Lecture 5 - Date : 18 May 2021

Reduction Formula

$$\circledast \int_0^{\pi/2} \sin^n x \; dx = \begin{cases} \dfrac{(n-1)}{n}\dfrac{(n-3)}{(n-2)}\dfrac{(n-5)}{(n-4)}\cdots\dfrac{1}{2}\times\dfrac{\pi}{2}, & \text{if } n \text{ is even} \\ \dfrac{(n-1)}{n}\dfrac{(n-3)}{(n-2)}\dfrac{(n-5)}{(n-4)}\cdots\dfrac{2}{3}, & \text{if } n \text{ is odd} \end{cases}$$

$$\circledast \int_0^{\pi/2} \cos^n x \; dx = \begin{cases} \dfrac{(n-1)}{n}\dfrac{(n-3)}{(n-2)}\dfrac{(n-5)}{(n-4)}\cdots\dfrac{1}{2}\times\dfrac{\pi}{2}, & \text{if } n \text{ is even} \\ \dfrac{(n-1)}{n}\dfrac{(n-3)}{(n-2)}\dfrac{(n-5)}{(n-4)}\cdots\dfrac{2}{3}, & \text{if } n \text{ is odd} \end{cases}$$

$$\circledast \int_0^{\pi/2} \sin^n x \cos^{\overset{m}{n}} x \; dx = \frac{(n-1)(n-3)(n-5)\ldots(m-1)(\overset{(m-3)}{\cancel{-3}})(m-5)\ldots}{(m+n)(m+n-2)(m+n-4)(m+n-6)\ldots}$$

and multiply by $\dfrac{\pi}{2}$ if both m and n are even.

Examples :- Q. Find $\int_0^{\pi/2} \sin^6 x \, dx$

Here $n=6$, even.

$$\therefore \int_0^{\pi/2} \sin^6 x \, dx = \frac{(6-1)(6-3)(6-5)}{6(6-2)(6-4)} \times \frac{\pi}{2}$$

$$= \frac{5\times3\times1}{6\times4\times2} \times \frac{\pi}{2} = \frac{5\pi}{32}$$

Q. Find $\int_0^{\pi/2} \cos^9 x \, dx$ Here $n=9$, odd

Ans:- $$\int_0^{\pi/2} \cos^9 x \, dx = \frac{(9-1)(9-3)(9-5)(9-7)}{9(9-2)(9-4)(9-6)}$$

$$= \frac{8}{9}\times\frac{6}{7}\times\frac{4}{5}\times\frac{2}{3} = \frac{128}{315}$$

Q. Find $\displaystyle\int_0^{\pi/2} \sin^{4=n} x \cos^{8=m} x \, dx$

Ans:- $\displaystyle\int_0^{\pi/2} \sin^{4} x \cos^{8} x \, dx$

$$= \frac{(4-1)(4-3)(8-1)(8-3)(8-5)(8-7)}{(12)(12-2)(12-4)(12-6)(12-8)(12-10)} \times \frac{\pi}{2}$$

$$= \frac{\cancel{3} \times 1 \times 7 \times \cancel{5} \times \cancel{3} \times 1}{4 \times \cancel{12} \times \underset{2}{\cancel{10}} \times 8 \times \underset{2}{\cancel{6}} \times 4 \times 2} \times \frac{\pi}{2} = \underline{\underline{\frac{7\pi}{2048}}}$$

Q. Find $\displaystyle I = \int_0^{\pi/2} \sin^{3=n} x \cos^{6=m} x \, dx$

Ans:- $$I = \frac{(3-1)(6-1)(6-3)(6-5)}{9(9-2)(9-4)(9-6)(9-8)}$$

$$= \frac{2 \times 5 \times 3 \times 1}{9 \times 7 \times 5 \times 3 \times 1} = \underline{\underline{\frac{2}{63}}}$$

MULTIPLE INTEGRALS

1. Evaluation of Double Integrals

$$\int_0^3 \int_1^2 xy(1+x+y)\,dy\,dx$$

Problem 1.1. *Evaluate*

$$\text{Let } I = \int_{x=0}^{x=3} \int_{y=1}^{y=2} xy\,(1+x+y)\,dydx$$

Ans:- then $I = \int_{x=0}^{3} \left(\int_{y=1}^{2} (xy + x^2y + xy^2)\,dy \right) dx$

$$= \int_{x=0}^{3} \left(x\frac{y^2}{2} + x^2\frac{y^2}{2} + x\frac{y^3}{3} \right)_{y=1}^{y=2} dx$$

$$= \int_{x=0}^{3} \left[\left(2x + 2x^2 + \frac{8}{3}x \right) - \left(\frac{x}{2} + \frac{x^2}{2} + \frac{x}{3} \right) \right] dx$$

$$= \int_{x=0}^{3} \left(\frac{3}{2}x + \frac{3}{2}x^2 + \frac{7}{3}x \right) dx$$

$$= \left(\frac{3}{2}\frac{x^2}{2} + \frac{3}{2}\frac{x^3}{3} + \frac{7}{3}\frac{x^2}{2} \right)_{x=0}^{3}$$

$$= \frac{123}{4} \quad //$$

OR:

$$I = \int_{y=1}^{2} \left(\int_{x=0}^{3} xy(1+x+y)\, dx \right) dy.$$

Problem 1.2. *Evaluate*

$$\int_{y=0}^{1}\int_{x=0}^{1} \frac{dxdy}{\sqrt{(1-x^2)(1-y^2)}}$$

Ans: Let $I = \int_{y=0}^{1}\int_{x=0}^{1} \frac{dxdy}{\sqrt{(1-x^2)(1-y^2)}}$

$$= \int_{y=0}^{1}\int_{x=0}^{1} \frac{dx\,dy}{\sqrt{1-x^2}\,\sqrt{1-y^2}}$$

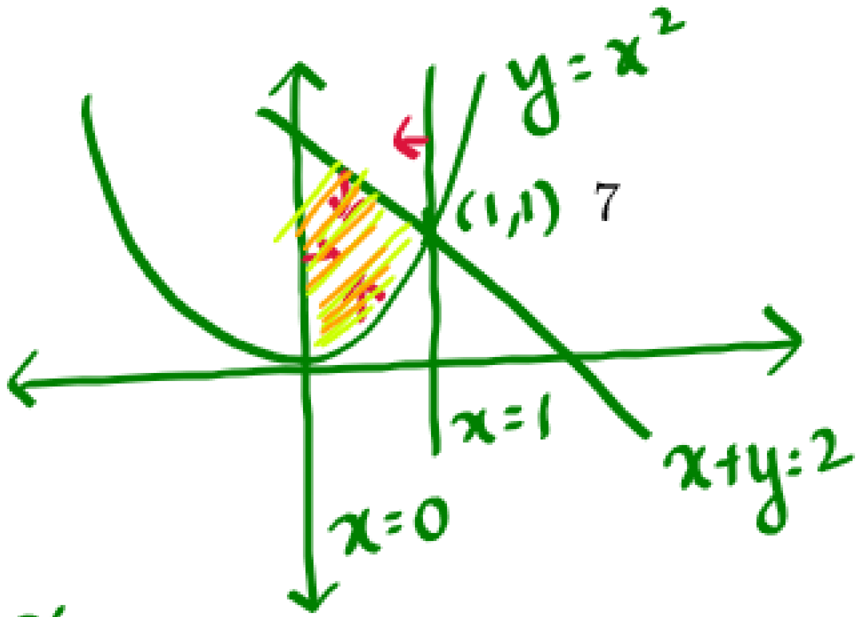
$$= \int_{y=0}^{1} \frac{1}{\sqrt{1-y^2}}\left(\int_{x=0}^{1} \frac{dx}{\sqrt{1-x^2}}\right) dy$$

$$= \int_{y=0}^{1} \frac{1}{\sqrt{1-y^2}} \left(\sin^{-1}x\right)_{x=0}^{1} dy$$

$$= \frac{\pi}{2}\int_{y=0}^{1} \frac{1}{\sqrt{1-y^2}}\, dy = \frac{\pi}{2}\left(\sin^{-1}y\right)_{y=0}^{1}$$

$$= \frac{\pi}{2} \times \frac{\pi}{2} = \underline{\frac{\pi^2}{4}}$$

$$\iint_R f(x,y)\,dx\,dy$$



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Problem 1.3. *Evaluate*

Ans:- Let $I = \displaystyle\int_0^1 \int_{x^2}^{2-x} xy\, dx\, dy$

$$= \int_{x=0}^{x=1} \left(\int_{y=x^2}^{y=2-x} xy\, dy \right) dx$$

$$= \int_{x=0}^{x=1} x \left(\frac{y^2}{2} \right)_{y=x^2}^{y=2-x} dx$$

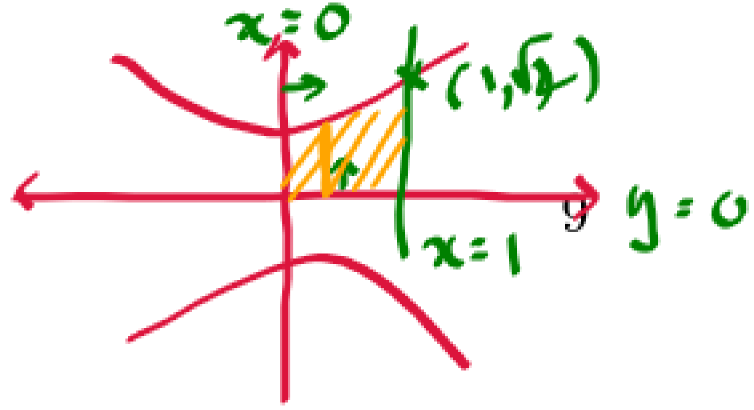
$$= \int_{x=0}^{1} \frac{x}{2} \left((2-x)^2 - x^4 \right) dx$$

$$= \frac{1}{2} \int_{x=0}^{1} x \left(4 - 4x + x^2 - x^4 \right) dx$$

$$= \frac{1}{2} \int_{x=0}^{1} \left(4x - 4x^2 + x^3 - x^5 \right) dx$$

$= \; ? \qquad =$ **Ans:** $\underline{\underline{3/8}}$

$x^2 - y^2 = -1$



Problem 1.4. *Evaluate*

Ans: Let $I = \displaystyle\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\ dy}{1+x^2+y^2}$

Then $$I = \int_{x=0}^{x=1}\left(\int_{y=0}^{y=\sqrt{1+x^2}} \frac{dy}{1+x^2+y^2}\right)dx$$

$$= \int_{x=0}^{x=1}\left(\int_{y=0}^{\sqrt{1+x^2}} \frac{dy}{\left(\sqrt{1+x^2}\right)^2+y^2}\right)dx$$

$$= \int_{x=0}^{x=1}\left[\frac{1}{\sqrt{1+x^2}}\tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right]_{y=0}^{\sqrt{1+x^2}} dx$$

$$= \int_{x=0}^{1}\left[\frac{1}{\sqrt{1+x^2}}\tan^{-1}(1) - 0\right]dx$$

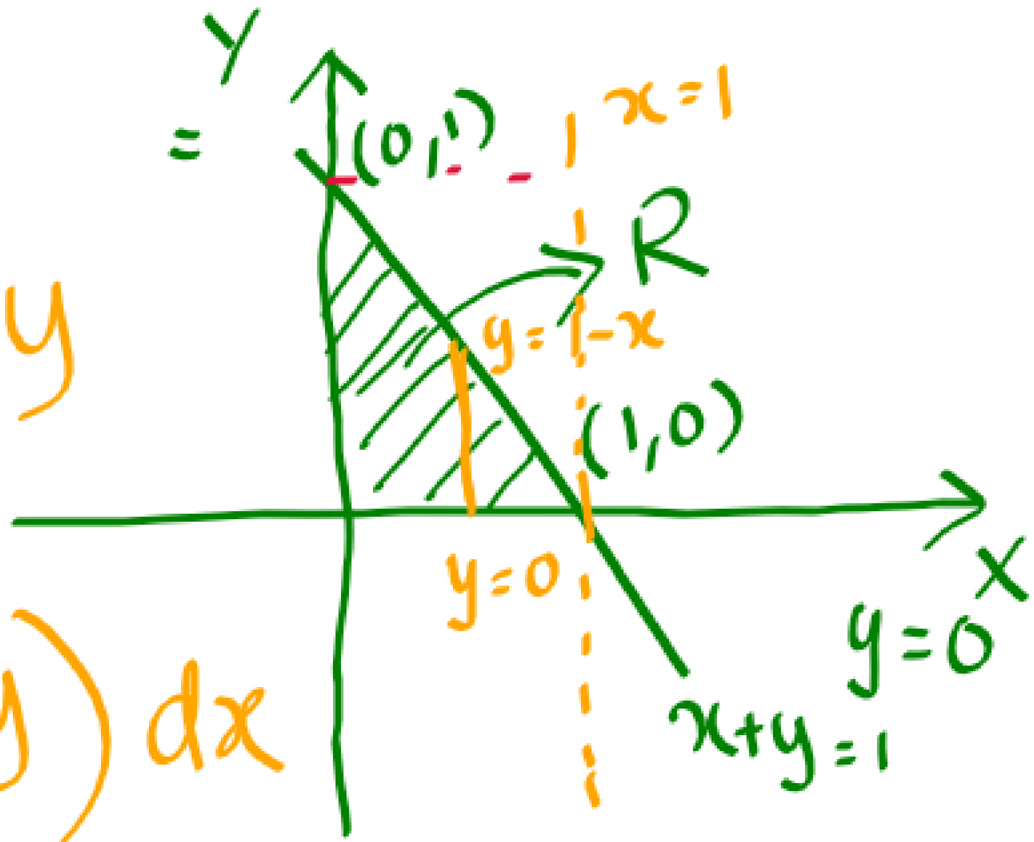
$$= \frac{\pi}{4}\int_{x=0}^{1}\frac{1}{\sqrt{1+x^2}} = \frac{\pi}{4}\left[\log\left(x+\sqrt{1+x^2}\right)\right]_{x=0}^{1}$$

$$= \frac{\pi}{4}\left[\log\left(\sqrt{2}+1\right)\right]$$

Problem 1.5. *Evaluate*

$$\iint_R e^{2x+3y}\ dx\ dy$$

over the region R bounded by the lines $x=0, y=0$ and $x+y=1$.



$$\text{Let } I = \iint_R e^{2x+3y}\,dx\,dy$$

$$= \int_{x=0}^{x=1} \left(\int_{y=0}^{y=1-x} e^{2x+3y}\,dy \right) dx$$

$$= \int_{x=0}^{1} e^{2x} \left(\int_{y=0}^{1-x} e^{3y}\,dy \right) dx$$

$$= \int_{x=0}^{1} e^{2x} \left(\frac{e^{3y}}{3} \right)_{y=0}^{1-x} dx$$

$$= \frac{1}{3} \int_{x=0}^{1} e^{2x} \left(e^{3-3x} - 1 \right) dx$$

$$= \frac{1}{3} \int_{x=0}^{1} \left[e^{3} \cdot e^{-x} - e^{2x} \right] dx$$

$$= \frac{1}{3}\left[e^3 \cdot \left(-e^{-x}\right) - \frac{e^{2x}}{2}\right]_{x=0}^{1}$$

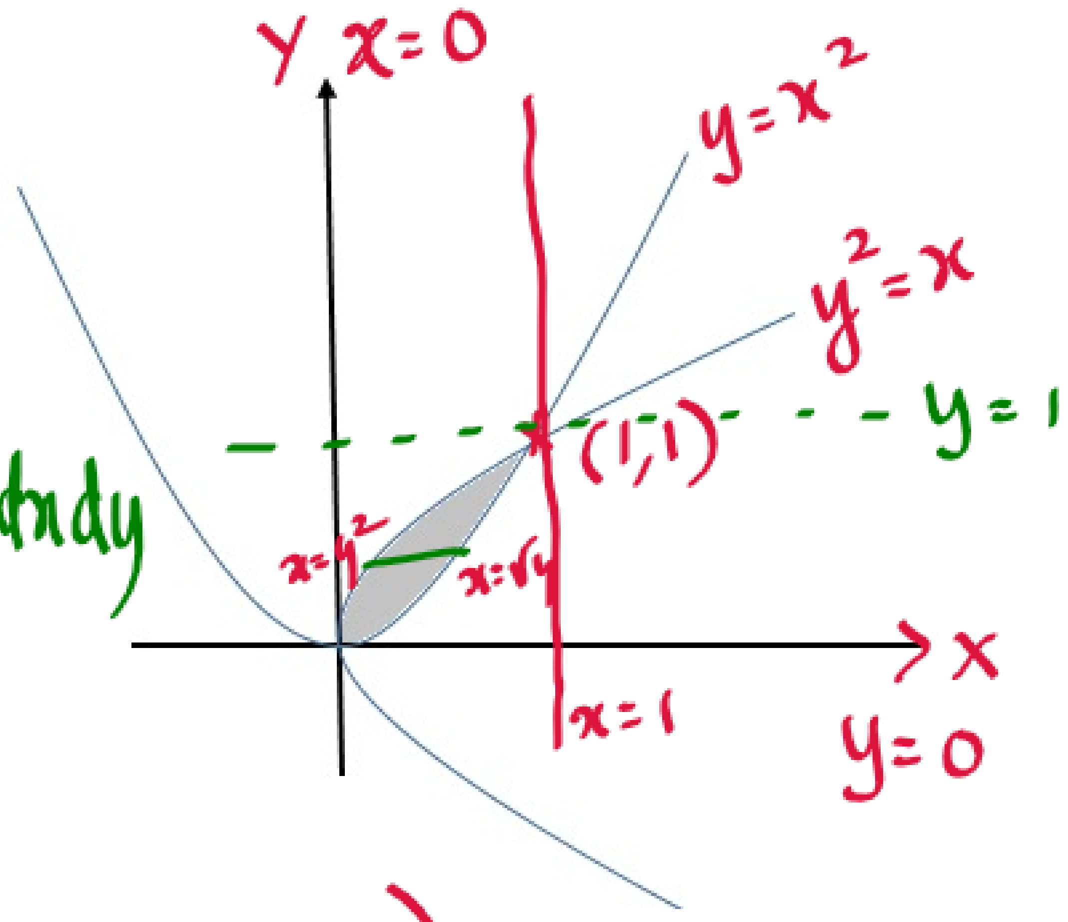
$$= \frac{1}{3}\left[-e^2 - \frac{e^2}{2} + e^3 + \frac{1}{2}\right]$$

$$= \frac{1}{6}\left[2e^3 - 3e^2 + 1\right]$$

Problem 1.6. *Evaluate*

$$\iint_A xy(x+y)\ dx\ dy$$

over the region A bounded by the curves $y = x^2$ and $y^2 = x$.



Ans:-

Let $I = \iint_A (x^2y + xy^2)\,dx\,dy$

$$= \int_{y=0}^{y=1}\left(\int_{x=y^2}^{x=\sqrt{y}} (x^2y + xy^2)\,dx\right)dy$$

$$= \int_{y=0}^{1}\left(y\,\frac{x^3}{3} + y^2\,\frac{x^2}{2}\right)_{x=y^2}^{x=\sqrt{y}} dy$$

$$= \int_{y=0}^{1}\left(\frac{y^{5/2}}{3} + \frac{y^3}{2} - \frac{y^7}{3} - \frac{y^6}{2}\right) dy$$

$$= \left(\frac{2}{21}y^{7/2} + \frac{y^4}{8} - \frac{y^8}{24} - \frac{y^7}{14}\right)_{y=0}^{1} = \underline{\underline{\frac{3}{28}}}$$

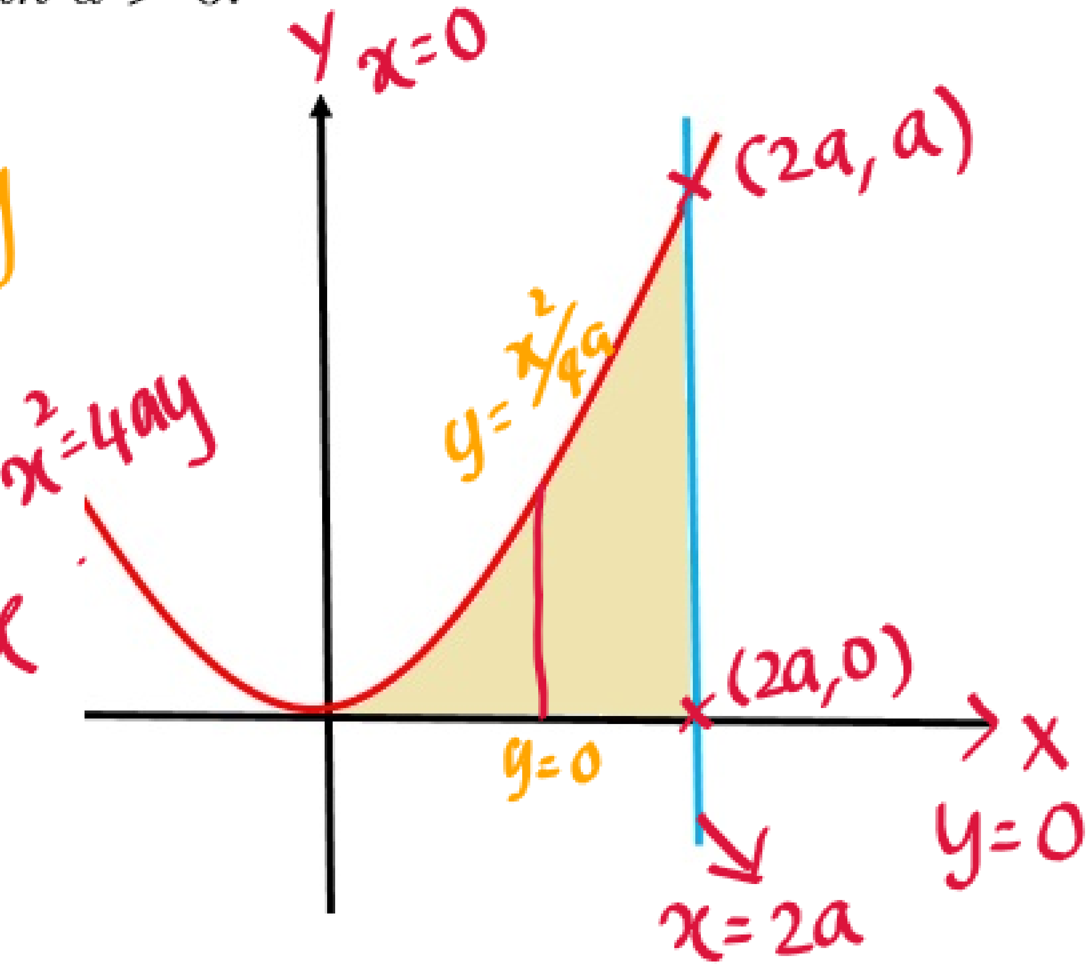
Problem 1.7. *Evaluate*

$$\iint_R xy\ dx\ dy$$

where the region R is bounded by the x-axis between $x = 0$ and $x = 2a$ and the curve $x^2 = 4ay$ with $a > 0$.

Ans: Let $I = \iint_R xy\, dx\, dy$

$$= \int_{x=0}^{x=2a} \left(\int_{y=0}^{y=x^2/4a} xy\ dy \right) dx$$



$$= \int_{x=0}^{2a} x \left(\frac{y^2}{2} \right)_{y=0}^{y=\frac{x^2}{4a}} dx$$

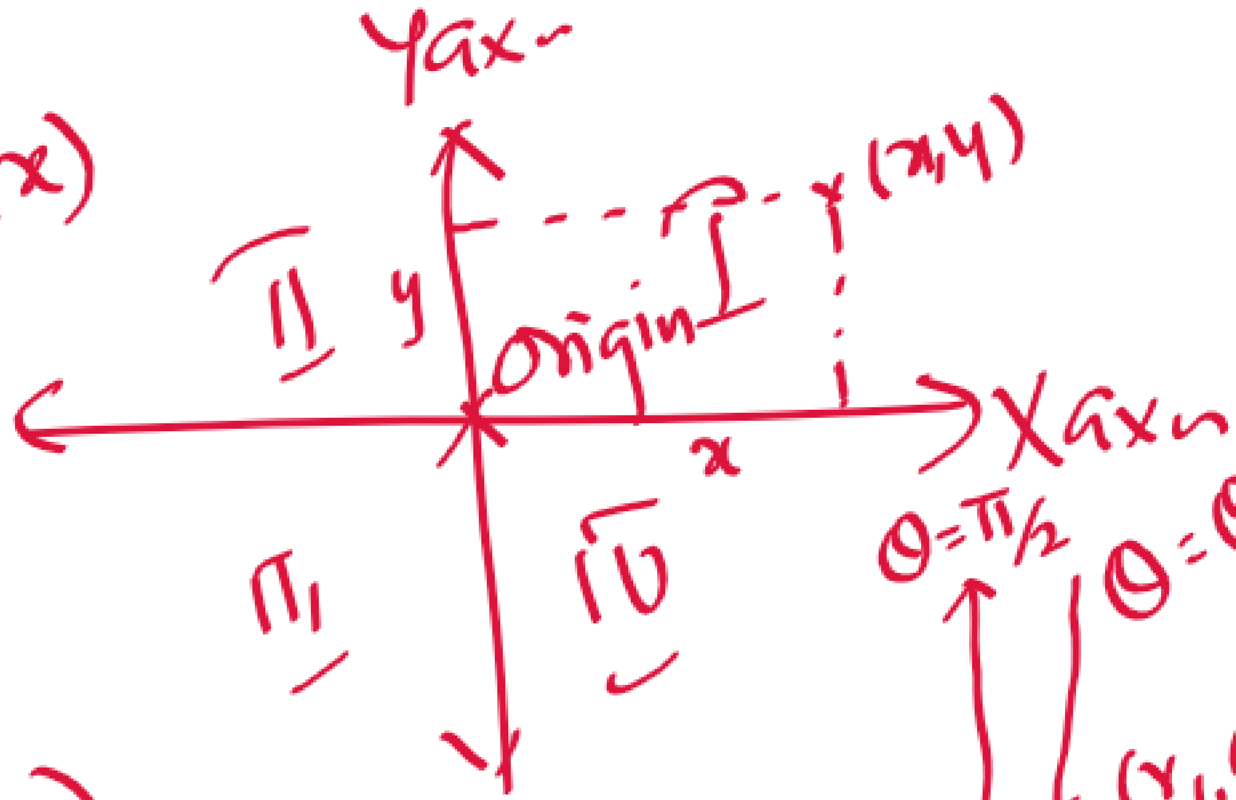
$$= \frac{1}{2} \int_{x=0}^{2a} x \left[\frac{x^4}{16a^2} \right] dx$$

$$= \frac{1}{32a^2} \int_{x=0}^{2a} x^5\ dx = \frac{1}{32a^2} \frac{(2a)^6}{6}$$

$$= \frac{a^4}{3}$$

POLAR CO-ORDINATES

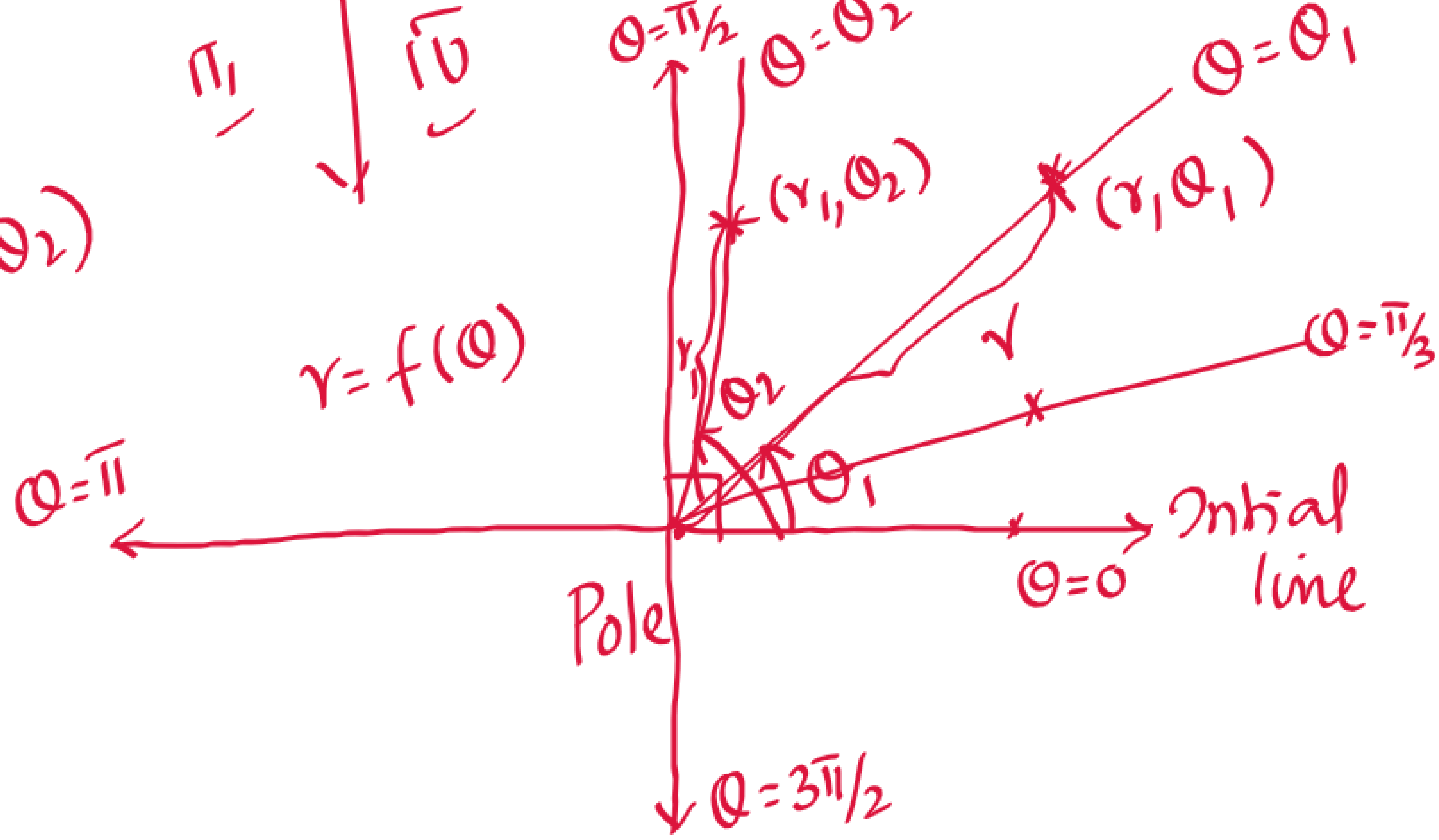
$y = f(x)$



(r, θ_1) $\quad \theta = \theta_1$

(r_1, θ_2)

$r = f(\theta)$



$r = 1 + \cos\theta$

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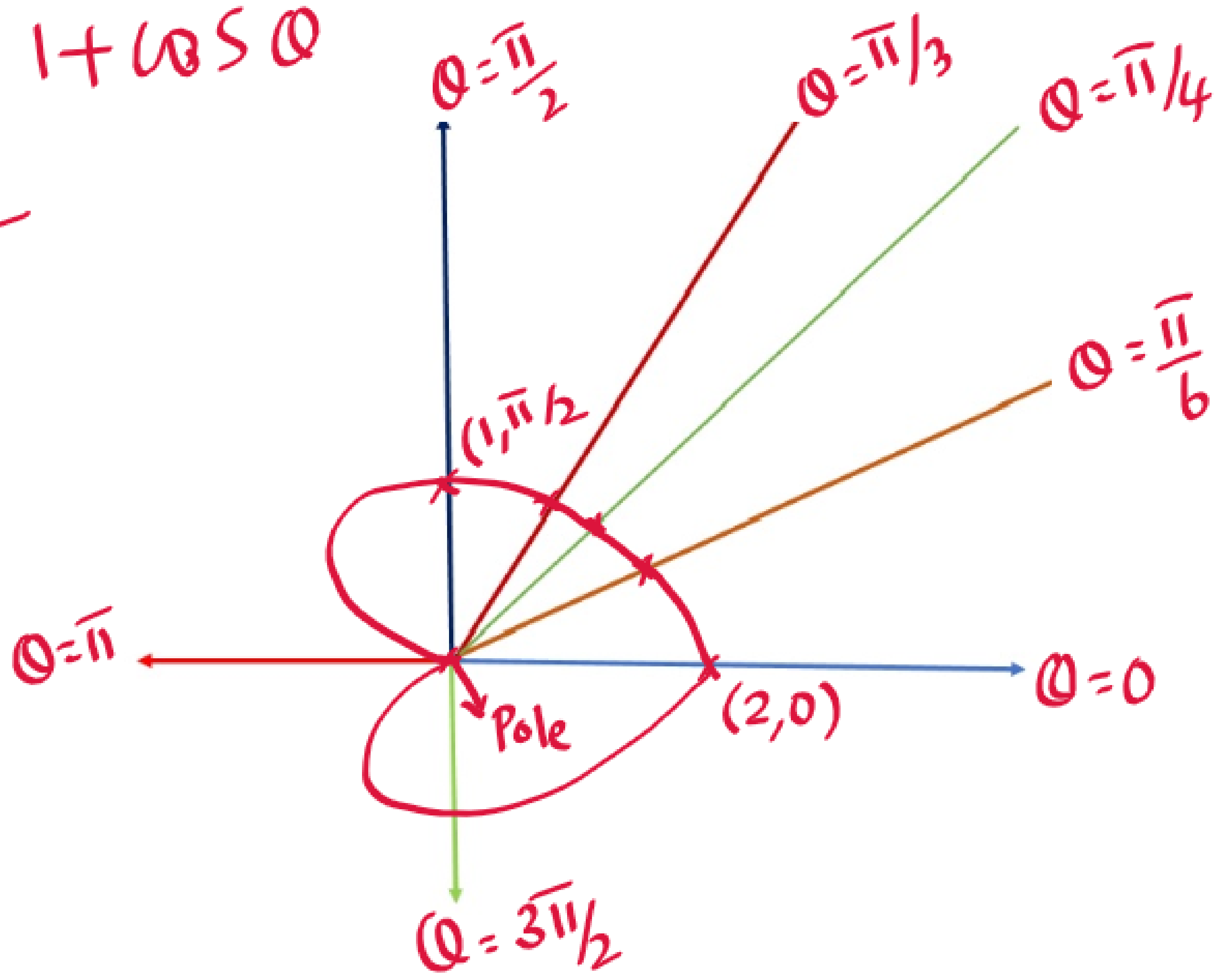
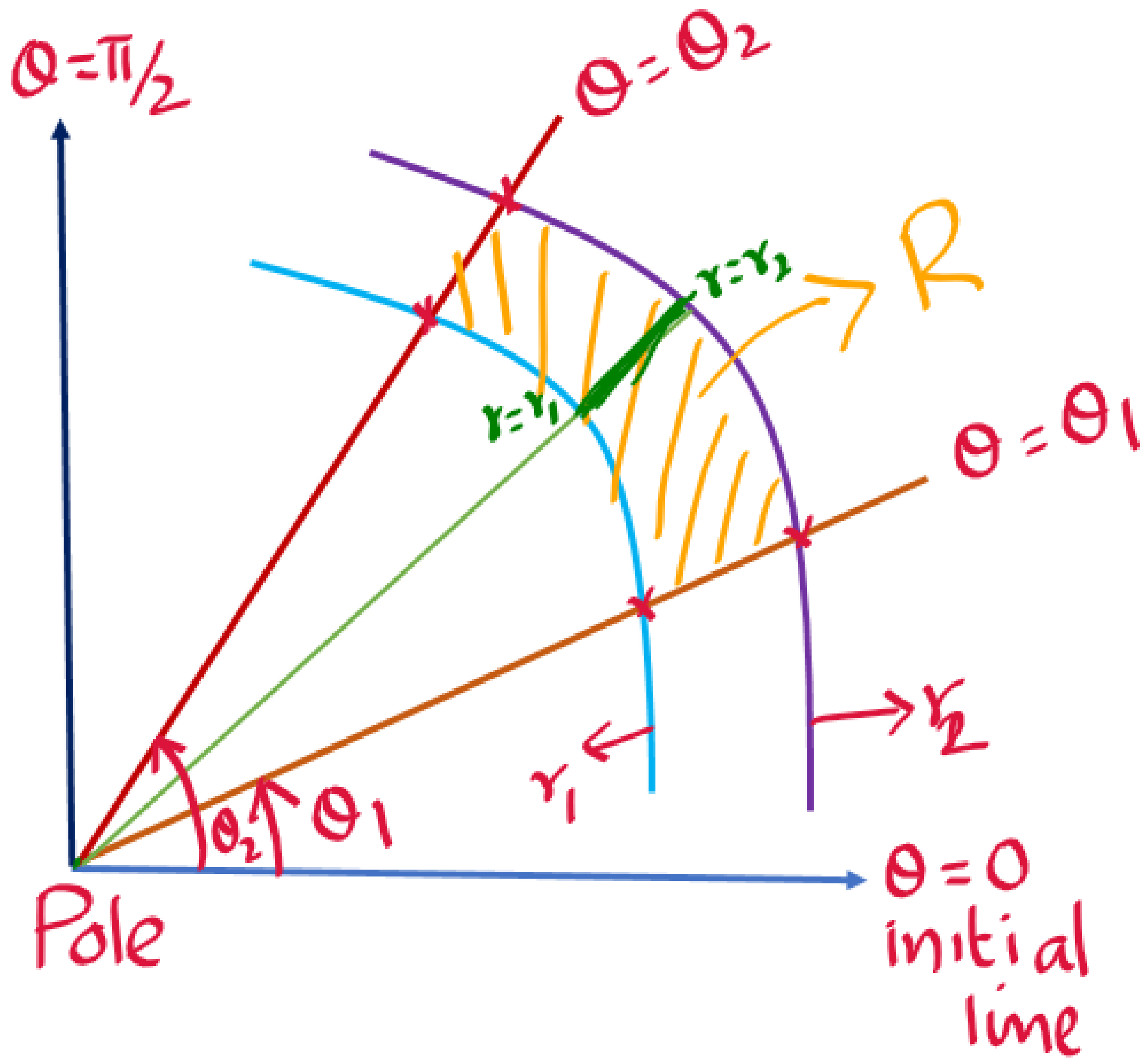


Any curve in polar coordinates is represented by $r = f(\theta)$.

2. Evaluation of double integrals in polar coordinates

Consider the two curves $r_1 = f(\theta)$ and $r_2 = g(\theta)$ and the lines $\theta = \theta_1$ and $\theta = \theta_2$.



$$\therefore \iint_R f(r,\theta)\, dr\, d\theta.$$

$$= \int_{\theta=\theta_1}^{\theta=\theta_2} \left(\int_{r=r_1}^{r=r_2} f(r,\theta)\, dr \right) d\theta$$

ie; this means, to evaluate $f(r,\theta)$ over the region bounded by the two lines $\theta = \theta_1$ and $\theta = \theta_2$ and the curves $r_1 = f(\theta)$ and $r_2 = g(\theta)$.

$$(x,y) \longmapsto (r\cos\theta, r\sin\theta)$$

Problem 2.1. *Evaluate*

$$\iint_A r^3 \ dr \ d\theta$$

ie; $x = r\cos\theta$
$y = r\sin\theta$

over the area A ~~is~~ bounded between the circles $r = 2\cos\theta$ *and* $r = 4\cos\theta$.

Ans:- we're, $r = 2\cos\theta$

$$\Rightarrow r^2 = 2r\cos\theta$$

$$\Rightarrow x^2+y^2 = 2x \Rightarrow x^2-2x+y^2=0$$

$$\Rightarrow (x^2-2x+1)+y^2 = 1$$

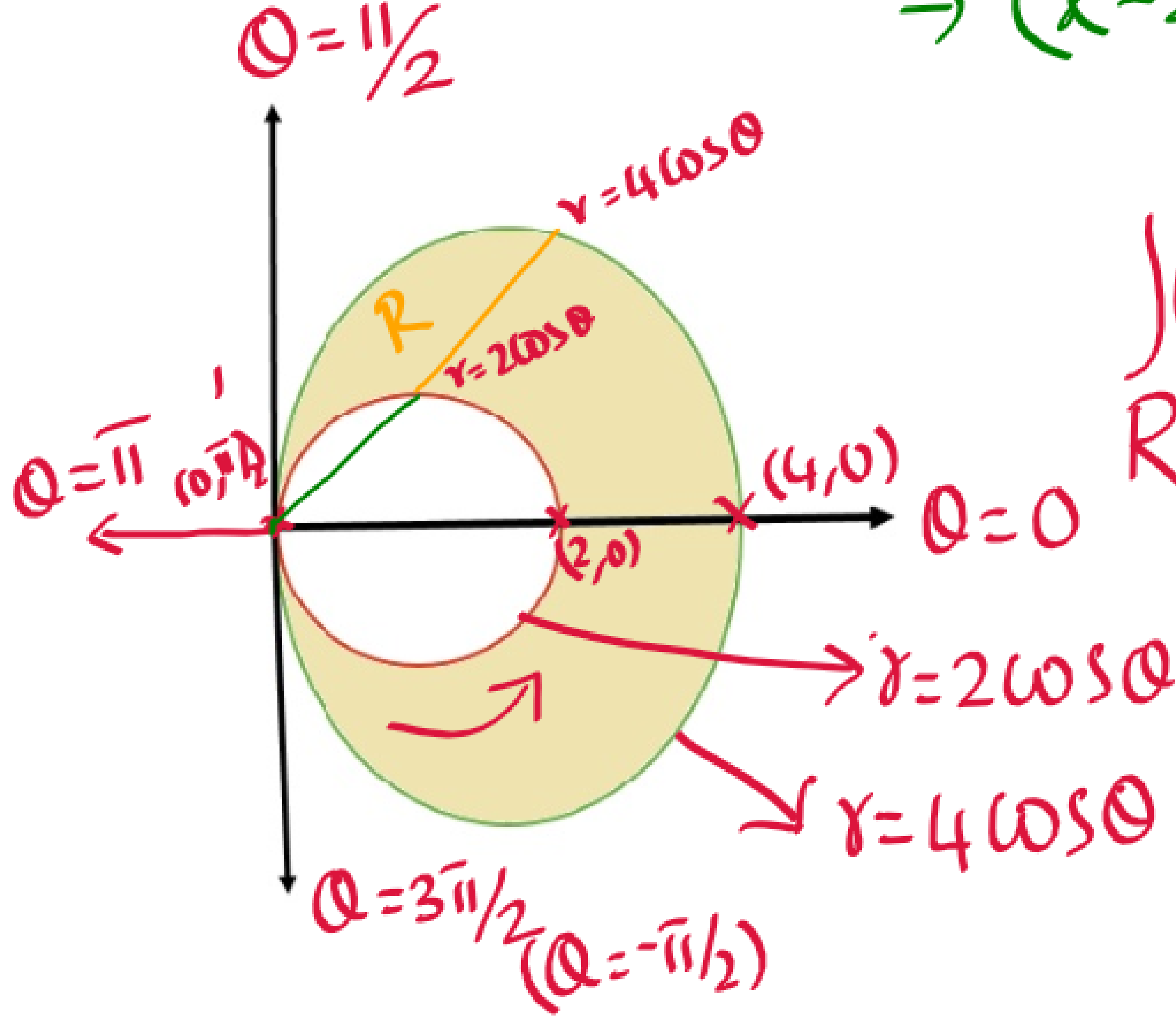
$$\Rightarrow (x-1)^2+(y-0)^2 = 1^2$$

|||ly $r = 4\cos\theta \Rightarrow r^2 = 4r\cos\theta$

$$\Rightarrow x^2+y^2 = 4x$$

$$\Rightarrow (x^2-4x+4)+y^2 = 4$$

$$\Rightarrow (x-2)^2+y^2 = 4$$



$$\iint_R r^3 dr\, d\theta = \int_{\theta=-\frac{\pi}{2}}^{\pi/2} \left(\int_{r_1=2\cos\theta}^{r=4\cos\theta} r^3 dr \right) d\theta$$

$$= \int_{\theta=-\pi/2}^{\pi/2} \left(\frac{r^4}{4}\right)_{r=2\cos\theta}^{r=4\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left(64\cos^4\theta - 4\cos^4\theta\right) d\theta$$

$$= 60 \int_{-\pi/2}^{\pi/2} \cos^4\theta \, d\theta$$

$$= 120 \int_{0}^{\pi/2} \cos^4\theta \, d\theta$$

$$= 120 \frac{(4-1)(4-3)}{4(4-2)} \times \frac{\pi}{2}$$

$$= \frac{45\pi}{2}$$

PRACTICE PROBLEMS

Q. Evaluate the following integrals:

(i) $\displaystyle\int_1^a \int_1^b \frac{dy\,dx}{xy}$ Ans: $\log(a)\log(b)$.

(ii) $\displaystyle\int_{x=0}^{1} \int_{y=0}^{1} \frac{x-y}{(x+y)^3}\,dy\,dx$ Ans: $\frac{1}{2}$.

(iii) $\displaystyle\int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx$ Ans:- $\frac{\pi a^3}{6}$.

(iv) $\displaystyle\int_0^1 \int_y^{y^2+1} x^2 y\,dy\,dx$ Ans:- $\frac{67}{120}$

Q. Evaluate $\displaystyle\iint_R (x+y)^2\,dx\,dy$ where R is the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Ans: $\frac{\pi}{4}(ab)(a^2+b^2)$.

Q. Let R be the region bounded by the circle $x^2+y^2=1$ in the first quadrant. Then evaluate, $\displaystyle\iint_R \frac{xy}{\sqrt{1-y^2}}\,dx\,dy$. Ans:- $\frac{1}{6}$.