Year 1 — Partial Differential Equations

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine. A note of acknowledgement to Dexter Chua, Ph.D. Harvard University for the template.

Catalog Description
Lecture, three hours; discussion, one hour. Prerequisites: courses 33A, 33B. Linear partial differential equations, boundary and initial value problems; wave equation, heat equation, and Laplace equation; separation of variables, eigenfunction expansions; selected topics, as method of characteristics for nonlinear equations.

Textbook Reading
Partial Differential Equations (2nd Edition), *W.A. Strauss*

Contact
This document is a summary of the notes that I have taken during lectures at UCLA; please note that this lecture note will not necessarily coincide with what you might learn. If you find any errors, don't hesitate to reach out to me below:

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1 Introduction

Prelude to be written.

2 Origin of Partial Differential Equations

2.1 Preliminaries

What is a Partial Differential Equation? A PDE is any equation involving an unknown multivariate function and its partial derivatives.

Example 1. In 2D, a general PDE with unknown function $u(x, y)$ reads

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y), u_{xx}(x, y), u_{xy}(x, y), u_{yy}(x, y), \dots) = 0$$

for some nonlinear function F.

The "order" of a PDE is the order of the highest order partial derivative in F appearing in F.

Example 2. (1st and 2nd Order PDE)

$$F(x, y, u, u_x, u_y) = 0 \qquad F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$$

We can continue for any nth order.

We call u a solution to a PDE if it solves such an equation. We now explore some classes of Partial Differential Equations that we will study more in-depth as the course progresses.

Example 3. (Transport Equations)

$$u_x + u_y = 0 \qquad \text{or} \qquad u_x + yu_y = 0$$

To solve this, we must use the method of characteristics (converting to an ODE to get a solution set).

Example 4. (Nonlinear Transport / Burger's Equations)

$$u_x + uu_y = 0$$

Example 5. (Laplace's Equation in 2D)

$$u_{xx} + u_{yy} = 0$$

The Inhomogeneous Laplace's Equation, also known as Poisson's Equation is

$$u_{xx} + u_{yy} = f$$

Laplace's equation is fundamental to the study of Partial Differential Equations as it forms the basis of electrostatics, Newtonian gravity and a myriad of other applications ranging from stationary probability measures involving Brownian motion to solution's of Laplace's equation described via real/imaginary parts of holomorphic functions.

Example 6. (Wave Equation)

$$u_{tt} = u_{xx}$$

A Nonlinear Wave Equation can be

$$u_{tt} = u_{xx} + u^3$$

Example 7. (Heat Equation)

$$u_t = u_{xx}$$

Example 8. (Schrödinger's Equation)

$$u_t - iu_{xx} = 0, i^2 = -1$$

Notice that the Heat equation and the Schrödinger Equation is very similar in their format, however, they display completely different behavior. In general, we can expect to see applications of Partial Differential Equations in classical mechanics, einstein field equations (boiled down to simply a nonlinear wave equation), quantum field theory (although more complex) and even machine learning (stochastic gradient descent and brownian motion).

We make a quick note on notation of Partial Derivatives. Thus far, we've used the notation,

$$u_x = \frac{\partial f}{\partial x} \qquad u_y = \frac{\partial f}{\partial y}$$

This is simply a choice and one can choose either way to represent partial derivatives. The 2nd order Partial Derivatives are written,

$$u_{xx} = \frac{\partial^2 f}{\partial x^2} \qquad u_{xy} = \frac{\partial^2 f}{\partial x \partial y} \qquad u_{yy} = \frac{\partial^2 f}{\partial y^2}$$

You might also expect to come across the notation: $D_1 f, D_{11} f, \ldots$. They're all valid methods of representing partial derivatives; however, for the purpose of simplicity, we stick to the notation we first began this class with although you will see the traditional version just as much.

For sake of notation, I will rewrite the above examples in the traditional sense,

Example 9. (Linear & Nonlinear Transport Equations)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \qquad \frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

Nonlinear Transport is as follows,

$$\frac{\partial u}{\partial x} + u(x, y)\frac{\partial u}{\partial y} = 0$$

Example 10. (Homogeneous and Inhomogeneous Laplace's Equation in 2D)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Inhomogeneous Laplace's Equation is as follows,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f$$

Example 11. (Linear and Nonlinear Wave Equation)

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

An example of a Nonlinear Wave Equation is as follows,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + u^3$$

Example 12. (Heat Equation)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Example 13. (Schrödinger's Equation)

$$\frac{\partial u}{\partial t} - i\frac{\partial^2 u}{\partial x^2}, i^2 = -1$$

Definition 1. (Linear) We say a partial differential equation is a homogenous linear equation fi it can be written as

$$Lu = 0$$

where Lu is a linear combination of partial derivatives of u.

In other words,

$$Lu = au + bu_x + cu_y + du_{xx} + eu_{xy} + fu_{yy} + \dots$$

for some coefficients $a = a(x,y), b = b(x,y), c = c(x,y), \dots$. Note that if u and v both solve $Lu = Lv = 0$, then for any $c_1, c_2 \in \mathbb{R}$, we've

$$L[c_1 u + c_2 v] = 0$$

To be precise, Lu is simply a linear homogeneous partial differential equation. We also have *linear inhomogeneous partial differential equations*,

$$L[u] = g(x,y)$$

A differential operator applied to a function, the forcing term. Here f is called a "source" term, "forcing" term or "inhomogeneous" term.

The superposition principle only to homogeneous equations. If you take a linear combination of homogeneous solutions, you'd get a homogeneous solution. The same is not true for inhomogeneous equations; in general, you can **not** take a linear combination of solutions to get another solution.

Recall that with ordinary differential equations (ODEs), a general solution has m arbitrary coefficients if the ODE is order m or a system of m first-order equations. A general theme of PDEs is that they may be thought of as *infinite dimensional systems of ODEs* with arbitrary functions in the general solution.

Example 14.

$$u_{xx}(x,y) = 0 \iff \frac{\partial^2 u}{\partial x^2}(x,y)$$

y simply plays the role of a parameter over here.

$$u_x(x,y) = f(y)$$
$$u(x,y) = xf(y) + g(y)$$

This is the *general solution*.

When we have an ordinary differential equation such $u^{''}(x) = 0$, the general solution is $u(x) = c_1 x + c_2$ with c_1, c_2 simply constant coefficients. However, in the case of a partial differential equation, we get arbitrary functions of y.

Example 15.

$$u_{xx} + u = 0 \iff \frac{\partial^2 u}{\partial x^2}(x, y) + u(x, y) = 0$$

Using the Ansatz, $u(x, y) = e^{\lambda x}$,

$$\lambda^2 + 1 = 0 \iff \lambda = \pm i$$

It follows that the general solution is some linear combination of $\sin x$ and $\cos x$. Hence,

$$u(x, y) = f(y)\cos x + g(y)\sin x$$

where f and g are arbitrary functions of y.

Example 16.

$$u_{xy} = 0 \iff \frac{\partial^2 u}{\partial x \partial y} = 0$$

Integrate once in x,

$$\begin{aligned} u_y(x, y) &= f(y) && \text{(constant in } x\text{)} \\ u(x, y) &= g(x) + \underbrace{\int f(y)\, dx}_{=:F(y)} = g(x) + F(y) \end{aligned}$$

2.2 First Order Equations with Constant Coefficients

2.3 First Order Equations with Variable Coefficients

We begin by understanding how variable coefficients change the general solution of first-order equations. This is also called the *method of characteristics.* Consider for instance,

$$u_x + y u_y = 0 \iff (1, v) \cdot \nabla u = 0$$

Assume $u(x, y)$ is constant along the graph of $y(x)$, i.e., $u(x, y(x))$ constant with respect to x. Then,

$$\begin{aligned} \frac{d}{dx}\left[u(x, y(x))\right] &= u_x(x, y(x)) + y'(x)\, u_y(x, y(x)) \\ &= (y'(x) - y(x))\, u_y(x, y(x)) \end{aligned}$$

3 Flows, Vibrations & Diffusions

3.1 Deriving the Transport Equation

3.2 Deriving the Heat Equation

3.2.1 Initial Value and Boundary Value Problems

3.3 Well-Posedness of PDEs and Second Order Equations