Groups

$(G, *)$ is said to be a group if $*$ satisfies

i) closure law
ii) Associative "
iii) Identity "
iv) Inverse "

ex:- ① $(\mathbb{Z}, +)$
② $(R-\{0\}, \cdot)$
③ $(M_{n\times n}, \cdot)$ $M_{n\times n} \to$ set of all invertible matrices of order n
$\cdot \to$ matrix $\times^n$
$e = I$

Subgroup

Let $(G, *)$ be a group, and H be a nonempty subset of G. Then H is said to be a subgroup of G if H itself forms a group under the same operation $*$

ex:- $(\mathbb{Z}, +)$ is a group
$\underbrace{(\mathbb{Z}_{2n}, +)}_{\text{set of all even integers}}$ is a subgroup of $(\mathbb{Z}, +)$ ↓ set of all integers

ex:- $(\mathbb{Z}, +)$ $(\mathbb{Q}, +)$ $(\mathbb{R}, +)$ $(\mathbb{C}, +)$ are gps
↳ Integers ↳ Rational ↳ real ↳ complex no

$\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$

ex:- $(\mathbb{Q}-\{0\}, \cdot)$ is a subgroup of $(R-\{0\}, \cdot)$ are gps

(4) $G = \{1, -1, i, -i\}$ and $* = \cdot$

$(G, \cdot)$ forms a group

$\cdot$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	$+i$
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

- closure ✓
- associative ✓
- 1 is identity elt
- $1^{-1} = 1$ $\quad (-1)^{-1} = 1$

  $i^{-1} = (-i)$

  $(-i)^{-1} = i$

consider $H = \{1, -1\}$, then $(H, \cdot)$ is a subgroup of $(G, \cdot)$

$H_2 = \{i, -i\}$ is not a subgp of $(G, \cdot)$

(5) $G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, ad - bc \neq 0 \right\}$ $(G, \cdot)$ is a group.

$H = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| \, ad \neq 0 \right\}$, then $(H, \cdot)$ is a subgp of $(G, \cdot)$

$H_2 = \left\{ \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \middle| \, -b \neq 0 \right\}$, $(H_2, \cdot)$ is not a subgp

Reason $\rightarrow$ $I \notin H_2$

$\therefore$ Identity law fails.

Note

For any gp $(G, *)$, there are always 2 trivial subgps i) $(\{e\}, *)$ ii) Itself

Theorem : A nonempty subset H of a group $(G, *)$ is a subgroup of G iff the follow cond^ns are satisfied i) $a*b \in H \quad \forall \ a,b \in H$ (closure)

ii) $a^{-1} \in H \quad \forall \ a \in H$ (inverse)

Proof :-

If $(H, *)$ is a subgp of $(G,*)$. Then by the defn of group (i) & (ii) are true

converse :- If the condns (i) & (ii) are true, we've to P.T $(H, *)$ is a subgroup

Since H is a nonempty subset, $\exists$ atleast elt $a \in H$,

- By (ii), we get $a^{-1} \in H$ → (inverse law holds)
- Let $a \in H$, $a^{-1} \in H$, By (i) $a*a^{-1} = e \in H$

$\therefore$ Identity law holds

- Closure law (i)
- Associative law holds

$\therefore (H, *)$ itself is a gp

$\therefore (H, *)$ is a subgp of $(G, *)$

Theorem: A nonempty subset H of a group $(G, *)$ is a subgroup of G iff $a * b^{-1} \in H$ $\forall$ $a, b \in H$

Proof

Let H be a subgp of G. (All 4 laws are true)

$\forall$ $a, b \in H$, $a * b \in H$ (By closure law)

$\forall$ $b \in H$, $b^{-1} \in H$ (Inverse law)

$\therefore$ $\forall$ $a, b \in H \Rightarrow a \in H$ & $b^{-1} \in H$

Now applying closure law $a * b^{-1} \in H$

Converse: Let $a * b^{-1} \in H$ for all $a, b \in H$

To P.T H is a subgp of G.

Since H is nonempty, $\exists$ atleast one elt $a \in H$

- $a \in H$, $a \in H$, then $a * a^{-1} \in H$

  $e \in H$ $\therefore$ Identity law holds

- $e \in H$, $a \in H$, then $e * a^{-1} \in H$

  $a^{-1} \in H$

  $\therefore$ Inverse law holds

- Associative law holds

- $a \in H$, $b^{-1} \in H$, $a * (b^{-1})^{-1} \in H$

  $a * b \in H$

  $\therefore$ closure law

$\therefore$ H is a subgp of G

① Let $(G, *)$ be a gp. Let H_1 & H_2 be 2 subgps of G. Check whether i) $H_1 \cap H_2$ is a subgp

ii) $H_1 \cup H_2$ is a subgp

Soln

$\boxed{\text{subgp} \Leftrightarrow a * b^{-1} \in H \ \forall\ a, b \in H}$

i) Given H_1 H_2 are subgps.

$\therefore e \in H_1$ and $e \in H_2$

$e \in H_1 \cap H_2$ & $H_1 \cap H_2$ is nonempty

Let $a, b \in H_1 \cap H_2 \Rightarrow a, b \in H_1$ and $a, b \in H_2$

$a * b^{-1} \in H_1$ ($\because H_1$ is a subgp)

$\Rightarrow a * b^{-1} \in H_2$ (as H_2 is a subgp)

$\therefore a * b^{-1} \in H_1 \cap H_2$

$\therefore \forall\ a, b \in H_1 \cap H_2$, $a * b^{-1} \in H_1 \cap H_2$

$\therefore H_1 \cap H_2$ is a subgp of $(G, *)$

ii) $H_1 \cup H_2$ is not a subgp of $(G, *)$

$(G, *) = (\mathbb{Z}, +)$

$H_1 = \mathbb{Z}_{2n} = \{ \ldots -6, -4, -2, 0, 2, 4, 6, \ldots \}$

$H_2 = \mathbb{Z}_{3n} = \{ \ldots -9, -6, -3, 0, 3, 6, 9, \ldots \}$

$H_1 \cup H_2 = \{ \ldots -9, -6, -4, -3, -2, 0, 2, 3 \ldots \}$

$2, 3 \in H_1 \cup H_2$

$2+3 = 5 \notin H_1 \cup H_2$

$\because$ closure law fails

$\therefore$ $H_1 \cup H_2$ is <u>not</u> a subgp

② Let $(H, \cdot)$ and $(K, \cdot)$ be two subgps of $(G, \cdot)$

Define $HK = \{hk \mid h \in H, k \in H\}$

P.T HK is a subgp of G iff $HK = KH$.

<u>Proof</u>

suppose HK is a subgp of G. Then we've to P.T $HK = KH$ (Prove $HK \subseteq KH$ & $KH \subseteq HK$)

let $x \in KH \Rightarrow x = kh$ where $k \in K$ & $h \in H$

$x^{-1} = (kh)^{-1} = h^{-1}k^{-1} \in HK$ where $h^{-1} \in H$, $k^{-1} \in K$

since HK is a subgp, if $x^{-1} \in HK$

$(x^{-1})^{-1} \in HK$ ($\because$ Inverse law)

$\Rightarrow x \in HK$

we showed, if $x \in KH$, then $x \in HK$

$\Rightarrow KH \subseteq HK$. ($\because$ all the elts of KH are elts of HK)

similarly we can prove $HK \subseteq KH$

$\therefore$ $HK = KH$

Converse :- Let $HK = KH$, we've to P.T HK is a subgp

since H & K are subgps, $e \in H$ & $e \in K$

$$\therefore \quad e \cdot e = e \in HK$$

$\therefore$ HK is nonempty

Let $a, b \in HK$

$a = h_1 k_1$ & $b = h_2 k_2$ where $h_1, h_2 \in H$, $k_1, k_2 \in K$

$$ab^{-1} = h_1 k_1 (h_2 k_2)^{-1}$$

$$= h_1 \underbrace{k_1 k_2^{-1} h_2^{-1}}$$

Now $k_1 k_2^{-1} h_2 \in KH = HK$

$k_1 k_2^{-1} h_2 = h_3 k_3$

for some $h_3 \in H$, $k_3 \in K$

$$= \underbrace{h_1 h_3} k_3$$

$$= h_4 k_3 \quad \text{where } h_4 = h_1 h_3 \in H$$

$ab^{-1} \in HK$

$\therefore$ HK is a subgp of G

Cosets :-

Let G be a group. H be a subgp of G. For any elt $a \in G$

the set $Ha = \{ha \mid h \in H, a \in G\} \longrightarrow$ right coset of H in G

$aH = \{ah \mid h \in H, a \in G\} \longrightarrow$ left coset of H in G.

ex:- $(\mathbb{Z}, +)$ is a gp

$(\mathbb{Z}_{2n}, +)$ is a subgp of $(\mathbb{Z}, +)$

$\mathbb{Z}_{2n} = \{ \dots -6, -4, -2, 0, 2, 4, 6, \dots \}$

$3 \in \mathbb{Z}$

$3 + H = \{ \dots -3, -1, 1, 3, 5, 7, 9, \dots \}$

left coset

$5 + H = \{ \dots -1, 1, 3, 5, 7, 9, 11, \dots \}$

$||$

$H + 5 = \{ \dots -1, 1, 3, 5, 7, 9, 11, \dots \}$

(2) $(G, \cdot)$, $G = \{1, -1, i, -i\}$

$H = \{1, -1\}$ & $(H, \cdot)$ is a subgp of $(G, \cdot)$

$Hi = \{i, -i\}$

$H_{-i} = \{-i, i\}$

Note

A left/right coset is a subset, need not be a subgroup of G.

Thm 1: Let G be a group and H be a subgroup. Then any 2 right cosets of H in G are either identical or disjoint

proof:-

Let Ha & Hb be two right cosets of H in G.
If Ha & Hb are disjoint, there is nothing to prove
If they are not disjoint, we must prove they are identical

Let Ha & Hb are not disjoint, ie $H_a \cap H_b \neq \phi$

Let $x \in Ha \cap Hb$

$\Rightarrow x \in Ha$ and $x \in Hb$

$\Rightarrow x = h_1 a$ and $x = h_2 b$ where $h_1, h_2 \in H$

$\Rightarrow b = h_2^{-1} x$

$\Rightarrow b = h_2^{-1} h_1 a$

Let $y \in Hb \Rightarrow y = h_3 b$ where $h_3 \in H$

$= \underbrace{h_3 h_2^{-1} h_1}\, a$

$= h_4 a$ where $h_4 = h_3 h_2^{-1} h_1 \in H$

$y \in Ha$

$Hb \subseteq Ha$

similarly, $Ha \subseteq Hb$

$\therefore Ha = Hb$ $\quad \therefore Ha$ & Hb are identical