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Matrix Algebra

* A matrix is a rectangular array of elements. The Elements can be symbolic expressions or numbers. Matrix is denoted by [A].

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

- Row i, of [A] has n elements & is $[a_{i1}\ a_{i2} \cdots a_{in}]$
- Column j of [A] has m elements & is $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$
- Each matrix has rows & columns and this defines the size of the matrix. If a Matrix [A] has m rows & n columns, the size of matrix is denoted by "$m \times n$". The matrix [A] may also be denoted by $[A]_{m \times n}$ to show that [A] is a matrix with m rows & n columns.

Types of Matrix:

1. Row Matrix: It has only single row (i.e)

$a_{m \times n}$, $m=1$ $n \in N$

eg: $[1\ 2\ 3\ 4]_{1 \times 4}$

2. Column Matrix: It has only single column (i.e)

$a_{m \times n}$, $m \in N$, $n=1$

eg: $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}_{4 \times 1}$

③ Sub-Matrix:

If some rows or columns of a Matrix are deleted the remaining matrix is called a submatrix of {A}

eg: $\{A\} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times3}$ Submatrix of $\{A\} = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}_{2\times2}$

④ Square Matrix:

If the no of rows m of a matrix is equal to the no of columns n of a matrix ($m=n$) then, the matrix is called a Square Matrix.

⊙ The entries $a_{11}, a_{22}, \ldots, a_{mn}$ are called diagonal elements of a Square matrix.

eg: $\{A\} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times3}$

① $a_{m\times n}$, $m=n$.

⑤ Upper Triangular Matrix:

⊙ $[A]_{m\times n}$, for which $a_{ij}=0$, $i>j$ is called upper Triangular Matrix.

⊙ That means the entries below diagonal entries are zero.

eg: $\{A\} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix}_{3\times3}$

⑥ Lower Triangular Matrix:

$\{A\}_{m\times n}$, $a_{ij}=0$ $i<j$ is Lower Δ^{le} Matrix. That means the entries above diagonal entries are zero.

eg: $\{A\} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 5 & 0 \\ 3 & 4 & 9 \end{bmatrix}_{3\times3}$

(7) Diagonal Matrix:

$A_{m\times n}$, $a_{ij}=0$, $i>j$ & $i<j$ then it is diagonal matrix. That means only diagonal Entries exist.

eg: $[A] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3\times 3}$.

(8) Identity Matrix:

$A_{m\times n}$, $a_{ij}=1$, $i\in N, j\in N$ than it is Identity Matrix. That means all entries in a Square matrix are equal to 1.

eg: $[A] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{3\times 3}$.

(9) Zero Matrix:

$A_{m\times n}$, $a_{ij}=0$ $\forall\, i,j\in N$. then it is zero Matrix. That means all entries are equal to zero.

eg: $[A] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$.

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(10) Tri-diagonal Matrix:

It is a square Matrix in which all elements not on the following are zero: the major diagonal, the diagonal above the major diagonal & diagonal below major diagonal.

eg: $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 5 & 0 & 0 \\ 7 & 2 & 8 & 0 \\ 0 & 6 & 3 & 10 \\ 0 & 0 & 9 & 4 \end{bmatrix}$

(11) Diagonally Dominant matrices:

A $n \times n$ square matrix $\{A\}$ is a diagonally dominant matrix if:

$$|a_{ii}| \geq \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}| \quad \forall\, i = 1, 2, \ldots, n$$

$$\& \ |a_{ii}| > \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}| \text{ for atleast one } i.$$

i.e, for each row, the absolute value of diagonal element is greater than or equal to the sum of the absolute values of the rest of elements of that row and that the inequality is strictly greater than for atleast one row.

Diagonally dominant matrices are important in ensuring convergence in iterative schemes of solving simultaneous linear eqn's.

Eg:

$$\left.\begin{aligned} 10x + 5y + 5z &= 10 \\ 7x + 10y + 3z &= 20 \\ x + 4y + 10z &= 30 \end{aligned}\right\} \quad \begin{aligned} a_{11}x + a_{12}y + a_{13}z &= 10 \\ a_{21}x + a_{22}y + a_{23}z &= 20 \\ a_{31}x + a_{32}y + a_{33}z &= 30 \end{aligned}$$

$$\begin{aligned} |a_{11}| &> |a_{12}| + |a_{13}| \\ |a_{22}| &> |a_{21}| + |a_{23}| \\ |a_{33}| &> |a_{31}| + |a_{32}| \end{aligned} \Rightarrow \begin{aligned} |10| &\geq |5| + |5| \\ |10| &\geq |7| + |3| \\ |10| &\geq |1| + |4| \end{aligned}$$

for atleast one only's ✓

Minor of a Matrix:-

* Let $A = (a_{ij})_{m \times n}$ is a matrix & B is its Sub-matrix of order r, then |B| the determinant is called an r-rowed minor of A.

* Let $A = (a_{ij})_{m \times n}$ be a matrix. A positive integer r is said to be the rank of matrix A if Matrix A have atleast one r-rowed minor which is different from zero, every (r+1) row minor of Matrix A is zero. Let $A = (a_{ij})_{m \times n}$ is a matrix & B is its Sub-matrix of order r then |B| the determinant is called an r- rowed minor of A

* Determinant of any Square Sub-matrix of matrix A is called minor of A.

eg:- $A = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}_{3 \times 3}$ minor:- $\begin{vmatrix} 4 & 2 \\ 2 & 3 \end{vmatrix}_{2 \times 2}$ $\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}_{2 \times 2}$ $\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix}_{2 \times 2}$

Rank of a Matrix:-

* Rank of a Matrix A is the positive integer "r" such that there exists atleast one r rowed square matrix with non vanishing determinant while every (r+1) or more rowed matrices having vanishing determinants.
* Thus Rank of a Matrix is the largest order of a non zero minor of matrix.
* The Rank of matrix is denoted by ρ(A).

* A matrix is said to be of rank "r" when it has atleast one nonzero minor of order "r"
(*) every minor of order higher than r vanishes

eg: $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}_{3\times3}$

$|A| = 1[20-12] - 2[5-4] + 3[6-8]$

$= 8 - 2 - 6 = 0.$

$\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}_{2\times2} = 2$ $\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}_{2\times2} = -1$ $\therefore \boxed{\rho(A) = 2}$

$|1|_{1\times1} = 1$ $|2|_{1\times1} = 2$

⇒ **Steps to find Rank of a Matrix by Minor Method:**

1. If a Matrix contains atleast one zero elements then $\rho(A) \geq 1$
2. The rank of the identity matrix is n.
3. If the rank of Matrix A is r, then there exists atleast one minor of order r which does not vanish. Every minor of matrix A of order (r+1) & higher order (if any) vanishes.
4. If A is a matrix of m×n, then $\rho(A) \leq \min\{m,n\}$
5. A square matrix A of order n has to inverse iff $\rho(A) = n$

Simple language: 3×3 $|A| \neq 0$ Rank = 3

$|A| = 0$, minor any one $\neq 0$ Rank = 2

$|A| = 0$, All minor = 0, element $\neq 0$ Rank = 1.

⇒ **Steps to find the Rank of Matrix by Echelon form:-**

① The first Element of every non zero row should be 1.

② The row in which every element is zero, then that row should be below the non zero rows.

③ Total no of zeros in the next non zero row should be more than the no of zeros in the previous non zero row.

The no of non zero rows in the echelon form gives the rank matrix.

→ By elementary operations, we can easily bring the given matrix to the echelon form.

Note:- The rank of matrix does not change if we perform the following elementary row operations are applied to the matrix

⇒ Transformations:-

① Two rows are interchanged $(R_i \leftrightarrow R_j)$

② A row is multiplied by a non-zero constant,

$(R_i \leftrightarrow KR_i)$ where $K \neq 0$

③ A constant multiple of another row is added to a given row.

$(R_i \longrightarrow R_i \pm KR_j)$, where $i \neq j$

Note:

① Rank of A & A^T are Same.

② Rank of null matrix is 0.

③ For a rectangular matrix A of order $m \times n$, rank of A $\leq \min(m,n)$ (ie) rank cannot exceed the smaller of m & n.

④ For a n-Square matrix, if rank = n then $|A| \neq 0$ (i.e) A is non-Singular.

⑤ For any Square matrix, if rank < n, then $|A| = 0$ (ie) A is Singular.

Ex: ① $A = \begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -1.5 \end{bmatrix}$

Sol: By minor method:

$|A|_{3\times 3} = 0 \quad \begin{vmatrix} 4 & 2 \\ 8 & 4 \end{vmatrix}_{2\times 2} = 0 \quad$ all $2\times 2 = 0 \quad |4| \neq 0$

$\therefore \rho(A) = 1.$

By Echelon form:

$\begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -1.5 \end{bmatrix} \xrightarrow{R_1 \to R_1/4} \begin{bmatrix} 1 & 1/2 & 3/4 \\ 8 & 4 & 6 \\ -2 & -1 & -1.5 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 2R_1]{R_2 \to R_2 - 8R_1}$

$\rho(A) = 1 \qquad \begin{bmatrix} 1 & 1/2 & 3/4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(2) Find the Rank of matrix $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}_{4\times 4}$

Sol: $R_2 \to R_2 - 2R_1$

$R_3 \to R_3 + R_1$

$R_4 \to R_4 - 2R_1$

$$\begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix} \xrightarrow[R_4 \to \frac{1}{3}R_4]{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore \rho(A) = 4$

(3) Determine the values of b such that rank $= 3$.

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ b & 2 & 2 & 2 \\ 9 & 9 & b & 3 \end{bmatrix}$$

Sol: $R_2 \to R_2 - 4R_1$, $R_4 \to R_4 - 9R_1$

$R_3 \to R_3 - bR_1$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 2-b & 2+b & 2 \\ 0 & 0 & b+9 & 3 \end{bmatrix} \xrightarrow[R_3 \leftrightarrow \frac{R_3}{3}]{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2-b & 2+b & 2 \\ 0 & 0 & \frac{b+9}{3} & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_4 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2-b & 2+b & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \frac{9+b}{3} & 1 \end{bmatrix} \xrightarrow{R_4 - R_3} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2-b & 2+b & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \frac{9+b}{3} - 1 & 0 \end{bmatrix}$$

$\rho(A) = 3$

$\frac{9+b}{3} - 1 = 0$

$b = -6$

Solving System of linear eqn's :

* A general set of m linear eqn in n unknowns

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

* $\boxed{Ax = B}$

A = Coefficient matrix

x = Soln vector B = Right hand side vector

* $Ax = B$
  - Consistent System
    - Unique soln
    - Infinite soln
  - Inconsistent System → No soln

* $Ax = 0$ → Homogeneous system

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

↓ Echelon form.

① $\rho(A)$ = no of unknowns : System has Trivial solⁿ. (all unknowns = 0).

② $\rho(A) <$ no of unknowns : System has infinite no of soln.

eg: For ①

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \rho(A) = n$$

$$\underset{3}{} \qquad \underset{3}{}$$

$x_1 + 2x_2 + 3x_3 = 0$; $x_2 = 0$; $x_3 = 0$

$\therefore x_1 = x_2 = x_3 = 0$

For ②

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \rho(A) < n$$

$x_1 + 2x_2 + 3x_3 = 0$; $x_2 = 0$; $\therefore$ Infinite

let $x_3 = K$ then $x_1 = -3K \; \forall K \in R$

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④ $Ax = B \rightarrow$ Non Homogeneous System

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$[A|B] \rightarrow$ agumented matrix + echelon form $= \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$

$\rho(A)$

$\rho(A|B)$

① $\rho(A) = \rho(A|B) =$ no of unknowns $\therefore$ System has unique soln.

② $\rho(A) = \rho(A|B) <$ no of unknowns $\therefore$ System has infinite soln.

③ $\rho(A) \neq \rho(A|B) \therefore$ System has no soln.

eg: for ①

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 1 & 0 & 22 \\ 0 & 0 & 1 & 12 \end{array}\right]$$

$x+2y+3z=10$
$y=22$
$z=12$

for ②

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 1 & 6 & 22 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x+2y+3z=10$
$y+6z=22$

let $z=k$, $y=22-6k$, $x=10-2(22-6k)-3k$

for ③

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 24 \\ 0 & 0 & 1 & 42/5 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$x_1+2x_2+3x_3=24$
$x_2=\frac{42}{5}$
No soln.

⊛ There in Homogeneous:

$0x_1+0x_2+0x_3=0 \Rightarrow 0=0$ So we took let $x_3=k$. Infinite soln

But in non homogeneous:

$0x_1+0x_2+0x_3=1 \Rightarrow 0\neq 1$ So, no soln

Examples:

① $x+y+z=11$
$2x-6y-z=0$
$3x-4y+2z=0$

② $2x_1-2x_2+4x_3+3x_4=9$
$x_1-x_2+2x_3+2x_4=6$
$2x_1-2x_2+x_3+2x_4=3$
$x_1-x_2+x_4=2$

① Sol:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 11 \\ 2 & -6 & -1 & 0 \\ 3 & -4 & 2 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 11 \\ 0 & -8 & -3 & -22 \\ 0 & -7 & -1 & 33 \end{array}\right] \xrightarrow{R_2 \to -\frac{1}{8}R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 11 \\ 0 & 1 & 3/8 & 22/8 \\ 0 & -7 & -1 & -33 \end{array}\right] \xrightarrow{R_3 \to R_3 + 7R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 11 \\ 0 & 1 & 3/8 & \frac{22}{8} \\ 0 & 0 & 13/8 & \frac{-55}{4} \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 11 \\ 0 & 1 & \frac{3}{8} & \frac{22}{8} \\ 0 & 0 & 1 & \frac{-55}{4}\times\frac{8}{13} \end{array}\right]$$

$\rho(A) = \rho(A|B) = n = 3 \to$ unique solⁿ

$x + y + z = 11$

$y + \frac{3}{8}z = \frac{22}{8}$

$z = \frac{-55 \times 2}{13}$

⇒ Gauss elimination method: (methods to solve system of eqⁿ's)

$$Ax = B$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Two steps: ① Forward Elimination
② Back Substitution.

⊛ If $Ax = B$ then write augmented matrix $[A|B]$ & change $[A|B]$ into "upper triangular matrix" & starting entries need not to be 1. & find solⁿ's.

Ex: (1) $\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$

Sol:

$$\downarrow$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

$25x_1 + 5x_2 + x_3 = 106.8$ — (1)

$-4.8x_2 - 1.56x_3 = -96.21$ — (2)

$0.7x_3 = 0.735$ — (3)

solve (1), (2) & (3)

(2)

$$[A|B] = \left[\begin{array}{cccc|c} 2 & -2 & 4 & 3 & 9 \\ 1 & -1 & 2 & 2 & 6 \\ 2 & -2 & 1 & 2 & 3 \\ 1 & -1 & 0 & 1 & 2 \end{array}\right]$$

Sol:

$$\downarrow R_1 \leftrightarrow R_4$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 2 \\ 1 & -1 & 2 & 2 & 6 \\ 2 & -2 & 1 & 2 & 3 \\ 2 & -2 & 4 & 3 & 9 \end{array}\right] \xrightarrow[\substack{R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 2R_1}]{R_2 \to R_2 - R_1} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 4 & 1 & 5 \end{array}\right]$$

$$\downarrow R_3 \leftrightarrow R_2$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right] \xleftarrow[\substack{R_4 \to R_4 - 4R_2 \\ R_4 \to R_4 - R_3}]{R_3 \to R_3 - 2R_2} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 4 & 1 & 5 \end{array}\right]$$

$\rho(A) \neq \rho(A|B)$

(3) Determine the values of a & b for which the system has : (i) no soln
(ii) Infinite no of soln
(iii) unique soln

$x+2y+3z=6$

$x+3y+5z=9$

$2x+5y+az=b$

Sol: $[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 1 & 3 & 5 & 9 \\ 2 & 5 & a & b \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & a-6 & b-12 \end{array}\right]$

$\downarrow R_3 \to R_3 - R_2$

$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & a-8 & b-15 \end{array}\right]$

(1) $\rho(A) \neq \rho(A|B)$

$a=8, b\neq 15$

(2) $\rho(A) = \rho(A|B) < n$

$2 = 2 < 3$

$a=8, b=15$

(3) $\rho(A) = \rho(A|B) = n = 3$

$a \neq 8$

because if $b=15$ also $\rho(A|B)=3$ as $a\neq 8$ is there in last row.

(4) $4x-2y+6z=8$

$x+y-3z=-1$

$15x-3y+9z=21$

(5) $x+y-3z+2w=0$

$2x-y+2z-3w=0$

$3x-2y+z-4w=0$

$-4x+y-3z+w=0$

Homogeneous Question only ~~echelon form~~ upper-triangle form for coefficient matrix

⑥ Solve the system by Gauss elimination

$$3x+3y+2z=1$$
$$x+2y=4$$
$$10y+3z=-2$$
$$2x-3y-z=5$$

Sol:

$$\left[\begin{array}{ccc|c} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 3 & 3 & 2 & 1 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$
$R_4 \to R_4 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{array}\right] \xrightarrow{R_2 \to -\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -\frac{2}{3} & 11/3 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{array}\right]$$

$$\xrightarrow[R_4 \to R_4 + 7R_2]{R_3 \to R_3 - 10R_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -2/3 & 11/3 \\ 0 & 0 & 29/3 & -116/3 \\ 0 & 0 & -17/3 & 68/3 \end{array}\right]$$

$R_4 \to R_4 + \frac{17}{29}R_3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -\frac{2}{3} & 11/3 \\ 0 & 0 & \frac{29}{3} & -116/3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x+2y=4$$
$$y-\frac{2}{3}z=\frac{11}{3}$$
$$\frac{29}{3}z=\frac{-116}{3}$$

$$\therefore x=2,\ y=1,\ z=-4$$

⇒ Gauss-Jordan method

#① $AX = B$

$A^{-1}AX = A^{-1}B$

$IX = A^{-1}B$

$\boxed{X = A^{-1}B}$

(*) $[A \mid I]$

$\downarrow$ By elementary row transformations.

$[I \mid A^{-1}]$

Simple steps of transformations:-

S-①: $a_{11} \to 1$

S-②: ~~R_2~~ By using R_1 change R_2 & R_3 to 0.

S-③: $a'_{22} \to 1$

S-④: By using R_2 change R_1 & R_3 to 0.

S-⑤: $a'_{33} \to 1$

S-⑥: By using R_3 change R_2 & R_1 to 0.

(*) To find unknowns:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B.$$

Ex: (1) Find the inverse of: $A = \begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

Sol: $[A|I] = \left[\begin{array}{ccc|ccc} 8 & 4 & 3 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$

$\downarrow R_3 \leftrightarrow R_1 \quad (\because a_{11} \to 1)$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 8 & 4 & 3 & 1 & 0 & 0 \end{array}\right]$$

$\downarrow R_2 \to R_2 - 2R_1$

$\downarrow R_3 \to R_3 - 8R_1 \quad (\because a_{21}, a_{31} \to 0)$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & -3 & -1 & 0 & 1 & -2 \\ 0 & -12 & -5 & 1 & 0 & -8 \end{array}\right]$$

$\downarrow R_2 \to R_2/-3 \quad (\because a_{22} \to 1)$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1/3 & 0 & -1/3 & 2/3 \\ 0 & -12 & -5 & 1 & 0 & -8 \end{array}\right]$$

$\downarrow R_1 \to R_1 - 2R_2$

$\downarrow R_3 \to R_3 + 12R_2 \quad (\because a_{12}, a_{32} \to 0)$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & 0 & 2/3 & -1/3 \\ 0 & 1 & 1/3 & 0 & -1/3 & 2/3 \\ 0 & 0 & -1 & 1 & -4 & 0 \end{array}\right]$$

$\downarrow R_3 \to -R_3 \quad (\because a_{33} \to 1)$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/3 & 0 & 2/3 & -1/3 \\ 0 & 1 & 1/3 & 0 & -1/3 & 2/3 \\ 0 & 0 & 1 & -1 & 4 & 0 \end{array}\right] \quad \left(\because \begin{array}{c} a_{13} \\ a_{23} \end{array} \to 0\right)$$

$$\Big\downarrow \begin{array}{l} R_1 \to R_1 - \frac{1}{3}R_3 \\ R_2 \to R_2 - \frac{1}{3}R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & -2/3 & -1/3 \\ 0 & 1 & 0 & 1/3 & -5/3 & 2/3 \\ 0 & 0 & 1 & -1 & 4 & 0 \end{array}\right]$$

$$\qquad I \qquad\qquad\qquad A^{-1}$$

(2) Solve the system of eq^n by Gauss Jordan Elimination.

(i) $2x_1 + 5x_2 + 2x_3 - 3x_4 = 3$

$3x_1 + 6x_2 + 5x_3 + 2x_4 = 2$

$4x_1 + 5x_2 + 14x_3 + 14x_4 = 11$

$5x_1 + 10x_2 + 8x_3 + 4x_4 = 4$

(ii) $x_1 + x_2 + x_3 + x_4 = 0$

$x_1 + x_2 + x_3 - x_4 = 4$

$x_1 + x_2 - x_3 + x_4 = -4$

$x_1 - x_2 + x_3 + x_4 = 2$

Sol (ii)

Gauss-Jordan Elim

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & -1 & 4 \\ 1 & 1 & -1 & 1 & -4 \\ 1 & -1 & 1 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow$$

$$I \qquad A^{-1}$$

$$\underbrace{\begin{bmatrix} -1/2 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & -1/2 \\ 1/2 & 0 & -1/2 & 0 \\ 1/2 & -1/2 & 0 & 0 \end{bmatrix}}_{A^{-1}} \underbrace{\begin{bmatrix} 0 \\ 4 \\ -4 \\ 2 \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}}_{X}$$

Ans

15/11/21

⇒ Iterative methods (approxm sol^n):

Gauss-Jacobi & Gauss-Seidal method

(*) First condition for both is the matrix should be diagonally dominant.

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

$|a_{ii}| \geq \sum_{i \neq j} |a_{ij}|$ ⇒ $|a_{11}| > |a_{12}| + |a_{13}|$

$|a_{22}| > |a_{21}| + |a_{23}|$

$|a_{33}| > |a_{31}| + |a_{32}|$

(*) If equal also ok but atleast one of the 3 conditions should be > only. (= => also diagonally dominant)

eg's: ①
$$\left.\begin{aligned} 12x + 5y + 3z &= 20 \\ 2x + 20y + 3z &= 12 \\ 3x + 4y - 20z &= 24 \end{aligned}\right\} \quad \begin{aligned} |12| &> |5| + |3| \\ |20| &> |2| + |3| \\ |-20| &> |3| + |4| \end{aligned}$$

② $3x + 4y - 20z = 24$ — ③

$2x + 20y + 3z = 12$ — ②

$12x + 5y + 3z = 20$ — ①

(*) If we observe in ex: ② in the question form they are not diagonally dominant but if we rearrange them as eq^n ①, ②, ③ they are diag domin. So, this we should do first in questions.

④ Gauss – Jacobi's method:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \quad -①$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \quad -②$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \quad -③$$

} diagonally dominant.

then write these eqn's as $x_1 = , x_2 = , x_3 =$

one thing while writing we have to consider ~~dia~~ diagonal elements variables only.

$$\Rightarrow \quad x_1 = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3\right]$$

$$x_2 = \frac{1}{a_{22}}\left\{b_2 - a_{21}x_1 - a_{23}x_3\right\}$$

$$x_3 = \frac{1}{a_{33}}\left\{b_3 - a_{31}x_1 - a_{32}x_2\right\}$$

Now we have to do iterations (i.e) approximation's.
for finding K+1 iteration of variables we have to take K^{th} knowing variable value.

$$\Rightarrow \quad x_1^{(K+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(K)} - a_{13}x_3^{(K)}\right]$$

$$x_2^{(K+1)} = \frac{1}{a_{22}}\left\{b_2 - a_{21}x_1^{(K)} - a_{23}x_3^{(K)}\right\}$$

$$x_3^{(K+1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(K)} - a_{32}x_2^{(K)}\right]$$

Initial approximation:

To find first iteration $k=0$.

we let $x_1^0 = 0, x_2^0 = 0, x_3^0 = 0$

& proceed & solve upto when the values come very near. (i.e) error should be very less.

Gauss-Seidals method:

⇒ Everything same as Gauss-Jacobi method but the differance is in Jacobi we used the term which we found in next iteration but in Seidals we use at next value onwards.

⇒ Let $x_1^0 = x_2^0 = x_3^0 = 0$.

$$\Rightarrow x_1^{(K+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(K)} - a_{13}x_3^{(K)}\right]$$

$$x_2^{(K+1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(K+1)} - a_{23}x_3^{K}\right]$$

$$x_3^{(K+1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(K+1)} - a_{32}(x_2)^{(K+1)}\right]$$

See we find x_1 & we use it in next x_2 & x_3 of same iteration.

⇒ By this no of iterations decrases & we can find answer in less no of iterations.

Absolute Relative Approximate Error:

$$|\varepsilon_a|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| \times 100$$

When this error becomes a minute value then it indicates our answer is approximately correct.

Ex ① Solve by G. Jacobi & G-Seidal method.

$$5x_1 - 2x_2 + 3x_3 = -1 \quad -①$$
$$-3x_1 + 9x_2 + x_3 = 2 \quad -②$$
$$2x_1 - x_2 - 7x_3 = 3 \quad -③$$

Sol:

$|5| \geq |-2| + |3|$

$|9| > |-3| + |1|$

$|-7| > |2| + |-1|$

G-Jacobi

$$x_1^{(K+1)} = \frac{1}{5}\left\{-1 + 2x_2^{(K)} - 3x_3^{(K)}\right\}$$

$$x_2^{(K+1)} = \frac{1}{9}\left\{2 + 3x_1^{(K)} - x_3^{(K)}\right\}$$

$$x_3^{(K+1)} = -\frac{1}{7}\left\{3 - 2x_1^{(K)} + x_2^{(K)}\right\}$$

let $x_1^0 = x_2^0 = x_3^0 = 0$.

	0th	Ist	IInd	IIIrd	IVth	Vth	VIth
x_1	0	-0.2	0.145	0.191	0.181	0.185	0.186
x_2	0	0.222	0.203	0.328	0.332	0.329	0.331
x_3	0	-0.428	-0.517	-0.416	-0.421	-0.424	-0.423

G-Seidal

$$x_1^{(K+1)} = \frac{1}{5}\left\{-1 + 2x_2^{(K)} - 3x_3^{(K)}\right\}$$

$$x_2^{(K+1)} = \frac{1}{9}\left\{2 + 3x_1^{(K+1)} - x_3^{(K)}\right\}$$

$$x_3^{(K+1)} = -\frac{1}{7}\left\{3 - 2x_1^{(K+1)} + x_2^{(K+1)}\right\}$$

	0th	Ist	IInd	IIIrd	IV
x_1	0	-0.2	0.167	0.191	0.185
x_2	0	0.155	0.334	0.333	0.333
x_3	0.	-0.507	-0.429	-0.421	-0.422

Jacobi: $\text{error} = \frac{0.186 - 0.185}{0.186} = 0.0053$

Seidal: $\text{error} = \frac{0.333 - 0.333}{0.333} = 0$

$\therefore x_1 = 0.18$

$x_2 = 0.33$

$x_3 = -0.42$

(2) $10x_1 - 2x_2 - x_3 - x_4 = 3$ — (1)

$-x_1 - x_2 + 10x_3 - 2x_4 = 27$ — (3)

$-x_1 - x_2 - 2x_3 + 10x_4 = -9$ — (4)

$-2x_1 + 10x_2 - x_3 - x_4 = 15$ — (2)

Sol: $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 0$.

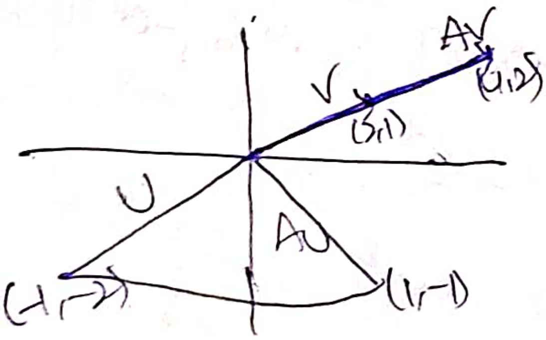
18/11/21

Eigen value & Eigen vector:

Eg: $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$ $U = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ $V = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$AU = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $AV = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$\Rightarrow \underset{A}{\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}} \underset{U}{\begin{bmatrix} -1 \\ -2 \end{bmatrix}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = AU$ ⊗ | $\underset{A}{\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}} \underset{V}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = AV = KV$



$A \times V = AV = KV$

K is eigen value

V is eigen vector

(*) we are taking 2 matrices A & X then:

$Ax = \lambda x$ for some scalar λ.

λ is eigen value

x is eigen vector.

$Ax - \lambda x = 0$

$\Rightarrow Ax - \lambda I x = 0 \Rightarrow \boxed{|A - \lambda I| = 0}$

* $Ax - \lambda x = 0$ → Homogeneous eq^n has 2 sol^{n}s → Trivial, NonTrivial

$Ax - \lambda I x = 0$

no change if we multiply identity matrix

$(A - \lambda I)x = 0$

→ Here $x \neq 0$ (NonTrivial) because it is variable there wouldn't be any meaning if $x = 0$.

→ for Trivial ⇒ $\boxed{|A - \lambda I| = 0}$

* Suppose there exist a scalar λ & non zero column matrix x of order n such that $Ax = \lambda x$, then λ is called eigen value of A & x is called corresponding eigen vector of matrix A.

⇒ $Ax = \lambda x$ — ①

$Ax = \lambda I x$

$\boxed{|A - \lambda I| = 0}$ → characteristic eq^n.

det ↳
$$\left| \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & \cdots & 0_{(n)} \\ 0 & 1 & \cdots & 0_{(n)} \\ \cdots & \cdots & & \cdots \\ 0_{(m)} & 0_{(m)} & \cdots & 1 \end{bmatrix} \right|$$

$$\Rightarrow \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (-1)^n \lambda^n + K_1 \lambda^{n-1} + K_2 \lambda^{n-2} + \cdots + K_n = 0.$$

(A) why we are taking $|A-\lambda I|=0$?

Ans: In previous page we saw one reason now it is another one.

As we know already it is Homogeneous eq^n

we find $A-\lambda I$ matrix

we have 2 possibilities!

① $\rho(A-\lambda I)=$ no of unknowns → Trivial sol^n

then the variable $X=0$

So this shouldn't happen because variable exists

(X) case.

② $\rho(A-\lambda I) <$ no of unknowns → Non Trivial sol^n
                                          ⇓
                                      Infinite sol^n

then variable $X \neq 0$

By observation if it is $n \times n$ matrix n variable exists (eg: 3×3 then 3 variable).

But we want $\rho(A-\lambda I)$ less (ie) $n-1$ (eg: 2)

So by minor method of finding rank:

det of $n=0$ & det of $n-1 \neq 0$ then Rank $= n-1$

So $\boxed{|A-\lambda I|=0}$

eg: $2x_1+3x_2+5x_3=0$ → $(2-\lambda)x_1+3x_2+5x_3=0$

$2x_1+4x_2+5x_3=0$ → $2x_1+(4-\lambda)x_2+5x_3=0$

$3x_1+5x_2+6x_3=0$ → $3x_1+5x_2+(6-\lambda)x_3=0$

$$\begin{bmatrix} 2 & 3 & 5 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{vmatrix} 2-\lambda & 3 & 5 \\ 2 & 4-\lambda & 5 \\ 3 & 5 & 6-\lambda \end{vmatrix} = 0$$

$\rho(A) < n$

$2 < 3$

$|A_{3\times3}| = 0$

⇒ Properties of eigen values & eigen vectors

① If A is real, its eigen values are real (or) Complex Conjugates in pairs.

→ Expanding the characteristic eqⁿ.

$$|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

☆ ② Sum of the diagonal elements of A = Trace of A = Sum of roots of the polynomial eqⁿ

☆☆ $\boxed{\text{Trace } A = \text{sum of all eigen values of } A}$

☆ ③ $|A|$ = product of eigen values of A

☆ ④ A & A^T has same eigen values.

⑤ If all the eigen values are non zero, then $|A| \neq 0$. ($\because |A|$ = product of all eigen values)

⑥ Similarly, if atleast one eigen value is zero then $|A| = 0$ ($\because |A|$ = product of all eigen values)

☆ ⑦ A^{-1} exists iff 0 is not an eigen value of A.

Eigen values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \cdots, \frac{1}{\lambda_n}$

{E: $Ax = \lambda x$

$A^{-1}Ax = A^{-1}\lambda x$

$Ix = A^{-1}\lambda x$

$x = \lambda A^{-1}x \Rightarrow A^{-1}x = \frac{1}{\lambda}x$ }

⑧ KA has eigen values $K\lambda$.

$\rightarrow$ (A has $\lambda_1, \lambda_2, \lambda_3$ eigenvalues then KA has $K\lambda_1, K\lambda_2, K\lambda_3$

$$K(A-\lambda I)X = 0 \Rightarrow |K(A-\lambda I)| = 0$$

$$\Rightarrow |KA - K\lambda I| = 0$$

⑨ A^m has eigen values λ^m

$\rightarrow$ $A = \lambda_1, \lambda_2, \lambda_3$

$A^2 = \lambda_1^2, \lambda_2^2, \lambda_3^2$

$A^m = \lambda_1^m, \lambda_2^m, \lambda_3^m$.

⑩ Characteristic vector cannot correspond to two distinct characteristic values.

$\rightarrow$ Suppose X_1 corresponds to λ_1 & λ_2 where $\lambda_1 \neq \lambda_2$ then

$(A-\lambda_1 I)X_1 = 0$ — ① $\quad (A-\lambda_2 I)X_1 = 0$ — ②

② − ① $\Rightarrow (\lambda_1 - \lambda_2) I X_1 = 0$

$X_1 = 0$ since $\lambda_1 \neq \lambda_2$

But ~~never~~ $X_1 \neq 0$ Hence a contradiction

$\rightarrow$ eg: $\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

For same A, for 2 values 1 & 2 we get 2 different eigen vectors.

Examples:

① Find the eigen values & corresponding eigenvectors of matrix: (i) $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}$

Sol: (i) $Ax = \lambda x$

$Ax = \lambda I x$

$(A - \lambda I)x = 0$

$|A - \lambda I| = 0$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

~~$\Rightarrow (8-\lambda)\{(7-\lambda)(3-\lambda)+6(6(3-\lambda)+8)+2(24-2(7-\lambda))\}$~~ -16

$$\Rightarrow \{(8-\lambda)((7-\lambda)(3-\lambda)-16)\} + (6(-6(3-\lambda)+8)) + (2(24-2(7-\lambda))) = 0$$

$$\Rightarrow -\lambda^3 + 18\lambda^2 - 45\lambda = 0$$

$$\Rightarrow \lambda(\lambda^2 - 18\lambda + 45) = 0$$

$\therefore \boxed{\lambda = 0, 3, 15}$

Check: $8+7+3$
$= 0+3+15 = 18$ ✓

$$Ax = \lambda x \Rightarrow \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Lengthly method

$\Rightarrow \lambda = 0 \Rightarrow Ax = 0x \Rightarrow$

$$8x_1 - 6x_2 + 2x_3 = 0x_1$$
$$-6x_1 + 7x_2 - 4x_3 = 0x_2$$
$$2x_1 - 4x_2 + 3x_3 = 0x_3$$

$\Downarrow$

Homogeneous System

find solⁿ.

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \xrightarrow{R_1/8} \begin{bmatrix} 1 & -\frac{6}{8} & \frac{2}{8} \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + 6R_1} \begin{bmatrix} 1 & -3/4 & 1/4 \\ 0 & 5/2 & -5/2 \\ 0 & -5/2 & 5/2 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & -3/4 & 1/4 \\ 0 & 5/2 & -5/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank < no of u
$2 < 3$
infinite soln

$\Rightarrow x_1 - 3/4x_2 + 1/4x_3 = 0$

$5/2x_2 - 5/2x_3 = 0$

let $x_3 = k$ $\Rightarrow x_2 = x_3 \Rightarrow x_2 = k$

$x_1 = \frac{3}{4}x_2 - \frac{1}{4}x_3 \Rightarrow x_1 = \frac{1}{2}k$

let & put $k = 1$ then: $\boxed{x_1 = \frac{1}{2}, x_2 = 1, x_3 = 1}$

$\therefore$ For $\lambda = 0$ eigenvalue, eigenvector is $\begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}$.

Another method:

$\Rightarrow \lambda = 3 \Rightarrow 8x_1 - 6x_2 + 2x_3 = 3x_1 \Rightarrow 5x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 + 7x_2 - 4x_3 = 3x_2 \quad -6x_1 + 4x_2 - 4x_3 = 0$

$2x_1 - 4x_2 + 3x_3 = 3x_3 \quad 2x_1 - 4x_2 - 0x_3 = 0$

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_2 \\ 5 & -6 & 2 & 5 & -6 & \\ -6 & 4 & -4 & -6 & 4 & \end{array} \Rightarrow \frac{x_1}{24-8} = \frac{x_2}{-12+20} = \frac{x_3}{20-36} = k$$

$\Rightarrow \frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16} = k \Rightarrow x_1 = 2k, x_2 = k, x_3 = -2k$

For $k=1$ $\therefore$ For $\lambda = 3 \Rightarrow \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ $\quad A\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

$\Rightarrow$ Similarly for $\lambda = 15 \Rightarrow \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

Rayleigh's Power method (Iterative method) :-

- In this method we find eigen values & eigen vectors (highest eigen value).
- First we will assume eigen vectors:-

$\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ as 0^{th} Iteration

$\Rightarrow A x^{(0)} = \lambda^{(1)} x^{(1)}$

we multiply marix A & x^0 then we will get another matrix

- In that matrix the highest value irrespective of sign (whether +ve (or) -ve) we take the values outside as common. {If it is -ve then -ve sign will be inside only}.

→ Now this value is 1^{st} Iterated λ & corresponding vector is x^1 1^{st} Iterated.

$\Rightarrow$ Next $A x^{(1)} = \lambda^{(2)} x^{(2)}$ - - - Continues.

$\Rightarrow$ we will stop when the Iterated eigen value & eigen vector are same then it is declared as answer.

→ In exam the assuming eigen vector is given & also they will tell upto which iteration we have to do.

$A :$ e.v $: \lambda$

$A^{-1} :$ e.v $: \frac{1}{\lambda}$

$\Rightarrow$ To find smallest value we should do:

$\frac{1}{\lambda}$ is smallest e.v of. A

$A^{-1} x^{(0)} = \lambda^{(1)} x^{(1)}$

$A^{-1} x^{(1)} = \lambda^{(2)} x^{(2)}$

The procedure is same we have to take common highest only but at last answer is smallest eigen value as we took A^{-1} in place of A.

Ex:

1. Find the largest eigen value & corresponding eigen vector with initial eigen vector $[1\ 0\ 0]^T$

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Sol:

$$A x^{(0)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \underset{\lambda^{(1)}}{2} \underset{x^{(1)}}{\begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix}}$$

$$A x^{(1)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 0 \\ 2 \end{bmatrix} = \underset{\lambda^{(2)}}{2.5} \underset{x^{(2)}}{\begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix}}$$

$$A x^{(2)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 0 \\ 2.6 \end{bmatrix} = \underset{\lambda^{(3)}}{2.8} \underset{x^{(3)}}{\begin{bmatrix} 1 \\ 0 \\ 0.93 \end{bmatrix}}$$

$$A x^{(3)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.93 \end{bmatrix} = \begin{bmatrix} 2.93 \\ 0 \\ 2.86 \end{bmatrix} = \underset{\lambda^{(4)}}{2.93} \underset{x^{(4)}}{\begin{bmatrix} 1 \\ 0 \\ 0.98 \end{bmatrix}}$$

$$A x^{(4)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.98 \end{bmatrix} = \begin{bmatrix} 2.98 \\ 0 \\ 2.96 \end{bmatrix} = \underset{\lambda^{(5)}}{2.98} \underset{x^{(5)}}{\begin{bmatrix} 1 \\ 0 \\ 0.99 \end{bmatrix}}$$

$$A x^{(5)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.99 \end{bmatrix} = \begin{bmatrix} 2.99 \\ 0 \\ 2.98 \end{bmatrix} = \underset{\lambda^{(6)}}{2.99} \underset{x^{(6)}}{\begin{bmatrix} 1 \\ 0 \\ 0.996 \end{bmatrix}}$$

$$A x^{(6)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.996 \end{bmatrix} = \begin{bmatrix} 2.996 \\ 0 \\ 2.992 \end{bmatrix} = \underset{\lambda^{(7)}}{2.996} \underset{x^{(7)}}{\begin{bmatrix} 1 \\ 0 \\ 0.99 \end{bmatrix}}$$

$\therefore \lambda^{(7)}$ is eigen value $x^{(7)}$ is eigen vector.