MERA

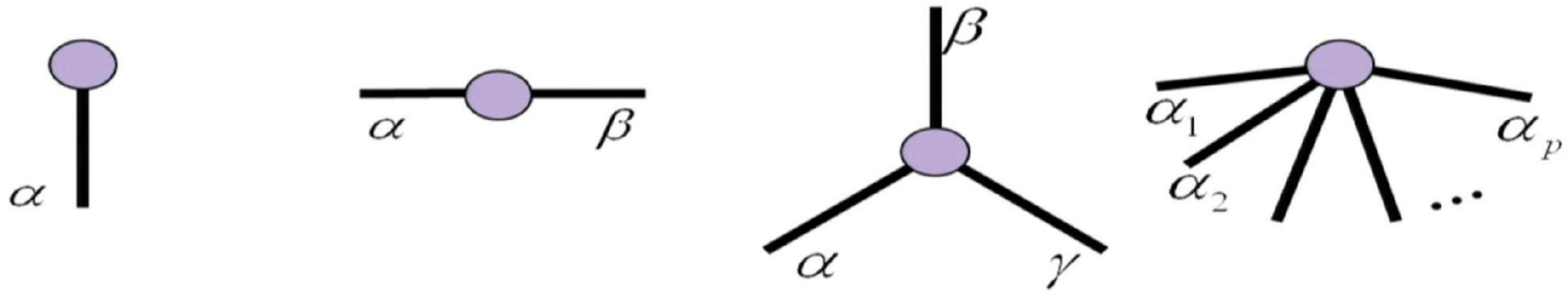
Abhilash, Daksh, Monu, Punyam
SPS, NISER

Building blocks?
Tensor Networks

Graphical representation of matrices/tensors

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \qquad B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}$$

- vector a_α
- matrix $b_{\alpha\beta}$
- rank 3 tensor $c_{\alpha\beta\gamma}$
- rank p tensor $t_{\alpha_1\alpha_2\cdots\alpha_p}$

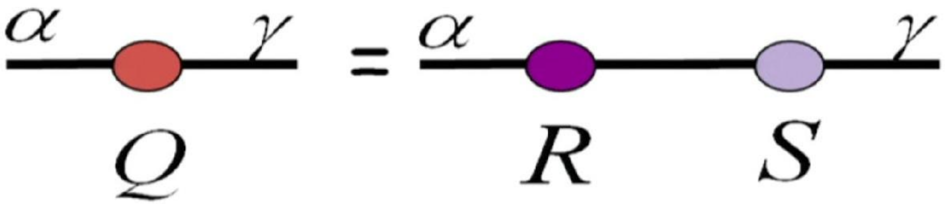


Graphical representation of matrices/tensors

- product of tensors (matrices)

$$Q = RS$$

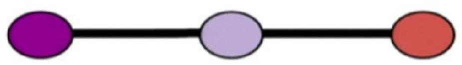
$$q_{\alpha\gamma} = \sum_{\beta} r_{\alpha\beta} s_{\beta\gamma}$$



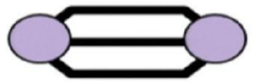
Graphical representation of matrices/tensors

- other examples:

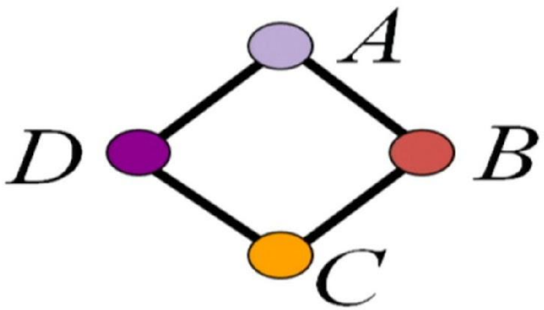
$$x^\dagger A y$$



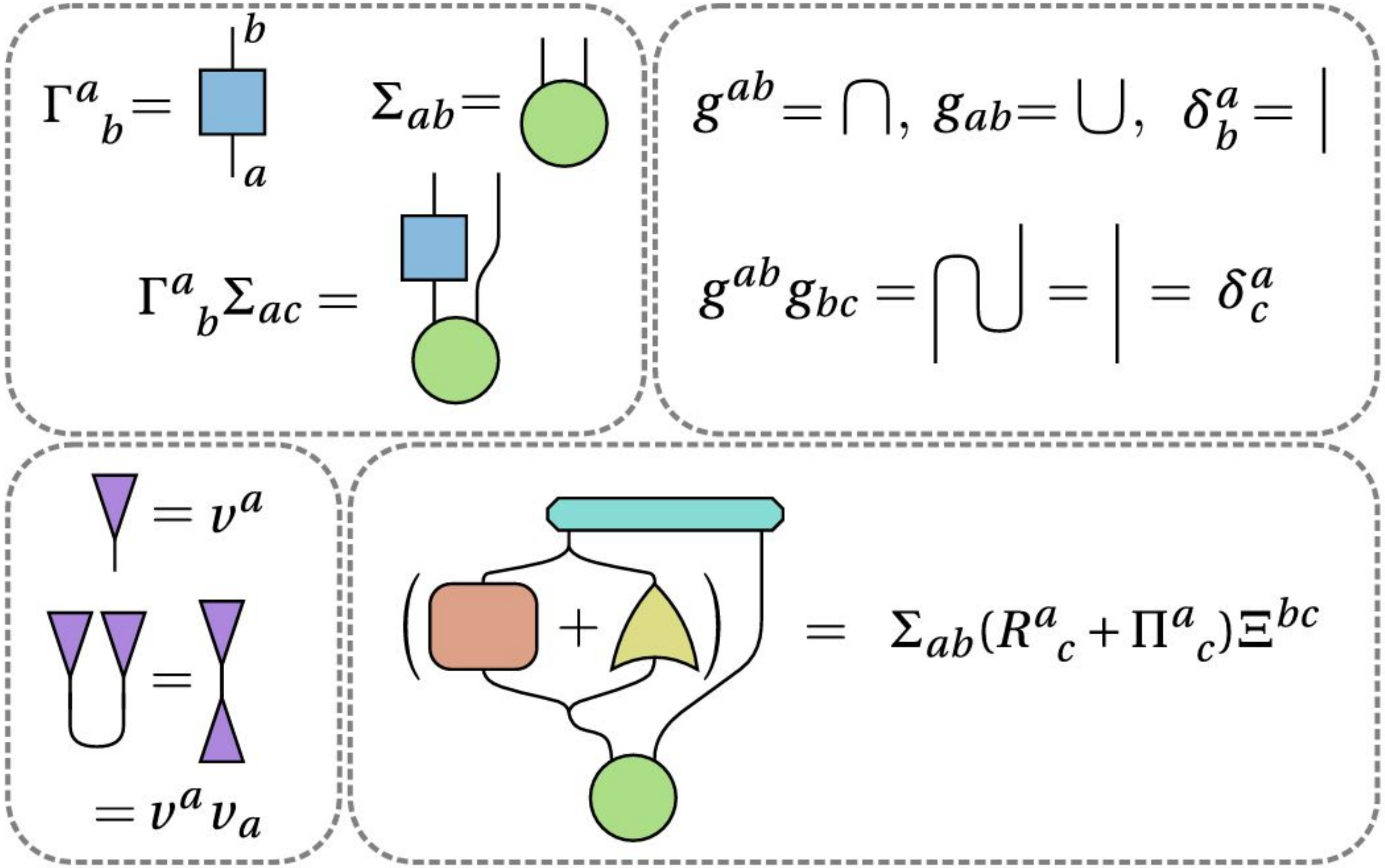
$$\sum_{\alpha\beta\gamma} c_{\alpha\beta\gamma} d_{\alpha\beta\gamma}$$



?



Summary



$\Gamma^a{}_b =$
b
a
$\Sigma_{ab} =$
$\Gamma^a{}_b \Sigma_{ac} =$
$g^{ab} = \cap, \; g_{ab} = \cup, \; \delta^a_b = |$
$g^{ab} g_{bc} = \; = \; = \delta^a_c$
$= v^a$
$=$
$= v^a v_a$
$+$
$= \Sigma_{ab}(R^a{}_c + \Pi^a{}_c)\Xi^{bc}$

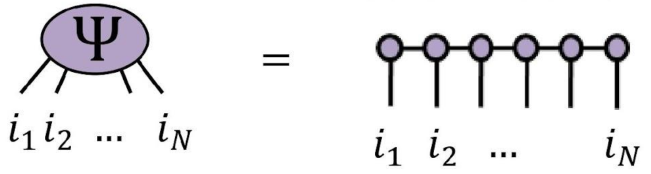
Tensor Networks

Many-body wave-function of N spins

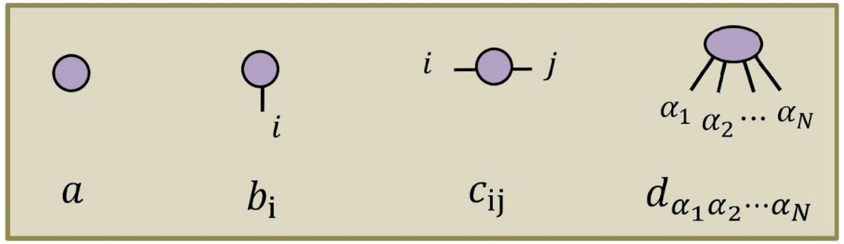
$$|\Psi\rangle = \sum_{i_1, i_2, \cdots, i_N} \Psi_{i_1 i_2 \cdots i_N} |i_1 i_2 \cdots i_N\rangle$$

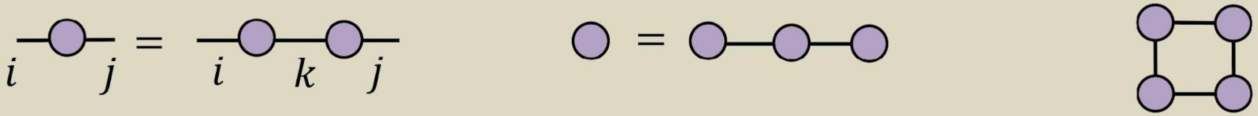
2^N parameters

tensor network



graphical notation



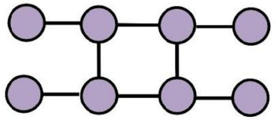


$$T_{ij} = \sum_k R_{ik} S_{kj}$$

$$a = \vec{y}^\dagger \cdot M \cdot \vec{x}$$

$$tr(ABCD)$$

why bother?



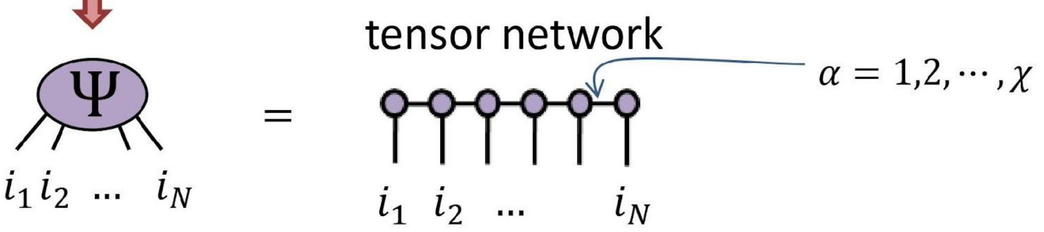
$$\sum_{ijklmnop} A_{ijk} B_{jlm} C_{nko} D_{kmr} x_i y_l z_n v_r$$

Tensor Networks

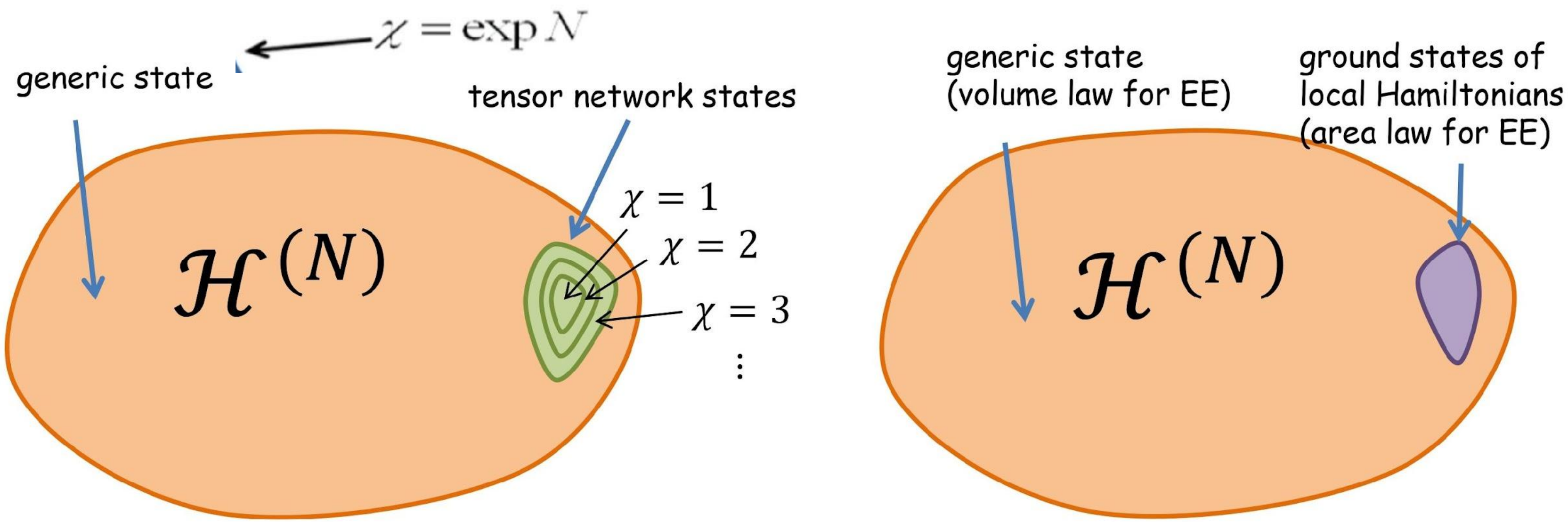
Many-body wave-function of N spins

$$|\Psi\rangle = \sum_{i_1, i_2, \cdots, i_N} \Psi_{i_1 i_2 \cdots i_N} |i_1 i_2 \cdots i_N\rangle$$

2^N parameters



2^N parameters	$\longrightarrow$	$O(N)$ parameters
inefficient		**efficient**



What makes a tensor network useful? (MPS)

- Efficiency (computational cost)

  1) Efficient representation of a many-body wavefunction

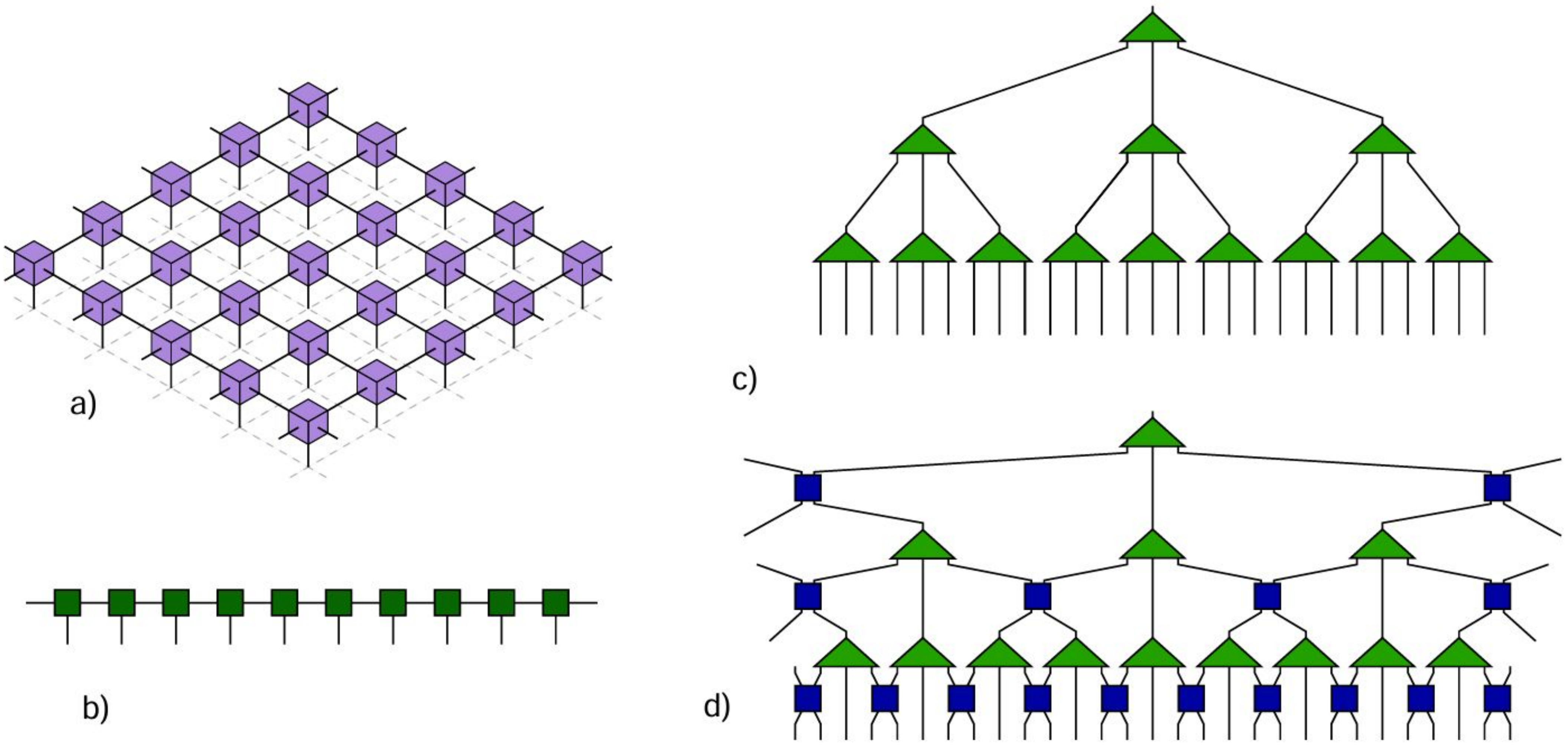
  2) Computation of expectation values

- Accurate approximation of many-body states (e.g. ground states)

  1) Entanglement entropy?

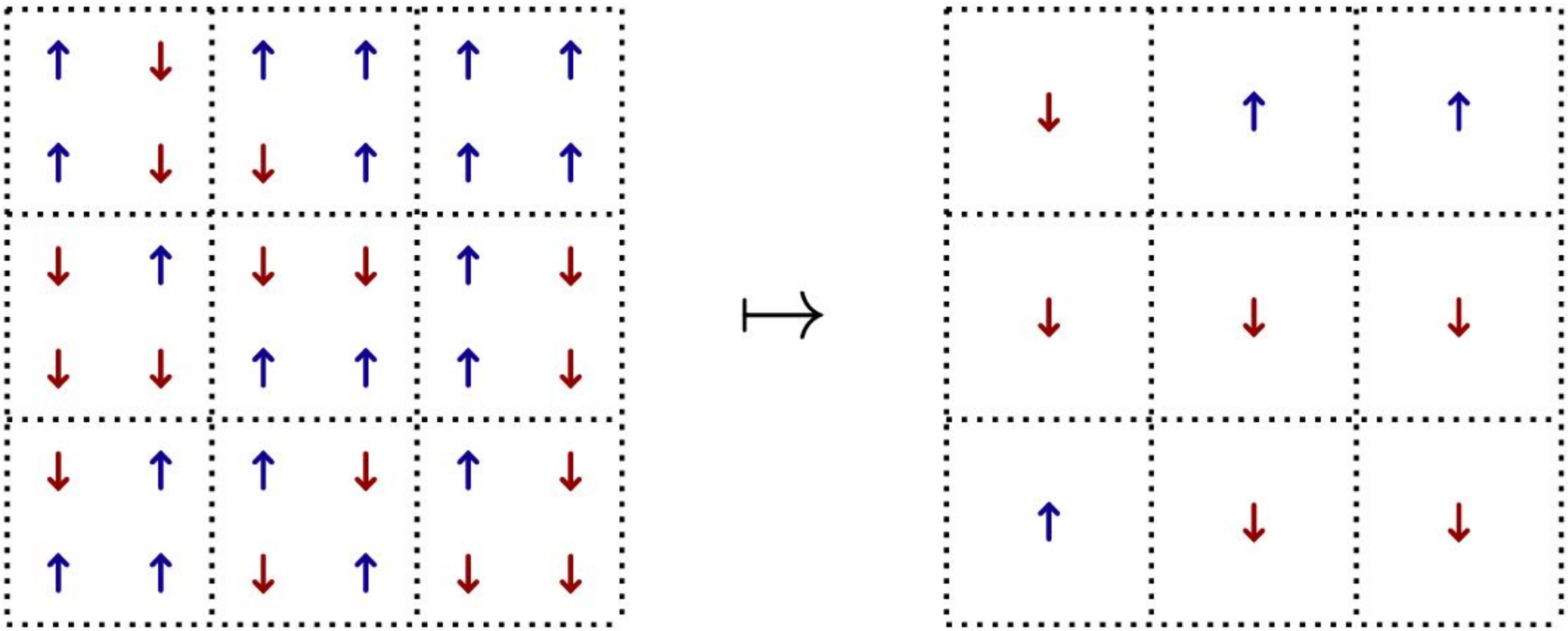
  2) Correlations?

Types of tensor networks



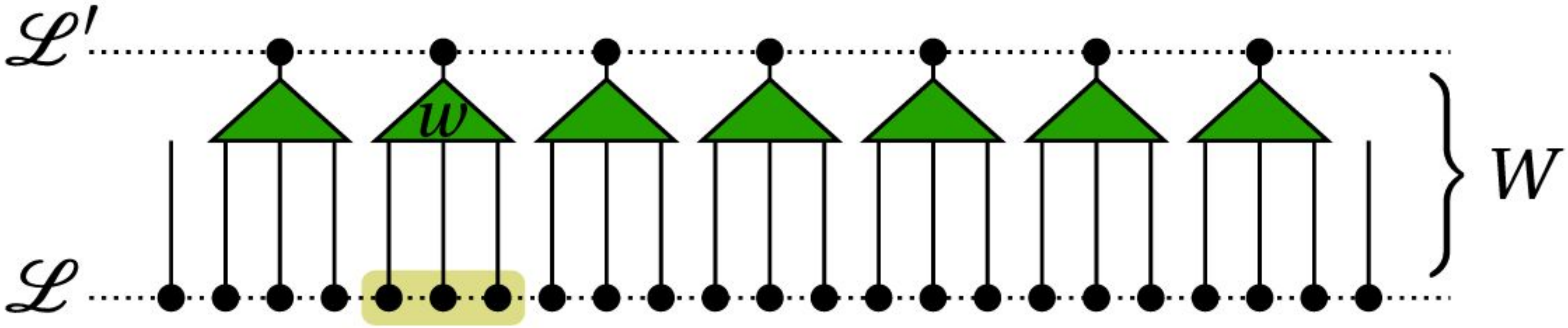
The Process.
Renormalization

Kadanoff's Block-spin tr.

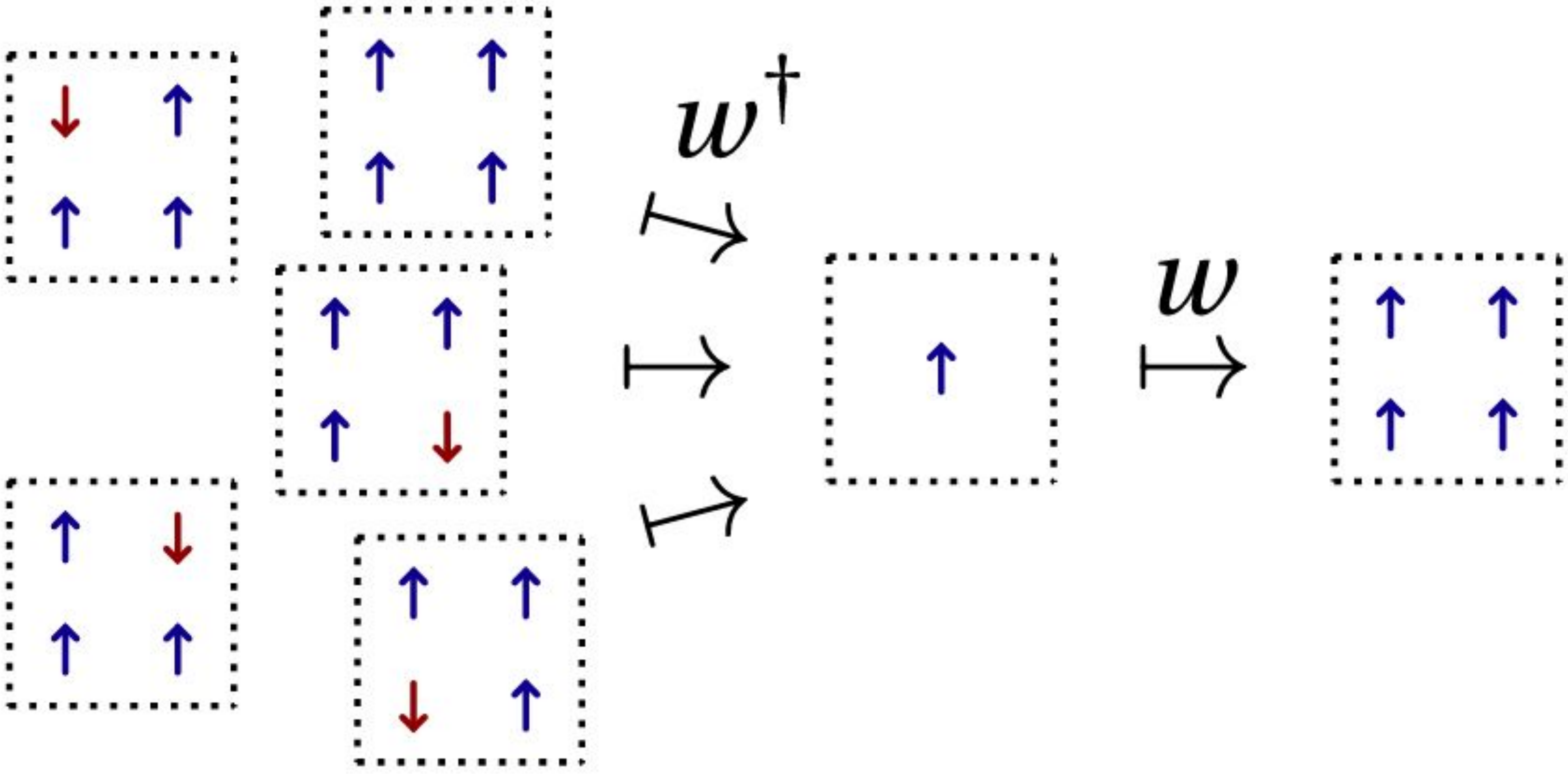


Coarse-graining of 1D lattice

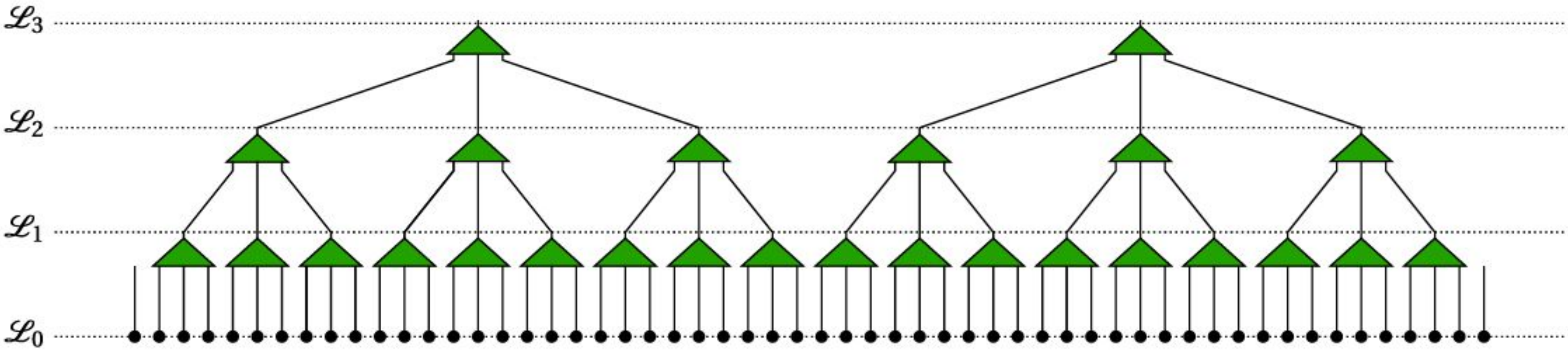
Coarse-graining of 1D lattice



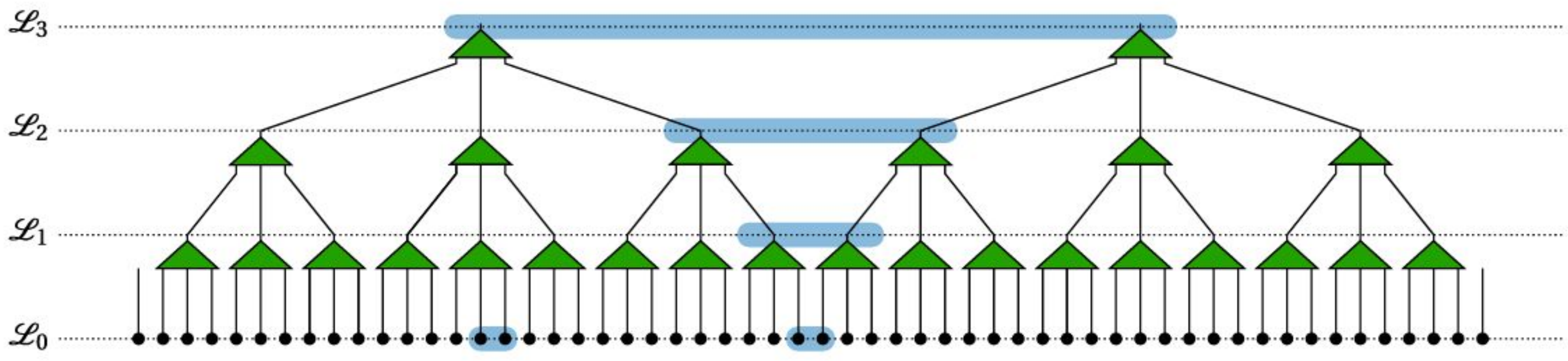
Coarse-graining of 1D lattice



Renormalized group flow

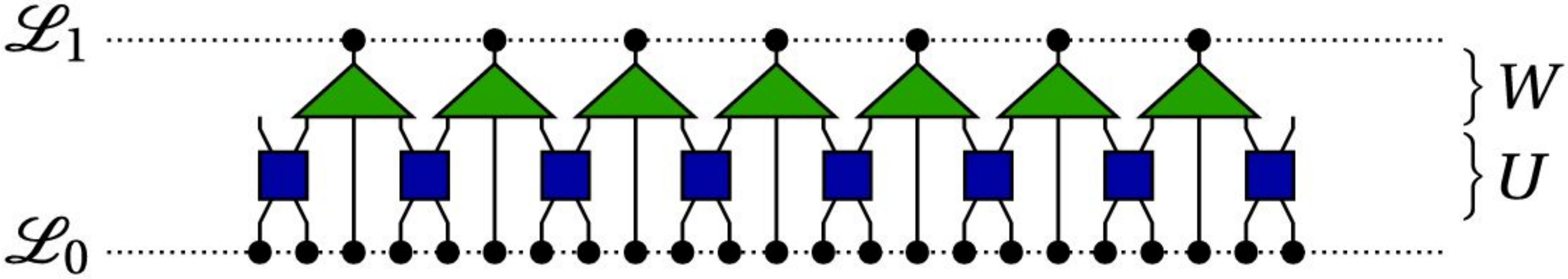


Renormalization group flow

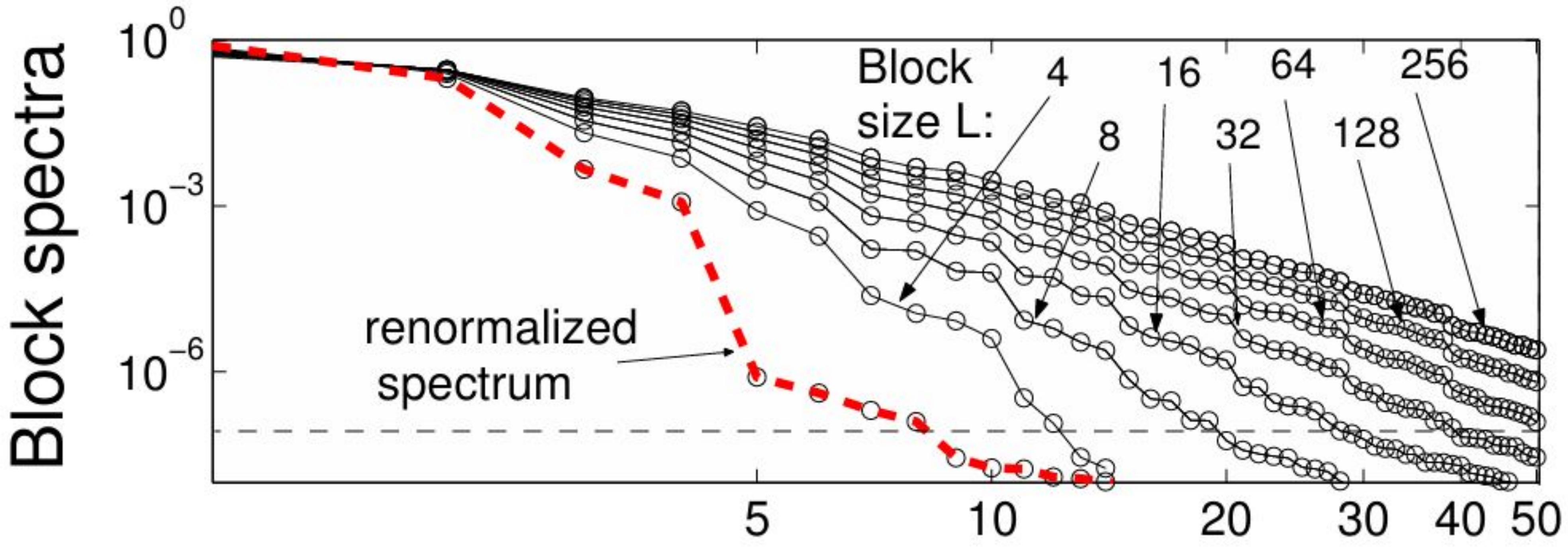


Entanglement Renormalization

Entanglement Renormalization



Impact



Block spectra
10^0
10^{-3}
10^{-6}
Block size L:
4
8
16
32
64
128
256
renormalized spectrum
5
10
20
30
40
50

Coarse-graining of 1D lattice

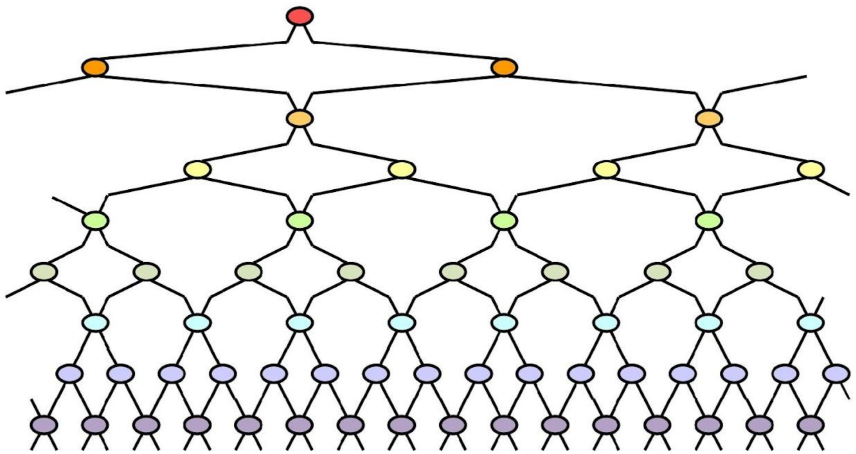
Coarse-graining of 1D lattice

The Ansatz
MERA

MERA

Multi-scale entanglement renormalization ansatz

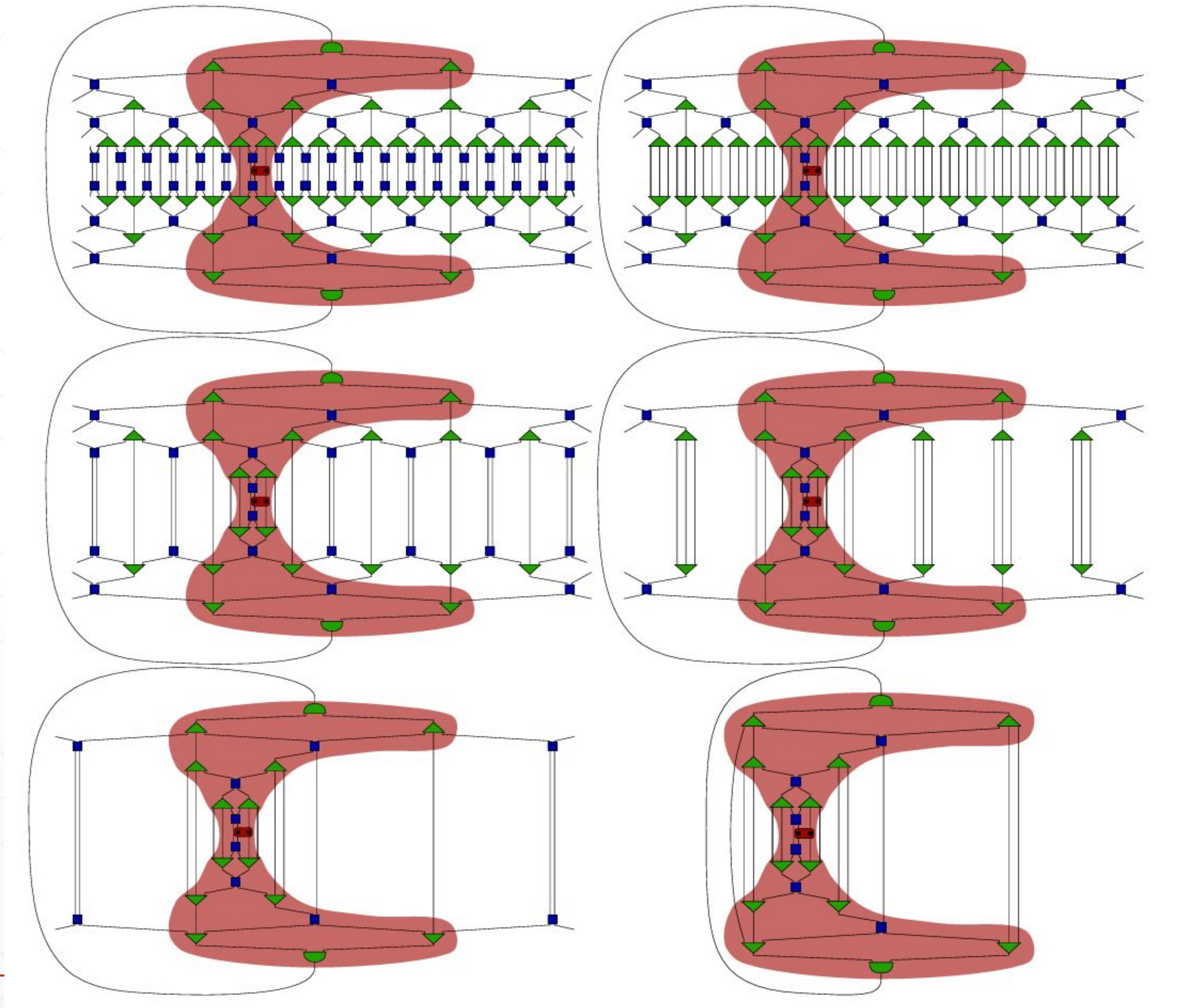
(MERA)



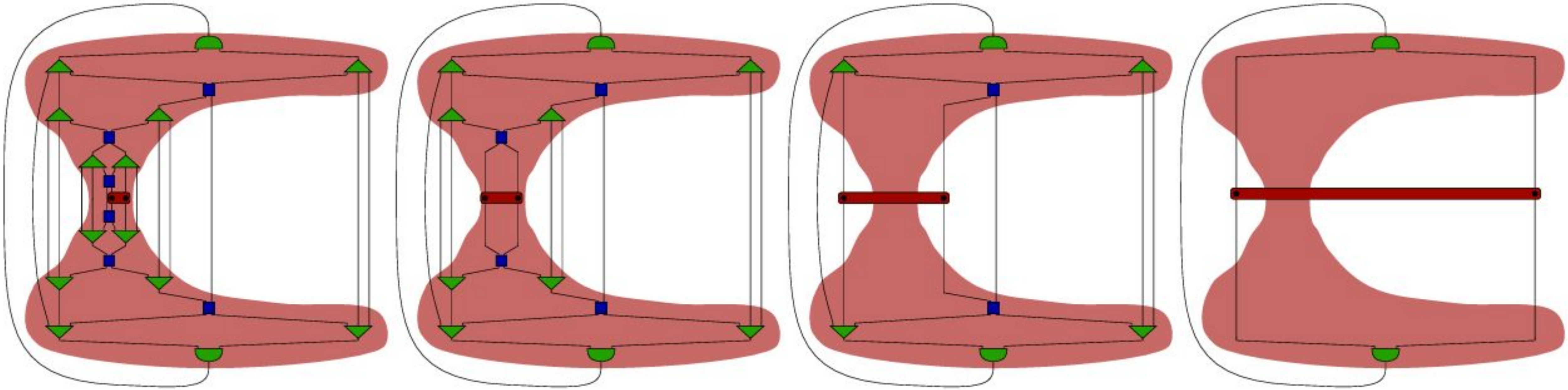
- Variational class of states for 1d systems, which extends in space and scale
- Variational parameters for different length scales stored in different tensors
- It is secretly a **quantum circuit** and an **RG transformation**

Observables

MERA

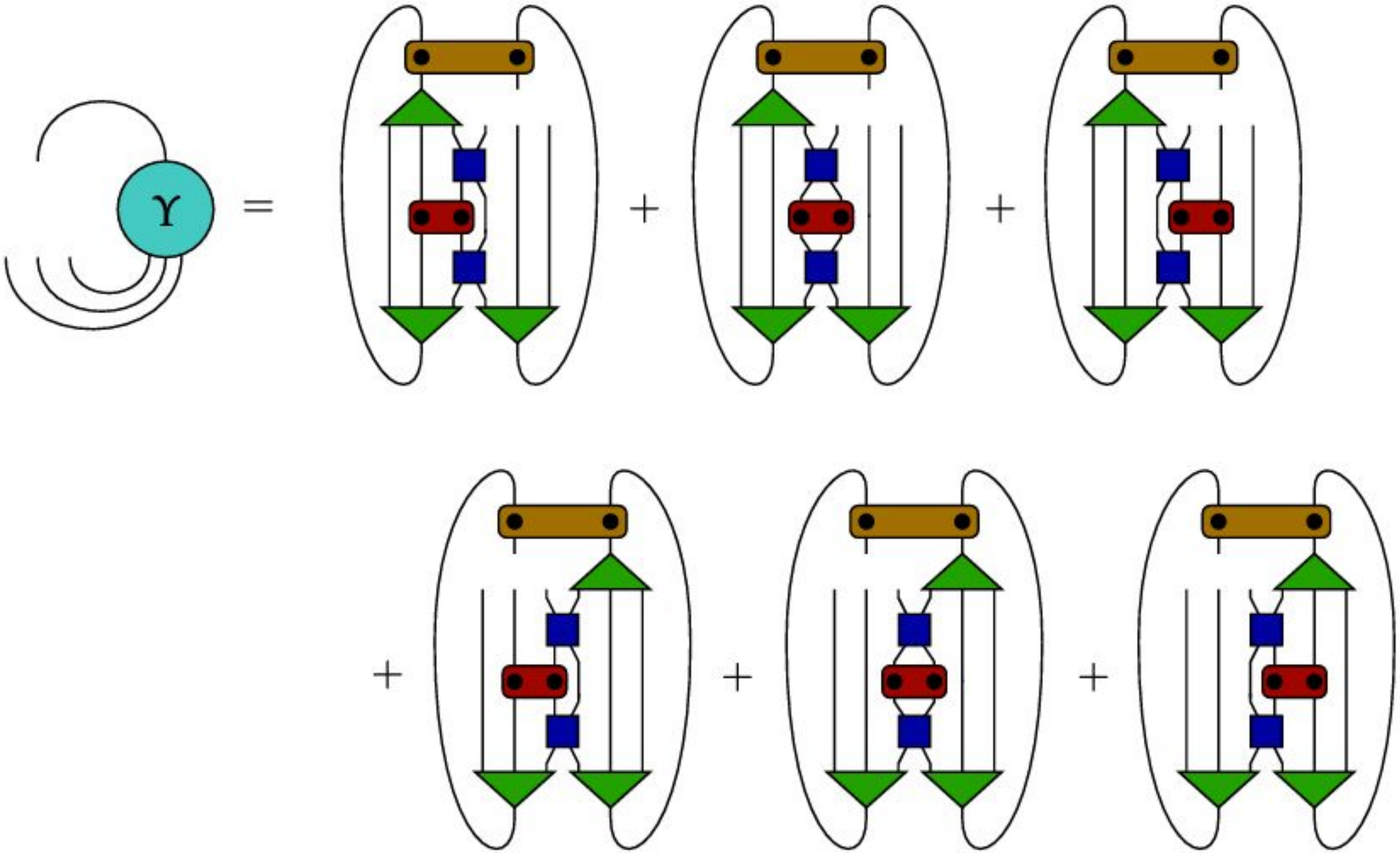


MERA



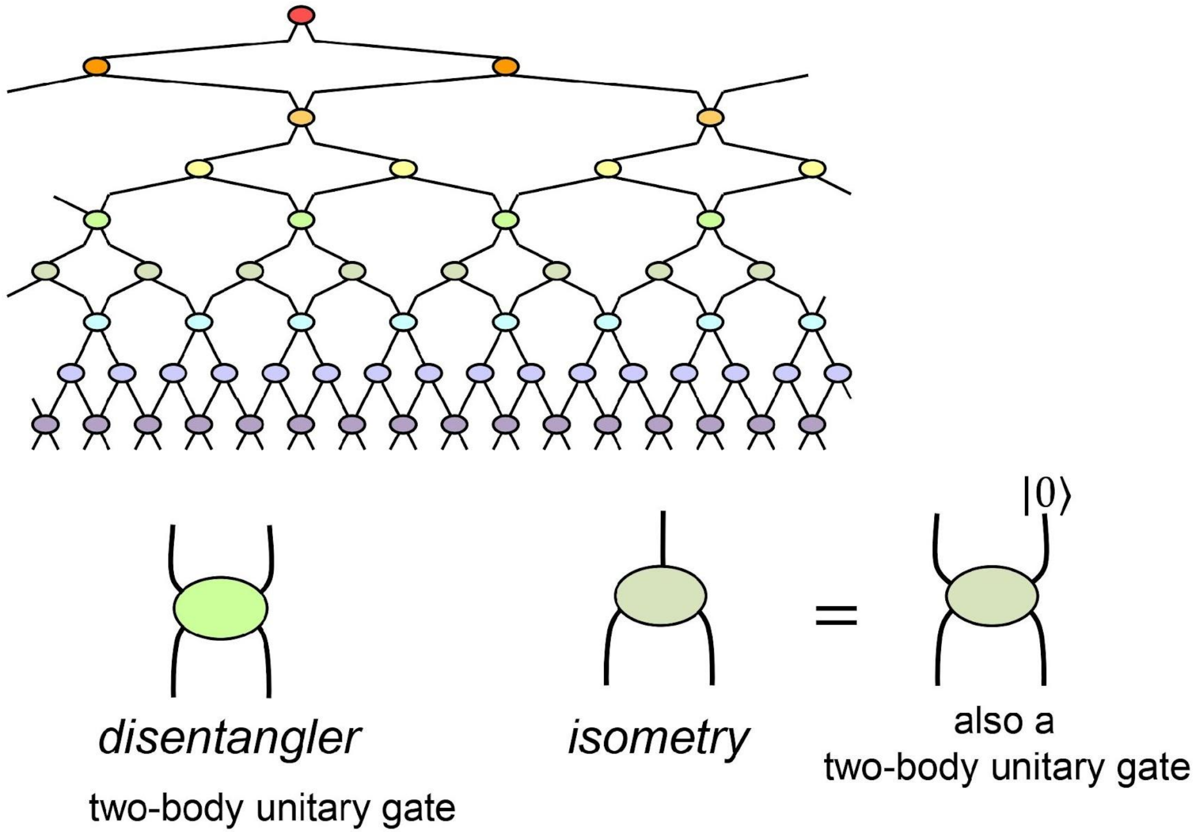
Optimization

MERA

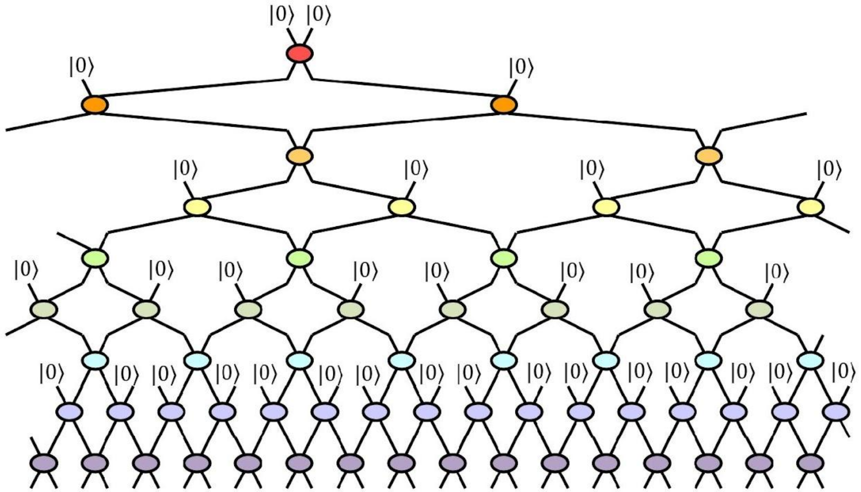


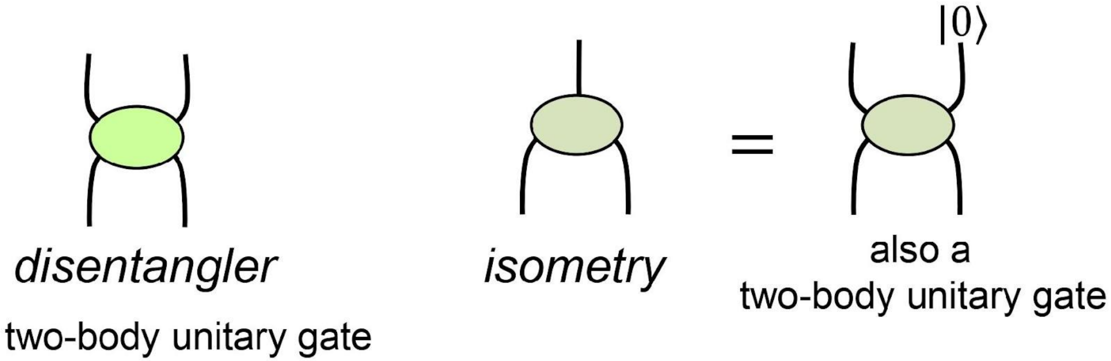
MERA

MERA as a quantum circuit



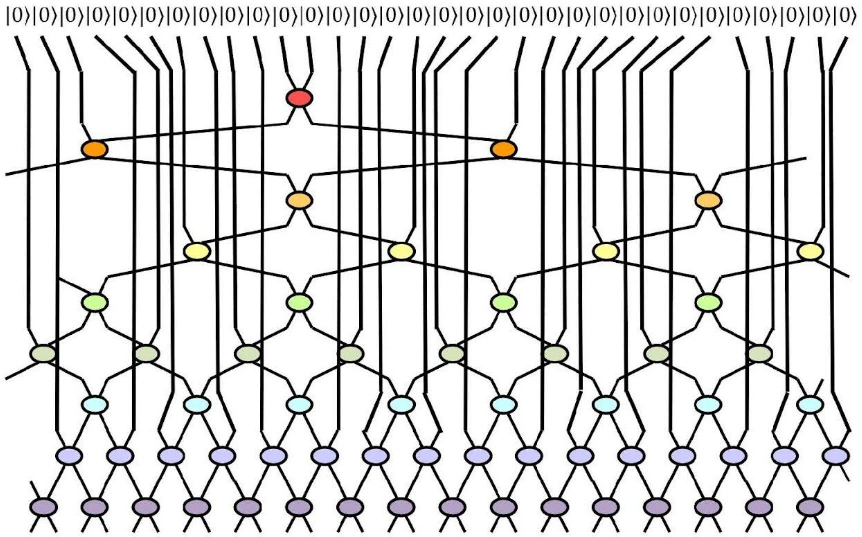
MERA as a quantum circuit

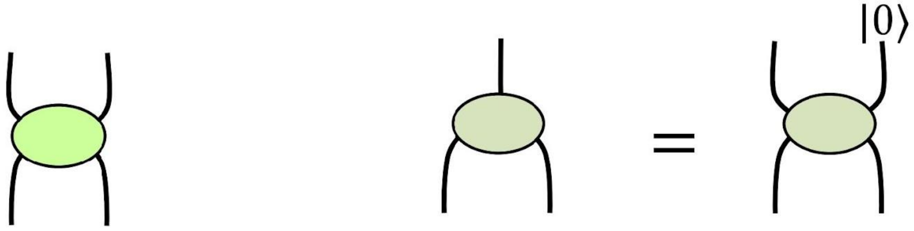




MERA

MERA as a quantum circuit





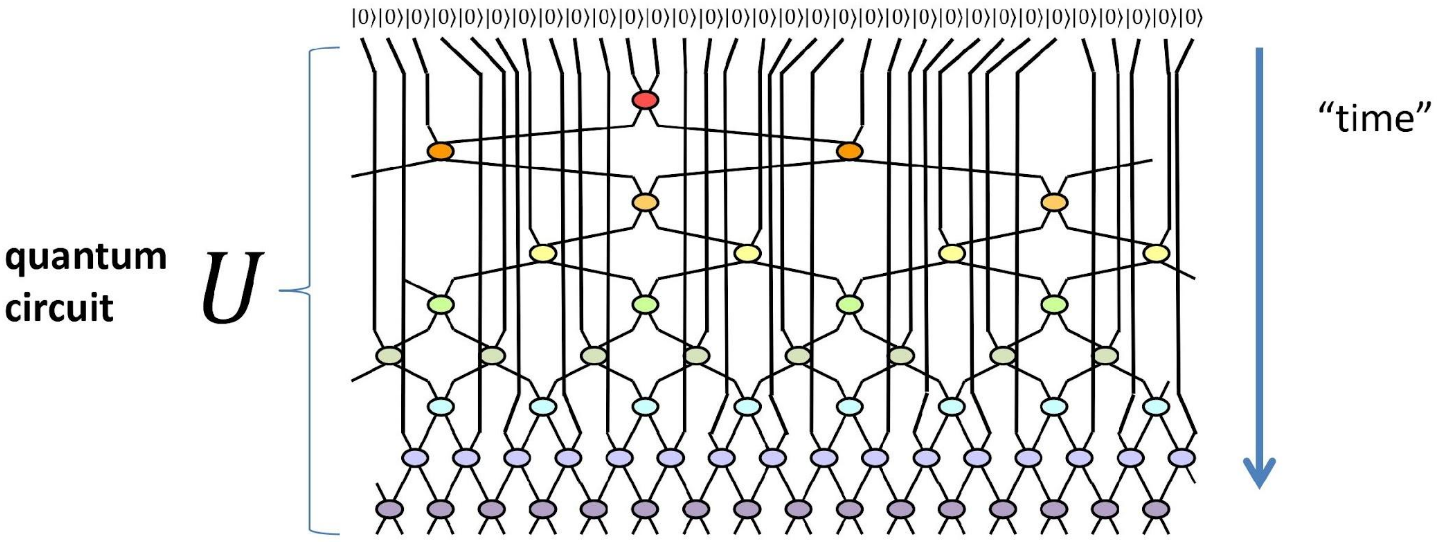
disentangler

two-body unitary gate

isometry

also a
two-body unitary gate

MERA as a quantum circuit

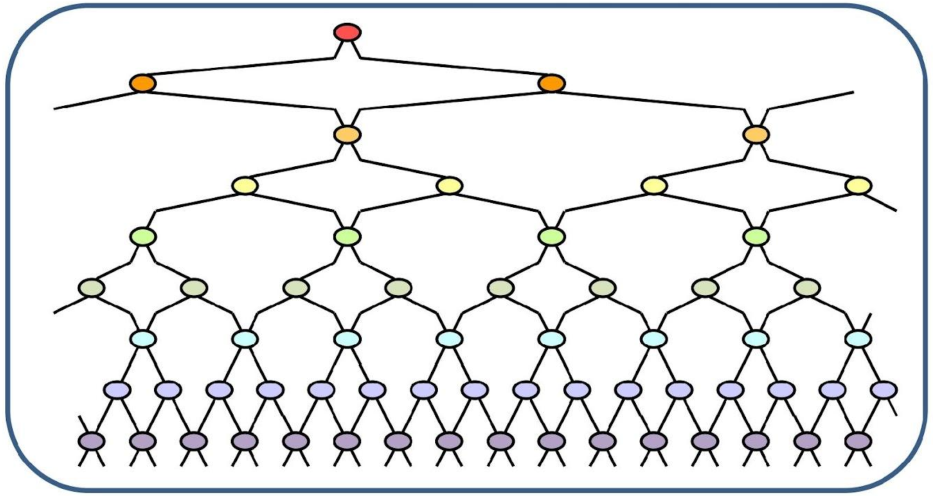


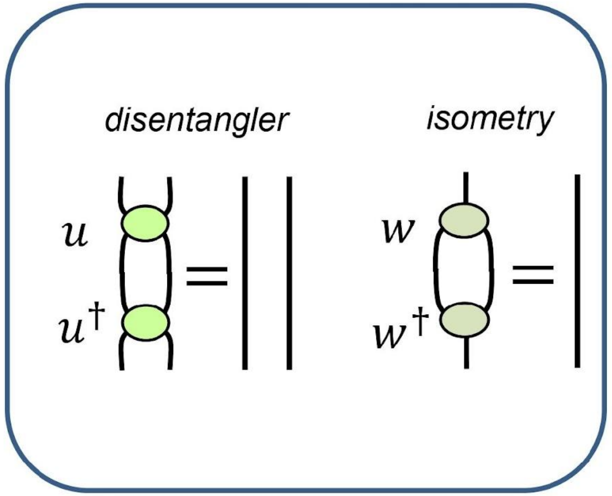
ground state ansatz $|\Psi\rangle = U\,|0\rangle^{\otimes N}$

Entanglement introduced by gates at different "times" (= length scales)

MERA

MERA = tensor network + isometric/unitary constraints





~ hyperbolic plane?

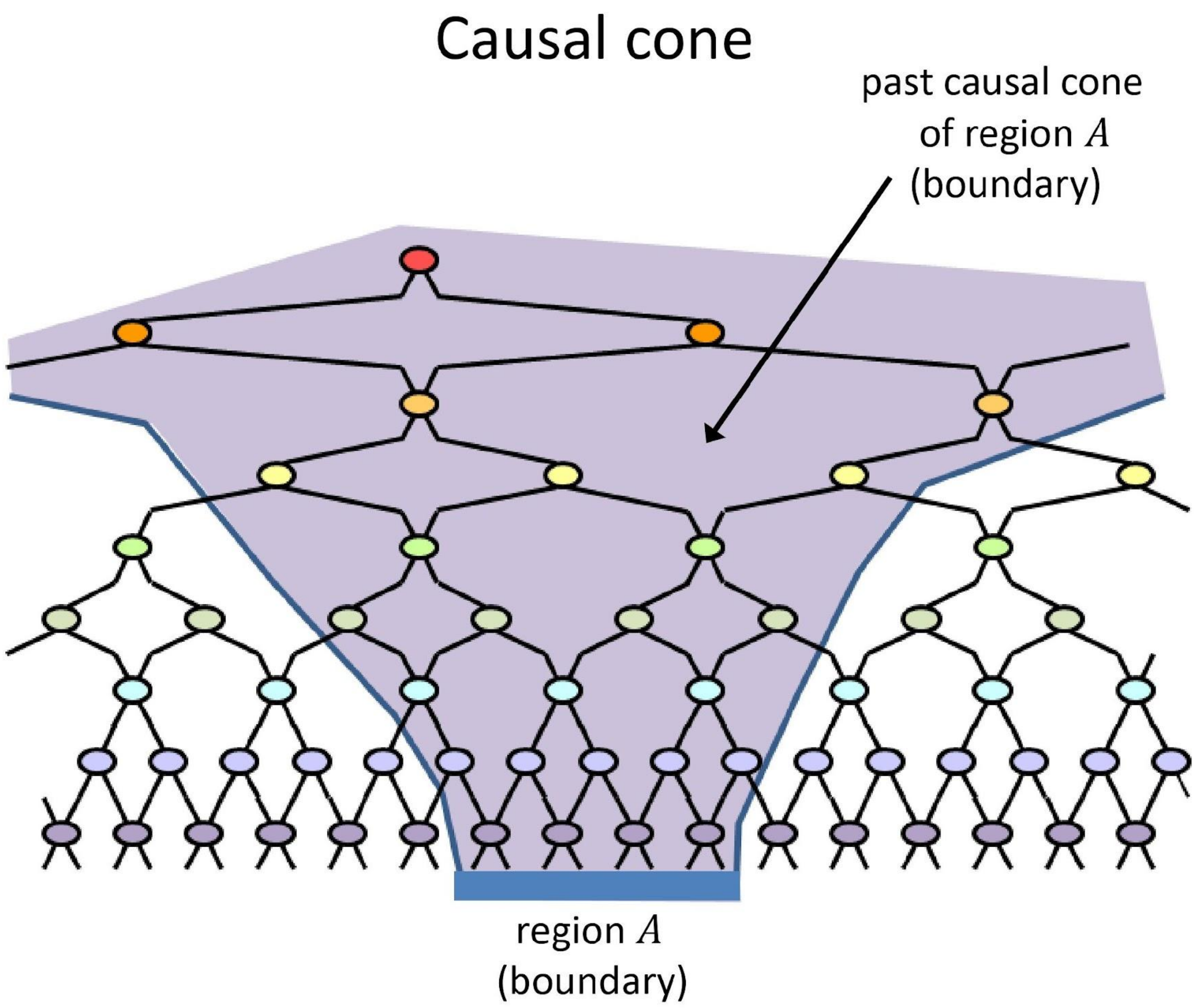
(Swingle 2009)

~ de Sitter space?

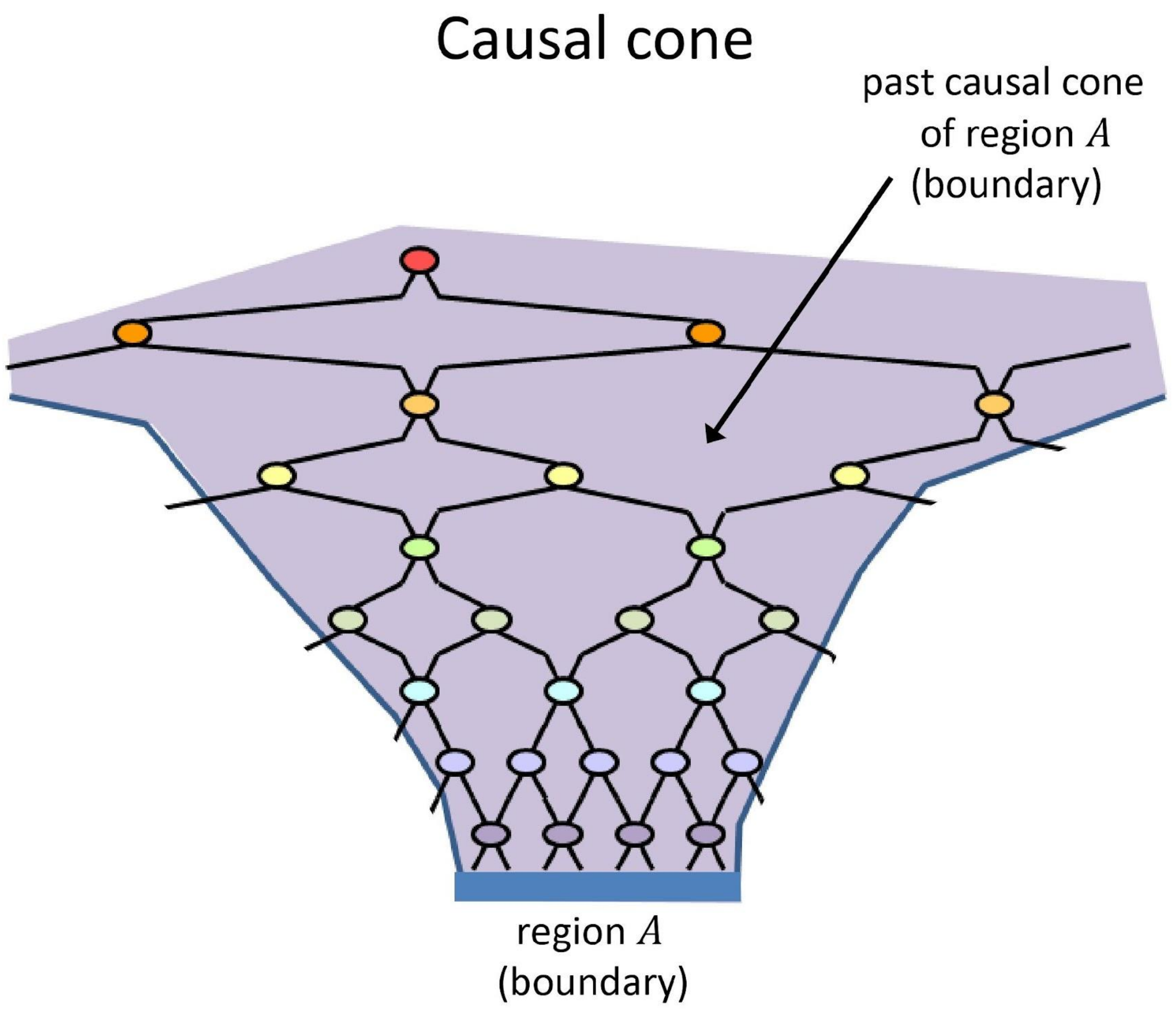
(Beny 2011, Czech 2015)

Causal structure

essential for many MERA properties
and computational efficiency


Causal cone
past causal cone
of region A
(boundary)
region A
(boundary)

MERA

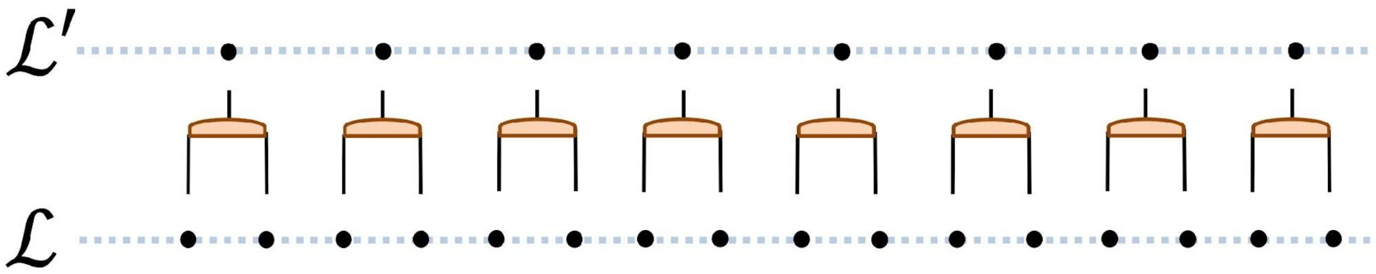

Causal cone
past causal cone
of region A
(boundary)
region A
(boundary)

MERA as RG Transformation

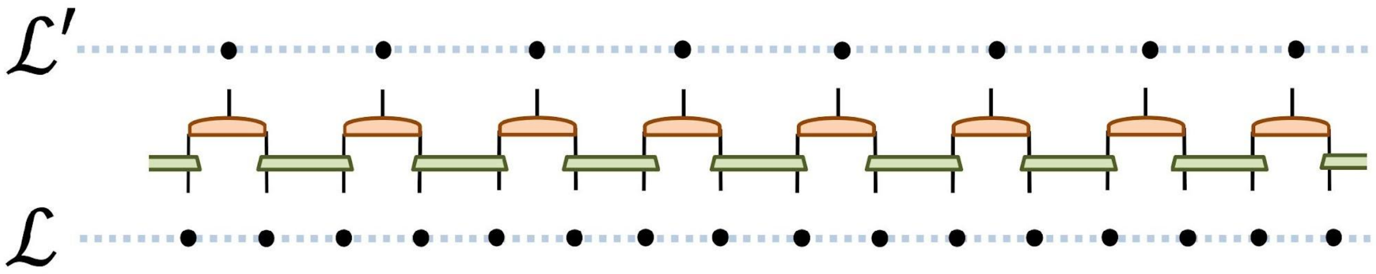
[Vidal 05]

Kadanoff (1966) + White (1992)

blocking variational optimization



Entanglement renormalization (2005)

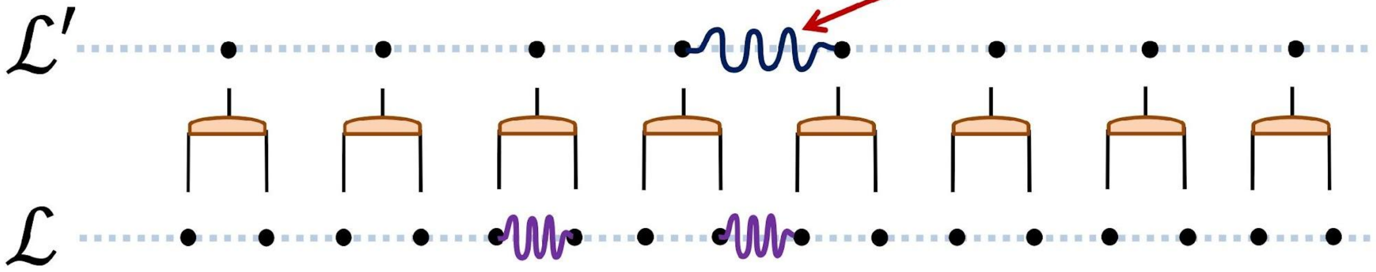


MERA as RG Transformation

[Vidal 05]

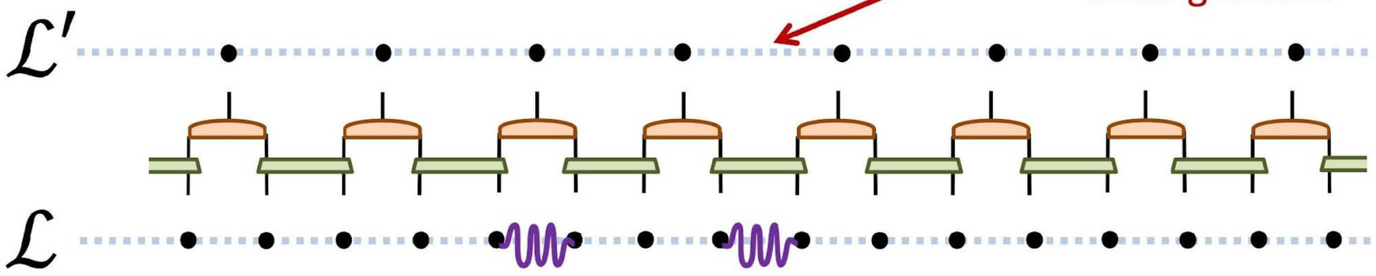
Kadanoff (1966) + White (1992)
blocking variational optimization

failure to remove ***some*** short-range entanglement !

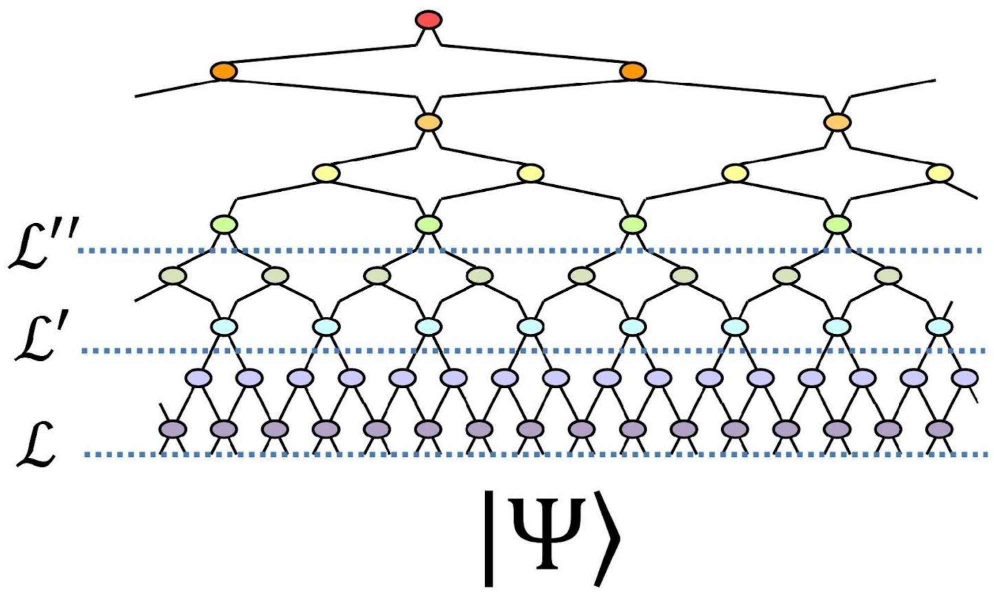


Entanglement renormalization (2005)

removal of ***all*** short-range entanglement

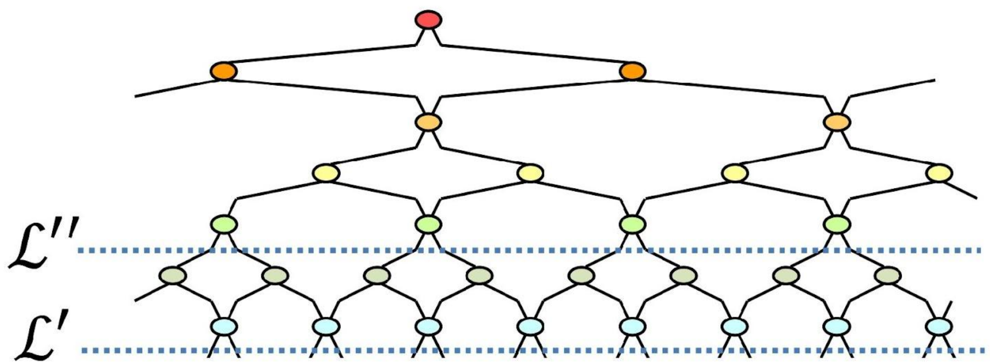


MERA as a sequence of ground state wave-functions



$$|\Psi\rangle \to |\Psi'\rangle \to |\Psi''\rangle \to \cdots$$

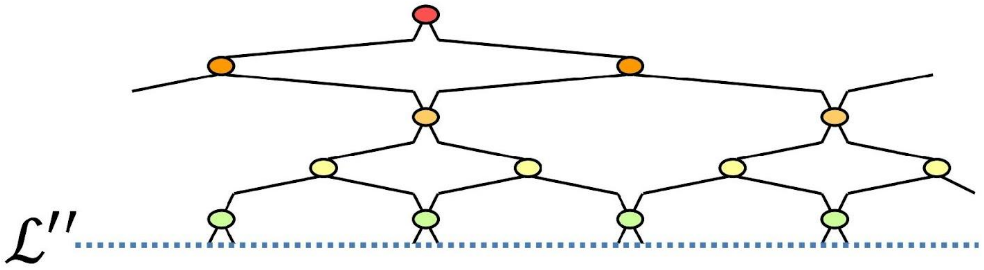
MERA as a sequence of ground state wave-functions



$$|\Psi'\rangle$$

$$|\Psi\rangle \to |\Psi'\rangle \to |\Psi''\rangle \to \cdots$$

MERA as a sequence of ground state wave-functions



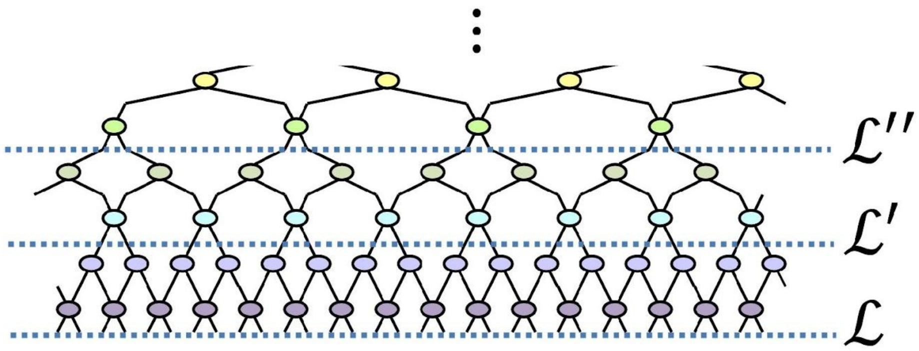
$$|\Psi''\rangle$$

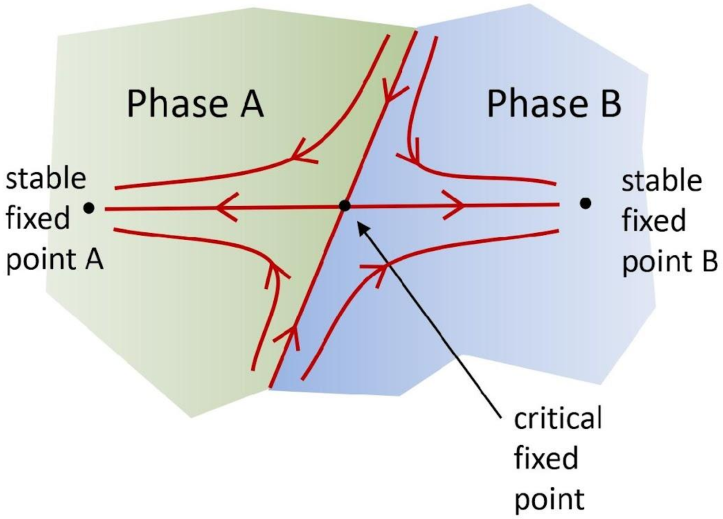
$$|\Psi\rangle \to |\Psi'\rangle \to |\Psi''\rangle \to \cdots$$

MERA

MERA defines an RG flow
in the space of wave-functions

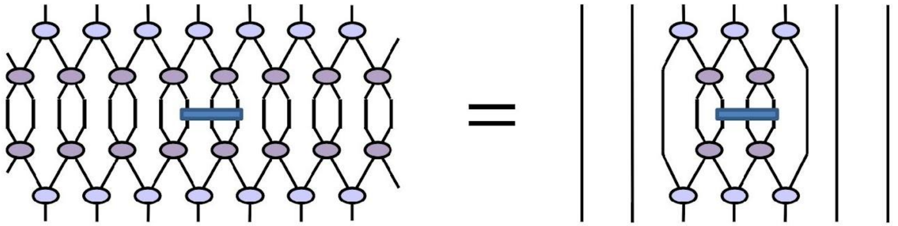
$$|\Psi\rangle \to |\Psi'\rangle \to |\Psi''\rangle \to \cdots$$





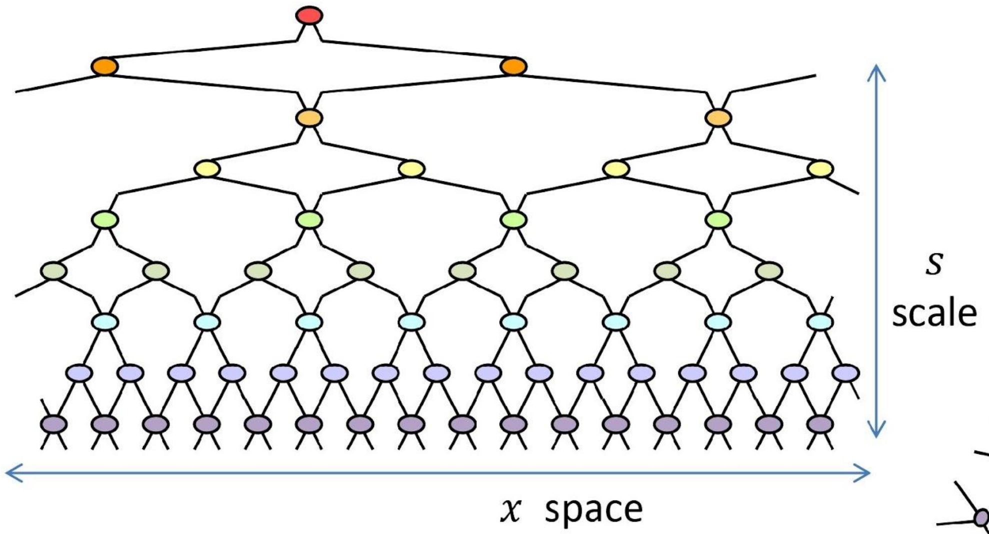
… and in the space of Hamiltonians

$$H \to H' \to H'' \to \cdots$$



local operators
are mapped into
local operators !

Entanglement entropy and correlations



- entanglement entropy

$$S_L \approx \log{(L)}$$

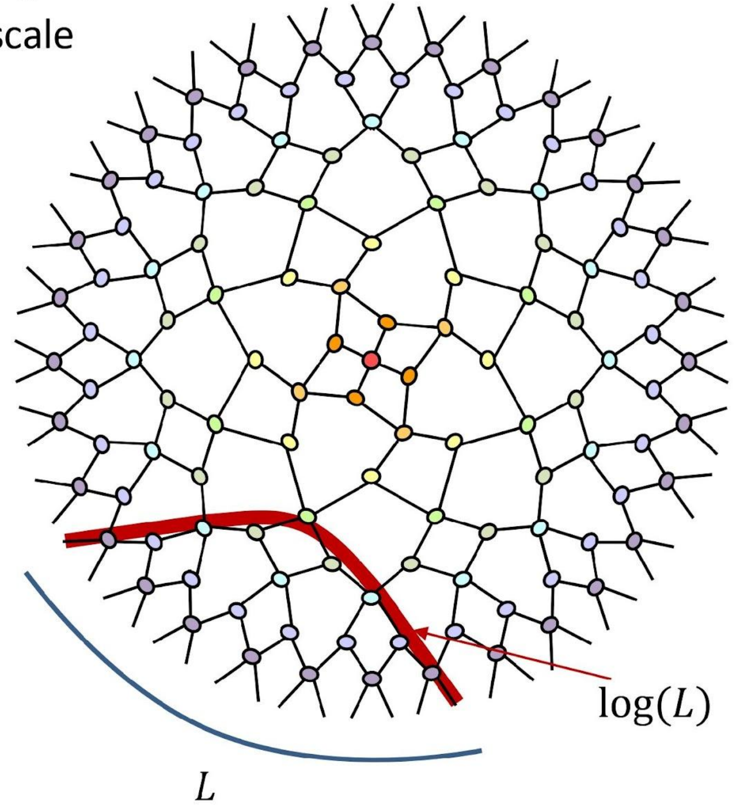
Computation of density matrix requires tracing out $\sim\log(L)$ indices

- two-point correlations

$$C(L) \approx L^{-2\Delta}$$

Geodesic distance $D \approx \log(L)$

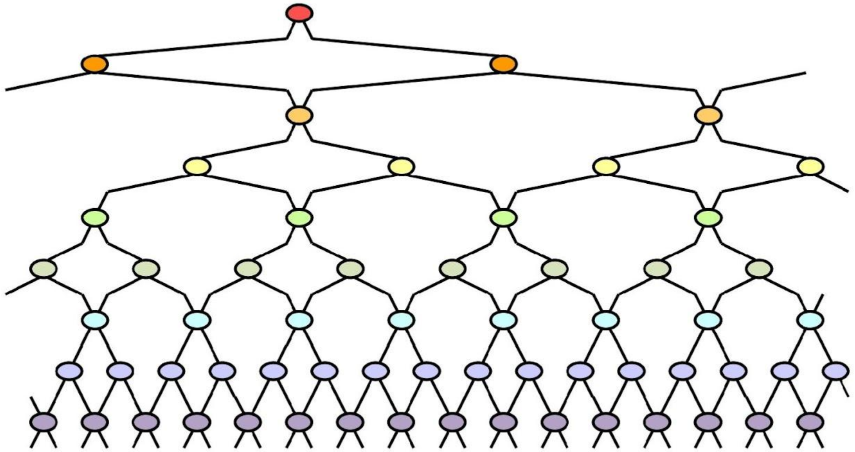
$$C(L) \approx e^{-D} = e^{-2\Delta\log(L)} = L^{-2\Delta}$$



MERA

Summary so far

MERA



- Variational parameters for different length scales

- It is secretly a **quantum circuit** → *"entanglement at different length scales"*

and an **RG transformation**

"removes short-range entanglement" $|\Psi\rangle \to |\Psi'\rangle$ $H \to H'$ *"preservation of locality"*

- Entanglement entropy and correlations as in 1+1 critical ground states

$S_L \approx \log(L)$ $C(L) \approx L^{-2\Delta}$

blah, blah, blah… However, does it work?

[Given lattice Hamiltonian H,
optimize variational parameters by energy minimization]

input

1D quantum Hamiltonian

- **on the lattice**
- **at a critical point**

output

Numerical determination of

- **central charge** c
- **scaling dimensions**
  and conformal spi
- **OPE coefficients** C_{c}

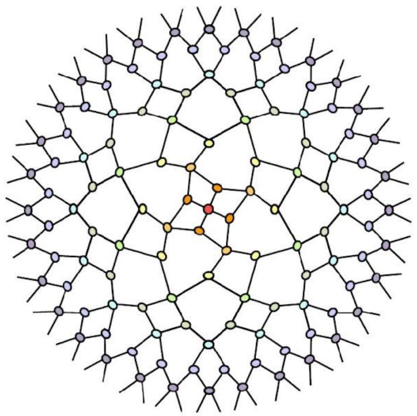
Also:
1+1 critical systems with

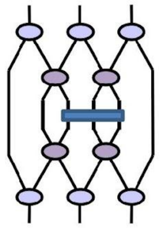
- ➢ impurities/defects
- ➢ boundaries
- ➢ interfaces

2+1 gapped phases with

- ➢ topological order
- ➢ frustrated spins
- ➢ interacting fermions

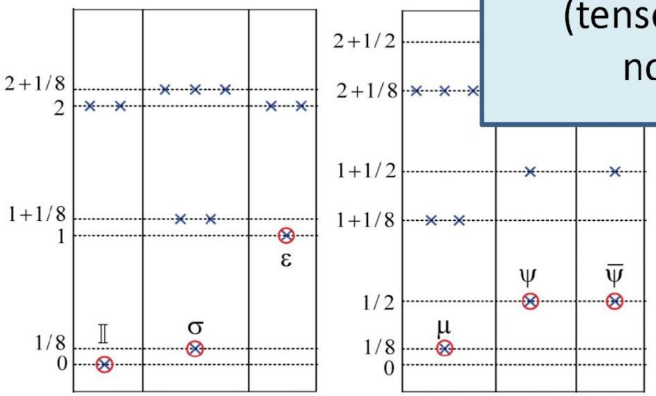
(tensor networks have no sign problem)





Pfeifer, Evenbly, Vidal 08

e.g. critical Ising model (appro



$\Delta_\varepsilon \approx 0.99993$

$\Delta_\mu \approx 0.125002$

$\Delta_\psi \approx 0.500001$

$\Delta_{\bar\psi} \approx 0.500001$

$C_{\epsilon\sigma\sigma} = \frac{1}{2}$ $\quad C_{\epsilon\mu\mu} = -\frac{1}{2}$

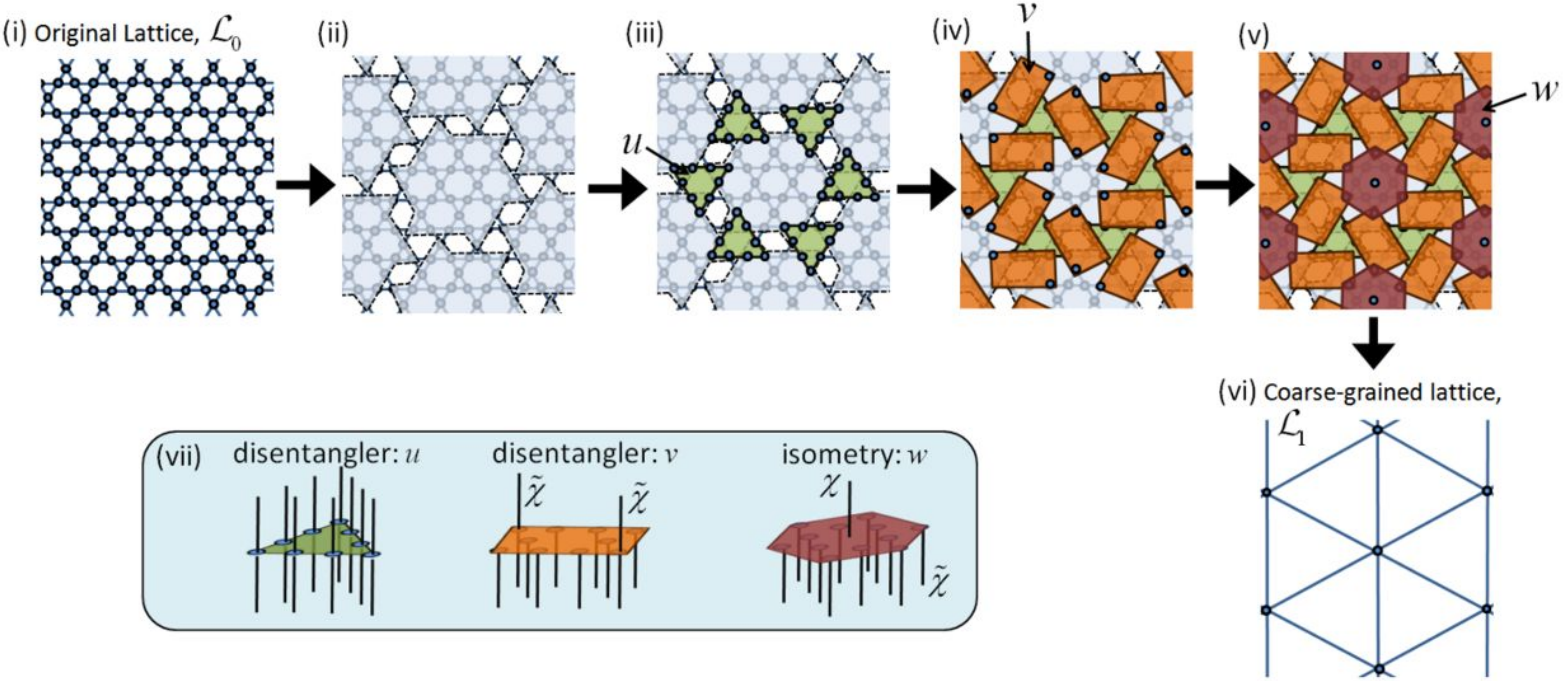
$C_{\epsilon\psi\bar\psi} = i$ $\quad C_{\epsilon\bar\psi\psi} = -i$

$C_{\psi\mu\sigma} = \frac{e^{-\frac{i\pi}{4}}}{\sqrt{2}}$ $\quad C_{\bar\psi\mu\sigma} = \frac{e^{\frac{i\pi}{4}}}{\sqrt{2}}$

$(\pm 6 \times 10^{-4})$

Results

Results

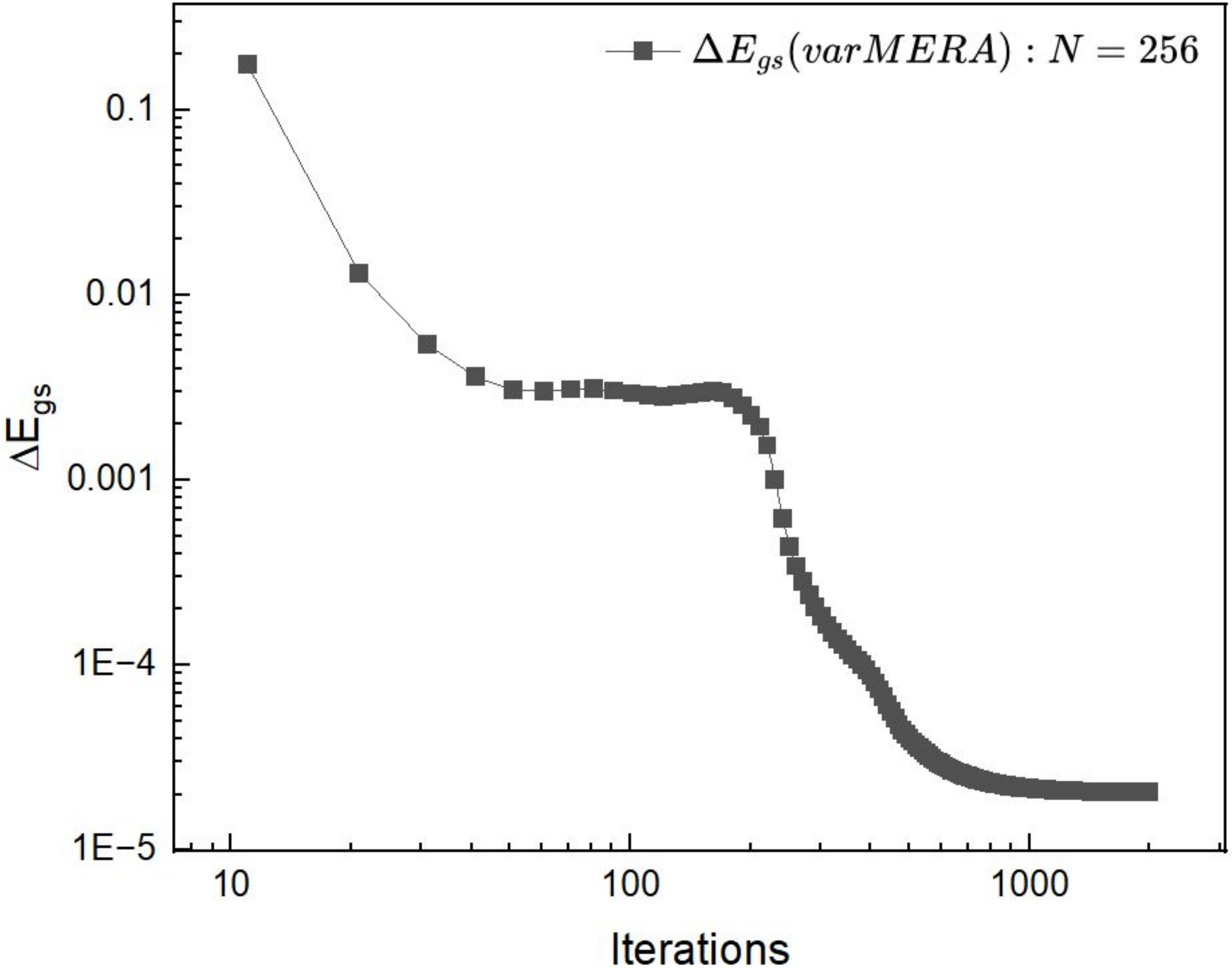


(i) Original Lattice, $\mathcal{L}_0$
(ii)
(iii)
u
(iv)
v
(v)
w
(vi) Coarse-grained lattice,
$\mathcal{L}_1$
(vii)
disentangler: u
disentangler: v
isometry: w
$\tilde{\chi}$
$\tilde{\chi}$
χ
$\tilde{\chi}$

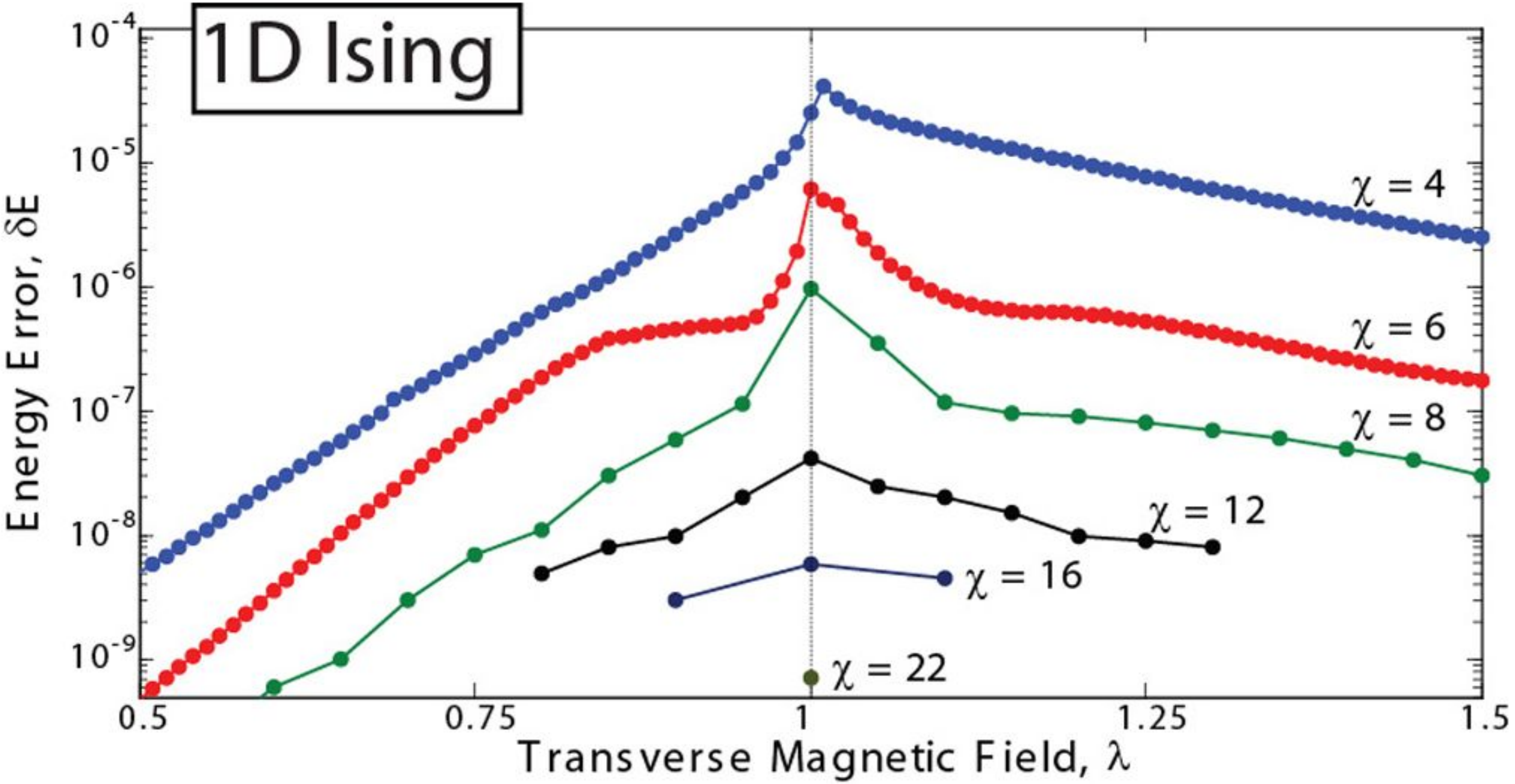
Results

Critical quantum 1D Ising Model in the presence of transverse magnetic field. Periodic boundary conditions

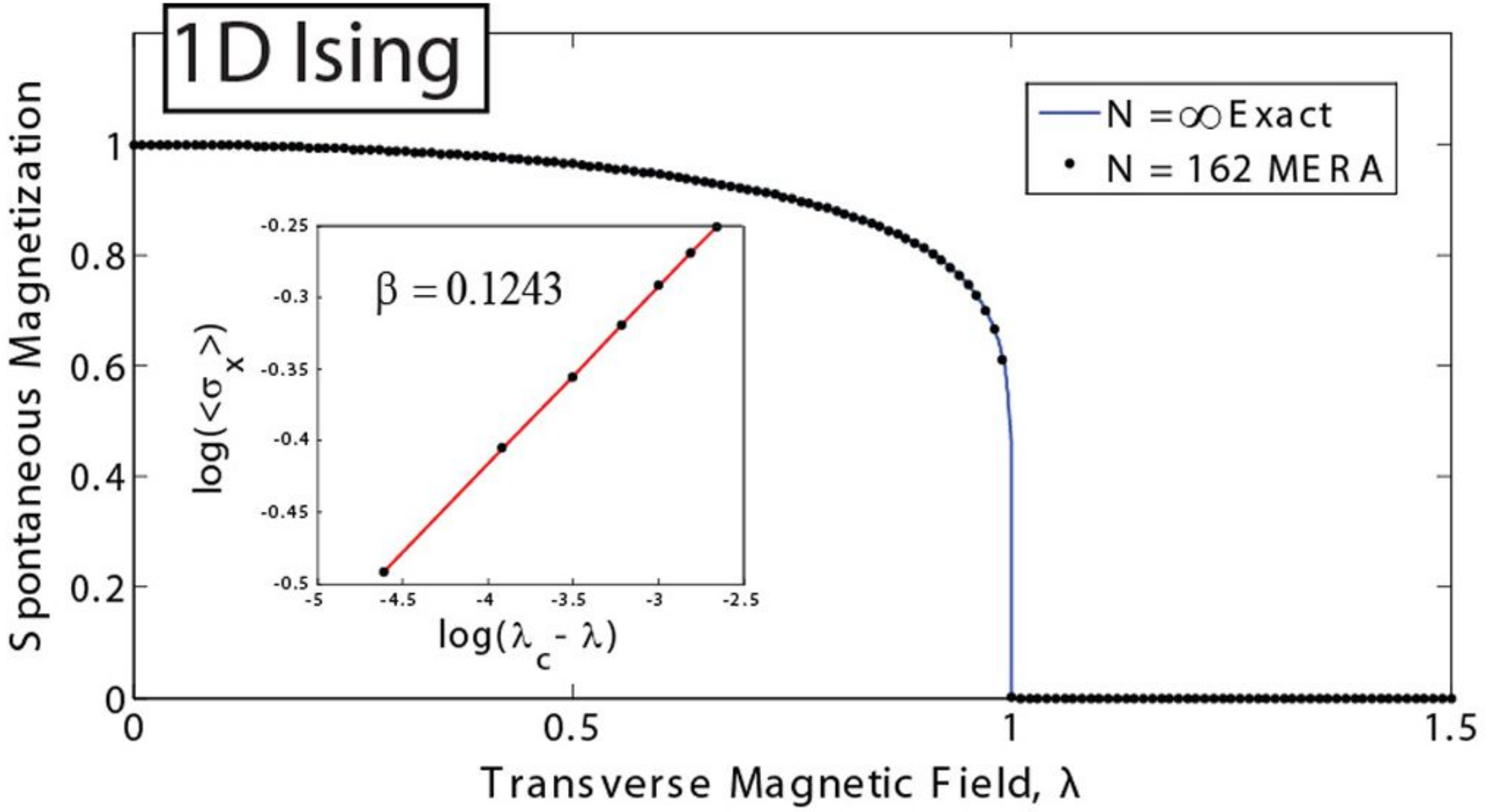
No. of sites $N = 256$



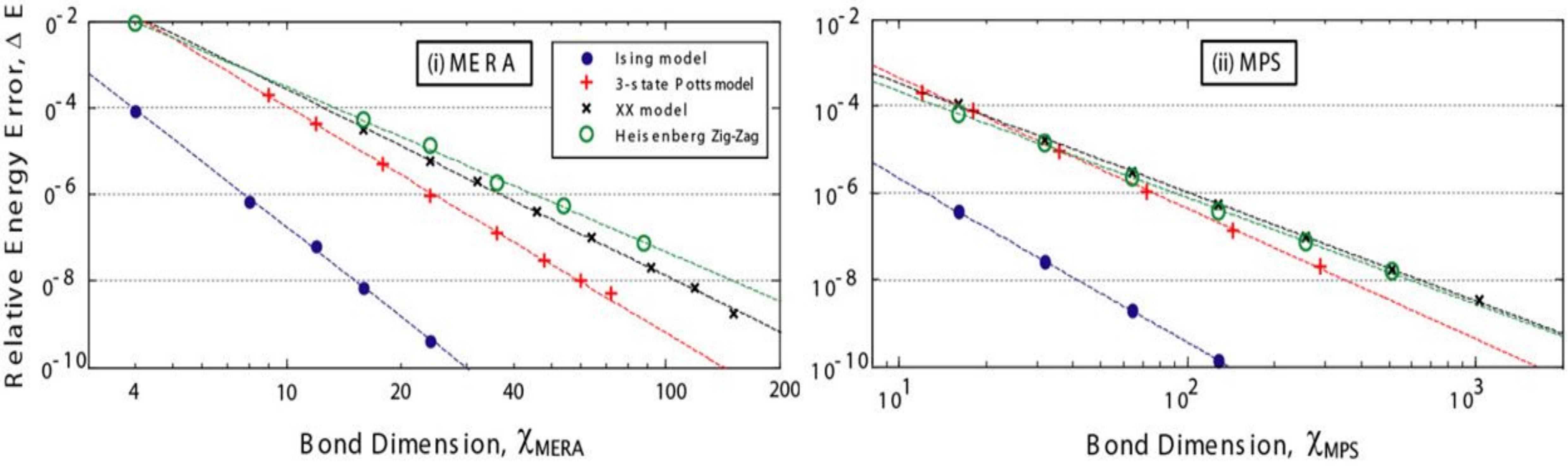
Results



1D Ising
Energy Error, δE
Transverse Magnetic Field, λ
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
0.5
0.75
1
1.25
1.5
χ = 4
χ = 6
χ = 8
χ = 12
χ = 16
χ = 22

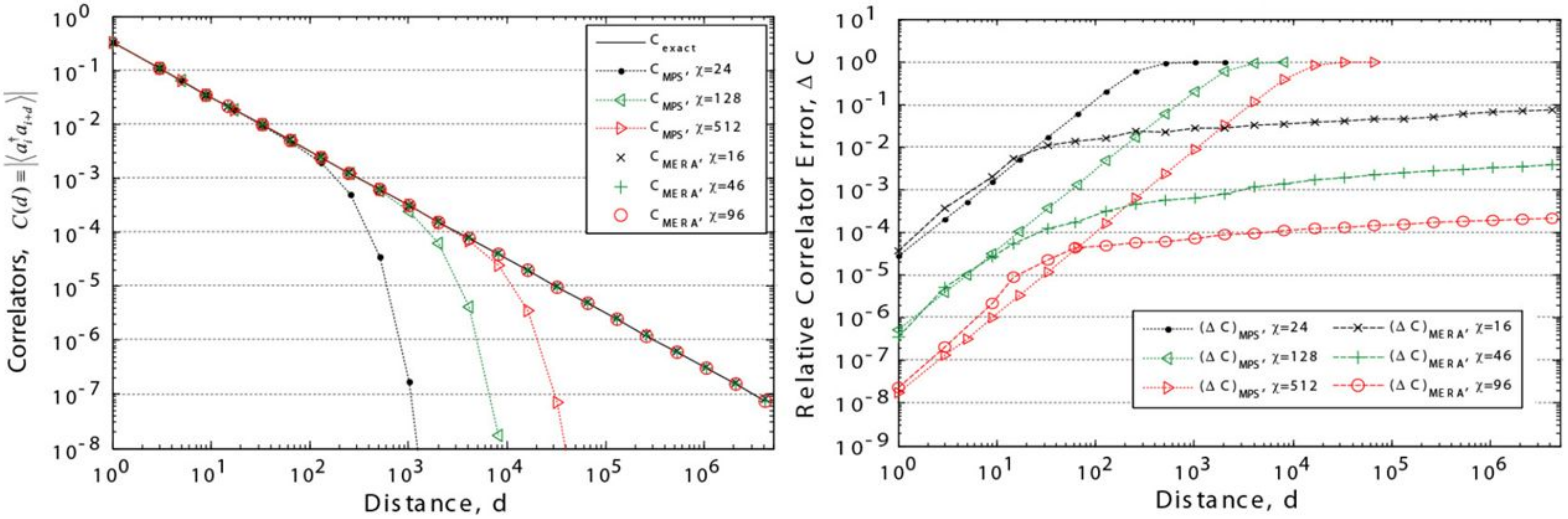

1D Ising
N = ∞ Exact
N = 162 MERA
Spontaneous Magnetization
Transverse Magnetic Field, λ
β = 0.1243
log(<σx>)
log(λc - λ)
1
0.8
0.6
0.4
0.2
0
0.5
1.5
-0.25
-0.3
-0.35
-0.4
-0.45
-0.5
-5
-4.5
-4
-3.5
-3
-2.5

Results



Relative Energy Error, Δ E
0^-2
0^-4
0^-6
0^-8
0^-10
(i) MERA
Ising model
3-state Potts model
XX model
Heisenberg Zig-Zag
4
10
20
40
100
200
Bond Dimension, χMERA
10^-2
10^-4
10^-6
10^-8
10^-10
(ii) MPS
10^1
10^2
10^3
Bond Dimension, χMPS

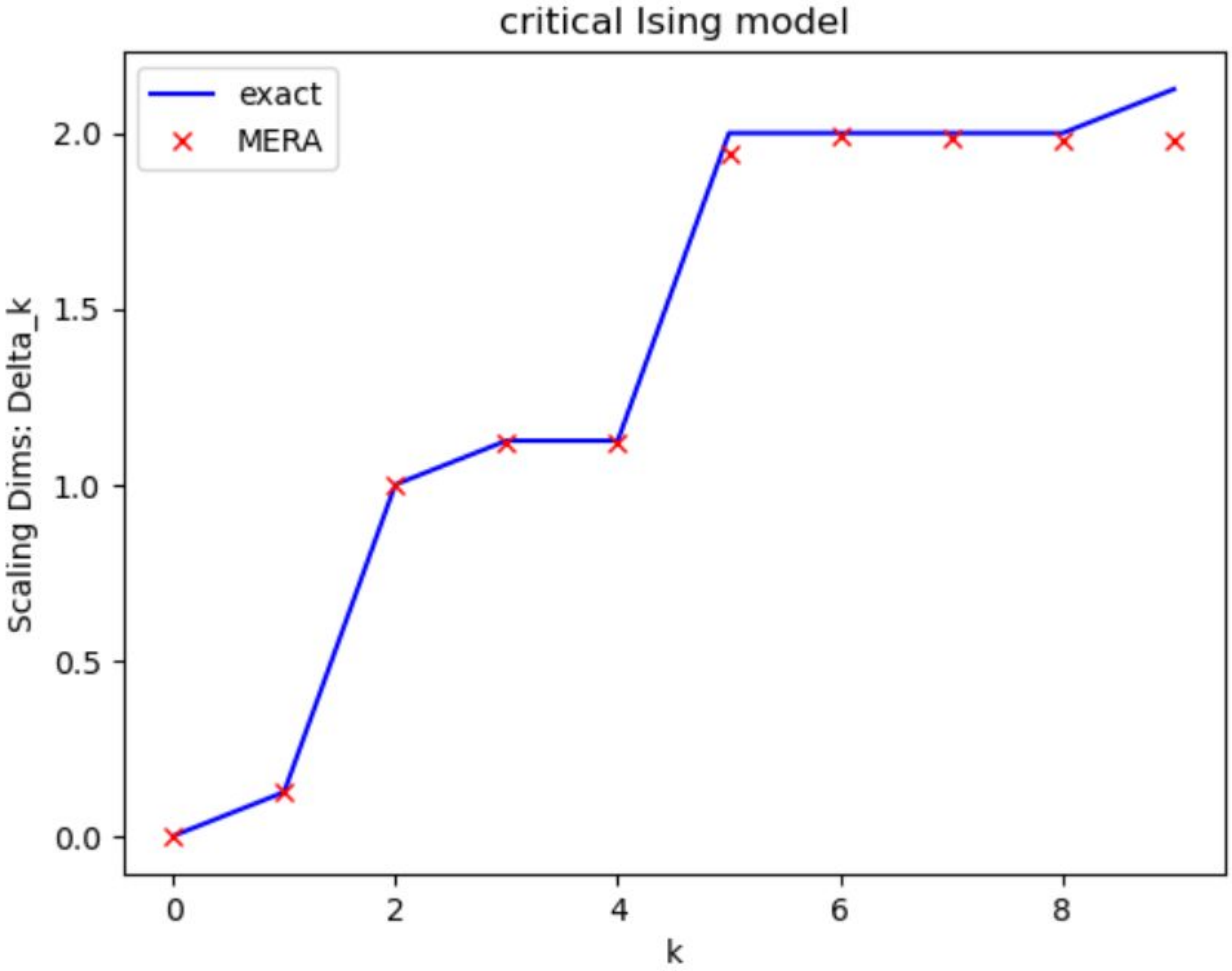
Results


Correlators, $C(d) \equiv |\langle a_i^\dagger a_{i+d} \rangle|$
Distance, d
C_{exact}
C_{MPS}, χ=24
C_{MPS}, χ=128
C_{MPS}, χ=512
C_{MERA}, χ=16
C_{MERA}, χ=46
C_{MERA}, χ=96
Relative Correlator Error, ΔC
$(\Delta C)_{MPS}$, χ=24
$(\Delta C)_{MPS}$, χ=128
$(\Delta C)_{MPS}$, χ=512
$(\Delta C)_{MERA}$, χ=16
$(\Delta C)_{MERA}$, χ=46
$(\Delta C)_{MERA}$, χ=96

Results

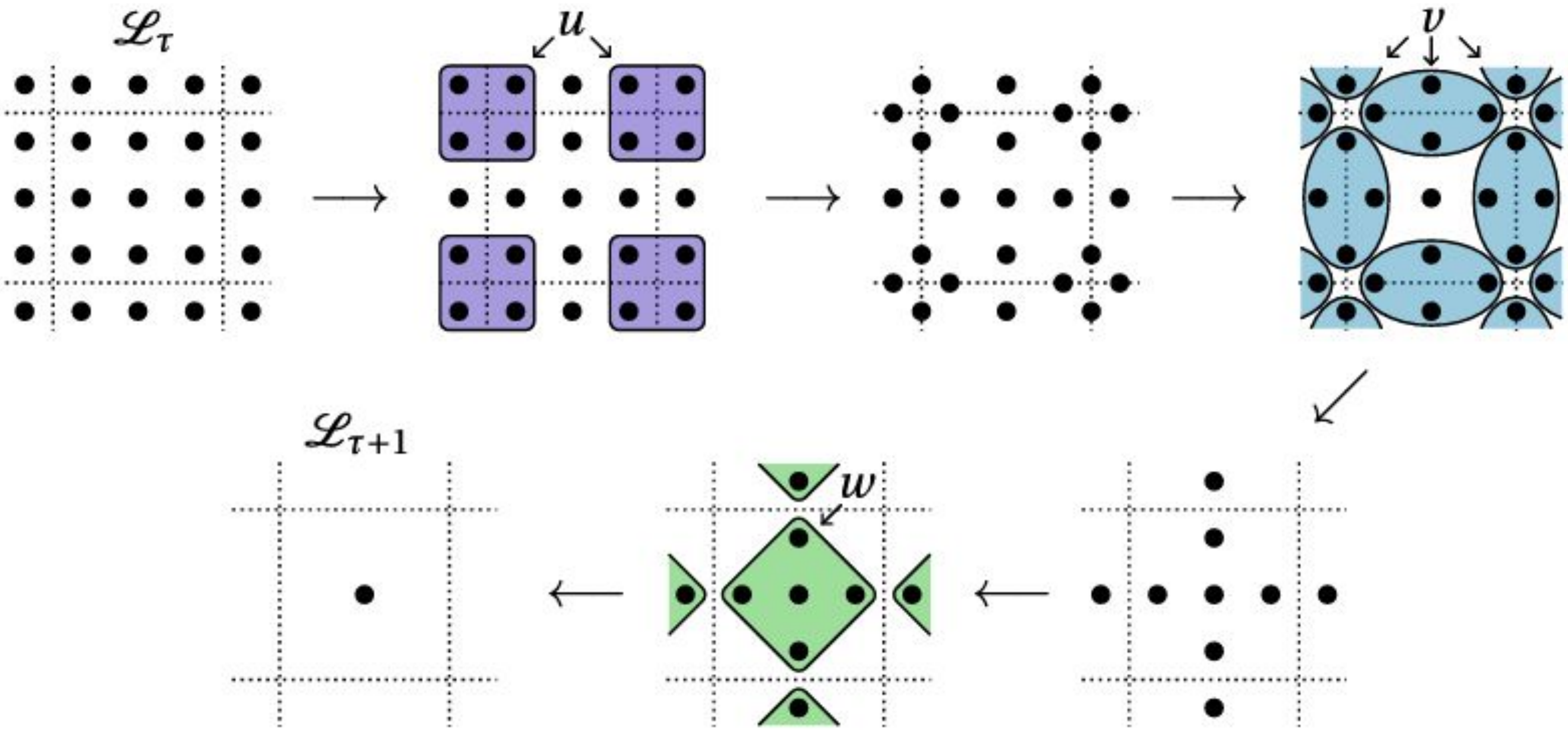
Critical quantum 1D Ising Model in the presence of transverse magnetic field. Periodic boundary conditions

No. of sites $N = 256$

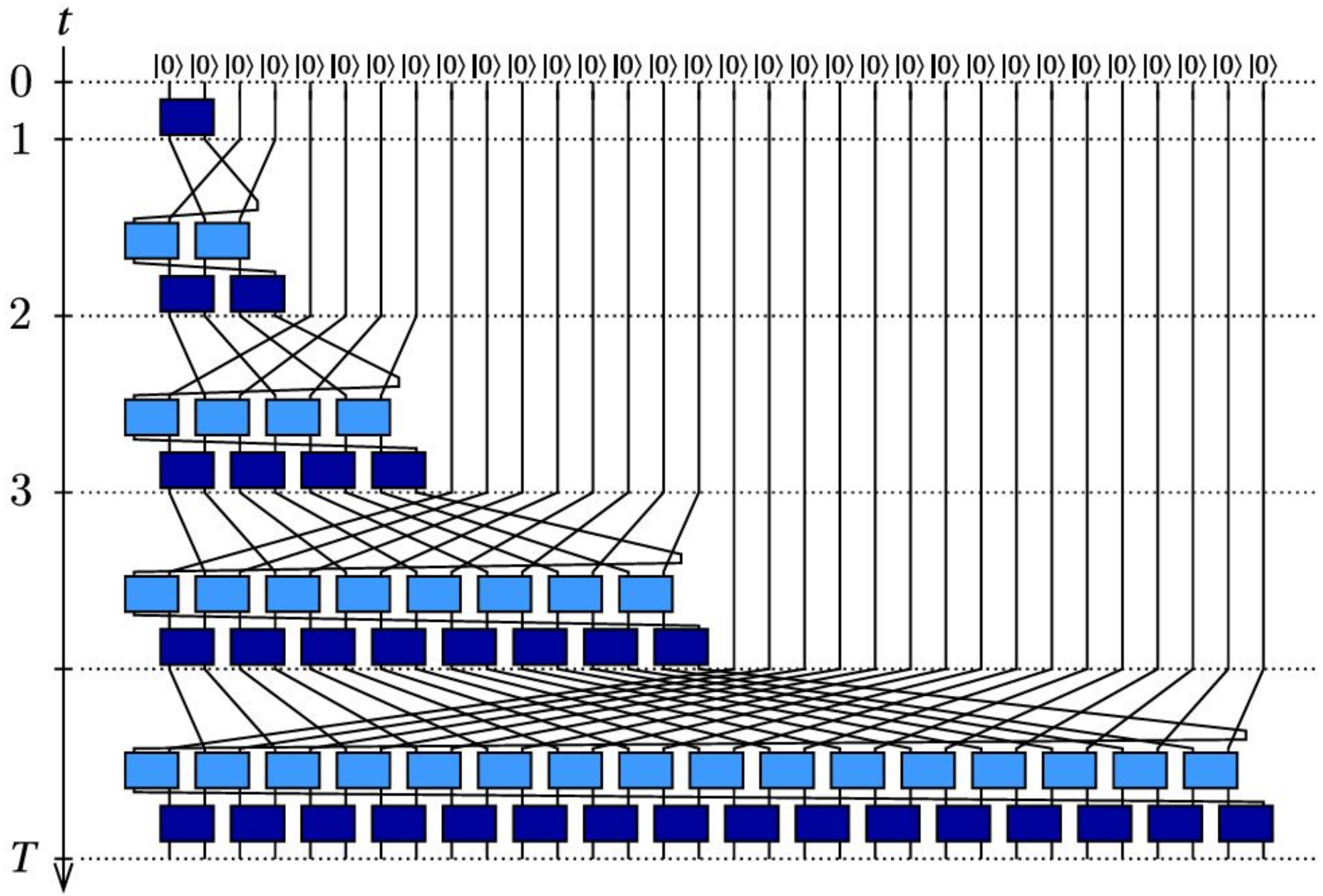


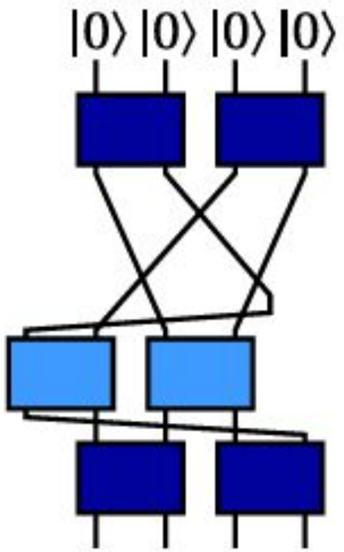
Extensions, applications

2-D MERA

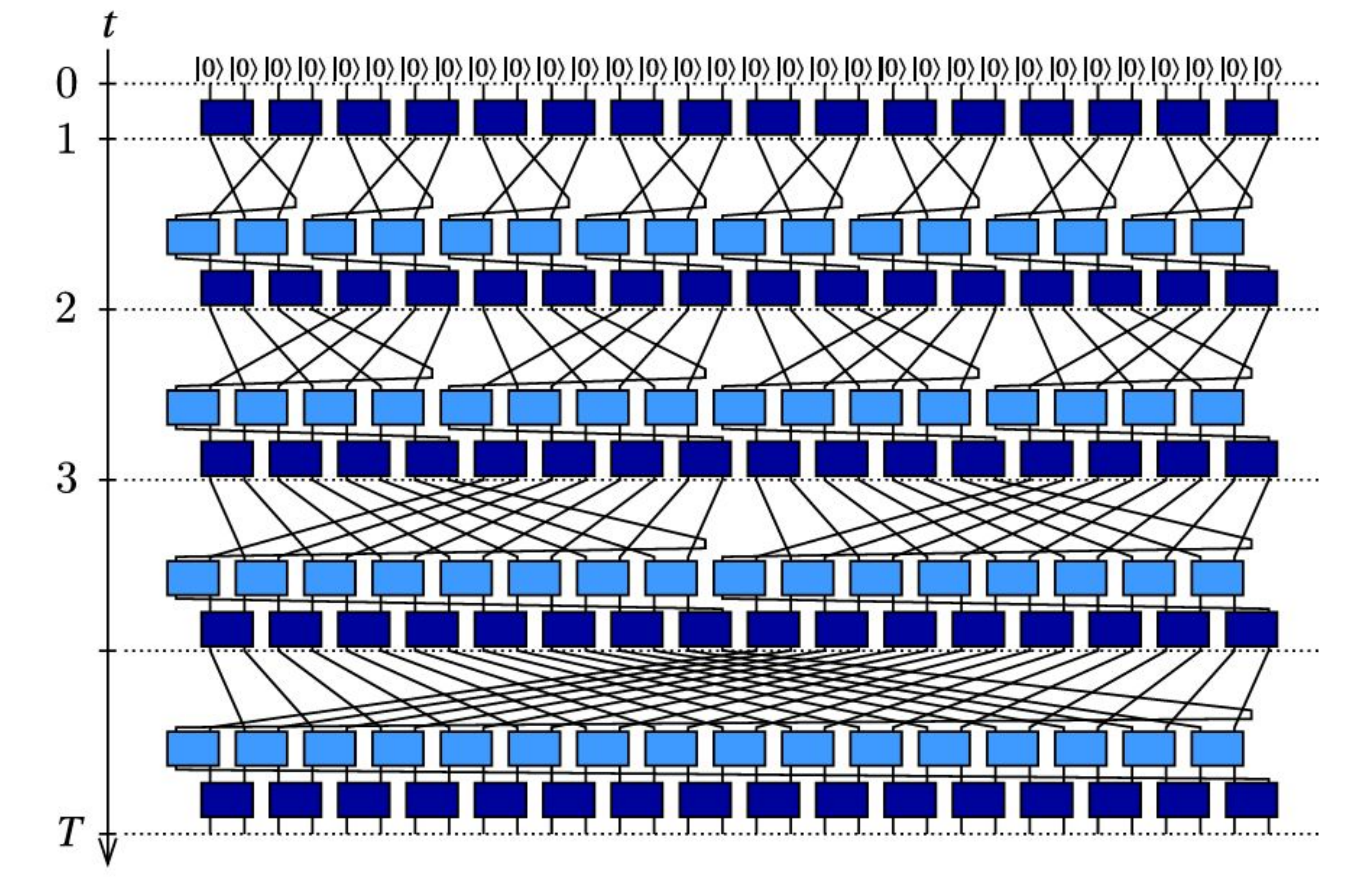


Branching MERA

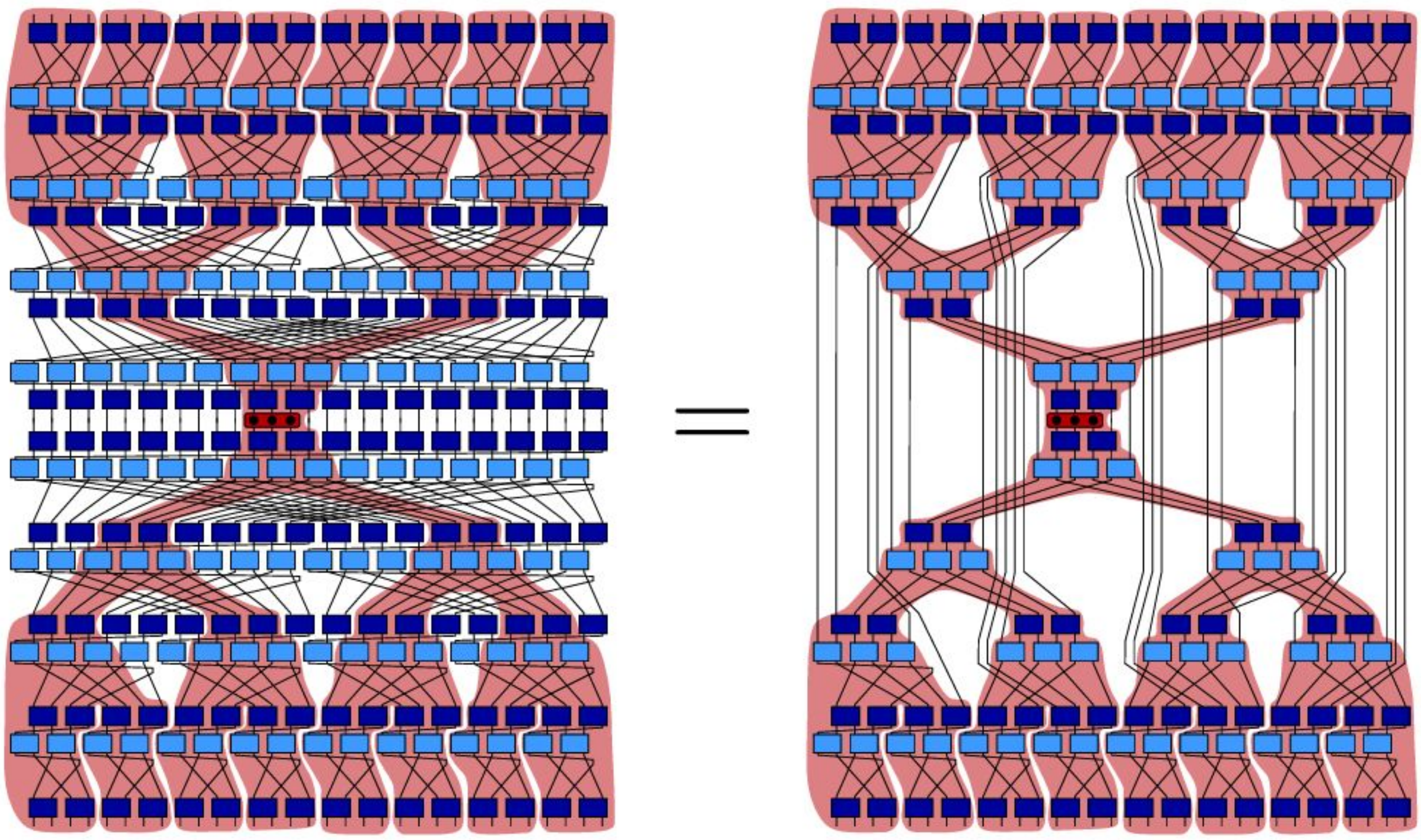




Branching MERA



Branching MERA



2-D MERA

2-D MERA

Thank You