= 18.56 \text{ kN}$$

The pins will be in a state of double shear (see § 1.15), the shear stress set up being given by

$$\tau = \frac{\text{load}}{2 \times \text{area}} = \frac{18.56 \times 10^3}{2 \times \frac{\pi}{4} \times 20^2 \times 10^{-6}}$$

$$= 29.5 \text{ MN/m}^2$$

Problems

2.1 (A). A power transmission cable consists of ten copper wires each of 1.6 mm diameter surrounding three steel wires each of 3 mm diameter. Determine the combined E for the compound cable and hence determine the extension of a 30 m length of the cable when it is being laid with a tension of 2 kN.

For steel, $E = 200 \text{ GN/m}^2$; for copper, $E = 100 \text{ GN/m}^2$.

[151.3 GN/m²; 9.6 mm.]

2.2 (A). If the maximum stress allowed in the copper of the cable of problem 2.1 is 60 MN/m², determine the maximum tension which the cable can support. [3.75 kN.]

2.3 (A). What will be the stress induced in a steel bar when it is heated from 15°C to 60°C, all expansion being prevented?

For mild steel, $E = 210 \text{ GN/m}^2$ and $\alpha = 11 \times 10^{-6}$ per °C.

[104 MN/m^{2°}.]

2.4 (A). A 75 mm diameter compound bar is constructed by shrinking a circular brass bush onto the outside of a 50 mm diameter solid steel rod. If the compound bar is then subjected to an axial compressive load of 160 kN determine the load carried by the steel rod and the brass bush and the compressive stress set up in each material.

For steel, $E = 210 \text{ GN/m}^2$; for brass, $E = 100 \text{ GN/m}^2$. [I. Struct. E.] [100.3, 59.7 kN; 51.1, 24.3 MN/m².]

2.5 (B). A steel rod of cross-sectional area 600 mm² and a coaxial copper tube of cross-sectional area 1000 mm² are firmly attached at their ends to form a compound bar. Determine the stress in the steel and in the copper when the temperature of the bar is raised by 80°C and an axial tensile force of 60 kN is applied.

For steel, $E = 200 \text{ GN/m}^2$ with $\alpha = 11 \times 10^{-6}$ per °C.

For copper, $E = 100 \text{ GN/m}^2$ with $\alpha = 16.5 \times 10^{-6}$ per °C.

[E.I.E.] [94.6, 3.3 MN/m².]

2.6 (B). A stanchion is formed by butt welding together four plates of steel to form a square tube of outside cross-section 200 mm × 200 mm. The constant metal thickness is 10 mm. The inside is then filled with concrete.

(a) Determine the cross-sectional area of the steel and concrete

(b) If E for steel is 200 GN/m² and this value is twenty times that for the concrete find, when the stanchion carries a load of 368.8 kN,

(i) The stress in the concrete

(ii) The stress in the steel

(iii) The amount the stanchion shortens over a length of 2m.

[C.G.] [2, 40 MN/m²; 40 mm]

CHAPTER 3

SHEARING FORCE AND BENDING MOMENT DIAGRAMS

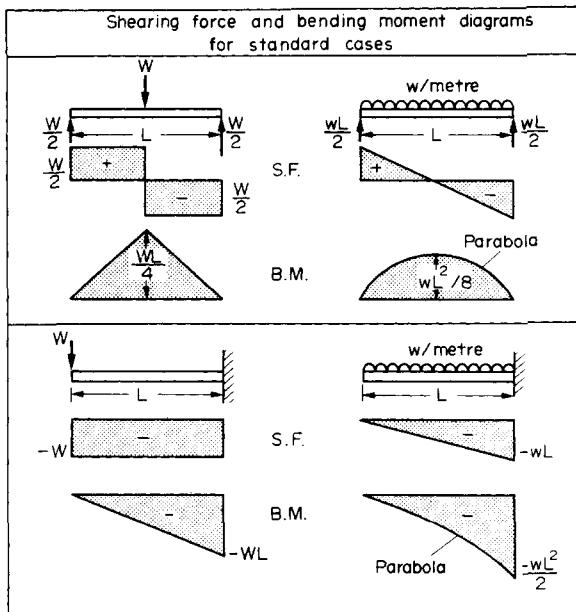
Summary

At any section in a beam carrying transverse loads the shearing force is defined as the algebraic sum of the forces taken on either side of the section.

Similarly, the bending moment at any section is the algebraic sum of the moments of the forces about the section, again taken on either side.

In order that the shearing-force and bending-moment values calculated on either side of the section shall have the same magnitude and sign, a convenient sign convention has to be adopted. This is shown in Figs. 3.1 and 3.2 (see page 42).

Shearing-force (S.F.) and bending-moment (B.M.) diagrams show the variation of these quantities along the length of a beam for any fixed loading condition.



3.1. Shearing force and bending moment

At every section in a beam carrying transverse loads there will be resultant forces on either side of the section which, for equilibrium, must be equal and opposite, and whose combined

action tends to shear the section in one of the two ways shown in Fig. 3.1a and b. *The shearing force (S.F.) at the section is defined therefore as the algebraic sum of the forces taken on one side of the section.* Which side is chosen is purely a matter of convenience but in order that the value obtained on both sides shall have the same magnitude and sign a convenient sign convention has to be adopted.

3.1.1. Shearing force (S.F.) sign convention

Forces upwards to the left of a section or downwards to the right of the section are positive. Thus Fig. 3.1a shows a positive S.F. system at X-X and Fig. 3.1b shows a negative S.F. system.

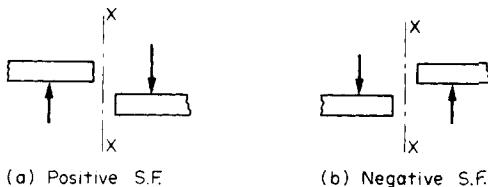


Fig. 3.1. S.F. sign convention.

In addition to the shear, every section of the beam will be subjected to bending, i.e. to a resultant B.M. which is the net effect of the moments of each of the individual loads. Again, for equilibrium, the values on either side of the section must have equal values. *The bending moment (B.M.) is defined therefore as the algebraic sum of the moments of the forces about the section, taken on either side of the section.* As for S.F., a convenient sign convention must be adopted.

3.1.2. Bending moment (B.M.) sign convention

Clockwise moments to the left and counterclockwise to the right are positive. Thus Fig. 3.2a shows a positive bending moment system resulting in *sagging* of the beam at X-X and Fig. 3.2b illustrates a negative B.M. system with its associated *hogging* beam.

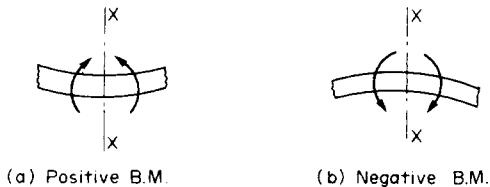


Fig. 3.2. B.M. sign convention.

It should be noted that whilst the above sign conventions for S.F. and B.M. are somewhat arbitrary and could be completely reversed, the systems chosen here are the only ones which yield the mathematically correct signs for slopes and deflections of beams in subsequent work and therefore are highly recommended.

Diagrams which illustrate the variation in the B.M. and S.F. values along the length of a beam or structure for any fixed loading condition are termed *B.M. and S.F. diagrams*. They are therefore graphs of B.M. or S.F. values drawn on the beam as a base and they clearly illustrate in the early design stages the positions on the beam which are subjected to the greatest shear or bending stresses and hence which may require further consideration or strengthening.

At this point it is imperative to note that there are two general forms of loading to which structures may be subjected, namely, concentrated and distributed loads. The former are assumed to act at a point and immediately introduce an oversimplification since all practical loading systems must be applied over a finite area. Nevertheless, for calculation purposes this area is assumed to be so small that the load can be justly assumed to act at a point. Distributed loads are assumed to act over part, or all, of the beam and in most cases are assumed to be equally or uniformly distributed; they are then termed uniformly distributed loads (u.d.l.). Occasionally, however, the distribution is not uniform but may vary linearly across the loaded portion or have some more complex distribution form.

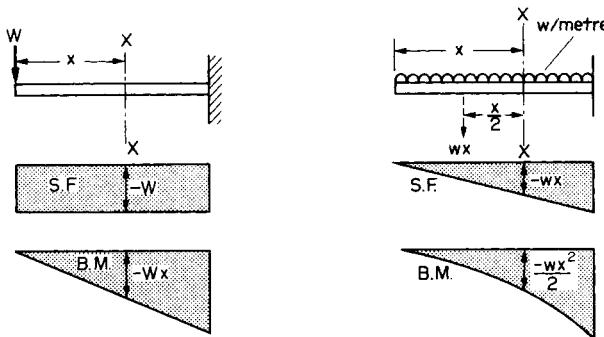


Fig. 3.3. S.F.-B.M. diagrams for standard cases.

Thus in the case of a cantilever carrying a concentrated load W at the end (Fig. 3.3), the S.F. at any section $X-X$, distance x from the free end, is $S.F. = -W$. This will be true whatever the value of x , and so the S.F. diagram becomes a rectangle. The B.M. at the same section $X-X$ is $-Wx$ and this will increase linearly with x . The B.M. diagram is therefore a triangle.

If the cantilever now carries a uniformly distributed load, the S.F. at $X-X$ is the net load to one side of $X-X$, i.e. $-wx$. In this case, therefore, the S.F. diagram becomes triangular, increasing to a maximum value of $-wL$ at the support. The B.M. at $X-X$ is obtained by treating the load to the left of $X-X$ as a concentrated load of the same value acting at the centre of gravity,

$$\text{i.e. } B.M. \text{ at } X-X = -wx \frac{x}{2} = -\frac{-wx^2}{2}$$

Plotted against x this produces the parabolic B.M. diagram shown.

3.2. S.F. and B.M. diagrams for beams carrying concentrated loads only

In order to illustrate the procedure to be adopted for the determination of S.F. and B.M. values for more complicated load conditions, consider the simply supported beam shown in

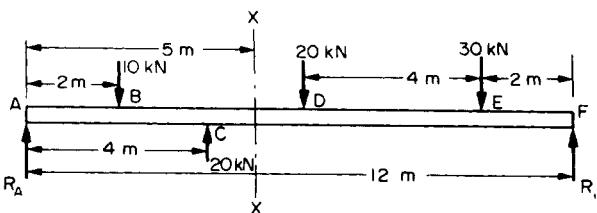


Fig. 3.4.

Fig. 3.4 carrying concentrated loads only. (The term *simply supported* means that the beam can be assumed to rest on knife-edges or roller supports and is free to bend at the supports without any restraint.)

The values of the reactions at the ends of the beam may be calculated by applying normal equilibrium conditions, i.e. by taking moments about *F*.

$$\text{Thus } R_A \times 12 = (10 \times 10) + (20 \times 6) + (30 \times 2) - (20 \times 8) = 120$$

$$R_A = 10 \text{ kN}$$

For vertical equilibrium

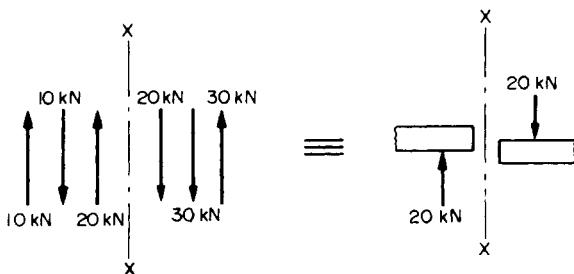
$$\text{total force up} = \text{total load down}$$

$$R_A + R_F = 10 + 20 + 30 - 20 = 40$$

$$R_F = 30 \text{ kN}$$

At this stage it is advisable to check the value of R_F by taking moments about *A*.

Summing up the forces on either side of *X-X* we have the result shown in Fig. 3.5. Using the sign convention listed above, the shear force at *X-X* is therefore +20 kN, i.e. the resultant force at *X-X* tending to shear the beam is 20 kN.

Fig. 3.5. Total S.F. at *X-X*.

Similarly, Fig. 3.6 shows the summation of the moments of the forces at *X-X*, the resultant B.M. being 40 kN m.

In practice only one side of the section is normally considered and the summations involved can often be completed by mental arithmetic. The complete S.F. and B.M. diagrams for the beam are shown in Fig. 3.7, and the B.M. values used to construct the diagram are derived on page 45.

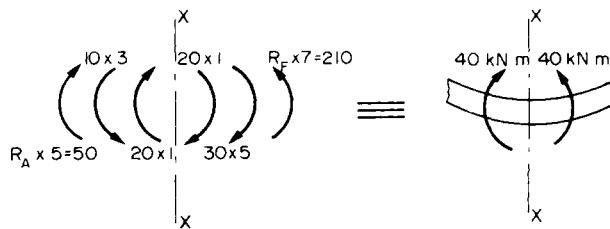


Fig. 3.6. Total B.M. at X-X.

B.M. at A	= 0
B.M. at B = $+(10 \times 2)$	= $+20 \text{ kN m}$
B.M. at C = $+(10 \times 4) - (10 \times 2)$	= $+20 \text{ kN m}$
B.M. at D = $+(10 \times 6) + (20 \times 2) - (10 \times 4) = +60 \text{ kN m}$	
B.M. at E = $+(30 \times 2)$	= $+60 \text{ kN m}$
B.M. at F	= 0

All the above values have been calculated from the moments of the forces to the *left* of each section considered except for E where forces to the right of the section are taken.

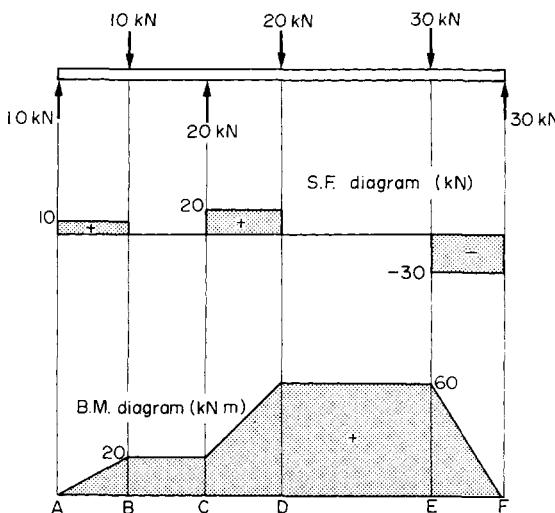


Fig. 3.7.

It may be observed at this stage that the S.F. diagram can be obtained very quickly when working from the left-hand side, since after plotting the S.F. value at the support all subsequent steps are in the direction of and equal in magnitude to the applied loads, e.g. 10 kN up at A, down 10 kN at B, up 20 kN at C, etc., with horizontal lines joining the steps to show that the S.F. remains constant between points of application of concentrated loads.

The S.F. and B.M. values at the left-hand support are determined by considering a section an infinitely small distance to the right of the support. The only load to the left (and hence the

S.F.) is then the reaction of 10 kN upwards, i.e. positive, and the bending moment = reaction \times zero distance = zero.

The following characteristics of the two diagrams are now evident and will be explained later in this chapter:

- between *B* and *C* the S.F. is zero and the B.M. remains constant;
- between *A* and *B* the S.F. is positive and the slope of the B.M. diagram is positive; vice versa between *E* and *F*;
- the difference in B.M. between *A* and *B* = 20 kN m = area of S.F. diagram between *A* and *B*.

3.3. S.F. and B.M. diagrams for uniformly distributed loads

Consider now the simply supported beam shown in Fig. 3.8 carrying a u.d.l. $w = 25 \text{ kN/m}$ across the complete span.

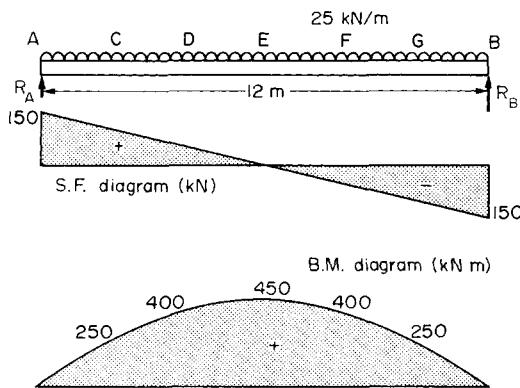


Fig. 3.8.

Here again it is necessary to evaluate the reactions, but in this case the problem is simplified by the symmetry of the beam. Each reaction will therefore take half the applied load,

i.e.

$$R_A = R_B = \frac{25 \times 12}{2} = 150 \text{ kN}$$

The S.F. at *A*, using the usual sign convention, is therefore $+150$ kN.

Consider now the beam divided into six equal parts 2 m long. The S.F. at any other point *C* is, therefore,

$$\begin{aligned} & 150 - \text{load downwards between } A \text{ and } C \\ & = 150 - (25 \times 2) = +100 \text{ kN} \end{aligned}$$

The whole diagram may be constructed in this way, or much more quickly by noticing that the S.F. at *A* is $+150$ kN and that between *A* and *B* the S.F. decreases uniformly, producing the required sloping straight line, shown in Fig. 3.7. Alternatively, the S.F. at *A* is $+150$ kN and between *A* and *B* this decreases gradually by the amount of the applied load (i.e. by $25 \times 12 = 300$ kN) to -150 kN at *B*.

When evaluating B.M.'s it is assumed that a u.d.l. can be replaced by a concentrated load of equal value acting at the middle of its spread. When taking moments about *C*, therefore, the portion of the u.d.l. between *A* and *C* has an effect equivalent to that of a concentrated load of $25 \times 2 = 50$ kN acting the centre of *AC*, i.e. 1 m from *C*.

$$\text{B.M. at } C = (R_A \times 2) - (50 \times 1) = 300 - 50 = 250 \text{ kN m}$$

Similarly, for moments at *D* the u.d.l. on *AD* can be replaced by a concentrated load of

$$25 \times 4 = 100 \text{ kN at the centre of } AD, \text{ i.e. at } C.$$

$$\text{B.M. at } D = (R_A \times 4) - (100 \times 2) = 600 - 200 = 400 \text{ kN m}$$

Similarly,

$$\text{B.M. at } E = (R_A \times 6) - (25 \times 6)3 = 900 - 450 = 450 \text{ kN m}$$

The B.M. diagram will be symmetrical about the beam centre line; therefore the values of B.M. at *F* and *G* will be the same as those at *D* and *C* respectively. The final diagram is therefore as shown in Fig. 3.8 and is parabolic.

Point (a) of the summary is clearly illustrated here, since the B.M. is a maximum when the S.F. is zero. Again, the reason for this will be shown later.

3.4. S.F. and B.M. diagrams for combined concentrated and uniformly distributed loads

Consider the beam shown in Fig. 3.9 loaded with a combination of concentrated loads and u.d.l.s.

Taking moments about *E*

$$(R_A \times 8) + (40 \times 2) = (10 \times 2 \times 7) + (20 \times 6) + (20 \times 3) + (10 \times 1) + (20 \times 3 \times 1.5)$$

$$8R_A + 80 = 420$$

$$R_A = 42.5 \text{ kN} (= \text{S.F. at } A)$$

$$\text{Now } R_A + R_E = (10 \times 2) + 20 + 20 + 10 + (20 \times 3) + 40 = 170$$

$$R_E = 127.5 \text{ kN}$$

Working from the left-hand support it is now possible to construct the S.F. diagram, as indicated previously, by following the direction arrows of the loads. In the case of the u.d.l.'s the S.F. diagram will decrease gradually by the amount of the total load until the end of the u.d.l. or the next concentrated load is reached. Where there is no u.d.l. the S.F. diagram remains horizontal between load points.

In order to plot the B.M. diagram the following values must be determined:

B.M. at <i>A</i>	=	0
B.M. at <i>B</i> = $(42.5 \times 2) - (10 \times 2 \times 1) = 85 - 20$	=	65 kN m
B.M. at <i>C</i> = $(42.5 \times 5) - (10 \times 2 \times 4) - (20 \times 3) = 212.5 - 80 - 60$	=	72.5 kN m
B.M. at <i>D</i> = $(42.5 \times 7) - (10 \times 2 \times 6) - (20 \times 5) - (20 \times 2)$		
$- (20 \times 2 \times 1) = 297.5 - 120 - 100 - 40 - 40 = 297.5 - 300 = - 2.5 \text{ kN m}$		
B.M. at <i>E</i> = $(- 40 \times 2)$ working from r.h.s.	=	- 80 kN m
B.M. at <i>F</i>	=	0

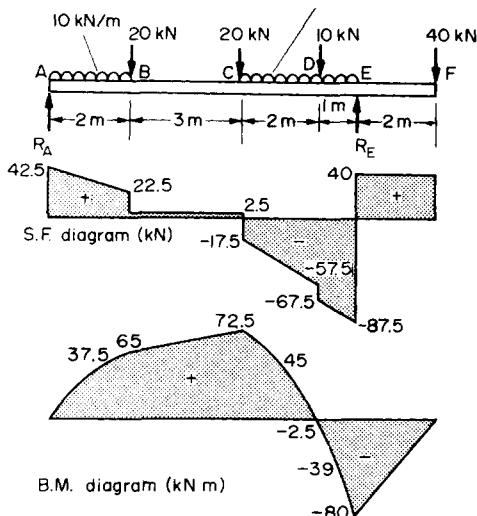


Fig. 3.9.

For complete accuracy one or two intermediate values should be obtained along each u.d.l. portion of the beam,

$$\text{e.g. } \begin{aligned} \text{B.M. midway between } A \text{ and } B &= (42.5 \times 1) - (10 \times 1 \times \frac{1}{2}) \\ &= 42.5 - 5 = 37.5 \text{ kN m} \end{aligned}$$

Similarly, B.M. midway between \$C\$ and \$D = 45 \text{ kN m}\$

$$\text{B.M. midway between } D \text{ and } E = -39 \text{ kN m}$$

The B.M. and S.F. diagrams are then as shown in Fig. 3.9.

3.5. Points of contraflexure

A point of contraflexure is a point where the curvature of the beam changes sign. It is sometimes referred to as a *point of inflexion* and will be shown later to occur at the point, or points, on the beam where the B.M. is zero.

For the beam of Fig. 3.9, therefore, it is evident from the B.M. diagram that this point lies somewhere between \$C\$ and \$D\$ (B.M. at \$C\$ is positive, B.M. at \$D\$ is negative). If the required point is a distance \$x\$ from \$C\$ then at that point

$$\begin{aligned} \text{B.M.} &= (42.5)(5+x) - (10 \times 2)(4+x) - 20(3+x) - 20x - \frac{20x^2}{2} \\ &= 212.5 + 42.5x - 80 - 20x - 60 - 20x - 20x - 10x^2 \\ &= 72.5 - 17.5x - 10x^2 \end{aligned}$$

Thus the B.M. is zero where

$$0 = 72.5 - 17.5x - 10x^2$$

i.e. where

$$x = 1.96 \text{ or } -3.7$$

Since the last answer can be ignored (being outside the beam), the point of contraflexure must be situated at 1.96 m to the right of C.

3.6. Relationship between shear force Q , bending moment M and intensity of loading w

Consider the beam AB shown in Fig. 3.10 carrying a uniform loading intensity (uniformly distributed load) of $w \text{ kN/m}$. By symmetry, each reaction takes half the total load, i.e., $wL/2$.

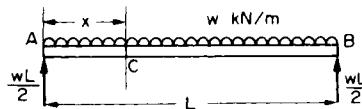


Fig. 3.10.

The B.M. at any point C, distance x from A, is given by

$$M = \frac{wL}{2}x - (wx)\frac{x}{2}$$

i.e.

$$M = \frac{1}{2}wLx - \frac{1}{2}wx^2$$

Differentiating,

$$\frac{dM}{dx} = \frac{1}{2}wL - wx$$

Now

$$\text{S.F. at } C = \frac{1}{2}wL - wx = Q \quad (3.1)$$

∴

$$\frac{dM}{dx} = Q \quad (3.2)$$

Differentiating eqn. (3.1),

$$\frac{dQ}{dx} = -w \quad (3.3)$$

These relationships are the basis of the rules stated in the summary, the proofs of which are as follows:

(a) The maximum or minimum B.M. occurs where $dM/dx = 0$

But

$$\frac{dM}{dx} = Q$$

Thus where S.F. is zero B.M. is a maximum or minimum.

(b) The slope of the B.M. diagram $= dM/dx = Q$.

Thus where Q = 0 the slope of the B.M. diagram is zero, and the B.M. is therefore constant.

(c) Also, since Q represents the slope of the B.M. diagram, it follows that *where the S.F. is positive the slope of the B.M. diagram is positive, and where the S.F. is negative the slope of the B.M. diagram is also negative.*

(d) The area of the S.F. diagram between any two points, from basic calculus, is

$$\int Q dx$$

But

$$\frac{dM}{dx} = Q \quad \text{or} \quad M = \int Q dx$$

i.e. the B.M. change between any two points is the area of the S.F. diagram between these points.

This often provides a very quick method of obtaining the B.M. diagram once the S.F. diagram has been drawn.

(e) With the chosen sign convention, when the B.M. is positive the beam is *sagging* and when it is negative the beam is *hogging*. Thus when the curvature of the beam changes from *sagging* to *hogging*, as at X-X in Fig. 3.11, or vice versa, the B.M. changes sign, i.e. becomes instantaneously zero. This is termed a *point of inflexion* or *contraflexure*. Thus a point of *contraflexure* occurs where the B.M. is zero.

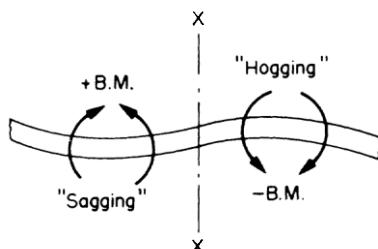


Fig. 3.11. Beam with point of contraflexure at X-X.

3.7. S.F. and B.M. diagrams for an applied couple or moment

In general there are two ways in which the couple or moment can be applied: (a) with horizontal loads and (b) with vertical loads, and the method of solution is different for each.

Type (a): couple or moment applied with horizontal loads

Consider the beam AB shown in Fig. 3.12 to which a moment $F.d$ is applied by means of horizontal loads at a point C, distance a from A.

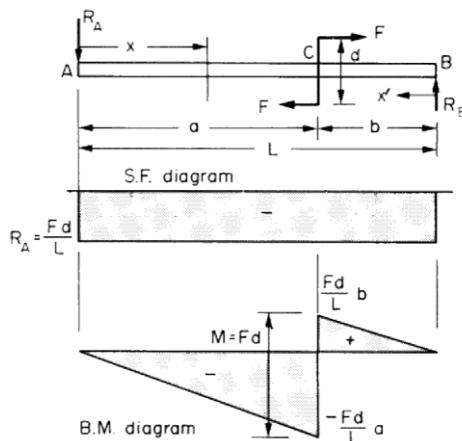


Fig. 3.12.

Since this will tend to lift the beam at A , R_A acts downwards.

Moments about B : $R_A \times L = Fd$

$$\therefore R_A = \frac{Fd}{L}$$

and for vertical equilibrium

$$R_B = R_A = \frac{Fd}{L}$$

The S.F. diagram can now be drawn as the horizontal loads have no effect on the vertical shear.

The B.M. at any section between A and C is

$$M = -R_A x = -\frac{Fd}{L} x$$

Thus the value of the B.M. increases linearly from zero at A to $\frac{-Fd}{L} a$ at C .

Similarly, the B.M. at any section between C and B is

$$M = -R_A x + Fd = R_B x' = \frac{Fd}{L} x'$$

i.e. the value of the B.M. again increases linearly from zero at B to $\frac{Fd}{L} b$ at C . The B.M. diagram is therefore as shown in Fig. 3.12.

Type (b): moment applied with vertical loads

Consider the beam AB shown in Fig. 3.13; taking moments about B :

$$R_A L = F(d+b)$$

$$\therefore R_A = \frac{F(d+b)}{L}$$

Similarly,

$$R_B = \frac{F(a-d)}{L}$$

The S.F. diagram can therefore be drawn as in Fig. 3.13 and it will be observed that in this case F does affect the diagram.

For the B.M. diagram an equivalent system is used. The offset load F is replaced by a moment and a force acting at C , as shown in Fig. 3.13. Thus

$$\text{B.M. between } A \text{ and } C = R_A x$$

$$= \frac{F(d+b)}{L} x$$

i.e. increasing linearly from zero to $\frac{F(d+b)}{L} a$ at C .

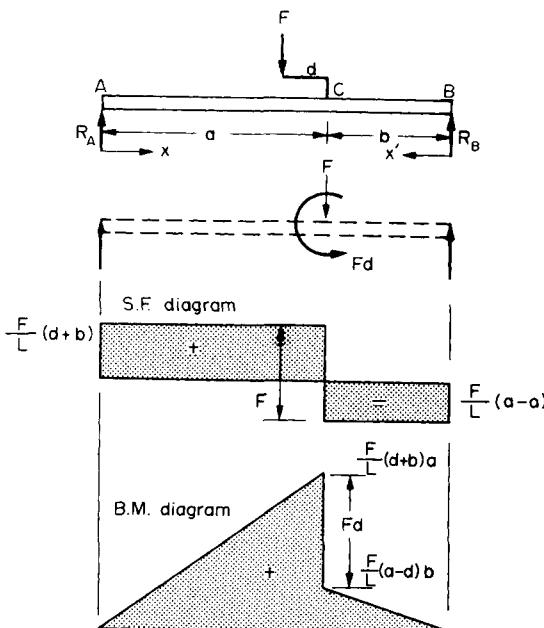


Fig. 3.13.

Similarly,

$$\begin{aligned} \text{B.M. between } C \text{ and } B &= R_B x' \\ &= \frac{F(a-d)}{L} x' \end{aligned}$$

i.e. increasing linearly from zero to $\frac{F(a-d)}{L} b$ at C.

The difference in values at C is equal to the applied moment Fd , as with type (a).

Consider now the beam shown in Fig. 3.14 carrying concentrated loads in addition to the applied moment of 30 kN m (which can be assumed to be of type (a) unless otherwise stated). The principle of superposition states that the total effect of the combined loads will be the same as the algebraic sum of the effects of the separate loadings, i.e. the final diagram will be the combination of the separate diagrams representing applied moment and those representing concentrated loads. The final diagrams are therefore as shown shaded, all values quoted being measured from the normal base line of each diagram. In each case, however, the applied-moment diagrams have been inverted so that the negative areas can easily be subtracted. Final values are now measured from the dotted lines: e.g. the S.F. and B.M. at any point G are as indicated in Fig. 3.14.

3.8. S.F. and B.M. diagrams for inclined loads

If a beam is subjected to inclined loads as shown in Fig. 3.15 each of the loads must be resolved into its vertical and horizontal components as indicated. The vertical components

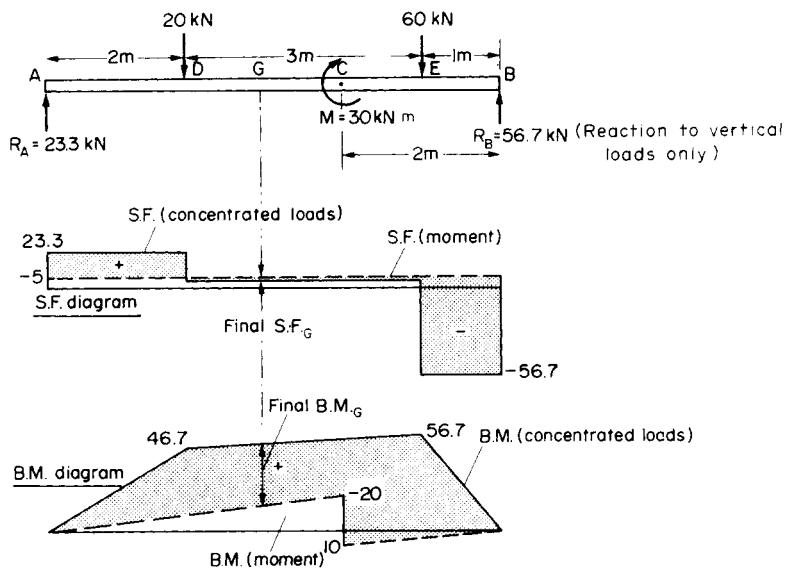


Fig. 3.14.

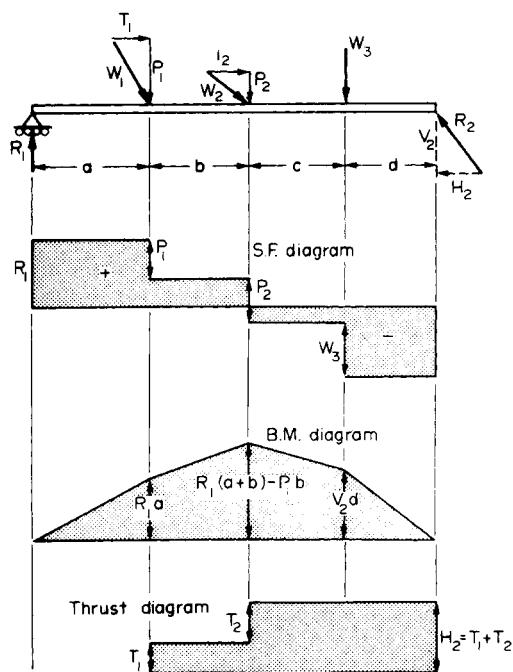


Fig. 3.15. S.F., B.M. and thrust diagrams for system of inclined loads.

yield the values of the vertical reactions at the supports and hence the S.F. and B.M. diagrams are obtained as described in the preceding sections. In addition, however, there must be a horizontal constraint applied to the beam at one or both reactions to bring the horizontal components of the applied loads into equilibrium. Thus there will be a horizontal force or *thrust diagram* for the beam which indicates the axial load carried by the beam at any point. If the constraint is assumed to be applied at the right-hand end the thrust diagram will be as indicated.

3.9. Graphical construction of S.F. and B.M. diagrams

Consider the simply supported beam shown in Fig. 3.16 carrying three concentrated loads of different values. The procedure to be followed for graphical construction of the S.F. and B.M. diagrams is as follows.

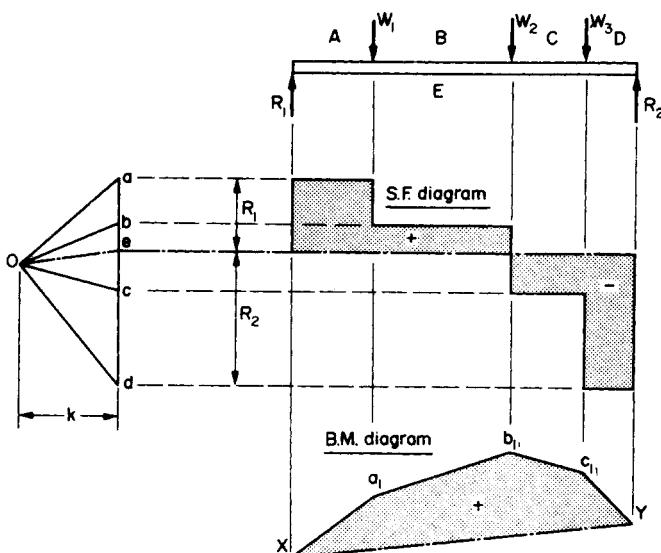


Fig. 3.16. Graphical construction of S.F. and B.M. diagrams.

- Letter the spaces between the loads and reactions A, B, C, D and E . Each force can then be denoted by the letters of the spaces on either side of it.
- To one side of the beam diagram construct a force vector diagram for the applied loads, i.e. set off a vertical distance ab to represent, in magnitude and direction, the force W_1 dividing spaces A and B to some scale, bc to represent W_2 and cd to represent W_3 .
- Select any point O , known as a *pole point*, and join Oa, Ob, Oc and Od .
- Drop verticals from all loads and reactions.
- Select any point X on the vertical through reaction R_1 and from this point draw a line in space A parallel to Oa to cut the vertical through W_1 in a_1 . In space B draw a line from a_1 parallel to ob , continue in space C parallel to Oc , and finally in space D parallel to od to cut the vertical through R_2 in Y .

- (f) Join XY and through the pole point O draw a line parallel to XY to cut the force vector diagram in e . The distance ea then represents the value of the reaction R_1 in magnitude and direction and de represents R_2 .
- (g) Draw a horizontal line through e to cut the vertical projections from the loading points and to act as the base line for the S.F. diagram. Horizontal lines from a in gap A , b in gap B , c in gap C , etc., produce the required S.F. diagram to the same scale as the original force vector diagram.
- (h) The diagram $Xa_1b_1c_1Y$ is the B.M. diagram for the beam, vertical distances from the inclined base line XY giving the bending moment at any required point to a certain scale.

If the original beam diagram is drawn to a scale $1\text{ cm} = L$ metres (say), the force vector diagram scale is $1\text{ cm} = W$ newton, and, if the horizontal distance from the pole point O to the vector diagram is k cm, then the scale of the B.M. diagram is

$$1\text{ cm} = kLW \text{ newton metre}$$

The above procedure applies for beams carrying concentrated loads only, but an approximate solution is obtained in a similar way for u.d.l.s. by considering the load divided into a convenient number of concentrated loads acting at the centres of gravity of the divisions chosen.

3.10. S.F. and B.M. diagrams for beams carrying distributed loads of increasing value

For beams which carry distributed loads of varying intensity as in Fig. 3.18 a solution can be obtained from eqn. (3.3) provided that the loading variation can be expressed in terms of the distance x along the beam span, i.e. as a function of x .

$$\frac{dQ}{dx} = -w = -f(x)$$

Integrating once yields the shear force Q in terms of a constant of integration A since

$$\frac{dM}{dx} = Q$$

Integration again yields an expression for the B.M. M in terms of A and a second constant of integration B . Known conditions of B.M. or S.F., usually at the supports or ends of the beam, yield the values of the constants and hence the required distributions of S.F. and B.M. A typical example of this type has been evaluated on page 57.

3.11. S.F. at points of application of concentrated loads

In the preceding sections it has been assumed that concentrated loads can be applied precisely at a point so that S.F. diagrams are shown to change value suddenly from one value to another, and sometimes one sign to another, at the loading points. It would appear from the S.F. diagrams drawn previously, therefore, that two possible values of S.F. exist at any one loading point and this is obviously not the case. In practice, loads can only be applied over

finite areas and the S.F. must change gradually from one value to another across these areas. The vertical line portions of the S.F. diagrams are thus highly idealised versions of what actually occurs in practice and should be replaced more accurately by lines slightly inclined to the vertical. All sharp corners of the diagrams should also be rounded. Despite these minor inaccuracies, B.M. and S.F. diagrams remain a highly convenient, powerful and useful representation of beam loading conditions for design purposes.

Examples

Example 3.1

Draw the S.F. and B.M. diagrams for the beam loaded as shown in Fig. 3.17, and determine (a) the position and magnitude of the maximum B.M., and (b) the position of any point of contraflexure.

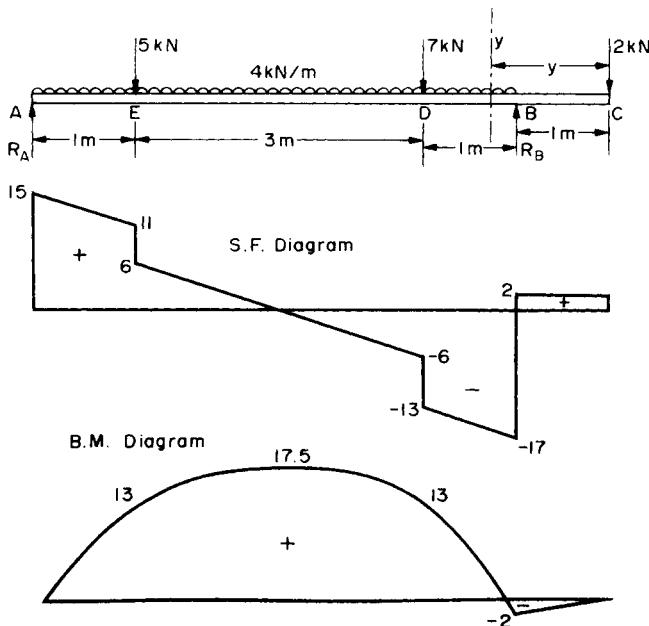


Fig. 3.17.

Solution

Taking the moments about A,

$$5R_B = (5 \times 1) + (7 \times 4) + (2 \times 6) + (4 \times 5) \times 2.5$$

$$\therefore R_B = \frac{5 + 28 + 12 + 50}{5} = 19 \text{ kN}$$

and since

$$\begin{aligned} R_A + R_B &= 5 + 7 + 2 + (4 \times 5) = 34 \\ R_A &= 34 - 19 = 15 \text{ kN} \end{aligned}$$

The S.F. diagram may now be constructed as described in §3.4 and is shown in Fig. 3.17.

Calculation of bending moments

$$\text{B.M. at } A \text{ and } C = 0$$

$$\text{B.M. at } B = -2 \times 1 = -2 \text{ kNm}$$

$$\text{B.M. at } D = -(2 \times 2) + (19 \times 1) - (4 \times 1 \times \frac{1}{2}) = +13 \text{ kNm}$$

$$\text{B.M. at } E = +(15 \times 1) - (4 \times 1 \times \frac{1}{2}) = +13 \text{ kNm}$$

The maximum B.M. will be given by the point (or points) at which dM/dx (i.e. the shear force) is zero. By inspection of the S.F. diagram this occurs midway between *D* and *E*, i.e. at 1.5 m from *E*.

$$\begin{aligned} \text{B.M. at this point} &= (2.5 \times 15) - (5 \times 1.5) - \left(4 \times 2.5 \times \frac{2.5}{2} \right) \\ &= +17.5 \text{ kNm} \end{aligned}$$

There will also be local maxima at the other points where the S.F. diagram crosses its zero axis, i.e. at point *B*.

Owing to the presence of the concentrated loads (reactions) at these positions, however, these will appear as discontinuities in the diagram; there will not be a smooth contour change. The value of the B.M.s at these points should be checked since the position of maximum stress in the beam depends upon the numerical maximum value of the B.M.; this does not necessarily occur at the mathematical maximum obtained above.

The B.M. diagram is therefore as shown in Fig. 3.17. Alternatively, the B.M. at any point between *D* and *E* at a distance of *x* from *A* will be given by

$$M_{xx} = 15x - 5(x - 1) - \frac{4x^2}{2} = 10x + 5 - 2x^2$$

The maximum B.M. position is then given where $\frac{dM}{dx} = 0$.

$$\frac{dM}{dx} = 10 - 4x = 0 \quad \therefore \quad x = 2.5 \text{ m}$$

i.e.

1.5 m from *E*, as found previously.

(b) Since the B.M. diagram only crosses the zero axis once there is only one point of contraflexure, i.e. between *B* and *D*. Then, B.M. at distance *y* from *C* will be given by

$$\begin{aligned} M_{yy} &= -2y + 19(y - 1) - 4(y - 1)\frac{1}{2}(y - 1) \\ &= -2y + 19y - 19 - 2y^2 + 4y - 2 = 0 \end{aligned}$$

The point of contraflexure occurs where B.M. = 0, i.e. where $M_{yy} = 0$,

$$\therefore 0 = -2y^2 + 21y - 21$$

i.e.

$$2y^2 - 21y + 21 = 0$$

Then

$$y = \frac{21 \pm \sqrt{(21^2 - 4 \times 2 \times 21)}}{4} = 1.12 \text{ m}$$

i.e. point of contraflexure occurs **0.12 m to the left of B**.**Example 3.2**

A beam *ABC* is 9 m long and supported at *B* and *C*, 6 m apart as shown in Fig. 3.18. The beam carries a triangular distribution of load over the portion *BC* together with an applied counterclockwise couple of moment 80 kN m at *B* and a u.d.l. of 10 kN/m over *AB*, as shown. Draw the S.F. and B.M. diagrams for the beam.

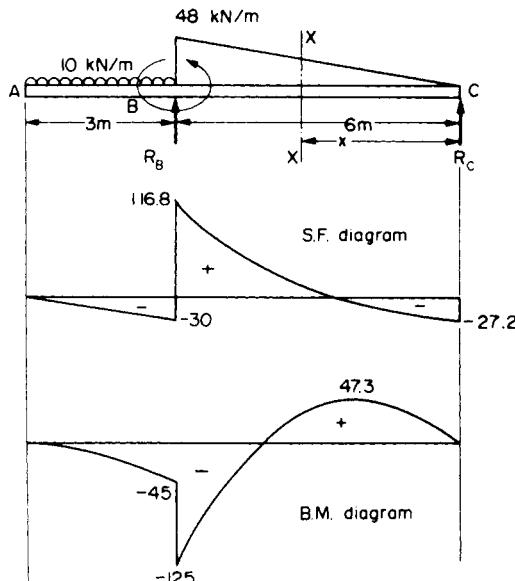


Fig. 3.18.

SolutionTaking moments about *B*,

$$(R_C \times 6) + (10 \times 3 \times 1.5) + 80 = (\frac{1}{2} \times 6 \times 48) \times \frac{1}{3} \times 6$$

$$6R_C + 45 + 80 = 288$$

$$R_C = 27.2 \text{ kN}$$

and

$$R_C + R_B = (10 \times 3) + (\frac{1}{2} \times 6 \times 48)$$

$$= 30 + 144 = 174$$

$$\therefore R_B = 146.8 \text{ kN}$$

At any distance x from C between C and B the shear force is given by

$$\text{S.F.}_{xx} = -\frac{1}{2}wx + R_C$$

and by proportions

$$\frac{w}{x} = \frac{48}{6} = 8$$

i.e.

$$w = 8x \text{ kN/m}$$

$$\begin{aligned}\therefore \text{S.F.}_{xx} &= -(R_C - \frac{1}{2} \times 8x \times x) \\ &= -R_C + 4x^2 \\ &= -27.2 + 4x^2\end{aligned}$$

The S.F. diagram is then as shown in Fig. 3.18.

Also

$$\begin{aligned}\text{B.M.}_{xx} &= -(\frac{1}{2}wx)\frac{x}{3} + R_C x \\ &= 27.2x - \frac{4x^3}{3}\end{aligned}$$

For a maximum value,

$$\frac{d(\text{B.M.})}{dx} = \text{S.F.} = 0$$

i.e., where

$$4x^2 = 27.2$$

or

$$x = 2.61 \text{ m from } C$$

$$\begin{aligned}\text{B.M.}_{\max} &= 27.2(2.61) - \frac{4}{3}(2.61)^3 \\ &= 47.3 \text{ kN m}\end{aligned}$$

$$\text{B.M. at } A \text{ and } C = 0$$

$$\text{B.M. immediately to left of } B = -(10 \times 3 \times 1.5) = -45 \text{ kN m}$$

At the point of application of the applied moment there will be a sudden change in B.M. of 80 kN m. (There will be no such discontinuity in the S.F. diagram; the effect of the moment will merely be reflected in the values calculated for the reactions.)

The B.M. diagram is therefore as shown in Fig. 3.18.

Problems

3.1 (A). A beam AB , 1.2 m long, is simply-supported at its ends A and B and carries two concentrated loads, one of 10 kN at C , the other 15 kN at D . Point C is 0.4 m from A , point D is 1 m from A . Draw the S.F. and B.M. diagrams for the beam inserting principal values. [9.17, -0.83, -15.83 kN; 3.67, 3.17 kNm.]

3.2 (A). The beam of question 3.1 carries an additional load of 5 kN upwards at point E , 0.6 m from A . Draw the S.F. and B.M. diagrams for the modified loading. What is the maximum B.M.?

[6.67, -3.33, 1.67, -13.33 kN; 2.67, 2, 2.67 kNm.]

3.3 (A). A cantilever beam AB , 2.5 m long is rigidly built in at A and carries vertical concentrated loads of 8 kN at B and 12 kN at C , 1 m from A . Draw S.F. and B.M. diagrams for the beam inserting principal values.

[-8, -20 kN; -11.2, -31.2 kNm.]

3.4 (A). A beam AB , 5 m long, is simply-supported at the end B and at a point C , 1 m from A . It carries vertical loads of 5 kN at A and 20 kN at D , the centre of the span BC . Draw S.F. and B.M. diagrams for the beam inserting principal values.

[$-5, 11.25, -8.75$ kN; $-5, 17.5$ kNm.]

3.5 (A). A beam AB , 3 m long, is simply-supported at A and B . It carries a 16 kN concentrated load at C , 1.2 m from A , and a u.d.l. of 5 kN/m over the remainder of the beam. Draw the S.F. and B.M. diagrams and determine the value of the maximum B.M.

[12.3, $-3.7, -12.7$ kN; 14.8 kNm.]

3.6 (A). A simply supported beam has a span of 4 m and carries a uniformly distributed load of 60 kN/m together with a central concentrated load of 40 kN. Draw the S.F. and B.M. diagrams for the beam and hence determine the maximum B.M. acting on the beam.

[S.F. 140, $\pm 20, -140$ kN; B.M. 0, 160, 0 kNm.]

3.7 (A). A 2 m long cantilever is built-in at the right-hand end and carries a load of 40 kN at the free end. In order to restrict the deflection of the cantilever within reasonable limits an upward load of 10 kN is applied at mid-span. Construct the S.F. and B.M. diagrams for the cantilever and hence determine the values of the reaction force and moment at the support.

[30 kN, 70 kNm.]

3.8 (A). A beam 4.2 m long overhangs each of two simple supports by 0.6 m. The beam carries a uniformly distributed load of 30 kN/m between supports together with concentrated loads of 20 kN and 30 kN at the two ends. Sketch the S.F. and B.M. diagrams for the beam and hence determine the position of any points of contraflexure.

[S.F. $-20, +43, -47, +30$ kN; B.M. $-12, 18.75, -18$ kNm; 0.313 and 2.553 m from l.h. support.]

3.9 (A/B). A beam $ABCDE$, with A on the left, is 7 m long and is simply supported at B and E . The lengths of the various portions are $AB = 1.5$ m, $BC = 1.5$ m, $CD = 1$ m and $DE = 3$ m. There is a uniformly distributed load of 15 kN/m between B and a point 2 m to the right of B and concentrated loads of 20 kN act at A and D with one of 50 kN at C .

(a) Draw the S.F. diagrams and hence determine the position from A at which the S.F. is zero.

(b) Determine the value of the B.M. at this point.

(c) Sketch the B.M. diagram approximately to scale, quoting the principal values.

[3.32 m; 69.8 kNm; 0, $-30, 69.1, 68.1, 0$ kNm.]

3.10 (A/B). A beam $ABCDE$ is simply supported at A and D . It carries the following loading: a distributed load of 30 kN/m between A and B ; a concentrated load of 20 kN at B ; a concentrated load of 20 kN at C ; a concentrated load of 10 kN at E ; a distributed load of 60 kN/m between D and E . Span $AB = 1.5$ m, $BC = CD = DE = 1$ m. Calculate the value of the reactions at A and D and hence draw the S.F. and B.M. diagrams. What are the magnitude and position of the maximum B.M. on the beam?

[41.1, 113.9 kN; 28.15 kNm; 1.37 m from A .]

3.11 (B). A beam, 12 m long, is to be simply supported at 2 m from each end and to carry a u.d.l. of 30 kN/m together with a 30 kN point load at the right-hand end. For ease of transportation the beam is to be jointed in two places, one joint being situated 5 m from the left-hand end. What load (to the nearest kN) must be applied to the left-hand end to ensure that there is no B.M. at the joint (i.e. the joint is to be a point of contraflexure)? What will then be the best position on the beam for the other joint? Determine the position and magnitude of the maximum B.M. present on the beam.

[114 kN, 1.6 m from r.h. reaction; 4.7 m from l.h. reaction; 43.35 kNm.]

3.12 (B). A horizontal beam AB is 4 m long and of constant flexural rigidity. It is rigidly built-in at the left-hand end A and simply supported on a non-yielding support at the right-hand end B . The beam carries uniformly distributed vertical loading of 18 kN/m over its whole length, together with a vertical downward load of 10 kN at 2.5 m from the end A . Sketch the S.F. and B.M. diagrams for the beam, indicating all main values.

[I. Struct. E.] [S.F. 45, $-10, -37.6$ kN; B.M. $-18.6, +36.15$ kNm.]

3.13 (B). A beam ABC , 6 m long, is simply-supported at the left-hand end A and at B 1 m from the right-hand end C . The beam is of weight 100 N/metre run.

(a) Determine the reactions at A and B .

(b) Construct to scales of 20 mm = 1 m and 20 mm = 100 N, the shearing-force diagram for the beam, indicating thereon the principal values.

(c) Determine the magnitude and position of the maximum bending moment. (You may, if you so wish, deduce the answers from the shearing force diagram without constructing a full or partial bending-moment diagram.)

[C.G.] [240 N, 360 N, 288 Nm, 2.4 m from A .]

3.14 (B). A beam $ABCD$, 6 m long, is simply-supported at the right-hand end D and at a point B 1 m from the left-hand end A . It carries a vertical load of 10 kN at A , a second concentrated load of 20 kN at C , 3 m from D , and a uniformly distributed load of 10 kN/m between C and D . Determine:

(a) the values of the reactions at B and D ,

(b) the position and magnitude of the maximum bending moment.

[33 kN, 27 kN, 2.7 m from D , 36.45 kNm.]

3.15 (B). A beam $ABCD$ is simply-supported at B and C with $AB = CD = 2$ m; $BC = 4$ m. It carries a point load of 60 kN at the free end A , a uniformly distributed load of 60 kN/m between B and C and an anticlockwise moment of

80 kN m in the plane of the beam applied at the free end *D*. Sketch and dimension the S.F. and B.M. diagrams, and determine the position and magnitude of the maximum bending moment.

[E.I.E.] [S.F. -60, +170, -70 kN; B.M. -120, +120.1, +80 kN m; 120.1 kN m at 2.83 m to right of *B*.]

3.16 (B). A beam *ABCDE* is 4.6 m in length and loaded as shown in Fig. 3.19. Draw the S.F. and B.M. diagrams for the beam, indicating all major values.

[I.E.I.] [S.F. 28.27, 7.06, -12.94, -30.94, +18, 0; B.M. 28.27, 7.06, 15.53, -10.8.]

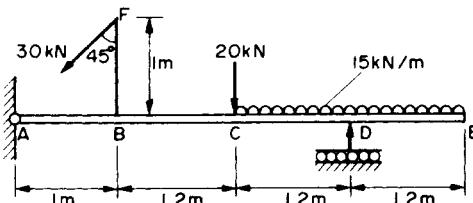


Fig. 3.19.

3.17 (B). A simply supported beam has a span of 6 m and carries a distributed load which varies in a linear manner from 30 kN/m at one support to 90 kN/m at the other support. Locate the point of maximum bending moment and calculate the value of this maximum. Sketch the S.F. and B.M. diagrams.

[U.L.] [3.25 m from l.h. end; 272 kN m.]

3.18 (B). Obtain the relationship between the bending moment, shearing force, and intensity of loading of a laterally loaded beam. A simply supported beam of span *L* carries a distributed load of intensity kx^2/L^2 , where *x* is measured from one support towards the other. Determine: (a) the location and magnitude of the greatest bending moment, (b) the support reactions.

[U. Birm.] [0.0394 kL^2 at 0.63 of span; $kL/12$ and $kL/4$.]

3.19 (B). A beam *ABC* is continuous over two spans. It is built-in at *A*, supported on rollers at *B* and *C* and contains a hinge at the centre of the span *AB*. The loading consists of a uniformly distributed load of total weight 20 kN on the 7 m span *AB* and a concentrated load of 30 kN at the centre of the 3 m span *BC*. Sketch the S.F. and B.M. diagrams, indicating the magnitudes of all important values.

[I.E.I.] [S.F. 5, -15, 26.67, -3.33 kN; B.M. 4.38, -35, +5 kN m.]

3.20 (B). A log of wood 225 mm square cross-section and 5 m in length is rendered impervious to water and floats in a horizontal position in fresh water. It is loaded at the centre with a load just sufficient to sink it completely. Draw S.F. and B.M. diagrams for the condition when this load is applied, stating their maximum values. Take the density of wood as 770 kg/m³ and of water as 1000 kg/m³.

[S.F. 0, ±0.285, 0 kN; B.M. 0, 0.356, 0 kN m.]

3.21 (B). A simply supported beam is 3 m long and carries a vertical load of 5 kN at a point 1 m from the left-hand end. At a section 2 m from the left-hand end a clockwise couple of 3 kN m is exerted, the axis of the couple being horizontal and perpendicular to the longitudinal axis of the beam. Draw to scale the B.M. and S.F. diagrams and mark on them the principal dimensions.

[I.Mech.E.] [S.F. 2.33, -2.67 kN; B.M. 2.33, -0.34, +2.67 kN m.]

CHAPTER 4

BENDING

Summary

The *simple theory of elastic bending* states that

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

where M is the applied bending moment (B.M.) at a transverse section, I is the second moment of area of the beam cross-section about the neutral axis (N.A.), σ is the stress at distance y from the N.A. of the beam cross-section, E is the Young's modulus of elasticity for the beam material, and R is the radius of curvature of the N.A. at the section.

Certain assumptions and conditions must obtain before this theory can strictly be applied: see page 64.

In some applications the following relationship is useful:

$$M = Z\sigma_{\max}$$

where $Z = I/y_{\max}$ and is termed the *section modulus*; σ_{\max} is then the stress at the maximum distance from the N.A.

The most useful standard values of the second moment of area I for certain sections are as follows (Fig. 4.1):

$$\text{rectangle about axis through centroid} = \frac{bd^3}{12} = I_{\text{N.A.}}$$

$$\text{rectangle about axis through side} = \frac{bd^3}{3} = I_{xx}$$

$$\text{circle about axis through centroid} = \frac{\pi D^4}{64} = I_{\text{N.A.}}$$

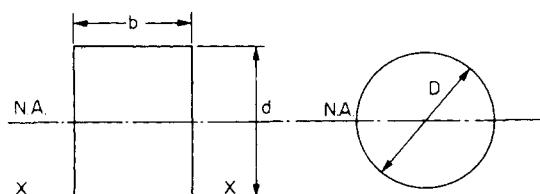


Fig. 4.1.

The *centroid* is the centre of area of the section through which the N.A., or axis of zero stress, is always found to pass.

In some cases it is convenient to determine the second moment of area about an axis other than the N.A. and then to use the *parallel axis theorem*.

$$I_{\text{N.A.}} = I_G + Ah^2$$

For *composite beams* one material is replaced by an equivalent width of the other material given by

$$t' = \frac{E}{E'} t$$

where E/E' is termed the *modular ratio*. The relationship between the stress in the material and its equivalent area is then given by

$$\sigma = \frac{E}{E'} \sigma'$$

For *skew loading of symmetrical sections* the stress at any point (x, y) is given by

$$\sigma = \frac{M_{xx}}{I_{xx}} y \pm \frac{M_{yy}}{I_{yy}} x$$

the angle of the N.A. being given by

$$\tan \theta = \pm \frac{M_{yy}}{M_{xx}} \frac{I_{xx}}{I_{yy}}$$

For *eccentric loading on one axis*,

$$\sigma = \frac{P}{A} \pm \frac{Pey}{I}$$

the N.A. being positioned at a distance

$$y_N = \pm \frac{I}{Ae}$$

from the axis about which the eccentricity is measured.

For *eccentric loading on two axes*,

$$\sigma = \frac{P}{A} \pm \frac{Ph}{I_{yy}} x \pm \frac{Pk}{I_{xx}} y$$

For *concrete or masonry rectangular or circular section columns*, the load must be retained within the middle third or middle quarter areas respectively.

Introduction

If a piece of rubber, most conveniently of rectangular cross-section, is bent between one's fingers it is readily apparent that one surface of the rubber is stretched, i.e. put into tension, and the opposite surface is compressed. The effect is clarified if, before bending, a regular set of lines is drawn or scribed on each surface at a uniform spacing and perpendicular to the axis

of the rubber which is held between the fingers. After bending, the spacing between the set of lines on one surface is clearly seen to increase and on the other surface to reduce. The thinner the rubber, i.e. the closer the two marked faces, the smaller is the effect for the same applied moment. The change in spacing of the lines on each surface is a measure of the strain and hence the stress to which the surface is subjected and it is convenient to obtain a formula relating the stress in the surface to the value of the B.M. applied and the amount of curvature produced. In order for this to be achieved it is necessary to make certain simplifying assumptions, and for this reason the theory introduced below is often termed the simple theory of bending. The assumptions are as follows:

- (1) The beam is initially straight and unstressed.
- (2) The material of the beam is perfectly homogeneous and isotropic, i.e. of the same density and elastic properties throughout.
- (3) The elastic limit is nowhere exceeded.
- (4) Young's modulus for the material is the same in tension and compression.
- (5) Plane cross-sections remain plane before and after bending.
- (6) Every cross-section of the beam is symmetrical about the plane of bending, i.e. about an axis perpendicular to the N.A.
- (7) There is no resultant force perpendicular to any cross-section.

4.1. Simple bending theory

If we now consider a beam initially unstressed and subjected to a constant B.M. along its length, i.e. pure bending, as would be obtained by applying equal couples at each end, it will bend to a radius R as shown in Fig. 4.2b. As a result of this bending the top fibres of the beam will be subjected to tension and the bottom to compression. It is reasonable to suppose, therefore, that somewhere between the two there are points at which the stress is zero. The locus of all such points is termed the *neutral axis*. The radius of curvature R is then measured to this axis. For symmetrical sections the N.A. is the axis of symmetry, but whatever the section the N.A. will always pass through the centre of area or centroid.

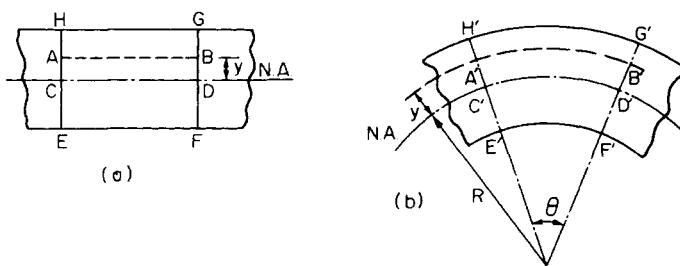


Fig. 4.2. Beam subjected to pure bending (a) before, and (b) after, the moment M has been applied.

Consider now two cross-sections of a beam, HE and GF , originally parallel (Fig. 4.2a). When the beam is bent (Fig. 4.2b) it is assumed that these sections remain plane; i.e. $H'E'$ and $G'F'$, the final positions of the sections, are still straight lines. They will then subtend some angle θ .

Consider now some fibre AB in the material, distance y from the N.A. When the beam is bent this will stretch to $A'B'$.

$$\text{Strain in fibre } AB = \frac{\text{extension}}{\text{original length}} = \frac{A'B' - AB}{AB}$$

But $AB = CD$, and, since the N.A. is unstressed, $CD = C'D'$.

$$\therefore \text{strain} = \frac{A'B' - C'D'}{C'D'} = \frac{(R+y)\theta - R\theta}{R\theta} = \frac{y}{R}$$

But $\frac{\text{stress}}{\text{strain}} = \text{Young's modulus } E$

$$\therefore \text{strain} = \frac{\sigma}{E}$$

Equating the two equations for strain,

$$\frac{\sigma}{E} = \frac{y}{R}$$

or $\frac{\sigma}{y} = \frac{E}{R}$ (4.1)

Consider now a cross-section of the beam (Fig. 4.3). From eqn. (4.1) the stress on a fibre at distance y from the N.A. is

$$\sigma = \frac{E}{R} y$$

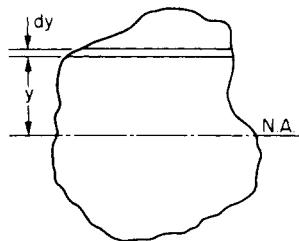


Fig. 4.3. Beam cross-section.

If the strip is of area δA the force on the strip is

$$F = \sigma \delta A = \frac{E}{R} y \delta A$$

This has a moment about the N.A. of

$$Fy = \frac{E}{R} y^2 \delta A$$

The total moment for the whole cross-section is therefore

$$M = \sum \frac{E}{R} y^2 \delta A$$

$$= \frac{E}{R} \Sigma y^2 \delta A$$

since E and R are assumed constant.

The term $\Sigma y^2 \delta A$ is called the *second moment of area* of the cross-section and given the symbol I .

$$\therefore M = \frac{E}{R} I \quad \text{and} \quad \frac{M}{I} = \frac{E}{R} \quad (4.2)$$

Combining eqns. (4.1) and (4.2) we have the bending theory equation

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R} \quad (4.3)$$

From eqn. (4.2) it will be seen that if the beam is of uniform section, the material of the beam is homogeneous and the applied moment is constant, the values of I , E and M remain constant and hence the radius of curvature of the bent beam will also be constant. Thus for pure bending of uniform sections, beams will deflect into circular arcs and for this reason the term *circular bending* is often used. From eqn. (4.2) the radius of curvature to which any beam is bent by an applied moment M is given by:

$$R = \frac{EI}{M}$$

and is thus directly related to the value of the quantity EI . Since the radius of curvature is a direct indication of the degree of flexibility of the beam (the larger the value of R , the smaller the deflection and the greater the rigidity) the quantity EI is often termed the *flexural rigidity* or *flexural stiffness* of the beam. The relative stiffnesses of beam sections can then easily be compared by their EI values.

It should be observed here that the above proof has involved the assumption of pure bending without any shear being present. From the work of the previous chapter it is clear that in most practical beam loading cases shear and bending occur together at most points. Inspection of the S.F. and B.M. diagrams, however, shows that when the B.M. is a maximum the S.F. is, in fact, always zero. It will be shown later that bending produces by far the greatest magnitude of stress in all but a small minority of special loading cases so that beams designed on the basis of the maximum B.M. using the simple bending theory are generally more than adequate in strength at other points.

4.2. Neutral axis

As stated above, it is clear that if, in bending, one surface of the beam is subjected to tension and the opposite surface to compression there must be a region within the beam cross-section at which the stress changes sign, i.e. where the stress is zero, and this is termed the *neutral axis*.

Further, eqn. (4.3) may be re-written in the form

$$\sigma = \frac{M}{I} y \quad (4.4)$$

which shows that at any section the stress is directly proportional to y , the distance from the N.A., i.e. σ varies linearly with y , the maximum stress values occurring in the outside surface of the beam where y is a maximum.

Consider again, therefore, the general beam cross-section of Fig. 4.3 in which the N.A. is located at some arbitrary position. The force on the small element of area is σdA acting perpendicular to the cross-section, i.e. parallel to the beam axis. The total force parallel to the beam axis is therefore $\int \sigma dA$.

Now one of the basic assumptions listed earlier states that when the beam is in equilibrium there can be no resultant force across the section, i.e. the tensile force on one side of the N.A. must exactly balance the compressive force on the other side.

$$\therefore \int \sigma dA = 0$$

Substituting from eqn. (4.1)

$$\int \frac{E}{R} y dA = 0 \quad \text{and hence} \quad \frac{E}{R} \int y dA = 0$$

This integral is the *first moment of area* of the beam cross-section about the N.A. since y is always measured from the N.A. Now the only first moment of area for the cross-section which is zero is that about an axis through the centroid of the section since this is the basic condition required of the centroid. It follows therefore that *the neutral axis must always pass through the centroid*.

It should be noted that this condition only applies with stresses maintained within the elastic range and different conditions must be applied when stresses enter the plastic range of the materials concerned.

Typical stress distributions in bending are shown in Fig. 4.4. It is evident that the material near the N.A. is always subjected to relatively low stresses compared with the areas most removed from the axis. In order to obtain the maximum resistance to bending it is advisable therefore to use sections which have large areas as far away from the N.A. as possible. For this reason beams with I- or T-sections find considerable favour in present engineering applications, such as girders, where bending plays a large part. Such beams have large moments of area about one axis and, provided that it is ensured that bending takes place about this axis, they will have a high resistance to bending stresses.

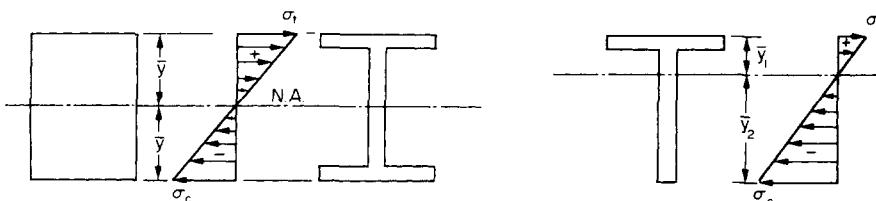


Fig. 4.4. Typical bending stress distributions.

4.3. Section modulus

From eqn. (4.4) the maximum stress obtained in any cross-section is given by

$$\sigma_{\max} = \frac{M}{I} y_{\max} \quad (4.5)$$

For any given allowable stress the maximum moment which can be accepted by a particular shape of cross-section is therefore

$$M = \frac{I}{y_{\max}} \sigma_{\max}$$

For ready comparison of the strength of various beam cross-sections this is sometimes written in the form

$$M = Z \sigma_{\max} \quad (4.6)$$

where $Z (= I/y_{\max})$ is termed the *section modulus*. The higher the value of Z for a particular cross-section the higher the B.M. which it can withstand for a given maximum stress.

In applications such as cast-iron or reinforced concrete where the properties of the material are vastly different in tension and compression two values of maximum allowable stress apply. This is particularly important in the case of unsymmetrical sections such as T-sections where the values of y_{\max} will also be different on each side of the N.A. (Fig. 4.4) and here two values of section modulus are often quoted,

$$Z_1 = I/y_1 \quad \text{and} \quad Z_2 = I/y_2 \quad (4.7)$$

each being then used with the appropriate value of allowable stress.

Standard handbooks† are available which list section modulus values for a range of girders, etc; to enable appropriate beams to be selected for known section modulus requirements.

4.4. Second moment of area

Consider the rectangular beam cross-section shown in Fig. 4.5 and an element of area dA , thickness dy , breadth B and distance y from the N.A. which by symmetry passes through the

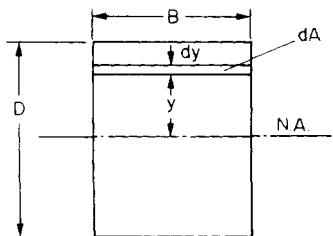


Fig. 4.5.

† *Handbook on Structural Steelwork*. BCSA/CONSTRADO. London, 1971, Supplement 1971, 2nd Supplement 1976 (in accordance with BS449, 'The use of structural steel in building'). *Structural Steelwork Handbook for Standard Metric Sections*. CONSTRADO. London, 1976 (in accordance with BS4848, 'Structural hollow sections').

centre of the section. The *second moment of area* I has been defined earlier as

$$I = \int y^2 dA$$

Thus for the rectangular section the second moment of area about the N.A., i.e. an axis through the centre, is given by

$$\begin{aligned} I_{\text{N.A.}} &= \int_{-D/2}^{D/2} y^2 B dy = B \int_{-D/2}^{D/2} y^2 dy \\ &= B \left[\frac{y^3}{3} \right]_{-D/2}^{D/2} = \frac{BD^3}{12} \end{aligned} \quad (4.8)$$

Similarly, the second moment of area of the rectangular section about an axis through the lower edge of the section would be found using the same procedure but with integral limits of 0 to D ,

$$I = B \left[\frac{y^3}{3} \right]_0^D = \frac{BD^3}{3} \quad (4.9)$$

These standard forms prove very convenient in the determination of $I_{\text{N.A.}}$ values for built-up sections which can be conveniently divided into rectangles. For *symmetrical sections* as, for instance, the I-section shown in Fig. 4.6,

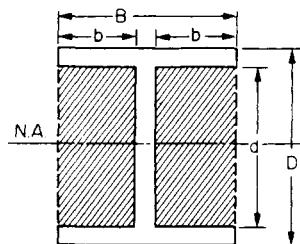


Fig. 4.6.

$$I_{\text{N.A.}} = I \text{ of dotted rectangle} - I \text{ of shaded portions}$$

$$= \frac{BD^3}{12} - 2 \left(\frac{bd^3}{12} \right) \quad (4.10)$$

It will be found that any symmetrical section can be divided into convenient rectangles with the N.A. running through each of their centroids and the above procedure can then be employed to effect a rapid solution.

For *unsymmetrical sections* such as the T-section of Fig. 4.7 it is more convenient to divide the section into rectangles with their *edges* in the N.A., when the second type of standard form may be applied.

$$I_{\text{N.A.}} = I_{ABCD} - I_{\substack{\text{shaded areas} \\ (\text{about } DC)}} + I_{EFGH} \quad (\text{about } HG)$$

(each of these quantities may be written in the form $BD^3/3$).

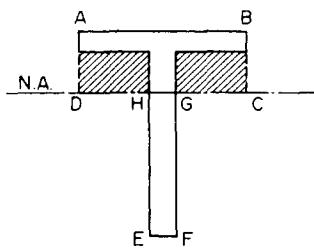


Fig. 4.7.

As an alternative procedure it is possible to determine the second moment of area of each rectangle about an axis through its own centroid ($I_G = BD^3/12$) and to "shift" this value to the equivalent value about the N.A. by means of the *parallel axis theorem*.

$$I_{\text{N.A.}} = I_G + Ah^2 \quad (4.11)$$

where A is the area of the rectangle and h the distance of its centroid G from the N.A. Whilst this is perhaps not so quick or convenient for sections built-up from rectangles, it is often the only procedure available for sections of other shapes, e.g. rectangles containing circular holes.

4.5. Bending of composite or flitched beams

(a) A composite beam is one which is constructed from a combination of materials. If such a beam is formed by rigidly bolting together two timber joists and a reinforcing steel plate, then it is termed a *flitched beam*.

Since the bending theory only holds good when a constant value of Young's modulus applies across a section it cannot be used directly to solve composite-beam problems where two different materials, and therefore different values of E , are present. The method of solution in such a case is to replace one of the materials by an *equivalent section* of the other.

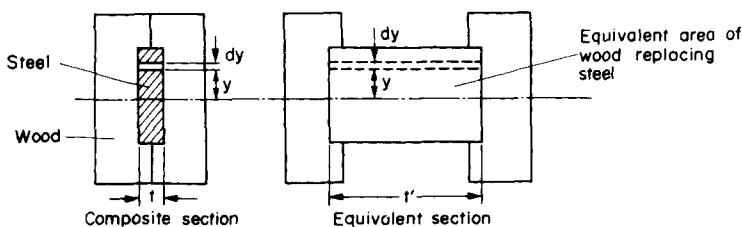


Fig. 4.8. Bending of composite or flitched beams: original beam cross-section and equivalent of uniform material (wood) properties.

Consider, therefore, the beam shown in Fig. 4.8 in which a steel plate is held centrally in an appropriate recess between two blocks of wood. Here it is convenient to replace the steel by an equivalent area of wood, retaining the same bending strength, i.e. the moment at any section must be the same in the equivalent section as in the original so that the force at any given dy in the equivalent beam must be equal to that at the strip it replaces.

$$\begin{aligned}\therefore \sigma t dy &= \sigma' t' dy \\ \sigma t &= \sigma' t' \\ \varepsilon E t &= \varepsilon' E' t'\end{aligned}\tag{4.12}$$

since

$$\frac{\sigma}{\varepsilon} = E$$

Again, for true similarity the strains must be equal,

$$\therefore \varepsilon = \varepsilon'$$

$$\therefore Et = E't' \quad \text{or} \quad \frac{t'}{t} = \frac{E}{E'}\tag{4.13}$$

i.e.

$$t' = \frac{E}{E'} t\tag{4.14}$$

Thus to replace the steel strip by an equivalent wooden strip the thickness must be multiplied by the modular ratio E/E' .

The equivalent section is then one of the same material throughout and the simple bending theory applies. The stress in the wooden part of the original beam is found directly and that in the steel found from the value at the same point in the equivalent material as follows:

$$\text{from eqn. (4.12)} \quad \frac{\sigma}{\sigma'} = \frac{t'}{t}$$

$$\text{and from eqn. (4.13)} \quad \frac{\sigma}{\sigma'} = \frac{E}{E'} \quad \text{or} \quad \sigma = \frac{E}{E'} \sigma'\tag{4.15}$$

$$\text{i.e.} \quad \text{stress in steel} = \text{modular ratio} \times \text{stress in equivalent wood}$$

The above procedure, of course, is not limited to the two materials treated above but applies equally well for any material combination. The wood and steel flitched beam was merely chosen as a convenient example.

4.6. Reinforced concrete beams – simple tension reinforcement

Concrete has a high compressive strength but is very weak in tension. Therefore in applications where tension is likely to result, e.g. bending, it is necessary to reinforce the concrete by the insertion of steel rods. The section of Fig. 4.9a is thus a compound beam and can be treated by reducing it to the equivalent concrete section, shown in Fig. 4.9b.

In calculations, the concrete is assumed to carry no tensile load; hence the gap below the N.A. in Fig. 4.9b. The N.A. is then fixed since it must pass through the centroid of the area assumed in this figure: i.e. moments of area about the N.A. must be zero.

Let t = tensile stress in the steel,

c = compressive stress in the concrete,

A = total area of steel reinforcement,

m = modular ratio, $E_{\text{steel}}/E_{\text{concrete}}$,

other symbols representing the dimensions shown in Fig. 4.9.

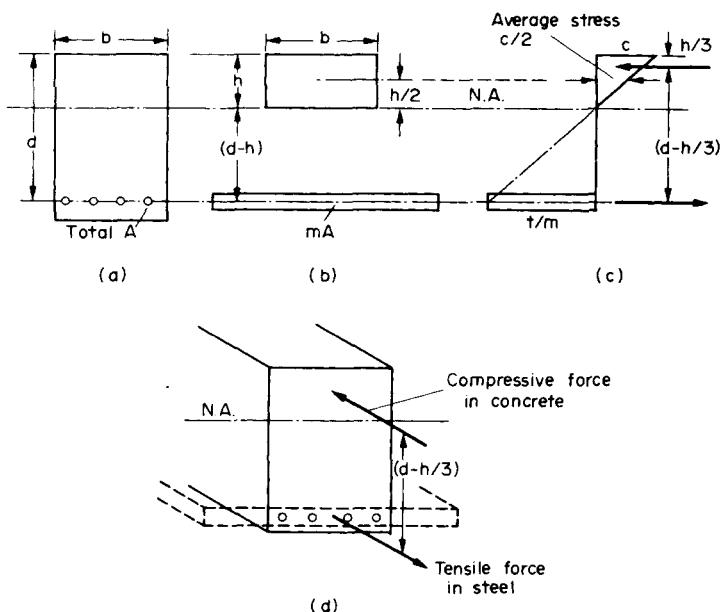


Fig. 4.9. Bending of reinforced concrete beams with simple tension reinforcement.

Then

$$bh \frac{h}{2} = mA(d - h) \quad (4.16)$$

which can be solved for h .

The moment of resistance is then the moment of the couple in Fig. 4.9c and d.

Therefore

moment of resistance (based on compressive forces)

$$M = (bh) \times \frac{c}{2} \times \left(d - \frac{h}{3} \right) = \frac{bhc}{2} \left(d - \frac{h}{3} \right) \quad (4.17)$$

area average stress lever arm

Similarly,

moment of resistance (based on tensile forces)

$$M = mA \times \frac{t}{m} \times \left(d - \frac{h}{3} \right) = At \left(d - \frac{h}{3} \right) \quad (4.18)$$

area stress lever arm

Both t and c are usually given as the maximum allowable values, which may or may not be reached at the same time. Equations (4.17) and (4.18) must both be worked out, therefore, and the lowest value taken, since the larger moment would give a stress greater than the allowed maximum stress in the other material.

In design applications where the dimensions of reinforced concrete beams are required which will carry a known B.M. the above equations generally contain too many unknowns,

and certain simplifications are necessary. It is usual in these circumstances to assume a *balanced* section, i.e. one in which the maximum allowable stresses in the steel and concrete occur simultaneously. There is then no wastage of materials, and for this reason the section is also known as an *economic* or *critical* section.

For this type of section the N.A. is positioned by proportion of the stress distribution (Fig. 4.9c).

Thus by similar triangles

$$\frac{c}{h} = \frac{t/m}{(d-h)}$$

$$mc(d-h) = th \quad (4.19)$$

Thus d can be found in terms of h , and since the moment of resistance is known this relationship can be substituted in eqn. (4.17) to solve for the unknown depth d .

Also, with a balanced section,

moment of resistance (compressive) = moment of resistance (tensile)

$$\begin{aligned} bh\frac{c}{2}\left(d - \frac{h}{3}\right) &= mA \frac{t}{m}\left(d - \frac{h}{3}\right) \\ \frac{bhc}{2} &= At \end{aligned} \quad (4.20)$$

By means of eqn. (4.20) the required total area of reinforcing steel A can thus be determined.

4.7. Skew loading (bending of symmetrical sections about axes other than the axes of symmetry)

Consider the simple rectangular-section beam shown in Fig. 4.10 which is subjected to a load inclined to the axes of symmetry. In such cases bending will take place about an inclined axis, i.e. the N.A. will be inclined at some angle θ to the XX axis and deflections will take place perpendicular to the N.A.

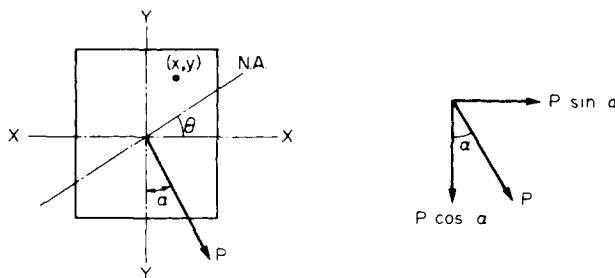


Fig. 4.10. Skew loading of symmetrical section.

In such cases it is convenient to resolve the load P , and hence the applied moment, into its components parallel with the axes of symmetry and to apply the simple bending theory to the resulting bending about both axes. It is thus assumed that simple bending takes place

simultaneously about both axes of symmetry, the total stress at any point (x, y) being given by combining the results of the separate bending actions algebraically using the normal conventions for the signs of the stress, i.e. tension-positive, compression-negative.

Thus

$$\sigma = \frac{M_{xx}}{I_x} y \pm \frac{M_{yy}}{I_{yy}} x \quad (4.21)$$

The equation of the N.A. is obtained by setting eqn. (4.21) to zero,

i.e.

$$\tan \theta = \frac{y}{x} = \pm \frac{M_{yy}}{M_{xx}} \frac{I_{xx}}{I_{yy}} \quad (4.22)$$

4.8. Combined bending and direct stress – eccentric loading

(a) Eccentric loading on one axis

There are numerous examples in engineering practice where tensile or compressive loads on sections are not applied through the centroid of the section and which thus will introduce not only tension or compression as the case may be but also considerable bending effects. In concrete applications, for example, where the material is considerably weaker in tension than in compression, any bending and hence tensile stresses which are introduced can often cause severe problems. Consider, therefore, the beam shown in Fig. 4.11 where the load has been applied at an eccentricity e from one axis of symmetry. The stress at any point is determined by calculating the bending stress at the point on the basis of the simple bending theory and combining this with the direct stress (load/area), taking due account of sign,

i.e.

$$\sigma = \frac{P}{A} \pm \frac{My}{I} \quad (4.23)$$

where

$$M = \text{applied moment} = Pe$$

∴

$$\sigma = \frac{P}{A} \pm \frac{Pey}{I} \quad (4.24)$$

The positive sign between the two terms of the expression is used when both parts have the same effect and the negative sign when one produces tension and the other compression.

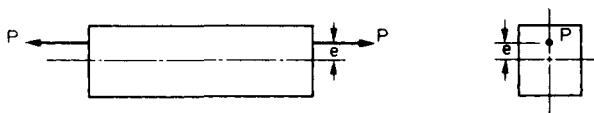


Fig. 4.11. Combined bending and direct stress – eccentric loading on one axis.

It should now be clear that any eccentric load can be treated as precisely equivalent to a direct load acting through the centroid plus an applied moment about an axis through the centroid equal to load \times eccentricity. The distribution of stress across the section is then given by Fig. 4.12.

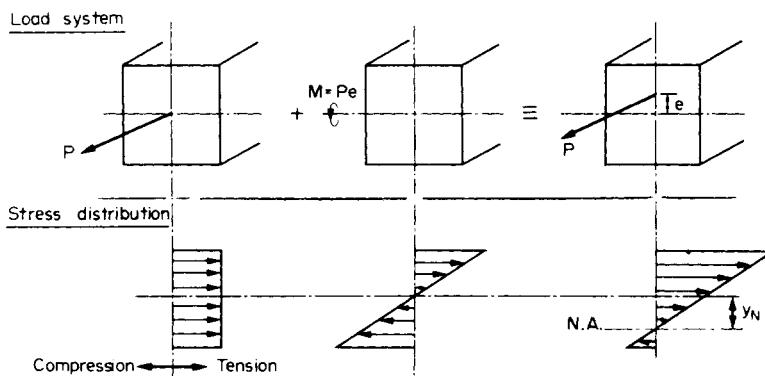


Fig. 4.12. Stress distributions under eccentric loading.

The *equation of the N.A.* can be obtained by setting σ equal to zero in eqn. (4.24),

i.e.
$$y = \pm \frac{I}{Ae} = y_N \quad (4.25)$$

Thus with the load eccentric to one axis the N.A. will be parallel to that axis and a distance y_N from it. The larger the eccentricity of the load the nearer the N.A. will be to the axis of symmetry through the centroid for given values of A and I .

(b) Eccentric loading on two axes

In some cases the applied load will not be applied on either of the axes of symmetry so that there will now be a direct stress effect plus simultaneous bending about both axes. Thus, for the section shown in Fig. 4.13, with the load applied at P with eccentricities of h and k , the total stress at any point (x, y) is given by

$$\sigma = \frac{P}{A} \pm \frac{Phx}{I_{yy}} \pm \frac{Pky}{I_{xx}} \quad (4.26)$$

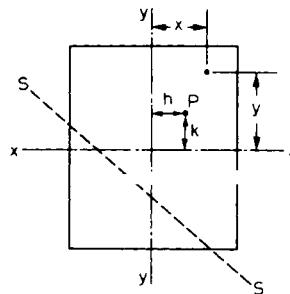


Fig. 4.13. Eccentric loading on two axes showing possible position of neutral axis SS.

Again the *equation of the N.A.* is obtained by equating eqn. (4.26) to zero, when

$$\frac{P}{A} \pm \frac{Phx}{I_{yy}} \pm \frac{Pky}{I_{xx}} = 0$$

or

$$\frac{Ahx}{I_{yy}} \pm \frac{Aky}{I_{xx}} = \pm 1 \quad (4.27)$$

This equation is a linear equation in x and y so that the N.A. is a straight line such as SS which may or may not cut the section.

4.9. "Middle-quarter" and "middle-third" rules

It has been stated earlier that considerable problems may arise in the use of cast-iron or concrete sections in applications in which eccentric loads are likely to occur since both materials are notably weaker in tension than in compression. It is convenient, therefore, that for rectangular and circular cross-sections, provided that the load is applied within certain defined areas, no tension will be produced whatever the magnitude of the applied compressive load. (Here we are solely interested in applications such as column and girder design which are principally subjected to compression.)

Consider, therefore, the rectangular cross-section of Fig. 4.13. The stress at any point (x, y) is given by eqn. (4.26) as

$$\sigma = \frac{P}{A} \pm \frac{Phx}{I_{yy}} \pm \frac{Pky}{I_{xx}}$$

Thus, with a compressive load applied, the most severe tension stresses are introduced when the last two terms have their maximum value and are tensile in effect,

$$\begin{aligned} \text{i.e. } \sigma &= \frac{P}{A} - \frac{Ph}{I_{yy}} \times \frac{b}{2} - \frac{Pk}{I_{xx}} \times \frac{d}{2} \\ &= \frac{P}{bd} - \frac{Phb}{2} \times \frac{12}{db^3} - \frac{Pkd}{2} \times \frac{12}{bd^3} \end{aligned}$$

For no tension to result in the section, σ must be equated to zero,

$$0 = \frac{1}{bd} - \frac{6h}{db^2} - \frac{6k}{bd^2}$$

$$\text{or } \frac{bd}{6} = dh + bk$$

This is a linear expression in h and k producing the line SS in Fig. 4.13. If the load is now applied in each of the other three quadrants the total limiting area within which P must be applied to produce zero tension in the section is obtained. This is the diamond area shown shaded in Fig. 4.14 with diagonals of $b/3$ and $d/3$ and hence termed the *middle third*.

For circular sections of diameter d , whatever the position of application of P , an axis of symmetry will pass through this position so that the problem reduces to one of eccentricity about a single axis of symmetry.

Now from eqn. (4.23)

$$\sigma = \frac{P}{A} + \frac{Pe}{I} y$$

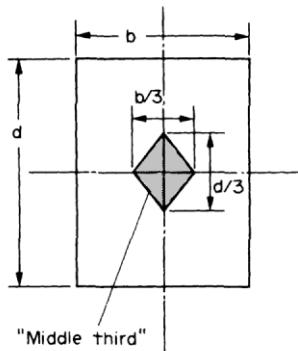


Fig. 4.14. Eccentric loading of rectangular sections – “middle third”.

Therefore for zero tensile stress in the presence of an eccentric compressive load

$$\frac{P}{A} = \frac{Pey}{A}$$

$$\frac{4}{\pi d^2} = e \times \frac{d}{2} \times \frac{64}{\pi d^4}$$

$$e = \frac{d}{8}$$

Thus the limiting region for application of the load is the shaded circular area of diameter $d/4$ (shown in Fig. 4.15) which is termed the *middle quarter*.

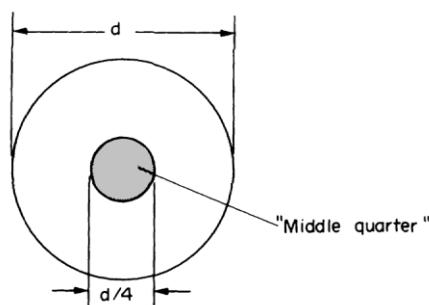


Fig. 4.15. Eccentric loading of circular sections – “middle quarter”.

4.10. Shear stresses owing to bending

It can be shown that any cross-section of a beam subjected to bending by transverse loads experiences not only direct stresses as given by the bending theory but also shear stresses. The magnitudes of these shear stresses at a particular section is always such that they sum up to the total shear force Q at that section. A full treatment of the procedures used to determine the distribution of the shear stresses is given in Chapter 7.

4.11. Strain energy in bending

For beams subjected to bending the total strain energy of the system is given by

$$U = \int_0^L \frac{M^2 ds}{2EI}$$

For uniform beams, or parts of beams, subjected to a constant B.M. M , this reduces to

$$U = \frac{M^2 L}{2EI}$$

In most beam-loading cases the strain energy due to bending far exceeds that due to other forms of loading, such as shear or direct stress, and energy methods of solution using *Castigliano* or *unit load* procedures based on the above equations are extremely powerful methods of solution. These are covered fully in Chapter 11.

4.12. Limitations of the simple bending theory

It has been observed earlier that the theory introduced in preceding sections is often termed the “simple theory of bending” and that it relies on a number of assumptions which either have been listed on page 64 or arise in the subsequent proofs. It should thus be evident that in practical engineering situations the theory will have certain limitations depending on the degree to which these assumptions can be considered to hold true. The following paragraphs give an indication of when some of the more important assumptions can be taken to be valid and when alternative theories or procedures should be applied.

Assumption: *Stress is proportional to the distance from the axis of zero stress (neutral axis), i.e. $\sigma = Ey/R = Ee$.*

Correct for homogeneous beams within the elastic range.

Incorrect (a) for loading conditions outside the elastic range when $\sigma \neq E$,

(b) for composite beams with different materials or properties when ‘equivalent sections’ must be used; see §4.5

Assumption: *Strain is proportional to the distance from the axis of zero strain, i.e. $\epsilon = y/R$.*

Correct for initially straight beams or, for engineering purposes, beams with $R/d > 10$ (where d = total depth of section).

Incorrect for initially curved beams for which special theories have been developed or to which correction factors $\sigma = K (My/I)$ may be applied.

Assumption: *Neutral axis passes through the centroid of the section.*

Correct for pure bending with no axial load.

Incorrect for combined bending and axial load systems such as eccentric loading. In such cases the loading effects must be separated, stresses arising from each calculated and the results superimposed – see §4.8

Assumption: *Plane cross-sections remain plane.*

- Correct** (a) for cross-sections at a reasonable distance from points of local loading or stress concentration (usually taken to be at least one-depth of beam),
 (b) when change of cross-section with length is gradual,
 (c) in the absence of end-condition spurious effects.

These conditions are known as ‘*St Venant’s principle*’.

- Incorrect** (a) for points of local loading;
 (b) at positions of stress concentration such as holes, keyways, fillets and other changes in geometry;
 (c) in regions of rapid change of cross-section.

In such cases appropriate *stress concentration factors*† must be applied or experimental stress/strain analysis techniques adopted.

Assumption: *The axis of the applied bending moment is coincident with the neutral axis.*

- Correct** when the axis of bending is a principal axis ($I_{xy} = 0$) e.g. on axis of symmetry

- Incorrect** for so-called *unsymmetrical bending* cases when the axis of the applied bending moment is not a principal axis.

In such cases the moment should be resolved into components about the principal axes.

Assumption: *Lateral contraction or expansion is not prevented.*

- Correct** when the beam can be considered narrow (i.e. width the same order as the depth).

- Incorrect** for wide beams or plates in which the width may be many times the depth. Special procedures apply for such cases.

It should now be evident that care is required in the application of “simple” theory and reference should be made where necessary to more advanced theories.

Examples

Example 4.1

An I-section girder, 200 mm wide by 300 mm deep, with flange and web of thickness 20 mm is used as a simply supported beam over a span of 7 m. The girder carries a distributed load of 5 kN/m and a concentrated load of 20 kN at mid-span. Determine: (a) the second moment of area of the cross-section of the girder, (b) the maximum stress set-up.

† *Stress Concentration Factors*, R. C. Peterson (Wiley & Sons).

Solution

(a) The second moment of area of the cross-section may be found in two ways.

Method 1 – Use of standard forms

For sections with symmetry about the N.A., use can be made of the standard I value for a rectangle about an axis through its centroid, i.e. $bd^3/12$. The section can thus be divided into convenient rectangles for each of which the N.A. passes through the centroid, e.g. in this case, enclosing the girder by a rectangle (Fig. 4.16).

$$\begin{aligned} I_{\text{girder}} &= I_{\text{rectangle}} - I_{\text{shaded portions}} \\ &= \left[\frac{200 \times 300^3}{12} \right] 10^{-12} - 2 \left[\frac{90 \times 260^3}{12} \right] 10^{-12} \\ &= (4.5 - 2.64) 10^{-4} = 1.86 \times 10^{-4} \text{ m}^4 \end{aligned}$$

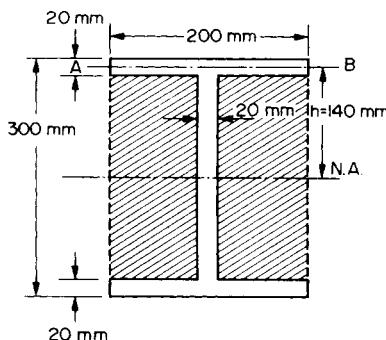


Fig. 4.16.

For sections without symmetry about the N.A., e.g. a T-section, a similar procedure can be adopted, this time dividing the section into rectangles with their edges in the N.A. and applying the standard $I = bd^3/3$ for this condition (see Example 4.2).

Method 2 – Parallel axis theorem

Consider the section divided into three parts—the web and the two flanges.

$$I_{\text{N.A. for the web}} = \frac{bd^3}{12} = \left[\frac{20 \times 260^3}{12} \right] 10^{-12}$$

$$I \text{ of flange about } AB = \frac{bd^3}{12} = \left[\frac{200 \times 20^3}{12} \right] 10^{-12}$$

Therefore using the parallel axis theorem

$$I_{\text{N.A. for flange}} = I_{AB} + Ah^2$$

where h is the distance between the N.A. and AB ,

$$I_{\text{N.A. for flange}} = \left[\frac{200 \times 20^3}{12} \right] 10^{-12} + [(200 \times 20)140^2] 10^{-12}$$

Therefore total $I_{\text{N.A.}}$ of girder

$$\begin{aligned} &= 10^{-12} \left\{ \left[\frac{20 \times 260^3}{12} \right] + 2 \left[\frac{200 \times 20^3}{12} \right] + 200 \times 20 \times 140^2 \right\} \\ &= 10^{-6} (29.3 + 0.267 + 156.8) \\ &= 1.86 \times 10^{-4} \text{ m}^4 \end{aligned}$$

Both methods thus yield the same value and are equally applicable in most cases. Method 1, however, normally yields the quicker solution.

(b) The maximum stress may be found from the simple bending theory of eqn. (4.4), i.e.

$$\sigma_{\max} = \frac{M_{\max} y_{\max}}{I}$$

Now the maximum B.M. for a beam carrying a u.d.l. is at the centre and given by $wL^2/8$. Similarly, the value for the central concentrated load is $WL/4$ also at the centre. Thus, in this case,

$$\begin{aligned} M_{\max} &= \frac{WL}{4} + \frac{WL^2}{8} = \left[\frac{20 \times 10^3 \times 7}{4} \right] + \left[\frac{5 \times 10^3 \times 7^2}{8} \right] \text{ N m} \\ &= (35.0 + 30.63)10^3 = 65.63 \text{ kN m} \\ \therefore \sigma_{\max} &= \frac{65.63 \times 10^3 \times 150 \times 10^{-3}}{1.9 \times 10^{-4}} = 51.8 \text{ MN/m}^2 \end{aligned}$$

The maximum stress in the girder is 52 MN/m^2 , this value being compressive on the upper surface and tensile on the lower surface.

Example 4.2

A uniform T-section beam is 100 mm wide and 150 mm deep with a flange thickness of 25 mm and a web thickness of 12 mm. If the limiting bending stresses for the material of the beam are 80 MN/m^2 in compression and 160 MN/m^2 in tension, find the maximum u.d.l. that the beam can carry over a simply supported span of 5 m.

Solution

The second moment of area value I used in the simple bending theory is that about the N.A. Thus, in order to determine the I value of the T-section shown in Fig. 4.17, it is necessary first to position the N.A.

Since this always passes through the centroid of the section we can take moments of area about the base to determine the position of the centroid and hence the N.A.

Thus

$$\begin{aligned} (100 \times 25 \times 137.5)10^{-9} + (125 \times 12 \times 62.5)10^{-9} &= 10^{-6} [(100 \times 25) + (125 \times 12)\bar{y}] \\ (343750 + 93750)10^{-9} &= 10^{-6}(2500 + 1500)\bar{y} \\ \bar{y} &= \frac{437.5 \times 10^{-6}}{4000 \times 10^{-6}} = 109.4 \times 10^{-3} = 109.4 \text{ mm.} \end{aligned}$$

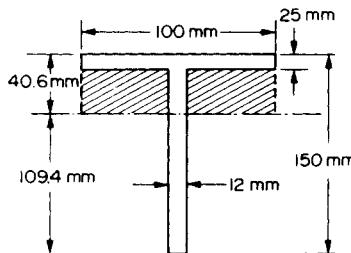


Fig. 4.17.

Thus the N.A. is positioned, as shown, a distance of 109.4 mm above the base. The second moment of area I can now be found as suggested in Example 4.1 by dividing the section into convenient rectangles with their edges in the neutral axis.

$$\begin{aligned} I &= \frac{1}{3}[(100 \times 40.6^3) - (88 \times 15.6^3) + (12 \times 109.4^3)] 10^{-12} \\ &= \frac{1}{3}(6.69 - 0.33 + 15.71) 10^{-6} = 7.36 \times 10^{-6} \text{ m}^4 \end{aligned}$$

Now the maximum compressive stress will occur on the upper surface where $y = 40.6$ mm, and, using the limiting compressive stress value quoted,

$$M = \frac{\sigma I}{y} = \frac{80 \times 10^6 \times 7.36 \times 10^{-6}}{40.6 \times 10^{-3}} = 14.5 \text{ kN m}$$

This suggests a maximum allowable B.M. of 14.5 kN m. It is now necessary, however, to check the tensile stress criterion which must apply on the lower surface,

$$\text{i.e. } M = \frac{\sigma I}{y} = \frac{160 \times 10^6 \times 7.36 \times 10^{-6}}{109.4 \times 10^{-3}} = 10.76 \text{ kN m}$$

The greatest moment that can therefore be applied to retain stresses within both conditions quoted is therefore $M = 10.76$ kN m.

But for a simply supported beam with u.d.l.,

$$\begin{aligned} M_{\max} &= \frac{wL^2}{8} \\ w &= \frac{8M}{L^2} = \frac{8 \times 10.76 \times 10^3}{5^2} \\ &= 3.4 \text{ kN/m} \end{aligned}$$

The u.d.l. must be limited to 3.4 kN m.

Example 4.3

A flitched beam consists of two 50 mm × 200 mm wooden beams and a 12 mm × 80 mm steel plate. The plate is placed centrally between the wooden beams and recessed into each so that, when rigidly joined, the three units form a 100 mm × 200 mm section as shown in Fig. 4.18. Determine the moment of resistance of the flitched beam when the maximum

bending stress in the timber is 12 MN/m^2 . What will then be the maximum bending stress in the steel?

For steel $E = 200 \text{ GN/m}^2$; for wood $E = 10 \text{ GN/m}^2$.

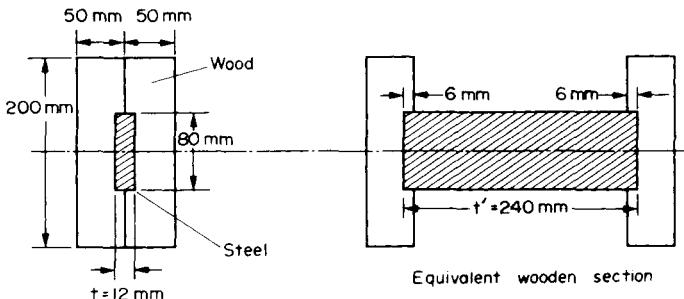


Fig. 4.18.

Solution

The flitched beam may be considered replaced by the equivalent wooden section shown in Fig. 4.18. The thickness t' of the wood equivalent to the steel which it replaces is given by eqn. (4.14),

$$t' = \frac{E}{E'} t = \frac{200 \times 10^9}{10 \times 10^9} \times 12 = 240 \text{ mm}$$

Then, for the equivalent section

$$\begin{aligned} I_{\text{N.A.}} &= 2 \left[\frac{50 \times 200^3}{12} \right] - 2 \left[\frac{6 \times 80^3}{12} \right] + \left[\frac{240 \times 80^3}{12} \right] 10^{-12} \\ &= (66.67 - 0.51 + 10.2) 10^{-6} = 76.36 \times 10^{-6} \text{ m}^4 \end{aligned}$$

Now the maximum stress in the timber is 12 MN/m^2 , and this will occur at $y = 100 \text{ mm}$; thus, from the bending theory,

$$M = \frac{\sigma I}{y} = \frac{12 \times 10^6 \times 76.36 \times 10^{-6}}{100 \times 10^{-3}} = 9.2 \text{ kN m}$$

The moment of resistance of the beam, i.e. the bending moment which the beam can withstand within the given limit, is 9.2 kN m .

The maximum stress in the steel with this moment applied is then determined by finding first the maximum stress in the equivalent wood at the same position, i.e. at $y = 40 \text{ mm}$.

Therefore maximum stress in equivalent wood

$$\sigma'_{\max} = \frac{My}{I} = \frac{9.2 \times 10^3 \times 40 \times 10^{-3}}{76.36 \times 10^{-6}} = 4.82 \times 10^6 \text{ N/m}^2$$

Therefore from eqn. (4.15), the maximum stress in the steel is given by

$$\begin{aligned}\sigma_{\max} &= \frac{E}{E'} \sigma'_{\max} = \frac{200 \times 10^9}{10 \times 10^9} \times 4.82 \times 10^6 \\ &= 96 \times 10^6 = 96 \text{ MN/m}^2\end{aligned}$$

Example 4.4

(a) A reinforced concrete beam is 240 mm wide and 450 mm deep to the centre of the reinforcing steel rods. The rods are of total cross-sectional area $1.2 \times 10^{-3} \text{ m}^2$ and the maximum allowable stresses in the steel and concrete are 150 MN/m^2 and 8 MN/m^2 respectively. The modular ratio (steel : concrete) is 16. Determine the moment of resistance of the beam.

(b) If, after installation, it is required to up-rate the service loads by 30 % and to replace the above beam with a second beam of increased strength but retaining the same width of 240 mm, determine the new depth and area of steel for tension reinforcement required.

Solution

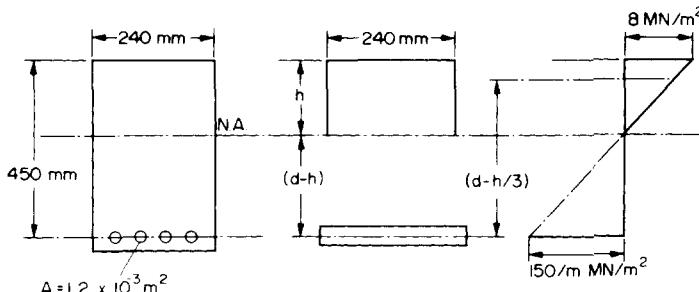


Fig. 4.19.

(a) From eqn. (4.16) moments of area about the N.A. of Fig. 4.19.

$$\begin{aligned}10^{-9} \left(240 \times h \times \frac{h}{2} \right) &= 16 \times 1.2 \times 10^{-3} (450 - h) 10^{-3} \\ 120h^2 &= (8640 - 19.2h) 10^3 \\ h^2 + 160h - 72000 &= 0\end{aligned}$$

From which

$$h = 200 \text{ mm}$$

Substituting in eqn. (4.17),

$$\begin{aligned}\text{moment of resistance (compressive)} &= (240 \times 200 \times 10^{-6}) \frac{8 \times 10^6}{2} (450 - 66.7) 10^{-3} \\ &= 73.6 \text{ kN m}\end{aligned}$$

and from eqn. (4.18)

$$\text{moment of resistance (tensile)} = (16 \times 1.2 \times 10^{-3}) \frac{150 \times 10^6}{16} (450 - 66.7) 10^{-3} \\ = 69.0 \text{ kN m}$$

Thus the safe moment which the beam can carry within *both* limiting stress values is **69 kN m**.

(b) For this part of the question the dimensions of the new beam are required and it is necessary to assume a *critical* or *economic* section. The position of the N.A. is then determined from eqn. (4.19) by consideration of the proportions of the stress distribution (i.e. assuming that the maximum stresses in the steel and concrete occur together).

Thus from eqn. (4.19)

$$\frac{h}{d} = \frac{1}{1 + \frac{t}{mc}} = \frac{1}{1 + \frac{150 \times 10^6}{16 \times 8 \times 10^6}} = 0.46$$

$$\text{From (4.17)} \quad M = \frac{bhc}{2} \left(d - \frac{h}{3} \right) = \frac{h}{2d} \left(1 - \frac{h}{3d} \right) c bd^2$$

Substituting for $\frac{h}{d} = 0.46$ and solving for d gives

$$d = 0.49 \text{ m}$$

$$\therefore h = 0.46 \times 0.49 = 0.225 \text{ m}$$

$$\therefore \text{From (4.20)} \quad A = \frac{0.24 \times 0.225 \times 8 \times 10^6}{2 \times 150 \times 10^6}$$

$$\text{i.e.} \quad A = 1.44 \times 10^{-3} \text{ m}^2$$

Example 4.5

(a) A rectangular masonry column has a cross-section $500 \text{ mm} \times 400 \text{ mm}$ and is subjected to a vertical compressive load of 100 kN applied at point P shown in Fig. 4.20. Determine the value of the maximum stress produced in the section. (b) Is the section *at any point* subjected to tensile stresses?

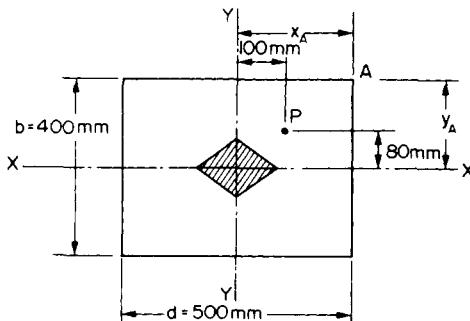


Fig. 4.20.

Solution

In this case the load is eccentric to both the XX and YY axes and bending will therefore take place simultaneously about both axes.

$$\text{Moment about } XX = 100 \times 10^3 \times 80 \times 10^{-3} = 8000 \text{ N m}$$

$$\text{Moment about } YY = 100 \times 10^3 \times 100 \times 10^{-3} = 10000 \text{ N m}$$

Therefore from eqn. (4.26) the maximum stress in the section will be compressive at point A since at this point the compressive effects of bending about both XX and YY add to the direct compressive stress component due to P ,

$$\begin{aligned} \text{i.e. } \sigma_{\max} &= - \left[\frac{P}{A} + \frac{M_{xx}y_A}{I_{xx}} + \frac{M_{yy}x_A}{I_{yy}} \right] \\ &= - \left[\frac{100 \times 10^3}{500 \times 400 \times 10^{-6}} + \frac{8000 \times 200 \times 10^{-3} \times 12}{(500 \times 400^3)10^{-12}} \right. \\ &\quad \left. + \frac{10000 \times 250 \times 10^{-3} \times 12}{(400 \times 500^3)10^{-12}} \right] \\ &= -(0.5 + 0.6 + 0.6)10^6 = -1.7 \text{ MN/m}^2 \end{aligned}$$

For the section to contain no tensile stresses, P must be applied within the middle third. Now since $b/3 = 133 \text{ mm}$ and $d/3 = 167 \text{ mm}$ it follows that the maximum possible values of the coordinates x or y for P are $y = \frac{1}{2} \times 133 = 66.5 \text{ mm}$ and $x = \frac{1}{2} \times 167 = 83.5 \text{ mm}$.

The given position for P lies outside these values so that tensile stresses will certainly exist in the section.

(The full middle-third area is in fact shown in Fig. 4.20 and P is clearly outside this area.)

Example 4.6

The crank of a motor vehicle engine has the section shown in Fig. 4.21 along the line AA . Derive an expression for the stress at any point on this section with the con-rod thrust P

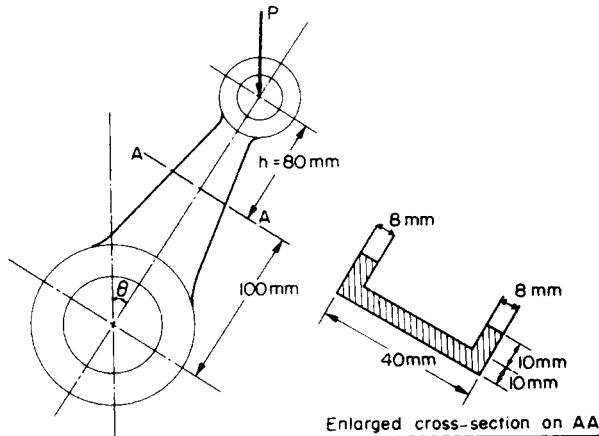


Fig. 4.21.

applied at some angle θ as shown. Hence, if the maximum tensile stress in the section is not to exceed 100 MN/m^2 , determine the maximum value of P which can be permitted with $\theta = 60^\circ$. What will be the distribution of stress along the section AA with this value of P applied?

Solution

Assuming that the load P is applied in the plane of the crank the stress at any point along the section AA will be the result of (a) a direct compressive load of magnitude $P \cos \theta$, and (b) a B.M. in the plane of the crank of magnitude $P \sin \theta \cdot h$; i.e. stress at any point along AA , distance s from the centre-line, is given by eqn. (4.26) as

$$\begin{aligned}\sigma &= \frac{P \cos \theta}{A} \pm \frac{(P \sin \theta \cdot h)}{I_{AA}} \cdot s \\ &= \frac{P \cos \theta}{A} \left[1 \pm \frac{hs \tan \theta}{k_{AA}^2} \right]\end{aligned}$$

where k_{AA} is the radius of gyration of the section AA about its N.A.

$$\text{Now } A = [(2 \times 20 \times 8) + (24 \times 10)] 10^{-6} = 560 \times 10^{-6} \text{ m}^2$$

$$\text{and } I_{\text{N.A.}} = \frac{1}{12} [20 \times 40^3 - 10 \times 24^3] 10^{-12} = 9.51 \times 10^{-8} \text{ m}^4$$

$$\begin{aligned}\therefore \sigma &= -P \left[\frac{0.5}{560 \times 10^{-6}} \pm \frac{0.866 \times 80 \times 10^{-3} s \times 10^{-3}}{9.51 \times 10^{-8}} \right] \\ &= -P [0.893 \pm 0.729s] 10^3 \text{ N/m}^2\end{aligned}$$

where s is measured in millimetres,

$$\begin{aligned}\text{i.e. } \text{maximum tensile stress} &= P[-0.893 + 0.729 \times 20] 10^3 \text{ N/m}^2 \\ &= 13.69P \text{ kN/m}^2\end{aligned}$$

In order that this stress shall not exceed 100 MN/m^2

$$100 \times 10^6 = 13.69 \times P \times 10^3$$

$$P = 7.3 \text{ kN}$$

With this value of load applied the direct stress on the section will be

$$-0.893P \times 10^3 = -6.52 \text{ MN/m}^2$$

and the bending stress at each edge

$$\pm 0.729 \times 20 \times 10^3 P = 106.4 \text{ MN/m}^2$$

The stress distribution along AA is then obtained as shown in Fig. 4.22.

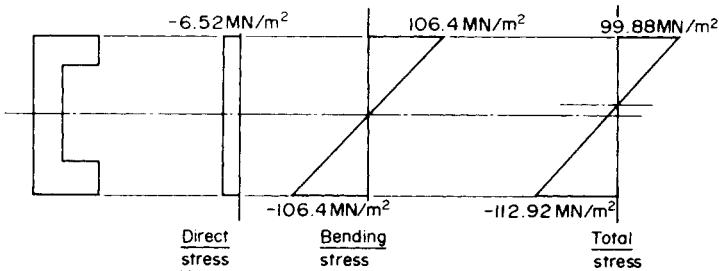


Fig. 4.22.

Problems

- 4.1 (A).** Determine the second moments of area about the axes XX for the sections shown in Fig. 4.23.
[15.69, 7.88, 41.15, 24; all $\times 10^{-6} \text{ m}^4$.]

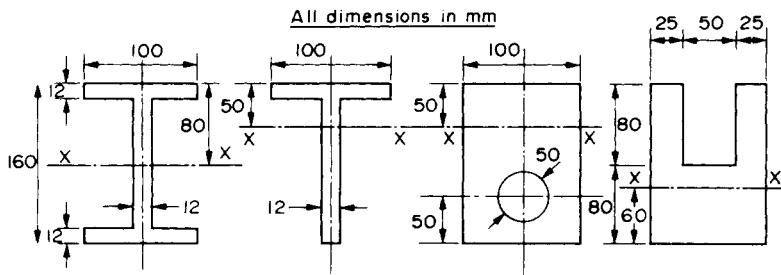


Fig. 4.23.

- 4.2 (A).** A rectangular section beam has a depth equal to twice its width. It is the same material and mass per unit length as an I-section beam 300 mm deep with flanges 25 mm thick and 150 mm wide and a web 12 mm thick. Compare the flexural strengths of the two beams. [8.59 : 1.]

- 4.3 (A).** A conveyor beam has the cross-section shown in Fig. 4.24 and it is subjected to a bending moment in the plane YY . Determine the maximum permissible bending moment which can be applied to the beam (a) for bottom flange in tension, and (b) for bottom flange in compression, if the safe stresses for the material in tension and compression are 30 MN/m^2 and 150 MN/m^2 respectively. [32.3, 84.8 kN m.]

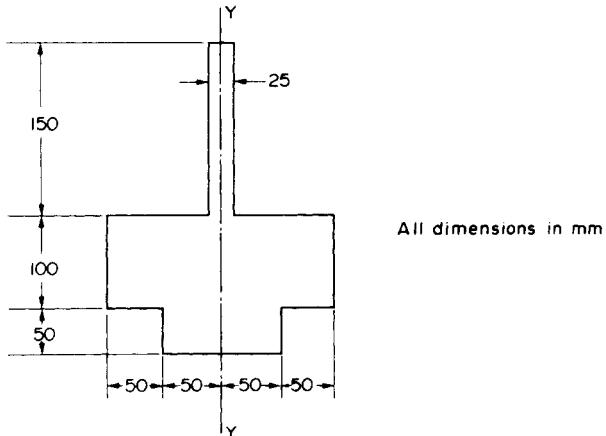


Fig. 4.24.

4.4 (A/B). A horizontal steel girder has a span of 3 m and is built-in at the left-hand end and freely supported at the other end. It carries a uniformly distributed load of 30 kN/m over the whole span, together with a single concentrated load of 20 kN at a point 2 m from the left-hand end. The supporting conditions are such that the reaction at the left-hand end is 65 kN.

- Determine the bending moment at the left-hand end and draw the B.M. diagram.
- Give the value of the maximum bending moment.
- If the girder is 200 mm deep and has a second moment of area of $40 \times 10^{-6} \text{ m}^4$ determine the maximum stress [I.Mech.E.] [40 kN m; 100 MN/m².]

4.5 (A/B). Figure 4.25 represents the cross-section of an extruded alloy member which acts as a simply supported beam with the 75 mm wide flange at the bottom. Determine the moment of resistance of the section if the maximum permissible stresses in tension and compression are respectively 60 MN/m² and 45 MN/m².

[I.E.I.] [2.62 kN m.]

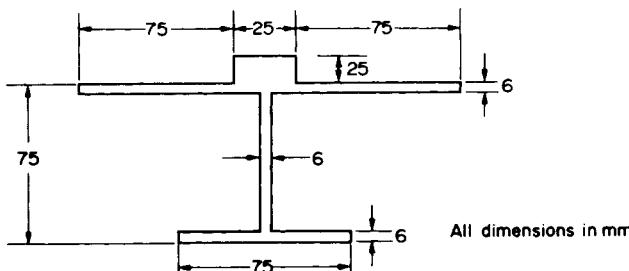


Fig. 4.25.

4.6 (A/B). A trolley consists of a pressed steel section as shown in Fig. 4.26. At each end there are rollers at 350 mm centres.

If the trolley supports a mass of 50 kg evenly distributed over the 350 mm length of the trolley calculate, using the data given in Fig. 4.26, the maximum compressive and tensile stress due to bending in the pressed steel section. State clearly your assumptions.

[C.G.] [14.8, 42.6 MN/m²]

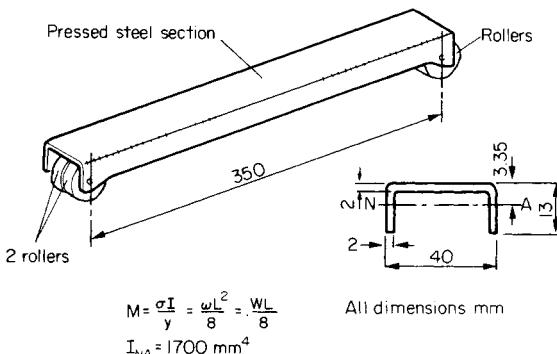


Fig. 4.26.

4.7 (A/B). The channel section of Fig. 4.21 is used as a simply-supported beam over a span of 2.8 m. The channel is used as a guide for a roller of an overhead crane gantry and can be expected to support a maximum load (taken to be a concentrated point load) of 40 kN. At what position of the roller will the bending moment of the channel be a maximum and what will then be the maximum tensile bending stress? If the maximum allowable stress for the material of the beam is 320 MN/m² what safety factor exists for the given loading condition.

[Centre, 79.5 MN/m², 4]

4.8 (A/B). A $120 \times 180 \times 15$ mm uniform I-section steel girder is used as a cantilever beam carrying a uniformly distributed load ω kN/m over a span of 2.4 m. Determine the maximum value of ω which can be applied before yielding of the outer fibres of the beam cross-section commences. In order to strengthen the girder, steel plates are attached to the outer surfaces of the flanges to double their effective thickness. What width of plate should be added (to the nearest mm) in order to reduce the maximum stress by 30%? The yield stress for the girder material is 320 MN/m². [35.5 kN/m, 67 mm]

4.9 (A/B). A 200 mm wide \times 300 mm deep timber beam is reinforced by steel plates 200 mm wide \times 12 mm deep on the top and bottom surfaces as shown in Fig. 4.27. If the maximum allowable stresses for the steel and timber are 120 MN/m² and 8 MN/m² respectively, determine the maximum bending moment which the beam can safely carry.

For steel $E = 200$ GN/m²; for timber $E = 10$ GN/m².

[I.Mech.E.] [103.3 kN m.]

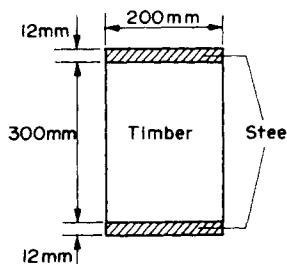


Fig. 4.27.

4.10 (A/B). A composite beam is of the construction shown in Fig. 4.28. Calculate the allowable u.d.l. that the beam can carry over a simply supported span of 7 m if the stresses are limited to 120 MN/m² in the steel and 7 MN/m² in the timber.

Modular ratio = 20.

[1.13 kN/m.]

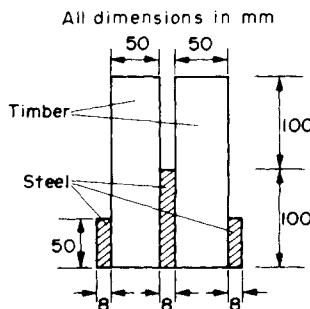


Fig. 4.28.

4.11 (A/B). Two bars, one of steel, the other of aluminium alloy, are each of 75 mm width and are rigidly joined together to form a rectangular bar 75 mm wide and of depth $(t_s + t_A)$, where t_s = thickness of steel bar and t_A = thickness of alloy bar.

Determine the ratio of t_s to t_A in order that the neutral axis of the compound bar is coincident with the junction of the two bars. ($E_s = 210$ GN/m²; $E_A = 70$ GN/m².)

If such a beam is 50 mm deep determine the maximum bending moment the beam can withstand if the maximum stresses in the steel and alloy are limited to 135 MN/m² and 37 MN/m² respectively. [0.577; 1.47 kN m.]

4.12 (A/B). A brass strip, 50 mm \times 12 mm in section, is riveted to a steel strip, 65 mm \times 10 mm in section, to form a compound beam of total depth 22 mm, the brass strip being on top and the beam section being symmetrical about the vertical axis. The beam is simply supported on a span of 1.3 m and carries a load of 2 kN at mid-span.

- (a) Determine the maximum stresses in each of the materials owing to bending.
 (b) Make a diagram showing the distribution of bending stress over the depth of the beam.

Take E for steel = 200 GN/m² and E for brass = 100 GN/m².

[U.L.] [$\sigma_b = 130 \text{ MN/m}^2$; $\sigma_s = 162.9 \text{ MN/m}^2$.]

- 4.13 (B).** A concrete beam, reinforced in tension only, has a rectangular cross-section, width 200 mm and effective depth to the tensile steel 500 mm, and is required to resist a bending moment of 70 kN m. Assuming a modular ratio of 15, calculate (a) the minimum area of reinforcement required if the stresses in steel and concrete are not to exceed 190 MN/m² and 8 MN/m² respectively, and (b) the stress in the non-critical material when the bending moment is applied.

[E.I.E.] [$0.916 \times 10^{-3} \text{ m}^2$; 177 MN/m².]

- 4.14 (B).** A reinforced concrete beam of rectangular cross-section, $b = 200 \text{ mm}$, d (depth to reinforcement) = 300 mm, is reinforced in tension only, the steel ratio, i.e. the ratio of reinforcing steel area to concrete area (neglecting cover), being 1%. The maximum allowable stresses in concrete and steel are 8 MN/m² and 135 MN/m² respectively. The modular ratio may be taken as equal to 15. Determine the moment of resistance capable of being developed in the beam.

[I.Struct.E.] [20.9 kN m.]

- 4.15 (B).** A rectangular reinforced concrete beam is 200 mm wide and 350 mm deep to reinforcement, the latter consisting of three 20 mm diameter steel rods. If the following stresses are not to be exceeded, calculate: (a) the maximum bending moment which can be sustained, and (b) the steel stress and the maximum concrete stress when the section is subjected to this maximum moment.

Maximum stress in concrete in bending not to exceed 8 MN/m².

Maximum steel stress not to exceed 150 MN/m².

Modular ratio $m = 15$.

[I.Struct.E.] [38.5 kN m; 138, 8 MN/m².]

- 4.16 (B).** A reinforced concrete beam has to carry a bending moment of 100 kN m. The maximum permissible stresses are 8 MN/m² and 135 MN/m² in the concrete and steel respectively. The beam is to be of rectangular cross-section 300 mm wide. Design a suitable section with "balanced" reinforcement if $E_{\text{steel}}/E_{\text{concrete}} = 12$.

[I.Mech.E.] [$d = 482.4 \text{ mm}$; $A = 1.782 \times 10^{-3} \text{ m}^2$.]

CHAPTER 5

SLOPE AND DEFLECTION OF BEAMS

Summary

The following relationships exist between loading, shearing force (S.F.), bending moment (B.M.), slope and deflection of a beam:

$$\text{deflection} = y \quad (\text{or } \delta)$$

$$\text{slope} = i \text{ or } \theta = \frac{dy}{dx}$$

$$\text{bending moment} = M = EI \frac{d^2y}{dx^2}$$

$$\text{shearing force} = Q = EI \frac{d^3y}{dx^3}$$

$$\text{loading} = w = EI \frac{d^4y}{dx^4}$$

In order that the above results should agree mathematically the sign convention illustrated in Fig. 5.4 must be adopted.

Using the above formulae the following standard values for *maximum slopes* and *deflections* of simply supported beams are obtained. (These assume that the beam is uniform, i.e. EI is constant throughout the beam.)

MAXIMUM SLOPE AND DEFLECTION OF SIMPLY SUPPORTED BEAMS

Loading condition	Maximum slope	Deflection (y)	Max. deflection (y_{\max})
Cantilever with concentrated load W at end	$\frac{WL^2}{2EI}$	$\frac{W}{6EI}[2L^3 - 3L^2x + x^3]$	$\frac{WL^3}{3EI}$
Cantilever with u.d.l. across the complete span	$\frac{wL^3}{6EI}$	$\frac{w}{24EI}[3L^4 - 4L^3x + x^4]$	$\frac{wL^4}{8EI}$
Simply supported beam with concentrated load W at the centre	$\frac{WL^2}{16EI}$	$\frac{Wx}{48EI}[3L^2 - 4x^2]$	$\frac{WL^3}{48EI}$
Simply supported beam with u.d.l. across complete span	$\frac{wL^3}{24EI}$	$\frac{wx}{24EI}[L^3 - 2Lx^2 + x^3]$	$\frac{5wL^4}{384EI}$
Simply supported beam with concentrated load W offset from centre (distance a from one end b from the other)	$0.062 \frac{WL^2}{EI}$	$\frac{Wa}{3EI} \left[\frac{L^2 - a^2}{3} \right]^{3/2}$	

Here L is the length of span, EI is known as the flexural rigidity of the member and x for the cantilevers is measured from the free end.

The determination of beam slopes and deflections by simple integration or Macaulay's methods requires a knowledge of certain conditions for various loading systems in order that the constants of integration can be evaluated. They are as follows:

- (1) Deflections at supports are assumed zero unless otherwise stated.
- (2) Slopes at built-in supports are assumed zero unless otherwise stated.
- (3) Slope at the centre of symmetrically loaded and supported beams is zero.
- (4) Bending moments at the free ends of a beam (i.e. those not built-in) are zero.

Mohr's theorems for slope and deflection state that if A and B are two points on the deflection curve of a beam and B is a point of zero slope, then

$$(1) \quad \text{slope at } A = \text{area of } \frac{M}{EI} \text{ diagram between } A \text{ and } B$$

For a uniform beam, EI is constant, and the above equation reduces to

$$\text{slope at } A = \frac{1}{EI} \times \text{area of B.M. diagram between } A \text{ and } B$$

N.B.—If B is not a point of zero slope the equation gives the change of slope between A and B .

$$(2) \quad \text{Total deflection of } A \text{ relative to } B = \text{first moment of area of } \frac{M}{EI} \text{ diagram about } A$$

For a uniform beam

$$\text{total deflection of } A \text{ relative to } B = \frac{1}{EI} \times \text{first moment of area of B.M. diagram about } A$$

Again, if B is not a point of zero slope the equation only gives the deflection of A relative to the tangent drawn at B .

Useful quantities for use with uniformly distributed loads are shown in Fig. 5.1.

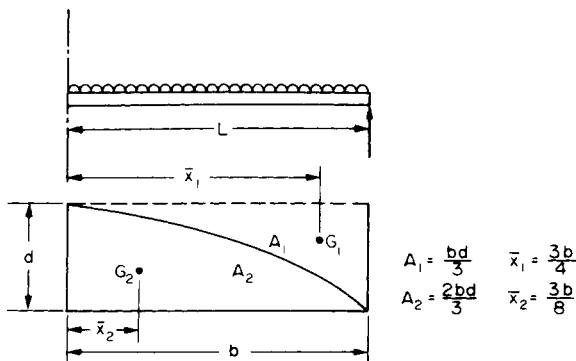


Fig. 5.1.

Both the straightforward integration method and Macaulay's method are based on the relationship $M = EI \frac{d^2y}{dx^2}$ (see § 5.2 and § 5.3).

Clapeyron's equations of three moments for continuous beams in its simplest form states that for any portion of a beam on three supports 1, 2 and 3, with spans between of L_1 and L_2 , the bending moments at the supports are related by

$$-M_1 L_1 - 2M_2 (L_1 + L_2) - M_3 L_2 = 6 \left[\frac{A_1 \bar{x}_1}{L_1} + \frac{A_2 \bar{x}_2}{L_2} \right]$$

where A_1 is the area of the B.M. diagram, assuming span L_1 simply supported, and \bar{x}_1 is the distance of the centroid of this area from the left-hand support. Similarly, A_2 refers to span L_2 , with \bar{x}_2 the centroid distance from the right-hand support (see Examples 5.6 and 5.7). The following standard results are useful for $\frac{6A\bar{x}}{L}$:

- (a) Concentrated load W , distance a from the nearest outside support

$$\frac{6A\bar{x}}{L} = \frac{Wa}{L} (L^2 - a^2)$$

- (b) Uniformly distributed load w

$$\frac{6A\bar{x}}{L} = \frac{wL^3}{4} \quad (\text{see Example 5.6})$$

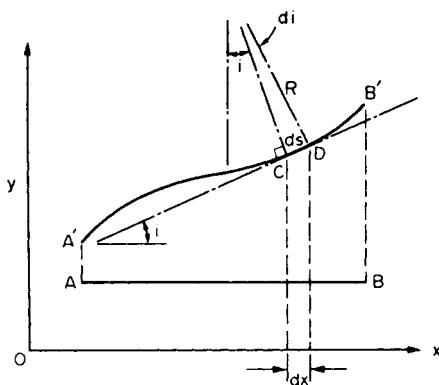
Introduction

In practically all engineering applications limitations are placed upon the performance and behaviour of components and normally they are expected to operate within certain set limits of, for example, stress or deflection. The stress limits are normally set so that the component does not yield or fail under the most severe load conditions which it is likely to meet in service. In certain structural or machine linkage designs, however, maximum stress levels may not be the most severe condition for the component in question. In such cases it is the limitation in the maximum deflection which places the most severe restriction on the operation or design of the component. It is evident, therefore, that methods are required to accurately predict the deflection of members under lateral loads since it is this form of loading which will generally produce the greatest deflections of beams, struts and other structural types of members.

5.1. Relationship between loading, S.F., B.M., slope and deflection

Consider a beam AB which is initially horizontal when unloaded. If this deflects to a new position $A'B'$ under load, the slope at any point C is

$$i = \frac{dy}{dx}$$

Fig. 5.2. Unloaded beam AB deflected to $A'B'$ under load.

This is usually very small in practice, and for small curvatures

$$ds = dx = Rdi \quad (\text{Fig. 5.2})$$

$$\therefore \frac{di}{dx} = \frac{1}{R}$$

But

$$i = \frac{dy}{dx}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{R} \quad (5.1)$$

Now from the simple bending theory

$$\frac{M}{I} = \frac{E}{R}$$

$$\therefore \frac{1}{R} = \frac{M}{EI}$$

Therefore substituting in eqn. (5.1)

$$M = EI \frac{d^2y}{dx^2} \quad (5.2)$$

This is the basic differential equation for the deflection of beams.

If the beam is now assumed to carry a distributed loading which varies in intensity over the length of the beam, then a small element of the beam of length dx will be subjected to the loading condition shown in Fig. 5.3. The parts of the beam on either side of the element $EFGH$ carry the externally applied forces, while reactions to these forces are shown on the element itself.

Thus for vertical equilibrium of $EFGH$,

$$\begin{aligned} Q - wdx &= Q - dQ \\ \therefore dQ &= wdx \end{aligned}$$

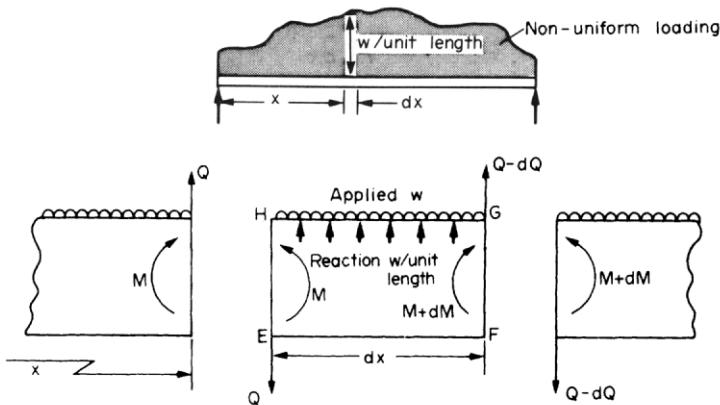


Fig. 5.3. Small element of beam subjected to non-uniform loading (effectively uniform over small length dx).

and integrating,

$$Q = \int w dx \quad (5.3)$$

Also, for equilibrium, moments about any point must be zero.
Therefore taking moments about F,

$$(M + dM) + wdx \frac{dx}{2} = M + Qdx$$

Therefore neglecting the square of small quantities,

$$dM = Qdx$$

and integrating,

$$M = \int Q dx$$

The results can then be summarised as follows:

$$\text{deflection} = y$$

$$\text{slope} = \frac{dy}{dx}$$

$$\text{bending moment} = EI \frac{d^2y}{dx^2}$$

$$\text{shear force} = EI \frac{d^3y}{dx^3}$$

$$\text{load distribution} = EI \frac{d^4y}{dx^4}$$

In order that the above results should agree algebraically, i.e. that positive slopes shall have the normal mathematical interpretation of the positive sign and that B.M. and S.F. conventions are consistent with those introduced earlier, it is imperative that the sign convention illustrated in Fig. 5.4 be adopted.

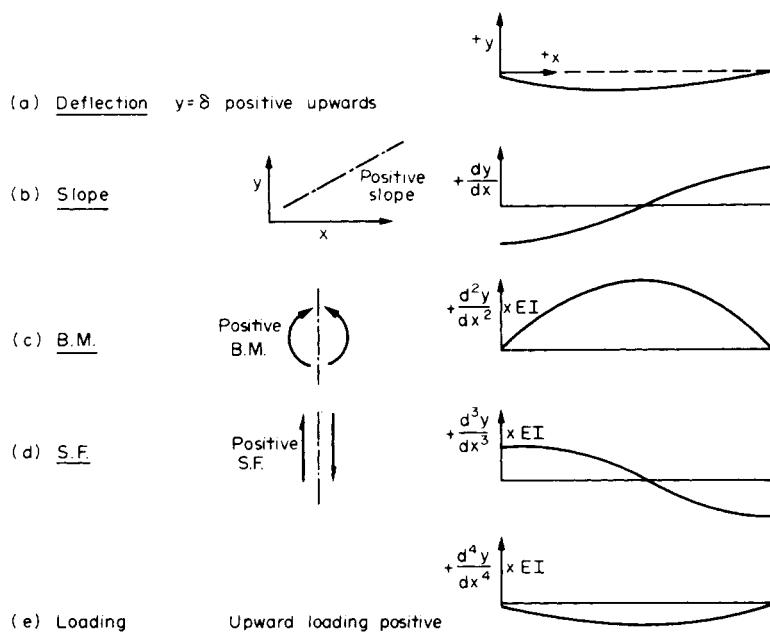


Fig. 5.4. Sign conventions for load, S.F., B.M., slope and deflection.

 N/η'

5.2. Direct integration method

If the value of the B.M. at any point on a beam is known in terms of x , the distance along the beam, and provided that the equation applies along the complete beam, then integration of eqn. (5.4a) will yield slopes and deflections at any point,

$$\text{i.e. } M = EI \frac{d^2y}{dx^2} \quad \text{and} \quad \frac{dy}{dx} = \int \frac{M}{EI} dx + A$$

$$\text{or } y = \int \left(\int \left(\frac{M}{EI} dx \right) dx + Ax + B \right)$$

where A and B are constants of integration evaluated from known conditions of slope and deflection for particular values of x .

(a) Cantilever with concentrated load at the end (Fig. 5.5)

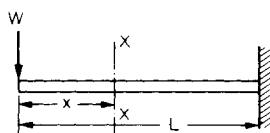


Fig. 5.5.

$$M_{xx} = EI \frac{d^2y}{dx^2} = -Wx$$

$$\therefore EI \frac{dy}{dx} = -\frac{Wx^2}{2} + A$$

assuming EI is constant.

$$EIy = -\frac{Wx^3}{6} + Ax + B$$

Now when

$$x = L, \quad \frac{dy}{dx} = 0 \quad \therefore A = \frac{WL^2}{2}$$

and when

$$x = L, y = 0 \quad \therefore B = \frac{WL^3}{6} - \frac{WL^2}{2} L = -\frac{WL^3}{3}$$

$$\therefore y = \frac{1}{EI} \left[-\frac{Wx^3}{6} + \frac{WL^2 x}{2} - \frac{WL^3}{3} \right] \quad (5.5)$$

This gives the deflection at all values of x and produces a maximum value at the tip of the cantilever when $x = 0$,

$$\text{i.e.} \quad \text{Maximum deflection} = y_{\max} = -\frac{WL^3}{3EI} \quad (5.6)$$

The negative sign indicates that deflection is in the negative y direction, i.e. downwards.

Similarly

$$\frac{dy}{dx} = \frac{1}{EI} \left[-\frac{Wx^2}{2} + \frac{WL^2}{2} \right] \quad (5.7)$$

and produces a maximum value again when $x = 0$.

$$\text{Maximum slope} = \left(\frac{dy}{dx} \right)_{\max} = \frac{WL^2}{2EI} \quad (\text{positive}) \quad (5.8)$$

(b) Cantilever with uniformly distributed load (Fig. 5.6)

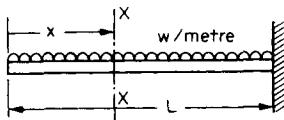


Fig. 5.6.

$$M_{xx} = EI \frac{d^2y}{dx^2} = -\frac{wx^2}{2}$$

$$EI \frac{dy}{dx} = -\frac{wx^3}{6} + A$$

$$EIy = -\frac{wx^4}{24} + Ax + B$$

Again, when

$$x = L, \quad \frac{dy}{dx} = 0 \quad \text{and} \quad A = \frac{wL^3}{6}$$

$$x = L, \quad y = 0 \quad \text{and} \quad B = \frac{wL^4}{24} - \frac{wL^4}{6} = -\frac{wL^4}{8}$$

$$\therefore y = \frac{1}{EI} \left[-\frac{wx^4}{24} + \frac{wL^3x}{6} - \frac{wL^4}{8} \right] \quad (5.9)$$

At $x = 0$,

$$y_{\max} = -\frac{wL^4}{8EI} \quad \text{and} \quad \left(\frac{dy}{dx} \right)_{\max} = \frac{wL^3}{6EI} \quad (5.10)$$

(c) Simply-supported beam with uniformly distributed load (Fig. 5.7)

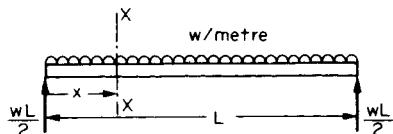


Fig. 5.7.

$$M_{xx} = EI \frac{d^2y}{dx^2} = \frac{wLx}{2} - \frac{wx^2}{2}.$$

$$EI \frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} + A$$

$$EIy = \frac{wLx^3}{12} - \frac{wx^4}{24} + Ax + B$$

$$\text{At } x = 0, \quad y = 0 \quad \therefore \quad B = 0$$

$$\text{At } x = L, \quad y = 0 \quad \therefore \quad 0 = \frac{wL^4}{12} - \frac{wL^4}{24} + AL$$

$$\therefore A = -\frac{wL^3}{24}$$

$$\therefore y = \frac{1}{EI} \left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24} \right] \quad (5.11)$$

In this case the maximum deflection will occur at the centre of the beam where $x = L/2$.

$$\begin{aligned} \therefore y_{\max} &= \frac{1}{EI} \left[\frac{wL}{12} \left(\frac{L^3}{8} \right) - \frac{w}{24} \left(\frac{L^4}{16} \right) - \frac{wL^3}{24} \left(\frac{L}{2} \right) \right] \\ &= -\frac{5wL^4}{384EI} \end{aligned} \quad (5.12)$$

Similarly $\left(\frac{dy}{dx} \right)_{\max} = \pm \frac{wL^3}{24EI}$ at the ends of the beam. (5.13)

(d) Simply supported beam with central concentrated load (Fig. 5.8)

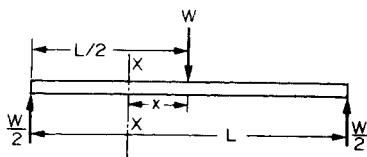


Fig. 5.8.

In order to obtain a single expression for B.M. which will apply across the complete beam in this case it is convenient to take the origin for x at the centre, then:

$$M_{xx} = EI \frac{d^2y}{dx^2} = \frac{W}{2} \left(\frac{L}{2} - x \right) = \frac{WL}{4} - \frac{Wx}{2}$$

$$EI \frac{dy}{dx} = \frac{WL}{4}x - \frac{Wx^2}{4} + A$$

$$EIy = \frac{WLx^2}{8} - \frac{Wx^3}{12} + Ax + B$$

At $x = 0, \frac{dy}{dx} = 0 \quad \therefore A = 0$

$$x = \frac{L}{2}, \quad y = 0 \quad \therefore 0 = \frac{WL^3}{32} - \frac{WL^3}{96} + B$$

$$\therefore B = -\frac{WL^3}{48}$$

$$\therefore y = \frac{1}{EI} \left[\frac{WLx^2}{8} - \frac{Wx^3}{12} - \frac{WL^3}{48} \right] \quad (5.14)$$

$$\therefore y_{\max} = -\frac{WL^3}{48EI} \quad \text{at the centre} \quad (5.15)$$

and

$$\left(\frac{dy}{dx} \right)_{\max} = \pm \frac{WL^2}{16EI} \quad \text{at the ends} \quad (5.16)$$

In some cases it is not convenient to commence the integration procedure with the B.M. equation since this may be difficult to obtain. In such cases it is often more convenient to commence with the equation for the loading at the general point XX on the beam. A typical example follows:

(e) Cantilever subjected to non-uniform distributed load (Fig. 5.9)

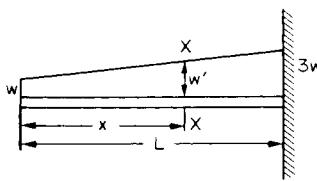


Fig. 5.9.

The loading at section XX is

$$w' = EI \frac{d^4 y}{dx^4} = - \left[w + (3w - w) \frac{x}{L} \right] = - w \left(1 + \frac{2x}{L} \right)$$

Integrating,

$$EI \frac{d^3 y}{dx^3} = - w \left(x + \frac{x^2}{L} \right) + A \quad (1)$$

$$EI \frac{d^2 y}{dx^2} = - w \left(\frac{x^2}{2} + \frac{x^3}{3L} \right) + Ax + B \quad (2)$$

$$EI \frac{dy}{dx} = - w \left(\frac{x^3}{6} + \frac{x^4}{12L} \right) + \frac{Ax^2}{2} + Bx + C \quad (3)$$

$$EIy = - w \left(\frac{x^4}{24} + \frac{x^5}{60L} \right) + \frac{Ax^3}{6} + \frac{Bx^2}{2} + Cx + D \quad (4)$$

Thus, before the slope or deflection can be evaluated, four constants have to be determined; therefore four conditions are required. They are:

At $x = 0$, S.F. is zero

$$\therefore \text{from (1)} \quad A = 0$$

At $x = 0$, B.M. is zero

$$\therefore \text{from (2)} \quad B = 0$$

At $x = L$, slope $dy/dx = 0$ (slope normally assumed zero at a built-in support)

$$\therefore \text{from (3)} \quad 0 = - w \left(\frac{L^3}{6} + \frac{L^3}{12} \right) + C$$

$$C = \frac{wL^3}{4}$$

At $x = L$, $y = 0$

$$\therefore \text{from (4)} \quad 0 = - w \left(\frac{L^4}{24} + \frac{L^4}{60} \right) + \frac{wL^4}{4} + D$$

$$\therefore D = - \frac{23wL^4}{120}$$

$$EIy = -\frac{wx^4}{24} - \frac{wx^5}{60L} + \frac{wL^3x}{4} - \frac{23wL^4}{120}$$

Then, for example, the deflection at the tip of the cantilever, where $x = 0$, is

$$y = -\frac{23wL^4}{120EI}$$

5.3. Macaulay's method

The simple integration method used in the previous examples can only be used when a single expression for B.M. applies along the complete length of the beam. In general this is not the case, and the method has to be adapted to cover all loading conditions.

Consider, therefore, a small portion of a beam in which, at a particular section *A*, the shearing force is *Q* and the B.M. is *M*, as shown in Fig. 5.10. At another section *B*, distance *a* along the beam, a concentrated load *W* is applied which will change the B.M. for points beyond *B*.

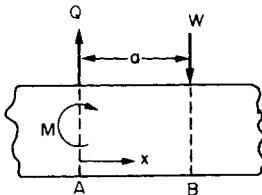


Fig. 5.10.

Between *A* and *B*,

$$M = EI \frac{d^2y}{dx^2} = M + Qx \quad (1)$$

$$\therefore EI \frac{dy}{dx} = Mx + Q \frac{x^2}{2} + C_1 \quad (2)$$

$$\text{and } EIy = M \frac{x^2}{2} + Q \frac{x^3}{6} + C_1 x + C_2 \quad (3)$$

Beyond *B*

$$M = EI \frac{d^2y}{dx^2} = M + Qx - W(x - a) \quad (4)$$

$$\therefore EI \frac{dy}{dx} = Mx + Q \frac{x^2}{2} - W \frac{x^2}{2} + Wax + C_3 \quad (5)$$

$$\text{and } EIy = M \frac{x^2}{2} + Q \frac{x^3}{6} - W \frac{x^3}{6} + Wa \frac{x^2}{2} + C_3 x + C_4 \quad (6)$$

Now for the same slope at *B*, equating (2) and (5),

$$Mx + Q \frac{x^2}{2} + C_1 = Mx + Q \frac{x^2}{2} - W \frac{x^2}{2} + Wax + C_3$$

But at $B, x = a$

$$\therefore C_1 = -\frac{Wa^2}{2} + Wa^2 + C_3$$

$$\therefore C_3 = C_1 - \frac{Wa^2}{2}$$

Substituting in (5),

$$\begin{aligned} EI \frac{dy}{dx} &= Mx + Q \frac{x^2}{2} - W \frac{x^2}{2} + Wax + C_1 - \frac{Wa^2}{2} \\ \therefore EI \frac{dy}{dx} &= Mx + Q \frac{x^2}{2} - \frac{W}{2}(x-a)^2 + C_1 \end{aligned} \quad (7)$$

Also, for the same deflection at B equating (3) and (6), with $x = a$

$$\begin{aligned} \frac{Ma^2}{2} + \frac{Qa^3}{6} + C_1 a + C_2 &= \frac{Ma^2}{2} + \frac{Qa^3}{6} - \frac{Wa^3}{6} + \frac{Wa^3}{2} + C_3 a + C_4 \\ \therefore C_1 a + C_2 &= -\frac{Wa^3}{6} + \frac{Wa^3}{2} + C_3 a + C_4 \\ &= -\frac{Wa^3}{6} + \frac{Wa^3}{2} + \left(C_1 - \frac{Wa^2}{2} \right) a + C_4 \\ \therefore C_4 &= C_2 + \frac{Wa^3}{6} \end{aligned}$$

Substituting in (6),

$$\begin{aligned} EIy &= M \frac{x^2}{2} + Q \frac{x^3}{6} - W \frac{x^3}{6} + Wa \frac{x^2}{2} \left(C_1 - \frac{Wa^2}{2} \right) x + W \frac{a^3}{6} + C_2 \\ &= M \frac{x^2}{2} + Q \frac{x^3}{6} - W \frac{(x-a)^3}{6} + C_1 x + C_2 \end{aligned} \quad (8)$$

Thus, inspecting (4), (7) and (8), we can see that the general method of obtaining slopes and deflections (i.e. integrating the equation for M) will still apply provided that the term $W(x-a)$ is integrated with respect to $(x-a)$ and not x . Thus, when integrated, the term becomes

$$W \frac{(x-a)^2}{2} \quad \text{and} \quad W \frac{(x-a)^3}{6}$$

successively.

In addition, since the term $W(x-a)$ applies only after the discontinuity, i.e. when $x > a$, it should be considered only when $x > a$ or when $(x-a)$ is positive. For these reasons such terms are conventionally put into square or curly brackets and called *Macaulay terms*.

Thus Macaulay terms must be (a) integrated with respect to themselves and (b) neglected when negative.

For the whole beam, therefore,

$$EI \frac{d^2y}{dx^2} = M + Qx - W[(x-a)]$$

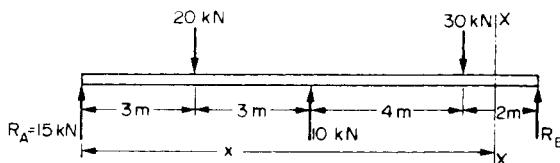


Fig. 5.11.

As an illustration of the procedure consider the beam loaded as shown in Fig. 5.11 for which the central deflection is required. Using the Macaulay method the equation for the B.M. at any general section XX is then given by

$$\text{B.M.}_{XX} = 15x - 20[(x-3)] + 10[(x-6)] - 30[(x-10)]$$

Care is then necessary to ensure that the terms inside the square brackets (Macaulay terms) are treated in the special way noted on the previous page.

Here it must be emphasised that all loads in the right-hand side of the equation are in units of kN (i.e. newtons $\times 10^3$). In subsequent working, therefore, it is convenient to carry through this factor as a denominator on the left-hand side in order that the expressions are dimensionally correct.

Integrating,

$$\frac{EI}{10^3} \frac{dy}{dx} = 15 \frac{x^2}{2} - 20 \left[\frac{(x-3)^2}{2} \right] + 10 \left[\frac{(x-6)^2}{2} \right] - 30 \left[\frac{(x-10)^2}{2} \right] + A$$

$$\text{and } \frac{EI}{10^3} y = 15 \frac{x^3}{6} - 20 \left[\frac{(x-3)^3}{6} \right] + 10 \left[\frac{(x-6)^3}{6} \right] - 30 \left[\frac{(x-10)^3}{6} \right] + Ax + B$$

where A and B are two constants of integration.

Now when $x = 0$, $y = 0$ $\therefore B = 0$

and when $x = 12$, $y = 0$

$$\begin{aligned} 0 &= \frac{15 \times 12^3}{6} - 20 \left[\frac{9^3}{6} \right] + 10 \left[\frac{6^3}{6} \right] - 30 \left[\frac{2^3}{6} \right] + 12A \\ &= 4320 - 2430 + 360 - 40 + 12A \end{aligned}$$

$$\therefore 12A = -4680 + 2470 = -2210$$

$$\therefore A = -184.2$$

The deflection at any point is given by

$$\frac{EI}{10^3} y = 15 \frac{x^3}{6} - 20 \left[\frac{(x-3)^3}{6} \right] + 10 \left[\frac{(x-6)^3}{6} \right] - 30 \left[\frac{(x-10)^3}{6} \right] - 184.2x$$

The deflection at mid-span is thus found by substituting $x = 6$ in the above equation, bearing in mind that the dimensions of the equation are kN m^3 .

N.B.—Two of the Macaulay terms then vanish since one becomes zero and the other negative and therefore neglected.

$$\begin{aligned} \therefore \text{central deflection} &= \frac{10^3}{EI} \left[\frac{15 \times 6^3}{6} - \frac{20 \times 3^3}{6} - 184.2 \times 6 \right] \\ &= -\frac{655.2 \times 10^3}{EI} \end{aligned}$$

With typical values of $E = 208 \text{ GN/m}^2$ and $I = 82 \times 10^{-6} \text{ m}^4$

$$\text{central deflection} = 38.4 \times 10^{-3} \text{ m} = 38.4 \text{ mm}$$

5.4. Macaulay's method for u.d.l.s

If a beam carries a uniformly distributed load over the complete span as shown in Fig. 5.12a the B.M. equation is

$$\text{B.M.}_{xx} = EI \frac{d^2y}{dx^2} = R_A x - \frac{wx^2}{2} - W_1[(x-a)] - W_2[(x-b)]$$

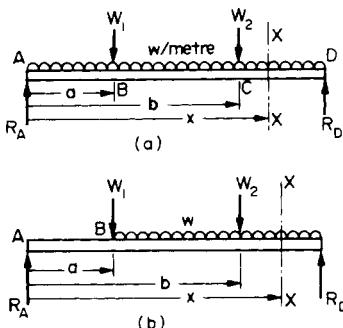


Fig. 5.12.

The u.d.l. term applies across the complete span and does not require the special treatment associated with the Macaulay terms. If, however, the u.d.l. starts at B as shown in Fig. 5.12b the B.M. equation is modified and the u.d.l. term becomes a Macaulay term and is written inside square brackets.

$$\text{B.M.}_{xx} = EI \frac{d^2y}{dx^2} = R_A x - W_1[(x-a)] - w \left[\frac{(x-a)^2}{2} \right] - W_2[(x-b)]$$

Integrating,

$$EI \frac{dy}{dx} = R_A \frac{x^2}{2} - W_1 \left[\frac{(x-a)^2}{2} \right] - w \left[\frac{(x-a)^3}{6} \right] - W_2 \left[\frac{(x-b)^2}{2} \right] + A$$

$$EIy = R_A \frac{x^3}{6} - W_1 \left[\frac{(x-a)^3}{6} \right] - w \left[\frac{(x-a)^4}{24} \right] - W_2 \left[\frac{(x-b)^3}{6} \right] + Ax + B$$

Note that Macaulay terms are integrated with respect to, for example, $(x-a)$ and they must be ignored when negative. Substitution of end conditions will then yield the values of the constants A and B in the normal way and hence the required values of slope or deflection.

It must be appreciated, however, that once a term has been entered in the B.M. expression it will apply across the complete beam. The modifications to the procedure required for cases when u.d.l.s. are applied over part of the beam only are introduced in the following theory.

5.5. Macaulay's method for beams with u.d.l. applied over part of the beam

Consider the beam loading case shown in Fig. 5.13a.

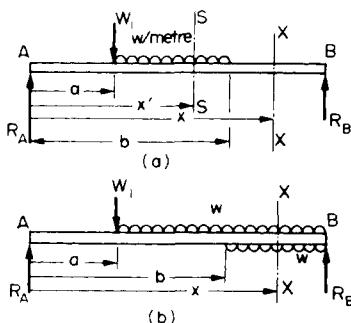


Fig. 5.13.

The B.M. at the section SS is given by the previously introduced procedure as

$$\text{B.M.}_{SS} = R_A x' - W_1[(x' - a)] - W \left[\frac{(x' - a)^2}{2} \right]$$

Having introduced the last (u.d.l.) term, however, it will apply for all values of x' greater than a , i.e. across the rest of the span to the end of the beam. (Remember, Macaulay terms are only neglected when they are negative, e.g. with $x' < a$.) The above equation is *NOT* therefore the correct equation for the load condition shown. The Macaulay method requires that this continuation of the u.d.l. be shown on the loading diagram and the required loading condition can therefore only be achieved by introducing an equal and opposite u.d.l. over the last part of the beam to cancel the unwanted continuation of the initial distributed load. This procedure is shown in Fig. 5.13b.

The correct B.M. equation for any general section XX is then given by

$$\text{B.M.}_{XX} = EI \frac{d^2y}{dx^2} = R_A x - W_1[(x - a)] - w \left[\frac{(x - a)^2}{2} \right] + w \left[\frac{(x - b)^2}{2} \right]$$

This type of approach can be adopted for any beam loading cases in which u.d.l.s are stopped or added to.

A number of examples are shown in Figs. 5.14–17. In each case the required loading system is shown first, followed by the continuation and compensating load system and the resulting B.M. equation.

5.6. Macaulay's method for couple applied at a point

Consider the beam AB shown in Fig. 5.18 with a moment or couple M applied at some point C. Considering the equilibrium of moments about each end in turn produces reactions of

$$R_A = \frac{M}{L} \quad \text{upwards,} \quad \text{and} \quad R_B = \frac{M}{L} \quad \text{downwards}$$

These equal and opposite forces then automatically produce the required equilibrium of vertical forces.

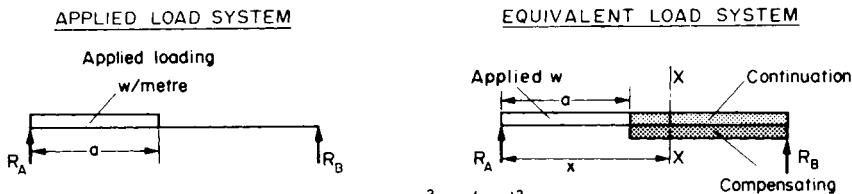


Fig. 5.14.

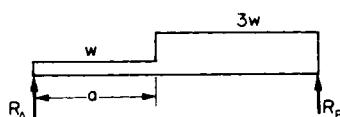


Fig. 5.15.

$$B.M._{xx} = R_A x - \frac{wx^2}{2} - 2w\left[\frac{(x-a)^2}{2}\right]$$

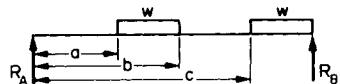


Fig. 5.16.

$$B.M._{xx} = R_A x - w\left[\frac{(x-a)^2}{2}\right] + w\left[\frac{(x-b)^2}{2}\right] - w\left[\frac{(x-c)^2}{2}\right]$$

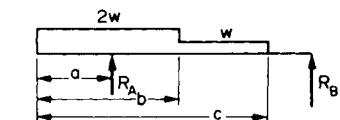
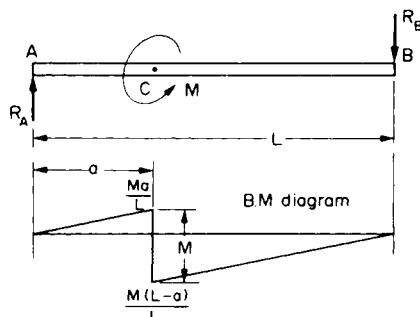


Fig. 5.17.

$$B.M._{xx} = -2w\frac{x^2}{2} + R_A[(x-a)] + w\left[\frac{(x-b)^2}{2}\right] + w\left[\frac{(x-c)^2}{2}\right]$$

Figs. 5.14, 5.15, 5.16 and 5.17. Typical equivalent load systems for Macaulay method together with appropriate B.M. expressions.

Fig. 5.18. Beam subjected to applied couple or moment M .

For sections between A and C the B.M. is $\frac{M}{L}x$.

For sections between C and B the B.M. is $\frac{Mx}{L} - M$.

The additional $(-M)$ term which enters the B.M. expression for points beyond C can be adequately catered for by the Macaulay method if written in the form

$$M[(x-a)^0]$$

This term can then be treated in precisely the same way as any other Macaulay term, integration being carried out with respect to $(x-a)$ and the term being neglected when x is less than a . The full B.M. equation for the beam is therefore

$$M_{xx} = EI \frac{d^2y}{dx^2} = \frac{Mx}{L} - M[(x-a)^0] \quad (5.17)$$

Then

$$EI \frac{dy}{dx} = \frac{Mx^2}{2L} - M[(x-a)] + A, \text{ etc.}$$

5.7. Mohr's "area-moment" method

In applications where the slope or deflection of beams or cantilevers is required at only one position the determination of the complete equations for slope and deflection at all points as obtained by Macaulay's method is rather laborious. In such cases, and in particular where loading systems are relatively simple, the Mohr moment-area method provides a rapid solution.

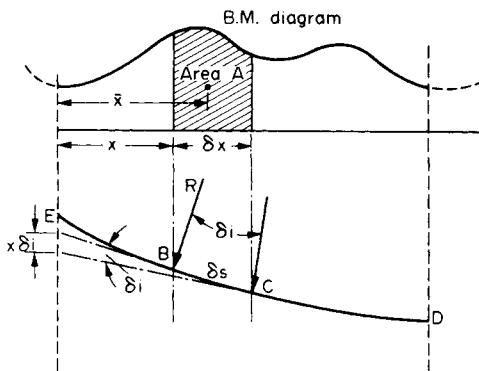


Fig. 5.19.

Figure 5.19 shows the deflected shape of part of a beam ED under the action of a B.M. which varies as shown in the B.M. diagram. Between any two points B and C the B.M. diagram has an area A and centroid distance \bar{x} from E . The tangents at the points B and C give an intercept of $x\delta_i$ on the vertical through E , where δ_i is the angle between the tangents.

Now

$$\delta s = R\delta i$$

and

$$\delta x = \delta_s$$

if slopes are small.

$$\therefore \delta i = \frac{\delta x}{R} = \frac{M}{EI} \delta x$$

$$\therefore \text{change of slope between } E \text{ and } D = i = \int_E^D \frac{M}{EI} dx$$

i.e. $\text{change of slope} = \text{area of } M/EI \text{ diagram between } E \text{ and } D \quad (5.18)$

N.B.—For a uniform beam (EI constant) this equals $\frac{1}{EI} \times \text{area of B.M. diagram}$.

Deflection at E resulting from the bending of $BC = x\delta i$

$$\begin{aligned} \therefore \text{total deflection resulting from bending of } ED &= \int x\delta i \\ &= \int_E^D \frac{Mx}{EI} dx \end{aligned}$$

The total deflection of E relative to the tangent at D is equal to the first moment of area of the M/EI diagram about E . (5.19)

Again, if EI is constant this equals $1/EI \times \text{first moment of area of the B.M. diagram about } E$.

The theorem is particularly useful when one point on the beam is a point of zero slope since the tangent at this point is then horizontal and deflections relative to the tangent are absolute values of vertical deflections. Thus if D is a point of zero slope the above equations yield the actual slope and deflection at E .

The Mohr area-moment procedure may be summarised in its most useful form as follows: if A and B are two points on the deflection curve of a beam, EI is constant and B is a point of zero slope, then Mohr's theorems state that:

- (1) **Slope at $A = 1/EI \times \text{area of B.M. diagram between } A \text{ and } B$.** (5.20)

- (2) **Deflection of A relative to $B = 1/EI \times \text{first moment of area of B.M. diagram between } A \text{ and } B \text{ about } A$.** (5.21)

In many cases of apparently complicated load systems the loading can be separated into a combination of several simple systems which, by the application of the principle of superposition, will produce the same results. This procedure is illustrated in Examples 5.4 and 5.5.

The Mohr method will now be applied to the standard loading cases solved previously by the direct integration procedure.

(a) Cantilever with concentrated load at the end

In this case B is a point of zero slope and the simplified form of the Mohr theorems stated above can be applied.

Slope at $A = \frac{1}{EI} [\text{area of B.M. diagram between } A \text{ and } B \text{ (Fig. 5.20)}]$

$$= \frac{1}{EI} \left[\frac{L}{2} WL \right] = \frac{WL^2}{2EI}$$

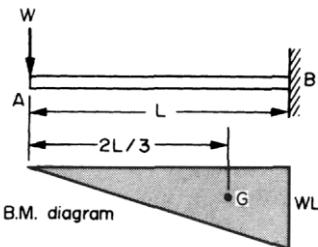


Fig. 5.20.

Deflection at A (relative to B)

$$\begin{aligned} &= \frac{1}{EI} [\text{first moment of area of B.M. diagram between } A \text{ and } B \text{ about } A] \\ &= \frac{1}{EI} \left[\left(\frac{L}{2} WL \right) \frac{2L}{3} \right] = \frac{WL^3}{3EI} \end{aligned}$$

(b) Cantilever with u.d.l.

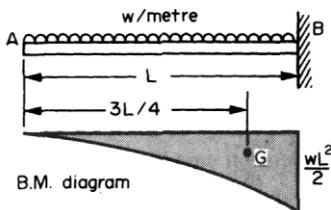


Fig. 5.21.

Again B is a point of zero slope.

slope at $A = \frac{1}{EI} [\text{area of B.M. diagram (Fig. 5.21)}]$

$$\begin{aligned} &= \frac{1}{EI} \left[\frac{1}{3} L \frac{wL^2}{2} \right] \\ &= \frac{wL^3}{6EI} \end{aligned}$$

Deflection at $A = \frac{1}{EI} [\text{moment of B.M. diagram about } A]$

$$= \frac{1}{EI} \left[\left(\frac{1}{3} L \frac{wL^2}{2} \right) \frac{3L}{4} \right] = \frac{wL^4}{8EI}$$

(c) Simply supported beam with u.d.l.

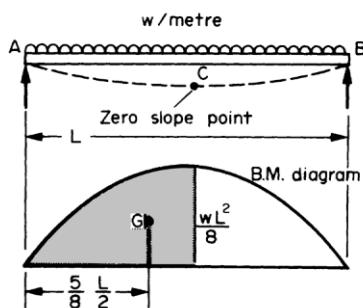


Fig. 5.22.

Here the point of zero slope is at the centre of the beam C . Working relative to C ,

$$\begin{aligned} \text{slope at } A &= \frac{1}{EI} [\text{area of B.M. diagram between } A \text{ and } C \text{ (Fig. 5.22)}] \\ &= \frac{1}{EI} \left[\frac{2wL^2}{3} \frac{L}{8} \frac{2}{2} \right] = \frac{wL^3}{24EI} \end{aligned}$$

Deflection of A relative to C (= central deflection relative to A)

$$\begin{aligned} &= \frac{1}{EI} [\text{moment of B.M. diagram between } A \text{ and } C \text{ about } A] \\ &= \frac{1}{EI} \left[\left(\frac{2wL^2}{3} \frac{L}{8} \frac{2}{2} \right) \left(\frac{5L}{16} \right) \right] = \frac{5wL^4}{384EI} \end{aligned}$$

(d) Simply supported beam with central concentrated load

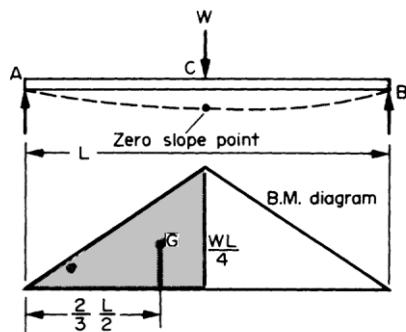


Fig. 5.23.

Again working relative to the zero slope point at the centre C ,

$$\text{slope at } A = \frac{1}{EI} [\text{area of B.M. diagram between } A \text{ and } C \text{ (Fig. 5.23)}]$$

$$= \frac{1}{EI} \left[\frac{1}{2} \frac{L}{2} \frac{WL}{4} \right] = \frac{WL^2}{16EI}$$

Deflection of A relative to C (= central deflection of C)

$$= \frac{1}{EI} [\text{moment of B.M. diagram between } A \text{ and } C \text{ about } A]$$

$$= \frac{1}{EI} \left[\left(\frac{1}{2} \frac{L}{2} \frac{WL}{4} \right) \left(\frac{2}{3} \frac{L}{2} \right) \right] = \frac{WL^3}{48EI}$$

5.8. Principle of superposition

The general statement for the principle of superposition asserts that the resultant stress or strain in a system subjected to several forces is the algebraic sum of their effects when applied separately. The principle can be utilised, however, to determine the deflections of beams subjected to complicated loading conditions which, in reality, are merely combinations of a number of simple systems. In addition to the simple standard cases introduced previously, numerous different loading conditions have been solved by various workers and their results may be found in civil or mechanical engineering handbooks or data sheets. Thus, the algebraic sum of the separate deflections caused by a convenient selection of standard loading cases will produce the total deflection of the apparently complex case.

It must be appreciated, however, that the principle of superposition is only valid whilst the beam material remains elastic and for small beam deflections. (Large deflections would produce unacceptable deviation of the lines of action of the loads relative to the beam axis.)

5.9. Energy method

A further, alternative, procedure for calculating deflections of beams or structures is based upon the application of strain energy considerations. This is introduced in detail in Chapter 11 and will not be considered further here.

5.10. Maxwell's theorem of reciprocal displacements

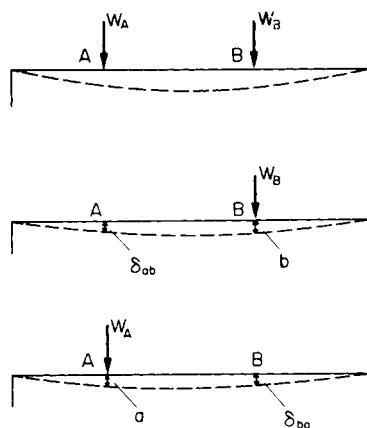
Consider a beam subjected to two loads W_A and W_B at points A and B respectively as shown in Fig. 5.24. Let W_A be gradually applied first, producing a deflection a at A .

$$\text{Work done} = \frac{1}{2} W_A a$$

When W_B is applied it will produce a deflection b at B and an additional deflection δ_{ab} at A (the latter occurring in the presence of a now constant load W_A).

$$\text{Extra work done} = \frac{1}{2} W_B b + W_A \delta_{ab}$$

$$\therefore \text{total work done} = \frac{1}{2} W_A a + \frac{1}{2} W_B b + W_A \delta_{ab}$$



δ_{ab} = deflection at A with load at B

δ_{ba} = deflection at B with load at A

Fig. 5.24. Maxwell's theorem of reciprocal displacements.

Similarly, if the loads were applied in reverse order and the load W_A at A produced an additional deflection δ_{ba} at B, then

$$\text{total work done} = \frac{1}{2} W_B b + \frac{1}{2} W_A a + W_B \delta_{ba}$$

It should be clear that, regardless of the order in which the loads are applied, the total work done must be the same. Inspection of the above equations thus shows that

$$W_A \delta_{ab} = W_B \delta_{ba}$$

If the two loads are now made equal, then

$$\delta_{ab} = \delta_{ba} \quad (5.22)$$

i.e. the deflection at A produced by a load at B equals the deflection at B produced by the same load at A. This is Maxwell's theorem of reciprocal displacements.

As a typical example of the application of this theorem to beams consider the case of a simply supported beam carrying a single concentrated load off-set from the centre (Fig. 5.25).

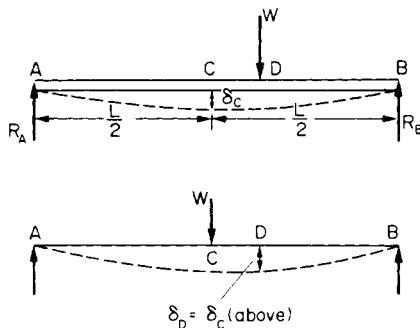
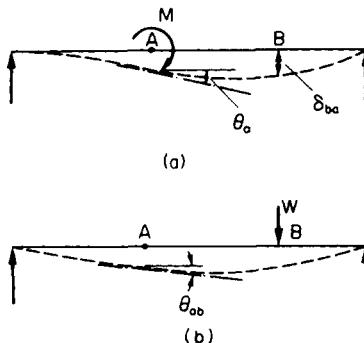


Fig. 5.25.

The central deflection of the beam for this loading condition would be given by the reciprocal displacement theorem as the deflection at D if the load is moved to the centre. Since the deflection equation for a central point load is one of the standard cases treated earlier the required deflection value can be readily obtained.

Maxwell's theorem of reciprocal displacements can also be applied if one or both of the loads are replaced by moments or couples. In this case it can be shown that the theorem is modified to the relevant one of the following forms (a), (b):

- (a) *The angle of rotation at A due to a concentrated force at B is numerically equal to the deflection at B due to a couple at A provided that the force and couple are also numerically equal (Fig. 5.26).*



$$\begin{aligned}\theta_a &= \text{slope at } A \text{ with moment (or load) at } A \\ \theta_{ab} &= \text{slope at } a \text{ with load at } B\end{aligned}$$

Fig. 5.26.

- (b) *The angle of rotation at A due to a couple at B is equal to the rotation at B due to the same couple applied at A (Fig. 5.27).*

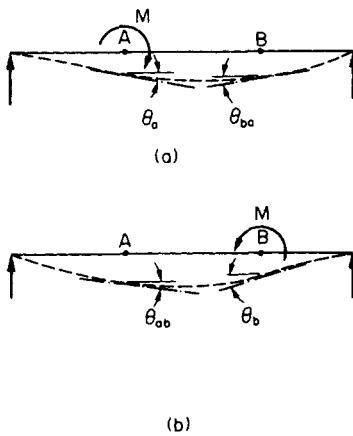


Fig. 5.27.

All three forms of the theorem are quite general in application and are not restricted to beam problems. Any type of component or structure subjected to bending, direct load, shear or torsional deformation may be considered provided always that linear elastic conditions prevail, i.e. Hooke's law applies, and deflections are small enough not to significantly affect the undeformed geometry.

5.11. Continuous beams – Clapeyron's “three-moment” equation

When a beam is supported on more than two supports it is termed *continuous*. In cases such as these it is not possible to determine directly the reactions at the three supports by the normal equations of static equilibrium since there are too many unknowns. An extension of Mohr's area–moment method is therefore used to obtain a relationship between the B.M.s at the supports, from which the reaction values can then be determined and the B.M. and S.F. diagrams drawn.

Consider therefore the beam shown in Fig. 5.28. The areas A_1 and A_2 are the “free” B.M. diagrams, treating the beam as simply supported over two separate spans L_1 and L_2 . In general the B.M.s at the three supports will not be zero as this diagram suggests, but will have some values M_1 , M_2 and M_3 . Thus a *fixing-moment diagram* must be introduced as shown, the actual B.M. diagram then being the algebraic sum of the two diagrams.

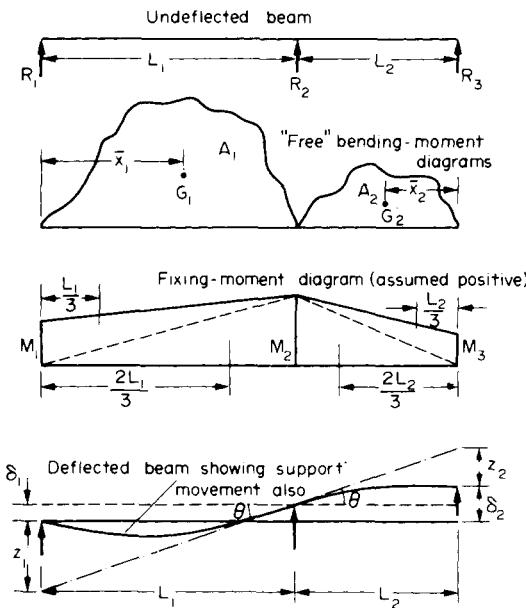


Fig. 5.28. Continuous beam over three supports showing “free” and “fixing” moment diagrams together with the deflected beam form including support movement.

The bottom figure shows the deflected position of the beam, the deflections δ_1 and δ_2 being relative to the left-hand support. If a tangent is drawn at the centre support then the intercepts at the end of each span are z_1 and z_2 and θ is the slope of the tangent, and hence the beam, at the centre support.

Now, assuming deflections are small,

$$\theta \text{ (radians)} = \frac{z_1 + \delta_1}{L_1} = \frac{z_2 + \delta_2 - \delta_1}{L_2}$$

$$\therefore \frac{z_1}{L_1} + \frac{\delta_1}{L_1} = \frac{z_2}{L_2} + \frac{(\delta_2 - \delta_1)}{L_2}$$

But from Mohr's area-moment method,

$$z = \frac{A\bar{x}}{EI}$$

where A is the area of the B.M. diagram over the span to which z refers.

$$\begin{aligned} z_1 &= -\frac{1}{EI_1} \left[A_1 \bar{x}_1 + \left(\frac{M_1 L_1}{2} \times \frac{L_1}{3} \right) + \left(\frac{M_2 L_1}{2} \times \frac{2L_1}{3} \right) \right] \\ &= -\frac{1}{EI_1} \left[A_1 \bar{x}_1 + \frac{M_1 L_1^2}{6} + \frac{M_2 L_1^2}{3} \right] \end{aligned}$$

$$\begin{aligned} \text{and } z_2 &= \frac{1}{EI_2} \left[A_2 \bar{x}_2 + \left(\frac{M_3 L_2}{2} \times \frac{L_2}{3} \right) + \left(\frac{M_2 L_2}{2} \times \frac{2L_2}{3} \right) \right] \\ &= \frac{1}{EI_2} \left[A_2 \bar{x}_2 + \frac{M_3 L_2^2}{6} + \frac{M_2 L_2^2}{3} \right] \end{aligned}$$

N.B.—Since the intercepts are in opposite directions, they are of opposite sign.

$$\begin{aligned} \therefore -\frac{\left[A_1 \bar{x}_1 + \frac{M_1 L_1^2}{6} + \frac{M_2 L_1^2}{3} \right]}{EI_1 L_1} + \frac{\delta_1}{L_1} &= \frac{\left[A_2 \bar{x}_2 + \frac{M_3 L_2^2}{6} + \frac{M_2 L_2^2}{3} \right]}{EI_2 L_2} + \frac{(\delta_2 - \delta_1)}{L_2} \\ \therefore -\frac{A_1 \bar{x}_1}{I_1 L_1} - \frac{M_1 L_1}{6I_1} - \frac{M_2 L_1}{3I_1} + \frac{E\delta_1}{L_1} &= \frac{A_2 \bar{x}_2}{I_2 L_2} + \frac{M_3 L_2}{6I_2} + \frac{M_2 L_2}{3I_2} + \frac{E(\delta_2 - \delta_1)}{L_2} \\ \therefore -\frac{M_1 L_1}{I_1} - 2M_2 \left[\frac{L_1}{I_1} + \frac{L_2}{I_2} \right] - \frac{M_3 L_2}{I_2} &= 6 \left[\frac{A_1 \bar{x}_1}{I_1 L_1} + \frac{A_2 \bar{x}_2}{I_2 L_2} \right] + 6E \left[\frac{(\delta_2 - \delta_1)}{L_2} - \frac{\delta_1}{L_1} \right] \end{aligned} \quad (5.23)$$

This is the full three-moment equation; it can be greatly simplified if the beam is uniform, i.e. $I_1 = I_2 = I$, as follows:

$$-M_1 L_1 - 2M_2 [L_1 + L_2] - M_3 L_2 = 6 \left[\frac{A_1 \bar{x}_1}{L_1} + \frac{A_2 \bar{x}_2}{L_2} \right] + 6EI \left[\frac{(\delta_2 - \delta_1)}{L_2} - \frac{\delta_1}{L_1} \right]$$

If the supports are on the same level, i.e. $\delta_1 = \delta_2 = 0$,

$$-M_1 L_1 - 2M_2 [L_1 + L_2] - M_3 L_2 = 6 \left[\frac{A_1 \bar{x}_1}{L_1} + \frac{A_2 \bar{x}_2}{L_2} \right] \quad (5.24)$$

This is the form in which Clapeyron's three-moment equation is normally used.

The following standard results for $\frac{6A\bar{x}}{L}$ are very useful:

(1) Concentrated loads (Fig. 5.29)

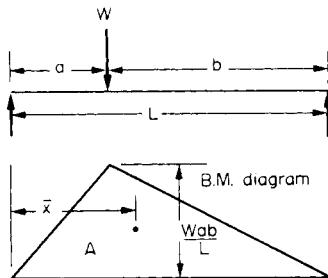


Fig. 5.29.

$$\begin{aligned}\frac{6A\bar{x}}{L} &= \frac{6}{L} \left[\frac{Wab}{L} \times \frac{a}{2} \times \frac{2a}{3} + \frac{Wab}{L} \times \frac{b}{2} \left(a + \frac{b}{3} \right) \right] \\ &= \frac{6Wab}{L^2} \left[\frac{a^2}{3} + \frac{ab}{2} + \frac{b^2}{6} \right] \\ &= \frac{Wab}{L^2} [2a^2 + 3ab + b^2] = \frac{Wab}{L^2} (2a + b)(a + b) \\ &= \frac{Wab}{L} (2a + b)\end{aligned}\quad (5.25)$$

But

$$b = L - a$$

$$\begin{aligned}\therefore \frac{6A\bar{x}}{L} &= \frac{Wa}{L} (L - a)(2a + L - a) \\ &= \frac{Wa}{L} (L^2 - a^2)\end{aligned}\quad (5.26)$$

(2) Uniformly distributed loads (Fig. 5.30)

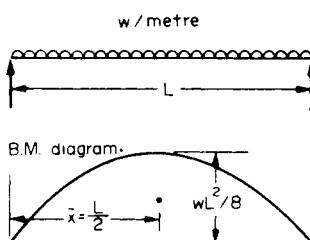


Fig. 5.30.

Here the B.M. diagram is a parabola for which

$$\text{area} = \frac{2}{3} \text{base} \times \text{height}$$

$$\begin{aligned}\therefore \frac{6A\bar{x}}{L} &= \frac{6}{L} \times \frac{2}{3} \times L \times \frac{wL^2}{8} \times \frac{L}{2} \\ &= \frac{wL^3}{4}\end{aligned}\quad (5.27)$$

5.12. Finite difference method

A numerical method for the calculation of beam deflections which is particularly useful for non-prismatic beams or for cases of irregular loading is the so-called *finite difference* method.

The basic principle of the method is to replace the standard differential equation (5.2) by its finite difference approximation, obtain equations for deflections in terms of moments at various points along the beam and solve these simultaneously to yield the required deflection values.

Consider, therefore, Fig. 5.31 which shows part of a deflected beam with the x axis divided into a series of equally spaced intervals. By convention, the ordinates are numbered with respect to the central ordinate B .

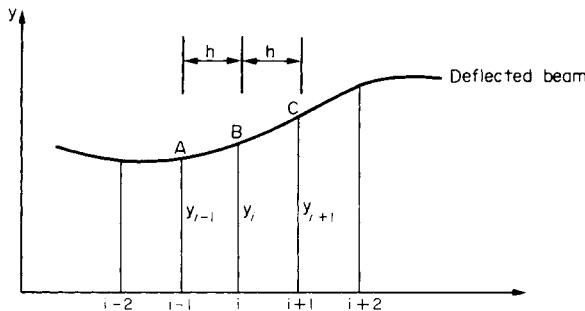


Fig. 5.31. Deflected beam curve divided into a convenient number of equally spaced intervals.

If the equation for the deflected curve of the beam is taken to be $y = f(x)$ then the first derivative dy/dx at B is the slope of the curve at B . Approximately (provided h is small) this can be taken to be the slope of the chord joining A and C so that:

$$\left(\frac{dy}{dx}\right)_i = \frac{(y_{i+1} - y_{i-1})}{2h} = \frac{1}{2h}(y_{i+1} - y_{i-1}) \quad (5.28)$$

The rate of change of the first derivative, i.e. the rate of change of the slope $\left(=\frac{d^2y}{dx^2}\right)$ is given in the same way approximately as the slope to the right of i minus the slope to the left of i divided by the interval between them.

$$\text{Thus: } \left(\frac{d^2y}{dx^2}\right)_i = \frac{\frac{(y_{i+1} - y_i)}{h} - \frac{(y_i - y_{i-1})}{h}}{h} = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1}) \quad (5.29)$$

Equations 5.28 and 5.29 are the *finite difference approximations* of the standard beam deflection differential equations and, because they are written in terms of ordinates on either side of the central point i , they are known as *central differences*. Alternative expressions which can be formed to contain only ordinates at, or to the right of i , or ordinates at, or to the left of i are known as forward and backward differences, respectively but these will not be considered here.

Now from eqn. (5.2)

$$M = EI \frac{d^2y}{dx^2}$$

∴ At position i , combining eqn. (5.2) and (5.29).

$$\left(\frac{M}{EI_i} \right) = \frac{1}{h^2} (y_{i+1} - 2y_i + y_{i-1}) \quad (5.30)$$

A solution for any of the deflection (y) values can then be obtained by applying the finite difference equation at a series of points along the beam and solving the resulting simultaneous equations – see Example 5.8.

The higher the number of points selected the greater the accuracy of solution but the more the number of equations which are required to be solved. The method thus lends itself to computer-assisted evaluation.

In addition to the solution of statically determinate beam problems of the type treated in Example 5.8 the method is also applicable to the analysis of statically indeterminate beams, i.e. those beam loading conditions with unknown (or redundant) quantities such as prop loads or fixing moments – see Example 5.9.

The method is similar in that the bending moment is written in terms of the applied loads and the redundant quantities and equated to the finite difference equation at selected points. Since each redundancy is usually associated with a known (or assumed) condition of slope or deflection, e.g. zero deflection at a propped support, there will always be sufficient equations to allow solution of the unknowns.

The principal advantages of the finite difference method are thus:

- (a) that it can be applied to statically determinate or indeterminate beams,
- (b) that it can be used for non-prismatic beams,
- (c) that it is amenable to computer solutions.

5.13. Deflections due to temperature effects

It has been shown in §2.3 that a uniform temperature increase t on an unconstrained bar of length L will produce an increase in length

$$\Delta L = \alpha Lt$$

where α is the coefficient of linear expansion of the material of the bar. Provided that the bar remains unconstrained, i.e. is free to expand, no stresses will result.

Similarly, in the case of a beam supported in such a way that longitudinal expansion can occur freely, no stresses are set up and there will be no tendency for the beam to bend. If, however, the beam is constrained then stresses will result, their values being calculated using

the procedure of §2.3 provided that the temperature change is uniform across the whole beam section.

If the temperature is not constant across the beam then, again, stresses and deflections will result and the following procedure must be adopted:

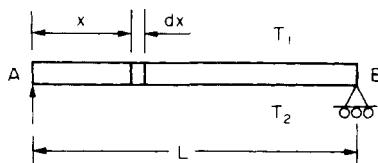


Fig. 5.32(a). Beam initially straight before application of temperature T_1 on the top surface and T_2 on the lower surface. (Beam supported on rollers at B to allow "free" lateral expansion).

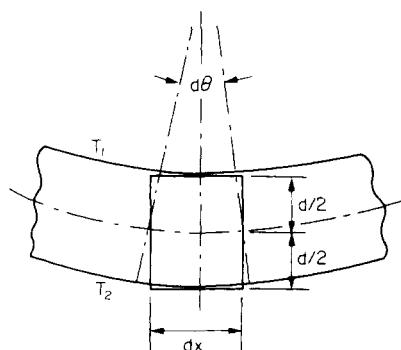


Fig. 5.32(b). Beam after application of temperatures T_1 and T_2 , showing distortions of element dx .

Consider the initially straight, simply-supported beam shown in Fig. 5.32(a) with an initial uniform temperature T_0 . If the temperature changes to a value T_1 on the upper surface and T_2 on the lower surface with, say, $T_2 > T_1$ then an element dx on the bottom surface will expand to $\alpha(T_2 - T_0).dx$ whilst the same length on the top surface will only expand to $\alpha(T_1 - T_0).dx$. As a result the beam will bend to accommodate the distortion of the element dx , the sides of the element rotating relative to one another by the angle $d\theta$, as shown in Fig. 5.32(b). For a depth of beam d :

$$d.d\theta = \alpha(T_2 - T_0).dx - \alpha(T_1 - T_0).dx$$

$$\text{or } \frac{d\theta}{dx} = \frac{\alpha(T_2 - T_1)}{d} \quad (5.31)$$

The differential equation gives the rate of change of slope of the beam and, since $\theta = dy/dx$,

$$\text{then } \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2y}{dx^2} = \frac{\alpha(T_2 - T_1)}{d}$$

Thus the standard differential equation for bending of the beam due to temperature gradient

across the beam section is:

$$\frac{d^2y}{dx^2} = \frac{\alpha(T_2 - T_1)}{d} \quad (5.32)$$

This is directly analogous to the standard deflection equation $\frac{d^2y}{dx^2} = \frac{M}{EI}$ so that integration of this equation in exactly the same way as previously for bending moments allows a solution for slopes and deflections produced by the thermal effects.

N.B. If the temperature gradient across the beam section is linear, the average temperature $\frac{1}{2}(T_1 + T_2)$ will occur at the mid-height position and, in addition to the bending, the beam will change in overall length by an amount $\alpha L[\frac{1}{2}(T_1 + T_2) - T_0]$ in the absence of any constraint.

Application to cantilevers

Consider the cantilever shown in Fig. 5.33 subjected to temperature T_1 on the top surface and T_2 on the lower surface. In the absence of external loads, and because the cantilever is free to bend, there will be no moment or reaction set up at the built-in end.

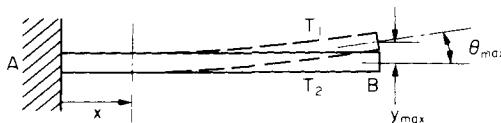


Fig. 5.33. Cantilever with temperature T_1 on the upper surface, T_2 on the lower surface ($T_2 > T_1$).

Applying the differential equation (5.32) we have:

$$\frac{d^2y}{dx^2} = \frac{\alpha(T_2 - T_1)}{d}.$$

Integrating:

$$\frac{dy}{dx} = \frac{\alpha(T_2 - T_1)}{d}x + C_1$$

But at $x = 0$, $\frac{dy}{dx} = 0$, $\therefore C_1 = 0$ and:

$$\frac{dy}{dx} = \frac{\alpha(T_2 - T_1)}{d}x = \theta$$

\therefore The slope at the end of the cantilever is:

$$\theta_{\max} = \frac{\alpha(T_2 - T_1)}{d}L \quad (5.33)$$

Integrating again to find deflections:

$$y = \frac{\alpha(T_2 - T_1)x^2}{d} \frac{L}{2} + C_2$$

and, since $y = 0$ at $x = 0$, then $C_2 = 0$, and:

$$y = \frac{\alpha(T_2 - T_1)}{2d} x^2$$

At the end of the cantilever, therefore, the deflection is:

$$y_{\max} = \frac{\alpha(T_2 - T_1)}{2d} L^2 \quad (5.34)$$

Application to built-in beams

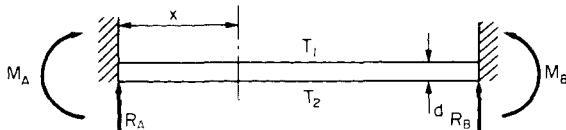


Fig. 5.34. Built-in beam subjected to thermal gradient with temperature T_1 on the upper surface, T_2 on the lower surface.

Consider the built-in beam shown in Fig. 5.34. Using the principle of superposition the differential equation for the beam is given by the combination of the equations for applied bending moment and thermal effects.

$$\text{For bending } EI \frac{d^2y}{dx^2} = M_A + R_A x.$$

$$\text{For thermal effects } \frac{d^2y}{dx^2} = \frac{\alpha(T_2 - T_1)}{d}$$

$$\therefore EI \frac{d^2y}{dx^2} = EI \frac{\alpha(T_2 - T_1)}{d}$$

\therefore The combined differential equation is:

$$EI \frac{d^2y}{dx^2} = M_A + R_A x + EI \frac{\alpha(T_2 - T_1)}{d}.$$

However, in the absence of applied loads and from symmetry of the beam:

$$R_A = R_B = 0,$$

and

$$M_A = M_B = M.$$

$$\therefore EI \frac{d^2y}{dx^2} = M + EI \frac{\alpha(T_2 - T_1)}{d}.$$

$$\text{Integrating: } EI \frac{dy}{dx} = Mx + EI \frac{\alpha(T_2 - T_1)}{d} x + C_1$$

$$\text{Now at } x = 0, \frac{dy}{dx} = 0 \quad \therefore C_1 = 0,$$

$$\text{and at } x = L, \frac{dy}{dx} = 0 \quad \therefore M = -EI \frac{\alpha(T_2 - T_1)}{d} \quad (5.35)$$

Integrating again to find the deflection equation we have:

$$EIy = M \cdot \frac{x^2}{2} + EI \frac{\alpha(T_2 - T_1)}{d} \cdot \frac{x^2}{2} + C_2$$

When $x = 0, y = 0 \therefore C_2 = 0$,

and, since $M = -EI \frac{\alpha(T_2 - T_1)}{d}$ then $y = 0$ for all values of x .

Thus a rather surprising result is obtained whereby the beam will remain horizontal in the presence of a thermal gradient. It will, however, be subject to residual stresses arising from the constraint on overall expansion of the beam under the average temperature $\frac{1}{2}(T_1 + T_2)$. i.e. from §2.3

$$\begin{aligned} \text{residual stress} &= E\alpha[\frac{1}{2}(T_1 + T_2)] \\ &= \frac{1}{2}E\alpha(T_1 + T_2). \end{aligned} \quad (5.36)$$

Examples

Example 5.1

(a) A uniform cantilever is 4 m long and carries a concentrated load of 40 kN at a point 3 m from the support. Determine the vertical deflection of the free end of the cantilever if $EI = 65 \text{ MN m}^2$.

(b) How would this value change if the same total load were applied but uniformly distributed over the portion of the cantilever 3 m from the support?

Solution

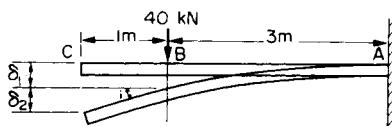


Fig. 5.35.

(a) With the load in the position shown in Fig. 5.35 the cantilever is effectively only 3 m long, the remaining 1 m being unloaded and therefore not bending. Thus, the standard equations for slope and deflections apply between points A and B only.

$$\text{Vertical deflection of } B = -\frac{WL^3}{3EI} = -\frac{40 \times 10^3 \times 3^3}{3 \times 65 \times 10^6} = -5.538 \times 10^{-3} \text{ m} = \delta_1$$

$$\text{Slope at } B = \frac{WL^2}{2EI} = \frac{40 \times 10^3 \times 3^2}{2 \times 65 \times 10^6} = 2.769 \times 10^{-3} \text{ rad} = i$$

Now BC remains straight since it is not subject to bending.

$$\therefore \delta_2 = -iL = -2.769 \times 10^{-3} \times 1 = -2.769 \times 10^{-3} \text{ m}$$

$$\therefore \text{vertical deflection of } C = \delta_1 + \delta_2 = -(5.538 + 2.769)10^{-3} = -8.31 \text{ mm}$$

The negative sign indicates a deflection in the negative y direction, i.e. downwards.

(b) With the load uniformly distributed,

$$w = \frac{40 \times 10^3}{3} = 13.33 \times 10^3 \text{ N/m}$$

Again using standard equations listed in the summary

$$\delta'_1 = -\frac{wL^4}{8EI} = \frac{13.33 \times 10^3 \times 3^4}{8 \times 65 \times 10^6} = -2.076 \times 10^{-3} \text{ m}$$

$$\text{and slope } i = \frac{wL^3}{6EI} = \frac{13.33 \times 10^3 \times 3^3}{6 \times 65 \times 10^6} = 0.923 \times 10^3 \text{ rad}$$

$$\therefore \delta'_2 = -0.923 \times 10^{-3} \times 1 = 0.923 \times 10^{-3} \text{ m}$$

$$\therefore \text{vertical deflection of } C = \delta'_1 + \delta'_2 = -(2.076 + 0.923)10^{-3} = -3 \text{ mm}$$

There is thus a considerable (63.9 %) reduction in the end deflection when the load is uniformly distributed.

Example 5.2

Determine the slope and deflection under the 50 kN load for the beam loading system shown in Fig. 5.36. Find also the position and magnitude of the maximum deflection.

$$E = 200 \text{ GN/m}^2; I = 83 \times 10^{-6} \text{ m}^4.$$

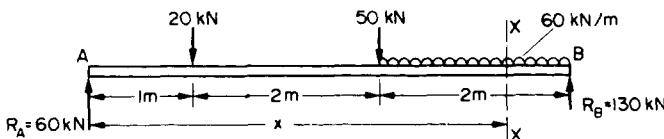


Fig. 5.36.

Solution

Taking moments about either end of the beam gives

$$R_A = 60 \text{ kN} \quad \text{and} \quad R_B = 130 \text{ kN}$$

Applying Macaulay's method,

$$BM_{xx} = \frac{EI}{10^3} \frac{d^2y}{dx^2} = 60x - 20[(x-1)] - 50[(x-3)] - 60 \left[\frac{(x-3)^2}{2} \right] \quad (1)$$

The load unit of kilonewton is accounted for by dividing the left-hand side of (1) by 10^3 and the u.d.l. term is obtained by treating the u.d.l. to the left of XX as a concentrated load of $60(x-3)$ acting at its mid-point of $(x-3)/2$ from XX .

Integrating (1),

$$\frac{EI}{10^3} \frac{dy}{dx} = \frac{60x^2}{2} - 20 \left[\frac{(x-1)^2}{2} \right] - 50 \left[\frac{(x-3)^2}{2} \right] - 60 \left[\frac{(x-3)^3}{6} \right] + A \quad (2)$$

and $\frac{EI}{10^3} y = \frac{60x^3}{6} - 20 \left[\frac{(x-1)^3}{6} \right] - 50 \left[\frac{(x-3)^3}{6} \right] - 60 \left[\frac{(x-3)^4}{24} \right] + Ax + B \quad (3)$

Now when $x = 0, y = 0 \therefore B = 0$

when $x = 5, y = 0 \therefore$ substituting in (3)

$$0 = \frac{60 \times 5^3}{6} - \frac{20 \times 4^3}{6} - \frac{50 \times 2^3}{6} - \frac{60 \times 2^4}{24} + 5A$$

$$0 = 1250 - 213.3 - 66.7 - 40 + 5A$$

$$\therefore 5A = -930 \quad A = -186$$

Substituting in (2),

$$\frac{EI}{10^3} \frac{dy}{dx} = \frac{60x^2}{2} - 20 \left[\frac{(x-1)^2}{2} \right] - 50 \left[\frac{(x-3)^2}{2} \right] - 60 \left[\frac{(x-3)^3}{6} \right] - 186$$

\therefore slope at $x = 3$ m (i.e. under the 50 kN load)

$$\begin{aligned} &= \frac{10^3}{EI} \left[\frac{60 \times 3^2}{2} - \frac{20 \times 2^2}{2} - 186 \right] = \frac{10^3 \times 44}{200 \times 10^9 \times 83 \times 10^{-6}} \\ &= 0.00265 \text{ rad} \end{aligned}$$

And, substituting in (3),

$$\frac{EI}{10^3} y = \frac{60 \times 3^3}{6} - 20 \left[\frac{(x-1)^3}{6} \right] - 50 \left[\frac{(x-3)^3}{6} \right] - 60 \left[\frac{(x-3)^4}{24} \right] - 186x$$

\therefore deflection at $x = 3$ m

$$\begin{aligned} &= \frac{10^3}{EI} \left[\frac{60 \times 3^3}{6} - \frac{20 \times 2^3}{6} - 186 \times 3 \right] \\ &= \frac{10^3}{EI} [270 - 26.67 - 558] = - \frac{10^3 \times 314.7}{200 \times 10^9 \times 83 \times 10^{-6}} \\ &= -0.01896 \text{ m} = -19 \text{ mm} \end{aligned}$$

In order to determine the maximum deflection, its position must first be estimated. In this case, as the slope is positive under the 50 kN load it is reasonable to assume that the maximum deflection point will occur somewhere between the 20 kN and 50 kN loads. For this position, from (2),

$$\begin{aligned} \frac{EI}{10^3} \frac{dy}{dx} &= \frac{60x^2}{2} - 20 \frac{(x-1)^2}{2} - 186 \\ &= 30x^2 - 10x^2 + 20x - 10 - 186 \\ &= 20x^2 + 20x - 196 \end{aligned}$$

But, where the deflection is a maximum, the slope is zero.

$$\therefore 0 = 20x^2 + 20x - 196$$

$$\therefore x = \frac{-20 \pm (400 + 15680)^{1/2}}{40} = \frac{-20 \pm 126.8}{40}$$

i.e. $x = 2.67 \text{ m}$

Then, from (3), the maximum deflection is given by

$$\begin{aligned}\delta_{\max} &= -\frac{10^3}{EI} \left[\frac{60 \times 2.67^3}{6} - \frac{20 \times 1.67^3}{6} - 186 \times 2.67 \right] \\ &= -\frac{10^3 \times 321.78}{200 \times 10^9 \times 83 \times 10^{-6}} = -0.0194 = -19.4 \text{ mm}\end{aligned}$$

In loading situations where this point lies within the portion of a beam covered by a uniformly distributed load the above procedure is cumbersome since it involves the solution of a cubic equation to determine x .

As an alternative procedure it is possible to obtain a reasonable estimate of the position of zero slope, and hence maximum deflection, by sketching the slope diagram, commencing with the slope at either side of the estimated maximum deflection position; slopes will then be respectively positive and negative and the point of zero slope thus may be estimated. Since the slope diagram is generally a curve, the accuracy of the estimate is improved as the points chosen approach the point of maximum deflection.

As an example of this procedure we may re-solve the final part of the question.

Thus, selecting the initial two points as $x = 2$ and $x = 3$,
when $x = 2$,

$$\frac{EI}{10^3} \frac{dy}{dx} = \frac{60 \times 2^2}{2} - \frac{20(1^2)}{2} - 186 = -76$$

when $x = 3$,

$$\frac{EI}{10^3} \frac{dy}{dx} = \frac{60 \times 3^2}{2} - \frac{20(2^2)}{2} - 186 = +44$$

Figure 5.37 then gives a first estimate of the zero slope (maximum deflection) position as $x = 2.63$ on the basis of a straight line between the above-determined values. Recognising the inaccuracy of this assumption, however, it appears reasonable that the required position can

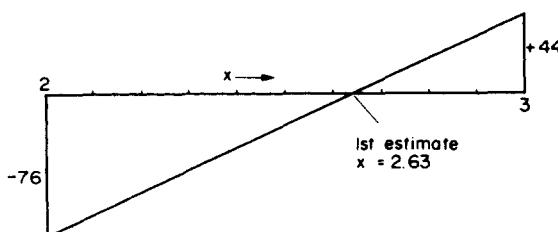


Fig. 5.37.

be more closely estimated as between $x = 2.5$ and $x = 2.7$. Thus, refining the process further, when $x = 2.5$,

$$\frac{EI}{10^3} \frac{dy}{dx} = \frac{60 \times 2.5^2}{2} - \frac{20 \times 1.5^2}{2} - 186 = -21$$

when $x = 2.7$,

$$\frac{EI}{10^3} \frac{dy}{dx} = \frac{60 \times 2.7^2}{2} - \frac{20 \times 1.7^2}{2} - 186 = +3.8$$

Figure 5.38 then gives the improved estimate of

$$x = 2.669$$

which is effectively the same value as that obtained previously.

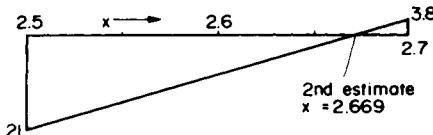


Fig. 5.38.

Example 5.3

Determine the deflection at a point 1 m from the left-hand end of the beam loaded as shown in Fig. 5.39a using Macaulay's method. $EI = 0.65 \text{ MN m}^2$.

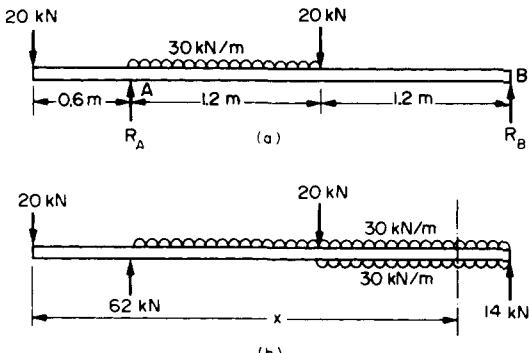


Fig. 5.39.

Solution

Taking moments about B

$$(3 \times 20) + (30 \times 1.2 \times 1.8) + (1.2 \times 20) = 2.4R_A$$

$$\therefore R_A = 62 \text{ kN} \quad \text{and} \quad R_B = 20 + (30 \times 1.2) + 20 - 62 = 14 \text{ kN}$$

Using the modified Macaulay approach for distributed loads over part of a beam introduced in § 5.5 (Fig. 5.39b),

$$\begin{aligned} M_{xx} &= \frac{EI}{10^3} \frac{d^2y}{dx^2} = -20x + 62[(x-0.6)] - 30\left[\frac{(x-0.6)^2}{2}\right] + 30\left[\frac{(x-1.8)^2}{2}\right] - 20[(x-1.8)] \\ \frac{EI}{10^3} \frac{dy}{dx} &= \frac{-20x^2}{2} + 62\left[\frac{(x-0.6)^2}{2}\right] - 30\left[\frac{(x-0.6)^3}{6}\right] + 30\left[\frac{(x-1.8)^3}{6}\right] \\ &\quad - 20\left[\frac{(x-1.8)^2}{2}\right] + A \\ \frac{EI}{10^3} y &= \frac{-20x^3}{6} + 62\left[\frac{(x-0.6)^3}{6}\right] - 30\left[\frac{(x-0.6)^4}{24}\right] + 30\left[\frac{(x-1.8)^4}{24}\right] \\ &\quad - 20\left[\frac{(x-1.8)^3}{6}\right] + Ax + B \end{aligned}$$

Now when $x = 0.6$, $y = 0$,

$$\begin{aligned} \therefore 0 &= -\frac{20 \times 0.6^3}{6} + 0.6A + B \\ 0.72 &= 0.6A + B \end{aligned} \tag{1}$$

and when $x = 3$, $y = 0$,

$$\begin{aligned} \therefore 0 &= -\frac{20 \times 3^3}{6} + \frac{62 \times 2.4^3}{6} - \frac{30 \times 2.4^4}{24} + \frac{30 \times 1.2^4}{24} - \frac{20 \times 1.2^3}{6} + 3A + B \\ &= -90 + 142.848 - 41.472 + 2.592 - 5.76 + 3A + B \\ -8.208 &= 3A + B \end{aligned} \tag{2}$$

$$(2) - (1)$$

$$-8.928 = 2.4A \quad \therefore A = -3.72$$

Substituting in (1),

$$B = 0.72 - 0.6(-3.72) \quad B = 2.952$$

Substituting into the Macaulay deflection equation,

$$\begin{aligned} \frac{EI}{10^3} y &= -\frac{20x^3}{6} + 62\left[\frac{(x-0.6)^3}{6}\right] - 30\left[\frac{(x-0.6)^4}{24}\right] + 30\left[\frac{(x-1.8)^4}{24}\right] \\ &\quad - 20\left[\frac{(x-1.8)^3}{6}\right] - 3.72x + 2.952 \end{aligned}$$

At $x = 1$

$$y = \frac{10^3}{EI} \left[-\frac{20}{6} + \frac{62}{6} \times 0.4^3 - \frac{30 \times 0.4^4}{24} - 3.72 \times 1 + 2.952 \right]$$

$$\begin{aligned}
 &= \frac{10^3}{EI} [-3.33 + 0.661 - 0.032 - 3.72 + 2.952] \\
 &= -\frac{10^3 \times 3.472}{0.65 \times 10^6} = -5.34 \times 10^{-3} \text{ m} = -5.34 \text{ mm}
 \end{aligned}$$

The beam therefore is deflected *downwards* at the given position.

Example 5.4

Calculate the slope and deflection of the beam loaded as shown in Fig. 5.40 at a point 1.6 m from the left-hand end. $EI = 1.4 \text{ MN m}^2$.

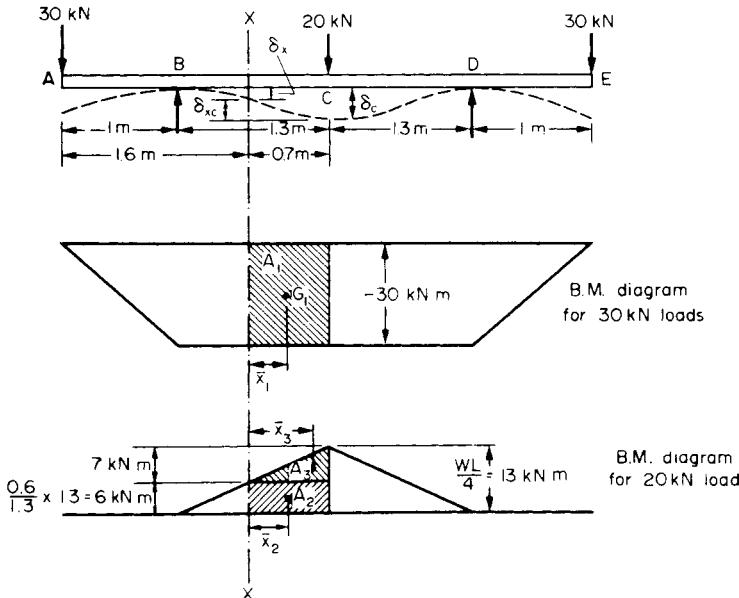


Fig. 5.40.

Solution

Since, by symmetry, the point of zero slope can be located at C a solution can be obtained conveniently using Mohr's method. This is best applied by drawing the B.M. diagrams for the separate effects of (a) the 30 kN loads, and (b) the 20 kN load as shown in Fig. 5.40.

Thus, using the zero slope position C as the datum for the Mohr method, from eqn. (5.20)

$$\begin{aligned}
 \text{slope at } X &= \frac{1}{EI} [\text{area of B.M. diagram between } X \text{ and } C] \\
 &= \frac{10^3}{EI} [(-30 \times 0.7) + (6 \times 0.7) + (\frac{1}{2} \times 7 \times 0.7)] \\
 &= \frac{10^3}{EI} [-21 + 4.2 + 2.45] = -\frac{14.35 \times 10^3}{1.4 \times 10^6} \\
 &= -10.25 \times 10^{-3} \text{ rad}
 \end{aligned}$$

and from eqn. (5.21)

deflection at X relative to the tangent at C

$$\begin{aligned}
 &= \frac{1}{EI} [\text{first moment of area of B.M. diagram between } X \text{ and } C \text{ about } X] \\
 \delta_{XC} &= \frac{10^3}{EI} \left[(-30 \times 0.7 \times 0.35) + (6 \times 0.7 \times 0.35) + (7 \times 0.7 \times \frac{1}{2} \times \frac{2}{3} \times 0.7) \right] \\
 &= \frac{10^3}{EI} [-7.35 + 1.47 + 1.143] = -\frac{10^3 \times 4.737}{1.4 \times 10^6} \\
 &= -3.38 \times 10^{-3} \text{ m} = -3.38 \text{ mm}
 \end{aligned}$$

This must now be subtracted from the deflection of C relative to the support B to obtain the actual deflection at X .

Now deflection of C relative to B

$$\begin{aligned}
 &= \text{deflection of } B \text{ relative to } C \\
 &= \frac{1}{EI} [\text{first moment of area of B.M. diagram between } B \text{ and } C \text{ about } B] \\
 &= \frac{10^3}{EI} [(-30 \times 1.3 \times 0.65) + (13 \times 1.3 \times \frac{1}{2} \times 1.3 \times \frac{2}{3})] \\
 &= \frac{10^3}{EI} [-25.35 + 7.323] = -\frac{18.027 \times 10^3}{1.4 \times 10^6} = -12.88 \times 10^{-3} = -12.88 \text{ mm}
 \end{aligned}$$

$$\therefore \text{required deflection of } X = -(12.88 - 3.38) = -9.5 \text{ mm}$$

Example 5.5

(a) Find the slope and deflection at the tip of the cantilever shown in Fig. 5.41.

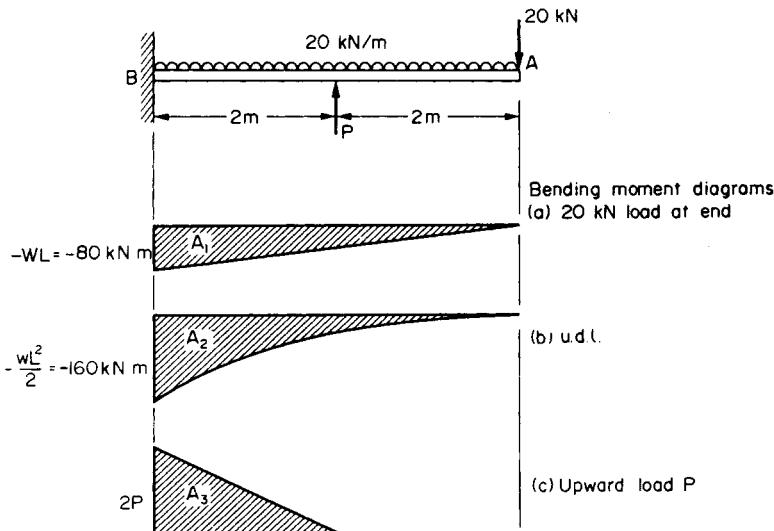


Fig. 5.41.

- (b) What load P must be applied upwards at mid-span to reduce the deflection by half?
 $EI = 20 \text{ MN m}^2$.

Solution

Here again the best approach is to draw separate B.M. diagrams for the concentrated and uniformly distributed loads. Then, since B is a point of zero slope, the Mohr method may be applied.

$$(a) \text{ Slope at } A = \frac{1}{EI} [\text{area of B.M. diagram between } A \text{ and } B]$$

$$\begin{aligned} &= \frac{1}{EI} [A_1 + A_2] = \frac{10^3}{EI} [\{\frac{1}{2} \times 4 \times (-80)\} + \{\frac{1}{3} \times 4 \times (-160)\}] \\ &= \frac{10^3}{EI} [-160 - 213.3] = \frac{373.3 \times 10^3}{20 \times 10^6} \\ &= 18.67 \times 10^{-3} \text{ rad} \end{aligned}$$

$$\text{Deflection of } A = \frac{1}{EI} [\text{first moment of area of B.M. diagram between } A \text{ and } B \text{ about } A]$$

$$\begin{aligned} &= \frac{10^3}{EI} \left[\left(\frac{-80 \times 4 \times \frac{2}{3} \times 4}{2} \right) + \left(\frac{-160 \times 4 \times \frac{3}{4} \times 4}{3} \right) \right] \\ &= \frac{-10^3}{EI} [426.6 + 640] = -\frac{1066.6 \times 10^3}{20 \times 10^6} = -53.3 \times 10^{-3} \text{ m} = -53 \text{ mm} \end{aligned}$$

- (b) When an extra load P is applied upwards at mid-span its effect on the deflection is required to be $\frac{1}{2} \times 53.3 = 26.67 \text{ mm}$. Thus

$$\begin{aligned} 26.67 \times 10^{-3} &= \frac{1}{EI} [\text{first moment of area of B.M. diagram for } P \text{ about } A] \\ &= \frac{10^3}{EI} [\frac{1}{2} \times 2P \times 2(2 + \frac{2}{3} \times 2)] \end{aligned}$$

$$\therefore P = \frac{26.67 \times 20 \times 10^6}{10^3 \times 6.66} = 80 \times 10^3 \text{ N}$$

The required load at mid-span is 80 kN.

Example 5.6

The uniform beam of Fig. 5.42 carries the loads indicated. Determine the B.M. at B and hence draw the S.F. and B.M. diagrams for the beam.

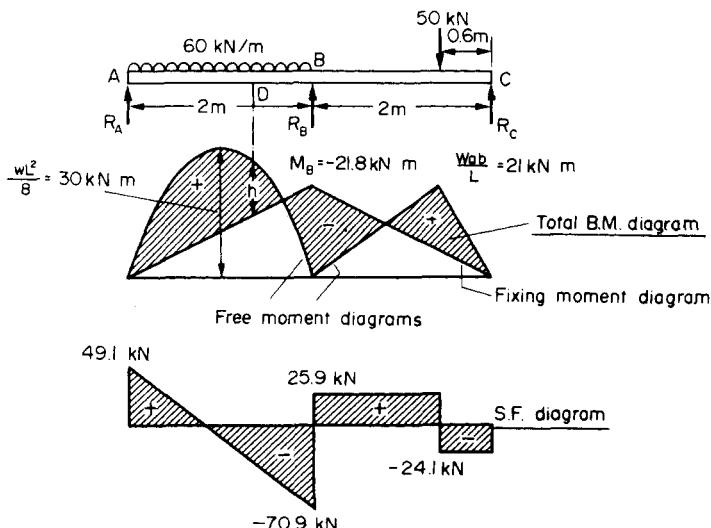


Fig. 5.42.

Solution

Applying the three-moment equation (5.24) to the beam we have,

$$-M_A L_1 - 2M_B(L_1 + L_2) - M_C L_2 = \frac{wL_1^3}{4} + \frac{Wa}{L_2}(L_2^2 - a^2)$$

(Note that the dimension a is always to the "outside" support of the particular span carrying the concentrated load.)

Now with A and C simply supported

$$M_A = M_C = 0$$

$$\therefore -2M_B(2+2) = \frac{60 \times 10^3 \times 2^3}{4} + \frac{50 \times 10^3 \times 0.6}{2} (2^2 - 0.6^2)$$

$$-8M_B = (120 + 54.6)10^3 = 174.6 \times 10^3$$

$$M_B = -21.8 \text{ kN m}$$

With the normal B.M. sign convention the B.M. at B is therefore -21.8 kN m .

Taking moments about B (forces to left),

$$2R_A - (60 \times 10^3 \times 2 \times 1) = -21.8 \times 10^3$$

$$R_A = \frac{1}{2}(-21.8 + 120)10^3 = 49.1 \text{ kN}$$

Taking moments about B (forces to right),

$$2R_C - (50 \times 10^3 \times 1.4) = -21.8 \times 10^3$$

$$R_C = \frac{1}{2}(-21.8 + 70) = 24.1 \text{ kN}$$

and, since the total load

$$= R_A + R_B + R_C = 50 + (60 \times 2) = 170 \text{ kN}$$

$$\therefore R_B = 170 - 49.1 - 24.1 = 96.8 \text{ kN}$$

The B.M. and S.F. diagrams are then as shown in Fig. 5.42. The fixing moment diagram can be directly subtracted from the free moment diagrams since M_B is negative. The final B.M. diagram is then as shown shaded, values at any particular section being measured from the fixing moment line as datum,

e.g.

$$\text{B.M. at } D = +h \text{ (to scale)}$$

Example 5.7

A beam $ABCDE$ is continuous over four supports and carries the loads shown in Fig. 5.43. Determine the values of the fixing moment at each support and hence draw the S.F. and B.M. diagrams for the beam.

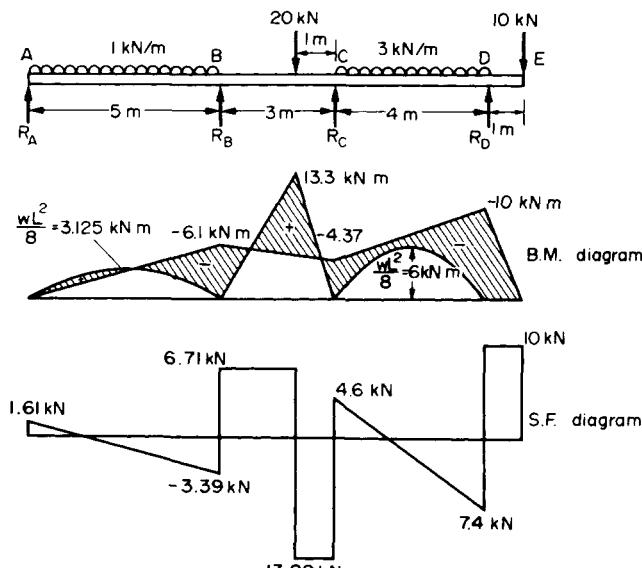


Fig. 5.43.

Solution

By inspection, $M_A = 0$ and $M_D = -1 \times 10 = -10 \text{ kN m}$

Applying the three-moment equation for the first two spans,

$$\begin{aligned}
 -M_A L_1 - 2M_B(L_1 + L_2) - M_C L_2 &= \frac{wL_1^3}{4} + \frac{wa}{L_2}(L_2^2 - a^2) \\
 0 - 2M_B(5 + 3) - 3M_C &= \left[\frac{1 \times 5^3}{4} + \frac{20 \times 1}{3}(3^2 - 1^2) \right] 10^3 \\
 -16M_B - 3M_C &= (31.25 + 53.33) 10^3 \\
 -16M_B - 3M_C &= 84.58 \times 10^3
 \end{aligned} \tag{1}$$

and, for the second and third spans,

$$\begin{aligned} -M_B L_2 - 2M_C(L_2 + L_3) - M_D L_3 &= \frac{w a}{L_2} (L_2^2 - a^2) + \frac{w L_3^3}{4} \\ -3M_B - 2M_C(3+4) - (-10 \times 10^3)4 &= \left[\frac{20 \times 2}{3} (3^2 - 2^2) + \frac{(3 \times 4^3)}{4} \right] 10^3 \\ -3M_B - 14M_C + (40 \times 10^3) &= (66.67 + 48)10^3 \\ -3M_B - 14M_C &= 74.67 \times 10^3 \end{aligned} \quad (2)$$

$$(2) \times 16/3 \quad -16M_B - 74.67M_C = 398.24 \times 10^3 \quad (3)$$

$$(3) - (1) \quad -71.67M_C = 313.66 \times 10^3$$

$$M_C = -4.37 \times 10^3 \text{ Nm}$$

Substituting in (1),

$$\begin{aligned} -16M_B - 3(-4.37 \times 10^3) &= 84.58 \times 10^3 \\ M_B &= -\frac{(84.58 - 13.11)10^3}{16} \\ &= -4.47 \text{ kN m} \end{aligned}$$

Moments about *B* (to left),

$$\begin{aligned} R_A \times 5 - \left(\frac{1 \times 10^3}{2} \times 5^2 \right) &= -4.47 \times 10^3 \\ 5R_A &= (-4.47 + 12.5)10^3 \\ R_A &= 1.61 \text{ kN} \end{aligned}$$

Moments about *C* (to left),

$$\begin{aligned} R_A \times 8 - (1 \times 10^3 \times 5 \times 5.5) + (R_B \times 3) - (20 \times 10^3 \times 1) &= -4.37 \times 10^3 \\ 3R_B &= -4.37 \times 10^3 + 27.5 \times 10^3 + 20 \times 10^3 - 8 \times 1.61 \times 10^3 \\ 3R_B &= 30.3 \times 10^3 \\ R_B &= 10.1 \text{ kN} \end{aligned}$$

Moments about *C* (to right),

$$\begin{aligned} (-10 \times 10^3 \times 5) + 4R_D - (3 \times 10^3 \times 4 \times 2) &= -4.37 \times 10^3 \\ 4R_D &= (-4.37 + 50 + 24)10^3 \\ R_D &= 17.4 \text{ kN} \end{aligned}$$

Then, since

$$\begin{aligned} R_A + R_B + R_C + R_D &= 47 \text{ kN} \\ 1.61 + 10.1 + R_C + 17.4 &= 47 \\ R_C &= 17.9 \text{ kN} \end{aligned}$$

This value should then be checked by taking moments to the right of *B*,

$$(-10 \times 10^3 \times 8) + 7R_D + 3R_C - (3 \times 10^3 \times 4 \times 5) - (20 \times 10^3 \times 2) = -4.47 \times 10^3$$

$$3R_C = (-4.47 + 40 + 60 + 80 - 121.8)10^3 = 53.73 \times 10^3$$

$$R_C = 17.9 \text{ kN}$$

The S.F. and B.M. diagrams for the beam are shown in Fig. 5.43.

Example 5.8

Using the finite difference method, determine the central deflection of a simply-supported beam carrying a uniformly distributed load ω over its complete span. The beam can be assumed to have constant flexural rigidity EI throughout.

Solution

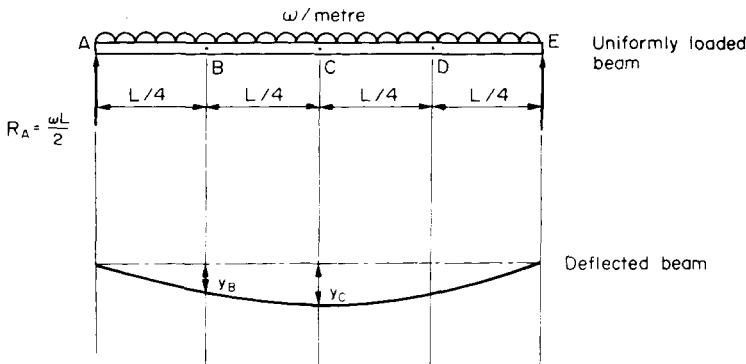


Fig. 5.44.

As a simple demonstration of the finite difference approach, assume that the beam is divided into only four equal segments (thus reducing the accuracy of the solution from that which could be achieved with a greater number of segments).

$$\text{Then, B.M. at } B = \frac{\omega L}{2} \times \frac{L}{4} - \frac{\omega L}{4} \cdot \frac{L}{8} = \frac{3\omega L^2}{32} = M_B$$

but, from eqn. (5.30):

$$\frac{M_B}{EI} = \frac{1}{h^2} (y_{i+1} - 2y_i + y_{i-1})$$

$$\therefore \frac{I}{EI} \left(\frac{3\omega L^2}{32} \right) = \frac{1}{(L/4)^2} (y_c - 2y_B + y_A)$$

and, since $y_A = 0$,

$$\frac{3\omega L^2}{512 EI} = y_c - 2y_B. \quad (1)$$

Similarly $\text{B.M. at } C = \frac{\omega L}{2} \cdot \frac{L}{2} - \frac{\omega L}{2} \cdot \frac{L}{4} = \frac{\omega L^2}{8} = M_c$.

and, from eqn. (5.30)

$$\frac{1}{EI} \left(\frac{\omega L^2}{8} \right) = \frac{1}{(L/4)^2} (y_B - 2y_c + y_D)$$

Now, from symmetry, $y_D = y_B$

$$\therefore \frac{\omega L^4}{128EI} = 2y_B - 2y_c \quad (2)$$

Adding eqns. (1) and (2);

$$-y_c = \frac{\omega L^4}{128EI} + \frac{3\omega L^4}{512EI}$$

$$\therefore y_c = \frac{-7\omega L^4}{512EI} = -0.0137 \frac{\omega L^4}{EI}$$

the negative sign indicating a downwards deflection as expected. This value compares with the "exact" value of:

$$y_c = \frac{5\omega L^4}{384EI} = -0.01302 \frac{\omega L^4}{EI}$$

a difference of about 5 %. As stated earlier, this comparison could be improved by selecting more segments but, nevertheless, it is remarkably accurate for the very small number of segments chosen.

Example 5.9

The statically indeterminate propped cantilever shown in Fig. 5.45 is propped at *B* and carries a central load *W*. It can be assumed to have a constant flexural rigidity *EI* throughout.

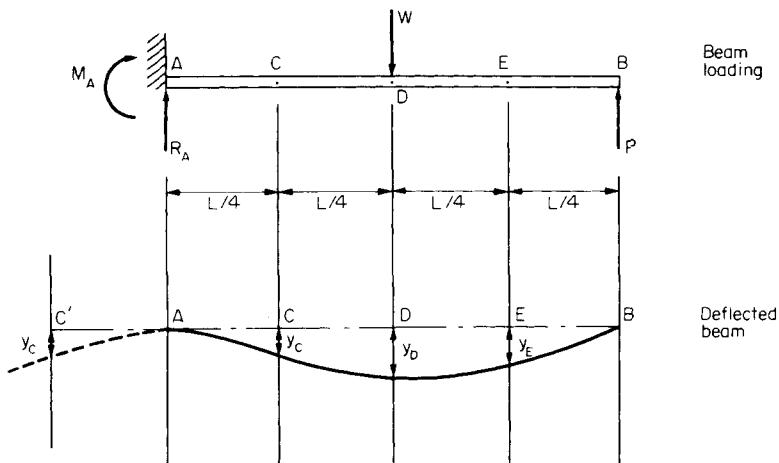


Fig. 5.45.

Determine, using a finite difference approach, the values of the reaction at the prop and the central deflection.

Solution

Whilst at first sight, perhaps, there appears to be a number of redundancies in the cantilever loading condition, in fact the problem reduces to that of a single redundancy, say the unknown prop load P , since with a knowledge of P the other "unknowns" M_A and R_A can be evaluated easily.

Thus, again for simplicity, consider the beam divided into four equal segments giving three unknown deflections y_C , y_D and y_E (assuming zero deflection at the prop B) and one redundancy. Four equations are thus required for solution and these may be obtained by applying the difference equation at four selected points on the beam:

From eqn. (5.30)

$$\text{B.M. at } E = M_E = \frac{PL}{4} = \frac{EI}{(L/4)^2} (y_B - 2y_E + y_D)$$

$$\text{but } y_B = 0 \quad y_D - 2y_E = \frac{PL^3}{64EI} \quad (1)$$

$$\text{B.M. at } D = M_D = \frac{PL}{2} = \frac{EI}{(L/4)^2} (y_E - 2y_D + y_C)$$

$$\therefore y_E - 2y_D + y_C = \frac{PL^3}{32EI} \quad (2)$$

$$\text{B.M. at } C = M_C = P \cdot \frac{3L}{4} - \frac{WL}{4} = \frac{EI}{(L/4)^2} (y_A - 2y_C + y_D)$$

But $y_A = 0$

$$\therefore y_D - 2y_C = \frac{3PL^3}{64} - \frac{WL^3}{64} \quad (3)$$

At point A it is necessary to introduce the mirror image of the beam giving point C' to the left of A with a deflection $y'_C = y_C$ in order to produce the fourth equation.

Then:

$$\text{B.M. at } A = M_A = PL - \frac{WL}{2} = \frac{EI}{(L/4)^2} (y'_C - 2y_A + y_C)$$

$$\text{and again since } y_A = 0 \quad y_C = \frac{PL^3}{32} - \frac{WL^3}{64} \quad (4)$$

Solving equations (1) to (4) simultaneously gives the required prop load:

$$P = \frac{7W}{22} = 0.318 \text{ W},$$

and the central deflection:

$$y_D = -\frac{17WL^3}{1408EI} = -0.0121 \frac{WL^3}{EI}$$

Problems

5.1 (A/B). A beam of length 10 m is symmetrically placed on two supports 7 m apart. The loading is 15 kN/m between the supports and 20 kN at each end. What is the central deflection of the beam?
 $E = 210 \text{ GN/m}^2$, $I = 200 \times 10^{-6} \text{ m}^4$.

[6.8 mm.]

5.2 (A/B). Derive the expression for the maximum deflection of a simply supported beam of negligible weight carrying a point load at its mid-span position. The distance between the supports is L , the second moment of area of the cross-section is I and the modulus of elasticity of the beam material is E .

The maximum deflection of such a simply supported beam of length 3 m is 4.3 mm when carrying a load of 200 kN at its mid-span position. What would be the deflection at the free end of a cantilever of the same material, length and cross-section if it carries a load of 100 kN at a point 1.3 m from the free end? [13.4 mm.]

5.3 (A/B). A horizontal beam, simply supported at its ends, carries a load which varies uniformly from 15 kN/m at one end to 60 kN/m at the other. Estimate the central deflection if the span is 7 m, the section 450 mm deep and the maximum bending stress 100 MN/m². $E = 210 \text{ GN/m}^2$. [U.L.] [21.9 mm.]

5.4 (A/B). A beam AB , 8 m long, is freely supported at its ends and carries loads of 30 kN and 50 kN at points 1 m and 5 m respectively from A . Find the position and magnitude of the maximum deflection.
 $E = 210 \text{ GN/m}^2$, $I = 200 \times 10^{-6} \text{ m}^4$.

[14.4 mm.]

5.5 (A/B). A beam 7 m long is simply supported at its ends and loaded as follows: 120 kN at 1 m from one end A , 20 kN at 4 m from A and 60 kN at 5 m from A . Calculate the position and magnitude of the maximum deflection. The second moment of area of the beam section is $400 \times 10^{-6} \text{ m}^4$ and E for the beam material is 210 GN/m^2 .

[9.8 mm at 3.474 m.]

5.6 (B). A beam $ABCD$, 6 m long, is simply-supported at the right-hand end D and at a point B 1 m from the left-hand end A . It carries a vertical load of 10 kN at A , a second concentrated load of 20 kN at C , 3 m from D , and a uniformly distributed load of 10 kN/m between C and D . Determine the position and magnitude of the maximum deflection if $E = 208 \text{ GN/m}^2$ and $I = 35 \times 10^{-6} \text{ m}^4$. [3.553 m from A , 11.95 mm.]

5.7 (B). A 3 m long cantilever ABC is built-in at A , partially supported at B , 2 m from A , with a force of 10 kN and carries a vertical load of 20 kN at C . A uniformly distributed load of 5 kN/m is also applied between A and B . Determine a) the values of the vertical reaction and built-in moment at A and b) the deflection of the free end C of the cantilever.

Develop an expression for the slope of the beam at any position and hence plot a slope diagram. $E = 208 \text{ GN/m}^2$ and $I = 24 \times 10^{-6} \text{ m}^4$. [20 kN, 50 kN m, -15 mm.]

5.8 (B). Develop a general expression for the slope of the beam of question 5.6 and hence plot a slope diagram for the beam. Use the slope diagram to confirm the answer given in question 5.6 for the position of the maximum deflection of the beam.

5.9 (B). What would be the effect on the end deflection for question 5.7, if the built-in end A were replaced by a simple support at the same position and point B becomes a full simple support position (i.e. the force at B is no longer 10 kN). What general observation can you make about the effect of built-in constraints on the stiffness of beams? [5.7 mm.]

5.10 (B). A beam AB is simply supported at A and B over a span of 3 m. It carries loads of 50 kN and 40 kN at 0.6 m and 2 m respectively from A , together with a uniformly distributed load of 60 kN/m between the 50 kN and 40 kN concentrated loads. If the cross-section of the beam is such that $I = 60 \times 10^{-6} \text{ m}^4$ determine the value of the deflection of the beam under the 50 kN load. $E = 210 \text{ GN/m}^2$. Sketch the S.F. and B.M. diagrams for the beam. [3.7 mm.]

5.11 (B). Obtain the relationship between the B.M., S.F., and intensity of loading of a laterally loaded beam.

A simply supported beam of span L carries a distributed load of intensity kx^2/L^2 where x is measured from one support towards the other. Find:

(a) the location and magnitude of the greatest bending moment;

(b) the support reactions.

[U.Birm.] [0.63L, 0.0393kL², kL/12, kL/4.]

5.12 (B). A uniform beam 4 m long is simply supported at its ends, where couples are applied, each 3 kN m in magnitude but opposite in sense. If $E = 210 \text{ GN/m}^2$ and $I = 90 \times 10^{-6} \text{ m}^4$ determine the magnitude of the deflection at mid-span.

What load must be applied at mid-span to reduce the deflection by half?

[0.317 mm, 2.25 kN.]

5.13 (B). A 500 mm × 175 mm steel beam of length 8 m is supported at the left-hand end and at a point 1.6 m from the right-hand end. The beam carries a uniformly distributed load of 12 kN/m on its whole length, an additional uniformly distributed load of 18 kN/m on the length between the supports and a point load of 30 kN at the right-hand end. Determine the slope and deflection of the beam at the section midway between the supports and also at the right-hand end. EI for the beam is $1.5 \times 10^8 \text{ Nm}^2$. [U.L.] [1.13 × 10⁻⁴, 3.29 mm, 9.7 × 10⁻⁴, 1.71 mm.]

5.14 (B). A cantilever, 2.6 m long, carrying a uniformly distributed load w along the entire length, is propped at its free end to the level of the fixed end. If the load on the prop is then 30 kN, calculate the value of w . Determine also the slope of the beam at the support. If any formula for deflection is used it must first be proved. $E = 210 \text{ GN/m}^2$; $I = 4 \times 10^{-6} \text{ m}^4$.

[U.E.I.] [30.8 kN/m, 0.014 rad.]

5.15 (B). A beam ABC of total length L is simply supported at one end A and at some point B along its length. It carries a uniformly distributed load of w per unit length over its whole length. Find the optimum position of B so that the greatest bending moment in the beam is as low as possible. [U.Birm.] [$L/2$.]

5.16 (B). A beam AB , of constant section, depth 400 mm and $I_{\max} = 250 \times 10^{-6} \text{ m}^4$, is hinged at A and simply supported on a non-yielding support at C . The beam is subjected to the given loading (Fig. 5.46). For this loading determine (a) the vertical deflection of B ; (b) the slope of the tangent to the bent centre line at C . $E = 80 \text{ GN/m}^2$.

[I.Struct.E.] [27.3 mm, 0.0147 rad.]

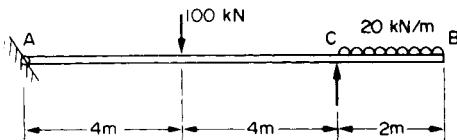


Fig. 5.46.

5.17 (B). A simply supported beam AB is 7 m long and carries a uniformly distributed load of 30 kN/m run. A couple is applied to the beam at a point C , 2.5 m from the left-hand end, A , the couple being clockwise in sense and of magnitude 70 kNm. Calculate the slope and deflection of the beam at a point D , 2 m from the left-hand end. Take $EI = 5 \times 10^7 \text{ Nm}^2$.

[E.M.E.U.] [5.78 × 10⁻³ rad, 16.5 mm.]

5.18 (B). A uniform horizontal beam ABC is 0.75 m long and is simply supported at A and B , 0.5 m apart, by supports which can resist upward or downward forces. A vertical load of 50 N is applied at the free end C , which produces a deflection of 5 mm at the centre of span AB . Determine the position and magnitude of the maximum deflection in the span AB , and the magnitude of the deflection at C . [E.I.E.] [5.12 mm (upwards), 20.1 mm.]

5.19 (B). A continuous beam ABC rests on supports at A , B and C . The portion AB is 2 m long and carries a central concentrated load of 40 kN, and BC is 3 m long with a u.d.l. of 60 kN/m on the complete length. Draw the S.F. and B.M. diagrams for the beam.

[-3.25, 148.75, 74.5 kN (Reactions); $M_B = -46.5 \text{ kN m}$.]

5.20 (B). State Clapeyron's theorem of three moments. A continuous beam $ABCD$ is constructed of built-up sections whose effective flexural rigidity EI is constant throughout its length. Bay lengths are $AB = 1 \text{ m}$, $BC = 5 \text{ m}$, $CD = 4 \text{ m}$. The beam is simply supported at B , C and D , and carries point loads of 20 kN and 60 kN at A and midway between C and D respectively, and a distributed load of 30 kN/m over BC . Determine the bending moments and vertical reactions at the supports and sketch the B.M. and S.F. diagrams.

[U.Birm.] [-20, -66.5, 0 kN m; 85.7, 130.93, 13.37 kN.]

5.21 (B). A continuous beam $ABCD$ is simply supported over three spans $AB = 1 \text{ m}$, $BC = 2 \text{ m}$ and $CD = 2 \text{ m}$. The first span carries a central load of 20 kN and the third span a uniformly distributed load of 30 kN/m. The central span remains unloaded. Calculate the bending moments at B and C and draw the S.F. and B.M. diagrams. The supports remain at the same level when the beam is loaded.

[1.36, -7.84 kN m; 11.36, 4.03, 38.52, 26.08 kN (Reactions).]

5.22 (B). A beam, simply supported at its ends, carries a load which increases uniformly from 15 kN/m at the left-hand end to 100 kN/m at the right-hand end. If the beam is 5 m long find the equation for the rate of loading and, using this, the deflection of the beam at mid-span if $E = 200 \text{ GN/m}^2$ and $I = 600 \times 10^{-6} \text{ m}^4$.

[$w = -(15 + 85x/L)$; 3.9 mm.]

5.23 (B). A beam 5 m long is firmly fixed horizontally at one end and simply supported at the other by a prop. The beam carries a uniformly distributed load of 30 kN/m run over its whole length together with a concentrated load of 60 kN at a point 3 m from the fixed end. Determine:

(a) the load carried by the prop if the prop remains at the same level as the end support;

(b) the position of the point of maximum deflection.

[B.P.] [82.16 kN; 2.075 m.]

5.24 (B/C). A continuous beam $ABCDE$ rests on five simple supports A , B , C , D and E . Spans AB and BC carry a u.d.l. of 60 kN/m and are respectively 2 m and 3 m long. CD is 2.5 m long and carries a concentrated load of 50 kN at 1.5 m from C . DE is 3 m long and carries a concentrated load of 50 kN at the centre and a u.d.l. of 30 kN/m. Draw the B.M. and S.F. diagrams for the beam.

[Fixing moments: 0, -44.91, -25.1, -38.95, 0 kN m. Reactions: 37.55, 179.1, 97.83, 118.5, 57.02 kN.]

CHAPTER 6

BUILT-IN BEAMS

Summary

The maximum bending moments and maximum deflections for built-in beams with standard loading cases are as follows:

MAXIMUM B.M. AND DEFLECTION FOR BUILT-IN BEAMS

Loading case	Maximum B.M.	Maximum deflection
Central concentrated load W	$\frac{WL}{8}$	$\frac{WL^3}{192EI}$
Uniformly distributed load w /metre (total load W)	$\frac{wL^2}{12} = \frac{WL}{12}$	$\frac{wL^4}{384EI} = \frac{WL^3}{384EI}$
Concentrated load W not at mid-span	$\frac{Wab^2}{L^2}$ or $\frac{Wa^2b}{L^2}$	$\frac{2Wa^3b^2}{3EI(L+2a)^2} \quad \text{at } x = \frac{2aL}{(L+2a)}$ $\left(\text{where } a < \frac{L}{2} \right)$ $= \frac{Wa^3b^3}{3EIL^3} \quad \text{under load}$
Distributed load w' varying in intensity between $x = x_1$ and $x = x_2$	$M_A = - \int_{x_1}^{x_2} \frac{w'(L-x)^2}{L^2} dx$ $M_B = - \int_{x_1}^{x_2} \frac{w'(L-x)x^2}{L^2} dx$	

Effect of movement of supports

If one end B of an initially horizontal built-in beam AB moves through a distance δ relative to end A , end moments are set up of value

$$M_A = -M_B = \frac{6EI\delta}{L^2}$$

and the reactions at each support are

$$R_A = -R_B = \frac{12EI\delta}{L^3}$$

Thus, in most practical situations where loaded beams sink at the supports the above values represent *changes* in fixing moment and reaction values, their directions being indicated in Fig. 6.6.

Introduction

When both ends of a beam are rigidly fixed the beam is said to be *built-in*, *encastered* or *encastré*. Such beams are normally treated by a modified form of Mohr's area-moment method or by Macaulay's method.

Built-in beams are assumed to have zero slope at each end, so that the total change of slope along the span is zero. Thus, from Mohr's first theorem,

$$\text{area of } \frac{M}{EI} \text{ diagram across the span} = 0$$

or, if the beam is uniform, EI is constant, and

$$\text{area of B.M. diagram} = 0 \quad (6.1)$$

Similarly, if both ends are level the deflection of one end relative to the other is zero. Therefore, from Mohr's second theorem:

$$\text{first moment of area of } \frac{M}{EI} \text{ diagram about one end} = 0$$

and, if EI is constant,

$$\text{first moment of area of B.M. diagram about one end} = 0 \quad (6.2)$$

To make use of these equations it is convenient to break down the B.M. diagram for the built-in beam into two parts:

- (a) that resulting from the loading, assuming simply supported ends, and known as the *free-moment diagram*;
- (b) that resulting from the end moments or fixing moments which must be applied at the ends to keep the slopes zero and termed the *fixing-moment diagram*.

6.1. Built-in beam carrying central concentrated load

Consider the centrally loaded built-in beam of Fig. 6.1. A_a is the area of the free-moment diagram and A_b that of the fixing-moment diagram.

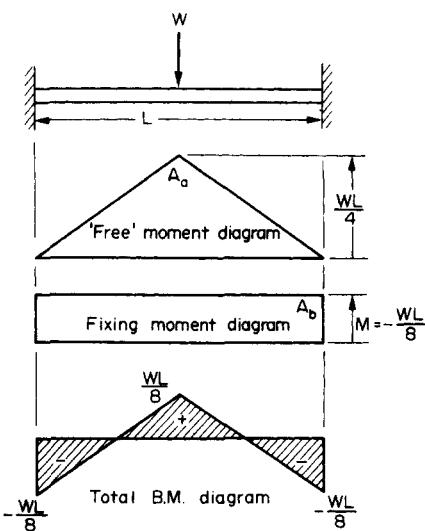


Fig. 6.1.

By symmetry the fixing moments are equal at both ends. Now from eqn. (6.1)

$$A_a + A_b = 0$$

$$\therefore \frac{1}{2} \times L \times \frac{WL}{4} = -ML$$

$$M = \frac{WL}{8} \quad (6.3)$$

The B.M. diagram is therefore as shown in Fig. 6.1, the maximum B.M. occurring at both the ends and the centre.

Applying Mohr's second theorem for the deflection at mid-span,

$$\begin{aligned} \delta &= \left[\text{first moment of area of B.M. diagram between centre and one end about the centre} \right] \times \frac{1}{EI} \\ &= \frac{1}{EI} \left[\frac{1}{2} \left(\frac{1}{2} \times \frac{WL}{4} \times L \right) \left(\frac{1}{3} \times \frac{L}{2} \right) + \left(\frac{ML}{2} \times \frac{L}{4} \right) \right] \\ &= \frac{1}{EI} \left[\frac{WL^3}{96} + \frac{ML^2}{8} \right] = \frac{1}{EI} \left[\frac{WL^3}{96} - \frac{WL^3}{64} \right] \\ &= -\frac{WL^3}{192EI} \quad (\text{i.e. downward deflection}) \end{aligned} \quad (6.4)$$

6.2. Built-in beam carrying uniformly distributed load across the span

Consider now the uniformly loaded beam of Fig. 6.2.

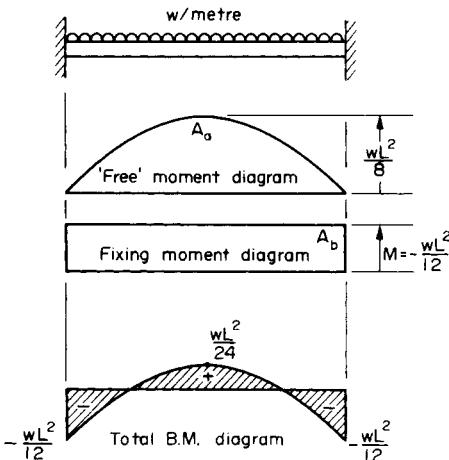


Fig. 6.2.

Again, for zero change of slope along the span,

$$A_a + A_b = 0$$

$$\therefore \frac{2}{3} \times \frac{wL^2}{8} \times L = -ML$$

$$\therefore M = \frac{wL^2}{12} \quad (6.5)$$

The deflection at the centre is again given by Mohr's second theorem as the moment of one-half of the B.M. diagram about the centre.

$$\begin{aligned} \therefore \delta &= \left[\left(\frac{2}{3} \times \frac{wL^2}{8} \times \frac{L}{2} \right) \left(\frac{3}{8} \times \frac{L}{2} \right) + \left(\frac{ML}{2} \times \frac{L}{4} \right) \right] \frac{1}{EI} \\ &= \frac{1}{EI} \left[\frac{3wL^4}{384} + \frac{ML^2}{8} \right] = \frac{1}{EI} \left[\frac{3wL^4}{384} - \frac{wL^4}{96} \right] \\ &= -\frac{wL^4}{384EI} \end{aligned} \quad (6.6)$$

The negative sign again indicates a downwards deflection.

6.3. Built-in beam carrying concentrated load offset from the centre

Consider the loaded beam of Fig. 6.3.

Since the slope at both ends is zero the change of slope across the span is zero, i.e. the total area between A and B of the B.M. diagram is zero (Mohr's theorem).

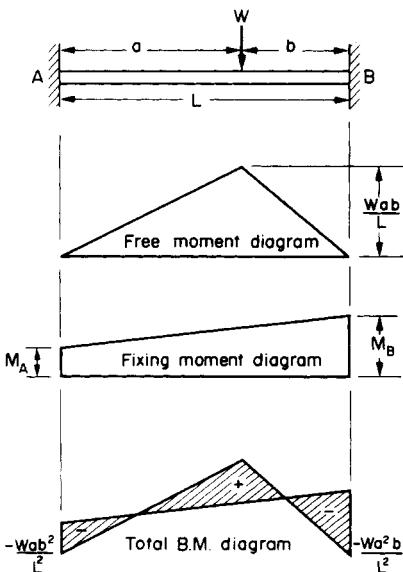


Fig. 6.3.

$$\therefore \left(\frac{1}{2} \times \frac{Wab}{L} \times L \right) + \frac{1}{2} (M_A + M_B) L = 0$$

$$M_A + M_B = - \frac{Wab}{L} \quad (1)$$

Also the deflection of *A* relative to *B* is zero; therefore the moment of the B.M. diagram between *A* and *B* about *A* is zero.

$$\therefore \left[\frac{1}{2} \times \frac{Wab}{L} \times a \right] \frac{2a}{3} + \left[\frac{1}{2} \times \frac{Wab}{L} \times b \right] \left(a + \frac{b}{3} \right) + \left(\frac{1}{2} M_A L \times \frac{L}{3} \right) + \left(\frac{1}{2} M_B L \times \frac{2L}{3} \right) = 0$$

$$\frac{L^2}{6} (M_A + 2M_B) + \frac{Wa^3b}{3L} + \frac{Wab^2}{3L} \left(a + \frac{b}{3} \right) = 0$$

$$M_A + 2M_B = - \frac{Wab}{L^3} [2a^2 + 3ab + b^2] \quad (2)$$

Subtracting (1),

$$M_B = - \frac{Wab}{L^3} [2a^2 + 3ab + b^2 - L^2]$$

but $L = a + b$,

$$\begin{aligned} \therefore M_B &= - \frac{Wab}{L^3} [2a^2 + 3ab + b^2 - a^2 - 2ab - b^2] \\ &= - \frac{Wab}{L^3} [a^2 + ab] = - \frac{Wa^2bL}{L^3} \\ &= - \frac{Wa^2b}{L^2} \end{aligned} \quad (6.7)$$

Substituting in (1),

$$\begin{aligned}
 M_A &= -\frac{Wab}{L} + \frac{Wa^2b}{L^2} \\
 &= -\frac{Wab(a+b)}{L^2} + \frac{Wa^2b}{L^2} \\
 &= -\frac{Wab^2}{L^2}
 \end{aligned} \tag{6.8}$$

6.4. Built-in beam carrying a non-uniform distributed load

Let w' be the distributed load varying in intensity along the beam as shown in Fig. 6.4. On a short length dx at a distance x from A there is a load of $w'dx$. Contribution of this load to M_A

$$\begin{aligned}
 &= -\frac{Wab^2}{L^2} \quad (\text{where } W = w'dx) \\
 &= -\frac{w'dx \times x(L-x)^2}{L^2}
 \end{aligned}$$

$$\therefore \text{total } M_A = -\int_0^L \frac{w'x(L-x)^2 dx}{L^2} \tag{6.9}$$

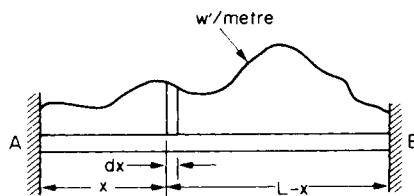


Fig. 6.4. Built-in (*encastré*) beam carrying non-uniform distributed load.

Similarly,

$$M_B = -\int_0^L \frac{w'(L-x)x^2}{L^2} dx \tag{6.10}$$

If the distributed load is across only part of the span the limits of integration must be changed to take account of this: i.e. for a distributed load w' applied between $x = x_1$ and $x = x_2$ and varying in intensity,

$$M_A = -\int_{x_1}^{x_2} \frac{w'x(L-x)^2}{L^2} dx \tag{6.11}$$

$$M_B = -\int_{x_1}^{x_2} \frac{w'(L-x)x^2}{L^2} dx \tag{6.12}$$

6.5. Advantages and disadvantages of built-in beams

Provided that perfect end fixing can be achieved, built-in beams carry smaller maximum B.M.s (and hence are subjected to smaller maximum stresses) and have smaller deflections than the corresponding simply supported beams with the same loads applied; in other words built-in beams are stronger and stiffer. Although this would seem to imply that built-in beams should be used whenever possible, in fact this is not the case in practice. The principal reasons are as follows:

- (1) The need for high accuracy in aligning the supports and fixing the ends during erection increases the cost.
- (2) Small subsidence of either support can set up large stresses.
- (3) Changes of temperature can also set up large stresses.
- (4) The end fixings are normally sensitive to vibrations and fluctuations in B.M.s, as in applications introducing rolling loads (e.g. bridges, etc.).

These disadvantages can be reduced, however, if hinged joints are used at points on the beam where the B.M. is zero, i.e. at *points of inflexion or contraflexure*. The beam is then effectively a central beam supported on two end cantilevers, and for this reason the construction is sometimes termed the *double-cantilever* construction. The beam is then free to adjust to changes in level of the supports and changes in temperature (Fig. 6.5).

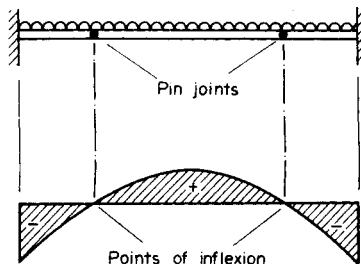


Fig. 6.5. Built-in beam using “double-cantilever” construction.

6.6. Effect of movement of supports

Consider a beam AB initially unloaded with its ends at the same level. If the slope is to remain horizontal at each end when B moves through a distance δ relative to end A , the moments must be as shown in Fig. 6.6. Taking moments about B

$$R_A \times L = M_A + M_B$$

and, by symmetry,

$$M_A = M_B = M$$

$$R_A = \frac{2M}{L}$$

Similarly,

$$R_B = \frac{2M}{L}$$

in the direction shown.

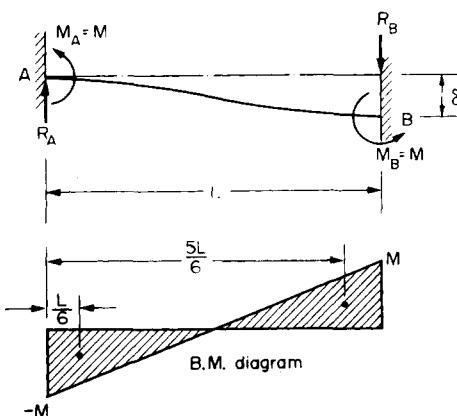


Fig. 6.6. Effect of support movement on B.M.s.

Now from Mohr's second theorem the deflection of A relative to B is equal to the first moment of area of the B.M. diagram about $A \times 1/EI$.

$$\begin{aligned}\therefore \delta &= \left[\left(-\frac{1}{2}M \times \frac{L}{2} \right) \frac{L}{6} + \left(\frac{1}{2}M \times \frac{L}{2} \right) \frac{5L}{6} \right] \frac{1}{EI} \\ &= \frac{ML^2}{24EI} (-1 + 5) = \frac{ML^2}{6EI}\end{aligned}\quad (6.13)$$

$$\therefore M = \frac{6EI\delta}{L^2} \quad \text{and} \quad R_A = R_B = \frac{12EI\delta}{L^3} \quad (6.14)$$

in the directions shown in Fig. 6.6.

These values will also represent the *changes* in the fixing moments and end reactions for a beam under load when one end sinks relative to the other.

Examples

Example 6.1

An encaustre beam has a span of 3 m and carries the loading system shown in Fig. 6.7. Draw the B.M. diagram for the beam and hence determine the maximum bending stress set up. The beam can be assumed to be uniform, with $I = 42 \times 10^{-6} \text{ m}^4$ and with an overall depth of 200 mm.

Solution

Using the *principle of superposition* the loading system can be reduced to the three cases for which the B.M. diagrams have been drawn, together with the fixing moment diagram, in Fig. 6.7.

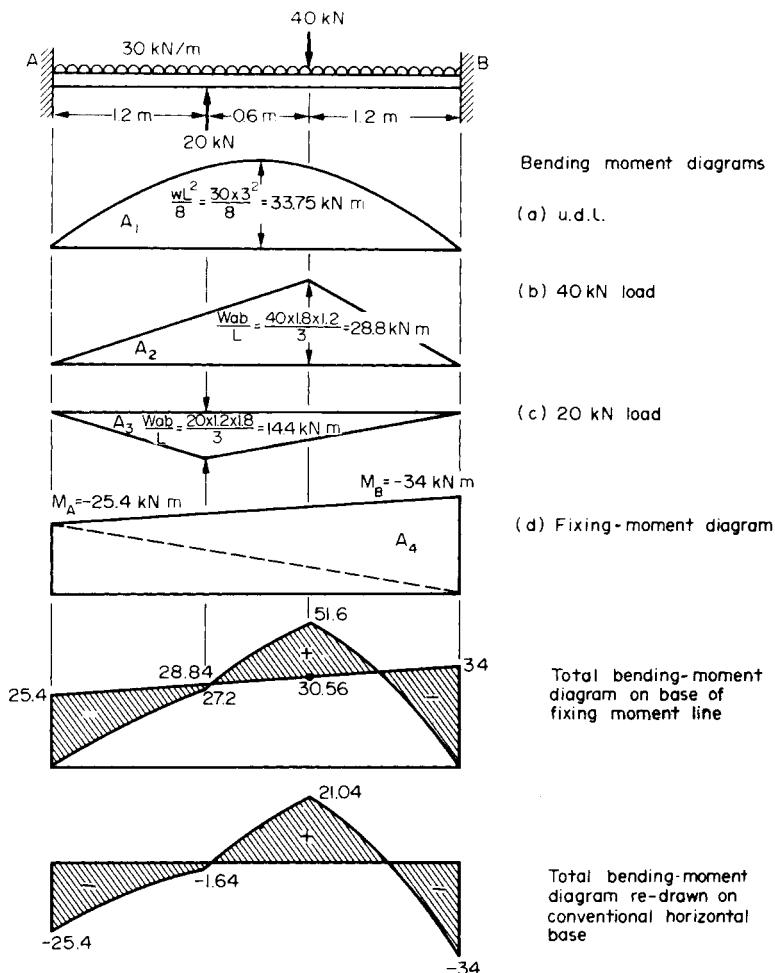


Fig. 6.7. Illustration of the application of the "principle of superposition" to Mohr's area-moment method of solution.

Now from eqn. (6.1)

$$A_1 + A_2 + A_4 = A_3$$

$$\left(\frac{2}{3} \times 33.75 \times 10^3 \times 3\right) + \left(\frac{1}{2} \times 28.8 \times 10^3 \times 3\right) + \left[\frac{1}{2}(M_A + M_B)3\right] = \left(\frac{1}{2} \times 14.4 \times 10^3 \times 3\right)$$

$$67.5 \times 10^3 + 43.2 \times 10^3 + 1.5(M_A + M_B) = 21.6 \times 10^3$$

$$M_A + M_B = -59.4 \times 10^3 \quad (1)$$

Also, from eqn. (6.2), taking moments of area about A,

$$A_1 \bar{x}_1 + A_2 \bar{x}_2 + A_4 \bar{x}_4 = A_3 \bar{x}_3$$

and, dividing areas A_2 and A_4 into the convenient triangles shown,

$$(67.5 \times 10^3 \times 1.5) + (\frac{1}{2} \times 28.8 \times 10^3 \times 1.8) \frac{2 \times 1.8}{3} + (\frac{1}{2} \times 28.8 \times 10^3 \times 1.2)(1.8 + \frac{1}{3} \times 1.2)$$

$$+ (\frac{1}{2} M_A \times 3 \times \frac{1}{3} \times 3) + (\frac{1}{2} M_B \times 3 \times \frac{2}{3} \times 3) = (\frac{1}{2} \times 14.4 \times 10^3 \times 1.2) \frac{2}{3} \times 1.2$$

$$+ (\frac{1}{2} \times 14.4 \times 10^3 \times 1.8) \left(1.2 + \frac{1.8}{3} \right)$$

$$(101.25 + 31.1 + 38.0)10^3 + 1.5 M_A + 3M_B = (6.92 + 23.3)10^3$$

$$1.5 M_A + 3M_B = -140 \times 10^3$$

$$\therefore M_A + 2M_B = -93.4 \times 10^3$$

$$(2) - (1),$$

$$M_B = -34 \times 10^3 \text{ N m} = -34 \text{ kN m}$$

and from (1),

$$M_A = -25.4 \times 10^3 \text{ N m} = -25.4 \text{ kN m}$$

The fixing moments are therefore negative and not positive as assumed in Fig. 6.7. The total B.M. diagram is then found by combining all the separate loading diagrams and the fixing moment diagram to produce the result shown in Fig. 6.7. It will be seen that the maximum B.M. occurs at the built-in end B and has a value of 34 kN m. This will therefore be the position of the maximum bending stress also, the value being determined from the simple bending theory

$$\sigma_{\max} = \frac{My}{I} = \frac{34 \times 10^3 \times 100 \times 10^{-3}}{42 \times 10^{-6}}$$

$$= 81 \times 10^6 = 81 \text{ MN/m}^2$$

Example 6.2

A built-in beam, 4 m long, carries combined uniformly distributed and concentrated loads as shown in Fig. 6.8. Determine the end reactions, the fixing moments at the built-in supports and the magnitude of the deflection under the 40 kN load. Take $EI = 14 \text{ MN m}^2$.

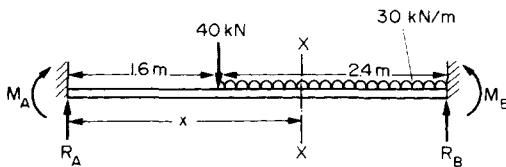


Fig. 6.8.

Solution

Using Macaulay's method (see page 106)

$$M_{xx} = \frac{EI}{10^3} \frac{d^2y}{dx^2} = M_A + R_A x - 40[x - 1.6] - 30 \left[\frac{(x - 1.6)^2}{2} \right]$$

Note that the unit of load of kilonewton is conveniently accounted for by dividing EI by 10^3 . It can then be assumed in further calculation that R_A is in kN and M_A in kNm.

Integrating,

$$\frac{EI}{10^3} \frac{dy}{dx} = M_A x + R_A \frac{x^2}{2} - \frac{40}{2} [(x - 1.6)^2] - \frac{30}{6} [(x - 1.6)^3] + A$$

and

$$\frac{EI}{10^3} y = M_A \frac{x^2}{2} + R_A \frac{x^3}{6} - \frac{40}{6} [(x - 1.6)^3] - \frac{30}{24} [(x - 1.6)^4] + Ax + B$$

Now, when $x = 0$, $y = 0$ $\therefore B = 0$

and when $x = 0$, $\frac{dy}{dx} = 0$ $\therefore A = 0$

When $x = 4$, $y = 0$

$$0 = M_A \times \frac{4^2}{2} + R_A \times \frac{4^3}{6} - \frac{40}{6} (2.4)^3 - \frac{30}{24} (2.4)^4$$

$$0 = 8M_A + 10.67 R_A - 92.16 - 41.47$$

$$133.6 = 8M_A + 10.67 R_A \quad N/\eta' \quad (1)$$

When $x = 4$, $\frac{dy}{dx} = 0$

$$0 = 4M_A + \frac{4^2}{2} R_A - \frac{40}{2} (2.4)^2 - \frac{30}{6} (2.4)^3$$

$$0 = 4M_A + 8R_A - 115.2 - 69.12$$

$$184.32 = 4M_A + 8R_A \quad (2)$$

Multiply (2) $\times 2$,

$$368.64 = 8M_A + 16R_A \quad (3)$$

(3) - (1),

$$235.04 = 5.33 R_A$$

$$R_A = \frac{235.04}{5.33} = 44.1 \text{ kN}$$

Now

$$R_A + R_B = 40 + (2.4 \times 30) = 112 \text{ kN}$$

\therefore

$$R_B = 112 - 44.1 = 67.9 \text{ kN}$$

Substituting in (2),

$$4M_A + 352.8 = 184.32$$

$$\therefore M_A = \frac{1}{4} (184.32 - 352.8) = -42.12 \text{ kN m}$$

i.e. M_A is in the opposite direction to that assumed in Fig. 6.8.

Taking moments about *A*,

$$M_B + 4R_B - (40 \times 1.6) - (30 \times 2.4 \times 2.8) - (-42.12) = 0$$

$$\therefore M_B = -(67.9 \times 4) + 64 + 201.6 - 42.12 = -48.12 \text{ kN m}$$

i.e. again in the opposite direction to that assumed in Fig. 6.8.

(Alternatively, and more conveniently, this value could have been obtained by substitution into the original Macaulay expression with $x = 4$, which is, in effect, taking moments about *B*. The need to take additional moments about *A* is then overcome.)

Substituting into the Macaulay deflection expression,

$$\frac{EI}{10^3}y = -42.1 \frac{x^2}{2} + \frac{44.1x^3}{6} - \frac{20}{3}[x - 1.6]^3 - \frac{5}{4}[x - 1.6]^4$$

Thus, under the 40 kN load, where $x = 1.6$ (and neglecting negative Macaulay terms),

$$\begin{aligned} y &= \frac{10^3}{EI} \left[\frac{-(42.1 \times 2.56)}{2} + \frac{(44.1 \times 4.1)}{6} - 0 - 0 \right] \\ &= -\frac{23.75 \times 10^3}{14 \times 10^6} = -1.7 \times 10^{-3} \text{ m} \\ &= -1.7 \text{ mm} \end{aligned}$$

The negative sign as usual indicates a deflection downwards.

Example 6.3

Determine the fixing moment at the left-hand end of the beam shown in Fig. 6.9 when the load varies linearly from 30 kN/m to 60 kN/m along the span of 4 m.

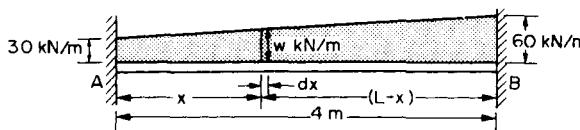


Fig. 6.9.

Solution

From § 6.4

$$M_A = - \int_0^L \frac{w'x(L-x)^2}{L^2} dx$$

Now $w' = \left(30 + \frac{30x}{4} \right) 10^3 = (30 + 7.5x) 10^3 \text{ N/m}$

$$\begin{aligned}
 \therefore M_A &= - \int_0^4 \frac{(30 + 7.5x)10^3 (4-x)^2}{4^2} x \, dx \\
 &= - \frac{10^3}{16} \int_0^4 (30 + 7.5x)(16 - 8x + x^2)x \, dx \\
 &= - \frac{10^3}{16} \int_0^4 (480x - 240x^2 + 30x^3 + 120x^2 - 60x^3 + 7.5x^4) \, dx \\
 &= - \frac{10^3}{16} \int_0^4 (480x - 120x^2 - 30x^3 + 7.5x^4) \, dx \\
 &= - \frac{10^3}{16} \left[\frac{480x^2}{2} - \frac{120x^3}{3} - \frac{30x^4}{4} + \frac{7.5x^5}{5} \right]_0^4 \\
 &= - \frac{10^3}{16} [240 \times 16 - 40 \times 64 - 30 \times 64 + 2.5 \times 1024] \\
 &= - 120 \times 10^3 \text{ N m}
 \end{aligned}$$

The required moment at A is thus 120 kN m in the opposite direction to that shown in Fig. 6.8.

Problems

6.1 (A/B). A straight beam $ABCD$ is rigidly built-in at A and D and carries point loads of 5 kN at B and C .

$$AB = BC = CD = 1.8 \text{ m}$$

If the second moment of area of the section is $7 \times 10^{-6} \text{ m}^4$ and Young's modulus is 210 GN/m^2 , calculate:

- (a) the end moments;
- (b) the central deflection of the beam.

[U.Birm.] [-6 kN m; 4.13 mm.]

6.2 (A/B). A beam of uniform section with rigidly fixed ends which are at the same level has an effective span of 10 m . It carries loads of 30 kN and 50 kN at 3 m and 6 m respectively from the left-hand end. Find the vertical reactions and the fixing moments at each end of the beam. Determine the bending moments at the two points of loading and sketch, approximately to scale, the B.M. diagram for the beam.

[41.12, 38.88 kN; -92, -90.9, 31.26, 64.62 kN m.]

6.3 (A/B). A beam of uniform section and of 7 m span is "fixed" horizontally at the same level at each end. It carries a concentrated load of 100 kN at 4 m from the left-hand end. Neglecting the weight of the beam and working from first principles, find the position and magnitude of the maximum deflection if $E = 210 \text{ GN/m}^2$ and $I = 190 \times 10^{-6} \text{ m}^4$.

[3.73 from l.h. end; 4.28 mm.]

6.4 (A/B). A uniform beam, built-in at each end, is divided into four equal parts and has equal point loads, each W , placed at the centre of each portion. Find the deflection at the centre of this beam and prove that it equals the deflection at the centre of the same beam when carrying an equal total load uniformly distributed along the entire length.

[U.C.L.I.] $\left[\frac{WL^3}{96EI} \right]$

6.5 (A/B). A horizontal beam of I-section, rigidly built-in at the ends and 7 m long, carries a total uniformly distributed load of 90 kN as well as a concentrated central load of 30 kN. If the bending stress is limited to 90 MN/m^2 and the deflection must not exceed 2.5 mm, find the depth of section required. Prove the deflection formulae if used, or work from first principles. $E = 210 \text{ GN/m}^2$. [U.L.C.I.] [583 mm.]

6.6 (A/B). A beam of uniform section is built-in at each end so as to have a clear span of 7 m. It carries a uniformly distributed load of 20 kN/m on the left-hand half of the beam, together with a 120 kN load at 5 m from the left-hand end. Find the reactions and the fixing moments at the ends and draw a B.M. diagram for the beam, inserting the principal values. [U.L.] [-105.4, -148 kN; 80.7, 109.3 kN m.]

6.7 (A/B). A steel beam of 10 m span is built-in at both ends and carries two point loads, each of 90 kN, at points 2.6 m from the ends of the beam. The middle 4.8 m has a section for which the second moment of area is $300 \times 10^{-6} \text{ m}^4$ and the 2.6 m lengths at either end have a section for which the second moment of area is $400 \times 10^{-6} \text{ m}^4$. Find the fixing moments at the ends and calculate the deflection at mid-span. Take $E = 210 \text{ GN/m}^2$ and neglect the weight of the beam. [U.L.] [$M_A = M_B = 173.2 \text{ kN m}$; 8.1 mm.]

6.8 (B.) A loaded horizontal beam has its ends securely built-in; the clear span is 8 m and $I = 90 \times 10^{-6} \text{ m}^4$. As a result of subsidence one end moves vertically through 12 mm. Determine the changes in the fixing moments and reactions. For the beam material $E = 210 \text{ GN/m}^2$. [21.26 kN m; 5.32 kN.]

CHAPTER 7

SHEAR STRESS DISTRIBUTION

Summary

The *shear stress* in a beam at any transverse cross-section in its length, and at a point a vertical distance y from the neutral axis, *resulting from bending* is given by

$$\tau = \frac{Q A \bar{y}}{I b} \quad \text{or} \quad \tau = \frac{Q}{I b} \int y dA$$

where Q is the applied vertical shear force at that section; A is the area of cross-section "above" y , i.e. the area between y and the outside of the section, which may be above or below the neutral axis (N.A.); \bar{y} is the distance of the centroid of area A from the N.A.; I is the second moment of area of the complete cross-section; and b is the breadth of the section at position y .

For **rectangular sections**,

$$\tau = \frac{6Q}{bd^3} \left[\frac{d^2}{4} - y^2 \right] \quad \text{with} \quad \tau_{\max} = \frac{3Q}{2bd} \quad \text{when} \quad y = 0$$

For **I-section beams** the *vertical shear* in the web is given by

$$\tau = \frac{Q}{2I} \left[\frac{h^2}{4} - y^2 \right] + \frac{Qb}{2It} \left[\frac{d^2}{4} - \frac{h^2}{4} \right]$$

with a *maximum* value of

$$\tau_{\max} = \frac{Qh^2}{8I} + \frac{Qb}{2It} \left[\frac{d^2}{4} - \frac{h^2}{4} \right]$$

The *maximum* value of the *horizontal shear* in the flanges is

$$\tau_{\max} = \frac{Qb}{4I} (d - t_1)$$

For **circular sections**

$$\tau = \frac{4Q}{3\pi R^2} \left[1 - \left(\frac{y}{R} \right)^2 \right]$$

with a *maximum* value of

$$\tau_{\max} = \frac{4Q}{3\pi R^2}$$

The *shear centre* of a section is that point, in or outside the section, through which load must be applied to produce zero twist of the section. Should a section have two axes of symmetry, the point where they cross is automatically the shear centre.

The shear centre of a channel section is given by

$$e = \frac{k^2 h^2 t}{4I}$$

Introduction

If a horizontal beam is subjected to vertical loads a shearing force (S.F.) diagram can be constructed as described in Chapter 3 to determine the value of the vertical S.F. at any section. This force tends to produce relative sliding between adjacent vertical sections of the beam, and it will be shown in Chapter 13, §13.2, that it is always accompanied by *complementary shears* which in this case will be horizontal. Since the concept of complementary shear is sometimes found difficult to appreciate, the following explanation is offered.

Consider the case of two rectangular-sectioned beams lying one on top of the other and supported on simple supports as shown in Fig. 7.1. If some form of vertical loading is applied the beams will bend as shown in Fig. 7.2, i.e. if there is negligible friction between the mating surfaces of the beams each beam will bend independently of the other and as a result the lower surface of the top beam will slide relative to the upper surface of the lower beam.

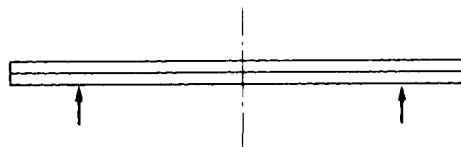


Fig. 7.1. Two beams (unconnected) on simple supports prior to loading.

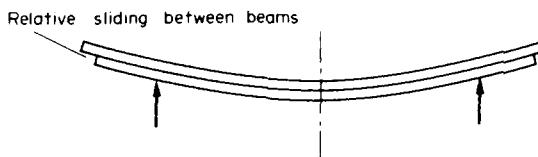


Fig. 7.2. Illustration of the presence of shear (relative sliding) between adjacent planes of a beam in bending.

If, therefore, the two beams are replaced by a single solid bar of depth equal to the combined depths of the initial two beams, then there must be some internal system of forces, and hence stresses, set up within the beam to prevent the above-mentioned sliding at the central fibres as bending takes place. Since the normal bending theory indicates that direct stresses due to bending are zero at the centre of a rectangular section beam, the prevention of sliding can only be achieved by horizontal shear stresses set up by the bending action.

Now on any element it will be shown in §13.2 that applied shears are always accompanied by complementary shears of equal value but opposite rotational sense on the perpendicular faces. Thus the *horizontal shears* due to bending are always associated with *complementary vertical shears* of equal value. For an element at either the top or bottom surface, however,

there can be no vertical shears if the surface is "free" or unloaded and hence the horizontal shear is also zero. It is evident, therefore, that, for beams in bending, shear stresses are set up both vertically and horizontally varying from some as yet undetermined value at the centre to zero at the top and bottom surfaces.

The method of determination of the remainder of the shear stress distribution across beam sections is considered in detail below.

7.1. Distribution of shear stress due to bending

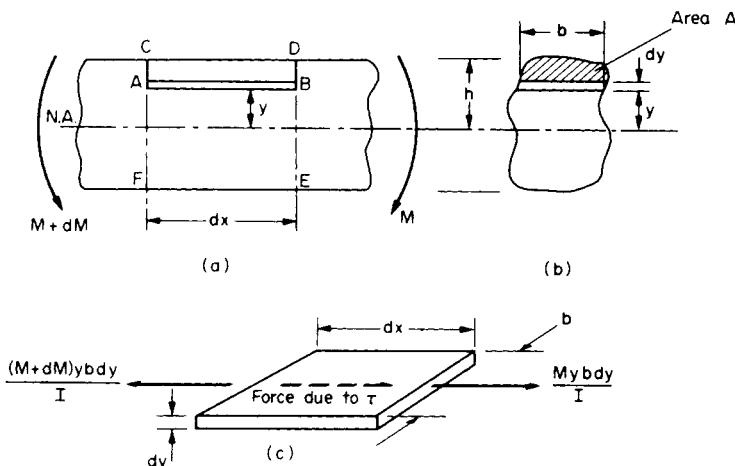


Fig. 7.3.

Consider the portion of a beam of length dx , as shown in Fig. 7.3a, and an element AB distance y from the N.A. Under any loading system the B.M. across the beam will change from M at B to $(M + dM)$ at A. Now as a result of bending,

$$\text{longitudinal stress } \sigma = \frac{My}{I}$$

$$\text{longitudinal stress at } A = \frac{(M + dM)y}{I}$$

and

$$\text{longitudinal stress at } B = \frac{My}{I}$$

$$\therefore \text{longitudinal force on the element at } A = \sigma A = \frac{(M + dM)y}{I} \times bdy$$

$$\text{and } \text{longitudinal force on the element at } B = \frac{My}{I} \times bdy$$

The force system on the element is therefore as shown in Fig. 7.3c with a net out-of-balance force to the left

$$\begin{aligned}
 &= \frac{(M + dM)y}{I} bdy - \frac{My}{I} bdy \\
 &= \frac{dM}{I} y bdy
 \end{aligned}$$

Therefore total out-of-balance force from all sections above height y

$$= \int_y^h \frac{dM}{I} y b dy$$

For equilibrium, this force is resisted by a shear force set up on the section of length dx and breadth b , as shown in Fig. 7.4.

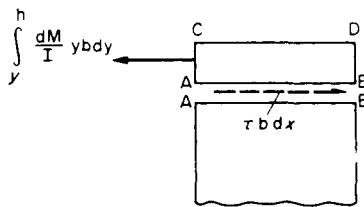


Fig. 7.4.

Thus if the shear stress is τ , then

$$\tau b dx = \frac{dM}{I} \int_y^h y b dy \quad (7.1)$$

But $\int_y^h y b dy =$ first moment of area of shaded portion of Fig. 7.3b about the N.A.
 $= A \bar{y}$

where A is the area of shaded portion and \bar{y} the distance of its centroid from the N.A.

Also $\frac{dM}{dx} =$ rate of change of the B.M.
 $= S.F. Q$ at the section

$$\therefore \tau = \frac{Q A \bar{y}}{I b} \quad (7.2)$$

or, alternatively,

$$\tau = \frac{Q}{I b} \int_y^h y dA \quad \text{where } dA = b dy \quad (7.3)$$

7.2. Application to rectangular sections

Consider now the rectangular-sectioned beam of Fig. 7.5 subjected at a given transverse cross-section to a S.F. Q .

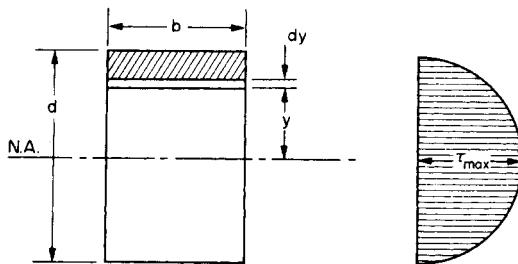


Fig. 7.5. Shear stress distribution due to bending of a rectangular section beam.

$$\tau = \frac{Q A \bar{y}}{I b}$$

$$\begin{aligned}
 &= \frac{Q}{I b} \times b \left(\frac{d}{2} - y \right) \times \left[y + \frac{\left(\frac{d}{2} - y \right)}{2} \right] \\
 &= \frac{Q \times 12}{bd^3 \times b} \times b \left(\frac{d}{2} - y \right) \frac{\left(\frac{d}{2} + y \right)}{2} \\
 &= \frac{6Q}{bd^3} \left[\frac{d^2}{4} - y^2 \right] \quad (\text{i.e. a parabola})
 \end{aligned} \tag{7.4}$$

Now

$$\tau_{\max} = \frac{6Q}{bd^3} \times \frac{d^2}{4} = \frac{3Q}{2bd} \quad \text{when } y = 0 \tag{7.5}$$

and

$$\text{average } \tau = \frac{Q}{bd}$$

∴

$$\tau_{\max} = \frac{3}{2} \times \tau_{\text{average}} \tag{7.6}$$

7.3. Application to I-section beams

Consider the I-section beam shown in Fig. 7.6.

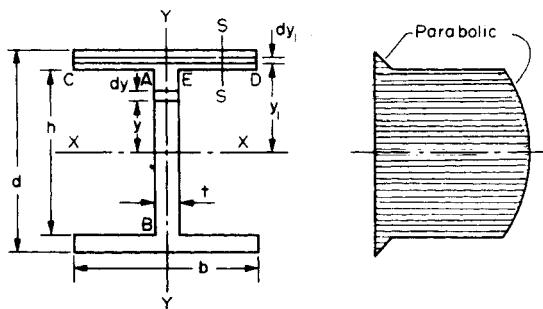


Fig. 7.6. Shear stress distribution due to bending of an I-section beam.

7.3.1. Vertical shear in the web

The distribution of shear stress due to bending at any point in a given transverse cross-section is given, in general, by eqn. (7.3)

$$\tau = \frac{Q}{Ib} \int_y^{d/2} y dA$$

In the case of the I-beam, however, the width of the section is not constant so that the quantity dA will be different in the web and the flange. Equation (7.3) must therefore be modified to

$$\begin{aligned} \tau &= \frac{Q}{It} \int_y^{h/2} ty dy + \frac{Q}{It} \int_{h/2}^{d/2} by_1 dy_1 \\ &= \frac{Q}{2I} \left[\frac{h^2}{4} - y^2 \right] + \frac{Qb}{2It} \left[\frac{d^2}{4} - \frac{h^2}{4} \right] \end{aligned}$$

As for the rectangular section, the first term produces a parabolic stress distribution. The second term is a constant and equal to the value of the shear stress at the top and bottom of the web, where $y = h/2$,

$$\text{i.e. } \tau_A = \tau_B = \frac{Qb}{2It} \left[\frac{d^2}{4} - \frac{h^2}{4} \right] \quad (7.7)$$

The *maximum shear* occurs at the N.A., where $y = 0$,

$$\tau_{\max} = \frac{Qh^2}{8I} + \frac{Qb}{2It} \left[\frac{d^2}{4} - \frac{h^2}{4} \right] \quad (7.8)$$

7.3.2. Vertical shear in the flanges

(a) Along the central section YY

The vertical shear in the flange where the width of the section is b is again given by eqn. (7.3) as

$$\begin{aligned} \tau &= \frac{Q}{Ib} \int_{y_1}^{d/2} y_1 dA \\ &= \frac{Q}{Ib} \int_{y_1}^{d/2} y_1 b dy_1 = \frac{Q}{I} \left[\frac{d^2}{8} - \frac{y_1^2}{2} \right] \end{aligned} \quad (7.9)$$

The *maximum value* is that at the bottom of the flange when $y_1 = h/2$,

$$\tau_{\max} = \frac{Q}{I} \left[\frac{d^2}{8} - \frac{h^2}{8} \right] = \frac{Q}{8I} [d^2 - h^2] \quad (7.10)$$

this value being considerably smaller than that obtained at the top of the web.

At the outside of the flanges, where $y_1 = d/2$, the vertical shear (and the complementary horizontal shear) are zero. At intermediate points the distribution is again parabolic producing the total stress distribution indicated in Fig. 7.6. As a close approximation, however, the distribution across the flanges is often taken to be linear since its effect is minimal compared with the values in the web.

(b) Along any other section SS, removed from the web

At the general section SS in the flange the shear stress at both the upper and lower edges must be zero. The distribution across the thickness of the flange is then the same as that for a rectangular section of the same dimensions.

The discrepancy between the values of shear across the free surfaces CA and ED and those at the web-flange junction indicate that the distribution of shear at the junction of the web and flange follows a more complicated relationship which cannot be investigated by the elementary analysis used here. Advanced elasticity theory must be applied to obtain a correct solution, but the values obtained above are normally perfectly adequate for general design work particularly in view of the following comments.

As stated above, the vertical shear stress in the flanges is very small in comparison with that in the web and is often neglected. Thus, in girder design, it is normally assumed that the web carries all the vertical shear. Additionally, the thickness of the web t is often very small in comparison with b such that eqns. (7.7) and (7.8) are nearly equal. The distribution of shear across the web in such cases is then taken to be uniform and equal to the total shear force Q divided by the cross-sectional area (th) of the web alone.

7.3.3. Horizontal shear in the flanges

The proof of §7.1 considered the equilibrium of an element in a vertical section of a component similar to element A of Fig. 7.9. Consider now a similar element B in the horizontal flange of the channel section (or I section) shown in Fig. 7.7.

The element has dimensions dz , t and dx comparable directly to the element previously treated of dy , b and dx . The proof of §7.1 can be applied in precisely the same way to this flange element giving an out-of-balance force on the element, from Fig. 7.9(b),

$$\begin{aligned} &= \frac{(M + dM)}{I} y \cdot t dz - \frac{My \cdot t dz}{I} \\ &= \frac{dM}{I} y \cdot t dz \end{aligned}$$

with a total out-of-balance force for the sections between z and L

$$= \int_z^L \frac{dM}{I} y \cdot t dz.$$

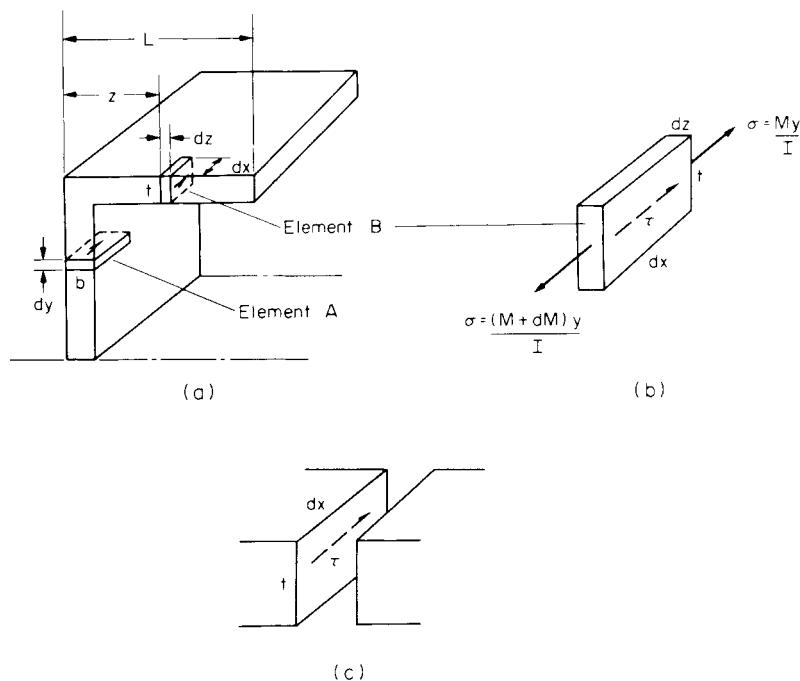


Fig. 7.7. Horizontal shear in flanges.

This force being reacted by the shear on the element shown in Fig. 7.9(c),

$$= \tau t dx$$

$$\therefore \tau t dx = \int_z^L \frac{dM}{I} \cdot y t dz$$

and $\tau = \frac{dM}{dx} \cdot \frac{1}{It} \int_z^L t dz \cdot y$

But $\int_z^L t dz \cdot y = A\bar{y}$ and $\frac{dM}{dx} = Q$.

$$\therefore \tau = \frac{QA\bar{y}}{It} \quad (7.11)$$

Thus the same form of expression is obtained to that of eqn (7.2) but with the breadth b of the web replaced by thickness t of the flange: I and y still refer to the N.A. and A is the area of the flange 'beyond' the point being considered.

Thus the horizontal shear stress distribution in the flanges of the I section of Fig. 7.8 can

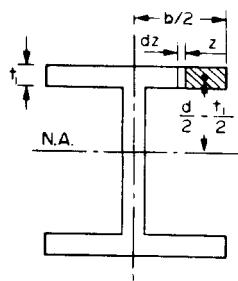


Fig. 7.8.

now be obtained from eqn. (7.11):

$$\tau = \frac{Q A \bar{y}}{I t}$$

with

$$\bar{y} = \frac{d}{2} - \frac{t_1}{2} = \frac{1}{2}(d - t_1)$$

$$A = t_1 dz$$

$$t = t_1$$

$$\text{Thus } \tau = \frac{Q}{I t_1} \int_0^{b/2} \frac{1}{2}(d - t_1) t_1 dz = \frac{Q}{2I} (d - t_1) [z]_0^{b/2}$$

The distribution is therefore linear from zero at the free ends of the flange to a maximum value of

$$\tau_{\max} = \frac{Qb}{4I} (d - t_1) \quad \text{at the centre} \quad (7.12)$$

7.4. Application to circular sections

In this case it is convenient to use the alternative form of eqn. (7.2), namely (7.1),

$$\tau b dx = \frac{dM}{I} \int_z^h b y dy$$

$$\tau = \frac{Q}{Ib} \int_z^h b y dy$$

Consider now the element of thickness dz and breadth b shown in Fig. 7.9.

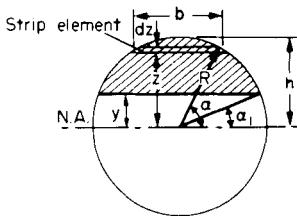


Fig. 7.9.

Now $b = 2R \cos \alpha$, $y = z = R \sin \alpha$ and $dz = R \cos \alpha d\alpha$ and, at section distance y from the N.A., $b = 2R \cos \alpha_1$,

$$\begin{aligned}
 \therefore \tau &= \frac{Q}{I \times 2R \cos \alpha_1} \int_{\alpha_1}^{\pi/2} 2R \cos \alpha R \sin \alpha R \cos \alpha d\alpha \\
 &= \frac{Q \times 4}{2R \cos \alpha_1 \times \pi R^4} \int_{\alpha_1}^{\pi/2} 2R^3 \cos^2 \alpha \sin \alpha d\alpha \quad \text{since } I = \frac{\pi R^4}{4} \\
 &= \frac{4Q}{\pi R^2 \cos \alpha_1} \left[-\frac{\cos^3 \alpha}{3} \right]_{\alpha_1}^{\pi/2} \\
 &= \frac{4Q}{3\pi R^2 \cos \alpha_1} \left[\cos^3 \alpha_1 \right]_{\pi/2}^{\alpha_1} = \frac{4Q \cos^2 \alpha_1}{3\pi R^2} \\
 &= \frac{4Q}{3\pi R^2} [1 - \sin^2 \alpha_1] = \frac{4Q}{3\pi R^2} \left[1 - \left(\frac{y}{R} \right)^2 \right]
 \end{aligned} \tag{7.13}$$

i.e. a parabola with its maximum value at $y = 0$.

Thus

$$\tau_{\max} = \frac{4Q}{3\pi R^2}$$

Now

$$\text{mean stress} = \frac{Q}{\pi R^2}$$

∴

$$\frac{\text{maximum shear stress}}{\text{mean shear stress}} = \frac{\frac{4Q}{3\pi R^2}}{\frac{Q}{\pi R^2}} = \frac{4}{3} \tag{7.14}$$

Alternative procedure

Using eqn. (7.2), namely $\tau = \frac{Q A \bar{y}}{I b}$, and referring to Fig. 7.9,

$$\frac{b}{2} = (R^2 - z^2)^{1/2} = R \cos \alpha \quad \text{and} \quad \sin \alpha = \frac{z}{R}$$

$$\begin{aligned}
 A\bar{y} \text{ for the shaded segment} &= \int_{R \sin \alpha}^R A\bar{y} \quad \text{for strip element} \\
 &= \int_{R \sin \alpha}^R b dz z \\
 &= 2 \int_{R \sin \alpha}^R (R^2 - z^2)^{1/2} z dz \\
 &= \frac{2}{3} [(R^2 - z^2)^{3/2}]_{R \sin \alpha}^R \\
 &= \frac{2}{3} [R^2(1 - \sin^2 \alpha)]^{3/2} \\
 &= \frac{2}{3} R^3 (\cos^2 \alpha)^{3/2} = \frac{2}{3} R^3 \cos^3 \alpha \\
 \therefore \tau &= \frac{Q A \bar{y}}{Ib} = \frac{Q \times \frac{2}{3} R^3 \cos^3 \alpha}{\frac{\pi R^4}{4} \times 2R \cos \alpha}
 \end{aligned}$$

since

$$I = \frac{\pi R^4}{4}$$

$$\begin{aligned}
 \therefore \tau &= \frac{4Q}{3\pi R^2} \cos^2 \alpha = \frac{4Q}{3\pi R^2} [1 - \sin^2 \alpha] \\
 &= \frac{4Q}{3\pi R^2} \left[1 - \left(\frac{y}{R} \right)^2 \right] \tag{7.13 bis.}
 \end{aligned}$$

7.5. Limitation of shear stress distribution theory

There are certain practical situations where eq. (7.2) leads to an incomplete solution and it is necessary to consider other conditions, such as equilibrium at a free surface, before a valid solution is obtained. Take, for example, the case of the bending of a bar or beam having a circular cross-section as shown in Fig. 7.10(a).

The shear stress distribution across the section owing to bending is given by eqn. (7.2) as:

$$\tau = \frac{Q}{3I} \left[\left(\frac{d}{2} \right)^2 - y^2 \right] = \tau_{yz} \tag{7.15}$$

At some horizontal section *AB*, therefore, the shear stresses will be as indicated in Fig. 7.10, and will be equal along *AB*, with a value given by eqn. (7.15).

Now, for an element at *A*, for example, there should be no component of stress normal to the surface since it is unloaded and equilibrium would not result. The vertical shear given by the equation above, however, clearly would have a component normal to the surface. A valid solution can only be obtained therefore if a secondary shear stress τ_{xz} is set up in the *z* direction which, together with τ_{yz} , produces a resultant shear stress tangential to the free surface—see Fig. 7.10(b).

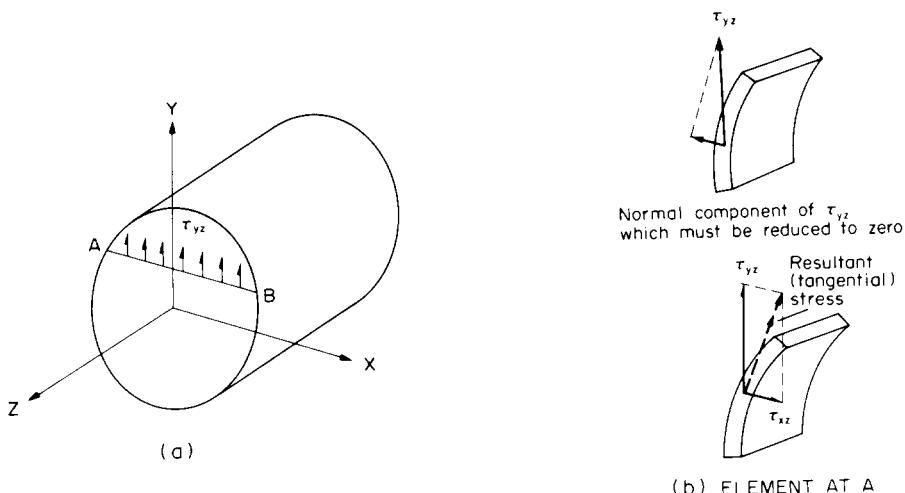


Fig. 7.10.

Solutions for the value of τ_{xz} and its effect on τ_{yz} are beyond the scope of this text† but the principal outlined indicates a limitation of the shear stress distribution theory which should be appreciated.

7.6. Shear centre

Consider the channel section of Fig. 7.11 under the action of a shearing force Q at a distance e from the centre of the web. The shearing stress at any point on the cross-section of the channel is then given by the equation $\tau = \frac{QA\bar{y}}{Ib}$. The distribution in the rectangular web

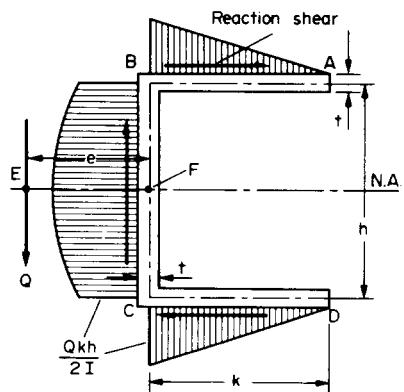


Fig. 7.11.

† I. S. Sokolnikoff, *Mathematical Theory of Elasticity*, 2nd edition (McGraw Hill, New York, 1956).

will be parabolic, as previously found, but will not reduce to zero at each end because of the presence of the flange areas.

When the stress in the flange is being determined the breadth b is replaced by the thickness t , but I and \bar{y} still refer to the N.A.

$$\text{Thus from eqn. 7.11, } \tau_B = \frac{Q A \bar{y}}{It} = \frac{Q \times kt}{It} \times \frac{h}{2} = \frac{Qkh}{2I}$$

and

$$\tau_A = 0 \quad \text{since area beyond } A = 0$$

Between A and B the distribution is linear, since τ is directly proportional to the distance along AB (Q , t , h and I all being constant). An exactly similar distribution will be obtained for CD .

The stresses in the flanges give rise to forces represented by

$$\text{average stress} \times \text{area} = \frac{1}{2} \times \frac{Qkh}{2I} \times kt = \frac{Qk^2ht}{4I}$$

These produce a torque about F which must equal the applied torque, stresses in the web producing forces which have no moment about F .

Equating torques about F for equilibrium:

$$\begin{aligned} Q \times e &= \frac{Qk^2ht}{4I} \times h \\ \therefore e &= \frac{k^2 h^2 t}{4I} \end{aligned} \tag{7.16}$$

Thus if a force acts on the axis of symmetry, distance e from the centre of the web, there will be no tendency for the section to twist since moments will be balanced. The point E is then termed the *shear centre* of the section.

The shear centre of a section is therefore defined as that point through which load must be applied for zero twist of the section. With loads applied at the shear centre of beam sections, stresses will be produced due to pure bending, and evaluation of the stresses produced will be much easier than would be the case if torsion were also present.

It should be noted that if a section has two axes of symmetry the point where they cross is automatically the shear centre.

Examples

Example 7.1

At a given position on a beam of uniform I-section the beam is subjected to a shear force of 100 kN. Plot a curve to show the variation of shear stress across the section and hence determine the ratio of the maximum shear stress to the mean shear stress.

Solution

Consider the I-section shown in Fig. 7.12. By symmetry, the centroid of the section is at

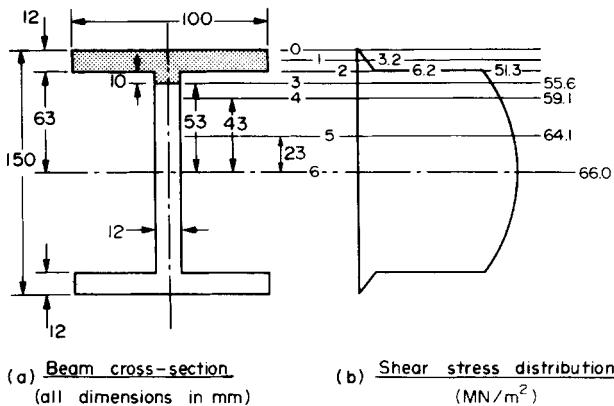


Fig. 7.12.

mid-height and the neutral axis passes through this position. The second moment of area of the section is then given by

$$I = \frac{100 \times 150^3 \times 10^{-12}}{12} - \frac{88 \times 126^3 \times 10^{-12}}{12}$$

$$= (28.125 - 14.67)10^{-6} = 13.46 \times 10^{-6} \text{ m}^4$$

The distribution of shear stress across the section is

$$\tau = \frac{Q A \bar{y}}{I b} = \frac{100 \times 10^3 A \bar{y}}{13.46 \times 10^{-6} b} = 7.43 \times 10^9 \frac{A \bar{y}}{b}$$

The solution of this equation is then best completed in tabular form as shown below. In this case, because of symmetry, only sections above the N.A. need be considered since a similar distribution will be obtained below the N.A.

It should be noted that two values of shear stress are required at section 2 to take account of the change in breadth at this section. The values of A and \bar{y} for sections 3, 4, 5 and 6 are those of a T-section beam and may be found as shown for section 3 (shaded area of Fig. 7.12a).

Section	$A \times 10^{-6}$ (m^2)	$\bar{y} \times 10^{-3}$ (m)	$b \times 10^{-3}$ (m)	$\tau = \frac{Q A \bar{y}}{I b}$ (MN/m^2)
0	0	—	—	—
1	$100 \times 6 = 600$	72	100	3.2
2	$100 \times 12 = 1200$	69	100	6.2
2	1200	69	12	51.3
3	1320	68	12	55.6
4	1440	66.3	12	59.1
5	1680	61.6	12	64.1
6	1956	54.5	12	66.0

For section 3:

Taking moments about the top edge (Fig. 7.12a),

$$(100 \times 12 \times 6)10^{-9} + (10 \times 12 \times 17)10^{-9} = (100 \times 12 + 10 \times 12)h \times 10^{-9}$$

where h is the centroid of the shaded T-section,

$$7200 + 2040 = (1200 + 120)h$$

$$h = \frac{9240}{1320} = 7 \text{ mm}$$

$$\bar{y}_3 = 75 - 7 = 68 \text{ mm}$$

The distribution of shear stress due to bending is then shown in Fig. 7.12b, giving a maximum shear stress of $\tau_{\max} = 66 \text{ MN/m}^2$.

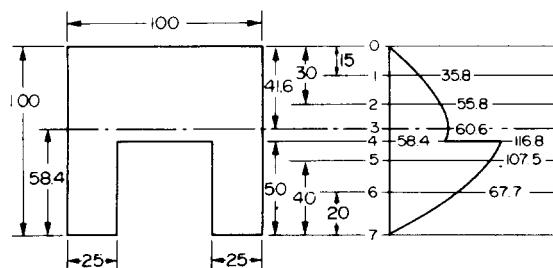
Now the mean shear stress across the section is:

$$\begin{aligned}\tau_{\text{mean}} &= \frac{\text{shear force}}{\text{area}} = \frac{100 \times 10^3}{3.912 \times 10^{-3}} \\ &= 25.6 \text{ MN/m}^2\end{aligned}$$

$$\frac{\text{max. shear stress}}{\text{mean shear stress}} = \frac{66}{25.6} = 2.58$$

Example 7.2

At a certain section a beam has the cross-section shown in Fig. 7.13. The beam is simply supported at its ends and carries a central concentrated load of 500 kN together with a load of 300 kN/m uniformly distributed across the complete span of 3 m. Draw the shear stress distribution diagram for a section 1 m from the left-hand support.



(a) Beam cross-section (mm) (b) Shear stress distribution (MN/m^2)

Fig. 7.13.

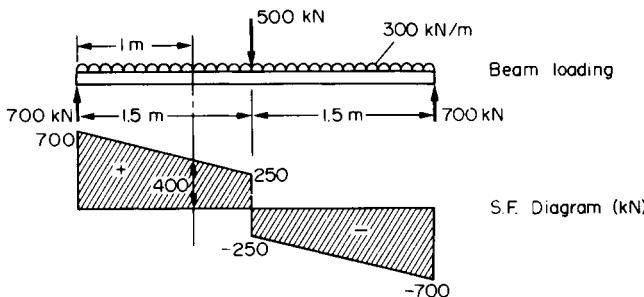


Fig. 7.14.

Solution

From the S.F. diagram for the beam (Fig. 7.14) it is evident that the S.F. at the section 1 m from the left-hand support is 400 kN,

i.e.

$$Q = 400 \text{ kN}$$

To find the position of the N.A. of the beam section of Fig. 7.13(a) take moments of area about the base.

$$(100 \times 100 \times 50)10^{-9} - (50 \times 50 \times 25)10^{-9} = (100 \times 100 - 50 \times 50)\bar{y} \times 10^{-9}$$

$$500000 - 62500 = (10000 - 2500)\bar{y}$$

$$\bar{y} = \frac{437500}{7500} = 58.4 \text{ mm}$$

Then

$$I_{\text{N.A.}} = \left[\frac{100 \times 41.6^3}{3} + 2 \left(\frac{25 \times 58.4^3}{3} \right) + \left(\frac{50 \times 8.4^3}{3} \right) \right] 10^{-12}$$

$$= (2.41 + 3.3 + 0.0099)10^{-6} = 5.72 \times 10^{-6} \text{ m}^4$$

$$\therefore \tau = \frac{QA\bar{y}}{Ib} = \frac{400 \times 10^3}{5.72 \times 10^{-6}} \frac{A\bar{y}}{b} = 7 \times 10^{10} \frac{A\bar{y}}{b}$$

Section	$A \times 10^{-6}$ (m^2)	$\bar{y} \times 10^{-3}$ (m)	$b \times 10^{-3}$ (m)	$\tau = 7 \times 10^4 \frac{A\bar{y}}{b}$ (MN/m^2)
0	0	—	—	0
1	1500	34.1	100	35.8
2	3000	26.6	100	55.8
3	4160	20.8	100	60.6
4	2500	33.4	100	58.4
4	2500	33.4	50	116.8
5	2000	38.4	50	107.5
6	1000	48.4	50	67.7
7	0	—	—	0

The shear stress distribution across the section is shown in Fig. 7.13b.

Example 7.3

Determine the values of the shear stress owing to bending at points A, B and C in the beam cross-section shown in Fig. 7.15 when subjected to a shear force of $Q = 140 \text{ kN}$. Hence sketch the shear stress distribution diagram.

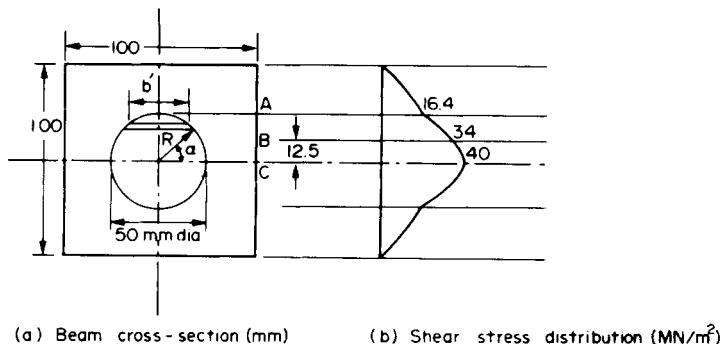


Fig. 7.15.

Solution

By symmetry the centroid will be at the centre of the section and

$$I_{\text{N.A.}} = \frac{100 \times 100^3}{12} \times 10^{-12} - \frac{\pi \times 50^4}{64} \times 10^{-12}$$

$$= (8.33 - 0.31)10^{-6} = 8.02 \times 10^{-6} \text{ m}^4$$

At A:

$$\tau_A = \frac{Q A \bar{y}}{I_b} = \frac{140 \times 10^3 \times (100 \times 25 \times 37.5)10^{-9}}{8.02 \times 10^{-6} \times 100 \times 10^{-3}} = 16.4 \text{ MN/m}^2$$

At B:

Here the required $A\bar{y}$ is obtained by subtracting $A\bar{y}$ for the portion of the circle above B from that of the rectangle above B. Now for the circle

$$A\bar{y} = \int_{12.5}^{25} b'y dy \quad (\text{Fig. 7.15a})$$

$$\text{But, when } y = 12.5, \quad \sin \alpha = \frac{12.5}{25} = \frac{1}{2} \quad \therefore \quad \alpha = \pi/6$$

$$\text{and when } y = 25, \quad \sin \alpha = \frac{25}{25} = 1 \quad \therefore \quad \alpha = \pi/2$$

$$\text{Also} \quad b' = 2R \cos \alpha, \quad y = R \sin \alpha \quad \text{and} \quad dy = R \cos \alpha d\alpha$$

$$\begin{aligned}
 \therefore \text{for circle portion, } A\bar{y} &= \int_{\pi/6}^{\pi/2} 2R \cos \alpha R \sin \alpha R \cos \alpha d\alpha \\
 &= \int_{\pi/6}^{\pi/2} 2R^3 \cos^2 \alpha \sin \alpha d\alpha \\
 &= 2R^3 \left[-\frac{\cos^3 \alpha}{3} \right]_{\pi/6}^{\pi/2} \\
 &= \frac{2 \times 25^3 \times 10^{-9}}{3} \left[\left(\frac{\sqrt{3}}{2} \right)^3 \right] = 6.75 \times 10^{-6} \text{ m}^3
 \end{aligned}$$

\therefore for complete section above *B*

$$\begin{aligned}
 A\bar{y} &= 100 \times 37.5 \times 31.25 \times 10^{-9} - 6.75 \times 10^{-6} \\
 &= 110.25 \times 10^{-6} \text{ m}^3
 \end{aligned}$$

and

$$b' = 2R \cos \pi/6 = 2 \times 25 \times 10^{-3} \times \sqrt{3}/2 = 43.3 \times 10^{-3} \text{ m}$$

$$\therefore b = (100 - 43.3)10^{-3} = 56.7 \times 10^{-3} \text{ m}$$

$$\therefore \tau_B = \frac{Q\bar{A}}{Ib} = \frac{140 \times 10^3 \times 110.25 \times 10^{-6}}{8.02 \times 10^{-6} \times 56.7 \times 10^{-3}} = 34 \text{ MN/m}^2$$

At C:

$$\begin{aligned}
 \bar{A}y \text{ for semicircle} &= 2R^3 \left[-\frac{\cos^3 \alpha}{3} \right]_0^{\pi/2} \\
 &= \frac{2 \times 25^3 \times 10^{-9}}{3} [-(0-1)] = 10.41 \times 10^{-6} \text{ m}^3
 \end{aligned}$$

$A\bar{y}$ for section above C

$$= (100 \times 50 \times 25)10^{-9} - 10.41 \times 10^{-6} = 114.59 \times 10^{-6} \text{ m}^3$$

and

$$b = (100 - 50)10^{-3} = 50 \times 10^{-3} \text{ m}$$

$$\therefore \tau_c = \frac{140 \times 10^3 \times 114.59 \times 10^{-6}}{8.02 \times 10^{-6} \times 50 \times 10^{-3}} = 40 \text{ MN/m}^2$$

The total shear stress distribution across the section is then sketched in Fig. 7.15b.

Example 7.4

A beam having the cross-section shown in Fig. 7.16 is constructed from material having a constant thickness of 1.3 mm. Through what point must vertical loads be applied in order that there shall be no twisting of the section? Sketch the shear stress distribution.

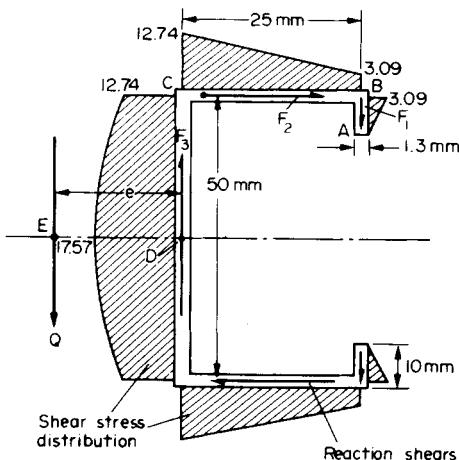


Fig. 7.16.

Solution

Let a load of QN be applied through the point E , distance e from the centre of the web.

$$\begin{aligned} I_{N.A.} &= \left[\frac{1.3 \times 50^3}{12} + 2\left(\frac{25 \times 1.3^3}{12} + 25 \times 1.3 \times 25^2\right) \right. \\ &\quad \left. + 2\left(\frac{1.3 \times 10^3}{12} + 1.3 \times 10 \times 20^2\right) \right] \times 10^{-12} \\ &= [1.354 + 2(0.00046 + 2.03) + 2(0.011 + 0.52)]10^{-8} \\ &= 6.48 \times 10^{-8} \text{ m}^4 \end{aligned}$$

Shear stress

$$\tau_A = \frac{QA\bar{y}}{Ib} = \frac{Q \times 0}{Ib} = 0$$

$$\tau_B = \frac{Q \times (10 \times 1.3 \times 20)10^{-9}}{6.48 \times 10^{-8} \times 1.3 \times 10^{-3}} = 3.09Q \text{ kN/m}^2$$

$$\tau_C = 3.09Q + \frac{Q(25 \times 1.3 \times 25)10^{-9}}{6.48 \times 10^{-8} \times 1.3 \times 10^{-3}}$$

$$= 3.09Q + 9.65Q = 12.74Q \text{ kN/m}^2$$

$$\tau_D = 12.74Q + \frac{Q(25 \times 1.3 \times 12.5)10^{-9}}{6.48 \times 10^{-8} \times 1.3 \times 10^{-3}}$$

$$= 12.74Q + 4.83Q = 17.57Q \text{ kN/m}^2$$

The shear stress distribution is then sketched in Fig. 7.16. It should be noted that whilst the distribution is linear along BC it is not strictly so along AB . For ease of calculation of the shear centre, however, it is usually assumed to be linear since the contribution of this region to

moments about D is small (the shear centre is the required point through which load must be applied to produce zero twist of the section).

Thus taking moments of forces about D for equilibrium,

$$\begin{aligned} Q \times e \times 10^{-3} &= 2F_1 \times 25 \times 10^{-3} + 2F_2 \times 25 \times 10^{-3} \\ &= 50 \times 10^{-3} \left[\frac{1}{2} \times 3.09 Q \times 10^3 \times (10 \times 1.3 \times 10^{-6}) \right. \\ &\quad \left. + 10^3 \frac{(3.09Q + 12.74Q)}{2} (25 \times 1.3 \times 10^{-6}) \right] \\ &= 13.866Q \times 10^{-3} \\ e &= 13.87 \text{ mm} \end{aligned}$$

Thus, loads must be applied through the point E , 13.87 mm to the left of the web centre-line for zero twist of the section.

Problems

7.1 (A/B). A uniform I-section beam has flanges 150 mm wide by 8 mm thick and a web 180 mm wide and 8 mm thick. At a certain section there is a shearing force of 120 kN. Draw a diagram to illustrate the distribution of shear stress across the section as a result of bending. What is the maximum shear stress? [86.7 MN/m².]

7.2 (A/B). A girder has a uniform T cross-section with flange 250 mm \times 50 mm and web 220 mm \times 50 mm. At a certain section of the girder there is a shear force of 360 kN.

Plot neatly to scale the shear-stress distribution across the section, stating the values:

- (a) where the web and the flange of the section meet;
- (b) at the neutral axis.

[B.P.] [7.47, 37.4, 39.6 MN/m².]

7.3 (A/B). A beam having an inverted T cross-section has an overall width of 150 mm and overall depth of 200 mm. The thickness of the crosspiece is 50 mm and of the vertical web 25 mm. At a certain section along the beam the vertical shear force is found to be 120 kN. Draw neatly to scale, using 20 mm spacing except where closer intervals are required, a shear-stress distribution diagram across this section. If the mean stress is calculated over the whole of the cross-sectional area, determine the ratio of the maximum shear stress to the mean shear stress.

[B.P.] [3.37.]

7.4 (A/B). The channel section shown in Fig. 7.17 is simply supported over a span of 5 m and carries a uniformly distributed load of 15 kN/m run over its whole length. Sketch the shearing-stress distribution diagram at the point of maximum shearing force and mark important values. Determine the ratio of the maximum shearing stress to the average shearing stress.

[B.P.] [3, 9.2, 9.3 MN/m²; 2.42.]

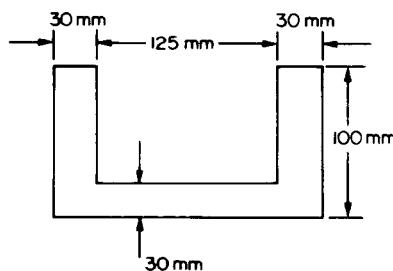


Fig. 7.17.

7.5 (A/B). Fig. 7.18 shows the cross-section of a beam which carries a shear force of 20 kN. Plot a graph to scale which shows the distribution of shear stress due to bending across the cross-section.

[I.Mech.E.] [21.7, 5.2, 5.23 MN/m².]

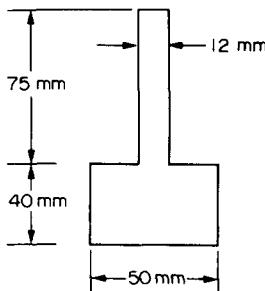


Fig. 7.18.

7.6 (B). Show that the difference between the maximum and mean shear stress in the web of an I-section beam is $\frac{Qh^2}{24I}$ where Q is the shear force on the cross-section, h is the depth of the web and I is the second moment of area of the cross-section about the neutral axis of bending. Assume the I-section to be built of rectangular sections, the flanges having width B and thickness t and the web a thickness b . Fillet radii are to be ignored. [I.Mech.E.]

7.7 (B). Deduce an expression for the shearing stress at any point in a section of a beam owing to the shearing force at that section. State the assumptions made.

A simply supported beam carries a central load W . The cross-section of the beam is rectangular of depth d . At what distance from the neutral axis will the shearing stress be equal to the mean shearing stress on the section?

[U.L.C.I.] [$d/\sqrt{12}$.]

7.8 (B). A steel bar rolled to the section shown in Fig. 7.19 is subjected to a shearing force of 200 kN applied in the direction YY. Making the usual assumptions, determine the average shearing stress at the sections A, B, C and D, and find the ratio of the maximum to the mean shearing stress in the section. Draw to scale a diagram to show the variation of the average shearing stress across the section.

[U.L.] [Clue: treat as equivalent section similar to that of Example 7.3.] [7.2, 12.3, 33.6, 43.8 MN/m², 3.93.]

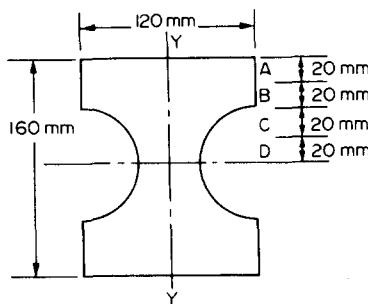


Fig. 7.19.

7.9 (C). Using customary notation, show that the shear stress over the cross-section of a loaded beam is given by $\tau = \frac{QA\bar{y}}{Ib}$.

The cross-section of a beam is an isosceles triangle of base B and height H , the base being arranged in a horizontal plane. Find the shear stress at the neutral axis owing to a shear force Q acting on the cross-section and express it in terms of the mean shear stress.

(The second moment of area of a triangle about its base is $BH^3/12$.)

[U.L.C.I.] $\left[\frac{8Q}{3BH}; \frac{4}{3}\tau_{\text{mean}} \right]$

7.10 (C). A hollow steel cylinder, of 200 mm external and 100 mm internal diameter, acting as a beam, is subjected to a shearing force $Q = 10 \text{ kN}$ perpendicular to the axis. Determine the mean shearing stress across the section and, making the usual assumptions, the average shearing stress at the neutral axis and at sections 25, 50 and 75 mm from the neutral axis as fractions of the mean value.

Draw a diagram to show the variation of average shearing stress across the section of the cylinder.

[U.L.] [0.425 MN/m²; 1.87, 1.65, 0.8, 0.47 MN/m².]

7.11 (C). A hexagonal-cross-section bar is used as a beam with its greatest dimension vertical and simply supported at its ends. The beam carries a central load of 60 kN. Draw a stress distribution diagram for a section of the beam at quarter span. All sides of the bar have a length of 25 mm.

($I_{\text{N.A.}}$ for triangle = $bh^3/36$ where b = base and h = height.)

[0, 9.2, 14.8, 25.9 MN/m² at 12.5 mm intervals above and below the N.A.]

CHAPTER 8

TORSION

Summary

For a solid or hollow shaft of uniform circular cross-section throughout its length, the theory of pure torsion states that

$$\frac{T}{J} = \frac{\tau}{R} = \frac{G\theta}{L}$$

where T is the applied external torque, constant over length L ;

J is the polar second moment of area of shaft cross-section

$$= \frac{\pi D^4}{32} \text{ for a solid shaft and } \frac{\pi(D^4 - d^4)}{32} \text{ for a hollow shaft;}$$

D is the outside diameter; R is the outside radius;

d is the inside diameter;

τ is the shear stress at radius R and is the maximum value for both solid and hollow shafts;

G is the modulus of rigidity (shear modulus); and

θ is the angle of twist in radians on a length L .

For very thin-walled hollow shafts

$$J = 2\pi r^3 t, \text{ where } r \text{ is the mean radius of the shaft wall and } t \text{ is the thickness.}$$

Shear stress and shear strain are related to the angle of twist thus:

$$\tau = \frac{G\theta}{L} R = G\gamma$$

Strain energy in torsion is given by

$$U = \frac{T^2 L}{2GJ} = \frac{GJ\theta^2}{2L} \left(= \frac{\tau^2}{4G} \times \text{volume for solid shafts} \right)$$

For a circular shaft subjected to combined bending and torsion the equivalent bending moment is

$$M_e = \frac{1}{2}[M + \sqrt{(M^2 + T^2)}]$$

and the equivalent torque is

$$T_e = \frac{1}{2}\sqrt{(M^2 + T^2)}$$

where M and T are the applied bending moment and torque respectively.
The power transmitted by a shaft carrying torque T at ω rad/s = $T\omega$.

8.1. Simple torsion theory

When a uniform circular shaft is subjected to a torque it can be shown that every section of the shaft is subjected to a state of pure shear (Fig. 8.1), the moment of resistance developed by the shear stresses being everywhere equal to the magnitude, and opposite in sense, to the applied torque. For the purposes of deriving a simple theory to describe the behaviour of shafts subjected to torque it is necessary to make the following basic assumptions:

- (1) The material is homogeneous, i.e. of uniform elastic properties throughout.
- (2) The material is elastic, following Hooke's law with shear stress proportional to shear strain.
- (3) The stress does not exceed the elastic limit or limit of proportionality.
- (4) Circular sections remain circular.
- (5) Cross-sections remain plane. (This is certainly not the case with the torsion of non-circular sections.)
- (6) Cross-sections rotate as if rigid, i.e. every diameter rotates through the same angle.

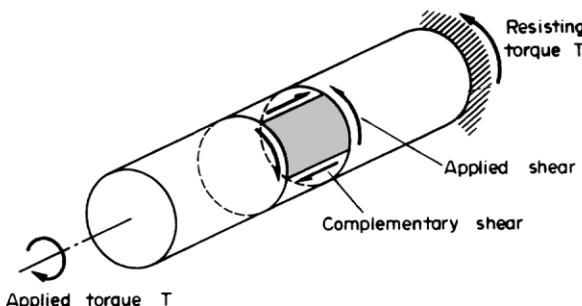


Fig. 8.1. Shear system set up on an element in the surface of a shaft subjected to torsion.

Practical tests carried out on circular shafts have shown that the theory developed below on the basis of these assumptions shows excellent correlation with experimental results.

(a) Angle of twist

Consider now the solid circular shaft of radius R subjected to a torque T at one end, the other end being fixed (Fig. 8.2). Under the action of this torque a radial line at the free end of the shaft twists through an angle θ , point A moves to B , and AB subtends an angle γ at the fixed end. This is then the angle of distortion of the shaft, i.e. *the shear strain*.

Since angle in radians = arc ÷ radius

$$\text{arc } AB = R\theta = L\gamma$$

$$\therefore \gamma = R\theta/L \quad (8.1)$$

From the definition of rigidity modulus

$$G = \frac{\text{shear stress } \tau}{\text{shear strain } \gamma}$$

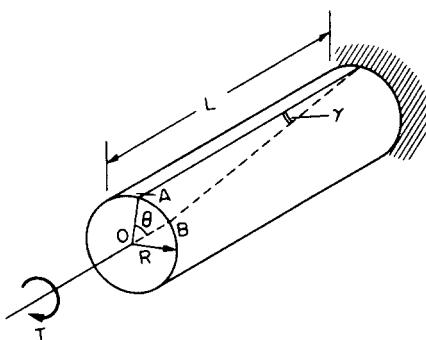


Fig. 8.2.

$$\therefore \gamma = \frac{\tau}{G} \quad (8.2)$$

where τ is the shear stress set up at radius R .

Therefore equating eqns. (8.1) and (8.2),

$$\begin{aligned} \frac{R\theta}{L} &= \frac{\tau}{G} \\ \frac{\tau}{R} &= \frac{G\theta}{L} \left(= \frac{\tau'}{r} \right) \end{aligned} \quad (8.3)$$

where τ' is the shear stress at any other radius r .

(b) Stresses

Let the cross-section of the shaft be considered as divided into elements of radius r and thickness dr as shown in Fig. 8.3 each subjected to a shear stress τ' .

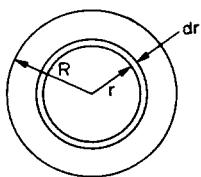


Fig. 8.3. Shaft cross-section.

The force set up on each element

$$\begin{aligned} &= \text{stress} \times \text{area} \\ &= \tau' \times 2\pi r dr \text{ (approximately)} \end{aligned}$$

This force will produce a moment about the centre axis of the shaft, providing a contribution to the torque

$$\begin{aligned} &= (\tau' \times 2\pi r dr) \times r \\ &= 2\pi\tau' r^2 dr \end{aligned}$$

The total torque on the section T will then be the sum of all such contributions across the section,

i.e.

$$T = \int_0^R 2\pi\tau' r^2 dr$$

Now the shear stress τ' will vary with the radius r and must therefore be replaced in terms of r before the integral is evaluated.

From eqn. (8.3)

$$\begin{aligned} \tau' &= \frac{G\theta}{L} r \\ \therefore T &= \int_0^R 2\pi \frac{G\theta}{L} r^3 dr \\ &= \frac{G\theta}{L} \int_0^R 2\pi r^3 dr \end{aligned}$$

The integral $\int_0^R 2\pi r^3 dr$ is called the *polar second moment of area* J , and may be evaluated as a standard form for solid and hollow shafts as shown in §8.2 below.

$$T = \frac{G\theta}{L} J$$

or

$$\frac{T}{J} = \frac{G\theta}{L}$$

(8.4)

Combining eqns. (8.3) and (8.4) produces the so-called simple theory of torsion:

$$\frac{T}{J} = \frac{\tau}{R} = \frac{G\theta}{L} \quad (8.5)$$

8.2. Polar second moment of area

As stated above the polar second moment of area J is defined as

$$J = \int_0^R 2\pi r^3 dr$$

For a solid shaft,

$$\begin{aligned} J &= 2\pi \left[\frac{r^4}{4} \right]_0^R \\ &= \frac{\pi R^4}{4} \quad \text{or} \quad \frac{\pi D^4}{32} \end{aligned} \quad (8.6)$$

For a hollow shaft of internal radius r ,

$$\begin{aligned} J &= 2\pi \int_r^R r^3 dr = 2\pi \left[\frac{r^4}{4} \right]_r^R \\ &= \frac{\pi}{2} (R^4 - r^4) \quad \text{or} \quad \frac{\pi}{32} (D^4 - d^4) \end{aligned} \quad (8.7)$$

For thin-walled hollow shafts the values of D and d may be nearly equal, and in such cases there can be considerable errors in using the above equation involving the difference of two large quantities of similar value. It is therefore convenient to obtain an alternative form of expression for the polar moment of area.

Now

$$\begin{aligned} J &= \int_0^R 2\pi r^3 dr = \Sigma (2\pi r dr)r^2 \\ &= \Sigma Ar^2 \end{aligned}$$

where $A (= 2\pi r dr)$ is the area of each small element of Fig. 8.3, i.e. J is the sum of the Ar^2 terms for all elements.

If a thin hollow cylinder is therefore considered as just one of these small elements with its wall thickness $t = dr$, then

$$\begin{aligned} J &= Ar^2 = (2\pi r t)r^2 \\ &= 2\pi r^3 t \text{ (approximately)} \end{aligned} \quad (8.8)$$

8.3. Shear stress and shear strain in shafts

The shear stresses which are developed in a shaft subjected to pure torsion are indicated in Fig. 8.1, their values being given by the simple torsion theory as

$$\tau = \frac{G\theta}{L} R$$

Now from the definition of the shear or rigidity modulus G ,

$$\tau = G\gamma$$

It therefore follows that the two equations may be combined to relate the shear stress and strain in the shaft to the angle of twist per unit length, thus

$$\tau = \frac{G\theta}{L} R = G\gamma \quad (8.9)$$

or, in terms of some internal radius r ,

$$\tau' = \frac{G\theta}{L} r = G\gamma \quad (8.10)$$

These equations indicate that the shear stress and shear strain vary linearly with radius and have their maximum value at the outside radius (Fig. 8.4). The applied shear stresses in the plane of the cross-section are accompanied by complementary stresses of equal value on longitudinal planes as indicated in Figs. 8.1 and 8.4. The significance of these longitudinal shears to material failure is discussed further in §8.10.

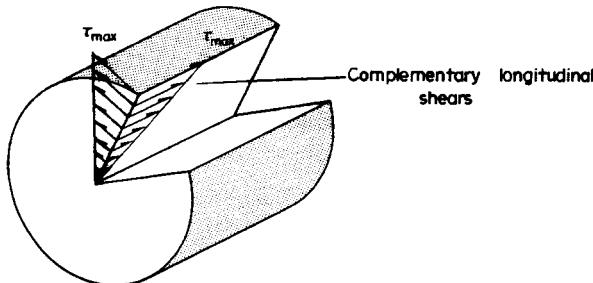


Fig. 8.4. Complementary longitudinal shear stress in a shaft subjected to torsion.

8.4. Section modulus

It is sometimes convenient to re-write part of the torsion theory formula to obtain the maximum shear stress in shafts as follows:

$$\frac{T}{J} = \frac{\tau}{R}$$

$$\therefore \tau = \frac{TR}{J}$$

With R the outside radius of the shaft the above equation yields the greatest value possible for τ (Fig. 8.4),

$$\text{i.e. } \tau_{\max} = \frac{TR}{J}$$

$$\therefore \tau_{\max} = \frac{T}{Z} \quad (8.11)$$

where $Z = J/R$ is termed the *polar section modulus*. It will be seen from the preceding section that:

for solid shafts,

$$Z = \frac{\pi D^3}{16} \quad (8.12)$$

and for hollow shafts,

$$Z = \frac{\pi(D^4 - d^4)}{16D} \quad (8.13)$$

8.5. Torsional rigidity

The angle of twist per unit length of shafts is given by the torsion theory as

$$\frac{\theta}{L} = \frac{T}{GJ}$$

The quantity GJ is termed the *torsional rigidity* of the shaft and is thus given by

$$GJ = \frac{T}{\theta/L} \quad (8.14)$$

i.e. the torsional rigidity is the torque divided by the angle of twist (in radians) per unit length.

8.6. Torsion of hollow shafts

It has been shown above that the maximum shear stress in a solid shaft is developed in the outer surface, values at other radii decreasing linearly to zero at the centre. It is clear, therefore, that if there is to be some limit set on the maximum allowable working stress in the shaft material then only the outer surface of the shaft will reach this limit. The material within the shaft will work at a lower stress and, particularly near the centre, will not contribute as much to the torque-carrying capacity of the shaft. In applications where weight reduction is of prime importance as in the aerospace industry, for instance, it is often found advisable to use hollow shafts.

The relevant formulae for hollow shafts have been introduced in §8.2 and will not be repeated here. As an example of the increased torque-to-weight ratio possible with hollow shafts, however, it should be noted for a hollow shaft with an inside diameter half the outside diameter that the maximum stress increases by 6% over that for a solid shaft of the same outside diameter whilst the weight reduction achieved is approximately 25%.

8.7. Torsion of thin-walled tubes

The torsion of thin-walled tubes of circular and non-circular cross-section is treated fully in *Mechanics of Materials 2*.[†]

8.8. Composite shafts – series connection

If two or more shafts of different material, diameter or basic form are connected together in such a way that each carries the same torque, then the shafts are said to be connected in series and the composite shaft so produced is therefore termed *series-connected* (Fig. 8.5) (see Example 8.3). In such cases the composite shaft strength is treated by considering each component shaft separately, applying the torsion theory to each in turn; the composite shaft will therefore be as weak as its weakest component. If relative dimensions of the various parts are required then a solution is usually effected by equating the torques in each shaft, e.g. for two shafts in series

$$T = \frac{G_1 J_1 \theta_1}{L_1} = \frac{G_2 J_2 \theta_2}{L_2} \quad (8.15)$$

[†]E. J. Hearn, *Mechanics of Materials 2*, 3rd edition (Butterworth-Heinemann, Oxford, 1997).

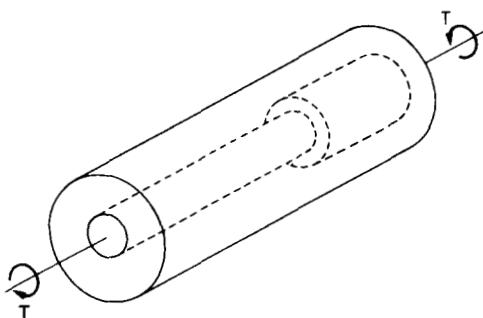


Fig. 8.5. "Series-connected" shaft – common torque.

In some applications it is convenient to ensure that the angles of twist in each shaft are equal, i.e. $\theta_1 = \theta_2$, so that for similar materials in each shaft

$$\frac{J_1}{L_1} = \frac{J_2}{L_2}$$

or

$$\frac{L_1}{L_2} = \frac{J_1}{J_2} \quad (8.16)$$

8.9. Composite shafts – parallel connection

If two or more materials are rigidly fixed together such that the applied torque is shared between them then the composite shaft so formed is said to be *connected in parallel* (Fig. 8.6).

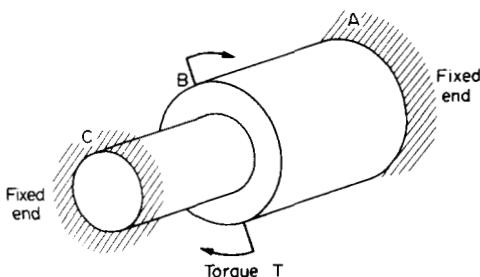


Fig. 8.6. "Parallel-connected" shaft – shared torque.

For parallel connection,

$$\text{total torque } T = T_1 + T_2 \quad (8.17)$$

In this case the angles of twist of each portion are equal and

$$\frac{T_1 L_1}{G_1 J_1} = \frac{T_2 L_2}{G_2 J_2} \quad (8.18)$$

i.e. for equal lengths (as is normally the case for parallel shafts)

$$\frac{T_1}{T_2} = \frac{G_1 J_1}{G_2 J_2} \quad (8.19)$$

Thus two equations are obtained in terms of the torques in each part of the composite shaft and these torques can therefore be determined.

The maximum stresses in each part can then be found from

$$\tau_1 = \frac{T_1 R_1}{J_1} \quad \text{and} \quad \tau_2 = \frac{T_2 R_2}{J_2}$$

8.10. Principal stresses

It will be shown in §13.2 that a state of pure shear as produced by the torsion of shafts is equivalent to a system of biaxial direct stresses, one stress tensile, one compressive, of equal value and at 45° to the shaft axis as shown in Fig. 8.7; these are then the principal stresses.

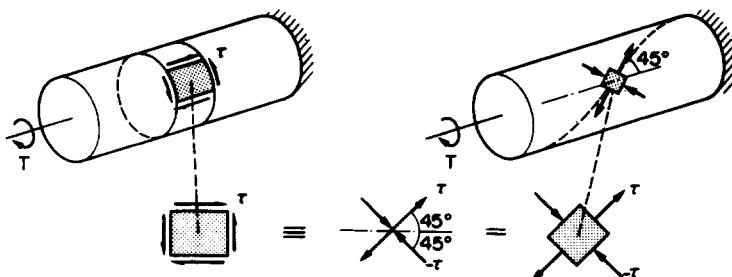


Fig. 8.7. Shear and principal stresses in a shaft subjected to torsion.

Thus shafts which are constructed from brittle materials which are notably weaker under direct stress than in shear (cast-iron, for example) will fail by cracking along a helix inclined at 45° to the shaft axis. This can be demonstrated very simply by twisting a piece of chalk to failure (Fig. 8.8a). Ductile materials, however, which are weaker in shear, fail on the shear planes at right angles to the shaft axis (Fig. 8.8b). In some cases, e.g. timber, failure occurs by cracking along the shear planes parallel to the shaft axis owing to the nature of the material with fibres generally parallel to the axis producing a weakness in shear longitudinally rather than transversely. The complementary shears of Fig. 8.4 then assume greater significance.

8.11. Strain energy in torsion

It will be shown in §11.4 that the strain energy stored in a solid circular bar or shaft subjected to a torque T is given by the alternative expressions

$$U = \frac{1}{2} T \theta = \frac{T^2 L}{2 G J} = \frac{G J \theta^2}{2 L} = \frac{\tau^2}{4 G} \times \text{volume} \quad (8.20)$$

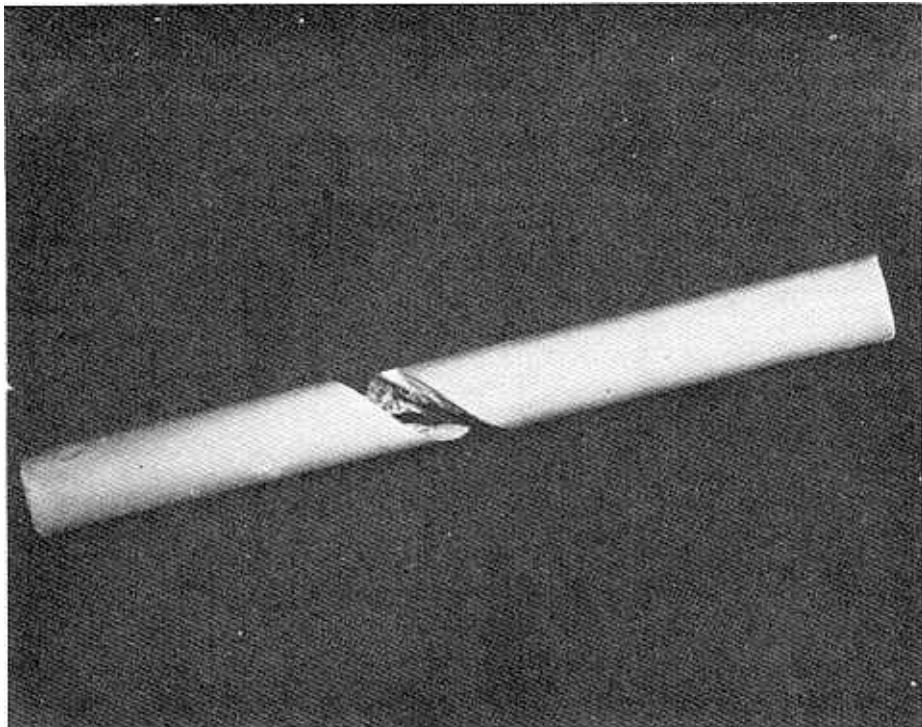


Fig. 8.8a. Typical failure of a brittle material (chalk) in torsion. Failure occurs on a 45° helix owing to the action of the direct tensile stresses produced at 45° by the applied torque.

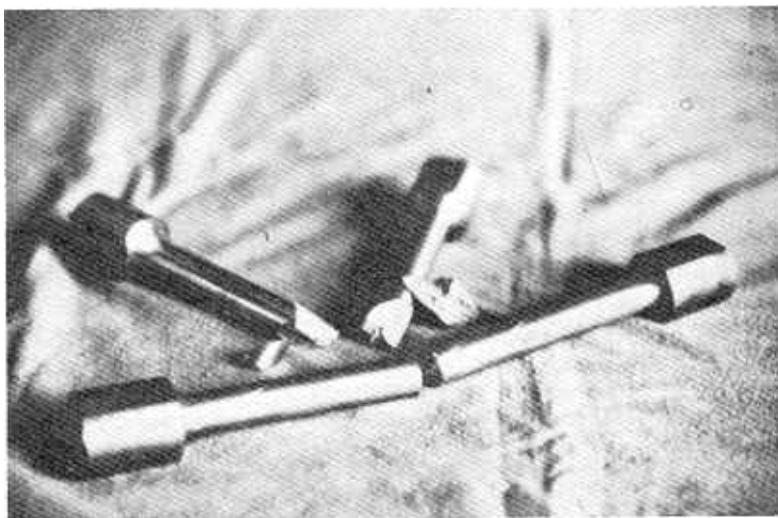


Fig. 8.8b. (Foreground) Failure of a ductile steel in torsion on a plane perpendicular to the specimen longitudinal axis. Scribed lines on the surface of the specimen which were parallel to the longitudinal axis before torque application indicate the degree of twist applied to the specimen. (Background) Equivalent failure of a more brittle, higher carbon steel in torsion. Failure again occurs on 45° planes but in this case, as often occurs in practice, a clean fracture into two pieces did not take place.

8.12. Variation of data along shaft length – torsion of tapered shafts

This section illustrates the procedure which may be adopted when any of the quantities normally used in the torsion equations vary along the length of the shaft. Provided the variation is known in terms of x , the distance along the shaft, then a solution can be obtained.

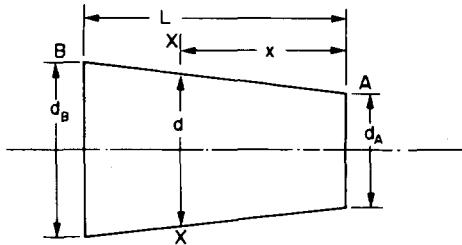


Fig. 8.9. Torsion of a tapered shaft.

Consider, therefore, the tapered shaft shown in Fig. 8.9 with its diameter changing linearly from d_A to d_B over a length L . The diameter at any section x from end A is then given by

$$d = d_A + (d_B - d_A) \frac{x}{L}$$

Provided that the angle of the taper is not too great, the simple torsion theory may be applied to an element at section XX in order to determine the angle of twist of the shaft, i.e. for the element shown,

$$\frac{Gd\theta}{dx} = \frac{T}{J_{XX}}$$

Therefore the total angle of twist of the shaft is given by

$$\theta = \int_0^L \frac{T}{GJ_{XX}} dx$$

Now $J_{XX} = \frac{\pi d^4}{32} = \frac{\pi}{32} \left[d_A + (d_B - d_A) \frac{x}{L} \right]^4$

Substituting and integrating,

$$\theta = \frac{32TL}{3\pi G} \left[\frac{1}{d_A^3} - \frac{1}{d_B^3} \right] \left[\frac{1}{d_B} - \frac{1}{d_A} \right] = \frac{32TL}{3\pi G} \left[\frac{d_A^2 + d_A d_B + d_B^2}{d_A^3 d_B^3} \right]$$

When $d_A = d_B = d$ this reduces to $\theta = \frac{32TL}{\pi G d^4}$ the standard result for a parallel shaft.

8.13. Power transmitted by shafts

If a shaft carries a torque T Newton metres and rotates at ω rad/s it will do work at the rate of

$$T\omega \text{ Nm/s (or joule/s).}$$

Now the rate at which a system works is defined as its power, the basic unit of power being the Watt (1 Watt = 1 Nm/s).

Thus, the power transmitted by the shaft:

$$= T\omega \text{ Watts.}$$

Since the Watt is a very small unit of power in engineering terms use is normally made of S.I. multiples, i.e. kilowatts (kW) or megawatts (MW).

8.14. Combined stress systems – combined bending and torsion

In most practical transmission situations shafts which carry torque are also subjected to bending, if only by virtue of the self-weight of the gears they carry. Many other practical applications occur where bending and torsion arise simultaneously so that this type of loading represents one of the major sources of complex stress situations.

In the case of shafts, bending gives rise to tensile stress on one surface and compressive stress on the opposite surface whilst torsion gives rise to pure shear throughout the shaft. An element on the tensile surface will thus be subjected to the stress system indicated in Fig. 8.10 and eqn. (13.11) or the Mohr circle procedure of §13.6 can be used to obtain the principal stresses present.

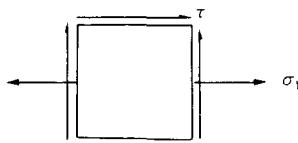


Fig. 8.10. Stress system on the surface of a shaft subjected to torque and bending.

Alternatively, the shaft can be considered to be subjected to *equivalent torques* or *equivalent bending moments* as described below.

8.15. Combined bending and torsion – equivalent bending moment

For shafts subjected to the simultaneous application of a bending moment M and torque T the *principal stresses* set up in the shaft can be shown to be equal to those produced by an *equivalent bending moment*, of a certain value M_e acting alone.

From the simple bending theory the maximum direct stresses set up at the outside surface of the shaft owing to the bending moment M are given by

$$\sigma = \frac{My_{\max}}{I} = \frac{MD}{2I}$$

Similarly, from the torsion theory, the maximum shear stress in the surface of the shaft is given by

$$\tau = \frac{TR}{J} = \frac{TD}{2J}$$

But for a circular shaft $J = 2I$,

$$\therefore \tau = \frac{TD}{4I}$$

The principal stresses for this system can now be obtained by applying the formula derived in §13.4,

i.e.

$$\sigma_1 \text{ or } \sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau^2]}$$

and, with $\sigma_y = 0$, the maximum principal stress σ_1 is given by

$$\begin{aligned} \sigma_1 &= \frac{1}{2}\left(\frac{MD}{2I}\right) + \frac{1}{2}\sqrt{\left[\left(\frac{MD}{2I}\right)^2 + 4\left(\frac{TD}{4I}\right)^2\right]} \\ &= \frac{1}{2}\left(\frac{D}{2I}\right)[M + \sqrt{(M^2 + T^2)}] \end{aligned}$$

Now if M_e is the bending moment which, acting alone, will produce the same maximum stress, then

$$\sigma_1 = \frac{M_e y_{\max}}{I} = \frac{M_e D}{2I}$$

$$\therefore \frac{M_e D}{2I} = \frac{1}{2}\left(\frac{D}{2I}\right)[M + \sqrt{(M^2 + T^2)}]$$

i.e. the equivalent bending moment is given by

$$M_e = \frac{1}{2}[M + \sqrt{(M^2 + T^2)}] \quad (8.21)$$

and it will produce the same maximum direct stress as the combined bending and torsion effects.

8.16. Combined bending and torsion – equivalent torque

Again considering shafts subjected to the simultaneous application of a bending moment M and a torque T the *maximum shear stress* set up in the shaft may be determined by the application of an *equivalent torque* of value T_e acting alone.

From the preceding section the principal stresses in the shaft are given by

$$\sigma_1 = \frac{1}{2}\left(\frac{D}{2I}\right)[M + \sqrt{(M^2 + T^2)}] = \frac{1}{2}\left(\frac{D}{J}\right)[M + \sqrt{(M^2 + T^2)}]$$

$$\text{and } \sigma_2 = \frac{1}{2}\left(\frac{D}{2I}\right)[M - \sqrt{(M^2 + T^2)}] = \frac{1}{2}\left(\frac{D}{J}\right)[M - \sqrt{(M^2 + T^2)}]$$

Now the maximum shear stress is given by eqn. (13.12)

$$\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_2) = \frac{1}{2}\left(\frac{D}{J}\right)\sqrt{(M^2 + T^2)}$$

But, from the torsion theory, the equivalent torque T_e will set up a maximum shear stress of

$$\tau_{\max} = \frac{T_e D}{2 J}$$

Thus if these maximum shear stresses are to be equal,

$$T_e = \sqrt{(M^2 + T^2)} \quad (8.22)$$

It must be remembered that the equivalent moment M_e and equivalent torque T_e are merely convenient devices to obtain the maximum principal direct stress or maximum shear stress, respectively, under the combined stress system. They should not be used for other purposes such as the calculation of power transmitted by the shaft; this depends solely on the torque T carried by the shaft (not on T_e).

8.17. Combined bending, torsion and direct thrust

Additional stresses arising from the action of direct thrusts on shafts may be taken into account by adding the direct stress due to the thrust σ_d to that of the direct stress due to bending σ_b , taking due account of sign. The complex stress system resulting on any element in the shaft is then as shown in Fig. 8.11 and may be solved to determine the principal stresses using Mohr's stress circle method of solution described in § 13.6.

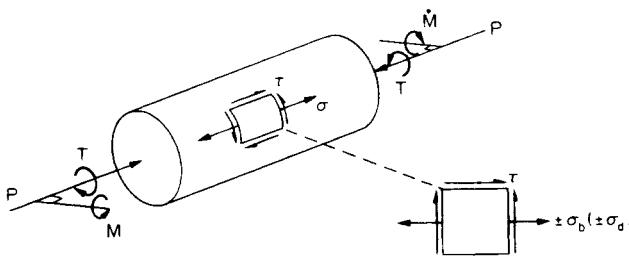


Fig. 8.11. Shaft subjected to combined bending, torque and direct thrust.

This type of problem arises in the service loading condition of marine propeller shafts, the direct thrust being the compressive reaction of the water on the propeller as the craft is pushed forward. This force then exists in combination with the torque carried by the shaft in doing the required work and any bending moments which exist by virtue of the self-weight of the shaft between bearings.

The compressive stress σ_d arising from the propeller reaction is thus superimposed on the bending stresses; on the compressive bending surface it will be additive to σ_b whilst on the "tensile" surface it will effectively reduce the value of σ_b , see Fig. 8.11.

8.18. Combined bending, torque and internal pressure

In the case of pressurised cylinders, direct stresses will be introduced in two perpendicular directions. These have been introduced in Chapters 9 and 10 as the radial and circumferential

stresses σ_y and σ_x . If the cylinder also carries a torque then shear stresses will be introduced, their value being calculated from the simple torsion theory of § 8.3. The stress system on an element will thus become that shown in Fig. 8.12.

If bending is present it will generally be on the x axis and will result in a modification to the value of σ_x . If the element is taken on the tensile surface of the cylinder then the bending stress σ_b will add to the value of σ_x , if on the compressive surface it must be subtracted from σ_x .

Once again a solution to such problems can be effected either by application of eqn. (13.11) or by a Mohr circle approach.

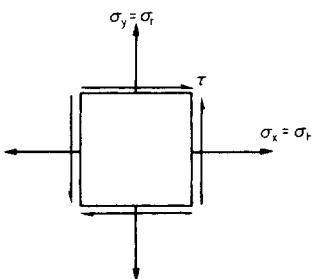


Fig. 8.12. Stress system under combined torque and internal pressure.

Examples

Example 8.1

(a) A solid shaft, 100 mm diameter, transmits 75 kW at 150 rev/min. Determine the value of the maximum shear stress set up in the shaft and the angle of twist per metre of the shaft length if $G = 80 \text{ GN/m}^2$.

(b) If the shaft were now bored in order to reduce weight to produce a tube of 100 mm outside diameter and 60 mm inside diameter, what torque could be carried if the same maximum shear stress is not to be exceeded? What is the percentage increase in power/weight ratio effected by this modification?

Solution

$$(a) \quad \text{Power} = T\omega \quad \therefore \text{torque } T = \frac{\text{power}}{\omega}$$

$$\therefore T = \frac{75 \times 10^3}{150 \times 2\pi/60} = 4.77 \text{ kN m}$$

From the torsion theory

$$\frac{T}{J} = \frac{\tau}{R} \quad \text{and} \quad J = \frac{\pi}{32} \times 100^4 \times 10^{-12} = 9.82 \times 10^{-6} \text{ m}^4$$

$$\therefore \tau_{\max} = \frac{TR_{\max}}{J} = \frac{4.77 \times 10^3 \times 50 \times 10^{-3}}{9.82 \times 10^{-6}} = 24.3 \text{ MN/m}^2$$

Also from the torsion theory

$$\begin{aligned}\theta &= \frac{TL}{GJ} = \frac{4.77 \times 10^3 \times 1}{80 \times 10^9 \times 9.82 \times 10^{-6}} = 6.07 \times 10^{-3} \text{ rad/m} \\ &= 6.07 \times 10^{-3} \times \frac{360}{2\pi} = 0.348 \text{ degrees/m}\end{aligned}$$

(b) When the shaft is bored, the polar moment of area J is modified thus:

$$J = \frac{\pi}{32} (D^4 - d^4) = \frac{\pi}{32} (100^4 - 60^4) 10^{-12} = 8.545 \times 10^{-6} \text{ m}^4$$

The torque carried by the modified shaft is then given by

$$T = \frac{\tau J}{R} = \frac{24.3 \times 10^6 \times 8.545 \times 10^{-6}}{50 \times 10^{-3}} = 4.15 \times 10^3 \text{ Nm}$$

Now, weight/metre of original shaft

$$= \frac{\pi}{4} (100)^2 \times 10^{-6} \times 1 \times \rho g = 7.854 \times 10^{-3} \rho g$$

where ρ is the density of the shaft material.

$$\begin{aligned}\text{Also, weight/metre of modified shaft} &= \frac{\pi}{4} (100^2 - 60^2) 10^{-6} \times 1 \times \rho g \\ &= 5.027 \times 10^{-3} \rho g\end{aligned}$$

$$\begin{aligned}\text{Power/weight ratio for original shaft} &= \frac{T\omega}{\text{weight/metre}} \\ &= \frac{4.77 \times 10^3 \omega}{7.854 \times 10^{-3} \rho g} = 6.073 \times 10^5 \frac{\omega}{\rho g}\end{aligned}$$

Power/weight ratio for modified shaft

$$= \frac{4.15 \times 10^3 \omega}{5.027 \times 10^{-3} \rho g} = 8.255 \times 10^5 \frac{\omega}{\rho g}$$

Therefore percentage increase in power/weight ratio

$$= \frac{(8.255 - 6.073)}{6.073} \times 100 = 36\%$$

Example 8.2

Determine the dimensions of a hollow shaft with a diameter ratio of 3:4 which is to transmit 60 kW at 200 rev/min. The maximum shear stress in the shaft is limited to 70 MN/m² and the angle of twist to 3.8° in a length of 4 m.

For the shaft material $G = 80 \text{ GN/m}^2$.

Solution

The two limiting conditions stated in the question, namely maximum shear stress and angle of twist, will each lead to different values for the required diameter. The larger shaft must then be chosen as the one for which neither condition is exceeded.

Maximum shear stress condition

Since power = $T\omega$ and $\omega = 200 \times \frac{2\pi}{60} = 20.94 \text{ rad/s}$

$$\text{then } T = \frac{60 \times 10^3}{20.94} = 2.86 \times 10^3 \text{ Nm}$$

From the torsion theory

$$J = \frac{TR}{\tau}$$

$$\therefore \frac{\pi}{32}(D^4 - d^4) = \frac{2.86 \times 10^3 \times D}{70 \times 10^6 \times 2}$$

But $d/D = 0.75$

$$\therefore \frac{\pi}{32}D^4(1 - 0.75^4) = 20.43 \times 10^{-6}D$$

$$D^3 = \frac{20.43 \times 10^{-6}}{0.0671} = 304.4 \times 10^{-6}$$

$$\therefore D = 0.0673 \text{ m} = 67.3 \text{ mm}$$

$$\text{and } d = 50.5 \text{ mm}$$

Angle of twist condition

Again from the torsion theory

$$J = \frac{TL}{G\theta}$$

$$\frac{\pi}{32}(D^4 - d^4) = \frac{2.86 \times 10^3 \times 4 \times 360}{80 \times 10^9 \times 3.8 \times 2\pi}$$

$$\frac{\pi}{32}D^4(1 - 0.75^4) = 2.156 \times 10^{-6}$$

$$D^4 = \frac{2.156 \times 10^{-6}}{0.0671} = 32.12 \times 10^{-6}$$

$$D = 0.0753 \text{ m} = 75.3 \text{ mm}$$

$$\text{and } d = 56.5 \text{ mm}$$

Thus the dimensions required for the shaft to satisfy both conditions are outer diameter 75.3 mm; inner diameter 56.5 mm.

Example 8.3

(a) A steel transmission shaft is 510 mm long and 50 mm external diameter. For part of its length it is bored to a diameter of 25 mm and for the rest to 38 mm diameter. Find the maximum power that may be transmitted at a speed of 210 rev/min if the shear stress is not to exceed 70 MN/m².

(b) If the angle of twist in the length of 25 mm bore is equal to that in the length of 38 mm bore, find the length bored to the latter diameter.

Solution

(a) This is, in effect, a question on *shafts in series* since each part is subjected to the same torque.

From the torsion theory

$$T = \frac{\tau J}{R}$$

and as the maximum stress and the radius at which it occurs (the outside radius) are the same for both shafts the torque allowable for a known value of shear stress is dependent only on the value of J . This will be least where the internal diameter is greatest since

$$J = \frac{\pi}{32} (D^4 - d^4)$$

$$\therefore \text{least value of } J = \frac{\pi}{32} (50^4 - 38^4) 10^{-12} = 0.41 \times 10^{-6} \text{ m}^4$$

Therefore maximum allowable torque if the shear stress is not to exceed 70 MN/m² (at 25 mm radius) is given by

$$T = \frac{70 \times 10^6 \times 0.41 \times 10^{-6}}{25 \times 10^{-3}} = 1.15 \times 10^3 \text{ Nm}$$

$$\begin{aligned} \text{Maximum power} &= T\omega = 1.15 \times 10^3 \times 210 \times \frac{2\pi}{60} \\ &= 25.2 \times 10^3 = 25.2 \text{ kW} \end{aligned}$$

(b) Let suffix 1 refer to the 38 mm diameter bore portion and suffix 2 to the other part. Now for shafts in series, eqn. (8.16) applies,

i.e.

$$\frac{J_1}{L_1} = \frac{J_2}{L_2}$$

$$\therefore \frac{L_2}{L_1} = \frac{J_2}{J_1} = \frac{\frac{\pi}{32} (50^4 - 25^4) 10^{-12}}{\frac{\pi}{32} (50^4 - 38^4) 10^{-12}} = 1.43$$

$$\therefore L_2 = 1.43 L_1$$

$$\text{But } L_1 + L_2 = 510 \text{ mm}$$

$$\therefore L_1(1 + 1.43) = 510$$

$$L_1 = \frac{510}{2.43} = 210 \text{ mm}$$

Example 8.4

A circular bar ABC , 3 m long, is rigidly fixed at its ends A and C . The portion AB is 1.8 m long and of 50 mm diameter and BC is 1.2 m long and of 25 mm diameter. If a twisting moment of 680 N m is applied at B , determine the values of the resisting moments at A and C and the maximum stress in each section of the shaft. What will be the angle of twist of each portion?

For the material of the shaft $G = 80 \text{ GN/m}^2$.

Solution

In this case the two portions of the shaft are *in parallel* and the applied torque is shared between them. Let suffix 1 refer to portion AB and suffix 2 to portion BC .

Since the angles of twist in each portion are equal and G is common to both sections,

then

$$\frac{T_1 L_1}{J_1} = \frac{T_2 L_2}{J_2}$$

$$\begin{aligned} \therefore T_1 &= \frac{J_1}{J_2} \times \frac{L_2}{L_1} \times T_2 = \frac{\frac{\pi}{32} \times 50^4}{\frac{\pi}{32} \times 25^4} \times \frac{1.2}{1.8} \times T_2 \\ &= \frac{16 \times 1.2}{1.8} T_2 = 10.67 T_2 \end{aligned}$$

$$\text{Total torque} = T_1 + T_2 = T_2(10.67 + 1) = 680$$

$$\therefore T_2 = \frac{680}{11.67} = 58.3 \text{ N m}$$

and

$$T_1 = 621.7 \text{ N m}$$

For portion AB ,

$$\tau_{\max} = \frac{T_1 R_1}{J_1} = \frac{621.7 \times 25 \times 10^{-3}}{\frac{\pi}{32} \times 50^4 \times 10^{-12}} = 25.33 \times 10^6 \text{ N/m}^2$$

For portion BC,

$$\tau_{\max} = \frac{T_2 R_2}{J_2} = \frac{58.3 \times 12.5 \times 10^{-3}}{\frac{\pi}{32} \times 25^4 \times 10^{-12}} = 19.0 \times 10^6 \text{ N/m}^2$$

$$\begin{aligned} \text{Angle of twist for each portion} &= \frac{T_1 L_1}{J_1 G} \\ &= \frac{621.7 \times 1.8}{\frac{\pi}{32} \times 50^4 \times 10^{-12} \times 80 \times 10^9} = 0.0228 \text{ rad} = 1.3 \text{ degrees} \end{aligned}$$

Problems

8.1 (A) A solid steel shaft *A* of 50 mm diameter rotates at 250 rev/min. Find the greatest power that can be transmitted for a limiting shearing stress of 60 MN/m² in the steel.

It is proposed to replace *A* by a hollow shaft *B*, of the same external diameter but with a limiting shearing stress of 75 MN/m². Determine the internal diameter of *B* to transmit the same power at the same speed.

[38.6 kW, 33.4 mm.]

8.2 (A) Calculate the dimensions of a hollow steel shaft which is required to transmit 750 kW at a speed of 400 rev/min if the maximum torque exceeds the mean by 20% and the greatest intensity of shear stress is limited to 75 MN/m². The internal diameter of the shaft is to be 80% of the external diameter. (The mean torque is that derived from the horsepower equation.)

[135.2, 108.2 mm.]

8.3 (A) A steel shaft 3 m long is transmitting 1 MW at 240 rev/min. The working conditions to be satisfied by the shaft are:

- (a) that the shaft must not twist more than 0.02 radian on a length of 10 diameters;
- (b) that the working stress must not exceed 60 MN/m².

If the modulus of rigidity of steel is 80 GN/m² what is

(i) the diameter of the shaft required;

(ii) the actual working stress;

(iii) the angle of twist of the 3 m length?

[B.P.] [150 mm; 60 MN/m²; 0.030 rad.]

8.4 (A) A hollow shaft has to transmit 6 MW at 150 rev/min. The maximum allowable stress is not to exceed 60 MN/m² nor the angle of twist 0.3° per metre length of shafting. If the outside diameter of the shaft is 300 mm find the minimum thickness of the hollow shaft to satisfy the above conditions. $G = 80 \text{ GN/m}^2$.

[61.5 mm.]

8.5 (A) A flanged coupling having six bolts placed at a pitch circle diameter of 180 mm connects two lengths of solid steel shafting of the same diameter. The shaft is required to transmit 80 kW at 240 rev/min. Assuming the allowable intensities of shearing stresses in the shaft and bolts are 75 MN/m² and 55 MN/m² respectively, and the maximum torque is 1.4 times the mean torque, calculate:

(a) the diameter of the shaft;

(b) the diameter of the bolts.

[B.P.] [67.2, 13.8 mm.]

8.6 (A) A hollow low carbon steel shaft is subjected to a torque of 0.25 MN m. If the ratio of internal to external diameter is 1 to 3 and the shear stress due to torque has to be limited to 70 MN/m² determine the required diameters and the angle of twist in degrees per metre length of shaft.

$G = 80 \text{ GN/m}^2$.

[I.Struct.E.] [264, 88 mm; 0.38°.]

8.7 (A) Describe how you would carry out a torsion test on a low carbon steel specimen and how, from data taken, you would find the modulus of rigidity and yield stress in shear of the steel. Discuss the nature of the torque-twist curve and compare it with the shear stress-shear strain relationship.

[U.Birm.]

8.8 (A/B) Opposing axial torques are applied at the ends of a straight bar ABCD. Each of the parts AB, BC and CD is 500 mm long and has a hollow circular cross-section, the inside and outside diameters being, respectively, AB 25 mm and 60 mm, BC 25 mm and 70 mm, CD 40 mm and 70 mm. The modulus of rigidity of the material is 80 GN/m² throughout. Calculate:

- (a) the maximum torque which can be applied if the maximum shear stress is not to exceed 75 MN/m²;
- (b) the maximum torque if the twist of *D* relative to *A* is not to exceed 2°.

[E.I.E.] [3.085 kN m, 3.25 kN m.]

8.9 (A/B). A solid steel shaft of 200 mm diameter transmits 5 MW at 500 rev/min. It is proposed to alter the horsepower to 7 MW and the speed to 440 rev/min and to replace the solid shaft by a hollow shaft made of the same type of steel but having only 80% of the weight of the solid shaft. The length of both shafts is the same and the hollow shaft is to have the same maximum shear stress as the solid shaft. Find:

- (a) the ratio between the torque per unit angle of twist per metre for the two shafts;
- (b) the external and internal diameters for the hollow shaft.

[I.Mech.E.] [2.085; 261, 190 mm.]

8.10 (A/B). A shaft ABC rotates at 600 rev/min and is driven through a coupling at the end A . At B a pulley takes off two-thirds of the power, the remainder being absorbed at C . The part AB is 1.3 m long and of 100 mm diameter; BC is 1.7 m long and of 75 mm diameter. The maximum shear stress set up in BC is 40 MN/m^2 . Determine the maximum stress in AB and the power transmitted by it, and calculate the total angle of twist in the length AC .
Take $G = 80 \text{ GN/m}^2$.

[I.Mech.E.] [16.9 MN/m²; 208 kW; 1.61°.]

8.11 (A/B). A composite shaft consists of a steel rod of 75 mm diameter surrounded by a closely fitting brass tube firmly fixed to it. Find the outside diameter of the tube such that when a torque is applied to the composite shaft it will be shared equally by the two materials.

$$G_S = 80 \text{ GN/m}^2; G_B = 40 \text{ GN/m}^2.$$

If the torque is 16 kN m, calculate the maximum shearing stress in each material and the angle of twist on a length of 4 m.
[U.L.] [98.7 mm; 96.6, 63.5 MN/m²; 7.38°.]

8.12 (A/B). A circular bar 4 m long with an external radius of 25 mm is solid over half its length and bored to an internal radius of 12 mm over the other half. If a torque of 120 N m is applied at the centre of the shaft, the two ends being fixed, determine the maximum shear stress set up in the surface of the shaft and the work done by the torque in producing this stress.
[2.51 MN/m²; 0.151 N m.]

8.13 (A/B). The shaft of Problem 8.12 is now fixed at one end only and the torque applied at the free end. How will the values of maximum shear stress and work done change?
[5.16 MN/m²; 0.603 N m.]

8.14 (B). Calculate the minimum diameter of a solid shaft which is required to transmit 70 kW at 600 rev/min if the shear stress is not to exceed 75 MN/m^2 . If a bending moment of 300 N m is now applied to the shaft find the speed at which the shaft must be driven in order to transmit the same horsepower for the same value of maximum shear stress.
[630 rev/min.]

8.15 (B). A solid shaft of 75 mm diameter and 4 m span supports a flywheel of weight 2.5 kN at a point 1.8 m from one support. Determine the maximum direct stress produced in the surface of the shaft when it transmits 35 kW at 200 rev/min.
[65.9 MN/m².]

8.16 (B). The shaft of Problem 12.15 is now subjected to an axial compressive end load of 80 kN, the other conditions remaining unchanged. What will be the magnitudes of the maximum principal stress in the shaft?
[84 MN/m².]

8.17 (B). A horizontal shaft of 75 mm diameter projects from a bearing, and in addition to the torque transmitted the shaft carries a vertical load of 8 kN at 300 mm from the bearing. If the safe stress for the material, as determined in a simple tension test, is 135 MN/m^2 find the safe torque to which the shaft may be subjected using as the criterion (a) the maximum shearing stress, (b) the maximum strain energy per unit volume. Poisson's ratio $\nu = 0.29$.
[U.L.] [5.05, 8.3 kN m.]

8.18 (B). A pulley subjected to vertical belt drive develops 10 kW at 240 rev/min, the belt tension ratio being 0.4. The pulley is fixed to the end of a length of overhead shafting which is supported in two self-aligning bearings, the centre line of the pulley overhanging the centre line of the left-hand bearing by 150 mm. If the pulley is of 250 mm diameter and weight 270 N, neglecting the weight of the shafting, find the minimum shaft diameter required if the maximum allowable stress intensity at a point on the top surface of the shaft at the centre line of the left-hand bearing is not to exceed 90 MN/m^2 direct or 40 MN/m^2 shear.
[50.5 mm.]

8.19 (B). A hollow steel shaft of 100 mm external diameter and 50 mm internal diameter transmits 0.6 MW at 500 rev/min and is subjected to an end thrust of 45 kN. Find what bending moment may safely be applied if the greater principal stress is not to exceed 90 MN/m^2 . What will then be the value of the smaller principal stress?
[City U.] [3.6 kN m; -43.1 MN/m².]

8.20 (B). A solid circular shaft is subjected to an axial torque T and to a bending moment M . If $M = kT$, determine in terms of k the ratio of the maximum principal stress to the maximum shear stress. Find the power transmitted by a 50 mm diameter shaft, at a speed of 300 rev/min when $k = 0.4$ and the maximum shear stress is 75 MN/m^2 .
[I.Mech.] [$1 + k/\sqrt{(k^2 + 1)}$; 57.6 kW.]

8.21 (B). (a) A solid circular steel shaft is subjected to a bending moment of 10 kN m and is required to transmit a maximum power of 550 kW at 420 rev/min. Assuming the shaft to be simply supported at each end and neglecting the shaft weight, determine the ratio of the maximum principal stress to the maximum shear stress induced in the shaft material.

(b) A 300 mm external diameter and 200 mm internal diameter hollow steel shaft operates under the following conditions:

power transmitted = 2280 kW; maximum torque = $1.2 \times$ mean torque; maximum bending moment = 11 kN m; maximum end thrust = 66 kN; maximum principal compressive stress = 40 MN/m².

Determine the maximum safe speed of rotation for the shaft. [1.625 : 1; 169 rev/min.]

8.22 (C). A uniform solid shaft of circular cross-section will drive the propeller of a ship. It will therefore necessarily be subject simultaneously to a thrust load and a torque. The magnitude of the thrust can be related to the magnitude of the torque by the simple relationship $N = KT$, where N denotes the magnitude of the thrust, T that of the torque and K is a constant. There will also be some bending moment on the shaft. Assuming that the design requirement is that the maximum shearing stress in the material shall nowhere exceed a certain value, denoted by τ , show that the maximum bending moment that can be allowed is given by the expression

$$\text{bending moment, } M = \left[\left(\frac{\tau \pi^2 r^6}{4T^2} - 1 \right)^{1/2} - \frac{Kr}{4} \right] T$$

where r denotes the radius of the shaft cross-section.

[City U.]

CHAPTER 9

THIN CYLINDERS AND SHELLS

Summary

The stresses set up in the walls of a *thin cylinder* owing to an internal pressure p are:

$$\text{circumferential or hoop stress } \sigma_H = \frac{pd}{2t}$$

longitudinal or axial stress $\sigma_L = \frac{pd}{4t}$

where d is the internal diameter and t is the wall thickness of the cylinder.

Then: longitudinal strain $\varepsilon_L = \frac{1}{E} [\sigma_L - v\sigma_H]$

$$\text{hoop strain } \varepsilon_H = \frac{1}{E} [\sigma_H - v\sigma_L]$$

$$\text{change of internal volume of cylinder under pressure} = \frac{pd}{4tE} [5 - 4v] V$$

$$\text{change of volume of contained liquid under pressure} = \frac{pV}{K}$$

where K is the bulk modulus of the liquid.

For thin rotating cylinders of mean radius R the tensile hoop stress set up when rotating at ω rad/s is given by $\sigma_H = \rho\omega^2 R^2$.

For thin spheres:

circumferential or hoop stress $\sigma_H = \frac{pd}{4t}$

$$\text{change of volume under pressure} = \frac{3pd}{4tE} [1 - v] V$$

Effects of end plates and joints—add “joint efficiency factor” η to denominator of stress equations above.

9.1. Thin cylinders under internal pressure

When a thin-walled cylinder is subjected to internal pressure, three mutually perpendicular principal stresses will be set up in the cylinder material, namely the circumferential or hoop

stress, the *radial* stress and the *longitudinal* stress. Provided that the ratio of thickness to inside diameter of the cylinder is less than 1/20, it is reasonably accurate to assume that the hoop and longitudinal stresses are constant across the wall thickness and that the magnitude of the radial stress set up is so small in comparison with the hoop and longitudinal stresses that it can be neglected. This is obviously an approximation since, in practice, it will vary from zero at the outside surface to a value equal to the internal pressure at the inside surface. For the purpose of the initial derivation of stress formulae it is also assumed that the ends of the cylinder and any riveted joints present have no effect on the stresses produced; in practice they will have an effect and this will be discussed later (§9.6).

9.1.1. Hoop or circumferential stress

This is the stress which is set up in resisting the bursting effect of the applied pressure and can be most conveniently treated by considering the equilibrium of half of the cylinder as shown in Fig. 9.1.

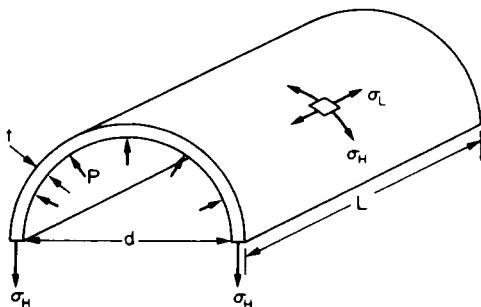


Fig. 9.1. Half of a thin cylinder subjected to internal pressure showing the hoop and longitudinal stresses acting on any element in the cylinder surface.

Total force on half-cylinder owing to internal pressure = $p \times$ projected area = $p \times dL$

Total resisting force owing to hoop stress σ_H set up in the cylinder walls

$$= 2\sigma_H \times Lt$$

∴

$$2\sigma_H Lt = pdL$$

$$\therefore \text{circumferential or hoop stress } \sigma_H = \frac{pd}{2t} \quad (9.1)$$

9.1.2. Longitudinal stress

Consider now the cylinder shown in Fig. 9.2.

Total force on the end of the cylinder owing to internal pressure

$$= \text{pressure} \times \text{area} = p \times \frac{\pi d^2}{4}$$

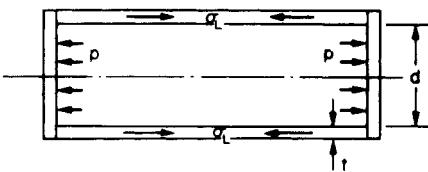


Fig. 9.2. Cross-section of a thin cylinder.

Area of metal resisting this force = πdt (approximately)

$$\therefore \text{stress set up} = \frac{\text{force}}{\text{area}} = p \times \frac{\pi d^2/4}{\pi dt} = \frac{pd}{4t}$$

i.e. $\text{longitudinal stress } \sigma_L = \frac{pd}{4t}$ (9.2)

9.1.3. Changes in dimensions

(a) Change in length

The change in length of the cylinder may be determined from the longitudinal strain, i.e. neglecting the radial stress.

$$\text{Longitudinal strain} = \frac{1}{E} [\sigma_L - v\sigma_H]$$

and

$$\text{change in length} = \text{longitudinal strain} \times \text{original length}$$

$$\begin{aligned} &= \frac{1}{E} [\sigma_L - v\sigma_H] L \\ &= \frac{pd}{4tE} [1 - 2v] L \end{aligned} \quad (9.3)$$

(b) Change in diameter

As above, the change in diameter may be determined from the strain on a diameter, i.e. the diametral strain.

$$\text{Diametral strain} = \frac{\text{change in diameter}}{\text{original diameter}}$$

Now the change in diameter may be found from a consideration of the circumferential change. The stress acting around a circumference is the hoop or circumferential stress σ_H giving rise to the circumferential strain ϵ_H .

$$\begin{aligned} \text{Change in circumference} &= \text{strain} \times \text{original circumference} \\ &= \epsilon_H \times \pi d \end{aligned}$$

$$\begin{aligned}\text{New circumference} &= \pi d + \pi d \varepsilon_H \\ &= \pi d (1 + \varepsilon_H)\end{aligned}$$

But this is the circumference of a circle of diameter $d(1 + \varepsilon_H)$

$$\begin{aligned}\therefore \text{New diameter} &= d(1 + \varepsilon_H) \\ \therefore \text{Change in diameter} &= d\varepsilon_H\end{aligned}$$

$$\text{Diametral strain } \varepsilon_D = \frac{d\varepsilon_H}{d} = \varepsilon_H$$

i.e. **the diametral strain equals the hoop or circumferential strain** (9.4)

$$\begin{aligned}\text{Thus} \quad \text{change in diameter} &= d\varepsilon_H = \frac{d}{E} [\sigma_H - v\sigma_L] \\ &= \frac{pd^2}{4tE} [2 - v]\end{aligned}\tag{9.5}$$

(c) Change in internal volume

$$\text{Change in volume} = \text{volumetric strain} \times \text{original volume}$$

From the work of §14.5, page 364.

$$\begin{aligned}\text{volumetric strain} &= \text{sum of three mutually perpendicular direct strains} \\ &= \varepsilon_L + 2\varepsilon_D \\ &= \frac{1}{E} [\sigma_L - v\sigma_H] + \frac{2}{E} [\sigma_H - v\sigma_L] \\ &= \frac{1}{E} [\sigma_L + 2\sigma_H - v(\sigma_H + 2\sigma_L)] \\ &= \frac{pd}{4tE} [1 + 4 - v(2 + 2)] \\ &= \frac{pd}{4tE} [5 - 4v]\end{aligned}$$

Therefore with original internal volume V

$$\text{change in internal volume} = \frac{pd}{4tE} [5 - 4v] V\tag{9.6}$$

9.2. Thin rotating ring or cylinder

Consider a thin ring or cylinder as shown in Fig. 9.3 subjected to a radial pressure p caused by the centrifugal effect of its own mass when rotating. The centrifugal effect on a unit length

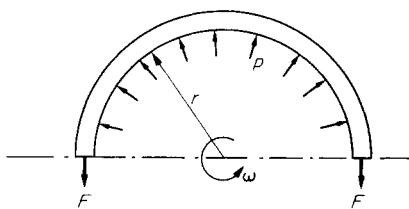


Fig. 9.3. Rotating thin ring or cylinder.

of the circumference is:

$$p = m\omega^2 r$$

Thus, considering the equilibrium of half the ring shown in the figure:

$$\begin{aligned} 2F &= p \times 2r \\ F &= pr \end{aligned}$$

where F is the hoop tension set up owing to rotation.

The cylinder wall is assumed to be so thin that the centrifugal effect can be taken to be constant across the wall thickness.

$$F = pr = m\omega^2 r^2$$

This tension is transmitted through the complete circumference and therefore is restricted by the complete cross-sectional area.

$$\therefore \text{hoop stress} = \frac{F}{A} = \frac{m\omega^2 r^2}{A}$$

where A is the cross-sectional area of the ring.

Now with unit length assumed, m/A is the mass of the ring material per unit volume, i.e. the density ρ .

$$\therefore \text{hoop stress} = \rho\omega^2 r^2 \quad (9.7)$$

9.3. Thin spherical shell under internal pressure

Because of the symmetry of the sphere the stresses set up owing to internal pressure will be two mutually perpendicular hoop or circumferential stresses of equal value and a radial stress. As with thin cylinders having thickness to diameter ratios less than 1 : 20, the radial stress is assumed negligible in comparison with the values of hoop stress set up. The stress system is therefore one of equal biaxial hoop stresses.

Consider, therefore, the equilibrium of the half-sphere shown in Fig. 9.4.

Force on half-sphere owing to internal pressure

$$= \text{pressure} \times \text{projected area}$$

$$= p \times \frac{\pi d^2}{4}$$

$$\text{Resisting force} = \sigma_H \times \pi dt \quad (\text{approximately})$$

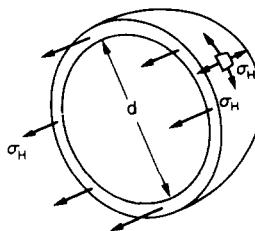


Fig. 9.4. Half of a thin sphere subjected to internal pressure showing uniform hoop stresses acting on a surface element.

$$\therefore p \times \frac{\pi d^2}{4} = \sigma_H \times \pi dt$$

or

$$\sigma_H = \frac{pd}{4t}$$

i.e. circumferential or hoop stress $= \frac{pd}{4t}$ (9.8)

9.3.1. Change in internal volume

As for the cylinder,

change in volume = original volume \times volumetric strain

but

volumetric strain = sum of three mutually perpendicular strains (in this case all equal)

$$= 3\varepsilon_D = 3\varepsilon_H$$

$$= \frac{3}{E} [\sigma_H - v\sigma_H]$$

$$= \frac{3pd}{4tE} [1 - v]$$

$$\therefore \text{change in internal volume} = \frac{3pd}{4tE} [1 - v] V \quad (9.9)$$

9.4. Vessels subjected to fluid pressure

If a fluid is used as the pressurisation medium the fluid itself will change in volume as pressure is increased and this must be taken into account when calculating the amount of fluid which must be pumped into the cylinder in order to raise the pressure by a specified amount, the cylinder being initially full of fluid at atmospheric pressure.

Now the *bulk modulus* of a fluid is defined as follows:

$$\text{bulk modulus } K = \frac{\text{volumetric stress}}{\text{volumetric strain}}$$

where, in this case, volumetric stress = pressure p

and volumetric strain = $\frac{\text{change in volume}}{\text{original volume}} = \frac{\delta V}{V}$

$$\therefore K = \frac{p}{\delta V/V} = \frac{pV}{\delta V}$$

i.e. change in volume of fluid under pressure = $\frac{pV}{K}$ (9.10)

The extra fluid required to raise the pressure must, therefore, take up this volume together with the increase in internal volume of the cylinder itself.

\therefore extra fluid required to raise cylinder pressure by p

$$= \frac{pd}{4tE} [5 - 4v] V + \frac{pV}{K} \quad (9.11)$$

Similarly, for spheres, the extra fluid required is

$$= \frac{3pd}{4tE} [1 - v] V + \frac{pV}{K} \quad (9.12)$$

9.5. Cylindrical vessel with hemispherical ends

Consider now the vessel shown in Fig. 9.5 in which the wall thickness of the cylindrical and hemispherical portions may be different (this is sometimes necessary since the hoop stress in the cylinder is twice that in a sphere of the same radius and wall thickness). For the purpose of the calculation the internal diameter of both portions is assumed equal. From the preceding sections the following formulae are known to apply:

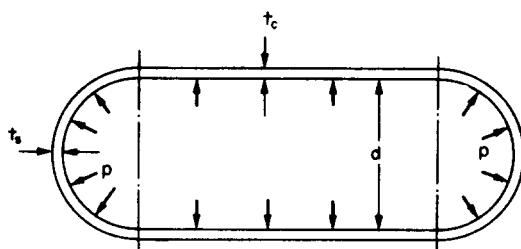


Fig. 9.5. Cross-section of a thin cylinder with hemispherical ends.

(a) For the cylindrical portion

$$\text{hoop or circumferential stress} = \sigma_{H_c} = \frac{pd}{2t_c}$$

$$\text{longitudinal stress} = \sigma_{L_c} = \frac{pd}{4t_c}$$

$$\begin{aligned}\therefore \text{hoop or circumferential strain} &= \frac{1}{E} [\sigma_{H_c} - v\sigma_{L_c}] \\ &= \frac{pd}{4t_c E} [2 - v]\end{aligned}$$

(b) *For the hemispherical ends*

$$\text{hoop stress} = \sigma_{H_s} = \frac{pd}{4t_s}$$

$$\begin{aligned}\therefore \text{hoop strain} &= \frac{1}{E} [\sigma_{H_s} - v\sigma_{H_s}] \\ &= \frac{pd}{4t_s E} [1 - v]\end{aligned}$$

Thus equating the two strains in order that there shall be no distortion of the junction,

$$\frac{pd}{4t_c E} [2 - v] = \frac{pd}{4t_s E} [1 - v]$$

$$\text{i.e. } \frac{t_s}{t_c} = \frac{(1 - v)}{(2 - v)} \quad (9.13)$$

With the normally accepted value of Poisson's ratio for general steel work of 0.3, the thickness ratio becomes

$$\frac{t_s}{t_c} = \frac{0.7}{1.7}$$

i.e. the thickness of the cylinder walls must be approximately 2.4 times that of the hemispherical ends for no distortion of the junction to occur. In these circumstances, because of the reduced wall thickness of the ends, the maximum stress will occur in the ends. For *equal maximum stresses* in the two portions the thickness of the cylinder walls must be twice that in the ends but some distortion at the junction will then occur.

9.6. Effects of end plates and joints

The preceding sections have all assumed uniform material properties throughout the components and have neglected the effects of endplates and joints which are necessary requirements for their production. In general, the strength of the components will be reduced by the presence of, for example, riveted joints, and this should be taken into account by the introduction of a *joint efficiency factor* η into the equations previously derived.

Thus, for *thin cylinders*:

$$\text{hoop stress} = \frac{pd}{2t\eta_L}$$

where η_L is the efficiency of the longitudinal joints,

$$\text{longitudinal stress} = \frac{pd}{4t\eta_C}$$

where η_C is the efficiency of the circumferential joints.

For *thin spheres*:

$$\text{hoop stress} = \frac{pd}{4t\eta}$$

Normally the joint efficiency is stated in percentage form and this must be converted into equivalent decimal form before substitution into the above equations.

9.7. Wire-wound thin cylinders

In order to increase the ability of thin cylinders to withstand high internal pressures without excessive increases in wall thickness, and hence weight and associated material cost, they are sometimes wound with high tensile steel tape or wire under tension. This subjects the cylinder to an initial hoop, compressive, stress which must be overcome by the stresses owing to internal pressure before the material is subjected to tension. There then remains at this stage the normal pressure capacity of the cylinder before the maximum allowable stress in the cylinder is exceeded.

It is normally required to determine the tension necessary in the tape during winding in order to ensure that the maximum hoop stress in the cylinder will not exceed a certain value when the internal pressure is applied.

Consider, therefore, the half-cylinder of Fig. 9.6, where σ_H denotes the hoop stress in the cylinder walls and σ_t the stress in the rectangular-sectioned tape. Let conditions before pressure is applied be denoted by suffix 1 and after pressure is applied by suffix 2.

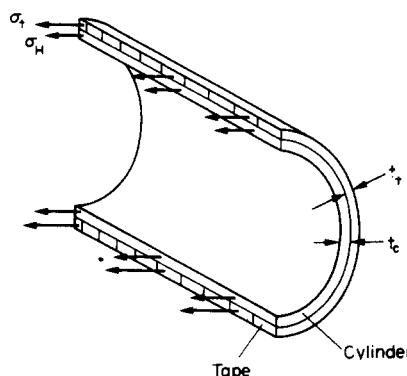


Fig. 9.6. Section of a thin cylinder with an external layer of tape wound on with a tension.

Now

$$\begin{aligned}\text{force owing to tape} &= \sigma_{t_1} \times \text{area} \\ &= \sigma_{t_1} \times 2Lt_t\end{aligned}$$

$$\text{resistive force in the cylinder material} = \sigma_{H_1} \times 2Lt_c$$

i.e. for equilibrium

$$\sigma_{t_1} \times 2Lt_t = \sigma_{H_1} \times 2Lt_c$$

or

$$\sigma_{t_1} \times t_t = \sigma_{H_1} \times t_c$$

so that the *compressive* hoop stress set up in the cylinder walls after winding and before pressurisation is given by

$$\sigma_{H_1} = \sigma_{t_1} \times \frac{t_t}{t_c} \quad (\text{compressive}) \quad (9.14)$$

This equation will be modified if wire of circular cross-section is used for the winding process in preference to rectangular-sectioned tape. The area carrying the stress σ_{t_1} will then be $2na$ where a is the cross-sectional area of the wire and n is the number of turns along the cylinder length.

After pressure has been applied another force is introduced

$$= \text{pressure} \times \text{projected area} = pdL$$

Again, equating forces for equilibrium of the half-cylinder,

$$pdL = (\sigma_{H_2} \times 2Lt_c) + (\sigma_{t_2} \times 2Lt_t) \quad (9.15)$$

where σ_{H_2} is the hoop stress in the cylinder after pressurisation and σ_{t_2} is the final stress in the tape after pressurisation.

Since the limiting value of σ_{H_2} is known for any given internal pressure p , this equation yields the value of σ_{t_2} .

Now the change in strain on the outside surface of the cylinder must equal that on the inside surface of the tape if they are to remain in contact.

$$\text{Change in strain in the tape} = \frac{\sigma_{t_2} - \sigma_{t_1}}{E_t}$$

where E_t is Young's modulus of the tape.

In the absence of any internal pressure originally there will be no longitudinal stress or strain so that the original strain in the cylinder walls is given by σ_{H_1}/E_c , where E_c is Young's modulus of the cylinder material. When pressurised, however, the cylinder will be subjected to a longitudinal strain so that the final strain in the cylinder walls is given by

$$\frac{1}{E_c} [\sigma_{H_2} - v\sigma_L] = \frac{1}{E_c} \left[\sigma_{H_2} - v \frac{pd}{4t_c} \right]$$

$$\therefore \text{change in strain on the cylinder} = \frac{1}{E_c} \left[\sigma_{H_2} - v \frac{pd}{4t_c} - \sigma_{H_1} \right]$$

$$\therefore \frac{1}{E_t} [\sigma_{t_2} - \sigma_{t_1}] = \frac{1}{E_c} \left[\sigma_{H_2} - v \frac{pd}{4t_c} - \sigma_{H_1} \right]$$

Thus with σ_{H_1} obtained in terms of σ_{t_1} from eqn. (9.14), p and σ_{H_2} known, and σ_{t_2} found from eqn. (9.15) the only unknown σ_{t_1} can be determined.

Examples

Example 9.1

A thin cylinder 75 mm internal diameter, 250 mm long with walls 2.5 mm thick is subjected to an internal pressure of 7 MN/m². Determine the change in internal diameter and the change in length.

If, in addition to the internal pressure, the cylinder is subjected to a torque of 200 N m, find the magnitude and nature of the principal stresses set up in the cylinder. $E = 200 \text{ GN/m}^2$, $\nu = 0.3$.

Solution

(a) From eqn. (9.5), change in diameter = $\frac{pd^2}{4tE} (2 - \nu)$

$$= \frac{7 \times 10^6 \times 75^2 \times 10^{-6}}{4 \times 2.5 \times 10^{-3} \times 200 \times 10^9} (2 - 0.3)$$

$$= 33.4 \times 10^{-6} \text{ m}$$

$$= 33.4 \mu\text{m}$$

(b) From eqn. (9.3), change in length = $\frac{pdL}{4tE} (1 - 2\nu)$

$$= \frac{7 \times 10^6 \times 75 \times 10^{-3} \times 250 \times 10^{-3}}{4 \times 2.5 \times 10^{-3} \times 200 \times 10^9} (1 - 0.6)$$

$$= 26.2 \mu\text{m}$$

(c) Hoop stress $\sigma_H = \frac{pd}{2t} = \frac{7 \times 10^6 \times 75 \times 10^{-3}}{2 \times 2.5 \times 10^{-3}}$

$$= 105 \text{ MN/m}^2$$

Longitudinal stress $\sigma_L = \frac{pd}{4t} = \frac{7 \times 10^6 \times 75 \times 10^{-3}}{2 \times 2.5 \times 10^{-3}}$

$$= 52.5 \text{ MN/m}^2$$

In addition to these stresses a shear stress τ is set up.

From the torsion theory,

$$\frac{T}{J} = \frac{\tau}{R} \quad \therefore \quad \tau = \frac{TR}{J}$$

Now $J = \frac{\pi}{32} \frac{(80^4 - 75^4)}{10^{12}} = \frac{\pi}{32} \frac{(41 - 31.6)}{10^6} = 0.92 \times 10^{-6} \text{ m}^4$

Then shear stress $\tau = \frac{200 \times 20 \times 10^{-3}}{0.92 \times 10^{-6}} = 4.34 \text{ MN/m}^2$.

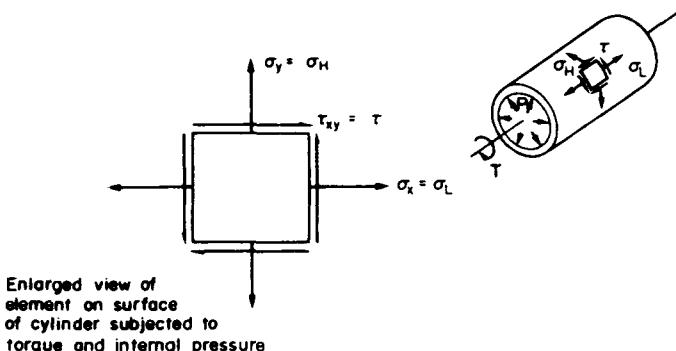


Fig. 9.7. Enlarged view of the stresses acting on an element in the surface of a thin cylinder subjected to torque and internal pressure.

The stress system then acting on any element of the cylinder surface is as shown in Fig. 9.7. The principal stresses are then given by eqn. (13.11),

$$\begin{aligned}\sigma_1 \text{ and } \sigma_2 &= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]} \\ &= \frac{1}{2}(105 + 52.5) \pm \frac{1}{2}\sqrt{[(105 - 52.5)^2 + 4(4.34)^2]} \\ &= \frac{1}{2} \times 157.5 \pm \frac{1}{2}\sqrt{(2760 + 75.3)} \\ &= 78.75 \pm 26.6\end{aligned}$$

Then

$$\sigma_1 = 105.35 \text{ MN/m}^2 \quad \text{and} \quad \sigma_2 = 52.15 \text{ MN/m}^2$$

The principal stresses are

$$105.4 \text{ MN/m}^2 \quad \text{and} \quad 52.2 \text{ MN/m}^2 \quad \text{both tensile.}$$

Example 9.2

A cylinder has an internal diameter of 230 mm, has walls 5 mm thick and is 1 m long. It is found to change in internal volume by $12.0 \times 10^{-6} \text{ m}^3$ when filled with a liquid at a pressure p . If $E = 200 \text{ GN/m}^2$ and $v = 0.25$, and assuming rigid end plates, determine:

- the values of hoop and longitudinal stresses;
- the modifications to these values if joint efficiencies of 45% (hoop) and 85% (longitudinal) are assumed;
- the necessary change in pressure p to produce a further increase in internal volume of 15%. The liquid may be assumed incompressible.

Solution

- (a) From eqn. (9.6)

$$\text{change in internal volume} = \frac{pd}{4tE} (5 - 4v)V$$

$$\text{original volume } V = \frac{\pi}{4} \times 230^2 \times 10^{-6} \times 1 = 41.6 \times 10^{-3} \text{ m}^3$$

Then change in volume = $12 \times 10^{-6} = \frac{p \times 230 \times 10^{-3} \times 41.6 \times 10^{-3}}{4 \times 5 \times 10^{-3} \times 200 \times 10^9}$ (5 - 1)

Thus
$$p = \frac{12 \times 10^{-6} \times 4 \times 5 \times 10^{-3} \times 200 \times 10^9}{230 \times 10^{-3} \times 41.6 \times 10^{-3} \times 4}$$

 $= 1.25 \text{ MN/m}^2$

Hence, hoop stress = $\frac{pd}{2t} = \frac{1.25 \times 10^6 \times 230 \times 10^{-3}}{2 \times 5 \times 10^{-3}}$
 $= 28.8 \text{ MN/m}^2$

$$\text{longitudinal stress} = \frac{pd}{4t} = 14.4 \text{ MN/m}^2$$

(b) Hoop stress, acting on the longitudinal joints (§9.6)

$$= \frac{pd}{2t\eta_L} = \frac{1.25 \times 10^6 \times 230 \times 10^{-3}}{2 \times 5 \times 10^{-3} \times 0.85}$$

 $= 33.9 \text{ MN/m}^2$

Longitudinal stress (acting on the circumferential joints)

$$= \frac{pd}{4t\eta_c} = \frac{1.25 \times 10^6 \times 230 \times 10^{-3}}{4 \times 5 \times 10^{-3} \times 0.45}$$

 $= 32 \text{ MN/m}^2$

(c) Since the change in volume is directly proportional to the pressure, the necessary 15% increase in volume is achieved by increasing the pressure also by 15%.

$$\begin{aligned} \text{Necessary increase in } p &= 0.15 \times 1.25 \times 10^6 \\ &= 1.86 \text{ MN/m}^2 \end{aligned}$$

Example 9.3

(a) A sphere, 1 m internal diameter and 6 mm wall thickness, is to be pressure-tested for safety purposes with water as the pressure medium. Assuming that the sphere is initially filled with water at atmospheric pressure, what extra volume of water is required to be pumped in to produce a pressure of 3 MN/m^2 gauge? For water, $K = 2.1 \text{ GN/m}^2$.

(b) The sphere is now placed in service and filled with gas until there is a volume change of $72 \times 10^{-6} \text{ m}^3$. Determine the pressure exerted by the gas on the walls of the sphere.

(c) To what value can the gas pressure be increased before failure occurs according to the maximum principal stress theory of elastic failure?

For the material of the sphere $E = 200 \text{ GN/m}^2$, $v = 0.3$ and the yield stress σ_y in simple tension = 280 MN/m^2 .

Solution

(a) Bulk modulus $K = \frac{\text{volumetric stress}}{\text{volumetric strain}}$

Now volumetric stress = pressure $p = 3 \text{ MN/m}^2$

and volumetric strain = change in volume \div original volume

i.e.

$$K = \frac{p}{\delta V/V}$$

$$\therefore \text{change in volume of water} = \frac{pV}{K} = \frac{3 \times 10^6}{2.1 \times 10^9} \times \frac{4\pi}{3} (0.5)^3 \\ = 0.748 \times 10^{-3} \text{ m}^3$$

(b) From eqn. (9.9) the change in volume is given by

$$\delta V = \frac{3pd}{4tE} (1 - v)V$$

$$\therefore 72 \times 10^{-6} = \frac{3p \times 1 \times \frac{4}{3}\pi(0.5)^3(1 - 0.3)}{4 \times 6 \times 10^{-3} \times 200 \times 10^9}$$

$$\therefore p = \frac{72 \times 10^{-6} \times 4 \times 6 \times 200 \times 10^6 \times 3}{3 \times 4\pi(0.5)^3 \times 0.7} \\ = 314 \times 10^3 \text{ N/m}^2 = 314 \text{ kN/m}^2$$

(c) The maximum stress set up in the sphere will be the hoop stress,

i.e. $\sigma_1 = \sigma_H = \frac{pd}{4t}$

Now, according to the maximum principal stress theory (see §15.2) failure will occur when the maximum principal stress equals the value of the yield stress of a specimen subjected to simple tension,

i.e. when

$$\sigma_1 = \sigma_y = 280 \text{ MN/m}^2$$

Thus

$$280 \times 10^6 = \frac{pd}{4t}$$

$$p = \frac{280 \times 10^6 \times 4 \times 6 \times 10^{-3}}{1} \\ = 6.72 \times 10^6 \text{ N/m}^2 = 6.7 \text{ MN/m}^2$$

The sphere would therefore yield at a pressure of 6.7 MN/m^2 .

Example 9.4

A closed thin copper cylinder of 150 mm internal diameter having a wall thickness of 4 mm is closely wound with a single layer of steel tape having a thickness of 1.5 mm, the tape being

wound on when the cylinder has no internal pressure. Estimate the tensile stress in the steel tape when it is being wound to ensure that when the cylinder is subjected to an internal pressure of 3.5 MN/m^2 the tensile hoop stress in the cylinder will not exceed 35 MN/m^2 . For copper, Poisson's ratio $\nu = 0.3$ and $E = 100 \text{ GN/m}^2$; for steel, $E = 200 \text{ GN/m}^2$.

Solution

Let σ , be the stress in the tape and let conditions before pressure is applied be denoted by suffix 1 and after pressure is applied by suffix 2.

Consider the half-cylinder shown (before pressure is applied) in Fig. 9.6 (see page 206):

$$\begin{aligned}\text{force owing to tension in tape} &= \sigma_{t_1} \times \text{area} \\ &= \sigma_{t_1} \times 1.5 \times 10^{-3} \times L \times 2\end{aligned}$$

$$\text{resistive force in the material of cylinder wall} = \sigma_{H_1} \times 4 \times 10^{-3} \times L \times 2$$

$$\therefore 2\sigma_{H_1} \times 4 \times 10^{-3} \times L = 2\sigma_{t_1} \times 1.5 \times 10^{-3} \times L$$

$$\therefore \sigma_{H_1} = \frac{1.5}{4} \sigma_{t_1} = 0.375 \sigma_{t_1} \text{ (compressive)} \quad (1)$$

After pressure is applied another force is introduced

$$\begin{aligned}&= \text{pressure} \times \text{projected area} \\ &= p(dL)\end{aligned}$$

Equating forces now acting on the half-cylinder,

$$pdL = (\sigma_{H_2} \times 2 \times 4 \times 10^{-3} \times L) + (\sigma_{t_2} \times 2 \times 1.5 \times 10^{-3} \times L)$$

$$\text{but } p = 3.5 \times 10^6 \text{ N/m}^2 \quad \text{and} \quad \sigma_{H_2} = 35 \times 10^6 \text{ N/m}^2$$

$$\therefore 3.5 \times 10^6 \times 150 \times 10^{-3} L = (35 \times 10^6 \times 2 \times 4 \times 10^{-3} L) + (\sigma_{t_2} \times 2 \times 1.5 \times 10^{-3} \times L)$$

$$\therefore 525 \times 10^6 = 280 \times 10^6 + 3\sigma_{t_2}$$

$$\therefore \sigma_{t_2} = \frac{(525 - 280)}{3} 10^6$$

$$\sigma_{t_2} = 82 \text{ MN/m}^2$$

The change in strain on the outside of the cylinder and on the inside of the tape must be equal:

$$\text{change in strain in tape} = \frac{\sigma_{t_2} - \sigma_{t_1}}{E_s}$$

$$\text{original strain in cylinder walls} = \frac{\sigma_{H_1}}{E_c}$$

(Since there is no pressure in the cylinder in the original condition there will be no longitudinal stress.)

Final strain in cylinder (after pressurising)

$$= \frac{\sigma_{H_2}}{E_c} - \frac{v\sigma_L}{E_c}$$

$$= \frac{1}{Ec} \left(\sigma_{H_2} - \frac{vpd}{4t} \right)$$

Then change in strain in cylinder

$$= \frac{1}{Ec} \left(\sigma_{H_2} - \frac{vpd}{4t} - \sigma_{H_1} \right)$$

Then $\frac{1}{E_s} (\sigma_{t_2} - \sigma_{t_1}) = \frac{1}{Ec} \left(\sigma_{H_2} - \frac{vpd}{4t} - \sigma_{H_1} \right)$

Substituting for σ_{H_1} from eqn. (1)

$$\frac{82 \times 10^6 - \sigma_{t_1}}{200 \times 10^9} = \frac{1}{100 \times 10^9} \left[35 \times 10^6 - \frac{0.3 \times 3.5 \times 10^6 \times 154 \times 10^{-3}}{4 \times 4 \times 10^{-3}} - 0.375 \sigma_{t_1} \right]$$

$$82 \times 10^6 - \sigma_{t_1} = 2(35 \times 10^6 - 10.1 \times 10^6 - 0.375 \sigma_{t_1})$$

$$= 49.8 \times 10^6 - 0.75 \sigma_{t_1}$$

Then $1.75 \sigma_{t_1} = (82.0 - 49.8)10^6$

$$\sigma_{t_1} = \frac{32.2 \times 10^6}{1.75}$$

$$= 18.4 \text{ MN/m}^2$$

Problems

9.1 (A). Determine the hoop and longitudinal stresses set up in a thin boiler shell of circular cross-section, 5 m long and of 1.3 m internal diameter when the internal pressure reaches a value of 2.4 bar (240 kN/m^2). What will then be its change in diameter? The wall thickness of the boiler is 25 mm. $E = 210 \text{ GN/m}^2$; $v = 0.3$.

[6.24, 3.12 MN/m²; 0.033 mm.]

9.2 (A). Determine the change in volume of a thin cylinder of original volume $65.5 \times 10^{-3} \text{ m}^3$ and length 1.3 m if its wall thickness is 6 mm and the internal pressure 14 bar (1.4 MN/m^2). For the cylinder material $E = 210 \text{ GN/m}^2$; $v = 0.3$.

[$17.5 \times 10^{-6} \text{ m}^3$.]

9.3 (A). What must be the wall thickness of a thin spherical vessel of diameter 1 m if it is to withstand an internal pressure of 70 bar (7 MN/m^2) and the hoop stresses are limited to 270 MN/m^2 ? [12.96 mm.]

9.4 (A/B). A steel cylinder 1 m long, of 150 mm internal diameter and plate thickness 5 mm, is subjected to an internal pressure of 70 bar (7 MN/m^2); the increase in volume owing to the pressure is $16.8 \times 10^{-6} \text{ m}^3$. Find the values of Poisson's ratio and the modulus of rigidity. Assume $E = 210 \text{ GN/m}^2$. [U.L.] [0.299; 80.8 GN/m^2 .]

9.5 (B). Define bulk modulus K , and show that the decrease in volume of a fluid under pressure p is pV/K . Hence derive a formula to find the extra fluid which must be pumped into a thin cylinder to raise its pressure by an amount p .

How much fluid is required to raise the pressure in a thin cylinder of length 3 m, internal diameter 0.7 m, and wall thickness 12 mm by 0.7 bar (70 kN/m^2)? $E = 210 \text{ GN/m}^2$ and $v = 0.3$ for the material of the cylinder and $K = 2.1 \text{ GN/m}^2$ for the fluid.

[$5.981 \times 10^{-3} \text{ m}^3$.]

9.6 (B). A spherical vessel of 1.7 m diameter is made from 12 mm thick plate, and it is to be subjected to a hydraulic test. Determine the additional volume of water which it is necessary to pump into the vessel, when the vessel is initially just filled with water, in order to raise the pressure to the proof pressure of 116 bar (11.6 MN/m^2). The bulk modulus of water is 2.9 GN/m^2 . For the material of the vessel, $E = 200 \text{ GN/m}^2$, $v = 0.3$.

[$26.14 \times 10^{-3} \text{ m}^3$.]

9.7 (B). A thin-walled steel cylinder is subjected to an internal fluid pressure of 21 bar (2.1 MN/m^2). The boiler is of 1 m inside diameter and 3 m long and has a wall thickness of 30 mm. Calculate the hoop and longitudinal stresses present in the cylinder and determine what torque may be applied to the cylinder if the principal stress is limited to 150 MN/m^2 .
 [35, 17.5 MN/m^2 ; 6 MN m .]

9.8 (B). A thin cylinder of 300 mm internal diameter and 12 mm thickness is subjected to an internal pressure p while the ends are subjected to an external pressure of $\frac{1}{2}p$. Determine the value of p at which elastic failure will occur according to (a) the maximum shear stress theory, and (b) the maximum shear strain energy theory, if the limit of proportionality of the material in simple tension is 270 MN/m^2 . What will be the volumetric strain at this pressure?
 $E = 210 \text{ GN/m}^2$; $\nu = 0.3$
 [21.6, 23.6 MN/m^2 , 2.289×10^{-3} , 2.5×10^{-3} .]

9.9 (C). A brass pipe has an internal diameter of 400 mm and a metal thickness of 6 mm. A single layer of high-tensile wire of diameter 3 mm is wound closely round it at a tension of 500 N. Find (a) the stress in the pipe when there is no internal pressure; (b) the maximum permissible internal pressure in the pipe if the working tensile stress in the brass is 60 MN/m^2 ; (c) the stress in the steel wire under condition (b). Treat the pipe as a thin cylinder and neglect longitudinal stresses and strains.
 $E_S = 200 \text{ GN/m}^2$; $E_B = 100 \text{ GN/m}^2$.

[U.L.] [27.8 , 3.04 MN/m^2 ; 104.8 MN/m^2 .]

9.10 (B). A cylindrical vessel of 1 m diameter and 3 m long is made of steel 12 mm thick and filled with water at 16°C . The temperature is then raised to 50°C . Find the stresses induced in the material of the vessel given that over this range of temperature water increases 0.006 per unit volume. (Bulk modulus of water = 2.9 GN/m^2 ; E for steel = 210 GN/m^2 and $\nu = 0.3$.) Neglect the expansion of the steel owing to temperature rise.

[663, 331.5 MN/m^2 .]

9.11 (C). A 3 m long aluminium-alloy tube, of 150 mm outside diameter and 5 mm wall thickness, is closely wound with a single layer of 2.5 mm diameter steel wire at a tension of 400 N. It is then subjected to an internal pressure of 70 bar (7 MN/m^2).

- (a) Find the stress in the tube before the pressure is applied.
- (b) Find the final stress in the tube.

$$E_A = 70 \text{ GN/m}^2, \nu_A = 0.28; E_S = 200 \text{ GN/m}^2$$

[-32 , 20.5 MN/m^2 .]

9.12 (B). (a) Derive the equations for the circumferential and longitudinal stresses in a thin cylindrical shell.

(b) A thin cylinder of 300 mm internal diameter, 3 m long and made from 3 mm thick metal, has its ends blanked off. Working from first principles, except that you may use the equations derived above, find the change in capacity of this cylinder when an internal fluid pressure of 20 bar is applied.
 $E = 200 \text{ GN/m}^2$; $\nu = 0.3$. [$201 \times 10^{-6} \text{ m}^3$.]

9.13 (A/B). Show that the tensile hoop stress set up in a thin rotating ring or cylinder is given by:

$$\sigma_H = \rho \omega^2 r^2.$$

Hence determine the maximum angular velocity at which the disc can be rotated if the hoop stress is limited to 20 MN/m^2 . The ring has a mean diameter of 260 mm.
 [3800 rev/min.]

CHAPTER 10

THICK CYLINDERS

Summary

The *hoop and radial stresses* at any point in the wall cross-section of a thick cylinder at radius r are given by the Lamé equations:

$$\text{hoop stress } \sigma_H = A + \frac{B}{r^2}$$

$$\text{radial stress } \sigma_r = A - \frac{B}{r^2}$$

With internal and external pressures P_1 and P_2 and internal and external radii R_1 and R_2 respectively, the *longitudinal stress* in a cylinder with closed ends is

$$\sigma_L = \frac{P_1 R_1^2 - P_2 R_2^2}{(R_2^2 - R_1^2)} = \text{Lamé constant } A$$

Changes in dimensions of the cylinder may then be determined from the following strain formulae:

$$\text{circumferential or hoop strain} = \text{diametral strain}$$

$$= \frac{\sigma_H}{E} - v \frac{\sigma_r}{E} - v \frac{\sigma_L}{E}$$

$$\text{longitudinal strain} = \frac{\sigma_L}{E} - v \frac{\sigma_r}{E} - v \frac{\sigma_H}{E}$$

For *compound tubes* the resultant hoop stress is the algebraic sum of the hoop stresses resulting from shrinkage and the hoop stresses resulting from internal and external pressures.

For *force and shrink fits* of cylinders made of *different materials*, the total interference or shrinkage allowance (on radius) is

$$[\varepsilon_{H_o} - \varepsilon_{H_i}]r$$

where ε_{H_o} and ε_{H_i} are the hoop strains existing in the outer and inner cylinders respectively at the common radius r . For cylinders of the *same material* this equation reduces to

$$\frac{r}{E} [\sigma_{H_o} - \sigma_{H_i}]$$

For a *hub or sleeve shrunk on a solid shaft* the shaft is subjected to constant hoop and radial stresses, each equal to the pressure set up at the junction. The hub or sleeve is then treated as a thick cylinder subjected to this internal pressure.

Wire-wound thick cylinders

If the internal and external radii of the cylinder are R_1 and R_2 respectively and it is wound with wire until its external radius becomes R_3 , the radial and hoop stresses in the wire at any radius r between the radii R_2 and R_3 are found from:

$$\text{radial stress} = \left(\frac{r^2 - R_1^2}{2r^2} \right) T \log_e \left(\frac{R_3^2 - R_1^2}{r^2 - R_1^2} \right)$$

$$\text{hoop stress} = T \left\{ 1 - \left(\frac{r^2 + R_1^2}{2r^2} \right) \log_e \left(\frac{R_3^2 - R_1^2}{r^2 - R_1^2} \right) \right\}$$

where T is the constant tension stress in the wire.

The hoop and radial stresses in the cylinder can then be determined by considering the cylinder to be subjected to an external pressure equal to the value of the radial stress above when $r = R_2$.

When an additional internal pressure is applied the final stresses will be the algebraic sum of those resulting from the internal pressure and those resulting from the wire winding.

Plastic yielding of thick cylinders

For initial yield, the internal pressure P_1 is given by:

$$P_1 = \frac{\sigma_y}{2R_2^2} [R_2^2 - R_1^2]$$

For yielding to a radius R_p ,

$$P_1 = \sigma_y \left[\log_e \frac{R_1}{R_p} - \frac{1}{2R_2} (R_2^2 - R_p^2) \right]$$

and for complete collapse,

$$P_1 = \sigma_y \left[\log_e \frac{R_1}{R_2} \right]$$

10.1. Difference in treatment between thin and thick cylinders – basic assumptions

The theoretical treatment of thin cylinders assumes that the hoop stress is constant across the thickness of the cylinder wall (Fig. 10.1), and also that there is no pressure gradient across the wall. Neither of these assumptions can be used for thick cylinders for which the variation of hoop and radial stresses is shown in Fig. 10.2, their values being given by the Lamé equations:

$$\sigma_H = A + \frac{B}{r^2} \quad \text{and} \quad \sigma_r = A - \frac{B}{r^2}$$

Development of the theory for thick cylinders is concerned with sections remote from the

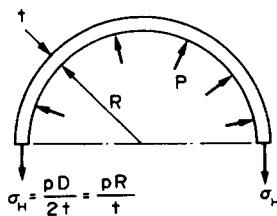


Fig. 10.1. Thin cylinder subjected to internal pressure.

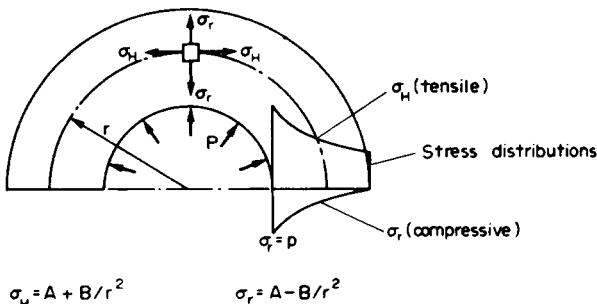


Fig. 10.2. Thick cylinder subjected to internal pressure.

ends since distribution of the stresses around the joints makes analysis at the ends particularly complex. For central sections the applied pressure system which is normally applied to thick cylinders is symmetrical, and all points on an annular element of the cylinder wall will be displaced by the same amount, this amount depending on the radius of the element. Consequently there can be no shearing stress set up on transverse planes and stresses on such planes are therefore principal stresses (see page 331). Similarly, since the radial shape of the cylinder is maintained there are no shears on radial or tangential planes, and again stresses on such planes are principal stresses. Thus, consideration of any element in the wall of a thick cylinder involves, in general, consideration of a mutually perpendicular, tri-axial, principal stress system, the three stresses being termed **radial**, **hoop** (tangential or circumferential) and **longitudinal** (axial) stresses.

10.2. Development of the Lamé theory

Consider the thick cylinder shown in Fig. 10.3. The stresses acting on an element of unit length at radius r are as shown in Fig. 10.4, the radial stress increasing from σ_r to $\sigma_r + d\sigma_r$ over the element thickness dr (*all stresses are assumed tensile*).

For radial equilibrium of the element:

$$(\sigma_r + d\sigma_r)(r + dr)d\theta \times 1 - \sigma_r \times rd\theta \times 1 = 2\sigma_H \times dr \times 1 \times \sin \frac{d\theta}{2}$$

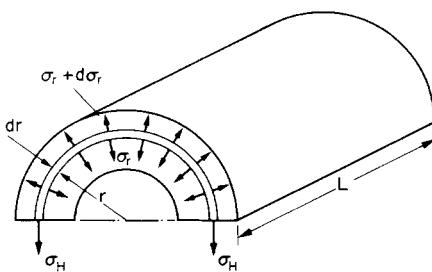


Fig. 10.3.

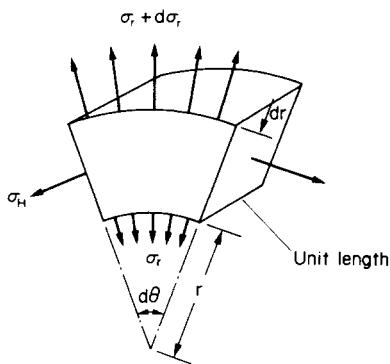


Fig. 10.4.

For small angles:

$$\sin \frac{d\theta}{2} \approx \frac{d\theta}{2} \text{ radian}$$

Therefore, neglecting second-order small quantities,

$$rd\sigma_r + \sigma_r dr = \sigma_H dr$$

$$\therefore \sigma_r + r \frac{d\sigma_r}{dr} = \sigma_H$$

or

$$\sigma_H - \sigma_r = r \frac{d\sigma_r}{dr} \quad (10.1)$$

Assuming now that plane sections remain plane, i.e. the longitudinal strain ε_L is constant across the wall of the cylinder,

$$\text{then } \varepsilon_L = \frac{1}{E} [\sigma_L - v\sigma_r - v\sigma_H]$$

$$= \frac{1}{E} [\sigma_L - v(\sigma_r + \sigma_H)] = \text{constant}$$

It is also assumed that the longitudinal stress σ_L is constant across the cylinder walls at points remote from the ends.

$$\therefore \sigma_r + \sigma_H = \text{constant} = 2A \quad (\text{say}) \quad (10.2)$$

Substituting in (10.1) for σ_H ,

$$2A - \sigma_r - \sigma_r = r \frac{d\sigma_r}{dr}$$

Multiplying through by r and rearranging,

$$2\sigma_r r + r^2 \frac{d\sigma_r}{dr} - 2Ar = 0$$

i.e.

$$\frac{d}{dr}(\sigma_r r^2 - Ar^2) = 0$$

Therefore, integrating,

$$\sigma_r r^2 - Ar^2 = \text{constant} = -B \text{ (say)}$$

$$\therefore \sigma_r = A - \frac{B}{r^2} \quad (10.3)$$

and from eqn. (10.2)

$$\sigma_H = A + \frac{B}{r^2} \quad (10.4)$$

The above equations yield the radial and hoop stresses at any radius r in terms of constants A and B . For any pressure condition there will always be two known conditions of stress (usually radial stress) which enable the constants to be determined and the required stresses evaluated.

10.3. Thick cylinder – internal pressure only

Consider now the thick cylinder shown in Fig. 10.5 subjected to an internal pressure P , the external pressure being zero.

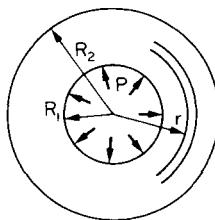


Fig. 10.5. Cylinder cross-section.

The two known conditions of stress which enable the Lamé constants A and B to be determined are:

$$\text{At } r = R_1 \quad \sigma_r = -P \quad \text{and} \quad \text{at } r = R_2 \quad \sigma_r = 0$$

N.B. – The internal pressure is considered as a negative radial stress since it will produce a radial compression (i.e. thinning) of the cylinder walls and the normal stress convention takes compression as negative.

Substituting the above conditions in eqn. (10.3),

$$-P = A - \frac{B}{R_1^2}$$

$$0 = A - \frac{B}{R_2^2}$$

i.e.

$$A = \frac{PR_1^2}{(R_2^2 - R_1^2)} \quad \text{and} \quad B = \frac{PR_1^2 R_2^2}{(R_2^2 - R_1^2)}$$

$$\therefore \text{radial stress } \sigma_r = A - \frac{B}{r^2}$$

$$\begin{aligned} &= \frac{PR_1^2}{(R_2^2 - R_1^2)} \left[1 - \frac{R_2^2}{r^2} \right] \\ &= \frac{PR_1^2}{(R_2^2 - R_1^2)} \left[\frac{r^2 - R_2^2}{r^2} \right] = -P \left[\frac{(R_2/r)^2 - 1}{k^2 - 1} \right] \end{aligned} \quad (10.5)$$

where k is the diameter ratio $D_2/D_1 = R_2/R_1$

and hoop stress $\sigma_H = \frac{PR_1^2}{(R_2^2 - R_1^2)} \left[1 + \frac{R_2^2}{r^2} \right]$

$$\begin{aligned} &= \frac{PR_1^2}{(R_2^2 - R_1^2)} \left[\frac{r^2 + R_2^2}{r^2} \right] = P \left[\frac{(R_2/r)^2 + 1}{k^2 - 1} \right] \end{aligned} \quad (10.6)$$

These equations yield the stress distributions indicated in Fig. 10.2 with maximum values of both σ_r and σ_H at the inside radius.

10.4. Longitudinal stress

Consider now the cross-section of a thick cylinder with closed ends subjected to an internal pressure P_1 and an external pressure P_2 (Fig. 10.6).

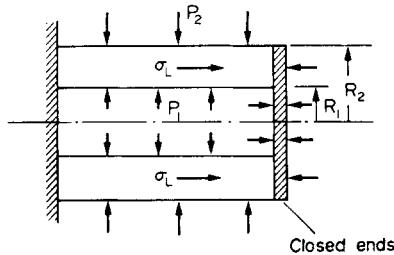


Fig. 10.6. Cylinder longitudinal section.

For horizontal equilibrium:

$$P_1 \times \pi R_1^2 - P_2 \times \pi R_2^2 = \sigma_L \times \pi (R_2^2 - R_1^2)$$

where σ_L is the longitudinal stress set up in the cylinder walls,

$$\therefore \text{longitudinal stress } \sigma_L = \frac{P_1 R_1^2 - P_2 R_2^2}{R_2^2 - R_1^2} \quad (10.7)$$

i.e. a constant.

It can be shown that the constant has the same value as the constant A of the Lame equations. This can be verified for the “internal pressure only” case of §10.3 by substituting $P_2 = 0$ in eqn. (10.7) above.

For combined internal and external pressures, the relationship $\sigma_L = A$ also applies.

10.5. Maximum shear stress

It has been stated in §10.1 that the stresses on an element at any point in the cylinder wall are principal stresses.

It follows, therefore, that the maximum shear stress at any point will be given by eqn. (13.12) as

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2}$$

i.e. *half the difference between the greatest and least principal stresses.*

Therefore, in the case of the thick cylinder, normally,

$$\tau_{\max} = \frac{\sigma_H - \sigma_r}{2}$$

since σ_H is normally tensile, whilst σ_r is compressive and both exceed σ_L in magnitude.

$$\begin{aligned} \therefore \tau_{\max} &= \frac{1}{2} \left[\left(A + \frac{B}{r^2} \right) - \left(A - \frac{B}{r^2} \right) \right] \\ \tau_{\max} &= \frac{B}{r^2} \end{aligned} \quad (10.8)$$

The greatest value of τ_{\max} thus normally occurs at the inside radius where $r = R_1$.

10.6. Change of cylinder dimensions

(a) Change of diameter

It has been shown in §9.3 that the diametral strain on a cylinder equals the hoop or circumferential strain.

$$\begin{aligned} \text{Therefore change of diameter} &= \text{diametral strain} \times \text{original diameter} \\ &= \text{circumferential strain} \times \text{original diameter} \end{aligned}$$

With the principal stress system of hoop, radial and longitudinal stresses, all assumed tensile, the circumferential strain is given by

$$\varepsilon_H = \frac{1}{E} [\sigma_H - v\sigma_r - v\sigma_L]$$

Thus the change of diameter at any radius r of the cylinder is given by

$$\Delta D = \frac{2r}{E} [\sigma_H - v\sigma_r - v\sigma_L] \quad (10.9)$$

(b) Change of length

Similarly, the change of length of the cylinder is given by

$$\Delta L = \frac{L}{E} [\sigma_L - \sigma_v - v\sigma_H] \quad (10.10)$$

10.7. Comparison with thin cylinder theory

In order to determine the limits of D/t ratio within which it is safe to use the simple thin cylinder theory, it is necessary to compare the values of stress given by both thin and thick cylinder theory for given pressures and D/t values. Since the maximum hoop stress is normally the limiting factor, it is this stress which will be considered.

From thin cylinder theory:

$$\sigma_H = P \frac{D}{2t}$$

$$\text{i.e. } \frac{\sigma_H}{P} = \frac{K}{2} \quad \text{where } K = D/t$$

For thick cylinders, from eqn. (10.6),

$$\begin{aligned} \sigma_H &= \frac{PR_1^2}{(R_2^2 - R_1^2)} \left[1 + \frac{R_2^2}{r^2} \right] \\ \text{i.e. } \sigma_{H_{\max}} &= P \frac{(R_1^2 + R_2^2)}{(R_2^2 - R_1^2)} \quad \text{at } r = R_1 \end{aligned} \quad (10.11)$$

Now, substituting for $R_2 = R_1 + t$ and $D = 2R_1$,

$$\begin{aligned} \sigma_{H_{\max}} &= \left[\frac{\frac{1}{2}D^2 + t(D+t)}{t(D+t)} \right] P \\ &= \left[\frac{D^2}{2t^2(D/t+1)} + 1 \right] P \\ \text{i.e. } \frac{\sigma_{H_{\max}}}{P} &= \frac{K^2}{2(K+1)} + 1 \end{aligned} \quad (10.12)$$

Thus for various D/t ratios the stress values from the two theories may be plotted and compared; this is shown in Fig. 10.7.

Also indicated in Fig. 10.7 is the percentage error involved in using the thin cylinder theory. It will be seen that the error will be held within 5% if D/t ratios in excess of 15 are used.

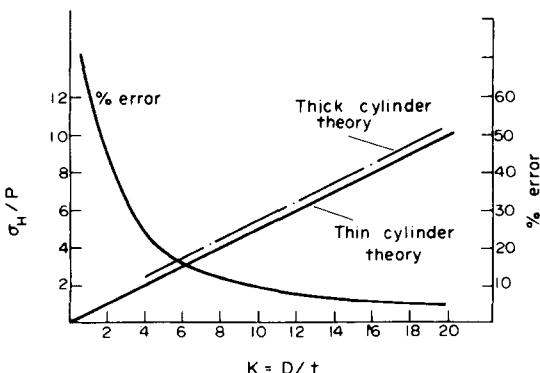


Fig. 10.7. Comparison of thin and thick cylinder theories for various diameter/thickness ratios.

However, if D is taken as the *mean* diameter for calculation of the thin cylinder values instead of the inside diameter as used here, the percentage error reduces from 5% to approximately 0.25% at $D/t = 15$.

10.8. Graphical treatment – Lamé line

The Lamé equations when plotted on stress and $1/r^2$ axes produce straight lines, as shown in Fig. 10.8.

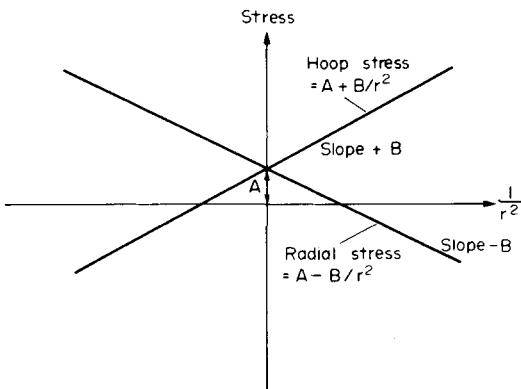


Fig. 10.8. Graphical representation of Lamé equations – Lamé line.

Both lines have exactly the same intercept A and the same magnitude of slope B , the only difference being the sign of their slopes. The two are therefore combined by plotting hoop stress values to the left of the σ axis (again against $1/r^2$) instead of to the right to give the single line shown in Fig. 10.9. In most questions one value of σ , and one value of σ_H , or alternatively two values of σ_r , are given. In both cases the single line can then be drawn.

When a thick cylinder is subjected to external pressure only, the radial stress at the inside radius is zero and the graph becomes the straight line shown in Fig. 10.10.

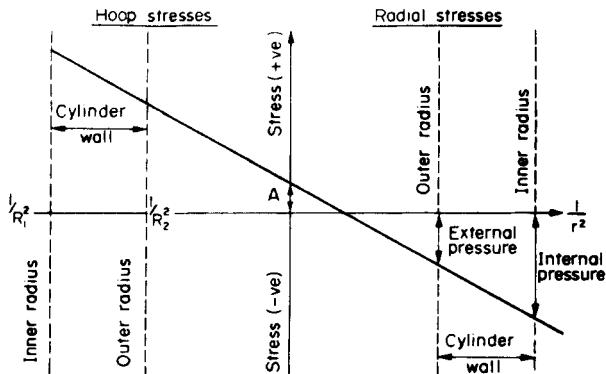


Fig. 10.9. Lamé line solution for cylinder with internal and external pressures.

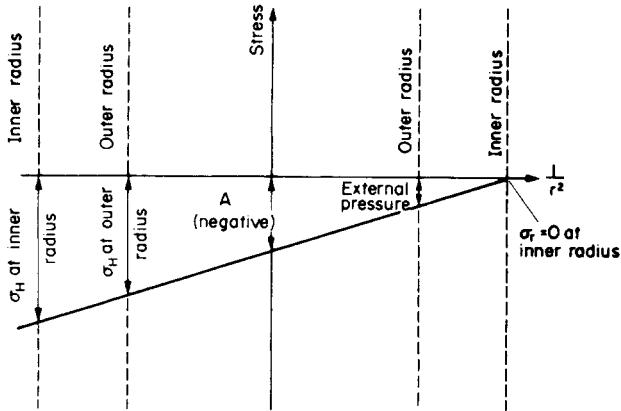


Fig. 10.10. Lamé line solution for cylinder subjected to external pressure only.

N.B.—From §10.4 the value of the *longitudinal stress* σ_L is given by the *intercept A on the σ axis*.

It is not sufficient simply to read off stress values from the axes since this can introduce appreciable errors. Accurate values must be obtained from proportions of the figure using similar triangles.

10.9. Compound cylinders

From the sketch of the stress distributions in Fig. 10.2 it is evident that there is a large variation in hoop stress across the wall of a cylinder subjected to internal pressure. The material of the cylinder is not therefore used to its best advantage. To obtain a more uniform hoop stress distribution, cylinders are often built up by shrinking one tube on to the outside of another. When the outer tube contracts on cooling the inner tube is brought into a state of

compression. The outer tube will conversely be brought into a state of tension. If this compound cylinder is now subjected to internal pressure the resultant hoop stresses will be the algebraic sum of those resulting from internal pressure and those resulting from shrinkage as drawn in Fig. 10.11; thus a much smaller total fluctuation of hoop stress is obtained. A similar effect is obtained if a cylinder is wound with wire or steel tape under tension (see §10.19).

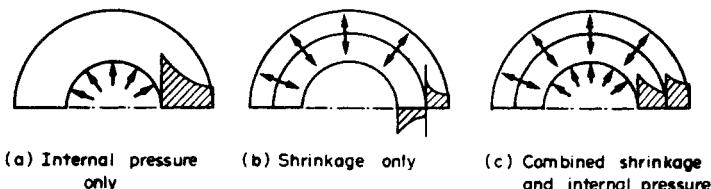


Fig. 10.11. Compound cylinders – combined internal pressure and shrinkage effects.

(a) Same materials

The method of solution for compound cylinders constructed from similar materials is to break the problem down into three separate effects:

- (a) shrinkage pressure only on the inside cylinder;
- (b) shrinkage pressure only on the outside cylinder;
- (c) internal pressure only on the complete cylinder (Fig. 10.12).

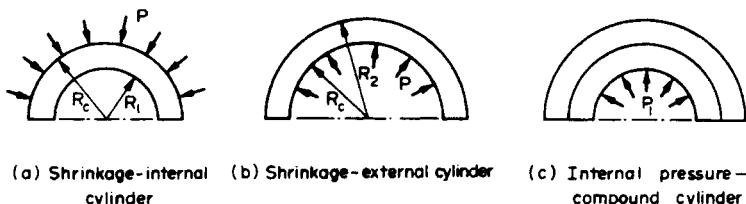


Fig. 10.12. Method of solution for compound cylinders.

For each of the resulting load conditions there are two known values of radial stress which enable the Lamé constants to be determined in each case.

i.e. condition (a) *shrinkage–internal cylinder*:

$$\text{At } r = R_1, \sigma_r = 0$$

$$\text{At } r = R_c, \sigma_r = -p \quad (\text{compressive since it tends to reduce the wall thickness})$$

condition (b) *shrinkage–external cylinder*:

$$\text{At } r = R_2, \sigma_r = 0$$

$$\text{At } r = R_c, \sigma_r = -p$$

condition (c) *internal pressure–compound cylinder*:

$$\text{At } r = R_2, \sigma_r = 0$$

$$\text{At } r = R_1, \sigma_r = -P_1$$

Thus for each condition the hoop and radial stresses at any radius can be evaluated and the principle of superposition applied, i.e. the various stresses are then combined algebraically to produce the stresses in the compound cylinder subjected to both shrinkage and internal pressure. In practice this means that the compound cylinder is able to withstand greater internal pressures before failure occurs or, alternatively, that a thinner compound cylinder (with the associated reduction in material cost) may be used to withstand the same internal pressure as the single thick cylinder it replaces.

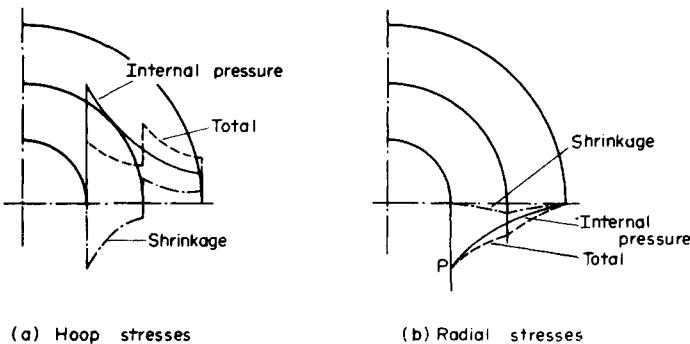


Fig. 10.13. Distribution of hoop and radial stresses through the walls of a compound cylinder.

(b) Different materials

(See §10.14.)

10.10. Compound cylinders – graphical treatment

The graphical, or Lamé line, procedure introduced in §10.8 can be used for solution of compound cylinder problems. The vertical lines representing the boundaries of the cylinder walls may be drawn at their appropriate $1/r^2$ values, and the solution for condition (c) of Fig. 10.12 may be carried out as before, producing a single line across both cylinder sections (Fig. 10.14a).

The graphical representation of the effect of shrinkage does not produce a single line, however, and the effect on each cylinder must therefore be determined by projection of known lines on the radial side of the graph to the respective cylinder on the hoop stress side, i.e. conditions (a) and (b) of Fig. 10.12 must be treated separately as indeed they are in the analytical approach. The resulting graph will then appear as in Fig. 10.14b.

The total effect of combined shrinkage and internal pressure is then given, as before, by the algebraic combination of the separate effects, i.e. the graphs must be added together, taking due account of sign to produce the graph of Fig. 10.14c. In practice this is the only graph which need be constructed, all effects being considered on the single set of axes. Again, *all values should be calculated from proportions of the figure*, i.e. by the use of similar triangles.

10.11. Shrinkage or interference allowance

In the design of compound cylinders it is important to relate the difference in diameter of the mating cylinders to the stresses this will produce. This difference in diameter at the

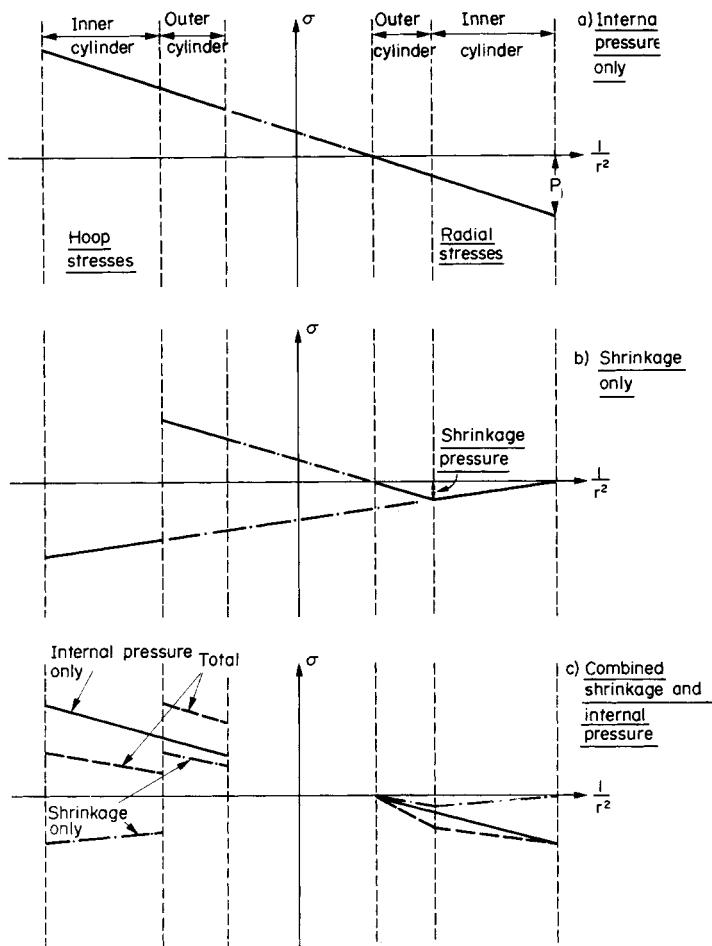


Fig. 10.14. Graphical (Lamé line) solution for compound cylinders.

"common" surface is normally termed the *shrinkage* or *interference allowance* whether the compound cylinder is formed by a shrinking or a force fit procedure respectively. Normally, however, the shrinking process is used, the outer cylinder being heated until it will freely slide over the inner cylinder thus exerting the required junction or shrinkage pressure on cooling.

Consider, therefore, the compound cylinder shown in Fig. 10.15, *the material of the two cylinders not necessarily being the same*.

Let the pressure set up at the junction of two cylinders owing to the force or shrink fit be p . Let the hoop stresses set up at the junction on the inner and outer tubes resulting from the pressure p be σ_{H_i} (compressive) and σ_{H_o} (tensile) respectively.

Then, if

$$\delta_o = \text{radial shift of outer cylinder}$$

and

$$\delta_i = \text{radial shift of inner cylinder (as shown in Fig. 10.15)}$$

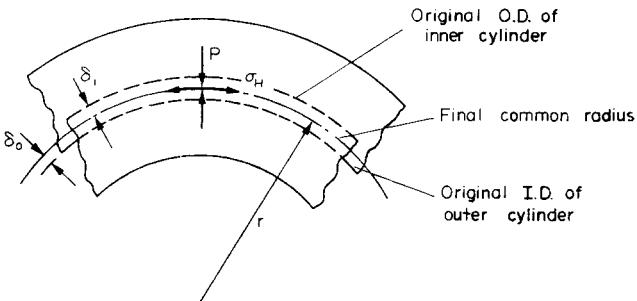


Fig. 10.15. Interference or shrinkage allowance for compound cylinders –
total interference = $\delta_o + \delta_i$.

since

$$\text{circumferential strain} = \text{diametral strain}$$

$$\text{circumferential strain at radius } r \text{ on outer cylinder} = \frac{2\delta_o}{2r} = \frac{\delta_o}{r} = \varepsilon_{H_o}$$

$$\text{circumferential strain at radius } r \text{ on inner cylinder} = \frac{2\delta_i}{2r} = \frac{\delta_i}{r} = -\varepsilon_{H_i}$$

(negative since it is a *decrease* in diameter).

$$\begin{aligned}\text{Total interference or shrinkage} &= \delta_o + \delta_i \\ &= r\varepsilon_{H_o} + r(-\varepsilon_{H_i}) \\ &= (\varepsilon_{H_o} - \varepsilon_{H_i})r\end{aligned}$$

Now assuming open ends, i.e. $\sigma_L = 0$,

$$\varepsilon_{H_o} = \frac{\sigma_{H_o}}{E_1} - \frac{v_1}{E_1}(-p) \quad \text{since } \sigma_{r_o} = -p$$

$$\text{and} \quad \varepsilon_{H_i} = \frac{\sigma_{H_i}}{E_2} - \frac{v_2}{E_2}(-p) \quad \text{since } \sigma_{r_i} = -p$$

where E_1 and v_1 , E_2 and v_2 are the elastic modulus and Poisson's ratio of the two tubes respectively.

Therefore total interference or shrinkage allowance (based on radius)

$$= \left[\frac{1}{E_1}(\sigma_{H_o} + v_1 p) - \frac{1}{E_2}(\sigma_{H_i} + v_2 p) \right] r \quad (10.13)$$

where r is the initial nominal radius of the mating surfaces.

N.B. σ_{H_i} , being compressive, will change the negative sign to a positive one when its value is substituted. Shrinkage allowances *based on diameter* will be twice this value, i.e. replacing radius r by diameter d .

Generally, however, the tubes are of the same material.

$$\therefore E_1 = E_2 = E \quad \text{and} \quad v_1 = v_2 = v$$

$$\therefore \text{Shrinkage allowance} = \frac{r}{E}(\sigma_{H_o} - \sigma_{H_i}) \quad (10.14)$$

The values of σ_{H_o} and σ_{H_i} may be determined graphically or analytically in terms of the shrinkage pressure p which can then be evaluated for any known shrinkage or interference allowance. Other stress values throughout the cylinder can then be determined as described previously.

10.12. Hub on solid shaft

The Lamé equations give

$$\sigma_H = A + \frac{B}{r^2} \quad \text{and} \quad \sigma_r = A - \frac{B}{r^2}$$

Since σ_H and σ_r cannot be infinite when $r = 0$, i.e. at the centre of the solid shaft, it follows that B must be zero since this is the only solution which can yield finite values for the stresses.

From the above equations, therefore, it follows that $\sigma_H = \sigma_r = A$ for all values of r .

Now at the outer surface of the shaft

$$\sigma_r = -p \quad \text{the shrinkage pressure.}$$

Therefore the hoop and radial stresses throughout a solid shaft are everywhere equal to the shrinkage or interference pressure and both are compressive. The maximum shear stress $= \frac{1}{2}(\sigma_1 - \sigma_2)$ is thus zero throughout the shaft.

10.13. Force fits

It has been stated that compound cylinders may be formed by *shrinking* or by *force-fit* techniques. In the latter case the interference allowance is small enough to allow the outer cylinder to be pressed over the inner cylinder with a large axial force.

If the interference pressure set up at the common surface is p , the normal force N between the mating cylinders is then

$$N = p \times 2\pi r L$$

where L is the axial length of the contact surfaces.

The friction force F between the cylinders which has to be overcome by the applied force is thus

$$F = \mu N$$

where μ is the coefficient of friction between the contact surfaces.

∴

$$\begin{aligned} F &= \mu(p \times 2\pi r L) \\ &= 2\pi\mu pr L \end{aligned} \tag{10.15}$$

With a knowledge of the magnitude of the applied force required the value of p may be determined.

Alternatively, for a known interference between the cylinders the procedure of § 10.11 may be carried out to determine the value of p which will be produced and hence the force F which will be required to carry out the press-fit operation.

10.14. Compound cylinder – different materials

The value of the shrinkage or interference allowance for compound cylinders constructed from cylinders of different materials is given by eqn. (10.13). The value of the shrinkage pressure set up owing to a known amount of interference can then be calculated as with the standard compound cylinder treatment, each component cylinder being considered separately subject to the shrinkage pressure.

Having constructed the compound cylinder, however, the treatment is different for the analysis of stresses owing to applied internal and/or external pressures. Previously the compound cylinder has been treated as a single thick cylinder and, e.g., a single Lamé line drawn across both cylinder walls for solution. In the case of cylinders of different materials, however, each component cylinder must be considered separately as with the shrinkage effects. Thus, for a known internal pressure P_1 which sets up a common junction pressure p , the Lamé line solution takes the form shown in Fig. 10.16.

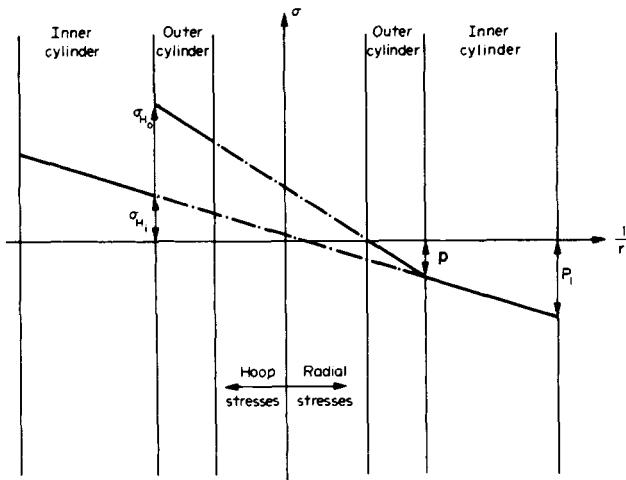


Fig. 10.16. Graphical solution for compound tubes of different materials.

For a full solution of problems of this type it is often necessary to make use of the equality of *diametral* strains at the common junction surface, i.e. to realise that for the cylinders to maintain contact with each other the diametral strains must be equal at the common surface.

Now

diametral strain = circumferential strain

$$= \frac{1}{E} [\sigma_H - v_o \sigma_r - v_i \sigma_L]$$

Therefore at the common surface, ignoring longitudinal strains and stresses,

$$\frac{1}{E} [\sigma_{H_o} - v_o \sigma_r] = \frac{1}{E_i} [\sigma_{H_i} - v_i \sigma_r] \quad (10.16)$$

where E_o and ν_o = Young's modulus and Poisson's ratio of outer cylinder,
 E_i and ν_i = Young's modulus and Poisson's ratio of inner cylinder,
 $\sigma_r = -p$ = radial stress at common surface,
and σ_{H_o} and σ_{H_i} = (as before) the hoop stresses at the common surface for the
outer and inner cylinders respectively.

10.15. Uniform heating of compound cylinders of different materials

When an initially unstressed compound cylinder constructed from two tubes of the same material is heated uniformly, all parts of the cylinder will expand at the same rate, this rate depending on the value of the coefficient of expansion for the cylinder material.

If the two tubes are of different materials, however, each will attempt to expand at different rates and *differential thermal stresses* will be set up as described in § 2.3. The method of treatment for such compound cylinders is therefore similar to that used for compound bars in the section noted.

Consider, therefore, two tubes of different material as shown in Fig. 10.17. Here it is convenient, for simplicity of treatment, to take as an example steel and brass for the two materials since the coefficients of expansion for these materials are known, the value for brass being greater than that for steel. Thus if the inner tube is of brass, as the temperature rises the brass will attempt to expand at a faster rate than the outer steel tube, the "free" expansions being indicated in Fig. 10.17a. In practice, however, when the tubes are joined as a compound cylinder, the steel will restrict the expansion of the brass and, conversely, the brass will force the steel to expand beyond its "free" expansion position. As a result a compromise situation is reached as shown in Fig. 10.17b, both tubes being effectively compressed radially (i.e. on their thickness) through the amounts shown. An effective increase p_i in "shrinkage" pressure is thus introduced.

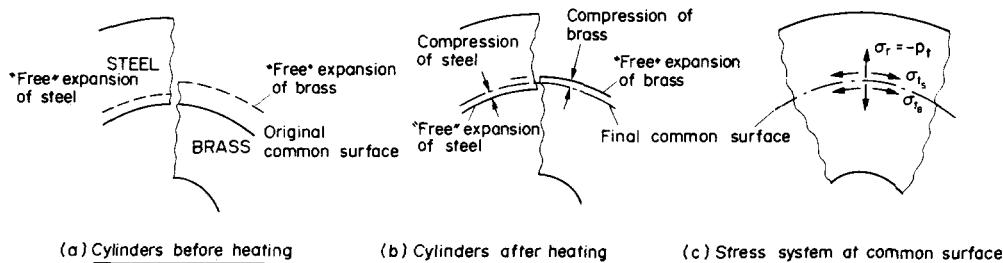


Fig. 10.17. Uniform heating of compound cylinders constructed from tubes of different materials - in this case, steel and brass.

p_i is the radial pressure introduced at the common interface by virtue of the differential thermal expansions.

Therefore, as for the compound bar treatment of § 2.3:

$$\text{compression of steel} + \text{compression of brass} = \text{difference in "free" lengths}$$

$$\begin{aligned} \varepsilon_s d + \varepsilon_B d &= (\alpha_B - \alpha_s) t d \\ &= (\alpha_B - \alpha_s)(T_2 - T_1)d \end{aligned} \quad (10.17)$$

where d is the initial nominal diameter of the mating surfaces, α_B and α_s are the coefficients of linear expansion for the brass and steel respectively, $t = T_2 - T_1$ is the temperature change, and ϵ_B and ϵ_s are the diametral strains in the two materials.

Alternatively, using a treatment similar to that used in the derivation of the thick cylinder "shrinkage fit" expressions

$$\Delta d = (\epsilon_{H_o} - \epsilon_{H_i})d = (\alpha_B - \alpha_s)td$$

ϵ_{H_i} being compressive, then producing an identical expression to that obtained above.

Now since diametral strain = circumferential or hoop strain

$$\epsilon_s = \frac{1}{E_s} [\sigma_{t_s} - v_s \sigma_r]$$

$$\epsilon_B = \frac{1}{E_B} [\sigma_{t_B} - v_B \sigma_r]$$

σ_{t_s} and σ_{t_B} being the hoop stresses set up at the common interface surfaces in the steel and brass respectively due to the differential thermal expansion and σ_r the effective increase in radial stress at the common junction surface caused by the same effect, i.e. $\sigma_r = -p_t$.

However, any radial pressure at the common interface will produce hoop tension in the outer cylinder but hoop compression in the inner cylinder. The expression for ϵ_B obtained above will thus always be negative when the appropriate stress values have been inserted. Since eqn. (10.17) deals with magnitudes of displacements only, it follows that a negative sign must be introduced to the value of ϵ_B before it can be substituted into eqn. (10.17).

Substituting for ϵ_s and ϵ_B in eqn. (10.17) with $\sigma_r = -p_t$

$$\frac{1}{E_s} [\sigma_{t_s} + v_s p_t] - \frac{1}{E_B} [\sigma_{t_B} + v_B p_t] = [a_B - a_s] [T_2 - T_1] \quad (10.18)$$

The values of σ_{t_s} and σ_{t_B} are found in terms of the radial stress p_t at the junction surfaces by calculation or by graphical means as shown in Fig. 10.18.

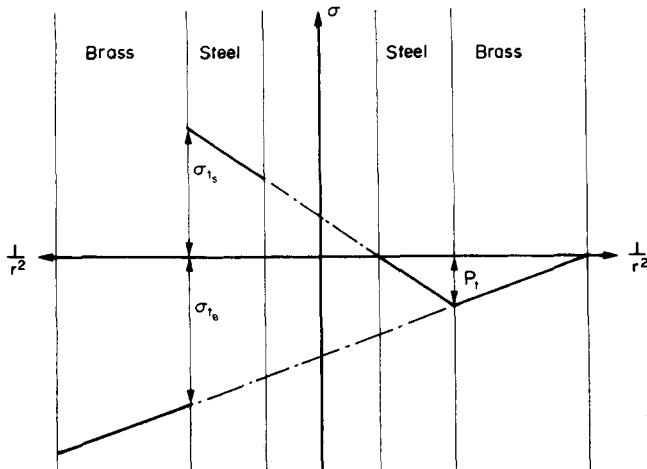


Fig. 10.18.

Substitution in eqn. (10.18) then yields the value of the “unknown” p_t and hence the other resulting stresses.

10.16. Failure theories – yield criteria

For thick cylinder design the Tresca (maximum shear stress) criterion is normally used for *ductile materials* (see Chapter 15), i.e. the maximum shear stress in the cylinder wall is equated to the maximum shear stress at yield in simple tension,

$$\tau_{\max} = \sigma_y / 2$$

Now the maximum shear stress is at the inside radius (§ 10.5) and is given by

$$\tau_{\max} = \frac{\sigma_H - \sigma_r}{2}$$

Therefore, for cylinder failure

$$\frac{\sigma_y}{2} = \frac{\sigma_H - \sigma_r}{2}$$

i.e.

$$\sigma_y = \sigma_H - \sigma_r$$

Here, σ_H and σ_r are the hoop and radial stresses at the inside radius and σ_y is the allowable yield stress of the material taking into account any safety factors which may be introduced by the company concerned.

For *brittle materials* such as cast iron the Rankine (maximum principal stress) theory is used. In this case failure is deemed to occur when

$$\sigma_y = \sigma_{H_{\max}}$$

10.17. Plastic yielding – “auto-fretage”

It has been shown that the most highly stressed part of a thick cylinder is at the inside radius. It follows, therefore, that if the internal pressure is increased sufficiently, yielding of the cylinder material will take place at this position. Fortunately the condition is not too serious at this stage since there remains a considerable bulk of elastic material surrounding the yielded area which contains the resulting strains within reasonable limits. As the pressure is increased further, however, plastic penetration takes place deeper and deeper into the cylinder wall and eventually the whole cylinder will yield.

If the pressure is such that plastic penetration occurs only partly into the cylinder wall, on release of that pressure the elastic outer zone attempts to return to its original dimensions but is prevented from doing so by the permanent deformation or “set” of the yielded material. The result is that the elastic material is held in a state of residual tension whilst the inside is brought into residual compression. This process is termed *auto-fretage* and it has the same effect as shrinking one tube over another without the necessary complications of the shrinking procedure, i.e. on subsequent loading cycles the cylinder is able to withstand a higher internal pressure since the compressive residual stress at the inside surface has to be overcome before this region begins to experience tensile stresses. For this reason gun barrels and other pressure vessels are often pre-stressed in this way prior to service.

A full theoretical treatment of the auto-frettage process is introduced in Chapter 18 together with associated plastic collapse theory.

10.18. Wire-wound thick cylinders

Consider a thick cylinder with inner and outer radii R_1 and R_2 respectively, wound with wire under tension until its external radius becomes R_3 . The resulting hoop and radial stresses developed in the cylinder will depend upon the way in which the tension T in the wire varies. The simplest case occurs when the tension in the wire is held constant throughout the winding process, and the solution for this condition will be introduced here. Solution for more complicated tension conditions will be found in more advanced texts and are not deemed appropriate for this volume. The method of solution, however, is similar.

(a) Stresses in the wire

Let the combined tube and wire be considered as a thick cylinder. The tension in the wire produces an "effective" external pressure on the tube and hence a compressive hoop stress.

Now for a thick cylinder subjected to an *external* pressure P the hoop and radial stresses are given by

$$\sigma_H = - \frac{PR_2^2}{(R_2^2 - R_1^2)} \left[\frac{r^2 + R_1^2}{r^2} \right]$$

and

$$\sigma_r = - \frac{PR_2^2}{(R_2^2 - R_1^2)} \left[\frac{r^2 - R_1^2}{r^2} \right]$$

i.e.

$$\sigma_H = \sigma_r \left[\frac{r^2 + R_1^2}{r^2 - R_1^2} \right]$$

If the initial tensile stress in the wire is T the final tensile hoop stress in the winding at any radius r is less than T by an amount equal to the compressive hoop stress set up by the effective "external" pressure caused by the winding,

i.e. final hoop stress in the winding at radius $r = T - \sigma_r \left[\frac{(r^2 + R_1^2)}{(r^2 - R_1^2)} \right]$ (10.19)

Using the same analysis outlined in § 10.2,

$$\sigma_H = \sigma_r + r \frac{d\sigma_r}{dr}$$

$$\therefore \sigma_r + r \frac{d\sigma_r}{dr} = T - \sigma_r \left[\frac{(r^2 + R_1^2)}{(r^2 - R_1^2)} \right]$$

$$\begin{aligned} \therefore r \frac{d\sigma_r}{dr} &= T - \sigma_r \left[\frac{r^2 + R_1^2 - r^2 + R_1^2}{(r^2 - R_1^2)} \right] \\ &= T - \left[\frac{2R_1^2 \sigma_r}{r^2 - R_1^2} \right] \end{aligned}$$

Multiplying through by $\frac{r}{(r^2 - R_1^2)}$ and rearranging,

$$\begin{aligned} \frac{r^2}{(r^2 - R_1^2)} \frac{d\sigma_r}{dr} + 2 \frac{R_1^2 r}{(r^2 - R_1^2)^2} \sigma_r &= \frac{Tr}{(r^2 - R_1^2)} \\ \therefore \frac{d}{dr} \left[\frac{r^2}{(r^2 - R_1^2)} \sigma_r \right] &= \frac{Tr}{(r^2 - R_1^2)} \\ \frac{r^2}{(r^2 - R_1^2)} \sigma_r &= \frac{T}{2} \log_e(r^2 - R_1^2) + A \end{aligned} \quad (10.20)$$

But $\sigma_r = 0$ when $r = R_3$,

$$\begin{aligned} \therefore 0 &= \frac{T}{2} \log_e(R_3^2 - R_1^2) + A \\ A &= -\frac{T}{2} \log_e(R_3^2 - R_1^2) \end{aligned}$$

Therefore substituting in eqn. (10.20),

$$\begin{aligned} \frac{r^2}{(r^2 - R_1^2)} \sigma_r &= \frac{T}{2} \log_e \frac{(r^2 - R_1^2)}{(R_3^2 - R_1^2)} \\ \therefore \sigma_r &= \frac{(r^2 - R_1^2)}{2r^2} T \log_e \frac{(r^2 - R_1^2)}{(R_3^2 - R_1^2)} \\ &= -\frac{(r^2 - R_1^2)}{2r^2} T \log_e \frac{(R_3^2 - R_1^2)}{(r^2 - R_1^2)} \end{aligned} \quad (10.21)$$

From eqn. (10.19),

$$\sigma_H = T - \sigma_r \left[\frac{r^2 + R_1^2}{r^2 - R_1^2} \right]$$

Therefore since the sign of σ_r has been taken into account in setting up eqn. (10.19)

$$\sigma_H = T \left[1 - \frac{(r^2 + R_1^2)}{2r^2} \log_e \frac{(R_3^2 - R_1^2)}{(r^2 - R_1^2)} \right] \quad (10.22)$$

Thus eqns. (10.21) and (10.22) give the stresses in the wire winding for all radii between R_2 and R_3 .

(b) Stresses in the tube

The stresses in the tube due to wire winding may be found from the normal thick cylinder expressions when it is considered subject to an external pressure P_2 at radius R_2 . The value of P_2 is that obtained from eqn. (10.21) with $r = R_2$.

If an additional internal pressure is applied to the wire-wound cylinder it may be treated as a single thick cylinder and the resulting stresses combined algebraically with those due to winding to obtain the resultant effect.

Examples

Example 10.1 (B)

A thick cylinder of 100 mm internal radius and 150 mm external radius is subjected to an internal pressure of 60 MN/m² and an external pressure of 30 MN/m². Determine the hoop and radial stresses at the inside and outside of the cylinder together with the longitudinal stress if the cylinder is assumed to have closed ends.

Solution (a): analytical

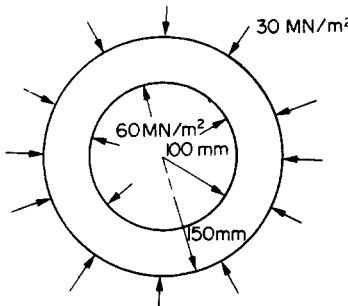


Fig. 10.19.

The internal and external pressures both have the effect of decreasing the thickness of the cylinder; the radial stresses at both the inside and outside radii are thus compressive, i.e. negative (Fig. 10.19).

$$\therefore \quad \text{at } r = 0.1 \text{ m}, \quad \sigma_r = -60 \text{ MN/m}^2$$

$$\text{and} \quad \text{at } r = 0.15 \text{ m}, \quad \sigma_r = -30 \text{ MN/m}^2$$

Therefore, from eqn. (10.3), with stress units of MN/m²,

$$-60 = A - 100B \quad (1)$$

$$\text{and} \quad -30 = A - 44.5B \quad (2)$$

Subtracting (2) from (1),

$$-30 = -55.5B$$

$$B = 0.54$$

$$\text{Therefore, from (1),} \quad A = -60 + 100 \times 0.54$$

$$A = -6$$

Therefore, at $r = 0.1$ m, from eqn. (10.4),

$$\begin{aligned} \sigma_H &= A + \frac{B}{r^2} = -6 + 0.54 \times 100 \\ &= 48 \text{ MN/m}^2 \end{aligned}$$

and at $r = 0.15 \text{ m}$,

$$\begin{aligned}\sigma_H &= -6 + 0.54 \times 44.5 = -6 + 24 \\ &= 18 \text{ MN/m}^2\end{aligned}$$

From eqn. (10.7) the longitudinal stress is given by

$$\begin{aligned}\sigma_L &= \frac{P_1 R_1^2 - P_2 R_2^2}{(R_2^2 - R_1^2)} = \frac{(60 \times 0.1^2 - 30 \times 0.15^2)}{(0.15^2 - 0.1^2)} \\ &= \frac{10^2(60 - 30 \times 2.25)}{1.25 \times 10^2} = -6 \text{ MN/m}^2 \quad \text{i.e. compressive}\end{aligned}$$

Solution (b): graphical

The graphical solution is shown in Fig. 10.20, where the boundaries of the cylinder are given by

$$\frac{1}{r^2} = 100 \text{ for the inner radius where } r = 0.1 \text{ m}$$

$$\frac{1}{r^2} = 44.5 \text{ for the outer radius where } r = 0.15 \text{ m}$$

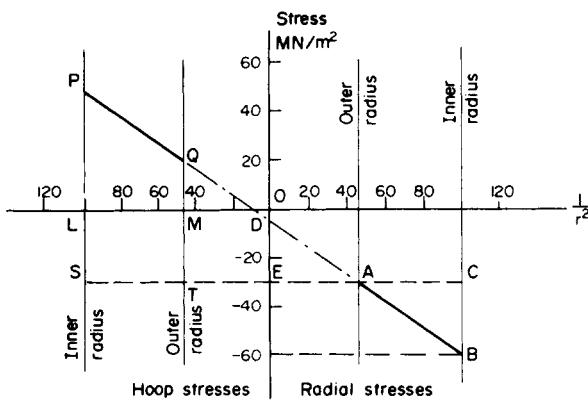


Fig. 10.20.

The two conditions which enable the Lamé line to be drawn are the same as those used above for the analytical solution,

i.e. $\sigma_r = -60 \text{ MN/m}^2 \text{ at } r = 0.1 \text{ m}$

$\sigma_r = -30 \text{ MN/m}^2 \text{ at } r = 0.15 \text{ m}$

The hoop stresses at these radii are then given by points P and Q on the graph. For complete accuracy these values should be calculated by proportions of the graph thus:

by similar triangles PAS and BAC

$$\frac{PS}{100 + 44.5} = \frac{CB}{100 - 44.5}$$

$$\therefore \frac{PL + LS}{144.5} = \frac{30}{55.5}$$

i.e. hoop stress at radius $r = 0.1$ m

$$= PL = \frac{30 \times 144.5}{55.5} - LS$$

$$= 78 - 30 = 48 \text{ MN/m}^2$$

Similarly, the hoop stress at radius $r = 0.15$ m is QM and given by the similar triangles QAT and BAC ,

i.e. $\frac{QM + MT}{44.5 + 44.5} = \frac{30}{55.5}$

$$QM = \frac{30 \times 89}{55.5} - 30$$

$$= 48 - 30 = 18 \text{ MN/m}^2$$

The longitudinal stress σ_L = the intercept on the σ axis (which is negative)

$$= DO = OE - DE = 30 - DE$$

Now $\frac{DE}{44.5} = \frac{30}{55.5}$

$$\therefore DE = 24$$

$$\therefore \sigma_L = 30 - 24 = 6 \text{ MN/m}^2 \text{ compressive}$$

Example 10.2 (B)

An external pressure of 10 MN/m^2 is applied to a thick cylinder of internal diameter 160 mm and external diameter 320 mm. If the maximum hoop stress permitted on the inside wall of the cylinder is limited to 30 MN/m^2 , what maximum internal pressure can be applied assuming the cylinder has closed ends? What will be the change in outside diameter when this pressure is applied? $E = 207 \text{ GN/m}^2$, $v = 0.29$.

Solution (a): analytical

The conditions for the cylinder are:

When $r = 0.08 \text{ m}$, $\sigma_r = -p$ and $\frac{1}{r^2} = 156$

when $r = 0.16 \text{ m}$, $\sigma_r = -10 \text{ MN/m}^2$ and $\frac{1}{r^2} = 39$

and when $r = 0.08 \text{ m}$, $\sigma_H = 30 \text{ MN/m}^2$

since the maximum hoop stress occurs at the inside surface of the cylinder.

Using the latter two conditions in eqns. (10.3) and (10.4) with units of MN/m^2 ,

$$-10 = A - 39B \quad (1)$$

$$30 = A + 156B \quad (2)$$

Subtracting (1) from (2),

$$40 = 195B \quad \therefore B = 0.205$$

Substituting in (1),

$$A = -10 + (39 \times 0.205)$$

$$= -10 + 8 \quad \therefore A = -2$$

Therefore, at $r = 0.08$, from eqn. (10.3),

$$\begin{aligned} \sigma_r &= -p = A - 156B \\ &= -2 - 156 \times 0.205 \\ &= -2 - 32 = -34 \text{ MN/m}^2 \end{aligned}$$

i.e. the allowable internal pressure is 34 MN/m^2 .

From eqn. (10.9) the change in diameter is given by

$$\Delta D = \frac{2r_0}{E} (\sigma_H - v\sigma_r - v\sigma_L)$$

Now at the outside surface

$$\begin{aligned} \sigma_r &= -10 \text{ MN/m}^2 \quad \text{and} \quad \sigma_H = A + \frac{B}{r^2} \\ &= -2 + (39 \times 0.205) \\ &= -2 + 8 = 6 \text{ MN/m}^2 \end{aligned}$$

$$\sigma_L = \frac{P_1 R_1^2 - P_2 R_2^2}{(R_2^2 - R_1^2)} = \frac{(34 \times 0.08^2 - 10 \times 0.16^2)}{(0.16^2 - 0.08^2)}$$

$$= \frac{(34 \times 0.64 - 10 \times 2.56)}{(2.56 - 0.64)} = \frac{21.8 - 25.6}{1.92}$$

$$= -\frac{3.8}{1.92} = 1.98 \text{ MN/m}^2 \text{ compressive}$$

$$\therefore \Delta D = \frac{0.32}{207 \times 10^9} [6 - 0.29(-10) - 0.29(-1.98)] 10^6$$

$$= \frac{0.32}{207 \times 10^3} (6 + 2.9 + 0.575)$$

$$= 14.7 \mu\text{m}$$

Solution (b): graphical

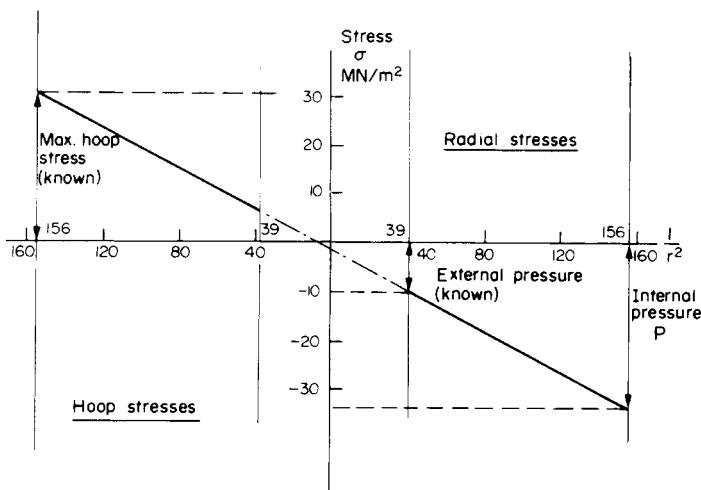


Fig. 10.21.

The graphical solution is shown in Fig. 10.21. The boundaries of the cylinder are as follows:

$$\text{for } r = 0.08 \text{ m}, \quad \frac{1}{r^2} = 156$$

$$\text{and for } r = 0.16 \text{ m}, \quad \frac{1}{r^2} = 39$$

The two fixed points on the graph which enable the line to be drawn are, therefore,

$$\sigma_r = -10 \text{ MN/m}^2 \text{ at } r = 0.16 \text{ m} \quad \text{and} \quad \sigma_H = 30 \text{ MN/m}^2 \text{ at } r = 0.08 \text{ m}$$

The allowable internal pressure is then given by the value of σ_r at $r = 0.08 \text{ m}$ ($\frac{1}{r^2} = 156$), i.e. 34 MN/m^2 .

Similarly, the hoop stress at the outside surface is given by the value of σ_H at $\frac{1}{r^2} = 39$, i.e. 6 MN/m^2 , and the longitudinal stress by the intercept on the σ axis, i.e. 2 MN/m^2 compressive.

N.B.—In practice all these values should be calculated by proportions.

Example 10.3 (B)

- (a) In an experiment on a thick cylinder of 100 mm external diameter and 50 mm internal diameter the hoop and longitudinal strains as measured by strain gauges applied to the outer

surface of the cylinder were 240×10^{-6} and 60×10^{-6} , respectively, for an internal pressure of 90 MN/m^2 , the external pressure being zero.

Determine the actual hoop and longitudinal stresses present in the cylinder if $E = 208 \text{ GN/m}^2$ and $\nu = 0.29$. Compare the hoop stress value so obtained with the theoretical value given by the Lame equations.

(b) Assuming that the above strain readings were obtained for a thick cylinder of 100 mm external diameter but unknown internal diameter calculate this internal diameter.

Solution

(a)

$$\varepsilon_H = \frac{1}{E} (\sigma_H - \nu \sigma_L) \quad \text{and} \quad \varepsilon_L = \frac{1}{E} (\sigma_L - \nu \sigma_H)$$

since $\sigma_r = 0$ at the outside surface of the cylinder for zero external pressure.

$$\therefore 240 \times 10^{-6} \times 208 \times 10^9 = \sigma_H - 0.29 \sigma_L = 50 \times 10^6 \quad (1)$$

$$60 \times 10^{-6} \times 208 \times 10^9 = \sigma_L - 0.29 \sigma_H = 12.5 \times 10^6 \quad (2)$$

$$(1) \times 0.29 \quad 0.29 \sigma_H - 0.084 \sigma_L = 14.5 \times 10^6 \quad (3)$$

$$(2) \quad \sigma_L - 0.29 \sigma_H = 12.5 \times 10^6$$

$$(3) + (2) \quad 0.916 \sigma_L = 27 \times 10^6$$

$$\sigma_L = 29.5 \text{ MN/m}^2$$

$$\text{Substituting in (2)} \quad 0.29 \sigma_H = 29.5 - 12.5 = 17 \times 10^6$$

$$\sigma_H = 58.7 \text{ MN/m}^2$$

The theoretical values of σ_H for an internal pressure of 90 MN/m^2 may be obtained from Fig. 10.22, the boundaries of the cylinder being given by $r = 0.05$ and $r = 0.025$,

$$\text{i.e. } \frac{1}{r^2} = 400 \text{ and } 1600 \text{ respectively}$$

$$\text{i.e. } \sigma_H = 60 \text{ MN/m}^2 \text{ theoretically}$$

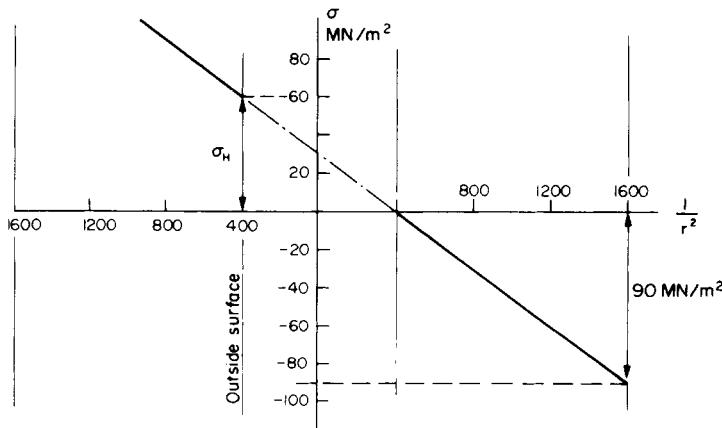


Fig. 10.22.

(b) From part (a) $\sigma_H = 58.7 \text{ MN/m}^2$ at $r = 0.05$
 and $\sigma_r = 0$ at $r = 0.05$
 $\therefore 58.7 = A + 400B$
 $0 = A - 400B$

Adding: $58.7 = 2A \quad \therefore A = 29.35$

and since $A = 400B \quad \therefore B = 0.0734$

Therefore for the internal radius R_1 where $\sigma_r = 90 \text{ MN/m}^2$

$$-90 = 29.35 - \frac{0.0734}{R_1^2}$$

$$R_1^2 = \frac{0.0734}{119.35} = 0.000615$$

$$= 6.15 \times 10^{-4}$$

$$\therefore R_1 = 2.48 \times 10^{-2} \text{ m} = 24.8 \text{ mm}$$

Internal diameter = 49.6 mm

For a graphical solution of part (b), see Fig. 10.23, where the known points which enable the Lame line to be drawn are, as above:

$$\sigma_H = 58.7 \text{ at } \frac{1}{r^2} = 400 \text{ and } \sigma_r = 0 \text{ at } \frac{1}{r^2} = 400$$

It is thus possible to determine the value of $1/R_1^2$ which will produce $\sigma_r = -90 \text{ MN/m}^2$.

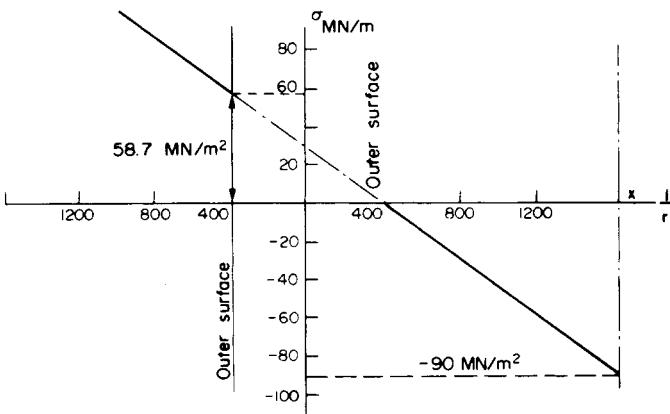


Fig. 10.23.

Let the required value of $\frac{1}{R_1^2} = x$

then by proportions $\frac{90}{x - 400} = \frac{58.7}{800}$

$$x - 400 = \frac{90 \times 800}{58.7} = 1225$$

$$\therefore x = 1625$$

$$\therefore R_1 = 24.8 \text{ mm}$$

i.e. required internal diameter = **49.6 mm**

Example 10.4 (B)

A compound cylinder is formed by shrinking a tube of 250 mm internal diameter and 25 mm wall thickness onto another tube of 250 mm external diameter and 25 mm wall thickness, both tubes being made of the same material. The stress set up at the junction owing to shrinkage is 10 MN/m^2 . The compound tube is then subjected to an internal pressure of 80 MN/m^2 . Compare the hoop stress distribution now obtained with that of a single cylinder of 300 mm external diameter and 50 mm thickness subjected to the same internal pressure.

Solution (a): analytical

A solution is obtained as described in §10.9, i.e. by considering the effects of shrinkage and internal pressure separately and combining the results algebraically.

Shrinkage only – outer tube

At $r = 0.15$, $\sigma_r = 0$ and at $r = 0.125$, $\sigma_r = -10 \text{ MN/m}^2$

$$\therefore 0 = A - \frac{B}{0.15^2} = A - 44.5B \quad (1)$$

$$-10 = A - \frac{B}{0.125^2} = A - 64B \quad (2)$$

$$\text{Subtracting (1) – (2), } 10 = 19.5B \quad \therefore B = 0.514$$

$$\text{Substituting in (1), } A = 44.5B \quad \therefore A = 22.85$$

$$\therefore \text{hoop stress at } 0.15 \text{ m radius} = A + 44.5B = 45.7 \text{ MN/m}^2$$

$$\text{hoop stress at } 0.125 \text{ m radius} = A + 64B = 55.75 \text{ MN/m}^2$$

Shrinkage only – inner tube

At $r = 0.10$, $\sigma_r = 0$ and at $r = 0.125$, $\sigma_r = -10 \text{ MN/m}^2$

$$\therefore 0 = A - \frac{B}{0.1^2} = A - 100B \quad (3)$$

$$-10 = A - \frac{B}{0.125^2} = A - 64B \quad (4)$$

$$\text{Subtracting (3) – (4), } 10 = -36B \quad \therefore B = -0.278$$

$$\text{Substituting in (3), } A = 100B \quad \therefore A = -27.8$$

∴ hoop stress at 0.125 m radius = $A + 64B = -45.6 \text{ MN/m}^2$

hoop stress at 0.10 m radius = $A + 100B = -55.6 \text{ MN/m}^2$

Considering internal pressure only (on complete cylinder)

At $r = 0.15$, $\sigma_r = 0$ and at $r = 0.10$, $\sigma_r = -80$

$$0 = A - 44.5B \quad (5)$$

$$-80 = A - 100B \quad (6)$$

Subtracting (5) – (6), $80 = 55.5B \quad \therefore B = 1.44$

From (5), $A = 44.5B \quad \therefore A = 64.2$

∴ At $r = 0.15 \text{ m}$, $\sigma_H = A + 44.5B = 128.4 \text{ MN/m}^2$

$r = 0.125 \text{ m}$, $\sigma_H = A + 64B = 156.4 \text{ MN/m}^2$

$r = 0.1 \text{ m}$, $\sigma_H = A + 100B = 208.2 \text{ MN/m}^2$

The resultant stresses for combined shrinkage and internal pressure are then:

outer tube: $r = 0.15 \quad \sigma_H = 128.4 + 45.7 = 174.1 \text{ MN/m}^2$

$r = 0.125 \quad \sigma_H = 156.4 + 55.75 = 212.15 \text{ MN/m}^2$

inner tube: $r = 0.125 \quad \sigma_H = 156.4 - 45.6 = 110.8 \text{ MN/m}^2$

$r = 0.1 \quad \sigma_H = 208.2 - 55.6 = 152.6 \text{ MN/m}^2$

Solution (b): graphical

The graphical solution is obtained in the same way by considering the separate effects of shrinkage and internal pressure as shown in Fig. 10.24.

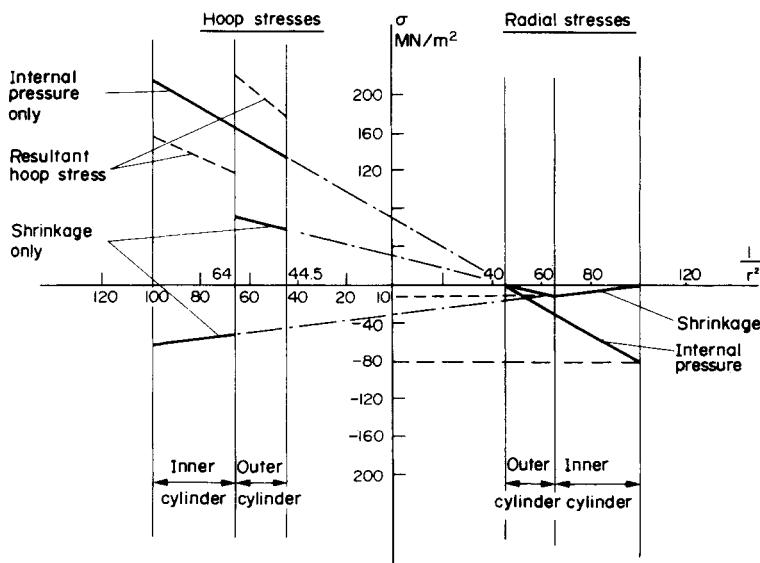


Fig. 10.24.

The final results are illustrated in Fig. 10.25 (values from the graph again being determined by proportion of the figure).

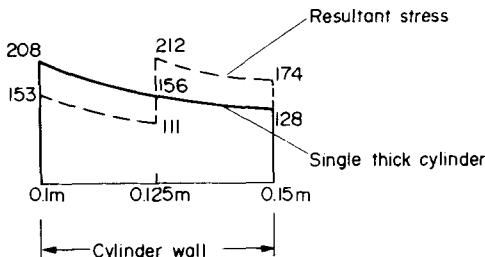


Fig. 10.25.

Example 10.5 (B)

A compound tube is made by shrinking one tube of 100 mm internal diameter and 25 mm wall thickness on to another tube of 100 mm external diameter and 25 mm wall thickness. The shrinkage allowance, *based on radius*, is 0.01 mm. If both tubes are of steel (with $E = 208 \text{ GN/m}^2$), calculate the radial pressure set up at the junction owing to shrinkage.

Solution

Let p be the required shrinkage pressure, then for the inner tube:

$$\text{At } r = 0.025, \sigma_r = 0 \quad \text{and} \quad \text{at } r = 0.05, \sigma_r = -p$$

$$0 = A - \frac{B}{0.025^2} = A - 1600B \quad (1)$$

$$-p = A - \frac{B}{0.05^2} = A - 400B \quad (2)$$

Subtracting (2) – (1),

$$-p = 1200B \quad \therefore B = -p/1200$$

$$\text{From (1),} \quad A = 1600B \quad \therefore A = -\frac{1600p}{1200} = -\frac{4p}{3}$$

Therefore at the common radius the hoop stress is given by eqn. (10.4),

$$\begin{aligned} \sigma_{H_i} &= A + B/0.05^2 \\ &= -\frac{4p}{3} + 400\left(-\frac{p}{1200}\right) = -\frac{5p}{3} = -1.67p \end{aligned}$$

For the outer tube:

$$\text{at } r = 0.05, \sigma_r = -p \quad \text{and} \quad \text{at } r = 0.075, \sigma_r = 0$$

$$-p = A - B/0.05^2 = A - 400B \quad (3)$$

$$0 = A - B/0.075^2 = A - 178B \quad (4)$$

Subtracting (4) – (3), $p = 222B \quad \therefore B = p/222$

From (4) $A = 178B \quad \therefore A = \frac{178p}{222}$

Therefore at the common radius the hoop stress is given by

$$\begin{aligned} \sigma_{H_o} &= A + \frac{B}{0.05^2} \\ &= \frac{178p}{222} + \frac{p}{222} \times 400 = \frac{578p}{222} = 2.6p \end{aligned}$$

Now from eqn. (10.14) the shrinkage allowance is

$$\begin{aligned} \frac{r}{E} [\sigma_{H_o} - \sigma_{H_i}] \\ \therefore 0.01 \times 10^{-3} = \frac{50 \times 10^{-3}}{208 \times 10^9} [2.6p - (-1.67p)] 10^6 \end{aligned}$$

where p has units of MN/m^2

$$\begin{aligned} \therefore 4.27p &= \frac{0.01 \times 208 \times 10^3}{50} = 41.6 \\ \therefore p &= 9.74 \text{ MN/m}^2 \end{aligned}$$

Hoop and radial stresses in the compound cylinder owing to shrinkage and/or internal pressure can now be determined if desired using the procedure of the previous example.

Once again a graphical solution could have been employed to obtain the values of σ_{H_o} and σ_{H_i} in terms of the unknown pressure p which is set off to some convenient distance on the $r = 0.05$, i.e. $1/r^2 = 400$, line.

Example 10.6 (B)

Two steel rings of radial thickness 30 mm, common radius 70 mm and length 40 mm are shrunk together to form a compound ring. It is found that the axial force required to separate the rings, i.e. to push the inside ring out, is 150 kN. Determine the shrinkage pressure at the mating surfaces and the shrinkage allowance. $E = 208 \text{ GN/m}^2$. The coefficient of friction between the junction surfaces of the two rings is 0.15.

Solution

Let the pressure set up between the rings be $p \text{ MN/m}^2$.

Then, normal force between rings = $p \times 2\pi r L = N$

$$\begin{aligned} &= p \times 10^6 \times 2\pi \times 70 \times 10^{-3} \times 40 \times 10^{-3} \\ &= 5600\pi p \text{ newtons.} \end{aligned}$$

\therefore friction force between rings = $\mu N = 0.15 \times 5600\pi p$

$$0.15 \times 5600\pi p = 150 \times 10^3$$

$$\begin{aligned} p &= \frac{150 \times 10^3}{0.15 \times 5600\pi} \\ &= 57 \text{ MN/m}^2 \end{aligned}$$

Now, for the inner tube:

$$\sigma_r = -57 \text{ at } r = 0.07 \quad \text{and} \quad \sigma_r = 0 \text{ at } r = 0.04$$

$$\therefore -57 = A - B/0.07^2 = A - 204B \quad (1)$$

$$0 = A - B/0.04^2 = A - 625B \quad (2)$$

$$\text{Subtracting (2) - (1),} \quad 57 = -421B \quad \therefore B = -0.135$$

$$\text{From (2),} \quad A = 625B \quad \therefore A = -84.5$$

Therefore at the common radius the hoop stress in the inner tube is given by

$$\sigma_{H_i} = A + \frac{B}{0.07^2} = A + 204B = -112.1 \text{ MN/m}^2$$

For the outer tube:

$$\sigma_r = -57 \text{ at } r = 0.07 \quad \text{and} \quad \sigma_r = 0 \text{ at } r = 0.1$$

$$\therefore -57 = A - 204B \quad (3)$$

$$0 = A - 100B \quad (4)$$

$$\text{Subtracting (4) - (3),} \quad 57 = 104B \quad \therefore B = 0.548$$

$$\text{From (4),} \quad A = 100B \quad \therefore A = 54.8$$

Therefore at the common radius the hoop stress in the outer tube is given by

$$\sigma_{H_o} = A + \frac{B}{0.07^2} = A + 204B = 166.8 \text{ MN/m}^2$$

$$\begin{aligned} \text{shrinkage allowance} &= \frac{r}{E} (\sigma_{H_o} - \sigma_{H_i}) \\ &= \frac{70 \times 10^{-3}}{208 \times 10^9} [166.8 - (-112.1)] 10^6 \\ &= \frac{70 \times 278.9}{208} \times 10^{-6} \text{ m} \\ &= 93.8 \times 10^{-6} = 0.094 \text{ mm} \end{aligned}$$

Example 10.7 (B)

- (a) A steel sleeve of 150 mm outside diameter is to be shrunk on to a solid steel shaft of 100 mm diameter. If the shrinkage pressure set up is 15 MN/m², find the initial difference between the inside diameter of the sleeve and the outside diameter of the shaft.

- (b) What percentage error would be involved if the shaft were assumed to be incompressible? For steel, $E = 208 \text{ GN/m}^2$; $\nu = 0.3$.

Solution

- (a) Treating the sleeve as a thick cylinder with internal pressure 15 MN/m^2 , at $r = 0.05$, $\sigma_r = -15 \text{ MN/m}^2$ and at $r = 0.075$, $\sigma_r = 0$

$$-15 = A - B/0.05^2 = A - 400B \quad (1)$$

$$0 = A - B/0.075^2 = A - 178B \quad (2)$$

Subtracting (2) – (1), $15 = 222B \quad \therefore B = 0.0676$

From (2), $A = 178B \quad \therefore A = 12.05$

Therefore the hoop stress in the sleeve at $r = 0.05 \text{ m}$ is given by

$$\begin{aligned}\sigma_{H_o} &= A + B/0.05^2 \\ &= A + 400B = 39 \text{ MN/m}^2\end{aligned}$$

The shaft will be subjected to a hoop stress which will be compressive and equal in value to the shrinkage pressure (see §10.12),

i.e. $\sigma_{H_i} = -15 \text{ MN/m}^2$

Thus the difference in radii or shrinkage allowance

$$\begin{aligned}&= \frac{r}{E} (\sigma_{H_o} - \sigma_{H_i}) = \frac{50 \times 10^{-3}}{208 \times 10^9} [39 + 15] 10^6 \\ &= \frac{50 \times 54}{208} \times 10^{-6} = 13 \times 10^{-6} \text{ m} \\ \therefore &\text{difference in diameters} = 0.026 \text{ mm}\end{aligned}$$

- (b) If the shaft is assumed incompressible the difference in diameters will equal the necessary change in diameter of the sleeve to fit the shaft. This can be found from the diametral strain, i.e. from eqn. (10.9)

$$\Delta D = \frac{2r}{E} (\sigma_H - \nu \sigma_r) \quad \text{assuming } \sigma_L = 0$$

$$\therefore \text{change of diameter} = \frac{100 \times 10^{-3}}{208 \times 10^9} [39 - 0.3(-15)] 10^6$$

$$= \frac{43.5}{208} \times 10^{-4} = 20.9 \times 10^{-6}$$

$$= 0.0209 \text{ mm}$$

$$\therefore \text{percentage error} = \frac{(0.026 - 0.0209)}{0.026} \times 100 = 19.6\%$$

Example 10.8 (C)

A thick cylinder of 100 mm external diameter and 50 mm internal diameter is wound with steel wire of 1 mm diameter, initially stressed to 20 MN/m² until the outside diameter is 120 mm. Determine the maximum hoop stress set up in the cylinder if an internal pressure of 30 MN/m² is now applied.

Solution

To find the stresses resulting from internal pressure only the cylinder and wire may be treated as a single thick cylinder of 50 mm internal diameter and 120 mm external diameter.

$$\text{Now } \sigma_r = -30 \text{ MN/m}^2 \text{ at } r = 0.025 \text{ and } \sigma_r = 0 \text{ at } r = 0.06$$

$$\therefore -30 = A - B/0.025^2 = A - 1600B \quad (1)$$

$$0 = A - B/0.06^2 = A - 278B \quad (2)$$

$$\text{Subtracting (2) - (1), } 30 = 1322B \quad \therefore B = 0.0227$$

$$\text{From (2), } A = 278B \quad \therefore A = 6.32$$

$$\therefore \text{hoop stress at 25 mm radius} = A + 1600B = 42.7 \text{ MN/m}^2$$

$$\text{hoop stress at 50 mm radius} = A + 400B = 15.4 \text{ MN/m}^2$$

The external pressure acting on the cylinder owing to wire winding is found from eqn. (10.21),

$$\text{i.e. } \sigma_r = -\frac{(r^2 - R_1^2)}{2r^2} T \log_e \frac{(R_3^2 - R_1^2)}{(r^2 - R_1^2)} = p$$

$$\therefore \text{where } r = R_2 = 0.05 \text{ m}, R_1 = 0.025 \text{ and } R_3 = 0.06$$

$$p = -\frac{(0.05^2 - 0.025^2)}{2 \times 0.05^2} T \log_e \frac{(0.06^2 - 0.025^2)}{(0.05^2 - 0.025^2)}$$

$$= -\frac{(25 - 6.25)}{50} 20 \log_e \frac{(36 - 6.25)}{(25 - 6.25)} \text{ MN/m}^2$$

$$= -\frac{18.75 \times 20}{50} \log_e \frac{29.75}{18.75}$$

$$= -7.5 \log_e 1.585 = -7.5 \times 0.4606$$

$$= -3.45 \text{ MN/m}^2$$

Therefore for wire winding only the stresses in the tube are found from the conditions

$$\sigma_r = -3.45 \text{ at } r = 0.05 \text{ and } \sigma_r = 0 \text{ at } r = 0.025$$

$$\therefore -3.45 = A - 400B$$

$$0 = A - 1600B$$

Subtracting,

$$-3.45 = 1200B \quad -B = -2.88 \times 10^{-3}$$

$$\therefore A = -4.6$$

$$\therefore \text{hoop stress at } 25 \text{ mm radius} = A + 1600B = -9.2 \text{ MN/m}^2$$

$$\text{hoop stress at } 50 \text{ mm radius} = A + 400B = -5.75 \text{ MN/m}^2$$

The resultant stresses owing to winding and internal pressure are, therefore:

$$\text{At } r = 25 \text{ mm}, \quad \sigma_H = 42.7 - 9.2 = 33.5 \text{ MN/m}^2$$

$$\text{At } r = 50 \text{ mm}, \quad \sigma_H = 15.4 - 5.75 = 9.65 \text{ MN/m}^2$$

Thus the maximum hoop stress is **33.5 MN/m²**

Example 10.9 (C)

A thick cylinder of internal and external radii 300 mm and 500 mm respectively is subjected to a gradually increasing internal pressure P . Determine the value of P when:

- (a) the material of the cylinder first commences to yield;
- (b) yielding has progressed to mid-depth of the cylinder wall;
- (c) the cylinder material suffers complete "collapse".

Take $\sigma_y = 600 \text{ MN/m}^2$.

Solution See Chapter 3 from *Mechanics of Materials 2*[†]

From eqn. (3.35) the initial yield pressure

$$\begin{aligned} &= \frac{\sigma_y}{2R_2^2} [R_2^2 - R_1^2] = \frac{600}{2 \times 0.5^2} [0.5^2 - 0.3^2] \\ &= \frac{600}{2 \times 25} [25 - 9] = 192 \text{ MN/m}^2 \end{aligned}$$

The pressure required to cause yielding to a depth $R_p = 40 \text{ mm}$ is given by eqn. (3.36)

$$\begin{aligned} \sigma_r &= \sigma_y \left[\log_e \frac{R_1}{R_p} - \frac{1}{2R_2^2} (R_2^2 - R_p^2) \right] \\ &= 600 \left[\log_e \frac{0.3}{0.4} - \frac{1}{2 \times 0.5^2} (0.5^2 - 0.4^2) \right] \\ &= -600 \left[\log_e 1.33 + \left(\frac{25 - 16}{50} \right) \right] \\ &= -600(0.2852 + 0.18) \\ &= -600 \times 0.4652 = -280 \text{ MN/m}^2 \end{aligned}$$

[†]E. J. Hearn, *Mechanics of Materials 2*, 3rd edition (Butterworth-Heinemann, Oxford, 1997).

i.e. the required internal pressure = **280 MN/m²**

For complete collapse from eqn. (3.34),

$$\begin{aligned} p &= -\sigma_y \log_e \frac{R_1}{R_2} = \sigma_y \log_e \frac{R_2}{R_1} \\ &= \sigma_y \log_e \frac{0.5}{0.3} \\ &= 600 \times \log_e 1.67 \\ &= 600 \times 0.513 = \mathbf{308 \text{ MN/m}^2} \end{aligned}$$

Problems

10.1 (B). A thick cylinder of 150 mm inside diameter and 200 mm outside diameter is subjected to an internal pressure of 15 MN/m². Determine the value of the maximum hoop stress set up in the cylinder walls.

[53.4 MN/m².]

10.2 (B). A cylinder of 100 mm internal radius and 125 mm external radius is subjected to an external pressure of 14 bar (1.4 MN/m²). What will be the maximum stress set up in the cylinder? [−7.8 MN/m².]

10.3 (B). The cylinder of Problem 10.2 is now subjected to an additional internal pressure of 200 bar (20 MN/m²). What will be the value of the maximum stress? [84.7 MN/m².]

10.4 (B). A steel thick cylinder of external diameter 150 mm has two strain gauges fixed externally, one along the longitudinal axis and the other at right angles to read the hoop strain. The cylinder is subjected to an internal pressure of 75 MN/m² and this causes the following strains:

- (a) hoop gauge: 455×10^{-6} tensile;
- (b) longitudinal gauge: 124×10^{-6} tensile.

Find the internal diameter of the cylinder assuming that Young's modulus for steel is 208 GN/m² and Poisson's ratio is 0.283. [B.P.] [96.7 mm.]

10.5 (B) A compound tube of 300 mm external and 100 mm internal diameter is formed by shrinking one cylinder on to another, the common diameter being 200 mm. If the maximum hoop tensile stress induced in the outer cylinder is 90 MN/m² find the hoop stresses at the inner and outer diameters of both cylinders and show by means of a sketch how these stresses vary with the radius. [90, 55.35; −92.4, 57.8 MN/m².]

10.6 (B). A compound thick cylinder has a bore of 100 mm diameter, a common diameter of 200 mm and an outside diameter of 300 mm. The outer tube is shrunk on to the inner tube, and the radial stress at the common surface owing to shrinkage is 30 MN/m².

Find the maximum internal pressure the cylinder can receive if the maximum circumferential stress in the outer tube is limited to 110 MN/m². Determine also the resulting circumferential stress at the outer radius of the inner tube. [B.P.] [79, −18 MN/m².]

10.7 (B). Working from first principles find the interference fit per metre of diameter if the radial pressure owing to this at the common surface of a compound tube is 90 MN/m², the inner and outer diameters of the tube being 100 mm and 250 mm respectively and the common diameter being 200 mm. The two tubes are made of the same material, for which $E = 200 \text{ GN/m}^2$. If the outside diameter of the inner tube is originally 200 mm, what will be the original inside diameter of the outer tube for the above conditions to apply when compound? [199.44 mm.]

10.8 (B). A compound cylinder is formed by shrinking a tube of 200 mm outside and 150 mm inside diameter on to one of 150 mm outside and 100 mm inside diameter. Owing to shrinkage the radial stress at the common surface is 20 MN/m². If this cylinder is now subjected to an internal pressure of 100 MN/m² (1000 bar), what is the magnitude and position of the maximum hoop stress? [164 MN/m² at inside of outer cylinder.]

10.9 (B). A thick cylinder has an internal diameter of 75 mm and an external diameter of 125 mm. The ends are closed and it carries an internal pressure of 60 MN/m². Neglecting end effects, calculate the hoop stress and radial stress at radii of 37.5 mm, 40 mm, 50 mm, 60 mm and 62.5 mm. Plot the values on a diagram to show the variation of these stresses through the cylinder wall. What is the value of the longitudinal stress in the cylinder?

[C.U.] [Hoop: 128, 116, 86.5, 70.2, 67.5 MN/m². Radial: −60, −48.7, −19, −2.9, 0 MN/m²; 33.8 MN/m².]

10.10 (B) A compound tube is formed by shrinking together two tubes with common radius 150 mm and thickness 25 mm. The shrinkage allowance is to be such that when an internal pressure of 30 MN/m² (300 bar) is applied the final maximum stress in each tube is to be the same. Determine the value of this stress. What must have been the difference in diameters of the tubes before shrinkage? $E = 210 \text{ GN/m}^2$. [83.1 MN/m²; 0.025 mm.]

10.11 (B) A steel shaft of 75 mm diameter is pressed into a steel hub of 100 mm outside diameter and 200 mm long in such a manner that under an applied torque of 6 kN m relative slip is just avoided. Find the interference fit, assuming a 75 mm common diameter, and the maximum circumferential stress in the hub. $\mu = 0.3$. $E = 210 \text{ GN/m}^2$ [0.0183 mm; 40.4 MN/m².]

10.12 (B) A steel plug of 75 mm diameter is forced into a steel ring of 125 mm external diameter and 50 mm width. From a reading taken by fixing in a circumferential direction an electric resistance strain gauge on the external surface of the ring, the strain is found to be 1.49×10^{-4} . Assuming $\mu = 0.2$ for the mating surfaces, find the force required to push the plug out of the ring. Also estimate the greatest hoop stress in the ring. $E = 210 \text{ GN/m}^2$. [I.Mech.E.] [65.6 kN; 59 MN/m².]

10.13 (B) A steel cylindrical plug of 125 mm diameter is forced into a steel sleeve of 200 mm external diameter and 100 mm long. If the greatest circumferential stress in the sleeve is 90 MN/m², find the torque required to turn the sleeve, assuming $\mu = 0.2$ at the mating surfaces. [U.L.] [19.4 kN m.]

10.14 (B) A solid steel shaft of 0.2 m diameter has a bronze bush of 0.3 m outer diameter shrunk on to it. In order to remove the bush the whole assembly is raised in temperature uniformly. After a rise of 100°C the bush can just be moved along the shaft. Neglecting any effect of temperature in the axial direction, calculate the original interface pressure between the bush and the shaft.

For steel: $E = 208 \text{ GN/m}^2$, $v = 0.29$, $\alpha = 12 \times 10^{-6}$ per °C.

For bronze: $E = 112 \text{ GN/m}^2$, $v = 0.33$, $\alpha = 18 \times 10^{-6}$ per °C.

[C.E.I.] [20.2 MN/m².]

10.15 (B) (a) State the Lamé equations for the hoop and radial stresses in a thick cylinder subjected to an internal pressure and show how these may be expressed in graphical form. Hence, show that the hoop stress at the outside surface of such a cylinder subjected to an internal pressure P is given by

$$\sigma_H = \frac{2PR_1^2}{(R_2^2 - R_1^2)}$$

where R_1 and R_2 are the internal and external radii of the cylinder respectively.

(b) A steel tube is shrunk on to another steel tube to form a compound cylinder 60 mm internal diameter, 180 mm external diameter. The initial radial compressive stress at the 120 mm common diameter is 30 MN/m². Calculate the shrinkage allowance. $E = 200 \text{ GN/m}^2$.

(c) If the compound cylinder is now subjected to an internal pressure of 25 MN/m² calculate the resultant hoop stresses at the internal and external surfaces of the compound cylinder. [0.0768 mm; -48.75, +54.25 MN/m².]

10.16 (B) A bronze tube, 60 mm external diameter and 50 mm bore, fits closely inside a steel tube of external diameter 100 mm. When the assembly is at a uniform temperature of 15°C the bronze tube is a sliding fit inside the steel tube, that is, the two tubes are free from stress. The assembly is now heated uniformly to a temperature of 115°C.

- (a) Calculate the radial pressure induced between the mating surfaces and the thermal circumferential stresses, in magnitude and nature, induced at the inside and outside surfaces of each tube. [10.9 MN/m²; 23.3, 12.3, -71.2, -60.3 MN/m².]
- (b) Sketch the radial and circumferential stress distribution across the combined wall thickness of the assembly when the temperature is 115°C and insert the numerical values. Use the tabulated data given below.

	Young's modulus (E)	Poisson's ratio (v)	Coefficient of linear expansion (α)
Steel	200 GN/m ²	0.3	$12 \times 10^{-6}/^\circ\text{C}$
Bronze	100 GN/m ²	0.33	$19 \times 10^{-6}/^\circ\text{C}$

10.17 (B) A steel cylinder, 150 mm external diameter and 100 mm internal diameter, just fits over a brass cylinder of external diameter 100 mm and bore 50 mm. The compound cylinder is to withstand an internal pressure of such a

magnitude that the pressure set up between the common junction surfaces is 30 MN/m^2 when the internal pressure is applied. The external pressure is zero. Determine:

- the value of the internal pressure;
- the hoop stress induced in the material of both tubes at the inside and outside surfaces.

Lamé's equations for thick cylinders may be assumed without proof, and neglect any longitudinal stress and strain.

For steel, $E = 207 \text{ GN/m}^2$ (2.07 Mbar) and $\nu = 0.28$.

For brass, $E = 100 \text{ GN/m}^2$ (1.00 Mbar) and $\nu = 0.33$.

Sketch the hoop and radial stress distribution diagrams across the combined wall thickness, inserting the peak values.
[B.P.] [123 MN/m^2 ; 125.4, 32.2 MN/m^2 ; 78.2, 48.2 MN/m^2 .]

10.18 (C). Assuming the Lamé equations for stresses in a thick cylinder, show that the radial and circumferential stresses in a solid shaft owing to the application of external pressure are equal at all radii.

A solid steel shaft having a diameter of 100 mm has a steel sleeve shrunk on to it. The maximum tensile stress in the sleeve is not to exceed twice the compressive stress in the shaft. Determine (a) the least thickness of the sleeve and (b) the maximum tensile stress in the sleeve after shrinkage if the shrinkage allowance, based on diameter, is 0.015 mm. $E = 210 \text{ GN/m}^2$.
[I.Mech.E.] [36.6 mm; 21 MN/m^2 .]

10.19 (C). A steel tube of internal radius 25 mm and external radius 40 mm is wound with wire of 0.75 mm diameter until the external diameter of the tube and wire is 92 mm. Find the maximum hoop stress set up within the walls of the tube if the wire is wound with a tension of 15 MN/m^2 and an internal pressure of 30 MN/m^2 (300 bar) acts within the tube.
[49 MN/m^2 .]

10.20 (C). A thick cylinder of 100 mm internal diameter and 125 mm external diameter is wound with wire until the external diameter is increased by 30 %. If the initial tensile stress in the wire when being wound on the cylinder is 135 MN/m^2 , calculate the maximum stress set up in the cylinder walls.
[- 144.5 MN/m^2 .]

CHAPTER 11

STRAIN ENERGY

Summary

The energy stored within a material when work has been done on it is termed the *strain energy* or *resilience*,

i.e. $\text{strain energy} = \text{work done}$

In general there are four types of loading which can be applied to a material:

1. Direct load (tension or compression)

$$\begin{aligned}\text{Strain energy } U &= \int \frac{P^2 ds}{2AE} \quad \text{or} \quad \frac{P^2 L}{2AE} \\ &= \frac{\sigma^2 AL}{2E} = \frac{\sigma^2}{2E} \times \text{volume of bar}\end{aligned}$$

2. Shear load

$$\begin{aligned}\text{Strain energy } U &= \int \frac{Q^2 ds}{2AG} \quad \text{or} \quad \frac{Q^2 L}{2AG} \\ &= \frac{\tau^2}{2G} \times AL = \frac{\tau^2}{2G} \times \text{volume of bar}\end{aligned}$$

3. Bending

$$\text{Strain energy } U = \int \frac{M^2 ds}{2EI} \quad \text{or} \quad \frac{M^2 L}{2EI} \quad \text{if } M \text{ is constant}$$

4. Torsion

$$\text{Strain energy } U = \int \frac{T^2 ds}{2GJ} \quad \text{or} \quad \frac{T^2 L}{2GJ} \quad \text{if } T \text{ is constant}$$

From 1 above, the strain energy or resilience when the tensile stress reaches the proof stress σ_p , i.e. the *proof resilience*, is

$$\frac{\sigma_p^2}{2E} \times \text{volume of bar}$$

and the *modulus of resilience* is

$$\frac{\sigma_p^2}{2E}$$

The *strain energy per unit volume* of a three-dimensional principal stress system is

$$U = \frac{1}{2E} [\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2v(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)]$$

The *volumetric* or “*dilatational*” strain energy per unit volume is then

$$\frac{(1-2\nu)}{6E} [(\sigma_1 + \sigma_2 + \sigma_3)^2]$$

and the *shear*, or “*distortional*”, strain energy per unit volume is

$$\frac{1}{12G} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

The *maximum instantaneous stress* in a uniform bar caused by a weight W falling through a distance h on to the bar is given by

$$\sigma = \frac{W}{A} \pm \sqrt{\left[\left(\frac{W}{A} \right)^2 + \frac{2WEh}{AL} \right]}$$

The *instantaneous extension* is then given by

$$\delta = \frac{\sigma L}{E}$$

If this is small compared to the height h , then

$$\sigma = \sqrt{\left(\frac{2WEh}{AL} \right)}$$

For any *shock-loaded system* the instantaneous deflection is given by

$$\delta = \delta_s \left[1 \pm \sqrt{\left(1 + \frac{2h}{\delta_s} \right)} \right]$$

where δ_s is the deflection under an equal static load.

Castigliano's first theorem for deflection states that:

If the total strain energy expressed in terms of the external loads is partially differentiated with respect to one of the loads the result is the deflection of the point of application of that load and in the direction of that load (see Examples 11.5 and 11.6):

i.e. Deflection in direction of $W = \frac{\partial U}{\partial W} = \delta$

In applications where bending provides practically all of the strain energy,

$$\delta = \frac{\partial}{\partial W} \int \frac{M^2 ds}{2EI} = \int \frac{M}{EI} \frac{\partial M}{\partial W} ds$$

This is sometimes written in the form

$$\delta = \int \frac{Mm}{EI} ds$$

where $m = \frac{\partial M}{\partial W}$ = the bending moment resulting from a unit load only in the place of W . This method of solution is then termed the *unit load method*.

Castigliano's theorem also applies to **angular movements**:

If the total strain energy expressed in terms of the external moments be partially differentiated with respect to one of the moments, the result is the angular deflection in radians of the point of application of that moment and in its direction

$$\theta = \int \frac{M}{EI} \frac{\partial M}{\partial M_i} ds$$

where M_i is the actual or imaginary moment at the point where θ is required.

Deflections due to shear

Beam loading	Shear deflection	
	Rectangular-section beam	I-section beam
Cantilever-concentrated end load W'	$\frac{6WL}{5AG}$	$\frac{WL}{AG}$
Cantilever-u.d.l.	$\frac{3wL^2}{5AG}$	$\frac{wL^2}{2AG} = \frac{WL}{2AG}$
Simply supported beam-central concentrated load W	$\frac{3WL}{10AG}$	$\frac{WL}{4AG}$
Simply supported beam-concentrated load dividing span into lengths a and b	$\frac{6Wab}{5AGL}$	
Simply supported beam-u.d.l.	$\frac{3wL^2}{20AG}$	$\frac{wL^2}{8AG} = \frac{WL}{8AG}$

Introduction

Energy is normally defined as the *capacity to do work* and it may exist in any of many forms, e.g. mechanical (potential or kinetic), thermal, nuclear, chemical, etc. The potential energy of a body is the form of energy which is stored by virtue of the work which has previously been done on that body, e.g. in lifting it to some height above a datum. Strain energy is a particular form of potential energy which is stored within materials which have been subjected to strain, i.e. to some change in dimension. The material is then capable of doing work, equivalent to the amount of strain energy stored, when it returns to its original unstrained dimension.

Strain energy is therefore defined as the energy which is stored within a material when work has been done on the material. Here it is assumed that the material remains elastic whilst work is done on it so that all the energy is recoverable and no permanent deformation occurs due to yielding of the material,

i.e. strain energy $U = \text{work done}$

Thus for a gradually applied load the work done in straining the material will be given by the shaded area under the load-extension graph of Fig. 11.1.

$$U = \frac{1}{2} P\delta$$

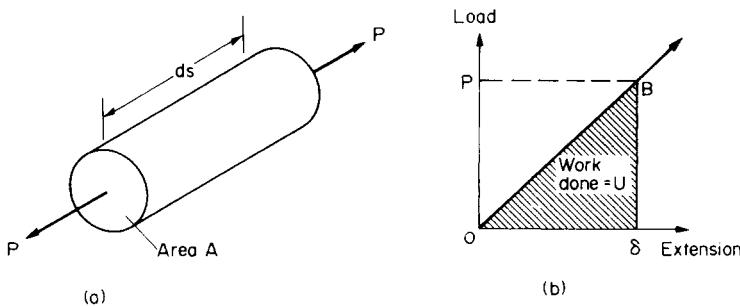


Fig. 11.1. Work done by a gradually applied load.

The strain energy per unit volume is often referred to as the *resilience*. The value of the resilience at the yield point or at the proof stress for non-ferrous materials is then termed the *proof resilience*.

The unshaded area above the line OB of Fig. 11.1 is called the *complementary energy*, a quantity which is utilised in some advanced energy methods of solution and is not considered within the terms of reference of this text.†

11.1. Strain energy – tension or compression

(a) Neglecting the weight of the bar

Consider a small element of a bar, length ds , shown in Fig. 11.1. If a graph is drawn of load against elastic extension the shaded area under the graph gives the work done and hence the strain energy,

i.e. strain energy $U = \frac{1}{2} P\delta$

$$\text{Now} \quad \text{Young's modulus } E = \frac{\text{stress}}{\text{strain}} = \frac{P}{A} \times \frac{ds}{\delta}$$

$$\delta = \frac{Pds}{AE}$$

∴ for the bar element $U = \frac{P^2 ds}{2AE}$

$$\therefore \text{total strain energy for a bar of length } L = \int_0^L \frac{P^2 ds}{2AE}$$

Thus, assuming that the area of the bar remains constant along the length,

$$U = \frac{P^2 L}{2AE} \quad (11.1)$$

[†] See H. Ford and J. M. Alexander, *Advanced Mechanics of Materials* (Longmans, London, 1963).

or, in terms of the stress $\sigma (= P/A)$,

$$U = \frac{\sigma^2 AL}{2E} = \frac{\sigma^2}{2E} \times \text{volume of bar} \quad (11.2)$$

i.e. strain energy, or resilience, *per unit volume* of a bar subjected to direct load, tensile or compressive

$$= \frac{\sigma^2}{2E} \quad (11.3)$$

or, alternatively,

$$= \frac{1}{2} \sigma \times \frac{\sigma}{E} = \frac{1}{2} \sigma \times \epsilon$$

i.e. **resilience = $\frac{1}{2}$ stress \times strain**

(b) Including the weight of the bar

Consider now a bar of length L mounted vertically, as shown in Fig. 11.2. At any section AB the total load on the section will be the external load P together with the weight of the bar material below AB .

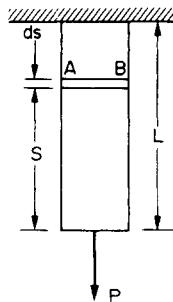


Fig. 11.2. Direct load – tension or compression.

Assuming a uniform cross-section of area A with density ρ ,

$$\text{load on section } AB = P \pm \rho g As$$

the positive sign being used when P is tensile and the negative sign when P is compressive. Thus, for a tensile force P the extension of the element ds is given by the definition of Young's modulus E to be

$$\begin{aligned} \delta &= \frac{\sigma ds}{E} \\ &= \frac{(P + \rho g As)}{AE} ds \end{aligned}$$

\therefore work done = $\frac{1}{2} \times \text{load} \times \text{extension}$

$$= \frac{1}{2}(P + \rho g As) \frac{(P + \rho g As)}{AE} ds$$

$$= \frac{P^2}{2AE} ds + \frac{P\rho g}{E} s ds + \frac{(\rho g)^2 A}{2E} s^2 ds$$

\therefore total strain energy or work done

$$\begin{aligned} &= \int_0^L \frac{P^2}{2AE} ds + \int_0^L \frac{P\rho g}{E} s ds + \int_0^L \frac{(\rho g)^2 A}{2E} s^2 ds \\ &= \frac{P^2 L}{2AE} + \frac{P\rho g L^2}{2E} + \frac{(\rho g)^2 A L^3}{6E} \end{aligned} \quad (11.4)$$

The last two terms are therefore the modifying terms to eqn. (11.1) to account for the body-weight effect of the bar.

11.2. Strain energy–shear

Consider the elemental bar now subjected to a shear load Q at one end causing deformation through the angle γ (the shear strain) and a shear deflection δ , as shown in Fig. 11.3.

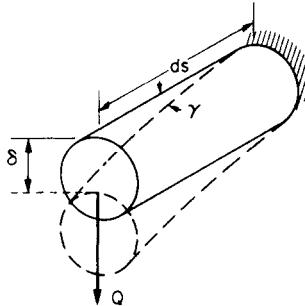


Fig. 11.3. Shear.

$$\text{Strain energy } U = \text{work done} = \frac{1}{2}Q\delta = \frac{1}{2}Q\gamma ds$$

Now

$$G = \frac{\text{shear stress}}{\text{shear strain}} = \frac{\tau}{\gamma} = \frac{Q}{\gamma A}$$

$$\therefore \gamma = \frac{Q}{AG}$$

$$\therefore \text{shear strain energy} = \frac{1}{2}Q \times \frac{Q}{AG} \times ds = \frac{Q^2}{2AG} ds$$

∴ total strain energy resulting from shear

$$= \int_0^L \frac{Q^2 ds}{2AG} = \frac{Q^2 L}{2AG} \quad (11.5)$$

or, in terms of the shear stress $\tau = (Q/A)$,

$$U = \frac{\tau^2 AL}{2G} = \frac{\tau^2}{2G} \times \text{volume of bar} \quad (11.6)$$

11.3. Strain energy – bending

Let the element now be subjected to a constant bending moment M causing it to bend into an arc of radius R and subtending an angle $d\theta$ at the centre (Fig. 11.4). The beam will also have moved through an angle $d\theta$.

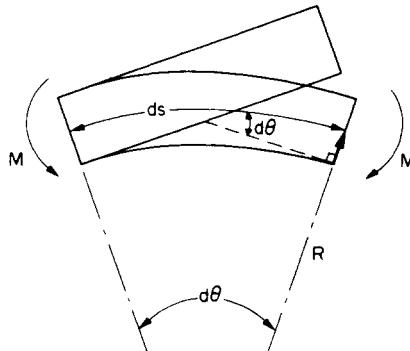


Fig. 11.4. Bending.

Strain energy = work done = $\frac{1}{2} \times \text{moment} \times \text{angle turned through}$ (in radians)

$$= \frac{1}{2} M d\theta$$

But $ds = R d\theta$ and $\frac{M}{I} = \frac{E}{R}$

$$\therefore d\theta = \frac{ds}{R} = \frac{M}{EI} ds$$

$$\therefore \text{strain energy} = \frac{1}{2} M \times \frac{M}{EI} ds = \frac{M^2 ds}{2EI}$$

Total strain energy resulting from bending,

$$U = \int_0^L \frac{M^2 ds}{2EI} \quad (11.7)$$

If the bending moment is constant this reduces to

$$U = \frac{M^2 L}{2EI}$$

11.4. Strain energy – torsion

The element is now considered subjected to a torque T as shown in Fig. 11.5, producing an angle of twist $d\theta$ radians.

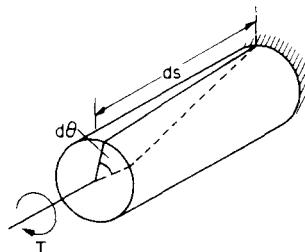


Fig. 11.5. Torsion.

$$\text{Strain energy} = \text{work done} = \frac{1}{2} T d\theta$$

But, from the simple torsion theory,

$$\frac{T}{J} = \frac{Gd\theta}{ds} \quad \text{and} \quad d\theta = \frac{T ds}{GJ}$$

∴ total strain energy resulting from torsion,

$$U = \int_0^L \frac{T^2 ds}{2GJ} = \frac{T^2 L}{2GJ} \quad (11.8)$$

since in most practical applications T is constant.

For a hollow circular shaft eqn. (11.8) still applies

i.e. $\text{Strain energy } U = \frac{T^2 L}{2GJ}$

Now, from the simple bending theory

$$\frac{T}{J} = \frac{\tau}{r} = \frac{\tau_{\max}}{R}$$

where R is the outer radius of the shaft and

$$J = \frac{\pi}{2} (R^4 - r^4)$$

$$T = \frac{\pi}{2R} \tau_{\max} (R^4 - r^4)$$

Substituting in the strain energy equation (11.8) we have:

$$\begin{aligned} U &= \frac{\left[\frac{\pi \tau_{\max}}{2R} (R^4 - r^4) \right]^2 L}{2G \frac{\pi}{2} (R^4 - r^4)} \\ &= \frac{\tau_{\max}^2 \pi (R^4 - r^4) L}{4G R^2} \\ &= \frac{\tau_{\max}^2 [R^2 + r^2]}{4G R^2} \times \text{volume of shaft} \end{aligned}$$

or

$$\text{Strain energy/unit volume} = \frac{\tau_{\max}^2 [R^2 + r^2]}{4G R^2} \quad (11.8a)$$

It should be noted that in the four types of loading case considered above the strain energy expressions are all identical in form,

i.e. strain energy $U = \frac{(\text{applied "load"})^2 \times L}{2 \times \text{product of two related constants}}$

the constants being related to the type of loading considered. In bending, for example, the relevant constants which appear in the bending theory are E and I , whilst for torsion G and J are more applicable. Thus the above standard equations for strain energy should easily be remembered.

11.5. Strain energy of a three-dimensional principal stress system

The reader is referred to §14.17 for the derivation of the following expression for the strain energy of a system of three principal stresses:

$$U = \frac{1}{2E} [\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \quad \text{per unit volume}$$

It is then shown in §14.17 that this total strain energy can be conveniently considered as made up of two parts:

- (a) the *volumetric* or *dilatational* strain energy;
- (b) the *shear* or *distortional* strain energy.

11.6. Volumetric or dilatational strain energy

This is the strain energy associated with a mean or hydrostatic stress of $\frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \bar{\sigma}$ acting equally in all three mutually perpendicular directions giving rise to no distortion, merely a change in volume.

Then from eqn. (14.22),

$$\text{volumetric strain energy} = \frac{(1 - 2\nu)}{6E} [(\sigma_1 + \sigma_2 + \sigma_3)^2] \quad \text{per unit volume}$$

11.7. Shear or distortional strain energy

In order to consider the general principal stress case it has been shown necessary, in §14.6, to add to the mean stress $\bar{\sigma}$ in the three perpendicular directions, certain so-called deviatoric stress values to return the stress system to values of σ_1 , σ_2 and σ_3 . These *deviatoric stresses* are then associated directly with change of shape, i.e. distortion, without change in volume and the strain energy associated with this mechanism is shown to be given by

$$\begin{aligned}\text{shear strain energy} &= \frac{1}{12G} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \quad \text{per unit volume} \\ &= \frac{1}{6G} [\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - (\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \quad \text{per unit volume}\end{aligned}$$

This equation is used as the basis of the Maxwell-von Mises theory of elastic failure which is discussed fully in Chapter 15.

11.8. Suddenly applied loads

If a load P is applied gradually to a bar to produce an extension δ the load-extension graph will be as shown in Fig. 11.1 and repeated in Fig. 11.6, the work done being given by $U = \frac{1}{2}P\delta$.

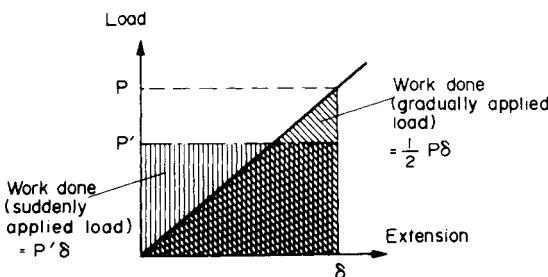


Fig. 11.6. Work done by a suddenly applied load.

If now a load P' is suddenly applied (i.e. applied with an instantaneous value, not gradually increasing from zero to P') to produce the same extension δ , the graph will now appear as a horizontal straight line with a work done or strain energy = $P'\delta$.

The bar will be strained by an equal amount δ in both cases and the energy stored must therefore be equal,

i.e.

$$P'\delta = \frac{1}{2}P\delta$$

or

$$P' = \frac{1}{2}P$$

Thus the suddenly applied load which is required to produce a certain value of instantaneous strain is half the equivalent value of static load required to perform the same function. It is then clear that vice versa a load P which is suddenly applied will produce twice the effect of the same load statically applied. Great care must be exercised, therefore, in the design

of, for example, machine parts to exclude the possibility of sudden applications of load since associated stress levels are likely to be doubled.

11.9. Impact loads – axial load application

Consider now the bar shown vertically in Fig. 11.7 with a rigid collar firmly attached at the end. The load W is free to slide vertically and is suspended by some means at a distance h above the collar. When the load is dropped it will produce a maximum instantaneous extension δ of the bar, and will therefore have done work (neglecting the mass of the bar and collar)

$$= \text{force} \times \text{distance} = W(h + \delta)$$

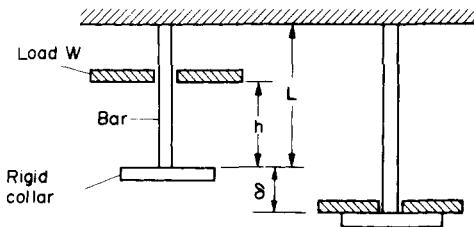


Fig. 11.7. Impact load – axial application.

This work will be stored as strain energy and is given by eqn. (11.2):

$$U = \frac{\sigma^2 AL}{2E}$$

where σ is the instantaneous stress set up.

$$\therefore \frac{\sigma^2 AL}{2E} = W(h + \delta) \quad (11.9)$$

If the extension δ is small compared with h it may be ignored and then, approximately,

$$\sigma^2 = 2WEh/AL$$

$$\text{i.e. } \sigma = \sqrt{\left(\frac{2WEh}{AL}\right)} \quad (11.10)$$

If, however, δ is not small compared with h it must be expressed in terms of σ , thus

$$E = \frac{\text{stress}}{\text{strain}} = \frac{\sigma L}{\delta} \quad \text{and} \quad \delta = \frac{\sigma L}{E}$$

Therefore substituting in eqn. (11.9)

$$\frac{\sigma^2 AL}{2E} = Wh + \frac{W\sigma L}{E}$$

$$\therefore \frac{\sigma^2 AL}{2E} - \sigma \frac{WL}{E} - Wh = 0$$

$$\sigma^2 - \frac{2W}{A}\sigma - \frac{2WEh}{AL} = 0$$

Solving by "the quadratic formula" and ignoring the negative sign,

$$\sigma = \frac{1}{2} \left\{ \frac{2W}{A} + \sqrt{\left[\left(\frac{2W}{A} \right)^2 + 4 \left(\frac{2WEh}{AL} \right) \right]} \right\}$$

i.e.

$$\sigma = \frac{W}{A} + \sqrt{\left[\left(\frac{W}{A} \right)^2 + \frac{2WEh}{AL} \right]} \quad (11.11)$$

This is the *accurate* equation for the *maximum* stress set up, and should always be used if there is any doubt regarding the relative magnitudes of δ and h .

Instantaneous extensions can then be found from

$$\delta = \frac{\sigma L}{E}$$

If the load is not dropped but *suddenly applied* from effectively zero height, $h = 0$, and eqn. (11.11) reduces to

$$\sigma = \frac{W}{A} + \frac{W}{A} = \frac{2W}{A}$$

This verifies the work of §11.8 and confirms that stresses resulting from suddenly applied loads are twice those resulting from statically applied loads of the same magnitude. Inspection of eqn. (11.11) shows that stresses resulting from impact loads of similar magnitude will be even higher than this and any design work in applications where impact loading is at all possible should always include a safety factor well in excess of two.

11.10. Impact loads – bending applications

Consider the beam shown in Fig. 11.8 subjected to a shock load W falling through a height h and producing an instantaneous deflection δ .

$$\text{Work done by falling load} = W(h + \delta)$$

In these cases it is often convenient to introduce an *equivalent static load* W_E defined as that load which, when gradually applied, produces the same deflection as the shock load

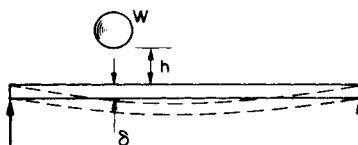


Fig. 11.8. Impact load – bending application.

which it replaces, then

$$\begin{aligned} \text{work done by equivalent static load} &= \frac{1}{2} W_E \delta \\ W(h + \delta) &= \frac{1}{2} W_E \delta \end{aligned} \quad (11.12)$$

Thus if δ is obtained in terms of W_E using the standard deflection equations of Chapter 5 for the support conditions in question, the above equation becomes a quadratic equation in one unknown W_E . Hence W_E can be determined and the required stresses or deflections can be found on the equivalent beam system using the usual methods for static loading, i.e. the dynamic load case has been reduced to the equivalent static load condition.

Alternatively, if W produces a deflection δ_s when applied statically then, by proportion,

$$\frac{W_E}{\delta} = \frac{W}{\delta_s} \quad \text{or} \quad W_E = \frac{\delta}{\delta_s} W$$

Substituting in eqn. (11.12)

$$\begin{aligned} W(h + \delta) &= \frac{1}{2} W \times \frac{\delta}{\delta_s} \times \delta \\ \therefore \quad \delta^2 - 2\delta_s \delta - 2\delta_s h &= 0 \\ \therefore \quad \delta &= \delta_s \pm \sqrt{(\delta_s + 2\delta_s h)} \\ \delta &= \delta_s \left[1 \pm \left(1 + \frac{2h}{\delta_s} \right)^{\frac{1}{2}} \right] \end{aligned} \quad (11.13)$$

The instantaneous deflection of any shock-loaded system is thus obtained from a knowledge of the static deflection produced by an equal load. Stresses are then calculated as before.

11.11. Castigliano's first theorem for deflection

Castigliano's first theorem states that:

If the total strain energy of a body or framework is expressed in terms of the external loads and is partially differentiated with respect to one of the loads the result is the deflection of the point of application of that load and in the direction of that load,

i.e. if U is the total strain energy, the deflection in the direction of load $W = \partial U / \partial W$.

In order to prove the theorem, consider the beam or structure shown in Fig. 11.9 with forces P_A , P_B , P_C , etc., acting at points A , B , C , etc.

If a , b , c , etc., are the deflections in the direction of the loads then the total strain energy of the system is equal to the work done.

$$U = \frac{1}{2} P_A a + \frac{1}{2} P_B b + \frac{1}{2} P_C c + \dots \quad (11.14)$$

N.B. *Limitations of theory.* The above simplified approach to impact loading suffers severe limitations. For example, the distribution of stress and strain under impact conditions will not strictly be the same as under static loading, and perfect elasticity of the bar will not be exhibited. These and other effects are discussed by Roark and Young in their advanced treatment of dynamic stresses: *Formulas for Stress & Strain*, 5th edition (McGraw Hill), Chapter 15.

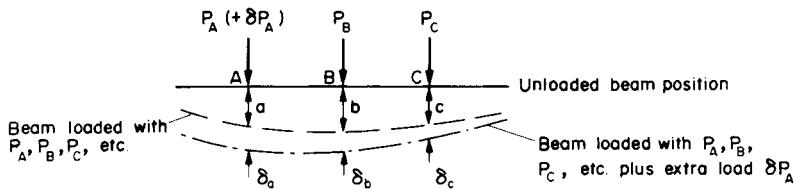


Fig. 11.9. Any beam or structure subjected to a system of applied concentrated loads $P_A, P_B, P_C \dots P_N$, etc.

If one of the loads, P_A , is now increased by an amount δP_A the changes in deflections will be δa , δb and δc , etc., as shown in Fig. 11.9.

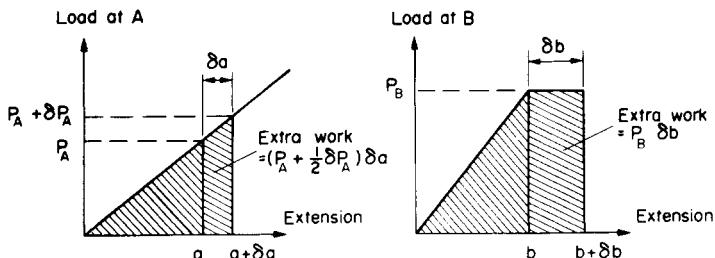


Fig. 11.10. Load-extension curves for positions A and B.

Extra work done at A (see Fig. 11.10)

$$= (P_A + \frac{1}{2}\delta P_A)\delta a$$

Extra work done at B, C, etc. (see Fig. 11.10)

$$= P_B \delta b, P_C \delta c, \text{etc.}$$

Increase in strain energy

$$= \text{total extra work done}$$

$$\therefore \delta U = P_A \delta a + \frac{1}{2} \delta P_A \delta a + P_B \delta b + P_C \delta c + \dots$$

and neglecting the product of small quantities

$$\delta U = P_A \delta a + P_B \delta b + P_C \delta c + \dots \quad (11.15)$$

But if the loads $P_A + \delta P_A$, P_B , P_C , etc., were applied gradually from zero the total strain energy would be

$$U + \delta U = \sum \frac{1}{2} \times \text{load} \times \text{extension}$$

$$U + \delta U = \frac{1}{2} (P_A + \delta P_A)(a + \delta a) + \frac{1}{2} P_B(b + \delta b) + \frac{1}{2} P_C(c + \delta c) + \dots$$

$$= \frac{1}{2} P_A a + \frac{1}{2} P_A \delta a + \frac{1}{2} \delta P_A a + \frac{1}{2} P_A \delta a + \frac{1}{2} P_B b + \frac{1}{2} P_B \delta b + \frac{1}{2} P_C c + \frac{1}{2} P_C \delta c + \dots$$

Neglecting the square of small quantities ($\frac{1}{2}\delta P_A \delta a$) and subtracting eqn. (11.14),

$$\delta U = \frac{1}{2} \delta P_A a + \frac{1}{2} P_A \delta a + \frac{1}{2} P_B \delta b + \frac{1}{2} P_C \delta c + \dots$$

or

$$2\delta U = \delta P_A a + P_A \delta a + P_B \delta b + P_C \delta c + \dots$$

Subtracting eqn. (11.15),

$$\delta U = \delta P_A a \quad \therefore \quad \frac{\delta U}{\delta P_A} = a$$

or, in the limit,

$$\frac{\partial U}{\partial P_A} = a$$

i.e. the partial differential of the strain energy U with respect to P_A gives the deflection under and in the direction of P_A . Similarly,

$$\frac{\partial U}{\partial P_B} = b \quad \text{and} \quad \frac{\partial U}{\partial P_C} = c, \text{ etc.}$$

In most beam applications the strain energy, and hence the deflection, resulting from end loads and shear forces are taken to be negligible in comparison with the strain energy resulting from bending (torsion not normally being present),

$$\begin{aligned} \therefore U &= \int \frac{M^2}{2EI} ds \\ \frac{\partial U}{\partial P} &= \frac{\partial U}{\partial M} \times \frac{\partial M}{\partial P} = \int \frac{2M}{2EI} ds \times \frac{\partial M}{\partial P} \\ \text{i.e. } \delta &= \frac{\partial U}{\partial P} = \int \frac{M}{EI} \frac{\partial M}{\partial P} ds \end{aligned} \quad (11.16)$$

which is the usual form of Castiglano's first theorem. The integral is evaluated as it stands to give the deflection under an existing load P , the value of the bending moment M at some general section having been determined in terms of P . If no general expression for M in terms of P can be obtained to cover the whole beam then the beam, and hence the integral limits, can be divided into any number of convenient parts and the results added. In cases where the deflection is required at a point or in a direction in which there is no load applied, an imaginary load P is introduced in the required direction, the integral obtained in terms of P and then evaluated with P equal to zero.

The above procedures are illustrated in worked examples at the end of this chapter.

11.12. "Unit-load" method

It has been shown in §11.11 that in applications where bending provides practically all of the total strain energy of a system

$$\delta = \int \frac{M}{EI} \frac{\partial M}{\partial W} ds$$

Now W is an applied concentrated load and M will therefore include terms of the form Wx , where x is some distance from W to the point where the bending moment (B.M.) is required plus terms associated with the other loads. The latter will reduce to zero when partially differentiated with respect to W since they do not include W .

Now

$$\frac{\partial}{\partial W} (Wx) = x = 1 \times x$$

i.e. the partial differential of the B.M. term containing W is identical to the result achieved if W is replaced by unity in the B.M. expression. Using this information the Castigiano expression can be simplified to remove the partial differentiation procedure, thus

$$\delta = \int \frac{Mm}{EI} ds \quad (11.17)$$

where m is the B.M. resulting from a *unit load only* applied at the point of application of W and in the direction in which the deflection is required. The value of M remains the same as in the standard Castigiano procedure and is therefore the B.M. due to the *applied load system, including W* .

This so-called “unit load” method is particularly powerful for cases where deflections are required at points where no external load is applied or in directions different from those of the applied loads. The method mentioned previously of introducing imaginary loads P and then subsequently assuming P is zero often gives rise to confusion. It is much easier to simply apply a unit load at the point, and in the direction, in which deflection is required regardless of whether external loads are applied there or not (see Example 11.6).

11.13. Application of Castigiano's theorem to angular movements

Castigiano's theorem can also be applied to angular rotations under the action of bending moments or torques. For the bending application the theorem becomes:

If the total strain energy, expressed in terms of the external moments, be partially differentiated with respect to one of the moments, the result is the angular deflection (in radians) of the point of application of that moment and in its direction,

i.e. $\theta = \int \frac{M}{EI} \frac{\partial M}{\partial M_i} ds \quad (11.18)$

where M_i is the imaginary or applied moment at the point where θ is required.

Alternatively the “unit-load” procedure can again be used, this time replacing the applied or imaginary moment at the point where θ is required by a “unit moment”. Castigiano's expression for slope or angular rotation then becomes

$$\theta = \int \frac{Mm}{EI} \cdot ds$$

where M is the bending moment at a general point due to the applied loads or moments and m is the bending moment at the same point due to the unit moment at the point where θ is required and in the required direction. See Example 11.8 for a simple application of this procedure.

11.14. Shear deflection

(a) Cantilever carrying a concentrated end load

In the majority of beam-loading applications the deflections due to bending are all that need be considered. For very short, deep beams, however, a secondary deflection, that due to

shear, must also be considered. This may be determined using the strain energy formulae derived earlier in this chapter.

For bending,

$$U_B = \int_0^L \frac{M^2 ds}{2EI}$$

For shear,

$$U_s = \int_0^L \frac{Q^2 ds}{2AG} = \frac{\tau^2}{2G} \times \text{volume}$$

Consider, therefore, the cantilever, of solid rectangular section, shown in Fig. 11.11.

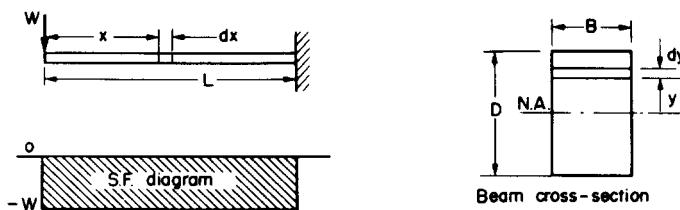


Fig. 11.11.

For the element of length dx

$$U_s = \int \frac{\tau^2}{2G} \times B dy dx$$

But

$$\tau = \frac{Q A \bar{y}}{I_b} \quad (\text{see } \S 7.1)$$

$$= Q \times \frac{B \left(\frac{D}{2} - y \right)}{I_b} \left[\frac{\left(\frac{D}{2} - y \right)}{2} + y \right]$$

$$= \frac{Q}{2I} \left(\frac{D^2}{4} - y^2 \right)$$

$$\therefore U_s = \frac{1}{2G} \int \left\{ \frac{Q}{2I} \left(\frac{D^2}{4} - y^2 \right) \right\}^2 B dx dy$$

$$= \frac{B dx}{2G} \int_{-D/2}^{D/2} \left\{ \frac{Q}{2I} \left(\frac{D^2}{4} - y^2 \right) \right\}^2 dy$$

$$= \frac{Q^2 B}{8GI^2} dx \left(\frac{D^5}{30} \right)$$

To obtain the total strain energy we must now integrate this along the length of the cantilever. In this case Q is constant and equal to W and the integration is simple.

$$\begin{aligned} U_s &= \int_0^L \frac{W^2 B}{8G I^2} \frac{D^5}{30} dx \\ &= \frac{W^2 B D^5}{8G I^2} \frac{L}{30} = \frac{W^2 B L D^5}{240 G} \left(\frac{12}{BD^3} \right)^2 \\ &= \frac{3 W^2 L}{5 A G} \end{aligned}$$

where $A = BD$.

Therefore deflection due to shear

$$\delta_s = \frac{\partial U_s}{\partial W} = \frac{6WL}{5AG} \quad (11.19)$$

Similarly, since $M = -Wx$

$$U_B = \int_0^L \frac{(-Wx)^2}{2EI} ds = \frac{W^2 L^3}{6EI}$$

Therefore deflection due to bending

$$\delta_B = \frac{\partial U}{\partial W} = \frac{WL^3}{3EI} \quad (11.20)$$

Comparison of eqns. (11.19) and (11.20) then yields the relationship between the shear and bending deflections. For very short beams, where the length equals the depth, the shear deflection is almost twice that due to bending. For longer beams, however, the bending deflection is very much greater than that due to shear and the latter can usually be neglected, e.g. for $L = 10D$ the deflection due to shear is less than 1% of that due to bending.

(b) Cantilever carrying uniformly distributed load

Consider now the same cantilever but carrying a uniformly distributed load over its complete length as shown in Fig. 11.12.

The shear force at any distance x from the free end

$$Q = wx$$

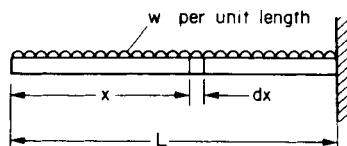


Fig. 11.12.

Therefore shear deflection over the length of the small element dx

$$= \frac{6}{5} \frac{(wx)}{AG} dx \quad \text{from (11.19)}$$

Therefore total shear deflection

$$\delta_s = \int_0^L \frac{6}{5} \frac{wx dx}{AG} = \frac{3wL^2}{5AG} \quad (11.21)$$

(c) *Simply supported beam carrying central concentrated load*

In this case it is convenient to treat the beam as two cantilevers each of length equal to half the beam span and each carrying an end load half that of the central beam load (Fig. 11.13). The required central deflection due to shear will equal that of the end of each cantilever, i.e. from eqn. (11.19), with $W = W/2$ and $L = L/2$,

$$\delta_s = \frac{6}{5AG} \left(\frac{W}{2} \times \frac{L}{2} \right) = \frac{3WL}{10AG} \quad (11.22)$$

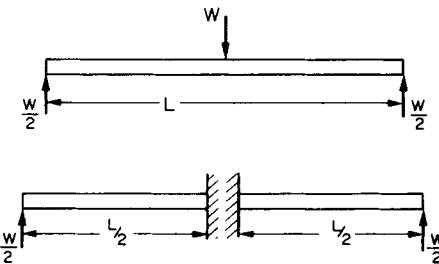


Fig. 11.13. Shear deflection of simply supported beam carrying central concentrated load-equivalent loading diagram.

(d) *Simply supported beam carrying a concentrated load in any position*

If the load divides the beam span into lengths a and b the reactions at each end will be Wa/L and Wb/L . The equivalent cantilever system is then shown in Fig. 11.14 and the shear

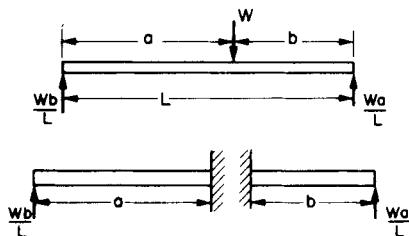


Fig. 11.14. Equivalent loading for offset concentrated load.

deflection under the load is equal to the end deflection of either cantilever and given by eqn. (11.19),

$$\delta_s = \frac{6}{5AG} \left(\frac{Wa}{L} \right) b \quad \text{or} \quad \delta_s = \frac{6}{5AG} \left(\frac{Wb}{L} \right) a$$

$$\therefore \delta_s = \frac{6Wab}{5AGL} \quad (11.23)$$

(e) *Simply supported beam carrying uniformly distributed load*

Using a similar treatment to that described above, the equivalent cantilever system is shown in Fig. 11.15, i.e. each cantilever now carries an end load of $wL/2$ in one direction and a uniformly distributed load w over its complete length $L/2$ in the opposite direction.

From eqns. (11.19) and (11.20)

$$\delta_s = \frac{6}{5AG} \left(\frac{wL}{2} \times \frac{L}{2} \right) - \frac{3}{5AG} w \left(\frac{L}{2} \right)^2$$

$$\delta_s = \frac{3wL^2}{20AG} \quad (11.24)$$

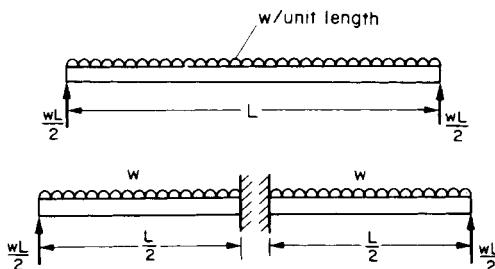


Fig. 11.15. Equivalent loading for uniformly loaded beam.

(f) *I-section beams*

If the shear force is assumed to be uniformly distributed over the web area A , a similar treatment to that described above yields the following approximate results:

cantilever with concentrated end load W

$$\delta_s = \frac{WL}{AG}$$

cantilever with uniformly distributed load w

$$\delta_s = \frac{wL^2}{2AG} = \frac{WL}{2AG}$$

simply supported beam with concentrated end load W

$$\delta_s = \frac{WL}{4AG}$$

simply supported beam with uniformly distributed load w

$$\delta_s = \frac{wL^2}{8AG} = \frac{WL}{8AG}$$

In the above expressions the effect of the flanges has been neglected and it therefore follows that the same formulae would apply for rectangular sections if it were assumed that the shear stress is evenly distributed across the section. The result of WL/AG for the cantilever carrying a concentrated end load is then directly comparable to that obtained in eqn. (11.19) taking full account of the variation of shear across the section, i.e. $6/5 (WL/AG)$. Since the shear strain $\gamma = \delta/L$ it follows that both the deflection and associated shear strain is underestimated by 20% if the shear is assumed to be uniform.

(g) Shear deflections at points other than loading points

In the case of simply supported beams, deflections at points other than loading positions are found by simple proportion, deflections increasing linearly from zero at the supports (Fig. 11.16). For cantilevers, however, if the load is not at the free end, the above remains true between the load and the support but between the load and the free end the beam remains horizontal, i.e. there is no shear deflection. This, of course, must not be confused with deflections due to bending when there will always be some deflection of the end of a cantilever whatever the position of loading.

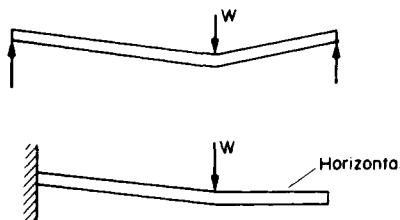


Fig. 11.16. Shear deflections of simply supported beams and cantilevers.
These must not be confused with bending deflections.

Examples

Example 11.1

Determine the diameter of an aluminium shaft which is designed to store the same amount of strain energy per unit volume as a 50 mm diameter steel shaft of the same length. Both shafts are subjected to equal compressive axial loads.

What will be the ratio of the stresses set up in the two shafts?

$$E_{\text{steel}} = 200 \text{ GN/m}^2; \quad E_{\text{aluminium}} = 67 \text{ GN/m}^2.$$

Solution

$$\text{Strain energy per unit volume} = \frac{\sigma^2}{2E}$$

Since the strain energy/unit volume in the two shafts is equal,

then

$$\frac{\sigma_A^2}{2E_A} = \frac{\sigma_S^2}{2E_S}$$

$$\therefore \frac{\sigma_A^2}{\sigma_S^2} = \frac{E_A}{E_S} = \frac{67}{200} = \frac{1}{3} \text{ (approximately)} \quad (1)$$

$$\therefore 3\sigma_A^2 = \sigma_S^2 \quad (2)$$

Now

$$\sigma = \frac{P}{\text{area}} \quad \text{where } P \text{ is the applied load}$$

Therefore from (1)

$$\left[\frac{P}{\frac{\pi}{4}D_A^2} \right]^2 \times \left[\frac{\frac{\pi}{4}D_S^2}{P} \right]^2 = \frac{1}{3}$$

$$\frac{D_S^4}{D_A^4} = \frac{1}{3}$$

$$\begin{aligned} D_A^4 &= 3 \times D_S^4 = 3 \times (50)^4 \\ &= 3 \times 625 \times 10^4 \end{aligned}$$

$$\therefore D_A = \sqrt[4]{1875 \times 10^4} = 65.8 \text{ mm}$$

The required diameter of the aluminium shaft is 65.8 mm.

From (2)

$$3\sigma_A^2 = \sigma_S^2$$

$$\frac{\sigma_S}{\sigma_A} = \sqrt{3}$$

Example 11.2

Two shafts are of the same material, length and weight. One is solid and 100 mm diameter, the other is hollow. If the hollow shaft is to store 25 % more energy than the solid shaft when transmitting torque, what must be its internal and external diameters?

Assume the same maximum shear stress applies to both shafts.

Solution

Let A be the solid shaft and B the hollow shaft. If they are the same weight and the same material their volume must be equal.

$$\therefore \frac{\pi}{4} D_A^2 \times L = \frac{\pi}{4} [D_B^2 - d_B^2] L$$

$$\therefore D_A^2 = D_B^2 - d_B^2 = \frac{100^2}{10^6} \text{ m}^2 = 10 \times 10^{-3} \text{ m}^2 \quad (1)$$

Now for the same maximum shear stress

$$\tau = \frac{Tr}{J} = \frac{TD}{2J}$$

i.e.

$$\frac{T_A D_A}{J_A} = \frac{T_B D_B}{J_B}$$

$$\therefore \frac{T_A}{T_B} = \frac{D_B J_A}{D_A J_B} \quad (2)$$

But the strain energy of $B = 1.25 \times$ strain energy of A .

$$\therefore \text{since } U = \frac{T^2 L}{2GJ}$$

then $\frac{T_B^2 L}{2GJ_B} = 1.25 \frac{T_A^2 L}{2GJ_A} \quad \text{or} \quad \frac{T_A^2}{T_B^2} = \frac{J_A}{1.25 J_B}$

Therefore substituting from (2),

$$\begin{aligned} \frac{D_B^2}{D_A^2} &= \frac{J_B}{1.25 J_A} \\ \therefore \frac{D_B^2}{D_A^2} &= \frac{\frac{\pi}{32} [D_B^4 - d_B^4]}{1.25 \frac{\pi}{32} D_A^4} = \frac{D_B^4 - d_B^4}{1.25 D_A^4} \\ D_B^2 &= \frac{D_B^4 - d_B^4}{1.25 D_A^2} \\ &= \frac{D_B^4 - (D_B^2 - 10 \times 10^{-3})^2}{1.25 \times 10 \times 10^{-3}} \end{aligned}$$

$$12.5 \times 10^{-3} D_B^2 = D_B^4 - D_B^2 + 20 \times 10^{-3} D_B^2 - 100 \times 10^{-6}$$

$$\therefore 7.5 \times 10^{-3} \times D_B^2 = 100 \times 10^{-6}$$

$$D_B^2 = \frac{100 \times 10^{-6}}{7.5 \times 10^{-3}} = 13.3 \times 10^{-3}$$

$$D_B = 115.47 \text{ mm}$$

$$d_B^2 = D_B^2 - D_A^2 = \frac{13.3}{10^3} - \frac{10}{10^3} = \frac{3.3}{10^3}$$

$$d_B = 57.74 \text{ mm}$$

The internal and external diameters of the hollow tube are therefore 57.7 mm and 115.5 mm respectively.

Example 11.3

- (a) What will be the instantaneous stress and elongation of a 25 mm diameter bar, 2.6 m long, suspended vertically, if a mass of 10 kg falls through a height of 300 mm on to a collar which is rigidly attached to the bottom end of the bar?

Take $g = 10 \text{ m/s}^2$.

(b) When used horizontally as a simply supported beam, a concentrated force of 1 kN applied at the centre of the support span produces a static deflection of 5 mm. The same load will produce a maximum bending stress of 158 MN/m².

Determine the magnitude of the instantaneous stress produced when a mass of 10 kg is allowed to fall through a height of 12 mm on to the beam at mid-span.

What will be the instantaneous deflection?

Solution

(a) From eqn. (11.9)

$$W\left(h + \frac{\sigma L}{E}\right) = \frac{\sigma^2}{2E} \times \text{volume} \quad (\text{Fig. 11.7})$$

$$\text{volume of bar} = \frac{1}{4}\pi \times \frac{25^2}{10^6} \times 2.6 = 12.76 \times 10^{-4}$$

$$\text{Then } 10 \times 10 \left(0.3 + \frac{2.6\sigma}{200 \times 10^9}\right) = \frac{\sigma^2 \times 12.76 \times 10^{-4}}{2 \times 200 \times 10^9}$$

$$\therefore 30 + \frac{1.3\sigma}{10^9} = \frac{\sigma^2}{313 \times 10^{12}}$$

$$\text{and } 30 \times 313 \times 10^{12} + \frac{1.3\sigma}{10^9} \times 313 \times 10^{12} = \sigma^2$$

$$\text{Then } \sigma^2 - 406.9 \times 10^3 \times \sigma - 9390 \times 10^{12} = 0$$

$$\begin{aligned} \sigma &= \frac{406.9 \times 10^3 \pm \sqrt{(166 \times 10^9 + 37560 \times 10^{12})}}{2} \\ &= \frac{406.9 \times 10^3 \pm 193.9 \times 10^6}{2} \\ &= 97.18 \text{ MN/m}^2 \end{aligned}$$

If the instantaneous deflection is ignored (the term $\sigma L/E$ omitted) in the above calculation a very small difference in stress is noted in the answer,

$$\text{i.e. } W(h) = \frac{\sigma^2 \times \text{volume}}{2E}$$

$$\therefore 100 \times 0.3 = \frac{\sigma^2 \times 12.76 \times 10^{-4}}{2 \times 200 \times 10^9}$$

$$\therefore \sigma^2 = \frac{30 \times 400 \times 10^9}{12.76 \times 10^{-4}} = 9404 \times 10^{12}$$

$$\therefore \sigma = 96.97 \text{ MN/m}^2$$

This suggests that if the deflection δ is small in comparison to h (the distance through which

the mass falls) it can, for all practical purposes, be ignored in the above calculation:

$$\text{deflection produced } (\delta) = \frac{\sigma L}{E} = \frac{97.18 \times 2.6 \times 10^6}{200 \times 10^9}$$

i.e. elongation of bar = 1.26 mm

(b) Consider the loading system shown in Fig. 11.8. Let W_E be the equivalent force that produces the same deflection and stress when gradually applied as that produced by the falling mass.

Then

$$\frac{W_E}{\delta_{\max}} = \frac{W_s}{\delta_s}$$

where W_s is a known load, gradually applied to the beam at mid-span, producing deflection δ_s and stress σ_s .

Then

$$\delta_{\max} = \frac{W_E \delta_s}{W_s} = \frac{W_E \times 5 \times 10^{-3}}{1 \times 10^3}$$

∴

$$\delta_{\max} = \frac{5}{10^6} W_E$$

Now

$$W(h + \delta_{\max}) = \frac{W_E}{2} \delta_{\max}$$

∴

$$100 \left[\frac{12}{10^3} + \frac{5 W_E}{10^6} \right] = \frac{W_E}{2} \times \frac{5 W_E}{10^6}$$

$$1.2 + \frac{500 W_E}{10^6} = \frac{2.5 W_E^2}{10^6}$$

$$\therefore W_E^2 - \frac{500 W_E}{2.5} - \frac{1.2 \times 10^6}{2.5} = 0$$

$$\text{and } W_E^2 - 200 W_E - 0.48 \times 10^6 = 0$$

$$\text{By factors, } W_E = 800 \text{ N or } -600 \text{ N}$$

∴

$$W_E = 800 \text{ N}$$

By proportion

$$\frac{\sigma_s}{W_s} = \frac{\sigma_{\max}}{W_E}$$

and the maximum stress is given by

$$\sigma_{\max} = \frac{\sigma_s}{W_s} \times W_E = \frac{158 \times 10^6 \times 800}{1 \times 10^3} = 126.4 \text{ MN/m}^2$$

And since

$$\frac{W_E}{\delta} = \frac{W_s}{\delta_s}$$

the deflection is given by

$$\begin{aligned}\delta &= \frac{W_E}{W_s} \times \delta_s \\ &= \frac{800 \times 5 \times 10^{-3}}{1 \times 10^3} = 4 \times 10^{-3} \\ &= 4 \text{ mm}\end{aligned}$$

Example 11.4

A horizontal steel beam of I-section rests on a rigid support at one end, the other end being supported by a vertical steel rod of 20 mm diameter whose upper end is rigidly held in a support 2.3 m above the end of the beam (Fig. 11.17). The beam is a $200 \times 100 \text{ mm B.S.B.}$ for which the relevant I -value is $23 \times 10^{-6} \text{ m}^4$ and the distance between its two points of support is 3 m. A load of 2.25 kN falls on the beam at mid-span from a height of 20 mm above the beam.

Determine the maximum stresses set up in the beam and rod, and find the deflection of the beam at mid-span measured from the unloaded position. Assume $E = 200 \text{ GN/m}^2$ for both beam and rod.

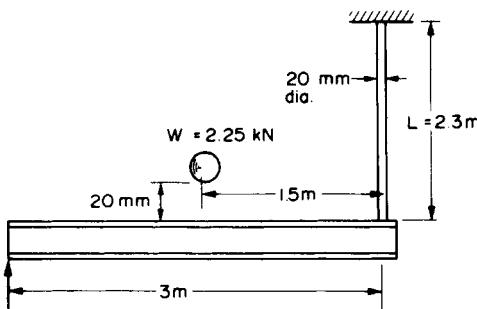


Fig. 11.17.

Solution

Let the shock load cause a deflection δ_B of the beam at the load position and an extension δ_R of the rod. Then if W_E is the equivalent static load which produces the deflection δ_B and P is the maximum tension in the rod,

$$\begin{aligned}\text{total strain energy} &= \frac{P^2 L_R}{2AE} + \frac{1}{2} W_E \delta_B \\ &= \text{work done by falling mass}\end{aligned}$$

Now the mass falls through a distance

$$h + \delta_B + \frac{\delta_R}{2}$$

where $\delta_R/2$ is the effect of the rod extension on the mid-point of the beam. (This assumes that the beam remains straight and rotates about the fixed support position.)

$$\therefore \text{work done by falling mass} = W \left(h + \delta_B + \frac{\delta_R}{2} \right)$$

If

P = reaction at one end of beam

then

$$P = \frac{W_E}{2}$$

$$\therefore W \left(h + \delta_B + \frac{\delta_R}{2} \right) = \frac{W_E^2 L_B}{8AE} + \frac{W_E \delta_B}{2} \quad (1)$$

$$\text{For a centrally loaded beam} \quad \delta = \frac{WL^3}{48EI}$$

$$\therefore \delta_B = \frac{W_E \times 3^3}{48 \times 200 \times 10^9 \times 23 \times 10^{-6}} = \frac{W_E}{8.18 \times 10^6} \quad (2)$$

$$\text{For an axially loaded rod} \quad \delta_R = \frac{WL}{AE}$$

$$\therefore \delta_R = \frac{W_E \times 2.3}{\frac{\pi}{4} \times 20^2 \times 10^{-6} \times 200 \times 10^9} = \frac{W_E}{27.3 \times 10^6} \quad (3)$$

Substituting (2) and (3) in (1),

$$2.25 \times 10^3 \left[\frac{20}{10^3} + \frac{W_E}{8.18 \times 10^6} + \frac{W_E}{54.6 \times 10^6} \right] = \frac{W_E^2 \times 2.3}{8(\frac{\pi}{4} \times 20^2 \times 10^{-6}) \times 200 \times 10^9} + \frac{W_E^2}{2 \times 8.18 \times 10^6}$$

$$45 + \frac{2.25 \times 10^3 W_E}{8.18 \times 10^6} + \frac{2.25 \times 10^3 W_E}{54.6 \times 10^6} = \frac{W_E^2 \times 2.3}{8 \times 314 \times 10^{-6} \times 200 \times 10^9} + \frac{W_E^2}{16.36 \times 10^6}$$

$$45 + 275 \times 10^{-6} W_E + 41.2 \times 10^{-6} W_E = 4.58 \times 10^{-9} W_E^2 + 61.1 \times 10^{-9} W_E^2$$

$$45 + 316.2 W_E \times 10^{-6} = 65.68 \times 10^{-9} W_E^2$$

$$\text{Then} \quad W_E^2 - \frac{316.2 \times 10^{-6}}{65.68 \times 10^{-9}} W_E - \frac{45}{65.68 \times 10^{-9}} = 0$$

$$\therefore W_E^2 - 4.8 \times 10^3 W_E - 685 \times 10^6 = 0$$

and

$$\begin{aligned}
 W_E &= \frac{4.8 \times 10^3 \pm \sqrt{(23 \times 10^6 + 2740 \times 10^6)}}{2} \\
 &= \frac{4.8 \times 10^3 \pm \sqrt{(2763 \times 10^6)}}{2} \\
 &= \frac{4.8 \times 10^3 \pm 52.59 \times 10^3}{2} \\
 &= \frac{57.3 \times 10^3}{2} \\
 &= 28.65 \times 10^3 \text{ N}
 \end{aligned}$$

$$\begin{aligned}
 \text{Maximum bending moment} &= \frac{W_E L}{4} \\
 &= \frac{28.65 \times 10^3 \times 3}{4} \\
 &= 21.5 \times 10^3 \text{ N}
 \end{aligned}$$

$$\begin{aligned}
 \text{Then maximum bending stress} &= \frac{My}{I} \\
 &= \frac{21.5 \times 10^3 \times 100 \times 10^{-3}}{23 \times 10^{-6}} \\
 &= 93.9 \times 10^6 \text{ N/m}^2
 \end{aligned}$$

$$\begin{aligned}
 \text{Maximum stress in rod} &= \frac{\frac{1}{2} W_E}{\text{area}} \\
 &= \frac{28.65 \times 10^3}{2 \times \frac{\pi}{4} \times 20^2 \times 10^{-6}} \\
 &= 45.9 \times 10^6 \text{ N/m}^2
 \end{aligned}$$

$$\begin{aligned}
 \text{Deflection of beam } \delta_B &= \frac{W_E}{8.18 \times 10^6} \\
 &= \frac{28.65 \times 10^3}{8.18 \times 10^6} \\
 &= 3.52 \times 10^{-3} \text{ m}
 \end{aligned}$$

This is the extension at mid-span and neglects the extension of the rod.

$$\begin{aligned}
 \text{Extension of rod} &= \frac{\sigma L}{E} = \frac{PL}{AE} = \frac{W_E L}{2AE} \\
 &= \frac{28.8 \times 10^3 \times 2.3}{2 \times 314 \times 10^{-6} \times 200 \times 10^9} \\
 &= 0.527 \times 10^{-3} \text{ m}
 \end{aligned}$$

Assuming, as stated earlier, that the beam remains straight and that the beam rotates about the fixed end, then the effect of the rod extension at the mid-span

$$= \frac{\delta_R}{2} = \frac{0.527 \times 10^{-3}}{2} = 0.264 \times 10^{-3} \text{ m}$$

$$\begin{aligned}\text{Then, total deflection at mid-span} &= \delta_B + \delta_R/2 \\ &= 3.52 \times 10^{-3} + 0.264 \times 10^{-3} \\ &= 3.784 \times 10^{-3} \text{ m}\end{aligned}$$

Example 11.5

Using Castigliano's first theorem, obtain the expressions for (a) the deflection under a single concentrated load applied to a simply supported beam as shown in Fig. 11.18, (b) the deflection at the centre of a simply supported beam carrying a uniformly distributed load.

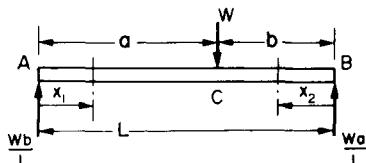


Fig. 11.18.

Solution

(a) For the beam shown in Fig. 11.18

$$\begin{aligned}\delta &= \int_B^A \frac{M}{EI} \frac{\partial M}{\partial W} ds \\ &= \int_A^C \frac{M}{EI} \frac{\partial M}{\partial W} ds + \int_C^B \frac{M}{EI} \frac{\partial M}{\partial W} ds \\ &= \frac{1}{EI} \int_0^a \frac{Wbx_1}{L} \times \frac{bx_1}{L} \times dx_1 + \frac{1}{EI} \int_0^b \frac{Wax_2}{L} \times \frac{ax_2}{L} \times dx_2 \\ &= \frac{Wb^2}{L^2 EI} \int_0^a x_1^2 dx_1 + \frac{Wa^2}{L^2 EI} \int_0^b x_2^2 dx_2 \\ &= \frac{Wb^2 a^3}{3L^2 EI} + \frac{Wa^2 b^3}{3L^2 EI} = \frac{Wa^2 b^2}{3L^2 EI} (a+b) = \frac{Wa^2 b^2}{3LEI}\end{aligned}$$

(b) For the u.d.l. beam shown in Fig. 11.19a an imaginary load P must be introduced at mid-span; then the mid-span deflection will be

$$\delta = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial W} ds = 2 \int_0^{L/2} \frac{M}{EI} \frac{\partial M}{\partial W} ds$$

but

$$M_{xx} = \frac{(wL + P)}{2}x - \frac{wx^2}{2} \quad \text{and} \quad \frac{\partial M}{\partial W} = \frac{x}{2}$$

Then

$$\delta = \frac{2}{EI} \int_0^{L/2} \left[\frac{(wL + P)}{2}x - \frac{wx^2}{2} \right] \frac{x}{2} dx$$

$$= \frac{1}{2EI} \int_0^{L/2} (wLx^2 - wx^3) dx \quad \text{since } P = 0$$

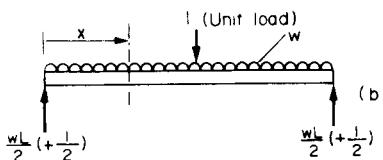
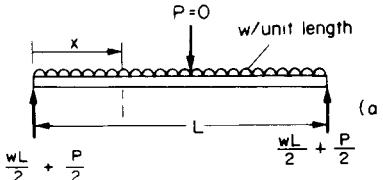


Fig. 11.19.

Alternatively, using a unit load applied vertically at mid-span (Fig. 11.19b),

$$\delta = \int_0^L \frac{Mm}{EI} ds = 2 \int_0^{L/2} \frac{Mm}{EI} ds$$

where

$$M = \frac{wL}{2} - \frac{wx^2}{2} \quad \text{and} \quad m = \frac{x}{2}$$

Then

$$\delta = \frac{2}{EI} \int_0^{L/2} \left(\frac{wLx}{2} - \frac{wx^2}{2} \right) \frac{x}{2} dx$$

$$= \frac{1}{2EI} \int_0^{L/2} (wLx^2 - wx^3) dx$$

as before. Thus, in each case,

$$\delta = \frac{w}{2EI} \left[\frac{Lx^3}{3} - \frac{x^4}{4} \right]_0^{L/2}$$

$$= \frac{wL^4}{2EI} \left[\frac{1}{24} - \frac{1}{64} \right]$$

$$= \frac{wL^4}{2EI} \left[\frac{8 - 3}{192} \right] = \frac{5WL^4}{384EI}$$

Example 11.6

Determine by the methods of unit load and Castigiano's first theorem, (a) the vertical deflection of point A of the bent cantilever shown in Fig. 11.20 when loaded at A with a vertical load of 600 N. (b) What will then be the horizontal movement of A?

The cantilever is constructed from 50 mm diameter bar throughout, with $E = 200 \text{ GN/m}^2$.

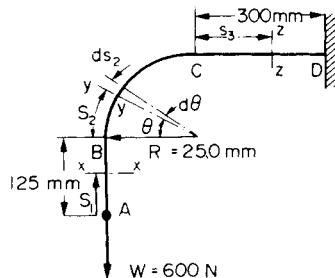


Fig. 11.20.

Solution

The total deflection of A can be considered in three parts, resulting from AB, BC, and CD. Since the question requires solution by two similar methods, they will be worked in parallel.

(a) For vertical deflection

Castigiano	Unit load
$\delta = \int \frac{M}{EI} \frac{\partial M}{\partial W} ds$	$\delta = \int \frac{Mm}{EI} ds$ where m = bending moment resulting from a unit load at A.

For AB $M_{xx} = 0$. Hence vertical deflection resulting from AB = 0 by both methods.

For CD

$$M_{zz} = W(0.25 + s_3)$$

$$\frac{\partial M}{\partial W} = 0.25 + s_3$$

$$\delta_{CD} = \int_0^{0.3} \frac{W(0.25 + s_3)(0.25 + s_3) ds_3}{EI}$$

$$M_{zz} = W(0.25 + s_3)$$

$$m = 1(0.25) + s_3$$

$$\therefore \delta_{CD} = \int_0^{0.3} \frac{W(0.25 + s_3)(0.25 + s_3) ds_3}{EI}$$

Thus the same equation is achieved by both methods.

Castigliano	Unit load
$\therefore \delta_{CD} = \frac{W}{EI} \int_0^{0.3} (0.0625 + 0.5s_3 + s_3^2) ds_3$ $= \frac{W}{EI} \left[0.0625s_3 + \frac{0.5s_3^2}{2} + \frac{s_3^3}{3} \right]_0^{0.3}$ $= \frac{W}{EI} [0.01875 + 0.0225 + 0.009]$ $= \frac{600}{EI} \times 0.05025 = \frac{30.15}{EI}$	

For BC	
$M_{yy} = W(0.25 - 0.25 \cos \theta)$	$M_{yy} = W(0.25 - 0.25 \cos \theta)$
$\frac{\partial M}{\partial W} = 0.25 - 0.25 \cos \theta$	$m = 1(0.25 - 0.25 \cos \theta)$
$ds_2 = 0.25 d\theta$	$ds_2 = 0.25 d\theta$

Once again the same equation for deflection is obtained

$$\text{i.e. } \delta_{BC} = \int_0^{\pi/2} \frac{W(0.25 - 0.25 \cos \theta)}{EI} (0.25 - 0.25 \cos \theta) 0.25 d\theta$$

$$= \frac{(0.25)^3 W}{EI} \int_0^{\pi/2} (1 - 2 \cos \theta + \cos^2 \theta) d\theta$$

$$\text{but } \cos^2 \theta = \frac{1 + \cos 2\theta}{2}$$

$$\therefore \delta_{BC} = \frac{(0.25)^3 W}{EI} \int_0^{\pi/2} \left(1 - 2 \cos \theta + \frac{1 + \cos 2\theta}{2} \right) d\theta$$

$$= \frac{(0.25)^3 W}{EI} \left[\theta - 2 \sin \theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{\pi/2}$$

$$= \frac{(0.25)^3 W}{EI} \left[\frac{\pi}{2} - 2 + \frac{\pi}{4} \right]$$

$$= \frac{(0.25)^3 \times 600}{EI} [\frac{3}{4}\pi - 2]$$

$$= \frac{3.34}{EI}$$

Total vertical deflection at A

$$= \frac{30.15 + 3.34}{EI} = \frac{33.49 \times 64 \times 10^{12}}{200 \times 10^9 \times \pi \times 50^4} = 0.546 \text{ mm}$$

Castigliano	Unit load
Again, working in parallel with Castigliano and unit load methods:-	
(b) For the horizontal deflection using Castigliano's method an imaginary load P must be applied horizontally since there is no external load in this direction at A (Fig. 11.21).	For the unit load method a unit load must be applied at A in the direction in which the deflection is required as shown in Fig. 11.22.
<p>Fig. 11.21.</p>	<p>Fig. 11.22.</p>
<p>Then $\delta_H = \int \frac{M}{EI} \frac{\partial M}{\partial P} ds$, with $P = 0$</p> <p>For AB</p> $M_{xx} = P \times s_1 + W \times 0 = Ps_1$ $\therefore \frac{\partial M}{\partial P} = s_1$ $\therefore \delta_{AB} = \int \frac{Ps_1}{EI} \times s_1 ds_1$ <p>but $P = 0$</p> $\therefore \delta_{AB} = 0$	<p>Then $\delta_H = \int \frac{Mm}{EI} ds$</p> $M_{xx} = W \times 0 = 0$ $m = 1 \times s_1$ $\therefore \delta_{AB} = 0$
<p>For BC</p> $M_{yy} = W(0.25 - 0.25 \cos \theta) + P(0.125 + 0.25 \sin \theta)$ $\frac{\partial M}{\partial P} = 0.125 + 0.25 \sin \theta$ $ds_2 = 0.25 d\theta$ $\therefore \delta_{BC} = \int_0^{\pi/2} \frac{W}{EI} (0.25 - 0.25 \cos \theta) \times (0.125 + 0.25 \sin \theta) 0.25 d\theta$ <p>since $P = 0$</p>	$M_{yy} = W(0.25 - 0.25 \cos \theta)$ $m = 1(0.125 + 0.25 \sin \theta)$ $ds_2 = 0.25 d\theta$ $\therefore \delta_{BC} = \int_0^{\pi/2} \frac{W}{EI} (0.25 - 0.25 \cos \theta) \times (0.125 - 0.25 \sin \theta) 0.25 d\theta$

Thus, once again, the same equation is obtained. This is always the case and there is little difference in the amount of work involved in the two methods.

$$\therefore \delta_{BC} = \frac{W \times 0.25^3}{EI} \int_0^{\pi/2} (1 - \cos \theta)(0.5 + \sin \theta) d\theta$$

$$= \frac{0.25^3 W}{EI} \int_0^{\pi/2} \left(0.5 - \frac{\cos \theta}{2} + \sin \theta - \sin \theta \cos \theta\right) d\theta$$

Castigliano	Unit load
but $\sin \theta \cos \theta = \frac{1}{2} \sin^2 \theta$ $\therefore \delta_{BC} = \frac{0.25^3 W}{EI} \int_0^{\pi/2} \left(\frac{1}{2} - \frac{\cos \theta}{2} + \sin \theta - \frac{\sin 2\theta}{2} \right) d\theta$ $= \frac{0.25^3 W}{EI} \left[\frac{\theta}{2} - \frac{\sin \theta}{2} - \cos \theta + \frac{\cos 2\theta}{4} \right]_0^{\pi/2}$ $= \frac{0.25^3 W}{EI} [(\frac{\pi}{4} - \frac{1}{2} - \frac{1}{4}) - (-1 + \frac{1}{4})]$ $= \frac{0.25^3 \times 600}{EI} \left(\frac{\pi}{4} \right) = \frac{7.36}{EI}$	

For CD, using unit load method,

$$M_{zz} = W(0.25 + s_3) \quad m = 1(0.125 + 0.25) = 0.375$$

$$\delta_{CD} = \frac{1}{EI} \int_0^{0.3} W(0.25 + s_3)(0.375) ds_3$$

$$= \frac{0.375 W}{EI} \int_0^{0.3} (0.25 + s_3) ds_3$$

$$= \frac{0.375 W}{EI} \left[0.25s_3 + \frac{s_3^2}{2} \right]_0^{0.3}$$

$$= \frac{0.375 W}{EI} [0.075 + 0.045]$$

$$= \frac{0.375 \times 600}{EI} \times (0.12) = \frac{27}{EI}$$

Therefore total horizontal deflection

$$= \frac{7.36 + 27}{EI} = \frac{34.36 \times 64 \times 10^{12}}{200 \times 10^9 \times \pi \times 50^4}$$

$$= 0.56 \text{ mm}$$

Example 11.7

The frame shown in Fig. 11.23 is constructed from rectangular bar 25 mm wide by 12 mm thick. The end A is constrained by guides to move in a vertical direction and carries a vertical load of 400 N. For the frame material $E = 200 \text{ GN/m}^2$.

Determine (a) the horizontal reaction at the guides, (b) the vertical deflection of A.

Solution

(a) Consider the frame of Fig. 11.23. If A were not constrained in guides it would move in some direction (shown dotted) which would have both horizontal and vertical components. If

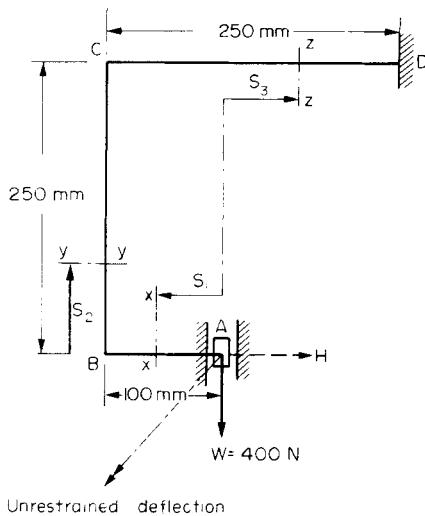


Fig. 11.23.

the horizontal movement is restricted by guides a horizontal reaction H must be set up as shown. Its value is determined by equating the horizontal deflection of A to zero,

$$\text{i.e. } \int \frac{M}{EI} \frac{\partial M}{\partial H} ds = 0$$

For AB

$$M_{xx} = Ws_1 \quad \text{and} \quad \frac{\partial M}{\partial H} = 0$$

\therefore

$$\delta_{AB} = 0$$

For BC

$$M_{yy} = 0.1W - Hs_2 \quad \text{and} \quad \frac{\partial M}{\partial H} = -s_2$$

$$\therefore \delta_{BC} = \int_0^{0.25} \frac{(0.1W - Hs_2)}{EI} (-s_2) ds_2$$

$$= \frac{1}{EI} \int_0^{0.25} (-0.1Ws_2 + Hs_2^2) ds_2$$

$$= \frac{1}{EI} \left[-\frac{0.1Ws_2^2}{2} + \frac{Hs_2^3}{3} \right]_0^{0.25}$$

$$= \frac{1}{EI} \left[-\frac{0.00625W}{2} + \frac{0.015625H}{3} \right]$$

$$= \frac{1}{EI \times 10^3} (-3.125W + 5.208H)$$

For CD

$$M_{zz} = Ws_3 + 0.25H \quad \text{and} \quad \frac{\partial M}{\partial H} = 0.25$$

$$\begin{aligned}\therefore \delta_{CD} &= \int_{-0.10}^{0.15} \frac{(Ws_3 + 0.25H)}{EI} 0.25 ds_3 \\ &= \frac{1}{EI} \int_{-0.10}^{0.15} (0.25Ws_3 + 0.0625H) ds_3 \\ &= \frac{1}{EI} \left[\frac{0.25Ws_3^2}{2} + 0.0625Hs_3 \right]_{-0.10}^{0.15} \\ &= \frac{1}{EI} \left\{ \left[\frac{0.25W}{2} \times 0.0225 + 0.0625H \times 0.15 \right] \right. \\ &\quad \left. - \left[\frac{0.25W}{2} \times 0.01 + 0.0625H(-0.1) \right] \right\} \\ &= \frac{1}{EI \times 10^3} \{ (1.25 \times 2.25W + 6.25 \times 1.5H) - (1.25W - 6.25H) \} \\ &= \frac{1}{EI \times 10^3} \{ (2.81W + 9.375H) - (1.25W - 6.25H) \} \\ &= \frac{1}{EI \times 10^3} (1.56W + 15.625H)\end{aligned}$$

Now the total horizontal deflection of $A = 0$

$$\therefore -3.125W + 5.208H + 1.56W + 15.625H = 0$$

$$-1.565W + 20.833H = 0$$

$$\therefore H = \frac{1.565 \times 400}{20.833} = 30 \text{ N}$$

Since a positive sign has been obtained, H must be in the direction assumed.

(b) For vertical deflection

$$\delta = \int \frac{M}{EI} \frac{\partial M}{\partial W} ds$$

For AB

$$M_{xx} = Ws_1 \quad \text{and} \quad \frac{\partial M}{\partial W} = s_1$$

$$\therefore \delta_{AB} = \int_0^{0.1} \frac{Ws_1 \times s_1}{EI} ds_1$$

$$= \frac{400}{EI} \left[\frac{s_1^3}{3} \right]_0^{0.1}$$

$$= \frac{0.4}{3EI} = \frac{0.133}{EI}$$

For BC $M_{yy} = W \times 0.1 - 30s_2$ and $\frac{\partial M}{\partial W} = 0.1$

$$\therefore \delta_{BC} = \int_0^{0.25} \frac{(0.1W - 30s_2)}{EI} \times 0.1 ds_2$$

$$= \frac{1}{EI} \int_0^{0.25} (0.01 \times 400 - 3s_2) ds_2$$

$$= \frac{1}{EI} \left[4s_2 - \frac{3s_2^2}{2} \right]_0^{0.25}$$

$$= \frac{1}{EI} \left[1 - \frac{3 \times 0.0625}{2} \right]$$

$$= \frac{0.906}{EI}$$

For CD

$$M_{zz} = Ws_3 + 0.25H \quad \text{and} \quad \frac{\partial M}{\partial W} = s_3$$

$$\therefore \delta_{CD} = \int_{-0.10}^{+0.15} \frac{(Ws_3 + 0.25H)}{EI} s_3 ds_3$$

$$= \frac{1}{EI} \int_{-0.1}^{+0.15} (Ws_3^2 + 0.25Hs_3) ds_3$$

$$= \frac{1}{EI} \left[\frac{400 \times s_3^3}{3} + \frac{0.25Hs_3^2}{2} \right]_{-0.1}^{0.15}$$

$$= \frac{1}{EI} \left[\frac{400}{3} (3.375 \times 10^{-3} + 1 \times 10^{-3}) + \frac{0.25 \times 30}{2} (22.5 \times 10^{-3} - 10 \times 10^{-3}) \right]$$

$$= \frac{1}{EI} \left[\frac{400}{3} \times 4.375 \times 10^{-3} + \frac{0.25 \times 30}{2} \times 12.5 \times 10^{-3} \right]$$

$$= \frac{1}{EI} [0.583 + 0.047]$$

$$= \frac{0.63}{EI}$$

Total vertical deflection of A

$$\begin{aligned} &= \frac{1}{EI} (0.133 + 0.906 + 0.63) \\ &= \frac{1.669}{EI} \\ &= \frac{1.669 \times 12 \times 10^{12}}{200 \times 10^9 \times 25 \times 12^3} = 2.32 \text{ mm} \end{aligned}$$

Example 11.8 (B)

Derive the equation for the slope at the free end of a cantilever carrying a uniformly distributed load over its full length.

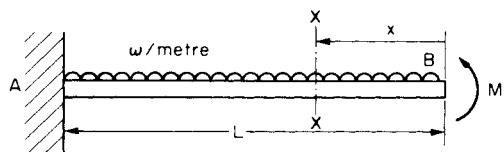


Fig. 11.24.

Solution (a)

Using Castigliano's procedure, apply an imaginary moment M_i in a positive direction at point B where the slope, i.e. rotation, is required.

BM at XX due to applied loading and imaginary couple

$$M = M_i - \frac{wx^2}{2}$$

$$\frac{\partial M}{\partial M_i} = 1$$

from Castigliano's theorem

$$\begin{aligned} \theta &= \int_0^L \frac{M}{EI} \cdot \frac{\partial M}{\partial M_i} \cdot dx \\ &= \frac{1}{EI} \int_0^L \left(M_i - \frac{wx^2}{2} \right) (1) dx \end{aligned}$$

which, with $M_i = 0$ in the absence of any applied moment at B , becomes

$$\theta = \frac{-w}{2EI} \int_0^L x^2 \cdot dx = \frac{wL^3}{6EI} \text{ radian}$$

The negative sign indicates that rotation of the free end is in the opposite direction to that taken for the imaginary moment, i.e. the beam will slope downwards at B as should have been expected.

Alternative solution (b)

Using the “unit-moment” procedure, apply a unit moment at the point B where rotation is required and since we know that the beam will slope downwards the unit moment can be applied in the appropriate direction as shown.

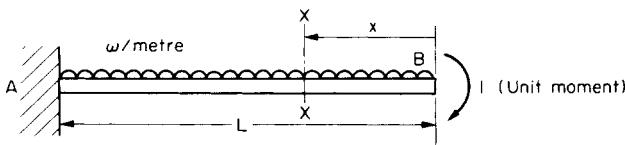


Fig. 11.25.

$$\text{B.M. at } XX \text{ due to applied loading} = M = -\frac{wx^2}{2}$$

$$\text{B.M. at } XX \text{ due to unit moment} = m = -1$$

The required rotation, or slope, is now given by

$$\begin{aligned}\theta &= \int_0^L \frac{Mm}{EI} \cdot dx \\ &= \frac{1}{EI} \int_0^L \left(-\frac{wx^2}{2} \right) (-1) dx. \\ &= \frac{w}{2EI} \int_0^L x^2 dx = \frac{wL^3}{6EI} \text{ radian.}\end{aligned}$$

The answer is thus the same as before and a positive value has been obtained indicating that rotation will occur in the direction of the applied unit moment (i.e. opposite to M_i in the previous solution).

Problems

11.1 (A). Define what is meant by “resilience” or “strain energy”. Derive an equation for the strain energy of a uniform bar subjected to a tensile load of P newtons. Hence calculate the strain energy in a 50 mm diameter bar, 4 m long, when carrying an axial tensile pull of 150 kN. $E = 208 \text{ GN/m}^2$. [110.2 N m.]

11.2 (A). (a) Derive the formula for strain energy resulting from bending of a beam (neglecting shear).

(b) A beam, simply supported at its ends, is of 4 m span and carries, at 3 m from the left-hand support, a load of 20 kN. If I is $120 \times 10^{-6} \text{ m}^4$ and $E = 200 \text{ GN/m}^2$, find the deflection under the load using the formula derived in part (a). [0.625 mm.]

11.3 (A) Calculate the strain energy stored in a bar of circular cross-section, diameter 0.2 m, length 2 m:

- when subjected to a tensile load of 25 kN,
- when subjected to a torque of 25 kNm,
- when subjected to a uniform bending moment of 25 kNm.

For the bar material $E = 208 \text{ GN/m}^2$, $G = 80 \text{ GN/m}^2$.

[0.096, 49.7, 38.2 N m.]

11.4 (A/B) Compare the strain energies of two bars of the same material and length and carrying the same gradually applied compressive load if one is 25 mm diameter throughout and the other is turned down to 20 mm diameter over half its length, the remainder being 25 mm diameter.

If both bars are subjected to pure torsion only, compare the torsional strain energies stored if the shear stress in both bars is limited to 75 MN/m².

[0.78, 2.22.]

11.5 (A/B) Two shafts, one of steel and the other of phosphor bronze, are of the same length and are subjected to equal torques. If the steel shaft is 25 mm diameter, find the diameter of the phosphor-bronze shaft so that it will store the same amount of energy per unit volume as the steel shaft. Also determine the ratio of the maximum shear stresses induced in the two shafts. Take the modulus of rigidity for phosphor bronze as 50 GN/m² and for steel as 80 GN/m².

[27.04 mm, 1.26.]

11.6 (A/B) Show that the torsional strain energy of a solid circular shaft transmitting power at a constant speed is given by the equation:

$$U = \frac{\tau^2}{4G} \times \text{volume.}$$

Such a shaft is 0.06 m in diameter and has a flywheel of mass 30 kg and radius of gyration 0.25 m situated at a distance of 1.2 m from a bearing. The flywheel is rotating at 200 rev/min when the bearing suddenly seizes. Calculate the maximum shear stress produced in the shaft material and the instantaneous angle of twist under these conditions. Neglect the shaft inertia. For the shaft material $G = 80 \text{ GN/m}^2$. [B.P.] [196.8 MN/m², 5.64°.]

11.7 (A/B) A solid shaft carrying a flywheel of mass 100 kg and radius of gyration 0.4 m rotates at a uniform speed of 75 rev/min. During service, a bearing 3 m from the flywheel suddenly seizes producing a fixation of the shaft at this point. Neglecting the inertia of the shaft itself determine the necessary shaft diameter if the instantaneous shear stress produced in the shaft does not exceed 180 MN/m². For the shaft material $G = 80 \text{ GN/m}^2$. Assume all kinetic energy of the shaft is taken up as strain energy without any losses.

[22.7 mm.]

11.8 (A/B) A multi-bladed turbine disc can be assumed to have a combined mass of 150 kg with an effective radius of gyration of 0.59 m. The disc is rigidly attached to a steel shaft 2.4 m long and, under service conditions, rotates at a speed of 250 rev/min. Determine the diameter of shaft required in order that the maximum shear stress set up in the event of sudden seizure of the shaft shall not exceed 200 MN/m². Neglect the inertia of the shaft itself and take the modulus of rigidity G of the shaft material to be 85 GN/m². [284 mm.]

11.9 (A/B) Develop from first principles an expression for the instantaneous stress set up in a vertical bar by a weight W falling from a height h on to a stop at the end of the bar. The instantaneous extension x may not be neglected.

A weight of 500 N can slide freely on a vertical steel rod 2.5 m long and 20 mm diameter. The rod is rigidly fixed at its upper end and has a collar at the lower end to prevent the weight from dropping off. The weight is lifted to a distance of 50 mm above the collar and then released. Find the maximum instantaneous stress produced in the rod. $E = 200 \text{ GN/m}^2$. [114 MN/m².]

11.10 (A/B) A load of 2 kN falls through 25 mm on to a stop at the end of a vertical bar 4 m long, 600 mm² cross-sectional area and rigidly fixed at its other end. Determine the instantaneous stress and elongation of the bar. $E = 200 \text{ GN/m}^2$. [94.7 MN/m², 1.9 mm.]

11.11 (A/B) A load of 2.5 kN slides freely on a vertical bar of 12 mm diameter. The bar is fixed at its upper end and provided with a stop at the other end to prevent the load from falling off. When the load is allowed to rest on the stop the bar extends by 0.1 mm. Determine the instantaneous stress set up in the bar if the load is lifted and allowed to drop through 12 mm on to the stop. What will then be the extension of the bar? [365 MN/m², 1.65 mm.]

11.12 (A/B) A bar of a certain material, 40 mm diameter and 1.2 m long, has a collar securely fitted to one end. It is suspended vertically with the collar at the lower end and a mass of 2000 kg is gradually lowered on to the collar producing an extension in the bar of 0.25 mm. Find the height from which the load could be dropped on to the collar if the maximum tensile stress in the bar is to be 100 MN/m². Take $g = 9.81 \text{ m/s}^2$. The instantaneous extension cannot be neglected. [U.L.] [3.58 mm]

11.13 (A/B) A stepped bar is 2 m long. It is 40 mm diameter for 1.25 m of its length and 25 mm diameter for the remainder. If this bar hangs vertically from a rigid structure and a ring weight of 200 N falls freely from a height of 75 mm on to a stop formed at the lower end of the bar, neglecting all external losses, what would be the maximum instantaneous stress induced in the bar, and the maximum extension? $E = 200 \text{ GN/m}^2$.

[99.3 MN/m², 0.615 mm.]

11.14 (B). A beam of uniform cross-section, with centroid at mid-depth and length 7 m, is simply supported at its ends and carries a point load of 5 kN at 3 m from one end. If the maximum bending stress is not to exceed 90 MN/m² and the beam is 150 mm deep, (i) working from first principles find the deflection under the load, (ii) what load dropped from a height of 75 mm on to the beam at 3 m from one end would produce a stress of 150 MN/m² at the point of application of the load? $E = 200 \text{ GN/m}^2$. [24 mm; 1.45 kN.]

11.15 (B). A steel beam of length 7 m is built in at both ends. It has a depth of 500 mm and the second moment of area is $300 \times 10^{-6} \text{ m}^4$. Calculate the load which, falling through a height of 75 mm on to the centre of the span, will produce a maximum stress of 150 MN/m². What would be the maximum deflection if the load were gradually applied? $E = 200 \text{ GN/m}^2$. [B.P.] [7.77 kN, 0.23 mm.]

11.16 (B). When a load of 20 kN is gradually applied at a certain point on a beam it produces a deflection of 13 mm and a maximum bending stress of 75 MN/m². From what height can a load of 5 kN fall on to the beam at this point if the maximum bending stress is to be 150 MN/m²? [U.L.] [78 mm.]

11.17 (B). Show that the vertical and horizontal deflections of the end B of the quadrant shown in Fig. 11.26 are, respectively,

$$\frac{WR^3}{EI} \left[\frac{3\pi}{4} - 2 \right] \quad \text{and} \quad \frac{WR^3}{2EI}.$$

What would the values become if W were applied horizontally instead of vertically?

$$\left[\frac{WR^3}{EI} \left(\frac{\pi}{4} \right); \frac{WR^3}{2EI} \right]$$

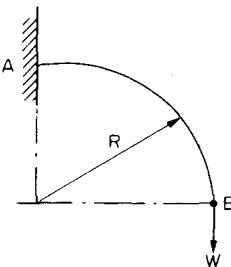


Fig. 11.26.

11.18 (B). A semi-circular frame of flexural rigidity EI is built in at A and carries a vertical load W at B as shown in Fig. 11.27. Calculate the magnitudes of the vertical and horizontal deflections at B and hence the magnitude and direction of the resultant deflection.

$$\left[\frac{3\pi}{2} \frac{WR^3}{EI}; 2 \frac{WR^3}{EI}; 5.12 \frac{WR^3}{EI} \text{ at } 23^\circ \text{ to vertical.} \right]$$

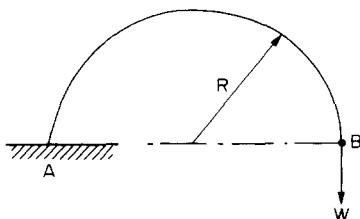


Fig. 11.27.

11.19 (B). A uniform cantilever, length L and flexural rigidity EI carries a vertical load W at mid-span. Calculate the magnitude of the vertical deflection of the free end.

$$\left[5 \frac{WL^3}{48EI} \right]$$

11.20 (B). A steel rod, of flexural rigidity EI, forms a cantilever ABC lying in a vertical plane as shown in Fig. 11.28. A horizontal load of P acts at C. Calculate:

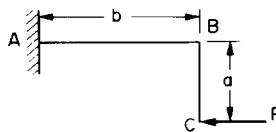


Fig. 11.28.

- (a) the horizontal deflection of C ;
 (b) the vertical deflection of C ;

(c) the slope at B .

Consider the strain energy resulting from bending only.

$$[\text{U.E.I.}] \left[\frac{Pa^2}{3EI} [a + 3b]; \frac{Pab^2}{2EI}; \frac{Pab}{EI} \right]$$

11.21 (B). Derive the formulae for the slope and deflection at the free end of a cantilever when loaded at the end with a concentrated load W . Use a strain energy method for your solution.

A cantilever is constructed from metal strip 25 mm deep throughout its length of 750 mm. Its width, however, varies uniformly from zero at the free end to 50 mm at the support. Determine the deflection of the free end of the cantilever if it carries uniformly distributed load of 300 N/m across its length. $E = 200 \text{ GN/m}^2$. [1.2 mm.]

11.22 (B). Determine the vertical deflection of point A on the bent cantilever shown in Fig. 11.29 when loaded at A with a vertical load of 25 N. The cantilever is built in at B , and EI may be taken as constant throughout and equal to 450 N m^2 . [B.P.] [0.98 mm.]

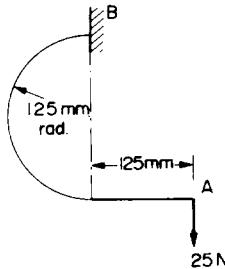


Fig. 11.29.

11.23 (B). What will be the horizontal deflection of A in the bent cantilever of Problem 11.22 when carrying the vertical load of 25 N? [0.56 mm.]

11.24 (B). A steel ring of mean diameter 250 mm has a square section 2.5 mm by 2.5 mm. It is split by a narrow radial saw cut. The saw cut is opened up farther by a tangential separating force of 0.2 N. Calculate the extra separation at the saw cut. $E = 200 \text{ GN/m}^2$. [U.E.I.] [5.65 mm.]

11.25 (B). Calculate the strain energy of the gantry shown in Fig. 11.30 and hence obtain the vertical deflection of the point C . Use the formula for strain energy in bending $U = \int \frac{M^2}{2EI} dx$, where M is the bending moment, E is Young's modulus, I is second moment of area of the beam section about axis XX . The beam section is as shown in Fig. 11.30. Bending takes place along AB and BC about the axis XX . $E = 210 \text{ GN/m}^2$. [U.L.C.I.] [53.9 mm.]

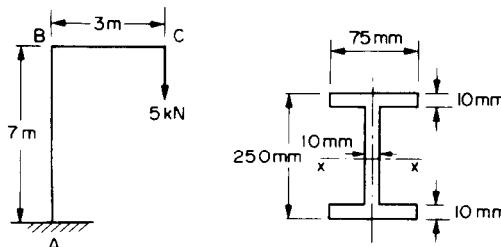


Fig. 11.30.

11.26 (B). A steel ring, of 250 mm diameter, has a width of 50 mm and a radial thickness of 5 mm. It is split to leave a narrow gap 5 mm wide normal to the plane of the ring. Assuming the radial thickness to be small compared with the radius of ring curvature, find the tangential force that must be applied to the edges of the gap to just close it. What will be the maximum stress in the ring under the action of this force? $E = 200 \text{ GN/m}^2$.

[I.Mech.E.] [28.3 N; 34 MN/m².]

11.27 (B). Determine, for the cranked member shown in Fig. 11.31:

- (a) the magnitude of the force P necessary to produce a vertical movement of P of 25 mm;
- (b) the angle, in degrees, by which the tip of the member diverges when the force P is applied.

The member has a uniform width of 50 mm throughout. $E = 200 \text{ GN/m}^2$. [B.P.] [6.58 kN; 4.1°.]

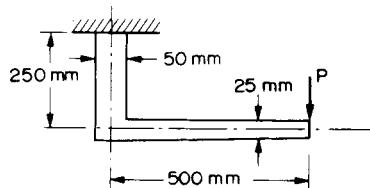


Fig. 11.31.

11.28 (C). A 12 mm diameter steel rod is bent to form a square with sides $2a = 500 \text{ mm}$ long. The ends meet at the mid-point of one side and are separated by equal opposite forces of 75 N applied in a direction perpendicular to the plane of the square as shown in perspective in Fig. 11.32. Calculate the amount by which they will be out of alignment. Consider only strain energy due to bending. $E = 200 \text{ GN/m}^2$. [38.3 mm.]

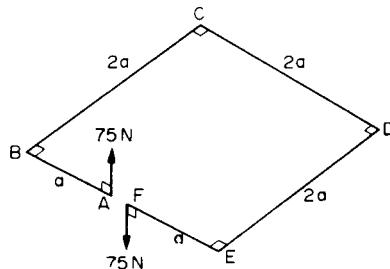


Fig. 11.32

11.29 (B/C). A state of two-dimensional plane stress on an element of material can be represented by the principal stresses σ_1 and σ_2 ($\sigma_1 > \sigma_2$). The strain energy can be expressed in terms of the strain energy per unit volume. Then:

- (a) working from first principles show that the strain energy per unit volume is given by the expression

$$\frac{1}{2E} (\sigma_1^2 + \sigma_2^2 - 2\nu\sigma_1\sigma_2)$$

for a material which follows Hooke's law where E denotes Young's modulus and ν denotes Poisson's ratio, and

- (b) by considering the relations between each of σ_x , σ_y , τ_{xy} , respectively and the principal stresses, where x and y are two other mutually perpendicular axes in the same plane, show that the expression

$$\frac{1}{2E} [\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y + 2(1+\nu)\tau_{xy}^2]$$

is identical with the expression given above.

[City U.]

CHAPTER 12

SPRINGS

Summary

Close-coiled springs

(a) Under axial load W

Maximum shear stress set up in the material of the spring

$$= \tau_{\max} = \frac{2WR}{\pi r^3} = \frac{8WD}{\pi d^3}$$

Total deflection of the spring for n turns

$$= \delta = \frac{4WR^3n}{Gr^4} = \frac{8WD^3n}{Gd^4}$$

where r is the radius of the wire and R the mean radius of the spring coils.

i.e. Spring rate = $\frac{W}{\delta} = \frac{Gd^4}{8nD^3}$

(b) Under axial torque T

$$\text{Maximum bending stress set up} = \sigma_{\max} = \frac{4T}{\pi r^3} = \frac{32T}{\pi d^3}$$

$$\text{Wind-up angle} = \theta = \frac{8TRn}{Er^4} = \frac{64TDn}{Ed^4}$$

$$\therefore \text{Torque per turn} = \frac{T}{\theta/2\pi} = \frac{\pi Ed^4}{32Dn}$$

The stress formulae given in (a) and (b) may be modified in practice by the addition of 'Wahl' correction factors.

Open-coiled springs

(a) Under axial load W

$$\text{Deflection } \delta = 2\pi n WR^3 \sec \alpha \left[\frac{\cos^2 \alpha}{GJ} + \frac{\sin^2 \alpha}{EI} \right]$$

$$\text{Angular rotation } \theta = 2\pi n WR^2 \sin \alpha \left[\frac{1}{GJ} - \frac{1}{EI} \right]$$

(b) Under axial torque T

$$\text{Wind-up angle } \theta = 2\pi n R T \sec \alpha \left[\frac{\sin^2 \alpha}{GJ} + \frac{\cos^2 \alpha}{EI} \right]$$

where α is the helix angle of the spring.

$$\text{Axial deflection } \delta = 2\pi n T R^2 \sin \alpha \left[\frac{1}{GJ} - \frac{1}{EI} \right]$$

Springs in series

$$\text{Stiffness } S = \frac{S_1 S_2}{(S_1 + S_2)}$$

Springs in parallel

$$\text{Stiffness } S = S_1 + S_2$$

Leaf or carriage springs

(a) *Semi-elliptic*

Under a central load W :

$$\text{maximum bending stress} = \frac{3WL}{2nbt^2}$$

$$\text{deflection } \delta = \frac{3WL^3}{8Enbt^3}$$

where L is the length of spring, b is the breadth of each plate, t is the thickness of each plate, and n is the number of plates.

$$\text{Proof load } W_p = \frac{8Enbt^3}{3L^3} \delta_p$$

where δ_p is the initial central “deflection”.

$$\text{Proof or limiting stress } \sigma_p = \frac{4tE}{L^2} \delta_p$$

(b) *Quarter-elliptic*

$$\text{Maximum bending stress} = \frac{6WL}{nbt^2}$$

$$\text{Deflection } \delta = \frac{6WL^3}{Enbt^3}$$

Plane spiral springs

$$\text{Maximum bending stress} = \frac{6Ma}{RBt^2}$$

or, assuming $a = 2R$,

$$\text{maximum bending stress} = \frac{12M}{Bt^2}$$

$$\text{wind-up angle } \theta = \frac{ML}{EI}$$

where M is the applied moment to the spring spindle, R is the radius of spring from spindle to pin, a is the maximum dimension of the spring from the pin, B is the breadth of the material of the spring, t is the thickness of the material of the spring, L is equal to $\frac{1}{2}(\pi n)(a+b)$, and b is the diameter of the spindle.

Introduction

Springs are energy-absorbing units whose function it is to store energy and to release it slowly or rapidly depending on the particular application. In motor vehicle applications the springs act as buffers between the vehicle itself and the external forces applied through the wheels by uneven road conditions. In such cases the shock loads are converted into strain energy of the spring and the resulting effect on the vehicle body is much reduced. In some cases springs are merely used as positioning devices whose function it is to return mechanisms to their original positions after some external force has been removed.

From a design point of view "good" springs store and release energy but do not significantly absorb it. Should they do so then they will be prone to failure.

Throughout this chapter reference will be made to strain energy formulae derived in Chapter 11 and it is suggested that the reader should become familiar with the equations involved.

12.1. Close-coiled helical spring subjected to axial load W

(a) Maximum stress

A close-coiled helical spring is, as the name suggests, constructed from wire in the form of a helix, each turn being so close to the adjacent turn that, for the purposes of derivation of formulae, the helix angle is considered to be so small that it may be neglected, i.e. each turn may be considered to lie in a horizontal plane if the central axis of the spring is vertical. Discussion throughout the subsequent section on both close-coiled and open-coiled springs will be limited to those constructed from wire of circular cross-section and of constant coil diameter.

Consider, therefore, one half-turn of a close-coiled helical spring shown in Fig. 12.1. Every cross-section will be subjected to a torque WR tending to twist the section, a bending moment tending to alter the curvature of the coils and a shear force W . Stresses set up owing to the shear force are usually insignificant and with close-coiled springs the bending stresses

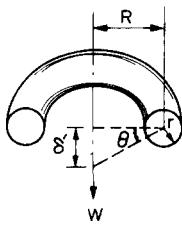


Fig. 12.1. Close-coiled helical spring subjected to axial load W .

are found to be negligible compared with the torsional stresses. Thus the maximum stress in the spring material may be determined to a good approximation using the torsion theory.

$$\tau_{\max} = \frac{Tr}{J} = \frac{WRr}{\pi r^4/2}$$

i.e.

$$\text{maximum stress} = \frac{2WR}{\pi r^3} = \frac{8WD}{\pi d^3} \quad (12.1)$$

(b) Deflection

Again, for one half-turn, if one cross-section twists through an angle θ relative to the other, then from the torsion theory

$$\theta = \frac{TL}{GJ} = \frac{WR(\pi R)}{G} \times \frac{2}{\pi r^4} = \frac{2WR^2}{Gr^4}$$

But

$$\delta' = R\theta = \frac{2WR^3}{Gr^4}$$

$$\therefore \text{total deflection } \delta = 2n\delta' = \frac{4WR^3n}{Gr^4} = \frac{8WD^3n}{Gd^4} \quad (12.2)$$

$$\text{Spring rate} = \frac{W}{\delta'} = \frac{Gd^4}{8nD^3}$$

12.2. Close-coiled helical spring subjected to axial torque T

(a) Maximum stress

In this case the material of the spring is subjected to pure bending which tends to reduce the radius R of the coils (Fig. 12.2). The bending moment is constant throughout the spring and equal to the applied axial torque T . The maximum stress may thus be determined from the bending theory

$$\sigma_{\max} = \frac{My}{I} = \frac{Tr}{\pi r^4/4}$$

i.e.

$$\text{maximum bending stress} = \frac{4T}{\pi r^3} = \frac{32T}{\pi d^3} \quad (12.3)$$

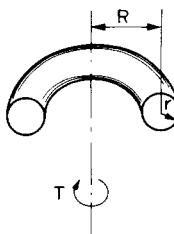


Fig. 12.2. Close-coiled helical spring subjected to axial torque T .

(b) *Deflection (wind-up angle)*

Under the action of an axial torque the deflection of the spring becomes the “wind-up angle” of the spring, i.e. the angle through which one end turns relative to the other. This will be equal to the total change of slope along the wire, which, according to Mohr’s area-moment theorem (see § 5.7), is the area of the M/EI diagram between the ends.

$$\therefore \theta = \int_0^L \frac{M dL}{EI} = \frac{TL}{EI}$$

where L = total length of the wire = $2\pi Rn$.

$$\therefore \theta = \frac{T 2\pi Rn}{E} \times \frac{4}{\pi r^4}$$

i.e. $\text{wind-up angle } \theta = \frac{8T Rn}{Er^4} \quad (12.4)$

N.B. The stress formulae derived above are slightly inaccurate in practice, particularly for small D/d ratios, since they ignore the higher stress produced on the inside of the coil due to the high curvature of the wire. “Wahl” correction factors are therefore introduced – see 307.

12.3. Open-coiled helical spring subjected to axial load W

(a) *Deflection*

In an open-coiled spring the coils are no longer so close together that the effect of the helix angle α can be neglected and the spring is subjected to comparable bending and twisting effects. The axial load W can now be considered as a direct load W acting on the spring at the mean radius R , together with a couple WR about AB (Fig. 12.3). This couple has a component about AX of $WR \cos \alpha$ tending to twist the section, and a component about AY

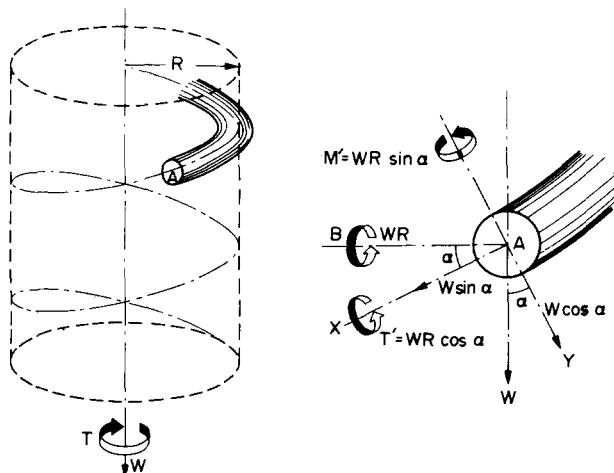


Fig. 12.3. Open-coiled helical spring.

of $WR \sin \alpha$ tending to reduce the curvature of the coils, i.e. a bending effect. Once again the shearing effect of W across the spring section is neglected as being very small in comparison with the other effects.

Thus

$$T' = WR \cos \alpha \quad \text{and} \quad M' = WR \sin \alpha$$

Now, the total strain energy, neglecting shear,

$$\begin{aligned} U &= \frac{T^2 L}{2GJ} + \frac{M^2 L}{2EI} \quad (\text{see §§ 11.3 and 11.4}) \\ &= \frac{L (WR \cos \alpha)^2}{2GJ} + \frac{L (WR \sin \alpha)^2}{2EI} \\ &= \frac{LW^2 R^2}{2} \left[\frac{\cos^2 \alpha}{GJ} + \frac{\sin^2 \alpha}{EI} \right] \end{aligned} \quad (12.5)$$

and this must equal the total work done $\frac{1}{2} W\delta$.

$$\therefore \frac{1}{2} W\delta = \frac{LW^2 R^2}{2} \left[\frac{\cos^2 \alpha}{GJ} + \frac{\sin^2 \alpha}{EI} \right]$$

From the helix form of Fig. 12.4

$$2\pi Rn = L \cos \alpha$$

$$\therefore L = 2\pi Rn \sec \alpha$$

$$\therefore \text{deflection } \delta = 2\pi n WR^3 \sec \alpha \left[\frac{\cos^2 \alpha}{GJ} + \frac{\sin^2 \alpha}{EI} \right] \quad (12.6)$$

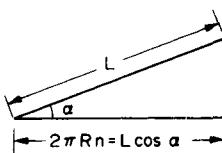


Fig. 12.4.

Since the stiffness of a spring S is normally defined as the value of W required to produce unit deflection,

$$\text{stiffness } S = \frac{W}{\delta}$$

$$\therefore \frac{1}{S} = \frac{\delta}{W} = 2\pi n R^3 \sec \alpha \left[\frac{\cos^2 \alpha}{GJ} + \frac{\sin^2 \alpha}{EI} \right] \quad (12.7)$$

Alternatively, the deflection in the direction of W is given by Castigliano's theorem (see § 11.11) as

$$\begin{aligned} \delta &= \frac{\partial U}{\partial W} = \frac{\partial}{\partial W} \left[\frac{LW^2 R^2}{2} \left(\frac{\cos^2 \alpha}{GJ} + \frac{\sin^2 \alpha}{EI} \right) \right] \\ &= LWR^2 \left[\frac{\cos^2 \alpha}{GJ} + \frac{\sin^2 \alpha}{EI} \right] \end{aligned}$$

and with $L = 2\pi R n \sec \alpha$

$$\delta = 2\pi n WR^3 \sec \alpha \left[\frac{\cos^2 \alpha}{GJ} + \frac{\sin^2 \alpha}{EI} \right] \quad (12.8)$$

This is the same equation as obtained previously and illustrates the flexibility and ease of application of Castigliano's energy theorem.

(b) Maximum stress

The material of the spring is subjected to combined bending and torsion, the maximum stresses in each mode of loading being determined from the appropriate theory.

From the bending theory

$$\sigma = \frac{My}{I} \quad \text{with} \quad M = WR \sin \alpha$$

and from the torsion theory

$$\tau = \frac{Tr}{J} \quad \text{with} \quad T = WR \cos \alpha$$

The principal stresses at any point can then be obtained analytically or graphically using the procedures described in § 13.4.

(c) Angular rotation

Consider an imaginary axial torque T applied to the spring, together with W producing an angular rotation θ of one end of the spring relative to the other.

The combined twisting moment on the spring cross-section is then

$$\bar{T} = WR \cos \alpha + T \sin \alpha$$

and the combined bending moment

$$\bar{M} = T \cos \alpha - WR \sin \alpha$$

The total strain energy of the system is then

$$\begin{aligned} U &= \frac{\bar{T}^2 L}{2GJ} + \frac{\bar{M}^2 L}{2EI} \\ &= \frac{(WR \cos \alpha + T \sin \alpha)^2 L}{2GJ} + \frac{(T \cos \alpha - WR \sin \alpha)^2 L}{2EI} \end{aligned}$$

Now from Castiglano's theorem the angle of twist in the direction of the axial torque T is given by $\theta = \frac{\partial U}{\partial T}$ and since $T = 0$ all terms including T may be ignored.

$$\begin{aligned} \therefore \theta &= \frac{2WR \cos \alpha \sin \alpha L}{2GJ} + \frac{(-2WR \sin \alpha \cos \alpha)L}{2EI} \\ &= WRL \cos \alpha \sin \alpha \left[\frac{1}{GJ} - \frac{1}{EI} \right] \end{aligned}$$

$$\text{i.e. } \theta = 2\pi n WR^2 \sin \alpha \left[\frac{1}{GJ} - \frac{1}{EI} \right] \quad (12.9)$$

12.4. Open-coiled helical spring subjected to axial torque T

(a) Wind-up angle

When an axial torque T is applied to an open-coiled helical spring it has components as shown in Fig. 12.5, i.e. a torsional component $T \sin \alpha$ about AX and a flexural (bending) component $T \cos \alpha$ about AY , the latter tending to increase the curvature of the coils.

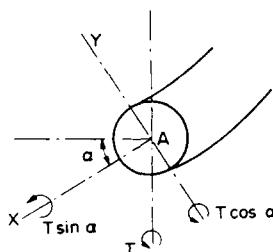


Fig. 12.5. Open-coiled helical spring subjected to axial torque T .

As for the close-coiled spring the total strain energy is given by

$$\begin{aligned}\text{strain energy } U &= \frac{T^2 L}{2GJ} + \frac{M^2 L}{2EI} \\ &= \frac{L}{2} \left[\frac{(T \sin \alpha)^2}{GJ} + \frac{(T \cos \alpha)^2}{EI} \right] \\ &= \frac{T^2 L}{2} \left[\frac{\sin^2 \alpha}{GJ} + \frac{\cos^2 \alpha}{EI} \right]\end{aligned}\quad (12.10)$$

and this is equal to the work done by T , namely, $\frac{1}{2} T \theta$, where θ is the angle turned through by one end relative to the other, i.e. the wind-up angle of the spring.

$$\therefore \frac{1}{2} T \theta = \frac{1}{2} T^2 L \left[\frac{\sin^2 \alpha}{GJ} + \frac{\cos^2 \alpha}{EI} \right]$$

and, with $L = 2\pi R n \sec \alpha$ as before,

$$\text{wind-up angle } \theta = 2\pi n R T \sec \alpha \left[\frac{\sin^2 \alpha}{GJ} + \frac{\cos^2 \alpha}{EI} \right] \quad (12.11)$$

(b) Maximum stress

The maximum stress in the spring material will be found by the procedure outlined in § 12.3(b) with a bending moment of $T \cos \alpha$ and a torque of $T \sin \alpha$ applied to the section.

(c) Axial deflection

Assuming an imaginary axial load W applied to the spring the total strain energy is given by eqn. (11.5) as

$$U = \frac{(WR \cos \alpha + T \sin \alpha)^2 L}{2GJ} + \frac{(T \cos \alpha - WR \sin \alpha)^2 L}{2EI}$$

Now from Castigliano's theorem the deflection in the direction of W is given by

$$\begin{aligned}\delta &= \frac{\partial U}{\partial W} \\ &= TRL \cos \alpha \sin \alpha \left[\frac{1}{GJ} - \frac{1}{EI} \right] \quad \text{when } W = 0 \\ \therefore \text{deflection } \delta &= 2\pi n TR^2 \sin \alpha \left[\frac{1}{GJ} - \frac{1}{EI} \right]\end{aligned}\quad (12.12)$$

12.5. Springs in series

If two springs of different stiffness are joined end-on and carry a common load W , they are said to be *connected in series* and the combined stiffness and deflection are given by the following equations.

$$\begin{aligned}\text{Deflection } &= \frac{W}{S} = \delta_1 + \delta_2 = \frac{W}{S_1} + \frac{W}{S_2} \\ &= W \left[\frac{1}{S_1} + \frac{1}{S_2} \right] \\ \therefore \quad &\frac{1}{S} = \frac{1}{S_1} + \frac{1}{S_2}\end{aligned}\tag{12.13}$$

and $\text{stiffness } S = \frac{S_1 S_2}{S_1 + S_2}$

$$\tag{12.14}$$

12.6. Springs in parallel

If two springs are joined in such a way that they have a common deflection δ they are said to be *connected in parallel*. In this case the load carried is shared between the two springs and

$$\text{total load } W = W_1 + W_2 \tag{1}$$

Now $\delta = \frac{W}{S} = \frac{W_1}{S_1} = \frac{W_2}{S_2}$

$$\tag{12.15}$$

so that $W_1 = \frac{S_1 W}{S}$ and $W_2 = \frac{S_2 W}{S}$

Substituting in eqn. (1)

$$\begin{aligned}W &= \frac{S_1 W}{S} + \frac{S_2 W}{S} \\ &= \frac{W}{S} \left[S_1 + S_2 \right]\end{aligned}$$

i.e. $\text{combined stiffness } S = S_1 + S_2$

$$\tag{12.16}$$

12.7. Limitations of the simple theory

Whilst the simple torsion theory can be applied successfully to bars with small curvature without significant error the theory becomes progressively more inappropriate as the curvatures increase and become high as in most helical springs. The stress and deflection equations derived in the preceding sections, are, therefore, slightly inaccurate in practice, particularly for small D/d ratios. For accurate assessment of stresses and deflections account should be taken of the influence of curvature and slope by applying factors due to Wahl[†] and Ancker and Goodier[‡]. These are discussed in Roark and Young[§] where the more accurate

[†] A. M. Wahl, *Mechanical Springs*, 2nd edn. (McGraw-Hill, New York 1963).

[‡] C. J. Ancker (Jr) and J. N. Goodier, "Pitch and curvature correction for helical springs", *ASME J. Appl. Mech.*, 25(4), Dec. 1958.

[§] R. J. Roark and W. C. Young, *Formulas for Stress and Strain*, 5th edn. (McGraw-Hill, Kogakusha, 1965).

expressions for circular, square and rectangular section springs are introduced. For the purposes of this text it is considered sufficient to indicate the use of these factors on circular section wire.

For example, Ancker and Goodier write the stress and deflection equations for circular section springs subjected to an axial load W in the following form (which can be related directly to eqns. (12.1) and (12.2)).

$$\text{Maximum stress } \tau_{\max} = K_1 \left(\frac{2WR}{\pi r^3} \right) = K_1 \left(\frac{8WD}{\pi d^3} \right)$$

$$\text{and deflection } \delta = K_2 \left(\frac{4WR^3n}{Gr^4} \right) = K_2 \left(\frac{8WD^3n}{Gd^4} \right)$$

$$\text{where } K_1 = \left[1 + \frac{5}{8} \left(\frac{d}{R} \right) + \frac{7}{32} \left(\frac{d}{R} \right)^2 \right]$$

$$\text{and } K_2 = \left[1 - \frac{3}{64} \left(\frac{d}{R} \right)^2 + \frac{(3+v)}{2(1+v)} (\tan \alpha)^2 \right]$$

where α is the pitch angle of the spring.

In an exactly similar way Wahl also proposes the introduction of correction factors which are related to the so-called spring index $C = D/d$.

Thus, for central load W :

$$\text{maximum stress } \tau_{\max} = K \left[\frac{8WD}{\pi d^3} \right]$$

$$\text{with } K = \frac{(4C-1)}{(4C-4)} + \frac{0.615}{C}$$

The British Standard for spring design, BS1726, quotes a simpler equation for K , namely:

$$K = \left[\frac{C+0.2}{C-1} \right]$$

The Standard also makes the point that the influence of the correction factors is often small in comparison with the uncertainty regarding what should be selected as the true number of working coils (depending on the method of support, etc).

Values of K for different ratios of spring index are given in Fig. 12.6 on page 308.

12.8. Extension springs – initial tension.

The preceding laws and formulae derived for compression springs apply equally to extension springs except that the latter are affected by initial tension. When springs are closely wound a force is required to hold the coils together and this can seldom be controlled to a greater accuracy than $\pm 10\%$. This does not increase the ultimate load capacity but must be included in the stress calculation. As an approximate guide, the initial tension obtained in hand-coiled commercial-quality springs is taken to be equivalent to the rate of the spring, although this can be far exceeded if special coiling methods are used.

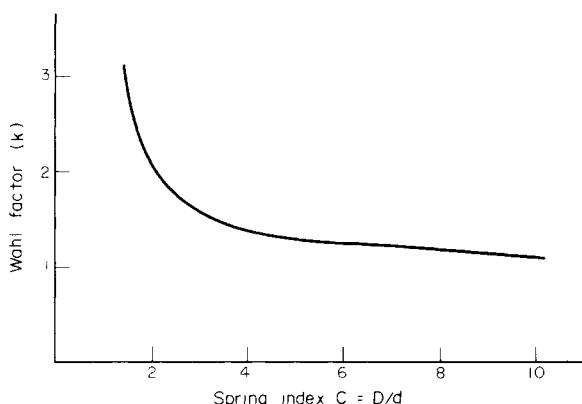


Fig. 12.6. Wahl correction factors for maximum shear stress.

12.9. Allowable stresses

As a rough approximation, the torsional elastic limit of commercial wire materials is taken to be 40 % of the tensile strength. This is applied equally to ferrous and non-ferrous materials such as phosphor bronze and brass.

Typical values of allowable stress for hard-drawn spring steel piano wire based on the above assumption are given in Table 12.1.[†] These represent the corrected stress and generally should not be exceeded unless exceptionally high grade materials are used.

TABLE 12.1. Allowable stresses for hard-drawn steel spring wire

Wire size SWG	Allowable stress (MN/m ²)	
	Compression/Extension	Torsion
44-39	1134	1409
48-35	1079	1340
34-31	1031	1272
30-28	983	1203
27-24	928	1169
23-18	859	1066
17-13	770	963
12-10	688	859
9-7	619	756
6-5	550	688
4-3	516	619

Care must be exercised in the application of the quoted values bearing in mind the presence of any irregularities in the form or clamping method and the duty the spring is to perform. For example the quoted values may be far too high for springs to operate at high frequency, particularly in the presence of stress raisers, when fatigue failure would soon result. Under

[†] Spring Design, *Engineering Materials And Design*, Feb. 1980.

such conditions a high-grade annealed spring steel suitably heat-treated should be considered.

A useful comparison of the above theories together with further ones due to Rover, Honegger, Göhner and Bergsträsser is given in the monograph† *Helical Springs*, which then goes on to consider the effect of pitch angle, failure considerations, vibration frequency and spring surge (speed of propagation of wave along the axis of a spring).

12.10. Leaf or carriage spring: semi-elliptic

The principle of using a beam in bending as a spring has been known for many years and widely used in motor-vehicle applications. If the beam is arranged as a simple cantilever, as in Fig. 12.7a, it is called a *quarter-elliptic* spring, and if as a simply supported beam with central load, as in Fig. 12.7b, it is termed a *half* or *semi-elliptic* spring. The latter will be discussed first.

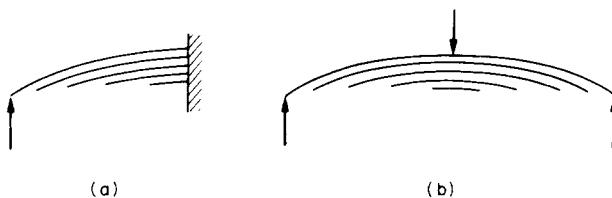


Fig. 12.7. (a) Quarter-elliptic, (b) semi-elliptic, carriage springs.

(a) Maximum stress

Consider the semi-elliptic leaf spring shown in Fig. 12.8. With a constant thickness t this design of spring gives a uniform stress throughout and is therefore economical in both material and weight.

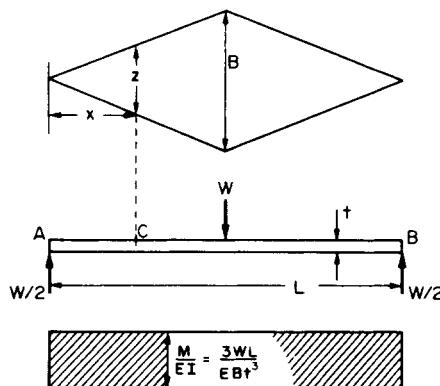


Fig. 12.8. Semi-elliptic leaf spring.

† J. R. Finniecome, *Helical Springs*. Mechanical World Monograph 56 (Emmott & Co., Manchester 1949).

By proportions

$$\frac{z}{x} = \frac{B}{L/2} \quad \therefore z = \frac{2Bx}{L}$$

$$\text{Bending moment at } C = \frac{Wx}{2} \quad \text{and} \quad I = \frac{zt^3}{12} = \frac{2Bxt^3}{12L}$$

Therefore from the bending theory the stress set up at any section is given by

$$\begin{aligned}\sigma &= \frac{My}{I} = \frac{Wx}{2} \times \frac{t}{2} \times \frac{12L}{2Bxt^3} \\ &= \frac{3WL}{2Bt^2}\end{aligned}$$

i.e. the bending stress in a semi-elliptic leaf spring is independent of x and equal to

$$\frac{3WL}{2Bt^2} \quad (12.17)$$

If the spring is constructed from strips and placed one on top of the other as shown in Fig. 12.9, uniform stress conditions are retained, since if the strips are cut along XX and replaced side by side, the equivalent leaf spring is obtained as shown.

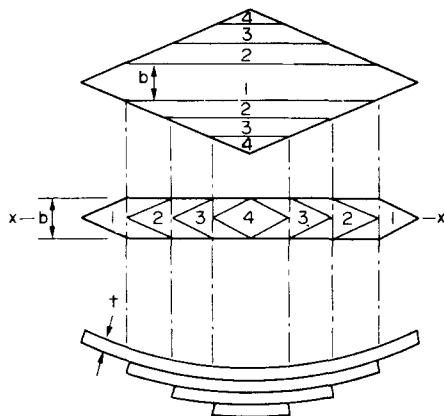


Fig. 12.9. Semi-elliptic carriage spring showing initial pre-forming.

Such a spring is then termed a *carriage spring* with n strips of width b , i.e. $B = nb$. Therefore the bending stress in a semi-elliptic carriage spring is

$$\frac{3WL}{2nbt^2} \quad (12.18)$$

The diamond shape of the leaf spring could also be obtained by varying the thickness, but this type of spring is difficult to manufacture and has been found unsatisfactory in practice.

(b) Deflection

From the simple bending theory

$$\frac{M}{I} = \frac{E}{R} \quad \therefore R = \frac{EI}{M}$$

$$R = E \times \frac{2Bxt^3}{12L} \times \frac{2}{Wx} = \frac{EBt^3}{3WL} \quad (12.19)$$

i.e. for a given spring and given load, R is constant and the spring bends into the arc of a circle.

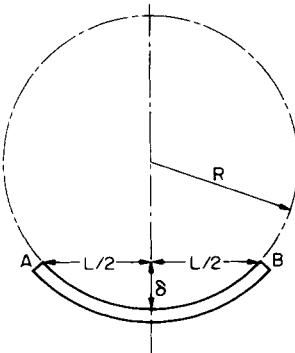


Fig. 12.10.

From the properties of intersecting chords (Fig. 12.10)

$$\delta(2R - \delta) = \frac{L}{2} \times \frac{L}{2}$$

Neglecting δ^2 as the product of small quantities

$$\begin{aligned} \delta &= \frac{L^2}{8R} \\ &= \frac{L^2}{8} \times \frac{3WL}{EBt^3} \end{aligned}$$

i.e. deflection of a semi-elliptic *leaf* spring

$$\delta = \frac{3WL^3}{8EBt^3} \quad (12.20)$$

But $B = nb$, so that the deflection of a semi-elliptic *carriage* spring is given by

$$\delta = \frac{3WL^3}{8Enbt^3} \quad (12.21)$$

(c) Proof load

The proof load of a leaf or carriage spring is the load which is required to straighten the plates from their initial preformed position. From eqn. 12.18 the maximum bending stress for

any given load W is

$$\sigma = \frac{3WL}{2nbt^2}$$

Thus if σ_p denotes the stress corresponding to the application of the proof load W_p

$$W_p = \frac{2nbt^2}{3L} \sigma_p \quad (12.22)$$

Now from eqn. (12.19) and inserting $B = nb$, the load W which would produce bending of a flat carriage spring to some radius R is given by

$$W = \frac{Enbt^3}{3RL}$$

Conversely, therefore, the load which is required to straighten a spring from radius R will be of the same value,

i.e. $W_p = \frac{Enbt^3}{3RL}$

Substituting for $R = \frac{L^2}{8\delta}$

$$\therefore \text{proof load } W_p = \frac{8Enbt^3}{3L^3} \delta_p \quad (12.23)$$

where δ_p is the initial central “deflection” of the spring.

Equating eqns. (12.22) and (12.23),

$$\frac{2nbt^2}{3L} \sigma_p = \frac{8Enbt^3}{3L^3} \delta_p$$

i.e. $\text{proof stress } \sigma_p = \frac{4tE}{L^2} \delta_p \quad (12.24)$

For a given spring material the limiting value of σ_p will be known as will the value of E . The above equation therefore yields the correct relationship between the thickness and initial curvature of the spring plates.

12.11. Leaf or carriage spring: quarter-elliptic

(a) Maximum stress

Consider the *quarter-elliptic* leaf and carriage springs shown in Fig. 12.11. In this case the equations for the semi-elliptic spring of the previous section are modified to

$$z = \frac{Bx}{L} \quad \text{and} \quad \text{B.M. at } C = Wx$$

$$\therefore I = \frac{zt^3}{12} = \frac{Bxt^3}{12L}$$

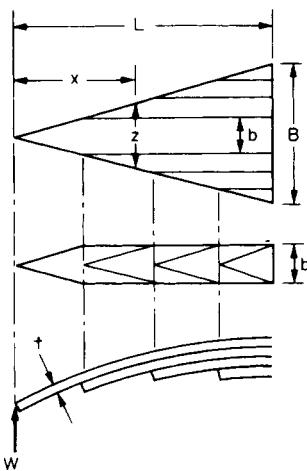


Fig. 12.11. Quarter-elliptic leaf and carriage springs.

Now

$$\sigma = \frac{My}{I} = \frac{Wxt}{2} \times \frac{12L}{Bxt^3} = \frac{6WL}{Bt^2}$$

Therefore the maximum bending stress for a quarter-elliptic *leaf* spring

$$= \frac{6WL}{Bt^2} \quad (12.25)$$

and the maximum bending stress for a quarter-elliptic *carriage* spring

$$= \frac{6WL}{nbt^2} \quad (12.26)$$

(b) Deflection

With B.M. at $C = Wx$ and replacing $L/2$ by L in the proof of §12.7(b),

$$\delta = \frac{L^2}{2R}$$

and

$$R = \frac{EI}{M} = \frac{E}{Wx} \times \frac{Bxt^3}{12L} = \frac{Ebt^3}{12WL}$$

$$\therefore \delta = \frac{L^2}{2} \times \frac{12WL}{EBt^3} = \frac{6WL^3}{EBt^3}$$

Therefore deflection of a quarter-elliptic *leaf* spring

$$= \frac{6WL^3}{EBt^3} \quad (12.27)$$

and deflection of a quarter-elliptic carriage spring

$$= \frac{6WL^3}{Enbt^3} \quad (12.28)$$

12.12. Spiral spring

(a) Wind-up angle

Spiral springs are normally constructed from thin rectangular-section strips wound into a spiral in one plane. They are often used in clockwork mechanisms, the winding torque or moment being applied to the central spindle and the other end firmly anchored to a pin at the outside of the spiral. Under the action of this central moment all sections of the spring will be subjected to uniform bending which tends to reduce the radius of curvature at all points.

Consider now the spiral spring shown in Fig. 12.12.

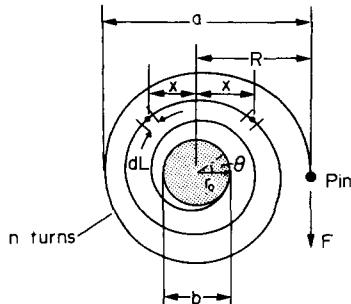


Fig. 12.12. Spiral spring.

Let M = winding moment applied to the spring spindle, R = radius of spring from spindle to pin, a = maximum dimension of the spring from the pin, B = breadth of the material of the spring, t = thickness of the material of the spring, and b = diameter of the spindle.

Assuming the polar equation of the spiral to be that of an Archimedean spiral,

$$r = r_0 + \left(\frac{A}{2\pi} \right) \theta \quad \text{where } A \text{ is some constant}$$

$$\text{When } \theta = 0, \quad r = r_0 = \frac{b}{2}$$

and for the n th turn, $\theta = 2n\pi$ and

$$r = \frac{a}{2} = \frac{b}{2} + \left(\frac{A}{2\pi} \right) 2n\pi$$

$$\therefore A = \frac{(a-b)}{2n}$$

i.e. the equation to the spiral is

$$r = \frac{b}{2} + \frac{(a-b)}{4\pi n} \theta \quad (12.29)$$

When a torque or winding couple M is applied to the spindle a resistive force F will be set up at the pin such that

$$\text{winding couple } M = F \times R$$

Consider now two small elements of material of length dl at distance x to each side of the centre line (Fig. 12.12).

For small deflections, from Mohr's area-moment method the change in slope between two points is

$$\left(\frac{M}{EI} \right) dL \quad (\text{see } \S 5.7)$$

For the portion on the left,

$$\text{change in slope} = d\theta_1 = \frac{F(R+x)dL}{EI}$$

and similarly for the right-hand portion,

$$\text{change in slope} = d\theta_2 = \frac{F(R-x)dL}{EI}$$

The sum of these changes in slope is thus

$$\begin{aligned} d\theta_1 + d\theta_2 &= \frac{F(R+x)dL}{EI} + \frac{F(R-x)dL}{EI} \\ &= \frac{2FRdL}{EI} \end{aligned}$$

If this is integrated along the length of the spring the result obtained will be twice the total change in slope along the spring, i.e. twice the angle of twist.

$$\therefore \text{angle of twist} = \frac{1}{2} \int_0^L \frac{2FRdL}{EI} = \frac{FRL}{EI} = \frac{ML}{EI}$$

where M is the applied winding moment and L the total length of the spring.

$$\begin{aligned} \text{Now } L &= \int_0^L dL = \int_0^{2\pi n} r d\theta = \int_0^{2\pi n} \frac{b}{2} + \frac{(a-b)}{4\pi n} \theta d\theta \\ &= \left[\frac{b\theta}{2} + \frac{(a-b)}{4\pi n} \frac{\theta^2}{2} \right]_0^{2\pi n} = \left[\frac{2nb\pi}{2} + \frac{(a-b)}{4\pi n} \frac{(2n\pi)^2}{2} \right] \\ &= \pi n \left[b + \frac{(a-b)}{2} \right] \\ &= \frac{\pi n}{2} \left[a + b \right] \end{aligned} \quad (12.30)$$

Therefore the wind-up angle of a spiral spring is

$$\theta = \frac{M}{EI} \left[\frac{\pi n}{2} (a + b) \right] \quad (12.31)$$

(b) Maximum stress

The maximum bending stress set up in the spring will be at the point of greatest bending moment, since the material of the spring is subjected to pure bending.

$$\text{Maximum bending moment} = F \times a$$

$$\therefore \text{maximum bending stress} = \frac{My}{I} = \frac{Fa(t/2)}{I}$$

But, for rectangular-section spring material of breadth B and thickness t ,

$$I = \frac{Bt^3}{12}$$

$$\therefore \sigma_{\max} = \frac{Fat}{2} \times \frac{12}{Bt^3} = \frac{6Fa}{Bt^2}$$

$$\text{Now the applied moment} \quad M = F \times R$$

$$\therefore \text{maximum bending stress } \sigma_{\max} = \frac{6Ma}{RBt^2} \quad (12.32)$$

or, assuming $a = 2R$,

$$\sigma_{\max} = \frac{12M}{Bt^2} \quad (12.33)$$

Examples

Example 12.1

A close-coiled helical spring is required to absorb 2.25×10^3 joules of energy. Determine the diameter of the wire, the mean diameter of the spring and the number of coils necessary if:

- (a) the maximum stress is not to exceed 400 MN/m^2 ;
- (b) the maximum compression of the spring is limited to 250 mm ;
- (c) the mean diameter of the spring can be assumed to be eight times that of the wire.

How would the answers change if appropriate Wahl factors are introduced?

For the spring material $G = 70 \text{ GN/m}^2$.

Solution

The spring is required to absorb 2.25×10^3 joules or 2.25 kNm of energy.

$$\therefore \text{work done} = \frac{1}{2} W\delta = 2.25 \times 10^3$$

But δ is limited to 250 mm.

$$\therefore \frac{1}{2} W \times 250 \times 10^{-3} = 2.25 \times 10^3$$

$$W = \frac{2.25 \times 10^3 \times 2}{250 \times 10^{-3}} = 18 \text{ kN}$$

Thus the maximum load which can be carried by the spring is 18 kN.

Now the maximum stress is not to exceed 400 MN/m²; therefore from eqn. (12.1),

$$\frac{2WR}{\pi r^3} = 400 \times 10^6$$

But $R = 8r$

$$\therefore \frac{2 \times 18 \times 10^3 \times 8r}{\pi r^3} = 400 \times 10^6$$

$$r^2 = \frac{2 \times 18 \times 10^3 \times 8}{\pi \times 400 \times 10^6} = 229 \times 10^{-6}$$

$$r = 15.1 \times 10^{-3} = 15.1 \text{ mm}$$

The required diameter of the wire, for practical convenience, is, therefore,

$$2 \times 15 = 30 \text{ mm}$$

and, since $R = 8r$, the required mean diameter of the coils is

$$8 \times 30 = 240 \text{ mm}$$

Now total deflection

$$\delta = \frac{4WR^3n}{Gr^4} = 250 \text{ mm}$$

$$n = \frac{250 \times 10^{-3} \times 70 \times 10^9 \times (15 \times 10^{-3})^4}{4 \times 18 \times 10^3 \times (120 \times 10^{-3})^3} \\ = 7.12$$

Again from practical considerations, the number of complete coils necessary = 7. (If 8 coils were chosen the maximum deflection would exceed 250 mm.)

The effect of introducing Wahl correction factors is determined as follows:

From the given data $C = D/d = 8 \quad \therefore$ From Fig. 12.6 $K = 1.184$.

Now $\tau_{\max} = K \left[\frac{8WD}{\pi d^3} \right] = K \left[\frac{2WR}{\pi r^3} \right] = 400 \times 10^6$

$$\therefore 400 \times 10^6 = \frac{1.184 \times 2 \times 18 \times 10^3 \times 8r}{\pi r^3}$$

$$\therefore r^2 = \frac{1.184 \times 2 \times 18 \times 10^3 \times 8}{\pi \times 400 \times 10^6} = 271.35 \times 10^{-6}$$

$$\therefore r = 16.47 \times 10^{-3} = 16.47 \text{ mm}$$

i.e. for practical convenience $d = 2 \times 16.5 = 33 \text{ mm}$,
and since $D = 8d$, $D = 8 \times 33 = 264 \text{ mm}$.

Total deflection $\delta = \frac{4WR^3n}{Gr^4} = 250 \text{ mm.}$

$$\therefore n = \frac{250 \times 10^{-3} \times 70 \times 10^9 \times (16.5 \times 10^{-3})^4}{4 \times 18 \times 10^3 \times (132 \times 10^{-3})^3}$$

$$= 7.83.$$

Although this is considerably greater than the value obtained before, the number of complete coils required remains at 7 if maximum deflection is strictly limited to 250 mm.

Example 12.2

A compression spring is required to carry a load of 1.5 kN with a limiting shear stress of 250 MN/m^2 . If the spring is to be housed in a cylinder of 70 mm diameter estimate the size of spring wire required. Use appropriate Wahl factors in your solution.

Solution

Maximum shear stress $\tau_{\max} = K \left[\frac{8WD}{\pi d^3} \right]$

i.e. $d^3 = \frac{8WDK}{\pi \tau_{\max}}$

$$\therefore d = \sqrt[3]{\frac{8 \times 1.5 \times 10^3 DK}{\pi \times 250 \times 10^6}}$$

$$= 2.481 \times 10^{-2} \sqrt[3]{DK} \quad (1)$$

Unfortunately, this cannot readily be solved for d since K is dependent on d , and D the mean diameter is not known except so far as its maximum value is limited to $(70 - d) \text{ mm}$.

If, therefore, as a first approximation, D is taken to be 70 mm and K is assumed to be 1, a rough order of magnitude is obtained for d from the above equation (1).

i.e. $d = 2.481 \times 10^{-2} (70 \times 10^{-3} \times 1)^{\frac{1}{3}}$

$$= 10.22 \text{ mm.}$$

It is now appropriate to apply a graphical solution to the determination of the precise value of d using assumed values of d close to the above rough value, reading the appropriate value of K from Fig. 12.6 and calculating the corresponding d value from eqn. (1).

Assumed d	D $(= 70 - d)$	C $(= D/d)$	K	Calculated d (from eqn. (1))
10	60	6.0	1.25	10.46
10.5	59.5	5.67	1.27	10.49
11.0	59	5.36	1.29	10.51
11.5	58.5	5.09	1.304	10.522

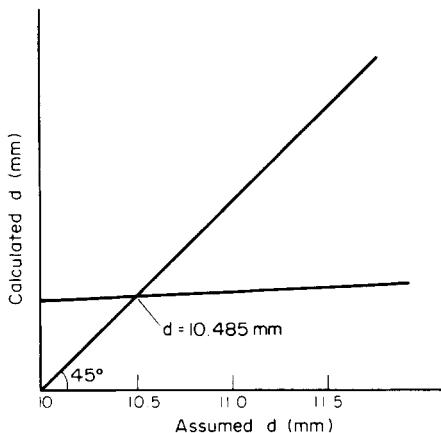


Fig. 12.13.

Plotting the assumed and calculated values gives the nearly horizontal line of Fig. 12.13.

The other line is that of the required solution, i.e. it represents all points along which the assumed and calculated d values are the same (i.e. at 45° to the axes). Thus, where this line crosses the previously plotted line is the required value of d , namely 10.485 mm.

The spring wire must therefore have a minimum diameter of **10.485 mm** and a mean diameter of $70 - 10.485 = \mathbf{59.51 \text{ mm}}$.

Example 12.3

A close-coiled helical spring, constructed from wire of 10 mm diameter and with a mean coil diameter of 50 mm, is used to join two shafts which transmit 1 kilowatt of power at 4000 rev/min. If the number of turns of the spring is 10 and the modulus of elasticity of the spring material is 210 GN/m^2 determine:

- the relative angle of twist between the two ends of the spring;
- the maximum stress set up in the spring material.

Solution

$$\text{Power} = T\omega = 1000 \text{ W}$$

$$T = \frac{1000 \times 60}{4000 \times 2\pi} = 2.39 \text{ N m}$$

Now the wind-up angle of the spring, from eqn. (12.4),

$$= \frac{8TRn}{Er^4}$$

$$\begin{aligned} \theta &= \frac{8 \times 2.39 \times 25 \times 10^{-3} \times 10}{210 \times 10^9 \times (5 \times 10^{-3})^4} \\ &= 0.036 \text{ radian} = 2.1^\circ \end{aligned}$$

The maximum stress is then given by eqn. (12.3),

$$\begin{aligned}\sigma_{\max} &= \frac{4T}{\pi r^3} = \frac{4 \times 2.39}{\pi \times (5 \times 10^{-3})^3} \\ &= 24.3 \times 10^6 = 24.3 \text{ MN/m}^2\end{aligned}$$

Example 12.4

Show that the ratio of extension per unit axial load to angular rotation per unit axial torque of a close-coiled helical spring is directly proportional to the square of the mean diameter, and hence that the constant of proportionality is $\frac{1}{4}(1 + v)$.

If Poisson's ratio $v = 0.3$, determine the angular rotation of a close-coiled helical spring of mean diameter 80 mm when subjected to a torque of 3 N m, given that the spring extends 150 mm under an axial load of 250 N.

Solution

From eqns. (12.2) and (12.4)

$$\delta = \frac{4WR^3n}{Gr^4} \quad \text{and} \quad \theta = \frac{8TRn}{Er^4}$$

$$\therefore \frac{\delta}{W} = \frac{4R^3n}{Gr^4} \quad \text{and} \quad \frac{\theta}{T} = \frac{8Rn}{Er^4}$$

$$\therefore \frac{\delta/W}{\theta/T} = \frac{4R^3n}{Gr^4} \times \frac{Er^4}{8Rn} = \frac{R^2E}{2G} = \frac{D^2E}{8G}$$

But $E = 2G(1 + v)$

$$\therefore \frac{\delta/W}{\theta/T} = \frac{D^2}{8} \times \frac{2G(1 + v)}{G} = \frac{1}{4}(1 + v)D^2 \quad (1)$$

Thus the ratio is directly proportional to D^2 and the constant of proportionality is $\frac{1}{4}(1 + v)$.

From eqn. (1)

$$\frac{T\delta}{W\theta} = \frac{1}{4}(1 + v)D^2$$

$$\begin{aligned}\therefore \frac{3 \times 150 \times 10^{-3}}{250 \times \theta} &= \frac{1}{4}(1 + 0.3)(80 \times 10^{-3})^2 \\ \theta &= \frac{3 \times 150 \times 10^{-3} \times 4}{250 \times 1.3 \times 6400 \times 10^{-6}} \\ &= 0.865 \text{ radian} = 49.6^\circ\end{aligned}$$

The required angle of rotation is 49.6° .

Example 12.5

- (a) Determine the load required to produce an extension of 8 mm on an open coiled helical spring of 10 coils of mean diameter 76 mm, with a helix angle of 20° and manufactured from

wire of 6 mm diameter. What will then be the bending and shear stresses in the surface of the wire? For the material of the spring, $E = 210 \text{ GN/m}^2$ and $G = 70 \text{ GN/m}^2$.

(b) What would be the angular twist at the free end of the above spring when subjected to an axial torque of 1.5 N m?

Solution

(a) From eqn. (12.6) the extension of an open-coiled helical spring is given by

$$\delta = 2\pi n W R^3 \sec \alpha \left[\frac{\cos^2 \alpha}{GJ} + \frac{\sin^2 \alpha}{EI} \right]$$

Now $I = \frac{\pi d^4}{64} = \frac{\pi \times (6 \times 10^{-3})^4}{64} = 63.63 \times 10^{-12} \text{ m}^4$

and $J = \frac{\pi d^4}{32} = 127.26 \times 10^{-12} \text{ m}^4$

$$\therefore 8 \times 10^{-3} = 2\pi \times 10 \times W \times (38 \times 10^{-3})^3 \sec 20^\circ \left[\frac{\cos^2 20^\circ}{70 \times 10^9 \times 127.26 \times 10^{-12}} + \frac{\sin^2 20^\circ}{210 \times 10^9 \times 63.63 \times 10^{-12}} \right]$$

$$= \frac{20\pi W \times 38^3 \times 10^{-9}}{0.9397} \left[\frac{(0.9397)^2}{8.91} + \frac{(0.342)^2}{13.36} \right]$$

$$= \frac{20\pi W \times 38^3 \times 10^{-9}}{0.9397} [0.1079]$$

$$\therefore W = \frac{8 \times 10^{-3} \times 0.9397}{20\pi \times 38^3 \times 10^{-9} \times 0.1079}$$

$$= 20 \text{ N}$$

The bending moment acting on the spring is

$$WR \sin \alpha = 20 \times 38 \times 10^{-3} \times 0.342$$

$$= 0.26 \text{ N m}$$

$$\therefore \text{bending stress} = \frac{My}{I} = \frac{0.26 \times 3 \times 10^{-3}}{63.63 \times 10^{-12}} = 12.3 \text{ MN/m}^2$$

Similarly, the torque on the spring material is

$$WR \cos \alpha = 20 \times 38 \times 10^{-3} \times 0.9397$$

$$= 0.714 \text{ N m}$$

$$\therefore \text{shear stress} = \frac{Tr}{J} = \frac{0.714 \times 3 \times 10^{-3}}{127.26 \times 10^{-12}}$$

$$= 16.8 \text{ MN/m}^2$$

(b) The wind-up angle of the spring under the action of an axial torque is given by eqn. (12.11):

$$\begin{aligned}\theta &= 2\pi nRT \sec \alpha \left[\frac{\sin^2 \alpha}{GJ} + \frac{\cos^2 \alpha}{EI} \right] \\ &= \frac{2\pi \times 10 \times 38 \times 10^{-3} \times 1.5}{0.9397} \left[\frac{(0.342)^2}{8.91} + \frac{(0.9397)^2}{13.36} \right] \\ &= \frac{2\pi \times 10 \times 38 \times 10^{-3} \times 1.5}{0.9397} [0.0792] \\ &= 0.302 \text{ radian} = 17.3^\circ\end{aligned}$$

Example 12.6

Calculate the thickness and number of leaves of a semi-elliptic carriage spring which is required to support a central load of 2 kN on a span of 1 m if the maximum stress is limited to 225 MN/m² and the central deflection to 75 mm. The breadth of each leaf can be assumed to be 100 mm.

For the spring material $E = 210 \text{ GN/m}^2$.

Solution

From eqn. (12.18),

$$\text{maximum stress} = \frac{3WL}{2nbt^2} = 225 \times 10^6$$

$$\therefore \frac{3 \times 2000 \times 1}{2 \times n \times 100 \times 10^{-3} t^2} = 225 \times 10^6$$

$$nt^2 = \frac{3 \times 2000}{2 \times 100 \times 10^{-3} \times 225 \times 10^6} = 0.133 \times 10^{-3}$$

And from eqn. (12.21),

$$\text{Deflection } \delta = \frac{3WL^3}{8Enbt^3}$$

$$\therefore 75 \times 10^{-3} = \frac{3 \times 2000 \times 1}{8 \times 210 \times 10^9 \times n \times 100 \times 10^{-3} \times t^3}$$

$$\begin{aligned}\therefore nt^3 &= \frac{3 \times 2000}{75 \times 10^{-3} \times 8 \times 210 \times 10^8} \\ &= 0.476 \times 10^{-6}\end{aligned}$$

$$\therefore \frac{nt^3}{nt^2} = t = \frac{0.476 \times 10^{-6}}{0.133 \times 10^{-3}}$$

$$t = 3.58 \times 10^{-3} = 3.58 \text{ mm}$$

and, since $nt^2 = 0.133 \times 10^{-3}$,

$$n = \frac{0.133 \times 10^{-3}}{(3.58 \times 10^{-3})^2} \\ = 10.38$$

The nearest whole number of leaves is therefore 10. However, with $n = 10$, the stress limit would be exceeded and this should be compensated for by increasing the thickness t in the ratio $\sqrt{\left(\frac{10.38}{10}\right)} = 1.02$,

i.e.

$$t = 3.65 \text{ mm}$$

Example 12.7

A flat spiral spring is pinned at the outer end and a winding couple is applied to a spindle attached at the inner end as shown in Fig. 12.11, with $a = 150 \text{ mm}$, $b = 40 \text{ mm}$ and $R = 75 \text{ mm}$. The material of the spring is rectangular in cross-section, 12 mm wide and 2.5 mm thick, and there are 5 turns. Determine:

- (a) the angle through which the spindle turns;
- (b) the maximum bending stress produced in the spring material when a torque of 1.5 N m is applied to the winding spindle.

For the spring material, $E = 210 \text{ GN/m}^2$.

Solution

(a)

$$\text{The angle of twist} = \frac{ML}{EI}$$

where

$$L = \frac{\pi n}{2} (a + b) \\ = \frac{\pi \times 5}{2} (150 + 40) 10^{-3} \\ = 1492.3 \times 10^{-3} \text{ m} = 1.492 \text{ m}$$

$$\therefore \text{angle of twist} = \frac{1.5 \times 1.492 \times 12}{210 \times 10^9 \times 12 \times 2.5^3 \times 10^{-12}} \\ = 0.682 \text{ radian} \\ = 39.1^\circ$$

$$\text{Maximum bending moment} = F \times a$$

where

$$\text{applied moment} = F \times R = 1.5 \text{ N m}$$

i.e.

$$F = \frac{1.5}{75 \times 10^{-3}} = 20 \text{ N}$$

\therefore maximum bending moment $= 20 \times 150 \times 10^{-3} = 3 \text{ N m}$

$$\begin{aligned}\therefore \text{maximum bending stress} &= \frac{My}{I} = \frac{3 \times (t/2)}{I} \\ &= \frac{3 \times 1.25 \times 10^{-3} \times 12}{12 \times 2.5^3 \times 10^{-12}} \\ &= 240 \times 10^6 = 240 \text{ MN/m}^2\end{aligned}$$

Problems

(Take $E = 210 \text{ GN/m}^2$ and $G = 70 \text{ GN/m}^2$ throughout)

12.1 (A/B). A close-coiled helical spring is to have a stiffness of 90 kN/m and to exert a force of 3 kN ; the mean diameter of the coils is to be 75 mm and the maximum stress is not to exceed 240 MN/m^2 . Calculate the required number of coils and the diameter of the steel rod from which the spring should be made.

[E.I.E.] [8, 13.5 mm.]

12.2 (A/B). A close-coiled helical spring is fixed at one end and subjected to axial twist at the other. When the spring is in use the axial torque varies from 0.75 N m to 3 N m , the working angular deflection between these torques being 35° . The spring is to be made from rod of circular section, the maximum permissible stress being 150 MN/m^2 . The mean diameter of the coils is eight times the rod diameter. Calculate the mean coil diameter, the number of turns and the wire diameter.

[B.P.] [48, 6 mm; 24.]

12.3 (A/B). A close-coiled helical compression spring made from round wire fits over the spindle of a plunger and has to work inside a tube. The spindle diameter is 12 mm and the tube is of 25 mm outside diameter and 0.15 mm thickness. The maximum working length of the spring has to be 120 mm and the minimum length 90 mm . The maximum force exerted by the spring has to be 350 N and the minimum force 240 N . If the shearing stress in the spring is not to exceed 600 MN/m^2 find:

- (a) the free length of the spring (i.e. before assembly);
- (b) the mean coil diameter;
- (c) the wire diameter;
- (d) the number of free coils.

[185.4, 18.3, 3 mm; 32.]

12.4 (A/B). A close-coiled helical spring of circular wire and mean diameter 100 mm was found to extend 45 mm under an axial load of 50 N . The same spring when firmly fixed at one end was found to rotate through 90° under a torque of 5.7 N m . Calculate the value of Poisson's ratio for the material.

[C.U.] [0.3.]

12.5 (B). Show that the total strain energy stored in an open-coiled helical spring by an axial load W applied together with an axial couple T is

$$U = (T \cos \theta - WR \sin \theta)^2 \frac{L}{2EI} + (WR \cos \theta + T \sin \theta)^2 \frac{L}{2GJ}$$

where θ is the helix angle and L the total length of wire in the spring, and the sense of the couple is in a direction tending to wind up the spring. Hence, or otherwise, determine the rotation of one end of a spring of helix angle 20° having 10 turns of mean radius 50 mm when an axial load of 25 N is applied, the other end of the spring being securely fixed. The diameter of the wire is 6 mm .

[B.P.] [26°]

12.6 (B). Deduce an expression for the extension of an open-coiled helical spring carrying an axial load W . Take α as the inclination of the coils, d as the diameter of the wire and R as the mean radius of the coils. Find by what percentage the axial extension is underestimated if the inclination of the coils is neglected for a spring in which $\alpha = 25^\circ$. Assume n and R remain constant.

[U.L.] [3.6%]

12.7 (B). An open-coiled spring carries an axial vertical load W . Derive expressions for the vertical displacement and angular twist of the free end. Find the mean radius of an open-coiled spring (angle of helix 30°) to give a vertical displacement of 23 mm and an angular rotation of the loaded end of 0.02 radian under an axial load of 40 N . The material available is steel rod of 6 mm diameter.

[U.L.] [182 mm.]

12.8 (B). A compound spring comprises two close-coiled helical springs having exactly the same initial length when unloaded. The outer spring has 16 coils of 12 mm diameter bar coiled to a mean diameter of 125 mm and the inner spring has 24 coils with a mean diameter of 75 mm. The working stress in each spring is to be the same. Find (a) the diameter of the steel bar for the inner spring and (b) the stiffness of the compound spring.

[I.Mech.E.] [6.48 mm; 7.33 kN/m.]

12.9 (B). A composite spring has two close-coiled helical springs connected in series; each spring has 12 coils at a mean diameter of 25 mm. Find the diameter of the wire in one of the springs if the diameter of wire in the other spring is 2.5 mm and the stiffness of the composite spring is 700 N/m. Estimate the greatest load that can be carried by the composite spring and the corresponding extension for a maximum shearing stress of 180 MN/m².

[U.L.] [44.2 N; 63.2 mm.]

12.10 (B). (a) Derive formulae in terms of load, leaf width and thickness, and number of leaves for the maximum deflection and maximum stress induced in a cantilever leaf spring. (b) A cantilever leaf spring is 750 mm long and the leaf width is to be 8 times the leaf thickness. If the bending stress is not to exceed 210 MN/m² and the spring is not to deflect more than 50 mm under a load of 5 kN, find the leaf thickness, the least number of leaves required, the deflection and the stress induced in the leaves of the spring.

[11.25 mm, say 12 mm; 9.4, say 10; 47 mm, 197.5 MN/m².]

12.11 (B). Make a sketch of a leaf spring showing the shape to which the ends of the plate should be made and give the reasons for doing this. A leaf spring which carries a central load of 9 kN consists of plates each 75 mm wide and 7 mm thick. If the length of the spring is 1 m, determine the least number of plates required if the maximum stress owing to bending is limited to 210 MN/m² and the maximum deflection must not exceed 30 mm. Find, for the number of plates obtained, the actual values of the maximum stress and maximum deflection and also the radius to which the plates should be formed if they are to straighten under the given load.

[U.L.] [14; 200 MN/m², 29.98 mm; 4.2 m.]

12.12 (B). A semi-elliptic laminated carriage spring is 1 m long and 75 mm wide with leaves 10 mm thick. It has to carry a central load of 6 kN with a deflection of 25 mm. Working from first principles find (a) the number of leaves, (b) the maximum induced stress.

[6; 200 MN/m².]

12.13 (B). A semi-elliptic leaf spring has a span of 720 mm and is built up of leaves 10 mm thick and 45 mm wide. Find the number of leaves required to carry a load of 5 kN at mid-span if the stress is not to exceed 225 MN/m², nor the deflection 12 mm. Calculate also the radius of curvature to which the spring must be initially bent if it must just flatten under the application of the above load.

[7; 6.17 m.]

12.14 (B). An open-coiled helical spring has 10 coils of 12 mm diameter steel bar wound with a mean diameter of 150 mm. The helix angle of the coils is 32°. Find the axial extension produced by a load of 250 N. Any formulae used must be established by the application of fundamental principles relating to this type of spring.

[U.L.] [49.7 mm.]

12.15 (B). An open-coiled spring carries an axial load W . Show that the deflection is related to W by

$$\delta = \frac{8 W n D^3}{G d^4} \times K$$

where K is a correction factor which allows for the inclination of the coils, n = number of effective coils, D = mean coil diameter, and d = wire diameter.

A close-coiled helical spring is wound from 6 mm diameter steel wire into a coil having a mean diameter of 50 mm. If the spring has 20 effective turns and the maximum shearing stress is limited to 225 MN/m², what is the greatest safe deflection obtainable?

[U.Birm.] [84.2 mm.]

12.16 (B/C). A flat spiral spring, as shown in Fig. 12.11, has the following dimensions: $a = 150$ mm, $b = 25$ mm, $R = 80$ mm. Determine the maximum value of the moment which can be applied to the spindle if the bending stress in the spring is not to exceed 150 MN/m². Through what angle does the spindle turn in producing this stress? The spring is constructed from steel strip 25 mm wide \times 1.5 mm thick and has six turns.

[0.75 N m, 48°.]

12.17 (B/C). A strip of steel of length 6 m, width 12 mm and thickness 2.5 mm is formed into a flat spiral around a spindle, the other end being attached to a fixed pin. Determine the couple which can be applied to the spindle if the maximum stress in the steel is limited to 300 MN/m². What will then be the energy stored in the spring?

[1.875 N m, 3.2 J.]

12.18 (B/C). A flat spiral spring is 12 mm wide, 0.3 mm thick and 2.5 m long. Assuming the maximum stress of 900 MN/m² to occur at the point of greatest bending moment, calculate the torque, the work stored and the number of turns to wind up the spring.

[U.L.] [0.081, 1.45 J; 5.68.]

CHAPTER 13

COMPLEX STRESSES

Summary

The normal stress σ and shear stress τ on oblique planes resulting from direct loading are

$$\sigma = \sigma_y \sin^2 \theta \quad \text{and} \quad \tau = \frac{1}{2}\sigma_y \sin 2\theta$$

The stresses on oblique planes owing to a complex stress system are:

$$\text{normal stress} = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\text{shear stress} = \frac{1}{2}(\sigma_x - \sigma_y) \sin 2\theta - \tau_{xy} \cos 2\theta$$

The *principal stresses* (i.e. the maximum and minimum direct stresses) are then

$$\sigma_1 = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}$$

$$\sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}$$

and these occur on planes at an angle θ to the plane on which σ_x acts, given by either

$$\tan 2\theta = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)} \quad \text{or} \quad \tan \theta = \frac{\sigma_p - \sigma_x}{\tau_{xy}}$$

where $\sigma_p = \sigma_1$, or σ_2 , the planes being termed *principal planes*. The principal planes are always at 90° to each other, and the *planes of maximum shear* are then located at 45° to them.

The *maximum shear stress* is

$$\tau_{\max} = \frac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]} = \frac{1}{2}(\sigma_1 - \sigma_2)$$

In problems where the principal stress in the third dimension σ_3 either is known or can be assumed to be zero, the true maximum shear stress is then

$$\frac{1}{2}(\text{greatest principal stress} - \text{least principal stress})$$

$$\text{Normal stress on plane of maximum shear} = \frac{1}{2}(\sigma_x + \sigma_y)$$

$$\text{Shear stress on plane of maximum direct stress (principal plane)} = 0$$

Most problems can be solved graphically by *Mohr's stress circle*. All questions which are capable of solution by this method have been solved both analytically and graphically.

13.1. Stresses on oblique planes

Consider the general case, shown in Fig. 13.1, of a bar under direct load F giving rise to stress σ_y vertically.

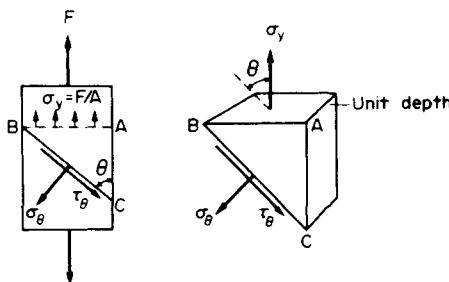


Fig. 13.1. Bar subjected to direct stress, showing stresses acting on any inclined plane.

Let the block be of *unit depth*; then considering the equilibrium of forces on the triangular portion *ABC*:

resolving forces perpendicular to *BC*,

$$\sigma_\theta \times BC \times 1 = \sigma_y \times AB \times 1 \times \sin \theta$$

But $AB = BC \sin \theta$,

$$\therefore \sigma_\theta = \sigma_y \sin^2 \theta \quad (13.1)$$

Now resolving forces parallel to *BC*,

$$\tau_\theta \times BC \times 1 = \sigma_y \times AB \times 1 \times \cos \theta$$

Again $AB = BC \sin \theta$,

$$\begin{aligned} \therefore \tau_\theta &= \sigma_y \sin \theta \cos \theta \\ &= \frac{1}{2} \sigma_y \sin 2\theta \end{aligned} \quad (13.2)$$

The stresses on the inclined plane, therefore, are not simply the resolutions of σ_y perpendicular and tangential to that plane. The direct stress σ_θ has a maximum value of σ_y when $\theta = 90^\circ$ whilst the shear stress τ_θ has a maximum value of $\frac{1}{2}\sigma_y$ when $\theta = 45^\circ$.

Thus any material whose yield stress in shear is less than half that in tension or compression will yield initially in shear under the action of direct tensile or compressive forces.

This is evidenced by the typical "cup and cone" type failure in tension tests of ductile specimens such as low carbon steel where failure occurs initially on planes at 45° to the specimen axis. Similar effects occur in compression tests on, for example, timber where failure is again due to the development of critical shear stresses on 45° planes.

13.2. Material subjected to pure shear

Consider the element shown in Fig. 13.2 to which shear stresses have been applied to the sides *AB* and *DC*. Complementary shear stresses of equal value but of opposite effect are then set up on sides *AD* and *BC* in order to prevent rotation of the element. Since the applied and complementary shears are of equal value on the *x* and *y* planes, they are both given the symbol τ_{xy} .

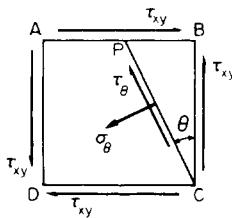


Fig. 13.2. Stresses on an element subjected to pure shear.

Consider now the equilibrium of portion *PBC*.

Resolving normal to *PC* assuming unit depth,

$$\begin{aligned}\sigma_\theta \times PC &= \tau_{xy} \times BC \sin \theta + \tau_{xy} \times PB \cos \theta \\ &= \tau_{xy} \times PC \cos \theta \sin \theta + \tau_{xy} \times PC \sin \theta \cos \theta \\ \therefore \sigma_\theta &= \tau_{xy} \sin 2\theta\end{aligned}\quad (13.3)$$

The maximum value of σ_θ is τ_{xy} when $\theta = 45^\circ$.

Similarly, resolving forces parallel to *PC*,

$$\begin{aligned}\tau_\theta \times PC &= \tau_{xy} \times PB \sin \theta - \tau_{xy} \times BC \cos \theta \\ &= \tau_{xy} \times PC \sin^2 \theta - \tau_{xy} \times PC \cos^2 \theta \\ \therefore \tau_\theta &= -\tau_{xy} \cos 2\theta\end{aligned}\quad (13.4)$$

The negative sign means that the sense of τ_θ is opposite to that assumed in Fig. 13.2.

The maximum value of τ_θ is τ_{xy} when $\theta = 0^\circ$ or 90° and it has a value of zero when $\theta = 45^\circ$, i.e. on the planes of maximum direct stress.

Further consideration of eqns. (13.3) and (13.4) shows that the system of pure shear stresses produces an equivalent direct stress system as shown in Fig. 13.3, one set compressive and one tensile, each at 45° to the original shear directions, and equal in magnitude to the applied shear.

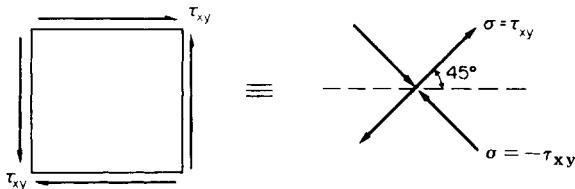


Fig. 13.3. Direct stresses due to shear.

This has great significance in the measurement of shear stresses or torques on shafts using strain gauges where the gauges are arranged to record the direct strains at 45° to the shaft axis.

Practical evidence of the theory is also provided by the failure of brittle materials in shear. A shaft of a brittle material subjected to torsion will fail under direct stress on planes at 45° to the shaft axis. (This can be demonstrated easily by twisting a piece of blackboard chalk in

one's hands; see Fig. 8.8a on page 185.) Tearing of a wet cloth when it is being wrung out is also attributed to the direct stresses introduced by the applied torsion.

13.3. Material subjected to two mutually perpendicular direct stresses

Consider the rectangular element of *unit depth* shown in Fig. 13.4 subjected to a system of two direct stresses, both tensile, at right angles, σ_x and σ_y .

For equilibrium of the portion *ABC*, resolving perpendicular to *AC*,

$$\begin{aligned}\sigma_\theta \times AC \times 1 &= \sigma_x \times BC \times 1 \times \cos \theta + \sigma_y \times AB \times 1 \times \sin \theta \\&= \sigma_x \times AC \cos^2 \theta + \sigma_y \times AC \sin^2 \theta \\ \therefore \sigma_\theta &= \frac{1}{2} \sigma_x (1 + \cos 2\theta) + \frac{1}{2} \sigma_y (1 - \cos 2\theta) \\ \text{i.e. } \sigma_\theta &= \frac{1}{2} (\sigma_x + \sigma_y) + \frac{1}{2} (\sigma_x - \sigma_y) \cos 2\theta\end{aligned}\quad (13.5)$$

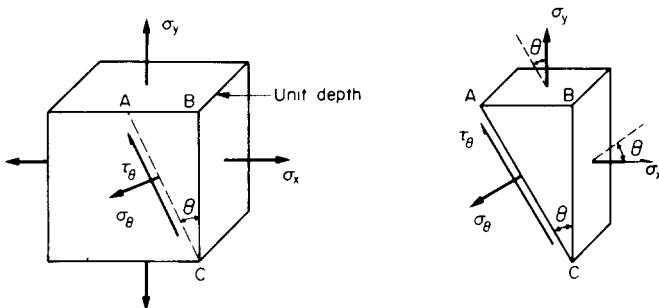


Fig. 13.4. Element from a material subjected to two mutually perpendicular direct stresses.

Resolving parallel to *AC*:

$$\begin{aligned}\tau_\theta \times AC \times 1 &= \sigma_x \times BC \times 1 \times \sin \theta - \sigma_y \times AB \times 1 \times \cos \theta \\ \tau_\theta &= \sigma_x \cos \theta \sin \theta - \sigma_y \cos \theta \sin \theta \\ \therefore \tau_\theta &= \frac{1}{2} (\sigma_x - \sigma_y) \sin 2\theta\end{aligned}\quad (13.6)$$

The maximum direct stress will equal σ_x or σ_y , whichever is the greater, when $\theta = 0$ or 90° .

The maximum shear stress in the plane of the applied stresses (see §13.8) occurs when $\theta = 45^\circ$,

$$\text{i.e. } \tau_{\max} = \frac{1}{2} (\sigma_x - \sigma_y) \quad (13.7)$$

13.4. Material subjected to combined direct and shear stresses

Consider the complex stress system shown in Fig. 13.5 acting on an element of material.

The stresses σ_x and σ_y may be compressive or tensile and may be the result of direct forces or bending. The shear stresses may be as shown or completely reversed and occur as a result of either shear forces or torsion.

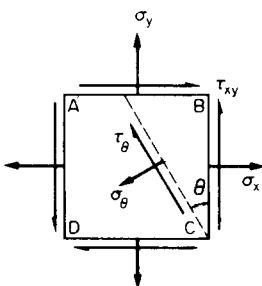


Fig. 13.5. Two-dimensional complex stress system.

The diagram thus represents a complete stress system for any condition of applied load in two dimensions and represents an addition of the stress systems previously considered in §§13.2 and 13.3.

The formulae obtained in these sections may therefore be combined to give

$$\sigma_\theta = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y) \cos 2\theta + \tau_{xy} \sin 2\theta \quad (13.8)$$

and

$$\tau_\theta = \frac{1}{2}(\sigma_x - \sigma_y) \sin 2\theta - \tau_{xy} \cos 2\theta \quad (13.9)$$

The *maximum and minimum stresses* which occur on any plane in the material can now be determined as follows:

For σ_θ to be a maximum or minimum $\frac{d\sigma_\theta}{d\theta} = 0$

Now $\sigma_\theta = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y) \cos 2\theta + \tau_{xy} \sin 2\theta$

$$\therefore \frac{d\sigma_\theta}{d\theta} = -(\sigma_x - \sigma_y) \sin 2\theta + 2\tau_{xy} \cos 2\theta = 0$$

$$\text{or } \tan 2\theta = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)} \quad (13.10)$$

$$\therefore \text{from Fig. 13.6} \quad \sin 2\theta = \frac{2\tau_{xy}}{\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}}$$

$$\cos 2\theta = \frac{(\sigma_x - \sigma_y)}{\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}}$$

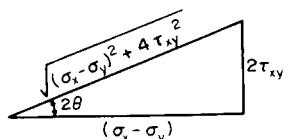


Fig. 13.6.

Therefore substituting in eqn. (13.8), the maximum and minimum direct stresses are given by

$$\sigma_1 \text{ or } \sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2} \frac{(\sigma_x - \sigma_y)(\sigma_x - \sigma_y)}{\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}} + \frac{\tau_{xy} \times 2\tau_{xy}}{\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}}$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]} \quad (13.11)$$

These are then termed the *principal stresses* of the system.

The solution of eqn. (13.10) yields two values of 2θ separated by 180° , i.e. two values of θ separated by 90° . Thus the two principal stresses occur on mutually perpendicular planes termed *principal planes*, and substitution for θ from eqn. (13.10) into the shear stress expression eqn. (13.9) will show that $\tau_\theta = 0$ on the principal planes.

The complex stress system of Fig. 13.5 can now be reduced to the equivalent system of principal stresses shown in Fig. 13.7.

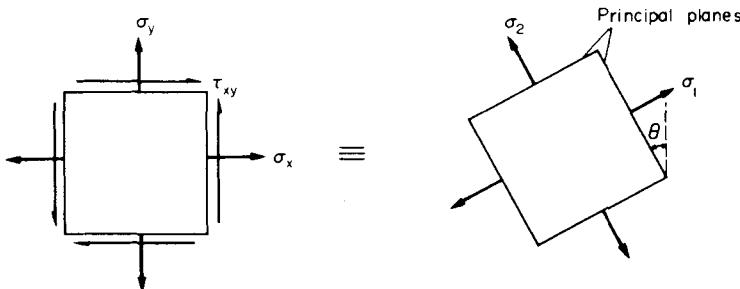


Fig. 13.7. Principal planes and stresses.

From eqn. (13.7) the maximum shear stress present in the system is given by

$$\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_2) \quad (13.12)$$

$$= \frac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]} \quad (13.13)$$

and this occurs on planes at 45° to the principal planes.

This result could have been obtained using a similar procedure to that used for determining the principal stresses, i.e. by differentiating expression (13.9), equating to zero and substituting the resulting expression for θ .

13.5. Principal plane inclination in terms of the associated principal stress

It has been stated in the previous section that expression (13.10), namely

$$\tan 2\theta = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$$

yields two values of θ , i.e. the inclination of the two principal planes on which the principal stresses σ_1 and σ_2 act. It is uncertain, however, which stress acts on which plane unless eqn. (13.8) is used, substituting one value of θ obtained from eqn. (13.10) and observing which one of the two principal stresses is obtained. The following alternative solution is therefore to be preferred.

Consider once again the equilibrium of a triangular block of material of unit depth (Fig. 13.8); this time AC is a principal plane on which a principal stress σ_p acts, and the shear stress is zero (from the property of principal planes).

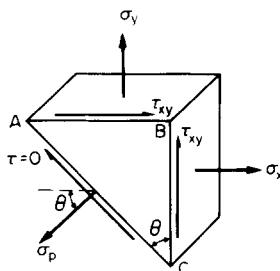


Fig. 13.8.

Resolving forces horizontally,

$$(\sigma_x \times BC \times 1) + (\tau_{xy} \times AB \times 1) = (\sigma_p \times AC \times 1) \cos \theta$$

$$\sigma_x + \tau_{xy} \tan \theta = \sigma_p$$

$$\therefore \tan \theta = \frac{\sigma_p - \sigma_x}{\tau_{xy}} \quad (13.14)$$

Thus we have an equation for the inclination of the principal planes *in terms of the principal stress*. If, therefore, the principal stresses are determined and substituted in the above equation, each will give the corresponding angle of the plane on which it acts and there can then be no confusion.

The above formula has been derived with two tensile direct stresses and a shear stress system, as shown in the figure; should any of these be reversed in action, then the appropriate minus sign must be inserted in the equation.

13.6. Graphical solution – Mohr's stress circle

Consider the complex stress system of Fig. 13.5 (p. 330). As stated previously this represents a complete stress system for any condition of applied load in two dimensions.

In order to find graphically the direct stress σ_θ and shear stress τ_θ on any plane inclined at θ to the plane on which σ_x acts, proceed as follows:

- (1) Label the block $ABCD$.
- (2) Set up axes for direct stress (as abscissa) and shear stress (as ordinate) (Fig. 13.9).
- (3) Plot the stresses acting on two adjacent faces, e.g. AB and BC , using the following sign conventions:

direct stresses: tensile, positive; compressive, negative;

shear stresses: tending to turn block clockwise, positive; tending to turn block counterclockwise, negative.

This gives two points on the graph which may then be labelled \overline{AB} and \overline{BC} respectively to denote stresses on these planes.

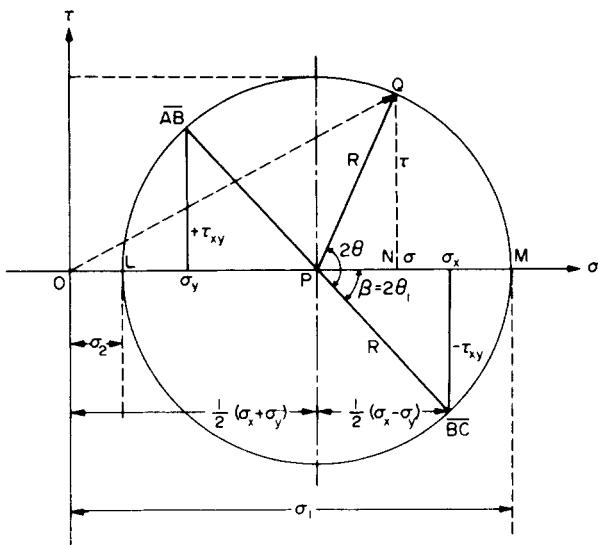


Fig. 13.9. Mohr's stress circle.

(4) Join \overline{AB} and \overline{BC} .

(5) The point P where this line cuts the σ axis is then the centre of Mohr's circle, and the line is the diameter; therefore the circle can now be drawn.

Every point on the circumference of the circle then represents a state of stress on some plane through C .

Proof

Consider any point Q on the circumference of the circle, such that PQ makes an angle 2θ with \overline{BC} , and drop a perpendicular from Q to meet the σ axis at N .

Coordinates of Q :

$$\begin{aligned} ON &= OP + PN = \frac{1}{2}(\sigma_x + \sigma_y) + R \cos(2\theta - \beta) \\ &= \frac{1}{2}(\sigma_x + \sigma_y) + R \cos 2\theta \cos \beta + R \sin 2\theta \sin \beta \end{aligned}$$

But

$$R \cos \beta = \frac{1}{2}(\sigma_x - \sigma_y) \quad \text{and} \quad R \sin \beta = \tau_{xy}$$

∴

$$ON = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y) \cos 2\theta + \tau_{xy} \sin 2\theta$$

On inspection this is seen to be eqn. (13.8) for the direct stress σ_θ on the plane inclined at θ to BC in Fig. 13.5.

Similarly,

$$\begin{aligned} QN &= R \sin(2\theta - \beta) \\ &= R \sin 2\theta \cos \beta - R \cos 2\theta \sin \beta \\ &= \frac{1}{2}(\sigma_x - \sigma_y) \sin 2\theta - \tau_{xy} \cos 2\theta \end{aligned}$$

Again, on inspection this is seen to be eqn. (13.9) for the shear stress τ_θ on the plane inclined at θ to BC .

Thus the coordinates of Q are the normal and shear stresses on a plane inclined at θ to BC in the original stress system.

N.B. – Single angle $BCPQ$ is 2θ on Mohr's circle and not θ , it is evident that *angles are doubled on Mohr's circle*. This is the only difference, however, as they are measured in the same direction and from the same plane in both figures (in this case counterclockwise from BC).

Further points to note are:

- (1) The direct stress is a maximum when Q is at M , i.e. OM is the length representing the maximum principal stress σ_1 and $2\theta_1$ gives the angle of the plane θ_1 from BC . Similarly, OL is the other principal stress.
- (2) The maximum shear stress is given by the highest point on the circle and is represented by the radius of the circle. This follows since shear stresses and complementary shear stresses have the same value; *therefore the centre of the circle will always lie on the σ axis midway between σ_x and σ_y* .
- (3) From the above point the direct stress on the plane of maximum shear must be midway between σ_x and σ_y , i.e. $\frac{1}{2}(\sigma_x + \sigma_y)$.
- (4) The shear stress on the principal planes is zero.
- (5) Since the resultant of two stresses at 90° can be found from the parallelogram of vectors as the diagonal, as shown in Fig. 13.10, the resultant stress on the plane at θ to BC is given by OQ on Mohr's circle.

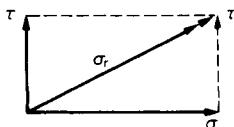


Fig. 13.10. Resultant stress (σ_r) on any plane.

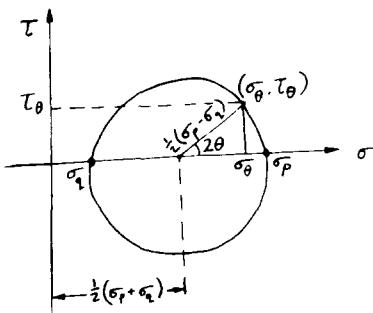
The graphical method of solution of complex stress problems using Mohr's circle is a very powerful technique since all the information relating to any plane within the stressed element is contained in the single construction. It thus provides a convenient and rapid means of solution which is less prone to arithmetical errors and is highly recommended.

With the growing availability and power of programmable calculators and microcomputers it may be that the practical use of Mohr's circle for the analytical determination of stress (and strain – see Chapter 14) values will become limited. It will remain, however, a highly effective medium for the teaching and understanding of complex stress systems.

A free-hand sketch of the Mohr circle construction, for example, provides a convenient mechanism for the derivation (by simple geometric relationships) of the principal stress equations (13.11) or of the equations for the shear and normal stresses on any inclined plane in terms of the principal stresses as shown in Fig. 13.11.

13.7. Alternative representations of stress distributions at a point

The way in which the stress at a point varies with the angle at which a plane is taken through the point may be better understood with the aid of the following alternative graphical representations.



$$\begin{aligned}\sigma_{\theta} &= \frac{1}{2}(\sigma_i + \sigma_p) + \frac{1}{2}(\sigma_p - \sigma_i) \cos 2\theta \\ \tau_{\theta} &= \frac{1}{2}(\sigma_p - \sigma_i) \sin 2\theta\end{aligned}$$

Fig. 13.11. Free-hand sketch of Mohr's stress circle.

Equations (13.8) and (13.9) give the values of the direct stress σ_{θ} and shear stress τ_{θ} on any plane inclined at an angle θ to the plane on which the direct stress σ_x acts within a two-dimensional complex stress system, viz:

$$\begin{aligned}\sigma_{\theta} &= \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y) \cos 2\theta + \tau_{xy} \sin 2\theta \\ \tau_{\theta} &= \frac{1}{2}(\sigma_x - \sigma_y) \sin 2\theta - \tau_{xy} \cos 2\theta\end{aligned}$$

(a) Uniaxial stresses

For the special case of a single uniaxial stress σ_x as in simple tension or on the surface of a beam in bending, $\sigma_y = \tau_{xy} = 0$ and the equations (13.8) and (13.9) reduce to

$$\sigma_{\theta} = \frac{1}{2}\sigma_x(1 + \cos 2\theta) = \sigma_x \cos^2 \theta.$$

N.B. If the single stress were selected as σ_y then the relationship would have reduced to that of eqn. (13.1), i.e.

$$\sigma_{\theta} = \sigma_y \sin^2 \theta.$$

Similarly:

$$\tau_{\theta} = \frac{1}{2}\sigma_x \sin 2\theta.$$

Plotting these equations on simple Cartesian axes produces the stress distribution diagrams of Fig. 13.12, both sinusoidal in shape with shear stress "shifted" by 45° from the normal stress.

Principal stresses σ_p and σ_i occur, as expected, at 90° intervals and the amplitude of the normal stress curve is given by the difference between the principal stress values. It should also be noted that shear stress is proportional to the derivative of the normal stress with respect to θ , i.e. τ_{θ} is a maximum where $d\sigma_{\theta}/d\theta$ is a maximum and τ_{θ} is zero where $d\sigma_{\theta}/d\theta$ is zero, etc.

Alternatively, plotting the same equations on polar graph paper, as in Fig. 13.13, gives an even more readily understood pictorial representation of the stress distributions showing a peak value of direct stress in the direction of application of the applied stress σ_x falling to zero

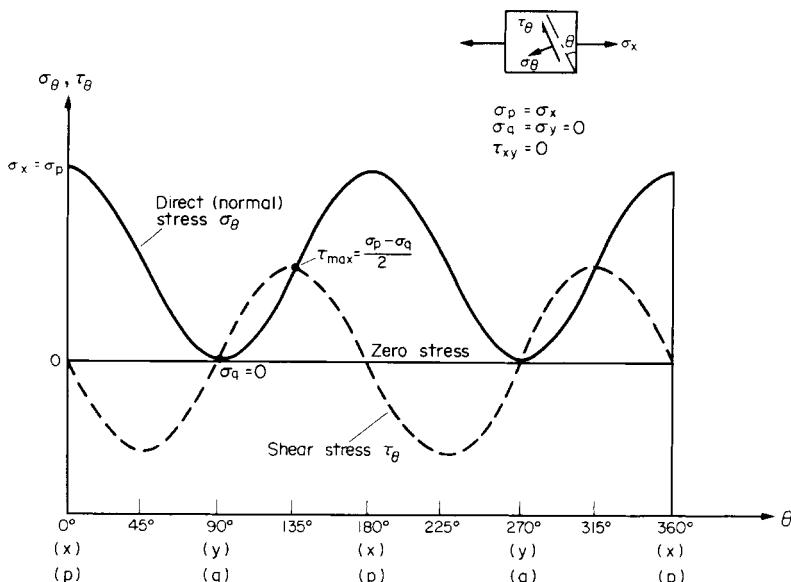


Fig. 13.12. Cartesian plot of stress distribution at a point under uniaxial applied stress.

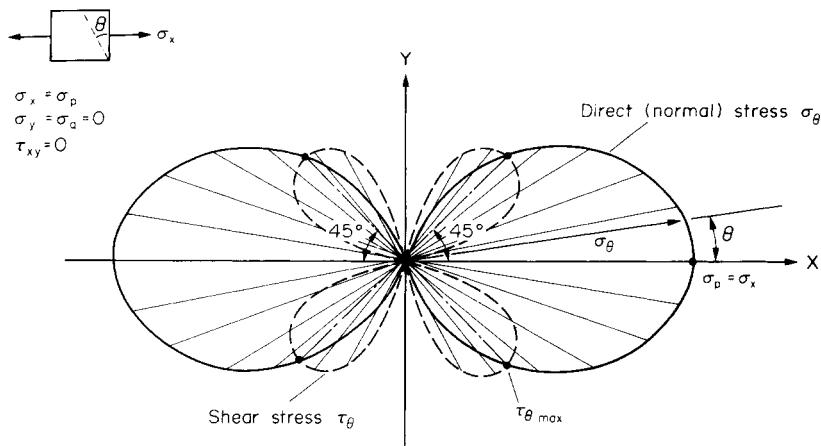


Fig. 13.13. Polar plot of stress distribution at a point under uniaxial applied stress.

in directions at right angles and maximum shearing stresses on planes at 45° with zero shear on the x and y (principal) axes.

(b) Biaxial stresses

In almost all modes of loading on structural members or engineering components the stresses produced are a maximum at the free (outside) surface. This is particularly evident for

the cases of pure bending or torsion as shown by the stress diagrams of Figs. 4.4 and 8.4, respectively, but is also true for other more complex combined loading situations with the major exception of direct bearing loads where maximum stress conditions can be sub-surface. Additionally, at free surfaces the stress normal to the surface is always zero so that the most severe stress condition often reduces, at worst, to a two-dimensional plane stress system within the surface of the component. It should be evident, therefore that the biaxial stress system is of considerable importance to practical design considerations.

The Cartesian plot of a typical bi-axial stress state is shown in Fig. 13.14 whilst Fig. 13.15 shows the polar plot of stresses resulting from the bi-axial stress system present on the surface of a thin cylindrical pressure vessel for which $\sigma_p = \sigma_H$ and $\sigma_q = \sigma_L = \frac{1}{2}\sigma_H$ with $\tau_{xy} = 0$.

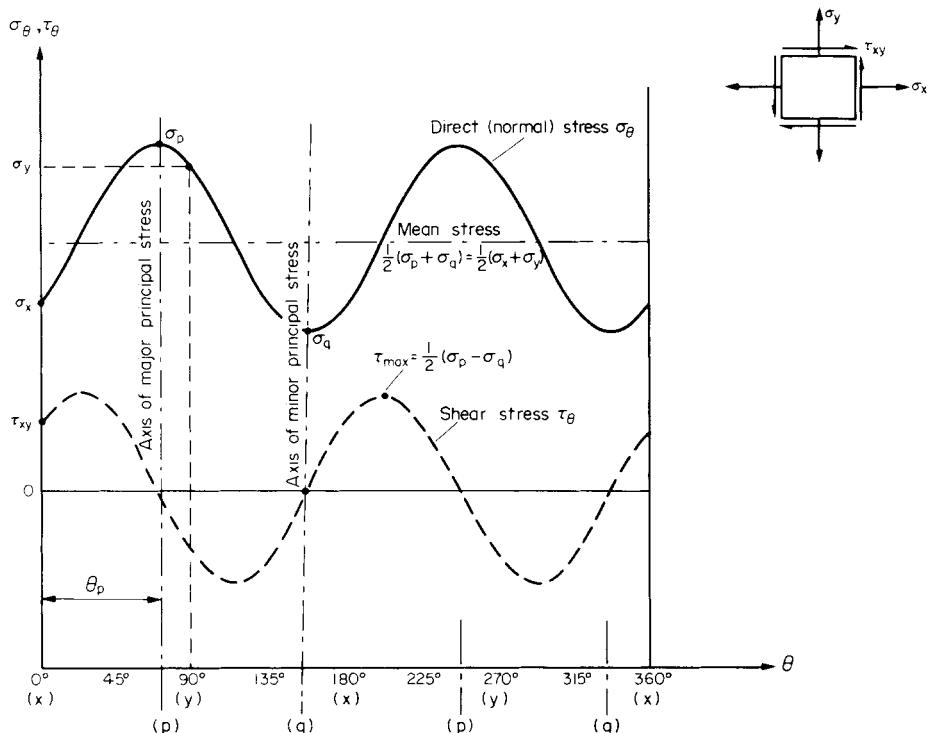


Fig. 13.14. Cartesian plot of stress distribution at a point under a typical biaxial applied stress system.

It should be noted that the whole of the information conveyed on these alternative representations is also available from the relevant Mohr circle which, additionally, is more amenable to quantitative analysis. They do not, therefore, replace Mohr's circle but are included merely to provide alternative pictorial representations which may aid a clearer understanding of the general problem of stress distribution at a point. The equivalent diagrams for strain are given in §14.16.

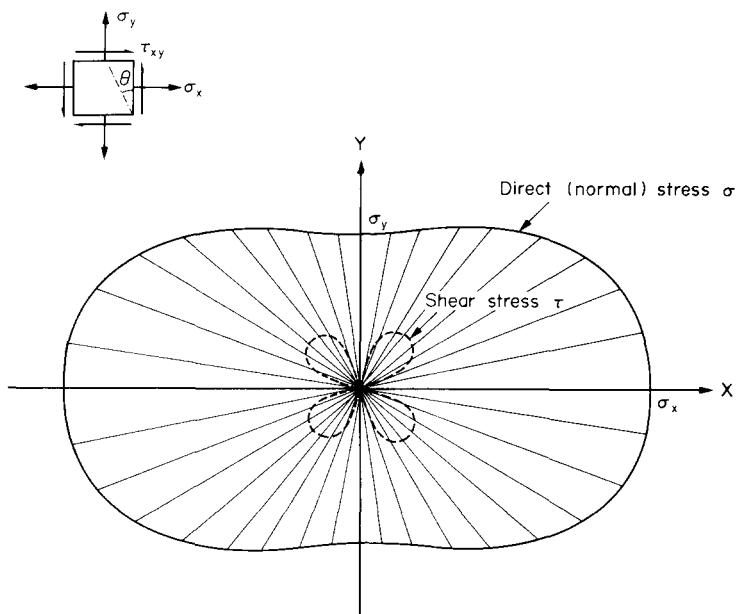


Fig. 13.15. Polar plot of stress distribution under typical biaxial applied stress system.

13.8. Three-dimensional stresses – graphical representation

Figure 13.16 shows the general *three-dimensional* state of stress at any point in a body, i.e. the body will be subjected to three mutually perpendicular direct stresses and three shear stresses.

Figure 13.17 shows a *principal element* at the same point, i.e. one in general rotated relative to the first until the stresses on the faces are principal stresses with no associated shear.

Figure 13.18 then represents true views on the various faces of the principal element, and for each two-dimensional stress condition so obtained a Mohr circle may be drawn. These

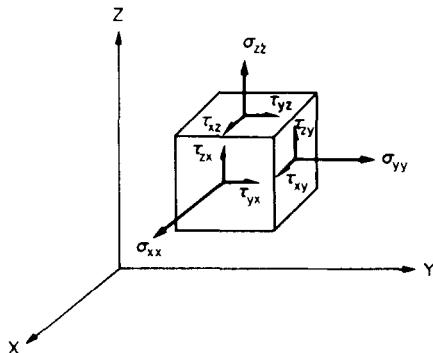


Fig. 13.16. Three-dimensional stress system.

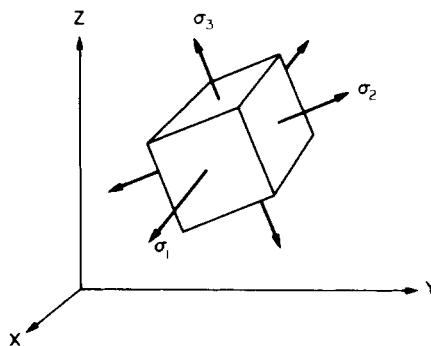


Fig. 13.17. Principal element.

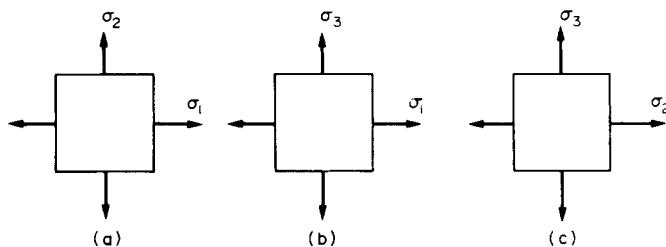


Fig. 13.18. True views on the various faces of the principal element.

can then be combined to produce the complete three-dimensional Mohr circle representation shown in Fig. 13.19.

The large circle between points σ_1 and σ_3 represents stresses on all planes through the point in question containing the σ_2 axis. Likewise the small circle between σ_2 and σ_3 represents

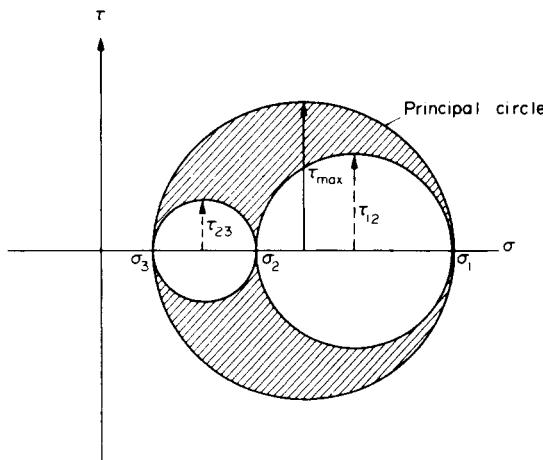


Fig. 13.19. Mohr circle representation of three-dimensional stress state showing the principal circle, the radius of which is equal to the greatest shear stress present in the system.

stresses on all planes containing the σ_1 axis and the circle between σ_1 and σ_2 all planes containing the σ_3 axis.

There are, of course, an infinite number of planes passing through the point which do not contain any of the three principal axes, but it can be shown that all such planes are represented by the shaded area between the circles. The procedure involved in the location of a particular point in the shaded area which corresponds to any given plane is covered in *Mechanics of Materials 2*.[†] In practice, however, it is often the maximum direct and shear stresses which will govern the elastic failure of materials. These are determined from the larger of the three circles which is thus termed the *principal circle* ($\tau_{\max} = \text{radius}$).

It is perhaps evident now that in many two-dimensional cases the maximum (greatest) shear stress value will be missed by not considering $\sigma_3 = 0$ and constructing the principal circle.

Consider the stress state shown in Fig. 13.20(a). If the principal stresses σ_1 , σ_2 and σ_3 all have non-zero values the system will be termed "three-dimensional"; if one of the principal stresses is zero the system is said to be "two-dimensional" and with two principal stresses zero a "uniaxial" stress condition is obtained. In all cases, however, it is necessary to consider all three principal stress values in the determination of the maximum shear stress since out-of-plane shear stresses will be dependent on all three values and one will be a maximum – see Fig. 13.20(b), (c) and (d).

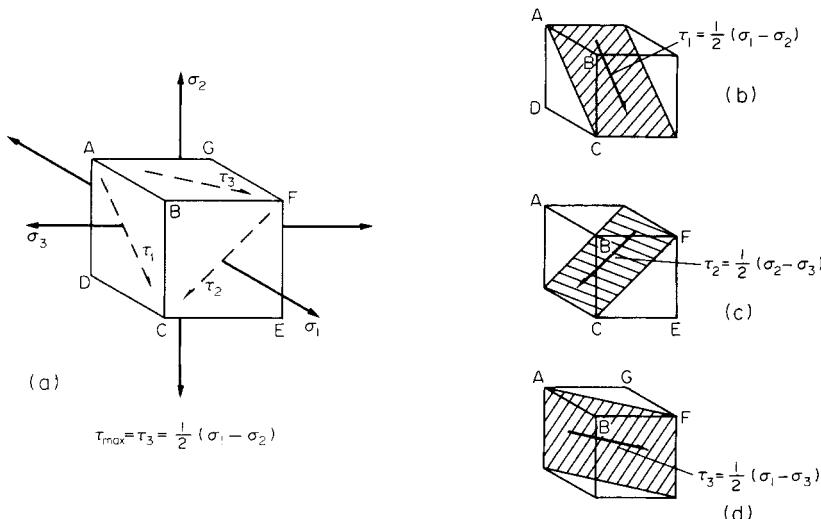


Fig. 13.20. Maximum shear stresses in a three-dimensional stress system.

Examples of the crucial effect of consideration of the third (zero) principal stress value in apparently "two-dimensional" stress states are given below:

(a) *Thin cylinder.*

An element in the surface of a thin cylinder subjected to internal pressure p will have principal stresses:

$$\sigma_1 = \sigma_H = pd/2t$$

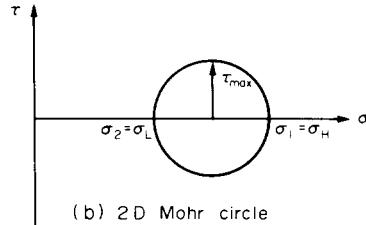
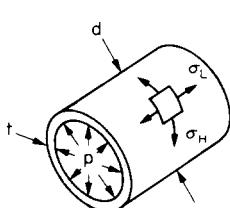
$$\sigma_2 = \sigma_L = pd/4t$$

[†]E. J. Hearn, *Mechanics of Materials 2*, 3rd edition (Butterworth-Heinemann, Oxford, 1997).

with the third, radial, stress σ_r , assumed to be zero – see Fig. 13.21(a).

A two-dimensional Mohr circle representation of the stresses in the element will give Fig. 13.21(b) with a maximum shear stress:

$$\begin{aligned}\tau_{\max} &= \frac{1}{2}(\sigma_1 - \sigma_2) \\ &= \frac{1}{2}\left(\frac{pd}{2t} - \frac{pd}{4t}\right) = \frac{pd}{8t}\end{aligned}$$



$$\begin{aligned}(a) \quad \sigma_H &= \frac{pd}{2t} \\ \sigma_L &= \frac{pd}{4t} \\ \tau_{\max} &= \frac{1}{2}(\sigma_1 - \sigma_2) = \frac{pd}{4t}\end{aligned}$$

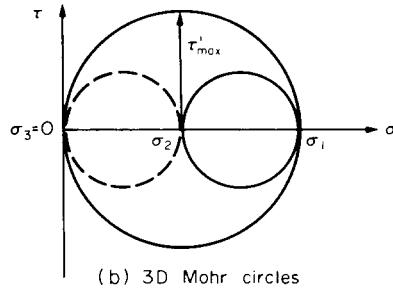


Fig. 13.21. Maximum shear stresses in a pressurised thin cylinder.

A three-dimensional Mohr circle construction, however, is shown in Fig. 13.21(c), the zero value of σ_3 producing a much larger principal circle and a maximum shear stress:

$$\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3) = \frac{1}{2}\left(\frac{pd}{2t} - 0\right) = \frac{pd}{4t}$$

i.e. twice the value obtained from the two-dimensional circle.

(b) Sphere

Consider now an element in the surface of a sphere subjected to internal pressure p as shown in Fig. 13.22(a). Principal stresses on the element will then be $\sigma_1 = \sigma_2 = \frac{pd}{4t}$ with $\sigma_r = \sigma_3 = 0$ normal to the surface.

The two-dimensional Mohr circle is shown in Fig. 13.22(b), in this case reducing to a point since σ_1 and σ_2 are equal. The maximum shear stress, which always equals the radius of Mohr's circle is thus zero and would seem to imply that, although the material of the vessel may well be ductile and susceptible to shear failure, no shear failure could ensue. However,

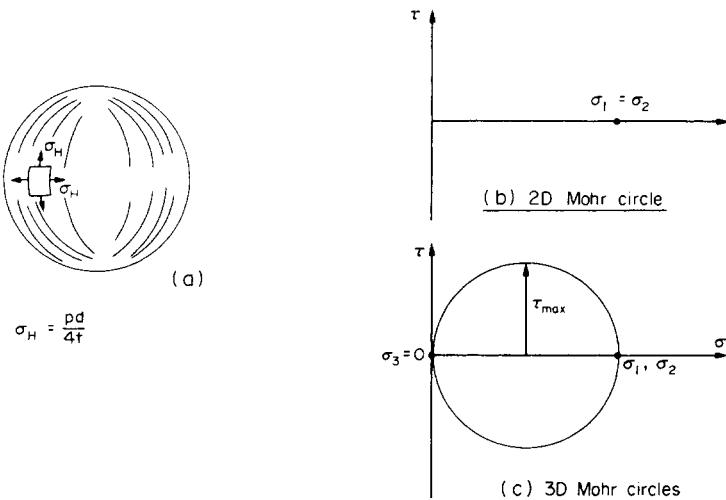


Fig. 13.22. Maximum shear stresses in a pressurised thin sphere.

this is far from the truth as will be evident when the full three-dimensional representation is drawn as in Fig. 13.22(c) with the third, zero, principal stress taken into account.

A maximum shear stress is now produced within the $\sigma_1 \sigma_3$ plane of value:

$$\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3) = pd/8t$$

The greatest value of τ can be obtained *analytically* by using the statement

$$\tau_{\max} = \frac{1}{2}(\text{greatest principal stress} - \text{least principal stress})$$

and considering separately the principal stress conditions as illustrated in Fig. 13.18.

Examples

Example 13.1 (A)

A circular bar 40 mm diameter carries an axial tensile load of 100 kN. What is the value of the shear stress on the planes on which the normal stress has a value of 50 MN/m² tensile?

Solution

$$\text{Tensile stress } \sigma_y = \frac{F}{A} = \frac{100 \times 10^3}{\pi \times (0.02)^2} = 79.6 \text{ MN/m}^2$$

Now the normal stress on an oblique plane is given by eqn. (13.1):

$$\sigma_\theta = \sigma_y \sin^2 \theta$$

$$50 \times 10^6 = 79.6 \times 10^6 \sin^2 \theta$$

$$\theta = 52^\circ 28'$$

The shear stress on the oblique plane is then given by eqn. (13.2):

$$\begin{aligned}\tau_\theta &= \frac{1}{2} \sigma_y \sin 2\theta \\ &= \frac{1}{2} \times 79.6 \times 10^6 \times \sin 104^\circ 56' \\ &= 38.6 \times 10^6\end{aligned}$$

The required shear stress is 38.6 MN/m^2 .

Example 13.2 (A/B)

Under certain loading conditions the stresses in the walls of a cylinder are as follows:

- (a) 80 MN/m^2 tensile;
- (b) 30 MN/m^2 tensile at right angles to (a);
- (c) shear stresses of 60 MN/m^2 on the planes on which the stresses (a) and (b) act; the shear couple acting on planes carrying the 30 MN/m^2 stress is clockwise in effect.

Calculate the principal stresses and the planes on which they act. What would be the effect on these results if owing to a change of loading (a) becomes compressive while stresses (b) and (c) remain unchanged?

Solution

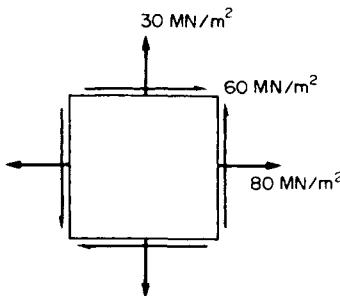


Fig. 13.23.

The principal stresses are given by the formula

$$\begin{aligned}\sigma_1 \quad \text{and} \quad \sigma_2 &= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]} \\ &= \frac{1}{2}(80 + 30) \pm \frac{1}{2}\sqrt{[(80 - 30)^2 + (4 \times 60^2)]} \\ &= 55 \pm 5\sqrt{(25 + 144)} \\ &= 55 \pm 65\end{aligned}$$

∴

$$\sigma_1 = 120 \text{ MN/m}^2$$

$$\text{and} \quad \sigma_2 = -10 \text{ MN/m}^2 \quad (\text{i.e. compressive})$$

The planes on which these stresses act can be determined from eqn. (13.14),

i.e. $\tan \theta_1 = \frac{\sigma_p - \sigma_x}{\tau_{xy}}$

$$\therefore \tan \theta_1 = \frac{120 - 80}{60} = 0.6667$$

$$\therefore \theta_1 = 33^\circ 41'$$

Also $\tan \theta_2 = \frac{-10 - 80}{60} = 1.50$

$$\therefore \theta_2 = -56^\circ 19' \text{ or } 123^\circ 41'$$

N.B. – The resulting angles are at 90° to each other as expected.

If the loading is now changed so that the 80 MN/m^2 stress becomes compressive:

$$\begin{aligned}\sigma_1 &= \frac{1}{2}(-80 + 30) + \frac{1}{2}\sqrt{(-80 - 30)^2 + (4 \times 60^2)} \\ &= -25 + 5\sqrt{121 + 144} \\ &= -25 + 81.5 = 56.5 \text{ MN/m}^2\end{aligned}$$

and $\sigma_2 = -25 - 81.5 = -106.5 \text{ MN/m}^2$

Then $\tan \theta_1 = \frac{56.5 - (-80)}{60} = 2.28$

$$\therefore \theta_1 = 66^\circ 19'$$

and $\theta_2 = 66^\circ 19' + 90 = 156^\circ 19'$

Mohr's circle solutions

In the first part of the question the stress system and associated Mohr's circle are as drawn in Fig. 13.24.

By measurement: $\sigma_1 = 120 \text{ MN/m}^2$ tensile

$\sigma_2 = 10 \text{ MN/m}^2$ compressive

and $\theta_1 = 34^\circ$ counterclockwise from BC

$\theta_2 = 124^\circ$ counterclockwise from BC

When the 80 MN/m^2 stress is reversed, the stress system is that in Fig. 13.25, giving Mohr's circle as drawn.

The required values are then:

$\sigma_1 = 56.5 \text{ MN/m}^2$ tensile

$\sigma_2 = 106.5 \text{ MN/m}^2$ compressive

$\theta_1 = 66^\circ 15'$ counterclockwise to BC

and $\theta_2 = 156^\circ 15'$ counterclockwise to BC

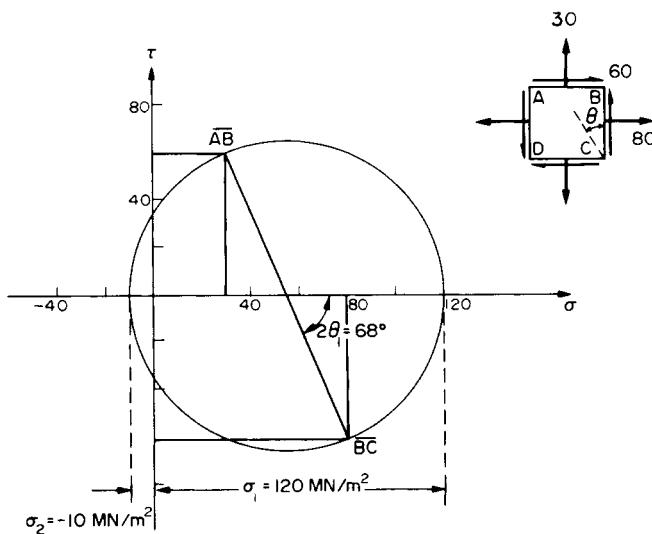


Fig. 13.24.

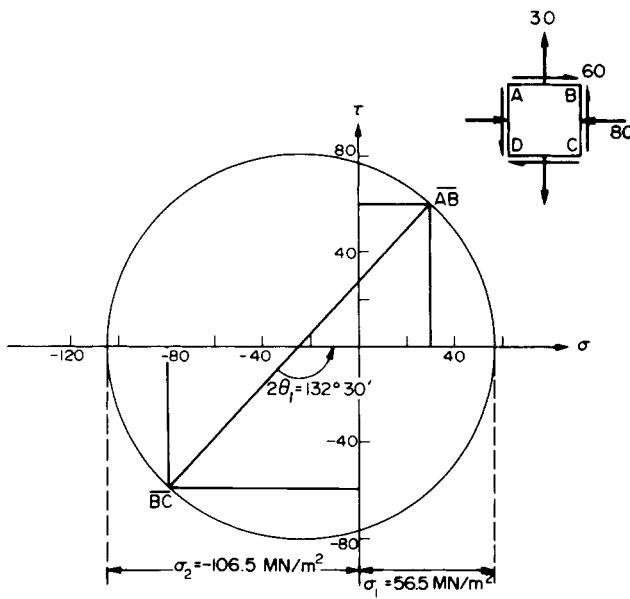


Fig. 13.25.

Example 13.3 (B)

A material is subjected to two mutually perpendicular direct stresses of 80 MN/m^2 tensile and 50 MN/m^2 compressive, together with a shear stress of 30 MN/m^2 . The shear couple acting on planes carrying the 80 MN/m^2 stress is clockwise in effect. Calculate

- (a) the magnitude and nature of the principal stresses;
- (b) the magnitude of the maximum shear stresses in the plane of the given stress system;
- (c) the direction of the planes on which these stresses act.

Confirm your answer by means of a Mohr's stress circle diagram, and from the diagram determine the magnitude of the normal stress on a plane inclined at 20° counterclockwise to the plane on which the 50 MN/m^2 stress acts.

Solution

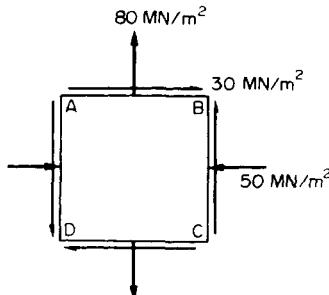


Fig. 13.26.

- (a) To find the principal stresses:

$$\begin{aligned}\sigma_1 \quad \text{and} \quad \sigma_2 &= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]} \\ &= \frac{1}{2}(-50 + 80) \pm \frac{1}{2}\sqrt{[-50 - 80]^2 + (4 \times 900)} \\ &= 5[3 \pm \sqrt{(169 + 36)}] = 5[3 \pm 14.31] \\ \therefore \quad \sigma_1 &= 86.55 \text{ MN/m}^2 \\ \sigma_2 &= -56.55 \text{ MN/m}^2\end{aligned}$$

The principal stresses are

86.55 MN/m^2 tensile and 56.55 MN/m^2 compressive

- (b) To find the maximum shear stress:

$$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{86.55 - (-56.55)}{2} = \frac{143.1}{2} = 71.6 \text{ MN/m}^2$$

Maximum shear stress = 71.6 MN/m^2

- (c) To find the directions of the principal planes:

$$\begin{aligned}\tan \theta_1 &= \frac{\sigma_p - \sigma_x}{\tau_{xy}} = \frac{86.55 - (-50)}{30} \\ &= \frac{136.55}{30} = 4.552\end{aligned}$$

$$\therefore \theta_1 = 77^\circ 36'$$

$$\therefore \theta_2 = 77^\circ 36' + 90^\circ = 167^\circ 36'$$

The principal planes are inclined at $77^\circ 36'$ to the plane on which the 50 MN/m^2 stress acts. The maximum shear planes are at 45° to the principal planes.

Mohr's circle solution

The stress system shown in Fig. 13.26 gives the Mohr's circle in Fig. 13.27.

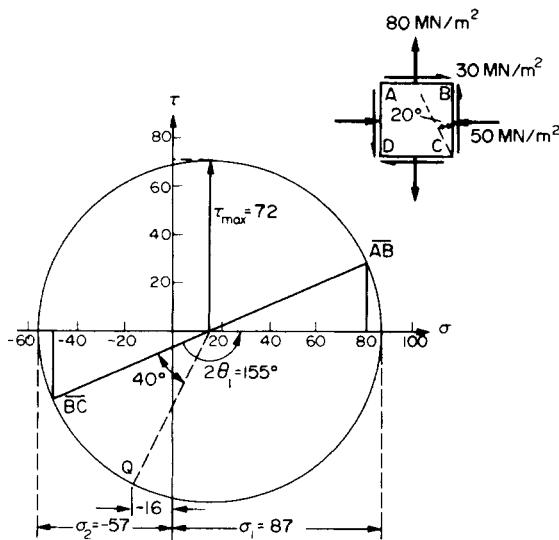


Fig. 13.27.

By measurement

$$\sigma_1 = 87 \text{ MN/m}^2 \text{ tensile}$$

$$\sigma_2 = 57 \text{ MN/m}^2 \text{ compressive}$$

$$\tau_{\max} = 72 \text{ MN/m}^2$$

and

$$\theta_1 = \frac{155^\circ}{2} = 77^\circ 30'$$

The direct or normal stress on a plane inclined at 20° counterclockwise to BC is obtained by measuring from BC on the Mohr's circle through $2 \times 20^\circ = 40^\circ$ in the same direction.

This gives

$$\sigma = 16 \text{ MN/m}^2 \text{ compressive}$$

Example 13.4 (B)

At a given section a shaft is subjected to a bending stress of 20 MN/m^2 and a shear stress of 40 MN/m^2 . Determine:

- the principal stresses;
- the directions of the principal planes;
- the maximum shear stress and the planes on which this acts;
- the tensile stress which, acting alone, would produce the same maximum shear stress;
- the shear stress which, acting alone, would produce the same maximum tensile principal stress.

Solution

(a) The bending stress is a direct stress and can be treated as acting on the x axis, so that $\sigma_x = 20 \text{ MN/m}^2$; since no other direct stresses are given, $\sigma_y = 0$.

$$\begin{aligned}\text{Principal stress } \sigma_1 &= \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]} \\ &= \frac{1}{2} \times 20 + \frac{1}{2}\sqrt{[20^2 + (4 \times 40^2)]} \\ &= 10 + 5\sqrt{(68)} = 10 + 5 \times 8.246 \\ &= 51.23 \text{ MN/m}^2\end{aligned}$$

and

$$\begin{aligned}\sigma_2 &= 10 - 41.23 \\ &= -31.23 \text{ MN/m}^2\end{aligned}$$

$$(b) \text{ Then } \tan \theta_1 = \frac{\sigma_p - \sigma_x}{\tau_{xy}} = \frac{51.23 - 20}{40} = \frac{31.23}{40} = 0.7808$$

$$\therefore \theta_1 = 37^\circ 59'$$

$$\text{and } \tan \theta_2 = \frac{-31.23 - 20}{40} = \frac{-51.23}{40} = -1.2808$$

$$\therefore \theta_2 = -52^\circ 1' \text{ or } 127^\circ 59'$$

both angles being measured counterclockwise from the plane on which the 20 MN/m^2 stress acts.

(c) Maximum shear stress

$$\begin{aligned}\tau_{\max} &= \frac{\sigma_1 - \sigma_2}{2} = \frac{51.23 - (-31.23)}{2} \\ &= \frac{82.46}{2} = 41.23 \text{ MN/m}^2\end{aligned}$$

This acts on planes at 45° to the principal planes,

i.e. **at $82^\circ 59'$ or $-7^\circ 1'$**

(d) Maximum shear stress

$$\tau_{\max} = \frac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}$$

Thus if a tensile stress is to act alone to give the same maximum shear stress ($\sigma_x = 0$ and $\tau_{xy} = 0$):

$$\text{maximum shear stress} = \frac{1}{2}\sqrt{(\sigma_x^2)} = \frac{1}{2}\sigma_x$$

$$41.23 = \frac{1}{2}\sigma_x$$

i.e.

$$\sigma_x = 82.46 \text{ MN/m}^2$$

The required tensile stress is 82.46 MN/m^2 .

(e) Principal stress

$$\sigma_1 = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}$$

Thus if a shear stress is to act alone to give the same principal stress ($\sigma_x = \sigma_y = 0$):

$$\sigma_1 = \frac{1}{2}\sqrt{(4\tau_{xy}^2)} = \tau_{xy}$$

$$51.23 = \tau_{xy}$$

The required shear stress is 51.23 MN/m^2 .

Mohr's circle solutions

(a), (b), (c) The stress system and corresponding Mohr's circle are as shown in Fig. 13.28. By measurement:

$$(a) \quad \sigma_1 \approx 51 \text{ MN/m}^2 \text{ tensile}$$

$$\sigma_2 \approx 31 \text{ MN/m}^2 \text{ compressive}$$

$$(b) \quad \theta_1 = \frac{76^\circ}{2} = 38^\circ$$

$$\theta_2 = 38^\circ + 90^\circ = 128^\circ$$

$$(c) \quad \tau_{\max} \approx 41 \text{ MN/m}^2$$

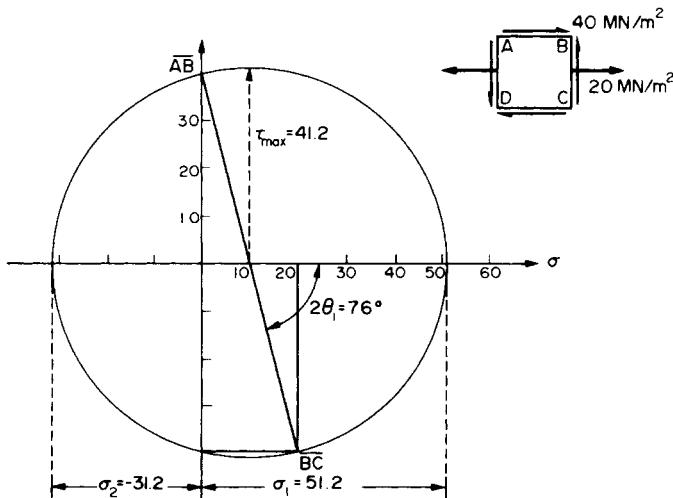


Fig. 13.28.

Angle of maximum shear plane

$$= \frac{166}{2} = 83^\circ$$

(d) If a tensile stress σ_x is to act alone to give the same maximum shear stress, then $\sigma_y = 0$, $\tau_{xy} = 0$ and $\tau_{\max} = 41 \text{ MN/m}^2$. The Mohr's circle therefore has a radius of 41 MN/m^2 and passes through the origin (Fig. 13.29).

Hence the required tensile stress is 82 MN/m^2 .

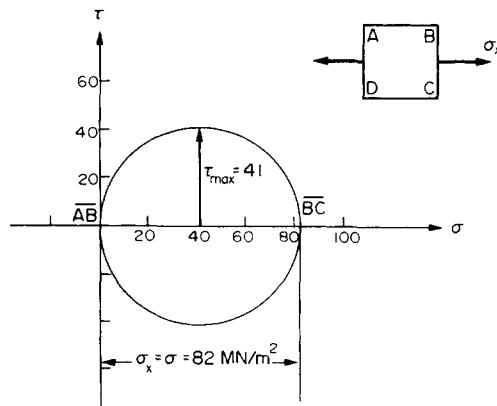


Fig. 13.29.

(e) If a shear stress is to act alone to produce the same principal stress, $\sigma_x = 0$, $\sigma_y = 0$ and $\sigma_1 = 51 \text{ MN/m}^2$. The Mohr's circle thus has its centre at the origin and passes through $\sigma = 51 \text{ MN/m}^2$ (Fig. 13.30).

Hence the required shear stress is 51 MN/m^2 .

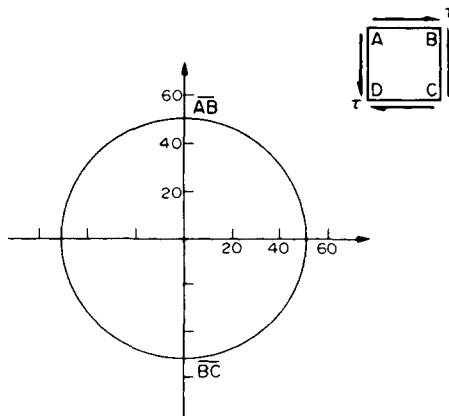


Fig. 13.30.

Example 13.5 (B)

At a point in a piece of elastic material direct stresses of 90 MN/m^2 tensile and 50 MN/m^2 compressive are applied on mutually perpendicular planes. The planes are also subjected to a shear stress. If the greater principal stress is limited to 100 MN/m^2 tensile, determine:

- the value of the shear stress;
- the other principal stress;
- the normal stress on the plane of maximum shear;
- the maximum shear stress.

Make a neat sketch showing clearly the positions of the principal planes and planes of maximum shear stress with respect to the planes of the applied stresses.

Solution

(a) Principal stress $\sigma_1 = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}$

This is limited to 100 MN/m^2 ; therefore shear stress τ_{xy} is given by

$$100 = \frac{1}{2}(90 - 50) + \frac{1}{2}\sqrt{[(90 + 50)^2 + 4\tau_{xy}^2]}$$

$$\therefore 200 = 40 + 10\sqrt{[14^2 + 4\tau_{xy}^2]}$$

$$\begin{aligned} \therefore \tau_{xy} &= \sqrt{\left(\frac{16^2 - 14^2}{0.04}\right)} = \sqrt{\left(\frac{256 - 196}{0.04}\right)} = \frac{\sqrt{60}}{0.2} \\ &= 38.8 \text{ MN/m}^2 \end{aligned}$$

The required shear stress is 38.8 MN/m^2 .

- (b) The other principal stress σ_2 is given by

$$\begin{aligned} \sigma_2 &= \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]} \\ &= \frac{1}{2}[(90 - 50) - 10\sqrt{(14^2 + 60)}] = \frac{40 - 10\sqrt{(256)}}{2} \\ &= \frac{40 - 160}{2} = -60 \text{ MN/m}^2 \end{aligned}$$

The other principal stress is 60 MN/m^2 compressive.

- (c) The normal stress on the plane of maximum shear

$$\begin{aligned} &= \frac{\sigma_1 + \sigma_2}{2} = \frac{100 - 60}{2} \\ &= 20 \text{ MN/m}^2 \end{aligned}$$

The required normal stress is 20 MN/m^2 tensile.

- (d) The maximum shear stress is given by

$$\begin{aligned} \tau_{\max} &= \frac{\sigma_1 - \sigma_2}{2} = \frac{100 + 60}{2} \\ &= 80 \text{ MN/m}^2 \end{aligned}$$

The maximum shear stress is 80 MN/m^2 .

In order to be able to draw the required sketch (Fig. 13.31) to indicate the relative positions of the planes on which the above stresses act, the angles of the principal planes are required. These are given by

$$\tan \theta = \frac{\sigma_p - \sigma_x}{\tau_{xy}} = \frac{100 - (-50)}{38.8}$$

$$= \frac{150}{38.8} = 3.87$$

$$\therefore \theta_1 = 75^\circ 30'$$

to the plane on which the 50 MN/m^2 stress acts.

The required sketch is then shown in Fig. 13.31.

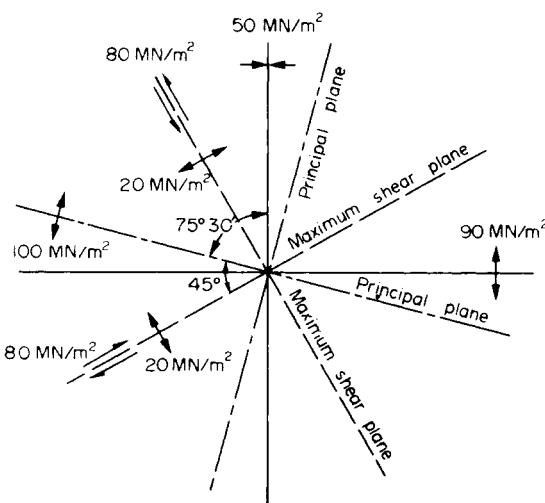


Fig. 13.31. Summary of principal planes and maximum shear planes.

Mohr's circle solution

The stress system is as shown in Fig. 13.32. The centre of the Mohr's circle is positioned midway between the two direct stresses given, and the radius is such that $\sigma_1 = 100 \text{ MN/m}^2$.

By measurement:

$$\tau = 39 \text{ MN/m}^2$$

$$\sigma_2 = 60 \text{ MN/m}^2 \text{ compressive}$$

$$\tau_{\max} = 80 \text{ MN/m}^2$$

$$\theta_1 = \frac{151}{2} = 75^\circ 30' \text{ to } BC, \text{ the plane on which the } 50 \text{ MN/m}^2 \text{ stress acts}$$

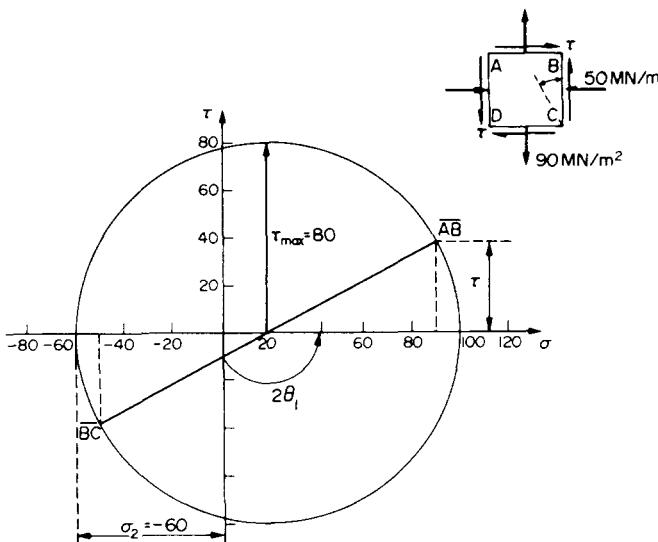


Fig. 13.32.

Example 13.6 (B)

In a certain material under load a plane AB carries a tensile direct stress of 30 MN/m^2 and a shear stress of 20 MN/m^2 , while another plane BC carries a tensile direct stress of 20 MN/m^2 and a shear stress. If the planes are inclined to one another at 30° and plane AC at right angles to plane AB carries a direct stress unknown in magnitude and nature, find:

- the value of the shear stress on BC ;
- the magnitude and nature of the direct stress on AC ;
- the principal stresses.

Solution

Referring to Fig. 13.33 let the shear stress on BC be τ and the direct stress on AC be σ_x , assumed tensile. Consider the equilibrium of the elemental wedge ABC . Assume this wedge to be of unit depth. A complementary shear stress equal to that on AB will be set up on AC .

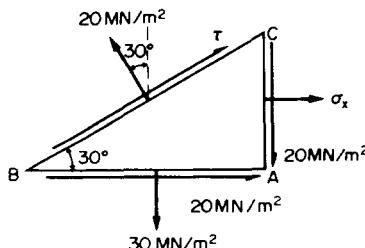


Fig. 13.33.

(a) To find τ , resolve forces vertically:

$$30 \times (AB \times 1) + 20 \times (AC \times 1) = 20 \times (BC \times 1) \cos 30^\circ + \tau \times (BC \times 1) \sin 30^\circ$$

Now

$$AB = BC \cos 30^\circ \quad \text{and} \quad AC = BC \sin 30^\circ$$

$$\therefore 30 \times BC \cos 30^\circ + 20 \times BC \sin 30^\circ = 20 \times BC \cos 30^\circ + \tau \times BC \sin 30^\circ$$

$$30 \frac{\sqrt{3}}{2} + 20 \times \frac{1}{2} = 20 \times \frac{\sqrt{3}}{2} + \tau \times \frac{1}{2}$$

$$30\sqrt{3} + 20 = 20\sqrt{3} + \tau$$

$$\therefore \tau = 10\sqrt{3} + 20 = 37.32 \text{ MN/m}^2$$

The required shear stress is 37.32 MN/m^2 .

(b) To find σ_x , resolve forces horizontally:

$$20 \times (AB \times 1) + \sigma_x \times (AC \times 1) + \tau \times (BC \times 1) \cos 30^\circ = 20 \times (BC \times 1) \sin 30^\circ$$

$$20 \times BC \cos 30^\circ + \sigma_x \times BC \sin 30^\circ + \tau \times BC \cos 30^\circ = 20 \times BC \sin 30^\circ$$

$$20 \times \frac{\sqrt{3}}{2} + \sigma_x \times \frac{1}{2} + \tau \times \frac{\sqrt{3}}{2} = 20 \times \frac{1}{2}$$

$$20\sqrt{3} + \sigma_x + \sqrt{3} \times 37.32 = 10$$

$$\therefore \sigma_x = 10 - \sqrt{3} \times 57.32 = 10 - 99.2$$

$$= -89.2 \text{ MN/m}^2, \text{ i.e. compressive}$$

(c) The principal stresses are now given by

$$\begin{aligned}\sigma_{1,2} &= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]} \\ &= \frac{1}{2}\{(-89.2 + 30) \pm \sqrt{[(-89.2 - 30)^2 + 4 \times 20^2]}\} \\ &= 5\{-5.92 \pm \sqrt{[(-11.92)^2 + 16]}\} \\ &= 5[-5.92 \pm \sqrt{158}] = 5[-5.92 \pm 12.57]\end{aligned}$$

$$\therefore \sigma_1 = 33.25 \text{ MN/m}^2$$

$$\sigma_2 = -92.45 \text{ MN/m}^2$$

The principal stresses are 33.25 MN/m^2 tensile and 92.45 MN/m^2 compressive.

Example 13.7 (B)

A hollow steel shaft of 100 mm external diameter and 50 mm internal diameter transmits 0.75 MW at 500 rev/min and is also subjected to an axial end thrust of 50 kN. Determine the maximum bending moment which can be safely applied in conjunction with the applied torque and thrust if the maximum compressive principal stress is not to exceed 100 MN/m^2 compressive. What will then be the value of:

- (a) the other principal stress;
- (b) the maximum shear stress?

Solution

The torque on the shaft may be found from

$$\text{power} = T \times \omega$$

$$\therefore T = \frac{0.75 \times 10^6 \times 60}{2\pi \times 500} = 14.3 \times 10^3 = 14.3 \text{ kN m}$$

The shear stress in the shaft at the surface is then given by the torsion theory

$$\begin{aligned}\frac{T}{J} &= \frac{\tau}{R} \\ \tau &= \frac{TR}{J} = \frac{14.3 \times 10^3 \times 50 \times 10^{-3} \times 2}{\pi(50^4 - 25^4)10^{-12}} \\ &= 0.78 \times 10^8 \\ &= 78 \text{ MN/m}^2\end{aligned}$$

The direct stress resulting from the end thrust is given by

$$\begin{aligned}\sigma_d &= \frac{\text{load}}{\text{area}} = \frac{-50 \times 10^3}{\pi(50^2 - 25^2)} 10^{-6} \\ &= -8.5 \times 10^6 \\ &= -8.5 \text{ MN/m}^2\end{aligned}$$

The bending moment to be applied will produce a direct stress in the same direction as σ_d . Thus the total stress in the x direction is

$$\sigma_x = \sigma_b + \sigma_d$$

the greatest value of σ_x being obtained where the bending stress is of the same sign as the end thrust or, in other words, compressive. The stress system is therefore as shown in Fig. 13.34.

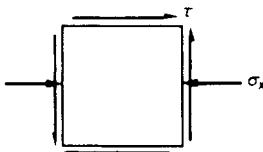


Fig. 13.34.

N.B. $\sigma_y = 0$; there is no stress in the y direction.

$$\sigma_1 = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}$$

Therefore substituting all stresses in units of MN/m^2 ,

$$-100 = \frac{1}{2}\sigma_x \pm \frac{1}{2}\sqrt{(\sigma_x^2 + 4\tau^2)}$$

$$\therefore -200 - \sigma_x = \pm \sqrt{(\sigma_x^2 + 4\tau^2)}$$

$$\begin{aligned}\therefore 4 \times 10^4 + 400\sigma_x + \sigma_x^2 &= \sigma_x^2 + 4\tau^2 \\ \therefore 400\sigma_x &= 4\tau^2 - 4 \times 10^4 \\ &= 24320 - 40000 \\ \therefore \sigma_x &= -39.2 \text{ MN/m}^2\end{aligned}$$

Therefore stress owing to bending

$$\begin{aligned}\sigma_b &= \sigma_x - \sigma_d = -39.2 - (-8.5) \\ &= -30.7 \text{ MN/m}^2 \quad (\text{i.e. compressive})\end{aligned}$$

But from bending theory

$$\begin{aligned}\sigma_b &= \frac{My}{I} \\ \therefore M &= \frac{30.7 \times 10^6 \times \pi(50^4 - 25^4)10^{-12}}{50 \times 10^{-3} \times 4} \\ &= 2830 \text{ N m} \\ &= 2.83 \text{ kN m}\end{aligned}$$

i.e. the bending moment which can be safely applied is **2.83 kN m**.

(a) The other principal stress

$$\begin{aligned}\sigma_2 &= \frac{1}{2}\sigma_x + \frac{1}{2}\sqrt{(\sigma_x^2 + 4\tau^2)} \\ &= -19.6 + \frac{1}{2}\sqrt{(39.2^2 + 24320)} \\ &= -19.6 + 80.5 \\ &= \mathbf{60.9 \text{ MN/m}^2 \text{ (tensile)}}\end{aligned}$$

(b) The maximum shear stress is given by

$$\begin{aligned}\tau_{\max} &= \frac{1}{2}(\sigma_1 - \sigma_2) \\ &= \frac{1}{2}(-100 - 60.9) \\ &= \mathbf{-80.45 \text{ MN/m}^2}\end{aligned}$$

i.e. the maximum shear stress is **80.45 MN/m²**.

Example 13.8

A beam of symmetrical I-section is simply supported at each end and loaded at the centre of its 3 m span with a concentrated load of 100 kN. The dimensions of the cross-section are: flanges 150 mm wide by 30 mm thick; web 30 mm thick; overall depth 200 mm.

For the transverse section at the point of application of the load, and considering a point at the top of the web where it meets the flange, calculate the magnitude and nature of the principal stresses. Neglect the self-mass of the beam.

Solution

At any section of the beam there will be two sets of stresses acting simultaneously:

(1) bending stresses

$$\sigma_b = \frac{My}{I}$$

(2) shear stresses

$$\tau = \frac{Q\bar{A}\bar{y}}{Ib}$$

together with their associated complementary shear stresses of the same value (Fig. 13.35a).

The stress system on any element of the beam can therefore be represented as in Fig. 13.36. The stress distribution diagrams are shown in Fig. 13.35b.

Bending stress

$$\sigma_b = \frac{My}{I}$$

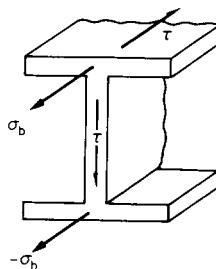
M = maximum bending moment

$$= \frac{WL}{4} = \frac{100 \times 10^3 \times 3}{4} = 75 \text{ kN m}$$

and

$$I = \frac{0.15 \times 0.2^3 - 0.12 \times 0.14^3}{12} \text{ m}^4$$

$$= 72.56 \times 10^{-6} \text{ m}^4$$



(a)

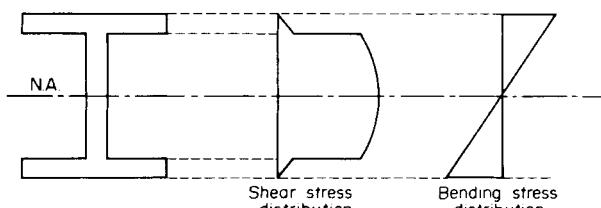


Fig. 13.35.

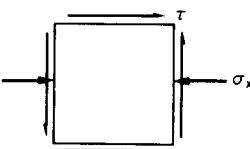


Fig. 13.36.

Therefore at the junction of web and flange

$$\begin{aligned}\sigma_b &= \frac{75 \times 10^3 \times 0.07}{72.56 \times 10^{-6}} \\ &= 72.35 \times 10^6 = 72.35 \text{ MN/m}^2 \text{ and is compressive}\end{aligned}$$

Shear stress

$$\begin{aligned}\tau &= \frac{Q A \bar{y}}{I b} \\ &= \frac{50 \times 10^3 \times (150 \times 30) \times 85 \times 10^{-9}}{72.56 \times 10^{-4} \times 30 \times 10^{-3}} \\ &= 8.79 \text{ MN/m}^2\end{aligned}$$

The principal stresses are then given by

$$\sigma_1 \quad \text{or} \quad \sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}$$

with

$$\sigma_x = -\sigma_b \quad \text{and} \quad \sigma_y = 0$$

$$\begin{aligned}\therefore \sigma_1 \quad \text{or} \quad \sigma_2 &= \frac{1}{2}(-72.35) \pm \frac{1}{2}\sqrt{[(-72.35)^2 + 4 \times 8.79^2]} \text{ MN/m}^2 \\ &= -36.2 \pm \sqrt{(5544)} \\ &= -36.2 \pm 74.5 \\ \therefore \sigma_2 &= -110.7 \text{ MN/m}^2 \\ \sigma_1 &= +38.3 \text{ MN/m}^2\end{aligned}$$

i.e. the principal stresses are 110.7 MN/m² compressive and 38.3 MN/m² tensile in the top of the web. At the bottom of the web the stress values obtained would be of the same value but of opposite sign.

Problems

13.1 (A). An axial tensile load of 10 kN is applied to a 12 mm diameter bar. Determine the maximum shearing stress in the bar and the planes on which it acts. Find also the value of the normal stresses on these planes.
[44.1 MN/m² at 45° and 135°; ± 44.2 MN/m².]

13.2 (A). A compressive member of a structure is of 25 mm square cross-section and carries load of 50 kN. Determine, from first principles, the normal, tangential and resultant stresses on a plane inclined at 60° to the axis of the bar.
[60, 34.6, 69.3 MN/m².]

13.3 (A). A rectangular block of material is subjected to a shear stress of 30 MN/m^2 together with its associated complementary shear stress. Determine the magnitude of the stresses on a plane inclined at 30° to the directions of the applied stresses, which may be taken as horizontal.

[26, 15 MN/m^2 .]

13.4 (A). A material is subjected to two mutually perpendicular stresses, one 60 MN/m^2 compressive and the other 45 MN/m^2 tensile. Determine the direct, shear and resultant stresses on a plane inclined at 60° to the plane on which the 45 MN/m^2 stress acts.

[18.75, 45.5, 49.2 MN/m^2 .]

13.5 (A/B). The material of Problem 13.4 is now subjected to an additional shearing stress of 10 MN/m^2 . Determine the principal stresses acting on the material and the maximum shear stress.

[46, -61, 53.5 MN/m^2 .]

13.6 (A/B). At a certain section in a material under stress, direct stresses of 45 MN/m^2 tensile and 75 MN/m^2 tensile act on perpendicular planes together with a shear stress τ acting on these planes. If the maximum stress in the material is limited to 150 MN/m^2 tensile determine the value of τ .

[88.7 MN/m^2 .]

13.7 (A/B). At a point in a material under stress there is a compressive stress of 200 MN/m^2 and a shear stress of 300 MN/m^2 acting on the same plane. Determine the principal stresses and the directions of the planes on which they act.

[216 MN/m^2 at 54.2° to 200 MN/m^2 plane; -416 MN/m^2 at 144.2° .]

13.8 (A/B). At a certain point in a material the following stresses act: a tensile stress of 150 MN/m^2 , a compressive stress of 105 MN/m^2 at right angles to the tensile stress and a shear stress clockwise in effect of 30 MN/m^2 . Calculate the principal stresses and the directions of the principal planes.

[153.5 , -108.5 MN/m^2 ; at 6.7° and 96.7° counterclockwise to 150 MN/m^2 plane.]

13.9 (B). The stresses across two mutually perpendicular planes at a point in an elastic body are 120 MN/m^2 tensile with 45 MN/m^2 clockwise shear, and 30 MN/m^2 tensile with 45 MN/m^2 counterclockwise shear. Find (i) the principal stresses, (ii) the maximum shear stress, and (iii) the normal and tangential stresses on a plane measured at 20° counterclockwise to the plane on which the 30 MN/m^2 stress acts. Draw sketches showing the positions of the stresses found above and the planes on which they act relative to the original stresses.

[138.6 , 11.4 , 63.6 , 69.5 , -63.4 MN/m^2 .]

13.10 (B). At a point in a strained material the stresses acting on planes at right angles to each other are 200 MN/m^2 tensile and 80 MN/m^2 compressive, together with associated shear stresses which may be assumed clockwise in effect on the 80 MN/m^2 planes. If the principal stress is limited to 320 MN/m^2 tensile, calculate:

- (a) the magnitude of the shear stresses;
- (b) the directions of the principal planes;
- (c) the other principal stress;
- (d) the maximum shear stress.

[219 MN/m^2 , 28.7 and 118.7° counterclockwise to 200 MN/m^2 plane; -200 MN/m^2 , 260 MN/m^2 .]

13.11 (B). A solid shaft of 125 mm diameter transmits 0.5 MW at 300 rev/min . It is also subjected to a bending moment of 9 kN m and to a tensile end load. If the maximum principal stress is limited to 75 MN/m^2 , determine the permissible end thrust. Determine the position of the plane on which the principal stress acts, and draw a diagram showing the position of the plane relative to the torque and the plane of the bending moment.

[61.4 kN ; 61° to shaft axis.]

13.12 (B). At a certain point in a piece of material there are two planes at right angles to one another on which there are shearing stresses of 150 MN/m^2 together with normal stresses of 300 MN/m^2 tensile on one plane and 150 MN/m^2 tensile on the other plane. If the shear stress on the 150 MN/m^2 planes is taken as clockwise in effect determine for the given point:

- (a) the magnitudes of the principal stresses;
- (b) the inclinations of the principal planes;
- (c) the maximum shear stress and the inclinations of the planes on which it acts;
- (d) the maximum strain if $E = 208 \text{ GN/m}^2$ and Poisson's ratio = 0.29.

[392.7 , 57.3 MN/m^2 ; 31.7° , 121.7° ; 167.7 MN/m^2 , 76.7° , 166.7° ; $1810 \mu\epsilon$.]

13.13 (B). A 250 mm diameter solid shaft drives a screw propeller with an output of 7 MW . When the forward speed of the vessel is 35 km/h the speed of revolution of the propeller is 240 rev/min . Find the maximum stress resulting from the torque and the axial compressive stress resulting from the thrust in the shaft; hence find for a point on the surface of the shaft (a) the principal stresses, and (b) the directions of the principal planes relative to the shaft axis. Make a diagram to show clearly the direction of the principal planes and stresses relative to the shaft axis.

[U.L.] [90.8 , 14.7 , 98.4 , -83.7 MN/m^2 ; 47° and 137° .]

13.14 (B). A hollow shaft is 460 mm inside diameter and 25 mm thick. It is subjected to an internal pressure of 2 MN/m^2 , a bending moment of 25 kN m and a torque of 40 kN m . Assuming the shaft may be treated as a thin cylinder, make a neat sketch of an element of the shaft, showing the stresses resulting from all three actions. Determine the values of the principal stresses and the maximum shear stress.

[21.5 , 11.8 , 16.6 MN/m^2 .]

13.15 (B). In a piece of material a tensile stress σ_1 and a shearing stress τ act on a given plane. Show that the principal stresses are always of opposite sign. If an additional tensile stress σ_2 acts on a plane perpendicular to that of σ_1 , find the condition that both principal stresses may be of the same sign. [U.L.] [$\tau = \sqrt{(\sigma_1 \sigma_2)}$.]

13.16 (B). A shaft 100 mm diameter is subjected to a twisting moment of 7 kN m, together with a bending moment of 2 kN m. Find, at the surface of the shaft, (a) the principal stresses, (b) the maximum shear stress.

$$[47.3, -26.9 \text{ MN/m}^2; 37.1 \text{ MN/m}^2.]$$

13.17 (B). A material is subjected to a horizontal tensile stress of 90 MN/m² and a vertical tensile stress of 120 MN/m², together with shear stresses of 75 MN/m², those on the 120 MN/m² planes being counterclockwise in effect. Determine:

- (a) the principal stresses;
- (b) the maximum shear stress;
- (c) the shear stress which, acting alone, would produce the same principal stress;
- (d) the tensile stress which, acting alone, would produce the same maximum shear stress.

$$[181.5, 28.5 \text{ MN/m}^2; 76.5 \text{ MN/m}^2; 181.5 \text{ MN/m}^2; 153 \text{ MN/m}^2.]$$

13.18 (B). Two planes *AB* and *BC* in an elastic material under load are inclined at 45° to each other. The loading on the material is such that the stresses on these planes are as follows:

On *AB*, 150 MN/m² direct stress and 120 MN/m² shear.

On *BC*, 80 MN/m² shear and a direct stress σ .

Determine the value of the unknown stress σ on *BC* and hence determine the principal stresses which exist in the material.

$$[190, 214, -74 \text{ MN/m}^2.]$$

13.19 (B). A beam of I-section, 500 mm deep and 200 mm wide, has flanges 25 mm thick and web 12 mm thick. It carries a concentrated load of 300 kN at the centre of a simply supported span of 3 m. Calculate the principal stresses set up in the beam at the point where the web meets the flange.

$$[83.4, -6.15 \text{ MN/m}^2.]$$

13.20 (B). At a certain point on the outside of a shaft which is subjected to a torque and a bending moment the shear stresses are 100 MN/m² and the longitudinal direct stress is 60 MN/m² tensile. Find, by calculation from first principles or by graphical construction which must be justified:

- (a) the maximum and minimum principal stresses;
- (b) the maximum shear stress;
- (c) the inclination of the principal stresses to the original stresses.

Summarize the answers clearly on a diagram, showing their relative positions to the original stresses.

$$[\text{E.M.E.U.}] [134.4, -74.4 \text{ MN/m}^2, 104.4 \text{ MN/m}^2; 35.5^\circ.]$$

13.21 (B). A short vertical column is firmly fixed at the base and projects a distance of 300 mm from the base. The column is of I-section, 200 mm deep by 100 mm wide, flanges 10 mm thick, web 6 mm thick.

An inclined load of 80 kN acts on the top of the column in the centre of the section and in the plane containing the central line of the web; the line of action is inclined at 30 degrees to the vertical. Determine the position and magnitude of the greatest principal stress at the base of the column.

$$[\text{U.L.}] [48 \text{ MN/m}^2 \text{ at junction of web and flange.}]$$

CHAPTER 14

COMPLEX STRAIN AND THE ELASTIC CONSTANTS

Summary

The relationships between the elastic constants are

$$E = 2G(1 + \nu) \quad \text{and} \quad E = 3K(1 - 2\nu)$$

Poisson's ratio ν being defined as the ratio of lateral strain to longitudinal strain and bulk modulus K as the ratio of volumetric stress to volumetric strain.

The strain in the x direction in a material subjected to three mutually perpendicular stresses in the x , y and z directions is given by

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E} = \frac{1}{E}(\sigma_x - \nu\sigma_y - \nu\sigma_z)$$

Similar equations apply for ε_y and ε_z .

Thus the principal strain in a given direction can be found in terms of the principal stresses, since

$$\varepsilon_1 = \frac{\sigma_1}{E} - \nu \frac{\sigma_2}{E} - \nu \frac{\sigma_3}{E} = \frac{1}{E}(\sigma_1 - \nu\sigma_2 - \nu\sigma_3)$$

For a two-dimensional stress system (i.e. $\sigma_3 = 0$), principal stresses can be found from known principal strains, since

$$\sigma_1 = \frac{(\varepsilon_1 + \nu\varepsilon_2)}{(1 - \nu^2)}E \quad \text{and} \quad \sigma_2 = \frac{(\varepsilon_2 + \nu\varepsilon_1)}{(1 - \nu^2)}E$$

When the linear strains in two perpendicular directions are known, together with the associated shear strain, or when three linear strains are known, the principal strains are easily determined by the use of Mohr's strain circle.

14.1. Linear strain for tri-axial stress state

Consider an element subjected to three mutually perpendicular tensile stresses σ_x , σ_y and σ_z as shown in Fig. 14.1.

If σ_y and σ_z were not present the strain in the x direction would, from the basic definition of Young's modulus E , be

$$\varepsilon_x = \frac{\sigma_x}{E}$$

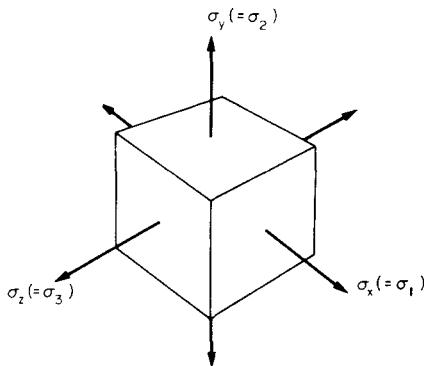


Fig. 14.1.

The effects of σ_y and σ_z in the x direction are given by the definition of Poisson's ratio ν to be

$$-\nu \frac{\sigma_y}{E} \quad \text{and} \quad -\nu \frac{\sigma_z}{E} \quad \text{respectively}$$

the negative sign indicating that if σ_y and σ_z are positive, i.e. tensile, then they tend to reduce the strain in the x direction.

Thus the total linear strain in the x direction is given by

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E}$$

i.e.

$$\varepsilon_x = \frac{1}{E} (\sigma_x - \nu \sigma_y - \nu \sigma_z) \quad (14.1)$$

Similarly the strains in the y and z directions would be

$$\varepsilon_y = \frac{1}{E} (\sigma_y - \nu \sigma_x - \nu \sigma_z)$$

$$\varepsilon_z = \frac{1}{E} (\sigma_z - \nu \sigma_x - \nu \sigma_y)$$

The three equations being known as the "generalised Hooke's Law" from which the simple uniaxial form of §1.5 is obtained (when two of the three stresses are reduced to zero).

14.2. Principal strains in terms of stresses

In the absence of shear stresses on the faces of the element shown in Fig. 14.1 the stresses σ_x , σ_y and σ_z are in fact principal stresses. Thus the principal strain in a given direction is obtained from the principal stresses as

$$\varepsilon_1 = \frac{1}{E} (\sigma_1 - \nu \sigma_2 - \nu \sigma_3)$$

or

$$\varepsilon_2 = \frac{1}{E} (\sigma_2 - v\sigma_1 - v\sigma_3) \quad (14.2)$$

or

$$\varepsilon_3 = \frac{1}{E} (\sigma_3 - v\sigma_1 - v\sigma_2)$$

14.3. Principal stresses in terms of strains – two-dimensional stress system

For a two-dimensional stress system, i.e. $\sigma_3 = 0$, the above equations reduce to

$$\varepsilon_1 = \frac{1}{E} (\sigma_1 - v\sigma_2)$$

and

$$\varepsilon_2 = \frac{1}{E} (\sigma_2 - v\sigma_1)$$

with

$$\varepsilon_3 = \frac{1}{E} (-v\sigma_1 - v\sigma_2)$$

∴

$$E\varepsilon_1 = \sigma_1 - v\sigma_2$$

$$E\varepsilon_2 = \sigma_2 - v\sigma_1$$

Solving these equations simultaneously yields the following values for the principal stresses:

$$\sigma_1 = \frac{E}{(1-v^2)} (\varepsilon_1 + v\varepsilon_2) \quad (14.3)$$

and

$$\sigma_2 = \frac{E}{(1-v^2)} (\varepsilon_2 + v\varepsilon_1)$$

14.4. Bulk modulus K

It has been shown previously that Young's modulus E and the shear modulus G are defined as the ratio of stress to strain under direct load and shear respectively. Bulk modulus is similarly defined as a ratio of stress to strain under uniform pressure conditions. Thus if a material is subjected to a uniform pressure (or volumetric stress) σ in all directions then

$$\text{bulk modulus} = \frac{\text{volumetric stress}}{\text{volumetric strain}}$$

i.e.

$$K = \frac{\sigma}{\varepsilon_v} \quad (14.4)$$

the volumetric strain being defined below.

14.5. Volumetric strain

Consider a rectangular block of sides x , y and z subjected to a system of equal direct stresses σ on each face. Let the sides be changed in length by δx , δy and δz respectively under stress (Fig. 14.2).

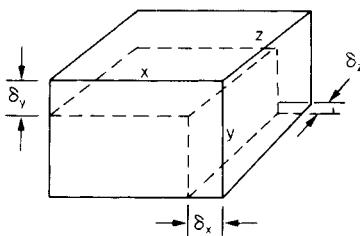


Fig. 14.2. Rectangular element subjected to uniform compressive stress on all faces producing decrease in size shown.

The volumetric strain is defined as follows:

$$\text{volumetric strain} = \frac{\text{change in volume}}{\text{original volume}} = \frac{\delta V}{V} = \frac{\delta V}{xyz}$$

The change in volume can best be found by calculating the volume of the strips to be cut off the original size of block to reduce it to the dotted block shown in Fig. 14.2.

Then

$$\delta V = xy\delta z + y(z - \delta z)\delta x + (x - \delta x)(z - \delta z)\delta y$$

strip at back strip at side strip on top

and neglecting the products of small quantities

$$\delta V = xy\delta z + yz\delta x + xz\delta y$$

$$\therefore \text{volumetric strain} = \frac{(xy\delta z + yz\delta x + xz\delta y)}{xyz} = \epsilon_v$$

$$\therefore \epsilon_v = \frac{\delta z}{z} + \frac{\delta x}{x} + \frac{\delta y}{y} = \epsilon_x + \epsilon_y + \epsilon_z \quad (14.5)$$

i.e. **volumetric strain = sum of the three mutually perpendicular linear strains**

14.6. Volumetric strain for unequal stresses

It has been shown above that the volumetric strain is the sum of the three perpendicular linear strains

$$\epsilon_v = \epsilon_x + \epsilon_y + \epsilon_z$$

Substituting for the strains in terms of stresses as given by eqn. (14.1),

$$\begin{aligned} \epsilon_v &= \frac{1}{E}(\sigma_x - v\sigma_y - v\sigma_z) + \frac{1}{E}(\sigma_y - v\sigma_x - v\sigma_z) \\ &\quad + \frac{1}{E}(\sigma_z - v\sigma_x - v\sigma_y) \\ \epsilon_v &= \frac{1}{E}(\sigma_x + \sigma_y + \sigma_z)(1 - 2v) \end{aligned} \quad (14.6)$$

It will be shown later that the following relationship applies between the elastic constants E , v and K ,

$$E = 3K(1 - 2v)$$

Thus the volumetric strain may be written in terms of the bulk modulus as follows:

$$\epsilon_v = \frac{(\sigma_x + \sigma_y + \sigma_z)}{3K} \quad (14.7)$$

This equation applies to solid bodies only and cannot be used for the determination of internal volume (or capacity) changes of hollow vessels. It may be used, however, for changes in cylinder wall volume.

14.7. Change in volume of circular bar

A simple application of eqn. (14.6) is to the determination of volume changes of circular bars under direct load.

Consider, therefore, a circular bar subjected to a direct stress σ applied axially as shown in Fig. 14.3.

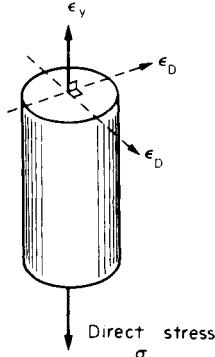


Fig. 14.3. Circular bar subjected to direct axial stress σ .

Here

$$\sigma_y = \sigma, \quad \sigma_x = 0 \quad \text{and} \quad \sigma_z = 0$$

Therefore from eqn. (14.6)

$$\epsilon_v = \frac{\sigma}{E}(1 - 2v) = \frac{\delta V}{V}$$

$$\therefore \text{change of volume} = \delta V = \frac{\sigma V}{E}(1 - 2v) \quad (14.8)$$

This formula could have been obtained from eqn. (14.5) with

$$\epsilon_y = \frac{\sigma}{E} \quad \text{and} \quad \epsilon_x = \epsilon_z = \epsilon_D = -v \frac{\sigma}{E}$$

then

$$\epsilon_v = \epsilon_x + \epsilon_y + \epsilon_z = \frac{\delta V}{V}$$

$$\therefore \delta V = V \left(\frac{\sigma}{E} - 2v \frac{\sigma}{E} \right) = \frac{\sigma V}{E}(1 - 2v)$$

14.8. Effect of lateral restraint

(a) Restraint in one direction only

Consider a body subjected to a two-dimensional stress system with a rigid lateral restraint provided in the y direction as shown in Fig. 14.4. Whilst the material is free to contract laterally in the x direction the “Poisson’s ratio” extension along the y axis is totally prevented.

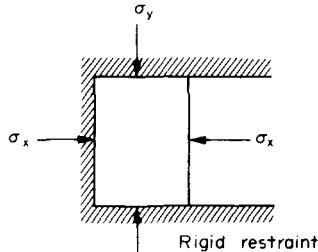


Fig. 14.4. Material subjected to lateral restraint in the y direction.

Therefore strain in the y direction with σ_x and σ_y both compressive, i.e. negative,

$$= \varepsilon_y = -\frac{1}{E} (\sigma_y - v\sigma_x) = 0$$

$$\therefore \sigma_y = v\sigma_x$$

Thus strain in the x direction

$$\begin{aligned} = \varepsilon_x &= -\frac{1}{E} (\sigma_x - v\sigma_y) \\ &= -\frac{1}{E} (\sigma_x - v^2\sigma_x) \\ &= -\frac{\sigma_x}{E} (1 - v^2) \end{aligned} \quad (14.9)$$

Thus the introduction of a lateral restraint affects the stiffness and hence the load-carrying capacity of the material by producing an **effective change of Young's modulus** from

$$E \text{ to } E/(1 - v^2)$$

(b) Restraint in two directions

Consider now a material subjected to a three-dimensional stress system σ_x , σ_y and σ_z with restraint provided in both the y and z directions. In this case,

$$\varepsilon_y = -\frac{1}{E} (\sigma_y - v\sigma_x - v\sigma_z) = 0 \quad (1)$$

$$\varepsilon_z = -\frac{1}{E} (\sigma_z - v\sigma_x - v\sigma_y) = 0 \quad (2)$$

and

From (1),

$$\sigma_y = v\sigma_x + v\sigma_z$$

$$\therefore \sigma_z = (\sigma_y - v\sigma_x) \frac{1}{v} \quad (3)$$

Substituting in (2),

$$\begin{aligned} \frac{1}{v}(\sigma_y - v\sigma_x) - v\sigma_x - v\sigma_y &= 0 \\ \therefore \sigma_y - v\sigma_x - v^2\sigma_x - v^2\sigma_y &= 0 \\ \sigma_y(1 - v^2) &= \sigma_x(v + v^2) \\ \sigma_y &= \sigma_x \frac{v(1 + v)}{(1 - v^2)} \\ &= \frac{\sigma_x v}{(1 - v)} \end{aligned}$$

and from (3),

$$\begin{aligned} \sigma_z &= \frac{1}{v} \left[\frac{v\sigma_x}{(1 - v)} - v\sigma_x \right] \\ &= \sigma_x \left[\frac{1 - (1 - v)}{(1 - v)} \right] = \frac{v\sigma_x}{(1 - v)} \\ \therefore \text{strain in } x \text{ direction} &= -\frac{\sigma_x}{E} + v\frac{\sigma_y}{E} + v\frac{\sigma_z}{E} \\ &= -\frac{\sigma_x}{E} \left[1 - \frac{v^2}{(1 - v)} - \frac{v^2}{(1 - v)} \right] \\ &= -\frac{\sigma_x}{E} \left[1 - \frac{2v^2}{(1 - v)} \right] \quad (14.10) \end{aligned}$$

Again Young's modulus E is effectively changed, this time to

$$E \left/ \left[1 - \frac{2v^2}{(1 - v)} \right] \right.$$

14.9. Relationship between the elastic constants E , G , K and v

(a) E , G and v

Consider a cube of material subjected to the action of the shear and complementary shear forces shown in Fig. 14.5 producing the strained shape indicated.

Assuming that the strains are small the angle ACB may be taken as 45° .

Therefore strain on diagonal OA

$$= \frac{BC}{OA} = \frac{AC \cos 45^\circ}{a\sqrt{2}} = \frac{AC}{a\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{AC}{2a}$$

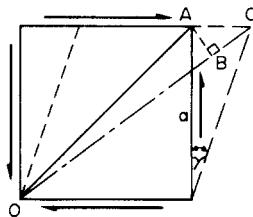


Fig. 14.5. Element subjected to shear and associated complementary shear.

But $AC = a\gamma$, where γ = angle of distortion or shear strain.

$$\therefore \text{strain on diagonal} = \frac{a\gamma}{2a} = \frac{\gamma}{2}$$

Now
$$\frac{\text{shear stress } \tau}{\text{shear strain } \gamma} = G$$

$$\therefore \gamma = \frac{\tau}{G}$$

$$\therefore \text{strain on diagonal} = \frac{\tau}{2G} \quad (1)$$

From §13.2 the shear stress system can be replaced by a system of direct stresses at 45° , as shown in Fig. 14.6. One set will be compressive, the other tensile, and both will be equal in value to the applied shear stresses.

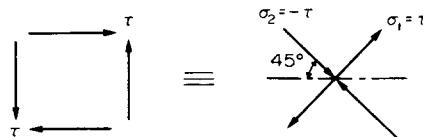


Fig. 14.6. Direct stresses due to shear.

Thus, from the direct stress system which applies along the diagonals:

$$\begin{aligned} \text{strain on diagonal} &= \frac{\sigma_1}{E} - \nu \frac{\sigma_2}{E} \\ &= \frac{\tau}{E} - \nu \frac{(-\tau)}{E} \\ &= \frac{\tau}{E}(1 + \nu) \end{aligned} \quad (2)$$

Combining (1) and (2),

$$\frac{\tau}{2G} = \frac{\tau}{E}(1 + \nu) \quad E = 2G(1 + \nu) \quad (14.11)$$

(b) E , K and ν

Consider a cube subjected to three equal stresses σ as in Fig. 14.7 (i.e. volumetric stress = σ).

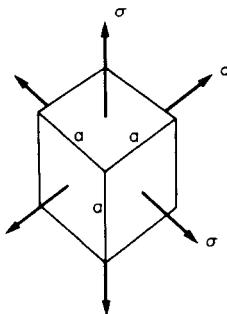


Fig. 14.7. Cubical element subjected to uniform stress σ on all faces ("volumetric" or "hydrostatic" stress).

$$\begin{aligned}\text{Total strain along one edge} &= \frac{\sigma}{E} - \nu \frac{\sigma}{E} - \nu \frac{\sigma}{E} \\ &= \frac{\sigma}{E}(1 - 2\nu)\end{aligned}$$

But

$$\text{volumetric strain} = 3 \times \text{linear strain} \quad (\text{see eqn. 14.5})$$

$$= \frac{3\sigma}{E}(1 - 2\nu) \quad (3)$$

By definition:

$$\text{bulk modulus } K = \frac{\text{volumetric stress}}{\text{volumetric strain}}$$

$$\text{volumetric strain} = \frac{\sigma}{K} \quad (4)$$

Equating (3) and (4),

$$\frac{\sigma}{K} = \frac{3\sigma}{E}(1 - 2\nu)$$

$$\therefore E = 3K(1 - 2\nu) \quad (14.12)$$

(c) G , K and ν

Equations (14.11) and (14.12) can now be combined to give the final relationship as follows:
From eqn. (14.11),

$$\nu = \frac{E}{2G} - 1$$

and from eqn. (14.12),

$$\nu = \frac{1}{2} - \frac{E}{6K}$$

Therefore, equating,

$$\frac{E}{2G} - 1 = \frac{1}{2} - \frac{E}{6K}$$

$$E \left[\frac{1}{2G} + \frac{1}{6K} \right] = \frac{3}{2}$$

$$\therefore E \left[\frac{6K + 2G}{12KG} \right] = \frac{3}{2}$$

i.e.

$$E = \frac{9KG}{(3K + G)} \quad (14.13)$$

14.10. Strains on an oblique plane

(a) Linear strain

Consider a rectangular block of material $OLMN$ as shown in the xy plane (Fig. 14.8). The strains along Ox and Oy are ϵ_x and ϵ_y , and γ_{xy} is the shearing strain.

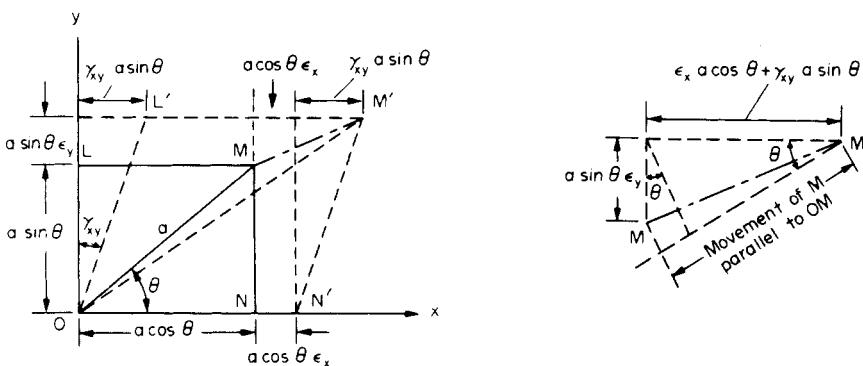


Fig. 14.8. Strains on an inclined plane.

Let the diagonal OM be of length a ; then $ON = a \cos \theta$ and $OL = a \sin \theta$, and the increases in length of these sides under strain are $\epsilon_x a \cos \theta$ and $\epsilon_y a \sin \theta$ (i.e. strain \times original length).

If M moves to M' , the movement of M parallel to the x axis is

$$\epsilon_x a \cos \theta + \gamma_{xy} a \sin \theta$$

and the movement parallel to the y axis is

$$\epsilon_y a \sin \theta$$

Thus the movement of M parallel to OM , which since the strains are small is practically coincident with MM' , is

$$(\epsilon_x a \cos \theta + \gamma_{xy} a \sin \theta) \cos \theta + (\epsilon_y a \sin \theta) \sin \theta$$

Then

$$\begin{aligned} \text{strain along } OM &= \frac{\text{extension}}{\text{original length}} \\ &= (\epsilon_x \cos \theta + \gamma_{xy} \sin \theta) \cos \theta + (\epsilon_y \sin \theta) \sin \theta \\ \therefore \epsilon_\theta &= \epsilon_x \cos^2 \theta + \epsilon_y \sin^2 \theta + \gamma_{xy} \sin \theta \cos \theta \\ \therefore \epsilon_\theta &= \frac{1}{2}(\epsilon_x + \epsilon_y) + \frac{1}{2}(\epsilon_x - \epsilon_y) \cos 2\theta + \frac{1}{2}\gamma_{xy} \sin 2\theta \end{aligned} \quad (14.14)$$

This is identical in form with the equation defining the direct stress on any inclined plane θ with ϵ_x and ϵ_y replacing σ_x and σ_y , and $\frac{1}{2}\gamma_{xy}$ replacing τ_{xy} , i.e. the shear stress is replaced by HALF the shear strain.

(b) Shear strain

To determine the shear strain in the direction OM consider the displacement of point P at the foot of the perpendicular from N to OM (Fig. 14.9).

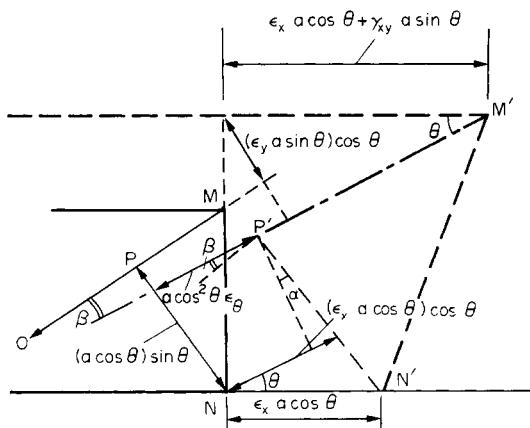


Fig. 14.9. Enlarged view of part of Fig. 14.8.

In the strained condition this point moves to P' .

Since

$$\text{strain along } OM = \epsilon_\theta$$

$$\text{extension of } OM = OM \cdot \epsilon_\theta$$

$$\text{extension of } OP = OP \cdot \epsilon_\theta$$

$$OP = (a \cos \theta) \cos \theta$$

But

$$\text{extension of } OP = a \cos^2 \theta \epsilon_\theta$$

\therefore

During straining the line PN rotates counterclockwise through a small angle α .

$$\begin{aligned}\alpha &= \frac{(\varepsilon_x a \cos \theta) \cos \theta - a \cos^2 \theta \varepsilon_\theta}{a \cos \theta \sin \theta} \\ &= (\varepsilon_x - \varepsilon_\theta) \cot \theta\end{aligned}$$

The line OM also rotates, but clockwise, through a small angle

$$\beta = \frac{(\varepsilon_x a \cos \theta + \gamma_{xy} a \sin \theta) \sin \theta - (\varepsilon_y a \sin \theta) \cos \theta}{a}$$

Thus the required shear strain γ_θ in the direction OM , i.e. the amount by which the angle OPN changes, is given by

$$\gamma_\theta = \alpha + \beta = (\varepsilon_x - \varepsilon_\theta) \cot \theta + (\varepsilon_x \cos \theta + \gamma_{xy} \sin \theta) \sin \theta - \varepsilon_y \sin \theta \cos \theta$$

Substituting for ε_θ from eqn. (14.14) gives

$$\gamma_\theta = 2(\varepsilon_x - \varepsilon_y) \cos \theta \sin \theta - \gamma_{xy} (\cos^2 \theta - \sin^2 \theta)$$

$$\therefore \frac{1}{2} \gamma_\theta = \frac{1}{2} (\varepsilon_x - \varepsilon_y) \sin 2\theta - \frac{1}{2} \gamma_{xy} \cos 2\theta$$

which again is similar in form to the expression for the shear stress τ on any inclined plane θ .

For consistency of sign convention, however (see §14.11 below), because OM' moves clockwise with respect to OM it is considered to be a negative shear strain, i.e.

$$\frac{1}{2} \gamma_\theta = -[\frac{1}{2} (\varepsilon_x - \varepsilon_y) \sin 2\theta - \frac{1}{2} \gamma_{xy} \cos 2\theta] \quad (14.15)$$

14.11. Principal strain – Mohr's strain circle

Since the equations for stress and strain on oblique planes are identical in form, as noted above, it is evident that Mohr's stress circle construction can be used equally well to represent strain conditions using the horizontal axis for linear strains and the vertical axis for half the shear strain. It should be noted, however, that angles given by Mohr's stress circle refer to the directions of the planes on which the stresses act and not to the direction of the stresses themselves. The directions of the stresses and hence the associated strains are therefore normal (i.e. at 90°) to the directions of the planes. Since angles are doubled in Mohr's circle construction it follows therefore that for true similarity of working a relative rotation of the axes of $2 \times 90^\circ = 180^\circ$ must be introduced. This is achieved by plotting positive shear strains vertically downwards on the strain circle construction as shown in Fig. 14.10.

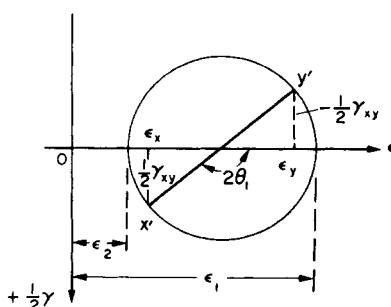


Fig. 14.10. Mohr's strain circle.

The sign convention adopted for strains is as follows:

Linear strains: extension positive
compression negative.

Shear strains:

The convention for shear strains is a little more difficult. The first subscript in the symbol γ_{xy} usually denotes the shear strain associated with that direction, i.e. with Ox . Similarly, γ_{yx} is the shear strain associated with Oy . If, under strain, the line associated with the first subscript moves counterclockwise with respect to the other line, the shearing strain is said to be positive, and if it moves clockwise it is said to be negative. It will then be seen that positive shear strains are associated with planes carrying positive shear stresses and negative shear strains with planes carrying negative shear stresses.

Thus,

$$\gamma_{xy} = -\gamma_{yx}$$

Mohr's circle for strains ϵ_x , ϵ_y and shear strain γ_{xy} (positive referred to x direction) is therefore constructed as for the stress circle with $\frac{1}{2}\gamma_{xy}$ replacing τ_{xy} and the axis of shear reversed, as shown in Fig. 14.10.

The maximum principal strain is then ϵ_1 at an angle θ_1 to ϵ_x in the same angular direction as that in Mohr's circle (Fig. 14.11).

Again, angles are doubled on Mohr's circle.

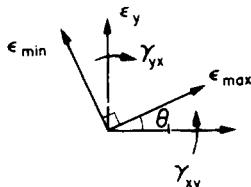


Fig. 14.11. Strain system at a point, including the principal strains and their inclination.

Strain conditions at any angle α to ϵ_x are found as in the stress circle by marking off an angle 2α from the point representing the x direction, i.e. x' . The coordinates of the point on the circle thus obtained are the strains required.

Alternatively, the principal strains may be determined *analytically* from eqn. (14.14), i.e.

$$\epsilon_\theta = \frac{1}{2}(\epsilon_x + \epsilon_y) + \frac{1}{2}(\epsilon_x - \epsilon_y)\cos 2\theta + \frac{1}{2}\gamma_{xy}\sin 2\theta$$

As for the derivation of the principal stress equations on page 331, the principal strains, i.e. the maximum and minimum values of strain, occur at values of θ obtained by equating $d\epsilon_\theta/d\theta$ to zero.

The procedure is identical to that of page 331 for the stress case and will not be repeated here. The values obtained are

$$\epsilon_1 \text{ or } \epsilon_2 = \frac{1}{2}(\epsilon_x + \epsilon_y) \pm \frac{1}{2}\sqrt{[(\epsilon_x - \epsilon_y)^2 + \gamma_{xy}^2]} \quad (14.16)$$

i.e. once again identical in form to the principal stress equation with ϵ replacing σ and $\frac{1}{2}\gamma$ replacing τ .

Similarly,

$$\frac{1}{2}\gamma_{\max} = \pm \frac{1}{2}\sqrt{[(\epsilon_x - \epsilon_y)^2 + \gamma_{xy}^2]} \quad (14.17)$$

14.12. Mohr's strain circle – alternative derivation from the general stress equations

The direct stress on any plane within a material inclined at an angle θ to the xy axes is given by eqn. (13.8) as:

$$\begin{aligned}\sigma_\theta &= \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta \\ \therefore \sigma_{\theta+90^\circ} &= \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos(2\theta + 180^\circ) + \tau_{xy}\sin(2\theta + 180^\circ) \\ &= \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta - \tau_{xy}\sin 2\theta\end{aligned}$$

Also, from eqn. (13.9),

$$\tau_\theta = \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta \quad (1)$$

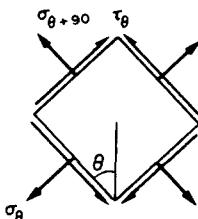


Fig. 14.12.

Now for the two-dimensional stress system shown in Fig. 14.12,

$$\begin{aligned}\varepsilon_\theta &= \frac{1}{E}(\sigma_\theta - v\sigma_{\theta+90^\circ}) \\ &= \frac{1}{E} \left\{ \left[\frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta \right] \right. \\ &\quad \left. - v \left[\frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta - \tau_{xy}\sin 2\theta \right] \right\} \\ &= \frac{1}{E} \left[\frac{1}{2}(1-v)(\sigma_x + \sigma_y) + \frac{1}{2}(1+v)(\sigma_x - \sigma_y)\cos 2\theta + (1+v)\tau_{xy}\sin 2\theta \right]\end{aligned}$$

But $\varepsilon_y = \frac{1}{E}(\sigma_y - v\sigma_x)$

and $\varepsilon_x = \frac{1}{E}(\sigma_x - v\sigma_y)$

from which $\sigma_y = \frac{E}{(1-v^2)}[\varepsilon_y + v\varepsilon_x]$

and $\sigma_x = \frac{E}{(1-v^2)}[\varepsilon_x + v\varepsilon_y]$

$\therefore \frac{1}{2}(\sigma_x + \sigma_y) = \frac{E(1+v)}{2(1-v^2)}(\varepsilon_y + \varepsilon_x)$

$$\text{and } \frac{1}{2}(\sigma_x - \sigma_y) = \frac{E(1-v)}{2(1-v^2)} (\varepsilon_x - \varepsilon_y)$$

$$\begin{aligned}\therefore \varepsilon_\theta &= \frac{1}{E} \left[\frac{E(1-v)(1+v)(\varepsilon_y + \varepsilon_x)}{2(1-v^2)} \right. \\ &\quad \left. + \frac{E(1+v)(1-v)(\varepsilon_x - \varepsilon_y)}{2(1-v^2)} \cos 2\theta + (1+v)\tau_{xy} \sin 2\theta \right] \\ &= \frac{1}{2}(\varepsilon_y + \varepsilon_x) + \frac{1}{2}(\varepsilon_x - \varepsilon_y) \cos 2\theta + \tau_{xy} \frac{(1+v)}{E} \sin 2\theta\end{aligned}$$

Now

$$\frac{\tau}{\gamma} = G \quad \therefore \quad \tau = G\gamma \quad \text{and} \quad E = 2G(1+v)$$

$$\therefore \tau_{xy} = \frac{E}{2(1+v)} \gamma_{xy}$$

$$\therefore \varepsilon_\theta = \frac{1}{2}(\varepsilon_x + \varepsilon_y) + \frac{1}{2}(\varepsilon_x - \varepsilon_y) \cos 2\theta + \frac{1}{2}\gamma_{xy} \sin 2\theta \quad (14.14)$$

Similarly, substituting for $\frac{1}{2}(\sigma_x - \sigma_y)$ and τ_{xy} in (1),

$$\tau_\theta = \frac{E(\varepsilon_x - \varepsilon_y)(1-v)}{2(1-v^2)} \sin 2\theta - \frac{E}{2(1+v)} \gamma_{xy} \cos 2\theta$$

But

$$\tau_\theta = \frac{E}{2(1+v)} \gamma_\theta$$

$$\therefore \frac{E}{2(1+v)} \gamma_\theta = \frac{E(\varepsilon_x - \varepsilon_y)}{2(1+v)} \sin 2\theta - \frac{E}{2(1+v)} \gamma_{xy} \cos 2\theta$$

$$\gamma_\theta = (\varepsilon_x - \varepsilon_y) \sin 2\theta - \gamma_{xy} \cos 2\theta$$

$$\therefore \frac{1}{2}\gamma_\theta = \frac{1}{2}(\varepsilon_x - \varepsilon_y) \sin 2\theta - \frac{1}{2}\gamma_{xy} \cos 2\theta$$

Again, for consistency of sign convention, since *OM* will move clockwise under strain, the above shear strain must be considered negative,

$$\text{i.e. } \frac{1}{2}\gamma_\theta = -[\frac{1}{2}(\varepsilon_x - \varepsilon_y) \sin 2\theta - \frac{1}{2}\gamma_{xy} \cos 2\theta] \quad (14.15)$$

Equations (14.14) and (14.15) are similar in form to eqns. (13.8) and (13.9) which are the basis of Mohr's circle solution for stresses provided that $\frac{1}{2}\gamma_{xy}$ is used in place of τ_{xy} and linear stresses σ are replaced by linear strains ε . These equations will therefore provide a graphical solution known as Mohr's strain circle if axes of ε and $\frac{1}{2}\gamma$ are used.

14.13. Relationship between Mohr's stress and strain circles

Consider now a material subjected to the two-dimensional principal stress system shown in Fig. 14.13a. The stress and strain circles are then as shown in Fig. 14.13(b) and (c).

For Mohr's stress circle (Fig. 14.13b),

$$OA \times \text{stress scale} = \frac{(\sigma_1 + \sigma_2)}{2}$$

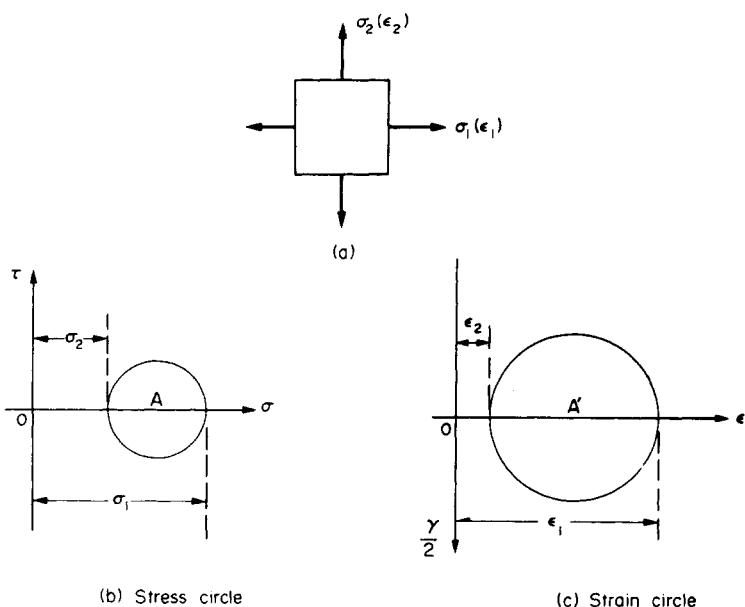


Fig. 14.13.

$$\therefore OA = \frac{(\sigma_1 + \sigma_2)}{2 \times \text{stress scale}} \quad (1)$$

$$\text{and} \quad \text{radius of stress circle} \times \text{stress scale} = \frac{1}{2}(\sigma_1 - \sigma_2) \quad (2)$$

For Mohr's strain circle (Fig. 14.13c),

$$OA' \times \text{strain scale} = \frac{(\epsilon_1 + \epsilon_2)}{2}$$

$$\text{But} \quad \epsilon_1 = \frac{1}{E} (\sigma_1 - v\sigma_2)$$

$$\text{and} \quad \epsilon_2 = \frac{1}{E} (\sigma_2 - v\sigma_1)$$

$$\begin{aligned} \therefore \epsilon_1 + \epsilon_2 &= \frac{1}{E} [(\sigma_1 + \sigma_2) - v(\sigma_1 + \sigma_2)] \\ &= \frac{1}{E} (\sigma_1 + \sigma_2)(1 - v) \end{aligned}$$

$$\therefore OA' = \frac{(\sigma_1 + \sigma_2)(1 - v)}{2E \times \text{strain scale}} \quad (3)$$

Thus, in order that the circles shall be concentric (Fig. 14.14),

$$OA = OA'$$

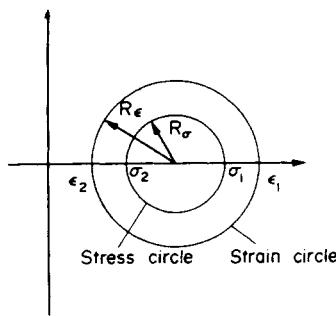


Fig. 14.14. Combined stress and strain circles.

Therefore from (1) and (3)

$$\frac{(\sigma_1 + \sigma_2)}{2 \times \text{stress scale}} = \frac{(\sigma_1 + \sigma_2)(1 - v)}{2E \times \text{strain scale}}$$

$$\therefore \text{stress scale} = \frac{E}{(1 - v)} \times \text{strain scale} \quad (14.18)$$

Now radius of strain circle \times strain scale

$$= \frac{1}{2}(\varepsilon_1 - \varepsilon_2)$$

$$= \frac{1}{2E} [(\sigma_1 - v\sigma_2) - (\sigma_2 - v\sigma_1)]$$

$$= \frac{1}{2E} (\sigma_1 - \sigma_2)(1 + v) \quad (4)$$

$$\therefore \frac{\text{radius of stress circle} \times \text{stress scale}}{\text{radius of strain circle} \times \text{strain scale}} = \frac{\frac{1}{2}(\sigma_1 - \sigma_2)}{\frac{1}{2E} (\sigma_1 - \sigma_2)(1 + v)}$$

$$= \frac{E}{(1 + v)} \quad (14.19)$$

$$\frac{\text{radius of stress circle}}{\text{radius of strain circle}} = \frac{E}{(1 + v)} \times \frac{\text{strain scale}}{\text{stress scale}}$$

i.e.

$$\frac{R_\sigma}{R_\epsilon} = \frac{E}{(1 + v)} \times \frac{(1 - v)}{E}$$

$$= \frac{(1 - v)}{(1 + v)} \quad (14.19)$$

In other words, provided suitable scales are chosen so that

$$\text{stress scale} = \frac{E}{(1 - v)} \times \text{strain scale}$$

the stress and strain circles will have the same centre. If the radius of one circle is known the radius of the other circle can then be determined from the relationship

$$\text{radius of stress circle} = \frac{(1-v)}{(1+v)} \times \text{radius of strain circle}$$

Other relationships for the stress and strain circles are shown in Fig. 14.15.

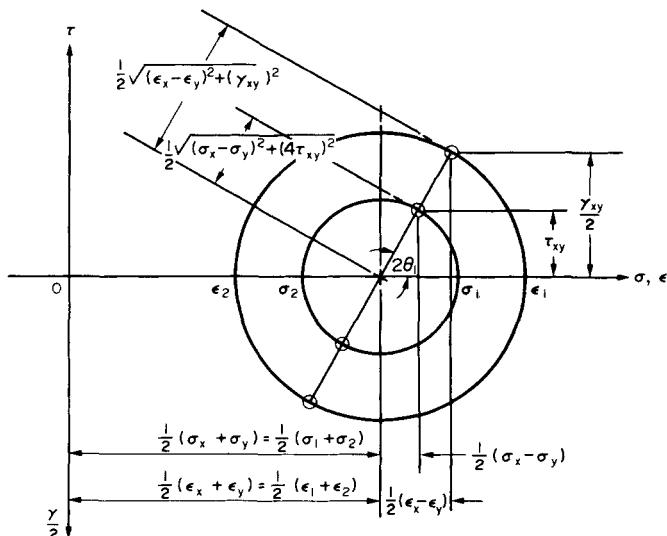


Fig. 14.15. Other relationships for Mohr's stress and strain circles.

14.14. Construction of strain circle from three known strains (McClintock method)—rosette analysis

In order to measure principal strains on the surface of engineering components the normal experimental technique involves the bonding of a strain gauge rosette at the point under consideration. This gives the values of strain in three known directions and enables Mohr's strain circle to be constructed as follows.

Consider the three-strain system shown in Fig. 14.16, the known directions of strain being at angles α_a , α_b and α_c to a principal strain direction (this being one of the primary requirements of such readings). The construction sequence is then:

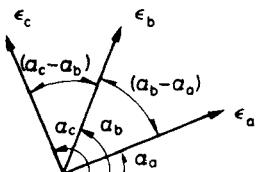


Fig. 14.16. System of three known strains.

(1) On a horizontal line da mark off the known strains ϵ_a , ϵ_b and ϵ_c to the same scale to give points a , b and c (see Fig. 14.18).

(2) From a , b and c draw perpendiculars to the line da .

(3) From a convenient point X on the perpendicular through b mark off lines corresponding to the known strain directions of ϵ_a and ϵ_c to intersect (projecting back if necessary) perpendiculars through c and a at C and A .

Note that these directions must be identical relative to Xb as they are relative to ϵ_b in Fig. 14.16,
i.e. XC is $\alpha_c - \alpha_b$ counterclockwise from Xb and XA is $\alpha_b - \alpha_a$ clockwise from Xb

(4) Construct perpendicular bisectors of the lines XA and XC to meet at the point Y , which is then the centre of Mohr's strain circle (Fig. 14.18).

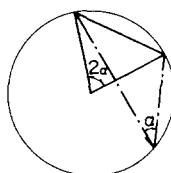


Fig. 14.17. Useful relationship for development of Mohr's strain circle (see Fig. 14.18).

(5) With centre Y and radius YA or YC draw the strain circle to cut Xb in the point B .

(6) The vertical shear strain axis can now be drawn through the zero of the strain scale da ; the horizontal linear strain axis passes through Y .

(7) Join points A , B and C to Y . These radii must then be in the same angular order as the original strain directions. As in Mohr's stress circle, however, angles between them will be double in value, as shown in Fig. 14.18.

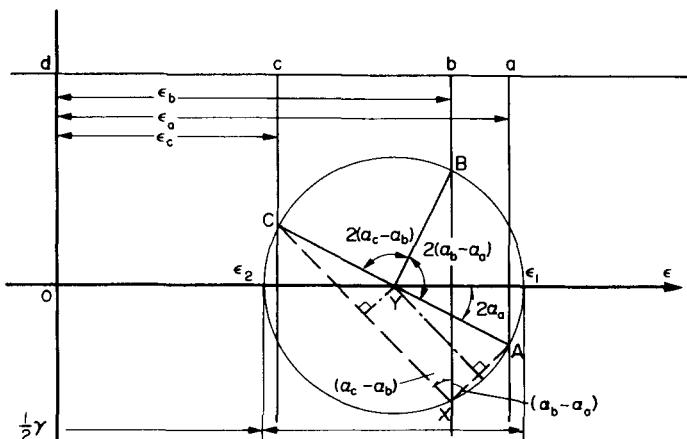


Fig. 14.18. Construction of strain circle from three known strains - McClintock construction.
(Strain gauge rosette analysis.)

The principal strains are then ϵ_1 and ϵ_2 as indicated. Principal stresses can now be determined either from the relationships

$$\sigma_1 = \frac{E}{(1-\nu^2)} [\epsilon_1 + \nu\epsilon_2] \quad \text{and} \quad \sigma_2 = \frac{E}{(1-\nu^2)} [\epsilon_2 + \nu\epsilon_1]$$

or by superimposing the stress circle using the relationships established in §14.13.

The above construction applies whatever the values of strain and whatever the angles between the individual gauges of the rosette. The process is simplified, however, if the rosette axes are arranged:

- (a) in sequence, in order of ascending or descending strain magnitude,
- (b) so that the included angle between axes of maximum and minimum strain is less than 180° .

For example, consider three possible results of readings from the rosette of Fig. 14.16 as shown in Fig. 14.19(i), (ii) and (iii).

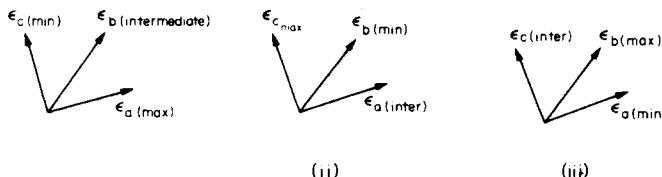


Fig. 14.19. Three possible orders of results from any given strain gauge rosette.

These may be rearranged as suggested above by projecting axes where necessary as shown in Fig. 14.20(i), (ii) and (iii).

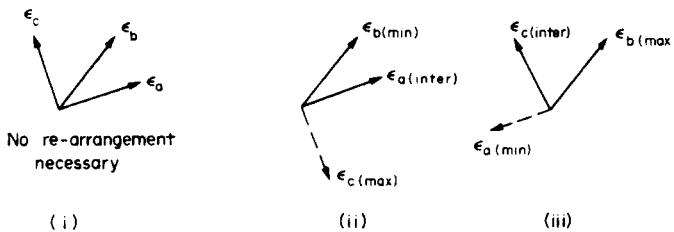
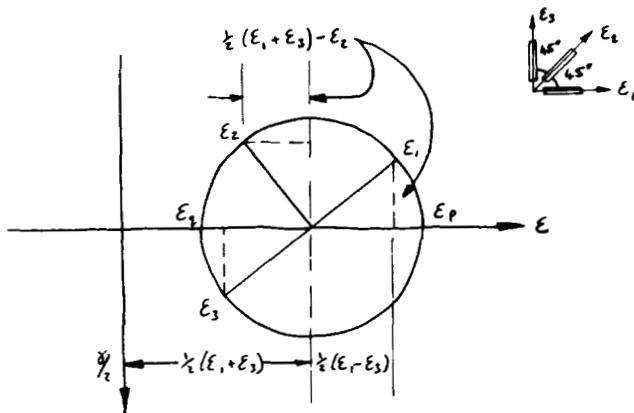


Fig. 14.20. Suitable rearrangement of Fig. 14.19 to facilitate the McClintock construction.

In all the above cases, the most convenient construction still commences with the starting point X on the vertical through the intermediate strain value, and will appear similar in form to the construction of Fig. 14.18.

Mohr's strain circle solution of rosette readings is strongly recommended because of its simplicity, speed and the ease with which principal stresses may be obtained by superimposing Mohr's stress circle. In addition, when one becomes familiar with the construction procedure, there is little opportunity for arithmetical error. As stated in the previous chapter, the advent of cheap but powerful calculators and microcomputers may reduce the effectiveness of Mohr's circle as a quantitative tool. It remains, however, a very powerful

medium for the teaching and understanding of complex stress and strain systems and a valuable "aide-memoire" for some of the complex formulae which may be required for solution by other means. For example Fig. 14.21 shows the use of a free-hand sketch of the Mohr circle given by rectangular strain gauge rosette readings to obtain, from simple geometry, the corresponding principal strain equations.



$$\begin{aligned} \epsilon_{p,1} &= \frac{1}{2}(\epsilon_1 + \epsilon_3) \pm \sqrt{\left[\frac{1}{2}(\epsilon_1 - \epsilon_3)\right]^2 + \left[\frac{1}{2}(\epsilon_1 + \epsilon_3) - \epsilon_2\right]^2} \\ &= \frac{1}{2}(\epsilon_1 + \epsilon_3) \pm \frac{1}{\sqrt{2}}\sqrt{(\epsilon_1 - \epsilon_2)^2 + (\epsilon_2 - \epsilon_3)^2} \end{aligned}$$

Fig. 14.21. Free-hand sketch of Mohr's strain circle.

14.15. Analytical determination of principal strains from rosette readings

The values of the principal strains associated with the three strain readings taken from a strain gauge rosette may be found by calculation using eqn. (14.14),

i.e. $\epsilon_\theta = \frac{1}{2}(\epsilon_x + \epsilon_y) + \frac{1}{2}(\epsilon_x - \epsilon_y)\cos 2\theta + \frac{1}{2}\gamma_{xy}\sin 2\theta$

This equation can be applied three times for the three values of θ of the rosette gauges. Thus with three known values of ϵ_θ for three known values of θ , three simultaneous equations will give the unknown strains ϵ_x , ϵ_y and γ_{xy} .

The principal strains can then be determined from eqn. (14.16).

$$\epsilon_1 \quad \text{or} \quad \epsilon_2 = \frac{1}{2}(\epsilon_x + \epsilon_y) \pm \frac{1}{2}\sqrt{[(\epsilon_x - \epsilon_y)^2 + \gamma_{xy}^2]}$$

The direction of the principal strain axes are then given by the equivalent strain expression to that derived for stresses [eqn. (13.10)],

i.e. $\tan 2\theta = \frac{\gamma_{xy}}{(\epsilon_x - \epsilon_y)} \quad (14.20)$

angles being given relative to the X axis.

The majority of rosette gauges in common use today are either rectangular rosettes with $\theta = 0^\circ$, 45° and 90° or delta rosettes with $\theta = 0^\circ$, 60° and 120° (Fig. 14.22).

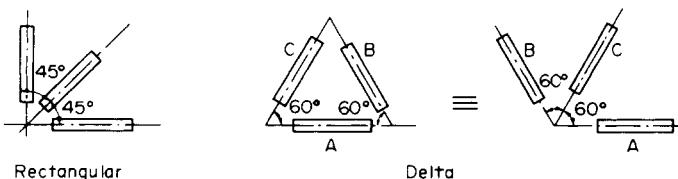


Fig. 14.22. Typical strain gauge rosette configurations.

In each case the calculations are simplified if the X axis is chosen to coincide with $\theta = 0$. Then, for both types of rosette, eqn. (14.14) reduces (for $\theta = 0$) to

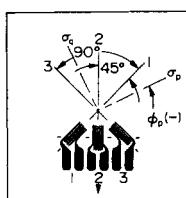
$$\varepsilon_0 = \frac{1}{2}(\varepsilon_x + \varepsilon_y) + \frac{1}{2}(\varepsilon_x - \varepsilon_y) = \varepsilon_x$$

and ε_x is obtained directly from the ε_0 strain gauge reading. Similarly, for the rectangular rosette ε_y is obtained directly from the ε_{90° reading.

If a large number of rosette gauge results have to be analysed, the calculation process may be computerised. In this context the relationship between the rosette readings and resulting principal stresses shown in Table 14.1 for three standard types of strain gauge rosette is recommended.

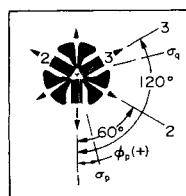
TABLE 14.1. Principal strains and stresses from strain gauge rosettes*
(Gauge readings = σ_1 , σ_2 and σ_3 ; Principal stresses = σ_p and σ_q .)

Rectangular (45°) Rosette—Arbitrarily oriented with respect to principal axes.



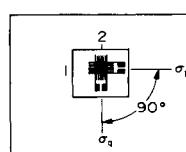
$$\begin{aligned}\epsilon_{p,q} &= \frac{\epsilon_1 + \epsilon_3 \pm \frac{1}{\sqrt{2}}\sqrt{(\epsilon_1 - \epsilon_2)^2 + (\epsilon_2 - \epsilon_3)^2}}{2} \\ \sigma_{p,q} &= \frac{E}{2} \left(\frac{\epsilon_1 + \epsilon_3 \pm \sqrt{2}}{1 - \nu} \sqrt{(\epsilon_1 - \epsilon_2)^2 + (\epsilon_2 - \epsilon_3)^2} \right) \\ \phi_{p,q} &= \frac{1}{2} \tan^{-1} \left(\frac{(\epsilon_2 - \epsilon_3) - (\epsilon_1 - \epsilon_2)}{\epsilon_1 - \epsilon_3} \right) \text{ (if } \epsilon_1 > \frac{\epsilon_1 + \epsilon_3}{2}, \phi_{p,q} = \phi_p \text{)} \\ &\quad \text{if } \epsilon_1 < \frac{\epsilon_1 + \epsilon_3}{2}, \phi_{p,q} = \phi_q \\ &\quad \text{if } \epsilon_1 = \frac{\epsilon_1 + \epsilon_3}{2}, \phi_p = \pm 45^\circ)\end{aligned}$$

Delta (equiangular) Rosette—Arbitrarily oriented with respect to principal axes.



$$\begin{aligned}\epsilon_{p,q} &= \frac{\epsilon_1 + \epsilon_2 + \epsilon_3 \pm \frac{\sqrt{2}}{3}\sqrt{(\epsilon_1 - \epsilon_2)^2 + (\epsilon_2 - \epsilon_3)^2 + (\epsilon_1 - \epsilon_3)^2}}{3} \\ \sigma_{p,q} &= \frac{E}{3} \left(\frac{\epsilon_1 + \epsilon_2 + \epsilon_3 \pm \sqrt{2}}{1 - \nu} \sqrt{(\epsilon_1 - \epsilon_2)^2 + (\epsilon_2 - \epsilon_3)^2 + (\epsilon_1 - \epsilon_3)^2} \right) \\ \phi_{p,q} &= \frac{1}{2} \tan^{-1} \left(\frac{\sqrt{3}(\epsilon_2 - \epsilon_3)}{(\epsilon_1 - \epsilon_2) + (\epsilon_1 - \epsilon_3)} \right) \text{ (if } \epsilon_1 > \frac{\epsilon_1 + \epsilon_2 + \epsilon_3}{3}, \phi_{p,q} = \phi_p \text{)} \\ &\quad \text{if } \epsilon_1 < \frac{\epsilon_1 + \epsilon_2 + \epsilon_3}{3}, \phi_{p,q} = \phi_q \\ &\quad \text{if } \epsilon_1 = \frac{\epsilon_1 + \epsilon_2 + \epsilon_3}{3}, \phi_p = \pm 45^\circ)\end{aligned}$$

Tee Rosette—Gage elements must be aligned with principal axes.



$$\begin{aligned}\epsilon_p &= \epsilon_1 \\ \epsilon_q &= \epsilon_2 \\ \sigma_p &= \frac{E}{1 - \nu^2} (\epsilon_1 + \nu \epsilon_2) \\ \sigma_q &= \frac{E}{1 - \nu^2} (\epsilon_2 + \nu \epsilon_1)\end{aligned}$$

Rosette Gage-Numbering Considerations

The equations at the left for calculating principal strains and stresses from rosette strain measurements assume that the gage elements are numbered in a particular manner. Improper numbering of the gage elements will lead to ambiguity in the interpretation of $\phi_{p,q}$; and, in the case of the rectangular rosette, can also cause errors in the calculated principal strains and stresses.

Treating the latter situation first, it is always necessary in a rectangular rosette that gage numbers 1 and 3 be assigned to the two mutually perpendicular gages. Any other numbering arrangement will produce incorrect principal strains and stresses.

Ambiguities in the interpretation of $\phi_{p,q}$ for both rectangular and delta rosettes can be eliminated by numbering the gage elements as follows:

In a rectangular rosette, Gage 2 must be 45° away from Gage 1; and Gage 3 must be 90° away, in the same direction. Similarly, in a delta rosette, Gages 2 and 3 must be 60° and 120° away respectively, in the same direction from Gage 1. By definition, $\phi_{p,q}$ is the angle from the axis of Gage 1 to the nearest principal axis. When $\phi_{p,q}$ is positive, the direction is the same as that of the gage numbering; and, when negative, the opposite.

* Reproduced with permission from Vishay Measurements Ltd wall chart.

14.16. Alternative representations of strain distributions at a point

Alternative forms of representation for the distribution of stress at a point were presented in §13.7; the directly equivalent representations for strain are given below.

The values of the direct strain ϵ_θ and shear strain γ_θ for any inclined plane θ are given by equations (14.14) and (14.15) as

$$\epsilon_\theta = \frac{1}{2}(\epsilon_x + \epsilon_y) + \frac{1}{2}(\epsilon_x - \epsilon_y)\cos 2\theta + \frac{1}{2}\gamma_{xy}\sin 2\theta$$

$$\frac{1}{2}\gamma_\theta = -[\frac{1}{2}(\epsilon_x - \epsilon_y)\sin 2\theta - \frac{1}{2}\gamma_{xy}\cos 2\theta]$$

Plotting these values for the uniaxial stress state on Cartesian axes yields the curves of Fig. 14.23 which can then be compared directly to the equivalent stress distributions of Fig. 13.12. Again the shear curves are "shifted" by 45° from the normal strain curves.

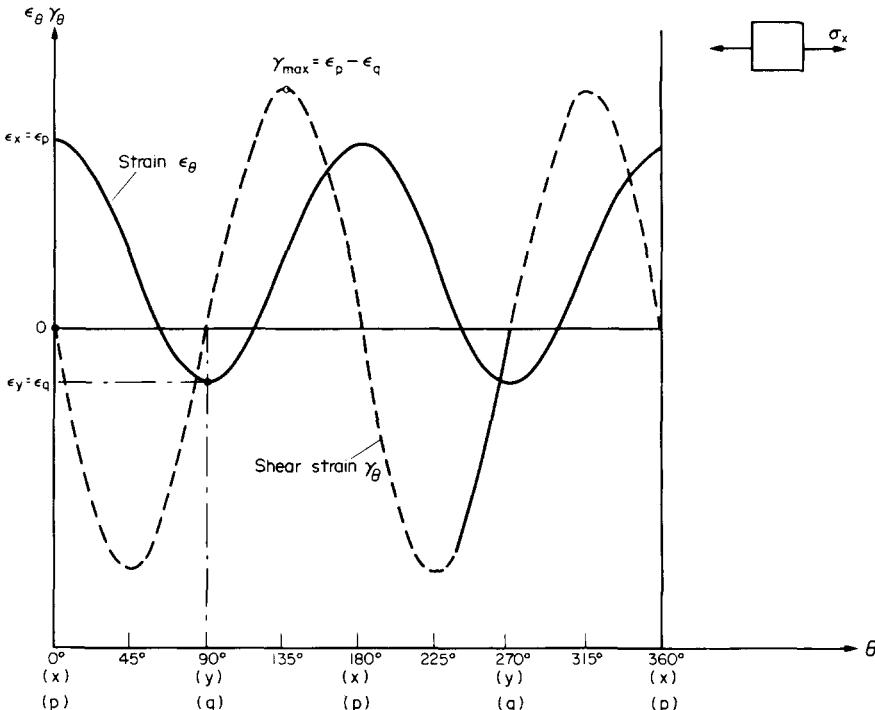


Fig. 14.23. Cartesian plot of strain distribution at a point under uniaxial applied stress.

Comparison with Fig. 13.12 shows that the normal stress and shear stress curves are each in phase with their respective normal strain and shear strain curves. Other relationships between the shear strain and normal strain curves are identical to those listed on page 335 for the normal stress and shear stress distributions.

The alternative polar strain representation for the uniaxial stress system is shown in Fig. 14.24 whilst the Cartesian and polar diagrams for the same biaxial stress systems used for Figs. 13.14 and 13.15 are shown in Figs. 14.25 and 14.26.

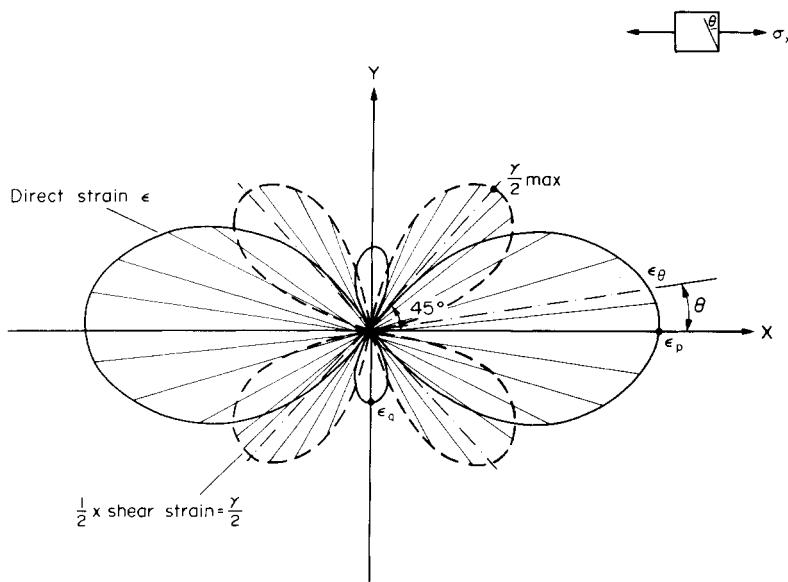


Fig. 14.24. Polar plot of strain distribution at a point under uniaxial applied stress.

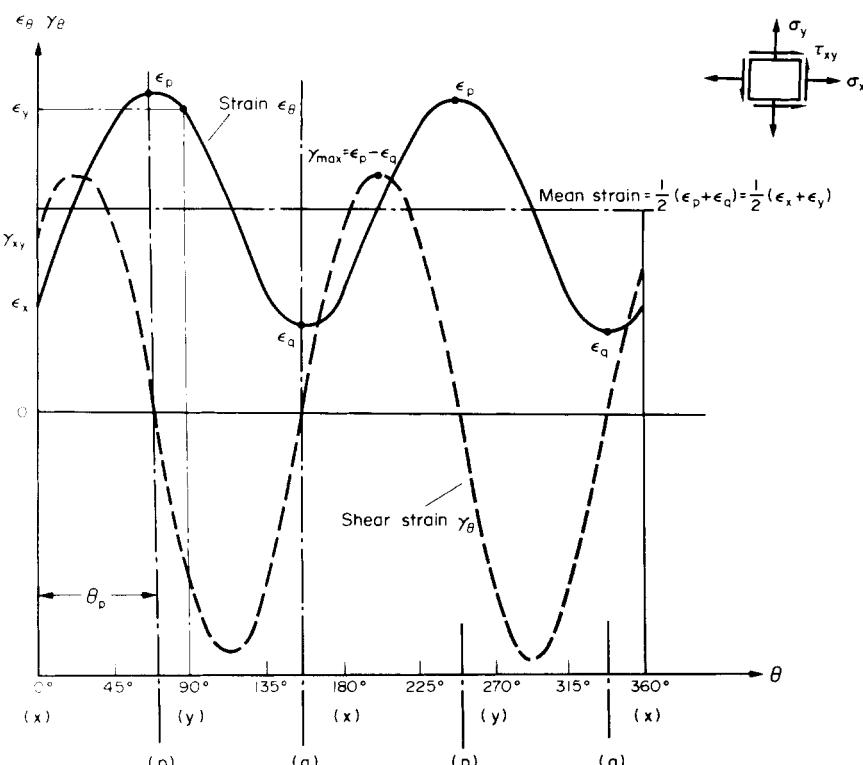


Fig. 14.25. Cartesian plot of strain distribution at a point under a typical biaxial applied stress system.

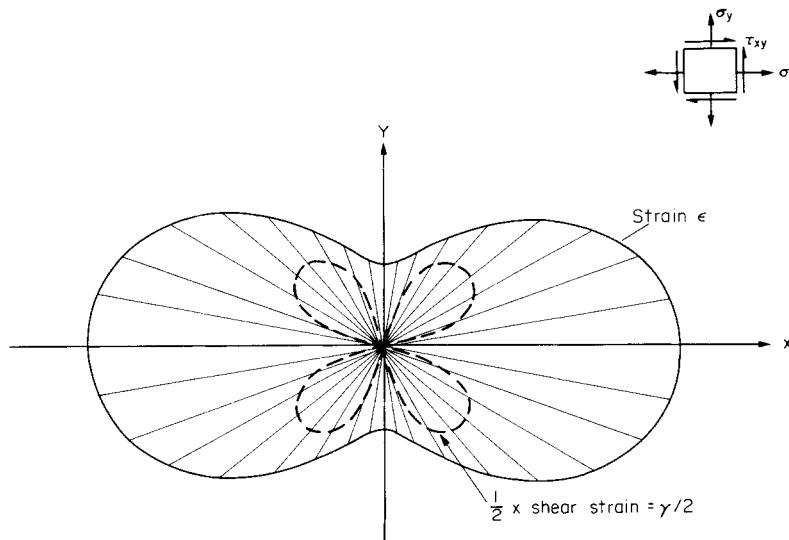


Fig. 14.26. Polar plot of strain distribution at a point under a typical biaxial applied stress system.

14.17. Strain energy of three-dimensional stress system

(a) Total strain energy

Any three-dimensional stress system may be reduced to three principal stresses σ_1 , σ_2 and σ_3 acting on a unit cube, the faces of which are principal planes and, therefore, by definition, subjected to zero shear stress. If the corresponding principal strains are ϵ_1 , ϵ_2 and ϵ_3 , then the total strain energy U_t per unit volume is equal to the total work done by the system and given by the equation

$$U_t = \frac{1}{2} \sigma \epsilon$$

since the stresses are applied gradually from zero (see page 258).

$$\therefore U_t = \frac{1}{2} \sigma_1 \epsilon_1 + \frac{1}{2} \sigma_2 \epsilon_2 + \frac{1}{2} \sigma_3 \epsilon_3$$

Substituting for the principal strains using eqn. (14.2),

$$U_t = \frac{1}{2E} [\sigma_1(\sigma_1 - v\sigma_2 - v\sigma_3) + \sigma_2(\sigma_2 - v\sigma_3 - v\sigma_1) + \sigma_3(\sigma_3 - v\sigma_2 - v\sigma_1)]$$

$$\therefore U_t = \frac{1}{2E} [\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2v(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \text{ per unit volume} \quad (14.21)$$

(b) Shear (or "distortion") strain energy

As above, consider the three-dimensional stress system reduced to principal stresses σ_1 , σ_2 and σ_3 acting on a unit cube as in Fig. 14.27. For convenience the principal stresses may be

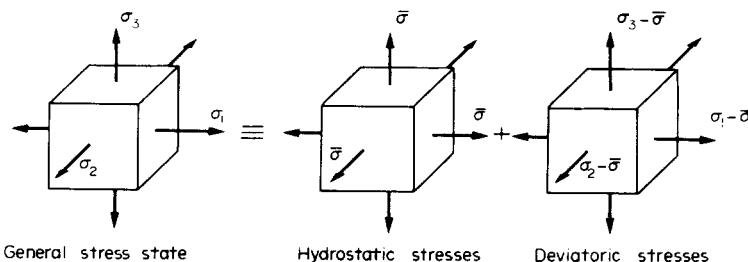


Fig. 14.27. Resolution of general three-dimensional principal stress state into "hydrostatic" and "deviatoric" components.

written in terms of a mean stress $\bar{\sigma} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$ and additional shear stress terms,

$$\text{i.e. } \sigma_1 = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) + \frac{1}{3}(\sigma_1 - \sigma_2) + \frac{1}{3}(\sigma_1 - \sigma_3)$$

$$\sigma_2 = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) + \frac{1}{3}(\sigma_2 - \sigma_1) + \frac{1}{3}(\sigma_2 - \sigma_3)$$

$$\sigma_3 = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) + \frac{1}{3}(\sigma_3 - \sigma_1) + \frac{1}{3}(\sigma_3 - \sigma_2)$$

The mean stress term may be considered as a *hydrostatic* tensile stress, equal in all directions, the strains associated with this giving rise to no distortion, i.e. the unit cube under the action of the hydrostatic stress alone would be strained into a cube. The hydrostatic stresses are sometimes referred to as the *spherical* or *dilatational* stresses.

The strain energy associated with the hydrostatic stress is termed the *volumetric strain energy* and is found by substituting

$$\sigma_1 = \sigma_2 = \sigma_3 = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

into eqn. (14.21),

$$\text{i.e. volumetric strain energy} = \frac{3}{2E} \left[\left(\frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \right)^2 \right] (1 - 2\nu)$$

$$\therefore U_v = \frac{(1 - 2\nu)}{6E} [(\sigma_1 + \sigma_2 + \sigma_3)^2] \text{ per unit volume} \quad (14.22)$$

The remaining terms in the modified principal stress equations are shear stress terms (i.e. functions of principal stress differences in the various planes) and these are the only stresses which give rise to distortion of the stressed element. They are therefore termed *distortional* or *deviatoric* stresses.

Now

total strain energy per unit volume = shear strain energy per unit volume + volumetric strain energy per unit volume

i.e.

$$U_t = U_s + U_v$$

Therefore shear strain energy per unit volume is given by:

$$U_s = U_t - U_v$$

$$\text{i.e. } U_s = \frac{1}{2E} [\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] - \frac{(1 - 2\nu)}{6E} [(\sigma_1 + \sigma_2 + \sigma_3)^2]$$

This simplifies to

$$U_s = \frac{(1+\nu)}{6E} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

and, since $E = 2G(1+\nu)$,

$$U_s = \frac{1}{12G} [\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - (\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \quad (14.23a)$$

or, alternatively,

$$U_s = \frac{1}{6G} [\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - (\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \quad (14.23b)$$

It is interesting to note here that even a uniaxial stress condition may be divided into hydrostatic (dilatational) and deviatoric (distortional) terms as shown in Fig. 14.28.

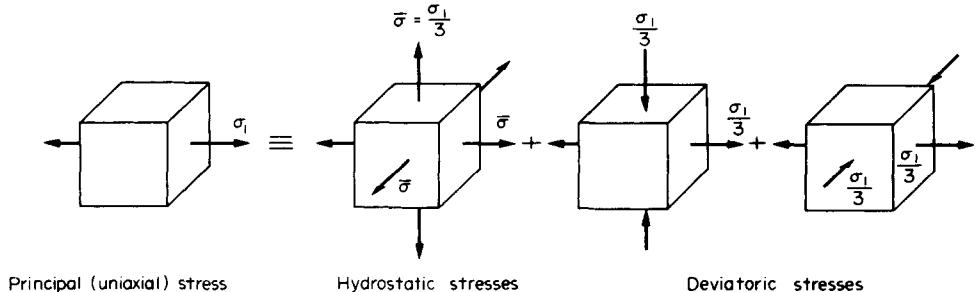


Fig. 14.28. Resolution of uniaxial stress into hydrostatic and deviatoric components.

Examples

Example 14.1

When a bar of 25 mm diameter is subjected to an axial pull of 61 kN the extension on a 50 mm gauge length is 0.1 mm and there is a decrease in diameter of 0.013 mm. Calculate the values of E , ν , G , and K .

Solution

$$\text{Longitudinal stress} = \frac{\text{load}}{\text{area}} = \frac{61 \times 10^3}{\frac{1}{4}\pi(0.025)^2} = 124.2 \text{ MN/m}^2$$

$$\text{Longitudinal strain} = \frac{\text{extension}}{\text{original length}} = \frac{0.1 \times 10^{-3}}{10^3 \times 50} = 2 \times 10^{-3}$$

$$\text{Young's modulus } E = \frac{\text{stress}}{\text{strain}} = \frac{124.2 \times 10^6}{2 \times 10^{-3}} = 62.1 \text{ GN/m}^2$$

$$\text{Lateral strain} = \frac{\text{change in diameter}}{\text{original diameter}} = \frac{0.013 \times 10^{-3}}{10^3 \times 25} = 0.52 \times 10^{-3}$$

$$\text{Poisson's ratio } (\nu) = \frac{\text{lateral strain}}{\text{longitudinal strain}} = \frac{0.52 \times 10^{-3}}{2 \times 10^{-3}} = 0.26$$

Now

$$E = 2G(1 + \nu) \quad \therefore \quad G = \frac{E}{2(1 + \nu)}$$

$$G = \frac{62.1 \times 10^9}{2(1 + 0.26)} = 24.6 \text{ GN/m}^2$$

Also

$$E = 3K(1 - 2\nu) \quad \therefore \quad K = \frac{E}{3(1 - 2\nu)}$$

$$K = \frac{62.1 \times 10^9}{3 \times 0.48} = 43.1 \text{ GN/m}^2$$

Example 14.2

A bar of mild steel 25 mm diameter twists 2 degrees in a length of 250 mm under a torque of 430 N m. The same bar deflects 0.8 mm when simply supported at each end horizontally over a span of 500 mm and loaded at the centre of the span with a vertical load of 1.2 kN. Calculate the values of E , G , K and Poisson's ratio ν for the material.

Solution

$$J = \frac{\pi}{32} D^4 = \frac{\pi}{32} (0.025)^4 = 0.0383 \times 10^{-6} \text{ m}^4$$

$$\text{Angle of twist} \quad \theta = 2 \times \frac{\pi}{180} = 0.0349 \text{ radian}$$

$$\text{From the simple torsion theory} \quad \frac{T}{J} = \frac{G\theta}{L} \quad \therefore \quad G = \frac{TL}{J\theta}$$

$$G = \frac{430 \times 250 \times 10^6}{0.0349 \times 10^3 \times 0.0383} = 80.3 \times 10^9 \text{ N/m}^2 \\ = 80.3 \text{ GN/m}^2$$

For a simply supported beam the deflection at mid-span with central load W is

$$\delta = \frac{WL^3}{48EI}$$

$$\text{Then} \quad E = \frac{WL^3}{48\delta I} \quad \text{and} \quad I = \frac{\pi}{64} D^4 = \frac{\pi}{64} (0.025)^4 = 0.0192 \times 10^{-6} \text{ m}^4$$

$$\therefore E = \frac{1.2 \times 10^3 \times (0.5)^3 \times 10^6 \times 10^3}{48 \times 0.0192 \times 0.8} = 203 \times 10^9 \text{ N/m}^2 \\ = 203 \text{ GN/m}^2$$

Now

$$E = 2G(1 + \nu) \quad \therefore \quad \nu = \frac{E}{2G} - 1$$

\therefore

$$\nu = \frac{203}{2 \times 80.3} - 1 = 0.268$$

Also

$$E = 3K(1 - 2\nu) \quad \therefore \quad K = \frac{E}{3(1 - 2\nu)}$$

$$K = \frac{203 \times 10^9}{3(1 - 0.536)} = 146 \times 10^9 \text{ N/m}^2$$

$$= 146 \text{ GN/m}^2$$

Example 14.3

A rectangular bar of metal 50 mm \times 25 mm cross-section and 125 mm long carries a tensile load of 100 kN along its length, a compressive load of 1 MN on its 50 \times 125 mm faces and a tensile load of 400 kN on its 25 \times 125 mm faces. If $E = 208 \text{ GN/m}^2$ and $\nu = 0.3$, find

- (a) the change in volume of the bar;
- (b) the increase required in the 1 MN load to produce no change in volume.

Solution

$$(a) \quad \begin{aligned} \sigma_x &= \frac{\text{load}}{\text{area}} = \frac{100 \times 10^3 \times 10^6}{50 \times 25} = 80 \text{ MN/m}^2 \\ \sigma_y &= \frac{400 \times 10^3 \times 10^6}{125 \times 25} = 128 \text{ MN/m}^2 \\ \sigma_z &= \frac{-1 \times 10^6 \times 10^6}{125 \times 50} = -160 \text{ MN/m}^2 \quad (\text{Fig. 14.29}) \end{aligned}$$

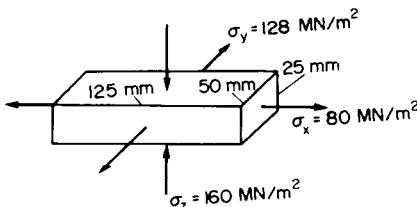


Fig. 14.29.

From §14.6

$$\begin{aligned} \text{change in volume} &= \frac{V}{E} (\sigma_x + \sigma_y + \sigma_z)(1 - 2\nu) \\ &= \frac{(125 \times 50 \times 25)}{208 \times 10^9} 10^{-9} [80 + 128 + (-160)] 10^6 \times 0.4 \\ &= \frac{125 \times 50 \times 25 \times 48 \times 0.4}{208 \times 10^{12}} \text{ m}^3 = 14.4 \text{ mm}^3 \end{aligned}$$

i.e. the bar increases in volume by 14.4 mm³.

(b) If the 1 MN load is to be changed, then σ_z will be changed; therefore the equation for the change in volume becomes

$$\text{change in volume} = 0 = \frac{(125 \times 50 \times 25)}{208 \times 10^9} 10^{-9} (80 + 128 + \sigma_z) 10^6 \times 0.4$$

Then

$$0 = 80 + 128 + \sigma_z$$

$$\sigma_z = -208 \text{ MN/m}^2$$

Now

$$\text{load} = \text{stress} \times \text{area}$$

$$\therefore \text{new load required} = -208 \times 10^6 \times 125 \times 50 \times 10^{-6}$$

$$= -1.3 \text{ MN}$$

Therefore the compressive load of 1 MN must be increased by 0.3 MN for no change in volume to occur.

Example 14.4

A steel bar *ABC* is of circular cross-section and transmits an axial tensile force such that the total change in length is 0.6 mm. The total length of the bar is 1.25 m, *AB* being 750 mm and 20 mm diameter and *BC* being 500 mm long and 13 mm diameter (Fig. 14.30). Determine for the parts *AB* and *BC* the changes in (a) length, and (b) diameter. Assume Poisson's ratio ν for the steel to be 0.3 and Young's modulus E to be 200 GN/m².

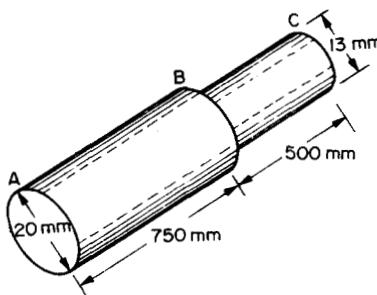


Fig. 14.30.

Solution

(a) Let the tensile force be P newtons.

Then

$$\text{stress in } AB = \frac{\text{load}}{\text{area}} = \frac{P}{\frac{1}{4}\pi(0.02)^2} = \frac{P}{100\pi} \text{ MN/m}^2$$

$$\text{stress in } BC = \frac{P}{\frac{1}{4}\pi(0.013)^2} = \frac{P}{42\pi} \text{ MN/m}^2$$

Then

$$\text{strain in } AB = \frac{\text{stress}}{E} = \frac{P \times 10^6}{100\pi \times 200 \times 10^9} = \frac{P}{20\pi} \times 10^{-6}$$

and

$$\text{strain in } BC = \frac{P \times 10^6}{42\pi \times 200 \times 10^9} = \frac{P}{8.4\pi} \times 10^{-6}$$

$$\text{change in length of } AB = \frac{P \times 10^{-6}}{20\pi} \times 750 \times 10^{-3} = 11.95P \times 10^{-9}$$

$$\text{change in length of } BC = \frac{P \times 10^{-6}}{8.4\pi} \times 500 \times 10^{-3} = 18.95P \times 10^{-9}$$

$$\text{total change in length} = (11.95P + 18.95P)10^{-9} = 0.6 \times 10^{-3}$$

$$\therefore P(11.95 + 18.95)10^{-9} = 0.6 \times 10^{-3}$$

$$\therefore P = \frac{0.6 \times 10^9}{10^3 \times 30.9} = 19.4 \text{ kN}$$

Then change in length of $AB = 19.4 \times 10^3 \times 11.95 \times 10^{-9}$

$$= 0.232 \times 10^{-3} \text{ m} = 0.232 \text{ mm}$$

and change in length of $BC = 19.4 \times 10^3 \times 18.95 \times 10^{-9}$

$$= 0.368 \times 10^{-3} = 0.368 \text{ mm}$$

(b) The lateral (in this case "diametral") strain can be found from the definition of Poisson's ratio ν .

$$\nu = \frac{\text{lateral strain}}{\text{longitudinal strain}}$$

$$\therefore \text{lateral strain} = \text{strain on the diameter} (= \text{diametral strain}) \\ = \nu \times \text{longitudinal strain}$$

$$\text{Lateral strain on } AB = \frac{\nu P \times 10^{-6}}{20\pi} = \frac{0.3 \times 19.4 \times 10^3}{20\pi \times 10^6} \\ = 92.7 \times 10^{-6} (= 92.7 \mu\epsilon) \text{ compressive}$$

$$\text{Lateral strain on } BC = \frac{\nu P \times 10^{-6}}{8.4 \times \pi} = \frac{0.3 \times 19.4 \times 10^3}{8.4\pi \times 10^6} \\ = 220.5 \times 10^{-6} (= 220.5 \mu\epsilon) \text{ compressive}$$

Then, change in diameter of $AB = 92.7 \times 10^{-6} \times 20 \times 10^{-3}$

$$= 1.854 \times 10^{-6} = 0.00185 \text{ mm}$$

and change in diameter of $BC = 220.5 \times 10^{-6} \times 13 \times 10^{-3}$

$$= 2.865 \times 10^{-6} = 0.00286 \text{ mm}$$

Both these changes are *decreases*.

Example 14.5

At a certain point a material is subjected to the following strains:

$$\epsilon_x = 400 \times 10^{-6}; \quad \epsilon_y = 200 \times 10^{-6}; \quad \gamma_{xy} = 350 \times 10^{-6} \text{ radian}$$

Determine the magnitudes of the principal strains, the directions of the principal strain axes and the strain on an axis inclined at 30° clockwise to the x axis.

Solution

Mohr's strain circle is as shown in Fig. 14.31.

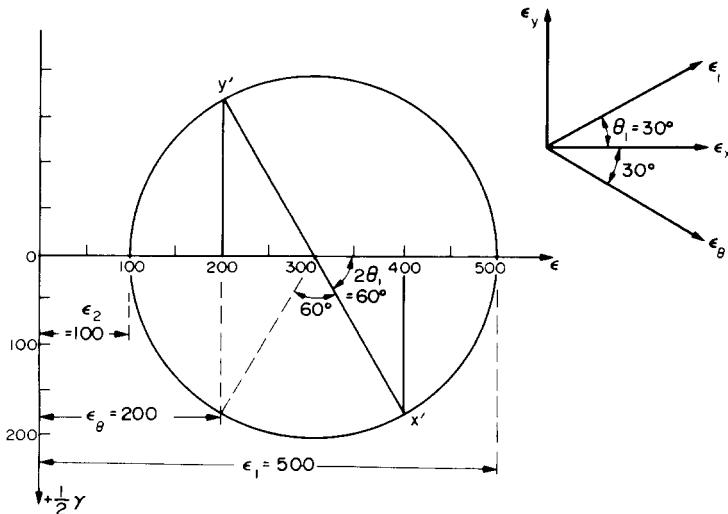


Fig. 14.31.

By measurement:

$$\epsilon_1 = 500 \times 10^{-6} \quad \epsilon_2 = 100 \times 10^{-6}$$

$$\theta_1 = \frac{60^\circ}{2} = 30^\circ \quad \theta_2 = 90^\circ + 30^\circ = 120^\circ$$

$$\epsilon_{30} = 200 \times 10^{-6}$$

the angles being measured counterclockwise from the direction of ϵ_x .

Example 14.6

A material is subjected to two mutually perpendicular strains, $\epsilon_x = 350 \times 10^{-6}$ and $\epsilon_y = 50 \times 10^{-6}$, together with an unknown shear strain γ_{xy} . If the principal strain in the material is 420×10^{-6} , determine:

- (a) the magnitude of the shear strain;
- (b) the other principal strain;
- (c) the direction of the principal strain axes;
- (d) the magnitudes of the principal stresses if $E = 200 \text{ GN/m}^2$ and $\nu = 0.3$.

Solution

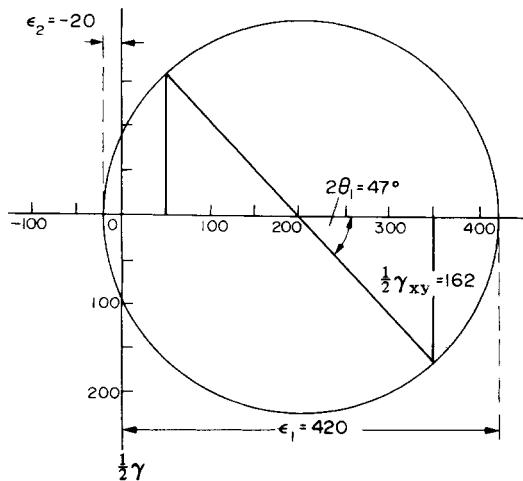


Fig. 14.32.

Mohr's strain circle is as shown in Fig. 14.32. The centre has been positioned half-way between ϵ_x and ϵ_y , and the radius is such that the circle passes through the ϵ axis at 420×10^{-6} . Then, by measurement:

- (a) Shear strain $\gamma_{xy} = 2 \times 162 \times 10^{-6} = 324 \times 10^{-6}$ radian.
- (b) Other principal strain $= -20 \times 10^{-6}$ (compressive).
- (c) Direction of principal strain $\epsilon_1 = \frac{47}{2} = 23^\circ 30'$.

Direction of principal strain $\epsilon_2 = 90^\circ + 23^\circ 30' = 113^\circ 30'$.

- (d) The principal stresses may then be determined from the equations

$$\sigma_1 = \frac{(\epsilon_1 + \nu\epsilon_2)}{1-\nu^2} E \quad \text{and} \quad \sigma_2 = \frac{(\epsilon_2 + \nu\epsilon_1)}{1-\nu^2} E$$

$$\therefore \sigma_1 = \frac{[420 + 0.3(-20)] 10^{-6} \times 200 \times 10^9}{1 - (0.3)^2}$$

$$= \frac{414 \times 200 \times 10^3}{0.91} = 91 \text{ MN/m}^2 \text{ tensile}$$

and

$$\sigma_2 = \frac{(-20 + 0.3 \times 420)10^{-6} \times 200 \times 10^9}{1 - (0.3)^2}$$

$$= \frac{106 \times 200 \times 10^3}{0.91} = 23.3 \text{ MN/m}^2 \text{ tensile}$$

Thus the principal stresses are 91 MN/m^2 and 23.3 MN/m^2 , both tensile.

Example 14.7

The following strain readings were recorded at the angles stated relative to a given horizontal axis:

$$\epsilon_a = -2.9 \times 10^{-5} \text{ at } 20^\circ$$

$$\epsilon_b = 3.1 \times 10^{-5} \text{ at } 80^\circ$$

$$\epsilon_c = -0.5 \times 10^{-5} \text{ at } 140^\circ$$

as shown in Fig. 14.33. Determine the magnitude and direction of the principal stresses. $E = 200 \text{ GN/m}^2$; $\nu = 0.3$.

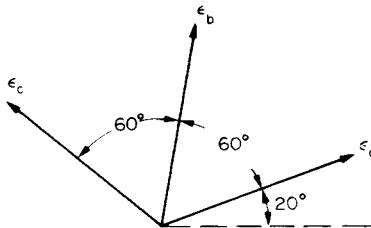


Fig. 14.33.

Solution

Consider now the construction shown in Fig. 14.34 giving the strain circle for the strain values in the question and illustrated in Fig. 14.33.

For a strain scale of $1 \text{ cm} = 1 \times 10^{-5}$ strain, in order to superimpose a stress circle concentric with the strain circle, the necessary scale is

$$1 \text{ cm} = \frac{E}{(1-\nu)} \times 1 \times 10^{-5} = \frac{200 \times 10^9}{0.7} \times 10^{-5}$$

$$= 2.86 \text{ MN/m}^2$$

Also

$$\text{radius of strain circle} = 3.5 \text{ cm}$$

$$\therefore \text{radius of stress circle} = 3.5 \times \frac{(1-\nu)}{(1+\nu)} = \frac{3.5 \times 0.7}{1.3}$$

$$= 1.886 \text{ cm}$$

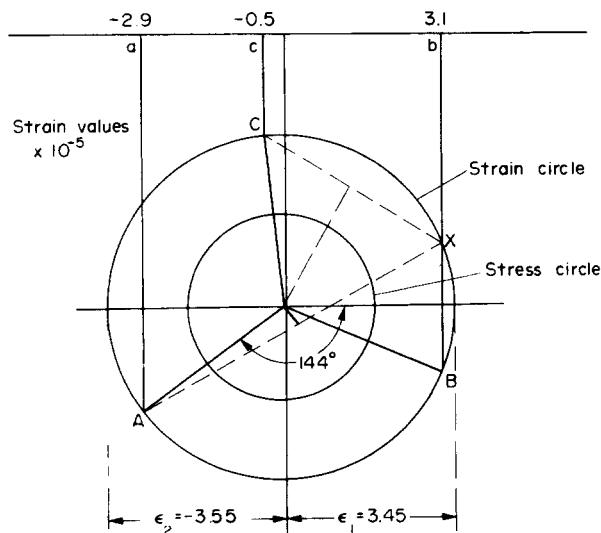


Fig. 14.34.

Superimposing the stress circle of radius 1.886 cm concentric with the strain circle, the principal stresses to a scale 1 cm = 2.86 MN/m² are found to be

$$\sigma_1 = 1.8 \times 2.86 \times 10^6 = 5.15 \text{ MN/m}^2$$

$$\sigma_2 = -2.0 \times 2.86 \times 10^6 = -5.72 \text{ MN/m}^2$$

The principal strains will be at an angle $\frac{144}{2} = 72^\circ$ and 162° counterclockwise from the direction of ε_a, i.e. 92° and 182° counterclockwise from the given horizontal axis. The principal stresses will therefore also be in these directions.

Example 14.8

A rectangular rosette of strain gauges on the surface of a material under stress recorded the following readings of strain:

gauge A	$+450 \times 10^{-6}$
gauge B, at 45° to A	$+200 \times 10^{-6}$
gauge C, at 90° to A	-200×10^{-6}

the angles being counterclockwise from A.

Determine:

- the magnitudes of the principal strains,
- the directions of the principal strain axes, both by calculation and by Mohr's strain circle.

Solution

If ε_1 and ε_2 are the principal strains and ε_θ is the strain in a direction at θ to the direction of ε_1 , then eqn. (14.14) may be rewritten as

$$\varepsilon_\theta = \frac{1}{2}(\varepsilon_1 + \varepsilon_2) + \frac{1}{2}(\varepsilon_1 - \varepsilon_2)\cos 2\theta$$

since $\gamma_{xy} = 0$ on principal strain axes. Thus if gauge A is at an angle θ to ε_1 :

$$450 \times 10^{-6} = \frac{1}{2}(\varepsilon_1 + \varepsilon_2) + \frac{1}{2}(\varepsilon_1 - \varepsilon_2)\cos 2\theta \quad (1)$$

$$200 \times 10^{-6} = \frac{1}{2}(\varepsilon_1 + \varepsilon_2) + \frac{1}{2}(\varepsilon_1 - \varepsilon_2)\cos(90^\circ + 2\theta) \quad (2)$$

$$\begin{aligned} -200 \times 10^{-6} &= \frac{1}{2}(\varepsilon_1 + \varepsilon_2) + \frac{1}{2}(\varepsilon_1 - \varepsilon_2)\cos(180^\circ + 2\theta) \\ &= \frac{1}{2}(\varepsilon_1 + \varepsilon_2) - \frac{1}{2}(\varepsilon_1 - \varepsilon_2)\cos 2\theta \end{aligned} \quad (3)$$

Adding (1) and (3),

$$250 \times 10^{-6} = \varepsilon_1 + \varepsilon_2 \quad (4)$$

Substituting (4) in (2),

$$200 \times 10^{-6} = 125 \times 10^{-6} + \frac{1}{2}(\varepsilon_1 - \varepsilon_2)\cos(90^\circ + 2\theta)$$

$$\therefore 75 \times 10^{-6} = -\frac{1}{2}(\varepsilon_1 - \varepsilon_2)\sin 2\theta \quad (5)$$

Substituting (4) in (1),

$$450 \times 10^{-6} = 125 \times 10^{-6} + \frac{1}{2}(\varepsilon_1 - \varepsilon_2)\cos 2\theta$$

$$\therefore 325 \times 10^{-6} = \frac{1}{2}(\varepsilon_1 - \varepsilon_2)\cos 2\theta \quad (6)$$

Dividing (5) by (6),

$$\frac{75 \times 10^{-6}}{325 \times 10^{-6}} = \frac{-\frac{1}{2}(\varepsilon_1 - \varepsilon_2)\sin 2\theta}{\frac{1}{2}(\varepsilon_1 - \varepsilon_2)\cos 2\theta}$$

$$\therefore \tan 2\theta = -0.231$$

$$\therefore 2\theta = -13^\circ \quad \text{or} \quad 180^\circ - 13^\circ = 167^\circ$$

$$\therefore \theta = 83^\circ 30'$$

Thus gauge A is $83^\circ 30'$ counterclockwise from the direction of ε_1 .

Therefore from (6),

$$325 \times 10^{-6} = \frac{1}{2}(\varepsilon_1 - \varepsilon_2)\cos 167^\circ$$

$$\therefore \varepsilon_1 - \varepsilon_2 = \frac{2 \times 325 \times 10^{-6}}{-0.9744} = -667 \times 10^{-6}$$

But from eqn. (4),

$$\varepsilon_1 + \varepsilon_2 = 250 \times 10^{-6}$$

Therefore adding,

$$2\varepsilon_1 = -417 \times 10^{-6} \quad \therefore \quad \varepsilon_1 = -208.5 \times 10^{-6}$$

and subtracting,

$$2\varepsilon_2 = 917 \times 10^{-6} \quad \therefore \quad \varepsilon_2 = 458.5 \times 10^{-6}$$

Thus the principal strains are -208×10^{-6} and 458.5×10^{-6} , the former being on an axis $83^\circ 30'$ clockwise from gauge A.

Alternatively, these results may be obtained using Mohr's strain circle as shown in Fig. 14.35. The circle has been drawn using the construction procedure of §14.14 and gives principal strains of

$\epsilon_1 = 458.5 \times 10^6$, tensile, at $6^\circ 30'$ counterclockwise from gauge A

$\epsilon_2 = 208.5 \times 10^6$, compressive, at $83^\circ 30'$ clockwise from gauge A

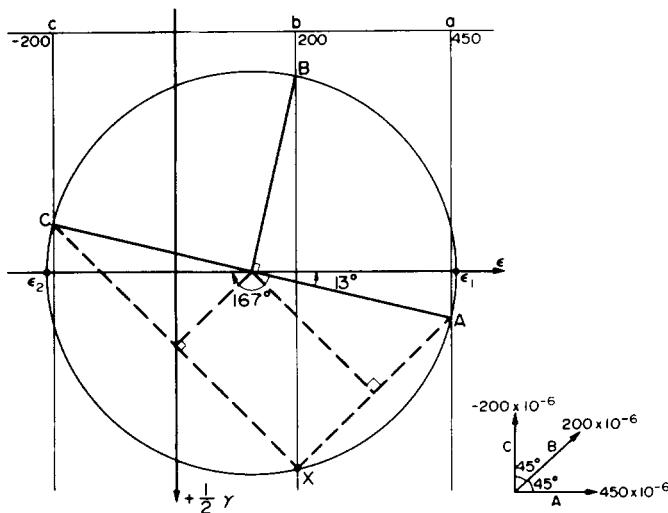


Fig. 14.35.

Problems

14.1 (A). A bar of 40 mm diameter carries a tensile load of 100 kN. Determine the longitudinal extension of a 50 mm gauge length and the contraction of the diameter.

Young's modulus $E = 210 \text{ GN/m}^2$ and Poisson's ratio $\nu = 0.3$.

[0.019, 0.0045 mm.]

14.2 (A). Establish the relationship between Young's modulus E , the modulus of rigidity G and the bulk modulus K in the form

$$E = \frac{9KG}{3K+G}$$

14.3 (A). The extension of a 100 mm gauge length of 14.33 mm diameter bar was found to be 0.15 mm when a tensile load of 50 kN was applied. A torsion specimen of the same specification was made with a 19 mm diameter and a 200 mm gauge length. On test it twisted 0.502 degree under the action of a torque of 45 N m. Calculate E , G , K and ν . [206.7, 80.9, 155 GN/m²; 0.278.]

14.4 (A). A rectangular steel bar of 25 mm × 12 mm cross-section deflects 6 mm when simply supported on its 25 mm face over a span of 1.2 m and loaded at the centre with a concentrated load of 126 N. If Poisson's ratio for the material is 0.28 determine the values of (a) the rigidity modulus, and (b) the bulk modulus. [82, 159 GN/m².]

14.5 (A). Calculate the changes in dimensions of a 37 mm × 25 mm rectangular bar when loaded with a tensile load of 600 kN.

Take $E = 210 \text{ GN/m}^2$ and $\nu = 0.3$.

[0.034, 0.023 mm.]

14.6 (A). A rectangular block of material $125 \text{ mm} \times 100 \text{ mm} \times 75 \text{ mm}$ carries loads normal to its faces as follows: 1 MN tensile on the $125 \times 100 \text{ mm}$ faces; 0.48 MN tensile on the $100 \times 75 \text{ mm}$ faces; zero load on the $125 \times 75 \text{ mm}$ faces.

If Poisson's ratio = 0.3 and $E = 200 \text{ GN/m}^2$, determine the changes in dimensions of the block under load. What is then the change in volume? [0.025, 0.0228, -0.022 mm; 270 mm^3 .]

14.7 (A). A rectangular bar consists of two sections, AB 25 mm square and 250 mm long and BC 12 mm square and 250 mm long. For a tensile load of 20 kN determine:

- the change in length of the complete bar;
- the changes in dimensions of each portion.

Take $E = 80 \text{ GN/m}^2$ and Poisson's ratio $\nu = 0.3$.

[0.534 mm; 0.434, -0.003, 0.1, -0.0063 mm.]

14.8 (A). A cylindrical brass bar is 50 mm diameter and 250 mm long. Find the change in volume of the bar when an axial compressive load of 150 kN is applied.

Take $E = 100 \text{ GN/m}^2$ and $\nu = 0.27$.

[172.5 mm 3 .]

14.9 (A). A certain alloy bar of 32 mm diameter has a gauge length of 100 mm. A tensile load of 25 kN produces an extension of 0.014 mm on the gauge length and a torque of 2.5 kN m produces an angle of twist of 1.63 degrees. Calculate E , G , K and ν . [222, 85.4, 185 GN/m 2 , 0.3.]

14.10 (A/B). Derive the relationships which exist between the elastic constants (a) E , G and ν , and (b) E , K and ν . Find the change in volume of a steel cube of 150 mm side immersed to a depth of 3 km in sea water.

Take E for steel = 210 GN/m^2 , $\nu = 0.3$ and the density of sea water = 1025 kg/m^3 .

[580 mm 3 .]

14.11 (B). Two steel bars have the same length and the same cross-sectional area, one being circular in section and the other square. Prove that when axial loads are applied the changes in volume of the bars are equal.

14.12 (B). Determine the percentage change in volume of a bar 50 mm square and 1 m long when subjected to an axial compressive load of 10 kN. Find also the restraining pressure on the sides of the bar required to prevent all lateral expansion.

For the bar material, $E = 210 \text{ GN/m}^2$ and $\nu = 0.27$.

[$0.876 \times 10^{-3}\%$, 1.48 MN/m^2 .]

14.13 (B). Derive the formula for longitudinal strain due to axial stress σ_x when all lateral strain is prevented.

A piece of material 100 mm long by 25 mm square is in compression under a load of 60 kN. Determine the change in length of the material if all lateral strain is prevented by the application of a uniform external lateral pressure of a suitable intensity.

For the material, $E = 70 \text{ GN/m}^2$ and Poisson's ratio $\nu = 0.25$.

[0.114 mm.]

14.14 (B). Describe briefly an experiment to find Poisson's ratio for a material.

A steel bar of rectangular cross-section 40 mm wide and 25 mm thick is subjected to an axial tensile load of 100 kN. Determine the changes in dimensions of the sides and hence the percentage decrease in cross-sectional area if $E = 200 \text{ GN/m}^2$ and Poisson's ratio = 0.3.

[-6×10^{-3} , -3.75×10^{-3} , -0.03%.]

14.15 (B). A material is subjected to the following strain system:

$$\varepsilon_x = 200 \times 10^{-6}; \varepsilon_y = -56 \times 10^{-6}; \gamma_{xy} = 230 \times 10^{-6} \text{ radian}$$

Determine:

- the principal strains;
- the directions of the principal strain axes;
- the linear strain on an axis inclined at 50° counterclockwise to the direction of ε_x .

[244, -100×10^{-6} ; 21° ; 163×10^{-6} .]

14.16 (B). A material is subjected to two mutually perpendicular linear strains together with a shear strain. Given that this system produces principal strains of 0.0001 compressive and 0.0003 tensile and that one of the linear strains is 0.00025 tensile, determine the magnitudes of the other linear strain and the shear strain.

[-50×10^{-6} , 265×10^{-6} .]

14.17 (C). A 50 mm diameter cylinder is subjected to an axial compressive load of 80 kN. The cylinder is partially enclosed by a well-fitted casing covering almost the whole length, which reduces the lateral expansion by half. Determine the ratio between the axial strain when the casing is fitted and that when it is free to expand in diameter. Take $\nu = 0.3$.

[0.871.]

14.18 (C). A thin cylindrical shell has hemispherical ends and is subjected to an internal pressure. If the radial change of the cylindrical part is to be equal to that of the hemispherical ends, determine the ratio between the thickness necessary in the two parts. Take $\nu = 0.3$.

[2.43 : 1.]

14.19 (B). Determine the values of the principal stresses present in the material of Problem 14.16. Describe an experimental technique by which the directions and magnitudes of these stresses could be determined in practice. For the material, take $E = 208 \text{ GN/m}^2$ and $\nu = 0.3$.

[61.6, 2.28 MN/m 2 .]

14.20 (B). A rectangular prism of steel is subjected to purely normal stresses on all six faces (i.e. the stresses are principal stresses). One stress is 60 MN/m² tensile, and the other two are denoted by σ_x and σ_y , and may be either tensile or compressive, their magnitudes being such that there is no strain in the direction of σ_y and that the maximum shearing stress in the material does not exceed 75 MN/m² on any plane. Determine the range of values within which σ_x may lie and the corresponding values of σ_y . Make sketches to show the two limiting states of stress, and calculate the strain energy per cubic metre of material in the two limiting conditions. Assume that the stresses are not sufficient to cause elastic failure. For the prism material $E = 208 \text{ GN/m}^2$; $v = 0.286$.

$$[\text{U.L.}] [-90 \text{ to } 210; -8.6, 77.2 \text{ MN/m}^2.]$$

For the following problems on the application of strain gauges additional information may be obtained in §21.2 (Vol. 2).

14.21 (A/B). The following strains are recorded by two strain gauges, their axes being at right angles: $\varepsilon_x = 0.00039$; $\varepsilon_y = -0.00012$ (i.e. one tensile and one compressive). Find the values of the stresses σ_x and σ_y acting along these axes if the relevant elastic constants are $E = 208 \text{ GN/m}^2$ and $v = 0.3$. $[80.9, -0.69 \text{ MN/m}^2.]$

14.22 (B). Explain how strain gauges can be used to measure shear strain and hence shear stresses in a material.

Find the value of the shear stress present in a shaft subjected to pure torsion if two strain gauges mounted at 45° to the axis of the shaft record the following values of strain: 0.00029; -0.00029. If the shaft is of steel, 75 mm diameter, $G = 80 \text{ GN/m}^2$ and $v = 0.3$, determine the value of the applied torque. $[46.4 \text{ MN/m}^2, 3.84 \text{ kN m.}]$

14.23 (B). The following strains were recorded on a rectangular strain rosette: $\varepsilon_a = 450 \times 10^{-6}$; $\varepsilon_b = 230 \times 10^{-6}$; $\varepsilon_c = 0$.

Determine:

(a) the principal strains and the directions of the principal strain axes;

(b) the principal stresses if $E = 200 \text{ GN/m}^2$ and $v = 0.3$.

$$[451 \times 10^{-6} \text{ at } 1^\circ \text{ clockwise from } A, -1 \times 10^{-6} \text{ at } 91^\circ \text{ clockwise from } A; 98, 29.5 \text{ MN/m}^2.]$$

14.24 (B). The values of strain given in Problem 14.23 were recorded on a 60° rosette gauge. What are now the values of the principal strains and the principal stresses?

$$[484 \times 10^{-6}, -27 \times 10^{-6}; 104 \text{ MN/m}^2, 25.7 \text{ MN/m}^2.]$$

14.25 (B). Describe briefly how you would proceed, with the aid of strain gauges, to find the principal stresses present on a material under the action of a complex stress system.

Find, by calculation, the principal stresses present in a material subjected to a complex stress system given that strain readings in directions at 0°, 45° and 90° to a given axis are $+240 \times 10^{-6}$, $+170 \times 10^{-6}$ and $+40 \times 10^{-6}$ respectively.

For the material take $E = 210 \text{ GN/m}^2$ and $v = 0.3$.

$$[59, 25 \text{ MN/m}^2.]$$

14.26 (B). Check the calculation of Problem 14.25 by means of Mohr's strain circle.

14.27 (B). A closed-ended steel pressure vessel of diameter 2.5 m and plate thickness 18 mm has electric resistance strain gauges bonded on the outer surface in the circumferential and axial directions. These gauges have a resistance of 200 ohms and a gauge factor of 2.09. When the pressure is raised to 9 MN/m² the change of resistance is 1.065 ohms for the circumferential gauge and 0.265 ohm for the axial gauge. Working from first principles calculate the value of Young's modulus and Poisson's ratio. $[\text{I.Mech.E.}] [0.287, 210 \text{ GN/m}^2.]$

14.28 (B). Briefly describe the mode of operation of electric resistance strain gauges, and a simple circuit for the measurement of a static change in strain.

The torque on a steel shaft of 50 mm diameter which is subjected to pure torsion is measured by a strain gauge bonded on its outer surface at an angle of 45° to the longitudinal axis of the shaft. If the change of the gauge resistance is 0.35 ohm in 200 ohms and the strain gauge factor is 2, determine the torque carried by the shaft. For the shaft material $E = 210 \text{ GN/m}^2$ and $v = 0.3$. $[\text{I.Mech.E.}] [3.47 \text{ kN.}]$

14.29 (A/B). A steel test bar of diameter 11.3 mm and gauge length 56 mm was found to extend 0.08 mm under a load of 30 kN and to have a contraction on the diameter of 0.00452 mm. A shaft of 80 mm diameter, made of the same quality steel, rotates at 420 rev/min. An electrical resistance strain gauge bonded to the outer surface of the shaft at an angle of 45° to the longitudinal axis gave a recorded resistance change of 0.189 Ω. If the gauge resistance is 100 Ω and the gauge factor is 2.1 determine the maximum power transmitted. $[650 \text{ kW.}]$

14.30 (B). A certain equiangular strain gauge rosette is made up of three separate gauges. After it has been installed it is found that one of the gauges has, in error, been taken from an odd batch; its gauge factor is 2.0, that of the other two being 2.2. As the three gauges appear identical it is impossible to say which is the rogue and it is decided to proceed with the test. The following strain readings are obtained using a gauge factor setting on the strain gauge equipment of 2.2:

Gauge direction	0°	60°	120°
Strain $\times 10^{-6}$	+1	-250	+200

Taking into account the various gauge factor values evaluate the greatest possible shear stress value these readings can represent.

For the specimen material $E = 207 \text{ GN/m}^2$ and $\nu = 0.3$.

[City U.] [44 MN/m².]

14.31 (B). A solid cylindrical shaft is 250 mm long and 50 mm diameter and is made of aluminium alloy. The periphery of the shaft is constrained in such a way as to prevent lateral strain. Calculate the axial force that will compress the shaft by 0.5 mm.

Determine the change in length of the shaft when the lateral constraint is removed but the axial force remains unaltered.

Calculate the required reduction in axial force for the non-constrained shaft if the axial strain is not to exceed 0.2 %

Assume the following values of material constants, $E = 70 \text{ GN/m}^2$; $\nu = 0.3$.

[C.E.I.] [370 kN; 0.673 mm; 95.1 kN.]

14.32 (C). An electric resistance strain gauge rosette is bonded to the surface of a square plate, as shown in Fig. 14.36. The orientation of the rosette is defined by the angle gauge A makes with the X direction. The angle between gauges A and B is 120° and between A and C is 120° . The rosette is supposed to be orientated at 45° to the X direction. To check this orientation the plate is loaded with a uniform tension in the X direction only (i.e. $\sigma_y = 0$), unloaded and then loaded with a uniform tension stress of the same magnitude, in the Y direction only (i.e. $\sigma_x = 0$), readings being taken from the strain gauges in both loading cases.

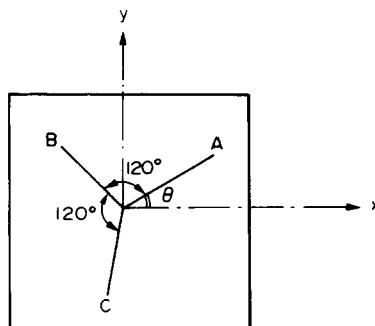


Fig. 14.36.

Denoting the greater principal strain in both loading cases by ε_1 , show that if the rosette is correctly orientated, then

(a) the strain shown by gauge A should be

$$\varepsilon_A = \frac{(1-\nu)}{2} \varepsilon_1$$

for both load cases, and

(b) that shown by gauge B should be

$$\varepsilon_B = \frac{(1-\nu)}{2} \varepsilon_1 + \frac{(1+\nu)}{2} \frac{\sqrt{3}}{2} \varepsilon_1 \quad \text{for the } \sigma_x \text{ case}$$

or

$$\varepsilon_B = \frac{(1-\nu)}{2} \varepsilon_1 - \frac{(1+\nu)}{2} \frac{\sqrt{3}}{2} \varepsilon_1 \quad \text{for the } \sigma_y \text{ case}$$

Hence obtain the corresponding expressions for ε_c .

[As for B , but reversed.]

CHAPTER 15

THEORIES OF ELASTIC FAILURE

Summary

TABLE 15.1

Theory	Value in tension test at failure	Value in complex stress system	Criterion for failure
Maximum principal stress (Rankine)	σ_y	σ_1	$\sigma_1 = \sigma_y$
Maximum shear stress (Guest-Tresca)	$\frac{1}{2}\sigma_y$	$\frac{1}{2}(\sigma_1 - \sigma_3)$	$\sigma_1 - \sigma_3 = \sigma_y$
Maximum principal strain (Saint-Venant)	$\frac{\sigma_y}{E}$	$\frac{\sigma_1}{E} - v\frac{\sigma_2}{E} - v\frac{\sigma_3}{E}$	$\sigma_1 - v\sigma_2 - v\sigma_3 = \sigma_y$
Total strain energy per unit volume (Haigh)	$\frac{\sigma_y^2}{2E}$	$\frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2v(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)]$	$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2v(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \sigma_y^2$
Shear strain energy per unit volume Distortion energy theory (Maxwell-Huber-von Mises)	$\frac{\sigma_y^2}{6G}$	$\frac{1}{12G}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$	$\frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] = \sigma_y^2$
Modified shear stress Internal friction theory (Mohr)			$\frac{\sigma_1}{\sigma_{y_i}} + \frac{\sigma_2}{\sigma_{y_c}} = 1$

Introduction

When dealing with the design of structures or components the physical properties of the constituent materials are usually found from the results of laboratory experiments which have only subjected the materials to the simplest stress conditions. The most usual test is the simple tensile test in which the value of the stress at yield or at fracture (whichever occurs first) is easily determined. The strengths of materials under complex stress systems are not generally known except in a few particular cases. In practice it is these complicated systems of stress which are more often encountered, and therefore it is necessary to have some basis for

determining allowable working stresses so that failure will not occur. Thus the function of the theories of elastic failure is to predict from the behaviour of materials in a simple tensile test when elastic failure will occur under *any* condition of applied stress.

A number of theoretical criteria have been proposed each seeking to obtain adequate correlation between estimated component life and that actually achieved under service load conditions for both brittle and ductile material applications. The five main theories are:

- (a) Maximum principal stress theory (Rankine).
- (b) Maximum shear stress theory (Guest-Tresca).
- (c) Maximum principal strain (Saint-Venant).
- (d) Total strain energy per unit volume (Haigh).
- (e) Shear strain energy per unit volume (Maxwell-Huber-von Mises).

In each case the value of the selected critical property implied in the title of the theory is determined for both the simple tension test and a three-dimensional complex stress system. These values are then equated to produce the so-called *criterion for failure* listed in the last column of Table 15.1.

In Table 15.1 σ_y is the stress at the yield point in the simple tension test, and σ_1 , σ_2 and σ_3 are the three principal stresses in the three-dimensional complex stress system in order of magnitude. Thus in the case of the maximum shear stress theory $\sigma_1 - \sigma_3$ is the greatest numerical difference between two principal stresses taking into account signs and the fact that one principal stress may be zero.

Each of the first five theories listed in Table 15.1 will be introduced in detail in the following text, as will a sixth theory, (f) **Mohr's modified shear stress theory**. Whereas the previous theories (a) to (e) assume equal material strength in tension and compression, the Mohr's modified theory attempts to take into account the additional strength of brittle materials in compression.

15.1. Maximum principal stress theory

This theory assumes that when the maximum principal stress in the complex stress system reaches the elastic limit stress in simple tension, failure occurs. The criterion of failure is thus

$$\sigma_1 = \sigma_y$$

It should be noted, however, that failure could also occur in compression if the least principal stress σ_3 were compressive and its value reached the value of the yield stress in compression for the material concerned before the value of σ_y was reached in tension. An additional criterion is therefore

$$\sigma_3 = \sigma_y \quad (\text{compressive})$$

Whilst the theory can be shown to hold fairly well for brittle materials, there is considerable experimental evidence that the theory should not be applied for ductile materials. For example, even in the case of the pure tension test itself, failure for ductile materials takes place not because of the direct stresses applied but in shear on planes at 45° to the specimen axis. Also, truly homogeneous materials can withstand very high hydrostatic pressures without failing, thus indicating that maximum direct stresses alone do not constitute a valid failure criteria for all loading conditions.

15.2. Maximum shear stress theory

This theory states that failure can be assumed to occur when the maximum shear stress in the complex stress system becomes equal to that at the yield point in the simple tensile test.

Since the maximum shear stress is half the greatest difference between two principal stresses the criterion of failure becomes

$$\frac{1}{2}(\sigma_1 - \sigma_3) = \frac{1}{2}(\sigma_y - 0)$$

i.e.

$$\sigma_1 - \sigma_3 = \sigma_y \quad (15.1)$$

the value of σ_3 being algebraically the smallest value, i.e. taking account of sign *and the fact that one stress may be zero*. This produces fairly accurate correlation with experimental results particularly for ductile materials, and is often used for ductile materials in machine design. The criterion is often referred to as the "Tresca" theory and is one of the widely used laws of plasticity.

15.3. Maximum principal strain theory

This theory assumes that failure occurs when the maximum strain in the complex stress system equals that at the yield point in the tensile test,

$$\text{i.e. } \frac{\sigma_1}{E} - v \frac{\sigma_2}{E} - v \frac{\sigma_3}{E} = \frac{\sigma_y}{E}$$

$$\sigma_1 - v\sigma_2 - v\sigma_3 = \sigma_y, \quad (15.2)$$

This theory is contradicted by the results obtained from tests on flat plates subjected to two mutually perpendicular tensions. The Poisson's ratio effect of each tension reduces the strain in the perpendicular direction so that according to this theory failure should occur at a higher load. This is not always the case. The theory holds reasonably well for cast iron but is not generally used in design procedures these days.

15.4. Maximum total strain energy per unit volume theory

The theory assumes that failure occurs when the total strain energy in the complex stress system is equal to that at the yield point in the tensile test.

From the work of §14.17 the criterion of failure is thus

$$\frac{1}{2E} [\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2v(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] = \frac{\sigma_y^2}{2E}$$

i.e.

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2v(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \sigma_y^2 \quad (15.3)$$

The theory gives fairly good results for ductile materials but is seldom used in preference to the theory below.

15.5. Maximum shear strain energy per unit volume (or distortion energy) theory

Section 14.17 again indicates how the strain energy of a stressed component can be divided into volumetric strain energy and shear strain energy components, the former being

associated with volume change and no distortion, the latter producing distortion of the stressed elements. This theory states that failure occurs when the maximum shear strain energy component in the complex stress system is equal to that at the yield point in the tensile test,

$$\text{i.e. } \frac{1}{12G} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] = \frac{\sigma_y^2}{6G} \quad (\text{eqn. (14.23a)})$$

$$\text{or } \frac{1}{6G} [\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - (\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] = \frac{\sigma_y^2}{6G}$$

$$\therefore (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_y^2 \quad (15.4)$$

This theory has received considerable verification in practice and is widely regarded as the most reliable basis for design, particularly when dealing with ductile materials. It is often referred to as the "von Mises" or "Maxwell" criteria and is probably the best theory of the five. It is also sometimes referred to as the **distortion energy** or **maximum octahedral shear stress theory**.

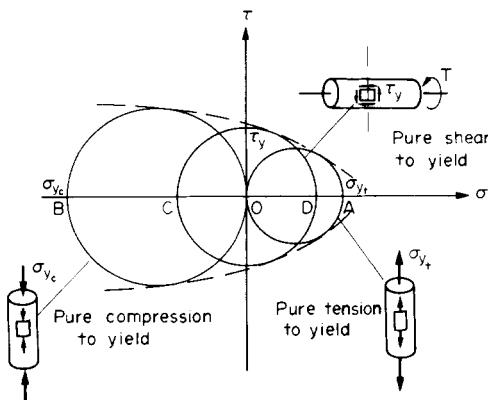
In the above theories it has been assumed that the properties of the material in tension and compression are similar. It is well known, however, that certain materials, notably concrete, cast iron, soils, etc., exhibit vastly different properties depending on the nature of the applied stress. For brittle materials this has been explained by Griffith,[†] who has introduced the principle of surface energy at microscopic cracks and shown that an existing crack will propagate rapidly if the available elastic strain energy release is greater than the surface energy of the crack.[‡] In this way Griffith indicates the greater seriousness of tensile stresses compared with compressive ones with respect to failure, particularly in fatigue environments. A further theory has been introduced by Mohr to predict failure of materials whose strengths are considerably different in tension and shear; this is introduced below.

15.6. Mohr's modified shear stress theory for brittle materials (sometimes referred to as the internal friction theory)

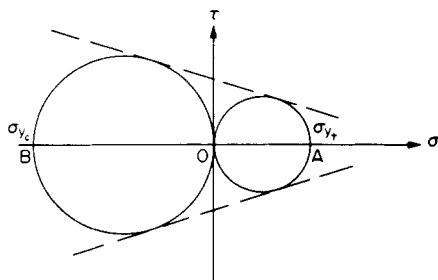
Brittle materials in general show little ability to deform plastically and hence will usually fracture at, or very near to, the elastic limit. Any of the so-called "yield criteria" introduced above, therefore, will normally imply fracture of a brittle material. It has been stated previously, however, that brittle materials are usually considerably stronger in compression than in tension and to allow for this Mohr has proposed a construction based on his stress circle in the application of the maximum shear stress theory. In Fig. 15.1 the circle on diameter OA is that for pure tension, the circle on diameter OB that for pure compression and the circle centre O and diameter CD is that for pure shear. Each of these types of test can be performed to failure relatively easily in the laboratory. An envelope to these curves, shown dotted, then represents the failure envelope according to the Mohr theory. A failure condition is then indicated when the stress circle for a particular complex stress condition is found to cut the envelope.

[†] A. A. Griffith, The phenomena of rupture and flow of solids, *Phil. Trans. Royal Soc.*, London, 1920.

[‡] J. F. Knott, *Fundamentals of Fracture Mechanics* (Butterworths, London), 1973.

Fig. 15.1. Mohr theory on σ - τ axes.

As a close approximation to this procedure Mohr suggests that only the pure tension and pure compression failure circles need be drawn with OA and OB equal to the yield or fracture strengths of the brittle material. Common tangents to these circles may then be used as the failure envelope as shown in Fig. 15.2. Circles drawn tangent to this envelope then represent the condition of failure at the point of tangency.

Fig. 15.2. Simplified Mohr theory on σ - τ axes.

In order to develop a theoretical expression for the failure criterion, consider a general stress circle with principal stresses of σ_1 and σ_2 . It is then possible to develop an expression relating σ_1 , σ_2 , the principal stresses, and σ_{y_t} , σ_{y_c} , the yield strengths of the brittle material in tension and compression respectively.

From the geometry of Fig. 15.3,

$$\frac{KL}{KM} = \frac{JL}{MH}$$

Now, in terms of the stresses,

$$KL = \frac{1}{2}(\sigma_1 + \sigma_2) - \sigma_1 + \frac{1}{2}\sigma_{y_t} = \frac{1}{2}(\sigma_{y_t} - \sigma_1 + \sigma_2)$$

$$KM = \frac{1}{2}\sigma_{y_t} + \frac{1}{2}\sigma_{y_c} = \frac{1}{2}(\sigma_{y_t} + \sigma_{y_c})$$

$$JL = \frac{1}{2}(\sigma_1 + \sigma_2) - \frac{1}{2}\sigma_{y_t} = \frac{1}{2}(\sigma_1 + \sigma_2 - \sigma_{y_t})$$

$$MH = \frac{1}{2}\sigma_{y_c} - \frac{1}{2}\sigma_{y_t} = \frac{1}{2}(\sigma_{y_c} - \sigma_{y_t})$$

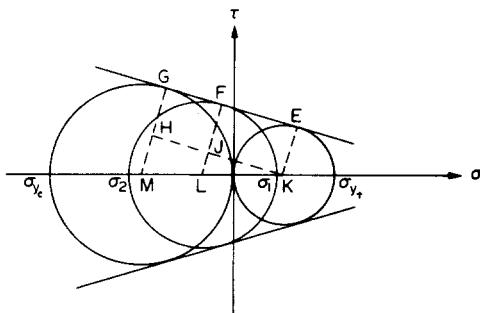


Fig. 15.3.

Substituting,

$$\frac{\sigma_{y_t} - \sigma_1 + \sigma_2}{\sigma_{y_t} + \sigma_c} = \frac{\sigma_1 + \sigma_2 - \sigma_{y_t}}{\sigma_{y_t} - \sigma_c}$$

Cross-multiplying and simplifying this reduces to

$$\frac{\sigma_1}{\sigma_{y_t}} + \frac{\sigma_2}{\sigma_c} = 1 \quad (15.5)$$

which is then the Mohr's modified shear stress criterion for brittle materials.

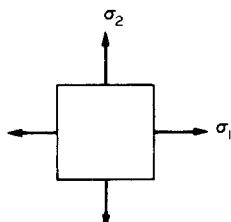
15.7. Graphical representation of failure theories for two-dimensional stress systems (one principal stress zero)

Having obtained the equations for the elastic failure criteria above in the general three-dimensional stress state it is relatively simple to obtain the corresponding equations when one of the principal stresses is zero.

Each theory may be represented graphically as described below, the diagrams often being termed *yield loci*.

(a) Maximum principal stress theory

For simplicity of treatment, ignore for the moment the normal convention for the principal stresses, i.e. $\sigma_1 > \sigma_2 > \sigma_3$ and consider the two-dimensional stress state shown in Fig. 15.4

Fig. 15.4. Two-dimensional stress state ($\sigma_3 = 0$).

where σ_3 is zero and σ_2 may be tensile or compressive as appropriate, i.e. σ_2 may have a value less than σ_3 for the purpose of this development.

The maximum principal stress theory then states that failure will occur when σ_1 or $\sigma_2 = \sigma_y$, or σ_{y_c} . Assuming $\sigma_{y_c} = \sigma_y = \sigma_y$, these conditions are represented graphically on σ_1 , σ_2 coordinates as shown in Fig. 15.5. If the point with coordinates (σ_1, σ_2) representing any complex two-dimensional stress system falls outside the square, then failure will occur according to the theory.

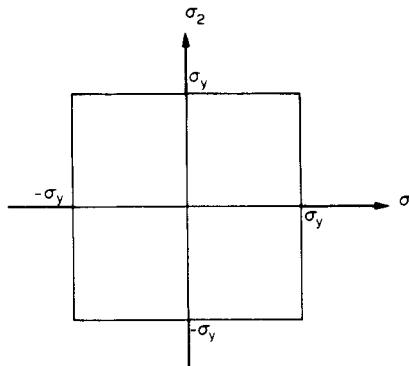


Fig. 15.5. Maximum principal stress failure envelope (locus).

(b) Maximum shear stress theory

For *like* stresses, i.e. σ_1 and σ_2 , both tensile or both compressive (first and third quadrants), the maximum shear stress criterion is

$$\frac{1}{2}(\sigma_1 - 0) = \frac{1}{2}\sigma_y \quad \text{or} \quad \frac{1}{2}(\sigma_2 - 0) = \frac{1}{2}\sigma_y$$

i.e.

$$\sigma_1 = \sigma_y \quad \text{or} \quad \sigma_2 = \sigma_y$$

thus producing the same result as the previous theory in the first and third quadrants.

For *unlike* stresses the criterion becomes

$$\frac{1}{2}(\sigma_1 - \sigma_2) = \frac{1}{2}\sigma_y$$

since consideration of the third stress as zero will not produce as large a shear as that when σ_2 is negative. Thus for the second and fourth quadrants,

$$\frac{\sigma_1}{\sigma_y} - \frac{\sigma_2}{\sigma_y} = 1 \quad \left(\text{or} \quad \frac{\sigma_2}{\sigma_y} - \frac{\sigma_1}{\sigma_y} = 1 \right)$$

These are straight lines and produce the failure envelope of Fig. 15.6. Again, any point outside the failure envelope represents a condition of potential failure.

(c) Maximum principal strain theory

For yielding in tension the theory states that

$$\sigma_1 - v\sigma_2 = \sigma_y$$

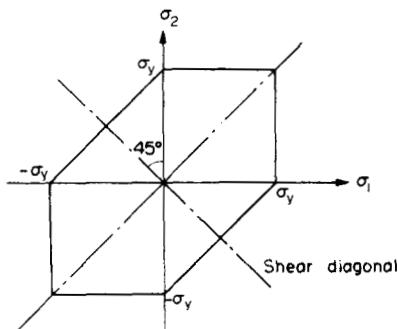


Fig. 15.6. Maximum shear stress failure envelope.

and for compressive yield, with σ_2 compressive,

$$\sigma_2 - \nu\sigma_1 = \sigma_y$$

Since this theory does not find general acceptance in any engineering field it is sufficient to note here, without proof, that the above equations produce the rhomboid failure envelope shown in Fig. 15.7.

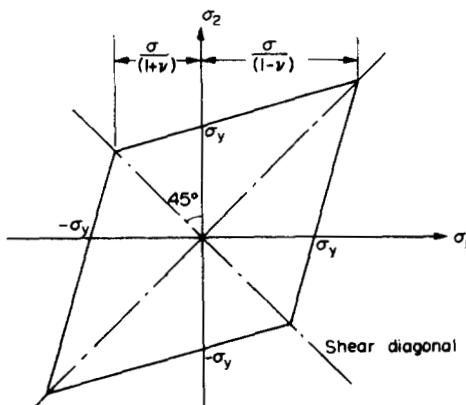


Fig. 15.7. Maximum principal strain failure envelope.

(d) Maximum strain energy per unit volume theory

With $\sigma_3 = 0$ this failure criterion reduces to

$$\sigma_1^2 + \sigma_2^2 - 2\nu\sigma_1\sigma_2 = \sigma_y^2$$

i.e.

$$\left(\frac{\sigma_1}{\sigma_y}\right)^2 + \left(\frac{\sigma_2}{\sigma_y}\right)^2 - 2\nu\left(\frac{\sigma_1}{\sigma_y}\right)\left(\frac{\sigma_2}{\sigma_y}\right) = 1$$

This is the equation of an ellipse with major and minor semi-axes

$$\frac{\sigma_y}{\sqrt{(1-\nu)}} \quad \text{and} \quad \frac{\sigma_y}{\sqrt{(1+\nu)}}$$

respectively, each at 45° to the coordinate axes as shown in Fig. 15.8.

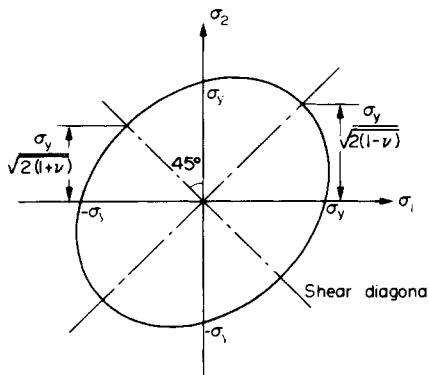


Fig. 15.8. Failure envelope for maximum strain energy per unit volume theory.

(e) Maximum shear strain energy per unit volume theory

With $\sigma_3 = 0$ the criteria of failure for this theory reduces to

$$\frac{1}{2}[(\sigma_1 - \sigma_2)^2 + \sigma_2^2 + \sigma_1^2] = \sigma_y^2$$

$$\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2 = \sigma_y^2$$

$$\left(\frac{\sigma_1}{\sigma_y}\right)^2 + \left(\frac{\sigma_2}{\sigma_y}\right)^2 - \left(\frac{\sigma_1}{\sigma_y}\right)\left(\frac{\sigma_2}{\sigma_y}\right) = 1$$

again an ellipse with semi-axes $\sqrt{2}\sigma_y$ and $\sqrt{\frac{2}{3}}\sigma_y$ at 45° to the coordinate axes as shown in Fig. 15.9. The ellipse will circumscribe the maximum shear stress hexagon.

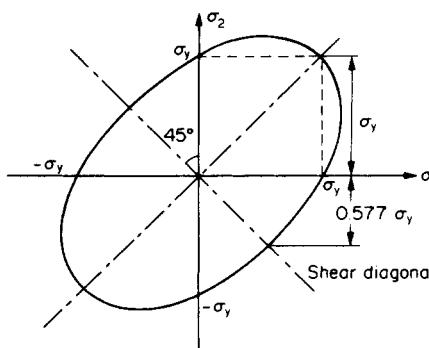


Fig. 15.9. Failure envelope for maximum shear strain energy per unit volume theory.

(f) Mohr's modified shear stress theory ($\sigma_{y_c} > \sigma_{y_t}$)

For the original formulation of the theory based on the results of pure tension, pure compression and pure shear tests the Mohr failure envelope is as indicated in Fig. 15.10.

In its simplified form, however, based on just the pure tension and pure compression results, the failure envelope becomes that of Fig. 15.11.

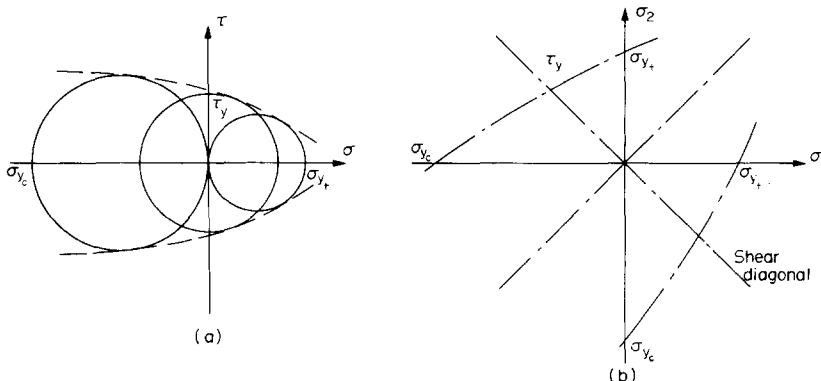


Fig. 15.10. (a) Mohr theory on σ - τ axes. (b) Mohr theory failure envelope on σ_1 - σ_2 axes.

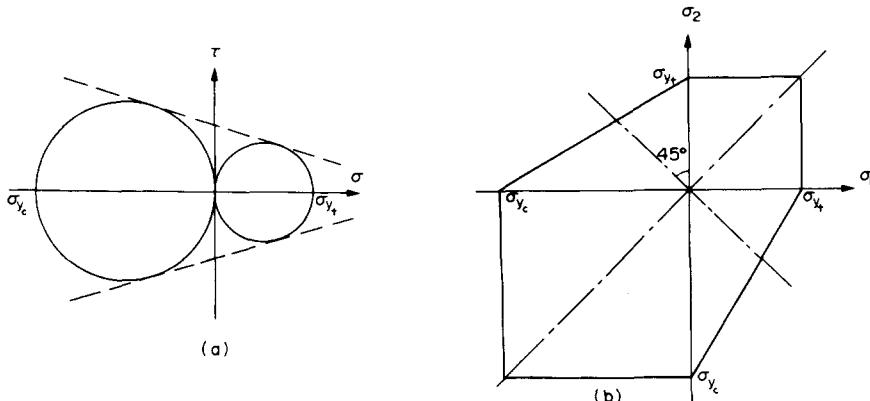


Fig. 15.11. (a) Simplified Mohr theory on σ - τ axes. (b) Failure envelope for simplified Mohr theory.

15.8. Graphical solution of two-dimensional theory of failure problems

The graphical representations of the failure theories, or yield loci, may be combined onto a single set of σ_1 and σ_2 coordinate axes as shown in Fig. 15.12. Inside any particular locus or failure envelope elastic conditions prevail whilst points outside the loci suggest that yielding or fracture will occur. It will be noted that in most cases the maximum shear stress criterion is the most conservative of the theories. The combined diagram is particularly useful since it allows experimental points to be plotted to give an immediate assessment of failure

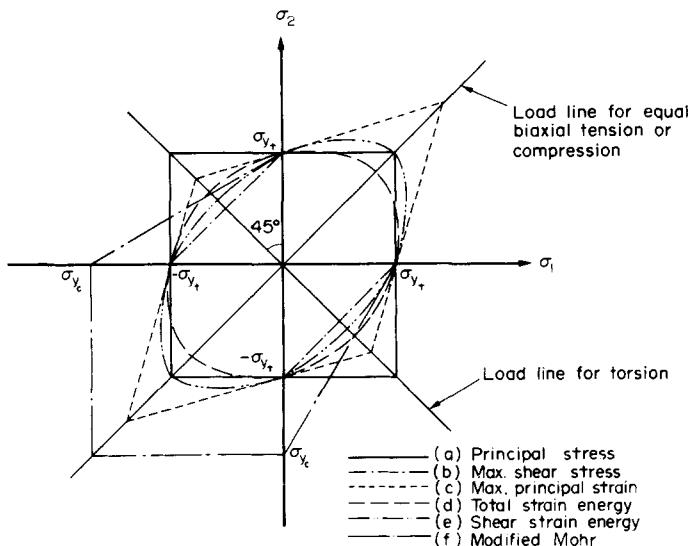


Fig. 15.12. Combined yield loci for the various failure theories.

probability according to the various theories. In the case of equal biaxial tension or compression for example $\sigma_1/\sigma_2 = 1$ and a so-called *load line* may be drawn through the origin with a slope of unity to represent this loading case. This line cuts the yield loci in the order of theories *d*; (*a, b, e, f*); and *c*. In the case of pure torsion, however, $\sigma_1 = \tau$ and $\sigma_2 = -\tau$, i.e. $\sigma_1/\sigma_2 = -1$. This load line will therefore have a slope of -1 and the order of yield according to the various theories is now changed considerably to (*b; e, f, d, c, a*). The load line procedure may be used to produce rapid solutions of failure problems as shown in Example 15.2.

15.9. Graphical representation of the failure theories for three-dimensional stress systems

15.9.1. Ductile materials

(a) Maximum shear strain energy or distortion energy (von Mises) theory

It has been stated earlier that the failure of most ductile materials is most accurately governed by the distortion energy criterion which states that, at failure,

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_y^2 = \text{constant}$$

In the special case where $\sigma_3 = 0$, this has been shown to give a yield locus which is an ellipse symmetrical about the shear diagonal. For a three-dimensional stress system the above equation defines the surface of a regular prism having a circular cross-section, i.e. a cylinder with its central axis along the line $\sigma_1 = \sigma_2 = \sigma_3$. The axis thus passes through the origin of the principal stress coordinate system shown in Fig. 15.13 and is inclined at equal angles to each

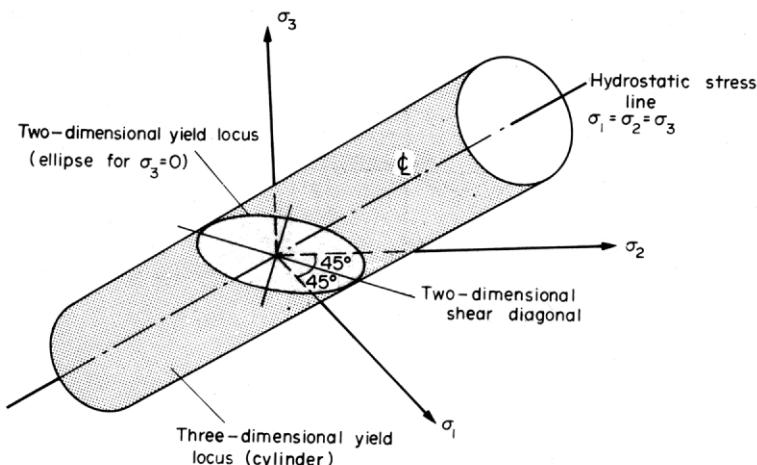


Fig. 15.13. Three-dimensional yield locus for Maxwell-von Mises distortion energy (shear strain energy per unit volume) theory.

axis. It will be observed that when $\sigma_3 = 0$ the failure condition reverts to the ellipse mentioned above, i.e. that produced by intersection of the (σ_1, σ_2) plane with the inclined cylinder.

The yield locus for the von Mises theory in a three-dimensional stress system is thus the *surface* of the inclined cylinder. Points within the cylinder represent safe conditions, points outside indicate failure conditions. It should be noted that the cylinder axis extends indefinitely along the $\sigma_1 = \sigma_2 = \sigma_3$ line, this being termed the *hydrostatic stress line*. It can be shown that hydrostatic stress alone cannot cause yielding and it is presumed that all other stress conditions which fall within the cylindrical boundary may be considered equally safe.

(b) Maximum shear stress (Tresca) theory

With a few exceptions, e.g. aluminium alloys and certain steels, the yielding of most ductile materials is adequately governed by the Tresca maximum shear stress condition, and because of its relative simplicity it is often used in preference to the von Mises theory. For the Tresca theory the three-dimensional yield locus can be shown to be a regular prism with hexagonal cross-section (Fig. 15.14). The central axis of this figure is again on the line $\sigma_1 = \sigma_2 = \sigma_3$ (the hydrostatic stress line) and again extends to infinity.

Points representing stress conditions plotted on the principal stress coordinate axes indicate safe conditions if they lie within the surface of the hexagonal cylinder. The two-dimensional yield locus of Fig. 15.6 is obtained as before by the intersection of the σ_1, σ_2 plane ($\sigma_3 = 0$) with this surface.

15.9.2. Brittle materials

Failure of brittle materials has been shown previously to be governed by the maximum principal tensile stress present in the three-dimensional stress system. This is thought to be

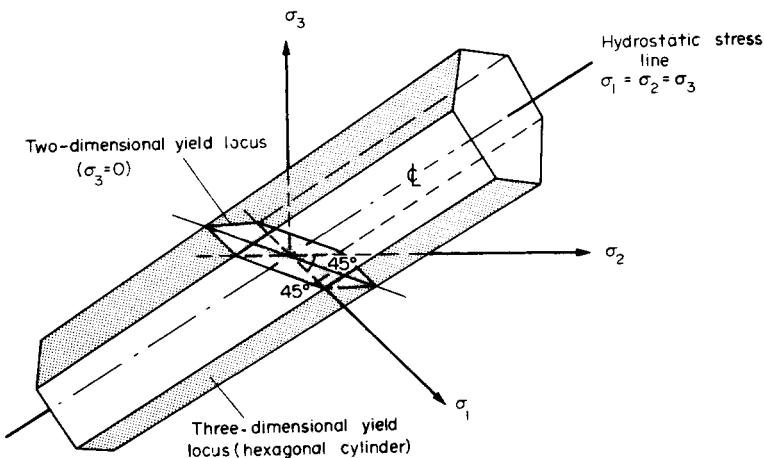


Fig. 15.14. Three-dimensional yield locus for Tresca (maximum shear stress) theory.

due to the microscopic cracks, flaws or discontinuities which are present in most brittle materials and which act as local stress raisers. These stress raisers, or *stress concentrations*, have a much greater adverse effect in tension and hence produce the characteristic weaker behaviour of brittle materials in tension than in compression.

Thus if the greatest tensile principal stress exceeds the yield stress then failure occurs, and such a simple condition does not require a graphical representation.

15.10. Limitations of the failure theories

It is important to remember that the theories introduced above are those of *elastic* failure, i.e. they relate to the "failure" which is assumed to occur under elastic loading conditions at an equivalent stage to that of yielding in a simple tensile test. If it is anticipated that loading conditions are such that the component may fail in service in a way which cannot easily be related to standard simple loading tests (e.g. under fatigue, creep, buckling, impact loading, etc.) then the above "classical" elastic failure theories should not be applied. A good example of this is the brittle fracture failure of steel under low temperature or very high strain rate (impact) conditions compared with simple ductile failure under normal ambient conditions. If any doubt exists about the relevance of the failure theories then, ideally, specially designed tests should be carried out on the component with loading conditions as near as possible to those expected in service. If, however, elastic failure can be assumed to be relevant it is necessary to consider which of the theories is the most appropriate for the material in question and for the service loading condition expected.

In most cases the Von Mises "distortion energy" theory is considered to be the most reliable and relevant theory with the following exceptions:

- For brittle materials the maximum principal stress or Mohr "internal friction" theories are most suitable. (It must be noted, however, that the former is definitely unsafe for ductile materials.) Some authorities also recommend the Mohr theory for extension of

the theories to ductile *fracture* consideration as opposed to ductile yielding as assumed in the elastic theories.

- (b) All theories produce similar results in loading situations where one principal stress is large compared to another. This can be readily appreciated from the graphical representations if a load-line is drawn with a very small positive or negative slope.
- (c) The greatest discrepancy between the theories is found in the second and fourth quadrants of the graphical representations where the principal stresses are of opposite sign but numerically equal.
- (d) For bi-axial stress conditions, the Mohr modified theory is often preferred, provided that reliable test data are available for tension, compression and torsion.
- (e) In most general bi-axial and tri-axial stress conditions the Tresca maximum shear stress theory is the most conservative (i.e. the safest) theory and this, together with its easily applied and simple formula, probably explains its widespread use in industry.
- (f) The St. Venant maximum principal strain and Haigh total strain energy per unit volume theories are now rarely, if ever, used in general engineering practice.

15.11. Effect of stress concentrations

Whilst stress concentrations have their most significant effect under fatigue loading conditions and impact situations, nevertheless, there are also some important considerations for static loading applications, namely:

- (a) In the presence of ductile yielding, stress concentrations are relatively unimportant since the yielding which will occur at the concentration, e.g. the tip of a notch, will merely redistribute the stresses and not necessarily lead to failure. If, however, there is only marginal ductility, or in the presence of low temperatures, then stress concentrations become more significant as the likelihood of brittle failure increases. It is wise, therefore, to keep stress concentration factors as low as possible.
- (b) For brittle materials like cast iron, internal stress concentrations arise within the material due to the presence of, e.g., flaws, impurities or graphite flakes. These produce stress increases at least as large as those given by surface stress concentrations which, therefore, may have little or no effect on failure. A cast iron bar with a small transverse hole, for example, may not fracture at the hole when a tensile load is applied!

15.12. Safety factors

When using elastic design procedures incorporating any of the failure theories introduced in this chapter it is normal to incorporate safety factors to take account of various imponderables which arise when one attempts to forecast accurately service loads or operating conditions or to make allowance for variations in material properties or behaviour from those assumed by the acceptance of "standard" values. "Ideal" application of the theories, i.e. a rigorous mathematical analysis, is thus rarely possible and the following factors indicate in a little more detail the likely sources of inaccuracy:

1. Whilst design may have been based up nominally static loading, changing service conditions or misuse by operators can often lead to dynamic, fluctuating or impact loading situations which will produce significant increases in maximum stress levels.

2. A precise knowledge of the mechanical properties of the material used in the design is seldom available. Standard elastic values found in reference texts assume ideal homogeneous and isotropic materials with equal "strengths" in all directions. This is rarely true in practice and the effect of internal flaws, inclusions or other weaknesses in the material may be quite significant.
3. The method of manufacture or construction of the component can have a significant effect on service life, particularly if residual stresses are introduced by, e.g., welding or straining beyond the elastic limit during the assembly stages.
4. Complex designs often give rise to difficult analysis problems which even after time-consuming and expensive theoretical procedures, at best yield only a reasonable estimate of maximum service stresses.

Despite these problems and the assumptions which are often required to overcome them, it has been shown that elastic design procedures can be made to agree with experimental results within a reasonable margin of error provided that appropriate safety factors are applied.

It has been shown in § 1.16 that alternative definitions are used for the safety factor depending upon whether it is based on the tensile strength of the material used or its yield strength, i.e., either

$$\text{safety factor, } n = \frac{\text{tensile strength}}{\text{allowable working stress}}$$

or safety factor, $n = \frac{\text{yield stress (or proof stress)}}{\text{allowable working stress}}$

Clearly, *it is important when quoting safety factors to state which definition has been used.*

Safety values vary depending on the type of industry and the area of application of the component being designed. National codes of practice (e.g. British Standards) or other external authority regulations often quote mandatory values to be applied and some companies produce their own guideline values.

Table 15.2 shows the way in which the various factors outlined above contribute to the overall factor of safety for some typical service conditions. These values are based on the yield stress of the materials concerned.

TABLE 15.2. Typical safety factors.

Application	(a) Nature of stress	(b) Nature of load	(c) Type of service	Overall safety factor (a) × (b) × (c)
Steelwork in buildings	1	1	2	2
Pressure vessels	1	1	3	3
Transmission shafts	3	1	2	6
Connecting rods	3	2	1.5	9

It should be noted, however, that the values given in the "type of service" column can be considered to be conservative and severe misuse or overload could increase these (and, hence, the overall factors) by as much as five times.

Recent legislative changes such as "Product Liability" and "Health and Safety at Work" will undoubtedly cause renewed concern that appropriate safety factors are applied, and may

lead to the adoption of higher values. Since this could well result in uneconomic utilisation of materials, such a trend would be regrettable and a move to enhanced product testing and service load monitoring is to be preferred.

15.13. Modes of failure

Before concluding this chapter, the first which looks at design procedures to overcome possible failure (in this case elastic overload), it is appropriate to introduce the reader to the many other ways in which components may fail in order that an appreciation is gained of the complexities often facing designers of engineering components. Sub-classification and a certain amount of cross-referencing does make the list appear to be formidably long but even allowing for these it is evident that the designer, together with his supporting materials and stress advisory teams, has an unenviable task if satisfactory performance and reliability of components is to be obtained in the most complex loading situations. The list below is thus a summary of the so-called “*modes (or methods) of failure*”

1. Mechanical overload/under-design
2. Elastic yielding – force and/or temperature induced.
3. Fatigue
 - high cycle
 - low cycle
 - thermal
 - corrosion
 - fretting
 - impact
 - surface
4. Brittle fracture
5. Creep
6. Combined creep and fatigue
7. Ductile rupture
8. Corrosion
 - direct chemical
 - galvanic
 - pitting
 - cavitation
 - stress
 - intergranular
 - crevice
 - erosion
 - hydrogen damage
 - selective leaching
 - biological
 - corrosion fatigue
9. Impact
 - fracture
 - fatigue

- deformation
- wear
- fretting
- 10. Instability
 - buckling
 - creep buckling
 - torsional instability
- 11. Wear
 - adhesive
 - abrasive
 - corrosive
 - impact
 - deformation
 - surface fatigue
 - fretting
- 12. Vibration
- 13. Environmental
 - thermal shock
 - radiation damage
 - lubrication failure
- 14. Contact
 - spalling
 - pitting
 - galling and seizure
- 15. Stress rupture
- 16. Thermal relaxation

Examples

Example 15.1

A material subjected to a simple tension test shows an elastic limit of 240 MN/m^2 . Calculate the factor of safety provided if the principal stresses set up in a complex two-dimensional stress system are limited to 140 MN/m^2 tensile and 45 MN/m^2 compressive. The appropriate theories of failure on which your answer should be based are:

- the maximum shear stress theory;
- the maximum shear strain energy theory.

Solution

(a) Maximum shear stress theory

This theory states that failure will occur when the maximum shear stress in the material equals the maximum shear stress value at the yield point in a simple tension test, i.e. when

$$\frac{1}{2}(\sigma_1 - \sigma_3) = \frac{1}{2}\sigma_y$$

or

$$\sigma_1 - \sigma_3 = \sigma_y$$

In this case the system is two-dimensional, i.e. the principal stress in one plane is zero. However, since one of the given principal stresses is a compressive one, it follows that the zero value is that of σ_2 since the negative value of σ_3 associated with the compressive stress will produce a numerically greater value of stress difference $\sigma_1 - \sigma_3$ and hence must be used in the above criterion.

Thus $\sigma_1 = 140 \text{ MN/m}^2$, $\sigma_2 = 0$ and $\sigma_3 = -45 \text{ MN/m}^2$.

Now with a factor of safety applied the design yield point becomes σ_y/n and this must replace σ_y in the yield criterion which then becomes

$$\frac{\sigma_y}{n} = \sigma_1 - \sigma_3$$

$$\therefore \frac{240}{n} = 140 - (-45) \quad \text{units of MN/m}^2 \text{ throughout}$$

$$n = \frac{240}{185} = 1.3$$

The required factor of safety is 1.3.

(b) Maximum shear strain energy theory

Once again equating the values of the quantity concerned in the tensile test and in the complex stress system,

$$\frac{\sigma_y^2}{6G} = \frac{1}{12G} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

$$\sigma_y^2 = \frac{1}{2} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

With the three principal stress values used above and with σ_y/n replacing σ_y ,

$$\left(\frac{240}{n} \right)^2 = \frac{1}{2} \{ (140 - 0)^2 + [0 - (-45)]^2 + (-45 - 140)^2 \}$$

$$\frac{5.76 \times 10^4}{n^2} = \frac{1}{2} [1.96 + 0.203 + 3.42] 10^4$$

$$n^2 = \frac{2 \times 5.76 \times 10^4}{5.583 \times 10^4} = 2.063$$

$$\therefore n = 1.44$$

The required factor of safety is now 1.44.

Example 15.2

A steel tube has a mean diameter of 100 mm and a thickness of 3 mm. Calculate the torque which can be transmitted by the tube with a factor of safety of 2.25 if the criterion of failure is (a) maximum shear stress; (b) maximum strain energy; (c) maximum shear strain energy. The elastic limit of the steel in tension is 225 MN/m² and Poisson's ratio ν is 0.3.

Solution

From the torsion theory

$$\frac{T}{J} = \frac{\tau}{R} \quad \therefore \quad \tau = \frac{TR}{J}$$

Now mean diameter of tube = 100 mm and thickness = 3 mm.

$$\therefore J = \pi dt \times r^2 = \frac{\pi d^3 t}{4} \quad (\text{approximately})$$

$$= \frac{\pi \times 0.1^3 \times 0.003}{4} = 2.36 \times 10^{-6} \text{ m}^4$$

$$\therefore \text{shear stress } \tau = \frac{T \times 51.5 \times 10^{-3}}{2.36 \times 10^{-6}} = (2.18 \times 10^4)T \text{ N/m}^2 \\ = 21.8T \text{ kN/m}^2$$

(a) Maximum shear stress

Torsion introduces pure shear onto elements within the tube material and it has been shown in § 13.2 that pure shear produces an equivalent principal direct stress system, one tensile and one compressive and both equal in value to the applied shear stress,

$$\text{i.e. } \sigma_1 = \tau, \quad \sigma_3 = -\tau \quad (\text{and } \sigma_2 = 0)$$

Thus for the maximum shear stress criterion, taking account of the safety factor,

$$\frac{\sigma_y}{n} = \sigma_1 - \sigma_3 = \tau - (-\tau)$$

$$\therefore \frac{225 \times 10^6}{2.25} = 2\tau = 2 \times 21.8T \times 10^3$$

$$\therefore T = \frac{100 \times 10^6}{2 \times 21.8 \times 10^3} = 2.3 \times 10^3 \text{ N m}$$

The torque which can be safely applied = 2.3 kN m.

(b) Maximum strain energy

From eqn. (15.3) the relevant criterion of failure is

$$\sigma_y^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)$$

Taking account of the safety factor

$$\left(\frac{225 \times 10^6}{2.25} \right)^2 = \tau^2 + 0 + (-\tau)^2 - 2 \times 0.3[\tau \times (-\tau)] \\ = 2.6\tau^2 \\ = 2.6(21.8 \times 10^3 T)^2$$

$$\therefore T = \frac{100 \times 10^6}{\sqrt{(2.6) \times 21.8 \times 10^3}} = 2.84 \times 10^3 \text{ N m}$$

The safe torque is now 2.84 kN m.

(c) *Maximum shear strain energy*

From eqn. (15.4) the criterion of failure is

$$\sigma_y^2 = \frac{1}{2} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

$$\therefore \left(\frac{225 \times 10^6}{2.25} \right)^2 = \frac{1}{2} \{ (\tau - 0)^2 + [0 - (-\tau)]^2 + (-\tau - \tau)^2 \} \\ = 3\tau^2$$

$$\therefore \tau = \frac{100 \times 10^6}{\sqrt{3}} = 21.8 \times 10^3 T$$

$$\therefore T = \frac{100 \times 10^6}{21.8 \times 10^3 \times \sqrt{3}} = 2.65 \times 10^3 \text{ N m}$$

The safe torque is now 2.65 kN m.

Example 15.3

A structure is composed of circular members of diameter d . At a certain position along one member the loading is found to consist of a shear force of 10 kN together with an axial tensile load of 20 kN. If the elastic limit in tension of the material of the members is 270 MN/m² and there is to be a factor of safety of 4, estimate the magnitude of d required according to (a) the maximum principal stress theory, and (b) the maximum shear strain energy per unit volume theory. Poisson's ratio $\nu = 0.283$.

Solution

The stress system at the point concerned is as shown in Fig. 15.15, the principal stress normal to the surface of the member being zero.

Now the direct stress along the axis of the bar is tensile, i.e. positive, and given by

$$\sigma_x = \frac{\text{load}}{\text{area}} = \frac{20}{\pi d^2/4} = \frac{80}{\pi d^2} \text{ kN/m}^2$$

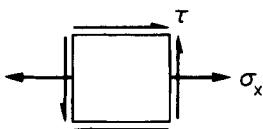


Fig. 15.15.

and the shear stress is

$$\tau = \frac{\text{shear load}}{\text{area}} = \frac{10}{\pi d^2/4} = \frac{40}{\pi d^2} \text{ kN/m}^2$$

The principal stresses are given by Mohr's circle construction (πd^2 being a common denominator) or from

$$\sigma_1 \quad \text{and} \quad \sigma_3 = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}$$

with σ_y zero,

$$\text{i.e. } \sigma_1 \quad \text{or} \quad \sigma_3 = \frac{1}{2} \left\{ \frac{80}{\pi d^2} \pm \sqrt{\left[\left(\frac{80}{\pi d^2} \right)^2 + 4 \left(\frac{40}{\pi d^2} \right)^2 \right]} \right\}$$

$$= \frac{40}{\pi d^2} (1 \pm \sqrt{2})$$

$$\therefore \sigma_1 = \frac{40 \times 2.414}{\pi d^2} = \frac{30.7}{d^2} \text{ kN/m}^2$$

$$\sigma_3 = -\frac{40 \times 0.414}{\pi d^2} = -\frac{5.27}{d^2} \text{ kN/m}^2$$

and

$$\sigma_2 = 0$$

Since the elastic limit in tension is 270 MN/m² and the factor of safety is 4, the working stress or effective yield stress is

$$\sigma_y = \frac{270}{4} = 67.5 \text{ MN/m}^2$$

(a) Maximum principal stress theory

Failure is assumed to occur when

$$\sigma_1 = \sigma_y$$

$$\therefore \frac{30.7 \times 10^3}{d^2} = 67.5 \times 10^6$$

$$\therefore d^2 = \frac{30.7}{67.5} \times 10^{-3} = 4.55 \times 10^{-4} \text{ m}^2$$

$$\therefore d = 2.13 \times 10^{-2} \text{ m} = 21.3 \text{ mm}$$

(b) Maximum shear strain energy

From eqn. (15.4) the criterion of failure is

$$2\sigma_y^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2$$

Therefore taking account of the safety factor

$$\begin{aligned} 2(67.5 \times 10^6)^2 &= \left[\left(\frac{30.7}{d^2} \right)^2 + \left(-\frac{5.27}{d^2} \right)^2 + \left(\frac{-5.27 - 30.7}{d^2} \right)^2 \right] \times 10^6 \\ &= \frac{2264 \times 10^6}{d^4} \end{aligned}$$

$$\therefore d^4 = \frac{1132 \times 10^6}{(67.5 \times 10^6)^2}$$

$$\therefore d^2 = \frac{33.6 \times 10^3}{67.5 \times 10^6} = 4.985 \times 10^{-4} \text{ m}^2$$

$$\therefore d = 22.3 \text{ mm}$$

Example 15.4

Assuming the formulae for the principal stresses and the maximum shear stress induced in a material owing to combined stresses and the fundamental formulae for pure bending, derive a formula in terms of the bending moment M and the twisting moment T for the equivalent twisting moment on a shaft subjected to combined bending and torsion for

- (a) the maximum principal stress criterion;
- (b) the maximum shear stress criterion.

Solution

The *equivalent torque*, or turning moment, is defined as that torque which, acting alone, will produce the same conditions of stress as the combined bending and turning moments.

At failure the stress produced by the equivalent torque T_E is given by the torsion theory

$$\frac{T}{J} = \frac{\tau}{R}$$

$$\therefore \tau_{\max} = \frac{T_E R}{J} = \frac{T_E \times D}{2J}$$

The direct stress owing to bending is

$$\sigma_x = \frac{My_{\max}}{I} = \frac{MD}{2I} = \frac{MD}{J}$$

and the shear stress due to torsion is

$$\tau = \frac{TD}{2J}$$

The principal stresses are then given by

$$\sigma_{1,3} = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]} \quad \text{with} \quad \sigma_y = 0 \quad \text{and} \quad \sigma_2 = 0$$

$$= \frac{1}{2}\left(\frac{MD}{J}\right) \pm \frac{1}{2}\sqrt{\left[\left(\frac{MD}{J}\right)^2 + 4\left(\frac{TD}{2J}\right)^2\right]}$$

$$= \frac{D}{2J}(M \pm \sqrt{[M^2 + T^2]})$$

$$\therefore \sigma_1 = \frac{D}{2J} (M + \sqrt{[M^2 + T^2]})$$

$$\sigma_3 = \frac{D}{2J} (M - \sqrt{[M^2 + T^2]})$$

(a) *For maximum principal stress criterion*

$$\frac{T_E D}{2J} = \sigma_1 = \frac{D}{2J} (M + \sqrt{[M^2 + T^2]})$$

$$\therefore T_E = M + \sqrt{(M^2 + T^2)}$$

(b) *For maximum shear stress criterion*

$$\frac{T_E D}{2J} = \frac{1}{2} (\sigma_1 - \sigma_3)$$

$$= \frac{1}{2} \left\{ \frac{D}{2J} (M + \sqrt{[M^2 + T^2]}) - \frac{D}{2J} (M - \sqrt{[M^2 + T^2]}) \right\}$$

$$T_E = \sqrt{(M^2 + T^2)}$$

Example 15.5

The test strengths of a material under pure compression and pure tension are $\sigma_{y_c} = 350 \text{ MN/m}^2$ and $\sigma_{y_t} = 300 \text{ MN/m}^2$. In a certain design of component the material may be subjected to each of the five biaxial stress states shown in Fig. 15.16. Assuming that failure is deemed to occur when yielding takes place, arrange the five stress states in order of diminishing factor of safety according to the maximum principal or normal stress, maximum shear stress, maximum shear strain energy (or distortion energy) and modified Mohr's (or internal friction) theories.

Solution

A graphical solution of this problem can be employed by constructing the combined yield loci for the criteria mentioned in the question. Since σ_1 the maximum principal stress is $+100 \text{ MN/m}^2$ in each of the stress states only half the combined loci diagram is required, i.e. the positive σ_1 half.

Here it must be remembered that for stress condition (e) pure shear is exactly equivalent to two mutually perpendicular direct stresses – one tensile, the other compressive, acting on 45° planes and of equal value to the applied shear, i.e. for condition (e) $\sigma_1 = 100 \text{ MN/m}^2$ and $\sigma_2 = -100 \text{ MN/m}^2$ (see §13.2).

It is now possible to construct the “load lines” for each stress state with slopes of σ_2/σ_1 . An immediate solution is then obtained by considering the intersection of each load line with the failure envelopes.

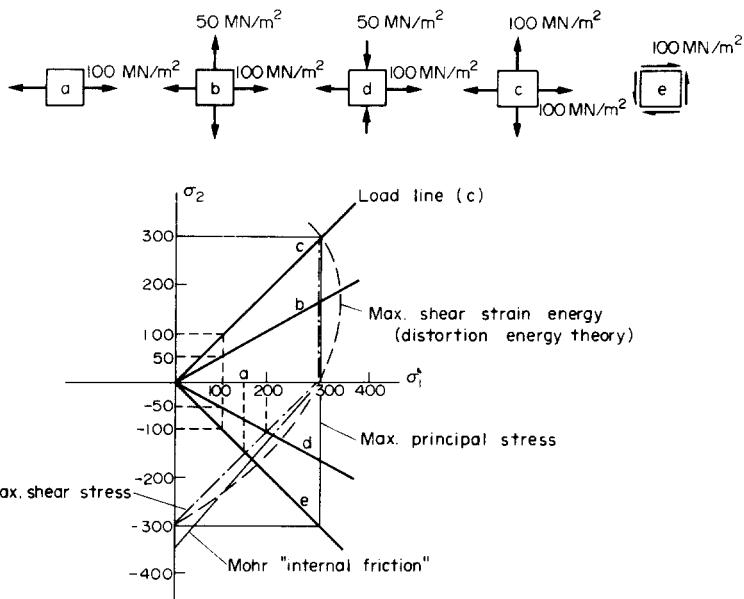


Fig. 15.16.

Maximum principal stress theory

All five load lines cut the failure envelope for this theory at $\sigma_1 = 300 \text{ MN/m}^2$. According to this theory, therefore, all the stress states will produce failure when the maximum direct stress reaches 300 MN/m^2 . Since the maximum principal stress present in each stress state is 100 MN/m^2 it therefore follows that the safety factor for each state according to the maximum principal stress theory is $\frac{300}{100} = 3$.

Maximum shear stress theory

The load lines a, b and c cut the failure envelope for this theory at $\sigma_1 = 300 \text{ MN/m}^2$ whilst d and e cut it at $\sigma_1 = 200 \text{ MN/m}^2$ and $\sigma_1 = 150 \text{ MN/m}^2$ respectively as shown in Fig. 15.16. The safety factors are, therefore,

$$a, b, c = \frac{300}{100} = 3, \quad d = \frac{200}{100} = 2, \quad e = \frac{150}{100} = 1.5$$

Maximum shear strain energy theory

In decreasing order, the factors of safety for this theory, found as before from the points where each load line crosses the failure envelope, are

$$b = \frac{347}{100} = 3.47, \quad a, c = \frac{300}{100} = 3, \quad d = \frac{227}{100} = 2.27, \quad e = \frac{173}{100} = 1.73$$

Mohr's modified or internal friction theory (with $\sigma_{y_c} = 350 \text{ MN/m}^2$)

In this case the safety factors are:

$$a, b, c = \frac{300}{100} = 3, \quad d = \frac{210}{100} = 2.1, \quad e = \frac{162}{100} = 1.62$$

Example 15.6

The cast iron used in the manufacture of an engineering component has tensile and compressive strengths of 400 MN/m^2 and 1.20 GN/m^2 respectively.

- If the maximum value of the tensile principal stress is to be limited to one-quarter of the tensile strength, determine the maximum value and nature of the other principal stress using Mohr's modified yield theory for brittle materials.
- What would be the values of the principal stresses associated with a maximum shear stress of 450 MN/m^2 according to Mohr's modified theory?
- At some point in a component principal stresses of 100 MN/m^2 tensile and 100 MN/m^2 compressive are found to be present. Estimate the safety factor with respect to initial yield using the maximum principal stress, maximum shear stress, distortion energy and Mohr's modified theories of elastic failure.

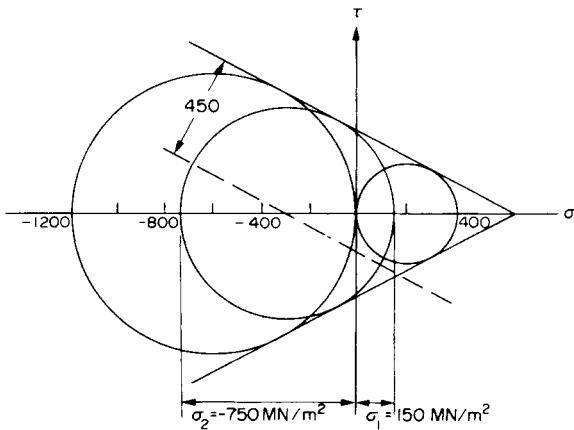


Fig. 15.17.

Solution

$$(a) \text{ Maximum principal stress} = \frac{400}{4} = 100 \text{ MN/m}^2$$

According to Mohr's theory

$$\frac{\sigma_1}{\sigma_{y_t}} + \frac{\sigma_2}{\sigma_{y_c}} = 1$$

$$\therefore \frac{100 \times 10^6}{400 \times 10^6} + \frac{\sigma_2}{-1.2 \times 10^9} = 1$$

$$\therefore \sigma_2 = -1.2 \times 10^9 (1 - \frac{1}{4}) = -900 \text{ MN/m}^2$$

- (b) In any Mohr circle construction the radius of the circle equals the maximum shear stress value. In order to answer this part of the question, therefore, it is necessary to draw the Mohr failure envelope on σ - τ axes as shown in Fig. 15.17 and to construct the circle which is tangential to the envelope and has a radius of 450 MN/m^2 . This is achieved by drawing a line parallel to the failure envelope and a distance of 450 MN/m^2 (to scale) from it. Where this line cuts the σ axis is then the centre of the required circle. The desired principal stresses are then, as usual, the extremities of the horizontal diameter of the circle.

Thus from Fig. 15.17

$$\sigma_1 = 150 \text{ MN/m}^2 \quad \text{and} \quad \sigma_2 = -750 \text{ MN/m}^2$$

- (c) The solution here is similar to that used for Example 15.5. The yield loci are first plotted for the given failure theories and the required safety factors determined from the points of intersection of the loci and the load line with a slope of $100/-100 = -1$.

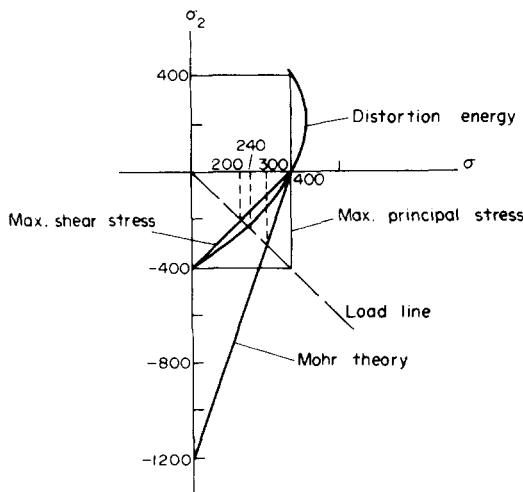


Fig. 15.18.

Thus from Fig. 15.18 the safety factors are:

$$\text{Maximum principal stress} = \frac{400}{100} = 4$$

$$\text{Maximum shear stress} = \frac{200}{100} = 2$$

Distortion energy $= \frac{240}{100} = 2.4$

Mohr theory $= \frac{300}{100} = 3$

Problems

15.1 (B). If the principal stresses at a point in an elastic material are 120 MN/m^2 tensile, 180 MN/m^2 tensile and 75 MN/m^2 compressive, find the stress at the limit of proportionality expected in a simple tensile test assuming:

- (a) the maximum shear stress theory;
- (b) the maximum shear strain energy theory;
- (c) the maximum principal strain theory.

Assume $\nu = 0.294$.

[$255, 230.9, 166.8 \text{ MN/m}^2$.]

15.2 (B). A horizontal shaft of 75 mm diameter projects from a bearing, and in addition to the torque transmitted the shaft carries a vertical load of 8 kN at 300 mm from the bearing. If the safe stress for the material, as determined in a simple tensile test, is 135 MN/m^2 , find the safe torque to which the shaft may be subjected using as the criterion (a) the maximum shearing stress; (b) the maximum strain energy. Poisson's ratio $\nu = 0.29$.

[U.L.] [$5.05, 6.3 \text{ kN m}$.]

15.3 (B). Show that the strain energy per unit volume of a material under a single direct stress is given by $\frac{1}{2}$ (stress \times strain). Hence show that for a material under the action of the principal stresses σ_1, σ_2 and σ_3 the strain energy per unit volume becomes

$$\frac{1}{2E} [\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3)]$$

A thin cylinder 1 m diameter and 3 m long is filled with a liquid to a pressure of 2 MN/m^2 . Assuming a yield stress for the material of 240 MN/m^2 in simple tension and a safety factor of 4, determine the necessary wall thickness of the cylinder, taking the maximum shear strain energy as the criterion of failure.

For the cylinder material, $E = 207 \text{ GN/m}^2$ and $\nu = 0.286$.

[14.4 mm .]

15.4 (B). An aluminium-alloy tube of 25 mm outside diameter and 22 mm inside diameter is to be used as a shaft. It is 500 mm long, in self-aligning bearings, and supports a load of 0.5 kN at mid-span. In order to find the maximum allowable shear stress a length of tube was tested in tension and reached the limit of proportionality at 21 kN . Assuming the criterion for elastic failure to be the maximum shear stress, find the greatest torque to which the shaft could be subjected.

[98.2 N m .]

15.5 (B). A bending moment of 4 kN m is found to cause elastic failure of a solid circular shaft. An exactly similar shaft is now subjected to a torque T . Determine the value of T which will cause failure of the shaft according to the following theories:

- (a) maximum principal stress;
- (b) maximum principal strain;
- (c) maximum shear strain energy. ($\nu = 0.3$.)

Which of these values would you expect to be the most reliable and why?

[$8, 6.15, 4.62 \text{ kN m}$.]

15.6 (B). A thin cylindrical pressure vessel with closed ends is required to withstand an internal pressure of 4 MN/m^2 . The inside diameter of the vessel is to be 500 mm and a factor of safety of 4 is required. A sample of the proposed material tested in simple tension gave a yield stress of 360 MN/m^2 .

Find the thickness of the vessel, assuming the criterion of elastic failure to be (a) the maximum shear stress, (b) the shear strain energy.

[E.M.E.U.] [$11.1, 9.62 \text{ mm}$.]

15.7 (B). Derive an expression for the strain energy stored in a material when subjected to three principal stresses.

A material is subjected to a system of three mutually perpendicular stresses as follows: f tensile, $2f$ tensile and f compressive. If this material failed in simple tension at a stress of 180 MN/m^2 , determine the value of f if the criterion of failure is:

- (a) maximum principal stress;
- (b) maximum shear stress;
- (c) maximum strain energy.

Take Poisson's ratio $\nu = 0.3$.

[$90, 60, 70 \text{ MN/m}^2$.]

15.8 (B). The external and internal diameters of a hollow steel shaft are 150 mm and 100 mm. A power transmission test with a torsion dynamometer showed an angle of twist of 0.13 degree on a 250 mm length when the speed was 500 rev/min. Find the power being transmitted and the torsional strain energy per metre length.

If, in addition to this torque, a bending moment of 15 kN m together with an axial compressive force of 80 kN also acted upon the shaft, find the value of the equivalent stress in simple tension corresponding to the maximum shear strain energy theory of elastic failure. Take $G = 80 \text{ GN/m}^2$.

[I.Mech.E.] [1.52 MW; 13.13 J/m; 113 MN/m².]

15.9 (B/C). A close-coiled helical spring has a wire diameter of 2.5 mm and a mean coil diameter of 40 mm. The spring is subjected to a combined axial load of 60 N and a torque acting about the axis of the spring. Determine the maximum permissible torque if (a) the material is brittle and ultimate failure is to be avoided, the criterion of failure is the maximum tensile stress, and the ultimate tensile stress is 1.2 GN/m^2 , (b) the material is ductile and failure by yielding is to be avoided, the criterion of failure is the maximum shear stress, and the yield in tension is 0.9 GN/m^2 .

[I.Mech.E.] [1.645, 0.68 N.m.]

15.10 (C). A closed-ended thick-walled steel cylinder with a diameter ratio of 2 is subjected to an internal pressure. If yield occurs at a pressure of 270 MN/m^2 find the yield strength of the steel used and the diametral strain at the bore at yield. Yield can be assumed to occur at a critical value of the maximum shear stress. It can be assumed that the stresses in a thick-walled cylinder are:

$$\text{hoop stress } \sigma_H = A + \frac{B}{r^2}$$

$$\text{radial stress } \sigma_r = A - \frac{B}{r^2}$$

$$\text{axial stress } \sigma_L = \frac{1}{2}(\sigma_H + \sigma_r)$$

where A and B are constants and r is any radius.

For the cylinder material $E = 210 \text{ GN/m}^2$ and $v = 0.3$.

[I.Mech.E.] [721 MN/m²; 2.4×10^{-3} .]

15.11 (C). For a certain material subjected to plane stress it is assumed that the criterion of elastic failure is the shear strain energy per unit volume. By considering co-ordinates relative to two axes at 45° to the principal axes, show that the limiting values of the two principal stresses can be represented by an ellipse having semi-diameters $\sigma_e\sqrt{2}$ and $\sigma_e\sqrt{\frac{3}{2}}$, where σ_e is the equivalent simple tension. Hence show that for a given value of the major principal stress the elastic factor of safety is greatest when the minor principal stress is half the major, both stresses being of the same sign.

[U.L.]

15.12 (C). A horizontal circular shaft of diameter d and second moment of area I is subjected to a bending moment $M \cos \theta$ in a vertical plane and to an axial twisting moment $M \sin \theta$. Show that the principal stresses at the ends of a vertical diameter are $\frac{1}{2}Mk(\cos \theta \pm 1)$, where

$$k = \frac{d}{2I}$$

If strain energy is the criterion of failure, show that

$$\tau_{\max} = \frac{\tau_0 \sqrt{2}}{[\cos^2 \theta(1-v) + (1+v)]^{\frac{1}{2}}}$$

where τ_{\max} = maximum shearing stress,

τ_0 = maximum shearing stress in the special case when $\theta = 0$,

v = Poisson's ratio.

[U.L.]

15.13 (C). What are meant by the terms "yield criterion" and "yield locus" as related to ductile metals and why, in general, are principal stresses involved?

Define the maximum shear stress and shear strain energy theories of yielding. Describe the three-dimensional loci and sketch the plane stress loci for the above theories.

[C.E.I.]

15.14 (B). The maximum shear stress theory of elastic failure is sometimes criticised because it makes no allowance for the magnitude of the intermediate principal stress. On these grounds a theory is preferred which predicts that yield will not occur provided that

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 < 2\sigma_y^2$$

What is the criterion of failure implied here?

Assuming that σ_1 and σ_3 are fixed and unequal, find the value of σ_2 which will be most effective in preventing failure according to this theory. If this theory is correct, by what percentage does the maximum shear stress theory underestimate the strength of a material in this case?

[City U.] [$\frac{1}{3}(\sigma_1 + \sigma_3)$; 13.4%.]

15.15 (B) The cast iron used in the manufacture of an engineering component has tensile and compressive strengths of 400 MN/m^2 and 1.20 GN/m^2 respectively.

- If the maximum value of the tensile principal stress is to be limited to one-quarter of the tensile strength, determine the maximum value and nature of the other principal stress using Mohr's modified yield theory for brittle materials.
- What would be the values of the principal stresses associated with a maximum shear stress of 450 MN/m^2 according to Mohr's modified theory?

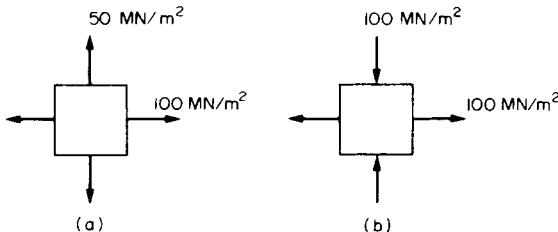


Fig. 15.19.

- Estimate the safety factor with respect to initial failure for the stress conditions of Fig. 15.19 using the maximum principal stress, maximum shear stress, distortion energy and Mohr's modified theories of elastic failure.

[B.P.] [-900 MN/m²; 150, -750 MN/m²; 4, 4, 4.7, 4 and 4, 2, 2.4, 3.]

15.16 (B). Show that for a material subjected to two principal stresses, σ_1 and σ_2 , the strain energy per unit volume is

$$\frac{1}{2E} (\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2)$$

A thin-walled steel tube of internal diameter 150 mm, closed at its ends, is subjected to an internal fluid pressure of 3 MN/m^2 . Find the thickness of the tube if the criterion of failure is the maximum strain energy. Assume a factor of safety of 4 and take the elastic limit in pure tension as 300 MN/m^2 . Poisson's ratio $\nu = 0.28$.

[I.Mech.E.] [2.95 mm]

15.17 (B). A circular shaft, 100 mm diameter is subjected to combined bending moment and torque, the bending moment being 3 times the torque. If the direct tension yield point of the material is 300 MN/m^2 and the factor of safety on yield is to be 4, calculate the allowable twisting moment by the three following theories of failure:

- Maximum principal stress theory
- Maximum shear stress theory
- Maximum shear strain energy theory.

[U.L.] [2.86, 2.79, 2.83 kNm]

15.18 (B). A horizontal shaft of 75 mm diameter projects from a bearing and, in addition to the torque transmitted, the shaft carries a vertical load of 8 kN at 300 mm from the bearing. If the safe stress for the material, as determined in a simple tension test, is 135 MN/m^2 find the safe torque to which the shaft may be subjected using as a criterion

- the maximum shearing stress,
- the maximum strain energy per unit volume.

Poisson's ratio $\nu = 0.29$.

[U.L.] [5.59, 8.3 kNm.]

CHAPTER 16

EXPERIMENTAL STRESS ANALYSIS

Introduction

We live today in a complex world of manmade structures and machines. We work in buildings which may be many storeys high and travel in cars and ships, trains and planes; we build huge bridges and concrete dams and send mammoth rockets into space. Such is our confidence in the modern engineer that we take these manmade structures for granted. We assume that the bridge will not collapse under the weight of the car and that the wings will not fall away from the aircraft. We are confident that the engineer has assessed the stresses within these structures and has built in sufficient strength to meet all eventualities.

This attitude of mind is a tribute to the competence and reliability of the modern engineer. However, the commonly held belief that the engineer has been able to calculate mathematically the stresses within the complex structures is generally ill-founded. When he is dealing with familiar design problems and following conventional practice, the engineer draws on past experience in assessing the strength that must be built into a structure. A competent civil engineer, for example, has little difficulty in selecting the size of steel girder that he needs to support a wall. When he departs from conventional practice, however, and is called upon to design unfamiliar structures or to use new materials or techniques, the engineer can no longer depend upon past experience. The mathematical analysis of the stresses in complex components may not, in some cases, be a practical proposition owing to the high cost of computer time involved. If the engineer has no other way of assessing stresses except by recourse to the nearest standard shape and hence analytical solution available, he builds in greater strength than he judges to be necessary (i.e. he incorporates a factor of safety) in the hope of ensuring that the component will not fail in practice. Inevitably, this means unnecessary weight, size and cost, not only in the component itself but also in the other members of the structure which are associated with it.

To overcome this situation the modern engineer makes use of experimental techniques of stress measurement and analysis. Some of these consist of "reassurance" testing of completed structures which have been designed and built on the basis of existing analytical knowledge and past experience: others make use of scale models to predict the stresses, often before final designs have been completed.

Over the past few years these *experimental stress analysis* or *strain measurement* techniques have served an increasingly important role in aiding designers to produce not only efficient but economic designs. In some cases substantial reductions in weight and easier manufacturing processes have been achieved.

A large number of problems where experimental stress analysis techniques have been of particular value are those involving fatigue loading. Under such conditions failure usually starts when a fatigue crack develops at some position of high localised stress and propagates

until final rupture occurs. As this often requires several thousand repeated cycles of load under service conditions, full-scale production is normally well under way when failure occurs. Delays at this stage can be very expensive, and the time saved by stress analysis techniques in locating the source of the trouble can far outweigh the initial cost of the equipment involved.

The main techniques of experimental stress analysis which are in use today are:

- (1) brittle lacquers
- (2) strain gauges
- (3) photoelasticity
- (4) photoelastic coatings

The aim of this chapter is to introduce the fundamental principles of these techniques, together with limited details of the principles of application, in order that the reader can appreciate (a) the role of the experimental techniques as against the theoretical procedures described in the other chapters, (b) the relative merits of each technique, and (c) the more specialised literature which is available on the techniques, to which reference will be made.

16.1. Brittle lacquers

The brittle-lacquer technique of experimental stress analysis relies on the failure by cracking of a layer of a brittle coating which has been applied to the surface under investigation. The coating is normally sprayed onto the surface and allowed to air- or heat-cure to attain its brittle properties. When the component is loaded, this coating will crack as its so-called *threshold strain* or *strain sensitivity* is exceeded. A typical crack pattern obtained on an engineering component is shown in Fig. 16.1. Cracking occurs where the strain is greatest,

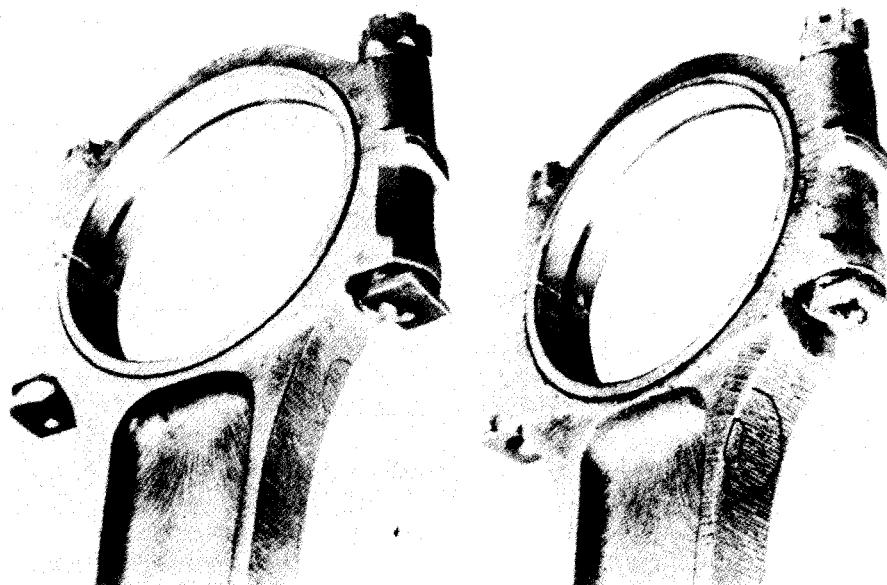


Fig. 16.1. Typical brittle-lacquer crack pattern on an engine con-rod. (Magnaflux Corporation.)

so that an immediate indication is given of the presence of stress concentrations. The cracks also indicate the directions of maximum strain at these points since they are always aligned at right angles to the direction of the maximum principal tensile strain. The method is thus of great value in determining the optimum positions in which to place strain gauges (see §16.2) in order to record accurately the measurements of strain in these directions.

The brittle-coating technique was first used successfully in 1932 by Dietrich and Lehr in Germany despite the fact that references relating to observation of the phenomenon can be traced back to Clarke's investigations of tubular bridges in 1850. The most important advance in brittle-lacquer technology, however, came in the United States in 1937–41 when Ellis, De Forrest and Stern produced a series of lacquers known as "Stresscoat" which, in a modified form, remain widely used in the world today.

There are many every-day examples of brittle coatings which can be readily observed by the reader to exhibit cracks indicating local yielding when the strain is sufficiently large, e.g. cellulose, vitreous or enamel finishes. Cellulose paints, in fact, are used by some engineering companies as a brittle lacquer on rubber models where the strains are quite large.

As an interesting experiment, try spraying a comb with several thin coats of hair-spray lacquer, giving each layer an opportunity to dry before application of the next coat. Finally, allow the whole coating several hours to fully cure; cracks should then become visible when the comb is bent between your fingers.

In engineering applications a little more care is necessary in the preparation of the component and application of the lacquer, but the technique remains a relatively simple and hence attractive one. The surface of the component should be relatively smooth and clean, standard solvents being used to remove all traces of grease and dirt. The lacquer can then be applied, the actual application procedure depending on the type of lacquer used. Most lacquers may be sprayed or painted onto the surface, spraying being generally more favoured since this produces a more uniform thickness of coating and allows a greater control of the thickness. Other lacquers, for example, are in wax or powder form and require pre-heating of the component surface in order that the lacquer will melt and run over the surface. Optimum coating thicknesses depend on the lacquer used but are generally of the order of 1 mm.

In order to determine the strain sensitivity of the lacquer, and hence to achieve an approximate idea of the strains existing in the component, it is necessary to coat calibration bars at the same time and in exactly the same manner as the specimen itself. These bars are normally simple rectangular bars which fit into the calibration jig shown in Fig. 16.2 to form a simple cantilever with an offset cam at the end producing a known strain distribution along the cantilever length. When the lacquer on the bar is fully cured, the lever on the cam is moved forward to depress the end of the bar by a known amount, and the position at which the cracking of the lacquer begins gives the strain sensitivity when compared with the marked strain scale. This enables quantitative measurements of strain levels to be made on the components under test since if, for example, the calibration sensitivity is shown to be 800 microstrain ($\text{strain} \times 10^{-6}$), then the strain at the point on the component at which cracks first appear is also 800 microstrain.

This type of quantitative measurement is generally accurate to no better than 10–20%, and brittle-lacquer techniques are normally used to locate the *positions of stress maxima*, the actual values then being determined by subsequent strain-gauge testing.

Loading is normally applied to the component in increments, held for a few minutes and released to zero prior to application of the next increment; the time interval between increments should be several times greater than that of the loading cycle. With this procedure

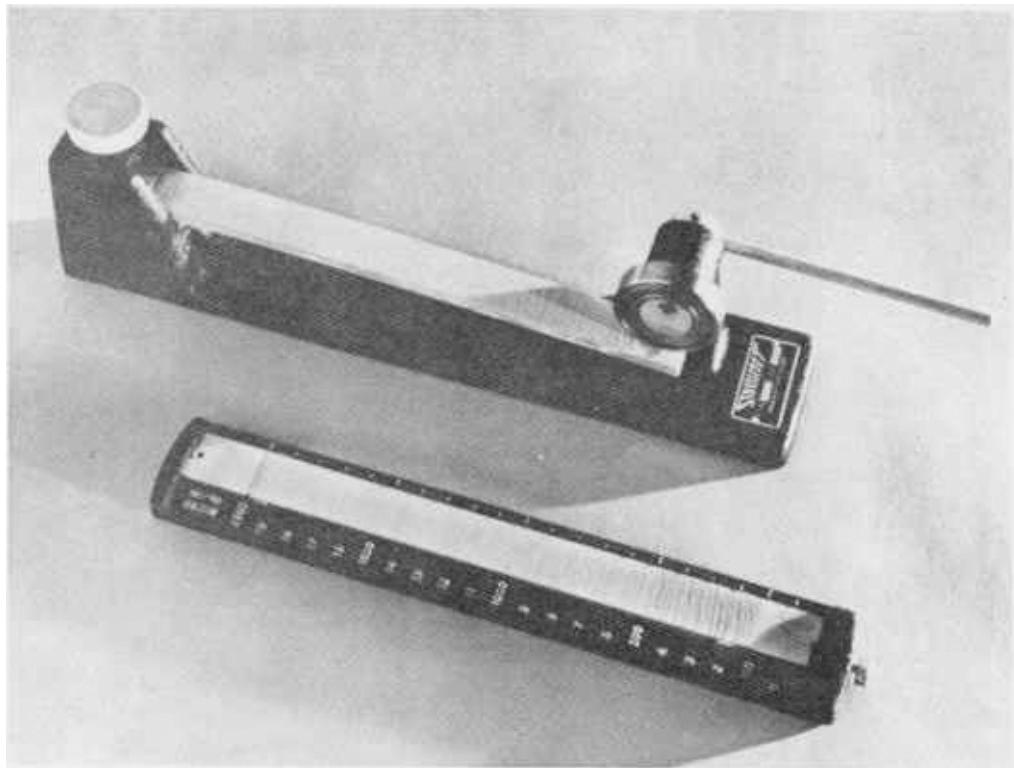


Fig. 16.2. (Top) Brittle-lacquer calibration bar in a calibration jig with the cam depressed to apply load. (Bottom) Calibration of approximately 100 microstrain. (Magna-flux Corporation.)

creep effects in the lacquer, where strain in the lacquer changes at constant load, are completely overcome. After each load application, cracks should be sought and, when located, encircled and identified with the load at that stage using a chinagraph pencil. This enables an accurate record of the development of strain throughout the component to be built up.

There are a number of methods which can be used to aid crack detection including (a) pre-coating the component with an aluminium undercoat to provide a background of uniform colour and intensity, (b) use of a portable torch which, when held close to the surface, highlights the cracks by reflection on the crack faces, (c) use of dye-etchants or special electrified particle inspection techniques, details of which may be found in standard reference texts.⁽³⁾

Given good conditions, however, and a uniform base colour, cracks are often visible without any artificial aid, viewing the surface from various angles generally proving sufficient.

Figures 16.3 and 16.4 show further examples of brittle-lacquer crack patterns on typical engineering components. The procedure is simple, quick and relatively inexpensive; it can be carried out by relatively untrained personnel, and immediate qualitative information, such as positions of stress concentration, is provided on the most complicated shapes.

Various types of lacquer are available, including a special ceramic lacquer which is

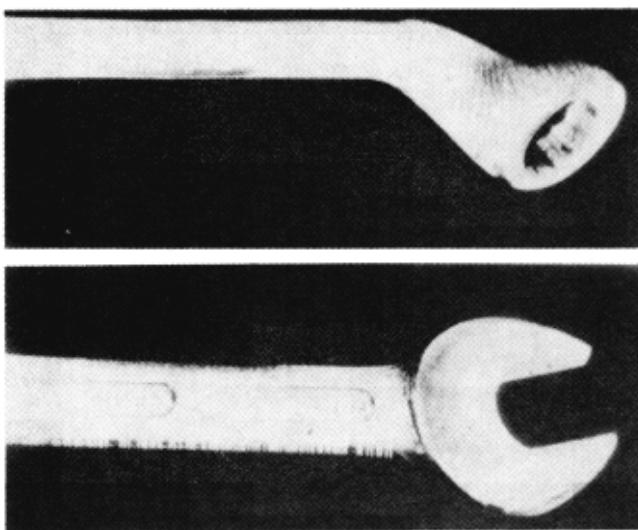


Fig. 16.3. Brittle-lacquer crack patterns on an open-ended spanner and a ring spanner. In the former the cracks appear at right angles to the maximum bending stress in the edge of the spanner whilst in the ring spanner the presence of torsion produces an inclination of the principal stress and hence of the cracks in the lacquer.

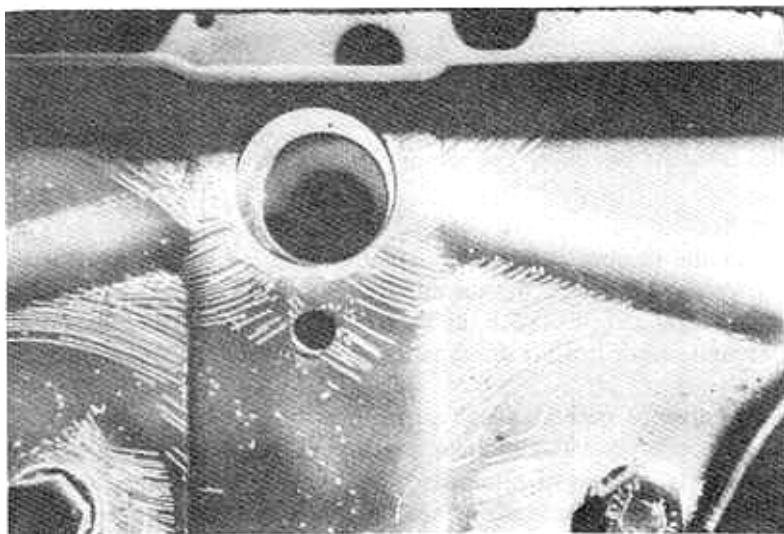


Fig. 16.4. Brittle-lacquer crack pattern highlighting the positions of stress concentration on a motor vehicle component. (Magnaflux Corporation.)

particularly useful for investigation under adverse environmental conditions such as in the presence of water, oil or heavy vibration.

Refinements to the general technique allow the study of residual stresses, compressive stress fields, dynamic situations, plastic yielding and miniature components with little

increased difficulty. For a full treatment of these and other applications, the reader is referred to ref. 3.

16.2. Strain gauges

The accurate assessment of stresses, strains and loads in components under working conditions is an essential requirement of successful engineering design. In particular, the location of peak stress values and stress concentrations, and subsequently their reduction or removal by suitable design, has applications in every field of engineering. The most widely used experimental stress-analysis technique in industry today, particularly under working conditions, is that of strain gauges.

Whilst a number of different types of strain gauge are commercially available, this section will deal almost exclusively with the electrical resistance type of gauge introduced in 1939 by Ruge and Simmons in the United States.

The *electrical resistance strain gauge* is simply a length of wire or foil formed into the shape of a continuous grid, as shown in Fig. 16.5, cemented to a non-conductive backing. The gauge is then bonded securely to the surface of the component under investigation so that any strain in the surface will be experienced by the gauge itself. Since the fundamental equation for the electrical resistance R of a length of wire is

$$R = \frac{\rho L}{A} \quad (16.1)$$

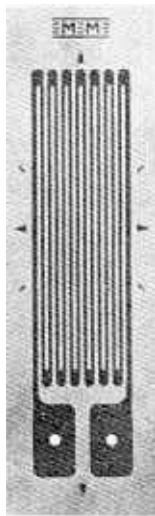


Fig. 16.5. Electric resistance strain gauge. (Welwyn Strain Measurement Ltd.)

where L is the length, A is the cross-sectional area and ρ is the *specific resistance* or *resistivity*, it follows that any change in length, and hence sectional area, will result in a change of resistance. Thus measurement of this resistance change with suitably calibrated equipment enables a direct reading of linear strain to be obtained. This is made possible by the

relationship which exists for a number of alloys over a considerable strain range between change of resistance and strain which may be expressed as follows:

$$\frac{\Delta R}{R} = K \times \frac{\Delta L}{L} \quad (16.2)$$

where ΔR and ΔL are the changes in resistance and length respectively and K is termed the *gauge factor*.

Thus

$$\text{gauge factor } K = \frac{\Delta R/R}{\Delta L/L} = \frac{\Delta R/R}{\varepsilon} \quad (16.3)$$

where ε is the strain. The value of the gauge factor is always supplied by the manufacturer and can be checked using simple calibration procedures if required. Typical values of K for most conventional gauges lie in the region of 2 to 2.2, and most modern strain-gauge instruments allow the value of K to be set accordingly, thus enabling strain values to be recorded directly.

The changes in resistance produced by normal strain levels experienced in engineering components are very small, and sensitive instrumentation is required. Strain-gauge instruments are basically *Wheatstone bridge* networks as shown in Fig. 16.6, the condition of balance for this network being (i.e. the galvanometer reading zero when)

$$R_1 \times R_3 = R_2 \times R_4 \quad (16.4)$$

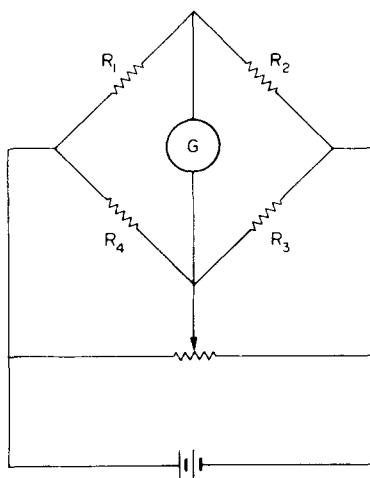


Fig. 16.6 Wheatstone bridge circuit.

In the simplest half-bridge wiring system, gauge 1 is the *active gauge*, i.e. that actually being strained. Gauge 2 is so-called *dummy gauge* which is bonded to an unstrained piece of metal similar to that being strained, its purpose being to cancel out any resistance change in R_1 that occurs due to temperature fluctuations in the vicinity of the gauges. Gauges 1 and 2 then represent the working half of the network – hence the name “half-bridge” system – and gauges 3 and 4 are standard resistors built into the instrument. Alternative wiring systems utilise one (*quarter-bridge*) or all four (*full-bridge*) of the bridge resistance arms.

16.3. Unbalanced bridge circuit

With the Wheatstone bridge initially balanced to zero any strain on gauge R_1 will cause the galvanometer needle to deflect. This deflection can be calibrated to read strain, as noted above, by including in the circuit an arrangement whereby gauge-factor adjustment can be achieved. Strain readings are therefore taken with the pointer off the zero position and the bridge is thus *unbalanced*.

16.4. Null balance or balanced bridge circuit

An alternative measurement procedure makes use of a variable resistance in one arm of the bridge to cancel any deflection of the galvanometer needle. This adjustment can then be calibrated directly as strain and readings are therefore taken with the pointer on zero, i.e. in the *balanced* position.

16.5. Gauge construction

The basic forms of wire and foil gauges are shown in Fig. 16.7. Foil gauges are produced by a printed-circuit process from selected melt alloys which have been rolled to a thin film, and these have largely superseded the previously popular wire gauge. Because of the increased area of metal in the gauge at the ends, the foil gauge is not so sensitive to strains at right angles to the direction in which the major axis of the gauge is aligned, i.e. it has a low transverse or cross-sensitivity – one of the reasons for its adoption in preference to the wire gauge. There are many other advantages of foil gauges over wire gauges, including better strain transmission from the substrate to the grid and better heat transmission from the grid to the substrate; as a result of which they are usually more stable. Additionally, the grids of foil gauges can be made much smaller and there is almost unlimited freedom of grid configuration, solder tab arrangement, multiple grid configuration, etc.

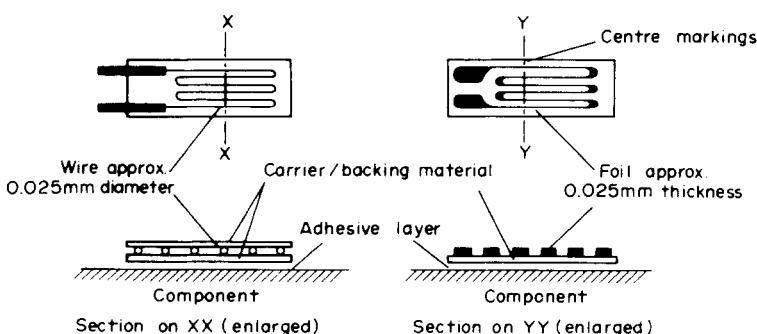


Fig. 16.7. Basic format of wire and foil gauges. (Merrow.)

Gage Patterns
Actual Size
 (Grids Run Vertically,

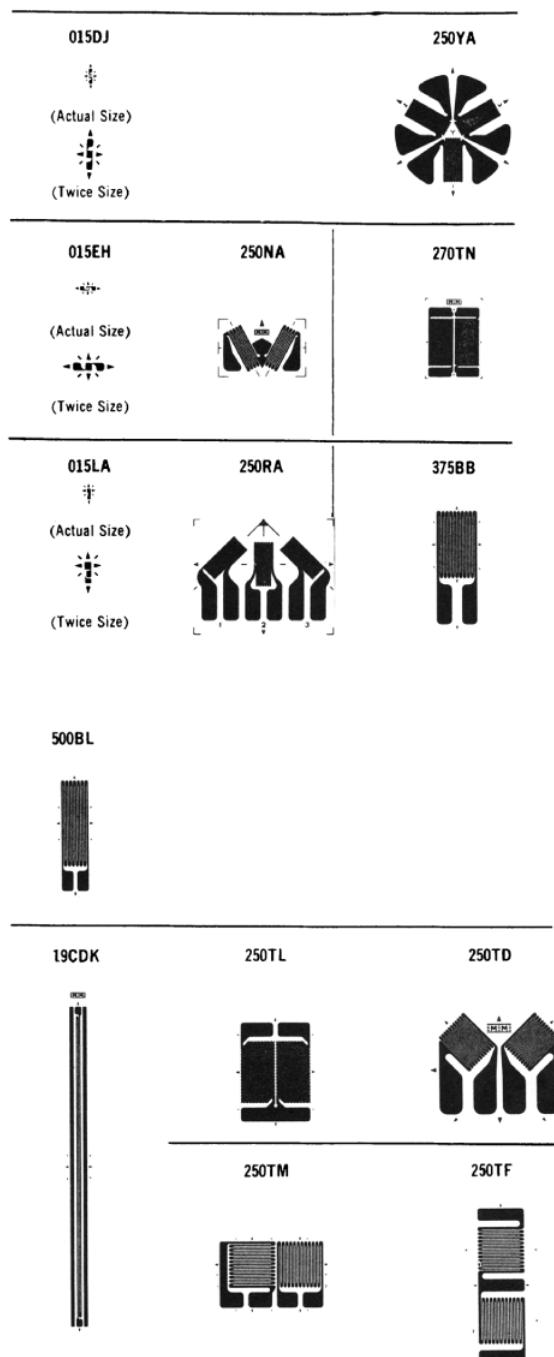


Fig. 16.8. Typical gauge sizes and formats. (Welwyn Strain Measurement Division.)

16.6. Gauge selection

Figure 16.8 shows but a few of the many types and size of gauge which are available. So vast is the available range that it is difficult to foresee any situation for which there is no gauge suitable. Most manufacturers' catalogues⁽¹³⁾ give full information on gauge selection, and any detailed treatment would be out of context in this section. Essentially, the choice of a suitable gauge incorporates consideration of physical size and form, resistance and sensitivity, operating temperature, temperature compensation, strain limits, flexibility of the gauge backing (and hence relative stiffness) and cost.

16.7. Temperature compensation

Unfortunately, in addition to strain, other factors affect the resistance of a strain gauge, the major one being temperature change. It can be shown that temperature change of only a few degrees completely dwarfs any readings due to the typical strains encountered in engineering applications. Thus it is vitally important that any temperature effects should be cancelled out, leaving only the mechanical strain required. This is achieved either by using the conventional dummy gauge, *half-bridge*, system noted earlier, or, alternatively, by the use of *self-temperature-compensated gauges*. These are gauges constructed from material which has been subjected to particular metallurgical processes and which produce very small (and calibrated) thermal output over a specified range of temperature when bonded onto the material for which the gauge has been specifically designed (see Fig. 16.9).

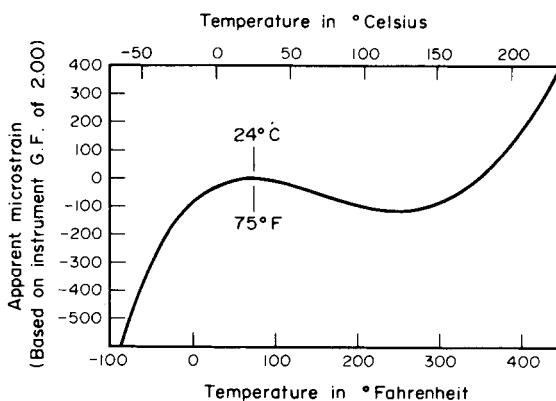


Fig. 16.9. Typical output from self-temperature-compensated gauge (Vishay).

In addition to the gauges, the lead-wire system must also be compensated, and it is normal practice to use the three-lead-wire system shown in Fig. 16.10. In this technique, two of the leads are in opposite arms of the bridge so that their resistance cancels, and the third lead, being in series with the power supply, does not influence the bridge balance. All leads must be of equal length and wound tightly together so that they experience the same temperature conditions.

In applications where a single self-temperature-compensated gauge is used in a quarter-bridge arrangement the three-wire circuit becomes that shown in Fig. 16.11. Again, only one of the current-carrying lead-wires is in series with the active strain gauge, the other is in series with the bridge completion resistor (occasionally still referred to as a "dummy") in the adjacent arm of the bridge. The third wire, connected to the lower solder tab of the active gauge, carries essentially no current but acts simply as a voltage-sensing lead. Provided the two lead-wires (resistance R_L) are of the same size and length and maintained at the same temperature (i.e. kept physically close to each other) then any resistance changes due to temperature will cancel.

16.8. Installation procedure

The quality and success of any strain-gauge installation is influenced greatly by the care and precision of the installation procedure and correct choice of the adhesive. The apparently mundane procedure of actually cementing the gauge in place is a critical step in the operation. Every precaution must be taken to ensure a chemically clean surface if perfect adhesion is to be attained. Full details of typical procedures and equipment necessary are given in refs 6 and

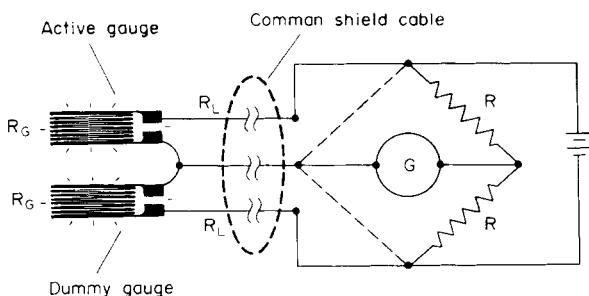


Fig. 16.10. Three-lead-wire system for half-bridge (dummy-active) operation.

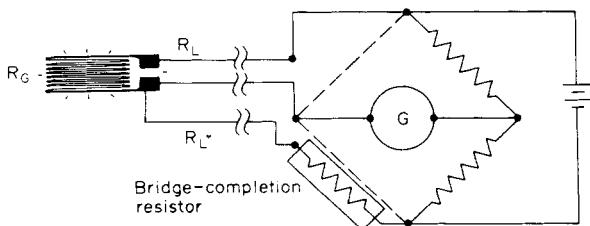


Fig. 16.11. Three-lead-wire system for quarter-bridge operation with single self-temperature-compensated gauge.

13, as are the methods which may be used to test the validity of the installation prior to recording measurements. Techniques for protection of gauge installations are also covered. Typical strain-gauge installations are shown in Figs. 16.12 and 16.13.

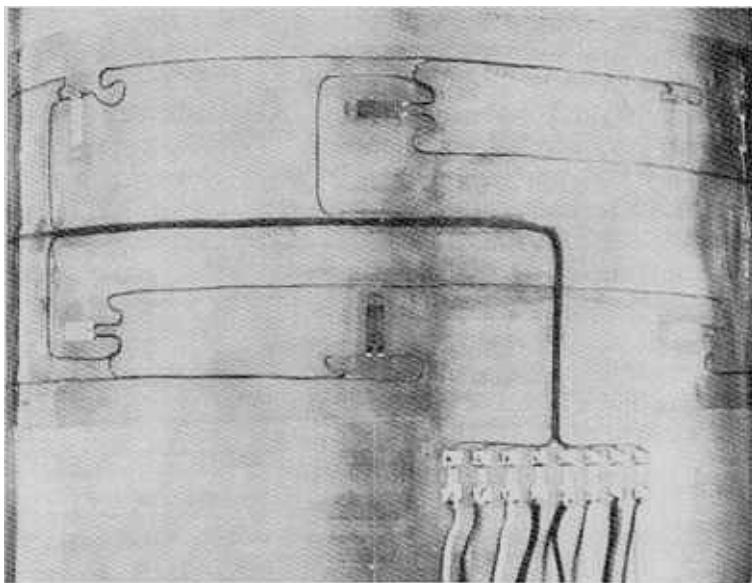


Fig. 16.12. Typical strain-gauge installation showing six of eight linear gauges bonded to the surface of a cylinder to record longitudinal and hoop strains. (Crown copyright.)

16.9. Basic measurement systems

(a) For direct strain

The standard procedure for the measurement of tensile or compressive direct strains utilises the *full-bridge* circuit of Fig. 16.14 in which not only are the effects of any bending eliminated but the sensitivity is increased by a factor of 2.6 over that which would be achieved using a single linear gauge.

Bearing in mind the balance requirement of the Wheatstone bridge, i.e. $R_1 R_3 = R_2 R_4$, each pair of gauges on either side of the equation will have an additive effect if their signs are similar or will cancel if opposite. Thus the opposite signs produced by bending cancel on both pairs whilst the similar signs of the direct strains are additive. The value 2.6 arises from twice the applied axial strain (R_1 and R_3) plus twice the Poisson's ratio strain (R_2 and R_4), assuming $\nu = 0.3$. The latter is compressive, i.e. negative, on the opposite side of the bridge from R_1 and R_3 , and hence is an added signal to that of R_1 and R_3 .

(b) Bending

Figure 16.15a shows the arrangement used to record bending strains independently of direct strains. It is normal to bond linear gauges on opposite surfaces of the component and to use the *half-bridge* system shown in Fig. 16.6; this gives a sensitivity of twice that which would be achieved with a single-linear gauge. Alternatively, it is possible to utilise again the Poisson strains as in §16.9(a) by bonding additional lateral gauges (i.e. perpendicular to the other

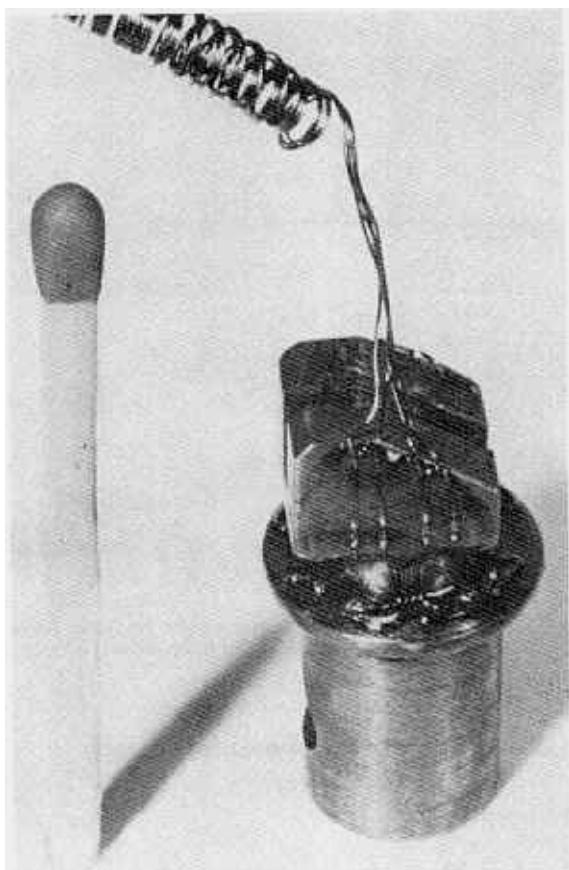


Fig. 16.13. Miniature strain-gauge installation. (Welwyn.)

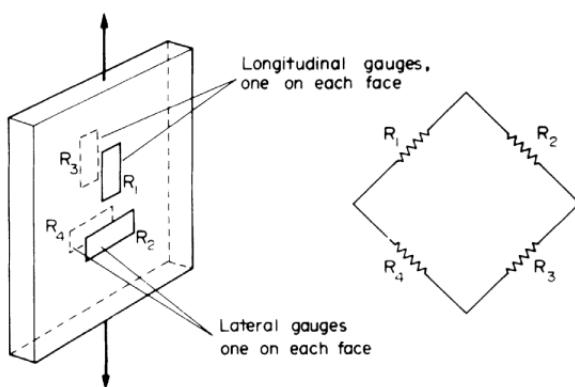


Fig. 16.14. "Full bridge" circuit arranged to eliminate any bending strains produced by unintentional eccentricities of loading in a nominal axial load application. The arrangement also produces a sensitivity 2.6 times that of a single active gauge. (Merrow.)

other gauges) on each surface and using a full-bridge circuit to achieve a sensitivity of 2.6. In this case, however, gauges R_2 and R_4 would be interchanged from the arrangement shown in Fig. 16.14 and would appear as in Fig. 16.15b.

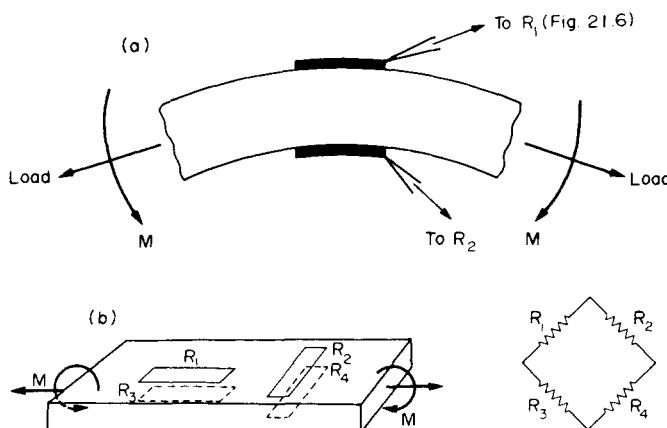


Fig. 16.15. (a) Determination of bending strains independent of end loads: "half-bridge" method. Sensitivity twice that of a single active gauge. (b) Determination of bending strains independent of end loads: "full-bridge" procedure. Sensitivity 2.6.

(c) Torsion

It has been shown that pure torsion produces direct stresses on planes at 45° to the shaft axis – one set tensile, the other compressive. Measurements of torque or shear stress using strain-gauge techniques therefore utilise gauges bonded at 45° to the axis in order to record the direct strains. Again, it is convenient to use a wiring system which automatically cancels unwanted signals, i.e. in this case the signals arising due to unwanted direct or bending strains which may be present. Once again, a full-bridge system is used and a sensitivity of four times that of a single gauge is achieved (Fig. 16.16).

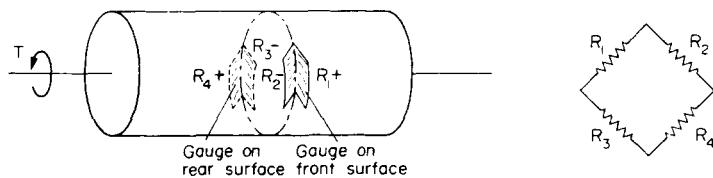


Fig. 16.16. Torque measurement using full-bridge circuit – sensitivity four times that of a single active gauge.

16.10. D.C. and A.C. systems

The basic Wheatstone bridge circuit shown in all preceding diagrams is capable of using either a direct current (d.c.) or an alternating current (a.c.) source; Fig. 16.6, for example,

shows the circuit excited by means of a standard battery (d.c.) source. Figure 16.17, however, shows a typical arrangement for a so-called a.c. *carrier frequency* system, the main advantage of this being that all unwanted signals such as noise are eliminated and a stable signal of gauge output is produced. The relative merits and disadvantages of the two types of system are outside the scope of this section but may be found in any standard reference text (refs. 4, 6, 7 and 13).

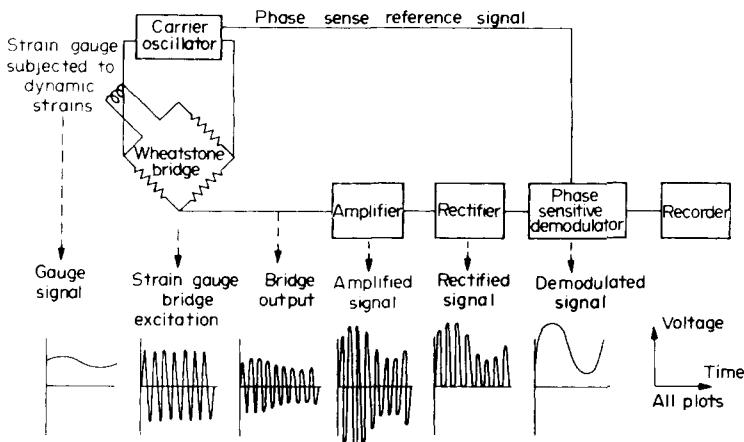


Fig. 16.17. Schematic arrangement of a typical carrier frequency system. (Merrow.)

16.11. Other types of strain gauge

The previous discussion has related entirely to the electrical resistance type of strain gauge and, indeed, this is by far the most extensively used type of gauge in industry today. It should be noted, however, that many other forms of strain gauge are available. They include:

- (a) **mechanical gauges or extensometers** using optical or mechanical lever systems;
- (b) **pneumatic gauges** using changes in pressure;
- (c) **acoustic gauges** using the change in frequency of a vibrating wire;
- (d) **semiconductor or piezo-resistive gauges** using the piezo-resistive effect in silicon to produce resistance changes;
- (e) **inductance gauges** using changes in inductance of, e.g., differential transformer systems;
- (f) **capacitance gauges** using changes in capacitance between two parallel or near-parallel plates.

Each type of gauge has a particular field of application in which it can compete on equal, or even favourable, terms with the electrical resistance form of gauge. None, however, are as versatile and generally applicable as the resistance gauge. For further information on each type of gauge the reader is referred to the references listed at the end of this chapter.

16.12. Photoelasticity

In recent years, photoelastic stress analysis has become a technique of outstanding importance to engineers. When polarised light is passed through a stressed transparent model, interference patterns or *fringes* are formed. These patterns provide immediate qualitative information about the general distribution of stress, positions of stress concentrations and of areas of low stress. On the basis of these results, designs may be modified to reduce or disperse concentrations of stress or to remove excess material from areas of low stress, thereby achieving reductions in weight and material costs. As photoelastic analysis may be carried out at the design stage, stress conditions are taken into account before production has commenced; component failures during production, necessitating expensive design modifications and re-tooling, may thus be avoided. Even when service failures do occur, photoelastic analysis provides an effective method of failure investigation and often produces valuable information leading to successful re-design. Typical photoelastic fringe patterns are shown in Fig. 16.18.

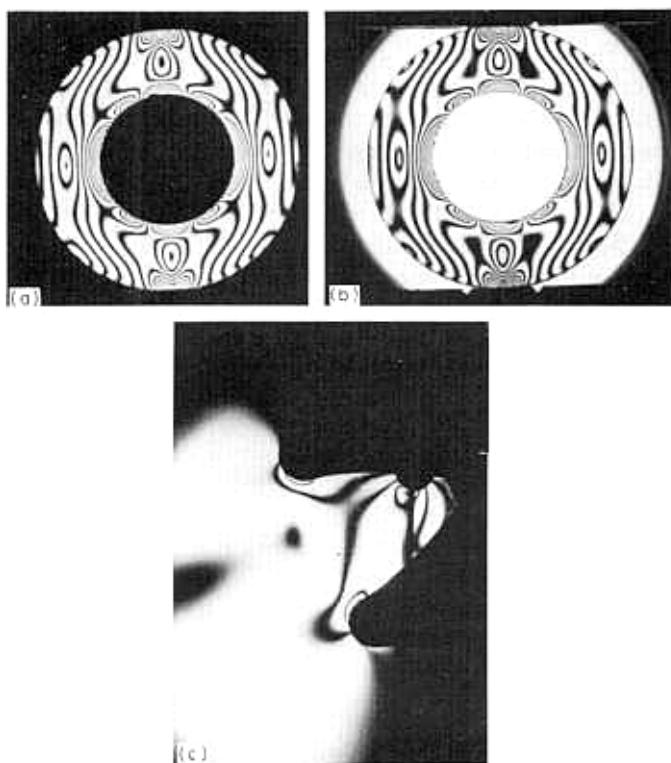


Fig. 16.18. Typical photoelastic fringe patterns. (a) Hollow disc subjected to compression on a diameter (dark field background). (b) As (a) but with a light field background. (c) Stress concentrations at the roots of a gear tooth.

Conventional or *transmission photoelasticity* has for many years been a powerful tool in the hands of trained stress analysts. However, untrained personnel interested in the technique have often been dissuaded from attempting it by the large volume of advanced mathematical

and optical theory contained in reference texts on the subject. Whilst this theory is, no doubt, essential for a complete understanding of the phenomena involved and of some of the more advanced techniques, it is important to accept that a wealth of valuable information can be obtained by those who are not fully conversant with all the complex detail. A major feature of the technique is that it allows one to effectively "look into" the component and pin-point flaws or weaknesses in design which are otherwise difficult or impossible to detect. Stress concentrations are immediately visible, stress values around the edge or boundary of the model are easily obtained and, with a little more effort, the separate principal stresses within the model can also be determined.

16.13. Plane-polarised light – basic polariscope arrangements

Before proceeding with the details of the photoelastic technique it is necessary to introduce the meaning and significance of *plane-polarised light* and its use in the equipment termed *polariscopes* used for photoelastic stress analysis. If light from an ordinary light bulb is passed through a polarising sheet or *polariser*, the sheet will act like a series of vertical slots so that the emergent beam will consist of light vibrating in one plane only: the plane of the slots. The light is then said to be *plane polarised*.

When directed onto an unstressed photoelastic model, the plane-polarised light passes through unaltered and may be completely extinguished by a second polarising sheet, termed an *analyser*, whose axis is perpendicular to that of the polariser. This is then the simplest form of polariscope arrangement which can be used for photoelastic stress analysis and it is termed a "crossed" set-up (see Fig. 16.19). Alternatively, a "parallel" set-up may be used in which the axes of the polariser and analyser are parallel, as in Fig. 16.20. With the model unstressed, the plane-polarised light will then pass through both the model and analyser unaltered and maximum illumination will be achieved. When the model is stressed in the parallel set-up, the resulting fringe pattern will be seen against a light background or "field", whilst with the crossed arrangement there will be a completely black or "dark field".

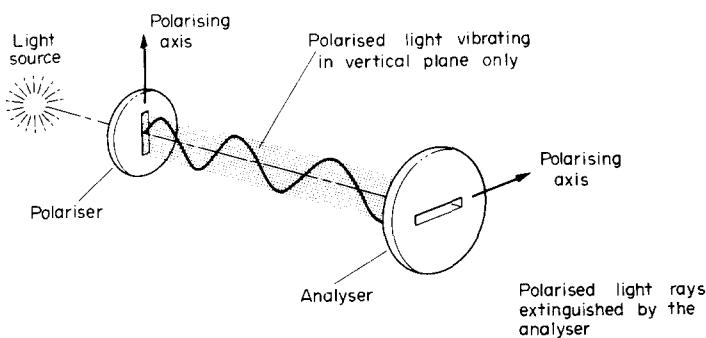


Fig. 16.19. "Crossed" set-up. Polariser and analyser arranged with their polarising axes at right angles; plane polarised light from the polariser is completely extinguished by the analyser. (Merrow.)

16.14. Temporary birefringence

Photoelastic models are constructed from a special class of transparent materials which exhibit a property known as *birefringence*, i.e. they have the ability to split an incident plane-

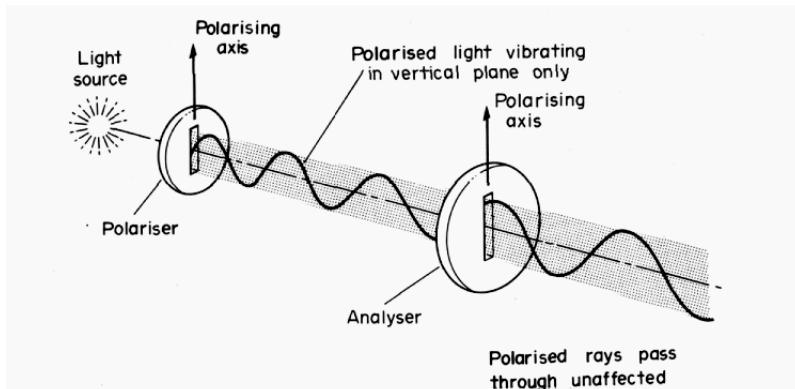


Fig. 16.20. "Parallel" set-up. Polariser and analyser axes parallel; plane-polarised light from the polariser passes through the analyser unaffected, producing a so-called "light field" arrangement. (Merrow.)

polarised ray into two component rays; they are *double refracting*. This property is only exhibited when the material is under stress, hence the qualified term "*temporary birefringence*", and the direction of the component rays always coincides with the directions of the principal stresses (Fig. 16.21). Further, the speeds of the rays are proportional to the magnitudes of the respective stresses in each direction, so that the rays emerging from the model are out of phase and hence produce interference patterns when combined.

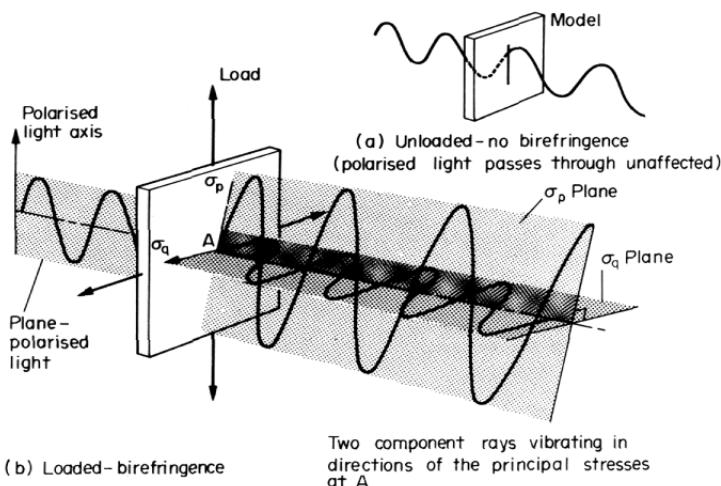


Fig. 16.21. Temporary birefringence. (a) Plane-polarised light directed onto an unstressed model passes through unaltered. (b) When the model is stressed the incident plane-polarised light is split into two component rays. The directions of the rays coincide with the directions of the principal stresses, and the speeds of the rays are proportional to the magnitudes of the respective stresses in their directions. The emerging rays are out of phase, and produce an interference pattern of fringes. (Merrow.)

16.15. Production of fringe patterns

When a model of an engineering component constructed from a birefringent material is stressed, it has been shown above that the incident plane-polarised light will be split into two component rays, the directions of which at any point coincide with the directions of the principal stresses at the point. The rays pass through the model at speeds proportional to the principal stresses in their directions and emerge out of phase. When they reach the analyser, shown in the crossed position in Fig. 16.22, only their horizontal components are transmitted and these will combine to produce interference fringes as shown in Fig. 16.23.

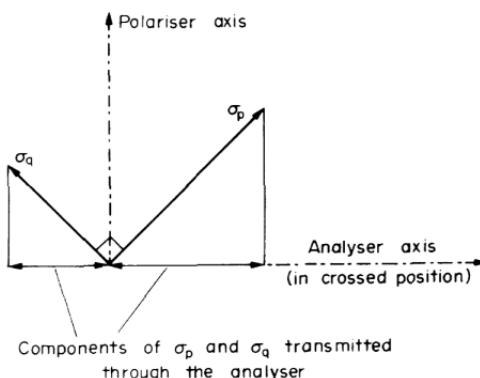
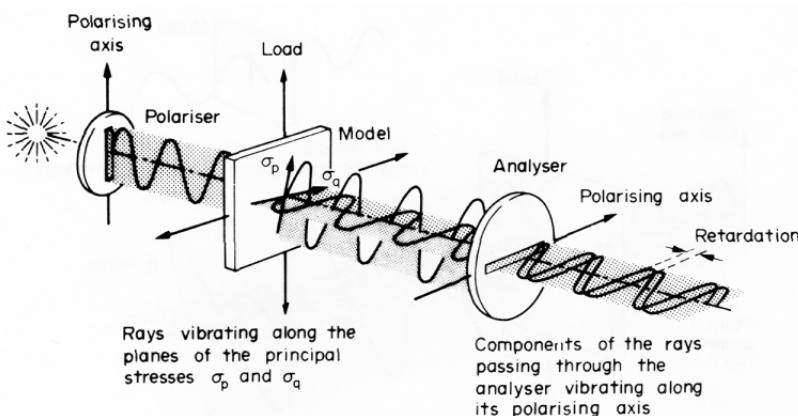


Fig. 16.22. Transmission through the analyser. (Merrow.)



The difference in speeds of the rays, and hence the amount of interference produced, is proportional to the difference in the principal stress values ($\sigma_p - \sigma_q$) at the point in question. Since the maximum shear stress in any two-dimensional stress system is given by

$$\tau_{\max} = \frac{1}{2}(\sigma_p - \sigma_q)$$

it follows that the interference or fringe pattern produced by the photoelastic technique will give an immediate indication of the variation of shear stress throughout the model. Only at a free, unloaded boundary of a model, where one of the principal stresses is zero, will the fringe pattern yield a direct indication of the principal direct stress (in this case the tangential boundary stress). However, since the majority of engineering failures are caused by fatigue cracks commencing at the point of maximum tensile stress at the boundary, this is not a severe limitation. Further discussion of the interpretation of fringe patterns is referred to the following section.

If the original light source is *monochromatic*, e.g. mercury green or sodium yellow, the fringe pattern will appear as a series of distinct black lines on a uniform green or yellow background. These black lines or fringes correspond to points where the two rays are exactly 180° out of phase and therefore cancel. If white light is used, however, each composite wavelength of the white light is cancelled in turn and a multicoloured pattern of fringes termed *isochromatics* is obtained.

Monochromatic sources are preferred for accurate quantitative photoelastic measurements since a high number of fringes can be clearly discerned at, e.g., stress concentration positions. With a white light source the isochromatics become very pale at high stress regions and clear fringe boundaries are no longer obtained. White light sources are therefore normally reserved for general qualitative assessment of models, for isolation of zero fringe order positions (i.e. zero shear stress) which appear black on the multicoloured background, and for the investigation of stress directions using *isoclinics*. These are defined in detail in §16.19.

16.16. Interpretation of fringe patterns

It has been stated above that the pattern of fringes achieved by the photoelastic technique yields:

(a) *A complete indication of the variation of shear stress throughout the entire model.* Since ductile materials will generally fail in shear rather than by direct stress, this is an important feature of the technique. At points where the fringes are most numerous and closely spaced, the stress is highest; at points where they are widely spaced or absent, the stress is low. With a white-light source such areas appear black, indicating zero shear stress, but it cannot be emphasised too strongly that this does not necessarily mean zero stress since if the values of σ_p and σ_q (however large) are equal, then $(\sigma_p - \sigma_q)$ will be zero and a black area will be produced. Extreme care must therefore be taken in the interpretation of fringe patterns. Generally, however, fringe patterns may be compared with contour lines on a map, where close spacing relates to steep slopes and wide spacing to gentle inclines. Peaks and valleys are immediately evident, and actual heights are readily determined by counting the contours and converting to height by the known scale factor. In an exactly similar way, photoelastic fringes are counted from the known zero (black) positions and the resulting number or order of fringe at the point in question is converted to stress by a calibration constant known as the *material fringe value*. Details of the calibration procedure will be given later.

(b) *Individual values of the principal stresses at free unloaded boundaries, one of these always being zero.* The particular relevance of this result to fatigue failures has been mentioned, and the use of photoelasticity to produce modifications to boundary profiles in order to reduce boundary stress concentrations and hence the likelihood of fatigue failures has been a major

use of the technique. In addition to the immediate indication of high stress locations, the photoelastic model will show regions of low stress from which material can be conveniently removed without weakening the component to effect a reduction in weight and material cost. Surprisingly, perhaps, a reduction in material at or near a high stress concentration can also produce a significant reduction in maximum stress. Re-design can be carried out on a “file-it-and-see” basis, models being modified or re-shaped within minutes in order to achieve the required distribution of stress.

Whilst considerable valuable qualitative information can be readily obtained from photoelastic models without any calculations at all, there are obviously occasions where the precise values of the stresses are required. These are obtained using the following basic equation of photoelasticity,

$$\sigma_p - \sigma_q = \frac{nf}{t} \quad (16.5)$$

where σ_p and σ_q are the values of the maximum and minimum principal stresses at the point under consideration, n is the fringe number or *fringe order* at the point, f is the *material fringe value* or *coefficient*, and t is the model thickness.

Thus with a knowledge of the material fringe value obtained by calibration as described below, the required value of $(\sigma_p - \sigma_q)$ at any point can be obtained readily by simply counting the fringes from zero to achieve the value n at the point in question and substitution in the above relatively simple expression.

Maximum shear stress or boundary stress values are then easily obtained and the application of one of the so-called *stress-separation* procedures will yield the separate value of the principal stress at other points in the model with just a little more effort. These may be of particular interest in the design of components using brittle materials which are known to be relatively weak under the action of direct stresses.

16.17. Calibration

The value of f , which, it will be remembered, is analogous to the height scale for contours on a survey map, is determined by a simple calibration experiment in which the known stress at some point in a convenient model is plotted against the fringe value at that point under various loads. One of the most popular loading systems is diametral compression of a disc, when the relevant equation for the stress at the centre is

$$\sigma_p - \sigma_q = \frac{8P}{\pi Dt} \quad (16.6)$$

where P is the applied load, D is the disc diameter and t is the thickness.

Thus, comparing with the photoelastic equation (16.5),

$$\frac{nf}{t} = \frac{8P}{\pi Dt}$$

The slope of the load versus fringe order graph is given by

$$\frac{P}{n} = f \times \frac{\pi D}{8} \quad (16.7)$$

Hence f can be evaluated.

16.18. Fractional fringe order determination – compensation techniques

The accuracy of the photoelastic technique is limited, among other things, to the accuracy with which the fringe order at the point under investigation can be evaluated. It is not sufficiently accurate to count to the nearest whole number of fringes, and precise determination of fractions of fringe order at points lying between fringes is required. Conventional methods for determining these fractions of fringe order are termed *compensation techniques* and allow estimation of fringe orders to an accuracy of one-fiftieth of a fringe. The two methods most often used are the Tardy and Senarmont techniques. Before either technique can be adopted, the directions of the polariser and analyser must be aligned with the directions of the principal stresses at the point. This is achieved by rotating both units together in the plane polariscope arrangement until an *isoclinic* (§16.19) crosses the point. In most modern polariscopes facilities exist to couple the polariser and analyser together in order to facilitate synchronous rotation. The procedure for the two techniques then varies slightly.

(a) *Tardy method*

Quarter-wave plates are inserted at 45° to the polariser and analyser as the dark field circular polariscope set-up of Fig. 16.24. Normal fringe patterns will then be visible in the absence of isoclinics.

(b) *Senarmont method*

The polariser and analyser are rotated through a further 45° retaining the dark field, thus moving the polarising axes at 45° to the principal stress directions at the point. Only one quarter-wave plate is then inserted between the model and the analyser and rotated to again achieve a dark field. The normal fringe pattern is then visible as with the Tardy method.

Thus, having identified the integral value n of the fringe order at the point, i.e. between 1 and 2, or 2 and 3, for instance, the fractional part can now be established for both methods in the same way.

The analyser is rotated on its own to produce movement of the fringes. In particular, the nearest *lower order* of fringe is moved to the point of interest and the angle θ moved by the analyser recorded.

The fringe order at the chosen point is then $n + \frac{\theta}{180}^\circ$.

N.B. – Rotation of the analyser in the opposite direction ϕ° would move the nearest *highest order* fringe ($n + 1$) back to the point. In this case the fringe order at the point would be

$$(n + 1) - \frac{\phi}{180}$$

It can be shown easily by trial that the sum of the two angles θ and ϕ is always 180° .

There is little to choose between the two methods in terms of accuracy; some workers prefer to use Tardy, others to use Senarmont.

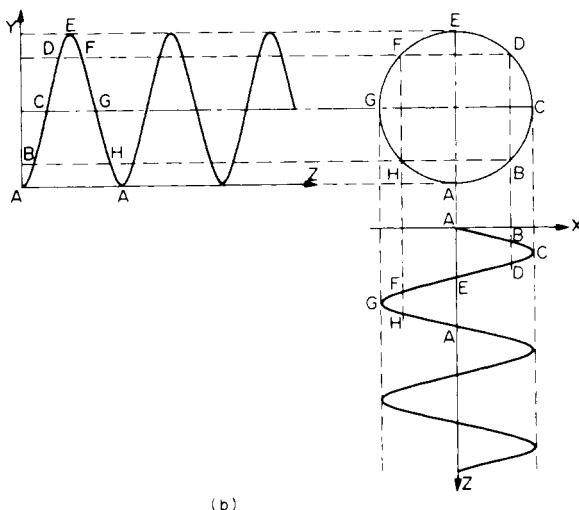
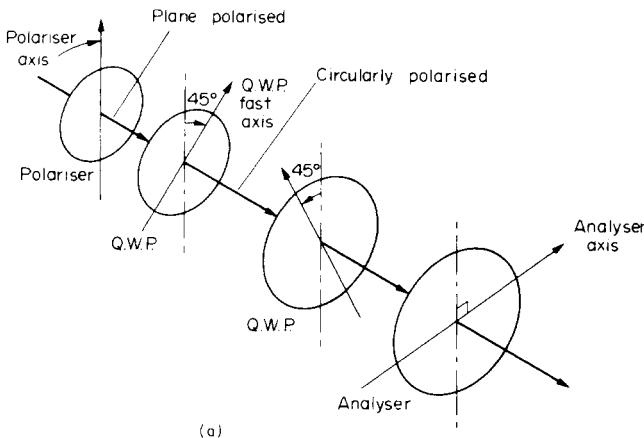


Fig. 16.24. (a) Circular polariscope arrangement. Isoclinics are removed optically by inserting quarter-wave plates (Q.W.P.) with optical axes at 45° to those of the polariser and analyser. Circularly polarised light is produced. (Merrow.) (b) Graphical construction for the addition of two rays at right angles a quarter-wavelength out of phase, producing resultant circular envelope, i.e. circularly polarised light.

16.19. Isoclinics – circular polarisation

If plane-polarised light is used for photoelastic studies as suggested in the preceding text, the fringes or isochromatics will be partially obscured by a set of black lines known as isoclinics (Fig. 16.25). With the coloured isochromatics of a white light source, these are easily identified, but with a monochromatic source confusion can easily arise between the black fringes and the black isoclinics.

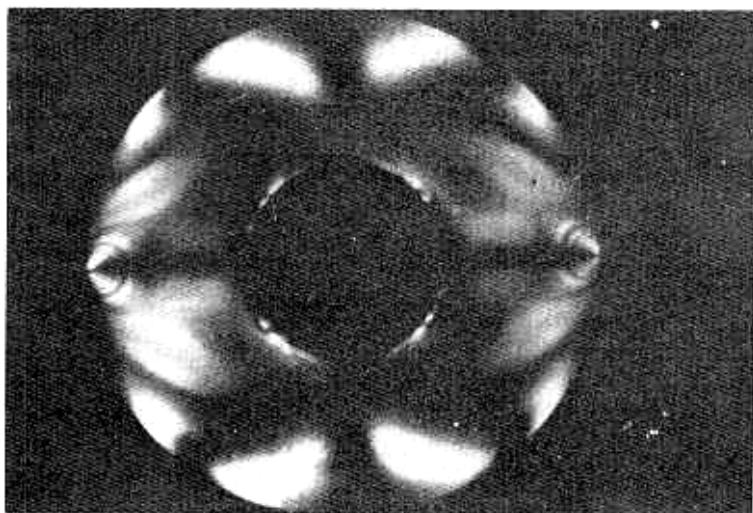


Fig. 16.25. Hollow disc subjected to diametral compression as in Fig. 16.18a but in this case showing the isoclinics superimposed.

It is therefore convenient to use a different optical system which eliminates the isoclinics but retains the basic fringe pattern. The procedure adopted is outlined below.

An *isoclinic* line is a locus of points at which the principal stresses have the same inclination; the 20° isoclinic, for example, passes through all points at which the principal stresses are inclined at 20° to the vertical and horizontal (Fig. 16.26). Thus isoclinics are not peculiar to photoelastic studies; it is simply that they have a particular relevance in this case and they are readily visualised. For the purpose of this introduction it is sufficient to note that they are used as the basis for construction of *stress trajectories* which show the directions of the principal stresses at all points in the model, and hence in the component. Further details may be found in the relevant standard texts.

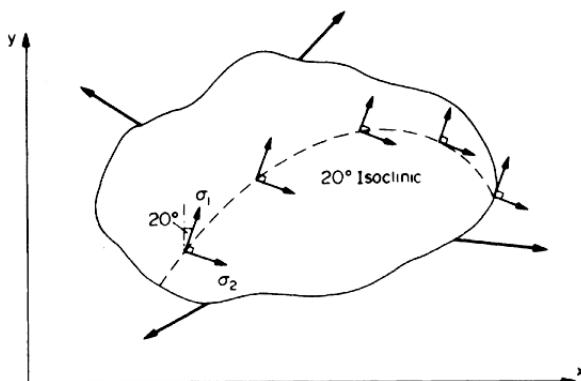


Fig. 16.26. The 20° isoclinic in a body subjected to a general stress system. The isoclinic is given by the locus of all points at which the principal stresses are inclined at 20° to the reference x and y axes.

To prevent the isoclinics interfering with the analysis of stress magnitudes represented by the basic fringe pattern, they are removed optically by inserting quarter-wave plates with their axes at 45° to those of the polariser and analyser as shown in Fig. 16.24. These eliminate all unidirectional properties of the light by converting it into *circularly polarised* light. The amount of interference between the component rays emerging from the model, and hence the fringe patterns, remains unchanged and is now clearly visible in the absence of the isoclinics.

16.20. Stress separation procedures

The photoelastic technique has been shown to provide principal stress difference and hence maximum shear stresses at all points in the model, boundary stress values and stress directions. It has also been noted that there are occasions where the separate values of the principal stresses are required at points other than at the boundary, e.g. in the design of components using brittle materials. In this case it is necessary to employ one of the many *stress separation* procedures which are available. It is beyond the scope of this section to introduce these in detail, and full information can be obtained if desired from standard texts.^(8, 9, 11) The principal techniques which find most application are (a) the oblique incidence method, and (b) the shear slope or "shear difference" method.

16.21. Three-dimensional photoelasticity

In the preceding text, reference has been made to models of uniform thickness, i.e. two-dimensional models. Most engineering problems, however, arise in the design of components which are three-dimensional. In such cases the stresses vary not only as a function of the shape in any one plane but also throughout the "thickness" or third dimension. Often a proportion of the more simple three-dimensional model or loading cases can be represented by equivalent two-dimensional systems, particularly if the models are symmetrical, but there remains a greater proportion which cannot be handled by the two-dimensional approach. These, however, can also be studied using the photoelastic method by means of the so-called *stress-freezing* technique.

Three-dimensional photoelastic models constructed from the same birefringent material introduced previously are loaded, heated to a critical temperature and cooled very slowly back to room temperature. It is then found that a fringe pattern associated with the elastic stress distribution in the component has been locked or "frozen" into the model. It is then possible to cut the model into thin slices of uniform thickness, each slice then being examined as if it were a two-dimensional model. Special procedures for model manufacture, slicing of the model and fringe interpretation are required, but these are readily obtained with practice.

16.22. Reflective coating technique⁽¹²⁾

A special adaptation of the photoelastic technique utilises a thin sheet of photoelastic material which is bonded onto the surface of a metal component using a special adhesive containing an aluminium pigment which produces a reflective layer. Polarised light is directed onto the photoelastic coating and viewed through an analyser after reflection off the metal surface using a *reflection polariscope* as shown in Fig. 16.27.

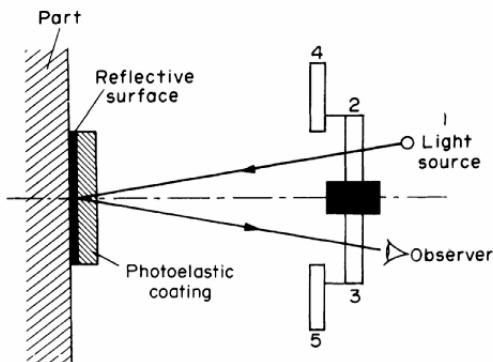


Fig. 16.27. Reflection polariscope principle and equipment.

A fringe pattern is observed which relates to the strain in the metal component. The technique is thus no longer a model technique and allows the evaluation of strains under loading conditions. Static and dynamic loading conditions can be observed, the latter with the aid of a stroboscope or high-speed camera, and the technique gives a full field view of the strain distribution in the surface of the component. Unlike the transmission technique, however, it gives no information as to the stresses *within* the material.

Standard photoelastic sheet can be used for bonding to flat components, but special casting techniques are available which enable the photoelastic material to be obtained in a partially polymerised, very flexible, stage, and hence allows it to be contoured or moulded around complex shapes without undue thickness changes. After a period has been allowed for

complete polymerisation to occur in the moulded position, the sheet is removed and bonded firmly back into place with the reflective adhesive.

The reflective technique is particularly useful for the observation of service loading conditions over wide areas of structure and is often used to highlight the stress concentration positions which can subsequently become the subject of detailed strain-gauge investigations.

16.23. Other methods of strain measurement

In addition to the widely used methods of experimental stress analysis or strain measurement covered above, there are a number of lesser-used techniques which have particular advantages in certain specialised conditions. These techniques can be referred to under the general title of grid methods, although in some cases a more explicit title would be "interference methods".

The standard **grid technique** consists of marking a grid, either mechanically or chemically, on the surface of the material under investigation and measuring the distortions of this grid under strain. A direct modification of this procedure, known as the "**replica**" **technique**, involves the firing of special pellets from a gun at the grid both before and during load. The surface of the pellets are coated with "Woods metal" which is heated in the gun prior to firing. Replicas of the undeformed and deformed grids are then obtained in the soft metal on contact with the grid-marked surface. These are viewed in a vernier comparison microscope to obtain strain readings.

A further modification of the grid procedure, known as the **moiré technique**, superimposes the deformed grid on an undeformed master (or vice versa). An interference pattern, known as **moiré fringes**, similar to those obtained when two layers of net curtain are superimposed, is produced and can be analysed to yield strain values.

X-rays can be used to obtain surface strain values from measurements of crystal lattice deformation. **Acoustoelasticity**, based on a principle similar to photoelasticity but using polarised ultrasonic sound waves, has been proposed but is not universally accepted to date. **Holography**, using the laser as a source of coherent light, and again relying on the interference obtained between holograms of deformed and undeformed components, has recently created considerable interest, but none of these techniques appear at the moment to represent a formidable challenge to the major techniques listed earlier.

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APPENDIX 1

**TYPICAL MECHANICAL AND PHYSICAL PROPERTIES
FOR ENGINEERING METALS**

Material	Young's modulus of elasticity E (GN/m ²)	Shear modulus G	"Elastic" limit σ_y	Shear yield strength τ_y	Tensile strength	Ultimate strength in shear	Percentage elongation (%)	Density (kg/m ³)	Linear coefficient of thermal expansion ($\times 10^{-6}/^{\circ}\text{C}$)
Aluminium alloy	69	26	230	—	390	240	23	2770	23
Brass	102	38	—	—	350	—	40	8350	18.9
Bronze	115	45	210	—	310	—	20	7650	18
Cast iron: Grey	90	41	—	—	210	—	8	7640	10.5
Malleable	170	83	248	166	370	330	12	7640	12
Low carbon (mild) steel	207	80	280	175	480	350	25	7800	11.7
Nickel-chrome steel	208	82	1200	650	1700	950	12	7800	11.7
Titanium	107	40	480	—	551	—	—	4507	9.5
Magnesium	45	17	262	—	379	165	—	1791	28.8

APPENDIX 2

TYPICAL MECHANICAL PROPERTIES OF NON-METALS

Material	Young's modulus of elasticity E (GN/m ²)	Tensile strength (MN/m ²)	Compressive strength (MN/m ²)	Elongation (maximum) %
Acetals	—	69	124	75
Cellulose acetate	1.4	41	207	20
Cellulose nitrate	1.4	48	138	40
Epoxy (glass filler)	—	145	234	—
Hard rubber	3.0	48	—	—
Melamine	8.0	55	227	0.7
Nylon filaments	4.1	340	—	—
Polycarbonate – unreinforced Makralon	2.3	70	83	100
Reinforced Makralon	6.0	90	—	8
Polyester (unfilled)	2.0	41	—	2
Polyethylene H.D.	—	28	22	100
Polyethylene L.D.	—	10	—	800
Polypropylene	—	34	510	250
Polystyrene	3.4	20	76	1.2
Polystyrene – impact resistant	1.4	38	41	80
P.T.F.E.	—	34	248	70
P.V.C. (rigid)	3.4	50–60	69	40
P.V.C. (plasticised)	—	20	0.7	200
Rubber (natural-vulcanised)	—	7–34	—	—
Silicones (elastomeric)	—	1.5–6	—	—
Timber	9.0	70	—	—
Urea (cellulose filler)	10.0	62	241	0.7

* Data taken in part from *Design Engineering Handbook on Plastics* (Product Journals Ltd).

APPENDIX 3

OTHER PROPERTIES OF NON-METALS*

Material	Chemical resistance					Max. useful temp. (°C)	
	Organic Solvents	Acids		Alkalies			
		Weak	Strong	Weak	Strong		
Acetal	×	×	00	×	×	90	
Acrylic	Varies	×	x-0	×	×	90	
Nylon 66	×	×	00	×	×	150	
Polycarbonate	Varies	×	0	x-0	00	120	
Polyethylene LD	×	×	x-00	×	×	90	
Polyethylene HD	×	×	x-0	×	×	120	
Polypropylene	×	×	x-0	×	×	150	
Polystyrene	Varies	×	x-0	×	×	95	
PTFE	×	×	x	×	×	240	
PVC	Varies	×	x-0	×	×	80	
Epoxy	×	×	x	×	0	430	
Melamine	×	×	00	×	0	100-200	
Phenolic	×	x-0	0-00	0-00	00		
Polyester/glass	x-0	0	00	0	00	250	
Silicone	x-0	x-0	00	0-00	00	180	
Urea	x-0	x-0	00	0-00	00	90	

x – Resistant, 0 – slightly attacked, 00 – markedly attacked

* Data taken from Design Engineering Handbook on Plastics. (Product Journals Ltd).

INDEX

- A.C. system 443
Acoustic gauge 444
Acoustoelasticity 456
Active gauge 436, 440, 442, 443
Allowable stresses 308
Area
 first moment of 67
 second moment of 62, 68
Auto fretting 233
Axis neutral 66
- Balanced circuit 437
Balanced section 72
Beams
 bending stress in 64
 built-in 140–147
 cantilever 97, 98, 101, 109, 110
 continuous 115
 curved 284
 shear stress in 154–166
- Bending
 circular 66
 moment 41–56
 of beams 62–79
 plus direct stress 74
 plus torsion 187
 simple theory of 64
- Birefringence 446
Boundary stress 429
Brittle materials 8, 402, 404, 412
Built-in beams 140–147
Bulk modulus 198, 202
- Calibration 433
Capacitance gauge 444
Carriage spring 309
Carrier frequency system 444
Castigliano 255, 266, 269, 303
Centroid 64, 70
Circular shafts 176–190
Clapeyron's "three-moment" equation 115
Close-coiled spring 299
Combined modulus 29
Complementary energy 257
Complex strain 361
Complex stress 326
Composite beam 70
Compound bars 27
Compound beams 70
Compound shafts 182–184
Compound tubes 224
Concrete 71
 columns 76, 77
 constants, elastic 3, 9, 361, 363, 367
Continuous beams 115
Contraflexure 48
Crack detection 433
Creep 15, 432
Criterion of failure 401
Critical section 72
Crossed set-up 446
Cross-sensitivity 437
Curvature, radius of 62
Cylinders
 compound 224
 plastic yielding 223
 thick 215
 thin 198
- Dark field 446
D.C. system 443
De Forrest 432
Deflections
 bending 92–123
 impact 264
 shear 269
 temperature effects 119
- Delta rosette 382
Deviatoric stress 263
Dilatational stresses 386
Direct integration method 97
Distortion energy 385
Distortional stresses 386
Double cantilever 146
Double integration method 89
Ductile materials 8, 14, 402, 404, 411
Ductility 4, 8
Dummy gauge 436, 438
Dye-etchant 433
Dynamic strain 434
- Eccentric loading 74
Economic section 73
Elastic constants 361, 367
Elastic limit 4
Elastic modulus 3
Elasticity 3
Elongation 8
Encastred (encastré) beams 140–147
Endurance limit 17
Energy method 112
Equivalent modulus 29
Equivalent moment 187
Equivalent torque 187, 188

- Factor
 load 13, 414
 of safety 12
 Failure envelope 407
 Failure modes 416
 Failure theories 401
 Fatigue 17
 Finite difference method 118
 Fixed-ended beams 140–147
 Fixing moment diagram 115, 142, 143
 Flitched beam 70
 Foil gauge 435, 437
 Force fits 229
 Free length 30–34
 Free-moment diagram 115, 142, 143
 Fringe order 449, 450
 Fringe pattern 445–456
 Fringe value 449, 450
 Frozen stress technique 454
 Full bridge circuit 436
- Gauge
 acoustic 444
 electric resistance 433–444
 strain 434–444
 Gauge factor 436
 Graphical procedure
 stress 332
 thick cylinders 223
 Grid technique 456
 Griffith 404
 Guest 401
- Haigh 401
 Half bridge 436
 Helical spring
 close-coiled 299
 open-coiled 301
 Hogging of beams 42
 Hollow shafts 182
 Holography 456
 Hooke's law 3
 Hydrostatic stress 386
 Hydrostatic stress line 413
 Hysteresis loop 6
- Impact loads and stresses 264
 Inclined loads 52
 Increasing loads 100
 Inductance gauge 444
 Inflection, point of 48, 146
 Interference 445, 448
 Interference allowance 226
 Isochromatic 449
 Isoclinic 452
- Joint efficiency 205
- Lamé line 223
 Lamé theory 217
 Lateral restraint 366
 Leaf spring 309
 Light field 446
- Limit of proportionality 4
 Load
 alternating 1
 dead 1
 fatigue 1
 fluctuating 1
 impact 1, 265
 line 411
 live 1
 shock 1
 static 1
 Load factor 13
- Macaulay's method 102
 Masonry columns 76
 Material fringe value 450
 Maximum principal strain 361–387, 404
 Maximum principal stress 330–342, 403
 Maximum shear plane 331
 Maximum shear stress 329, 331, 403
 Maxwell failure theory 401, 404
 Maxwell reciprocal displacement 112
 McClintock method 378
 Modulus
 bulk 198, 203, 363
 combined 29
 elasticity 3
 equivalent 29
 rigidity 12, 13
 section 62
 Young's 3
 Modular ratio 71
 Mohr's modified shear stress theory 404
 Mohr's strain circle 372
 Mohr's stress circle 332, 335
 Mohr's theory for slope and deflection 108, 140, 141
 Moiré 456
 Moment
 bending 41–56, 62–79
 fixing 115, 142, 143
 of area 66, 68
 of resistance 73
 Moments, equation of three 115
 Monochromatic light 449
 Movement of supports 146
- Necking 5
 Neutral axis 64, 66
 Neutral surface 67
 Null balance 437
- Oblique plane
 strain on 370
 stress on 326
 Octahedral shear stress 404
 Open-coiled springs 301
- Parallel axis theorem 70
 Parallel connection of shafts 183
 Parallel set-up 446
 Percentage elongation 8
 Percentage reduction in area 8
 Permanent set 5

- Photoelastic coating 454
 Photoelasticity 445
 reflection 454
 transmission 445
 Piezo-resistive gauge 444
 Plane polarisation 446
 Plastic deformation 5
 Pneumatic gauge 444
 Poisson's ratio 9–11
 Polar section modulus 181
 Polar second moment of area 179
 Polariscope 446
 Polariser 446
 Pole point 54
 Principal planes 331
 Principal strain 362, 372
 Principal stress 331
 Principle of superposition 34, 52, 112
 Proof load 311
 Proof stress 5
 Proof stress (springs) 312
 Proportionality, limit of 4
 Propped cantilever 130
 Pure shear 327

 Quarter bridge 436
 Quarter-wave plates 451, 452

 Rankine 401
 Rectangular rosette 382
 Reflection polariscope 454
 Reflective coating 454
 Relation between M , Q , and w 94
 Replica technique 456
 Resilience 257
 modulus of 254
 proof 254, 257
 Resistivity 435
 Rosette strain gauge analysis 378, 381
 Ruge and Simmons 435

 Safety factor 13, 414, 430
 Sagging 42
 Saint-Venant 401
 Second moment of area 66, 68
 Semi-conductor gauges 444
 Senarmont 45
 Series-connected shafts 182
 Shafts, torsion of 176–190
 Shear
 complementary 155, 327
 double 12
 Shear centre 165
 Shear deflection 269
 Shear distribution 156
 Shear force 11, 41–56, 154–166
 Shear strain 11, 180, 371
 Shear strain energy 259, 385
 Shear stress 11, 180, 326
 in bending 77, 154
 in torsion 176
 Shells, thin 202–206
 Shrinkage allowance 226

 Shrink-fit cylinders 224, 226
 Specific resistance 435
 Spiral spring 314
 Spring
 carriage 309
 close-coiled 299
 helical 299
 in parallel 306
 in series 305
 open-coiled 301
 quarter-elliptic 312
 semi-elliptic 309
 spiral 317
 stiffness 306
 Stern 598
 Stiffness of springs 306
 Strain
 complex 361
 diametral 200
 direct 2
 initial 16
 lateral 9
 micro 2
 principal 362, 372
 rosette 378, 381
 shear 11, 180, 371
 threshold 431
 volumetric 201, 202
 Strain circle 372
 Strain energy
 dilatational 262
 distortional 263
 in bending 260
 in curved members 284
 in direct stress 257
 in shear 263
 in springs 302, 304
 in torsion 184, 261
 shear 263, 385
 total 385
 volumetric 263, 385
 Strain gauge 378, 434
 Strength, tensile 5
 Stress
 bending 65–79
 boundary 449
 circumferential 199
 complementary shear 155, 237
 complex 326
 deviatoric 263
 direct 2
 hoop 199, 202
 hydrostatic 386
 longitudinal 199
 proof 5, 312
 radial 198
 separation 454
 shear 11, 180, 326
 spherical 386
 three-dimensional 338
 two-dimensional 10, 326
 ultimate tensile 5
 volumetric 363

- working 12
yield 4, 401
Stress concentration 14, 413, 515
Stress concentration factor 14
Stress freezing technique 454
Stress trajectory 453
Stresscoat 432
Superposition, principle of 34, 52, 112
- Tapered shaft 186
Tardy compensaion 451
Temperature compensation 438
Temperature stresses 30–34, 231, 439
Temporary birefringence 446
Tensile strength 5
Tensile test 4
Theory of failure 401–417
Thermal stresses 231
Thick cylinders 215–251
Thin cylinders 198–207
Threshold strain 431
Thrust diagram 53
Torsion
 of shafts 176
Torsional rigidity 182
Toughness 14
Transverse sensitivty 437
- Tresca theory 401, 403, 412
Twist, angle of 176–190
- Unbalanced bridge 437
Unit load method 268
- Volume change 201, 203, 364, 365
Volumetric strain 363, 369
Volumetric strain energy 263, 385
von Mises theory 401, 404
- Wheatstone bridge 436
Wind-up angle 301, 305
Wire gauge 435
Wire-wound thick cylinder 194
Wire-wound thin cylinder 206
- X-rays 456
- Yield criteria 401
Yield loci
 two-dimensional 406
 three-dimensional 412
Yield point 5
Yield stress 4, 401
Yield theories 401–417
Young's modulus 3, 361

MECHANICS OF MATERIALS 2

*An Introduction to the Mechanics of Elastic and
Plastic Deformation of Solids and Structural Materials*

THIRD EDITION

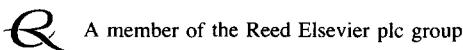
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Butterworth-Heinemann
Linacre House, Jordan Hill, Oxford OX2 8DP
225 Wildwood Avenue, Woburn, MA 01801-2041
A division of Reed Educational and Professional Publishing Ltd



OXFORD AUCKLAND BOSTON
MELBOURNE NEW DELHI

First published 1977
Reprinted with corrections 1980, 1981, 1982
Second edition 1985
Reprinted with corrections 1989
Reprinted 1992, 1995, 1996
Third edition 1997
Reprinted 1999

© E.J. Hearn 1977, 1985, 1997

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British Library Cataloguing in Publication Data

Hearn, E. J. (Edwin John)

Mechanics of materials. – 3rd ed.

1. An introduction to the mechanics of elastic and plastic deformation of solids and structural materials

1. Strength of materials 2. Strains and stress

3. Deformations (Mechanics) 4. Elasticity

I. Title

620.1'12

ISBN 0 7506 3266 6

Library of Congress Cataloguing in Publication Data

Hearn, E. J. (Edwin John)

Mechanics of materials I: an introduction to the mechanics of elastic and plastic deformation of solids and structural materials/E. J. Hearn. – 3rd ed.

p. cm.

Includes bibliographical references and index.

ISBN 0 7506 3266 6

1. Strength of materials. I. Title

TA405.H3

620.1'123-dc21

96-49967

CIP

Typeset by Laser Words, Madras, India
Printed and bound in Great Britain

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CONTENTS

Introduction	xv
Notation	xvii
1 Unsymmetrical Bending	1
Summary	1
Introduction	2
1.1 Product second moment of area	3
1.2 Principal second moments of area	4
1.3 Mohr's circle of second moments of area	6
1.4 Land's circle of second moments of area	7
1.5 Rotation of axes: determination of moments of area in terms of the principal values	8
1.6 The ellipse of second moments of area	9
1.7 Momental ellipse	11
1.8 Stress determination	11
1.9 Alternative procedure for stress determination	11
1.10 Alternative procedure using the momental ellipse	13
1.11 Deflections	15
Examples	16
Problems	24
2 Struts	28
Summary	28
Introduction	30
2.1 Euler's theory	31
2.2 Equivalent strut length	35
2.3 Comparison of Euler theory with experimental results	36
2.4 Euler "validity limit"	37
2.5 Rankine or Rankine–Gordon formula	38
2.6 Perry–Robertson formula	39
2.7 British Standard procedure (BS 449)	41
2.8 Struts with initial curvature	41
2.9 Struts with eccentric load	42
2.10 Laterally loaded struts	46
2.11 Alternative procedure for any strut-loading condition	48

2.12	<i>Struts with unsymmetrical cross-section</i>	49
	<i>Examples</i>	50
	<i>Problems</i>	56
3	Strains Beyond the Elastic Limit	61
	<i>Summary</i>	61
	<i>Introduction</i>	62
3.1	<i>Plastic bending of rectangular-sectioned beams</i>	64
3.2	<i>Shape factor – symmetrical sections</i>	65
3.3	<i>Application to I-section beams</i>	67
3.4	<i>Partially plastic bending of unsymmetrical sections</i>	67
3.5	<i>Shape factor – unsymmetrical sections</i>	69
3.6	<i>Deflections of partially plastic beams</i>	69
3.7	<i>Length of yielded area in beams</i>	69
3.8	<i>Collapse loads – plastic limit design</i>	71
3.9	<i>Residual stresses after yielding: elastic-perfectly plastic material</i>	73
3.10	<i>Torsion of shafts beyond the elastic limit – plastic torsion</i>	75
3.11	<i>Angles of twist of shafts strained beyond the elastic limit</i>	77
3.12	<i>Plastic torsion of hollow tubes</i>	77
3.13	<i>Plastic torsion of case-hardened shafts</i>	79
3.14	<i>Residual stresses after yield in torsion</i>	79
3.15	<i>Plastic bending and torsion of strain-hardening materials</i>	80
	(a) <i>Inelastic bending</i>	80
	(b) <i>Inelastic torsion</i>	83
3.16	<i>Residual stresses – strain-hardening materials</i>	84
3.17	<i>Influence of residual stresses on bending and torsional strengths</i>	84
3.18	<i>Plastic yielding in the eccentric loading of rectangular sections</i>	85
3.19	<i>Plastic yielding and residual stresses under axial loading with stress concentrations</i>	86
3.20	<i>Plastic yielding of axially symmetric components</i>	87
	(a) <i>Thick cylinders – collapse pressure</i>	87
	(b) <i>Thick cylinders – “auto-frettage”</i>	89
	(c) <i>Rotating discs</i>	94
	<i>Examples</i>	96
	<i>Problems</i>	109
4	Rings, Discs and Cylinders Subjected to Rotation and Thermal Gradients	117
	<i>Summary</i>	117
4.1	<i>Thin rotating ring or cylinder</i>	118
4.2	<i>Rotating solid disc</i>	119
4.3	<i>Rotating disc with a central hole</i>	122
4.4	<i>Rotating thick cylinders or solid shafts</i>	124
4.5	<i>Rotating disc of uniform strength</i>	125

4.6	<i>Combined rotational and thermal stresses in uniform discs and thick cylinders</i>	126
	<i>Examples</i>	129
	<i>Problems</i>	136
5	Torsion of Non-Circular and Thin-Walled Sections	141
	<i>Summary</i>	141
5.1	<i>Rectangular sections</i>	142
5.2	<i>Narrow rectangular sections</i>	143
5.3	<i>Thin-walled open sections</i>	143
5.4	<i>Thin-walled split tube</i>	145
5.5	<i>Other solid (non-tubular) shafts</i>	145
5.6	<i>Thin-walled closed tubes of non-circular section (Bredt–Batho theory)</i>	147
5.7	<i>Use of “equivalent J” for torsion of non-circular sections</i>	149
5.8	<i>Thin-walled cellular sections</i>	150
5.9	<i>Torsion of thin-walled stiffened sections</i>	151
5.10	<i>Membrane analogy</i>	152
5.11	<i>Effect of warping of open sections</i>	153
	<i>Examples</i>	154
	<i>Problems</i>	160
6	Experimental Stress Analysis	166
	<i>Introduction</i>	166
6.1	<i>Brittle lacquers</i>	167
6.2	<i>Strain gauges</i>	171
6.3	<i>Unbalanced bridge circuit</i>	173
6.4	<i>Null balance or balanced bridge circuit</i>	173
6.5	<i>Gauge construction</i>	173
6.6	<i>Gauge selection</i>	175
6.7	<i>Temperature compensation</i>	175
6.8	<i>Installation procedure</i>	176
6.9	<i>Basic measurement systems</i>	177
6.10	<i>D.C. and A.C. systems</i>	179
6.11	<i>Other types of strain gauge</i>	180
6.12	<i>Photoelasticity</i>	181
6.13	<i>Plane-polarised light – basic polariscope arrangements</i>	182
6.14	<i>Temporary birefringence</i>	183
6.15	<i>Production of fringe patterns</i>	184
6.16	<i>Interpretation of fringe patterns</i>	185
6.17	<i>Calibration</i>	186
6.18	<i>Fractional fringe order determination – compensation techniques</i>	187
6.19	<i>Isoclinics–circular polarisation</i>	188
6.20	<i>Stress separation procedures</i>	190
6.21	<i>Three-dimensional photoelasticity</i>	190

6.22	<i>Reflective coating technique</i>	190
6.23	<i>Other methods of strain measurement</i>	192
	<i>Bibliography</i>	192
7	Circular Plates and Diaphragms	193
	<i>Summary</i>	193
	A. CIRCULAR PLATES	195
7.1	<i>Stresses</i>	195
7.2	<i>Bending moments</i>	197
7.3	<i>General equation for slope and deflection</i>	198
7.4	<i>General case of a circular plate or diaphragm subjected to combined uniformly distributed load q (pressure) and central concentrated load F</i>	199
7.5	<i>Uniformly loaded circular plate with edges clamped</i>	200
7.6	<i>Uniformly loaded circular plate with edges freely supported</i>	202
7.7	<i>Circular plate with central concentrated load F and edges clamped</i>	203
7.8	<i>Circular plate with central concentrated load F and edges freely supported</i>	205
7.9	<i>Circular plate subjected to a load F distributed round a circle</i>	206
7.10	<i>Application to the loading of annular rings</i>	208
7.11	<i>Summary of end conditions</i>	208
7.12	<i>Stress distributions in circular plates and diaphragms subjected to lateral pressures</i>	209
7.13	<i>Discussion of results – limitations of theory</i>	211
7.14	<i>Other loading cases of practical importance</i>	212
	B. BENDING OF RECTANGULAR PLATES	213
7.15	<i>Rectangular plates with simply supported edges carrying uniformly distributed loads</i>	213
7.16	<i>Rectangular plates with clamped edges carrying uniformly distributed loads</i>	214
	<i>Examples</i>	215
	<i>Problems</i>	218
8	Introduction to Advanced Elasticity Theory	220
8.1	<i>Types of stress</i>	220
8.2	<i>The cartesian stress components: notation and sign convention</i>	220
8.2.1	<i>Sign conventions</i>	221
8.3	<i>The state of stress at a point</i>	221
8.4	<i>Direct, shear and resultant stresses on an oblique plane</i>	224
8.4.1	<i>Line of action of resultant stress</i>	226
8.4.2	<i>Line of action of normal stress</i>	227

8.4.3	<i>Line of action of shear stress</i>	227
8.4.4	<i>Shear stress in any other direction on the plane</i>	227
8.5	<i>Principal stresses and strains in three dimensions – Mohr's circle representation</i>	228
8.6	<i>Graphical determination of the direction of the shear stress τ_n on an inclined plane in a three-dimensional principal stress system</i>	229
8.7	<i>The combined Mohr diagram for three-dimensional stress and strain systems</i>	230
8.8	<i>Application of the combined circle to two-dimensional stress systems</i>	232
8.9	<i>Graphical construction for the state of stress at a point</i>	234
8.10	<i>Construction for the state of strain on a general strain plane</i>	235
8.11	<i>State of stress–tensor notation</i>	235
8.12	<i>The stress equations of equilibrium</i>	236
8.13	<i>Principal stresses in a three-dimensional cartesian stress system</i>	242
8.13.1	<i>Solution of cubic equations</i>	242
8.14	<i>Stress invariants – Eigen values and Eigen vectors</i>	243
8.15	<i>Stress invariants</i>	244
8.16	<i>Reduced stresses</i>	246
8.17	<i>Strain invariants</i>	247
8.18	<i>Alternative procedure for determination of principal stresses</i>	247
8.18.1	<i>Evaluation of direction cosines for principal stresses</i>	248
8.19	<i>Octahedral planes and stresses</i>	249
8.20	<i>Deviatoric stresses</i>	251
8.21	<i>Deviatoric strains</i>	253
8.22	<i>Plane stress and plane strain</i>	254
8.22.1	<i>Plane stress</i>	255
8.22.2	<i>Plane strain</i>	255
8.23	<i>The stress–strain relations</i>	256
8.24	<i>The strain–displacement relationships</i>	257
8.25	<i>The strain equations of transformation</i>	259
8.26	<i>Compatibility</i>	261
8.27	<i>The stress function concept</i>	263
8.27.1	<i>Forms of Airy stress function in Cartesian coordinates</i>	265
8.27.2	<i>Case 1 – Bending of a simply supported beam by a uniformly distributed loading</i>	267
8.27.3	<i>The use of polar coordinates in two dimensions</i>	271
8.27.4	<i>Forms of stress function in polar coordinates</i>	272
8.27.5	<i>Case 2 – Axi-symmetric case: solid shaft and thick cylinder radially loaded with uniform pressure</i>	273
8.27.6	<i>Case 3 – The pure bending of a rectangular section curved beam</i>	273
8.27.7	<i>Case 4 – Asymmetric case $n = 1$. Shear loading of a circular arc cantilever beam</i>	274
8.27.8	<i>Case 5 – The asymmetric cases $n \geq 2$ – stress concentration at a circular hole in a tension field</i>	276

8.27.9 <i>Other useful solutions of the biharmonic equation</i>	279
<i>Examples</i>	283
<i>Problems</i>	290
9 Introduction to the Finite Element Method	300
<i>Introduction</i>	300
9.1 <i>Basis of the finite element method</i>	300
9.2 <i>Applicability of the finite element method</i>	302
9.3 <i>Formulation of the finite element method</i>	303
9.4 <i>General procedure of the finite element method</i>	303
9.4.1 <i>Identification of the appropriateness of analysis by the finite element method</i>	303
9.4.2 <i>Identification of the type of analysis</i>	305
9.4.3 <i>Idealisation</i>	305
9.4.4 <i>Discretisation of the solution region</i>	305
9.4.5 <i>Creation of the material model</i>	312
9.4.6 <i>Node and element ordering</i>	312
9.4.7 <i>Application of boundary conditions</i>	316
9.4.8 <i>Creation of a data file</i>	317
9.4.9 <i>Computer, processing, steps</i>	318
9.4.10 <i>Interpretation and validation of results</i>	318
9.4.11 <i>Modification and re-run</i>	319
9.5 <i>Fundamental arguments</i>	319
9.5.1 <i>Equilibrium</i>	319
9.5.2 <i>Compatibility</i>	321
9.5.3 <i>Stress-strain law</i>	322
9.5.4 <i>Force/displacement relation</i>	322
9.6 <i>The principle of virtual work</i>	323
9.7 <i>A rod element</i>	324
9.7.1 <i>Formulation of a rod element using fundamental equations</i>	324
9.7.2 <i>Formulation of a rod element using the principle of virtual work equation</i>	328
9.8 <i>A simple beam element</i>	334
9.8.1 <i>Formulation of a simple beam element using fundamental equations</i>	334
9.8.2 <i>Formulation of a simple beam element using the principle of virtual work equation</i>	339
9.9 <i>A simple triangular plane membrane element</i>	343
9.9.1 <i>Formulation of a simple triangular plane membrane element using the principle of virtual work equation</i>	344
9.10 <i>Formation of assembled stiffness matrix by use of a dof. correspondence table</i>	347
9.11 <i>Application of boundary conditions and partitioning</i>	349

9.12	<i>Solution for displacements and reactions</i>	349
	<i>Bibliography</i>	350
	<i>Examples</i>	350
	<i>Problems</i>	375
10	Contact Stress, Residual Stress and Stress Concentrations	381
	<i>Summary</i>	381
10.1	<i>Contact stresses</i>	382
	<i>Introduction</i>	382
10.1.1	<i>General case of contact between two curved surfaces</i>	385
10.1.2	<i>Special case 1 – Contact of parallel cylinders</i>	386
10.1.3	<i>Combined normal and tangential loading</i>	388
10.1.4	<i>Special case 2 – Contacting spheres</i>	389
10.1.5	<i>Design considerations</i>	390
10.1.6	<i>Contact loading of gear teeth</i>	391
10.1.7	<i>Contact stresses in spur and helical gearing</i>	392
10.1.8	<i>Bearing failures</i>	393
10.2	<i>Residual stresses</i>	394
	<i>Introduction</i>	394
10.2.1	<i>Reasons for residual stresses</i>	395
(a)	<i>Mechanical processes</i>	395
(b)	<i>Chemical treatment</i>	397
(c)	<i>Heat treatment</i>	398
(d)	<i>Welds</i>	400
(e)	<i>Castings</i>	401
10.2.2	<i>The influence of residual stress on failure</i>	402
10.2.3	<i>Measurement of residual stresses</i>	402
	<i>The hole-drilling technique</i>	404
	<i>X-ray diffraction</i>	407
10.2.4	<i>Summary of the principal effects of residual stress</i>	408
10.3	<i>Stress concentrations</i>	408
	<i>Introduction</i>	408
10.3.1	<i>Evaluation of stress concentration factors</i>	413
10.3.2	<i>St. Venant's principle</i>	420
10.3.3	<i>Theoretical considerations of stress concentrations due to concentrated loads</i>	422
(a)	<i>Concentrated load on the edge of an infinite plate</i>	422
(b)	<i>Concentrated load on the edge of a beam in bending</i>	423
10.3.4	<i>Fatigue stress concentration factor</i>	423
10.3.5	<i>Notch sensitivity</i>	424
10.3.6	<i>Strain concentration – Neuber's rule</i>	425
10.3.7	<i>Designing to reduce stress concentrations</i>	426
(a)	<i>Fillet radius</i>	427
(b)	<i>Keyways or splines</i>	427

<i>(c) Grooves and notches</i>	429
<i>(d) Gear teeth</i>	430
<i>(e) Holes</i>	431
<i>(f) Oil holes</i>	431
<i>(g) Screw threads</i>	431
<i>(h) Press or shrink fit members</i>	433
10.3.8 <i>Use of stress concentration factors with yield criteria</i>	434
10.3.9 <i>Design procedure</i>	434
<i>References</i>	435
<i>Examples</i>	437
<i>Problems</i>	442
11 Fatigue, Creep and Fracture	443
<i>Summary</i>	443
11.1 <i>Fatigue</i>	446
<i>Introduction</i>	446
11.1.1 <i>The S/N curve</i>	446
11.1.2 <i>P/S/N curves</i>	449
11.1.3 <i>Effect of mean stress</i>	451
11.1.4 <i>Effect of stress concentration</i>	453
11.1.5 <i>Cumulative damage</i>	454
11.1.6 <i>Cyclic stress-strain</i>	455
11.1.7 <i>Combating fatigue</i>	458
11.1.8 <i>Slip bands and fatigue</i>	460
11.2 <i>Creep</i>	462
<i>Introduction</i>	462
11.2.1 <i>The creep test</i>	462
11.2.2 <i>Presentation of creep data</i>	465
11.2.3 <i>The stress-rupture test</i>	466
11.2.4 <i>Parameter methods</i>	467
11.2.5 <i>Stress relaxation</i>	470
11.2.6 <i>Creep-resistant alloys</i>	471
11.3 <i>Fracture mechanics</i>	472
<i>Introduction</i>	472
11.3.1 <i>Energy variation in cracked bodies</i>	473
(a) <i>Constant displacement</i>	474
(b) <i>Constant loading</i>	474
11.3.2 <i>Linear elastic fracture mechanics (L.E.F.M.)</i>	475
(a) <i>Griffith's criterion for fracture</i>	475
(b) <i>Stress intensity factor</i>	477
11.3.3 <i>Elastic-plastic fracture mechanics (E.P.F.M.)</i>	481
11.3.4 <i>Fracture toughness</i>	483
11.3.5 <i>Plane strain and plane stress fracture modes</i>	484
11.3.6 <i>General yielding fracture mechanics</i>	484
11.3.7 <i>Fatigue crack growth</i>	486
11.3.8 <i>Crack tip plasticity under fatigue loading</i>	488

11.3.9 <i>Measurement of fatigue crack growth</i>	489
<i>References</i>	490
<i>Examples</i>	491
<i>Problems</i>	503
12 Miscellaneous topics	509
12.1 <i>Bending of beams with initial curvature</i>	509
12.2 <i>Bending of wide beams</i>	515
12.3 <i>General expression for stresses in thin-walled shells subjected to pressure or self-weight</i>	517
12.4 <i>Bending stresses at discontinuities in thin shells</i>	518
12.5 <i>Viscoelasticity</i>	521
<i>References</i>	527
<i>Examples</i>	527
<i>Problems</i>	527
Appendix 1. Typical mechanical and physical properties for engineering metals	534
Appendix 2. Typical mechanical properties of non-metals	535
Appendix 3. Other properties of non-metals	536
Index	537

INTRODUCTION

This text is a revised and extended third edition of the highly successful text initially published in 1977 intended to cover the material normally contained in degree and honours degree courses in mechanics of materials and in courses leading to exemption from the academic requirements of the Engineering Council. It should also serve as a valuable reference medium for industry and for post-graduate courses. Published in two volumes, the text should also prove valuable for students studying mechanical science, stress analysis, solid mechanics or similar modules on Higher Certificate, Higher Diploma or equivalent courses in the UK or overseas and for appropriate NVQ* programmes.

The study of mechanics of materials is the study of the behaviour of solid bodies under load. The way in which they react to applied forces, the deflections resulting and the stresses and strains set up within the bodies, are all considered in an attempt to provide sufficient knowledge to enable any component to be designed such that it will not fail within its service life.

Typical components considered in detail in the first volume, *Mechanics of Materials 1*, include beams, shafts, cylinders, struts, diaphragms and springs and, in most simple loading cases, theoretical expressions are derived to cover the mechanical behaviour of these components. Because of the reliance of such expressions or certain basic assumptions, the text also includes a chapter devoted to the important experimental stress and strain measurement techniques in use today with recommendations for further reading.

Building upon the fundamentals established in *Mechanics of Materials 1*, this book extends the scope of material covered into more complex areas such as unsymmetrical bending, loading and deflection of struts, rings, discs, cylinders plates, diaphragms and thin walled sections. There is a new treatment of the Finite Element Method of analysis, and more advanced topics such as contact and residual stresses, stress concentrations, fatigue, creep and fracture are also covered.

Each chapter of both books contains a summary of essential formulae which are developed within the chapter and a large number of worked examples. The examples have been selected to provide progression in terms of complexity of problem and to illustrate the logical way in which the solution to a difficult problem can be developed. Graphical solutions have been introduced where appropriate. In order to provide clarity of working in the worked examples there is inevitably more detailed explanation of individual steps than would be expected in the model answer to an examination problem.

All chapters conclude with an extensive list of problems for solution by students together with answers. These have been collected from various sources and include questions from past examination papers in imperial units which have been converted to the equivalent SI values. Each problem is graded according to its degree of difficulty as follows:

* National Vocational Qualifications.

- A Relatively easy problem of an introductory nature.
- A/B Generally suitable for first-year studies.
- B Generally suitable for second or third-year studies.
- C More difficult problems generally suitable for third-year studies.

Gratitude is expressed to the following examination boards, universities and colleges who have kindly given permission for questions to be reproduced:

City University	C.U.
East Midland Educational Union	E.M.E.U.
Engineering Institutions Examination	E.I.E. and C.E.I.
Institution of Mechanical Engineers	I.Mech.E.
Institution of Structural Engineers	I.Struct.E.
Union of Educational Institutions	U.E.I.
Union of Lancashire and Cheshire Institutes	U.L.C.I.
University of Birmingham	U.Birm.
University of London	U.L.

Both volumes of the text together contain 150 worked examples and more than 500 problems for solution, and whilst it is hoped that no errors are present it is perhaps inevitable that some errors will be detected. In this event any comment, criticism or correction will be gratefully acknowledged.

The symbols and abbreviations throughout the text are in accordance with the latest recommendations of BS 1991 and PD 5686†

As mentioned above, graphical methods of solution have been introduced where appropriate since it is the author's experience that these are more readily accepted and understood by students than some of the more involved analytical procedures; substantial time saving can also result. Extensive use has also been made of diagrams throughout the text since in the words of the old adage "a single diagram is worth 1000 words".

Finally, the author is indebted to all those who have assisted in the production of this text; to Professor H. G. Hopkins, Mr R. Brettell, Mr R. J. Phelps for their work associated with the first edition, to Dr A. S. Tooth¹, Dr N. Walker², Mr R. Winters² for their contributions to the second edition and to Dr M. Daniels³ for the extended treatment of the Finite Element Method which is the major change in this third edition. Thanks also go to the publishers for their advice and assistance, especially in the preparation of the diagrams and editing and to Dr. C. C. Perry (USA) for his most valuable critique of the first edition.

E. J. HEARN

† Relevant Standards for use in Great Britain: BS 1991; PD 5686: Other useful SI Guides: *The International System of Units*, N.P.L. Ministry of Technology, H.M.S.O. (Britain). Mechty, *The International System of Units (Physical Constants and Conversion Factors)*, NASA, No SP-7012, 3rd edn. 1973 (U.S.A.) *Metric Practice Guide*, A.S.T.M. Standard E380-72 (U.S.A.).

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NOTATION

<i>Quantity</i>	<i>Symbol</i>	<i>SI Unit</i>
Angle	$\alpha, \beta, \theta, \gamma, \phi$	rad (radian)
Length	L, s	m (metre) mm (millimetre)
Area	A	m^2
Volume	V	m^3
Time	t	s (second)
Angular velocity	ω	rad/s
Velocity	v	m/s
Weight	W	N (newton)
Mass	m	kg (kilogram)
Density	ρ	kg/m^3
Force	F or P or W	N
Moment	M	Nm
Pressure	P	Pa (Pascal) N/m^2 bar ($= 10^5 N/m^2$) N/m^2
Stress	σ	N/m^2
Strain	ε	—
Shear stress	τ	N/m^2
Shear strain	γ	—
Young's modulus	E	N/m^2
Shear modulus	G	N/m^2
Bulk modulus	K	N/m^2
Poisson's ratio	ν	—
Modular ratio	m	—
Power	—	W (watt)
Coefficient of linear expansion	α	$m/m^\circ C$
Coefficient of friction	μ	—
Second moment of area	I	m^4
Polar moment of area	J	m^4
Product moment of area	I_{xy}	m^4
Temperature	T	$^\circ C$
Direction cosines	l, m, n	—
Principal stresses	$\sigma_1, \sigma_2, \sigma_3$	N/m^2
Principal strains	$\varepsilon_1, \varepsilon_2, \varepsilon_3$	—
Maximum shear stress	τ_{max}	N/m^2
Octahedral stress	σ_{oct}	N/m^2

Quantity	Symbol	SI Unit
Deviatoric stress	σ'	N/m ²
Deviatoric strain	ϵ'	—
Hydrostatic or mean stress	$\bar{\sigma}$	N/m ²
Volumetric strain	Δ	—
Stress concentration factor	K	—
Strain energy	U	J
Displacement	δ	m
Deflection	δ or y	m
Radius of curvature	ρ	m
Photoelastic material fringe value	f	N/m ² /fringe/m
Number of fringes	n	—
Body force stress	X, Y, Z F_R, F_θ, F_Z	N/m ³
Radius of gyration	k	m
Slenderness ratio	L/k	—
Gravitational acceleration	g	m/s ²
Cartesian coordinates	x, y, z	—
Cylindrical coordinates	r, θ, z	—
Eccentricity	e	m
Number of coils or leaves of spring	n	—
Equivalent J or effective polar moment of area	J_{eq} or J_E	m ⁴
Autofrettage pressure	P_A	N/m ² or bar
Radius of elastic-plastic interface	R_p	m
Thick cylinder radius ratio R_2/R_1	K	—
Ratio elastic–plastic interface radius to internal radius of thick cylinder R_p/R_1	m	—
Resultant stress on oblique plane	p_n	N/m ²
Normal stress on oblique plane	σ_n	N/m ²
Shear stress on oblique plane	τ_n	N/m ²
Direction cosines of plane	l, m, n	—
Direction cosines of line of action of resultant stress	l', m', n'	—
Direction cosines of line of action of shear stress	l_s, m_s, n_s	—
Components of resultant stress on oblique plane	p_{xn}, p_{yn}, p_{zn}	N/m ²
Shear stress in any direction ϕ on oblique plane	τ_ϕ	N/m ²
Invariants of stress	I_1 I_2 I_3	N/m ² $(N/m^2)^2$ $(N/m^2)^3$
Invariants of reduced stresses	J_1, J_2, J_3	—
Airy stress function	ϕ	—

<i>Quantity</i>	<i>Symbol</i>	<i>SI Unit</i>
'Operator' for Airy stress function biharmonic equation	∇	—
Strain rate	$\dot{\epsilon}$	s^{-1}
Coefficient of viscosity	η	—
Retardation time (creep strain recovery)	t'	s
Relaxation time (creep stress relaxation)	t''	s
Creep contraction or lateral strain ratio	$J(t)$	—
Maximum contact pressure (Hertz)	p_0	N/m^2
Contact formulae constant	Δ	$(N/m^2)^{-1}$
Contact area semi-axes	a, b	m
Maximum contact stress	$\sigma_c = -p_0$	N/m^2
Spur gear contact formula constant	K	N/m^2
Helical gear profile contact ratio	m_p	—
Elastic stress concentration factor	K_t	—
Fatigue stress concentration factor	K_f	—
Plastic flow stress concentration factor	K_p	—
Shear stress concentration factor	K_{t_s}	—
Endurance limit for n cycles of load	S_n	N/m^2
Notch sensitivity factor	q	—
Fatigue notch factor	K_f	—
Strain concentration factor	K_ε	—
Griffith's critical strain energy release	G_c	—
Surface energy of crack face	γ	Nm
Plate thickness	B	m
Strain energy	U	Nm
Compliance	C	mN^{-1}
Fracture stress	σ_f	N/m^2
Stress Intensity Factor	K or K_1	$N/m^{3/2}$
Compliance function	Y	—
Plastic zone dimension	r_p	m
Critical stress intensity factor	K_{IC}	$N/m^{3/2}$
"J" Integral	J	—
Fatigue crack dimension	a	m
Coefficients of Paris Erdogan law	C, m	—
Fatigue stress range	σ_r	N/m^2
Fatigue mean stress	σ_m	N/m^2
Fatigue stress amplitude	σ_a	N/m^2
Fatigue stress ratio	R_s	—
Cycles to failure	N_f	—
Fatigue strength for N cycles	σ_N	N/m^2
Tensile strength	σ_{TS}	N/m^2
Factor of safety	F	—

<i>Quantity</i>	<i>Symbol</i>	<i>SI Unit</i>
Elastic strain range	$\Delta\varepsilon_e$	—
Plastic strain range	$\Delta\varepsilon_p$	—
Total strain range	$\Delta\varepsilon_t$	—
Ductility	D	—
Secondary creep rate	ε_s^0	s^{-1}
Activation energy	H	Nm
Universal Gas Constant	R	J/kgK
Absolute temperature	T	°K
Arrhenius equation constant	A	—
Larson–Miller creep parameter	P_1	
Sherby–Dorn creep parameter	P_2	
Manson–Haford creep parameter	P_3	
Initial stress	σ_i	N/m^2
Time to rupture	t_r	s
Constants of power law equation	β, n	—

CHAPTER 1

UNSYMMETRICAL BENDING

Summary

The second moments of area of a section are given by

$$I_{xx} = \int y^2 dA \quad \text{and} \quad I_{yy} = \int x^2 dA$$

The product second moment of area of a section is defined as

$$I_{xy} = \int xy dA$$

which reduces to $I_{xy} = Ahk$ for a rectangle of area A and centroid distance h and k from the X and Y axes.

The *principal second moments of area* are the maximum and minimum values for a section and they occur about the principal axes. *Product second moments of area about principal axes are zero.*

With a knowledge of I_{xx} , I_{yy} and I_{xy} for a given section, the principal values may be determined using either Mohr's or Land's circle construction.

The following relationships apply between the second moments of area about different axes:

$$I_u = \frac{1}{2}(I_{xx} + I_{yy}) + \frac{1}{2}(I_{xx} - I_{yy}) \sec 2\theta$$

$$I_v = \frac{1}{2}(I_{xx} + I_{yy}) - \frac{1}{2}(I_{xx} - I_{yy}) \sec 2\theta$$

where θ is the angle between the U and X axes, and is given by

$$\tan 2\theta = \frac{2I_{xy}}{(I_{yy} - I_{xx})}$$

Then

$$I_u + I_v = I_{xx} + I_{yy}$$

The second moment of area about the neutral axis is given by

$$I_{N.A.} = \frac{1}{2}(I_u + I_v) + \frac{1}{2}(I_u - I_v) \cos 2\alpha_u$$

where α_u is the angle between the neutral axis (N.A.) and the U axis.

Also

$$I_{xx} = I_u \cos^2 \theta + I_v \sin^2 \theta$$

$$I_{yy} = I_v \cos^2 \theta + I_u \sin^2 \theta$$

$$I_{xy} = \frac{1}{2}(I_v - I_u) \sin 2\theta$$

$$I_{xx} - I_{yy} = (I_u - I_v) \cos 2\theta$$

Stress determination

For skew loading and other forms of bending about principal axes

$$\sigma = \frac{M_u v}{I_u} + \frac{M_v u}{I_v}$$

where M_u and M_v are the components of the applied moment about the U and V axes.

Alternatively, with $\sigma = Px + Qy$

$$M_{xx} = PI_{xy} + QI_{xx}$$

$$M_{yy} = -PI_{yy} - QI_{xy}$$

Then the inclination of the N.A. to the X axis is given by

$$\tan \alpha = -\frac{P}{Q}$$

As a further alternative,

$$\sigma = \frac{M' n}{I_{N.A.}}$$

where M' is the component of the applied moment about the N.A., $I_{N.A.}$ is determined either from the momental ellipse or from the Mohr or Land constructions, and n is the perpendicular distance from the point in question to the N.A.

Deflections of unsymmetrical members are found by applying standard deflection formulae to bending about either the principal axes or the N.A. taking care to use the correct component of load and the correct second moment of area value.

Introduction

It has been shown in Chapter 4 of *Mechanics of Materials 1*[†] that the simple bending theory applies when bending takes place about an axis which is perpendicular to a plane of symmetry. If such an axis is drawn through the centroid of a section, and another mutually perpendicular to it also through the centroid, then these axes are principal axes. Thus a plane of symmetry is automatically a principal axis. Second moments of area of a cross-section about its principal axes are found to be maximum and minimum values, while the product second moment of area, $\int xy dA$, is found to be zero. All plane sections, whether they have an axis of symmetry or not, have two perpendicular axes about which the product second moment of area is zero. *Principal axes are thus defined as the axes about which the product second moment of area is zero.* Simple bending can then be taken as bending which takes place about a principal axis, moments being applied in a plane parallel to one such axis.

In general, however, moments are applied about a convenient axis in the cross-section; the plane containing the applied moment may not then be parallel to a principal axis. Such cases are termed “unsymmetrical” or “asymmetrical” bending.

The most simple type of unsymmetrical bending problem is that of “skew” loading of sections containing at least one axis of symmetry, as in Fig. 1.1. This axis and the axis

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

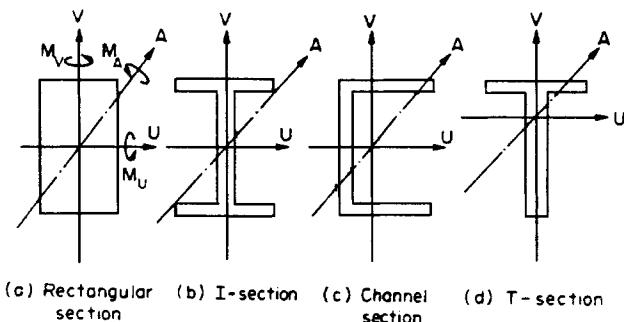


Fig. 1.1. Skew loading of sections containing one axis of symmetry.

perpendicular to it are then principal axes and the term skew loading implies load applied at some angle to these principal axes. The method of solution in this case is to resolve the applied moment M_A about some axis A into its components about the principal axes. Bending is then assumed to take place simultaneously about the two principal axes, the total stress being given by

$$\sigma = \frac{M_u v}{I_u} + \frac{M_v u}{I_v}$$

With at least one of the principal axes being an axis of symmetry the second moments of area about the principal axes I_u and I_v can easily be determined.

With unsymmetrical sections (e.g. angle-sections, Z-sections, etc.) the principal axes are not easily recognized and the second moments of area about the principal axes are not easily found except by the use of special techniques to be introduced in §§1.3 and 1.4. In such cases an easier solution is obtained as will be shown in §1.8. Before proceeding with the various methods of solution of unsymmetrical bending problems, however, it is advisable to consider in some detail the concept of principal and product second moments of area.

1.1. Product second moment of area

Consider a small element of area in a plane surface with a centroid having coordinates (x, y) relative to the X and Y axes (Fig. 1.2). The second moments of area of the surface about the X and Y axes are defined as

$$I_{xx} = \int y^2 dA \quad \text{and} \quad I_{yy} = \int x^2 dA \quad (1.1)$$

Similarly, the product second moment of area of the section is defined as follows:

$$I_{xy} = \int xy dA \quad (1.2)$$

Since the cross-section of most structural members used in bending applications consists of a combination of rectangles the value of the product second moment of area for such sections is determined by the addition of the I_{xy} value for each rectangle (Fig. 1.3),

i.e. $I_{xy} = Ahk$ (1.3)

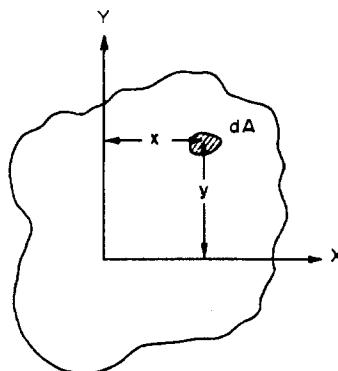


Fig. 1.2.

where h and k are the distances of the centroid of each rectangle from the X and Y axes respectively (taking account of the normal sign convention for x and y) and A is the area of the rectangle.

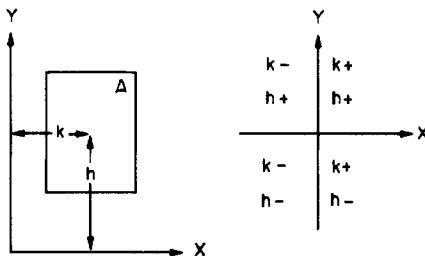


Fig. 1.3.

1.2. Principal second moments of area

The principal axes of a section have been defined in the introduction to this chapter. Second moments of area about these axes are then termed principal values and these may be related to the standard values about the conventional X and Y axes as follows.

Consider Fig. 1.4 in which GX and GY are any two mutually perpendicular axes inclined at θ to the principal axes GV and GU . A small element of area A will then have coordinates (u, v) to the principal axes and (x, y) referred to the axes GX and GY . The area will thus have a product second moment of area about the principal axes given by $uv dA$.
 \therefore total product second moment of area of a cross-section

$$\begin{aligned} I_{uv} &= \int uv dA \\ &= \int (x \cos \theta + y \sin \theta)(y \cos \theta - x \sin \theta) dA \end{aligned}$$

$$\begin{aligned}
 &= \int (xy \cos^2 \theta + y^2 \sin \theta \cos \theta - x^2 \cos \theta \sin \theta - xy \sin^2 \theta) dA \\
 &= (\cos^2 \theta - \sin^2 \theta) \int xy dA + \sin \theta \cos \theta \left[\int y^2 dA - \int x^2 dA \right] \\
 &= I_{xy} \cos 2\theta + \frac{1}{2}(I_{xx} - I_{yy}) \sin 2\theta
 \end{aligned}$$

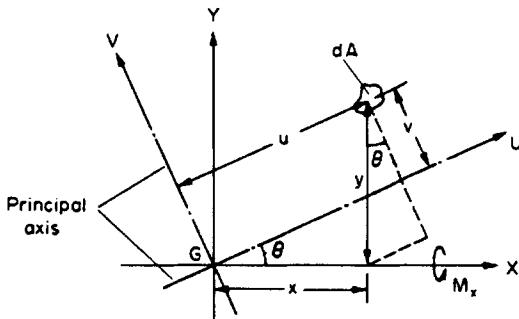


Fig. 1.4.

Now for principal axes the product second moment of area is zero.

$$\begin{aligned}
 0 &= I_{xy} \cos 2\theta + \frac{1}{2}(I_{xx} - I_{yy}) \sin 2\theta \\
 \tan 2\theta &= \frac{-2I_{xy}}{(I_{xx} - I_{yy})} = \frac{2I_{xy}}{(I_{yy} - I_{xx})}
 \end{aligned} \tag{1.4}$$

This equation, therefore, gives the direction of the principal axes.

To determine the second moments of area about these axes,

$$\begin{aligned}
 I_u &= \int v^2 dA = \int (y \cos \theta - x \sin \theta)^2 dA \\
 &= \cos^2 \theta \int y^2 dA + \sin^2 \theta \int x^2 dA - 2 \cos \theta \sin \theta \int xy dA \\
 &= I_{xx} \cos^2 \theta + I_{yy} \sin^2 \theta - I_{xy} \sin 2\theta
 \end{aligned} \tag{1.5}$$

Substituting for I_{xy} from eqn. (1.4),

$$\begin{aligned}
 I_u &= \frac{1}{2}(1 + \cos 2\theta)I_{xx} + \frac{1}{2}(1 - \cos 2\theta)I_{yy} - \frac{1}{2} \frac{\sin^2 2\theta}{\cos 2\theta} (I_{yy} - I_{xx}) \\
 &= \frac{1}{2}(1 + \cos 2\theta)I_{xx} + \frac{1}{2}(1 - \cos 2\theta)I_{yy} - \frac{1}{2} \left[\frac{(1 - \cos^2 2\theta)}{\cos 2\theta} (I_{yy} - I_{xx}) \right] \\
 &= \frac{1}{2}(1 + \cos 2\theta)I_{xx} + \frac{1}{2}(1 - \cos 2\theta)I_{yy} - \frac{1}{2} \sec 2\theta (I_{yy} - I_{xx}) + \frac{1}{2} \cos 2\theta (I_{yy} - I_{xx}) \\
 &= \frac{1}{2}(I_{xx} + I_{yy}) + (I_{xx} - I_{yy}) \cos 2\theta - (I_{yy} - I_{xx}) \sec 2\theta + (I_{yy} - I_{xx}) \cos 2\theta
 \end{aligned}$$

i.e.

$$I_u = \frac{1}{2}(I_{xx} + I_{yy}) + \frac{1}{2}(I_{xx} - I_{yy}) \sec 2\theta \quad (1.6)$$

Similarly,

$$\begin{aligned} I_v &= \int u^2 dA = \int (x \cos \theta + y \sin \theta)^2 dA \\ &= \frac{1}{2}(I_{xx} + I_{yy}) - \frac{1}{2}(I_{xx} - I_{yy}) \sec 2\theta \end{aligned} \quad (1.7)$$

N.B.—Adding the above expressions,

$$I_u + I_v = I_{xx} + I_{yy}$$

Also from eqn. (1.5),

$$\begin{aligned} I_u &= I_{xx} \cos^2 \theta + I_{yy} \sin^2 \theta - I_{xy} \sin 2\theta \\ &= \frac{1}{2}(1 + \cos 2\theta)I_{xx} + \frac{1}{2}(1 - \cos 2\theta)I_{yy} - I_{xy} \sin 2\theta \\ I_u &= \frac{1}{2}(I_{xx} + I_{yy}) + \frac{1}{2}(I_{xx} - I_{yy}) \cos 2\theta - I_{xy} \sin 2\theta \end{aligned} \quad (1.8)$$

Similarly,

$$I_v = \frac{1}{2}(I_{xx} + I_{yy}) - \frac{1}{2}(I_{xx} - I_{yy}) \cos 2\theta + I_{xy} \sin 2\theta \quad (1.9)$$

These equations are then identical in form with the complex-stress eqns. (13.8) and (13.9)[†] with I_{xx} , I_{yy} , and I_{xy} replacing σ_x , σ_y and τ_{xy} and Mohr's circle can be drawn to represent I values in exactly the same way as Mohr's stress circle represents stress values.

1.3. Mohr's circle of second moments of area

The construction is as follows (Fig. 1.5):

- (1) Set up axes for second moments of area (horizontal) and product second moments of area (vertical).
- (2) Plot the points A and B represented by (I_{xx}, I_{xy}) and $(I_{yy}, -I_{xy})$.
- (3) Join AB and construct a circle with this as diameter. *This is then the Mohr's circle.*
- (4) Since the principal moments of area are those about the axes with a zero product second moment of area they are given by the points where the circle cuts the horizontal axis.

Thus OC and OD are the principal second moments of area I_v and I_u .

The point A represents values on the X axis and B those for the Y axis. Thus, in order to determine the second moment of area about some other axis, e.g. the N.A., at some angle α counterclockwise to the X axis, construct a line from G at an angle 2α counterclockwise to GA on the Mohr construction to cut the circle in point N . The horizontal coordinate of N then gives the value of $I_{N.A.}$

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

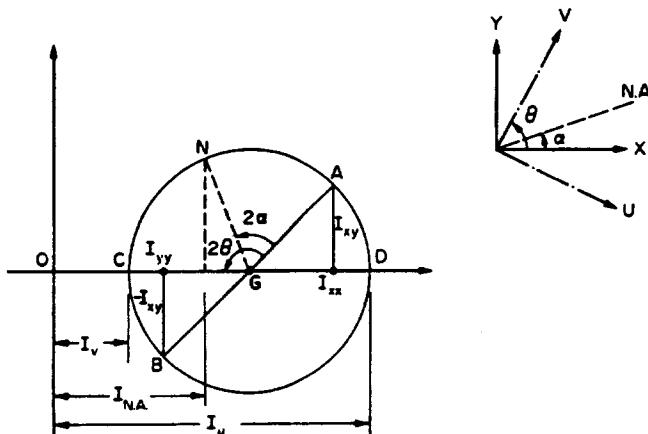


Fig. 1.5. Mohr's circle of second moments of area.

The procedure is therefore identical to that for determining the direct stress on some plane inclined at α to the plane on which σ_x acts in Mohr's stress circle construction, i.e. angles are DOUBLED on Mohr's circle.

1.4. Land's circle of second moments of area

An alternative graphical solution to the Mohr procedure has been developed by Land as follows (Fig. 1.6):

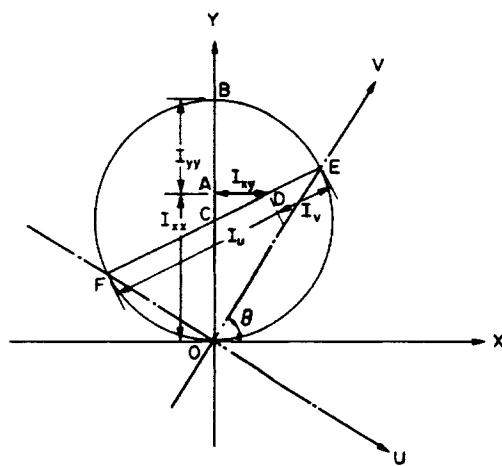


Fig. 1.6. Land's circle of second moments of area.

- (1) From O as origin of the given XY axes mark off lengths $OA = I_{xx}$ and $AB = I_{yy}$ on the vertical axis.

- (2) Draw a circle with OB as diameter and centre C . This is then Land's circle of second moment of area.
- (3) From point A mark off $AD = I_{xy}$ parallel with the X axis.
- (4) Join the centre of the circle C to D , and produce, to cut the circle in E and F . Then $ED = I_v$ and $DF = I_u$ are the principal moments of area about the principal axes OV and OU the positions of which are found by joining OE and OF . The principal axes are thus inclined at an angle θ to the OX and OY axes.

1.5. Rotation of axes: determination of moments of area in terms of the principal values

Figure 1.7 shows any plane section having coordinate axes XX and YY and principal axes UU and VV , each passing through the centroid O . Any element of area dA will then have coordinates (x, y) and (u, v) , respectively, for the two sets of axes.

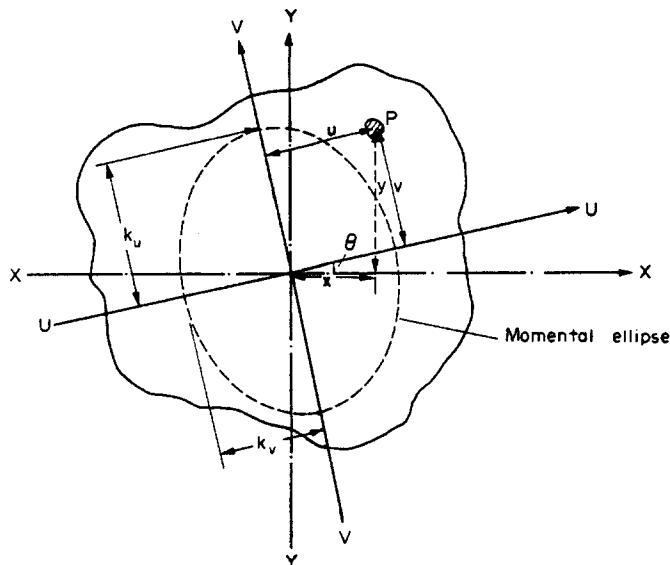


Fig. 1.7. The momental ellipse.

Now

$$\begin{aligned} I_{xx} &= \int y^2 dA = \int (v \cos \theta + u \sin \theta)^2 dA \\ &= \int v^2 \cos^2 \theta dA + \int 2uv \sin \theta \cos \theta dA + \int u^2 \sin^2 \theta dA \end{aligned}$$

But UU and VV are the principal axes so that $I_{uv} = \int uv dA$ is zero.

$$\therefore I_{xx} = I_u \cos^2 \theta + I_v \sin^2 \theta \quad (1.10)$$

Similarly,

$$\begin{aligned} I_{yy} &= \int x^2 dA = \int (u \cos \theta - v \sin \theta)^2 dA \\ &= \int u^2 \cos^2 \theta dA - \int 2uv \sin \theta \cos \theta dA + \int v^2 \sin^2 \theta dA \end{aligned}$$

and with $\int uv dA = 0$

$$I_{yy} = I_v \cos^2 \theta + I_u \sin^2 \theta \quad (1.11)$$

Also

$$\begin{aligned} I_{xy} &= \int xy dA = \int (u \cos \theta - v \sin \theta)(v \cos \theta + u \sin \theta) dA \\ &= \int [uv(\cos^2 \theta - \sin^2 \theta) + (u^2 - v^2) \sin \theta \cos \theta] dA \\ &= I_{uv} \cos 2\theta + \frac{1}{2}(I_v - I_u) \sin 2\theta \quad \text{and} \quad I_{uv} = 0 \\ \therefore I_{xy} &= \frac{1}{2}(I_v - I_u) \sin 2\theta \end{aligned} \quad (1.12)$$

From eqns. (1.10) and (1.11)

$$\begin{aligned} I_{xx} - I_{yy} &= I_u \cos^2 \theta + I_v \sin^2 \theta - I_v \cos^2 \theta - I_u \sin^2 \theta \\ &= (I_u - I_v) \cos^2 \theta - (I_u - I_v) \sin^2 \theta \\ I_{xx} - I_{yy} &= (I_u - I_v) \cos 2\theta \end{aligned} \quad (1.13)$$

Combining eqns. (1.12) and (1.13) gives

$$\tan 2\theta = \frac{2I_{xy}}{I_{yy} - I_{xx}} \quad (1.14)$$

and combining eqns. (1.10) and (1.11) gives

$$I_{xx} + I_{yy} = I_u + I_v \quad (1.15)$$

Substitution into eqns. (1.10) and (1.11) then yields

$$I_u = \frac{1}{2}[I_{xx} + I_{yy} + (I_{xx} - I_{yy}) \sec 2\theta] \quad (1.16) \text{ as (1.6)}$$

$$I_v = \frac{1}{2}[I_{xx} + I_{yy} - (I_{xx} - I_{yy}) \sec 2\theta] \quad (1.17) \text{ as (1.7)}$$

1.6. The ellipse of second moments of area

The above relationships can be used as the basis for construction of the moment of area ellipse proceeding as follows:

- (1) Plot the values of I_u and I_v on two mutually perpendicular axes and draw concentric circles with centres at the origin, and radii equal to I_u and I_v (Fig. 1.8).
- (2) Plot the point with coordinates $x = I_u \cos \theta$ and $y = I_v \sin \theta$, the value of θ being given by eqn. (1.14).

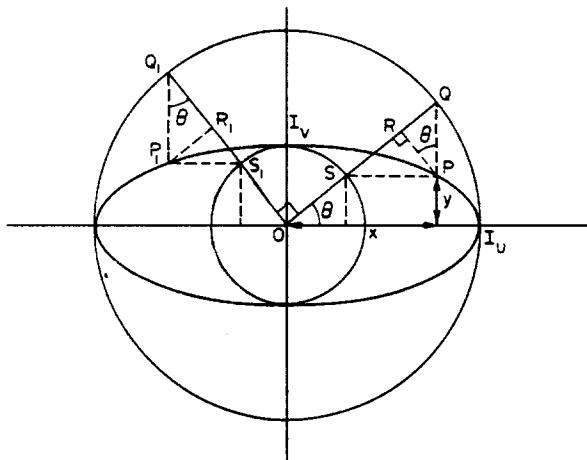


Fig. 1.8. The ellipse of second moments of area.

It then follows that

$$\frac{x^2}{(I_u)^2} + \frac{y^2}{(I_v)^2} = 1$$

This equation is the locus of the point P and represents the equation of an ellipse – the *ellipse of second moments of area*.

- (3) Draw OQ at an angle θ to the I_u axis, cutting the circle through I_v in point S and join SP which is then parallel to the I_u axis. Construct a perpendicular to OQ through P to meet OQ in R .

Then

$$\begin{aligned} OR &= OQ - RQ \\ &= I_u - (I_u \sin \theta - I_v \sin \theta) \sin \theta \\ &= I_u - (I_u - I_v) \sin^2 \theta \\ &= I_u \cos^2 \theta + I_v \sin^2 \theta \\ &= I_{xx} \end{aligned}$$

Similarly, repeating the process with OQ_1 perpendicular to OQ gives the result

$$OR_1 = I_{yy}$$

Further,

$$\begin{aligned} PR &= PQ \cos \theta \\ &= (I_u \sin \theta - I_v \sin \theta) \cos \theta \\ &= \frac{1}{2}(I_u - I_v) \sin 2\theta = I_{xy} \end{aligned}$$

Thus the construction shown in Fig. 1.8 can be used to determine the second moments of area and the product second moment of area about any set of perpendicular axis at a known orientation to the principal axes.

1.7. Momental ellipse

Consider again the general plane surface of Fig. 1.7 having radii of gyration k_u and k_v about the U and V axes respectively. An ellipse can be constructed on the principal axes with semi-major and semi-minor axes k_u and k_v , respectively, as shown.

Thus the perpendicular distance between the axis UU and a tangent to the ellipse which is parallel to UU is equal to the radius of gyration of the surface about UU . Similarly, the radius of gyration k_v is the perpendicular distance between the tangent to the ellipse which is parallel to the VV axis and the axis itself. Thus if the radius of gyration of the surface is required about any other axis, e.g. the N.A., then it is given by the distance between the N.A. and the tangent AA which is parallel to the N.A. (see Fig. 1.11). Thus

$$k_{\text{N.A.}} = h$$

The ellipse is then termed the *momental ellipse* and is extremely useful in the solution of unsymmetrical bending problems as described in §1.10.

1.8. Stress determination

Having determined both the values of the principal second moments of area I_u and I_v and the inclination of the principal axes U and V from the equations listed below,

$$I_u = \frac{1}{2}(I_{xx} + I_{yy}) + \frac{1}{2}(I_{xx} - I_{yy}) \sec 2\theta \quad (1.16)$$

$$I_v = \frac{1}{2}(I_{xx} + I_{yy}) - \frac{1}{2}(I_{xx} - I_{yy}) \sec 2\theta \quad (1.17)$$

and

$$\tan 2\theta = \frac{2I_{xy}}{(I_{yy} - I_{xx})} \quad (1.14)$$

the stress at any point is found by application of the simple bending theory simultaneously about the principal axes,

$$\text{i.e. } \sigma = \frac{M_v u}{I_v} + \frac{M_u v}{I_u} \quad (1.18)$$

where M_v and M_u are the moments of the applied loads about the V and U axes, e.g. if loads are applied to produce a bending moment M_x about the X axis (see Fig. 1.14), then

$$M_v = M_x \sin \theta$$

$$M_u = M_x \cos \theta$$

the maximum value of M_x , and hence M_u and M_v , for cantilevers such as that shown in Fig. 1.10, being found at the root of the cantilever. The maximum stress due to bending will then occur at this position.

1.9. Alternative procedure for stress determination

Consider any unsymmetrical section, represented by Fig. 1.9. The assumption is made initially that the stress at any point on the unsymmetrical section is given by

$$\sigma = Px + Qy \quad (1.19)$$

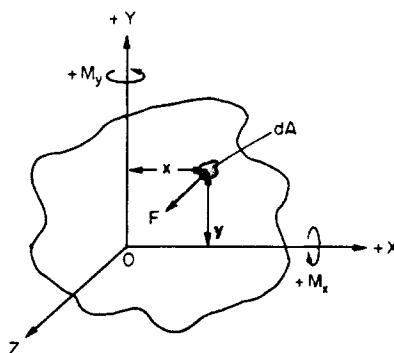


Fig. 1.9. Alternative procedure for stress determination.

where P and Q are constants; in other words it is assumed that bending takes place about the X and Y axes at the same time, stresses resulting from each effect being proportional to the distance from the respective axis of bending.

Now let there be a tensile stress σ on the element of area dA . Then

$$\text{force } F \text{ on the element} = \sigma dA$$

the direction of the force being parallel to the Z axis. The moment of this force about the X axis is then $\sigma dA y$.

$$\begin{aligned}\therefore \text{total moment} &= M_x = \int \sigma dA y \\ &= \int (Px + Qy)y dA = \int Pxy dA + \int Qy^2 dA\end{aligned}$$

Now, by definition,

$$I_{xx} = \int y^2 dA, \quad I_{yy} = \int x^2 dA \quad \text{and} \quad I_{xy} = \int xy dA$$

the latter being termed the product second moment of area (see §1.1):

$$\therefore M_x = PI_{xy} + QI_{xx} \quad (1.20)$$

Similarly, considering moments about the Y axis,

$$\begin{aligned}\therefore M_y &= - \int \sigma dA x = - \int (Px + Qy)x dA \\ \therefore M_y &= -PI_{yy} - QI_{xy} \quad (1.21)\end{aligned}$$

The sign convention used above for bending moments is the *corkscrew rule*. A positive moment is the direction in which a corkscrew or screwdriver has to be turned in order to produce motion of a screw in the direction of positive X or Y , as shown in Fig. 1.9. Thus with a knowledge of the applied moments and the second moments of area about any two perpendicular axes, P and Q can be found from eqns. (1.20) and (1.21) and hence the stress at any point (x, y) from eqn. (1.19).

Since stresses resulting from bending are zero on the N.A. the equation of the N.A. is

$$Px + Qy = 0$$

$$\frac{y}{x} = -\frac{P}{Q} = \tan \alpha_{N.A.} \quad (1.22)$$

where $\alpha_{N.A.}$ is the inclination of the N.A. to the X axis.

If the unsymmetrical member is drawn to scale and the N.A. is inserted through the centroid of the section at the above angle, the points of maximum stress can be determined quickly by inspection as the points most distant from the N.A., e.g. for the angle section of Fig. 1.10, subjected to the load shown, the maximum tensile stress occurs at R while the maximum compressive stress will arise at either S or T depending on the value of α .

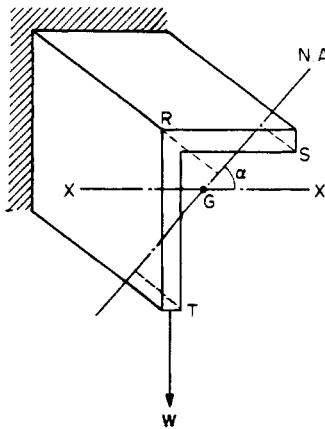


Fig. 1.10.

1.10. Alternative procedure using the momental ellipse

Consider the unsymmetrical section shown in Fig. 1.11 with principal axes UU and VV . Any moment applied to the section can be resolved into its components about the principal axes and the stress at any point found by application of eqn. (1.18).

For example, if vertical loads only are applied to the section to produce moments about the OX axis, then the components will be $M \cos \theta$ about UU and $M \sin \theta$ about VV . Then

$$\text{stress at } P = \frac{M \cos \theta}{I_u} v - \frac{M \sin \theta}{I_v} u \quad (1.23)$$

the value of θ having been obtained from eqn. (1.14).

Alternatively, however, the problem may be solved by realising that the N.A. and the plane of the external bending moment are conjugate diameters of an ellipse[†] – the *momental*

[†] *Conjugate diameters of an ellipse:* two diameters of an ellipse are conjugate when each bisects all chords parallel to the other diameter.

Two diameters $y = m_1 x$ and $y = m_2 x$ are conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $m_1 m_2 = -\frac{b^2}{a^2}$.

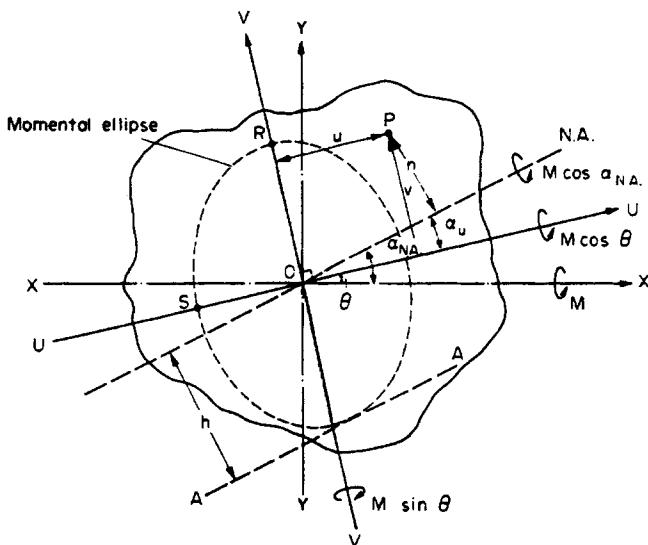


Fig. 1.11. Determination of stresses using the momental ellipse.

ellipse. The actual plane of resultant bending will then be perpendicular to the N.A., the inclination of which, relative to the *U* axis (α_u), is obtained by equating the above formula for stress at *P* to zero,

$$\text{i.e. } \frac{M \cos \theta}{I_u} v = \frac{M \sin \theta}{I_v} u$$

$$\begin{aligned} \text{so that } \tan \alpha_u &= \frac{v}{u} = \frac{I_u}{I_v} \tan \theta \\ &= \frac{k_u^2}{k_v^2} \tan \theta \end{aligned} \quad (1.24)$$

where k_u and k_v are the radii of gyration about the principal axes and hence the semi-axes of the momental ellipse.

The N.A. can now be added to the diagram to scale. The second moment of area of the section about the N.A. is then given by Ah^2 , where h is the perpendicular distance between the N.A. and a tangent *AA* to the ellipse drawn parallel to the N.A. (see Fig. 1.11 and §1.7).

The bending moment about the N.A. is $M \cos \alpha_{\text{N.A.}}$ where $\alpha_{\text{N.A.}}$ is the angle between the N.A. and the axis *XX* about which the moment is applied.

The stress at *P* is now given by the simple bending formula

$$\sigma = \frac{M \cos \alpha_{\text{N.A.}}}{I_{\text{N.A.}}} n \quad (1.25)$$

the distance n being measured perpendicularly from the N.A. to the point *P* in question.

As for the procedure introduced in §1.7, this method has the advantage of immediate indication of the points of maximum stress once the N.A. has been drawn. The solution does, however, involve the use of principal moments of area which must be obtained by calculation or graphically using Mohr's or Land's circle.

1.11. Deflections

The deflections of unsymmetrical members in the directions of the principal axes may always be determined by application of the standard deflection formulae of §5.7.[†]

For example, the deflection at the free end of a cantilever carrying an end-point-load is

$$\frac{WL^3}{3EI}$$

With the appropriate value of I and the correct component of the load perpendicular to the principal axis used, the required deflection is obtained.

Thus $\delta_v = \frac{W_u L^3}{3EI_u}$ and $\delta_u = \frac{W_v L^3}{3EI_v}$ (1.26)

where W_u and W_v are the components of the load *perpendicular* to the U and V principal axes respectively.

The total resultant deflection is then given by combining the above values vectorially as shown in Fig. 1.12,

i.e. $\delta = \sqrt{(\delta_u^2 + \delta_v^2)}$

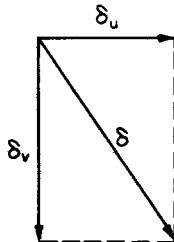


Fig. 1.12.

Alternatively, since bending always occurs about the N.A., the deflection equation can be written in the form

$$\delta = \frac{W' L^3}{3EI_{N.A.}} \quad (1.27)$$

where $I_{N.A.}$ is the second moment of area about the N.A. and W' is the component of the load perpendicular to the N.A. The value of $I_{N.A.}$ may be found either graphically using Mohr's circle or the momental ellipse, or by calculation using

$$I_{N.A.} = \frac{1}{2}[(I_u + I_v) + (I_u - I_v) \cos 2\alpha_u] \quad (1.28)$$

where α_u is the angle between the N.A. and the principal U axis.

[†] E.J. Hearn, *Mechanics of Materials I*, Butterworth-Heinemann, 1997.

Examples

Example 1.1

A rectangular-section beam 80 mm × 50 mm is arranged as a cantilever 1.3 m long and loaded at its free end with a load of 5 kN inclined at an angle of 30° to the vertical as shown in Fig. 1.13. Determine the position and magnitude of the greatest tensile stress in the section. What will be the vertical deflection at the end? $E = 210 \text{ GN/m}^2$.

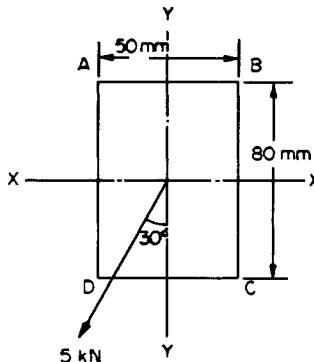


Fig. 1.13.

Solution

In the case of symmetrical sections such as this, subjected to skew loading, a solution is obtained by resolving the load into its components parallel to the two major axes and applying the bending theory simultaneously to both axes, i.e.

$$\sigma = \frac{M_{xx}y}{I_{xx}} \pm \frac{M_{yy}x}{I_{yy}}$$

Now the most highly stressed areas of the cantilever will be those at the built-in end where

$$M_{xx} = 5000 \cos 30^\circ \times 1.3 = 5629 \text{ Nm}$$

$$M_{yy} = 5000 \sin 30^\circ \times 1.3 = 3250 \text{ Nm}$$

The stresses on the short edges AB and DC resulting from bending about XX are then

$$\frac{M_{xx}}{I_{xx}}y = \frac{5629 \times 40 \times 10^{-3} \times 12}{50 \times 80^3 \times 10^{-12}} = 105.5 \text{ MN/m}^2$$

tensile on AB and compressive on DC .

The stresses on the long edges AD and BC resulting from bending about YY are

$$\frac{M_{yy}}{I_{yy}}x = \frac{3250 \times 25 \times 10^{-3} \times 12}{80 \times 50^3 \times 10^{-12}} = 97.5 \text{ MN/m}^2$$

tensile on BC and compressive on AD .

The maximum tensile stress will therefore occur at point B where the two tensile stresses add, i.e.

$$\text{maximum tensile stress} = 105.5 + 97.5 = 203 \text{ MN/m}^2$$

The deflection at the free end of the cantilever is then given by

$$\delta = \frac{WL^3}{3EI}$$

Therefore deflection vertically (i.e. along the YY axis) is

$$\delta_v = \frac{(W \cos 30^\circ)L^3}{3EI_{xx}} = \frac{5000 \times 0.866 \times 1.3^3 \times 12}{3 \times 210 \times 10^9 \times 50 \times 80^3 \times 10^{-12}}$$

$$= 0.0071 = 7.1 \text{ mm}$$

Example 1.2

A cantilever of length 1.2 m and of the cross section shown in Fig. 1.14 carries a vertical load of 10 kN at its outer end, the line of action being parallel with the longer leg and arranged to pass through the shear centre of the section (i.e. there is no twisting of the section, see §7.5[†]). Working from first principles, find the stress set up in the section at points A, B and C, given that the centroid is located as shown. Determine also the angle of inclination of the N.A.

$$I_{xx} = 4 \times 10^{-6} \text{ m}^4, \quad I_{yy} = 1.08 \times 10^{-6} \text{ m}^4$$

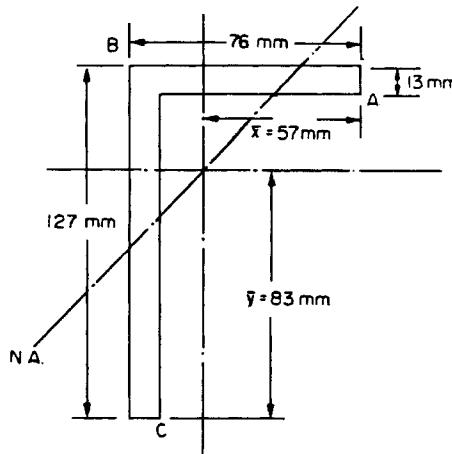


Fig. 1.14.

Solution

The product second moment of area of the section is given by eqn. (1.3).

$$I_{xy} = \Sigma A h k$$

$$= \{76 \times 13(\frac{1}{2} \times 76 - 19)(44 - \frac{1}{2} \times 13)$$

$$+ 114 \times 13[-(83 - \frac{1}{2} \times 114)][-(19 - \frac{1}{2} \times 13)]\}10^{-12}$$

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

$$= (0.704 + 0.482)10^{-6} = 1.186 \times 10^{-6} \text{ m}^4$$

From eqn. (1.20)

$$M_x = PI_{xy} + QI_{xx} = 10\,000 \times 1.2 = 12\,000$$

i.e.

$$1.186P + 4Q = 12\,000 \times 10^6 \quad (1)$$

Since the load is vertical there will be no moment about the Y axis and eqn. (1.21) gives

$$M_y = -PI_{yy} - QI_{xy} = 0$$

\therefore

$$-1.08P - 1.186Q = 0$$

\therefore

$$\frac{P}{Q} = -\frac{1.186}{1.08} = -1.098$$

But the angle of inclination of the N.A. is given by eqn. (1.22) as

$$\tan \alpha_{\text{N.A.}} = -\frac{P}{Q} = 1.098$$

i.e.

$$\alpha_{\text{N.A.}} = 47^\circ 41'$$

Substituting $P = -1.098Q$ in eqn. (1),

$$1.186(-1.098Q) + 4Q = 12\,000 \times 10^6$$

$$\therefore Q = \frac{12\,000 \times 10^6}{2.69} = 4460 \times 10^6$$

\therefore

$$P = -4897 \times 10^6$$

If the N.A. is drawn as shown in Fig. 1.14 at an angle of $47^\circ 41'$ to the XX axis through the centroid of the section, then this is the axis about which bending takes place. The points of maximum stress are then obtained by inspection as the points which are the maximum perpendicular distance from the N.A.

Thus B is the point of maximum tensile stress and C the point of maximum compressive stress.

Now from eqn (1.19) the stress at any point is given by

$$\sigma = Px + Qy$$

$$\therefore \text{stress at } A = -4897 \times 10^6(57 \times 10^{-3}) + 4460 \times 10^6(31 \times 10^{-3}) \\ = -141 \text{ MN/m}^2 \text{ (compressive)}$$

$$\text{stress at } B = -4897 \times 10^6(-19 \times 10^{-3}) + 4460 \times 10^6(44 \times 10^{-3}) \\ = 289 \text{ MN/m}^2 \text{ (tensile)}$$

$$\text{stress at } C = -4897 \times 10^6(-6 \times 10^{-3}) + 4460 \times 10^6(-83 \times 10^{-3}) \\ = -341 \text{ MN/m} \text{ (compressive)}$$

Example 1.3

(a) A horizontal cantilever 2 m long is constructed from the Z-section shown in Fig. 1.15. A load of 10 kN is applied to the end of the cantilever at an angle of 60° to the horizontal as

shown. Assuming that no twisting moment is applied to the section, determine the stresses at points A and B. ($I_{xx} \times 48.3 \times 10^{-6} \text{ m}^4$, $I_{yy} = 4.4 \times 10^{-6} \text{ m}^4$.)

(b) Determine the principal second moments of area of the section and hence, by applying the simple bending theory about each principal axis, check the answers obtained in part (a).

(c) What will be the deflection of the end of the cantilever? $E = 200 \text{ GN/m}^2$.

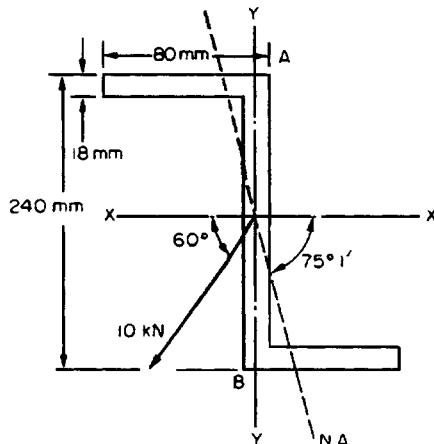


Fig. 1.15.

Solution

(a) For this section I_{xy} for the web is zero since its centroid lies on both axes and hence h and k are both zero. The contributions to I_{xy} of the other two portions will be negative since in both cases either h or k is negative.

$$\begin{aligned}\therefore I_{xy} &= -2(80 \times 18)(40 - 9)(120 - 9)10^{-12} \\ &= -9.91 \times 10^{-6} \text{ m}^4\end{aligned}$$

Now, at the built-in end,

$$M_x = +10000 \sin 60^\circ \times 2 = +17320 \text{ Nm}$$

$$M_y = -10000 \cos 60^\circ \times 2 = -10000 \text{ Nm}$$

Substituting in eqns. (1.20) and (1.21),

$$\begin{aligned}17320 &= PI_{xy} + QI_{xx} = (-9.91P + 48.3Q)10^{-6} \\ -10000 &= -PI_{yy} - QI_{xy} = (-4.4P + 9.91Q)10^{-6}\end{aligned}$$

$$\therefore 1.732 \times 10^{10} = -9.91P + 48.3Q \quad (1)$$

$$-1 \times 10^{10} = -4.4P + 9.91Q \quad (2)$$

$$(1) \times \frac{4.4}{9.91},$$

$$0.769 \times 10^{10} = -4.4P + 21.45Q \quad (3)$$

(3) – (2),

$$1.769 \times 10^{10} = 11.54Q$$

$$\therefore Q = 1533 \times 10^6$$

and substituting in (2) gives

$$P = 5725 \times 10^6$$

The inclination of the N.A. relative to the X axis is then given by

$$\tan \alpha_{\text{N.A.}} = -\frac{P}{Q} = -\frac{5725}{1533} = -3.735$$

$$\alpha_{\text{N.A.}} = -75^\circ 1'$$

This has been added to Fig. 1.15 and indicates that the points A and B are on either side of the N.A. and equidistant from it. Stresses at A and B are therefore of equal magnitude but opposite sign.

Now

$$\sigma = Px + Qy$$

$$\therefore \text{stress at A} = 5725 \times 10^6 \times 9 \times 10^{-3} + 1533 \times 10^6 \times 120 \times 10^{-3} \\ = 235 \text{ MN/m}^2 \text{ (tensile)}$$

Similarly,

$$\text{stress at B} = 235 \text{ MN/m}^2 \text{ (compressive)}$$

(b) The principal second moments of area may be found from Mohr's circle as shown in Fig. 1.16 or from eqns. (1.6) and (1.7),

$$\text{i.e. } I_u, I_v = \frac{1}{2}(I_{xx} + I_{yy}) \pm \frac{1}{2}(I_{xx} - I_{yy}) \sec 2\theta$$

$$\text{with } \tan 2\theta = \frac{2I_{xy}}{I_{yy} - I_{xx}} = \frac{-2 \times 9.91 \times 10^{-6}}{(4.4 - 48.3)10^{-6}} \\ = 0.451$$

$$\therefore 2\theta = 24^\circ 18', \theta = 12^\circ 9'$$

$$\therefore I_u, I_v = \frac{1}{2}[(48.3 + 4.4) \pm (48.3 - 4.4)1.0972]10^{-6} \\ = \frac{1}{2}[52.7 \pm 48.17]10^{-6}$$

$$\therefore I_u = 50.43 \times 10^{-6} \text{ m}^4$$

$$I_v = 2.27 \times 10^{-6} \text{ m}^4$$

The required stresses can now be obtained from eqn. (1.18).

$$\sigma = \frac{M_v u}{I_v} + \frac{M_u v}{I_u}$$

Now

$$M_u = 10000 \sin(60^\circ - 12^\circ 9') \times 2$$

$$= 10000 \sin 47^\circ 51' \times 2 = 14828 \text{ Nm}$$

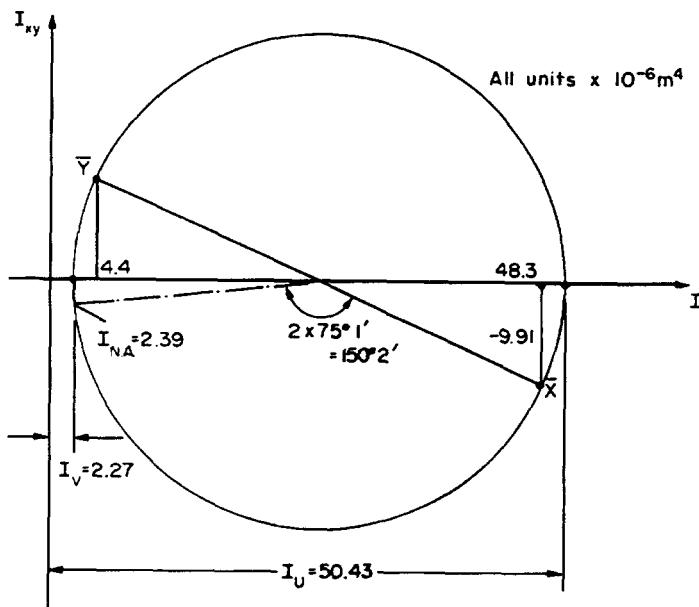


Fig. 1.16.

and

$$M_v = 10000 \cos 47^\circ 51' \times 2 = 13422 \text{ Nm}$$

and, for A ,

$$\begin{aligned} u &= x \cos \theta + y \sin \theta = (9 \times 0.9776) + (120 \times 0.2105) \\ &= 34.05 \text{ mm} \end{aligned}$$

$$\begin{aligned} v &= y \cos \theta - x \sin \theta = (120 \times 0.9776) - (9 \times 0.2105) \\ &= 115.4 \text{ mm} \end{aligned}$$

$$\begin{aligned} \therefore \sigma &= \frac{14828 \times 115.4 \times 10^{-3}}{50.43 \times 10^{-6}} + \frac{13422 \times 34.05 \times 10^{-3}}{2.27 \times 10^{-6}} \\ &= 235 \text{ MN/m}^2 \text{ as before.} \end{aligned}$$

(c) The deflection at the free end of a cantilever is given by

$$\delta = \frac{WL^3}{3EI}$$

Therefore component of deflection perpendicular to the V axis

$$\begin{aligned} \delta_v &= \frac{W_v L^3}{3EI_v} = \frac{10000 \cos 47^\circ 51' \times 2^3}{3 \times 200 \times 10^9 \times 2.27 \times 10^{-6}} \\ &= 39.4 \times 10^{-3} = 39.4 \text{ mm} \end{aligned}$$

and component of deflection perpendicular to the U axis

$$\delta_u = \frac{W_u L^3}{3EI_u} = \frac{10\,000 \sin 47^\circ 51' \times 2^3}{3 \times 200 \times 10^9 \times 50.43 \times 10^{-6}} \\ = 1.96 \times 10^{-3} = 1.96 \text{ mm}$$

The total deflection is then given by

$$= \sqrt{(\delta_u^2 + \delta_v^2)} = 10^{-3} \sqrt{(39.4^2 + 1.96^2)} = 39.45 \times 10^{-3} \\ = 39.45 \text{ mm}$$

Alternatively, since bending actually occurs about the N.A., the deflection can be found from

$$\delta = \frac{W_{\text{N.A.}} L^3}{3EI_{\text{N.A.}}}$$

its direction being normal to the N.A.

From Mohr's circle of Fig. 1.16, $I_{\text{N.A.}} = 2.39 \times 10^{-6} \text{ m}^4$

$$\therefore \delta = \frac{10\,000 \sin(30^\circ + 14^\circ 59') \times 2^3}{3 \times 200 \times 10^9 \times 2.39 \times 10^{-6}} = 39.44 \times 10^{-3} \\ = 39.44 \text{ mm}$$

Example 1.4

Check the answer obtained for the stress at point B on the angle section of Example 1.2 using the momental ellipse procedure.

Solution

The semi-axes of the momental ellipse are given by

$$k_u = \sqrt{\frac{I_u}{A}} \quad \text{and} \quad k_v = \sqrt{\frac{I_v}{A}}$$

The ellipse can then be constructed by setting off the above dimensions on the principal axes as shown in Fig. 1.17 (The inclination of the N.A. can be determined as in Example 1.2 or from eqn. (1.24).) The second moment of area of the section about the N.A. is then obtained from the momental ellipse as

$$I_{\text{N.A.}} = Ah^2$$

Thus for the angle section of Fig. 1.14

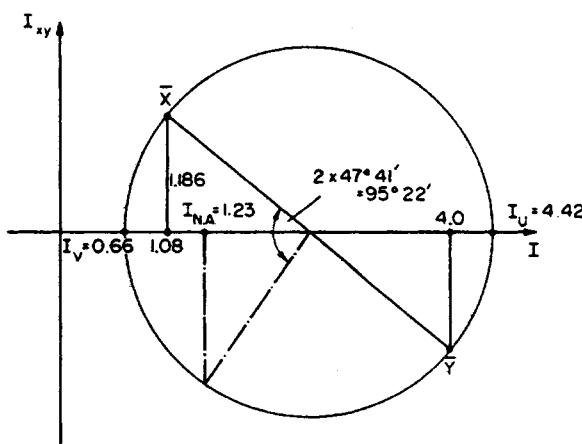
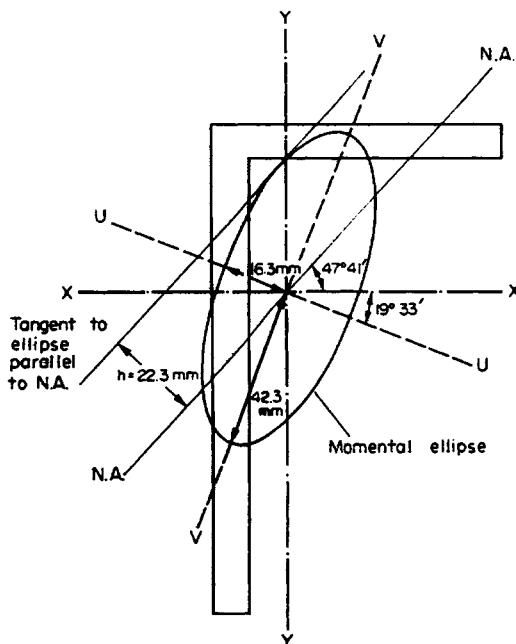
$$I_{xy} = 1.186 \times 10^{-6} \text{ m}^4, \quad I_{xx} = 4 \times 10^{-6} \text{ m}^4, \quad I_{yy} = 1.08 \times 10^{-6} \text{ m}^4$$

The principal second moments of area are then given by Mohr's circle of Fig. 1.18 or from the equation

$$I_u, I_v = \frac{1}{2}[(I_{xx} + I_{yy}) \pm (I_{xx} - I_{yy}) \sec 2\theta]$$

where

$$\tan 2\theta = \frac{2I_{xy}}{(I_{yy} - I_{xx})} = \frac{2 \times 1.186 \times 10^{-6}}{(1.08 - 4)10^{-6}} = -0.8123$$



$$\therefore 2\theta = -39^\circ 5', \quad \theta = -19^\circ 33'$$

and

$$\sec 2\theta = -1.2883$$

$$\begin{aligned} \therefore I_u, I_v &= \frac{1}{2}[(4 + 1.08) \pm (4 - 1.08)(-1.2883)]10^{-6} \\ &= \frac{1}{2}[5.08 \pm 3.762]10^{-6} \end{aligned}$$

$$I_u = 4.421 \times 10^{-6}, I_v = 0.659 \times 10^{-6} \text{ m}^4$$

and

$$A = [(76 \times 13) + (114 \times 13)]10^{-6} = 2.47 \times 10^{-3} \text{ m}^2$$

$$\therefore k_u = \sqrt{\left(\frac{4.421 \times 10^{-6}}{2.47 \times 10^{-3}}\right)} = 0.0423 = 42.3 \text{ mm}$$

$$k_v = \sqrt{\left(\frac{0.659 \times 10^{-6}}{2.47 \times 10^{-3}}\right)} = 0.0163 = 16.3 \text{ mm}$$

The momental ellipse can now be constructed as described above and drawn in Fig. 1.17 and by measurement

$$h = 22.3 \text{ mm}$$

Then

$$I_{N.A.} = Ah^2 = 2.47 \times 10^{-3} \times 22.3^2 \times 10^{-6}$$

$$= 1.23 \times 10^{-6} \text{ m}^4$$

(This value may also be obtained from Mohr's circle of Fig. 1.18.)

The stress at *B* is then given by

$$\sigma = \frac{M_{N.A.} n}{I_{N.A.}}$$

where

$$n = \text{perpendicular distance from } B \text{ to the N.A.}$$

$$= 44 \text{ mm}$$

and

$$M_{N.A.} = 10000 \cos 47^\circ 41' \times 1.2 = 8079 \text{ Nm}$$

$$\therefore \text{stress at } B = \frac{8079 \times 44 \times 10^{-3}}{1.23 \times 10^{-6}} = 289 \text{ MN/m}^2$$

This confirms the result obtained with the alternative procedure of Example 1.2.

Problems

1.1 (B). A rectangular-sectioned beam of 75 mm × 50 mm cross-section is used as a simply supported beam and carries a uniformly distributed load of 500 N/m over a span of 3 m. The beam is supported in such a way that its long edges are inclined at 20° to the vertical. Determine:

- (a) the maximum stress set up in the cross-section;
- (b) the vertical deflection at mid-span.

E = 208 GN/m².

[17.4 MN/m²; 1.76 mm.]

1.2 (B). An I-section girder 1.3 m long is rigidly built in at one end and loaded at the other with a load of 1.5 kN inclined at 30° to the web. If the load passes through the centroid of the section and the girder dimensions are: flanges 100 mm × 20 mm, web 200 mm × 12 mm, determine the maximum stress set up in the cross-section. How does this compare with the maximum stress set up if the load is vertical?

[18.1, 4.14 MN/m².]

1.3 (B). A 75 mm × 75 mm × 12 mm angle is used as a cantilever with the face *AB* horizontal, as shown in Fig. 1.19. A vertical load of 3 kN is applied at the tip of the cantilever which is 1 m long. Determine the stress at *A*, *B* and *C*.

[196.37, -207 MN/m².]

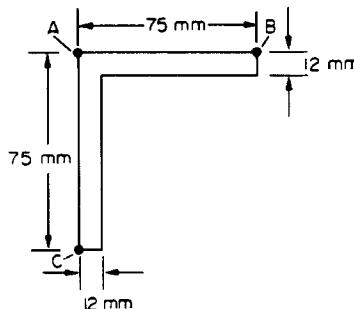


Fig. 1.19.

1.4 (B). A cantilever of length 2 m is constructed from 150 mm × 100 mm by 12 mm angle and arranged with its 150 mm leg vertical. If a vertical load of 5 kN is applied at the free end, passing through the shear centre of the section, determine the maximum tensile and compressive stresses set up across the section.

[B.P.] [169, - 204 MN/m².]

1.5 (B). A 180 mm × 130 mm × 13 mm unequal angle section is arranged with the long leg vertical and simply supported over a span of 4 m. Determine the maximum central load which the beam can carry if the maximum stress in the section is limited to 90 MN/m². Determine also the angle of inclination of the neutral axis.

$$I_{xx} = 12.8 \times 10^{-6} \text{ m}^4, I_{yy} = 5.7 \times 10^{-6} \text{ m}^4.$$

What will be the vertical deflection of the beam at mid-span? $E = 210 \text{ GN/m}^2$. [8.73 kN, 41.6°, 7.74 mm.]

1.6 (B). The unequal-leg angle section shown in Fig. 1.20 is used as a cantilever with the 130 mm leg vertical. The length of the cantilever is 1.3 m. A vertical point load of 4.5 kN is applied at the free end, its line of action passing through the shear centre.

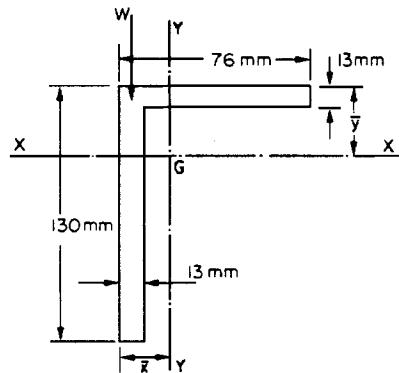


Fig. 1.20.

The properties of the section are as follows:

$$\bar{x} = 19 \text{ mm}, \bar{y} = 45 \text{ mm}, I_{xx} = 4 \times 10^{-6} \text{ m}^4, I_{yy} = 1.1 \times 10^{-6} \text{ m}^4, I_{xy} = 1.2 \times 10^{-6} \text{ m}^4.$$

Determine:

- the magnitude of the principal second moments of area together with the inclination of their axes relative to XX;
- the position of the neutral plane ($N-N$) and the magnitude of I_{NN} ;
- the end deflection of the centroid G in magnitude, direction and sense.

Take $E = 207 \text{ GN/m}^2$ (2.07 Mbar).

[$444 \times 10^{-8} \text{ m}^4$, $66 \times 10^{-8} \text{ m}^4$, $-19^\circ 51'$ to XX , $47^\circ 42'$ to XX , $121 \times 10^{-8} \text{ m}^4$, 8.85 mm at $-42^\circ 18'$ to XX .]

1.7 (B). An extruded aluminium alloy section having the cross-section shown in Fig. 1.21 will be used as a cantilever as indicated and loaded with a single concentrated load at the free end. This load F acts in the plane of the cross-section but may have any orientation within the cross-section. Given that $I_{xx} = 101.2 \times 10^{-8} \text{ m}^4$ and $I_{yy} = 29.2 \times 10^{-8} \text{ m}^4$:

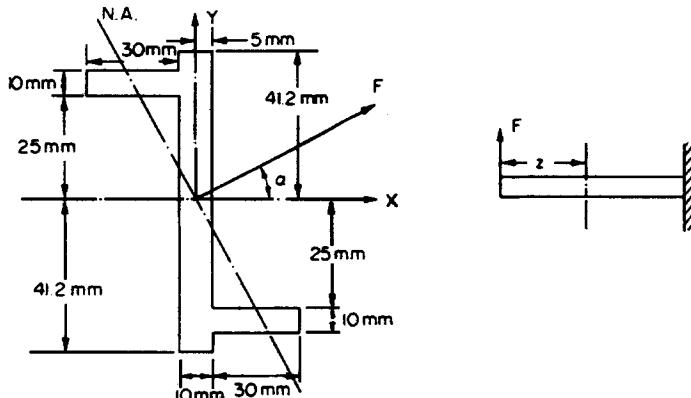


Fig. 1.21.

- (a) determine the values of the principal second moments of area and the orientation of the principal axes;
- (b) for such a case that the neutral axis is orientated at -45° to the X -axis, as shown, find the angle α of the line of action of F to the X -axis and hence determine the numerical constant K in the expression $\sigma = KFz$, which expresses the magnitude of the greatest bending stress at any distance z from the free end.

[City U.] [116.1×10^{-8} , 14.3×10^{-8} , 22.5° , -84° , 0.71×10^5 .]

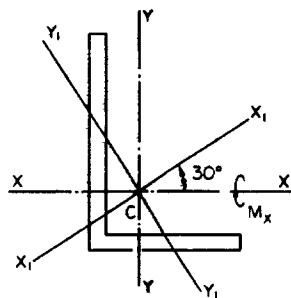


Fig. 1.22.

1.8 (B). A beam of length 2 m has the unequal-leg angle section shown in Fig. 1.22 for which $I_{xx} = 0.8 \times 10^{-6} \text{ m}^4$, $I_{yy} = 0.4 \times 10^{-6} \text{ m}^4$ and the angle between $X - X$ and the principal second moment of area axis $X_1 - X_1$ is 30° . The beam is subjected to a constant bending moment (M_x) of magnitude 1000 Nm about the $X - X$ axis as shown.

Determine:

- (a) the values of the principal second moments of area I_{x_1} and I_{y_1} respectively;
- (b) the inclination of the N.A., or line of zero stress ($N - N$) relative to the axis $X_1 - X_1$ and the value of the second moment of area of the section about $N - N$, that is I_N ;

(c) the magnitude, direction and sense of the resultant maximum deflection of the centroid C .

For the beam material, Young's modulus $E = 200 \text{ GN/m}^2$. For a beam subjected to a constant bending moment M , the maximum deflection δ is given by the formula

$$\delta = \frac{ML^2}{8EI}$$

[1×10^{-6} , $0.2 \times 10^{-6} \text{ m}^4$, $-70^\circ 54'$ to X_1X_1 , $0.2847 \times 10^{-6} \text{ m}^4$, 6.62 mm, 90° to N.A.]

CHAPTER 2

STRUTS

Summary

The allowable stresses and end loads given by Euler's theory for struts with varying end conditions are given in Table 2.1.

Table 2.1.

End condition	Fixed-free	Pinned-pinned (or rounded)	Fixed-pinned	Fixed-fixed
Euler load P_e	$\frac{\pi^2 EI}{4L^2}$	$\frac{\pi^2 EI}{L^2}$	$\frac{2\pi^2 EI}{L^2}$	$\frac{4\pi^2 EI}{L^2}$
	or, writing $I = Ak^2$, where k = radius of gyration			
	$\frac{\pi^2 EA}{4(L/k)^2}$	$\frac{\pi^2 EA}{(L/k)^2}$	$\frac{2\pi^2 EA}{(L/k)^2}$	$\frac{4\pi^2 EA}{(L/k)^2}$
Euler stress σ_e	$\frac{\pi^2 E}{4(L/k)^2}$	$\frac{\pi^2 E}{(L/k)^2}$	$\frac{2\pi^2 E}{(L/k)^2}$	$\frac{4\pi^2 E}{(L/k)^2}$

Here L is the length of the strut and the term L/k is known as the *slenderness ratio*.

Validity limit for Euler formulae

$$L/k = \sqrt{\left(\frac{C\pi^2 E}{\sigma_y}\right)}$$

where C is a constant depending on the end condition of the strut.

Rankine-Gordon Formula

$$\sigma = \frac{\sigma_y}{1 + a(L/k)^2}$$

where $a = (\sigma_y/\pi^2 E)$ theoretically but is usually found by experiment. Typical values are given in Table 2.2.

Table 2.2.

Material	Compressive yield stress (MN/m ²)	a	
		Pinned ends	Fixed ends
Mild steel	315	1/7500	1/30 000
Cast iron	540	1/1600	1/64 000
Timber	35	1/3000	1/12 000

N.B. The value of a for pinned ends is always four times that for fixed ends

Perry–Robertson Formula

$$N\sigma = \frac{[\sigma_y + (\eta + 1)\sigma_e]}{2} - \sqrt{\left\{ \left[\frac{\sigma_y + (\eta + 1)\sigma_e}{2} \right]^2 - \sigma_y \sigma_e \right\}}$$

where η is a constant depending on the material.

For a brittle material

$$\eta = 0.015L/k$$

For a ductile material

$$\eta = 0.3 \left(\frac{L}{100k} \right)^2$$

These values will be modified for eccentric loading conditions. The Perry–Robertson formula is the basis of BS 449 as shown in §2.7.

Struts with initial curvature

$$\text{Maximum deflection } \delta_{\max} = \left[\frac{P_e}{(P_e - P)} \right] C_0$$

$$\text{Maximum stress } \sigma_{\max} = \frac{P}{A} \pm \left[\frac{PP_e}{(P_e - P)} \right] \frac{C_0 h}{I}$$

where C_0 is the initial central deflection and h is the distance of the highest strained fibre from the neutral axis (N.A.).

Smith–Southwell formula for eccentrically loaded struts

With pinned ends the maximum stress reached in the strut is given by

$$\sigma_{\max} = \sigma \left[1 + \frac{eh}{k^2} \sec \frac{L}{2} \sqrt{\left(\frac{\sigma}{Ek^2} \right)} \right]$$

or

$$\sigma_{\max} = \sigma \left[1 + \frac{eh}{k^2} \sec \frac{1}{2}\pi \sqrt{\left(\frac{\sigma}{\sigma_E} \right)} \right]$$

where e is the eccentricity of loading, h is the distance of the highest strained fibre from the N.A., k is the minimum radius of gyration of the cross-section, and σ is the applied load/cross-sectional area.

Since the required allowable stress σ cannot be obtained directly from this equation a solution is obtained graphically or by trial and error.

With other end conditions the value L in the above formula should be replaced by the appropriate *equivalent strut* length (see §2.2).

Webb's approximation for the Smith–Southwell formula

$$\sigma_{\max} = \frac{P}{A} \left[1 + \frac{eh}{k^2} \left(\frac{P_e + 0.26P}{P_e - P} \right) \right]$$

Laterally loaded struts

(a) *Central concentrated load*

$$\text{Maximum deflection} = \frac{W}{2nP} \left[\tan \frac{nL}{2} - \frac{nL}{2} \right]$$

$$\text{maximum bending moment (B.M.)} = \frac{W}{2n} \tan \frac{nL}{2}$$

(b) *Uniformly distributed load*

$$\text{Maximum deflection} = \frac{w}{n^2 P} \left[\left(\sec \frac{nL}{2} - 1 \right) - \frac{n^2 L^2}{8} \right]$$

$$\text{maximum B.M.} = \frac{w}{n^2} \left(\sec \frac{nL}{2} - 1 \right)$$

Introduction

Structural members which carry compressive loads may be divided into two broad categories depending on their relative lengths and cross-sectional dimensions. Short, thick members are generally termed *columns* and these usually fail by *crushing* when the yield stress of the material in compression is exceeded. Long, slender columns or *struts*, however, fail by *buckling* some time before the yield stress in compression is reached. The buckling occurs owing to one or more of the following reasons:

- (a) the strut may not be perfectly straight initially;
- (b) the load may not be applied exactly along the axis of the strut;
- (c) one part of the material may yield in compression more readily than others owing to some lack of uniformity in the material properties throughout the strut.

At values of load below the buckling load a strut will be in stable equilibrium where the displacement caused by any lateral disturbance will be totally recovered when the disturbance is removed. At the buckling load the strut is said to be in a state of neutral equilibrium, and theoretically it should then be possible to gently deflect the strut into a simple sine wave provided that the amplitude of the wave is kept small. This can be demonstrated quite simply using long thin strips of metal, e.g. a metal rule, and gentle application of compressive loads.

Theoretically, it is possible for struts to achieve a condition of unstable equilibrium with loads exceeding the buckling load, any slight lateral disturbance then causing failure by buckling; this condition is never achieved in practice under static load conditions. Buckling occurs immediately at the point where the buckling load is reached owing to the reasons stated earlier.

The above comments and the contents of this chapter refer to the *elastic* stability of struts only. It must also be remembered that struts can also fail plastically, and in this case the failure is irreversible.

2.1. Euler's theory

(a) Strut with pinned ends

Consider the axially loaded strut shown in Fig. 2.1 subjected to the crippling load P_e producing a deflection y at a distance x from one end. Assume that the ends are either pin-jointed or rounded so that there is no moment at either end.

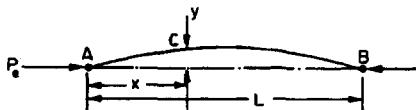


Fig. 2.1. Strut with axial load and pinned ends.

$$\text{B.M. at } C = EI \frac{d^2y}{dx^2} = -P_e y$$

$$EI \frac{d^2y}{dx^2} + P_e y = 0$$

$$\frac{d^2y}{dx^2} + \frac{P_e}{EI} y = 0$$

i.e. in operator form, with $D \equiv d/dx$,

$$(D^2 + n^2)y = 0, \quad \text{where } n^2 = P_e/EI$$

This is a second-order differential equation which has a solution of the form

$$y = A \cos nx + B \sin nx$$

$$\text{i.e. } y = A \cos \sqrt{\left(\frac{P_e}{EI}\right)}x + B \sin \sqrt{\left(\frac{P_e}{EI}\right)}x$$

$$\text{Now at } x = 0, y = 0 \quad \therefore A = 0$$

$$\text{and at } x = L, y = 0 \quad \therefore B \sin L\sqrt{\left(\frac{P_e}{EI}\right)} = 0$$

$$\therefore \text{either } B = 0 \text{ or } \sin L\sqrt{\left(\frac{P_e}{EI}\right)} = 0$$

If $B = 0$ then $y = 0$ and the strut has not yet buckled. Thus the solution required is

$$\sin L\sqrt{\left(\frac{P_e}{EI}\right)} = 0 \quad \therefore L\sqrt{\left(\frac{P_e}{EI}\right)} = \pi$$

$$\therefore P_e = \frac{\pi^2 EI}{L^2} \quad (2.1)$$

It should be noted that other solutions exist for the equation

$$\sin L \sqrt{\left(\frac{P}{EI}\right)} = 0 \quad \text{i.e.} \quad \sin nL = 0$$

The solution chosen of $nL = \pi$ is just one particular solution; the solutions $nL = 2\pi$, 3π , 5π , etc., are equally as valid mathematically and they do, in fact, produce values of P_e which are equally valid for modes of buckling of the strut different from that of the simple bow of Fig. 2.1. Theoretically, therefore, there are an infinite number of values of P_e , each corresponding with a different mode of buckling. The value selected above is the so-called *fundamental* mode value and is the lowest critical load producing the single-bow buckling condition. The solution $nL = 2\pi$ produces buckling in two half-waves, 3π in three half-waves, etc., as shown in Fig. 2.2. If load is applied sufficiently quickly to the strut, it is possible to pass through the fundamental mode and to achieve at least one of the other modes which are theoretically possible. In practical loading situations, however, this is rarely achieved since the high stress associated with the first critical condition generally ensures immediate collapse. The buckling load of a strut with pinned ends is, therefore, for all practical purposes, given by eqn. (2.1).

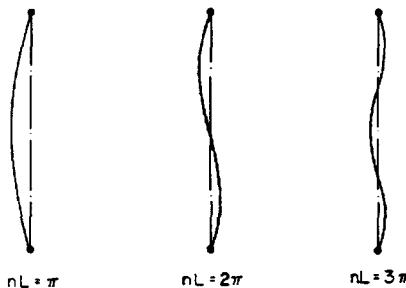


Fig. 2.2. Strut failure modes.

(b) One end fixed, the other free

Consider now the strut of Fig. 2.3 with the origin at the fixed end.

$$\text{B.M. at } C = EI \frac{d^2y}{dx^2} = +P(a - y)$$

$$\therefore \frac{d^2y}{dx^2} + \frac{Py}{EI} = \frac{Pa}{EI}$$

$$\therefore (D^2 + n^2)y = n^2a \quad (2.2)$$



Fig. 2.3. Fixed-free strut.

N.B.—It is always convenient to arrange the diagram and origin such that the differential equation is achieved in the above form since the solution will then always be of the form

$$y = A \cos nx + B \sin nx + (\text{particular solution})$$

The *particular solution* is a particular value of y which satisfies eqn. (2.2), and in this case can be shown to be $y = a$.

$$\therefore y = A \cos nx + B \sin nx + a$$

Now when $x = 0$, $y = 0$

$$A = -a$$

when $x = 0$, $dy/dx = 0$

$$B = 0$$

$$\therefore y = -a \cos nx + a$$

But when $x = L$, $y = a$

$$a = -a \cos nL + a$$

$$0 = \cos nL$$

The fundamental mode of buckling in this case therefore is given when $nL = \frac{1}{2}\pi$.

$$\therefore L \sqrt{\left(\frac{P}{EI}\right)} = \frac{\pi}{2}$$

$$\text{or } P_e = \frac{\pi^2 EI}{4L^2} \quad (2.3)$$

(c) Fixed ends

Consider the strut of Fig. 2.4 with the origin at the centre.

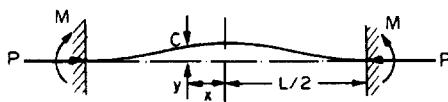


Fig. 2.4. Strut with fixed ends.

In this case the B.M. at C is given by

$$EI \frac{d^2 y}{dx^2} = M - Py$$

$$\frac{d^2 y}{dx^2} + \frac{P}{EI} y = \frac{M}{EI}$$

$$(D^2 + n^2)y = M/EI$$

Here the particular solution is

$$y = \frac{M}{n^2 EI} = \frac{M}{P}$$

$$\therefore y = A \cos nx + B \sin nx + M/P$$

Now when $x = 0$, $dy/dx = 0 \therefore B = 0$

and when $x = \frac{1}{2}L$, $y = 0 \quad \therefore A = \frac{-M}{P} \sec \frac{nL}{2}$

$$\therefore y = -\frac{M}{P} \sec \frac{nL}{2} \cos nx + \frac{M}{P}$$

But when $x = \frac{1}{2}L$, dy/dx is also zero,

$$\therefore 0 = \frac{nM}{P} \sec \frac{nL}{2} \sin \frac{nL}{2}$$

$$0 = \frac{nM}{P} \tan \frac{nL}{2}$$

The fundamental buckling mode is then given when $nL/2 = \pi$

$$\therefore \frac{L}{2} \sqrt{\left(\frac{P}{EI}\right)} = \pi$$

or $P_e = \frac{4\pi^2 EI}{L^2}$ (2.4)

(d) One end fixed, the other pinned

In order to maintain the pin-joint on the horizontal axis of the unloaded strut, it is necessary in this case to introduce a vertical load F at the pin (Fig. 2.5). The moment of F about the built-in end then balances the fixing moment.

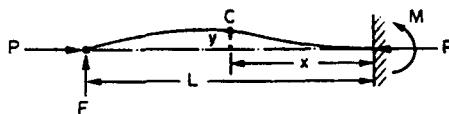


Fig. 2.5. Strut with one end pinned, the other fixed.

With the origin at the built-in end the B.M. at C is

$$EI \frac{d^2y}{dx^2} = -Py + F(L-x)$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{F}{EI}(L-x)$$

$$(D^2 + n^2)y = \frac{F}{EI}(L-x)$$

The particular solution is

$$y = \frac{F}{n^2 EI} (L - x) = \frac{F}{P} (L - x)$$

The full solution is therefore

$$y = A \cos nx + B \sin nx + \frac{F}{P} (L - x)$$

$$\text{When } x = 0, y = 0, \quad \therefore A = -\frac{FL}{P}$$

$$\text{When } x = 0, dy/dx = 0, \quad \therefore B = \frac{F}{nP}$$

$$\begin{aligned} y &= -\frac{FL}{P} \cos nx + \frac{F}{nP} \sin nx + \frac{F}{P} (L - x) \\ &= \frac{F}{nP} [-nL \cos nx + \sin nx + n(L - x)] \end{aligned}$$

But when $x = L$, $y = 0$

$$\begin{aligned} \therefore \quad nL \cos nL &= \sin nL \\ \tan nL &= nL \end{aligned}$$

The lowest value of nL (neglecting zero) which satisfies this condition and which therefore produces the fundamental buckling condition is $nL = 4.5$ radians.

$$\therefore L \sqrt{\left(\frac{P}{EI}\right)} = 4.5$$

$$\text{or} \quad P_e = \frac{20.25EI}{L^2} \quad (2.5)$$

or, approximately

$$P_e = \frac{2\pi^2 EI}{L^2} \quad (2.6)$$

2.2. Equivalent strut length

Having derived the result for the buckling load of a strut with pinned ends the Euler loads for other end conditions may all be written in the same form,

$$\text{i.e.} \quad P_e = \frac{\pi^2 EI}{l^2} \quad (2.7)$$

where l is the *equivalent length* of the strut and can be related to the actual length of the strut depending on the end conditions. The equivalent length is found to be the length of a simple bow (half sine-wave) in each of the strut deflection curves shown in Fig. 2.6. The buckling load for each end condition shown is then readily obtained.

The use of the equivalent length is not restricted to the Euler theory and it will be used in other derivations later.

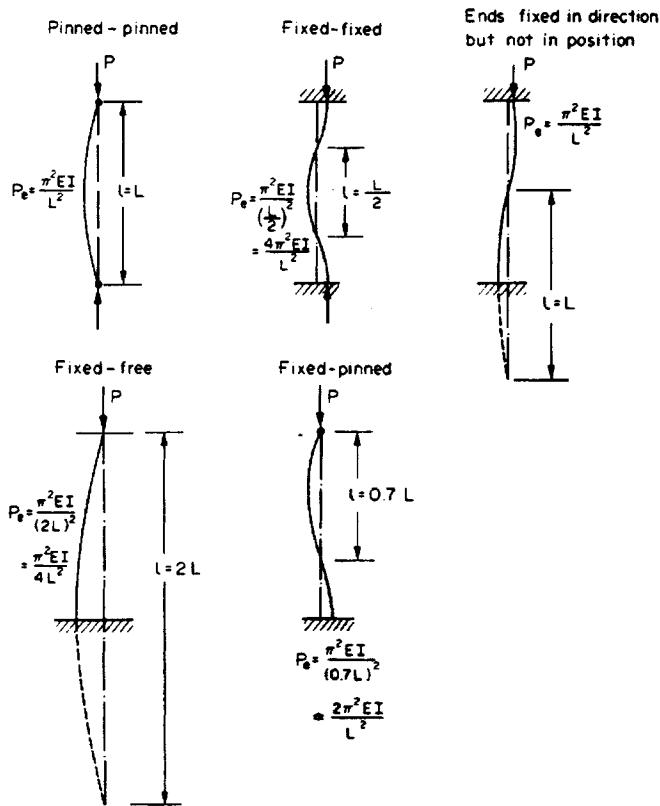


Fig. 2.6. "Equivalent length" of struts with different end conditions. In each case l is the length of a single bow.

2.3. Comparison of Euler theory with experimental results (see Fig. 2.7)

Between $L/k = 40$ and $L/k = 100$ neither the Euler results nor the yield stress are close to the experimental values, each suggesting a critical load which is in excess of that which is actually required for failure—a very unsafe situation! Other formulae have therefore been derived to attempt to obtain closer agreement between the actual failing load and the predicted value in this particular range of slenderness ratio.

(a) Straight-line formula

$$P = \sigma_y A [1 - n(L/k)] \quad (2.8)$$

the value of n depending on the material used and the end condition.

(b) Johnson parabolic formula

$$P = \sigma_y A [1 - b(L/k)^2] \quad (2.9)$$

the value of b depending also on the end condition.

Neither of the above formulae proved to be very successful, and they were replaced by:

(c) Rankine-Gordon formula

$$\frac{1}{P_R} = \frac{1}{P_e} + \frac{1}{P_c} \quad (2.10)$$

where P_e is the Euler buckling load and P_c is the crushing (compressive yield) load = $\sigma_y A$. This formula has been widely used and is discussed fully in §2.5.

2.4. Euler "validity limit"

From the graph of Fig. 2.7 and the comments above, it is evident that the Euler theory is unsafe for small L/k ratios. It is useful, therefore, to determine the limiting value of L/k below which the Euler theory should not be applied; this is termed the *validity limit*.

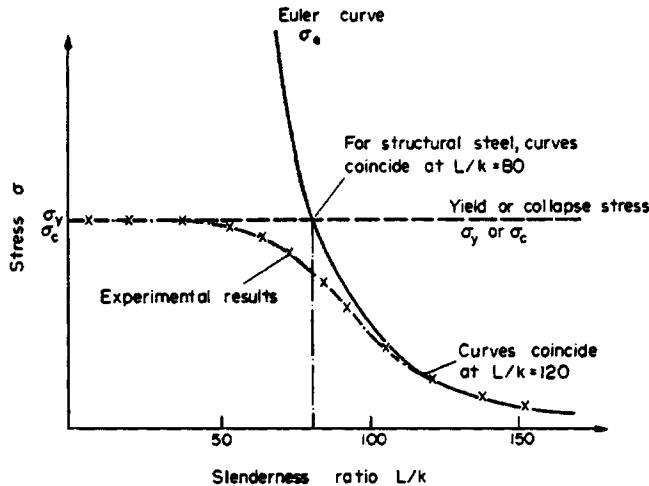


Fig. 2.7. Comparison of experimental results with Euler curve.

The validity limit is taken to be the point where the Euler σ_e equals the yield or crushing stress σ_y , i.e. the point where the strut load

$$P = \sigma_y A$$

Now the Euler load can be written in the form

$$P_e = C \frac{\pi^2 EI}{L^2} = C \frac{\pi^2 E A k^2}{L^2}$$

where C is a constant depending on the end condition of the strut.

Therefore in the limiting condition

$$\sigma_y A = C \frac{\pi^2 E A k^2}{L^2}$$

$$\frac{L}{k} = \sqrt{\left(\frac{C\pi^2 E}{\sigma_y}\right)}$$

The value of this expression will vary with the type of end condition; as an example, low carbon steel struts with pinned ends give $L/k \approx 80$.

2.5. Rankine or Rankine–Gordon formula

As stated above, the Rankine formula is a combination of the Euler and crushing loads for a strut

$$\frac{1}{P_R} = \frac{1}{P_e} + \frac{1}{P_c}$$

For very short struts P_e is very large; $1/P_e$ can therefore be neglected and $P_R = P_c$. For very long struts P_e is very small and $1/P_e$ is very large so that $1/P_c$ can be neglected. Thus $P_R = P_e$.

The Rankine formula is therefore valid for extreme values of L/k . It is also found to be fairly accurate for the intermediate values in the range under consideration. Thus, re-writing the formula in terms of stresses,

$$\frac{1}{\sigma A} = \frac{1}{\sigma_e A} + \frac{1}{\sigma_y A}$$

i.e.

$$\frac{1}{\sigma} = \frac{1}{\sigma_e} + \frac{1}{\sigma_y} = \frac{\sigma_e + \sigma_y}{\sigma_e \sigma_y}$$

$$\sigma = \frac{\sigma_e \sigma_y}{\sigma_e + \sigma_y} = \frac{\sigma_y}{[1 + (\sigma_y/\sigma_e)]}$$

For a strut with both ends pinned

$$\sigma_e = \frac{\pi^2 E}{(L/k)^2}$$

∴

$$\sigma = \frac{\sigma_y}{1 + \frac{\sigma_y}{\pi^2 E} \left(\frac{L}{k}\right)^2}$$

i.e. Rankine stress

$$\sigma_R = \frac{\sigma_y}{1 + a(L/k)^2} \quad (2.11)$$

where $a = \sigma_y/\pi^2 E$, theoretically, but having a value normally found by experiment for various materials. This will take into account other types of end condition.

Therefore Rankine load

$$P_R = \frac{\sigma_y A}{1 + a(L/k)^2} \quad (2.12)$$

Typical values of a for use in the Rankine formula are given in Table 2.3.

However, since the values of a are not exactly equal to the theoretical values, the Rankine loads for long struts will not be identical to those estimated by the Euler theory as suggested earlier.

Table 2.3.

Material	σ_y or σ_c (MN/m ²)	a	
		Pinned ends	Fixed ends
Low carbon steel	315	1/7500	1/30 000
Cast iron	540	1/1600	1/64 000
Timber	35	1/3000	1/12 000

N.B. a for pinned ends = $4 \times (a$ for fixed ends)

2.6. Perry–Robertson formula

The Perry–Robertson proof is based on the assumption that any imperfections in the strut, through faulty workmanship or material or eccentricity of loading, can be allowed for by giving the strut an initial curvature. For ease of calculation this is assumed to be a cosine curve, although the actual shape assumed has very little effect on the result.

Consider, therefore, the strut AB of Fig. 2.8, of length L and pin-jointed at the ends. The initial curvature y_0 at any distance x from the centre is then given by

$$y_0 = C_0 \cos \frac{\pi x}{L}$$

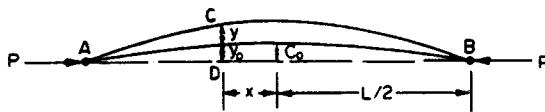


Fig. 2.8. Strut with initial curvature.

If a load P is now applied at the ends, this deflection will be increased to $y + y_0$.

$$\therefore BM_c = EI \frac{d^2 y}{dx^2} = -P \left(y + C_0 \cos \frac{\pi x}{L} \right)$$

$$\therefore \frac{d^2 y}{dx^2} + \frac{P}{EI} \left(y + C_0 \cos \frac{\pi x}{L} \right) = 0$$

the solution of which is

$$y = A \sin \sqrt{\left(\frac{P}{EI}\right)} x + B \cos \sqrt{\left(\frac{P}{EI}\right)} x + \left[\left(\frac{PC_0}{EI} \cos \frac{\pi x}{L} \right) / \left(\frac{\pi^2}{L^2} - \frac{P}{EI} \right) \right]$$

where A and B are the constants of integration.

Now when $x = \pm L/2$, $y = 0$

$$\therefore A = B = 0$$

$$\therefore y = \left[\left(\frac{PC_0}{EI} \cos \frac{\pi x}{L} \right) / \left(\frac{\pi^2}{L^2} - \frac{P}{EI} \right) \right] = \left[\left(PC_0 \cos \frac{\pi x}{L} \right) / \left(\frac{\pi^2 EI}{L^2} - P \right) \right]$$

Therefore dividing through, top and bottom, by A ,

$$y = \left[\left(\frac{P}{A} C_0 \cos \frac{\pi x}{L} \right) \Big/ \left(\frac{\pi^2 EI}{L^2 A} - \frac{P}{A} \right) \right]$$

But $P/A = \sigma$ and $(\pi^2 EI)/(L^2 A) = \sigma_e$ (the Euler stress for pin-ended struts)

$$\therefore y = \frac{\sigma}{(\sigma_e - \sigma)} C_0 \cos \frac{\pi x}{L}$$

Therefore total deflection at any point is given by

$$\begin{aligned} y + y_0 &= \left[\frac{\sigma}{(\sigma_e - \sigma)} \right] C_0 \cos \frac{\pi x}{L} + C_0 \cos \frac{\pi x}{L} \\ &= \left[\frac{\sigma_e}{(\sigma_e - \sigma)} \right] C_0 \cos \frac{\pi x}{L} \end{aligned} \quad (2.13)$$

$$\therefore \text{Maximum deflection (when } x = 0) = \left[\frac{\sigma_e}{(\sigma_e - \sigma)} \right] C_0 \quad (2.14)$$

$$\therefore \text{maximum B.M.} = P \left[\frac{\sigma_e}{(\sigma_e - \sigma)} \right] C_0 \quad (2.15)$$

$$\therefore \text{maximum stress owing to bending} = \frac{My}{I} = \frac{P}{I} \left[\frac{\sigma_e}{(\sigma_e - \sigma)} \right] C_0 h$$

where h is the distance of the outside fibre from the N.A. of the strut.

Therefore the maximum stress owing to combined bending and thrust is given by

$$\begin{aligned} \sigma_{\max} &= \frac{P}{I} \left[\frac{\sigma_e}{(\sigma_e - \sigma)} \right] C_0 h + \frac{P}{A} \\ &= \frac{P}{Ak^2} \left[\frac{\sigma_e}{(\sigma_e - \sigma)} \right] C_0 h + \frac{P}{A} \\ &= \sigma \left[\frac{\eta \sigma_e}{(\sigma_e - \sigma)} + 1 \right] \text{ where } \eta = \frac{C_0 h}{k^2} \end{aligned} \quad (2.16)$$

If $\sigma_{\max} = \sigma_y$, the compressive yield stress for the material of the strut, the above equation when solved for σ gives

$$\sigma = \frac{[\sigma_y + (\eta + 1)\sigma_e]}{2} - \sqrt{\left\{ \left[\frac{\sigma_y + (\eta + 1)\sigma_e}{2} \right]^2 - \sigma_y \sigma_e \right\}} \quad (2.17)$$

This is the Perry–Robertson formula required. If the material is brittle, however, and failure is likely to occur in tension, then the sign between the two square-bracketed terms becomes positive and σ_y is the *tensile* yield strength.

2.7. British Standard procedure (BS 449)

With a *load factor* N applied, the Perry–Robertson equation becomes

$$N\sigma = \frac{[\sigma_y + (\eta + 1)\sigma_e]}{2} - \sqrt{\left\{ \left[\frac{\sigma_y + (\eta + 1)\sigma_e}{2} \right]^2 - \sigma_y \sigma_e \right\}} \quad (2.18)$$

With values for steel of $\sigma_y = 225 \text{ MN/m}^2$, $E = 200 \text{ GN/m}^2$, $N = 1.7$ and $\eta = 0.3(L/100k)^2$, the above equation gives the graph shown in Fig. 2.9. This graph then indicates the basis of design using BS449: 1959 (amended 1964). Allowable values are provided in the standard, however, in tabular form.

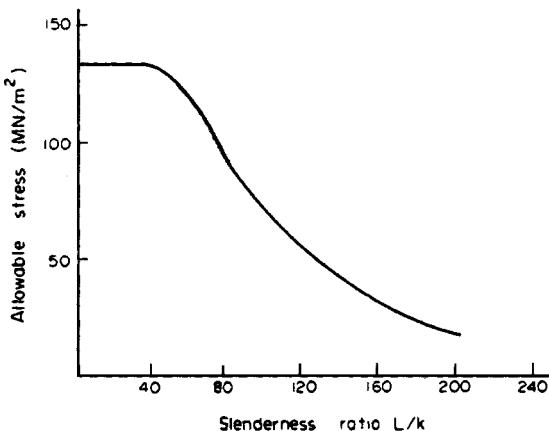


Fig. 2.9. Graph of allowable stress as given in BS 449: 1964 (in tabulated form) against slenderness ratio.

If, however, design is based on the *safety factor* method instead of the *load factor* method, then N is omitted and σ_y/n replaces σ_y in the formula, where n is the safety factor.

2.8. Struts with initial curvature

In §2.6 the Perry–Robertson equation was derived on the assumption that strut imperfections could be allowed for by giving the strut an initial curvature. This proof applies equally well, of course, for struts which have genuine initial curvatures and, provided the curvature is small, the precise shape of the curve has little effect on the end result.

Thus for an initial curvature with a central deflection C_0 ,

$$\text{maximum deflection} = \left[\frac{\sigma_e}{(\sigma_e - \sigma)} \right] C_0 = \left[\frac{P_e}{(P_e - P)} \right] C_0 \quad (2.19)$$

$$\text{maximum B.M.} = P \left[\frac{\sigma_e}{(\sigma_e - \sigma)} \right] C_0 = \left[\frac{PP_e}{(P_e - P)} \right] C_0 \quad (2.20)$$

and

$$\sigma_{\max} = \frac{P}{A} \pm \left[\frac{P\sigma_e}{(\sigma_e - \sigma)} \right] \frac{hC_0}{I}$$

$$= \frac{P}{A} \pm \left[\frac{PP_e}{(P_e - P)} \right] \frac{hC_0}{I} \quad (2.21)$$

where h is the distance from the N.A. to the outside fibres of the strut.

2.9. Struts with eccentric load

For eccentric loading at the ends of a strut Ayrton and Perry suggest that the Perry–Robertson formula can be modified by replacing C_0 by $(C_0 + 1.2e)$ where e is the eccentricity.

Then

$$\eta' = \eta + 1.2 \frac{eh}{k^2} \quad (2.22)$$

and η' replaces η in the original Perry–Robertson equation.

(a) Pinned ends – the Smith–Southwell formula

For a more fundamental treatment consider the strut loaded as shown in Fig. 2.10 carrying a load P at an eccentricity e on one principal axis. In this case there is strictly no ‘buckling’ load as previously described since the strut will bend immediately load is applied, bending taking place about the other principal axis.

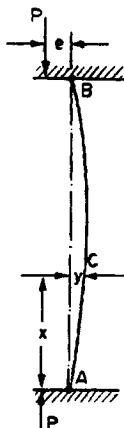


Fig. 2.10. Strut with eccentric load (pinned ends)

Applying a similar procedure to that used previously

$$\text{B.M. at } C = -P(y + e)$$

$$EI \frac{d^2y}{dx^2} = -P(y + e)$$

$$\frac{d^2y}{dx^2} + n^2(y + e) = 0$$

where

$$n = \sqrt{(P/EI)}$$

This is a second-order differential equation, the solution of which is as follows:

$$y = A \sin nx + B \cos nx - e$$

Now when $x = 0$, $y = 0$

$$\therefore B = e$$

and when $x = \frac{L}{2}$, $\frac{dy}{dx} = 0$

$$\therefore 0 = nA \cos n\frac{L}{2} - ne \sin n\frac{L}{2}$$

$$\therefore A = e \tan \frac{nL}{2}$$

$$\therefore y + e = e \tan \frac{nL}{2} \sin nx + e \cos nx$$

\therefore maximum deflection, when

$$x = L/2 \text{ and } y = \delta, \text{ is}$$

$$\begin{aligned} \delta + e &= e \frac{\sin^2 \frac{nL}{2}}{\cos \frac{nL}{2}} + e \cos \frac{nL}{2} \\ &= e \frac{\left(\sin^2 \frac{nL}{2} + \cos^2 \frac{nL}{2} \right)}{\cos \frac{nL}{2}} = e \sec \frac{nL}{2} \end{aligned} \quad (2.23)$$

$$\therefore \text{maximum B.M.} = P(\delta + e) = Pe \sec \frac{nL}{2} \quad (2.24)$$

$$\therefore \text{maximum stress owing to bending} = \frac{My}{I} = Pe \sec \frac{nL}{2} \times \frac{h}{I}$$

where h is the distance from the N.A. to the highest stressed fibre.

Therefore the total maximum compressive stress owing to combined bending and thrust, assuming a *ductile* material[†], is given by

$$\begin{aligned} \sigma_{\max} &= \frac{P}{A} + \left(Pe \sec \frac{nL}{2} \right) \frac{h}{I} \\ &= \sigma \left[1 + \frac{eh}{k^2} \sec \frac{nL}{2} \right] \\ &= \sigma \left[1 + \frac{eh}{k^2} \sec \frac{L}{2} \sqrt{\left(\frac{P}{EI} \right)} \right] \end{aligned} \quad (2.25)$$

[†] For a brittle material which is relatively weak in tension it is the maximum tensile stress which becomes the criterion of failure and the bending and direct stress components are opposite in sign.

$$\text{i.e. } \sigma_{\max} = \sigma \left[1 + \frac{eh}{k^2} \sec \frac{L}{2} \sqrt{\left(\frac{\sigma}{Ek^2} \right)} \right] \quad (2.26)$$

This formula is known as the Smith–Southwell formula.

Unfortunately, since $\sigma = P/A$, the above equation represents a function of P (the required unknown) which can only be solved by trial and error or graphically. A good approximation however, is obtained as shown below:

Webb's approximation

$$\begin{aligned} \text{From above} \quad \sigma_{\max} &= \sigma \left[1 + \frac{eh}{k^2} \sec \frac{nL}{2} \right] \\ \text{Let} \quad \frac{nL}{2} &= \theta \\ \text{Then} \quad \theta &= \frac{L}{2} \sqrt{\left(\frac{P}{EI} \right)} = \frac{\pi}{2} \sqrt{\left(\frac{L^2}{\pi^2 EI} \frac{P}{EI} \right)} = \frac{\pi}{2} \sqrt{\left(\frac{P}{P_e} \right)} \end{aligned} \quad (2.25)(\text{bis})$$

Now for θ between 0 and $\pi/2$,

$$\sec \theta \approx \frac{1 + 0.26 \left(\frac{2\theta}{\pi} \right)^2}{1 - \left(\frac{2\theta}{\pi} \right)^2} = \frac{1 + 0.26 \frac{P}{P_e}}{1 - \frac{P}{P_e}} = \frac{P_e + 0.26P}{P_e - P}$$

Therefore substituting in eqn. (2.25)

$$\begin{aligned} \sigma_{\max} &= \sigma \left[1 + \frac{eh}{k^2} \left(\frac{P_e + 0.26P}{P_e - P} \right) \right] \\ &= \frac{P}{A} \left[1 + \frac{eh}{k^2} \left(\frac{P_e + 0.26P}{P_e - P} \right) \right] \end{aligned} \quad (2.27)$$

where σ_{\max} is the maximum allowable stress in the strut material, P_e is the Euler buckling load for axial loading, and P is the maximum allowable value of the eccentric load.

The above equation can be re-written into a more readily observed quadratic equation in P , thus:

$$P^2 \left[1 - 0.26 \frac{eh}{k^2} \right] - P \left[P_e \left(1 + \frac{eh}{k^2} \right) + \sigma_{\max} A \right] + \sigma_{\max} A P_e = 0 \quad (2.28)$$

For any given eccentric load condition P is the only unknown and the equation can be readily solved.

(b) *One end fixed, the other free*

Consider the strut shown in Fig. 2.11.

$$BM_c = EI \frac{d^2 y}{dx^2} = P(e_0 - y)$$

$$\therefore \frac{d^2y}{dx^2} + n^2 y = n^2 e_0$$

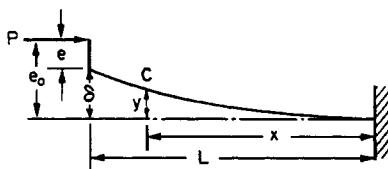


Fig. 2.11. Strut with eccentric load (one end fixed, the other free)

The solution of the expression is

$$y = A \cos nx + B \sin xn + e_0$$

$$\text{At } x = 0, y = 0 \quad \therefore A + e_0 = 0 \text{ or } A = -e_0$$

$$\text{At } x = 0, dy/dx = 0 \quad \therefore B = 0$$

$$\therefore y = -e_0 \cos nx + e_0$$

$$\text{Now at } x = L, y = \delta$$

$$\begin{aligned} \therefore \delta &= -e_0 \cos nL + e_0 \\ &= e_0(1 - \cos nL) \\ &= (\delta + e)(1 - \cos nL) \\ &= \delta - \delta \cos nL + e - e \cos nL \end{aligned}$$

$$\therefore \delta \cos nL = e - e \cos nL$$

$$\therefore \delta = e(\sec nL - 1)$$

$$\text{or } \delta + e = e \sec nL$$

This is the same form of solution as that obtained previously for pinned ends with L replaced by $2L$, i.e. the Smith–Southwell formula will apply in this case provided that the equivalent length of the strut ($l = 2L$) is used in place of L .

Thus the Smith–Southwell formula can be written in the form

$$\sigma_{\max} = \sigma \left[1 + \frac{eh}{k^2} \sec \frac{l}{2} \sqrt{\left(\frac{\sigma}{Ek^2} \right)} \right] \quad (2.29)$$

the value of the equivalent length l to be used for any given end condition being given by the diagrams of Fig. 2.6, §2.2.

The exception to this rule, however, is the case of fixed ends where the only effect of eccentricity of loading is to increase the fixing moments within the supports at each end; there will be no effect on the deflection or stress in the strut itself. Thus, eccentricity of loading can be neglected in the case of fixed-ended struts – an important factor since most practical struts can be considered to be of this type.

2.10. Laterally loaded struts

(a) Central concentrated load

With the origin at the centre of the strut as shown in Fig. 2.12,

$$\text{B.M. at } C = EI \frac{d^2y}{dx^2} = -Py - \frac{W}{2} \left(\frac{L}{2} - x \right)$$

$$\frac{d^2y}{dx^2} + n^2 y = -\frac{W}{2EI} \left(\frac{L}{2} - x \right)$$

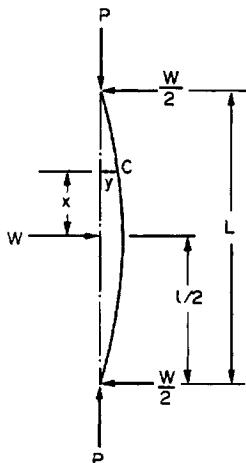


Fig. 2.12.

The solution of this equation is similar to that of §2.1(d),

i.e. $y = A \cos nx + B \sin nx - \frac{W}{2P} \left(\frac{L}{2} - x \right)$

Now when $x = 0$, $dy/dx = 0$ $\therefore B = -\frac{W}{2nP}$

and when $x = L/2$, $y = 0$ $\therefore A = \frac{W}{2nP} \tan \frac{nL}{2}$

$$\therefore y = \frac{W}{2nP} \left[\tan \frac{nL}{2} \cos nx - \sin nx - n \left(\frac{L}{2} - x \right) \right]$$

The maximum deflection occurs where x is zero,

i.e. $y_{\max} = \frac{W}{2nP} \left[\tan \frac{nL}{2} - \frac{nL}{2} \right] \quad (2.30)$

The maximum B.M. acting on the strut is at the same position and is given by

$$M_{\max} = -Py_{\max} - \frac{WL}{2} \frac{L}{2}$$

$$= -\frac{W}{2n} \tan \frac{nL}{2} \quad (2.31)$$

(b) Uniformly distributed load

Consider now the uniformly loaded strut of Fig. 2.13 with the origin again selected at the centre but y measured from the maximum deflected position.

$$\begin{aligned} \text{B.M.}_c &= EI \frac{d^2y}{dx^2} = P(\delta - y) + \frac{wL}{2} \left(\frac{L}{2} - x \right) - \frac{w}{2} \left(\frac{L}{2} - x \right)^2 \\ &= P\delta - Py + \frac{w}{2} \left(\frac{L^2}{4} - x^2 \right) \\ \therefore \frac{d^2y}{dx^2} + n^2y &= \frac{w}{2EI} \left(\frac{L^2}{4} - x^2 \right) + n^2\delta \end{aligned}$$

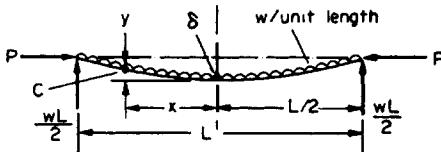


Fig. 2.13.

The solution of this equation is

$$y = A \cos nx + B \sin nx - \frac{w}{2P} \left(\frac{L^2}{4} - x^2 \right) + \delta + \frac{2w}{2n^2P}$$

$$\text{i.e. } y - \delta = A \cos nx + B \sin nx - \frac{w}{2P} \left(\frac{L^2}{4} - x^2 - \frac{2}{n^2} \right)$$

When $x = 0$, $dy/dx = 0 \quad \therefore B = 0$

$$\text{When } x = L/2, y = \delta \quad \therefore A = \frac{w}{n^2P} \sec \frac{nL}{2}$$

$$\therefore y - \delta = \frac{w}{n^2P} \left[\left(\sec \frac{nL}{2} \cos nx - 1 \right) - n^2 \left(\frac{L^2}{8} - \frac{x^2}{2} \right) \right]$$

Thus the maximum deflection δ , when $y = 0$ and $x = 0$, is given by

$$\delta = y_{\max} = \frac{w}{n^2P} \left[\left(\sec \frac{nL}{2} - 1 \right) - \frac{n^2L^2}{8} \right] \quad (2.32)$$

and the maximum B.M. is

$$M_{\max} = P\delta + \frac{wL^2}{8} = \frac{w}{n^2} \left(\sec \frac{nL}{2} - 1 \right) \quad (2.33)$$

In the case of a member carrying a tensile load (i.e. a *tie*) together with a uniformly distributed load, the above procedure applies with the sign for P reversed. The relevant differential expression then becomes

$$\frac{d^2y}{dx^2} - n^2y = \frac{w}{2EI} \left[\frac{L^2}{4} - x^2 \right] + n^2\delta$$

i.e. $(D^2 - n^2)y$ in place of $(D^2 + n^2)y$ as usual.

The solution of this equation involves hyperbolic functions but remains of identical form to that obtained previously,

i.e.

$$M = A \cosh nx + B \sinh nx + \text{etc.}$$

giving

$$M_{\max} = \frac{w}{n^2} \left(\operatorname{sech} \frac{nL}{2} - 1 \right)$$

2.11. Alternative procedure for any strut-loading condition

If deflections are not the primary interest and only the B.M.'s and hence maximum stress are required, it is convenient to commence the analysis with a differential expression for the B.M. M .

This is most easily achieved by considering the moment divided into two parts:

- (a) that due to the end load P ;
- (b) that due to any transverse load (M').

Thus

$$\text{total moment } M = -Py + M'$$

Differentiating twice, $\frac{d^2M}{dx^2} + P \frac{d^2y}{dx^2} = \frac{d^2M'}{dx^2}$

But $P \frac{d^2y}{dx^2} = \frac{P}{EI} \left(EI \frac{d^2y}{dx^2} \right) = n^2M$

$$\therefore \frac{d^2M}{dx^2} + n^2M = \frac{d^2M'}{dx^2}$$

The general solution will be of the form

$$M = A \cos nx + B \sin nx + \text{particular solution}$$

Now for zero transverse load (or for any concentrated load) (d^2M'/dx^2) is zero, the particular solution is also zero, and the solution for the above expression is in the form

$$M = A \cos nx + B \sin nx$$

Thus, for an **eccentrically loaded strut** (Smith–Southwell):

$$\text{shear force} = \frac{dM}{dx} = 0 \quad \text{when } x = 0 \quad \therefore B = 0$$

and

$$M = Pe \quad \text{when } x = \frac{1}{2}L \quad \therefore A = Pe \sec \frac{nL}{2}$$

Therefore substituting, $M = Pe \sec \frac{nL}{2} \cos nx$

and $M_{\max} = Pe \sec \frac{nL}{2}$ as before

For a **central concentrated load** (see Fig. 2.12)

$$M' = \frac{W}{2} \left(\frac{L}{2} - x \right)$$

$$\therefore \frac{d^2M'}{dx^2} = 0 \quad \text{and the particular solution } = 0$$

$$\therefore M = A \cos nx + B \sin nx$$

$$\text{Shear force } = \frac{dM}{dx} = \frac{W}{2} \quad \text{when } x = 0 \quad \therefore B = \frac{W}{2n}$$

$$\text{and } M = 0 \quad \text{when } x = \frac{1}{2}L \quad \therefore A = -\frac{W}{2n} \tan \frac{nL}{2}$$

$$\therefore M = -\frac{W}{2n} \left[\tan \frac{nL}{2} \cos nx + \sin nx \right]$$

$$\text{and } M_{\max} = -\frac{W}{2n} \tan \frac{nL}{2} \quad \text{as before}$$

For a **uniformly distributed lateral load** (see Fig. 2.13)

$$M' = \frac{w}{2} \left[\frac{L^2}{4} - x^2 \right] \quad (\text{see page 47})$$

$$\therefore \frac{d^2M'}{dx^2} = -w$$

$$\text{Hence } \frac{d^2M}{dx^2} + n^2 M = -w \quad \text{and the particular integral is } \frac{w}{n^2}$$

$$\therefore M = A \cos nx + B \sin nx - w/n^2$$

$$\text{Now when } x = 0, dM/dx = 0 \quad \therefore B = 0$$

$$\text{and when } x = L/2, M = 0 \quad \therefore A = \frac{w}{n^2} \sec \frac{nL}{2}$$

$$\therefore M = \frac{w}{n^2} \left[\sec \frac{nL}{2} \cos nx - 1 \right]$$

$$\text{and } M_{\max} = \frac{w}{n^2} \left[\sec \frac{nL}{2} - 1 \right] \quad \text{as before}$$

2.12. Struts with unsymmetrical cross-sections

The formulae derived in the preceding paragraphs have assumed that buckling takes place about an axis of symmetry. Loading is then normally applied to produce bending on the

strongest or major principal axis (that about which I has a maximum value) so that buckling is assumed to occur about the minor axis. It is also assumed that the end conditions allow rotation in this direction and this is normally achieved by loading through ball ends.

For sections with only one axis of symmetry, e.g. channel or T-sections, the shear centre is not coincident with the centroid and torsional effects are often introduced. These may, in some cases, affect the failure condition of the strut. Certainly, in the case of totally unsymmetrical sections, the loading condition always involves considerable torsion and the theoretical buckling load has little relevance. One popular form of section which falls in this category is the unequal-leg angle section.

Some sections, e.g. cruciform sections, are subject to both flexural and torsional buckling and the reader is referred to more advanced texts for the methods of treatment in such cases.

A special form of failure is associated with hollow low carbon steel columns with small thickness to diameter ratios when the strut is found to *crinkle*, i.e. the material forms into folds when the direct stress is approximately equal to the yield stress. Southwell has investigated this problem and produced the formula

$$\sigma = E \frac{t}{R} \left[\frac{1}{3(1 - \nu^2)} \right]^{1/2}$$

where σ is the stress causing yielding, R is the mean radius of the column and t is the thickness. It should be noted, however, that this type of failure is not common since very small t/R ratios of the order of 1/400 are required before crinkling can occur.

Examples

Example 2.1

Two 300 mm \times 120 mm I-section joists are united by 12 mm thick plates as shown in Fig. 2.14 to form a 7 m long stanchion. Given a factor of safety of 3, a compressive yield stress of 300 MN/m² and a constant a of 1/7500, determine the allowable load which can be carried by the stanchion according to the Rankine–Gordon formulae.

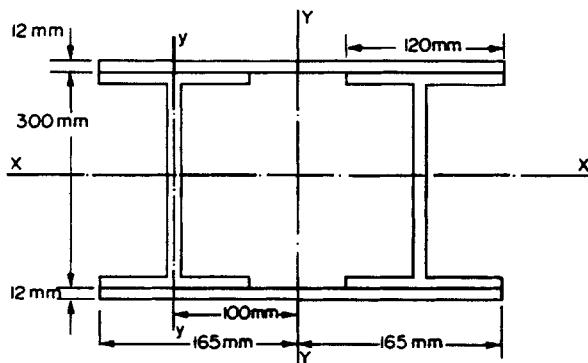


Fig. 2.14.

The relevant properties of each joist are:

$$I_{xx} = 96 \times 10^{-6} \text{ m}^4, \quad I_{yy} = 4.2 \times 10^{-6} \text{ m}^4, \quad A = 6 \times 10^{-3} \text{ m}^2$$

Solution

For the strut of Fig. 2.14:

$$\begin{aligned} I_{xx} \text{ for joists} &= 2 \times 96 \times 10^{-6} = 192 \times 10^{-6} \text{ m}^4 \\ I_{xx} \text{ for plates} &= 0.33 \times \frac{0.324^3}{12} - \frac{0.33 \times 0.300^3}{12} \\ &= \frac{0.33}{12}[0.034 - 0.027] = 192.5 \times 10^{-6} \text{ m}^4 \\ \therefore \text{total } I_{xx} &= (192 + 192.5)10^{-6} = 384.5 \times 10^{-6} \text{ m}^4 \end{aligned}$$

From the parallel axis theorem:

$$\begin{aligned} I_{yy} \text{ for joists} &= 2(4.2 \times 10^{-6} + 6 \times 10^{-3} \times 0.1^2) \\ &= 128.4 \times 10^{-6} \text{ m}^4 \end{aligned}$$

and $I_{yy} \text{ for plates} = 2 \times 0.012 \times \frac{0.33^3}{12} = 71.9 \times 10^{-6} \text{ m}^4$

$$\therefore \text{total } I_{yy} = 200.3 \times 10^{-6} \text{ m}^4$$

Now the smallest value of the Rankine–Gordon stress σ_R is given when k , and hence I , is a minimum.

$$\therefore \text{smallest } I = I_{yy} = 200.3 \times 10^{-6} = Ak^2$$

$$\text{total area } A = 2 \times 6 \times 10^{-3} + 2 \times 0.33 \times 12 \times 10^{-3} = 19.92 \times 10^{-3}$$

$$\therefore 19.92 \times 10^{-3}k^2 = 200.3 \times 10^{-6}$$

$$\therefore k^2 = \frac{200.3 \times 10^{-6}}{19.92 \times 10^{-3}} = 10.05 \times 10^{-3}$$

$$\therefore \left(\frac{L}{k}\right)^2 = \frac{7^2}{10.05 \times 10^{-3}} = 4.9 \times 10^3$$

and $\sigma_R = \frac{\sigma_y}{1 + a \left(\frac{L}{k}\right)^2} = \frac{300 \times 10^6}{1 + \frac{4.9 \times 10^3}{7500}}$

$$\therefore = \frac{300 \times 10^6}{1.653} = 181.45 \text{ MN/m}^2$$

$$\therefore \text{allowable load} = \sigma_R \times A = 181.45 \times 10^6 \times 19.92 \times 10^{-3} = 3.61 \text{ MN}$$

With a factor of safety of 3 the maximum permissible load therefore becomes

$$P_{\max} = \frac{3.61 \times 10^6}{3} = \mathbf{1.203 \text{ MN}}$$

Example 2.2

An 8 m long column is constructed from two 400 mm × 250 mm I-section joists joined as shown in Fig. 2.15. One end of the column is arranged to be fixed and the other free and a load equal to one-third of the Euler load is applied. Determine the load factor provided if the Perry–Robertson formula is used as the basis for design.

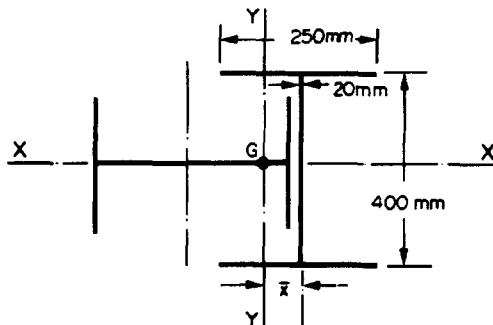


Fig. 2.15.

For each joist:

$$I_{\max} = 213 \times 10^{-6} \text{ m}^4, \quad I_{\min} = 9.6 \times 10^{-6} \text{ m}^4, \quad A = 8.4 \times 10^{-3} \text{ m}^2,$$

with web and flange thicknesses of 20 mm. For the material of the joist, $E = 208 \text{ GN/m}^2$ and $\sigma_y = 270 \text{ MN/m}^2$.

Solution

To find the position of the centroid G of the built-up section take moments of area about the centre line of the vertical joist.

$$2 \times 8.4 \times 10^{-3} \bar{x} = 8.4 \times 10^{-3} (200 + 10) 10^{-3}$$

$$\bar{x} = \frac{210}{2} \times 10^{-3} = 105 \text{ mm}$$

Now

$$I_{xx} = (213 + 9.6) 10^{-6} = 222.6 \times 10^{-6} \text{ m}^4$$

and

$$I_{yy} = [213 + 8.4(210 - 105)^2] 10^{-6} + [9.6 + 8.4 \times 105^2] 10^{-6}$$

i.e. greater than I_{xx} .

$$\therefore \text{least } I = 222.6 \times 10^{-6} \text{ m}^4$$

$$\therefore \text{least } k^2 = \frac{222.6 \times 10^{-6}}{2 \times 8.4 \times 10^{-3}} = 13.25 \times 10^{-3}$$

Now Euler load for fixed-free ends

$$\begin{aligned} &= \frac{\pi^2 EI}{4L^2} = \frac{\pi^2 \times 208 \times 10^9 \times 222.6 \times 10^{-6}}{4 \times 8^2} \\ &= 1786 \times 10^3 = 1.79 \text{ MN} \end{aligned}$$

Therefore actual load applied to the column

$$= \frac{1.79}{3} = 0.6 \text{ MN}$$

i.e.

$$\text{actual stress} = \frac{\text{load}}{\text{area}} = \frac{0.6 \times 10^6}{2 \times 8.4 \times 10^{-3}}$$

$$= 35.7 \text{ MN/m}^2$$

The Perry–Robertson constant is

$$\eta = 0.3 \left(\frac{L}{100k} \right)^2 = 0.3 \left(\frac{8^2}{10^4 \times 13.25 \times 10^{-3}} \right)$$

$$= 0.144$$

and

$$N\sigma = \frac{(\sigma_y + 1.144\sigma_e)}{2} - \sqrt{\left\{ \left[\frac{(\sigma_y + 1.144\sigma_e)}{2} \right]^2 - \sigma_y \sigma_e \right\}}$$

But

$$\sigma_y = 270 \text{ MN/m}^2 \text{ and } \sigma_e = \frac{1.79 \times 10^6}{2 \times 8.4 \times 10^{-3}} = 106.5 \text{ MN/m}^2$$

i.e. in units of MN/m²:

$$\therefore N\sigma = \frac{(270 + 121.8)}{2} - \sqrt{\left\{ \left[\frac{270 + 121.8}{2} \right]^2 - 270 \times 106.5 \right\}}$$

$$= 196 - 98 = 98$$

$$\therefore \text{load factor } N = \frac{98}{35.7} = 2.75$$

Example 2.3

Determine the maximum compressive stress set up in a 200 mm × 60 mm I-section girder carrying a load of 100 kN with an eccentricity of 6 mm from the critical axis of the section (see Fig. 2.16). Assume that the ends of the strut are pin-jointed and that the overall length is 4 m.

Take $I_{yy} = 3 \times 10^{-6} \text{ m}^4$, $A = 6 \times 10^{-3} \text{ m}^2$, $E = 207 \text{ GN/m}^2$.

Solution

Normal stress on the section

$$\sigma = \frac{P}{A} = \frac{100 \times 10^3}{6 \times 10^{-3}} = \frac{100}{6} \text{ MN/m}^2$$

$$I = Ak^2 = 3 \times 10^{-6} \text{ m}^4$$

$$\therefore k^2 = \frac{3 \times 10^{-6}}{6 \times 10^{-3}} = 5 \times 10^{-4} \text{ m}^2$$

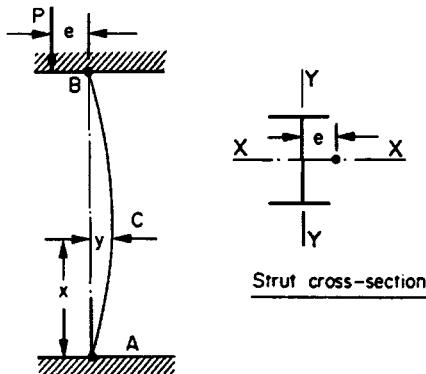


Fig. 2.16.

Now from eqn. (2.26)

$$\sigma_{\max} = \sigma \left[1 + \frac{eh}{k^2} \sec \frac{L}{2} \sqrt{\left(\frac{\sigma}{Ek^2} \right)} \right]$$

∴ with $e = 6 \text{ mm}$ and $h = 30 \text{ mm}$

$$\begin{aligned}\sigma_{\max} &= \frac{100}{6} \left[1 + \frac{30 \times 6 \times 10^{-6}}{5 \times 10^{-4}} \sec 2 \sqrt{\left(\frac{100 \times 10^6 \times 10^4}{6 \times 207 \times 10^9 \times 5} \right)} \right] \\ &= \frac{100}{6} [1 + 0.36 \sec 2 \sqrt{(0.161)}] \\ &= \frac{100}{6} [1 + 0.36 \times 1.44] = 25.3 \text{ MN/m}^2\end{aligned}$$

Example 2.4

A horizontal strut 2.5 m long is constructed from rectangular section steel, 50 mm wide by 100 mm deep, and mounted with pinned ends. The strut carries an axial load of 120 kN together with a uniformly distributed lateral load of 5 kN/m along its complete length. If $E = 200 \text{ GN/m}^2$ determine the maximum stress set up in the strut.

Check the result using the approximate Perry method with

$$M_{\max} = M_0 \left[\frac{P_e}{P_e - P} \right]$$

Solution

From eqn. (2.34)

$$M_{\max} = \frac{w}{n^2} \left(\sec \frac{nL}{2} - 1 \right)$$

$$\begin{aligned}\text{where } n^2 &= \frac{P}{EI} = \frac{120 \times 10^3 \times 12}{200 \times 10^9 \times 50 \times 100^3 \times 10^{-12}} \\ &= 0.144\end{aligned}$$

$$\begin{aligned}\therefore \frac{nL}{2} &= \frac{2.5}{2} \sqrt{(0.144)} = 0.474 \text{ radian} \\ \therefore M_{\max} &= \frac{5 \times 10^3}{0.144} (\sec 0.474 - 1) \\ &= 34.7 \times 10^3 (1.124 - 1) = 4.3 \times 10^3 \text{ Nm}\end{aligned}$$

The maximum stress due to the axial load and the eccentricity caused by bending is then given by

$$\begin{aligned}\sigma_{\max} &= \frac{P}{A} + \frac{My}{I} \\ &= \frac{120 \times 10^3}{(0.1 \times 0.05)} + \frac{4.34 \times 10^3 \times 0.05 \times 12}{(50 \times 100^3)10^{-12}} \\ &= 24 \times 10^6 + 51.6 \times 10^6 \\ &= \mathbf{75.6 \text{ MN/m}^2}\end{aligned}$$

Using the approximate Perry method,

$$M_{\max} = M_0 \left[\frac{P_e}{P_e - P} \right]$$

where $M_0 = \text{B.M. due to lateral load only} = \frac{wL^2}{8}$

But $P_e = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 \times 200 \times 10^9}{2.5^2} \times \frac{(50 \times 100^3)10^{-12}}{12}$
 $= 1.316 \text{ MN}$

$$\begin{aligned}\therefore M_{\max} &= \frac{wL^2}{8} \left[\frac{P_e}{P_e - P} \right] \\ &= \frac{5 \times 10^3 \times 2.5^2}{8} \left[\frac{1316 \times 10^3}{(1316 - 120)10^3} \right] \\ &= 4.3 \times 10^3 \text{ Nm}\end{aligned}$$

In this case, therefore, the approximate method yields the same answer for maximum B.M. as the full solution. The maximum stress will then also be equal to that obtained above, i.e. **75.6 MN/m²**.

Example 2.5

A hollow circular steel strut with its ends fixed in position has a length of 2 m, an outside diameter of 100 mm and an inside diameter of 80 mm. Assuming that, before loading, there is an initial sinusoidal curvature of the strut with a maximum deflection of 5 mm, determine the maximum stress set up due to a compressive end load of 200 kN. $E = 208 \text{ GN/m}^2$.

Solution

The assumed sinusoidal initial curvature may be expressed alternatively in the complementary cosine form

$$y_0 = \delta_0 \cos \frac{\pi x}{L} \quad (\text{Fig. 2.17})$$

Now when P is applied, y_0 increases to y and the central deflection increases from $\delta_0 = 5 \text{ mm}$ to δ .

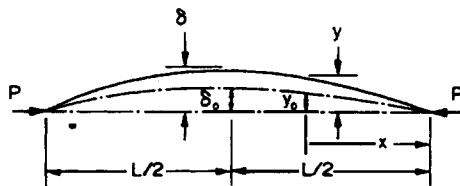


Fig. 2.17.

For the above initial curvature it can be shown that

$$\delta = \left[\frac{P_e}{P_e - P} \right] \delta_0$$

$$\therefore \text{maximum B.M.} = P\delta_0 \left[\frac{P_e}{P_e - P} \right]$$

where P_e for ends fixed in direction only = $\frac{\pi^2 EI}{L^2}$

$$I = \frac{\pi}{64} (0.1^4 - 0.08^4) = \frac{\pi}{64} (1 - 0.41) 10^{-4} = 2.896 \times 10^{-6} \text{ m}^4$$

$$\therefore P_e = \frac{\pi^2 \times 208 \times 10^9 \times 2.89 \times 10^{-6}}{4} = 1.486 \text{ MN}$$

$$\therefore \text{maximum B.M.} = 200 \times 10^3 \times 5 \times 10^{-3} \left[\frac{1486 \times 10^3}{(1486 - 200)10^3} \right] = 1.16 \text{ kN m}$$

$$\begin{aligned} \therefore \text{maximum stress} &= \frac{P}{A} + \frac{My}{I} = \frac{200 \times 10^3 \times 4}{\pi(0.1^2 - 0.08^2)} + \frac{1.16 \times 10^3 \times 0.05}{2.89 \times 10^{-6}} \\ &= 70.74 \times 10^6 + 20.07 \times 10^6 \\ &= 90.8 \text{ MN/m}^2 \end{aligned}$$

Problems

2.1 (A/B). Compare the crippling loads given by the Euler and Rankine-Gordon formulae for a pin-jointed cylindrical strut 1.75 m long and of 50 mm diameter. (For Rankine-Gordon use $\sigma_y = 315 \text{ MN/m}^2$; $a = 1/7500$; $E = 200 \text{ GN/m}^2$.) [197.7, 171 kN.]

2.2 (A/B). In an experiment an alloy rod 1 m long and of 6 mm diameter, when tested as a simply supported beam over a length of 750 mm, was found to have a maximum deflection of 5.8 mm under the action of a central load of 5 N.

- (a) Find the Euler buckling load when this rod is tested as a strut, pin-jointed and guided at both ends.
 (b) What will be the central deflection of this strut when the material reaches a yield stress of 240 MN/m^2 ?

$$\text{(Clue: maximum stress} = \frac{P}{A} \pm \frac{My}{I} \text{ where } M = P \times \delta_{\max}.) \quad [74.8 \text{ N; } 67 \text{ mm.}]$$

2.3 (B) A steel strut is built up of two T-sections riveted back to back to form a cruciform section of overall dimensions $150 \text{ mm} \times 220 \text{ mm}$. The dimensions of each T-section are $150 \text{ mm} \times 15 \text{ mm} \times 110 \text{ mm}$ high. The ends of the strut are rigidly secured and its effective length is 7 m. Find the maximum safe load that this strut can carry with a factor of safety of 5, given $\sigma_y = 315 \text{ MN/m}^2$ and $a = 1/30000$ in the Rankine–Gordon formula.

[192 kN.]

2.4 (B). State the assumptions made when deriving the Euler formula for a strut with pin-jointed ends. Derive the Euler crippling load for such a strut—the general equation of bending and also the solution of the differential equation may be assumed.

A straight steel rod 350 mm long and of 6 mm diameter is loaded axially in compression until it buckles. Assuming that the ends are pin-jointed, find the critical load using the Euler formula. Also calculate the maximum central deflection when the material reaches a yield stress of 300 MN/m^2 compression. Take $E = 200 \text{ GN/m}^2$.

[1.03 kN; 5.46 mm.]

2.5 (B). A steel stanchion 5 m long is to be built of two I-section rolled steel joists 200 mm deep and 150 mm wide flanges with a 350 mm wide \times 20 mm thick plate riveted to the flanges as shown in Fig. 2.18. Find the spacing of the joists so that for an axially applied load the resistance to buckling may be the same about the axes XX and YY . Find the maximum allowable load for this condition with ends pin-jointed and guided, assuming $a = 1/7500$ and $\sigma_y = 315 \text{ MN/m}^2$ in the Rankine formula.

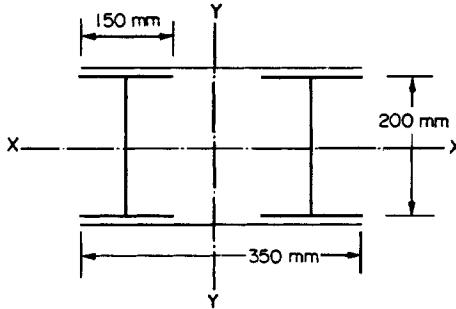


Fig. 2.18.

If the maximum working stress in compression σ for this strut is given by $\sigma = 135[1 - 0.005 L/k] \text{ MN/m}^2$, what factor of safety must be used with the Rankine formula to give the same result? For each R.S.J. $A = 6250 \text{ mm}^2$, $k_x = 85 \text{ mm}$, $k_y = 35 \text{ mm}$.

[180.6 mm, 6.23 MN; 2.32.]

2.6 (B). A stanchion is made from two $200 \text{ mm} \times 75 \text{ mm}$ channels placed back to back, as shown in Fig. 2.19, with suitable diagonal bracing across the flanges. For each channel $I_{xx} = 20 \times 10^{-6} \text{ m}^4$, $I_{yy} = 1.5 \times 10^{-6} \text{ m}^4$, the cross-sectional area is $3.5 \times 10^{-3} \text{ m}^2$ and the centroid is 21 mm from the back of the web.

At what value of p will the radius of gyration of the whole cross-section be the same about the X and Y axes? The strut is 6 m long and is pin-ended. Find the Euler load for the strut and discuss briefly the factors which cause the actual failure load of such a strut to be less than the Euler load. $E = 210 \text{ GN/m}^2$.

[163.6 mm; 2.3 MN.]

2.7 (B). In tests it was found that a tube 2 m long, 50 mm outside diameter and 2 mm thick when used as a pin-jointed strut failed at a load of 43 kN. In a compression test on a short length of this tube failure occurred at a load of 115 kN.

- (a) Determine whether the value of the critical load obtained agrees with that given by the Euler theory.
 (b) Find from the test results the value of the constant a in the Rankine–Gordon formula. Assume $E = 200 \text{ GN/m}^2$.

[Yes; 1/7080.]

2.8 (B). Plot, on the same axes, graphs of the crippling stresses for pin-ended struts as given by the Euler and Rankine–Gordon formulae, showing the variation of stress with slenderness ratio

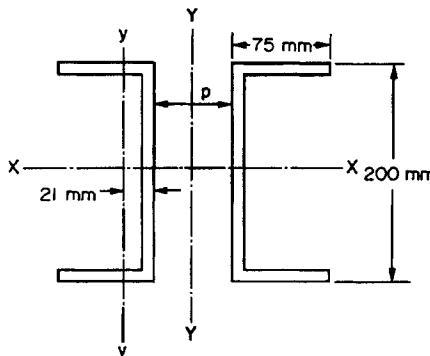


Fig. 2.19.

For the Euler formula use L/k values from 80 to 150, and for the Rankine formula L/k from 0 to 150, with $\sigma_y = 315 \text{ MN/m}^2$ and $a = 1/7500$.

From the graphs determine the values of the stresses given by the two formulae when $L/k = 130$ and the slenderness ratio required by both formulae for a crippling stress of 135 MN/m^2 . $E = 210 \text{ GN/m}^2$.

[122.6 MN/m^2 , 96.82 MN/m^2 ; 124,100.]

2.9 (B/C). A timber strut is $75 \text{ mm} \times 75 \text{ mm}$ square-section and is 3 m high. The base is rigidly built-in and the top is unrestrained. A bracket at the top of the strut carries a vertical load of 1 kN which is offset 150 mm from the centre-line of the strut in the direction of one of the principal axes of the cross-section. Find the maximum stress in the strut at its base cross-section if $E = 9 \text{ GN/m}^2$. [I.Mech.E.] [2.3 MN/m^2 .]

2.10 (B/C). A slender column is built-in at one end and an eccentric load is applied at the free end. Working from first principles find the expression for the maximum length of column such that the deflection of the free end does not exceed the eccentricity of loading. [I.Mech.E.] [$\sec^{-1} 2/\sqrt{(P/EI)}$.]

2.11 (B/C). A slender column is built-in one end and an eccentric load of 600 kN is applied at the other (free) end. The column is made from a steel tube of 150 mm o.d. and 125 mm i.d. and it is 3 m long. Deduce the equation for the deflection of the free end of the beam and calculate the maximum permissible eccentricity of load if the maximum stress is not to exceed 225 MN/m^2 . $E = 200 \text{ GN/m}^2$. [I.Mech.E.] [4 mm.]

2.12 (B). A compound column is built up of two $300 \text{ mm} \times 125 \text{ mm}$ R.S.J.s arranged as shown in Fig. 2.20. The joists are braced together; the effects of this bracing on the stiffness may, however, be neglected. Determine the safe height of the column if it is to carry an axial load of 1 MN. Properties of joists: $A = 6 \times 10^{-3} \text{ m}^2$; $k_{yy} = 27 \text{ mm}$; $k_{xx} = 125 \text{ mm}$.

The allowable stresses given by BS449: 1964 may be found from the graph of Fig. 2.9.

[8.65 m.]

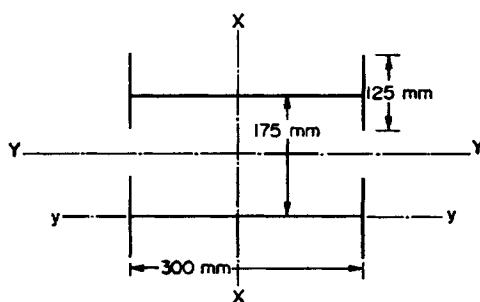


Fig. 2.20.

2.13 (B). A 10 mm long column is constructed from two 375 mm × 100 mm channels placed back to back with a distance h between their centroids and connected together by means of narrow batten plates, the effects of which may be ignored. Determine the value of h at which the section develops its maximum resistance to buckling.

Estimate the safe axial load on the column using the Perry–Robertson formula (a) with a load factor of 2, (b) with a factor of safety of 2. For each channel $I_{xx} = 175 \times 10^{-6}$ m⁴, $I_{yy} = 7 \times 10^{-6}$ m⁴, $A = 6.25 \times 10^{-3}$ m², $E = 210$ GN/m² and yield stress = 300 MN/m². Assume $\eta = 0.003 L/k$ and that the ends of the column are effectively pinned. [328 mm; 1.46, 1.55 MN.]

2.14 (B). (a) Compare the buckling loads that would be obtained from the Rankine–Gordon formula for two identical steel columns, one having both ends fixed, the other having pin-jointed ends, if the slenderness ratio is 100.

(b) A steel column, 6 m high, of square section 120 mm × 120 mm, is designed using the Rankine–Gordon expression to be used as a strut with both ends pin-jointed.

The values of the constants used were $a = 1/7500$, and $\sigma_c = 300$ MN/m². If, in service, the load is applied axially but parallel to and a distance x from the vertical centroidal axis, calculate the maximum permissible value of x . Take $E = 200$ GN/m². [7.4; 0.756 m.]

2.15 (B). Determine the maximum compressive stress set up in a 200 mm × 60 mm I-section girder carrying a load of 100 kN with an eccentricity of 6 mm. Assume that the ends of the strut are pin-jointed and that the overall length is 4 m.

Take $I = 3 \times 10^{-6}$ m⁴; $A = 6 \times 10^{-3}$ m² and $E = 207$ GN/m².

[25.4 MN/m².]

2.16 (B). A slender strut, initially straight, is pinned at each end. It is to be subjected to an eccentric compressive load whose line of action is parallel to the original centre-line of the strut.

(a) Prove that the central deflection y of the strut, relative to its initial centre-line, is given by the expression

$$y = e \left[\sec \frac{1}{2} \sqrt{\left(\frac{PL^2}{EI} \right)} - 1 \right]$$

where P is the applied load, L is the effective length of the strut, e is the eccentricity of the line of action of the load from the initially straight strut axis and EI is the flexural rigidity of the strut cross-section.

(b) Using the above formula, and assuming that the strut is made of a ductile material, show that, for a maximum compressive stress, σ , the value of P is given by the expression

$$P = \frac{\sigma A}{\frac{he}{k^2} \sec \frac{1}{2} \sqrt{\left(\frac{PL^2}{EI} \right)} + 1}$$

the symbols A , h and k having their usual meanings.

(c) Such a strut, of constant tubular cross-section throughout, has an outside diameter of 64 mm, a principal second moment of area of 52×10^{-8} m⁴ and a cross-sectional area of 12.56×10^{-4} m². The effective length of the strut is 2.5 m. If $P = 120$ kN and $\sigma = 300$ MN/m², determine the permissible value of e . Take $E = 200$ GN/m². [B.P.] [6.25 mm.]

2.17 (C). A strut of length L has each end fixed in an elastic material which can exert a restraining moment μ per radian. Prove that the critical load P is given by the equation

$$P + \mu \sqrt{\left(\frac{P}{EI} \right)} \tan \frac{L}{2} \sqrt{\left(\frac{P}{EI} \right)} = 0$$

The designed buckling load of a 1 m long strut, assuming the ends to be rigidly fixed, was 2.5 kN. If, during service, the ends were found to rotate with each mounting exerting a restraining moment of 1 kN m per radian, show that the buckling load decreases by 20%. [C.E.I.]

2.18 (C). A uniform elastic bar of circular cross-section and of length L , free at one end and rigidly built-in at the other end, is subjected to a single concentrated load P at the free end. In general the line of action of P may be at an angle θ to the axis of the bar ($0 < \theta < \pi/2$) so that the bar is simultaneously compressed and bent. For this general case:

(a) Show that the deflection at the free end is given by

$$\delta = \tan \theta \left\{ \left(\frac{\tan mL}{m} - L \right) \right\}$$

(b) Hence show that as $\theta \rightarrow \pi/2$, then $\delta \rightarrow PL^3/3EI$

(c) Show that when $\theta = 0$ no deflection unless P has certain particular values.

Note that in the above, m^2 denotes $P \cos \theta/EI$.

The following expression may be used in part (b) where appropriate:

$$\tan \alpha = \alpha + \frac{\alpha^3}{3} + \frac{2\alpha^5}{15} \quad [\text{City U.}]$$

2.19 (C). A slender strut of length L is encastré at one end and pin-jointed at the other. It carries an axial load P and a couple M at the pinned end. If its flexural rigidity is EI and $P/EI = n$, show that the magnitude of the couple at the fixed end is

$$M \left[\frac{nL - \sin nL}{nL \cos nL - \sin nL} \right]$$

What is the value of this couple when (a) P is one-quarter the Euler critical load and (b) P is zero?

[U.L.] [0.571 M, 0.5 M.]

2.20 (C). An initially straight strut of length L has lateral loading w per metre and a longitudinal load P applied with an eccentricity e at both ends.

If the strut has area A , second moment of area I , section modulus Z and the end moments and lateral loading have opposing effects, find an expression for the central bending moment and show that the maximum stress at the centre will be equal to

$$\frac{P}{A} + \frac{\left(Pe - \frac{wEI}{P} \right) \sec \frac{L}{2} \sqrt{\left(\frac{P}{EI} \right) + \frac{wEI}{P}}}{Z} \quad [\text{U.L.}]$$

CHAPTER 3

STRAINS BEYOND THE ELASTIC LIMIT

Summary

For rectangular-sectioned beams strained up to and beyond the elastic limit, i.e. for *plastic bending*, the bending moments (B.M.) which the beam can withstand at each particular stage are:

$$\text{maximum elastic moment} \quad M_E = \frac{BD^2}{6} \sigma_y$$

$$\text{partially plastic moment} \quad M_{PP} = \frac{B\sigma_y}{12} [3D^2 - d^2]$$

$$\text{fully plastic moment} \quad M_{FP} = \frac{BD^2}{4} \sigma_y$$

where σ_y is the stress at the elastic limit, or *yield stress*.

$$\text{Shape factor } \lambda = \frac{\text{fully plastic moment}}{\text{maximum elastic moment}}$$

For I-section beams:

$$M_E = \sigma_y \left[\frac{BD^3}{12} - \frac{bd^3}{12} \right] \frac{2}{D}$$

$$M_{FP} = \sigma_y \left[\frac{BD^2}{4} - \frac{bd^2}{4} \right]$$

The position of the neutral axis (N.A.) for fully plastic unsymmetrical sections is given by:

$$\text{area of section above or below N.A.} = \frac{1}{2} \times \text{total area of cross-section}$$

Deflections of partially plastic beams are calculated on the basis of the elastic areas only.

In plastic limit or ultimate collapse load procedures the normal elastic safety factor is replaced by a load factor as follows:

$$\text{load factor} = \frac{\text{collapse load}}{\text{allowable working load}}$$

For **solid shafts**, radius R , strained up to and beyond the elastic limit in shear, i.e. for *plastic torsion*, the torques which can be transmitted at each stage are

$$\text{maximum elastic torque} \quad T_E = \frac{\pi R^3}{2} \tau_y$$

$$\text{partially plastic torque} \quad T_{PP} = \frac{\pi \tau_y}{6} [4R^3 - R_1^3] \quad (\text{yielding to radius } R_1)$$

fully plastic torque

$$T_{FP} = \frac{2\pi R^3}{3} \tau_y$$

where τ_y is the shear stress at the elastic limit, or shear yield stress. Angles of twist of partially plastic shafts are calculated on the basis of the elastic core only.

For **hollow shafts**, inside radius R_1 , outside radius R yielded to radius R_2 ,

$$T_{PP} = \frac{\pi \tau_y}{6R_2} [4R^3 R_2 - R_2^4 - 3R_1^4]$$

$$T_{FP} = \frac{2\pi \tau_y}{3} [R^3 - R_1^3]$$

For **eccentric loading** of rectangular sections the fully plastic moment is given by

$$M_{FP} = \frac{BD^2}{4} \sigma_y - \frac{P^2 N^2}{4B\sigma_y}$$

where P is the axial load, N the load factor and B the width of the cross-section.

The maximum allowable moment is then given by

$$M = \frac{BD^2}{4N} \sigma_y - \frac{P^2 N}{4B\sigma_y}$$

For a **solid rotating disc**, radius R , the collapse speed ω_p is given by

$$\omega_p^2 = \frac{3\sigma_y}{\rho R^2}$$

where ρ is the density of the disc material.

For **rotating hollow discs** the collapse speed is found from

$$\omega_p^2 = \frac{3\sigma_y}{\rho} \left[\frac{R_2 - R_1}{R_2^3 - R_1^3} \right]$$

Introduction

When the design of components is based upon the elastic theory, e.g. the simple bending or torsion theory, the dimensions of the components are arranged so that the maximum stresses which are likely to occur under service loading conditions do not exceed the allowable working stress for the material in either tension or compression. The allowable working stress is taken to be the yield stress of the material divided by a convenient safety factor (usually based on design codes or past experience) to account for unexpected increase in the level of service loads. If the maximum stress in the component is likely to exceed the allowable working stress, the component is considered unsafe, yet it is evident that complete failure of the component is unlikely to occur even if the yield stress is reached at the outer fibres provided that some portion of the component remains elastic and capable of carrying load, i.e. the strength of a component will normally be much greater than that assumed on the basis of initial yielding at any position. To take advantage of the inherent additional

strength, therefore, a different design procedure is used which is often referred to as *plastic limit design*. The revised design procedures are based upon a number of basic assumptions about the material behaviour.

Figure 3.1 shows a typical stress-strain curve for annealed low carbon steel indicating the presence of both upper and lower yield points and strain-hardening characteristics.

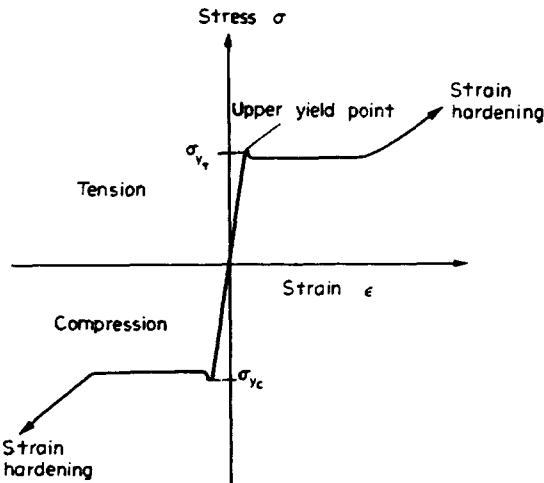


Fig. 3.1. Stress-strain curve for annealed low-carbon steel indicating upper and lower yield points and strain-hardening characteristics.

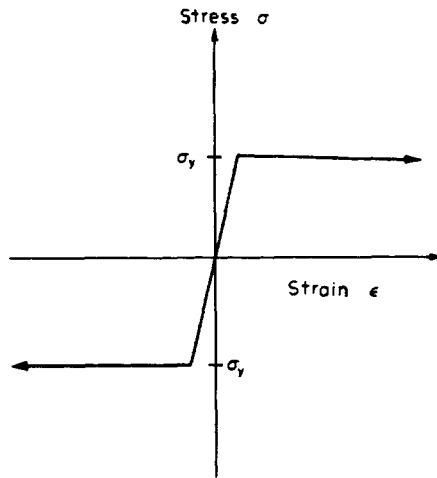


Fig. 3.2. Assumed stress-strain curve for plastic theory – no strain-hardening, equal yield points, $\sigma_{yt} = \sigma_{yc} = \sigma_y$.

Figure 3.2 shows the assumed material behaviour which:

- ignores the presence of upper and lower yields and suggests only a single yield point;
- takes the yield stress in tension and compression to be equal;

- (c) assumes that yielding takes place at constant strain thereby ignoring any strain-hardening characteristics. Thus, once the material has yielded, stress is assumed to remain constant throughout any further deformation.

It is further assumed, despite assumption (c), that transverse sections of beams in bending remain plane throughout the loading process, i.e. strain is proportional to distance from the neutral axis.

It is now possible on the basis of the above assumptions to determine the moment which must be applied to produce:

- the maximum or limiting elastic conditions in the beam material with yielding just initiated at the outer fibres;
- yielding to a specified depth;
- yielding across the complete section.

The latter situation is then termed a fully plastic state, or “*plastic hinge*”. Depending on the support and loading conditions, one or more plastic hinges may be required before complete collapse of the beam or structure occurs, the load required to produce this situation then being termed the *collapse load*. This will be considered in detail in §3.6.

3.1. Plastic bending of rectangular-sectioned beams

Figure 3.3(a) shows a rectangular beam loaded until the yield stress has just been reached in the outer fibres. The beam is still completely elastic and the bending theory applies, i.e.

$$M = \frac{\sigma I}{y}$$

$$\therefore \text{maximum elastic moment} = \sigma_y \times \frac{BD^3}{12} \times \frac{2}{D}$$

$$M_E = \frac{BD^2}{6} \sigma_y \quad (3.1)$$

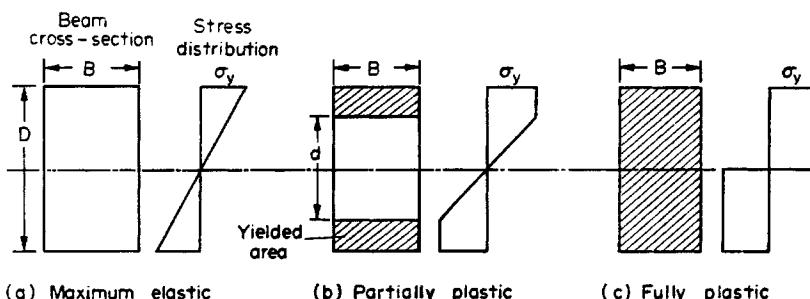


Fig. 3.3. Plastic bending of rectangular-section beam.

If loading is then increased, it is assumed that instead of the stress at the outside increasing still further, more and more of the section reaches the yield stress σ_y . Consider the stage shown in Fig. 3.3(b).

Partially plastic moment,

$$M_{PP} = \text{moment of elastic portion} + \text{total moment of plastic portion}$$

$$\therefore M_{PP} = \frac{Bd^2}{6}\sigma_y + 2 \left\{ \sigma_y \times B \left[\frac{D}{2} - \frac{d}{2} \right] \left[\frac{1}{2} \left(\frac{D}{2} - \frac{d}{2} \right) + \frac{d}{2} \right] \right\}$$

stress area moment arm

$$\begin{aligned} M_{PP} &= \sigma_y \left[\frac{Bd^2}{6} + \frac{B}{4}(D-d)(D+d) \right] \\ &= \frac{B\sigma_y}{12}[2d^2 + 3(D^2 - d^2)] = \frac{B\sigma_y}{12}[3D^2 - d^2] \end{aligned} \quad (3.2)$$

When loading has been continued until the stress distribution is as in Fig. 3.3(c) (assumed), the beam with collapse. The moment required to produce this fully plastic state can be obtained from eqn. (3.2), since d is then zero, i.e.

$$\text{fully plastic moment, } M_{FP} = \frac{B\sigma_y}{12} \times 3D^2 = \frac{BD^2}{4}\sigma_y \quad (3.3)$$

This is the moment therefore which produces a plastic hinge in a rectangular-section beam.

3.2. Shape factor – symmetrical sections

The shape factor is defined as the ratio of the moments required to produce fully plastic and maximum elastic states:

$$\text{shape factor } \lambda = \frac{M_{FP}}{M_E} \quad (3.4)$$

It is a factor which gives a measure of the increase in strength or load-carrying capacity which is available beyond the normal elastic design limits for various shapes of section, e.g. for the *rectangular section* above,

$$\text{shape factor} = \frac{BD^2}{4}\sigma_y / \frac{BD^2}{6}\sigma_y = 1.5$$

Thus rectangular-sectioned beams can carry 50% additional moment to that which is required to produce initial yielding at the edge of the beam section before a fully plastic hinge is formed. (It will be shown later that even greater strength is available beyond this stage depending on the support conditions used.) It must always be remembered, however, that should the stresses exceed the yield at any time during service there will be some associated *permanent set* or deflection when load is removed, and consideration should be given to whether or not this is acceptable. Bearing in mind, however, that normal design office practice involves the use of a safety factor to take account of abnormalities of loading, it should be evident that even at this stage considerable advantages are obtained by application of this factor to the fully plastic condition rather than the limiting elastic case. It is then

possible to arrange for all normal loading situations to be associated with elastic stresses in the beam, the additional strength in the partially plastic condition being used as the safety margin to take account of unexpected load increases.

Figure 3.4 shows the way in which moments build up with increasing depth or penetration of yielding and associated radius of curvature as the beam bends.

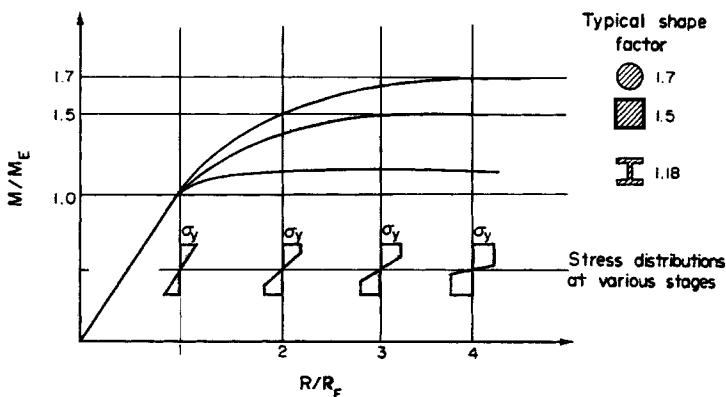


Fig. 3.4. Variation of moment of resistance of beams of various cross-section with depth of plastic penetration and associated radius of curvature.

Here the moment M carried by the beam at any particular stage and its associated radius of curvature R are considered as ratios of the values at the maximum elastic or initial yield condition. It will be noticed that at large curvature ratios, i.e. high plastic penetrations, the values of M/M_E approach the shape factor of the sections indicated, e.g. 1.5 for the rectangular section.

Shape factors of other symmetrical sections such as the I-section beam are found as follows (Fig. 3.5).

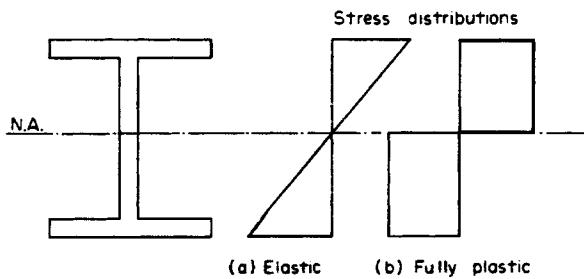


Fig. 3.5. Plastic bending of symmetrical (I-section) beam.

First determine the value of the maximum elastic moment M_E by applying the simple bending theory

$$\frac{M}{I} = \frac{\sigma}{y}$$

with y the maximum distance from the N.A. (the axis of symmetry passing through the centroid) to an outside fibre and $\sigma = \sigma_y$, the yield stress.

Then, in the fully plastic condition, the stress will be uniform across the section at σ_y and the section can be divided into any convenient number of rectangles of area A and centroid distance h from the neutral axis.

Then

$$M_{FP} = \sum (\sigma_y A) h \quad (3.5)$$

The shape factor M_{FP}/M_E can then be determined.

3.3. Application to I-section beams

When the B.M. applied to an I-section beam is just sufficient to initiate yielding in the extreme fibres, the stress distribution is as shown in Fig. 3.5(a) and the value of the moment is obtained from the simple bending theory by subtraction of values for convenient rectangles.

i.e.

$$\begin{aligned} M_E &= \frac{\sigma I}{y} \\ &= \sigma_y \left[\frac{BD^3}{12} - \frac{bd^3}{12} \right] \frac{2}{D} \end{aligned}$$

If the moment is then increased to produce full plasticity across the section, i.e. a plastic hinge, the stress distribution is as shown in Fig. 3.5(b) and the value of the moment is obtained by applying eqn. (3.3) to the same convenient rectangles considered above.

$$M_{FP} = \sigma_y \left[\frac{BD^2}{4} - \frac{bd^2}{4} \right]$$

The value of the shape factor can then be obtained as the ratio of the above equations M_{FP}/M_E . A typical value of shape factor for commercial rolled steel joists is 1.18, thus indicating only an 18% increase in "strength" capacity using plastic design procedures compared with the 50% of the simple rectangular section.

3.4. Partially plastic bending of unsymmetrical sections

Consider the T-section beam shown in Fig. 3.6. Whilst stresses remain within the elastic limit the position of the N.A. can be obtained in the usual way by taking moments of area

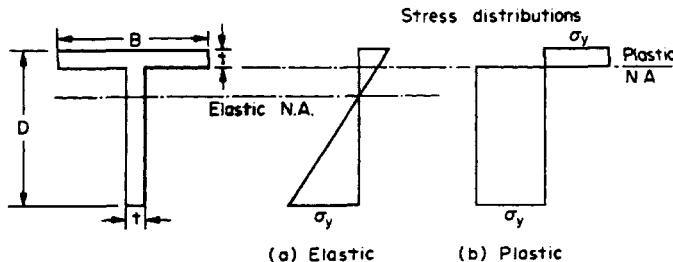


Fig. 3.6. Plastic bending of unsymmetrical (T-section) beam.

about some convenient axis as described in Chapter 4.[†] A typical position of the elastic N.A. is shown in the figure. Application of the simple blending theory about the N.A. will then yield the value of M_E as described in the previous paragraph.

Whatever the state of the section, be it elastic, partially plastic or fully plastic, equilibrium of forces must always be maintained, i.e. at any section the tensile forces on one side of the N.A. must equal the compressive forces on the other side.

$$\sum \text{stress} \times \text{area above N.A.} = \sum \text{stress} \times \text{area below N.A.}$$

In the *fully plastic* condition, therefore, when the stress is equal throughout the section, the above equation reduces to

$$\sum \text{areas above N.A.} = \sum \text{areas below N.A.} \quad (3.6)$$

and in the special case shown in Fig. 3.5 the N.A. will have moved to a position coincident with the lower edge of the flange. Whilst this position is peculiar to the particular geometry chosen for this section it is true to say that for all unsymmetrical sections the N.A. will move from its normal position when the section is completely elastic as plastic penetration proceeds. In the ultimate stage when a plastic hinge has been formed the N.A. will be positioned such that eqn. (3.6) applies, or, often more conveniently,

$$\text{area above or below N.A.} = \frac{1}{2} \text{ total area} \quad (3.7)$$

In the partially plastic state, as shown in Fig. 3.7, the N.A. position is again determined by applying equilibrium conditions to the forces above and below the N.A. The section is divided into convenient parts, each subjected to a force = average stress \times area, as indicated, then

$$F_1 + F_2 = F_3 + F_4 \quad (3.8)$$

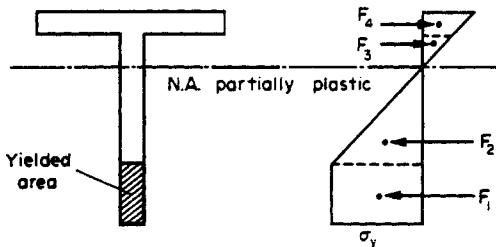


Fig. 3.7. Partially plastic bending of unsymmetrical section beam.

and this is an equation in terms of a single unknown \bar{y}_p , which can then be determined, as can the independent values of F_1 , F_2 , F_3 and F_4 .

The sum of the moments of these forces about the N.A. then yields the value of the partially plastic moment M_{PP} . Example 3.2 describes the procedure in detail.

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

3.5. Shape factor – unsymmetrical sections

Whereas with symmetrical sections the position of the N.A. remains constant as the axis of symmetry through the centroid, in the case of unsymmetrical sections additional work is required to take account of the movement of the N.A. position. However, having determined the position of the N.A. in the fully plastic condition using eqn. (3.6) or (3.7), the procedure outlined in §3.2 can then be followed to evaluate shape factors of unsymmetrical sections – see Example 3.2.

3.6. Deflections of partially plastic beams

Deflections of partially plastic beams are normally calculated on the assumption that the yielded areas, having yielded, offer no resistance to bending. Deflections are calculated therefore on the basis of the elastic core only, i.e. by application of simple bending theory and/or the standard deflection equations of Chapter 5[†] to the elastic material only. Because the second moment of area I of the central core is proportional to the fourth power of d , and I appears in the denominator of deflection formulae, deflections increase rapidly as d approaches zero, i.e. as full plasticity is approached.

If an experiment is carried out to measure the deflection of beams as loading, and hence B.M., is increased, the deflection graph for simply supported end conditions will appear as shown in Fig. 3.8. Whilst the beam is elastic the graph remains linear. The initiation of yielding in the outer fibres of the beam is indicated by a slight change in slope, and when plastic penetration approaches the centre of the section deflections increase rapidly for very small increases in load. For rectangular sections the ratio M_{FP}/M_E will be 1.5 as determined theoretically above.

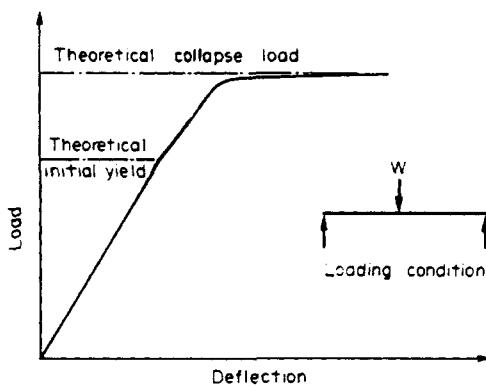


Fig. 3.8. Typical load-deflection curve for plastic bending.

3.7. Length of yielded area in beams

Consider a simply supported beam of rectangular section carrying a central concentrated load W . The B.M. diagram will be as shown in Fig. 3.9 with a maximum value of $WL/4$ at

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

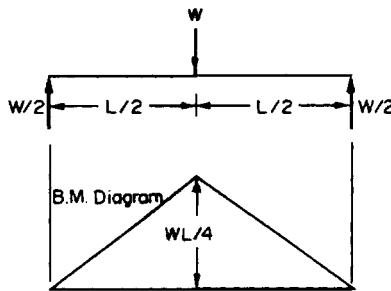


Fig. 3.9.

the centre. If loading is increased, yielding will commence therefore at the central section when $(WL/4) = (BD^2/6)\sigma_y$, and will gradually penetrate from the outside fibres towards the N.A. As this proceeds with further increase in loads, the B.M. at points away from the centre will also increase, and in some other positions near the centre it will also reach the value required to produce the initial yielding, namely $BD^2\sigma_y/6$. Thus, when full plasticity is achieved at the central section with a load W_p , there will be some other positions on either side of the centre, distance x from the supports, where yielding has just commenced at the outer fibres; between these two positions the beam will be in some elastic-plastic state. Now at distance x from the supports:

$$\text{B.M.} = W_p \frac{x}{2} = \frac{2}{3} M_{FP} = \frac{2}{3} \frac{W_p L}{4}$$

$$\therefore x = \frac{L}{3}$$

The central third of the beam span will be affected therefore by plastic yielding to some depth. At any general section within this part of the beam distance x' from the supports the B.M. will be given by

$$\text{B.M.} = W_p \frac{x'}{2} = \frac{B\sigma_y}{12} [3D^2 - d^2] \quad (1)$$

$$\text{Now since } \frac{BD^2}{4}\sigma_y = W_p \frac{L}{4} \quad \sigma_y = \frac{W_p L}{BD^2}$$

Therefore substituting in (1),

$$W_p \frac{x'}{2} = \frac{B}{12} [3D^2 - d^2] \frac{W_p L}{BD^2}$$

$$x' = \frac{(3D^2 - d^2)}{6D^2} L$$

$$x' = \frac{L}{2} \left[1 - \frac{d^2}{3D^2} \right]$$

This is the equation of a parabola with

$$x' = L/2 \text{ when } d = 0 \text{ (i.e. fully plastic section)}$$

and $x' = L/3$ when $d = D$ (i.e. section elastic)

The yielded portion of the beam is thus as indicated in Fig. 3.10.

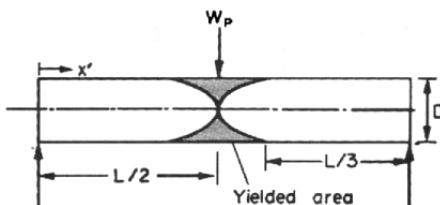


Fig. 3.10. Yielded area in beam carrying central point load.

Other beam support and loading cases may be treated similarly. That for a simply supported beam carrying a uniformly distributed load produces linear boundaries to the yielded areas as shown in Fig. 3.11.

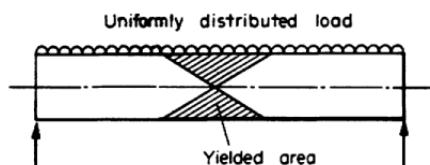


Fig. 3.11. Yielded area in beam carrying uniformly distributed load.

3.8. Collapse loads – plastic limit design

Having determined the moment required to produce a plastic hinge for the shape of beam cross-section used in any design it is then necessary to decide from a knowledge of the support and loading conditions how many such hinges are required before complete collapse of the beam or structure takes place, and to calculate the corresponding load. Here it is necessary to consider a plastic hinge as a pin-joint and to decide how many pin-joints are required to convert the structure into a "mechanism". If there are a number of points of "local" maximum B.M., i.e. peaks in the B.M. diagram, the first plastic hinge will form at the numerical maximum; if further plastic hinges are required these will occur successively at the next highest value of maximum B.M. etc. It is assumed that when a plastic hinge has developed at any cross-section the moment of resistance at that point remains constant until collapse of the whole structure takes place owing to the formation of the required number of further plastic hinges.

Consider, therefore, the following loading cases.

(a) Simply supported beam or cantilever

Whatever the loading system there will only be one point of maximum B.M. and plastic collapse will occur with **one** plastic hinge at this point (Fig. 3.12).

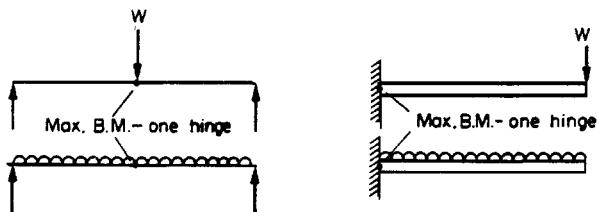


Fig. 3.12.

(b) Propped cantilever

In the case of propped cantilevers, i.e. cantilevers carrying opposing loads, the B.M. diagram is as shown in Fig. 3.13. The maximum B.M. then occurs at the built-in support and a plastic hinge forms first at this position. Due to the support of the prop, however, the beam does not collapse at this stage but requires another plastic hinge before complete failure or collapse occurs. This is formed at the other local position of maximum B.M., i.e. at the prop position, the moments at the support remaining constant until that at the prop also reaches the value required to form a plastic hinge.

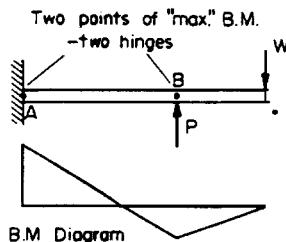


Fig. 3.13.

Collapse therefore occurs when $M_A = M_B = M_{FP}$, and thus **two** plastic hinges are required.

(c) Built-in beam

In this case there are three positions of local maximum B.M., two of them being at the supports, and **three** plastic hinges are required for collapse (Fig. 3.14).

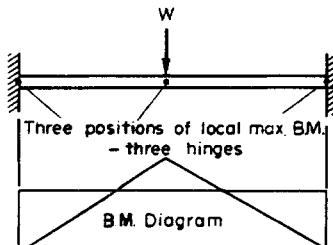


Fig. 3.14.

Other structures may require even more plastic hinges depending on their particular support conditions and degree of redundancy, but these need not be considered here. It should be evident, however, that there is now even more strength or load-carrying capacity available beyond that suggested by the shape factor, i.e. with a knowledge of the yield stress and hence the maximum elastic moment for any particular cross-section, the shape factor determines the increase in moment required to produce a fully plastic section or plastic hinge; depending on the support and loading conditions it may then be possible to increase the moment beyond this value until a sufficient number of plastic hinges have been formed to produce complete collapse. In order to describe the increased strength available using this "plastic limit" or "collapse load" procedure a *load factor* is introduced defined as

$$\text{load factor} = \frac{\text{collapse load}}{\text{allowable working load}} \quad (3.9)$$

This is completely different from, and must not be confused with, the safety factor, which is a factor to be applied to the yield stress in simple *elastic* design procedures.

3.9. Residual stresses after yielding: elastic-perfectly plastic material

Reference to the results of simple tensile or proof tests detailed in §1.7[†] shows that when materials are loaded beyond the yield point the resulting deformation does not disappear completely when load is removed and the material is subjected to permanent deformation or so-called *permanent set* (Fig. 3.15). In bending applications, therefore, when beams may be subjected to moments producing partial plasticity, i.e. part of the beam section remains elastic whilst the outer fibres yield, this permanent set associated with the yielded areas prevents those parts of the material which are elastically stressed from returning to their unstressed state when load is removed. *Residual stress* are therefore produced. In order to determine the magnitude of these residual stresses it is normally assumed that the unloading process, from either partially plastic or fully plastic states, is completely elastic (see Fig. 3.15). The

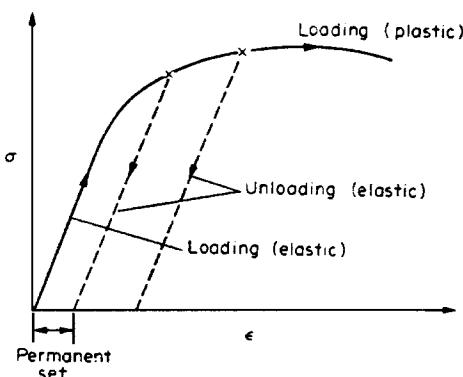


Fig. 3.15. Tensile test stress-strain curve showing elastic unloading process from any load condition.

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

unloading stress distribution is therefore linear and it can be subtracted graphically from the stress distribution in the plastic or partially plastic state to obtain the residual stresses.

Consider, therefore, the rectangular beam shown in Fig. 3.16 which has been loaded to its fully plastic condition as represented by the stress distribution rectangles $oabc$ and $odef$. The bending stresses which are then superimposed during the unloading process are given by the line goh and are opposite to sign. Subtracting the two distributions produces the shaded areas which then indicate the residual stresses which remain after unloading the plastically deformed beam. In order to quantify these areas, and the values of the residual stresses, it should be observed that the loading and unloading moments must be equal, i.e. the moment of the force due to the rectangular distribution $oabc$ about the N.A. must equal the moment of the force due to the triangular distribution oag .

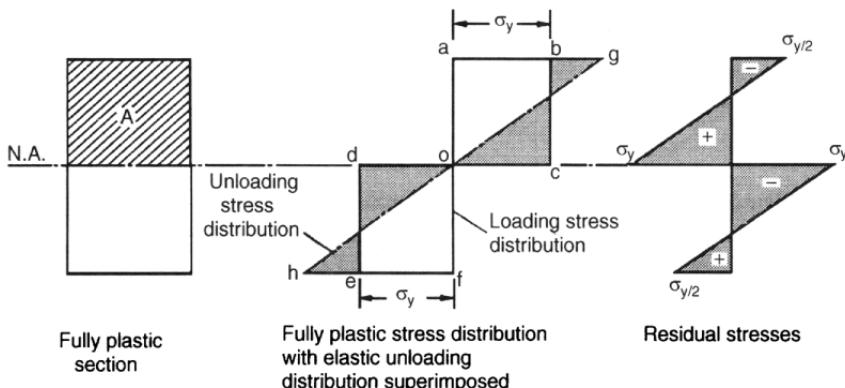


Fig. 3.16. Residual stresses produced after unloading a rectangular-section beam from a fully plastic state.

Now, moment due to $oabc$

$$\begin{aligned} &= \text{stress} \times \text{area} \times \text{moment arm} \\ &= ab \times A \times oa/2 \end{aligned}$$

and moment due to oag

$$\begin{aligned} &= \text{average stress} \times \text{area} \times \text{moment arm} \\ &= ag/2 \times A \times 2oa/3 \end{aligned}$$

Equating these values of moment yields

$$ag = \frac{3}{2}ab$$

$$\text{Now } ab = \text{yield stress } \sigma_y \quad \therefore ag = 1\frac{1}{2}\sigma_y$$

Thus the residual stresses at the outside surfaces of the beam $= \frac{1}{2}\sigma_y$. The maximum residual stresses occur at the N.A. and are equal to the yield stress. The complete residual stress distribution is shown in Fig. 3.16.

In loading cases where only partial plastic bending has occurred in the beam prior to unloading the stress distributions obtained, using a similar procedure to that outlined above, are shown in Fig. 3.17. Again, the unloading process is assumed elastic and the line goh in

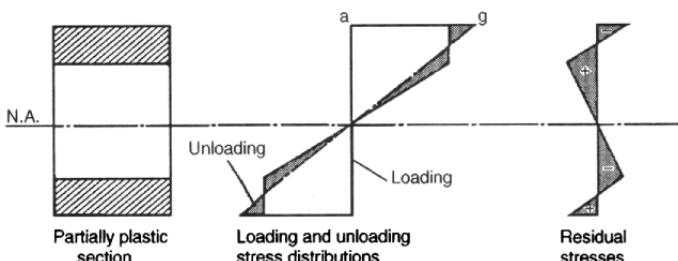


Fig. 3.17. Residual stress produced after unloading a rectangular-section beam from a partially plastic state.

this case is positioned such that the moments of the loading and unloading stress distributions are once more equal, i.e. the stress at the outside fibre ag is determined by considering the plastic moment M_{pp} applied to the beam assuming it to be elastic; thus

$$ag = \sigma = \frac{M_y}{I} = \frac{M_{pp} D}{I / 2}$$

Whereas in the previous case the maximum residual stress occurs at the centre of the beam, in this case it may occur either at the outside or at the inner boundary of the yielded portion depending on the depth of plastic penetration. There is no residual stress at the centre of the beam.

Because of the permanent set mentioned above and the resulting stresses, beams which have been unloaded from plastic or partially plastic states will be deformed from their original shape. The straightening moment which is required at any section to return the beam to its original position is that which is required to remove the residual stresses from the elastic core (see Example 3.3).

The residual or permanent radius of curvature R after load is removed can be found from

$$\frac{1}{R} = \frac{1}{R_E} - \frac{1}{R_P} \quad (3.10)$$

where R_P is the radius of curvature in the plastic condition and R_E is the elastic spring-back, calculated by applying the simple bending theory to the complete section with a moment of M_{pp} or M_{fp} as the case may be.

3.10. Torsion of shafts beyond the elastic limit – plastic torsion

The method of treatment of shafts subjected to torques sufficient to initiate yielding of the material is similar to that used for plastic bending of beams (§3.1), i.e. it is usual to assume a stress-strain curve for the shaft material of the form shown in Fig. 3.2, the stress being proportional to strain up to the elastic limit and constant thereafter. It is also assumed that plane cross-sections remain plane and that any radial line across the section remains straight.

Consider, therefore, the cross-section of the shaft shown in Fig. 3.18(a) with its associated shear stress distribution. Whilst the shaft remains elastic the latter remains linear, and as the torque increases the shear stress in the outer fibres will eventually reach the yield stress in shear of the material τ_y . The torque at this point will be the maximum that the shaft can withstand whilst it is completely elastic.

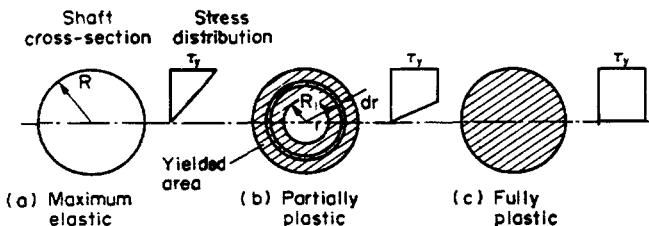


Fig. 3.18. Plastic torsion of a circular shaft.

From the torsion theory

$$\frac{T}{J} = \frac{\tau}{r}$$

Therefore maximum elastic torque

$$\begin{aligned} T_E &= \frac{\tau_y J}{R} = \frac{\tau_y \pi R^4}{R} \frac{\pi}{2} \\ &= \frac{\pi R^3}{2} \tau_y \end{aligned} \quad (3.11)$$

If the torque is now increased further it is assumed that, instead of the stress in the outer fibre increasing beyond τ_y , more and more of the material will yield and take up the stress τ_y , giving the stress distribution shown in Fig. 3.18(b). Consider the case where the material has yielded to a radius R_1 , then:

Partially plastic torque

$$T_{PP} = \text{torque owing to elastic core} + \text{torque owing to plastic portion}$$

The first part is obtained directly from eqn. (3.11) with R_1 replacing R ,

$$\text{i.e. } \frac{\pi R_1^3}{2} \tau_y$$

For the second part consider an element of radius r and thickness dr , carrying a stress τ_y , (see Fig. 3.18(b)),

$$\text{force on element} = \tau_y \times 2\pi r dr$$

$$\text{contribution to torque} = \text{force} \times \text{radius}$$

$$= (\tau_y \times 2\pi r dr)r$$

$$= 2\pi r^2 dr \tau_y$$

$$\therefore \text{total contribution} = \int_{R_1}^R \tau_y 2\pi r^2 dr$$

$$= 2\pi \tau_y \left[\frac{r^3}{3} \right]_{R_1}^R$$

$$= \frac{2\pi \tau_y}{3} [R^3 - R_1^3]$$

Therefore, partially plastic torque

$$\begin{aligned} T_{PP} &= \frac{\pi R_1^3}{2} \tau_y + \frac{2\pi}{3} \tau_y [R^3 - R_1^3] \\ &= \frac{\pi \tau_y}{6} [4R^3 - R_1^3] \end{aligned} \quad (3.12)$$

In Fig. 3.18(c) the torque has now been increased until the whole cross-section has yielded, i.e. become plastic. The torque required to reach this situation is then easily determined from eqn. (3.12) since $R_1 = 0$.

$$\begin{aligned} \therefore \text{fully plastic torque } T_{FP} &= \frac{\pi \tau_y}{6} \times 4R^3 \\ &= \frac{2\pi}{3} R^3 \tau_y \end{aligned} \quad (3.13)$$

There is thus a considerable torque capacity beyond that required to produce initial yield, the ratio of fully plastic to maximum elastic torques being

$$\begin{aligned} \frac{T_{FP}}{T_E} &= \frac{2\pi R^3}{3} \tau_y \times \frac{2}{\pi R^3} \tau_y \\ &= \frac{4}{3} \end{aligned}$$

The fully plastic torque for a solid shaft is therefore 33% greater than the maximum elastic torque. As in the case of beams this can be taken account of in design procedures to increase the allowable torque which can be carried by the shaft or it may be treated as an additional safety factor. In any event it must be remembered that should stresses in the shaft at any time exceed the yield point for the material, then some permanent deformation will occur.

3.11. Angles of twist of shafts strained beyond the elastic limit

Angles of twist of shafts in the partially plastic condition are calculated on the basis of the elastic core only, thus assuming that once the outer regions have yielded they no longer offer any resistance to torque. This is in agreement with the basic assumption listed earlier that radial lines remain straight throughout plastic torsion, i.e. $\theta_{PP} = \theta_E$ for the core.

For the elastic core, therefore,

$$\begin{aligned} \frac{\tau_y}{R_1} &= \frac{G\theta}{L} \\ \text{i.e. } \theta_{PP} &= \frac{\tau_y L}{R_1 G} \end{aligned} \quad (3.14)$$

3.12. Plastic torsion of hollow tubes

Consider the hollow tube of Fig. 3.19 with internal radius R_1 and external radius R subjected to a torque sufficient to produce yielding to a radius R_2 . The torque carried by the

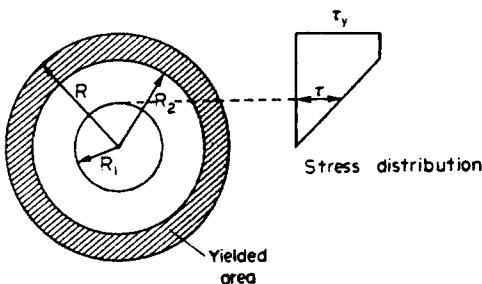


Fig. 3.19. Plastic torsion of a hollow shaft.

equivalent partially plastic solid shaft, i.e. ignoring the central hole, is given by eqn. (3.12) with R_2 replacing R_1 as

$$\frac{\pi\tau_y}{6}[4R^3 - R_2^3]$$

The torque carried by the hollow tube can then be determined by subtracting from the above the torque which would be carried by a solid shaft of diameter equal to the central hole and subjected to a shear stress at its outside fibre equal to τ . i.e. from eqn. (3.11) torque on imaginary shaft

$$= \frac{\pi R_1^3}{2} \tau$$

but by proportions of the stress distribution diagram

$$\tau = \frac{R_1}{R_2} \tau_y$$

Therefore torque on imaginary shaft equal in diameter to the hollow core

$$= \frac{\pi R_1^4}{2R_2} \tau_y$$

Therefore, partially plastic torque for the hollow tube

$$\begin{aligned} T_{PP} &= \frac{\pi\tau_y}{6}[4R^3 - R_2^3] - \frac{\pi}{2} \frac{R_1^4}{R_2} \tau_y \\ &= \frac{\pi\tau_y}{6R_2} [4R^3 R_2 - R_2^4 - 3R_1^4] \end{aligned} \quad (3.15)$$

The fully plastic torque is then obtained when $R_2 = R_1$,

$$\text{i.e. } T_{FP} = \frac{\pi\tau_y}{6R_1} [4R^3 R_1 - 4R_1^4] = \frac{2\pi\tau_y}{3} [R^3 - R_1^3] \quad (3.16)$$

This equation could also have been obtained by adaptation of eqn. (3.13), subtracting a fully plastic core of diameter equal to the central hole.

As an aid in visualising the stresses and torque capacities of members loaded to the fully plastic condition an analogy known as the *sand-heap analogy* has been introduced. Whilst full details have been given by Nadai[†] it is sufficient for the purpose of this text to note that

[†] A. Nadai, *Theory of Flow and Fracture of Solids*, Vol. 1, 2nd edn., McGraw-Hill, New York, 1950.

if dry sand is poured on to a raised flat surface having the same shape as the cross-section of the member under consideration, the sand heap will assume a constant slope, e.g. a cone on a circular disc and a pyramid on a square base. The volume of the sand heap, and hence its weight, is then found to be directly proportional to the fully plastic torque which would be carried by that particular shape of cross-section. Thus by calibration, i.e. with a knowledge of the fully plastic torque for a circular shaft, direct comparison of the weight of appropriate sand heaps yields an immediate indication of the fully plastic torque of some other more complicated section.

3.13. Plastic torsion of case-hardened shafts

Consider now the case-hardened shaft shown in Fig. 3.20. Whilst it is often assumed in such cases that the shear-modulus is the same for the material of the case and core, this is certainly not the case for the yield stresses; indeed, there is often a considerable difference, the value for the case being generally much larger than that for the core. Thus, when the shaft is subjected to a torque sufficient to initiate yielding at the outside fibres, the normal triangular elastic stress distribution required to maintain straight radii must be modified, since this would imply that some of the core material is stressed beyond its yield stress. Since the basic assumption used throughout this treatment is that stress remains constant at the yield stress for any increase in strain, it follows that the stress distribution must be as indicated in Fig. 3.20. The shaft thus contains at this stage a plastic region sandwiched between two elastic layers. Torques for each portion must be calculated separately, therefore, and combined to yield the partially plastic torque for the case-hardened shaft. (Example 3.5.)

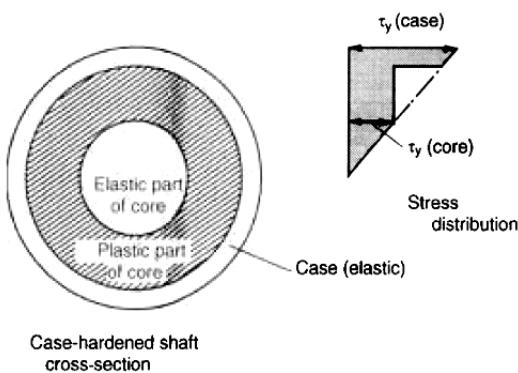


Fig. 3.20. Plastic torsion of a case-hardened shaft.

3.14. Residual stresses after yield in torsion

If shafts are stressed at any time beyond their elastic limit to a partially plastic state as described previously, a permanent deformation will remain when torque is removed. Associated with this plastic deformation will be a system of residual stresses which will affect the strength of the shaft in subsequent loading cycles. The magnitudes of the residual stresses are determined using the method described in detail for beams strained beyond the

elastic limit on page 73, i.e. the removal of torque is assumed to be a completely elastic process so that the associated stress distribution is linear. The residual stresses are thus obtained by subtracting the elastic unloading stress distribution from that of the partially plastic loading condition. Now, from eqn. (3.12), partially plastic torque = T_{PP} .

Therefore elastic torque to be applied during unloading = T_{PP} .

The stress τ' at the outer fibre of the shaft which would be achieved by this torque, assuming elastic material, is given by the torsion theory

$$\frac{T}{J} = \frac{\tau}{R}, \quad \text{i.e. } \tau' = \frac{T_{PP}R}{J}$$

Thus, for a solid shaft the residual stress distribution is obtained as shown in Fig. 3.21.

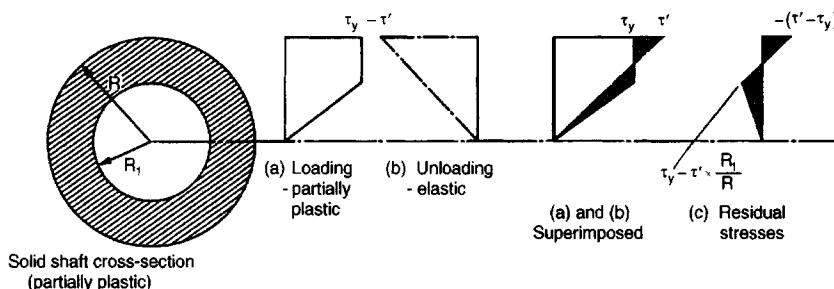


Fig. 3.21. Residual stresses produced in a solid shaft after unloading from a partially plastic state.

Similarly, for hollow shafts, the residual stress distribution will be as shown in Fig. 3.22.

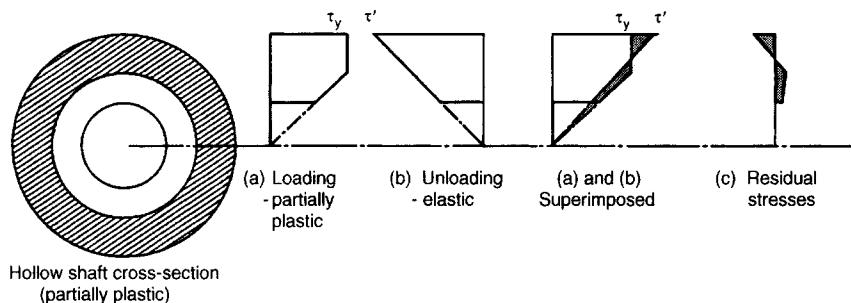


Fig. 3.22. Residual stresses produced in a hollow shaft after unloading from a partially plastic state.

3.15. Plastic bending and torsion of strain-hardening materials

(a) Inelastic bending

Whilst the material in this case no longer follows Hooke's law it is necessary to assume that cross-sections of the beam remain plane during bending so that strains remain proportional to distance from the neutral axis.

Consider, therefore, the rectangular section beam shown in Fig. 3.23(b) with its neutral axis positioned at a distance h_1 from the lower surface and h_2 from the upper surface. Bearing in mind the assumption made in the preceding paragraph we can now locate the neutral axis position by the usual equilibrium conditions.

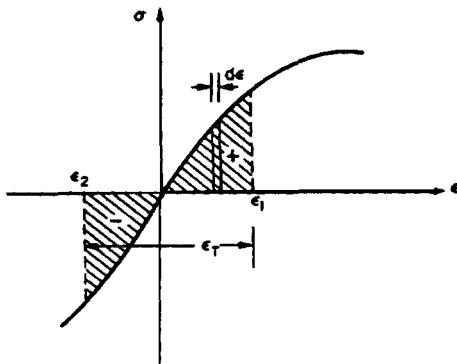


Fig. 3.23(a). Stress-strain curve for a beam in bending constructed from a strain-hardening material.

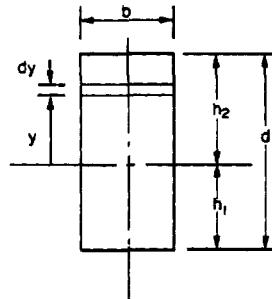


Fig. 3.23(b).

i.e. Since the sum of forces normal to any cross-section must always be zero then:

$$\int \sigma dA = \int_{-h_1}^{h_2} \sigma \cdot b dy = 0$$

But, from eqn (4.1)[†]

$$y = R \frac{\sigma}{E} = R\varepsilon \quad \therefore dy = R d\varepsilon$$

$$\therefore \int_{-h_1}^{h_2} \sigma b R d\varepsilon = 0$$

$$\text{or } \int_{\varepsilon_1}^{\varepsilon_2} \sigma b R d\varepsilon = 0.$$

where ε_1 and ε_2 are the strains in the top and bottom surfaces of the beam, respectively. They are also indicated on Fig. 3.23(a).

Since b and R are constant then the position of the neutral axis must be such that:

$$\int_{\varepsilon_1}^{\varepsilon_2} \sigma d\varepsilon = 0 \tag{3.17}$$

i.e. the total area under the $\sigma-\varepsilon$ curve between ε_1 and ε_2 must be zero. This is achieved by marking the length ε_T on the horizontal axis of Fig. 3.23(a) in such a way as to make the positive and negative areas of the diagram equal. This identifies the appropriate values for ε_1 and ε_2 with:

$$\varepsilon_T = |\varepsilon_1 + \varepsilon_2| = \frac{h_1}{R} + \frac{h_2}{R} = \frac{1}{R}(h_1 + h_2)$$

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

i.e. $\epsilon_T = \frac{d}{R}$ (3.18)

Because strains have been assumed linear with distance from the neutral axis the position of the N.A. is then obtained by simple proportions:

$$\frac{h_1}{h_2} = \frac{\epsilon_1}{\epsilon_2} \quad (3.19)$$

The value of the applied bending moment M is then given by the sum of the moments of forces above and below the neutral axis.

i.e. $M = \int \sigma dA \cdot y = \int_{-h_1}^{h_2} \sigma \cdot b dy \cdot y$

and, since $dy = R \cdot d\epsilon$ and $y = Re$.

$$M = \int_{\epsilon_1}^{\epsilon_2} \sigma b \cdot R^2 \epsilon d\epsilon = R^2 b \int_{\epsilon_1}^{\epsilon_2} \sigma \epsilon d\epsilon.$$

Substituting, from eqn. (3.18), $R = d/\epsilon_T$:

$$M = \frac{bd^2}{\epsilon_T^2} \int_{\epsilon_1}^{\epsilon_2} \sigma \epsilon d\epsilon. \quad (3.20)$$

The integral part of this expression is the first moment of area of the shaded parts of Fig. 3.23(a) about the vertical axis and evaluation of this integral allows the determination of M for any assumed value of ϵ_T .

An alternative form of the expression is obtained by multiplying the top and bottom of the expression by $12R$ using $R = d/\epsilon_T$ for the numerator,

i.e. $M = 12 \frac{(d/\epsilon_T)}{12R} \left[\frac{bd^2}{\epsilon_T^2} \int_{\epsilon_1}^{\epsilon_2} \sigma \epsilon d\epsilon \right] = \frac{1}{R} \cdot \frac{bd^3}{12} \cdot \frac{12}{\epsilon_T^3} \int_{\epsilon_1}^{\epsilon_2} \sigma \epsilon d\epsilon.$

which can be reduced to a form similar to the standard bending eqn. (4.3)[†] $M = EI/R$

i.e. $M = \frac{E_r I}{R} \quad (3.21)$

with E_r known as the *reduced modulus* and given by:

$$E_r = \frac{12}{\epsilon_T^3} \int_{\epsilon_1}^{\epsilon_2} \sigma \epsilon d\epsilon. \quad (3.22)$$

The appropriate value of the reduced modulus E_r for any particular curvature is best obtained from a curve of E_r against ϵ_T . This is constructed rather laboriously by determining the relevant values of ϵ_1 and ϵ_2 for a set of assumed ϵ_T values using the condition of equal positive and negative areas for each ϵ_T value and then evaluating the integral of eqn. (3.22). Having found E_r , the value of the bending moment for any given curvature R is found from eqn. (3.21).

It is sometimes useful to remember that, because strains are linear with distance from the neutral axis, the distribution of bending stresses across the beam section will take exactly the same form as that of the stress-strain diagram of Fig. 3.23(a) turned through 90° with

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1977.

ε_T replaced by the beam depth d . The position of the neutral axis indicated by eqn. (3.19) is then readily observed.

(b) Inelastic torsion

A similar treatment can be applied to the torsion of shafts constructed from materials which exhibit strain hardening characteristics. Figure 3.24 shows the shear stress–shear strain curve for such a material.

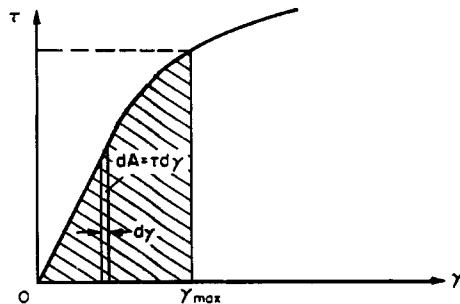


Fig. 3.24. Shear stress–shear strain curve for torsion of materials exhibiting strain-hardening characteristics.

Once again it is necessary to assume that cross-sections of the shaft remain plane and that the radii remain straight under torsion. The shear strain at any radius r is then given by eqn. (8.9)[†] as:

$$\gamma = \frac{r\theta}{L}$$

For a shaft of radius R the maximum shearing strain is thus

$$\gamma_{\max} = \frac{R\theta}{L}$$

the corresponding shear stress being given by the relevant ordinate of Fig. 3.24.

Now the torque T has been shown in §8.1[†] to be given by:

$$T = \int_0^R 2\pi r^2 \tau' dr$$

where τ' is the shear stress at any general radius r .

Now, since $\gamma = \frac{r\theta}{L}$ then $d\gamma = \frac{\theta}{L} \cdot dr$

and, substituting for r and dr , we have:

$$\begin{aligned} T &= \int_0^{\gamma_{\max}} 2\pi \left(\frac{\gamma L}{\theta} \right)^2 \tau' \frac{L}{\theta} \cdot d\gamma \\ &= \frac{2\pi L^3}{\theta^3} \int_0^{\gamma_{\max}} \tau' \gamma^2 d\gamma \end{aligned} \quad (3.23)$$

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

The integral part of the expression is the second moment of area of the shaded portion of Fig. 3.24 about the vertical axis. Thus, determination of this quantity for a given y_{\max} value yields the corresponding value of the applied torque T .

As for the case of inelastic bending, the form of the shear stress-strain curve, Fig. 3.24, is identical to the shear stress distribution across the shaft section with the γ axis replaced by radius r .

3.16. Residual stresses – strain-hardening materials

The procedure for determination of residual stresses arising after unloading from given stress states is identical to that described in §3.9 and §3.14.

For example, it has been shown previously that the stress distribution across a beam section in inelastic bending will be similar to that shown in Fig. 3.23(a) with the beam depth corresponding to the strain axis. Application of the elastic unloading stress distribution as described in §3.9 will then yield the residual stress distribution shown in Fig. 3.25. The same procedure should be adopted for residual stresses in torsion situations, reference being made to §3.14.

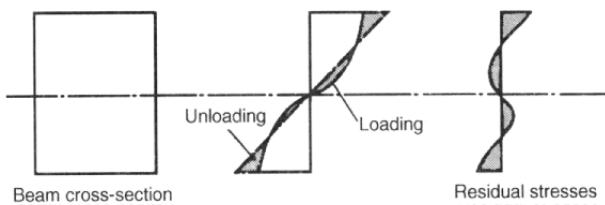


Fig. 3.25. Residual stresses produced in a beam constructed from a strain-hardening material.

3.17. Influence of residual stresses on bending and torsional strengths

The influence of residual stresses on the future loading of members has been summarised by Juvinal[†] into the following rule:

An overload causing yielding produces residual stresses which are favourable to future overloads in the same direction and unfavourable to future overloads in the opposite direction.

This suggests that the residual stresses represent a favourable stress distribution which has to be overcome by any further load system before any adverse stress can be introduced into the member or structure. This principle is taken advantage of by spring manufacturers, for example, who intentionally yield springs in the direction of anticipated service loads as part of the manufacturing process. A detailed discussion of residual stress can be found in the *Handbook of Experimental Stress Analysis* of Hetényi.[‡]

[†] R. C. Juvinal, *Engineering Considerations of Stress, Strain and Strength*, McGraw-Hill, 1967.

[‡] M. Hetényi, *Handbook of Experimental Stress Analysis*, John Wiley, 1966.

3.18. Plastic yielding in the eccentric loading of rectangular sections

When a column or beam is subjected to an axial load and a B.M., as in the application of eccentric loads, the elastic stress distribution is as shown in Fig. 3.25(a), the N.A. being displaced from the centroidal axis of the section. As the load increases the yield stress will be reached on one side of the section first as shown in Fig. 3.26(b) and, as in the case of the partially plastic bending of unsymmetrical sections in §3.4, the N.A. will move as plastic penetration proceeds. In the limiting case, when plasticity has spread across the complete section, the N.A. will be situated at a distance h from the centroidal axis (the axis through the centroid of the section) (Fig. 3.26(c)). The precise position of the N.A. is related to the excess of the total tensile force over the total compressive force, i.e. to the area shown shaded in Fig. 3.26(c). In simple bending, for example, there is no resultant force across the section and the shaded area reduces to zero. Thus, the magnitude of the axial load for full plasticity as given by the shaded area

$$= P_{FP} = 2h \times B \times \sigma_y$$

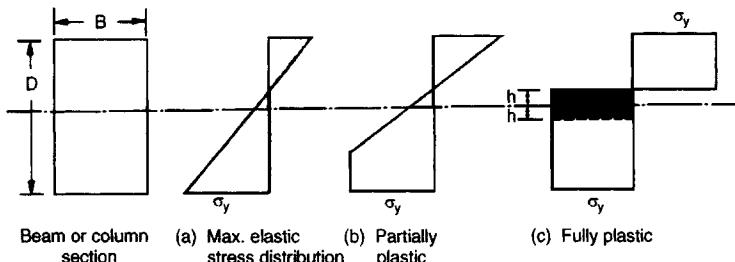


Fig. 3.26. Plastic yielding of eccentrically loaded rectangular section.

where B is the width of the section,

i.e.

$$h = \frac{P_{FP}}{2B\sigma_y} \quad (3.24)$$

The fully plastic load is sometimes written in terms of a load factor N defined as

$$\text{load factor } N = \frac{\text{fully plastic load}}{\text{axial load}} = \frac{P_{FP}}{P}$$

then

$$h = \frac{PN}{2B\sigma_y} \quad (3.25)$$

The fully plastic moment on the section is given by the difference in the moments produced by the stress distributions of Fig. 3.27,

i.e.

$$M_{FP} = \frac{BD^2}{4}\sigma_y - 2(Bh\sigma_y)\frac{h}{2}$$

$$M_{FP} = \frac{BD^2}{4}\sigma_y - Bh^2\sigma_y$$

$$= \frac{BD^2}{4}\sigma_y - \frac{P^2N^2}{4B\sigma_y} \quad (3.26)$$

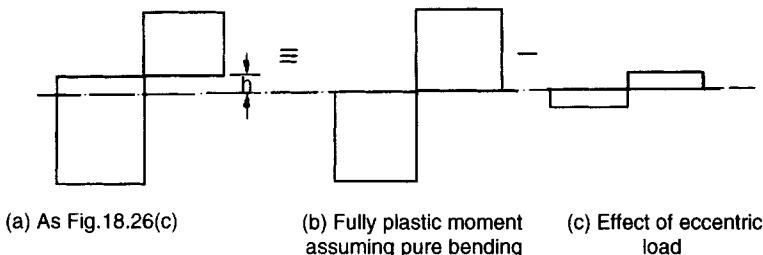


Fig. 3.27.

The fully plastic moment required in eccentric load conditions is therefore reduced from that in the simple bending case by an amount depending on the values of the load, yield stress, section shape and load factor.

The maximum allowable working moment for a single plastic hinge in eccentric loading situations with a load factor N is therefore given by

$$M = \frac{M_{FP}}{N} = \frac{BD^2}{4N} \sigma_y - \frac{P^2 N}{4B \sigma_y} \quad (3.27)$$

3.19. Plastic yielding and residual stresses under axial loading with stress concentrations

If a bar with uniform cross-section is loaded beyond its yield point in pure tension (or compression), the bar will experience permanent deformation when load is removed but no residual stresses will be created since all of the material in the cross-section will have yielded simultaneously and all will return to the same unloaded condition. If, however, stress concentrations such as notches, keyways, holes, etc., are present in the bar, these will result in local stress increases or *stress concentrations*, and the material will yield at these positions before the rest of the cross-section. If the local stress concentration factor is K then the maximum stress in the section with an axial load P is given by

$$\sigma_{\max} = K(P/A)$$

where P/A is the mean stress across the section assuming no stress concentration is present.

When the load has been increased to a value P_y , just sufficient to initiate yielding at the root of the notch or other stress concentration, the stress distribution will be as shown in Fig. 3.28(a). Since equilibrium considerations require the mean stress across the section to equal P_y/A it follows that the stress at the centre of the section must be less than P_y/A .

If the load is now increased to P_2 , yielding will continue at the root of the notch and plastic penetration will proceed towards the centre of the section. At some stage the stress distribution will appear as in Fig. 3.28(b) with a mean stress value of P_2/A . If the load is then removed the residual stresses may be obtained using the procedure of §§3.9 and 3.14, i.e. by superimposing an elastic stress distribution of opposite sign but equal moment value (shown dotted in Fig. 3.28(b)). The resulting residual stress distribution would then be similar to that shown in Fig. 3.28(c).

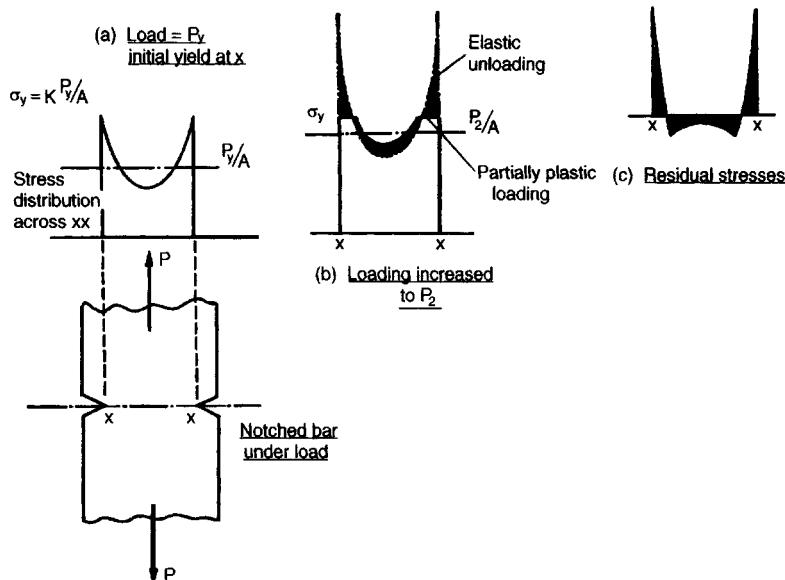


Fig. 3.28. Residual stresses at stress concentrations.

Whilst subsequent application of loads above the value of P_2 will cause further yielding, no yielding will be caused by the application of loads up to the value of P_2 however many times they are applied. With a sufficiently high value of stress concentration factor it is possible to produce a residual stress distribution which exceeds the compressive yield stress at the root of the notch, i.e. the material will be stressed from tensile yield to compressive yield throughout one cycle. Provided that further cycles remain within these limits, the component will not experience additional yielding, and it can be considered safe in, for example, high strain, low-cycle fatigue conditions.

3.20. Plastic yielding of axially symmetric components[†]

(a) Thick cylinders under internal pressure – collapse pressure

Consider the thick cylinder shown in Fig. 3.29 subjected to an internal pressure P_1 of sufficient magnitude to produce yielding to a radius R_p .

Now for ductile materials, from §10.17,[‡] yield is deemed to occur when

$$\sigma_y = \sigma_H - \sigma_r. \quad (3.28)$$

but from eqn. (10.2),[‡], the equilibrium equation,

$$\sigma_H - \sigma_r = r \frac{d\sigma_r}{dr}$$

[†] J. Heyman, *Proc. I.Mech.E.* **172** (1958). W.R.D. Manning, *High Pressure Engineering*, Bulleid Memorial Lecture, 1963, University of Nottingham.

[‡] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

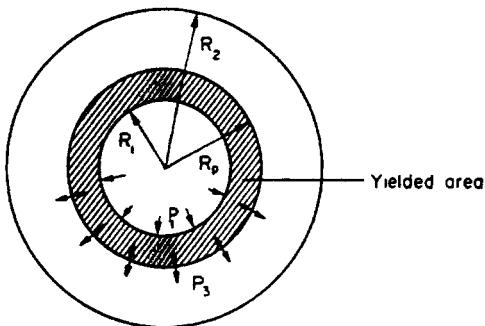


Fig. 3.29.

$$\therefore \sigma_r = r \frac{d\sigma_r}{dr}$$

$$\frac{d\sigma_r}{dr} = \frac{\sigma_y}{r}$$

Integrating: $\sigma_r = \sigma_y \log_e r + \text{constant}$

Now $\sigma_r = -P_3$ at $r = R_p$

$\therefore \text{constant} = -P_3 - \sigma_y \log_e R_p$

$\therefore \sigma_r = \sigma_y \log_e r - P_3 - \sigma_y \log_e R_p$

i.e. $\sigma_r = \sigma_y \log_e \frac{r}{R_p} - P_3 \quad (3.29)$

and from eqn. (3.28)

$$\sigma_H = \sigma_y + \sigma_r$$

$\therefore \sigma_H = \sigma_y \left(1 + \log_e \frac{r}{R_p} \right) - P_3 \quad (3.30)$

These equations thus yield the hoop and radial stresses *throughout the plastic zone* in terms of the radial pressure at the elastic-plastic interface P_3 . The *numerical* value of P_3 may be determined as follows (the sign has been allowed for in the derivation of eqn. (3.29)).

At the stage where plasticity has penetrated partly through the cylinder walls the cylinder may be considered as a compound cylinder with the inner tube plastic and the outer tube elastic, the latter being subjected to an internal pressure P_3 . From eqns. (10.5) and (10.6)[†] the hoop and radial stresses *in the elastic portion* are therefore given by

$$\sigma_r = \frac{P_3 R_p^2}{(R_2^2 - R_p^2)} \left[\frac{R_p^2 - R_2^2}{R_p^2} \right]$$

$$\sigma_H = \frac{P_3 R_p^2}{(R_2^2 - R_p^2)} \left[\frac{R_p^2 + R_2^2}{R_p^2} \right]$$

and

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

i.e. the maximum shear stress is

$$\frac{\sigma_H - \sigma_r}{2} = \frac{P_3 R_p^2}{2(R_2^2 - R_p^2)} \left[\frac{2R_2^2}{R_p^2} \right] = \frac{P_3 R_2^2}{(R_2^2 - R_p^2)}$$

Again, applying the Tresca yield criterion,

$$\frac{\sigma_y}{2} = \frac{P_3 R_2^2}{(R_2^2 - R_p^2)}$$

i.e. the radial pressure at the elastic interface is

$$P_3 = \frac{\sigma_y}{2R_2^2} [R_2^2 - R_p^2] \quad (3.31)$$

Thus from eqns. (3.29) and (3.30) the stresses in the plastic zone are given by

$$\sigma_r = \sigma_y \left[\log_e \frac{r}{R_p} - \frac{1}{2R_2^2} (R_2^2 - R_p^2) \right] \quad (3.32)$$

and $\sigma_H = \sigma_y \left[\left(1 + \log_e \frac{r}{R_p} \right) - \frac{1}{2R_2^2} (R_2^2 - R_p^2) \right]$ (3.33)

The pressure required for complete plastic “collapse” of the cylinder is given by eqn. (3.29) when $r = R_1$ and $R_p = R_2$ with $P_3 = P_2 = 0$ (at the outside edge).

For “collapse” $\sigma_r = -P_1 = \sigma_y \log_e \frac{R_1}{R_2}$ (3.34)

With a knowledge of this collapse pressure the design pressure can be determined by dividing it by a suitable load factor as described in §3.8.

The *pressure at initial yield* is found from eqn. (3.31) when $R_p = R_1$,

i.e. initial yield pressure $= \frac{\sigma_y}{2R_2^2} [R_2^2 - R_1^2]$ (3.35)

Finally, the *internal pressure required to cause yielding to a radius R_p* is given by eqn. (3.32) when $r = R_1$,

i.e. $\sigma_r = -P_1 = \sigma_y \left[\log_e \frac{R_1}{R_p} - \frac{1}{2R_2^2} (R_2^2 - R_p^2) \right]$ (3.36)

(b) Thick cylinders under internal pressure (“auto-fretage”)

When internal pressure is applied to thick cylinders it has been shown that maximum tensile stresses are set up at the inner surface of the bore. If the internal pressure is increased sufficiently, yielding of the cylinder material will take place at this position and the working safety factor n according to the Tresca theory will be given by

$$\sigma_H - \sigma_r = \sigma_y/n$$

where σ_H and σ_r are the hoop and radial stresses at the bore.

Fortunately, the condition is not too serious at this stage since there remains a considerable bulk of elastic material surrounding the yielded area which contains the resulting strains

within reasonable limits. As the pressure is increased further, however, plastic penetration takes place deeper and deeper into the cylinder wall and eventually the whole cylinder will yield. Fatigue life of the cylinder will also be heavily dependent upon the value of the maximum tensile stress at the bore so that any measures which can be taken to reduce the level of this stress will be beneficial to successful operation of the cylinder. Such methods include the use of compound cylinders with force or shrink fits and/or external wire winding; the largest effect is obtained, however, with a process known as "autofrettage".

If the pressure inside the cylinder is increased beyond the initial yield value so that plastic penetration occurs only partly into the cylinder wall then, on release of the pressure, the elastic zone attempts to return to its original dimensions but is prevented from doing so by the permanent deformation or "set" of the yielded material. The result is that residual stresses are introduced, the elastic material being held in a state of residual tension whilst the inside layers are brought into residual compression. On subsequent loading cycles, therefore, the cylinder is able to withstand a higher internal pressure since the compressive residual stress at the bore has to be overcome before this region begins to experience tensile stresses. The autofrettage process has the same effect as shrinking one tube over another without the complications of the shrinking process. With careful selection of cylinder dimensions and autofrettage pressure the resulting residual compressive stresses can significantly reduce or even totally eliminate tensile stresses which would otherwise be achieved at the bore under working conditions. As a result the fatigue life and the safety factor at the bore are considerably enhanced and for this reason gun barrels and other pressure vessels are often pre-stressed in this way prior to service.

Care must be taken in the design process, however, since the autofrettage process introduces a secondary critical stress region at the position of the elastic/plastic interface of the autofrettage pressure loading condition. This will be discussed further below.

The autofrettage pressure required for yielding to any radius R_p is given by the High Pressure Technology Association (HPTA) code of practice[†] as

$$P_A = \frac{\sigma_y}{2} \left[\frac{K^2 - m^2}{K^2} \right] + \sigma_y \log_e m \quad (3.37)$$

where $K = R_2/R_1$ and $m = R_p/R_1$, where R_1 the internal radius and R_2 the external radius.

This is simply a modified form of eqn. (3.36) developed in the preceding section.

The maximum allowable autofrettage pressure is then given as that which will produce yielding to the geometric mean radius $R_p = \sqrt{R_1 R_2}$.

Stress distribution under autofrettage pressure loading

From eqns. (3.32) and (3.33) the stresses in the plastic zone at any radius r are given by:

$$\sigma_r = \sigma_y \left[\log_e \left(\frac{r}{R_p} \right) - \frac{1}{2} \left(1 - \frac{R_p^2}{R_2^2} \right) \right] \quad (3.38)$$

$$\sigma_H = \sigma_y \left[1 + \log_e \left(\frac{r}{R_p} \right) - \frac{1}{2} \left(1 - \frac{R_p^2}{R_2^2} \right) \right] \quad (3.39)$$

[†] High Pressure Safety Code. High Pressure Technology Association, 1975.

Also in §3.20 (a) it has been shown that stresses at any radius r in the elastic zone are obtained in terms of the radial pressure $P_3 = P_p$ set up at the elastic-plastic interface with:

$$\sigma_r = \frac{P_p R_p^2}{(R_2^2 - R_p^2)} \left[1 - \frac{R_2^2}{r^2} \right] \quad (3.40)$$

$$\sigma_H = \frac{P_p \cdot R_p^2}{(R_2^2 - R_p^2)} \left[1 + \frac{R_2^2}{r^2} \right] \quad (3.41)$$

with

$$P_p = \frac{\sigma_y}{2R_2^2} [R_2^2 - R_p^2] \quad (3.31)(bis)$$

The above equations yield hoop and radial stress distributions throughout the cylinder wall typically of the form shown in Fig. 3.30.

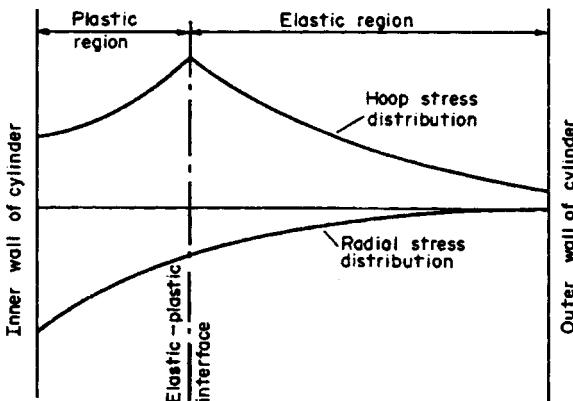


Fig. 3.30. Stress distributions under autofrettage pressure.

Residual stress distributions

Residual stress after unloading can then be obtained using the procedure introduced in §3.9 of elastic unloading, i.e. the autofrettage loading pressure is assumed to be removed (applied in a negative sense) elastically across the whole cylinder, the unloading elastic stress distribution being given by eqns. (10.5) and (10.6)[†] as:

$$\sigma_r = P_A \left[\frac{1 - \left(\frac{R_2}{r} \right)^2}{K^2 - 1} \right] \quad (3.42)$$

$$\sigma_H = P_A \left[\frac{1 + \left(\frac{R_2}{r} \right)^2}{K^2 - 1} \right] \quad \text{with } K = R_2/R_1 \quad (3.43)$$

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

Superposition of these distributions on the previous loading distributions allows the two curves to be subtracted for both hoop and radial stresses and produces residual stresses of the form shown in Figs. 3.31 and 3.32.

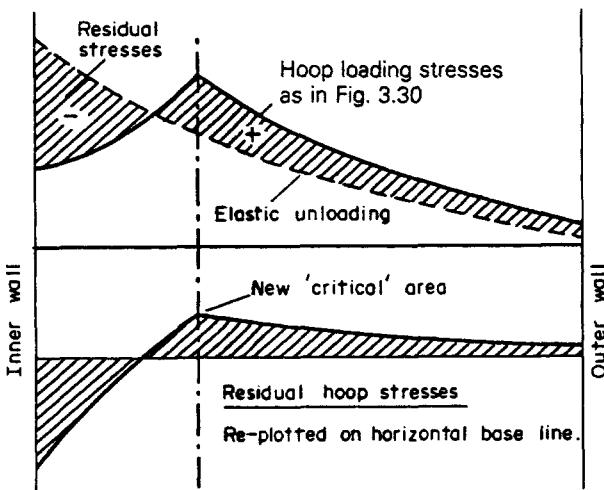


Fig. 3.31. Determination of residual hoop stresses by elastic unloading.

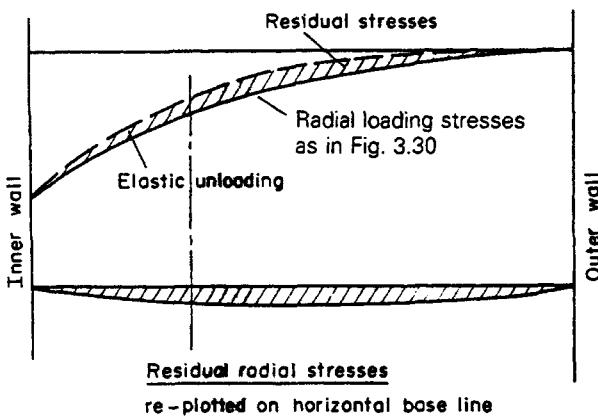


Fig. 3.32. Determination of residual radial stresses by elastic unloading.

Working stress distributions

Finally, if the stress distributions due to an elastic internal working pressure P_w are superimposed on the residual stress state then the final working stress state is produced as in Figs. 3.33 and 3.34.

The elastic working stresses are given by eqns. (3.42) and (3.43) with P_A replaced by P_w . Alternatively a Lamé line solution can be adopted. The final stress distributions show that

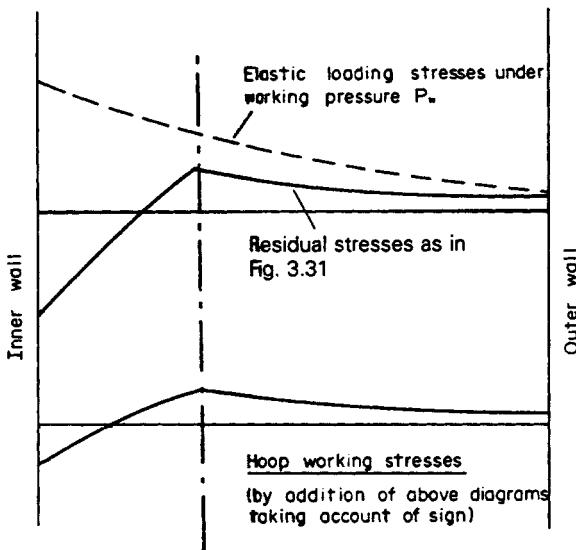


Fig. 3.33. Evaluation of hoop working stresses.

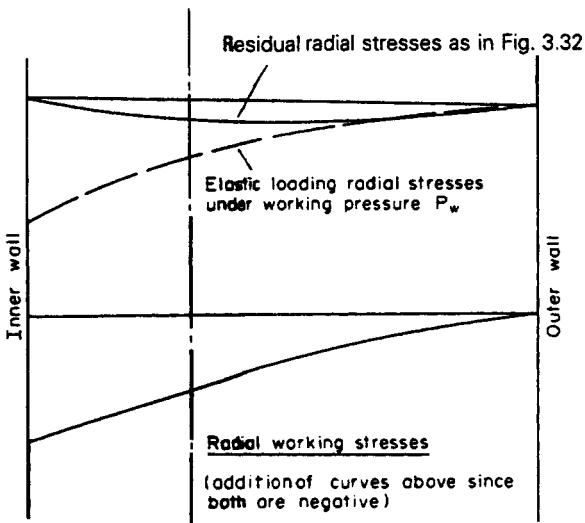


Fig. 3.34. Evaluation of radial working stresses.

the maximum tensile stress, instead of being at the bore as in the plain cylinder, is now at the elastic/plastic interface position. Application of the Tresca maximum shear stress failure criterion:

i.e.

$$\sigma_H - \sigma_r = \sigma_y/n$$

also indicates the elastic/plastic interface as now more critical than the internal bore – see Fig. 3.35.

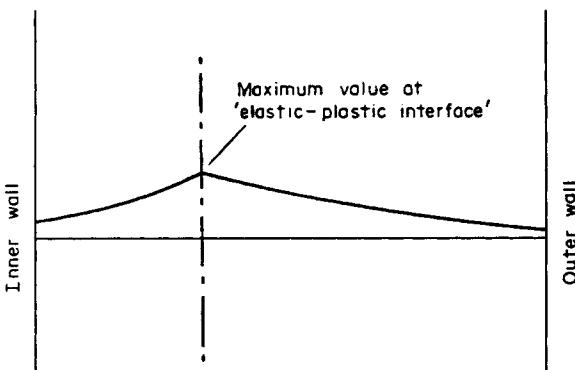


Fig. 3.35. Distribution of maximum shear stress = $\frac{1}{2}(\sigma_\theta - \sigma_r)$ by combination of Figs. 3.33 and 3.34.

Effect of axial stresses and end restraint

Depending on the end conditions which can be assumed for the cylinder during both the autofrettage process and its normal working condition a further complication can arise since the axial stresses σ_z which are produced can affect the application of the Tresca criterion.

Strictly, Tresca requires the use of the greatest difference in the principal stresses which, if σ_z is zero, = $\sigma_H - \sigma_r$. If, however, σ_z has a value it must be used in conjunction with σ_H and σ_r to produce the greatest difference.

The procedure used above to determine residual hoop and radial stresses and subsequent working stresses should therefore be repeated for axial stresses with values in the plastic region being found as suggested by Franklin and Morrison[†] from:

$$\sigma_z = P_A \frac{(1 - 2\nu)}{(K^2 - 1)} + \nu(\sigma_H - \sigma_r) \quad (3.44)$$

and axial stresses under elastic conditions being given by eqn. (10.7)[‡] with $P_2 = 0$ and $P_1 = P_A$ or P_W as required.

(c) Rotating discs

It will be shown in Chapter 4 that the centrifugal forces which act on rotating discs produce two-dimensional tensile stress systems. At any given radius the hoop or circumferential stress is always greater than, or equal to, the radial stress, the maximum values occurring at the inside radius. It follows, therefore, that yielding will first occur at the inside surface when the speed of rotation has increased sufficiently to make the circumferential stress equal to the tensile yield stress. With further increase of speed, plastic penetration will gradually proceed towards the centre of the disc and eventually complete plastic collapse will occur.

[†] G.J. Franklin and J.L.M. Morrison, Autofrettage of cylinders: reduction of pressure/external expansion curves and calculation of residual stresses. *Proc. J. Mech. E.* **174** (35) 1960.

[‡] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

Now for a *solid disc* the equilibrium eqn. (4.1) derived on page 120 is, with $\sigma_H = \sigma_y$,

$$\sigma_y - \sigma_r - r \frac{d\sigma_r}{dr} = \rho r^2 \omega^2$$

$$\therefore \sigma_r + r \frac{d\sigma_r}{dr} = \sigma_y - \rho r^2 \omega^2$$

$$\text{Integrating, } r\sigma_r = r\sigma_y - \rho \frac{r^3 \omega^2}{3} + A \quad (1)$$

Now since the stress cannot be infinite at the centre where $r = 0$, then A must be zero.

$$\therefore r\sigma_r = r\sigma_y - \rho \frac{r^3 \omega^2}{3}$$

Now at $r = R$, i.e. at the outside of the disc, $\sigma_r = 0$.

$$\therefore R\sigma_y = \rho \frac{R^3 \omega^2}{3}$$

i.e. the collapse speed ω_p is given by

$$\omega_p^2 = \frac{3\sigma_y}{\rho R^2} \quad (3.45)$$

For a *disc with a central hole*, (1) still applies, but in this case the value of the constant A is determined from the condition

$$\sigma_r = 0 \text{ at } r = R_1 \text{ the inside radius}$$

$$\text{i.e. } A = \rho \frac{R_1^3 \omega^2}{3} - R_1 \sigma_y$$

Again, $\sigma_r = 0$ at the outside surface where $r = R$.

Substituting in (1),

$$0 = R\sigma_y - \rho \frac{R^3 \omega^2}{3} + \rho \frac{R_1^3 \omega^2}{3} - R_1 \sigma_y$$

$$0 = \sigma_y(R - R_1) - \rho \frac{\omega^2}{3}(R^3 - R_1^3)$$

i.e. the collapse speed ω_p is given by

$$\omega_p^2 = \frac{3\sigma_y}{\rho} \frac{(R - R_1)}{(R^3 - R_1^3)} \quad (3.46)$$

If a rotating disc is stopped after only partial penetration, residual stresses are set up similar to those discussed in the case of thick cylinders under internal pressure (auto-fretting). Their values may be determined in precisely the same manner as that described in earlier sections, namely, by calculating the elastic stress distribution at an appropriate higher speed and subtracting this from the partially plastic stress distribution. Once again, favourable compressive residual stresses are set up on the surface of the central hole which increases the stress range – and hence the speed limit – available on subsequent cycles. This process is sometimes referred to as *overspeeding*.

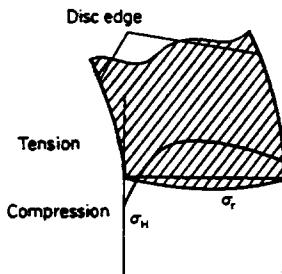


Fig. 3.36. Residual stresses produced after plastic yielding ("overspeeding") of rotating disc with a central hole.

A typical residual stress distribution is shown in Fig. 3.36.

Examples

Example 3.1

(a) A rectangular-section steel beam, 50 mm wide by 20 mm deep, is used as a simply supported beam over a span of 2 m with the 20 mm dimension vertical. Determine the value of the central concentrated load which will produce initiation of yield at the outer fibres of the beam.

(b) If the central load is then increased by 10% find the depth to which yielding will take place at the centre of the beam span.

(c) Over what length of beam will yielding then have taken place?

(d) What are the maximum deflections for each load case?

For steel σ_y in simple tension and compression = 225 MN/m² and $E = 206.8$ GN/m².

Solution

(a) From eqn. (3.1) the B.M. required to initiate yielding is

$$\frac{BD^2}{6}\sigma_y = \frac{50 \times 20^2 \times 10^{-9}}{6} \times 225 \times 10^6 = 750 \text{ N m}$$

But the maximum B.M. on a beam with a central point load is $WL/4$, at the centre.

$$\therefore \frac{W \times 2}{4} = 750$$

i.e.

$$W = 1500 \text{ N}$$

The load required to initiate yielding is 1500 N.

(b) If the load is increased by 10% the new load is

$$W' = 1500 + 150 = 1650 \text{ N}$$

The maximum B.M. is therefore increased to

$$M' = \frac{W'L}{4} = \frac{1650 \times 2}{4} = 825 \text{ Nm}$$

and this is sufficient to produce yielding to a depth d , and from eqn. (3.2),

$$M_{pp} = \frac{B\sigma_y}{12} [3D^2 - d^2] = 825 \text{ Nm}$$

$$\therefore 825 = \frac{50 \times 10^{-3} \times 225 \times 10^6}{12} [3 \times 2^2 - d^2] 10^{-4}$$

where d is the depth of the elastic core in centimetres,

$$\therefore 8.8 = 12 - d^2$$

$$d^2 = 3.2 \text{ and } d = 1.79 \text{ cm}$$

$$\therefore \text{depth of yielding} = \frac{1}{2}(D - d) = \frac{1}{2}(20 - 17.9) = 1.05 \text{ mm}$$

(c) With the central load at 1650 N the yielding will have spread from the centre as shown in Fig. 3.37. At the extremity of the yielded region, a distance x from each end of the beam, the section will just have yielded at the extreme surface fibres, i.e. the moment carried at this section will be the maximum elastic moment and given by eqn. (3.1) – see part (a) above.

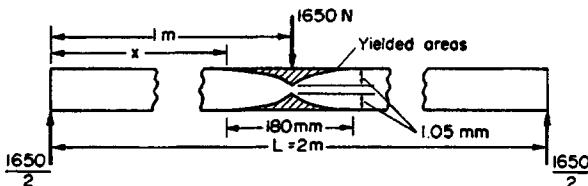


Fig. 3.37.

Now the B.M. at the distance x from the support is

$$\frac{1650x}{2} = \frac{BD^2}{6} \sigma_y = 750$$

$$\therefore x = \frac{2 \times 750}{1650} = 0.91 \text{ m}$$

Therefore length of beam over which yielding has occurred

$$= 2 - 2 \times 0.91 = 0.18 \text{ m} = 180 \text{ mm}$$

(d) For $W = 1500 \text{ N}$ the beam is completely elastic and the maximum deflection, at the centre, is given by the standard form of eqn. (5.15)[†]:

$$\delta = \frac{WL^3}{48EI} = \frac{1500 \times 2^3 \times 12}{48 \times 206.8 \times 10^9 \times 50 \times 20^3 \times 10^{-12}} \\ = 0.0363 \text{ m} = 36.3 \text{ mm}$$

With $W = 1650 \text{ N}$ and the beam partially plastic, deflections are calculated on the basis of the elastic core only,

$$\text{i.e. } \delta = \frac{W'L^3}{48EI'} = \frac{1650 \times 2^3 \times 12}{48 \times 206.8 \times 10^9 \times 50 \times 17.9^3 \times 10^{-12}} \\ = 0.0556 \text{ m} = 55.6 \text{ mm}$$

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

Example 3.2

(a) Determine the “shape factor” of a T-section beam of dimensions 100 mm \times 150 mm \times 12 mm as shown in Fig. 3.38.

(b) A cantilever is to be constructed from a beam with the above section and is designed to carry a uniformly distributed load over its complete length of 2 m. Determine the maximum u.d.l. that the cantilever can carry if yielding is permitted over the lower part of the web to a depth of 25 mm. The yield stress of the material of the cantilever is 225 MN/m².

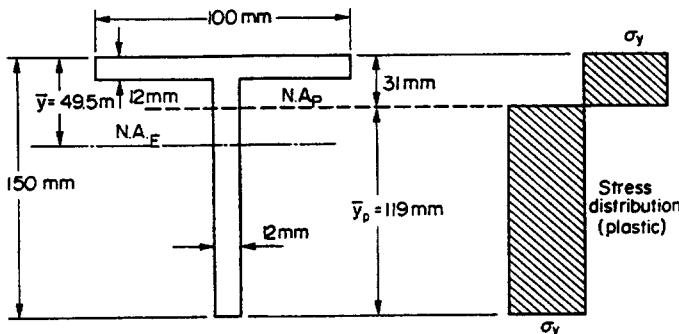


Fig. 3.38.

Solution

(a) Shape factor =
$$\frac{\text{fully plastic moment}}{\text{maximum elastic moment}}$$

To determine the maximum moment carried by the beam while completely elastic we must first determine the position of the N.A.

Take moments of area about the top edge (see Fig. 3.38):

$$(100 \times 12 \times 6) + (138 \times 12 \times 81) = [(100 \times 12) + (138 \times 12)] \bar{y}$$

$$7200 + 134136 = (1200 + 1656) \bar{y}$$

$$\therefore \bar{y} = 49.5 \text{ mm}$$

$$\begin{aligned} I_{NA} &= \left[\frac{100 \times 49.5^3}{3} + \frac{12 \times 100.5^3}{3} - \frac{88 \times 37.5^3}{3} \right] 10^{-12} \text{ m}^4 \\ &= \frac{1}{3}[121.29 + 121.81 - 46.4] 10^{-7} \\ &= 6.56 \times 10^{-6} \text{ m}^4 \end{aligned}$$

Now from the simple bending theory the moment required to produce the yield stress at the edge of the section (in this case the lower edge), i.e. the maximum elastic moment, is

$$M_E = \frac{\sigma I}{y_{\max}} = \sigma_y \times \frac{6.56 \times 10^{-6}}{100.5 \times 10^{-3}} = 0.065 \times 10^{-3} \sigma_y$$

When the section becomes fully plastic the N.A. is positioned such that

$$\text{area below N.A.} = \text{half total area}$$

i.e. if the plastic N.A. is a distance \bar{y}_p above the base, then

$$\bar{y}_p \times 12 = \frac{1}{2}(1200 + 1656)$$

$$\therefore \bar{y}_p = 119 \text{ mm}$$

The fully plastic moment is then obtained by considering the moments of forces on convenient rectangular parts of the section, each being subjected to a uniform stress σ_y ,

$$\begin{aligned} \text{i.e. } M_{FP} &= \left[\sigma_y(100 \times 12)(31 - 6) + \sigma_y(31 - 12) \times 12 \times \frac{(31 - 12)}{2} \right. \\ &\quad \left. + \sigma_y(119 \times 12) \frac{119}{2} \right] 10^{-9} \\ &= \sigma_y(30000 + 2166 + 84966)10^{-9} \\ &= 0.117 \times 10^{-3} \sigma_y \end{aligned}$$

$$\therefore \text{shape factor} = \frac{M_{FP}}{M_E} = \frac{0.117 \times 10^{-3}}{0.065 \times 10^{-3}} = 1.8$$

(b) For this part of the question the load on the cantilever is such that yielding has progressed to a depth of 25 mm over the lower part of the web. It has been shown in §3.4 that whilst plastic penetration proceeds, the N.A. of the section moves and is always positioned by the rule:

$$\text{compressive force above N.A.} = \text{tensile force below N.A.}$$

Thus if the partially plastic N.A. is positioned a distance y above the extremity of the yielded area as shown in Fig. 3.39, the forces exerted on the various parts of the section may be established (proportions of the stress distribution diagram being used to determine the various values of stress noted in the figure).

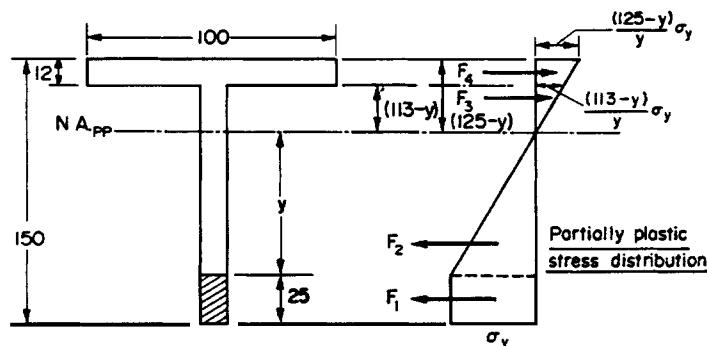


Fig. 3.39.

Force on yielded area

$$\begin{aligned}F_1 &= \text{stress} \times \text{area} \\&= 225 \times 10^6 (12 \times 25 \times 10^{-6}) \\&= 67.5 \text{ kN}\end{aligned}$$

Force on elastic portion of web below N.A.

$$\begin{aligned}F_2 &= \text{average stress} \times \text{area} \\&= \frac{225 \times 10^6}{2} (12 \times y \times 10^{-6}) \\&= 1.35y \text{ kN}\end{aligned}$$

where y is in millimetres.

Force in web above N.A.

$$\begin{aligned}F_3 &= \text{average stress} \times \text{area} \\&= \frac{(113 - y)}{2y} (225 \times 10^6) (113 - y) 12 \times 10^{-6} \\&= 1.35 \frac{(113 - y)^2}{y} \text{ kN}\end{aligned}$$

Force in flange

$$\begin{aligned}F_4 &= \text{average stress} \times \text{area} \\&= \frac{1}{2} \left[\frac{(113 - y)}{y} + \frac{(125 - y)}{y} \right] (225 \times 10^6) 100 \times 12 \times 10^{-6} \text{ approximately} \\&= \frac{(238 - 2y)}{2y} 225 \times 10^6 \times 100 \times 12 \times 10^{-6} \\&= 135 \frac{(238 - 2y)}{y} \text{ kN}\end{aligned}$$

Now for the resultant force across the section to be zero,

$$F_1 + F_2 = F_3 + F_4$$

$$67.5 + 1.35y = \frac{1.35(113 - y)^2}{y} + \frac{135(238 - 2y)}{y}$$

$$\therefore 67.5y + 1.35y^2 = 17.24 \times 10^3 - 305y + 1.35y^2 + 32.13 \times 10^3 - 270y$$

$$642.5y = 49370$$

$$y = 76.8 \text{ mm}$$

Substituting back,

$$F_1 = 67.5 \text{ kN} \quad F_2 = 103.7 \text{ kN}$$

$$F_3 = 23 \text{ kN} \quad F_4 = 148.1 \text{ kN}$$

The moment of resistance of the beam can now be obtained by taking the moments of these forces about the N.A. Here, for ease of calculation, it is assumed that F_4 acts at the mid-point of the web. This, in most cases, is sufficiently accurate for practical purposes.

$$\begin{aligned}\text{Moment of resistance} &= \left\{ F_1(y + 12.5) + F_2 \left(\frac{2y}{3} \right) + F_3 \left[\frac{2}{3}(113 - y) \right] \right. \\ &\quad \left. + F_4[(113 - y) + 6] \right\} 10^{-3} \text{ kNm} \\ &= (6030 + 5312 + 554 + 6243)10^{-3} \text{ kNm} \\ &= 18.14 \text{ kNm}\end{aligned}$$

Now the maximum B.M. present on a cantilever carrying a u.d.l. is $wL^2/2$ at the support

$$\therefore \frac{wL^2}{2} = 18.15 \times 10^3$$

The maximum u.d.l. which can be carried by the cantilever is then

$$w = \frac{18.15 \times 10^3 \times 2}{4} = 9.1 \text{ kN/m}$$

Example 3.3

(a) A steel beam of rectangular section, 80 mm deep by 30 mm wide, is simply supported over a span of 1.4 m and carries a u.d.l. w . If the yield stress of the material is 240 MN/m², determine the value of w when yielding of the beam material has penetrated to a depth of 20 mm from each surface of the beam.

(b) What will be the magnitudes of the residual stresses which remain when load is removed?

(c) What external moment must be applied to the unloaded beam in order to return it to its undeformed (straight) position?

Solution

(a) From eqn. (3.2) the partially plastic moment carried by a rectangular section is given by

$$M_{pp} = \frac{B\sigma_y}{12}[3D^2 - d^2]$$

Thus, for the simply supported beam carrying a u.d.l., the maximum B.M. will be at the centre of the span and given by

$$\begin{aligned}BM_{max} &= \frac{wL^2}{8} = \frac{B\sigma_y}{12}[3D^2 - d^2] \\ \therefore w &= \frac{8 \times 30 \times 10^{-3} \times 240 \times 10^6}{1.4^2 \times 12} [3 \times 80^2 - 40^2]10^{-6} \\ &= 43.1 \text{ kN/m}\end{aligned}$$

(b) From the above working

$$M_{pp} = \frac{B\sigma_y}{12}[3D^2 - d^2] = \frac{wL^2}{8}$$

$$= 43.1 \times 10^3 \times \frac{1.4^2}{8} = 10.6 \text{ kNm}$$

During the unloading process a moment of equal value but opposite sense is applied to the beam assuming it to be completely elastic. Thus the equivalent maximum elastic stress σ' introduced at the outside surfaces of the beam by virtue of the unloading is given by the simple bending theory with $M = M_{pp} = 10.6 \text{ kNm}$,

i.e.

$$\sigma' = \frac{My}{I} = \frac{10.6 \times 10^3 \times 40 \times 10^{-3} \times 12}{30 \times 80^3 \times 10^{-12}}$$

$$= 0.33 \times 10^9 = 330 \text{ MN/m}^2$$

The unloading, elastic stress distribution is then linear from zero at the N.A. to $\pm 330 \text{ MN/m}^2$ at the outside surfaces, and this may be subtracted from the partially plastic loading stress distribution to yield the residual stresses as shown in Fig. 3.40.

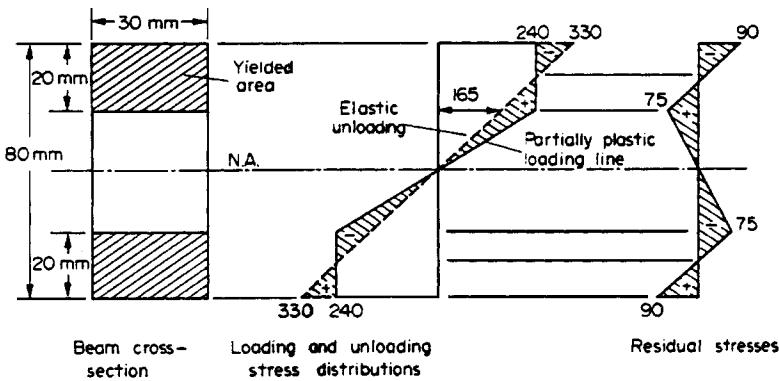


Fig. 3.40.

(c) The residual stress distribution of Fig. 3.40 indicates that the central portion of the beam, which remains elastic throughout the initial loading process, is subjected to a residual stress system when the beam is unloaded from the partially plastic state. The beam will therefore be in a deformed state. In order to remove this deformation an external moment must be applied of sufficient magnitude to return the elastic core to its unstressed state. The required moment must therefore introduce an elastic stress distribution producing stresses of $\pm 75 \text{ MN/m}^2$ at distances of 20 mm from the N.A. Thus, applying the bending theory,

$$M = \frac{\sigma I}{y} = \frac{75 \times 10^6}{20 \times 10^{-3}} \times \frac{30 \times 80^3 \times 10^{-12}}{12}$$

$$= 4.8 \text{ kNm}$$

Alternatively, since a moment of 10.6 kNm produces a stress of 165 MN/m² at 20 mm from the N.A., then, by proportion, the required moment is

$$M = 10.6 \times \frac{75}{165} = 4.8 \text{ kNm}$$

Example 3.4

A solid circular shaft, of diameter 50 mm and length 300 mm, is subjected to a gradually increasing torque T . The yield stress in shear for the shaft material is 120 MN/m² and, up to the yield point, the modulus of rigidity is 80 GN/m².

- (a) Determine the value of T and the associated angle of twist when the shaft material first yields.
- (b) If, after yielding, the stress is assumed to remain constant for any further increase in strain, determine the value of T when the angle of twist is increased to twice that at yield.

Solution

(a) For this part of the question the shaft is elastic and the simple torsion theory applies,

$$\begin{aligned} \text{i.e. } T &= \frac{\tau J}{R} = \frac{120 \times 10^6}{25 \times 10^{-3}} \times \frac{\pi(25 \times 10^{-3})^4}{2} = 2950 \\ &\quad = 2.95 \text{ kNm} \\ \theta &= \frac{\tau L}{GR} = \frac{120 \times 10^6 \times 300 \times 10^{-3}}{80 \times 10^9 \times 25 \times 10^{-3}} = 0.018 \text{ radian} \\ &\quad = 1.03^\circ \end{aligned}$$

If the torque is now increased to double the angle of twist the shaft will yield to some radius R_1 . Applying the torsion theory to the elastic core only,

$$\begin{aligned} \theta &= \frac{\tau L}{GR} \\ \text{i.e. } 2 \times 0.018 &= \frac{120 \times 10^6 \times 300 \times 10^{-3}}{80 \times 10^9 \times R_1} \\ \therefore R_1 &= \frac{120 \times 10^6 \times 300 \times 10^{-3}}{2 \times 0.018 \times 80 \times 10^9} = 0.0125 = 12.5 \text{ mm} \end{aligned}$$

Therefore partially plastic torque, from eqn. (3.12),

$$\begin{aligned} &= \frac{\pi \tau_y}{6} [4R^3 - R_1^3] \\ &= \frac{\pi \times 120 \times 10^6}{6} [4 \times 25^3 - 12.5^3] 10^{-9} \\ &\quad \cdots \cdots \cdots \end{aligned}$$

Example 3.5

A 50 mm diameter steel shaft is case-hardened to a depth of 2 mm. Assuming that the inner core remains elastic up to a yield stress in shear of 180 MN/m² and that the case can also be assumed to remain elastic up to failure at the shear stress of 320 MN/m², calculate:

- the torque required to initiate yielding at the outside surface of the case;
- the angle of twist per metre length at this stage.

Take $G = 85 \text{ GN/m}^2$ for both case and core whilst they remain elastic.

Solution

Since the modulus of rigidity G is assumed to be constant throughout the shaft whilst elastic, the angle of twist θ will be constant.

The stress distribution throughout the shaft cross-section at the instant of yielding of the outside surface of the case is then as shown in Fig. 3.41, and it is evident that whilst the failure stress of 320 MN/m² has only just been reached at the outside of the case, the yield stress of the core of 180 NM/m² has been exceeded beyond a radius r producing a fully plastic annulus and an elastic core.

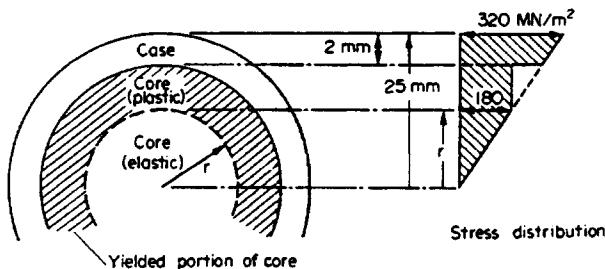


Fig. 3.41.

By proportions, since $G_{\text{case}} = G_{\text{core}}$, then

$$\left(\frac{\tau}{r}\right)_{\text{case}} = \left(\frac{\tau}{r}\right)_{\text{core}}$$

$$\frac{180}{r} = \frac{320}{25}$$

$$\therefore r = \frac{180}{320} \times 25 = 14.1 \text{ mm}$$

The shaft can now be considered in three parts:

- A solid elastic core of 14.1 mm external radius;
- A fully plastic cylindrical region between $r = 14.1 \text{ mm}$ and $r = 23 \text{ mm}$;
- An elastic outer cylinder of external diameter 50 mm and thickness 2 mm.

$$\begin{aligned} \text{Torque on elastic core} &= \frac{\tau_y J}{R} = \frac{180 \times 10^6}{14.1 \times 10^{-3}} \times \frac{\pi(14.1 \times 10^{-3})^4}{2} \\ &= 793 \text{ Nm} = 0.793 \text{ kNm} \end{aligned}$$

$$\begin{aligned}
 \text{Torque on plastic section} &= 2\pi\tau_y \int_{r_1}^{r_2} r^2 dr \\
 &= \frac{2\pi \times 180 \times 10^6}{3} [23^3 - 14.1^3] 10^{-9} \\
 &= \frac{2\pi \times 180 \times 10^6 \times 9364 \times 10^{-9}}{3} \\
 &= 3.53 \text{ kNm} \\
 \text{Torque on elastic outer case} &= \frac{\tau_y J}{r} = \frac{320 \times 10^6}{25 \times 10^{-3} \pi} \left[\frac{25^4 - 23^4}{2} \right] 10^{-12} \\
 &= 2.23 \text{ kNm}
 \end{aligned}$$

Therefore total torque required = $(0.793 + 3.53 + 2.23)10^3$
= 6.55 kNm

Since the angle of twist is assumed constant across the whole shaft its value may be determined by application of the simple torsion theory to either the case or the elastic core.

$$\begin{aligned}
 \text{For the case: } \frac{\theta}{L} &= \frac{\tau}{GR} = \frac{320 \times 10^6}{85 \times 10^9 \times 25 \times 10^{-3}} \\
 &= 0.15 \text{ rad} = 8.6^\circ
 \end{aligned}$$

Example 3.6

A hollow circular bar of 100 mm external diameter and 80 mm internal diameter (Fig. 3.42) is subjected to a gradually increasing torque T . Determine the value of T :

- (a) when the material of the bar first yields;
- (b) when plastic penetration has occurred to a depth of 5 mm;
- (c) when the section is fully plastic.

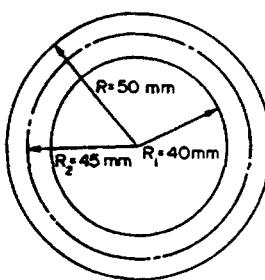


Fig. 3.42.

The yield stress in shear of the shaft material is 120 MN/m^2 .

Determine the distribution of the residual stresses present in the shaft when unloaded from conditions (b) and (c).

Solution

(a) Maximum elastic torque from eqn. (3.11)

$$\begin{aligned} &= \frac{\pi \tau_y}{2R} [R^4 - R_1^4] = \frac{\pi \times 120 \times 10^6}{2 \times 50 \times 10^{-3}} (625 - 256) 10^{-8} \\ &= 13900 \text{ Nm} = \mathbf{13.9 \text{ kNm}} \end{aligned}$$

(b) Partially plastic torque, from eqns. (3.11) and (3.13),

$$\begin{aligned} &= \frac{\pi \tau_y}{2R_2} [R_2^4 - R_1^4] + \frac{2\pi \tau_y}{3} [R^3 - R_2^3] \\ &= \frac{\pi \times 120 \times 10^6}{2 \times 45 \times 10^{-3}} (4.5^4 - 256) 10^{-8} + \frac{2\pi \times 120 \times 10^6}{3} (125 - 91) 10^{-6} \\ &= 6450 + 8550 = 15000 \text{ Nm} = \mathbf{15 \text{ kNm}} \end{aligned}$$

(c) Fully plastic torque from eqn. (3.16) or eqn. (3.13)

$$\begin{aligned} &= \frac{2\pi \tau_y}{3} [R^3 - R_1^3] \\ &= \frac{2\pi \times 120 \times 10^6}{3} [125 - 64] 10^{-6} = 15330 = \mathbf{15.33 \text{ kNm}} \end{aligned}$$

In order to determine the residual stresses after unloading, the unloading process is assumed completely elastic.

Thus, unloading from condition (b) is equivalent to applying a moment of 15 kNm of opposite sense to the loading moment on a complete elastic bar. The effective stress introduced at the outer surface by this process is thus given by the simple torsion theory

$$\begin{aligned} \frac{T}{J} &= \frac{\tau}{R} \\ \text{i.e. } \tau &= \frac{TR}{J} = \frac{15 \times 10^3 \times 50 \times 10^{-3} \times 2}{\pi \times (50^4 - 40^4) 10^{-12}} \\ &= \frac{15 \times 10^3 \times 50 \times 10^{-3} \times 2}{\pi (5^4 - 4^4) 10^{-8}} \\ &= 129 \text{ MN/m}^2 \end{aligned}$$

The unloading stress distribution is then linear, from zero at the centre of the bar to 129 MN/m² at the outside. This can be subtracted from the partially plastic loading stress distribution as shown in Fig. 3.43 to produce the residual stress distribution shown.

Similarly, unloading from the fully plastic state is equivalent to applying an elastic torque of 15.33 kNm of opposite sense. By proportion, from the above calculation,

$$\text{equivalent stress at outside of shaft on unloading} = \frac{15.33}{15} \times 129 = 132 \text{ MN/m}^2$$

Subtracting the resulting unloading distribution from the fully plastic loading one gives the residual stresses shown in Fig. 3.44.

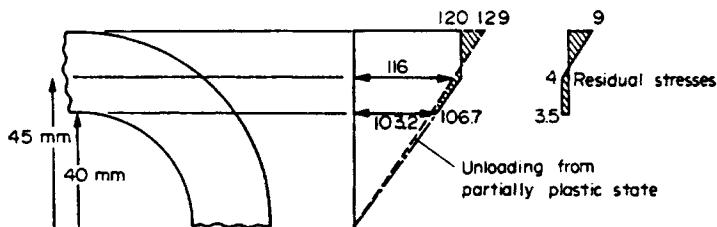


Fig. 3.43.

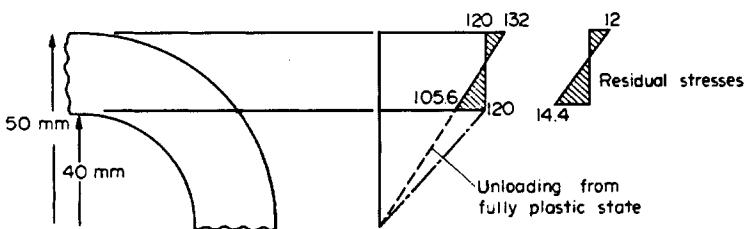


Fig. 3.44.

Example 3.7

(a) A thick cylinder, inside radius 62.5 mm and outside radius 190 mm, forms the pressure vessel of an isostatic compacting press used in the manufacture of ceramic components. Determine, using the Tresca theory of elastic failure, the safety factor on initial yield of the cylinder when an internal working pressure P_W of 240 MN/m² is applied.

(b) In view of the relatively low value of the safety factor which is achieved at this working pressure the cylinder is now subjected to an autofrettage pressure P_A of 580 MN/m². Determine the residual stresses produced at the bore of the cylinder when the autofrettage pressure is removed and hence determine the new value of the safety factor at the bore when the working pressure $P_W = 240$ MN/m² is applied.

The yield stress of the cylinder material is $\sigma_y = 850$ MN/m² and axial stresses may be ignored.

Solution

(a) Plain cylinder – working conditions $K = 190/62.5 = 9.24$
From eqn (10.5)[†]

$$\begin{aligned}\sigma_{rr} &= -P \left[\frac{(R_2/r)^2 - 1}{K^2 - 1} \right] = \frac{-240}{8.24} \left[\frac{0.19^2}{r^2} - 1 \right] \\ &= -240 \text{ MN/m}^2 \text{ at the bore surface } (r = 0.0625 \text{ mm})\end{aligned}$$

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

and from eqn. (10.6)[†]

$$\begin{aligned}\sigma_{\theta\theta} &= P \left[\frac{(R_2/r)^2 + 1}{K^2 - 1} \right] = \frac{240}{8.24} \left[\frac{0.19^2}{r^2} + 1 \right] \\ &= 298.3 \text{ MN/m}^2 \text{ at the bore surface}\end{aligned}$$

Thus, assuming axial stress will be the intermediate stress (σ_2) value, the critical stress conditions for the cylinder at the internal bore are $\sigma_1 = 298.3 \text{ MN/m}^2$ and $\sigma_3 = -240 \text{ MN/m}^2$.
 \therefore Applying the Tresca theory of failure ($\sigma_1 - \sigma_3 = \sigma_y/n$)

$$\text{Safety factor } n = \frac{850}{298.3 - (-240)} = 1.58$$

(b) Autofrettage conditions

From eqn 3.37 the radius R_p of the elastic/plastic interface under autofrettage pressure of 580 MN/m² will be given by:

$$\begin{aligned}P_A &= \frac{\sigma_y}{2} \left[\frac{K^2 - m^2}{K^2} \right] + \sigma_y \log_e m \\ \therefore 580 \times 10^6 &= \frac{850 \times 10^6}{2} \left[\frac{3.04^2 - m^2}{3.04^2} \right] + 850 \times 10^6 \log_e m\end{aligned}$$

By trial and error:

m	$850 \log_e m$	$\frac{850}{2} \left[\frac{3.04^2 - m^2}{3.04^2} \right]$	P_A
1.6	399.5	307.3	706.8
1.4	286.0	334.8	620.8
1.3	223.0	347.3	570.3
1.33	242.4	343.6	585.6
1.325	239.2	344.2	583.4

\therefore to a good approximation $m = 1.325 = R_p/R_1$

$$\therefore R_p = 1.325 \times 62.5 = 82.8 \text{ mm}$$

\therefore From eqns 3.38 and 3.39 stresses in the plastic zone are:

$$\begin{aligned}\sigma_{rr} &= 850 \times 10^6 \left[\log_e \left(\frac{r}{82.8} \right) - \frac{1}{2 \times 190^2} (190^2 - 82.8^2) \right] \\ &= 850 \times 10^6 [\log_e(r/82.8) - 0.405]\end{aligned}$$

and

$$\sigma_{\theta\theta} = 850 \times 10^6 [\log_e(r/82.8) + 0.595]$$

\therefore At the bore surface where $r = 62.5 \text{ mm}$ the stresses due to autofrettage are:

$$\sigma_{rr} = -580 \text{ MN/m}^2 \text{ and } \sigma_{\theta\theta} = 266.7 \text{ MN/m}^2.$$

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

Residual stresses are then obtained by *elastic unloading* of the autofrettage pressure, i.e. by applying $\sigma_{rr} = +580 \text{ MN/m}^2$ at the bore in eqns (10.5) and (10.6)[†]; i.e. by proportions:

$$\sigma_{rr} = 580 \text{ MN/m}^2 \text{ and } \sigma_{\theta\theta} = -298.3 \times \frac{580}{240} = -721 \text{ MN/m}^2.$$

Giving residual stresses at the bore of:

$$\sigma'_{rr} = 580 - 580 = 0$$

$$\sigma'_{\theta\theta} = 266.7 - 721 = -453 \text{ MN/m}^2$$

Working stresses are then obtained by the addition of elastic loading stresses due to an internal working pressure of 240 MN/m^2

i.e. from part (a) $\sigma_{rr} = -240 \text{ MN/m}^2, \sigma_{\theta\theta} = 298.3 \text{ MN/m}^2$

\therefore final working stresses are:

$$\sigma_{rr_w} = 0 - 240 = -240 \text{ MN/m}^2$$

$$\sigma_{\theta\theta_w} = 298.3 - 454.3 = -156 \text{ MN/m}^2.$$

\therefore New safety factor according to Tresca theory

$$n = \frac{850}{-156 - (-240)} = 10.1.$$

N.B. It is unlikely that the Tresca theory will give such a high value in practice since the axial working stress (ignored in this calculation) may well become the major principal stress σ_1 in the working condition and increase the magnitude of the denominator to reduce the resulting value of n .

Problems

- 3.1 (A/B). Determine the shape factors for the beam cross-sections shown in Fig. 3.45, in the case of section (c) treating the section both with and without the dotted area. [1.23, 1.81, 1.92, 1.82.]

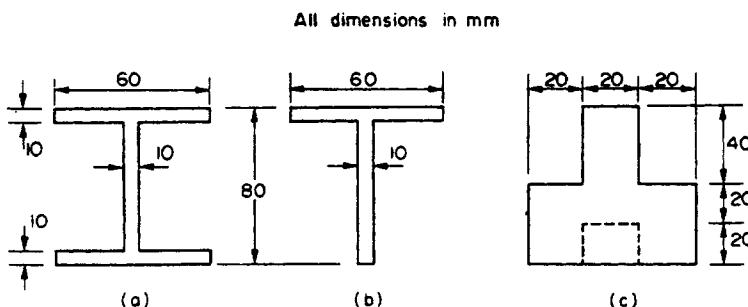


Fig. 3.45.

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

3.2 (B). A 50 mm × 20 mm rectangular-section beam is used simply supported over a span of 2 m. If the beam is used with its long edges vertical, determine the value of the central concentrated load which must be applied to produce initial yielding of the beam material.

If this load is then increased by 10% determine the depth to which yielding will take place at the centre of the beam span.

Over what length of beam has yielding taken place?

What are the maximum deflections for each load case? Take $\sigma_y = 225 \text{ MN/m}^2$ and $E = 206.8 \text{ GN/m}^2$.
[1.5 kN; 1.05 mm; 180 mm; 36.3, 55.5 mm.]

3.3 (B). A steel bar of rectangular section 72 mm × 30 mm is used as a simply supported beam on a span of 1.2 m and loaded at mid-span. If the yield stress is 280 MN/m² and the long edges of the section are vertical, find the loading when yielding first occurs.

Assuming that a further increase in load causes yielding to spread inwards towards the neutral axis, with the stress in the yielded part remaining at 280 MN/m², find the load required to cause yielding for a depth of 12 mm at the top and bottom of the section at mid-span, and find the length of beam over which yielding has occurred.

[24.2 kN; 31 kN; 0.264 m.]

3.4 (B). A 300 mm × 125 mm I-beam has flanges 13 mm thick and web 8.5 mm thick. Calculate the shape factor and the moment of resistance in the fully plastic state. Take $\sigma_y = 250 \text{ MN/m}^2$ and $I_{xx} = 85 \times 10^{-6} \text{ m}^4$.

[1.11, 141 kNm.]

3.5 (B). Find the shape factor for a 150 mm × 75 mm channel in pure bending with the plane of bending perpendicular to the web of the channel. The dimensions are shown in Fig. 3.46 and $Z = 21 \times 10^{-6} \text{ m}^3$.

[2.2.]

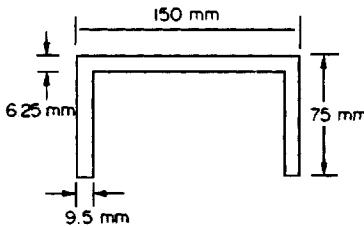


Fig. 3.46.

3.6 (B). A cantilever is to be constructed from a 40 mm × 60 mm T-section beam with a uniform thickness of 5 mm. The cantilever is to carry a u.d.l. over its complete length of 1 m. Determine the maximum u.d.l. that the cantilever can carry if yielding is permitted over the lower part of the web to a depth of 10 mm. $\sigma_y = 225 \text{ MN/m}^2$.
[2433 N/m.]

3.7 (B). A 305 mm × 127 mm symmetrical I-section has flanges 13 mm thick and a web 5.4 mm thick. Treating the web and flanges as rectangles, calculate the bending moment of resistance of the cross-section (a) at initial yield, (b) for full plasticity of the flanges only, and (c) for full plasticity of the complete cross-section. Yield stress in simple tension and compression = 310 MN/m². What is the shape factor of the cross-section?

[167, 175.6, 188.7 kNm; 1.13.]

3.8 (B). A steel bar of rectangular section 80 mm by 40 mm is used as a simply supported beam on a span of 1.4 m and point-loaded at mid-span. If the yield stress of the steel is 300 MN/m² in simple tension and compression and the long edges of the section are vertical, find the load when yielding first occurs.

Assuming that a further increase in load causes yielding to spread in towards the neutral axis with the stress in the yielded part remaining constant at 300 MN/m², determine the load required to cause yielding for a depth of 10 mm at the top and bottom of the section at mid-span and find the length of beam over which yielding at the top and bottom faces will have occurred.

[U.L.] [36.57, 44.6 kN; 0.232 m.]

3.9 (B). A straight bar of steel of rectangular section, 76 mm wide by 25 mm deep, is simply supported at two points 0.61 m apart. It is subjected to a uniform bending moment of 3 kNm over the whole span. Determine the depth of beam over which yielding will occur and make a diagram showing the distribution of bending stress over the full depth of the beam. Yield stress of steel in tension and compression = 280 MN/m².

Estimate the deflection at mid-span assuming $E = 200 \text{ GN/m}^2$ for elastic conditions. [5.73, 44.4 mm.]

3.10 (B). A symmetrical I-section beam of length 6 m is simply supported at points 1.2 m from each end and is to carry a u.d.l. $w \text{ kN/m}$ run over its entire length. The second moment of area of the cross-section about the neutral

axis parallel to the flanges is 6570 cm^4 and the beam cross-section dimensions are: flange width and thickness, 154 mm and 13 mm respectively, web thickness 10 mm, overall depth 254 mm.

- Determine the value of w to just cause initial yield, stating the position of the transverse section in the beam length at which it occurs.
- By how much must w be increased to ensure full plastic penetration of the flanges only, the web remaining elastic?

Take the yield stress of the beam material in simple tension and compression as 340 MN/m^2 .

[B.P.] [195, 20 kN/m.]

3.11 (B). A steel beam of rectangular cross-section, 100 mm wide by 50 mm deep, is bent to the arc of a circle until the material just yields at the outer fibres, top and bottom. Bending takes place about the neutral axis parallel to the 100 mm side. If the yield stress for the steel is 330 MN/m^2 in simple tension and compression, determine the applied bending moment and the radius of curvature of the neutral layer. $E = 207 \text{ GN/m}^2$.

Find how much the bending moment has to be increased so that the stress distribution is as shown in Fig. 3.47.

[I.Mech.E.] [13.75 kNm; 15.7 m; 16.23 kNm.]

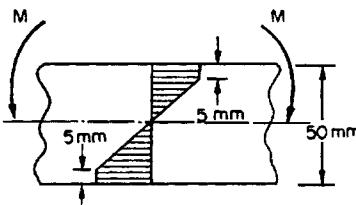


Fig. 3.47.

3.12 (B). A horizontal steel cantilever beam, 2.8 m long and of uniform I-section throughout, has the following cross-sectional dimensions: flanges 150 mm \times 25 mm, web 13 mm thick, overall depth 305 mm. It is fixed at one end and free at the other.

- Determine the intensity of the u.d.l. which the beam has to carry across its entire length in order to produce fully developed plasticity of the cross-section.
- What is the value of the shape factor of the cross-section?
- Determine the length of the beam along the top and bottom faces, measured from the fixed end, over which yielding will occur due to the load found in (a).

Yield stress of steel = 330 MN/m^2 .

[106.2 kN/m; 1.16; 0.2 m.]

3.13 (B). A rectangular steel beam, 60 mm deep by 30 mm wide, is supported on knife-edges 2m apart and loaded with two equal point loads at one-third of the span from each end. Find the load at which yielding just begins, the yield stress of the material in simple tension and compression being 300 MN/m^2 .

If the loads are increased to 25% above this value, estimate how far the yielding penetrates towards the neutral axis, assuming that the yield stress remains constant.

[U.L.] [8.1 kN; 8.79 mm.]

3.14 (B). A steel bar of rectangular section, 72 mm deep by 30 mm wide, is used as a beam simply supported at each end over a span of 1.2 m and loaded at mid-span with a point load. The yield stress of the material is 280 MN/m^2 . Determine the value of the load when yielding first occurs.

Find the load to cause an inward plastic penetration of 12 mm at the top and bottom of the section at mid-span. Also find the length, measured along the top and bottom faces, over which yielding has occurred, and the residual stresses present after unloading.

[U.L.] [24.2 kN; 31 kN; 0.26 m, $\mp 79, \pm 40.7 \text{ MN/m}^2$.]

3.15 (B). A symmetrical I-section beam, 300 mm deep, has flanges 125 mm wide by 13 mm thick and a web 8.5 mm thick. Determine:

- the applied bending moment to cause initial yield;
- the applied bending moment to cause full plasticity of the cross-section;
- the shape factor of the cross-section.

Take the yield stress = 250 MN/m^2 and assume $I = 85 \times 10^6 \text{ mm}^4$.

[$141 \times 10^6 \text{ N mm}$; $156 \times 10^6 \text{ N mm}$; 1.11.]

3.16 (B). A rectangular steel beam AB , 20 mm wide by 10 mm deep, is placed symmetrically on two knife-edges C and D , 0.5 m apart, and loaded by applying equal loads at the ends A and B . The steel follows a linear stress/strain law ($E = 200 \text{ GN/m}^2$) up to a yield stress of 300 MN/m^2 ; at this constant stress considerable plastic deformation occurs. It may be assumed that the properties of the steel are the same in tension and compression.

Calculate the bending moment on the central part of the beam CD when yielding commences and the deflection at the centre relative to the supports.

If the loads are increased until yielding penetrates half-way to the neutral axis, calculate the new value of the bending moment and the corresponding deflection. [U.L.] [100 Nm, 9.375 mm; 137.5 Nm, 103 mm.]

3.17 (B). A steel bar of rectangular material, 75 mm \times 25 mm, is used as a simply supported beam on a span of 2 m and is loaded at mid-span. The 75 mm dimension is placed vertically and the yield stress for the material is 240 MN/m². Find the load when yielding first occurs.

The load is further increased until the bending moment is 20% greater than that which would cause initial yield. Assuming that the increased load causes yielding to spread inwards towards the neutral axis, with the stress in the yielded part remaining at 240 MN/m², find the depth at the top and bottom of the section at mid-span to which the yielding will extend. Over what length of the beam has yielding occurred?

[B.P.] [11.25 kN; 8.45 mm; 0.33 m.]

3.18 (B). The cross-section of a beam is a channel, symmetrical about a vertical centre line. The overall width of the section is 150 mm and the overall depth 100 mm. The thickness of both the horizontal web and each of the vertical flanges is 12 mm. By comparing the behaviour in both the elastic and plastic range determine the shape factor of the section. Work from first principles in both cases. [1.806.]

3.19 (B). The T-section beam shown in Fig. 3.48 is subjected to increased load so that yielding spreads to within 50 mm of the lower edge of the flange. Determine the bending moment required to produce this condition. $\sigma_y = 240 \text{ MN m}^2$. [44 kN m.]

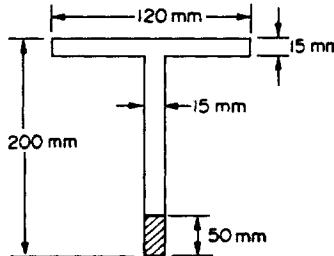


Fig. 3.48.

3.20 (B). A steel beam of I-section with overall depth 300 mm, flange width 125 mm and length 5 m, is simply supported at each end and carries a uniformly distributed load of 114 kN/m over the full span. Steel reinforcing plates 12 mm thick are welded symmetrically to the outside of the flanges producing a section of overall depth 324 mm. If the plate material is assumed to behave in an elastic-ideally plastic manner, determine the plate width necessary such that yielding has just spread through each reinforcing plate at mid-span under the given load.

Determine also the positions along the reinforcing plates at which the outer surfaces have just reached the yield point. At these sections what is the horizontal shearing stress at the interfaces of the reinforcing plates and the flanges?

Take the yield stress $\sigma_y = 300 \text{ MN/m}^2$ and the second moment of area of the basic I-section to be $80 \times 10^{-6} \text{ m}^4$. [C.E.I.] [175 mm; 1.926 m; 0.94 MN/m².]

3.21 (B). A horizontal cantilever is propped at the free end to the same level as the fixed end. It is required to carry a vertical concentrated load W at any position between the supports. Using the normal assumption of plastic limit design, determine the least favourable position of the load. (Note that the calculation of bending moments under elastic conditions is not required.)

Hence calculate the maximum permissible value of W which may be carried by a rectangular-section cantilever with depth d equal to twice the width over a span L . Assume a load factor of n and a yield stress for the beam material σ_y . [0.586 L from built-in end; $d^3\sigma_y / 1.371 Ln$.]

3.22 (B). (a) Sketch the idealised stress-strain diagram which is used to establish a quantitative relationship between stress and strain in the plastic range of a ductile material. Include the effect of strain-hardening.

(b) Neglecting strain-hardening, sketch the idealised stress-strain diagram and state, in words, the significance of any alteration you make in the diagram shown for part (a) when calculations are made, say, for pure bending beyond the yield point.

(c) A steel beam of rectangular cross-section, 200 mm wide \times 100 mm deep, is bent to the arc of a circle, bending taking place about the neutral axis parallel to the 200 mm side.

Determine the bending moment to be applied such that the stress distribution is as shown in (i) Fig. 3.49(a) and (ii) Fig. 3.49(b).

Take the yield stress of steel in tension and compression as 250 MN/m^2 .

[B.P.] [98.3, 125 kN m.]

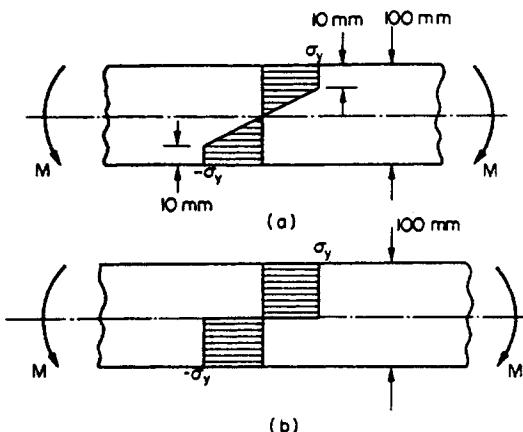


Fig. 3.49.

3.23 (B). (a) A rectangular section beam is 80 mm wide, 120 mm deep and is simply supported at each end over a span of 4 m. Determine the maximum uniformly distributed load that the beam can carry:

- if yielding of the beam material is permitted to a depth of 40 mm;
- before complete collapse occurs.

(b) What residual stresses would be present in the beam after unloading from condition (a) (i)?

(c) What external moment must be applied to the beam to hold the deformed bar in a straight position after unloading from condition (a) (i)?

The yield stress of the material of the beam = 280 MN/m^2 .

[B.P.] [38.8, 40.3 kN/m; $\pm 123, \pm 146 \text{ MN/m}^2$; 84.3 kN m.]

3.24 (C). A rectangular beam 80 mm wide and 20 mm deep is constructed from a material with a yield stress in tension of 270 MN/m^2 and a yield stress in compression of 300 MN/m^2 . If the beam is now subjected to a pure bending moment find the value required to produce:

- initial yield;
- initial yield on the compression edge;
- a fully plastic section.

[1.44; 1.59, 2.27 kN m.]

3.25 (C). Determine the load factor of a propped cantilever carrying a concentrated load W at the centre.

Allowable working stress = 150 MN/m^2 , yield stress = 270 MN/m^2 . The cantilever is of I-section with dimensions $300 \text{ mm} \times 80 \text{ mm} \times 8 \text{ mm}$.

[2.48.]

3.26 (C). A $300 \text{ mm} \times 100 \text{ mm}$ beam is carried over a span of 7 m the ends being rigidly built in. Find the maximum point load which can be carried at 3 m from one end and the maximum working stress set up.

Take a load factor of 1.8 and $\sigma_y = 240 \text{ MN/m}^2$.

$I = 85 \times 10^{-6} \text{ m}^4$ and the shape factor = 1.135.

[100 kN; 172 MN/m^2 .]

3.27 (C). A $300 \text{ mm} \times 125 \text{ mm}$ I-beam is carried over a span of 20 m the ends being rigidly built in. Find the maximum point load which can be carried at 8 m from one end and the maximum working stress set up. Take a load factor of 1.8 and $\sigma_y = 250 \text{ MN/m}^2$; $Z = 56.6 \times 10^{-5} \text{ m}^3$ and shape factor $\lambda = 1.11$.

[36 kN; 183 MN/m^2 .]

3.28 (C). Determine the maximum intensity of loading that can be sustained by a simply supported beam, 75 mm wide \times 100 mm deep, assuming perfect elastic-plastic behaviour with a yield stress in tension and compression of 135 MN/m^2 . The beam span is 2 m.

What will be the distribution of residual stresses in the beam after unloading?

[$50.6 \text{ kN/m}; 67, 135, -67 \text{ MN/m}^2$.]

3.29 (C). A short column of 0.05 m square cross-section is subjected to a compressive load of 0.5 MN parallel to but eccentric from the central axis. The column is made from elastic – perfectly plastic material which has a yield stress in tension or compression of 300 MN/m². Determine the value of the eccentricity which will result in the section becoming just fully plastic. Also calculate the residual stress at the outer surfaces after elastic unloading from the fully plastic state.

[10.4 mm; 250, 150 MN/m².]

3.30 (C). A rectangular beam 75 mm wide and 200 mm deep is constructed from a material with a yield stress in tension of 270 MN/m² and a yield stress in compression of 300 MN/m². If the beam is now subjected to a pure bending moment, determine the value of the moment required to produce (a) initial yield, (b) initial yield on the compression edge, (c) a fully plastic section.

[135, 149.2, 213.2 kN m.]

3.31 (C). Figure 3.50 shows the cross-section of a welded steel structure which forms the shell of a gimbal frame used to support the ship-to-shore transport platform of a dock installation. The section is symmetrical about the vertical centre-line with a uniform thickness of 25 mm throughout.

As a preliminary design study what would you assess as the maximum bending moment which the section can withstand in order to prevent:

- (a) initial yielding at any point in the structure if the yield stress for the material is 240 MN/m²,
- (b) complete collapse of the structure?

What would be the effect of adverse weather conditions which introduce instantaneous loads approaching, but not exceeding that predicted in (b). Quantify your answers where possible.

State briefly the factors which you would consider important in the selection of a suitable material for such a structure.

[309.3 kN m; 423 kN m; local yielding, residual stress max = 279 MN/m².]

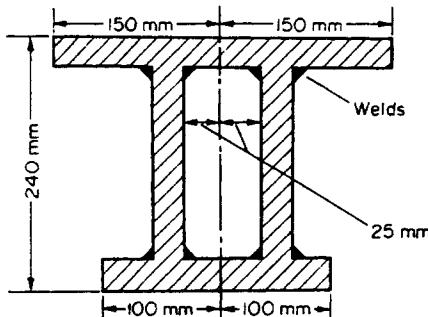


Fig. 3.50.

3.32 (B). A solid shaft 40 mm diameter is made of a steel the yield point of which in shear is 150 MN/m². After yielding, the stress remains constant for a very considerable increase in strain. Up to the yield point the modulus of rigidity $G = 80 \text{ GN/m}^2$. If the length of the shaft is 600 mm calculate:

- (a) the angle of twist and the twisting moment when the shaft material first yields;
- (b) the twisting moment when the angle of twist is increased to twice that at yield.

[3.32°; 1888, 2435 Nm.]

3.33 (B). A solid steel shaft, 76 mm diameter and 1.53 m long, is subjected to pure torsion. Calculate the applied torque necessary to cause initial yielding if the material has a yield stress in pure tension of 310 MN/m². Adopt the Tresca criterion of elastic failure.

(b) If the torque is increased to 10% above that at first yield, determine the radial depth of plastic penetration. Also calculate the angle of twist of the shaft at this increased torque. Up to the yield point in shear, $G = 83 \text{ GN/m}^2$.

(c) Calculate the torque to be applied to cause the cross-section to become fully plastic.

[13.36 kNm; 4.26 mm, 0.085 rad; 17.8 kNm.]

3.34 (B). A hollow steel shaft having outside and inside diameters of 32 mm and 18 mm respectively is subjected to a gradually increasing axial torque. The yield stress in shear is reached at the surface when the torque is 1 kNm, the angle of twist per metre length then being 7.3°. Find the magnitude of the yield shear stress.

If the torque is increased to 1.1 kN m, calculate (a) the depth to which yielding will have penetrated, and (b) the angle of twist per metre length.

State any assumptions made and prove any special formulae used.

[U.L.] [172.7 MN/m²; 1.8 mm; 8.22°.]

3.35 (B). A hollow shaft, 50 mm diameter and 25 mm bore, is made of steel with a yield stress in shear of 150 MN/m² and a modulus of rigidity of 83 GN/m². Calculate the torque and the angle of twist when the material first yields, if the shaft has a length of 2 m.

On the assumption that the yield stress, after initial yield, then remains constant for a considerable increase in strain, calculate the depth of penetration of plastic yield for an increase in torque of 10% above that at initial yield. Determine also the angle of twist of the shaft at the increased torque.

[U.L.] [3.45 kN m; 8.29°; 2.3 mm, 9.15°.]

3.36 (C). A steel shaft of length 1.25 m has internal and external diameters of 25 mm and 50 mm respectively. The shear stress at yield of the steel is 125 MN/m². The shear modulus of the steel is 80 GN/m². Determine the torque and overall twist when (a) yield first occurs, (b) the material has yielded outside a circle of diameter 40 mm, and (c) the whole section has just yielded. What will be the residual stresses after unloading from (b) and (c)?

[2.88, 3.33, 3.58 kN m; 0.0781, 0.0975, 0.1562 rad, (a) 19.7, -9.2, -5.5 MN/m², (b) 30.6, -46.75 MN/m².]

3.37 (B). A shaft having a diameter of 90 mm is turned down to 87 mm for part of its length. If a torque is applied to the shaft of sufficient magnitude just to produce yielding at the surface of the shaft in the unturned part, determine the depth of yielding which would occur in the turned part. Find also the angle of twist per unit length in the turned part to that in the unturned part of the shaft.

[U.L.] [5.3 mm; 1.18.]

3.38 (B). A steel shaft, 90 mm diameter, is solid for a certain distance from one end but hollow for the remainder of its length with an inside diameter of 38 mm. If a pure torque is transmitted from one end of the shaft to the other of such a magnitude that yielding just occurs at the surface of the solid part of the shaft, find the depth of yielding in the hollow part of the shaft and the ratio of the angles of twist per unit length for the two parts of the shaft.

[U.L.] [1.5 mm; 1.0345:1.]

3.39 (B). A steel shaft of solid circular cross-section is subjected to a gradually increasing torque. The diameter of the shaft is 76 mm and it is 1.22 m long. Determine for initial yield conditions in the outside surface of the shaft (a) the angle of twist of one end relative to the other, (b) the applied torque, and (c) the total resilience stored.

Assume a yield in shear of 155 MN/m² and a shear modulus of 85 GN/m². If the torque is increased to a value 10% greater than that at initial yield, estimate (d) the depth of penetration of plastic yielding and (e) the new angle of twist.

[B.P.] [3.35°; 13.4 kN m; 391 J; 4.3 mm; 3.8°.]

3.40 (B). A solid steel shaft, 50 mm diameter and 1.22 m long, is transmitting power at 10 rev/s.

- (a) Determine the power to be transmitted at this speed to cause yielding of the outer fibres of the shaft if the yield stress in shear is 170 MN/m².
- (b) Determine the increase in power required to cause plastic penetration to a radial depth of 6.5 mm, the speed of rotation remaining at 10 rev/s. What would be the angle of twist of the shaft in this case? G for the steel is 82 GN/m².

[B.P.] [262 kW, 52 kW, 7.83°.]

3.41 (B). A marine propulsion shaft of length 6 m and external diameter 300 mm is initially constructed from solid steel bar with a shear stress at yield of 150 MN/m².

In order to increase its power/weight ratio the shaft is machined to convert it into a hollow shaft with internal diameter 260 mm, the outer diameter remaining unchanged.

Compare the torques which may be transmitted by the shaft in both its initial and machined states:

- (a) when yielding first occurs,
- (b) when the complete cross-section has yielded.

If, in service, the hollow shaft is subjected to an unexpected overload during which condition (b) is achieved, what will be the distribution of the residual stresses remaining in the shaft after torque has been removed?

[795 kN m, 346 kN m, 1060 kN m, 370 kN m; -10.2, +11.2 MN/m².]

3.42 (C). A solid circular shaft 100 mm diameter is in an elastic-plastic condition under the action of a pure torque of 24 kN m. If the shaft is of steel with a yield stress in shear of 120 MN/m² determine the depth of the plastic zone in the shaft and the angle of twist over a 3 m length. Sketch the residual shear stress distribution on unloading. $G = 85$ GN/m².

[0.95 mm; 4.95°.]

3.43 (C). A column is constructed from elastic - perfectly plastic material and has a cross-section 60 mm square. It is subjected to a compressive load of 0.8 MN parallel to the central longitudinal axis of the beam but eccentric from it. Determine the value of the eccentricity which will produce a fully plastic section if the yield stress of the column material is 280 MN/m².

What will be the values of the residual stresses at the outer surfaces of the column after unloading from this condition?

[7 mm; 213, -97 MN/m².]

3.44 (C). A beam of rectangular cross-section with depth d is constructed from a material having a stress-strain diagram consisting of two linear portions producing moduli of elasticity E_1 in tension and E_2 in compression.

Assuming that the beam is subjected to a positive bending moment M and that cross-sections remain plane, show that the strain on the outer surfaces of the beam can be written in the form

$$\varepsilon_1 = \frac{d}{R} \left[\frac{\sqrt{E_2}}{\sqrt{E_1} + \sqrt{E_2}} \right]$$

where R is the radius of curvature.

Hence derive an expression for the bending moment M in terms of the elastic moduli, the second moment of area I of the beam section and R the radius of curvature.

$$\left[M = \frac{4E_1 E_2 I}{R(\sqrt{E_1} + \sqrt{E_2})^2} \right]$$

3.45 (C). Explain what is meant by the term "autofrettage" as applied to thick cylinder design. What benefits are obtained from autofrettage and what precautions should be taken in its application?

(b) A thick cylinder, inside radius 62.5 mm and outside radius 190 mm, forms the pressure vessel of an isostatic compacting press used in the manufacture of sparking plug components. Determine, using the Tresca theory of elastic failure, the safety factor on initial yield of the cylinder when an internal working pressure P_w of 240 MN/m² is applied.

(c) In view of the relatively low value of safety factor which is achieved at this working pressure, the cylinder is now subjected to an autofrettage pressure of $P_A = 580$ MN/m².

Determine the residual stresses produced at the bore of the cylinder when the autofrettage pressure is removed and hence determine the new value of the safety factor at the bore when the working pressure P_w is applied.

The yield stress of the cylinder material $\sigma_y = 850$ MN/m² and axial stresses may be ignored.

3.46 (C). A thick cylinder of outer radius 190 mm and radius ratio $K = 3.04$ is constructed from material with a yield stress of 850 MN/m² and tensile strength 1 GN/m². In order to prepare it for operation at a working pressure of 248 MN/m² it is subjected to an initial autofrettage pressure of 584 MN/m².

Ignoring axial stresses, compare the safety factors against initial yielding of the bore of the cylinder obtained with and without the autofrettage process.

[1.53, 8.95.]

3.47 (C). What is the maximum autofrettage pressure which should be applied to a thick cylinder of the dimensions given in problem 3.46 in order to achieve yielding to the geometric mean radius?

Determine the maximum hoop and radial residual stresses produced by the application and release of this pressure and plot the distributions of hoop and radial residual stress across the cylinder wall.

[758 MN/m²; -55.2 MN/m²; -8.5 MN/m².]

CHAPTER 4

RINGS, DISCS AND CYLINDERS SUBJECTED TO ROTATION AND THERMAL GRADIENTS

Summary

For *thin rotating rings and cylinders* of mean radius R , the tensile hoop stress set up is given by

$$\sigma_H = \rho\omega^2 R^2$$

The radial and hoop stresses at any radius r in a *disc of uniform thickness* rotating with an angular velocity ω rad/s are given by

$$\begin{aligned}\sigma_r &= A - \frac{B}{r^2} - (3 + \nu) \frac{\rho\omega^2 r^2}{8} \\ \sigma_H &= A + \frac{B}{r^2} - (1 + 3\nu) \frac{\rho\omega^2 r^2}{8}\end{aligned}$$

where A and B are constants, ρ is the density of the disc material and ν is Poisson's ratio.

For a *solid disc* of radius R these equations give

$$\begin{aligned}\sigma_r &= (3 + \nu) \frac{\rho\omega^2}{8} (R^2 - r^2) \\ \sigma_H &= \frac{\rho\omega^2}{8} [(3 + \nu)R^2 - (1 + 3\nu)r^2]\end{aligned}$$

At the centre of the solid disc these equations yield the maximum stress values

$$\sigma_{H_{\max}} = \sigma_{r_{\max}} = (3 + \nu) \frac{\rho\omega^2 R^2}{8}$$

At the outside radius,

$$\sigma_r = 0$$

$$\sigma_H = (1 - \nu) \frac{\rho\omega^2 R^2}{4}$$

For a *disc with a central hole*,

$$\begin{aligned}\sigma_r &= (3 + \nu) \frac{\rho\omega^2}{8} \left[R_1^2 + R_2^2 - \frac{R_1^2 R_2^2}{r^2} - r^2 \right] \\ \sigma_H &= \frac{\rho\omega^2}{8} \left[(3 + \nu) \left(R_1^2 + R_2^2 + \frac{R_1^2 R_2^2}{r^2} \right) - (1 + 3\nu)r^2 \right]\end{aligned}$$

the maximum stresses being

$$\sigma_{H_{\max}} = \frac{\rho\omega^2}{4} [(3 + \nu)R_2^2 + (1 - \nu)R_1^2] \quad \text{at the centre}$$

and $\sigma_{r_{\max}} = (3 + \nu) \frac{\rho\omega^2}{8} [R_2 - R_1]^2 \quad \text{at } r = \sqrt{(R_1 R_2)}$

For *thick cylinders* or *solid shafts* the results can be obtained from those of the corresponding disc by replacing

$$\nu \text{ by } \nu/(1 - \nu),$$

e.g. hoop stress at the centre of a rotating solid shaft is

$$\sigma_H = \left[3 + \frac{\nu}{(1 - \nu)} \right] \frac{\rho\omega^2 r^2}{8}$$

Rotating thin disc of uniform strength

For uniform strength, i.e. $\sigma_H = \sigma_r = \sigma$ (constant over plane of disc), the disc thickness must vary according to the following equation:

$$t = t_0 e^{(-\rho\omega^2 r^2)/(2\sigma)}$$

4.1. Thin rotating ring or cylinder

Consider a thin ring or cylinder as shown in Fig. 4.1 subjected to a radial pressure p caused by the centrifugal effect of its own mass when rotating. The centrifugal effect on a unit length of the circumference is

$$p = m\omega^2 r$$

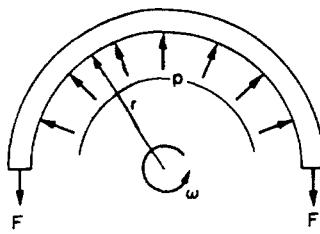


Fig. 4.1. Thin ring rotating with constant angular velocity ω .

Thus, considering the equilibrium of half the ring shown in the figure,

$$2F = p \times 2r \quad (\text{assuming unit length})$$

$$F = pr$$

where F is the hoop tension set up owing to rotation.

The cylinder wall is assumed to be so thin that the centrifugal effect can be assumed constant across the wall thickness.

$$\therefore F = \text{mass} \times \text{acceleration} = m\omega^2 r^2 \times r$$

This tension is transmitted through the complete circumference and therefore is resisted by the complete cross-sectional area.

$$\therefore \text{hoop stress} = \frac{F}{A} = \frac{m\omega^2 r^2}{A}$$

where A is the cross-sectional area of the ring.

Now with unit length assumed, m/A is the mass of the material per unit volume, i.e. the density ρ .

$$\therefore \text{hoop stress} = \rho\omega^2 r^2$$

4.2. Rotating solid disc

(a) General equations

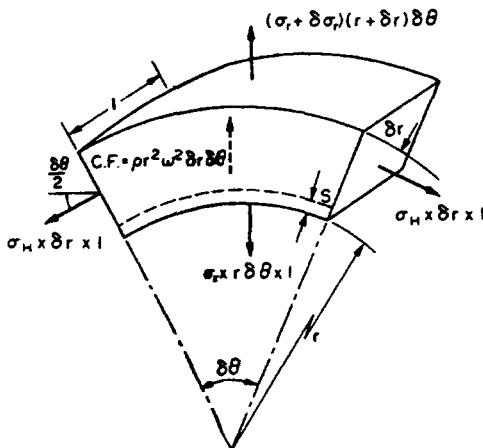


Fig. 4.2. Forces acting on a general element in a rotating solid disc.

Consider an element of a disc at radius r as shown in Fig. 4.2. Assuming unit thickness:

$$\text{volume of element} = r \delta\theta \times \delta r \times 1 = r \delta\theta \delta r$$

$$\text{mass of element} = \rho r \delta\theta \delta r$$

Therefore centrifugal force acting on the element

$$\begin{aligned} &= m\omega^2 r \\ &= \rho r \delta\theta \delta r \omega^2 r = \rho r^2 \omega^2 \delta\theta \delta r \end{aligned}$$

Now for equilibrium of the element radially

$$2\sigma_H \delta r \sin \frac{\delta\theta}{2} + \sigma_r r \delta\theta - (\sigma_r + \delta\sigma_r)(r + \delta r) \delta\theta = \rho r^2 \omega^2 \delta\theta \delta r$$

If $\delta\theta$ is small,

$$\sin \frac{\delta\theta}{2} = \frac{\delta\theta}{2} \text{ radian}$$

Therefore in the limit, as $\delta r \rightarrow 0$ (and therefore $\delta\sigma_r \rightarrow 0$) the above equation reduces to

$$\sigma_H - \sigma_r - r \frac{d\sigma_r}{dr} = \rho r^2 \omega^2 \quad (4.1)$$

If there is a radial movement or “shift” of the element by an amount s as the disc rotates, the radial strain is given by

$$\varepsilon_r = \frac{ds}{dr} = \frac{1}{E}(\sigma_r - \nu\sigma_H) \quad (4.2)$$

Now it has been shown in §9.1.3(a)[†] that the diametral strain is equal to the circumferential strain.

$$\therefore \frac{s}{r} = \frac{1}{E}(\sigma_H - \nu\sigma_r) \quad (4.3)$$

$$s = \frac{1}{E}(\sigma_H - \nu\sigma_r)$$

$$\text{Differentiating, } \frac{ds}{dr} = \frac{1}{E}(\sigma_H - \nu\sigma_r) + \frac{r}{E} \left[\frac{d\sigma_H}{dr} - \frac{\nu d\sigma_r}{dr} \right] \quad (4.4)$$

Equating eqns. (4.2) and (4.4) and simplifying,

$$(\sigma_H - \sigma_r)(1 + \nu) + r \frac{d\sigma_H}{dr} - \nu r \frac{d\sigma_r}{dr} = 0 \quad (4.5)$$

Substituting for $(\sigma_H - \sigma_r)$ from eqn. (4.1),

$$\left(r \frac{d\sigma_r}{dr} + \rho r^2 \omega^2 \right) (1 + \nu) + r \frac{d\sigma_H}{dr} - \nu r \frac{d\sigma_r}{dr} = 0$$

$$\frac{d\sigma_H}{dr} + \frac{d\sigma_r}{dr} = -\rho r \omega^2 (1 + \nu)$$

Integrating,

$$\sigma_H + \sigma_r = -\frac{\rho r^2 \omega^2}{2} (1 + \nu) + 2A \quad (4.6)$$

where $2A$ is a convenient constant of integration.

Subtracting eqn. (4.1),

$$2\sigma_r + r \frac{d\sigma_r}{dr} = -\frac{\rho r^2 \omega^2}{2} (3 + \nu) + 2A$$

$$\text{But } 2\sigma_r + r \frac{d\sigma_r}{dr} = \frac{d}{dr} [(r^2 \sigma_r)] \times \frac{1}{r}$$

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

$$\frac{d}{dr}(r^2\sigma_r) = r \left[-\frac{\rho r^2 \omega^2}{2} (3 + \nu) + 2A \right]$$

$$r^2\sigma_r = -\frac{\rho r^4 \omega^2}{8} (3 + \nu) + \frac{2Ar^2}{2} - B$$

where $-B$ is a second convenient constant of integration,

$$\sigma_r = A - \frac{B}{r^2} - (3 + \nu) \frac{\rho \omega^2 r^2}{8} \quad (4.7)$$

and from eqn. (4.5),

$$\sigma_H = A + \frac{B}{r^2} - (1 + 3\nu) \frac{\rho \omega^2 r^2}{8} \quad (4.8)$$

For a solid disc the stress at the centre is given when $r = 0$. With r equal to zero the above equations will yield infinite stresses whatever the speed of rotation unless B is also zero,

i.e. $B = 0$ and hence $B/r^2 = 0$ gives the only finite solution.

Now at the outside radius R the radial stress must be zero since there are no external forces to provide the necessary balance of equilibrium if σ_r were not zero.

Therefore from eqn. (4.7),

$$\sigma_r = 0 = A - (3 + \nu) \frac{\rho \omega^2 R^2}{8}$$

$$\therefore A = (3 + \nu) \frac{\rho \omega^2 R^2}{8}$$

Substituting in eqns. (4.7) and (4.8) the hoop and radial stresses at any radius r in a solid disc are given by

$$\sigma_H = (3 + \nu) \frac{\rho \omega^2 R^2}{8} - (1 + 3\nu) \frac{\rho \omega^2 r^2}{8}$$

$$= \frac{\rho \omega^2}{8} [(3 + \nu)R^2 - (1 + 3\nu)r^2] \quad (4.9)$$

$$\sigma_r = (3 + \nu) \frac{\rho \omega^2 R^2}{8} - (3 + \nu) \frac{\rho \omega^2 r^2}{8}$$

$$= (3 + \nu) \frac{\rho \omega^2}{8} [R^2 - r^2] \quad (4.10)$$

(b) Maximum stresses

At the *centre* of the disc, where $r = 0$, the above equations yield equal values of hoop and radial stress which may also be seen to be the maximum stresses in the disc, i.e. maximum hoop and radial stress (at the centre)

$$= (3 + \nu) \frac{\rho \omega^2 R^2}{8} \quad (4.11)$$

At the *outside* of the disc, at $r = R$, the equations give

$$\sigma_r = 0 \quad \text{and} \quad \sigma_H = (1 - \nu) \frac{\rho \omega^2 R^2}{4} \quad (4.12)$$

The complete distributions of radial and hoop stress across the radius of the disc are shown in Fig. 4.3.

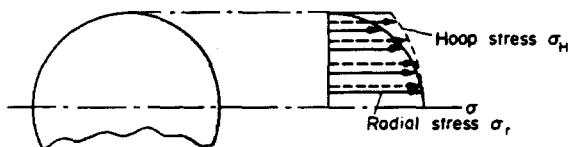


Fig. 4.3. Hoop and radial stress distributions in a rotating solid disc.

4.3. Rotating disc with a central hole

(a) General equations

The general equations for the stresses in a rotating hollow disc may be obtained in precisely the same way as those for the solid disc of the previous section,

$$\begin{aligned} \text{i.e.} \quad \sigma_r &= A - \frac{B}{r^2} - (3 + \nu) \frac{\rho \omega^2 r^2}{8} \\ \sigma_H &= A + \frac{B}{r^2} - (1 + 3\nu) \frac{\rho \omega^2 r^2}{8} \end{aligned}$$

The only difference to the previous treatment is the conditions which are required to evaluate the constants A and B since, in this case, B is not zero.

The above equations are similar in form to the Lamé equations for pressurised thick rings or cylinders with modifying terms added. Indeed, should the condition arise in service where a rotating ring or cylinder is also pressurised, then the pressure and rotation boundary conditions may be substituted simultaneously to determine appropriate values of the constants A and B .

However, returning to the rotation only case, the required boundary conditions are zero radial stress at both the inside and outside radius,

$$\text{i.e. at } r = R_1, \quad \sigma_r = 0$$

$$\therefore 0 = A - \frac{B}{R_1^2} - (3 + \nu) \frac{\rho \omega^2 R_1^2}{8}$$

$$\text{and at } r = R_2, \quad \sigma_r = 0$$

$$\therefore 0 = A - \frac{B}{R_2^2} - (3 + \nu) \frac{\rho \omega^2 R_2^2}{8}$$

Subtracting and simplifying,

$$B = (3 + \nu) \frac{\rho \omega^2 R_1^2 R_2^2}{8}$$

and

$$A = (3 + \nu) \frac{\rho \omega^2 (R_1^2 + R_2^2)}{8}$$

Substituting in eqns. (4.7) and (4.8) yields the final equation for the stresses

$$\sigma_r = (3 + \nu) \frac{\rho \omega^2}{8} \left[R_1^2 + R_2^2 - \frac{R_1^2 R_2^2}{r^2} - r^2 \right] \quad (4.13)$$

$$\sigma_H = \frac{\rho \omega^2}{8} \left[(3 + \nu) \left(R_1^2 + R_2^2 + \frac{R_1^2 R_2^2}{r^2} \right) - (1 + 3\nu)r^2 \right] \quad (4.14)$$

(b) Maximum stresses

The *maximum hoop stress* occurs at the inside radius where $r = R_1$,

$$\begin{aligned} \text{i.e. } \sigma_{H_{\max}} &= \frac{\rho \omega^2}{8} [(3 + \nu)(R_1^2 + R_2^2 + R_2^2) - (1 + 3\nu)R_1^2] \\ &= \frac{\rho \omega^2}{4} [(3 + \nu)R_2^2 + (1 - \nu)R_1^2] \end{aligned} \quad (4.15)$$

As the value of the inside radius approaches zero the maximum hoop stress value approaches

$$\frac{\rho \omega^2}{4} (3 + \nu) R_2^2$$

This is **twice** the value obtained at the centre of a solid disc rotating at the same speed. Thus the drilling of even a very small hole at the centre of a solid disc will double the maximum hoop stress set up owing to rotation.

At the outside of the disc when $r = R_2$

$$\sigma_{H_{\min}} = \frac{\rho \omega^2}{4} [(3 + \nu)R_1^2 + (1 - \nu)R_2^2]$$

The *maximum radial stress* is found by consideration of the equation

$$\sigma_r = (3 + \nu) \frac{\rho \omega^2}{8} \left[R_1^2 + R_2^2 - \frac{R_1^2 R_2^2}{r^2} - r^2 \right] \quad (4.13)(\text{bis})$$

This will be a maximum when $\frac{d\sigma_r}{dr} = 0$,

$$\text{i.e. when } 0 = \frac{d}{dr} \left[R_1^2 + R_2^2 - \frac{R_1^2 R_2^2}{r^2} - r^2 \right]$$

$$0 = R_1^2 R_2^2 \frac{2}{r^3} - 2r$$

$$r^4 = R_1^2 R_2^2$$

$$r = \sqrt{(R_1 R_2)} \quad (4.16)$$

Substituting for r in eqn. (4.13).

$$\begin{aligned}\sigma_{r_{\max}} &= (3 + \nu) \frac{\rho \omega^2}{8} [R_1^2 + R_2^2 - R_1 R_2 - R_1 R_2] \\ &= (3 + \nu) \frac{\rho \omega^2}{8} [R_2 - R_1]^2\end{aligned}\quad (4.17)$$

The complete radial and hoop stress distributions are indicated in Fig. 4.4.

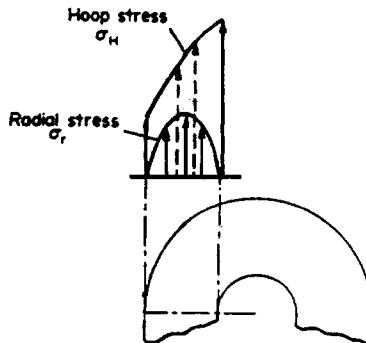


Fig. 4.4. Hoop and radial stress distribution in a rotating hollow disc.

4.4. Rotating thick cylinders or solid shafts

In the case of rotating thick cylinders the longitudinal stress σ_L must be taken into account and the longitudinal strain is assumed to be constant. Thus, writing the equations for the strain in three mutually perpendicular directions (see §4.2),

$$\varepsilon_L = \frac{1}{E} (\sigma_L - \nu \sigma_H - \nu \sigma_r) \quad (4.18)$$

$$\varepsilon_r = \frac{1}{E} (\sigma_r - \nu \sigma_H - \nu \sigma_L) = \frac{ds}{dr} \quad (4.19)$$

$$\varepsilon_H = \frac{1}{E} (\sigma_H - \nu \sigma_r - \nu \sigma_L) = \frac{s}{r} \quad (4.20)$$

From eqn. (4.20)

$$Es = r [\sigma_H - \nu (\sigma_r + \sigma_L)]$$

Differentiating,

$$E \frac{ds}{dr} = r \left[\frac{d\sigma_H}{dr} - \nu \frac{d\sigma_r}{dr} - \nu \frac{d\sigma_L}{dr} \right] + 1 [\sigma_H - \nu \sigma_r - \nu \sigma_L]$$

Substituting for $E(ds/dr)$ in eqn. (4.19),

$$\begin{aligned}\sigma_r - \nu \sigma_H - \nu \sigma_L &= r \left[\frac{d\sigma_H}{dr} - \nu \frac{d\sigma_r}{dr} - \nu \frac{d\sigma_L}{dr} \right] + \sigma_H - \nu \sigma_r - \nu \sigma_L \\ 0 &= (\sigma_H - \sigma_r)(1 + \nu) + r \frac{d\sigma_H}{dr} - \nu r \frac{d\sigma_r}{dr} - \nu r \frac{d\sigma_L}{dr}\end{aligned}$$

Now, since ε_L is constant, differentiating eqn. (4.18),

$$\frac{d\sigma_L}{dr} = \nu \left[\frac{d\sigma_H}{dr} + \frac{d\sigma_r}{dr} \right]$$

$$\therefore 0 = (\sigma_H - \sigma_r)(1 + \nu) + r(1 - \nu^2) \frac{d\sigma_H}{dr} - \nu r(1 + \nu) \frac{d\sigma_r}{dr}$$

Dividing through by $(1 + \nu)$,

$$0 = (\sigma_H - \sigma_r) + r(1 - \nu) \frac{d\sigma_H}{dr} - \nu r \frac{d\sigma_r}{dr}$$

But the general equilibrium equation will be the same as that obtained in §4.2, eqn. (4.1),

i.e. $\sigma_H - \sigma_r - r \frac{d\sigma_r}{dr} = \rho \omega^2 r^2$

Therefore substituting for $(\sigma_H - \sigma_r)$,

$$0 = \rho \omega^2 r^2 + r \frac{d\sigma_r}{dr} + r(1 - \nu) \frac{d\sigma_H}{dr} - \nu r \frac{d\sigma_r}{dr}$$

$$0 = \rho \omega^2 r^2 + r(1 - \nu) \left[\frac{d\sigma_H}{dr} + \frac{d\sigma_r}{dr} \right]$$

$$\therefore \frac{d\sigma_H}{dr} + \frac{d\sigma_r}{dr} = -\frac{\rho \omega^2 r}{(1 - \nu)}$$

Integrating,

$$\sigma_H + \sigma_r = -\frac{\rho \omega^2 r^2}{2(1 - \nu)} + 2A$$

where $2A$ is a convenient constant of integration. This equation can now be compared with the equivalent equation of §4.2, when it is evident that similar results for σ_H and σ_r can be obtained if $(1 + \nu)$ is replaced by $1/(1 - \nu)$ or, alternatively, if ν is replaced by $\nu/(1 - \nu)$, see §8.14.2. **Thus hoop and radial stresses in rotating thick cylinders can be obtained from the equations for rotating discs provided that Poisson's ratio ν is replaced by $\nu/(1 - \nu)$** , e.g. the stress at the centre of a rotating solid shaft will be given by eqn. (4.11) for a solid disc modified as stated above,

i.e. $\sigma_H = \left[3 + \frac{\nu}{(1 - \nu)} \right] \frac{\rho \omega^2 R^2}{8}$ (4.21)

4.5. Rotating disc of uniform strength

In applications such as turbine blades rotating at high speeds it is often desirable to design for constant stress conditions under the action of the high centrifugal forces to which they are subjected.

Consider, therefore, an element of a disc subjected to equal hoop and radial stresses,

i.e. $\sigma_H = \sigma_r = \sigma$ (Fig. 4.5)

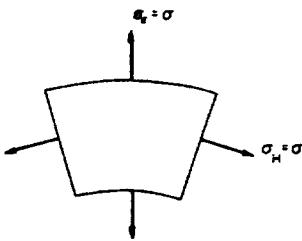


Fig. 4.5. Stress acting on an element in a rotating disc of uniform strength.

The condition of equal stress can only be achieved, as in the case of uniform strength cantilevers, by varying the thickness. Let the thickness be t at radius r and $(t + \delta t)$ at radius $(r + \delta r)$.

Then centrifugal force on the element

$$\begin{aligned} &= \text{mass} \times \text{acceleration} \\ &= (\rho t r \delta \theta \delta r) \omega^2 r \\ &= \rho t \omega^2 r^2 \delta \theta \delta r \end{aligned}$$

The equilibrium equation is then

$$\rho t \omega^2 r^2 \delta \theta \delta r + \sigma(r + \delta r) \delta \theta (t + \delta t) = 2\sigma t \delta r \sin \frac{1}{2}\delta \theta + \sigma_r t \delta \theta$$

i.e. in the limit

$$\begin{aligned} \sigma t dr &= \rho \omega^2 r^2 t dr + \sigma t dr + \sigma r dt \\ \therefore \sigma r dt &= -\rho \omega^2 r^2 t dr \\ \therefore \frac{dt}{dr} &= -\frac{\rho \omega^2 r t}{\sigma} \end{aligned}$$

Integrating,

$$\log_e t = -\frac{\rho \omega^2 r^2}{2\sigma} + \log_e A$$

where $\log_e A$ is a convenient constant.

$$t = A e^{(-\rho \omega^2 r^2)/(2\sigma)}$$

where $r = 0$

$$t = A = t_0$$

i.e. for uniform strength the thickness of the disc must vary according to the following equation,

$$t = t_0 e^{(-\rho \omega^2 r^2)/(2\sigma)} \quad (4.22)$$

4.6. Combined rotational and thermal stresses in uniform discs and thick cylinders

If the temperature of any component is raised uniformly then, provided that the material is free to expand, expansion takes place without the introduction of any so-called thermal or temperature stresses. In cases where components, e.g. discs, are subjected to thermal

gradients, however, one part of the material attempts to expand at a faster rate than another owing to the difference in temperature experienced by each part, and as a result stresses are developed. These are analogous to the differential expansion stresses experienced in compound bars of different materials and treated in §2.3.[†]

Consider, therefore, a disc initially unstressed and subjected to a temperature rise T . Then, for a radial movement s of any element, eqns. (4.2) and (4.3) may be modified to account for the strains due to temperature thus:

$$\frac{ds}{dr} = \frac{1}{E}(\sigma_r - v\sigma_H + E\alpha T) \quad (4.23)$$

and

$$\frac{s}{r} = \frac{1}{E}(\sigma_H - v\sigma_r + E\alpha T) \quad (4.24)$$

where α is the coefficient of expansion of the disc material (see §2.3)[†]

From eqn. (4.24),

$$\frac{ds}{dr} = \frac{1}{E} \left[(\sigma_H - v\sigma_r + E\alpha T) + r \left(\frac{d\sigma_H}{dr} - v \frac{d\sigma_r}{dr} + E\alpha \frac{dT}{dr} \right) \right]$$

Therefore from eqn. (4.23),

$$\begin{aligned} \frac{1}{E}(\sigma_r - v\sigma_H + E\alpha T) &= \frac{1}{E} \left[(\sigma_H - v\sigma_r + E\alpha T) - r \left(\frac{d\sigma_H}{dr} - v \frac{d\sigma_r}{dr} + E\alpha \frac{dT}{dr} \right) \right] \\ \therefore (\sigma_H - \sigma_r)(1 + v) + r \frac{d\sigma_H}{dr} - vr \frac{d\sigma_r}{dr} + E\alpha r \frac{dT}{dr} &= 0 \end{aligned} \quad (4.25)$$

but, from the equilibrium eqn. (4.1),

$$\sigma_H - \sigma_r - r \frac{d\sigma_r}{dr} = \rho r^2 \omega^2$$

Therefore substituting for $(\sigma_H - \sigma_r)$ in eqn. (4.25),

$$\begin{aligned} (1 + v) \left(\rho r^2 \omega^2 + r \frac{d\sigma_r}{dr} \right) + r \frac{d\sigma_H}{dr} - vr \frac{d\sigma_r}{dr} + E\alpha r \frac{dT}{dr} &= 0 \\ (1 + v)\rho r^2 \omega^2 + r \frac{d\sigma_r}{dr} + r \frac{d\sigma_H}{dr} + E\alpha r \frac{dT}{dr} &= 0 \\ \frac{d\sigma_H}{dr} + \frac{d\sigma_r}{dr} &= -(1 + v)\rho r \omega^2 - E\alpha \frac{dT}{dr} \end{aligned}$$

Integrating,

$$\sigma_H + \sigma_r = -(1 + v) \frac{\rho r^2 \omega^2}{2} - E\alpha T + 2A \quad (4.26)$$

where, again, $2A$ is a convenient constant.

Subtracting eqn. (4.1),

$$2\sigma_r + r \frac{d\sigma_r}{dr} = -\frac{\rho r^2 \omega^2}{2}(3 + v) - E\alpha T + 2A$$

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

But $2\sigma_r + r \frac{d\sigma_r}{dr} = \frac{d}{dr} \left[(r^2 \sigma_r) \times \frac{1}{r} \right]$

$$\therefore \frac{d}{dr} (r^2 \sigma_r) = r \left[-\frac{\rho r^2 \omega^2}{2} (3 + \nu) - E\alpha T + 2A \right]$$

Integrating, $r^2 \sigma_r = -\frac{\rho r^4 \omega^2}{8} (3 + \nu) - E\alpha \int Tr dr + \frac{2Ar^2}{2} - B$

where, as in eqn. (4.7), $-B$ is a second convenient constant of integration.

$$\therefore \sigma_r = A - \frac{B}{r^2} - \frac{\rho r^2 \omega^2}{8} (3 + \nu) - \frac{Ea}{r^2} \int Tr dr \quad (4.27)$$

Then, from eqn. (4.26),

$$\sigma_H = A + \frac{B}{r^2} - (1 + 3\nu) \frac{\rho r^2 \omega^2}{8} - E\alpha T + \frac{Ea}{r^2} \int Tr dr \quad (4.28)$$

i.e. the expressions obtained for the hoop and radial stresses are those of the standard Lamé equations for simple pressurisation with (a) modifying terms for rotational effects as obtained in previous sections of this chapter, and (b) modifying terms for thermal effects.

A solution to eqns. (4.27) and (4.28) for discs may thus be obtained provided that the way in which T varies with r is known. Because of the form of the equations it is clear that, if required, pressure, rotational and thermal effects can be considered simultaneously and the appropriate values of A and B determined.

For thick cylinders with an axial length several times the outside diameter the above plane stress equations may be modified to the equivalent plane strain equations (see §8.14.2) by replacing ν by $\nu/(1 - \nu)$, E by $E/(1 - \nu^2)$ and α by $(1 + \nu)\alpha$.

i.e.

$E\alpha$ becomes $E\alpha(1 - \nu)$

In the absence of rotation the equations simplify to

$$\sigma_r = A - \frac{B}{r^2} - \frac{Ea}{r^2} \int Tr dr \quad (4.29)$$

$$\sigma_H = A + \frac{B}{r^2} + \frac{Ea}{r^2} \int Tr dr - E\alpha T \quad (4.30)$$

With a linear variation of temperature from $T = 0$ at $r = 0$,

i.e. with

$$T = Kr$$

$$\sigma_r = A - \frac{B}{r^2} - \frac{EaKr}{3} \quad (4.31)$$

$$\sigma_H = A + \frac{B}{r^2} - 2 \frac{EaKr}{3} \quad (4.32)$$

With a steady heat flow, for example, in the case of thick cylinders when $E\alpha$ becomes $E\alpha/(1 - \nu)$ —see p. 125.

$$\frac{rdT}{dr} = \text{constant} = b$$

$$\therefore \frac{dT}{dr} = \frac{b}{r} \quad \text{and} \quad T = a + b \log_e r$$

and the equations become

$$\sigma_r = A - \frac{B}{r^2} - \frac{EaT}{2(1-\nu)} \quad (4.33)$$

$$\sigma_H = A + \frac{B}{r^2} - \frac{EaT}{2(1-\nu)} - \frac{Eab}{2(1-\nu)} \quad (4.34)$$

In practical applications where the temperature is higher on the inside of the disc or thick cylinder than the outside, the thermal stresses are tensile on the outside surface and compressive on the inside. They may thus be considered as favourable in pressurised thick cylinder applications where they will tend to reduce the high tensile stresses on the inside surface due to pressure. However, in the chemical industry, where endothermic reactions may be contained within the walls of a thick cylinder, the reverse situation applies and the two stress systems add to provide a potentially more severe stress condition.

Examples

Example 4.1

A steel ring of outer diameter 300 mm and internal diameter 200 mm is shrunk onto a solid steel shaft. The interference is arranged such that the radial pressure between the mating surfaces will not fall below 30 MN/m² whilst the assembly rotates in service. If the maximum circumferential stress on the inside surface of the ring is limited to 240 MN/m², determine the maximum speed at which the assembly can be rotated. It may be assumed that no relative slip occurs between the shaft and the ring.

For steel, $\rho = 7470 \text{ kg/m}^3$, $\nu = 0.3$, $E = 208 \text{ GN/m}^2$.

Solution

From eqn. (4.7)

$$\sigma_r = A - \frac{B}{r^2} - \frac{(3+\nu)}{8} \rho \omega^2 r^2 \quad (1)$$

Now when $r = 0.15$, $\sigma_r = 0$

$$\therefore 0 = A - \frac{B}{0.15^2} - \frac{3.3}{8} \rho \omega^2 (0.15)^2 \quad (2)$$

Also, when $r = 0.1$, $\sigma_r = -30 \text{ MN/m}^2$

$$\therefore -30 \times 10^6 = A - \frac{B}{0.1^2} - \frac{3.3}{8} \rho \omega^2 (0.1)^2 \quad (3)$$

$$(2)-(3), \quad 30 \times 10^6 = B(100 - 44.4) - \frac{3.3}{8} \rho \omega^2 (0.0225 - 0.01)$$

$$\therefore B = \frac{30 \times 10^6}{55.6} + 3.3 \times \frac{0.0125 \times 7470}{8 \times 55.6} \omega^2$$

$$B = 0.54 \times 10^6 + 0.693 \omega^2$$

and from (3),

$$\begin{aligned} A &= 100(0.54 \times 10^6 + 0.693\omega^2) + \frac{3.3 \times 7470 \times 0.01\omega^2}{8} - 30 \times 10^6 \\ &= 54 \times 10^6 + 69.3\omega^2 + 30.8\omega^2 - 30 \times 10^6 \\ &= 24 \times 10^6 + 100.1\omega^2 \end{aligned}$$

But since the maximum hoop stress at the inside radius is limited to 240 MN/m², from eqn. (4.8)

$$\sigma_H = A + \frac{B}{r^2} - \frac{(1+3\nu)}{8}\rho\omega^2 r^2$$

i.e.

$$240 \times 10^6 = (24 \times 10^6 + 100.1\omega^2) + \frac{(0.54 \times 10^6 + 0.693\omega^2)}{0.1^2} - \frac{1.9}{8} \times 7470 \times 0.01\omega^2$$

$$240 \times 10^6 = 78 \times 10^6 + 169.3\omega^2 - 17.7\omega^2$$

$$\therefore 151.7\omega^2 = 162 \times 10^6$$

$$\omega^2 = \frac{162 \times 10^6}{151.7} = 1.067 \times 10^6$$

$$\omega = 1033 \text{ rad/s} = \mathbf{9860 \text{ rev/min}}$$

Example 4.2

A steel rotor disc which is part of a turbine assembly has a uniform thickness of 40 mm. The disc has an outer diameter of 600 mm and a central hole of 100 mm diameter. If there are 200 blades each of mass 0.153 kg pitched evenly around the periphery of the disc at an effective radius of 320 mm, determine the rotational speed at which yielding of the disc first occurs according to the maximum shear stress criterion of elastic failure.

For steel, $E = 200 \text{ GN/m}^2$, $\nu = 0.3$, $\rho = 7470 \text{ kg/m}^3$ and the yield stress σ_y in simple tension = 500 MN/m².

Solution

$$\text{Total mass of blades} = 200 \times 0.153 = 30.6 \text{ kg}$$

$$\text{Effective radius} = 320 \text{ mm}$$

$$\text{Therefore centrifugal force on the blades} = m\omega^2 r = 30.6 \times \omega^2 \times 0.32$$

$$\text{Now the area of the disc rim} = \pi d t = \pi \times 0.6 \times 0.004 = 0.024\pi \text{m}^2$$

The centrifugal force acting on this area thus produces an effective radial stress acting on the outside surface of the disc since the blades can be assumed to produce a uniform loading around the periphery.

Therefore radial stress at outside surface

$$= \frac{30.6 \times \omega^2 \times 0.32}{0.024\pi} = 130\omega^2 \text{ N/m}^2 \quad (\text{tensile})$$

Now eqns. (4.7) and (4.8) give the general form of the expressions for hoop and radial stresses set up owing to rotation,

$$\text{i.e. } \sigma_r = A - \frac{B}{r^2} - \frac{(3 + \nu)}{8} \rho \omega^2 r^2 \quad (1)$$

$$\sigma_H = A + \frac{B}{r^2} - \frac{(1 + 3\nu)}{8} \rho \omega^2 r^2 \quad (2)$$

When $r = 0.05$,

$$\sigma_r = 0$$

$$\therefore 0 = A - 400B - \frac{3.3}{8} \rho \omega^2 (0.05)^2 \quad (3)$$

When $r = 0.3$,

$$\sigma_r = +130\omega^2$$

$$\therefore 130\omega^2 = A - 11.1B - \frac{3.3}{8} \rho \omega^2 (0.3)^2 \quad (4)$$

$$(4)-(3), \quad 130\omega^2 = 388.9B - \frac{3.3}{8} \rho \omega^2 (9 - 0.25)10^{-2}$$

$$130\omega^2 = 388.9B - 270\omega^2$$

$$B = \frac{(130 + 270)}{388.9} \omega^2 = 1.03\omega^2$$

Substituting in (3),

$$\begin{aligned} A &= 412\omega^2 + \frac{3.3}{8} \times 7470(0.05)^2\omega^2 \\ &= 419.7\omega^2 = 420\omega^2 \end{aligned}$$

Therefore substituting in (2) and (1), the stress conditions at the inside surface are

$$\sigma_H = 420\omega^2 + 412\omega^2 - 4.43\omega^2 = 827\omega^2$$

with

$$\sigma_r = 0$$

and at the outside

$$\sigma_H = 420\omega^2 + 11.42\omega^2 - 159\omega^2 = 272\omega^2$$

with

$$\sigma_r = 130\omega^2$$

The most severe stress conditions therefore occur at the inside radius where the maximum shear stress is greatest

$$\text{i.e. } \tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{827\omega^2 - 0}{2}$$

Now the maximum shear stress theory of elastic failure states that failure is assumed to occur when this stress equals the value of τ_{\max} at the yield point in simple tension,

$$\text{i.e. } \tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_y - 0}{2} = \frac{\sigma_y}{2}$$

Thus, for failure according to this theory,

$$\frac{\sigma_y}{2} = \frac{827\omega^2}{2}$$

i.e.

$$827\omega^2 = \sigma_y = 500 \times 10^6$$

$$\therefore \omega^2 = \frac{500}{827} \times 10^6 = 0.604 \times 10^6$$

$$\omega = 780 \text{ rad/s} = \mathbf{7450 \text{ rev/min}}$$

Example 4.3

The cross-section of a turbine rotor disc is designed for uniform strength under rotational conditions. The disc is keyed to a 60 mm diameter shaft at which point its thickness is a maximum. It then tapers to a minimum thickness of 10 mm at the outer radius of 250 mm where the blades are attached. If the design stress of the shaft is 250 MN/m² at the design speed of 12 000 rev/min, what is the required maximum thickness? For steel $\rho = 7470 \text{ kg/m}^3$.

Solution

From eqn. (4.22) the thickness of a uniform strength disc is given by

$$t = t_0 e^{(-\rho\omega^2 r^2)/(2\sigma)} \quad (1)$$

where t_0 is the thickness at $r = 0$.

Now at $r = 0.25$,

$$\frac{\rho\omega^2 r^2}{2\sigma} = \frac{7470}{2 \times 250 \times 10^6} \left(12000 \times \frac{2\pi}{60} \right)^2 \times 0.25^2 = 1.47$$

and at $r = 0.03$,

$$\begin{aligned} \frac{\rho\omega^2 r^2}{2\sigma} &= \frac{7470}{2 \times 250 \times 10^6} \left(12000 \times \frac{2\pi}{60} \right)^2 \times 0.03^2 \\ &= 1.47 \times \frac{9 \times 10^{-4}}{625 \times 10^{-4}} = 0.0212 \end{aligned}$$

But at $r = 0.25$, $t = 10 \text{ mm}$

Therefore substituting in (1),

$$\begin{aligned} 0.01 &= t_0 e^{-1.47} = 0.2299 t_0 \\ t_0 &= \frac{0.01}{0.2299} = 0.0435 \text{ m} = 43.5 \text{ mm} \end{aligned}$$

Therefore at $r = 0.03$

$$\begin{aligned} t &= 0.0435 e^{-0.0212} = 0.0435 \times 0.98 \\ &= 0.0426 \text{ m} = \mathbf{42.6 \text{ mm}} \end{aligned}$$

Example 4.4

(a) Derive expressions for the hoop and radial stresses developed in a solid disc of radius R when subjected to a thermal gradient of the form $T = Kr$. Hence determine the position

and magnitude of the maximum stresses set up in a steel disc of 150 mm diameter when the temperature rise is 150°C. For steel, $\alpha = 12 \times 10^{-6}$ per °C and $E = 206.8 \text{ GN/m}^2$.

(b) How would the values be changed if the temperature at the centre of the disc was increased to 30°C, the temperature rise across the disc maintained at 150°C and the thermal gradient now taking the form $T = a + br$?

Solution

(a) The hoop and radial stresses are given by eqns. (4.29) and (4.30) as follows:

$$\sigma_r = A - \frac{B}{r^2} - \frac{\alpha E}{r^2} \int Tr dr \quad (1)$$

$$\sigma_H = A + \frac{B}{r^2} + \frac{\alpha E}{r^2} \int Tr dr - \alpha ET \quad (2)$$

In this case

$$\int Tr dr = K \int r^2 dr = \frac{Kr^3}{3}$$

the constant of integration being incorporated into the general constant A .

$$\therefore \sigma_r = A - \frac{B}{r^2} - \frac{\alpha E Kr}{3} \quad (3)$$

$$\sigma_H = A + \frac{B}{r^2} + \frac{\alpha E Kr}{3} - \alpha E Kr \quad (4)$$

Now in order that the stresses at the centre of the disc, where $r = 0$, shall not be infinite, B must be zero and hence B/r^2 is zero. Also $\sigma_r = 0$ at $r = R$.

Therefore substituting in (3),

$$0 = A - \frac{\alpha E Kr}{3} \text{ and } A = \frac{\alpha E Kr}{3}$$

Substituting in (3) and (4) and rearranging,

$$\sigma_r = \frac{\alpha E K}{3} (R - r)$$

$$\sigma_H = \frac{\alpha E K}{3} (R - 2r)$$

The variation of both stresses with radius is linear and they will both have maximum values at the centre where $r = 0$.

$$\begin{aligned} \therefore \sigma_{r_{\max}} &= \sigma_{H_{\max}} = \frac{\alpha E Kr}{3} \\ &= \frac{12 \times 10^{-6} \times 206.8 \times 10^9 \times K \times 0.075}{3} \end{aligned}$$

Now $T = Kr$ and T must therefore be zero at the centre of the disc where r is zero. Thus, with a known temperature rise of 150°C, it follows that the temperature at the outside radius must be 150°C.

$$\therefore 150 = K \times 0.075$$

$$\therefore K = 2000^\circ/\text{m}$$

i.e.

$$\sigma_{r_{\max}} = \sigma_{H_{\max}} = \frac{12 \times 10^{-6} \times 206.8 \times 10^9 \times 2000 \times 0.075}{3}$$

$$= 124 \text{ MN/m}^2$$

(b) With the modified form of temperature gradient,

$$\int Tr dr = \int (a + br)r dr = \int (ar + br^2) dr$$

$$= \frac{ar^2}{2} + \frac{br^3}{3}$$

Substituting in (1) and (2),

$$\sigma_r = A - \frac{B}{r^2} - \frac{\alpha E}{r^2} \left[\frac{ar^2}{2} + \frac{br^3}{3} \right] \quad (5)$$

$$\sigma_H = A + \frac{B}{r^2} + \frac{\alpha E}{r^2} \left[\frac{ar^2}{2} + \frac{br^3}{3} \right] - \alpha ET \quad (6)$$

Now

$$T = a + br$$

Therefore at the inside of the disc where $r = 0$ and $T = 30^\circ C$,

$$30 = a + b(0) \quad (7)$$

and

$$a = 30$$

At the outside of the disc where $T = 180^\circ C$,

$$180 = a + b(0.075) \quad (8)$$

$$(8) - (7) \quad 150 = 0.075b \quad \therefore b = 2000$$

Substituting in (5) and (6) and simplifying,

$$\sigma_r = A - \frac{B}{r^2} - \alpha E(15 + 667r) \quad (9)$$

$$\sigma_H = A + \frac{B}{r^2} + \alpha E(15 + 667r) - \alpha ET \quad (10)$$

Now for finite stresses at the centre,

$$B = 0$$

Also, at $r = 0.075$,

$$\sigma_r = 0 \text{ and } T = 180^\circ C$$

Therefore substituting in (9),

$$0 = A - 12 \times 10^{-6} \times 206.8 \times 10^9 (15 + 667 \times 0.075)$$

$$0 = A - 12 \times 206.8 \times 10^3 \times 65$$

$$\therefore A = 161.5 \times 10^6$$

From (9) and (10) the maximum stresses will again be at the centre where $r = 0$,

i.e. $\sigma_{r_{\max}} = \sigma_{H_{\max}} = A - \alpha ET = 124 \text{ MN/m}^2$, as before.

N.B. The same answers would be obtained for any linear gradient with a temperature difference of 150°C. Thus a solution could be obtained with the procedure of part (a) using the form of distribution $T = Kr$ with the value of T at the outside taken to be 150°C (the value at $r = 0$ being automatically zero).

Example 4.5

An initially unstressed short steel cylinder, internal radius 0.2 m and external radius 0.3 m, is subjected to a temperature distribution of the form $T = a + b \log_e r$ to ensure constant heat flow through the cylinder walls. With this form of distribution the radial and circumferential stresses at any radius r , where the temperature is T , are given by

$$\sigma_r = A - \frac{B}{r^2} - \frac{\alpha ET}{2(1-\nu)}$$

$$\sigma_H = A + \frac{B}{r^2} - \frac{\alpha ET}{2(1-\nu)} - \frac{E\alpha b}{2(1-\nu)}$$

If the temperatures at the inside and outside surfaces are maintained at 200°C and 100°C respectively, determine the maximum circumferential stress set up in the cylinder walls. For steel, $E = 207 \text{ GN/m}^2$, $\nu = 0.3$ and $\alpha = 11 \times 10^{-6}$ per °C.

Solution

$$T = a + b \log_e r$$

$$\therefore 200 = a + b \log_e 0.2 = a + b(0.6931 - 2.3026)$$

$$200 = a - 1.6095 b \quad (1)$$

$$\text{also } 100 = a + b \log_e 0.3 = a + b(1.0986 - 2.3026)$$

$$100 = a - 1.204 b \quad (2)$$

$$(2) - (1), \quad 100 = -0.4055 b$$

$$b = -246.5 = -247$$

$$\begin{aligned} \text{Also } \frac{E\alpha}{2(1-\nu)} &= \frac{207 \times 10^9 \times 11 \times 10^{-6}}{2(1-0.29)} \\ &= 1.6 \times 10^6 \end{aligned}$$

Therefore substituting in the given expression for radial stress,

$$\sigma_r = A - \frac{B}{r^2} - 1.6 \times 10^6 T$$

At $r = 0.3$, $\sigma_r = 0$ and $T = 100$

$$0 = A - \frac{B}{0.09} - 1.6 \times 10^6 \times 100 \quad (3)$$

At $r = 0.2$, $\sigma_r = 0$ and $T = 200$

$$0 = A - \frac{B}{0.04} - 1.6 \times 10^6 \times 200 \quad (4)$$

$$(4) - (3), \quad 0 = B(11.1 - 25) - 1.6 \times 10^8$$

$$B = -11.5 \times 10^6$$

and from (4),

$$\begin{aligned} A &= 25B + 3.2 \times 10^8 \\ &= (-2.88 + 3.2)10^8 = 0.32 \times 10^8 \end{aligned}$$

substituting in the given expression for hoop stress,

$$\sigma_H = 0.32 \times 10^8 - \frac{11.5 \times 10^6}{r^2} - 1.6 \times 10^6 T + 1.6 \times 10^6 \times 247$$

$$\text{At } r = 0.2, \quad \sigma_H = (0.32 - 2.88 - 3.2 + 3.96)10^8 = -180 \text{ MN/m}^2$$

$$\text{At } r = 0.3, \quad \sigma_H = (0.32 - 1.28 - 1.6 + 3.96)10^8 = +140 \text{ MN/m}^2$$

The maximum tensile circumferential stress therefore occurs at the outside radius and has a value of 140 MN/m². The maximum compressive stress is 180 MN/m² at the inside radius.

Problems

Unless otherwise stated take the following material properties for steel:

$$\rho = 7470 \text{ kg/m}^3; \quad v = 0.3; \quad E = 207 \text{ GN/m}^2$$

4.1 (B). Determine equations for the hoop and radial stresses set up in a solid rotating disc of radius R commencing with the following relationships:

$$\sigma_r = A - \frac{B}{r^2} - (3 + v) \frac{\rho \omega^2 r^2}{8}$$

$$\sigma_H = A + \frac{B}{r^2} - (1 + 3v) \frac{\rho \omega^2 r^2}{8}$$

Hence determine the maximum stress and the stress at the outside of a 250 mm diameter disc which rotates at 12 000 rev/min. [76, 32.3 MN/m².]

4.2 (B). Determine from first principles the hoop stress at the inside and outside radius of a thin steel disc of 300 mm diameter having a central hole of 100 mm diameter, if the disc is made to rotate at 5000 rev/min. What will be the position and magnitude of the maximum radial stress?

$$[38.9, 12.3 \text{ MN/m}^2; 87 \text{ mm rad}; 8.4 \text{ MN/m}^2.]$$

4.3 (B). Show that the tensile hoop stress set up in a thin rotating ring or cylinder is given by

$$\sigma_H = \rho \omega^2 r^2$$

Hence determine the maximum angular velocity at which the disc can be rotated if the hoop stress is limited to 20 MN/m². The ring has a mean diameter of 260 mm. [3800 rev/min.]

4.4 (B). A solid steel disc 300 mm diameter and of small constant thickness has a steel ring of outer diameter 450 mm and the same thickness shrunk onto it. If the interference pressure is reduced to zero at a rotational speed of 3000 rev/min, calculate

- (a) the radial pressure at the interface when stationary;
- (b) the difference in diameters of the mating surfaces of the disc and ring before assembly.

The radial and circumferential stresses at radius r in a ring or disc rotating at ω rad/s are obtained from the following relationships:

$$\sigma_r = A - \frac{B}{r^2} - (3 + v) \frac{\rho \omega^2 r^2}{8}$$

$$\sigma_H = A + \frac{B}{r^2} - (1 + 3\nu) \frac{\rho\omega^2 r^2}{8} \quad [8.55 \text{ MN/m}^2, 0.045 \text{ mm.}]$$

4.5 (B). A steel rotor disc of uniform thickness 50 mm has an outer rim of diameter 800 mm and a central hole of diameter 150 mm. There are 200 blades each of weight 2 N at an effective radius of 420 mm pitched evenly around the periphery. Determine the rotational speed at which yielding first occurs according to the maximum shear stress criterion.

Yield stress in simple tension = 750 MN/m².

The basic equations for radial and hoop stresses given in Example 4.4 may be used without proof.

[7300 rev/min.]

4.6 (B). A rod of constant cross-section and of length $2a$ rotates about its centre in its own plane so that each end of the rod describes a circle of radius a . Find the maximum stress in the rod as a function of the peripheral speed V . $[\frac{1}{2}(\rho\omega^2 a^2).]$

4.7 (B). A turbine blade is to be designed for constant tensile stress σ under the action of centrifugal force by varying the area A of the blade section. Consider the equilibrium of an element and show that the condition is

$$A = A_h e^{[-\rho\omega^2(r^2 - r_h^2)]/(2\sigma)}$$

where A_h and r_h are the cross-sectional area and radius at the hub (i.e. base of the blade).

4.8 (B). A steel turbine rotor of 800 mm outside diameter and 200 mm inside diameter is 50 mm thick. The rotor carries 100 blades each 200 mm long and of mass 0.5 kg. The rotor runs at 3000 rev/min. Assuming the shaft to be rigid, calculate the expansion of the inner bore of the disc due to rotation and hence the initial shrinkage allowance necessary. [0.14 mm.]

4.9 (B). A steel disc of 750 mm diameter is shrunk onto a steel shaft of 80 mm diameter. The interference on the diameter is 0.05 mm.

(a) Find the maximum tangential stress in the disc at standstill.

(b) Find the speed in rev/min at which the contact pressure is zero.

(c) What is the maximum tangential stress at the speed found in (b)? [65 MN/m²; 3725; 65 MN/m².]

4.10 (B). A flat steel turbine disc of 600 mm outside diameter and 120 mm inside diameter rotates at 3000 rev/min at which speed the blades and shrouding cause a tensile rim loading of 5 MN/m². The maximum stress at this speed is to be 120 MN/m². Find the maximum shrinkage allowance on the diameter when the disc is put on the shaft. [0.097 mm.]

4.11 (B). Find the maximum permissible speed of rotation for a steel disc of outer and inner radii 150 mm and 70 mm respectively if the outer radius is not to increase in magnitude due to centrifugal force by more than 0.03 mm. [7900 rev/min.]

4.12 (B). The radial and hoop stresses at any radius r for a disc of uniform thickness rotating at an angular speed ω rad/s are given respectively by

$$\sigma_r = A - \frac{B}{r^2} - (3 + \nu) \frac{\rho\omega^2 r^2}{8}$$

$$\sigma_H = A + \frac{B}{r^2} - (1 + 3\nu) \frac{\rho\omega^2 r^2}{8}$$

where A and B are constants, ν is Poisson's ratio and ρ is the density of the material. Determine the greatest values of the radial and hoop stresses for a disc in which the outer and inner radii are 300 mm and 150 mm respectively.

Take $\omega = 150$ rad/s, $\nu = 0.304$ and $\rho = 7470 \text{ kg/m}^3$. [U.L.] [1.56, 13.2 MN/m².]

4.13 (B). Derive an expression for the tangential stress set up when a thin hoop, made from material of density $\rho \text{ kg/m}^3$, rotates about its polar axis with a tangential velocity of $v \text{ m/s}$.

What will be the greatest value of the mean radius of such a hoop, made from flat mild-steel bar, if the maximum allowable tensile stress is 45 MN/m² and the hoop rotates at 300 rev/min?

Density of steel = 7470 kg/m³. [2.47 m.]

4.14 (C). Determine the hoop stresses at the inside and outside surfaces of a long thick cylinder inside radius = 75 mm, outside radius = 225 mm, which is rotated at 4000 rev/min.

Take $\nu = 0.3$ and $\rho = 7470 \text{ kg/m}^3$. [57.9, 11.9 MN/m².]

4.15 (C). Calculate the maximum principal stress and maximum shear stress set up in a thin disc when rotating at 12000 rev/min. The disc is of 300 mm outside diameter and 75 mm inside diameter.

Take $\nu = 0.3$ and $\rho = 7470 \text{ kg/m}^3$. [221, 110.5 MN/m².]

4.16 (B). A thin-walled cylindrical shell made of material of density ρ has a mean radius r and rotates at a constant angular velocity of ω rad/s. Assuming the formula for centrifugal force, establish a formula for the circumferential (hoop) stress induced in the cylindrical shell due to rotation about the longitudinal axis of the cylinder and, if necessary, adjust the derived expression to give the stress in MN/m².

A drum rotor is to be used for a speed of 3000 rev/min. The material is steel with an elastic limit stress of 248 MN/m² and a density of 7.8 Mg/m³. Determine the mean diameter allowable if a factor of safety of 2.5 on the elastic limit stress is desired. Calculate also the expansion of this diameter (in millimetres) when the shell is rotating.

For steel, $E = 207$ GN/m².

[I.Mech.E.] [0.718 m; 0.344 mm.]

4.17 (B). A forged steel drum, 0.524 m outside diameter and 19 mm wall thickness, has to be mounted in a machine and spun about its longitudinal axis. The centrifugal (hoop) stress induced in the cylindrical shell is not to exceed 83 MN/m². Determine the maximum speed (in rev/min) at which the drum can be rotated.

For steel, the density = 7.8 Mg/m³.

[3630.]

4.18 (B). A cylinder, which can be considered as a thin-walled shell, is made of steel plate 16 mm thick and is 2.14 m internal diameter. The cylinder is subjected to an internal fluid pressure of 0.55 MN/m² gauge and, at the same time, rotated about its longitudinal axis at 3000 rev/min. Determine:

- the hoop stress induced in the wall of the cylinder due to rotation;
- the hoop stress induced in the wall of the cylinder due to the internal pressure;
- the factor of safety based on an ultimate stress of the material in simple tension of 456 MN/m².

Steel has a density of 7.8 Mg/m³.

[89.5, 36.8 MN/m²; 3.6]

4.19 (B). The "bursting" speed of a cast-iron flywheel rim, 3 m mean diameter, is 850 rev/min. Neglecting the effects of the spokes and boss, and assuming that the flywheel rim can be considered as a thin rotating hoop, determine the ultimate tensile strength of the cast iron. Cast iron has a density of 7.3 Mg/m³.

A flywheel rim is to be made of the same material and is required to rotate at 400 rev/min. Determine the maximum permissible mean diameter using a factor of safety of 8.

[U.L.C.I.] [2.25 mm]

4.20 (B). An internal combustion engine has a cast-iron flywheel that can be considered to be a uniform thickness disc of 230 mm outside diameter and 50 mm inside diameter. Given that the ultimate tensile stress and density of cast iron are 200 N/mm² and 7180 kg/m³ respectively, calculate the speed at which the flywheel would burst. Ignore any stress concentration effects and assume Poisson's ratio for cast iron to be 0.25.

[C.E.I.] [254.6 rev/s.]

4.21 (B). A thin steel circular disc of uniform thickness and having a central hole rotates at a uniform speed about an axis through its centre and perpendicular to its plane. The outside diameter of the disc is 500 mm and its speed of rotation is 81 rev/s. If the maximum allowable direct stress in the steel is not to exceed 110 MN/m² (11.00 h bar), determine the diameter of the central hole.

For steel, density $\rho = 7800$ kg/m³ and Poisson's ratio $\nu = 0.3$.

Sketch diagrams showing the circumferential and radial stress distribution across the plane of the disc indicating the peak values and state the radius at which the maximum radial stress occurs.

[B.P.] [264 mm.]

4.22 (B). (a) Prove that the differential equation for radial equilibrium in cylindrical coordinates of an element in a uniform thin disc rotating at ω rad/s and subjected to principal direct stresses σ_r and σ_θ is given by the following expression:

$$\sigma_\theta - \sigma_r - r \frac{d\sigma_r}{dr} = \rho \omega^2 r^2$$

(b) A thin solid circular disc of uniform thickness has an outside diameter of 300 mm. Using the maximum shear strain energy per unit volume theory of elastic failure, calculate the rotational speed of the disc to just cause initiation of plastic yielding if the yield stress of the material of the disc is 300 MN/m², the density of the material is 7800 kg/m³ and Poisson's ratio for the material is 0.3.

[B.P.] [324 rev/s.]

Thermal gradients

4.23 (C). Determine expressions for the stresses developed in a hollow disc subjected to a temperature gradient of the form $T = Kr$. What are the maximum stresses for such a case if the internal and external diameters of the cylinder are 80 mm and 160 mm respectively?

$\alpha = 12 \times 10^{-6}$ per °C and $E = 206.8$ GN/m².

The temperature at the outside radius is -50°C .

[-34.5, 27.6 MN/m².]

4.24 (C). Calculate the maximum stress in a solid magnesium alloy disc 60 mm diameter when the temperature rise is linear from 60°C at the centre to 90°C at the outside.

$$\alpha = 7 \times 10^{-6} \text{ per } ^\circ\text{C} \text{ and } E = 105 \text{ GN/m}^2.$$

$$[7.4 \text{ MN/m}^2.]$$

4.25 (C). Calculate the maximum compressive and tensile stresses in a hollow steel disc, 100 mm outer diameter and 20 mm inner diameter when the temperature rise is linear from 100°C at the inner surface to 50°C at the outer surface.

$$\alpha = 10 \times 10^{-6} \text{ per } ^\circ\text{C} \text{ and } E = 206.8 \text{ GN/m}^2.$$

$$[-62.9, +40.3 \text{ MN/m}^2.]$$

4.26 (C). Calculate the maximum tensile and compressive stresses in a hollow copper cylinder 20 mm outer diameter and 10 mm inner diameter when the temperature rise is linear from 0°C at the inner surface to 100°C at the outer surface.

$$\alpha = 16 \times 10^{-6} \text{ per } ^\circ\text{C} \text{ and } E = 104 \text{ GN/m}^2.$$

$$[142, -114 \text{ MN/m}^2.]$$

4.27 (C). A hollow steel disc has internal and external diameters of 0.2 m and 0.4 m respectively. Determine the circumferential thermal stresses set up at the inner and outer surfaces when the temperature at the outside surface is 100°C. A temperature distribution through the cylinder walls of the form $T = Kr$ may be assumed, i.e. when $r = \text{zero}$, $T = \text{zero}$.

$$\text{For steel, } E = 207 \text{ GN/m}^2 \text{ and } \alpha = 11 \times 10^{-6} \text{ per } ^\circ\text{C}.$$

What is the significance of (i) the first two terms of the stress eqns. (4.29) and (4.30), (ii) the remaining terms?

Hence comment on the relative magnitude of the maximum hoop stresses obtained in a high pressure vessel which is used for (iii) a chemical action which is exothermic, i.e. generating heat, (iv) a chemical reaction which is endothermic, i.e. absorbing heat.

$$[63.2, -50.5 \text{ MN/m}^2.]$$

4.28 (C). In the previous problem sketch the thermal hoop and radial stress variation diagrams across the wall thickness of the disc inserting the numerical value of the hoop stresses at the inner, mean and outer radii, and also the maximum radial stress, inserting the radius at which it occurs.

$$[\sigma_{\text{mean}} = -2.78 \text{ MN/m}^2, \sigma_{r_{\text{max}}} = 9.65 \text{ MN/m}^2.]$$

4.29 (C). A thin uniform steel disc, 254 mm outside diameter with a central hole 50 mm diameter, rotates at 10000 rev/min. The temperature gradient varies linearly such that the difference of temperature between the inner and outer (hotter) edges of the plate is 46°C. For the material of the disc, $E = 205 \text{ GN/m}^2$, Poisson's ratio = 0.3 and the coefficient of linear expansion = 11×10^{-6} per °C. The density of the material is 7700 kg/m³.

Calculate the hoop stresses induced at the inner and outer surfaces.

$$[176-12.1 \text{ MN/m}^2.]$$

4.30 (C). An unloaded steel cylinder has internal and external diameters of 204 mm and 304 mm respectively. Determine the circumferential thermal stresses at the inner and outer surfaces where the steady temperatures are 200°C and 100°C respectively.

$$\text{Take } E = 207 \text{ GN/m}^2, \alpha = 11 \times 10^{-6} \text{ per } ^\circ\text{C} \text{ and Poisson's ratio} = 0.29.$$

The temperature distribution through the wall thickness may be regarded as follows:

$$T = a + b \log_e r, \text{ where } a \text{ and } b \text{ are constants}$$

With this form of temperature distribution, the radial and circumferential thermal stresses at radius r where the temperature is T are obtained from

$$\sigma_r = A - \frac{B}{r^2} - \frac{E\alpha T}{2(1-\nu)} \quad \text{and} \quad \sigma_H = A + \frac{B}{r^2} - \frac{E\alpha T}{2(1-\nu)} - \frac{Eab}{2(1-\nu)}$$

$$[-255, 196 \text{ MN/m}^2.]$$

4.31 (C). Determine the hoop stresses at the inside and outside surfaces of a long thick cylinder which is rotated at 4000 rev/min. The cylinder has an internal radius of 80 mm and an external radius of 250 mm and is constructed from steel, the relevant properties of which are given above.

How would these values be modified if, under service conditions, the temperatures of the inside and outside surfaces reached maximum levels of 40°C and 90°C respectively?

A linear thermal gradient may be assumed.

$$\text{For steel } \alpha = 11 \times 10^{-6} \text{ per } ^\circ\text{C}.$$

$$[71.4, 18.9, 164.5, -46.8 \text{ MN/m}^2.]$$

4.32 (C). (a) Determine the wall thickness required for a high pressure cylindrical vessel, 800 mm diameter, in order that yielding shall be prevented according to the Tresca criterion of elastic failure when the vessel is subjected to an internal pressure of 450 bar.

(b) Such a vessel is now required to form part of a chemical plant and to contain exothermic reactions which produce a maximum internal temperature of 120°C at a reaction pressure of 450 bar, the outer surface being cooled to an "ambient" temperature of 20°C. In the knowledge that such a thermal gradient condition will introduce

additional stresses to those calculated in part (a) the designer proposes to increase the wall thickness by 20% in order that, once again, yielding shall be prevented according to the Tresca theory. Is this a valid proposal?

You may assume that the thermal gradient is of the form $T = a + br^2$ and that the modifying terms to the Lamé expressions to cover thermal gradient conditions are

for radial stress:

$$-\frac{\alpha E}{r^2} \int Tr dr$$

for hoop stress:

$$\frac{\alpha E}{r^2} \int Tr dr - \alpha ET.$$

For the material of the vessel, $\sigma_y = 280 \text{ MN/m}^2$, $\alpha = 12 \times 10^{-6}$ per $^\circ\text{C}$ and $E = 208 \text{ GN/m}^2$.

[52 mm; No-design requires $\sigma_y = 348 \text{ MN/m}^2$.]

CHAPTER 5

TORSION OF NON-CIRCULAR AND THIN-WALLED SECTIONS

Summary

For torsion of *rectangular sections* the maximum shear stress τ_{\max} and angle of twist θ are given by

$$\tau_{\max} = \frac{T}{k_1 db^2}$$

$$\frac{\theta}{L} = \frac{T}{k_2 db^3 G}$$

k_1 and k_2 being two constants, their values depending on the ratio d/b and being given in Table 5.1.

For *narrow rectangular sections*, $k_1 = k_2 = \frac{1}{3}$.

Thin-walled open sections may be considered as combinations of narrow rectangular sections so that

$$\begin{aligned}\tau_{\max} &= \frac{T}{\Sigma k_1 db^2} = \frac{3T}{\Sigma db^2} \\ \frac{\theta}{L} &= \frac{T}{\Sigma k_2 db^3 G} = \frac{3T}{G \Sigma db^3}\end{aligned}$$

The relevant formulae for other non-rectangular, non-tubular solid shafts are given in Table 5.2.

For *thin-walled closed sections* the stress at any point is given by

$$\tau = \frac{T}{2At}$$

where A is the area enclosed by the median line or mean perimeter and t is the thickness. The maximum stress occurs at the point where t is a minimum.

The angle of twist is then given by

$$\theta = \frac{TL}{4A^2 G} \int \frac{ds}{t}$$

which, for *tubes of constant thickness*, reduces to

$$\frac{\theta}{L} = \frac{Ts}{4A^2 Gt} = \frac{\tau s}{2AG}$$

where s is the length or perimeter of the median line.

Thin-walled cellular sections may be solved using the concept of constant shear flow $q(= \tau t)$, bearing in mind that the angles of twist of all cells or constituent parts are assumed equal.

5.1. Rectangular sections

Detailed analysis of the torsion of non-circular sections which includes the warping of cross-sections is beyond the scope of this text. For *rectangular shafts*, however, with longer side d and shorter side b , it can be shown by experiment that the maximum shearing stress occurs at the centre of the longer side and is given by

$$\tau_{\max} = \frac{T}{k_1 db^2} \quad (5.1)$$

where k_1 is a constant depending on the ratio d/b and given in Table 5.1 below.

Table 5.1. Table of k_1 and k_2 values for rectangular sections in torsion^(a).

d/b	1.0	1.5	1.75	2.0	2.5	3.0	4.0	6.0	8.0	10.0	∞
k_1	0.208	0.231	0.239	0.246	0.258	0.267	0.282	0.299	0.307	0.313	0.333
k_2	0.141	0.196	0.214	0.229	0.249	0.263	0.281	0.299	0.307	0.313	0.333

^(a) S. Timoshenko, *Strength of Materials*, Part I, *Elementary Theory and Problems*, Van Nostrand, New York.

The essential difference between the shear stress distributions in circular and rectangular members is illustrated in Fig. 5.1, where the shear stress distribution along the major and minor axes of a rectangular section together with that along a "radial" line to the corner of the section are indicated. The maximum shear stress is shown at the centre of the longer side, as noted above, and the stress at the corner is zero.

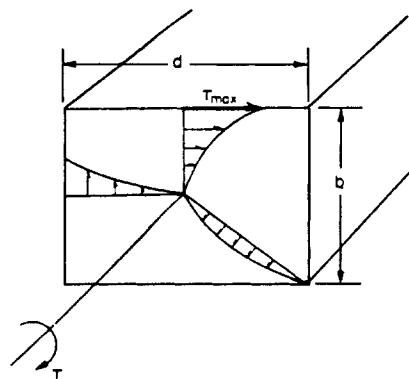


Fig. 5.1. Shear stress distribution in a solid rectangular shaft.

The angle of twist per unit length is given by

$$\frac{\theta}{L} = \frac{T}{k_2 db^3 G} \quad (5.2)$$

k_2 being another constant depending on the ratio d/b and also given in Table 5.1.

In the absence of Table 5.1, however, it is possible to reduce the above equations to the following *approximate* forms:

$$\tau_{\max} = \frac{T}{db^2} \left[3 + 1.8 \frac{b}{d} \right] = \frac{T}{db^3} [3d + 1.8b] \quad (5.3)$$

and $\theta = \frac{42TLJ}{GA^4} = \frac{42TLJ}{Gd^4b^4}$ (5.4)

where A is the cross-sectional area of the section ($= bd$) and $J = (bd/12)(b^2 + d^2)$.

5.2. Narrow rectangular sections

From Table 5.1 it is evident that as the ratio d/b increases, i.e. the rectangular section becomes longer and thinner, the values of constants k_1 and k_2 approach 0.333. Thus, for narrow rectangular sections in which $d/b > 10$ both k_1 and k_2 are assumed to be 1/3 and eqns. (5.1) and (5.2) reduce to

$$\tau_{\max} = \frac{3T}{db^2} \quad (5.5)$$

$$\frac{\theta}{L} = \frac{3T}{db^3G} \quad (5.6)$$

5.3. Thin-walled open sections

There are many cases, particularly in civil engineering applications, where rolled steel or extruded alloy sections are used where some element of torsion is involved. In most cases the sections consist of a combination of rectangles, and the relationships given in eqns. (5.1) and (5.2) can be adapted with reasonable accuracy provided that:

- (a) the sections are “open”, i.e. angles, channels, T-sections, etc., as shown in Fig. 5.2;
- (b) the sections are thin compared with the other dimensions.

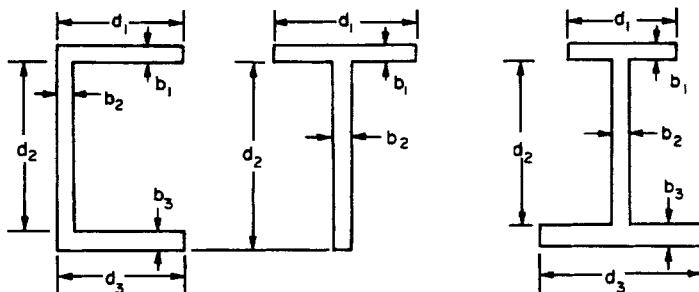


Fig. 5.2. Typical thin-walled open sections.

For such sections eqns. (5.1) and (5.2) may be re-written in the form

$$\tau_{\max} = \frac{T}{k_1 db^2} = \frac{T}{Z'} \quad (5.7)$$

and

$$\frac{\theta}{L} = \frac{T}{k_2 db^3 G} = \frac{T}{J_{\text{eq}} G} \quad (5.8)$$

where Z' is the torsion section modulus

$$\begin{aligned} &= Z' \text{ web} + Z' \text{ flanges} = k_1 d_1 b_1^2 + k_1 d_2 b_2^2 + \dots \text{etc.} \\ &= \sum k_1 db^2 \end{aligned}$$

and J_{eq} is the "effective" polar moment of area or "equivalent J" (see §5.7)

$$\begin{aligned} &= J_{\text{eq}} \text{ web} + J_{\text{eq}} \text{ flanges} = k_2 d_1 b_1^3 + k_2 d_2 b_2^3 + \dots \text{etc.} \\ &= \sum k_2 db^3 \end{aligned}$$

i.e.

$$\tau_{\max} = \frac{T}{\sum k_1 db^2} \quad (5.9)$$

and

$$\frac{\theta}{L} = \frac{T}{G \sum k_2 db^3} \quad (5.10)$$

and for d/b ratios in excess of 10, $k_1 = k_2 = \frac{1}{3}$, so that

$$\tau_{\max} = \frac{3T}{\sum db^2} \quad (5.11)$$

$$\frac{\theta}{L} = \frac{3T}{G \sum db^3} \quad (5.12)$$

To take account of the stress concentrations at the fillets of such sections, however, Timoshenko and Young[†] suggest that the maximum shear stress as calculated above is multiplied by the factor

$$\left[1 + \frac{b}{4a} \right]$$

(Figure 5.3). This has been shown to be fairly reliable over the range $0 < a/b < 0.5$. In the event of sections containing limbs of different thicknesses the largest value of b should be used.

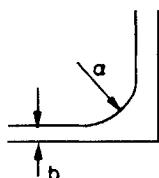


Fig. 5.3.

[†] S. Timoshenko and A.D. Young, *Strength of Materials*, Van Nostrand, New York, 1968 edition.

5.4. Thin-walled split tube

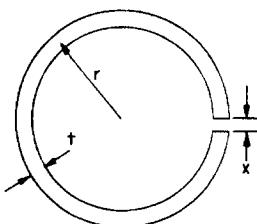
The thin-walled split tube shown in Fig. 5.4 is considered to be a special case of the thin-walled open type of section considered in §5.3. It is therefore treated as an equivalent rectangle with a longer side d equal to the circumference (less the gap), and a width b equal to the thickness.

Then

$$\tau_{\max} = \frac{T}{k_1 db^2}$$

and

$$\frac{\theta}{L} = \frac{T}{k_2 db^3 G}$$



$$d = \text{mean circumference} = 2\pi r$$

Fig. 5.4. Thin tube with longitudinal split.

where k_1 and k_2 for thin-walled tubes are usually equal to $\frac{1}{3}$.

It should be noted here that the presence of even a very small cut or gap in a thin-walled tube produces a torsional stiffness (torque per unit angle of twist) very much smaller than that for a complete tube of the same dimensions.

5.5. Other solid (non-tubular) shafts

Table 5.2 (see p. 146) indicates the relevant formulae for maximum shear stress and angle of twist of other standard non-circular sections which may be encountered in practice.

Approximate angles of twist for other solid cross-sections may be obtained by the substitution of an elliptical cross-section of the same area A and the same polar second moment of area J . The relevant equation for the elliptical section in Table 5.2 may then be applied.

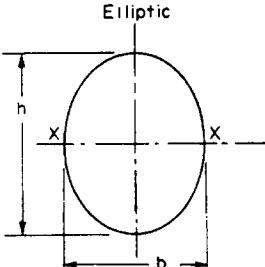
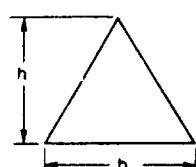
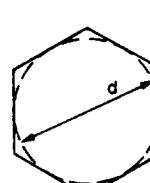
Alternatively, a very powerful procedure which applies for all solid sections, however irregular in shape, utilises a so-called "inscribed circle" procedure described in detail by Roark[†]. The procedure is equally applicable to thick-walled standard T , I and channel sections and is outlined briefly below:

Inscribed circle procedure

Roark shows that the maximum shear stress which is set up when any solid section is subjected to torque occurs at, or very near to, one of the points where the largest circle which

[†] R.J. Roark and W.C. Young, *Formulas for Stress & Strain*, 5th edn. McGraw-Hill, Kogakusha.

Table 5.2^(a).

Cross-section	Maximum shear stress	Angle of twist per unit length
Elliptic 	$\frac{16T}{\pi b^2 h}$ at end of minor axis XX where $J = \frac{\pi}{64} [bh^3 + hb^3]$ and A is the area of cross-section $= \pi b h / 4$	$\frac{4\pi^2 TJ}{A^4 G}$
Equilateral triangle 	$\frac{20T}{b^3}$ at the middle of each side	$\frac{46.2T}{b^4 G}$
Regular hexagon 	$\frac{T}{0.217 Ad}$	$\frac{T}{0.133 Ad^2 G}$

where d is the diameter of inscribed circle and A is the cross-sectional area

^(a) From S. Timoshenko, *Strength of Materials*, Part II, *Advanced Theory and Problems*, Van Nostrand, New York, p. 235. Approximate angles of twist for other solid cross-sections may be obtained by the substitution of an equivalent elliptical cross-section of the same area A and the same polar second moment of area J . The relevant equation for the elliptical section in Table 5.2 may then be applied.

can be constructed within the cross-section touches the section boundary – see Fig. 5.5. Normally it occurs at the point where the curvature of the boundary is algebraically the least, convex curvatures being taken as positive and concave or re-entrant curvatures negative.

The maximum shear stress is then obtained from either:

$$\tau_{\max} = \left(\frac{G\theta}{L} \right) C \quad \text{or} \quad \tau_{\max} = \left(\frac{\tau}{K} \right) C$$

where, for positive curvatures (i.e. straight or convex boundaries),

$$C = \frac{D}{\pi^2 D^4} \left[1 + 0.15 \left(\frac{\pi^2 D^4}{16A^2} - \frac{D}{2r} \right) \right]$$

with D = diameter of the largest inscribed circle,

r = radius of curvature of boundary at selected position (positive),

A = cross-sectional area of section,

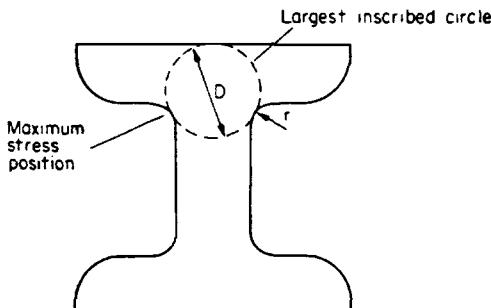


Fig. 5.5. Inscribed circle stress evaluation procedure.

or, for negative curvatures (concave or re-entrant boundaries):

$$C = \frac{D}{1 + \frac{\pi^2 D^4}{16A^2}} \left[1 + \left\{ 0.118 \log_e \left(1 - \frac{D}{2r} \right) - 0.238 \frac{D}{2r} \right\} \tanh \frac{2\phi}{\pi} \right]$$

with ϕ = angle through which a tangent to the boundary rotates in travelling around the re-entrant position (radians) and r being taken as negative.

For standard thick-walled open sections such as T , I , Z , angle and channel sections Roark also introduces formulae for angles of twist based upon the same inscribed circle procedure parameters.

5.6. Thin-walled closed tubes of non-circular section (Bredt–Batho theory)

Consider the thin-walled closed tube shown in Fig. 5.6 subjected to a torque T about the Z axis, i.e. in a transverse plane. Both the cross-section and the wall thickness around the periphery may be irregular as shown, but for the purposes of this simplified treatment it must be assumed that the thickness does not vary along the length of the tube. Then, if τ is the shear stress at B and τ' is the shear stress at C (where the thickness has increased to t') then, from the equilibrium of the complementary shears on the sides AB and CD of the element shown, it follows that

$$\tau t dz = \tau' t' dz$$

$$\tau t = \tau' t'$$

i.e. the product of the shear stress and the thickness is constant at all points on the periphery of the tube. This constant is termed the *shear flow* and denoted by the symbol q (shear force per unit length).

Thus $q = \tau t = \text{constant}$ (5.13)

The quantity q is termed the shear flow because if one imagines the inner and outer boundaries of the tube section to be those of a channel carrying a flow of water, then, provided that the total quantity of water in the system remains constant, the quantity flowing past any given point is also constant.

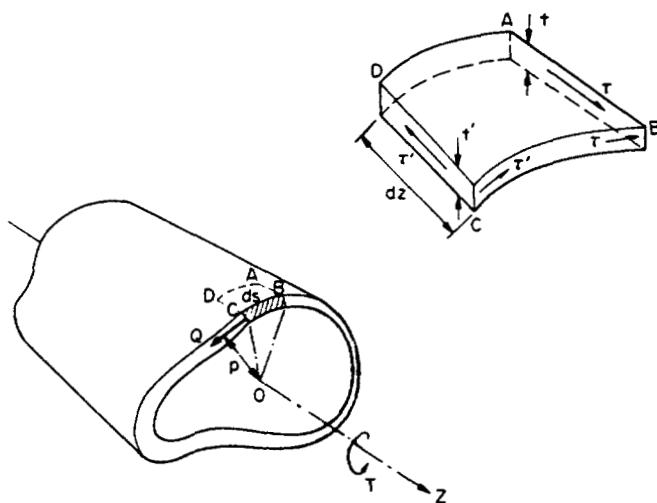


Fig. 5.6. Thin-walled closed section subjected to axial torque.

At any point, then, the shear force Q on an element of length ds is $Q = \tau t ds = q ds$ and the shear stress is q/t .

Consider now, therefore, the element BC subjected to the shear force $Q = q ds = \tau t ds$.

The moment of this force about O

$$= dT = Qp$$

where p is the perpendicular distance from O to the force Q .

$$\therefore dT = q ds p$$

Therefore the moment, or torque, for the whole section

$$= \int qp ds = q \int p ds$$

But the area $COB = \frac{1}{2}$ base \times height $= \frac{1}{2}p ds$

i.e. $dA = \frac{1}{2}p ds$ or $2dA = p ds$

$$\therefore \text{torque } T = 2q \int dA$$

$$T = 2qA \quad (5.14)$$

where A is the area enclosed within the median line of the wall thickness.

Now, since

$$q = \tau t$$

it follows that

$$T = 2\tau t A$$

or

$$\tau = \frac{T}{2At} \quad (5.15)$$

where t is the thickness at the point in question.

It is evident, therefore, that *the maximum shear stress* in such cases *occurs at the point of minimum thickness*.

Consider now an axial strip of the tube, of length L , along which the thickness and hence the shear stress is constant. The shear strain energy *per unit volume* is given by

$$U = \int \frac{\tau^2}{2G}$$

Thus, with thickness t , width ds and hence $V = tL ds$

$$\begin{aligned} U &= \int \frac{\tau^2}{2G} tL ds \\ &= \int \left(\frac{T}{2At} \right)^2 \frac{tL}{2G} ds \\ &= \frac{T^2 L}{8A^2 G} \int \frac{ds}{t} \end{aligned}$$

But the energy stored equals the work done $= \frac{1}{2}T\theta$.

$$\therefore \frac{1}{2}T\theta = \frac{T^2 L}{8A^2 G} \int \frac{ds}{t}$$

The angle of twist of the tube is therefore given by

$$\theta = \frac{TL}{4A^2 G} \int \frac{ds}{t}$$

For *tubes of constant thickness* this reduces to

$$\theta = \frac{TLs}{4A^2 Gt} = \frac{\tau Ls}{2AG} \quad (5.16)$$

where s is the perimeter of the median line.

The above equations must be used with care and do not apply to cases where there are abrupt changes in thickness or re-entrant corners.

For closed sections which have constant thickness over specified lengths but varying from one part of the perimeter to another:

$$\frac{\theta}{L} = \frac{T}{4A^2 G} \left[\frac{s_1}{t_1} + \frac{s_2}{t_2} + \frac{s_3}{t_3} + \dots \text{etc.} \right]$$

5.7. Use of “equivalent J ” for torsion of non-circular sections

The simple torsion theory for circular sections can be written in the form:

$$\frac{\theta}{L} = \frac{T}{GJ}$$

and, as stated on page 143, it is often convenient to express the twist of non-circular sections in similar form:

i.e.

$$\frac{\theta}{L} = \frac{T}{GJ_{eq}}$$

where J_{eq} is the "equivalent J' " or "effective polar moment of area" for the section in question.

Thus, for open sections:

$$\frac{\theta}{L} = \frac{T}{\Sigma k_2 db^3 G} = \frac{T}{GJ_{eq}}$$

with $J_{eq} = \Sigma k_2 db^3$ ($= \frac{1}{3} \Sigma db^3$ for $d/b > 10$).

Similarly, for square tubes of closed section:

$$\frac{\theta}{L} = \frac{TLs}{4A^2 Gt} = \frac{T}{G[4A^2 t/s]} = \frac{T}{GJ_{eq}}$$

and $J_{eq} = 4A^2 t/s$.

The torsional stiffness of any section, i.e. the ratio of torque divided by angle of twist per unit length, is then directly given by the value of GJ or GJ_{eq} i.e.

$$\text{Stiffness} = \frac{T}{\theta/L} = GJ \text{ (or } GJ_{eq}).$$

5.8. Thin-walled cellular sections

The Bredt–Batho theory developed in the previous section may be applied to the solution of problems involving cellular sections of the type shown in Fig. 5.7.

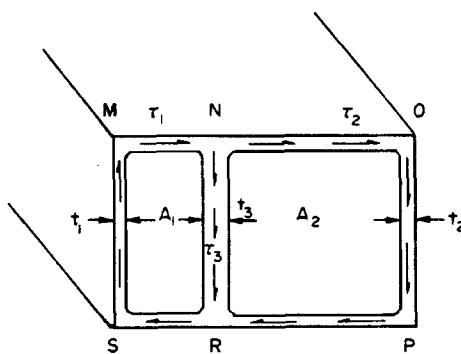


Fig. 5.7. Thin-walled cellular section.

Assume the length $RSMN$ is of constant thickness t_1 and subjected therefore to a constant shear stress τ_1 . Similarly, $NOPR$ is of thickness t_2 and stress τ_2 with NR of thickness t_3 and stress τ_3 .

Considering the equilibrium of complementary shear stresses on a longitudinal section at N , it follows that

$$\tau_1 t_1 = \tau_2 t_2 + \tau_3 t_3 \quad (5.17)$$

Alternatively, this equation may be obtained considering the arrows shown to be directions of shear flow $q (= \tau t)$. At N the flow q_1 along MN divides into q_2 along NO and q_3 along NR ,

$$\text{i.e.} \quad q_1 = q_2 + q_3$$

$$\text{or} \quad \tau_1 t_1 = \tau_2 t_2 + \tau_3 t_3 \quad (\text{as before})$$

The total torque for the section is then found as the sum of the torques on the two cells by application of eqn. (5.14) to the two cells and adding the result,

$$\text{i.e.} \quad T = 2q_1 A_1 + 2q_2 A_2$$

$$T = 2(\tau_1 t_1 A_1 + \tau_2 t_2 A_2) \quad (5.18)$$

Also, since the angle of twist will be common to both cells, applying eqn. (5.16) to each cell gives

$$\theta = \frac{L}{2G} \left(\frac{\tau_1 s_1 + \tau_3 s_3}{A_1} \right) = \frac{L}{2G} \left(\frac{\tau_2 s_2 - \tau_3 s_3}{A_2} \right)$$

where s_1 , s_2 and s_3 are the median line perimeters $RSMN$, $NOPR$ and NR respectively.

The negative sign appears in the final term because the shear flow along NR for this cell opposes that in the remainder of the perimeter.

$$\therefore \frac{2G\theta}{L} = \frac{1}{A_1}(\tau_1 s_1 + \tau_3 s_3) = \frac{1}{A_2}(\tau_2 s_2 - \tau_3 s_3) \quad (5.19)$$

5.9. Torsion of thin-walled stiffened sections

The stiffness of any section has been shown above to be given by its value of GJ or GJ_{eq} .

Consider, therefore, the rectangular polymer extrusion of simple symmetrical cellular constructions shown in Fig. 5.8(a). The shear flow in each cell is indicated.

At A

$$q_1 = q_2 + q_3.$$

But because of symmetry q_1 must equal q_3 $\therefore q_2 = 0$;

i.e., for a symmetrical cellular thin-walled member there is no shear carried by the central web and therefore as far as stiffness of the section is concerned the web can be ignored.

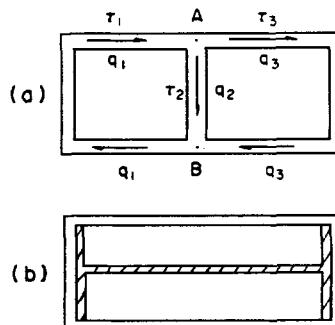


Fig. 5.8(a). Polymer cellular section with symmetrical cells. (b) Polymer cell with central web removed but reinforced by steel I section.

∴ Stiffness of complete section, from eqn. (5.16)

$$= GJ_E = \frac{4A^2t}{s}G$$

where A and s are the area and perimeter of the complete section.

Now since G of the polymer is likely to be small, the stiffness of the section, and its resistance to applied torque, will be low. It can be reinforced by metallic insertions such as that of the I section shown in Fig. 5.8(b).

For the I section, from eqn. (5.8)

$$GJ_E = G\Sigma k_2 db^3$$

and the value represents the increase in stiffness presented by the compound section.

Stress conditions for limiting twist per unit lengths are then given by:

For the tube

$$T = GJ_E(\theta/L) = 2At\tau$$

$$\therefore (\theta/L)_{\max} = \frac{2At}{GJ_E} \cdot \tau_{\max}$$

and for the I section

$$T = GJ_E(\theta/L) = (\Sigma k_2 db^3 G)\theta/L$$

or

$$T = (\Sigma k_1 db^2)\tau$$

$$\therefore (\theta/L)_{\max} = \frac{\Sigma k_1 d}{Gb\Sigma k_2 d} \cdot \tau_{\max}$$

Usually (but not always) this would be considerably greater than that for the polymer tube, making the tube the controlling design factor.

5.10. Membrane analogy

It has been stated earlier that the mathematical solution for the torsion of certain solid and thin-walled sections is complex and beyond the scope of this text. In such cases it is extremely fortunate that an analogy exists known as the *membrane analogy*, which provides a very convenient mental picture of the way in which stresses build up in such components and allows experimental determination of their values.

It can be shown that the mathematical solution for elastic torsion problems involving partial differential equations is identical in form to that for a thin membrane lightly stretched over a hole. The membrane normally used for visualisation is a soap film. Provided that the hole used is the same shape as the cross-section of the shaft in question and that air pressure is maintained on one side of the membrane, the following relationships exist:

- the torque carried by the section is equal to twice the volume enclosed by the membrane;
- the shear stress at any point in the section is proportional to the slope of the membrane at that point (Fig. 5.9);
- the direction of the shear stress at any point in the section is always at right angles to the slope of the membrane at the same point.

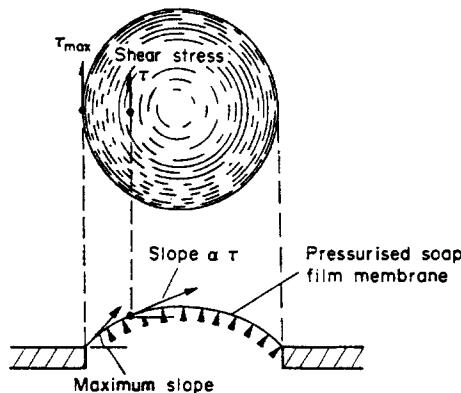


Fig. 5.9. Membrane analog.

Application of the above rules to the open sections of Fig. 5.2 shows that each section will carry approximately the same torque at the same maximum shear stress since the volumes enclosed by the membranes and the maximum slopes of the membranes are approximately equal in each case.

The membrane analogy is particularly powerful in the study of the comparative torsional properties of different sections without the need for detailed calculations. For example, it should be evident from the volume relationship (a) above that if two cross-sections have the same area, that which is nearer to circular will be the stronger in torsion since it will produce the greatest enclosed volume.

The analogy also helps to support the theory used for thin-walled open sections in §5.3 when thin rectangular sections are taken to have the same torsional stiffness be they left as a single rectangle or bent into open tubes, angle sections, channel sections, etc.

From the slope relationship (b) the greatest shear stresses usually occur at the boundary of the thickest parts of the section. They are usually high at positions where the boundary is sharply concave but low at the ends of outstanding flanges.

5.11. Effect of warping of open sections

In the preceding paragraphs it has been assumed that the torque is applied at the ends of the member and that all sections are free to warp. In practice, however, there are often cases where one or more sections of a member are constrained in some way so that cross-sections remain plane, i.e. warping is prevented. Whilst this has little effect on the angle of twist of certain solid cross-sections, e.g. rectangular or elliptical sections where the length is significantly greater than the section dimensions, it may have a considerable effect on the twist of open sections. In the latter case the constraint of warping is often accompanied by considerable bending of the flanges. Detailed treatment of warping is beyond the scope of this text[†] and it is sufficient to note here that when warping is restrained, angles of twist are generally reduced and hence torsional stiffnesses increased.

[†] S. Timoshenko and J.N. Goodier, *Theory of Elasticity*, McGraw-Hill, New York.

Examples

Example 5.1

A rectangular steel bar 25 mm wide and 38 mm deep is subjected to a torque of 450 Nm. Estimate the maximum shear stress set up in the material of the bar and the angle of twist, using the experimentally derived formulae stated in §5.1.

What percentage error would be involved in each case if the approximate equations are used?

For steel, take $G = 80 \text{ GN/m}^2$.

Solution

The maximum shear stress is given by eqn. (5.1):

$$\tau_{\max} = \frac{T}{k_1 db^2}$$

In this case $d = 38 \text{ mm}$, $b = 25 \text{ mm}$, i.e. $d/b = 1.52$ and k_1 for d/b of 1.5 = 0.231.

$$\therefore \tau_{\max} = \frac{450}{0.231 \times 38 \times 10^{-3} \times (25 \times 10^{-3})^2} = 82 \text{ MN/m}^2$$

The angle of twist per unit length is given by eqn. (5.2):

$$\frac{\theta}{L} = \frac{T}{k_2 db^3 G}$$

and from the tables, for $d/b = 1.5$, k_2 is 0.196.

$$\begin{aligned} \therefore \theta &= \frac{450}{0.196 \times 38 \times 10^{-3} \times (25 \times 10^{-3})^3 \times 80 \times 10^9} \\ &= 0.0483 \text{ rad/m} \\ &= 2.77 \text{ degrees/m} \end{aligned}$$

Approximately

$$\begin{aligned} \tau_{\max} &= \frac{T}{db^2} (3 + 1.8b/d) \\ &= \frac{450}{38 \times 10^{-3} \times (25 \times 10^{-3})^2} \left(3 + 1.8 \times \frac{25}{38} \right) \\ &= \frac{450}{2.375 \times 10^{-5}} (3 + 1.184) = 79.3 \text{ MN/m}^2 \end{aligned}$$

Therefore percentage error

$$= \left(\frac{79.3 - 82.02}{82.02} \right) 100 = -3.3\%$$

Again, approximately,

$$\theta = \frac{42TJ}{GA^4} \text{ per metre}$$

Now

$$J = I_{xx} + I_{yy} = \frac{bd^3}{12} + \frac{db^3}{12} = \frac{bd}{12}(d^2 + b^2)$$

$$= \frac{25 \times 38(25^2 + 38^2)}{12 \times 10^{12}} = 0.1638 \times 10^{-6} \text{ m}^4$$

$$\therefore \theta = \frac{42 \times 450 \times 0.164 \times 10^{-6}}{80 \times 10^9 \times (25 \times 38 \times 10^{-6})^4} = 0.0476 \text{ rad/m}$$

$$= 2.73 \text{ degrees/m}$$

$$\text{Percentage error} = \left(\frac{2.73 - 2.77}{2.77} \right) 100 = -1.44\%$$

Example 5.2

Compare the torsional stiffness of the following cross-sections which can be assumed to be of unit length. Compare also the maximum shear stresses set up in each case:

- (a) a hollow tube 40 mm mean diameter and 2 mm wall thickness;
- (b) the same tube with a 2 mm wide saw-cut along its length;
- (c) a rectangular solid bar, side ratio 4 to 1, having the same cross-sectional area as that enclosed by the mean diameter of the hollow tube;
- (d) an equal-leg angle section having the same perimeter and thickness as the tube;
- (e) a square box section having the same perimeter and thickness as the tube.

Solution

- (a) In the case of the *closed hollow tube* we can apply the standard torsion equation

$$\frac{T}{J} = \frac{G\theta}{L} = \frac{\tau}{r}$$

together with the simplified formula for the polar moment of area J of thin tubes,

$$J = 2\pi r^3 t$$

$$\therefore \text{torsional stiffness} = \frac{T}{\theta} = \frac{GJ}{L} = \frac{2\pi \times (20 \times 10^{-3})^3 \times 2 \times 10^{-3} G}{1}$$

$$= 100.5 \times 10^{-9} G$$

$$\text{maximum shear stress} = \frac{TR}{J} = \frac{20 \times 10^{-3} \times T}{2\pi \times (20 \times 10^{-3})^3 \times 2 \times 10^{-3}}$$

$$= 0.198 \times 10^6 T$$

(b) *Tube with split*

From the work of §5.4,

$$\text{angle of twist/unit length} = \frac{\theta}{L} = \frac{T}{k_2 db^3 G} = \frac{T}{k_2 (2\pi r - x) t^3 G}$$

$$\begin{aligned}\therefore \text{torsional stiffness} &= \frac{T}{\theta} = \frac{k_2(2\pi r - x)t^3 G}{L} \\ &= \frac{0.333[2\pi \times 20 \times 10^{-3} - 2 \times 10^{-3}](2 \times 10^{-3})^3 G}{1} \\ &= 0.333(125.8 - 2)8 \times 10^{-12} G \\ &= \mathbf{329.8 \times 10^{-12} G} \\ \text{Maximum shear stress} &= \frac{T}{k_1 db^2} \\ &= \frac{T}{0.333 \times 123.8 \times 10^{-3} \times (2 \times 10^{-8})^2} \\ &= \mathbf{6.06 \times 10^6 T}\end{aligned}$$

i.e. splitting the tube along its length has reduced the stiffness by a factor of approximately **300**, the maximum stress increasing by approximately **30** times.

(c) *Rectangular bar*

$$\text{Area of hollow tube} = \text{area of bar}$$

$$= \pi \times (20 \times 10^{-3})^2$$

$$\therefore 4b^2 = 8\pi \times 10^{-4}$$

$$b^2 = 2\pi \times 10^{-4}$$

$$\therefore b = 2.5 \times 10^{-2} \text{ m} = 25 \text{ mm}$$

$$\therefore d = 4b = 100 \text{ mm}$$

$$d/b \text{ ratio} = 4$$

$$\therefore k_1 = 0.282 \quad \text{and} \quad k_2 = 0.281$$

Therefore from eqn. (5.2),

$$\frac{\theta}{L} = \frac{T}{k_2 db^3 G}$$

$$\begin{aligned}\therefore \frac{T}{\theta} &= 0.281 \times 10 \times 10^{-2} \times (2.5 \times 10^{-2})^3 G \\ &= 43.9 \times 10^{-8} G \\ &= \mathbf{439 \times 10^{-9} G}\end{aligned}$$

From eqn. (5.1),

$$\begin{aligned}\tau_{\max} &= \frac{T}{k_1 db^2} = \frac{T}{0.282 \times 10 \times 10^{-2} \times (2.5 \times 10^{-2})^2} \\ &= \mathbf{0.057 \times 10^6 T}\end{aligned}$$

(d) Equal-leg angle section

$$\begin{aligned}\text{Perimeter of angle} &= \text{perimeter of tube} \\ &= 2\pi \times 20 \times 10^{-3} \text{ m}\end{aligned}$$

$$\therefore \text{Length of side } d = 20\pi \times 10^{-3} \text{ m}$$

Therefore applying eqn. (5.12),

$$\begin{aligned}\frac{\theta}{L} &= \frac{3T}{G\Sigma db^3} \\ &= \frac{3T}{2G \times 20\pi \times 10^{-3} \times (2 \times 10^{-3})^3} \\ \frac{T}{\theta} &= (2G \times 20\pi \times 8 \times 10^{-12})/3 \\ &= \mathbf{0.335 \times 10^{-9} G}\end{aligned}$$

And from eqn. (5.11)

$$\begin{aligned}\tau_{\max} &= \frac{3T}{\Sigma db^2} \\ &= \frac{3T}{2 \times 20\pi \times 10^{-3} \times (2 \times 10^{-3})^2} \\ &= \mathbf{5.97 \times 10^6 T}\end{aligned}$$

(e) Square box section (closed)

$$\text{Perimeter } s = \text{tube perimeter} = 2\pi \times 20 \times 10^{-3} \text{ m}$$

$$\therefore \text{side length} = \frac{2\pi \times 20 \times 10^{-3}}{4} = \pi \times 10^{-2} \text{ m}$$

Therefore area enclosed by median line

$$= A = (\pi \times 10^{-2})^2$$

From eqn. (5.16),

$$\begin{aligned}\theta &= \frac{TLs}{4A^2Gt} \\ \therefore \frac{T}{\theta} &= \frac{4 \times (\pi \times 10^{-2})^4 G \times 2 \times 10^{-3}}{1 \times 2\pi \times 20 \times 10^{-3}} \\ &= \mathbf{62 \times 10^{-9} G}\end{aligned}$$

From eqn. (5.15)

$$\begin{aligned}\tau_{\max} &= \frac{T}{2At} = \frac{T}{2 \times (\pi \times 10^{-2})^2 \times 2 \times 10^{-3}} \\ &= \mathbf{0.253 \times 10^6 T}\end{aligned}$$

Example 5.3

A thin-walled member 1.2 m long has the cross-section shown in Fig. 5.10. Determine the maximum torque which can be carried by the section if the angle of twist is limited to 10° . What will be the maximum shear stress when this maximum torque is applied? For the material of the member $G = 80 \text{ GN/m}^2$.

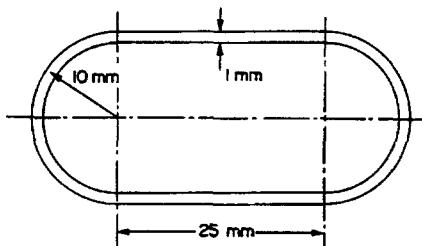


Fig. 5.10.

Solution

This problem is of the type considered in §5.6, a solution depending upon the length of, and the area enclosed by, the median line.

$$\begin{aligned} \text{Now, } \quad \text{perimeter of median line } &= s = (2 \times 25 + 2\pi \times 10) \text{ mm} \\ &= 112.8 \text{ mm} \end{aligned}$$

$$\begin{aligned} \text{area enclosed by median} &= A = (20 \times 25 + \pi \times 10^2) \text{ mm}^2 \\ &= 814.2 \text{ mm}^2 \end{aligned}$$

$$\text{From eqn (5.16), } \theta = \frac{TLs}{4A^2 Gt}$$

$$\therefore \frac{10 \times 2\pi}{360} = \frac{T \times 1.2 \times 112.8 \times 10^{-3}}{4(814.2 \times 10^{-6})^2 \times 80 \times 10^9 \times 1 \times 10^{-3}}$$

i.e. maximum torque possible,

$$\begin{aligned} T &= \frac{20\pi \times 4 \times 814.2^2 \times 80 \times 10^{-6}}{360 \times 1.2 \times 112.8 \times 10^{-3}} \\ &= 273 \text{ Nm} \end{aligned}$$

$$\begin{aligned} \text{From eqn. (5.15), } \tau_{\max} &= \frac{T}{2At} \\ &= \frac{273}{2 \times 814.2 \times 10^{-6} \times 1 \times 10^{-3}} \\ &= 168 \times 10^6 = 168 \text{ MN/m}^2 \end{aligned}$$

The maximum stress produced is 168 MN/m^2 .

Example 5.4

The median dimensions of the two cells shown in the cellular section of Fig. 5.6 are $A_1 = 20 \text{ mm} \times 40 \text{ mm}$ and $A_2 = 50 \text{ mm} \times 40 \text{ mm}$ with wall thicknesses $t_1 = 2 \text{ mm}$, $t_2 = 1.5 \text{ mm}$

and $t_3 = 3$ mm. If the section is subjected to a torque of 320 Nm, determine the angle of twist per unit length and the maximum shear stress set up. The section is constructed from a light alloy with a modulus of rigidity $G = 30$ GN/m².

Solution

$$\text{From eqn. (5.18), } 320 = 2(\tau_1 \times 2 \times 20 \times 40 + \tau_2 \times 1.5 \times 50 \times 40)10^{-9} \quad (1)$$

From eqn. (5.19),

$$2 \times 30 \times 10^9 \times \theta = \frac{1}{20 \times 40 \times 10^{-6}} [\tau_1(40 + 2 \times 20)10^{-3} + \tau_3 \times 40 \times 10^{-3}] \quad (2)$$

$$\text{and } 2 \times 30 \times 10^9 \times \theta = \frac{1}{50 \times 40 \times 10^{-6}} [\tau_2(40 + 2 \times 50)10^{-3} - \tau_3 \times 40 \times 10^{-3}] \quad (3)$$

Equating (2) and (3),

$$\text{From eqn. (5.17), } 2\tau_1 = 1.5\tau_2 + 3\tau_3 \quad (4)$$

$$\frac{1}{8}[80\tau_1 + 40\tau_3] = \frac{1}{20}[140\tau_2 - 40\tau_3]$$

Multiply through by 40,

$$\begin{aligned} 400\tau_1 + 200\tau_3 &= 280\tau_2 - 80\tau_3 \\ 40\tau_1 &= 28\tau_2 - 28\tau_3 \end{aligned} \quad (5)$$

$$(5) \times 60/28 \quad 85.7\tau_1 = 60\tau_2 - 60\tau_3 \quad (6)$$

But, from (4), multiplied by 20,

$$40\tau_1 = 30\tau_2 + 60\tau_3 \quad (7)$$

$$(6) + (7), \quad 125.7\tau_1 = 90\tau_2 \quad (8)$$

and from (1),

$$320 = (3.2\tau_1 + 6\tau_2)10^{-6}$$

$$320 \times 10^6 = 3.2\tau_1 + 6\tau_2 \quad (9)$$

substituting for τ_2 from (8),

$$\begin{aligned} 320 \times 10^6 &= 3.2\tau_1 + 6 \times \frac{125.7}{90}\tau_1 \\ &= 3.2\tau_1 + 8.4\tau_1 \\ \therefore \tau_1 &= \frac{320 \times 10^6}{11.6} = 27.6 \times 10^6 = 27.6 \text{ MN/m}^2 \end{aligned}$$

From (8),

$$\tau_2 = \frac{125.7}{90} \times 27.6 = 38.6 \text{ MN/m}^2$$

From (4),

$$\begin{aligned} \tau_3 &= \frac{1}{3}(2 \times 27.6 - 1.5 \times 38.6) \\ &= \frac{1}{3}(55.2 - 57.9) = \frac{1}{3} \times (-2.7) = -0.9 \text{ MN/m}^2 \end{aligned}$$

The negative sign indicates that the direction of shear flow in the wall of thickness t_3 is reversed from that shown in Fig. 5.6.

The maximum shear stress present in the section is thus **38.6 MN/m²** in the 1.5 mm wall thickness.

From (3),

$$\begin{aligned} 2 \times 30 \times 10^9 \times \theta &= \frac{(140t_2 - 40t_3)}{50 \times 40 \times 10^{-3}} \\ &= \frac{140 \times 38.6 \times 10^6 - 40(-0.9 \times 10^6)}{50 \times 40 \times 10^{-3} \times 2 \times 30 \times 10^9} \\ &= \frac{(5.40 + 0.036)}{120} \text{ radian} \\ &= \frac{5.440}{120} \times \frac{360}{2\pi} = 2.6^\circ \end{aligned}$$

The angle of twist of the section is 2.6° .

Problems

5.1 (A). A 40 mm \times 20 mm rectangular steel shaft is subjected to a torque of 1 kNm. What will be the approximate position and magnitude of the maximum shear stress set up in the shaft? Determine also the corresponding angle of twist per metre length of the shaft.

For the bar material $G = 80 \text{ GN/m}^2$.

[254 MN/m²; 9.78°/m.]

5.2 (B). An extruded light alloy angle section has dimensions 80 mm \times 60 mm \times 4 mm and is subjected to a torque of 20 Nm. If $G = 30 \text{ GN/m}^2$ determine the maximum shear stress and the angle of twist per unit length. How would the former answer change if one considered the stress concentration effect at the fillet owing to a fillet radius of 10 mm?

[27.6 MN/m²; 13.2°/m; 30.4 MN/m².]

5.3 (B). Compare the torsional rigidities of the following sections:

- (a) a hollow tube 30 mm outside diameter and 1.5 mm thick; $[2.7 \times 10^{-8} G.]$
- (b) the same tube split along its length with a 1 mm gap; $[0.0996 \times 10^{-9} G.]$
- (c) an equal leg angle section having the same perimeter and thickness as (b); $[0.0996 \times 10^{-9} G.]$
- (d) a square box section with side length 30 mm and 1.5 mm wall thickness; $[3.48 \times 10^{-8} G.]$
- (e) a rectangular solid bar, side ratio 2.5 to 1, having the same metal cross-sectional area as the hollow tube. $[1.79 \times 10^{-9} G.]$

Compare also the maximum stresses arising in each case.

$[0.522 \times 10^6 T; 15 \times 10^6 T; 15 \times 10^6 T; 0.41 \times 10^6 T; 4.05 \times 10^6 T.]$

5.4 (B). The spring return of an interlocking device for a cold room door is to be made of a rectangular strip of spring steel loaded in torsion. The width of the strip cannot be greater than 10 mm and the effective length 100 mm. Calculate the thickness of the strip if the torque is to be 15 Nm at an angle of 10° and if the torsion yield stress of 420 MN/m² is not to be exceeded at this angle. Take G as 83 GN/m^2 .

Assume $k_1 = k_2 = \frac{1}{3}$.

[3.27 mm.]

5.5 (B). A thin-walled member of 2 m long has the section shown in Fig. 5.11. Determine the torque that can be applied and the angle of twist achieved if the maximum shear stress is limited to 30 MN/m². $G = 250 \text{ GN/m}^2$.
[42.85 Nm; 0.99°.]

5.6 (B). A steel sheet, 400 mm wide by 2 mm thick, is to be formed into a hollow section by bending through 360° and butt-welding the long edges together. The shape may be (a) circular, (b) square, (c) a rectangle 140 mm \times 60 mm. Assume a median length of 400 mm in each case (i.e. no stretching) and square corners for non-circular sections. The allowable shearing stress is 90 MN/m². For each of the shapes listed determine the magnitude of the maximum permissible torque and the angles of twist per metre length if $G = 80 \text{ GN/m}^2$.

[4.58, 3.6, 3.01 kNm; $1^\circ, 1^\circ 17', 1^\circ 31'$.]

5.7 (B). Figure 5.12 represents the cross-section of an aircraft fuselage made of aluminium alloy. The sheet thicknesses are: 1 mm from A to B and C to D ; 0.8 mm from B to C and 0.7 mm from D to A . For a maximum torque of 5000 Nm determine the magnitude of the maximum shear stress and the angle of twist/metre length.
 $G = 30 \text{ GN/m}^2$.

[50 MN/m²; 0.0097 rad.]

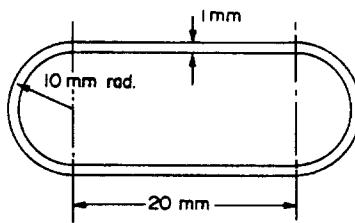


Fig. 5.11.

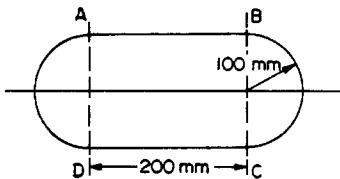


Fig. 5.12.

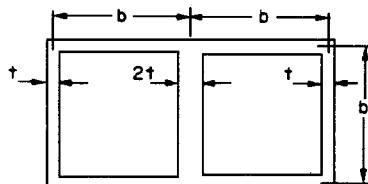


Fig. 5.13.

5.8 (B/C). Show that for the symmetrical section shown in Fig. 5.13 there is no stress in the central web. Show also that the shear stress in the remainder of the section has a value of $T/4tb^2$.

5.9 (C). A washing machine agitator of the cross-section shown in Fig. 5.14 acts as a torsional member subjected to a torque T . The central tube is 100 mm internal diameter and 12 mm thick; the rectangular bars are 50 mm \times 18 mm section. Assuming that the total torque carried by the member is given by

$$T = T_{\text{tube}} + 4T_{\text{bar}}$$

determine the maximum value of T which the shaft can carry if the maximum stress is limited to 80 MN/m².

(Hint: equate angles of twist of tube and bar.)

[19.1 kNm.]

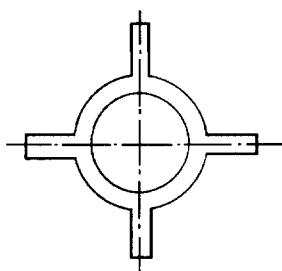


Fig. 5.14.

5.10 (C). The cross-section of an aeroplane elevator is shown in Fig. 5.15. If the elevator is 2 m long and constructed from aluminium alloy with $G = 30 \text{ GN/m}^2$, calculate the total angle of twist of the section and the magnitude of the shear stress in each part for an applied torque of 40 Nm.

[0.0169°; 3.43, 2.58, $1.15 \times 10^5 \text{ N/m}^2$.]

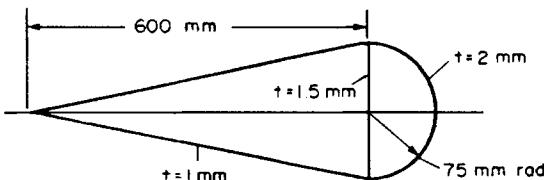


Fig. 5.15.

5.11 (B/C). Develop a relationship between torque and angle of twist for a closed uniform tube of thin-walled non-circular section and use this to derive the twist per unit length for a strip of thin rectangular cross-section.

Use the above relationship to show that, for the same torque, the ratio of angular twist per unit length for a closed square-section tube to that for the same section but opened by a longitudinal slit and free to warp is approximately $4t^2/3b^2$, where t , the material thickness, is much less than the mean width b of the cross-section. [C.E.I.]

5.12 (C). A torsional member used for stirring a chemical process is made of a circular tube to which is welded four rectangular strips as shown in Fig. 5.16. The tube has inner and outer diameters of 94 mm and 100 mm respectively, each strip is 50 mm × 18 mm, and the stirrer is 3 m in length.

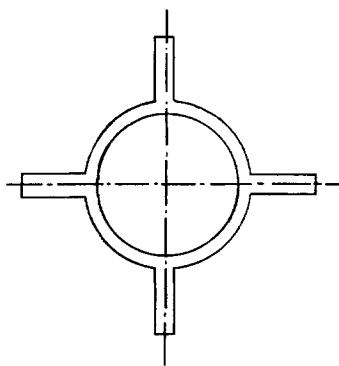


Fig. 5.16.

If the maximum shearing stress in any part of the cross-section is limited to 56 MN/m^2 , neglecting any stress concentration, calculate the maximum torque which can be carried by the stirrer and the resulting angle of twist over the full length.

For torsion of rectangular sections the torque T is related to the maximum shearing stress, τ_{\max} , and angle of twist, θ , in radians per unit length, as follows:

$$T = k_1 b d^2 \tau_{\max} = k_2 b d^3 G \theta$$

where b is the longer and d the shorter side of the rectangle and in this case, $k_1 = 0.264$, $k_2 = 0.258$ and $G = 83 \text{ GN/m}^2$. [C.E.I.] [2.83 kNm, 2.4°.]

5.13 (C). A long tube is subjected to a torque of 200 Nm. The tube has the double-cell, thin-walled, effective cross-section illustrated in Fig. 5.17. Assuming that no buckling occurs and that the twist per unit length of the tube is constant, determine the maximum shear stresses in each wall of the tube.

[C.E.I.] [0.76, 1.01, 0.19 MN/m^2 .]

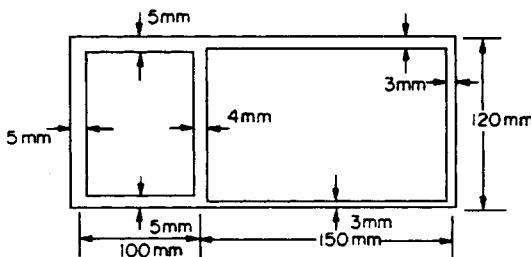


Fig. 5.17.

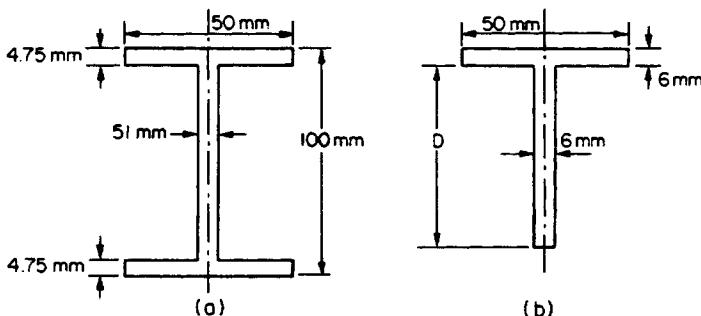


Fig. 5.18.

5.14 (B/C). An I-section has the dimensions shown in Fig. 5.18(a), and is subjected to an axial torque. Find the maximum value of the torque if the shear stress in the material is limited to 56 MN/m^2 and the twist per metre length is limited to 9° . Assume the modulus of rigidity G for the material is 82 GN/m^2 .

If the I-section is replaced by a T-section made of the same material and transmits the same torque, what will be the limb length, D , of the T-section and the angle of twist per metre length? Assume the T-section is subjected to the same limiting conditioning as the I-section and that it has the dimensions shown in Fig. 5.18(b). For narrow rectangular sections assume k values of $\frac{1}{3}$ in the formulae for torque and angle of twist.

[B.P.] [0.081 m; $6.5^\circ/\text{m.}$]

5.15 (B/C). (a) An aluminium sheet, 600 mm wide and 4 mm thick, is to be formed into a hollow section tube by bending through 360° and butt-welding the long edges together. The cross-section shape may be either circular or square.

Assuming a median length of 600 mm in each case, i.e. assuming no stretching occurs, determine the maximum torque that can be carried and the resulting angle of twist per metre length in each case.

Maximum allowable shearing stress = 65 MN/m^2 , shear modulus $G = 40 \text{ GN/m}^2$.

(b) What would be the effect on the stiffness per metre length of each type of section of a narrow saw-cut through the tube wall along the length of the tube? In the case of the square section assume that the cut is taken along the centre of one face.

[B.P.] [14.9 kNm, 0.975° ; 11.7 kNm, 1.24° ; reduction 1690 times, reduction 1050 times.]

5.16 (B). The two sections shown in Fig. 5.19 are under consideration for an engineering application which includes both bending and applied torque. Make a critical comparison of the strengths of the two sections under the two modes of loading and make a recommendation as to the section which should be adopted. The material to be used is to be the same for both sections.

The rectangular section torsion constants k_1 and k_2 may be found in terms of the section d/b ratio from Table 3.1.
[Tubular]

5.17 (B). Compare the angles of twist of the following sections when each is subjected to the same torque of 3 kNm;

- circular tube, 80 mm outside diameter, 6 mm thick;
- square tube, 52 mm side length (median dimension), 6 mm thick;

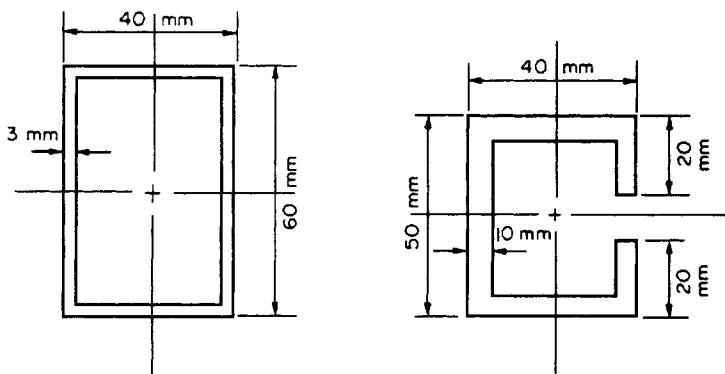


Fig. 5.19.

- (c) circular tube as (a) but with additional four rectangular fins 80 mm long by 15 mm wide symmetrically placed around the tube periphery.

All sections have the same length of 2 m and $G = 80 \text{ GN/m}^2$

[0.039 rad; 0.088 rad; 0.038 rad]

To what maximum torque can sections (a), (b) and (c) be subjected if the maximum shear stress is limited to 100 MN/m^2 ?

[4.8 kNm; 3.24 kNm; 5.7 kNm]

What maximum angle of twist can be accepted by tube (c) for the same limiting shear stress? [0.0625 rad]

5.18 (B). Figure 5.20 shows part of the stirring mechanism for a chemical process, consisting of a circular stainless-steel tube of length 2 m, outside diameter 75 mm and wall thickness 6 mm, welded onto a square mild-steel tube of length 1.5 m. Four blades of rectangular section stainless-steel, 100 mm \times 15 mm, are welded along the full length of the stainless-steel tube as shown.

(a) Select a suitable section for the square tube from the available stock list below so that when the maximum allowable shear stress of 58 MN/m^2 is reached in the stainless-steel, the shear stress in the mild steel of the square tube does not exceed 130 MN/m^2 .

Section	Dimension	Wall thickness	Torsion constant (J equiv)
1	50 \times 50 mm	5 mm	476 000 mm^4
2	60 \times 60 mm	4 mm	724 000 mm^4
3	70 \times 70 mm	3.6 mm	1080 000 mm^4

(b) Having selected an appropriate mild steel tube, determine how much the entire mechanism will twist during operation at a constant torque of 3 kNm.

The shear modulus of stainless steel is 78 GN/m^2 and of mild steel is 83 GN/m^2 . Neglect the effect of any stress concentration.

[50 mm \times 50 mm; 0.152 rad]

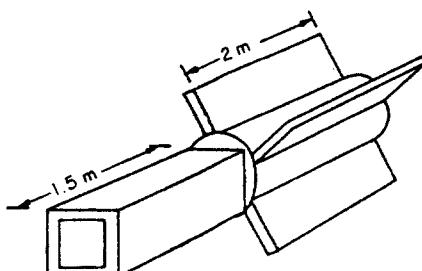


Fig. 5.20.

5.19 (B). Figure 5.21 shows the cross-section of a thin-walled fabricated service conduit used for the protection of long runs of electrical wiring in a production plant. The lower plate AB may be removed for inspection and re-cabling purposes.

Owing to the method by which the conduit is supported and the weight of pipes/wires that it carries, the section is subjected to a torque of 130 Nm. With plate AB assumed in position, determine the maximum shear stress set up in the walls of the conduit. What will be the angle of twist per unit length?

By consideration of maximum stress levels and angles of twist, establish whether the section design is appropriate for the removal of plate AB for maintenance purposes assuming that the same torque remains applied. If modifications are deemed to be necessary suggest suitable measures.

For the conduit material $G = 80 \text{ GN/m}^2$ and maximum allowable shear stress $= 180 \text{ MN/m}^2$.

[167 MN/m²; 39°/m]

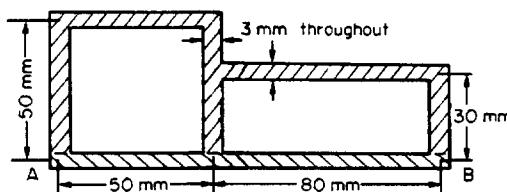


Fig. 5.21.

5.20 (B). (a) Figure 5.22 shows the cross-section of a thin-walled duct which forms part of a fluid transfer system. The wire mesh, FC, through which sediment is allowed to pass, may be assumed to contribute no strength

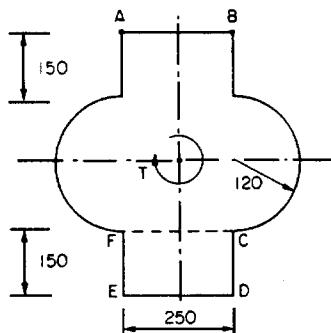


Fig. 5.22. All dimensions (mm) may be taken as median dimensions.

to the section. Owing to the method of support, the weight of the fluid and duct introduces a torque to the section which may be assumed uniform.

If the maximum shear stress in the duct material is limited to 150 MN/m^2 ; determine the maximum torque which can be tolerated and the angle of twist per metre length when this maximum torque is applied. For the duct material $G = 85 \text{ GN/m}^2$.

[432.6 kNm; 0.516°/m]

(b) In order to facilitate cleaning and inspection of the duct, plates AB and ED are removable. What would be the effect on the results of part (a) if plate AB were inadvertently left off over part of the duct length after inspection?

[5.12 kNm; 12.6°/m]

5.21 (C). Figure 5.8 shows a polymer extrusion of wall thickness 4 mm. The section is to be stiffened by the insertion of an aluminium I section as shown, the centre web of the polymer extrusion having been removed. The I section wall thickness is also 4 mm.

If $G = 3.3 \text{ GN/m}^2$ for the polymer and 70 GN/m^2 for the aluminium, what increase in stiffness is achieved? What increase in torque is allowable, if the design is governed by maximum allowable stresses of 5 MN/m^2 and 100 MN/m^2 in the polymer and aluminium respectively?

[258%, 7.4%]

CHAPTER 6

EXPERIMENTAL STRESS ANALYSIS

Introduction

We live today in a complex world of manmade structures and machines. We work in buildings which may be many storeys high and travel in cars and ships, trains and planes; we build huge bridges and concrete dams and send mammoth rockets into space. Such is our confidence in the modern engineer that we take these manmade structures for granted. We assume that the bridge will not collapse under the weight of the car and that the wings will not fall away from the aircraft. We are confident that the engineer has assessed the stresses within these structures and has built in sufficient strength to meet all eventualities.

This attitude of mind is a tribute to the competence and reliability of the modern engineer. However, the commonly held belief that the engineer has been able to calculate mathematically the stresses within the complex structures is generally ill-founded. When he is dealing with familiar design problems and following conventional practice, the engineer draws on past experience in assessing the strength that must be built into a structure. A competent civil engineer, for example, has little difficulty in selecting the size of steel girder that he needs to support a wall. When he departs from conventional practice, however, and is called upon to design unfamiliar structures or to use new materials or techniques, the engineer can no longer depend upon past experience. The mathematical analysis of the stresses in complex components may not, in some cases, be a practical proposition owing to the high cost of computer time involved. If the engineer has no other way of assessing stresses except by recourse to the nearest standard shape and hence analytical solution available, he builds in greater strength than he judges to be necessary (i.e. he incorporates a factor of safety) in the hope of ensuring that the component will not fail in practice. Inevitably, this means unnecessary weight, size and cost, not only in the component itself but also in the other members of the structure which are associated with it.

To overcome this situation the modern engineer makes use of experimental techniques of stress measurement and analysis. Some of these consist of "reassurance" testing of completed structures which have been designed and built on the basis of existing analytical knowledge and past experience: others make use of scale models to predict the stresses, often before final designs have been completed.

Over the past few years these *experimental stress analysis* or *strain measurement* techniques have served an increasingly important role in aiding designers to produce not only efficient but economic designs. In some cases substantial reductions in weight and easier manufacturing processes have been achieved.

A large number of problems where experimental stress analysis techniques have been of particular value are those involving fatigue loading. Under such conditions failure usually starts when a fatigue crack develops at some position of high localised stress and propagates until final rupture occurs. As this often requires several thousand repeated cycles of load under service conditions, full-scale production is normally well under way when failure

occurs. Delays at this stage can be very expensive, and the time saved by stress analysis techniques in locating the source of the trouble can far outweigh the initial cost of the equipment involved.

The main techniques of experimental stress analysis which are in use today are:

- (1) brittle lacquers
- (2) strain gauges
- (3) photoelasticity
- (4) photoelastic coatings

The aim of this chapter is to introduce the fundamental principles of these techniques, together with limited details of the principles of application, in order that the reader can appreciate (a) the role of the experimental techniques as against the theoretical procedures described in the other chapters, (b) the relative merits of each technique, and (c) the more specialised literature which is available on the techniques, to which reference will be made.

6.1. Brittle lacquers

The brittle-lacquer technique of experimental stress analysis relies on the failure by cracking of a layer of a brittle coating which has been applied to the surface under investigation. The coating is normally sprayed onto the surface and allowed to air- or heat-cure to attain its brittle properties. When the component is loaded, this coating will crack as its so-called *threshold strain* or *strain sensitivity* is exceeded. A typical crack pattern obtained on an engineering component is shown in Fig. 6.1. Cracking occurs where the strain is

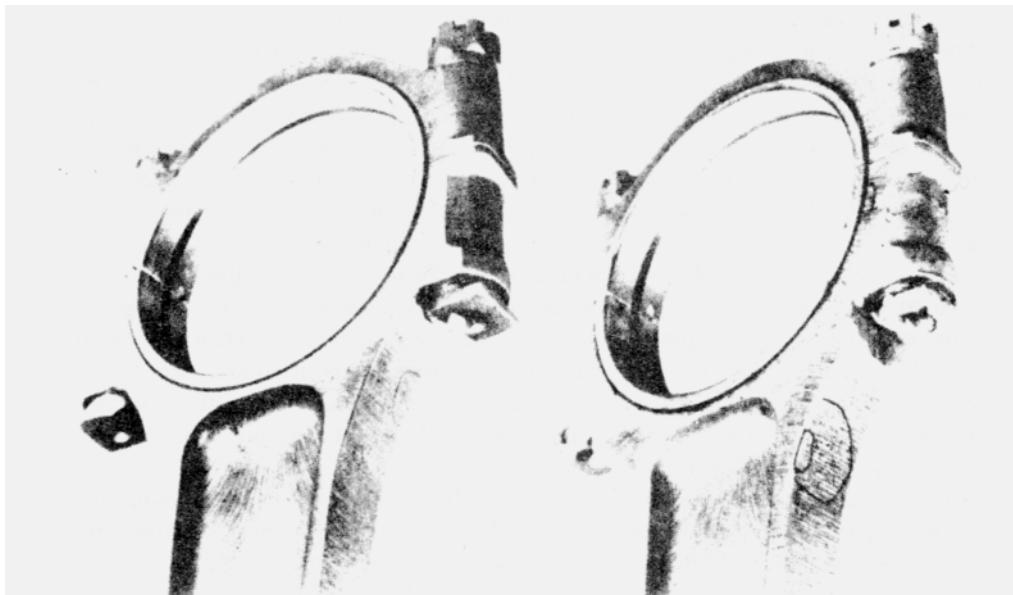


Fig. 6.1. Typical brittle-lacquer crack pattern on an engine con-rod. (Magnaflux Corporation.)

greatest, so that an immediate indication is given of the presence of stress concentrations. The cracks also indicate the directions of maximum strain at these points since they are always aligned at right angles to the direction of the maximum principal tensile strain. The method is thus of great value in determining the optimum positions in which to place strain gauges (see §6.2) in order to record accurately the measurements of strain in these directions.

The brittle-coating technique was first used successfully in 1932 by Dietrich and Lehr in Germany despite the fact that references relating to observation of the phenomenon can be traced back to Clarke's investigations of tubular bridges in 1850. The most important advance in brittle-lacquer technology, however, came in the United States in 1937–41 when Ellis, De Forrest and Stern produced a series of lacquers known as "Stresscoat" which, in a modified form, remain widely used in the world today.

There are many every-day examples of brittle coatings which can be readily observed by the reader to exhibit cracks indicating local yielding when the strain is sufficiently large, e.g. cellulose, vitreous or enamel finishes. Cellulose paints, in fact, are used by some engineering companies as a brittle lacquer on rubber models where the strains are quite large.

As an interesting experiment, try spraying a comb with several thin coats of hair-spray lacquer, giving each layer an opportunity to dry before application of the next coat. Finally, allow the whole coating several hours to fully cure; cracks should then become visible when the comb is bent between your fingers.

In engineering applications a little more care is necessary in the preparation of the component and application of the lacquer, but the technique remains a relatively simple and hence attractive one. The surface of the component should be relatively smooth and clean, standard solvents being used to remove all traces of grease and dirt. The lacquer can then be applied, the actual application procedure depending on the type of lacquer used. Most lacquers may be sprayed or painted onto the surface, spraying being generally more favoured since this produces a more uniform thickness of coating and allows a greater control of the thickness. Other lacquers, for example, are in wax or powder form and require pre-heating of the component surface in order that the lacquer will melt and run over the surface. Optimum coating thicknesses depend on the lacquer used but are generally of the order of 1 mm.

In order to determine the strain sensitivity of the lacquer, and hence to achieve an approximate idea of the strains existing in the component, it is necessary to coat calibration bars at the same time and in exactly the same manner as the specimen itself. These bars are normally simple rectangular bars which fit into the calibration jig shown in Fig. 6.2 to form a simple cantilever with an offset cam at the end producing a known strain distribution along the cantilever length. When the lacquer on the bar is fully cured, the lever on the cam is moved forward to depress the end of the bar by a known amount, and the position at which the cracking of the lacquer begins gives the strain sensitivity when compared with the marked strain scale. This enables quantitative measurements of strain levels to be made on the components under test since if, for example, the calibration sensitivity is shown to be 800 microstrain ($\text{strain} \times 10^{-6}$), then the strain at the point on the component at which cracks first appear is also 800 microstrain.

This type of quantitative measurement is generally accurate to no better than 10–20%, and brittle-lacquer techniques are normally used to locate the *positions* of stress maxima, the actual values then being determined by subsequent strain-gauge testing.

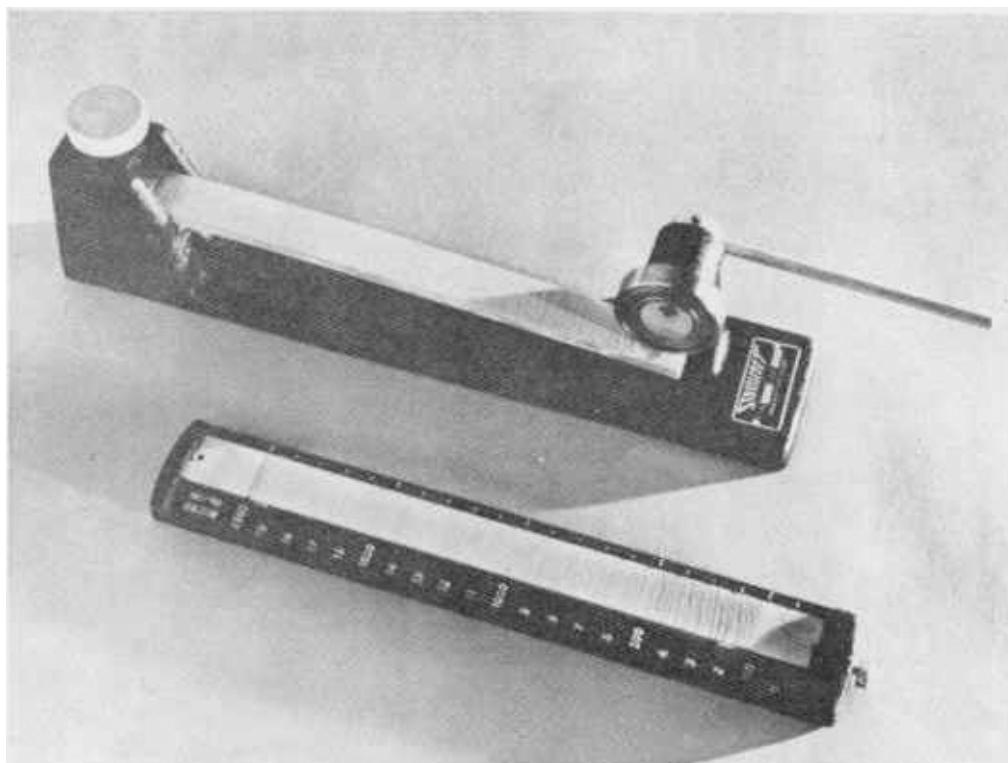


Fig. 6.2. (Top) Brittle-lacquer calibration bar in a calibration jig with the cam depressed to apply load. (Bottom) Calibration of approximately 100 microstrain. (Magnaflux Corporation.)

Loading is normally applied to the component in increments, held for a few minutes and released to zero prior to application of the next increment; the time interval between increments should be several times greater than that of the loading cycle. With this procedure *creep* effects in the lacquer, where strain in the lacquer changes at constant load, are completely overcome. After each load application, cracks should be sought and, when located, encircled and identified with the load at that stage using a chinagraph pencil. This enables an accurate record of the development of strain throughout the component to be built up.

There are a number of methods which can be used to aid crack detection including (a) pre-coating the component with an aluminium undercoat to provide a background of uniform colour and intensity, (b) use of a portable torch which, when held close to the surface, highlights the cracks by reflection on the crack faces, (c) use of dye-etchants or special electrified particle inspection techniques, details of which may be found in standard reference texts.⁽³⁾

Given good conditions, however, and a uniform base colour, cracks are often visible without any artificial aid, viewing the surface from various angles generally proving sufficient.

Figures 6.3 and 6.4 show further examples of brittle-lacquer crack patterns on typical engineering components. The procedure is simple, quick and relatively inexpensive; it can be carried out by relatively untrained personnel, and immediate qualitative information, such as positions of stress concentration, is provided on the most complicated shapes.

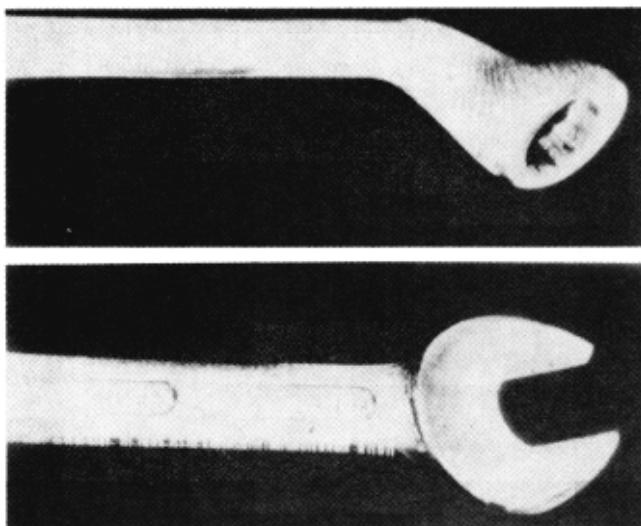


Fig. 6.3. Brittle-lacquer crack patterns on an open-ended spanner and a ring spanner. In the former the cracks appear at right angles to the maximum bending stress in the edge of the spanner whilst in the ring spanner the presence of torsion produces an inclination of the principal stress and hence of the cracks in the lacquer.

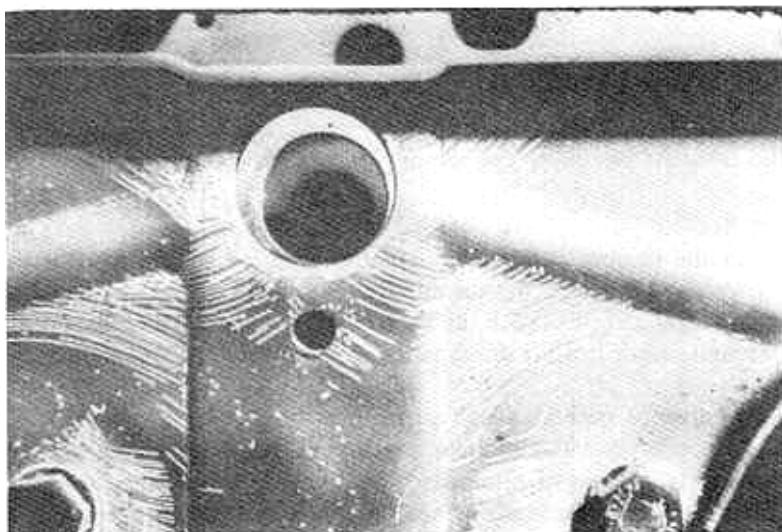


Fig. 6.4. Brittle-lacquer crack pattern highlighting the positions of stress concentration on a motor vehicle component. (Magnaflux Corporation.)

Various types of lacquer are available, including a special ceramic lacquer which is particularly useful for investigation under adverse environmental conditions such as in the presence of water, oil or heavy vibration.

Refinements to the general technique allow the study of residual stresses, compressive stress fields, dynamic situations, plastic yielding and miniature components with little increased difficulty. For a full treatment of these and other applications, the reader is referred to ref. 3.

6.2. Strain gauges

The accurate assessment of stresses, strains and loads in components under working conditions is an essential requirement of successful engineering design. In particular, the location of peak stress values and stress concentrations, and subsequently their reduction or removal by suitable design, has applications in every field of engineering. The most widely used experimental stress-analysis technique in industry today, particularly under working conditions, is that of strain gauges.

Whilst a number of different types of strain gauge are commercially available, this section will deal almost exclusively with the electrical resistance type of gauge introduced in 1939 by Ruge and Simmons in the United States.

The *electrical resistance strain gauge* is simply a length of wire or foil formed into the shape of a continuous grid, as shown in Fig. 6.5, cemented to a non-conductive backing. The gauge is then bonded securely to the surface of the component under investigation so that any strain in the surface will be experienced by the gauge itself. Since the fundamental equation for the electrical resistance R of a length of wire is

$$R = \frac{\rho L}{A} \quad (6.1)$$

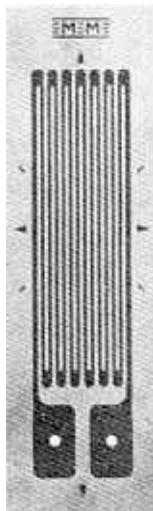


Fig. 6.5. Electric resistance strain gauge. (Welwyn Strain Measurement Ltd.)

where L is the length, A is the cross-sectional area and ρ is the *specific resistance* or *resistivity*, it follows that any change in length, and hence sectional area, will result in a change of resistance. Thus measurement of this resistance change with suitably calibrated equipment enables a direct reading of linear strain to be obtained. This is made possible by the relationship which exists for a number of alloys over a considerable strain range between change of resistance and strain which may be expressed as follows:

$$\frac{\Delta R}{R} = K \times \frac{\Delta L}{L} \quad (6.2)$$

where ΔR and ΔL are the changes in resistance and length respectively and K is termed the *gauge factor*.

Thus

$$\text{gauge factor } K = \frac{\Delta R/R}{\Delta L/L} = \frac{\Delta R/R}{\epsilon} \quad (6.3)$$

where ϵ is the strain. The value of the gauge factor is always supplied by the manufacturer and can be checked using simple calibration procedures if required. Typical values of K for most conventional gauges lie in the region of 2 to 2.2, and most modern strain-gauge instruments allow the value of K to be set accordingly, thus enabling strain values to be recorded directly.

The changes in resistance produced by normal strain levels experienced in engineering components are very small, and sensitive instrumentation is required. Strain-gauge instruments are basically *Wheatstone bridge* networks as shown in Fig. 6.6, the condition of balance for this network being (i.e. the galvanometer reading zero when)

$$R_1 \times R_3 = R_2 \times R_4 \quad (6.4)$$

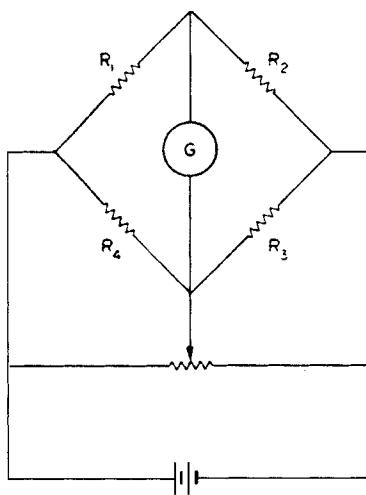


Fig. 6.6. Wheatstone bridge circuit.

In the simplest half-bridge wiring system, gauge 1 is the *active gauge*, i.e. that actually being strained. Gauge 2 is so-called *dummy gauge* which is bonded to an unstrained piece of metal

similar to that being strained, its purpose being to cancel out any resistance change in R_1 that occurs due to temperature fluctuations in the vicinity of the gauges. Gauges 1 and 2 then represent the working half of the network – hence the name “half-bridge” system – and gauges 3 and 4 are standard resistors built into the instrument. Alternative wiring systems utilise one (*quarter-bridge*) or all four (*full-bridge*) of the bridge resistance arms.

6.3. Unbalanced bridge circuit

With the Wheatstone bridge initially balanced to zero any strain on gauge R_1 will cause the galvanometer needle to deflect. This deflection can be calibrated to read strain, as noted above, by including in the circuit an arrangement whereby gauge-factor adjustment can be achieved. Strain readings are therefore taken with the pointer off the zero position and the bridge is thus *unbalanced*.

6.4. Null balance or balanced bridge circuit

An alternative measurement procedure makes use of a variable resistance in one arm of the bridge to cancel any deflection of the galvanometer needle. This adjustment can then be calibrated directly as strain and readings are therefore taken with the pointer on zero, i.e. in the *balanced* position.

6.5. Gauge construction

The basic forms of wire and foil gauges are shown in Fig. 6.7. Foil gauges are produced by a printed-circuit process from selected melt alloys which have been rolled to a thin film, and these have largely superseded the previously popular wire gauge. Because of the increased area of metal in the gauge at the ends, the foil gauge is not so sensitive to strains at right angles to the direction in which the major axis of the gauge is aligned, i.e. it has a low transverse or cross-sensitivity – one of the reasons for its adoption in preference to the wire gauge. There are many other advantages of foil gauges over wire gauges, including better strain transmission from the substrate to the grid and better heat transmission from the grid

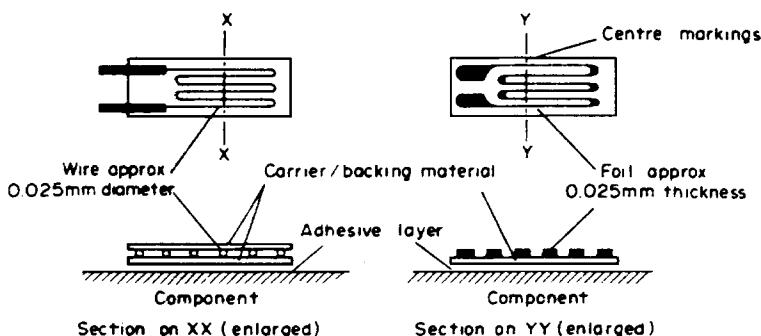


Fig. 6.7. Basic format of wire and foil gauges. (Merrow)

Gage Patterns
Actual Size
 (Grids Run Vertically.)

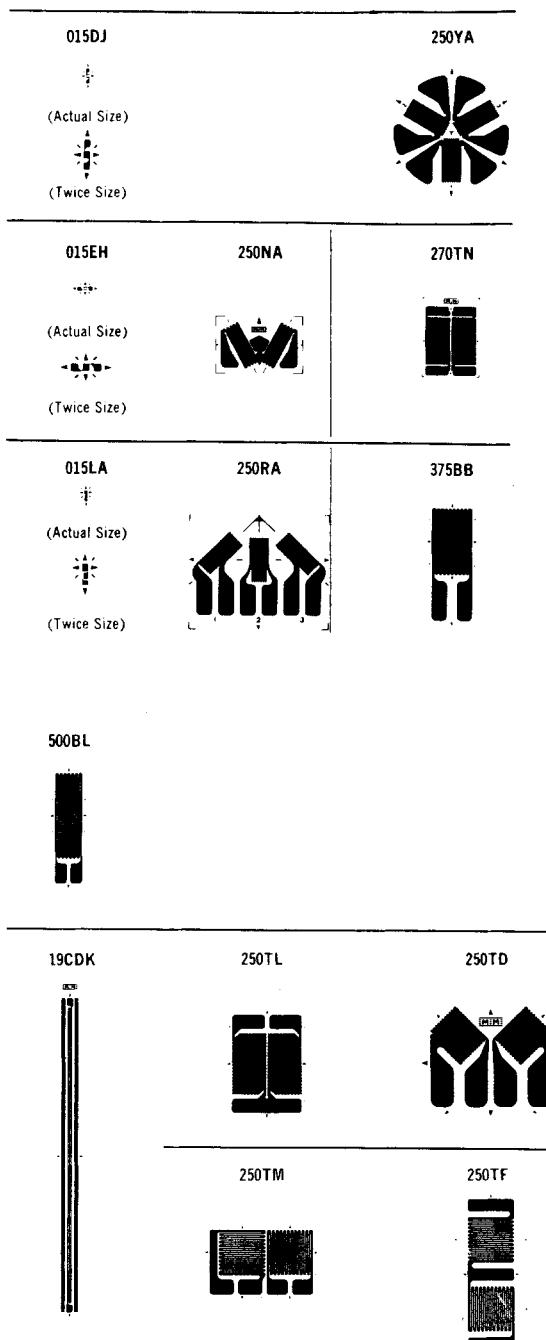


Fig. 6.8. Typical gauge sizes and formats. (Welwyn Strain Measurement Division)

to the substrate; as a result of which they are usually more stable. Additionally, the grids of foil gauges can be made much smaller and there is almost unlimited freedom of grid configuration, solder tab arrangement, multiple grid configuration, etc.

6.6. Gauge selection

Figure 6.8 shows but a few of the many types and size of gauge which are available. So vast is the available range that it is difficult to foresee any situation for which there is no gauge suitable. Most manufacturers' catalogues⁽¹³⁾ give full information on gauge selection, and any detailed treatment would be out of context in this section. Essentially, the choice of a suitable gauge incorporates consideration of physical size and form, resistance and sensitivity, operating temperature, temperature compensation, strain limits, flexibility of the gauge backing (and hence relative stiffness) and cost.

6.7. Temperature compensation

Unfortunately, in addition to strain, other factors affect the resistance of a strain gauge, the major one being temperature change. It can be shown that temperature change of only a few degrees completely dwarfs any readings due to the typical strains encountered in engineering applications. Thus it is vitally important that any temperature effects should be cancelled out, leaving only the mechanical strain required. This is achieved either by using the conventional dummy gauge, *half-bridge*, system noted earlier, or, alternatively, by the use of *self-temperature-compensated gauges*. These are gauges constructed from material which has been subjected to particular metallurgical processes and which produce very small (and calibrated) thermal output over a specified range of temperature when bonded onto the material for which the gauge has been specifically designed (see Fig. 6.9).

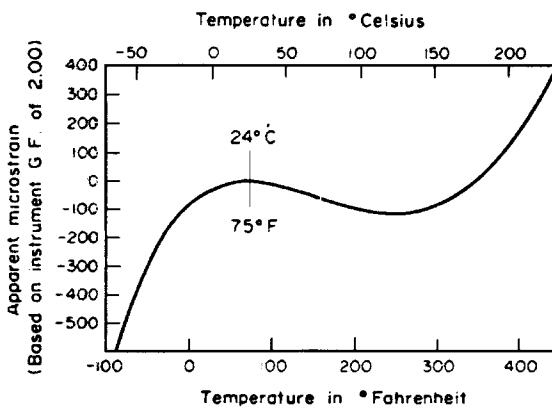


Fig. 6.9. Typical output from self-temperature-compensated gauge (Vishay)

In addition to the gauges, the lead-wire system must also be compensated, and it is normal practice to use the three-lead-wire system shown in Fig. 6.10. In this technique, two of the

leads are in opposite arms of the bridge so that their resistance cancels, and the third lead, being in series with the power supply, does not influence the bridge balance. All leads must be of equal length and wound tightly together so that they experience the same temperature conditions.

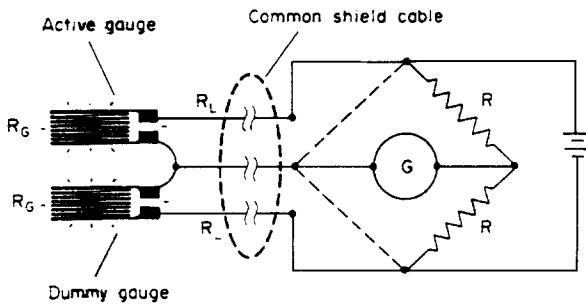


Fig. 6.10. Three-lead wire system for half-bridge (dummy-active) operation.

In applications where a single self-temperature-compensated gauge is used in a quarter-bridge arrangement the three-wire circuit becomes that shown in Fig. 6.11. Again, only one of the current-carrying lead-wires is in series with the active strain gauge, the other is in series with the bridge completion resistor (occasionally still referred to as a "dummy") in the adjacent arm of the bridge. The third wire, connected to the lower solder tab of the active gauge, carries essentially no current but acts simply as a voltage-sensing lead. Provided the two lead-wires (resistance R_L) are of the same size and length and maintained at the same temperature (i.e. kept physically close to each other) then any resistance changes due to temperature will cancel.

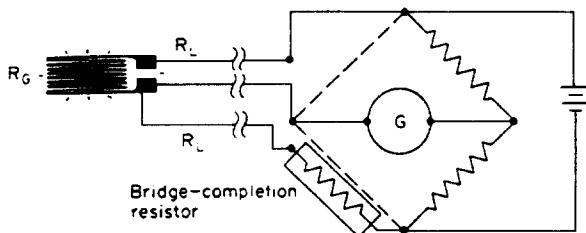


Fig. 6.11. Three-lead-wire system for quarter-bridge operation with single self-temperature-compensated gauge.

6.8. Installation procedure

The quality and success of any strain-gauge installation is influenced greatly by the care and precision of the installation procedure and correct choice of the adhesive. The apparently mundane procedure of actually cementing the gauge in place is a critical step in the operation. Every precaution must be taken to ensure a chemically clean surface if perfect adhesion is to be attained. Full details of typical procedures and equipment necessary are given in refs 6 and 13, as are the methods which may be used to test the validity of the installation prior to

recording measurements. Techniques for protection of gauge installations are also covered. Typical strain-gauge installations are shown in Figs. 6.12 and 6.13.

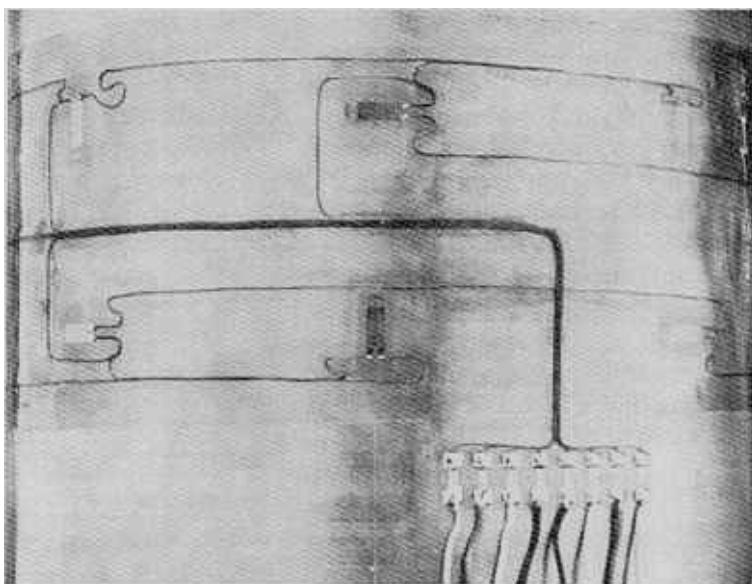


Fig. 6.12. Typical strain-gauge installation showing six of eight linear gauges bonded to the surface of a cylinder to record longitudinal and hoop strains. (Crown copyright.)

6.9. Basic measurement systems

(a) For direct strain

The standard procedure for the measurement of tensile or compressive direct strains utilises the *full-bridge* circuit of Fig. 6.14 in which not only are the effects of any bending eliminated but the sensitivity is increased by a factor of 2.6 over that which would be achieved using a single linear gauge.

Bearing in mind the balance requirement of the Wheatstone bridge, i.e. $R_1, R_3 = R_2R_4$, each pair of gauges on either side of the equation will have an additive effect if their signs are similar or will cancel if opposite. Thus the opposite signs produced by bending cancel on both pairs whilst the similar signs of the direct strains are additive. The value 2.6 arises from twice the applied axial strain (R_1 and R_3) plus twice the Poisson's ratio strain (R_2 and R_4), assuming $\nu = 0.3$. The latter is compressive, i.e. negative, on the opposite side of the bridge from R_1 and R_3 , and hence is an added signal to that of R_1 and R_3 .

(b) Bending

Figure 5.15(a) shows the arrangement used to record bending strains independently of direct strains. It is normal to bond linear gauges on opposite surfaces of the component and to use the *half-bridge* system shown in Fig. 6.6; this gives a sensitivity of twice that which would be

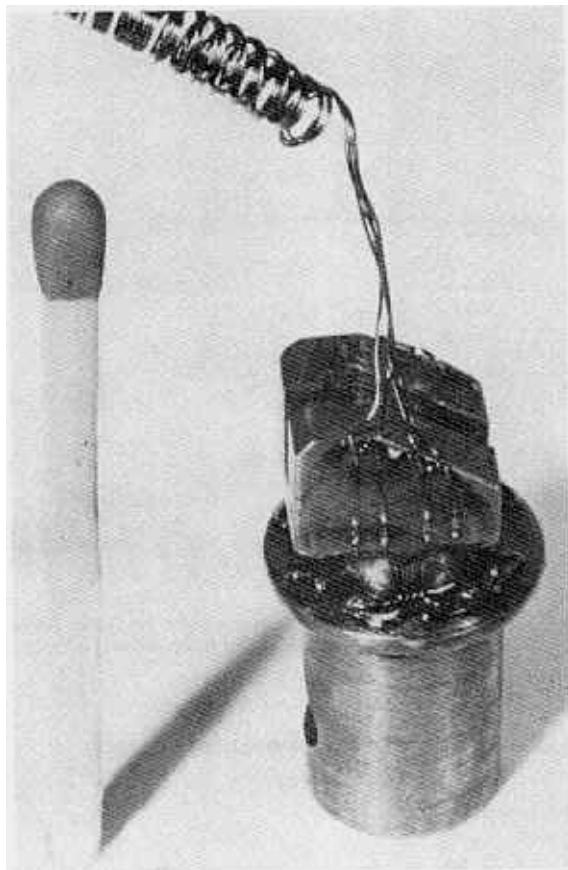


Fig. 6.13. Miniature strain-gauge installation. (Welwyn.)

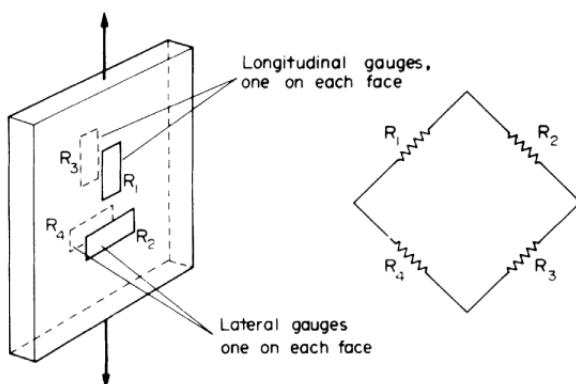


Fig. 6.14. "Full bridge" circuit arranged to eliminate any bending strains produced by unintentional eccentricities of loading in a nominal axial load application. The arrangement also produces a sensitivity 26 times that of a single active gauge. (Merrow.)

achieved with a single-linear gauge. Alternatively, it is possible to utilise again the Poisson strains as in §6.9(a) by bonding additional lateral gauges (i.e. perpendicular to the other gauges) on each surface and using a full-bridge circuit to achieve a sensitivity of 2.6. In this case, however, gauges R_2 and R_4 would be interchanged from the arrangement shown in Fig. 6.14 and would appear as in Fig. 6.15(b).

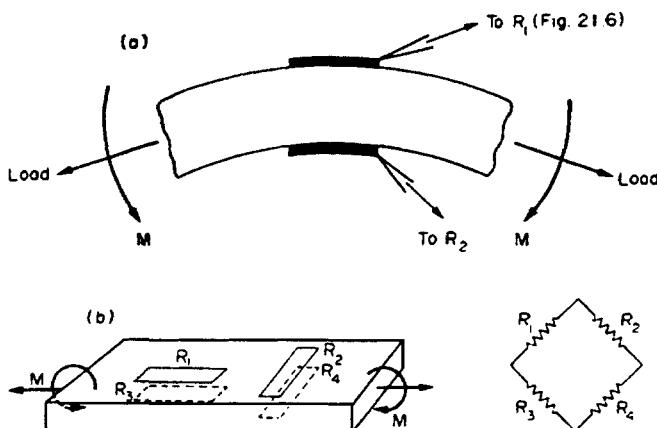


Fig. 6.15. (a) Determination of bending strains independent of end loads: "half-bridge" method. Sensitivity twice that of a single active gauge. (b) Determination of bending strains independent of end loads: "full-bridge" procedure. Sensitivity 2.6.

(c) Torsion

It has been shown that pure torsion produces direct stresses on planes at 45° to the shaft axis – one set tensile, the other compressive. Measurements of torque or shear stress using strain-gauge techniques therefore utilise gauges bonded at 45° to the axis in order to record the direct strains. Again, it is convenient to use a wiring system which automatically cancels unwanted signals, i.e. in this case the signals arising due to unwanted direct or bending strains which may be present. Once again, a full-bridge system is used and a sensitivity of four times that of a single gauge is achieved (Fig. 6.16).

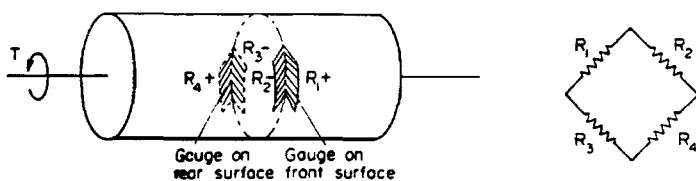


Fig. 6.16. Torque measurement using full-bridge circuit-sensitivity four times that of a single active gauge.

6.10. D.C. and A.C. systems

The basic Wheatstone bridge circuit shown in all preceding diagrams is capable of using either a direct current (d.c.) or an alternating current (a.c.) source; Fig. 6.6, for example,

shows the circuit excited by means of a standard battery (d.c.) source. Figure 6.17, however, shows a typical arrangement for a so-called a.c. *carrier frequency* system, the main advantage of this being that all unwanted signals such as noise are eliminated and a stable signal of gauge output is produced. The relative merits and disadvantages of the two types of system are outside the scope of this section but may be found in any standard reference text (refs. 4, 6, 7 and 13).

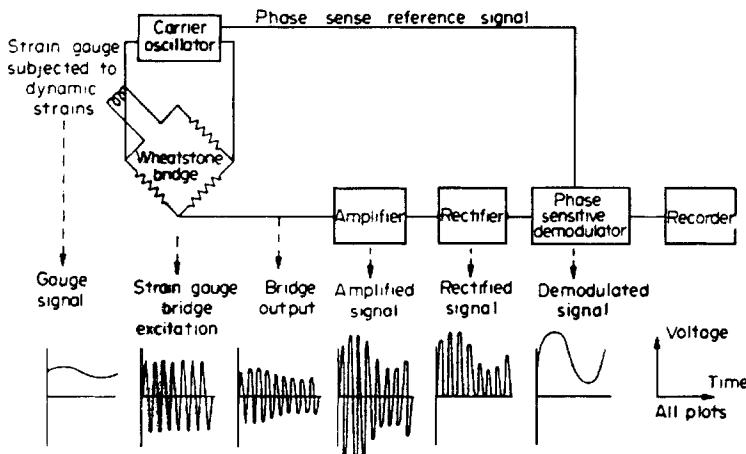


Fig. 6.17. Schematic arrangement of a typical carrier frequency system. (Merrow.)

6.11. Other types of strain gauge

The previous discussion has related entirely to the electrical resistance type of strain gauge and, indeed, this is by far the most extensively used type of gauge in industry today. It should be noted, however, that many other forms of strain gauge are available. They include:

- (a) **mechanical gauges or extensometers** using optical or mechanical lever systems;
- (b) **pneumatic gauges** using changes in pressure;
- (c) **acoustic gauges** using the change in frequency of a vibrating wire;
- (d) **semiconductor or piezo-resistive gauges** using the piezo-resistive effect in silicon to produce resistance changes;
- (e) **inductance gauges** using changes in inductance of, e.g., differential transformer systems;
- (f) **capacitance gauges** using changes in capacitance between two parallel or near-parallel plates.

Each type of gauge has a particular field of application in which it can compete on equal, or even favourable, terms with the electrical resistance form of gauge. None, however, are as versatile and generally applicable as the resistance gauge. For further information on each type of gauge the reader is referred to the references listed at the end of this chapter.

6.12. Photoelasticity

In recent years, photoelastic stress analysis has become a technique of outstanding importance to engineers. When polarised light is passed through a stressed transparent model, interference patterns or *fringes* are formed. These patterns provide immediate qualitative information about the general distribution of stress, positions of stress concentrations and of areas of low stress. On the basis of these results, designs may be modified to reduce or disperse concentrations of stress or to remove excess material from areas of low stress, thereby achieving reductions in weight and material costs. As photoelastic analysis may be carried out at the design stage, stress conditions are taken into account before production has commenced; component failures during production, necessitating expensive design modifications and re-tooling, may thus be avoided. Even when service failures do occur, photoelastic analysis provides an effective method of failure investigation and often produces valuable information leading to successful re-design, typical photoelastic fringe patterns are shown in Fig. 6.18.

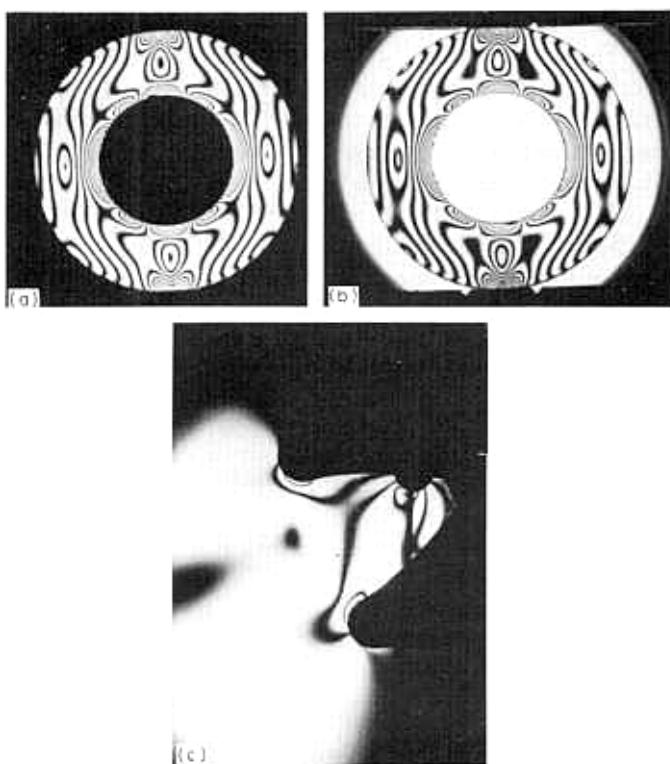


Fig. 6.18. Typical photoelastic fringe patterns. (a) Hollow disc subjected to compression on a diameter (dark field background). (b) As (a) but with a light field background. (c) Stress concentrations at the roots of a gear tooth.

Conventional or *transmission photoelasticity* has for many years been a powerful tool in the hands of trained stress analysts. However, untrained personnel interested in the technique have often been dissuaded from attempting it by the large volume of advanced mathematical

and optical theory contained in reference texts on the subject. Whilst this theory is, no doubt, essential for a complete understanding of the phenomena involved and of some of the more advanced techniques, it is important to accept that a wealth of valuable information can be obtained by those who are not fully conversant with all the complex detail. A major feature of the technique is that it allows one to effectively "look into" the component and pin-point flaws or weaknesses in design which are otherwise difficult or impossible to detect. Stress concentrations are immediately visible, stress values around the edge or boundary of the model are easily obtained and, with a little more effort, the separate principal stresses within the model can also be determined.

6.13. Plane-polarised light – basic polariscope arrangements

Before proceeding with the details of the photoelastic technique it is necessary to introduce the meaning and significance of *plane-polarised light* and its use in the equipment termed *polariscopes* used for photoelastic stress analysis. If light from an ordinary light bulb is passed through a polarising sheet or *polariser*, the sheet will act like a series of vertical slots so that the emergent beam will consist of light vibrating in one plane only: the plane of the slots. The light is then said to be *plane polarised*.

When directed onto an unstressed photoelastic model, the plane-polarised light passes through unaltered and may be completely extinguished by a second polarising sheet, termed an *analyser*, whose axis is perpendicular to that of the polariser: This is then the simplest form of polariscope arrangement which can be used for photoelastic stress analysis and it is termed a "*crossed*" set-up (see Fig. 6.19). Alternatively, a "*parallel*" set-up" may be used in which the axes of the polariser and analyser are parallel, as in Fig. 6.20. With the model unstressed, the plane-polarised light will then pass through both the model and analyser unaltered and maximum illumination will be achieved. When the model is stressed in the parallel set-up, the resulting fringe pattern will be seen against a light background or "field", whilst with the crossed arrangement there will be a completely black or "dark field".

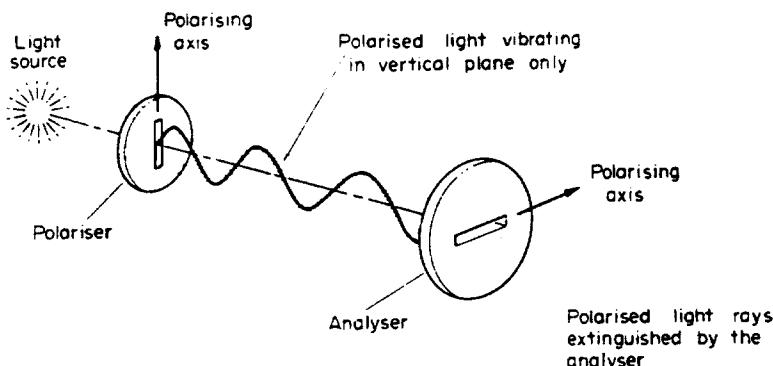


Fig. 6.19. "Crossed" set-up. Polariser and analyser arranged with their polarising axes at right angles; plane polarised light from the polariser is completely extinguished by the analyser. (Merrow.)

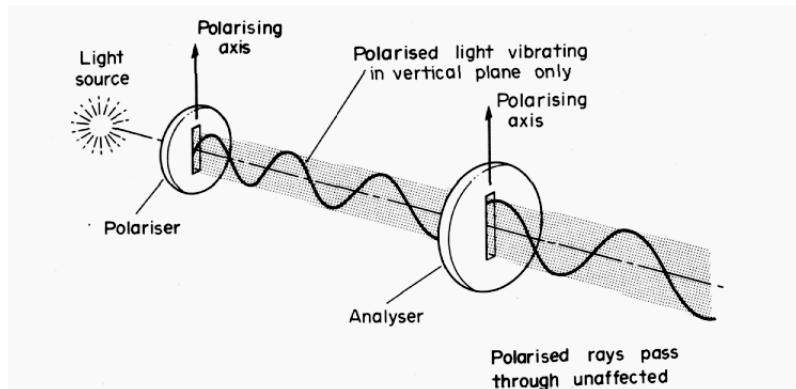


Fig. 6.20. "Parallel" set-up. Polariser and analyser axes parallel; plane-polarised light from the polariser passes through the analyser unaffected, producing a so-called "light field" arrangement. (Merrow.)

6.14. Temporary birefringence

Photoelastic models are constructed from a special class of transparent materials which exhibit a property known as *birefringence*, i.e. they have the ability to split an incident plane-polarised ray into two component rays; they are *double refracting*. This property is only exhibited when the material is under stress, hence the qualified term "*temporary birefringence*", and the direction of the component rays always coincides with the directions of the principal stresses (Fig. 6.21). Further, the speeds of the rays are proportional to the magnitudes of the respective stresses in each direction, so that the rays emerging from the model are out of phase and hence produce interference patterns when combined.

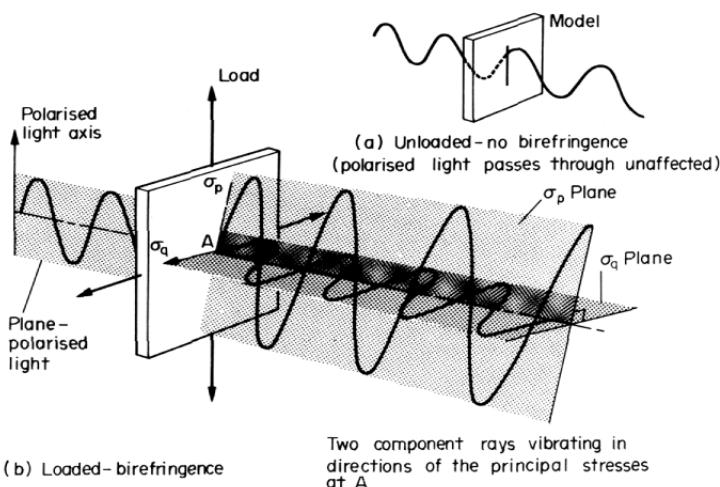


Fig. 6.21. Temporary birefringence. (a) Plane-polarised light directed onto an unstressed model passes through unaltered. (b) When the model is stressed the incident plane-polarised light is split into two component rays. The directions of the rays coincide with the directions of the principal stresses, and the speeds of the rays are proportional to the magnitudes of the respective stresses in their directions. The emerging rays are out of phase, and produce an interference pattern of fringes. (Merrow.)

6.15. Production of fringe patterns

When a model of an engineering component constructed from a birefringent material is stressed, it has been shown above that the incident plane-polarised light will be split into two component rays, the directions of which at any point coincide with the directions of the principal stresses at the point. The rays pass through the model at speeds proportional to the principal stresses in their directions and emerge out of phase. When they reach the analyser, shown in the crossed position in Fig. 6.22, only their horizontal components are transmitted and these will combine to produce interference fringes as shown in Fig. 6.23.

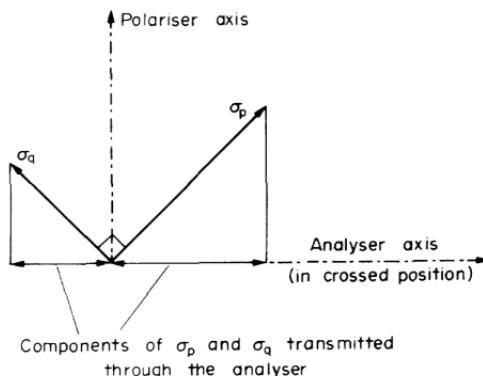


Fig. 6.22. Transmission through the analyser. (Merrow.)

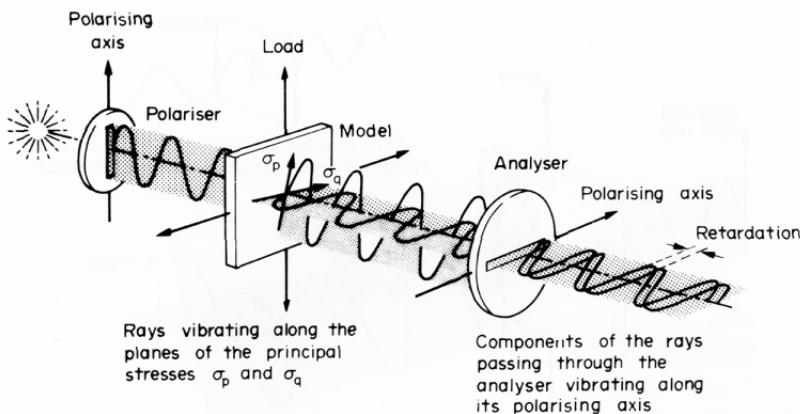


Fig. 6.23. Loaded model viewed in a plane polariscope arrangement with a "crossed set-up"

The difference in speeds of the rays, and hence the amount of interference produced, is proportional to the difference in the principal stress values ($\sigma_p - \sigma_q$) at the point in question. Since the maximum shear stress in any two-dimensional stress system is given by

$$\tau_{\max} = \frac{1}{2}(\sigma_p - \sigma_q)$$

it follows that the interference or fringe pattern produced by the photoelastic technique will give an immediate indication of the variation of shear stress throughout the model. Only at a free, unloaded boundary of a model, where one of the principal stresses is zero, will the fringe pattern yield a direct indication of the principal direct stress (in this case the tangential boundary stress). However, since the majority of engineering failures are caused by fatigue cracks commencing at the point of maximum tensile stress at the boundary, this is not a severe limitation. Further discussion of the interpretation of fringe patterns is referred to the following section.

If the original light source is *monochromatic*, e.g. mercury green or sodium yellow, the fringe pattern will appear as a series of distinct black lines on a uniform green or yellow background. These black lines or fringes correspond to points where the two rays are exactly 180° out of phase and therefore cancel. If white light is used, however, each composite wavelength of the white light is cancelled in turn and a multicoloured pattern of fringes termed *isochromatics* is obtained.

Monochromatic sources are preferred for accurate quantitative photoelastic measurements since a high number of fringes can be clearly discerned at, e.g., stress concentration positions. With a white light source the isochromatics become very pale at high stress regions and clear fringe boundaries are no longer obtained. White light sources are therefore normally reserved for general qualitative assessment of models, for isolation of zero fringe order positions (i.e. zero shear stress) which appear black on the multicoloured background, and for the investigation of stress directions using *isoclinics*. These are defined in detail in §6.19.

6.16. Interpretation of fringe patterns

It has been stated above that the pattern of fringes achieved by the photoelastic technique yields:

(a) *A complete indication of the variation of shear stress throughout the entire model.* Since ductile materials will generally fail in shear rather than by direct stress, this is an important feature of the technique. At points where the fringes are most numerous and closely spaced, the stress is highest; at points where they are widely spaced or absent, the stress is low. With a white-light source such areas appear black, indicating zero shear stress, but it cannot be emphasised too strongly that this does not necessarily mean zero stress since if the values of σ_p and σ_q (however large) are equal, then $(\sigma_p - \sigma_q)$ will be zero and a black area will be produced. Extreme care must therefore be taken in the interpretation of fringe patterns. Generally, however, fringe patterns may be compared with contour lines on a map, where close spacing relates to steep slopes and wide spacing to gentle inclines. Peaks and valleys are immediately evident, and actual heights are readily determined by counting the contours and converting to height by the known scale factor. In an exactly similar way, photoelastic fringes are counted from the known zero (black) positions and the resulting number or order of fringe at the point in question is converted to stress by a calibration constant known as the *material fringe value*. Details of the calibration procedure will be given later.

(b) *Individual values of the principal stresses at free unloaded boundaries, one of these always being zero.* The particular relevance of this result to fatigue failures has been mentioned, and the use of photoelasticity to produce modifications to boundary profiles in order to reduce boundary stress concentrations and hence the likelihood of fatigue failures has been a major use of the technique. In addition to the immediate indication of high stress

locations, the photoelastic model will show regions of low stress from which material can be conveniently **removed** without weakening the component to effect a reduction in weight and material cost. Surprisingly, perhaps, a reduction in material at or near a high stress concentration can also produce a significant reduction in maximum stress. Re-design can be carried out on a “file-it-and-see” basis, models being modified or re-shaped within minutes in order to achieve the required distribution of stress.

Whilst considerable valuable qualitative information can be readily obtained from photoelastic models without any calculations at all, there are obviously occasions where the precise values of the stresses are required. These are obtained using the following basic equation of photoelasticity,

$$\sigma_p - \sigma_q = \frac{nf}{t} \quad (6.5)$$

where σ_p and σ_q are the values of the maximum and minimum principal stresses at the point under consideration, n is the fringe number or *fringe order* at the point, f is the *material fringe value* or *coefficient*, and t is the model thickness.

Thus with a knowledge of the material fringe value obtained by calibration as described below, the required value of $(\sigma_p - \sigma_q)$ at any point can be obtained readily by simply counting the fringes from zero to achieve the value n at the point in question and substitution in the above relatively simple expression.

Maximum shear stress or boundary stress values are then easily obtained and the application of one of the so-called *stress-separation* procedures will yield the separate value of the principal stress at other points in the model with just a little more effort. These may be of particular interest in the design of components using brittle materials which are known to be relatively weak under the action of direct stresses.

6.17. Calibration

The value of f , which, it will be remembered, is analogous to the height scale for contours on a survey map, is determined by a simple calibration experiment in which the known stress at some point in a convenient model is plotted against the fringe value at that point under various loads. One of the most popular loading systems is diametral compression of a disc, when the relevant equation for the stress at the centre is

$$\sigma_p - \sigma_q = \frac{8P}{\pi Dt} \quad (6.6)$$

where P is the applied load, D is the disc diameter and t is the thickness.

Thus, comparing with the photoelastic eqn. (6.1),

$$\frac{nf}{t} = \frac{8P}{\pi Dt}$$

The slope of the load versus fringe order graph is given by

$$\frac{P}{n} = f \times \frac{\pi D}{8} \quad (6.7)$$

Hence f can be evaluated.

6.18. Fractional fringe order determination – compensation techniques

The accuracy of the photoelastic technique is limited, among other things, to the accuracy with which the fringe order at the point under investigation can be evaluated. It is not sufficiently accurate to count to the nearest whole number of fringes, and precise determination of fractions of fringe order at points lying between fringes is required. Conventional methods for determining these fractions of fringe order are termed *compensation techniques* and allow estimation of fringe orders to an accuracy of one-fiftieth of a fringe. The two methods most often used are the Tardy and Senarmont techniques. Before either technique can be adopted, the directions of the polariser and analyser must be aligned with the directions of the principal stresses at the point. This is achieved by rotating both units together in the plane polariscope arrangement until an *isoclinic* (§6.19) crosses the point. In most modern polariscopes facilities exist to couple the polariser and analyser together in order to facilitate synchronous rotation. The procedure for the two techniques then varies slightly.

(a) Tardy method

Quarter-wave plates are inserted at 45° to the polariser and analyser as the dark field circular polariscope set-up of Fig. 6.24. Normal fringe patterns will then be visible in the absence of isoclinics.

(b) Senarmont method

The polariser and analyser are rotated through a further 45° retaining the dark field, thus moving the polarising axes at 45° to the principal stress directions at the point. Only one quarter-wave plate is then inserted between the model and the analyser and rotated to again achieve a dark field. The normal fringe pattern is then visible as with the Tardy method.

Thus, having identified the integral value n of the fringe order at the point, i.e. between 1 and 2, or 2 and 3, for instance, the fractional part can now be established for both methods in the same way.

The analyser is rotated on its own to produce movement of the fringes. In particular, the nearest *lower order* of fringe is moved to the point of interest and the angle θ moved by the analyser recorded.

The fringe order at the chosen point is then $n + \frac{\theta}{180^\circ}$.

N.B.–Rotation of the analyser in the opposite direction ϕ° would move the nearest *highest order* fringe ($n + 1$) back to the point. In this case the fringe order at the point would be

$$(n + 1) - \frac{\phi}{180^\circ}$$

It can be shown easily by trial that the sum of the two angles θ and ϕ is always 180°

There is little to choose between the two methods in terms of accuracy; some workers prefer to use Tardy, others to use Senarmont.

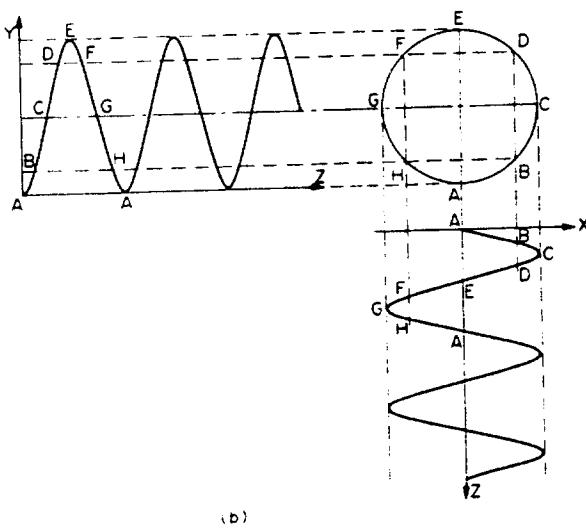
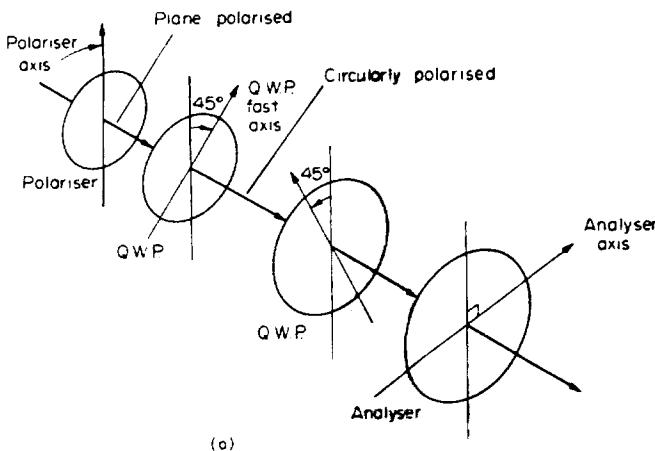


Fig. 6.24. (a) Circular polariscope arrangement. Isoclines are removed optically by inserting quarter-wave plates (Q.W.P.) with optical axes at 45° to those of the polariser and analyser. Circularly polarised light is produced. (Merrow.) (b) Graphical construction for the addition of two rays at right angles a quarter-wavelength out of phase, producing resultant circular envelope, i.e. circularly polarised light.

6.19. Isoclines – circular polarisation

If plane-polarised light is used for photoelastic studies as suggested in the preceding text, the fringes or isochromatics will be partially obscured by a set of black lines known as isoclinics (Fig. 6.25). With the coloured isochromatics of a white light source, these are easily identified, but with a monochromatic source confusion can easily arise between the black fringes and the black isoclinics.

It is therefore convenient to use a different optical system which eliminates the isoclinics but retains the basic fringe pattern. The procedure adopted is outlined below.

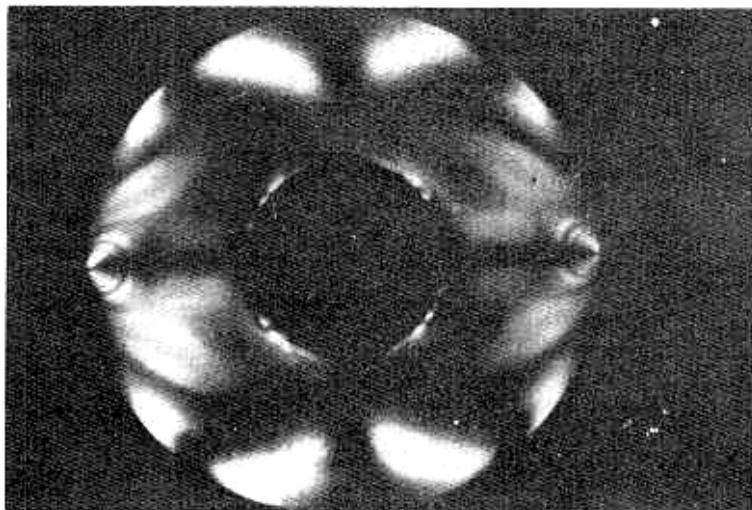


Fig. 6.25. Hollow disc subjected to diametral compression as in Fig. 6.18(a) but in this case showing the isoclinics superimposed.

An *isoclinic* line is a locus of points at which the principal stresses have the same inclination; the 20° isoclinic, for example, passes through all points at which the principal stresses are inclined at 20° to the vertical and horizontal (Fig. 6.26). Thus isoclinics are not peculiar to photoelastic studies; it is simply that they have a particular relevance in this case and they are readily visualised. For the purpose of this introduction it is sufficient to note that they are used as the basis for construction of *stress trajectories* which show the directions of the principal stresses at all points in the model, and hence in the component. Further details may be found in the relevant standard texts.

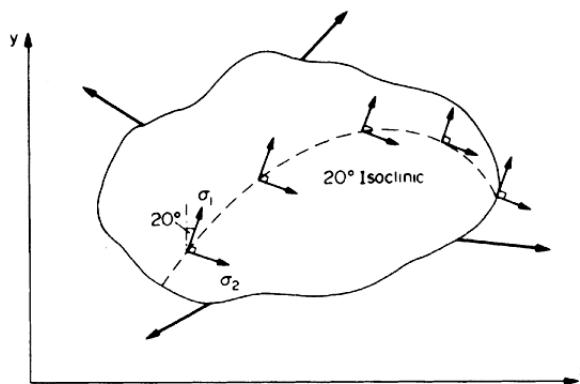


Fig. 6.26. The 20° isoclinic in a body subjected to a general stress system. The isoclinic is given by the locus of all points at which the principal stresses are inclined at 20° to the reference x and y axes.

To prevent the isoclinics interfering with the analysis of stress magnitudes represented by the basic fringe pattern, they are removed optically by inserting quarter-wave plates with

their axes at 45° to those of the polariser and analyser as shown in Fig. 6.24. These eliminate all unidirectional properties of the light by converting it into *circularly polarised* light. The amount of interference between the component rays emerging from the model, and hence the fringe patterns, remains unchanged and is now clearly visible in the absence of the isoclinics.

6.20. Stress separation procedures

The photoelastic technique has been shown to provide principal stress difference and hence maximum shear stresses at all points in the model, boundary stress values and stress directions. It has also been noted that there are occasions where the separate values of the principal stresses are required at points other than at the boundary, e.g. in the design of components using brittle materials. In this case it is necessary to employ one of the many *stress separation* procedures which are available. It is beyond the scope of this section to introduce these in detail, and full information can be obtained if desired from standard texts.^(8,9,11) The principal techniques which find most application are (a) the oblique incidence method, and (b) the shear slope or "shear difference" method.

6.21. Three-dimensional photoelasticity

In the preceding text, reference has been made to models of uniform thickness, i.e. two-dimensional models. Most engineering problems, however, arise in the design of components which are three-dimensional. In such cases the stresses vary not only as a function of the shape in any one plane but also throughout the "thickness" or third dimension. Often a proportion of the more simple three-dimensional model or loading cases can be represented by equivalent two-dimensional systems, particularly if the models are symmetrical, but there remains a greater proportion which cannot be handled by the two-dimensional approach. These, however, can also be studied using the photoelastic method by means of the so-called *stress-freezing* technique.

Three-dimensional photoelastic models constructed from the same birefringent material introduced previously are loaded, heated to a critical temperature and cooled very slowly back to room temperature. It is then found that a fringe pattern associated with the elastic stress distribution in the component has been locked or "frozen" into the model. It is then possible to cut the model into thin slices of uniform thickness, each slice then being examined as if it were a two-dimensional model. Special procedures for model manufacture, slicing of the model and fringe interpretation are required, but these are readily obtained with practice.

6.22. Reflective coating technique⁽¹²⁾

A special adaptation of the photoelastic technique utilises a thin sheet of photoelastic material which is bonded onto the surface of a metal component using a special adhesive containing an aluminium pigment which produces a reflective layer. Polarised light is directed onto the photoelastic coating and viewed through an analyser after reflection off the metal surface using a *reflection polariscope* as shown in Fig. 6.27.

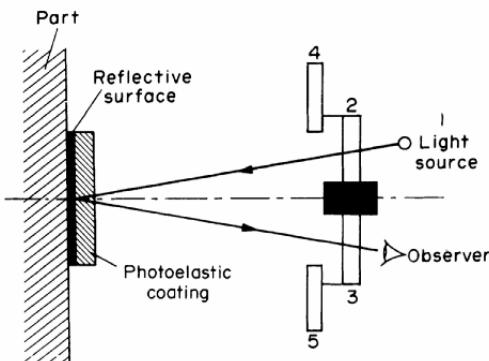


Fig. 6.27. Reflection polariscope principle and equipment.

A fringe pattern is observed which relates to the strain in the metal component. The technique is thus no longer a model technique and allows the evaluation of strains under loading conditions. Static and dynamic loading conditions can be observed, the latter with the aid of a stroboscope or high-speed camera, and the technique gives a full field view of the strain distribution in the surface of the component. Unlike the transmission technique, however, it gives no information as to the stresses *within* the material.

Standard photoelastic sheet can be used for bonding to flat components, but special casting techniques are available which enable the photoelastic material to be obtained in a partially polymerised, very flexible, stage, and hence allows it to be contoured or moulded around

complex shapes without undue thickness changes. After a period has been allowed for complete polymerisation to occur in the moulded position, the sheet is removed and bonded firmly back into place with the reflective adhesive.

The reflective technique is particularly useful for the observation of service loading conditions over wide areas of structure and is often used to highlight the stress concentration positions which can subsequently become the subject of detailed strain-gauge investigations.

6.23. Other methods of strain measurement

In addition to the widely used methods of experimental stress analysis or strain measurement covered above, there are a number of lesser-used techniques which have particular advantages in certain specialised conditions. These techniques can be referred to under the general title of grid methods, although in some cases a more explicit title would be "interference methods".

The standard **grid technique** consists of marking a grid, either mechanically or chemically, on the surface of the material under investigation and measuring the distortions of this grid under strain. A direct modification of this procedure, known as the "**replica**" **technique**, involves the firing of special pellets from a gun at the grid both before and during load. The surface of the pellets are coated with "Woods metal" which is heated in the gun prior to firing. Replicas of the undeformed and deformed grids are then obtained in the soft metal on contact with the grid-marked surface. These are viewed in a vernier comparison microscope to obtain strain readings.

A further modification of the grid procedure, known as the **moiré technique**. superimposes the deformed grid on an undeformed master (or vice versa). An interference pattern, known as **moiré fringes**, similar to those obtained when two layers of net curtain are superimposed, is produced and can be analysed to yield strain values.

X-rays can be used to obtain surface strain values from measurements of crystal lattice deformation. **Acoustoelasticity**, based on a principle similar to photoelasticity but using polarised ultrasonic sound waves, has been proposed but is not universally accepted to date. **Holography**, using the laser as a source of coherent light, and again relying on the interference obtained between holograms of deformed and undeformed components, has recently created considerable interest, but none of these techniques appear at the moment to represent a formidable challenge to the major techniques listed earlier.

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CHAPTER 7

CIRCULAR PLATES AND DIAPHRAGMS

Summary

The slope and deflection of circular plates under various loading and support conditions are given by the fundamental deflection equation

$$\frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{dy}{dr} \right) \right] = -\frac{Q}{D}$$

where y is the deflection at radius r ; dy/dr is the slope θ at radius r ; Q is the applied load or shear force per unit length, usually given as a function of r ; D is a constant termed the “flexural stiffness” or “flexural rigidity” $= Et^3/[12(1 - v^2)]$ and t is the plate thickness.

For applied uniformly distributed load (i.e. pressure q) the equation becomes

$$\frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{dy}{dr} \right) \right] = -\frac{qr}{2D}$$

For central concentrated load F

$$Q = \frac{F}{2\pi r} \text{ and the right-hand-side becomes } -\frac{F}{2\pi r D}$$

For axisymmetric non-uniform pressure (e.g. impacting gas or water jet)

$$q = K/r \text{ and the right-hand-side becomes } -K/2D$$

The *bending moments per unit length* at any point in the plate are:

$$M_r = M_{xy} = D \left[\frac{d\theta}{dr} + v \frac{\theta}{r} \right]$$

$$M_z = M_{yz} = D \left[v \frac{d\theta}{dr} + \frac{\theta}{r} \right]$$

Similarly, the *radial and tangential stresses* at any radius r are given by:

$$\text{radial stress } \sigma_r = \frac{Eu}{(1 - v^2)} \left[\frac{d\theta}{dr} + v \frac{\theta}{r} \right]$$

$$\text{tangential stress } \sigma_z = \frac{Eu}{(1 - v^2)} \left[v \frac{d\theta}{dr} + \frac{\theta}{r} \right]$$

Alternatively,

$$\sigma_r = \frac{12u}{t^3} M_r \quad \text{and} \quad \sigma_z = \frac{12u}{t^3} M_z$$

For a **circular plate**, radius R , *freely supported* at its edge and subjected to a *load* F distributed around a circle radius R_1

$$\text{and } \begin{aligned} y_{\max} &= \frac{F}{8\pi D} \left[\frac{(3+\nu)}{2(1+\nu)} (R^2 - R_1^2) - R_1^2 \log_e \frac{R}{R_1} \right] \\ \sigma_{r_{\max}} &= \frac{3F}{4\pi t^2} \left[2(1+\nu) \log_e \frac{R}{R_1} + (1-\nu) \frac{(R^2 - R_1^2)}{R^2} \right] \\ &= \sigma_{z_{\max}} \end{aligned}$$

Table 7.1. Summary of maximum deflections and stresses.

Loading condition	Maximum deflection (y_{\max})	Maximum stresses	
		$\sigma_{r_{\max}}$	$\sigma_{z_{\max}}$
Uniformly loaded, edges clamped	$\frac{3qR^4}{16Et^3}(1-\nu^2)$	$\frac{3qR^2}{4t^2}$	$\frac{3qR^2}{8t^2}(1+\nu)$
Uniformly loaded, edges freely supported	$\frac{3qR^4}{16Et^3}(5+\nu)(1-\nu)$	$\frac{3qR^2}{8t^2}(3+\nu)$	$\frac{3qR^2}{8t^2}(3+\nu)$
Central load F , edges clamped	$\frac{3FR^2}{4\pi Et^3}(1-\nu^2)$	$\frac{3F}{2\pi t^2}$	$\frac{3\nu F}{2\pi t^2}$
Central load F , edges freely supported	$\frac{3FR^2}{4\pi Et^3}(3+\nu)(1-\nu)$	From $\frac{3F}{2\pi t^2}(1+\nu) \log_e \frac{R}{r}$	From $\frac{3F}{2\pi t^2} \left[(1+\nu) \log_e \frac{R}{r} + (1-\nu) \right]$

For an **annular ring**, *freely supported* at its outside edge, with total *load* F applied around the inside radius R_1 , the maximum stress is tangential at the inside radius,

$$\text{i.e. } \sigma_{z_{\max}} = \frac{3F(1+\nu)}{\pi t^2} \left[\frac{R^2}{(R-R_1)} \log_e \frac{R}{R_1} \right]$$

If the outside edge is *clamped* the maximum stress becomes

$$\sigma_{\max} = \frac{3F}{2\pi t^2} \left[\frac{(R^2 - R_1^2)}{R^2} \right]$$

For **thin membranes** subjected to *uniform pressure* q the maximum deflection is given by

$$y_{\max} = 0.662 R \left[\frac{qR}{Et} \right]^{1/3}$$

For **rectangular plates** subjected to *uniform loads* the maximum deflection and bending moments are given by equations of the form

$$y_{\max} = \alpha \frac{qb^4}{Et^3}$$

$$M = \beta qb^2$$

the constants α and β depending on the method of support and plate dimensions. Typical values are listed later in Tables 7.3 and 7.4.

A. CIRCULAR PLATES

7.1. Stresses

Consider the portion of a thin plate or diaphragm shown in Fig. 7.1 bent to a radius R_{XY} in the XY plane and R_{YZ} in the YZ plane. The relationship between stresses and strains in a three-dimensional strain system is given by eqn. (7.2),[†]

$$\varepsilon_x = \frac{1}{E} [\sigma_x - \nu \sigma_y - \nu \sigma_z]$$

$$\varepsilon_z = \frac{1}{E} [\sigma_z - \nu \sigma_x - \nu \sigma_y]$$

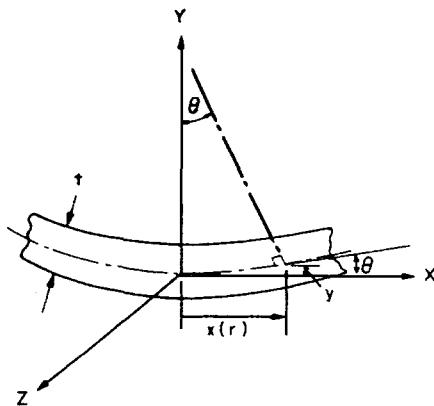


Fig. 7.1.

Now for thin plates, provided deflections are restricted to no greater than half the plate thickness,[‡] the direct stress in the Y direction may be assumed to be zero and the above equations give

$$\sigma_x = \frac{E}{(1 - \nu^2)} [\varepsilon_x + \nu \varepsilon_z] \quad (7.1)$$

$$\sigma_z = \frac{E}{(1 - \nu^2)} [\varepsilon_z + \nu \varepsilon_x] \quad (7.2)$$

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

[‡] S. Timoshenko, *Theory of Plates and Shells*, 2nd edn., McGraw-Hill, 1959.

If u is the distance of any fibre from the neutral axis, then, for pure bending in the XY and YZ planes,

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R} \quad \text{and} \quad \frac{\sigma}{E} = \frac{u}{R} = \varepsilon$$

$$\therefore \varepsilon_x = \frac{u}{R_{XY}} \quad \text{and} \quad \varepsilon_z = \frac{u}{R_{YZ}}$$

Now $\frac{1}{R} = \frac{d^2y}{dx^2}$ and, for small deflections, $\frac{du}{dx} = \tan \theta = \theta$ (radians).

$$\therefore \frac{1}{R_{XY}} = \frac{d^2y}{dx^2} = \frac{d\theta}{dx}$$

$$\text{and} \quad \varepsilon_x = u \frac{d\theta}{dx} \quad (= \text{radial strain}) \quad (7.3)$$

Consider now the diagram Fig. 7.2 in which the radii of the concentric circles through C_1 and D_1 on the unloaded plate increase to $[(x + dx) + (\theta + d\theta)u]$ and $[x + u\theta]$, respectively, when the plate is loaded.

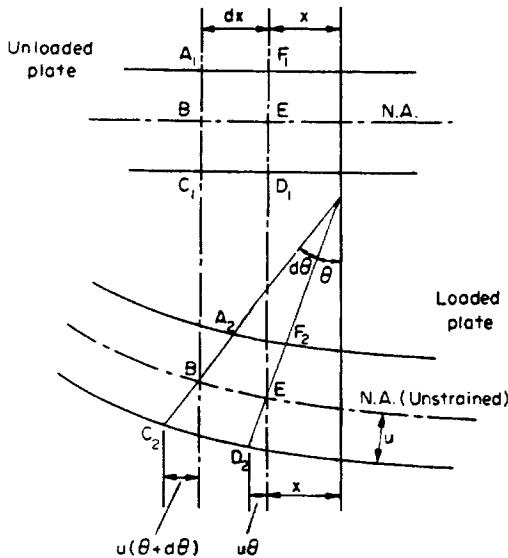


Fig. 7.2.

Circumferential strain at D_2

$$\begin{aligned} \varepsilon_z &= \frac{2\pi(x + u\theta) - 2\pi x}{2\pi x} \\ &= \frac{u\theta}{x} \quad (= \text{circumferential strain}) \end{aligned} \quad (7.4)$$

Substituting eqns. (7.3) and (7.4) in eqns. (7.1) and (7.2) yields

$$\sigma_x = \frac{E}{(1 - v^2)} \left[u \frac{d\theta}{dx} + v \frac{u\theta}{x} \right]$$

i.e.

$$\sigma_x = \frac{Eu}{(1 - v^2)} \left[\frac{d\theta}{dx} + v \frac{\theta}{x} \right] \quad (7.5)$$

Similarly,

$$\sigma_z = \frac{Eu}{(1 - v^2)} \left[\frac{\theta}{x} + v \frac{d\theta}{dx} \right] \quad (7.6)$$

Thus we have equations for the stresses in terms of the slope θ and rate of change of slope $d\theta/dx$. We shall now proceed to evaluate the bending moments in the two planes in similar form and hence to the procedure for determination of θ and $d\theta/dx$ from a knowledge of the applied loading.

7.2. Bending moments

Consider the small section of plate shown in Fig. 7.3, which is of unit length. Defining the moments M as *moments per unit length* and applying the simple bending theory,

$$M = \frac{\sigma I}{y} = \frac{\sigma}{u} \left[\frac{1 \times t^3}{12} \right] = \frac{\sigma t^3}{12u}$$

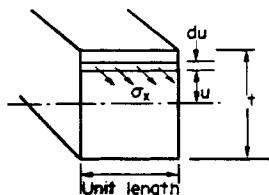


Fig. 7.3.

Substituting eqns. (7.5) and (7.6),

$$M_{XY} = \frac{Et^3}{12(1 - v^2)} \left[\frac{d\theta}{dx} + v \frac{\theta}{x} \right] \quad (7.7)$$

Now $\frac{Et^3}{12(1 - v^2)}$ = D is a constant and termed the *flexural stiffness*

so that

$$M_{XY} = D \left[\frac{d\theta}{dx} + v \frac{\theta}{x} \right] \quad (7.8)$$

and, similarly,

$$M_{YZ} = D \left[\frac{\theta}{x} + v \frac{d\theta}{dx} \right] \quad (7.9)$$

It is now possible to write the stress equations in terms of the applied moments,

i.e.

$$\sigma_x = M_{XY} \frac{12u}{t^3} \quad (7.10)$$

$$\sigma_z = M_{YZ} \frac{12u}{t^3} \quad (7.11)$$

7.3. General equation for slope and deflection

Consider now Fig. 7.4 which shows the forces and moments per unit length acting on a small element of the plate subtending an angle $\delta\phi$ at the centre. Thus M_{XY} and M_{YZ} are the moments per unit length in the two planes as described above and Q is the shearing force per unit length in the direction OY .

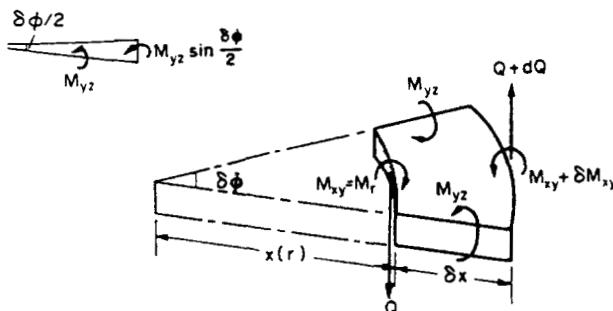


Fig. 7.4. Small element of circular plate showing applied moments and forces per unit length.

For equilibrium of moments in the radial XY plane, taking moments about the outside edge,

$$(M_{XY} + \delta M_{XY})(x + \delta x)\delta\phi - M_{XY}x\delta\phi - 2M_{YZ}\delta x \sin \frac{1}{2}\delta\phi + Qx\delta\phi\delta x = 0$$

which, neglecting squares of small quantities, reduces to

$$M_{XY}\delta x + \delta M_{XY}x - M_{YZ}\delta x + Qx\delta x = 0$$

In the limit, therefore,

$$M_{XY} + x \frac{dM_{XY}}{dx} - M_{YZ} = -Qx$$

Substituting eqns. (7.8) and (7.9), and simplifying

$$\frac{d^2\theta}{dx^2} + \frac{1}{x} \frac{d\theta}{dx} - \frac{\theta}{x^2} = -\frac{Q}{D}$$

This may be re-written in the form

$$\frac{d}{dx} \left[\frac{1}{x} \frac{d(x\theta)}{dx} \right] = -\frac{Q}{D} \quad (7.12)$$

This is then the general equation for slopes and deflections of circular plates or diaphragms. Provided that the applied loading Q is known as a function of x the expression can be treated

in a similar manner to the equation

$$M = EI \frac{d^2 y}{dx^2}$$

used in the Macaulay beam method, i.e. it may be successively integrated to determine θ , and hence y , in terms of constants of integration, and these can then be evaluated from known end conditions of the plate.

It will be noted that the expressions have been derived using cartesian coordinates (X , Y and Z). For circular plates, however, it is convenient to replace the variable x with the general radius r when the equations derived above may be re-written as follows:

$$\frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{dy}{dr} \right) \right] = -\frac{Q}{D} \quad (7.13)$$

radial stress

$$\sigma_r = \frac{Eu}{(1-\nu^2)} \left[\frac{d\theta}{dr} + \nu \frac{\theta}{r} \right] \quad (7.14)$$

tangential stress

$$\sigma_\theta = \frac{Eu}{(1-\nu^2)} \left[\nu \frac{d\theta}{dr} + \frac{\theta}{r} \right] \quad (7.15)$$

moments

$$M_r = D \left[\frac{d\theta}{dr} + \nu \frac{\theta}{r} \right] \quad (7.16)$$

$$M_\theta = D \left[\nu \frac{d\theta}{dr} + \frac{\theta}{r} \right] \quad (7.17)$$

In the case of applied uniformly distributed loads, i.e. pressures q , the effective shear load Q per unit length for use in eqn. (7.13) is found as follows.

At any radius r , for equilibrium,

$$Q \times 2\pi r = q \times \pi r^2$$

i.e.

$$Q = \frac{qr}{2}$$

Thus for applied pressures eqn. (7.13) may be re-written

$$\frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{dy}{dr} \right) \right] = -\frac{qr}{2D} \quad (7.18)$$

7.4. General case of a circular plate or diaphragm subjected to combined uniformly distributed load q (pressure) and central concentrated load F

For this general case the equivalent shear Q per unit length is given by

$$Q \times 2\pi r = q \times \pi r^2 + F$$

$$\therefore Q = \frac{qr}{2} + \frac{F}{2\pi r}$$

Substituting in eqn. (7.18)

$$\frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{dy}{dr} \right) \right] = \left[-\frac{qr}{2} - \frac{F}{2\pi r} \right] \frac{1}{D}$$

Integrating,

$$\begin{aligned} \frac{1}{r} \frac{d}{dr} \left(r \frac{dy}{dr} \right) &= -\frac{1}{D} \int \left[\frac{qr}{2} + \frac{F}{2\pi r} \right] dr \\ &= -\frac{1}{D} \left[\frac{qr^2}{4} + \frac{Fr}{2\pi} \log_e r \right] + C_1 \end{aligned}$$

$$\therefore \frac{d}{dr} \left(r \frac{dy}{dr} \right) = -\frac{1}{D} \left[\frac{qr^3}{4} + \frac{Fr}{2\pi} \log_e r \right] + C_1 r$$

$$\text{Integrating, } r \frac{dy}{dr} = -\frac{1}{D} \left[\frac{qr^4}{16} + \frac{F}{2\pi} \left\{ \frac{r^2}{2} \log_e r - \frac{r^2}{4} \right\} \right] + \frac{C_1 r^2}{2} + C_2$$

$$\therefore \text{slope } \theta = \frac{dy}{dr} = -\frac{qr^3}{16D} - \frac{Fr}{8\pi D} [2 \log_e r - 1] + C_1 \frac{r}{2} + \frac{C_2}{r} \quad (7.19)$$

Integrating again and simplifying,

$$\text{deflection } y = -\frac{qr^4}{64D} - \frac{Fr^2}{8\pi D} [\log_e r - 1] + C_1 \frac{r^2}{4} + C_2 \log_e r + C_3 \quad (7.20)$$

The values of the constants of integration will be determined from known end conditions of the plate; slopes and deflections at any radius can then be evaluated. As an example of the procedure used it is now convenient to consider a number of standard loading cases and to determine the maximum deflections and stresses for each.

7.5. Uniformly loaded circular plate with edges clamped

The relevant fundamental equation for this loading condition has been shown to be

$$\frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{dy}{dr} \right) \right] = -\frac{qr}{2D}$$

$$\text{Integrating, } \frac{1}{r} \frac{d}{dr} \left(r \frac{dy}{dr} \right) = -\frac{qr^2}{4D} + C_1$$

$$\frac{d}{dr} \left(r \frac{dy}{dr} \right) = -\frac{qr^3}{4D} + C_1 r$$

$$\text{Integrating, } r \frac{dy}{dr} = -\frac{qr^4}{16D} + C_1 \frac{r^2}{2} + C_2$$

$$\therefore \text{slope } \theta = \frac{dy}{dr} = -\frac{qr^3}{16D} + C_1 \frac{r}{2} + \frac{C_2}{r} \quad (7.21)$$

Integrating,

$$\text{deflection } y = \frac{-qr^4}{64D} + \frac{C_1 r^2}{4} + C_2 \log_e r + C_3 \quad (7.22)$$

Now if the slope θ is not to be infinite at the centre of the plate, $C_2 = 0$.

Taking the origin at the centre of the deflected plate, $y = 0$ when $r = 0$.

Therefore, from eqn. (7.22), $C_3 = 0$.

At the outside, clamped edge where $r = R$, $\theta = dy/dr = 0$.

Therefore substituting in the slope eqn. (6.21),

$$-\frac{qR^3}{16D} + \frac{C_1 R}{2} = 0$$

$$\therefore C_1 = \frac{qR^2}{8D}$$

The maximum deflection of the plate will be at the centre, but since this has been used as the origin the deflection equation will yield $y = 0$ at $r = 0$; indeed, this was one of the conditions used to evaluate the constants. We must therefore determine the equivalent amount by which the end supports are assumed to move up relative to the "fixed" centre.

Substituting $r = R$ in the deflection eqn. (7.22) yields

$$\text{maximum deflection} = -\frac{qR^4}{64D} + \frac{qR^4}{32D} = \frac{qR^4}{64D}$$

The positive value indicates, as usual, upwards deflection of the ends relative to the centre, i.e. along the positive y direction. The central deflection of the plate is thus, as expected, in the same direction as the loading, along the negative y direction (downwards).

Substituting for D ,

$$\begin{aligned} y_{\max} &= \frac{qR^4}{64} \left[\frac{12(1 - \nu^2)}{Et^3} \right] \\ &= \frac{3qR^4}{16Et^3}(1 - \nu^2) \end{aligned} \quad (7.23)$$

Similarly, from eqn. (7.21),

$$\begin{aligned} \text{slope } \theta &= -\frac{qr^3}{16D} + \frac{qR^2 r}{16D} = -\frac{qr}{16D}[r^2 - R^2] \\ \therefore \frac{d\theta}{dr} &= -\frac{3qr^2}{16D} + \frac{qR^2}{16D} = -\frac{q}{16D}[3r^2 - R^2] \end{aligned}$$

Now, from eqn. (7.14)

$$\begin{aligned} \sigma_r &= \frac{Eu}{(1 - \nu^2)} \left[\frac{d\theta}{dr} + \nu \frac{\theta}{r} \right] \\ &= \frac{Eu}{(1 - \nu^2)} \left[-\frac{qr^2}{16D}(3 + \nu) + \frac{qR^2}{16D}(1 + \nu) \right] \end{aligned}$$

The maximum stress for the clamped edge condition will thus be obtained at the edge where $r = R$ and at the surface of the plate where $u = t/2$,

$$\text{i.e. } \sigma_{r_{\max}} = \frac{E}{(1-\nu^2)} \frac{t}{2} \frac{2qR^2}{16D} = \frac{3qR^2}{4t^2} \quad (7.24)$$

N.B.-It is not possible to determine the maximum stress by equating $d\sigma_r/dr$ to zero since this only gives the point where the slope of the σ_r curve is zero (see Fig. 7.7). The value of the stress at this point is not as great as the value at the edge.

Similarly,

$$\begin{aligned} \sigma_z &= \frac{Eu}{(1-\nu^2)} \left[\frac{\theta}{r} + \nu \frac{d\theta}{dr} \right] \\ &= \frac{Eu}{(1-\nu^2)} \left[-\frac{qr^2}{16D}(3\nu+1) + \frac{qR^2}{16D}(1+\nu) \right] \end{aligned}$$

Unlike σ_r , this has a maximum value when $r = 0$, i.e. at the centre.

$$\begin{aligned} \sigma_{z_{\max}} &= \frac{E}{(1-\nu^2)} \frac{t}{2} \frac{qR^2}{16D}(1+\nu) \\ &= \frac{3qR^2}{8t^2}(1+\nu) \end{aligned} \quad (7.25)$$

7.6. Uniformly loaded circular plate with edges freely supported

Since the loading, and hence fundamental equation, is the same as for §7.4, the slope and deflection equations will be of the same form, i.e. eqns (7.21) and (7.22) will apply. Further, the constants C_2 and C_3 will again be zero for the same reasons as before and only one new condition to solve for the constant C_1 is required.

Here we must make use of the fact that the bending moment is always zero at any free support,

$$\text{i.e. at } r = R, \quad M_r = 0$$

Therefore from eqn. (7.16),

$$\begin{aligned} D \left[\frac{d\theta}{dr} + \nu \frac{\theta}{r} \right] &= 0 \\ \therefore \frac{d\theta}{dr} &= -\nu \frac{\theta}{r} \end{aligned}$$

Substituting from eqn. (7.21) with $r = R$ and $C_2 = 0$,

$$\begin{aligned} -\frac{3qR^2}{16D} + \frac{C_1}{2} &= -\nu \left[-\frac{qr^2}{16D} + \frac{C_1}{2} \right] \\ \therefore C_1 &= \frac{qR^2}{8D} \left[\frac{(3+\nu)}{(1+\nu)} \right] \end{aligned}$$

The maximum deflection is at the centre and again equal to the deflection of the supports relative to the centre.

Substituting for the constants with $r = R$ in eqn. (7.22),

$$\begin{aligned}\text{maximum deflection} &= -\frac{qR^4}{64D} + \frac{qR^2}{8D} \frac{(3+\nu)}{(1+\nu)} \frac{R^2}{4} \\ &= \frac{qR^4}{64D} \left[\frac{(5+\nu)}{(1+\nu)} \right]\end{aligned}$$

i.e. substituting for D ,

$$y_{\max} = \frac{3qR^4}{16Et^3} (5+\nu)(1-\nu) \quad (7.26)$$

With $\nu = 0.3$ this value is approximately *four times* that for the clamped edge condition.

As before, the stresses are obtained from eqns. (7.14) and (7.15) by substituting for $d\theta/dr$ and θ/r from eqn. (7.21),

$$\sigma_r = \frac{Eu}{(1-\nu^2)} \left[-\frac{qr^2}{16D}(3+\nu) + \frac{qR^2}{16D}(3+\nu) \right]$$

This gives a maximum stress at the centre where $r = 0$

$$\begin{aligned}\sigma_{r_{\max}} &= \frac{E}{(1-\nu^2)} \frac{t}{2} \frac{qR^2}{16D} (3+\nu) \\ &= \frac{3qR^2}{8t^2} (3+\nu)\end{aligned}$$

Similarly, $\sigma_{z_{\max}} = \frac{3qR^2}{8t^2} (3+\nu)$ also at the centre

i.e. for a uniformly loaded circular plate with edges freely supported,

$$\sigma_{r_{\max}} = \sigma_{z_{\max}} = \frac{3qR^2}{8t^2} (3+\nu) \quad (7.27)$$

7.7. Circular plate with central concentrated load F and edges clamped

For a central concentrated load,

$$Q \times 2\pi r = F$$

$$\therefore Q = \frac{F}{2\pi r}$$

The fundamental equation for slope and deflection is, therefore,

$$\frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{dy}{dr} \right) \right] = -\frac{F}{2\pi r D}$$

$$\text{Integrating, } \frac{1}{r} \frac{d}{dr} \left(r \frac{dy}{dr} \right) = -\frac{F}{2\pi D} \log_e r + C_1$$

$$\frac{d}{dr} \left(r \frac{dy}{dr} \right) = -\frac{Fr}{2\pi D} \log_e r + C_1 r$$

Integrating,

$$r \frac{dy}{dr} = -\frac{F}{2\pi D} \left[\frac{r^2}{2} \log_e r - \frac{r^2}{4} \right] + \frac{C_1 r^2}{2} + C_2$$

∴

$$\theta = \frac{dy}{dr} = -\frac{F}{2\pi D} \left[\frac{r}{2} \log_e r - \frac{r}{4} \right] + C_1 \frac{r}{2} + \frac{C_2}{r} \quad (7.28)$$

Integrating,

$$y = -\frac{Fr^2}{8\pi D} [\log_e r - 1] + \frac{C_1 r^2}{4} + C_2 \log_e r + C_3 \quad (7.29)$$

Again, taking the origin at the centre of the deflected plate as shown in Fig. 7.5, the following conditions apply:

For a non-infinite slope at the centre $C_2 = 0$ and at $r = 0$, $y = 0$, ∴ $C_3 = 0$.

Also, at $r = R$, slope $\theta = dy/dr = 0$.

Therefore from eqn. (7.28),

$$\frac{F}{2\pi D} \left[\frac{R}{2} \log_e R - \frac{R}{4} \right] = \frac{C_1 R}{2}$$

$$\therefore \frac{F}{\pi D} \left[\frac{\log_e R}{2} - \frac{1}{4} \right] = C_1$$

The maximum deflection will be at the centre and again equivalent to that obtained when $r = R$, i.e. from eqn. (7.29),

$$\begin{aligned} \text{maximum deflection} &= -\frac{FR^2}{8\pi D} [\log_e R - 1] + \frac{FR^2}{4\pi D} \left[\frac{\log_e R}{2} - \frac{1}{4} \right] \\ &= \frac{FR^2}{16\pi D} [-2 \log_e R + 2 + 2 \log_e R - 1] \\ &= \frac{FR^2}{16\pi D} \end{aligned}$$

Substituting for D ,

$$\begin{aligned} y_{\max} &= \frac{FR^2}{16\pi} \frac{12(1 - v^2)}{Et^3} \\ &= \frac{3FR^2}{4\pi Et^3} (1 - v^2) \end{aligned} \quad (7.30)$$

Again substituting for $d\theta/dr$ and θ/r from eqn. (7.28) into eqns (7.14) and (7.15) yields

$$\sigma_{r_{\max}} = \frac{3F}{2\pi t^2} \quad (7.31)$$

$$\sigma_{z_{\max}} = \frac{3vF}{2\pi t^2} \quad (7.32)$$

7.8. Circular plate with central concentrated load F and edges freely supported

The fundamental equation and hence the slope and deflection expressions will be as for the previous section (§7.7),

$$\text{i.e. } \theta = \frac{dy}{dr} = -\frac{F}{2\pi D} \left[\frac{r}{2} \log_e r - \frac{r}{4} \right] + \frac{C_1 r}{2} \quad (7.33)$$

$$y = -\frac{Fr^2}{8\pi D} [\log_e r - 1] + \frac{C_1 r^2}{4} \quad (7.34)$$

constants C_2 and C_3 being zero as before.

As for the uniformly loaded plate with freely supported edges, the constant C_1 is determined from the knowledge that the bending moment M_r is zero at the free support,

$$\text{i.e. at } r = R, \quad M_r = 0$$

Therefore from eqn. (7.16),

$$D \left[\frac{d\theta}{dr} + \nu \frac{\theta}{r} \right] = 0 \quad \text{and} \quad \frac{d\theta}{dr} = -\nu \frac{\theta}{r}$$

and, substituting from eqn. (7.33) with $r = R$,

$$\begin{aligned} -\frac{F}{8\pi D} [2 \log_e R - 1] + \frac{C_1}{2} &= -\frac{\nu F}{8\pi D} [2 \log_e R - 1] - \frac{\nu C_1}{2} \\ \therefore \frac{C_1}{2} (1 + \nu) &= \frac{F}{8\pi D} [2(1 + \nu) \log_e R - (1 - \nu)] \\ C_1 &= \frac{F}{4\pi D} \left[2 \log_e R + \frac{(1 - \nu)}{(1 + \nu)} \right] \end{aligned}$$

As before, the maximum deflection is at the centre and equivalent to that obtained with $r = R$.

Substituting in eqn. (7.34),

$$\begin{aligned} \text{maximum deflection} &= \frac{FR^2}{8\pi D} [\log_e R - 1] + \frac{FR^2}{16\pi D} \left[2 \log_e R + \frac{(1 - \nu)}{(1 + \nu)} \right] \\ &= \frac{FR^2}{16\pi D} \frac{(3 + \nu)}{(1 + \nu)} \end{aligned}$$

Substituting for D

$$y_{\max} = \frac{3FR^2}{4\pi Et^3} (3 + \nu)(1 - \nu) \quad (7.35)$$

For $\nu = 0.3$ this is approximately 2.5 times that for the clamped edge condition.

From eqn. (7.14),

$$\sigma_r = \frac{Eu}{(1 - \nu^2)} \left[\frac{d\theta}{dr} + \nu \frac{\theta}{r} \right]$$

Substituting for $d\theta/dr$ and θ/r as above,

$$\sigma_r = \frac{Eu}{(1 - \nu^2)} \left[\frac{F}{4\pi D} (1 + \nu) \log_e \frac{R}{r} \right]$$

$$= \frac{3F}{2\pi t^2} (1 + \nu) \log_e \frac{R}{r} \quad (7.36)$$

Thus the radial stress σ_r will be zero at the edge and will rise to a maximum value (theoretically infinite) at the centre. However, in practice, load cannot be applied strictly at a point but must contact over a finite area. Provided this area is known the maximum stress can be calculated.

Similarly, from eqn. (7.15)

$$\sigma_z = \frac{Eu}{(1 - \nu^2)} \left[\nu \frac{d\theta}{dr} + \frac{\theta}{r} \right]$$

and, again substituting for $d\theta/dr$ and θ/r ,

$$\sigma_z = \frac{3F}{2\pi t^2} \left[(1 + \nu) \log_e \frac{R}{r} + (1 - \nu) \right] \quad (7.37)$$

7.9. Circular plate subjected to a load F distributed round a circle

Consider the circular plate of Fig. 7.5 subjected to a total load F distributed round a circle of radius R_1 . A solution is obtained to this problem by considering the plate as consisting of two parts $r < R_1$ and $r > R_1$, bearing in mind that the values of θ , y and M_r must be the same for both parts at the common radius $r = R_1$.

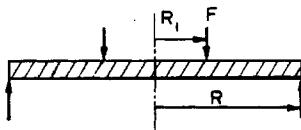


Fig. 7.5. Solid circular plate subjected to total load F distributed around a circle of radius R_1 .

Thus, for $r < R_1$, we have a plate with zero distributed load and zero central concentrated load,

i.e.

$$q = F = 0$$

Therefore from eqn. (7.20),

$$y = \frac{C_1 r^2}{4} + C_2 \log_e r + C_3$$

and from eqn. (7.19)

$$\theta = \frac{dy}{dr} = \frac{C_1 r}{2} + \frac{C_2}{r}$$

For non-infinite slope at the centre, $C_2 = 0$ and with the axis for deflections at the centre of the plate, $y = 0$ when $r = 0$, $\therefore C_3 = 0$.

Therefore for the inner portion of the plate

$$y = \frac{C_1 r^2}{4} \quad \text{and} \quad \theta = \frac{dy}{dr} = \frac{C_1 r}{2}$$

For the outer portion of the plate $r > R_1$ and eqn. (22.20) reduces to

$$y = -\frac{Fr^2}{8\pi D}[\log_e r - 1] + \frac{C'_1 r^2}{4} + C'_2 \log_e r + C'_3 \quad (7.38)$$

and from eqn. (7.19)

$$\theta = \frac{dy}{dr} = -\frac{Fr}{8\pi D}[2 \log_e r - 1] + \frac{C'_1 r}{2} + \frac{C'_2}{r} \quad (7.39)$$

Equating these values with those obtained for the inner portions,

$$\frac{C_1 R_1^2}{4} = -\frac{FR_1^2}{8\pi D}[\log_e R_1 - 1] + \frac{C'_1 R_1^2}{4} + C'_2 \log_e R_1 + C'_3$$

and

$$\frac{C_1 R_1}{2} = -\frac{FR_1}{8\pi D}[2 \log_e R_1 - 1] + \frac{C'_1 R_1}{2} + \frac{C'_2}{R_1}$$

Similarly, from (7.16), equating the values of M_r at the common radius R_1 yields

$$-\frac{F}{8\pi D}[2(1+\nu)\log_e R_1 + (1-\nu)] + \frac{C'_1}{2}(1+\nu) - \frac{C'_2(1-\nu)}{R_1^2} = \frac{C_1}{2}(1+\nu)$$

Further, with $M_r = 0$ at $r = R$, the outside edge, from eqn. (7.16)

$$-\frac{F}{8\pi D}[2(1+\nu)\log_e R + (1-\nu)] + \frac{C'_1}{2}(1+\nu) - \frac{C'_2(1-\nu)}{R^2} = 0 \quad (7.40)$$

There are thus four equations with four unknowns C_1 , C'_1 , C'_2 and C'_3 and a solution using standard simultaneous equation procedures is possible. Such a solution yields the following values:

$$C'_1 = \frac{F}{4\pi D} \left[2 \log_e R + \frac{(1-\nu)(R^2 - R_1^2)}{(1+\nu) R^2} \right]$$

$$C'_2 = -\frac{FR_1^2}{8\pi D}$$

$$C'_3 = \frac{FR_1^2}{8\pi D}[\log_e R_1 - 1]$$

The central deflection is found, as before, from the deflection of the edge, $r = R$, relative to the centre.

Substituting in eqn. (7.38) yields

$$y_{\max} = \frac{F}{8\pi D} \left[\frac{(3+\nu)}{2(1+\nu)}(R^2 - R_1^2) - R_1^2 \log_e \frac{R}{R_1} \right] \quad (7.41)$$

The maximum radial bending moment and hence radial stress occurs at $r = R_1$, giving

$$\sigma_{r_{\max}} = \frac{3F}{4\pi t^2} \left[2(1+\nu) \log_e \frac{R}{R_1} + (1-\nu) \frac{(R^2 - R_1^2)}{R^2} \right] \quad (7.42)$$

It can also be shown similarly that the maximum tangential stress is of equal value to the maximum radial stress.

7.10. Application to the loading of annular rings

The general eqns. (7.38) and (7.39) derived above apply also for annular rings with a total load F applied around the inner edge of radius R_1 as shown in Fig. 7.6.

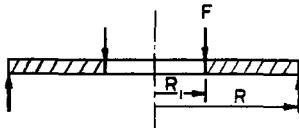


Fig. 7.6. Annular ring with total load F distributed around inner radius.

Here, however, the radial bending M_r is zero at both $r = R_1$ and $r = R$. Thus, applying the condition of eqn. (7.40) for both these radii yields

$$-\frac{F}{8\pi D}[2(1+\nu)\log_e R + (1-\nu)] + \frac{C_1}{2}(1+\nu) - \frac{C_2}{R^2}(1-\nu) = 0$$

and

$$-\frac{F}{8\pi D}[2(1+\nu)\log_e R_1 + (1-\nu)] + \frac{C_1}{2}(1+\nu) - \frac{C_2}{R_1^2}(1-\nu) = 0$$

Subtracting to eliminate C_1 gives

$$C_2 = \frac{F}{4\pi D} \left[\frac{(1+\nu)}{(1-\nu)} \frac{R^2 R_1^2}{(R^2 - R_1^2)} \log_e \frac{R}{R_1} \right]$$

and hence

$$C_1 = \frac{F}{4\pi D} \left[\frac{2(R^2 \log_e R - R_1^2 \log_e R_1)}{(R^2 - R_1^2)} + \frac{(1-\nu)}{(1+\nu)} \right]$$

It can then be shown that the maximum stress set up is the tangential stress at $r = R_1$ of value

$$\sigma_{z_{\max}} = \frac{3F(1+\nu)}{\pi t^2} \left[\frac{R^2}{(R^2 - R_1^2)} \right] \log_e \frac{R}{R_1} \quad (7.43)$$

If the **outside edge of the plate is clamped** instead of freely supported the maximum stress becomes

$$\sigma_{\max} = \frac{3F}{2\pi t^2} \left[\frac{(R^2 - R_1^2)}{R^2} \right]$$

7.11. Summary of end conditions

Axes can be selected to move with the plate as shown in Fig. 7.7(a) or stay at the initial, undeflected position Fig. 7.7(b).

For the former case, i.e. axes origin at the centre of the deflected plate, the end conditions which should be used for solution of the constants of integration are:

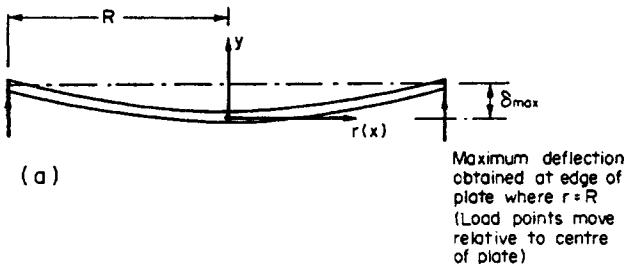


Fig. 7.7(a). Origin of reference axes taken to move with the plate.

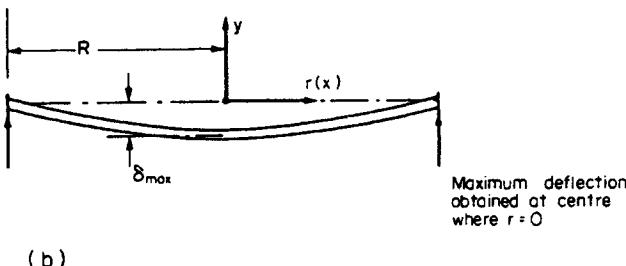


Fig. 7.7(b). Origin of reference axes remaining in the undeflected plate position.

Edges freely supported:

- (i) Slope θ and deflection y non-infinite at the centre. $\therefore C_2 = 0$.
- (ii) At $x = 0, y = 0$ giving $C_3 = 0$.
- (iii) At $x = R, M_{xy} = 0$; hence C_1 .

The maximum deflection is then that given at $x = R$.

Edges clamped:

- (i) Slope θ and deflection y non-infinite at the centre. $\therefore C_2 = 0$.
- (ii) At $x = 0, y = 0 \quad \therefore C_3 = 0$.
- (iii) At $x = R, \frac{dy}{dx} = 0$; hence C_1 .

Again the maximum deflection is that given at $x = R$.

7.12. Stress distributions in circular plates and diaphragms subjected to lateral pressures

It is now convenient to consider the stress distribution in plates subjected to lateral, uniformly distributed loads or pressures in more detail since this represents the loading condition encountered most often in practice.

Figures 7.8(a) and 7.8(b) show the radial and tangential stress distributions on the lower surface of a thin plate subjected to uniform pressure as given by the equations obtained in §§7.5 and 7.6.

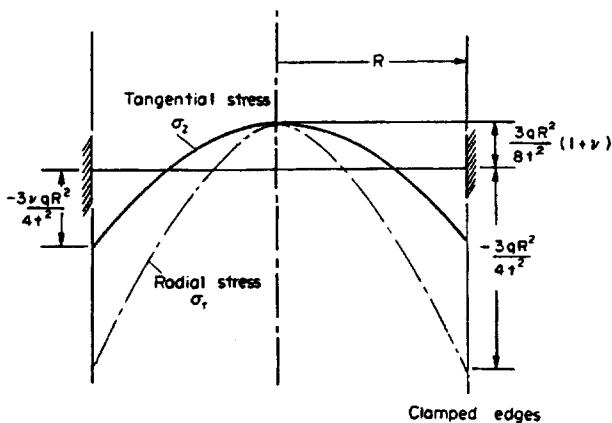


Fig. 7.8(a). Radial and tangential stress distributions in circular plates with clamped edges.

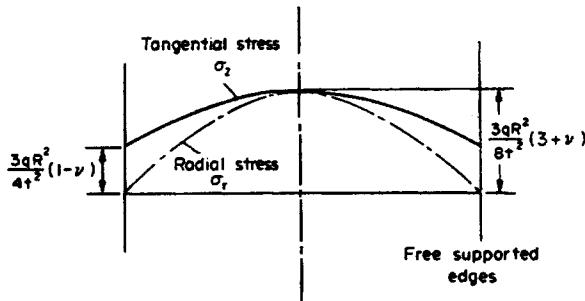


Fig. 7.8(b). Radial and tangential stress distributions in circular plates with freely supported edges.

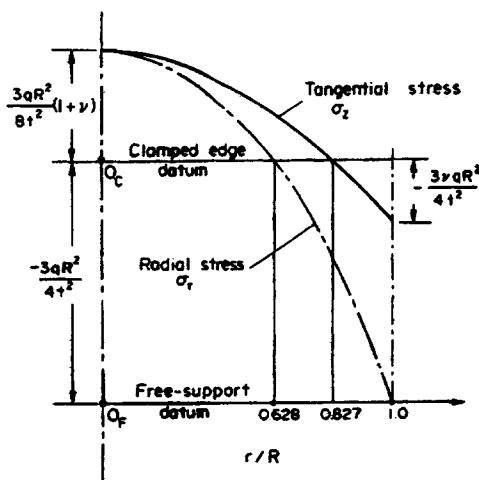


Fig. 7.9. Comparison of bending stresses in circular plates for clamped and freely supported edge conditions.

The two figures may be combined on to common axes as in Fig. 7.9 to facilitate comparison of the stress distributions for freely supported and clamped-edge conditions. Then if ordinates are measured from the horizontal axis through origin O_c , the curves give the values of radial and tangential stress for clamped-edge conditions.

Alternatively, measuring the ordinates from the horizontal axis passing through origin O_F in Fig. 7.9, i.e. adding to the clamped-edges stresses the constant value $\frac{3}{4}qR^2/t^2$, we obtain the stresses for a simply supported edge condition. The combined diagram clearly illustrates that a more favourable stress distribution is obtained when the edges of a plate are clamped.

7.13. Discussion of results – limitations of theory

The results of the preceding paragraphs are summarised in Table 7.1 at the start of the chapter. From this table the following approximate relationships are seen to apply:

- (1) The maximum deflection of a uniformly loaded circular plate with freely supported edges is approximately four times that for the clamped-edge condition.
- (2) Similarly, for a central concentrated load, the maximum deflection in the freely supported edge condition is 2.5 times that for clamped edges.
- (3) With clamped edges the maximum deflection for a central concentrated load is four times that for the equivalent u.d.l. (i.e. $F = q \times \pi R^2$) and the maximum stresses are doubled.
- (4) With freely supported edges, the maximum deflection for a central concentrated load is 2.5 times that for the equivalent u.d.l.

It must be remembered that the theory developed in this chapter has been based upon the assumption that deflections are small in comparison with the thickness of the plate. If deflections exceed half the plate thickness, then stretching of the middle surface of the plate must be considered. Under these conditions deflections are no longer proportional to the loads applied, e.g. for circular plates with clamped edges deflections δ can be determined from the equation

$$\delta + 0.58 \frac{\delta^3}{t^2} = \frac{qR^4}{64D} \quad (7.44)$$

For very thin diaphragms or membranes subjected to uniform pressure, stresses due to stretching of the middle surface may far exceed those due to bending and under these conditions the central deflection is given by

$$y_{\max} = 0.0662 R \left[\frac{qR}{Et} \right]^{1/3} \quad (7.45)$$

In the design of circular plates subjected to central concentrated loading, the maximum tensile stress on the lower surface of the plate is of prime interest since the often higher compressive stresses in the upper surface are generally much more localised. Local yielding of ductile materials in these regions will not generally affect the overall deformation of the plate provided that the lower surface tensile stresses are kept within safe limits. The situation is similar for plates constructed from brittle materials since their compressive strengths far exceed their strength in tension so that a limit on the latter is normally a safe design procedure. The theory covered in this text has involved certain simplifying assumptions; a full treatment of the problem shows that the limiting tensile stress is more accurately given

by the equation

$$\sigma_{r_{\max}} = \frac{F}{t^2} (1 + v) (0.485 \log_e R/t + 0.52) \quad (7.46)$$

7.14. Other loading cases of practical importance

In addition to the standard cases covered in the previous sections there are a number of other loading cases which are often encountered in practice; these are illustrated in Fig. 7.10[†]. The method of solution for such cases is introduced briefly below.[‡]

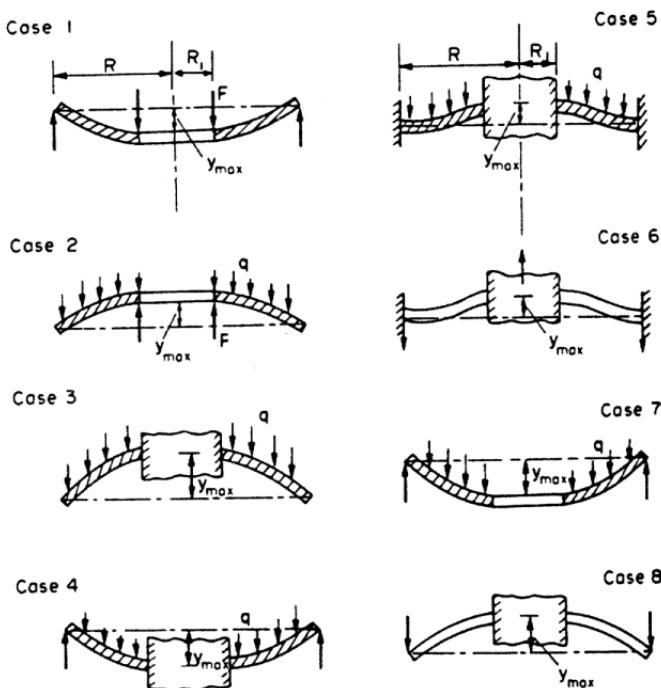


Fig. 7.10. Circular plates and diaphragms: various loading cases encountered in practice.

In all the cases illustrated the maximum stress is obtained from the following standard form of equations:

For *uniformly distributed loads*

$$\sigma_{\max} = k_1 \frac{qR^2}{t^2}$$

For *loads concentrated around the edge of the central hole*,

$$\sigma_{\max} = \frac{k_1 F}{t^2}$$

[†] S. Timoshenko, *Strength of Materials*, Part II, *Advanced Theory and Problems*, Van Nostrand.

[‡] A.M. Wahl and G. Lobo, *Trans. ASME* 52 (1929).

Similarly, the **maximum deflections** in each case are given by the following equations:
For *uniformly distributed loads*,

$$y_{\max} = k_2 \frac{qR^4}{Et^3} \quad (7.49)$$

For *loads concentrated around the central hole*,

$$y_{\max} = k_2 \frac{FR^2}{Et^3} \quad (7.50)$$

The values of the factors k_1 and k_2 for the loading cases of Fig. 7.10 are given in Table 7.2, assuming a Poisson's ratio ν of 0.3.

Table 7.2. Coefficients k_1 and k_2 for the eight cases shown in Fig. 7.10^(a).

$\frac{R}{R_1}$	1.25		1.5		2		3		4		5	
Case	k_1	k_2	k_1	k_2	k_1	k_2	k_1	k_2	k_1	k_2	k_1	k_2
1	1.10	0.341	1.26	0.519	1.48	0.672	1.88	0.734	2.17	0.724	2.34	0.704
2	0.66	0.202	1.19	0.491	2.04	0.902	3.34	1.220	4.30	1.300	5.10	1.310
3	0.135	0.00231	0.410	0.0183	1.04	0.0938	2.15	0.293	2.99	0.448	3.69	0.564
4	0.122	0.00343	0.336	0.0313	0.74	0.1250	1.21	0.291	1.45	0.417	1.59	0.492
5	0.090	0.00077	0.273	0.0062	0.71	0.0329	1.54	0.110	2.23	0.179	2.80	0.234
6	0.115	0.00129	0.220	0.0064	0.405	0.0237	0.703	0.062	0.933	0.092	1.13	0.114
7	0.592	0.184	0.976	0.414	1.440	0.664	1.880	0.824	2.08	0.830	2.19	0.813
8	0.227	0.00510	0.428	0.0249	0.753	0.0877	1.205	0.209	1.514	0.293	1.745	0.350

^(a) S. Timoshenko, *Strength of Materials*, Part II, *Advanced Theory and Problems*, Van Nostrand, p. 113.

B. BENDING OF RECTANGULAR PLATES

The theory of bending of rectangular plates is beyond the scope of this text and will not be introduced here. The standard formulae obtained from the theory,[†] however, may be presented in simple form and are relatively easy to apply. The results for the two most frequently used loading conditions are therefore summarised below.

7.15. Rectangular plates with simply supported edges carrying uniformly distributed loads

For a rectangular plate length d , shorter side b and thickness t , the *maximum deflection* is found to occur at the centre of the plate and given by

$$y_{\max} = \alpha \frac{qb^4}{Et^3} \quad (7.51)$$

the value of the factor α depending on the ratio d/b and given in Table 7.3.

[†] S. Timoshenko, *Theory of Plates and Shells*, 2nd edn., McGraw-Hill, New York, 1959.

Table 7.3. Constants for uniformly loaded rectangular plates with simply supported edges^(a).

d/b	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7
α	0.0443	0.0530	0.0616	0.0697	0.0770	0.0843	0.0906	0.0964
β_1	0.0479	0.0553	0.0626	0.0693	0.0753	0.0812	0.0862	0.0908
β_2	0.0479	0.0494	0.0501	0.0504	0.0506	0.0500	0.0493	0.0486
d/b	1.8	1.9	2.0	3.0	4.0	5.0	∞	
α	0.1017	0.1064	0.1106	0.1336	0.1400	0.1416	0.1422	
β_1	0.0948	0.0985	0.1017	0.1189	0.1235	0.1246	0.1250	
β_2	0.0479	0.0471	0.0464	0.0404	0.0384	0.0375	0.0375	

(a) S. Timoshenko, *Theory of Plates and Shells*, 2nd edn., McGraw-Hill, New York, 1959.

The *maximum bending moments*, per unit length, also occur at the centre of the plate and are given by

$$M_{XY_{\max}} = \beta_1 qb^2 \quad (7.52)$$

$$M_{YZ_{\max}} = \beta_2 qb^2 \quad (7.53)$$

the factors β_1 and β_2 being given in Table 7.4 for an assumed value of Poisson's ratio ν equal to 0.3.

It will be observed that for length ratios d/b in excess of 3 the values of the factors α , β_1 , and β_2 remain practically constant as also will the corresponding maximum deflections and bending moments.

7.16. Rectangular plates with clamped edges carrying uniformly distributed loads

Here again the *maximum deflection* takes place at the centre of the plate, the value being given by an equation of similar form to eqn. (7.51) for the simply-supported edge case but with different values of α ,

i.e. $y_{\max} = \alpha \frac{qb^4}{Et^3}$

The *bending moment* equations are also similar in form, the *numerical maximum occurring at the middle of the longer side* and given by

$$M_{\max} = \beta qb^2$$

Typical values for α and β are given in Table 7.4. In this case values are practically constant for $d/b > 2$.

Table 7.4. Constants for uniformly loaded rectangular plates with clamped edges.^(a)

d/b	1.00	1.25	1.50	1.75	2.00	∞
α	0.0138	0.0199	0.0240	0.0264	0.0277	0.0284
β	0.0513	0.0665	0.0757	0.0806	0.0829	0.0833

(a) S. Timoshenko, *Theory of Plates and Shells*, 2nd edn., McGraw-Hill, New York, 1959.

It will be observed, by comparison of the values of the factors in Tables 7.3 and 7.4, that when the edges of a plate are clamped the maximum deflection is considerably reduced from the freely supported condition but the maximum bending moments, and hence maximum stresses, are not greatly affected.

Examples

Example 7.1

A circular flat plate of diameter 120 mm and thickness 10 mm is constructed from steel with $E = 208 \text{ GN/m}^2$ and $\nu = 0.3$. The plate is subjected to a uniform pressure of 5 MN/m^2 on one side only. If the plate is clamped at the edges determine:

- the maximum deflection;
- the position and magnitude of the maximum radial stress.

What percentage change in the results will be obtained if the edge conditions are changed such that the plate can be assumed to be freely supported?

Solution

- From eqn. (7.23) the maximum deflection with clamped edges is given by

$$\begin{aligned} y_{\max} &= \frac{3qR^4}{16Et^3}(1 - \nu^2) \\ &= \frac{3 \times 5 \times 10^6 \times (60 \times 10^{-3})^4(1 - 0.3^2)}{16 \times 208 \times 10^9 \times (10 \times 10^{-3})^3} \\ &= 0.053 \times 10^{-3} = \mathbf{0.053 \text{ mm}} \end{aligned}$$

- From eqn. (7.24) the maximum radial stress occurs at the outside edge and is given by

$$\begin{aligned} \sigma_{r_{\max}} &= \frac{3qR^2}{4t^2} \\ &= \frac{3 \times 5 \times 10^6 \times (60 \times 10^{-3})^2}{4 \times (10 \times 10^{-3})^2} \\ &= 135 \times 10^6 = \mathbf{135 \text{ MN/m}^2} \end{aligned}$$

When the edges are freely supported, eqn. (7.26) gives

$$\begin{aligned} y'_{\max} &= \frac{3qR^4}{16Et^3}(5 + \nu)(1 - \nu) \\ &= \frac{(5 + \nu)(1 - \nu)}{(1 - \nu^2)} y_{\max} \\ &= \frac{(5.3 \times 0.7)}{0.91} \times 0.053 = \mathbf{0.216 \text{ mm}} \end{aligned}$$

and eqn. (7.27) gives

$$\begin{aligned}\sigma'_{r_{\max}} &= \frac{3qR^2}{8t^2}(3+\nu) \\ \sigma'_{r_{\max}} &= \frac{(3+\nu)}{2}\sigma_{r_{\max}} \\ &= \frac{3.3}{2} \times 135 = 223 \text{ MN/m}^2\end{aligned}$$

Thus the percentage increase in maximum deflection

$$= \frac{(0.216 - 0.053)}{0.053} 100 = 308\%$$

and the percentage increase in maximum radial stress

$$= \frac{(223 - 135)}{135} 100 = 65\%$$

Example 7.2

A circular disc 150 mm diameter and 12 mm thickness is clamped around the periphery and built into a piston of diameter 60 mm at the centre. Assuming that the piston remains rigid, determine the maximum deflection of the disc when the piston carries a load of 5 kN. For the material of the disc $E = 208 \text{ GN/m}^2$ and $\nu = 0.3$.

Solution

From eqn. (7.29) the deflection of the disc is given by

$$y = \frac{-Fr^2}{8\pi D} [\log_e r - 1] + \frac{C_1 r^2}{4} + C_2 \log_e r + C_3 \quad (1)$$

and from eqn. (7.28)

$$\text{slope } \theta = \frac{-Fr}{8\pi D} [2 \log_e r - 1] + \frac{C_1 r}{2} + \frac{C_2}{r} \quad (2)$$

Now slope = 0 at $r = 0.03 \text{ m}$.

Therefore from eqn. (2)

$$0 = \frac{-5000 \times 0.03}{8\pi D} [2 \log_e 0.03 - 1] + 0.015C_1 + 33.3C_2$$

But

$$D = \frac{Et^3}{12(1-\nu^2)} = \frac{208 \times 10^9 \times (12 \times 10^{-3})^3}{12(1-0.09)}$$

$$= \frac{208 \times 1728}{12 \times 0.91} = 32900$$

$$0 = \frac{-5000 \times 0.03}{8\pi \times 32900} [2(-3.5066) - 1] + 0.015C_1 + 33.3C_2$$

$$\therefore -1.45 \times 10^{-3} = 0.015C_1 + 33.3C_2 \quad (3)$$

Also the slope = 0 at $r = 0.075$.

Therefore from eqn. (2) again,

$$\begin{aligned} 0 &= \frac{-5000 \times 0.075}{8\pi \times 32900} [2 \log_e 0.075 - 1] + 0.0375C_1 + \frac{C_2}{0.075} \\ &= -4.54 \times 10^{-4}[2(-2.5903) - 1] + 0.0375C_1 + 13.33C_2 \\ &\quad - 2.8 \times 10^{-3} = 0.0375C_1 + 13.33C_2 \end{aligned} \quad (4)$$

$$(3) \times \frac{0.0375}{0.015},$$

$$-3.625 \times 10^{-3} = 0.0375C_1 + 83.25C_2 \quad (5)$$

$$(5) - (4),$$

$$-0.825 \times 10^{-3} = 69.92C_2$$

$$\therefore C_2 = -11.8 \times 10^{-6}$$

Substituting in (5),

$$\begin{aligned} -3.625 \times 10^{-3} &= 0.0375C_1 - 9.82 \times 10^{-4} \\ C_1 &= -\frac{(3.625 - 0.982)}{0.0375} 10^{-3} \\ &= -7.048 \times 10^{-2} \end{aligned}$$

Now taking $y = 0$ at $r = 0.075$, from eqn. (1)

$$\begin{aligned} 0 &= \frac{-5000 \times (0.075)^2}{8\pi \times 32900} [\log_e 0.075 - 1] - \frac{7.048 \times 10^{-2}}{4} (0.075)^2 \\ &\quad - 11.8 \times 10^{-6} \log_e 0.075 + C_3 \\ &= -3.4 \times 10^{-5}(-3.5903) - 99.1 \times 10^{-6} + 30.6 \times 10^{-6} + C_3 \\ &= 10^{-6}(122 - 99.1 + 30.6) + C_3 \\ \therefore C_3 &= -53.5 \times 10^{-6} \end{aligned}$$

Therefore deflection at $r = 0.03$ is given by eqn. (1),

$$\begin{aligned} \delta_{\max} &= \frac{-5000 \times (0.03)^2}{8\pi \times 32900} [\log_e 0.03 - 1] - \frac{7.048 \times 10^{-2}}{4} (0.03)^2 \\ &\quad - 11.8 \times 10^{-6} \log_e 0.03 - 53.5 \times 10^{-6} \\ &= 10^{-6}[+24.5 - 15.9 + 41.4 - 53.5] = -3.5 \times 10^{-6} \text{ m} \end{aligned}$$

Problems

In the following examples assume that

$$\frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{dy}{dr} \right) \right] = -\frac{Q}{D} \quad \text{or} \quad = -\frac{qr}{2D}$$

with conventional notations.

Unless otherwise stated, $E = 207 \text{ GN/m}^2$ and $\nu = 0.3$.

7.1 (B/C). A circular flat plate of 120 mm diameter and 6.35 mm thickness is clamped at the edges and subjected to a uniform lateral pressure of 345 kN/m^2 . Evaluate (a) the central deflection, (b) the position and magnitude of the maximum radial stress. [1.45 $\times 10^{-5}$ m, 23.1 MN/m 2 ; $r = 60 \text{ mm}$.]

7.2 (B/C). The plate of Problem 7.1 is subjected to the same load but is simply supported round the edges. Calculate the central deflection. [58 $\times 10^{-6}$ m.]

7.3 (B/C). An aluminium plate diaphragm is 500 mm diameter and 6 mm thick. It is clamped around its periphery and subjected to a uniform pressure q of 70 kN/m^2 . Calculate the values of maximum bending stress and deflection.

Take $Q = qr/2$, $E = 70 \text{ GN/m}^2$ and $\nu = 0.3$.

[91, 59.1 MN/m 2 ; 3.1 mm.]

7.4 (B/C). A circular disc of uniform thickness 1.5 mm and diameter 150 mm is clamped around the periphery and built into a piston, diameter 50 mm, at the centre. The piston may be assumed rigid and carries a central load of 450 N. Determine the maximum deflection. [0.21 mm.]

7.5 (C). A circular steel plate 5 mm thick, outside diameter 120 mm, inside diameter 30 mm, is clamped at its outer edge and loaded by a ring of edge moments $M_r = 8 \text{ kN/m}$ of circumference at its inner edge. Calculate the deflection at the inside edge. [4.68 mm.]

7.6 (C). A solid circular steel plate 5 mm thick, 120 mm outside diameter, is clamped at its outer edge and loaded by a ring of loads at $r = 20 \text{ mm}$. The total load on the plate is 10 kN. Calculate the central deflection of the plate. [0.195 mm.]

7.7 (C). A pressure vessel is fitted with a circular manhole 600 mm diameter, the cover of which is 25 mm thick. If the edges are clamped, determine the maximum allowable pressure, given that the maximum principal strain in the cover plate must not exceed that produced by a simple direct stress of 140 MN/m^2 . [1.19 MN/m 2 .]

7.8 (B/C). The crown of a gas engine piston may be treated as a cast-iron diaphragm 300 mm diameter and 10 mm thick, clamped at its edges. If the gas pressure is 3 MN/m^2 , determine the maximum principal stresses and the central deflection.

$\nu = 0.3$ and $E = 100 \text{ GN/m}^2$.

[506, 329 MN/m 2 ; 2.59 mm.]

7.9 (B/C). How would the values for Problem 7.8 change if the edges are released from clamping and freely supported? [835,835 MN/m 2 ; 10.6 mm.]

7.10 (B/C). A circular flat plate of diameter 305 mm and thickness 6.35 mm is clamped at the edges and subjected to a uniform lateral pressure of 345 kN/m^2 .

Evaluate: (a) the central deflection, (b) the position and magnitude of the maximum radial stress.

[6.1 $\times 10^{-4}$ m; 149.2 MN/m 2 .]

7.11 (B/C). The plate in Problem 7.10 is subjected to the same load, but simply supported round the edges. Evaluate the central deflection. [24.7 $\times 10^{-4}$ m.]

7.12 (B/C). The flat end-plate of a 2 m diameter container can be regarded as clamped around its edge. Under operating conditions the plate will be subjected to a uniformly distributed pressure of 0.02 MN/m^2 . Calculate from first principles the required thickness of the end plate if the bending stress in the plate should not exceed 150 MN/m^2 . For the plate material $E = 200 \text{ GN/m}^2$ and $\nu = 0.3$. [C.E.I.] [10 mm.]

7.13 (C). A cylinder head valve of diameter 38 mm is subjected to a gas pressure of 1.4 MN/m^2 . It may be regarded as a uniform thin circular plate simply supported around the periphery. Assuming that the valve stem applies a concentrated force at the centre of the plate, calculate the movement of the stem necessary to lift the valve from its seat. The flexural rigidity of the valve is 260 Nm and Poisson's ratio for the material is 0.3. [C.E.I.] [0.067 mm.]

7.14 (C). A diaphragm of light alloy is 200 mm diameter, 2 mm thick and firmly clamped around its periphery before and after loading. Calculate the maximum deflection of the diaphragm due to the application of a uniform pressure of 20 kN/m^2 normal to the surface of the plate.

Determine also the value of the maximum radial stress set up in the material of the diaphragm.

Assume $E = 70 \text{ GN/m}^2$ and Poisson's ratio $\nu = 0.3$.

[B.P.] [0.61 mm; 37.5 MN/m².]

7.15 (C). A thin plate of light alloy and 200 mm diameter is firmly clamped around its periphery. Under service conditions the plate is to be subjected to a uniform pressure p of 20 kN/m² acting normally over its whole surface area.

Determine the required minimum thickness t of the plate if the following design criteria apply:

- (a) the maximum deflection is not to exceed 6 mm;
- (b) the maximum radial stress is not to exceed 50 MN/m².

Take $E = 70 \text{ GN/m}^2$ and $\nu = 0.3$.

[B.P.] [1.732 mm.]

7.16 (C). Determine equations for the maximum deflection and maximum radial stress for a circular plate, radius R , subjected to a distributed pressure of the form $q = K/r$. Assume simply supported edge conditions:

$$\left[\delta_{\max} = \frac{-KR^3(4+\nu)}{36D(1+\nu)}, \sigma_{\max} = \frac{EtRK(2+\nu)}{12D(1-\nu^2)} \right]$$

7.17 (C). The cover of the access hole for a large steel pressure vessel may be considered as a circular plate of 500 mm diameter which is firmly clamped around its periphery. Under service conditions the vessel operates with an internal pressure of 0.65 MN/m².

Determine the minimum thickness of plate required in order to achieve the following design criteria:

- (a) the maximum deflection is limited to 5 mm;
- (b) the maximum radial stress is limited to 200 MN/m².

For the steel, $E = 208 \text{ GN/m}^2$ and $\nu = 0.3$.

You may commence your solution on the assumption that the deflection y at radius r for a uniform circular plate under the action of a uniform pressure q is given by:

$$\frac{d}{dr} \left[\frac{1}{r} \cdot \frac{d}{dr} \left(r \cdot \frac{dy}{dr} \right) \right] = -\frac{qr}{2D}$$

where D is the "flexural stiffness" of the plate.

[9.95 mm.]

7.18 (C). A circular plate, 300 mm diameter and 5 mm thick, is built-in at its periphery. In order to strengthen the plate against a concentrated central axial load P the plate is stiffened by radial ribs and a prototype is found to have a stiffness of 11300 N per mm central deflection.

- (a) Check that the equation:

$$y = \frac{Pr^2}{8\pi D} \left[l_n \left(\frac{r}{R_2} \right) - \frac{1}{2} \right] + \frac{PR_2^2}{16\pi D}$$

satisfies the boundary conditions for the *unstiffened* plate.

- (b) Hence determine the stiffness of the plate without the ribs in terms of central deflection and calculate the relative stiffening effect of the ribs.
- (c) What additional thickness would be required for an unstiffened plate to produce the same effect? For the plate material $E = 200 \text{ GN/m}^2$ and $\nu = 0.28$.

[5050 N/mm; 124%; 1.54 mm.]

CHAPTER 8

INTRODUCTION TO ADVANCED ELASTICITY THEORY

8.1. Type of stress

Any element of material may be subjected to three independent types of stress. Two of these have been considered in detail previously, namely *direct stresses* and *shear stresses*, and need not be considered further here. The third type, however, has not been specifically mentioned previously although it has in fact been present in some of the loading cases considered in earlier chapters; these are the so-called *body-force stresses*. These body forces arise by virtue of the bulk of the material, typical examples being:

- (a) gravitational force due to a component's own weight: this has particular significance in civil engineering applications, e.g. dam and chimney design;
- (b) centrifugal force, depending on radius and speed of rotation, with particular significance in high-speed engine or turbine design;
- (c) magnetic field forces.

In many practical engineering applications the only body force present is the gravitational one, and in the majority of cases its effect is minimal compared with the other applied forces due to mechanical loading. In such cases it is therefore normally neglected. In high-speed dynamic loading situations such as the instances quoted in (b) above, however, the centrifugal forces far exceed any other form of loading and are therefore the primary factor for consideration.

Unlike direct and shear stresses, body force stresses are defined as **force per unit volume**, and particular note must be taken of this definition in relation to the proofs of formulae which follow.

8.2. The cartesian stress components: notation and sign convention

Consider an element of material subjected to a complex stress system in three dimensions. Whatever the type of applied loading the resulting stresses can always be reduced to the nine components, i.e. three direct and six shear, shown in Fig. 8.1.

It will be observed that in this case a modified notation is used for the stresses. This is termed the double-suffix notation and it is particularly useful in the detailed study of stress problems since it indicates both the direction of the stress *and* the plane on which it acts.

The *first* suffix gives the *direction* of the stress.

The *second* suffix gives the *direction of the normal of the plane* on which the stress acts. Thus, for example,

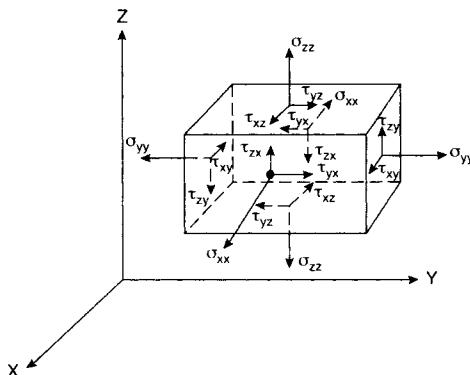


Fig. 8.1. The cartesian stress components.

σ_{xx} is the stress in the X direction on the X facing face (i.e. a *direct* stress). Common suffices therefore always indicate that the stress is a direct stress. Similarly, σ_{xy} is the stress in the X direction on the Y facing face (i.e. a *shear* stress). Mixed suffices always indicate the presence of shear stresses and thus allow the alternative symbols σ_{xy} or τ_{xy} . Indeed, the alternative symbol τ is not strictly necessary now since the suffices indicate whether the stress σ is a direct one or a shear.

8.2.1. Sign conventions

(a) *Direct stresses*. As always, direct stresses are assumed positive when tensile and negative when compressive.

(b) *Shear stresses*. Shear stresses are taken to be positive if they act in a positive cartesian (X , Y or Z) direction whilst acting on a plane whose outer normal points also in a positive cartesian direction.

Thus positive shear is assumed with + direction and + facing face.

Alternatively, positive shear is also given with – direction and – facing face (a double negative making a positive, as usual).

A careful study of Fig. 8.1 will now reveal that all stresses shown are positive in nature.

The *cartesian stress components* considered here relate to the three mutually perpendicular axes X , Y and Z . In certain loading cases, notably those involving axial symmetry, this system of components is inconvenient and an alternative set known as *cylindrical components* is used. These involve the variables, radius r , angle θ and axial distance z , and will be considered in detail later.

8.3. The state of stress at a point

Consider any point Q within a stressed material, the nine cartesian stress components at Q being known. It is now possible to determine the normal, direct and resultant stresses which act on any plane through Q whatever its inclination relative to the cartesian axes. Suppose one such plane ABC has a normal n which makes angles n_x , n_y and n_z with the YZ , XZ and XY planes respectively as shown in Figs. 8.2 and 8.3. (Angles between planes

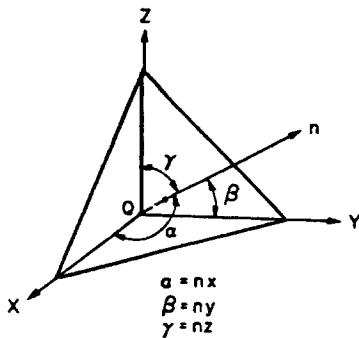


Fig. 8.2.

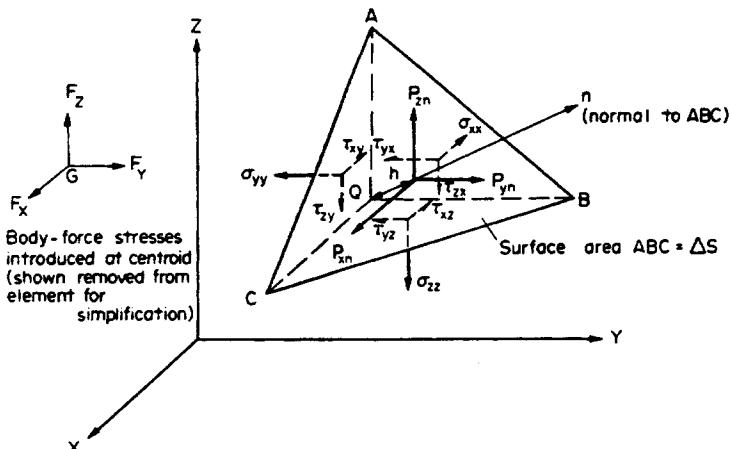


Fig. 8.3. The state of stress on an inclined plane through any given point in a three-dimensional cartesian stress system.

ABC and YZ are given by the angle between the normals to both planes n and x , etc.) For convenience, let the plane ABC initially be some perpendicular distance h from Q so that the cartesian stress components actually acting at Q can be shown on the sides of the tetrahedron element $ABCQ$ so formed (Fig. 8.3). In the derivation below the value of h will be reduced to zero so that the equations obtained will relate to the condition when ABC passes through Q .

In addition to the cartesian components, the unknown components of the stress on the plane ABC , i.e. p_{xn} , p_{yn} and p_{zn} , are also indicated, as are the body-force field stress components which act at the centre of gravity of the tetrahedron. (To improve clarity of the diagram they are shown displaced from the element.)

Since body-force stresses are defined as forces/unit volume, the components in the X , Y and Z directions are of the form

$$F \times \Delta S \frac{h}{3}$$

where $\Delta Sh/3$ is the volume of the tetrahedron. If the area of the surface ABC , i.e. ΔS , is assumed small then all stresses can be taken to be uniform and the component of force in

the X direction due to σ_{xx} is given by

$$\sigma_{xx} \Delta S \cos nx$$

Stress components in the other axial directions will be similar in form.

Thus, for equilibrium of forces in the X direction,

$$p_{xn} \Delta S + F_x \Delta S \frac{h}{3} = \sigma_{xx} \Delta S \cos nx + \tau_{xy} \Delta S \cos ny + \tau_{xz} \Delta S \cos nz$$

As $h \rightarrow 0$ (i.e. plane ABC passes through Q), the second term above becomes very small and can be neglected. The above equation then reduces to

$$p_{xn} = \sigma_{xx} \cos nx + \tau_{xy} \cos ny + \tau_{xz} \cos nz \quad (8.1)$$

Similarly, for equilibrium of forces in the y and z directions,

$$p_{yn} = \sigma_{yy} \cos ny + \tau_{yx} \cos nx + \tau_{yz} \cos nz \quad (8.2)$$

$$p_{zn} = \sigma_{zz} \cos nz + \tau_{zx} \cos nx + \tau_{zy} \cos ny \quad (8.3)$$

The resultant stress p_n on the plane ABC is then given by

$$p_n = \sqrt{(p_{xn}^2 + p_{yn}^2 + p_{zn}^2)} \quad (8.4)$$

The normal stress σ_n is given by resolution perpendicular to the face ABC ,

i.e.

$$\sigma_n = p_{xn} \cos nx + p_{yn} \cos ny + p_{zn} \cos nz \quad (8.5)$$

and, by Pythagoras' theorem (Fig. 8.4), the shear stress τ_n is given by

$$\tau_n = \sqrt{(p_n^2 - \sigma_n^2)} \quad (8.6)$$

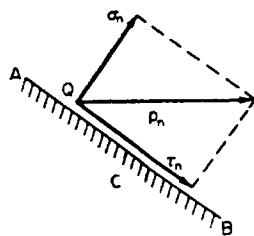


Fig. 8.4. Normal, shear and resultant stresses on the plane ABC .

It is often convenient and quicker to define the line of action of the resultant stress p_n by the *direction cosines*

$$l' = \cos(p_n x) = p_{xn} / p_n \quad (8.7)$$

$$m' = \cos(p_n y) = p_{yn} / p_n \quad (8.8)$$

$$n' = \cos(p_n z) = p_{zn} / p_n \quad (8.9)$$

The direction of the plane ABC being given by other direction cosines

$$l = \cos nx, \quad m = \cos ny, \quad n = \cos nz$$

It can be shown by simple geometry that

$$l^2 + m^2 + n^2 = 1 \quad \text{and} \quad (l')^2 + (m')^2 + (n')^2 = 1$$

Equations (8.1), (8.2) and (8.3) may now be written in two alternative ways.

(a) *Using the common symbol σ for stress and relying on the double suffix notation to discriminate between shear and direct stresses:*

$$p_{xn} = \sigma_{xx} \cos nx + \sigma_{xy} \cos ny + \sigma_{xz} \cos nz \quad (8.10)$$

$$p_{yn} = \sigma_{yx} \cos nx + \sigma_{yy} \cos ny + \sigma_{yz} \cos nz \quad (8.11)$$

$$p_{zn} = \sigma_{zx} \cos nx + \sigma_{zy} \cos ny + \sigma_{zz} \cos nz \quad (8.12)$$

In each of the above equations the first suffix is common throughout, the second suffix on the right-hand-side terms are in the order x, y, z throughout, and in each case the cosine term relates to the second suffix. These points should aid memorisation of the equations.

(b) *Using the direction cosine form:*

$$p_{xn} = \sigma_{xx} l + \sigma_{xy} m + \sigma_{xz} n \quad (8.13)$$

$$p_{yn} = \sigma_{yx} l + \sigma_{yy} m + \sigma_{yz} n \quad (8.14)$$

$$p_{zn} = \sigma_{zx} l + \sigma_{zy} m + \sigma_{zz} n \quad (8.15)$$

Memory is again aided by the notes above, but in this case it is the direction cosines, l, m and n which relate to the appropriate second suffices x, y and z .

Thus, provided that the direction cosines of a plane are known, together with the cartesian stress components at some point Q on the plane, the direct, normal and shear stresses on the plane at Q may be determined using, firstly, eqns. (8.13–15) and, subsequently, eqns. (8.4–6).

Alternatively the procedure may be carried out graphically as will be shown in §8.9.

8.4. Direct, shear and resultant stresses on an oblique plane

Consider again the oblique plane ABC having direction cosines l, m and n , i.e. these are the cosines of the angle between the normal to plane and the x, y, z directions.

In general, the resultant stress on the plane p_n will not be normal to the plane and it can therefore be resolved into two alternative sets of components.

- (a) In the co-ordinate directions giving components p_{xn} , p_{yn} and p_{zn} , as shown in Fig. 8.5, with values given by eqns. (8.13), (8.14) and (8.15).
- (b) Normal and tangential to the plane as shown in Fig. 8.6, giving components, of σ_n (normal or direct stress) and τ_n (shear stress) with values given by eqns. (8.5) and (8.6).

The value of the resultant stress can thus be obtained from either of the following equations:

$$p_n^2 = \sigma_n^2 + \tau_n^2 \quad (8.16)$$

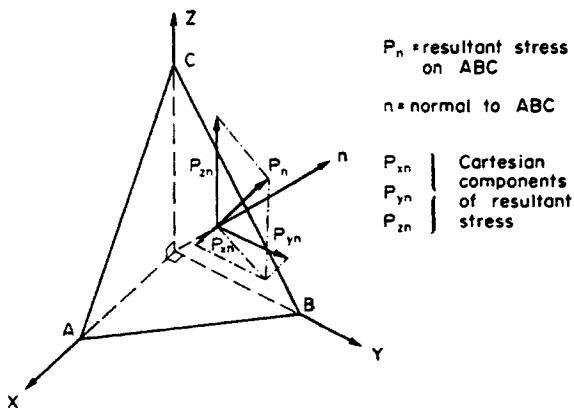


Fig. 8.5. Cartesian components of resultant stress on an inclined plane.

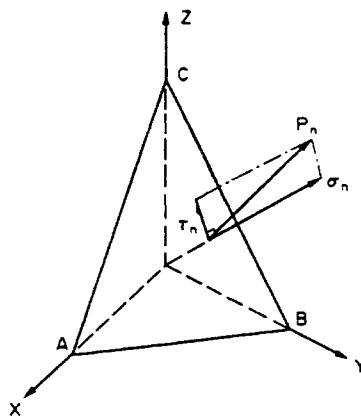


Fig. 8.6. Normal and tangential components of resultant stress on an inclined plane.

or

$$p_n^2 = p_{xn}^2 + p_{yn}^2 + p_{zn}^2 \quad (8.17)$$

these being alternative forms of eqns. (8.6) and (8.4) respectively.

From eqn. (8.5) the normal stress on the plane is given by:

$$\sigma_n = p_{xn} \cdot l + p_{yn} \cdot m + p_{zn} \cdot n$$

But from eqns. (8.13), (8.14) and (8.15)

$$p_{xn} = \sigma_{xx} \cdot l + \sigma_{xy} \cdot m + \sigma_{xz} \cdot n$$

$$p_{yn} = \sigma_{yx} \cdot l + \sigma_{yy} \cdot m + \sigma_{yz} \cdot n$$

$$p_{zn} = \sigma_{zx} \cdot l + \sigma_{zy} \cdot m + \sigma_{zz} \cdot n$$

\therefore Substituting into eqn (8.5) and using the relationships $\sigma_{xy} = \sigma_{yx}$; $\sigma_{xz} = \sigma_{zx}$ and $\sigma_{yz} = \sigma_{zy}$ which will be proved in §8.12

$$\sigma_n = \sigma_{xx} \cdot l^2 + \sigma_{yy} \cdot m^2 + \sigma_{zz} \cdot n^2 + 2\sigma_{xy} \cdot lm + 2\sigma_{yz} \cdot mn + 2\sigma_{xz} \cdot ln. \quad (8.18)$$

and from eqn. (8.6) the shear stress on the plane will be given by

$$\tau_n^2 = p_{xn}^2 + p_{yn}^2 + p_{zn}^2 - \sigma_n^2 \quad (8.19)$$

In the particular case where plane ABC is a principal plane (i.e. no shear stress):

$$\sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0$$

and

$$\sigma_{xx} = \sigma_1, \quad \sigma_{yy} = \sigma_2 \quad \text{and} \quad \sigma_{zz} = \sigma_3$$

the above equations reduce to:

$$\sigma_n = \sigma_1 \cdot l^2 + \sigma_2 \cdot m^2 + \sigma_3 \cdot n^2 \quad (8.20)$$

and since

$$p_{xn} = \sigma_1 \cdot l \quad p_{yn} = \sigma_2 \cdot m \quad \text{and} \quad p_{zn} = \sigma_3 \cdot n$$

$$\tau_n^2 = \sigma_1^2 \cdot l^2 + \sigma_2^2 \cdot m^2 + \sigma_3^2 \cdot n^2 - \sigma_n^2 \quad (8.21)$$

8.4.1. Line of action of resultant stress

As stated above, the resultant stress p_n is generally not normal to the plane ABC but inclined to the x , y and z axes at angles θ_x , θ_y and θ_z – see Fig. 8.7.

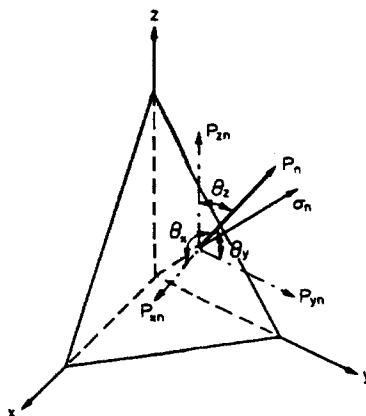


Fig. 8.7. Line of action of resultant stress.

The components of p_n in the x , y and z directions are then

$$\left. \begin{aligned} p_{xn} &= p_n \cdot \cos \theta_x \\ p_{yn} &= p_n \cdot \cos \theta_y \\ p_{zn} &= p_n \cdot \cos \theta_z \end{aligned} \right\} \quad (8.22)$$

and the direction cosines which define the line of actions of the resultant stress are

$$\left. \begin{aligned} l' &= \cos \theta_x = p_{xn} / p_n \\ m' &= \cos \theta_y = p_{yn} / p_n \\ n' &= \cos \theta_z = p_{zn} / p_n \end{aligned} \right\} \quad (8.23)$$

8.4.2. Line of action of normal stress

By definition the normal stress is that which acts normal to the plane, i.e. the line of action of the normal stress has the same direction cosines as the normal to plane viz: l , m and n .

8.4.3. Line of action of shear stress

As shown in §8.4 the resultant stress p_n can be considered to have two components; one normal to the plane (σ_n) and one along the plane (the shear stress τ_n) – see Fig. 8.6.

Let the direction cosines of the line of action of this shear stress be l_s , m_s and n_s .

The alternative components of the resultant stress, p_{xn} , p_{yn} and p_{zn} , can then either be obtained from eqn (8.22) or by resolution of the normal and shear components along the x , y and z directions as follows:

$$\left. \begin{aligned} p_{xn} &= \sigma_n \cdot l + \tau_n \cdot l_s \\ p_{yn} &= \sigma_n \cdot m + \tau_n \cdot m_s \\ p_{zn} &= \sigma_n \cdot n + \tau_n \cdot n_s \end{aligned} \right\} \quad (8.24)$$

Thus the direction cosines of the line of action of the shear stress τ_n are:

$$\left. \begin{aligned} l_s &= \frac{p_{xn} - l \cdot \sigma_n}{\tau_n} \\ m_s &= \frac{p_{yn} - m \cdot \sigma_n}{\tau_n} \\ n_s &= \frac{p_{zn} - n \cdot \sigma_n}{\tau_n} \end{aligned} \right\} \quad (8.25)$$

8.4.4. Shear stress in any other direction on the plane

Let ϕ be the angle between the direction of the shear stress τ_n and the required direction. Then, since the angle between any two lines in space is given by,

$$\cos \phi = l_s \cdot l_\phi + m_s \cdot m_\phi + n_s \cdot n_\phi \quad (8.26)$$

where l_ϕ , m_ϕ , n_ϕ are the direction cosines of the new shear stress direction, it follows that the required magnitude of the shear stress on the “ ϕ ” plane will be given by

$$\tau_\phi = \tau_n \cdot \cos \phi \quad (8.27)$$

Alternatively, resolving the components of the resultant stress (p_{xn} , p_{yn} and p_{zn}) along the new direction we have:

$$\tau_\phi = p_{xn} \cdot l_\phi + p_{yn} \cdot m_\phi + p_{zn} \cdot n_\phi \quad (8.28)$$

and substituting eqns. (8.13), (8.14) and (8.15)

$$\begin{aligned} \tau_\phi &= \sigma_{xx} \cdot ll_\phi + \sigma_{yy} \cdot mm_\phi + \sigma_{zz} \cdot nn_\phi + \sigma_{xy}(lm_\phi + l_\phi \cdot m) \\ &\quad + \sigma_{xz}(ln_\phi + nl_\phi) + \sigma_{yz}(mn_\phi + nm_\phi) \end{aligned} \quad (8.29)$$

Whilst eqn. (8.28) has been derived for the shear stress τ_ϕ it will, in fact, apply equally for any type of stress (i.e. shear or normal) which acts on the plane ABC in the ϕ direction.

In the case of the shear stress, however, its line of action must always be perpendicular to the normal to the plane so that

$$ll_\phi + mm_\phi + nn_\phi = 0.$$

In the case of a normal stress the relationship between the direction cosines is simply

$$l = l_\phi, m = m_\phi \text{ and } n = n_\phi$$

since the stress and the normal to the plane are in the same direction. Eqn. (8.29) then reduces to that found previously, viz. eqn. (8.18).

8.5. Principal stresses and strains in three dimensions – Mohr's circle representation

The procedure used for constructing Mohr's circle representation for a three-dimensional principal stress system has previously been introduced in §13.7[†]. For convenience of reference the resulting diagram is repeated here as Fig. 8.8. A similar representation for a three-dimensional principal strain system is shown in Fig. 8.9.

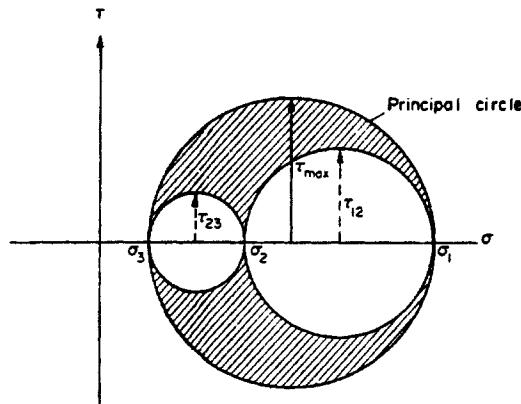


Fig. 8.8. Mohr circle representation of three-dimensional stress state showing the principal circle, the radius of which is equal to the greatest shear stress present in the system.

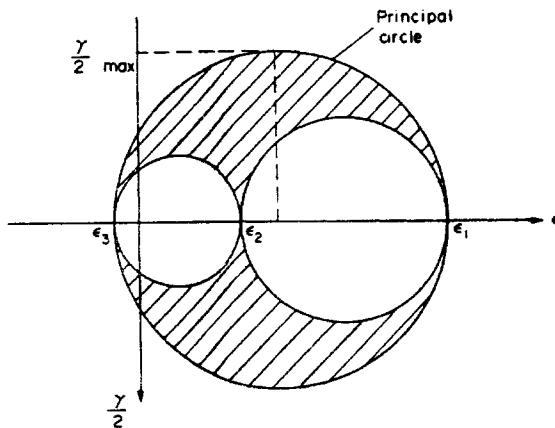


Fig. 8.9. Mohr representation for a three-dimensional principal strain system.

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1977.

In both cases the **principal circle** is indicated, the radius of which gives the maximum shear stress and *half* the maximum shear strain, respectively, in the three-dimensional system.

This form of representation utilises different diagrams for the stress and strain systems. An alternative procedure uses a single *combined diagram* for both cases and this is described in detail §§8.6 and 8.7.

8.6. Graphical determination of the direction of the shear stress τ_n on an inclined plane in a three-dimensional principal stress system

As before, let the inclined plane have direction cosines l, m and n . A true representation of this plane is given by constructing a so-called "true shape triangle" the ratio of the lengths of its sides being the ratio of the direction cosines—Fig. 8.10.

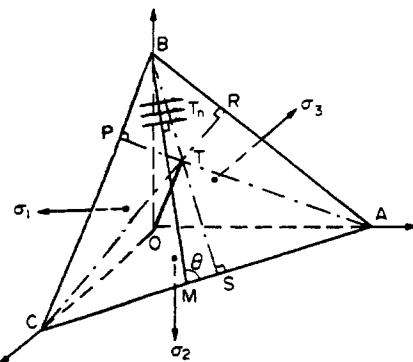


Fig. 8.10. Graphical determination of direction of shear stress on an inclined plane.

If lines are drawn perpendicular to each side from the opposite vertex, meeting the sides at points P, R and S , they will intersect at point T the "orthocentre". This is also the point through which the normal to the plane from O passes.

If σ_1, σ_2 and σ_3 are the three principal stresses then point M is positioned on AC such that

$$\frac{CM}{CA} = \frac{(\sigma_2 - \sigma_3)}{(\sigma_1 - \sigma_2)}$$

The required direction of the shear stress is then perpendicular to the line BD .

The equivalent procedure on the Mohr circle construction is as follows (see Fig. 8.11).

Construct the three stress circles corresponding to the three principal stresses σ_1, σ_2 and σ_3 . Set off line AB at an angle $\alpha = \cos^{-1} l$ to the left of the vertical through A .

Set off line CB at an angle $\gamma = \cos^{-1} n$ to the right of the vertical through C to meet AB at B .

Mark the points where these lines cut the principal circle R and P respectively.

Join AP and CR to cut at point T .

Join BT and extend to cut horizontal axis AC at S .

With point M the σ_2 position, join BM .

The required shear stress direction is then perpendicular to the line BM .

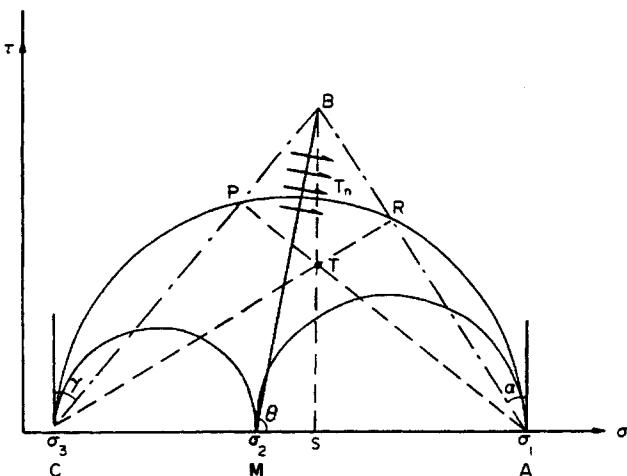


Fig. 8.11. Mohr circle equivalent procedure to that of Fig. 8.10.

8.7. The combined Mohr diagram for three-dimensional stress and strain systems

Consider any three-dimensional stress system with principal stresses σ_1 , σ_2 and σ_3 (all assumed tensile). Principal strains are then related to the principal stresses as follows:

$$\begin{aligned} \varepsilon_1 &= \frac{1}{E}(\sigma_1 - \nu\sigma_2 - \nu\sigma_3), \text{ etc.} \\ E\varepsilon_1 &= \sigma_1 - \nu(\sigma_2 + \sigma_3) \\ &= \sigma_1 - \nu(\sigma_1 + \sigma_2 + \sigma_3) + \nu\sigma_1 \end{aligned} \quad (1)$$

Now the *hydrostatic, volumetric* or *mean stress* $\bar{\sigma}$ is defined as

$$\bar{\sigma} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

Therefore substituting in (1),

$$E\varepsilon_1 = \sigma_1(1 + \nu) - 3\nu\bar{\sigma} \quad (2)$$

But the volumetric stress $\bar{\sigma}$ may also be written in terms of the bulk modulus,

i.e. bulk modulus $K = \frac{\text{volumetric stress}}{\text{volumetric strain}}$

and

volumetric strain = sum of the three linear strains

$$= \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \Delta$$

$$\therefore K = \frac{\bar{\sigma}}{\Delta}$$

but

$$E = 3K(1 - 2\nu)$$

$$\therefore \bar{\sigma} = \Delta K = \Delta \frac{E}{3(1 - 2\nu)}$$

Substituting in (2),

$$E\varepsilon_1 = \sigma_1(1 + \nu) - \frac{3\nu\Delta E}{3(1 - 2\nu)}$$

and, since $E = 2G(1 + \nu)$,

$$\begin{aligned} 2G(1 + \nu)\varepsilon_1 &= \sigma_1(1 + \nu) - \frac{\nu\Delta 2G(1 + \nu)}{(1 - 2\nu)} \\ \therefore \quad \sigma_1 &= 2G \left[\varepsilon_1 + \frac{\nu\Delta}{(1 - 2\nu)} \right] \end{aligned}$$

But, mean strain

$$\bar{\varepsilon} = \frac{1}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) = \frac{1}{3}\Delta$$

$$\therefore \quad \sigma_1 = 2G \left[\varepsilon_1 + \frac{3\nu\bar{\varepsilon}}{(1 - 2\nu)} \right] \quad (8.30)$$

Alternatively, re-writing eqn. (8.16) in terms of ε_1 ,

$$\varepsilon_1 = \frac{\sigma_1}{2G} - \frac{3\nu}{(1 - 2\nu)}\bar{\varepsilon}$$

But

$$\bar{\varepsilon} = \frac{\Delta}{3} = \frac{\bar{\sigma}}{3K}$$

But

$$E = 2G(1 + \nu) = 3K(1 - 2\nu)$$

i.e.

$$3K = 2G \frac{(1 + \nu)}{(1 - 2\nu)}$$

∴

$$\bar{\varepsilon} = \frac{\bar{\sigma}(1 - 2\nu)}{2G(1 + \nu)}$$

∴

$$\varepsilon_1 = \frac{\sigma_1}{2G} + \frac{\bar{\sigma}}{2G} \frac{(1 - 2\nu)}{(1 + \nu)}$$

i.e.

$$\varepsilon_1 = \frac{1}{2G} \left[\sigma_1 - \frac{3\nu\bar{\sigma}}{(1 + \nu)} \right] \quad (8.31)$$

In the above derivation the cartesian stresses σ_{xx} , σ_{yy} and σ_{zz} could have been used in place of the principal stresses σ_1 , σ_2 and σ_3 to yield more general expressions but of identical form. It therefore follows that the stress and associated strain in *any* given direction within a complex three-dimensional stress system is given by eqns. (8.30) and (8.31) which must satisfy the three-dimensional Mohr's circle construction.

Comparison of eqns. (8.30) and (8.31) indicates that

$$2G\varepsilon_1 = \sigma_1 - \frac{3\nu}{(1 + \nu)}\bar{\sigma}$$

Thus, having constructed the three-dimensional Mohr's *stress* circle representations, the equivalent *strain* values may be obtained simply by reference to a new axis displaced a distance $(3\nu/(1 + \nu))\bar{\sigma}$ as shown in Fig. 8.12 bringing the new axis origin to O' .

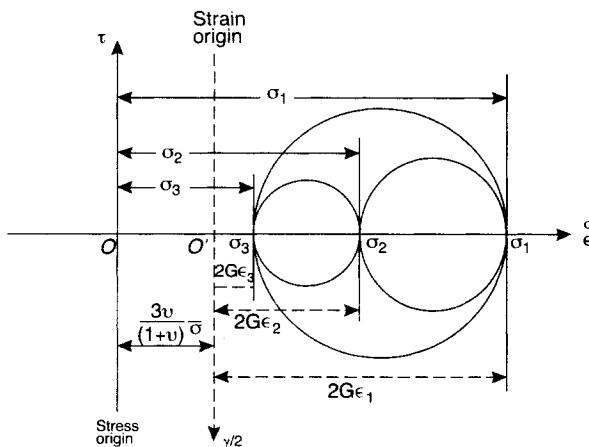


Fig. 8.12. The “combined Mohr diagram” for three-dimensional stress and strain systems.

Distances from the new axis to any principal stress value, e.g. σ_1 , will then be $2G$ times the corresponding ϵ_1 principal strain value,

i.e.

$$O'\sigma_1 \div 2G = \epsilon_1$$

Thus the same circle construction will apply for both stresses and strains provided that:

- (a) the shear strain axis is offset a distance $\frac{3v}{(1+v)}\bar{\sigma}$ to the right of the shear stress axis;
- (b) a scale factor of $2G$, [$= E/(1+\nu)$], is applied to measurements from the new axis.

8.8. Application of the combined circle to two-dimensional stress systems

The procedure of §14.13[†] uses a common set of axes and a common centre for Mohr's stress and strain circles, each having an appropriate radius and scale factor. An alternative procedure utilises the combined circle approach introduced above where a single circle can be used in association with two different origins to obtain both stress and strain values.

As in the above section the relationship between the stress and strain scales is

$$\frac{\text{stress scale}}{\text{strain scale}} = \frac{E}{(1+\nu)} = 2G$$

This is in fact the condition for both the stress and strain circles to have the same radius[‡] and should not be confused with the condition required in §14.13[†] of the alternative approach for the two circles to be concentric, when the ratio of scales is $E/(1-\nu)$.

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

[‡] For equal radii of both the stress and strain circles

$$\frac{(\sigma_1 - \sigma_2)}{2 \times \text{stress scale}} = \frac{(\epsilon_1 - \epsilon_2)}{2 \times \text{strain scale}}$$

$$\frac{\text{stress scale}}{\text{strain scale}} = \frac{(\sigma_1 - \sigma_2)}{(\epsilon_1 - \epsilon_2)} = \frac{(\sigma_1 - \sigma_2)}{(\sigma_1 - \sigma_2)} \frac{E}{(1+\nu)} = \frac{E}{(1+\nu)}$$

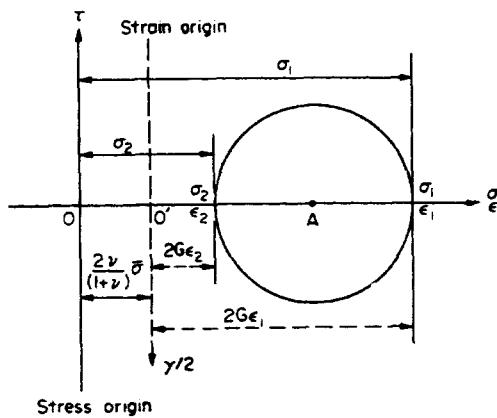


Fig. 8.13. Combined Mohr diagram for two-dimensional stress and strain systems.

With reference to Fig. 8.13 the two origins must then be positioned such that

$$OA = \frac{(\sigma_1 + \sigma_2)}{2 \times \text{stress scale}}$$

$$O'A = \frac{(\varepsilon_1 + \varepsilon_2)}{2 \times \text{strain scale}}$$

$$\begin{aligned} \therefore \frac{OA}{O'A} &= \frac{(\sigma_1 + \sigma_2)}{(\varepsilon_1 + \varepsilon_2)} \times \frac{\text{strain scale}}{\text{stress scale}} \\ &= \frac{(\sigma_1 + \sigma_2)}{(\varepsilon_1 + \varepsilon_2)} \times \frac{(1 + \nu)}{E} \end{aligned}$$

But

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2)$$

$$\varepsilon_2 = \frac{1}{E}(\sigma_2 - \nu\sigma_1)$$

$$\therefore \varepsilon_1 + \varepsilon_2 = \frac{1}{E}(\sigma_1 + \sigma_2)(1 - \nu)$$

$$\therefore \frac{OA}{O'A} = \frac{(\sigma_1 + \sigma_2)}{(\sigma_1 + \sigma_2)(1 - \nu)} \frac{E}{(1 - \nu)} \frac{(1 + \nu)}{E} = \frac{(1 + \nu)}{(1 - \nu)}$$

Thus the distance between the two origins is given by

$$\begin{aligned} OO' &= OA - O'A = OA - \frac{(1 - \nu)}{(1 + \nu)} OA \\ &= \frac{(\sigma_1 + \sigma_2)}{2} \left[1 - \frac{(1 - \nu)}{(1 + \nu)} \right] \\ &= \frac{(\sigma_1 + \sigma_2)(2\nu)}{2(1 + \nu)} = \frac{\nu}{(1 + \nu)} (\sigma_1 + \sigma_2) \\ &= \frac{2\nu}{(1 + \nu)} \bar{\sigma} \end{aligned} \tag{8.32}$$

where $\bar{\sigma}$ is the mean stress in the two-dimensional stress system $= \frac{1}{2}(\sigma_1 + \sigma_2)$ = position of centre of stress circle.

The relationship is thus identical in form to the three-dimensional equivalent with 2 replacing 3 for the two-dimensional system.

Again, therefore, the *single-circle construction applies for both stresses and strain provided that the axes are offset by the appropriate amount and a scale factor for strains of $2G$ is applied.*

8.9. Graphical construction for the state of stress at a point

The following procedure enables the determination of the direct (σ_n) and shear (τ_n) stresses at any point on a plane whose direction cosines are known and, in particular, on the *octahedral planes* (see §8.19).

The construction procedure for Mohr's circle representation of three-dimensional stress systems has been introduced in §8.4. Thus, for a given state of stress producing principal stress σ_1 , σ_2 and σ_3 , Mohr's circles are as shown in Fig. 8.8.

For a given plane S characterised by direction cosines l , m and n the remainder of the required construction proceeds as follows (Fig. 8.14). (Only half the complete Mohr's circle representation is shown since this is sufficient for the execution of the construction procedure.)

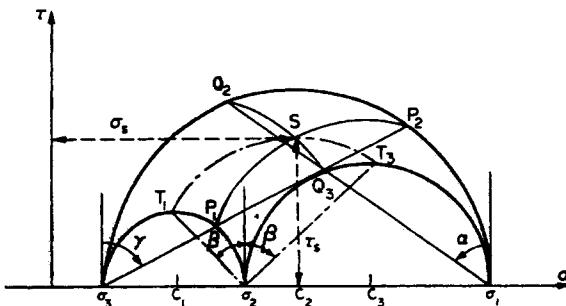


Fig. 8.14. Graphical construction for the state of stress on a general stress plane.

- (1) Set off angle $\alpha = \cos^{-1} l$ from the vertical at σ_1 to cut the circles in Q_2 and Q_3 .
- (2) With centre C_1 (centre of σ_2 , σ_3 circle) draw arc Q_2Q_3 .
- (3) Set off angle $\gamma = \cos^{-1} n$ from the vertical at σ_3 to cut the circles at P_1 and P_2 .
- (4) With centre C_3 (centre of σ_1 , σ_2 circle) draw arc P_1P_2 .
- (5) The position S representing the required plane is then given by the point where the two arcs Q_2Q_3 and P_1P_2 intersect. *The stresses on this plane are then σ_s and τ_s as shown.* Careful study of the above construction procedure shows that the suffices of points considered in each step always complete the grouping 1, 2, 3. This should aid memorisation of the procedure.
- (6) As a check on the accuracy of the drawing, set off angles $\beta = \cos^{-1} m$ on either side of the vertical through σ_2 to cut the $\sigma_2\sigma_3$ circle in T_1 and the $\sigma_1\sigma_2$ circle in T_3 .

- (7) With centre C_2 (centre of the $\sigma_1\sigma_3$ circle) draw arc T_1T_3 which should then pass through S if all steps have been carried out correctly and the diagram is accurate. The construction is very much easier to follow if all steps connected with points P , Q and T are carried out in different colours.

8.10. Construction for the state of strain on a general strain plane

The construction detailed above for determination of the state of stress on a general stress plane applies equally to the determination of *strains* when the symbols σ_1 , σ_2 and σ_3 are replaced by the principal *strain* values ϵ_1 , ϵ_2 and ϵ_3 .

Thus, having constructed the three-dimensional Mohr representation of the principal strains as described in §8.4, the general plane is located as described above and illustrated in Fig. 8.15.

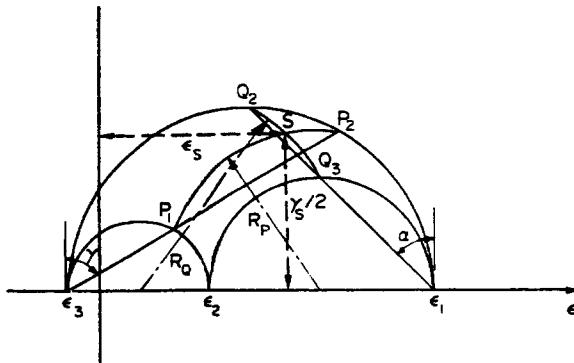


Fig. 8.15. Graphical construction for the state of strain on a general strain plane.

8.11. State of stress–tensor notation

The state of stress equations for any three-dimensional system of cartesian stress components have been obtained in §8.3 as:

$$\begin{aligned} p_{xn} &= \sigma_{xx} \cdot l + \sigma_{xy} \cdot m + \sigma_{xz} \cdot n \\ p_{yn} &= \sigma_{yx} \cdot l + \sigma_{yy} \cdot m + \sigma_{yz} \cdot n \\ p_{zn} &= \sigma_{zx} \cdot l + \sigma_{zy} \cdot m + \sigma_{zz} \cdot n \end{aligned}$$

The cartesian stress components within this equation can then be remembered conveniently in *tensor notation* as:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \text{ (general stress tensor)} \quad (8.33)$$

For a *principal stress system*, i.e. no shear, this reduces to:

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \text{ (principal stress tensor)} \quad (8.34)$$

and a special case of this is the so-called “*hydrostatic*” stress system with equal principal stresses in all three directions, i.e. $\sigma_1 = \sigma_2 = \sigma_3 = \bar{\sigma}$, and the tensor becomes:

$$\begin{bmatrix} \bar{\sigma} & 0 & 0 \\ 0 & \bar{\sigma} & 0 \\ 0 & 0 & \bar{\sigma} \end{bmatrix} \text{ (hydrostatic stress tensor)} \quad (8.35)$$

As shown in §23.16 it is often convenient to divide a general stress into two parts, one due to a hydrostatic stress $\bar{\sigma} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$, the other due to shearing deformations.

Another convenient tensor notation is therefore that for pure shear, i.e. $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0$ giving the tensor:

$$\begin{bmatrix} 0 & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & 0 & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & 0 \end{bmatrix} \text{ (pure shear tensor)} \quad (8.36)$$

The general stress tensor (8.33) is then the combination of the hydrostatic stress tensor and the pure shear tensor.

i.e. *General three-dimensional stress state* = *hydrostatic stress state* + *pure shear state*.

This approach is utilised in other sections of this text, notably: §8.16, §8.19 and §8.20.

It therefore follows that an alternative method of presentation of a *pure shear state of stress* is, in tensor form:

$$\begin{bmatrix} (\sigma_1 - \bar{\sigma}) & 0 & 0 \\ 0 & (\sigma_2 - \bar{\sigma}) & 0 \\ 0 & 0 & (\sigma_3 - \bar{\sigma}) \end{bmatrix} \quad (8.37)$$

N.B.: It can be shown that the condition for a state of stress to be one of pure shear is that the first stress invariant is zero.

i.e.

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 0 \quad (\text{see 8.15})$$

8.12. The stress equations of equilibrium

(a) In cartesian components

In all the previous work on complex stress systems it has been assumed that the stresses acting on the sides of any element are constant. In many cases, however, a general system of direct, shear and body forces, as encountered in practical engineering applications, will produce stresses of variable magnitude throughout a component. Despite this, however, the distribution of these stresses must always be such that overall equilibrium both of the component, and of any element of material within the component, is maintained, and it is a consideration of the conditions necessary to produce this equilibrium which produces the so-called *stress equations of equilibrium*.

Consider, therefore, a body subjected to such a general system of forces resulting in the cartesian stress components described in §8.2 together with the body-force stresses F_x ,

F_y and F_z . The element shown in Fig. 8.16 then displays, for simplicity, only the stress components in the X direction together with the body-force stress components. It must be realised, however, that similar components act in the Y and Z directions and these must be considered when deriving equations for equilibrium in these directions: they, of course, have no effect on equilibrium in the X direction.

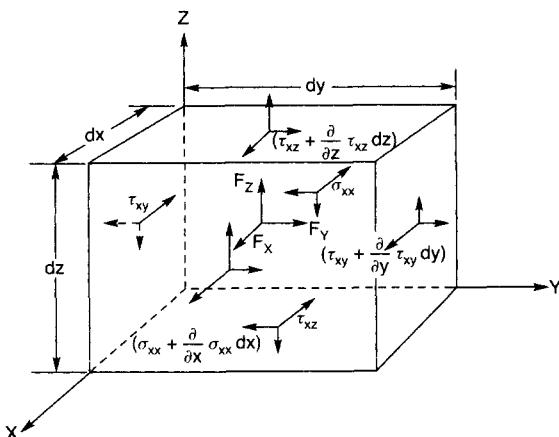


Fig. 8.16. Small element showing body force stresses and other stresses in the X direction only.

It will be observed that on each pair of opposite faces the stress changes in magnitude in the following manner,

$$\text{e.g.} \quad \text{stress on one face} = \sigma_{xx}$$

$$\begin{aligned} \text{stress on opposite face} &= \sigma_{xx} + \text{change in stress} \\ &= \sigma_{xx} + \text{rate of change} \times \text{distance between faces} \end{aligned}$$

Now the rate of change of σ_{xx} with x is given by $\partial\sigma_{xx}/\partial x$, partial differentials being used since σ_{xx} may well be a function of y and z as well as of x .

Therefore

$$\text{stress on opposite face} = \sigma_{xx} + \frac{\partial\sigma_{xx}}{\partial x} dx$$

Multiplying by the area $dy dz$ of the face on which this stress acts produces the force in the X direction.

Thus, for equilibrium of forces in the X direction,

$$\begin{aligned} &\left[\sigma_{xx} + \frac{\partial}{\partial x} \sigma_{xx} dx - \sigma_{xx} \right] dy dz + \left[\tau_{xy} + \frac{\partial}{\partial y} \tau_{xy} dy - \tau_{xy} \right] dx dz \\ &+ \left[\tau_{xz} + \frac{\partial}{\partial z} \tau_{xz} dz - \tau_{xz} \right] dx dy + F_x dx dy dz = 0 \end{aligned}$$

(The body-force term being defined as a stress per unit volume is multiplied by the volume ($dx dy dz$) to obtain the corresponding force.)

Dividing through by $dx dy dz$ and simplifying,

$$\left. \begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x &= 0 \\ \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + F_y &= 0 \\ \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + F_z &= 0 \end{aligned} \right\} \quad (8.38)$$

and in the Z direction,

these equations being termed the *general stress equations of equilibrium*.

Bearing in mind the comments of §8.2, the symbol τ in the above equations may be replaced by σ , the mixed suffix denoting the fact that it is a shear stress, and the above equations can be remembered quite easily using a similar procedure to that used in §8.2 based on the suffices, i.e. first suffices and body-force terms are constant for each horizontal row and in the normal order x , y and z .

	X	Y	Z	
X	$\frac{\partial \sigma_{xx}}{\partial x}$	$\frac{\partial \sigma_{xy}}{\partial y}$	$\frac{\partial \sigma_{xz}}{\partial z}$	$+F_x = 0$
Y	$\frac{\partial \sigma_{yx}}{\partial x}$	$\frac{\partial \sigma_{yy}}{\partial y}$	$\frac{\partial \sigma_{yz}}{\partial z}$	$+F_y = 0$
Z	$\frac{\partial \sigma_{zx}}{\partial x}$	$\frac{\partial \sigma_{zy}}{\partial y}$	$\frac{\partial \sigma_{zz}}{\partial z}$	$+F_z = 0$

The above equations have been derived by consideration of equilibrium of *forces* only, and this does not represent a complete check on the equilibrium of the system. This can only be achieved by an additional consideration of the *moments of the forces* which must also be in balance.

Consider, therefore, the element shown in Fig. 8.17 which, again for simplicity, shows only the stresses which produce moments about the Y axis. For convenience the origin of the cartesian coordinates has in this case been chosen to coincide with the centroid of the element. In this way the direct stress and body-force stress terms will be eliminated since the forces produced by these will have no moment about axes through the centroid.

It has been assumed that shear stresses τ_{xy} , τ_{yz} and τ_{xz} act on the coordinate planes passing through G so that they will each increase and decrease on either side of these planes as described above.

Thus, for equilibrium of moments about the Y axis,

$$\begin{aligned} &\left[\tau_{xz} + \frac{\partial(\tau_{xz})}{\partial z} \frac{dz}{2} \right] dx dy \frac{dz}{2} + \left[\tau_{xz} - \frac{\partial(\tau_{xz})}{\partial z} \frac{dz}{2} \right] dx dy \frac{dz}{2} \\ &- \left[\tau_{zx} + \frac{\partial(\tau_{zx})}{\partial x} \frac{dx}{2} \right] dy dz \frac{dx}{2} - \left[\tau_{zx} - \frac{\partial(\tau_{zx})}{\partial x} \frac{dx}{2} \right] dy dz \frac{dx}{2} = 0 \end{aligned}$$

Dividing through by $(dx dy dz)$ and simplifying, this reduces to

$$\tau_{xz} = \tau_{zx}$$

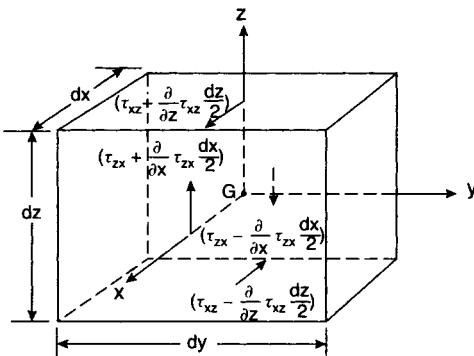


Fig. 8.17. Element showing only stresses which contribute to a moment about the Y axis.

Similarly, by consideration of the equilibrium of moments about the X and Z axes,

$$\tau_{zy} = \tau_{yz}$$

$$\tau_{xy} = \tau_{yx}$$

Thus the shears and complementary shears on adjacent faces are equal as in the simple two-dimensional case. The nine cartesian stress components thus reduce to six independent values,

i.e.
$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$

(b) In cylindrical coordinates

The equations of equilibrium derived above in cartesian components are very useful for components and stress systems which can easily be referred to a set of three mutually perpendicular axes. There are many cases, however, e.g. those components with *axial symmetry*, where other coordinate axes prove far more convenient. One such set of axes is the *cylindrical coordinate system* with variables r , θ and z as shown in Fig. 8.18.

Consider, therefore, the equilibrium in a *radial* direction of the element shown in Fig. 8.19(a). Again, for simplicity, only those stresses which produce force components in this direction are indicated. It must be observed, however, that in this case the $\sigma_{\theta\theta}$ terms will also produce components in the radial direction as shown by Fig. 8.19(b). The body-force stress components are denoted by F_R , F_Z and F_θ .

Therefore, resolving forces radially,

$$\begin{aligned} & \left[\sigma_{rr} + \frac{\partial}{\partial r}(\sigma_{rr}) dr \right] (r + dr) d\theta dz - \sigma_{rr} r d\theta dz + \left[\sigma_{r\theta} + \frac{\partial}{\partial \theta}(\sigma_{r\theta}) d\theta \right] dr dz \cos \frac{d\theta}{2} \\ & - \sigma_{\theta\theta} dr dz \cos \frac{d\theta}{2} + \left[\left(\sigma_{rz} + \frac{\partial(\sigma_{rz})}{\partial z} dz \right) - \sigma_{rz} \right] \left(r + \frac{dr}{2} \right) d\theta dr \\ & - \sigma_{\theta\theta} dr dz \sin \frac{d\theta}{2} - \left[\sigma_{\theta\theta} + \frac{\partial}{\partial \theta}(\sigma_{\theta\theta}) d\theta \right] dr dz \sin \frac{d\theta}{2} + F_R r dr d\theta dz = 0 \end{aligned}$$

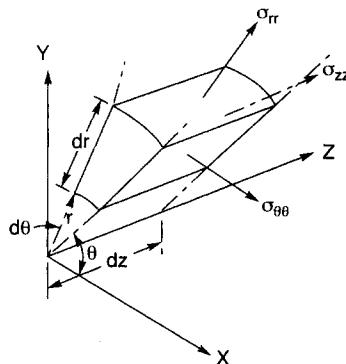


Fig. 8.18. Cylindrical coordinates.

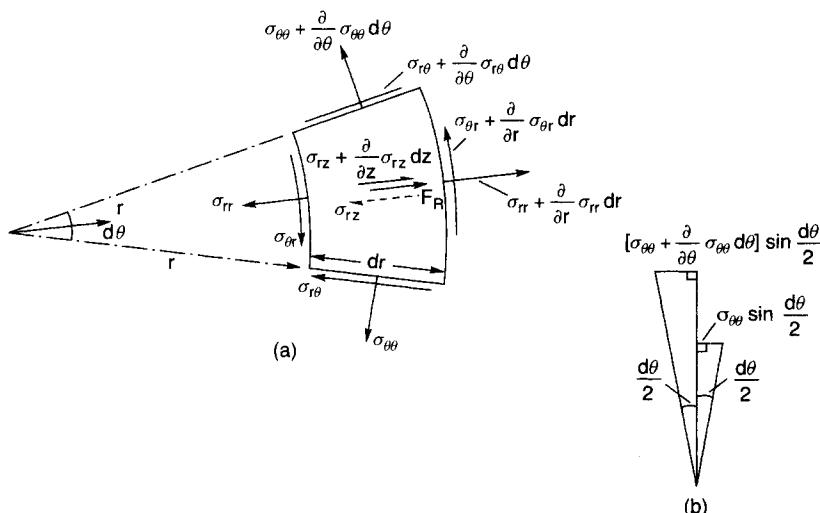


Fig. 8.19. (a) Element showing stresses which contribute to equilibrium in the radial and circumferential directions. (b) Radial components of hoop stresses.

With $\cos \frac{d\theta}{2} \approx 1$ and $\sin \frac{d\theta}{2} \approx \frac{d\theta}{2}$, this equation reduces to

$$\frac{\partial}{\partial r}(\sigma_{rr}) + \frac{1}{r} \frac{\partial}{\partial \theta}(\sigma_{r\theta}) + \frac{\partial}{\partial z}(\sigma_{rz}) + \frac{(\sigma_{rr} - \sigma_{\theta\theta})}{r} + F_R = 0 \quad \left. \right\}$$

Similarly, in the θ direction, the relevant equilibrium equation reduces to

$$\frac{\partial}{\partial r}(\sigma_{r\theta}) + \frac{1}{r} \frac{\partial}{\partial \theta}(\sigma_{\theta\theta}) + \frac{\partial}{\partial z}(\sigma_{\theta z}) + \frac{2\sigma_{r\theta}}{r} + F_\theta = 0 \quad \left. \right\} \quad (8.39)$$

and in the Z direction (Fig. 8.20)

$$\frac{\partial}{\partial r}(\sigma_{rz}) + \frac{1}{r} \frac{\partial}{\partial \theta}(\sigma_{\theta z}) + \frac{\partial}{\partial z}(\sigma_{zz}) + \frac{\sigma_{rz}}{r} + F_z = 0 \quad \left. \right\}$$

These are, then, the *stress equations of equilibrium in cylindrical coordinates* and in their most general form. Clearly these are difficult to memorise and, fortunately, very few problems arise in which the equations in this form are required. In many cases *axial symmetry* exists and circular sections remain concentric and circular throughout loading, i.e. $\sigma_{r\theta} = 0$.

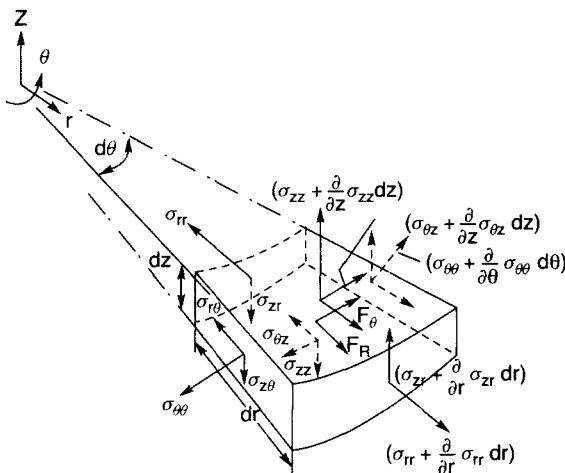


Fig. 8.20. Element indicating additional stresses which contribute to equilibrium in the axial (z) direction.

Thus for **axial symmetry** the equations reduce to

$$\left. \begin{aligned} \frac{\partial}{\partial r}(\sigma_{rr}) + \frac{\partial}{\partial z}(\sigma_{rz}) + \frac{(\sigma_{rr} - \sigma_{\theta\theta})}{r} + F_R &= 0 \\ \frac{1}{r} \frac{\partial(\sigma_{\theta\theta})}{\partial \theta} + \frac{\partial(\sigma_{\theta z})}{\partial z} + F_\theta &= 0 \\ \frac{\partial}{\partial r}(\sigma_{rz}) + \frac{1}{r} \frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial(\sigma_{zz})}{\partial z} + \frac{\sigma_{rz}}{r} + F_z &= 0 \end{aligned} \right\} \quad (8.40)$$

Further simplification applies in cases where the **coordinate axes** can be selected to **coincide with principal stress directions** as in the case of thick cylinders subjected to uniform pressure or thermal gradients. In such cases there will be no shear, and in the absence of body forces the equations reduce to the relatively simple forms

$$\left. \begin{aligned} \frac{\partial}{\partial r}(\sigma_{rr}) + \frac{(\sigma_{rr} - \sigma_{\theta\theta})}{r} &= 0 \\ \frac{\partial(\sigma_{\theta\theta})}{\partial \theta} &= 0 \\ \frac{\partial(\sigma_{zz})}{\partial z} &= 0 \end{aligned} \right\} \quad (8.41)$$

8.13. Principal stresses in a three-dimensional cartesian stress system

As an alternative to the graphical Mohr's circle procedures the principal stresses in three-dimensional complex stress systems can be determined analytically as follows.

The equations for the state of stress at a point derived in §8.3 may be combined to give the equation

$$\begin{aligned} \sigma_n^3 - (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})\sigma_n^2 + (\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{xx}\sigma_{zz} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)\sigma_n \\ - (\sigma_{xx}\sigma_{yy}\sigma_{zz} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{zx}^2 - \sigma_{zz}\tau_{xy}^2 + 2\tau_{xy}\tau_{yz}\tau_{zx}) = 0 \end{aligned} \quad (8.42)$$

With a knowledge of the cartesian stress components this cubic equation can be solved for σ_n to produce the three principal stress values required. A general procedure for the solution of cubic equations is given below.

8.13.1. Solution of cubic equations

Consider the cubic equation

$$x^3 + ax^2 + bx + c = 0 \quad (1)$$

Substituting, $x = y - a/3$ (2)

with $p = b - a^2/3$ (3)

and $q = c - \frac{ab}{3} + \frac{2a^3}{27}$ (4)

we obtain the modified equation

$$y^3 + py + q = 0 \quad (5)$$

Substituting, $y = rz$ (6)

$$z^3 + \frac{pz}{r^2} + \frac{q}{r^3} = 0 \quad (7)$$

Now consider the standard trigonometric identity

$$\cos 3\theta = 4\cos^3 \theta - 3\cos \theta \quad (8)$$

Rearranging and substituting $z = \cos \theta$, (9)

$$z^3 - \frac{3z}{4} - \frac{1}{4}\cos 3\theta = 0 \quad (10)$$

(7) and (10) are of similar form and will be identical provided that

$$r = \sqrt{-\frac{4p}{3}} \quad (11)$$

and $\cos 3\theta = -\frac{4q}{r^3}$ (12)

Three values of θ may be obtained to satisfy (12),

i.e. $\theta, \theta + 120^\circ$ and $\theta + 240^\circ$

Then, from (9), three corresponding values of z are obtained, namely

$$z_1 = \cos \theta^\circ$$

$$z_2 = \cos(\theta + 120^\circ)$$

$$z_3 = \cos(\theta + 240^\circ)$$

(6) then yields appropriate values of y and hence the required values of x via (2).

8.14. Stress invariants; Eigen values and Eigen vectors

Consider the special case of the “stress at a point” tetrahedron Fig. 8.3 where plane ABC is a principal plane subjected to a principal stress σ_p and, by definition, zero shear stress. The normal stress is thus coincident with the resultant stress and both equal to σ_p .

If the direction cosines of σ_p (and hence of the principal plane) are l_p, m_p, n_p then:

$$p_{xn} = \sigma_p \cdot l_p$$

$$p_{yn} = \sigma_p \cdot m_p$$

$$p_{zn} = \sigma_p \cdot n_p$$

i.e. substituting in eqns. (8.13), (8.14) and (8.15) we have:

$$\sigma_p \cdot l_p = \sigma_{xx} \cdot l_p + \sigma_{xy} \cdot m_p + \sigma_{xz} \cdot n_p$$

$$\sigma_p \cdot m_p = \sigma_{yx} \cdot l_p + \sigma_{yy} \cdot m_p + \sigma_{yz} \cdot n_p$$

$$\sigma_p \cdot n_p = \sigma_{zx} \cdot l_p + \sigma_{zy} \cdot m_p + \sigma_{zz} \cdot n_p$$

or

$$\left. \begin{aligned} \mathbf{0} &= (\sigma_{xx} - \sigma_p)l_p + \sigma_{xy} \cdot m_p + \sigma_{xz} \cdot n_p \\ \mathbf{0} &= \sigma_{yx}l_p + (\sigma_{yy} - \sigma_p)m_p + \sigma_{yz} \cdot n_p \\ \mathbf{0} &= \sigma_{zx}l_p + \sigma_{zy} \cdot m_p + (\sigma_{zz} - \sigma_p)n_p \end{aligned} \right\} \quad (8.43)$$

Considering eqn. (8.43) as a set of three homogeneous linear equations in unknowns l_p, m_p and n_p , the direction cosines of the principal plane, one possible solution, viz. $l_p = m_p = n_p = 0$, can be dismissed since $l^2 + m^2 + n^2 = 1$ must always be maintained. The only other solution which gives real values for the direction cosines is that obtained by equating the determinant of the R.H.S. to zero:

$$\text{i.e. } \begin{vmatrix} (\sigma_{xz} - \sigma_p) & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & (\sigma_{yy} - \sigma_p) & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & (\sigma_{zz} - \sigma_p) \end{vmatrix} = 0$$

Evaluating the determinant yields the so-called “characteristic equation”

$$\begin{aligned} \sigma_p^3 - (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})\sigma_p^2 + [(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx}) - (\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2)]\sigma_p \\ - [\sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{xy}\sigma_{yz}\sigma_{zx} - (\sigma_{xx}\sigma_{yz}^2 + \sigma_{yy}\sigma_{zx}^2 + \sigma_{zz}\sigma_{xy}^2)] = 0 \end{aligned} \quad (8.44)$$

Thus, for any given set of cartesian stress components in three dimensions a solution of this cubic equation is required before principal stress value can be determined; a graphical solution is not possible.

Eigen values

The solutions for the principal stresses σ_1 , σ_2 and σ_3 from the characteristic equation are known as the **Eigen values** whilst the associated direction cosines l_p , m_p and n_p are termed the **Eigen vectors**.

One procedure for solution of the cubic characteristic equation is given in §8.10.

8.15. Stress invariants

If, for the same applied stress system, the stress components had been given relative to some other set of cartesian co-ordinates x' , y' and z' , the above equation would still apply (with x' replacing x , y' replacing y and z' replacing z) and would still produce the same principal stress values. It follows, therefore, that whatever axis system is chosen the coefficients of the various terms of the characteristics equation must have the same values, i.e. they are “non-varying quantities” or “*invariant*”.

The equation can thus be re-written in the form:

$$\sigma_p^3 - I_1\sigma_p^2 - I_2\sigma_p - I_3 = 0 \quad (8.45)$$

with
$$\left. \begin{aligned} I_1 &= \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \\ I_2 &= (\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2) - (\sigma_{xx} \cdot \sigma_{yy} + \sigma_{yy} \cdot \sigma_{zz} + \sigma_{zz} \cdot \sigma_{xx}) \\ I_3 &= \sigma_{xx} \cdot \sigma_{yy} \cdot \sigma_{zz} + 2\sigma_{xy} \cdot \sigma_{yz} \cdot \sigma_{zx} - \sigma_{xx} \sigma_{yz}^2 - \sigma_{yy} \sigma_{zx}^2 - \sigma_{zz} \sigma_{xy}^2 \end{aligned} \right\} \quad (8.46)$$

the three quantities I_1 , I_2 and I_3 being termed the **stress invariants**.

If the reference axes selected are the principal stress axes in the system then all shear components reduce to zero and the equations (8.46) reduce to:

$$\left. \begin{aligned} I_1 &= \sigma_1 + \sigma_2 + \sigma_3 &= \Sigma\sigma_p \\ I_2 &= -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) &= \Sigma\sigma_p^2 \\ I_3 &= \sigma_1\sigma_2\sigma_3 &= \Sigma\sigma_p^3 \end{aligned} \right\} \quad (8.47)$$

The first and second invariants are particularly important in development of the theory of plasticity since it is assumed that:

- (a) I_1 has no influence on initial yielding
- (b) $I_2 = \text{constant}$ can be taken as an important criterion of yielding.

For biaxial stress conditions, i.e. $\sigma_3 = 0$, the third stress invariant vanishes and the others reduce to

$$\left. \begin{aligned} I_1 &= \sigma_1 + \sigma_2 \\ I_2 &= \sigma_1\sigma_2 \end{aligned} \right\} \quad (8.48)$$

or, in the xy plane, from eqn. (8.46)

$$\left. \begin{aligned} I_1 &= \sigma_{xx} + \sigma_{yy} \\ I_2 &= \sigma_{xx}\sigma_{yy} - \sigma_{xy}^2 \end{aligned} \right\} \quad (8.49)$$

Now from eqn. (13.11)[†] the principal stresses in a two-dimensional stress system are given by:

$$\begin{aligned}\sigma_{1,2} &= \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) \pm \frac{1}{2}[(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2]^{\frac{1}{2}} \\ &= \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) \pm \frac{1}{2}[(\sigma_{xx} + \sigma_{yy})^2 - 4\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2]^{\frac{1}{2}}\end{aligned}$$

which is the general solution of the following quadratic equation:

$$\sigma_p^2 = (\sigma_{xx} + \sigma_{yy})\sigma_p + (\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2) = 0$$

i.e.

$$\sigma_p^2 - I_1\sigma_p + I_2 = 0 \quad (8.50)$$

The graphical solution of this equation is as follows (see Fig. 8.21):

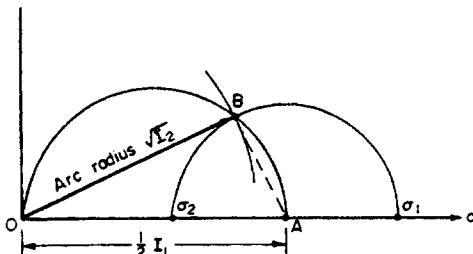


Fig. 8.21. Graphical determination of principal stresses in a two-dimensional stress system from known stress invariant I values (solution for positive I_2 value)

- On a horizontal (direct stress) axis mark off a length $OA = \frac{1}{2}I_1$.
- Draw semi-circle on OA as diameter.
- With centre O draw arc OB , radius $\sqrt{I_2}$, to cut the semi-circle at B .
- With centre A and radius AB draw semi-circle to cut stress axis at σ_1 and σ_2 the required principal stress values.

N.B. If I_2 is negative (see §8.46), algebraically $\sqrt{I_2} > \frac{1}{2}I_1$ and the line OB cannot cut the semi-circle on OA as diameter and no solution can be obtained. In this case an alternative construction is required – see Fig. 8.22.

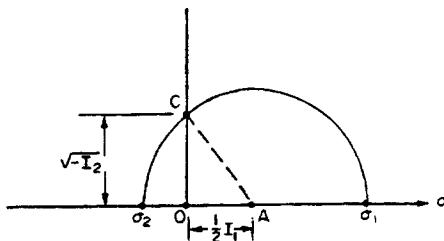


Fig. 8.22. As Fig. 8.21 but for negative I_2 value.

- Again mark off length $OA = \frac{1}{2}I_1$.
- Erect perpendicular at O of length $OC = \sqrt{-I_2}$.

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1977.

- (iii) With centre A and radius AC draw a circle to cut OA (produced as necessary) at σ_1 and σ_2 the required principal stress values.

Returning to a three-dimensional principal stress system a further interesting graphical relationship is obtained from the 3D Mohr circle construction – see Fig. 8.23.*

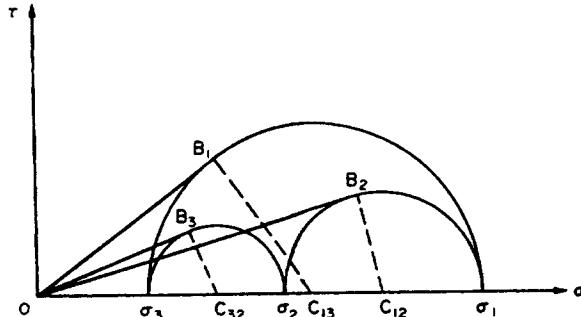


Fig. 8.23. Stress invariants for a three-dimensional stress system in terms of tangents to the Mohr stress circles
 $I_1 = \sigma_1 + \sigma_2 + \sigma_3$, $I_2 = OB_1^2 + OB_2^2 + OB_3^2$, $I_3 = OB_1 \cdot OB_2 \cdot OB_3$.

The three stress invariants are given in Fig. 8.23 in terms of the tangents to the three circles from the origin 0 as:

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 = OB_1^2 + OB_2^2 + OB_3^2$$

$$I_3 = \sigma_1\sigma_2\sigma_3 = OB_1 \times OB_2 \times OB_3$$

8.16. Reduced stresses

An alternative form of the cubic characteristic equation is obtained if a “hydrostatic stress” of $I_1/3$ is subtracted from the original stress system to produce “reduced stresses” $\sigma' = \sigma - I_1/3$.

Thus, replacing σ_p by $(\sigma' + I_1/3)$ in eqn. (8.45) we have:

$$\sigma'^3 - \left(\frac{I_1^2 + 3I_2}{3} \right) \sigma' - \left(\frac{2I_1^3 + 9I_1I_2 + 27I_3}{27} \right) = 0$$

or

$$\sigma'^3 - J_1\sigma'^2 - J_2\sigma' - J_3 = 0 \quad (8.51)$$

with

$$J_1 = 0$$

$$J_2 = \frac{1}{3}[I_1^2 + 3I_2]$$

$$J_3 = \frac{1}{27}[2I_1^3 + 9I_1I_2 + 27I_3]$$

* M.G. Derrington and W. Johnson, *The Defect of Mohr's Circle for Three-Dimensional Stress States*.

The terms J_1 , J_2 and J_3 are termed the *invariants of reduced stress* and, again, have special significance in the consideration of yielding of metals and associated plastic theory.

It will be shown in §8.20 that the hydrostatic stress component does not affect the yield of metals and

$$\text{hydrostatic stress} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3}I_1$$

It therefore follows that first stress invariant I_1 also has no significance on yielding and since the principal stress system can be written, as above, in terms of reduced stresses $\sigma' = (\sigma - 1/3 I_1)$ it also follows that it must be the reduced stress components which influence yielding.

(N.B.: “Reduced stresses” are synonymous with the deviatoric stresses introduced in §8.20.)

Other useful relationships which can be derived from the above equations are:

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 6J_2 \quad (8.52)$$

$$\text{and } (\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) = 2I_1^2 + 6I_2 \quad (8.53)$$

The left-hand sides of both equations are thus, in themselves, invariant and are useful in further considerations of strain energy, yielding and failure.

For example, the shear strain energy theory of elastic failure uses the criterion:

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_y^2 = \text{constant}$$

which, from eqn. (8.52), can be simply re-written as

$$J_2 = \text{constant.}$$

N.B.: It should be remembered that eqns. (8.52) and (8.53) are merely different ways of presenting the same information since:

$$6J_2 = 2I_1^2 + 6I_2.$$

8.17. Strain invariants

It has been shown in §14.10[†] that the basic transformation equations for stress and strain have identical form provided that ϵ is used in place of σ and $\gamma/2$ in place of τ . The equations derived above for the stress invariants will therefore apply equally for strain conditions provided that the same rules are followed.

8.18. Alternative procedure for determination of principal stresses (eigen values)

An alternative solution to the characteristic cubic equation expressed in stress invariant format, viz. eqn. (8.45), is as follows:

Given the basic equation:

$$\sigma_p^3 - I_1\sigma_p^2 - I_2\sigma_p - I_3 = 0 \quad (8.45)\text{bis}$$

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

the stress invariants may be calculated from:

$$\begin{aligned} I_1 &= \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \\ I_2 &= -(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx}) + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 \\ I_3 &= \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{zx}^2 - \sigma_{zz}\tau_{xy}^2 \end{aligned} \quad (8.46)\text{bis}$$

and the required principal stresses obtained from[‡]:

$$\begin{aligned} \sigma_{p_1} &= 2S \cos(\alpha/3) + I_1/3 \\ \sigma_{p_2} &= 2S \cos[(\alpha/3) + 120^\circ] + I_1/3 \\ \sigma_{p_3} &= 2S \cos[(\alpha/3 + 240^\circ)] + I_1/3 \end{aligned} \quad (8.54)$$

with

$$S = (R/3)^{1/2} \quad \text{and} \quad \alpha = \cos^{-1}(-Q/2T)$$

and

$$R = \frac{1}{3}I_1^2 - I_2$$

$$Q = \frac{1}{3}I_1I_2 - I_3 - \frac{2}{27}I_1^3$$

$$T = \left(\frac{1}{27}R^3 \right)^{1/2}$$

After calculation of the three principal stress values, they can be placed in their normal conventional order of magnitude, viz. σ_1 , σ_2 and σ_3 .

The procedure is, in effect, the same as that of §8.13 but carried out in terms of the stress invariants.

8.18.1. Evaluation of direction cosines for principal stresses (eigen vectors)

Having determined the three principal stress values for a given three-dimensional complex stress state using the procedures of §8.13.1 or §8.18, above, a complete solution of the problem generally requires a determination of the directions in which these stresses act—as given by their respective direction cosines or eigen vector values.

The relationship between a particular principal stress σ_p and the cartesian stress components is given by eqn (8.43)

i.e.

$$(\sigma_{xx} - \sigma_p)l + \tau_{xy} \cdot m + \tau_{xz} \cdot n = 0$$

$$\tau_{xy} \cdot l + (\sigma_{yy} - \sigma_p)m + \tau_{yz} \cdot n = 0$$

$$\tau_{xz} \cdot l + \tau_{yz} \cdot m + (\sigma_{zz} - \sigma_p)n = 0$$

If one of the known principal stress values, say σ_1 , is substituted in the above equations together with the given cartesian stress components, three equations result in the three unknown direction cosines for that principal stress i.e. l_1 , m_1 and n_1 .

However, only two of these are independent equations and the additional identity $l_1^2 + m_1^2 + n_1^2 = 1$ is required in order to evaluate l_1m_1 and n_1 .

[‡] E.E. Messal, "Finding true maximum shear stress", *Machine Design*, Dec. 1978.

The procedure can then be repeated substituting the other principal stress values σ_2 and σ_3 , in turn, to produce eigen vectors for these stresses but it is tedious and an alternative matrix approach is recommended as follows:

Equation (8.43) above can be expressed in matrix form, thus:

$$\begin{bmatrix} (\sigma_{xx} - \sigma_p) & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & (\sigma_{yy} - \sigma_p) & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & (\sigma_{zz} - \sigma_p) \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = 0$$

with cofactors of the determinant on the elements of the first row of:

$$a = \begin{vmatrix} (\sigma_{yy} - \sigma_p) & \tau_{yz} \\ \tau_{yz} & (\sigma_{zz} - \sigma_p) \end{vmatrix}$$

$$b = - \begin{vmatrix} \tau_{xy} & \tau_{yz} \\ \tau_{xz} & (\sigma_{zz} - \sigma_p) \end{vmatrix}$$

$$c = \begin{vmatrix} \tau_{xy} & (\sigma_{yy} - \sigma_p) \\ \tau_{xz} & \tau_{yz} \end{vmatrix}$$

with the direction cosines or eigen vectors of the principal stresses given by:

$$l_p = ak \quad m_p = bk \quad n_p = ck$$

with

$$k = \frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

thus satisfying the identity $l_p^2 + m_p^2 + n_p^2 = 1$.

Substitution of any principal stress value, again say σ_1 , into the above equations together with the given cartesian stress components allows solution of the determinants and yields values for a_1 , b_1 and c_1 , hence k_1 and hence l_1 , m_1 and n_1 , the desired eigen vectors. The process can then be repeated for the other principal stress values $\sigma_2 + \sigma_3$.

8.19. Octahedral planes and stresses

Any complex three-dimensional stress system produces three mutually perpendicular principal stresses σ_1 , σ_2 , and σ_3 . Associated with this stress state are so-called *octahedral planes* each of which cuts across the corners of a principal element such as that shown in Fig. 8.24 to produce the octahedron (8-sided figure) shown in Fig. 8.25. The stresses acting on the octahedral planes have particular significance.

The normal stresses acting on each of the octahedral planes are equal in value and tend to compress or enlarge the octahedron without distorting its shape. They are thus said to be *hydrostatic* stresses and have values given by

$$\sigma_{\text{oct}} = \frac{1}{3}[\sigma_1 + \sigma_2 + \sigma_3] = \bar{\sigma} \quad (8.55)$$

Similarly, the shear stresses acting on each of the octahedral planes are also identical and tend to distort the octahedron without changing its volume.

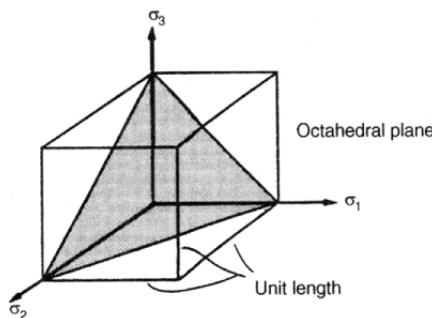


Fig. 8.24. Cubical principal stress element showing one of the octahedral planes.

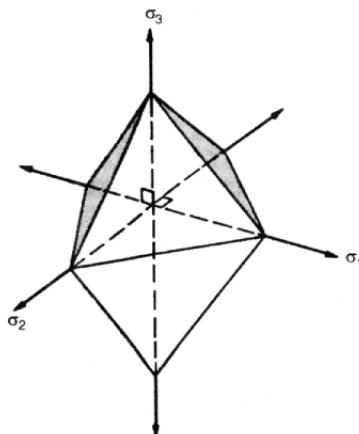


Fig. 8.25. Principal stress system showing the eight octahedral planes forming an octahedron.

The value of the *octahedral shear stresses*[†] is given by

$$\begin{aligned}\tau_{\text{oct}} &= \frac{1}{3} - [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{1/2} \\ &= \frac{2}{3} - [\tau_{12}^2 + \tau_{23}^2 + \tau_{13}^2]^{1/2}\end{aligned}$$

τ_{12} , τ_{23} and τ_{13} being the maximum shear stresses in the 1–2, 2–3 and 1–3 planes respectively.

Thus the general state of stress may be represented on octahedral planes as shown in Fig. 8.26, the *direction cosines* of the octahedral planes being given by

$$l = m = n = \pm 1/\sqrt{1^2 + 1^2 + 1^2} = \pm 1/\sqrt{3} \quad (8.58)$$

The values of the octahedral shear and direct stresses may also be obtained by the graphical construction of §8.9 since they are represented by a point in the shaded area of the three-dimensional Mohr's circle construction of Figs. 8.8 and 8.9.

[†] A.J. Durelli, E.A. Phillips and C.H. Tsao. *Analysis of Stress and Strain*, chap. 3, p. 26, McGraw-Hill, New York, 1958.

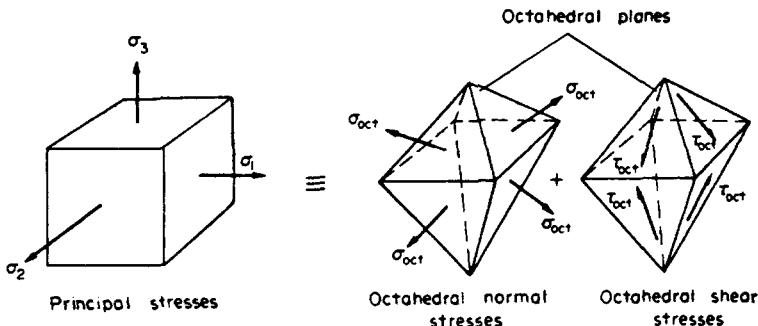


Fig. 8.26. Representation of a general state of stress on the octahedral planes.

The octahedral shear stress has a particular significance in relation to the elastic failure of materials. Whilst its value is always smaller than the greatest numerical (principal) shear stress, it nevertheless has a value which is influenced by all three principal stress values and has been shown to be a reliable criterion for predicting yielding under complex loading conditions.

The **maximum octahedral shear stress theory of elastic failure** thus assumes that yield or failure under complex stress conditions will occur when the octahedral shear stress has a value equal to that obtained in the simple tensile test at yield.

Now for uniaxial tension, $\sigma_2 = \sigma_3 = 0$ and $\sigma_1 = \sigma_y$ and from eqn. (8.56)

$$\tau_{\text{oct}} = \frac{\sqrt{2}}{3} \sigma_y$$

Therefore the criterion of failure becomes

$$\frac{\sqrt{2}}{3} \sigma_y = \frac{1}{3} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{1/2}$$

$$\text{i.e. } 2\sigma_y^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \quad (8.59)$$

This is clearly the same criterion as that referred to earlier as the Maxwell/von Mises *distortion* or *shear strain energy* theory.

8.20. Deviatoric stresses

It is sometimes convenient to consider stresses with reference to some false zero, i.e. to measure their values above or below some selected datum stress value, and not their absolute values. This is particularly useful in advanced analysis using the theory of plasticity.

The selected datum stress $\bar{\sigma}$ or "false zero" is taken to be that stress which produces only a change in volume. This is the stress which acts equally in all directions and is referred to earlier (page 251) as the *hydrostatic* or *dilatational* stress. This is defined in terms of the principal stresses or the cartesian stresses as follows:

$$\bar{\sigma} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) \quad (8.60)$$

i.e. $\bar{\sigma}$ = mean of the three principal stress values.

The principal stresses in any three-dimensional complex stress system may now be written in the form

$$\begin{aligned}\sigma_1 &= \text{mean stress} + \text{deviation from the mean} \\ &= \text{hydrostatic stress} + \text{deviatoric stress}\end{aligned}$$

Thus the additional terms required to make up any stress value from the datum to the absolute value are termed the *deviatoric stresses* and written with a prime superscript,

i.e. $\sigma_1 = \bar{\sigma} + \sigma'$, etc.

Cartesian stresses σ_{xx} , σ_{yy} and σ_{zz} can now be referred to the new datum as follows:

$$\left. \begin{aligned}\sigma'_{xx} &= \sigma_{xx} - \bar{\sigma} = \frac{1}{3}(2\sigma_{xx} - \sigma_{yy} - \sigma_{zz}) \\ \sigma'_{yy} &= \sigma_{yy} - \bar{\sigma} = \frac{1}{3}(2\sigma_{yy} - \sigma_{xx} - \sigma_{zz}) \\ \sigma'_{zz} &= \sigma_{zz} - \bar{\sigma} = \frac{1}{3}(2\sigma_{zz} - \sigma_{xx} - \sigma_{yy})\end{aligned}\right\} \quad (8.61)$$

All the above values then represent deviatoric stresses.

It may be observed that the system used for representing stresses in terms of the datum stress and the deviation from the datum is, in effect, a consideration of the normal and shear stresses respectively, on the octahedral planes, since the octahedral and deviatoric planes are equally inclined to all three axes ($l = m = n = \pm 1/\sqrt{3}$) and the selected datum stress

$$\bar{\sigma} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

is also the octahedral normal stress value.

As stated earlier when discussing octahedral stresses, this has a particular relevance to the yield behaviour of materials.

Whilst any detailed study of the theory of plasticity is beyond the scope of this text, the fundamental requirements of the theory should be understood. These are:

- (a) the volume of material remains constant under plastic deformation;
- (b) the hydrostatic stress component $\bar{\sigma}$ does not cause yielding of the material;
- (c) the hydrostatic stress component $\bar{\sigma}$ does not influence the point at which yielding occurs.

From these points it is clear that it is therefore the **deviatoric or octahedral shear stresses which must govern the yield behaviour of materials**. This is supported by the accuracy of the octahedral shear stress (distortion energy) theory and, to a lesser extent, the maximum shear stress theory, in predicting the elastic failure of *ductile* materials. Both theories involve stress differences, i.e. shear stresses, and are therefore independent of the hydrostatic stress as indicated by (b) above.

The representation of a principal stress system in terms of the octahedral and deviatoric stresses may thus be shown as in Fig. 8.27.

It should now be clear that the terms *hydrostatic*, *volumetric*, *mean*, *dilatational* and *octahedral normal stresses* all indicate the same quantity.

The standard elastic stress-strain relationships of eqn. (8.71)

$$\begin{aligned}\varepsilon_{xx} &= \frac{1}{E}[\sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz}] \\ \varepsilon_{yy} &= \frac{1}{E}[\sigma_{yy} - \nu\sigma_{xx} - \nu\sigma_{zz}] \\ \varepsilon_{zz} &= \frac{1}{E}[\sigma_{zz} - \nu\sigma_{xx} - \nu\sigma_{yy}]\end{aligned}$$

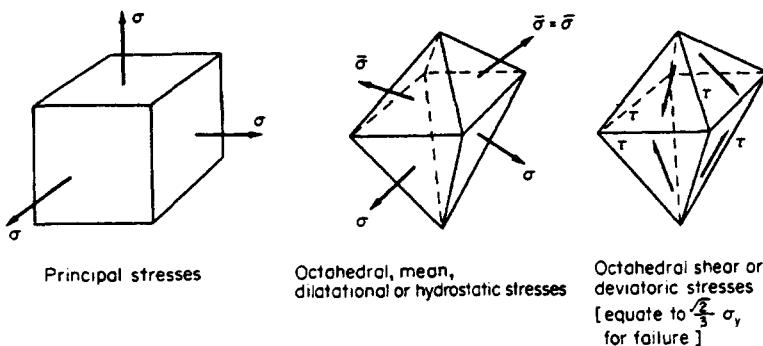


Fig. 8.27. Representation of a principal stress system in terms of octahedral and deviatoric stresses.

may be re-written in a form which distinguishes between those parts which contribute only to a change in volume and those producing a change of shape.

Thus, for a hydrostatic or mean stress $\sigma_m = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$ and remembering the relationship between the elastic constants $E = 2G(1 + \nu)$

$$\left. \begin{aligned} \varepsilon_{xx} &= \frac{1}{E}(1-2\nu)\sigma_m + \frac{1}{2G}(\sigma_{xx} - \sigma_m) \\ \varepsilon_{yy} &= \frac{1}{E}(1-2\nu)\sigma_m + \frac{1}{2G}(\sigma_{yy} - \sigma_m) \\ \varepsilon_{zz} &= \frac{1}{E}(1-2\nu)\sigma_m + \frac{1}{2G}(\sigma_{zz} - \sigma_m) \end{aligned} \right\} \quad (8.62)$$

with $\gamma_{xy} = \tau_{xy}/2G$; $\gamma_{yz} = \tau_{yz}/2G$; $\gamma_{zx} = \tau_{zx}/2G$.

The terms $(\sigma_{xx} - \sigma_m)$, $(\sigma_{yy} - \sigma_m)$ and $(\sigma_{zz} - \sigma_m)$ are the *deviatoric* components of stress.

The volumetric strain ε_m associated with the hydrostatic or mean stress σ_m is then:

$$\varepsilon_m = \frac{\sigma_m}{K} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

where K is the bulk modulus.

8.21. Deviatoric strains

As for the deviatoric stresses the *deviatoric strains* are also defined with reference to some selected “false zero” or datum value.

$$\bar{\varepsilon} = \frac{1}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \quad (8.63)$$

= mean of the three principal strain values.

Thus, referred to the new datum, the principal strain values become

$$\varepsilon'_1 = \varepsilon_1 - \bar{\varepsilon} = \varepsilon_1 - \frac{1}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$$

$$\therefore \left. \begin{aligned} \varepsilon'_1 &= \frac{1}{3}(2\varepsilon_1 - \varepsilon_2 - \varepsilon_3) \\ \varepsilon'_2 &= \frac{1}{3}(2\varepsilon_2 - \varepsilon_1 - \varepsilon_3) \\ \varepsilon'_3 &= \frac{1}{3}(2\varepsilon_3 - \varepsilon_1 - \varepsilon_2) \end{aligned} \right\} \quad (8.64)$$

Similarly,

and these are the so-called *deviatoric strains*. It may now be observed that the following relationship applies:

$$\varepsilon'_1 + \varepsilon'_2 + \varepsilon'_3 = 0$$

It can also be shown that the deviatoric strains are related to the principal strains as follows:

$$(\varepsilon'_1)^2 + (\varepsilon'_2)^2 + (\varepsilon'_3)^2 = \frac{1}{3}[(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2] \quad (8.66)$$

8.22. Plane stress and plane strain

If a body consists of two parallel planes a constant thickness apart and bounded by any closed surface as shown in Fig. 8.28, it is said to be a *plane body*. Associated with this type of body there is a particular class of problems within the general theory of elasticity which are termed *plane elastic* problems, and these allow a number of simplifying assumptions in their treatment.

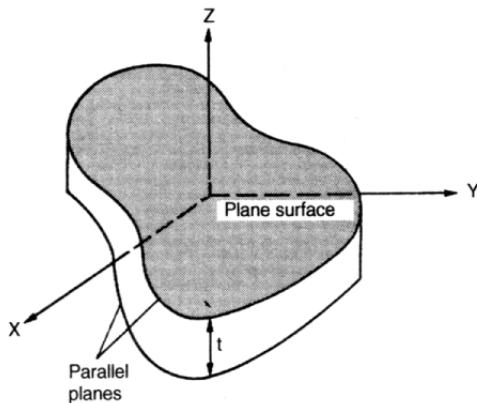


Fig. 8.28. A plane element.

In order to qualify for these simplifications, however, there are a number of restrictions which must be placed on the applied load system:

- (1) no loads may be applied to the top and bottom plane surfaces (in practice there is often a uniform stress in the Z direction on the planes but this can always be reduced to zero by superimposing a suitable stress σ_{zz} of opposite sign);
- (2) the loads on the lateral boundaries (and the surface shears) must be in the plane of the body and must be uniformly distributed across the thickness;
- (3) similarly, body forces in the X and Y directions must be uniform across the thickness and the body force in the Z direction must be zero.

There is no limitation on the thickness of the plane body and, indeed, the thickness serves as a means of classification within the general type of problem. Normally a *plane stress* approach is applied to members which are relatively thin in relation to their other dimensions, whereas *plane strain* methods are employed for relatively thick members. The terms plane stress and plane strain are defined in detail below.

The plane elastic type of problem may thus be defined as one in which stresses and strains do not vary in the Z direction. Additionally, lines parallel to the Z axis remain straight and parallel to the axis throughout loading.

i.e. $\gamma_{zx} = \gamma_{zy} = 0$

(The problem of torsion provides an exception to this rule.)

8.22.1. Plane stress

A plane stress problem is taken to be one in which σ_{zz} is zero. As stated above, cases where a uniform stress is applied to the plane surfaces can easily be reduced to this condition by application of a suitable σ_{zz} stress of opposite sign. Shear components in the Z direction must also be zero.

i.e. $\tau_{zx} = \tau_{zy} = 0$ (8.67)

Under these conditions the stress equations of equilibrium in cartesian coordinates reduce to

$$\left. \begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + Y &= 0 \end{aligned} \right\} \quad (8.68)$$

The following stress and strain relationships then apply:

$$\begin{aligned} \varepsilon_{xx} &= \frac{1}{E}(\sigma_{xx} - v\sigma_{yy}) & \varepsilon_{yy} &= \frac{1}{E}(\sigma_{yy} - v\sigma_{xx}) \\ \sigma_{xx} &= \frac{E}{(1-v^2)}[\varepsilon_{xx} + v\varepsilon_{yy}] & \sigma_{yy} &= \frac{E}{(1-v^2)}[\varepsilon_{yy} + v\varepsilon_{xx}] \\ \tau_{xy} &= G\gamma_{xy} = \frac{E}{2(1+v)}\gamma_{xy} \end{aligned}$$

Plane stress systems are often referred to as *two-dimensional* or *bi-axial* stress systems, a typical example of which is the case of thin plates loaded at their edges with forces applied in the plane of the plate.

8.22.2. Plane strain

Plane strain problems are normally defined as those in which the strains in the Z direction are zero. Again, problems with a uniform strain in the Z direction at all points on the plane surface can be reduced to the above case by the addition of a suitable uniform stress σ_{zz} , the additional lateral strains and displacements so introduced being easily calculated.

Thus

$$\varepsilon_{zz} = \gamma_{yz} = \gamma_{zx} = 0 \quad (8.69)$$

Also, from the basic assumptions of plane elastic problems,

$$\tau_{zy} = \tau_{zx} = 0$$

The equations of stress equilibrium in this case reduce to

$$\left. \begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + Y &= 0 \end{aligned} \right\} \quad (8.70)$$

The stress-strain relations are then as follows:

$$\begin{aligned} \varepsilon_{xx} &= \frac{(1 - \nu^2)}{E} \left[\sigma_{xx} - \frac{\nu}{(1 - \nu)} \sigma_{yy} \right] \quad \text{and} \quad \varepsilon_{yy} = \frac{(1 - \nu^2)}{E} \left[\sigma_{yy} - \frac{\nu}{(1 - \nu)} \sigma_{xx} \right] \\ \sigma_{xx} &= \frac{E(1 - \nu)}{(1 + \nu)(1 - 2\nu)} \left[\varepsilon_{xx} + \frac{\nu}{(1 - \nu)} \varepsilon_{yy} \right] \\ \sigma_{yy} &= \frac{E(1 - \nu)}{(1 + \nu)(1 - 2\nu)} \left[\varepsilon_{yy} + \frac{\nu}{(1 - \nu)} \varepsilon_{xx} \right] \end{aligned}$$

Also

$$\tau_{xy} = G\gamma_{xy}$$

It should be noted that the plane strain equations can be derived simply from the plane stress equations by replacing

$$\nu \text{ by } \frac{\nu}{(1 - \nu)} \quad \text{and} \quad E \text{ by } \frac{E}{(1 - \nu^2)}$$

A typical example of plane strain is the pressurisation of long cylinders where the above equations give accurate results, particularly in the middle portion of the cylinder, whether the end conditions are free, partially fixed or rigidly fixed.

An example of the transfer of a plane stress to a corresponding plane strain solution is given when the relevant equations for the hoop and radial stresses present in rotating thick cylinders are readily obtained from those of rotating thin discs by use of the substitution $\nu/(1 - \nu)$ in place of ν (see §4.4).

8.23. The stress-strain relations

The following formulae form a useful summary of the relationships which exist between the stresses and strains in a general three-dimensional stress system.

(a) Strains in terms of stress

$$\left. \begin{aligned} \varepsilon_{xx} &= \frac{1}{E} [\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})] & \gamma_{xy} &= \frac{2(1 + \nu)}{E} \tau_{xy} = \frac{\tau_{xy}}{G} \\ \varepsilon_{yy} &= \frac{1}{E} [\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})] & \gamma_{yz} &= \frac{2(1 + \nu)}{E} \tau_{yz} = \frac{\tau_{yz}}{G} \\ \varepsilon_{zz} &= \frac{1}{E} [\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})] & \gamma_{zx} &= \frac{2(1 + \nu)}{E} \tau_{zx} = \frac{\tau_{zx}}{G} \end{aligned} \right\} \quad (8.71)$$

(b) *Stresses in terms of strains*

$$\left. \begin{aligned} \sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)} [\varepsilon_{xx} + \nu(\varepsilon_{yy} + \varepsilon_{zz} - \varepsilon_{xx})] & \tau_{xy} &= \frac{E}{2(1+\nu)} \gamma_{xy} = G\gamma_{xy} \\ \sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)} [\varepsilon_{yy} + \nu(\varepsilon_{xx} + \varepsilon_{zz} - \varepsilon_{yy})] & \tau_{yz} &= \frac{E}{2(1+\nu)} \gamma_{yz} = G\gamma_{yz} \\ \sigma_{zz} &= \frac{E}{(1+\nu)(1-2\nu)} [\varepsilon_{zz} + \nu(\varepsilon_{xx} + \varepsilon_{yy} - \varepsilon_{zz})] & \tau_{zx} &= \frac{E}{2(1+\nu)} \gamma_{zx} = G\gamma_{zx} \end{aligned} \right\} \quad (8.72)$$

with

$$E = 2G(1+\nu) \quad \text{and} \quad E = 3K(1-2\nu)$$

hence

$$K = \frac{2G(1+\nu)}{3(1-2\nu)}$$

(c) *For biaxial stress conditions:*

(a) *Strains in terms of stresses*

$$\begin{aligned} \varepsilon_{xx} &= \frac{1}{E} [\sigma_{xx} - \nu\sigma_{yy}] \\ \varepsilon_{yy} &= \frac{1}{E} [\sigma_{yy} - \nu\sigma_{xx}] \quad \text{and} \quad \gamma_{xy} = \frac{2(1+\nu)}{E} \tau_{xy} \\ \varepsilon_{zz} &= -\frac{\nu}{E} [\sigma_{xx} + \sigma_{yy}] \end{aligned}$$

(b) *Stresses in terms of strains*

$$\begin{aligned} \sigma_{xx} &= \frac{E}{(1-\nu^2)} [\varepsilon_{xx} + \nu\varepsilon_{yy}] \quad \text{and} \quad \tau_{xy} = \frac{E}{2(1+\nu)} \gamma_{xy} \\ \sigma_{yy} &= \frac{E}{(1-\nu^2)} [\varepsilon_{yy} + \nu\varepsilon_{xx}] \end{aligned}$$

Equivalent expressions apply for polar coordinates with r , θ and z replacing x , y and z respectively.

8.24. The strain-displacement relationships

Consider the deformation of a cubic element of material as load is applied. Any corner of the element, e.g. P , will then move to some position P' , the movement having components u , v and w in the X , Y and Z directions respectively as shown in Fig. 8.29. Other points in the cube will also be displaced but generally by different amounts.

The movement in the X direction will be given by

$$u = \left(\frac{\partial u}{\partial x} \right) \delta x$$

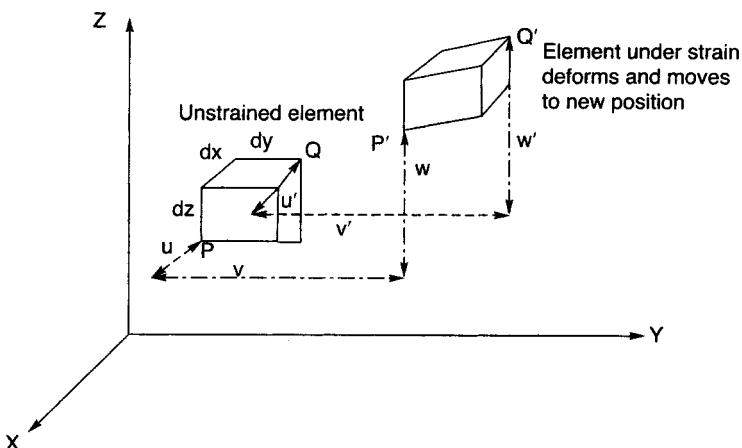


Fig. 8.29. Deformation of a cubical element under load.

The *strain* in the *X* direction will then be

$$\varepsilon_{xx} = \frac{\text{change in length}}{\text{original length}} = \frac{\left(\frac{\partial u}{\partial x}\right) \delta x}{\delta x}$$

i.e.

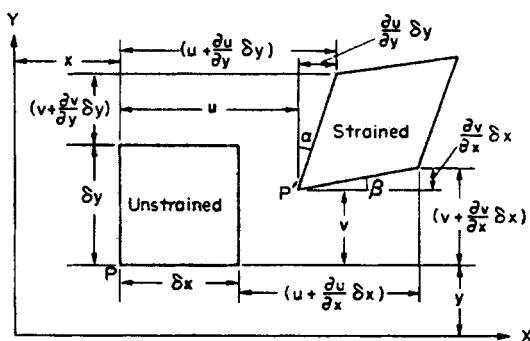
$$\varepsilon_{xx} = \frac{\partial u}{\partial x}$$

Similarly,

$$\left. \begin{aligned} \varepsilon_{yy} &= \frac{\partial v}{\partial y} \\ \varepsilon_{zz} &= \frac{\partial w}{\partial z} \end{aligned} \right\}$$

(8.73)

Consider now Fig. 8.30 which shows the deformations in the *XY* plane enlarged.

Fig. 8.30. Deformations under load in the *XY* plane.

Shear strains are defined as angles of deformation or changes in angles between two perpendicular segments. Thus γ_{xy} is the change in angle between two perpendicular segments

in the XY plane as load is applied,

$$\text{i.e. } \gamma_{xy} = \alpha + \beta = \frac{\left(\frac{\partial v}{\partial x}\right) \delta x}{\delta x} + \frac{\left(\frac{\partial u}{\partial y}\right) \delta y}{\delta y}$$

$$\therefore \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \quad \left. \begin{array}{l} \\ \end{array} \right\} \quad (8.74)$$

$$\text{Similarly, } \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \quad \left. \begin{array}{l} \\ \end{array} \right\} \quad (8.74)$$

$$\text{and } \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \quad \left. \begin{array}{l} \\ \end{array} \right\} \quad (8.74)$$

Summary of the strain-displacement equations

(a) *In cartesian coordinates* with displacements u , v and w along x , y and z respectively.

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \quad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \quad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \quad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

(b) *In polar coordinates* with displacements u_r , u_θ and u_z along r , θ and z respectively: these equations become:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}$$

$$\varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r} \cdot \frac{\partial u_\theta}{\partial \theta}$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

$$\text{with } \gamma_{r\theta} = \frac{1}{r} \cdot \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}$$

$$\gamma_{\theta z} = \frac{1}{r} \cdot \frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}$$

$$\gamma_{zr} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}$$

8.25. The strain equations of transformation

Using the experimental or theoretical procedures described in earlier sections it is possible to derive the values of the direct and shear stresses acting at a point on a body. These are normally obtained with reference to some convenient set of X , Y coordinates which, for

example, may be parallel to the edges of the component considered. Sometimes, however, it may be more convenient to refer the values obtained to some other set of axes $X'Y'$ at an angle θ to the original axes.

In this case the two-dimensional versions of eqns. (8.73) and (8.74) apply equally well to the new axes (Fig. 8.31),

$$\text{i.e. } \varepsilon_{x'x'} = \frac{\partial u'}{\partial x'} \quad \varepsilon_{y'y'} = \frac{\partial v'}{\partial y'} \quad \text{and} \quad \gamma_{x'y'} = \frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'}$$

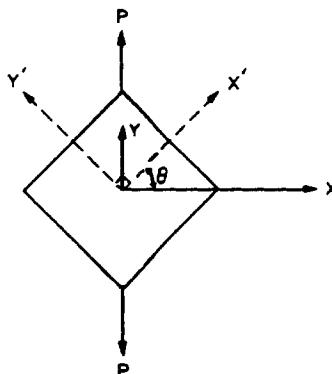


Fig. 8.31. Alternative coordinates to which strains may be referred.

Now, using the partial differentiation chain rule,

$$\begin{aligned} \frac{\partial u'}{\partial x'} &= \left[\frac{\partial}{\partial x} \frac{\partial x}{\partial x'} + \frac{\partial}{\partial y} \frac{\partial y}{\partial x'} \right] u' \\ &= \left[\cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y} \right] (u \cos \theta + v \sin \theta) \\ &= \cos^2 \theta \frac{\partial u}{\partial x} + \sin^2 \theta \frac{\partial v}{\partial y} + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \sin \theta \cos \theta \end{aligned}$$

$$\therefore \varepsilon_{x'x'} = \varepsilon_{xx} \cos^2 \theta + \varepsilon_{yy} \sin^2 \theta + \gamma_{xy} \sin \theta \cos \theta$$

Or, in terms of the double angle 2θ ,

$$\varepsilon_{x'x'} = \frac{1}{2}(\varepsilon_{xx} + \varepsilon_{yy}) + \frac{1}{2}(\varepsilon_{xx} - \varepsilon_{yy}) \cos 2\theta + \frac{1}{2}\gamma_{xy} \sin 2\theta \quad (8.75)$$

This is the same as eqn. (14.14) obtained in §14.10[†] for the normal strain on any plane in terms of the coordinate strains. Indeed, the above represents an alternative proof for what are really similar requirements.

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

8.26. Compatibility

Equations (8.73) and (8.74) relate the six components of strain (three direct and three shear) to the equivalent displacements under a three-dimensional stress system. If, however, the situation arises where the six strain components are known, as they could well be following some theoretical or experimental strain analysis, then the above equations represent three in excess of that required for solution of the three unknown displacements (three unknowns require only three equations for solution). Thus, unless the solution obtained from any three equations satisfies the other three equations, then the values cannot be accepted as a valid solution. Certain specific relations must therefore be satisfied before a valid solution is obtained and these are termed *the compatibility relations*.

The problem can be considered physically as follows: consider a body divided into a large number of small cubic elements. When load is applied the elements deform and simple measurements of length and angle changes will yield the direct and shear strains in each element. These can be summated to produce the overall component strains if required. If, however, the deformed elements are separated and provided in their deformed shapes as a jigsaw puzzle, the puzzle can only be completed, i.e. the elements fully assembled without voids or discontinuities, if each element is correctly strained or deformed. The procedure used to check this condition then represents the compatibility equations. The compatibility relationships in terms of strain are derived as follows:

$$\begin{aligned}\varepsilon_{xx} &= \frac{\partial u}{\partial x} \quad \therefore \quad \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = \frac{\partial^3 u}{\partial x \partial y^2} \\ \varepsilon_{yy} &= \frac{\partial v}{\partial y} \quad \therefore \quad \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^3 v}{\partial x^2 \partial y}\end{aligned}$$

But

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Therefore differentiating once with respect to x and once with respect to y ,

$$\begin{aligned}\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} &= \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial x^2 \partial y} \\ \text{i.e. } \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} &= \left. \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} \right\} \\ \text{Similarly, } \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} &= \left. \frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} \right\} \\ \text{and } \frac{\partial^2 \gamma_{zx}}{\partial z \partial x} &= \left. \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} \right\}\end{aligned}\tag{8.76}$$

These are three of the compatibility equations.

It can also be shown† that a further three compatibility relationships apply, namely

† A.E.H. Love, *Treatise on the Mathematical Theory of Elasticity*, 4th edn., Dover Press, New York, 1944.

$$\left. \begin{aligned} 2 \frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} &= \frac{\partial}{\partial x} \left[\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{yz}}{\partial x} \right] \\ 2 \frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} &= \frac{\partial}{\partial y} \left[\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{yz}}{\partial x} \right] \\ 2 \frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} &= \frac{\partial}{\partial z} \left[-\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{yz}}{\partial x} \right] \end{aligned} \right\} \quad (8.77)$$

The compatibility equations can also be written in terms of stress as follows:
Consider the first of the strain compatibility relationships given in eqn. (8.41).

i.e.

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

For plane strain conditions (and a similar derivation shows that the equation derived is equally appropriate for plane stress) we have:

$$\begin{aligned} \varepsilon_{xx} &= \frac{1}{E} (\sigma_{xx} - v \sigma_{yy}) \\ \varepsilon_{yy} &= \frac{1}{E} (\sigma_{yy} - v \sigma_{xx}) \end{aligned}$$

and

$$\gamma_{xy} = \frac{2(1+v)}{E} \tau_{xy}$$

Substituting:

$$\frac{1}{E} \frac{\partial^2 \sigma_{xx}}{\partial y^2} - \frac{v}{E} \frac{\partial^2 \sigma_{yy}}{\partial y^2} + \frac{1}{E} \frac{\partial^2 \sigma_{yy}}{\partial x^2} - \frac{v}{E} \frac{\partial^2 \sigma_{xx}}{\partial x^2} = \frac{2(1+v)}{E} \frac{\partial^2 \tau_{xy}}{\partial x \partial y} \quad (1)$$

Now from the equilibrium equations assuming plane stress and zero body force stresses we have:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \quad (2)$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \quad (3)$$

Differentiating (2) with respect to x and (3) with respect to y and adding we have:

$$\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} = -2 \frac{\partial^2 \tau_{xy}}{\partial x \partial y} \quad (4)$$

Eliminating τ_{xy} between eqns (4) and (1) we obtain:

$$\frac{1}{E} \frac{\partial^2 \sigma_{xx}}{\partial y^2} - \frac{v}{E} \frac{\partial^2 \sigma_{yy}}{\partial y^2} + \frac{1}{E} \frac{\partial^2 \sigma_{yy}}{\partial x^2} - \frac{v}{E} \frac{\partial^2 \sigma_{xx}}{\partial x^2} = -\frac{(1+v)}{E} \frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2}$$

i.e.

$$\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial x^2} + \frac{\partial^2 \sigma_{xx}}{\partial y^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} = 0$$

or

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] (\sigma_x + \sigma_y) = 0 \quad (8.78)$$

A similar development for cylindrical coordinates yields the stress equation of compatibility

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial}{\partial r} \right] (\sigma_{rr} + \sigma_{\theta\theta}) = 0 \quad (8.79)$$

which in the case of axial symmetry (where stresses are independent of θ) reduces to:

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right] (\sigma_{rr} + \sigma_{\theta\theta}) = 0. \quad (8.80)$$

8.27. The stress function concept

From the earlier work of this chapter it should now be evident that in elastic stress analysis there are generally fifteen unknown quantities to be determined; six stresses, six strains and three displacements. These are functions of the independent variables x , y and z (in cartesian coordinates) or r , θ and z (in cylindrical polar coordinates). A quick look at the governing equations presented earlier in the chapter will convince the reader that the equations are difficult to solve for these unknowns, except for a number of relatively simple problems.

In order to extend the range of useful solutions several techniques are available. In the first instance one may make certain assumptions about the physical problem in an effort to simplify the equations. For example, are the loading and boundary conditions such that:

- (i) the plane stress assumption is adequate – as in a thin-walled pressure vessel? or,
- (ii) does plane strain exist – as in the case of a pressurised thick cylinder?

If we can convince ourselves that these assumptions are valid we reduce the three-dimensional problem to the two-dimensional case.

Having simplified the governing differential equations one must then devise techniques to solve, or further reduce, their complexity. One such concept was that proposed by Sir George B. Airy.[†] His approach was to assume that the stresses in the two-dimensional problem σ_{xx} , σ_{yy} and τ_{xy} could be described by a single function of x and y . This function ϕ is referred to as a “stress function” (later the “Airy stress function”) and it appears to be the first time that such a concept was used. Airy’s approach was later generalised for the three-dimensional case by Clerk Maxwell.[‡]

Airy proposed that the stresses be derived from a particular function ϕ such that:

$$\left. \begin{aligned} \sigma_{xx} &= \frac{\partial^2 \phi}{\partial y^2} \\ \sigma_{yy} &= \frac{\partial^2 \phi}{\partial x^2} \\ \tau_{xy} &= -\frac{\partial^2 \phi}{\partial x \partial y} \end{aligned} \right\} \quad (8.81)$$

[†] G.B. Airy, *Brit. Assoc. Advancement of Sci. Rep.* 1862; *Phil. Trans. Roy. Soc.* **153** (1863), 49–80

[‡] J.C. Maxwell *Edinburgh Roy. Soc. Trans.*, **26** (1872), 1–40.

It should be noted that these equations satisfy the two-dimensional versions of equilibrium equations (8.38):

$$\text{i.e. } \left. \begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} &= 0 \end{aligned} \right\} \quad (8.82)$$

It is also necessary that the stress function ϕ must not only satisfy the equilibrium conditions of the problem but must also satisfy the compatibility relationships, i.e. eqn. 8.76. For the two-dimensional case these reduce to:

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} \quad (8.76)\text{bis.}$$

This equation can be written in terms of stress using the appropriate constitutive (stress-strain) relations. To illustrate the procedure the *plane strain* case will be considered. In this the relevant equations are:

$$\left. \begin{aligned} \varepsilon_{xx} &= \frac{(1-v^2)}{E} \left[\sigma_{xx} - \frac{v}{(1-v)} \sigma_{yy} \right] \\ \varepsilon_{yy} &= \frac{(1-v^2)}{E} \left[\sigma_{yy} - \frac{v}{(1-v)} \sigma_{xx} \right] \\ \gamma_{xy} &= \frac{\tau_{xy}}{G} = \frac{2(1+v)}{E} \tau_{xy} \end{aligned} \right\} \quad (8.83)$$

By substituting these into the compatibility equation (8.76) the following is obtained:

$$2 \frac{\partial^2 \tau_{xy}}{\partial x \partial y} = \frac{\partial^2}{\partial y^2} [(1-v)\sigma_{xx} - v\sigma_{yy}] + \frac{\partial^2}{\partial x^2} [(1-v)\sigma_{yy} - v\sigma_{xx}]$$

From the equilibrium eqn. (8.70) we get:

$$2 \frac{\partial^2 \tau_{xy}}{\partial x \partial y} = - \left(\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} \right) - \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right)$$

Combining these equations to eliminate the shear stress τ_{xy} , gives:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\sigma_{xx} + \sigma_{yy}) = -\frac{1}{(1-v)} \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) \quad (8.84)$$

A similar equation can be obtained for the *plane stress* case, namely:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\sigma_{xx} + \sigma_{yy}) = -(1+v) \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) \quad (8.85)$$

If the body forces X and Y have constant values the same equation holds for both plane stress and plane strain, namely:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\sigma_{xx} + \sigma_{yy}) = \nabla^2 (\sigma_{xx} + \sigma_{yy}) = 0 \quad (8.86)$$

This equation is known as the “*Laplace differential equation*” or the “*harmonic differential equation*.” The function ($\sigma_{xx} + \sigma_{yy}$) is referred to as a “harmonic” function. It is interesting to note that the Laplace equation, which of course incorporates all the previous equations, does not contain the elastic constants of the material. **This is an important conclusion for the experimentalist since, providing there exists geometric similarity, material isotropy and linearity and similar applied loading of both model and prototype, then the stress distribution per unit load will be identical in each.** The stress function, previously defined, must satisfy the ‘Laplace equation’ (8.86). Thus:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = 0$$

or,

$$\frac{\partial^4 \phi}{\partial x^4} + \frac{2\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \quad (8.87)$$

Alternatively, this can be re-written in the form

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 \phi = 0$$

or abbreviated to

$$\nabla^4 \phi = 0 \quad (8.88)$$

indicating that the stress function must be a biharmonic function. Equation (8.87) is often referred to as the “*biharmonic equation*” with ϕ known as the “*Airy stress function*”.

It is worth noting, at this point in the development, that the problem of plane strain, or plane stress, has been reduced to seeking a solution of the biharmonic equation (8.87) such that the stress components satisfy the boundary conditions of the problem.

Thus, provided that a suitable polynomial expression in x and y (or r and θ) is used for the stress function ϕ then both equilibrium and compatibility are automatically assured. Consideration of the boundary conditions associated with any particular stress system will then yield the appropriate coefficients of the various terms of the polynomial and a complete solution is obtained.

8.27.1. Forms of Airy stress function in Cartesian coordinates

The stress function concept was developed by Airy initially to investigate the bending theory of straight rectangular beams. It was thus natural that a rectangular cartesian coordinate system be used. As an introduction to this topic, therefore, forms of stress function in cartesian coordinates will be explored and applied to a number of fairly simple beam problems. It is hoped that the reader will gain confidence in using the approach and be able to tackle a range of more interesting problems where cylindrical polars (r, θ) is an appropriate alternative coordinate system.

(a) The eqns. (8.81) which define the stress function imply that the most simple function of ϕ to produce a stress is $\phi = Ax^2$, since the lower orders when differentiated twice give a zero result. Substituting this into eqns. (8.81) gives:

$$\sigma_{xx} = 0, \quad \sigma_{yy} = 2A \quad \text{and} \quad \tau_{xy} = 0$$

Thus a stress function of the form $\phi = Ax^2$ can be used to describe a condition of constant stress $2A$ in the y direction over the entire region of a component, e.g. uniform tension or

compression testing

$$(b) \quad \phi = By^3.$$

For this stress function

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = 6By$$

with σ_{yy} and τ_{xy} zero.

Thus σ_{xx} is a linear function of vertical dimension y , a situation typical of beam bending.

$$(c) \quad \phi = Ax^2 + Bxy + Cy^2.$$

In this case

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = 2C \quad (\text{a constant})$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = 2A \quad (\text{a constant})$$

$$\tau_{xy} = \frac{\partial^2 \phi}{\partial x \partial y} = -B \quad (\text{a constant})$$

and the stress function is suitable for any uniform plane stress state.

$$(d) \quad \phi = Ax^3 + Bx^2y + Cxy^2 + Dy^3.$$

Then

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = 2Cx + 6Dy$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = 6Ax + 2By$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -2Bx - 2Cy$$

and all stresses may be seen to vary linearly with x and y .

For the particular case where $A = B = C = 0$ the situation resolves itself into that of case (b) i.e. suitable for pure bending.

For many problems an extension of the above function to a comprehensive polynomial expression is found to be rather useful. An appropriate technique is to postulate a general form which will adequately represent the applied loading and boundary conditions. The form of this could be:

$$\begin{aligned} \phi = & Ax^2 + Bxy + Cy^2 + Dx^3 + Ex^2y + Fxy^2 + Gy^3 \\ & + Hx^4 + Jx^3y + Kx^2y^2 + Lxy^3 + My^4 + Nx^5 + Px^4y \\ & + Qx^3y^2 + Rx^2y^3 + Sxy^4 + Ty^5 + \dots \end{aligned} \quad (8.89)$$

Any term containing x or y up to the third power will automatically satisfy the biharmonic equation $\nabla^4(\phi) = 0$. However, terms containing x^4 or y^4 , or higher powers, will appear in

the biharmonic equation. Relations of the associated coefficients can thereby be found which will satisfy $\nabla^4(\phi) = 0$.

Although beyond the scope of the present text, it is worth noting that the polynomial approach has severe limitations when applied to cases with discontinuous loads on the boundary. For such cases, a stress function in the form of a trigonometric series – a Fourier series for example – should be used.

8.27.2. Case 1 – Bending of a simply supported beam by a uniformly distributed loading

An end-supported beam of length $2L$, depth $2d$ and unit width is loaded with a uniformly distributed load $w/\text{unit length}$ as shown in Fig. 8.32. From the work of Chapter 4† the reader will be aware of the solution of this problem using the simple bending theory sometimes known as “engineers bending”. Using this simple approach it is possible to obtain values for the longitudinal stress σ_{xx} and the shear stress τ_{xy} . However, the stress function provides the stress analyst with information about *all* the two-dimensional stresses and thereby the regions of applicability where the more straightforward methods can be used with confidence. The boundary conditions of this problem are:

- (i) at $y = +d$; $\sigma_{yy} = 0$ for all values of x ,
- (ii) at $y = -d$; $\sigma_{yy} = -w$ for all values of x ,
- (iii) at $y = \pm d$; $\tau_{xy} = 0$ for all values of x .

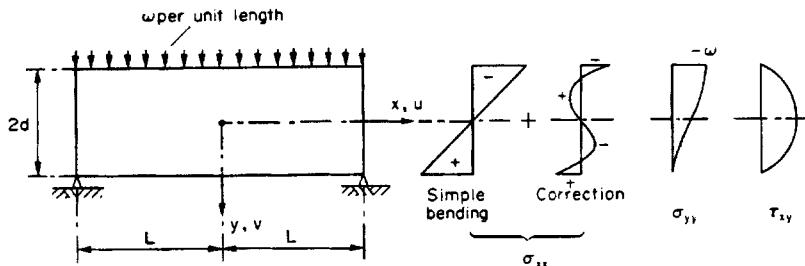


Fig. 8.32. The bending of a simply supported beam by a uniformly distributed load $w/\text{unit length}$.

The overall equilibrium requirements are: –

- (iv) $\int_{-d}^d \sigma_{xx} y \cdot dy = w(L^2 - x^2)/2$ for the equilibrium of moments at any position x ,
- (v) $\int_{-d}^d \sigma_{xx} dy = 0$ for the equilibrium of forces at any position x .

The biharmonic equation:

- (vi) $\nabla^4(\phi) = 0$ must also be satisfied.

To deal with these conditions it is necessary to use the 5th-order polynomial as given in eqn. (8.89) containing eighteen coefficients A to T .

† E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

From eqn. (8.81)

$$\left. \begin{aligned} \sigma_{xx} &= \frac{\partial^2 \phi}{\partial y^2} = 2C + 2Fx + 6Gy + 2Kx^2 + 6Lxy + 12My^2 + 2Qx^3 + 6Rx^2y \\ &\quad + 12Sxy^2 + 20Ty^3 \\ \sigma_{yy} &= \frac{\partial^2 \phi}{\partial x^2} = 2A + 6Dx + 2Ey + 12Hx^2 + 6Jxy + 2Ky^2 + 20Nx^3 + 12Px^2y \\ &\quad + 6Qxy^2 + 2Ry^3 \\ \tau_{xy} &= -\frac{\partial^2 \phi}{\partial x \partial y} = -[B + 2Ex + 2Fy + 3Jx^2 + 4Kxy + 3Ly^2 + 4Px^3 + 6Qx^2y \\ &\quad + 6Rxy^2 + 4Sy^3] \end{aligned} \right\} \quad (8.90)$$

Using the conditions (i) to (vi) it is possible to set up a series of algebraic equations to determine the values of the eighteen coefficients A to T . Since these conditions must be satisfied for all x values it is appropriate to equate the coefficients of the x terms, for example x^3 , x^2 , x and the constants, on both sides of the equations. In the case of the biharmonic equation, condition (vi), all x and y values must be satisfied. This procedure gives the following results:

$$\begin{array}{lll} A = -w/4 & G = (wL^2/8d^3) - w/20d & N = 0 \\ B = 0 & H = 0 & P = 0 \\ C = 0 & J = 0 & Q = 0 \\ D = 0 & K = 0 & R = -w/8d^3 \\ E = 3w/8d & L = 0 & S = 0 \\ F = 0 & M = 0 & T = w/40d^3 \end{array}$$

The stress function ϕ can thus be written:

$$\phi = -\frac{w}{4}x^2 + \frac{3w}{8d}x^2y + \left(\frac{wL^2}{8d^3} - \frac{w}{20d} \right)y^3 - \frac{w}{8d^3}x^2y^3 + \frac{5}{40d^3}y^5 \quad (8.91)$$

The values for the stresses follow using eqn. (8.90) with $I = 2d^3/3$

$$\left. \begin{aligned} \sigma_{xx} &= \frac{w(L^2 - x^2)y}{2I} + \frac{w}{2I} \left(-\frac{2}{5}d^2y + \frac{2}{3}y^3 \right) \\ \sigma_{yy} &= -\frac{w}{2I} \left(\frac{2}{3}d^3 - d^2y + \frac{y^3}{3} \right) \\ \tau_{xy} &= -\frac{wx}{2I}(d^2 - y^2) \end{aligned} \right\} \quad (8.92a-c)$$

These stresses are plotted in Fig. 8.32. The longitudinal stress σ_{xx} consists of two parts. The first term $w(L^2 - x^2)y/2I$ is that given by simple bending theory ($\sigma_{xx} = My/I$). The second term may be considered as a correction term which arises because of the effect of the σ_{yy} compressive stress between the longitudinal fibres. The term is independent of x and therefore constant along the beam. It thus has a value on the ends of the beam given by $x = \pm L$. The expression for σ_{xx} in eqn. (8.92a) is, therefore, only an exact solution

if normal forces on the end exist and are distributed in such a manner as to produce the σ_{xx} values given by eqn. (8.92a) at $x = \pm L$. That is as shown by the correction term in Fig. 8.32. However, conditions (iv) and (v) have guaranteed that forces and moments are in equilibrium at the ends $x = \pm L$ and thus, from Saint-Venant's principle, one could conclude that at distance larger than, say, the depth of the beam, the stress distribution given by eqn. (8.92a) is accurate even when the ends are free. Such correction stresses are, however, of small magnitude compared with the simple bending terms when the span of the beam is large in comparison with its depth.

The equation for the shear stress (8.92c) predicts a parabolic distribution of τ_{xy} on every section x . This implies that at the ends $x = \pm L$ the beam must be supported in such a way that these shear stresses are developed. The values predicted by eqn. (8.92c) coincide with the simple solution. The σ_{yy} stress decreases from its maximum on the top surface to zero at the bottom edge. This again is of small magnitude compared to σ_{xx} in a thin beam type component. However, these stresses can be of importance in a deep beam, or a slab arrangement.

Derivation of the displacements in the beam

From the strain displacement relations, the constitutive relations and the derived stresses it is possible to obtain the displacements in the beam. Although this approach is not really part of the stress function concept, it is included for interest at this point in the development. The procedure is as follows:

$$\left. \begin{aligned} \epsilon_{xx} &= \frac{\partial v}{\partial x} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy}) \\ \epsilon_{yy} &= \frac{\partial v}{\partial y} = \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx}) \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \tau_{xy}/G \end{aligned} \right\} \quad (8.93a-c)$$

Substituting for σ_{xx} and σ_{yy} from eqns (8.92a,b) and integrating (8.93a,b) the following is obtained:

$$u = \frac{w}{2EI} \left[\left(L^2x - \frac{x^3}{3} \right) y + \left(\frac{2}{3}y^3 - \frac{2}{5}d^2y \right) x + vx \left(\frac{1}{3} \cdot y^3 - d^3y + \frac{2}{3}d^3 \right) \right] + u_0(y) \quad (8.94)$$

where $u_0(y)$ is a function of y ,

$$v = -\frac{w}{2EI} \left[\frac{y^4}{12} - \frac{d^2y^2}{2} + \frac{2d^3y}{3} + \frac{v}{2}(L^2 - x^2)y^2 + \frac{v}{6}y^4 - \frac{v}{5}d^2y \right] + v_0(x) \quad (8.95)$$

where $v_0(x)$ is a function of x .

From eqns (8.92c) and (8.93c)

$$\gamma_{xy} = -\frac{w(1+\nu)}{EI}(d^2 - y^2)x \quad (8.96)$$

Differentiating u with respect to y and v with respect to x and adding as in eqn. (8.93c) one can equate the result to the right hand side of eqn. 8.96. After simplifying:

$$\frac{w}{2EI} \left[L^2x - \frac{x^3}{3} - \frac{2}{5}xd^2 - vxd^2 \right] + \frac{\partial u_0(y)}{\partial y} + \frac{\partial v_0(x)}{\partial x} = -\frac{w(1+\nu)}{EI} d^2x \quad (8.97)$$

In eqn. (8.97) some terms are functions of x alone and some are functions of y alone. There is no constant term. Denoting the functions of x and y by $F(x)$ and $G(y)$ respectively, we have:

$$F(x) = \frac{w}{2EI} \left[L^2x - \frac{x^3}{3} - \frac{2}{5}xd^2 - vxd^2 \right] + \frac{w(1+\nu)d^2x}{EI} + \frac{\partial v_0(x)}{\partial x}$$

$$G(y) = \frac{\partial u_0(y)}{\partial y}.$$

Equation (8.97) is thus written

$$F(x) + G(y) = 0$$

If such an equation is to apply for all values of x and y then the functions $F(x)$ and $G(y)$ must themselves be constants and they must be equal in value but opposite in sign. That is in this case, $F(x) = A_1$ and $G(y) = -A_1$.

$$\text{Thus: } \frac{\partial u_0(y)}{\partial y} = -A_1 \quad \therefore u_0(y) = -A_1 y + B_1 \quad (8.98)$$

$$\frac{\partial v_0(x)}{\partial x} = -\frac{w}{2EI} \left[L^2x - \frac{x^3}{3} - \frac{2}{5}xd^2 - vxd^2 + 2(1+\nu)d^2x \right] + A_1$$

$$\therefore v_0(x) = -\frac{w}{2EI} \left[L^2 \frac{x^2}{2} - \frac{x^4}{12} - \frac{1}{5}x^2d^2 - \frac{\nu}{2}x^2d^2 + (1+\nu)d^2x^2 \right] + A_1x + C_1 \quad (8.99)$$

Using the boundary conditions of the problem:

at $x = 0, y = 0, u = 0$: substituting eqn. (8.98) into (8.94) gives $B_1 = 0$,

at $x = 0, y = 0, v = \delta$: substituting eqn. (8.99) into (8.95) gives $C_1 = \delta$,

at $x = 0, y = 0, \frac{\partial v}{\partial x} = 0$ thus $A_1 = 0$.

Thus:

$$\left. \begin{aligned} u &= \frac{w}{2EI} \left[\left(L^2x - \frac{x^3}{3} \right) y + x \left(\frac{2}{3}y^2 - \frac{2}{5}d^2y \right) + vx \left(\frac{y^3}{3} - d^2y + \frac{2}{3}d^3 \right) \right] \\ v &= -\frac{w}{2EI} \left[\frac{y^4}{12} - \frac{d^2y^2}{2} + \frac{2}{3}d^3y + v(L^2 - x^2)\frac{y^2}{2} + \frac{\nu}{6}y^4 - \frac{\nu}{5}d^2y \right. \\ &\quad \left. + L^2 \frac{x^2}{2} - \frac{x^4}{12} - \frac{d^2x^2}{5} + \left(1 + \frac{\nu}{2} \right) d^2x^2 \right] + \delta \end{aligned} \right\} \quad (8.100)$$

To determine the vertical deflection of the central axis we put $y = 0$ in the above equation, that is:

$$v_{y=0} = \delta - \frac{w}{2EI} \left[L^2 \frac{x^2}{2} - \frac{x^4}{12} - \frac{d^2x^2}{5} + \left(1 + \frac{\nu}{2} \right) d^2x^2 \right]$$

Using the fact that $v = 0$ at $x = \pm L$ we find that the central deflection δ is given by:

$$\delta = \frac{5}{24} \frac{wL^4}{EI} \left[1 + \frac{d^2}{L^2} \frac{12}{5} \left(\frac{4}{5} + \frac{v}{2} \right) \right] \quad (8.101)$$

The first term is the central deflection predicted by the simple bending theory. The second term is the correction to include deflection due to shear. As indicated by the form of eqn. (8.101) the latter is small when the span/depth ratio is large, but is more significant for deep beams. By combining equations (8.100) and (8.101) the displacements u and v can be obtained at any point (x, y) in the beam.

8.27.3. The use of polar coordinates in two dimensions

Many engineering components have a degree of axial symmetry, that is they are either rotationally symmetric about a central axis, as in a circular ring, disc and thick cylinder, or contain circular holes which dominate the stress field, or yet again are made up from parts of hollow discs, like a curved bar. In such cases it is advantageous to use cylindrical polar coordinates (r, θ, z) , where r and θ are measured from a fixed origin and axis, respectively and z is in the axial direction. The equilibrium equations for this case are given in eqns. (8.40) and (8.41).

The form of applied loading for these components need not be restricted to the simple rotationally symmetric cases dealt with in earlier chapters. In fact the great value of the stress function concept is that complex loading patterns can be adequately represented by the use of either $\cos n\theta$ and/or $\sin n\theta$, where n is the harmonic order.

A two-dimensional stress field $(\sigma_{rr}, \sigma_{\theta\theta}, \tau_{r\theta})$ is again used for these cases. That is plane stress or plane strain is assumed to provide an adequate approximation of the three-dimensional problem. The next step is to transform the biharmonic eqn. (8.87) to the relevant polar form, namely:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \phi = 0 \quad (8.102)$$

The stresses σ_{rr} , $\sigma_{\theta\theta}$ and $\tau_{r\theta}$ are related to the stress function ϕ in a similar manner to σ_{xx} and σ_{yy} . The resulting values are:

$$\left. \begin{aligned} \sigma_{rr} &= \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \\ \sigma_{\theta\theta} &= \frac{\partial^2 \phi}{\partial r^2} \\ \tau_{r\theta} &= \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} - \frac{1}{r} \frac{\partial^2 \phi}{\partial r \partial \theta} = -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \phi}{\partial \theta} \right) \end{aligned} \right\} \quad (8.103)$$

The derivation of these from the corresponding cartesian coordinate values is a worthwhile exercise for a winter evening.

8.27.4. Forms of stress function in polar coordinates

In cylindrical polars the stress function is, in general, of the form:

$$\phi = f(r) \cos n\theta \quad \text{or} \quad \phi = f(r) \sin n\theta \quad (8.104)$$

where $f(r)$ is a function of r alone and n is an integer.

In exploring the form of ϕ in polars one can avoid the somewhat tedious polynomial expression used for the cartesian coordinates, by considering the following three cases:

- (a) *The axi-symmetric case when $n = 0$* (independent of θ), $\phi = f(r)$. Here the biharmonic eqn. (8.102) reduces to:

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right)^2 \phi = 0 \quad (8.105)$$

and the stresses in eqn. (8.103) to:

$$\sigma_{rr} = \frac{1}{r} \frac{d\phi}{dr}, \quad \sigma_{\theta\theta} = \frac{d^2\phi}{dr^2}, \quad \tau_{r\theta} = 0 \quad (8.106)$$

Equation (8.105) has a general solution:

$$\phi = Ar^2 \ln r + Br^2 + C \ln r + D \quad (8.107)$$

- (b) *The asymmetric case $n = 1$*

$$\phi = f_1(r) \sin \theta \quad \text{or} \quad \phi = f_1(r) \cos \theta.$$

Equation (8.102) has the solution for

$$f_1(r) = A_1 r^3 + B_1/r + C_1 r + D_1 r \ln r \quad (8.108)$$

i.e. $\phi = (A_1 r^3 + B_1/r + C_1 r + D_1 r \ln r) \sin \theta \quad (\text{or } \cos \theta)$

- (c) *The asymmetric cases $n \geq 2$.*

$$\phi = f_n(r) \sin n\theta \quad \text{or} \quad \phi = f_n(r) \cos n\theta$$

$$f_n(r) = A_n r^n + B_n r^{-n} + C_n r^{n+2} + D_n r^{-n+2} \quad (8.109)$$

i.e. $\phi = (A_n r^n + B_n r^{-n} + C_n r^{n+2} + D_n r^{-n+2}) \sin n\theta \quad (\text{or } \cos n\theta)$

Other useful solutions are $\phi = Cr \sin \theta$ or $\phi = Cr \cos \theta$ (8.110)

In the above A , B , C and D are constants of integration which enable formulation of the various problems.

As in the case of the cartesian coordinate system these stress functions must satisfy the compatibility relation embodied in the biharmonic equation (8.102). Although the reader is assured that they are satisfactory functions, checking them is always a beneficial exercise.

In those cases when it is not possible to adequately represent the form of the applied loading by a single term, say $\cos 2\theta$, then a Fourier series representation using eqn. (8.109) can be used. Details of this are given by Timoshenko and Goodier.[†]

[†] S. Timoshenko and J.N. Goodier, *Theory of Elasticity*, McGraw-Hill, 1951.

In the presentation that follows examples of these cases are given. It will be appreciated that the scope of these are by no means exhaustive but a number of worthwhile solutions are given to problems that would otherwise be intractable. Only the stress values are presented for these cases, although the derivation of the displacements is a natural extension.

8.27.5. Case 2 – Axi-symmetric case: solid shaft and thick cylinder radially loaded with uniform pressure

This obvious case will be briefly discussed since the Lamé equations which govern this problem are so well known and do provide a familiar starting point.

Substituting eqn. (8.107) into the stress equations (8.106) results in

$$\left. \begin{aligned} \sigma_{rr} &= A(1 + 2 \ln r) + 2B + C/r^2 \\ \sigma_{\theta\theta} &= A(3 + 2 \ln r) + 2B - C/r^2 \\ \tau_{r\theta} &= 0 \end{aligned} \right\} \quad (8.111)$$

When a *solid shaft* is loaded on the external surface, the constants A and C must vanish to avoid the singularity condition at $r = 0$. Hence $\sigma_{rr} = \sigma_{\theta\theta} = 2B$. That is uniform tension, or compression over the cross section.

In the case of the *thick cylinder*, three constants, A , B , and C have to be determined. The constant A is found by examining the form of the tangential displacement v in the cylinder. The expression for this turns out to be a multi-valued expression in θ , thus predicting a different displacement every time θ is increased to $\theta + 2\pi$. That is every time we scan one complete revolution and arrive at the same point again we get a different value for v . To avoid this difficulty we put $A = 0$. Equations (8.111) are thus identical in form to the Lamé eqns. (10.3 and 10.4).[†] The two unknown constants are determined from the applied load conditions at the surface.

8.27.6. Case 3 – The pure bending of a rectangular section curved beam

Consider a circular arc curved beam of narrow rectangular cross-section and unit width, bent in the plane of curvature by end couples M (Fig. 8.33). The beam has a constant cross-section and the bending moment is constant along the beam. In view of this one would expect that the stress distribution will be the same on each radial cross-section, that is, it will be independent of θ . The axi-symmetric form of ϕ , as given in eqn. (8.107), can thus be used:-

$$\text{i.e. } \phi = Ar^2 \ln r + Br^2 + C \ln r + D$$

The corresponding stress values are those of eqns (8.111)

$$\begin{aligned} \sigma_{rr} &= A(1 + 2 \ln r) + 2B + C/r^2 \\ \sigma_{\theta\theta} &= A(3 + 2 \ln r) + 2B - C/r^2 \\ \tau_{r\theta} &= 0 \end{aligned}$$

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

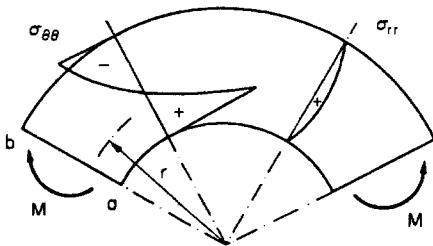


Fig. 8.33. Pure bending of a curved beam.

The boundary conditions for the curved beam case are:

- (i) $\sigma_{rr} = 0$ at $r = a$ and $r = b$ (a and b are the inside and outside radii, respectively);
- (ii) $\int_a^b \sigma_{\theta\theta} = 0$, for the equilibrium of forces, over any cross-section;
- (iii) $\int_a^b \sigma_{\theta\theta} r dr = -M$, for the equilibrium of moments, over any cross-section;
- (iv) $\tau_{r\theta} = 0$, at the boundary $r = a$ and $r = b$.

Using these conditions the constants A , B and C can be determined. The final stress equations are as follows:

$$\left. \begin{aligned} \sigma_{rr} &= \frac{4M}{Q} \left(\frac{a^2 b^2}{r^2} \ln \frac{b}{a} - a^2 \ln \frac{r}{a} - b^2 \ln \frac{b}{r} \right) \\ \sigma_{\theta\theta} &= \frac{4M}{Q} \left(-\frac{a^2 b^2}{r^2} \ln \frac{b}{a} - a^2 \ln \frac{r}{a} - b^2 \ln \frac{b}{r} + b^2 - a^2 \right) \\ \tau_{r\theta} &= 0 \end{aligned} \right\} \quad (8.112)$$

$$\text{where } Q = 4a^2 b^2 \left(\ln \frac{b}{a} \right)^2 - (b^2 - a^2)^2$$

The distributions of these stresses are shown on Fig. 8.33. Of particular note is the nonlinear distribution of the $\sigma_{\theta\theta}$ stress. This predicts a higher inner fibre stress than the simple bending ($\sigma = My/I$) theory.

8.27.7. Case 4. Asymmetric case $n = 1$. Shear loading of a circular arc cantilever beam

To illustrate this form of stress function the curved beam is again selected; however, in this case the loading is a shear loading as shown in Fig. 8.34.

As previously the beam is of narrow rectangular cross-section and unit width. Under the shear loading P the bending moment at any cross-section is proportional to $\sin \theta$ and, therefore it is reasonable to assume that the circumferential stress $\sigma_{\theta\theta}$ would also be associated with $\sin \theta$. This points to the case $n = 1$ and a stress function given in eqn. (8.108).

$$\text{i.e. } \phi = (A_1 r^3 + B_1/r + C_1 r + D_1 r \ln r) \sin \theta \quad (8.113)$$

Using eqns. (8.103) the three stresses can be written

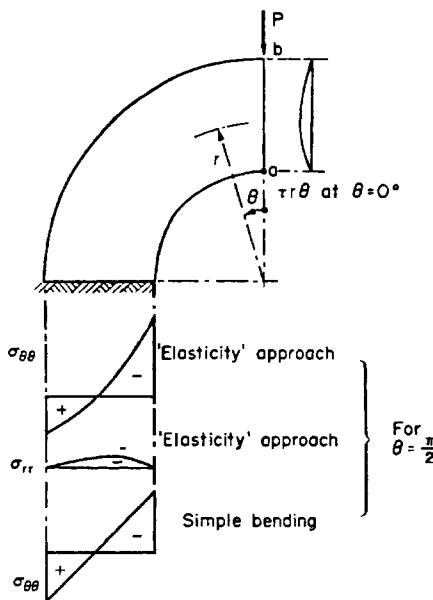


Fig. 8.34. Shear loading of a curved cantilever.

$$\left. \begin{aligned} \sigma_{rr} &= (2A_1r - 2B_1/r^3 + D_1/r) \sin \theta \\ \sigma_{\theta\theta} &= (6A_1r + 2B_1/r^3 + D_1/r) \sin \theta \\ \tau_{r\theta} &= -(2A_1r - 2B_1/r^3 + D_1/r) \cos \theta \end{aligned} \right\} \quad (8.114)$$

The boundary conditions are:

- (i) $\sigma_{rr} = \tau_{r\theta} = 0$, for $r = a$ and $r = b$.
- (ii) $\int_a^b \tau_{r\theta} dr = P$, for equilibrium of vertical forces at $\theta = 0$.

Using these conditions the constants A_1 , B_1 and D_1 can be determined. The final stress values are:

$$\left. \begin{aligned} \sigma_{rr} &= \frac{P}{S} \left(r + \frac{a^2 b^2}{r^3} - \frac{a^2 + b^2}{r} \right) \sin \theta \\ \sigma_{\theta\theta} &= \frac{P}{S} \left(3r - \frac{a^2 b^2}{r^3} - \frac{a^2 + b^2}{r} \right) \sin \theta \\ \tau_{r\theta} &= -\frac{P}{S} \left(r + \frac{a^2 b^2}{r^3} - \frac{a^2 + b^2}{r} \right) \cos \theta \end{aligned} \right\} \quad (8.115)$$

where $s = a^2 - b^2 + (a^2 + b^2) \ln b/a$.

It is noted from these equations that at the load point $\theta = 0$,

$$\left. \begin{aligned} \sigma_{rr} &= \sigma_{\theta\theta} = 0 \\ \tau_{r\theta} &= -\frac{P}{S} \left(r + \frac{a^2 b^2}{r^3} - \frac{a^2 + b^2}{r} \right) \end{aligned} \right\} \quad (8.116)$$

As in the previous cases the load P must be applied to the cantilever according to eqn. (8.116) – see Fig. 8.34.

$$\left. \begin{aligned} \text{At the fixed end, } \theta = \frac{\pi}{2}; \quad \sigma_{rr} &= \frac{P}{S} \left(r + \frac{a^2 b^2}{r^3} - \frac{a^2 + b^2}{r} \right) \\ \sigma_{\theta\theta} &= \frac{P}{S} \left(3r - \frac{a^2 b^2}{r^3} - \frac{a^2 + b^2}{r} \right) \\ \tau_{r\theta} &= 0 \end{aligned} \right\} \quad (8.117)$$

The distributions of these stresses are shown in Fig. 8.34. They are similar to that for the pure moment application. The simple bending ($\sigma = My/I$) result is also shown. As in the previous case it is noted that the simple approach underestimates the stresses on the inner fibre.

8.27.8. Case 5—The asymmetric cases $n \geq 2$ —stress concentration at a circular hole in a tension field

The example chosen to illustrate this category concerns the derivation of the stress concentration due to the presence of a circular hole in a tension field. A large number of stress concentrations arise because of geometric discontinuities—such as holes, notches, fillets, etc., and the derivation of the peak stress values, in these cases, is clearly of importance to the stress analyst and the designer.

The distribution of stress round a small circular hole in a flat plate of unit thickness subject to a uniform tension σ_{xx} , in the x direction was first obtained by Prof. G. Kirsch in 1898.[†] The width of the plate is considered large compared with the diameter of the hole as shown in Fig. 8.35. Using the Saint-Venant's[‡] principle the small central hole will not affect the

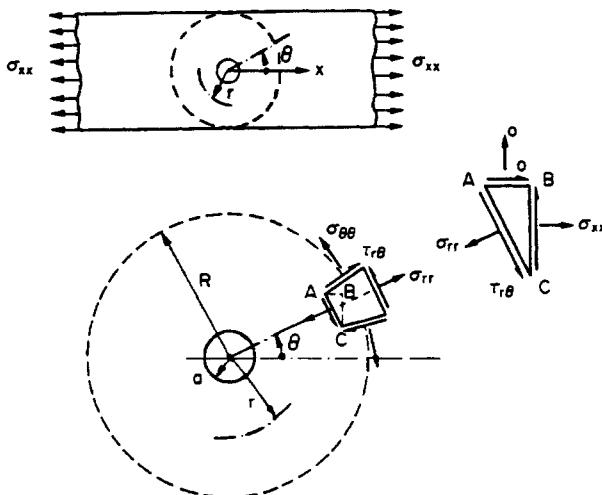


Fig. 8.35. Elements in a stress field some distance from a circular hole.

[†] G. Kirsch Verein Deutscher Ingenieure (V.D.I.) Zeitschrift, **42** (1898), 797–807.

[‡] B. de Saint-Venant, Mem. Acad. Sc. Savants E'trangers, **14** (1855), 233–250.

stress distribution at distances which are large compared with the diameter of the hole—say the width of the plate. Thus on a circle of large radius R the stress in the x direction, on $\theta = 0$ will be σ_{xx} . Beyond the circle one can expect that the stresses are effectively the same as in the plate without the hole.

Thus at an angle θ , equilibrium of the element ABC , at radius $r = R$, will give

$$\sigma_{rr} \cdot AC = \sigma_{xx} BC \cos \theta, \quad \text{and since, } \cos \theta = BC/AC$$

$$\sigma_{rr} = \sigma_{xx} \cos^2 \theta,$$

or
$$\sigma_{rr} = \frac{\sigma_{xx}}{2} (1 + \cos 2\theta).$$

Similarly, $\tau_{r\theta} \cdot AC = -\sigma_{xx} BC \sin \theta$

$$\therefore \tau_{r\theta} = -\sigma_{xx} \cos \theta \sin \theta = -\frac{\sigma_{xx}}{2} \sin 2\theta.$$

Note the sign of $\tau_{r\theta}$ indicates a direction opposite to that shown on Fig. 8.35.

Kirsch noted that the total stress distribution at $r = R$ can be considered in two parts:

(a) a constant radial stress $\sigma_{xx}/2$

(b) a condition varying with 2θ , that is; $\sigma_{rr} = \frac{\sigma_{xx}}{2} \cos 2\theta$, $\tau_{r\theta} = -\frac{\sigma_{xx}}{2} \sin 2\theta$.

The final result is obtained by combining the distributions from (a) and (b). *Part (a), shown in Fig. 8.36*, can be treated using the Lamé equations; The boundary conditions are:

$$\text{at } r = a \quad \sigma_{rr} = 0$$

$$r = R \quad \sigma_{rr} = \sigma_{xx}/2$$

Using these in the Lamé equation, $\sigma_{rr} = A + B/r^2$

gives,
$$A = \frac{\sigma_{xx}}{2} \left(\frac{R^2}{R^2 - a^2} \right) \quad \text{and} \quad B = -\frac{\sigma_{xx}}{2} \left(\frac{R^2 a^2}{R^2 - a^2} \right)$$

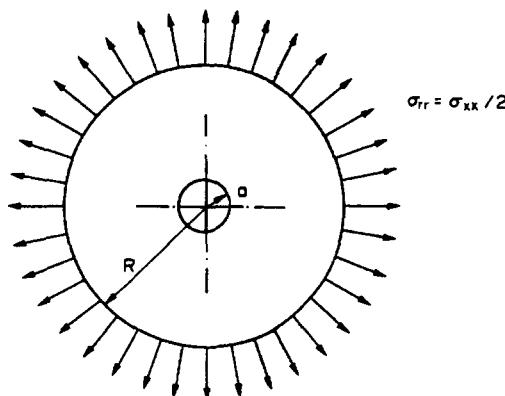


Fig. 8.36. A circular plate loaded at the periphery with a uniform tension.

When $R \gg a$ these can be modified to $A = \frac{\sigma_{xx}}{2}$ and $B = -\frac{\sigma_{xx}}{2}a^2$

Thus

$$\left. \begin{aligned} \sigma_{rr} &= \frac{\sigma_{xx}}{2} \left(1 - \frac{a^2}{r^2} \right) \\ \sigma_{\theta\theta} &= \frac{\sigma_{xx}}{2} \left(1 + \frac{a^2}{r^2} \right) \\ \tau_{r\theta} &= 0 \end{aligned} \right\} \quad (8.118)$$

Part (b), shown in Fig 8.37 is a new case with normal stresses varying with $\cos 2\theta$ and shear stresses with $\sin 2\theta$.

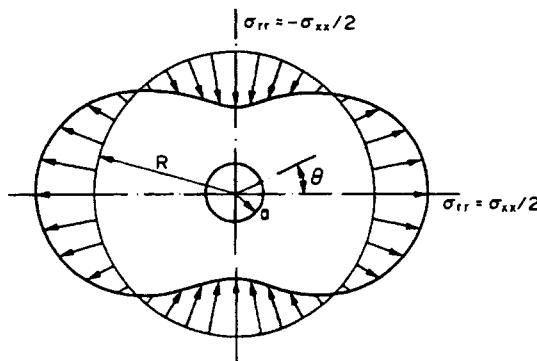


Fig. 8.37. A circular plate loaded at the periphery with a radial stress $= \frac{\sigma_{xx}}{2} \cos 2\theta$ (shown above) and a shear stress $= -\frac{\sigma_{xx}}{2} \sin 2\theta$.

This fits into the category of $n = 2$ with a stress function eqn. (8.109);

i.e. $\phi = (A_2 r^2 + B_2/r^2 + C_2 r^4 + D_2) \cos 2\theta \quad (8.119)$

Using eqns. (8.103) the stresses can be written:

$$\left. \begin{aligned} \sigma_{rr} &= -(2A_2 + 6B_2/r^4 + 4D_2/r^2) \cos 2\theta \\ \sigma_{\theta\theta} &= (2A_2 + 6B_2/r^4 + 12C_2 r^2) \cos 2\theta \\ \tau_{r\theta} &= (2A_2 - 6B_2/r^4 + 6C_2 r^2 - 2D_2/r^2) \sin 2\theta \end{aligned} \right\} \quad (8.120)$$

The four constants are found such that σ_{rr} and $\tau_{r\theta}$ satisfy the boundary conditions:

at $r = a$, $\sigma_{rr} = \tau_{r\theta} = 0$

at $r = R \rightarrow \infty$, $\sigma_{rr} = \frac{\sigma_{xx}}{2} \cos 2\theta$, $\tau_{r\theta} = -\frac{\sigma_{xx}}{2} \sin 2\theta$

From these,

$$A_2 = -\sigma_{xx}/4, \quad B_2 = -\sigma_{xx}a^4/4$$

$$C_2 = 0, \quad D_2 = \sigma_{xx}a^2/2$$

Thus:

$$\left. \begin{aligned} \sigma_{rr} &= \frac{\sigma_{xx}}{2} \left(1 - \frac{4a^2}{r^2} + \frac{3a^4}{r^4} \right) \cos 2\theta \\ \sigma_{\theta\theta} &= -\frac{\sigma_{xx}}{2} \left(1 + \frac{3a^4}{r^4} \right) \cos 2\theta \\ \tau_{r\theta} &= -\frac{\sigma_{xx}}{2} (1 + 2a^2/r^2 - 3a^4/r^4) \sin 2\theta \end{aligned} \right\} \quad (8.121)$$

The sum of the stresses given by eqns. (8.120) and (8.121) is that proposed by Kirsch. At the edge of the hole σ_{rr} and $\tau_{r\theta}$ should be zero and this can be verified by substituting $r = a$ into these equations.

The distribution of $\sigma_{\theta\theta}$ round the hole, i.e. $r = a$, is obtained by combining eqns. (8.120) and (8.121):

i.e.

$$\sigma_{\theta\theta} = \sigma_{xx}(1 - 2 \cos 2\theta) \quad (8.122)$$

and is shown on Fig. 8.38(a).

When $\theta = 0$; $\sigma_{\theta\theta} = -\sigma_{xx}$ and when $\theta = \frac{\pi}{2}$; $\sigma_{\theta\theta} = 3\sigma_{xx}$.

The stress concentration factor (S.C.F) defined as Peak stress/Average stress, gives an S.C.F. = 3 for this case.

The distribution across the plate from point A ($\theta = \frac{\pi}{2}$) is:

$$\sigma_{\theta\theta} = \frac{\sigma_{xx}}{2} \left(2 + \frac{a^2}{r^2} + \frac{3a^4}{r^4} \right) \quad (8.123)$$

This is shown in Fig. 8.38(b), which indicates the rapid way in which $\sigma_{\theta\theta}$ approaches σ_{xx} as r increases. Although the solution is based on the fact that $R \gg a$, it can be shown that even when $R = 4a$, that is the width of the plate is four times the diameter of the hole, the error in the S.C.F. is less than 6%.

Using the stress distribution derived for this case it is possible, using superposition, to obtain S.C.F. values for a range of other stress fields where the circular hole is present, see problem No. 8.52 for solution at the end of this chapter.

A similar, though more complicated, analysis can be carried out for an elliptical hole of major diameter $2a$ across the plate and minor diameter $2b$ in the stress direction. In this case the S.C.F. = $1 + 2a/b$ (see also §8.3). Note that for the circular hole $a = b$, and the S.C.F. = 3, as above.

8.27.9. Other useful solutions of the biharmonic equation

(a) Concentrated line load across a plate

The way in which an elastic medium responds to a concentrated line of force is the final illustrative example to be presented in this section. In practice it is neither possible to apply a genuine line load nor possible for the plate to sustain a load without local plastic deformation. However, despite these local perturbations in the immediate region of the load, the rest of the plate behaves in an elastic manner which can be adequately represented by the governing equations obtained earlier. It is thus possible to use the techniques developed above to analyse the concentrated load problem.

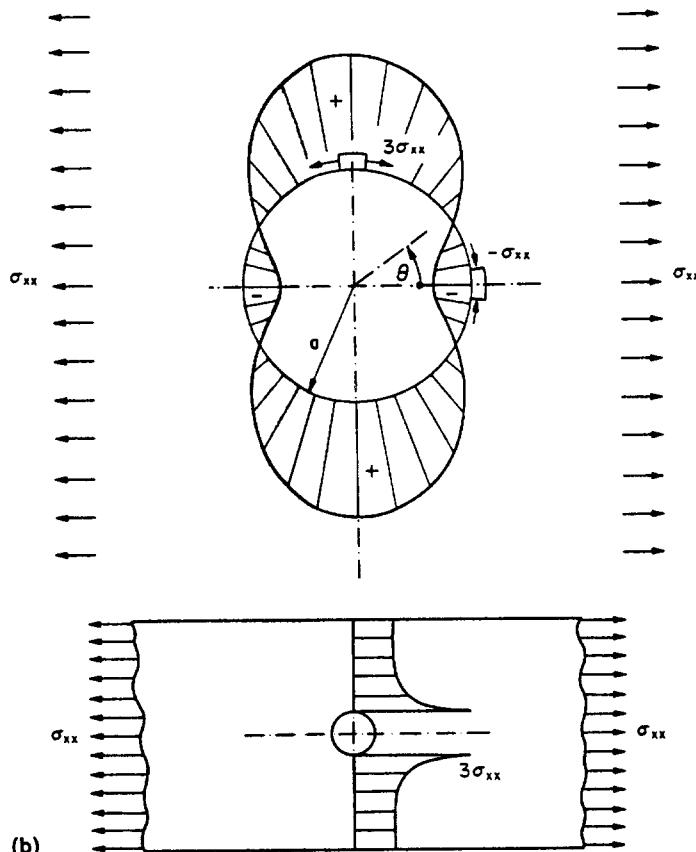


Fig. 8.38. (a) Distribution of circumferential stress $\sigma_{\theta\theta}$ round the hole in a tension field; (b) distribution of circumferential stress $\sigma_{\theta\theta}$ across the plate.

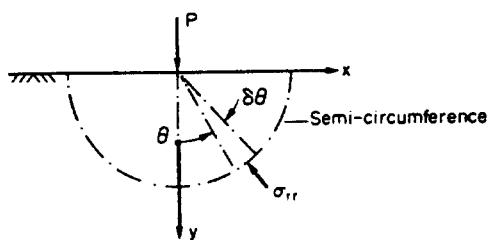


Fig. 8.39. Concentrated load on a semi-infinite plate.

Consider a force P per unit width of the plate applied as a line load normal to the surface – see Fig. 8.39. The plate will be considered as equivalent to a semi-infinite solid, that is, one that extends to infinity in the x and y directions below the horizon, $\theta = \pm \frac{\pi}{2}$. The plate is assumed to be of unit width. It is convenient to use cylindrical polars again for this problem.

Using Boussinesq's solutions[†] for a semi-infinite body, Alfred-Aimé Flamant obtained (in 1892)[‡] the stress distribution for the present case. He showed that on any semi-circumference round the load point the stress is entirely radial, that is: $\sigma_{\theta\theta} = \tau_{r\theta} = 0$ and σ_{rr} will be a principal stress. He used a stress function of the type given in eqn. (8.110), namely: $\phi = Cr\theta \sin\theta$ which predicts stresses:

$$\sigma_{rr} = \frac{2C}{r} \cos\theta, \quad \sigma_{\theta\theta} = \tau_{r\theta} = 0$$

Applying overall equilibrium to this case it is noted that the resultant vertical force over any semi-circle, of radius r , must equal the applied force P :

$$P = - \int_{-\pi/2}^{\pi/2} (\sigma_{rr} \cdot r d\theta) \cos\theta = - \int_{-\pi/2}^{\pi/2} (2C \cos^2\theta) d\theta = -C\pi$$

Thus $\phi = -\frac{Pr\theta}{\pi} \sin\theta$

and $\sigma_{rr} = -\frac{2P \cos\theta}{\pi r}$ (8.124)

This can be transformed into x and y coordinates:

$$\left. \begin{aligned} \sigma_{yy} &= \sigma_{rr} \cos^2\theta \\ \sigma_{xx} &= \sigma_{rr} \sin^2\theta \\ \tau_{xy} &= \sigma_{rr} \sin\theta \cos\theta \end{aligned} \right\} \quad (8.125)$$

See also §8.3.3 for further transformation of these equations.

This type of solution can be extended to consider the wedge problem, again subject to a line load as shown in Figs. 8.40(a) and (b).

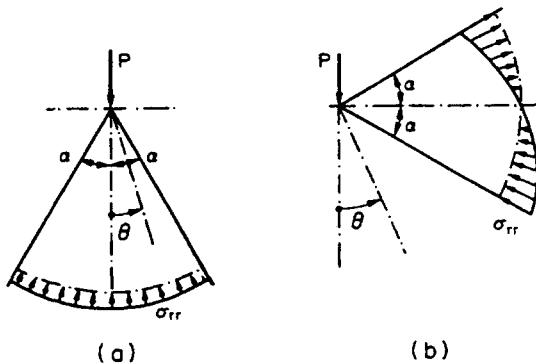


Fig. 8.40. Forces on a wedge.

[†] J. Boussinesq, *Application de potentiels à l'étude de l'équilibre*! Paris, 1885; also *Comptes Rendus Acad Sci.*, **114** (1892), 1510–1516.

[‡] Flamant AA *Comptes Rendus Acad. Sci.*, **114** (1892), 1465–1468.

(b) The wedge subject to an axial load – Figure 8.40(a)

For this case,

$$P = - \int_{-\alpha}^{\alpha} (\sigma_{rr} \cdot r d\theta) \cos \theta$$

$$P = - \int_{-\alpha}^{\alpha} 2C \cdot \cos^2 \theta d\theta$$

$$P = -C(2\alpha + \sin 2\alpha)$$

Thus,

$$\sigma_{rr} = -\frac{2P \cos \theta}{r(2\alpha + \sin 2\alpha)} \quad (8.126)$$

(c) The wedge subject to a normal end load – Figure 8.40(a)

Here,

$$P = - \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} (\sigma_{rr} \cdot r d\theta) \cos \theta$$

$$P = - \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} 2C \cdot \cos^2 \theta d\theta$$

$$P = -C(2\alpha - 2 \sin 2\alpha).$$

Thus,

$$\sigma_{rr} = -\frac{2p \cos \theta}{r(2\alpha - \sin 2\alpha)} \quad (8.127)$$

From a combination of these cases any inclination of the load can easily be handled.

(d) Uniformly distributed normal load on part of the surface – Fig. 8.41

The result for σ_{rr} obtained in eqn. (8.124) can be used to examine the case of a uniformly distributed normal load q per unit length over part of a surface-say $\theta = \frac{\pi}{2}$. It is required to find the values of the normal and shear stresses (σ_{xx} , σ_{yy} , τ_{xy}) at the point A situated as indicated in Fig. 8.41. In this case the load is divided into a series of discrete lengths δx over which the load is δP , that is $\delta P = q \delta x$. To make use of eqn. (8.124) we must transform this into polars (r , θ). That is

$$dx = r d\theta / \cos \theta. \quad \text{Thus, } dP = q \cdot r d\theta / \cos \theta \quad (8.128)$$

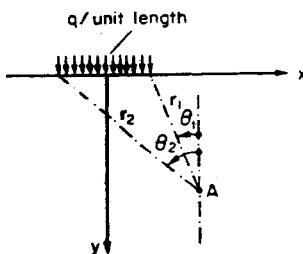


Fig. 8.41. A distributed force on a semi-infinite plate.

Then from eqn. (8.124)

$$d\sigma_{rr} = -\frac{2}{\pi r} dP \cos \theta$$

$$\text{Substituting eqn. (8.128): } d\sigma_{rr} = -\frac{2}{\pi r} \cdot q \cdot r d\theta = -\frac{2q}{\pi} d\theta$$

$$\text{Making use of eqns. (8.125): } d\sigma_{yy} = -\frac{2q}{\pi} \cos^2 \theta d\theta$$

$$d\sigma_{xx} = -\frac{2q}{\pi} \sin^2 \theta d\theta$$

$$d\tau_{xy} = -\frac{2q}{\pi} \sin \theta \cos \theta$$

The total stress values at the point A due to all the discrete loads over θ_1 to θ_2 can then be written,

$$\left. \begin{aligned} \sigma_{yy} &= -\frac{2q}{\pi} \int_{\theta_1}^{\theta_2} \cos^2 \theta d\theta \\ &= -\frac{q}{2\pi} [2(\theta_2 - \theta_1) + (\sin 2\theta_2 - \sin 2\theta_1)] \\ \sigma_{xx} &= -\frac{q}{2\pi} [2(\theta_2 - \theta_1) - (\sin 2\theta_2 - \sin 2\theta_1)] \\ \tau_{xy} &= -\frac{q}{2\pi} [\cos 2\theta_1 - \cos 2\theta_2] \end{aligned} \right\} \quad (8.129)$$

Closure

The stress function concept described above was developed over 100 years ago. Despite this, however, the ideas contained are still of relevance today in providing a series of classical solutions to otherwise intractable problems, particularly in the study of plates and shells.

Examples

Example 8.1

At a point in a material subjected to a three-dimensional stress system the cartesian stress coordinates are:

$$\sigma_{xx} = 100 \text{ MN/m}^2 \quad \sigma_{yy} = 80 \text{ MN/m}^2 \quad \sigma_{zz} = 150 \text{ MN/m}^2$$

$$\sigma_{xy} = 40 \text{ MN/m}^2 \quad \sigma_{yz} = -30 \text{ MN/m}^2 \quad \sigma_{zx} = 50 \text{ MN/m}^2$$

Determine the normal, shear and resultant stresses on a plane whose normal makes angles of 52° with the X axis and 68° with the Y axis.

Solution

The direction cosines for the plane are as follows:

$$l = \cos 52^\circ = 0.6157$$

$$m = \cos 68^\circ = 0.3746$$

and, since $l^2 + m^2 + n^2 = 1$,

$$\begin{aligned} n^2 &= 1 - (0.6157^2 + 0.3746^2) \\ &= 1 - (0.3791 + 0.1403) = 0.481 \\ \therefore n &= 0.6935 \end{aligned}$$

Now from eqns. (8.13–15) the components of the resultant stress on the plane in the X, Y and Z directions are given by

$$p_{xn} = \sigma_{xx}l + \sigma_{xy}m + \sigma_{xz}n$$

$$p_{yn} = \sigma_{yy}m + \sigma_{yx}l + \sigma_{yz}n$$

$$p_{zn} = \sigma_{zz}n + \sigma_{zx}l + \sigma_{zy}m$$

$$p_{xn} = (100 \times 0.6157) + (40 \times 0.3746) + (50 \times 0.6935) = 111.2 \text{ MN/m}^2$$

$$p_{yn} = (80 \times 0.3746) + (40 \times 0.6157) + (-30 \times 0.6935) = 33.8 \text{ MN/m}^2$$

$$p_{zn} = (150 \times 0.6935) + (50 \times 0.6157) + (-30 \times 0.3746) = 123.6 \text{ MN/m}^2$$

Therefore from eqn. (8.4) the resultant stress p_n is given by

$$\begin{aligned} p_n &= \left[p_{xn}^2 + p_{yn}^2 + p_{zn}^2 \right]^{1/2} = [111.2^2 + 33.8^2 + 123.6^2]^{1/2} \\ &= 169.7 \text{ MN/m}^2 \end{aligned}$$

The normal stress σ_n is given by eqn. (8.5),

$$\begin{aligned} \sigma_n &= p_{xn}l + p_{yn}m + p_{zn}n \\ &= (111.2 \times 0.6157) + (33.8 \times 0.3746) + (123.6 \times 0.6935) \\ &= 166.8 \text{ MN/m}^2 \end{aligned}$$

and the shear stress τ_n is found from eqn. (8.6),

$$\begin{aligned} \tau_n &= \sqrt{(p_n^2 - \sigma_n^2)} = \sqrt{(28798 - 27830)^{1/2}} \\ &= 31 \text{ MN/m}^2 \end{aligned}$$

Example 8.2

Show how the equation of equilibrium in the radial direction of a cylindrical coordinate system can be reduced to the form

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{(\sigma_{rr} - \sigma_{\theta\theta})}{r} = 0$$

for use in applications involving long cylinders of thin uniform wall thickness.

Hence show that for such a cylinder of internal radius R_0 , external radius R and wall thickness T (Fig. 8.42) the radial stress σ_{rr} at any thickness t is given by

$$\sigma_{rr} = -p \frac{R_0}{T} \frac{(T-t)}{(R_0+t)}$$

where p is the internal pressure, the external pressure being zero.

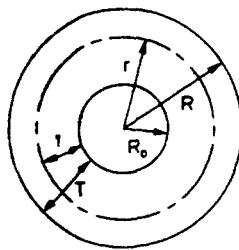


Fig. 8.42.

For thin-walled cylinders the circumferential stress $\sigma_{\theta\theta}$ can be assumed to be independent of radius.

What will be the equivalent expression for the circumferential stress?

Solution

The relevant equation of equilibrium is

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{(\sigma_{rr} - \sigma_{\theta\theta})}{r} + F_r = 0$$

Now for long cylinders plane strain conditions may be assumed,

i.e.

$$\frac{\partial \sigma_{rz}}{\partial z} = 0$$

By symmetry, the stress conditions are independent of θ ,

$$\frac{\partial \sigma_{r\theta}}{\partial \theta} = 0$$

and, in the absence of body forces,

$$F_r = 0$$

Thus the equilibrium equation reduces to

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{(\sigma_{rr} - \sigma_{\theta\theta})}{r} = 0$$

Since $\sigma_{\theta\theta}$ is independent of r this equation can be conveniently rearranged as follows:

$$\sigma_{rr} + r \frac{\partial \sigma_{rr}}{\partial r} = \sigma_{\theta\theta}$$

$$\frac{\partial}{\partial r}(r\sigma_{rr}) = \sigma_{\theta\theta}$$

Integrating,

$$r\sigma_{rr} = \sigma_{\theta\theta}r + C \quad (1)$$

Now at $r = R$, $\sigma_{rr} = 0$

$$\therefore \text{substituting in (1),} \quad 0 = R\sigma_{\theta\theta} + C$$

$$\therefore C = -R\sigma_{\theta\theta} \quad (2)$$

Also at $r = R_0$, $\sigma_{rr} = -p$,

$$\begin{aligned}\therefore -R_0 p &= R_0 \sigma_{\theta\theta} + C \\ &= -(R - R_0) \sigma_{\theta\theta} \\ \therefore \sigma_{\theta\theta} &= \frac{R_0 p}{(R - R_0)}\end{aligned}\quad (3)$$

Substituting in (1),

$$\begin{aligned}r\sigma_{rr} &= \sigma_{\theta\theta}r - R\sigma_{\theta\theta} = -(R - r)\sigma_{\theta\theta} \\ \therefore \sigma_{rr} &= -\frac{(R - r)}{r} \times \frac{R_0 p}{(R - R_0)} \\ \sigma_{rr} &= -\frac{(T - t)}{r} p \frac{R_0}{T} \\ &= -\frac{pR_0}{T} \frac{(T - t)}{(R_0 + t)}\end{aligned}$$

and from (3)

$$\sigma_{\theta\theta} = \frac{R_0 p}{(R - R_0)} = \frac{R_0 p}{T}$$

Example 8.3

A three-dimensional complex stress system has principal stress values of 280 MN/m^2 , 50 MN/m^2 and -120 MN/m^2 . Determine (a) analytically and (b) graphically:

- (i) the limiting value of the maximum shear stress;
- (ii) the values of the octahedral normal and shear stresses.

Solution (a): Analytical

- (i) The limiting value of the maximum shear stress is the greatest value obtained in any plane of the three-dimensional system. In terms of the principal stresses this is given by

$$\begin{aligned}\tau_{\max} &= \frac{1}{2}(\sigma_1 - \sigma_3) \\ &= \frac{1}{2}[280 - (-120)] = 200 \text{ MN/m}^2\end{aligned}$$

- (ii) The octahedral normal stress is given by

$$\begin{aligned}\sigma_{\text{oct}} &= \frac{1}{3} [\sigma_1 + \sigma_2 + \sigma_3] \\ &= \frac{1}{3} [280 + 50 + (-120)] = 70 \text{ MN/m}^2\end{aligned}$$

- (iii) The octahedral shear stress is

$$\begin{aligned}\tau_{\text{oct}} &= \frac{1}{3} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{1/2} \\ &= \frac{1}{3} [(280 - 50)^2 + (50 + 120)^2 + (-120 - 280)^2]^{1/2}\end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{3} [52900 + 28900 + 160000]^{1/2} \\
 &= 163.9 \text{ MN/m}^2
 \end{aligned}$$

Solution (b): Graphical

- (i) The graphical solution is obtained by constructing the three-dimensional Mohr's representation of Fig. 8.43. The limiting value of the maximum shear stress is then equal to the radius of the principal circle.

i.e.

$$\tau_{\max} = 200 \text{ MN/m}^2$$

- (ii) The direction cosines of the octahedral planes are

$$l = m = n = \frac{1}{\sqrt{3}} = 0.5774$$

i.e.

$$\alpha = \beta = \gamma = \cos^{-1} 0.5774 = 54^\circ 52'$$

The values of the normal and shear stresses on these planes are then obtained using the procedures of §8.7.

By measurement,

$$\sigma_{\text{oct}} = 70 \text{ MN/m}^2$$

$$\tau_{\text{oct}} = 164 \text{ MN/m}^2$$

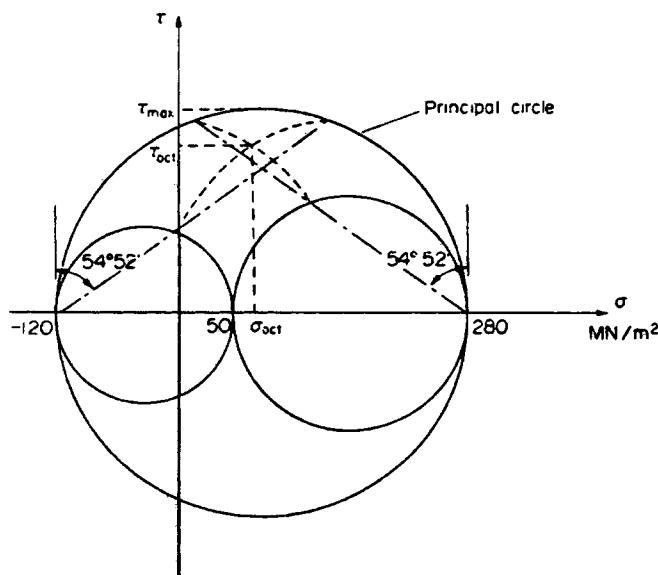


Fig. 8.43.

Example 8.4

A rectangular strain gauge rosette bonded at a point on the surface of an engineering component gave the following readings at peak load during test trials:

$$\epsilon_0 = 1240 \times 10^{-6}, \epsilon_{45} = 400 \times 10^{-6}, \epsilon_90 = 200 \times 10^{-6}$$

Determine the magnitude and direction of the principal stresses present at the point, and hence construct the full three-dimensional Mohr representations of the stress and strain systems present. $E = 210 \text{ GN/m}^2$, $\nu = 0.3$.

Solution

The two-dimensional Mohr's strain circle representing strain conditions in the plane of the surface at the point in question is drawn using the procedure of §14.14[†] (Fig. 8.44).

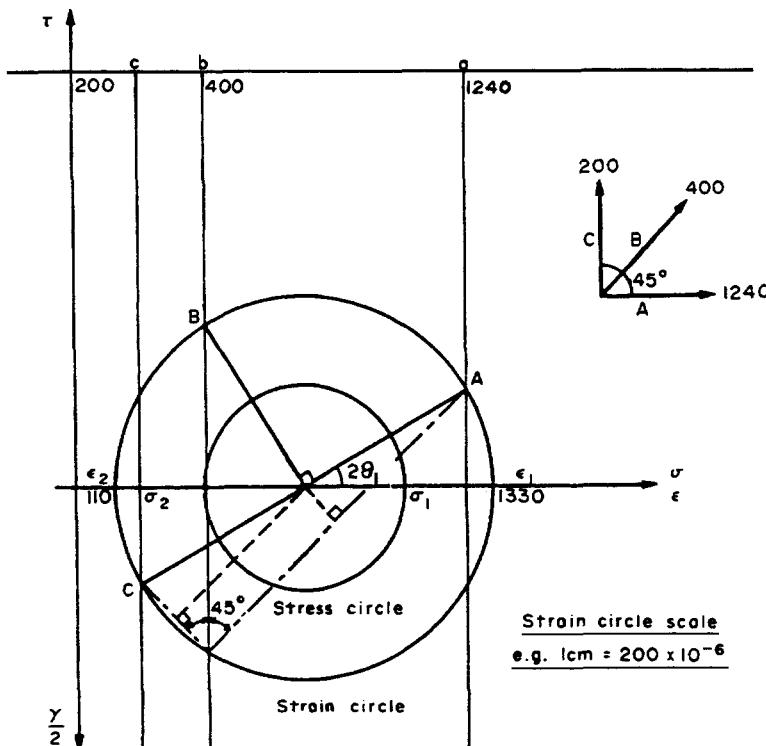


Fig. 8.44.

This establishes the values of the principal strains in the surface plane as $1330 \mu\epsilon$ and $110 \mu\epsilon$.

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

The relevant two-dimensional stress circle can then be superimposed as described in §14.13 using the relationships:

$$\begin{aligned}\text{radius of stress circle} &= \frac{(1 - \nu)}{(1 + \nu)} \times \text{radius of strain circle} \\ &= \frac{0.7}{1.3} \times 3.05 = 1.64 \text{ cm} \\ \text{stress scale} &= \frac{E}{(1 - \nu)} \times \text{strain scale} \\ &= \frac{210 \times 10^9}{0.7} \times 200 \times 10^{-6} \\ &= 60 \text{ MN/m}^2\end{aligned}$$

i.e. 1 cm on the stress diagram represents 60 MN/m².

The two principal stresses in the plane of the surface are then:

$$\begin{aligned}\sigma_1 (&= 5.25 \text{ cm}) &= 315 \text{ MN/m}^2 \\ \sigma_2 (&= 2.0 \text{ cm}) &= 120 \text{ MN/m}^2\end{aligned}$$

The third principal stress, normal to the free (unloaded) surface, is zero,

i.e.

$$\sigma_3 = 0$$

The directions of the principal stresses are also obtained from the stress circle. With reference to the 0° gauge direction,

$$\begin{aligned}\sigma_1 \text{ lies at } \theta_1 &= 15^\circ \text{ clockwise} \\ \sigma_2 \text{ lies at } (15^\circ + 90^\circ) &= 105^\circ \text{ clockwise}\end{aligned}$$

with σ_3 **normal to the surface** and hence to the plane of σ_1 and σ_2 .

N.B. – These angles are the directions of the principal stresses (and strains) and they do not refer to the directions of the plane on which the stresses act, these being normal to the above directions.

It is now possible to determine the value of the third principal strain, i.e. that normal to the surface. This is given by eqn. (14.2) as

$$\begin{aligned}\varepsilon_3 &= \frac{1}{E} [\sigma_3 - \nu\sigma_1 - \nu\sigma_2] \\ &= \frac{1}{210 \times 10^9} [0 - 0.3(315 + 120)] 10^6 \\ &= -621 \times 10^{-6} = -621 \mu\varepsilon\end{aligned}$$

The complete Mohr's three-dimensional stress and strain representations can now be drawn as shown in Figs. 8.45 and 8.46.

[†] E.J. Hearn, *Mechanics of Materials I*, Butterworth-Heinemann, 1997.

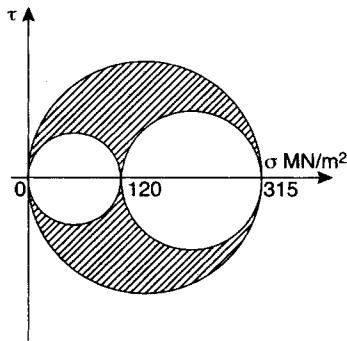


Fig. 8.45. Mohr stress circles.

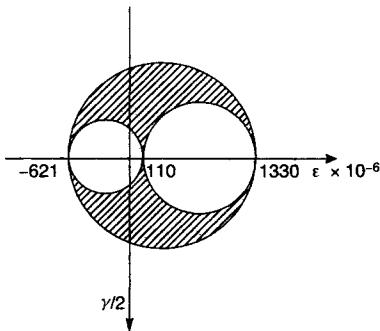


Fig. 8.46. Mohr strain circles.

Problems

8.1 (B). Given that the following strains exist at a point in a three-dimensional system determine the equivalent stresses which act at the point.

Take $E = 206 \text{ GN/m}^2$ and $\nu = 0.3$.

$$\varepsilon_{xx} = 0.0010 \quad \gamma_{xy} = 0.0002$$

$$\varepsilon_{yy} = 0.0005 \quad \gamma_{zx} = 0.0008$$

$$\varepsilon_{zz} = 0.0007 \quad \gamma_{yz} = 0.0010$$

$$[420, 340, 372, 15.8, 63.4, 79.2 \text{ MN/m}^2.]$$

8.2 (B). The following cartesian stresses act at a point in a body subjected to a complex loading system. If $E = 206 \text{ GN/m}^2$ and $\nu = 0.3$, determine the equivalent strains present.

$$\sigma_{xx} = 225 \text{ MN/m}^2 \quad \sigma_{yy} = 75 \text{ MN/m}^2 \quad \sigma_{zz} = 150 \text{ MN/m}^2$$

$$\tau_{xy} = 110 \text{ MN/m}^2 \quad \tau_{yz} = 50 \text{ MN/m}^2 \quad \tau_{zx} = 70 \text{ MN/m}^2$$

$$[764.6, 182, 291, 1388, 631, 883.5, \text{all } \times 10^{-6}.]$$

8.3 (B). Does a uniaxial stress field produce a uniaxial strain condition? Repeat Problem 8.2 for the following stress field:

$$\sigma_{xx} = 225 \text{ MN/m}^2$$

$$\sigma_{yy} = \sigma_{zz} = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0$$

$$[\text{No}; 1092, -327.7, -327.7, 0, 0, 0, \text{all } \times 10^{-6}.]$$

8.4 (C). The state of stress at a point in a body is given by the following equations:

$$\begin{aligned}\sigma_{xx} &= ax + by^2 + cz^3 & \tau_{xy} &= l + mz \\ \sigma_{yy} &= dx + ey^2 + fz^3 & \tau_{yz} &= ny + pz \\ \sigma_{zz} &= gx + hy^2 + kz^3 & \tau_{zx} &= qx^2 + sz^2\end{aligned}$$

If equilibrium is to be achieved what equations must the body-force stresses X , Y and Z satisfy?

$$[-(a + 2sz); -(p + 2ey); -(n + 2qx + 3kz^2).]$$

8.5 (C). At a point the state of stress may be represented in standard form by the following:

$$\begin{array}{lll}(3x^2 + 3y^2 - z) & (z - 6xy - \frac{3}{4}) & (x + y - \frac{3}{2}) \\ (z - 6xy - \frac{3}{4}) & 3y^2 & 0 \\ (x + y - \frac{3}{2}) & 0 & (3x + y - z + \frac{5}{4})\end{array}$$

Show that, if body forces are neglected, equilibrium exists.

8.6 (C). The plane stress distribution in a flat plate of unit thickness is given by:

$$\begin{aligned}\sigma_{xx} &= yx^3 - 2axy + by \\ \sigma_{yy} &= xy^3 - 2x^3y \\ \sigma_{xy} &= -\frac{3}{2}x^2y^2 + ay^2 + \frac{x^4}{2} + c\end{aligned}$$

Show that, in the absence of body forces, equilibrium exists. The load on the plate is specified by the following boundary conditions:

$$\begin{aligned}&\text{At } x = \pm \frac{w}{2}, \quad \sigma_{xy} = 0 \\ &\text{At } x = -\frac{w}{2}, \quad \sigma_{xx} = 0\end{aligned}$$

where w is the width of the plate.

If the length of the plate is L , determine the values of the constants a, b and c and determine the total load on the edge of the plate, $x = w/2$.

$$[B.P.] \left[\frac{3w^2}{8}, -\frac{w^3}{4}, -\frac{w^4}{32}, -\frac{w^3L^2}{4} \right]$$

8.7 (C). Derive the stress equations of equilibrium in cylindrical coordinates and show how these may be simplified for plane strain conditions.

A long, thin-walled cylinder of inside radius R and wall thickness T is subjected to an internal pressure p . Show that, if the hoop stresses are assumed independent of radius, the radial stress at any thickness t is given by

$$\sigma_{rr} = \frac{pR}{(R+t)} \left[\frac{t}{T} - 1 \right]$$

8.8 (B). Prove that the following relationship exists between the direction cosines:

$$l^2 + m^2 + n^2 = 1$$

8.9 (C). The six cartesian stress components are given at a point P for three different loading cases as follows (all MN/m^2):

	Case 1	Case 2	Case 3
σ_{xx}	100	100	100
σ_{yy}	200	200	-200
σ_{zz}	300	100	100
τ_{xy}	0	300	200
τ_{yz}	0	100	300
τ_{zx}	0	200	300

Determine for each case the resultant stress at P on a plane through P whose normal is coincident with the X axis.
 $[100, 374, 374 \text{ MN/m}^2.]$

8.10 (C). At a point in a material the stresses are:

$$\begin{aligned}\sigma_{xx} &= 37.2 \text{ MN/m}^2 & \sigma_{yy} &= 78.4 \text{ MN/m}^2 & \sigma_{zz} &= 149 \text{ MN/m}^2 \\ \sigma_{xy} &= 68.0 \text{ MN/m}^2 & \sigma_{yz} &= -18.1 \text{ MN/m}^2 & \sigma_{zx} &= 32 \text{ MN/m}^2\end{aligned}$$

Calculate the shear stress on a plane whose normal makes an angle of 48° with the X axis and 71° with the Y axis.
 $[41.3 \text{ MN/m}^2.]$

8.11 (C). At a point in a stressed material the cartesian stress components are:

$$\begin{aligned}\sigma_{xx} &= -40 \text{ MN/m}^2 & \sigma_{yy} &= 80 \text{ MN/m}^2 & \sigma_{zz} &= 120 \text{ MN/m}^2 \\ \sigma_{xy} &= 72 \text{ MN/m}^2 & \sigma_{yz} &= 46 \text{ MN/m}^2 & \sigma_{zx} &= 32 \text{ MN/m}^2\end{aligned}$$

Calculate the normal, shear and resultant stresses on a plane whose normal makes an angle of 48° with the X axis and 61° with the Y axis.
 $[135, 86.6, 161 \text{ MN/m}^2.]$

8.12 (C). Commencing from the equations defining the state of stress at a point, derive the general stress relationship for the normal stress on an inclined plane:

$$\sigma_n = \sigma_{xx}l^2 + \sigma_{zz}n^2 + \sigma_{yy}m^2 + 2\sigma_{xy}lm + 2\sigma_{yz}mn + 2\sigma_{zx}ln$$

Show that this relationship reduces for the plane stress system ($\sigma_{zz} = \sigma_{zx} = \sigma_{zy} = 0$) to the well-known equation

$$\sigma_n = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \frac{1}{2}(\sigma_{xx} - \sigma_{yy})\cos 2\theta + \sigma_{xy} \sin 2\theta$$

where $\cos \theta = l$.

8.13 (C). At a point in a material a resultant stress of value 14 MN/m^2 is acting in a direction making angles of 43° , 75° and $50^\circ 53'$ with the coordinate axes X , Y and Z .

- (a) Find the normal and shear stresses on an oblique plane whose normal makes angles of $67^\circ 13'$, 30° and $71^\circ 34'$, respectively, with the same coordinate axes.
- (b) If $\sigma_{xy} = 1.5 \text{ MN/m}^2$, $\sigma_{yz} = -0.2 \text{ MN/m}^2$ and $\sigma_{xz} = 3.7 \text{ MN/m}^2$ determine σ_{xx} , σ_{yy} and σ_{zz} .
 $[10, 9.8, 19.9, 3.58, 23.5 \text{ MN/m}^2.]$

8.14 (C). Three principal stresses of 250 , 100 and -150 MN/m^2 act in a direction X , Y and Z respectively. Determine the normal, shear and resultant stresses which act on a plane whose normal is inclined at 30° to the Z axis, the projection of the normal on the XY plane being inclined at 55° to the XZ plane.
 $[-75.2, 134.5, 154.1 \text{ MN/m}^2.]$

8.15 (C). The following cartesian stress components exist at a point in a body subjected to a three-dimensional complex stress system:

$$\begin{aligned}\sigma_{xx} &= 97 \text{ MN/m}^2 & \sigma_{yy} &= 143 \text{ MN/m}^2 & \sigma_{zz} &= 173 \text{ MN/m}^2 \\ \sigma_{xy} &= 0 & \sigma_{yz} &= 0 & \sigma_{zx} &= 102 \text{ MN/m}^2\end{aligned}$$

Determine the values of the principal stresses present at the point.
 $[233.8, 143.2, 35.8 \text{ MN/m}^2.]$

8.16 (C). A certain stress system has principal stresses of 300 MN/m^2 , 124 MN/m^2 and 56 MN/m^2 .

- (a) What will be the value of the maximum shear stress?
- (b) Determine the values of the shear and normal stresses on the octahedral planes.
- (c) If the yield stress of the material in simple tension is 240 MN/m^2 , will the above stress system produce failure according to the distortion energy and maximum shear stress criteria?
 $[122 \text{ MN/m}^2; 104, 160 \text{ MN/m}^2; \text{No, Yes.}]$

8.17 (C). A pressure vessel is being tested at an internal pressure of 150 atmospheres ($1 \text{ atmosphere} = 1.013 \text{ bar}$). Strains are measured at a point on the inside surface adjacent to a branch connection by means of an equiangular strain rosette. The readings obtained are:

$$\varepsilon_0 = 0.23\% \quad \varepsilon_{+120} = 0.145\% \quad \varepsilon_{-120} = 0.103\%$$

Draw Mohr's circle to determine the magnitude and direction of the principal strains. $E = 208 \text{ GN/m}^2$ and $\nu = 0.3$. Determine also the octahedral normal and shear strains at the point.

[0.235%, 0.083%, -0.142%, 9°28'; $\epsilon_{\text{oct}} = 0.0589\%$, $\gamma_{\text{oct}} = 0.310\%$.]

8.18 (C). At a point in a stressed body the principal stresses in the X , Y and Z directions are:

$$\sigma_1 = 49 \text{ MN/m}^2 \quad \sigma_2 = 27.5 \text{ MN/m}^2 \quad \sigma_3 = -6.3 \text{ MN/m}^2$$

Calculate the resultant stress on a plane whose normal has direction cosines $l = 0.73$, $m = 0.46$, $n = 0.506$. Draw Mohr's stress plane for the problem to check your answer. [38 MN/m².]

8.19 (C). For the data of Problem 8.18 determine graphically, and by calculation, the values of the normal and shear stresses on the given plane.

Determine also the values of the octahedral direct and shear stresses. [30.3, 23 MN/m²; 23.4, 22.7 MN/m².]

8.20 (C). During tests on a welded pipe-tee, internal pressure and torque are applied and the resulting distortion at a point near the branch gives rise to shear components in the r , θ and z directions.

A rectangular strain gauge rosette mounted at the point in question yields the following strain values for an internal pressure of 16.7 MN/m²:

$$\epsilon_0 = 0.0013 \quad \epsilon_{45} = 0.00058 \quad \epsilon_{90} = 0.00187$$

Use the Mohr diagrams for stress and strain to determine the state of stress on the octahedral plane. $E = 208 \text{ GN/m}^2$ and $\nu = 0.29$.

What is the direct stress component on planes normal to the direction of zero extension?

[$\sigma_{\text{oct}} = 310 \text{ MN/m}^2$; $\tau_{\text{oct}} = 259 \text{ MN/m}^2$; 530 MN/m².]

8.21 (C). During service loading tests on a nuclear pressure vessel the distortions resulting near a stress concentration on the inside surface of the vessel give rise to shear components in the r , θ and z directions. A rectangular strain gauge rosette mounted at the point in question gives the following strain values for an internal pressure of 5 MN/m².

$$\epsilon_0 = 150 \times 10^{-6}, \epsilon_{45} = 220 \times 10^{-6} \text{ and } \epsilon_{90} = 60 \times 10^{-6}$$

Use the Mohr diagrams for stress and strain to determine the principal stresses and the state of stress on the octahedral plane at the point. For the material of the pressure vessel $E = 210 \text{ GN/m}^2$ and $\nu = 0.3$.

[B.P.] [52.5, 13.8, -5 MN/m²; $\sigma_{\text{oct}} = 21 \text{ MN/m}^2$, $\tau_{\text{oct}} = 24 \text{ MN/m}^2$.]

8.22 (C). From the construction of the Mohr strain plane show that the ordinate $\frac{1}{2}\gamma$ for the case of $\alpha = \beta = \gamma$ (octahedral shear strain) is

$$\frac{1}{3}[(\epsilon_1 - \epsilon_2)^2 + (\epsilon_2 - \epsilon_3)^2 + (\epsilon_3 - \epsilon_1)^2]^{1/2}$$

8.23 (C). A stress system has three principal values:

$$\sigma_1 = 154 \text{ MN/m}^2 \quad \sigma_2 = 113 \text{ MN/m}^2 \quad \sigma_3 = 68 \text{ MN/m}^2$$

(a) Find the normal and shear stresses on a plane with direction cosines of $l = 0.732$, $m = 0.521$ with respect to the σ_1 and σ_2 directions.

(b) Determine the octahedral shear and normal stresses for this system. Check numerically.

[126, 33.4 MN/m²; 112, 35.1 MN/m².]

8.24 (C). A plane has a normal stress of 63 MN/m² inclined at an angle of 38° to the greatest principal stress which is 126 MN/m². The shear stress on the plane is 92 MN/m² and a second principal stress is 53 MN/m². Find the value of the third principal stress and the angle of the normal of the plane to the direction of stress.

[-95 MN/m²; 60°.]

8.25 (C). The normal stress σ_n on a plane has a direction cosine l and the shear stress on the plane is τ_n . If the two smaller principal stresses are equal show that

$$\sigma_1 = \sigma_n + \frac{\tau_n}{l} \sqrt{(1 - l^2)} \quad \text{and} \quad \sigma_2 = \sigma_3 = \sigma_n - \frac{1}{\sqrt{(1 - l^2)}}$$

If $\tau_n = 75 \text{ MN/m}^2$, $\sigma_n = 36 \text{ MN/m}^2$ and $l = 0.75$, determine, graphically σ_1 and σ_2 . [102, -48 MN/m².]

8.26 (C). If the strains at a point are $\epsilon = 0.0063$ and $\gamma = 0.00481$, determine the value of the maximum principal strain ϵ_1 if it is known that the strain components make the following angles with the three principal strain

directions:

$$\begin{aligned} \text{For } \varepsilon : \quad \alpha &= 38.5^\circ & \beta &= 56^\circ & \gamma &= \text{positive} \\ \text{For } \gamma : \quad \alpha' &= 128^\circ 32' & \beta' &= 45^\circ 10' & \gamma' &= \text{positive} \end{aligned} \quad [0.0075.]$$

8.27 (C). What is meant by the term deviatoric strain as related to a state of strain in three dimensions? Show that the sum of three deviatoric strains ε'_1 , ε'_2 and ε'_3 is zero and also that they can be related to the principal strains ε_1 , ε_2 and ε_3 as follows:

$$\varepsilon_1'^2 + \varepsilon_2'^2 + \varepsilon_3'^2 = \frac{1}{3}[(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2] \quad [\text{C.E.I.}]$$

8.28 (C). The readings from a rectangular strain gauge rosette bonded to the surface of a strained component are as follows:

$$\varepsilon_0 = 592 \times 10^{-6} \quad \varepsilon_{45} = 308 \times 10^{-6} \quad \varepsilon_{90} = -432 \times 10^{-6}$$

Draw the full three-dimensional Mohr's stress and strain circle representations and hence determine:

- (a) the principal strains and their directions;
- (b) the principal stresses;
- (c) the maximum shear stress.

Take $E = 200 \text{ GN/m}^2$ and $\nu = 0.3$.

$$[640 \times 10^{-6}, -480 \times 10^{-6}; \text{ at } 12^\circ \text{ and } 102^\circ \text{ to } A, 109, -63.5, 86.25 \text{ MN/m}^2]$$

8.29 (C). For a rectangular beam, unit width and depth $2d$, simple beam theory gives the longitudinal stress $\sigma_{xx} = CMy/I$ where

y = ordinate in depth direction (+ downwards)

M = BM in yx plane (+ sagging)

The shear force is Q and the shear stress τ_{xy} is to be taken as zero at top and bottom of the beam.

$\sigma_{yy} = 0$ at the bottom and $\sigma_{yy} = -w/\text{unit length}$, i.e. a distributed load, at the top.

$$\sigma_{zz} = \sigma_{zx} = \sigma_{zy} = 0$$

Using the equations of equilibrium in cartesian coordinates and without recourse to beam theory, find the distribution of σ_{yy} and σ_{xy} .

$$[\text{U.L.}] \left[\sigma_{yy} = \frac{w}{2I} \left(d^2y - \frac{y^3}{3} - \frac{2d^3}{3} \right), \quad \sigma_{xy} = -\frac{Q}{2I}(d^2 - y^2). \right]$$

8.30 (C). Determine whether the following strain fields are compatible:

(a) $\varepsilon_{xx} = 2x^2 + 3y^2 + z + 1$	(b) $\varepsilon_{xx} = 3y^2 + xy$
$\varepsilon_{yy} = 2y^2 + x^2 + 3z + 2$	$\varepsilon_{yy} = 2y + 4z + 3$
$\varepsilon_{zz} = 3x + 2y + z^2 + 1$	$\varepsilon_{zz} = 3zx + 2xy + 3yz + 2$
$\gamma_{xy} = 8xy$	$\gamma_{xy} = 6xy$
$\gamma_{yz} = 0$	$\gamma_{yz} = 2x$
$\gamma_{zx} = 0$	$\gamma_{zx} = 2y$
[Yes]	[No]

8.31 (C). The normal stress σ_n on a plane has a direction cosine l and the shear stress on the plane is τ . If the two smaller principal stresses are equal show that

$$\sigma_1 = \sigma_n + \frac{\tau}{l} \sqrt{(1 - l^2)} \quad \text{and} \quad \sigma_2 = \sigma_3 = \sigma_n - \frac{\tau l}{\sqrt{(1 - l^2)}}$$

8.32 (C). (i) A long thin-walled cylinder of internal radius R_0 , external radius R and wall thickness T is subjected to an internal pressure p , the external pressure being zero. Show that if the circumferential stress ($\sigma_{\theta\theta}$) is independent of the radius r then the radial stress (σ_{rr}) at any thickness t is given by

$$\sigma_{rr} = -p \frac{R_0}{T} \frac{(T - t)}{(R_0 + t)}$$

The relevant equation of equilibrium which may be used is:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{(\sigma_{rr} - \sigma_{\theta\theta})}{r} + F_r = 0$$

- (ii) Hence determine an expression for $\sigma_{\theta\theta}$ in terms of T .
 (iii) What difference in approach would you adopt for a similar treatment in the case of a thick-walled cylinder?
 [B.P.] $[R_0 p/T]$

8.33 (C). Explain what is meant by the following terms and discuss their significance:

- (a) Octahedral planes and stresses.
- (b) Hydrostatic and deviatoric stresses.
- (c) Plastic limit design.
- (d) Compatibility.
- (e) Principal and product second moments of area.

[B.P.]

8.34 (C). At a point in a stressed material the cartesian stress components are:

$$\begin{aligned}\sigma_{xx} &= -40 \text{ MN/m}^2 & \sigma_{yy} &= 80 \text{ MN/m}^2 & \sigma_{zz} &= 120 \text{ MN/m}^2 \\ \sigma_{xy} &= 72 \text{ MN/m}^2 & \sigma_{xz} &= 32 \text{ MN/m}^2 & \sigma_{yz} &= 46 \text{ MN/m}^2\end{aligned}$$

Calculate the normal, shear and resultant stresses on a plane whose normal makes an angle of 48° with the X axis and 61° with the Y axis.
 [B.P.] $[135.3, 86.6, 161 \text{ MN/m}^2]$

8.35 (C). The Cartesian stress components at a point in a three-dimensional stress system are those given in problem 8.33 above.

- (a) What will be the directions of the normal and shear stresses on the plane making angles of 48° and 61° with the X and Y axes respectively?

$$[l'm'n' = 0.1625, 0.7010, 0.6914; l_s m_s n_s = -0.7375, 0.5451, 0.4053]$$

- (b) What will be the magnitude of the shear stress on the octahedral planes where $l = m = n = 1/\sqrt{3}$?

$$[10.7 \text{ MN/m}^2]$$

8.36 (C). Given that the cartesian stress components at a point in a three-dimensional stress system are:

$$\begin{aligned}\sigma_{xx} &= 20 \text{ MN/m}^2, & \sigma_{yy} &= 5 \text{ MN/m}^2, & \sigma_{zz} &= -50 \text{ MN/m}^2 \\ \tau_{xy} &= 0, & \tau_{yz} &= 20 \text{ MN/m}^2, & \tau_{zx} &= -40 \text{ MN/m}^2\end{aligned}$$

- (a) Determine the stresses on planes with direction cosines 0.8165, 0.4082 and 0.4082 relative to the X , Y and Z axes respectively.
 [-14.2, 46.1, 43.8 MN/m 2]

- (b) Determine the shear stress on these planes in a direction with direction cosines of 0, -0.707 , 0.707 .

$$[39 \text{ MN/m}^2]$$

8.37 (C). In a finite element calculation of the stresses in a steel component, the stresses have been determined as follows, with respect to the reference directions X , Y and Z :

$$\begin{aligned}\sigma_{xx} &= 10.9 \text{ MN/m}^2 & \sigma_{yy} &= 51.9 \text{ MN/m}^2 & \sigma_{zz} &= -27.8 \text{ MN/m}^2 \\ \tau_{xy} &= -41.3 \text{ MN/m}^2 & \tau_{yz} &= -8.9 \text{ MN/m}^2 & \tau_{zx} &= 38.5 \text{ MN/m}^2\end{aligned}$$

It is proposed to change the material from steel to unidirectional glass-fibre reinforced polyester, and it is important that the direction of the fibres is the same as that of the maximum principal stress, so that the tensile stresses perpendicular to the fibres are kept to a minimum.

Determine the values of the three principal stresses, given that the value of the intermediate principal stress is 3.9 MN/m^2 .
 [-53.8; 3.9; 84.9 MN/m 2]

Compare them with the safe design tensile stresses for the glass-reinforced polyester of: parallel to the fibres, 90 MN/m^2 ; perpendicular to the fibres, 10 MN/m^2 .

Then take the direction cosines of the major principal stress as $l = 0.569$, $m = -0.781$, $n = 0.256$ and determine the maximum allowable misalignment of the fibres to avoid the risk of exceeding the safe design tensile stresses. (Hint: compression stresses can be ignored.)
 [15.9°]

8.38 (C). The stresses at a point in an isotropic material are:

$$\begin{aligned}\sigma_{xx} &= 10 \text{ MN/m}^2 & \sigma_{yy} &= 25 \text{ MN/m}^2 & \sigma_{zz} &= 50 \text{ MN/m}^2 \\ \tau_{xy} &= 15 \text{ MN/m}^2 & \tau_{yz} &= 10 \text{ MN/m}^2 & \tau_{zx} &= 20 \text{ MN/m}^2\end{aligned}$$

Determine the magnitudes of the maximum principal normal strain and the maximum principal shear strain at this point, if Young's modulus is 207 GN/m^2 and Poisson's ratio is 0.3.
 [280 $\mu\epsilon$; 419 $\mu\epsilon$]

8.39 (C). Determine the principal stresses in a three-dimensional stress system in which:

$$\begin{array}{lll} \sigma_{xx} = 40 \text{ MN/m}^2 & \sigma_{yy} = 60 \text{ MN/m}^2 & \sigma_{zz} = 50 \text{ MN/m}^2 \\ \sigma_{xz} = 30 \text{ MN/m}^2 & \sigma_{xy} = 20 \text{ MN/m}^2 & \sigma_{yz} = 10 \text{ MN/m}^2 \end{array}$$

$$[90 \text{ MN/m}^2, 47.3 \text{ MN/m}^2, 12.7 \text{ MN/m}^2]$$

8.40 (C). If the stress tensor for a three-dimensional stress system is as given below and one of the principal stresses has a value of 40 MN/m² determine the values of the three eigen vectors.

$$\begin{bmatrix} 30 & 10 & 10 \\ 10 & 0 & 20 \\ 10 & 20 & 0 \end{bmatrix}$$

$$[0.816, 0.408, 0.408]$$

8.41 (C). Determine the values of the stress invariants and the principal stresses for the cartesian stress components given in Problem 8.2.

$$[450; 423.75; 556.25; 324.8; 109.5; 15.6 \text{ MN/m}^2]$$

8.42 (C). The stress tensor for a three-dimensional stress system is given below. Determine the magnitudes of the three principal stresses and determine the eigen vectors of the major principal stress.

$$\begin{bmatrix} 80 & 15 & 10 \\ 15 & 0 & 25 \\ 10 & 25 & 0 \end{bmatrix}$$

$$[85.3, 19.8, -25.1 \text{ MN/m}^2, 0.9592, 0.2206, 0.1771.]$$

8.43 (C). A hollow steel shaft is subjected to combined torque and internal pressure of unknown magnitudes. In order to assess the strength of the shaft under service conditions a rectangular strain gauge rosette is mounted on the outside surface of the shaft, the centre gauge being aligned with the shaft axis. The strain gauge readings recorded from this gauge are shown in Fig. 8.47.

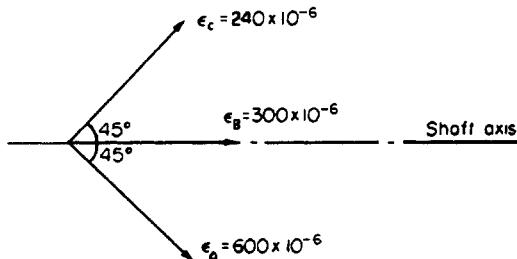


Fig. 8.47.

If E for the steel = 207 GN/m² and $\nu = 0.3$, determine:

- (a) the principal strains and their directions;
- (b) the principal stresses.

Draw complete Mohr's circle representations of the stress and strain systems present and hence determine the maximum shear stresses and maximum shear strain.

$$[636 \times 10^{-6} \text{ at } 16.8^\circ \text{ to } A, -204 \times 10^{-6} \text{ at } 106.8^\circ \text{ to } A, -360 \times 10^{-6} \text{ perp. to plane}; 159, -90, 0 \text{ MN/m}^2; 79.5 \text{ MN/m}^2, 996 \times 10^{-6}.]$$

8.44 (C). At a certain point in a material a resultant stress of 40 MN/m² acts in a direction making angles of 45°, 70° and 60° with the coordinate axes X , Y and Z . Determine the values of the normal and shear stresses on an oblique plane through the point given that the normal to the plane makes angles of 80°, 54° and 38° with the same coordinate axes.

If $\sigma_{xy} = 25 \text{ MN/m}^2$, $\sigma_{xz} = 18 \text{ MN/m}^2$ and $\sigma_{yz} = -10 \text{ MN/m}^2$, determine the values of σ_{xx} , σ_{yy} and σ_{zz} which act at the point.

$$[28.75, 27.7 \text{ MN/m}^2; -3.5, 29.4, 28.9 \text{ MN/m}^2.]$$

8.45 (C). The plane stress distribution in a flat plate of unit thickness is given by

$$\sigma_{xx} = x^3y - 2y^3x$$

$$\sigma_{yy} = y^3x - 2pxy + qx$$

$$\sigma_{xy} = \frac{y^4}{2} - \frac{3}{2}x^2y^2 + px^2 + s$$

If body forces are neglected, show that equilibrium exists.

The dimensions of the plate are given in Fig. 8.48 and the following boundary conditions apply:

$$\text{at } y = \pm \frac{b}{2} \quad \sigma_{xy} = 0$$

$$\text{and} \quad \text{at } y = -\frac{b}{2} \quad \sigma_{yy} = 0$$

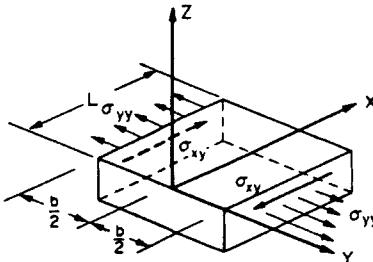


Fig. 8.48.

Determine:

(a) the values of the constants p , q and s ;

(b) the total load on the edge $y = \pm b/2$.

$$[\text{B.P.}] \left[\frac{3b^2}{8}, \frac{-b^3}{4}, \frac{-b^4}{32}, \frac{b^3L^2}{4} \right]$$

8.46 (C). Derive the differential equation in cylindrical coordinates for radial equilibrium without body force of an element of a cylinder subjected to stresses σ_r , σ_θ .

A steel tube has an internal diameter of 25 mm and an external diameter of 50 mm. Another tube, of the same steel, is to be shrunk over the outside of the first so that the shrinkage stresses just produce a condition of yield at the inner surfaces of each tube. Determine the necessary difference in diameters of the mating surfaces before shrinking and the required external diameter of the outer tube. Assume that yielding occurs according to the maximum shear stress criterion and that no axial stresses are set up due to shrinking. The yield stress in simple tension or compression = 420 MN/m² and $E = 208$ GN/m². [C.E.I.] [0.126 mm, 100 mm.]

8.47 (C). For a particular plane strain problem the strain displacement equations in cylindrical coordinates are:

$$\varepsilon_r = \frac{\partial u}{\partial r}, \quad \varepsilon_\theta = \frac{u}{r}, \quad \varepsilon_z = \gamma_{r\theta} = \gamma_{\theta z} = \gamma_{zr} = 0$$

Show that the appropriate compatibility equation in terms of stresses is

$$\nu r \frac{\partial \sigma_r}{\partial r} - (1 - \nu)r \frac{\partial \sigma_\theta}{\partial r} + \sigma_r - \sigma_\theta = 0$$

where ν is Poisson's ratio.

State the nature of a problem that the above equations can represent.

[C.E.I.]

8.48 (C). A bar length L , depth d , thickness t is simply supported and loaded at each end by a couple C as shown in Fig. 8.49. Show that the stress function $\phi = Ay^3$ adequately represents this problem. Determine the value of the coefficient A in terms of the given symbols. [$A = 2C/td^3$]

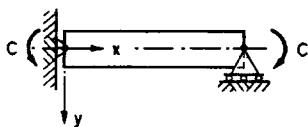


Fig. 8.49.

8.49 (C). A cantilever of unit width and depth $2d$ is loaded with forces at its free end as shown in Fig. 8.50. The stress function which satisfies the loading is found to be of the form:

$$\phi = ay^2 + by^3 + cxy^3 + exy$$

where the coordinates are as shown.

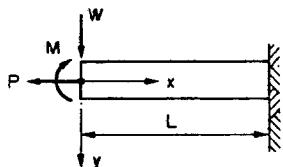


Fig. 8.50.

Determine the value of the constants a , b , c and e and hence show that the stresses are:

$$\sigma_{xx} = P/2d + 3My/2d^3 - 3Wxy/2d^3,$$

$$\sigma_{yy} = 0$$

$$\tau_{xy} = 3Wy^2/4d^3 - 3W/4d.$$

8.50 (C). A cantilever of unit width length L and depth $2a$ is loaded by a linearly distributed load as shown in Fig. 8.51, such that the load at distance x is qx per unit length. Proceeding from the sixth order polynomial derive the 25 constants using the boundary conditions, overall equilibrium and the biharmonic equation. Show that the stresses are:

$$\sigma_{xx} = \frac{qx^3y}{4a^3} + \frac{q}{4a^3} \left(-2xy^3 + \frac{6}{5}a^2xy \right)$$

$$\sigma_{yy} = -q\frac{x}{2} + qx \left(\frac{y^3}{4a^3} - \frac{3y}{4a} \right)$$

$$\tau_{xy} = \frac{3qx^2}{8a^3}(a^2 - y^2) - \frac{q}{8a^3}(a^4 - y^4) + \frac{3q}{20a}(a^2 - y^2)$$

Examine the state of stress at the free end ($x = 0$) and comment on the discrepancy of the shear stress. Compare the shear stress obtained from elementary theory, for $L/2a = 10$, with the more rigorous approach with the additional terms.

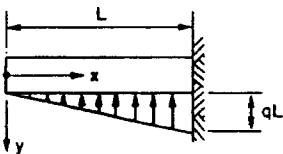


Fig. 8.51.

8.51 (C). Determine if the expression $\phi = (\cos^3 \theta)/r$ is a permissible Airy stress function, that is, make sure it satisfies the biharmonic equation. Determine the radial and shear stresses (σ_{rr} and $\tau_{r\theta}$) and sketch these on the periphery of a circle of radius a .

$$\left[\sigma_{rr} = \frac{2}{r^3} \cos \theta (3 - 5 \cos^2 \theta), \tau_{r\theta} = -\frac{6}{r^3} \cos^2 \theta \sin \theta. \right]$$

8.52 (C). The stress concentration factor due to a small circular hole in a flat plate subject to tension (or compression) in one direction is three. By superposition of the Kirsch solutions determine the stress concentration factors due to a hole in a flat plate subject to (a) pure shear, (b) two-dimensional hydrostatic tension. Show that the same result for case (b) can be obtained by considering the Lamé solution for a thick cylinder under external tension when the outside radius tends to infinity. [(a) 4; (b) 2.]

8.53 (C). Show that $\phi = Cr^2(\alpha - \theta + \sin \theta \cos \theta - \tan \alpha \cos^2 \theta)$ is a permissible Airy stress function and derive expressions for the corresponding stresses σ_{rr} , $\sigma_{\theta\theta}$ and $\tau_{r\theta}$.

These expressions may be used to solve the problem of a tapered cantilever beam of thickness carrying a uniformly distributed load $q/\text{unit length}$ as shown in Fig. 8.52.

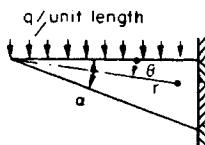


Fig. 8.52.

Show that the derived stresses satisfy every boundary condition along the edges $\theta = 0^\circ$ and $\theta = \alpha$. Obtain a value for the constant C in terms of q and α and thus show that:

$$\sigma_{rr} = \frac{qr}{t(\tan \alpha - \alpha)} \quad \text{when } \theta = 0^\circ$$

Compare this value with the longitudinal bending stress at $\theta = 0^\circ$ obtained from the simple bending theory when $\alpha = 5^\circ$ and $\alpha = 30^\circ$. What is the percentage error when using simple bending?

$$\left[C = \frac{q}{2r(\tan \alpha - \alpha)}, \quad -0.2\% \text{ and } -7.6\% \text{ (simple bending is lower)} \right]$$

CHAPTER 9

INTRODUCTION TO THE FINITE ELEMENT METHOD

Introduction

So far in this text we have studied the means by which components can be analysed using so-called Mechanics of Materials approaches whereby, subject to making simplifying assumptions, solutions can be obtained by hand calculation. In the analysis of complex situations such an approach may not yield appropriate or adequate results and calls for other methods. In addition to experimental methods, numerical techniques using digital computers now provide a powerful alternative. Numerical techniques for structural analysis divides into three areas; the long established but limited capability *finite difference* method, the finite element method (developed from earlier structural matrix methods), which gained prominence from the 1950s with the advent of digital computers and, emerging over a decade later, the boundary element method. Attention in this chapter will be confined to the most popular finite element method and the coverage is intended to provide

- an insight into some of the basic concepts of the finite element method (fem.), and, hence, some basis of finite element (fe.), practice,
- the theoretical development associated with some relatively simple elements, enabling analysis of applications which can be solved with the aid of a simple calculator, and
- a range of worked examples to show typical applications and solutions.

It is recommended that the reader wishing for further coverage should consult the many excellent specialist texts on the subject.¹⁻¹⁰ This chapter does require some knowledge of matrix algebra, and again, students are directed to suitable texts on the subject.¹¹

9.1. Basis of the finite element method

The fem. is a numerical technique in which the governing equations are represented in matrix form and as such are well suited to solution by digital computer. The solution region is represented, (discretised), as an assemblage (mesh), of small sub-regions called *finite elements*. These elements are connected at discrete points (at the extremities (corners), and in some cases also at intermediate points), known as nodes. Implicit with each element is its displacement function which, in terms of parameters to be determined, defines how the displacements of the nodes are interpolated over each element. This can be considered as an extension of the Rayleigh-Ritz process (used in Mechanics of Machines for analysing beam vibrations⁶). Instead of approximating the entire solution region by a single assumed displacement distribution, as with the Rayleigh-Ritz process, displacement distributions are assumed for each element of the assemblage. When applied to the analysis of a continuum (a solid or fluid through which the behavioural properties vary continuously), the discretisation becomes

an assemblage of a number of elements each with a limited, i.e. *finite* number of degrees of freedom (dof). The *element* is the basic “building unit”, with a predetermined number of dof., and can take various forms, e.g. one-dimensional rod or beam, two-dimensional membrane or plate, shell, and solid elements, see Fig. 9.1.

In stress applications, implicit with each element type is the nodal force/displacement relationship, namely the element stiffness property. With the most popular *displacement formulation* (discussed in §9.3), analysis requires the assembly and solution of a set of

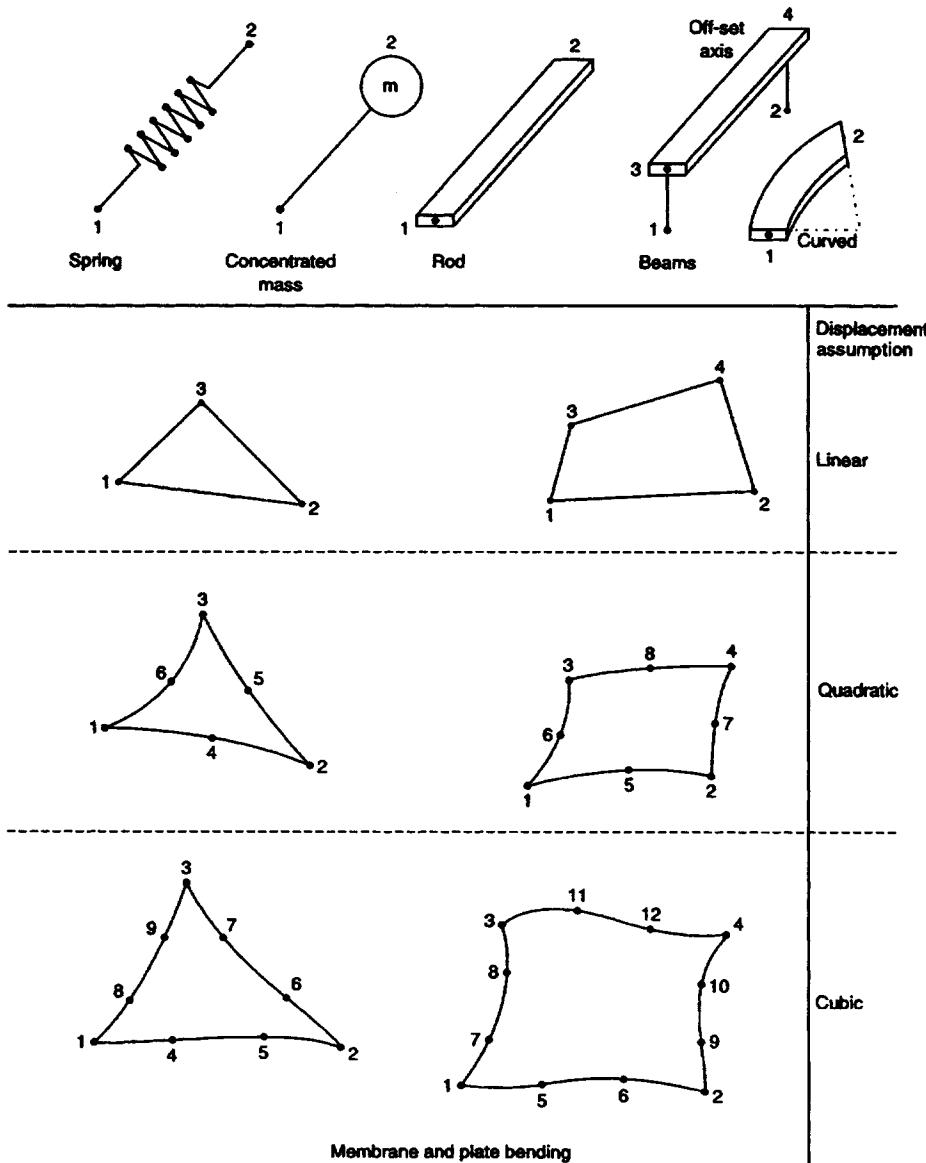


Fig. 9.1(a). Examples of element types with nodal points numbered.

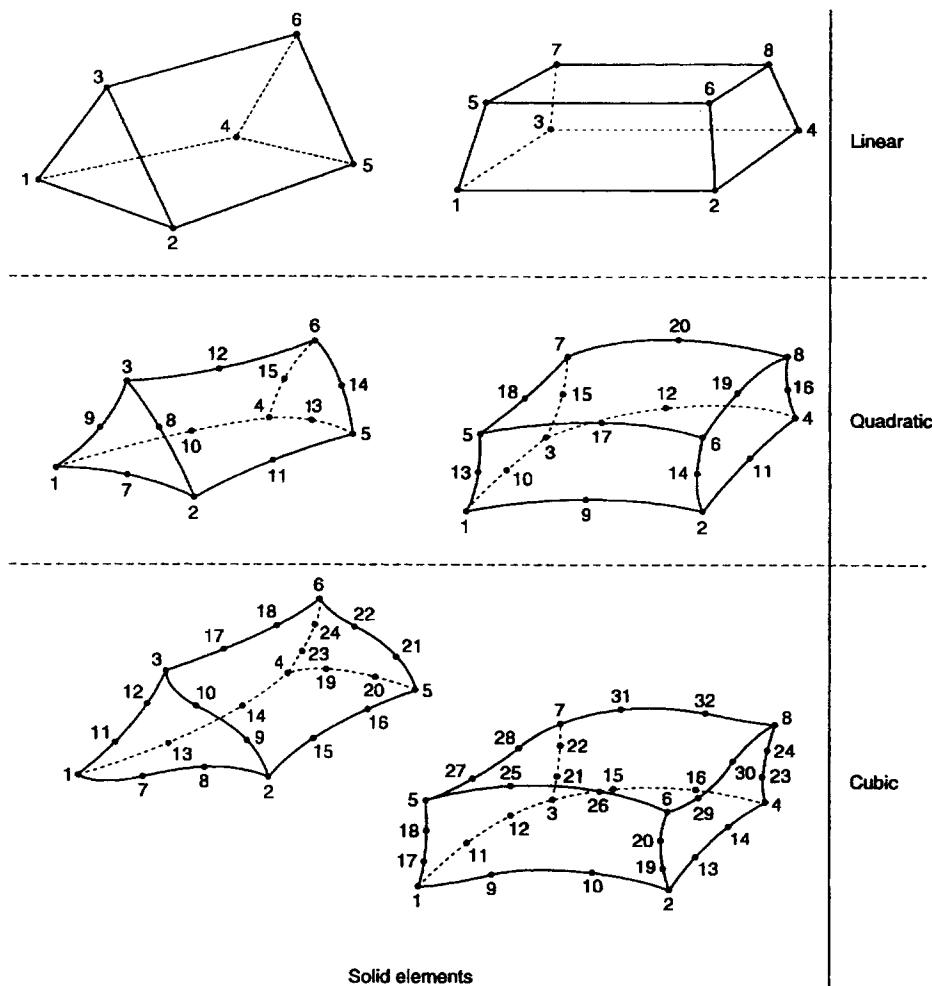


Fig. 9.1(b). Examples of element types with nodal points numbered.

simultaneous equations to provide the displacements for every node in the model. Once the displacement field is determined, the strains and hence the stresses can be derived, using strain-displacement and stress-strain relations, respectively.

9.2. Applicability of the finite element method

The fem. emerged essentially from the aerospace industry where the demand for extensive structural analyses was, arguably, the greatest. The general nature of the theory makes it applicable to a wide variety of boundary value problems (i.e. those in which a solution is required in a region of a body subject to satisfying prescribed boundary conditions, as encountered in equilibrium, eigenvalue and propagation or transient applications). Beyond the basic linear elastic/static stress analysis, finite element analysis (fea.), can provide solutions

to non-linear material and/or geometry applications, creep, fracture mechanics, free and forced vibration. Furthermore, the method is not confined to solid mechanics, but is applied successfully to other disciplines such as heat conduction, fluid dynamics, seepage flow and electric and magnetic fields. However, attention in this text will be restricted to linearly elastic static stress applications, for which the assumption is made that the displacements are sufficiently small to allow calculations to be based on the undeformed condition.

9.3. Formulation of the finite element method

Even with restriction to solid mechanics applications, the fem. can be formulated in a variety of ways which broadly divides into ‘differential equation’, or ‘variational’ approaches. Of the differential equation approaches, the most important, most widely used and most extensively documented, is the *displacement, or stiffness, based fem.* Due to its simplicity, generality and good numerical properties, almost all major general purpose analysis programmes have been written using this formulation. Hence, only the displacement based fem. will be considered here, but it should be realised that many of the concepts are applicable to other formulations.

In §9.7, 9.8 and 9.9 the theory using the displacement method will be developed for a rod, simple beam and triangular membrane element, respectively. Before this, it is appropriate to consider here, a brief overview of the steps required in a fe. linearly elastic static stress analysis. Whilst it can be expected that there will be detail differences between various packages, the essential procedural steps will be common.

9.4. General procedure of the finite element method

The basic steps involved in a fea. are shown in the flow diagram of Fig. 9.2. Only a simple description of these steps is given below. The reader wishing for a more in-depth treatment is urged to consult some of numerous texts on the subject, referred to in the introduction.

9.4.1. Identification of the appropriateness of analysis by the finite element method

Engineering components, except in the simplest of cases, frequently have non-standard features such as those associated with the geometry, material behaviour, boundary conditions, or excitation (e.g. loading), for which classical solutions are seldom available. The analyst must therefore seek alternative approaches to a solution. One approach which can sometimes be very effective is to simplify the application grossly by making suitable approximations, leading to Mechanics of Materials solutions (the basis of the majority of this text). Allowance for the effects of local disturbances, e.g. rapid changes in geometry, can be achieved through the use of design charts, which provide a means of *local enhancement*. In current practice, many design engineers prefer to take advantage of high speed, large capacity, digital computers and use numerical techniques, in particular the fem. The range of application of the fem. has already been noted in §9.2. The versatility of the fem. combined with the avoidance; or reduction in the need for prototype manufacture and testing offer significant benefits. However, the purchase and maintenance of suitable fe. packages, provision of a computer platform with adequate performance and capacity, application of a suitably

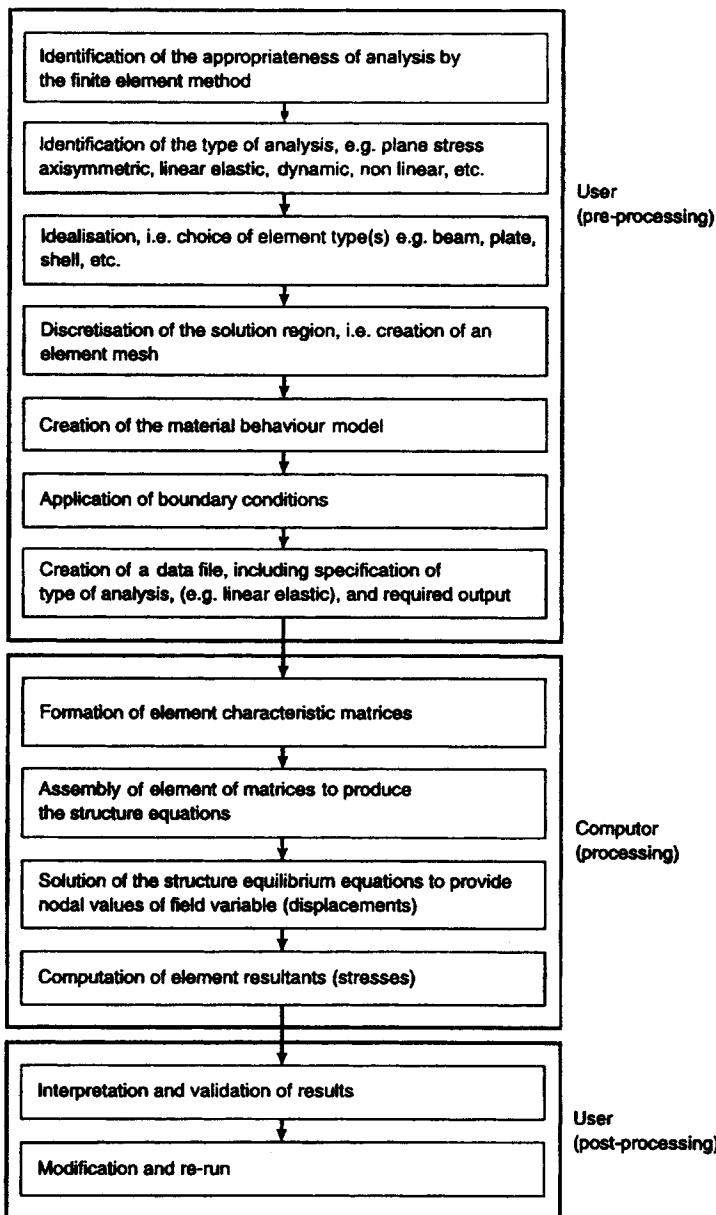


Fig. 9.2. Basic steps in the finite element method.

trained and experienced analyst and time for data preparation and processing should not be underestimated when selecting the most appropriate method. Experimental methods such as those described in Chapter 6 provide an effective alternative approach.

It is desirable that an analyst has access to all methods, i.e. analytical, numerical and experimental, and to not place reliance upon a single approach. This will allow essential validation of one technique by another and provide a degree of confidence in the results.

9.4.2. Identification of the type of analysis

The most appropriate type(s) of analysis to be employed needs to be identified in order that the component behaviour can best be represented. The assumption of either plane stress or plane strain is a common example. The high cost of a full three-dimensional analysis can be avoided if the assumption of both geometric and load symmetry can be made. If the application calls for elastic stress analysis, then the system equations will be linear and can be solved by a variety of methods, Gaussian elimination, Choleski factorisation or Gauss-Seidel procedure.⁵

For large displacement or post-yield material behaviour applications the system equations will be non-linear and iterative solution methods are required, such as that of Newton-Raphson.⁵

9.4.3. Idealisation

Commercially available finite element packages usually have a number of different elements available in the element library. For example, one such package, HKS ABAQUS¹² has nearly 400 different element variations. Examples of some of the commonly used elements have been given in Fig. 9.1.

Often the type of element to be employed will be evident from the physical application. For example, rod and beam elements can represent the behaviour of frames, whilst shell elements may be most appropriate for modelling a pressure vessel. Some regions which are actually three-dimensional can be described by only one or two independent coordinates, e.g. pistons, valves and nozzles, etc. Such regions can be idealised by using axisymmetric elements. Curved boundaries are best represented by elements having mid-side (or intermediate) nodes in addition to their corner nodes. Such elements are of higher *order* than linear elements (which can only represent straight boundaries) and include quadratic and cubic elements. The most popular elements belong to the so-called *isoparametric* family of elements, where the same parameters are used to define the geometry as define the displacement variation over the element. Therefore, those isoparametric elements of quadratic order, and above, are capable of representing curved sides and surfaces.

In situations where the type of elements to be used may not be apparent, the choice could be based on such considerations as

- (a) number of dof.,
- (b) accuracy required,
- (c) computational effort,
- (d) the degree to which the physical structure needs to be modelled.

Use of the elements with a quadratic displacement assumption are generally recommended as the best compromise between the relatively low cost but inferior performance of linear elements and the high cost but superior performance of cubic elements.

9.4.4. Discretisation of the solution region

This step is equivalent to replacing the actual structure or continuum having an infinite number of dof. by a system having a finite number of dof. This process, known as

discretisation, calls for engineering judgement in order to model the region as closely as necessary. Having selected the element type, discretisation requires careful attention to *extent of the model* (i.e. location of model boundaries), *element size and grading*, *number of elements*, and factors influencing the *quality of the mesh*, to achieve adequately accurate results consistent with avoiding excessive computational effort and expense. These aspects are briefly considered below.

Extent of model

Reference has already been made above to applications which are axisymmetric, or those which can be idealised as such. Generally, advantage should be taken of geometric and loading symmetry wherever it exists, whether it be plane or axial. Appropriate boundary conditions need to be imposed to ensure the reduced portion is representative of the whole. For example, in the analysis of a semi-infinite tension plate with a central circular hole, shown in Fig. 9.3, only a quadrant need be modelled. However, in order that the quadrant is representative of the whole, respective v and u displacements must be prevented along the x and y direction symmetry axes, since there will be no such displacements in the full model/component.

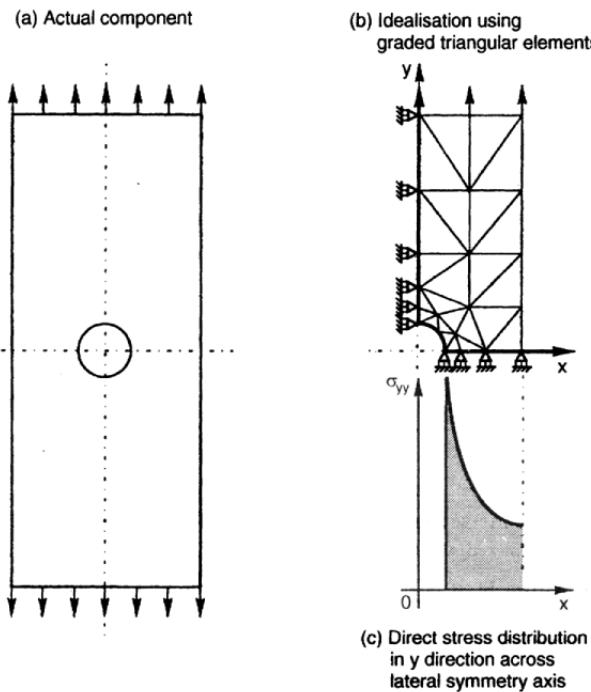


Fig. 9.3. Finite element analysis of a semi-infinite tension plate with a central circular hole, using triangular elements.

Further, it is known that disturbances to stress distributions due to rapid changes in geometry or load concentrations are only local in effect. Saint-Venant's principle states that the effect of stress concentrations essentially disappear within relatively small distances (approximately

equal to the larger lateral dimension), from the position of the disturbance. Advantage can therefore be taken of this principle by reducing the necessary extent of a finite element model. A rule-of-thumb is that a model need only extend to one-and-a-half times the larger lateral dimension from a disturbance, see Fig. 9.4.

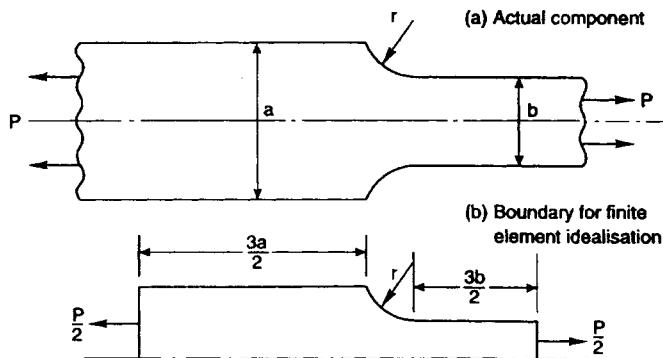


Fig. 9.4. Idealisation of a shouldered tension strip.

Element size and grading

The relative size of elements directly affects the quality of the solution. As the element size is reduced so the accuracy of solution can be expected to increase since there is better representation of the field variable, e.g. displacement, and/or better representation of the geometry. However, as the element size is reduced, so the number of elements increases with the accompanying penalty of increased computational effort. Needlessly small elements in regions with little variation in field variable or geometry will be wasteful. Equally, in regions where the stress variation is not of primary interest then a locally coarse mesh can be employed providing it is sufficiently far away from the region of interest and that it still provides an accurate stiffness representation. Therefore, element sizes should be graded in order to take account of anticipated stress/strain variations and geometry, and the results required. The example of stress analysis of a semi-infinite tension plate with a central circular hole, Fig. 9.3, serves to illustrate how the size of the elements can be graded from small-size elements surrounding the hole (where both the stress/strain and geometry are varying the most), to become coarser with increasing distance from the hole.

Number of elements

The number of elements is related to the previous matter of element size and, for a given element type, the number of elements will determine the total number of dof. of the model, and combined with the relative size determines the *mesh density*. An increase in the number of elements can result in an improvement in the accuracy of the solution, but a limit will be reached beyond which any further increase in the number of elements will not significantly improve the accuracy. This matter of *convergence* of solution is clearly important, and with experience a near optimal mesh may be achievable. As an alternative to increasing the number of elements, improvements in the model can be obtained by increasing the element order. This alternative form of *enrichment* can be performed manually (by substituting elements),

or can be performed automatically, e.g. the commercial package RASNA has this capability. Clearly, any increase in the number of elements (or element order), and hence dof., will require greater computational effort, will put greater demands on available computer memory and increase cost.

Quality of the mesh

The quality of the fe. predictions (e.g. of displacements, temperatures, strains or stresses), will clearly be affected by the performance of the model and its constituent elements. The factors which determine quality¹³ will now be explored briefly, namely

- (a) coincident elements,
- (b) free edges,
- (c) poorly positioned “midside” nodes,
- (d) interior angles which are too extreme,
- (e) warping, and
- (f) distortion.

(a) Coincident elements

Coincident elements refer to two or more elements which are overlaid and share some of the nodes, see Fig. 9.5. Such coincident elements should be deleted as part of cleaning-up of a mesh.

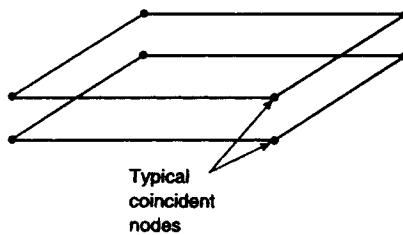


Fig. 9.5. Coincident elements.

(b) Free edges

A free edge should only exist as a model boundary. Neighbouring elements should share nodes along common inter-element boundaries. If they do not, then a free edge exists and will need correction, see Fig. 9.6.

(c) Poorly positioned “midside” nodes

Displacing an element’s “midside” node from its mid-position will cause distortion in the mapping process associated with high order elements; and in extreme cases can significantly degrade an element’s performance. There are two aspects to “midside” node displacement, namely, the relative position between the corner nodes, and the node’s offset from a straight

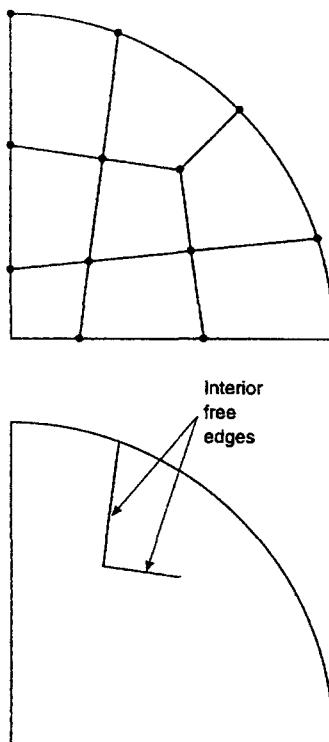
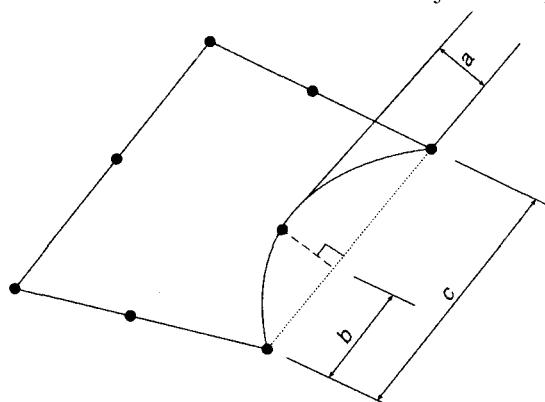


Fig. 9.6. Free edges.

line joining the corner nodes, see Fig. 9.7. The midside node's relative position should ideally be 50% of the side length for a parabolic element and 33.3% for a cubic element. An example of the effect of displacement of the “midside” node to the 25% position, is reported for a parabolic element¹⁴ to result in a 15% error in the major stress prediction.



$$\text{Percent displacement} = 100 \frac{b}{c}$$

$$\text{Offset} = a/c$$

Fig. 9.7. “Midside” node displacement.

(d) Interior angles which are too extreme

Interior angles which are excessively small or large will, like displaced “midside” nodes, cause distortion in the mapping process. A re-entrant corner (i.e. an interior angle greater than 180°), see Fig. 9.8, will cause failure in the mapping as the *Jacobian matrix* (relating the derivatives with respect to curvilinear (r,s), coordinates, to those with respect to cartesian (x,y), coordinates), will not have an inverse (i.e. its determinate will be zero). For quadrilateral elements the ideal interior angle is 90° , and for triangular elements it is 60° .

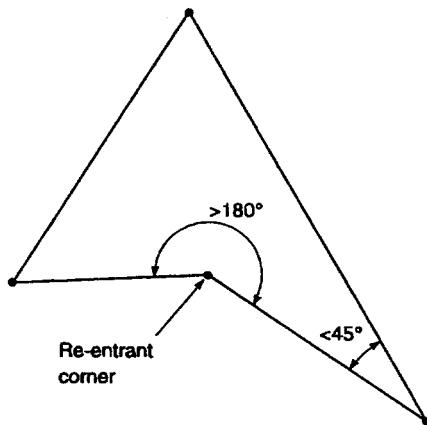


Fig. 9.8. Extreme interior angles.

(e) Warping

Warping refers to the deviation of the face of a “planar” element from being planar, see Fig. 9.9. The analogy of the three-legged milking stool (which is steady no-matter how uneven the surface is on which it is placed), to the triangular element serves to illustrate an advantage of this element over its quadrilateral counterpart.

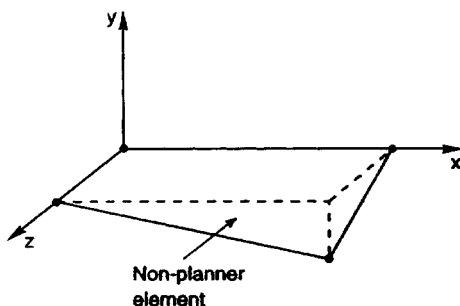


Fig. 9.9. Warping.

(f) Distortion

Distortion is the deviation of an element from its ideal shape, which corresponds to that in curvilinear coordinates. SDRC I-DEAS¹³ gives two measures, namely

- (1) the departure from the basic element shape which is known as *distortion*, see Fig. 9.10.

Ideally, for a quadrilateral element, with regard to distortion, the shape should be a rectangle, and

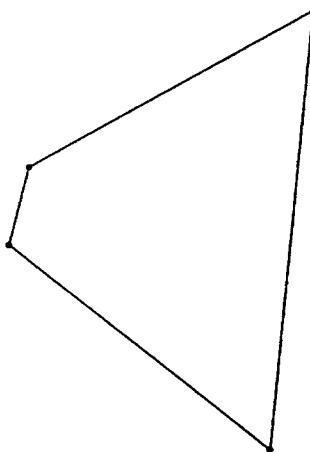


Fig. 9.10. Distortion.

- (2) the amount of elongation suffered by an element which is known as *stretch*, or *aspect ratio distortion*, see Fig. 9.11. Ideally, for a quadrilateral element, with regard to stretch, the shape should be square.

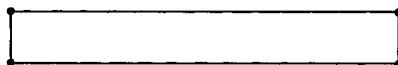


Fig. 9.11. Stretch.

Whilst small amounts of deviation of an element's shape from that of the parent element can, and must, be tolerated, unnecessary and excessive distortions and stretch, etc. must be avoided if degraded results are to be minimised. High order elements in gradually varying strain fields are most tolerant of shape deviation, whilst low order elements in severe strain fields are least tolerant.⁵ There are automatic means by which element shape deviation can be measured, using information derived from the Jacobian matrix. Errors in a solution and the rate of convergence can be judged by computing so-called energy norms derived from successive solutions.⁷ However, it is left to the judgement of the user to establish the degree of shape deviation which can be tolerated. Most packages offer quality checking facilities, which allows the user to interrogate the shape deviation of all, or a selection of, elements. I-DEAS provides a measure of element quality using a value with a range of -1 to $+1$, (where

+1 is the target value corresponding to zero distortion, and stretch, etc.). Negative values, which arise for example, from re-entrant corners, referred to above, will cause an attempted solution to fail, and hence need to be rectified. A distortion value above 0.7 can be considered acceptable, but errors will be incurred with any value below 1.0. However, circumstances may dictate acceptance of elements with a distortion value below 0.7. Similarly, as a rule-of-thumb, a stretch value above 0.5 can be considered acceptable, but again, errors will be incurred with any value below 1.0. Companies responsible for analyses may issue guidelines for quality, an example of which is shown in Table 9.1.

Table 9.1. Example of element quality guidelines.

Element	Interior angle°	Warpage	Distortion	Stretch
Triangle	30–90	N/A	0.35	0.3
Quadrilateral	45–135	0.2	0.60	0.3
Wedge	30–90	N/A	0.35	0.3
Tetrahedron	30–90	N/A	0.10	0.1
Hexahedron	45–135	0.2	0.5	0.3

9.4.5. Creation of the material model

The least material data required for a stress analysis is the empirical elastic modulus for the component under analysis describing the relevant stress/strain law. For a dynamic analysis, the material density must also be specified. Dependent upon the type of analysis, other properties may be required, including Poisson's ratio for two- and three-dimensional models and the coefficient of thermal expansion for thermal analyses. For analyses involving non-linear material behaviour then, as a minimum, the yield stress and yield criterion, e.g. von Mises, need to be defined. If the material within an element can be assumed to be isotropic and homogeneous, then there will be only one value of each material property. For non-isotropic material, i.e. orthotropic or anisotropic, then the material properties are direction and spatially dependent, respectively. In the extreme case of anisotropy, 21 independent values are required to define the material matrix.⁵

9.4.6. Node and element ordering

Before moving on to consider boundary conditions, it is appropriate to examine node and element ordering and its effect on efficiency of solution by briefly exploring the methods used. The formation of the element characteristic matrices (to be considered in §9.7, 9.8 and 9.9), and the subsequent solution are the two most computationally intensive steps in any fe. analysis. The computational effort and memory requirements of the solution are affected by the method employed, and are considered below.

It will be seen in Section 9.7, and subsequently, that the displacement based method involves the assembly of the *structural*, or *assembled*, *stiffness matrix* [K], and the load and displacement column matrices, $\{P\}$ and $\{p\}$, respectively, to form the governing equation for stress analysis $\{P\} = [K]\{p\}$. With reference to §9.7, and subsequently, two features of the fem. will be seen to be that the assembled stiffness matrix [K], is sparsely populated and is symmetric. Advantage can be taken of this in reducing the storage requirements of the

computer. Two solution methods are used, namely, *banded* or *frontal*, the choice of which is dependent upon the number of dof. in the model.

Banded method of solution

The banded method is appropriate for small to medium size jobs (i.e. up to 10 000 dof.). By carefully ordering the dof. the assembled stiffness matrix [K], can be banded with non-zero terms occurring only on the leading diagonal. Symmetry permits only half of the band to be stored, but storage requirements can still be high. It is advantageous therefore to minimise the *bandwidth*. A comparison of different node numbering schemes is provided by Figs. 9.12 and 9.13 in which a simple model comprising eight triangular linear elements is considered, and for further simplicity the nodal contributions are denoted as shaded squares, the empty squares denoting zeros.

The semi-bandwidth can be seen to depend on the node numbering scheme and the number of dof. per node and has a direct effect on the storage requirements and computational effort.

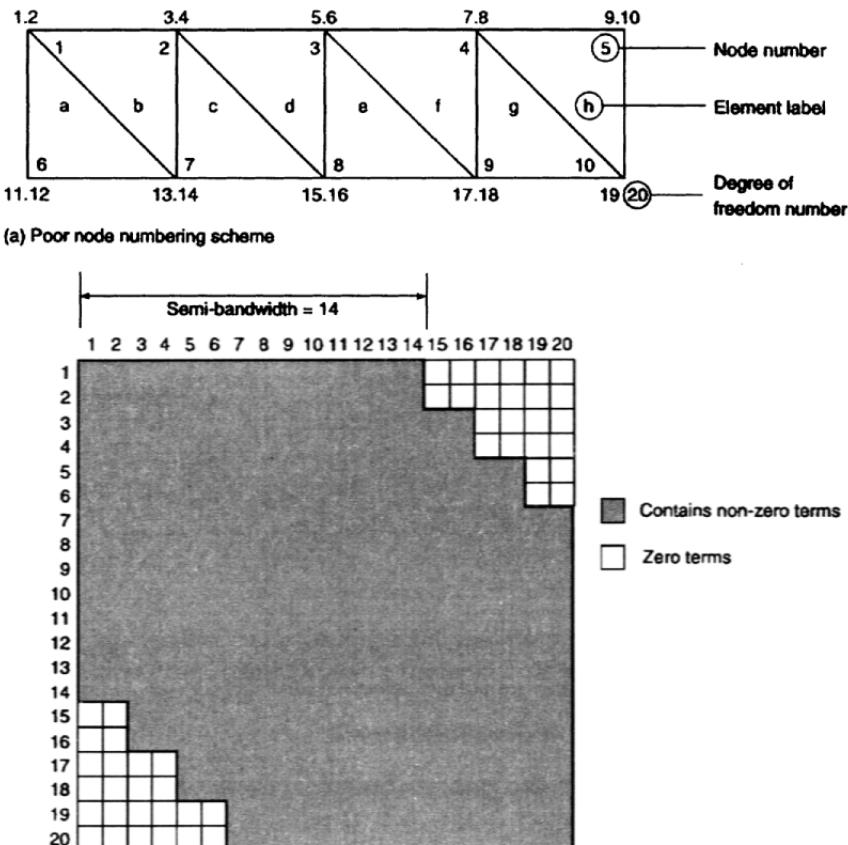
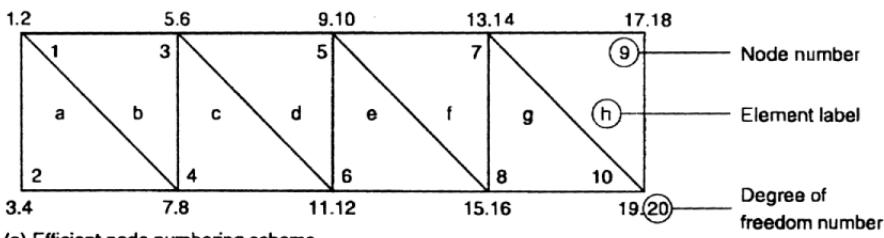


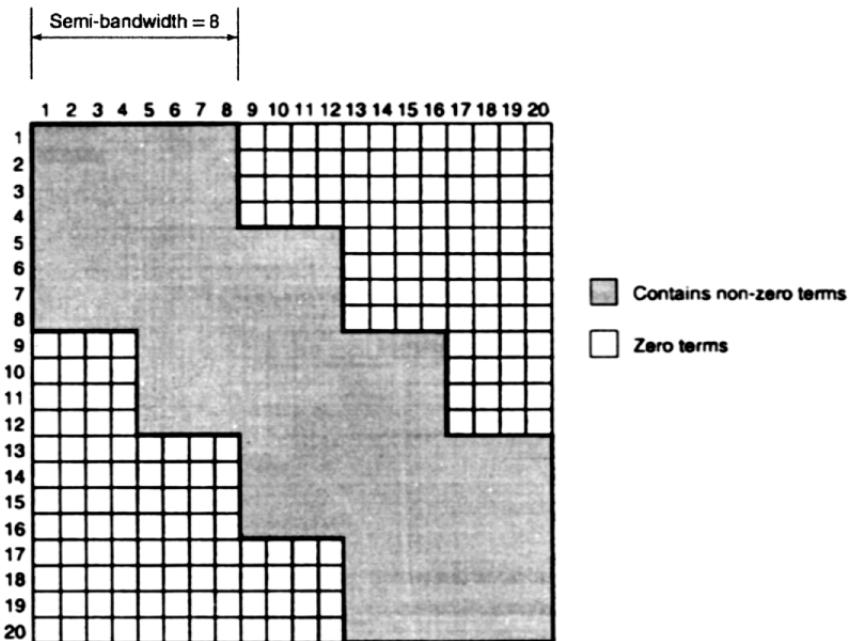
Fig. 9.12. Structural stiffness matrix corresponding to poor node ordering.

For a given number of dof. per node, which is generally fixed for each assemblage, the bandwidth can be minimised by using a proper node numbering scheme.

With reference to Figs. 9.12 and 9.13 there are a total of 20 dof. in the model (i.e. 10 nodes each with an assumed 2 dof.), and if the symmetry and bandedness is not taken advantage of, storage of the entire matrix would require $20^2 = 400$ locations. For the efficiently numbered model with a semi-bandwidth of 8, see Fig. 9.13, taking advantage of the symmetry and bandedness, the storage required for the upper, or lower, half-band is only $8 \times 20 = 160$ locations.



(a) Efficient node numbering scheme



(b) Structural stiffness matrix with non-zero terms closely banded

Fig. 9.13. Structural stiffness matrix corresponding to efficient node ordering.

From observation of Figs. 9.12 and 9.13 it can be deduced that the semi-bandwidth can be calculated from

$$\text{semi-bandwidth} = f(d + 1)$$

where f is the number of dof. per node and d is the maximum largest difference in the node numbers for all elements in the assemblage. This expression is applicable to any type of finite element. It follows that to minimise the bandwidth, d must be minimised and this is achieved by simply numbering the nodes across the shortest dimension of the region.

For large jobs the capacity of computer memory can be exceeded using the above banded method, in which case a frontal solution is used.

Frontal method of solution

The frontal method is appropriate for medium to large size jobs (i.e. greater than 10 000 dof.). To illustrate the method, consider the simple two-dimensional mesh shown in Fig. 9.14. Nodal contributions are assembled in element order. With reference to Fig. 9.14, with the assembly of element number 1 terms (i.e. contributions from nodes 1, 2, 6 and 7), all information relating to node number 1 will be complete since this node is not common to any other element. Thus the dofs. for node 1 can be eliminated from the set of system equations. Element number 2 contributions are assembled next, and the system matrix will now contain contributions from nodes 2, 3, 6, 7 and 8. At this stage the dofs. for node number 2 can be eliminated. Further element contributions are merged and at each stage any nodes which do not appear in later elements are *reduced out*. The solution thus proceeds as a front through the system. As, for example, element number 14 is assembled, dofs. for the nodes indicated by line B are required, see Fig. 9.14. After eliminations which follow assembly of element number 14, dofs. associated with line C are needed. The solution front has thus moved from line A to C.

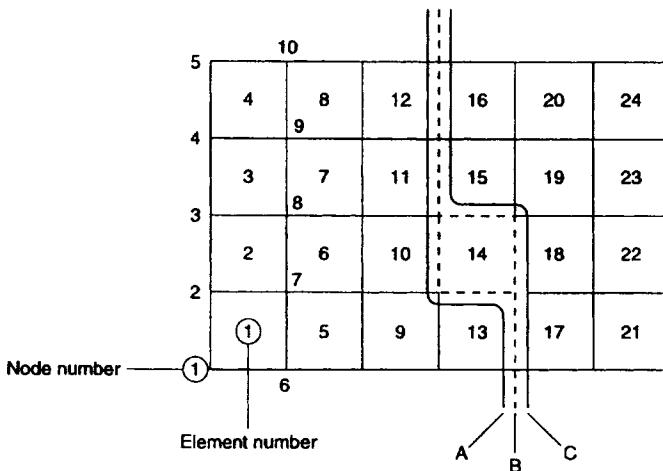


Fig. 9.14. Frontal method of solution.

To minimise memory requirements, which is especially important for jobs with large numbers of dof., the instantaneous width, i.e. *front size*, of the stiffness matrix during merging should be kept as small as possible. This is achieved by ensuring that elements are selected for merging in a specific order. Figures 9.15(a) and (b) serve to illustrate badly and well ordered elements, respectively, for a simple two-dimensional application. Front ordering facilities are

available with some fe. packages which will automatically re-order the elements to minimise the front size.

4	9	13	7	2
15	11	5	10	14
1	6	12	8	3

(a) Badly ordered elements

3	6	9	12	15
2	5	8	11	14
1	4	7	10	13

(b) Well ordered elements

Fig. 9.15. Examples of element ordering for frontal method.

9.4.7. Application of boundary conditions

Having created a mesh of finite elements and before the job is submitted for solution, it is necessary to enforce conditions on the boundaries of the model. Dependent upon the application, these can take the form of

- restraints,
- constraints,
- structural loads,
- heat transfer loads, or
- specification of active and inactive dof.

Attention will be restricted to a brief consideration of restraints and structural loads, which are sufficient conditions for a simple stress analysis. The reader wishing for further coverage is again urged to consult the many specialist texts.^{1–10}

Restraints

Restraints, which can be applied to individual, or groups of nodes, involve defining the displacements to be applied to the possible six dof., or perhaps defining a temperature. As an example, reference to Fig. 9.3(b) shows the necessary restraints to impose symmetry conditions. It can be assumed that the elements chosen have only 2 dof. per node, namely u and v translations, in the x and y directions, respectively. The appropriate conditions are

along the x -axis, $v = 0$, and
along the y -axis, $u = 0$.

The usual symbol, representing a frictionless roller support, which is appropriate in this case, is shown in Fig. 9.16(a), and corresponds to zero normal displacement, i.e. $\delta_n = 0$, and zero tangential shear stress, i.e. $\tau_t = 0$, see Fig. 9.16(b).

In a static stress analysis, unless sufficient restraints are applied, the system equations (see §9.4.5), cannot be solved, since an inverse will not exist. The physical interpretation of this is that the loaded body is free to undergo unlimited *rigid body motion*. Restraints must be



(a) Symbolic representation

(b) Actual restraint

Fig. 9.16. Boundary node with zero shear traction and zero normal displacement.

chosen to be sufficient, but not to create rigidity which does not exist in the actual component being modelled. This important matter of appropriate restraints can call for considerable engineering judgement, and the choice can significantly affect the behaviour of the model and hence the validity of the results.

Structural loads

Structural loads, which are applied to nodes can, through usual program facilities, be specified for application to groups of nodes, or to an entire model, and can take the form of loads, temperatures, pressures or accelerations. At the program level, only nodal loads are admissible, and hence when any form of distributed load needs to be applied, the nodal equivalent loads need to be computed, either manually or automatically. One approach is to simply define a set of statically equivalent loads, with the same resultant forces and moments as the actual loads. However, the most accurate method is to use kinematically equivalent loads⁵ as simple statically equivalent loads do not give a satisfactory solution for other than the simplest element interpolation. Figure 9.17 illustrates the case of an element with a quadratic displacement interpolation. Here the distributed load of total value W , is replaced by three nodal loads which produce the same work done as that done by the actual distributed load.

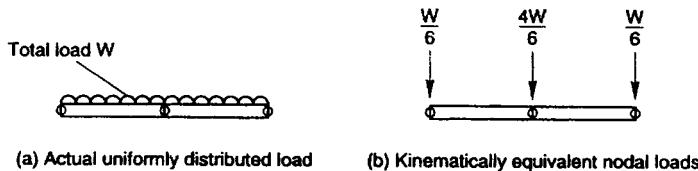


Fig. 9.17. Structural load representation.

9.4.8. Creation of a data file

The data file, or *deck*, will need to be in precisely the format required by the particular program being used; although essentially all programs will require the same basic model data, i.e. nodal coordinates, element type(s) and connectivity, material properties and boundary conditions. The type of solution will need to be specified, e.g. linear elastic, normal modes, etc. The required output will also need to be specified, e.g. deformations, stresses, strains, strain energy, reactions, etc. Much of the tedium of producing a data file is removed if automatic data preparation is available. Such an aid is beneficial with regard to minimising

the possibility of introducing data errors. The importance of avoiding errors cannot be over-emphasised, as the validity of the output is clearly dependent upon the correctness of the data. Any capability of a program to detect errors is to be welcomed. However, it should be realised that it is impossible for a program to detect all forms of error, e.g. incorrect but possible coordinates, incorrect physical or material properties, incompatible units, etc., can all go undetected. The user must, therefore, take every possible precaution to guard against errors. Displays of the mesh, including “shrunken” or “exploded” element views to reveal absent elements, restraints and loads should be scrutinised to ensure correctness before the solution stage is entered; the material and physical properties should also be examined.

9.4.9. Computer, processing, steps

The steps performed by the computer can best be followed by means of applications using particular elements, and this will be covered in §9.7 and subsequent sections.

9.4.10. Interpretation and validation of results

The numerical output following solution is often provided to a substantial number of decimal places which gives an aura of precision to the results. The user needs to be mindful that the fem. is numerical and hence is approximate. There are many potential sources of error, and a responsibility of the analyst is to ensure that errors are not significant. In addition to approximations in the model, significant errors can arise from round-off and truncation in the computation.

There are a number of checks that should be routine procedure following solution, and these are given below

- Ensure that any warning messages, given by the program, are pursued to ensure that the results are not affected. Error messages will usually accompany a failure in solution and clearly, will need corrective action.
- An obvious check is to examine the deformed geometry to ensure the model has behaved as expected, e.g. Poisson effect has occurred, slope continuity exists along axes of symmetry, etc.
- Ensure that equilibrium has been satisfied by checking that the applied loads and moments balance the reactions. Excessive out of balance indicates a poor mesh.
- Examine the smoothness of stress contours. Irregular boundaries indicate a poor mesh.
- Check inter-element stress discontinuities (stress jumps), as these give a measure of the quality of model. Large discontinuities indicate that the elements need to be enhanced.
- On traction-free boundaries the principal stress normal to the boundary should be zero. Any departure from this gives an indication of the quality to be expected in the other principal stress predictions for this point.
- Check that the directions of the principal stresses agree with those expected, e.g. normal and tangential to traction-free boundaries and axes of symmetry.

Results should always be assessed in the light of common-sense and engineering judgement. Manual calculations, using appropriate simplifications where necessary, should be carried out for comparison, as a matter of course.

9.4.11. Modification and re-run

Clearly, the need for design modification and subsequent fe. re-runs depends upon the particular circumstances. The computational burden may prohibit many re-runs. Indeed, for large jobs, (which may involve many thousands of dof. or many increments in the case of non-linear analyses), re-runs may not be feasible. The approach in such cases may be to run several exploratory crude models to gain some initial understanding how the component behaves, and hence aid final modelling.

9.5. Fundamental arguments

Regardless of the type of structure to be analysed, irrespective of whether the loading is static or dynamic, and whatever the material behaviour may be, there are only three types of argument to be invoked, namely, *equilibrium*, *compatibility* and *stress/strain law*. Whilst these arguments will be found throughout this text it is worthwhile giving them some explicit attention here as a sound understanding will help in following the theory of the fem. in the proceeding sections of this chapter.

9.5.1. Equilibrium

External nodal equilibrium

Static equilibrium requires that, with respect to some orthogonal coordinate system, the reactive forces and moments must balance the externally applied forces and moments. In fea. this argument extends to all nodes in the model. With reference to Fig. 9.18, some nodes may be subjected to applied forces and moments, (node number 4), and others may be support points (node numbers 1 and 6). There may be other nodes which appear to be neither of these (node numbers 2, 3, and 5), but are in fact nodes for which the applied force, or moment, is zero, whilst others provide support in one or two orthogonal directions and are loaded (or have zero load), in the remaining direction(s) (node number 6). Hence, for each node and with respect to appropriate orthogonal directions, satisfaction of external equilibrium requires

$$\text{external loads or reactions} = \text{summation of internal, element, loads}$$

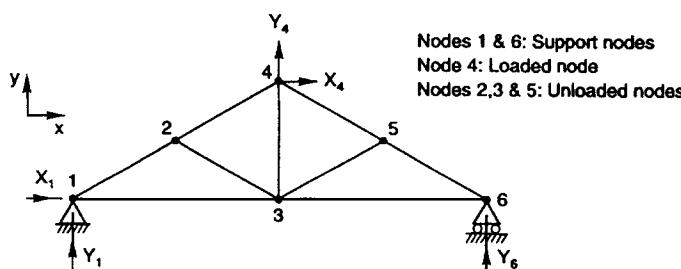


Fig. 9.18. Structural framework.

Then, for the j th node

$$\{P_j\} = \sum_{e=1}^m \{S^{(e)}\}$$

where the summation is of the internal loads at node j from all m elements joined at node j .

Use of this relation can be illustrated by considering the simple frame, idealised as planar with pin-joints and discretised as an assemblage of three elements, as shown in Fig. 9.19. The nodal force column matrix for the structure is

$$\{P\} = \{P_1 P_2 P_3\} = \{X_1 Y_1 X_2 Y_2 X_3 Y_3\}$$

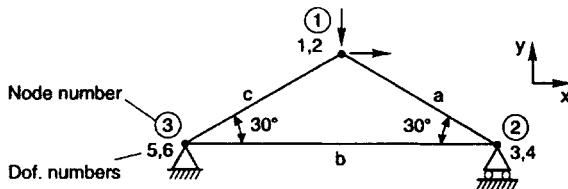


Fig. 9.19. Simple pin-jointed plane frame.

and the element force column matrix for the structure is

$$\{S\} = \{\{S^{(a)}\} \{S^{(b)}\} \{S^{(c)}\}\} = \{U_1 V_1 U_2 V_2, U_2 V_2 U_3 V_3, U_1 V_1 U_3 V_3\}$$

It follows from the above that external, nodal, equilibrium for the structure is satisfied by forming the relationship between the nodal and element forces as

$$\begin{bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \\ X_3 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{bmatrix}$$

Or, more concisely,

$$\{P\} = [a]^T \{S\} \quad (9.1)$$

which relates the nodal forces $\{P\}$ to the element forces $\{S\}$ for the whole structure.

Internal element equilibrium

Internal equilibrium can be explained most easily by considering an axial force element. For static equilibrium, the axial forces at each end will be equal in magnitude and opposite

in direction. If the element is pin-ended and has a uniform cross-sectional area, A , then for equilibrium within the element

$$A\sigma_x = U, \quad (9.2)$$

in which the axial stress σ_x is taken to be constant over the cross-section.

9.5.2. Compatibility

External nodal compatibility

The physical interpretation of external compatibility is that any displacement pattern is not accompanied with voids or overlaps occurring between previously continuous members. In fact, this requirement is usually only satisfied at the nodes. Often it is only the displacement field which is continuous at the nodes, and not an element's first or higher order displacement derivatives. Figure 9.20 shows quadratically varying displacement fields for two adjoining quadrilateral elements and serves to illustrate these limitations.

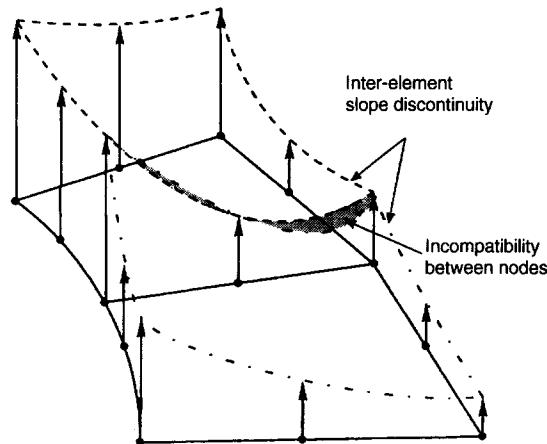


Fig. 9.20. Quadrilateral elements with quadratically varying displacement fields.

External, nodal, displacement compatibility will be shown to be automatically satisfied by a system of nodal displacements. For the simple frame shown in Fig. 9.19, the nodal displacement column matrix is

$$\{p\} = \{p_1 p_2 p_3\} = \{u_1 v_1, u_2 v_2, u_3 v_3\}$$

and the element displacement column matrix is

$$\{s\} = \{\{s^{(a)}\} \{s^{(b)}\} \{s^{(c)}\}\} = \{u_1 v_1 u_2 v_2, u_2 v_2 u_3 v_3, u_1 v_1 u_3 v_3\}$$

It follows from the above that external, nodal, compatibility is satisfied by forming the relationship between the element and nodal displacements as

$$\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \bar{u}_2 \\ v_2 \\ u_3 \\ v_3 \\ \bar{u}_1 \\ v_1 \\ u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

Or, more concisely, $\{s\} = [a]\{p\}$ (9.3)

which relates all the element displacements $\{s\}$ to the nodal displacements $\{p\}$ for the whole structure.

Internal element compatibility

Again, for simplicity consider an axial force element. For the displacement within such an element not to introduce any voids or overlaps the displacement along the element, u , needs to be a continuous function of position, x . The compatibility condition is satisfied by

$$\partial u / \partial x = \varepsilon_x \quad (9.4)$$

9.5.3. Stress/strain law

Assuming for simplicity the material behaviour to be homogeneous, isotropic and linearly elastic, then Hooke's law applies giving, for a one-dimensional stress system in the absence of thermal strain,

$$\varepsilon_x = \sigma_x / E \quad (9.5)$$

in which E is the empirical modulus of elasticity.

9.5.4. Force/displacement relation

Combining eqns. (9.2), (9.4) and (9.5) and taking u to be a function of x only, gives

$$U/A = \sigma_x = \sigma_x E = E du/dx$$

Or,

$$U dx = AE du$$

Integrating, and taking $u(0) = u_i$ and $u(L) = u_j$, corresponding to displacements at nodes i and j of an axial force element of length L , gives the force/displacement relationship

$$U = AE(u_j - u_i)/L \quad (9.6)$$

in which $(u_j - u_i)$ denotes the deformation of the element. Thus the force/displacement relationship for an axial force element has been derived from equilibrium, compatibility and stress/strain arguments.

9.6. The principle of virtual work

In the previous section the three basic arguments of equilibrium, compatibility and constitutive relations were invoked and, in the subsequent sections, it will be seen how these arguments can be used to derive rod and simple beam element equations. However, some situations, for example, may require elements of non-uniform cross-section or representation of complex geometry, and are not amenable to solution by this approach. In such situations, alternative approaches using energy principles are used, which allow the field variables to be represented by approximating functions whilst still satisfying the three fundamental arguments. Amongst the number of energy principles which can be used, the one known as the *principle of virtual work* will be considered here.

The equation of the principle of virtual work

Virtual work is produced by perturbing a system slightly from an equilibrium state. This can be achieved by allowing small, kinematically possible displacements, which are not necessarily real, and hence are *virtual displacements*. In the following brief treatment the corresponding equation of virtual work is derived by considering the linearly elastic, uniform cross-section, axial force element in Fig. 9.21. For a more rigorous treatment the reader is referred to Ref. 8 (p. 350). In Fig. 9.21 the nodes are shown detached to distinguish between the nodal and element quantities.

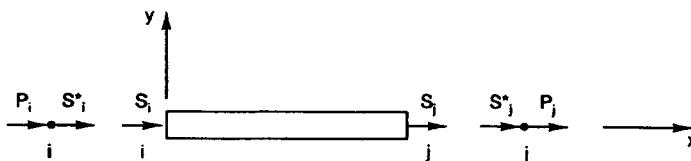


Fig. 9.21. Axial force element, shown with detached nodes.

Giving the nodal points virtual displacements \bar{u}_i and \bar{u}_j , the virtual work for the two nodal points is

$$\bar{W} = (P_i + S_i^*)\bar{u}_i + (P_j + S_j^*)\bar{u}_j \quad (9.7)$$

This virtual work must be zero since the two nodal points are rigid bodies. It follows, since the virtual displacements are arbitrary and independent, that

$$P_i + S_i^* = 0 \quad \text{or} \quad P_i = -S_i^*$$

and

$$P_j + S_j^* = 0 \quad \text{or} \quad P_j = -S_j^*$$

which, for the single element, are the equations of external equilibrium.

Applying Newton's third law, the forces between the nodes and element are related as

$$S_i^* = -S_i \quad \text{and} \quad S_j^* = -S_j \quad (9.8)$$

Substituting eqns. (9.8) into eqn. (9.7) gives

$$\begin{aligned} \bar{W} &= 0 = (P_i - S_i)\bar{u}_i + (P_j - S_j)\bar{u}_j \\ &= (P_i\bar{u}_i + P_j\bar{u}_j) - (S_i\bar{u}_i + S_j\bar{u}_j) \end{aligned} \quad (9.9)$$

The first quantity $(P_i\bar{u}_i + P_j\bar{u}_j)$, to first order approximation assuming linearly elastic behaviour represents the virtual work done by the applied external forces, denoted as \bar{W}_e . The second quantity, $(S_i\bar{u}_i + S_j\bar{u}_j)$, again to first order approximation represents the virtual work done by element internal forces, denoted as \bar{W}_i . Hence, eqn. (9.9) can be abbreviated to

$$0 = \bar{W}_e - \bar{W}_i \quad (9.10)$$

which is the equation of the principle of virtual work for a deformable body.

The external virtual work will be found from the product of external loads and corresponding virtual displacements, recognising that no work is done by reactions since they are associated with suppressed dof. The internal virtual work will be given by the strain energy, expressed using real stress and virtual strain (arising from virtual displacements), as

$$\bar{W}_i = \int_v \bar{\epsilon}\sigma dv \quad (9.11)$$

which, for the case of a prismatic element with constant stress and strain over the volume, becomes

$$\bar{W}_i = \bar{\epsilon}\sigma AL$$

9.7. A rod element

The formulation of a rod element will be considered using two approaches, namely the use of fundamental equations, based on equilibrium, compatibility and constitutive (i.e. stress/strain law), arguments and use of the principle of virtual work equation.

9.7.1. Formulation of a rod element using fundamental equations

Consider the structure shown in Fig. 9.18, for which the deformations (derived from the displacements), member forces, stresses and reactions are required. Idealising the structure such that all the members and loads are taken to be planar, and all the joints to act as frictionless hinges, i.e. pinned, and hence incapable of transmitting moments, the corresponding behaviour can be represented as an assemblage of rod finite elements. A typical element is shown in Fig. 9.22, for which the physical and material properties are taken to be constant throughout the element. Changes in properties, and load application are only admissible at nodal positions, which occur only at the extremities of the elements. Each node is considered to have two translatory freedoms, i.e. two dof., namely u and v displacements in the element x and y directions, respectively.

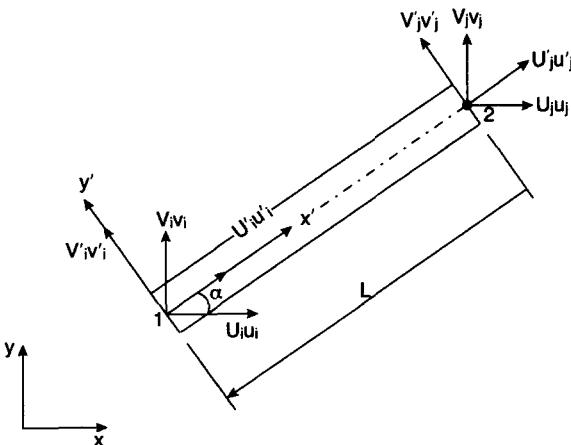


Fig. 9.22. An axial force rod element.

Element stiffness matrix in local coordinates

Quantities in the element coordinate directions are denoted with a prime ('), to distinguish from those with respect to the global coordinates. Displacements in the local element x' direction will cause an elongation of the element of $u'_j - u'_i$, with corresponding strain, $(u'_j - u'_i)/L$. Assuming Hookean behaviour, the element loads in the positive, local, x' direction will hence be given as

$$U'_i = AE(u'_i - u'_j)/L \quad \text{and} \quad U'_j = AE(u'_j - u'_i)/L$$

which are force/displacement relations similar to eqn. (9.6), and hence satisfy internal element equilibrium, compatibility and the appropriate stress/strain law.

In isolation, the element will not have any stiffness in the local y' direction. However, stiffness in this direction will arise from assembly with other elements with different inclinations. The element force/displacement relation can now be written in matrix form, as

$$\begin{bmatrix} U'_i \\ V'_i \\ U'_j \\ V'_j \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u'_i \\ v'_i \\ u'_j \\ v'_j \end{bmatrix} \quad (9.12)$$

$$\text{Or, more concisely, } \{S'^{(e)}\} = [k'^{(e)}]\{s'^{(e)}\} \quad (9.13)$$

from which the *element stiffness* matrix with respect to local coordinates is:

$$[k'^{(e)}] = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (9.14)$$

Element stress matrix in local coordinates

For a pin-jointed frame the only significant stress will be axial. Hence, with respect to local coordinates, the axial stress for a rod element will be given as:

$$\sigma^{(e)} = \frac{E}{L} [-1 \quad 0 \quad 1 \quad 0] \begin{bmatrix} u'_i \\ v'_i \\ u'_j \\ v'_j \end{bmatrix} \quad (9.15)$$

Or, more concisely,

$$\sigma^{(e)} = [H'^{(e)}] \{s'^{(e)}\} \quad (9.16)$$

from which the stress matrix with respect to local coordinates is:

$$[H'^{(e)}] = \frac{E}{L} [-1 \quad 0 \quad 1 \quad 0] \quad (9.17)$$

Transformation of displacements and forces

To enable assembly of contributions from each constituent element meeting at each joint, it is necessary to transform the force/displacement relationships to some global coordinate system, by means of a transformation matrix $[T^{(e)}]$. This matrix is derived by establishing the relationship between the displacements (or forces), in local coordinates x' , y' , and those in global coordinates x , y . Note that the element inclination α , is taken to be positive when acting clockwise viewed from the origin along the positive z -axis, and is measured from the positive global x -axis. With reference to Fig. 9.23, for node i ,

$$u'_i = u_i \cos \alpha + v_i \sin \alpha, \quad \text{and} \quad v'_i = -u_i \sin \alpha + v_i \cos \alpha.$$

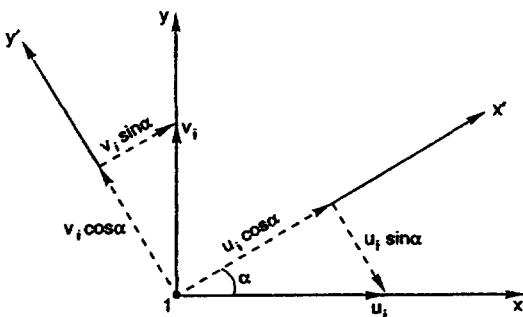


Fig. 9.23. Transformation of displacements.

Similarly, for node j ,

$$u'_j = u_j \cos \alpha + v_j \sin \alpha, \quad \text{and} \quad v'_j = -u_j \sin \alpha + v_j \cos \alpha.$$

Writing in matrix form the above becomes:

$$\begin{bmatrix} u'_i \\ v'_i \\ u'_j \\ v'_j \end{bmatrix} = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 & 0 \\ -\sin \alpha & \cos \alpha & 0 & 0 \\ 0 & 0 & \cos \alpha & \sin \alpha \\ 0 & 0 & -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{bmatrix}$$

Or, more concisely,

$$\{S'^{(e)}\} = [T^{(e)}] \{s^{(e)}\} \quad (9.18)$$

in which the transformation matrix is:

$$[T^{(e)}] = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 & 0 \\ -\sin \alpha & \cos \alpha & 0 & 0 \\ 0 & 0 & \cos \alpha & \sin \alpha \\ 0 & 0 & -\sin \alpha & \cos \alpha \end{bmatrix} \quad (9.19)$$

The same transformation enables the relationship between member loads in local and global coordinates to be written as:

$$\{S'^{(e)}\} = [T^{(e)}]\{S^{(e)}\} \quad (9.20)$$

Expressing eqn. (9.20) in terms of the member loads with respect to the required global coordinates, we obtain:

$$\{S^{(e)}\} = [T^{(e)}]^{-1}\{S'^{(e)}\}$$

Substituting from eqn. (9.13) gives:

$$\{S^{(e)}\} = [T^{(e)}]^{-1}[k'^{(e)}]\{s'^{(e)}\}$$

Further, substituting from eqn. (9.18) gives:

$$\{S^{(e)}\} = [T^{(e)}]^{-1}[k'^{(e)}][T^{(e)}]\{s^{(e)}\}$$

It can be shown, by equating work done in the local and global coordinates systems, that

$$[T^{(e)}]^T = [T^{(e)}]^{-1}$$

(This property of the transformation matrix, $[T^{(e)}]$, whereby the inverse equals the transpose is known as orthogonality.) Hence, element loads are given by:

$$\{S^{(e)}\} = [T^{(e)}]^T[k'^{(e)}][T^{(e)}]\{s^{(e)}\}$$

Or, more simply

$$\{S^{(e)}\} = [k'^{(e)}]\{s^{(e)}\} \quad (9.21)$$

in which the element stiffness matrix in global coordinates is

$$[k^{(e)}] = [T^{(e)}]^T[k'^{(e)}][T^{(e)}] \quad (9.22)$$

Element stiffness matrix in global coordinates

Substituting from eqns. (9.14) and (9.19) into eqn. (9.22), transposing the transformation matrix and performing the triple matrix product gives the element stiffness matrix in global coordinates as:

$$[k^{(e)}] = \frac{AE}{L} \begin{bmatrix} \cos^2 \alpha & \cos \alpha \sin \alpha & -\cos^2 \alpha & -\cos \alpha \sin \alpha \\ -\sin \alpha \cos \alpha & \sin^2 \alpha & -\sin \alpha \cos \alpha & -\sin^2 \alpha \\ -\cos^2 \alpha & -\cos \alpha \sin \alpha & \cos^2 \alpha & \cos \alpha \sin \alpha \\ -\sin \alpha \cos \alpha & -\sin^2 \alpha & \sin \alpha \cos \alpha & \sin^2 \alpha \end{bmatrix} \quad (9.23)$$

Element stress matrix in global coordinates

The element stress matrix found in local coordinates, eqn. (9.17), can be transformed to global coordinates by substituting from eqn. (9.18) into eqn. (9.16) to give

$$\sigma^{(e)} = [H'^{(e)}][T^{(e)}]\{s^{(e)}\} \quad (9.24)$$

in which the element stress matrix in global coordinates is

$$[H^{(e)}] = [H'^{(e)}][T^{(e)}] \quad (9.25)$$

Substituting from eqn. (9.17) and (9.19) into eqn. (9.25) gives the element stress matrix in global coordinates as

$$[H^{(e)}] = E^{(e)}[-\cos \alpha \ -\sin \alpha \ \cos \alpha \ \sin \alpha]^{(e)}/L^{(e)} \quad (9.26)$$

Formation of structural governing equation and assembled stiffness matrix

With reference to § 9.5, external nodal equilibrium is satisfied by relating the nodal loads, $\{P\}$, to the element loads, $\{S\}$, via

$$\{P\} = [a]^T \{S\} \quad (9.1)$$

Similarly, external, nodal, compatibility is satisfied by relating the element displacements, $\{s\}$, to the nodal displacements, $\{p\}$, via

$$\{s\} = [a]\{p\} \quad (9.3)$$

Substituting from eqn. (9.3) into eqn. (9.21) for all elements in the structure, gives:

$$\{S\} = [k][a]\{p\} \quad (9.27)$$

in which $[k]$ is the *unassembled stiffness* matrix. Premultiplying the above by $[a]^T$ and substituting from eqn. (9.1) gives:

$$\{P\} = [a]^T[k][a]\{p\}$$

Or, more simply

$$\{P\} = [K]\{p\} \quad (9.28)$$

which is the *structural governing equation* for static stress analysis, relating the nodal forces $\{P\}$ to the nodal displacements $\{p\}$ for all the nodes in the structure, in which the *structural, or assembled stiffness* matrix

$$[K] = [a]^T[k][a] \quad (9.29)$$

9.7.2. Formulation of a rod element using the principle of virtual work equation

Here, the principle of virtual work approach, described in § 9.6, will be used to formulate the equations for an axial force rod element. As described, the approach permits the displacement field to be represented by approximating functions, known as *interpolation* or *shape functions*, a brief description of which follows.

Shape functions

As the name suggests shape functions describe the way in which the displacements are interpolated throughout the element and often take the form of polynomials, which will be complete to some degree. The terms required to form complete linear, quadratic and cubic, etc., polynomials are given by Pascal's triangle and tetrahedron for two- and three dimensional elements, respectively. As well as completeness, there are other considerations

to be made when choosing polynomial terms to ensure the element is well behaved, and the reader is urged to consult detailed texts.⁶ One consideration, which will become apparent, is that the total number of terms in an interpolation polynomial should be equal to the number of dof. of the element.

Consider the axial force rod element shown in Fig. 9.24, for which the local and global axes have been taken to coincide. The purpose is to simplify the appearance of the equations by avoiding the need for the prime in denoting local coordinate dependent quantities. This element has only two nodes and each is taken to have only an axial dof. The total of only two dof. for this element limits the displacement interpolation function to a linear polynomial, namely

$$u(x) = \alpha_1 + \alpha_2 x$$

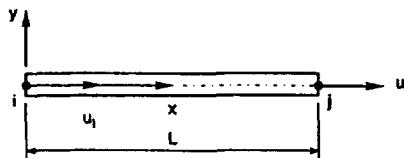


Fig. 9.24. Axial force rod element aligned with global x -axis.

where α_1 and α_2 , to be determined, are known as *generalised coefficients*, and are dependent on the nodal displacements and coordinates.

Writing in matrix form

$$u(x) = [1 \quad x] \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

Or, more concisely,

$$u(x) = [x]\{\alpha\} \quad (9.30)$$

At the nodal points, $u(0) = u_i$ and $u(L) = u_j$.

Substituting into eqn. (9.30) gives

$$\begin{bmatrix} u_i \\ u_j \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

Or, more concisely

$$\{u\} = [A]\{\alpha\}$$

The column matrix of generalised coefficients, $\{\alpha\}$, is obtained by evaluating

$$\{\alpha\} = [A]^{-1}\{u\} \quad (9.31)$$

for which the required inverse of matrix $[A]$, i.e. $[A]^{-1}$ is obtained using standard matrix inversion whereby

$$[A]^{-1} = \text{adj } [A] / \det [A]$$

in which

$$\text{adj } [A] = [C]^T, \text{ where } [C] \text{ is the cofactor matrix of } [A]$$

i.e.

$$\text{adj } [A] = \begin{bmatrix} L & -1 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} L & 0 \\ -1 & 1 \end{bmatrix}$$

and $\det [A] = 1 \times L - 0 \times 1 = L$

Hence,

$$[A]^{-1} = \frac{1}{L} \begin{bmatrix} L & 0 \\ -1 & 1 \end{bmatrix}$$

Substituting eqn. (9.31) into eqn. (9.30) and utilising the above result for $[A]^{-1}$, gives

$$\begin{aligned} u(x) &= [1 \quad x] \frac{1}{L} \begin{bmatrix} L & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_i \\ u_j \end{bmatrix} \\ &= \frac{1}{L} [L - x, x] \begin{bmatrix} u_i \\ u_j \end{bmatrix} = [1 - x/L, x/L] \begin{bmatrix} u_i \\ u_j \end{bmatrix} \\ &= [N]\{u\} \end{aligned} \quad (9.32)$$

in which $[N]$ is the matrix of shape functions. In this case, $N_1 = 1 - x/L$ and $N_2 = x/L$, and hence vary linearly over the element, as shown in Fig. 9.25. Note that the shape functions have the value unity at the node corresponding to the nodal displacement being interpolated and zero at all other nodes (in this case at the only other node), and is a property of all shape functions.

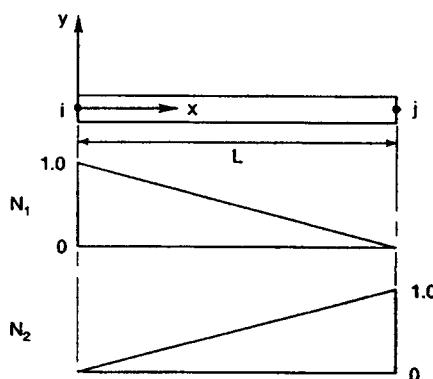


Fig. 9.25. Shape functions for the axial force rod element.

Element stiffness matrix in local coordinates

Consider the axial force element shown in Fig. 9.24. The only strain present will be a direct strain in the axial direction and is given by eqn. (9.4) as

$$\varepsilon_x = \varepsilon = \partial u / \partial x$$

Substituting from eqn. (9.32), gives

$$\varepsilon = \partial [N]\{u\} / \partial x = [B]\{u\} \quad (9.33)$$

and, taking the virtual strain to have a similar form to the real strain, gives

$$\bar{\varepsilon} = [B]\{\bar{u}\} \quad (9.34)$$

where

$$[B] = \partial[N]/\partial x$$

In the present case of the two-node linear rod element, eqn. (9.32) shows $[N] = \frac{1}{L}[L-x, x]$, and hence

$$[B] = \frac{1}{L}[-1 \ 1] \quad (9.35)$$

Note that in this case the derivative matrix $[B]$ contains only constants and does not involve functions of x and hence the strain, given by eqn. (9.33), will be constant along the length of the rod element.

Assuming Hookean behaviour and utilising eqn. (9.33)

$$\sigma = E[B]\{u\} \quad (9.36)$$

It follows for the linear rod element that the stress will be constant and is given as

$$\sigma = \frac{E}{L}[-1 \ 1] \begin{bmatrix} u_i \\ u_j \end{bmatrix} = \frac{E}{L}(u_j - u_i)$$

in which $(u_j - u_i)/L$ is the strain.

Substituting the expression for virtual strain from eqn. (9.34) and the real stress from eqn. (9.36) into eqn. (9.11) gives the internal virtual work as

$$\bar{W}_i = \int_v \bar{\varepsilon} \sigma dv = \int_v \{\bar{u}\}^T [B]^T E[B]\{u\} dv$$

Since the real and virtual displacements are constant, they can be taken outside the integral, to give

$$\bar{W}_i = \{\bar{u}\}^T \int_v [B]^T E[B] dv \{u\} \quad (9.37)$$

The external virtual work will be given by

$$\bar{W}_e = \{\bar{u}\}^T \{P\} \quad (9.38)$$

Substituting from eqns. (9.37) and (9.38) into the equation of the principle of virtual work, eqn. (9.10) gives

$$0 = \{\bar{u}\}^T \{P\} - \{\bar{u}\}^T \int_v [B]^T E[B] dv \{u\}$$

$$= \{\bar{u}\}^T (\{P\} - \int_v [B]^T E[B] dv \{u\})$$

The virtual displacements, $\{\bar{u}\}$, are arbitrary and nonzero, and hence the quantity in parentheses must be zero,

$$\text{i.e. } \{P\} = \int_v [B]^T E[B] dv \{u\} = [k'^{(e)}]\{u\} \quad (9.39)$$

where

$$[k'^{(e)}] = \int_v [B]^T E[B] dv \quad (9.40)$$

Evaluating the element stiffness matrix $[k'^{(e)}]$ for the linear rod element by substituting from eqns. (9.35) gives

$$\begin{aligned} [k'^{(e)}] &= \int_v \frac{1}{L} \begin{bmatrix} -1 \\ 1 \end{bmatrix} E \frac{1}{L} [-1 \ 1] dv \\ &= \frac{E}{L^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_v dv \end{aligned}$$

For a prismatic element $\int_v dv = AL$, and the element stiffness matrix becomes

$$[k'^{(e)}] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (9.41)$$

Expanding the force/displacement eqn. (9.39) to include terms associated with the y -direction, requires the insertion of zeros in the stiffness matrix of eqn. (9.41) and hence becomes identical to eqn. (9.14).

Element stress matrix in local coordinates

For the case of a linear rod element, substituting from eqn. (9.35) into eqn. (9.36) gives the element stress as

$$\sigma^{(e)} = \frac{E}{L} [-1 \ 1] \{u\} \quad (9.42)$$

Again, by inserting zeros in the matrix, to accommodate terms associated with the y -direction, eqn. (9.42) becomes identical to eqn. (9.15).

Transformation of element stiffness and stress matrices to global coordinates

The element stiffness and stress matrices obtained above can be transformed from local to global coordinates using the procedures of § 9.7.1 to give the results previously obtained, namely the stiffness matrix of eqn. (9.23) and stress matrix of eqn. (9.26).

Formation of structural governing equation and assembled stiffness matrix

Section 9.7.1 has covered the combination of individual element stiffness contributions, necessary to analyse an assemblage of elements representing a complete framework. Equilibrium and compatibility arguments were used to form the structural governing eqn. (9.28) and hence the assembled stiffness matrix, eqn. (9.29). Now, the alternative principle of virtual work will be used to derive the same equations.

Eqn. (9.37) gives the element internal virtual work in local coordinates as

$$W_i^{(e)} = \{u'^{(e)}\}^T \int_v [B^{(e)}]^T E^{(e)} [B^{(e)}] dv \{u'^{(e)}\}$$

Summing all such contributions for the entire structure of m elements, gives

$$\bar{W}_i = \sum_{e=1}^m (\{\bar{u}'^{(e)}\}^T \int_v [B^{(e)}]^T E^{(e)} [B^{(e)}] dv \{u'^{(e)}\}) \quad (9.43)$$

Summing the contributions over all n nodes, the external virtual work will be given by

$$\bar{W}_e = \sum_{i=1}^n \bar{p}_i P_i = \{\bar{p}\}^T \{P\} \quad (9.44)$$

where $\{\bar{p}\}$ is the column matrix of all nodal virtual displacements for the structure and $\{P\}$ is the column matrix of all nodal forces. Substituting from eqns. (9.43) and (9.44) into the equation of the principle of virtual work, eqn. (9.10) gives

$$0 = \{\bar{p}\}^T \{P\} - \sum_{e=1}^m (\{\bar{u}'^{(e)}\}^T \int_v [B^{(e)}]^T E^{(e)} [B^{(e)}] dv \{u'^{(e)}\}) \quad (9.45)$$

Relating the virtual displacements in local and global coordinates via the transformation matrix $[T^{(e)}]$, gives

$$\{\bar{u}'^{(e)}\} = [T^{(e)}] \{\bar{p}^{(e)}\} \quad \text{and} \quad \{\bar{u}'^{(e)}\}^T = \{\bar{p}^{(e)}\}^T [T^{(e)}]^T$$

Summing the contributions and recalling $\{\bar{p}\}$ denotes the nodal displacements for the entire structure, gives

$$\sum_{e=1}^m \{\bar{u}'^{(e)}\}^T = \{\bar{p}\}^T \sum_{e=1}^m [T^{(e)}]^T \quad \text{and} \quad \sum_{e=1}^m \{\bar{u}'^{(e)}\} = \sum_{e=1}^m ([T^{(e)}]) \{p\}$$

Hence, eqn. (9.45) can be re-written as

$$\begin{aligned} \{\bar{p}\}^T \{P\} &= \{\bar{p}\}^T \left(\sum_{e=1}^m [T^{(e)}]^T \int_v [B^{(e)}]^T E^{(e)} [B^{(e)}] dv [T^{(e)}] \right) \{p\} \\ &= \{\bar{p}\}^T \sum_{e=1}^m [k^{(e)}] \{p\} = \{\bar{p}\}^T [K] \{p\} \end{aligned} \quad (9.46)$$

where

$$[k^{(e)}] = [T^{(e)}]^T \int_v [B^{(e)}]^T E^{(e)} [B^{(e)}] dv [T^{(e)}]$$

and the assembled stiffness matrix

$$[K] = \sum_{e=1}^m [k^{(e)}] \quad (9.47)$$

It follows from eqn. (9.46) since $\{\bar{p}\}$ is arbitrary and non-zero, that

$$\{P\} = [K] \{p\}$$

which is the structural governing equation and the same as eqn. (9.28), and implies nodal force equilibrium.

9.8. A simple beam element

As with the previous treatment of the rod element, the two approaches using fundamental equations and the principle of virtual work will be employed to formulate the necessary equations for a simple beam element.

9.8.1. Formulation of a simple beam element using fundamental equations

Consider the case, similar to §9.7.1, in which the deformations, member stresses and reactions are required for planar frames, excepting that the member joints are now taken to be rigid and hence capable of transmitting moments. The behaviour of such frames can be represented as an assemblage of beam finite elements. A typical simple beam element is shown in Fig. 9.26, the physical and material properties of which are taken to be constant throughout the element. As with the previous rod element, changes in properties and load application are only admissible at nodal positions. In addition to u and v translatory freedoms, each node has a rotational freedom, θ , about the z axis, giving three dof. per node. Hence, axial, shear and flexural deformations will be represented, whilst torsional deformations which are inappropriate for planar frames will be ignored.

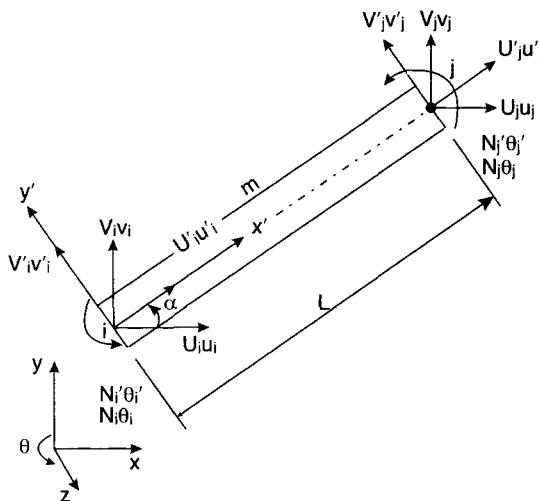


Fig. 9.26. A simple beam element.

Element stiffness matrix in local coordinates

The differential equation of flexure appropriate to a beam element can be written as

$$d^2v'/dx'^2 = N'/EI \quad (9.48)$$

in which v' denotes the displacement in the local y' direction, N' is the element moment, E is the modulus of elasticity and I is the relevant second moment of area of the beam. The first derivative of the moment with respect to distance x' along a beam is known to give the

shear force, V' ,

$$\text{i.e.} \quad dN'/dx' = V' \quad (9.49)$$

Similarly, the first derivative of the shear force will give the loading intensity, ω' ,

$$\text{i.e.} \quad dV'/dx' = \omega' \quad (9.50)$$

Differentiating eqn. (9.48) and utilising eqn. (9.49) gives

$$d^3v'/dx'^3 = V'/EI \quad (9.51)$$

Differentiating again and utilising eqn. (9.50) gives

$$d^4v'/dx'^4 = \omega'/EI \quad (9.52)$$

Integrating eqn. (9.52), recalling that loads can only be applied at the nodes, and hence $\omega' = 0$, gives

$$d^3v'/dx'^3 = C_1 = V'/EI, \quad (\text{from eqn. 9.51}) \quad (9.53)$$

Further integration gives

$$d^2v'/dx'^2 = C_1x' + C_2 = N'/EI, \quad (\text{from eqn. 9.48}) \quad (9.54)$$

$$dv'/dx' = C_1x'^2/2 + C_2x' + C_3 = \theta' \quad (9.55)$$

$$\text{and } v' = C_1x'^3/6 + C_2x'^2/2 + C_3x' + C_4 \quad (9.56)$$

With reference to Fig. 9.26, it can be seen that

$$v'(0) = v'_i, \quad v'(L) = v'_j, \quad dv'/dx'(0) = \theta'_i, \quad dv'/dx'(L) = \theta'_j$$

$$\text{It follows from eqn. (9.56) that } v'_i = C_4 \quad (9.57)$$

$$\text{from eqn. (9.55) } \theta'_i = C_3 \quad (9.58)$$

from eqn. (9.56)

$$v'_j = C_1L^3/6 + C_2L^2/2 + C_3L + C_4 = C_1L^3/6 + C_2L^2/2 + \theta'_iL + v'_i \quad (9.59)$$

and from eqn. (9.55)

$$\theta'_j = C_1L^2/2 + C_2L + C_3 = C_1L^2/2 + C_2L + \theta'_i \quad (9.60)$$

An expression for C_2 can now be obtained by multiplying eqn. (9.60) throughout by $L/3$ and subtracting the result from eqn. (9.59), (to eliminate C_1), to give

$$v'_j - \theta'_iL/3 = C_2(L^2/2 - L^2/3) + \theta'_i(L - L/3) + v'_i = C_2L^2/6 + \theta'_i2L/3 + v'_i$$

$$\begin{aligned} \text{Rearranging, } C_2 &= 6(v'_j - v'_i)/L^2 - 6(\theta'_iL/3 + \theta'_i2L/3)/L \\ &= 6(-v'_i + v'_j)/L^2 - 2(2\theta'_i + \theta'_j)/L \end{aligned} \quad (9.61)$$

Rearranging eqn. (9.60) and substituting from eqn. (9.61) gives

$$\begin{aligned} C_1 &= (2/L^2)[(\theta'_j - \theta'_i) - 6(-v'_i + v'_j)/L + 2(2\theta'_i + \theta'_j)] \\ &= 12(v'_i - v'_j)/L^3 + 6(\theta'_i + \theta'_j)/L^2 \end{aligned} \quad (9.62)$$

Substituting for constant C_1 from eqn. (9.62) into eqn. (9.53) gives shear force

$$V' = EIC_1 = 12EI(v'_i - v'_j)/L^3 + 6EI(\theta'_i + \theta'_j)/L^2 \quad (9.63)$$

Substituting for constants C_1 and C_2 from eqns. (9.62) and (9.63) into eqn. (9.54) gives the moment

$$\begin{aligned} N' &= EI(C_1x' + C_2) \\ &= 6EI(2x' - L)(v'_i - v'_j)/L^3 + 6EIx'(\theta'_i + \theta'_j)/L^2 - 2EI(2\theta'_i + \theta'_j)/L \end{aligned} \quad (9.64)$$

Note that the shear force, eqn. (9.63) is independent of distance x' along the beam, i.e. constant, whilst the moment, eqn. (9.64) is linearly dependent on distance x' , consistent with a beam subjected to concentrated forces.

It follows that the nodal shear force and moments are given as

$$V'(0) = V'(L) = 12EI(v'_i - v'_j)/L^3 + 6EI(\theta'_i + \theta'_j)/L^2 \quad (9.65)$$

$$N'(0) = 6EI(-v'_i + v'_j)/L^2 - 2EI(2\theta'_i + \theta'_j)/L \quad (9.66)$$

$$N'(L) = 6EI(v'_i - v'_j)/L^2 + 2EI(\theta'_i + 2\theta'_j)/L \quad (9.67)$$

The shear force and moments given by eqns. (9.65)–(9.67) use a Mechanics of Materials sign convention, namely, a positive shear force produces a clockwise couple and a positive moment produces sagging. To conform with the sign convention shown in Fig. 9.26, the following changes are required:

$$V'_i = -V'_j = V'(0)$$

$$N'_i = -N'(0)$$

$$N'_j = N'(L)$$

Writing in matrix form, eqns. (9.65)–(9.67) become

$$\begin{bmatrix} V'_i \\ N'_i \\ V'_j \\ N'_j \end{bmatrix} = \frac{EI}{L} \begin{bmatrix} 12/L^2 & 6/L & -12/L^2 & 6/L \\ 6/L & 4 & -6/L & 2 \\ -12/L^2 & -6/L & 12/L^2 & -6/L \\ 6/L & 2 & -6/L & 4 \end{bmatrix} \begin{bmatrix} v'_i \\ \theta'_i \\ v'_j \\ \theta'_j \end{bmatrix} \quad (9.68)$$

Combining eqn. (9.68) with eqn. (9.12) gives the matrix equation relating element axial and shear forces and moments to the element displacements as

$$\begin{bmatrix} U'_i \\ V'_i \\ N'_i \\ U'_j \\ V'_j \\ N'_j \end{bmatrix} = \begin{bmatrix} AE/L & 0 & 0 & -AE/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ -AE/L & 0 & 0 & AE/L & 0 & 0 \\ 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L \end{bmatrix} \begin{bmatrix} u'_i \\ v'_i \\ \theta'_i \\ u'_j \\ v'_j \\ \theta'_j \end{bmatrix} \quad (9.69)$$

from which the element stiffness matrix with respect to local coordinates is:

$$[k^{(e)}] = \begin{bmatrix} AE/L & 0 & 0 & -AE/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ -AE/L & 0 & 0 & AE/L & 0 & 0 \\ 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L \end{bmatrix} \quad (9.70)$$

Element stress matrix in local coordinates

Only bending and axial stresses will be considered, shear stresses being taken as insignificant. The points for stress calculation will be the extreme top and bottom fibres at each end of the element, which will always include the maximum stress point. With reference to Fig. 9.27, the beam element stress matrix will be

$$\{\sigma^{(e)}\} = \{\sigma_i^{\text{top}} \sigma_i^{\text{btm}} \sigma_j^{\text{top}} \sigma_j^{\text{btm}}\}$$

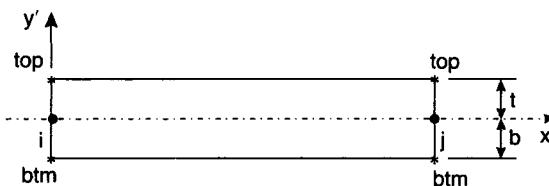


Fig. 9.27. Locations for beam element stress calculation.

Relating these stresses to the internal loads gives

$$\begin{bmatrix} \sigma_i^{\text{top}} \\ \sigma_i^{\text{btm}} \\ \sigma_j^{\text{top}} \\ \sigma_j^{\text{btm}} \end{bmatrix} = \begin{bmatrix} -1/A & 0 & t/I & 0 & 0 & 0 \\ -1/A & 0 & -b/I & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/A & 0 & -t/I \\ 0 & 0 & 0 & 1/A & 0 & b/I \end{bmatrix} \begin{bmatrix} U'_i \\ V'_i \\ N'_i \\ U'_j \\ V'_j \\ N'_j \end{bmatrix} \quad (9.71)$$

Or, more concisely,

$$\{\sigma^{(e)}\} = [h^{(e)}][S'^{(e)}] \quad (9.72)$$

Substituting for the element loads column matrix using a relation of the form of eqn. (9.13) gives

$$\{\sigma^{(e)}\} = [h^{(e)}][k'^{(e)}]\{s'^{(e)}\} = [H'^{(e)}]\{s'^{(e)}\}$$

which is the same form as eqn. (9.16) and $[H'^{(e)}]$ is the stress matrix with respect to local coordinates. Evaluating $[h^{(e)}][k'^{(e)}]$ gives the stress matrix as

$$[H'^{(e)}] = \frac{E}{L} \begin{bmatrix} -1 & 6t/L & 4t & 1 & 6t/L & 2t \\ -1 & -6b/L & -4b & 1 & -6b/L & -2b \\ -1 & -6t/L & -2t & 1 & -6t/L & -4t \\ -1 & 6b/L & 2b & 1 & 6b/L & 4b \end{bmatrix} \quad (9.73)$$

Transformation of displacements and loads

Relations of similar form to those of eqns. (9.18)–(9.23) but with additional rotational dof. terms, previously not included in the rod element transformation, will enable the above element stiffness and stress matrices to be transformed from local to global coordinates. The expanded form of eqn. (9.18) for the beam element will be given as

$$\begin{bmatrix} u_i \\ v'_i \\ \theta'_i \\ u'_j \\ v'_j \\ \theta'_j \end{bmatrix} = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 & 0 & 0 & 0 \\ -\sin \alpha & \cos \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos \alpha & \sin \alpha & 0 \\ 0 & 0 & 0 & -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_i \\ v_i \\ \theta_i \\ u_j \\ v_j \\ \theta_j \end{bmatrix}$$

in which the transformation matrix is:

$$[T^{(e)}] = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 & 0 & 0 & 0 \\ -\sin \alpha & \cos \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos \alpha & \sin \alpha & 0 \\ 0 & 0 & 0 & -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (9.74)$$

Element stiffness matrix in global coordinates

$$[k^{(e)}] = \frac{E}{L} \begin{bmatrix} A \cos^2 \alpha + (12I \sin^2 \alpha / L^2), & A \sin^2 \alpha + (12I \cos^2 \alpha / L^2), & 4I, \\ (A - 12I/L^2) \cos \alpha \sin \alpha, & (6I \cos \alpha)/L, & \\ -(6I \sin \alpha)/L, & (6I \cos \alpha)/L, & \\ -A \cos^2 \alpha - (12I \sin^2 \alpha / L^2), & (-A + 12I/L^2) \cos \alpha \sin \alpha, & (6I \sin \alpha)/L, \\ (-A + 12I/L^2) \cos \alpha \sin \alpha, & -A \sin^2 \alpha - (12I \cos^2 \alpha / L^2), & -(6I \cos \alpha)/L, \\ -(6I \sin \alpha)/L, & (6I \cos \alpha)/L, & 2I, \\ & & \\ & & \text{symmetric} \\ A \cos^2 \alpha + (12I \sin^2 \alpha / L^2), & A \sin^2 \alpha + (12I \cos^2 \alpha / L^2), & 4I, \\ (A - 12I/L^2) \cos \alpha \sin \alpha, & -(6I \cos \alpha)/L, & \\ (6I \sin \alpha)/L, & & \end{bmatrix} \quad (9.75)$$

Element stress matrix in global coordinates

Substituting from eqns. (9.73) and (9.74) into eqn. (9.25) gives the element stress matrix in global coordinates as:

$$[H^{(e)}] = \frac{E}{L} \begin{bmatrix} -\cos \alpha - 6t \sin(\alpha)/L & -\sin \alpha + 6t \cos(\alpha)/L & 4t \\ -\cos \alpha + 6b \sin(\alpha)/L & -\sin \alpha - 6b \cos(\alpha)/L & -4b \\ -\cos \alpha + 6t \sin(\alpha)/L & -\sin \alpha - 6t \cos(\alpha)/L & -2t \\ -\cos \alpha - 6b \sin(\alpha)/L & -\sin \alpha + 6b \cos(\alpha)/L & 2b \\ & & \\ \cos \alpha + 6t \sin(\alpha)/L & \sin \alpha - 6t \cos(\alpha)/L & 2t \\ \cos \alpha - 6b \sin(\alpha)/L & \sin \alpha + 6b \cos(\alpha)/L & -2b \\ \cos \alpha - 6t \sin(\alpha)/L & \sin \alpha + 6t \cos(\alpha)/L & -4t \\ \cos \alpha + 6b \sin(\alpha)/L & \sin \alpha - 6b \cos(\alpha)/L & 4b \end{bmatrix} \quad (9.76)$$

Formation of structural governing equation and assembled stiffness matrix

Whilst the beam element matrices include rotational dof. terms, not present in the rod element matrices, the procedures of Section 9.7.1 still apply, and lead to the structural governing equation

$$\{P\} = [K]\{p\} \quad (9.28)$$

and the assembled stiffness matrix

$$[K] = [a]^T[k][a] \quad (9.29)$$

9.8.2. Formulation of a simple beam element using the principle of virtual work equation

As Section 9.7.2 the principle of virtual work equation will be invoked, this time to formulate the equations for a simple beam element.

Consider the simple beam element shown in Fig. 9.28, for which the local and global axes have again been taken to coincide to avoid need for the prime and hence to simplify the appearance of the equations. The two nodes are each taken to have only normal and rotational dof. The terms associated with the omitted axial dof. have already been derived for the linear rod element in §9.7 and will be incorporated once the other terms have been derived. The total of four dof. for this beam element permits the displacement to be interpolated by a fourth order polynomial, namely

$$v(x) = \alpha_1 + \alpha_2 x/L + \alpha_3 x^2/L^2 + \alpha_4 x^3/L^3$$

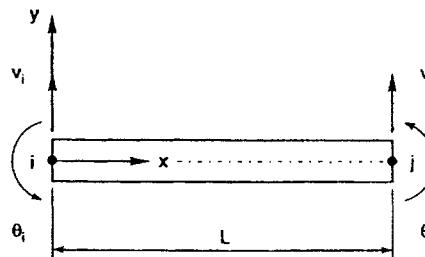


Fig. 9.28. Simple beam element, aligned with global x -axis.

where α_1 to α_4 are generalised coefficients to be determined. Utilisation of eqns. (9.48) and (9.49) shows this polynomial will provide for a linearly varying moment and constant shear force and hence will enable an exact solution for beams subjected to concentrated loads.

Writing in matrix form,

$$v(x) = [1, x/L, x^2/L^2, x^3/L^3] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}$$

or, more concisely

$$v(x) = [x]\{\alpha\} \quad (9.77)$$

At the nodal points, $v(0) = v_i$; $v(L) = v_j$;
and $dv/dx(0) = \theta_i$; $dv/dx(L) = \theta_j$

Substituting into eqn. (9.77) gives

$$\begin{bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/L & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1/L & 2/L & 3/L \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}$$

Or, more concisely,

$$\{v\} = [A]\{\alpha\}$$

Expressing in terms of the column matrix of generalised coefficients, $\{\alpha\}$, gives

$$\{\alpha\} = [A]^{-1}\{v\} \quad (9.78)$$

Evaluation of eqn. (9.78) requires the inverse of matrix $[A]$ obtained from

$$\begin{aligned} \text{adj } [A] = [C]^T &= \begin{bmatrix} 1/L^2 & 0 & -3/L^2 & 2/L^2 \\ 0 & 1/L & -2/L & 1/L \\ 0 & 0 & 3/L^2 & -2/L^2 \\ 0 & 0 & -1/L & 1/L \end{bmatrix}^T \\ &= \begin{bmatrix} 1/L^2 & 0 & 0 & 0 \\ 0 & 1/L & 0 & 0 \\ -3/L^2 & -2/L & 3/L^2 & -1/L \\ 2/L^2 & 1/L & -2/L^2 & 1/L \end{bmatrix} \end{aligned}$$

$$\text{and } \det [A] = 1(1/L)(1 \times 3/L - 1 \times 2/L) = 1/L^2$$

$$\begin{aligned} \text{Hence, } [A]^{-1} &= L^2 \begin{bmatrix} 1/L^2 & 0 & 0 & 0 \\ 0 & 1/L & 0 & 0 \\ -3/L^2 & 2/L & 3/L^2 & -1/L \\ 2/L^2 & 1/L & -2/L^2 & 1/L \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & L & 0 & 0 \\ -3 & -2L & 3 & -L \\ 2 & L & -2 & L \end{bmatrix} \end{aligned}$$

Substituting eqn. (9.78) into eqn. (9.77) and utilising the above result for $[A]^{-1}$, gives

$$\begin{aligned} v(x) &= [1, x/L, x^2/L^2, x^3/L^3] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & L & 0 & 0 \\ -3 & -2L & 3 & -L \\ 2 & L & -2 & L \end{bmatrix} \begin{bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{bmatrix} \\ &= [1 - 3x^2/L^2 + 2x^3/L^3, x - 2x^2/L + x^3/L^2, 3x^2/L^2 - 2x^3/L^3, -x^2/L + x^3/L^2] \begin{bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{bmatrix} \end{aligned} \quad (9.79)$$

which has the same form as eqn. (9.32), in this case with shape functions

$$N_1 = 1 - 3x^2/L^2 + 2x^3/L^3$$

$$N_2 = x - 2x^2/L + x^3/L^2$$

$$N_3 = 3x^2/L^2 - 2x^3/L^3$$

$$N_4 = -x^2/L + x^3/L^2$$

the variation of which is shown in Fig. 9.29.

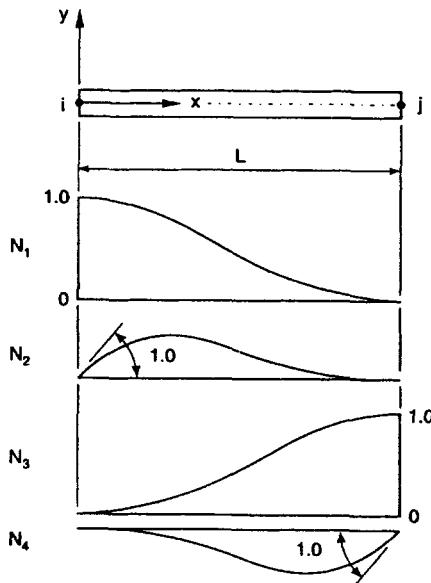


Fig. 9.29. Shape functions for a simple beam element.

Element stiffness matrix in local coordinates

Longitudinal bending stress is given by simple bending theory as

$$\sigma = My/I$$

in which, from the differential equation of flexure, eqn. (9.48),

$$M = EI d^2v/dx^2$$

to give

$$\sigma = E y d^2v/dx^2 \quad (9.80)$$

Substituting eqn. (9.79) into eqn. (9.80) gives

$$\begin{aligned} \sigma &= E y [(12x/L^3 - 6/L^2)(6x/L^2 - 4/L)(-12x/L^3 + 6/L^2)(6x/L^2 - 2/L)] \begin{bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{bmatrix} \\ &= E[B]\{u\} \end{aligned} \quad (9.81)$$

which has the same form as eqn. (9.36), except in this case

$$[B] = yd^2[N]/dx^2 \quad (9.82)$$

Assuming Hookean behaviour, the longitudinal bending strain is obtained using eqn. (9.81), as

$$\varepsilon = \sigma/E = [B]\{u\}$$

which has the same form as eqn. (9.33).

Taking the virtual longitudinal strain to be given in a form similar to the real strain, i.e.

$$\varepsilon = [B]\{u\} \quad (9.83)$$

Substituting the real stress from eqn. (9.81) and the virtual strain from eqn. (9.83) into the equation of the principle of virtual work, (9.10), gives

$$0 = \{u\}^T \{P\} - \int_v \{u\}^T [B]^T E[B] \{u\} dv$$

The real and virtual displacements, being constant, can be taken outside the integral, to give

$$= \{u\}^T (\{P\} - \int_v [B]^T E[B] dv \{u\})$$

The virtual displacements, $\{u\}$, are arbitrary and nonzero, hence

$$\{P\} = \int_v [B]^T E[B] dv \{u\} = [k'(e)]\{u\}$$

where $[k'(e)] = \int_v [B]^T E[B] dv$, which is an identical result to eqn. (9.40).

Substituting from eqn. (9.82) gives

$$\begin{aligned} [k'(e)] &= \int_v y(d^2[N]/dx^2)^T E y(d^2[N]/dx^2) dv \\ &= EI \int_0^L (d^2[N]/dx^2)^T (d^2[N]/dx^2) dx \\ &= EI \int_0^L \left[\begin{array}{c} \frac{6}{L^2} \left(2\frac{x}{L} - 1 \right) \\ \frac{2}{L} \left(3\frac{x}{L} - 2 \right) \\ \frac{6}{L^2} \left(-2\frac{x}{L} + 1 \right) \\ \frac{2}{L} \left(3\frac{x}{L} - 1 \right) \end{array} \right] \\ &\quad \times \left[\frac{6}{L^2} \left(2\frac{x}{L} - 1 \right) \frac{2}{L} \left(3\frac{x}{L} - 2 \right) \frac{6}{L^2} \left(-2\frac{x}{L} + 1 \right) \frac{2}{L} \left(3\frac{x}{L} - 1 \right) \right] dx \quad (9.84) \end{aligned}$$

The following gives examples of evaluating the integrals of eqn. (9.84) for two elements of the stiffness matrix, the rest are obtained by the same procedure.

$$\begin{aligned} k_{11} &= EI \int_0^L \left[\frac{6}{L^2} \left(2\frac{x}{L} - 1 \right) \frac{6}{L^2} \left(2\frac{x}{L} - 1 \right) \right] dx = \frac{36EI}{L^4} \int_0^L \left(4\frac{x^2}{L^2} - 4\frac{x}{L} + 1 \right) dx \\ &= \frac{36EI}{L^4} \left[\frac{4x^3}{3L^2} - 2\frac{x^2}{L} + x \right] = \frac{36EI}{L^3} \left[\frac{4}{3} - 2 + 1 \right] = \frac{12EI}{L^3} \end{aligned}$$

$$\text{and } k_{12} = k_{21} = EI \int_0^L \left[\frac{6}{L^2} \left(2\frac{x}{L} - 1 \right) \frac{2}{L} \left(3\frac{x}{L} - 2 \right) \right] dx = \frac{12EI}{L^3} \int_0^L \left(6\frac{x^2}{L^2} - 7\frac{x}{L} + 2 \right) dx \\ = \frac{12EI}{L^3} \left[\frac{2x^3}{L^2} - \frac{7x^2}{2L} + 2x \right] = \frac{12EI}{L^2} \left[2 - \frac{7}{2} + 2 \right] = \frac{6EI}{L^2}$$

Evaluation of all the integrals of eqn. (9.84) leads to the beam element flexural stiffness matrix

$$[k^{(e)}] = \frac{EI}{L} \begin{bmatrix} 12/L^2 & 6/L & -12/L^2 & 6/L \\ 6/L & 4 & -6/L & 2 \\ -12/L^2 & -6/L & 12/L^2 & -6/L \\ 6/L & 2 & -6/L & 4 \end{bmatrix}$$

which is identical to the stiffness matrix of eqn. (9.68) derived using fundamental equations. The same arguments made in Section 9.8.1 apply with regard to including axial terms to give the force/displacement relation, eqn. (9.69), and corresponding element stiffness matrix, eqn. (9.70).

Element stress matrix in local coordinates

Bending and axial stresses are obtained using the same relations as those in §9.8.1.

Transformation of element stiffness and stress matrices to global coordinates

The element stiffness and stress matrices are transformed from local to global coordinates using the procedures of §2.4.8.1 to give the stiffness matrix of eqn. (9.75) and stress matrix of eqn. (9.76).

Formation of structural governing equation and assembled stiffness matrix

The theorem of virtual work used in §9.7.2 to formulate rod element assemblages applies to the present beam elements. It follows, therefore, that the assembled stiffness matrix will be given by eqn. (9.47). The displacement column matrices will, for beams, include rotational dof., not present for rod elements. Further, at the nodes, moment equilibrium, as well as force equilibrium, is now implied by eqn. (9.28).

9.9. A simple triangular plane membrane element

The common occurrence of thin-walled structures merits devoting attention here to their analysis. Many applications are designed on the basis of in-plane loads only with resistance arising from membrane action rather than bending. Whilst thin plates can be curved to resist normal loads by membrane action, for simplicity only planar applications will be considered here. Membrane elements can have three or four edges, which can be straight or curvilinear, however, attention will be restricted here to the simplest, triangular, membrane element.

Unlike the previous rod and beam element formulations, with which displacement fields can be represented exactly and derived from fundamental arguments, the displacement fields represented by two-dimensional elements can only be approximate, and need to be derived using an energy principle. Here, the principle of virtual work will be invoked to derive the membrane element equations.

9.9.1. Formulation of a simple triangular plane membrane element using the principle of virtual work equation

With reference to Fig. 9.30, each node of the triangular membrane element has two dof., namely u and v displacements in the global x and y directions, respectively. The total of six dof. for the element limits the u and v displacement to linear interpolation. Hence

$$u(x, y) = \alpha_1 + \alpha_2x + \alpha_3y$$

and

$$v(x, y) = \alpha_4 + \alpha_5x + \alpha_6y$$

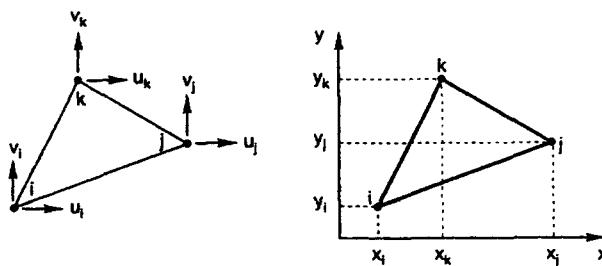


Fig. 9.30. Triangular plane membrane element.

or, in matrix form

$$\begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{bmatrix} \quad (9.85)$$

At the nodal points,

$$u(x_i, y_i) = u_i, \quad v(x_i, y_i) = v_i,$$

$$u(x_j, y_j) = u_j, \quad v(x_j, y_j) = v_j,$$

and

$$u(x_k, y_k) = u_k, \quad v(x_k, y_k) = v_k$$

Substituting into eqn. (9.85) gives

$$\begin{bmatrix} u_i \\ u_j \\ u_k \\ v_i \\ v_j \\ v_k \end{bmatrix} = \begin{bmatrix} 1 & x_i & y_i & 0 & 0 & 0 \\ 1 & x_j & y_j & 0 & 0 & 0 \\ 1 & x_k & y_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_i & y_i \\ 0 & 0 & 0 & 1 & x_j & y_j \\ 0 & 0 & 0 & 1 & x_k & y_k \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{bmatrix} \quad (9.86)$$

Or, more concisely, $\{p\} = [A]\{\alpha\}$

Similar to the previous sections, the column matrix of generalised coefficients, $\{\alpha\}$, is obtained by evaluating

$$\{\alpha\} = [A]^{-1}\{p\} \quad (9.87)$$

where, by arranging the dof. in the above sequence enables suitable partitioning of $[A]$ and minimises the effort required to obtain the inverse. Unlike the previous treatment of the rod and beam element, the evaluation of $[A]^{-1}$ is delayed until Example 9.5. The result, however, is given by eqn. (9.88). It is hoped this departure will enable the element formulation to be more easily assimilated.

$$[A]^{-1} = \frac{1}{2a} \begin{bmatrix} x_2y_3 - x_2y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 & 0 & 0 & 0 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 & 0 & 0 & 0 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ 0 & 0 & 0 & y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ 0 & 0 & 0 & x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \quad (9.88)$$

where a is equal to the area of the element.

Substituting from eqn. (9.87) into eqn. (9.85), utilising the result for $[A]^{-1}$, i.e. eqn. (9.88), and writing concisely the result of the matrix multiplication, gives

$$\begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} = \begin{bmatrix} N_i & N_j & N_k & 0 & 0 & 0 \\ 0 & 0 & 0 & N_i & N_j & N_k \end{bmatrix} \begin{bmatrix} u_i \\ u_j \\ u_k \\ v_k \\ v_j \\ v_k \end{bmatrix}$$

where the shape functions are given as

$$\begin{aligned} N_i &= \frac{1}{2a}[x_jy_k - x_ky_j + (y_j - y_k)x + (x_k - x_j)y] \\ N_j &= \frac{1}{2a}[x_ky_i - x_iy_k + (y_k - y_i)x + (x_i - x_k)y] \\ N_k &= \frac{1}{2a}[x_iy_j - x_jy_i + (y_i - y_j)x + (x_j - x_i)y] \end{aligned} \quad (9.89)$$

Note that the shape functions of eqns. (9.89) are linear in x and y . Further, evaluation of eqns. (9.89) shows that shape function $N_i(x_i, y_i) = 1$ and $N_i(x, y) = 0$ at nodes j and k , and at all points on the line joining these nodes. Similarly, $N_j(x_j, y_j) = 1$ and $N_k(x_k, y_k) = 1$, and equal zero at, and on the line between, the other nodes.

Formulation of element stiffness matrix

For plane stress analysis, the strain/displacement relations are

$$\varepsilon_{xx} = \partial u / \partial x, \quad \varepsilon_{yy} = \partial v / \partial y, \quad \varepsilon_{xy} = \partial u / \partial y + \partial v / \partial x$$

where ε_{xx} and ε_{yy} are the direct strains parallel to the x and y axes, respectively, and ε_{xy} is the shear strain in the xy plane. Writing in matrix form gives

$$\{\varepsilon\} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Substituting from eqn. (9.85) and performing the partial differentiation, the above becomes

$$\{\varepsilon\} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{bmatrix}$$

Substituting from eqn. (9.87) gives

$$\{\varepsilon\} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} [A]^{-1} \begin{bmatrix} u_i \\ u_j \\ u_k \\ v_i \\ v_j \\ v_k \end{bmatrix}$$

Or, more concisely,

$$\{\varepsilon\} = [B]\{u\} \quad (9.90)$$

where

$$[B] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} [A]^{-1} \quad (9.91)$$

Note that matrix $[B]$ is independent of position within the element with the consequence that the strain, and hence the stress, will be constant throughout the element.

For plane stress analysis ($\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$) with isotropic material behaviour, the stress/strain relations in matrix form are

$$\{\sigma\} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1-v^2)} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \{\varepsilon\}$$

Or, more concisely,

$$\{\sigma\} = [D]\{\varepsilon\} \quad (9.92)$$

where σ_{xx} and σ_{yy} are the direct stresses parallel to x and y axes, respectively, σ_{xy} is the shear stress in the xy plane, and $[D]$ is known as the *elasticity matrix*.

Following the same arguments used in the rod and beam formulations, namely, taking the expression for virtual strain to have a similar form to the real strain, eqn. (9.90), and substituting this and the expression for real stress, eqn. (9.92), into the equation of the principle of virtual work, (9.10), gives the element stiffness matrix as

$$[k^{(e)}] = \int_v [B]^T [D] [B] dv \quad (9.93)$$

The only departure of eqn. (9.93) from the previous expressions is the replacement of the modulus of elasticity, E , by the elasticity matrix $[D]$, due to the change from a one- to a two-dimensional stress system.

Recalling, for the present case that the displacement fields are linearly varying, then matrix $[B]$ is independent of the x and y coordinates. The assumption of isotropic homogeneous material means that matrix $[D]$ is also independent of coordinates. It follows, assuming a constant thickness, t , throughout the element, of area, a , eqn. (9.93) can be integrated to give

$$[k^{(e)}] = at [B]^T [D] [B] \quad (9.94)$$

Element stress matrix

The expression for the element direct and shear stresses is obtained by substituting from eqn. (9.90) into eqn. (9.92), to give

$$\{\sigma^{(e)}\} = [D][B]\{u\}$$

or, more fully,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = [D][B] \begin{bmatrix} u_i \\ u_j \\ u_k \\ v_i \\ v_j \\ v_k \end{bmatrix} \quad (9.95)$$

These stresses are with respect to the global coordinate axes and are taken to act at the element centroid.

Formation of structural governing equation and assembled stiffness matrix

As Sections 9.7.2 and 9.8.2, the structural governing equation is given by eqn. (9.28) and the assembled stiffness matrix by eqn. (9.47).

9.10. Formation of assembled stiffness matrix by use of a dof. correspondence table

Element stiffness matrices given, for example, by eqn. (9.23), are formed for each element in the structure being analysed, and are combined to form the assembled stiffness matrix $[K]$. Where nodes are common to more than one element, the assembly process requires that appropriate stiffness contributions from all such elements are summed for each node. Execution of finite element programs will enable assembly of the element stiffness contributions by utilising, for example, eqn. (9.29) deriving matrix $[a]$, and hence $[a]^T$, from the connectivity information provided by the element mesh. Alternatively, eqn. (9.47) can be used, the matrix summation requiring that all element stiffness matrices, $[k^{(e)}]$, are of the same order as the assembled stiffness matrix $[K]$. However, by efficient “housekeeping” only those rows and columns containing the non-zero terms need be stored.

For the purpose of performing hand calculations, the tedium of evaluating the triple matrix product of eqn. (9.29) can be avoided by summing the element stiffness contributions according to eqn. (9.47). The procedure to be adopted follows, and uses a so-called *dof. correspondence table*. Consider assembly of the element stiffness contributions for the

simple pin-jointed plane frame idealised as three rod elements, shown in Fig. 9.19. The element stiffness matrices in global coordinates can be illustrated as:

$$[k^{(a)}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$[k^{(b)}] = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

$$[k^{(c)}] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix}$$

The procedure is as follows:

- Label a diagram of the frame with dof. numbers in node number sequence.
- Construct a dof. correspondence table, entering a set of dof. numbers for each node of every element. For the rod element there will be two dof. in each set, namely, u and v displacements, and two sets per element, one for each node. The sequence of the sets must correspond to progression along the local axis direction, i.e. along each positive x' direction. This is essential to maintain consistency with the element matrices, above, the terms of which have been shown in eqn. (9.23) to involve angle α , the value of which will correspond to the inclination of the element at the end chosen as the origin of its local axis. The sequence shown in Table 9.2 corresponds to $\alpha_a = 330^\circ$, $\alpha_b = 180^\circ$ and $\alpha_c = 210^\circ$. The u and v dof. sequence within each set must be maintained.
- Choose an element for which the stiffness contributions are to be assembled.
- Assemble by either rows or columns according to the dof. correspondence table.
- Repeat for the remaining elements until all are assembled.

Table 9.2. Dof. correspondence table for assembly of structural stiffness matrix, [K].

Row and/or column in element stiffness matrix, $[k^{(e)}]$	Row and/or column in assembled stiffness matrix, [K]		
	element a	element b	element c
1	1	3	1
2	2	4	2
3	3	5	5
4	4	6	6

For example, choosing to assemble element b contributions by rows, then the first and the “element b” columns of the dof. table, Table 9.2, are used. Start by inserting in row 3, columns 3, 4, 5, 6 of structural stiffness matrix [K], the stiffness contributions respectively from row 1, columns 1, 2, 3, 4 of element stiffness matrix, $[k^{(b)}]$. Repeat for the remaining rows 4, 5, 6, inserting in columns 3, 4, 5, 6 of [K], the respective contributions from rows

2, 3, 4, columns 1, 2, 3, 4 of $[k^{(b)}]$. Repeat for remaining elements a and c, to give finally:

$$[K] = \begin{bmatrix} a_{11} + c_{11} & a_{12} + c_{12} & a_{13} & a_{14} & c_{13} & c_{14} \\ a_{21} + c_{21} & a_{22} + c_{22} & a_{23} & a_{24} & c_{23} & c_{24} \\ a_{31} & a_{32} & a_{33} + b_{11} & a_{34} + b_{12} & b_{13} & b_{14} \\ a_{41} & a_{42} & a_{43} + b_{21} & a_{44} + b_{22} & b_{23} & b_{24} \\ c_{31} & c_{32} & b_{31} & b_{32} & b_{33} + c_{33} & b_{34} + c_{34} \\ c_{41} & c_{42} & b_{41} & b_{42} & b_{43} + c_{43} & b_{44} + c_{44} \end{bmatrix}$$

The above assembly procedure is generally applicable to any element, albeit with detail changes. In the case of the simple beam element, with its rotational, as well as translational dof., reference to § 9.8 shows that the element stiffness matrix is of order 6×6 , and hence there will be two additional rows in the dof. correspondence table. A similar argument holds for the triangular membrane element, with its three nodes each having 2 dof. The Examples at the end of this chapter illustrate the assembly for rod, beam and membrane elements.

9.11. Application of boundary conditions and partitioning

With reference to §9.4.7, before the governing eqn. (9.28) can be solved to yield the unknown displacements, appropriate restraints need to be imposed. At some nodes the displacements will be prescribed, for example, at a fixed node the nodal displacements will be zero. Hence, some of the nodal displacements will be unknown, $\{p_\alpha\}$, and some will be prescribed, $\{p_\beta\}$. Following any necessary rearrangement to collect together equations relating to unknown, and those relating to prescribed, displacements, eqn. (9.28) can be partitioned into

$$\begin{bmatrix} \{P_\alpha\} \\ \{P_\beta\} \end{bmatrix} = \begin{bmatrix} [K_{\alpha\alpha}]^{-1} & [K_{\alpha\beta}] \\ [K_{\beta\alpha}] & [K_{\beta\beta}] \end{bmatrix} \begin{bmatrix} \{p_\alpha\} \\ \{p_\beta\} \end{bmatrix} \quad (9.96)$$

It will be found that where the loads are known, $\{P_\alpha\}$, [i.e. prescribed nodal forces (and moments, in beam applications)], the corresponding displacements will be unknown, $\{p_\alpha\}$, and where the displacements are known, $\{p_\beta\}$, (i.e. prescribed nodal displacements), the forces, $\{P_\beta\}$, (and moments, in beam applications), usually the reactions, will be unknown.

9.12. Solution for displacements and reactions

A solution for the unknown nodal displacements, $\{p_\alpha\}$, is obtained from the upper partition of eqn. (9.96)

$$\{P_\alpha\} = [K_{\alpha\alpha}]\{p_\alpha\} + [K_{\alpha\beta}]\{p_\beta\}$$

Rearranging

$$[K_{\alpha\alpha}]\{p_\alpha\} = \{P_\alpha\} - [K_{\alpha\beta}]\{p_\beta\}$$

To obtain a solution for the unknown nodal displacements, $\{p_\alpha\}$, it is only necessary to invert the submatrix $[K_{\alpha\alpha}]$. Pre-multiplying the above equation by $[K_{\alpha\alpha}]^{-1}$ (and using the matrix relation, $[K_{\alpha\alpha}]^{-1}[K_{\alpha\alpha}] = [I]$, the unit matrix), will yield the values of the unknown nodal displacements as

$$\{p_\alpha\} = [K_{\alpha\alpha}]^{-1}\{P_\alpha\} - [K_{\alpha\alpha}]^{-1}[K_{\alpha\beta}]\{p_\beta\} \quad (9.97)$$

If all the prescribed displacements are zero, i.e. $\{p_\beta\} = \{0\}$, the above reduces to

$$\{p_\alpha\} = [K_{\alpha\alpha}]^{-1}\{P_\alpha\} \quad (9.98)$$

The unknown reactions, $\{P_\beta\}$, can be found from the lower partition of eqn. (9.96)

$$\{P_\beta\} = [K_{\beta\alpha}]\{p_\alpha\} + [K_{\beta\beta}]\{p_\beta\} \quad (9.99)$$

Again, if all the prescribed displacements are zero, the above reduces to

$$\{P_\beta\} = [K_{\beta\alpha}]\{p_\alpha\} \quad (9.100)$$

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Examples

Example 9.1

Figure 9.31 shows a planar steel support structure, all three members of which have the same axial stiffness, such that $AE/L = 20 \text{ MN/m}$ throughout. Using the displacement based finite element method and treating each member as a rod:

- (a) assemble the necessary terms in the structural stiffness matrix;
- (b) hence, determine, with respect to the global coordinates (i) the nodal displacements, and (ii) the reactions, showing the latter on a sketch of the structure and demonstrating that equilibrium is satisfied.

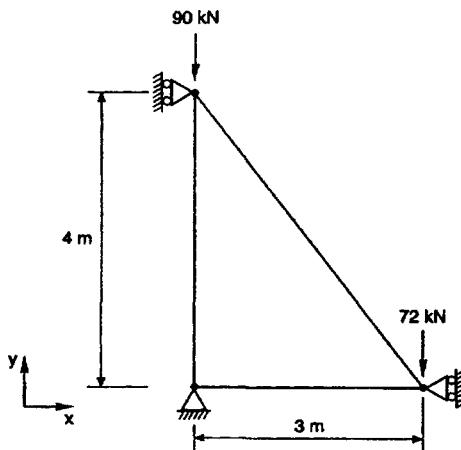


Fig. 9.31.

Solution

(a) Figure 9.32 shows suitable node, dof. and element labelling. Lack of symmetry prevents any advantage being taken to reduce the calculations. None of the members are redundant and hence the stiffness contributions of all three members need to be included.

All three elements will have the same stiffness matrix scalar,

$$\text{i.e. } (AE/L)^{(a)} = (AE/L)^{(b)} = (AE/L)^{(c)} = AE/L$$

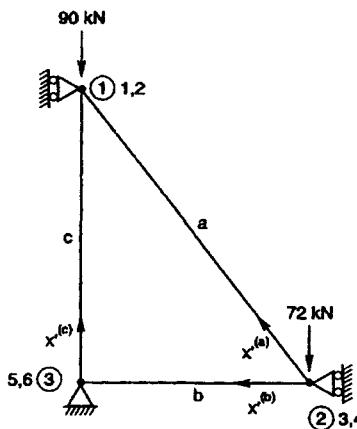


Fig. 9.32.

With reference to §9.7, the element stiffness matrix with respect to global coordinates is given by

$$[k^{(e)}] = \left(\frac{AE}{L} \right)^{(e)} \begin{bmatrix} \cos^2 \alpha & \sin^2 \alpha & \text{symmetric} \\ \sin \alpha \cos \alpha & -\sin^2 \alpha & \cos^2 \alpha \\ -\cos^2 \alpha & -\sin \alpha \cos \alpha & \sin^2 \alpha \\ -\sin \alpha \cos \alpha & -\sin^2 \alpha & \sin \alpha \cos \alpha & \sin^2 \alpha \end{bmatrix}^{(e)}$$

Evaluating the stiffness matrix for each element:

Element a

$$\alpha^{(a)} = -\tan^{-1}(4/3), \quad \cos \alpha^{(a)} = -0.6, \quad \sin \alpha^{(a)} = 0.8$$

$$[k^{(a)}] = \frac{AE}{L} \begin{bmatrix} 0.36 & -0.48 & -0.36 & 0.48 \\ -0.48 & 0.64 & 0.48 & -0.64 \\ -0.36 & 0.48 & 0.36 & -0.48 \\ 0.48 & -0.64 & -0.48 & 0.64 \end{bmatrix}$$

Element b

$$\alpha^{(b)} = 180^\circ, \quad \cos \alpha^{(b)} = -1, \quad \sin \alpha^{(b)} = 0$$

$$[k^{(b)}] = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Element c

$$\alpha^{(c)} = 90^\circ, \quad \cos \alpha^{(c)} = 0, \quad \sin \alpha^{(c)} = 1$$

$$[k^{(c)}] = \frac{AE}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

The structural stiffness matrix can now be assembled using a dof. correspondence table, (ref. §9.10). Observation of the highest dof. number, i.e. 6, gives the order (size), of the structural stiffness matrix, i.e. 6×6 . The structural governing equations and hence the required structural stiffness matrix are therefore given as

Row/ column in [$k^{(e)}$]	Row/column in [K]		
	a	b	c
1	3	3	5
2	4	4	6
3	1	5	1
4	2	6	2

$$\left[\begin{array}{c|ccccc|c} X_1 & 0.36 & -0.48 & -0.36 & 0.48 & & & u_1 \\ Y_1 & 0 & 0 & 0.48 & -0.64 & & & v_1 \\ X_2 & -0.48 & 0.64 & 0.48 & -0.64 & & & u_2 \\ Y_2 & 0 & 1 & 0 & 0 & & & v_2 \\ X_3 & -0.36 & 0.48 & 0.36 & -0.48 & & & u_3 \\ Y_3 & -0.48 & -0.64 & -0.48 & 0.64 & & & v_3 \end{array} \right] = \frac{AE}{L} \left[\begin{array}{cccc|cc} & & & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & -1 \\ & & & & 0 & 0 \\ & & & & -1 & 0 \\ & & & & 0 & 0 \end{array} \right]$$

(b) (i) Rearranging and partitioning, with $u_1 = u_2 = u_3 = v_3 = 0$, (i.e. $\{p_\beta\} = 0$)

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} -90 \times 10^3 \\ -72 \times 10^3 \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 1.64 & -0.64 \\ -0.64 & 0.64 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \text{ i.e. } \{P_\alpha\} = [K_{\alpha\alpha}] \{p_\alpha\}$$

Inverting $[K_{\alpha\alpha}]$ to enable a solution for the displacements using $\{p_\alpha\} = [K_{\alpha\alpha}]^{-1} \{P_\alpha\}$

$$\text{adj } [K_{\alpha\alpha}] = \frac{AE}{L} \begin{bmatrix} 0.64 & 0.64 \\ 0.64 & 1.64 \end{bmatrix} \quad \text{and} \quad \det[K_{\alpha\alpha}] = 0.64(AE/L)^2$$

$$\text{Then } [K_{\alpha\alpha}]^{-1} = \frac{L}{AE} \begin{bmatrix} 1 & 1 \\ 1 & 2.5625 \end{bmatrix} \text{ Check: } \frac{L}{AE} \begin{bmatrix} 1 & 1 \\ 1 & 2.5625 \end{bmatrix} \frac{AE}{L} \begin{bmatrix} 1.64 & -0.64 \\ -0.64 & 0.64 \end{bmatrix} = [I]$$

The required displacements are found from

$$\{p_\alpha\} = [K_{\alpha\alpha}]^{-1} \{P_\alpha\}$$

$$\text{Substituting, } \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{L}{AE} \begin{bmatrix} 1 & 1 \\ 1 & 2.5625 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = -5 \times 10^{-8} \begin{bmatrix} 1 & 1 \\ 1 & 2.5625 \end{bmatrix} \begin{bmatrix} 90.10^3 \\ 72.10^3 \end{bmatrix} \\ = \begin{bmatrix} -8.10 \\ -13.73 \end{bmatrix}_{\text{mm}}$$

The required nodal displacements are therefore $v_1 = -8.10$ mm and $v_2 = -13.73$ mm.

(ii) With reference to §9.12, nodal reactions are obtained from

$$\{P_\beta\} = [K_{\beta\alpha}] \{p_\alpha\}$$

Substituting gives

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ Y_3 \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} -0.48 & 0.48 \\ 0.48 & -0.48 \\ 0 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 2 \times 10^7 \begin{bmatrix} -0.48 & 0.48 \\ 0.48 & -0.48 \\ 0 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -8.10 \times 10^{-3} \\ -13.725 \times 10^{-3} \end{bmatrix} \\ = \begin{bmatrix} -54 \\ 54 \\ 0 \\ 162 \end{bmatrix}_{\text{kN}}$$

The required nodal reactions are therefore $X_1 = -54$ kN, $X_2 = 54$ kN, $X_3 = 0$ and $Y_3 = 162$ kN.

Representing these reactions together with the applied forces on a sketch of the structure, Fig. 9.33, and considering force and moment equilibrium, gives

$$\sum F_x = (54 - 54) \text{ kN} = 0$$

$$\sum F_y = (162 - 90 - 72) \text{ kN} = 0$$

$$\sum M_3 = (54 \times 4 - 73 \times 3) \text{ kNm} = 0$$

Hence, equilibrium is satisfied by the system of forces.

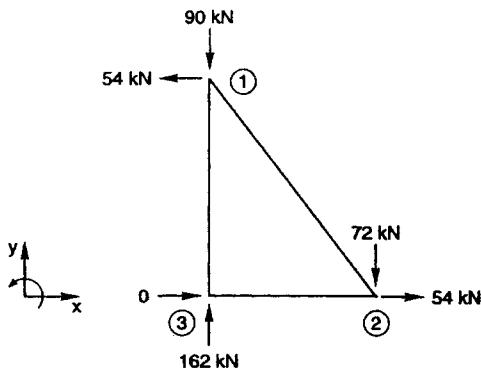


Fig. 9.33.

Example 9.2

Figure 9.34 shows the members and idealised support conditions for a roof truss. All three members of which are steel and have the same cross-sectional area such that $AE = 12$ MN throughout. Using the displacement based finite element method, treating the truss as a pin-jointed plane frame and each member as a rod:

- assemble the necessary terms in the structural stiffness matrix;
- hence, determine the nodal displacements with respect to the global coordinates, for the condition shown in Fig. 9.34.
- If, under load, the left support sinks by 5 mm, determine the resulting new nodal displacements, with respect to the global coordinates.

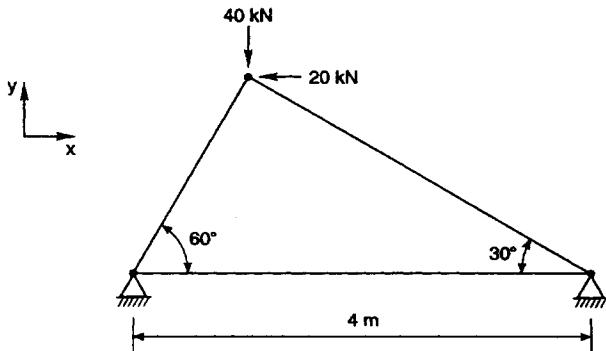


Fig. 9.34.

Solution

- Figure 9.35 shows suitable node, dof. and element labelling. Lack of symmetry prevents any advantage being taken to reduce the calculations. However, since both ends of the horizontal member are fixed it is redundant therefore and does not need to be considered further.

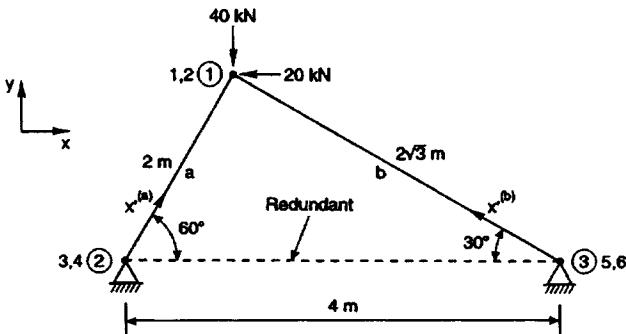


Fig. 9.35.

All three members have the same AE , hence

$$(AE)^{(a)} = (AE)^{(b)} = AE$$

With reference to §9.7, the element stiffness matrix with respect to global coordinates is given by

$$[k^{(e)}] = \left(\frac{AE}{L} \right)^{(e)} \begin{bmatrix} \cos^2 \alpha & & & \\ \sin \alpha \cos \alpha & \sin^2 \alpha & \text{symmetric} & \\ -\cos^2 \alpha & -\sin \alpha \cos \alpha & \cos^2 \alpha & \\ -\sin \alpha \cos \alpha & -\sin^2 \alpha & \sin \alpha \cos \alpha & \sin^2 \alpha \end{bmatrix}^{(e)}$$

Evaluating the stiffness matrix for both elements:

Element a

$$L^{(a)} = 2\text{m}, \quad \alpha^{(a)} = 60^\circ, \quad \cos \alpha^{(a)} = 1/2, \quad \sin \alpha^{(a)} = \sqrt{3}/2$$

$$[k^{(a)}] = \frac{AE}{8} \begin{bmatrix} 1 & \sqrt{3} & -1 & -\sqrt{3} \\ \sqrt{3} & 3 & -\sqrt{3} & -3 \\ -1 & -\sqrt{3} & 1 & \sqrt{3} \\ -\sqrt{3} & -3 & \sqrt{3} & 3 \end{bmatrix}$$

Element b

$$L^{(b)} = 2\sqrt{3}\text{m}, \quad \alpha^{(b)} = 150^\circ, \quad \cos \alpha^{(b)} = -\sqrt{3}/2, \quad \sin \alpha^{(b)} = 1/2$$

$$[k^{(b)}] = \frac{AE}{8} \begin{bmatrix} \sqrt{3} & -1 & -\sqrt{3} & 1 \\ -1 & 1/\sqrt{3} & 1 & -1/\sqrt{3} \\ -\sqrt{3} & 1 & \sqrt{3} & -1 \\ 1 & -1/\sqrt{3} & -1 & 1/\sqrt{3} \end{bmatrix}$$

The structural stiffness matrix can now be multi-assembled using a dof. correspondence table, (ref. §9.10), and will be of order 6×6 . Only the upper sub-matrices need to be completed, i.e. $[K_{\alpha\alpha}]$ and $[K_{\alpha\beta}]$, since the reactions are not required in this example. The necessary structural governing equations and hence the required structural stiffness matrix are therefore given as

Row/ column in $[k^{(e)}]$	Row/column in $[K]$	
	a	b
1	3	5
2	4	6
3	1	1
4	2	2

$$\begin{bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \\ X_3 \\ Y_3 \end{bmatrix} = \frac{AE}{8} \begin{bmatrix} 1 & \sqrt{3} & -1 & -\sqrt{3} & -\sqrt{3} & 1 \\ \sqrt{3} & 3 & -1 & -\sqrt{3} & -3 & -1/\sqrt{3} \\ -1 & -1/\sqrt{3} & 1 & 1 & -1 & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

(b) Corresponding to $u_2 = v_2 = u_3 = v_3 = 0$, the partitioned equations reduce to:

$$\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = \begin{bmatrix} -20 \times 10^3 \\ -40 \times 10^3 \end{bmatrix} = \frac{AE}{8} \begin{bmatrix} 2.7321 & 0.7321 \\ 0.7321 & 3.5774 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} \text{ i.e. } \{P_\alpha\} = [K_{\alpha\alpha}] \{p_\alpha\}$$

Inverting $[K_{\alpha\alpha}]$ to enable a solution for the displacements from $\{p_\alpha\} = [K_{\alpha\alpha}]^{-1} \{P_\alpha\}$

$$\text{adj } [K_{\alpha\alpha}] = \left(\frac{AE}{8}\right)^2 \begin{bmatrix} 3.5774 & -0.7321 \\ -0.7321 & 2.7321 \end{bmatrix} \text{ and } \det [K_{\alpha\alpha}] = \left(\frac{AE}{8}\right)^2 9.2378$$

Then $[K_{\alpha\alpha}]^{-1}$

$$= \frac{8}{AE} \begin{bmatrix} 0.3873 & -0.07925 \\ -0.07925 & 0.2958 \end{bmatrix} \text{ Check: } \frac{8}{AE} \begin{bmatrix} 0.3873 & -0.07925 \\ -0.07925 & 0.2958 \end{bmatrix} \frac{8}{AE} \begin{bmatrix} 2.7321 & 0.7321 \\ 0.7321 & 3.5774 \end{bmatrix} = [I]$$

Hence, the required displacements are given by

$$\{p_\alpha\} = [K_{\alpha\alpha}]^{-1} \{P_\alpha\}$$

$$\text{Substituting, } \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \frac{8}{12 \times 10^6} \begin{bmatrix} 0.3873 & -0.07925 \\ -0.07925 & 0.2958 \end{bmatrix} \begin{bmatrix} -20 \cdot 10^3 \\ -40 \cdot 10^3 \end{bmatrix} = \begin{bmatrix} -3.05 \\ -6.83 \end{bmatrix}_{\text{mm}}$$

The required nodal displacements are therefore $u_1 = -3.05$ mm and $v_1 = -6.83$ mm.

(c) With reference to §9.12, for non-zero prescribed displacements, i.e. $\{p_\beta\} \neq \{0\}$, the full partition of the governing equation is required, namely,

$$\{P_\alpha\} = [K_{\alpha\alpha}]\{p_\alpha\} + [K_{\alpha\beta}]\{p_\beta\}$$

Rearranging for the unknown displacements

$$\{p_\alpha\} = [K_{\alpha\alpha}]^{-1}\{P_\alpha\} - [K_{\alpha\alpha}]^{-1}[K_{\alpha\beta}]\{p_\beta\}$$

Evaluating

$$[K_{\alpha\alpha}]^{-1}[K_{\alpha\beta}] = \begin{bmatrix} 0.3873 & -0.07925 \\ -0.07925 & 0.2958 \end{bmatrix} \begin{bmatrix} -1 & -1.7321 & -1.7321 & 1 \\ -1.7321 & -3 & 1 & -0.5773 \end{bmatrix}$$

$$= \begin{bmatrix} -0.25 & -0.4331 & -0.75 & 0.4331 \\ -0.4331 & -0.75 & 0.4331 & -0.25 \end{bmatrix}$$

$$\text{and } [K_{\alpha\alpha}]^{-1}[K_{\alpha\beta}]\{p_\beta\} = \begin{bmatrix} -0.25 & -0.4331 & -0.75 & 0.4331 \\ -0.4331 & -0.75 & 0.4331 & -0.25 \end{bmatrix} \begin{bmatrix} 0 \\ -5 \times 10^{-3} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.1655 \\ 3.75 \\ 0 \\ 0 \end{bmatrix}_{\text{mm}}$$

Recalling from part (b) that

$$[K_{\alpha\alpha}]^{-1}\{P_\alpha\} = \begin{bmatrix} -3.05 \\ -6.83 \end{bmatrix}_{\text{mm}}$$

and substituting into the above rearranged governing equation,

$$\text{i.e. } \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} -3.05 \\ -6.83 \end{bmatrix} - \begin{bmatrix} 2.1655 \\ 3.75 \end{bmatrix} = \begin{bmatrix} -5.22 \\ -10.58 \end{bmatrix}_{\text{mm}}$$

yields the required new nodal displacements, namely, $u_1 = -5.22$ mm and $v_1 = -10.58$ mm.

Example 9.3

A steel beam is supported and loaded as shown in Fig. 9.36. The relevant second moments of area are such that $I^{(a)} = 2I^{(b)} = 2 \times 10^{-5} \text{ m}^4$ and Young's modulus E for the beam material = 200 GN/m². Using the displacement based finite element method and representing each member by a simple beam element:

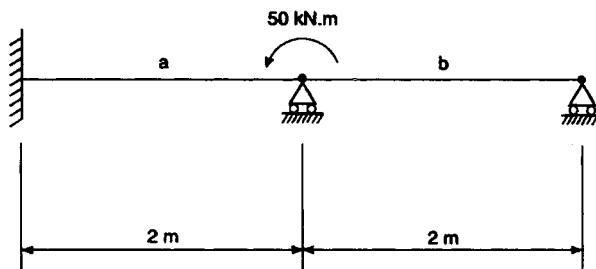


Fig. 9.36.

- (a) determine the nodal displacements;
- (b) hence, determine the nodal reactions, representing these on a sketch of the deformed geometry. Show that both force and moment equilibrium is satisfied.

Solution

(a) Figure 9.37 shows suitable node, dof. and element labelling. Lack of symmetry prevents any advantage being taken to reduce the calculations. There are no redundant members.

Employing two beam finite elements, (which is the least number in this case), both elements will have the same E/L , i.e.

$$(E/L)^{(a)} = (E/L)^{(b)} = E/L$$

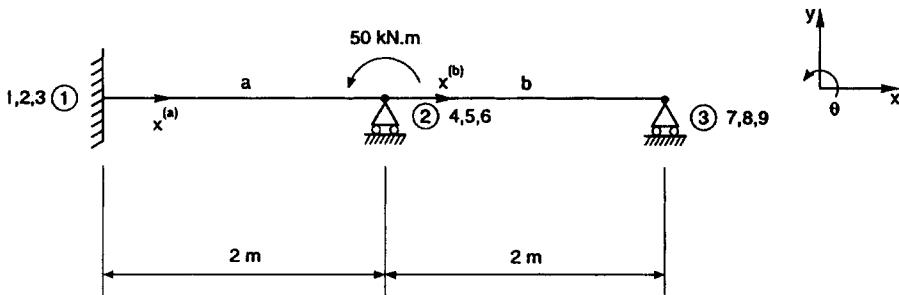


Fig. 9.37.

However, the second moments of area will be different, such that

$$I^{(a)} = 2I \text{ and } I^{(b)} = I$$

and will be the only difference between the two element stiffness matrices.

With reference to §9.8 and in the absence of axial forces, each element stiffness matrix with respect to local coordinates is given as

$$[k^{(e)}] = \left(\frac{EI}{L} \right)^{(e)} \begin{bmatrix} 12/L^2 & 6/L & -12/L^2 & 6/L \\ 6/L & 4 & -6/L & 2 \\ -12/L^2 & -6/L & 12/L^2 & -6/L \\ 6/L & 2 & -6/L & 4 \end{bmatrix} \text{ symmetric}$$

The above local coordinate element stiffness matrix will, in this case, be identical to that with respect to global coordinates since the local and global axes coincide.

Substituting for both elements:

Element a

Recalling $I^{(a)} = 2I$

$$[k^{(a)}] = \left(\frac{EI}{L} \right)^{(a)} \begin{bmatrix} 24/L^2 & 12/L & -24/L^2 & 12/L \\ 12/L & 8 & -12/L & 4 \\ -24/L^2 & -12/L & 24/L^2 & -12/L \\ 12/L & 4 & -12/L & 8 \end{bmatrix} \text{ symmetric}$$

Element b

Recalling $I^{(b)} = I$

$$[k^{(b)}] = \frac{EI}{L} \begin{bmatrix} 12/L^2 & & & \\ 6/L & 4 & & \text{symmetric} \\ -12/L^2 & -6/L & 12/L^2 & \\ 6/L & 2 & -6/L & 4 \end{bmatrix}$$

The structural stiffness matrix can now be assembled. A dof. correspondence table can be used as an aid to assembly. However, observation of the relatively simple element connectivity, shows that the stiffness contributions for element *a* will occupy the upper left 4×4 locations, whilst those for element *b* will occupy the lower right 4×4 locations of the 6×6 structural stiffness matrix. The reduced structural stiffness matrix is due to the omission of axial terms, otherwise the matrix would have been of order 9×9 . Hence, completing only those columns needed for the solution, gives

$$\begin{bmatrix} Y_1 \\ M_1 \\ Y_2 \\ M_2 \\ Y_3 \\ M_3 \end{bmatrix} = \frac{EI}{L} \begin{bmatrix} & & & 12/L & & \\ & & & 4 & & \\ & & & -12/L & & \\ & & & 6/L & 6/L & \\ & & & 8 & 4 & 2 \\ & & & -6/L & -6/L & \\ & & & 2 & & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{bmatrix}$$

No need to complete these columns

Corresponding to $v_1 = \theta_1 = v_2 = \theta_2 = v_3 = \theta_3 = 0$ (by omitting axial terms it has already been taken that $u_1 = u_2 = u_3 = 0$), the partitioned equations reduce to

$$\begin{bmatrix} M_2 \\ M_3 \end{bmatrix} = \frac{EI}{L} \begin{bmatrix} 12 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} \theta_2 \\ \theta_3 \end{bmatrix} \text{ i.e. } \{P_\alpha\} = [K_{\alpha\alpha}] \{p_\alpha\}$$

Inverting $[K_{\alpha\alpha}]$ to enable a solution for the displacements from $\{p_\alpha\} = [K_{\alpha\alpha}]^{-1} \{P_\alpha\}$

$$\text{where } \text{adj } [K_{\alpha\alpha}] = \frac{EI}{L} \begin{bmatrix} 4 & -2 \\ -2 & 12 \end{bmatrix} \text{ and } \det [K_{\alpha\alpha}] = 44(EI/L)^2$$

$$\text{Then, } [K_{\alpha\alpha}]^{-1} = \frac{L}{44EI} \begin{bmatrix} 4 & -2 \\ -2 & 12 \end{bmatrix} \text{ Check } \frac{L}{44EI} \begin{bmatrix} 4 & -2 \\ -2 & 12 \end{bmatrix} \frac{EI}{L} \begin{bmatrix} 12 & 2 \\ 2 & 4 \end{bmatrix} = [I]$$

The required displacements are found from

$$\{p_\alpha\} = [K_{\alpha\alpha}]^{-1} \{P_\alpha\}$$

$$\begin{aligned} \text{Substituting } \begin{bmatrix} \theta_2 \\ \theta_3 \end{bmatrix} &= \frac{L}{44EI} \begin{bmatrix} 4 & -2 \\ -2 & 12 \end{bmatrix} \begin{bmatrix} M_2 \\ M_3 \end{bmatrix} \\ &= \frac{2}{44 \times 200 \times 10^9 \times 1 \times 10^{-5}} \begin{bmatrix} 4 & -2 \\ -2 & 12 \end{bmatrix} \begin{bmatrix} 5 \times 10^4 \\ 0 \end{bmatrix} \\ &= 2.2727 \times 10^{-4} \begin{bmatrix} 20 \\ -10 \end{bmatrix} = \begin{bmatrix} 4.545 \cdot 10^{-3} \\ -2.273 \cdot 10^{-3} \end{bmatrix}_{\text{rad}} = \begin{bmatrix} 0.260 \\ -0.130 \end{bmatrix}_{\text{deg}} \end{aligned}$$

The required nodal displacements are therefore $\theta_2 = 0.26^\circ$ and $\theta_3 = -0.13^\circ$.

(b) With reference to §9.12, nodal reactions are obtained from

$$\{P_\alpha\} = [K_{\alpha\beta}]\{p_\alpha\}$$

$$\begin{aligned} \text{Substituting gives } \begin{bmatrix} Y_1 \\ M_1 \\ Y_2 \\ Y_3 \end{bmatrix} &= \frac{EI}{L} \begin{bmatrix} 12/L & 0 \\ 4 & 0 \\ -6/L & 6/L \\ -6/L & -6/L \end{bmatrix} \begin{bmatrix} \theta_2 \\ \theta_3 \end{bmatrix} \\ &= \frac{200 \times 10^9 \times 1 \times 10^{-5}}{2} \begin{bmatrix} 6 & 0 \\ 4 & 0 \\ -3 & 3 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} 4.545 \times 10^{-3} \\ -2.273 \times 10^{-3} \end{bmatrix} \\ &= \begin{bmatrix} 27.27 \text{ kN} \\ 18.18 \text{ kNm} \\ -20.45 \text{ kN} \\ -6.82 \text{ kN} \end{bmatrix} \end{aligned}$$

The required nodal reactions are therefore $Y_1 = 27.27 \text{ kN}$, $M_1 = 18.18 \text{ kNm}$, $Y_2 = -20.45 \text{ kN}$ and $Y_3 = -6.82 \text{ kN}$.

Representing these reactions together with the applied moment on a sketch of the deformed beam, Fig. 9.38, and considering force and moment equilibrium, gives

$$\Sigma F_y = (27.27 - 20.45 - 6.82) \text{ kN} = 0$$

$$\Sigma M_1 = (18.18 + 50 - 20.45 \times 2 - 6.82 \times 4) \text{ kNm} = 0$$

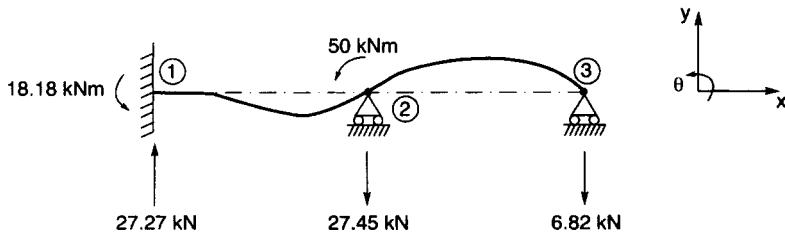


Fig. 9.38.

Example 9.4

The vehicle engine mounting bracket shown in Fig. 9.39 is made from uniform steel channel section for which Young's modulus, $E = 200 \text{ GN/m}^2$. It can be assumed for both

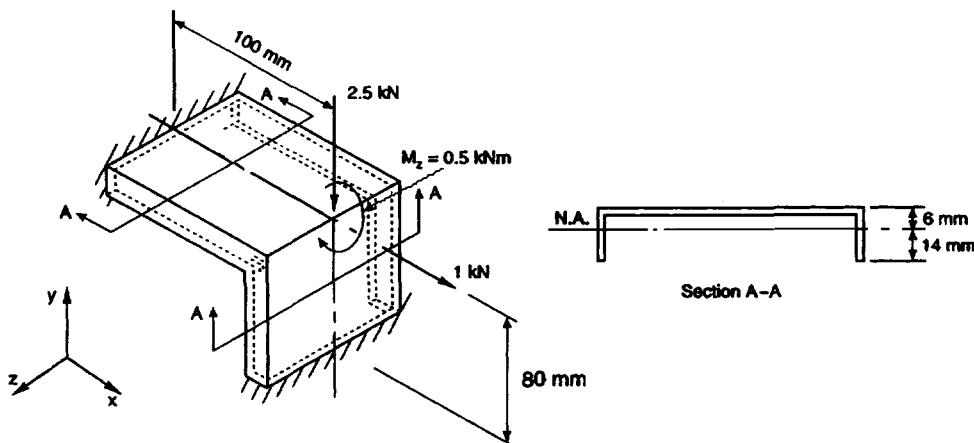


Fig. 9.39.

channels that the relevant second moment of area, $I = 2 \times 10^{-8} \text{ m}^4$ and cross-sectional area, $A = 4 \times 10^{-4} \text{ m}^2$. The bracket can be idealised as two beams, the common junction of which can be assumed to be infinitely stiff and the other ends to be fully restrained. Using the displacement based finite element method, and representing the constituent members as simple beam elements:

- (a) assemble the necessary terms in the structural stiffness matrix;
- (b) hence, determine for the condition shown in Fig. 9.39 (i) the nodal displacements with respect to the global coordinates, and (ii) the combined axial and bending extreme fibre stresses at the built-in ends and at the common junction.

Solution

- (a) Figure 9.40 shows suitable node, dof, and element labelling. The structure does not have symmetry or redundant members. The least number of beam elements will be used to

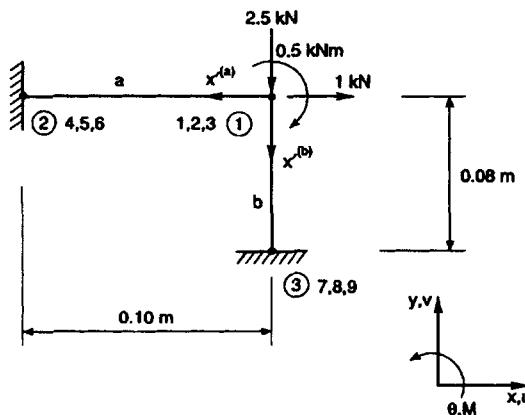


Fig. 9.40.

minimise the hand calculations which, in this example, is two.
Both elements will have the same A , E and I ,

i.e.

$$(A, E, I)^{(a)} = (A, E, I)^{(b)} = A, E, I,$$

but will have different lengths, i.e. $L^{(a)}$ and $L^{(b)}$.

With reference to §9.8, the element stiffness matrix inclusive of axial terms and in global coordinates is appropriate, namely:

$$[k^{(e)}] = \left(\frac{E}{L}\right)^{(e)} \begin{bmatrix} A \cos^2 \alpha + (12I \sin^2 \alpha)/L^2, & (A - 12I/L^2) \cos \alpha \sin \alpha, & A \sin^2 \alpha + (12I \cos^2 \alpha)/L^2, & 4I, \\ -(6I \sin \alpha)L, & (6I \cos \alpha)/L, & -A \cos^2 \alpha - (12I \sin^2 \alpha)/L^2, & (6I \sin \alpha)/L, \\ -A \cos^2 \alpha - (12I \sin^2 \alpha)/L^2, & -(A - 12I/L^2) \cos \alpha \sin \alpha, & A \sin^2 \alpha + (12I \cos^2 \alpha)/L^2, & -(6I \cos \alpha)/L, \\ -(6I \sin \alpha)/L, & (6I \cos \alpha)/L, & -(6I \cos \alpha)/L, & 2I \end{bmatrix}$$

$$\begin{bmatrix} A \cos^2 \alpha + (12I \sin^2 \alpha)/L^2, & (A - 12I/L^2) \cos \alpha \sin \alpha, & A \sin^2 \alpha + (12I \cos^2 \alpha)/L^2, & 4I, \\ (A - 12I/L^2) \cos \alpha \sin \alpha, & A \sin^2 \alpha + (12I \cos^2 \alpha)/L^2, & -(6I \cos \alpha)/L, & 4I \\ (6I \sin \alpha)/L, & -(6I \cos \alpha)/L, & (6I \sin \alpha)/L, & \end{bmatrix}$$

Evaluating, for both elements, only those stiffness terms essential for the analysis:

Element a

$$L^{(a)} = 0.1m, \alpha^{(a)} = 180^\circ, \cos \alpha^{(a)} = -1, \sin \alpha^{(a)} = 0$$

$$[k^{(a)}] = \frac{E}{L^{(a)}} \begin{bmatrix} A & 0 & 0 & (a) \\ 0 & 12I/L^2 & -6I/L & \\ 0 & -6I/L & 4I & \\ \hline & & & \end{bmatrix} = E \times 10^{-4} \begin{bmatrix} 40 & 0 & 0 & \\ 0 & 2.4 & -0.12 & \\ 0 & -0.12 & 8 \times 10^{-3} & \\ \hline & & & \end{bmatrix}$$

No need to complete these rows and columns, for these examples

Element b

$$L^{(b)} = 0.08m, \alpha^{(b)} = 270^\circ, \cos \alpha^{(b)} = 0, \sin \alpha^{(b)} = -1$$

$$[k^{(b)}] = \frac{E}{L^{(b)}} \begin{bmatrix} 12I/L^2 & 0 & 6I/L & (b) \\ 0 & A & 0 & \\ 6I/L & 0 & 4I & \\ \hline & & & \end{bmatrix} = E \times 10^{-4} \begin{bmatrix} 4.6875 & 0 & 0.1875 & \\ 0 & 50 & 0 & \\ 0.1875 & 0 & 10.10^{-3} & \\ \hline & & & \end{bmatrix}$$

The structural stiffness matrix can now be assembled. Whilst the structure has a total of 9 dof., only 3 are active, the remaining 6 dof. are suppressed corresponding to the statement in the question regarding the ends being fully restrained. The node numbering adopted in Fig. 9.40 simplifies the stiffness assembly, whereby the first 3×3 submatrix terms for both elements are assembled in the first 3×3 locations of the structural stiffness matrix; these being the only terms associated with the active dofs. It follows that rearrangement is unnecessary, prior to partitioning. The necessary structural governing equations and hence the required structural stiffness matrix are therefore given as

$$\begin{bmatrix} X_1 \\ Y_1 \\ M_1 \end{bmatrix} = E \times 10^{-4} \begin{bmatrix} 40 & 0 & 0 \\ 4.6875 & 0 & 0.1875 \\ 0 & 2.4 & -0.12 \\ 0 & 50 & 0 \\ 0 & -0.12 & 8 \times 10^{-3} \\ 0.1875 & 0 & 10 \times 10^{-3} \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \end{bmatrix}$$

$= E \times 10^{-4}$

These submatrices are not required, for this example.

(b) (i) Corresponding to $u_2 = v_2 = \theta_2 = u_3 = v_3 = \theta_3 = 0$, the partitioned equations reduce to

$$\begin{bmatrix} X_1 \\ Y_1 \\ M_1 \end{bmatrix} = E \times 10^{-4} \begin{bmatrix} 44.6875 & 0 & 0.1875 \\ 0 & 52.4 & -0.12 \\ 0.1875 & -0.12 & 0.018 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \end{bmatrix}$$

$$= 10^7 \begin{bmatrix} 89.375 & 0 & 0.375 \\ 0 & 104.8 & -0.24 \\ 0.375 & -0.24 & 0.036 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \end{bmatrix}$$

i.e. $\{P_\alpha\} = [K_{\alpha\alpha}]\{p_\alpha\}$

Inverting $[K_{\alpha\alpha}]$ to enable a solution for the displacements from $\{p_\alpha\} = [K_{\alpha\alpha}]^{-1}\{P_\alpha\}$

where $\text{adj } [K_{\alpha\alpha}] = 10^{14} \begin{bmatrix} 3.7152 & -0.09 & -39.3 \\ -0.09 & 3.0769 & 21.45 \\ -39.3 & 21.45 & 9366.5 \end{bmatrix}$

and $\det [K_{\alpha\alpha}] = 10^{21} \{89.375[104.8 \times 0.036$

$$- (-0.24)(-0.24)] - 0 + 0.375(0 - 0.375 \times 104.8)\}$$

$$= 317.3085 \times 10^{21}$$

Then $[K_{\alpha\alpha}]^{-1} = 10^{-10} \begin{bmatrix} 11.7085 & -0.2836 & -123.8542 \\ -0.2836 & 9.6969 & 67.5998 \\ -123.8542 & 67.5998 & 29518.59 \end{bmatrix}$

The required displacements are found from

$$p_\alpha = [K_{\alpha\alpha}]^{-1} \{P_\alpha\}$$

Substituting $\begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \end{bmatrix} = 10^{-10} \begin{bmatrix} 11.7085 & -0.2836 & -123.8542 \\ -0.2836 & 9.6969 & 67.5998 \\ -123.8542 & 67.5998 & 29518.59 \end{bmatrix} 10^3 \begin{bmatrix} 1 \\ -2.5 \\ -0.5 \end{bmatrix}$

$$= \begin{bmatrix} 7.434 \times 10^{-6} \text{ m} \\ -5.833 \times 10^{-6} \text{ m} \\ -1.505 \times 10^{-3} \text{ rad} \end{bmatrix}$$

The required nodal displacements are therefore $u_1 = 7.434 \times 10^{-6} \text{ m}$, $v_1 = -5.833 \times 10^{-6} \text{ m}$ and $\theta_1 = -1.505 \times 10^{-3} \text{ rad}$.

(b) (ii) With reference to §9.8, the element stress matrix in global coordinates is given as

$$[H^{(e)}] = \frac{E}{L} \begin{bmatrix} -\cos \alpha - 6t \sin(\alpha)/L & -\sin \alpha + 6t \cos(\alpha)/L & 4t & \cos \alpha + 6t \sin(\alpha)/L & \sin \alpha - 6t \cos(\alpha)/L & 2t \\ -\cos \alpha + 6b \sin(\alpha)/L & -\sin \alpha - 6b \cos(\alpha)/L & -4b & \cos \alpha - 6b \sin(\alpha)/L & \sin \alpha + 6b \cos(\alpha)/L & -2b \\ -\cos \alpha + 6t \sin(\alpha)/L & -\sin \alpha - 6t \cos(\alpha)/L & -2t & \cos \alpha - 6t \sin(\alpha)/L & \sin \alpha + 6t \cos(\alpha)/L & -4t \\ -\cos \alpha - 6b \sin(\alpha)/L & -\sin \alpha + 6b \cos(\alpha)/L & 2b & \cos \alpha + 6b \sin(\alpha)/L & \sin \alpha - 6b \cos(\alpha)/L & 4b \end{bmatrix}$$

Evaluating, for both elements, only those terms essential for the analysis:

Element a

$t^{(a)} = 14 \times 10^{-3} \text{ m}$, $b^{(a)} = 6 \times 10^{-3} \text{ m}$, and recalling from part (a) $L^{(a)} = 0.1 \text{ m}$, $\alpha^{(a)} = 180^\circ$, $\cos \alpha^{(a)} = -1$, $\sin \alpha^{(a)} = 0$

$$[H^{(a)}] = \frac{200 \times 10^9}{0.1} \begin{bmatrix} 1 & -0.84 & 56 \times 10^{-3} & | & | & | \\ 1 & 0.36 & -24 \times 10^{-3} & | & | & | \\ 1 & 0.84 & -28 \times 10^{-3} & | & | & | \\ 1 & -0.36 & 12 \times 10^{-3} & | & | & | \end{bmatrix}$$

No need to complete these columns for this example

With reference to §9.7, the element stresses are obtained from

With reference to §9.7, the element stresses are obtained from

$$\{\sigma^{(e)}\} = [H^{(e)}]\{s^{(e)}\}$$

where, for element a, the displacement column matrix is

$$\{s^{(a)}\} = \{u_1 \ v_1 \ \theta_1 \ u_2 \ v_2 \ \theta_2\} \text{ in which } u_2 = v_2 = \theta_2 = 0 \text{ in this example.}$$

Substituting for element a and letting superscript i denote extreme inner fibres and superscript o denote extreme outer fibres, gives

$$\begin{bmatrix} \sigma_1^i \\ \sigma_1^o \\ \sigma_2^i \\ \sigma_2^o \end{bmatrix} = 2 \times 10^{12} \begin{bmatrix} 1 & -0.84 & 56 \times 10^{-3} & | & | & | \\ 1 & 0.36 & -24 \times 10^{-3} & | & | & | \\ 1 & 0.84 & -28 \times 10^{-3} & | & | & | \\ 1 & -0.36 & 12 \times 10^{-3} & | & | & | \end{bmatrix} \begin{bmatrix} 7.434 \times 10^{-6} \\ -5.833 \times 10^{-6} \\ 1.505 \times 10^{-3} \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -143.89 \times 10^6 \\ 82.91 \times 10^6 \\ 89.35 \times 10^6 \\ -17.05 \times 10^6 \end{bmatrix}$$

The required element stresses are therefore $\sigma_1^i = 143.89 \text{ MN/m}^2 (C)$, $\sigma_1^o = 82.91 \text{ MN/m}^2 (T)$, $\sigma_2^i = 89.35 \text{ MN/m}^2 (T)$ and $\sigma_2^o = 17.05 \text{ MN/m}^2 (C)$.

Element b

$t^{(b)} = 6 \times 10^{-3} \text{ m}$, $b^{(b)} = 14 \times 10^{-3} \text{ m}$, and recalling from part (a) $L^{(b)} = 0.08 \text{ m}$, $\alpha^{(b)} = 270^\circ$, $\cos \alpha^{(b)} = 0$, $\sin \alpha^{(b)} = -1$,

$$[H^{(b)}] = \frac{200 \times 10^9}{0.08} \begin{bmatrix} 0.45 & 1 & 24 \times 10^{-3} & | & | & | \\ -1.05 & 1 & -56 \times 10^{-3} & | & | & | \\ -0.45 & 1 & -12 \times 10^{-3} & | & | & | \\ 1.05 & 1 & 28 \times 10^{-3} & | & | & | \end{bmatrix}$$

Again, the element stresses are obtained from

$$\{\sigma^{(e)}\} = [H^{(e)}]\{s^{(e)}\}$$

where, for element b , the displacement column matrix is

$$\{s^{(b)}\} = \{u_1 \ v_1 \ \theta_1 \ u_3 \ v_3 \ \theta_3\} \text{ in which } u_3 = v_3 = \theta_3 = 0 \text{ in this example.}$$

Substituting for element b gives

$$\begin{bmatrix} \sigma_1^o \\ \sigma_1^i \\ \sigma_3^o \\ \sigma_3^i \end{bmatrix} = 2.5 \times 10^{12} \begin{bmatrix} 0.45 & 1 & 24 \times 10^{-3} & | & | & | \\ -1.05 & 1 & -56 \times 10^{-3} & | & | & | \\ -0.45 & 1 & -12 \times 10^{-3} & | & | & | \\ 1.05 & 1 & 28 \times 10^{-3} & | & | & | \end{bmatrix} \begin{bmatrix} 7.434 \times 10^{-6} \\ -5.833 \times 10^{-6} \\ -1.505 \times 10^{-3} \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -96.52 \times 10^6 \\ 176.60 \times 10^6 \\ 22.20 \times 10^6 \\ -100.42 \times 10^6 \end{bmatrix}_{\text{N/m}^2}$$

The required element stresses are therefore $\sigma_1^i = 176.60 \text{ MN/m}^2 (T)$, $\sigma_1^o = 96.52 \text{ MN/m}^2 (C)$, $\sigma_3^i = 100.42 \text{ MN/m}^2 (C)$ and $\sigma_3^o = 22.20 \text{ MN/m}^2 (T)$.

Example 9.5

Derive the stiffness matrix in global coordinates for a three-node triangular membrane element for plane stress analysis. Assume that the elastic modulus, E , and thickness, t , are

constant throughout, and that the displacement functions are

$$u(x, y) = \alpha_1 + \alpha_2x + \alpha_3y$$

$$v(x, y) = \alpha_4 + \alpha_5x + \alpha_6y$$

Solution

With reference to §9.9 and with respect to the node labelling shown in Fig. 9.41, matrix $[A]$ will be given as:

$$[A] = \left[\begin{array}{ccc|ccc} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{array} \right] = \left[\begin{array}{c|c} A_{\alpha\alpha} & A_{\alpha\beta} \\ \hline A_{\beta\alpha} & A_{\beta\beta} \end{array} \right]$$

Then $[A]^{-1} = \begin{bmatrix} [A_{\alpha\alpha}]^{-1} & 0 \\ 0 & [A_{\beta\beta}]^{-1} \end{bmatrix}$ where $[A_{\alpha\alpha}]^{-1} = [A_{\beta\beta}]^{-1}$, in this case

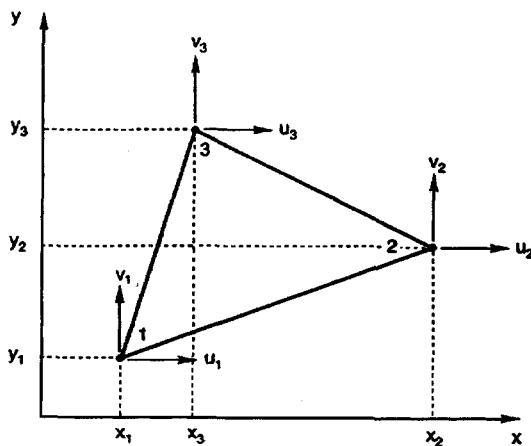


Fig. 9.41.

Obtaining the inverse of the partition

$$\begin{aligned} \text{adj } [A_{\alpha\alpha}] &= [C_{\alpha\alpha}]^T = \begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix}^T \\ &= \begin{bmatrix} x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{and } \det[A_{\alpha\alpha}] &= (x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1) \\ &= x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \\ &= 2 \times \text{area of element} = 2a, (\text{see following derivation}) \end{aligned}$$

Then $[A_{\alpha\alpha}]^{-1} = \frac{\text{adj}[A_{\alpha\alpha}]}{\det[A_{\alpha\alpha}]} = \frac{1}{2a} \begin{bmatrix} x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix}$

Hence, $[A]^{-1} = \frac{1}{2a} \begin{bmatrix} x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 & 0 & 0 & 0 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 & 0 & 0 & 0 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ 0 & 0 & 0 & y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ 0 & 0 & 0 & x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix}$

Area of element

With reference to Fig. 9.42, area of triangular element,

$$\begin{aligned} a &= \text{area of enclosing rectangle} - (\text{area of triangles b, c and d}) \\ &= (x_2 - x_1)(y_3 - y_1) - (1/2)(x_2 - x_1)(y_2 - y_1) - (1/2)(x_2 - x_3)(y_3 - y_2) \\ &\quad - (1/2)(x_3 - x_1)(y_3 - y_1) \\ &= x_2y_3 - x_2y_1 - x_1y_3 + x_1y_1 - (1/2)[x_2y_2 - x_2y_1 - x_1y_2 + x_1y_1] \\ &\quad + (x_2y_3 - x_2y_2 - x_3y_3 + x_3y_2) + (x_3y_3 - x_3y_1 - x_1y_3 + x_1y_1)] \\ &= (1/2)(x_2y_3 - x_2y_1 - x_1y_3 + x_1y_2 - x_3y_2 + x_3y_1) \\ &= (1/2)[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] \end{aligned}$$

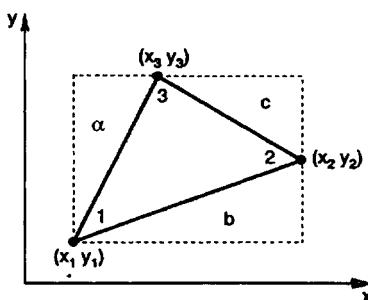


Fig. 9.42.

§9.9 gives matrix $[B]$ as

$$[B] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} [A]^{-1}$$

Substituting for $[A]^{-1}$ from above and evaluating the product gives

$$[B] = \frac{1}{2a} \begin{bmatrix} y_2 - y_3 & y_3 - y_1 & y_1 - y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 & y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \end{bmatrix}$$

The required element stiffness matrix can now be found by substituting into the relation

$$[k] = \text{at } [B]^T [D] [B]$$

$$= \frac{at}{2a} \begin{bmatrix} y_{23} & 0 & x_{32} \\ y_{31} & 0 & x_{13} \\ y_{12} & 0 & x_{21} \\ 0 & x_{32} & y_{23} \\ 0 & x_{13} & y_{31} \\ 0 & x_{21} & y_{12} \end{bmatrix} \frac{E}{1 - v^2} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & (1-v)/2 \end{bmatrix} \frac{1}{2a} \begin{bmatrix} y_{23} & y_{31} & y_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{32} & x_{13} & x_{21} \\ x_{32} & x_{13} & x_{21} & y_{23} & y_{31} & y_{12} \end{bmatrix}$$

where the abbreviation y_{23} denotes $y_2 - y_3$, etc.

Choosing to evaluate the product $[D][B]$ first, gives

$$[k] = \frac{Et}{4a(1-v^2)} \begin{bmatrix} y_{23} & 0 & x_{32} \\ y_{31} & 0 & x_{13} \\ y_{12} & 0 & x_{21} \\ 0 & x_{32} & y_{23} \\ 0 & x_{13} & y_{31} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} y_{23} & y_{31} & y_{12} & vx_{32} & vx_{13} & vx_{21} \\ vy_{23} & vy_{31} & vy_{12} & x_{32} & x_{13} & x_{21} \\ \frac{(1-v)}{2}x_{32} & \frac{(1-v)}{2}x_{13} & \frac{(1-v)}{2}x_{21} & \frac{(1-v)}{2}y_{23} & \frac{(1-v)}{2}y_{31} & \frac{(1-v)}{2}y_{12} \end{bmatrix}$$

Completing the matrix multiplication, reversing the sequence of some of the coordinates so that all subscripts are in descending order, gives the required element stiffness matrix as

$$[k] = \frac{Et}{4a(1-v^2)} \begin{bmatrix} y_{32}^2 + x_{32}^2(1-v)/2, & & & \\ -y_{32}y_{31} - x_{31}x_{32}(1-v)/2, & y_{31}^2 + x_{31}^2(1-v)/2, & & \\ y_{21}y_{32} + x_{21}x_{32}(1-v)/2, & -y_{21}y_{31} - x_{21}x_{31}(1-v)/2, & y_{21}^2 + x_{21}^2(1-v)/2, & \\ -vx_{32}y_{32} - y_{32}x_{32}(1-v)/2, & vx_{32}y_{31} + y_{32}x_{31}(1-v)/2, & -vx_{32}y_{21} - y_{32}x_{21}(1-v)/2, & \\ vx_{31}y_{32} + y_{31}x_{32}(1-v)/2, & -vx_{31}y_{31} - y_{31}x_{31}(1-v)/2, & vx_{31}y_{21} + y_{31}x_{21}(1-v)/2, & \\ -vx_{21}y_{32} - y_{21}x_{32}(1-v)/2, & vx_{21}y_{31} + y_{21}x_{31}(1-v)/2, & -vx_{21}y_{21} - y_{21}x_{21}(1-v)/2, & \end{bmatrix}$$

Symmetric

Example 9.6

- (a) Evaluate the element stiffness matrix, in global coordinates, for the three-node triangular membrane element, labelled a in Fig. 9.43. Assume plane stress conditions, Young's modulus, $E = 200 \text{ GN/m}^2$, Poisson's ratio, $\nu = 0.3$, thickness, $t = 1 \text{ mm}$, and the same displacement functions as Example 9.5.

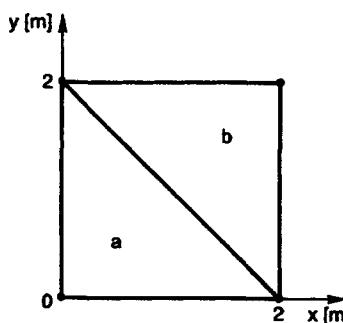


Fig. 9.43.

(b) Evaluate the element stiffness matrix for element b , assuming the same material properties and thickness as element a . Hence, evaluate the assembled stiffness matrix for the continuum.

Solution

(a) Figure 9.44 shows suitable node labelling for a single triangular membrane element. The resulting element stiffness matrix from the previous Example, 9.5, can be utilised. A specimen evaluation of an element stiffness term is given below for k_{11} . The rest are obtained by following the same procedure.

$$\begin{aligned}
 k_{11} &= \frac{Et}{4a(1-\nu^2)} [y_{32}^2 + x_{32}^2(1-\nu)/2] \\
 &= \frac{Et}{4a(1-\nu^2)} [(y_3 - y_2)^2 + (x_3 - x_2)^2(1-\nu)/2] \\
 \text{Substituting} \quad &= \frac{200 \times 10^9 \times 1 \times 10^{-3}}{4 \times 2(1 - 0.3^2)} [(2 - 0)^2 + (0 - 2)^2(1 - 0.3)/2] \\
 &= 14.835 \times 10^7 \text{ N/m}
 \end{aligned}$$

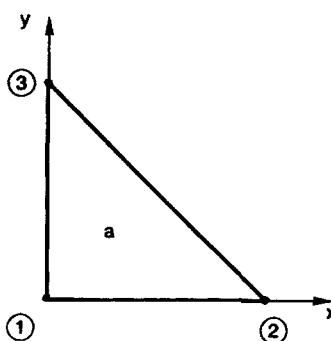


Fig. 9.44.

Evaluation of all the terms leads to the required triangular membrane element stiffness matrix for element *a*, namely

$$[k^{(a)}] = 10^7 [\text{N/m}] \begin{bmatrix} 14.835 & & & & \\ -10.989 & 10.989 & & & \text{symmetric} \\ -3.846 & 0 & 3.846 & & \\ 7.143 & -3.297 & -3.846 & 14.835 & \\ -3.846 & 0 & 3.846 & -3.846 & 3.846 \\ -3.297 & 3.297 & 0 & -10.989 & 0 & 10.989 \end{bmatrix}$$

(b) Element *b* can temporarily also be labelled with node numbers 1, 2 and 3, as element *a*. To avoid confusion, this is best done with the elements shown “exploded”, as in Fig. 9.45. The alternative is to re-number the subscripts in the element stiffness matrix result from Example 9.5.

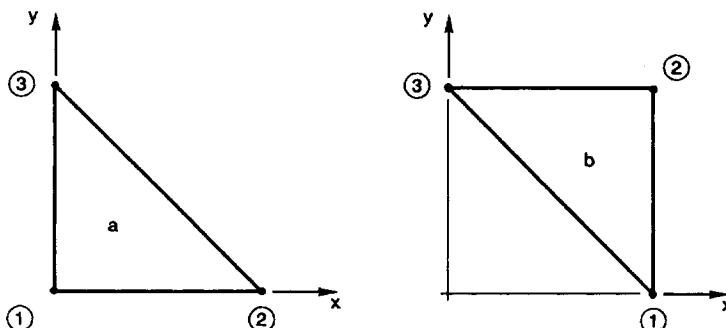


Fig. 9.45.

Performing the evaluations similar to part (a) leads to the required stiffness matrix for element *b*, namely

$$[k^{(b)}] = 10^7 [\text{N/m}] \begin{bmatrix} 3.846 & & & & \\ -3.846 & 14.835 & & & \text{symmetric} \\ 0 & -10.989 & 10.989 & & \\ 0 & -3.297 & 3.297 & 10.989 & \\ -3.846 & 7.143 & -3.297 & -10.989 & 14.835 \\ 3.846 & -3.846 & 0 & 0 & -3.846 & 3.846 \end{bmatrix}$$

With reference to §9.10, the structural stiffness matrix can now be assembled using a dof. correspondence table. The order of the structural stiffness matrix will be 8×8 , corresponding to four nodes, each having 2 dof. The dof. sequence, $u_1, u_2, u_3, v_1, v_2, v_3$, adopted for the convenience of inverting matrix $[A]$, covered in §9.9, can be converted to the more usual sequence, i.e. $u_1, v_1, u_2, v_2, u_3, v_3$, with the aid a dof. correspondence table. Whilst this re-sequencing is optional, the converted sequence is likely to result in less rearrangement of rows and columns, prior to partitioning the assembled stiffness matrix, than would otherwise be needed.

If row and column interchanges are to be avoided in solving the following Example, 9.7, and therefore save some effort, then the dof. labelling of Fig. 9.46 is recommended. This implies the final node numbering, also shown.

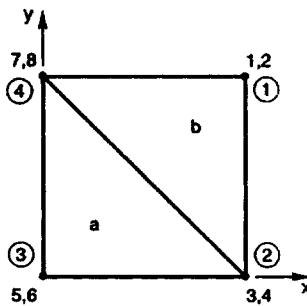


Fig. 9.46.

The dof. correspondence table will be as follows:

Row/column in $[k^{(e)}]$		1	2	3	4	5	6
Row/column in $[K]$	a	5	3	7	6	4	8
	b	3	1	7	4	2	8

Assembling the structural stiffness matrix, gives

$$[K] = 10^7 [\text{N/m}^2] \begin{bmatrix} -14.835 & 7.143 & -3.846 & -3.297 & 0 & 0 & -10.989 & -3.846 \\ 7.143 & 14.835 & -3.846 & -10.989 & 0 & 0 & -3.297 & -3.846 \\ -3.846 & -3.846 & 3.846 & 0 & -10.989 & -3.297 & 0 & 3.846 \\ -3.297 & -10.989 & 0 & 10.989 & 0 & 0 & 3.297 & 0 \\ 0 & 0 & 3.846 & -3.846 & -3.846 & 3.846 & 0 & 0 \\ -10.989 & -10.989 & -3.846 & 14.835 & 7.143 & -3.846 & -3.297 & -10.989 \\ -3.297 & -3.297 & -3.846 & 7.143 & 14.835 & -3.846 & 0 & -10.989 \\ 0 & 0 & 3.846 & -3.846 & -3.846 & 3.846 & 0 & 0 \\ -10.989 & -3.297 & 0 & 3.297 & 0 & 10.989 & 0 & 0 \\ 3.297 & 0 & 0 & -3.297 & -10.989 & 0 & 0 & 10.989 \\ -3.846 & -3.846 & 3.846 & 0 & 0 & 0 & 0 & 3.846 \end{bmatrix}$$

Summing the element stiffness contributions, and writing the structural governing equations, gives the result as

$$\begin{bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \\ X_3 \\ Y_3 \\ X_4 \\ Y_4 \end{bmatrix} = 10^7 [\text{N/m}] \begin{bmatrix} 14.835 & 7.143 & -3.846 & -3.297 & 0 & 0 & -10.989 & -3.846 \\ 7.143 & 14.835 & -3.846 & -10.989 & 0 & 0 & -3.297 & -3.846 \\ -3.846 & -3.846 & 14.835 & 0 & -10.989 & -3.297 & 0 & 7.143 \\ -3.297 & -10.989 & 0 & 14.835 & -3.846 & -3.846 & 7.143 & 0 \\ 0 & 0 & -10.989 & -3.846 & 14.835 & 7.143 & -3.846 & -3.297 \\ 0 & 0 & -3.297 & -3.846 & 7.143 & 14.835 & -3.846 & -10.989 \\ -10.989 & -3.297 & 0 & 7.143 & -3.846 & -3.846 & 14.835 & 0 \\ -3.846 & -3.846 & 7.143 & 0 & -3.297 & -10.989 & 0 & 14.835 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

i.e. $\{P\} = [K]\{p\}$

where $[K]$ is the required assembled stiffness matrix.

Example 9.7

Figure 9.47 shows a 1 mm thick sheet of steel, one edge of which is fully restrained whilst the opposite edge is subjected to a uniformly distributed tension of total value 40 kN. For the material Young's modulus, $E = 200 \text{ GN/m}^2$ and Poisson's ratio, $\nu = 0.3$, and plane stress condition can be assumed.

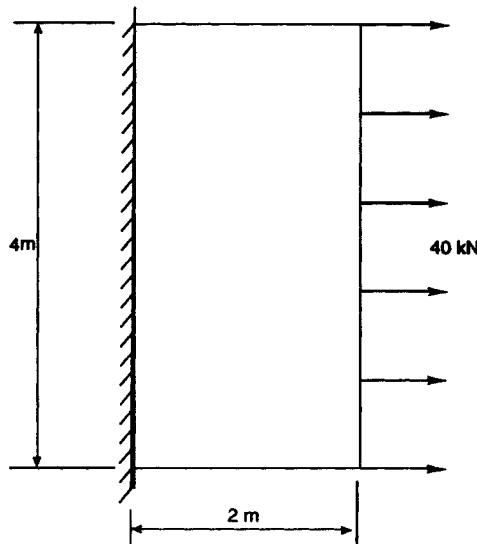


Fig. 9.47.

- (a) Taking advantage of any symmetry, using two triangular membrane elements and hence the assembled stiffness matrix derived for the previous Example, 9.6, determine the nodal displacements in global coordinates.
 (b) Determine the corresponding element principal stresses and their directions and illustrate these on a sketch of the continuum.

Solution

(a) Advantage can be taken of the single symmetry by modelling only half of the continuum. Figure 9.48 shows suitable node and dof. labelling, and division of the upper half of the continuum into two triangular membrane elements. Reference to the previous Example, 9.6, will reveal that the assembled stiffness matrix derived in answering this question can, conveniently, be utilised in solving the current example.

To simulate the clamped edge, dofs. 5 to 8 need to be suppressed, i.e. $u_3 = v_3 = u_4 = v_4 = 0$. Additionally, whilst node number 2 should be unrestrained in the x -direction, freedom in the y -direction needs to be suppressed to simulate the symmetry condition, i.e. $v_2 = 0$. Applying these boundary conditions and hence partitioning the structural stiffness matrix result from Example 9.6, gives the reduced equations as

$$\begin{bmatrix} X_1 \\ Y_1 \\ X_2 \end{bmatrix} = 10^7 [\text{N/m}] \begin{bmatrix} 14.835 & 7.143 & -3.846 \\ 7.143 & 14.835 & -3.846 \\ -3.846 & -3.846 & 14.835 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \end{bmatrix} \text{ i.e. } \{P_\alpha\} = [K_{\alpha\alpha}]\{p_\alpha\}$$

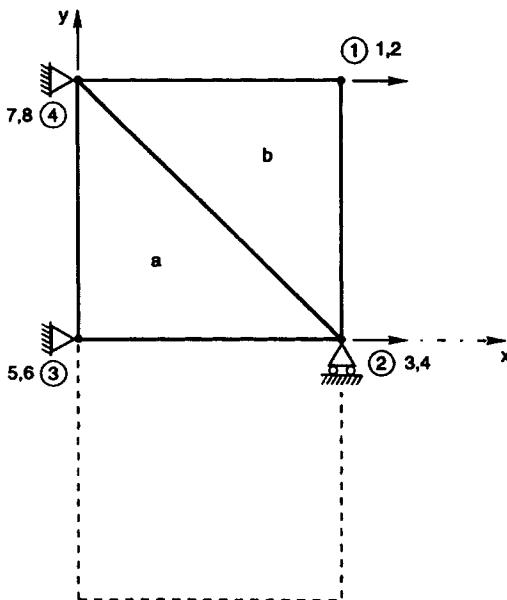


Fig. 9.48.

Inverting $[K_{\alpha\alpha}]$ to enable a solution for the displacements from $\{p_\alpha\} = [K_{\alpha\alpha}]^{-1}\{P_\alpha\}$

$$\text{where } \text{adj } [K_{\alpha\alpha}] = 10^{14} \begin{bmatrix} 205.286 & -91.175 & 29.583 \\ -91.175 & 205.286 & 29.583 \\ 29.583 & 29.583 & 169.055 \end{bmatrix}$$

$$\text{and } \det [K_{\alpha\alpha}] = 10^{21}[14.835(205.286) - 7.143(91.175) - 3.846(29.583)] = 2280.4 \times 10^{21}$$

$$\text{Then } [K_{\alpha\alpha}]^{-1} = 10^{-10} \begin{bmatrix} 90.03 & -39.98 & 12.97 \\ -39.98 & 90.03 & 12.97 \\ 12.97 & 12.97 & 74.13 \end{bmatrix}$$

With reference to §9.4.7, the nodal load column matrix corresponding to a uniformly distributed load of 10 kN/m, will be given by

$$\{P_\alpha\} = \begin{bmatrix} X_1 \\ Y_1 \\ X_2 \end{bmatrix} = 10^3 \begin{bmatrix} 10 \\ 0 \\ 10 \end{bmatrix}_{[N]}$$

Hence, the nodal displacements are found from

$$\{p_\alpha\} = [K_{\alpha\alpha}]^{-1}\{P_\alpha\}$$

Substituting,

$$\begin{bmatrix} u_1 \\ v_1 \\ u_2 \end{bmatrix} = 10^{-10} \begin{bmatrix} 90.03 & -39.98 & 12.97 \\ -39.98 & 90.03 & 12.97 \\ 12.97 & 12.97 & 74.13 \end{bmatrix} 10^3 \begin{bmatrix} 10 \\ 0 \\ 10 \end{bmatrix} = 10^{-6} \begin{bmatrix} 103 \\ -27 \\ 87 \end{bmatrix}_m = \begin{bmatrix} 0.103 \\ -0.027 \\ 0.087 \end{bmatrix}_{mm}$$

The required nodal displacements are therefore $u_1 = 0.103$ mm, $v_1 = -0.027$ mm and $u_2 = 0.087$ mm.

(b) With reference to §9.9, element direct and shearing stresses are found from

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = [D][B] \begin{bmatrix} u_i \\ u_j \\ u_k \\ v_i \\ v_j \\ v_k \end{bmatrix}$$

where, from Example 9.5,

$$[D][B] = \frac{E}{2a(1-v^2)}$$

$$\begin{bmatrix} y_{23} & y_{31} & y_{12} & vx_{32} & vx_{13} & vx_{21} \\ vy_{23} & vy_{31} & vy_{12} & x_{32} & x_{13} & x_{21} \\ \frac{(l-v)}{2}x_{32} & \frac{(l-v)}{2}x_{13} & \frac{(l-v)}{2}x_{21} & \frac{(l-v)}{2}y_{23} & \frac{(l-v)}{2}y_{31} & \frac{(l-v)}{2}y_{12} \end{bmatrix}$$

Evaluating the stresses for each element:

Element a

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{200 \times 10^9}{2 \times 2(1-0.3^2)} \begin{bmatrix} -2 & 2 & 0 & -0.6 & 0 & 0.6 \\ -0.6 & 0.6 & 0 & -2 & 0 & 2 \\ -0.7 & 0 & 0.7 & -0.7 & 0.7 & 0 \end{bmatrix} \begin{bmatrix} 87 \times 10^{-6} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 9.56 \times 10^6 \\ 2.87 \times 10^6 \\ 0 \end{bmatrix}_{\text{N/m}^2}$$

The required principal stresses for element *a* are therefore $\sigma_1 = 9.56$ MN/m² (T) and $\sigma_2 = 2.87$ MN/m² (T), and are illustrated in Fig. 9.49.

Element b

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{200 \times 10^9}{2 \times 2(1-0.3^2)} \begin{bmatrix} 0 & 2 & -2 & -0.6 & 0.6 & 0 \\ 0 & 0.6 & -0.6 & -2 & 2 & 0 \\ -0.7 & 0.7 & 0 & 0 & 0.7 & -0.7 \end{bmatrix} \begin{bmatrix} 87 \times 10^{-6} \\ 103 \times 10^{-6} \\ 0 \\ 0 \\ -27 \times 10^{-6} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 10.43 \times 10^6 \\ 0.43 \times 10^6 \\ 2.92 \times 10^6 \end{bmatrix}_{\text{N/m}^2}$$

The principal stresses are found from

$$\sigma_1, \sigma_2 = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) \pm \frac{1}{2}\sqrt{[(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2]}$$

Substituting gives

$$\begin{aligned}\sigma_1, \sigma_2 &= \left\{ \frac{1}{2}(10.43 + 0.43) \pm \frac{1}{2}\sqrt{[(10.43 - 0.43)^2 + 4 \times 2.92^2]} \right\} 10^6 \text{ N/m}^2 \\ &= (5.43 \pm 5.79) 10^6 \text{ N/m}^2\end{aligned}$$

giving $\sigma_1 = 11.22 \text{ MN/m}^2$ (T) and $\sigma_2 = 0.36 \text{ MN/m}^2$ (C)

The directions are found from

$$\theta = \frac{1}{2} \tan^{-1} [2\sigma_{xy}/(\sigma_{xx} - \sigma_{yy})]$$

substituting gives

$$\theta = \frac{1}{2} \tan^{-1} [2 \times 2.92 / (10.43 - 0.43)] = 15.14^\circ$$

The required principal stresses for element *b* are therefore $\sigma_1 = 11.22 \text{ MN/m}^2$ (T) and $\sigma_2 = 0.36 \text{ MN/m}^2$ (C) and are illustrated in Fig. 9.49.

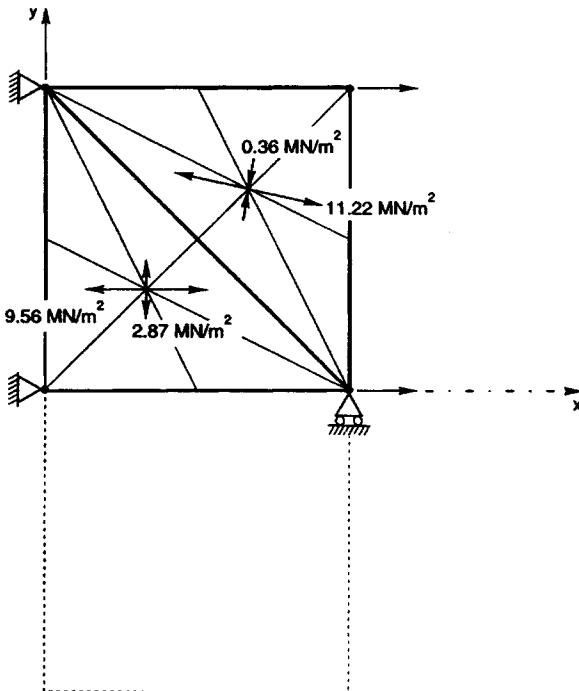


Fig. 9.49.

Problems

9.1 Figure 9.50 shows a support structure in the form of a pin-jointed plane frame, all three members of which are steel, of the same uniform cross-sectional area and length, such that $AE/L = 200 \text{ kN/m}$, throughout.

- (a) Using the displacement based finite element method and treating each member as a rod, determine the nodal displacements with respect to global coordinates for the frame shown in Fig. 9.50.

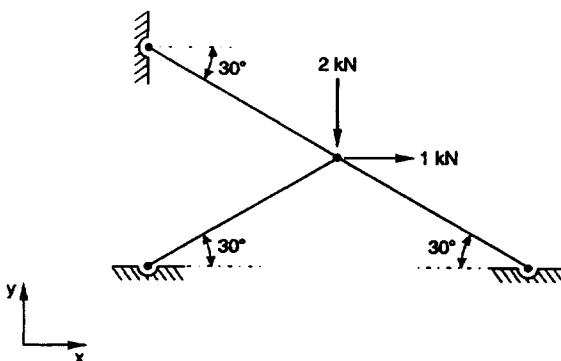


Fig. 9.50.

(b) Hence, determine the nodal reactions.

$$[-0.387, -13.557 \text{ mm}, -1.116, 0.644, 1.233, 0.712, -1.116, 0.644 \text{ kN}]$$

9.2 Figure 9.51 shows a roof truss, all members of which are made from steel, and have the same cross-sectional area, such that $AE = 10 \text{ MN}$, throughout. For the purpose of analysis the truss can be treated as a pin-jointed plane frame. Using the displacement based finite element method, taking advantage of any symmetry and redundancies and treating each member as a rod element, determine the nodal displacements with respect to global coordinates.

$$[0.516, -2.280, -2.313 \text{ mm}]$$

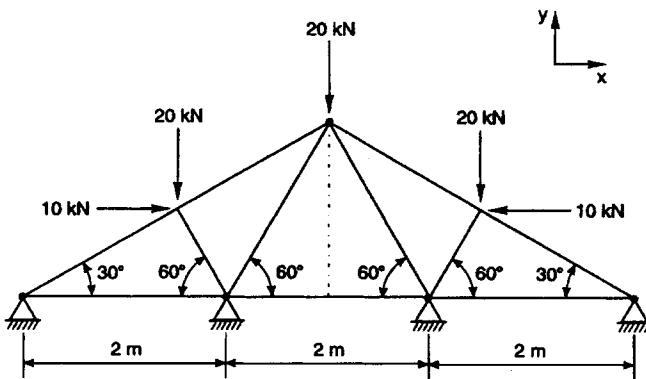


Fig. 9.51.

9.3 A recovery vehicle towing jib is shown in Fig. 9.52. It can be assumed that the jib can be idealised as a pin-jointed plane frame, with all three members made from steel, of the same uniform cross-sectional area, such that $AE = 40 \text{ MN}$, throughout. Using the displacement based finite element method and treating each member as a rod, determine the maximum load P which can be exerted whilst limiting the resultant maximum deflection to 10 mm.

$$[36.5 \text{ kN}]$$

9.4 A hoist frame, arranged as shown in Fig. 9.53, comprises uniform steel members, each 1m long for which $AE = 200 \text{ MN}$, throughout.

- Using the displacement based finite element method and assuming the frame members to be planar and pin-jointed, determine the nodal displacements with respect to global coordinates for the frame loaded as shown.
- Hence, determine the corresponding nodal reactions.

$$[0.5, 0.75, -0.722 \text{ mm}, 0, 86.6, -25.0, -43.3 \text{ kN}]$$

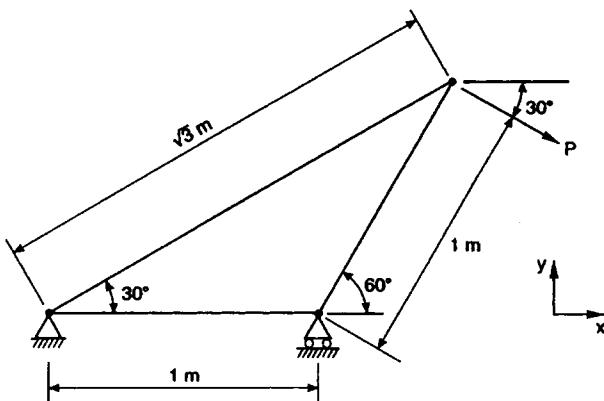


Fig. 9.52.

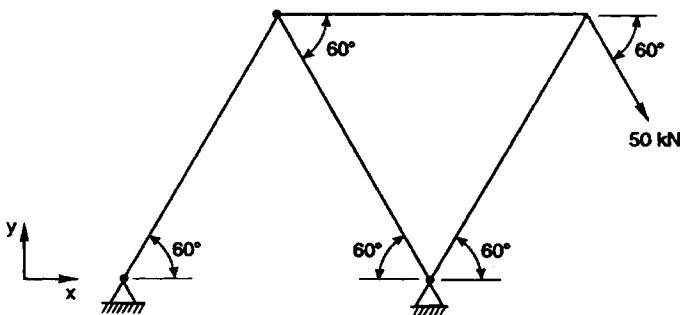


Fig. 9.53.

9.5 Figure 9.54 shows a towing "bracket" for a motor vehicle. Both members are made from uniform cylindrical section steel tubing, of outside diameter 40 mm, cross-sectional area $2.4 \times 10^{-4} \text{ m}^2$, relevant second moment of area $4.3 \times 10^{-8} \text{ m}^4$ and Young's modulus 200 GN/m². Using the displacement based finite element method, a simple beam element representation and assuming both members and load are coplanar:

- (a) assemble the necessary terms in the structural stiffness matrix;
- (b) hence, for the idealisation shown in Fig. 9.54, determine (i) the nodal displacements with respect to global coordinates, and (ii) the resultant maximum stresses at the built-in ends and at the common junction.

$$[0.421 \text{ mm}, 0.103^\circ, 131.34, 106.77, 127.20, 110.91 \text{ MN/m}^2]$$

9.6 A stepped steel shaft supports a pulley, as shown in Fig. 9.55, is rigidly built-in at one end and is supported in a bearing at the position of the step. The bearing provides translational but not rotational restraint. Young's modulus for the material is 200 GN/m².

- (a) Using the displacement based finite element method obtain expressions for the nodal displacements in global coordinates, using a two-beam model.
- (b) Given that, because of a design requirement, the angular misalignment of the bearing cannot exceed 0.5°, determine the maximum load, P , that can be exerted on the pulley.
- (c) Sketch the deformed geometry of the beam.

$$[4.985 \times 10^{-8} P, -2.905 \times 10^{-8} P, 8.475 \times 10^{-7} P, 175 \text{ kN}]$$

9.7 Figure 9.56 shows a chassis out-rigger which acts as a body support for an all-terrain vehicle. The outrigger is constructed from steel channel section rigidly welded at the out-board edge and similarly welded to the vehicle chassis. For the channel material, Young's modulus, $E = 200 \text{ GN/m}^2$, relevant second moment of area, $I = 2 \times 10^{-9} \text{ m}^4$ and cross-sectional area, $A = 4 \times 10^{-5} \text{ m}^2$. Using the displacement based finite element method, and representing the constituent members as simple beam elements

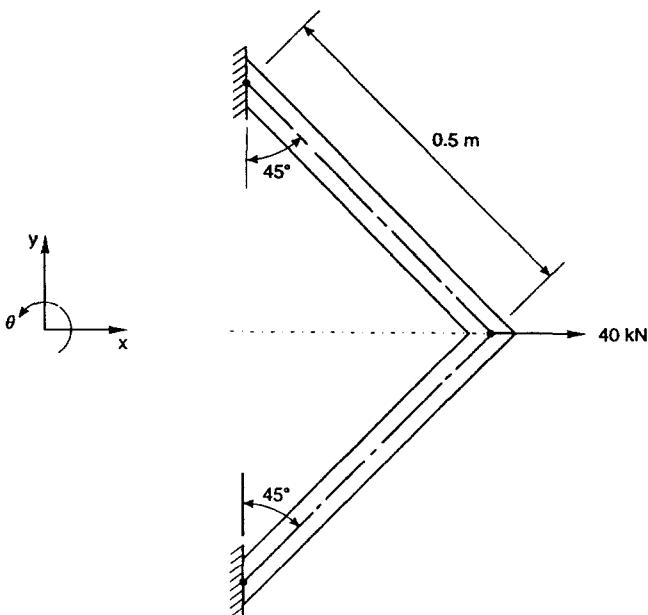


Fig. 9.54.

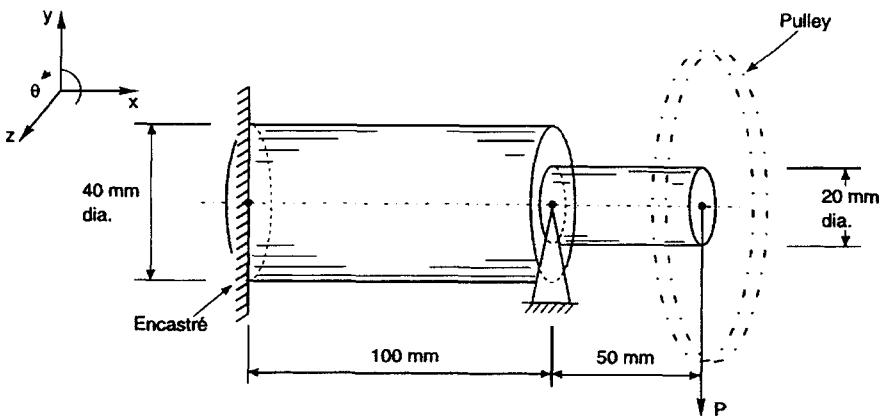


Fig. 9.55.

- (a) determine the nodal displacements with respect to global coordinates.
 (b) A modal analysis reveals that, to avoid resonance, the vertical stiffness of the out-rigger needs to be increased. Assuming only one of the members is to be stiffened, state which member and whether it should be the cross-sectional area or the second moment of area which should be increased, for most effect.

[0.1155, -0.4418 mm, -0.322°, inclined member's csa.]

9.8 The plane frame shown in Fig. 9.57 forms part of a steel support structure. The three members are rigidly connected at the common junction and are built-in at their opposite ends. All three members can be assumed to be axially rigid, and of constant cross-sectional area, A , relevant second moment of area, I , and Young's modulus, E . Using the displacement based finite element method and representing each member as a simple beam:

- (a) show that the angular displacement of the common node, due to application of the moment, M , is given by $ML/8EI$;

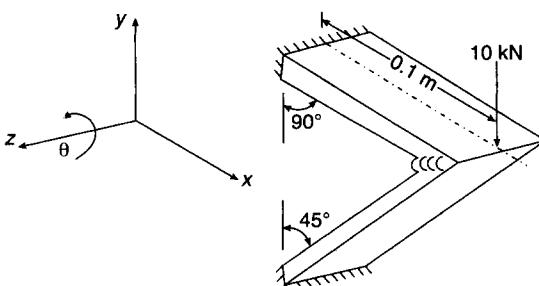


Fig. 9.56.

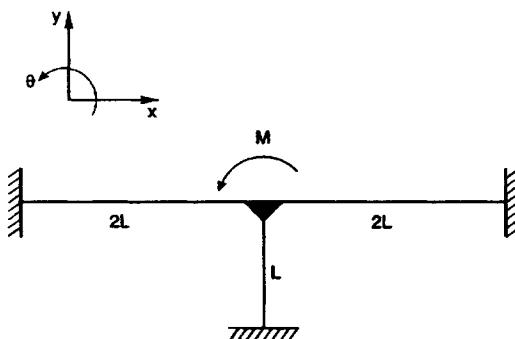


Fig. 9.57.

- (b) determine all nodal reaction forces and moments due to this moment, and represent these reactions on a sketch of the frame. Show that both force and moment equilibrium is satisfied.
 (c) If, due to a manufacturing defect, the joint at the lower end of the vertical member undergoes an angular displacement of $ML/4EI$, whilst all other properties remain unchanged, obtain a new expression for the angular displacement at the common junction.

$[0, -3M/16L, M/8, -3M/4L, 0, 0, 3M/16L, M/8, 3M/4L, 0, M/4, ML/16EI]$

9.9 Using the displacement based finite element method and a three-node triangular membrane element representation, determine the nodal displacements in global coordinates for the continuum shown in Fig. 9.58. Take advantage of any symmetry, assume plane stress conditions and use only two elements in the discretisation. For the material assume Young's modulus, $E = 200 \text{ GN/m}^2$ and Poisson's ratio, $\nu = 0.3$.

$[-3.00 \times 10^{-6}, 10.01 \times 10^{-6}, -3.00 \times 10^{-6}, 10.01 \times 10^{-6} \text{ m}]$

9.10 A crude lifting device is fabricated from a triangular sheet of steel, 6 mm thick, as shown in Fig. 9.59. Assume for the material Young's modulus, $E = 200 \text{ GN/m}^2$ and Poisson's ratio, $\nu = 0.3$, and that plane stress conditions are appropriate.

- (a) Taking advantage of any symmetry, ignoring any instability and using only a single three-node triangular membrane element representation, use the displacement based finite element method to predict the nodal displacements in global coordinates.
 (b) Determine the corresponding element principal stresses and their directions, and show these on a sketch of the element.

$[-0.05, -0.17, -0.60 \text{ mm}, 134.85 \text{ MN/m}^2 (\text{T}) \text{ at } 31.7^\circ \text{ from } x\text{-direction}, 51.50 \text{ MN/m}^2 (\text{C})]$

9.11 The web of a support structure, fabricated from steel sheet 1 mm thick, is shown in Fig. 9.60. Assume for the material Young's modulus, $E = 207 \text{ GN/m}^2$ and Poisson's ratio, $\nu = 0.3$, and that plane stress conditions are appropriate.

- (a) Neglecting any stiffening effects of adjoining members and any instability and using only a single three-node triangular membrane element representation, use the displacement based finite element method to predict the nodal displacements with respect to global coordinates.

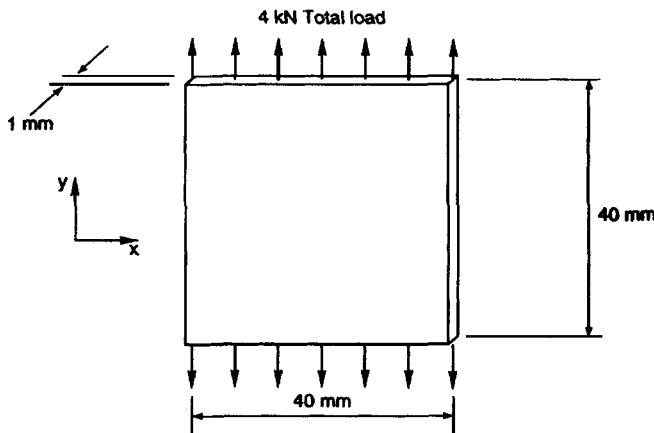


Fig. 9.58.

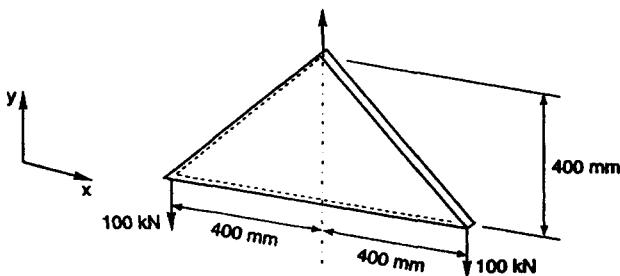


Fig. 9.59.

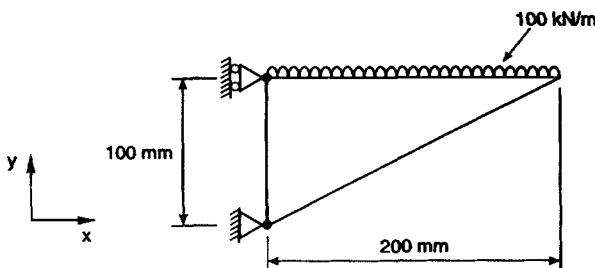


Fig. 9.60.

- (b) Determine the corresponding element principal stresses and their directions, and show these on a sketch of the web.

[0.058, -0.60, -0.10 mm, 123.6 MN/m² (T) at -31.7° from x -direction, 323.6 MN/m² (C)]

- 9.12 Derive the stiffness matrix in global coordinates for a three-node triangular membrane element for plane strain conditions. Assume the displacement functions are the same as those of Example 9.5.

CHAPTER 10

CONTACT STRESS, RESIDUAL STRESS AND STRESS CONCENTRATIONS

Summary

The maximum pressure p_0 or compressive stress σ_c at the centre of contact between two curved surfaces is:

$$p_0 = -\sigma_c = \frac{3P}{2\pi ab}$$

where a and b are the major and minor axes of the Hertzian contact ellipse and P is the total load.

For *contacting parallel cylinders* of length L and radii R_1 and R_2 ,

$$\text{maximum compressive stress, } \sigma_c = -0.591 \sqrt{\frac{P \left(\frac{1}{R_1} + \frac{1}{R_2} \right)}{L \Delta}} = -p_0$$

$$\text{with } \Delta = \frac{1}{E_1} [1 - \nu_1^2] + \frac{1}{E_2} [1 - \nu_2^2]$$

and the maximum shear stress, $\tau_{\max} = 0.295 p_0$ at a depth of $0.786b$ beneath the surface, with:

contact width,

$$b = 1.076 \sqrt{\frac{P \Delta}{L \left(\frac{1}{R_1} + \frac{1}{R_2} \right)}}$$

For *contacting spheres* of radii R_1 and R_2

$$\text{maximum compressive stress, } \sigma_c = -0.62 \sqrt[3]{\frac{P}{\Delta^2} \left[\frac{1}{R_1} + \frac{1}{R_2} \right]^2} = -p_0$$

maximum shear stress, $\tau_{\max} = 0.31 p_0$ at a depth of $0.5b$ beneath the surface with:

contact width (circular)

$$b = 0.88 \sqrt[3]{\frac{P \Delta}{\left(\frac{1}{R_1} + \frac{1}{R_2} \right)}}$$

For a *sphere on a flat surface* of the same material

$$\text{maximum compressive stress, } \sigma_c = -0.62 \sqrt[3]{\frac{PE^2}{4R^2}}$$

For a *sphere in a spherical seat* of the same material

$$\text{maximum compressive stress, } \sigma_c = -0.62 \sqrt[3]{PE^2 \left[\frac{R_2 - R_1}{R_1 R_2} \right]^2}$$

For *spur gears*

$$\text{maximum contact stress, } \sigma_c = -0.475\sqrt{K}$$

with

$$K = \frac{W}{F_w d} \left[\frac{m+1}{m} \right]$$

with W = tangential driving load; F_w = face width; d = pinion pitch diameter; m = ratio of gear teeth to pinion teeth.

For *helical gears*

$$\text{maximum contact stress, } \sigma_c = -C \sqrt{\frac{K}{m_p}}$$

where m_p is the profile contact ratio and C a constant, both given in Table 10.2.

$$\text{Elastic stress concentration factor } K_t = \frac{\text{maximum stress, } \sigma_{\max}}{\text{nominal stress, } \sigma_{\text{nom}}}$$

$$\text{Fatigue stress concentration factor } K_f = \frac{S_n \text{ for the unnotched material}}{S_n \text{ for notched material}}$$

with S_n the endurance limit for n cycles of load.

$$\text{Notch sensitivity factor } q = \frac{K_f - 1}{K_t - 1}$$

or, in terms of a significant linear dimension (e.g. fillet radius) R and a material constant a

$$q = \frac{1}{(1 + a/R)}$$

$$\text{Strain concentration factor } K_\varepsilon = \frac{\text{max. strain at notch}}{\text{nominal strain at notch}}$$

Stress concentration factor K_p in presence of plastic flow is related to K_ε by Neuber's rule

$$K_p K_\varepsilon = K_t^2$$

10.1. Contact Stresses

Introduction

The design of components subjected to contact, i.e. local compressive stress, is extremely important in such engineering applications as bearings, gears, railway wheels and rails, cams, pin-jointed links, etc. Whilst in most other types of stress calculation it is usual to neglect local deflection at the loading point when deriving equations for stress distribution in general bodies, in contact situations, e.g. the case of a circular wheel on a flat rail, such an assumption

would lead to infinite values of compressive stress (load \div "zero" area = infinity). This can only be avoided by local deflection, even yielding, of the material under the load to increase the bearing area and reduce the value of the compressive stress to some finite value.

Contact stresses between curved bodies in compression are often termed "Hertzian" contact stresses after the work on the subject by Hertz⁽¹⁾ in Germany in 1881. This work was concerned primarily with the evaluation of the maximum compressive stresses set up at the mating surfaces for various geometries of contacting body but it formed the basis for subsequent extension of consideration by other workers of stress conditions within the whole contact zone both at the surface and beneath it. It has now been shown that the strength and load-carrying capacity of engineering components subjected to contact conditions is not completely explained by the Hertz equations by themselves, but that further consideration of the following factors is an essential additional requirement:

(a) Local yielding and associated residual stresses

Yield has been shown to initiate sub-surface when the contact stress approaches $1.2 \sigma_y$ (σ_y being the yield stress of the contacting materials) with so-called "uncontained plastic flow" commencing when the stress reaches $2.8 \sigma_y$. Only at this point will material "escape" at the sides of the contact region. The ratio of loads to produce these two states is of the order of 350 although tangential (sliding) forces will reduce this figure significantly.

Unloading from any point between these two states produces a thin layer of residual tension at the surface and a sub-surface region of residual compression parallel to the surface. The residual stresses set up during an initial pass or passes of load can inhibit plastic flow in subsequent passes and a so-called "shakedown" situation is reached where additional plastic flow is totally prevented. Maximum contact pressure for shakedown is given by Johnson⁽¹⁴⁾ as $1.6 \sigma_y$.

(b) Surface shear loading caused by mutual sliding of the mating surfaces

Pure rolling of parallel cylinders has been considered by Radzimovsky⁽⁵⁾ whilst the effect of tangential shear loading has been studied by Deresiewicz⁽¹⁵⁾, Johnson⁽¹⁶⁾, Lubkin⁽¹⁷⁾, Mindlin⁽¹⁸⁾, Tomlinson⁽¹⁹⁾ and Smith and Liu⁽²⁰⁾.

(c) Thermal stresses and associated material property changes resulting from the heat set up by sliding friction. (Local temperatures can rise to some 500°F above ambient).

A useful summary of the work carried out in this area is given by Lipson & Juvinal⁽²¹⁾.

(d) The presence of lubrication – particularly hydrodynamic lubrication – which can greatly modify the loading and resulting stress distribution

The effects of hydrodynamic lubrication on the pressure distribution at contact (see Fig. 10.1) and resulting stresses have been considered by a number of investigators including Meldahl⁽²²⁾, M'Ewen⁽⁴⁾, Dowson, Higginson and Whitaker⁽²³⁾, Crook⁽²⁴⁾, Dawson⁽²⁵⁾ and

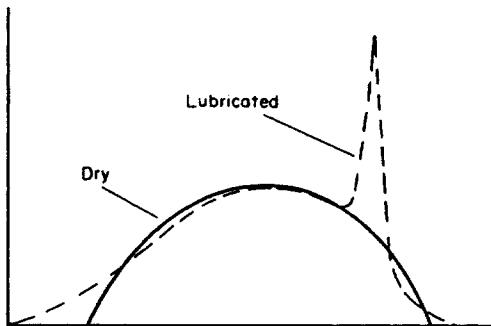


Fig. 10.1. Comparison of pressure distributions under dry and lubricated contact conditions.

Scott⁽²⁶⁾. One important conclusion drawn by Dowson *et al.* is that at high load and not excessive speeds hydrodynamic pressure distribution can be taken to be basically Hertzian except for a high spike at the exit side.

(e) The presence of residual stresses at the surface of e.g. hardened components and their distribution with depth

In discussion of the effect of residually stressed layers on contact conditions, Sherratt⁽²⁷⁾ notes that whilst the magnitude of the residual stress is clearly important, the depth of the residually stressed layer is probably even more significant and the biaxiality of the residual stress pattern also has a pronounced effect. Considerable dispute exists even today about the origin of contact stress failures, particularly of surface hardened gearing, and the aspect is discussed further in §10.1.6 on gear contact stresses.

Muro⁽²⁸⁾, in X-ray studies of the residual stresses present in hardened steels due to rolling contact, identified a compressive residual stress peak at a depth corresponding to the depth of the maximum shear stress – a value related directly to the applied load. He therefore concluded that residual stress measurement could form a useful load-monitoring tool in the analysis of bearing failures.

Detailed consideration of these factors and even of the Hertzian stresses themselves is beyond the scope of this text. An attempt will therefore be made to summarise the essential formulae and behaviour mechanisms in order to provide an overall view of the problem without recourse to proof of the various equations which can be found in more advanced treatments such as those referred to below:-

The following special cases attracted special consideration:

- (i) *Contact of two parallel cylinders* – principally because of its application to roller bearings and similar components. Here the Hertzian contact area tends towards a long narrow rectangle and complete solutions of the stress distribution are available from Belajef⁽²⁾, Foppl⁽³⁾, M'Ewen⁽⁴⁾ and Radzimovsky⁽⁵⁾.
- (ii) *Spur and helical gears* – Buckingham⁽⁶⁾ shows that the above case of contacting parallel cylinders can be used to fair accuracy for the contact of spur gears and whilst Walker⁽⁷⁾ and Wellauer⁽⁸⁾ show that helical gears are more accurately represented by contacting conical frustra, the parallel cylinder case is again fairly representative.

- (iii) *Circular contact* – as arising in the case of contacting spheres or crossed cylinders. Full solutions are available by Foppl⁽³⁾, Huber⁽⁹⁾ Morton and Close⁽¹⁰⁾ and Thomas and Hoersch⁽¹¹⁾.
- (iv) *General elliptical contact*. Work on this more general case has been extensive and complete solutions exist for certain selected axes, e.g. the axes of the normal load. Authors include Belajef⁽²⁾, Fessler and Ollerton⁽¹²⁾, Thomas and Heorsch⁽¹¹⁾ and Ollerton⁽¹³⁾.

Let us now consider the principal cases of contact loading:-

10.1.1. General case of contact between two curved surfaces

In his study of this general contact loading case, assuming elastic and isotropic material behaviour, Hertz showed that the intensity of pressure between the contacting surfaces could be represented by the elliptical (or, rather, semi-ellipsoid) construction shown in Fig. 10.2.

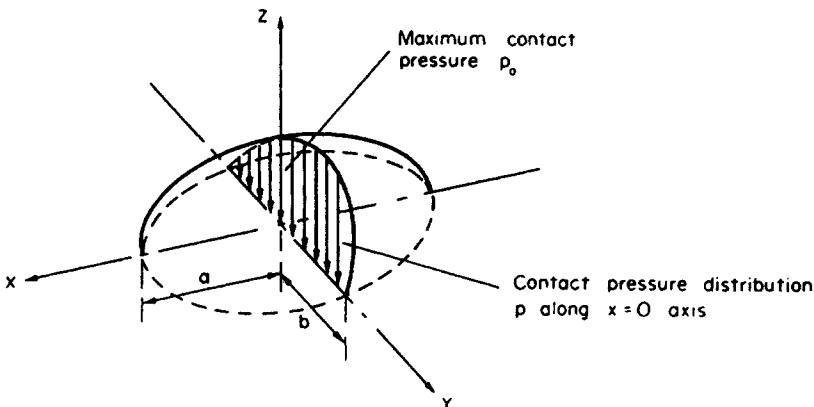


Fig. 10.2. Hertzian representation of pressure distribution between two curved bodies in contact.

If the maximum pressure at the centre of contact is denoted by p_0 then the **pressure at any other point within the contact region** was shown to be given by

$$p = p_0 \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \quad (10.1)$$

where a and b are the major and minor semi-axes, respectively.

The **total contact load** is then given by the volume of the semi-ellipsoid,

$$\text{i.e. } P = \frac{2}{3} \pi a b p_0 \quad (10.2)$$

with the **maximum pressure** p_0 therefore given in terms of the applied load as

$$p_0 = \frac{3P}{2\pi ab} = \text{maximum compressive stress } \sigma_c \quad (10.3)$$

For any given contact load P it is necessary to determine the value of a and b before the maximum contact stress can be evaluated. These are found analytically from equations suggested by Timoshenko and Goodier⁽²⁹⁾ and adapted by Lipson and Juvinal⁽²¹⁾.

i.e.

$$a = m \left[\frac{3P\Delta}{4A} \right]^{1/3} \quad \text{and} \quad b = n \left[\frac{3P\Delta}{4A} \right]^{1/3}$$

with

$$\Delta = \frac{1}{E_1} [1 - v_1^2] + \frac{1}{E_2} [1 - v_2^2]$$

a function of the elastic constants E and v of the contacting bodies and

$$A = \frac{1}{2} \left[\frac{1}{R_1} + \frac{1}{R'_1} + \frac{1}{R_2} + \frac{1}{R'_2} \right]$$

with R and R' the maximum and minimum radii of curvature of the unloaded contact surfaces in two perpendicular planes.

For flat-sided wheels R_1 will be the wheel radius and R'_1 will be infinite. Similarly for railway lines with head radius R_2 the value of R'_2 will be infinite to produce the flat length of rail.

$$B = \frac{1}{2} \left[\left(\frac{1}{R_1} - \frac{1}{R'_1} \right)^2 + \left(\frac{1}{R_2} - \frac{1}{R'_2} \right)^2 + 2 \left(\frac{1}{R_1} - \frac{1}{R'_1} \right) \left(\frac{1}{R_2} - \frac{1}{R'_2} \right) \cos 2\psi \right]^{1/2}$$

with ψ the angle between the planes containing curvatures $1/R_1$ and $1/R_2$.

Convex surfaces such as a sphere or roller are taken to be positive curvatures whilst internal surfaces of ball races are considered to be negative.

m and n are also functions of the geometry of the contact surfaces and their values are shown in Table 10.1 for various values of the term $\alpha = \cos^{-1}(B/A)$.

Table 10.1.

α degrees	20	30	35	40	45	50	55	60	65	70	75	80	85	90
m	3.778	2.731	2.397	2.136	1.926	1.754	1.611	1.486	1.378	1.284	1.202	1.128	1.061	1.000
n	0.408	0.493	0.530	0.567	0.604	0.641	0.678	0.717	0.759	0.802	0.846	0.893	0.944	1.000

10.1.2. Special case 1 – Contact of parallel cylinders

Consider the two parallel cylinders shown in Fig. 10.3(a) subjected to a contact load P producing a rectangular contact area of width $2b$ and length L . The contact stress distribution is indicated in Fig. 10.3(b).

The elliptical pressure distribution is given by the two-dimensional version of eqn (10.1)

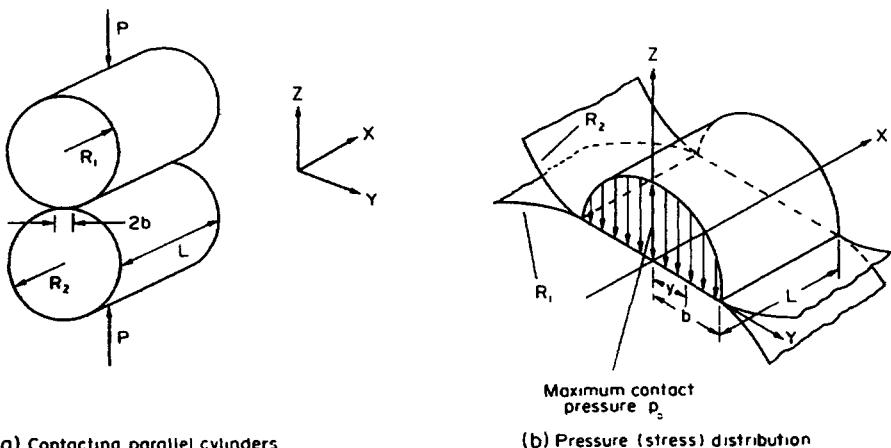
i.e.

$$p = p_0 \sqrt{1 - \frac{y^2}{b^2}} \quad (10.5)$$

The total load P is then the volume of the prism

i.e.

$$P = \frac{1}{2}\pi b L p_0 \quad (10.6)$$



(a) Contacting parallel cylinders

(b) Pressure (stress) distribution

Fig. 10.3. (a) Contact of two parallel cylinders; (b) stress distribution for contacting parallel cylinders.

and the **maximum pressure or maximum compressive stress**

$$p_0 = \sigma_c = \frac{2P}{\pi b L} \quad (10.7)$$

The **contact width** can be related to the geometry of the contacting surfaces as follows:-

$$b = 1.076 \sqrt{\frac{P\Delta}{L\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}} \quad (10.8)$$

giving the **maximum compressive stress** as:

$$\sigma_c = -p_0 = -0.591 \sqrt{\frac{P\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}{L\Delta}} \quad (10.9)$$

(For a flat plate R_2 is infinite, for a cylinder in a cylindrical bearing R_2 is negative). **Stress conditions at the surface on the load axis** are then:

$$\sigma_z = \sigma_c = -p_0$$

$$\sigma_y = -p_0$$

$$\sigma_x = -2vp_0 \quad (\text{along cylinder length})$$

The **maximum shear stress** is:

$$\tau_{\max} = 0.295p_0 \approx 0.3p_0$$

occurring at a depth beneath the surface of $0.786 b$ and on planes at 45° to the load axis.

In cases such as gears, bearings, cams, etc. which (as will be discussed later) can be likened to the contact of parallel cylinders, this shear stress will reduce gradually to zero as the rolling load passes the point in question and rise again to its maximum value as the next

load contact is made. However, this will not be the greatest reversal of shear stress since there is another shear stress on planes parallel and perpendicular to the load axes known as the "alternating" or "reversing" shear stress, at a depth of $0.5 b$ and offset from the load axis by $0.85 b$, which has a maximum value of $0.256 p_0$ which changes from positive to negative as the load moves across contact.

The maximum shear stress on 45° planes thus varies between zero and $0.3 p_0$ (approx) with an alternating component of $0.15 p_0$ about a mean of $0.15 p_0$. The **maximum alternating shear stress**, however, has an alternating component of $0.256 p_0$ about a mean of zero – see Fig. 10.4. The latter is therefore considerably more significant from a fatigue viewpoint.

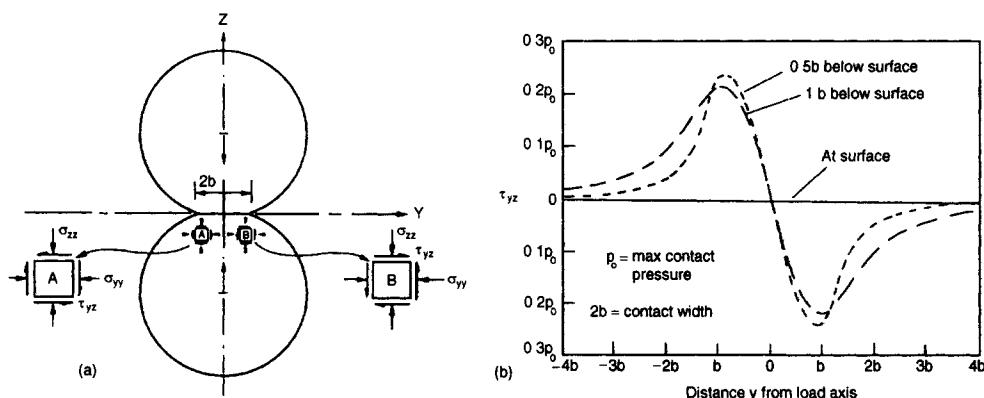


Fig. 10.4. Maximum alternating stress variation beneath contact surfaces.

N.B.: The above formulae assume the length of the cylinders to be very large in comparison with their radii. For short cylinders and/or cylinder/plate contacts with widths less than six times the contact area (or plate thickness less than six times the depth of the maximum shear stress) actual stresses can be significantly greater than those estimated by the given equations.

10.1.3. Combined normal and tangential loading

In normal contact conditions between contacting cylinders, gears, cams, etc. friction will be present reacting the sliding (or tendency to slide) of the mating surfaces. This will affect the stresses which are set up and it is usual in such cases to take the usual relationship between normal and tangential forces in the presence of friction

viz.

$$F = \mu R \quad \text{or} \quad q = \mu p_0$$

where q is the tangential pressure distribution, assumed to be of the same form as that of the normal pressure. Smith and Liu⁽²⁰⁾ have shown that with such an assumption:

- (a) A shear stress now exists on the surface at the contact point introducing principal stresses which are different from σ_x , σ_y and σ_z of the normal loading case.
- (b) The maximum shear stress may exist either at the surface or beneath it depending on whether μ is greater than or less than $1/9$ respectively.

- (c) The stress range in the y direction is increased by almost 90% on the normal loading value and there is also a reversal of sign. A useful summary of stress distributions in graphical form is given by Lipson and Juvinal⁽²¹⁾.

10.1.4. Special case 2 – Contacting spheres

For contacting spheres, eqns. (10.9) and (10.8) become

Maximum compressive stress (normal to surface)

$$\sigma_c = -p_0 = -0.62 \sqrt[3]{\frac{P}{\Delta^2} \left[\frac{1}{R_1} + \frac{1}{R_2} \right]^2} \quad (10.10)$$

with a maximum value of

$$\sigma_c = -1.5P/\pi a^2 \quad (10.11)$$

Contact dimensions (circular)

$$a = b = 0.88 \sqrt{\frac{P \Delta}{\left[\frac{1}{R_1} + \frac{1}{R_2} \right]}} \quad (10.12)$$

As for the cylinder, if contact occurs between one sphere and a flat surface then R_2 is infinite, and if the sphere contacts inside a spherical seating then R_2 is negative.

The other two **principal stresses in the surface plane** are given by:

$$\sigma_x = \sigma_y = -\frac{(1+2\nu)}{2} p_0 \quad (10.13)$$

For steels with Poisson's ratio $\nu = 0.3$ the **maximum shear stress** is then:

$$\tau_{\max} \simeq 0.31 p_0 \quad (10.14)$$

at a depth of half the radius of the contact surface.

The **maximum tensile stress** set up within the contact zone occurs at the edge of the contact zone in a radial direction with a value of:

$$\sigma_{t_{\max}} = \frac{(1-2\nu)}{3} p_0 \quad (10.15)$$

The circumferential stress at the same point is equal in value, but compressive, whilst the stress normal to the surface is effectively zero since contact has ended. With equal and opposite principal stresses in the plane of the surface, therefore, the **material is effectively in a state of pure shear**.

The **maximum octahedral shear stress** which is also an important value in consideration of elastic failure, occurs at approximately the same depth below the surface as the maximum shear stress. Its value may be obtained from eqn (8.24) by substituting the appropriate values of σ_x , σ_y and σ_z found from Fig. 10.5 which shows their variation with depth beneath the surface.

The **relative displacement, e , of the centres of the two spheres** is given by:

$$e = 0.77 \sqrt[3]{P^2 \left(\frac{1}{E_1} + \frac{1}{E_2} \right)^2 \left(\frac{1}{R_1} + \frac{1}{R_2} \right)} \quad (10.16)$$

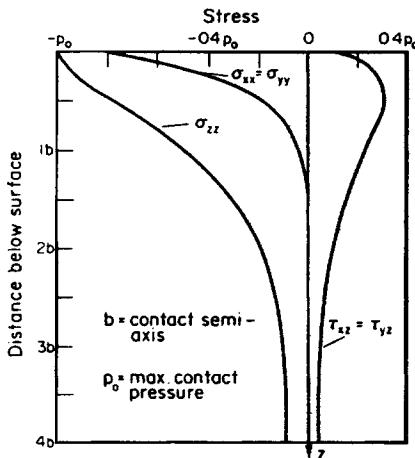


Fig. 10.5. Variation of stresses beneath the surface of contacting spheres.

For a *sphere contacting a flat surface* of the same material $R_2 = \infty$ and $E_1 = E_2 = E$. Substitution in eqns. (10.10) and (10.16) then yields

maximum compressive stress

$$\sigma_c = -0.62 \sqrt[3]{\frac{PE^2}{4R_1^2}} \quad (10.17)$$

and relative displacement of centres

$$e = 1.54 \sqrt[3]{\frac{P^2}{2E^2 R_1}} \quad (10.18)$$

For a *sphere on a spherical seat* of the same material

$$\sigma_c = -0.62 \sqrt[3]{\frac{PE^2}{4R_1 R_2}} \left[\frac{R_2 - R_1}{R_1 R_2} \right]^2 \quad (10.19)$$

with

$$e = 1.54 \sqrt[3]{\frac{P^2}{2E^2}} \left[\frac{R_2 - R_1}{R_1 R_2} \right] \quad (10.20)$$

For other, more general, loading cases the reader is referred to a list of formulae presented by Roark and Young⁽³³⁾.

10.1.5. Design considerations

It should be evident from the preceding sections that the maximum Hertzian compressive stress is not, in itself, a valid criteria of failure for contacting members although it can be used as a valid design guide provided that more critical stress states which have a more direct influence on failure can be related directly to it. It has been shown, for example, that alternating shear stresses exist beneath the surface which are probably critical to fatigue life

but these can be expressed as a simple proportion of the Hertzian pressure p_0 so that p_0 can be used as a simple index of contact load severity.

The contact situation is complicated under real service loading conditions by the presence of e.g. residual stresses in hardened surfaces, local yielding and associated additional residual stresses, friction forces and lubrication, thermal stresses and dynamic (including shock) load effects.

The failure of brittle materials under contact conditions correlates more closely with the maximum tensile stress at the surface rather than sub-surface shear stresses, whilst for static or very slow rolling operations failure normally arises as a result of excessive plastic flow producing indentation ("brinelling") of the surface. In both cases, however, the Hertzian pressure remains a valuable design guide or reference.

By far the greatest number of failures of contacting components remains the surface or sub-surface fatigue initiated type variably known as "pitting", "spalling", "onion-peel spalling" or "flaking". The principal service areas in which this type of failure occurs are gears and bearings.

10.1.6. Contact loading of gear teeth

Figure 10.6 shows the stress conditions which prevail in the region of a typical gear tooth contact. Immediately at the contact point, or centre of contact, there is the usual position of maximum compressive stress (p_0). Directly beneath this, and at a depth of approximately one-third of the contact width, is the maximum shear stress τ_{\max} acting on planes at 45° to the load axis. Between these two positions lies the maximum alternating or reversed shear stress τ_{alt} acting on planes perpendicular and parallel to the surface. Whilst τ_{alt} is numerically smaller than τ_{\max} it alternates between positive and negative values as the tooth proceeds

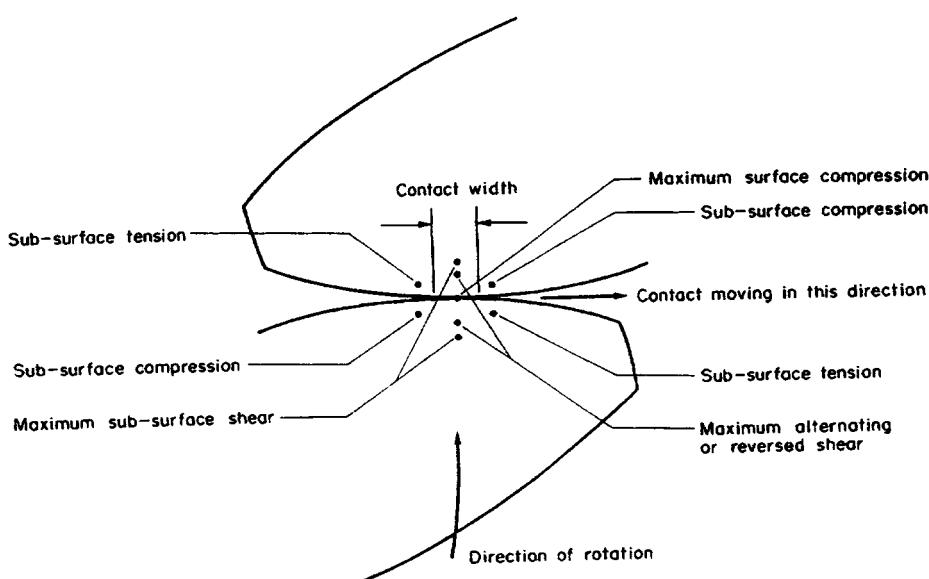


Fig. 10.6. Stress conditions in the region of gear tooth contact.

through mesh giving a stress range greater than that of τ_{\max} which ranges between a single value and zero. It is argued by many that, for this reason, τ_{alt} is probably more significant to fatigue life than τ_{\max} – particularly if its depth relates closely to that of peak residual stresses or case–core junctions of hardened gears.

As the gears rotate there is a combination of rolling and sliding motions, the latter causing additional surface stresses not shown in Fig. 8.6. Ahead of the contact area there is a narrow band of compression and behind the contact area a narrow band of tension. A single point on the surface of a gear tooth therefore passes through a complex variety of stress conditions as it goes through its meshing cycle. Both the surface and alternating stress change sign and other sub-surface stresses change from zero to their maximum value. Add to these fatigue situations the effects of residual stress, lubrication, thermal stresses and dynamic loading and it is not surprising that gears may fail in one of a number of ways either at the surface or sub-surface.

The majority of gear tooth failures are surface failures due to “pitting”, “spalling”, “flaking”, “wear”, etc. the three former modes referring to the fracture and shedding of pieces of various size from the surface. Considerable speculation and diverse views exist even among leading workers as to the true point of origin of some of these failures and considerable evidence has been produced of, apparently, both surface and sub-surface crack initiation. The logical conclusion would therefore seem to be that both types of initiation are possible depending on precisely the type of loading and contact conditions.

A strong body of opinion supports the suggestion of Johnson^(14,16) and Almen⁽³⁰⁾ who attribute contact stress failures to local plastic flow at inclusions or flaws in the material, particularly in situations where a known overload has occurred at some time prior to failure. The overload is sufficient to produce the initial plastic flow and successive cycles then extend the region of plasticity and crack propagation commences. Dawson⁽²⁵⁾ and Akaoka⁽³¹⁾ found evidence of sub-surface cracks running parallel to the surface, some breaking through to the surface, others completely unconnected with it. These were attributed to the fatigue action of the maximum alternating (reversed) shear stress. Undoubtedly, from the evidence presented by other authors, cracks can also initiate at the surface probably producing a “pitting” type of failure, i.e. smaller depth of damage. These cracks are suggested to initiate at positions of maximum tensile stress in the contact surface and subsequent propagation is then influenced by the presence (or otherwise) of lubricant.

In the case of helical gears, three-dimensional photoelastic tests undertaken by the author⁽³²⁾ indicate that maximum sub-surface stresses are considerably greater than those predicted by standard design procedures based on Hertzian contact and uniform loading along the contact line. Considerable non-uniformity of load was demonstrated which, together with dynamic effects, can cause maximum loads and stresses many times above the predicted nominal values. The tests showed the considerable benefit to be gained on the load distribution and resulting maximum stress values by the use of tip and end relief of the helical gear tooth profile.

10.1.7. Contact stresses in spur and helical gearing

Whilst the radius of an involute gear tooth will change slightly across the width of contact with a mating tooth it is normal to ignore this and take the contact of spur gear teeth as equivalent to the contact of parallel cylinders with the same radius of curvature at the point of contact. The Hertzian eqns. (10.8) and (10.9) can thus be applied to **spur gears** and,

for typical steel elastic constant values of $\nu = 0.3$ and $E = 206.8 \text{ GN/m}^2$, the **maximum contact stress** becomes

$$\sigma_c = -p_o = -0.475\sqrt{K} \text{ MN/m}^2 \quad (10.21)$$

where

$$K = \frac{W}{F_w d} \left[\frac{m+1}{m} \right]$$

with W = tangential driving load = pinion torque \div pinion pitch radius

F_w = face width

d = pinion pitch diameter

m = ratio of gear teeth to pinion teeth; the pinion taken to be the smaller of the two mating teeth.

For helical gears, the **maximum contact stress** is given by

$$\sigma_c = -p_o = -C \sqrt{\frac{K}{m_p}} \quad (10.22)$$

where K is the same factor as for spur gears

m_p is the profile contact ratio

C is a constant

the values of m_p and C being found in Table 10.2, for various helix angles and pressure angles.

Table 10.2. Typical values of C and m_p for helical gears.

Pressure angle	Spur		15° Helix		30° Helix		45° Helix	
	C	m_p	C	m_p	C	m_p	C	m_p
14½°	0.546	2.10	0.528	2.01	0.473	1.71	0.386	1.26
17½°	0.502	1.88	0.485	1.79	0.435	1.53	0.355	1.13
20°	0.474	1.73	0.458	1.65	0.410	1.41	0.335	1.05
25°	0.434	1.52	0.420	1.45	0.376	1.25	0.307	0.949

10.1.8. Bearing failures

Considerable care is necessary in the design of bearings when selecting appropriate ball and bearing race radii. If the radii are too similar the area of contact is large and excessive wear and thermal stress (from frictional heating) results. If the radii are too dissimilar then the contact area is very small, local compressive stresses become very high and the load capacity of the bearing is reduced. As a compromise between these extremes the radius of the race is normally taken to be between 1.03 and 1.08 times the ball radius.

Fatigue life tests and service history then indicate that the life of ball bearings varies approximately as the cube of the applied load whereas, for roller bearings, a 10/3 power relationship is more appropriate. These relationships can only be used as a rough "rule of

thumb", however, since commercially produced bearings, even under nominally similar and controlled production conditions, are notorious for the wide scatter of fatigue life results.

As noted previously, the majority of bearing failures are by spalling of the surface and most of the comments given in §10.1.6 relating to gear failures are equally relevant to bearing failures.

10.2. Residual Stresses

Introduction

It is probably true to say that all engineering components contain stresses (of variable magnitude and sign) before being subjected to service loading conditions owing to the history of the material prior to such service. These stresses, produced as a result of mechanical working of the material, heat treatment, chemical treatment, joining procedure, etc., are termed *residual stresses* and they can have a very significant effect on the fatigue life of components. These residual stresses are "locked into" the component in the absence of external loading and represent a datum stress over which the service load stresses are subsequently superimposed. If, by fortune or design, the residual stresses are of opposite sign to the service stresses then part of the service load goes to reduce the residual stress to zero before the combined stress can again rise towards any likely failure value; such residual stresses are thus extremely beneficial to the strength of the component and significantly higher fatigue strengths can result. If, however, the residual stresses are of the same sign as the applied stress, e.g. both tensile, then a smaller service load is required to produce failure than would have been the case for a component with a zero stress level initially; the strength and fatigue life in this case is thus reduced. Thus, both the magnitude and sign of residual stresses are important to fatigue life considerations, and methods for determining these quantities are introduced below.

It should be noted that whilst preceding chapters have been concerned with situations where it has been assumed that stresses are zero at zero load this is not often the case in practice, and great care must be exercised to either fully evaluate the levels of residual stress present and establish their effect on the strength of the design, or steps must be taken to reduce them to a minimum.

Bearing in mind that most loading applications in engineering practice involve fatigue to a greater or less degree it is relevant to note that surface residual stresses are the most critical as far as fatigue life is concerned since, almost invariably, fatigue cracks form at the surface. The work of §11.1.3 indicates that whilst tensile mean stresses promote fatigue crack initiation and propagation, compressive mean stresses are beneficial in that they impede fatigue failure. Compressive residual stresses are thus generally to be preferred (and there is not always a choice of course) if fatigue lives of components are to be enhanced. Indeed, compressive stresses are often deliberately introduced into the surface of components, e.g. by chemical methods which will be introduced below, in order to increase fatigue lives. There are situations, however, where compressive residual stress can be most undesirable; these include potential buckling situations where compressive surface stresses could lead to premature buckling failure, and operating conditions where the service loading stresses are also compressive. In the latter case the combined service and residual stresses may reach a sufficiently high value to exceed yield in compression and produce local plasticity on the first cycle of loading. On unloading, tensile residual stress "pockets" will be formed and

these can act as local stress concentrations and potential fatigue crack initiation positions. Such a situation arises in high-temperature applications such as steam turbines and nuclear plant, and in contact load applications.

Whilst it has been indicated above that tensile residual stresses are generally deleterious to fatigue life there are again exceptions to this “rule”, and very significant ones at that! It is now quite common to deliberately overload structures and components during proof testing to produce plastic flow at discontinuities and other stress concentrations to reduce their stress concentration effect on subsequent loading cycles. Other important techniques which involve the deliberate overloading of components in order to produce residual stress distribution favourable to subsequent loading cycles include “autofrettage” of thick cylinders (see §3.20(a)), “overspeeding” of rotating discs (see §3.20(b)) and pre-stressing of springs (see §3.8).

Whilst engineers have been aware of residual stresses for many years it is only recently that substantial efforts have been made to investigate their magnitudes and distributions with depth in components and hence their influence on performance and service life. This is probably due to the conservatism of old design procedures which generally incorporated sufficiently large safety factors to mask the effects of residual stresses on component integrity. However, with current drives for economy of manufacture coupled with enhanced product safety and reliability, design procedures have become far more stringent and residual stress effects can no longer be ignored. Principally, the designer needs to consider the effect of residual stress on structural or component failure but there is also need for detailed consideration of distortion and stability factors which are also closely related to residual stress levels.

10.2.1. Reasons for residual stresses

Residual stresses generally arise when conditions in the outer layer of a material differ from those internally. This can arise by one of three principal mechanisms: (a) mechanical processes, (b) chemical treatment, (c) heat treatment, although other mechanism are also discussed in the subsequent text.

(a) Mechanical processes

The most significant mechanical processes which induce surface residual stresses are those which involve plastic yielding and hence “cold-working” of the material such as rolling, shot-peening and forging. Practically all other standard machining procedures such as grinding, turning, polishing, etc., also involve local yielding (to a lesser extent perhaps) and also induce residual stresses. Reference should also be made to §3.9 and §3.10 which indicate how residual stresses can be introduced due to bending or torsion beyond the elastic limit.

Cold working

Shot peening is a very popular method for the introduction of favourable compressive residual stresses in the surface of components in order to increase their fatigue life. It is a process whereby small balls of iron or steel shot are bombarded at the component surface at high velocity from a rotating nozzle or wheel. It is applicable virtually to all metals and all

component geometries and so is probably the most versatile of all the mechanical working processes. The bombardment tends to compress the surface layer and thus laterally try to expand it. This lateral expansion at the surface is resisted by the core material and residual compression results, its magnitude depending on the size of shot used and the peening velocity. Typically, residual stresses of the order of half the yield strength of the material are readily obtained, with peak values slightly sub-surface. However, special procedures such as "strain peening" which bombard the surface whilst applying external tensile loads can produce residuals approaching the full yield strength.

The major benefit of shot peening arises in areas of small fillet radii, notches or other high stress gradient situations and on poor surface finishes such as those obtained after rough machining or decarburisation. It is widely used in machine parts produced from high-strength steels and on gears, springs, structural components, engine con-rods and other motor vehicle components when fatigue lives have been shown to have been increased by factors in excess of 100%.

A number of different peening procedures exist in addition to standard shot peening with spherical shot, e.g. needle peening (bombardment by long needles with rounded ends), hammer peening (surface indented with radius tool), roller-burnishing (rolling of undersized hole to required diameter), roto peening (impact of shot-coated flexible flaps). Figure 10.7 shows a typical residual stress distribution produced by shot peening, the maximum residual stress attainable being given by the following "rule of thumb" estimate

$$\sigma_m \approx 500 + (0.2 \times \text{tensile strength})$$

for steels with a tensile strength between 650 MN/m² and 2 GN/m².

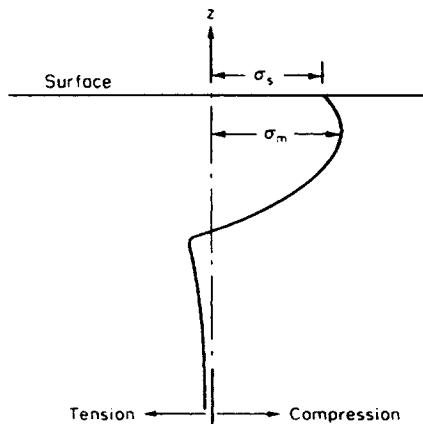


Fig. 10.7. Typical residual stress distribution with depth for the shot-peening process.

For lower-strength steels and alloys σ_m can initially reach the yield stress or 0.1% proof stress but this will fade under cyclic loading.

Cold rolling of threads, crankpins and axles relies on similar principles to those outlined above with, in this case, continuous pressure of the rollers producing controlled amounts of cold working. Further examples of cold working are the bending of pipes and conduits, cold

shaping of brackets and clips and cold drawing of bars and tubes – sometimes of complex cross-section.

In some of the above applications the stress gradient into the material can be quite severe and a measurement technique which can produce results over reasonable depth is essential if residual stress–fatigue life relationships and mechanisms are to be fully understood.

Machining

It has been mentioned above that plastic deformation is almost invariably present in any machining process and the extent of the plastically deformed layer, and hence of the residually stressed region, will depend on the depth of cut, sharpness of tool, rates of speed and feed and the machineability of the material. With sharp tools, the heat generated at the tip of the tool will not have great influence and the residually stressed layer is likely to be compressive and relatively highly localised near the surface. With blunt tools or multi-tipped tools, particularly grinding, much more heat will be generated and if cooling is not sufficient this will produce thermally induced compressive stresses which can easily exceed the tensile stresses applied by the mechanical action of the tool. If they are large enough to exceed yield then tensile residual stresses may arise on cooling and care may need to be exercised in the type and level of service stress to which the component is then subjected. The depth of the residually stressed layer will depend upon the maximum temperatures reached during the machining operation and upon the thermal expansion coefficient of the material but it is likely that it will exceed that due to machining plastic deformation alone.

Residual stresses in manufactured components can often be very high; in grinding, for example, it is quite possible for the tensile residual stresses to produce cracking, particularly sub-surface, and etching techniques are sometimes employed after the grinding of e.g. bearings to remove a small layer on the surface in order to check for grinding damage. Distortion is another product of high residual stresses, produced particularly in welding and other heat treatment processes.

(b) Chemical treatment

The principal chemical treatments which are used to provide components with surface residual stress layers favourable to subsequent service fatigue loading conditions are nitriding, tufftriding and carburising.

Nitriding

Nitriding is a process whereby certain alloy steels are heated to about 550°C in an ammonia atmosphere for periods between 10 and 100 hours. Nitrides form in the surface of the steel with an associated volume increase. The core material resists this expansion and, as a result, residual compressive stresses are set up which can be very high (see Fig. 10.8). The surface layer, which typically is of the order of 0.5 mm thick, is extremely hard and the combination of this with the high surface residual compressive stresses make nitrided components exceptionally resistant to stress concentrations such as surface notches; fatigue lives of nitrided components are thus considerably enhanced over those of the parent material.

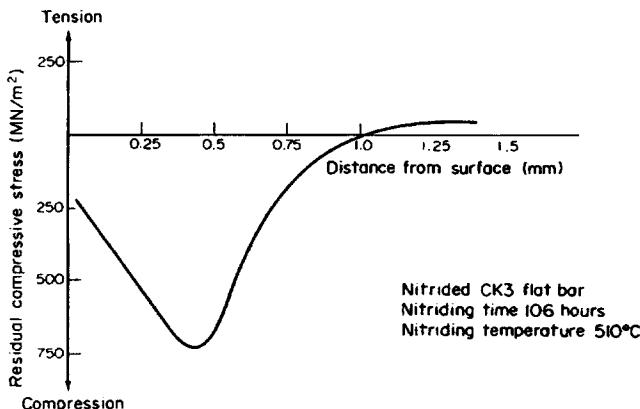


Fig. 10.8. Typical residual stress results for nitrided steel bar using the X-ray technique.

Minimal distortion or warping is produced by the nitriding process and no quenching is required.

Tufftriding

A special version of nitriding known as “tough nitriding” or, simply, “Tufftriding” consists of the heating of steel in a molten cyanide salt bath for approximately 90 minutes to allow nitrogen to diffuse into the steel surface and combine with the iron carbide formed in the outer skin when carbon is also released from the cyanide bath. The product of this combination is carbon-bearing epsilon-iron-nitride which forms a very tough but thin, wear-resistant layer, typically 0.1 mm thick. The process is found to be particularly appropriate for plain medium-carbon steels with little advantage over normal nitriding for the higher-strength alloy steels.

Carburising

Introduction of carbon into surface layers to produce so-called carburising may be carried out by solid, liquid or gaseous media. In each case the parent material contained in the selected medium such as charcoal, liquid sodium cyanide plus soda ash, or neutral gas enriched with propane, is heated to produce diffusion of the carbon into the surface. The depth of hardened case resulting varies from, typically, 0.25 mm on small articles to 0.37 mm on bearings (see Fig. 10.9).

(c) Heat treatment

Unlike chemical treatments, heat treatment procedures do not alter the chemical composition at the surface but simply modify the metallurgical structure of the parent material. Principal heat treatment procedures which induce favourable residual stress layers are induction hardening and flame hardening, although many other processes can also be considered within this category such as flame cutting, welding, quenching and even hot rolling or

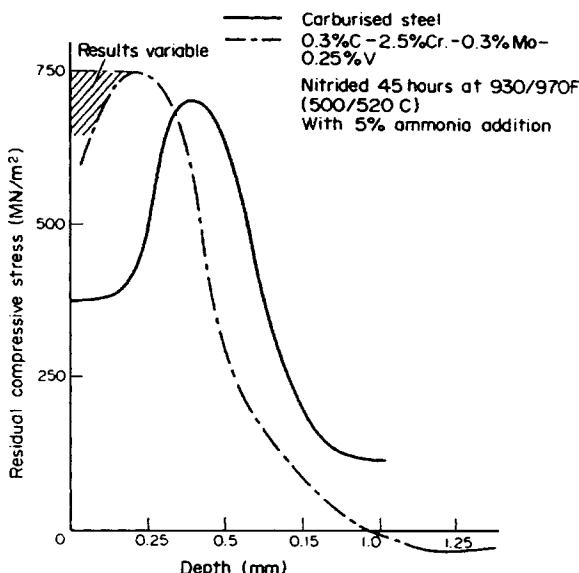


Fig. 10.9. Comparison of residual stress pattern present in nitrided chromium-molybdenum-vanadium steel and in a carburised steel - results obtained using the X-ray technique.

forging. In the two latter cases, however, chemical composition effects are included since carbon is removed from the surface by oxidation. This "decarburising" process produces surface layers with physical properties generally lower than those of the core and it is thus considered as a weakening process.

Returning to the more conventional heat treatment processes of flame and induction hardening, these again have a major effect at the surface where temperature gradients are the most severe. They produce both surface hardening and high compressive residual stresses with associated fatigue life improvements of up to 100%. There is some evidence of weakening at the case to core transition region but the process remains valuable for components with sharp stress gradients around their profile or in the presence of surface notches.

In both cases the surface is heated above some critical temperature and rapidly cooled and it is essential that the parent material has sufficient carbon or alloys to produce the required hardening by quenching. Heating either takes place under a gas flame or by electric induction heating caused by eddy currents generated in the surface layers. Typically, flame hardening is used for such components as gears and cams whilst induction hardening is applied to crankshaft journals and universal joints.

Many other components ranging from small shafts and bearings up to large forgings, fabricated structures and castings are also subjected to some form of heat treatment. Occasionally this may take the form of simple stress-relief operations aiming to reduce the level of residual stresses produced by prior manufacturing processes. Often, however, the treatment may be applied in order to effect some metallurgical improvement such as the normalising of large castings and forgings to improve their high-temperature creep characteristics or the surface hardening of gears, shafts and bearings. The required phase change of such processes usually entails the rapid cooling of components from some elevated temperature and it is this cooling which induces thermal gradients and, if these are sufficiently large (i.e. above yield), residual

stresses. The component surfaces tend to cool more rapidly, introducing tensile stresses in the outer layers which are resisted by the greater bulk of the core material and result in residual compressive stress. As stated earlier, the stress gradient with depth into the material will depend upon the temperatures involved, the coefficient of thermal expansion of the material and the method of cooling. Particularly severe stress gradients can be produced by rapid quenching in water or oil.

Differential thermal expansion is another area in which residual stresses—or stress systems which can be regarded as residual stresses—arise. In cases where components constructed from materials with different coefficients of linear expansion are subjected to uniform temperature rise, or in situations such as heat exchangers or turbine casings where one material is subjected to different temperatures in different areas, free expansions do not take place. One part of the component attempts to expand at a faster rate and is constrained from doing so by an adjacent part which is either cooler or has a lower coefficient of expansion. Residual stresses will most definitely occur on cooling if the differential expansion stresses at elevated temperatures exceed yield.

When dealing with the quenching of heated parts, as mentioned above, a simple rule is useful to remember: "*What cools last is in tension*". Thus the surface which generally cools first ends up in residual biaxial compression whilst the inner core is left in a state of triaxial tension. An exception to this is the quenching of normal through-hardened components when residual tensile stresses are produced at the surface unless a special process introduced by the General Motors Corporation of the U.S.A. termed "Marstressing" is used. This probably explains why surface-hardened parts generally have a much greater fatigue life than corresponding through-hardened items.

Should residual tensile stresses be achieved in a surface and be considered inappropriate then they can be relieved by tempering, although care must be taken to achieve the correct balance of ductility and strength after completion of the tempering process.

(d) Welds

One of the most common locations of fatigue failures resulting from residual stresses is at welded joints. Any weld junction can be considered to have three different regions; (a) the parent metal, (b) the weld metal, and (c) the heat-affected-zone (H.A.Z.), each with their own different physical properties including expansion coefficients. Residual stresses are then produced by the restraint of the parent metal on the shrinkage of the hot weld metal when it cools, and by differences in phase transformation behaviour of the three regions.

The magnitude and distribution of the residual stresses will depend upon the degree of preheat of the surfaces prior to welding, the heat input during welding, the number of weld passes, the match of the parent and weld metal and the skill of the operator. Even though the residuals can often be reduced by subsequent heat treatment this is not always effective owing to the different thermal expansions of the three zones. Differences between other physical properties in the three regions can also mean that failures need not always be associated with the region or part which is most highly stressed. Generally it is the heat-affected zones which contain sharp peaks of residual stress.

In welded structures, longitudinal shrinkages causes a weld and some parent material on either side to be in a state of residual tension often as high as the yield stress. This is balanced in the remainder of the cross-section by a residual compression which, typically, varies between 20 and 100 MN/m². When service load compressive stresses are applied to

the members, premature yielding occurs in the regions of residual compressive stress, the stiffness of the member is reduced and there is an increased tendency for the component to buckle. In addition to this longitudinal "tendon force" effect there are also transverse effects in welds known as "pull-in" and "wrap-up" effects (see Fig. 10.10) again dictated by the level of residual stress set up.

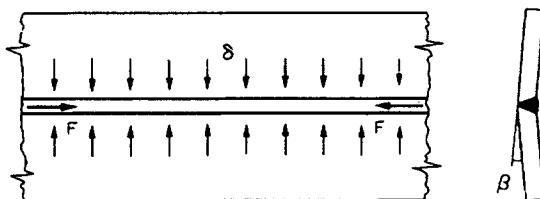


Fig. 10.10. The three basic parameters used to describe global weld shrinkage: F = tendon force; δ = pull-in; β = wrap-up.

The control of distortion is a major problem associated with large-scale welding. This can sometimes be minimised by clamping parts during welding to some pre-form curvatures or templates so that on release, after welding, they spring back to the required shape. Alternatively, components can be stretched or subjected to heat in order to redistribute the residual stress pattern and remove the distortion. In both cases, care needs to be exercised that unfavourable compressive stresses are not set up in regions which are critical to buckling failure.

(e) Castings

Another common problem area involving thermal effects and associated residual stresses is that of large castings. Whilst the full explanation for the source of residuals remains unclear (even after 70 years of research) it is clear that at least two mechanisms exist. Firstly, there are the physical restrictions placed on contraction of the casting, as it cools, by the mould itself and the differential thermal effects produced by different rates of cooling in different sections of e.g. different thickness. Secondly, there are metallurgical effects which arise largely as a result of differential cooling rates. Metal phase transformations and associated volume changes therefore occur at different positions at different times and rates. It is also suspected that different rates of cooling through the transformation range may create different material structures with different thermal coefficients. It is likely of course that the residual stress distribution produced will be as a result of a combination of these, and perhaps other, effects. It is certainly true, however, that whatever the cause there is frequently a need to subject large castings to some form of stress-relieving operation and any additional process such as this implies additional cost. It is therefore to be hoped that recent advances in measurement techniques, notably in the hole drilling method, will lead to a substantially enhanced understanding of residual stress mechanisms and to the development, for example, of suitable casting procedures which may avoid the need for additional stress-relieving operations.

10.2.2. The influence of residual stress on failure

It has been shown that residual stresses can be accommodated within the elastic failure theories quite simply by combining the residual and service load stresses (taking due account of sign) and inserting the combined stress value into the appropriate yield criterion. This is particularly true for ductile materials when both the Von Mises distortion energy theory and the Tresca maximum shear stress theory produce good correlation with experimental results. Should yielding in fact occur, there will normally be a change in the residual stress magnitude (usually a reduction) and distribution. The reduction of residual stress in this way is known as "fading".

It is appropriate to mention here another type of failure phenomenon which is related directly to residual stress termed "*stress corrosion cracking*". This occurs in metals which are subjected to corrosive environments whilst stressed, the cracks appearing in the surface layers.

Another source of potential failure is that of residual stress systems induced by the assembly of components with an initial lack of fit. This includes situations where the lack of fit is by deliberate design, e.g. shrinking or force-fit of compound cylinders or hubs on shafts and those where insufficient clearance or tolerances have been specified on mating components which, therefore, have to be forced together on assembly. This is, of course, a totally different situation to most of the cases listed above where residual stresses arise within a single member; it can nevertheless represent a potentially severe situation.

In contact loading situations such as in gearing or bearings, consideration should be given to the relationship between the distribution of residual stress with depth in the, typically, hardened surface and the depth at which the peak alternating shear stress occurs under the contact load. It is possible that the coincidence of the peak alternating value with the peak residual stress could explain the sub-surface initiation of cracks in spalling failures of such components. The hardness distribution with depth should also be considered in a similar way to monitor the strength/stress ratio, the lowest value of which can also initiate failure.

10.2.3. Measurement of residual stresses

The following methods have been used for residual stress investigations:

- (1) Progressive turning or boring – Sach's method⁽³⁴⁾
- (2) Sectioning
- (3) Layer removal – Rosenthal and Newton⁽³⁵⁾
 - Waisman and Phillips⁽³⁶⁾
- (4) Hole-drilling
 - Mather⁽³⁷⁾
 - Bathgate⁽³⁸⁾
 - Procter and Beaney^(39, 40, 41)
- (5) Trepanning or ring method – Milbradt⁽⁴²⁾
- (6) Chemical etch – Waisman and Phillips⁽³⁶⁾
- (7) Stresscoat brittle lacquer drilling – Durelli and Tsao⁽⁴³⁾
- (8) X-ray – French and Macdonald⁽⁴⁴⁾
 - Kirk^(45, 46)
 - Andrews et al⁽⁴⁷⁾
- (9) Magnetic method – Abuki and Culley⁽⁴⁸⁾

- (10) Hardness studies – Sines and Carlson⁽⁴⁹⁾
- (11) Ultrasonics – Noranha and West⁽⁵⁰⁾
 - Kino⁽⁵¹⁾
- (12) Modified layer removal – Hearn and Golsby⁽⁵²⁾
 - Spark machining – Denton⁽⁵³⁾
- (13) Photoelasticity
 - Hearn and Golsby⁽⁵²⁾

Of these techniques, the most frequently applied are the layer removal (either mechanically or chemically), the hole-drilling and the X-ray measurement procedures. Occasionally the larger scale sectioning of a component after, e.g. initially coating the surface with a photoelastic coating, a brittle lacquer or marking a grid, is useful for the semi-quantitative assessment of the type and level of residual stresses present. In each case the relaxed stresses are transferred to the coating or grid and are capable of interpretation. In the case of the brittle lacquer method the surface is coated with a layer of a brittle lacquer such as "Tenslac" or "Stress coat" and, after drying, is then drilled with a small hole at the point of interest. The relieved residual stresses, if of sufficient magnitude, will then produce a crack pattern in the lacquer which can be readily evaluated in terms of the stress magnitude and type.

The layer removal, progressive turning or boring, trepanning, chemical etch, modified layer removal and hole-drilling methods all rely on basically the same principle. The component is either machined, etched or drilled in stages so that the residual stresses are released producing relaxation deformations or strains which can be measured by mechanical methods or electrical resistance strain gauges and, after certain corrections, related to the initial residual stresses. Apart from the hole-drilling technique which is discussed in detail below, the other techniques of metal removal type are classed as destructive since the component cannot generally be used after the measurement procedure has been completed.

Most layer removal techniques rely on procedures for metal removal which themselves introduce or affect the residual stress distribution and associated measurement by the generation of heat or as a result of mechanical working of the surface – or both. Conventional machining procedures including grinding, milling and polishing all produce significant effects. Of the 'mechanical' processes, spark erosion has been shown to be the least damaging process and the only one to have an acceptably low effect on the measured stresses. Regrettably, however, it is not always available and it may prove impractical in certain situations, e.g. site measurement. In such cases, either chemical etching procedures are used or, if these too are impractical, then standard machining techniques have to be employed with suitable corrections applied to the results.

X-ray techniques are well established and will also be covered in detail below; they are, however, generally limited to the measurement of strains at, or very near to, the surface and require very sophisticated equipment if reasonably accurate results are to be achieved.

Ultrasonic and magneto-elastic methods until recently have not received much attention despite the promise which they show. Grain orientation and other metallurgical inhomogeneities affect the velocity and attenuation of ultrasonic waves and further development of the technique is required in order to effectively separate these effects from the changes due to residual stress. A sample of stress-free material is also required for calibration of the method for quantitative results. Considerable further development is also required in the case of the magneto-elastic procedure which relies on the changes which occur in magnetic flux densities in ferromagnetic materials with changing stress.

The attempts to relate residual stress levels to the hardness of surfaces again appear to indicate considerable promise since they would give an alternative non-destructive technique which is simple to apply and relatively inexpensive. Unfortunately, however, the proposals do not seem to have achieved acceptance to date and do not therefore represent any significant challenge to the three "popular" methods.

Let us now consider in greater detail the two most popular procedures, namely hole-drilling and X-ray methods.

The hole-drilling technique

The hole-drilling method of measurement of residual stresses was initially proposed by Mathar⁽³⁷⁾ in 1933 and involves the drilling of a small hole (i.e. small diameter and depth) normal to the surface at the point of interest and measurement of the resulting local surface deformations or strains. The radial stress at the edge of the hole must be zero from simple equilibrium conditions so that local redistribution of stress or "relaxation" must occur. At the time the technique was first proposed, the method of measurement of the relaxations was by mechanical extensometers and the accuracy of the technique was limited. Subsequent workers, and particularly those in recent years,⁽³⁸⁻⁴¹⁾ have used electric resistance strain gauges and much more refined procedures of hole drilling metal removal as described below.

The particular advantages of the hole drilling technique are that it is accurate, can be made portable and is the least "destructive" of the metal removal techniques, the small holes involved generally not preventing further use of the component under test—although care should be exercised in any such decision and may depend upon the level of stress present. Stress values are obtained at a point and their variation with depth can also be established. This is important with surface-hardening chemical treatments such as nitriding or carburising where substantial stress variation and stress reversals can take place beneath the surface – see Fig. 10.9.

Whichever method of hole drilling is proposed, the procedure now normally adopted is the bonding of a three-element strain gauge rosette at the point under investigation and the drilling of a hole at the gauge centre in order to release the residual stresses and allowing the recording of the three strains ε_1 , ε_2 and ε_3 in the three gauge element directions. Beaney⁽³⁹⁾ then quotes the formula which may be used for evaluation of the principal residual stresses σ_1 and σ_2 in the following form:

$$\frac{\sigma_1}{\sigma_2} \left\{ = -\frac{1}{K_1} \cdot \frac{E}{2} \left\{ \frac{(\varepsilon_1 + \varepsilon_3)}{1 - \nu(K_2/K_1)} \pm \frac{1}{1 + \nu(K_2/K_1)} \sqrt{(\varepsilon_3 - \varepsilon_1)^2 + [(\varepsilon_1 + \varepsilon_3)^2 - 2\varepsilon_2]^2} \right\} \right\}$$

Values of K_1 and K_2 are found by calibration, the value of K_1 and hence the "sensitivity" depending on the geometry of the hole and the position of the gauges relative to the hole—close control of these parameters are therefore important. For hole depths of approx. one hole diameter little error is introduced for steels by assuming the "modified" Poissons ratio term $\nu(K_2/K_1)$ to be constant at 0.3.

It should be noted that the drilled hole will act as a stress raiser with a stress concentration factor of at least 2. Thus, if residual stress levels are over half the yield stress of the material in question then some local plasticity will arise at the edge of the hole and the above formula will over-estimate the level of stress. However, the over-estimation is predictable and can be calibrated and in any case is negligible for residuals up to 70% of the yield stress.

Scaramangas *et al.*⁽⁵⁵⁾ show how simple correction factors can be applied to allow for variations of stress with depth, for the effects of surface preparation when mounting the gauges and for plastic yielding at the hole edge.

Methods of hole-drilling: (a) high-speed drill or router

Until recently, the 'standard' method of hole-drilling has been the utilisation of a small diameter, high-speed, tipped drill (similar to that used by dentists) fitted into a centring device which can be accurately located over the gauge centre using a removable eyepiece and fixed rigidly to the surface (see Fig. 10.11). Having located the fixture accurately over the required drilling position using cross-hairs the eye-piece is then removed and replaced by the drilling head. Since flat-bottomed holes were assumed in the derivation of the theoretical expressions it is common to use end-milling cutters of between $\frac{1}{8}$ in and $\frac{1}{4}$ in (3 mm to 6 mm) diameter. Unfortunately, the drilling operation itself introduces machining stresses into the component, of variable magnitude depending on the speed and condition of the tool,

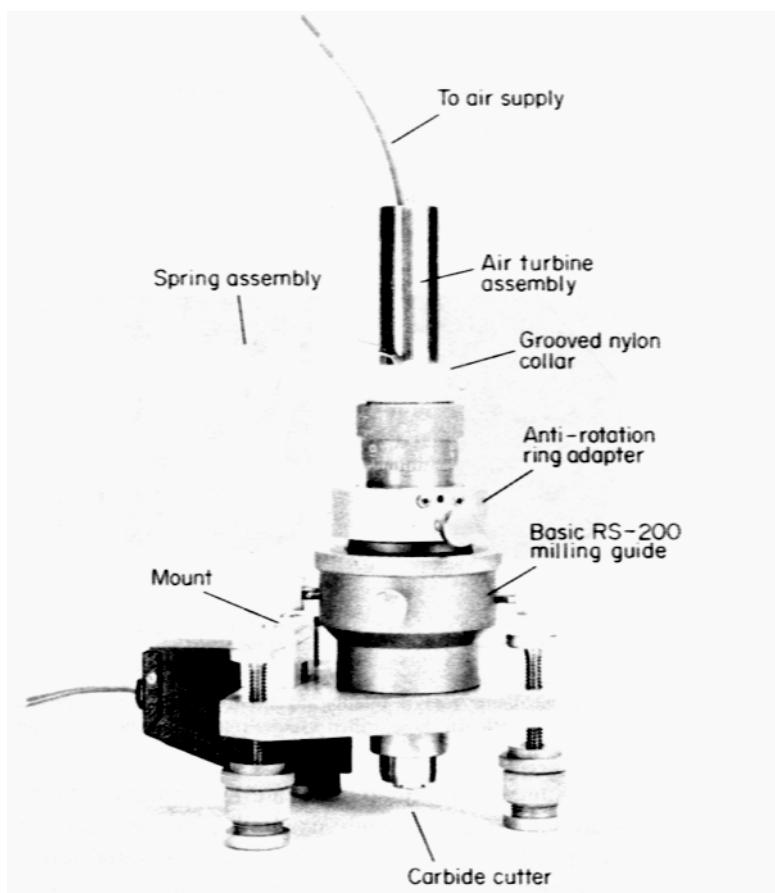


Fig. 10.11. Equipment used for the hole-drilling technique of residual stress measurement.

and these cannot readily be separated. Unless drilling is very carefully controlled, therefore, errors can arise in the measured strain values and the alternative "stress-free" machining technique outlined below is recommended.

(b) Air-abrasive machining

In this process the conventional drilling head is replaced by a device which directs a stream of air containing fine abrasive particles onto the surface causing controlled erosion of the material – see Fig. 10.12(a). The type of hole produced – see Fig. 10.12(b) – does not have a rectangular axial section but can be trepanned to produce axisymmetric, parallel-sided, holes

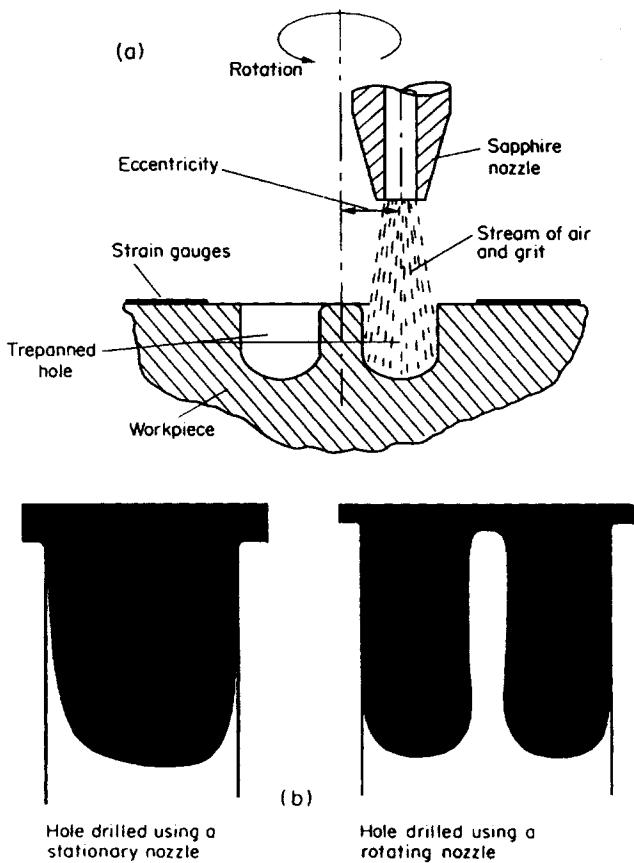


Fig. 10.12. (a) Air abrasive hole machining using a rotating nozzle; (b) types of holes produced: (i) hole drilled using a stationary nozzle, (ii) hole drilled using a rotating nozzle (trepanning)

with repeatable accuracy. Square-sided holes often aid this repeatability, but rotation of the nozzle on an eccentric around the gauge centre axis will produce a circular trepanned hole.

The optical device supplied with the "drilling" unit allows initial alignment of the unit with respect to the gauge centre and also measurement of the hole depth to enable stress distributions with depth to be plotted. It is also important to ascertain that the drilled depth is

at least 1.65 mm since this has been shown to be the minimum depth necessary to produce full stress relaxation in most applications.

Whilst the strain gauge rosettes originally designed for this work were for hole sizes of 1.59 mm, larger holes have now been found to give greater sensitivity and accuracy. Optimum hole size is now stated to be between 2 and 2.2 mm.

X-ray diffraction

The X-ray technique is probably the most highly developed non-destructive measurement technique available today. Unfortunately, however, although semi-portable units do exist, it is still essentially a laboratory tool and the high precision equipment involved is rather expensive. Because the technique is essentially concerned with the measurement of stresses in the surface it is important that the very thin layer which is examined is totally representative of the conditions required.

Two principal X-ray procedures are in general use: the *diffractometer approach* and the *film technique*. In the first of these a diffractometer is used to measure the relative shift of X-ray diffraction lines produced on the irradiated surface. The individual crystals within any polycrystalline material are made up of families of identical planes of atoms, with a fairly uniform interplanar spacing d . The so-called lattice strain normal to the crystal planes is then $\Delta d/d$ and at certain angles of incidence (known as Bragg angles) X-ray beams will be diffracted from a given family of planes as if they were being reflected. The diffraction is governed by the Bragg equation:

$$n\lambda = 2d \sin \theta \quad (10.23)$$

where n is an integer corresponding to the order of diffraction

λ is the wavelength of the X-ray radiation

θ is the angle of incidence of the crystal plane.

Any change in applied or residual stress caused, e.g. by removal of a layer of material from the rear face of the specimen, will produce a change in the angle of reflection obtained by differentiating the above equation

i.e.
$$\Delta\theta = -\frac{\Delta d}{d} \cdot \tan \theta \quad (10.24)$$

This equation relates the change in angle of incidence to the lattice strain at the surface. Typically, $\Delta\theta$ ranges between 0.3° and 0.02° depending on the initial value of θ used.

Two experimental procedures can be used to evaluate the stresses in the surface: (a) the $\sin^2 \psi$ technique and (b) the two-exposure technique, and full details of these procedures are given by Kirk⁽⁴⁵⁾. Certain problems exist in the interpretation of the results, such as the elastic anisotropy which is exhibited by metals with respect to their crystallographic directions. The appropriate values of E and v have thus to be established by separate X-ray experiments on tensile test bars or four-point bending beams.

In applications where successive layers of the metal are removed in order to determine the values of sub-surface stresses (to a very limited depth), the material removal produces a redistribution of the stresses to be measured and corrections have to be applied. Standard formulae⁽⁵⁴⁾ exist for this process.

An alternative method for determination of the Bragg angle θ is to use the film technique with a so-called "back reflection" procedure. Here the surface is coated with a thin layer of

powder from a standard substance such as silver which gives a diffraction ring near to that from the material under investigation. Measurements of the ring diameter and film-specimen distance are then used in either a single-exposure or double-exposure procedure to establish the required stress values.⁽⁴⁷⁾

The X-ray technique is valid only for measurement in materials which are elastic, homogeneous and isotropic. Fortunately most polycrystalline metals satisfy this requirement to a fair degree of accuracy but, nevertheless, it does represent a constraint on wider application of the technique.

10.2.4. Summary of the principal effects of residual stress

- (1) Considerable improvement in the fatigue life of components can be obtained by processes which introduce residual stress of appropriate sign in the surface layer. Compressive residual stresses are particularly beneficial in areas of potential fatigue (tensile stress) failure.
- (2) Pre-loading of components beyond yield in the same direction as future service loading will produce residual stress systems which strengthen the components.
- (3) Surface-hardened components have a greater fatigue resistance than through-hardened parts.
- (4) Residual stresses have their greatest influence on parts which are expected to undergo high numbers of loading cycles (i.e. low strain-high cycle fatigue); they are not so effective under high strain-low cycle fatigue conditions.
- (5) Considerable benefit can be obtained by local strengthening procedures at, e.g. stress concentrations, using shot peening or other localised procedures.
- (6) Failures always occur at positions where the ratio of strength to stress is least favourable. This is particularly important in welding applications.
- (7) Consideration of the influence of residual stresses must be part of the design process for all structures and components.

10.3. Stress concentrations

Introduction

In practically all the other chapters of this text loading conditions and components have been analysed in which stresses have been assumed, or shown, to be uniform or smoothly varying. In practice, however, this rarely happens owing to the presence of grooves, fillets, threads, holes, keyways, points of concentrated loading, material flaws, etc. In each of these cases, and many others too numerous to mention, the stress at the "discontinuity" is likely to be significantly greater than the assumed or nominally calculated figure and such discontinuities are therefore termed *stress raisers or stress concentrations*.

Most failures of structural members or engineering components occur at stress concentrations so that it is important that designers understand their significance and the magnitude of their effect since it is practically impossible to design any component without some form of stress raiser. In fatigue loading conditions, for example, virtually all failures occur at stress

concentrations and it is therefore necessary to be able to develop a procedure which will take them into account during design strength calculations.

Geometric discontinuities such as holes, sharp fillet radii, keyways, etc. are probably the most prevalent causes of failure and typical examples of failure are shown in Figs. 10.13 and 10.14.

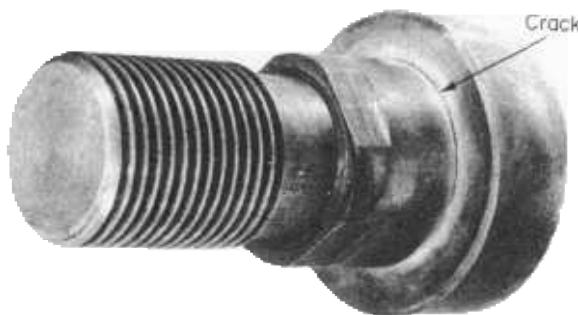


Fig. 10.13. Combined bending and tension fatigue load failure at a sharp fillet radius stress concentration position on a large retaining bolt of a heavy-duty extrusion press.

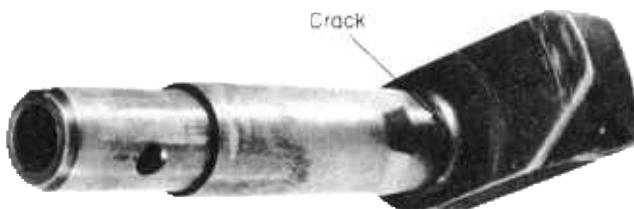


Fig. 10.14. Another failure emanating from a sharp fillet stress concentration position.

In order to be able to understand the stress concentration mechanism consider the simple example of the tensile bar shown in Figs. 10.15(a) and 10.15(b). In Fig. 10.15(a) the bar is solid and the tensile stress is nominally uniform at $\sigma = P/A = (P/bt)$ across the section.

In Fig. 10.15(b), however, the bar is drilled with a transverse hole of diameter d . Away from the hole the stress remains uniform across the section at $\sigma = P/bt$ and, using a similar calculation, the stress at the section through the centre of the hole should be $\sigma_{\text{nom}} = P/(b - d)t$ and uniform.

The sketch shows, however, that the stress at the edge of the hole is, in fact, much greater than this, indeed it is nearly 3 times as great (depending on the diameter d).

The ratio of the actual maximum stress σ_{max} and the nominal value σ_{nom} is then termed the *stress concentration factor* for the hole.

$$\text{Stress concentration factor } K_t = \frac{\text{maximum stress}}{\text{nominal stress}} = \frac{\sigma_{\text{max}}}{\sigma_{\text{nom}}} \quad (10.25)$$

For a small hole $K_t \approx 3$.

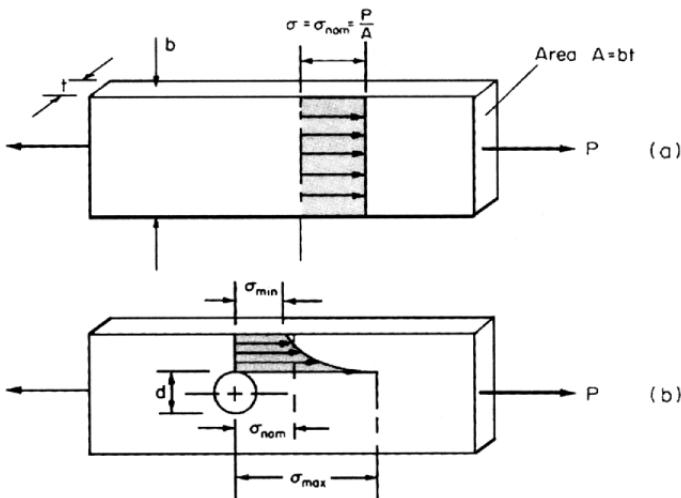


Fig. 10.15. Stress concentration effect of a hole in a tension bar.

It should also be observed that whilst the stress local to the hole is greater than the nominal stress, at distances greater than about one hole diameter away from the edge of the hole the stresses are less than the nominal value. This must be true from simple equilibrium condition since the sum of (stress \times area) across the section must balance the applied force; if the stress is greater than the nominal or average stress at one point it must therefore be less in another.

It should be evident that even had a safety factor of, say, 2.8 been used in the stress calculations for the tensile bar in question the bar would have failed since the stress concentration factor exceeds this and it is important not to rely on safety factors to cover stress concentration effects which can generally be estimated quite well, as will be discussed later. Safety factors should be reserved for allowing for uncertainties in service load conditions which cannot be estimated or anticipated with any confidence.

The cause of the stress concentration phenomenon is perhaps best understood by the use of a few analogies; firstly, that of the flow of liquid through a channel. It can be shown that the distribution of stress through a material is analogous to that of fluid flow through a channel, the cross-section of which varies in the same way as that of the material cross-section. Thus Fig. 10.16 shows the experimentally obtained flow lines for a fluid flowing round a pin of diameter d in a channel of width b , i.e. the same geometry as that of the tensile bar. It will be observed that the flow lines crowd together as the fluid passes the pin and the velocity of flow increases significantly in order that the same quantity of fluid can pass per second through the reduced gap. This is directly analogous to the increased stress at the hole in the tensile bar. Any other geometrical discontinuity will have a similar effect see Fig. 10.17.

An alternative analogy is to consider the bar without the hole as a series of stretched rubber bands parallel to each other as are the flow lines of Fig. 10.16(a). Again inserting a pin to represent the hole in the bar produces a distortion of the bands and pressure on the pin at its top and bottom diameter extremities – again directly analogous to the increased “pressure” or stress felt by the bar at the edge of the hole.

It is appropriate to mention here that the stress concentration factor calculation of eqn. (10.25) only applies while stresses remain in the elastic range. If stresses are increased



Fig. 10.16. Flow lines (a) without and (b) with discontinuity.

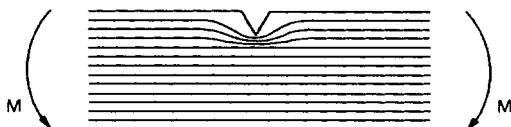


Fig. 10.17. Flow lines around a notch in a beam subjected to bending.

beyond the elastic region then local yielding takes place at the stress concentration and stresses will be redistributed as a result. In most cases this can reduce the level of the maximum stress which would be estimated by the stress concentration factor calculation. In the case of a notch or sharp-tipped crack, for example, the local plastic region forms to blunt the crack tip and reduce the stress-concentration effect for subsequent load increases. This local yielding represents a limiting factor on the maximum realistic value of stress concentration factor which can be obtained for most structural engineering materials. For very brittle materials such as glass, however, the high stress concentrations associated with very sharp notches or scratches can readily produce fracture in the absence of any significant plasticity. This, after all, is the principle of glass cutting!

The ductile flow or local yielding at stress concentrations is termed a *notch-strengthening effect* and stress concentration factors, although defined in the same way, become *plastic stress concentration factors* K_p . For most ductile materials, as the maximum stress in the component is increased up to the maximum tensile strength of the material, the value of K_p tends towards unity thus indicating that the *static* strength of the component has not been reduced significantly by the presence of the stress concentration. This is not the case for impact, fatigue or brittle fracture conditions where stress concentrations play a very significant part.

In complete contrast, stress concentrations of the types mentioned above are relatively inconsequential to the strength of heterogeneous brittle materials such as cast iron because of the high incidence of "natural" internal stress raisers within even the un-notched material, e.g. internal material flaws or impurities.

It has been shown above that the magnitude of the local increase in stress in the tensile bar caused by the stress concentration, i.e. the value of the stress concentration factor, is related to the geometry of both the bar and the hole since both b and d appear in the calculation of eqn (10.25).

Figure 10.18 shows the way in which the stress concentration value changes with different hole/bar geometries. It will be noted that the most severe effect (when related to the nominal area left after drilling the hole) is obtained when the hole diameter is smallest, producing a stress concentration factor (s.c.f.) of approximately 3. Whilst this is the largest s.c.f. value it does not mean, of course, that the bar is weaker the smaller the hole. Clearly a very large hole leaves very little material to carry the tensile load and the nominal stress will increase to produce failure. It is the combination of the nominal stress and the stress concentration factor which gives the value of the maximum stress that eventually produces

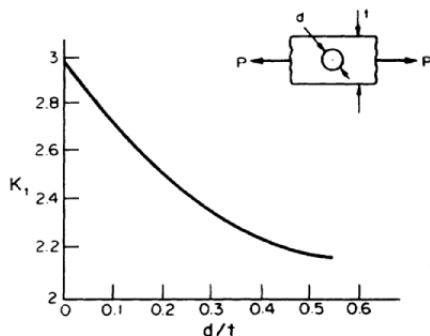


Fig. 10.18. Variation of elastic stress concentration factor K_t for a hole in a tensile bar with varying d/t ratios.

failure – both must therefore be considered.

i.e. Maximum stress = nominal stress \times stress concentration factor.

If load on the bar is increased sufficiently then failure will occur, the crack emanating from the peak stress position at the edge of the hole across the section to the outside (see Fig. 10.19).



(a)



(b)

Fig. 10.19. Tensile bar loaded to destruction – crack initiates at peak stress concentration position at the hole edge.

Other geometric factors will affect the stress-concentration effect of discontinuities such as the hole, e.g. its shape. Figure 10.20 shows the effect of various hole shapes on the s.c.f. achieved in the tensile plate for which it can be shown that, approximately, $K_t = 1 + 2(A/B)$ where A and B are the major and minor axis dimensions of the elliptical holes perpendicular and parallel to the axis of the applied stress respectively. When $A = B$, the ellipse becomes the circular hole considered previously and $K_t \approx 3$.

For large values of B , i.e. long elliptical slots parallel to the applied stress axis, stress concentration effects are reduced below 3 but for large A values, i.e. long elliptical slots perpendicular to the stress axis, s.c.f.'s rise dramatically and the potentially severe effect of slender slots or cracks such as this can readily be seen.

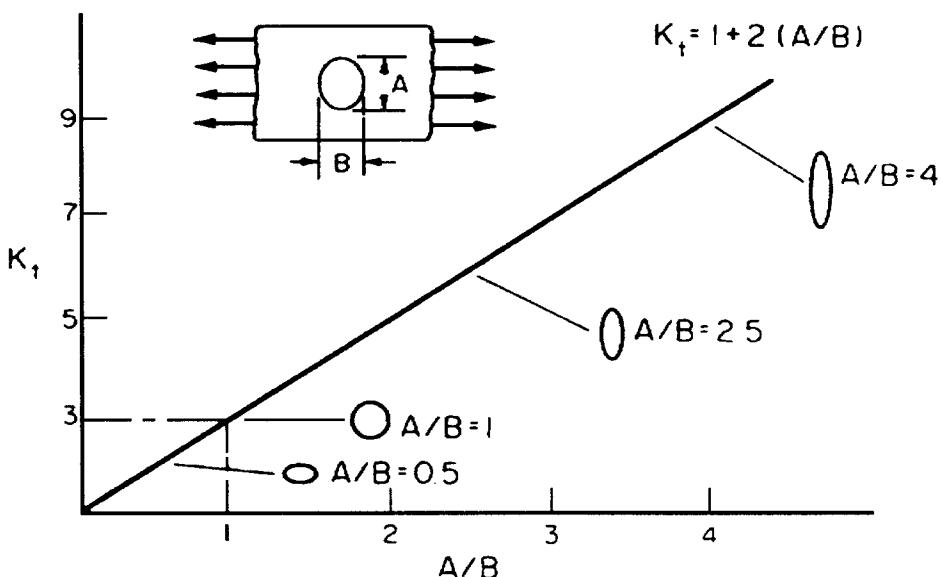


Fig. 10.20. Effect of shape of hole on the stress concentration factor for a bar with a transverse hole.

This is, of course, the theory of the perforated toilet paper roll which should tear at the perforation every time—which only goes to prove that theory very rarely applies perfectly in every situation!! (Closer consideration of the mode of loading and material used in this case helps to defend the theory, however.)

10.3.1. Evaluation of stress concentration factors

As stated earlier, the majority of the work in this text is devoted to consideration of stress situations where stress concentration effects are not present, i.e. to the calculation of nominal stresses. Before resulting stress levels can be applied to design situations, therefore, it is necessary for the designer to be able to estimate or predict the stress concentration factors associated with his particular design geometry and nominal stresses. In some cases these have been obtained analytically but in most cases graphs have been produced for standard geometric discontinuity configurations using experimental test procedures such as photoelasticity, or more recently, using finite element computer analysis.

Figures 10.21 to 10.30 give stress concentration factors for fillets, grooves and holes under various types of loading based upon a highly recommended reference volume⁽⁵⁷⁾. Many other geometrical forms and loading conditions are considered in this and other reference texts⁽⁶⁰⁾ but for non-standard cases the application of the photoelastic technique is also highly recommended (see §6.12).

The reference texts give stress concentration factors not only for two-dimensional plane stress situations such as the tensile plate but also for triaxial stress systems such as the common case of a shaft with a transverse hole or circumferential groove subjected to tension, bending or torsion.

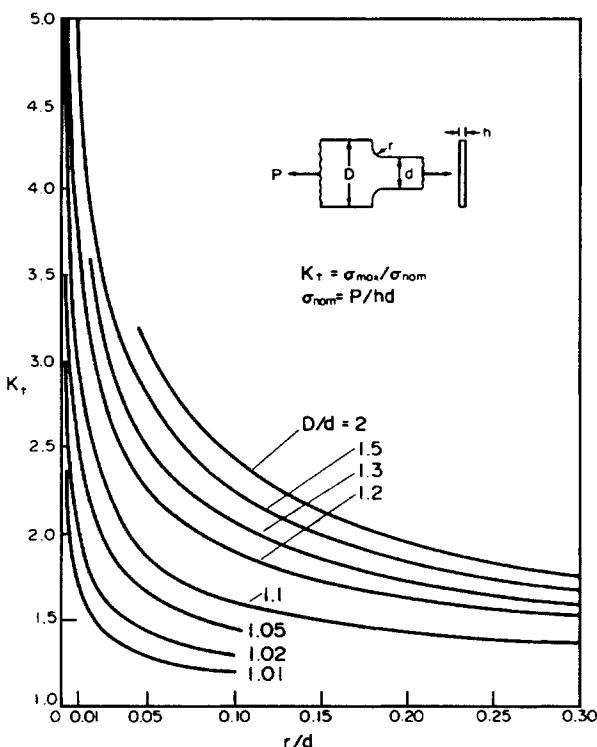


Fig. 10.21. Stress concentration factor K_t for a stepped flat tension bar with shoulder fillets.

Figures 10.31, 10.32 and 10.34 indicate the ease with which stress concentration positions can be identified within photoelastic models as the points at which the fringes are greatest in number and closest together. It should be noted that:

- (1) Stress concentration factors are different for a single geometry subjected to different types of loading. Appropriate K_t values must therefore be obtained for each type of loading. Figure 10.33 shows the way in which the stress concentration factors associated with a groove in a circular bar change with the type of applied load.
- (2) Care must be taken that stress concentration factors are applied to nominal stresses calculated on the same basis as that of the s.c.f. calculation itself, i.e. the same cross-sectional area must be used—usually the net section left after the concentration has been removed. In the case of the tensile bar of Fig. 10.15 for example, σ_{nom} has been taken as $P/(b - d)t$. An alternative system would have been to base the nominal stress σ_{nom} upon the full ‘un-notched’ cross-sectional area i.e. $\sigma_{nom} = P/t$. Clearly, the stress concentration factors resulting from this approach would be very different, particularly as the size of the hole increases.
- (3) In the case of combined loading, the stress calculated under each type of load must be multiplied by its own stress concentration factor. In combined bending and axial load, for example, the bending stress ($\sigma_b = My/I$) should be multiplied by the bending s.c.f. and the axial stress ($\sigma_d = P/A$) multiplied by the s.c.f. in tension.

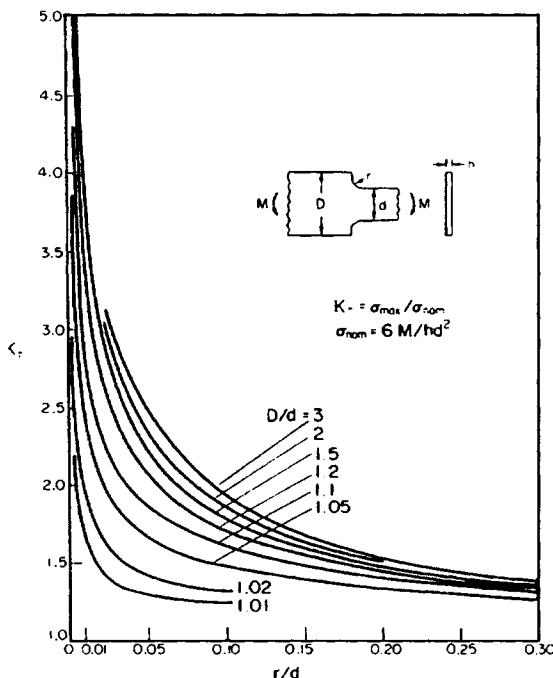


Fig. 10.22. Stress concentration factor K_t for a stepped flat tension bar with shoulder fillets subjected to bending.

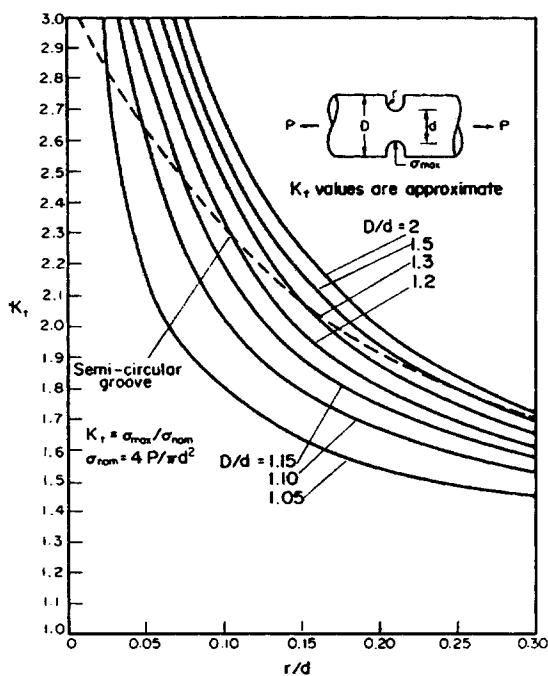


Fig. 10.23. Stress concentration factor K_t for a round tension bar with a U groove.

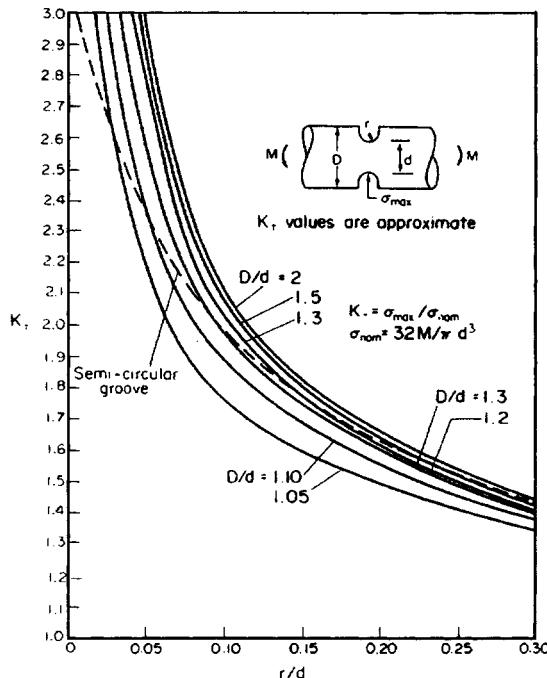


Fig. 10.24. Stress concentration factor K_t for a round bar with a U groove subjected to bending.

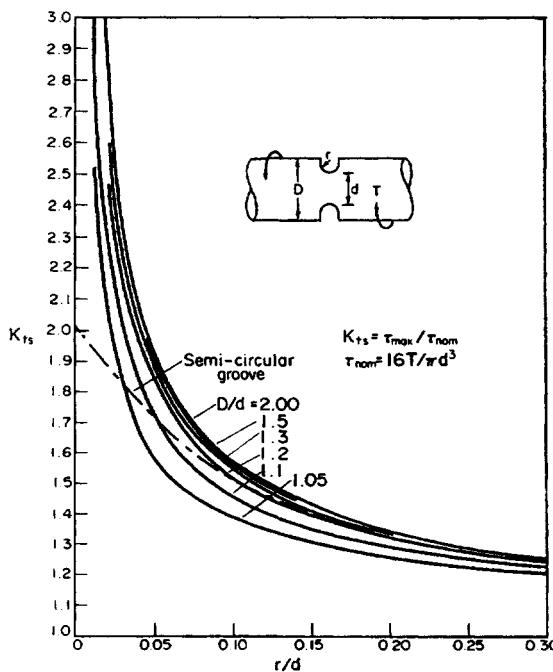


Fig. 10.25. Stress concentration factor K_t for a round bar with a U groove subjected to torsion.

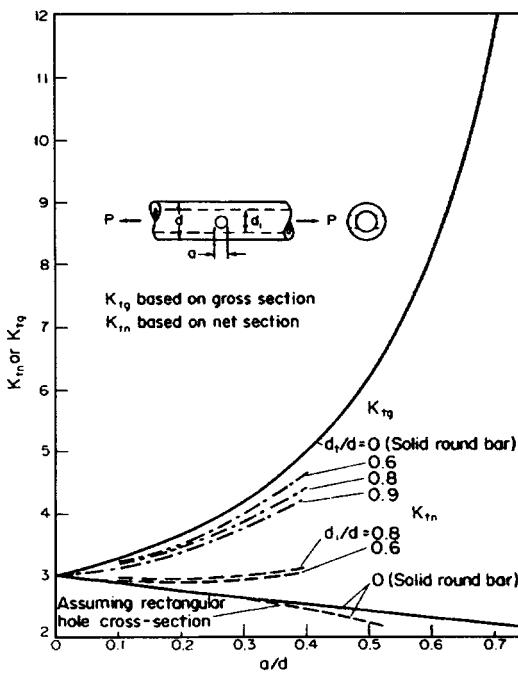


Fig. 10.26. Stress concentration factor K_t for a round bar or tube with a transverse hole subjected to tension.

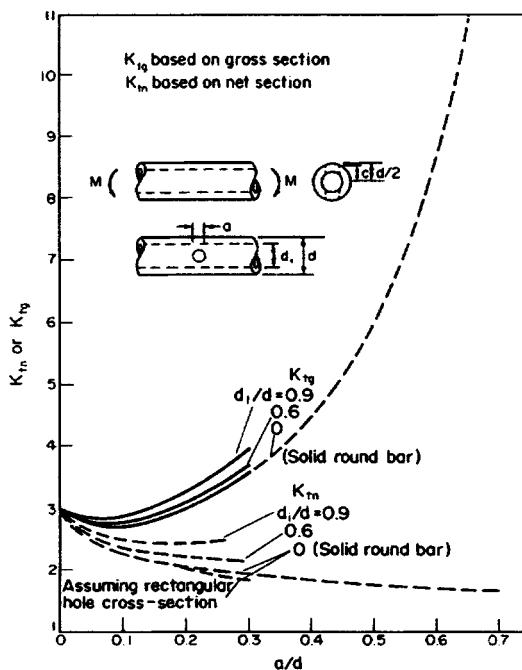


Fig. 10.27. Stress concentration factor K_t for a round bar or tube with a transverse hole subjected to bending.

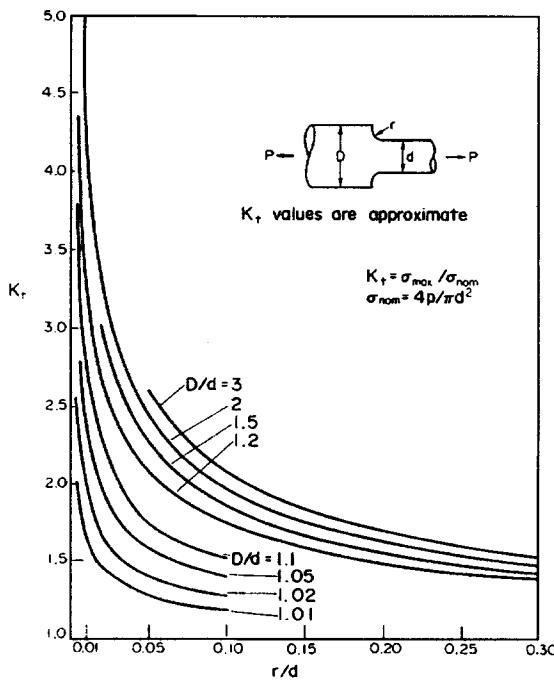


Fig. 10.28. Stress concentration factor K_t for a round bar with shoulder fillet subjected to tension.

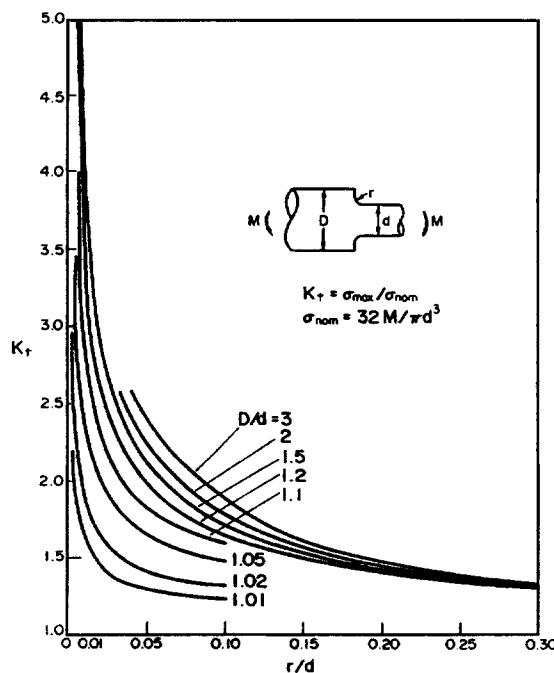


Fig. 10.29. Stress concentration factor K_t for a stepped round bar with shoulder fillet subjected to bending.

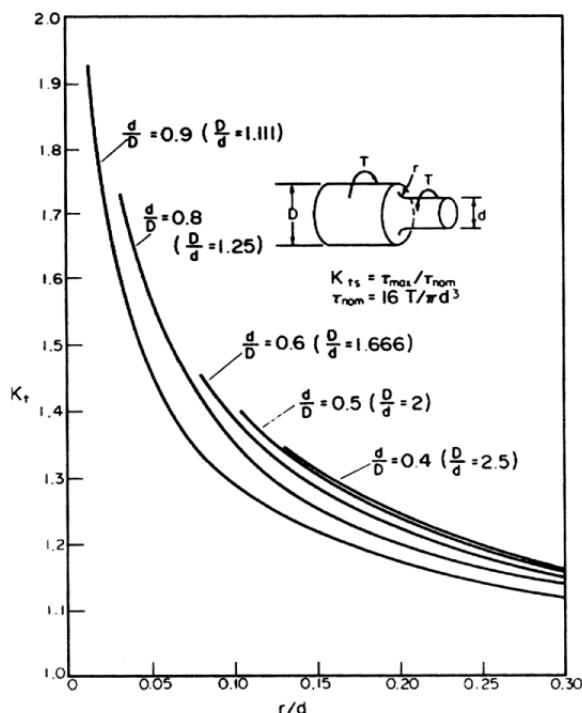


Fig. 10.30. Stress concentration factor K_t for a stepped round bar with shoulder fillet subjected to torsion.

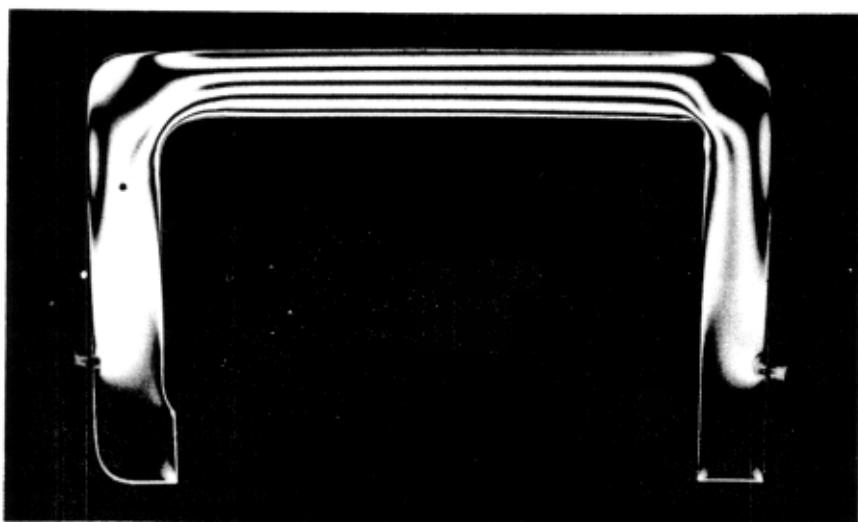


Fig. 10.31. Photoelastic fringe pattern of a portal frame showing stress concentration at the corner blend radii (different blend radii produce different stress concentration factors)

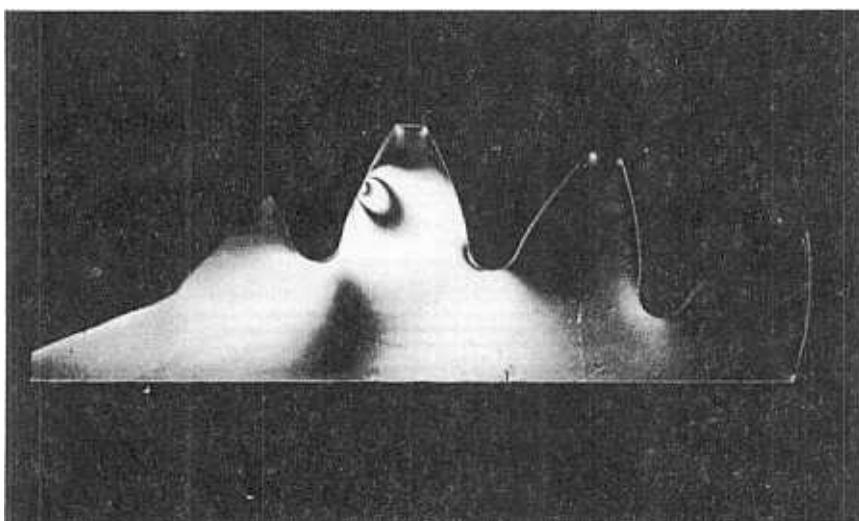
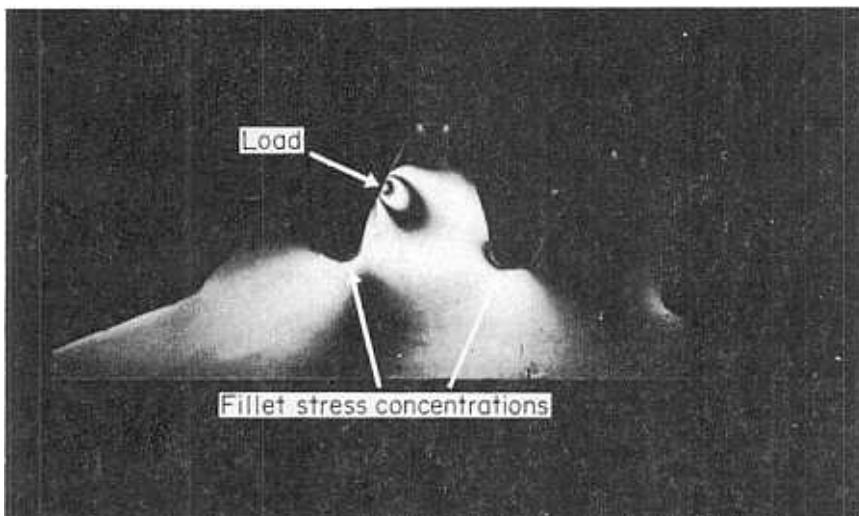


Fig. 10.32. Photoelastic fringe pattern of stress distribution in a gear tooth showing stress concentration at the loading point on the tooth flank and at the root fillet radii (higher concentration on the compressive fillet). Refer also to Fig. 10.45.

10.3.2. Saint-Venant's principle

The general problem of stress concentration was studied analytically by Saint-Venant who produced the following statement of principle: "If the forces acting on a small area of a body are replaced by a statically equivalent system of forces acting on the same area, there will be considerable changes in the local stress distribution but the effect at distances large compared with the area on which the forces act will be negligible". The effect of this principle is best demonstrated with reference to the photoelastic fringe pattern obtained in a model of a beam subjected to four-point bending, i.e. bending into a circular arc between the central

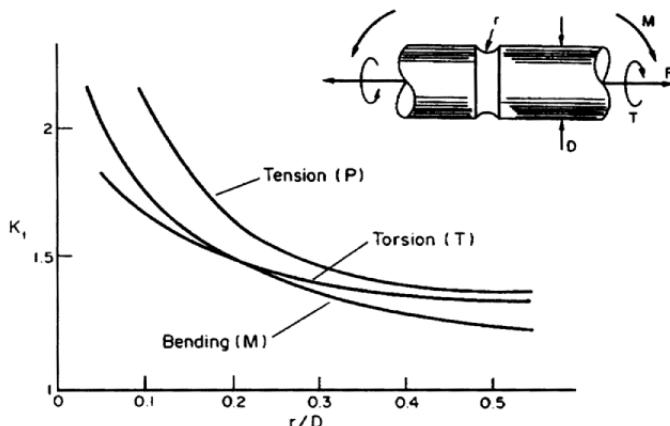


Fig. 10.33. Variation of stress concentration factors for a grooved shaft depending on the type of loading.

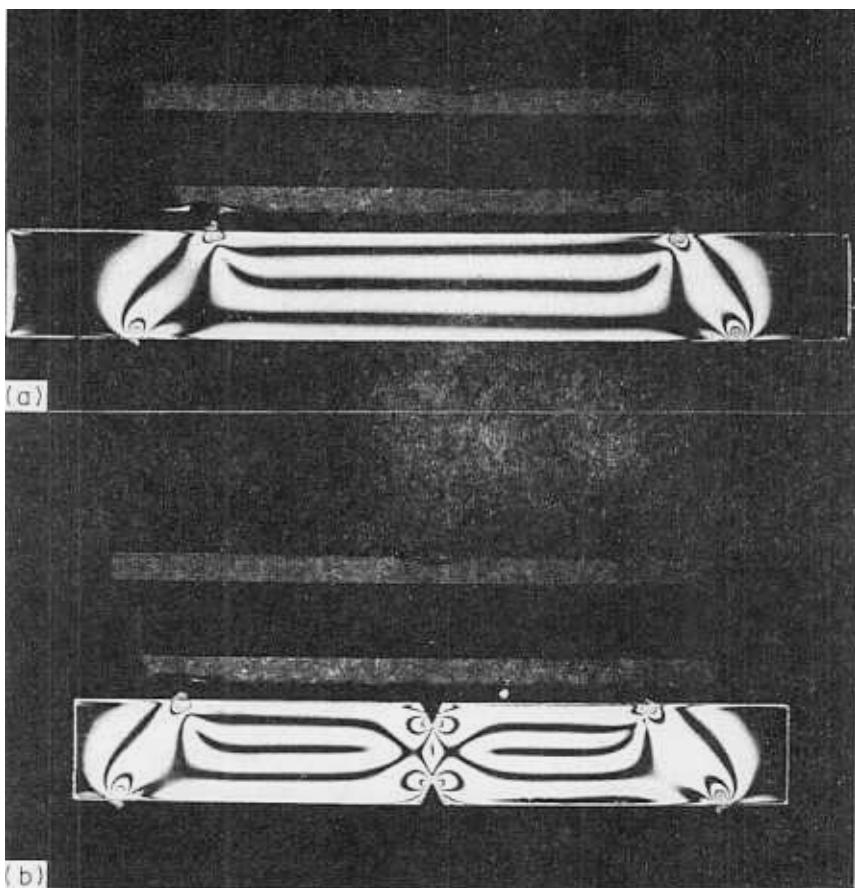


Fig. 10.34. (a) Photoelastic fringe pattern in a model of a beam subjected to four-point bending (i.e. circular arc bending between central supports); (b) as above but with a central notch.

supports – see Fig. 10.34(a). If the moment could have been applied by some other means so as to avoid the contact at the loading points then the fringe pattern would have been a series of parallel fringes, the centre one being the neutral axis. The stress concentrations due to the loading points are clearly visible as is the effect of these on the distribution of the fringes and hence stress. In particular, note the curvature of the neutral axis towards the inner loading points and the absence of the expected parallel fringe distribution both near to and outside the loading points. However, for points at least one depth of beam away from the stress concentrations (St. Venant) the fringe pattern is unaffected, the parallel fringes remain undisturbed and simple bending theory applies. If either the beam length is reduced or further stress concentrations (such as the notch of Fig. 10.34(b)) are introduced so that every part of the beam is within “one depth” of a stress concentration then at no point will simple theory apply and analysis of the fringe pattern is required for stress evaluation – there is no simple analytical procedure.

Similarly, in a round tension bar the stresses at the ends will be dependent upon the method of gripping or load application but within the main part of the bar, at least one diameter away from the loading point, stresses can again be obtained from simple theory. To the other extreme comes the case of a screw thread. The maximum s.c.f. arises at the first contacting thread at the plane of the bearing face of the head or nut and up to 70% of the load is carried by the first two or three threads. In such a case, simple theory cannot be applied anywhere within the component and the reader is referred to the appropriate B.S. Code of Practice and/or the work of Brown and Hickson⁽⁵⁹⁾.

10.3.3. Theoretical considerations of stress concentrations due to concentrated loads

A full treatment of the local stress distribution at points of application of concentrated load is beyond the scope of this text. Two particular cases will be introduced briefly, however, in order that the relevant useful equations can be presented.

(a) Concentrated load on the edge of an infinite plate

Work by St. Venant, Boussinesq and Flamant (see §8.7.9) has led to the development of a theory based upon the replacement of the concentrated load by a radial distribution of loads around a semi-circular groove (which replaces the local area of yielding beneath the concentrated load) (see Fig. 10.35). Elements in the material are then, according to Flamant, subjected to a radial compression of

$$\sigma_r = \frac{2P \cos \theta}{\pi b r} \quad \text{with } b = \text{width of plate} \quad (10.26)$$

This produces element cartesian stresses of:

$$\sigma_{xx} = \sigma_r \sin^2 \theta = -\frac{2P \cos \theta \sin^2 \theta}{\pi b r} = -\frac{2Px^2 y}{\pi b (x^2 + y^2)^2} \quad (10.27)$$

$$\sigma_{yy} = \sigma_r \cos^2 \theta = \frac{-2P \cos^3 \theta}{\pi b r} = \frac{-2Py^3}{\pi b (x^2 + y^2)^2} \quad (10.28)$$

$$\tau_{xy} = \sigma_r \sin \theta \cos \theta = -\frac{2P \sin \theta \cos^2 \theta}{\pi b r} = -\frac{2Pxy^2}{\pi b (x^2 + y^2)^2} \quad (10.29)$$

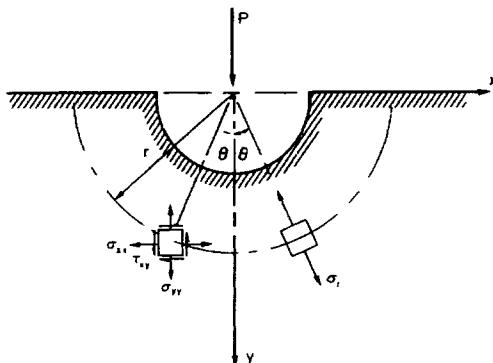


Fig. 10.35. Elemental stresses due to concentrated load P on the edge of an infinite plate.

(b) Concentrated load on the edge of a beam in bending

In this case a similar procedure is applied but, with a finite beam, consideration must be given to the horizontal forces set up within the groove which result in longitudinal stresses additional to the bending effects.

The total stress across the vertical section through the loading point (or groove) is then given by the so-called "Wilson-Stokes equation".

$$\sigma_{xx} = \frac{P}{\pi bd} \pm \left[\frac{L}{4} - \frac{d}{2\pi} \right] \frac{2Py}{bd^3} \quad (10.30)$$

where d is the depth of the beam, b the breadth and L the span.

This form of expression can be shown to indicate that the maximum longitudinal stresses set up are, in fact, less than those obtained from the simple bending theory alone (in the absence of the stress concentration).

10.3.4. Fatigue stress concentration factor

As noted above, the plastic flow which develops at positions of high stress concentration in ductile materials has a stress-relieving effect which significantly nullifies the effect of the stress raiser under static load conditions. Even under cyclic or fatigue loading there is a marked reduction in stress concentration effect and this is recognised by the use of a fatigue stress concentration factor K_f .

In the absence of any stress concentration (i.e. for $K_f = 1$) materials exhibit an "*endurance limit*" or "*fatigue limit*" – a defined stress amplitude below which the material can withstand an indefinitely large (sometimes infinite) number of repeated load cycles. This is often referred to as the un-notched fatigue limit – see Fig. 10.36.

For a totally brittle material in which the elastic stress concentration factor K_f might be assumed to have its full effect, e.g. $K_f = 2$, the fatigue life or notched endurance limit would be reduced accordingly. For materials with varying plastic flow capabilities, the effect of stress-raisers produces notched endurance limits somewhere between the un-notched value and that of the 'theoretical' value given by the full K_f – see Fig. 10.36, i.e. the fatigue stress concentration factor lies somewhere between the full K_f value and unity.

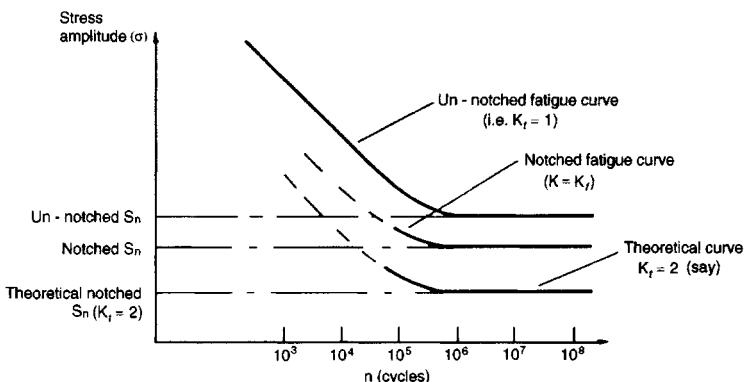


Fig. 10.36. Notched and un-notched fatigue curves.

If the endurance limit for a given number of cycles, n , is denoted by S_n then the fatigue stress concentration factor is defined as:

$$K_f = \frac{S_n \text{ for unnotched material}}{S_n \text{ for notched material}} \quad (10.31)$$

K_f is sometimes referred to by the alternative titles of "fatigue strength reduction factor" or, simply, the "fatigue notch factor".

The value of K_f is normally obtained from fatigue tests on identical specimens both with and without the notch or stress-raiser for which the stress concentration effect is required.

It is well known (and discussed in detail in Chapter 11) that the fatigue life of components is affected by a great number of variables such as mean stress, stress range, environment, size effect, surface condition, etc., and many different approaches have been proposed to allow realistic estimations of life under real working conditions as opposed to the controlled laboratory conditions under which most fatigue tests are carried out. One approach which is relevant to the present discussion is that proposed by Lipson & Juvinal⁽⁶⁰⁾ which utilises fatigue stress concentration factors, K_f , suitably modified by various coefficients to take account of the above-mentioned variables.

10.3.5. Notch sensitivity

A useful relationship between the elastic stress concentration factor K_t and the fatigue notch factor K_f introduces a *notch sensitivity* q defined as follows:

$$q = \frac{K_f - 1}{K_t - 1} \text{ or, in shear, } q = \frac{K_{fs} - 1}{K_{ts} - 1}$$

which may be re-written in terms of the fatigue notch factor as:

$$K_f = 1 + q(K_t - 1) \text{ with } 0 \leq q \leq 1 \quad (10.32)$$

It will be seen that, at the extreme values of q , eqn. (10.32) is valid since when $q = 1$ the full effect of the elastic stress concentration factor K_t applies and $K_f = K_t$; similarly when $q = 0$ and full ductility applies there is, in effect, no stress concentration and $K_f = 1$ with the material behaving in an unnotched fashion.

The value of the notch sensitivity for stress raisers with a significant linear dimension (e.g. fillet radius) R and a material constant "a" is given by:

$$q = \frac{1}{\left(1 + \frac{a}{R}\right)} \quad (10.33)$$

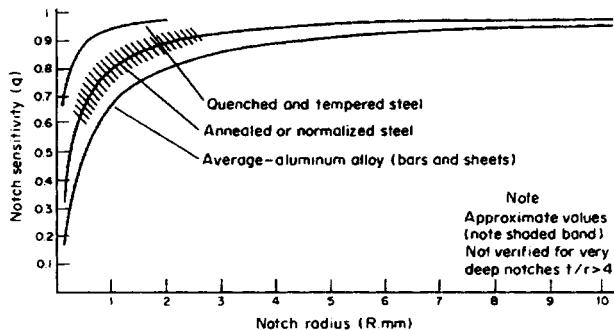


Fig. 10.37. Average fatigue notch sensitivity q for various notch radii and materials.

Typically, $a = 0.01$ for annealed or normalised steel, 0.0025 for quenched and tempered steel and 0.02 for aluminium alloy. However, values of "a" are not readily available for a wide range of materials and reference should be made to graphs of q versus R given by both Peterson⁽⁵⁷⁾ and Lipson and Juvinal⁽⁶⁰⁾.

The stress and strain distribution in a tensile bar containing a "through-hole" concentration are shown in Fig. 10.38 where the elastic stress concentration factor predictions are compared with those taking into account local yielding and associated stress redistribution.

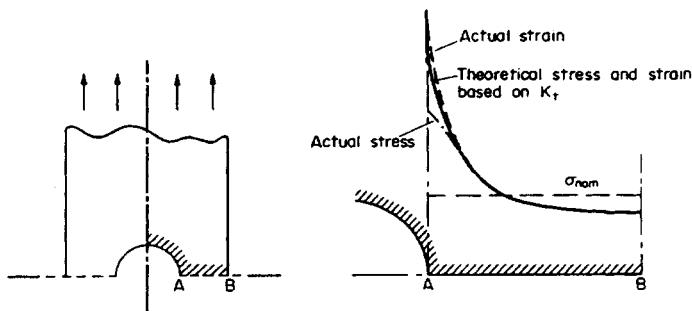


Fig. 10.38. Effect of a local yielding and associated stress re-distribution on the stress and strain concentration at the edge of a hole in a tensile bar.

10.3.6. Strain concentration – Neuber's rule

Within the elastic range, the concentration factor expressed in terms of strain rather than stress is equal to the stress concentration factor K_t . In the presence of plastic flow, however, the elastic stress concentration factor is reduced to the plastic factor K_p but local strains clearly exceed those predicted by elastic considerations – see Fig. 10.39.

A strain concentration factor can thus be defined as:

$$K_e = \frac{\text{maximum strain at the notch}}{\text{nominal strain at the notch}}$$

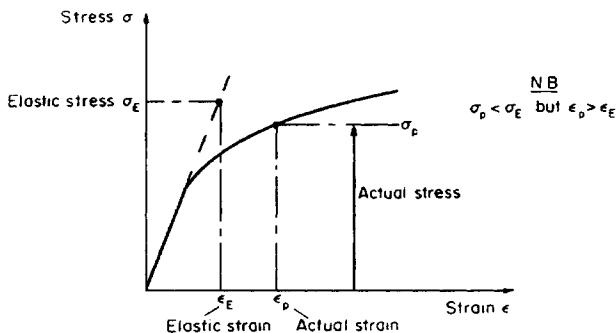


Fig. 10.39. Comparison of elastic and plastic stresses and strains.

the value of K_e increasing as the value of K_p decreases. One attempt to relate the two factors is known as "Neuber's Rule", viz.

$$K_p K_e = K_t^2 \quad (10.34)$$

It is appropriate here to observe that recent research in the fatigue behaviour of materials indicates that the strain range of fatigue loading may be more readily related to fatigue life than the stress range which formed the basis of much early fatigue study. This is said to be particularly true of low-cycle fatigue where, in particular, the plastic strain range is shown to be critical.

10.3.7. Designing to reduce stress concentrations

From the foregoing discussion it should now be evident that stress concentrations are critical to the life of engineering components and that fatigue failures, for example, almost invariably originate at such positions. It is essential, therefore, for any design to be successful that detailed consideration is given to the reduction of stress concentration effects to an absolute minimum.

One important rule in this respect is concerned with the initial placement of the stress concentration. Assuming that some freedom exists as to the position of e.g. oil-holes, keyways, grooves, etc., then it is essential that these be located at positions where the nominal stress is as low as possible. The resultant magnitude of stress concentration factor \times nominal stress is then also a minimum for a particular geometry of stress raiser.

In situations where no flexibility exists as to the position of the stress raiser then one of the procedures outlined below should be considered. In many cases a qualitative assessment of the benefits, or otherwise, of design changes is readily obtained by sketching the lines of stress flow through the component as in Fig. 10.17. Sharp changes in flow direction indicate high stress concentration factors, smooth changes in flow direction are the optimum solution.

The following standard stress concentration situations are common in engineering applications and procedures for reduction of the associated stress concentration factors are introduced for each case. The procedures, either individually or in combination, can then often be applied to produce beneficial stress reduction in other non-standard design situations.

(a) Fillet radius

Probably the most common form of stress concentration is that arising at the junction of two parts of a component of different shape, diameter, or other dimension. In almost every shaft, spindle, or axle design, for example, the component consists of a number of different diameter sections connected by shoulders and associated fillets.

If Fig. 10.40(a) is taken to be either the longitudinal section of a shaft or simply a flat plate, then the transition from one dimension to another via the right-angle junction is exceptionally bad design since the stress concentration associated with the sharp corner is exceedingly high. In practice, however, either naturally due to the fact that the machining tool has a finite radius, or by design, the junction is formed via a fillet radius and the wise designer employs the highest possible radius of fillet consistent with the function of the component in order to keep the s.c.f. as low as possible. Whilst, historically, circular arcs have generally been used for fillets, other types of blend geometry have been shown to produce even further reduction of s.c.f. notably elliptical and streamline fillets⁽⁶¹⁾, the latter following similar contours to those of a fluid when it flows out of a hole in the bottom of a tank. Fig. 10.41 shows the effect of elliptical fillets on the s.c.f. values.

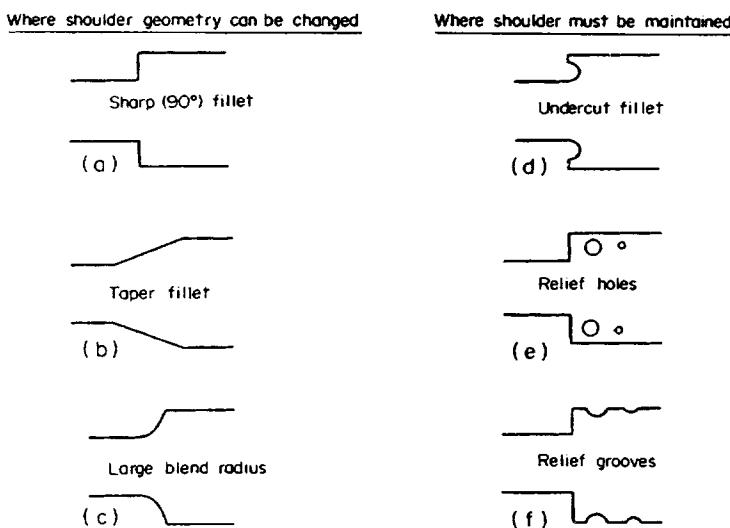


Fig. 10.40. Various methods for reduction of stress concentration factor at the junction of two parts of a component of different depth/diameter.

There are occasions, however, where the perpendicular faces at the junction need to be maintained and only a relatively small fillet radius can be allowed e.g. for retention of bearings or wheel hubs. A number of alternative solutions for reduction of the s.c.f.'s are shown in Fig. 10.40(d) to (f) and Fig. 10.42.

(b) Keyways or splines

It is common to use keyways or splines in shaft applications to provide transfer of torque between components. Gears or pulleys are commonly keyed to shafts, for example, by square

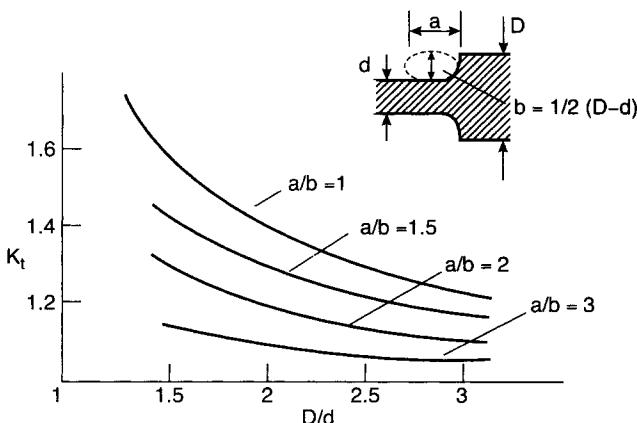


Fig. 10.41. Variation of elliptical fillet stress concentration factor with ellipse geometry.

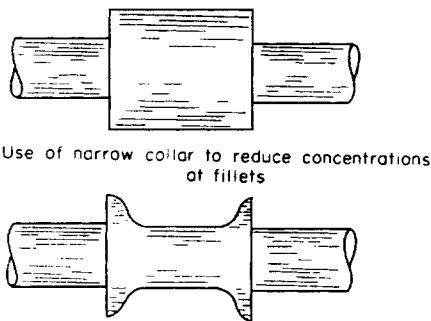


Fig. 10.42. Use of narrow collar to reduce stress concentration at fillet radii in shafts.

keys with side dimensions approximately equal to one-quarter of the shaft diameter with the depth of the keyway, therefore, one-eighth of the shaft diameter.

Analytical solutions for such a case have been carried out by both Leven⁽⁶³⁾ and Neuber⁽⁶⁵⁾ each considering the keyway without a key present. Neuber gives the following formula for stress concentration factor (based on shear stresses):

$$K_{t_s} = 1 + \sqrt{\frac{h}{r}} \quad (10.35)$$

where h = keyway depth and r = radius at the base of the groove or keyway (see Fig. 10.43). For a semi-circular groove $K_{t_s} = 2$.

Leven, considering the square keyway specifically, observes that the s.c.f. is a function of the keyway corner radius and the shaft diameter. For a practical corner radius of about one-tenth the keyway depth $K_{t_s} \approx 3$.

If fillet radii cannot be reduced then s.c.f.'s can be reduced by drilling holes adjacent to the keyway as shown in Fig. 10.43(b).

The presence of a key and its associated fit (or lack of) has a significant effect on the stress distribution and no general solution exists. Each situation strictly requires its own solution via practical testing such as photoelasticity.

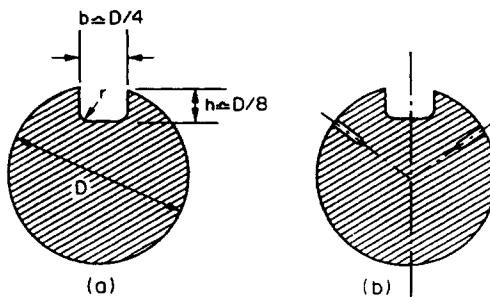


Fig. 10.43. Key-way dimensions and stress reduction procedure.

(c) Grooves and notches

Circumferential grooves or notches (particularly U-shaped notches) occur frequently in engineering design in such applications as C-ring retainer grooves, oil grooves, shoulder or grinding relief grooves, seal retainers, etc; even threads may be considered as multi-groove applications.

Most of the available s.c.f. data available for grooves or notches refers to U-shaped grooves and circular fillet radii and covers both plane stress and three-dimensional situations such as shafts with circumferential grooves. In general, the higher the blend radius, the lower the s.c.f.; the optimum value being $K_t = 2$ for a semi-circular groove as calculated by Neuber's equation (10.35) above.

Some data exists for other forms of groove such as V notches and hyperbolic fillets but, particularly in bending and tension, the latter have little advantage over circular arcs and V notches only show significant advantage for included angles greater than 120° . In cases where s.c.f. data for a particular geometry of notch are not readily available recourse can be made to standard factor data for plates with a central hole.

Stress concentrations at notches and grooves can be reduced by the "metal removal – stiffness reduction" technique utilising any procedure which improves the stress flow, e.g. multiple notches of *U* grooves or selected hole drilling as shown in Fig. 10.44. Reductions of the order of 30% can be obtained.

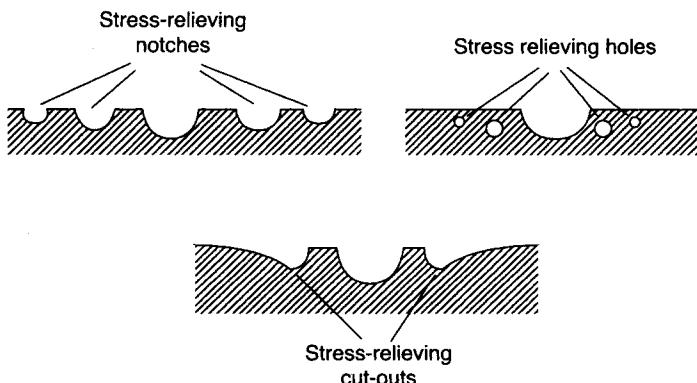


Fig. 10.44. Various procedures for the reduction of stress concentrations at notches or grooves.

This procedure of introducing secondary stress concentrations deliberately to reduce the local stiffness of the material adjacent to a stress concentration is a very powerful stress reduction technique. In effect, it causes more of the stiffer central region of the component to carry the load and persuades the stress lines to follow a path removed from the effect of the single, sharp concentration. Figures 10.40(d) to (f), 10.42 and 10.43 are all examples of the application of this technique, sometimes referred to as an "interference effect" the individual concentrations interfering with each other to mutual advantage.

(d) Gear teeth

The full analysis of the stress distribution in gear teeth is a highly complex problem. The reader is only referred in this section to the stress concentrations associated with the fillet radii at the base of the teeth – see Fig. 10.45.

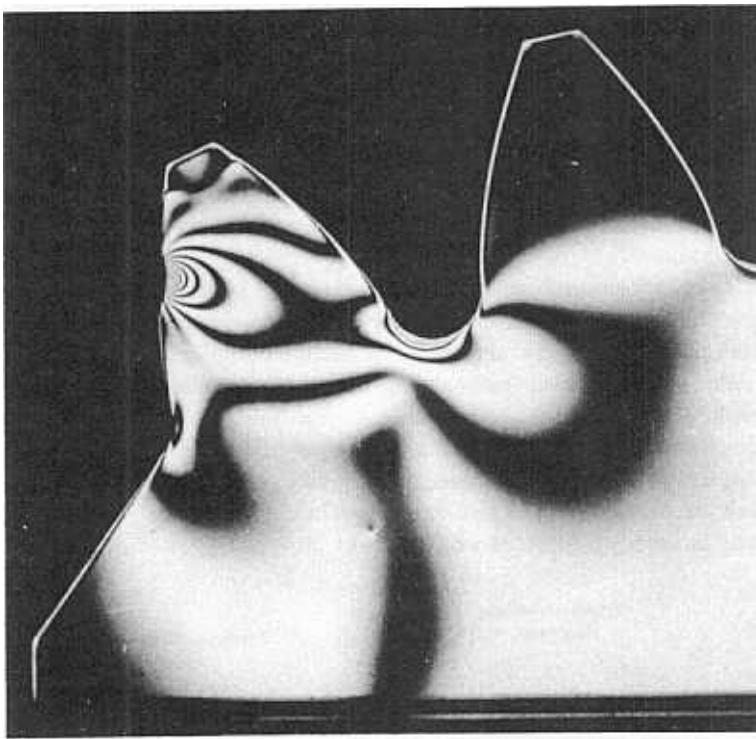


Fig. 10.45. Stress concentration at root fillet of gear tooth.

The loading on the tooth produces both direct stress and bending components on the root section and Dolan and Broghammer⁽⁶⁸⁾ in early studies of the problem gave the following formula for the combined stress concentration effect (for 20° pressure angle gears)

$$K_t = 0.18 + \frac{1}{\left(\frac{r}{t}\right)^{0.15} \left(\frac{h}{t}\right)^{0.45}}$$

Later work by Jacobson⁽⁶⁹⁾, again for 20° pressure angle gears, produced a series of charts of strength factors and more recently Hearn^(66,67) has carried out photoelastic studies of both two-dimensional involute tooth forms and three-dimensional helical gears which introduce new considerations of stress concentration factors, notably their variation in both magnitude and position as the load moves up and down the tooth flank.

(e) Holes

From much of the previous discussion it should now be evident that holes represent very significant stress raisers, be they in two-dimensional plates or three-dimensional bars. Fortunately, a correspondingly high amount of information and data is available, e.g. Peterson⁽⁵⁷⁾, covering almost every foreseeable geometry and loading situation. This includes not only individual holes but rows and groups of holes, pin-joints, internally pressurised holes and intersecting holes.

(f) Oil holes

The use of transverse and longitudinal holes as passages for lubricating oil is common in shafting, gearing, gear couplings and other dynamic mechanisms. Occasionally similar holes are also used for the passage of cooling fluids.

In the case of circular shafts, no problem arises when longitudinal holes are bored through the centre of the shaft since the nominal torsional stress at this location is very small and the effect on the overall strength of the shaft is minimal. A transverse hole, however, is a significant source of stress concentration in any mode of loading, i.e. bending, torsion or axial load, and the relevant s.c.f. values must be evaluated from standard reference texts^(57, 60). Whatever the type of loading, the value of K_t increases as the size of the hole increases for a given shaft diameter, with minimum values for very small holes of 2 for torsion and 3 for bending and tension.

In cases of combined loading, a conservative estimate⁽⁵⁸⁾ of the stress concentration may be obtained from values of K_t given by either Peterson⁽⁵⁷⁾ or Lipson⁽⁶⁰⁾ for an infinite plate containing a transverse hole and subjected to an equivalent biaxial stress condition.

One procedure for the reduction of the stress concentration at the point where transverse holes cut the surface of shafts is shown in Fig. 10.46.

(g) Screw threads

Again the stress distribution in screw threads is extremely complex, values of the stress concentration factors associated with each thread being dependent upon the tooth form, the fit between the nut and the bolt, the nut geometry, the presence or not of a bolt shank and the load system applied. Pre-tensioning also has a considerable effect. However, from numerous photoelastic studies carried out by the author and others^(59, 61, 62) it is clear that the greatest stress most often occurs at the first mating thread, generally at the mating face of the head of the nut with the bearing surface, with practically all the load shared between the first few threads. (One estimate of the source of bolt failures shows 65% in the thread at the nut face compared with 20% at the end of the thread and 15% directly under the bolt

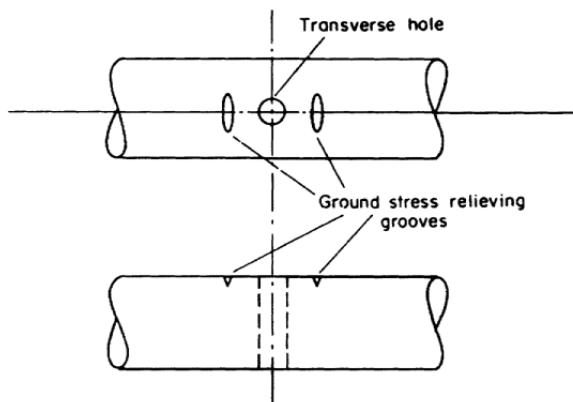


Fig. 10.46. Procedure for reduction of stress concentration at exit points of transverse holes in shafts.

head). Alternative designs of nut geometry can be introduced to spread the load distribution a little more evenly as shown in Figs. 10.47 and tapering of the thread is a very effective load-distribution mechanism.

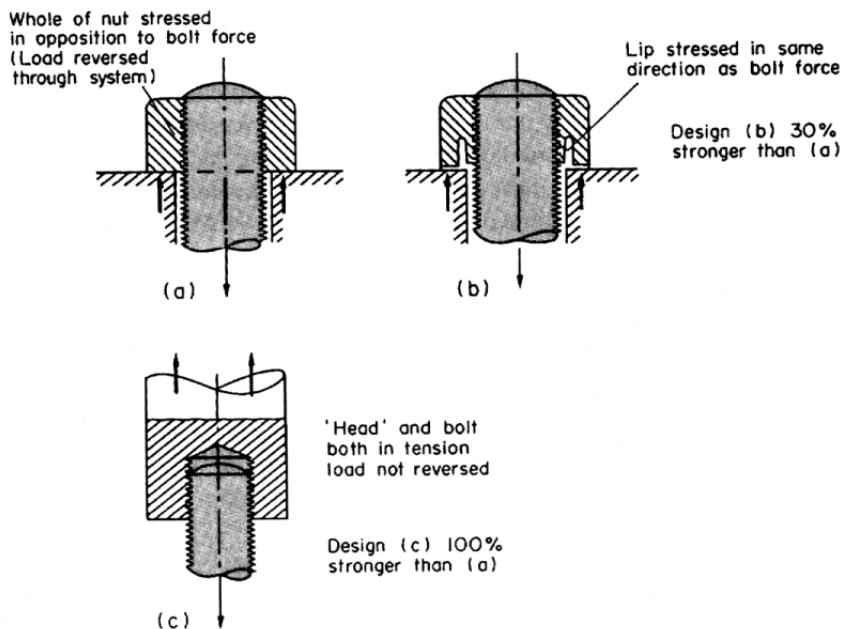


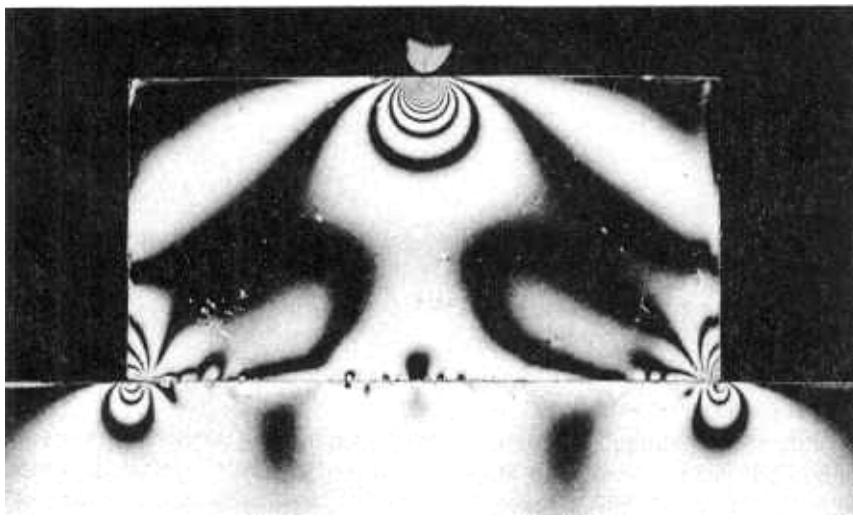
Fig. 10.47. Alternative bolt/nut designs for reduction of stress concentrations.

Reduction in diameter of the bolt shank and a correspondingly larger fillet radius under the bolt head also produces a substantial improvement as does the use of a material with a lower modulus of elasticity for the nut compared with the bolt; fatigue tests have shown strength improvements of between 35 and 60% for this technique.

Stress concentration data for various nut and bolt configurations are given by Hetenyi⁽⁶²⁾, again based on photoelastic studies. As an example of the severity of loading at the first thread, stress concentration factors of the order of 13 are readily obtained in conventional nut designs and even using the modified designs noted above s.c.f.'s of up to 9 are quite common. It is not perhaps surprising, therefore, that one of the most common causes of machinery or plant failure is that of stud or bolt fracture.

(h) Press or shrink fit members

There are some applications where discontinuity of component profile caused by two contacting members represents a substantial stress raiser effectively as great as a right-angle fillet. These include shrink or press-fit applications such as collars, gears, wheels, pulleys, etc., mounted on their drive shafts and even simple compressive loading of rectangular faces on wider support plates – see Fig. 10.48(a).



(a)

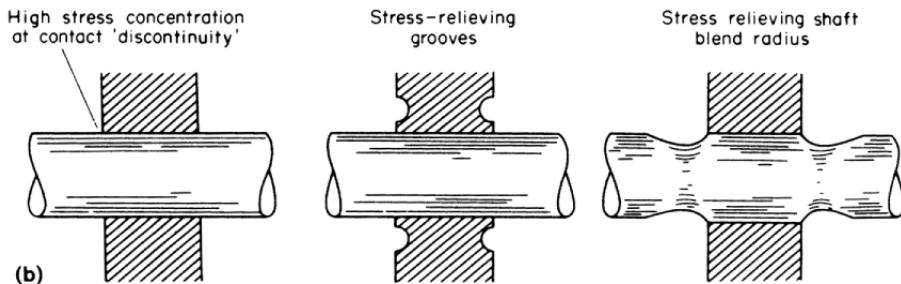


Fig. 10.48. (a) Photoelastic fringe pattern showing stress concentrations produced at contact discontinuities such as the loading of rectangular plates on a flat surface (equivalent to cross-section of cylindrical roller bearing on its support surface); (b) the reduction of stress concentration at press and shrink fits.

Significant stress concentration reductions can be obtained by introducing stress-relieving grooves or a blending fillet (or taper) in the press-fit member or the shaft – see Fig. 10.48.

10.3.8. Use of stress concentration factors with yield criteria

Whilst stress concentration factors are defined in terms of the maximum individual stress at the stress raiser it could be argued that, since stress conditions there are normally biaxial, it would be more appropriate to express them in terms of some “equivalent stress” employing one of the yield criteria introduced in Chapter 15.[†]

Since the maximum shear strain energy (distortion energy) theory of Von Mises is usually considered to be the most applicable to both static and dynamic conditions in ductile materials then, for a biaxial state the Von Mises equivalent stress can be defined as:

$$\sigma_e = \sqrt{\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2} \quad (10.36)$$

and, since there is always a direct relationship between σ_1 and σ_2 within the elastic range for biaxial states (i.e. $\sigma_1 = k\sigma_2$) then

$$\begin{aligned} \sigma_e &= \left[\sigma_1^2 - \frac{\sigma_1^2}{k} - \frac{\sigma_1^2}{k^2} \right]^{1/2} \\ &= \sigma_1 \left[1 - \left(\frac{1}{k} \right) - \left(\frac{1}{k^2} \right) \right]^{1/2} \end{aligned}$$

Then the stress concentration factor expressed in terms of this equivalent stress will be

$$K_e = \frac{\sigma_e}{\sigma_{\text{nom}}} = \frac{\sigma_1}{\sigma_{\text{nom}}} \left[1 - \left(\frac{1}{k} \right) - \left(\frac{1}{k^2} \right) \right]^{1/2} \quad (10.37)$$

Except for the special case of equal bi-axial stress conditions when $\sigma_1 = \sigma_2$ and $K = 1$ the value of K_e is always less than K_t .

A full treatment of the design procedures to be adopted for both ductile and brittle materials incorporating both yield criteria (Von Mises and Mohr) and stress concentration factors is carried out by Peterson⁽⁵⁷⁾ with consideration of static, alternating and combined static and alternating stress conditions.

10.3.9. Design procedure

The following procedure should be adopted for the design of components in order that the effect of stress concentration is minimised and for the component to operate safely and reliably throughout its intended service life.

- (1) Prepare a draft design incorporating the principal features and requirements of the component. The dimensions at this stage will be obtained with reference to the nominal stresses calculated on the basis of known or estimated service loads.
- (2) Identify the potential stress concentration locations.

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

- (3) Undertake the procedures outlined in §10.3.7 to reduce the stress concentration factors at these locations by:
 - (a) streamlining the design where possible to avoid sharp changes in geometry and producing gradual fillet transitions between adjacent parts of different shape and size.
 - (b) If fillet changes cannot be effected owing to design constraints, of e.g. bearing surfaces, undertake other modifications to the design to produce smoother "flow" of the stresses through the component.
 - (c) Where appropriate, reduce the stiffness of the material adjacent to the stress concentration positions to allow greater flexibility and a reduction in the associated stress concentration factor. This is probably best achieved by removal of material as discussed earlier.
- (4) Evaluate the stress concentration factors for the modified design using standard tables^(57, 60) or experimental test procedures such as photoelasticity. Depending on the material and the loading conditions either K_t or K_f may be appropriate.
- (5) Ensure that the maximum stress in the component taking into account both the stress concentration factors and an additional safety factor to account for service uncertainties, does not exceed the safe working stress for the material concerned.

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Examples

Example 10.1

- (a) Two parallel steel cylinders of radii 50mm and 100mm are brought into contact under a load of 2 kN. If the cylinders have a common length of 150 mm and elastic constants of $E = 208 \text{ GN/m}^3$ and $\nu = 0.3$ determine the value of the maximum contact pressure. What will then be the magnitude and position of the maximum shear stress?
- (b) How would the values change if the larger cylinder were replaced by a flat surface?

Solution (a)

For contacting parallel cylinders eqn. (10.9) gives the value of the maximum contact pressure (or compressive stress) as

$$\sigma_c = -p_0 = -0.591 \sqrt{\frac{P}{L\Delta} \left(\frac{1}{R_1} + \frac{1}{R_2} \right)}$$

where

$$\begin{aligned} \Delta &= \frac{1}{E_1}[1 - \nu_1^2] + \frac{1}{E_2}[1 - \nu_2^2] \\ &= \frac{2}{E}[1 - \nu^2] \text{ for similar materials} \\ &= \frac{2 \times 0.91}{208 \times 10^9} \end{aligned}$$

\therefore Max. contact pressure

$$\begin{aligned} p_0 &= 0.591 \sqrt{\frac{2 \times 10^3 \times 208 \times 10^9}{150 \times 10^{-3} \times 2 \times 0.91} \left(\frac{1}{50} + \frac{1}{100} \right) 10^3} \\ &= 0.591 \times 21.38 \times 10^7 \\ &= 126.4 \text{ MN/m}^2 \end{aligned}$$

Maximum shear stress
occurring at a depth

$$= 0.295 p_0 = 37.3 \text{ MN/m}^2$$

$$\mathbf{d = 0.786b}$$

with (from eqn. (10.8))

$$\begin{aligned} b &= 1.076 \sqrt{\frac{P\Delta}{L\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}} \\ &= 1.076 \sqrt{\frac{2 \times 10^3 \times 2 \times 0.91}{150 \times 10^{-3} \times 208 \times 10^9 \times 30}} \\ &= 1.076 \times 0.624 \times 10^{-4} \\ &= 0.067 \text{ mm} \end{aligned}$$

\therefore Depth of max shear stress = $0.786 \times 0.067 = 0.053 \text{ mm}$

(b) Replacing the 100 mm cylinder by a flat surface makes $\frac{1}{R_2} = 0$ and

$$\begin{aligned} \text{contact pressure } p_0 &= 0.591 \sqrt{\frac{2 \times 10^3 \times 208 \times 10^9}{150 \times 10^{-3} \times 2 \times 0.91} \left(\frac{1}{50}\right) 10^3} \\ &= 0.591 \times 17.48 \times 10^7 \\ &= 103.2 \text{ MN/m}^2 \end{aligned}$$

with max shear stress = $0.295 \times 103.2 = 30.4 \text{ MN/m}^2$

and

$$b = 0.082 \text{ mm}$$

\therefore Depth of max shear stress = $0.786 \times 0.082 = 0.064 \text{ mm}$.

Example 10.2

- (a) What will be the maximum compressive stress set up when two spur gears transmit a torque of 250 N m? One gear has 150 teeth on a pitch circle diameter of 200 mm whilst the second gear has 200 teeth. Both gears have a common face-width of 200 mm. Assume $E = 208 \text{ GN/m}^2$ and $\nu = 0.3$ for both gears.
- (b) How will this value change if the spur gears are replaced by helical gears of $17\frac{1}{2}^\circ$ pressure angle and 30° helix?

Solution (a)

(a) From eqn. (10.21) the maximum compressive stress at contact is

$$\sigma_c = -p_0 = -0.475\sqrt{K}$$

with

$$\begin{aligned} K &= \frac{W}{F_w d} \left[\frac{m+1}{m} \right] \\ &= \frac{250}{100 \times 10^{-3}} \times \frac{1}{200 \times 10^{-3} \times 200 \times 10^{-3}} \left[\frac{\frac{200}{150} + 1}{\frac{200}{150}} \right] \\ &= 109375 \end{aligned}$$

$$\therefore \sigma_c = -0.475\sqrt{109375} = -157.1 \text{ MN/m}^2$$

the negative sign indicating a compressive stress value.

Solution (b)

For the helical gears, eqn. (10.22) gives

$$\sigma_c = -p_0 = C \sqrt{\frac{K}{m_p}}$$

and for the given pressure angle and helix values Table 10.2 gives values of

$$C = 0.435 \text{ and } m_p = 1.53$$

$$\therefore \sigma_c = -0.435 \sqrt{\frac{109375}{1.52}} = -116.3 \text{ MN/m}^2$$

Example 10.3

A rectangular bar with shoulder fillet is subjected to a uniform bending moment of 100 Nm. Its dimensions are as follows (see Fig. 10.22) $D = 50 \text{ mm}$; $d = 25 \text{ mm}$; $r = 2.5 \text{ mm}$; $h = 10 \text{ mm}$.

Determine the maximum stress present in the bar for static load conditions. How would the value change if (a) the moment were replaced by a tensile load of 20 kN, (b) the moment and the tensile load are applied together.

Solution

For applied moment

From simple bending theory, nominal stress (related to smaller part of the bar) is:

$$\begin{aligned} \sigma_{\text{nom}} &= \frac{My}{I} = M \times \frac{d}{2} \times \frac{12}{hd^3} = \frac{6M}{hd^2} \\ &= \frac{6 \times 100}{10 \times 10^{-3} \times (25 \times 10^{-3})^2} = 96 \text{ MN/m}^2 \end{aligned}$$

Now from Fig. 10.22 the elastic stress concentration factor for $D/d = 2$ and $r/d = 0.1$ is:

$$K_t = 1.85$$

$$\therefore \text{Maximum stress} = 1.85 \times 96 = 177.6 \text{ MN/m}^2.$$

(a) For tensile load

Again for smallest part of the bar

$$\sigma_{\text{nom}} = \frac{P}{hd} = \frac{20 \times 10^3}{10 \times 10^{-3} \times 25 \times 10^{-3}} = 80 \text{ MN/m}^2$$

and from Fig. 10.2, $K_t = 2.44$

$$\therefore \text{Maximum stress} = 2.44 \times 80 = 195.2 \text{ MN/m}^2.$$

(b) For combined bending and tensile load

Since the maximum stresses arising from both the above conditions will be direct stresses in the fillet radius then the effects may be added directly, i.e. the most adverse stress condition will arise in the bending tensile fillet when the maximum stress due to combined tension and bending will be:

$$\begin{aligned}\sigma_{\max} &= K_t \sigma_{b,\text{nom}} + K'_t \sigma_{d,\text{nom}} \\ &= 177.6 + 195.2 = 372.8 \text{ MN/m}^2\end{aligned}$$

Example 10.4

A semi-circular groove of radius 3 mm is machined in a 50 mm diameter shaft which is then subjected to the following combined loading system:

- (a) a direct tensile load of 50 kN,
- (b) a bending moment of 150 Nm,
- (c) a torque of 320 Nm.

Determine the maximum value of the stress produced by each loading separately and hence estimate the likely maximum stress value under the combined loading.

Solution

For the shaft dimensions given, $D/d = 50/(50 - 6) = 1.14$ and $r/d = 3/44 = 0.068$.

(a) For tensile load

$$\text{Nominal stress } \sigma_{\text{nom}} = \frac{P}{A} = \frac{50 \times 10^3}{\pi \times (22 \times 10^{-3})^2} = 32.9 \text{ MN/m}^2.$$

From Fig. 10.23

$$K_t = 2.51$$

$$\therefore \text{Maximum stress} = 2.51 \times 32.9 = 82.6 \text{ MN/m}^2.$$

(b) For bending

$$\text{Nominal stress } \sigma_{\text{nom}} = \frac{32M}{\pi d^3} = \frac{32 \times 150}{\pi \times (44 \times 10^{-3})^3} = 18 \text{ MN/m}^2$$

and from Fig. 10.24,

$$K_t = 2.24$$

$$\therefore \text{Maximum stress} = 2.24 \times 18 = 40.3 \text{ MN/m}^2.$$

(c) For torsion

$$\text{Nominal stress } \tau_{\text{nom}} = \frac{16T}{\pi d^3} = \frac{16 \times 320}{\pi \times (44 \times 10^{-3})^3} = 19.1 \text{ MN/m}^2$$

and from Fig. 10.25,

$$K_{t_s} = 1.65$$

$$\therefore \text{Maximum stress} = 1.65 \times 19.1 = 31.5 \text{ MN/m}^2$$

(d) For the combined loading the direct stresses due to bending and tension add to give a total maximum direct stress of $82.6 + 40.3 = 122.9 \text{ MN/m}^2$ which will then act in conjunction with the shear stress of 31.5 MN/m^2 as shown on the element of Fig. 10.49.

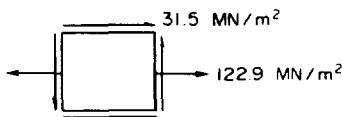


Fig. 10.49.

Then either by Mohrs circle or the use of eqn. (13.11)[†] the maximum principal stress will be

$$\sigma_1 = 130.5 \text{ MN/m}^2.$$

With a maximum shear stress of $\tau_{\max} = 69 \text{ MN/m}^2$.

Example 10.5

Estimate the bending strength of the shaft shown in Fig. 10.50 for two materials

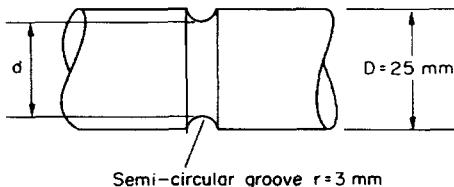


Fig. 10.50.

- (a) Normalised 0.4% C steel with an unnotched endurance limit of 206 MN/m^2
- (b) Heat-treated $3\frac{1}{2}\%$ Nickel steel with an unnotched endurance limit of 480 MN/m^2 .

Solution

From the dimension of the figure

$$\frac{D}{d} = \frac{25}{19} = 1.316 \quad \text{and} \quad \frac{r}{d} = \frac{3}{19} = 0.158$$

$$\therefore \text{From Fig. 10.24} \quad K_t = 1.75$$

From Fig. 10.37 for notch radius of 3 mm

$$q = 0.93 \text{ for normalised steel}$$

$$q = 0.97 \text{ for nickel steel (heat-treated)}$$

\therefore From eqn. (10.32) for the normalised steel

$$K_f = 1 + q(K_t - 1)$$

$$= 1 + 0.93(1.75 - 1) = 1.698$$

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

and the fatigue strength

$$\sigma_f = \frac{206}{1.698} = 12.3 \text{ MN/m}^2$$

and for the nickel steel

$$K_f = 1 + 0.97(1.75 - 1) = 1.728$$

and the fatigue strength

$$\sigma_f = \frac{480}{1.728} = 277.8 \text{ MN/m}^2$$

N.B. Safety factors should then be applied to these figures to allow for service loading conditions, etc.

Problems

10.1 (B). Two parallel steel cylinders of radii 100 mm and 150 mm are required to operate under service conditions which produce a maximum load capacity of 3000 N. If the cylinders have a common length of 200 mm and, for steel, $E = 208 \text{ GN/m}^2$ and $\nu = 0.3$ determine:

- (a) the maximum contact stress under peak load;
- (b) the maximum shear stress and its location also under peak load.

[99.9 MN/m²; 29.5 MN/m²; 0.075 mm]

10.2 (B). How would the answers for problem 10.1 change if the 150 mm radius cylinder were replaced by a flat steel surface?

[77.4 MN/m²; 22.8 MN/m²; 0.097 mm]

10.3 (B). The 150 mm cylinder of problem 10.1 is now replaced by an aluminium cylinder of the same size. What percentage change of results is obtained?

For aluminium $E = 70 \text{ GN/m}^2$ and $\nu = 0.27$. [-29.5%; -29.5%; +41.9%]

10.4 (B). A railway wheel of 400 mm radius exerts a force of 4500 N on a horizontal rail with a head radius of 300 mm. If $E = 208 \text{ GN/m}^2$ and $\nu = 0.3$ for both the wheel and rail determine the maximum contact pressure and the area of contact.

[456 MN/m²; 14.8 mm²]

10.5 (B). What will be the contact area and maximum compressive stress when two steel spheres of radius 200 mm and 150 mm are brought into contact under a force of 1 kN? Take $E = 208 \text{ GN/m}^2$ and $\nu = 0.3$.

[751 MN/m²; 2.01 mm²]

10.6 (B). Determine the maximum compressive stress set up in two spur gears transmitting a pinion torque of 160 Nm. The pinion has 100 teeth on a pitch circle diameter of 130 mm; the gear has 200 teeth and there is a common face-width of 130 mm. Take $E = 208 \text{ GN/m}^2$ and $\nu = 0.3$.

[222 MN/m²]

10.7 (B). Assuming the data of problem 10.6 now relate to a pair of helical gears of 30 helix and 20° pressure angle what will now be the maximum compressive stress?

[161.4 MN/m²]

CHAPTER 11

FATIGUE, CREEP AND FRACTURE

Summary

Fatigue loading is generally defined by the following parameters

$$\text{stress range, } \sigma_r = 2\sigma_a$$

$$\text{mean stress, } \sigma_m = \frac{1}{2}(\sigma_{\max} + \sigma_{\min})$$

$$\text{alternating stress amplitude, } \sigma_a = \frac{1}{2}(\sigma_{\max} - \sigma_{\min})$$

When the mean stress is not zero

$$\text{stress ratio, } R_s = \frac{\sigma_{\min}}{\sigma_{\max}}$$

The *fatigue strength* σ_N for N cycles under zero mean stress is related to that σ_a under a condition of mean stress σ_m by the following alternative formulae:

$$\sigma_a = \sigma_N[1 - (\sigma_m/\sigma_{TS})] \quad (\text{Goodman})$$

$$\sigma_a = \sigma_N[1 - (\sigma_m/\sigma_{TS})^2] \quad (\text{Geber})$$

$$\sigma_a = \sigma_N[1 - (\sigma_m/\sigma_y)] \quad (\text{Soderberg})$$

where σ_{TS} = tensile strength and σ_y = yield strength of the material concerned. Applying a factor of safety F to the Soderberg relationship gives

$$\sigma_a = \frac{\sigma_N}{F} \left[1 - \left(\frac{(\sigma_m \cdot F)}{\sigma_y} \right) \right]$$

Theoretical elastic stress concentration factor for elliptical crack of major and minor axes A and B is

$$K_t = 1 + 2A/B$$

The relationship between any given number of cycles n at one particular stress level to that required to break the component at the same stress level N is termed the “*stress ratio*” (n/N). *Miner’s law* then states that for cumulative damage actions at various stress levels:

$$\frac{n_1}{N_1} + \frac{n_2}{N_2} + \frac{n_3}{N_3} + \dots + \text{etc.} = 1$$

The *Coffin–Manson law* relating the plastic strain range $\Delta\epsilon_p$ to the number of cycles to failure N_f is:

$$\Delta\epsilon_p = K(N_f)^{-b}$$

$$\Delta\epsilon_p = \left(\frac{N_f}{D} \right)^{-b}$$

or

where D is the ductility, defined in terms of the reduction in area r during a tensile test as

$$D = l_n \left(\frac{1}{1 - r} \right)$$

The total strain range = elastic + plastic strain ranges

i.e.

$$\Delta \varepsilon_t = \Delta \varepsilon_e + \Delta \varepsilon_p$$

the elastic range being given by *Basquin's law*

$$\Delta \varepsilon_e = \frac{3.5 \sigma_{TS}}{E} \cdot N_f^{-0.12}$$

Under creep conditions the *secondary creep rate* $\dot{\varepsilon}_s^0$ is given by the *Arrhenius equation*

$$\dot{\varepsilon}_s^0 = A e^{\left(-\frac{H}{RT} \right)}$$

where H is the activation energy, R the universal gas constant, T the absolute temperature and A a constant.

Under increasing stress the power law equation gives the secondary creep rate as

$$\dot{\varepsilon}_s^0 = \beta \sigma^n$$

with β and n both being constants.

The latter two equations can then be combined to give

$$\dot{\varepsilon}_s^0 = K \sigma^n e^{\left(-\frac{H}{RT} \right)}$$

The *Larson–Miller parameter* for life prediction under creep conditions is

$$P_1 = T (\log_{10} t_r + C)$$

The *Sherby–Dorn parameter* is

$$P_2 = \log_{10} t_r - \frac{\alpha}{T}$$

and the *Manson–Haford parameter*

$$P_3 = \frac{T - T_a}{\log_{10} t_r - \log_{10} t_a}$$

where t_r = time to rupture and T_a and $\log_{10} t_a$ are the coordinates of the point at which graphs of T against $\log_{10} t_r$ converge. C and α are constants.

For *stress relaxation under constant strain*

$$\frac{1}{\sigma^{n-1}} = \frac{1}{\sigma_i^{n-1}} + \beta E(n-1)t$$

where σ is the instantaneous stress, σ_i the initial stress, β and n the constants of the power law equation, E is Young's modulus and t the time interval.

Griffith predicts that fracture will occur at a fracture stress σ_f given by

$$\sigma_f^2 = \frac{2bE\gamma}{\pi a(1-\nu^2)} \quad \text{for plane strain}$$

or $\sigma_f^2 = \frac{2bE\gamma}{\pi a}$ for plane stress

where $2a$ = initial crack length (in an infinite sheet)

b = sheet thickness

γ = surface energy of crack faces.

Irwin's expressions for the cartesian components of stress at a crack tip are, in terms of polar coordinates;

$$\begin{aligned}\sigma_{yy} &= \frac{K}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left[1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right] \\ \sigma_{xx} &= \frac{K}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left[1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right] \\ \sigma_{xy} &= \frac{K}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{3\theta}{2}\end{aligned}$$

where K is the *stress intensity factor* = $\sigma\sqrt{\pi a}$

or, for an edge-crack in a semi-infinite sheet

$$K = 1.12\sigma\sqrt{\pi a}$$

For *finite size components* with cracks generally growing from a free surface the *stress intensity factor* is modified to

$$K = \sigma Y \sqrt{a}$$

where Y is a *compliance function* of the form

$$Y = A \left(\frac{a}{W} \right)^{1/2} - B \left(\frac{a}{W} \right)^{3/2} + C \left(\frac{a}{W} \right)^{5/2} - D \left(\frac{a}{W} \right)^{7/2} + E \left(\frac{a}{W} \right)^{9/2}$$

In terms of load P , thickness b and width W

$$K = \frac{P}{bW^{1/2}} \cdot Y$$

For *elastic-plastic conditions* the plastic zone size is given by

$$r_p = \frac{K^2}{\pi\sigma_y^2} \quad \text{for plane stress}$$

and $r_p = \frac{K^2}{3\pi\sigma_y^2} \quad \text{for plane strain}$

r_p being the extent of the plastic zone along the crack axis measured from the crack tip.

Mode II crack growth is described by the *Paris-Erdogan Law*

$$\frac{da}{dN} = C(\Delta K)^m$$

where C and m are material coefficients.

11.1. Fatigue

Introduction

Fracture of components due to fatigue is the most common cause of service failure, particularly in shafts, axles, aircraft wings, etc., where cyclic stressing is taking place. With static loading of a ductile material, plastic flow precedes final fracture, the specimen necks and the fractured surface reveals a fibrous structure, but with fatigue, the crack is initiated from points of high stress concentration on the surface of the component such as sharp changes in cross-section, slag inclusions, tool marks, etc., and then spreads or propagates under the influence of the load cycles until it reaches a critical size when fast fracture of the remaining cross-section takes place. The surface of a typical fatigue-failed component shows three areas, the small point of initiation and then, spreading out from this point, a smaller glass-like area containing shell-like markings called “*arrest lines*” or “*conchoidal markings*” and, finally, the crystalline area of rupture.

Fatigue failures can and often do occur under loading conditions where the fluctuating stress is below the tensile strength and, in some materials, even below the elastic limit. Because of its importance, the subject has been extensively researched over the last one hundred years but even today one still occasionally hears of a disaster in which fatigue is a prime contributing factor.

11.1.1. The S/N curve

Fatigue tests are usually carried out under conditions of rotating – bending and with a zero mean stress as obtained by means of a Wohler machine.

From Fig. 11.1, it can be seen that the top surface of the specimen, held “cantilever fashion” in the machine, is in tension, whilst the bottom surface is in compression. As the specimen rotates, the top surface moves to the bottom and hence each segment of the surface moves continuously from tension to compression producing a stress-cycle curve as shown in Fig. 11.2.

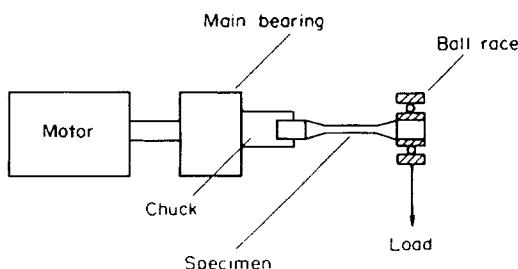


Fig. 11.1. Single point load arrangement in a Wohler machine for zero mean stress fatigue testing.

In order to understand certain terms in common usage, let us consider a stress-cycle curve where there is a positive tensile mean stress as may be obtained using other types of fatigue machines such as a Haigh “push-pull” machine.

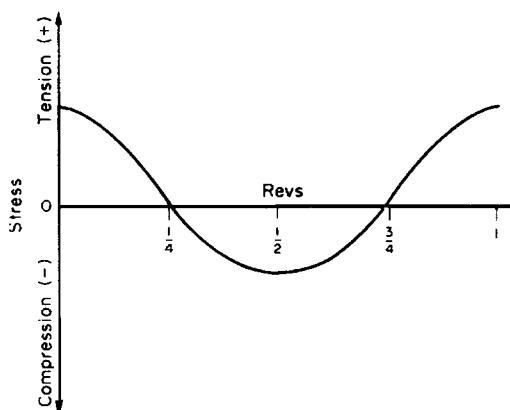


Fig. 11.2. Simple sinusoidal (zero mean) stress fatigue curve, "reversed-symmetrical".

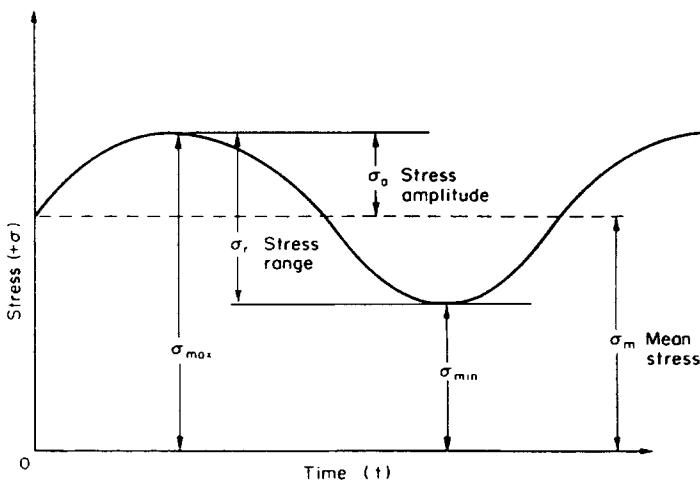


Fig. 11.3. Fluctuating tension stress cycle producing positive mean stress.

The stress-cycle curve is shown in Fig. 11.3, and from this diagram it can be seen that:

$$\text{Stress range, } \sigma_r = 2\sigma_a. \quad (11.1)$$

$$\text{Mean stress, } \sigma_m = \frac{\sigma_{\max} + \sigma_{\min}}{2} \quad (11.2)$$

$$\text{Alternating stress amplitude, } \sigma_a = \frac{\sigma_{\max} - \sigma_{\min}}{2} \quad (11.3)$$

If the mean stress is not zero, we sometimes make use of the "stress ratio" R_s where

$$R_s = \frac{\sigma_{\min}}{\sigma_{\max}} \quad (11.4)$$

The most general method of presenting the results of a fatigue test is to plot a graph of the stress amplitude as ordinate against the corresponding number of cycles to failure as

abscissa, the amplitude being varied for each new specimen until sufficient data have been obtained. This results in the production of the well-known *S/N curve* – Fig. 11.4.

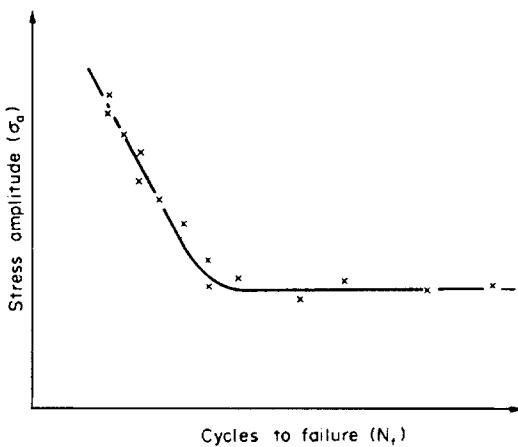


Fig. 11.4. Typical S/N curve fatigue life curve.

In using the S/N curve for design purposes it may be advantageous to express the relationship between σ_a and N_f , the number of cycles to failure. Various empirical relationships have been proposed but, provided the stress applied does not produce plastic deformation, the following relationship is most often used:

$$\sigma_r^a N_f = K \quad (11.5)$$

Where a is a constant which varies from 8 to 15 and K is a second constant depending on the material – see Example 11.1.

From the S/N curve the “fatigue limit” or “endurance limit” may be ascertained. The “fatigue limit” is the stress condition below which a material may endure an infinite number of cycles prior to failure. Ferrous metal specimens often produce S/N curves which exhibit fatigue limits as indicated in Fig. 11.5(a). The “fatigue strength” or “endurance limit”, is the stress condition under which a specimen would have a fatigue life of N cycles as shown in

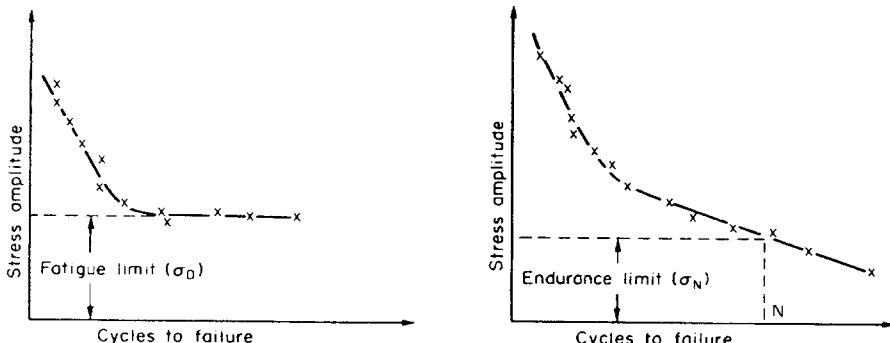


Fig. 11.5. S/N curve showing (a) fatigue limit, (b) endurance limit.

Fig. 10.5(b). Non-ferrous metal specimens show this type of curve and hence components made from aluminium, copper and nickel, etc., must always be designed for a finite life.

Another important fact to note is that the results of laboratory experiments utilising plain, polished, test pieces cannot be applied directly to structures and components without modification of the intrinsic values obtained. Allowance will have to be made for many differences between the component in its working environment and in the laboratory test such as the surface finish, size, type of loading and effect of stress concentrations. These factors will reduce the intrinsic (i.e. plain specimen) fatigue strength value thus,

$$\sigma'_N = \frac{\sigma_N}{K_f} [C_a \cdot C_b \cdot C_c] \quad (11.6)$$

where σ'_N is the “modified fatigue strength” or “modified fatigue limit”, σ_N is the intrinsic value, K_f is the fatigue strength reduction factor (see § 11.1.4) and C_a , C_b and C_c are factors allowing for size, surface finish, type of loading, etc.

The types of fatigue loading in common usage include direct stress, where the material is repeatedly loaded in its axial direction; plane bending, where the material is bent about its neutral plane; rotating bending, where the specimen is being rotated and at the same time subjected to a bending moment; torsion, where the specimen is subjected to conditions which produce reversed or fluctuating torsional stresses and, finally, combined stress conditions, where two or more of the previous types of loading are operating simultaneously. It is therefore important that the method of stressing and type of machine used to carry out the fatigue test should always be quoted.

Within a fairly wide range of approximately 100 cycles/min to 6000 cycles/min, the effect of speed of testing (i.e. frequency of load cycling) on the fatigue strength of metals is small but, nevertheless, frequency may be important, particularly in polymers and other materials which show a large hysteresis loss. Test details should, therefore, always include the frequency of the stress cycle, this being chosen so as not to affect the result obtained (depending upon the material under test) the form of test piece and the type of machine used. Further details regarding fatigue testing procedure are given in BS3518: Parts 1 to 5.

Most fatigue tests are carried out at room temperature but often tests are also carried out at elevated or sub-zero temperatures depending upon the expected environmental operating conditions. At low temperatures the fatigue strength of metals show no deterioration and may even show a slight improvement, however, with increase in temperature, the fatigue strength decreases as creep effects are added to those of fatigue and this is revealed by a more pronounced effect of frequency of cycling and of mean stress since creep is both stress- and time-dependent.

When carrying out elevated temperature tests in air, oxidation of the sample may take place producing a condition similar to corrosion fatigue. Under the action of the cyclic stress, protective oxide films are cracked allowing further and more severe attack by the corrosive media. Thus fatigue and corrosion together ensure continuous propagation of cracks, and materials which show a definite fatigue limit at room temperature will not do so at elevated temperatures or at ambient temperatures under corrosive conditions – see Fig. 11.6.

11.1.2. P/S/N curves

The fatigue life of a component as determined at a particular stress level is a very variable quantity so that seemingly identical specimens may give widely differing results. This scatter

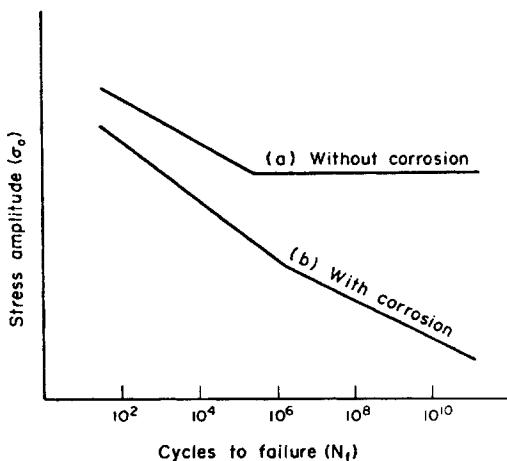


Fig. 11.6. The effect of corrosion on fatigue life. S/N Curve for (a) material showing fatigue limit; (b) same material under corrosion conditions.

arises from many sources including variations in material composition and heterogeneity, variations in surface finish, variations in axiality of loading, etc.

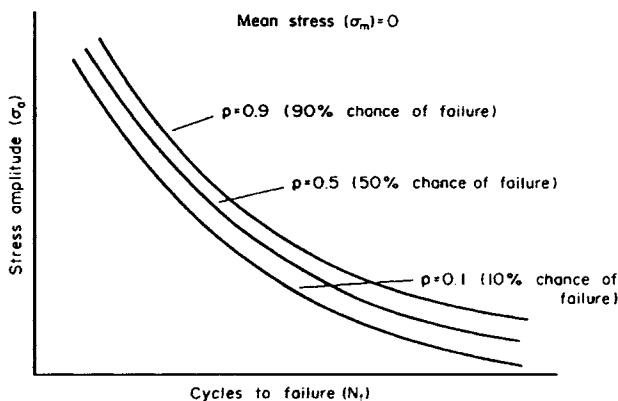


Fig. 11.7. P/S/N curves indicating percentage chance of failure for given stress level after known number of cycles (zero mean stress)

To overcome this problem, a number of test pieces should be tested at several different stresses and then an estimate of the life at a particular stress level for a given probability can be made. If the probability of 50% chance of failure is required then a P/S/N curve can be drawn through the median value of the fatigue life at the stress levels used in the test. It should be noted that this 50% ($p = 0.5$) probability curve is the curve often displayed in textbooks as the S/N curve for a particular material and if less probability of failure is required then the fatigue limit value will need to be reduced.

11.1.3. Effect of mean stress

If the fatigue test is carried out under conditions such that the mean stress is tensile (Fig. 11.3), then, in order that the specimen will fail in the same number of cycles as a similar specimen tested under zero mean stress conditions, the stress amplitude in the former case will have to be reduced. The fact that an increasing tensile mean stress lowers the fatigue or endurance limit is important, and all S/N curves should contain information regarding the test conditions (Fig. 11.8).

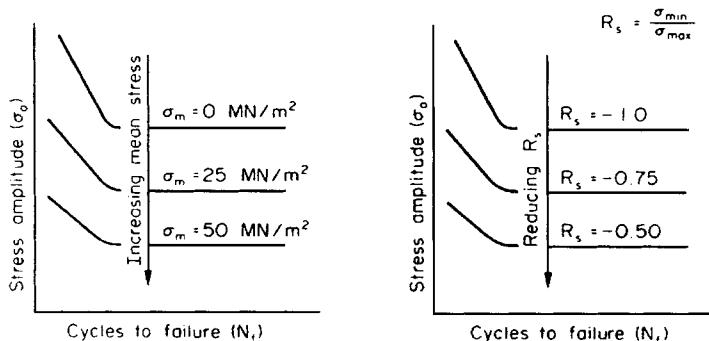


Fig. 11.8. Effect of mean stress on the S/N curve expressed in alternative ways.

A number of investigations have been made of the quantitative effect of *tensile mean stress* resulting in the following equations:

$$\text{Goodman}^{(1)} \quad \sigma_a = \sigma_N \left[1 - \left(\frac{\sigma_m}{\sigma_{TS}} \right) \right] \quad (11.7)$$

$$\text{Geber}^{(2)} \quad \sigma_a = \sigma_N \left[1 - \left(\frac{\sigma_m}{\sigma_{TS}} \right)^2 \right] \quad (11.8)$$

$$\text{Soderberg}^{(3)} \quad \sigma_a = \sigma_N \left[1 - \left(\frac{\sigma_m}{\sigma_y} \right) \right] \quad (11.9)$$

where σ_N = the fatigue strength for N cycles under zero mean stress conditions.

σ_a = the fatigue strength for N cycles under condition of mean stress σ_m .

σ_{TS} = tensile strength of the material.

σ_y = yield strength of the material.

The above equations may be shown in graphical form (Fig. 11.9) and in actual practice it has been found that most test results fall within the envelope formed by the parabolic curve of Geber and the straight line of Goodman. However, because the use of Soderberg gives an additional margin of safety, this is the equation often preferred – see Example 11.2.

Even when using the Soderberg equation it is usual to apply a factor of safety F to both the alternating and the steady component of stress, in which case eqn. (11.9) becomes:

$$\sigma_a = \frac{\sigma_N}{F} \left(1 - \frac{\sigma_m \times F}{\sigma_y} \right) \quad (11.10)$$

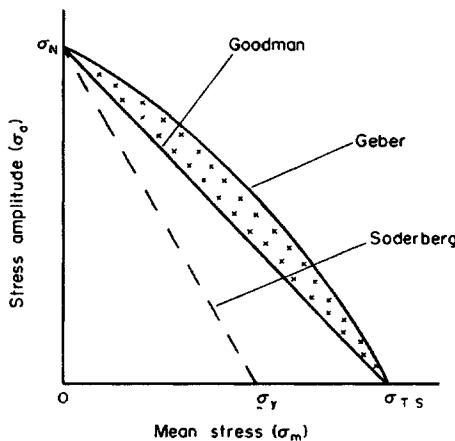


Fig. 11.9. Amplitude/mean stress relationships as per Goodman, Geber and Soderberg.

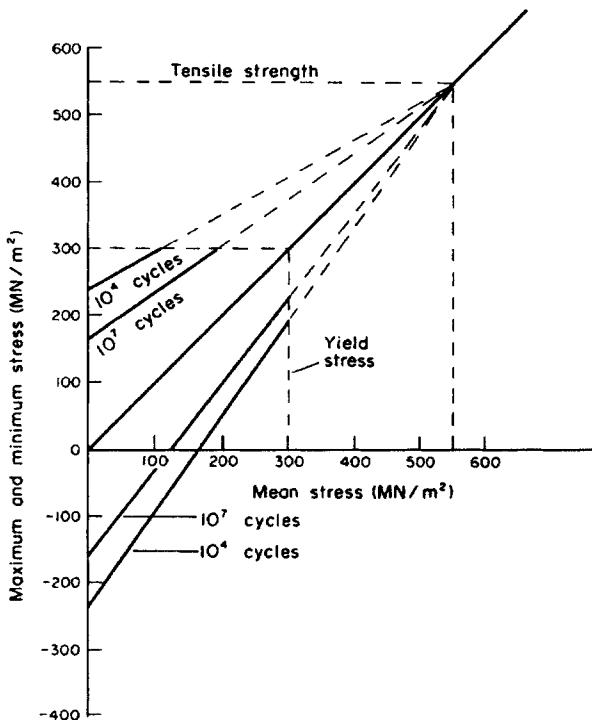


Fig. 11.10. Smith diagram.

The interrelationship of mean stress and alternating stress amplitude is often shown in diagrammatic form frequently collectively called Goodman diagrams. One example is shown in Fig. 11.10, and includes the experimentally derived curves for endurance limits of a specific steel. This is called a Smith diagram. Many alternative forms of presentation of data are possible including the Haigh diagram shown in Fig. 11.11, and when understood

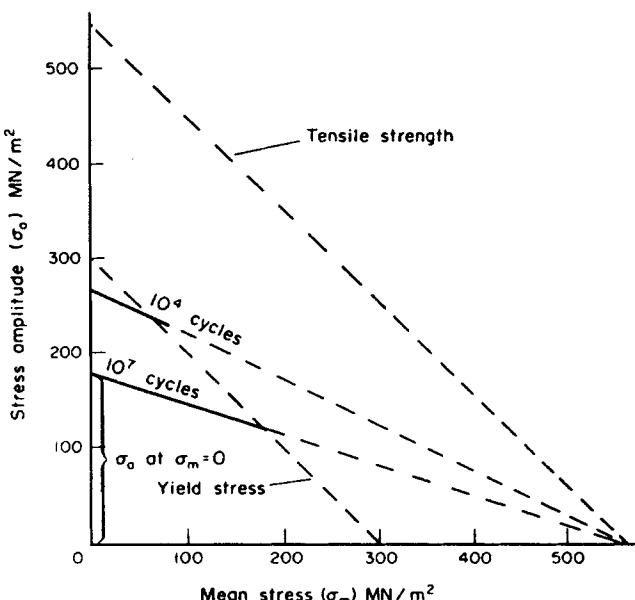


Fig. 11.11. Haigh diagram.

by the engineer these diagrams can be used to predict the fatigue life of a component under a particular stress regime. If the reader wishes to gain further information about the use of these diagrams it is recommended that other texts be consulted.

The effect of a *compressive mean stress* upon the life of a component is not so well documented or understood as that of a tensile mean stress but in general most materials do not become any worse and may even show an improved performance under a compressive mean stress. In calculations it is usual therefore to take the mean stress as zero under these conditions.

11.1.4. Effect of stress concentration

The influence of stress concentration (see §10.3) can be illustrated by consideration of an elliptical crack in a plate subjected to a tensile stress. Provided that the plate is very large, the "theoretical stress concentration" factor K_t is given by:

$$K_t = 1 + \frac{2A}{B} \quad (11.11)$$

where "A" and "B" are the crack dimensions as shown in Fig. 11.12.

If the crack is perpendicular to the direction of stress, then A is large compared with B and hence K_t will be large. If the crack is parallel to the direction of stress, then A is very small compared with B and hence $K_t = 1$. If the dimensions of A and B are equal such that the crack becomes a round hole, then $K_t = 3$ and a maximum stress of $3\sigma_{\text{nom}}$ acts at the sides of the hole.

The effect of sudden changes of section, notches or defects upon the fatigue performance of a component may be indicated by the "fatigue notch" or "fatigue strength reduction" factor

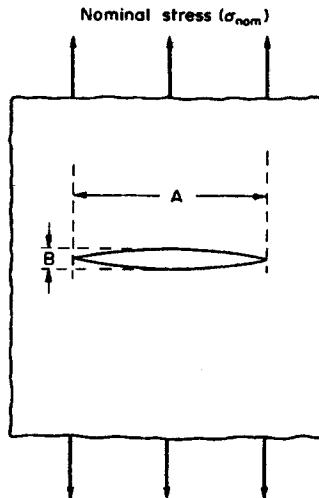


Fig. 11.12. Elliptical crack in semi-infinite plate.

K_f , which is the ratio of the stress amplitude at the fatigue limit of an un-notched specimen, to that of a notched specimen under the same loading conditions.

K_f is always less than the static theoretical stress concentration factor referred to above because under the compressive part of a tensile-compressive fatigue cycle, a fatigue crack is unlikely to grow. Also the ratio of K_f/K_t decreases as K_t increases, sharp notches having less effect upon fatigue life than would be expected. The extent to which the stress concentration effect under fatigue conditions approaches that for static conditions is given by the "notch sensitivity factor" q , and the relationship between them may be simply expressed by:

$$q = \frac{K_f - 1}{K_t - 1} \quad (11.12)$$

thus q is always less than 1. See also §10.3.5.

Notch sensitivity is a very complex factor depending not only upon the material but also upon the grain size, a finer grain size resulting in a higher value of q than a coarse grain size. It also increases with section size and tensile strength (thus under some circumstances it is possible to decrease the fatigue life by increasing tensile strength!) and, as has already been mentioned, it depends upon the severity of notch and type of loading.

In dealing with a ductile material it is usual to apply the factor K_f only to the fluctuating or alternating component of the applied stress. Equation (11.10) then becomes:

$$\sigma_a = \frac{\sigma_N}{F \cdot K_f} \left[1 - \left(\frac{\sigma_m \cdot F}{\sigma_y} \right) \right] \quad (11.13)$$

A typical application of this formula is given in Example 11.3.

11.1.5. Cumulative damage

In everyday, true-life situations, for example a car travelling over varying types of roads or an aeroplane passing through various weather conditions on its flight, stresses will not generally be constant but will vary according to prevailing conditions.

Several attempts have been made to predict the fatigue strength for such variable stresses using S/N curves for constant mean stress conditions. Some of the predictive methods available are very complex but the simplest and most well known is "Miner's Law."

Miner⁽⁷⁾ postulated that whilst a component was being fatigued, internal damage was taking place. The nature of the damage is difficult to specify but it may help to regard damage as the slow internal spreading of a crack, although this should not be taken too literally. He also stated that the extent of the damage was directly proportional to the number of cycles for a particular stress level, and quantified this by adding, "*The fraction of the total damage occurring under one series of cycles at a particular stress level, is given by the ratio of the number of cycles actually endured n to the number of cycles N required to break the component at the same stress level*". The ratio n/N is called the "cycle ratio" and Miner proposed that failure takes place when the sum of the cycle ratios equals unity.

i.e. when

$$\Sigma n/N = 1$$

or

$$\frac{n_1}{N_1} + \frac{n_2}{N_2} + \frac{n_3}{N_3} + \dots + \text{etc} = 1 \quad (11.14)$$

If equation (11.14) is merely treated as an algebraic expression then it should be unimportant whether we put n_3/N_3 before n_1/N_1 etc., but experience has shown that the order of application of the stress is a matter of considerable importance and that the application of a higher stress amplitude first has a more damaging effect on fatigue performance than the application of an initial low stress amplitude. Thus the cycle ratios rarely add up to 1, the sum varying between 0.5 and 2.5, but it does approach unity if the number of cycles applied at any given period of time for a particular stress amplitude is kept relatively small and frequent changes of stress amplitude are carried out, i.e. one approaches random loading conditions. A simple application of Miner's rule is given in Example 11.4.

11.1.6. Cyclic stress-strain

Whilst many components such as axle shafts, etc., have to withstand an almost infinite number of stress reversals in their lifetime, the stress amplitudes are relatively small and usually do not exceed the elastic limit. On the other hand, there are a growing number of structures such as aeroplane cabins and pressure vessels where the interval between stress cycles is large and where the stresses applied are very high such that plastic deformation may occur. Under these latter conditions, although the period in time may be long, the number of cycles to failure will be small and in recent years interest has been growing in this "low cycle fatigue".

If, during fatigue testing under these high stress cycle conditions, stress and strain are continually monitored, a hysteresis loop develops characteristic of each cycle.

Figure 11.13 shows typical loops under constant stress amplitude conditions, each loop being displaced to the right for the sake of clarity. It will be observed that with each cycle, because of work hardening, the width of the loop is reduced, eventually the loop narrowing to a straight line under conditions of total elastic deformation.

The relationship between the loop width W and the number of cycles N is given by:

$$W = AN^{-h} \quad (11.15)$$

where A is a constant and h the measure of the rate of work hardening.

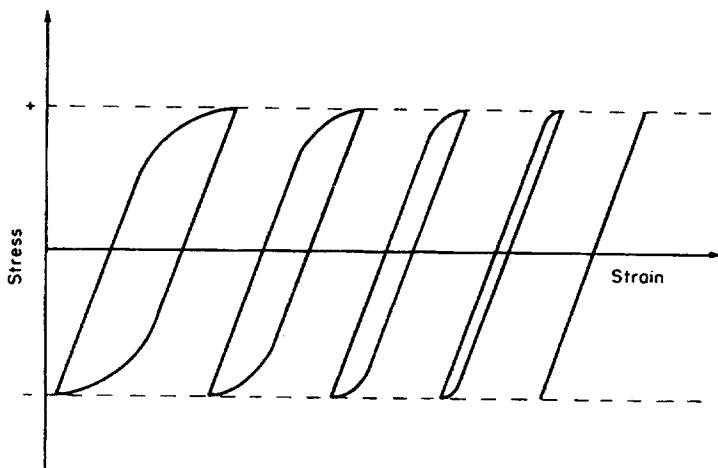


Fig. 11.13. Cyclic stress-strain under constant stress conditions – successive loading loops displaced to right for clarity. Hysteresis effects achieved under low cycle, high strain (constant stress amplitude) fatigue.

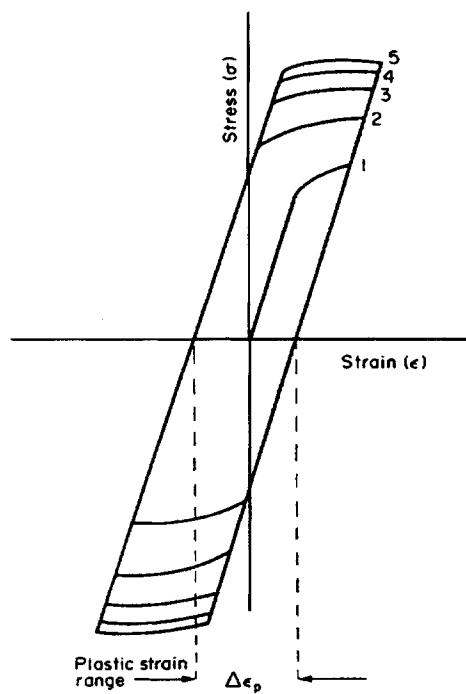


Fig. 11.14. Cyclic stress-strain under constant strain amplitude conditions.

If instead of using constant *stress* amplitude conditions, one uses constant *strain* amplitude conditions then the form of loop is indicated in Fig. 11.14. Under these conditions the stress range increases with the number of cycles but the extent of the increase reduces with each cycle such that after about 20% of the life of the component the loop becomes constant.

If now a graph is drawn (using logarithmic scales) of the plastic strain range against the number of cycles to failure a straight line results (Fig. 11.15). From this graph we obtain the following equation for the plastic strain range $\Delta\epsilon_p$ which is known as the Coffin–Manson Law.⁽⁸⁾

$$\Delta\epsilon_p = K(N_f)^{-b} \quad (11.16)$$

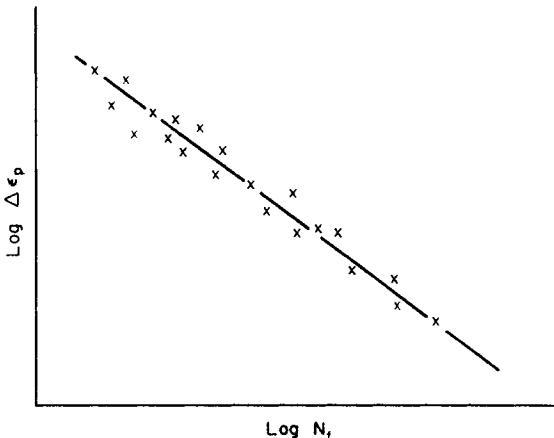


Fig. 11.15. Relationship between plastic strain and cycles to failure in low cycle fatigue.

The value of b varies between 0.5 and 0.6 for most metals, whilst the constant K can be related to the ductility of the metal. Equation (11.16) can also be expressed as:

$$\Delta\epsilon_p = \left(\frac{N_f}{D}\right)^{-b} \quad (11.17)$$

where D is the ductility as determined by the reduction in area r in a tensile test.

i.e.

$$D = l_n \left(\frac{1}{1 - r} \right)$$

In many applications, the total strain range may be known but it may be difficult to separate it into plastic and elastic components; thus a combined equation may be more useful.

$$\Delta\epsilon_t = \Delta\epsilon_e + \Delta\epsilon_p$$

Where $\Delta\epsilon_t$, $\Delta\epsilon_e$ and $\Delta\epsilon_p$ stand for total, elastic and plastic strain ranges respectively. Relationships between $\Delta\epsilon_p$ and N_f are given above but $\Delta\epsilon_e$ may be related to N_f by the following modified form of *Basquin's Law*.⁽⁹⁾

$$\Delta\epsilon_e = 3.5 \times \frac{\sigma_{TS}}{E} \times N_f^{-0.12} \quad (11.18)$$

If a graph is plotted (Fig. 11.16) of strain range against number of cycles to failure, it can be seen that the beginning part of the curve closely fits the slope of Coffin's equation while the latter part fits the modified Basquin's equation, the cross-over point being at about 10^5

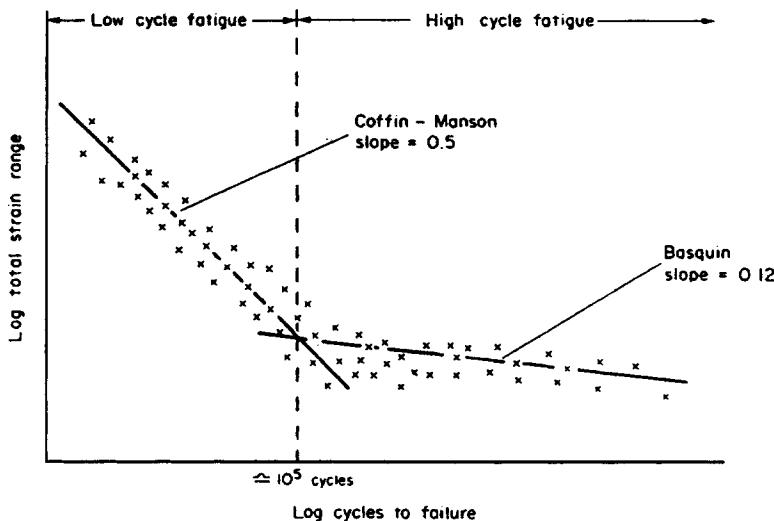


Fig. 11.16. Relationship between total strain and cycles to failure in low and high cycle fatigue.

cycles. Therefore, it can be said that up to this figure fatigue performance is a function of the material's ductility, whilst for cycles in excess of this, life is a function of the strength of the material.

11.1.7. Combating fatigue

When selecting a material for use under fatigue conditions it may be better to select one which shows a fatigue limit, e.g. steel, rather than one which exhibits an endurance limit, e.g. aluminium. This has the advantage of enabling the designer to design for an infinite life provided that the working stresses are kept to a suitably low level, whereas if the latter material is selected then design must be based upon a finite life.

In general, for most steels, the fatigue limit is about 0.5 of the tensile strength, therefore, by selecting a high-strength material the allowable working stresses may be increased. Figure 11.17.

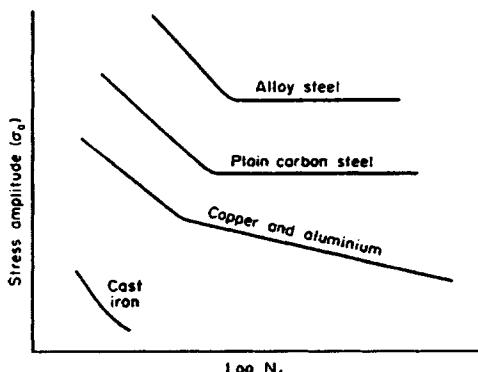


Fig. 11.17. Relative performance of various materials under fatigue conditions.

Following on the above, any process that increases tensile strength should raise the fatigue limit and one possible method of accomplishing this with steels is to carry out heat treatment. The general effect of heat treatment on a particular steel is shown in Fig. 11.18.

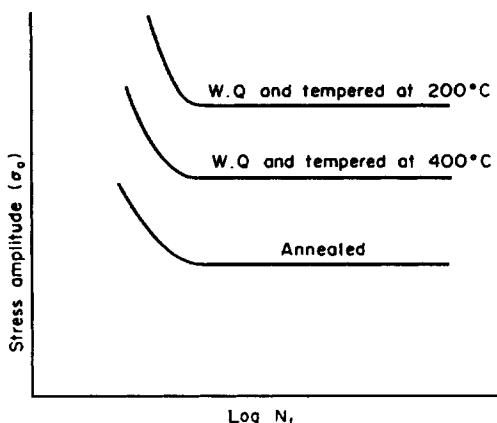


Fig. 11.18. Effect of heat treatment upon the fatigue limit of steel.

Sharp changes in cross-section will severely reduce the fatigue limit (see §10.3.4), and therefore generous radii can be used to advantage in design. Likewise, surface finish will also have a marked effect and it must be borne in mind that fatigue data obtained in laboratory tests are often based upon highly polished, notch-free, samples whilst in practice the component is likely to have a machined surface and many section changes. The sensitivity of a material to notches tends to increase with increase in tensile strength and decrease with increase in plasticity, thus, in design situations, a compromise between these opposing factors must be reached.

Figure 11.19 shows the fatigue limits of typical steels in service expressed as a percentage of the fatigue limits obtained for the same steels in the laboratory and it will be noticed that

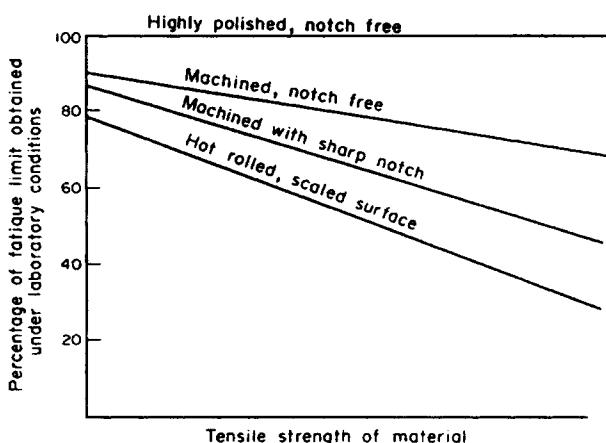


Fig. 11.19. Effect of surface conditions on the fatigue strength of materials.

the fatigue limit of a low-strength steel is not affected to the same extent as the high-strength steel, i.e. the former is less notch-sensitive (another factor to be taken account of when looking at the relative cost of the basic material). However, it must be pointed out that it may be poor economy to overspecify surface finish, particularly where stress levels are relatively low.

Because fatigue cracks generally initiate at the surface of a component under tensile stress conditions, certain processes, both chemical and mechanical, which introduce residual surface compressive stresses may be utilised to improve fatigue properties (see §10.2). However, the extent of the improvement is difficult to assess quantitatively at this juncture of time. Among the chemical treatments, the two most commonly employed are *carburising* and *nitriding* which bring about an expansion of the lattice at the metal surface by the introduction of carbon and nitrogen atoms respectively. Figure 11.20 shows the effect upon fatigue limit.

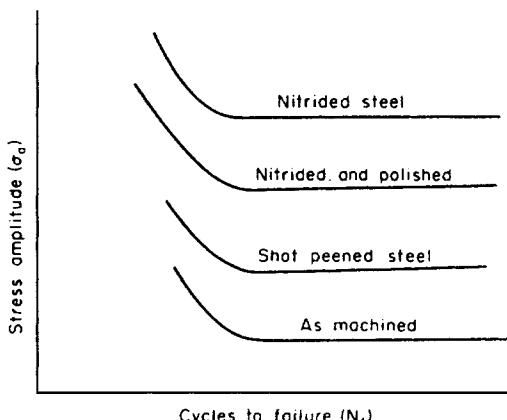


Fig. 11.20. Effect of processes which introduce surface residual stresses upon the fatigue strength of a steel.

The most popular mechanical method of improving fatigue limits is *shot peening*, the surface of the material being subjected to bombardment by small pellets or shot of suitable material. In this manner, compressive residual stresses are induced but only to a limited depth, roughly 0.25 mm. Other mechanical methods involve improving fatigue properties around holes by pushing through balls which are slightly over-sized – a process called “*ballising*,” and the use of balls or a roller to cold work shoulders on fillets – a process called “*rolling*”.

11.1.8. Slip bands and fatigue

The onset of fatigue is usually characterised by the appearance on the surface of the specimen of slip bands which, after about 5% of the fatigue life, become permanent and cannot be removed by electropolishing. With increase in the number of load cycles these bands deepen until eventually a crack is formed.

Using electron microscopical techniques Forsyth⁽¹⁰⁾ observed *extrusions* and *intrusions* from well-defined slip bands and Cottrell⁽¹¹⁾ proposed a theory of cross-slip or slip on alternate slip planes whereby, during the tensile half of the stress cycle, slip occurs on each plane in turn to produce two surface steps which on the compressive half of the cycle are

converted into an intrusion and an extrusion (see Fig. 11.21). Although an intrusion is only very small, being approximately $1\text{ }\mu\text{m}$ deep, it nevertheless can act as a stress raiser and initiate the formation of a true fatigue crack.

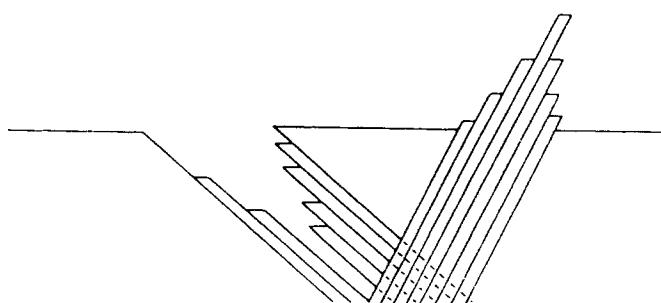


Fig. 11.21. Diagrammatic representation of the formation of intrusions and extrusions.

Fatigue endurance is commonly divided into two periods: (i) the “*crack initiation*” period; (ii) the “*crack growth*” or “*propagation*” period. It is now accepted that the fatigue crack is initiated by the deepening of the slip band grooves by dislocation movement into crevices and finally cracks, but this makes it very difficult to distinguish between crack initiation and crack propagation and therefore a division of the fatigue based upon mode of crack growth is often more convenient.

Initially the cracks will form in the surface grains and develop along the active slip plane as mentioned briefly above. These cracks are likely to be aligned with the direction of maximum shear within the component, i.e. at 45° to the maximum tensile stress. This is often referred to as *Stage I growth* and is favoured by zero mean stress and low cyclic stress conditions.

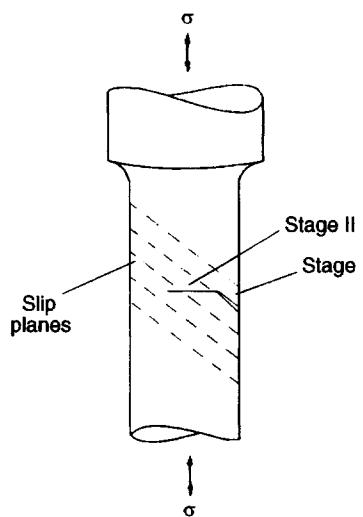


Fig. 11.22. Stage I and II fatigue crack propagation.

At some point, usually when the crack encounters a grain boundary. Stage I is replaced by *Stage II growth* in which the crack is normal to the maximum principal tensile stress. This stage is favoured by a tensile mean stress and high cyclic stress conditions. Close examination of the fractured surface shows that over that part associated with Stage II, there are a large number of fine lines called “*striations*”, each line being produced by one fatigue cycle and by measuring the distance between a certain number of striations the fatigue crack growth rate can be calculated.

Once the fatigue crack has reached some critical length such that the energy for further growth can be obtained from the elastic energy of the surrounding metal, catastrophic failure takes place. This final fracture area is rougher than the fatigue growth area and in mild steel is frequently crystalline in appearance. Sometimes it may show evidence of plastic deformation before final separation occurred. Further discussion of fatigue crack growth is introduced in §11.3.7.

11.2. Creep

Introduction

Creep is the time-dependent deformation which accompanies the application of stress to a material. At room temperatures, apart from the low-melting-point metals such as lead, most metallic materials show only very small creep rates which can be ignored. With increase in temperature, however, the creep rate also increases and above approximately $0.4 T_m$, where T_m is the melting point on the Kelvin scale, creep becomes very significant. In high-temperature engineering situations related to gas turbine engines, furnaces and steam turbines, etc., deformation caused by creep can be very significant and must be taken into account.

11.2.1. The creep test

The creep test is usually carried out at a constant temperature and under constant load conditions rather than at constant stress conditions. This is acceptable because it is more representative of service conditions. A typical creep testing machine is shown in Fig. 11.23. Each end of the specimen is screwed into the specimen holder which is made of a creep-resisting alloy and thermocouples and accurate extensometers are fixed to the specimen in order to measure temperature and strain. The electric furnace is then lowered into place and when all is ready and the specimen is at the desired temperature, the load is applied by adding weights to the lower arm and readings are taken at periodic intervals of extension against time. It is important that accurate control of temperature is possible and to facilitate this the equipment is often housed in a temperature-controlled room.

The results from the creep test are plotted in graphical form to produce a typical curve as shown in Fig. 11.24. After the initial extension *OA* which is produced as soon as the test load is applied, and which is not part of the creep process proper (but which nevertheless should not be ignored), the curve can be divided into three stages. In the first or *primary* stage *AB*, the movement of dislocations is very rapid, any barriers to movement caused by work-hardening being overcome by the recovery processes, albeit at a decreasing rate. Thus the initial *creep strain rate* is high but it rapidly decreases to a constant value. In the

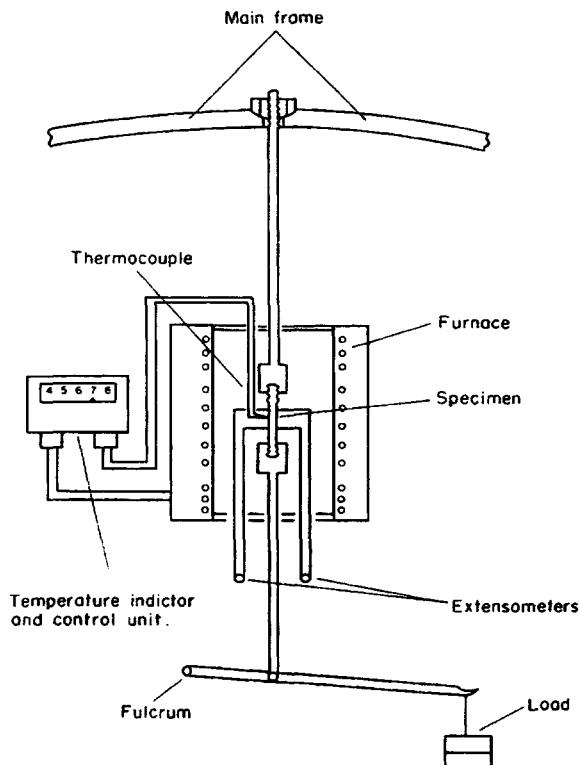


Fig. 11.23. Schematic diagram of a typical creep testing machine.

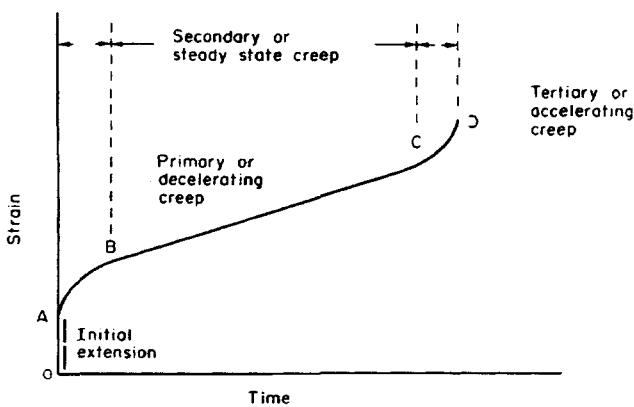


Fig. 11.24. Typical creep curve.

secondary stage BC, the work-hardening process of “dislocation pile-up” and “entanglement” are balanced by the recovery processes of “dislocation climb” and “cross-slip”, to give a straight-line relationship and the slope of the graph in this steady-state portion of the curve is equal to the secondary creep rate. Since, generally, the primary and tertiary stages occur quickly, it is the secondary creep rate which is of prime importance to the design engineer.

The third or *tertiary* stage *CD* coincides with the formation of internal voids within the specimen and this leads to "necking", causing the stress to increase and rapid failure to result.

The shape of the creep curve for any material will depend upon the temperature of the test and the stress at any time since these are the main factors controlling the work-hardening and recovery processes. With increase in temperature, the creep rate increases because the softening processes such as "dislocation climb" can take place more easily, being diffusion-controlled and hence a thermally activated process.

It is expected, therefore, that the creep rate is closely related to the *Arrhenius equation*, viz.:

$$\dot{\epsilon}_s^0 = A e^{-H/RT} \quad (11.19)$$

where $\dot{\epsilon}_s^0$ is the *secondary creep rate*, H is the *activation energy* for creep for the material under test, R is the universal gas constant, T is the absolute temperature and A is a constant. It should be noted that both A and H are not true constants, their values depending upon stress, temperature range and metallurgical variables.

The secondary creep rate also increases with increasing stress, the relationship being most commonly expressed by the *power law equation*:

$$\dot{\epsilon}_s^0 = \beta \sigma^n \quad (11.20)$$

where β and n are constants, the value of n usually varying between 3 and 8.

Equations (11.19) and (11.20) may be combined to give:

$$\dot{\epsilon}_s^0 = K \sigma^n e^{-H/RT} \quad (11.21)$$

Figure 11.25 illustrates the effect of increasing stress or temperature upon the creep curve and it can be seen that increasing either of these two variables results in a similar change of creep behaviour, that is, an increase in the secondary or minimum creep rate, a shortening of the secondary creep stage, and the earlier onset of tertiary creep and fracture.

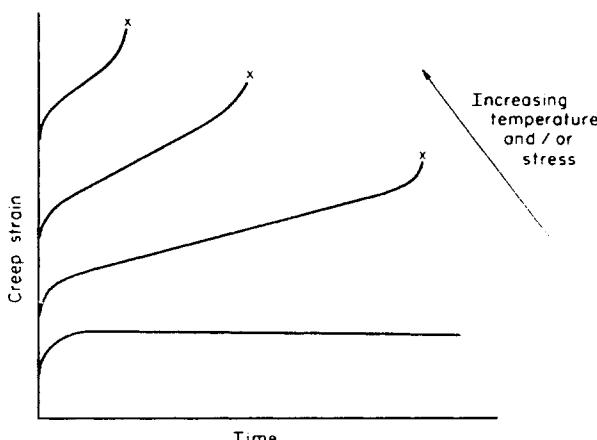


Fig. 11.25. Creep curves showing effect of increasing temperature or stress.

11.2.2. Presentation of creep data

When dealing with problems in which creep is important, the design engineer may wish to know whether the creep strain over the period of expected life of the component is tolerable, or he may wish to know the value of the maximum operating stress if the creep strain is not to exceed a specified figure over a given period of time.

In order to assist in the answering of these questions, creep data are often published in other forms than the standard strain-time curve. Figure 11.26 shows a number of fixed strain curves presented in the form of an *isometric stress-time* diagram which relates strain, stress and time for a fixed, specified, temperature and material, while Fig. 11.27 is an *isometric strain-time diagram*.

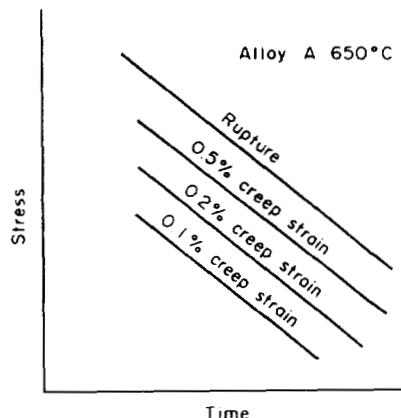


Fig. 11.26. Isometric stress-time diagrams.

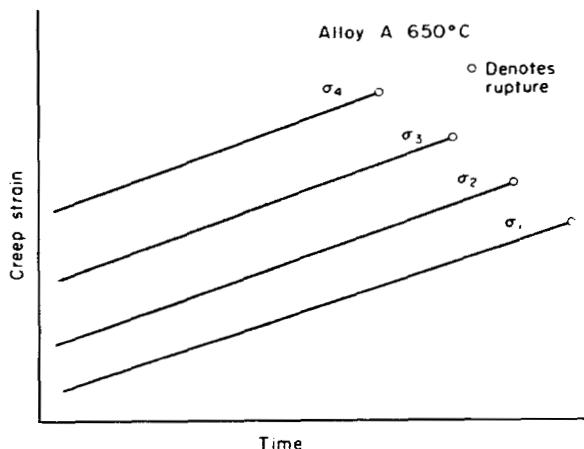


Fig. 11.27. Isometric strain-time diagram.

Sometimes, instead of presenting data relating to a fixed temperature, the strain may be constant and curves of equal time called *isochronous* stress–temperature curves. Fig. 11.28 may be given. Such curves can be used for comparing the properties of various alloys and Fig. 11.29 shows relations for a creep strain of 0.2% in 3000 hours. Such information might be applicable to an industrial gas turbine used intermittently.

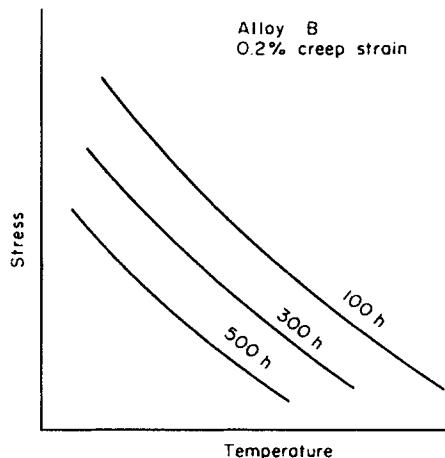


Fig. 11.28. Isochronous stress–temperature curves.

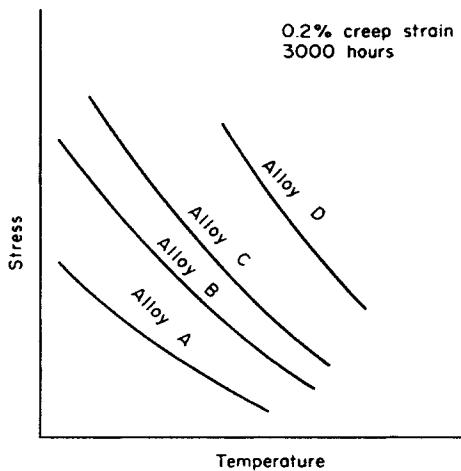


Fig. 11.29. 3000 hours, 0.2% creep–strain curves for various alloys.

11.2.3. The stress–rupture test

Where creep strain is the important design factor and fracture may be expected to take a very long time, the test is often terminated during the steady state of creep when sufficient

information has been obtained to produce a sufficiently accurate value of the secondary creep rate. Where life is the important design parameter, then the test is carried out to destruction and this is known as a *stress-rupture test*.

Because the total strain in a rupture test is much higher than in a creep test, the equipment can be less sophisticated. The loads used are generally higher, and thus the time of test shorter, than for creep. The percentage elongation and percentage reduction in area at fracture may be determined but the principal information obtained is the time to failure at a fixed temperature under nominal stress conditions.

A graph (Fig. 11.30), is plotted of time to rupture against stress on a log-log basis, and often a straight line results for each test temperature. Any change in slope of this stress-rupture line may be due to change in the mechanism of creep rupture within the material.

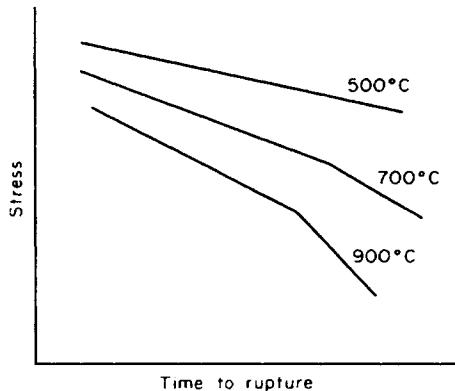


Fig. 11.30. Stress-rupture time curves at various temperatures.

11.2.4. Parameter methods

Very often, engineers have to confirm to customers that a particular component will withstand usage at elevated temperatures for a particular life-time which, in the case of furnace equipment or steam applications, may be a considerable number of years. It is impracticable to test such a component, for example, for twenty years before supplying the customer and therefore some method of extrapolation is required. The simplest method is to test at the temperature of proposed usage, calculate the minimum creep rate and assume that this will continue for the desired life-time and then ascertain whether the creep strain is acceptable. The obvious disadvantage of this method is that it does not allow for tertiary creep and sudden failure (which the creep curve shows will take place at some time in the future but at a point which cannot be determined because of time limitations).

In order to overcome this difficulty a number of workers have proposed methods involving accelerated creep tests, whereby the test is carried out at a higher temperature than that used in practice and the results used to predict creep-life or creep-strain over a longer period of time at a lower temperature.

The most well-known method is that of *Larson and Miller* and is based upon the Arrhenius equation (eqn. 11.19) which can be rewritten, in terms of \log_{10} as in eqn. (11.22) or in terms

of l_n without the constant 0.4343.

$$\log_{10} t_r = \log_{10} G + 0.4343 \cdot \frac{H}{R} \cdot \frac{1}{T} \quad (11.22)$$

where t_r is the time to rupture, G is a constant, T is the **absolute temperature**, R is the universal gas constant, and H is the activation energy for creep and is assumed to be stress-dependent.

$$\therefore \log_{10} t_r + C = m \cdot \frac{1}{T}$$

where m is a function of stress.

$$\therefore T(\log_{10} t_r + C) = m$$

this can be re-written as:

$$P_1 = f(\sigma)$$

where the *Larson–Miller parameter*

$$P_1 = T(\log_{10} t_r + C) \quad (11.23)$$

the value of the constant C can be obtained from the intercept when $\log_{10} t_r$ is plotted against $1/T$. For ferrous metals it usually lies between 15 and 30. If a test is carried out under a certain value of stress and temperature, the value of t_r can be determined and, if repeated for other stress and temperature values, the results can be plotted on a *master curve* (Fig. 11.31).

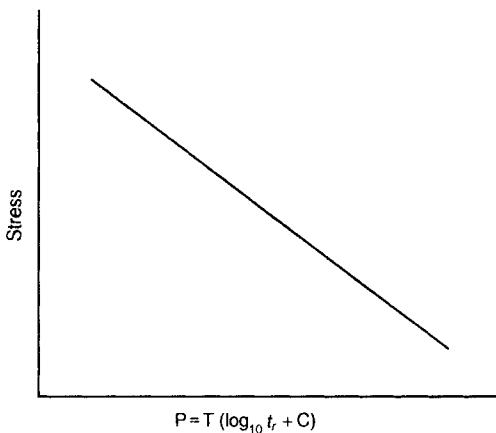


Fig. 11.31. Larson–Miller master curve.

The value of the parameter P is the same for a wide variety of combinations of t_r and temperature, ranging from short times and high temperatures representing test conditions, to long times at lower temperatures representing service conditions.

Results obtained by other workers, notably Sherby and Dorn⁽¹⁴⁾, suggest that G in the above equation (eqn. 11.22) is not a true constant but varies with stress whilst E is essentially constant. If $0.4343 E/R$ in eqn. (11.22) is replaced by α and $\log_{10} G$ by ϕ then eqn. (11.22)

can be written as:

$$\log_{10} t_r - \frac{\alpha}{T} = \phi$$

or

$$P_2 = f(\sigma)$$

where the *Sherby-Dorn parameter*

$$P_2 = \log_{10} t_r - \frac{\alpha}{T} \quad (11.24)$$

the constant α being determined from the common slope of a plot of $\log_{10} t_r$ versus $1/T$. After a series of creep tests, a master curve can then be plotted and used in the same manner as for the Larson-Miller parameter.

Another parameter was suggested by Manson and Haferd⁽¹³⁾ who found that, for a given material under different stress and temperature conditions, a family of lines was obtained which intersected at a point when $\log t_r$ was plotted against T . The family of lines of this kind could be represented by the equation:

$$T - T_a = m(\log_{10} t_r - \log_{10} t_a) \quad (11.25)$$

where the slope m is a function of stress and T_a and $\log_{10} t_a$ are the coordinates of the converging point (Fig. 11.32).

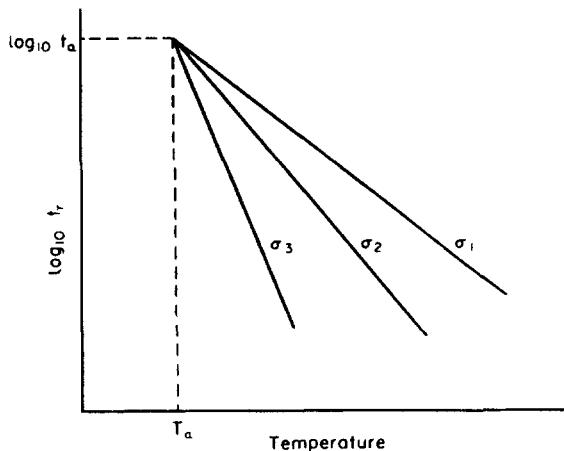


Fig. 11.32. Manson and Haferd curves.

The *Manson-Haferd parameter* can then be stated as:

$$P_3 = \frac{T - T_a}{\log_{10} t_r - \log_{10} t_a} \quad (11.26)$$

where $P_3 = m = f(\sigma)$.

A master curve can then be plotted in a similar manner to the other methods.

When using any of the above methods, certain facts should be borne in mind with regard to their limitations. Firstly, since the different methods give slightly differing results, this casts

doubts on the validity of all three methods and, in general, although the Manson–Haferd parameter has been found to produce more accurate predictions, it is difficult to determine the exact location of the point of convergence of the lines. Secondly, if one is using higher test temperatures than operating temperatures then the mechanisms of creep may be different and unrelated one to the other. Thirdly, the mechanism of creep failure may change with temperature; at lower temperatures, failure is usually transcrystalline whilst at higher temperatures it is intercrystalline, the change-over point being the *equi-cohesive temperature*. Results above this temperature are difficult to correlate with those obtained below this temperature.

11.2.5. Stress relaxation

So far we have been concerned with the study of material behaviour under constant loading or *constant stress* conditions where increase in strain is taking place and may eventually lead to failure. However, there are important engineering situations involving cylinder-head bolts, rivets in pressure vessels operating at elevated temperatures, etc., where we consider the *strain* to be *constant* and then we need to evaluate the decrease in stress which may take place. This time-dependent decrease in stress under constant strain conditions is called “*stress relaxation*”.

Consider two plates held together by a bolt deformed by a stress σ_i producing an initial strain ε_i which is all elastic.

Then

$$\varepsilon_i = \varepsilon_e = \sigma_i/E \quad (1)$$

At elevated temperatures and under conditions of steady-state creep, this bolt will tend to elongate at a rate ε^0 dictated by the power law:

$$\varepsilon^0 = \frac{d\varepsilon_c}{dt} = \beta\sigma^n \quad (2)$$

and, assuming the thickness of the plates remain constant, the strain caused by creep ε_c simply reduces the elastic part ε_e of the initial strain,

$$\text{i.e. } \varepsilon_e = \varepsilon_i - \varepsilon_c \quad (3)$$

But, since the creep strain decreases the elastic component of the initial strain, a corresponding decrease in stress must also result from eqn. (1).

Since ε_i is constant, if we differentiate eqn. (3) with respect to time we obtain:

$$\frac{d\varepsilon_e}{dt} = -\frac{d\varepsilon_c}{dt} \quad (4)$$

but $\varepsilon_e = \sigma E$ where σ is the instantaneous stress, therefore the LHS of eqn. (4) can be replaced by $(1/E) \cdot (d\sigma/dt)$ whilst, from eqn. (2), the RHS of eqn. (4) can be replaced by $\beta\sigma^n$.

Therefore, eqn. (4) can be rewritten:

$$\frac{1}{E} \cdot \frac{d\sigma}{dt} = -\beta\sigma^n \quad (5)$$

$$\therefore \int \frac{d\sigma}{\sigma^n} = -E\beta \int dt$$

$$\therefore -\frac{1}{(n-1)\sigma^{n-1}} = -E\beta t + C \quad (6)$$

To find C , consider the time $t = 0$, when the stress would be the initial stress σ_i

Then $C = -\frac{1}{(n-1)\sigma_i^{n-1}}$ (7)

substituting for C in eqn. (6), multiplying through by $(n-1)$ and re-arranging, gives:

$$\frac{1}{\sigma^{n-1}} = \frac{1}{\sigma_i^{n-1}} + \beta E(n-1)t \quad (11.27)$$

11.2.6. Creep-resistant alloys

The time-dependent deformation called "creep", as with all deformation processes, is largely dependent upon dislocation movement and, therefore, the development of alloys with a high resistance to creep involves producing a material in which movement of dislocations only takes place with difficulty.

Since creep only becomes an engineering problem above about $0.4 \times$ melting point temperature on the Kelvin scale T_m , the higher the melting point of the major alloy constituent the better. However, there are practical limitations; for instance, some high-melting-point metals e.g. tungsten (M.Pt 3377° C) are difficult to machine, some Molybdenum (M.Pt 2607° C) form volatile oxides and some others, e.g. Osmium (M.Pt 3027 C) are very expensive and therefore Nickel (M.Pt 1453) and Cobalt (M.Pt 1492° C) are used extensively at the moment.

The movement of dislocations will be hindered to a greater extent in an alloy rather than in a pure metal and alloying elements such as chromium and cobalt are added therefore to produce a solid-solution causing "*solid-solution-hardening*". Best results are obtained by rising an alloying element whose atomic size and valency are largely different from those of the parent metal, but this limits the amount that may be added. Also, the greater the amount of alloying element, the lower is likely to be the melting range of the alloy. Thus, the benefits of solution-hardening which hinders the dislocation movement may be outweighed at higher temperatures by a close approach to the solidus temperature.

Apart from solution-hardening, most creep-resisting alloys are further strengthened by *precipitation hardening* which uses carbides, oxides, nickel-titanium-aluminium, and nitride particles to block dislocation movement. Further deformation can then only take place by the dislocation rising above or "climbing" over the precipitate in its path and this is a diffusion-controlled process. Thus, metals with a low rate of self-diffusion e.g. face-centred-metals such as nickel are preferred to body-centred-metals.

Finally, *cold-working* is another method of increasing the high-temperature strength of an alloy and hindering dislocation movement but, since cold work lowers the re-crystallization temperature, for best results it is limited to about 15–20%. The use of alloying elements which raise the re-crystallization temperature in these circumstances will be beneficial.

All the methods above have their limitations. In solid-solution-hardening, a temperature increase will produce a corresponding increase in the mobility of the solute atoms which tend to lock the dislocations, thus making dislocation movement easier. With precipitation hardening, the increase in temperature may produce "*over-ageing*", resulting in a coarsening of the precipitate or even a complete solution of the precipitate, both effects resulting in

a softening and decrease in creep resistance. It may be possible, however, to arrange for a second precipitate to form which may strengthen the alloy. The effects of cold work are completely nullified when the temperature rises above the re-crystallisation temperature, hence the application of this technique is very limited.

At room temperature, grain boundaries are normally stronger than the grain material but, with increase in temperature, the strength of the boundary decreases at a faster rate than does the strength of the grain interior such that above the "equi-cohesive temperature" (Fig. 11.33), a coarse-grain material will have higher strength than a fine-grain material since the latter is associated with an increase in the amount of grain boundary region. It should be noted that T_e is not fixed, but dependent upon stress, being higher at high stresses than at low stresses.

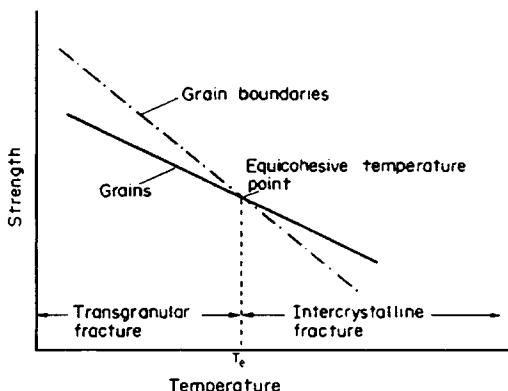


Fig. 11.33. Diagram showing concept of equi-cohesive temperature (T_e)

It can be shown also that creep rate is inversely proportional to the square of the grain size and dictated by the following formula:

$$\text{Creep rate} = \frac{K \sigma D}{d^2 T} \quad (11.28)$$

where K is a constant, σ is the applied stress, D is the coefficient of self diffusion, d is the grain size and T is the temperature. Thus, *grain boundary strengthening*, by the introduction of grain boundary carbide precipitates which help to prevent grain boundary sliding, and the control of grain size are important. Better still, the component may be produced from a single crystal such as the RB211 intermediate pressure turbine blade.

Apart from high creep resistance, alloys for use at high temperatures generally require other properties such as high oxidation resistance, toughness, high thermal fatigue resistance and low density, the importance of these factors depending upon the application of the material, and it is doubtful if any single test would provide a simple or accurate index of the qualities most desired.

11.3. Fracture mechanics

Introduction

The use of stress analysis in modern design procedures ensures that in normal service very few engineering components fail because they are overloaded. However, weakening of the

component by such mechanisms as corrosion or fatigue-cracking may produce a catastrophic fracture and in some instances, such as in the design of motorcycle crash helmets, the fracture properties of the component are the most important consideration. The study of how materials fracture is known as *fracture mechanics* and the resistance of a material to fracture is colloquially known as its “*toughness*”.

No structure is entirely free of defects and even on a microscopic scale these defects act as stress-raisers which initiate the growth of cracks. The theory of fracture mechanics therefore assumes the pre-existence of cracks and develops criteria for the catastrophic growth of these cracks. The designer must then ensure that no such criteria can be met in the structure.

In a stressed body, a crack can propagate in a combination of the three opening modes shown in Fig. 11.34. *Mode I* represents opening in a purely tensile field while *modes II* and *III* are in-plane and anti-plane shear modes respectively. The most commonly found failures are due to cracks propagating predominantly in mode I, and for this reason materials are generally characterised by their resistance to fracture in that mode. The theories examined in the following sections will therefore consider mode I only but many of the conclusions will also apply to modes II and III.

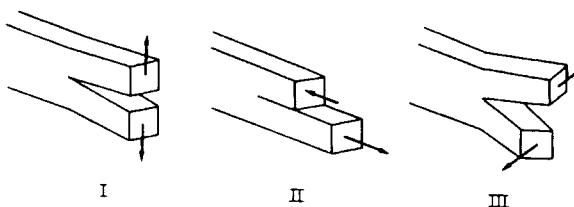


Fig. 11.34. The three opening modes, associated with crack growth: mode I—tensile; mode II—in-plane shear; mode III—anti-plane shear.

11.3.1. Energy variation in cracked bodies

A basic premise in thermodynamic theory is that a system will move to a state where the free energy of the system is lower. From this premise a simple criterion for crack growth can be formulated. It is assumed that a crack will only grow if there is a decrease in the free energy of the system which comprises the cracked body and the loading mechanism. The first usable criterion for fracture was developed from this assumption by Griffith⁽¹⁵⁾, whose theory is described in detail in §11.3.2.

For a clearer understanding of Griffith's theory it is necessary to examine the changes in stored elastic energy as a crack grows. Consider, therefore, the simple case of a strip containing an edge crack of length a under uniaxial tension as shown in Fig. 11.35. If load W is applied gradually, the load points will move a distance x and the strain energy, U , stored in the body will be given by

$$U = \frac{1}{2} Wx$$

for purely elastic deformation.

The load and displacement are related by the “*compliance*” C ,

i.e.
$$x = CW \quad (11.29)$$

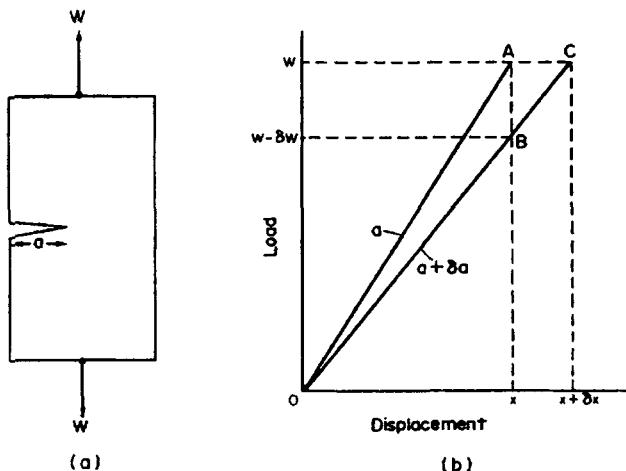


Fig. 11.35. (a) Cracked body under tensile load W ; (b) force-displacement curves for a body with crack lengths a and $a + da$.

The compliance is itself a function of the crack length but the exact relationship varies with the geometry of the cracked body. However, if the crack length increases, the body will become less stiff and the compliance will increase.

There are two limiting conditions to be considered depending on whether the cracked body is maintained at (a) constant displacement or (b) constant loading. Generally a crack will grow with both changing loads and displacement but these two conditions represent the extreme constraints.

(a) Constant displacement

Consider the case shown in Fig. 11.35(b). If the body is taken to be perfectly elastic then the load-displacement relationship will be linear. With an initial crack length a loading will take place along the line OA . If the crack extends a small distance δa while the points of application of the load remain fixed, there will be a small increase in the compliance resulting in a decrease in the load of δW . The load and displacement are then given by the point B . The change in stored energy will then be given by

$$\delta U_x = \frac{1}{2}(W - \delta W)x - \frac{1}{2}Wx$$

$$\delta U_x = -\frac{1}{2}\delta Wx \quad (11.30)$$

(b) Constant loading

In this case, if the crack again extends a small distance δa the loading points must move through an additional displacement δx in order to keep the load constant. The load and displacement are then represented by the point C .

There would appear to be an increase in stored energy given by

$$\begin{aligned}\delta U &= \frac{1}{2}W(x + \delta x) - \frac{1}{2}Wx \\ &= \frac{1}{2}W\delta x\end{aligned}$$

However, the load has supplied an amount of energy

$$= W \delta x$$

This has to be obtained from external sources so that there is a total reduction in the potential energy of the system of

$$\begin{aligned}\delta U_W &= \frac{1}{2} W \delta x - W \delta x \\ \delta U_W &= -\frac{1}{2} W \delta x\end{aligned}\quad (11.31)$$

For infinitesimally small increases in crack length the compliance C remains essentially constant so that

$$\delta x = C \delta W$$

Substituting in eqn. (11.31)

$$\delta U_W = -\frac{1}{2} W C \delta W = -\frac{1}{2} x \delta W$$

Comparison with eqn. (11.30) shows that, for small increases in crack length,

$$\delta U_w = \delta U_x$$

It is therefore evident that for small increases in crack length there is a similar decrease in potential energy no matter what the loading conditions. For large changes in crack length there is no equality but, generally, we are interested in the onset of crack growth since for monotonic (continuously increasing) loading catastrophic failure commonly follows crack initiation.

If there is a decrease in potential energy when a crack grows then there must be an energy requirement for the production of a crack – otherwise all cracked bodies would fracture instantaneously. The following section examines the most commonly used fracture criterion based on a net decrease in energy.

11.3.2. Linear elastic fracture mechanics (L.E.F.M.)

(a) Griffith's criterion for fracture

Griffith's thermodynamics approach was the first to produce a usable theory of fracture mechanics.⁽¹²⁾ His theoretical model shown in Fig. 11.36 was of an infinite sheet under a remotely applied uniaxial stress σ and containing a central crack of length $2a$. The preceding section has shown that when a crack grows there is a decrease in potential energy. Griffith, by a more mathematically rigorous treatment, was able to show that if that decrease in energy is greater than the energy required to produce new crack faces then there will be a net decrease in energy and the crack will propagate. For an increase in crack length of δa .

$$\delta U = 2yb \delta a$$

γ is the surface energy of the crack faces;

b is the thickness of the sheet.

At the onset of crack growth, δa is small and we have

$$\frac{dU}{da} = 2b\gamma$$

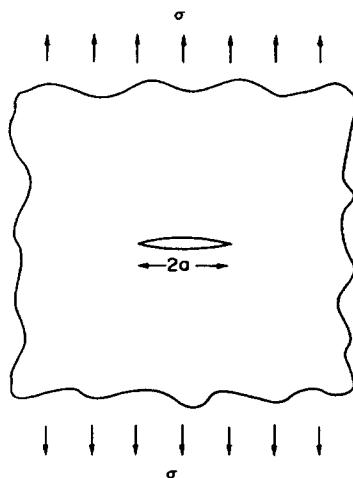


Fig. 11.36. Mathematical model for Griffith's analysis.

The expression on the left-hand side of the above equation is termed the “*critical strain energy release*” (with respect to crack length) and is usually denoted as G_c , i.e., at the onset of fracture,

$$G_c = \frac{\partial U}{\partial a} = 2b\gamma \quad (11.32)$$

This is the *Griffith criterion for fracture*.

Griffith's analysis gives G_c in terms of the fracture stress σ_f

$$G_c = \frac{\sigma_f^2 \pi a}{E} \quad \text{in plane stress} \quad (11.33)$$

$$G_c = \frac{\sigma_f^2 \pi a}{E} (1 - \nu^2) \quad \text{in plane strain.} \quad (11.34)$$

For finite bodies and those with edge cracks, correction factors must be introduced. Usually this involves replacing the factor π by some dimensionless function of the cracked body's geometry.

From eqns. (11.32) and (11.34) we can predict that, for *plane strain*, the fracture stress should be given by

$$\sigma_f^2 = \frac{2bE\gamma}{\pi a(1 - \nu^2)} \quad (11.35)$$

or, for *plane stress*:

$$\sigma_f^2 = \frac{2bE\gamma}{\pi a}$$

Griffith tested his theory on inorganic glasses and found a reasonable correlation between predicted and observed values of fracture stress. However, inorganic glasses are extremely brittle and when more ductile materials are examined it is found that the predicted values are far less than those observed. It is now known that even in apparently brittle fractures a ductile material will produce a localised plastic zone at the crack tip which effectively

blunts the crack. This has not prevented some workers measuring G_c experimentally and using it as a means of comparing materials but it is then understood that the energy required to propagate the crack includes the energy to produce the plastic zone.

(b) *Stress intensity factor*

Griffith's criterion is an energy-based theory which ignores the actual stress distribution near the crack tip. In this respect the theory is somewhat inflexible. An alternative treatment of the elastic crack was developed by Irwin⁽¹⁶⁾, who used a similar mathematical model to that employed by Griffith except in this case the remotely applied stress is biaxial – (see Fig. 11.37). Irwin's theory obtained expressions for the stress components near the crack tip. The most elegant expression of the stress field is obtained by relating the cartesian components of stress to polar coordinates based at the crack tip as shown in Fig. 11.38.

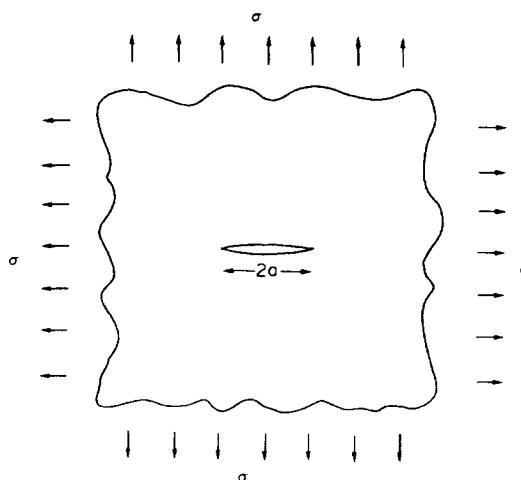


Fig. 11.37. Mathematical model for Irwin's analysis.

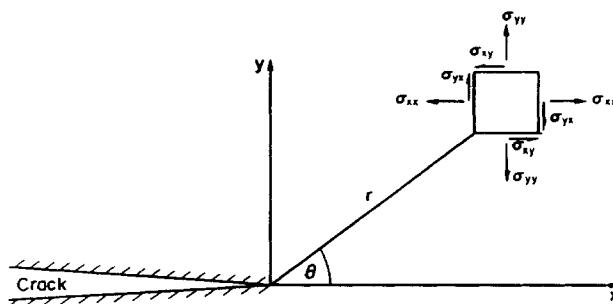


Fig. 11.38. Coordinate system for stress components in Irwin's analysis.

Then we have:

$$\left. \begin{aligned} \sigma_{yy} &= \frac{K}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left[1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right] \\ \sigma_{xx} &= \frac{K}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left[1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right] \\ \sigma_{xy} &= \frac{K}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{3\theta}{2} \end{aligned} \right\} \quad (11.36)$$

With, for plane stress,

$$\sigma_{zz} = 0$$

or, for plane strain,

$$\sigma_{zz} = v(\sigma_{xx} + \sigma_{yy})$$

with

$$\sigma_{zx} = \sigma_{zy} = 0 \quad \text{for both cases.}$$

The expressions on the right-hand side of the above equations are the first terms in series expansions but for regions near the crack tip where $r/a \gg 1$ the other terms can be neglected.

It is evident that each stress component is a function of the parameter K and the polar coordinates determining the point of measurement. The parameter K , which is termed the "stress intensity factor", therefore uniquely determines the stress field near the crack tip. If we base our criterion for fracture on the stresses near the crack tip then we are implying that the value of K determines whether the crack will propagate or not. The stress intensity factor K is simply a function of the remotely applied stress and crack length.

If more than one crack opening mode is to be considered then K sometimes carries the suffix I, II or III corresponding to the three modes shown in Fig. 11.34. However since this text is restricted to consideration of mode I crack propagation only, the formulae have been simplified by adopting the symbol K without its suffix. Other texts may use the full symbol K_I in development of similar formulae.

For Irwin's model, K is given by

$$K = \sigma \sqrt{\pi a} \quad (11.37a)$$

For an edge crack in a semi-infinite sheet

$$K = 1.12 \sigma \sqrt{\pi a} \quad (11.37b)$$

To accommodate different crack geometries a flaw shape parameter Q is sometimes introduced thus

$$K = \sigma \sqrt{\frac{\pi a}{Q}} \quad (11.37c)$$

or, for an edge crack

$$K = 1.12 \sigma \sqrt{\frac{\pi a}{Q}} \quad (11.37d)$$

Values of Q for various aspect (depth to width) ratios of crack can be obtained from standard texts*, but, typically, they range from 1.0 for an aspect ratio of zero to 2.0 for an aspect ratio of 0.4.

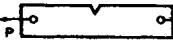
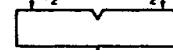
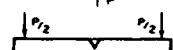
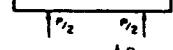
* Knott and Elliot, *Worked Examples in Fracture Mechanics*, Inst. met.

Normal manufactured components are of finite size and generally cracks grow from a free surface. However, for regions near the crack tip it is found that eqn (11.36) is a good approximation to the stress field if the stress intensity factor is modified to

$$K = \sigma Y \sqrt{a} \quad (11.38)$$

where a is the length of the edge crack and Y is a dimensionless correction factor often termed a "compliance function". Y is a polynomial of the ratio a/W where W is the uncracked width in the crack plane (see Table 11.1).

Table 11.1. Table of compliance functions (Y).

Specimen geometry	Specimen nomenclature	Equation for K	Compliance function constants				
			A	B	C	D	E
	Single edge notched (S.E.N.)	$K = \frac{P}{bW^{1/2}} \cdot Y$	1.99	0.41	18.70	38.48	53.85
	Three-point bend ($L = 4W$)	$K = \frac{3PL}{bW^{3/2}} \cdot Y$	1.93	3.07	14.53	25.11	25.80
	Four-point bend	$K = \frac{3PL}{bW^{3/2}} \cdot Y$	1.99	2.47	12.97	23.17	24.80
	Compact tension (C.T.S.)	$K = \frac{P}{bW^{1/2}} \cdot Y$	29.60	185.50	655.70	1017.0	638.90

It is common practice to express K in the directly measurable quantities of load P , thickness b , and width W . The effect of crack length is then totally incorporated into Y .

i.e.

$$K = \frac{P}{bW^{1/2}} \cdot Y \quad (11.39)$$

In practice, values of K can be determined for any geometry and for different types of loading. Table 11.1 gives the expressions derived for common laboratory specimen geometries.

Photoelastic determination of stress intensity factors

The stress intensity factor K defined in §11.3.2 is an important and useful parameter because it uniquely describes the stress field around a crack tip under tension. In a two-dimensional system the stress field around the crack tip has been defined, in polar co-ordinates, by eqns. (11.36) as

$$\left. \begin{aligned} \sigma_{xx} &= \frac{K}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left[1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right] \\ \sigma_{yy} &= \frac{K}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left[1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right] \\ \tau_{xy} &= \frac{K}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{3\theta}{2} \end{aligned} \right\} \quad (11.36)\text{bis}$$

with the crack plane running along the line $\theta = \pi$, see Fig. 11.38.

From §6.15 photoelastic fringes or “isochromatics” are contours of equal maximum shear stress τ_{\max} and from Mohr circle proportions, or from eqn. (13.13),[†]

$$\tau_{\max} = \frac{1}{2} \sqrt{(\sigma_{xx} - \sigma_{yy})^2 + 4\tau_{xy}^2}$$

Substituting eqns. (11.36) gives

$$\tau_{\max} = \frac{K \sin \theta}{2\sqrt{2\pi r}} \quad (11.40)$$

If, therefore, a photoelastic model is constructed of the particular crack geometry under consideration (see Fig. 11.39) a plot of τ_{\max} against $1/\sqrt{r}$ from the crack tip will produce a straight line graph from the slope of which K can be evaluated.

It is usual to record fringe order (and hence τ_{\max} values) along the line $\theta = 90^\circ$ when r will be at a maximum r_{\max} . Then:

$$\tau_{\max} = \frac{K}{2\sqrt{2\pi}} \cdot \frac{1}{\sqrt{r_{\max}}}$$

i.e.

$$\text{slope of graph} = \frac{K}{2\sqrt{2\pi}}$$

Correction factors

Equations (11.36), above, were derived for mathematical convenience for an infinite sheet containing a central crack of length $2a$ loaded under biaxial tension. For these conditions it is found that

$$K = \sigma \sqrt{\pi a}$$

In order to allow for the fact that the loading in the photoelastic test is uniaxial and that the model is of finite, limited size, a correction factor needs to be applied to the above K value in order that the “theoretical” value can be obtained and compared with the result obtained from the photoelastic test. This is normally written in terms of the function (a/w) where a = edge crack length and w = plate width. Then:

$$K = \sigma \sqrt{a} \left[1.99 - 0.41 \left(\frac{a}{w} \right) + 18.7 \left(\frac{a}{w} \right)^2 - 38.48 \left(\frac{a}{w} \right)^3 + 53.85 \left(\frac{a}{w} \right)^4 \right]$$

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

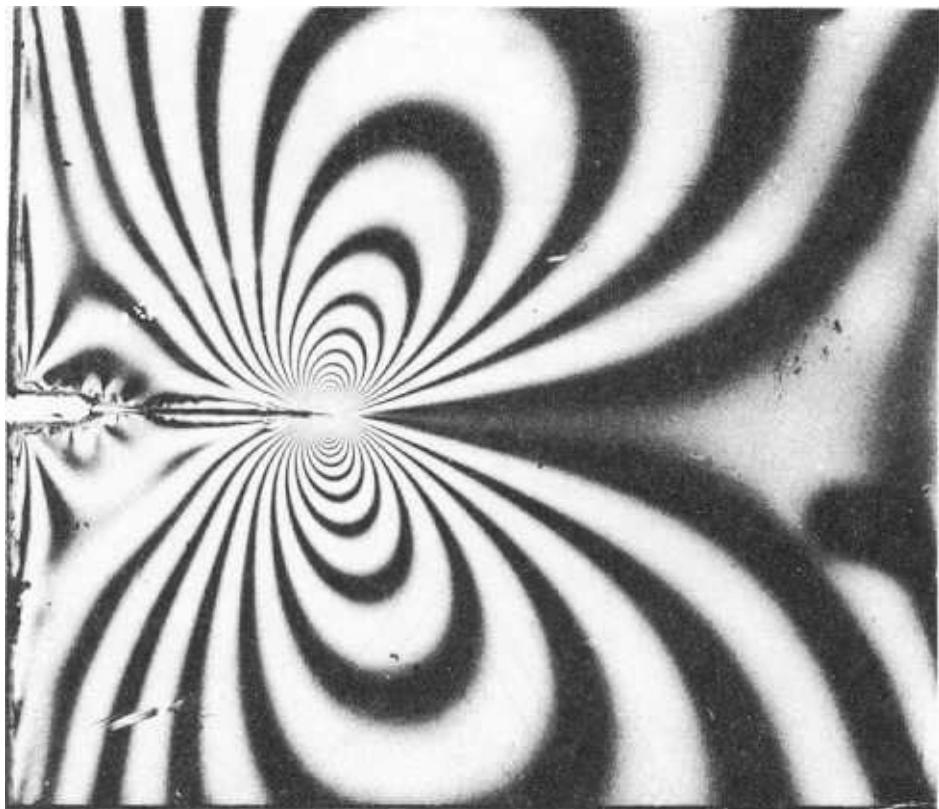


Fig. 11.39. Photoelastic fringe pattern associated with a particular crack geometry (relevant stress component nomenclature given in Fig. 11.38)

11.3.3. Elastic–plastic fracture mechanics (E.P.F.M.)

Irwin's description of the stress components near an elastic crack can be summarised as

$$\text{Stress} \propto \frac{K}{r^{1/2}} \times \text{a function of } \theta \text{ -- see eqn. .36}$$

which implies that each stress component rises to infinity as the crack tip is approached and as r nears zero.

In particular, the vertical stress in the crack plane where $\theta = 0$ is given by

$$\sigma_{yy} = \frac{K}{(2\pi x)^{1/2}} \quad (11.41)$$

which is represented by the dotted line shown in Fig. 11.40(a).

In a ductile material then, at some point the stress will exceed the yield stress and the material will yield. By following Knott's analysis⁽¹⁷⁾ we can estimate the extent of the plastic deformation.

If we consider *plane stress conditions* then σ_{yy} is the maximum and σ_{zz} the minimum ($= 0$) principal stress. Then, by the Tresca criterion, the material will shear in the yz plane

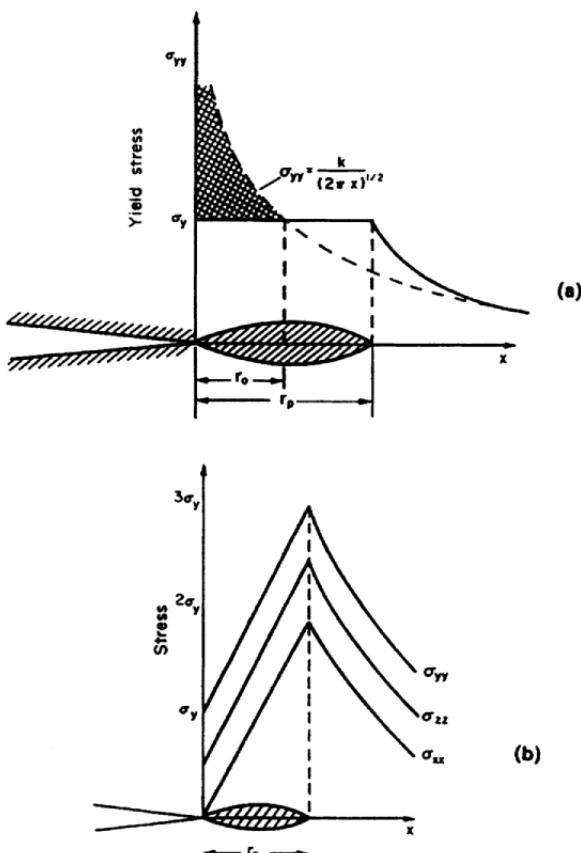


Fig. 11.40. Schematic representation of crack tip plasticity in; (a) plane stress; (b) plane strain.

and at 45 to the y and z axes when

$$\sigma_{yy} - \sigma_{zz} = \sigma_y$$

or

$$\sigma_{yy} = \sigma_y$$

where σ_y is the yield stress in uniaxial tension.

At some distance r_0 from the crack tip $\sigma_{yy} = \sigma_y$ as shown in Fig. 11.40(a). By simple integration, the area under the curve between the crack tip and r_0 is equal to $2\sigma_y r_0$. The shaded area in the figure must therefore have an area $\sigma_y r_0$. It is conventional to assume that the higher stress levels associated with the shaded area are redistributed so that the static zone extends a distance r_p where

$$r_p = 2r_0 = \frac{K^2}{\pi\sigma_y^2} \quad (11.42)$$

Figure 11.40(a) shows the behaviour of $\sigma_{yy}(x, 0)$, i.e. the variation of σ_{yy} with x at $y = 0$, for the case of plane stress. This is only a first approximation, but the estimate of the plastic zone size differs only by a small numerical factor from refined treatments. In the case of plane strain σ_{zz} is non-zero and σ_{xx} is the smallest principal stress in the vicinity of the

crack tip. Again, by using the Tresca criterion, we have

$$\sigma_{yy} - \sigma_{xx} = \sigma_y$$

or

$$\sigma_{yy} = \sigma_y + \sigma_{xx}$$

In the plastic zone the difference between σ_{yy} and σ_{xx} must be maintained at σ_y . As x increases from the crack tip σ_{xx} rises from zero and so σ_{yy} must rise above σ_y . The normal stress σ_{zz} must also increase and a schematic representation is shown in Fig. 11.40(b). The stress configuration is one of triaxial tension and the constraints on the material produce stresses higher than the uniaxial yield stress. The maximum stress under these conditions is often conveniently taken to be $3\sigma_y$. The plane strain plastic zone size is therefore taken to be one-third of the plane stress plastic zone

i.e.

$$r_p = \frac{K^2}{3\pi\sigma_y^2} \quad (11.43)$$

In both states of stress it is seen that the square of the stress intensity factor determines the size of the plastic zone. This would seem paradoxical as K is derived from a perfectly elastic model. However, if the plastic zone is small, the elastic stress field in the region around the plastic zone will still be described by eqn. (11.43). The plasticity is then termed “well-contained” or “ K -controlled” and we have an *elastic-plastic stress distribution*. A typical criterion for well-contained plasticity is that the plastic zone size should be less than one-fiftieth of the uncracked specimen ligament.

11.3.4. Fracture toughness

A fracture criterion for brittle and elastic-plastic cracks can be based on functions of the elastic stress components near the crack tip. No matter what function is assumed it is implied that K reaches some critical value since each stress component is uniquely determined by K . In other words the crack will become unstable when K reaches a value K_{IC} , the *Critical stress intensity factor* in mode I. K_{IC} is now almost universally denoted as the “*fracture toughness*”, and is used extensively to classify and compare materials which fracture under plane strain conditions.

The fracture toughness is measured by increasing the load on a pre-cracked laboratory specimen which usually has one of the geometries shown in Table 11.1. When the onset of crack growth is detected then the load at that point is used to calculate K_{IC} .

In brittle materials, the onset of crack growth is generally followed by a catastrophic failure whereas ductile materials may withstand a period of stable crack growth before the final fracture. The start of the stable growth is usually detected by changes in the compliance of the specimen and a clip-gauge mounted across the mouth of the crack produces a sensitive method of detecting changes in compliance. It is important that the crack is sharp and that its length is known. In soft materials a razor edge may suffice, but in metals the crack is generally grown by fatigue from a machined notch. The crack length can be found after the final fracture by examining the fracture surfaces when the boundary between the two types of growth is usually visible. Typical values of the fracture toughness of some common materials are given in Table 11.2.

Table 11.2. Typical K_{IC} values.

Material	K_{IC} (MN/m ^{3/2})
Concrete (dependent on mix and void content)	0.1–0.15
Epoxy resin	0.5–2.0
Polymethylmethacrylate	2–3
Aluminium	20–30
Low alloy steel	40–60

11.3.5. Plane strain and plane stress fracture modes

Generally, in plane stress conditions, the plastic zone crack tip is produced by shear deformation through the thickness of the specimen. Such deformation is enhanced if the thickness of the specimen is reduced. If, however, the specimen thickness is increased then the additional constraint on through-thickness yielding produces a triaxial stress distribution so that approximate plane strain deformation occurs with shear in the xy plane. There is usually a transition from plane stress to plane strain conditions as the thickness is increased. As K_{IC} values are generally quoted for plane strain, it is important that this condition prevails during fracture toughness testing.

A well-established criterion for plane strain conditions is that the thickness B should obey the following:

$$B \geq 2.5 \frac{(K_{IC})^2}{\sigma_y^2} \quad (11.44)$$

It should be noted that, even on the thickest specimens, a region of plane stress yielding is always present on the side surfaces because no triaxial stress can exist there. The greater plasticity associated with the plane stress deformation produces the characteristic “shear lips” often seen on the edges of fracture surfaces. In some instances the plane stress regions on the surfaces may be comparable in size with nominally plane strain regions and a mixed-mode failure is observed. However, many materials show a definite transition from plane stress to plane strain.

11.3.6. General yielding fracture mechanics

When the extent of plasticity which accompanies the growth of a crack becomes comparable with the crack length and the specimen dimensions we cannot apply linear elastic fracture mechanics (LEFM) and other theories have to be sought. It is beyond the scope of this book to review all the possible attempts to provide a unified theory. We will, however, examine the J integral developed by Rice⁽¹⁸⁾ because this has found the greatest favour in recent years amongst researchers in this field. In its simplest form the J integral can be defined as

$$J = \frac{\partial U^*}{\partial a} \quad (11.45)$$

where the asterisk denotes that this energy release rate includes both linear elastic and non-linear elastic strain energies. For linear elasticity J is equivalent to G .

The theory of the J integral was developed for non-linear elastic behaviour but, in the absence of any rival theory, the J integral is also used when the extent of plasticity produces a non-linear force-displacement curve.

As the crack propagates and the crack tip passes an element of the material, the element will partially unload. In cases of general yielding the elements adjacent to the crack tip will have been plastically deformed and will not unload reversibly, and the strain energy released will not be as great as for reversible non-linear elastic behaviour. At the initiation of growth no elements will have unloaded so that if we are looking for a criterion for crack growth then the difference between plastic and nonlinear elastic deformation may not be significant. By analogy with Griffith's definition of G_c in eqn. (11.32) we can define

$$J_c = \left(\frac{\partial U^*}{\partial a} \right)_c \quad (11.46)$$

the critical strain energy release rate for crack growth. Here the energy required to extend the crack is dominated by the requirement to extend the plastic zone as the crack grows. The surface energy of the new crack faces is negligible in comparison. Experiments on mild steel⁽¹⁶⁾ show that J_c is reasonably constant for the initiation of crack growth in different specimen geometries.

Plastic deformation in many materials is a time-dependent process so that, at normal rates of loading, the growth of cracks through structures with gross yielding can be stable and may be arrested by removing the load.

Calculation of J

Several methods of calculating J exist, but the simplest method using normal laboratory equipment is that developed by Begley and Landes.⁽¹⁹⁾ Several similar specimens of any suitable geometry are notched or pre-cracked to various lengths. The specimens are then extended while the force-displacement curves are recorded. Two typical traces where the crack length a_2 is greater than a_1 are shown in Fig. 11.41. At any one displacement x , the area under the $W-x$ curve gives U^* . For any given displacement, a graph can be plotted of

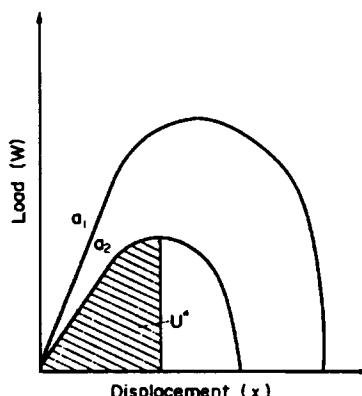


Fig. 11.41. Force-displacement curves for cracked bodies exhibiting general yielding (crack length $a_1 < a_2$)

U^* against crack length (Fig. 11.42). The slopes of these curves give J for any given combination of crack length and displacement, and can be plotted as a function of displacement (Fig. 11.43). By noting the displacement at the onset of crack growth, J_c can be assessed.

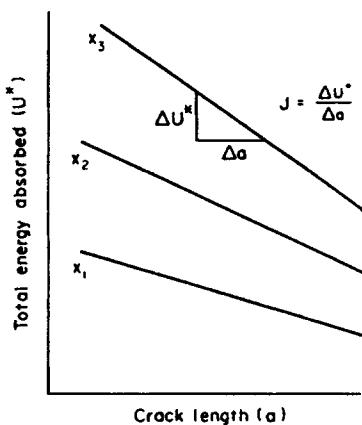


Fig. 11.42. Total energy absorbed as a function of crack length and at constant displacement ($x_3 > x_2 > x_1$)

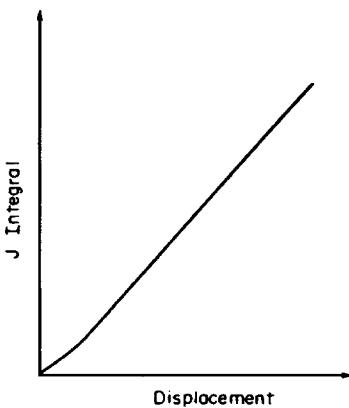


Fig. 11.43. The J integral as a function of displacement.

11.3.7. Fatigue crack growth

The failure of engineering components most commonly occurs at stress levels far below the maximum design stress. Also, components become apparently more likely to fail as their service life increases. This phenomenon, commonly termed fatigue, see §11.1, involves the growth of small defects into macroscopic cracks which grow until K_{IC} is exceeded and catastrophic failure occurs. One of the earliest observations of fatigue failure was that the amplitude of fluctuations in the applied stress had a greater influence on the fatigue life of

a component than the mean stress level. In fact if there is no fluctuation in loading then fatigue failure cannot occur, whatever magnitude of static stress is applied.

As stated earlier, fatigue failure is generally considered to be a three-stage process as shown schematically in Fig. 11.44. *Stage I* involves the initiation of a crack from a defect and the subsequent growth of the crack along some favourably orientated direction in the microstructure. Eventually the crack will become sufficiently large that the microstructure has a reduced effect on the crack direction and the crack will propagate on average in a plane normal to the maximum principal stress direction. This is *stage II* growth which has attracted the greatest attention because it is easier to quantify than the initiation stage. When the crack has grown so that K_{IC} is approached the crack accelerates more rapidly until K_{IC} is exceeded and a final catastrophic failure occurs. This accelerated growth is classified as *stage III*.

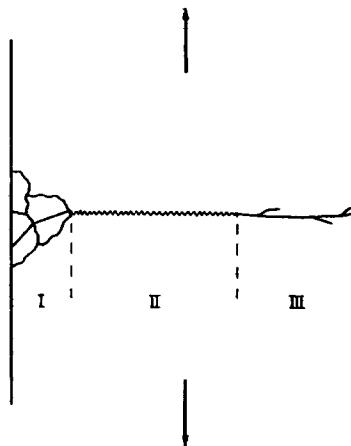


Fig. 11.44. Schematic representation of the three stages of fatigue crack growth.

The rate of growth of a fatigue crack is described in terms of the increase in crack length per load cycle, da/dN . This is related to the amplitude of the stress intensity factor, ΔK , during the cycle. If the amplitude of the applied stress remains constant then, as the crack grows, ΔK will increase. Such conditions produce growth-rate curves of the type shown in Fig. 11.45. Three distinct sections, which corresponds to the three stages of growth, can be seen.

There is a minimum value of ΔK below which the crack will not propagate. This is termed the *threshold value* or ΔK_{th} and is usually determined when the growth rate falls below 10^{-7} mm/cycle or, roughly, one atomic spacing. Growth rates of 10^{-9} mm/cycle can be detected but at this point we are measuring the average increase produced by a few areas of localised growth over the whole crack front. To remove any possibility of fatigue failure in a component it would be necessary to determine the maximum defect size, assume it was a sharp crack, and then ensure that variations in load do not produce ΔK_{th} .

Usually this would result in an over-strong component and it is necessary in many applications to assume that some fatigue crack growth will take place and assess the lifetime of the component before failure can occur. Only sophisticated detection techniques can resolve

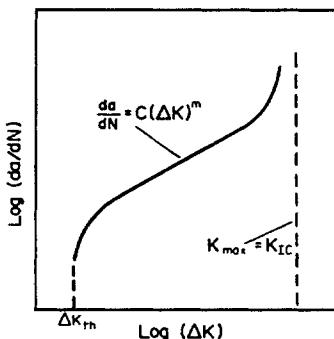


Fig. 11.45. Idealised crack growth rate plot for a constant load amplitude.

cracks in the initiation stage and generally it is assumed that the lifetime of fatigue cracks is the number of cycles endured in stage II.

For many materials stage II growth is described by the *Paris-Erdogan Law*⁽²⁰⁾.

$$\frac{da}{dN} = C(\Delta K)^m \quad (11.47)$$

C and m are material coefficients. Usually m lies between 2 and 7 but values close to 4 are generally found. This simple relationship can be used to predict the lifetime of a component if the stress amplitude remains approximately constant and the maximum crack size is known. If the stress amplitude varies, then the growth rate may depart markedly from the simple power law. Complications such as fatigue crack closure (effectively the wedging open of the crack faces by irregularities on the crack faces) and single overloads can reduce the crack growth rate drastically. Small changes in the concentration of corrosive agents in the environment can also produce very different results.

Stage III growth is usually a small fraction of the total lifetime of a fatigue crack and often neglected in the assessment of the maximum number of load cycles.

Since we are considering ΔK as the controlling parameter, only brittle materials or those with well-contained plasticity can be treated in this manner. When the plastic deformation becomes extensive we need another parameter. Attempts have been made to fit growth-rate curves to ΔJ the amplitude of the J integral. However while non-linear elastic and plastic behaviour may be conveniently merged in monotonic loading, in cyclic loading there are large differences in the two types of deformation. The non-linear elastic material has a reversible stress-strain relationship, while large hysteresis is seen when plastic material is stressed in the opposite sense. As yet the use of ΔJ has not been universally accepted but, on the other hand, no other suitable parameter has been developed.

11.3.8. Crack tip plasticity under fatigue loading

As a cracked body is loaded, a plastic zone will grow at the crack tip as described in §11.3.3. When the maximum load is reached and the load is subsequently decreased, the deformation of the plastic zone will not be entirely reversible. The elastic regions surrounding the plastic zone will attempt to return to their original displacement as the load is reduced. However, the plastic zone will act as a type of inclusion which the relaxing elastic material

then loads in compression. The greatest plastic strain on the increasing part of the load cycle is near the crack tip, and is therefore subjected to the lightest compressive stresses when the load decreases. At a sufficiently high load amplitude the material near the crack tip will yield in compression. A "reverse" plastic zone is produced inside the material which has previously yielded in tension. Figure 11.46 shows schematically the configuration of crack tip plasticity and the variation in vertical stress, in plane stress conditions, at the minimum load of the cycle.

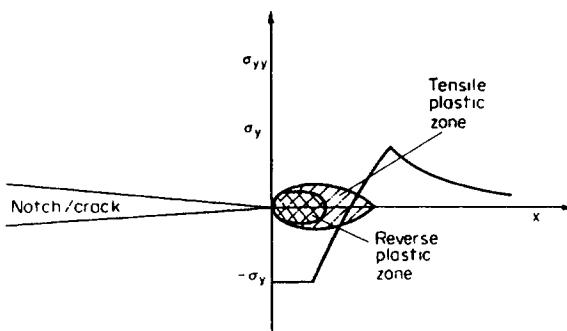


Fig. 11.46. Crack tip plasticity at the minimum load of the load cycle (plane stress conditions)

The material adjacent to the crack tip is therefore subjected to alternating plastic strains which lead to cumulative plastic damage and a weakening of the structure so that the crack can propagate. In metallic materials, striations on the fracture surface show the discontinuous nature of the crack propagation, and in many cases it can be assumed that the crack grows to produce a striation during each load cycle. Polymeric materials, however, can only show striations which occur after several thousands of load cycles.

11.3.9 Measurement of fatigue crack growth

In order to evaluate the fatigue properties of materials SN curves can be constructed as described in §11.1.1, or growth-rate curves drawn as shown in Fig. 11.45. Whilst non-destructive testing techniques can be used to detect fatigue cracks, e.g. ultrasonic detection methods to find flaws above a certain size or acoustic emission to determine whether cracks are propagating, growth-rate analysis requires more accurate measurement of crack length. Whilst a complete coverage of the many procedures available is beyond the scope of this text it is appropriate to introduce the most commonly used technique for metal fatigue studies, namely the D.C. potential drop method.

Essentially a large constant current (~ 30 amps) is passed through the specimen. As the crack grows the potential field in the specimen is disturbed and this disturbance is detected by a pair of potential probes, usually spot-welded on either side of the crack mouth. For single-edge notched (SEN) tensile and bend specimens theoretical solutions exist to relate the measured voltage to the crack length. In compact tension specimens (CTS) empirical calibrations are usually performed prior to the actual tests. Fig. 11.47 shows a block diagram of the potential drop technique. The bulk of the signal is "backed off" by the voltage source so that small changes in crack length can be detected. As the measured voltage is generally

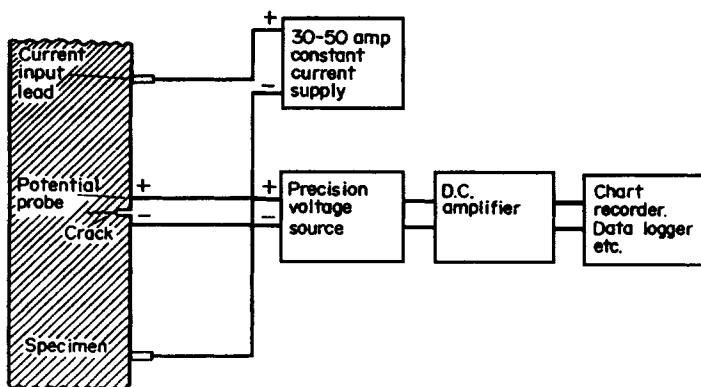


Fig. 11.47. Block diagram of the D.C. potential drop system for crack length measurement.

of the order of microvolts (steel and titanium) or nanovolts (aluminium), sensitive and stable amplifiers and voltage sources are required. A constant temperature environment is also desirable. If adequate precautions are taken, apparent increases in crack length of 10^{-9} mm can be detected in some materials.

If the material under test is found to be insensitive to loading frequency and a constant loading amplitude is required, the most suitable testing machine is probably one which employs a resonance principle. Whilst servo-hydraulic machines can force vibrations over a wider range of frequencies and produce intricate loading patterns, resonance machines are generally cheaper and require less maintenance. Each type of machine is usually provided with a cycle counter and an accurate load cell so that all the parameters necessary to generate the growth rate curve are readily available.

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Examples

Example 11.1

The fatigue behaviour of a specimen under alternating stress conditions with zero mean stress is given by the expression:

$$\sigma_r^a \cdot N_f = K$$

where σ_r is the range of cyclic stress, N_f is the number of cycles to failure and K and a are material constants.

It is known that $N_f = 10^6$ when $\sigma_r = 300 \text{ MN/m}^2$ and $N_f = 10^8$ when $\sigma_r = 200 \text{ MN/m}^2$.

Calculate the constants K and a and hence the life of the specimen when subjected to a stress range of 100 MN/m^2 .

Solution

Taking logarithms of the given expression we have:

$$a \log \sigma_r + \log N_f = \log K \quad (1)$$

Substituting the two given sets of condition for N_f and σ_r :

$$2.4771a + 6.0000 = \log K \quad (2)$$

$$2.3010a + 8.0000 = \log K \quad (3)$$

$$\therefore (3) - (2)$$

$$-0.1761a + 2.0000 = 0$$

$$\begin{aligned} a &= \frac{2.0000}{0.1761} \\ &= 11.357 \end{aligned}$$

Substituting in eqn. (2)

$$\begin{aligned} 11.357 \times 2.4771 + 6.000 &= \log K \\ &= 34.1324 \end{aligned}$$

$$\therefore K = 1.356 \times 10^{33}$$

Hence, for stress range of 100 MN/m^2 , from eqn (1):

$$11.357 \times 2.0000 + \log N_f = 34.1324$$

$$22.714 + \log N_f = 34.1324$$

$$\log N_f = 11.4184$$

$$N_f = 262.0 \times 10^9 \text{ cycles}$$

Example 11.2

A steel bolt 0.003 m^2 in cross-section is subjected to a static mean load of 178 kN. What value of completely reversed direct fatigue load will produce failure in 10^7 cycles? Use the Soderberg relationship and assume that the yield strength of the steel is 344 MN/m^2 and the stress required to produce failure at 10^7 cycles under zero mean stress conditions is 276 MN/m^2 .

Solution

From eqn. (11.9) of Soderberg

$$\sigma_a = \sigma_N \left[1 - \left(\frac{\sigma_m}{\sigma_y} \right) \right]$$

$$\begin{aligned}\text{Now, mean stress } \sigma_m \text{ on bolt} \\ &= \frac{178 \times 10^{-3}}{3 \times 10^{-3}} \\ &= 59.33 \text{ MN/m}^2\end{aligned}$$

$$\begin{aligned}\therefore \sigma_a &= 276 \left(1 - \frac{59.33}{344} \right) \\ &= 276(1 - 0.172) \\ &= 276 \times 0.828 \\ &= 228.5 \text{ MN/m}^2\end{aligned}$$

$$\begin{aligned}\text{Load} &= 228.5 \times 0.003 \text{ MN} \\ &= 0.6855 \text{ MN} \\ &= \mathbf{685.5 \text{ kN}}$$

Example 11.3

A stepped steel rod, the smaller section of which is 50 mm in diameter, is subjected to a fluctuating direct axial load which varies from +178 kN to -178 kN.

If the theoretical stress concentration due to the reduction in section is 2.2, the notch sensitivity factor is 0.97, the yield strength of the material is 578 MN/m^2 and the fatigue limit under rotating bending is 347 MN/m^2 , calculate the factor of safety if the fatigue limit in tension-compression is 0.85 of that in rotating bending.

Solution

From eqn. (11.12)

$$q = \frac{K_f - 1}{K_t - 1}$$

$$\begin{aligned}\therefore K_f &= q(K_t - 1) + 1 \\ &= 0.97(2.2 - 1) + 1 \\ &= 2.16\end{aligned}$$

But

$$\begin{aligned}\sigma_{\max} &= \frac{178 \times 4}{\pi \times (0.05)^2} \\ &= 90642 \text{ kN/m}^2 \\ &= 90.64 \text{ MN/m}^2\end{aligned}$$

$$\therefore \sigma_{\min} = -90.64 \text{ MN/m}^2 \text{ and } \sigma_{\text{mean}} = 0$$

\therefore Under direct stress conditions

$$\begin{aligned}\sigma_N &= 0.85 \times 347 \\ &= 294.95 \text{ MN/m}^2\end{aligned}$$

From eqn. (11.13)

$$\sigma_a = \frac{\sigma_N}{K_f F} \left(1 - \frac{\sigma_m \times F}{\sigma_y} \right)$$

\therefore With common units of MN/m²:

$$90.64 = \frac{294.95}{2.16 \times F} \left(1 - \frac{0 \times F}{578} \right)$$

$$\therefore F = \frac{294.95}{2.16 \times 90.64}$$

$$F = 1.5$$

Example 11.4

The values of the endurance limits at various stress amplitude levels for low-alloy constructional steel fatigue specimens are given below:

σ_a (MN/m ²)	N_f (cycles)
550	1 500
510	10 050
480	20 800
450	50 500
410	125 000
380	275 000

A similar specimen is subjected to the following programme of cycles at the stress amplitudes stated; 3 000 at 510 MN/m², 12 000 at 450 MN/m² and 80 000 at 380 MN/m², after which the sample remained unbroken. How many additional cycles would the specimen withstand at 480 MN/m² prior to failure? Assume zero mean stress conditions.

Solution

From Miner's Rule, eqn. (11.14), with X the required number of cycles:

$$\frac{n_1}{N_1} + \frac{n_2}{N_2} + \frac{n_3}{N_3} + \dots \text{ etc} = 1.$$

$$\therefore \frac{3000}{10050} + \frac{12000}{50500} + \frac{80000}{275000} + \frac{X}{20800} = 1$$

$$0.2985 + 0.2376 + 0.2909 + \frac{X}{20\,800} = 1$$

$$\frac{X}{20\,800} = 1 - 0.8270$$

$$\therefore X = 3598 \text{ cycles.}$$

Example 11.5

The blades in a steam turbine are 200 mm long and they elastically extend in operation by 0.02 mm. If the initial clearance between the blade tip and the housing is 0.075 mm and it is required that the final clearance be not less than 0.025 mm, calculate:

- (i) the maximum percentage creep strain that can be allowed in the blades,
- (ii) the minimum creep strain rate if the blades are to operate for 10 000 hours before replacement.

Solution

$$\begin{aligned}\text{Permissible creep extension} &= \text{Initial clearance} - (\text{Final clearance} + \text{Elastic extension}) \\ &= 0.075 - (0.025 + 0.02) \\ &= 0.03 \text{ mm} \\ \therefore \text{Max. percentage creep strain} &= \frac{0.03}{200} \times 100 \\ &= 0.015\% \\ \therefore \text{Min. creep rate} &= \frac{0.015}{10\,000} = 1.5 \times 10^{-6}\%/\text{h.}\end{aligned}$$

Example 11.6

The following secondary creep strain rates were obtained when samples of lead were subjected to a constant stress of 1.3 MN/m².

Temperature (°C)	Minimum creep rate (ε_s^0) (s ⁻¹)
33	8.71×10^{-5}
29	4.98×10^{-5}
27	3.42×10^{-5}

Assuming that the material complies with the Arrhenius equation, calculate the activation energy for creep of lead. Molar gas constant, $R = 8.314 \text{ J/mol K}$.

Solution

Construct a table as shown below:

°C	K	$\frac{1}{T} \times 10^{-3}$	ε_s^0	$\ln \varepsilon_s^0$
33	306	3.27	8.71×10^{-5}	-9.3485
29	302	3.31	4.98×10^{-5}	-9.9156
27	300	3.33	3.42×10^{-5}	-10.2833

The creep rate is related to temperature by eqn. (11.19):

$$\varepsilon_s^0 = Ae^{-H/RT}$$

Hence we can plot $I_n \varepsilon_s^0$ against $\frac{1}{T}$ (as in Fig. 11.48)

From the graph.

$$\text{Slope} = 15.48 \times 10^3$$

But

$$H = \text{slope} \times R$$

$$\begin{aligned} \therefore H &= 15.48 \times 10^3 \times 8.314 \\ &= 128.7 \text{ kJ/mol.} \end{aligned}$$

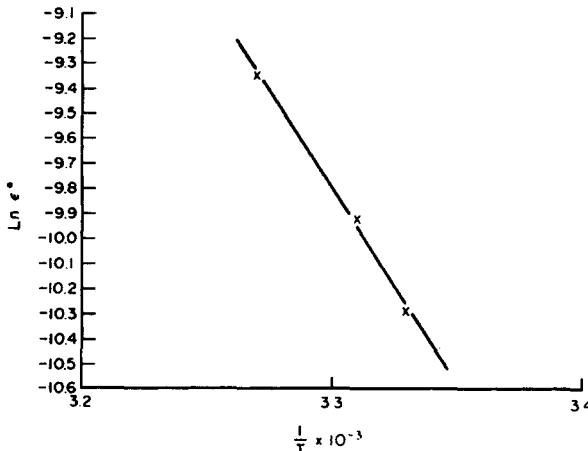


Fig. 11.48.

Example 11.7

An alloy steel bar 1500 mm long and 2500 mm² in cross-sectional area is subjected to an axial tensile load of 8.9 kN at an operating temperature of 600°C. Determine the value of creep elongation in 10 years using the relationship $\varepsilon_s^0 = \beta\sigma^n$ if, for 600°C, $\beta = 26 \times 10^{-12}$ h⁻¹(N/mm²)⁻⁶, and $n = 6.0$.

Solution

$$\text{Applied stress } \sigma = \frac{P}{A} = \frac{8900}{2500} = 3.56 \text{ N/mm}^2$$

$$\text{Duration of test} = 10 \times 365 \times 24 = 87600 \text{ hours}$$

\therefore From eqn. (11.20)

$$\begin{aligned} \therefore \varepsilon &= 26 \times 10^{-12} \times 87600 \times (3.56)^6 \\ &= 26 \times 10^{-12} \times 87600 \times 2036 \\ &= 4.637 \times 10^{-3} \end{aligned}$$

Since the member is 1500 mm long,

$$\text{total elongation} = 1500 \times 4.637 \times 10^{-3} = 6.96 \text{ mm.}$$

Example 11.8

Creep tests carried out on an alloy steel at 600°C produced the following data:

Stress (kN/m ²)	Minimum creep rate (% / 10 000 h)
10.2	0.4
13.8	1.2
25.5	10.0

A rod, 150 mm long and 625 mm² in cross-section, made of a similar steel and operating at 600°C, is not to creep more than 3.2 mm in 10 000 hours. Calculate the maximum axial load which can be applied.

Solution

$$\% \text{ Creep strain} = \frac{3.2}{150} \times 100 = 2.13\% / 10 000 \text{ h}$$

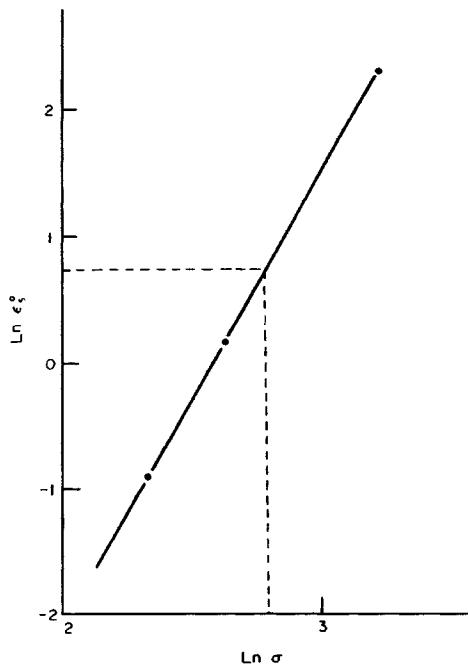


Fig. 11.49.

Since secondary creep rate is related to stress by the eqn. (11.20):

$$\varepsilon_s^0 = \beta\sigma^n,$$

a graph may be plotted of $l_n\varepsilon_s^0$ against $l_n\sigma$.

From the given data:

l_n stress	$l_n\varepsilon_s^0$
2.3224	-0.9163
2.6247	0.1823
3.2387	2.3026

Producing the straight line graph of Fig. 11.49.

For % creep strain rate 2.13%, $l_n\varepsilon_s^0 = 0.7561$.

∴ From graph, $l_n\sigma = 2.78$ and $\sigma = 16.12 \text{ kN/m}^2$.

If the cross sectional area of the rod is 625 mm^2

then

$$\begin{aligned} \text{load} &= 16120 \times 625 \times 10^{-6} \\ &= 10 \text{ N.} \end{aligned}$$

Example 11.9

The lives of Nimonic 90 turbine blades tested under varying conditions of stress and temperature are set out in the table below.

Stress (MN/m^2)	Temperature ($^\circ\text{C}$)	Life (h)
180	750	3 000
180	800	500
300	700	5 235
350	650	23 820

Use the information given to produce a master curve based upon the Larson–Miller parameter, and thus calculate the expected life of a blade when subjected to a stress of 250 MN/m^2 and a temperature of 750°C .

Solution

(i) To calculate C: from eqn. (11.23), inserting *absolute* temperatures:

$$T_1(l_n t_r + C) = T_2(l_n t_r + C)$$

$$1023(l_n 3000 + C) = 1073(l_n 500 + C)$$

$$1023(8.0064 + C) = 1073(6.2146 + C)$$

$$8190.5 + 1023C = 6668.27 + 1073C$$

$$1522.23 = 50C$$

$$C = 30.44.$$

(ii) To determine P values

Again, from eqn. (11.23):

$$\begin{aligned}
 P_1 &= [T(l_n t_r + C)] \\
 &= 1023(l_n 3000 + 30.44) \\
 &= 39\,330 \\
 P_2 &= 1073(l_n 500 + 30.44) \\
 &= 1073(6.2146 + 30.44) \\
 &= 39\,330 \\
 P_3 &= 973(l_n 5235 + 30.44) \\
 &= 973(8.5632 + 30.44) \\
 &= 37\,950 \\
 P_4 &= 923(l_n 23\,820 + 30.44) \\
 &= 923(10.0783 + 30.44) \\
 &= 37\,398.
 \end{aligned}$$

Plotting the master curve as per Fig. 11.31 we have the graph shown in Fig. 11.50.

From Fig. 11.50, when the stress equals 250 MN/m^2 the appropriate parameter $P = 38\,525$
 \therefore For the required temperature of 750°C ($= 1023^\circ$ absolute)

$$38\,525 = 1023(l_n t_r + 30.44)$$

$$38\,525 = 1023l_n t_r + 31\,144$$

$$l_n t_r = 7.219$$

$$t_r = 1365 \text{ hours.}$$

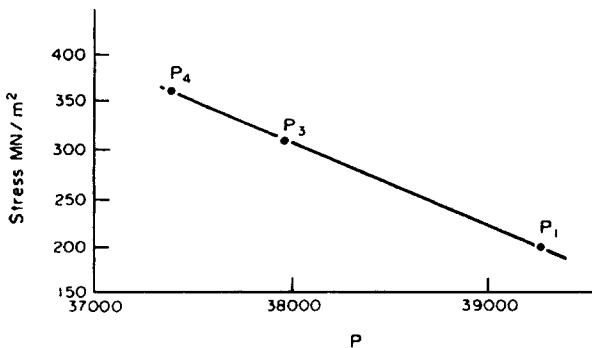


Fig. 11.50.

Example 11.10

The secondary creep rate in many metals may be represented by the equation

$$\dot{\varepsilon}^0 = \beta\sigma^n.$$

A steel bolt clamping two rigid plates together is held at a temperature of 1000°C. If n is 3.0 and $\dot{\varepsilon}^0 = 0.7 \times 10^{-9}\text{h}^{-1}$ at 28 MN/m^2 , calculate the stress remaining in the bolt after 9000 h if the bolt is initially tightened to a stress of 70 MN/m^2 .

Solution

From eqn. (11.20) $\dot{\varepsilon}^0 = \beta\sigma^n$

$$\begin{aligned} \therefore 0.7 \times 10^{-9} &= \beta(28 \times 10^6)^3 \\ \therefore \beta &= \frac{0.7 \times 10^{-9}}{21952 \times 10^{18}} \\ &= 3.189 \times 10^{-32} \end{aligned}$$

Using eqn. (11.27) for stress relaxation

$$\begin{aligned} \frac{1}{\sigma^{n-1}} &= \frac{1}{\sigma_i^{n-1}} + \beta E(n-1)t \\ \therefore \frac{1}{\sigma^2} &= \frac{1}{(70 \times 10^6)^2} + 3.189 \times 10^{-32} \times 200 \\ &\quad \times 10^9 \times 2 \times 900 \\ \frac{1}{\sigma^2} &= 10^{-16} \times 3.188 \\ \sigma &= 10^8 \times \sqrt{\frac{1}{3.188}} \\ &= 10^8 \times 0.56 \end{aligned}$$

i.e. stress in bolt = **56 MN/m²**

Example 11.11

A steel tie in a girder bridge has a rectangular cross-section 200 mm wide and 20 mm deep.

Inspection reveals that a fatigue crack has grown from the shorter edge and in a direction approximately normal to the edge. The crack has grown 23 mm across the width on one face and 25 mm across the width on the opposite face.

If K_{IC} for the material is $55 \text{ MN/m}^{3/2}$ estimate the greatest tension that the tie can withstand.

(Assume that the expression for K in a SEN specimen is applicable here.)

Solution

Since the crack length is not small compared with the width of the girder we need to calculate the compliance function.

Hence

$$a/W = 24/200 = 0.12$$

Then, from Table 11.1

$$\begin{aligned} Y &= 1.99(0.12)^{1/2} - 0.41(0.12)^{3/2} + 18.70(0.12)^{5/2} \\ &\quad - 38.48(0.12)^{7/2} + 53.85(0.12)^{9/2} \\ &= 0.689 - 0.017 + 0.093 - 0.023 + 0.004 \\ &= 0.745 \end{aligned}$$

Also, from eqn. (11.39),

$$K = \frac{PY}{BW^{1/2}}$$

At the onset of fracture $K = K_{IC}$

$$\therefore 55 \times 10^6 = \frac{P \times 0.745}{0.02 \times (0.2)^{1/2}}$$

Hence

failure load $P = 660$ kN.

Example 11.12

A thin cylinder has a diameter of 1.5 m and a wall thickness of 100 mm. The working internal pressure of the cylinder is 15 MN/m² and K_{IC} for the material is 38 MN/m^{3/2}. Estimate the size of the largest flaw that the cylinder can contain. (Assume that for this physical configuration $K = \sigma\sqrt{\pi a}$.)

Non-destructive testing reveals that no flaw above 10 mm exists in the cylinder. If, in the Paris–Erdogan formula, $C = 3 \times 10^{-12}$ (for K in MN/m^{3/2}) and $m = 3.8$, estimate the number of pressurisation cycles that the cylinder can safely withstand.

Solution

Assume that the flaw is sharp, of length $2a$, and perpendicular to the hoop stress.

Then from §9.1.1[†] hoop stress

$$\sigma = \frac{Pd}{2t} = \frac{15 \times 1.5}{2 \times 0.1} = 112.5 \text{ MN/m}^2$$

From eqn. (11.37a), at the point of fracture

$$K = K_{IC} = \sigma\sqrt{\pi a}$$

$$38 = 112.5\sqrt{\pi a}$$

Hence

$$a = 64.3 \text{ mm}$$

From eqn. (11.47)

$$\frac{da}{dN} = C(\Delta K)^m$$

and for pressurisation from zero $\Delta K = K_{max}$

$$\therefore \frac{da}{\Delta K^m} = C \cdot dN$$

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

$$\begin{aligned}
 N &= \frac{1}{C} \int_a^{a_{\max}} \frac{da}{(112.5\sqrt{\pi a})^{3.8}} \\
 &= 16.4 \left[\frac{-0.9}{a^{0.9}} \right]_{0.005}^{0.0643} \\
 &= 14.8(-11.88 + 1178) \\
 &= \mathbf{1156 \text{ cycles.}}
 \end{aligned}$$

Example 11.13

In a laboratory fatigue test on a CTS specimen of an aluminium alloy the following crack length measurements were taken.

Crack length (mm)	21.58	22.64	23.68	24.71	25.72	27.37	28.97	29.75
Cycles	3575	4255	4593	4831	5008	5273	5474	5514

The specimen has an effective width of 50.0 mm.

Load amplitude = 3 kN. Specimen thickness = 25.0 mm.

Construct a growth rate curve in order to estimate the constants C and m in the Paris–Erdogan equation

$$da/dN = C(\Delta K)^m$$

Use the expression given in Table 11.1 to evaluate ΔK .

Use the three-point method to evaluate da/dN .

i.e. at point n ;

$$\left(\frac{da}{dN} \right)_n = \frac{a_{n+1} - a_{n-1}}{N_{n+1} - N_{n-1}}$$

Solution

Equation (11.47) gives the Paris–Erdogan law as

$$da/dN = C(\Delta K)^m$$

$$\log(da/dN) = \log C + m \log(\Delta K)$$

From Table 11.1 we can calculate the amplitude of the stress intensity factor from the equation

$$\begin{aligned}
 \Delta K &= \frac{\Delta P}{BW^{1/2}} \left[29.6 \left(\frac{a}{W} \right)^{1/2} - 185.5 \left(\frac{a}{W} \right)^{3/2} + 655.7 \left(\frac{a}{W} \right)^{5/2} \right. \\
 &\quad \left. - 1017 \left(\frac{a}{W} \right)^{7/2} + 638.9 \left(\frac{a}{W} \right)^{9/2} \right]
 \end{aligned}$$

The crack growth rate is most easily found by using the three-point method. The crack growth rate at the point n is calculated as the slope of the straight line joining the $(n + 1)$ th and the $(n - 1)$ th points.

$$\left(\frac{da}{dN} \right)_n = \frac{a_{n+1} - a_{n-1}}{N_{n+1} - N_{n-1}}$$

From these calculations we obtain the following results

Crack length	21.58	22.64	23.68	24.71	25.72	27.37	28.97	29.75
Cycles	3575	4355	4593	4831	5008	5273	5474	5514
ΔK (MN/m ^{3/2})	4.13	4.38	4.65	4.84	5.26	5.86	6.57	6.97
$\log_{10}(\Delta K)$	0.616	0.642	0.668	0.685	0.721	0.769	0.818	0.843
$\frac{da}{dN} \times 10^6$ m/cycle		2.06	4.34	4.91	6.02	6.97	8.38	
$\log_{10}(da/dN)$		-5.68	-5.36	-5.31	-5.22	-5.15	-5.07	

A log-log plot of da/dN versus ΔK is shown in Fig. 11.51.

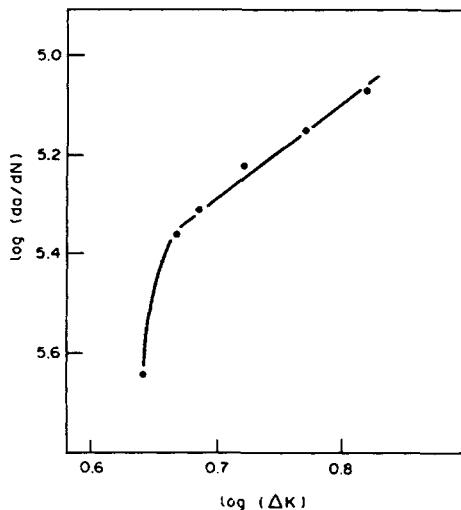


Fig. 11.51.

The first point is not close to the best straight line fit to the other points. The fatigue crack is normally initiated at a high stress amplitude in order to produce a uniform crack front. Although the stress amplitude is reduced gradually to the desired value the initial crack growth is through a crack tip plastic zone associated with the previous loading. The crack is then "blunted" by the larger plastic zone until the crack has grown through it. There is therefore a justification for ignoring this point.

A "least-squares" fit to the remaining points gives

$$\text{Slope} = 2.67 = m$$

$$\text{Intercept} = -7.57 = \log C$$

$$C = 2.69 \times 10^{-8}$$

The Paris-Erdogan equation (11.47) then becomes

$$\frac{da}{dN} = 2.69 \times 10^{-8} (\Delta K)^{2.67}$$

(For ΔK in MN/m^{3/2})

Problems

11.1 (B). (a) Write a short account of the microscopical aspects of fatigue crack initiation and growth.

(b) A fatigue crack is considered to have been initiated when the surface crack length has reached 10^{-3} mm. The percentage of cycle lifetime required to reach this stage may be calculated from the equation:

$$1000N_i = \sqrt{2.02} \times (N_f)^{\sqrt{2.02}}$$

Where N_i is the number of cycles to initiate the crack and N_f is the total number of cycles to failure.

(i) Determine the cyclic lifetime of two specimens, one having a N_i/N_f ratio of 0.01 corresponding to a stress range $\Delta\sigma_1$ and the other having a N_i/N_f ratio of 0.99 corresponding to a stress range $\Delta\sigma_2$.

(ii) If the crack at failure is 1 mm deep, determine the mean crack propagation rate of $\Delta\sigma_1$ and the mean crack nucleation rate at $\Delta\sigma_2$.

[103 cycles, 5.66×10^6 cycles, 9.794×10^{-3} mm/cycle, 17.83×10^{-9} mm/cycle]

11.2 (B). (a) "Under fatigue conditions it may be stated that for less than 1000 cycles, life is a function of ductility and for more than 10,000 cycles life is a function of strength." By consideration of cyclic strain-stress behaviour, show on what grounds this statement is based.

(b) In a tensile test on a steel specimen, the fracture stress was found to be 520 MN/m^2 and the reduction in area 25%.

Calculate: (i) the plastic strain amplitude to cause fracture in 100 cycles; (ii) the stress amplitude to cause fracture in 10^6 cycles.

[0.0268; 173.4 MN/m^2]

11.3 (B). An aluminium cantilever beam, 0.762 m long by 0.092 m wide and 0.183 m deep, is subjected to an end downwards fluctuating load which varies from a minimum value P_{\min} of 8.9 kN to some maximum value P_{\max} .

The material has a fatigue strength for complete stress reversal σ_N of 206.7 MN/m^2 and a static yield strength σ_y of 275.6 MN/m^2 .

By consideration of the Soderberg equation, derive an expression for P_{\max} and show that it is equal to:

$$\frac{2I\sigma_N}{y(1+p)L} + \frac{(1-p)}{(1+p)}P_{\min}$$

where $p = \sigma_N/\sigma_y$ and y is the distance of the extreme fibres from the neutral axis of bending and I is the second moment of area of the beam section. Determine the minimum value of P_{\max} which will produce failure of the beam.

[159 kN]

11.4 (B). (a) Explain the meaning of the term "stress concentration" and discuss its significance in relation to the fatigue life of metallic components.

(b) A member made of steel has the size and shape indicated in Fig. 11.52.

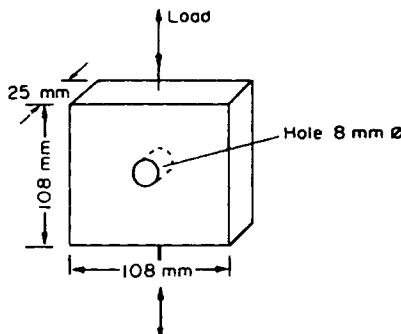


Fig. 11.52.

The member is subjected to a fluctuating axial load that varies from a minimum value of $P/2$ to a maximum value of P . Use the Soderberg equation to determine the value of P that will produce failure in 10^6 cycles.

Assume:

Yield strength of steel = 420 MN/m²

Fatigue strength of steel = 315 MN/m² for 10⁶ cycles

Notch sensitivity factor = 0.9

Static stress concentration factor = 3.0

[0.62 MN]

11.5 (B). (a). Explain briefly the concepts of survival probability and cumulative damage in respect of fatigue of structural components.

(b) The loading spectrum on an aluminium alloy component is given below for every 100 000 cycles. Also shown is the fatigue life at each stress level.

Determine an expected fatigue life based upon Miner's Hypothesis of damage.

Stress amplitude MN/m ²	Number of cycles in each 10 ⁵ cycles	Fatigue life (N _f)
340	3000	80 000
290	8000	330 000
240	15 000	1000 000
215	34 000	3000 000
190	40 000	35000 000

[11.21 × 10⁵ cycles]

11.6 (B). A stress analysis reveals that at a point in a steel part the stresses are σ_{xx} = 105, σ_{yy} = 35, σ_{zz} = -20, τ_{xy} = 56, τ_{yz} = 10, τ_{xz} = 25 MN/m². These stresses are cyclic, oscillating about a mean stress of zero. The fatigue strength of the steel is 380 MN/m². Determine the safety factor against fatigue failure. Assume that the fatigue could initiate at a small internal flaw located at the point in question, and assume a stress-concentration factor of 2. Ignore the effect of triaxial stresses on fatigue.

[1.35]

11.7 (B). (a) Briefly discuss how the following factors would affect the fatigue life of a component:

- (i) surface finish,
- (ii) surface treatment,
- (iii) surface shape.

(b) An aluminium airframe component was tested in the laboratory under an applied stress which varied sinusoidally about a mean stress of zero. The component failed under a stress range of 280 MN/m² after 10⁵ cycles and under a stress range of 200 MN/m² after 10⁷ cycles. Assuming that the fatigue behaviour can be represented by:

$$A\sigma(N_f)^a = C$$

where *a* and *C* are constants, find the number of cycles to failure for a component subjected to a stress range of 150 MN/m².

(c) After the component has already endured an estimated 4 × 10⁸ cycles at a stress range of 150 MN/m², it is decided that its failure life should be increased by 4 × 10⁸ cycles. Find the decrease in stress range necessary to achieve this additional life.

You may assume a simple cumulative damage law of the form:

$$\sum \frac{N_i}{N_f} = 1 \quad [15.2 \text{ MN/m}^2]$$

11.8 (B). (a) Write an account of the effect of mean stress upon the fatigue life of a metallic component. Include within your account a brief discussion of how mean stress may be allowed for in fatigue calculations.

(b) A thin-walled cylindrical vessel 160 mm internal diameter and with a wall thickness of 10 mm is subjected to an internal pressure that varies from a value of -P/4 to P. The fatigue strength of the material at 10⁸ cycles is 235 MN/m² and the tensile yield stress is 282 MN/m². Using the octahedral shear theory, determine a nominal allowable value for *P* such that failure will not take place in less than 10⁸ cycles.

[36.2 MN/m²]

11.9 (B). (a) The Manson-Haferd creep parameter method was developed on an entirely empirical basis, whilst those of Larson-Miller and Skerby-Dorn are based upon the well known Arrhenius equation. Compare all three extrapolation methods and comment on the general advantages and disadvantages of applying these methods in practice.

(b) The following table was produced from the results of creep tests carried out on specimens of Nimonic 80 A.

Stress (MN/m ²)	Temperature (°C)	Time to rupture (h)
180	752	56
180	502	1000
300	452	316
300	317	3160

Use the information given to produce a master curve based upon the Manson–Haferd parameter and thus estimate the expected life of a material when subjected to a stress of 250 MN/m² and a temperature of 400°C.

[1585 hours]

11.10 (B). (a) Briefly describe the generally desirable characteristics of a material for use at high temperatures.

(b) A cylindrical tube in a chemical plant is subjected to an internal pressure of 6 MN/m² which leads to a circumferential stress in the tube wall. The tube is required to withstand this stress at a temperature of 575°C for 9 years.

A designer has specified tubes of 40 mm bore and 2 mm wall thickness made from a stainless steel and the manufacturer's specification for this alloy gives the following information at $\sigma = 200 \text{ MN/m}^2$.

Temp (°C) ε/S	500 1.0×10^{-6}	550 2.1×10^{-6}	600 4.3×10^{-6}	650 7.7×10^{-6}	700 1.4×10^{-5}

Given that the effect of stress and temperature upon creep rate can be considered by the following equation:

$$\dot{\varepsilon}^\circ = A\sigma^6 e^{-\Delta H/RT}$$

and that failure of the tube will take place at a strain of 0.01, with the aid of a graph, calculate whether the tube will fulfil its design life function.

[No, $\varepsilon_9 = 2.07$]

11.11 (B). (a) By consideration of the Arrhenius equation, discuss the theoretical basis of the Larson–Miller parameter as applied to creep data and compare it with other well known alternative parameters.

(b) From the figures given below, determine the expected life at 650°C of an alloy steel when subjected to a stress of 205 MN/m².

Stress (MN/m ²)	Temperature (°C)	Life (h)
205	700	1000
205	720	315

[22 330 hours]

11.12 (B). A cylindrical polymer component is produced at constant pressure by expanding a smaller cylinder into a cylindrical mould. The initial polymer cylinder has length 1200 mm, internal diameter 20 mm and wall thickness 5 mm; and the mould has diameter 100 mm and length 1250 cm. Show that, neglecting end effects, the diameter of the cylindrical portion (which may be considered thin) will increase without any change of length until the material touches the mould walls. Hence determine the time taken for the plastic material to reach the mould walls under an internal pressure of 10 kN/m², if the uniaxial creep equation for the polymer is $d\varepsilon/dt = 32\sigma$, where t is the time in seconds and σ is the stress in N/mm².

The Levy–von Mises equations which govern the behaviour of the polymer are of the form

$$d\varepsilon_e = \frac{d\varepsilon_e}{\sigma_e} [\sigma_1 - \frac{1}{2}(\sigma_2 + \sigma_3)]$$

where σ_e is the equivalent stress and ε_e is the equivalent strain.

[1 second]

11.13 (B). (a) Explain, briefly, the meaning and importance of the term "stress relaxation" as applied to metallic materials.

(b) A pressure vessel is used for a chemical process operating at a pressure of 1 MN/m² and a temperature of 425°C. One of the ends of the vessel has a 400 mm diameter manhole placed at its centre and the cover plate is held in position by twenty steel bolts of 25 mm diameter spaced equally around the flanges.

Tests on the bolt steel indicate that $n = 4$ and $\varepsilon^\circ = 8.1 \times 10^{-10}/\text{h}$ at 21 MN/m^2 and 425°C .

Assuming that the stress in a bolt at any time is given by the equation:

$$\frac{1}{\sigma_t^{n-1}} = \frac{1}{\sigma_i^{n-1}} + \beta E(n-1)t$$

and that the secondary creep rate can be represented by the relationship $\dot{\varepsilon}^\circ = \beta\sigma^n$, with E for the bolt steel = 200 GN/m^2 , calculate:

- (i) the initial tightening stress in the bolts so that after 10 000 hours of creep relaxation there is still a safety factor of 2 against leakage,
- (ii) the total time for leakage to occur.

[30.7 MN/m^2 ; $177.2 \times 10^3 \text{ hours}$]

11.14 (B). (a) The lives of Nimonic 90 turbine blades tested under varying conditions of stress and temperature are set out in the table below:

Stress (MN/m^2)	Temperature ($^\circ\text{C}$)	Life (h)
180	750	3000
180	800	500
300	700	5235
350	650	23 820

Use the information given to produce a master curve based upon the Larson–Miller parameter, and thus calculate the expected life of a blade when subjected to a stress of 250 MN/m^2 and a temperature of 750°C .

(b) Discuss briefly the advantages and disadvantages of using parametric methods to predict creep data compared with the alternative method of using standard creep strain–time curves.

[1365 hours]

11.15 (B). (a) A support bracket is to be made from a steel to be selected from the table below. It is important that, if overloading occurs, yielding takes place before fracture. The thickness of the section is 10 mm and the maximum possible surface crack size that could have escaped non-destructive inspection is 10% of the section thickness. Select a suitable tempering temperature from the list. Assume that this is equivalent to an edge crack in a wide plate and assume an associated flaw shape parameter of 1.05.

Tempering temperature ($^\circ\text{C}$)	Yield strength (MN/m^2)	K_{IC} ($\text{MN/m}^{3/2}$)
480	1207	98.9
425	1413	95.4
370	1586	41.8

[425°C]

11.16 (B). A high-speed steel circular saw of 300 mm diameter and 3.0 mm thickness has a fatigue crack of length 26 mm running in a radial direction from the spindle hole. Assume that this is an edge crack in a semi-infinite plate. The proof strength of this steel is 1725 MN/m^2 and its fracture toughness is $23 \text{ MN/m}^{3/2}$. The tangential stress component adjacent to the spindle hole during operation has been calculated as follows:

Periphery temperature relative to centre ($^\circ\text{C}$)	Tangential stress (MN/m^2)
10	27
30	50
50	72

First estimate the size of the plastic zone in order to decide whether plane strain or plane stress conditions predominate. Then estimate at what periphery temperature the saw is likely to fail by fast fracture.

[$0.0226 \text{ mm}; 50^\circ\text{C}$]

What relative safety factor (on tangential stress) can be gained by using a carbide-tipped lower-strength but tougher steel saw of yield strength 1200 MN/m^2 and fracture toughness $99 \text{ MN/m}^{3/2}$, in the case of the same size of fatigue crack?

[4.3]

11.17 (B). (a) When a photoelastic model similar to the one shown in Fig. 11.39 is stressed the fifth fringe is found to have a maximum distance of 2.2 mm from the crack tip. If the fringe constant is $11 \text{ N/mm}^2/\text{fringe-mm}$

and the model thickness is 5 mm, determine the value of the stress-intensity factor (in N/m^{3/2}) under this applied load. Discuss any important errors which could be associated with this measurement. [1.293 MN/m^{3/2}]

(b) Suggest a way of checking to ensure that the stresses are purely mode I (i.e. those tending to "open" the crack) and that there is no superimposed mode II component (i.e. the tendency to shear the crack along its plane, as shown in Figure 11.34).

11.18 (B). (a) The stresses near the crack tip of a specimen containing a through-thickness crack loaded in tension perpendicular to the crack plane are given by the following equations

$$\sigma_{xx} = \frac{K_I \cos \frac{\theta}{2}}{\sqrt{2\pi r}} \left[1 - \sin \frac{\theta}{2} \cdot \sin \frac{3\theta}{2} \right]$$

$$\sigma_{yy} = \frac{K_I \cos \frac{\theta}{2}}{\sqrt{2\pi r}} \left[1 + \sin \frac{\theta}{2} \cdot \sin \frac{3\theta}{2} \right]$$

$$\sigma_{xy} = \frac{K_I \cos \frac{\theta}{2}}{\sqrt{2\pi r}} \left[\sin \frac{\theta}{2} \cdot \cos \frac{3\theta}{2} \right]$$

where K_I is the stress intensity factor, r is the distance from the crack tip and θ is the angle measured from the projected line of the crack in the uncracked region.

Using the proportions of Mohr's circle, or otherwise, show that the maximum shear stress near the crack tip is given by:

$$\tau_{\max} = \frac{K_I \sin \theta}{2\sqrt{2\pi r}}$$

(b) Sketch the photoelastic fringe pattern which would be expected from a model of this loading case.

(c) In such a fringe pattern the fourth fringe occurs at a distance of 1.45 mm from the crack tip. If the material fringe constant is 10.5 kN/m²/fringe/m and the model thickness is 5 mm determine the value of K_I under the given applied load. What error is associated with this measurement? [0.8 MN/m^{3/2}]

11.19 (B). (a) Write a short essay on the application of fracture mechanics to the problem of crack growth in components subjected to alternating loading conditions.

(b) After two years service a wide panel of an aluminium alloy was found to contain a 5 mm long edge crack orientated normal to the applied stress. The panel was designed to withstand one start-up/shut-down cycle per day for 20 years (assume 250 operating days in a year), the cyclic stress range being 0 to 70 MN/m².

If the fracture toughness of the alloy is 35 MNm^{-3/2} and the cyclic growth rate of the crack is represented by the equation:

$$\frac{da}{dN} = 3.3 \times 10^{-9} (\Delta K)^{3.0}$$

calculate whether the panel will meet its design life expectancy. (Assume $K_I = \sigma \sqrt{\pi a}$).

[No – 15.45 years]

11.20 (B). (a) Differentiate between the terms "stress concentration factor" and "stress intensity factor".

(b) A cylindrical pressure vessel of 7.5 m diameter and 40 mm wall thickness is to operate at a working pressure of 5.1 MN/m². The design assumes that failure will take place by fast fracture from a crack and to prevent this the total number of loading cycles must not exceed 3000.

The fracture toughness of the sheet is 200 MN/m^{3/2} and the growth of the crack may be represented by the equation:

$$\frac{da}{dN} = A(\Delta K)^4$$

Where $A = 2.44 \times 10^{-14}$ and K is the stress intensity factor. Find the minimum pressure to which the vessel must be tested before use to guarantee against fracture in under 3000 cycles. [8.97 MN/m²]

11.21 (B). (a) Write a short essay on application of fracture mechanics to the problem of crack growth in components subjected to alternating loading conditions.

(b) Connecting rods for an engine are to be made of S.G. iron for which $K_{IC} = 25 \text{ MNm}^{-3/2}$. NDT will detect cracks or flaws of length $2a$ greater than 2 mm, and rods with flaws larger than this are rejected.

Independent tests on the material show that cracks grow at a rate such that

$$\frac{da}{dN} = 2 \times 10^{-15} (\Delta K_I)^3 \text{ m cycle}^{-1}$$

The minimum cross-sectional area of the rod is 0.01 m^2 , its section is circular and the maximum tensile load in service is 1 MN.

Assuming that $K_I = \sigma\sqrt{\pi a}$ and the engine runs at 1000 rev/min calculate whether the engine will meet its design requirement of 20,000 h life. [$N_f = 4.405 \times 10^9$ and engine will survive]

11.22 (B). (a) By considering constant load conditions applied to a thin semi-infinite sheet and also the elastic energy in the material surrounding an internal crack of unit width and length $2c$, derive an expression for the stress when the crack propagates spontaneously.

(b) A pipeline is made from a steel of Young's modulus $2.06 \times 10^{11} \text{ Nm}^{-2}$ and surface energy 1.1 J m^{-2} .

Calculate the critical half-length of a Griffith crack for a stress of $6.2 \times 10^6 \text{ Nm}^{-2}$, assuming that all the supplied energy is used for forming the fracture surface.

$$\left[\sqrt{\frac{2\gamma E}{\pi c}}, \text{ } 3.752 \text{ mm} \right]$$

11.23 (B). (a) Outline a method for determining the plane strain fracture toughness K_{IC} , indicating any criteria to be met in proving the result valid.

(b) For a standard tension test piece, the stress intensity factor K_I is given by:

$$K_I = \frac{P}{BW^{1/2}} \left[29.6 \left(\frac{a}{W} \right)^{1/2} - 32.04 \left(\frac{a}{W} \right)^{3/2} \right]$$

the symbols having their usual meaning.

Using the DC potential drop crack detection procedure the load at crack initiation P was found to be 14.6 kN. Calculate K_{IC} for the specimen if $B = 25 \text{ mm}$, $W = 50 \text{ mm}$ and $a = 25 \text{ mm}$.

(c) If $\sigma_1 = 340 \text{ MN/m}^2$, calculate the minimum thickness of specimen which could be used still to give a valid K_{IC} value. [$25.07 \text{ MNm}^{-3/2}$, 13.6 mm]

CHAPTER 12

MISCELLANEOUS TOPICS

12.1. Bending of beams with initial curvature

The bending theory derived and applied in *Mechanics of Materials 1* was concerned with the bending of initially straight beams. Let us now consider the modifications which are required to this theory when the beams are initially curved before bending moments are applied. The problem breaks down into two classes:

- (a) initially curved beams where the depth of cross-section can be considered small in relation to the initial radius of curvature, and
- (b) those beams where the depth of cross-section and initial radius of curvature are approximately of the same order, i.e. deep beams with high curvature.

In both cases similar assumptions are made to those for straight beams even though some will not be strictly accurate if the initial radius of curvature is small.

(a) Initially curved slender beams

Consider now Fig. 12.1, with Fig. 12.1 (a) showing the initial curvature of the beam before bending, with radius R_1 , and Fig. 12.1 (b) the state after the bending moment M has been applied to produce a new radius of curvature R_2 . In both figures the radii are measured to the neutral axis.

The strain on any element $A'B'$ a distance y from the neutral axis will be given by:

$$\begin{aligned}\text{strain on } A'B' = \varepsilon &= \frac{A'B' - AB}{AB} \\ &= \frac{(R_2 + y)\theta_2 - (R_1 + y)\theta_1}{(R_1 + y)\theta_1} \\ &= \frac{R_2\theta_2 + y\theta_2 - R_1\theta_1 - y\theta_1}{(R_1 + y)\theta_1}\end{aligned}$$

Since there is no strain on the neutral axis in either figure $CD = C'D'$ and $R_1\theta_1 = R_2\theta_2$.

$$\therefore \varepsilon = \frac{y\theta_2 - y\theta_1}{(R_1 + y)\theta_1} = \frac{y(\theta_2 - \theta_1)}{(R_1 + y)\theta_1}$$

and, since $\theta_2 = R_1\theta_1/R_2$,

$$\varepsilon = \frac{y\theta_1 \left(\frac{R_1}{R_2} - 1 \right)}{(R_1 + y)\theta_1} = \frac{y(R_1 - R_2)}{R_2(R_1 + y)} \quad (12.1)$$

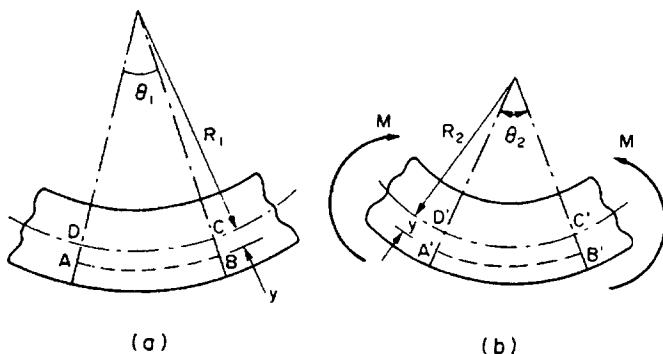


Fig. 12.1. Bending of beam with initial curvature (a) before bending, (b) after bending to new radius of curvature R_2 .

For the case of slender beams with y small in comparison with R_1 (i.e. when y can be neglected in comparison with R_1), the equation reduces to:

$$\varepsilon = y \frac{(R_1 - R_2)}{R_2 R_1} = y \left[\frac{1}{R_2} - \frac{1}{R_1} \right] \quad (12.2)$$

The strain is thus directly proportional to y the distance from the neutral axis and, as for the case of straight beams, the stress and strain distribution across the beam section will be linear and the neutral axis will pass through the centroid of the section. Equation (12.2) can therefore be incorporated into a modified form of the “simple bending theory” thus:

$$\frac{M}{I} = \frac{\sigma}{y} = E \left[\frac{1}{R_2} - \frac{1}{R_1} \right] \quad (12.3)$$

For initially straight beams R_1 is infinite and eqn. (12.2) reduces to:

$$\varepsilon = \frac{y}{R_2} = \frac{y}{R}$$

(b) Deep beams with high initial curvature (i.e. small radius of curvature)

For deep beams where y can no longer be neglected in comparison with R_1 eqn. (12.1) must be fully applied. As a result, the strain distribution is no longer directly proportional to y and hence the stress and strain distributions across the beam section will be non-linear as shown in Fig. 12.2 and the neutral axis will not pass through the centroid of the section.

From eqn. (12.1) the stress at any point in the beam cross-section will be given by:

$$\sigma = E\varepsilon = \frac{Ey(R_1 - R_2)}{R_2(R_1 + y)} \quad (12.4)$$

For equilibrium of transverse forces across the section in the absence of applied end load $\int \sigma dA$ must be zero.

$$\therefore \int \frac{Ey(R_1 - R_2)}{R_2(R_1 + y)} dA = \frac{E(R_1 - R_2)}{R_2} \int \frac{y}{(R_1 + y)} \cdot dA = 0 \quad (12.5)$$

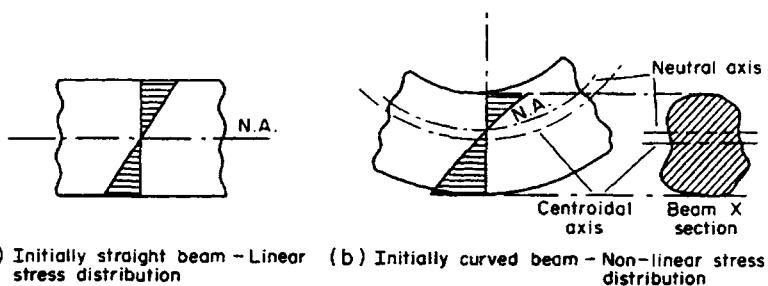


Fig. 12.2. Stress distributions across beams in bending. (a) Initially straight beam linear stress distribution; (b) initially curved deep beam-non-linear stress distribution.

i.e.

$$\int \frac{y}{(R_1 + y)} \cdot dA = 0 \quad (12.6)$$

Unlike the case of bending of straight beams, therefore, it will be seen by inspection that the above integral no longer represents the first moment of area of the section about the centroid. Thus, *the centroid and the neutral axis can no longer coincide*.

The bending moment on the section will be given by:

$$M = \int \sigma \cdot dA \cdot y = \frac{E(R_1 - R_2)}{R_2} - \frac{y_2}{(R_1 + y)} \cdot dA \quad (12.7)$$

but

$$\begin{aligned} \int \frac{y^2}{(R_1 + y)} \cdot dA &= \int \frac{y[(R_1 + y) - R_1]dA}{(R_1 + y)} \\ &= \int y \cdot dA - R_1 \int \frac{y \cdot dA}{(R_1 + y)} \end{aligned}$$

and from eqn. (12.5) the second integral term reduces to zero for equilibrium of transverse forces.

$$\therefore \int \frac{y_2}{(R_1 + y)} \cdot dA = \int y \cdot dA = A\bar{y} = Ah$$

where h is the distance of the neutral axis from the centroid axis, see Fig. 12.3. Substituting in eqn. (12.7) we have:

$$M = \frac{E(R_1 - R_2)}{R_2} \cdot hA \quad (12.8)$$

From eqn. (12.4)

$$\begin{aligned} \frac{\sigma}{y}(R_1 + y) &= \frac{E}{R_2}(R_1 - R_2) \\ \therefore M &= \frac{\sigma}{y}(R_1 + y)hA \end{aligned} \quad (12.9)$$

i.e.

$$\frac{\sigma}{y} = \frac{M}{hA(R_1 + y)} \quad (12.10)$$

or

$$\sigma = \frac{My}{hA(R_1 + y)} = \frac{My}{hAR_0} \quad (12.11)$$

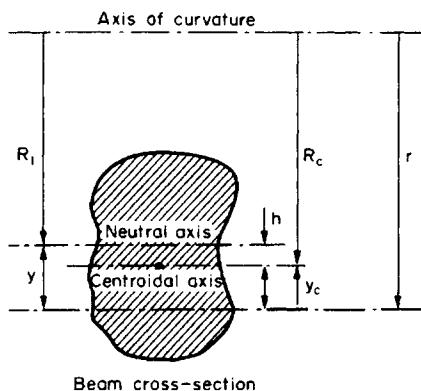


Fig. 12.3. Relative positions of neutral axis and centroidal axis.

On the opposite side of the neutral axis, where y will be negative, the stress becomes:

$$\sigma = -\frac{My}{hA(R_1 - y)} = -\frac{My}{hAR_i} \quad (12.12)$$

These equations show that the stress distribution follows a hyperbolic form. Equation (12.12) can be seen to be similar in form to the "simple bending" equation[†].

$$\frac{\sigma}{y} = \frac{M}{I}$$

with the term $hA(R_1 + y)$ replacing the second moment of area I .

Thus in order to be able to calculate stresses in deep-section beams with high initial curvature, it is necessary to evaluate h and R_1 , i.e. to locate the position of the neutral axis relative to the centroid or centroidal axis. This was shown above to be given by the condition:

$$\int \frac{y}{(R_1 + y)} \cdot dA = 0.$$

Now fibres distance y from the neutral axis will be some distance y_c from the centroidal axis as shown in Figs. 12.3 and 12.4 such that, in relation to the axis of curvature,

$$R_1 + y = R_c + y_c$$

with

$$y = y_c + h$$

∴ from eqn. (12.5)

$$\int \frac{(y_c + h)}{(R_c + y_c)} \cdot dA = 0$$

Re-writing

$$y_c + h = (R_c + y_c) - R_c + h = (R_c + y_c) - (R_c - h).$$

[†] Timoshenko and Roark both give details of correction factors which may be applied for standard cross-sectional shapes to be used in association with the simple straight beam equation. (S. Timoshenko, *Theory of Plates and Shells*, McGraw Hill, New York; R. J. Roark and W.C. Young, *Formulas for Stress and Strain*, McGraw Hill, New York).

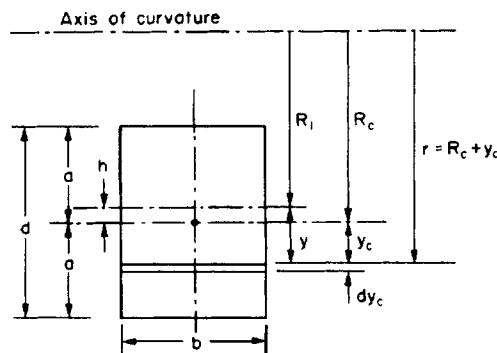


Fig. 12.4.

$$\begin{aligned} \int \frac{(y_c + h)}{(R_c + y_c)} &= \int \frac{(R_c + y_c)}{(R_c + y_c)} \cdot dA - (R_c - h) \int \frac{1}{(R_c + y_c)} \cdot dA \\ &= A - (R_c - h) \int \frac{1}{(R_c + y_c)} \cdot dA = 0 \\ \therefore h = R_c - \frac{A}{\int \frac{dA}{(R_c + y_c)}} &= R_c - \frac{A}{\int \frac{dA}{r}} \end{aligned} \quad (12.13)$$

and $R_1 = R_c - h = \frac{A}{\int \frac{dA}{(R_c + y_c)}} = \frac{A}{\int \frac{dA}{r}}$ (12.14)

Examples 12.1 and 12.2 show how the theory may be applied and Table 12.1 gives some useful equations for $\int \frac{dA}{r}$ for standard shapes of beam cross-section.

Note

Before applying the above theory for bending of initially curved members it is perhaps appropriate to consider the benefits to be gained over that of an approximate solution using the simple bending theory.

Provided that the curvature is not large then the simple theory is reasonably accurate; for example, for a radius to beam depth ratio R_c/d of as low as 5 the error introduced in the maximum stress value is only of the order of 7%. The error then rises steeply, however, as curvature increases to a figure of approx. 30% at $R_c/d = 1.5$.

(c) Initially curved beams subjected to bending and additional direct load

In many practical engineering applications such as chain links, crane hooks, G-clamps etc., the component cross-sections will be subjected to both bending and additional direct load, whereas the equations derived in the previous sections have all been derived on the assumption of pure bending only. It is therefore necessary in such cases to obtain a solution by the application of the principle of superposition i.e. by resolving the loading system into

Table 12.1. Values of $\int \frac{dA}{r}$ for curved bars.

Cross-section	$\int \frac{dA}{r}$
(a) Rectangle	$b \log_e \left(\frac{R_0}{R_i} \right)$ (N.B. The two following cross-sections are simply produced by the addition of terms of this form for each rectangular portion)
(b) T-section	$b_1 \log_e \left(\frac{R_i + d_1}{R_i} \right) + b_2 \log_e \left(\frac{R_0}{R_i + d_1} \right)$
(c) I-beam	$b_1 \log_e \left(\frac{R_i + d_1}{R_i} \right) + b_2 \log_e \left(\frac{R_0 - d_3}{R_i + d_1} \right) + b_3 \log_e \left(\frac{R_0}{R_0 - d_3} \right)$
(d) Trapezoid	$\left[\frac{(b_1 R_0 - b_2 R_i)}{d} \log_e \left(\frac{R_0}{R_i} \right) \right] - b_1 + b_2$
(e) Triangle	As above (d) with $b_2 = 0$ As above (d) with $b_1 = 0$
Circle	$2\pi \{ (R_i + R) - [(R_i + R)^2 - R^2]^{1/2} \}$

its separate bending, normal (and perhaps shear) loads on the section and combining the stress values obtained from the separate stress calculations. Normal and bending stresses may be added algebraically and combined with the shearing stresses using two- or three-dimensional complex stress equations or Mohr's circle.

Care must always be taken to consider the direction in which the moment is applied. In the derivation of the equations in the previous sections it has been shown acting in a direction to increase the initial curvature of the beam (Fig. 12.1) producing tensile bending stresses on the outside (convex) surface and compression on the inner (concave) surface. In the practical cases mentioned above, however, e.g. the chain link or crane hook, the moment which is usually applied will tend to straighten the beam and hence reduce its curvature. In these cases, therefore, tensile stresses will be set up on the inner surface and these will add to the tensile stresses produced by the direct load across the section to produce a maximum tensile (and potentially critical) stress condition on this surface – see Fig. 12.5.

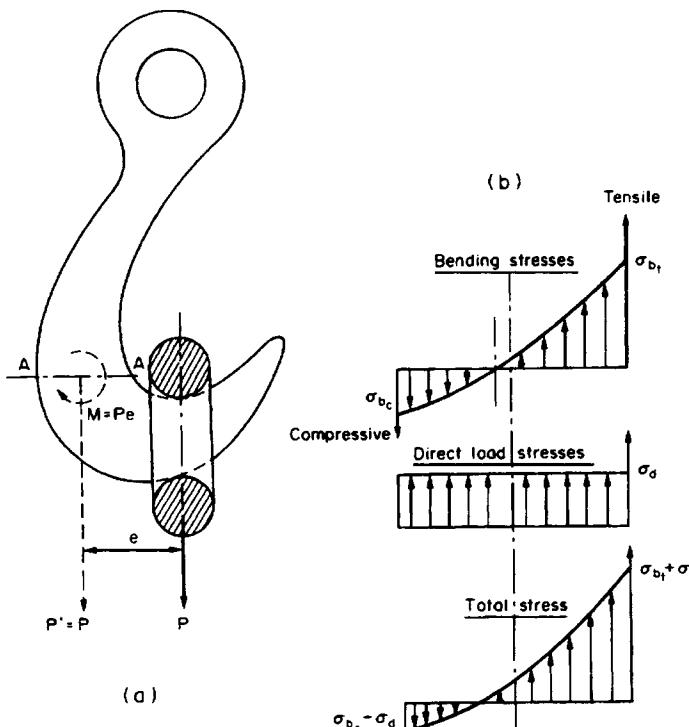


Fig. 12.5. Loading of a crane hook. (a) Load effect on section AA is direct load $P' = P$ plus moment $M = Pe$; (b) stress distributions across the section AA.

12.2. Bending of wide beams

The equations derived in *Mechanics of Materials 1* for the stress and deflection of beams subjected to bending relied on the assumption that the beams were narrow in relation to their depths in order that expansions or contractions in the lateral (z) direction could take place relatively freely.

For beams that are very wide in comparison with their depth – see Fig. 12.6 – lateral deflections are constrained, particularly towards the centre of the beam, and such beams become stiffer than predicted by the simple theory and deflections are correspondingly reduced. In effect, therefore, the bending of narrow beams is a plane stress problem whilst that of wide beams becomes a plane strain problem – see §8.22.

For the beam of Fig. 12.6 the strain in the z direction is given by eqn. (12.6) as:

$$\varepsilon_z = \frac{1}{E}(\sigma_z - \nu\sigma_x - \nu\sigma_y).$$

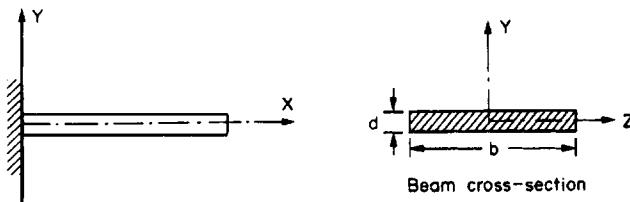


Fig. 12.6. Bending of wide beams ($b \gg d$)

Now for thin beams $\sigma_y = 0$ and, for total constraint of lateral (z) deformation at $z = 0$, $\varepsilon_z = 0$.

$$\therefore 0 = \frac{1}{E}(\sigma_z - \nu\sigma_x)$$

$$\text{i.e. } \sigma_z = \nu\sigma_x$$

Thus, the strain in the longitudinal x direction will be:

$$\begin{aligned} \varepsilon_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y - \nu\sigma_z) \\ &= \frac{1}{E}(\sigma_x - 0 - \nu(\nu\sigma_x)) \\ &= \frac{1}{E}(1 - \nu^2)\sigma_x \end{aligned} \quad (12.15)$$

$$= \frac{(1 - \nu^2)}{E} \cdot \frac{My}{I} \quad (12.16)$$

Compared with the narrow beam case where $\varepsilon_x = \sigma_x/E$ there is thus a reduction in strain by the factor $(1 - \nu^2)$ and this can be introduced into the deflection equation to give:

$$\frac{d^2y}{dx^2} = (1 - \nu^2) \frac{M}{EI} \quad (12.17)$$

Thus, all the formulae derived in Book 1 including those of the summary table, may be used for wide beams *provided that they are multiplied by $(1 - \nu^2)$* .

12.3. General expression for stresses in thin-walled shells subjected to pressure or self-weight

Consider the general shell or “surface of revolution” of arbitrary (but thin) wall thickness shown in Fig. 12.7 subjected to internal pressure. The stress system set up will be three-dimensional with stresses σ_1 (hoop) and σ_2 (meridional) in the plane of the surface and σ_3 (radial) normal to that plane. Strictly, all three of these stresses will vary in magnitude through the thickness of the shell wall but provided that the thickness is less than approximately one-tenth of the major, i.e. smallest, radius of curvature of the shell surface, this variation can be neglected as can the radial stress (which becomes very small in comparison with the hoop and meridional stresses).

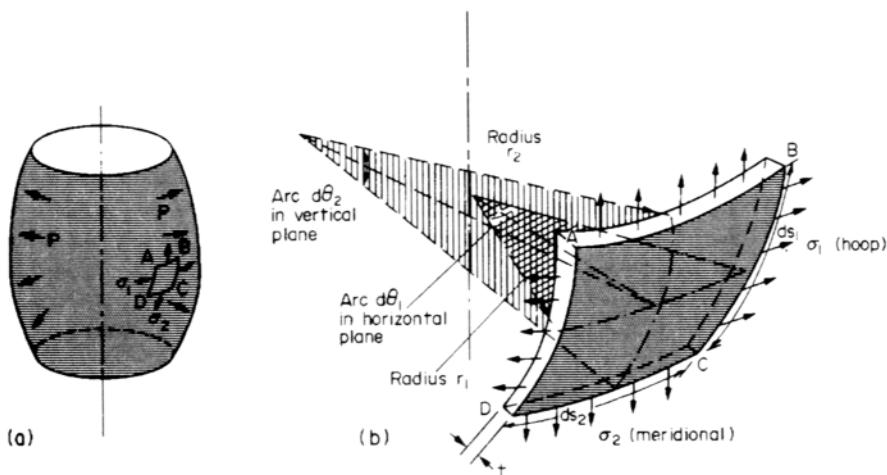


Fig. 12.7. (a) General surface of revolution subjected to internal pressure p ; (b) element of surface with radii of curvature r_1 and r_2 in two perpendicular planes.

Because of this limitation on thickness, which makes the system statically determinate, the shell can be considered as a membrane with little or no resistance to bending. The stresses set up on any element are thus only the so-called “membrane stresses” σ_1 and σ_2 mentioned above, no additional bending stresses being required.

Consider, therefore, the equilibrium of the element ABCD shown in Fig. 12.7(b) where r_1 is the radius of curvature of the element in the horizontal plane and r_2 is the radius of curvature in the vertical plane.

The forces on the “vertical” and “horizontal” edges of the element are $\sigma_1 t ds_1$ and $\sigma_2 t ds_2$, respectively, and each are inclined relative to the radial line through the centre of the element, one at an angle $d\theta_1/2$ the other at $d\theta_2/2$.

Thus, resolving forces along the radial line we have, for an internal pressure p :

$$2(\sigma_1 t ds_1 \cdot \sin \frac{d\theta_1}{2} + \sigma_2 t ds_2 \cdot \sin \frac{d\theta_2}{2}) = p \cdot ds_1 \cdot ds_2$$

Now for small angles $\sin d\theta/2 = d\theta/2$ radians

$$\therefore 2 \left(\sigma_1 t ds_1 \cdot \frac{d\theta_1}{2} + \sigma_2 t ds_2 \cdot \frac{d\theta_2}{2} \right) = p ds_1 \cdot ds_2$$

Also $ds_1 = r_2 d\theta_2$ and $ds_2 = r_1 d\theta_1$

$$\therefore \sigma_1 t ds_1 \cdot \frac{ds_2}{r_1} + \sigma_2 t ds_2 \cdot \frac{ds_1}{r_2} = p \cdot ds_1 \cdot ds_2$$

and dividing through by $ds_1 \cdot ds_2 \cdot t$ we have:

$$\frac{\sigma_1}{r_1} + \frac{\sigma_2}{r_2} = \frac{p}{t} \quad (12.18)$$

For a general shell of revolution, σ_1 and σ_2 will be unequal and a second equation is required for evaluation of the stresses set up. In the simplest application, i.e. that of the sphere, however, $r_1 = r_2 = r$ and symmetry of the problem indicates that $\sigma_1 = \sigma_2 = \sigma$. Equation (12.18) thus gives:

$$\sigma = \frac{pr}{2t}$$

In some cases, e.g. concrete domes or dishes, the self-weight of the vessel can produce significant stresses which contribute to the overall failure consideration of the vessel and to the decision on the need for, and amount of, reinforcing required. In such cases it is necessary to consider the vertical equilibrium of an element of the dome in order to obtain the required second equation and, bearing in mind that self-weight does not act radially as does applied pressure, eqn. (12.18) has to be modified to take into account the vertical component of the forces due to self-weight.

Thus for a dome of subtended arc 2θ with a force per unit area q due to self-weight, eqn. (12.18) becomes:

$$\frac{\sigma_1}{r_1} + \frac{\sigma_2}{r_2} = \pm \frac{q \cos \theta}{t} \quad (12.19)$$

Combining this equation with one obtained from vertical equilibrium considerations yields the required values of σ_1 and σ_2 .

12.4. Bending stresses at discontinuities in thin shells

It is normally assumed that thin shells subjected to internal pressure show little resistance to bending so that only membrane (direct) stresses are set up. In cases where there are changes in geometry of the shell, however, such as at the intersection of cylindrical sections with hemispherical ends, the "incompatibility" of displacements caused by the membrane stresses in the two sections may give rise to significant local bending effects. At times these are so severe that it is necessary to introduce reinforcing at the junction locations.

Consider, therefore, such a situation as shown in Fig. 12.8 where both the cylindrical and hemispherical sections of the vessel are assumed to have uniform and equal thickness membrane stresses in the cylindrical portion are

$$\sigma_1 = \sigma_H = \frac{pr}{t} \quad \text{and} \quad \sigma_2 = \sigma_L = \frac{pr}{2t}$$

whilst for the hemispherical ends

$$\sigma_1 = \sigma_2 = \sigma_H = \frac{pr}{t}.$$

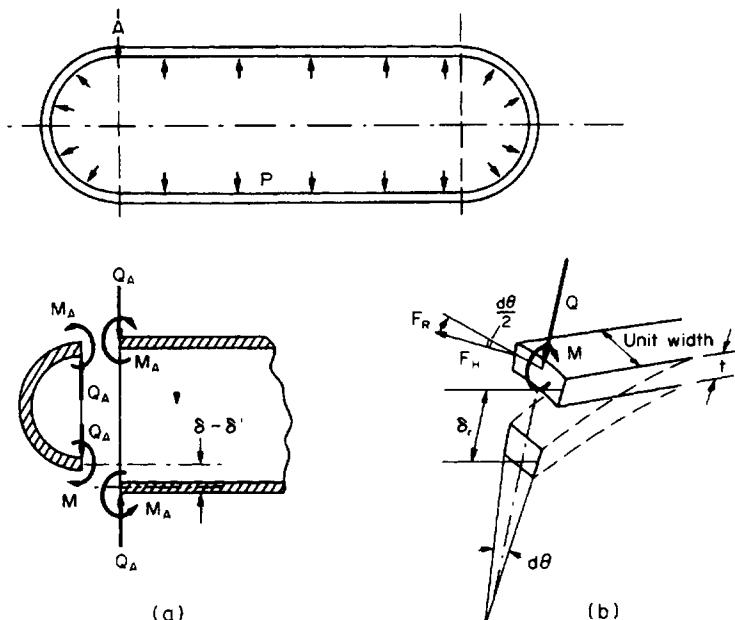


Fig. 12.8. Loading conditions at discontinuities in thin shells.

The radial displacements set up by these stress systems are; for the cylinder:

$$\delta = \frac{r}{E}(\sigma_H - v\sigma_L) = \frac{pr^2}{2tE}(2-v)$$

and for the hemispherical ends:

$$\delta' = \frac{r}{E}(\sigma_H - v\sigma_H) = \frac{pr^2}{2tE}(1-v).$$

There will thus be a difference in deformation radially of:

$$\begin{aligned}\delta - \delta' &= \frac{pr^2}{2tE}[(2-v) - 2(1-v)] \\ &= \frac{vpr^2}{2tE}\end{aligned}$$

which can only be reacted by the introduction of shear forces and moments as shown in Fig. 12.8(a) where Q = shear force and M = moment, both per unit length.

Because of the total symmetry of the cylinder about its axis we may now consider bending of a small element of the cylinder of unit width as shown in Fig. 12.8 (b).

The shear stress Q produces inward bending of the elemental strip through a radial displacement δ_r and a compressive hoop or circumferential strain given by:

$$\epsilon_H = \frac{\delta_r}{r}$$

with a corresponding hoop stress:

$$\sigma_H = \frac{E\delta_r}{r}$$

This stress sets up a force in the circumferential direction of

$$F_H = \sigma_H \times A = \frac{E\delta_r}{r} \times t \times 1.$$

This force has an outward radial component from both sides of the element of:

$$\begin{aligned} F_R &= 2F_H \sin \frac{d\theta}{2} = 2F_H \frac{d\theta}{2} = \frac{2E\delta_r}{r} \frac{t d\theta}{2} \\ &= \frac{E\delta_r t d\theta}{r} \end{aligned}$$

and since the strip is of unit width, $r d\theta = 1$

$$\therefore F_R = \frac{E\delta_r t}{r^2}$$

This force can be considered as a distributed load along the strip (since equal values will apply to all other unit lengths) and will act in opposition to the mis-match displacements caused by the membrane stresses.

If the strip were considered to be a simple beam then, the differential equation of bending would be:

$$\frac{EI d^4 y}{dx^4} = -\frac{E\delta_r t}{r^2}$$

but, as for the case of the deformation of circular plates in 7.2, the restraint on distortion produced by adjacent strips needs to be allowed for by replacing EI by the plate stiffness constant or flexural rigidity

$$D = \frac{Et^3}{12(1-\nu^2)} :$$

$$\begin{aligned} \text{i.e. } D \frac{d^4 y}{dx^4} &= -\frac{E\delta_r t}{r^2} \\ &= - \left[D \times 12 \frac{(1-\nu^2)}{Et^3} \right] \frac{E\delta_r t}{r^2} \\ &= -4D\beta^4 \delta_r = -4D\beta^4 y \end{aligned} \tag{1}$$

$$\text{where } \beta^4 = \frac{3(1-\nu^4)}{r^2 t^2} \text{ and } y = \delta_r.$$

The solution to eqn. (1) is of the form:

$$y = \delta_r = e^{\beta x} (A_1 \cos \beta x + A_2 \sin \beta x) + e^{-\beta x} (A_3 \cos \beta x + A_4 \sin \beta x) \tag{2}$$

Now as $x \rightarrow \infty$, $\delta_r \rightarrow \infty$ and $A_1 = A_2 = 0$.

At $x = 0$, $M = M_A$ and $D \frac{d^2 y}{dx^2} = -M_A$.

At $x = 0$, $Q = Q_A$ and $D \frac{dy^3}{dx^3} = -Q_A$

Substituting these conditions into equation (2) gives:

$$A_3 = \frac{1}{2\beta^3 D} (Q_A - \beta M_A)$$

and

$$A_4 = \frac{M_A}{2\beta^2 D}$$

Substituting back into eqn. (2) we have:

$$y = \delta_r = \frac{e^{-\beta x}}{2\beta^3 D} [Q_A \cos \beta x - M_A \beta (\cos \beta x - \sin \beta x)] \quad (12.20)$$

which is the equation of a heavily damped oscillation, showing that significant values of σ_r , i.e. significant bending, will only be obtained at points local to the cylinder-end intersection. Any stiffening which is desired need, therefore, only to be local to the "joint".

In the special case where the material and the thickness are uniform throughout there will be no moment set up at the intersection A since the shear force Q_A will produce equal slopes and deflections in both the cylinder and the hemispherical end.

Bending stresses can be obtained from the normal relationship:

$$M = D \frac{d^2 y}{dx^2}$$

i.e by differentiating equation (12.20) twice and by substitution of appropriate boundary conditions to determine the unknowns. For cases where the thickness is not constant throughout, and M therefore has a value, the conditions are:

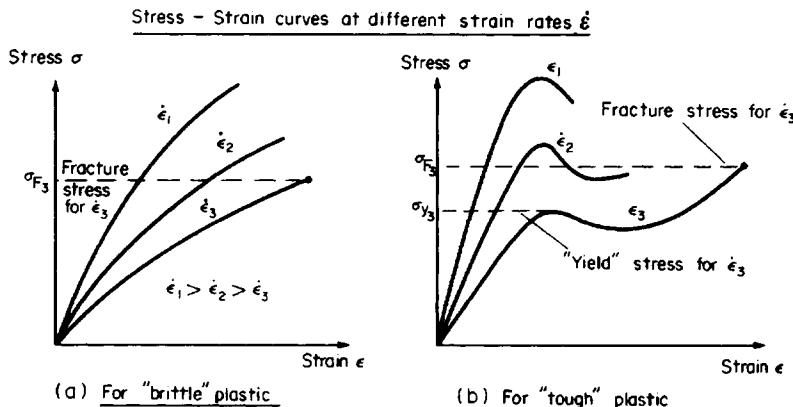
- (a) the sum of the deflections of the cylinder and the end at A must be zero,
- (b) the slope or angle of rotation of the two parts at A must be equal.

12.5. Viscoelasticity

Certain materials, e.g., rubbers and plastics, exhibit behaviour which combines the characteristics of a viscous liquid and an elastic solid and the term which is used to describe this behaviour is "viscoelasticity". In the case of the elastic solid which follows Hooke's law (a "Hookean" solid) stress is linearly related to strain. For so-called "Newtonian" viscous liquids, however, stress is proportional to strain rate. If, therefore, a tensile test is carried out on a viscoelastic material the resulting stress-strain diagram will depend significantly on the rate of straining $\dot{\epsilon}$, as shown in Fig. 12.9. Further, whilst the material may well recover totally from its strained position after release of loading it may do so along a different line from the loading line and stress will not be proportional to strain even within this "elastic" range.

One starting point for the mathematical consideration of the behaviour of viscoelastic materials is the derivation of a linear differential equation which, in its most general form, can be written as:

$$A\sigma = B\varepsilon$$

Fig. 12.9. Stress-strain curves at different strain rates $\dot{\epsilon}$.

with A and B linear differential operators with respect to time, or as:

$$A_0\sigma + A_1 \frac{d\sigma}{dt} + A_2 \frac{d^2\sigma}{dt^2} + \dots = B_0\varepsilon + B_1 \frac{d\varepsilon}{dt} + B_2 \frac{d^2\varepsilon}{dt^2} + \dots \quad (12.21).$$

In most cases this equation can be simplified to two terms on either side of the expression, the first relating to stress (or strain) the second to its first differential. This will be shown below to be equivalent to describing viscoelastic behaviour by mechanical models composed of various configurations of springs and dashpots. The simplest of these models contain one spring and one dashpot only and are due to Voigt/Kelvin and Maxwell.

(a) Voigt-Kelvin Model

The behaviour of Hookean solids can be simply represented by a spring in which stress is directly and linearly related to strain,

i.e.

$$\sigma_s = E\varepsilon_s$$

The Newtonian liquid, however, needs to be represented by a dashpot arrangement in which a piston is moved through the Newtonian fluid. The constant of proportionality relating stress to strain rate is then the coefficient of viscosity η of the fluid.

i.e.

$$\sigma_D = \eta\dot{\varepsilon}_D \quad (12.22)$$

In order to represent a viscoelastic material, therefore, it is necessary to consider a suitable combination of spring and dashpot. One such arrangement, known as the *Voigt-Kelvin model*, combines the spring and dashpot in parallel as shown in Fig. 12.10.

The response of this model, i.e. the relationship between stress σ , strain ε and strain rate $\dot{\varepsilon}$ is given by:

$$\sigma = \sigma_s + \sigma_D$$

and since the strain is common to both parts of the parallel model $\varepsilon_s = \varepsilon_D = \varepsilon$

$$\sigma = E\varepsilon + \eta\dot{\varepsilon} \quad (12.23)$$

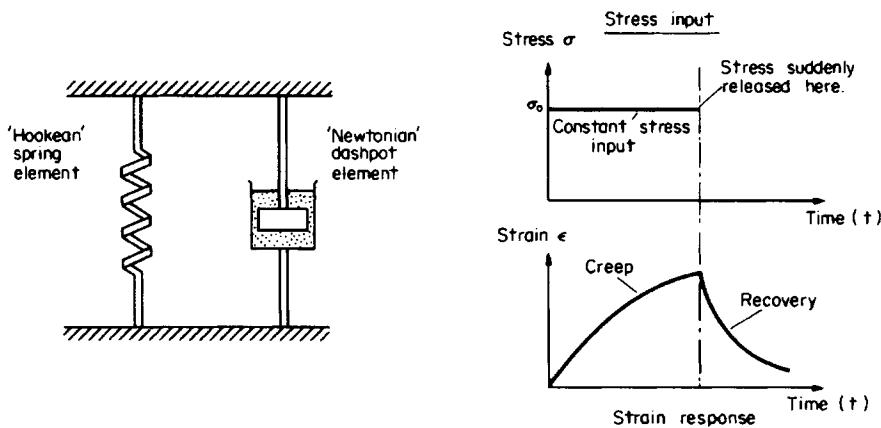


Fig. 12.10. Voigt–Kelvin spring/dashpot model with elements in parallel.

with the stress σ , in effect, shared between the two components of the model (the spring and the dashpot) as for any system of components in parallel.

The inclusion of the strain rate term $\dot{\epsilon}$ makes the stress response time-dependent and this represents the principal difference in behaviour from that of elastic solids.

If a stress σ_0 is applied to the model, held constant for a time t and then released the strain response will be that indicated in Fig. 12.10. The first part of the response, i.e. the change in strain at constant stress is termed the *creep* of the material, the second part, when stress is removed, is termed the *recovery*.

For *stress relaxation*, i.e. relaxation of stress at constant strain

$$\epsilon = \text{constant} \quad \text{and} \quad \frac{d\epsilon}{dt} = 0$$

Equation (12.23) then gives

$$\sigma = E\epsilon$$

indicating that, according to the Voigt–Kelvin model, the material behaves as an elastic solid under these conditions—clearly an inaccurate representation of viscoelastic behaviour in general.

For creep under constant stress $\sigma = \sigma_0$, however, eqn. (12.23) now gives;

$$\sigma_0 = E\epsilon + \eta \frac{d\epsilon}{dt}$$

from which it can be shown that

$$\epsilon = \frac{\sigma_0}{E} [1 - e^{-Et/\eta}] \quad (12.24)$$

In the special case where $\sigma = \sigma_0 = 0$, the so-called “recovery” stage, this reduces to:

$$\epsilon = \epsilon_0 e^{-Et/\eta} = \epsilon_0 e^{-t/t'} \quad (12.25)$$

and this equation indicates that the strain recovers exponentially with time, with t' a characteristic time constant known as the “retardation time”.

(b) Maxwell model

An alternative model for viscoelastic behaviour proposed by Maxwell again uses a combination of a spring and dashpot but this time in series as shown in Fig. 12.11.

Whereas in the Voigt–Kelvin (parallel) model the stress is shared between the components, in the Maxwell (series) model the stress is common to both elements.

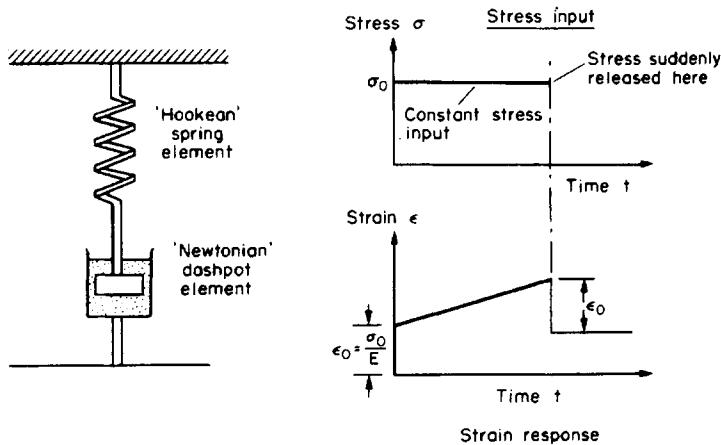


Fig. 12.11. Maxwell model with elements in series.

The strain, however, will be the sum of the strains of the two parts, i.e., the strain of the spring ϵ_S plus the strain of the dashpot ϵ_D

$$\therefore \epsilon = \epsilon_S + \epsilon_D$$

Differentiating:

$$\dot{\epsilon} = \dot{\epsilon}_S + \dot{\epsilon}_D \quad (1)$$

$$\text{Now } \sigma_S = E\epsilon_S \quad \therefore \quad \dot{\epsilon}_S = \frac{\dot{\sigma}_S}{E}$$

$$\text{and } \sigma_D = \eta \dot{\epsilon}_D \quad \therefore \quad \dot{\epsilon}_D = \frac{\dot{\sigma}_D}{\eta}$$

Now, for the series model,

$$\sigma_S = \sigma_D = \sigma$$

\therefore substituting in (1) we obtain the basic response equation for the Maxwell model.

$$\dot{\epsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta} \quad (12.26)$$

The response of this model to a stress σ_0 held constant over a time t and released, is shown in Fig. 12.11.

Let us now consider the response of the Maxwell model to the "standard" relaxation and recovery stages as was carried out previously for the Voigt–Kelvin model.

For stress relaxation $d\sigma/dt = 0$, and from eqn. (12.26)

$$0 = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}$$

i.e.

$$\frac{d\sigma}{\sigma} = -\frac{E}{\eta} \cdot dt$$

If, at $t = 0$, $\sigma = \sigma_0$, the initial stress, this equation can be integrated to yield

$$\sigma = \sigma_0 e^{-Et/\eta} = \sigma_0 e^{-t/t''} \quad (12.27)$$

This is analogous to the strain “recovery” equation (12.25) showing that, in this case, stress relaxes from its initial value σ_0 exponentially with time dependent upon the relaxation time t'' .

For the creep recovery stage from a constant level of stress, $d\sigma/dt = 0$ and eqn. (12.26) gives

$$\dot{\epsilon} = \frac{\sigma}{\eta} \quad (12.28)$$

the basic equation of pure Newtonian flow. Generally, however, the creep behaviour of viscoelastic materials is far more complex and, once again, the model does not adequately represent both recovery and relaxation situations. More accurate model representations can only be obtained, therefore, by suitable combinations of the Voigt–Kelvin and Maxwell models (see Figs. 12.12 and 12.13).

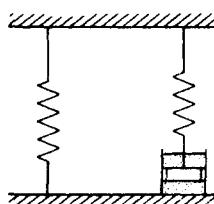


Fig. 12.12. The “standard linear solid” model.

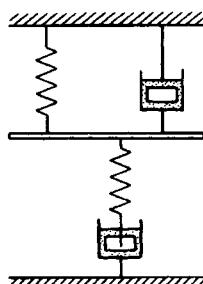


Fig. 12.13. Maxwell and Voigt–Kelvin models in series.

(c) Linear and non-linear viscoelasticity

Both the Voigt-Kelvin and Maxwell models represent so-called *linear viscoelasticity* (which must not be interpreted as meaning that stress is proportional to strain as indicated earlier). Linear viscoelasticity is said to occur when, as a result of a series of creep tests at constant stress levels, the ratios of strain to stress are plotted against time either in the form:

$$\varepsilon = \sigma f(t) \text{ or } \varepsilon = f_1(\sigma) f_2(t).$$

The strain to stress ratio in such tests is termed the *creep compliance*.

Neither the Voigt-Kelvin nor the Maxwell model, will fully represent the behaviour of polymers although the combination of the two, in series, as shown in Fig. 12.13, will give a reasonable approximation of polymer linear viscoelastic behaviour. Unfortunately, however, the range of strain over which linear viscoelasticity is exhibited by polymers is very small.

Non-linear viscoelasticity occurs when the creep compliance-time curve follows an equation of the form:

$$\varepsilon = f'(\sigma, t)$$

This form of viscoelasticity can only be modelled using non-linear springs and dashpots, and the analysis of such systems can become extremely complex.

A convenient approximate solution^(1,2) for the design of components constructed from polymers employs the use of "isochronous" stress-strain curves and a "secant modulus" $E_s(t)$. If a series of creep tests are carried out to produce a set of strain-time curves at various stress levels a number of constant time sections can be taken through the curves to enable isochronous (constant time) stress-strain diagrams to be plotted in Fig. 12.14. Such results may be obtained under tensile, compressive or shear loading. Alternatively these data may be obtained from manufacturers' data sheets. One of these isochronous curves can then be selected on the basis of the known lifetime requirement of the component and used for the determination of the secant modulus.

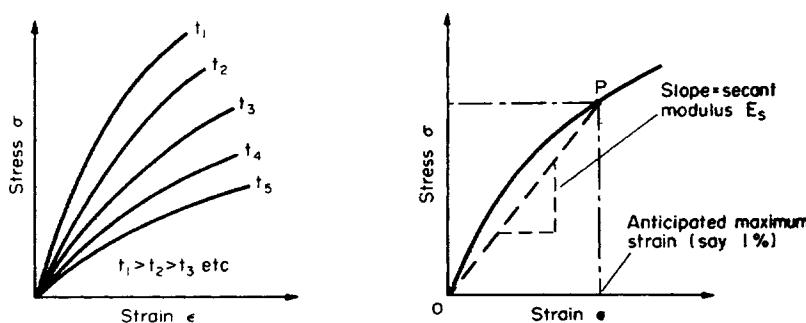


Fig. 12.14. Use of isochronous curves for design.

Defining a point P on the isochronous curve, be it either the expected maximum stress or strain (usually taken as 1%), allows a straight line to be drawn from P to the origin O , the slope of which gives the secant modulus. As stated above, this modulus may be as a result of tension, compression or shear and the appropriate value can then be used to replace E and G in the standard elastic formulae derived in other chapters of this text. If such formulae also

contains Poisson's ratio ν this must also be replaced by its equivalent under creep conditions, the so-called "creep contraction" or "lateral strain ratio" $v(t)$. See Example 12.2.

References

1. Benham, P. P. and McCommend, D., "Approximate creep analysis for thermoplastic beams and struts", *J.S.A.*, 6, 1, 1971.
2. Benham, P. P. and McCommend, D., "A study of design stress analysis problems for thermoplastics using time dependent data", *Plastics and Polymers*, Oct. 1969.

Examples

Example 12.1

The gantry shown in Fig. 12.15 is constructed from 100 mm \times 50 mm rectangular cross-section and, under service conditions, supports a maximum load P of 20 kN. Determine the maximum distance d at which P can be safely applied if the maximum tensile and compressive stresses for the material used are limited to 30 MN/m² and 100 MN/m² respectively.

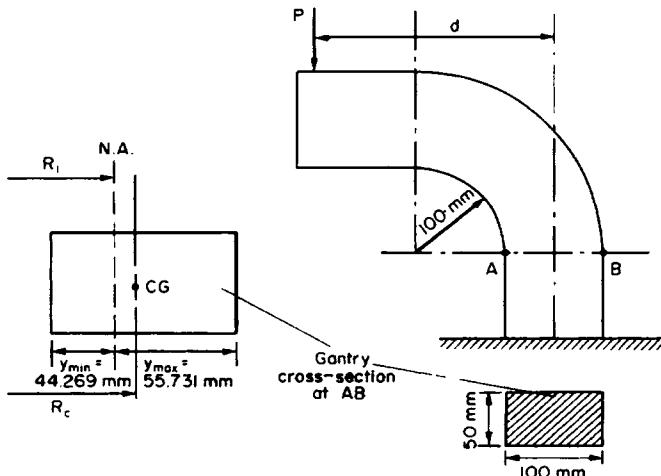


Fig. 12.15.

How would this value change if the cross-section were circular, but of the same cross-sectional area?

Solution

For the gantry and cross-section of Fig. 12.15 the following values are obtained by inspection:

$$R_c = 150 \text{ mm} \quad R_i = 100 \text{ mm} \quad R_o = 200 \text{ mm} \quad b = 50 \text{ mm}.$$

∴ From Table 12.1(a)

$$\int \frac{dA}{r} = b \log_e \left(\frac{R_o}{R_i} \right) = 50 \log_e \left(\frac{200}{100} \right) \\ = 34.6574 \text{ mm}$$

$$\therefore R_1 = \frac{A}{\int \frac{dA}{r}} = \frac{50 \times 100}{34.6574} = 144.269 \text{ mm}$$

$$\therefore h = R_c - R_1 = 150 - 144.269 = 5.731 \text{ mm.}$$

$$\text{Direct stress (compressive) due to } P = \frac{P}{A} = \frac{20 \times 10^3}{(100 \times 50)10^{-6}} = 4 \text{ MN/m}^2$$

Thus, for maximum tensile stress of 30 MN/m² to be reached at B the bending stress (tensile) must be $30 + 4 = 34 \text{ MN/m}^2$.

$$\text{Now } y_{\max} = 50 + 5.731 \\ = 55.731 \text{ at } B$$

$$\text{and bending stress at } B = \frac{My}{hA(R_1 + y)} = 34 \text{ MN/m}^2,$$

$$\therefore \frac{(40 \times 10^3 d) \times 55.731 \times 10^{-3}}{(5.731 \times 10^{-3})(50 \times 100 \times 10^{-6})(200 \times 10^{-3})} = 34 \times 10^6 \\ \therefore d = 174.69 \text{ mm.}$$

For maximum compressive stress of 100 MN/m² at A the compressive bending stress must be limited to $100 - 4 = 96 \text{ MN/m}^2$ in order to account for the additional direct load effect.

$$\therefore \text{At } A, \text{ with } y_{\min} = 50 - 5.731 = 44.269$$

$$\text{bending stress} = \frac{(20 \times 10^3)44.269 \times 10^{-3}}{(5.731 \times 10^{-3})(50 \times 100 \times 10^{-6})(100 \times 10^{-3})} = 96 \times 10^6 \\ \therefore d = 310.7 \text{ mm.}$$

The critical condition is therefore on the tensile stress at B and the required maximum value of d is **174.69 mm**.

If a circular section were used of radius R and of equal cross-sectional area to the rectangular section then $\pi R^2 = 100 \times 50$ and $R = 39.89 \text{ mm}$.

∴ From Table 12.1 assuming R_c remains at 150 mm

$$\int \frac{dA}{r} = 2\pi\{(R_i + R) - \sqrt{(R_1 + R)^2 - R^2}\} \\ = 2\pi\{150 - \sqrt{150^2 - 39.89^2}\} \\ = 2\pi \times 5.4024 = 33.944 \text{ mm.}$$

$$\therefore R_1 = \frac{A}{\int \frac{dA}{r}} = \frac{50 \times 100}{33.944} = 147.301 \text{ mm,}$$

with $h = R_c - R_1 = 150 - 147.3 = 2.699$ mm.

\therefore For critical tensile stress at B with $y = 39.894 + 2.699 = 42.593$.

$$\frac{My}{hA(R_1 + y)} = 34 \text{ MN/m}^2.$$

$$\therefore \frac{(20 \times 10^3 \times d)(42.593 \times 10^{-3})}{2.699 \times 10^{-3} \times (\pi \times 39.894^2 \times 10^{-6})(150 + 39.894)} = 34 \times 10^6$$

$$d = 102.3$$

i.e. Use of the circular section reduces the limit of d within which the load P can be applied.

Example 12.2

A constant time section of 1000 h taken through a series of strain-time creep curves obtained for a particular polymer at various stress levels yields the following isochronous stress-strain data.

$\sigma(\text{kN/m}^2)$	1.0	2.25	3.75	5.25	6.54	7.85	9.0
$\varepsilon(%)$	0.23	0.52	0.85	1.24	1.68	2.17	2.7

The polymer is now used to manufacture:

- (a) a disc of thickness 6 mm, which is to rotate at 500 rev/min continuously,
- (b) a diaphragm of the same thickness which is to be subjected to a uniform lateral pressure of 16 N/m² when clamped around its edge.

Determine the radius required for each component in order that a limiting stress of 6 kN/m² is not exceeded after 1000 hours of service. Hence find the maximum deflection of the diaphragm after this 1000 hours of service.

The lateral strain ratio for the polymer may be taken as 0.45 and its density as 1075 kN/m³.

Solution

- (a) From eqn. (4.11) the maximum stress at the centre of a solid rotating disc is given by:

$$\sigma_{r_{\max}} = \sigma_{\theta_{\max}} = (3 + \nu) \frac{\rho \omega^2 R^2}{8}$$

For the limiting stress condition, therefore, with Poissons ratio ν replaced by the lateral strain ratio:

$$6 \times 10^3 = 3.45 \times 1075 \times \frac{(500 \times 2\pi)^2}{60} \times \frac{R^2}{8}$$

From which

$$R^2 = 0.00472$$

and

$$R = 0.0687 \text{ m} = 68.7 \text{ mm.}$$

(b) For the diaphragm with clamped edges the maximum stress is given by eqn. (22.24) as:

$$\sigma_{r_{\max}} = \frac{3qR^2}{4t^2}$$

$$6 \times 10^3 = \frac{3 \times 16 \times R^2}{4 \times (6 \times 10^{-3})^2}$$

From which $R = 0.134 \text{ m} = 134 \text{ mm}$.

The maximum deflection of the diaphragm is then given in Table 7.1 as:

$$\delta_{\max} = \frac{3qR^4}{16Et^3}(1 - v^2)$$

Here it is necessary to replace Young's modulus E by the secant modulus obtained from the isochronous curve data and Poisson's ratio by the lateral strain ratio.

The 1000 hour isochronous curve has been plotted from the given data in Fig. 12.16 producing a secant modulus of 405 kN/m^2 at the stated limiting stress of 6 kN/m^2 ; this being the slope of the line from the origin to the 6 kN/m^2 point on the isochronous curve.

$$\therefore \delta_{\max} = \frac{3 \times 16 \times (134 \times 10^{-3})^4 \times (1 - 0.45^2)}{16 \times 405 \times 10^3 \times (6 \times 10^{-3})^3}$$

$$= 0.0088 \text{ m} = 8.8 \text{ mm.}$$

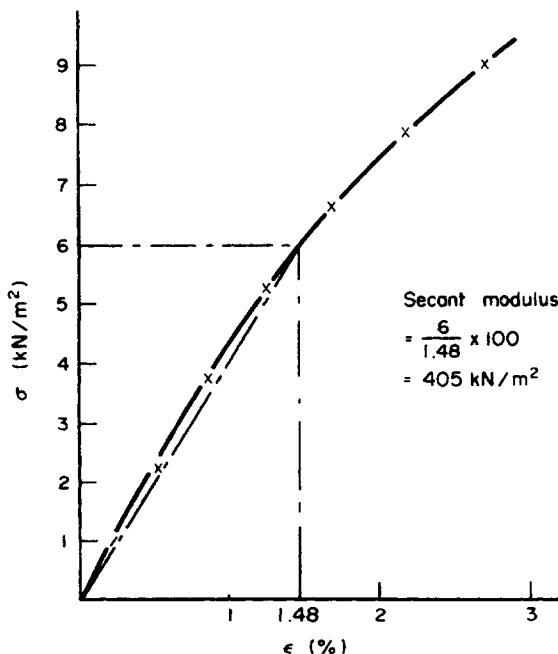


Fig. 12.16.

Problems

12.1 (B). The bracket shown in Fig. 12.17 is constructed from material with $50 \text{ mm} \times 25 \text{ mm}$ rectangular cross-section and it supports a vertical load of 10 kN at C . Determine the magnitude of the stresses set up at A and B .

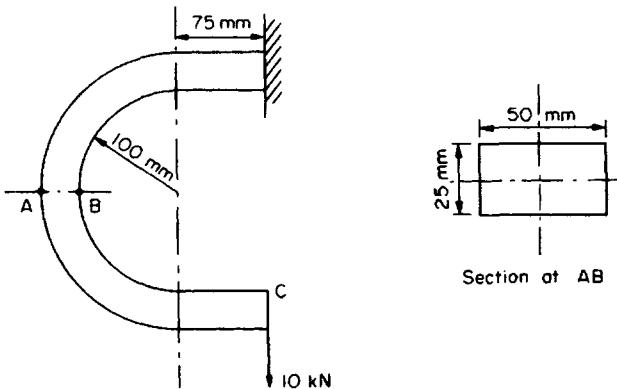


Fig. 12.17.

What percentage error would be obtained if the simple bending theory were applied?

[-161.4 MN/m^2 , $+169.4 \text{ MN/m}^2$, 19%, 13.6%]

12.2 (B). A crane hook is constructed from trapezoidal cross-section material. At the critical section AB the dimensions are as shown in Fig. 12.18. The hook supports a vertical load of 25 kN with a line of action 40 mm from B on the inside face. Calculate the values of the stresses at points A and B taking into account both bending and direct load effects across the section.

[129.2 MN/m^2 , -80.3 MN/m^2]

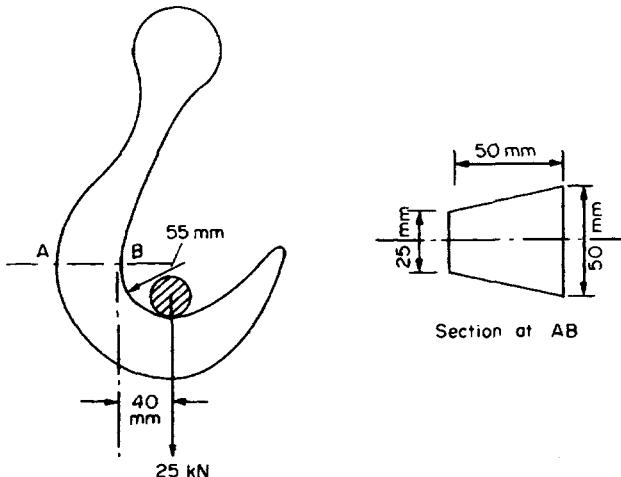


Fig. 12.18.

12.3 (B). A G-clamp is constructed from I-section material as shown in Fig. 12.19. Determine the maximum stresses at the central section AB when a clamping force of 2 kN is applied.

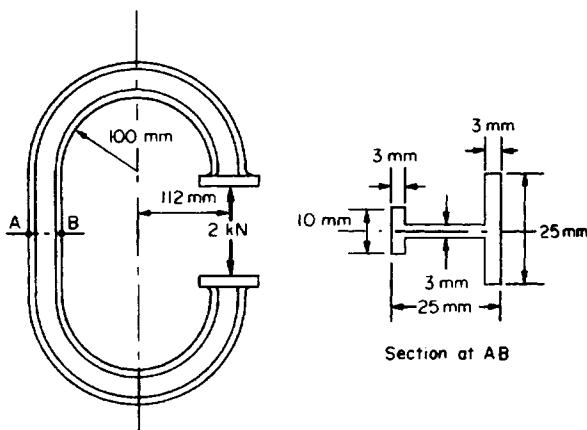


Fig. 12.19.

How do these values compare with those which would be obtained using simple bending theory applied to a straight beam of the same cross-section?

[267 MN/m², -347.5 MN/m², 240 MN/m², 380.7 MN/m²]

12.4 (B). Part of the frame of a machine tool can be considered to be of the form shown in Fig. 12.20. A decision is required whether to construct the frame from T or rectangular section material of the dimensions shown.

Compare the critical stresses set up at section AB for each of the cross-sections when the frame is subjected to a peak load of 5 kN and discuss the results obtained in relation to the decision required.

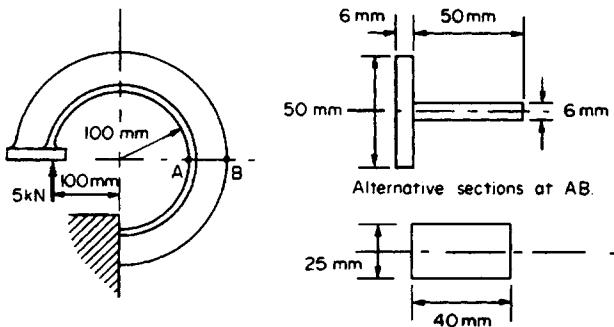


Fig. 12.20.

Plot diagrams of the stress distribution across AB for each cross-sectional shape.

[122.7 MN/m², -198 MN/m², 193.4 MN/m², -143.2 MN/m²]

12.5 (B). (a) By consideration of the Maxwell model, derive an expression for the internal stress after time t of a polymer held under constant strain conditions and hence show that the relaxation time is equal to η/G where η is the coefficient of viscosity and G is the shear modulus.

(b) A shear stress of 310 MN/m² is applied to a polymer which is then held under fixed strain conditions. After 1 year the internal stress decreases to a value of 207 MN/m². Calculate the value to which the stress will fall after 2 years, assuming the polymer behaves according to the Maxwell model. $[\tau = \tau_0 e^{-Gt/\eta}; 138 \text{ MN/m}^2]$

12.6 (B). (a) Spring and dashpot arrangements are often used to represent the mechanical behaviour of polymers. Analyse the mathematical stress strain relationship for the Maxwell and Kelvin-Voigt models under conditions of (i) constant stress, (ii) constant strain, (iii) recovery, and draw the appropriate strain-time, stress-time diagrams, commenting upon their suitability to predict behaviour of real polymers.

- (b) Maxwell and Kelvin–Voigt models are to be set up to simulate the behaviour of a plastic. The elastic and viscous constants for the Kelvin–Voigt model are $2 \times 10^9 \text{ N/m}^2$ and $100 \times 10^9 \text{ Ns/m}^2$ respectively and the viscous constant for the Maxwell model is $272 \times 10^9 \text{ Ns/m}^2$. Calculate a value for the elastic constant for the Maxwell model if both models are to predict the same strain after 100 seconds when subjected to the same stress.

$$[15.45 \times 10^9 \text{ N/m}^2]$$

- 12.7 (B).** The model shown in Fig. 12.21 is frequently used to simulate the mechanical behaviour of polymers:

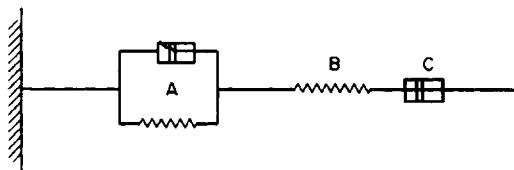


Fig. 12.21.

- (a) With reference to Figure 12.21, state what components of total strain the elements A, B and C represent.
 (b) Sketch a typical strain–time graph for the model when the load F is applied and then removed. Clearly label those parts of the graph corresponding to the strain components ε_1 , ε_2 and ε_3 .
 (c) A certain polymer may be modelled on such a system by using the following constants for the elements:

Dashpot A: viscosity = 10^6 Ns/m^2

Dashpot B: viscosity = $100 \times 10^6 \text{ Ns/m}^2$

Spring A: shear modulus = $50 \times 10^3 \text{ N/m}^2$

Spring B: shear modulus = 10^9 N/m^2

This polymer is subjected to a direct stress of $6 \times 10^3 \text{ N/m}^2$ for 30 seconds ONLY.

Determine the strain in the polymer after 30 seconds, 60 seconds and 2000 seconds.

$$[3.17 \times 10^{-2}, 0.75 \times 10^{-2}, 0.06 \times 10^{-2}]$$

- 12.8 (C).** For each of the following typical engineering components and loading situations sketch and dimension the components and allocate appropriate loadings. As a preliminary step towards finite element analysis of each case, select and sketch a suitable analysis region, specify complete boundary conditions and add an appropriate element mesh. Make use of symmetry and St. Venant's criteria wherever possible.

- (a) A shelf support bracket welded to a vertical upright.
 (b) An engine con-rod with particular attention paid to shoulder fillet radii for weight reduction purposes (see Fig. 6.1)
 (c) A washing machine agitator cross-section (see Fig. 5.14), bar-tube fillet radii and relative thicknesses of particular concern.
 (d) The extruded alloy section of Fig. 1.21. Model to be capable of consideration of varying lines of action of applied force.
 (e) A circular pipe flange used to connect two internally pressurised pipes. Model to be capable of including the effect of bolt tensions and external moments on the joint. You may assume that the pipe is free to expand axially.
 (f) A C.T.S. (compact test specimen) for brittle fracture compliance testing. Stress distributions at the crack tip are required.
 (g) A square storage hopper fabricated from thin rectangular plates welded together and supported by means of welded angle around the upper edge. It may be assumed that the hopper is full with an equivalent hydrostatic pressure p throughout. The supporting frame can be assumed rigid.
 (h) A four-point beam bending test rig with plastic beam mounted on steel pads over steel knife edges. The degree of indentation of the plastic and deformation of the steel pad are required.
 (i) Thick cylinder with flat ends and sharp fillet radii subjected to internal pressure. The model should be capable of assessing the effect of different end plate thicknesses.
 (j) A pressurised thick cylinder containing a 45° nozzle entry. Stress concentrations at the nozzle entry are required.

APPENDIX 1

TYPICAL MECHANICAL AND PHYSICAL PROPERTIES FOR ENGINEERING METALS

Material	Young's modulus of elasticity E (GN/m ²)	Shear modulus G	"Elastic" limit σ_y	Shear yield strength τ_y	Tensile strength	Ultimate strength in shear	Percentage elongation (%)	Density (Kg/m ³)	Linear coefficient of thermal expansion ($\times 10^{-6}$ /°C)
Aluminium alloy	69	26	230	—	390	240	23	2770	23
Brass	102	38	—	—	350	—	40	8350	18.9
Bronze	115	45	210	—	310	—	20	7650	18
Cast iron: Grey	90	41	—	—	210	—	8	7640	10.5
Malleable	170	83	248	166	370	330	12	7640	12
Low carbon (mild) steel	207	80	280	175	480	350	25	7800	11.7
Nickel-chrome steel	208	82	1200	650	1700	950	12	7800	11.7
Titanium	107	40	480	—	551	—	—	4507	9.5
Magnesium	45	17	262	—	379	165	—	1791	28.8

APPENDIX 2

TYPICAL MECHANICAL PROPERTIES OF NON-METALS

Material	Young's modulus of elasticity E (GN/m ²)	Tensile strength (MN/m ²)	Compressive strength (MN/m ²)	Elongation (maximum) %
Acetals	—	69	124	75
Cellulose acetate	1.4	41	207	20
Cellulose nitrate	1.4	48	138	40
Epoxy (glass filler)	—	145	234	—
Hard rubber	3.0	48	—	—
Melamine	8.0	55	227	0.7
Nylon filaments	4.1	340	—	—
Polycarbonate – unreinforced				
Makralon	2.3	70	83	100
Reinforced Makralon	6.0	90	—	8
Polyester (unfilled)	2.0	41	—	2
Polyethylene H.D.	—	28	22	100
Polyethylene L.D.	—	10	—	800
Polypropylene	—	34	510	250
Polystyrene	3.4	20	76	1.2
Polystyrene – impact resistant	1.4	38	41	80
P.T.F.E.	—	34	248	70
P.V.C. (rigid)	3.4	50–60	69	40
P.V.C. (plasticised)	—	20	0.7	200
Rubber (natural-vulcanised)	—	7–34	—	—
Silicones (elastomeric)	—	1.5–6	—	—
Timber	9.0	70	—	—
Urea (Cellulose filler)	10.0	62	241	0.7

* Data taken in part from *Design Engineering Handbook on Plastics* (Product Journals Ltd).

APPENDIX 3

OTHER PROPERTIES OF NON-METALS*

Material	Chemical resistance					Max useful temp. (C)	
	Organic Solvents	Acids		Alkalis			
		Weak	Strong	Weak	Strong		
Acetal	x	x	00	x	x	90	
Acrylic	Varies	x	x-0	x	x	90	
Nylon 66	x	x	00	x	x	150	
Polycarbonate	Varies	x	0	x-0	00	120	
Polyethylene LD	x	x	x-00	x	x	90	
Polyhethylen HD	x	x	x-0	x	x	120	
Polypropylene	x	x	x-0	x	x	150	
Polystyrene	Varies	x	x-0	x	x	95	
PTFE	x	x	x	x	x	240	
PVC	Varies	x	x-0	x	x	80	
Epoxy	x	x	x	x	0	430	
Melamine	x	x	00	x	0	100-200	
Phenolic	x	x-0	0-00	0-00	00		
Polyester/glass	x-0	0	00	0	00	200	
Silicone	x-0	x-0	00	0-00	00	250	
Urea	x-0	x-0	00	0-00	00	180	
		x - Resistant, 0 - Slightly attacked, 00 - markedly attacked					

*Data taken from Design Engineering Handbook on Plastics. (Product Journals Ltd)

INDEX

- AC system 179
Acoustic gauge 180
Acoustoelasticity 192
Active gauge 172, 176, 178–9
Airy stress function 263, 265
Allowable working load 73
Alternating stress amplitude 447
Annular rings 208
Area, principal moments of 4
Area, product moment of 3
Area, second moment of 3–10
Auto fretting 89
Ayton and Perry 42
- Balanced circuit 173
Banded method 313
Basquin's law 457
Beams, curved 509
Bending, of beams 509–16
Bending, unsymmetrical 1
Birefringence 183
Body force stress 220
Boundary condition, application 316, 349
Boundary stress 185
Bredt–Batho theory 147, 150
Brittle lacquers 167
Buckling of struts 30
Built-up girders 52
- Calibration 69, 169, 186
Capacitance gauge 180
Carrier frequency system 180
Cartesian stress 220
Case-hardened shafts 79
Circular plates 193
Circular polarisation 188
Clarke 168
Coffin–Manson Law 457
Collapse 64
Columns 30
Combined circle (Mohr) 228
Combined diagram 229
Compatibility 261, 321
Compensation 187
Conjugate diameters 13
Contact stress 382
Corkscrew rule 12
Correspondence table 347
Crack detection 169
Crack tip plasticity 482, 488
- Creep 169, 462
Crinkling of struts 50
Crossed set-up 182
Cross-sensitivity 173
Crushing stress 37
Cyclic stress–strain 455
Cylinders, residual stresses in 87
Cylindrical components 239
- Dark field 182
DC system 179
De Forrest 168
Deflections, unsymmetrical members 15
Deviatoric stress 251
Diaphragms 193
Directions cosine 223
Disc
 circular, solid 119
 plastic yielding of 94
 rotating 117
 uniform strength 125
 with central hole 122
Discretisation 305
Distortion 311
Dummy gauge 172, 175–6
Dye-etchant 169
Dynamic strain 171
- Eccentric loading
 of struts 42
 to collapse 85
Ellipse
 momental 11
 of second moments of area 9
Ellis 168
Endurance limit 423
Equilibrium 319
Equilibrium
 cartesian coordinates 236
 cylindrical coordinates 239
Equivalent length 35
Equivalent length (of Struts) 35
Euler theory 31
Extensometers 180
- Factor, load 41, 65, 166
Fatigue 446
Fatigue crack growth 486, 489
Finite element analysis 302

- Finite element mesh 308
 Finite element method 300
 Fixed-ended struts 33
 Flexural stiffness 193, 197
 Foil gauge 171, 173
 Fracture mechanics 472
 Fringe order 185–6
 Fringe pattern 181–2, 184–192, 420, 481
 Fringe value 185–6
 Frontal method 315
 Frozen stress technique 190
 Full bridge circuit 173
- Gauge**
 acoustic 180
 electrical resistance 171–180
 strain 171–80
 factor 172
Girder, built-up 50, 52
Graphical procedure, stress 228–9
Grid technique 192
Griffith's criterion 475
- Half-bridge** 172–3
Hetényi 84
Holography 192
Hydrostatic stress 249, 251
- Idealisation** 305
Inductance gauge 180
Initial curvature 41
Interference 181, 184
Isochromatic 185
Isoclinic 188
- Johnson parabolic formula** 36
- Land's circle of moments of area** 7
Larson-Miller parameter 468
Laterally loaded struts 46
Light field 182
Load factor 41, 73
- Manson-Haferd parameter** 469
Material fringe value 185–6
Maximum compressive stress 381, 385, 387
Mean stress 252, 451
Membrane 152
Modulus, reduced 82
Mohr's circle of second moments of area 6
Mohr's strain circle 228
Mohr's stress circle 228
Moiré 192
Momental ellipse 11, 13
Monochromatic light 185
- Neuber's rule** 425
Notch sensitivity 424
Null balance 173
- Oblique plane, stress on** 224
Octahedral planes 249
Octahedral shear stress 250
Octahedral stress 249
Overspeeding 95
- Parallel set-up** 183
Partitioning 349
Perry-Robertson 29, 39
Photoelastic coating 190
Photoelasticity
 reflection 190
 transmission 181
Piezo-resistive gauge 180
Plane polarisation 182
Plane strain 254
Plane stress 254
Plastic bending 64
Plastic deformation
 discs 94
 thick cylinder 87
Plastic hinge 71
Plastic limit design 71
Plastic torsion 75, 79
Pneumatic gauge 180
Polariscope 182
Polariser 182
Principal axes 2
Principal second moments of area 4
Principal strain 228
Principal stress 228, 247
Product second moment of area 3
- Quarter-bridge** 173
Quarter-wave plates 187
- Radial stress** 117–23, 125–8, 193–9
Radius of gyration 28
Rankine-Gordon theory 38
Rectangular plates 213
Reduced modulus 82
Reflection polariscope 190
Reflective coating 190
Refraction 183
Replica technique 192
Residual stresses 73, 79, 84, 86, 95–6, 394
Resistivity 172
Rotating cylinders 124
Rotating discs
 and rings 117
 collapse 94
Rotating hollow discs 122
Rotating uniform strength discs 125
Ruge and Simmons 171

- Safety factor 41, 65, 166
 Saint-Venant 420
 Sand heap analogy 78
 Second moment of area 2–10
 ellipse of 9
 principle 4
 product 3
 Semi-conductor gauges 180
 Senarmont 187
 Shape factor 65
 unsymmetrical sections 67, 69
 Shear flow 147
 Shear stress 224
 Sherby–Dorn parameter 469
 Skew loading 2
 Slenderness ration 28
 Smith–Southwell theory 42
 Southwell 42
 Specific resistance 172
 Stern 168
 Straight line formula 36
 Strain
 deviatoric 251, 253
 invariants 247
 threshold 167
 Strain circle 228
 Strain gauge 171
 Strain hardening 63–4, 80
 Stress
 body force 220
 boundary 185
 cartesian 220
 concentration 86
 concentration factor 408
 concentration factor (evaluation) 413
 crushing 37
 deviatoric 246, 251
 direct 224
 equations of equilibrium 236
 freezing technique 190
 hydrostatic 249–51
 invariants 243–4
 mean 252, 447, 451
 radial 193, 199
 range 447
 relaxation 470
 separation 190
 shear 224
 tangential 193, 199
 three-dimensional 220
 trajectory 189
 yield 61
 Stress concentration, effects 453
 Stress intensity factor 477
 stresscoat 168
 Struts 28
 Tangential stress 193, 199
 Tardy compensation 187
 Temperature stresses 126
 Temporary birefringence 183
 Tensor notation 235
 Thermal stresses 126
 Thin membranes 194
 Thin shells 517–18
 Threshold strain 167
 Timoshenko 142, 144, 146, 212, 214
 Torsion of cellular sections 150
 Torsion of non-circular sections 141
 Torsion of open sections 143, 150
 Torsion of rectangular sections 142
 Torsion of square closed or closed section 141
 Torsion of thin-walled closed sections 147
 Torsion of thin-walled stiffened sections 151
 Torsion section modulus 144
 Torsional rigidity 150
 Toughness 473
 Transformation 259, 326
 Transverse sensitivity 173
 Triangular plane membrane element 343
 Twist, angle of 141–7, 149–52
 Unbalanced bridge 173
 Uniform strength discs 125
 Unsymmetrical bending 1
 Unsymmetrical, section struts 49
 Validity limit (Euler) 37
 Virtual work 323
 Viscoelasticity 521
 Voight–Kelvin 522
 Warping 153, 310
 Webb's approximation 44
 Wheatstone bridge 172
 Wilson–Stokes equation 423
 Wire gauge 171
 X-rays 192
 Yield stress 61