Eg: Show that $Z_5 = (G, \oplus_5) = \{0, 1, 2, 3, 4\}$ forms a group under addition modulo 5.

$G \oplus_n = \{0, 1, \ldots n-1\}$

$\cancel{\otimes_5}$ $\cancel{\odot_5}$

$\oplus_5$	0	1	2	3	4
$\boxed{0}$	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$1 \oplus_5 4 = \frac{5}{5}$ $R = 0$

$2 \oplus_5 3 = \frac{5}{5}$ $R = 0$

$2 \oplus_5 4 = \frac{6}{5}$ $R = 1$

$(3 \oplus_5 0) \oplus_5 2$

→ In this table, all the elements are in G

i.e $a \oplus_5 b \in G$, $\forall a, b \in G$

→ closure satisfies.

→ $\parallel^{ly}$ associative law satisfies.

i.e, $\forall a, b, c \in G$, $(a \oplus_5 b) \oplus_5 c = a \oplus_5 (b \oplus_5 c)$

→ Identity element is '0'

→ Inverse law, $\bar{0}^1 = 0$, $\bar{1}^1 = 4$

$\bar{2}^1 = 3$, $\bar{3}^1 = 2$

$\bar{4}^1 = 1$

∴ Therefore $(G, \oplus_5)$ is a group

P.T the set $G = \{1, 2, 3, 4, 5, 6\}$ is abelian group of order 6 w.r.to multiplication modulo 7

$\left[\begin{array}{l} \otimes_7 \\ \odot_7 \end{array}\right.$

$\otimes_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

$2 \otimes_7 4 = \frac{8}{7}$

$R = 1$

$2 \otimes_7 5 = \frac{10}{7}$

$R = 3$

→ From the table,

$a \otimes_7 b \in G \quad , \quad \forall a, b \in G$

→ IIIly associative holds.

i.e., $(a \otimes_7 b) \otimes_7 c = a \otimes_7 (b \otimes_7 c)$

$\forall a, b, c \in G$

→ Identity element is 1.

→ Inverse: $1^{-1} = 1 \quad , \quad 2^{-1} = 4 \quad , \quad 3^{-1} = 5$

$4^{-1} = 2 \qquad 5^{-1} = 3$

$6^{-1} = 6$

→ Here $a \otimes_7 b = b \otimes_7 a \quad \forall a, b \in G$

i.e., $2 \otimes_7 5 = 3$

$5 \otimes_7 2 = 3$

$\boxed{\therefore G \text{ is abelian}}$

Q: A non-empty subset H of a group $(G, *)$ is a subgroup of G iff the following are satisfied:

✓ (i) $a * b \in H$, $\forall\, a, b \in H$

✓ ii) $a^{-1} \in H$, $\forall\, a \in H$

Ans: Let H be a non-empty subset of G.

($\Rightarrow$) Suppose H is a subgroup of G, then (i) and (ii) follows by definition

($\Leftarrow$) Suppose (i) and (ii) holds in H

We must prove that H is a subgroup of G.

— Closure property follows from (i)

— Inverse property follows from (ii)

— Elements of H are also elements of G $(H \subset G)$

$\therefore$ associative follows from G.

Since $H \neq \phi$, there is an element $a \in H$

By hypothesis $a^{-1} \in H$ (by (ii))

By (i), $a * a^{-1} \in H$ $\qquad a*b \in H$

$e \in H$, identity axiom satisfied.

$\Rightarrow H$ is a subgroup of G.

Q: A nonempty subset H of a group $(G,*)$ is a subgroup of G iff $a*b^{-1} \in H$, $\forall\, a,b \in H$.

Ans:

Let H be a non-empty subset of G.

($\Rightarrow$) Let H be a subgroup of G

$\forall\, a, b \in H$, $a*b \in H$ (closure)

Also, $a \in H$, $b^{-1} \in H$

$\therefore a*b^{-1} \in H$ (closure)

$\Leftarrow$ Let $a*b^{-1} \in H$, $\forall\, a, b \in H$ — (1)

To prove that H is a subgroup of G.

Since $H \neq \phi$, there is an element $a \in H$

$a \in H$, $a \in H$ $\Rightarrow$ $a*a^{-1} \in H$ (by (1))

$e \in H$

$\therefore$ identity axiom holds

$\forall\, a \in H$, We know that $e \in H$

$e*a^{-1} \in H$ by (1)

$a^{-1} \in H$

$\therefore$ inverse axiom holds.

Since H is a subset of G, associative follows.

Let $a, b \in H$, as $b \in H$, $b^{-1} \in H$

we have, $a*(b^{-1})^{-1} \in H$ by (1)

$a*b \in H$ (closure law holds)

$\therefore (H,*)$ forms a group

$\Rightarrow$ H is a subgroup of $(G,*)$.