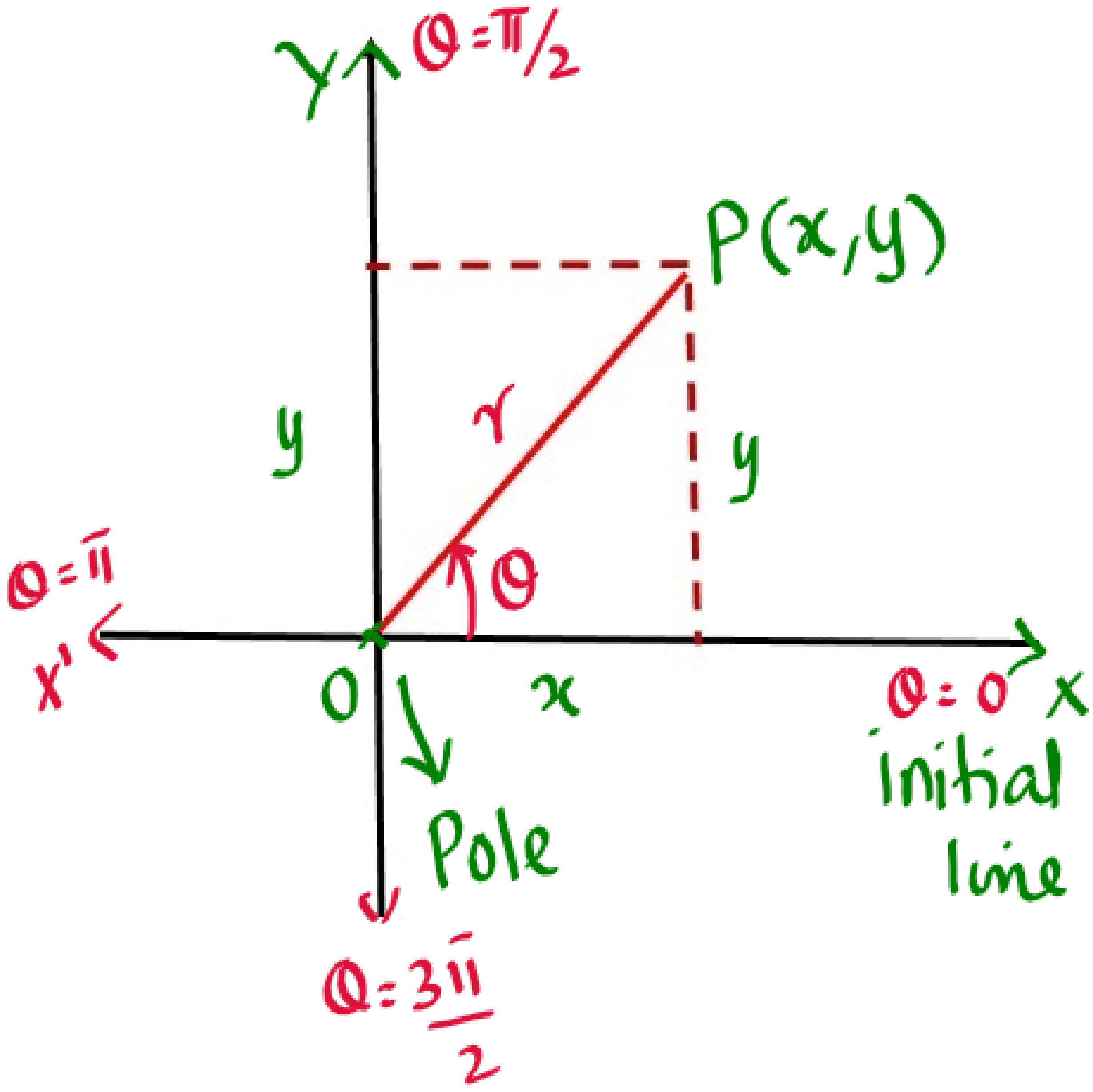
Lecture 6 - Date : 21 May 2021

Convert cartesian coordinates to polar coordinates

$$(x, y) \longrightarrow (r, \theta)$$



$$\cos\theta = \frac{x}{r}$$

$$\Rightarrow x = r\cos\theta$$

$$\sin\theta = \frac{y}{r}$$

$$\Rightarrow y = r\sin\theta$$

$$\Rightarrow x^2 + y^2 = r^2$$

$$\Rightarrow r = \sqrt{x^2+y^2}$$

$$\tan\theta = y/x$$

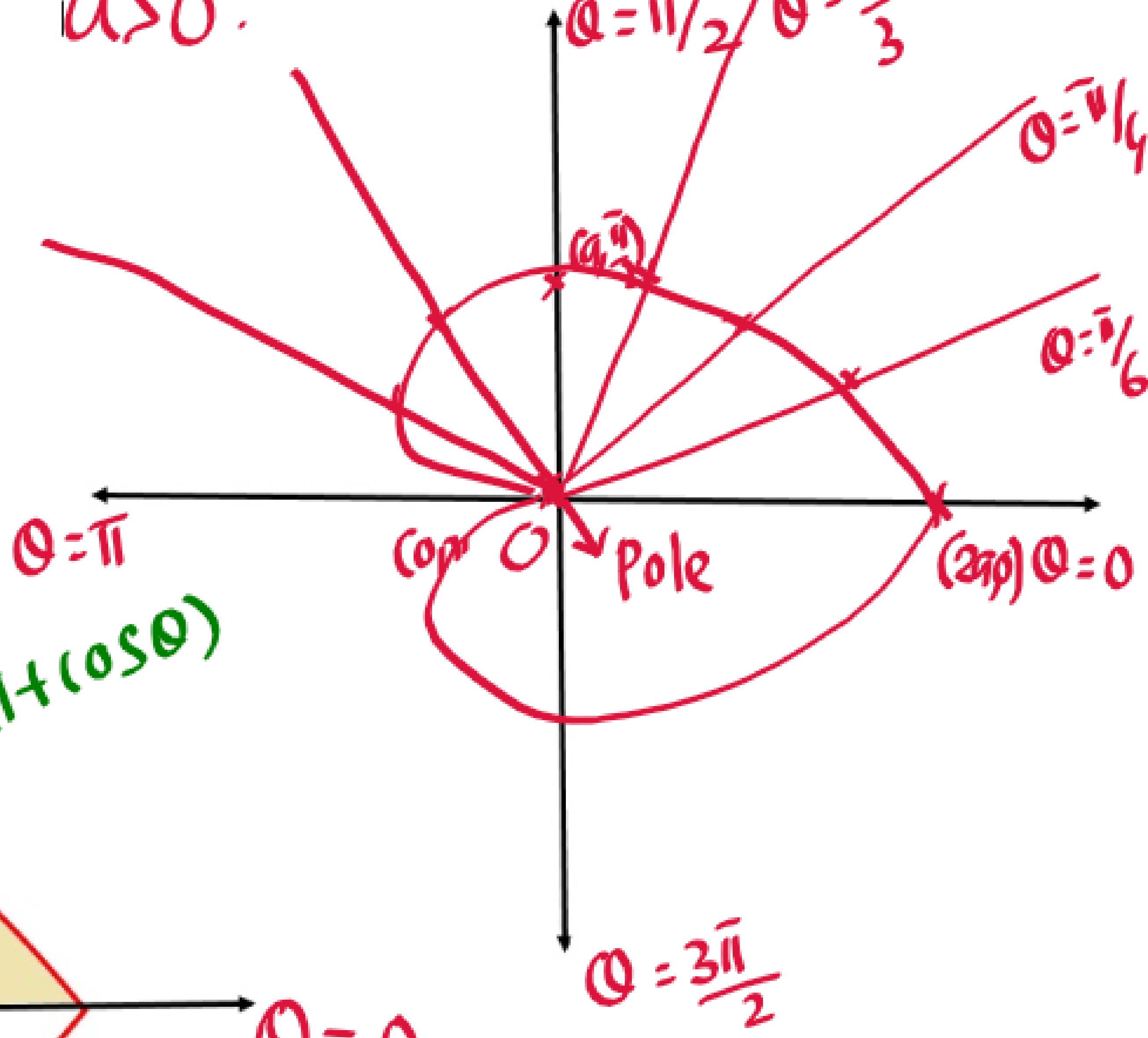
$$\theta = \tan^{-1}(y/x)$$

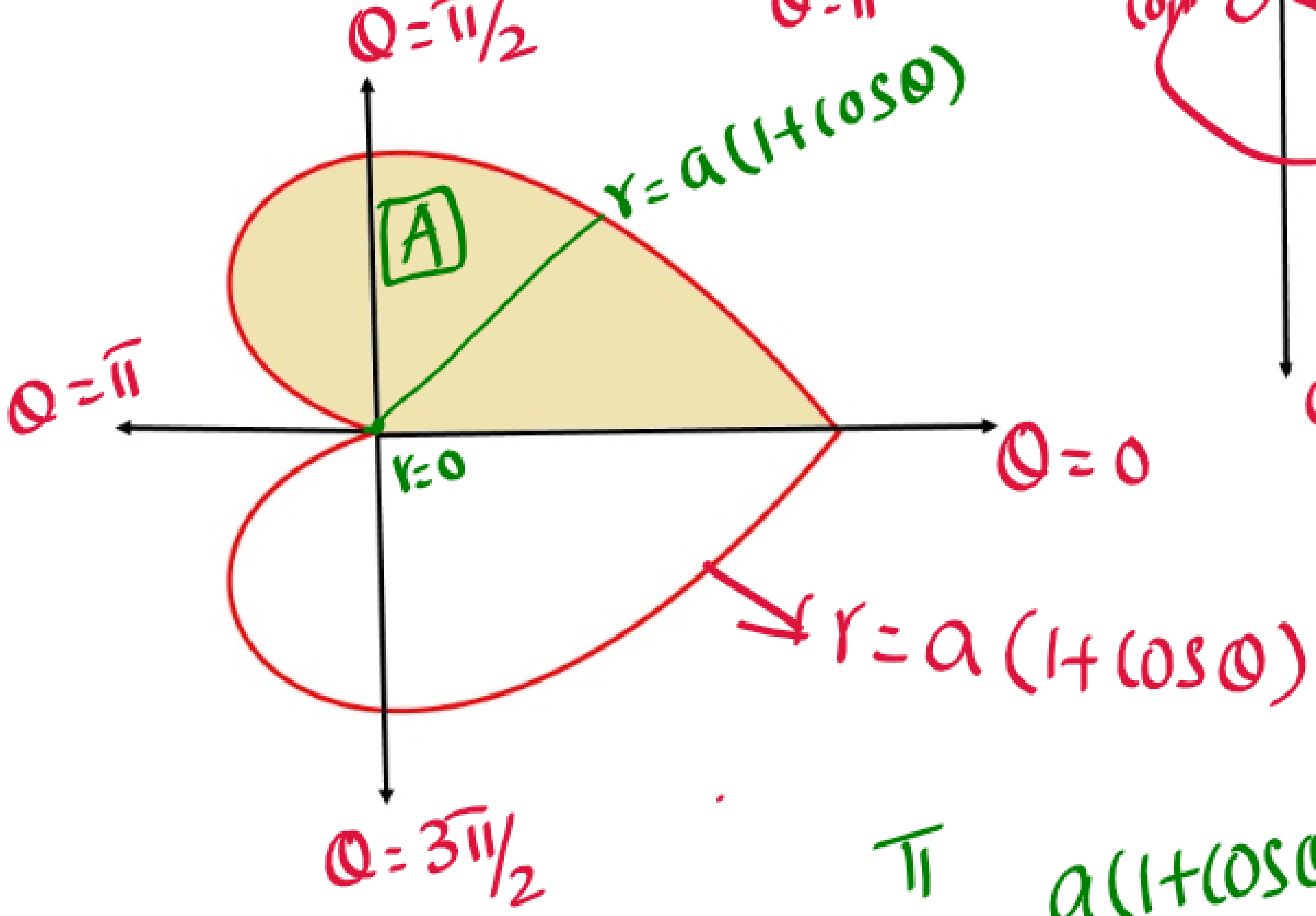
1. Evaluation of double integrals in polar coordinates

Problem 1.1. *Evaluate*

$$\iint_A r\sin\theta \; dr \; d\theta$$

over the area A ~~is~~ bounded by the cardiod $r = a(1+\cos\theta)$ *above the initial line.* where $a>0$.





$$\therefore \iint_A r\sin\theta \, dr\, d\theta = \int_{\theta=0}^{\pi}\left(\int_{r=0}^{a(1+\cos\theta)} r\sin\theta \, dr\right) d\theta$$

$$= \int_{\theta=0}^{\pi} \sin\theta \left(\frac{r^2}{2}\right)_{r=0}^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \sin\theta \; a^2 (1+\cos\theta)^2 \, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi} 2\sin\theta/2 \cos\theta/2 \; \left(2\cos^2(\theta/2)\right)^2 d\theta$$

$$= 4a^2 \int_0^{\pi} \sin(\theta/2)\cos^5(\theta/2)\, d\theta$$

$$= 4a^2 \int_0^{\pi/2} \sin t \cos^5 t \cdot 2\,dt$$

$$= 8a^2 \int_0^{\pi/2} \sin t \cos^5 t \, dt$$

put $\theta/2 = t$

$\Rightarrow d\theta = 2dt$

$\theta = 0 \Rightarrow t = 0$

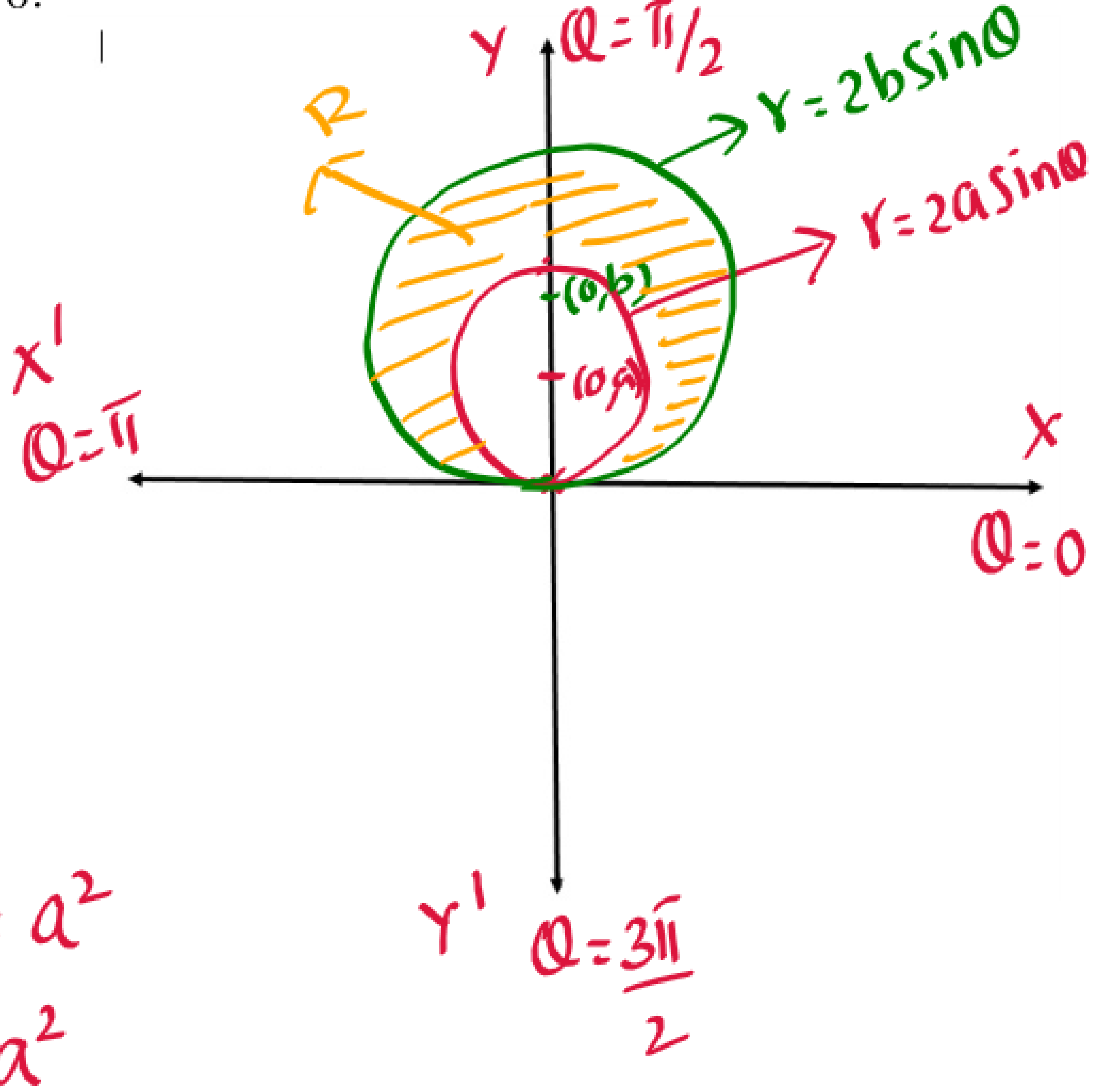
$\theta = \pi \Rightarrow t = \pi/2$

$$= 8a^2 \cdot \frac{1 \cdot 4 \cdot 2}{6 \cdot 4 \cdot 2} = \frac{4a^2}{3}$$

Problem 1.2. *Evaluate*

$$\iint_R r^3 \, dr \, d\theta$$

over the region R ~~is~~ bounded between the circles $r = 2a\sin\theta$ *and* $r = 2b\sin\theta$ *where* $b > a > 0$.



Ans:

$r = 2a\sin\theta$

$\Rightarrow r^2 = 2ar\sin\theta$

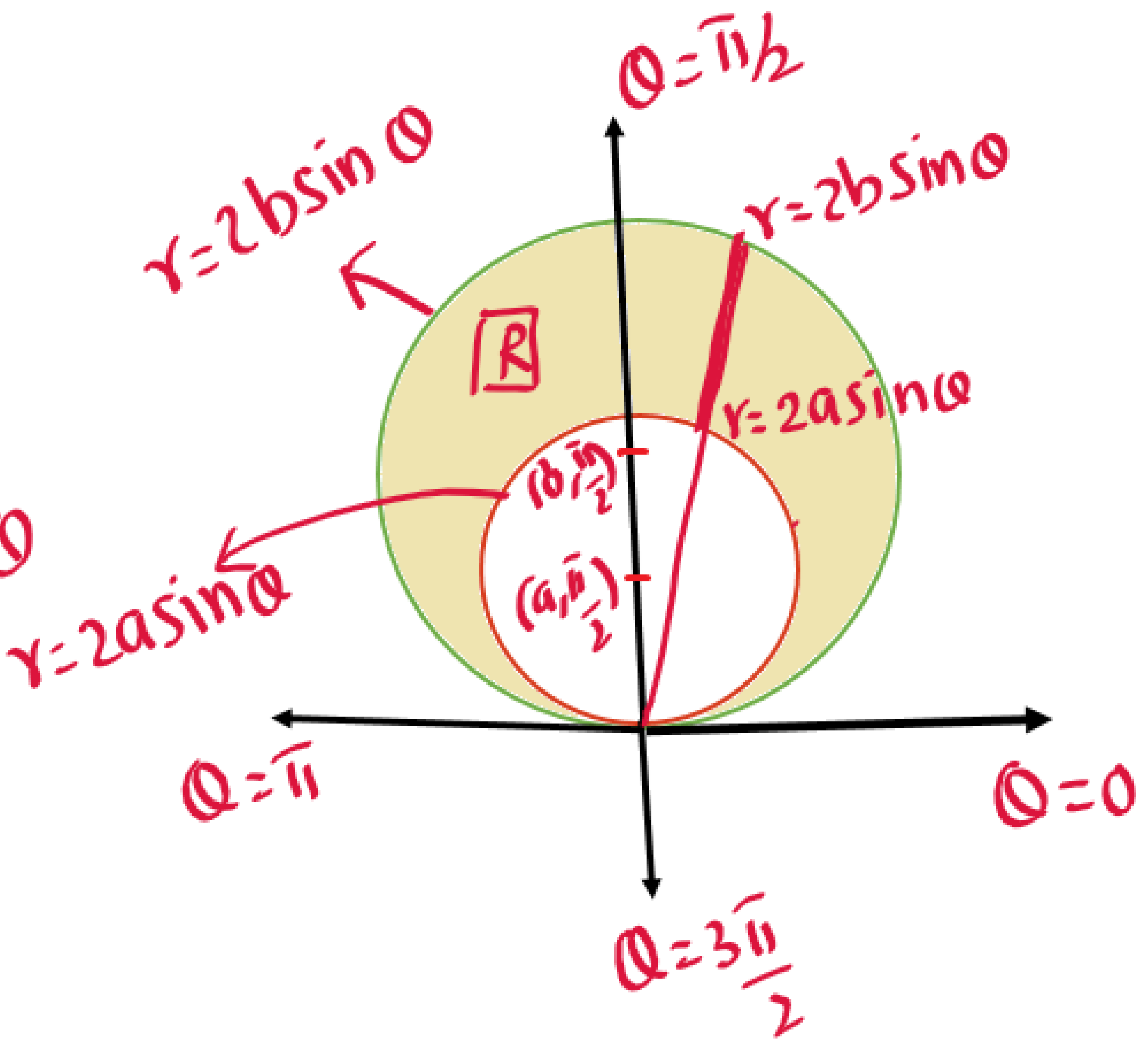
$\Rightarrow x^2 + y^2 = 2ay$

$\Rightarrow x^2 + y^2 - 2ay = 0$

$\Rightarrow x^2 + y^2 - 2ay + a^2 = a^2$

$\Rightarrow (x-0)^2 + (y-a)^2 = a^2$

$r = 2b\sin\theta \Rightarrow (x-0)^2 + (y-b)^2 = b^2$



On R:-

r : $2a\sin\theta$ to $2b\sin\theta$

θ : 0 to π

$$\therefore \iint_R r^3\,dr\,d\theta = \int_{\theta=0}^{\pi}\left(\int_{r=2a\sin\theta}^{2b\sin\theta} r^3\,dr\right)d\theta$$

$$= \int_{\theta=0}^{\pi}\left(\frac{r^4}{4}\right)_{r=2a\sin\theta}^{r=2b\sin\theta} d\theta$$

$$= \frac{1}{4}\int_{\theta=0}^{\pi}\left(2^4 b^4 \sin^4\theta - 2^4 a^4 \sin^4\theta\right) d\theta$$

$$= 4(b^4 - a^4)\int_0^{\pi} \sin^4\theta\,d\theta$$

$$= 4(b^4 - a^4)\cdot 2\int_0^{\pi/2} \sin^4(\theta)\,d\theta.$$

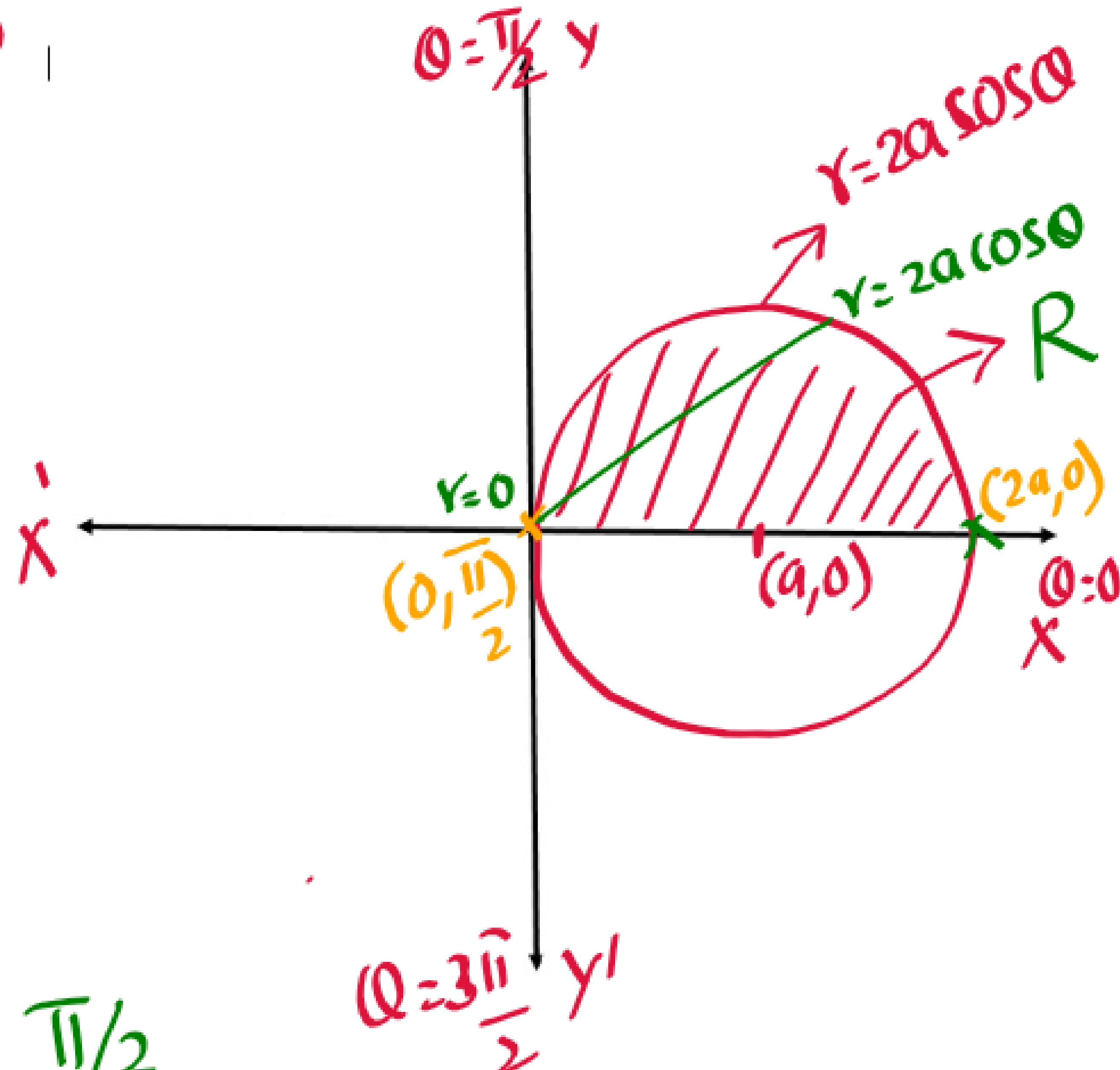
$$\left(\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \text{ if } f(2a-x) = f(x)\right)$$

$$= 8(b^4 - a^4)\,\frac{3\cdot 1}{4\cdot 2}\times\frac{\pi}{2} = \underline{\underline{\frac{3\pi}{2}(b^4 - a^4)}}$$

Problem 1.3. *Evaluate*

$$\iint_R r^2 \cos^2\theta \; dr \; d\theta$$

over the region R is the region bounded by the curve $r = 2a\cos\theta$ above the initial line. ; $a > 0$



Ans:- $r = 2a\cos\theta$

$\Rightarrow r^2 = 2a\, r\cos\theta$

$\Rightarrow x^2 + y^2 = 2ax$

$\Rightarrow (x-a)^2 + (y-0)^2 = a^2$

$$\therefore \iint_R r^2\cos^2\theta \, dr\, d\theta = \int_{\theta=0}^{\pi/2}\left(\int_{r=0}^{2a\cos\theta} r^2\cos^2\theta\right) dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \cos^2\theta \left(\frac{r^3}{3}\right)_0^{2a\cos\theta} d\theta$$

$$= \frac{8a^3}{3}\int_{\theta=0}^{\pi/2} \cos^2\theta \cdot \cos^3\theta \; d\theta$$

$$= \frac{8a^3}{3}\int_0^{\pi/2} \cos^5\theta \, d\theta$$

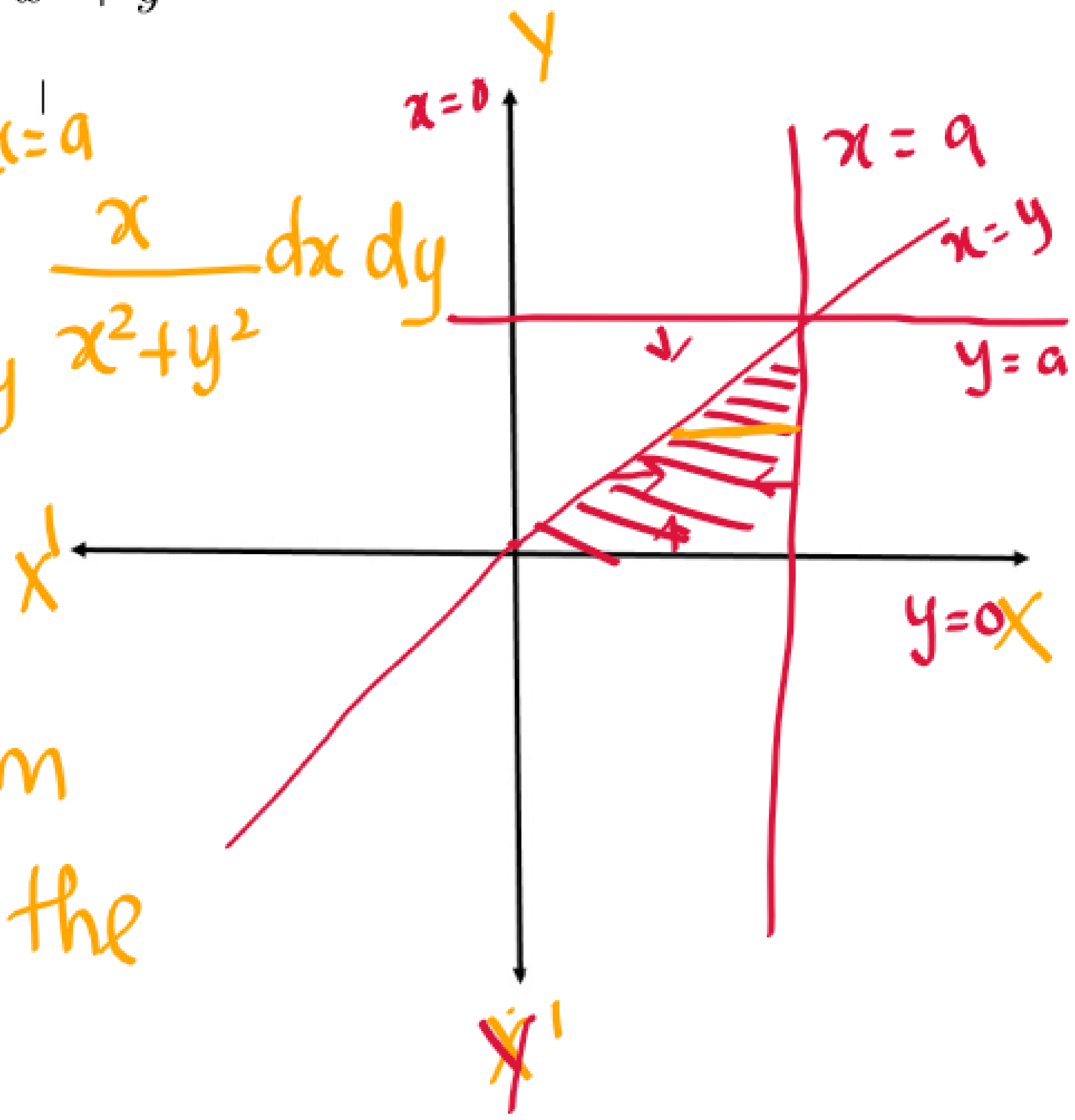
$$= \frac{8a^3}{3} \cdot \frac{4 \cdot 2}{5 \cdot 3} = \frac{64}{45} a^3$$

Problem 2.1. *Change the order of integration and evaluate*

$$\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$$

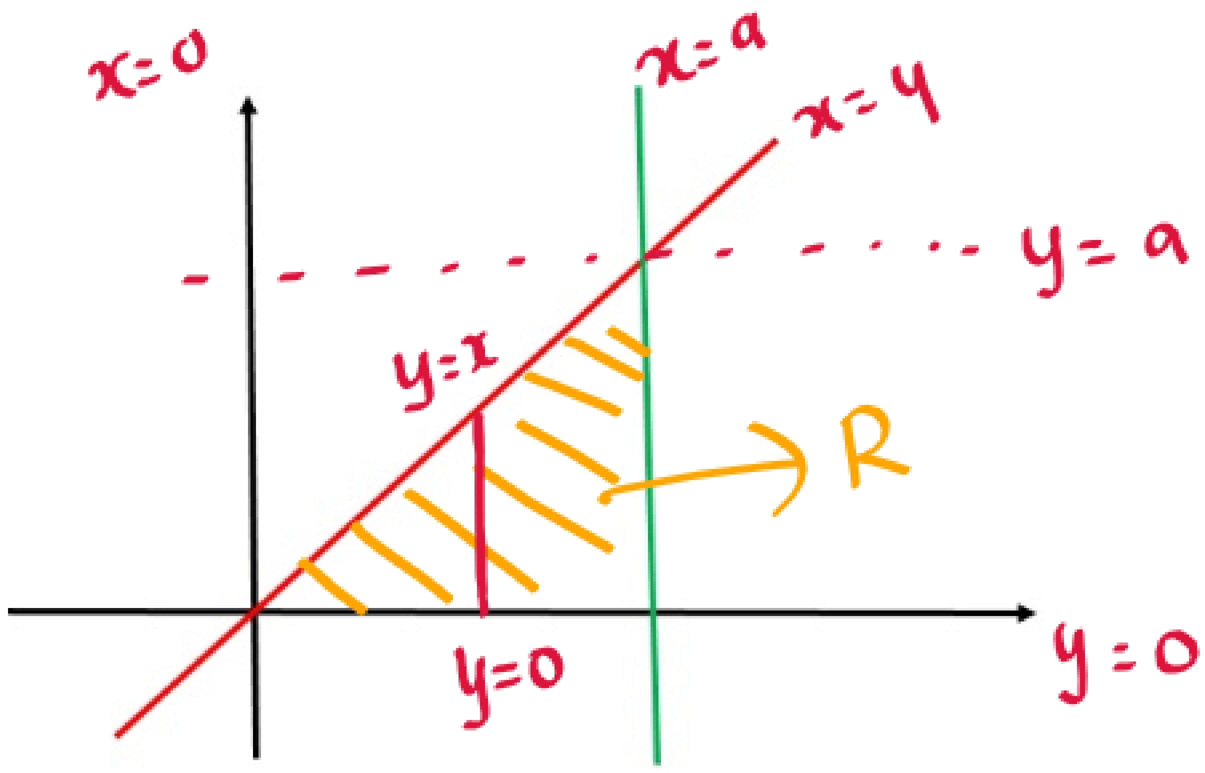
where $a > 0$.

Ans: Given $I = \int_{y=0}^{a} \int_{x=y}^{x=a} \frac{x}{x^2+y^2}\, dx\, dy$



Here, x varies from the line $x=y$ to the line $x=a$

y varies from the line $y=0$ to line $y=a$



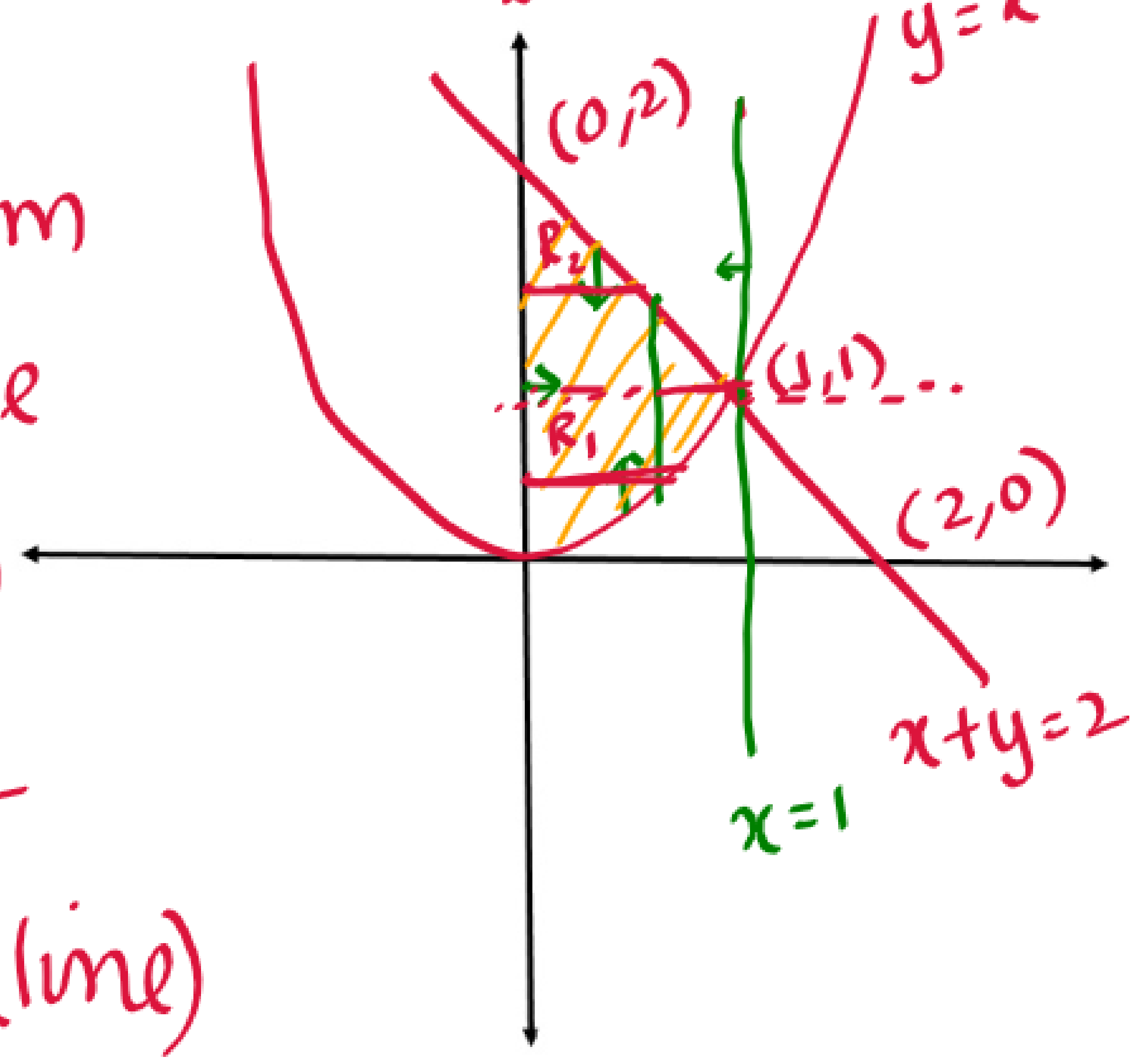
$$\therefore I = \int_{x=0}^{a} \left(\int_{y=0}^{x} \frac{x}{x^2+y^2}\, dy \right) dx = \int_{x=0}^{a} x \left(\frac{1}{x} \tan^{-1}(y/x) \right)_{y=0}^{x} dx$$

.

$$= \int_{x=0}^{a} \frac{\pi}{4}\,dx = \frac{\pi}{4}\left(x\right)_0^a = \underline{\underline{\frac{\pi a}{4}}}$$

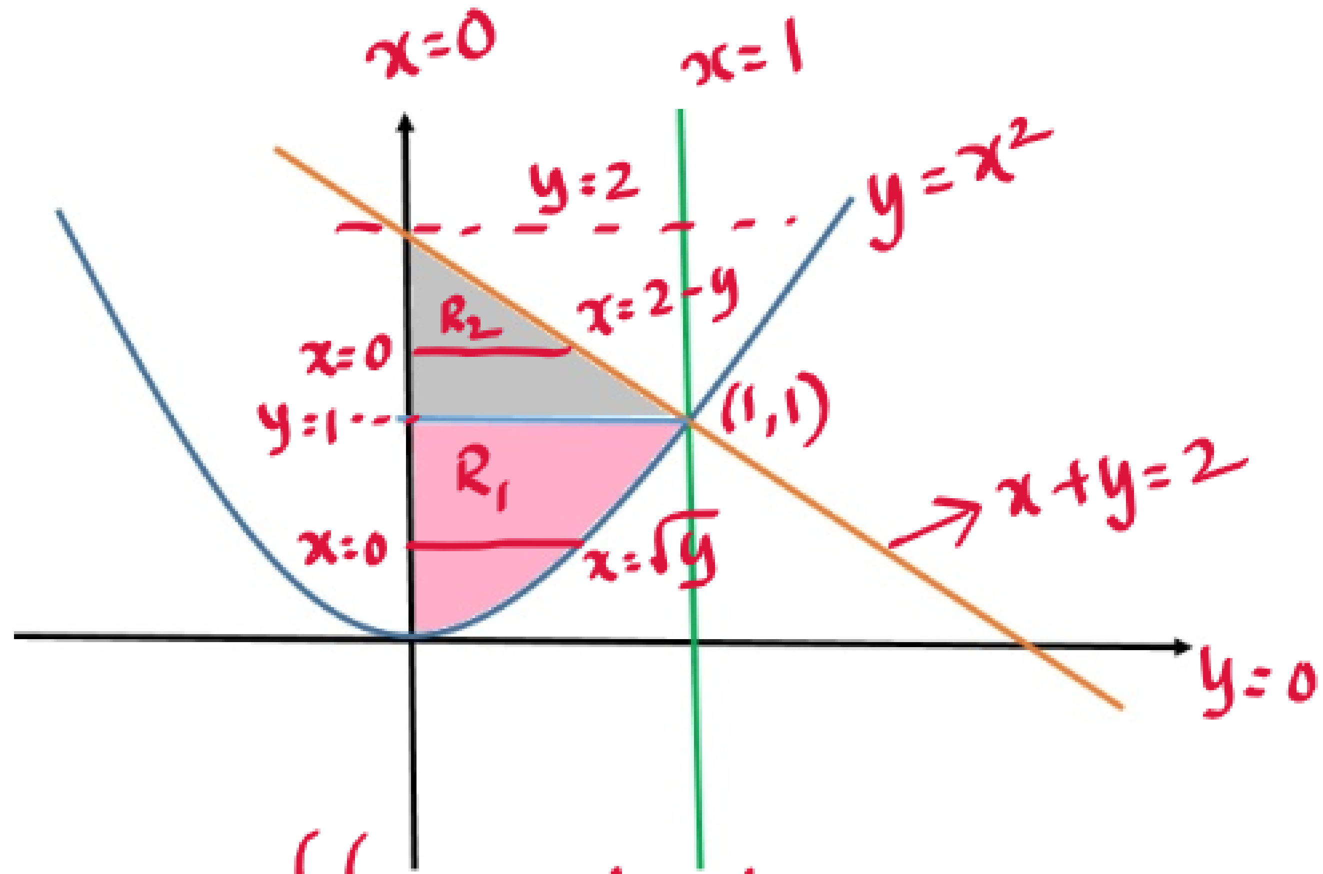
Problem 2.2. *Change the order of integration and evaluate*

Let $I = \int_{x=0}^{1} \int_{y=x^2}^{2-x} xy \, dy \, dx.$



Ans: Here, y varies from the boundary of the curve $y=x^2$ (parabola) to the boundary of the curve $y=2-x$ (line)

x varies from the line $x=0$ to the line $x=1$



$\therefore I = \iint_{R_1} xy \, dx \, dy + \iint_{R_2} xy \, dx \, dy$

$= \int_{y=0}^{1} y \left(\int_{x=0}^{\sqrt{y}} x \, dx \right) dy + \int_{y=1}^{2} y \left(\int_{x=0}^{x=2-y} x \, dx \right) dy$

$$= \int_{y=0}^{1} y \left(\frac{x^2}{2}\right)_0^{\sqrt{y}} dy + \int_{y=1}^{2} y \left(\frac{x^2}{2}\right)_{x=0}^{x=2-y} dy$$

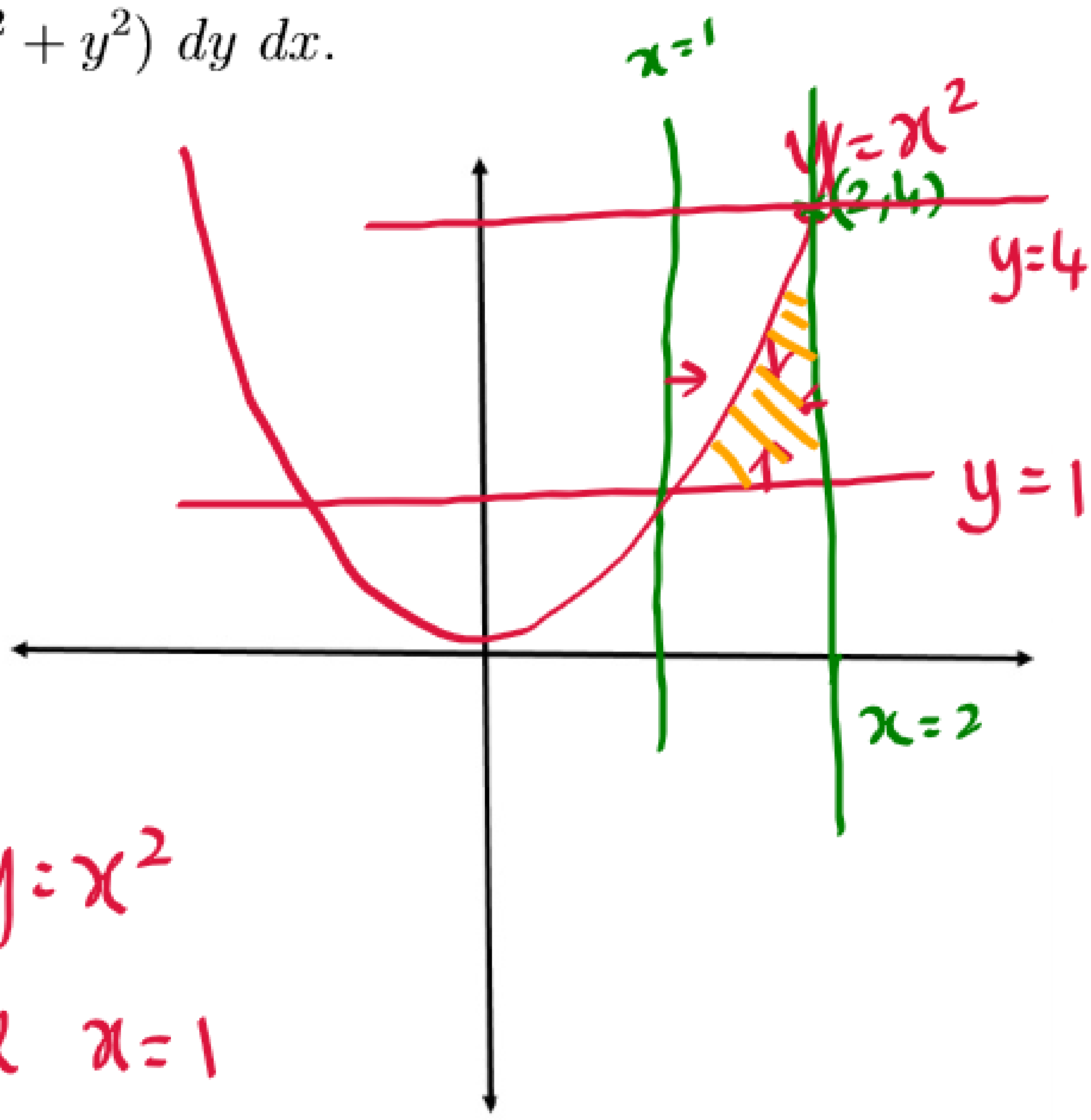
$$= \int_{y=0}^{1} \frac{y^2}{2}\, dy + \frac{1}{2}\int_{y=1}^{2} y\,(2-y)^2\, dy$$

$$= \,? \quad (\text{Ex})$$

$$= \frac{3}{8} \quad \underline{\underline{(\text{Ans})}}$$

Problem 2.3. *Change the order of integration and evaluate*

Let $I = \int_{x=1}^{2} \int_{y=1}^{x^2} (x^2 + y^2)\, dy\, dx.$

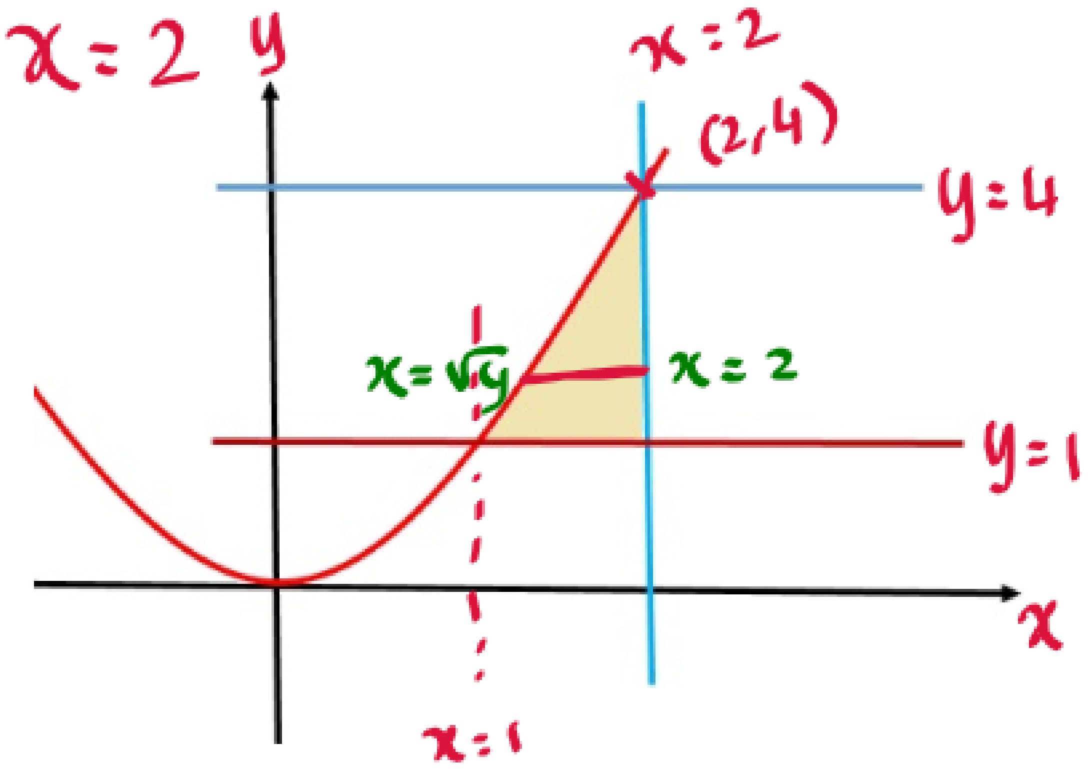


Ans:-

Here,

y : varies from the line y=1 to the parabola $y=x^2$

x varies from line x=1 to x=2



$$\therefore I = \int_{y=1}^{y=4} \int_{x=\sqrt{y}}^{x=2} (x^2+y^2)\, dx\, dy$$

$$= \int_{y=1}^{4} \left(\frac{x^3}{3} + xy^2\right)_{x=\sqrt{y}}^{x=2} dy = \int_{y=1}^{4} \left[\left(\frac{8}{3} + 2y^2\right) - \left(\frac{y^{3/2}}{3} + y^{5/2}\right)\right] dy$$

$$= (Ex)$$

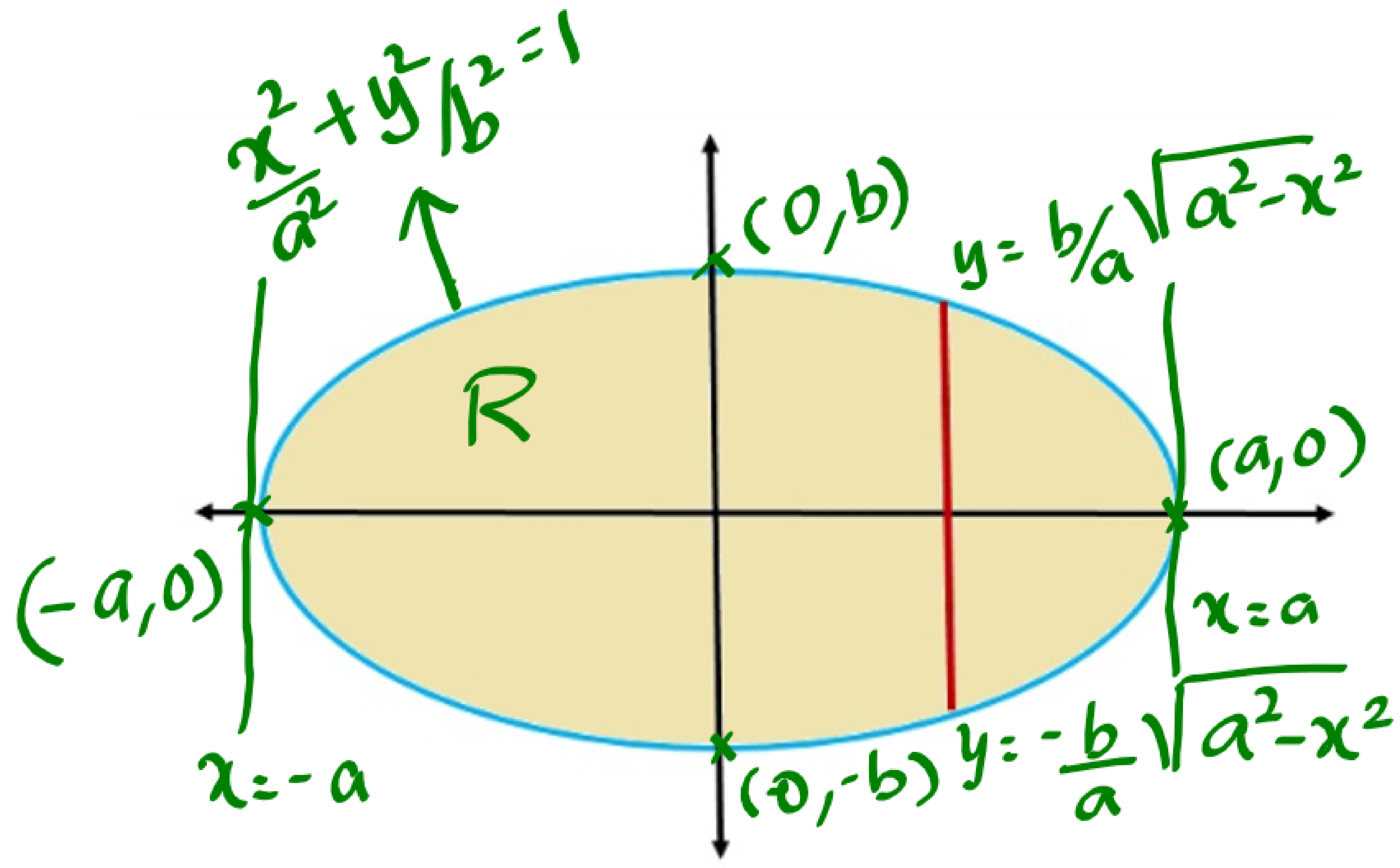
$$= \frac{1006}{105} \text{ (Ans)}$$

Q. Change the order of integration and evaluate

$$\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$$

<u>Ans</u>:- $1-\frac{1}{\sqrt{2}}$

Q. Evaluate $\iint_R (x+y)^2\,dx\,dy$ where R is the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.



Ans:

Let $I=\iint_R (x+y)^2\,dx\,dy$

then

$$I=\int_{x=-a}^{x=a}\left(\int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{y=\frac{b}{a}\sqrt{a^2-x^2}}(x^2+y^2+2xy)\,dy\right)dx$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\Rightarrow \frac{y^2}{b^2}=\frac{1}{a^2}(a^2-x^2)$$

$$\Rightarrow y^2=\frac{b^2}{a^2}(a^2-x^2)$$

$$\Rightarrow y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

$$=\int_{x=-a}^{a}\int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{y=\frac{b}{a}\sqrt{a^2-x^2}}(x^2+y^2)\,dy\,dx+\int_{x=-a}^{a}\int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{y=\frac{b}{a}\sqrt{a^2-x^2}}2xy\,dy\,dx$$

$$=2\int_{x=-a}^{a}\int_{0}^{y=\frac{b}{a}\sqrt{a^2-x^2}}(x^2+y^2)\,dy\,dx+0$$

$$=2\int_{x=-a}^{a}\left(x^2y+\frac{y^3}{3}\right)_{y=0}^{y=b/a\sqrt{a^2-x^2}}dx$$

$$=2\int_{x=-a}^{a}\left[x^2\frac{b}{a}\sqrt{a^2-x^2}+\frac{1}{3}\frac{b^3}{a^3}(a^2-x^2)^{3/2}\right]dx$$

$$=4\int_0^a\frac{b}{a}x^2\sqrt{a^2-x^2}\,dx+\frac{4}{3}\frac{b^3}{a^3}\int_0^a(a^2-x^2)^{3/2}\,dx$$

put $x = a\sin\theta \Rightarrow dx = a\cos\theta\, d\theta$

when $x = 0 \Rightarrow \theta = 0$

when $x = a \Rightarrow \theta = \pi/2$

$$\therefore I = \frac{4b}{a}\int_0^{\pi/2} a^3\sin^2\theta \cdot a\cos^2\theta\, d\theta + \frac{4b^3}{3a^3}\int_0^{\pi/2} a^4\cos^4\theta\, d\theta$$

$$= 4a^3b\int_0^{\pi/2}\sin^2\theta\cos^2\theta\, d\theta + \frac{4b^3a}{3}\int_0^{\pi/2}\cos^4\theta\, d\theta.$$

$$= 4a^3b\,\frac{1\cdot 1}{4\cdot 2}\times\frac{\pi}{2} + \frac{4ab^3}{3}\cdot\frac{3\cdot 1}{4\cdot 2}\times\frac{\pi}{2}$$

$$= \frac{\pi a^3b}{4} + \frac{\pi ab^3}{4} = \frac{\pi ab}{4}\left(a^2+b^2\right)$$