$$N(a_1'a_2'\cdots a_n') = N - \Sigma N(a_i) + \Sigma N(a_i a_j) - \Sigma N(a_i a_j a_k) + \cdots + (-1)^n N(a_1 a_2 \cdots a_n)$$

$$= n! - nC_1(n-1)! + nC_2(n-2)! - \cdots + (-1)^n$$

$$= n! - n! + \frac{n(n-1)}{2}(n-2)! - \frac{n(n-1)(n-2)}{3!}(n-3)! + \cdots + (-1)^n$$

$$= \frac{n!}{2!} - \frac{n!}{3!} + \frac{n!}{4!} - \cdots + (-1)^n \frac{n!}{n!}$$ ✓

$$= n!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + (-1)^n \frac{1}{n!}\right)$$

Eg: For $n = 6$, Ans is $6!\left(\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{6!}\right) = \underline{265}$

For $n = 4$, Ans is $4!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) = 9$

<u>Note</u>: $N(a_1'a_2'\cdots a_n') = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$

For large n,

$$= n!\,e^{-1} = \frac{n!}{e}$$

Show that the proportion of the permutations of $\{1,2,\cdots,n\}$ which contains no consecutive pair $(i, i+1)$ for any i is approximately $\frac{n+1}{ne}$.

(assume n is large)

Soln: a_i: property that $(i, i+1)$ occurs consecutively. $1 \le i \le n-1$

total permutations $N = n!$

$$N(a_i) = (n-1)!$$

$$\_\ \_\ (1\ 2)\ \_\ \_\ \_$$

$$N(a_i a_j) = (n-2)!$$

$$\_\ (j\ \ j+1)\ \_\ (i\ \ i+1)\ \_\ \_\ \_$$

$$N(a_1 a_5) \qquad N(a_1 a_2)$$

$$(1 2 3) \qquad (1 2)\ (5 6) \qquad (n-4)$$

$$N(a_i a_j a_k) = (n-3)!$$

$$\vdots$$

$$N(a_1 a_2 \cdots a_{n-1}) = 1$$

$$1\,2\,3 \cdots n$$

$$N(a_1' a_2' \cdots a_{n-1}') = N - \sum_{i=1}^{n-1} N(a_i) + \sum_{i<j} N(a_i a_j) - \cdots$$

$$a_1\ a_2 \cdots a_{n-1}$$

As $n \to \infty$

$$= n! - ({}^{n-1}C_1)(n-1)! + {}^{n-1}C_2 \cdot (n-2)! - {}^{n-1}C_3 \times (n-3)! + \cdots$$

$$= n! - (n-1)(n-1)! + \frac{(n-1)(n-2)}{2}(n-2)! - \frac{(n-1)(n-2)(n-3)}{3!}(n-3)! + \cdots$$

$$(n-1)!\left\{n-(n-1)+\frac{(n-2)}{2}-\frac{(n-3)}{3!}+\cdots\right\}$$

$$(n-1)!\left\{1+\left(\frac{n}{2}-1\right)-\left(\frac{n}{3!}-\frac{3}{3!}\right)+\left(\frac{n}{4!}-\frac{4}{4!}\right)-\cdots\right\}$$

$$(n-1)!\left\{\left(1-1+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\cdots\right)+n\left(\frac{1}{2}-\frac{1}{3!}+\frac{1}{4!}\cdots\right)\right\}$$

$$(n-1)!\left\{\frac{1}{e}+n\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}\cdots\right)\right\}$$

$$(n-1)!\left(\frac{1}{e}+\frac{n}{e}\right)=\frac{(n+1)(n-1)!}{e}$$

$$\text{proportion of permutations}=\frac{\frac{(n+1)(n-1)!}{e}}{n!}$$

1234	2134	3124	4123
1243	2143	3142	4132
1324	2314	3214	4213
1342	2341	3241	4231
1423	2413	3412	4312
1432	2431	3421	4321

$$=\frac{n+1}{ne}$$

Ex: $n=4$

$N=4!$

$N(a_1)=6$ [1234 1243 3124 3412 4123 4312]

$N(a_1a_2)=2$ [123 together, 1234, 4123

$N(a_1a_3)=2$ [12 together & 34 together 1234 , 3412]

$N(a_1a_2a_3)=1$ 1234 , $24-3\times3!+3\times2!-1=11$

Partitions and Composition of integers

A positive integer n can be represented as $n = a_1 + a_2 + \cdots + a_k$, (sum of one or more +ve integers) where $a_i > 0$ is an integer.

Ordered division of an integer is Composition

Unordered division of an integer is Partition.

Let $n = 5$

Various partitions of 5 are

5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1

Various compositions of 5 are

5, 4+1, 1+4, 3+2, 2+3, 3+1+1, 1+3+1, 1+1+3,

2+2+1, 2+1+2, 1+2+2, 2+1+1+1, 1+2+1+1, 1+1+2+1

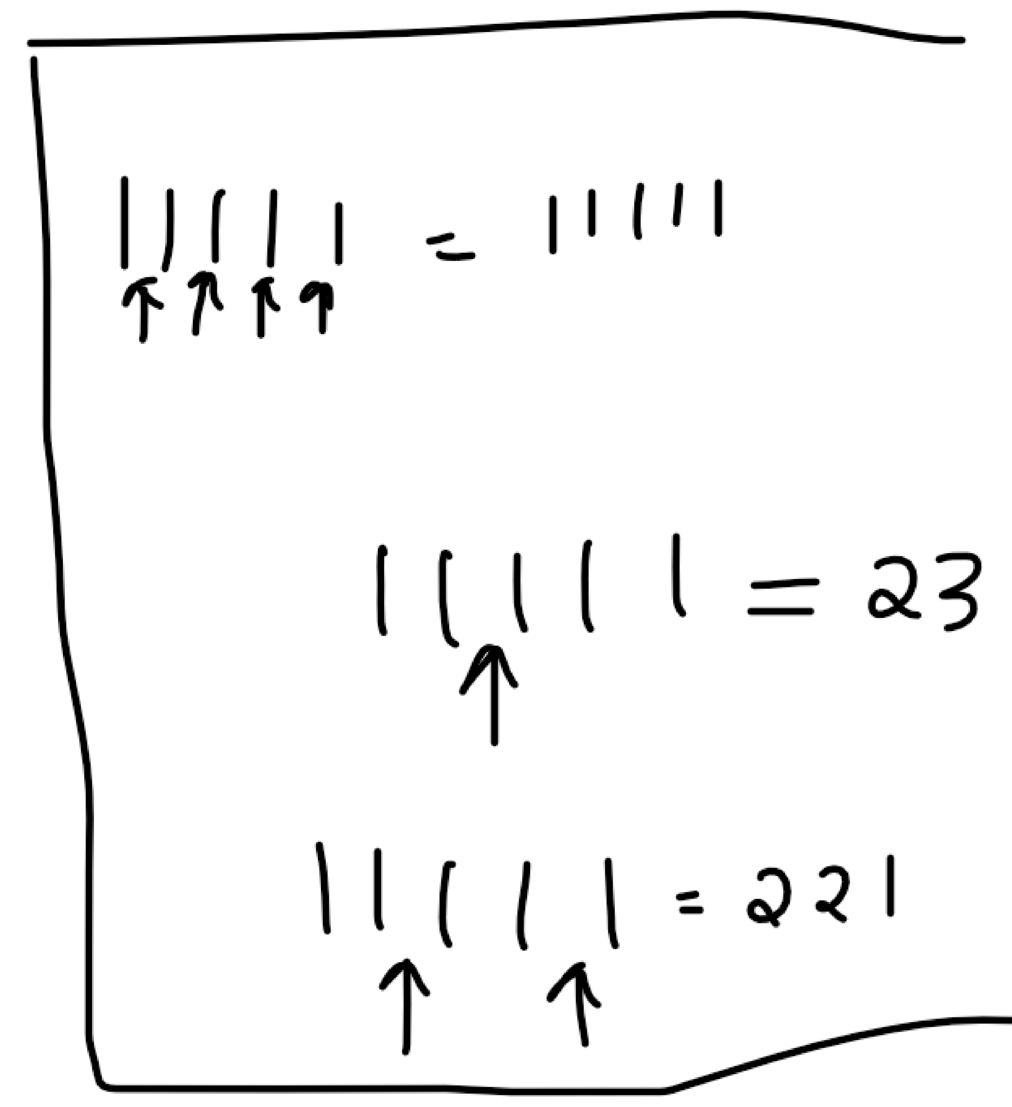
1+1+1+2, 1+1+1+1+1.

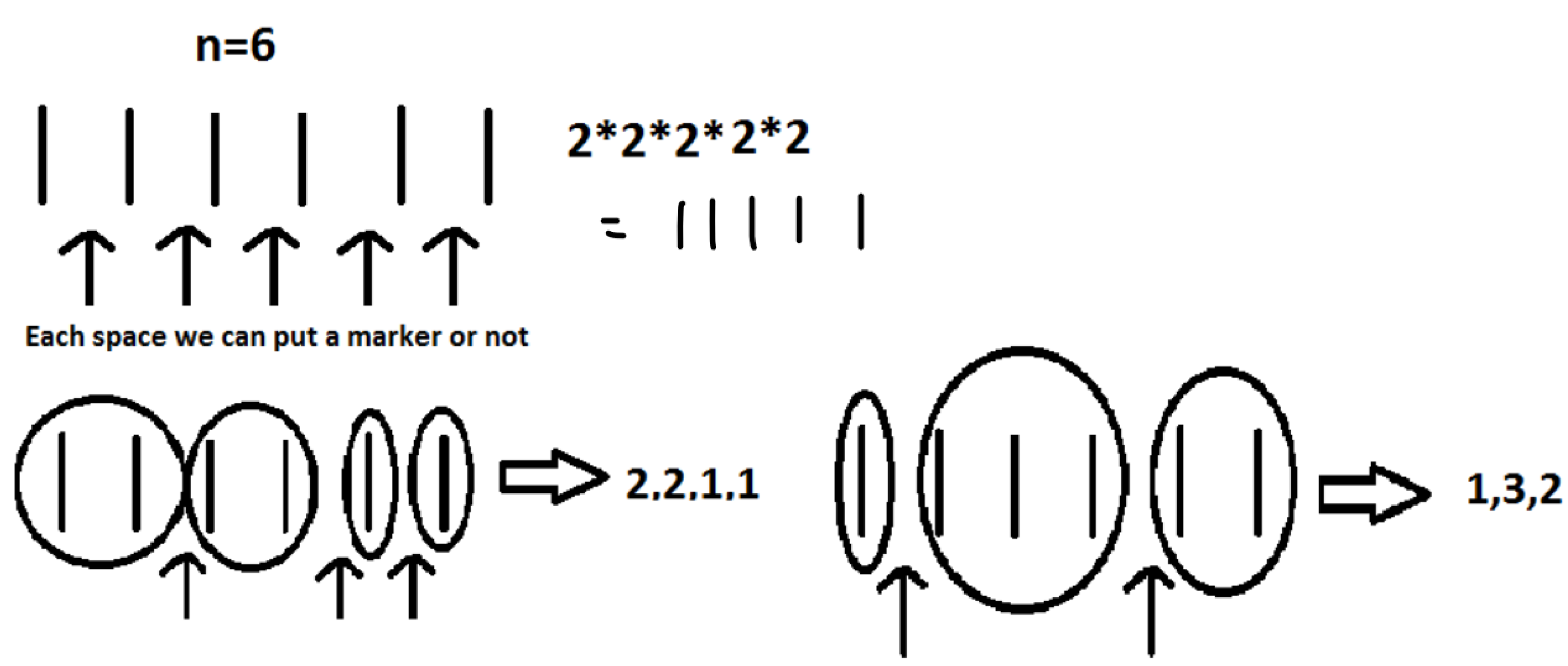
Partition & Composition say 2+1+1+1 can be written as 2,1,1,1 or $2, 1^3$

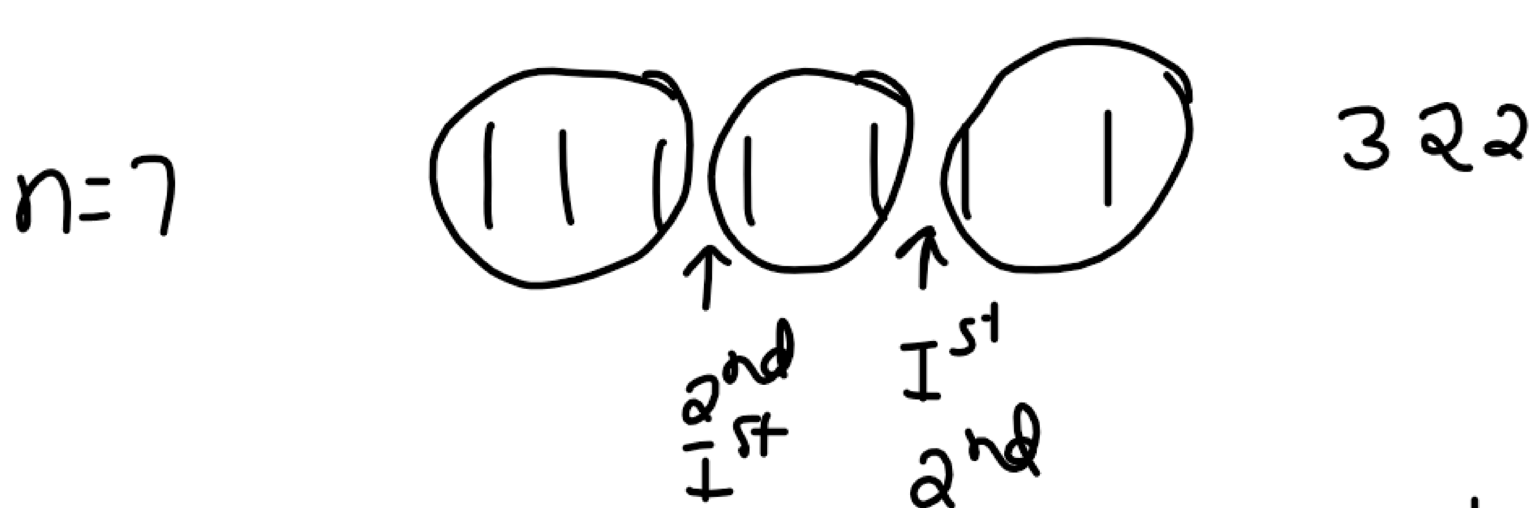
Enumerator for compositions

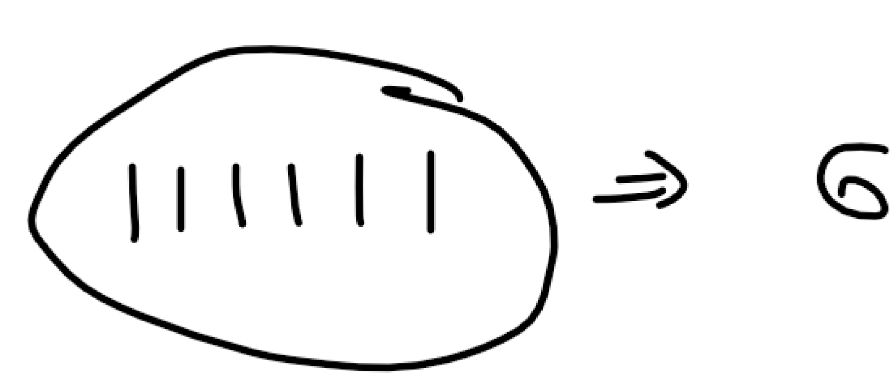
Consider a $+^{ve}$ integer n.
Consider n one's in a row.

If there is no restriction on the number of parts, we may or may not put a marker in any one of the $(n-1)$ spaces available, to form groups. This can be done is $\underline{\underline{2^{n-1}}}$ ways.









If we restrict the compositions to have exactly m parts, then $(m-1)$ markers are needed to form m groups and the number of placing $(m-1)$ markers in the $(n-1)$ spaces between the n ones is $\underline{\underline{{}^{n-1}C_{m-1}}}$.

$n=5$, $m=2$, Ans: ${}^4C_1 = 4$ ✓

$\Rightarrow (1,4)\ (4\ 1)\ (3\ 2)\ (2\ 3)$

$m=3$, ${}^4C_2 = 6$

$\Rightarrow (3\ 1\ 1)(1\ 1\ 3)\ (1\ 3\ 1)\ (2\ 2\ 1)\ (1\ 2\ 2)\ (2\ 1\ 2)$

Generating function for composition

Let $C_m(x)$ be the enumerator for composition of n with exactly m parts. $C_m(x)$ is the coeff of x^n in the series number of composition of n with m parts.

Each part of any composition can be one, two or three or any greater number.

Thus the factor in the enumerator must contain each of these powers x i.e. $(x + x^2 + x^3 + \cdots + x^k + \cdots)$

Since there are m parts, the GF is the product of m such factors

i.e., $C_m(x) = (x + x^2 + \cdots)^m$

$= x^m (1 + x + x^2 + \cdots)^m$

$= x^m (1-x)^{-m}$

coeff of x^n in $x^m (1-x)^{-m}$ is also equal to coeff of x^{n-m} in $(1-x)^{-m}$

$= \left(1 + {}^{m}C_1 x + {}^{m+1}C_2 x^2 + {}^{m+2}C_3 x^3 + \dots\right)$

coeff of x^{n-m} is ${}^{m+(n-m)-1}C_{n-m} = {}^{n-1}C_{n-m}$

$= {}^{n-1}C_{m-1}$

The generating function for composition of n with <u>no</u> restriction on number of parts is equal to

$$C(x) = \sum_{m=1}^{\infty} C_m(x)$$

$$= \sum_{m=1}^{\infty} x^m (1-x)^{-m}$$

$$= \sum_{m=1}^{\infty} \left(\frac{x}{1-x}\right)^m$$

Substituting $\frac{x}{1-x} = t$ we get $\Rightarrow \sum_{m=1}^{\infty} t^m$

$= t + t^2 + t^3 + \cdots \quad \cdot\cdot$

$= t(1 + t + t^2 + \cdots)$

$= t(1-t)^{-1}$

or $\quad t = \dfrac{x}{1-x} \quad \Longrightarrow \quad \dfrac{x}{1-2x}$

$= x(1-2x)^{-1} = x(1 + 2x + (2x)^2 + \cdots)$

coeff of x^n is $\underline{\underline{2^{n-1}}}$

Problems

Q1: How many compositions of n with m parts are there when zero as a part is allowed?

Soln: Consider a composition of n with m parts with zero parts allowed.
Add one to each part, then it will represent a composition of $n+m$ with m parts

$\therefore$ Ans is $^{n+m-1}C_{m-1}$

$n=5$
$m=3$

050	161
500	611
410	521
231	342
005	116

Example: Let $n=5$, $m=2$

Then (1 4) (4 1) (2 3) (3 2) (0 5) (5 0)

Adding 1 we get

(2 5) (5 2) (3 4) (4 3) (1 6) (6 1)

Composition of $5+2$ with 2 parts is $^6C_1 = \underline{6}$

Problem :

Q2) How many ways can an examiner assign 30 marks to 8 question so that no question receives less than 2 marks ?

Soln : Assign 1 mark to each of the 8 questions . Remaining 22 marks to 8 questions s.t each question gets at least 1 mark is equivalent to number of composition of 22 with exactly 8 parts and zero parts not allowed

$$= {}^{22-1}C_{8-1} = {}^{21}C_7$$

Or assign 2 marks to each question

Remaining questions 14 marks with exactly 8 with zero parts allowed is

$$ {}^{14+8-1}C_{8-1} = {}^{21}C_7$$