ENGINEERING MATHEMATICS-III

MAT 2155

- Permutation and Combinations
- Lattice Theory
- Graph Theory
- Group Theory
- Propositional and Predicate calculus

Permutation and combination

Reference:

An introduction to computational Combinatorics by E S Page and L B Wilson

Applied Combinatorics by Alan Tucker

Elements of Discrete Mathematics by C L Liu and D P Mohapatra

Permutation and combination

Addition Principle (OR):

Suppose that procedure 1 can be performed in m different ways, and procedure 2 can be performed in n different ways, and suppose that these procedures can not be performed simultaneously. Then number of ways in which one of these procedure can be performed is $(m + n)$.

Multiplication Principle (And):

Suppose that procedure 1 can be performed in m different ways, and procedure 2 can be performed in n different ways. Then number of ways in which these two procedure can happen in sequence is mn.

Pigeonhole Principle:

If $(n+1)$ pigeons are to be distributed in n pigeonholes, then there are at least one pigeonhole which contains at least 2 pigeons.

Permutation (Arrangement):

Let $a_1, a_2, a_3, \dots, a_n$ be n distinct elements.

An r −permutation of the above n elements is an ordered selection of r of the above elements.

Combination (Selection):

An r −combination of n object is an unordered selection of r- objects from given n objects.

Example: Suppose there are four objects *a, b, c* and *d* and selections of them are made two at a time.

The combinations without repetition are six in number, namely *ab, ac, ad, bc, bd, cd* and there are ten with repetition, which are *aa, ab, ac, ad, bb, bc, bd, cc, cd, dd.*

The twelve permutations without repetition are *ab, ac, ad, ba, bc, bd, ca, cb, cd, da, db, dc* and the sixteen with repetition are

aa, ab, ac, ad, ba, bb, bc, bd, ca, cb, cc, cd, da, db, dc, dd.

The Enumeration for Permutation and Combinations

Permutation:

Case 1: Consider n distinct objects

Suppose that, we arrange r of the objects without repetition.

One object can be chosen in n ways, and after this is done,

2nd object can be chosen in $(n-1)$ ways, 3rd object can be chosen in $(n-2)$ ways, …, and r^{th} object can be chosen in $\big(n-(r-1)\big)$ ways.

Hence by multiplication principle, the number of r −permutations of n objects is

$$n\,(n-1)(n-2)\ldots(n-r+1) = {}^{n}P_{r} = \frac{n!}{(n-r)!} = P(n,r)$$

- There are n! permutations of n objects (taken all at a time)

Case 2: Consider n distinct objects

Suppose that, we arrange r of the objects with repetition.

One object can be chosen in n ways, and again the 2nd object can be chosen in n ways, 3rd object can be chosen in n ways, …, and r^{th} object can also be chosen in n ways.

By multiplication principle, the number of r −permutations of n objects with unlimited repetition is n^r.

Case 3: Suppose that, we have n distinct objects. m_1 are of the one kind, m_2 are the 2nd kind, …, m_k are of k^{th} kind, so that $\sum_{i=1}^{k} m_i = n$, then number of permutations of the objects taken all at a time is $\dfrac{n!}{m_1!\ m_2!\ \dots m_k!}$

Case 4: Suppose that we have n distinct objects.

The number of circular permutations of these objects is $(n-1)!$

- If clockwise and anticlockwise permutations are indistinguishable. Then number of circular permutations would be $\frac{(n-1)!}{2}$

Combination:

Case 1: The number of r −combinations of n objects without repetition is denoted by

$$C(n,r) = {}^{n}C_{r} = \frac{n!}{(n-r)!\,r!}$$

Case 2: Consider a combination of n objects in which we select r of them with repetition.

Consider n distinct objects (1, 2, … ,n)

Let one r-combination of n objects in which repetition is allowed be $(c_1, c_2, \dots, c_r), and\ c_1 \leq c_2 \leq \cdots \leq c_r$.

Define $d_1, d_2, \dots, d_r$ as follows:

$$d_1 = c_1 + 0, \quad d_2 = c_2 + 1, \quad d_3 = c_3 + 2, \dots, d_i = c_i + (i-1), \dots, d_r = c_r + (r-1)$$

Observe that whatever $c_1, c_2, \ldots, c_r;\ \ d_1, d_2, \ldots, d_r\ \ are\ all\ distinct$.

The number of sets of *d*’s is the number of r −combinations without repetition of the objects 1*,* 2*, . . . , n* + *r* − 1 since the largest d is d_r when c_r has its maximum value n. Thus the number of sets of *d*’s is $^{n+r-1}C_r$ and this is equal to the number of *r*-combinations of n objects with unlimited repetition.

Distributions: A **distribution** is defined as a separation of a set of objects into a number of classes.

For example, the assignment of the objects to cells (or boxes); problems about distributions are very closely related to problems of permutations and combinations.

Distribution of distinct objects is equivalent to arrangements (Permutation) and distribution of identical objects is equivalent to selections (Combination).

Case 1: Consider distribution of r $-$distinct objects to n distinct cells such that each cell has at most one object.

Number of such distribution $= P(n, r),\ n \geq r$.

(since the first object may be assigned to any of the n cells, the second object to one of the n−1 remaining cells etc.)

Alternatively if $r \geq n$, then there are $P(r, n)$ ways.

(since the object assigned to the first cell may be done in r ways, the object assigned to the second cell in $(r-1)$ ways etc.)

Case 2: Consider distribution of r −distinct objects into n distinct cells such that each cell can hold any number of objects.

Number of such distributions $= n^r$

(This is true whether n is larger or smaller than r since the first object can be assigned to any one of the n cells and so can the second and the other objects.)

Case 3: Consider distribution of r −identical objects into n cells such that each cell has at most <u>one</u> object.

Number of such distribution $= C(n,r),\ n \geq r.$

$$= 1,\ r \geq n$$

Case 4: Consider distribution of r −identical objects into n distinct cells such that each cell can hold any number of objects.

Number of such distribution $= C(n+r+1,\ r)$

Problems:

1. Find the number of ways in which 3 exams can be scheduled in a 5 day period such that:

(i) No two exams are scheduled on the day

(ii) There are no restrictions on number of exams conducted on a day.

2. Determine the number of 4 digit numbers that contains distinct digits.

3. How many odd integers between 100 and 999 have distinct digits?

4. How many of the first thousand positive integers have distinct digits?

5. Find the number of permutations of the word ‘INSTITUTION’

(i) How many of these begin with ‘I’and end with ‘N’?

(ii) How many permutations are there with 3 T’s are not together?

(iii) How many of these begin with ‘I’?

6. How many ways 3 integers be selected from 3n consecutive integers so that the sum is a multiple of 3?

7. If 5 men A, B, C, D, E intend to speak at a meeting, in how many orders can they do so without B speaking before A? How many orders are there in which A speaks immediately before B?

8. How many positive integers less than 1 million can be formed using 7s, 8s, 9s only?

How many using 0s, 8s, and 9s only?

9. How many number of ways to choose 3 days out of 7 days (with repetition allowed)?

10. Determine the number of ways to seat 5 boys in a row of 12 chairs

11. In how many ways can a lady wear 5 rings on her fingers (not on the thumb) on her right hand?

12. Find the sum of all 4 digit numbers that can be obtained by using the digits 1, 2, 3, 4 once in each?

13. Find the sum of all 4 digit numbers that can be obtained by using the digits 1, 2, 3, 4 where any digit can occur any number of times.

14. A committee of k people has to be chosen from a set of 7 women and 4 men. How many ways are there to form the committee if:

(i) Committee has 5 people, 3 women and 2 men

(ii) Committee can be any +ve size but must have equal number of men and women

(iii) Committee has 4 people and one of them must be Mr. X

(iv) Committee has 4 people and at least 2 are women

(v) Committee has 4 people and 2 of each sex, Mr. X and Mrs. X can not be on the same committee

15. How many ways can 5 different messages be delivered by 3 messengers if no messenger is left unemployed? The order in which a messenger delivers his massages is immaterial.

16. (i) In how many ways can two integers be selected from the integers 1, 2, 3, …, 100 so that their difference is exactly seven?

(ii) In how many ways can two integers be selected from the integers 1, 2, 3, …, 100 if the difference is to be seven or less.

17. How many points of intersection are formed by n lines drawn in a plane if no two are parallel and no three concurrent? Into how many regions is the plane divided?

Principle of Inclusion and Exclusion (Sieve method)

Consider N objects and the two properties say $a \text{ and } b$.

Let $N(a) \text{and } N(b)$ denote the number of objects having the properties $a \text{ and } b$ respectively.

Let $N(ab)$ denote the number of objects having both the properties $a \text{ and } b$.

Then the number of objects without the properties $a \text{ and } b$ is given by

$$N(a' \, b') = N - N(a) - N(b) + N(ab)$$

Consider N objects and the r properties say $a_1, a_2, \dots, a_r$

Let $N(a_i)$ denote the number of objects having the properties a_i.

Let $N\left(a_i a_j\right)$ denote the number of objects having the properties $a_i \; and \; a_j$ & so on

Let $N(a_1 a_2 \dots a_r)$ denote the number of objects having properties $a_1, a_2, \dots, a_r$

Then the Principle of inclusion and exclusion states that the number of objects having none of these properties is

$$N(a_1' a_2' \dots a_r') = N - N(a_1) - N(a_2) - \cdots - N(a_r)$$
$$+N(a_1 a_2) + N(a_1 a_3) + \cdots + N(a_{r-1} a_r)$$
$$-N(a_1 a_2 a_3) - \cdots - N(a_{r-2}\, a_{r-1} a_r)$$
$$+ \cdots + (-1)^r (N(a_1 a_2 \dots a_r)$$

1. How many of the first 1000 integers are not divisible by 2, 3, 5 or 7 ?

2. Suppose there are 100 students in a school and there are 40 students taking each language French, Latin and German, 20 students are taking only French, 20 only Latin and 15 only German. In addition, 10 students are taking French and Latin. How many students are taking all the three languages and how many are not taking any of the languages?

Derangements:

Derangement is a permutation of objects such that none of the object is in its own position.

Example: 231, 312 are the derangements od 123.

1. How many permutations of n distinct elements $1, 2, \ldots, n$ are there in which the element k is not in the k^{th} position?

2. What is the probability that if n people randomly reach into a dark closet to retrieve their hats, no person will pick his own hat?

3). Show that the proportion of the permutations of $1, 2, \ldots, n$ which contains no consecutive pair $(i, i+1)$ for any i is approximately $\frac{n+1}{ne}$.

4). 7 people enter a lift. The lift stops only at 3 floors (unspecified). At each of the 3 floors, no one enter the lift but at least one person leaves the lift. After 3 stops, the lift is empty. In how many ways can this happen?

Practice Questions:

1. How many different strings can be formed using 2 A's, 3 B's, 2 C's, and 1 E, once each? In how many of these strings are all the vowels non-adjacent?

Ans: $8!/(2!\,3!\,2!)$ and $(5!\,/\,3!\,2!) \times {}^{6}C_{2} \times 4$

2. How many ways are there to distribute 4 identical oranges and 6 distinct apples (each of different variety) into 5 distinct boxes such that each box gets exactly two fruits?

Ans: 8100

3. How many times is the digit 5 written when listing all numbers from 1 to 100000?

Ans: 50, 000

4. How many integers between 1 and 6300 are neither divisible by 3 nor by 5?

Ans: 3360

5. If no three diagonals of a convex decagon meet at the same point inside the decagon, into how many line segments are the diagonals divided by their intersections?

Ans: 455