Interval Sequence Realizability in Directed Graphs

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Abstract

Degree sequence realizability deals with the following question: Given a sequence D of n integers (integer-pairs), does there exist an n vertex graph (digraph) whose degree (indegree-outdegree) sequence is D? In a seminal work, Erdős-Gallai [EG60] gave a characterization for n-length sequences realizable by undirected graphs, and Fulkerson-Chen-Anstee [Ans82, Che66, Ful60] studied the analogous problem for directed graphs.

A natural extension of degree realizability problem is the *interval realizability problem*, wherein, instead of exact degrees we are given an interval range for degree of each vertex. The interval realizability problem is well studied for undirected graphs [CDZ00, BNCPR20], however, nothing is known for digraphs.

In this paper we address the problem of interval realizability for directed graphs, and obtain the following results.

- We present a characterization for interval-pair sequences $S = \{([a_i, b_i], [c_i, d_i])\}_{i=1}^{n}$ for which there exists an n-vertex directed graph whose i^{th} vertex has indegree in the range $[a_i, b_i]$ and outdegree in the range $[c_i, d_i]$, for $1 \leqslant i \leqslant n$.
- We provide a linear time algorithm for verifying realizability of any n-length interval-pair sequence $S = \{([a_i, b_i], [c_i, d_i])\}_{i=1}^{n}$.

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1 Introduction

Degree sequence realizability is a fundamental problems in graph theory that has been studied extensively in the past six decades. A sequence of n positive integers, $D = (d_1, \ldots, d_n)$, is said to be *graphic* if there exists an n vertex graph G such that the i^{th} vertex in G has degree d_i. The problem was first studied by Erdős and Gallai [EG60], who gave a characterization for integer sequences that are graphic. Havel and Hakimi [Hak62, Hav55] presented a recursive characterization that, given a sequence D of integers, either computes (in optimal time) a realizing graph, or proves that the sequence is non-graphic. The analogous characterization problem for bipartite graphs was solved independently by Gale [Gal57] and Ryser [Rys57] using network flows.

A natural variant of the degree realization problem requires the realizing graph to be directed. Formally, a sequence of integer pairs $D = \{(p_i, q_i)\}_{i=1}^n$ is said to be *realizable* (or *digraphic*) if there exists a directed graph G whose i^{th} vertex has indegree p_i and outdegree q_i.

Kleitman and Wang [KW73] provided a recursive characterization for verifying digraph realizability. Fulkerson-Chen-Anstee [Ful60, Che66, Ans82] provided the following characterization for realizability of integer-pair sequences.

Theorem 1 (Fulkerson-Chen-Anstee) *An integer-pair sequence $D = \{(p_i, q_i)\}_{i=1}^n$ of non-negative integer pairs satisfying $p_1 \geqslant \cdots \geqslant p_n$ is digraphic if and only if $\sum_i p_i = \sum_i q_i$ and the following holds for $k \in [1, n]$,*

$$\sum_{i=1}^{k} p_i \;\leqslant\; \sum_{i=1}^{k} \min(q_i, k-1) \;+\; \sum_{i=k+1}^{n} \min(q_i, k)\;.$$

We consider a generalization of the degree sequence problem where instead of specifying the precise degrees, we are given a *range* (or interval) of possible degree values for each vertex. This is referred as *interval realizability problem*. In 2000, the interval realizability problem was studied for general undirected graphs by Cai, Deng, and Zang [CDZ00]. Around a decade later, Garg, Goel, and Tripathi [GGT11] provided a characterization for interval realizability problem for bipartite graphs. However, to the best of our knowledge, nothing is known for directed graphs.

Consider an interval-pair sequence S defined as $S = \big\{\big([a_i, b_i], [c_i, d_i]\big)\big\}_{i=1}^n$. The sequence S is said to be *realizable* (or *digraphic*) if there exists an integer-pair sequence $D = \{(p_i, q_i)\}_{i=1}^n$ such that $a_i \leqslant p_i \leqslant b_i$ and $c_i \leqslant q_i \leqslant d_i$ for $1 \leqslant i \leqslant n$ and D is digraphic.

In this paper, we present a characterization for digraphic interval-pair sequences. Furthermore, we develop a linear time algorithm for verifying if a given interval-pair sequence is realizable or not. Below we summarize our results.

1. *Characterization.* Let $S = \big\{\big([a_i, b_i], [c_i, d_i]\big)\big\}_{i=1}^n$ be an interval-pair sequence over non-negative integers such that $a_1 \geqslant \cdots \geqslant a_n$, and let σ be any permutation of $[1, n]$ satisfying $c_{\sigma(1)} \geqslant \cdots \geqslant c_{\sigma(n)}$. Then S is digraph realizable if and only if the following holds for $1 \leqslant k \leqslant n$,

$$\sum_{i=1}^{k} a_i \;\leqslant\; \sum_{i=1}^{k} \min(d_i, k-1) \;+\; \sum_{i=k+1}^{n} \min(d_i, k)$$

$$\sum_{i=1}^{k} c_{\sigma(i)} \;\leqslant\; \sum_{\substack{i\in[1,n] \\ s.t.\ \sigma^{-1}(i)\leqslant k}} \min(b_i, k-1) \;+\; \sum_{\substack{i\in[1,n] \\ s.t.\ \sigma^{-1}(i)> k}} \min(b_i, k)$$

2. *Verification.* For any n-length interval-pair sequence $S = \big\{\big([a_i, b_i], [c_i, d_i]\big)\big\}_{i=1}^n$ we can verify if there exists a digraphic sequence $D = \{(p_i, q_i)\}_{i=1}^n$ satisfying $a_i \leqslant p_i \leqslant b_i$ and $c_i \leqslant q_i \leqslant d_i$, for each $i \in [1, n]$, in $O(n)$ time.

In order to prove our characterization, we provide a constructive algorithm that given a sequence $S = \{([a_i, b_i], [c_i, d_i])\}_{i=1}^n$ finds a realizing digraph in polynomial time. Our algorithm is inspired by the work of Garg, Goel, and Tripathi [GGT11] on the interval realizability problem for bipartite graphs.

1.1 Related Work

Hartung and Nichterlein [HN15] showed that the problem of realizing degree sequences for directed acyclic graphs is NP-complete. Tripathi, Venugopalan, West [TVW10] provided a constructive proof of Erdös and Gallai's [EG60] characterization for general undirected graphs.

For undirected interval realizability problem, Bar-Noy et al. [BNCPR20] presented an algorithm that for any integer $n \geqslant 1$ and any n length interval sequence S, computes a graphic sequence D realizing S, if it exists, in $O(n \log n)$ time. Rechner [Rec17] studied the interval realizability problem for bipartite graphs and gave an algorithm to compute a bipartite graph realizing a given bipartition-degree-sequence in optimal time. Recently, Bar-Noy et al. [BBPR22] studied a variant of the bigraphic degree realization problem, wherein, instead of two lists we are given a single list of degrees.

Over the years, various extensions of the degree realization problems were studied as well, cf. [AT94, WK73]. The *Subgraph Realization problem* considers the restriction that the realizing graph must be a subgraph (*factor*) of some fixed input graph. For an interesting line of work on graph factors, we refer the reader to [Tut81, Ans85, HHKL90, GY14].

2 Preliminaries

A *sequence* is defined to be an n-element list whose entries are non-negative integers. An *integer-pair sequence* is a sequence of n integer-pairs $D = \{(p_i, q_i)\}_{i=1}^n$, such that p_i and q_i are non-negative, for $i \in [1, n]$. An *interval-pair sequence* is a sequence of n pairs of intervals $S = \{([a_i, b_i], [c_i, d_i])\}_{i=1}^n$ such that a_i, b_i, c_i, and d_i are non-negative integers, for each $i \in [1, n]$. We say that an integer pair-sequence $D = \{(p_i, q_i)\}_{i=1}^n$ *realizes* $S = \{([a_i, b_i], [c_i, d_i])\}_{i=1}^n$ if D is digraphic and it satisfies the condition that $a_i \leqslant p_i \leqslant b_i$ and $c_i \leqslant q_i \leqslant d_i$, for each $i \in [1, n]$. Given any two n-length sequences $X = (x_i)$ and $Y = (y_i)$, we say that $X \leqslant Y$ if $x_i \leqslant y_i$, for $i \in [1, n]$. Let $G = (V, E)$ be a directed graph, and let $v \in V$ be a vertex in G. We use notation $\text{OUTN}_G(v)$ to denote the set of all the out-neighbours of v in G; similarly, $\text{INN}_G(v)$ denotes the set of all the in-neighbours of v in G. Further, $\text{INDEG}_G(v)$ denotes the indegree of v in G, and $\text{OUTDEG}_G(v)$ denotes the outdegree of v in G. We drop the subscripts when the graph G is clear from the context.

3 Characterizing Realizable Interval-Pair Sequences

In this section, we present a characterization for digraph realizability of interval-pair sequences.

Theorem 2 *Let $S = \{([a_i, b_i], [c_i, d_i])\}_{i=1}^n$ be an interval-pair sequence such that $a_1 \geqslant \cdots \geqslant a_n$, and let σ be any permutation of $[1, n]$ satisfying $c_{\sigma(1)} \geqslant \cdots \geqslant c_{\sigma(n)}$. Then S is digraph realizable if and only if the following holds for $1 \leqslant k \leqslant n$,*

$$\sum_{i=1}^{k} a_i \leqslant \sum_{i=1}^{k} \min(d_i, k-1) + \sum_{i=k+1}^{n} \min(d_i, k) \tag{1}$$

$$\sum_{i=1}^{k} c_{\sigma(i)} \leqslant \sum_{\substack{i \in [1,n] \\ s.t.\ \sigma^{-1}(i) \leqslant k}} \min(b_i, k-1) + \sum_{\substack{i \in [1,n] \\ s.t.\ \sigma^{-1}(i) > k}} \min(b_i, k) \tag{2}$$

Proof: Let us first define some notations used in our proof. For any $k, j \in [1, n]$, we use notation $\text{OUT}_k(v_j)$ to denote the number of out-neighbours of v_j that lie in the set $\{v_1, \ldots, v_k\}$. Similarly, we use notation $\text{IN}_k(v_j)$ denote the number of in-neighbours of v_j that lie in the set $\{v_{\sigma(1)}, \ldots, v_{\sigma(k)}\}$, for $k, j \in [1, n]$.

We now prove the necessity. Let $S = \big\{\big([a_i, b_i], [c_i, d_i]\big)\big\}_{i=1}^{n}$ be an interval-pair sequence such that $a_1 \geqslant \cdots \geqslant a_n \geqslant 0$ and σ be a permutation of $[1, n]$ satisfying $c_{\sigma(1)} \geqslant \cdots \geqslant c_{\sigma(n)} \geqslant 0$. Further, let G be a directed graph with vertices $v_1, \ldots, v_n$ such that for $i \in [1, n]$, we have $a_i \leqslant \text{INDEG}(v_i) \leqslant b_i$ and $c_i \leqslant \text{OUTDEG}(v_i) \leqslant d_i$.

Recall $\text{IN}_k(v_j)$ is the number of in-neighbours of v_j that lie in the set $\{v_{\sigma(1)}, \ldots, v_{\sigma(k)}\}$. Now because the edge (v_i, v_i) does not exist for $i \in [1, n]$, we have,

$$
\begin{aligned}
\text{IN}_k(v_j) &\leqslant \min\big(\text{INDEG}(v_j),\ k-1\big) && \text{if} \quad \sigma^{-1}(j) \leqslant k \\
\text{IN}_k(v_j) &\leqslant \min\big(\text{INDEG}(v_j),\ k\big) && \text{if} \quad \sigma^{-1}(j) \geqslant k+1
\end{aligned}
$$

Thus, we get

$$
\sum_{j=1}^{n} \text{IN}_k(v_j) \leqslant \sum_{\substack{j \in [1,n] \\ s.t.\ \sigma^{-1}(j) \leqslant k}} \min(\text{INDEG}(v_j), k-1) + \sum_{\substack{j \in [1,n] \\ s.t.\ \sigma^{-1}(j) > k}} \min(\text{INDEG}(v_j), k).
$$

Observe $\sum_{i=1}^{k} \text{OUTDEG}(v_{\sigma(i)}) = \sum_{j=1}^{n} \text{IN}_k(v_j)$. This together with the fact that $c_i \leqslant \text{OUTDEG}(v_i)$ and $\text{INDEG}(v_i) \leqslant b_i$ for $1 \leqslant i \leqslant n$, gives

$$
\sum_{i=1}^{k} c_{\sigma(i)} \leqslant \sum_{\substack{j \in [1,n] \\ s.t.\ \sigma^{-1}(j) \leqslant k}} \min(b_j, k-1) + \sum_{\substack{j \in [1,n] \\ s.t.\ \sigma^{-1}(j) > k}} \min(b_j, k)
$$

We can prove the inequality in Eq. 2 similarly. We provide the proof for completeness. We have, $\text{OUT}_k(v_j) \leqslant \min(\text{OUTDEG}(v_j), k-1)$ for $j \leqslant k$, and $\text{OUT}_k(v_j) \leqslant \min(\text{OUTDEG}(v_j), k)$ for $j \geqslant k+1$. So,

$$
\begin{aligned}
\sum_{i=1}^{k} a_i \leqslant \sum_{i=1}^{k} \text{INDEG}(v_i) &= \sum_{j=1}^{n} \text{OUT}_k(v_j) \\
&\leqslant \sum_{j=1}^{k} \min(\text{OUTDEG}(v_j), k-1) + \sum_{j=k+1}^{n} \min(\text{OUTDEG}(v_j), k) \\
&\leqslant \sum_{j=1}^{k} \min(d_j, k-1) + \sum_{j=k+1}^{n} \min(d_j, k).
\end{aligned}
$$

We next prove the sufficiency. Let $S = \big\{\big([a_i, b_i], [c_i, d_i]\big)\big\}_{i=1}^{n}$ be a interval-pair sequence, and σ be a permutation of $[1, n]$ such that $a_1 \geqslant \cdots \geqslant a_n$ and $c_{\sigma(1)} \geqslant \cdots \geqslant c_{\sigma(n)}$. Moreover assume that S satisfies the conditions in Eq. 1 and Eq. 2.

We will provide algorithmic construction of the required graph. This will involve two stages.

Stage 1: Here we will provide construction of a graph G such that $\text{OUTDEG}(v_i) = c_i$ and $\text{INDEG}(v_i) \leqslant b_i$, for $i \in [1, n]$. We start with an empty graph, and throughout Stage 1 maintain the invariant that $\text{OUTDEG}(v_i) \leqslant c_i$ and $\text{INDEG}(v_i) \leqslant b_i$, for $i \in [1, n]$.

Let $r \in [1,n]$ be the largest index such that $\text{OUTDEG}(v_{\sigma(i)}) = c_{\sigma(i)}$ for each $i < r$, and $\text{OUTDEG}(v_{\sigma(r)}) < c_{\sigma(r)}$.

Note that, if $\forall j \in [1,n]$ we have,

$$\begin{aligned}
\text{IN}_r(v_j) &= \min(b_j, r-1) \qquad \text{for } \sigma^{-1}(j) \leqslant r \\
\text{IN}_r(v_j) &= \min(b_j, r) \qquad \text{for } \sigma^{-1}(j) > r
\end{aligned}$$

then,

$$\sum_{\substack{j\in[1,n] \\ s.t.\ \sigma^{-1}(j)\leqslant r}} \min(b_j, r-1) \;+\; \sum_{\substack{j\in[1,n] \\ s.t.\ \sigma^{-1}(j)>r}} \min(b_j, r) \;=\; \sum_{j=1}^{n} \text{IN}_r(v_j) \;=\; \sum_{i=1}^{r} \text{OUTDEG}(v_{\sigma(i)}) < \sum_{i=1}^{r} c_{\sigma(i)}$$

which contradicts Eq. 2. Hence there exists a j satisfying $\text{IN}_r(v_j) < \min(b_j, r-1)$ if $\sigma^{-1}(j) \leqslant r$, and $\text{IN}_r(v_j) < \min(b_j, r)$ if $\sigma^{-1}(j) > r$.

Let ℓ $(\in [1,r])$ be the largest index such that $\sigma(\ell) \neq j$ and $(v_{\sigma(\ell)}, v_j)$ is not an edge in G. Such an ℓ must exists by definition of j. Further, if $\ell < r$ then there must exists a vertex (say v) in $\text{OUTN}_G(v_{\sigma(l)}) \setminus \big(\text{OUTN}_G(v_{\sigma(r)}) \cup \{v_{\sigma(r)}\}\big)$ as $\text{OUTDEG}(v_{\sigma(l)}) = c_{\sigma(l)} \geqslant c_{\sigma(r)} > \text{OUTDEG}(v_{\sigma(r)})$. We consider the following cases:

Case 1. $\text{INDEG}(v_j) = b_j$
Since $\text{IN}_r(v_j) < b_j$, there must exists a k $(> r)$ such that $(v_{\sigma(k)}, v_j)$ is an edge in G. If $\ell = r$ then we replace $(v_{\sigma(k)}, v_j)$ with $(v_{\sigma(\ell)}, v_j)$, thereby increasing $\text{IN}_r(v_j)$ by 1. If $\ell < r$ then we replace edges $(v_{\sigma(l)}, v)$ and $(v_{\sigma(k)}, v_j)$ respectively with $(v_{\sigma(r)}, v)$ and $(v_{\sigma(l)}, v_j)$.

Case 2. $\text{INDEG}(v_j) < b_j$
If $l = r$ then we add edge $(v_{\sigma(\ell)}, v_j)$ to G. If $\ell < r$ then we replace $(v_{\sigma(\ell)}, v)$ with $(v_{\sigma(\ell)}, v_j)$ and add edge $(v_{\sigma(r)}, v)$ to G. In both scenarios $\text{IN}_r(v_j)$ is incremented by 1.

In both cases above we increment $\text{IN}_r(v_j)$ by 1, without affecting $\text{IN}_r(w)$ for $w \neq v_j$. Moreover, we increment outdegree of $v_{\sigma(r)}$ by at most 1. We keep incrementing $\text{IN}_r(v_j)$, for different v_j's until $\text{OUTDEG}(v_{\sigma(r)})$ becomes $v_{\sigma(r)}$. Once $\text{OUTDEG}(v_{\sigma(r)}) = c_{\sigma(r)}$ then we increment r and handle subsequent indices in a similar manner.

Stage 2: Now we have a graph with $\text{OUTDEG}(v_{\sigma(i)}) = c_{\sigma(i)}$ (or $\text{OUTDEG}(v_i) = c_i$) and $\text{INDEG}(v_i) \leqslant b_i$, for each $i \in [1,n]$. In this stage we will obtain the required graph. We maintain the invariants that for $1 \leqslant i \leqslant n$, $\text{OUTDEG}(v_i) \leqslant c_i$ and $\text{INDEG}(v_i) \leqslant b_i$.

Let $r \in [1,n]$ be the largest index such that $\text{INDEG}(v_i) \geqslant a_i$ for each $i < r$, and $\text{INDEG}(v_r) < a_r$. Now we have the following two cases:

Case 1. $\text{INDEG}(v_j) = a_j$ for each $j < r$
Note that if for all $i \in [1,n]$ we have,

$$\begin{aligned}
\text{OUT}_r(v_i) &= \min(d_i, r-1) \qquad \text{for } i \leqslant r \\
\text{OUT}_r(v_i) &= \min(d_i, r) \qquad \text{for } i > r
\end{aligned}$$

then,

$$\sum_{i=1}^{r} \min(d_i, r-1) + \sum_{i=r+1}^{n} \min(d_i, r) = \sum_{i=1}^{n} \text{OUT}_r(v_i) = \sum_{i=1}^{r} \text{INDEG}(v_i) < \sum_{i=1}^{r} a_i$$

which contradicts Eq. 1. Hence there exists an index i such that $\text{OUT}_r(v_i) < \min(d_i, r-1)$ if $i \leqslant r$, and $\text{OUT}_r(v_i) < \min(d_i, r)$ if $i > r$.

Let ℓ ($\in [1, r]$) be the largest index such that $\ell \neq i$ and (v_i, v_ℓ) is not an edge in G. Such an ℓ must exists by definition of i. Further, if $\ell < r$ then there must exists a vertex (say u) in $\text{INN}_G(v_l) \setminus \big(\text{INN}_G(v_r) \cup \{v_r\}\big)$ as $\text{INDEG}(v_l) = a_l \geqslant a_r > \text{INDEG}(v_r)$. We consider the following cases:

- Case 1.1 $\text{OUTDEG}(v_i) = d_i$
  Since $\text{OUT}_r(v_i) < d_i$, there must exists a k ($> r$) such that (v_i, v_k) is an edge in G. If $\ell = r$ then we replace (v_i, v_k) with (v_i, v_ℓ), thereby increasing $\text{OUT}_r(v_i)$ by 1. If $\ell < r$ then we replace edges (u, v_ℓ) and (v_i, v_k) respectively with (u, v_r) and (v_i, v_ℓ).

- Case 1.2 $\text{OUTDEG}(v_i) < d_i$
  If $l = r$ then we add edge (v_i, v_ℓ) to G. If $\ell < r$ then we replace (u, v_ℓ) with (v_i, v_ℓ) and add edge (u, v_r) to G. In both scenarios $\text{OUT}_r(v_j)$ is incremented by 1.

Case 2. There exists a $j < r$ such that $\text{INDEG}(v_j) > a_j$
Since $\text{INDEG}(v_j) > a_j \geqslant a_r > \text{INDEG}(v_r)$ hence $\exists\, u$ in $\text{INN}_G(v_j) \setminus \big(\text{INN}_G(v_r) \cup \{v_r\}\big)$. Here we replace (u, v_j) with (u, v_r).

With this we have our required graph G. This completes the proof for the digraph-realizability of interval-pair sequences. □

4 Verifying Realizability of Interval-Pair Sequences

We present here a linear time algorithm for efficiently verifying if a given interval-pair sequence $S = \big\{\big([a_i, b_i], [c_i, d_i]\big)\big\}_{i=1}^n$ is realizable. Without loss of generality, we assume $a_1 \geqslant \cdots \geqslant a_n$. We first perform bucket sort in $O(n)$ time to find a permutation σ of $[1, n]$ such that $c_{\sigma(1)} \geqslant \cdots \geqslant c_{\sigma(n)}$. We will present a linear time algorithm to check if the following holds for $1 \leqslant k \leqslant n$,

$$\begin{aligned}
\sum_{i=1}^{k} a_i &\leqslant \sum_{i=1}^{k} \min(d_i, k-1) + \sum_{i=k+1}^{n} \min(d_i, k) \\
\sum_{i=1}^{k} c_{\sigma(i)} &\leqslant \sum_{\substack{i \in [1,n] \\ s.t.\ \sigma^{-1}(i) \leqslant k}} \min(b_i, k-1) + \sum_{\substack{i \in [1,n] \\ s.t.\ \sigma^{-1}(i) > k}} \min(b_i, k)
\end{aligned}$$

Let A, B, C, D be four n length sequences such that

$$\begin{aligned}
A_k &= \sum_{i=1}^{k} a_i & D_k &= \sum_{i=1}^{k} \min(d_i, k-1) + \sum_{i=k+1}^{n} \min(d_i, k) \\
C_k &= \sum_{i=1}^{k} c_{\sigma(i)} & B_k &= \sum_{\substack{i \in [1,n] \\ s.t.\ \sigma^{-1}(i) \leqslant k}} \min(b_i, k-1) + \sum_{\substack{i \in [1,n] \\ s.t.\ \sigma^{-1}(i) > k}} \min(b_i, k)
\end{aligned}$$

Then, by Theorem 2, the sequence S is realizable if and only if $A \leqslant D$ and $B \leqslant C$.

For developing the $O(n)$ algorithm, we first need a recursive relation for B_k and D_k. We do this as follows.

$$\begin{aligned}
B_k - B_{k-1} &= \sum_{\substack{i\in[1,n] \\ s.t.\ \sigma^{-1}(i)\leqslant k-1}} (\min(b_i, k-1) - \min(b_i, k-2)) \;+\; \min(b_{\sigma(k)}, k-1) \\
&\quad + \sum_{\substack{i\in[1,n] \\ s.t.\ \sigma^{-1}(i)> k}} (\min(b_i, k) - \min(b_i, k-1)) \;-\; \min(b_{\sigma(k)}, k-1) \\
&= \sum_{\substack{i\in[1,n] \\ s.t.\ \sigma^{-1}(i)\leqslant k-1 \\ s.t.\ b_i\geqslant k-1}} 1 \;+\; \sum_{\substack{i\in[1,n] \\ s.t.\ \sigma^{-1}(i)> k \\ and\ \ b_i\geqslant k}} 1 \\
&= \sum_{\substack{i\in[1,n]\setminus\{\sigma(k)\} \\ s.t.\ b_i\geqslant k}} 1 \;+\; \sum_{\substack{i\in[1,n] \\ s.t.\ \sigma^{-1}(i)\leqslant k-1 \\ and\ \ b_i=k-1}} 1
\end{aligned}$$

Algorithm 1: Verifying realizability of Interval-Pair Sequence S.

Let $S = \big\{\big([a_i, b_i], [c_i, d_i]\big)\big\}_{i=1}^{n}$ be the input sequence satisfying $a_1 \geqslant \cdots \geqslant a_n$;
Compute a permutation σ of $[1, n]$ such that $c_{\sigma(1)} \geqslant \cdots \geqslant c_{\sigma(n)}$;
Set $X, Y, \bar{X}, \bar{Y}, A, B, C, D = \{0\}_{i=1}^{n}$;
Set $A_1 = a_1, C_1 = c_{\sigma(1)}$;
Set $B_1 = \sum_{\substack{i\in[1,n] \\ s.t.\ \sigma^{-1}(i)>1}} \min(b_i, 1)$ and $D_1 = \sum_{i\in[2,n]} \min(d_i, 1)$;
for $i = 1$ **to** n **do**
  if $(\sigma^{-1}(i) \leqslant b_i)$ **then** $Y_{b_i} = Y_{b_i} + 1$;
  if $(i \leqslant d_i)$ **then** $\bar{Y}_{d_i} = \bar{Y}_{d_i} + 1$;
  $X_{b_i} = X_{b_i} + 1$ and $\bar{X}_{d_i} = X_{d_i} + 1$;
end
for $i = (n-1)$ **to** 1 **do**
  $X_i = X_i + X_{i+1}$ and $\bar{X}_i = \bar{X}_i + \bar{X}_{i+1}$;
end
for $k = 2$ **to** n **do**
  $A_k = A_{k-1} + a_k$ and $C_k = C_{k-1} + c_{\sigma(k)}$;
  if $(d_k \geqslant k)$ **then** $D_k = D_{k-1} + \bar{X}_k + \bar{Y}_{(k-1)} - 1$;
  else $D_k = D_{k-1} + \bar{X}_k + \bar{Y}_{(k-1)}$;
  if $(b_{\sigma(k)} \geqslant k)$ **then** $B_k = B_{k-1} + X_k + Y_{(k-1)} - 1$;
  else $B_k = B_{k-1} + X_k + Y_{(k-1)}$;
end
if $(A \leqslant D$ **and** $B \leqslant C)$ **then** Return "Realizable";
else Return "Not-Realizable";

Now we define sequences X and Y as follows,

$$X_k := \sum_{\substack{i\in[1,n] \\ s.t.\ b_i\geqslant k}} 1\,, \qquad \text{and} \qquad Y_k := \sum_{\substack{i\in[1,n] \\ s.t.\ \sigma^{-1}(i)\leqslant k \\ and\ \ b_i=k}} 1$$

so,

$$B_k - B_{k-1} = \begin{cases} X_k + Y_{(k-1)} - 1 & \text{if } b_{\sigma(k)} \geqslant k, \\ X_k + Y_{(k-1)} & otherwise. \end{cases}$$

Now similarly for D_k, we have

$$\begin{aligned} D_k - D_{k-1} &= \sum_{i=1}^{k-1}(\min(d_i, k-1) - \min(b_i, k-2)) + \sum_{i=k+1}^{n}(\min(b_i, k) - \min(b_i, k-1)) \\ &= \sum_{\substack{i\in[1,k-1] \\ s.t.\ d_i \geqslant k-1}} 1 + \sum_{\substack{i\in[k+1,n] \\ s.t.\ d_i \geqslant k}} 1 \\ &= \sum_{\substack{i\in[1,n]\setminus\{k\} \\ s.t.\ d_i \geqslant k}} 1 + \sum_{\substack{i\in[1,k-1] \\ s.t.\ d_i = k-1}} 1 \end{aligned}$$

We define $\bar{X}$ and $\bar{Y}$ as follows,

$$\bar{X}_k := \sum_{\substack{i\in[1,n] \\ s.t.\ d_i \geqslant k}} 1\,, \quad \text{and} \quad \bar{Y}_k := \sum_{\substack{i\in[1,k] \\ s.t.\ d_i = k}} 1$$

so,

$$D_k - D_{k-1} = \begin{cases} \bar{X}_k + \bar{Y}_{(k-1)} - 1 & \text{if } d_k \geqslant k, \\ \bar{X}_k + \bar{Y}_{(k-1)} & otherwise. \end{cases}$$

In Algorithm 1 we present the steps to verify realizability of the input interval-pair sequence S using sequences $X, \bar{X}, Y, \bar{Y}$.

It is easy to verify that Algorithm 1 takes linear time, and correctly computes A, B, C, D. So the following theorem is immediate.

Theorem 3 *For any n-length interval-pair sequence $S = \big\{\big([a_i, b_i], [c_i, d_i]\big)\big\}_{i=1}^n$ we can verify if there exists a digraphic sequence $D = \{(p_i, q_i)\}_{i=1}^n$ satisfying $a_i \leqslant p_i \leqslant b_i$ and $c_i \leqslant q_i \leqslant d_i$, for each $i \in [1, n]$, in $O(n)$ time.*

References

[Ans82] Richard Anstee. Properties of a class of (0,1)-matrices covering a given matrix. *Can. J. Math.*, pages 438–453, 1982.

[Ans85] Richard Anstee. An algorithmic proof of tutte's f-factor theorem. *J. Algorithms*, 6(1):112–131, 1985.

[AT94] Martin Aigner and Eberhard Triesch. Realizability and uniqueness in graphs. *Discrete Mathematics*, 136:3–20, 1994.

[BBPR22] Amotz Bar-Noy, Toni Böhnlein, David Peleg, and Dror Rawitz. On realizing a single degree sequence by a bipartite graph (invited paper). In Artur Czumaj and Qin Xin, editors, *18th Scandinavian Symposium and Workshops on Algorithm Theory, SWAT 2022, June 27-29, 2022, Tórshavn, Faroe Islands*, volume 227 of *LIPIcs*, pages 1:1–1:17. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022.

[BNCPR20] Amotz Bar-Noy, Keerti Choudhary, David Peleg, and Dror Rawitz. Efficiently realizing interval sequences. *SIAM Journal on Discrete Mathematics*, 34(4):2318–2337, 2020.

[CDZ00] Mao-cheng Cai, Xiaotie Deng, and Wenan Zang. Solution to a problem on degree sequences of graphs. *Discrete Mathematics*, 219(1-3):253–257, 2000.

[Che66] Wai-Kai Chen. On the realization of a (p,s)-digraph with prescribed degrees. *Journal of the Franklin Institute*, 281(5):406 – 422, 1966.

[EG60] Paul Erdös and Tibor Gallai. Graphs with prescribed degrees of vertices [hungarian]. *Matematikai Lapok*, 11:264–274, 1960.

[Ful60] D.R. Fulkerson. Zero-one matrices with zero trace. *Pacific J. Math.*, 12:831 – 836, 1960.

[Gal57] David Gale. A theorem on flows in networks. *Pacific Journal of Mathematics*, 7:1073–1082, 1957.

[GGT11] Ankit Garg, Arpit Goel, and Amitabha Tripathi. Constructive extensions of two results on graphic sequences. *Discrete Applied Mathematics*, 159(17):2170–2174, 2011.

[GY14] Jiyun Guo and Jianhua Yin. A variant of Niessen's problem on degree sequences of graphs. *Discrete Mathematics and Theoretical Computer Science*, Vol. 16 no. 1 (in progress)(1):287–292, May 2014. Graph Theory.

[Hak62] S. Louis Hakimi. On realizability of a set of integers as degrees of the vertices of a linear graph –I. *SIAM J. Appl. Math.*, 10(3):496–506, 1962.

[Hav55] V. Havel. A remark on the existence of finite graphs [in Czech]. *Casopis Pest. Mat.*, 80:477–480, 1955.

[HHKL90] Katherine Heinrich, Pavol Hell, David G. Kirkpatrick, and Guizhen Liu. A simple existence criterion for g ¡ f–factors, with applications to [a, b]–factors. *Discrete Mathematics*, 85:313–317, 1990.

[HN15] Sepp Hartung and André Nichterlein. Np-hardness and fixed-parameter tractability of realizing degree sequences with directed acyclic graphs. *SIAM J. Discrete Math.*, 29(4):1931–1960, 2015.

[KW73] Daniel J. Kleitman and D. L. Wang. Algorithms for constructing graphs and digraphs with given valences and factors. *Discrete Mathematics*, 6(1):79–88, 1973.

[Rec17] Steffen Rechner. An optimal realization algorithm for bipartite graphs with degrees in prescribed intervals. *arXiv preprint arXiv:1708.05520*, 2017.

[Rys57] H. J. Ryser. Combinatorial properties of matrices of zeros and ones. *Canadian Journal of Mathematics*, 9:371–377, 1957.

[Tut81] W. T. Tutte. Graph factors. *Combinatorica*, 1(1):79–97, Mar 1981.

[TVW10] Amitabha Tripathi, Sushmita Venugopalan, and Douglas B. West. A short constructive proof of the erdos-gallai characterization of graphic lists. *Discrete Mathematics*, 310(4):843–844, 2010.

[WK73] D.L. Wang and D.J. Kleitman. On the existence of n-connected graphs with prescribed degrees ($n > 2$). *Networks*, 3:225–239, 1973.