Зад. 1 Сагымбаев А. БФ3201

1) $2 \le |z-i| \le 4$

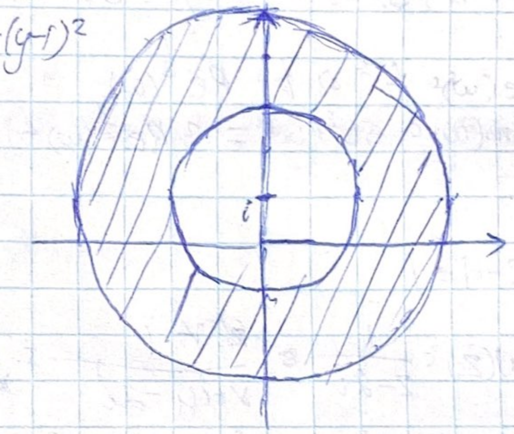
$z = x+iy$ $\quad |z-i| = |x+iy-i| = \sqrt{x^2+(y-1)^2}$

$\sqrt{x^2+(y-1)^2} \ge 2$ $\quad \sqrt{x^2+(y-1)^2} \le 4$

$x^2+(y-1)^2 \ge 4$ $\quad x^2+(y-1)^2 \le 16$

центр в т. $z_0 = i$

$S = \pi R_1^2 - \pi R_2^2 = 12\pi$



2) $|z-4i| + |z+4i| = 10$

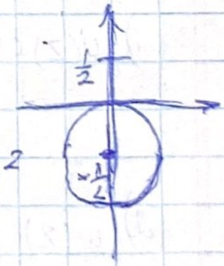
$|x+iy-4i| + |x+iy+4i| = 10$

$\sqrt{x^2+(y-4)^2} + \sqrt{x^2+(y+4)^2} = 10$ – эллипс

центр в $z_0 = 0$; Большая полуось $a = 5$

3) $\mathrm{Im}\frac{1}{z} = 1$

$\mathrm{Im}\left(\frac{z^*}{|z|}\right) = \frac{-y}{x^2+y^2} = 1 \Rightarrow x^2+y^2+y+\frac{1}{4} = \left(\frac{1}{2}\right)^2 = x^2+\left(y+\frac{1}{2}\right)^2$



Зад. 2

$1+2\varepsilon+3\varepsilon^2+\ldots+n\varepsilon^{n-1} = \frac{d}{d\varepsilon}(\varepsilon+\varepsilon^2+\ldots+\varepsilon^n) = \frac{d\left(\frac{\varepsilon(\varepsilon^n-1)}{\varepsilon-1}\right)}{d\varepsilon} = \frac{(1-(n+1)\varepsilon^n)(1-\varepsilon)+(\varepsilon-\varepsilon^{n+1})}{(1-\varepsilon)^2} =$

$= \frac{(n(\varepsilon-1)-1)\varepsilon^n+1}{(1-\varepsilon)^2}$

Зад 3

1) $Im(z)=1 \qquad z=t+i$

$w = z^3+3z-i = (t+i)^3+3t+3i-i = t^3+3t^2i+i = t^3+i(3t^2+1)$

$Re(w)=t^3 \Rightarrow t = Re^{\frac{1}{3}}(w)$

$Im(w) = 3t^2+1 = 3\,Re^{\frac{2}{3}}(w)+1$ – ответ

2) $|z-1|=1$

$w(z) = \frac{1}{z-2i} = \frac{1}{x+iy-2i} = \frac{1}{x+i(y-2)} = \frac{x-i(y-2)}{x^2+(y-2)^2} = \frac{x}{x^2+(y-2)^2} + i\frac{2-y}{x^2+(y-2)^2}$

$x^2+(y-1)^2=1 \Rightarrow x^2 = 1-(y-1)^2$

$x^2+(y-2)^2 = 1-(y-1)^2+(y-2)^2 = 4-2y = 2(2-y)$

$Im(w) = \frac{2-y}{x^2+(y-2)^2} = \frac{2-y}{2(2-y)} = \frac{1}{2}$ – ответ

Зад. 4

1) $w(z) = x^2+y^2 = u+iv \Rightarrow \frac{\partial u}{\partial x} = 2x \quad \frac{\partial v}{\partial y} = 0$

$\frac{\partial v}{\partial x} = 0 \quad \frac{\partial u}{\partial y} = 2y$ – усл К-Р не выполнятся

2) $w(z) = x^2-y^2+2ixy = u+iv$

$\frac{\partial u}{\partial x} = 2x \quad \frac{\partial v}{\partial y} = 2x$

$\frac{\partial u}{\partial y} = -2y \quad \frac{\partial v}{\partial x} = 2y$ – усл. К-Р выполняется

3) $w(z) = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$

$\frac{\partial u}{\partial x} = \frac{1(x^2+y^2)-x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad \frac{\partial v}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}$ – усл. К-Р выполняется

$\frac{\partial u}{\partial y} = \frac{2xy}{(x^2+y^2)^2} \quad \frac{\partial v}{\partial x} = -\frac{2xy}{(x^2+y^2)^2}$

Зад. 5

1) $|f| = e^{r^2\cos 2\varphi}$.

Пусть $f = e^{w(z)}$

$w(z) = \ln f = \ln(|f|\cdot e^{i\arg f}) = \overset{=u}{\ln|f|} + i\overset{=v}{\arg f} = x^2-y^2+2ixy = (x+iy)^2 = z^2$

$u = r^2\cos 2\varphi = r^2(\cos^2\varphi - \sin^2\varphi) = x^2-y^2$

$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \Rightarrow v = 2xy + \varphi(y)$

$\frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x} \Rightarrow v = 2xy$

$f = e^{w(z)} = e^{z^2}$ - Ответ

2) $\operatorname{Arg} f = xy$

$f = |f|e^{i\arg f} = e^{w(z)} \Rightarrow w(z) = \overset{=u}{\ln|f|} + i\overset{=v}{\arg f}$

$\frac{\partial v}{\partial y} = x = \frac{\partial u}{\partial x} \Rightarrow u = \frac{x^2}{2} + \psi(x)$

$\frac{\partial v}{\partial x} = y = -\frac{\partial u}{\partial y} \Rightarrow u = -\frac{y^2}{2} + \varphi(y)$

$\Rightarrow w(z) = \frac{x^2}{2} - \frac{y^2}{2} + xy = \frac{1}{2}z^2 \Rightarrow f(z) = e^{\frac{1}{2}z^2}$ - Ответ

Зад. 7

1) $\int_C z\,dz = \int i e^{2i\varphi}\,d\varphi = i\frac{1}{2i}e^{2i\varphi}\Big|_0^{2\pi} = \frac{1}{2}(e^{4\pi i}-1) = 0$

2) $\int_C z^*\,dz = \int_0^{2\pi} i e^{-i\varphi}\cdot e^{i\varphi}\,d\varphi = i\int_0^{2\pi} d\varphi = 2\pi i$

Зад. 8

1) $\int \frac{y\,dx - x\,dy}{x^2+y^2} = \cancel{\iint \frac{\left(\frac{\partial y}{\partial y} - \frac{\partial x}{\partial x}\right)dx\,dy}{x^2+y^2}} = \int_{-1}^{1} \sin\varphi\, d\cos\varphi - \cos\varphi\, d\sin\varphi = -\int_0^{2\pi} \sin^2\varphi + \cos^2\varphi\, d\varphi = -2\pi$

$x = \cos\varphi \quad y = \sin\varphi$

д) $x = 2 + \cos\varphi$

$$\int_0^{2\pi} \frac{-\sin^2\varphi - (2+\cos\varphi)\cos\varphi}{4\cos\varphi+5} d\varphi = \int_0^{2\pi} \frac{1+2\cos\varphi}{5+4\cos\varphi} d\varphi = -\int_0^{2\pi}\left(\frac{1}{2} - \frac{3}{2(5+4\cos\varphi)}\right)d\varphi = -\int_0^{2\pi}\frac{d\varphi}{2} +$$

$$+\frac{3}{2}\int \frac{1}{5+4\cos\varphi}d\varphi = -\pi + \frac{3}{2}\cdot\frac{2}{3}\arctan\left(\frac{\tan(\frac{x}{2})}{3}\right)\Big|_0^{2\pi} = -\pi + \pi = 0$$

Зад. 10

$y(1) = 0 \qquad y'(z) = \frac{1}{2z} = \frac{1}{2re^{i\varphi}} = \frac{\partial y}{re^{i\varphi}d\varphi} \Rightarrow \frac{\partial y}{\partial\varphi} = \frac{i}{2}$

1) $y(-1) - y(1) = \int\limits_0 y'(z)dz = \int_0^{\pi} \frac{i}{2}d\varphi = \frac{i\pi}{2} = y(-1)$ - ответ

2) $y(-1) = \int_{-\pi} \frac{i}{2}d\varphi = -\frac{i\pi}{2}$ - ответ

Зад. 11

$$\frac{1+2z^2}{z^3+z^5} = \frac{1+2z^2}{z^3(1+z^2)} = \left(\frac{1}{z^3} + \frac{2}{z}\right)(1-z^2) = \frac{1}{z^3} + \frac{2}{z} - \frac{1}{z} - 2z = \frac{1}{z^3} + \frac{0}{z^2} + \frac{1}{z}$$ - ответ

Зад. 12

$$f(z) = \frac{1}{z(e^z-1)} \approx \frac{1}{z(1+z+\frac{z^2}{2}-1)} = \frac{1}{z^2(1+\frac{z}{2})} = \frac{1}{z^2}\left(1-\frac{z}{2}\right) = \frac{1}{z^2} - \frac{1}{2z}$$ - ответ

Зад. 13

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z}\sum_{n=0}^{\infty} z^n = -\sum_{n=0}^{\infty} z^{n-1}$$

Зад. 15

$$f(z) = \frac{1}{\sin z} + \frac{2z}{z^2-\pi^2}$$

$$\sin z = -(x-\pi) + \frac{(x-\pi)^3}{3!} = (\pi-x)\left(1 - \frac{(\pi-x)^2}{3!}\right)$$

$$f(z) = \frac{1}{\pi-x}\left(1 + \frac{(\pi-x)^2}{3!}\right) + \frac{2}{\pi^2}z\left(1+\frac{z^2}{\pi^2}\right) = \frac{1}{\pi-x} + \frac{\pi-x}{3!} + \frac{2z}{\pi^2} + \frac{2z^3}{\pi^4}$$

Зад. 17 $f(z) = z e^{\frac{1}{z}} e^{-\frac{1}{z^2}} = z\sum\frac{1}{z^n n!}\sum\left(-\frac{1}{z^{2n}n!}\right)$

Зад. 16

1) $f(z) = \frac{\sin z}{1-\tan z}$

$z = \frac{\pi}{4} + \varepsilon$

$$\frac{\sin z}{1-\tan z} = \frac{\sin(\frac{\pi}{4}+\varepsilon)}{1-\frac{1+\tan\varepsilon}{1-\tan\varepsilon}} = -\frac{\sin(\frac{\pi}{4}+\varepsilon)(1-\tan\varepsilon)}{2\tan\varepsilon}$$

2) $$\frac{e^{\frac{c}{z-a}}}{e^{\frac{z}{a}}-1} = -\sum \frac{c^n}{(z-a)^n}\frac{1}{n!}\sum\left(e^{\frac{z}{a}}\right)^n = -\sum \frac{c^n}{(z-a)^n n!}\sum\left(\sum\left(\frac{z}{a}\right)^n\frac{1}{n!}\right)$$

Зад. 18

1) $$\int_C \frac{z e^z}{\tan z^2}dz = i\int_0^{2\pi}\frac{z^2 e^z}{\tan z^2}d\varphi \approx i\int_0^{2\pi}\frac{z^2}{z^2}e^z d\varphi = i\int_0^{2\pi} e^{e^{i\varphi}}d\varphi \approx i\int_0^{2\pi}\left(1+e^{i\varphi}+\frac{e^{2i\varphi}}{2}\right)d\varphi =$$

$$= i\int_0^{2\pi}d\varphi + i\underbrace{\int_0^{2\pi}e^{i\varphi}d\varphi}_{=0} + \frac{i}{2}\underbrace{\int_0^{2\pi}e^{2i\varphi}d\varphi}_{=0} = 2\pi i$$

2) $$\int_C e^{-\frac{1}{z}}\sin\left(\frac{1}{z}\right)dz \approx \int e^{-\frac{1}{z}}\left(\frac{1}{z}-\frac{1}{z^3 3!}\right)dz = i\int_0^{2\pi}\left(1-\frac{1}{z}+\frac{1}{2z^2}\right)\frac{z}{z}\left(1-\frac{1}{z^2 3!}\right)d\varphi =$$

$$= i\int \left|1-\frac{1}{z}+\frac{1}{z^2 3}+\frac{1}{3!\,z^3}\right|d\varphi = 2\pi i$$

3) $$\int_C \frac{e^z}{z^n}dz = \int_C \sum\frac{z^k}{z^n k!}dz = \int_C \ldots + \frac{1}{z(n-1)!} + \ldots dz = i\int_0^{2\pi}\frac{1}{(n-1)!}d\varphi = \frac{2\pi i}{(n-1)!}$$