Year 1 — Ordinary Differential Equations

Based on lectures by M. Zhou
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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine. A note of acknowledgement to Dexter Chua, Ph.D. Harvard University for the template.

Catalog Description
Lecture, three hours; discussion, one hour. Requisites: courses 33A, 33B. Selected topics in differential equations. Laplace transforms, existence and uniqueness theorems, Fourier series, separation of variable solutions to partial differential equations, Sturm-Liouville theory, calculus of variations, two point boundary value problems, Green's functions. P/NP or letter grading.

Textbook Reading
Differential Equations with Applications and Historical Notes (3rd Edition), *G. Simmons*

Contact
This document is a summary of the notes that I have taken during lectures at UCLA; please note that this lecture note will not necessarily coincide with what you might learn. If you find any errors, don't hesitate to reach out to me below:

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2.1 Homogeneous Linear Equations with Constant Coefficients

What's a differential equation? A simple answer to this question would go along the lines, "an equation that relates one or more unknown functions and their derivatives".

Example 1. $y'(x) = 0 \implies y(x) = y_0$

Example 2. $y'(x) = \lambda y(x) \implies y(x) = y_0 e^{\lambda x}$

Definition 1. (Homogeneous Linear Equation) A *homogeneous linear equation* of the nth order is of the form,

$$a_n(x)\,y^{(n)}(x) + a_{n-1}(x)\,y^{(n-1)}(x) + \cdots + a_1(x)\,y'(x) + a_0(x)\,y(x) = 0$$

This equation has $n+1$ terms.

The reason we call it *homogeneous* is because $y(x) = 0$ is a solution to the homogeneous differential equation. Linear Algebra guarantees that the set of all solutions to the above forms a linear space.

If $y_1(x), y_2(x)$ are solutions to the homogenous equation, then any arbitrary linear combination is also a solution, i.e., $c_1 y_1(x) + c_2 y_2(x)\,\forall c_1, c_2 \in R$.

We're now equipped to deal with an example of a 2nd order homogeneous equation with constant coefficients.

Example 3. Find the general solution to the Ordinary Differential Equation (ODE):

$$y''(x) + py'(x) + qy(x) = 0$$

We begin by noting that the set of all solutions to this differential equation is of dimension 2 and so, we can find two linearly independent solutions.

We can guess that $y(x) = e^{mx}$ for some m. So $y'(x) = me^{mx}$ and $y''(x) = m^2 e^{mx}$. When we plug in,

$$m^2 e^{mx} + pme^{mx} + eq^{mx} = 0 \implies e^{mx}\left(m^2 + pm + q\right) = 0$$

We know that since $e^{mx} \neq 0$, it must be that $m^2 + pm + q = 0$. As this expression doesn't depend on x, we only need need to solve algebraically for m.

If $p^2 - 4q > 0$, then $m_1, m_2 = \frac{1}{2}\left(-p \pm \sqrt{p^2 - 4q}\right)$ so we get 2 solutions to the ODE,

$$y_1(x) = e^{m_1 x} \quad \text{and} \quad y_2(x) = e^{m_2 x}$$

As a result, the general solution (the set of all the solutions that satisfy the ODE) is a linear combination of the two solutions

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

Where $m_1, m_2 = -\frac{1}{2}\left(p \pm \sqrt{p^2 - 4q}\right)$ whenever $p^2 - 4q > 0$.

If $p^2 - 4q < 0$, the algebraic equation $m^2 + pm + q$ has 2 complex solutions. Recall Euler's identity,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

While θ can be anything, for now, we restrict $\theta \in \mathbb{R}$. Therefore, the solution is

$$m_1, m_2 = -\frac{1}{2}\left(p \pm \sqrt{4q - p^2}i\right) =: a \pm bi$$

where $a = -\frac{1}{2}p, b = -\frac{1}{2}\sqrt{4q - p^2}$ so

$$\begin{aligned} y(x) = e^{m_1 x} \text{ or } y(x) = e^{m_2 x} = e^{(a \pm bi)x} &= e^{ax} \cdot e^{\pm ibx} \\ &= e^{ax}\left(\cos(\pm bx) + i\sin(\pm bx)\right) \\ &= e^{ax}\left(\cos(bx) \pm i\sin(bx)\right) \end{aligned}$$

Since the ODE has coefficients $\in \mathbb{R}, p, q \in \mathbb{R}$, we want a linear combination such that the solution is real.

$$y_1(x) = e^{ax}\cos bx \qquad y_2(x) = e^{ax}\sin bx$$

So, the general solution is

$$y(x) = e^{ax}\left(c_1\cos bx + c_2\sin bx\right)$$

If $p^2 - 4q = 0 \iff p^2 = 4q$, we have only one real solution, that is, $m_1 = m_2 = -\frac{p}{2}$. One solution is $y_1(x) = e^{mx}$. Assume that $y_2(x) = V(x)e^{mx}$.

Plugging in,

$$\left(V(x)e^{mx}\right)'' + p\left(V(x)e^{mx}\right)' + qV(x)e^{mx} = 0$$

$$\begin{aligned} V''(x)e^{mx} + 2V'(x)me^{mx} + V(x)m^2e^{mx} + pV'(x)e^{mx} + \\ pV(x)me^{mx} + qV(x)e^{mx} = 0 \end{aligned}$$

Reorganizing the terms,

$$V(x)\underbrace{\left(m^2 + pm + q\right)}_{=0}e^{mx} + V'(x)\underbrace{(2m + p)}_{=0}e^{mx} + V''(x)e^{mx} = 0$$

$$V''(x)e^{mx} = 0 \implies V''(x) = e^{mx}$$

It follows that $V(x)$ must be a linear equation $V(x) = cx + c'$. We get another solution,

$$y_2(x) = xe^{mx}$$

Therefore, the general solution is

$$y(x) = c_1e^{mx} + c_2xe^{mx}$$

2.2 Cauchy-Euler's Equation

3 Laplace Transform

3.1 Preliminaries

Before we begin to define what the *laplace transform* is, we must first understand the notion of a *transform*, which, naturally requires us to understand what is a *function*. This seems rather trivial, no less, useless to define. We have been working with functions for a better part of the undergrad curricula and certainly been exposed to linear transformations fairly recently. However, we must define the necessary details regardless.

Definition 2. (Function) A *function* $f(x)$ is a mapping

$$\begin{aligned} f : X &\longrightarrow Y \\ x &\longmapsto f(x) \end{aligned}$$

One could easily say that a function takes in a number and spits out another number. Given that we now know what a function is, we can now define a *transform.*

Definition 3. (Transform Operator) A *transform operator* is a mapping

$$\begin{aligned} T : X &\longrightarrow Y \\ f(x) &\longmapsto g(x) \end{aligned}$$

Just as how we defined a function, one can say that a transform operator takes in a function and spits out another function. Consider the following familiar examples,

Example 4.

$$\begin{array}{rcr l} \text{Differentiation} & D & f(x) \longmapsto & f'(x) \\ \text{Integration} & I & I[f](x) \longmapsto & \displaystyle\int_0^x f(t)\, dt \end{array}$$

Definition 4. (Linear Transformation) Let V denote any $\mathbb{F}$-vector space. A transform T is *linear* if

$$T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)\, \forall \alpha, \beta \in \mathbb{F}, f, g \in V$$

Example 5. (Integral Transform)

$$T[f](p) = \int_a^b K(p, x) f(x)\, dx$$

is a function of p, denoted by $F(p)$

We call $K(p, x)$ the *integration kernel.* A special case of the integral transform is when $a = 0, b = +\infty, K(p, x) = e^{-px}$. In this case, we call the integral transform the *Laplace Transform.*

3.2 Computing Laplace Transforms

Definition 5. (Laplace Transform) The *laplace transform* of a function f is given by

$$\mathcal{L}\left(f\left(x\right)\right) = L\left[f\right]\left(p\right) = \int_0^\infty e^{-px} f\left(x\right)\, dx$$

We can now compute the laplace transform for some common functions.

Example 6. Given $f\left(x\right) = 1$,

$$\begin{aligned}
\mathcal{L}\left(1\right) = \int_0^\infty e^{-px}\, dx &= \lim_{\alpha\to\infty} \int_0^\alpha e^{-px}\, dx \\
&= \lim_{\alpha\to\infty} \left(\frac{e^{-px}}{-p}\right)\Bigg|_{x=0}^{\alpha} \\
&= \lim_{\alpha\to\infty} \left(\frac{e^{-p\alpha}}{-p} - \frac{e^0}{-p}\right) \\
&= \frac{e^{-\infty}}{-p} + \frac{1}{p} \\
&= \frac{1}{p}
\end{aligned}$$

From now, we will assume that the limit exists and drop the limit. The reader, however, should verify that the limit exists in each case.

Example 7. Given $f\left(x\right) = x$,

$$\begin{aligned}
\mathcal{L}\left(x\right) &= \int_0^\infty e^{-px} x\, dx \\
&= \frac{xe^{-px}}{-p}\Bigg|_{x=0}^{\infty} + \int_0^\infty \frac{e^{-px}}{p}\, dx \\
&= 0 + \frac{1}{p}\left(\frac{e^{-px}}{-p}\right)\Bigg|_{x=0}^{\infty} \\
&= \frac{1}{p^2} \qquad (p > 0)
\end{aligned}$$

We now assume that the reader is familiar with basic integration techniques such as integration by parts.

Example 8. Given $f\left(x\right) = x^2$,

$$\begin{aligned}
\mathcal{L}\left(x^2\right) &= \int_0^\infty e^{-px} x^2\, dx \\
&= \frac{2}{p^3}
\end{aligned}$$

In general, $f\left(x\right) = x^n, n \in \mathbb{N}$,

$$L\left[f\right]\left(p\right) = \int_0^\infty e^{-px} x^n\, dx = \frac{n!}{p^{n+1}}$$

by induction. To show this,

$$\int_0^\infty e^{-px}x^n\,dx = \left(-\frac{1}{p}e^{-px}x^n\right)\Big|_0^\infty - \int_0^\infty -\frac{1}{p}e^{-px}nx^{n-1}\,dx$$
$$= 0 + \frac{n}{p}\underbrace{\int_0^\infty e^{-px}x^{n-1}\,dx}_{L[x^{n-1}](p)} = \frac{n}{p}\cdot\frac{(n-1)!}{p^n} = \frac{n!}{p^{n+1}}$$

Recall that the Laplace transform is simply a special type of an integral transform. Since we know that an integral transform is linear, we can compute Laplace transforms for any polynomial.

Example 9. Given $f(x) = x^2 + 6x + 9$, we can apply the Laplace operator to get,

$$\begin{aligned}\mathcal{L}\left(x^2+6x+9\right) &= \mathcal{L}\left(x^2\right) + 4\mathcal{L}(x) + 9\mathcal{L}(1)\\ &= \frac{2}{p^3} + 4\cdot\frac{1}{p^2} + 9\cdot\frac{1}{p}\\ &= \frac{2}{p^3} + \frac{4}{p^2} + \frac{9}{p}\\ &= \frac{2+4p+9p^2}{p^3}\end{aligned}$$

Example 10. Given $f(x) = e^{ax}$,

$$\begin{aligned}\mathcal{L}\left(e^{ax}\right) &= \int_0^\infty e^{-px}\cdot e^{ax}\,dx\\ &= \int_0^\infty e^{-(p-a)x}\,dx\\ &= \frac{1}{p-a} \qquad (p > a)\end{aligned}$$

We now compute the Laplace transform of trigonometric functions. Instead of directly evaluating for sin and cos, we will consider an alternate method. One could call this cheating but it's just as valid of a method to derive the Laplace transforms of trigonometric functions. Recall that

$$e^{i\alpha x} = \cos(\alpha x) + i\sin(\alpha x)$$

We comute the Laplace transform of $e^{i\alpha x}$ and take the real and imaginary parts. It follows,

$$\mathcal{L}\left(e^{i\alpha x}\right) = \int_0^\infty e^{-px}\cdot e^{i\alpha x}\,dx = \int_0^\infty e^{-(p-i\alpha)x}\,dx = \frac{1}{p-i\alpha}$$

Simplify as,

$$\frac{p+i\alpha}{(p-i\alpha)(p+i\alpha)} = \frac{p+i\alpha}{p^2+\alpha^2} = \frac{p}{p^2+\alpha^2} + i\frac{\alpha}{p^2+\alpha^2}$$

Hence,

$$\begin{aligned}\mathcal{L}(\sin ax) &= \frac{a}{p^2+a^2}\\ \mathcal{L}(\cos ax) &= \frac{p}{p^2+a^2}\end{aligned}$$

Example 11.

$$\begin{aligned}\mathcal{L}\left(2x+3\right) &= 2\mathcal{L}\left(x\right)+3\mathcal{L}\left(1\right)\\ &= 2\cdot\frac{1}{p^2}+3\cdot\frac{1}{p}\\ &= \frac{2}{p^2}+\frac{3}{p}\end{aligned}$$

What if we are given a Laplace transform of a function $f\left(x\right)$ and want to recover said $f\left(x\right)$? Consider the following example.

Example 12. Given $\mathcal{L}\left(f\left(x\right)\right)=\frac{1}{p^4+p^2}$,

$$\begin{aligned}\frac{1}{p^4+p^2} &= \frac{1}{p^2\left(p^2+1\right)}\\ &= \frac{\left(p^2+1\right)-p^2}{p^2\left(p^2+1\right)}\\ &= \frac{1}{p^2}-\frac{1}{p^2+1}\\ &= \mathcal{L}\left(x\right)-\mathcal{L}\left(\sin x\right)\\ &= \mathcal{L}\left(x-\sin x\right)\end{aligned}$$

Hence, $f\left(x\right)=x-\sin x$

4 Existence and Uniqueness

5 Fourier Series

6 Introduction to Partial Differential Equations

7 Calculus of Variations