Year 1 — Integration & Infinite Series

Based on lectures by D. Beers
Notes taken by Aryan Dalal

MATH 31B, Fall Quarter 2024, UCLA

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine. A note of acknowledgement to Dexter Chua, Ph.D. Harvard University for the LaTeXtemplate.

Catalog Description

Lecture, three hours; discussion, one hour. Requisite: course 31A with grade of C– or better. Not open for credit to students with credit for course 3B. Transcendental functions; methods and applications of integration; sequences and series. P/NP or letter grading.

Textbook Reading

N/A

Comments

N/A

Contact

This document is a summary of the notes that I have taken during lectures at UCLA; please note that this lecture note will not necessarily coincide with what you might learn. If you find any errors, don't hesitate to reach out to me below:

Aryan Dalal, aryandalal@ucla.edu

Contents

Part 1: Functions

1 Exponential Functions

In MATH 31A, the power rule was introduced to evaluate the derivative of x raised to the power of a. That is,

$$\frac{d}{dx}\left(x^a\right) = ax^{a-1}$$

$\forall\ x$ where $a \in \mathbb{R}$. And so, it follows that the power-rule for integration is

$$\int x^a\, dx = \frac{1}{a}x^{a+1} + C$$

for $a \neq 1$. We now want to understand how to evaluate the integral $\int \frac{1}{x}\, dx$ for which, we must introduce new functions.

Definition. *Exponential Function*

An *exponential function* is of the form

$$f\left(x\right) = b^x$$

for $b > 0, b \neq 1$.

If $b = 1$, $f\left(x\right) = 1^x = 1$. Therefore, b must be positive and it's called the *base*. Let's explore a few examples.

$$\text{Exponential Functions} \begin{cases} f\left(x\right) = 1.5^x \\ g\left(x\right) = \left(\dfrac{1}{2}\right)^x \end{cases}$$

However, $h\left(x\right) = \left(-1\right)^x$ is not an exponential function. Exponential functions have some nice properties which are crucial for performing arithmetic on exponentials.

Properties of Exponential Functions

(1) $b^0 = 1$
(2) $b^{x+y} = b^x b^y$
(3) $b^{xy} = \left(b^x\right)^y = \left(b^y\right)^x$
(4) $b^{-x} = \frac{1}{b^x} = \left(\dfrac{1}{b}\right)^x$

Remark b^{x^y} means $b^{\left(x^y\right)}$, *not* $\left(b^x\right)^y$. Many confuse b^{2x} with b^{x^2}. $b^{2x} = b^{x+x} = b^x \cdot b^x = \left(b^x\right)^2$ whereas $b^{x^2} = b^{x \cdot x} = \left(b^x\right)^x$.

You should note that the domain and range for exponentional functions are $\left(-\infty, \infty\right)$ and $\left(0, \infty\right)$ respectively.
If $b > 1$, then the function is increasing and

$$\lim_{x\to\infty} b^x = \infty \qquad \lim_{x\to-\infty} b^x = 0$$

If $b < 1$, then the function is decreasing and

$$\lim_{x\to\infty} b^x = 0 \qquad \lim_{x\to-\infty} b^x = \infty$$

1.1 Derivatives of Exponential Functions

By the limit definition of a derivative that you may have seen in AP Calculus/IB Mathematics or MATH 31A,

$$\begin{aligned}
\frac{d}{dx}\left(b^x\right) &= \lim_{h\to 0} \frac{b^{x+h} - b^x}{h} \\
&= \lim_{h\to 0} \frac{b^x \cdot b^h - b^x}{h} \\
&= \lim_{h\to 0} \frac{b^x \left(b^h - 1\right)}{h} \\
&= b^x \lim_{h\to 0} \frac{b^h - 1}{h} = b^x m\left(b\right)
\end{aligned}$$

Let's call the expression $\lim_{h\to 0} \dfrac{b^h - 1}{h}$ as $m\left(b\right)$. The significance of this shall be shown very soon. We know that the limit $m\left(b\right)$ eixsts, m is continuous and increasing and it's immediate by inspection that $m\left(3\right) > 1$, $m\left(2.5\right) < 1$. It follows that $\exists n$ $2.5 < n < 3$ s.t. $m\left(x\right) = 1$.

The approximation $m\left(b\right)$ is actually the number e (Euler's number) with an approximate value $e \approx 2.718\ldots$. Thus,

$$\begin{aligned}
\frac{d}{dx}\left(e^x\right) &= e^x \cdot m\left(e\right) \\
&= e^x
\end{aligned}$$

$$\int e^x \, dx = e^x + C$$

We will soon see how we can use our knowledge on exponential functions to evaluate integrals such as:

Example Given $I = \int xe^{2x^2}$, evaluate the integral.

Solution. Let $u = 2x^2 \implies du = 4x\,dx$

$$\begin{aligned}
\int xe^{2x^2}\, dx &= \frac{1}{4}\int e^u \, dx \\
&= \frac{1}{4} e^{2x^2}
\end{aligned}$$

2 Inverse Functions

Definition. *Inverse Function*

If f is a function with domain D and range R, and there exists a function g such that

$$\begin{aligned} f(g(x)) &= (f \circ g)(x) = x \qquad \forall x \in R \\ g(f(x)) &= (g \circ f)(x) = x \qquad \forall x \in D \end{aligned}$$

then g is called the **inverse** of f, denoted $f^{-1}(x)$.

Note, $f^{-1}(x)$ is **not** $\dfrac{1}{f(x)}$. $(f(x))^{-1} = \dfrac{1}{f(x)} \neq f^{-1}(x)$.

Consider the following example,

Table 1: f

Domain	Range
5	1
6	2
7	3

Table 2: f^{-1}

Domain	Range
1	5
2	6
3	7

To find an inverse of $y = f(x)$, we want to inter-change x with y and then, solve for y in terms of x.

Note that sometimes, it may not be possible to solve for y in terms elementary functions and so, finding inverses can be hard. However, assuming you have a nice enough function that has an equally nice enough inverse, then $y = f^{-1}(x)$ is the inverse of $f(x)$.

Example Find the inverse of $f(x) = 2x^3 + 18$
Solution. Interchange x and y, so,

$$x = 2y^3 + 18$$

Solving for y,

$$2y^3 = 18 - x \iff y^3 = 9 - \frac{1}{2}x \iff y = \left(9 - \frac{1}{2}x\right)^{1/3}$$

So,

$$f^{-1}(x) = \left(9 - \frac{1}{2}x\right)^{1/3}$$

is the inverse of f.

Note that it's not always possible to write out an inverse. (Consider $f(x) = x^5 + 2x + 2$. Give it a try if you think you can find an inverse.)

Definition *Injective*

Let f be a function with domain D. f is said to be **injective** if $\forall a, b \in D$, if $f(a) = f(b) \implies a = b$.
Equivalently, $a \neq b \implies f(a) \neq f(b)$.

By the above definition, we intend to say that for a function f with domain D, f is one-to-one if for every value c, there is at most one solution $f(x) = c$ for all x in D.
Many tend to use the statements, 'one-to-one function' and 'one-to-one correspondence' interchangeably; some may notice that the latter refers to bijective functions.

By the **Horizontal Line Test**, if every horizontal line intersects the graph of a function f at most once, then f is one-to-one.

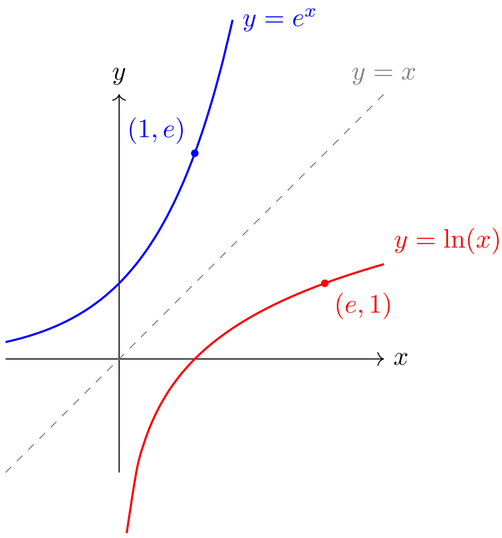
Theorem.

A function f is invertible if and only iff f is one-to-one. In addition,

$$\begin{aligned} \text{domain}\left(f^{-1}\right) &= \text{range}(f) \\ \text{range}\left(f^{-1}\right) &= \text{domain}(f) \end{aligned}$$

2.1 Graphing Inverse Functions

To graph inverse functions, we want to draw the function f, draw the line $y = x$ and reflect f along the line to achieve f^{-1}. Consider the below example for $f(x) = e^x$.



Theorem.

If $f(x)$ is differentiable and has an inverse, then

$$g(b) = \frac{1}{f'(g(b))}$$

where $g(x) = f^{-1}(x)$ and $f'(g(b)) \neq 0$.

Proof. (Sketch)

$$\begin{aligned} f(g(x)) &= x \\ \frac{d}{dx}[f(g(x))] &= \frac{d}{dx}[x] \\ g'(x)\, f'(g(x)) &= 1 \\ g'(x) &= \frac{1}{f'(g(x))} \end{aligned}$$

Note that this is useful in cases where we don't understand g.

Example

$$f(x) = x^5 + 2x + 2$$

We want $\left(f^{-1}\right)'(5)$. We know that $f'(x) = 5x^4 + 2$, so,

$$\left(f^{-1}\right)'(5) = \frac{1}{f'(f^{-1}(5))} = \frac{1}{f'(1)} = 7$$

3 Logarithms

Definition. *Logarithm.*

The function $\log_b(x)$ is defined as the inverse function of the exponential function $f(x) = b^x$.

In particular, the inverse of e^x is written $\ln(x)$ (sometimes just $\log x$) and is called the *natural logarithm.*

$$\ln x = \log_e x$$

Properties of Logarithms

(1) $\log_b(1) = 0$

(2) $\log_b(x) + \log_b(y) = \log_b(xy)$

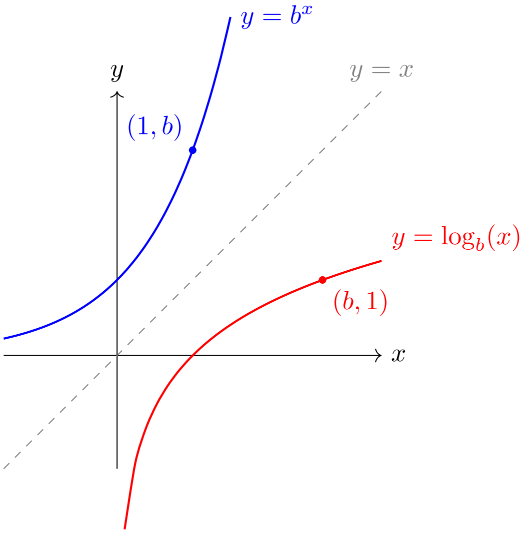
(3) $\log_b(x) - \log_b(y) = \log_b\left(\frac{x}{y}\right)$

(4) $\log_b(x^a) = a\log_b(x)$

(5) $\log_b(x) = \frac{\ln(x)}{\ln(b)}$

3.1 Graphs of Log Functions

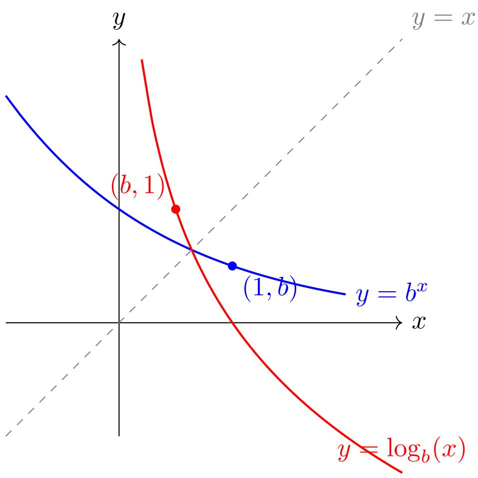
For $b > 1$,



It follows that

$$\lim_{x\to\infty} \log_b(x) = \infty \qquad \lim_{x\to 0^+} \log_b(x) = -\infty$$

For $0 < b < 1$,



It follows that

$$\lim_{x \to \infty} \log_b(x) = -\infty \qquad \lim_{x \to 0^+} \log_b(x) = \infty$$

3.2 Derivatives of Log Functions

We want to write $g(x) = \ln x$, $f(x) = e^x$ so that $g(x) = f^{-1}(x)$. Thus,

$$\begin{aligned}\frac{d}{dx}\ln(x) = g'(x) &= \frac{1}{f'(g(x))} \\ &= \frac{1}{f'(\ln(x))} \\ &= \frac{1}{e^{\ln x}} \\ &= \frac{1}{x}\end{aligned}$$

Thus,

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

We will now use this new information to derive a general expression for the derivative of $\log_b(x)$.

$$\begin{aligned}\frac{d}{dx}\log_b(x) &= \frac{d}{dx}\frac{\ln x}{\ln b} \\ &= \frac{1}{\ln b}\frac{d}{dx}\ln x \\ &= \frac{1}{\ln b}\cdot\frac{1}{x} \\ &= \frac{1}{x\ln b}\end{aligned}$$

Thus,

$$\frac{d}{dx}\log_b x = \frac{1}{x\ln b}$$

Continuing, we will use the above information to derive a general expression for the derivative of b^x. We set $g(x) = b^x, f(x) = \log_b x$ so $g(x) = f^{-1}(x)$. It follows that,

$$\begin{aligned}\frac{d}{dx}[b^x] = g'(x) &= \frac{1}{f'(g(x))}\\ &= \frac{1}{f'(b^x)}\\ &= \frac{1}{\frac{1}{b^x \ln b}} = b^x \ln b\end{aligned}$$

Thus,

$$\frac{d}{dx}b^x = b^x \ln b$$

As such,

Summary of Derivatives

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{d}{dx}\log_b x = \frac{1}{x \ln b}$$

$$\frac{d}{dx}b^x = b^x \ln b$$

We will now use the above to solve some more complex problems in differentiation.

Example Differentiate $f(x) = x^{\sin x}$.
Solution. We know that $x = e^{\ln x}$, so,

$$f(x) = x^{\sin x} = \left(e^{\ln x}\right)^{\sin x} = e^{\ln x \sin x}$$

Then,

$$\begin{aligned}f'(x) &= (\ln x \sin x)' e^{\ln x \sin x}\\ &= \left(\frac{1}{x}\sin x + \ln x \cos x\right) e^{\ln x \sin x}\end{aligned}$$

$$\begin{aligned}
&= \left(\frac{1}{x}\sin x + \ln x \cos x\right)\left(e^{\ln x}\right)^{\sin x} \\
&= \left(\frac{1}{x}\sin x + \ln x \cos x\right) x^{\sin x}
\end{aligned}$$

Example Differentiate $f(x) = \dfrac{2x^3 + x + 2}{\left(x^2+1\right)^2 (x+1)^3}$

Solution. Taking the natural log of both sides,

$$\begin{aligned}
\ln(f(x)) &= \ln\left(\frac{2x^3 + x + 2}{\left(x^2+1\right)^2 (x+1)^3}\right) \\
&= \ln\left(2x^3 + x + 2\right) - \ln\left(\left(x^2+1\right)^2 (x+1)^3\right) \\
&= \ln\left(2x^3 + x + 2\right) - 2\ln\left(x^2+1\right) - 3\ln(x+1)
\end{aligned}$$

Taking the derivative of both sides,

$$[\ln(f(x))]' = f'(x) \cdot \frac{1}{f(x)} = \frac{6x^2+1}{2x^3+x+2} - 2 \cdot \frac{2x}{x^2+1} - 3 \cdot \frac{1}{x+1}$$

Solving for $f'(x)$,

$$\begin{aligned}
f'(x) &= f(x)\left[\frac{6x^2+1}{2x^3+x+2} - 2\frac{2x}{x^2+1} - 3\frac{1}{x+1}\right] \\
&= \frac{2x^3 + x + 2}{\left(x^2+1\right)^2 (x+1)^3}\left[\frac{6x^2+1}{2x^3+x+2} - \frac{4x}{x^2+1} - \frac{3}{x+1}\right]
\end{aligned}$$

Let's consider taking the integral of $\frac{1}{x}$. We know that $\frac{d}{dx}\ln x = \frac{1}{x}$ and so,

$$\int \frac{1}{x}\, dx = \ln|x| + C$$

Note the absolute value. If $x < 0$, $\ln x$ is not defined but $\ln(|x|) = \ln(-x)$ is defined and

$$\frac{d}{dx}\ln(-x) = -1 \cdot \frac{1}{-x} = \frac{1}{x}$$

Example Evaluate $I = \int \frac{x}{x^2+2}\, dx$.

Solution. Let $u = x^2 + 2 \implies du = 2x\, dx \implies \frac{du}{2} = x\, dx$. So,

$$
\begin{aligned}
I = \int \frac{x}{u}\,dx = \int \frac{1}{2u}\,du = \frac{1}{2}\int \frac{1}{u}\,du \\
&= \frac{1}{2}\left(\ln|u|\right) \\
&= \frac{1}{2}\ln|x^2+2| + C \\
&= \frac{1}{2}\ln\left(x^2+2\right) + C
\end{aligned}
$$

4 L'hôpital's rule & Indeterminate Forms

If $f(x) = \frac{\sin x}{x}$, then $f(0)$ is not defined. Why? We would be performing division by 0. Recall that we say $\lim_{x\to a} f(x)$ exists if $f(x)$ is very close to a finite number L as x approaches a. From MATH 31A, we know that $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$ using *squeeze*. We will see why this limit is significant.
Sometimes plugging in $x = a$ to $\lim_{x\to a} f(x)$ gives $0 \cdot \infty$ or $\infty - \infty$. For example, plugging 0 to $\lim_{x\to 0} \frac{\sin x}{x}$ gives $\frac{1}{0} \cdot \sin 0$ but this does not make sense since the expression we have received is indeterminate. In general, we can avoid indeterminate forms by performing *conjugation*, *simplification* or *algebraic manipulation*. We can also use a technique known as *L'hôpital's Rule*.

Theorem. *L'Hôpital's Rule*

Let $f(x)$ and $g(x)$ be differentiable on an open interval containing a and $f(a)$ and $g(a)$:

$$\lim_{x\to a} f(x) \text{ and } \lim_{x\to a} g(x)$$

take on the following pair of values respectively

$$0 \text{ and } 0, \infty \text{ and } \infty, -\infty \text{ and } \infty, \infty \text{ and } -\infty, -\infty \text{ and } -\infty$$

i.e., either both go to 0 or both go to $\pm\infty$. In addition, assume $\lim_{x\to a} g(x) \neq 0$, then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Note that like u-substitution, it is entirely possible that we may need to play L'hôpital's rule more than once to receive an expression that's not indeterminate. Consider the limit

$$\lim_{x\to -\infty} \frac{e^x}{x}$$

The quotient is of the the form $\dfrac{\infty}{\infty}$ so we can apply L'hôpital's here. As such,

$$\lim_{x\to -\infty} \frac{e^x}{x} = \lim_{x\to -\infty} \frac{(e^x)'}{(x)'} = \lim_{x\to -\infty} \frac{e^x}{1} = \lim_{x\to -\infty} e^x = 0$$

Consider another limit

$$\lim_{x\to 0^+} x\ln x$$

This limit is of the form $0\cdot-\infty$ so we can't directly apply L'hôpital's here. However, we can manipulate the expression to achieve an indeterminate form compatible with L'hôpital's. See below.

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$$

We now have an indeterminate expression of the form $\frac{-\infty}{\infty}$.

$$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{(\ln x)'}{\left(\frac{1}{x}\right)'} = \lim_{x\to 0^+} = \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{x}{-1} = 0$$

Remark For indeterminate forms $0\cdot\infty$ and $\infty-\infty$, we want to manipulate expressions so as to achieve a $\frac{0}{0}$ or $\frac{\infty}{\infty}$ indeterminate form and try L'Hôpital's.

4.1 Further Indeterminate Forms

We know that we can algebraically manipulate expressions of the form $\infty-\infty$ and $0\cdot\infty$ so that they're compatible with L'Hôpital's. How should we go about with $1^\infty, \infty^0$ and 0^0? We will perform a *change of base*.

Example Find $\lim_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x$.

Solution. Any time we have a function of the form $f(x)^{g(x)}$ in a limit, it is a good idea to replace $f(x)$ with $e^{\ln f(x)}$. So,

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty}\left[e^{\ln\left(1+\frac{1}{x}\right)}\right]^x = \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)}$$

Simply evaluating the power,

$$\begin{aligned}\lim_{x\to} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\ln(1)}{\frac{1}{x}} &= \lim_{x\to\infty}\frac{-\frac{1}{x^2}\cdot\frac{1}{1+1/x}}{-\frac{1}{x^2}} \\ &= \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = 1\end{aligned}$$

So,

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^1 = e$$

It's important to study the limit properly before deciding to use a particular technique. That is to say, if a limit appears to be of an indeterminate form but you make an incorrect assumption, L'Hôpital's will yield an incorrect answer. Consider the following,

Example Find $\lim_{x\to 0} \dfrac{x}{x+1}$
Solution. Simply plugging in, we get

$$\lim_{x\to 0} \frac{x}{x+1} = \frac{0}{0+1} = 0$$

An **incorrect** answer would be

$$\lim_{x\to 0} \frac{x}{x+1} = \lim_{x\to 0} = \frac{(x)'}{(x+1)'} = \lim_{x\to 0} \frac{1}{1} = 1$$

Example Find $\lim_{x\to 0+} \sin x^{\sin x}$
Solution. $\sin x = e^{\ln \sin x}$, so,

$$\begin{aligned}
\lim_{x\to 0^+} \sin x^{\sin x} &= \lim_{x\to 0^+} \left(e^{\ln \sin x}\right)^{\sin x} \\
&= \lim_{x\to 0^+} e^{\sin x \ln \sin x} \\
&= e^{\lim_{x\to 0^+} \sin x \ln \sin x}
\end{aligned}$$

Simply evaluating the power,

$$\begin{aligned}
\lim_{x\to 0^+} \sin x \ln \sin x &= \lim_{x\to 0^+} \frac{\ln \sin x}{\frac{1}{\sin x}} \\
&= \lim_{x\to 0^+} \frac{\frac{1}{\sin x} \cdot \cos x}{-\frac{1}{(\sin x)^2} \cdot \cos x} \\
&= \lim_{x\to 0^+} \frac{\sin x}{-1} = 0
\end{aligned}$$

Thus,

$$\lim_{x\to 0^+} \sin x^{\sin x} = e^0 = 1$$

Try evaluating the limit

$$\lim_{x\to\infty} (x - \ln x)$$

Hint: You can express x as $x = \ln(e^x)$.

5 Trigonometric Functions in Calculus

From MATH 31A, you should know

$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}\tan x = \sec^2 x$$
$$\frac{d}{dx}\sec x = \sec x \tan x$$

There are also,

$$\frac{d}{dx}\csc x = -\csc x \cot x$$
$$\frac{d}{dx}\cot x = -\csc^2 x$$

However, the above two will not be used frequently in this class. What's important to know is that you really just need to understand the derivatives for $\sin x$ and $\cos x$ and you should be able to derive the derivatives for the other four trigonometric functions. $\sec x, \csc x, \cot x$ are simply reciprocals of $\sin x, \cos x, \tan x$ respectively and $\tan x$ is the quotient $\frac{\sin x}{\cos x}$. Try deriving the derivatives of $\tan, \sec, \csc$ and $\cot$ with only the derivatives of $\sin$ and $\cos$.

5.1 Graphs of Inverses and their Properties

From the horizontal-line test, we know that $\sin x$ is not invertible across its entire domain. However, we can still invert $\sin x$ locally by restricting its domain, that is to say, we want to restrict the domain of sin so that

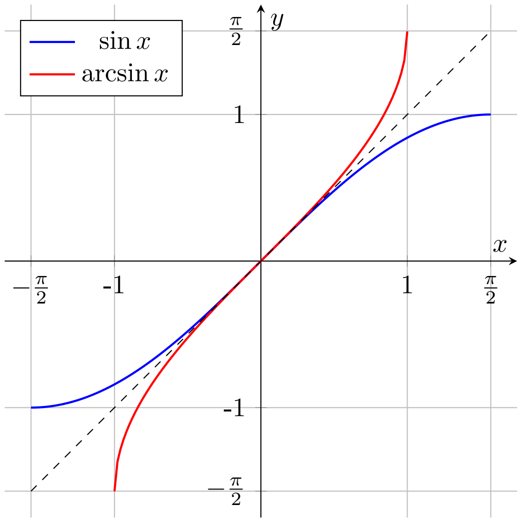
$$\frac{d}{dx}\sin x > 0$$

when $\cos x > 0$ which happens in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$. It follows that $\sin x$ is *increasing* on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Now that sin has been restricted, we get an inverse function.

Definition.

$\theta = \sin^{-1} x$ or $\arcsin x$ is the unique angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin\theta = x$.

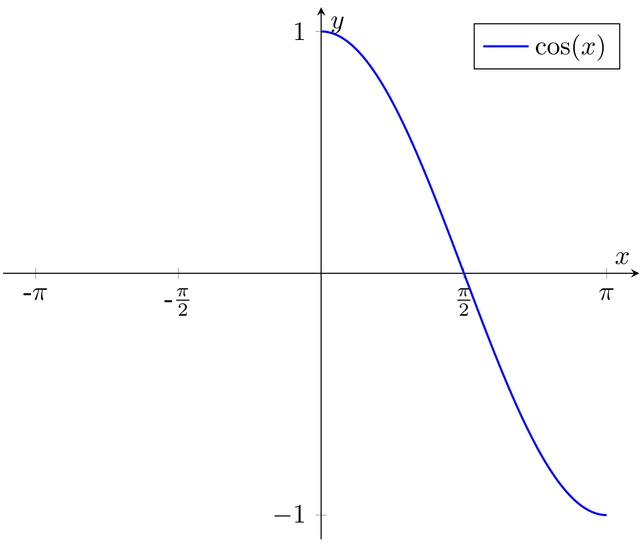
$$f\colon\ [-1,1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], f(x) = \sin^{-1}(x)$$



Example Find $\tan(\arcsin x)$.
Solution. Constructing a triangle with a side x and hypotenuse 1, we yield the other side non-hypotenuse side as $\sqrt{1-x^2}$, this is the side adjacent to θ. As such,

$$\tan(\arcsin x) = \frac{x}{\sqrt{1-x^2}}$$

Likewise with $\sin x$, $\cos x$ is not invertible across its entire domain but can be locally invertible on some interval. This inveral is $[0, \pi]$.

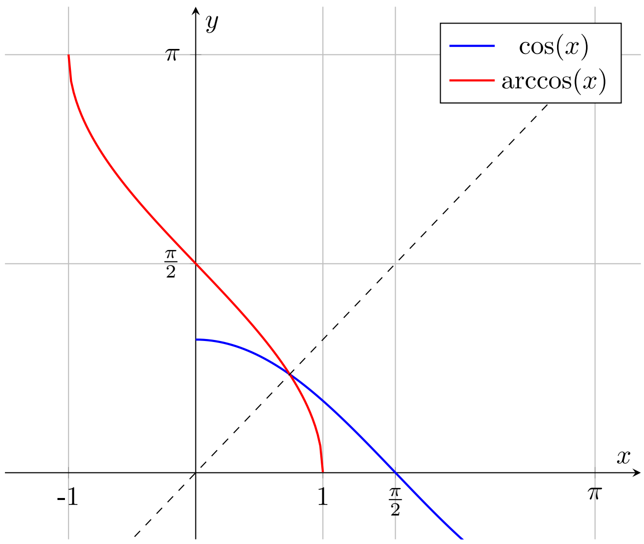


Definition.

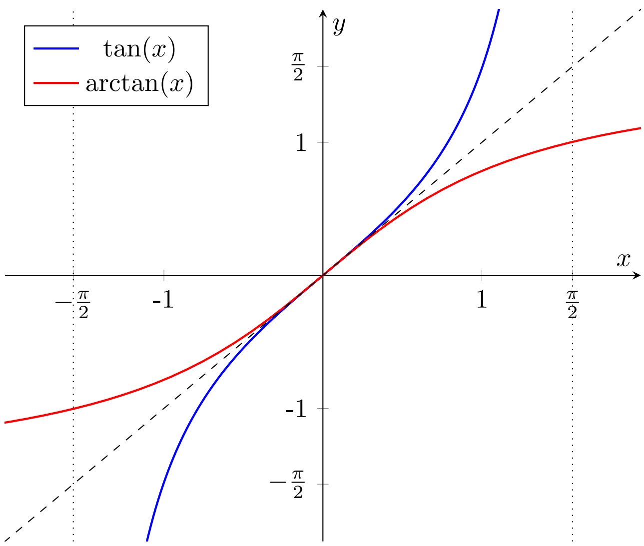
$\arccos x$ denoted the inverse of $\cos x$ when restricted to $[0, \pi]$.

$$f\colon\ [-1, 1] \longrightarrow [0, \pi]\,, f\,(x) = \arccos x$$

And so, the graph of the inverse is as follows:



With the similar idea, you can predict that $\tan x$ is also not invertible across its domain, it is only locally invertible to some interval. This interval is $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.



5.2 Derivative and Integrals

Let $g(x) = \arcsin x$ and $f(x) = \sin x$ so that $g(x) = f^{-1}(x)$, then,

$$\frac{d}{dx}[\arcsin x] = g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\cos(\arcsin x)}$$

$\cos(\arcsin x)$ is $\cos\theta$ for θ in a triangle with hypotenuse 1, opposite side x and adjacent side $\sqrt{1-x^2}$, so, $\cos\theta = \sqrt{1-x^2}$. Plugging in $\cos(\arcsin x) = \sqrt{1-x^2}$, we get

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

With similar procedure, we have

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\arccos x] = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}$$

Their integrals are,

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

$$\int -\frac{1}{\sqrt{1-x^2}}\,dx = \arccos x + C$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

Consider the integral

$$I = \int_0^{\sqrt{3}} \frac{5}{\sqrt{4-x^2}}\,dx$$

There are two methods to solve this integral. We can perform a *trigonometric substitution* or a u-substitution. In essence, a trigonometric substitution is just a u-substitution but is a lot faster. I will show both. Let's first do a u-substitution.

$$\int \frac{5}{\sqrt{4-x^2}}\,dx = 4\int \frac{1}{\sqrt{4\left(1-\frac{x^2}{4}\right)}} = \frac{5}{2}\int \frac{1}{\sqrt{1-\frac{x^2}{4}}}\,dx$$

Let $u = x/2 \implies du = \frac{1}{2}\,dx \implies dx = 2\,du$, thus,

$$\frac{5}{2}\frac{2}{\sqrt{1-u^2}}\,du = 5\int \frac{1}{\sqrt{1-u^2}}\,du = 5\arcsin(u) + C = 5\arcsin\left(\frac{x}{2}\right) + C$$

Hence,

$$5\arcsin\left(\frac{x}{2}\right)\Big|_0^{\sqrt{3}} = \frac{5\pi}{3}$$

Let's now do a direct trigonometric substitution. Notice that $4-x^2$ is very similar to $1-\sin^2 x$ which we know is equal to $\cos^2 x$. This should be apparent from high-school trigonometry. As such, we can let $x = 2\sin\theta \implies dx = 2\cos\theta\,d\theta$.

$$\int \frac{5}{\sqrt{4-x^2}}\,dx = 5\int \frac{1}{\sqrt{4-(2\sin\theta)^2}}2\cos\theta\,d\theta = \frac{5}{2}\int \frac{1}{\sqrt{\cos^2\theta}}2\cos\theta\,d\theta = 5\int d\theta$$

Since $x = 2\sin\theta \implies \frac{x}{2} = \sin\theta \implies \theta = \arcsin\left(\frac{x}{2}\right)$ and so,

$$5\theta + C = 5\arcsin\left(\frac{x}{2}\right)\Big|_0^{\sqrt{3}} = \frac{5\pi}{3}$$

While it may appear that trigonometric substitution is a longer process, in general, it's a more direct way of solving integrals that involve inverses of trigonometric functions.

5.3 Hyperbolic Trigonometric Functions

Definition.

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

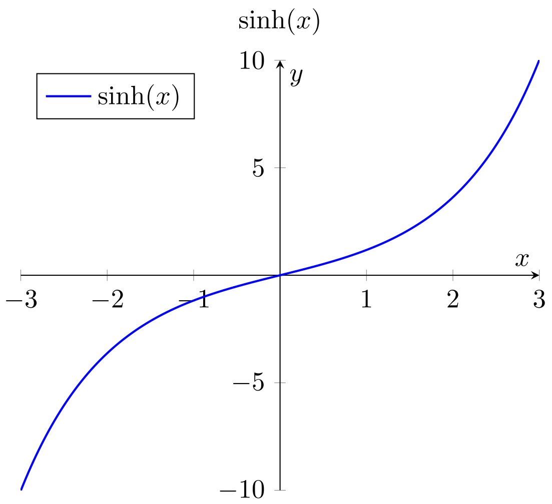
are the *hyperbolic* sine and cosine.

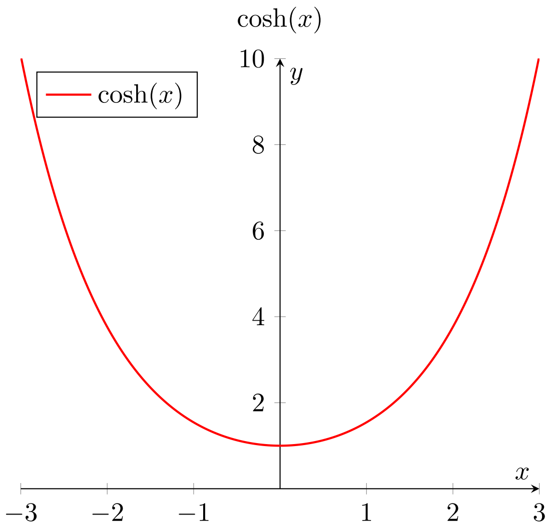
sinh is an odd function while cosh is an even function.

We have

$$\frac{d}{dx}\sinh x = \cosh x \qquad \frac{d}{dx}\cosh x = \sinh x$$

$$\int \cosh x = \sinh x + C \qquad \int \sinh x = \cosh x + C$$





Don't confuse the derivatives of $\cos x$ and $\cosh x$. The derivative of $\cos x$ is $-\sin x$, however, derivative of $\cosh x$ is $\sinh x$. Also note that there are more hyperbolic trigonometric functions, however, for this class, only these two are necessary.

Part 2: Integration

6 Techniques of Integration

In MATH 31A, we introduced the notion of evaluating integrals for simple functions. We will extend this idea by introducing more complex functions and studying different methods to solve them.

6.1 Integration by Parts

Let $u(x), v(x)$ be functions of x. Then,

$$\begin{aligned} \frac{d}{dx}[uv] &= u\frac{dv}{dx} + v\frac{du}{dx} \\ \int \frac{d}{dx}[uv]\,dx &= \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx \\ uv &= \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx \end{aligned}$$

Let $dv = \dfrac{dv}{dx}\,dx,\ du = \dfrac{du}{dx}\,dx$, then,

$$\begin{aligned} uv &= \int u\,dv + \int v\,du \\ \int u\,dv &= uv - \int v\,du \end{aligned}$$

We must choose functions u and v such that we're able to evaluate the above equation. How does one choose a good candidate for u and v? In general, the idea is differentiate more complex functions and integrate easier functions. This, however, is not a very rigorous suggestion; indeed, there's no mathematical per se that will reveal the best choices for u and v, however, we can generalize and recommend that the best candidates for u are,

Logarithms → Inverse Trig. → Polynomials → Exponentials → Trig. (Including Hyperbolic)

from left to right (LIPET for short). For the sake of problem-solving however, it's impractical to memorize this order, rather, you should evaluate the appropriate choices for u and v on a case-by-case basis following the integral.

Example Given $I = \int x\cos x\,dx$, evaluate the integral.
Solution. Let $u = x \implies du = dx,\ dv = \cos x\,dx \implies v = \sin x$,

$$\begin{aligned} \int x\cos x\,dx &= x\sin x - \int \sin x\,dx \\ &= x\sin x - (-\cos x + C) \\ &= x\sin x + \cos x + C \end{aligned}$$

Example Given $I = \int xe^{2x}\,dx$, evaluate the integral.
Solution. Let $u = x \implies du = dx$, $dv = e^{2x}\,dx$, $v = \dfrac{1}{2}e^{2x}$,

$$\begin{aligned}\int xe^{2x}\,dx &= \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx \\ &= \frac{1}{2}xe^{2x} - \frac{1}{2}\left(\frac{2^{2x}}{2}\right) \\ &= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C\end{aligned}$$

Example Given $I = \int x^{-\frac{1}{2}} \ln x\,dx$, evaluate the integral.
Solution. Let $u = \ln x$, $du = \dfrac{1}{x}\,dx$, $dv = x^{-\frac{1}{2}}\,dx \implies v = 2x^{\frac{1}{2}}$,

$$\begin{aligned}\int x^{-\frac{1}{2}} \ln x\,dx &= 2x^{\frac{1}{2}} \ln x - 2\int 2x^{\frac{1}{2}} \frac{1}{x}\,dx \\ &= 2x^{\frac{1}{2}} \ln x - 4x^{\frac{1}{2}} + C\end{aligned}$$

We know the derivative of $\ln x$, it's $\frac{1}{x}$. But what about $\int \ln x$? We will now derive the integral of the natural log using a clever approach to Integration by Parts.

Example Derive $\int \ln x\,dx$ using Integration by Parts.
Solution. Let $u = \ln x \implies du = \frac{1}{x}\,dx$, $dv = dx \implies v = x$,

$$\begin{aligned}\int \ln x\,dx &= x\ln x - \int dx \\ &= x\ln x - x + C\end{aligned}$$

6.2 u-Substitution V.S Integration by Parts

When given an integral, how should you know which technique to utilize? At first, it might appear that one should try both techniques to see which method provides a more structured approach to reaching the answer, however, this takes time. Consider the following,

$$\int \frac{\ln(\sqrt{x})}{\sqrt{x}}\,dx$$

Let $u = \sqrt{x}$. Immediately, we see that the derivative of $\sqrt{x}$ gives us $2\sqrt{x}$ which can be used to cancel the $\sqrt{x}$ in the denominator and thus, simplify the expression. Utilizing Integration By Parts in this scenario may make the process more complicated and therefore, more susceptible to computational error.

$$\begin{aligned}\int \frac{\ln \sqrt{x}}{\sqrt{x}}\,dx &= \int \frac{\ln u}{\sqrt{x}} \cdot 2\sqrt{x}\,du \\ &= 2\int \ln u\,du \\ &= 2\left(u \ln u - u\right) \\ &= 2\sqrt{x}\ln\sqrt{x} - 2\sqrt{x} + C\end{aligned}$$

How should we know when to apply Integration By Parts? This technique may prove useful when the integral is a product of two elementary functions such that the integral can be easier to solve. Consider the following,

$$\int x^2 \cos^{-1}(x)\; dx$$

This integral is a product of two elementary functions, namely x^2 and $\cos^{-1}(x)$ and so, let's see how Integration By Parts makes this integral easier to solve.

Let $u = \cos^{-1}(x) \implies du = -\dfrac{1}{\sqrt{1-x^2}}\,dx$, $dv = x^2\,dx \implies v = \frac{1}{3}x^3\,dx$,

$$\int x^2 \cos^{-1}\; dx = \frac{1}{3}x^3\cos^{-1}(x) + \int \frac{x^3}{3\sqrt{1-x^2}}\,dx$$

Let $u = 1 - x^2 \implies du = -2x\,dx \implies dx = -\frac{1}{2x}\,du$ and $x^2 = 1 - u \implies -x^2 = u - 1$, so,

$$\begin{aligned}\int x^2 \cos^{-1}\; dx &= \frac{1}{3}x^3\cos^{-1}(x) + \int \frac{x^3}{3\sqrt{u}} \cdot \frac{-1}{2x}\,du \\ &= \frac{1}{3}x^3\cos^{-1}(x) + \int \frac{-x^2}{6\sqrt{u}}\,du \\ &= \frac{1}{3}x^3\cos^{-1}(x) + \frac{1}{6}\int \frac{u-1}{\sqrt{u}}\,du \\ &= \frac{1}{3}x^3\cos^{-1}(x) + \frac{1}{6}\int \sqrt{u} - \frac{1}{\sqrt{u}}\,du \\ &= \frac{1}{3}x^3\cos^{-1}(x) + \frac{1}{6}\left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right) \\ &= \frac{1}{3}x^3\cos^{-1}(x) + \frac{1}{9}\left(1-x^2\right)^{3/2} - \frac{1}{3}\left(1-x^2\right)^{1/2} + C\end{aligned}$$

Notice that we had to use Integration By Parts and u-Substitution. It follows that sometimes, to solve the integral, you will have to use both techniques.

Let's consider another example where repeated Integration By Parts will be involved.

$$\int x^2 e^x\,dx$$

Let $u = x^2 \implies du = 2x\,dx$, $dv = e^x\,dx \implies v = e^x$,

$$\int x^2 e^x\,dx = x^2 e^x - \int 2xe^x\,dx$$

Let $u = 2x \implies du = 2\,dx$, $dv = e^x\,dx \implies v = e^x$,

$$\begin{aligned}\int x^2 e^x\,dx &= x^2e^x - \left(2xe^x - \int e^x\,dx\right)\\ &= x^2e^x - 2xe^x + e^x + C\end{aligned}$$

In some problems, you might have to perform Integration By Parts more than two times. It's easy to propagate errors when repeated Integration By Parts is necessary, and so, it's important to proofread each step of your working.

Sometimes, you might encounter problems such as the following,

$$I = \int e^x \sin x\,dx$$

Let $u = e^x \implies du = e^x\,dx$, $dv = \cos x\,dx \implies v = \sin x$,

$$\int e^x \sin x\,dx = -e^x\cos x - \int -e^x\cos x\,dx$$

Let $u = e^x \implies du = e^x\,dx$, $dv = \cos x\,dx \implies v = \sin x$,

$$\int e^x \sin x\,dx = -e^x\cos x + \left(e^x\sin x - \int e^x\sin x\,dx\right)$$

It appears that we've recovered the integral we were trying to solve. However, the only step remaining now is a matter of algebraic manipulation.

Since $I = \int e^x \sin x\,dx$

$$I = -e^x\cos x + e^x\sin x - I$$

and so,

$$\begin{aligned}I + I &= -e^x\cos x + e^x\sin x\,dx\\ 2I &= -e^x\cos x + e^x\sin x\,dx\\ 2\int e^x\sin x\,dx &= -e^x\cos x + e^x\sin x\,dx\\ \int e^x\sin x\,dx &= \frac{-e^x\cos x + e^x\sin x}{2}\\ &= \frac{e^x}{2}\left(\sin x - \cos x\right) + C\end{aligned}$$

6.3 Methods of Partial Fractions

Partial Fractions is another technique of integration that ultimately aims to decompose complex integrals into simpler expressions that can be evaluated with applications of standard integrals. We ultimately want to use the *method of partial fractions* to solve integrals of the form

$$\int \frac{P(x)}{Q(x)}\, dx$$

where P and Q are polynomials.

6.3.1 Partial Fraction Decomposition

Partial Fraction Decomposition is a way fo splitting the expression $\frac{P(x)}{Q(x)}$ into simpler expressions. Consider the following,

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

Note that from the above example, you will see that the integer 1 is also a polynomial; $1 = 1x^0$. We see that the above expression with two polynomials is broken down into a simpler expression since

$$\frac{1}{x} - \frac{1}{x+1} = \frac{x+1}{x(x+1)} - \frac{x}{x(x+1)} = \frac{x+1-x}{x(x+1)} = \frac{1}{x(x+1)}$$

Remark The **degree** of a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is n.

Example $5x^7 + x + 3$ has degree 7.

Note that Practial Fraction Decomposition simplifies $\frac{P(x)}{Q(x)}$ when degree(P) $<$ degree(Q).

6.3.2 Distinct Linear PFD

Suppose $Q(x) = c(x-a_1)(x-a_2)\cdots(x-a_n)$ where a_i for $i \in [1,n]$ are all distinct, then,

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_n}{x-a_n}$$

if degree(P) $< n$.

Example Find the Partial Fraction Decomposition of $\frac{1}{x(x+2)}$.
Solution. We want to express $\frac{1}{x(x+2)} = \frac{A_1}{x} + \frac{A_2}{x+2}$, Then,

$$1 = A_1 (x+2) + A_2 x$$

For $x = 0$,

$$1 = 2A_1 \implies A_1 = \frac{1}{2}$$

For $x = -2$,

$$1 = -2A_2 \implies A_2 = -\frac{1}{2}$$

Thus,

$$\frac{1}{x(x+2)} = \frac{\frac{1}{2}}{x} + \frac{-\frac{1}{2}}{x+2}$$

You will note that there are many different methods of Partial Fraction Decomposition, however, all of them ultimately want to reduce the expression into something simpler.

Example Find the Partial Fraction Decomposition of $\frac{x+2}{x(x-1)(x+1)}$
Solution. We want to express

$$\frac{x+2}{x(x-1)(x+1)} = \frac{A_1}{x} + \frac{A_2}{x-1} + \frac{A_3}{x+1}$$

Then,

$$x + 2 = A_1 (x-1)(x+1) + A_2 x (x+1) + A_3 x (x-1)$$

For $x = 0$,

$$2 = -A_1 \implies A_1 = -2$$

For $x = 1$,

$$3 = 2A_2 \implies A_2 = \frac{3}{2}$$

For $x = -1$,

$$1 = A_3 (-1)(-2) \implies 1 = 2A_3 \implies A_3 = \frac{1}{2}$$

Thus,

$$\frac{x+2}{x(x-1)(x+1)} = \frac{-2}{x} + \frac{\frac{3}{2}}{x-1} + \frac{\frac{1}{2}}{x+1}$$

6.3.3 Repeated Linear Factor

If $(x - a_i)^n$ appears in $Q(x)$, then we should have

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(x - a_i)^n} = \frac{A_1}{x - a_1} + \frac{A_2}{(x - a_i)^2} + \cdots + \frac{A_n}{(x - a_i)^n}$$

Example Find the Partial Fraction Decomposition of $\dfrac{1}{x^2(x-1)}$

Solution. We want to express

$$\frac{1}{x^2(x-1)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x-1}$$

Then,

$$1 = A_1 x(x-1) + A_2(x-1) + A_3 x^2$$

For $x = 0$,

$$1 = A_1 x(x-1) + A_2(x-1) + A_3 x^2$$

For $x = 1$,

$$A_3 = 1$$

For $x = -1$,

$$1 = A_1(-1)(-2) + A_2(-2) + A_3 \implies 2A_1 = -2 \implies A_1 = -1$$

Thus,

$$\frac{1}{x^2(x-1)} = \frac{-1}{x} + \frac{-1}{x^2} + \frac{1}{x-1}$$

Example Evaluate the integral

$$I = \int \frac{1}{(x+4)^2(x-1)}$$

Solution. The Partial Fraction Decomposition of the integrand is,

$$\frac{1}{(x+4)^2(x-1)} = \frac{A_1}{x+4} + \frac{A_2}{(x+4)^2} + \frac{A_3}{x-1}$$
$$1 = A_1(x+4)(x-1) + A_2(x-1) + A_3(x+4)^2$$

For $x = -4$,

$$1 = -5A_2 \implies A_2 = -\frac{1}{5}$$

For $x = 1$,

$$1 = 25A_3 \implies A_3 = \frac{1}{25}$$

For $x = 0$,

$$
\begin{aligned}
1 &= A_1 (4)(-1) - A_2 + 4^2 A_3 \\
&= -4A - 1 - A_2 + 16A_3 \\
&= -4A_1 + \frac{1}{5} + \frac{16}{25} \\
&= -4A_1 + \frac{21}{25} \implies A_1 = \frac{-1}{25}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\int \frac{1}{(x+4)^2 (x-1)} &= \int -\frac{1}{25} \frac{1}{x+4} - \frac{1}{5} \frac{1}{(x+4)^2} + \frac{1}{25} \frac{1}{x-1} \, dx \\
&= -\frac{1}{25} \ln|x+4| + \frac{1}{5} \frac{1}{x+4} + \frac{1}{25} \ln|x-1| + C
\end{aligned}
$$

6.3.4 Reducible Quadratics

Recall that a *quadratic polynomial* is of the form $Q(x) = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}, a \neq 0$. Thus, polynomials such as $f(x) = 2x^2 - 1, g(x) = \frac{1}{2}x^2$ are quadratic.
A quadratic $P(x)$ is **reducible** if it has any real roots:

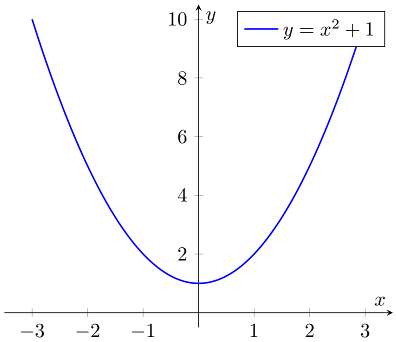
Example $x^2 = x \cdot x$ and $x^2 - 1 = (x+1)(x-1)$ are reducible.

Note that whenever you come across a reducible quadratic, you must factorize the quadratic and then apply the method of Partial Fraction Decomposition.

Example $\dfrac{1}{x^2 - 1} = \dfrac{1}{(x+1)(x-1)} = 1 - \dfrac{2}{x+1}$

6.3.5 Irreducible Quadratics

A quadratic $ax^2 + bx + c$ is called **irreducible** if it has no real roots.



This is to say, the function x^2+1 is always above the x-axis and so, $x^2+1>0$. Note that the graph of an irreducible quadratic doesn't necessarily have to be >0, if we flip the sign of x^2+1, then $-\left(x^2+1\right)<0$.
If we have an irreducible factor, then,

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{ax^2+bx+c} = \frac{A_1x+A_2}{ax^2+bx+c}$$

Remark By creating a system of linear equations, using Gaussian elimination can yield solutions of A_i for $i \in [1, n]$.

Example Evaluate the integral,

$$I = \int \frac{1+x}{x\left(x^2+1\right)}\,dx$$

Solution. The Partial Fraction Decomposition of the integrand is,

$$\begin{aligned} \frac{1+x}{x\left(x^2+1\right)} &= \frac{A_1}{x} + \frac{A_2x+A_3}{x^2+1} \\ 1+x = A_1\left(x^2+1\right) + A_2x^2 + A_3x \end{aligned}$$

For $x=0$,

$$1 = A_1 \iff A_1 = 1$$

For $x=1$,

$$2 = 2A_1 + A_2 + A_3 \implies A_2 + A_3 = 0 \tag{1}$$

For $x=-1$,

$$0 = 2A_1 + A_2 - A_3 \implies A_2 - A_3 = -2 \tag{2}$$

From (1) and (2),

$$\begin{cases} A_2 + A_3 = 0 \\ A_2 - A_3 = -2 \end{cases}$$

$2A_2 = -2 \implies A_2 = -1, A_3 = 1$ Thus,

$$\begin{aligned} \int \frac{1}{x\left(x^2+1\right)}\,dx &= \frac{1}{x} + \frac{1-x}{x^2+1}\,dx \\ &= \ln|x| + \int \frac{1}{x^2+1}\,dx - \int \frac{x}{x^2+1}\,dx \\ &= \ln|x| + \arctan x - \int \frac{x}{x^2+1}\,dx \end{aligned}$$

Let $u = x^2 \implies du = 2x\,dx \implies dx = \frac{1}{2x}\,du$,

$$\int \frac{x}{x^2+1}\,dx = \int \frac{x}{u+1}$$

$$
\begin{aligned}
&= \frac{1}{2}\int \frac{1}{u+1}\,du \\
&= \frac{1}{2}\ln|u+1| + C \\
&= \frac{1}{2}\ln|x^2+1| + C
\end{aligned}
$$

Hence,

$$\int \frac{1}{x^2\left(x+1\right)}\,dx = \ln|x| + \arctan x - \frac{1}{2}\ln|x^2+1|$$

6.3.6 Polynomial Long Division

If degree(P) $\geq$ degree(Q), we must peform *polynomial long division.*

$$\frac{P\left(x\right)}{Q\left(x\right)} = B(x) + \frac{R\left(x\right)}{Q\left(x\right)}$$

where $B\left(x\right)$ is the *quotient* polynomial, $R\left(x\right)$ is the *remainder* polynomial such that degree(R) $<$ degree(Q).

Example Evaluate the integral,

$$I = \int \frac{x^3-x-1}{x^2+x}\,dx$$

Solution. By Polynomial Long Division,

$$\frac{x^3-x+1}{x^2+x} = x-1+\frac{1}{x\left(x+1\right)}$$

So, we now want to express

$$
\begin{aligned}
\frac{1}{x\left(x+1\right)} &= \frac{A_1}{x} + \frac{A_2}{x+1} \\
1 &= A_1\left(x+1\right) + A_2 x
\end{aligned}
$$

For $x=0$,

$$A_1 = 1$$

For $x=-1$,

$$A_2 = -1$$

Thus,

$$
\begin{aligned}
\int \frac{x^3-x-1}{x^2+x}\,dx &= \int x-1+\frac{1}{x}-\frac{1}{x+1}\,dx \\
&= \frac{1}{2}x^2 - x + \ln|x| - \ln|x+1| + C
\end{aligned}
$$

7 Numerical Integration

There are functions whose integral we cannot evaluate in terms of elementary functions. A rather important integral

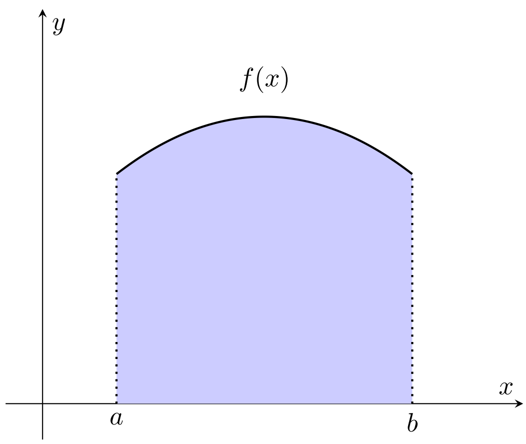
$$\int e^{-x^2}\,dx$$

cannot be expressed in elementary functions yet when evaluated on the domain $(-\infty, \infty)$,

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

by a clever use of the Jacobian transformation to convert from cartesian to polar coordinates. While this is not relevant for MATH 31B, we must find an alternate way to evaluate these special kinds of integrals.

We will try to *approximate* integrals. Recall that $\int_a^b f(x)$ has a geometric intepretation,



where

$$\int_a^b f(x)\,dx = \text{signed area of shaded region}$$

7.1 Midpoint Rule

Let $N \geq 1$ bean integer. We divide the interval $[a, b]$ into N subintervals $[x_i, x_{i+1}]$ of length $\dfrac{b-a}{N}$. We will now take the midpoint of each subinterval such that

$$c_i = \frac{x_{i-1} + x_i}{2}$$

Next, we will compute area of each midpoint rectangle

$$f(c_i) \cdot \frac{b-a}{N}$$

and sum the areas of the individual rectangles. It follows that

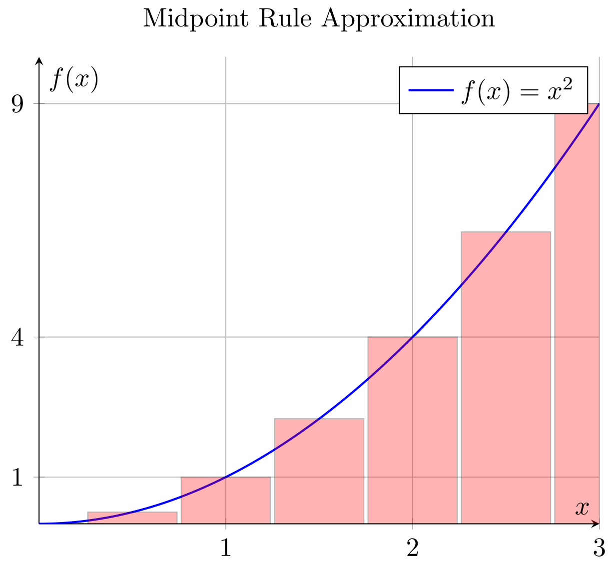
$$\mathcal{M}_n = f\left(c_1\right) \cdot \frac{b-a}{N} + f\left(c_2\right) \cdot \frac{b-a}{N} + \cdots + f\left(c_N\right) \cdot \frac{b-a}{N}$$

Let $\Delta x = \dfrac{b-a}{N}$, we have

$$\mathcal{M}_n = \Delta x \left(f\left(c_1\right) + \cdots + f\left(c_N\right)\right) = \Delta x \left(\sum_{i=1}^{N} f\left(c_i\right)\right)$$

Geometrically, the midpoint rule is

Midpoint Rule Approximation



$\mathcal{M}_n$ is the estimate of $\int_a^b f\left(x\right)\, dx$ using the midpoint rule and N subintervals. In general, the smaller the subintervals, the more accurate the estimation.

Example $f\left(x\right) = x^3$. Compute $\mathcal{M}_4$ for $\int_0^8 f\left(x\right) dx$
Solution. We divide $[0, 8]$ into 4 subintervals

$$\{[0,2], [2,4], [4,6], [6,8]\}$$

We then compute the midpoints of each subinterval as follows

$$\{1, 3, 5, 7\}$$

Then,

$$\mathcal{M}_4 = 2\left(f\left(1\right) + f\left(3\right) + f\left(5\right) + f\left(7\right)\right) = 2\left(1^3 + 3^3 + 5^3 + 7^3\right) = 992$$

7.2 Trapezoidal Rule

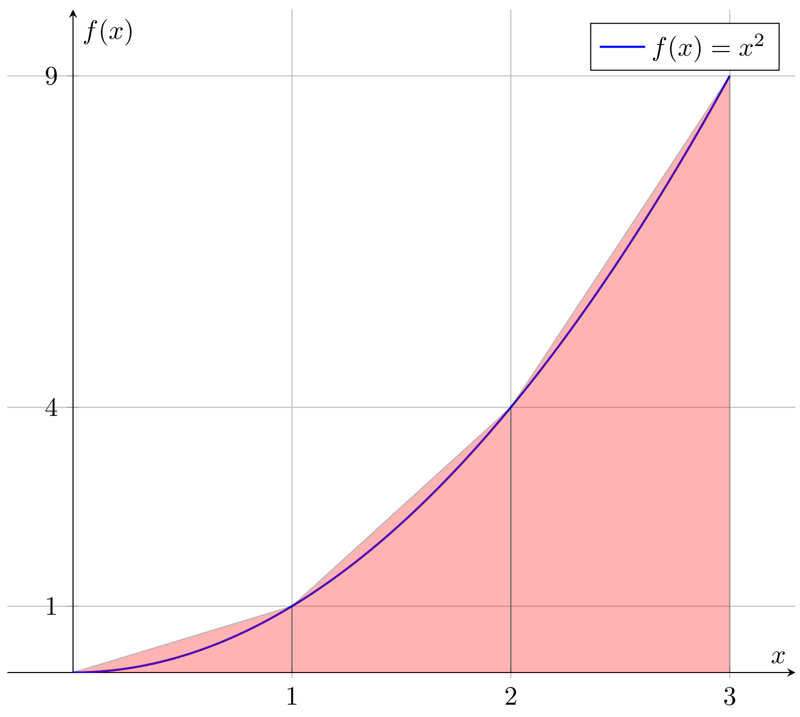
In the previous subsection, we used the *midpoint rule* to approximate $\int_a^b f(x)\,dx$, we will now see one two other ways to approximate.
Let $N \geq 1$ be an integer. Divide $[a, b]$ **evenly** into N subintervals $[x_i, x_{i+1}]$, We connect $(x_i, f(x_i))$ with $(x_{i+1}, f(x_{i+1}))$ with a straight line. Then, we compute the (signed) area of each *trapezoid*

$$\frac{f(x_i) + f(x_{i+1})}{2} \cdot \frac{b-a}{N}$$

and sum the areas of the individual trapezoids to achieve

$$\begin{aligned}\mathcal{T}_N &= \frac{1}{2} \cdot \frac{b-a}{N} \cdot ((f(x_0) + f(x_1)) + (f(x_1) + f(x_2)) + \cdots + (f(x_{N-1}) + f(x_N))) \\ &= \frac{1}{2}\Delta x\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{N-1}) + 2f(x_N)\right]\end{aligned}$$

Trapezoidal Rule Approximation



In general, the *midpoint rule* tends to serve as a more accurate representation of definie integrals compared to the *trapezoidal rule*.

7.3 Simpson's Rule

Let $N \geq 2$ be an **even** integer. We evenly divide $[a, b]$ into N subintervals $[x_i, x_{i+1}]$. For m even, find the parabola touching $(x_m, f(x_m)), (x_{m+1}, f(x_{m+1}))$

and $(x_{m+2}, f(x_{m+2}))$. We then calculate the signed area under the curve between x_m and x_{m+2}. As usual, we then sum over even m and thus,

$$\mathcal{S}_n = \frac{1}{3}\Delta x\left[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{N-2}) + 4f(x_{N-1}) + f(x_N)\right]$$

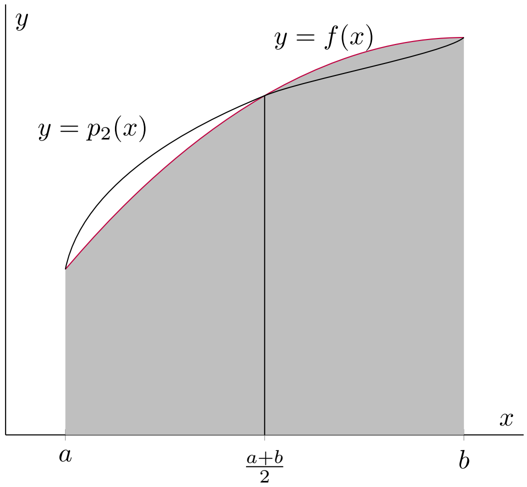


Figure 1: TikZ graph from TeX Stackexchange

For example,

$$\mathcal{S}_2 = \frac{1}{3}\Delta x\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$$\mathcal{S}_4 = \frac{1}{3}\Delta x\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$$

7.4 Error Bounds

Definition.

$$\text{error}(\mathcal{M}_n) = \left|\int_a^b f(x)\ dx - \mathcal{M}_n\right|$$

$$\text{error}(\mathcal{T}_n) = \left|\int_a^b f(x)\ dx - \mathcal{T}_n\right|$$

$$\text{error}(\mathcal{S}_n) = \left|\int_a^b f(x)\ dx - \mathcal{S}_n\right|$$

Theorem.

Suppose f'' exists and is continuous. Let K_2 be a number such that $|f''(x) \leq K_2 \ \forall x \in [a,b]$. Then

$$\text{error}(\mathcal{M}_n) \leq \frac{K_2 (b-a)^3}{24N^2}$$

$$\text{error}(\mathcal{T}_n) \leq \frac{K_2 (b-a)^3}{12N^2}$$

Recall that the midpoint rule tends to be a more accurate estimation of definite integrals. The error values above show that $\mathcal{M}_n$ is half of the error values of $\mathcal{T}_n$.

Theorem.

Suppose $f'''' = f^{(4)}$ exists and is continuous. Let K_4 be a number such that $|f^{(4)}(x) \leq K_4 \ \forall x \in [a,b]$. Then

$$\text{error}(\mathcal{S}_n) \leq \frac{K_4 (b-a)^5}{180N^4}$$

Example Let $f(x) = \ln x$. Find $K_2 = \max_{10 \leq x \leq 20} |f''(x)|$ and use K_2 to deduce a bound on error($\mathcal{M}_5$) for the integral $\int_{10}^{20} \ln x \, dx$.
Solution. We have $f(x) = \ln x$, so,

$$f'(x) = \frac{1}{x} \implies f''(x) = -\frac{1}{x^2} \implies |f''(x)| = \frac{1}{x^2}$$

We know that

$$\frac{d}{dx}\left(\frac{1}{x^2}\right) = -\frac{2}{x^2} < 0$$

if $x > 0$ so $|f''(x)|$ is decreasing between 10 and 20, hence,

$$K_2 = \max_{10 \leq x \leq 20} |f''(x)| = |f''(10)| = \frac{1}{10^2} = \frac{1}{100}$$

Thus,

$$\text{error}(\mathcal{M}_5) = \frac{K_2 (b-a)^3}{24N^2} = \frac{\frac{1}{100}(20-10)^3}{24 \cdot 5^2} = \frac{1}{60}$$

8 Specific Applications of Integration: Arc Length & Surface Area of Revolution

8.1 Arc Length

Let $f(x)$ be a function on $[a,b]$.

Theorem.

Assume $f'(x)$ exists and is continuous. Then the **arc length** of $f(x)$ over $[a,b]$ is given by

$$S = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Arc Length means the length of the graph.

8.2 Surface Area of Revolution

Let $f(x) \geq 0$ on $[a,b]$. We want to find the surface area of the surface obtained by revolving the graph of $f(x)$ about the x-axis.

Theorem.

Let $f(x) \geq 0$ on $[a,b]$. Suppose $f'(x)$ exists and is continuous. Then,

$$S = 2\pi \int_a^b f\left(x\sqrt{1 + (f'(x))^2}\right) dx$$

Example Find the surface area of a spherical cap of height h and radius R.

Solution. Given $S = 2\pi \int_{R-h}^{R} f(x) \sqrt{1 + (f'(x))^2}$ dx

We have $f\left(\sqrt{R^2 - x^2}\right)$ and so, $f'(x) = \dfrac{-x}{\sqrt{R^2 - x^2}}$. Hence,

$$\begin{aligned}
S &= 2\pi \int_{R-h}^{R} \sqrt{R^2 - x^2} \cdot \sqrt{1 + \frac{x^2}{R^2 - x^2}}\, dx \\
&= 2\pi \int_{R-h}^{R} \sqrt{R^2 - x^2} \cdot \sqrt{\frac{R^2 - x^2 + x^2}{R^2 - x^2}}\, dx \\
&= 2\pi \int_{R-h}^{R} \sqrt{R^2 - x^2} \cdot \frac{R}{\sqrt{R^2 - x^2}}\, dx \\
&= 2\pi \int_{R-h}^{R} R\, dx \\
&= 2\pi R x \Big|_{R-h}^{R} \\
&= 2\pi R h
\end{aligned}$$

9 Improper Integrals

An improper integral can be of two types. Type 1: domain of integration is unbounded. Type 2: Integrand is unbounded. Improper integrals may also be a combination of both, that is to say, the domain and the integrand can both be unbounded.

9.1 Type I Improper Integrals

Definition.

Suppose that the domain of integration is unbounded. Then,

$$\int_a^\infty f(x)\ dx = \lim_{R\to\infty}\int_a^R f(x)\ dx$$

if the limit exists. If the limit does not exist, then the integral *diverges*, otherwise, it *converges*.

Example Does the integral $\int_1^\infty \frac{1}{x^3}$ converge or diverge?
Solution.

$$\begin{aligned}\int_1^\infty \frac{1}{x^3}\,dx &= \lim_{R\to\infty}\int_1^R \frac{1}{x^3}\,dx\\ &= \lim_{R\to\infty}\left[\frac{-1}{2x^2}\right]_1^R\\ &= \lim_{R\to\infty}\left(-\frac{-1}{2R^2}+\frac{1}{2}\right)\\ &= \frac{1}{2}\end{aligned}$$

Hence,

$$\int_1^\infty \frac{1}{x^3}\,dx$$

converges.

Example Does the integral $\int_1^\infty \frac{1}{x^{0.5}}\,dx$ converge or diverge?
Solution.

$$\begin{aligned}\lim_1^\infty \frac{1}{x^{0.5}}\,dx &= \lim_{R\to\infty}\int_1^R \frac{1}{x^{0.5}}\,dx \qquad (1)\\ &= \lim_{R\to\infty}\left[2x^{0.5}\right]_1^R\\ &= \lim_{R\to\infty}\left(2R^{0.5}-2\right)\\ &= \infty\end{aligned}$$

So, the limit diverges, hence, the integral diverges.

Example Does the integral $\int_1^\infty \frac{1}{x}\,dx$ converge or diverge?

Solution.

$$\begin{aligned}\int_1^\infty \frac{1}{x}\,dx &= \lim_{R\to\infty}\int_1^R \frac{1}{x}\,dx\\ &= \lim_{R\to\infty} \ln x\Big|_1^R\\ &= \lim_{R\to\infty} \ln R - \ln 1\\ &= \infty\end{aligned}$$

So, the limit diverges, hence, the integral diverges.

These examples will now follow an important theorem.

Theorem. *(p-test)*

Let $a > 0$. Then,

$$\int_a^\infty \frac{1}{x^p}\,dx = \begin{cases}\dfrac{a^{1-p}}{p-1} & \text{if } p > 1\\ \infty & \text{if } 0 < p \le 1\end{cases}$$

i.e., the integral converges if $p > 1$ and diverges if $0 < p \le 1$.

Definition.

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{0} f(x)\,dx + \lim_{R\to\infty}\int_0^R f(x)\,dx$$

Do not make the mistake of

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx$$

This is **incorrect**. To emphasize the difference in solutions that can occur should you make the mistake of choosing the incorrect bounds for improper integrals, consider the following example.

Example Evaluate the integral $\lim_{-\infty}^{\infty} x\,dx$.
Solution. We will use the **correct** approach to solving this integral first.

$$\begin{aligned}\int_{-\infty}^{\infty} x\,dx &= \lim_{R\to\infty}\int_{-R}^{0} x\,dx + \lim_{R\to\infty}\int_{0}^{R} x\,dx\\ &= \lim_{R\to\infty}\left[\frac{1}{2}R^2\right] + \lim_{R\to\infty}\left[-\frac{1}{2}R^2\right]\end{aligned}$$

Both of these limits diverge and so, the integral diveges. Let's see the solution with the **incorrect** approach.

$$\begin{aligned}\int_{-\infty}^{\infty} x\,dx &= \lim_{R\to\infty}\int_{-R}^{R} x\,dx\\ &= \left[\frac{1}{2}R^2 - \frac{1}{2}R^2\right] = 0\end{aligned}$$

We can see how the two approaches yield two different solutions. It's important to be careful when solving improper integrals.

Theorem. *(Comparison Test)*

Assume $f(x) \geq g(x) \geq 0\ \forall x \geq a$. Then,
(1) If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges.
(2) If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges.
This is also true for $\int_{-\infty}^{a}$.

The intuition behind the comparison theorem is that if a larger function is able to converge, it follows that since the smaller function already takes on values much faster than the larger one, it should also converge. We can see this very clearly with the p-test. If $\frac{1}{x^2}$ is able to converge, $\frac{1}{x^3}$, a function that is smaller and is compatible with the comparison test converges faster. Likewise, if we know that $\frac{1}{x}$ doesn't converge fast enough and so, diverges, we can conclude that a larger function $\frac{1}{x^{0.5}}$ will also not converge.

Example Does $\int_1^\infty \dfrac{1}{x^2+\ln x}\,dx$ converge?
Solution.

$$0 \leq \frac{1}{x^2+\ln x} \leq \frac{1}{x^2}$$

$\forall x \geq 1$. We know that

$$\int_1^\infty \frac{1}{x^2}\,dx$$

converges by the p-test. So the comparison test implies

$$\int_1^\infty \frac{1}{x^2 + \ln x}\, dx$$

converges.

In general, you want to find which parts of f are most important for convergence of f and then, find a suitable function to compare with f.

Example Does the integral $\int_3^\infty \dfrac{1}{x^{5/2}+1}\, dx$ converge or diverge?

Solution. Notice that the expression $\frac{1}{x^{5/2}+1}$ is almost like $\frac{1}{x^2}$ for x large. The $+1$ has negligible effect on the function. Notice that on $[1, \infty)$ for a number K,

$$\frac{1}{x^{5/2}+1} \leq \frac{K}{x^{5/2}} \iff x^{5/2} \leq K\left(x^{5/2}+1\right)$$

true for $K = 1$. So,

$$\int_1^\infty \frac{1}{x^{5/2}+1}\, dx$$

converges by comparison with $\int \dfrac{1}{x^{5/2}}\, dx$ which converges by the p-test.

Try checking whether the integral

$$\int_1^\infty \frac{1}{x + e^{2x}}\, dx$$

converges or diverges. *Hint: Since* $\dfrac{1}{x}$ *diverges by p-test, it's not the suitable comparison function.*

9.2 Type II Improper Integrals

Definition.

Let $f(x)$ be continuous on $(a, b]$ and $\lim_{x\to a^+} f(x) = \pm\infty$. We define

$$\int_a^b f(x)\, dx = \lim_{R\to a^+} \int_R^a f(x)\, dx$$

if the limit exists. The integral *diverges* if the limit does not exist, otherwise, *converges*.

Example Does $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converge or diverge?

Solution.

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}}\,dx &= \lim_{R\to 0^+} \int_R^1 \frac{1}{\sqrt{x}}\,dx \\ &= \lim_{R\to 0^+} [2]_R^1 \\ &= \lim_{R\to 0^+} 2 - 2\sqrt{R} \\ &= 2\end{aligned}$$

Hence, the integral converges.

We will now study integrals with bounds $[0, a)$.

Theorem. *(p-test)*

Let $a > 0$, then,

$$\int_0^a \frac{1}{x^p} = \begin{cases} \dfrac{a^{1-p}}{1-p} & \text{if } 0 < p < 1 \\ \text{diverges} & \text{if } P \geq 1 \end{cases}$$

Do note that we are going from 0 to a this time, not a to ∞. Also note that in both cases, when $p = 1$, the integral diverges.

Theorem. *(Comparison Test)*

If $f(x) \geq g(x) \geq 0$, then,

(1) If $\int_a^b f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges.

(2) If $\int_a^b f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges.

All this says is that the comparison test also works for integrals with bounded domains.

Example Does the integral $\int_0^1 \frac{1}{(x+x^2)^{1/3}}\,dx$ converge or diverge?

Solution. As $x \to 0^+$, x is bigger than x^2, ideally we want to forget the x^2 term. By the p-test, $\int_0^1 \frac{1}{x^{1/3}}\,dx$ converges. So we suspect the integral converges as well. We want to find a number K such that on $(0, 1]$

$$\frac{1}{(x+x^2)^{1/3}} \leq \frac{K}{x^{1/3}} \iff x^{1/3} \leq K\left(x+x^2\right)^{1/3}$$

This is true when $K = 1$ so,

$$\int_0^1 \frac{1}{(x+x^2)^{1/3}}\, dx$$

converges.

As $x \to \infty$,

$$e^x >> x^p \text{ (p large) } >> x^p \text{ (p small) } >> \ln x, \log_b(x) >> \text{constants}$$

As $x \to 0^+$,

$$x^p \text{ (p small)} >> x^p \text{ (p large) } >> \ln x, \log_b(x) >> \text{constants}$$

Example Does the integral $\int_0^1 \dfrac{1}{2x^2+5x}\, dx$ converge or diverge?

Solution. Near 0, this function behaves like $\frac{1}{5x}$ which is divergent. So we suspect the integral diverges. On $(0, 1]$, for K such that

$$\frac{1}{2x^2+5x} \geq \frac{K}{5x} \iff 5x \geq K\left(2x^2+5x\right)$$

Note that up to this point, we have always had the above equaly true if $K = 1$; however, this is **false** if $K = 1$ but **true** if $K = \frac{1}{2}$. By p-test,

$$\int_0^1 \frac{\frac{1}{2}}{5x}\, dx = \frac{1}{10}\int_0^1 \frac{1}{x}\, dx$$

diverges. So, by comparison with the above,

$$\int_0^1 \frac{1}{2x^2+5x}\, dx$$

diverges.

Part 3: Sequences & Series

10 Sequences

We will now begin with studying sequences and series.

Definition. *Sequence*

A **sequence** is an ordered list of infinitely many numbers. It is denoted by $\{a_n\}_{n=1}^{\infty}$.
The value a_n is called the *term* and n is called an *index*.

When there is a formula for every a_n as a function of n, a_n is called the *general term*.
Interestingly, understanding sequences is very important to the study of analysis. If you will be taking MATH 32AH or MATH 131/131AH in the future, understanding sequences is pivotal.

Example Consider the following examples,

$$\left\{1, \frac{1}{1}, \frac{1}{4}, \frac{1}{8}, \ldots\right\} = \left\{\frac{1}{2^n}_{n=0}^{\infty}\right\}$$

$$\{1, 2, 3, \ldots\} = \{n\}_{n=1}^{\infty}$$

$$\{0, 1, 2, 3, \ldots\} = \{n\}_{n=0}^{\infty}$$

Note that the first index is usually 0 or 1, however, this is not a hard requirement.

A sequence can be generated by a function $f(x)$, that is to say, we can represent functions as simply an ordered list of numbers.

Example If $f(x) = \ln x$, we can define $a_n = f(n) = \ln n$.

$$a_1 = \ln 1 = 0$$
$$a_2 = \ln 2$$
$$a_3 = \ln 3$$

Definition. *Convergence*

The sequence $\{a_n\}_{n=1}^{\infty}$ is **convergent** to a limit L (a finite number) if for any number $\varepsilon > 0$, there exists N such that

$$|a_n - L| < \varepsilon$$

for every $n > N$. Otherwise, $\{a_n\}_{n=1}^{\infty}$ is **divergent**.

A quick note on notation, **converge** can be written as

$$\lim a_n = L \quad \text{or} \quad \lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \longrightarrow L \, (\text{as } n \longrightarrow \infty)$$

In lay terms, a sequence *converges* to L if it gets closer and closer to L. We can rewrite the definition of the convergence to be more precise (and by extension, more rigorous). As many of you will study calculus on $\mathbb{R}^n$ in the future, writing precise statements will be important.

Definition. *Sequence (2nd Version)*

$a_\nu \longrightarrow L$ means $\forall \varepsilon > 0$, $\exists \nu_0 > 0$ such that (s.t.) $\nu > \nu_0 \implies |a_\nu - L| < \varepsilon$.

Theorem.

Let $f(x)$ be a function and $a_n = f(n)$.

(i) If $\lim_{x\to\infty} f(x)$ exists, then the sequence $\{a_n\}_{n=1}^{\infty}$ is convergent to $\lim_{x\to\infty} f(x)$

(ii) If $\lim_{x\to\infty} f(x) = \pm\infty$, then the sequence $\{a_n\}_{n=1}^{\infty}$ is divergent (a_n diverges to) $\pm\infty$.

Example Does $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ diverge?

Solution. If $f(x) = \frac{1}{x}$, then $a_n = f(n)$, and $\lim_{x\to\infty} \frac{1}{x} = 0$ so $\{a_n\}_{n=1}^{\infty}$ converges to 0.

Example Does $\{a_n\}_{n=1}^{\infty} = \left\{\frac{n + \ln(n)}{n^2}\right\}_{n=1}^{\infty}$ converge?

Solution. Let $f(x) = \frac{x + \ln(x)}{x^2}$. So, $a_n = f(n)$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x + \ln x}{x^2} = (\text{L'Hopital's}) \lim_{x\to\infty} \frac{1 + \frac{1}{x}}{2x} = \frac{1}{\infty} = 0$$

So $\{a_n\}_{n=1}^{\infty}$ converges to 0.

10.1 Geometric Sequences

$\{cr^n\}_{n=1}^{\infty}$ is called a *geometric sequence* for $c \neq 0$ a constant.

$$\{cr^n\}_{n=0}^{\infty} = \{c, cr, cr^2, cr^3, \cdots\}$$

where r is called the *common ratio*.

Theorem.

If $-1 < r \leq 1$, then the sequence is *convergent*. If $r \geq -1$ or $r > 1$, the sequence is divergent.

The intitution behind this theorem is that if $r = 1$, the sequence is constant. If $-1 < r < 1$, the sequence gets smaller and smaller towards 0, otherwise, the sequence diverges.

Theorem. *(Squeeze Theorem)*

Let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}, \{c_n\}_{n=1}^{\infty}$ be sequences and $a_n \leq b_n \leq c_n$ $\forall n > M$, where M is some integer. If

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$$

then,

$$\lim_{n\to\infty} b_n = L$$

Example Determine if $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$ converges.

Solution. We know that

$$-1 \leq (-1)^n \leq 1$$

$\forall n$. So,

$$-\frac{1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n}$$

Hence, since

$$\lim_{x\to\infty} -\frac{1}{x} = \lim_{x\to\infty} \frac{1}{x} = 0$$

By Squeeze,

$$\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$$

converges to 0.

Theorem.

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two convergent sequences with

$$\lim_{n\to\infty} a_n = L$$
$$\lim_{n\to\infty} b_n = M$$

Then,

(1) $\lim_{n\to\infty} a_n \pm b_n = L \pm M$
(2) $\lim_{n\to\infty} a_n b_n = LM$
(3) $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{L}{M}$ if $M \neq 0$
(4) $\lim_{n\to\infty} ca_n = cL$ for any constant c

Remark If $\{d_n\}_{n=1^{\infty}}$ is divergent, so is

(1) $\{d_n + c\}_{n=1}^{\infty}$ where c is a constant

(2) $\{cd_n\}_{n=1}^{\infty}$ where $c \neq 0$ is a constant

Theorem.

If $f(x)$ is continuous and $\lim_{n\to\infty} a_n = L$, then $\{f(a_n)\}_{n=1}^{\infty}$ is convergent and converges to $f(L)$.

Does $\left\{\ln\left(\frac{12n+2}{4n-9}\right)\right\}_{n=3}^{\infty}$ converge?

Solution. Let $f(x) = \ln x$. $a_n = \frac{2n+2}{4n-9}$. So,

$$f(a_n) = \ln\left(\frac{12n+2}{4n-9}\right)$$

Let $g(x) = \frac{12x+2}{4x-9}$. So,

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{12x+2}{4x-9} = \lim_{x\to\infty} \frac{12}{4} = 3$$

By now, you should notice when L'Hôpital's has been applied. Continuing,

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} g(x) = 3$$

Thus,

$$\lim_{n\to\infty} \ln\left(\frac{12n+2}{4n-9}\right) = \ln\left(\lim_{n\to\infty} a_n\right) = \ln(3)$$

Another way, perhaps even simpler method to get $\lim_{n\to\infty} a_n$ is as follows

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{12n+2}{4n-9} = \lim_{n\to\infty} \frac{12+\frac{2}{n}}{4-\frac{9}{n}} = \frac{12}{4} = 3$$

Either method is acceptable. The example is simply there to demonstrate the theorem.

Theorem.

If $\{|a_n|\}_{n=1}^{\infty}$ is convergent and converging to 0, then $\{a_n\}_{n=1}^{\infty}$ is convergent and converging to 0.

Proof. We know that

$$-|a_n| \leq a_n \leq |a_n|$$

$\forall n$. We also know that

$$\lim_{n\to\infty} -|a_n| = \lim_{n\to\infty} |a_n| = 0$$

Thus, by squeeze, $\lim_{n\to\infty} a_n = 0$. $\square$

11 Series

A series is of the form

$$\sum_{n=a}^{N} a_n = a_1 + a_2 + \cdots + a_N$$

In particlar, $n = a$ signifies the starting index and N signifies when the indexing stops. Consider the following:

$$\begin{aligned}
\sum_{n=1}^{5}(2n-1) &= (2\cdot 1-1)+(2\cdot 2-1)+(2\cdot 3-1)+(2\cdot 4-1)+(2\cdot 5-1) \\
&= 1+3+5+7+9 \\
&= 25
\end{aligned}$$

Example

$$\sum_{n=1}^{\infty} n = 1+2+3+4+5+\cdots$$

Definition.

A **series** is a sum of infinitely many terms and is written

$$\sum_{n=k}^{\infty} a_n = a_k + a_{k+1} + a_{k+2} + \cdots$$

Example

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

is a series.

For the above example, let's see what happens if we add up the first few terms.

$$\begin{aligned}
1 &= 1 \\
1 + \frac{1}{2} &= 1.5 \\
1 + \frac{1}{2} + \frac{1}{3} &\approx 1.83 \\
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} &\approx 2.08
\end{aligned}$$

If we form a sequence of these sums,

$$\{1, 1.5, 1.83, 2.08, \cdots\}$$

we will soon show that the sequence diverges. If we do the same for the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$,

$$\begin{aligned} 1 &= 1 \\ 1 + \frac{1}{2} &= \frac{3}{2} \\ 1 + \frac{1}{2} + \frac{1}{4} &= \frac{7}{4} \\ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} &= \frac{15}{8} \end{aligned}$$

If we a form a sequence of these sums,

$$\left\{1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \cdots\right\}$$

it appears that it converges to 2. We will show it does in fact converge to 2.

Definition.

A series $\sum_{n=k}^{N} a_n$ is *convergent* (to L) if the partial sums $S_n = \sum_{n=K}^{N} a_n$ are convergent (to L) as a sequence. Otherwise, the series is called *divergent*.

Example The following is an example of a **Telescoping Series**.
Does $\sum_{n=1}^{\infty} \dfrac{1}{n(n+1)}$ converge?
Solution. We will use the method of Partial Fraction Decomposition.

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} \implies 1 = A(x+1) + B(x)$$

It's immediate that $A = 1, B = -1$, so,

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

The parial sums are

$$\begin{aligned} S_N = \sum_{n=1}^{N} \frac{1}{n(n+1)} &= \sum_{n=1}^{N} \frac{1}{n} - \frac{1}{n+1} \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right) \\ &= 1 - \frac{1}{N+1} \end{aligned}$$

$$\lim_{n\to\infty} S_N = \lim_{N\to\infty} 1 - \frac{1}{N+1} = 1$$

So, $\sum_{n=1}^{\infty} \dfrac{1}{n(n+1)} = 1$. In particular, the sum converges.

Theorem.

If $\sum_{n=k}^{\infty} a_n$ and $\sum_{n=k}^{\infty} b_n$ are convergent series, and c is a constant, then
(1) $\sum_{n=k}^{\infty} a_n \pm b_n = \sum_{n=k}^{\infty} a_n \pm \sum_{n=k}^{\infty} b_n$
(2) $\sum_{n=k}^{\infty} ca_n = c\sum_{n=k}^{\infty} a_n$

11.1 Geometric Series

Definition.

A *geometric series* is of the form

$$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + cr^3 + \cdots$$

where $c \neq 0$ and r is called the *common ratio.*

Recall that if $r \neq 1$,

$$1 + r + r^2 + \cdots + r^N = \sum_{n=0}^{N} r^n = \frac{1 - r^{N+1}}{1 - r}$$

Theorem.

Let $c \neq 0$. Then

$$\sum_{n=0}^{\infty} cr^n = \begin{cases} \text{convergent, equal to } \dfrac{c}{1-r} & \text{if } |r| < 1 \\ \text{divergent} & \text{if } |r| \geq 1 \end{cases}$$

Example Solve for the sum $\sum_{n=0}^{\infty} \dfrac{1}{2^n}$

Solution.

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1 - \frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

since $r = \frac{1}{2} \implies |r| < 1$. Thus, the series converges.

Example Solve for the sum $\sum_{n=3}^{\infty} 7\left(-\dfrac{3}{4}\right)^n$

Solution.

$$\begin{aligned}\sum_{n=3}^{\infty} 7\left(-\frac{3}{4}\right)^n &= 7\left(-\frac{3}{4}\right)^3 + 7\left(-\frac{3}{4}\right)^4 + 7\left(-\frac{3}{4}\right)^5 + \cdots \\ &= \sum_{n=0}^{\infty} 7\left(-\frac{3}{4}\right)^n\left(-\frac{3}{4}\right)^n\end{aligned}$$

$$= \frac{7\left(-\frac{3}{4}\right)^n}{1-\left(-\frac{3}{4}\right)}$$
$$= -\frac{27}{16}$$

Since $r = -\frac{3}{4} \implies |r| < 1$. Thus, the series converges.

Theorem. *(Divergence Test)*

If $\{a_n\}_{n=1}^{\infty}$ is divergent or $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=k}^{\infty} a_n$ diverges.

Do note that $\lim_{n\to\infty} a_n = 0$, this information tells us nothing valuable.

Example $\sum_{n=1}^{\infty} 1 - \dfrac{1}{n}$ diverges because $\lim_{n\to\infty} \left(1 - \dfrac{1}{n}\right) = 1 \neq 0$.

Example The divergence test **fails** to tell if $\sum_{n=1}^{\infty} \frac{1}{n}$ converges or not because $\lim_{n\to\infty} \frac{1}{n} = 0$.

We will see later that the sum in fact diverges.

11.2 Positive Series

Definition.

A *positive series* is of the form

$$\sum_{n=k}^{\infty} a_n$$

where $a_n > 0 \forall n$.

Positive series are similar to type I improper integrals.

Theorem. *(Integral Test)*

Let $a_n = f(n)$. If $f(x)$ is *positive*, *decreasing*, and *continuous* for $x \geq 1$, then

(1) If $\int_1^{\infty} f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
(2) If $\int_1^{\infty} f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example Does $\sum_{n=1}^{\infty} \dfrac{1}{n^2+1}$ converge or diverge?

Solution. Let $f(x) = \frac{1}{x^2+1}$. Then $a_n = f(n) = \frac{1}{n^2+1}$. We know that $f(x) > 0$ since $x^2 + 1 > 0$. We also know that $f(x)$ is strictly decreasing because $f'(x) = \frac{-2x}{(x^2+1)^2} < 0$ when $x \geq 1$. We also know that $f(x)$ is continuous. Thus,

$$\int_1^{\infty} \frac{1}{x^2+1}\, dx$$

by comparison test yields

$$0 \leq \frac{1}{x^2+1} \leq \frac{1}{x^2}$$

By p-test, we know that $\int_1^{\infty} \frac{1}{x^2}\, dx$ converges and so, by comparison, $\int_1^{\infty} \frac{1}{x^2+1}\, dx$ converges. Thus, by integral test, $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.

Theorem. *(p-series/p-test)*

If $p > 1$, then

$$\sum_{n=k}^{\infty} \frac{1}{n^p}$$

is convergent. If $0 < p \leq 1$, then

$$\sum_{n=k}^{\infty} \frac{1}{n^p}$$

is divergent.

As claimed earlier, the above theorem suggests that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Theorem. *(Comparison Test)*

If $0 \leq a_n \leq b_n$ $\forall n = 1, 2, 3, \cdots$. Then
(1) If $\sum_{n=k}^{\infty} b_n$ is convergent, then $\sum_{n=k}^{\infty} a_n$ is convergent
(2) If $\sum_{n=k}^{\infty} a_n$ is divergent, then $\sum_{n=k}^{\infty} b_n$ is divergent

Note that finding good comparisons is similar to the comparison test for type I improper integrals.

Example Does $\sum_{n=2}^{\infty} \frac{1}{n^2+\ln n}$ converge or diverge?
Solution. By comparison,

$$0 \leq \frac{1}{n^2+\ln n} \leq \frac{1}{n^2}$$

$\forall n \geq 2$. By p-test,

$$\sum_{n=2}^{\infty} \frac{1}{n^2}$$

is convergent. Thus,

$$\sum_{n=2}^{\infty} \frac{1}{n^2+\ln n}$$

is convergent.

Theorem. *(Limit Comparison Test)*

Let $\sum_{n=k}^{\infty} a_n$ and $\sum_{n=k}^{\infty} b_n$ be two positive series. Assume $L = \lim_{n\to\infty} \frac{a_n}{b_n}$ exist (or is ∞).

(1) If $0 < L < \infty$, then $\sum_{n=k}^{\infty} a_n$ and $\sum_{n=k}^{\infty}$ either both converge or diverge.
(2) If $L = \infty$ and $\sum_{n=k}^{\infty} a_n$ converges, then $\sum_{n=k}^{\infty} b_n$ converges
(3) If $L = 0$ and $\sum_{n=k}^{\infty} b_n$ converges, then $\sum_{n=k}^{\infty} a_n$ converges.

The Limit Comparison Test is often an easier alternative to the Comparison Test.

Example Does $\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^3+n}$ converge or diverge?
Solution. Let $a_n = \frac{n^2+n+1}{n^3+n}$. Let $b_n = \frac{n^2}{n^3} = \frac{1}{n}$. Then,

$$\begin{aligned} L = \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{n^2+n+1}{n^3+n}}{\frac{n^2}{n^3}} = \lim_{n\to\infty} \frac{\frac{n^2+n+1}{n^2}}{\frac{n^3+n}{n^3}} \\ &= \lim_{n\to\infty} \frac{1+\frac{1}{n}+\frac{1}{n^2}}{1+\frac{1}{n^2}} \\ &= 1 \end{aligned}$$

Since $0 < L < \infty$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-test, then so does $\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^3+n}$ diverge.

Example Does $\sum_{n=1}^{\infty} \dfrac{2^n}{3^n - \ln n}$ converge or diverge?

Solution. Let $a_n = \dfrac{2^n}{3^n - \ln n}$. Let $b_n = \dfrac{2^n}{3^n}$. Then,

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2^n}{3^n - \ln n}}{\frac{2^n}{e^n}} = \lim_{n\to\infty} \frac{1}{1 - \frac{\ln n}{3^n}} \\ &= \frac{1}{1 - \lim_{n\to\infty} \frac{\ln n}{3^n}} = \frac{1}{1 - \lim_{n\to\infty} \frac{\frac{1}{n}}{3^n \ln 3}} = \frac{1}{1 - \lim_{n\to\infty} \frac{1}{n3^n \ln n}} \\ &= \frac{1}{1-0} = 1 \end{aligned}$$

We know that $\sum_{n=1}^{\infty} \left(\frac{2^n}{3^n}\right) = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ converges since $|r| = \frac{2}{3} < 1$. Thus, by the limit comparison test, $\sum_{n=1}^{\infty} \frac{2^n}{3^n - \ln n}$ also converges.

11.3 Absolute and Conditional Convergence

What happens when a series is not positive? For example, does $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converge?

Definition.

A series $\sum_{n=k} a_n$ is called *absolutely convergent* if the series $\sum_{n=k}^{\infty} |a_n|$ is convergent.

Example $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is not absolutely convergent because $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ which diverges by p-test.

Example $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutely convergent because $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges by p-test.

Theorem.

Absolutely convergent series converge.

Definition.

A series $\sum_{n=k}^{\infty} a_n$ is called *conditionally convergent* if it is convergent $\sum_{n=k}^{\infty} |a_n|$ is divergent.

Remark Divergent, conditionally convergent, and absolutely convergent are mutually exclusive notions. A series can only be one of these. Both conditionally and absolutely convergent series converge.

Also note that a positive series $\sum_{n=k}^{\infty} a_n$ cannot be conditionally convergent. If $\sum_{n=k}^{\infty} a_n$ converges and is a positive series, then $\sum_{n=k}^{\infty} |a_n| = \sum_{n=k}^{\infty} a_n$ converges, so the series is absolutely convergent, not conditionally.

11.4 Alternating Series Test

Assume that $\{a_n\}_{n=k}^{\infty}$ is a sequence satisfying

(1) $a_1 > a_2 > a_3 > \cdots > 0$

(2) $\lim_{n\to\infty} a_n = 0$

Then $\sum_{n=k}^{\infty} (-1)^{n-1} a_n$ and $\sum_{n=k}^{\infty} (-1)^{n} a_n$ converge.

Example Does $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converge or diverge?
Solution. Set $a_n = \frac{1}{n}$. We know that $a_1 > a_2 > a_3 > \cdots > 0$ and $\lim_{n\to\infty} \frac{1}{n} = 0$. So, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by Alternating Series Test.

Example Does $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ absolutely converge, conditionally converge, or diverge?
Solution. First, let's show that the series is convergent. Let $a_n = \frac{1}{n \ln n}$. We know that $a_n > 0$ and $a_n > a_{n+1}$ since $\frac{1}{n \ln n} > \frac{1}{(n+1)\ln(n+1)}$. We also know that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n \ln n} = 0$. So $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges by Alternating Series Test. Now we must check if $\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{n \ln n}\right| = \sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges to determine if the series absolutely convergent. Let $f(x) = \frac{1}{x \ln x}$ for $x \geq 2$. We see that $f(x)$ is positive, decreasing and continuous. So, $\int_2^{\infty} f(x)\, dx$ and $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ either both converge or diverge by Integral test.

$$\int_2^{\infty} f(x)\, dx = \int_2^{\infty} \frac{1}{x \ln x}\, dx$$

Let $u = \ln x \implies du = \frac{1}{x}\, dx \implies dx = x\, du$. So,

$$\begin{aligned}
\int \frac{1}{x \ln x} &= \int \frac{1}{xu} \cdot x\, du \\
&= \int \frac{1}{u}\, du \\
&= \ln|u| = \ln|\ln x| + C
\end{aligned}$$

So,

$$\begin{aligned}
\int_2^{\infty} \frac{1}{x \ln x}\, dx &= \lim_{R\to\infty} \int_2^{R} \frac{1}{x \ln x}\, dx \\
&= \lim_{R\to\infty} \left(\ln|\ln x|\right)\Big|_2^R
\end{aligned}$$

$$= \lim_{R\to\infty} (\ln|\ln R| - \ln|\ln 2|) = \infty$$

So $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges by integral test. Hence, $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ is conditionally convergent.

The above example should demonstrate how questions may involve topics from integration as well in sequences & series.

11.5 Root & Ratio Tests

Theorem. *Ratio Test*

For any series $\sum_{n=}^{\infty} a_n$, assume that

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

exists or is ∞.

(1) If $0 \le \rho \le 1$, then $\sum_{n=k}^{\infty} a_n$ is absolutely convergent.
(2) If $1 < \rho \le \infty$, then $\sum_{n=k}^{\infty} a_n$ is divergent.
(3) If $\rho = 1$, the ratio test is inconclusive.

The intuition behind the ratio test is that if $0 \le \rho < 1$ then the series behaves like a convergent geometric series. If $1 < \rho \le \infty$, the series behaves like a divergent geometric series.

We will now introduce the concept of factorials very briefly. For positive integers n,

$$n! = n\,(n-1)\,(n-2)\,(n-3)\cdot\cdots\cdot 3\cdot 2\cdot 1$$

Do note that $0! = 1$. If you're interested in reading why, take a look at the Gamma Function.

Example Does $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converge or diverge?
Solution.

$$\begin{aligned} \rho &= \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \\ &= \lim_{n\to\infty} \frac{\left|\frac{2^{n+1}}{(n+1)!}\right|}{\left|\frac{2^n}{n!}\right|} \\ &= \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n} \\ &= \lim_{n\to\infty} \frac{2}{n+1} \\ &= 0 \end{aligned}$$

So the ratio test suggests that the series is absolutely convergent.

Theorem. *Root Test*

For any series $\sum_{n=k}^{\infty} a_n$ $(k \geq 1)$, assume

$$L = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$

exists (or is ∞). Then

(1) If $0 \leq L < 1$, then $\sum_{n=k}^{\infty} a_n$ is absolutely convergent.
(2) If $1 < L \leq \infty$, then $\sum_{n=k}^{\infty} a_n$ is divergent.
(3) If $L = 1$, the root test is inconclusive.

You will notice that the ratio test and the root test are quite similar. The conditions should be easy to remember.

Example Does $\sum_{n=1}^{\infty} \left(\dfrac{n}{3n+5}\right)^n$ converge or diverge?

Solution. Let $a_n = \left(\frac{n}{3n+5}\right)^n$.

$$\begin{aligned} L &= \lim_{n\to\infty} |a_n|^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \left|\left(\frac{n}{3n+5}\right)^n\right|^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{n}{3n+5} \\ &= \lim_{n\to\infty} \frac{1}{3} = \frac{1}{3} < 1 \end{aligned}$$

Hence, the series converges.

If $a_n = f(n)^{g(n)}$, the root test is a good idea to use.

12 Power Series

Definition.

A *power series* is an infinite sum of functions of the form $\sum_{n=0}^{\infty} a_n (x-c)^n$ where c is called the *center* and a_n's are constants.

Example

$$\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!} = \frac{1}{0!} + \frac{(x-1)}{1!} + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \cdots$$

is a power series with $c = 1$, $a_n = \frac{1}{n!}$.

Note that a power series never has terms for negative n. But it might have some $a_n = 0$ for $n \geq 0$.

Example

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

is a power series where $a_0 = 0, a_1 = 1, a_2 = 0, a_3 = -\dfrac{1}{3}, \cdots$.

Example

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is a power series. Let $x = 1$, so $f(1) = \sum_{n=0}^{\infty} \dfrac{1^n}{n!} = \sum_{n=0}^{\infty} \dfrac{1}{n!} = \sum_{n=0}^{\infty} a_n$.

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty} \frac{1}{(n+1)!} \cdot \frac{n!}{1} = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

By the ratio test, $0 \leq \rho < 1$ implies $\sum_{n=1}^{\infty} a_n = f(1)$ is absolutely convergent.

This is a rather long method, we will now introduce a rather faster method to check for which x a power series converges.

Theorem.

For any power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ there exists a number $R \geq 0$ which is the *radius of convergence*, such that

(1) If $0 \leq R < \infty$, then the power series converges absolutely for $x \in (c-R, c+R)$ and diverges for $x \in (-\infty, c-R) \cup (c+R, \infty)$

(2) If $R = \infty$, then the power series is absolutely convergent $\forall x$.

Note that to find the radius of convergence, you should always use the ratio test.

Example

$$\sum_{n=0}^{\infty} \frac{(x-5)^n}{9^n}$$

Solution. Let $b_n = \frac{(x-5)^n}{9^n}$. So,

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{|b_{n+1}|}{|b_n|} \\
&= \lim_{n\to\infty} \frac{\left|\frac{(x-5)^{n+1}}{9^{n+1}}\right|}{\left|\frac{(x-5)^n}{9^n}\right|} \\
&= \lim_{n\to\infty} \frac{|x-5|^{n+1}}{9^{n+1}} \cdot \frac{9^n}{|x-5|^n} \\
&= \frac{|x-5|}{9}
\end{aligned}$$

For the series to be absolutely convergent,

$$0 \le \rho < 1 \iff 0 \le \frac{|x-5|}{9} < 1 \iff 0 \le |x-5| < 9 \iff |x-5| < 9$$

For the series to be divergent,

$$\rho > 1 \iff |x-5| > 9$$

It follows that $x \in (-4, 14)$ and $R = 9$ is the radius of convergence.

Example Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.
Solution. Set $b_n = \frac{x^n}{n!} \implies b_{n+1} = \frac{x^{n+1}}{(n+1)!}$. So,

$$\rho = \lim_{n\to\infty} \frac{\left|\frac{x^{n+1}}{(n+1)!}\right|}{\left|\frac{x^n}{n!}\right|} = \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0$$

So $\rho = 0$ regardless of x. Hence $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is absolutely convergent $\forall x$.
So $R = \infty$.

Definition.

The *interval of convergence* of a power series is the set of x at which the power series is convergent.

When finding the interval of convergence, we must check if the power series converges at the endpoints $c - R$ and $c + R$ after finding R. If $R = \infty$, it's not necessary to check for endpoints because we know the series is convergent $\forall x$.

Example Find the interval of convergence for $\sum_{n=0}^{\infty} \dfrac{x^n}{2^n\sqrt{n+1}}$.

Solution. Set $b_n = \dfrac{x^n}{2^n\sqrt{n+1}}$ so $b_{n+1} = \dfrac{x^{n+1}}{2^{n+1}\sqrt{n+2}}$.

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{|b_{n+1}|}{|b_n|} = \lim_{n\to\infty} \frac{\left|\frac{x^{n+1}}{2^{n+1}\sqrt{n+2}}\right|}{\left|\frac{x^n}{2^n\sqrt{n+1}}\right|} = \lim_{n\to\infty} \frac{|x|^{n+1}}{2^{n+1}\sqrt{n+2}} \cdot \frac{2^n\sqrt{n+1}}{|x|^n} \\
&= \lim_{n\to\infty} \frac{|x|\sqrt{n+2}}{2\sqrt{n+1}} = \lim_{n\to\infty} \frac{|x|}{2}\sqrt{\frac{n+2}{n+1}} = \frac{|x|}{2} \lim_{n\to\infty} \sqrt{\frac{n+2}{n+1}} \\
&= \frac{|x|}{2}\sqrt{\lim_{n\to\infty} \frac{n+2}{n+1}} = \frac{|x|}{2}\sqrt{\lim_{n\to\infty} \frac{1}{1}} = \frac{|x|}{2}\sqrt{1} = \frac{|x|}{2}
\end{aligned}$$

The series converges absolutely if

$$0 \le \rho < 1 \iff 0 \le \frac{|x|}{2} < 1 \iff |x| < 2 \iff -2 < x < 2$$

The series diverges if

$$p > 1 \iff \frac{|x|}{2} > 1 \iff |x| > 2 \iff x < -2 \text{ or } x > 2$$

So $R = 2$. If $x = 2$, the series is $\sum_{n=0}^{\infty} \dfrac{2^n}{2^n\sqrt{n+2}} = \sum_{n=0}^{\infty} \dfrac{1}{\sqrt{n+2}}$. Set $c_n = \frac{1}{\sqrt{n+2}}, d_n = \frac{1}{\sqrt{n}}$.

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{c_n}{d_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+2}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{\sqrt{n+2}} \cdot \frac{\sqrt{n}}{1} \\
&= \lim_{n\to\infty} \sqrt{\frac{n}{n+2}} = \sqrt{\lim_{n\to\infty} \frac{n}{n+1}} = \sqrt{\lim_{n\to\infty} \frac{1}{1}} = \sqrt{1} = 1
\end{aligned}$$

By the p-test, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} d_n$ diverges. So $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ diverges by the limit comparison test. Therefore $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$ also diverges. Hence, the power series diverges at $x = 2$. If $x = -2$, the series is

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{2^n\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

Set $c_n = \frac{1}{\sqrt{n+1}}$.

(1) c_n is positive and decreasing since $\sqrt{n+1}$ is increasing.

(2) $\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$.

By Alternating Series Test,

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

converges. Hence, the interval of convergence is $[-2, 2)$.

For $|x| < 1$,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$$

We know that for $|x| \geq 1$, sum diverges. In particular, the interval of convergence of $\sum_{n=0}^{\infty} x_n$ is $(-1, 1)$. We can use this to write other functions as series and vice-versa.

Example

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} \left(x^2\right)^n = \sum_{n=0}^{\infty} x^{2n}$$

when $|x^2| < 1 \iff |x| < 1$.

Example

$$\sum_{n=0}^{\infty} \frac{(x-5)^n}{9^n} = \sum_{n=0}^{\infty} \left(\frac{x-5}{9}\right)^n = \frac{1}{1 - \frac{x-5}{9}} = \frac{9}{14-x}$$

So, if $|x-5| < 9 \implies \sum_{n=0}^{\infty} \dfrac{(x-5)^n}{9^n} = \dfrac{9}{14-x}$

Theorem. *Term-by-term Differentiation and Integration*

Assume $F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ has radius of convergence $R > 0$. Then F is differentiable on $(c-R, c+R)$, and

$$F'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$$

$$\int F(x)\, dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-c)^{n+1}$$

Moreover, the series on the right hand side of the above expressions both have radius of convergence R but the interval of convergence may change.

Example Find the series for $\frac{1}{(1-x)^2}$.

Solution. We know that $\frac{d}{dx}\left(\frac{1}{1-x}\right) = \dfrac{1}{(1-x)^2}$, so, for $|x| < 1$,

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + \cdots$$

Example Find the series representation of $\arctan x$.
Solution. We know that $\int \frac{1}{1+x^2}\,dx = \arctan x + C$. For $|x| < 1, |-x^2| < 1$, so,

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} \left(-x^2\right)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - \cdots$$

So $\arctan x = C + \sum_{n=0}^{\infty} \dfrac{(-1)^n}{2n+1} x^{2n+1}$. Setting $x = 0$, we have

$$C = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (0)^{2n+1} = \arctan 0 = 0$$

Hence,

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

for $|x| < 1$.

13 Taylor Polynomials

Definition. *(Taylor Polynomial)*

The n^{th} *Taylor polynomial* of $f(x)$ at $x = a$ is defined

$$T_n(x) = f(a) + f'(a)\frac{(x-a)^1}{1!} + f''(a)\frac{(x-a)^2}{2!} + \cdots + f^{(n)}(x)\frac{(x-a)^n}{n!}$$

where $f^{(n)}(a) = n^{\text{th}}$ derivative of f at $x = a$.

We note the notation $f^{(0)}(x) = f(x)$, then,

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

Definition. *(Maclaurin Polynomial)*

The *Maclaurin polynomial* of order n of $f(x)$ is

$$\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

Remark The Maclaurin polynomial is a special case of the Taylor polynomial for $a = 0$.

Example Find the Maclaurin polynomial $T_2(x)$ for $f(x) = \sin x$
Solution.

$$T_2(x) = \sum_{k=0}^{2} \frac{f^{(k)}(0)}{k!} x^k = \frac{f(0)}{0!} + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2$$

$$= f(0) + f'(0)x + \frac{f''(0)}{2} x^2$$

For $f(0) = \sin 0 = 0, f'(0) = \cos 0 = 1, f''(0) = -\sin 0 = 0$. So,

$$T_2(x) = 0 + 1 \cdot x + \frac{0}{2} \cdot x = x$$

Note that to find Taylor polynomials of order n, we need to compute

$$f(a), f'(a), \cdots, f^{(n)}(a)$$

Sometimes, a pattern emerges in the derivatives. You may notice this in the case when the function is trigonometric in nature.

Example Find the n^{th} Taylor polynomial of $f(x) - e^x$ at $x = a$.
Solution.

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

$$\begin{aligned} f(x) &= e^x & f(a) &= e^a \\ f'(x) &= e^x & f'(a) &= e^a \\ f''(x) &= e^x & \Longrightarrow \quad f''(a) &= e^a \\ &\vdots \vdots & &\vdots \vdots \\ f^{(k)}(x) &= e^x & f^{(k)}(a) &= e^a \end{aligned}$$

So,

$$T_n(x) = \sum_{k=0}^{n} \frac{e^a}{k!} (x-a)^k = e^a + e^a \cdot \frac{(x-a)}{1!} + e^a \cdot \frac{(x-a)^2}{2!} + \cdots + \frac{e^a}{n!} (x-a)^n$$

Example Find the n^{th} Maclaurin polynomial of $f(x) - \sin x$.
Solution.

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)(0)}}{k!} x$$

$$\begin{aligned}
f(x) &= \sin x & f(0) &= \sin 0 = 0 \\
f'(x) &= \cos x & f'(0) &= \cos 0 = 1 \\
f''(x) &= -\sin x & f''(0) &= -\sin 0 = 0 \\
f'''(x) &= -\cos x & f'''(0) &= -\cos 0 = -1 \\
f^{(4)}(x) &= \sin x & f^{(4)}(0) &= \sin 0 = 0 \\
f^{(5)}(x) &= \cos x & f^{(5)}(0) &= \cos 0 = 1 \\
f^{(6)}(x) &= -\sin x & f^{(6)}(0) &= -\sin 0 = 0
\end{aligned}$$

$$T_3(x) = 0 + 1 \cdot \frac{x}{1!} + 0 - \frac{1}{3!}x^3 = x - \frac{x^3}{3!} = x - \frac{x^3}{6}$$

$$\begin{aligned}
T_5(x) = 0 + 1 \cdot \frac{x}{1!} + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \\
&= x - \frac{1}{6}x63 + \frac{1}{120}x^5
\end{aligned}$$

In general,

$$T_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n}}{(2n+1)!}$$

13.1 Error Bound

Let K be a number such that

$$K \geq |f^{(n+1)}(u)|$$

for any u between a and x. Then,

$$|f(x) - T_n(x)| \leq K \cdot \frac{|x-a|^{n+1}}{(n+1)!}$$

Example Let $f(x) = \sin x$. Find the largest value of $|f'''(x)|$. Decude from this an error bound on $|f(0.1) - T_2(0.1)|$, where $T_2(x)$ is the 2^{nd} Maclaurin polynomial of f.
Solution. We know that $f'(x) = \cos x, f''(x) = -\sin x$ and $f'''(x) = -\cos x$. For any x, $|f'''(x)| = |\cos x| \leq 1$. And $|f'''(0)| = |\cos 0| = 1$, so $K = 1$ is the larget value attained by $|f'''(x)|$.

Setting $K = 1$, we see

$$|f'''(u) \leq K$$

for any u between a and x. Thus,

$$|f(0.1) - T_2(0.1)| \leq K\frac{|0.1-0|^{2+1}}{(2+1)!} = 1 \cdot \frac{0.1^3}{3!} = \frac{1}{6000}$$

Example Let $f(x) = e^x$ and $T_n(x)$ be the n^{th} Maclaurin polynomial of e^x. Find the maximum of $f^{(n+1)}(u)$ for u between 0 and 1 and deduce an upper bound on $|f(1) - T_n(1)|$.
Solution. We have $f'(x) = e^x, f''(x) = e^x, \cdots, f^{(n+1)(x)=e^x}$. So, $|f^{(n+1)(x)} = |e^x| = e^x$.

$$\frac{d}{dx}\left(|f^{(n+1)}(x)|\right) = \frac{d}{dx}(e^x) = e^x > 0$$

So $|f^{(n+1)}(x)|$ is increasing. So between $a = 0$ and 1, $|f^{(n+1)(u)}$ is largest at 1, i.e., $|f^{(n+1)}(1)| = e^1 = e$ is an upper bound on $f^{(n+1)(u)}$ for $u \in [0,1]$. So set $K = e$, hence,

$$|f(x) - T_n(x)| \leq K\frac{|1-0|^{n+1}}{(n+1)!} = \frac{e}{(n+1)!}$$

13.2 Taylor & Maclaurin Series

Definition.

If f is infinitely differentiable at $x = c$, then the *Taylor series* for f at $x = c$ is defined

$$T(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots = \sum_{k=0}^{\infty}\frac{f^{(k)}(c)}{k!}(x-c)^k$$

Given f, its Taylor series is a power series, of which, we know how to find its interval of convergence.

Definition.

The *Maclaurin series* of f is the Taylor series of f centered at $x = 0$.

Example Find the Maclaurin series of $f(x) = \cos x$.
Solution. We know that $f'(x) = \cos x, f''(x) = -\sin x, f'''(x) = -\cos x, f^{(4)}(x) = \sin x$ and this pattern repeats. Thus, we also know that $f(0) = 1, f'(0) = 0, f''(0) = -1, f'''(0) = 0$ and repeat. So,

$$T(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

The followiing are standard Taylor series expansions:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^3}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{5!} + \cdots$$

These are true $\forall x$.

Theorem.

If $f(x)$ can be represented by a power series centered at $x = c$ in an interval $|x - c|R$ (R positive), then this power series *is* the Taylor series of f about $x = c$.

Note this means the power series for $\arctan x, \frac{1}{1-x}, \ln x, \ln(1-x), \ln(1+x)$ *are* also their own Taylor series for $R = 1$.

Example Find the Taylor series of xe^{-x^2} about $x = 0$.
Solution. Note that in this example, we will not use the formula $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$. For any y,

$$e^y = \sum_{k=0}^{\infty} \frac{e^y}{k!}$$

Set $y = -x^2$, so,

$$e^{-x^2} = \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!}$$

So,

$$xe^{-x^2} = x\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!} = \sum_{l=0}^{\infty} \frac{(-1)^k x^{2k+1}}{k!}$$

for any x. So,

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{k!}$$

is the Taylor series for xe^{-x^2} centered at $x = 0$.

Example Find the Maclaurin series of $x^3 \arctan x$.
Solution. We know that

$$\arctan y = \sum_{k=0}^{\infty} \frac{(-1)^k e^{2k+1}}{2k+1}$$

for $|y| < 1$. Set $y = 3x$, so,

$$\arctan 3x = \sum_{k=0}^{\infty} \frac{(-1)^k (3x)^{2k+1}}{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k 3^{2k+1} x^{2k+1}}{2k+1}$$

if $|3x| < 1 \iff |x| < \frac{1}{3}$. So,

$$x^3 \arctan 3x = x^3 \sum_{k=0}^{\infty} \frac{(-1)^k 3^{2k+1} x^{2k+1}}{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k 3^{2k+1} x^{2k+4}}{2k+1}$$

Hence, the Maclaurin series of $x^3 \arctan x$ is

$$T(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} x^{2k+4}$$

Example Find the Taylor series for $f(x) = \frac{1}{x-2}$ centered at $x = 4$.
Solution. We want express the series in terms of $(x-4)$ and so, we must perform algebraic manipulation.

$$\frac{1}{x-2} = \frac{1}{2+(x-4)}$$

It appears that can manipulate more for the expression to take the shape of a geometric series, that is,

$$\begin{aligned} \frac{1}{x-2} = \frac{1}{2+(x-4)} &= \frac{1}{2} \cdot \frac{1}{1+\frac{1}{2}(x-4)} \\ &= \frac{1}{2} \cdot \frac{1}{1-\left(-\frac{1}{2}(x-4)\right)} \end{aligned}$$

Thus,

$$\begin{aligned}
\frac{1}{2} \cdot \frac{1}{1-\left(-\frac{1}{2}(x-4)\right)} &= \frac{1}{2} \sum_{k=0}^{\infty} \left(-\frac{1}{2}(x-4)\right)^k \\
&= \frac{1}{2} \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k (x-4)^k \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}} (x-4)^k
\end{aligned}$$

for $|x-4| < \dfrac{1}{2}$. So, the Taylor series of $f(x)$ centered at $x = 4$ is

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}} (x-4)^k$$

Part 4: Fall ‘24 Midterm and Final

14 Selected Problems: Midterm 1 and 2

Question 1. Simplify the following expressions. (25 Points, 5 points each)

(a) $\left(e^4\right)\left(e^-2\right)$

(b) $\ln\left(e^2\right)$

(c) $\log_{10}\left(\frac{1}{1000}\right)$

(d) $e^{\ln(\sin x)}$

(e) $\tan\left(\sin^{-1}(x)\right)$

Note: This question in Midterm 1 of Fall 2024 Math 31B didn't require any working but for the sake of practice, I would advise writing the working required.

Question 2. Answer the following questions about inverse functions. (25 Points)

(a) (12 Points) Let $h(x) = x^3 + 1$. This function h has an inverse function h^{-1} (you do not have to justify it). Find a formula for $h^{-1}(x)$.

(b) (13 Points) Let $f(x) = x^7 + x + 7$. This function has an inverse f^{-1} (you do not have to justify it.) Compute $\left(f^{-1}\right)'(7)$.

Question 3. Compute the following. (25 Points)

(a) (12 Points) $\lim_{x\to\infty}\left(1+\dfrac{1}{x^2}\right)^2$

(b) (13 Points) $\dfrac{d}{dx}\left(x^{\cosh x}\right)$

Question 4. Evaluate the following integrals. (25 Points)

(a) (12 Points) $\int_0^1 \dfrac{x}{x^4+1}\,dx$

(b) (13 Points) $\int_0^\pi e^{2x}\sin(x)\,dx$

The following problems are from Midterm 2 (Week 5 onwards).

Question 1. Determine the convergence of the following integrals, sequences, and sums (you do not need to show any working). (25 Points, 5 points each).

(a) $\int_1^\infty \dfrac{1}{x}\,dx$

(b) $\int_0^5 \dfrac{1}{x^{9/10}}\,dx$

(c) $\{(-1.5)^n\}_{n=1}^\infty$

(d) $\sum_{n=0}^\infty (-1)^n$

(e) $\sum_{n=1}^\infty \left(\dfrac{4}{5}\right)^n$

Question 2. Determine whether the following sequences and series converge or diverge. If the sequence or series converges, **compute the value it converges to**. (25 Points)

(a) (12 Points) Sequence $\left\{\dfrac{(-1)^n}{\ln(n)}\right\}_{n=2}^{\infty}$

(b) (13 Points) Series $\sum_{n=3}^{\infty} 2\left(\dfrac{2}{3}\right)^n$

Question 3. Answer the following questions. (25 Points)

(a) (12 Points) Compute $\int \dfrac{1}{(x-3)(x-2)}\, dx$

(b) (13 Points) Let $f(x) = x^3$. Find the surface area of the shape given by revolving the graph of f on the interval $[0,1]$ around the x-axis.

Question 4. Determine the convergence or divergence of the following improper integrals. You **do not** need to compute the value of the integral if it converges. (25 Points)

(a) (12 Points) $\int_0^{\infty} \dfrac{1}{e^x + x}\, dx$

(b) (13 Points) $\int_0^1 \dfrac{1}{\sqrt{x^3 + \frac{1}{2}x^4}}$

Note that I unfortunately don't have access to the final; however, the type of questions you can expect to receive should be similar to the style seen above. It goes without saying that the final will test everything that has been lectured.