NUMERICAL METHODS I

Introduction:

Most of the problems of engineering, physical and economical sciences can be formulated in terms of system of equations, ordinary or partial differential equations. In majority of the cases, the solutions to these problems in analytical form are non-existent or difficult. In all such cases, numerical methods provide approximate solutions, practical for analysis. Numerical methods do not strive for exactness. Instead, they yield approximations with specified degree of accuracy. The early disadvantage of the several numbers of computations involved has been removed through high speed computation using computers, giving results which are accurate, reliable and fast.

Interpolation

Introduction:

Suppose that a function $y = f(x)$ is given. For a set of values of x in the domain, we can tabulate the corresponding values of y. The central problem of interpolation is the converse of this:

Given a set of tabular values (x_0, y_0), (x_1, y_1), ..., (x_n, y_n) satisfying the relation $y = f(x)$ where the explicit nature of $f(x)$ is not known, it is required to find a simpler function, say $\phi(x)$, which approximates $f(x)$, such that $f(x)$ and $\phi(x)$ agree at the set of tabulated points. Such a process of approximation of an unknown function by a known function within the range where it is defined, such that both functions assume same values at the given set of tabulated points is called interpolation. The extrapolation is the process of approximating the unknown function by a function at a point outside the range of definition. If $\phi(x)$ is a polynomial, then the process is called polynomial interpolation and $\phi(x)$ is called the interpolating polynomial.

Weierstrass Theorem: If *f(x)* is continuous in $x_0 \leq x \leq x_n$, then given any ε> 0, there exists a polynomial *P(x)* such that $|f(x) - P(x)| < \varepsilon$ for all x in (x_0, x_n).

This approximation theorem justifies the polynomial approximation.

There are two main uses of these approximating polynomials. The first use is to reconstruct the function *f(x)* when it is not given explicitly and only values of *f(x)* and/ or its certain order derivatives are given at a set of distinct points called *nodes* or *tabular points*. The second use is to perform the required operations which were intended for *f(x),* like determination of roots, differentiation and integration etc. can be carried out using the approximating polynomial *P(x)*. The approximating polynomial *P(x)* can be used to predict the value of *f(x)* at a non-tabular point.

Remark: Through two distinct points, we can construct a unique polynomial of degree 1(straight line). Through three distinct points, we can construct a unique polynomial of degree atmost two (a parabola or a straight line). In general, through $n+1$ distinct points, we can construct a unique polynomial of degree atmost n.

Note: *The interpolation polynomial fitting a given data is unique*.

We consider x to be an independent variable and $y = f(x)$ as a function of x where the explicit nature of y is not known, but a set of values $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ satisfying $y_i = f(x_i)$, $i = 0, 1, 2, \ldots, n$ are given.

Interpolation with equally spaced points:

To obtain interpolating polynomials, we use finite differences.

Finite differences

Suppose that

$$x_i = x_0 + ih,\ h > 0,\ i = 1, 2, \ldots, n.$$

i.e., the values of x are equally spaced.

The following are the three types of finite differences

1. Forward Differences
2. Backward Differences
3. Central Differences

Forward Differences:

The differences $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ are called first order forward differences of y and are respectively denoted by $\Delta y_0, \Delta y_1, \Delta y_2, \ldots, \Delta y_n$. We call '$\Delta$',the *forward difference operator*. The differences of the first order forward differences are called second order forward differences and are denoted by $\Delta^2 y_0,\ \Delta^2 y_1,\ \Delta^2 y_2,\ \ldots,\ \Delta^2 y_{n-1}$.Thus

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - 2y_1 + y_0,\quad \Delta^2 y_1 = \Delta y_2 - \Delta y_1 = y_3 - 2y_2 + y_1$$

and so on. Similarly, one can find the rth order forward differences recursively as,

$$\Delta^r y_k = \Delta^{r-1} y_{k+1} - \Delta^{r-1} y_k,\ r = 1, 2, \ldots$$

These differences can be tabulated and the table so obtained is called forward difference table and it is read diagonally downwards.

x	Y	Δ	Δ^2	Δ^3	Δ^4
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

Backward differences:

The differences $y_1 - y_0$, $y_2 - y_1, \ldots, y_n - y_{n-1}$ when denoted by $\nabla y_1, \nabla y_2, \ldots, \nabla y_n$, are called first order backward differences of y. The operator ∇ is called the *backward difference operator*. The differences of the first order differences are called second order differences and are denoted by $\nabla^2 y_2, \nabla^2 y_3, \ldots, \nabla^2 y_n$.

Similarly one can define the r^{th} order backward differences,

$$\nabla^r y_k = \nabla^{r-1} y_k - \nabla^{r-1} y_{k-1}, \quad r = 1, 2, \ldots$$

The relation between Δ and ∇ is given by $\nabla y_r = \Delta y_{r-1}$.

The backward difference table is as shown below and it is read diagonally upwards.

x	y	∇	∇^2	∇^3	∇^4
x_0	y_0				
		∇y_1			
x_1	y_1		$\nabla^2 y_2$		
		∇y_2		$\nabla^3 y_3$	
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$
		∇y_3		$\nabla^3 y_4$	
x_3	y_3		$\nabla^2 y_4$		
		∇y_4			
x_4	y_4				

Other notations for $y = f(x)$ are

1. $\Delta y_x = f(x + h) - f(x)$,
2. $\nabla y_x = f(x) - f(x - h)$

where h is the difference between two consecutive x values.

Shift Operator:

The shift operator, denoted by E, is defined by the equation $Ey_k = y_{k+1}$. The operator shifts the functional value y_k to the next higher value y_{k+1}.

Again operating by E, we get $E^2y_k = E(Ey_k) = E(y_{k+1}) = y_{k+2}$. In general

$$E^r y_k = y_{k+r}.$$

The inverse shift operator, denoted by E^{-1}, is defined by the equation $E^{-1}y_k = y_{k-1}$.

In general

$$E^{-r}y_k = y_{k-r}.$$

Properties of finite differences:

1. Linearity Property:

For any two constants a and b and for any two functions $f(x)$ and $g(x)$, we have, $\Delta(a\, f(x) \pm b\, g(x)) = a\, \Delta f(x) \pm b\, \Delta g(x)$

2. Differences of Polynomial: The first order difference of a polynomial of degree n is a polynomial of degree $n-1$. Hence n^{th} difference of a polynomial of degree n is a constant.

Proof:

Consider

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n,\ a_0 \neq 0.$$

Then,

$$\begin{aligned}\Delta f(x) &= f(x+h) - f(x) \\ &= a_0\left((x+h)^n - x^n\right) + a_1\left((x+h)^{n-1} - x^{n-1}\right) + a_2\left((x+h)^{n-2} - x^{n-2}\right) + \ldots \\ &= a_0nhx^{n-1} + lower\ \deg ree\ terms\end{aligned}$$

which is a polynomial of degree $n-1$. Similarly,

$$\Delta^2 f(x) = a_0n(n-1)h^2x^{n-2} + lower\ \deg ree\ terms$$

which is a polynomial of degree $n-2$.

Continuing in this manner, we arrive at

$\Delta^n f(x) = a_0 n! h^n$,

which is a constant. Hence n^{th} difference is a constant.

Remark: $(n + 1)^{th}$ and higher order differences of a polynomial of degree n are zero.

Remark: Converse of the above is also true. i.e., if for a function $f(x)$ the n^{th} order finite difference is a constant and $(n+1)^{th}$ ordered finite difference is zero then $f(x)$ is a polynomial of degree n.

3. $1 + \Delta = E$.

4. $1 - \nabla = E^{-1}$.

4. $\Delta = E\nabla = \nabla E$

5. $$\Delta^k y_r = \sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} y_{k+i} \qquad \ldots\ldots.(1)$$

Proof: Since

$$E = 1 + \Delta, \; we\, have\, \Delta = E - 1$$

$$\Rightarrow \Delta^k = (1 - E)^k = \sum_{i=1}^{k} (-1)^i \binom{k}{i} E^i$$

$$\therefore \Delta^k y_r = \left(\sum_{i=1}^{k} (-1)^i \binom{k}{i} E^i \right) y_r = \sum_{i=1}^{k} (-1)^i \binom{k}{i} E^i y_r = \sum_{i=1}^{k} (-1)^i \binom{k}{i} y_{k+i}$$

6. $\Delta^k y_r = \nabla^k y_{k+r}$

Example: Find the missing term from the following table:

x	0	1	2	3	4
y	1	3	9	–	81

Explain why the result differs from the actual value $3^3 = 27$.

Solution: There are 4 tabulated values, that is, (0, 1), (1, 3), (2, 9) and (4, 81) are given, therefore its 4^{th} forward difference must be zero.

By constructing difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		2			
1	3		4		
		6		a–19	
2	9		a–15		124 – 4a
		a–9		105–3a	
3	a (say)		90–2a		
		81 –a			
4	81				

Now we put $\Delta^4 y = 0$, we have 124 – 4a = 0. This implies that a = 31

Thus y = 31 at x = 3.

Observation of the given values of y indicates that $y = 3^x$.

Putting x = 3 in 3^x, we get 27. This does not tally with the value obtained y = 3, reason is that 3^x is not a polynomial in x but we have assumed y = f(x) to be a polynomial of degree 3.

Exercise

1. Find the missing points in the following table:

x	45	50	55	60	65
y	3	-	2	-	-2.4

(**Ans:** $y(50) = 2.925$, $y(60) = 0.225$)

2. Find the missing values in the following table:

X	0	1	2	3	4	5	6
Y	5	11	22	40	-	140	-

3. Given $u_0 = 10$, $u_1 = 6$, $u_3 = 26$ and $u_5 = 130$. Find u_2 and u_4.

Newton-Gregory Forward Difference Interpolation Formula

Statement: Given a set of points (x_0, y_0), (x_1, y_1),…,(x_n, y_n) satisfying $y = f(x)$, where explicit nature of y is not known and values of x are equally spaced, the n^{th} degree polynomial $y_n(x)$ such that $f(x)$ and $y_n(x)$ agree at the tabulated points is given by

$$y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \ldots + \frac{p(p-1)(p-2)\ldots(p-n+1)}{n!}\Delta^n y_0 \qquad \ldots\ldots..(2)$$

where $x = x_0 + ph$.

Proof: Since $y_n(x)$ is a polynomial of degree n, it can be written as

$$y_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \ldots + a_n(x-x_0)(x-x_1)\ldots(x-x_{n-1}) \quad \ldots.(3)$$

$y_n(x_0) = y_0$ gives $a_0 = y_0$.

$y_n(x_1) = y_1$ gives $a_1 = \dfrac{\Delta y_0}{h}$.

$y_n(x_2) = y_2$ gives $a_2 = \dfrac{\Delta^2 y_0}{2!\, h^2}$.

$\vdots$

$y_n(x_n) = y_n$ gives $a_n = \dfrac{\Delta^n y_0}{n! h^n}$.

Set $x = x_0 + ph$. Then,

$x - x_0 = ph$, $x - x_1 = (p-1)h$, …, $x - x_{n-1} = (p-n+1)h$.

Substituting in (3) and simplifying we get (2).

The formula,

$$y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \ldots + \frac{p(p-1)\ldots(p-n+1)}{n!}\Delta^n y_0,$$

is called Newton- Gregory forward difference interpolation formula and it is useful for interpolating near the beginning of a set of tabular values.

Newton-Gregory Backward Difference Interpolation Formula

Statement: Given a set of points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ satisfying $y = f(x)$, where explicit nature of y is not known and values of x are equally spaced, the n^{th} degree polynomial $y_n(x)$ such that $f(x)$ and $y_n(x)$ agree at the tabulated points is given by

$$y_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \ldots + \frac{p(p+1)(p+2)\ldots(p+n-1)}{n!}\nabla^n y_n \quad \ldots\ldots.(4)$$

where $x = x_n + ph$.

Proof: Since $y_n(x)$ is a polynomial of degree n, it can be written as

$$y_n(x) = a_0 + a_1(x-x_n) + a_2(x-x_n)(x-x_{n-1}) + \ldots + a_n(x-x_n)(x-x_{n-1})\ldots(x-x_1) \ \ldots.(5)$$

$y_n(x_n) = y_n$ gives $a_0 = y_n$.

$y_n(x_{n-1}) = y_{n-1}$ gives $a_1 = \dfrac{\nabla y_n}{h}$.

$y_n(x_{n-2}) = y_{n-2}$ gives $a_2 = \dfrac{\nabla^2 y_n}{2!\,h^2}$.

$\vdots$

$y_n(x_1) = y_1$ gives $a_n = \dfrac{\nabla^n y_0}{n!h^n}$.

Set $x = x_n + ph$. Then,

$x - x_n = ph$, $x - x_{n-1} = (p+1)h$, …, $x - x_1 = (p+n-1)h$.

Substituting in (5) and simplifying we get (4).

The formula,

$$y_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \ldots + \frac{p(p+1)\ldots(p+n-1)}{n!}\nabla^n y_n$$

is called Newton- Gregory backward difference interpolation formula and it is useful for interpolating near the end of a set of tabular values.

Example 1: Find the cubic polynomial which takes the following values $y(0) = 1$, $y(1) = 0$, $y(2) = 1$ and $y(3) = 10$. Hence or otherwise, obtain y (0.5).

Here $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$

and $y_0 = 1$, $y_1 = 0$, $y_2 = 1$ $y_3 = 10$

Ans:

We form the difference table

x	y	Δ	Δ^2	Δ^3
$x_0 = 0$	$\mathbf{y_0 = 1}$			
		$\mathbf{\Delta y_0 = -1}$		
$x_1 = 1$	$y_1 = 0$		$\mathbf{\Delta^2 y_0 = 2}$	
		$\Delta y_1 = 1$		$\mathbf{\Delta^3 y_0 = 6}$
$x_2 = 2$	$y_2 = 1$		$\Delta^2 y_1 = 8$	
		$\Delta y_2 = 9$		
$x_3 = 3$	$y_3 = 10$			

From the above table we have,

$x_0 = 0$, $y_0 = 1$, $\Delta y_0 = -1$, $\Delta^2 y_0 = 2$, $\Delta^3 y_0 = 6$

Using Newton forward difference formula

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 \text{ where } x = x_0 + ph$$

Here $h = 1$, $x_0 = 0$, therefore $p = x$

$$y(x) = 1 + x(-1) + \frac{x(x-1)}{1.2}2 + \frac{x(x-1)(x-2)}{1.2.3}6$$

$$= 1 - x + (x^2 - x) + (x^3 - 3x^2 + 2x)$$

$$y(x) = x^3 - 2x^2 + 1$$

which is the polynomial from which we obtained the above tabular values.

To compute y (0.5)

Here $x_0 + ph = x = 0.5$

$p = 0.5$ since $x_0 = 0$ and $h = 1$

$$y(0.5) = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$= 1 + 0.5(-1) + \frac{(0.5)(0.5-1)}{2}\ 2 + \frac{(0.5)(0.5-1)(0.5-2)}{6}\ 6$$

$$= 6.25$$

which is the same value as that obtained by substituting $x = 0.5$ in the cubic polynomial.

Example 2: Using Newton's forward difference formula, find the sum

$$S_n = 1^3 + 2^3 + 3^3 + \ldots + n^3.$$

Solution: $\Delta S_n = (n+1)^3,\ \Delta^2 S_n = 3n^2 + 9n + 7,\ \Delta^3 S_n = 6n + 12,\ \Delta^4 S_n = 6.$

Thus, S_n is a polynomial of degree 4 in 'n'.

Further, $S_1 = 1,\ \Delta S_1 = 8,\ \Delta^2 S_1 = 19,\ \Delta^3 S_1 = 18,\ \Delta^4 S_1 = 6.$ Hence,

$$S_n = 1 + (n-1)(8) + \ldots + \frac{(n-1)(n-2)(n-3)(n-4)}{24}(6) = \left[\frac{n(n+1)}{2}\right]^2.$$

Example 3: From the following table, estimate the number of students who obtained marks between 40 and 45.

Marks	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
No. of students	31	42	51	35	31

Ans:

First we prepare the cumulative frequency table, as follows,

Marks less than (x)	40	50	60	70	80
No. of students y	31	73	124	159	190

Now the difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4
40	**31**				
		42			
50	73		**9**		
		51		**–25**	
60	124		–16		**37**
		35		– 12	
70	159		– 4		
		31			
80	190				

We shall find the number of students with marks less than 45.

From the above table we have,

$$y_0 = 31,\ \Delta y_0 = 42,\ \Delta^2 y_0 = 9,\ \Delta^3 y_0 = -25,\ \Delta^4 y_0 = 37$$

Taking $x_0 = 40$, $x = 45$

We have

$$x = x_0 + ph$$

$$45 = 40 + p10$$

$$p = 0.5$$

Using Newton's forward interpolation formula, we get

$$y(x) = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$y(45) = 31 + 0.5(42) + \frac{(0.5)(0.5-1)}{2!}(9) + \frac{(0.5)(0.5-1)(0.5-2)}{3!}(-25) + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{4!}(37)$$

$$= 31 + 21 - 1.125 - 1.5625 - 1.4453 = 47.87.$$

Therefore $y(45) = 47.87$

The number of students with marks less than 45 is 47.87, that is 48. But the number of students with marks less than 40 is 31.

Hence the number of students getting marks between 40 and 45 = 48 – 31 = 17 students.

Example 4: The table below gives the values of tan x for $0.10 \leq x \leq 0.30$:

X	0.10	0.15	0.20	0.25	0.30
y=tan x	0.1003	0.1511	0.2027	0.2553	0.3093

Find: (i) *tan* 0.12 (ii) *tan* 0.26 (iii) *tan* 0.40 and (iv) *tan* 0.50.

Solution: The difference table is

X	*tan x*	Δ	Δ^2	Δ^3	Δ^4
0.10	0.1003				
		0.0508			
0.15	0.1511		0.0008		
		0.0516		0.0002	
0.20	0.2027		0.0010		0.0002
		0.0526		0.0004	
0.25	0.2553		0.0014		
		0.0540			
0.30	0.3093				

(i) p = 0.4, *tan* (0.12) = 0.1205. (Actual Value : 0.1205)

(ii) p = -0.8, *tan* (0.26) = 0.2662. (Actual Value : 0.2660)

(iii) p = 2, *tan* (0.40) = 0.4241. (Extrapolation, Actual Value: 0.4227).

(iv) p = 4, *tan* (0.50) = 0.5543. (Extrapolation, Actual Value : 0.5463).

Remark: Comparison of the computed and actual values shows that in the first two cases (i.e., in interpolation) the results obtained are fairly accurate whereas in the last-two cases (i.e., in extrapolation) the errors are quite considerable. The example therefore demonstrates the important result that if a tabulated function is other than a polynomial, then extrapolation very far from the table limits.

Exercise

1. The population in decennial census was as under. Estimate the population for the year 1955:

Year	1921	1931	1941	1951	1961
Population	46	66	81	93	101

2. The probability integral $p = \sqrt{\frac{2}{\pi}}\int_0^x \exp\left(-\frac{1}{2}t^2\right)dt$ has the following values:

x	1.00	1.05	1.10	1.15	1.20	1.25
p	0.682689	0.706282	0.728668	0.749856	0.769861	0.788700

Calculate p for $x = 1.235$.

3. The values of the elliptic integral $K(m) = \int_0^{\frac{\pi}{2}}\left(1 - m\sin^2\theta\right)^{-1/2} d\theta$ for certain equidistant values of m are given below. Determine $K(0.25)$.

m	0.20	0.22	0.24	0.26	0.28	0.30
$K(m)$	1.659624	1.669850	1.680373	1.691208	1.702374	1.713889

4. From the following table, find y when $x = 1.45$.

x	1.0	1.2	1.4	1.6	1.8	2.0
y	0.0	-0.112	-0.016	0.336	0.992	2.0

5. Evaluate *sin* (0.197) from the following table:

x	0.15	0.17	0.19	0.21	0.23
$\sin x$	0.14944	0.16918	0.18886	0.20846	0.22798

6. Given the table of values:

x	150	152	154	156
$y=\sqrt{x}$	12.247	12.329	12.410	12.490

Evaluate $\sqrt{155}$.

7. From the following table find the number of students who obtained less than 45 marks.

Marks	< 40	40 - 50	50 - 60	60 – 70	70 - 80
No. of students	31	42	51	35	31

[Hint: Apply NFIF for cumulative frequency**]**

8. The population of a town in the decennial census was as given below. Estimate the population for the year 1895 and 1925.

Year: x	1891	1901	1911	1921	1931
Population: y (in thousands)	46	66	81	93	101

9. The following data gives the melting point of an alloy of lead and zinc, where t is the temperature in degree centigrade and p is the percentage of lead in the alloy.

p	40	50	60	70	80	90
t	184	204	226	250	276	304

Using Newton's interpolation formula, find the melting point of the alloy containing 84 percent of lead.

10. Construct Newton's forward interpolation polynomial for the following data.

x	4	6	8	10
y	1	3	8	16

11. Apply Newton's backward difference formula to the data below, to obtain a polynomial of degree 4 in x:

x	1	2	3	4	5
y	1	-1	1	-1	1

Interpolation with unevenly spaced points:

Consider a set of values (x_0, y_0), (x_1, y_1), $(x_2, y_2), \ldots, (x_n, y_n)$ satisfying $y_i = f(x_i)$, $i = 0, 1, 2, \ldots, n$, where values of x not necessarily be equally spaced. In such cases, we use the following interpolation methods.

1. Lagrange's Interpolation
2. Newton's divided differences method

Lagrange's Interpolation

Consider a set of values (x_0, y_0), (x_1, y_1), $(x_2, y_2), \ldots, (x_n, y_n)$ satisfying $y_i = f(x_i)$, $i = 0, 1, 2, \ldots, n$, where values of x not necessarily be equally spaced. Let $y_n(x)$ be the n^{th} degree polynomial such that $y_n(x)$ and $f(x)$ agree at the tabulated values.

Since $y_n(x)$ is a polynomial of degree n, it can be put in the form,

$$\begin{aligned} y_n(x) &= a_0(x-x_1)(x-x_2)\ldots(x-x_n) + a_1(x-x_0)(x-x_2)\ldots(x-x_n) + \ldots \\ &\quad + a_{n-1}(x-x_0)(x-x_1)\ldots(x-x_{n-1}) \qquad \ldots\ldots(6) \end{aligned}$$

$y_n(x_0) = y_0$ gives $a_0 = \dfrac{1}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} y_0$

$y_n(x_1) = y_1$ gives $a_1 = \dfrac{1}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} y_1$

⋮

$y_n(x_n) = y_n$ gives $a_n = \dfrac{1}{(x_n - x_0)(x_n - x_1)...(x_n - x_{n\text{-}1})} y_n$

Substituting in (6), we obtain

$$y_n(x) = \frac{(x-x_0)(x-x_2)...(x-x_n)}{(x_0-x_1)(x_0-x_2)...(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)...(x-x_n)}{(x_1-x_0)(x_1-x_2)...(x_1-x_n)} y_1 + ...$$
$$+ \frac{(x-x_0)(x-x_1)...(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)...(x_n-x_{n-1})} y_n.$$

This is called Lagrange's interpolation formula.

Example1: Find Lagrange's interpolation polynomial fitting the points y(1) = -3, y(3) = 0, y(4) = 30, y(6) = 132. Hence find y(5).

Solution: The given data can be arranged as:

x	1	3	4	6
y= f(x)	– 3	0	30	132

Using Lagrange's interpolation formula, we have y = f (x) =

$$\frac{(x-3)(x-4)(x-6)}{(1-3)(1-4)(1-6)}(-3) + \frac{(x-1)(x-4)(x-6)}{(3-1)(3-4)(3-6)}(0)$$

$$+ \frac{(x-1)(x-3)(x-6)}{(4-1)(4-3)(4-6)}(30) + \frac{(x-1)(x-3)(x-4)}{(6-1)(6-3)(6-4)}(132)$$

$$= \frac{x^3-13x^2+54x-72}{-30}(-3) + \frac{x^3-11x^2+34x-24}{6}(0)$$

$$+ \frac{x^3-10x^2+27x-18}{-6}(30) + \frac{x^3-8x^2+19x-12}{30}(132)$$

On simplification, we get y(x) = $\frac{1}{2}(-x^3+27x^2-92x+60)$, which is the required Lagrange's interpolation polynomial. Now y(5) = 75.

Example 2: Use Lagrange's interpolation formula to fit a polynomial for the data :

x	0	1	3	4
y	– 12	0	6	12

Hence estimate y at $x = 2$

Solution:

By data $x_0 = 0$, $x_1 = 1$, $x_2 = 3$ $x_3 = 4$

$y_0 = -12$, $y_1 = 0$ $y_2 = 6$ $y_3 = 12$

We have $y = f(x) = \dfrac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\, y_0 + \dfrac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\, y_1$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\, y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\, y_3$$

that is, $y = f(x) = \dfrac{(x-1)(x-3)(x-4)}{(-1)(-3)(-4)}(-12) + 0$

$$+ \frac{x(x-1)(x-4)}{(3)(2)(-1)}\, 6 + \frac{x(x-1)(x-3)}{(4)(3)(1)}\, 12$$

$$y = f(x) = x^3 - 7x^2 + 18x - 12$$

Now $f(2) = 2^3 - 7(2)^2 + 18(2) - 12 = 4$

Therefore $f(2) = 4$.

Putting $x = 2.7$, we get $log\, 2.7 \approx y(2.7) = 0.9932518.$

Example 3: The function $y = \sin x$ is tabulated below

x	0	π/4	π/2
y	0	0.70711	1.0

Using Lagrange's interpolation formula, find the value of $sin(\pi/6)$.

Solution:

$$\sin(\pi/6) \approx \frac{(\pi/6-0)(\pi/6-\pi/2)}{(\pi/4-0)(\pi/4-\pi/2)}(0.70711)+\frac{(\pi/6-0)(\pi/6-\pi/4)}{(\pi/2-0)(\pi/2-\pi/4)}(1)$$
$$= 0.51743.$$

Example 4: Using Lagrange's interpolation formula, find *y* as polynomial in *x*from the following table:

x	0	1	3	4
y	-12	0	12	24

Solution: Since $y = 0$ when $x = 1$, it follows that $(x–1)$ is a factor.

Let $y(x) = (x\text{-}1)R(x)$. Then $R(x)=y(x)/(x–1)$. We now tabulate the values of x and $R(x)$.

x	0	3	4
$R(x)$	12	6	8

Applying Lagrange's formula, we get $R(x) = x^2 - 5x + 12$.

Hence the required polynomial approximation toy(x) is given by

$$y(x) = (x-1)(x^2-5x + 12).$$

Example 5: Given the values

x	5	7	11	13	17
y	150	392	1452	2366	5202

Evaluate y(9) using Lagrange's interpolation formula

Solution: .

Given $x_0 = 5$, $x_1 = 7$, $x_2 = 11$, $x_3 = 13$, $x_4 = 17$

$y_0 = 150$, $y_1 = 392$, $y_2 = 1452$, $y_3 = 2366$, $y_4 = 5202$.

The Lagrange's interpolation formula for $n = 5$ is

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} y_0$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} y_2$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} y_3$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} y_4$$

Put $x = 9$ we get

$$y(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150$$

$$= \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392$$

$$= \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-17)(11-13)(11-17)} \times 1452$$

$$= \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2366$$

$$= \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202 = 810$$

Therefore $y(9) = 810$.

Inverse interpolation using Lagrange's Method:

Suppose that a data $(x_i, f(x_i)), i = 0, 1, ..., n,$ is given. In interpolation, we predict the value of the ordinate *f*(*x*) at a non-tabular point *x*. In many applications, we require the value of the abscissa *x* for a given value of the ordinate *f*(*x*). For this problem, we consider the given data as $(f(x_i), x_i),\ i = 0, 1, ..., n$ and construct the interpolation polynomial. This procedure is called inverse interpolation. The

Lagrange's inverse interpolation formula is obtained by interchanging the roles of x and y and it is given by

$$x = \frac{(y-y_1)(y-y_2)...(y-y_n)}{(y_0-y_1)(y_0-y_2)...(y_0-y_n)}x_0 + \frac{(y-y_0)(y-y_2)...(y-y_n)}{(y_1-y_0)(y_1-y_2)...(y_1-y_n)}x_1 + ...$$

Example 1: If $y_1 = 4,\ y_3 = 12,\ y_4 = 19$ and $y_x = 7,$ find x.

Solution:

$$x = \frac{(-5)(-12)}{(-8)(-15)}(1) + \frac{(3)(-12)}{(8)(-7)}(3) + \frac{(3)(-5)}{(15)(7)}(4)$$
$$= 1.86.$$

Exercise

1. Applying Lagrange's formula, find a cubic polynomial which approximates the following data:

x	-2	-1	2	3
$y(x)$	-12	-8	3	5

2. Given the data points (1,-3),(3,9),(4,30) and (6,132) satisfying the function $y = f(x)$, compute $f(5)$.
3. Given the table values

x	50	52	54	56
$\sqrt[3]{x}$	3.684	3.732	3.779	3.825

Use Lagrange's formula to find x when $\sqrt[3]{x} = 3.756.$

4. Find a real root of $f(t) = 0$, if $f(-1) = 2,\ f(2) = -2,\ f(5) = 4$ and $f(7) = 8.$
5. Use Lagrange's formula to find the value of $f(8)$ given

x	4	5	7	10	11	13
f (x)	48	100	294	900	1210	2028

Remark: In Lagrange's method, if an additional point is added to the existing data, we need to do the computations all over again. The n^{th} degree Lagrange polynomial obtained earlier will be of no use. This is the disadvantage of the Lagrange interpolation. However, Lagrange interpolation is a fundamental result and is used in proving many theoretical results of interpolation.

Newton's divided difference formula:

To overcome the disadvantage of Lagrange's method, Newton defined, what are known as divided differences, and derived an interpolation formula using these differences.

Divided differences:

If $(x_0,\ y_0),(x_1,\ y_1),(x_2,\ y_2),\ldots,(x_n,\ y_n)$ be the given points. Then the first order divided difference for the arguments x_0 and x_1 is defined by the relation

$$[x_0,\ x_1]=\frac{y_1-y_0}{x_1-x_0}$$

Similarly,

$$[x_1,\ x_2]=\frac{y_2-y_1}{x_2-x_1},\ [x_2,\ x_3]=\frac{y_3-y_2}{x_3-x_2},\ \ldots,\ [x_{n-1},\ x_n]=\frac{y_n-y_{n-1}}{x_n-x_{n-1}}.$$

The second order divided differences are defined as

$$[x_0,x_1,x_2]=\frac{[x_1,\ x_2]-[x_0,\ x_1]}{x_2-x_0},\ [x_1,x_2,x_3]=\frac{[x_2,\ x_3]-[x_1,\ x_2]}{x_3-x_1},\ \ldots,$$

$$[x_{n-2},x_{n-1},\ x_n]=\frac{[x_{n-1},\ x_n]-[x_{n-2},\ x_{n-1}]}{x_n-x_{n-2}}.$$

The third order divided differences are defined as

$$[x_0,x_1,x_2,x_3]=\frac{[x_1,x_2,x_3]-[x_0,x_1,x_2]}{x_3-x_0},\ \ [x_1,x_2,x_3,x_4]=\frac{[x_2,x_3,x_4]-[x_1,x_2,x_3]}{x_4-x_1},\ldots,$$

$$[x_{n-3}, x_{n-2}, x_{n-1}, x_n] = \frac{[x_{n-2}, x_{n-1}, x_n] - [x_{n-3}, x_{n-2}, x_{n-1}]}{x_n - x_{n-3}}.$$

Similarly higher order divided differences are also defined.

Divided difference table:

x	y	First Order Differences	Second Order Differences		n^{th} Order Differences
x_0	y_0				
		$[x_0, x_1]$			
x_1	y_1		$[x_0, x_1, x_2]$		
		$[x_1, x_2]$			
x_2	y_2		$[x_1, x_2, x_3]$		
		$[x_2, x_3]$			
x_3	y_3	.	.		
.	.			$\dots\ \dots$	$[x_0, x_1, \dots, x_n]$
.	.	.	.		
.	.	.	.		
		$[x_{n-2}, x_{n-1}]$	$[x_{n-3}, x_{n-2}, x_{n-1}]$		
x_{n-1}	y_{n-1}				
		$[x_{n-1}, x_n]$	$[x_{n-2}, x_{n-1}, x_n]$		
x_n	y_n				

Note: The divided differences are symmetric and independent of the order of the arguments. That is,

$$[x_0, x_1] = [x_1, x_0],$$

$$[x_0, x_1, x_2] = [x_2, x_1, x_0] = [x_0, x_2, x_1] = [x_1, x_2, x_0] = [x_1, x_0, x_2] = [x_2, x_0, x_1]$$

and similarly for higher order differences.

Newton's Divided Difference Interpolation Formula:

Let $y_0, y_1, \ldots, y_n$ be the values of $y = f(x)$ corresponding to the arguments $x_0, x_1, \ldots, x_n$. Then from the definition of divided differences, we have

$$[x_0, x_1] = \frac{y - y_0}{x - x_0} \Rightarrow y = y_0 + (x - x_0)[x, x_0] \quad \ldots\ldots\ldots(1)$$

Again

$$[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1} \Rightarrow [x, x_0] = [x_0, x_1] + (x - x_1)[x, x_0, x_1].$$

On using this in (1)

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1] \quad \ldots\ldots\ldots\ldots(2)$$

Also, from

$$[x, x_0, x_1, x_2] = \frac{[x, x_0, x_1] - [x_0, x_1, x_2]}{x - x_2}$$

$$\Rightarrow [x, x_0, x_1] = [x_0, x_1, x_2] + (x - x_2)[x, x_0, x_1, x_2]$$

Using this in (2), we get

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2]$$
$$+ (x - x_0)(x - x_1)(x - x_2)[x, x_0, x_1, x_2].$$

Proceeding in this manner, we arrive at

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2]$$
$$+ (x - x_0)(x - x_1)(x - x_2)[x, x_0, x_1, x_2] + \ldots$$
$$+ (x - x_0)(x - x_1)(x - x_2)\ldots(x - x_{n-1})[x_0, x_1, \ldots, x_n]$$
$$+ (x - x_0)(x - x_1)(x - x_2)\ldots(x - x_n)[x, x_0, x_1, \ldots, x_n]$$

This is called Newton's divided difference interpolation formula, the last term being the remainder term after (n + 1) terms.

Example 1: Given the values

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

Evaluate $f(8)$ using Newton's divided difference formula.

Solution:

x	$f(x)$	First Order Differences	Second Order Differences	Third order Differences	Fourth order Differences
4	48				
		52			
			15		
5	100				
		97		1	
			21		0
7	294			1	
		202			
10	900		27		0
		310		1	
11	1210		33		
		409			
13	2028				

Using divided difference formula, we get

$$f(x) = 48 + 52(x-4) + 15(x-4)(x-5) + 1(x-4)(x-5)(x-7)$$

$$f(8) = 448$$

Example 2: From the following table find f(x) and hence f(6) using Newton's interpolation formula:

x	1	2	7	8
$f(x)$	1	5	5	4

Solution:

x	y	First Order Differences	Second Order Differences	Third order Differences
1	1			
		4		
2	5		-2/3	
		0		1/14
7	5		-1/6	
		-1		
8	4			

On using divided difference formula, we get

$$f(x) = 1 + 4(x-1) + (-2/3)(x-1)(x-2) + 1/14(x-1)(x-2)(x-7)$$

$$= \frac{1}{42}(3x^3 - 58x^2 + 321x - 224)$$

$f(6) = 6.2380.$

Example 3: Find the equation y=f(x) of least degree and passing through the points (-1, -21), (1, 15), (2, 12), (3, 3). Find also y at x=0.

Solution:

x	y	First Order Differences	Second Order Differences	Third order Differences
-1	-21			
		18		
1	15		-7	
		-3		1
2	12		-3	
		-9		
3	3			

$$y = f(x) = -21 + (x+1)(18) + (x+1)(x-1)(-7) + (x+1)(x-1)(x-2)1$$
$$= x^3 - 9x^2 + 17x + 6$$

$y(0) = 6.$

Example 4: Find the Newton's divided differences polynomials for the data and also find $f(2.5)$.

X	-3	-1	0	3	5
$f(x)$	-30	-22	-12	330	3458

X	$f(x)$	First Order Differences	Second Order Differences	Third order Differences	Fourth order Differences
–3	–30				
		4			
–1	–22		2		
		10		4	
0	–12		26		5
		114			
3	330			44	
			290		
		1564			
5	3458				

On using Newton's divided difference polynomial, we have

$$y = f(x) = -30 + (x+3)4 + (x+3)(x+1)2 + (x+3)(x+1)(x-0)4$$

$$+(x+3)(x+1)x(x-3).$$

$$y = f(x) = 5x^4 + 9x^3 - 27x^2 - 21x - 12.$$

When $x = 2.5$, $y = 102.6785$.

Exercise

1. Fit an interpolating polynomial for the data $u_{10} = 355, u_0 = -5, u_8 = -21, u_1 = -14, u_4 = -125$ by using Newton's interpolation formula and hence evaluate u_2.

2. Construct the interpolation polynomial for the data given below using Newton's general interpolation formula for the divided differences

x	2	4	5	6	8	10
y	10	96	196	350	868	1746

3. Find $f(4.5)$ by using suitable interpolation

x	–1	0	2	5	10
$f(x)$	–2	–1	7	124	999

4. Fit a polynomial to the data (-4, 1245), (-1, 33), (0, 5), (2, 9), (5, 1335). Hence find $f(1)$ and $f(7)$

Numerical Differentiation:

Let $y_0, y_1, \ldots, y_n$ be the values of a function $y = f(x)$ corresponding to $x_0, x_1, \ldots, x_n$, the process of computing successive derivatives at some particular value of independent variable x is known as numerical differentiation.

The approximate values of these derivatives are obtained by differentiating an appropriate interpolation formula. If x is nearer to x_0 or nearer to x_n we use Newton's forward or backward interpolation formula provided the values $x_0, x_1, \ldots, x_n$ are equidistant. If the values $x_0, x_1, \ldots, x_n$ are at unequal intervals we use Newton's general interpolation formula.

Numerical Differentiation Using Newton's Forward Difference Formula

Let the given values of x be equidistant with step length h > 0. Newton's forward difference polynomial is given by

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots$$

$$\frac{dy}{dx} = \frac{dy}{dp}\frac{dp}{dx}$$

$$= \frac{1}{h}\left\{\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{24}\Delta^4 y_0 + \dots\right\}$$

……….(1)

Differentiating again,

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{\Delta^2 y_0 + (p-1)\Delta^3 y_0 + \frac{6p^2-18p+11}{12}\Delta^4 y_0 \dots\right\}$$

………(2)

In a similar way the higher order derivatives can be computed. The expressions (1) and (2) take a simpler form when the derivative is required at $x = x_0$, because at this point $p = 0$. Hence we get

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left\{\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \dots\right\}$$

and

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left\{\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \dots\right\}$$

The given values of x are equidistant and the given x is near to x_n, we use the backward difference table and consider Newton's backward interpolation formula.

$$y = f(x_n + ph) = y_n + p\nabla y_n + \frac{p(p+1)}{2}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \dots$$

Differentiating with respect to p, we obtain

$$\frac{dy}{dx}=\frac{1}{h}\left\{\nabla y_n+\frac{2p+1}{2}\nabla^2 y_n+\frac{3p^2+6p+2}{3!}\nabla^3 y_n+\ldots\right\} \qquad \ldots\ldots \qquad (3)$$

Again on differentiation, we have

$$\frac{d^2y}{dx^2}=\frac{1}{h^2}\left\{\nabla^2 y_n+(p+1)\nabla^3 y_n+\ldots\right\} \qquad \ldots\ldots.(4)$$

The expressions (3) and (4) take a simpler form when the derivative is required at $x = x_n$, because at this point $p = 0$. Hence we get

$$\left(\frac{dy}{dx}\right)_{x=x_n}=\frac{1}{h}\left\{\nabla y_n+\frac{1}{2}\nabla^2 y_n+\frac{1}{3}\nabla^3 y_n+\frac{1}{4}\nabla^4 y_n+\ldots\right\}$$

and

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n}=\frac{1}{h^2}\left\{\nabla^2 y_n+\nabla^3 y_n+\frac{11}{12}\nabla^4 y_n+\frac{5}{6}\nabla^5 y_n+\ldots\right\}$$

Example 1:

Given that

x	50	51	52	53	54	55	56
y	3.684	3.7084	3.7325	3.7563	3.7798	3.803	3.8259

find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 50$ and $x = 56$.

x	y	Δ	Δ^2	Δ^3
50	3.684			
		0.0244		
51	3.7084		-0.0003	
		0.0241		0
52	3.7325		-0.0003	
		0.0238		0
53	3.7563			
			-0.0003	
		0.0235		
54	3.7798			0
			-0.0003	
		0.0232		
55	3.803			0
			-0.0003	
		0.0229		
56	3.8259			

(a) At x = 50, h = 1,

$$\frac{dy}{dx} = \frac{1}{1}\left\{0.0244 - \frac{1}{2}(-0.0003) + \frac{1}{3}(0)\right\} = 0.02455.$$

$$\frac{d^2y}{dx^2} = 1[-0.0003] = -0.0003$$

(b) At x = 56, h = 1,

$$\frac{dy}{dx} = \frac{1}{1}\{0.0229 + 0.5\,(-0.0003) + 0\} = 0.02275.$$

$$\frac{d^2y}{dx^2} = 1[-0.0003] = -0.0003$$

Example 2: Given that

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250

Compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at x = 1.2 and x = 2.0.

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.0	2.7183						
		0.6018					
1.2	3.3201		0.1333				
		0.7351		0.0294			
1.4	4.0552		0.1627		0.0067		
		0.8978		0.0361		0.0013	
1.6	4.9530						
			0.1988		0.0080		0.0001
		1.0966					
1.8	6.0496			0.0441		0.0014	
			0.2429		0.0094		
		1.3395					
2.0	7.3891			0.0535			
			0.2964				
		1.6359					
2.2	9.0250						

(a) At x = 1.2, h = 0.2,

$$\frac{dy}{dx} = \frac{1}{0.2}\left\{0.7351 - \frac{1}{2}(0.1627) + \frac{1}{3}(0.0361) - \frac{1}{4}(0.0080) + \frac{1}{5}(0.0014)\right\}$$
$$= 3.3205$$

$$\frac{d^2y}{dx^2} = \frac{1}{(0.2)^2}\left\{0.1627 - 0.0361 + \frac{11}{12}(0.0080) - \frac{5}{6}(0.0014)\right\} = 3.318.$$

(b) At x=2.0, h = 0.2,

$$\frac{dy}{dx} = \frac{1}{0.2}\left\{1.3395 + \frac{1}{2}(0.2429) + \frac{1}{3}(0.0441) + \frac{1}{4}(0.0080) + \frac{1}{5}(0.0013)\right\}$$
$$= 7.3896$$

$$\frac{d^2y}{dx^2} = \frac{1}{(0.2)^2}\left\{0.2429 + 0.0441 + \frac{11}{12}(0.0080) - \frac{5}{6}(0.0013)\right\} = 7.3854.$$

Example 3: The population of a certain town is given below. Find the rate of growth of population in 1931 and 1971:

$Year$	1931	1941	1951	1961	1971
$Population$ (in 1000's)	40.62	60.8	79.95	103.56	132.65

x	y	Δ	Δ^2	Δ^3	Δ^4
1931	40.62				
		20.18			
1941	60.8		−1.03		
		19.15		5.49	
1951	79.95		4.46		−4.47
		23.61		1.02	
1961	103.56		5.48		
		29.09			
1971	132.65				

(i) At x=1931, h = 10,

$$\frac{dy}{dx} = \frac{1}{10}\left\{20.18 - \frac{1}{2}(-1.03) + \frac{1}{3}(5.49) - \frac{1}{4}(-4.47)\right\} = 2.36425$$

(ii) At x=1971, h = 10,

$$\frac{dy}{dx} = \frac{1}{10}\left\{29.09 + \frac{1}{2}(5.48) + \frac{1}{3}(1.02) + \frac{1}{4}(-4.47)\right\} = 3.10525$$

Exercise

1. A rod is rotating in a plane. The following table gives the angle θ in radians through which the rod has turned for various values of the time t seconds.

t	0	0.2	0.4	0.6	0.8	1.0	1.2
θ	0	0.12	0.49	1.12	2.02	3.2	4.67

Calculate the angular velocity and angular acceleration of the rod when t = 0.2 sec and t = 1.2 sec.

1. The following data gives corresponding values of pressure and specific volume of a superheated steam

v	2	4	6	8	10
p	105	42.7	25.3	16.7	13

Find the rate of change of pressure with respect to volume when $v = 2$.

2. Given the following table of values of x and y

x	1.00	1.05	1.10	1.15	1.20	1.25	1.30
y	1.0000	1.0247	1.0488	1.0723	1.0954	1.1180	1.1401

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1.00$.

4. A function is given according to the table below.

x	1.5	2.0	2.5	3.0	3.5	4.0
y	3.375	7.000	13.625	24.000	38.875	59.000

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx}$ at $x = 1.5$ and $x = 3.5$.

5. Given that

x	1.96	1.98	2.00	2.02	2.04
y	0.7825	0.7739	0.7651	0.7563	0.7473

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx}$ at $x = 1.98$ and $x = 2.04$.

6. From the following table of values of x and y:

x	1.00	1.05	1.10	1.15	1.20	1.25	1.30
y	1.0000	1.0247	1.0488	1.0723	1.0954	1.1180	1.1401

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx}$ at $x = 1.00$ and $x = 1.25$.

NUMERICAL INTEGRATION

Consider the definite integral

$$I = \int_a^b y\, dx$$

where y is known to be a function of x. If the function $y = f(x)$ is not known explicitly, or the function cannot be integrated by analytical methods, then we use numerical integration. The process of evaluating a definite integral from a set of tabulated values of the integrand $f(x)$ is called numerical integration. Given a set of data points $(x_0, y_0), (x_1, y_1), ..., (x_n, y_n)$ of a function $y = f(x)$, to evaluate the value of the definite integral $I = \int_a^b y dx$, we replace $f(x)$ by an interpolating polynomial $\phi(x)$ and on integration we obtain an approximate value of the definite integral. Thus we can obtain different integration formulae depending upon the type of interpolation formula used.

Newton-Cotes quadrature formula:

This formula is obtained by using Newton's forward difference interpolation formula.

We divide the interval $[a, b]$ into n equal subintervals, each of width h, such that $a = x_0 < x_1 < ... < x_n = b$. Then, $x_k = x_0 + kh,\ k = 1, 2, \ldots, n$.

Approximating y by Newton's forward difference formula, we obtain

$$I = \int_a^b y\, dx = \int_{x_0}^{x_n} \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 + ... \right] dx$$

Since $x=x_0+ph, dx=hdp$, when $x=x_0,\ p=0$ and when $x=x_n,\ p=n$, the above integral becomes

$$I=h\int_0^n\left[y_0+p\Delta y_0+\frac{p(p-1)}{2}\Delta^2 y_0+\frac{p(p-1)(p-2)}{6}\Delta^3 y_0+\ldots\right]dp$$

which gives, on simplification

$$\int_{x_0}^{x_n} ydx=nh\left[y_0+\frac{n}{2}\Delta y_0+\frac{n(2n-3)}{12}\Delta^2 y_0+\frac{n(n-2)^2}{24}\Delta^3 y_0+\ldots\right] \qquad (1)$$

This is known as Newton-Cotes quadrature formula. From this general formula we can obtain different integration formulae by taking n = 1, 2, 3, … .

Trapezoidal rule:

Put n = 1 in (1). Then all differences higher than the first will become zero and we obtain

$$\int_{x_0}^{x_1} ydx=h\left[y_0+\frac{1}{2}\Delta y_0\right]=h\left[y_0+\frac{1}{2}(y_1-y_0)\right]=\frac{h}{2}[y_0+y_1]$$

Similarly, for the interval $[x_1,x_2]$,

$$\int_{x_1}^{x_2} ydx=\frac{h}{2}[y_1+y_2]$$

$$\vdots$$

for the last interval $[x_{n-1},x_n]$ we have

$$\int_{x_{n-1}}^{x_n} ydx=\frac{h}{2}[y_{n-1}+y_n]$$

Summing up, we obtain

$$\int_{x_0}^{x_n} ydx=\frac{h}{2}\left[y_0+2(y_1+y_2+\ldots+y_{n-1})+y_n\right]$$

which is known as the trapezoidal rule.

Geometrical significance

The curve $y = f(x)$ is replaced by n straight line segments joining the points $(x_0, y_0)$and(x_1, y_1);$(x_1, y_1)$and(x_2, y_2);...;$(x_{n-1}, y_{n-1})$and(x_n, y_n). Then the area bounded by the curve $y = f(x)$, the ordinates $x = x_0$ and $x = x_n$ and the X-axis is approximately equal to the sum of the areas of the n trapeziums thus obtained.

Note: The error in the trapezoidal formula is

$$E = -\frac{(b-a)}{12} h^2 y''(\bar{x}) = O(h^2)$$

where $y''(\bar{x})$ denotes the largest value of the second derivative.

Simpson's 1/3 – rule:

Put n =2 in (1), we get

$$\int_{x_0}^{x_2} y dx = 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right] = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

Similarly, $\int_{x_2}^{x_4} y dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right]$

⋮

and finally $\int_{x_{n-2}}^{x_n} y dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right]$

Summing up, we obtain

$$\int_{x_0}^{x_n} y dx = \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + ... + y_{n-1}) + 2(y_2 + y_4 + ... + y_{n-2}) + y_n\right]$$

which is Simpson's 1/3 – rule.

Remark: While applying Simpson's 1/3 – rule, the given interval must be divided into even number of equal subintervals.

Note: The error in Simpson's 1/3 rule is

$$E = -\frac{(b-a)}{180} h^4 y^{iv}(\bar{x})$$

where $y^{iv}(\bar{x})$ denotes the largest value of the fourth derivative.

Simpson's 3/8 – rule:

Put $n = 3$ in (1), we obtain

$$\int_{x_0}^{x_3} y\,dx = 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right] = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

Similarly, $\displaystyle\int_{x_3}^{x_6} y\,dx = \frac{3h}{8}\left[y_3 + 3y_4 + 3y_5 + y_6\right]$

$$\vdots$$

and finally, $\displaystyle\int_{x_{n-3}}^{x_n} y\,dx = \frac{3h}{8}\left[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right]$

Summing up, we obtain

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-2} + y_{n-1}) + 2(y_3 + y_6 + \cdots + y_{n-3})\right]$$

Remark: Simpson's 3/8 – rule can be applied only if the number of sub-intervals is a multiple of 3.

Note: The error in Simpson's 3/8 – rule is

$$E = -\frac{3(b-a)}{80} h^4 y^{iv}(\bar{x})$$

where $y^{iv}(\bar{x})$ denotes the largest value of the fourth derivative.

Examples:

1. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using

(i) Trapezoidal rule (ii) Simpson's 1/3 rule (iii) Simpson's 3/8 – rule

by dividing the interval into six equal subintervals.

Solution:

Given n = 6, hence $h = \frac{b-a}{n} = \frac{6-0}{6} = 1$.

x	0	1	2	3	4	5	6
y	1 y_0	0.5 y_1	0.2 y_2	0.1 y_3	0.0588 y_4	0.0385 y_5	0.027 y_6

(i) By Trapezoidal rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{2}\left[(1+0.027)+2(0.5+0.2+0.1+0.0588+0.0385)\right] = 1.4108$$

(ii) By Simpson's 1/3 – rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{1}{3}\left[(1+0.027)+4(0.5+0.1+0.0385)+2(0.2+0.0588)\right] = 1.3662$$

(iii) By Simpson's 3/8 – rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3\right]$$

$$= \frac{3}{8}\left[(1+0.027)+3(0.5+0.2+0.0588+0.0385)+2(0.1)\right] = 1.3571$$

2. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Trapezoidal rule with h = 0.2. Hence obtain an approximate value of π.

x	0	0.2	0.4	0.6	0.8	1
y	1 y_0	0.96154 y_1	0.86207 y_2	0.73529 y_3	0.60976 y_4	0.5000 y_5

By Trapezoidal rule

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[\left(y_0+y_5\right)+2\left(y_1+y_2+y_3+y_4\right)\right]$$

$$= \frac{0.2}{2}\left[\left(1+0.5\right)+2\left(0.96154+0.86207+0.73529+0.60976\right)\right] = 0.783732$$

By actual integration,

$$\int_0^1 \frac{dx}{1+x^2} = (\tan^{-1} x)_0^1 = \frac{\pi}{4}$$

Hence $\frac{\pi}{4} \approx 0.783732$

$\pi \approx 3.13493$

3. A solid of revolution is formed by rotating about the x-axis the area between X-axis, the lines x=0 and x = 1 and a curve through the points with the following coordinates:

X	0.00	0.25	0.50	0.75	1.00
Y	1.0000	0.9896	0.9589	0.9089	0.8415

Estimate the volume of the solid formed.

Solution:

Volume $V = \pi \int_0^1 y^2 dx$

X	0.00	0.25	0.50	0.75	1.00
y^2	1.0000	0.9793	0.9195	0.8261	0.7081

$h = 0.25$.

Using Simpson's 1/3 rule,

$$V = \frac{\pi \times 0.25}{3}\left[1+4(0.9793+0.8261)+2(0.9195)+0.7081\right] = 2.8192.$$

4. A curve passes through the points (1, 2), (1.5, 2.4), (2, 2.7), (2.5, 2.8), (3, 3), (3.5, 2.6), (4, 2.1). Obtain the area bounded by the curve, X-axis, x=1 and x=4. Also find the volume of solid of revolution obtained by revolving this area about the X-axis.

Soln: Area = $\int_1^4 y dx$ where h = 0.5

$$= \frac{0.5}{3}\left[2+2.1+2(2.7+3)+4(2.4+2.8+2.6)\right] = 7.7833\, sq.units$$

Volume =

$$\frac{\pi \times 0.5}{3}\left[2^2+(2.1)^2+2\left(2.7^2+3^2\right)+4\left(2.4^2+2.8^2+2.6^2\right)\right] = 64.13\, cub.units$$

5. Compute the value of $\int_{0.2}^{1.4}\left(\sin x - \log x + e^x\right)dx$ using Simpson's 3/8th rule.

Solution:

Let $y = \sin x - \log x + e^x$ *and* $h = 0.2$, $n = 6$

The values of y are as given below:

x	0.2	0.4	0.6	0.8	1.00	1.2	1.4
y	3.0295	2.7975	2.8976	3.1660	3.5597	4.0698	4.4042

By Simpson's 3/8th rule, we have

$$\int_{0.2}^{1.4} \left(\sin x - \log x + e^x\right) dx = \frac{3h}{8}\left[\left(y_0 + y_6\right) + 3\left(y_1 + y_2 + y_4 + y_5\right) + 2y_3\right]$$

$$= \frac{3\times 0.2}{8}\left[(3.0295 + 4.4042) + 3(2.7975 + 2.8976 + 3.5597 + 4.0698) + 2(3.1660)\right]$$

$$= 4.0304$$

Exercise

1. Given that

x	4.0	4.2	4.4	4.6	4.8	5.00	5.2
$y = \log x$	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

Evaluate $\int_{4}^{5.2} y\,dx$ by

(i) Trapezoidal rule (ii) Simpson's 1/3 rule (iii) Simpson's 3/8 – rule

2. Evaluate $\int_{0}^{\frac{\pi}{2}} \sqrt{\cos\theta}\, d\theta$ using Simpson's 1/3 rule by taking 9 ordinates.

3. A reservoir discharging water trough sluices at a depth h below the water surface has a surface area A for various values of h as given below:

h (ft)	10	11	12	13	14
A (sq.ft)	950	1070	1200	1350	1530

If t denotes time t in minutes, the rate of flow of the surface is given by $\frac{dh}{dt} = -48\sqrt{\frac{h}{A}}$. Estimate the time taken for the water level to fall from 14 to 10ft above the sluices.

4. The velocities of a car (running on a straight road) at intervals of 2 minutes are given below.

Time in minutes	0	2	4	6	8	10	12
Velocity in km/hr	0	22	30	27	18	7	0

Apply Simpson's rule to find the distance covered by the car.

5. A curve is drawn to pass through the points given by the following table:

x	1	1.5	2	2.5	3	3.5	4
y	2	2.4	2.7	2.8	3	2.6	2.1

Estimate the area bounded by the curve, X-axis and the lines $x = 1$, $x = 4$.

6. A solid of revolution is formed by rotating about the x-axis, the area between the x-axis, the lines x=0 and x=1 and a curve through the points with the following co-ordinates:

x	0	0.25	0.50	0.75	1.00
y	1	0.9896	0.9589	0.9089	0.8419

Estimate the volume of the solid formed using Simpson's rule.