Composition → ordered

Partition ⟶ unordered

Compositions :- ① No of compositⁿs of an integer 'n' $= 2^n$

② No of compositⁿs of 'n' into 'm' parts ⇒ ${}^{n-1}C_{m-1}$

gf : case ① :- $\left(\frac{x}{1-2x}\right)$

case ② : $\left\{\frac{x}{1-x}\right\}^m$

③ No of compositions of an integer 'n' into 'm' parts when zero parts are allowed

$= {}^{m+n-1}C_{m-1}$

Partitions

No of partitions of an integer 'n' ⟶

$(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\cdots\cdots$

* gf to obtain the change for Rs-100/- interms of Rs-5/- Rs-2/- and Rs-1/- :- No of partⁿs of 100 in which each part is either 5 or 2 or 1

$(1-x)^{-1}(1-x^2)^{-1}(1-x^5)^{-1}$, pick the coeff of x^{100}

* No of partitⁿs of 'n' in which no part is > 10

$(1-x)^{-1}(1-x^2)^{-1}\cdots\cdots(1-x^{10})^{-1}$

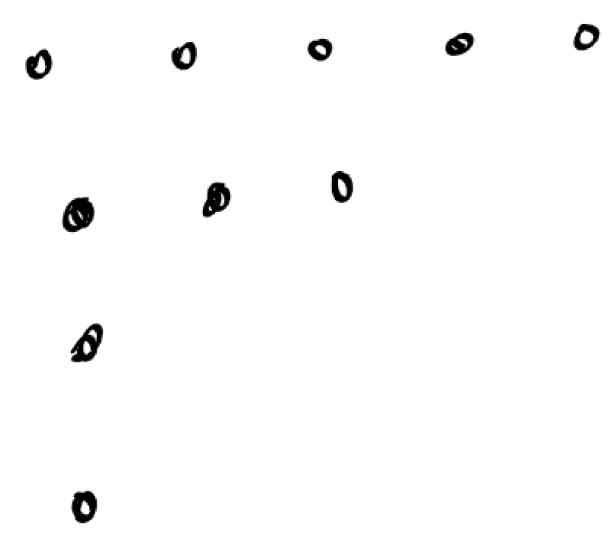
③ gf for partitions of 'n' in which no part occurs more than thrice and the Largest part is 5.

$$(1+x+x^2+x^3)(1+x^2+x^4+x^8)\cdots\cdots(1+x^5+x^{10}+x^{15})$$

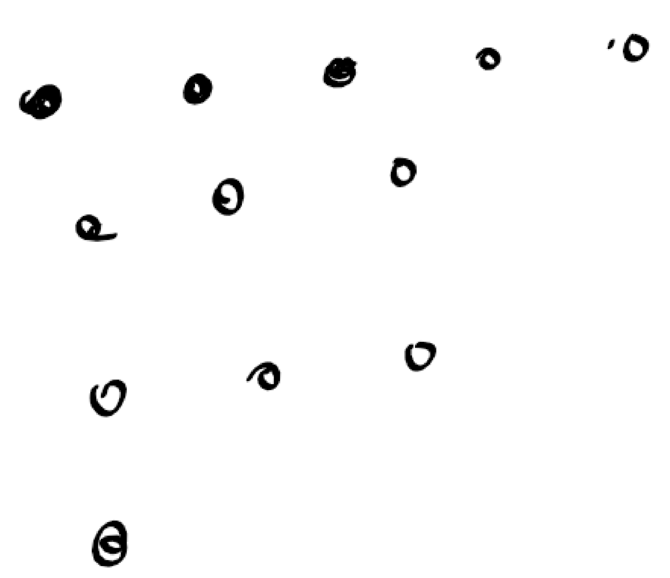
Ferrer's Graph

Partitions are represented by an array of dots known as Ferrers graph.

10 ⇒ 5 3 1 1



12 ⇒ 5 3 3 1



4 3 3 1 1

consider a partition of 'n'

① Total no of dots in the en e graph = n

② There is one r w corresp to every part

③ Upper row always contains atleast as many as dots as low rows

④ Ro are aligned on the left

Conjugate of a partition :-

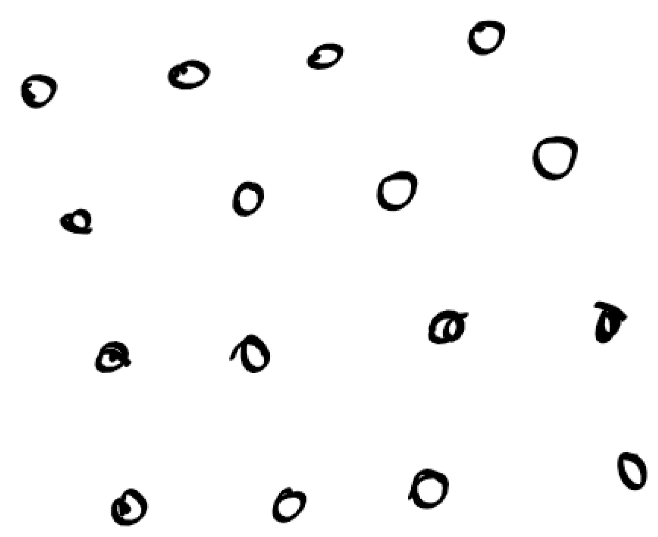
The partition obtained by reading the Ferrers graph by columns (columnwise)

ex:- Conjugate of (5331) is (43311)

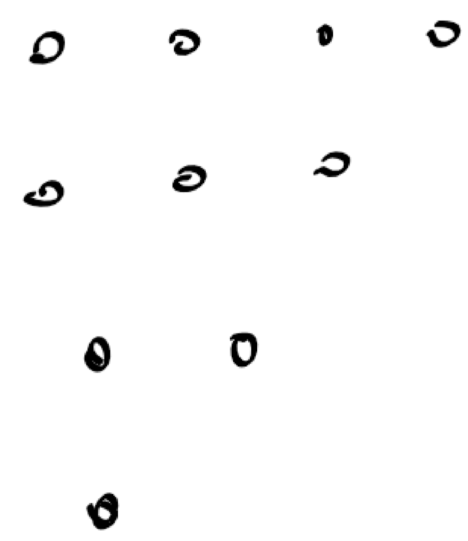
conjugate of (5311) is (42211)

Self conjugate: A partition is said to self conjugate if Ferrers graph is same by rows & cols.

ex: $\Rightarrow$ $16 = 4444$



ex: $10 = 4321$

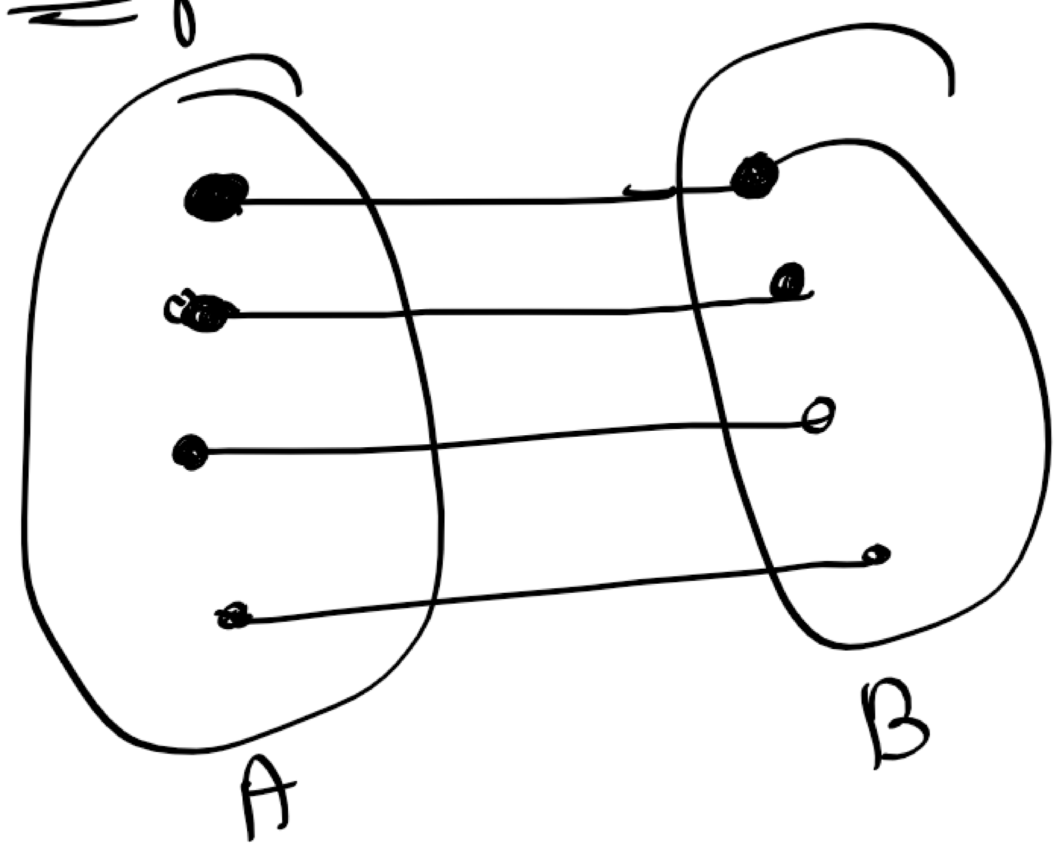


Theorem

The number of partitions of an integer 'n' with no part greater than 'k' is equal to the no of partitions of 'n' with atmost 'k' parts

$$\left\{ \begin{array}{l} \text{no of partitions} \\ \text{with no part} > k \end{array} \right. = \begin{array}{l} \text{no of partitiotion} \\ \text{with atmost } k \text{ parts} \end{array}$$

Proof:-



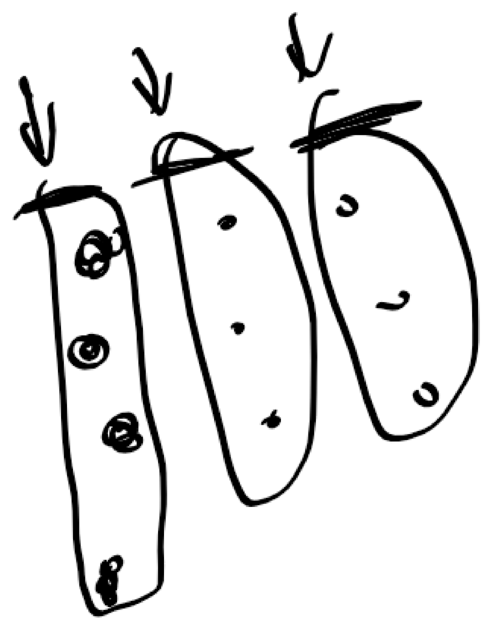
$n = 10$, $k = 3$

LHS $\Rightarrow$ $10 = 3331$
$= 22222$
$= 33211$
$= 111111\cdots1$

RHS $\Rightarrow$ $10 = 55$
$= 433$
$= 631$
$\Rightarrow 721$
$\Rightarrow 811$

To PoT $|A| = |B|$, we can show that there is bijectⁿ (one onto)

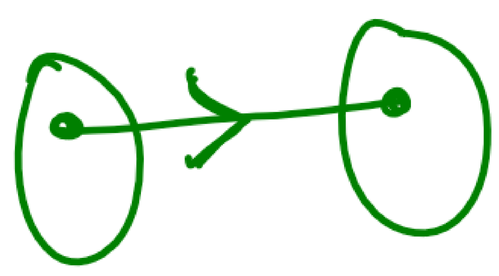
$(3331) \longrightarrow (433)$



433

Consider a partition of the integer 'n' in which no part is $> k$ and consider its Ferrers graph representation. The no of dots in each row $\leq k$. But, if we read the partition columnwise, the no of parts is $\leq k$.

Thus for every partition of 'n' in which no part is $> k$, there exists a partition (when we read columnwise) with atmost 'k' parts.

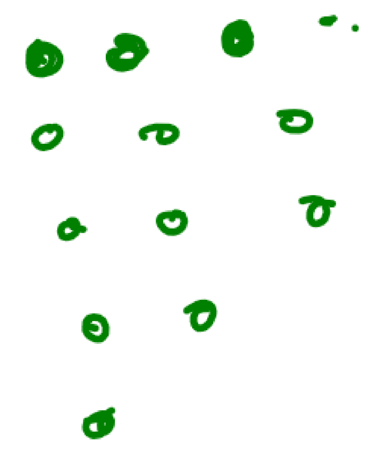


$n=12$ $k=4$

$12 \Rightarrow$

LHS $\Rightarrow$ 1 3 3 3 2 1

5 4 3



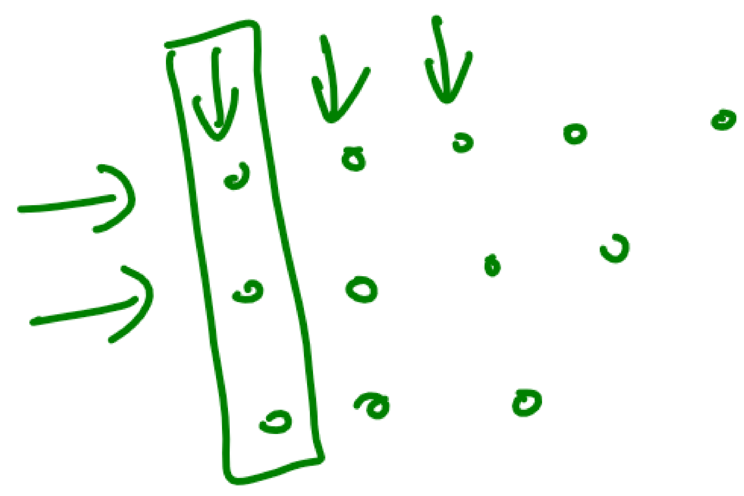
<u>converse</u>:

consider a partition of 'n' with atmost k parts. consider the Ferrers graph. The no of rows in the Ferrers graph is $\leq k$. If we read the same partition columnwise, we get a partition in which the size of any column is $\leq k$ ($\because$ no of rows $\leq k$ in the original)

Thus, for every partition of n with atmost k parts, there exists a partition with no part $> k$

12 : $k=4$

5 4 3



3 3 3 2 1

Thus there is bijection

Thus the no of partition of n with no part $> k$ is equal to the no of partitions with atmost k parts.

② Show that the no of partitions of 'n' is equal to the no of partitions of '2n' into exactly 'n' parts

Soln

LHS ⇒ All partitions of n

RHS ⇒ All partitions of '2n' into 'n' parts

ex:- partition 10 :- $\left(\begin{matrix} 10 \\ 19, 28, \ldots \\ \\ 1111\ldots1 \end{matrix} \right)$

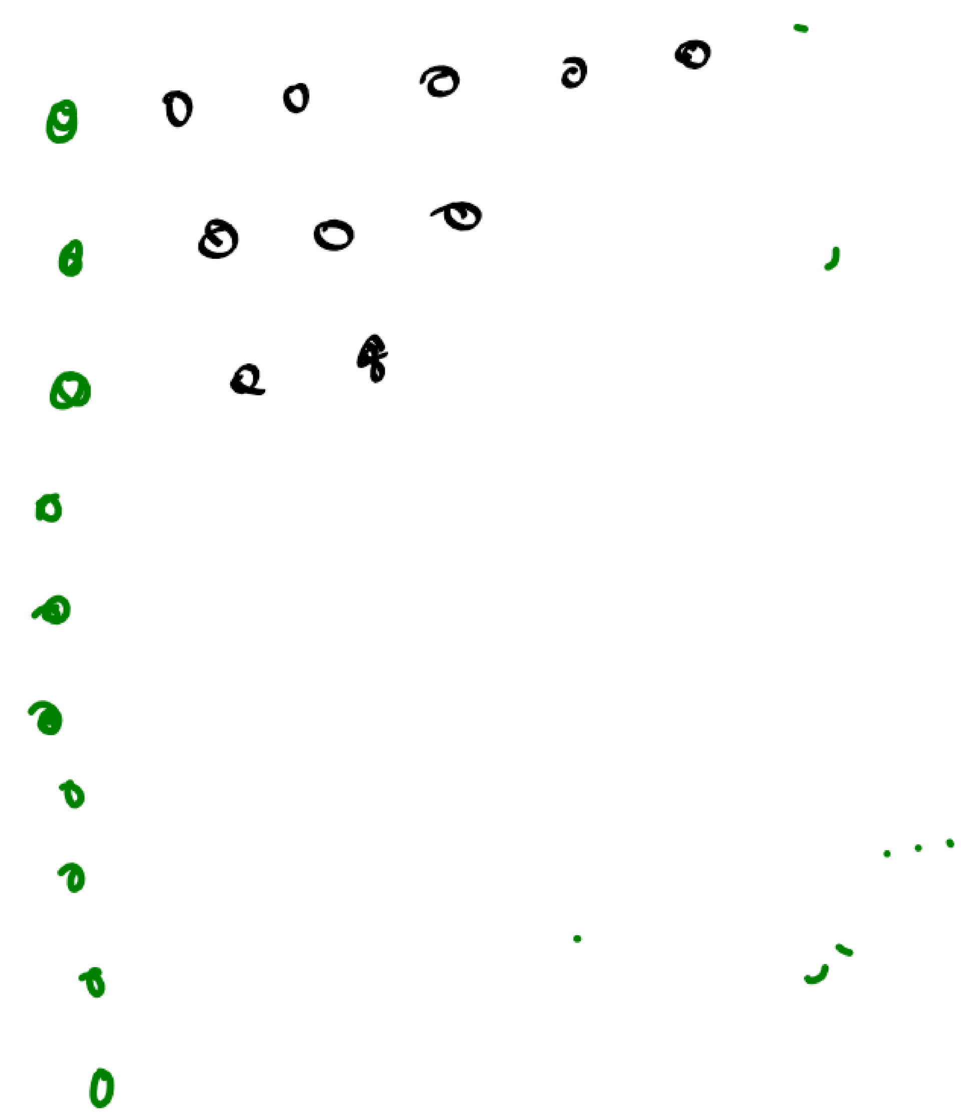
n

↓

2n

partitn of 20 into 10 parts : $\left(\begin{matrix} 2\,2\,2\,2\,2\,2\,2\,2\,2\,2 \\ 3\,2\,2\,2\,2\,2\,2\,2\,2\,1 \\ 3\,3\,2\,2\,2\,2\,2\,2\,1\,1 \\ \\ \end{matrix} \right)$

10 ⇒ 5 3 2



(5 3 2) ⇒ (6 4 3 1 1 1 1 1 1 1)

of 10   of 20 into exactly 10 parts

Consider a partition of n and its Ferrers graph. Add a col of 'n' dots on the left. Now this new graph corresps to a partition of (2n) into exactly (n parts)

For every partn of n, there exists a partition for (2n) which has exactly n parts.

Conversely, Consider a partition (2n) into n parts. Then the Ferrers graph repn has exactly n rows. Thus the left most col has exactly n dots. On deleting this

left most column, we get a partition of $(2n-n)=n$.
Thus for every partition of $(2n)$ into n parts, there exists a partition of n.

Thus there is a bijection

Hence the proof