Problems on Algebraic operations of Lattice -Lattice theory

Theorem 0.1

If the meet operation is distributive over join operation in a lattice, then the join operation is also distributive over meet operation and vice versa.

Proof.

Given

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \tag{1}$$

To prove

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \tag{2}$$

□

Proof continues...

Consider

$$
\begin{aligned}
(a \vee b) \wedge (a \vee c) =& [(a \vee b) \wedge a] \vee [(a \vee b) \wedge c] \qquad (applying\ (1)) \\
=& a \vee [(a \vee b) \wedge c] \qquad (absorption\ law) \\
=& a \vee [c \wedge (a \vee b)] \qquad (commutative\ law) \\
=& a \vee [(c \wedge a) \vee (c \wedge b)] \qquad (Distributive\ law) \\
=& [a \vee (c \wedge a)] \vee (c \wedge b) \qquad (associate\ law) \\
=& a \vee (c \wedge b) \qquad (absorption) \\
=& a \vee (b \wedge c) \qquad (commutative)
\end{aligned}
$$

By principle of duality, we obtain that if join is distributive over meet then meet also distributive over join operation.

Problems

1) Let a and b be two elements in a lattice $(A, \leq)$. Show that $a \wedge b = b$ if and only if $a \vee b = a$.

Solution: Let

$$a \wedge b = b \tag{3}$$

To prove $a \vee b = a$

$$\begin{aligned} a \vee (a \wedge b) =& a \qquad (Absorption) \\ a \vee b =& a \qquad (by (3)) \end{aligned}$$

Let

$$a \vee b = a \tag{4}$$

To prove $a \wedge b = b$

$$\begin{aligned} b \wedge (a \vee b) =& b \qquad (Absorption) \\ b \vee a =& b \qquad (by (4)) \\ a \vee b =& b \qquad (Commutative) \end{aligned}$$

2) Let a, b, c be elements in a lattice $(A, \leq)$. Show that if $a \leq b$, then $a \vee (b \wedge c) \leq b \wedge (a \vee c)$.

Solution: First we show that $a \leq b \wedge (a \vee c)$ and $b \wedge c \leq b \wedge (a \vee c)$.
Given $a \leq b$ and by Theorem 1, $a \leq a \vee c$.
By Theorem 2,

$$\begin{aligned} a \wedge a &\leq b \wedge (a \vee c) \\ a &\leq b \wedge (a \vee c) \end{aligned} \tag{5}$$

We know that $b \leq b$ and $c \leq a \vee c$.
By Theorem 2,

$$b \wedge c \leq b \wedge (a \vee c) \tag{6}$$

From equation (5) and (6),

$$\begin{aligned} a \vee (b \wedge c) &\leq [b \wedge (a \vee c)] \vee [b \wedge (a \vee c)] \\ a \vee (b \wedge c) &\leq b \wedge (a \vee c) \qquad (\textit{By idempotent law}) \end{aligned}$$

3) Let a, b, c be elements in a Lattice $(A, \leq)$. Show that

1. $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$.
2. $(a \wedge b) \vee (a \wedge c)] \leq a \wedge (b \vee c)$.

Solution: We know that $a \leq a \vee b$ (Thm 1)
and $a \leq a \vee c$.
From Theorem 2,

$$
\begin{aligned}
a \wedge a &\leq (a \vee b) \wedge (a \vee c) \\
a &\leq (a \vee b) \wedge (a \vee c) \qquad (7)
\end{aligned}
$$

We know that $b \leq a \vee b$ and $c \leq a \vee c$.
From Theorem 2,

$$b \wedge c \leq (a \vee b) \wedge (a \vee c) \qquad (8)$$

Using equation (7) and (8) and Theorem 2,

$$
\begin{aligned}
a \vee (b \wedge c) &\leq [(a \vee b) \wedge (a \vee c)] \vee [(a \vee b) \wedge (a \vee c)] \\
a \vee (b \wedge c) &\leq [(a \vee b) \wedge (a \vee c)] \qquad \textit{(idempotent law)}
\end{aligned}
$$

4) Let $(A, \vee, \wedge)$ be an algebraic system, where $\wedge$ and $\vee$ are binary operations satisfying absorption property. Show that $\wedge$ and $\vee$ also satisfies idempotent law.

Solution: Given for all $a, b \in A$,

$$\begin{aligned} a \vee (a \wedge b) =& a \quad and \\ a \wedge (a \vee b) =& a \end{aligned}$$

Then to prove $a \vee a = a$ and $a \wedge a = a$.

Consider

$$\begin{aligned} a \vee a =& a \vee (a \wedge (a \vee b)) \\ a \vee a =& a \qquad (Absorption) \end{aligned}$$

Consider

$$\begin{aligned} a \wedge a =& a \wedge (a \vee (a \wedge b)) \\ a \wedge a =& a \qquad (Absorption) \end{aligned}$$

5) Let $(A, \leq)$ be a distributive lattice. Show that if $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ for some $a \in A$, then $x = y$.

Solution: Consider $x \vee (a \wedge x) = x$ (Absorption)

$x \vee (a \wedge y) = x$ ($a \wedge x = a \wedge y$ and $a \vee x = a \vee y$)

$(x \vee a) \wedge (x \vee y) = x$ (Distribution)

$(a \vee x) \wedge (x \vee y) = x$ (Commutative)

$(a \vee y) \wedge (x \vee y) = x$

$y \vee (a \wedge x) = x$ (Distribution)

$y \vee (a \wedge y) = x$

$y = x$ (Absorption).

6) Show that a lattice is distributive if and only if for any elements a, b, c in the lattice, $(a \vee b) \wedge c \leq a \vee (b \wedge c)$.
Solution: Assume that lattice is distributive. Then

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \tag{9}$$

We know that $c \leq a \vee c$ and $a \vee b \leq a \vee b$.
$(a \vee b) \wedge c \leq (a \vee b) \wedge (a \vee c)$ (Thm 2)
$(a \vee b) \wedge c \leq a \vee (b \wedge c)$ (From eqn (9))
Conversely, suppose

$$(a \vee b) \wedge c \leq a \vee (b \wedge c), \tag{10}$$

to prove $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.
$(a \vee b) \wedge (a \vee c) \leq a \vee (b \wedge (a \vee c))$ (by eqn (10))
$(a \vee b) \wedge (a \vee c) \leq a \vee ((a \vee c) \wedge b)$ (by commutative law)
$(a \vee b) \wedge (a \vee c) \leq a \vee (a \vee (c \wedge b)$ (by eqn (10))
$(a \vee b) \wedge (a \vee c) \leq (a \vee a) \vee (c \wedge b)$ (by associative)

$$(a \vee b) \wedge (a \vee c) \leq a \vee (c \wedge b) \qquad (by\ idempotent) \tag{11}$$

From Problem (3), we have

$$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c). \tag{12}$$

From eqns (11) and (12), we get

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

7) Let $(A, \vee, \wedge)$ be an algebraic system where $\vee$ and $\wedge$ are binary operations satisfying the commutative, associative and absorption laws.
Define a binary operations $\leq$ on A as follows:
for all $a, b \in A$, $a \leq b$ if and only if $a \wedge b = a$.

a) Show that $\leq$ is partial ordering relation.
b) Show that $a \vee b$ is least upper bound of a and b in $(A, \leq)$.
c) Show that $a \wedge b$ is greatest lower bound of a and b in $(A, \leq)$.

Solution: a) From problem 4, we have proved that if $\vee$ and $\wedge$ satisfies absorption law, then $\vee$ and $\wedge$ also satisfies idempotent law.
By idempotent law, we have $a \wedge a = a \implies a \leq a \implies '' \leq `` $ is reflexive.
To prove antisymmetry,
If $a \leq b$ and $b \leq a$,
we have,

$$a \wedge b = a \tag{13}$$
$$b \wedge a = b \tag{14}$$

By commutative law, equation (14) becomes

$$a \wedge b = b \tag{15}$$

From eqns (13) and (15), $a = b \implies '' \leq ``$ is antisymmetric.

To prove transitive law,
if $a \leq b$ and $b \leq c$, then $a \wedge b = a$ and $b \wedge c = b$.
$a = a \wedge b = a \wedge (b \wedge c)$
$= (a \wedge b) \wedge c$
$= a \wedge c \implies a \leq c \implies '' \leq$ “ is transitive.
Therefore $'' \leq$ “ is Partial ordering relation.
b) To prove $a \vee b$ is lub of a and b.
First we will show that $a \vee b$ is upper bound of a and b.
From absorption law,

$$a \wedge (a \vee b) = a \implies a \leq a \vee b \tag{16}$$

Similarly,

$$b \wedge (a \vee b) = b \implies b \leq a \vee b \tag{17}$$

From equations (16) and (17), $a \vee b$ is an upper bound of a and b.

Suppose d is an other upper bound of a and b, i.e, $a \leq d$ and $b \leq d$.
Then we should prove $a \vee b \leq d$.
Given $a \wedge d = a$, $b \wedge d = b$.
To prove $(a \vee b) \wedge d = a \vee b$.
$a \vee b = (a \vee b) \wedge ((a \vee b) \wedge d) \qquad$ (absorption).
$a \vee b = (a \vee b) \wedge ((a \vee (b \wedge d)) \wedge d) \qquad$ (because $b = b \wedge d$).
$a \vee b = (a \vee b) \wedge ((a \vee [(b \wedge d) \wedge d]) \qquad$ (associative).
$a \vee b = (a \vee b) \wedge (a \vee d) \qquad$ (absorption).
$a \vee b = (a \vee b) \wedge ((a \wedge d) \vee d) \qquad$ (because $a = a \wedge d$).
$a \vee b = (a \vee b) \wedge d \qquad$ (absorption).