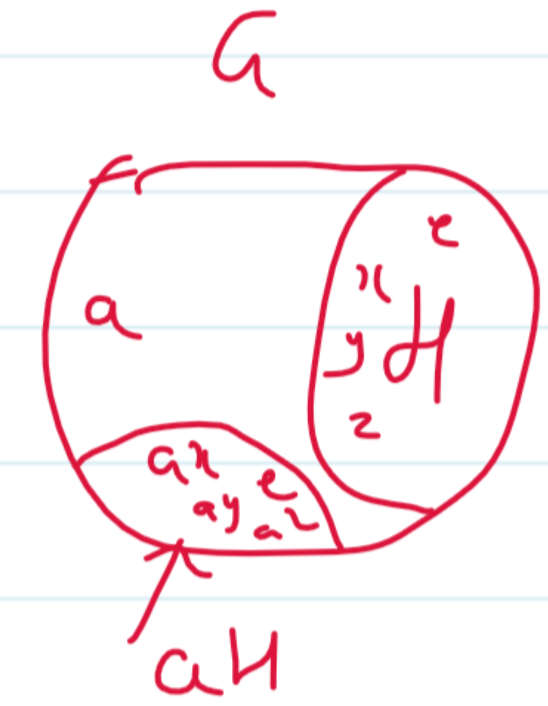
Coset :

Let $(G, \circ)$ be a group. Let H be a subgroup of G.
Let $a \in G$, then

$Ha = \{ ha \mid h \in H \}$ is right coset of H in G.

and

$aH = \{ ah \mid h \in H \}$ is left coset of H in G.



Example :

$G = \{ 1 \quad -1 \quad i \quad -i \}$

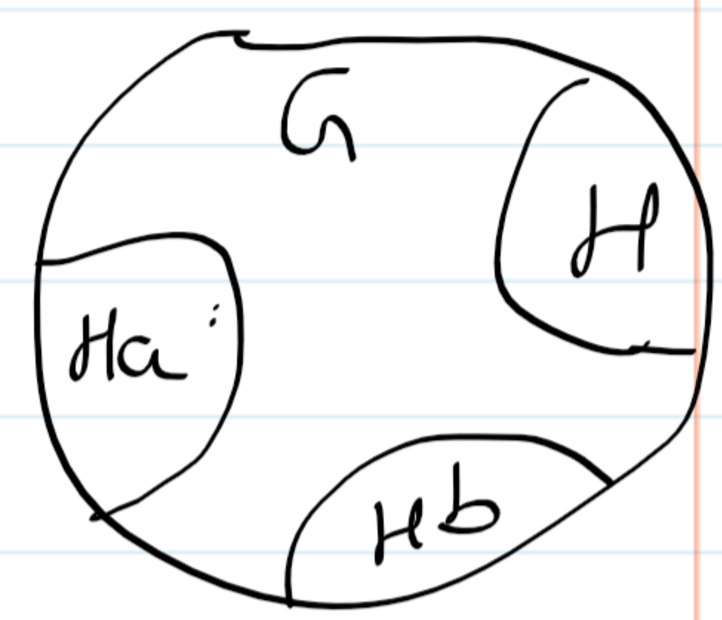
$H = \{ 1 \quad -1 \}$ subgroup.

$Hi = \{ 1 \times i, \ -1 \times i \} = \{ i, -i \} \neq H$ also Hi is not a subgroup of G.

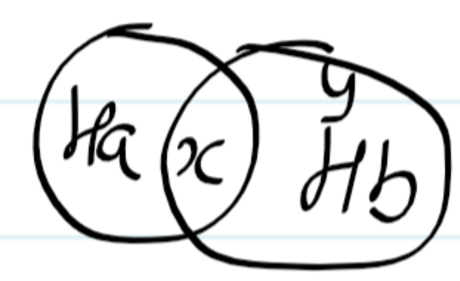
$1 \cdot H = \{ 1 \quad -1 \} = H$

A coset is a non empty subset of G.
But it not need be a subgroup of G.

Theorem 1: Let G be a group and let H be a subgroup of G. Then any 2 right cosets of H in G are either identical or disjoint.



Proof: Let Ha and Hb be 2 right cosets of H in G.
If they are disjoint, then there is nothing to prove.
Suppose they are not disjoint, we must prove that they are identical.



Let $x \in Ha \cap Hb$

$\Rightarrow x \in Ha, \quad x \in Hb$

$x = h_1 a, \quad x = h_2 b$

for some $h_1, h_2 \in H$

To prove $Ha = Hb$
$Ha \subseteq Hb$
$Hb \subseteq Ha$

$x = h_2 b$

$h_2^{-1} \cdot x = b$

$h_2^{-1}(h_1 a) = b \quad - (1)$

Let $y \in Hb$ — To show $y \in Ha$

$y = hb$ for, $h \in H$

$= h(h_2^{-1} h_1 a)$ — from (1)

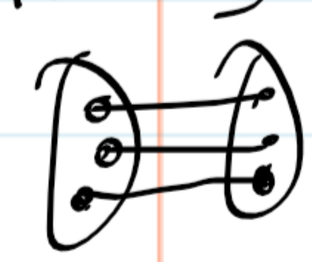
$= (h h_2^{-1} h_1) a$ — $h h_2^{-1} h_1 = h_3 \in H$

$y = h_3 a \Rightarrow h_3 a \in Ha \Rightarrow y \in Ha$

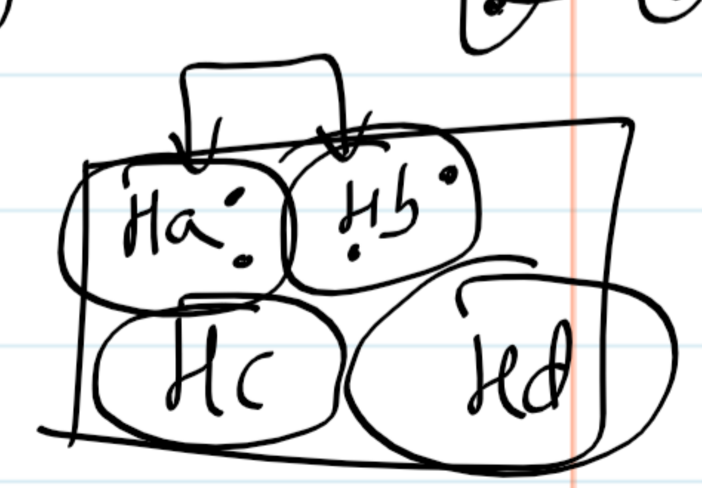
$\Rightarrow Hb \subseteq Ha \quad - (2)$

By a similar argument, we can show $Ha \subseteq Hb \quad - (3)$. From (2) & (3) we get

$f: A \to B$



$$Ha = Hb$$



Theorem 2: Any 2 right cosets of a subgroup H in G are in one to one correspondance with each other.

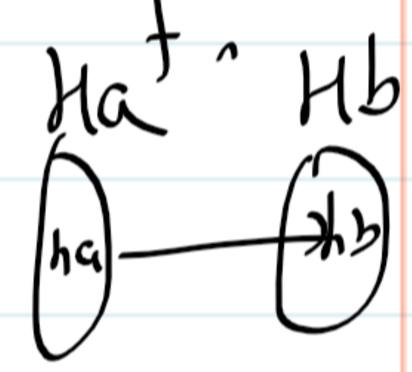
[Let A and B be 2 sets. Let $f: A \longrightarrow B$ be a function. If f is one-to-one and onto, then A and B are said to be in one to one correspondance with each other. Also, if A and B are finite, then $n(A) = n(B)$] $n(A)$: number of elements in A.

Proof: Let H be a subgroup of G. Let $a, b \in G$, $a \neq b$. Then Ha and Hb are 2 distinct right cosets of H in G.

Define a function $f: Ha \longrightarrow Hb$ by

$f(ha) = hb$, for $h \in H$

To check whether f is well defined



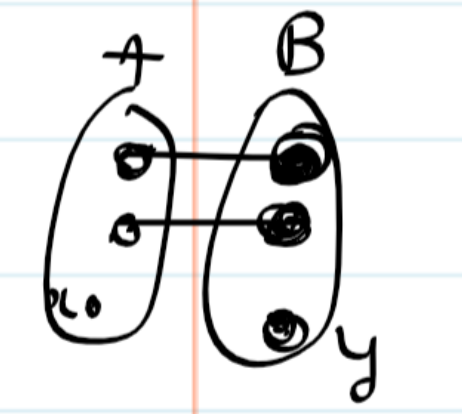
suppose $h_1 a = h_2 a$

$\Rightarrow h_1 = h_2$ right cancellation

$h_1 b = h_2 b$

$f(h_1 a) = f(h_2 a) \Rightarrow$ well defined.

To show



f is one one: suppose $f(h_1 a) = f(h_2 a)$

$h_1, h_2 \in H$

$h_1 b = h_2 b$ right cancellation

$h_1 = h_2$

$h_1 a = h_2 a \Rightarrow$ one one

To show

f is onto: Let $hb \in Hb$

$\Rightarrow h \in H$

$h.a \in Ha$

$f: X \to Y$
for every $y \in Y$, $\exists$ a atleast $x \in X$ s.t $f(x) = y$
onto

$\Rightarrow$ For any $hb \in Hb$, there exist a $ha \in Ha$ s.t $f(ha) = hb \Rightarrow$ onto

$\Rightarrow$ Any 2 right cosets of H in G are in one to one correspondance with each other.

If H is finite, $O(Ha) = O(Hb)$ here

$O(Ha) \to$ order of the set Ha.

Theorem 3: Any 2 left cosets of H in G have the same number of elements or any two left cosets of a subgroup H in G are in one-to-one correspondance with each other.

Proof: Let aH and bH be 2 left cosets of H in G.

Define a mapping $f: aH \longrightarrow bH$ given by $f(ah) = bh$ for all $h \in H$. f is well defined.

To prove f is one one:

For $f(ah_1) = f(ah_2)$

$$bh_1 = bh_2 \quad , \quad h_1, h_2 \in H$$

$$h_1 = h_2 \quad \text{by left cancellation}$$

$$\Rightarrow ah_1 = ah_2$$

$$\Rightarrow f \text{ is one one.}$$

To prove f is onto:

$$bh \in bH$$

$$\Rightarrow h \in H$$

$$\Rightarrow ah \in aH$$

So for any given $bh \in bH$, there exist a $ah \in aH$ s.t $f(ah) = bh$

$\Rightarrow f$ is onto

$\Rightarrow$ Any 2 left cosets are in one to one correspondance with each other.

So aH and bH have same number of elements.

Theorem 4: Suppose H is a subgroup of G. For $a \in G$, $aH = H$ if and only if $a \in H$.
($Ha = H$ iff $a \in H$)

Proof: Let $a \in H$. To prove $aH = H$

To show $\boxed{aH \subseteq H \quad H \subseteq aH}$

Let $x \in aH$

$x = ah$, for some $h \in H$

As $a \in H$ & $h \in H$ $\Rightarrow$ by closure property we get
$ah \in H \Rightarrow x \in H \Rightarrow aH \subseteq H$. $-(1)$

Consider $h \in H$ $\qquad a \in H,\ h \in H,\ a^{-1} \in H$

$h = aa^{-1}h = a(a^{-1}h) \in aH$ $\qquad a^{-1}h \in H$

$H \subseteq aH \quad -(2)$

$\Rightarrow \quad H = aH$

Conversely, if $aH = H$ to prove $a \in H$.

$a = a.e \in aH = H$, as $e \in H$

$\Rightarrow \quad a \in H$

Definition: The number of distinct right cosets (left cosets) of H in G is called index of H & denoted by $i_G(H)$.

Theorem 5: Suppose H is a subgroup of G. The number of distinct left cosets of H in G is equal to the number of distinct right cosets of H in G.

Or

If H is a subgroup of G, then there exist one to one correspondance between the set of left cosets of H in G and the set of right cosets of H in G.

Proof: Let $L = \{aH \mid a \in G\}$ & $R = \{Ha \mid a \in G\}$

Define a mapping $f : L \to R$ s.t

$f(aH) = Ha^{-1}$ for all $a \in G$.

If aH is a left coset of H, then Ha^{-1} is a right coset

To show f is well defined:

suppose $aH = bH$

$$b^{-1}aH = H$$

$$b^{-1}a \in H$$

by Th 4: $xH = H$, $Hx = H$ $\Rightarrow x \in H$

$$(b^{-1}a)^{-1} \in H$$

$$a^{-1}b \in H$$

$$Ha^{-1}b = H \longrightarrow \text{from Th 4:}$$

$$Ha^{-1} = Hb^{-1}$$

$$f(aH) = f(bH)$$

well defined.

To show one one: If $f(aH) = f(bH)$ to prove $aH = bH$

$$\Rightarrow Ha^{-1} = Hb^{-1}$$

$$Ha^{-1}a = Hb^{-1}a$$

$$H = Hb^{-1}a \quad \text{by Th 4}$$

$$b^{-1}a \in H$$

$$(b^{-1}a)^{-1} \in H$$

$$a^{-1}b \in H \quad \text{Th 4}$$

$$a^{-1}bH = H \Rightarrow aa^{-1}bH = aH \Rightarrow bH = aH$$

$\Rightarrow$ One One.

f is onto : For any $Ha \in R$, there exist $a^{-1}H \in L$ s.t
$f(a^{-1}H) = H(a^{-1})^{-1} = Ha$

$\Rightarrow$ On to

$\Rightarrow$ There is a one to one correspondance