$$\iint_{R_{xy}} f(x,y)\,dx\,dy = \iint_{R'_{uv}} f(\phi(u,v),\psi(u,v))\,|J|\,du\,dv$$

LECTURE 8 - DATE : **28 MAY 2021**

1. PROBLEMS ON CHANGE OF VARIABLES

Problem 1.1. *Evaluate*

Let $$I = \int_{x=0}^{2}\int_{y=0}^{\sqrt{2x-x^2}} \frac{x}{x^2+y^2}\,dy\,dx.$$

by changing to polar coordinates.

Ans: Here y varies from the line $y=0$ to the curve $y=\sqrt{2x-x^2}$ ie; $x^2+y^2=2x$

$\Rightarrow x^2-2x+1+y^2=1$

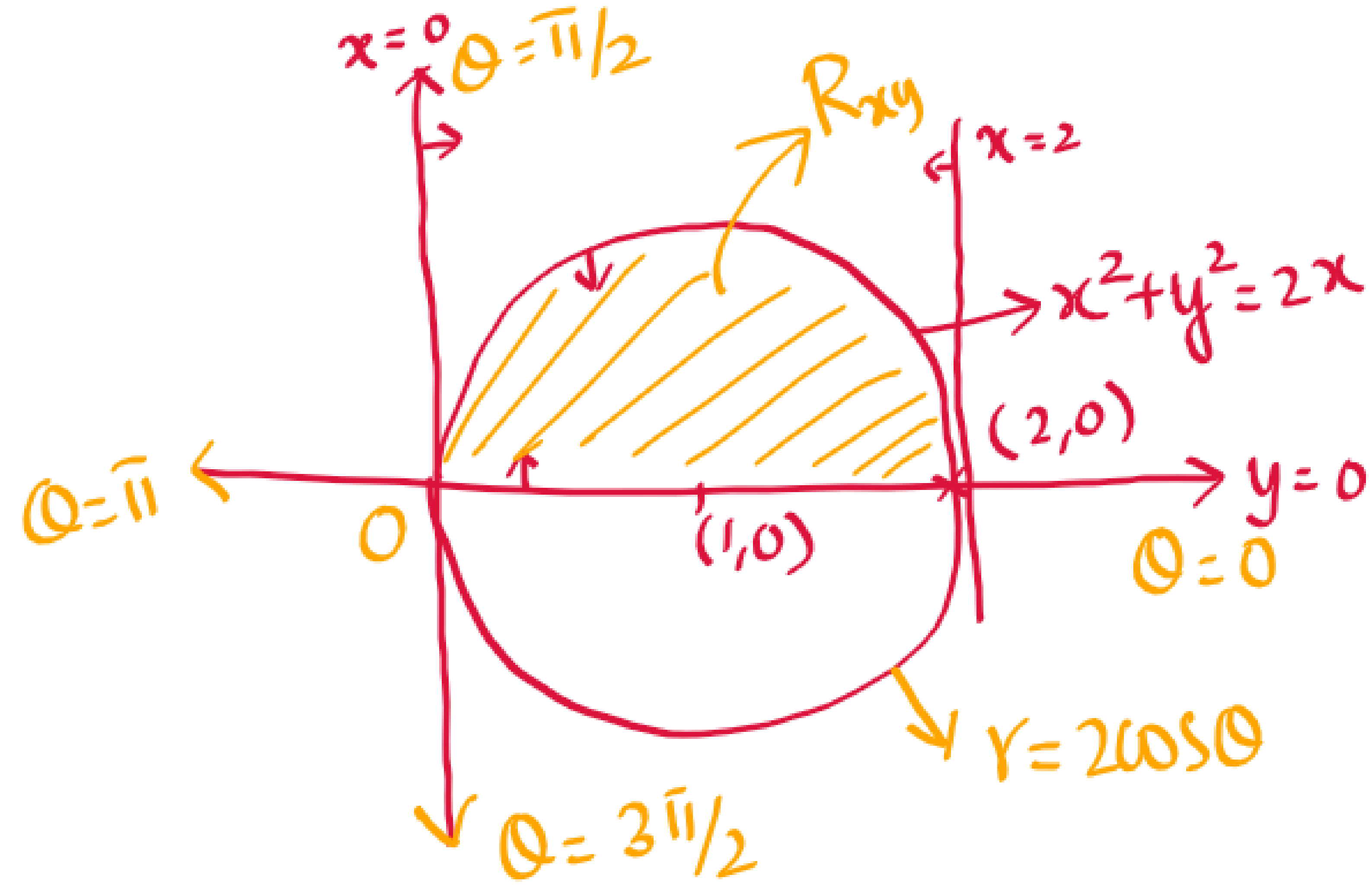
$\Rightarrow (x-1)^2+y^2=1$

x varies from the line $x=0$ to line $x=2$

By changing to polar coordinates put $x=r\cos\theta$

$y=r\sin\theta$

$dx\,dy = r\,dr\,d\theta$



$x^2+y^2=2x$

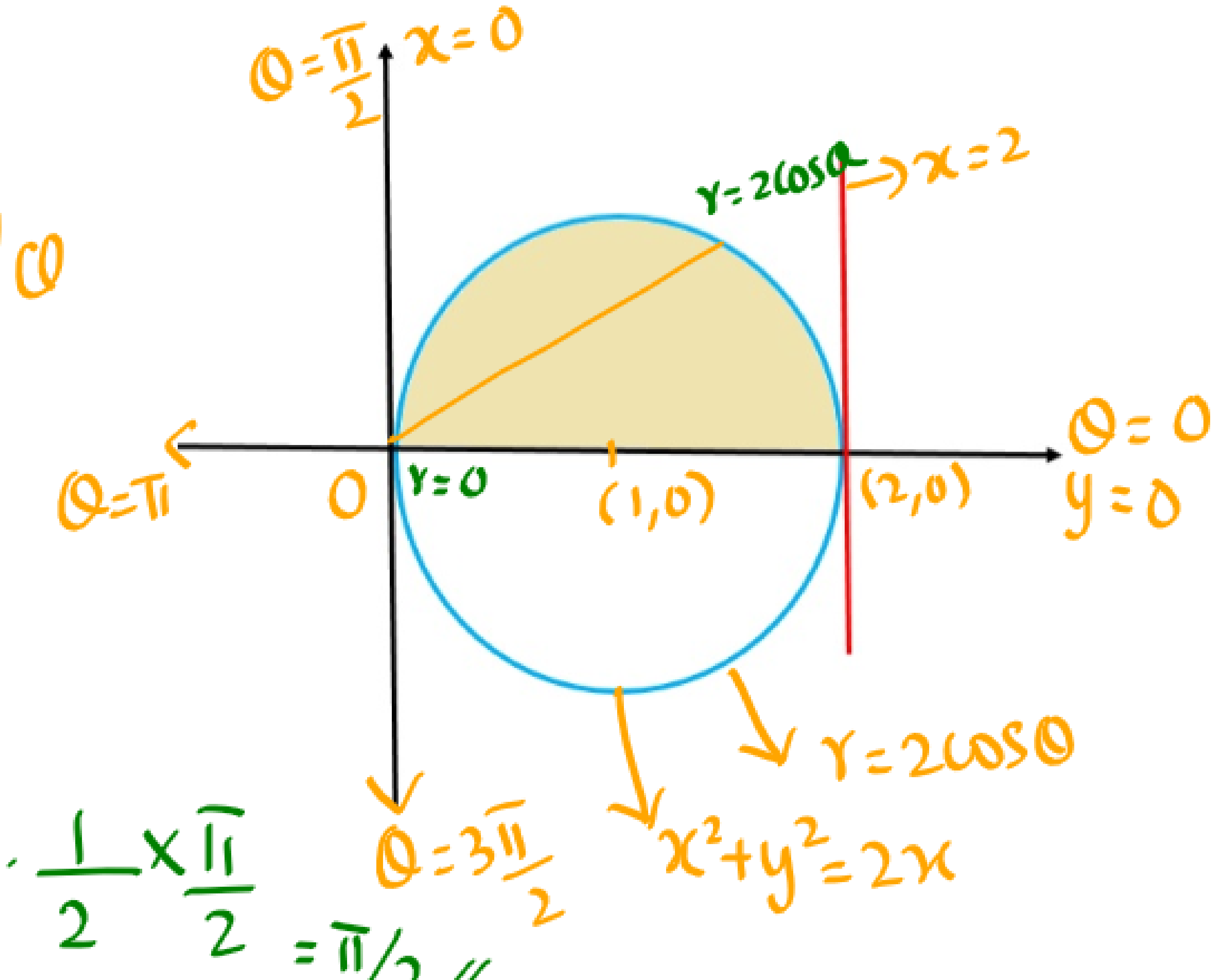
$\Rightarrow r^2=2r\cos\theta$

$\Rightarrow r=2\cos\theta$

$$\therefore I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{2\cos\theta} \frac{r\cos\theta}{r^2}\cdot r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2} \cos\theta\,(r)_0^{2\cos\theta}\,d\theta$$

$$= 2\int_0^{\pi/2} \cos^2\theta\,d\theta = 2\cdot\frac{1}{2}\times\frac{\pi}{2} = \pi/2$$



Problem 1.2. *Evaluate*

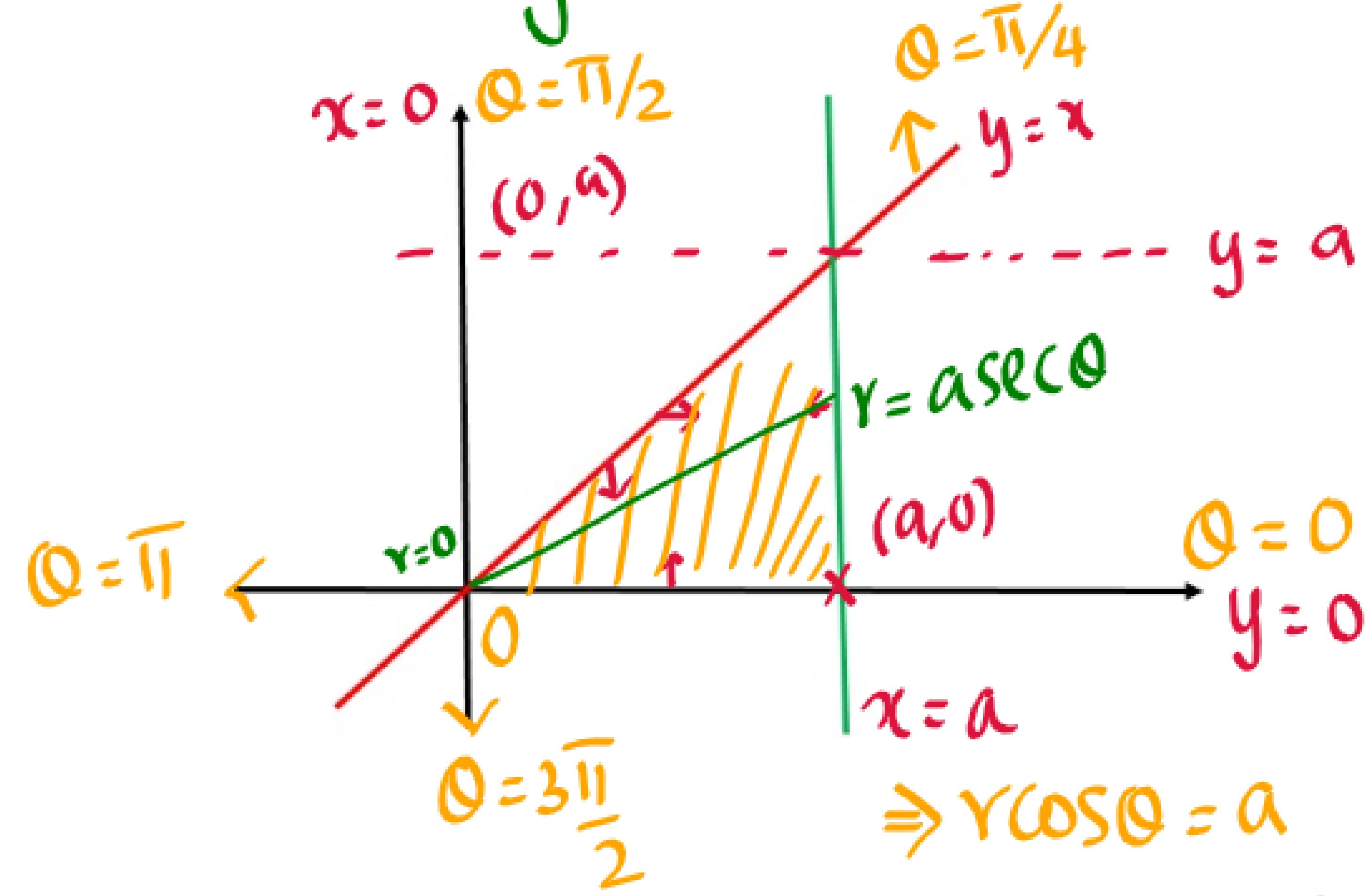
Let $I = \int_{y=0}^{a}\int_{x=y}^{a} \frac{x}{x^2+y^2}\,dx\,dy.$

by changing to polar coordinates. ; $a>0$

Ans:- x varies from the line $x=y$ to the line $x=a$
y varies from the line $y=0$ to the line $y=a$

By changing to polar coordinates $\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\}$

$dx\,dy = r\,dr\,d\theta$



$\Rightarrow r\cos\theta = a$

$\Rightarrow r = a\sec\theta$ ✓

$x=y \Rightarrow r\cos\theta = r\sin\theta \Rightarrow \tan\theta = 1$

$\Rightarrow \theta = \pi/4$

$$\therefore I = \int_{\theta=0}^{\pi/4}\int_{r=0}^{r=a\sec\theta} \frac{r\cos\theta}{r^2}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/4} \cos\theta\,(r)_0^{a\sec\theta}\,d\theta = a\int_0^{\pi/4} d\theta = \frac{\pi a}{4}$$

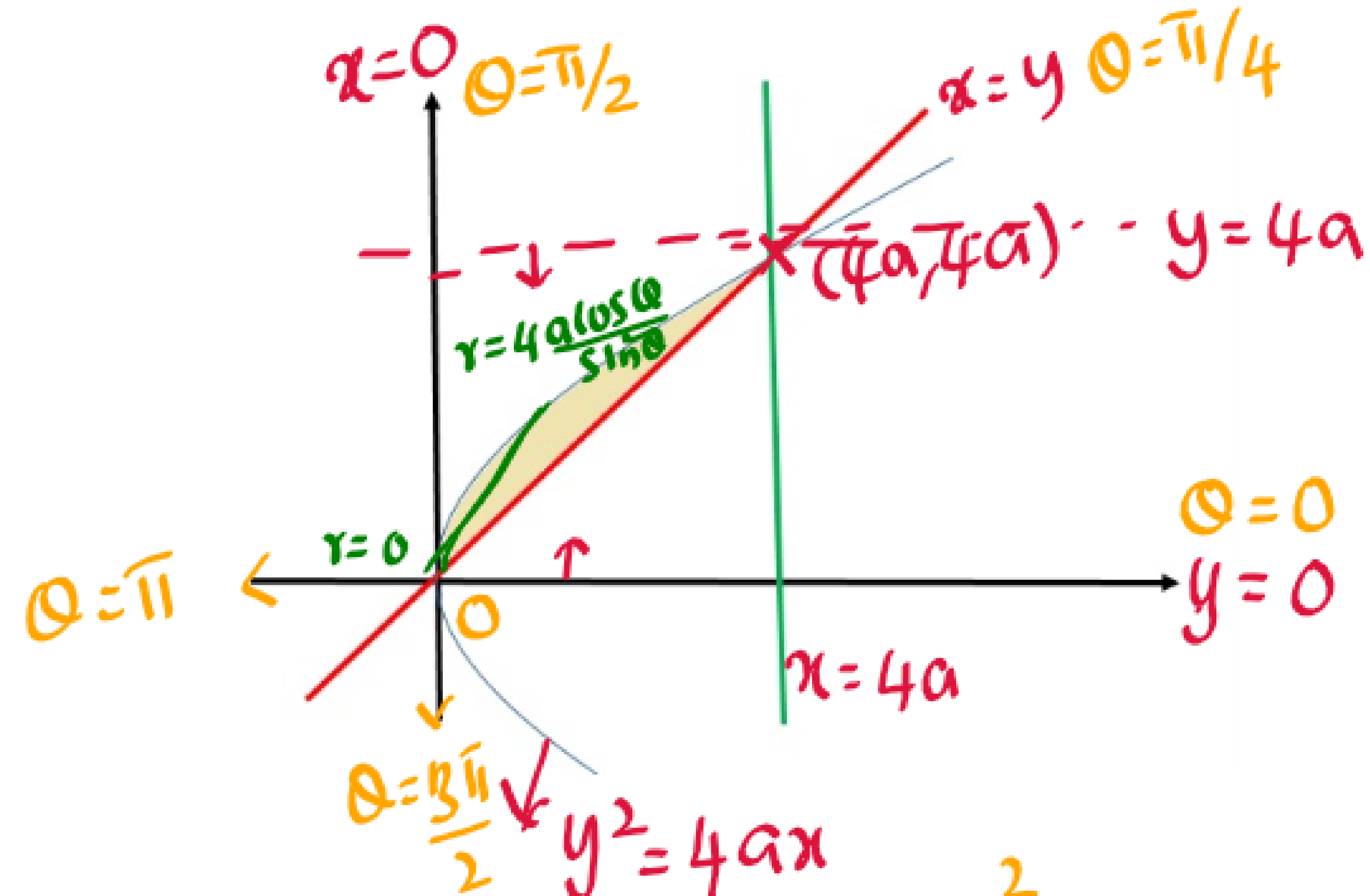
Problem 1.3. *Evaluate*

Let $I = \displaystyle\int_{y=0}^{4a} \int_{x=y^2/4a}^{y} \frac{x^2 - y^2}{x^2 + y^2}\, dx\, dy.$; $a>0$

by changing to polar coordinates.

Ans: Here x varies from the curve $x = y^2/4a$ i.e; $y^2 = 4ax$ to the line $x = y$

y varies from the line $y=0$ to line $y = 4a$.



By changing to polar coordinates $x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$\Rightarrow r^2\sin^2\theta = 4ar\cos\theta$

$\Rightarrow r = \dfrac{4a\cos\theta}{\sin^2\theta}$

$$\therefore I = \int_{\theta=\pi/4}^{\pi/2} \int_{r=0}^{r=\frac{4a\cos\theta}{\sin^2\theta}} (\cos^2\theta - \sin^2\theta)\, r\, dr\, d\theta$$

Also, $\dfrac{x^2-y^2}{x^2+y^2} = \dfrac{r^2(\cos^2\theta - \sin^2\theta)}{r^2}$

$= \cos^2\theta - \sin^2\theta$

$$= \int_{\theta=\pi/4}^{\pi/2} (\cos^2\theta - \sin^2\theta) \left(\frac{r^2}{2}\right)_0^{\frac{4a\cos\theta}{\sin^2\theta}} d\theta$$

$$= \frac{1}{2}\int_{\theta=\pi/4}^{\pi/2} (\cos^2\theta - \sin^2\theta)\, \frac{16a^2\cos^2\theta}{\sin^4\theta}\, d\theta \quad = 8a^2 \int_{\theta=\pi/4}^{\pi/2} (-2\sin^2\theta + 1)\, \frac{\cos^2\theta}{\sin^4\theta}\, d\theta$$

$$= 8a^2 \int_{\theta=\pi/4}^{\pi/2} \left(\frac{\cos^2\theta}{\sin^4\theta} - 2\cot^2\theta \right) d\theta$$

$$= 8a^2 \left[\int_{\theta=\pi/4}^{\pi/2} \frac{\sec^2\theta}{\tan^4\theta}\, d\theta - 2 \int_{\theta=\pi/4}^{\pi/2} (\operatorname{cosec}^2\theta - 1)\, d\theta \right]$$

$$= 8a^2 \left[\left(\frac{-1}{3\tan^3\theta} \right)_{\theta=\pi/4}^{\pi/2} - 2 \left[-\cot\theta - \theta \right]_{\theta=\pi/4}^{\pi/2} \right]$$

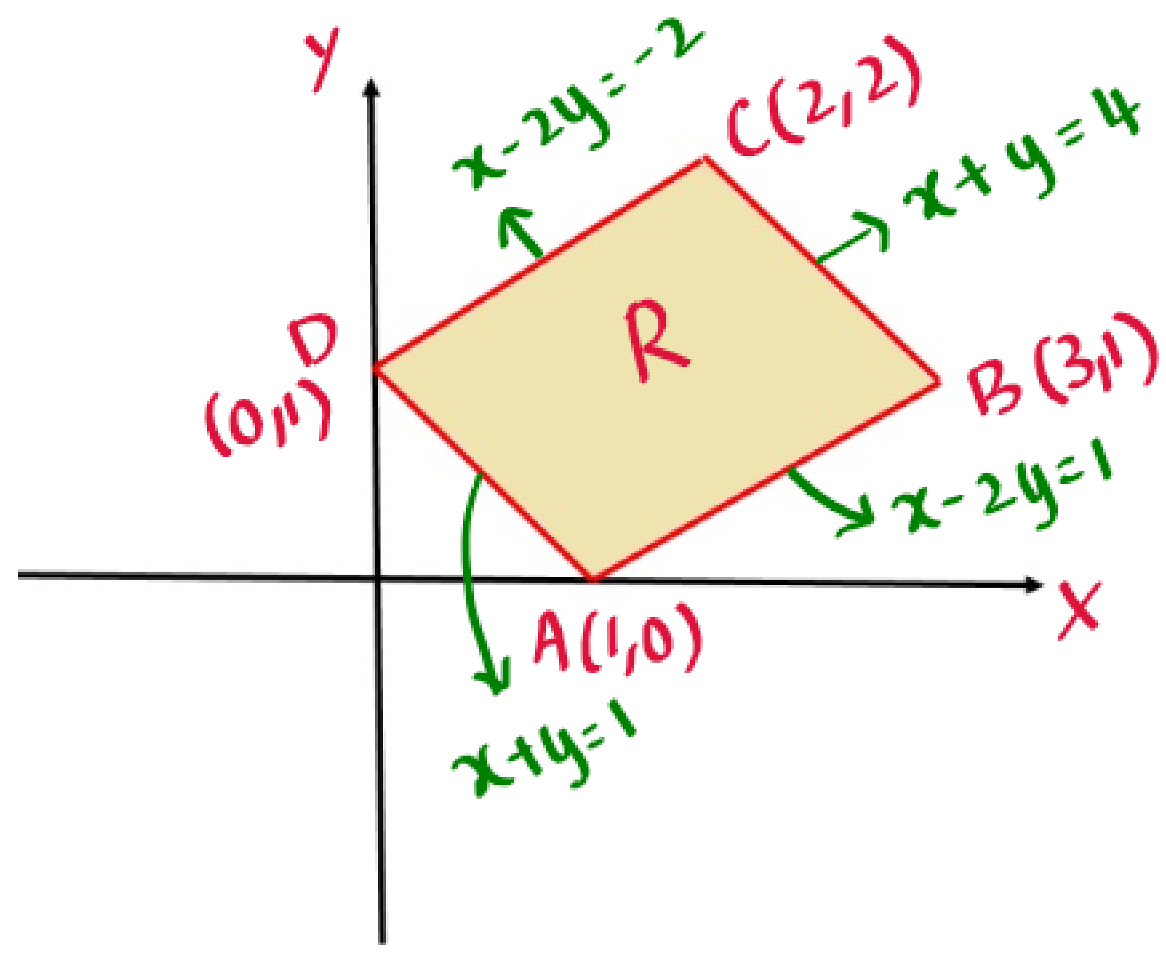
$$= 8a^2 \left[0 + \frac{1}{3} - 2 \left[0 - \frac{\pi}{2} + 1 + \pi/4 \right] \right]$$

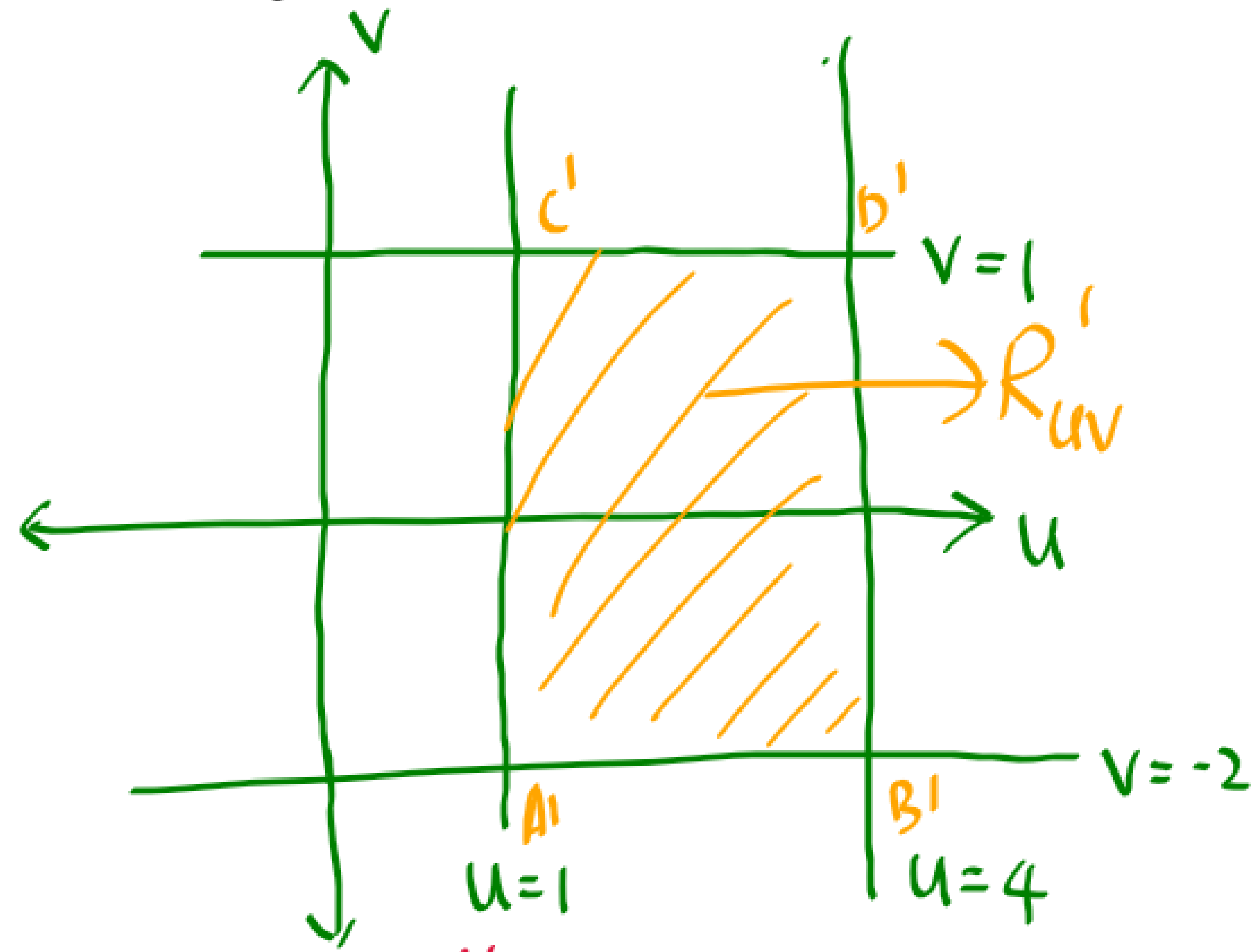
$$= 8a^2 \left[\frac{1}{3} + \pi - 2 - \frac{\pi}{2} \right] = 8a^2 \left[\frac{\pi}{2} - \frac{5}{3} \right]$$

Problem 1.4. *Evaluate*

$$\iint_R (x+y)^2 \, dx \, dy.$$

where R *is the parallelogram in the xy plane with vertices* $(1,0),(3,1),(2,2),(0,1)$ *using the transformation* $u=x+y$ *and* $v=x-2y$.





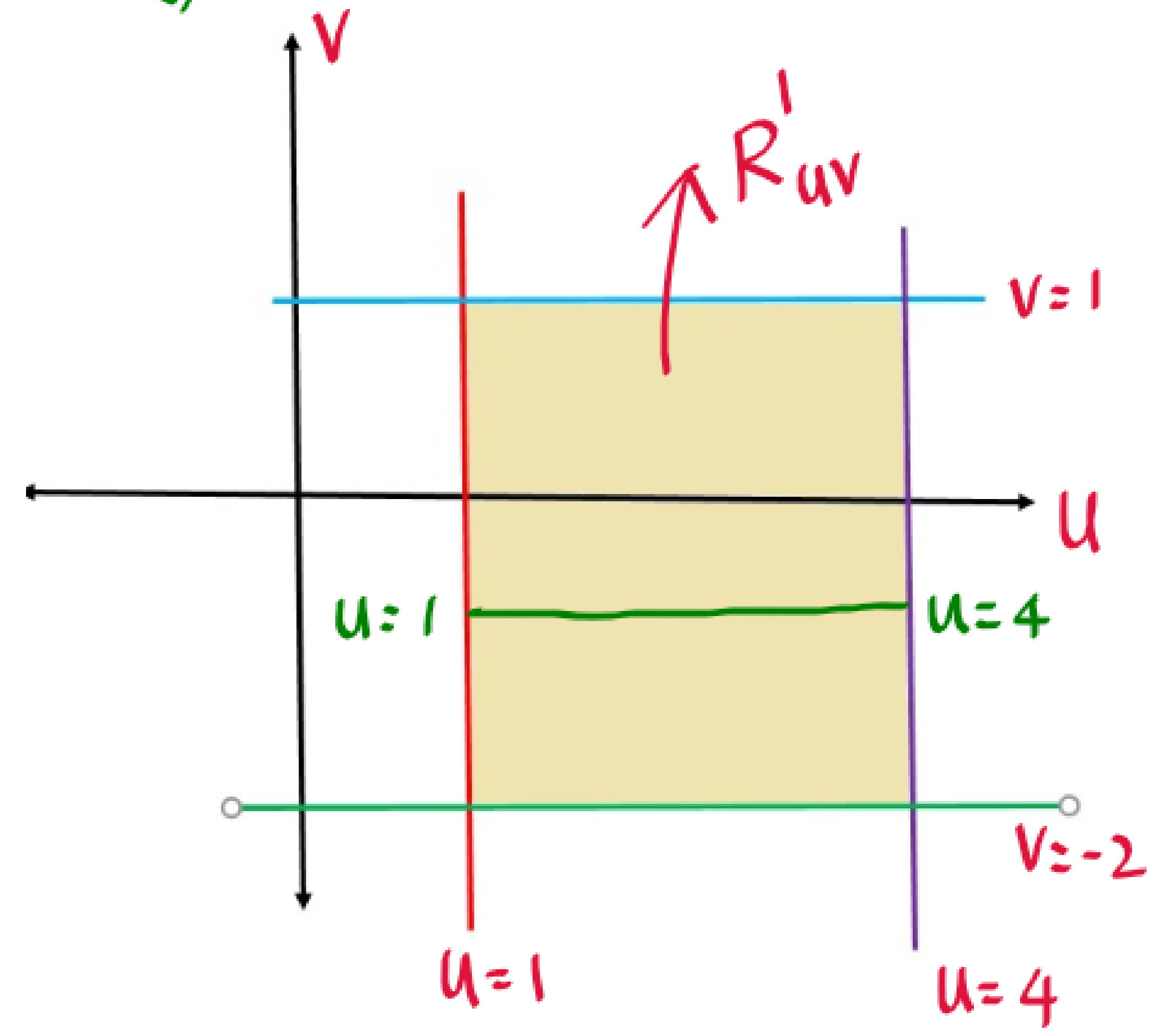
Given $u=x+y$ & $v=x-2y$.

<u>On AD</u> : $x+y=1 \Rightarrow u=1$

<u>On AB</u> : $x-2y=1 \Rightarrow v=1$

<u>On CB</u> : $x+y=4 \Rightarrow u=4$

<u>On CD</u> : $x-2y=-2 \Rightarrow v=-2$



$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{-1}{3} \end{vmatrix} = \frac{-1}{3}$$

$u = x+y$

$v = x-2y$

$x = \frac{1}{3}(2u+v)$

$y = \frac{1}{3}(u-v)$

$\therefore |J| = \frac{1}{3}$ $\quad \therefore I = \iint_{R'_{uv}} u^2 \cdot \frac{1}{3} \, du \, dv = \frac{1}{3} \int_{v=-2}^{1} \int_{u=1}^{4} u^2 \, du \, dv = \underline{\underline{21}}$

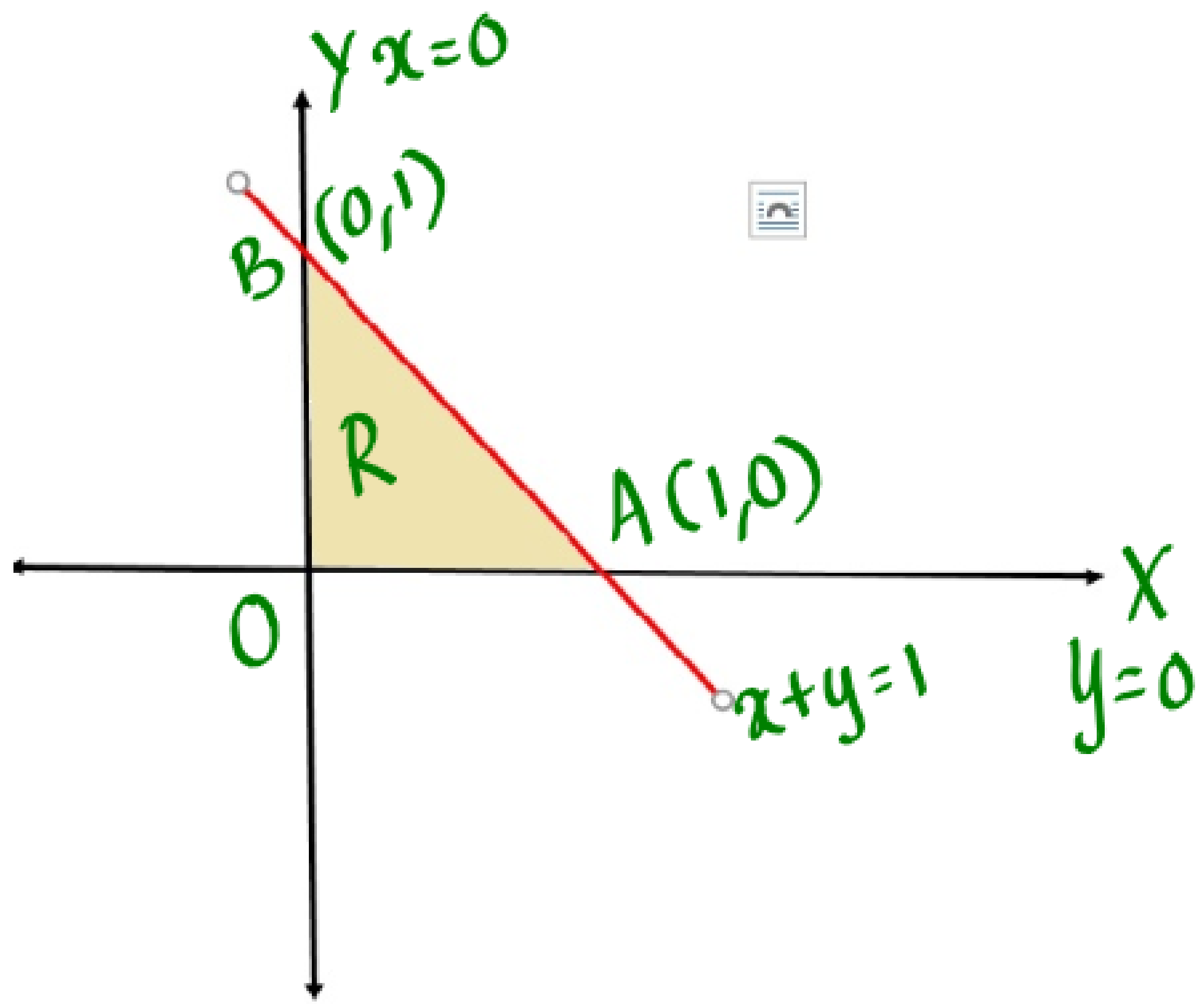
$$\int_{x=a}^{b}\int_{y=c}^{d} f(x)\,g(y)\,dy\,dx = \int_{x=a}^{b} f(x)\,dx \int_{y=c}^{d} g(y)\,dy$$

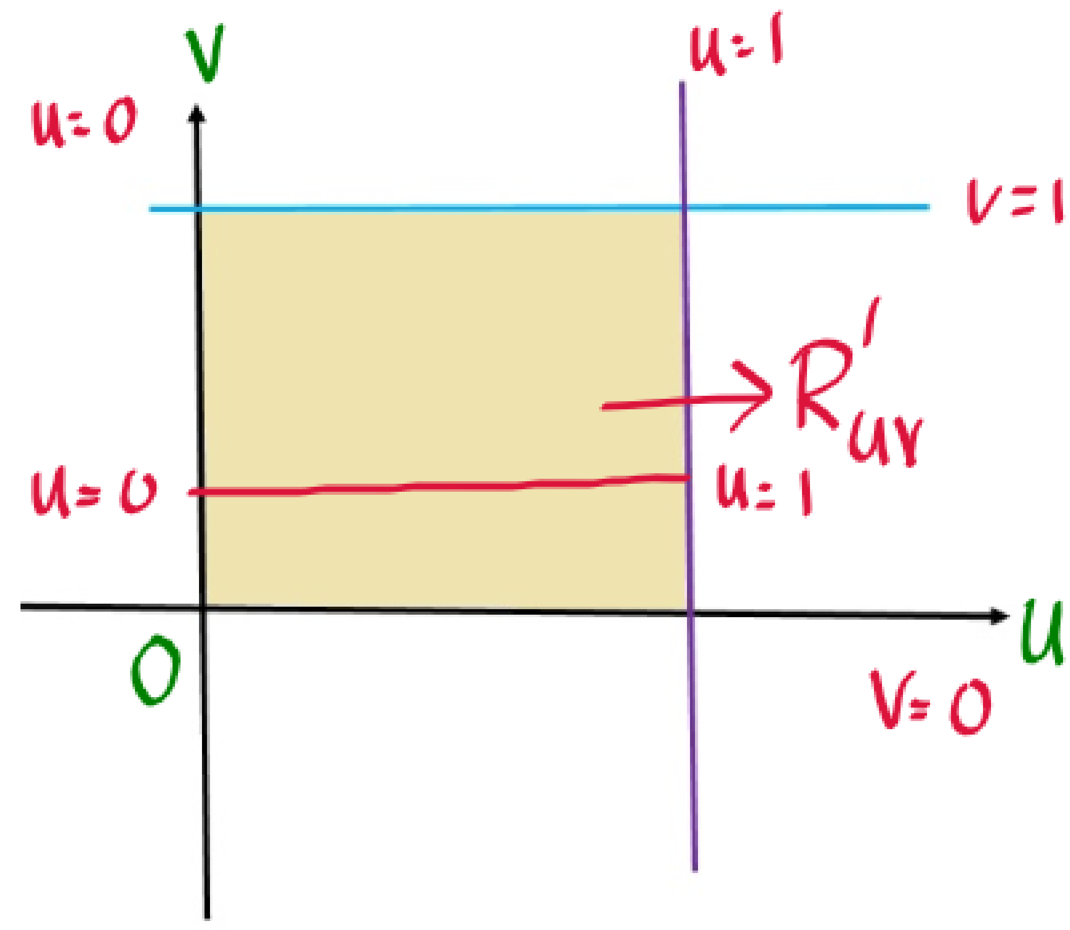
Problem 1.5. *Evaluate*

Let $I = \iint_D xy\sqrt{1-x-y}\,dx\,dy.$

where D *is the region bounded by* $x = 0, y = 0, x + y = 1$ *using the transformation* $x + y = u$ *and* $y = uv$.

Ans:





Given $x+y=u$ and $y=uv$ Here $x = u-uv$

when $y=0 \Rightarrow uv=0 \Rightarrow u=0$ or $v=0$

$x=0 \Rightarrow u=y \Rightarrow u=uv \Rightarrow u(1-v)=0 \Rightarrow u=0,\ v=1$

when $x+y=1 \Rightarrow u=1$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u$$

On R'_{uv} ; $|J| = |u| = u$

$$\therefore\ I = \iint_{R'_{uv}} (u-uv)\,uv\,\sqrt{1-u+uv-uv}\ (u)\,du\,dv$$

$$= \int_{v=0}^{1}\int_{u=0}^{1} u^3 v(1-v)\,(1-u)^{1/2}\,du\,dv$$

$$= \int_{v=0}^{1} v(1-v)\,dv \int_{u=0}^{1} u^3(1-u)^{1/2}\,du$$

put $u = \cos^2\theta$

$\Rightarrow \frac{du}{d\theta} = -2\cos\theta\sin\theta$

when $u=0 \Rightarrow \theta = \pi/2$

$u=1 \Rightarrow \theta = 0$

$$= \left(\frac{v^2}{2} - \frac{v^3}{3}\right)_0^1 \int_{\theta=0}^{\pi/2} \cos^6\theta \sin\theta \, {}^{-}2\sin\theta\cos\theta\,d\theta$$

$$= \frac{-1}{6} \times 2 \int_{\pi/2}^{0} \cos^7\theta \sin^2\theta\,d\theta$$

$$= \frac{1}{3} \int_0^{\pi/2} \sin^2\theta \cos^7\theta\,d\theta$$

$$= \frac{1}{3} \quad \frac{1 \times 6 \times 4 \times 2}{9 \times 7 \times 5 \times 3} = \frac{16}{945}$$

PRACTICE ~~PRACTICAL~~ PROBLEMS

Problem 1.6. *Evaluate*

$$\int_0^1 \int_0^{1-x} e^{y/x+y}\, dx\, dy.$$

using the transformation $x + y = u$ *and* $y = uv$. *Ans:* $\boxed{\frac{1}{2}(e-1)}$

Problem 1.7. *Evaluate*

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} y^2\sqrt{x^2+y^2}\, dy\, dx.$$

by changing to polar coordinates where $a > 0$. *Ans:* $\boxed{\frac{\pi a^5}{20}}$

Problem 1.8. *Evaluate*

$$\int_0^{2a} \int_0^{\sqrt{ax-x^2}} x^2\, dy\, dx.$$

by changing to polar coordinates where $a > 0$. *Ans:* $\boxed{\frac{5\pi a^4}{8}}$

Problem 1.9. *Evaluate*

$$\iint_R \sqrt{a^2-x^2-y^2}\, dx\, dy.$$

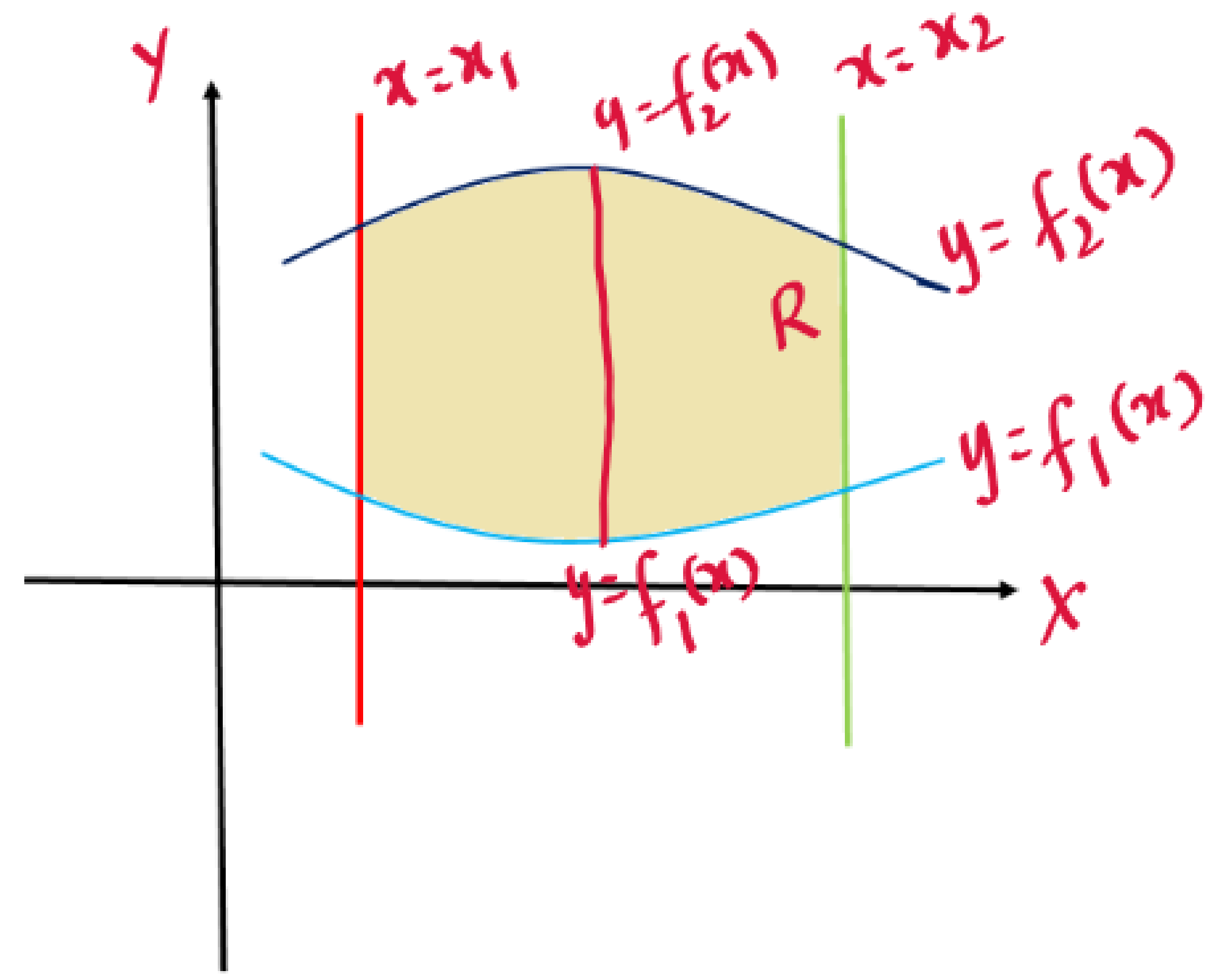
over the semicircle $x^2 + y^2 = ax$ *in the first quadrant, by changing to polar coordinates where* $a > 0$. *Ans:* $\boxed{\frac{a^3}{3}\left(\frac{\pi}{2} - \frac{2}{3}\right)}$.

2. Area of a region using double integrals

In cartesian coordinates

Area of the region R enclosed by the curves $y = f_1(x), y = f_2(x)$ and the ordinates $x = x_1$ and $x = x_2$ is given by,

$$Area(R) = \int_{x_1}^{x_2} \int_{y=f_1(x)}^{y=f_2(x)} dy\, dx$$

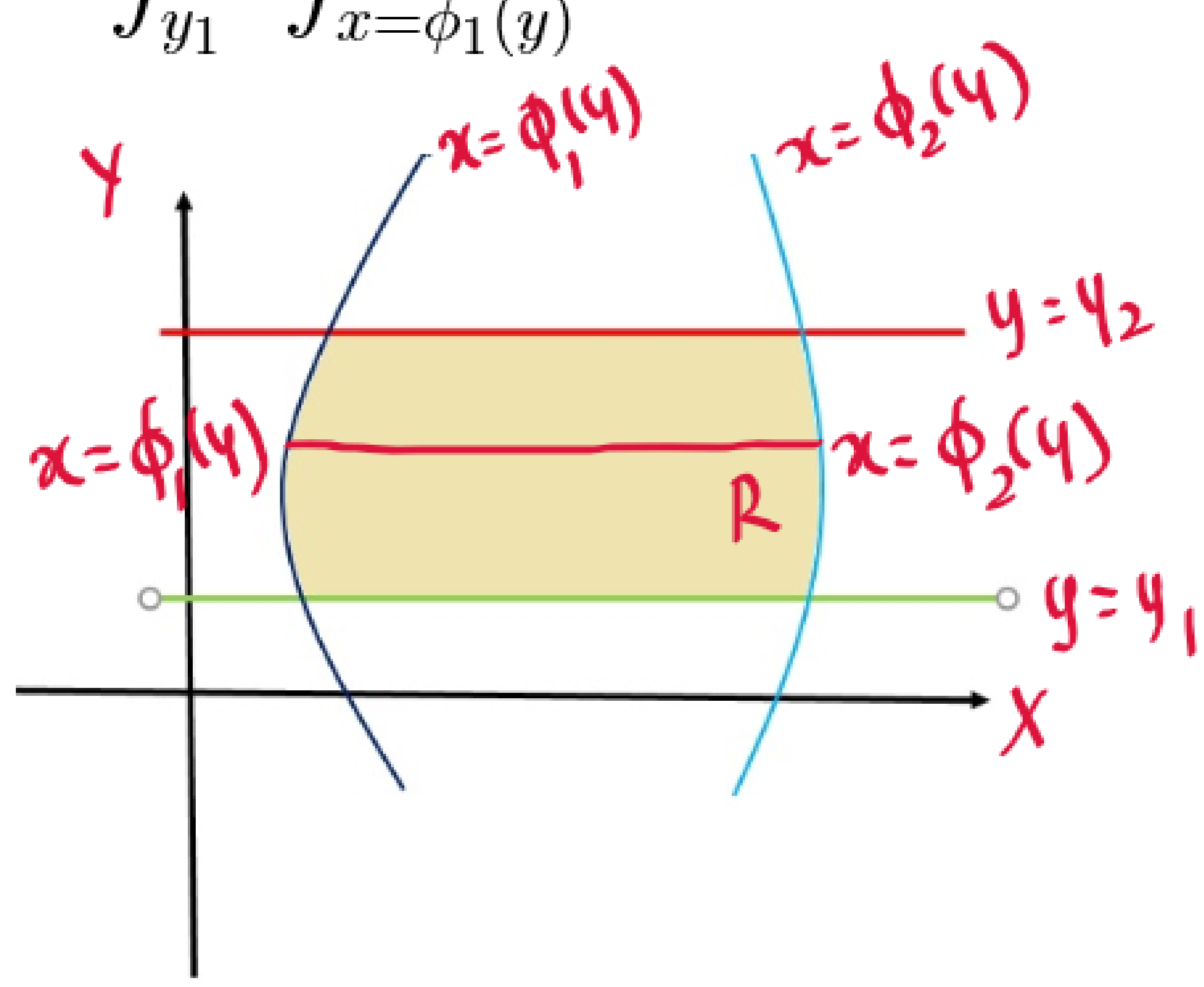


Area of the region R enclosed by the curves $x = \phi_1(y), x = \phi_2(y)$ and the lines $y = y_1$ and $y = y_2$ is given by,

$$Area(R) = \int_{y_1}^{y_2} \int_{x=\phi_1(y)}^{x=\phi_2(y)} dx\, dy$$

$$Area(R) = \iint_{R_{xy}} dx\,dy$$

$$Area(R) = \iint_{R_{r\theta}} r\,dr\,d\theta$$



⊛ The area of a region R in the polar coordinates is given by,

$$Area(R) = \iint_R r\, dr\, d\theta$$

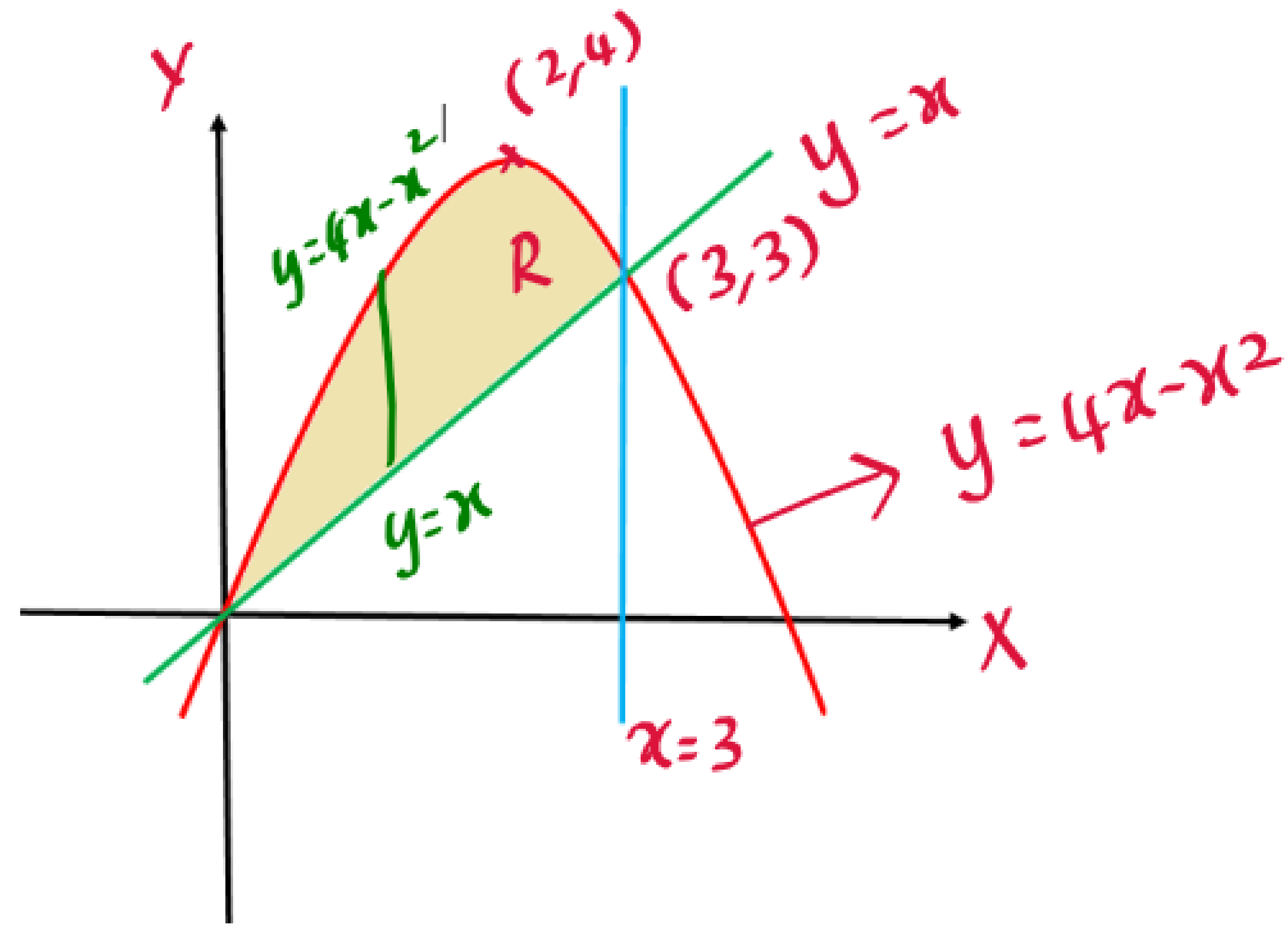
Problem 2.1. *Using double integration, find the area lying betwween the curve $y = 4x - x^2$ and the line $y = x$.*

Ans:-

$y = 4x - x^2$

$\Rightarrow x^2 - 4x = -y$

$\Rightarrow (x-2)^2 = -(y-4)$



$$Area(R) = \iint_R dx\,dy$$

$$= \int_{x=0}^{3}\int_{y=x}^{y=4x-x^2} dy\; dx = \int_{x=0}^{3} (4x - x^2 - x)\,dx = \int_0^3 (3x - x^2)\,dx$$

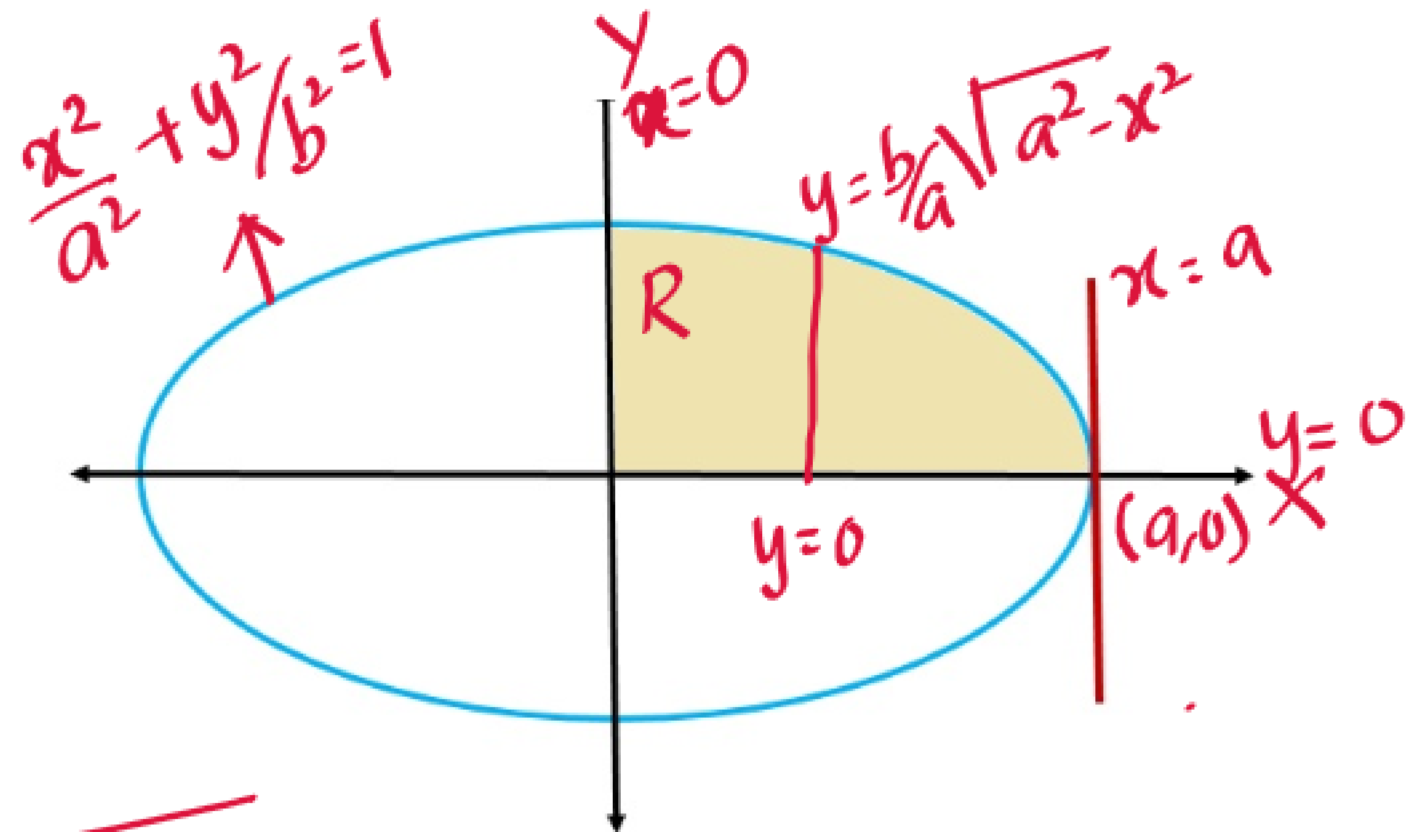
$= 9/2$ sq. units

Problem 2.2. *Using double integration, find the area of a plate in the form of the curve* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *in the first quadrant.*

Ans:

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2) \Leftarrow$$

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$

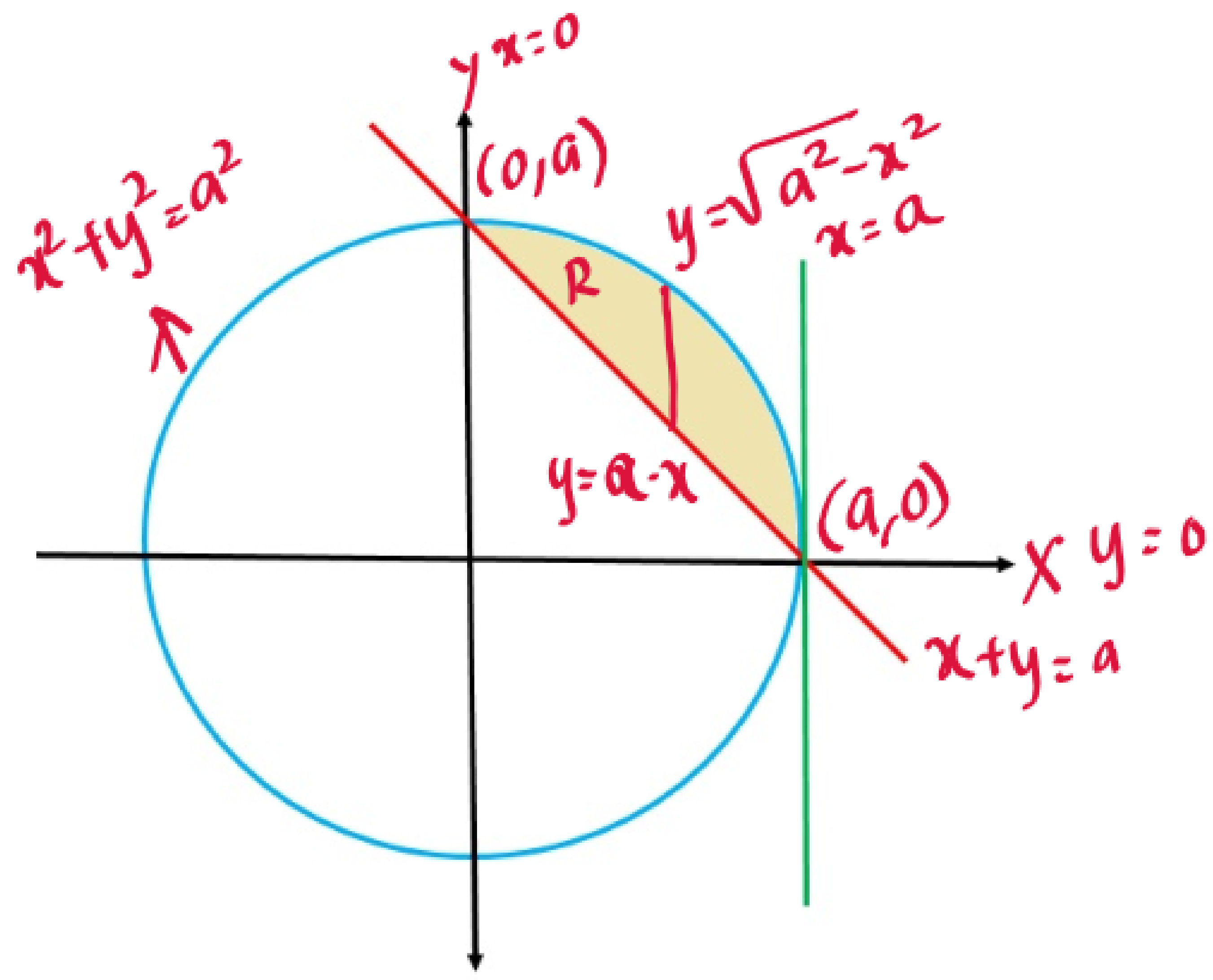


$$\therefore \text{Req'd area} = \int_{x=0}^{x=a} \int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} dy\,dx = \text{Ex.}$$

$$= \frac{\pi ab}{4} \text{ sq. units}$$

Problem 2.3. *Using double integration, find the area lying between the circle* $x^2 + y^2 = a^2$ *and the line* $x + y = a$ *in the first quadrant, where* $a > 0$.

Ans:-



$$\text{Area}(R) = \iint_R dy\,dx$$

$$= \int_{x=0}^{a} \int_{y=a-x}^{y=\sqrt{a^2-x^2}} dy\,dx$$

$$= \int_{x=0}^{a} \left(y\right)_{a-x}^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \sqrt{a^2-x^2}\,dx - \int_0^a (a-x)\,dx$$

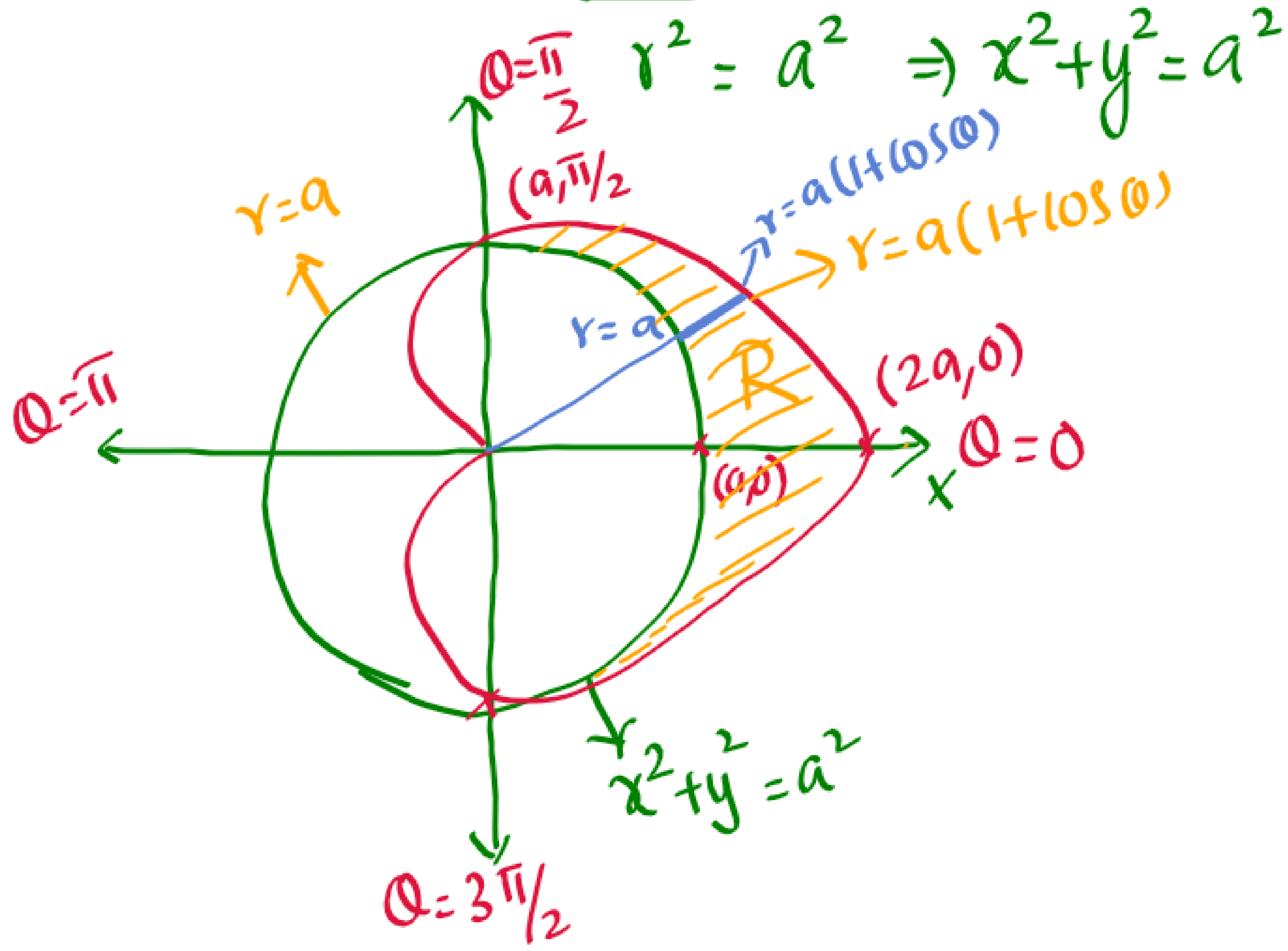
$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}(x/a)$$

$$= \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}(x/a)\right]_{x=0}^{a} - \left(ax - \frac{x^2}{2}\right)_0^a$$

$$= \left(0 + \frac{a^2}{2}\cdot\frac{\pi}{2} - 0\right) - \left(a^2 - \frac{a^2}{2}\right) = \frac{a^2\pi}{4} - \frac{a^2}{2}$$

$$= \frac{a^2}{4}(\pi - 2) \text{ Sq. units}$$

Problem 2.4. *Using double integration, find the area lying inside the cardiod* $r = a(1+\cos\theta)$ *and outside the circle* $r = a$ *where* $a > 0$.



$\therefore$ Req'd area $= 2 \times \iint r\,dr\,d\theta$

Region above the initial line

$$= 2 \times \int_{\theta=0}^{\pi/2} \int_{r=a}^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2 \times \int_{\theta=0}^{\pi/2} \left(\frac{r^2}{2}\right)_a^{a(1+\cos\theta)} d\theta$$

$$= a^2 \int_{\theta=0}^{\pi/2} \left((1+\cos\theta)^2 - 1\right) d\theta$$

$$= a^2 \int_{\theta=0}^{\pi/2} (\cos^2\theta + 2\cos\theta)\,d\theta = a^2 \int_0^{\pi/2} \cos^2\theta\,d\theta + 2a^2 \int_0^{\pi/2} \cos\theta\,d\theta$$

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$$= a^2\left(\frac{1}{2} \times \pi/2\right) + 2a^2\left(\sin\theta\right)_0^{\pi/2}.$$

$$= \frac{a^2\pi}{4} + 2a^2 \doteq \frac{a^2}{4}\underline{\underline{(\pi+8)}}\ \text{Sq. units.}$$

Problem 2.5. *Using double integration, find the area common to the circles* $r = a\cos\theta$ *and* $r = a\sin\theta$ *where* $a > 0$.

Hint:

$$\iint_R r\,dr\,d\theta$$

$$= \iint_{R_1} r\,dr\,d\theta + \iint_{R_2} r\,dr\,d\theta$$

$$= \int_0^{\pi/4}\int_{r=0}^{a\sin\theta} r\,dr\,d\theta + \int_{\pi/4}^{\pi/2}\int_{r=0}^{a\cos\theta} r\,dr\,d\theta$$