Problem :

Q2) How many ways can an examiner assign 30 marks to 8 question so that no question receives less than 2 marks ?

Soln : Assign 1 mark to each of the 8 questions. Remaining 22 marks to 8 questions s.t each question gets at least 1 mark is equivalent to number of composition of 22 with exactly 8 parts and zero parts not allowed

$= {}^{22-1}C_{8-1} = {}^{21}C_7$

Or assign 2 marks to each question Remaining 14 marks with exactly 8 questions with zero parts allowed is

${}^{14+8-1}C_{8-1} = {}^{21}C_7$

Generating function for unrestricted partitions

Let P_n be the no. of unrestricted partitions of n so that the G.F is

$$P(x) = P_0 + P_1 x + P_2 x^2 + \cdots + P_n x^n + \cdots$$

consider the polynomial $1 + x + x^2 + \cdots + x^k + \cdots$

The appearance of x^k can be interpreted as the existence of just k one's in a partition of the integer n.

Similarly, the polynomial $1 + x^2 + (x^2)^2 + \cdots + (x^2)^k + \cdots$ is concerned with twos in the partition, & coeff of x^{2k} represents the case of just k twos in the partition.

In general, $1 + x^r + x^{2r} + \cdots + (x^r)^k + \cdots$ can represent the r's in the partition.

The G.F for partition should contain one factor for 1's and one factor for two's & so on.

$$\therefore G.F = (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^3 + x^6 + \cdots)\cdots$$

$$= (1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\cdots$$

coeff of x^n will give no. of unrestricted partition of n.

$n = 5$

$$(\underbrace{1+x+x^2+x^3+x^4+x^5}_{1's})(\underbrace{1+x^2+x^4}_{2's})(1+x^3)(1+x^4)(1+x^5)$$

$\uparrow 2's \quad \uparrow 3's \quad \uparrow 4's \quad \uparrow 5's$

coeff of x^5 :

$1.1.1.1.x^5 \Longrightarrow 5$ ✓

$1\,x^2\,x^3\,1\,1 \Longrightarrow 2\,3$

$x^5.1.1.1.1 \Longrightarrow 11111$

$x^2.1.x^3.1.1 \Longrightarrow 113$

$x\,1.1.x^4.1 \Longrightarrow 14$

Q1: Find the number of ways that can be obtained as a change for ₹10 in terms of ₹5, ₹2 and ₹1.

<u>Soln</u>: GF: $(1+x+x^2+\cdots+x^{10})(1+x^2+\cdots+x^{10})(1+x^5+x^{10})$

$\uparrow$ ₹1 $\quad \uparrow$ ₹2 $\quad \uparrow$ ₹5

coeff of x^{10} :

$1.1.x^{10} \longrightarrow (5\,5)$

$x.\,x^4\,x^5 \longrightarrow (1\,2\,2\,5)$

$x^3\,x^2\,x^5 \longrightarrow (1\,1\,1\,2\,5)$

$x^5\,1\,x^5 \longrightarrow (1\,1\,1\,1\,1\,5)$ ...

$\vdots \quad 1 \qquad 2\,1^8 \qquad 2^5$

$2^2\,1^6 \qquad 1^4\,2^3$

$2^4\,1^2 \qquad 1^{10}$

...

Ans: <u>10</u>

Q2: Prove that the number of partitions of 'n' in which no integer occurs more than twice as a part is equal to the number of partitions of 'n' into parts not divisible by 3.

Soln: G.F for partition of 'n' in which no integer occurs more than twice is

$$G_1(x) = (1+x+x^2)(1+x^2+x^4)(1+x^3+x^6)\cdots$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ 1's & 2's & 3's \end{array}$$

G.F for partition of 'n' in which no part is divisible by 3 is

$$G_2(x) = (1+x+x^2+\cdots)(1+x^2+x^4+\cdots)(1+x^4+x^8+\cdots)\cdots$$

$$\begin{array}{ccc} \uparrow & & \\ 1's & 2's & 4's \end{array}$$

$$G_2(x) = (1-x)^{-1}(1-x^2)^{-1}(1-x^4)^{-1}(1-x^5)^{-1}(1-x^7)^{-1}\cdots$$

consider $G_1(x) = (1+x+x^2)(1+x^2+x^4)(1+x^3+x^6)(1+x^4+x^8)\cdots$

$$G_1(x) = \frac{(1-x)(1+x+x^2)}{(1-x)} \cdot \frac{(1-x^2)(1+x^2+x^4)}{(1-x^2)} \cdot \frac{(1-x^3)(1+x^3+x^6)}{(1-x^3)}\cdots$$

$$= \frac{\cancel{(1-x^3)}}{(1-x)}\frac{\cancel{(1-x^6)}}{(1-x^2)}\frac{\cancel{(1-x^9)}}{\cancel{(1-x^3)}}\frac{\cancel{(1-x^{12})}}{(1-x^4)}\frac{\cancel{(1-x^{15})}}{(1-x^5)}\frac{\cancel{(1-x^{18})}}{\cancel{(1-x^6)}}$$

$$= (1-x)^{-1}(1-x^2)^{-1}(1-x^4)^{-1}\cdots = G_2(x)$$

Q3. Show that the number of partitions of 'n' in which every part is odd is equal to the number of partitions of 'n' with unequal (or distinct) parts.

Soln: G.F. for partition of 'n' in which every part is odd

$$G_1(x) = \underset{1's}{(1-x)^{-1}}\ \underset{3's}{(1-x^3)^{-1}}\ \underset{5's}{(1-x^5)^{-1}} \cdots$$

GF for partition of 'n' with distinct part is

$$G_2(x) = \underset{1's}{(1+x)}\ \underset{2's}{(1+x^2)}\ \underset{3's}{(1+x^3)}\ \underset{4's}{(1+x^4)} \cdots$$

$$= \frac{(1-x)(1+x)}{(1-x)} \cdot \frac{(1-x^2)(1+x^2)}{(1-x^2)} \cdot \frac{(1-x^3)(1+x^3)}{(1-x^3)} \cdots$$

$$= \frac{\cancel{(1-x^2)}\,\cancel{(1-x^4)}\,(1-x^6)\cdots}{(1-x)\,\cancel{(1-x^2)}\,(1-x^3)\,\cancel{(1-x^4)}\cdots}$$

$$= (1-x)^{-1}(1-x^3)^{-1}\cdots = G_1(x)$$

Example ③ $n=7$: No. of partition of 7 in which all parts are odd : 7, 115, 1111111, 11113, 133 is $\underline{\underline{5}}$

Partition of 7 with distinct parts : 7, 124, 25, 34, 16 is $\underline{\underline{5}}$

Example ② : $n=7$. No. of partition of 7 in which no integer occurs more than twice as a part : 7, 61, 511, 43, 421, 331, 322, 52, 3211 is $\underline{\underline{9}}$.

no. of partition of 7 into parts not divisible by 3

are 7, 511, 421, 52, 2221, 4111, 22111, 1111111,

211111 is $\underline{\underline{9}}$ ~~21111~~

Ferrers graph

It is a graph to represent a partition by an array of dots.

It has the following properties.

(i) There is one row for each part

(ii) The number of dots in a row is the size of that part

(iii) An upper row always contains at least as many dots as a lower row.

(iv) The rows are alligned on the left.

Example : Consider the partition 5322

• • • • •
• • •
• •
• •

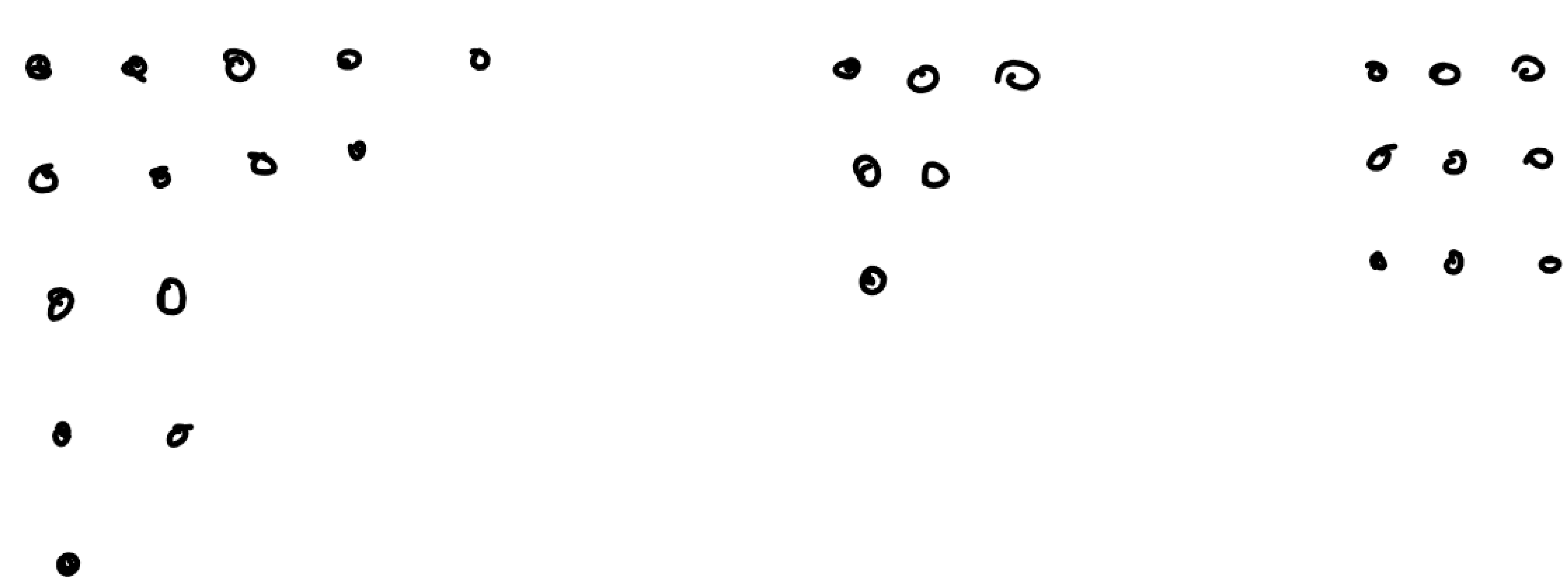
Ferrers graph representation of 5322 ✓

Conjugate Partition : The partition obtained by reading the Ferrers graph by column is called conjugate Partition.

In the above ferrers graph the conjugate partition of 5322 is 44211.

Self conjugate : A partition whose ferrers graph reads the same by rows and columns is called self conjugate.

Eg : 54221 ✓ 321 333



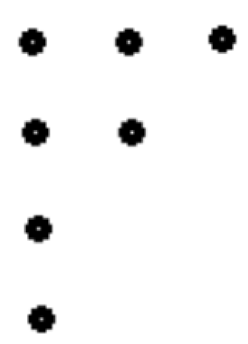
Problems :

Q1. Show that the number of partitions of n is equal to number of partitions of 2n with exactly n parts.

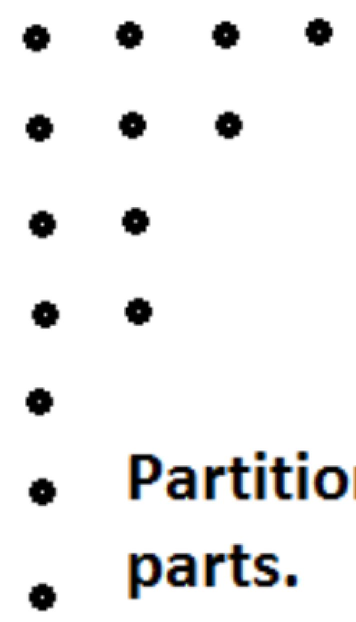
Soln : Consider a ferrers graph of a partition of n. Add a column of n dots on the left of the graph. Then the graph represents a partition of 2n with exactly n parts.

Consider a partition of 2n with exactly n parts. Then the leftmost column contains n dots. Eliminating the I^{st} column results in a partition of n.

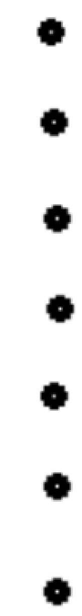
Thus, for every partition of n dots there corresponds a partition of 2n with exactly n parts & vice versa. Hence, no. of partition of n is equal to no. of partition of 2n with exactly n parts.



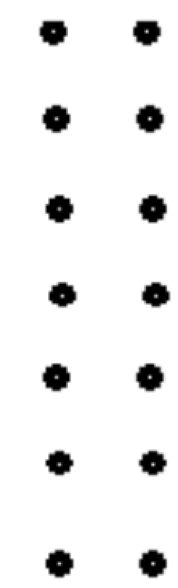
Partition of n=7



Partition of 14 with 7 parts.



Partition of n=7



Partition of 14 with 7 parts.