Math 136 Notes

Zihao Sun

Spring 2025

Contents

Lecture 1: March 31st

1.1 Introduction to Partial Differential Equations (PDEs)

Definition 1.1. A **Partial Differential Equation** (PDE) is any equation involving an unknown multivariable function and its partial derivatives.

Example 1.2. In 2D, a general PDE with unknown function $u(x,y)$ can be expressed as:

$$0 = F(x, y, u(x,y), u_x(x,y), u_y(x,y), u_{xx}(x,y), u_{xy}(x,y), u_{yy}(x,y) \ldots) \tag{1}$$

for some nonlinear function F.

Notation. In this course, we use u_x to denote $\frac{\partial u}{\partial x}$, u_{xy} to denote $\frac{\partial^2 u}{\partial x \partial y}$, and so on.

Definition 1.3. The **order** of a PDE is defined as the highest order of the derivatives present in the equation.

- A first-order PDE: $F(x, y, u, u_x, u_y) = 0$.
- A second-order PDE: $F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$.
- A third-order PDE: $F(x, y, u, u_x, u_y, \ldots, u_{xyy}, u_{yyy}) = 0$.

Definition 1.4. We call a function $u(x,y)$ a **solution** of a PDE if it satisfies the equation for all (x,y) in the domain of interest.

Example 1.5. We're now giving a few classical examples of PDEs:

1. The *transport equation*:
$$u_x + u_y = 0 \quad \text{or} \quad u_x + y u_y = 0$$
2. *Non-linear transport equation* (also known as *Burgers' equation*):
$$u_x + u u_y = 0$$
3. *Laplace's equation*:
$$u_{xx} + u_{yy} = 0 \quad \text{or} \quad u_{xx} + u_{yy} = f \quad (\textit{Poisson's equation})$$
This is useful in electrostatics, fluid flow, and heat conduction.
4. *Wave equation*:
$$u_{tt} - c^2 u_{xx} = 0$$
5. *Heat equation*:
$$u_t - k u_{xx} = 0$$

Lecture 2: April 2nd

2.1 Cont'd. on Introduction to PDE

Definition 2.1. We say that a PDE is **linear** if it can be written in the form

$$\mathcal{L}u = f.$$

where $\mathcal{L}$ is a linear operator of that is a linear combination of the partial derivatives of u with coefficients that are functions of the independent variables.

$$\mathcal{L}u = a(x_1, \ldots, x_n)u + b(x_1, \ldots, x_n)u_{x_1} + c(x_1, \ldots, x_n)u_{x_2} + \ldots$$

and f is a function of the same variables as u

Remark. The linearity, in this case, means that if u_1 and u_2 are solutions to the PDE, then $c_1u_1 + c_2u_2$ is also a solution for any constants c_1 and c_2.

Definition 2.2. A PDE is called **homogeneous** if $f = 0$, and **non-homogeneous** if $f \neq 0$.

Recall that an ODE has its general solution in the form of multiple constants. In contrast, a PDE has its general solution with infinite dimensions of functions. For a PDE, we deal with functional spaces, thus the solution space is actually a space of functions instead of a space of numbers.

For simple PDEs, we can find the solution by using the simple **method of integration**, as we can integrate the PDE with respect to one of the variables in ODE.

Example 2.3. Solve the following PDE:

$$u_{xx}(x, y) = 0$$

We can do the same thing as separating variables for ODEs. By integrating with respect to x:

$$u_x(x, y) = f(y) \implies u(x, y) = xf(y) + g(y)$$

where $f(y)$ and $g(y)$ are arbitrary functions of y.

Example 2.4. Solve the following PDE:

$$u_{xx}(x, y) + u(x, y) = 0$$

We can guess out the specific solution $e^{\lambda x}$. where λ is the solution of the characteristic equation: $\lambda^2 + 1 = 0$. Thus, we have $\lambda = i$ and $-i$. The general solution is:

$$u(x, y) = Ae^{ix} + Be^{-ix} = A\cos x + B\sin x$$

where A and B are arbitrary functions of y.

Example 2.5. Solve the following PDE:

$$u_{xy}(x, y) = 0$$

We can integrate with respect to x:

$$u_y(x, y) = f(y) \implies u(x, y) = f(x) + \int f(y)dy = f(x) + g(y)$$

where $f(x)$ and $g(y)$ are arbitrary functions of x and y respectively.

2.2 First Order Linear PDEs with Constant Coefficients

Definition 2.6. We the general form of a 2-dimensional first-order linear homogeneous PDE with constant coefficients is:

$$au_x + bu_y = 0, \quad a, b \in \mathbb{R}$$

where u is a function of x and y. This is equivalent

$$(a, b) \cdot \nabla u = 0$$

where $\nabla u = (u_x, u_y)$ is the gradient of u.

We first try to find the solution with **visualization**.

Solution. We do a projection of (x, y) onto the line orthogonal to $bx - ay = 0$. We know that any point following on this line must produce the same value output for any specific solution u. Then, we can guess the general solution as:

$$u(x, y) = f(bx - ay),$$

where $f \in \mathcal{C}^1$ is an arbitrary continuously differentiable function. We can verify by plugging it back into the PDE:

$$u_x = f'(bx - ay)b, \quad u_y = f'(bx - ay)(-a)$$

and thus

$$au_x + bu_y = abf'(bx - ay) - abf'(bx - ay) = 0$$

This is a valid solution. $\square$

We can also do solve this PDE by using the **change of coordinate method**: for a linear PDE, the coordinate method picks a change of coordinate system $x'(x, y)$ and $y'(x, y)$ such that the PDE becomes a simpler one. calculate the partial derivatives with respect to the new coordinates, and then plug them back into the PDE, with the new function

$$\tilde{u}(x'(x, y), y'(x, y)) = u(x, y)$$

that is defined to have the same output as $u(x, y)$ but in the new coordinates.

Solution. We can define a change of coordinate mapping from (x, y) to (x', y') as follows:

$$\begin{cases} x' = & ax + by \\ y' = & bx - ay \end{cases} \iff \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, by the partial differential equation, we know that if we define a new function

$$\tilde{u}(x', y') = \tilde{u}(x'(x, y), y'(x, y)) = u(x, y)$$

with same output but in the new coordinates, we can write the equation as:

$$\tilde{u}(x', y') = u(x, y) = 0$$

By the chain rule, we know that:

$$u_x = \frac{\partial \tilde{u}}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial \tilde{u}}{\partial y'}\frac{\partial y'}{\partial x} = a\tilde{u}_{x'} + b\tilde{u}_{y'}$$
$$u_y = \frac{\partial \tilde{u}}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial \tilde{u}}{\partial y'}\frac{\partial y'}{\partial y} = b\tilde{u}_{x'} - a\tilde{u}_{y'}$$

Then, we can plug these into the equation:

$$au_x + bu_y = a(a\tilde{u}_{x'} + b\tilde{u}_{y'}) + b(b\tilde{u}_{x'} - a\tilde{u}_{y'}) = (a^2 + b^2)\tilde{u}_{x'} = 0$$

Thus, we know that $\tilde{u}$ solves the equation $\tilde{u}_{x'} = 0$. In this way, we can downgrade the equation to one-variable case.

$$\tilde{u}(x', y') = f(y') \implies u(x, y) = f(bx - ay)$$

where f is an arbitrary function of y'. $\square$

Remark. Sometimes, for constant coefficients, the change of coordinates method is more efficient. For example, if we have $u_x + u_y + u = 0$, we can use the change of coordinates $x' = x + y$ and $y' = x - y$ to get the new PDE $u_{x'} + u = 0$. This is much easier to solve than the original PDE.

The solution is unique up to the arbitrary function f that is continuously differentiable. Here, we can call the parallel lines $bx - ay = c$ the **characteristic curves** of the PDE. We'll discuss them in the next lecture.

Lecture 3: April 4th

Today, we'll go on discussing solving the first order linear homogenous equations with general coefficients. For such PDEs with constant coefficients, we have the solutions characterized by the lines $bx - ay = C$. We can generalize this to the case by finding the characteristic curves.

3.1 Solving First Order Linear Homogenous PDEs

Definition 3.1. The first order linear homogenous PDE is defined as:

$$a(x, y)u_x + b(x, y)u_y = 0,$$

where $a(x, y)$ and $b(x, y)$ are continuous functions in the domain of interest.

We first try to give an example of solving a specific first order linear homogenous PDE.

Example 3.2. For example, consider the equation:

$$u_x + yu_y = 0 \iff (1, y)\nabla u = 0$$

The key idea is to consider $u(x, y)$ constant along the graph of $y(x)$, i.e. $u(x, y(x))$ is constant with respect to x. Then,

$$\frac{d}{dx}[u(x, y(x))] = u_x(x, y(x)) + u_y(x, y(x))\frac{dy}{dx} = (y'(x) - y(x))u_y(x, y(x)) = 0$$

So, to satisfy the equations, we require that $y'(x) = y(x)$. Any curve that satisfies this equation is a characteristic curve. The solution to the equation is constant along these curves. The general solution for the characteristic curves is given by:

$$y(x) = Ce^x$$

where C is a constant. The solution to the equation is then given by:

$$u(x, Ce^x) = f(C)$$

Since $y = Ce^x$, we can express C in terms of y as $C = ye^{-x}$. Thus, the solution can be expressed as:

$$u(x, y) = f(ye^{-x})$$

This means that the solution is constant along the characteristic curves defined by the equation $y = Ce^x$.

Definition 3.3. The **characteristic curves** are defined as the curves along which the solution to the PDE is constant. They are given by the solution to the ODE:

$$y'(x) = \frac{b(x, y(x))}{a(x, y(x))}$$

as the set of continuously differentiable functions $f \in \mathcal{C}^1$.

Theorem 3.4 (Method of Characteristics)**.** *If we have a first order linear homogenous equation of the form*

$$a(x, y)u_x + b(x, y)u_y = 0,$$

then can write the solution to the PDE as a function of the characteristic curves

$$u(x, y) = f(x, C)$$

where C is the constant in the characteristic curve $y = y_C(x)$. If the equation is homogeneous, then the solution is independent of x and can be expressed as:

$$u(x, y) = f(C)$$

where C is the constant in the characteristic curve $y = y_C(x)$.

Remark. The method of characteristics changes the PDE into a system of ODEs. Instead of slicing the domain into small rectangles, we slice it into small curves. We then decide which curve to take from C and which point to take from x.

Lecture 4: April 7th

Today, we'll give some examples of common PDEs and their solutions.

4.1 Transport Equation

Definition 4.1. In the simplest case, the **transport equation** of a scalar function $u(x, t)$ in one dimension with no external forces is given by

$$u_t + cu_x = 0$$

where $c \in \mathbb{R}$ is a constant representing the speed of transport.

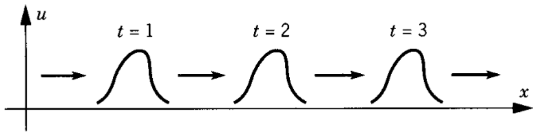


Figure 1: Wave traveling with speed c

Proposition 4.2. *The general solution of the transport equation is given by*

$$u(x,t) = f(x - ct)$$

where f is an arbitrary function.

The solution is a wave that travels with speed c in the positive x direction if $c > 0$ and in the negative x direction if $c < 0$.

Proof. We try to derive this equation with physical intuition. Consider the total amount of particles in a small interval $[0, b]$ at time $t = 0$. The amount of particles in the interval $[0, b]$ at time $t = 0$ is given by

$$\int_0^b u(t,x)\,dx.$$

Since we have constant transport at speed c, we know that

$$M = \int_0^b u(t,x)\,dx = \int_{ch}^{b+ch} u(t+h,x)\,dx = \int_0^b u(t+h,x+ch)\,dx$$

This means that

$$u(t+h, x+ch) = u(t,x) \text{ for } x \in [0,b]$$

When narrowing the change of h to 0, we have

$$\lim_{h\to 0} \frac{u(t+h,x+ch) - u(t,x)}{h} = 0 \implies \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

□

4.2 Vibrating String: String Equation

Consider a string of length $l > 0$, constant density $\rho > 0$, and tension vector $T(x,t)$ representing how much internal force is acted in the longitudinal and transverse directions. Let $u(x,t)$ be the height of the string relative to the equilibrium position. We try to find the equation of motion for the string.

Definition 4.3. The **string equation** of a vibrating string with fixed ends and no external forces is given by

$$u_{tt} = c^2 u_{xx}$$

where $c^2 = T/\rho$ represents the wave speed of the string, with T being the tension and ρ being the density of the string.

To do this, we consider a small segment of the string $[x_0, x_1]$ and apply Newton's second law. Since the segment is short enough, we can assume that this small segment of the string is straight line segment.

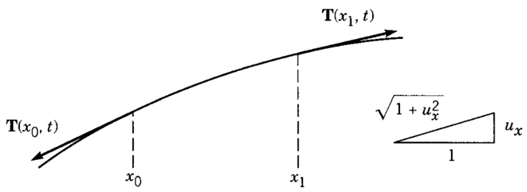


Figure 2: Vibrating string

Proof. We first calculate the longitudinal and transerse force acting on this segment. By pythagorean theorem, we have the total displacement of the string given by

$$\sqrt{u_x^2(x_1 - x_0)^2 + (x_2 - x_1)^2} = \sqrt{u_x^2 + 1}(x_1 - x_0)$$

Therefore, we can write the tension exerted by other parts on the rope as

$$T(x,t) = (\frac{T}{\sqrt{1+u_x^2}}\bigg|_{x_0}^{x_1}, \frac{Tu_x}{\sqrt{1+u_x^2}}\bigg|_{x_0}^{x_1})$$

Then, we apply the Newton's second law to the segment of the string. As the particles are moving only in the vertical direction, thus we can derive

$$\frac{T}{\sqrt{1+u_x^2}}\bigg|_{x_0}^{x_1} = 0 \quad \text{and} \quad \frac{Tu_x}{\sqrt{1+u_x^2}}\bigg|_{x_0}^{x_1} = \int_{x_0}^{x_1} \rho u_{tt}\, dx$$

Then, by assumming $\|u_x\| \ll 1$, we can linearize the equation to get

$$\sqrt{1+u_x^2} \approx 1 + o(u_x^2)$$

Moreover, let us assume a homogeneous tension $T(x,t)$, then the linearity becomes

$$\frac{Tu_x}{\sqrt{1+u_x^2}}\bigg|_{x_0}^{x_1} \approx Tu_x\bigg|_{x_0}^{x_1}$$

Hence, a linearized version of the equation becomes

$$T \cdot u_x \Big|_{x_0}^{x_1} = \int_{x_0}^{x_1} \rho u_{tt}\, dx$$

In particular, if we take $x_1 - x_0 = h \to 0$, we have

$$\lim_{h\to 0} \frac{T \cdot (u_x(x_0 + h, t) - u_x(x_0, t))}{h} = \lim_{h\to 0} \frac{1}{h} \int_{x_0}^{x_0+h} \rho u_{tt}\, dx$$

Thus we get the wave equation

$$Tu_{xx} = \rho u_{tt}$$

□

Remark. The variational forms of the wave equation is given by

(i) **Damped Wave Equation**:

$$u_{tt} - c^2 u_{xx} + r u_t = 0$$

where $r > 0$ is the damping coefficient representing the energy loss.

(ii) **Elastic Force**:

$$u_{tt} = c^2 u_{xx} + ku,$$

where $k > 0$ is the elastic coefficient. Here k represents the restorative force of the material.

(iii) **Inhomogeneous Wave Equation**:

$$u_{tt} = c^2 u_{xx} + f(x, t)$$

where $f(x, t)$ is a function representing the external force acting on the string.

Lecture 5: April 9th

5.1 Vibrating Drumhead: Wave Equation

In today's lecture, we'll generalize the idea of vibration of a string to the vibration of a drumhead. We'll end up with the wave equation, which looks very similar to the string vibration equation we derived in the last lecture.

Definition 5.1. The **wave equation** is a second-order partial differential equation that describes the propagation of waves, such as sound or light, in a medium with no external forces acting on it. It is given by

$$u_{tt} = c^2 \Delta u,$$

where u is the displacement of the wave, c is the speed of the wave, and Δ is the **Laplace operator** defined as

$$\Delta = \nabla \cdot \nabla = \mathrm{tr}(\nabla^2)$$

Proof. Suppose we have a closed domain D in $\mathbb{R}^2$ representing a part of the membrane, and we have a function $u(x, y, t)$ that describes the displacement of the drumhead at time t. We also define $n(x, y)$ as the unit normal vector to the boundary ∂D representing the direction of tension acting on any point on the boundary.

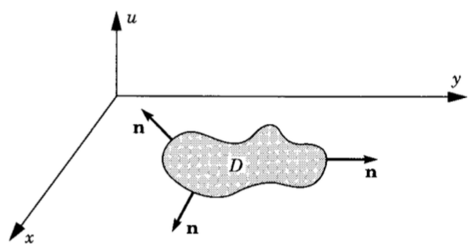


Figure 4

Then, we try to derive this equation. By the Newton's second law, we have $F = ma$, where F is the force acting on the drumhead, m is the mass of the drumhead, and a is the acceleration of the drumhead. Since the internal tensions cancels and only the tension on the boundary is taken into consideration, we can model the force exerted on the membrane in D by

$$F = \int_{\partial D} T \frac{\partial u}{\partial n}\, ds = \int_D \rho u_{tt}\, dx\, dy = ma$$

where ds is the arclength parametrization. By the divergence theorem, we can rewrite the left-hand side as

$$\int_{\partial D} T \frac{\partial u}{\partial n}\, ds = \int_{\partial D} T \cdot N \cdot \nabla u\, ds = \int_D \nabla \cdot (T \nabla u)\, dx\, dy$$

where N is the unit normal vector along the boundary ∂D. Since D is arbitary, we can drop the integral and get

$$\nabla \cdot (T \nabla u) = \rho u_{tt}$$

In particular, if we assume that the tension T is constant, we can rewrite the equation as

$$u_{tt} = T\nabla \cdot \nabla u = T\Delta u = T(u_{xx} + u_{yy}).$$

Generally, in higher dimensions, for example, sound waves or Electromagnetic waves, we can write the wave equation as

$$u_{tt} = c^2 \Delta u = c^2(u_{xx} + u_{yy} + u_{zz}).$$

$\square$

Lecture 6: April 11th

6.1 Diffusion and Heat Equation

We try to model the diffusion of a chemical in a one-dimensional rod. Notice that this is different from the transport equation: as in the transport equation, the wave shape is preserved, meaning that the *internal effects between the components are neglected*. In the diffusion equation, we consider the internal effects between the components, which means that the wave shape is not preserved.

Definition 6.1. The **diffusion equation** on a one-dimensional rod with no external forces and initial stability is given by

$$u_t = ku_{xx},$$

where k is a constant representing the thermal diffusivity of the material. We can also generalize it with some random function $k(x,t)$ to

$$u_t(x,t) = \frac{\partial}{\partial x}\left(k(x,t)u_x(x,t)\right)$$

where $k(x,t)$ is a function of space and time.

Proof. To derive this equation, we consider a thin rod filled with a chemical placed in vaccuum (no external heat exchange). Take the section of length $L = x_1 - x_0$. We consider how the temperature distribution of the system evolves in time.

We consider a continuous temperature distribution $u(x,t)$, where x is the position along the rod and t is time. We know that the total kinetic energy of the system is given by

$$M(t) = \int_{x_0}^{x_1} u(x,t)\,dx.$$

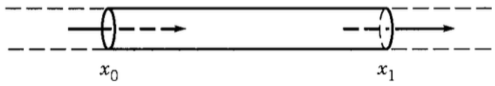


Figure 5

By **Fick's Law of Diffusion**, the rate of motion (total flux of boundary) is proportional to the concentration gradient (change in total amount of kinetic energy). Therefore,

$$\frac{dM}{dt}(t) = k(u_x(x_1,t) - u_x(x_0,t)) = k\int_{x_0}^{x_1} u_{xx}(x,t)\,dx,$$

where k is a constant. Since the equation above is true for all $[x_0, x_1]$, we can derive the one-dimensional heat equation, or diffusion equation:

$$u_t(x,t) = ku_{xx}(x,t).$$

$\square$

For higher dimensions (3D), we can generalize it into the heat equation.

Definition 6.2. The **heat equation** in 3D with no external forces and initial stability is given by

$$u_t = k\Delta u = k(u_{xx} + u_{yy} + u_{zz}),$$

where k is a constant representing the thermal diffusivity of the material. We can also interprete this as the probablity density function of the Brownian motion.

Proof. By the Fick's law, we know

$$\frac{d}{dt}\iiint_D u(x,y,z,t)\,dx\,dy\,dz = k\oiint_{\partial D} \mathbf{n}\cdot\nabla u(x,y,z,t)\,dS,$$

where $\mathbf{n}$ is the outward normal vector to the boundary ∂D of the domain D (Note that ∇ only counts the partial derivatives of x, y, and z). By the Divergence theorem, we can derive the diffusion equation in 3D:

$$\iiint_D u_t\,dx\,dy\,dz = k\iiint_D \nabla^2 u(x,y,z,t)\,dx\,dy\,dz.$$

Thus, since the region D is selected arbitrarily, we know that

$$u_t = k\nabla^2 u = k\Delta u.$$

$\square$

Remark. We can see that there are Laplacian operators in the wave and heat equations. For both equations, the stationary status are solutions to Laplace's equation.

Definition 6.3. The **Laplace's equation** represents a steady state of the diffusion equation (no time dependence):

$$\Delta u = u_{xx} + u_{yy} + u_{zz} = 0.$$

Its inhomogeneous form is called the **Poisson's equation**. It represents the effect of external forces on the system:

$$u_t = k\Delta u + f(x, y, z).$$

The solutions to Laplace's equation are involved in many physical phenomena, such as the potential flow of fluids, electrostatics, and heat conduction

6.2 Boundary Conditions and Initial Conditions

Recall for an n-th order ODE, or equivalently a system of n first-order ODEs, we need n initial conditions to have a unique solution.

Example 6.4. The ODE

$$y''(t) = -ky(t)$$

has the general solution

$$y(t) = A\cos(\sqrt{k}t) + B\sin(\sqrt{k}t).$$

To find the constants A and B, we need two initial conditions, such as $y(0) = y_0$ and $y'(0) = v_0$. Then we can find the unique solution

$$y(t) = y_0\cos(\sqrt{k}t) + \frac{v_0}{\sqrt{k}}\sin(\sqrt{k}t).$$

For PDEs, we must impose both initial conditions and boundary conditions. In general, we have the rule that *the number of initial conditions must be equal to top order of time derivatives, and the number of boundary conditions must be equal to the top order of spatial derivatives.*

Definition 6.5. The **initial condition** is the condition that specifies the value of the function at the initial time $t = t_0$.

Example 6.6. For example, for the diffusion equation, we have the initial condition

$$u(x, y, z, t_0) = f(x, y, z),$$

where $f(x, y, z)$ is a function of the initial temperature distribution of the system.

Definition 6.7. The **boundary condition** is the condition that specifies the value of the function at the boundary of the domain. A boundary condition is called **homogeneous** if the specified function is zero, and **inhomogeneous** if the specified function is non-zero.

Definition 6.8 (Types of Boundary Conditions)**.** There are three types of inhomogeneous boundary conditions:

1. **Dirichlet boundary condition**: the value of the function u is specified at the boundary.
2. **Neumann boundary condition**: the value of normal derivative $\partial u/\partial \mathbf{n}$ (flux) is specified.
3. **Robin boundary condition**: a linear combination of Dirichlet and Neumann boundary conditions.

Example 6.9. For a one-dimensional PDE with the variable $u(x, t)$ with boundary points $[x_0, x_1]$, we can have the following boundary conditions:

1. Dirichlet boundary condition:

$$u(x_0, t) = f_0(t), \quad u(x_1, t) = f_1(t).$$

2. Neumann boundary condition:

$$u_x(x_0, t) = g_0(t), \quad u_x(x_1, t) = g_1(t).$$

3. Robin boundary condition:

$$u(x_0, t) + \alpha(t)u_x(x_0, t) = f_0(t), \quad u(x_1, t) + \beta(t)u_x(x_1, t) = f_1(t).$$

Example 6.10. For the heat equation with insulated ends (boundary), we have its Neumann boundary condition:

$$u_x(x_0, t) = 0, \quad u_x(x_1, t) = 0.$$

Lecture 7: April 14th

7.1 Examples of Boundary and Initial Conditions

Example 7.1. We consider the hear equation with homogeneous Dirichlet boundary conditions and an initial condition at $t = 0$.

$$\begin{cases} u_t = u_{xx} \\ u(x,0) = \phi(x) \\ u(0,t) = u(L,t) = 0 \end{cases}$$

Then, we can derive a more concrete model to replace the homogeneous boundary conditions with Newton's law of cooling.

Example 7.2. We consider the heat equation with a inhomogeneous Robin boundary condition with constant coefficients.

$$\begin{cases} u_t = u_{xx} \\ u(x,0) = \phi(x) \\ u_x(0,t) = -\alpha(u(0,t) - f(t)) \\ u_x(L,t) = -\alpha(u(L,t) - g(t)) \end{cases}$$

where α is a constant and $f(t)$ and $g(t)$ are given functions.

Example 7.3. This is the wave equation with homogeneous Neumann boundary conditions and an initial condition at $t = 0$.

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in [0,L], t \geq 0 \\ u(x,0) = \phi(x), & x \in [0,L] \\ u_t(x,0) = \psi(x), & x \in [0,L] \\ u_x(0,t) = u_x(L,t) = 0, & t \geq 0 \end{cases}$$

This models the sound wave in a closed tube.

Remark. The Neumann boundary condition usually means that the boundary is insulated, while the Dirichlet boundary condition means that the boundary is kept at a fixed temperature. The Robin boundary condition is a combination of the two, where the heat flux is proportional to the temperature difference between the boundary and the surrounding medium.

What about the unbounded domains with "conditions at infinity"? For example, consider a semi-infinite insulated metal rod with a heat source at the origin.

Example 7.4. The heat equation with a heat source at the origin and insulated boundary condition at infinity has the following form:

$$\begin{cases} u_t = ku_{xx}, & x \in [0, \infty), t \geq 0 \\ u_x(0,t) = 0, & t \geq 0 \\ \lim_{x\to\infty} u_x(x,t) = 0, & t \geq 0 \\ u(x,0) = \phi(x), & t \geq 0 \end{cases}$$

Then, we can consider the boundary conditions for higher dimensions.

Definition 7.5 (Boundary Conditions in Higher Dimensions)**.** Consider the boundary conditions for higher dimensional problems. For example, consider the equation $u(x,t)$ with $\vec{x} \in D \subseteq \mathbb{R}^n$ and $t \in [0,T]$. The boundary conditions can be of the form:

1. Dirichlet boundary conditions:

$$u(\vec{x},t) = f(\vec{x},t), \quad \forall \vec{x} \in \partial D, t \in [0,T]$$

2. Neumann boundary conditions:

$$\frac{\partial u}{\partial n}(\vec{x},t) = \mathbf{n} \cdot \nabla u(\vec{x},t) = f(\vec{x},t), \quad \forall \vec{x} \in \partial D, t \in [0,T]$$

3. Robin boundary conditions:

$$\frac{\partial u}{\partial n}(\vec{x},t) + \alpha u(\vec{x},t) = f(\vec{x},t), \quad \forall \vec{x} \in \partial D, t \in [0,T]$$

where n is the outward normal vector to the boundary ∂D and α is a constant.

Example 7.6. The Laplace equation with Dirichlet boundary conditions on a disk in $\mathbb{R}^2$.

$$\begin{cases} u_{xx} + u_{yy} = 0, & x^2 + y^2 \leq 1 \\ u(x,y) = f(x,y), & x^2 + y^2 = 1 \end{cases}$$

Lecture 8: April 16th

8.1 Well-posedness of Problems in PDE

Definition 8.1 (Well-posedness)**.** A problem is said to be **well-posed** if it satisfies the following three conditions:

1. **Existence**: There exists a solution satisfying all the conditions.
2. **Uniqueness**: The solution is unique in the sense that if u_1 and u_2 are two solutions to the same problem, then $u_1 = u_2$.
3. **Stability**: The solution depends continuously on the initial and boundary data. More precisely, if u_1 and u_2 are two solutions to the same problem with different initial or boundary data, then there exists a constant C such that
$$\|u_1 - u_2\| \le C \left\| \vec{\phi}_1 - \vec{\phi}_2 \right\|$$
for all $t \in [0, T]$, where $\vec{\phi}_1$ and $\vec{\phi}_2$ are the initial or boundary data for u_1 and u_2, respectively.

If any of these conditions is violated, the problem is said to be **ill-posed**.

Remark. The stability condition is crucial for the well-posedness of a problem. It ensures that small changes in the initial or boundary data lead to small changes in the solution. This can also be expressed in terms of convergence.

Example 8.2 (Illustration of Stability in ODEs)**.** For a system of first-order ODEs: $\vec{x}' = \vec{f}(\vec{x})$,
$$\begin{cases} x_1' = f_1(x_1, x_2, \cdots, x_n) \\ x_2' = f_2(x_1, x_2, \cdots, x_n) \\ \vdots \\ x_n' = f_n(x_1, x_2, \cdots, x_n) \end{cases}$$
Suppose we have a sequence of solutions to the above
$$\vec{x}^{(m)}(t) = (x_1^{(m)}(t), x_2^{(m)}(t), \cdots, x_n^{(m)}(t))$$
with initial conditions
$$\vec{x}^{(m)}(0) = \vec{\phi}^{(m)} = (\phi_1^{(m)}, \phi_2^{(m)}, \cdots, \phi_n^{(m)}).$$
The stability condition can be expressed as
$$\lim_{m\to\infty} \phi^{(m)} = \vec{\phi} \in \mathbb{R}^n$$
Then, for each $t \in \mathbb{R}$, we have $\lim_{m\to\infty} x^{(m)}(t) = x(t)$, where $x(t)$ is the solution to the problem with initial condition $\vec{\phi} = x(0)$.

Definition 8.3. A problem is said to be **stable** if the solutions depend continuously on the initial and boundary data. More precisely, if $\vec{\phi}^{(m)}$ is a sequence of initial or boundary conditions (necessarily in functional spaces) that converges to $\vec{\phi}$, then the corresponding sequence of solutions $\vec{x}^{(m)}(t)$ converges to the solution $\vec{x}(t)$ with initial or boundary data $\vec{\phi}$.

$$\lim_{m\to\infty} \vec{x}^{(m)}(t) = \vec{x}(t)$$

where $\vec{x}(t)$ is the solution to the problem with initial or boundary data $\vec{\phi}$.

Remark. In $\mathbb{R}^n$, there is only one useful notion of convergence, which is the convergence in the norm (different definitions of convergence are equivalent). For instance, for a sequence of vectors $\vec{\phi}^{(1)}, \vec{\phi}^{(2)}, \vec{\phi}^{(3)}, \dots \in \mathbb{R}^n$, we can define the **convergence in the norm** as

$$\lim_{m\to\infty} \vec{\phi}^{(m)} = \vec{\phi} \in \mathbb{R}^n \iff \left\| \vec{\phi}^{(m)} - \vec{\phi} \right\| \to 0$$

However, for a infinite dimensional functional space, for example

$$C^0(\Omega) \coloneqq \{ f \colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}, f \text{ continuously differentiable} \},$$

there are many different notions of convergence. For $\varphi = f^{(m)} \in C^0(\Omega)$, we can define the **pointwise convergence** as

$$\lim_{m\to\infty} f^{(m)}(x) = f(x), \forall x \in \Omega$$

There is also a stronger sense of convergence, called **uniform convergence**, which is defined as $f^{(m)} \to f$ uniformly on Ω if

$$\lim_{m\to\infty} \max_{x\in\Omega} \left\| f^{(m)}(x) - f(x) \right\| \to 0$$

Pointwise convergence is weaker than uniform convergence, as uniformly convergent functions must be continuously differentiable, while pointwise convergent functions may not be.

Example 8.4 (Ill-posed unstable problem)**.** Consider the Laplacian Equation

$$u_{xx} + u_{yy} = 0$$

over the domain $D = \{-\infty < x < \infty, y > 0\}$. Consider a sequence of solutions $u_m(x, y)$ with boundary conditions

$$u_m(x, 0) = 0, \quad \frac{\partial u_m}{\partial y}(x, 0) = e^{-\sqrt{m}} \sin(nx)$$

Note that there's a pointwise and uniform convergence of the boundary conditions as

$$\lim_{m\to\infty} u_m(x,0) = 0, \quad \lim_{m\to\infty} \frac{\partial u_m}{\partial y}(x,0) = 0.$$

We can find the solutions as

$$u_m(x,y) = \frac{e^{-\sqrt{m}}}{\sqrt{m}} \sin(nx)\sinh(ny).$$

By plugging in, it's evident that $u_m(x,y)$ satisfies the PDE with boundary conditions. However, for all $y > 0$, we have

$$\lim_{m\to\infty} u_m(x,y) = \lim_{m\to\infty} \frac{e^{-\sqrt{m}}}{\sqrt{m}} \sin(nx)\sinh(ny) = \infty$$

as $\sinh(ny) \to \infty$ as $m \to \infty$. Thus, the solution is not stable, and the problem is ill-posed.

In contrast, if add some extra conditions to the problem, we can have a well-posed problem.

Example 8.5 (Well-posed problem)**.** Consider the PDE problem

$$\begin{cases} u_t t = c^2 u_{xx}, & t \geq 0, x \in \mathbb{R} \\ u(0,x) = \phi(x), & x \in \mathbb{R} \\ u_t(0,x) = \psi(x), & x \in \mathbb{R} \\ \lim_{t\to\infty} u(t,x) = 0, & t \geq 0, x \in \mathbb{R} \end{cases}$$

where $\phi(x)$ and $\psi(x)$ are continuous functions. The solution to this problem is given by the D'Alembert formula

$$u(t,x) = \frac{1}{2}\left(\phi(x-ct) + \phi(x+ct) + \int_{x-ct}^{x+ct} \psi(s)ds\right)$$

The solution is stable, as we can see that the solution depends continuously on the initial and boundary data. For example, if we have a sequence of solutions $u_m(t,x)$ with initial conditions $\phi_m(x)$ and $\psi_m(x)$, then we can show that

$$\lim_{m\to\infty} u_m(t,x) = u(t,x)$$

for all $t \geq 0$ and $x \in \mathbb{R}$. Thus, the problem is well-posed.

Lecture 9: April 18th

9.1 Categorizing 2nd order PDEs with Constant Coefficients

We can classify 2nd order PDEs into three types based on linear combinations. There are three cannonical forms of 2nd order PDEs:

Definition 9.1. All second order constant coefficient PDEs with two independent variables can be written as

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0$$

where a_{ij} are functions of x and y.

Remark. Notice that the coefficient a_{12} is multiplied by 2 because we have $u_{xy} = u_{yx}$. This is a convention to make the notation more consistent.

Theorem 9.2. *By choosing appropriate coordinates, i.e. doing linear transformations, we can transform the PDE in the above definition into one of the following* ***cannonical forms****:*

1. ***Elliptic****: If $a_{12}^2 - a_{11}a_{22} < 0$, then we can transform the second order constant coefficient PDEs with two independent variables into the form*

$$u_{xx} + u_{yy} + a_1u_x + a_2u_y + a_0u = 0$$

   like the ***Laplace equation*** *with lower order terms.*

2. ***Hyperbolic****: If $a_{12}^2 - a_{11}a_{22} > 0$, then we can transform the PDE into the form*

$$u_{xx} - u_{yy} + a_1u_x + a_2u_y + a_0u = 0$$

   like the ***wave equation*** *with lower order terms.*

3. ***Parabolic****: If $a_{12}^2 - a_{11}a_{22} = 0$, then we can transform the PDE into the form*

$$u_{xx} + a_1u_x + a_0u = 0$$

   like the ***heat equation*** *with lower order terms.*

Proof. We can assume that at least one of the coefficients a_{12}, a_{11}, a_{22} is non-zero. Let's first consider the case when $a_{11} \neq 0$. By rescaling the variables, we can assume that $a_{11} = 1$. Then we can write the PDE as

$$u_{xx} + a_{12}u_{xy} + a_{22}u_{yy} = \left(\frac{\partial}{\partial x} + a_{12}\frac{\partial}{\partial y}\right)^2 u + (a_{22} - a_{12}^2)\left(\frac{\partial}{\partial y}\right)^2 u$$

Thus, by defining the new coordinate system

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a_{12} & b \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

we can transform the PDE by making the substitution $u(x,y) = v(\xi,\eta)$, where v is the new function in the new coordinate system. Since

$$v_\xi = u_x \frac{\partial x}{\partial \xi} + u_y \frac{\partial y}{\partial \xi} = u_x + a_{12} u_y$$

$$v_\eta = u_x \frac{\partial x}{\partial \eta} + u_y \frac{\partial y}{\partial \eta} = b u_y$$

we can find the second derivatives as follows:

$$v_{\xi\xi} = \left(\frac{\partial}{\partial x} + a_{12} \frac{\partial}{\partial y} \right)^2 u = u_{xx} + 2a_{12} u_{xy} + a_{12}^2 u_{yy}$$

$$v_{\eta\eta} = \left(b \frac{\partial}{\partial y} \right)^2 u = b^2 u_{yy}$$

Therefore, we can write the PDE as

$$u_{xx} + a_{12} u_{xy} + a_{22} u_{yy} = v_{\xi\xi} + (a_{22} - a_{12}^2) \frac{1}{b^2} v_{\eta\eta}$$

By choosing $b = \sqrt{|a_{22} - a_{12}^2|}$, we can transform the PDE into the form

$$v_{\xi\xi} \pm v_{\eta\eta} = 0$$

where the sign depends on the sign of $a_{22} - a_{12}^2$ (which is actually $a_{11}a_{22} - a_{12}^2$). If $a_{22} - a_{12}^2 < 0$, then we have the elliptic case. If $a_{22} - a_{12}^2 > 0$, then we have the hyperbolic case. If $a_{22} - a_{12}^2 = 0$ take $b = 1$, then we have the parabolic case.

$$v_{xx} + 2a_{12} v_{xy} + a_{22} v_{yy} = v_{\xi\xi} + (a_{22} - a_{12}^2) v_{\eta\eta} = v_{\xi\xi}$$

□

Remark. For other cases as $a_{11} = 0$ and $a_{22} \neq 0$, by symmetry we can derive the same results. If $a_{22} = a_{11} = 0$, then the PDE is

$$2a_{12} u_{xy} + a_1 u_x + a_2 u_y + a_0 = 0$$

this is equivalent to

$$v_{\xi\xi} - v_{\eta\eta} + a_1' v_\xi + a_2' v_\eta + a_0' = 0$$

with coordinate transformation

$$\xi = x + y, \quad \eta = x - y$$

In this case, $a_{12}^2 > a_{11}a_{22} = 0$ and is hyperbolic.

9.2 Non-constant Coefficient 2nd-order PDEs

Theorem 9.3. *We can categorize 2nd order PDEs with non-constant coefficients into three types based on the sign of the* ***determinant***

$$D = a_{12}^2 - a_{11}a_{22}$$

with $D = 0$ being the parabolic case, $D < 0$ being the elliptic case, and $D > 0$ being the hyperbolic case.

For non-constant coefficients, in general, we cannot transform the PDE into the cannonical form (i.e. heat, wave, or Laplace) with change of coordinates. However, we can still comment on its geometry (hyperbolic, elliptic, parabolic) by looking at the sign of the determinant.

Example 9.4. For the PDE

$$xu_{xx} + yu_{yy} = 0$$

This is an elliptic PDE when $a_{11}a_{22} = xy > 0$ and hyperbolic when $a_{11}a_{22} < 0$. If $x = 0$ or $y = 0$, then the PDE is parabolic.

Lecture 10: April 21st

10.1 Classification of 2nd order PDEs in Higher Dimensions

Consider a constant coefficient, linear, second order PDE in n dimensions, where $n > 2$:

$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i} + au = 0$$

We can use a linear change of variables to transform the PDE into a canonical form:

$$\xi = Bx, \text{ where } \xi_i = \sum_{j=1}^{n} B_{ij} x_j$$

Let $v(\xi) = u(x)$, then

$$\frac{\partial \xi_j}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\sum_{k=1}^{n} B_{jk} x_k\right) = \sum_{k=1}^{n} B_{jk}\delta_{ik} = B_{ji}$$

where δ_{ik} is the Kronecker delta. Thus, we have

$$\frac{\partial u}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial v}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_i} = \sum_{j=1}^{n} B_{ji}\frac{\partial v}{\partial \xi_j} \iff \nabla u = B^T \nabla v$$

Therefore, we can write the second derivative as:

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{l=1}^{n} B_{li}\frac{\partial}{\partial \xi_l}\left(\sum_{k=1}^{n} B_{kj}\frac{\partial v}{\partial \xi_k}\right) = \sum_{l,k=1}^{n} B_{li}B_{kj}\frac{\partial^2 v}{\partial \xi_l \partial \xi_k}$$

Summing over i and j, we have:

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{i,j=1}^{n} a_{ij}\sum_{l,k=1}^{n} B_{li}B_{kj}\frac{\partial^2 v}{\partial \xi_l \partial \xi_k} = \sum_{l,k=1}^{n}\left(\sum_{i,j=1}^{n} a_{ij}B_{li}B_{kj}\right)\frac{\partial^2 v}{\partial \xi_l \partial \xi_k}$$

Writing this in matrix form, we have the second derivative term of u as:

$$\sum_{l,k=1}^{n} \left(B^T A B\right)_{lk}\frac{\partial^2 v}{\partial \xi_l \partial \xi_k}$$

where $A = (a_{ij})$ is the symmetric matrix of coefficients of the second order derivatives. Thus, we can diagonalize the matrix A using an orthonormal basis of eigenvectors. In particular, we can choose B to be the change of basis matrix, i.e. $B^T = B^{-1}$, such that:

$$A = BDB^T, \text{ where } D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

This gives us:

$$\sum_{l,k=1}^{n} \left(B^T A B\right)_{lk}\frac{\partial^2 v}{\partial \xi_l \partial \xi_k} = \sum_{l,k=1}^{n} D_{lk}\frac{\partial^2 v}{\partial \xi_l \partial \xi_k} = \sum_{l=1}^{n} \lambda_l v_{\xi_l \xi_l}$$

Therefore, we can write the PDE in the form:

$$\sum_{l=1}^{n} \lambda_l v_{\xi_l \xi_l} + \sum_{i=1}^{n} a_i v_{\xi_i} + av = 0$$

and define the cannonical form of the PDE as follows:

Definition 10.1. We say a second order PDE is

1. **elliptic** if all eigenvalues λ_i have the same sign (all positive or all negative);

2. **hyperbolic** if all but one eigenvalue λ_i have the same sign and all are non-zero;
3. **parabolic** if only one eigenvalue λ_i is zero and all others are all non-zero and have the same sign;
4. **ultra-hyperbolic** if at least 2 eigenvalues are positive and at least 2 are negative and all are non-zero.

Example 10.2. - The Laplace equation $u_{xx} + u_{yy} + u_{zz} = 0$ is elliptic.

- $u_{xx} + u_{yy} - u_{zz} = 0$ is hyperbolic.
- $u_t = u_{xx} + u_{yy}$ is parabolic.
- $u_{tt} + uss = 2u_{xx} + 2u_{yy}$ is ultra-hyperbolic.

10.2 1D Wave Equation

In this subsection, we're to explore the solution of the wave equation. We can actually solve the wave equation with the simple method of characteristics (disregarding the boundary and initial conditions).

Proposition 10.3. *Consider the wave equation in 1D:*

$$u_{tt} - c^2 u_{xx} = 0 \textit{ for } x \in \mathbb{R}, t > 0$$

Then, its general solution is given by:

$$u(x,t) = f(x+ct) + g(x-ct)$$

where f and g are arbitrary twice differentiable functions.

Proof. The original wave equation is equivalent to the operator form

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)u = 0 \iff \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = 0$$

Since

$$\frac{\partial}{\partial t} - c\frac{\partial}{\partial x} = (c,1)\cdot\nabla_{x,t} \text{ and } \frac{\partial}{\partial t} + c\frac{\partial}{\partial x} = (-c,1)\cdot\nabla_{x,t}$$

we can make the change of variables:

$$\xi = x + ct, \eta = x - ct$$

and write $v(\xi, \eta) = u(x, t)$, then

$$u_{xx} = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)^2 v = v_{\xi\xi} + 2v_{\xi\eta} + v_{\eta\eta}$$

$$u_{tt} = \left(c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta}\right)^2 v = c^2 v_{\xi\xi} - 2c^2 v_{\xi\eta} + c^2 v_{\eta\eta}$$

Plugging this into the wave equation gives:

$$c^2 v_{\xi\xi} - 2c^2 v_{\xi\eta} + c^2 v_{\eta\eta} - c^2 v_{\xi\xi} - 2v_{\xi\eta} - v_{\eta\eta} = 0 \iff v_{\xi\eta} = 0$$

Thus, we have the general solution of the wave equation as:

$$v(\xi, \eta) = f(\xi) + g(\eta) = f(x + ct) + g(x - ct)$$

where f and g are arbitrary twice differentiable functions. This is the D'Alembert solution to the wave equation. $\square$

Remark. The solution is actually a linear combination of two waves, one traveling along $x + ct$ and the other along $x - ct$. The wave travels at *speed* c in both directions. We can also think about the solutions as the linear combination of the solutions of the two first order equations:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \text{ and } \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = 0$$

Lecture 11: April 24th

11.1 Wave Equations

Last time, we covered the general solution to wave equation in one dimension. Today, we will look at the wave equation with boundary and initial conditions.

Theorem 11.1 (D'Alembert's Solution). *Given a wave equation with the following boundary and initial conditions:*

$$\begin{cases} u_{tt} = c^2 u_{xx}, & \text{for } x \in (0, L), t > 0 \\ u(x, 0) = \phi(x), & \text{for } x \in (0, L) \\ u_t(x, 0) = \psi(x), & \text{for } x \in (0, L) \end{cases}$$

where ϕ and ψ are continuous functions, the solution to the wave equation is given by:

$$u(x, t) = \frac{\phi(x + ct) + \phi(x - ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds$$

where c is the speed of the wave.

Proof. We already know the general solution to the wave equation is given by:

$$u(x,t) = f(x+ct) + g(x-ct)$$

where f and g are arbitrary functions. We will now apply the boundary conditions to find the specific solution for our problem. We can calculate

$$u(x,0) = f(x) + g(x) = \phi(x)$$

and

$$u_t(x,0) = cf'(x) - cg'(x) = \psi(x)$$

Then, we can express f and g in terms of ϕ and ψ:

$$f'(x) = \frac{1}{2}(\phi'(x) + \frac{1}{c}\psi(x))$$

Integrating these equations gives us:

$$f(x) = \frac{1}{2}\left(\phi(x) + \frac{1}{c}\int_0^x \psi(s)ds\right) + C_1$$

Since $g(x) = \phi(x) - f(x)$, we can find g as well:

$$g(x) = \frac{1}{2}\left(\phi(x) - \frac{1}{c}\int_0^x \psi(s)ds\right) - C_1$$

By the general solution, we can write the solution to the wave equation as:

$$u(x,t) = \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right) + \frac{1}{2c}\int_0^{x+ct} \psi(s)ds - \frac{1}{2c}\int_0^{x-ct} \psi(s)ds$$

Rearranging this gives us:

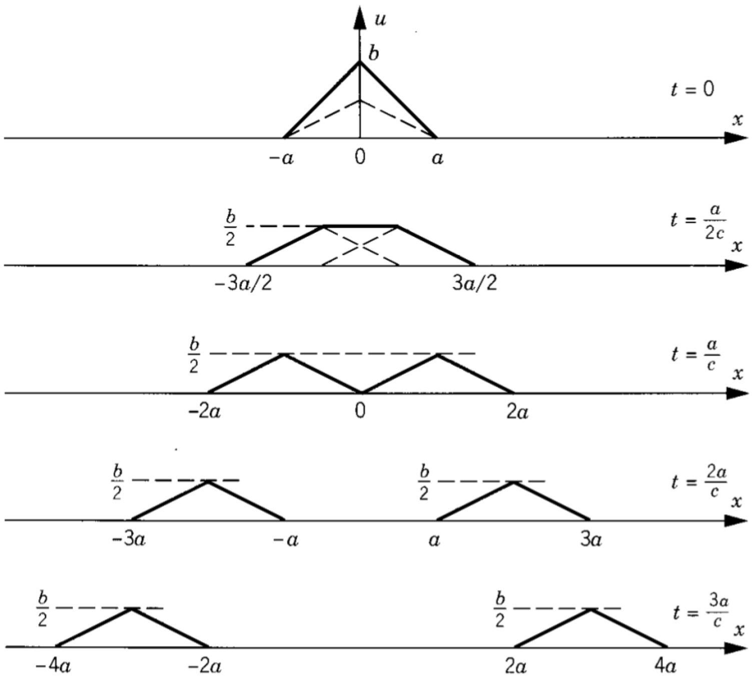
$$u(x,t) = \frac{\phi(x+ct) + \phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds$$

□

Example 11.2 (Plucked String)**.** Consider a plucked string travelling with speed c with the following boundary and initial conditions:

$$\phi(x) = \begin{cases} b - \dfrac{b\,|x|}{L}, & \text{for } |x| < a \\ 0, & \text{for } |x| \geq a \end{cases} \quad \text{and} \quad \psi(x) = 0$$

We can split it into different cases, which is described in the image above.



Lecture 12: April 28th

12.1 Energy of a System

Proposition 12.1 (Conservation of Energy in Wave Equation)**.** *Consider the wave equation*

$$u_{tt} = \frac{T}{\rho} u_{xx}$$

where T is the tension and ρ is the density. Then, it has a conserved energy

$$E(t) = \frac{1}{2} \int_{-\infty}^{\infty} \rho u_t^2(x,t) + T u_x^2(x,t)\, dx$$

where the first term is the kinetic energy and the second term is the potential energy.

Proof. We are to prove that

$$\frac{d}{dt} E(t) = 0$$

Therefore, we can differentiate the energy with respect to time

$$\frac{d}{dt} E(t) = \frac{1}{2} \int_{-\infty}^{\infty} \rho \frac{\partial}{\partial t} u_t^2(x,t) + T \frac{\partial}{\partial t} u_x^2(x,t)\, dx$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \rho u_t(x,t)u_{tt}(x,t) + Tu_x(x,t)u_{xt}(x,t)\,dx \\
&= \int_{-\infty}^{\infty} Tu_{xx}(x,t)u_t(x,t) + Tu_x(x,t)u_{xt}(x,t)\,dx \\
&= \int_{-\infty}^{\infty} T\frac{\partial}{\partial x}\left(u_x(x,t)u_t(x,t)\right)\,dx \\
&= \lim_{L\to\infty} T\int_{-L}^{L} \frac{\partial}{\partial x}\left(u_x(x,t)u_t(x,t)\right)\,dx \\
&= \lim_{L\to\infty} T\left[u_x(x,t)u_t(x,t)\right]_{-L}^{L}
\end{aligned}
$$

Assuming that the energy is bounded, the u_x and u_t vanish at infinity. This means that

$$\lim_{L\to\infty} \nabla_{x,t}u(x,t) = 0$$

we have

$$\frac{d}{dt}E(t) = 0$$

This means that the energy is conserved. □

Remark. This *energy method* is a common technique in PDEs. The idea is to multiply the equation by a function and integrate over the domain. This gives you an energy identity. We're also taking advantage of the fact the system has a bounded energy. This means that all the identity has to *vanish at infinity.*

The energy conservation law is important for hyperbolic systems. We call the energy system "Hamiltonian" if the energy is conserved. The wave equation is an example of a time-reversible system, as you can reverse the time to get another equation.

12.2 Causality of the Wave Equation

Recall that the D'Alembert formula for a wave travelling with speed c is given by

$$u(x,t) = \frac{\phi(x+ct)+\phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

where

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x).$$

Here $u(x_0,t_0)$ only depends on the information of its previous state in the light cone, called the **domain of dependence**.

$$\left\{(x,t)\in\mathbb{R}^2 :\ |x-x_0| \le x\,|t-t_0|\right\}.$$

This means that

1. The wave equation is **time-reversible**: if you reverse the time, you get the same equation. If you change from $u(x,t)$ to $u(x,-t)$, it also satisfies the wave equation.
2. The wave equation is **casual**: each point (x_0, t_0) in the wave equation can only be influenced by points in the past light cone. Nothing propogates with a speed greater than c, which is also called the **Principle of Causality**.

12.3 Heat Equation

In contrast, the heat equation does not conserve energy and is not time-reversible. (In fact, the heat equation has no conservation of entropy.) Moreover, the heat equation has an infinite speed of propagation of information.

Proposition 12.2. *For the heat equation, the energy is not conserved.*

Proof. Assuming there's no heat loss, the total mass is conserved.

$$\frac{d}{dt}M(t) = \frac{d}{dt}\int_{-\infty}^{\infty} u(x,t)\,dx = 0.$$

The dissipation of energy identity is given by

$$\begin{aligned}
\frac{d}{dt}E(t) =& \frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} u(x,t)^2\,dx\right) \\
=& \int_{-\infty}^{\infty} u(x,t)u_t(x,t)\,dx \\
=& \kappa\int_{-\infty}^{\infty} u_{xx}(x,t)u(x,t)\,dx \\
=& \kappa\int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left(u_x(x,t)u(x,t)\right)\,dx - \kappa\int_{-\infty}^{\infty} (u_x(x,t))^2\,dx \\
=& \kappa\lim_{L\to\infty}\left[u_x(x,t)u(x,t)\right]_{-L}^{L} - \kappa\int_{-\infty}^{\infty} (u_x(x,t))^2\,dx
\end{aligned}$$

Assuming that the decay of u and u_x as $L \to \infty$, we have

$$\frac{d}{dt}E(t) = -\kappa\int_{-\infty}^{\infty} (u_x(x,t))^2\,dx = -D(t)$$

is called the energy dissipation rate. This is called the entrophy identity. □

Consider a solution to the heat equation at $M = 1$ and $u(x,0) = \phi(x) > 0$. Then, we can interpret $u(x,t)$ as the probability density function.

Question 12.3. Show that

$$S(t) = -\int_{-\infty}^{\infty} u(x,t)\log(u(x,t))dx$$

is increasing in time. This is called the entropy identity.

Lecture 13: April 30th

13.1 Maximum Principle

A powerful test for analyzing parabolic and elliptic equations is the maximum principle. Roughly speaking, solutions to these equations cannot attain maximums at interior points unless the solution is constant.

Theorem 13.1 (Maximum Principle)**.** *Let $u(x,t)$ be a solution to the heat equation*

$$u_t = \kappa u_{xx}, \kappa > 0$$

over the domain $(x,t) \in [0,L] \times [0,T]$ for some $L, T > 0$. Then, the maximum and minimum of $u(x,t)$ is attained on the boundary of the domain, i.e.

$$(0 \times [0,T]) \cup (l \times [0,T]) \cup ([0,L] \times 0)$$

Remark. Note that the maximum and minimum of the solution is not necessarily attained at the top boundary. It's a *stronger* condition that the maximum and minimum are attained at the top boundary, where the function is *constant.*

Proof. (Method of Substitution) We'll only prove for the maximum, as we can take the negative of the function to get the minimum. Let M be the maximum of $u(x,t)$ over the left, right, and bottom boundaries. We want to show that

$$\max_{x\in[0,L]\times[0,T]} u(x,t) \leq M$$

Consider the substitution as a perturbation of the maximum

$$v(x,t) = u(x,t) - \varepsilon x^2$$

where $\varepsilon > 0$ is a small constant. Then, we have

$$v_t = u_t = \kappa u_{xx} = \kappa v_{xx} - 2\varepsilon\kappa$$

So v satisfies the inhomogeneous heat equation

$$v_t = \kappa v_{xx} - 2\varepsilon\kappa$$

Now, suppose v attains a maximum at some interiror point $(x_0, t_0) \in (0, L)\times(0, T)$. Then

$$v_t = 0, \quad v_x = 0, \quad v_{xx} \leq 0$$

However, the heat inequality implies that

$$0 = v_t(x_0, t_0) = \kappa v_{xx}(x_0, t_0) - 2\varepsilon\kappa \leq -2\varepsilon\kappa$$

This is a contradiction, since $\varepsilon > 0$ and $\kappa > 0$. Thus, v cannot attain a maximum at an interior point. Therefore, the maximum of u must be attained on the boundary. Similarly, if u attains a minimum at the upper boundary (x_0, T), then

$$v_t(x_0, t_0) \geq 0 \text{ and } v_{x_0,t_0} \leq 0.$$

There would be a contradiction, unless the function is constant and both inequality achieve equality. Thus, the maximum and minimum of u must be attained on the three sides of boundary. In particular,

$$\max_{\{x\in[0,L]\times[0,T]\}u(x,t)} \leq \max_{\{x\in[0,L]\times[0,T]\}v(x,t)} \leq \max_{\{x\in x=0\cup x=L\cup t=0\}} v(x,t) \leq M - \varepsilon x^2$$

On the other hand, we can take the limit as $\varepsilon \to 0$ to get

$$\max_{x\in[0,L]\times[0,T]} u(x,t) \leq M$$

□

Remark. This lemma shows that the heat equation is not well-posed in regions $t < 0$. We can see from the following proof that we can only attain uniquenesss in $t > 0$.

We can use this to show the uniqueness of the solution to the heat equation.

Theorem 13.2 (Uniqueness of Solution)**.** *Suppose $u(x,t)$ is a solution to the heat equation with the same initial and boundary conditions. Then, $u(x,t)$ is unique.*

Proof. Suppose we have two solutions $u_1(x,t)$ and $u_2(x,t)$ to the heat equation with the same initial and boundary conditions.

$$\begin{cases} (u_j)_t = \kappa(u_j)_{xx}, & 0 < x < L, 0 < t < T \\ u_j(x,0) = \phi(x), & 0 < x < L \\ u_j(0,t) = \psi(t), & 0 < t < T \\ u_j(L,t) = g(t), & 0 < t < T \end{cases}$$

Then, we can define the difference

$$w(x,t) = u_1(x,t) - u_2(x,t)$$

Then, w satisfies the heat equation

$$\begin{cases} w_t = \kappa w_{xx}, & 0 < x < L, 0 < t < T \\ w(x,0) = 0, & 0 < x < L \\ w(0,t) = 0, & 0 < t < T \\ w(L,t) = 0, & 0 < t < T \end{cases}$$

Thus, w satisfies the minimum principle, vanishing on $\{x \in x = 0 \cup x = L \cup t = 0\}$, with

$$\min_{x \in [0,L] \times [0,T]} w(x,t) \geq 0$$

Similarly, by the maximum principle, we have its maximum is also bounded by 0:

$$\max_{x \in [0,L] \times [0,T]} w(x,t) \leq 0$$

Thus, we know that $w(x,t) = 0$ for all $(x,t) \in [0,L] \times [0,T]$. Therefore, $u_1(x,t) = u_2(x,t)$ for all $(x,t) \in [0,L] \times [0,T]$. $\square$

Lecture 14: May 2nd

14.1 Solution to Heat Equation On Whole Line

We want to find the solution formula for the heat equation with the initial and boundary conditions

$$\begin{cases} u_t = \kappa u_{xx}, & t > 0, x \in \mathbb{R} \\ u(x,0) = \phi(x), & x \in \mathbb{R} \\ \lim_{x \to \pm\infty} u(x,t) = 0, & t > 0 \end{cases}$$

Under some special conditions, we can solve the heat equation using the Guassian kernel.

Definition 14.1. The **Gaussian kernel** is defined as

$$S(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} e^{-\frac{x^2}{4\kappa t}}$$

for $t > 0$ and $x \in \mathbb{R}$.

There's also a special function that we need to define, which is the **error function**:

Definition 14.2. The **error function** is defined as

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy$$

Proposition 14.3. *The key property we need to know about the error function is that*

$$\lim_{x\to\infty} \mathrm{Erf}(x) = 1, \quad \lim_{x\to-\infty} \mathrm{Erf}(x) = -1$$

and that the error function is an odd function.

Claim 14.4. Here are some properties of the solution to the heat equation that would help us find the solution.

1. Any translation of the solution $u(x - y, t)$ is also a solution to the heat equation for fixed y.
2. Any derivative of the solution u_x, u_t are also solutions to the heat equation.
3. The **integral** (or **convolution**) of the solution $S(x, t)$ is also a solution to the heat equation.

$$v(x, t) = \int_{-\infty}^{\infty} S(x - y, t)\phi(y)dy$$

   for any function $\phi(y)$, as long as the integral converges.
4. The **dilated** function $u(\sqrt{a}x, at)$ is also a solution to the heat equation.

Proposition 14.5. *The solution to the heat equation with the initial condition*

$$\phi(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

is given by the the integral of the Gaussian kernel

$$Q(x, t) = \int_{-\infty}^{x} S(y)dy = \frac{1}{2} + \frac{1}{2}\,\mathrm{Erf}\left(\frac{x}{\sqrt{4\kappa t}}\right)$$

where $S(y)$ is the Gaussian kernel.

Notation. We denote the solution to the special initial condition as $Q(x, t)$, in constrast to the general solution $u(x, t)$.

Proof. By the properties of dilation, we know that there's an intrinsic relation between the variables x and t in the solution. Thus, we can redefine the solution as a one-variable function of p

$$Q(x,t) = g(p), \quad p = \frac{x}{\sqrt{4\kappa t}}$$

We can also find the derivative of $Q(x,t)$ with respect to t and x:

$$\begin{aligned} Q_t &= \frac{dg}{dp}\frac{\partial p}{\partial t} = -\frac{1}{2t}\frac{x}{\sqrt{4\kappa t}}g'(p) \\ Q_x &= \frac{dg}{dp}\frac{\partial p}{\partial x} = \frac{1}{\sqrt{4\kappa t}}g'(p) \\ Q_{xx} &= \frac{dQ_x}{dp}\frac{\partial p}{\partial x} = \frac{1}{4\kappa t}g''(p) \end{aligned}$$

Thus, we can convert the heat equation into a one-variable ODE:

$$0 = Q_t - \kappa Q_{xx} = \frac{1}{t}\left(-\frac{1}{2}pg'(p) - \frac{1}{4}g''(p)\right)$$

Thus, we only need to solve the ODE

$$g'' + 2pg' = 0$$

This equation is easily solvable by integration:

$$Q(x,t) = g(p) = c_1 \int_0^{x/\sqrt{4\kappa t}} e^{-y^2}dy + c_2$$

where c_1 and c_2 are constants. We can find the constants by using the initial condition. Since the heat equation is only defined for $t > 0$, we can take the initial condition as $t \to 0^+$. For $x > 0$, we have

$$1 = \lim_{t\to 0^+} Q(x,t) = c_1 \int_0^{\infty} e^{-y^2}dy + c_2 = c_1\frac{\sqrt{\pi}}{2} + c_2$$

and for $x < 0$, we have

$$0 = \lim_{t\to 0^+} Q(x,t) = c_1 \int_0^{-\infty} e^{-y^2}dy + c_2 = -c_1\frac{\sqrt{\pi}}{2} + c_2$$

We can determine the constants by solving the system of equations. Thus, we have $c_1 = 1/\sqrt{\pi}$ and $c_2 = 1/2$. Therefore, we have

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4\kappa t}} e^{-y^2}dy = \frac{1}{2} + \frac{1}{2}\,\mathrm{Erf}\left(\frac{x}{\sqrt{4\kappa t}}\right)$$

This is the solution to the heat equation with the initial condition $\phi(x) = 1$ for $x > 0$ and $\phi(x) = 0$ for $x < 0$. $\square$

Corollary 14.6. *We can also find the relation between the error function and the Gaussian kernel:*

$$S(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} e^{-\frac{x^2}{4\kappa t}} = \frac{1}{2}\frac{\partial}{\partial x} \operatorname{Erf}\left(\frac{x}{\sqrt{4\kappa t}}\right) = Q_x(x,t)$$

where $Q(x,t)$ is the solution to the heat equation with the initial condition $\phi(x) = 1$ for $x > 0$ and $\phi(x) = 0$ for $x < 0$.

Proof. We can easily find that the Gaussian kernel $S(x,t)$ is actually the partial derivative of the solution $Q(x,t)$ with respect to x:

$$S(x,t) = Q_x(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} e^{-\frac{x^2}{4\kappa t}}$$

Thus, this gives us the relation between the Gaussian kernel and the error function. □

Remark. We can also have a useful fact that

$$S_x(x,t) = -\frac{1}{2\kappa t} S(x,t)$$

Then, we can use the properties of convolution to find the solution to the heat equation with the initial condition $\phi(x)$.

Theorem 14.7. *The solution to the heat equation with the initial condition $\phi(x)$ is given by the convolution of the Gaussian kernel $S(x,y)$ and the initial condition $\phi(x)$:*

$$u(x,t) = \int_{-\infty}^{\infty} \phi(y) S(x-y,t)\, dy = \int_{-\infty}^{\infty} \phi(x-y) S(y,t)\, dy = \phi * S(\cdot,t)$$

Remark. The proof would be given in Section 16

Lecture 15: May 5th

15.1 Application of the Solution to the Heat Equation

Last time, we covered the solution to the heat equation with a constant coefficient $\kappa \in \mathbb{R}$.

$$\begin{cases} u_t = \kappa u_{xx}, & x \in \mathbb{R}, t > 0, \\ u(x,0) = \phi(x), & x \in \mathbb{R} \end{cases}$$

We found the solution to be

$$u(x,t) = \int_{-\infty}^{\infty} \phi(y)S(x-y,t)\,dy = \int_{-\infty}^{\infty} \phi(x-y)S(y,t)\,dy = \phi * S(\cdot,t)$$

where $S(x,t)$ is the Gaussian kernel given by

$$S(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} e^{-\frac{x^2}{4\kappa t}} = Q_x(x,t)$$

with

$$Q(x,t) = \int_{-\infty}^{x} S(y,t)\,dy = \frac{1}{2} + \frac{1}{2}\operatorname{Erf}\left(\frac{x}{\sqrt{4\kappa t}}\right)$$

Example 15.1. Consider the initial condition

$$\phi(x) = \begin{cases} 0, & x < -a \\ b, & -a \le x \le a \\ 0, & x > a \end{cases}$$

Then, we can compute its solution as

$$\begin{aligned}
u(x,t) &= \int_{-\infty}^{\infty} \phi(y)S(x-y,t)\,dy \\
&= \int_{-a}^{a} bS(x-y,t)\,dy \\
&= b\left(\int_{\infty}^{a} S(x-y,t)\,dy - \int_{\infty}^{-a} S(x-y,t)\,dy\right) \\
&= b\left(-\int_{-\infty}^{x-a} S(x-y,t)\,d(x-y) + \int_{-\infty}^{x+a} S(x-y,t)\,d(x-y)\right) \\
&= b\left(\int_{x-a}^{-\infty} S(y,t)\,dy - \int_{x+a}^{-\infty} S(y,t)\,dy\right) \\
&= b\left(\int_{-\infty}^{x+a} S(y,t)\,dy - \int_{-\infty}^{x-a} S(y,t)\,dy\right) \\
&= b\left(Q(x+a,t) - Q(x-a,t)\right) \\
&= \frac{b}{2}\left(\operatorname{Erf}\left(\frac{x+a}{\sqrt{4\kappa t}}\right) - \operatorname{Erf}\left(\frac{x-a}{\sqrt{4\kappa t}}\right)\right)
\end{aligned}$$

This is the solution to the heat equation with the given initial condition.

Claim 15.2. Any piecewise constant initial condition has its solution given by $\operatorname{Erf}(x/\sqrt{4\kappa t})$ and its translation.

Example 15.3. Consider the initial condition $\phi(x) = e^{ax}$, then we can compute its solution as

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} e^{ay} S(x-y,t)\,dy \\ &= \int_{-\infty}^{\infty} \frac{e^{ay} e^{-\frac{(x-y)^2}{4\kappa t}}}{\sqrt{4\pi\kappa t}}\,dy \\ &= \frac{1}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{4\kappa t} + \frac{xy}{2\kappa t} - \frac{y^2}{4\kappa t} + ay}\,dy \end{aligned}$$

We can see the exponent as a quadratic form in y:

$$-\frac{x^2 + y^2 - 2xy - 4\kappa t a y}{4\kappa t} = -\frac{1}{4\kappa t}\left(x + 2\kappa t a - y\right)^2 + a^2\kappa t + ax$$

If we define

$$p = \frac{x + 2\kappa t a - y}{2\sqrt{\kappa t}}$$

then,we can calculate $u(x,t)$ as

$$u(x,t) = \int_{-\infty}^{\infty} \frac{e^{-p^2}}{\sqrt{\pi}} e^{a^2\kappa t + ax}\,dp = e^{a^2\kappa t + ax}$$

This is the solution to the heat equation with the given initial condition.

15.2 Uniqueness of the Solution

Theorem 15.4. *Let $\phi(x)$ be piecewise continuous and assume that $|\phi(x)| < e^{Mx^2}$ for some $M > 0$. Then, the unique solution on $\mathbb{R} \times (0, 1/4\kappa M)$ is given by*

$$u(x,t) = \int_{-\infty}^{\infty} S(y,t)\phi(x-y)\,dy$$

Remark. For this theorem, we assume that $\lim_{t\to 0^+} u(x,t) = \phi(x)$ at all points of continuity. At points of discontinuity, we assume that the limit is the average of the left and right limits.

$$\lim_{t\to 0^+} u(x,t) = \frac{1}{2}\lim_{h\to 0^+}\left(\phi(x-h) + \phi(x+h)\right)$$

Here is a famous counterexample to the uniqueness of the solution to the heat equation when there's no limit on the growth of the initial condition.

Example 15.5 (Tychonoff's Example)**.** Let $\alpha > 0$ and consider the following set of infinitely differentiable functions that are not analytic (i.e. does not have a Taylor expansion at $t = 0$):

$$g(t) = \begin{cases} e^{-\frac{1}{t^\alpha}} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

Then, we can construct solutions from the set of g to the heat equation $\kappa = 1$, as

$$u(x,t) = \sum_{k=0}^{\infty} \frac{g^{(k)}}{(2k)!} x^{2k}$$

with $u(x,0) \equiv 0$ but $u(x,t) \not\equiv 0$ for $t > 0$. Since we can choose any α for $g(t)$, we can construct infinitely many solutions to the heat equation with the same initial condition. This shows that the solution is not unique.

Lecture 16: May 7th

16.1 Errata on Uniqueness of Solution for Heat Equation

Proposition 16.1. *One only has uniqueness for solutions $u(x,t)$ that obey*

$$|u(x,t)| < Ce^{Rx^2}, t > 0, R > 0.$$

over the domain $(x,t) \in \mathbb{R} \times (0,T)$.

Example 16.2. This theorem also shows that the Tychonoff's example is not a solution to the heat equation. The Tychonoff's example is

$$u(x,t) = \sum_{k=0}^{\infty} \frac{g^{(k)}}{(2k)!} x^{2k}$$

where $g(x) \in C^\infty(\mathbb{R})$ is a smooth function. Here, for each $t > 0$,

$$|u(x,t)| \geq \delta e^{\frac{\delta x^2}{t}} \text{ for some } \delta > 0.$$

This is not a solution to the heat equation because it does not satisfy the uniqueness condition.

16.2 Convolution and Solutions to the Heat Equation

Then, we're going to show that our convolution solution solves for any initial value problem.

Proposition 16.3. *Given $\phi \in C^\infty(\mathbb{R})$, we have that*

$$S(\cdot,t) * \phi = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy$$

is a solution to the heat equation with initial condition ϕ over the domain $(x,t) \in \mathbb{R} \times \mathbb{R}^+$. Also,

$$\lim_{t \to 0+} u(x,t) = \phi(x)$$

for all $x \in \mathbb{R}$.

Proof. To see that it solves the heat equation, we can differentiate it with respect to t and x:

$$\frac{\partial}{\partial t}S(x,t) = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}S(x-y,t)\phi(y)dy$$

and

$$\frac{\partial^2}{\partial x^2}S(x,t) = \int_{-\infty}^{\infty} \frac{\partial^2}{\partial x^2}S(x-y,t)\phi(y)dy.$$

Since $S(x,t)$ is a solution to the heat equation, we have that

$$\frac{\partial}{\partial t}S(x,t) = \frac{\partial^2}{\partial x^2}S(x,t).$$

Thus we've shown that $S(\cdot,t) * \phi$ is a solution to the heat equation. On the other hand, to see that it satisfies the initial condition, we can fix x. Then, we have

$$\begin{aligned} u(x,t) =& \int_{-\infty}^{\infty} S(y,t)\phi(x-y)dy \\ =& \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{y^2}{4kt}}\phi(x-y)dy \end{aligned}$$

If we apply the change of variables

$$p = \frac{y}{\sqrt{4kt}} \implies dp = \frac{dy}{\sqrt{4kt}}$$

We can evaluate this as

$$u(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-p^2}\phi\left(x - p\sqrt{4kt}\right)dp$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} \left(\phi(x - p\sqrt{4kt}) - \phi(x) \right) dp + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} \phi(x) dp \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} \left(\phi(x - p\sqrt{4kt}) - \phi(x) \right) dp + \phi(x)
\end{aligned}
$$

Thus, we by having the limit $t \to 0^+$, we have the first term goes to 0

$$
\lim_{t \to 0^+} u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} \lim t \to 0^+ \left(\phi(x - p\sqrt{4kt}) - \phi(x) \right) dp + \phi(x) = \phi(x)
$$

Thus, we have that

$$
\lim_{t \to 0^+} u(x,t) = \phi(x)
$$

for all $x \in \mathbb{R}$. □

Actually, all the calculations above can be done with the convolution identity $\delta(x)$.

Definition 16.4. The **Dirac delta function** $\delta(x)$ is defined as

$$
\delta(x) = \begin{cases} 0 & x \neq 0 \\ +\infty & x = 0 \end{cases}
$$

and it satisfies the property that

$$
\int_{-\infty}^{\infty} f(x-y)\delta(y)dy = f(x)
$$

for all f.

With this definition, we can see that the second part of the proof can be easily done with

$$
\lim_{t \to 0^+} S(x,t) = \delta(x)
$$

Lecture 17: May 9th

17.1 Method of Reflections on Heat Equation

Up until now, we've been considering the heat equation on the whole line, with no boundary conditions. Now, we consider the heat equation on the half line, with boundary conditions. There are two extra boundary conditions we can consider: Dirichlet and Neumann.

Consider the Heat Equation over $x \in (0, \infty)$ instead of $x \in \mathbb{R}$ that is define on the half line with Dirichlet boundary condition.

$$\begin{cases} u_t = u_{xx} & x > 0, t > 0 \\ u(0,t) = 0 & t > 0 \\ u(x,0) = \phi(x) & x \geq 0 \end{cases}$$

We can use an add reflection of $\phi(x)$ to the left side so that the extension doesn't affect the solution on the right half.

$$\phi_{\text{odd}}(x) = \begin{cases} \phi(x) & x > 0 \\ -\phi(-x) & x < 0 \\ 0 & x = 0 \end{cases}$$

Then, our solution can be written as

$$u(x,t) = \int_{\infty}^{\infty} S(x-y,t)\phi_{\text{odd}}(y)dy$$

where $S(x,t)$ is the heat kernel. Indeed, we can check that

$$\begin{aligned} u_t = & \int_{\infty}^{\infty} S_t(x-y,t)\phi_{\text{odd}}(y)dy = k\int_{\infty}^{\infty} S_{xx}(x-y,t)\phi_{\text{odd}}(y)dy \\ =& k\left(\frac{\partial}{\partial x}\right)^2 \int_{\infty}^{\infty} S(x-y,t)\phi_{\text{odd}}(y)dy \\ =& k u_{xx}(x,t) \end{aligned}$$

Moreover, as $t \to 0^+$, we have

$$u(x,t) = \phi_{\text{odd}}(x) = \phi(x) \text{ for } x > 0$$

For boundary condition, we can check that

$$\begin{aligned} u(x,t) &= \int_0^{\infty} S(x-y,t)\phi(y)dy + \int_{-\infty}^{0} S(x-y,t)\phi(-y)dy \\ &= \int_0^{\infty} S(x-y,t)\phi(y)dy - \int_0^{\infty} S(x+y,t)\phi(y)dy \\ &= \int_0^{\infty} \left(S(x-y,t) - S(x+y,t)\right)\phi(y)dy \end{aligned}$$

When $x = 0$, we have

$$u(0,t) = \int_0^{\infty} \left(S(-y,t) - S(y,t)\right)\phi(y)dy = 0$$

This is because $S(-y,t) = S(y,t)$ and $\phi(y)$ is odd.

Remark. Note that for the odd extension, we require that $\phi(x)$ is odd. This means that we need a ***homogeneous boundary condition***. That is to say, if $u(0,t) \neq 0$, we have to make a change of variable to make it odd by minusing the offset of $u(0,t)$.

Example 17.1. Solve the heat equation with the initial condition $\phi(x) = 0$.

Solution. We have

$$\begin{aligned}
u(x,t) =& \int_0^\infty \left(S(x-y,t) - S(x+y,t)\right)\phi(y)dy \\
=& \int_{-\infty}^x S(z,t)dz - \int_x^\infty S(z,t)dz \\
=& \int_{-\infty}^x S(z,t)dz - 1 + \int_{-\infty}^x S(z,t)dz \\
=& 2\int_{-\infty}^x S(z,t)dz - 1 \\
=& 2\left(\frac{1}{2} + \frac{1}{2}\operatorname{Erf}\left(\frac{x}{\sqrt{4kt}}\right)\right) - 1 \\
=& \operatorname{Erf}\left(\frac{x}{\sqrt{4kt}}\right)
\end{aligned}$$

□

Then, we can consider the Heat Equation on the half-line with Neumann boundary condition. (This can be understood as having a cap that stops the heat from escaping.) Note that we also require that the ***boundary condition is homogeneous***.

$$\begin{cases} u_t = u_{xx} & x > 0, t > 0 \\ u_x(0,t) = 0 & t > 0 \\ u(x,0) = \phi(x) & x \geq 0 \end{cases}$$

We can use an even reflection of $\phi(x)$ to the left side so that the extension doesn't affect the solution on the right half.

$$\phi_{\text{even}}(x) = \begin{cases} \phi(x) & x > 0 \\ \phi(-x) & x < 0 \\ \phi(0) & x = 0 \end{cases}$$

Then, our solution can be written as

$$u(x,t) = \int_\infty^\infty S(x-y,t)\phi_{\text{even}}(y)dy$$

where $S(x,t)$ is the heat kernel. We can rewrite it as

$$u(x,t) = \int_0^\infty (S(x-y,t) + S(x+y,t))\,\phi(y)dy$$

Example 17.2. Solve the heat equation with the initial condition $\phi(x) \equiv 1$.

Solution. The solution can be simply given by

$$u(x,t) = \int_{-\infty}^\infty S(x-y,t)dy$$

□

17.2 Method of Reflections on Wave Equation

Consider the Wave Equation over $x \in (0,\infty)$ instead of $x \in \mathbb{R}$ that is define on the half line with boundary condition. (This can be understood as having a wall at $x = 0$)

$$\begin{cases} u_{tt} = c^2 u_{xx} & x > 0, t \in \mathbb{R} \\ u(x,0) = \phi(x) & x > 0 \\ u_t(x,0) = \psi(x) & x > 0 \\ u(0,t) = 0 & t \in \mathbb{R} \end{cases}$$

To solve this, we can add odd reflections to D'Alembert's solution.

$$u(x,t) = \frac{1}{2}\left(\phi_{\text{odd}}(x-ct) + \phi_{\text{odd}}(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{odd}}(y)dy$$

When $x \geq ct$, the formula is the same as the one for the whole line. For $x < ct$, we can use the fact that $\phi_{\text{odd}}(x) = -\phi_{\text{odd}}(-x)$ and $\psi_{\text{odd}}(x) = -\psi_{\text{odd}}(-x)$ to get

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(\phi(x+ct) - \phi(ct-x)\right) + \frac{1}{2c}\int_{x-ct}^{0} -\psi(-s)ds + \frac{1}{2c}\int_0^{x+ct} \psi(s)ds \\ &= \frac{1}{2}\left(\phi(x+ct) - \phi(ct-x)\right) + \frac{1}{2c}\int_0^{x+ct} \psi(s)ds - \frac{1}{2c}\int_0^{x-ct} \psi(s)ds \\ &= \frac{1}{2}\left(\phi(x+ct) - \phi(ct-x)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds \end{aligned}$$

Thus, we can combine the two cases to get

$$u(x,t) = \frac{1}{2}\left(\phi(x+ct) + \operatorname{sgn}(x-ct)\phi(|x-ct|)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{odd}}(s)ds$$

Remark. Note that here, we still need $\psi_{\text{odd}}(x)$ to be the odd extension of $\psi(x)$, since its domain would certainly go beyond the half line.

Lecture 18: May 12th

18.1 Diffusion with a Source

We're trying to solve the inhomogeneous diffusion equation with a source term. The equation is given by:

$$\begin{cases} u_t - ku_{xx} = f(x,t), & x \in \mathbb{R}, t > 0 \\ u(x,0) = \phi(x), \end{cases}$$

where $f(x,t)$ is the source term and $\phi(x)$ is the initial condition.

The idea is similar to solving the multi-dimensional first-order ODEs. Looks as the following example:

Proposition 18.1 (Duhamel's Principle)**.** *The solution of the first-order multi-dimensional linear ODE*

$$y' = Ay + f(t), \quad y(0) = y_0, \quad A \in \mathbb{R}^{n\times n},$$

is given by:

$$y(t) = e^{At}y_0 + \int_0^t e^{A(t-s)}f(s)\,ds.$$

where e^{At} is defined as the matrix exponential. The first term is the solution of the homogeneous equation, and the second term is the particular solution.

Lemma 18.2 (Leibniz Rule)**.** *We have the following rule for the derivative of an integral:*

$$\frac{\partial}{\partial t}\int_0^t F(z,t)\,dz = F(t,t) + \int_0^t F_t(z,t)\,dz,$$

Proof. If we define $F(r,t) = \int_0^r F(z,t)\,dz$, then

$$\frac{\partial}{\partial t}G(t,t) = G_r(t,t) + G_t(t,t) = F(t,t) + \int_0^t F_t(z,t)\,dz.$$

□

Theorem 18.3. *The solution of the inhomogeneous diffusion equation with a source term is given by:*

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy + \int_0^t\int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)\,dy\,ds,$$

where the first term is the solution of the homogeneous equation and the second term is the particular solution.

Proof. Then, we use the Duhamel's Pinciple to solve the inhomogeneous diffusion equation. We already know that the first term solves the homogeneous equation. Therefore, it suffices to check that the second term solves the inhomogeneous equation, i.e., the case when $\phi \equiv 0$. Let the second term be denoted as

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} S(x-y, t-s) f(y,s)\, dy\, ds,$$

By the previous lemma, we have:

$$u_t(x,t) = \int_{-\infty}^{\infty} S(x-y, 0^+) f(y,t)\, dy + \int_0^t \int_{-\infty}^{\infty} \frac{\partial}{\partial t} S(x-y, t-s) f(y,s)\, dy\, ds$$

Since $S(x-y, 0^+) = \delta(x-y)$ being the Dirac delta function, and given that $S(x-y, t-s)$ is the fundamental solution of the diffusion equation, we have:

$$\begin{aligned} u_t(x,t) =& f(x,t) + \int_0^t \int_{-\infty}^{\infty} \kappa S_x x(x-y, t-s) f(y,s)\, dy\, ds \\ =& f(x,t) + \kappa \left(\frac{\partial}{\partial x}\right)^2 \int_0^t \int_{-\infty}^{\infty} S(x-y, t-s) f(y,s)\, dy\, ds \\ =& f(x,t) + \kappa u_{xx}(x,t). \end{aligned}$$

This shows that the second term solves the inhomogeneous equation. □

For the heat equation, we can think of u as a solution to an infinite-dimensional function space.

$$u_t = Au + f, \quad A = \kappa \left(\frac{\partial}{\partial x}\right)^2.$$

where A is an operator on the space of functions. Therefore, we can write the solution formula as a solution to an ODE in fuunctional space.

Corollary 18.4 (Duhamel's Formula)**.** *The Duhamel's Formula for inhomogeneous heat equation is given by:*

$$u(x,t) \left[\mathcal{S}(t)\phi\right](x) + \int_0^t \left[\mathcal{S}(t-s)f\right](t,x)\, ds,$$

where $\mathcal{S}(t)$ is the operator that takes a function and convolves it with the fundamental solution of the diffusion equation

$$\mathcal{S}(t)v = \int_{-\infty}^{\infty} S(x-y, t) v(y)\, dy, \quad v \colon \mathbb{R} \to \mathbb{R}$$

for an appropriate function.

Remark. We can also conclude that the solution to ODE satisfies $\mathcal{S}(t) = e^{At}$.

Then, we see how to use the Duhamel's formula to solve the inhomogeneous diffusion equation on the half-line.

Example 18.5. Consider the heat equation on the half-line with a source term. Notice that we also need to specify the boundary condition. The equation is given by:

$$\begin{cases} u_t - ku_{xx} = f(x,t), & x > 0, t > 0 \\ u(0,t) = g(t), & t > 0 \\ u(x,0) = \phi(x), & x > 0 \end{cases}$$

where $g(t)$ is the boundary condition.

Proof. To solve this, we work with the "shifted" solution with a homogeneous boundary condition, so that we can use an odd extension a Dirichlet boundary condition.

$$v(x,t) = u(x,t) - g_o(t), \quad g_o(t) = \begin{cases} g(t), & t > 0 \\ 0, & t = 0 \\ -g(-t), & t < 0 \end{cases}$$

This solves the set of equations:

$$\begin{cases} v_t - kv_{xx} = f(x,t) - g_o'(t), & x > 0, t > 0 \\ v(0,t) = 0 \\ v(x,0) = \phi(x) - g_o(0) \end{cases}$$

We can apply the Duhamel's formula to this equation.

$$\begin{aligned} v(x,t) = &\int_0^\infty \left[S(x-y,t) - S(x+y,t)\right] \left(\phi(y) - g(0)\right)\, dy \\ &+ \int_0^t \int_0^\infty \left[S(x-y,t-s) - S(x+y,t-s)\right] \left(f(y,s) - g'(s)\right)\, dy\, ds \end{aligned}$$

□

We can also deal with the Neumann boundary condition with the *method of substraction.*

Example 18.6. Consider the heat equation on the half-line with a source term. The equation is given by:

$$\begin{cases} u_t - ku_{xx} = f(x,t), & x > 0, t > 0 \\ u_x(0,t) = g(t) \\ u(x,0) = \phi(x), & x > 0 \end{cases}$$

where $g(t)$ is the boundary condition.

Solution. We provide a sketch of the proof. We can substract the boundary condition from the solution.

$$v(x,t) = u(x,t) - g_e(t)x$$

This solves the set of equations:

$$\begin{cases} v_t - kv_{xx} = f(x,t) - g_e'(t) \\ v_x(0,t) = 0 \\ v(x,0) = \phi(x) - g_e(0)x \end{cases}$$

We then apply the even extension to the solution and the Duhamel's formula. $\square$

Lecture 19: May 16th

19.1 Wave Equation with a Source

Last time we covered the heat equation with a source. We will now look at the wave equation with a source. The wave equation under the Dirichlet boundary condition is given by:

$$\begin{cases} u_{tt} = c^2 u_{xx} + f(x,t), \quad x \in \mathbb{R}, t > 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

We'll show that the solution formula is given by the following theorem.

Theorem 19.1. *The solution of the wave equation with a source is given by*

$$u(x,t) = \frac{1}{2}\left(\phi(x-ct) + \phi(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds$$

where the last term is understood in the sense of distributions.

Remark. This formula can be understood as integrating the fuction $f(x,t)$ over the triangular region with vertices $(x-ct, 0)$, $(x+ct, 0)$, and (x,t).

$$\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds = \int_{\triangle} f(y,s)dyds$$

Proof. By linearity, it suffices to check that when $\phi = \psi \equiv 0$, the solution formula holds. Then, we have

$$u(x,t) = \frac{1}{2}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds$$

and by Lemma 18.2, we have

$$u_t(x,t) = \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dy\Bigg|_{s=t} + \frac{1}{2c}\int_0^t \frac{\partial}{\partial t}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds$$

The first term is zero because $x - c(t-t) = x + c(t-t) = x$. For the second term, if we assume that $F(y,s) = \int f(y,s)dy$, we have

$$\frac{\partial}{\partial t}\left(F(x+c(t-s),s) - F(x-c(t-s),s)\right) = f(x+c(t-s),s)\cdot c - f(x-c(t-s),s)\cdot(-c)$$

Thus, we have

$$u_t(x,t) = \frac{1}{2}\int_0^t \left(f(x+c(t-s),s) + f(x-c(t-s),s)\right) ds$$

Similarly, we can compute $u_{tt}(x,t)$:

$$\begin{aligned} u_{tt}(x,t) =& \frac{1}{2}\left(f(x,t) + f(x,t)\right) + \frac{1}{2}\int_0^t \left(cf_1(x+c(t-s),s) - f_1(x-c(t-s),s)\right) ds \\ =& f(x,t) + \frac{c}{2}\int_0^t \left(cf_1(x+(t-s),s) - f_1(x-c(t-s),s)\right) ds \end{aligned}$$

where $f_1(x,t) = \frac{\partial}{\partial x} f(x,t)$. We can compute also the partial derivative with respect to x:

$$u_x(x,t) = \frac{1}{2c}\int_0^t f(x+c(t-s),s)\,ds - \frac{1}{2c}\int_0^t f(x-c(t-s),s)\,ds$$

and

$$u_{xx}(x,t) = \frac{1}{2c}\int_0^t f_1(x+c(t-s),s)\,ds - \frac{1}{2c}\int_0^t f_1(x-c(t-s),s)\,ds$$

Thus, we can check that

$$u_{tt} - c^2 u_{xx} = f(x,t)$$

and we have the solution formula. $\square$

We can also have another interpretation of the solution formula with superposition.

Proposition 19.2. *Suppose that $f(x,t)$ were a perfectly localized pulse at some point $t = t_0$, i.e.,*

$$f(x,t) = \delta(t - t_0) \cdot F(x)$$

for some function $F(x,t)$ and the Kronecker delta function $\delta(t - t_0)$ so that we have the following wave equation:

$$\begin{cases} u_{tt} = c^2 u_{xx} + \delta(t - t_0) \cdot F(x), & x \in \mathbb{R}, t > 0 \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$$

Then, its solution also solves the following wave equation with a source.

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R}, t > t_0 \\ u(x,t_0) = 0 \\ u_t(x,t_0) = F(x) \end{cases}$$

Proof. We pick $u(x,t)$ to be the solution of the wave equation with a source. Clearly, by the kronecker delta function, we have that $u_{tt} = c^2 u_{xx}$ for $t > t_0$. Then, for u_t, we have

$$u_t(x,t) = u_t(x,t_0) + \int_{t_0-\epsilon}^{t} c^2 u_{xx}(x,s) + F(x)\delta(s - t_0) ds$$

The first term is zero because $u(x,t_0) = 0$ and as the Kronecker delta function is an identity for convolution, we have

$$u_t(x,t) = \int_{t_0-\epsilon}^{t} c^2 u_{xx}(x,s) ds + F(x)$$

Thus, as $t \to t_0$ and $\epsilon \to 0$, we have

$$u_t(x,t_0) = F(x)$$

and that

$$\lim_{t \to t_0^+} u(x,t) = 0$$

This shows that the solution of the wave equation with a source is also a solution of the wave equation without a source. $\square$

Lecture 20: May 19th

20.1 Wave Equation with Impulses

Last time we proved that the problem

$$\begin{cases} u_{tt} = c^2 u_{xx} + \delta(t - t_0)F(x), & x \in \mathbb{R}, t > 0 \\ u(x, 0) = u_t(x, 0) = 0 \end{cases}$$

is equivalent to the problem

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R}, t > t_0 \\ u(x, t_0) = 0 \\ u_t(x, t_0) = F(x) \end{cases}$$

Hence its solution is given by

$$u(x,t) = \frac{1}{2c} \int_{x+(t-t_0)c}^{x-(t-t_0)c} F(y)\, dy = \frac{1}{2c} \int_0^t \int_{x+(t-s)c}^{x-(t-s)c} \delta(s - t_0) F(y)\, dy\, ds$$

Today, we're using this to prove the general solution of the wave equation with a source term with the superposition principle.

Claim 20.1 (Superposition principle)**.** Using the formula for a single impulse, we can write the solution of the wave equation with a source term as an infinite sum of impulses.

Now, we give the proof with the principle of superposition.

Proof. More generally, given N impulses at times $t_1, \ldots, t_N$,

$$f(x,t) = \sum_{i=1}^{N} \delta(t - t_i) F_i(x)$$

where $\delta(t)$ is the solution operator of the wave equation. We can write by linearity

$$\begin{aligned} u(x,t) &= \sum_{i=1}^{N} \frac{1}{2c} \int_{x+(t-t_i)c}^{x-(t-t_i)c} F_i(y)\, dy \\ &= \sum_{i=1}^{N} \frac{1}{2c} \int_0^t \int_{x+(t-s)c}^{x-(t-s)c} \delta(s - t_i) F_i(y)\, dy\, ds \end{aligned}$$

Now if the impulses taken in a dense way such that $f(x,t) = \sum_{i=1}^{N} S(t-t_i)F_i(x)$ is a smooth function, we can take the limit as $N \to \infty$ and write

$$f(x,t) = \int_0^t \delta(t-s) f(x,s)\, ds$$

Hence, by takeing such a *linear superposition* of impulses $\{\delta(t-s)\}_{s\in[0,t]}$, we can write

$$\begin{aligned} u(x,t) =& \frac{1}{2c}\int_0^t \int_{x+(t-s)c}^{x-(t-s)c} \left(\int_0^t \delta(s-r) f(y,r)\, dr\right) dy\, ds \\ =& \frac{1}{2c}\int_0^t \int_{x+(t-s)c}^{x-(t-s)c} f(y,s)\, dy\, ds \end{aligned}$$

$\square$

20.2 Separation of Variable of Dirichlet Boundary Condition

Proposition 20.2. *The solution for heat equation with homogeneous Dirichlet boundary conditions*

$$\begin{cases} u_t = \kappa u_{xx}, & x \in [0,L], t > 0 \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = \phi(x) \end{cases}$$

is given by

$$u(x,t) = \sum_{k=1}^{\infty} e^{-\kappa\left(\frac{k\pi}{L}\right)^2 t} b_k \sin\left(\frac{k\pi x}{L}\right)$$

where ϕ is written as

$$\phi(x) = \sum_{k=1}^{\infty} b_j \sin\left(\frac{k\pi x}{L}\right), \quad b_k = \frac{2}{L}\int_0^L \phi(y) \sin\left(\frac{k\pi y}{L}\right) dy$$

Proof. A good guess is that the solution is an infinite sum of products of a function of x and a function of t (or a tensor product in functional spaces)

$$u(x,t) = X(x)T(t)$$

We can plug this into the equation and get

$$X(x)T'(t) = \kappa X''(x)T(t) \implies \frac{1}{\kappa}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda$$

Thus, we look for all possible solutions to the following system of equations

$$\begin{cases} T'(t) = \kappa\lambda T(t), & t > 0 \\ X''(x) = \lambda X(x), & x \in (0, L) \end{cases}$$

that satisfy the boundary conditions

$$0 = T(t)X(t) = T(t)X(t) = T(t)X(L), \quad t > 0 \iff X(0) = X(L) = 0$$

(a) For the positive case $\lambda > 0$, we can write

$$X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$$

Plugging in the boundary conditions, we get

$$\begin{cases} A + B = 0 \\ Ae^{\sqrt{\lambda}L} + Be^{-\sqrt{\lambda}L} = 0 \end{cases}$$

This gives us $A = 0$ and $B = 0$.

(b) For the zero case $\lambda = 0$, we have

$$X(x) = A + B, \quad X''(x) = 0$$

Plugging in the boundary conditions, we get

$$A = 0, \quad B = 0$$

(c) For the negative case $\lambda < 0$, we can write

$$\begin{cases} X''(x) = -\xi^2 X(x) \\ X(x) = A\cos(\xi x) + B\sin(\xi x) \end{cases}$$

Plugging in the boundary conditions, we get

$$\begin{cases} A = X(0) = 0 \\ B\sin(\xi L) = X(L) = 0 \end{cases}$$

If $\sin(\xi L) \neq 0$, we have $A = B = 0$. Otherwise, there must be

$$\xi L = n\pi, \quad n \in \mathbb{Z}.$$

Up to scaling, we have a non-trivial solution

$$X_k(x) = \sin\left(\frac{k\pi x}{L}\right), \quad k = 1, 2, \ldots$$

Here, we only consider non-negative integers k because the sine function is odd and the sine function is odd. The correspinding solution for $T(t)$ is

$$T_k'(t) = -\kappa\xi^2 T_k(t) = -\kappa\left(\frac{k\pi}{L}\right)^2 T_k(t).$$

Up to rescaling, we can write

$$T_k(t) = e^{-\kappa\left(\frac{k\pi}{L}\right)^2 t}$$

Hence, the solution is given by

$$u_k(x,t) = T_k(t)X_k(x) = e^{-\kappa\left(\frac{k\pi}{L}\right)^2 t}\sin\left(\frac{k\pi x}{L}\right)$$

By linearity, we can take any linear combination of the initial conditions in the given form

$$\phi(x) = \sum_{j=1}^{N} b_j \sin\left(\frac{j\pi x}{L}\right).$$

Then we can solve using

$$u(x,t) = \sum_{j=1}^{N} b_j e^{-\frac{j^2\pi^2\alpha t}{L^2}} \sin\left(\frac{j\pi x}{L}\right).$$

for any coefficients b_j. By the maximal principle, this is the unique solution to the initial data. Then, we can take $N \to \infty$ as an infinite collection of Fourier modes.

$$u(x,t) = \sum_{j=1}^{\infty} b_j e^{-\frac{j^2\pi^2\alpha t}{L^2}} \sin\left(\frac{j\pi x}{L}\right).$$

This satisfies the Heat Equation with the initial data given by the Fourier series.

$$u(x,0) = \sum_{j=1}^{\infty} b_j \sin\left(\frac{j\pi x}{L}\right).$$

$\square$

Lecture 21: May 21st

21.1 Fourier sine series and Basis

Last time, we gave the general solution to the heat equation on the interval $[0, L]$ with boundary conditions $u(0, t) = u(L, t) = 0$ for all $t \geq 0$. In fact, this requires that the boundary condition $\phi(x)$ be expanded in a Fourier sine series. Today, we discuss the Fourier sine series and how to write it.

In fact, every function can be written as a Fourier series.

Definition 21.1. The **Hilbert space** is an inner product space plus a complete metric over continuous functions. In particular, $L^2([0, L])$ represents the set of continuous functions $f : [0, L] \to \mathbb{R}$, with the inner product defined by

$$\langle f, g\rangle = \frac{2}{L}\int_0^L f(x)g(x)\,dx.$$

This inner product space induces the L^2 norm

$$\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\frac{2}{L}\int_0^L f(x)^2\,dx}.$$

and a metric

$$d(f, g) = \|f - g\| = \sqrt{\frac{2}{L}\int_0^L (f(x) - g(x))^2\,dx}.$$

Proposition 21.2. *The Hilbert space admits an* ***orthonormal basis*** *consisting of the functions*

$$\left\{X_j(x) = \sin\left(\frac{j\pi x}{L}\right) \mid j \in \mathbb{N}\right\},$$

Proof. We'll accept the fact that this is a basis, and show that the functions X_j are orthonormal. We can check that it's an orthonormal basis by checking the inner product when $j = k$ and $j \neq k$. When $j \neq k$, we have

$$\begin{aligned}
\langle X_j, X_k\rangle =& \frac{2}{L}\int_0^L \sin\left(\frac{j\pi x}{L}\right)\sin\left(\frac{k\pi x}{L}\right)\,dx\\
=& \frac{2}{L}\int_0^L \frac{1}{2}\left(\cos\left(\frac{(j-k)\pi x}{L}\right) - \cos\left(\frac{(j+k)\pi x}{L}\right)\right)\,dx\\
=& \frac{1}{L}\left[\frac{L}{(j-k)\pi}\sin\left(\frac{(j-k)\pi x}{L}\right) - \frac{L}{(j+k)\pi}\sin\left(\frac{(j+k)\pi x}{L}\right)\right]_0^L
\end{aligned}$$

$$= \frac{1}{(j-k)\pi} \sin((j-k)\pi) - \frac{1}{(j+k)\pi} \sin((j+k)\pi) = 0.$$

When $j = k$, we have

$$\langle X_j, X_j \rangle = \frac{2}{L} \int_0^L \sin^2\left(\frac{j\pi x}{L}\right) dx = \frac{1}{2}\left[x - \frac{L}{2j\pi} \sin\left(\frac{2j\pi x}{L}\right)\right]_0^L = 1.$$

Thus, the functions X_j are orthonormal. □

Remark. The Hilber space is a **complete metric space**, meaning that every Cauchy sequence converges to a limit in the space. This is important for the Fourier series, as it allows us to approximate every function in the space by a series of the linear combinations of the basis functions X_j.

Proposition 21.3. *From this we can write the coefficients of the Fourier sine series of every function in the Hilbert space $L^2([0, L])$ as*

$$\langle \phi, X_j \rangle = b_j \sum_{j=1}^{\infty} b_j \sin\left(\frac{j\pi x}{L}\right) = b_j \langle X_j, X_j \rangle$$

where the coefficients b_j are given by

$$b_j = \langle \phi, X_j \rangle = \frac{2}{L} \int_0^L \phi(x) \sin\left(\frac{j\pi x}{L}\right) dx.$$

Remark. Note that for other Hilbert spaces over different intervals, the inner product and coefficients will change. For example, for the interval $[-L, L]$, the inner product would be

$$\langle f, g \rangle = \frac{1}{L} \int_{-L}^{L} f(x) g(x)\, dx.$$

This would be discussed in a later lecture.

Lecture 22: May 23rd

22.1 Examples on Heat Equation with Dirichlet Boundary Conditions

In lecture 21, we covered the general solution to the heat equation over a finite domain.

$$u(x,t) = \sum_{j=1}^{\infty} b_j e^{-\kappa\left(\frac{j\pi}{L}\right)^2 t} \sin\left(\frac{j\pi x}{L}\right)$$

with the initial condition

$$\phi(x) = \sum_{j=1}^{\infty} b_j X_j(x), \quad X_j(x) = \sin\left(\frac{j\pi x}{L}\right)$$

and the coefficients

$$b_j = \langle \phi, X_j \rangle = \frac{2}{L}\int_0^L \phi(x)\sin\left(\frac{j\pi x}{L}\right) dx.$$

In general, given such an initial condition $\phi(x)$, this series will converge to $\phi(x)$ at least everywhere under appropriate conditions on $\phi(x)$ (e.g., piecewise continuous).

Example 22.1. For the initial condition $\phi(x) \equiv 1$ representing a metal rod with constant temperature 1 at $t = 0$, we can compute the coefficients b_j as follows:

$$b_j = \frac{2}{L}\int_0^L 1 \cdot \sin\left(\frac{j\pi x}{L}\right) dx = -\frac{2}{j\pi}\left((-1)^j - 1\right) = \frac{2}{j\pi}\left(1 - (-1)^j\right).$$

Then, we can write the solution as

$$v(x,t) = \sum_{j=1}^{\infty} \frac{2}{j\pi}\left(1 - (-1)^j\right)\sin\left(\frac{j\pi x}{L}\right) e^{-\kappa\left(\frac{j\pi}{L}\right)^2 t}.$$

By ignoring the terms where j is even (since they contribute zero), we can simplify this to

$$v(x,t) = \sum_{j=1}^{\infty} \frac{4}{(2j-1)\pi}\sin\left(\frac{(2j-1)\pi x}{L}\right) e^{-\kappa\left(\frac{(2j-1)\pi}{L}\right)^2 t}.$$

Discussing the order of the solution formula, we know

$$v(x,t) = \frac{4}{\pi}\sin \pi x e^{-\kappa\pi^2 t} + \frac{4}{3\pi}\sin 3\pi x e^{-9\kappa\pi^2 t} + \ldots = \frac{4}{\pi}\sin \pi x e^{-\kappa\pi^2 t} + O\left(e^{-9\kappa\pi^2 t}\right).$$

22.2 Wave Equation with Neumann Boundary Conditions

Proposition 22.2. *The solution to the wave equation with Neumann boundary conditions over the interval* $[0, L]$*:*

$$\begin{cases} u_{tt} = c^2 u_{xx}, x \in (0,L), t > 0, \\ u_x(0,t) = u_x(L,t) = 0, \quad t > 0, \\ u(x,0) = \phi(x), \\ u_t(x,0) = \psi(x). \end{cases}$$

is given by

$$u(x,t) = \frac{c_0 + d_0 t}{2} + \sum_{j=1}^{\infty} \left(c_j \cos\left(\frac{j\pi ct}{L}\right) + d_j \sin\left(\frac{j\pi ct}{L}\right)\right) \cos\left(\frac{j\pi x}{L}\right).$$

where

$$c_0 = \frac{1}{L}\int_0^L \phi(x)dx, \quad d_0 = \frac{1}{L}\int_0^L \psi(x)dx,$$

and for $j \geq 1$*,*

$$c_j = \frac{2}{L}\int_0^L \phi(x)\cos\left(\frac{j\pi x}{L}\right)dx, \quad d_j = \frac{2}{L}\int_0^L \psi(x)\cos\left(\frac{j\pi x}{L}\right)dx.$$

Proof. We're still looking for solutions of the form

$$u(x,t) = X(x)T(t).$$

By plugging this into the wave equation, we get

$$X(x)T''(t) = c^2 X''(x)T(t).$$

this is equivalent to

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = \lambda.$$

Then we have two ordinary differential equations:

$$\begin{cases} T''(t) - \lambda c^2 T(t) = 0, \\ X''(x) - \lambda X(x) = 0. \end{cases}$$

Then we look for $X(x)$ satisfying the boundary conditions $X(0) = X(L) = 0$. We need to discuss the positivity and negativity of λ.

1. If $\lambda = 0$, then $X''(x) = 0$ gives $X(x) = A + Bx$. The boundary conditions imply $A = 0$ and $B = 0$, so $X(x) \equiv 0$.
2. If $\lambda > 0$, then $X''(x) - \lambda X(x) = 0$ has the general solution

   $$X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}.$$

   and thus

   $$\begin{cases} X'(0) = A\sqrt{\lambda} - B\sqrt{\lambda} = 0, \\ X'(L) = A\sqrt{\lambda}e^{\sqrt{\lambda}L} - B\sqrt{\lambda}e^{-\sqrt{\lambda}L} = 0. \end{cases}$$

   This system of equations shows that $A = 0$ and $B = 0$, leading to $X(x) \equiv 0$.

3. If $\lambda < 0$, let $\lambda = -\mu^2$ for some $\mu > 0$. By letting $\xi = \sqrt{-\lambda}$, we have

$$X''(x) = -\xi^2 X(x).$$

This has the general solution

$$X(x) = A\cos(\xi x) + B\sin(\xi x).$$

The boundary conditions give us

$$\begin{cases} X'(0) = -A\sin(\xi 0) + B\xi\cos(\xi 0) = B\xi = 0, \\ X'(L) = -A\xi\sin(\xi L) + B\xi\cos(\xi L) = -A\xi\sin(\xi L) = 0. \end{cases}$$

From the first equation we know that $B = 0$. The second equation then gives us either $A = 0$ or $\sin(\xi L) = 0$. If $A \neq 0$, then $\sin(\xi L) = 0$ implies

$$\xi = \frac{j\pi}{L}, \quad j = \mathbb{Z} \setminus \{0\}.$$

Up to rescaling, we can get

$$X_j(x) = \cos\left(\frac{j\pi x}{L}\right), \quad j = 1, 2, \ldots$$

Note that we also have a trivial solution

$$X_0(x) = \frac{1}{2}$$

when $\xi = 0$. On the other hand, we try to find the corresponding $T(t)$:

$$T_0''(t) = 0 \implies T_0(t) = A_0 + B_0 t.$$

and

$$T_j''(t) = -c^2\left(\frac{j\pi}{L}\right)^2 T_j(t) \implies T_j(t) = A_j\cos\left(\frac{j\pi ct}{L}\right) + B_j\sin\left(\frac{j\pi ct}{L}\right).$$

We can conclude with the general solution

$$u(x,t) = \frac{c_0 + d_0 t}{2} + \sum_{j=1}^{\infty}\left(c_j\cos\left(\frac{j\pi ct}{L}\right) + d_j\sin\left(\frac{j\pi ct}{L}\right)\right)\cos\left(\frac{j\pi x}{L}\right).$$

□

Lecture 23: May 28th

23.1 Fourier Cosine Series

Last time, we covered the ways to solve the wave equation with Neumann boundary conditions. Its general solution can be represented as

$$u(x,t) = \frac{C_0 + D_0 t}{2} + \sum_{n=1}^{\infty} \left(C_n \cos\left(\frac{n\pi x}{L}\right) + D_n \sin\left(\frac{n\pi x}{L}\right) \right) \cos\left(\frac{n\pi ct}{L}\right)$$

Proposition 23.1. *The set of functions*

$$\left\{ \frac{1}{2}, \cos\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \dots \right\}$$

forms a complete basis for the Hilbert space of square integrable functions on the interval $[0, L]$*, i.e.,* $L^2([0, L])$*. This series is known as the **Fourier cosine series**.*

In fact, with the cosine series, we can write almost every solution as a Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \text{ with } a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx.$$

Note that

$$a_0 = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{0\cdot\pi x}{L}\right) dx = \frac{2}{L}\int_0^L f(x) dx = \text{average}(f).$$

Then, we try to solve for the initial conditions given the boundary conditions. We have

$$\phi(x) = u(x,0) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{L}\right)$$

and

$$\psi(x) = u_t(x,0) = \frac{D_0}{2} + \sum_{n=1}^{\infty} D_n \frac{n\pi c}{L} \cos\left(\frac{n\pi x}{L}\right).$$

Example 23.2. For the initial conditions $\phi(x) = 1 + \cos(\pi x/L)$ and $\psi(x) = 1 + \cos(3\pi x/L)$, we have

$$C_0 = 2, C_1 = 1, C_2 = 0, \cdots$$

$$D_0 = 0, D_1 = 0, D_2 = 0, D_3 = \frac{l}{3\pi c}, D_4 = 0, \cdots$$

Thus, the solution is

$$u(x,t) = 1 + t + \cos\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi ct}{L}\right) + \frac{L}{3\pi c}\sin\left(\frac{3\pi x}{L}\right)\cos\left(\frac{3\pi ct}{L}\right).$$

Remark. Note that this method doesn't work for initial data conditions of the form $\sin\left(\frac{\alpha x}{L}\right)$, $\alpha \notin \mathbb{Z}$. For such initial data, we must use the integral formula to get the coefficients.

Example 23.3. For the initial conditions $\phi(x) = \sin(\pi x/L)$ and $\psi(x) = 0$. We have

$$C_0 = \frac{2}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right) dx = \frac{2}{L}\left[-\frac{L^2}{\pi}\cos\left(\frac{\pi x}{L}\right)\right]_0^L = \frac{2}{\pi}.$$

When $n \neq 1$, we have

$$\begin{aligned}
C_n &= \frac{2}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{\pi x}{L}\right) dx \\
&= \frac{2}{L}\int_0^L \frac{1}{2}\left(\sin\left(\frac{(n+1)\pi x}{L}\right) - \sin\left(\frac{(n-1)\pi x}{L}\right)\right) dx \\
&= \frac{1}{L}\left[-\frac{L}{(n+1)\pi}\cos\left(\frac{(n+1)\pi x}{L}\right) + \frac{L}{(n-1)\pi}\cos\left(\frac{(n-1)\pi x}{L}\right)\right]_0^L \\
&= (\frac{1}{(n+1)\pi} - \frac{1}{(n-1)\pi})(1-(-1)^n) \\
&= \frac{2}{\pi}\frac{1-(-1)^n}{n^2-1}.
\end{aligned}$$

Similarly, since $\psi(x) = 0$, we have $D_n = 0$ for all n. Thus, the solution is

$$u(x,t) = \frac{1}{\pi} + \sum_{n=2}^{\infty} \frac{2}{\pi}\frac{1-(-1)^n}{n^2-1}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right).$$

Lecture 24: May 30th

24.1 Wrapping up Separation of Variables

Definition 24.1. For separtion of variables, we call the constant λ_n in the ODE the **eigenvalue** and the corresponing function $X_n(x)$ that satisfies the ODE with corresponding eigenvalue λ_n the **eigenfunction**.

We give a wrap up of the general solution of heat and wave equations in one dimension over the finite interval $[0, L]$ with Dirichlet and Neumann boundary conditions.

<table>
<tr><th></th><th>Dirichlet</th><th>Neumann</th></tr>
<tr><td rowspan="2">Heat</td><td>$T_n(t) = e^{-\kappa\left(\frac{n\pi}{L}\right)^2 t}$</td><td>$T_0(t) = 1, \quad T_n(t) = e^{-\kappa\left(\frac{n\pi}{L}\right)^2 t}$</td></tr>
<tr><td>$X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$</td><td>$X_0(x) = \frac{1}{2}, \quad X_n(x) = \cos\left(\frac{n\pi x}{L}\right)$</td></tr>
<tr><td rowspan="2">Wave</td><td colspan="2">$T_0(t) = \frac{C_0 + D_0 t}{2}, \quad T_n(t) = C_n \cos\left(\frac{n\pi ct}{L}\right) + D_n \sin\left(\frac{n\pi ct}{L}\right)$</td></tr>
<tr><td>$X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$</td><td>$X_0(x) = \frac{1}{2}, \quad X_n(x) = \cos\left(\frac{n\pi x}{L}\right)$</td></tr>
</table>

Example 24.2. Consider the Schrodinger equation representing an electron in a one-dimensional box of length L with Dirichlet boundary conditions:

$$\begin{cases} iu_t = u_{xx}, \\ u(0,t) = u(L,t) = 0, \\ u(x,0) = \phi(x). \end{cases}$$

The general solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{i\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right),$$

where the coefficients b_n are determined by the initial condition $\phi(x)$ using the formula

$$b_n = \frac{2}{L}\int_0^L \phi(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

24.2 Mixed Boundary Conditions

For mixed boundary conditions, there might be both Neumann and Dirichlet conditions at different boundaries. For example, consider the following set of conditions:

$$\begin{cases} u_t = \kappa u_{xx}, \\ u_x(0,t) = u(L,t) = 0, \\ u(x,0) = \phi(x). \end{cases}$$

This represents a rod of length L where the left end is insulated (Neumann condition) and the right end is held at a fixed temperature $t = 0$ (Dirichlet condition). We still use the separation of variables method to find the solution:

$$X'' = \lambda X, \quad X'(0) = X(L) = 0.$$

Then, there are no non-trivial solutions for $\lambda \leq 0$, when $\lambda = -\xi^2$ we have

$$X(x) = A\cos(\xi x) + B\sin(\xi x), \quad X'(x) = -A\xi\sin(\xi x) + B\xi\cos(\xi x).$$

For the Neumann condition at $x = 0$, we have $X'(0) = 0$, which gives $B = 0$. For the Dirichlet condition at $x = L$, we have $X(L) = 0$, leading to $\cos(\xi L) = 0$. Thus, $\xi = \frac{(2n+1)\pi}{2L}$ for $n \in \mathbb{Z}$. The eigenfunctions are then given by

$$X_n(x) = \cos\left(\frac{(2n+1)\pi x}{2L}\right).$$

The time-dependent part is given by

$$T_n(t) = e^{-\kappa\left(\frac{(2n+1)\pi}{2L}\right)^2 t}.$$

Thus, the general solution for this mixed boundary condition is

$$u(x,t) = \sum_{n=0}^{\infty} b_n e^{-\kappa\left(\frac{(2n+1)\pi}{2L}\right)^2 t} \cos\left(\frac{(2n+1)\pi x}{2L}\right),$$

24.3 Robin Boundary Conditions

Robin boundary conditions are a combination of Dirichlet and Neumann conditions. For example, consider the following boundary conditions:

$$\begin{cases} u_x(0,t) - \alpha_0 u(0,t) = 0, \\ u_x(L,t) + \alpha_L u(L,t) = 0, \end{cases}$$

where α_0 and α_L are constants. Note that we have different signs for the constants at the two boundaries. If we apply a separation of variables approach, we know that the eigenfunctions satisfy the equation

$$\begin{cases} X''(x) = \lambda X(x), \\ X'(0) - \alpha_0 X(0) = 0, \\ X'(L) + \alpha_L X(L) = 0. \end{cases}$$

This would lead to the following characteristic equation:

$$\tan(\lambda L) = \frac{(\alpha_L + \lambda L)\lambda}{\lambda^2 - \alpha_0 \alpha_L}.$$

On one hand, we can have non-trivial solution in the case $\lambda = \gamma_0^2 > 0$, leading to

$$X_0(x) = \cosh(\gamma_0 x) + \frac{\alpha_0}{\gamma_0}\sinh(\gamma_0 x).$$

On the other hand, for $\lambda = -\beta_n^2 < 0$, with $n = 1, 2, \ldots$, we might find

$$X_n(x) = \cos(\beta_n x) + \frac{\alpha_0}{\beta_n}\sin(\beta_n x)$$

However, this infinite series $\gamma_0, \beta_1, \beta_2, \ldots$ might not be analytically solvable, and we might need to use numerical methods to find the eigenvalues and eigenfunctions. We know that the time-dependent part is given by

$$T_n(t) = \begin{cases} e^{-\lambda\kappa t}, & \text{for diffusion,} \\ A_n \cos(\sqrt{\lambda_n} ct) + B_n \sin(\sqrt{\lambda_n} ct), & \text{for wave.} \end{cases}$$

Thus, the general solution for the Robin boundary conditions is given by

$$u(x,t) = A_0 X_0(x) T_0(t) + \sum_{n=1}^{\infty} A_n X_n(x) T_n(t),$$

However, the determination of the coefficients A_0, A_n depends on the initial conditions and might require numerical methods to solve.

24.4 Periodic Boundary Conditions

Periodic boundary conditions are often used in problems where the domain is cyclic, such as in simulations of physical systems with periodic structures. For example, consider the following boundary conditions:

$$\begin{cases} u(-L,t) = u(L,t), \\ u_x(-L,t) = u_x(L,t). \end{cases}$$

In this case, we can use the Fourier series approach to find the solution. The eigenfunctions are given by

$$\begin{cases} X''(x) = \lambda(x) \\ X(-L) = X(L) = 0, \\ X'(-L) = X'(L) = 0. \end{cases}$$

If $\lambda > 0$, we have no non-trivial solutions. If $\lambda = 0$, we have

$$X_0(x) = \frac{1}{2}$$

and if $\lambda < 0$, we have

$$X_n(x) = A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right).$$

Lecture 25: June 2nd

25.1 Full Fourier Series

Definition 25.1. For the Hilbert space

$$L^2([-l,l]) \coloneqq \{f\colon [-l,l] \to \mathbb{R} \mid \langle f, f\rangle < \infty\}$$

representing the functions that are square integrable on the interval $(-l,l)$, we define the inner product as

$$\langle f, g\rangle = \frac{1}{l}\int_{-l}^{l} f(x)g(x)\,dx.$$

Proposition 25.2. *The set of functions*

$$\left\{\frac{1}{2}\right\} \cup \left\{\cos\frac{n\pi x}{l}\right\}_{n\in\mathbb{Z}} \cup \left\{\sin\frac{n\pi x}{l}\right\}_{n\in\mathbb{Z}}$$

forms a complete orthogonal basis for the Hilbert space $L^2([-l,l])$.

Proof. We can check that

$$\left\langle \sin\frac{n\pi x}{l}, \sin\frac{m\pi x}{l}\right\rangle = \left\langle \cos\frac{n\pi x}{l}, \cos\frac{m\pi x}{l}\right\rangle = \delta_{n,m}, \quad \left\langle \sin\frac{n\pi x}{l}, \cos\frac{m\pi x}{l}\right\rangle = 0.$$

$$\left\langle \frac{1}{2}, \sin\frac{n\pi x}{l}\right\rangle = \left\langle \frac{1}{2}, \cos\frac{n\pi x}{l}\right\rangle = 0, \quad \left\langle \frac{1}{2}, \frac{1}{2}\right\rangle = \frac{1}{2}.$$

□

Proposition 25.3. *By the orthogonality of the basis, we can compute the coefficients of the fourier series can be computed as follows:*

$$a_0 = \frac{\left\langle f, \frac{1}{2}\right\rangle}{\left\langle \frac{1}{2}, \frac{1}{2}\right\rangle} = \frac{\left\langle f, \frac{1}{2}\right\rangle}{\frac{1}{2}} = \langle f, 1\rangle$$

$$a_n = \left\langle f, \cos\frac{n\pi x}{l}\right\rangle = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx, \quad n = 0, 1, 2, \ldots$$

$$b_n = \left\langle f, \sin\frac{n\pi x}{l}\right\rangle = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx., \quad n = 1, 2, \ldots$$

Remark. This partially explains why we have the factor of $\frac{1}{l}$ in the definition of the inner product, since we want to normalize the coefficients to be independent of the length of the interval. This also explains why we have the factor of $\frac{1}{2}$ in the definition of the inner product for the constant function $\frac{1}{2}$, since we want to normalize the constant term to be independent of the length of the interval as $\left\langle \frac{1}{2}, \frac{1}{2}\right\rangle = \frac{1}{2}$

Definition 25.4. The **full fourier series** of a function $f \in L^2([-l, l])$ is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}.$$

25.2 Complex Fourier Series

Definition 25.5. The **Complex Hilbert space** is the inner product space defined over the complex numbers $\mathbb{C}$ with a complete orthonormal set of functions:

$$\left\{e^{i\frac{n\pi x}{l}}\right\}_{n\in\mathbb{Z}} = \left\{\cos\frac{n\pi x}{l} + i\sin\frac{n\pi x}{l}\right\}_{n\in\mathbb{Z}}.$$

and the complete inner product is given by

$$\langle f, g\rangle = \frac{1}{2l}\int_{-l}^{l} f(x)\overline{g(x)}\,dx.$$

Proposition 25.6. *We have the coefficients of the complex fourier series can be computed as follows:*

$$c_n = \begin{cases} \frac{1}{2}(a_n + ib_n), & n > 0 \\ \frac{1}{2}a_0, & n = 0 \\ \frac{1}{2}(a_{-n} - ib_{-n}), & n < 0 \end{cases}$$

where a_n and b_n are from the full fourier series over the real numbers $\mathbb{R}$.

Proof. We know that the coefficients of the fourier series can be computed as

$$c_n = \left\langle f, e^{i\frac{n\pi x}{l}}\right\rangle = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-i\frac{n\pi x}{l}}\,dx$$

$$
\begin{aligned}
&= \frac{1}{2l}\int_{-l}^{l} f(x)\left(\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right)dx \\
&= \frac{1}{2l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx - i\frac{1}{2l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx
\end{aligned}
$$

In particular, if f is real-valued, then $c_{-n} = \overline{c_n}$. □

Corollary 25.7. *If f is even and real-valued, then $b_n = 0$ for all n, and if f is odd and real-valued, then $a_n = 0$ for all n.*

In this way, we can see that the sine and cosine series over $[0, L]$ introduced in the previous sections are simply a special case of the full fourier series in the Hilbert space $L^2([-l, l])$.

Example 25.8. Given a function $f\colon (0, l) \to \mathbb{R}$, we can take an **odd periodic extension** of f to the interval $(-l, l)$, which is defined as

$$
f_{\text{odd}}(x) = \begin{cases} 0, & x < 0 \\ f(x), & x \in (0, l) \\ 0, & x = 0 \\ -f(-x), & x \in (-l, 0) \\ 0, & x = l \\ f(x - 2l), & x > l \end{cases}
$$

Then, the full fourier series of f_{odd} is just the sine series of f, since the odd extension of f is zero at the endpoints, and thus the cosine terms vanish. In particular, we have

$$
f_{\text{odd}}(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, \quad b_n = \frac{1}{l}\int_{-l}^{l} f_{\text{odd}}(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_{0}^{l} f(x)\sin\frac{n\pi x}{l}\,dx.
$$

Similarly, if we take an **even periodic extension** of f to the interval $(-l, l)$, which is defined as

$$
f_{\text{even}}(x) = \begin{cases} 0, & x < 0 \\ f(x), & x \in (0, l) \\ 0, & x = 0 \\ f(-x), & x \in (-l, 0) \\ 0, & x = l \\ f(x - 2l), & x > l \end{cases}
$$

Then, the coefficients of the cosine terms in the fourier series is preserved, and the sine terms vanish, since the even extension of f is zero at the endpoints. In particular, we have

$$f_{\text{even}}(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}, \quad a_n = \frac{1}{l}\int_{-l}^{l} f_{\text{even}}(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx.$$

Remark. In the later part of the course, we will assume that the complete fourier series is defined on the real line $\mathbb{R}$. If we want to define the fourier series on a finite interval, we can simply take the periodic extension of the function to the real line. In this case, we have to also make sure that series converges uniformly on the interval, which gives rise to the following theorem:

Lecture 26: June 4th

26.1 Convergence of Fourier Series

Last time, we discussed the general form of Fourier series and how to compute the Fourier coefficients. We also looked at the convergence of Fourier series under certain conditions. We use the following notation for the Fourier series of a function f that is $2l$-periodic:

$$S_n(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{l}\right) + b_n \sin\left(\frac{2\pi n x}{l}\right) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n x}{l}},$$

with $a_n, b_n \in \mathbb{R}$ and $c_n \in \mathbb{C}$ would be its corresponding Fourier series.

Today, we'll discuss the convergence of Fourier series. There are three main types of convergence we will consider: pointwise convergence, uniform convergence, and L_2 convergence. Each type of convergence has its own set of conditions and implications for the Fourier series.

Definition 26.1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. Suppose we have a sequence of functions $\{f_n\}$, then there are three types of convergence we can consider:

1. **Pointwise Convergence:** The sequence $\{f_n\}$ converges pointwise to a function f if for every continuous point $x \in \mathbb{R}$, we have

$$\lim_{n\to\infty} f_n(x) = \lim_{h\to 0^+} \frac{f(x+h) + f(x-h)}{2}.$$

   This means that for each fixed x, the sequence converges to the value of $f(x)$.

2. **Uniform Convergence:** The sequence $\{f_n\}$ converges uniformly to a continuous function f if

$$\lim_{n\to\infty} \max_{x\in[-l,l]} |f_n(x) - f(x)| = 0.$$

This means that the convergence is uniform across the interval, and the maximum difference between $f_n(x)$ and $f(x)$ goes to zero as n increases.

3. L_2 **Convergence:** The sequence $\{f_n\}$ converges in the L_2 sense to a function f if

$$\lim_{n\to\infty} \int_{-l}^{l} |f_n(x) - f(x)|^2 \, dx = 0.$$

This means that the integral of the square of the difference between $f_n(x)$ and $f(x)$ goes to zero as n increases.

Theorem 26.2 (Uniform Convergence Theorem)**.** *If a sequence of functions $\{f_n\}$ converges uniformly to a function f, then f is continuous if each f_n is continuous.*

Then, there's the following theorem that enables us to quickly check the convergence of Fourier series:

Theorem 26.3. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function that is $2l$-periodic.*

1. *(Pointwise Convergence) If f is bounded and piecewise continuous, with bounded and piecewise continuous first derivative, then the fourier series converges pointwise to*

$$g(x) = \lim_{h\to 0^+} \frac{f(x+h) + f(x-h)}{2}.$$

Note that $f = g$ only at all points of continuity of f.

2. *(Uniform Convergence) Suppose the series of Fourier coefficients $\{a_n\}$ and $\{b_n\}$ are absolutely convergent, i.e.,*

$$\sum_{n=-\infty}^{\infty} |a_n| < \infty \quad \textit{and} \quad \sum_{n=-\infty}^{\infty} |b_n| < \infty.$$

Then the Fourier series converges uniformly to f, meaning that

$$\lim_{n\to\infty} \max_{x\in[-l,l]} |f(x) - S_n(x)| = 0.$$

3. *(L_2 Convergence) Suppose that the coefficients $\{a_n\}$ and $\{b_n\}$ are square summable, i.e.,*

$$\sum_{n=0}^{\infty} \|c_n\|^2 = \frac{a_0^2}{2} \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) < \infty.$$

Then we have convergence in the L_2 sense, meaning that

$$\frac{1}{l} \int_{-l}^{l} \|f(x) - S_n(x)\|^2 \, dx = \sum_{k=n+1}^{\infty} a_k^2 + b_k^2 = \sum_{k=n+1}^{\infty} \|c_k\|^2 \overset{n\to\infty}{\longrightarrow} 0.$$

Corollary 26.4 (Uniform Convergence Criterion)**.** *If f is continuous and $2l$-periodic, then a sufficient condition for uniform convergence is that $f(x)$ is twice continuously differentiable and that the second derivative is bounded and piecewise continuous.*

Proof. This is because

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos\left(\frac{2\pi n x}{l}\right) dx = \frac{l}{n^2\pi^2} \int_{-l}^{l} f''(x) \cos\left(\frac{2\pi n x}{l}\right) dx,$$

Hence, if f'' is bounded and piecewise continuous, then

$$|a_n| \le \frac{c}{n^2} \max_{x\in\mathbb{R}} |f''(x)|$$

The result for b_n is similar and thus we have

$$\sum_{n=1}^{\infty} |a_n| + |b_n| < \infty.$$

□

Remark. The third condition is the weakest, while the second condition is the strongest. The third condition is referred to as L_2 convergence, which is a weaker form of convergence than pointwise convergence. In context, it generalizes a sequence of functions $\{f_n\}$ to a function such that

$$\lim_{n\to\infty} \int_{-l}^{l} |f_n(x) - f(x)|^2 \, dx = 0.$$

But it's most possible that $\lim_{n\to\infty} f_n(x) \neq f(x)$ almost everywhere. We can check the Parseval's identity, which states that if f is square integrable, then

$$\frac{1}{l} \int_{-l}^{l} |f(x)|^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \le 2 \sum_{n=-\infty}^{\infty} |c_n|^2.$$

In particular, any piecewise continuous function f would have a finite L_2 integral, as

$$\int_{-l}^{l} |f(x)|^2 \, dx \le 2l \max_{x\in[-l,l]} |f(x)|^2 < \infty.$$

Lecture 27: June 6th

Today we first bring up a corollary on the convergence of Fourier sine series.

Corollary 27.1. *Let $f\colon (0,l) \to K$. And let b_n be the Fourier sine coefficients of f defined as above. Consider*

$$S_n(x) = \sum_{n=1}^{N} b_n \sin\left(\frac{n\pi x}{l}\right).$$

1. *Assuming f and f' are piecewise continuous and odd on $(0,l)$, then $S_n(x)$ converges pointwise to $f(x)$ for all $x \in (0,l)$.*

$$\lim_{n\to\infty} S_n(x) = \begin{cases} f(x) & \text{if } x \in (0,l) \\ 0 & \text{if } x = 0 \text{ or } x = l \end{cases}$$

2. *Assume that the Fourier sine coefficients b_n are absolutely summable, i.e.,*

$$\sum_{n=1}^{\infty} |b_n| < \infty.$$

Then $S_n(x)$ converges uniformly to $f(x)$ on $(0,l)$. In particular, this is true when f, f', and f'' are all continuous, f is odd on $(0,l)$, and $\lim_{x\to 0^+} f(x) = 0$ and $\lim_{x\to l^-} f(x) = 0$.

$$\lim_{n\to\infty} S_n(x) = f(x) \text{ uniformly on } (0,l).$$

3. *If the fourier sine coefficients b_n are square summable, i.e.,*

$$\sum_{n=1}^{\infty} b_n^2 < \infty,$$

then $S_n(x)$ converges to $f(x)$ in the mean square sense, i.e.,

$$\lim_{n\to\infty} \int_0^l |S_n(x) - f(x)|^2 \, dx = 0.$$

Proof. The derivation from the previous theorem comes from an odd periodic extension of f to $(-l,l)$.

$$f_{\text{odd}}(x) = \begin{cases} -f(2l-x) & \text{if } x \in (l,2l) \\ f(x) & \text{if } x \in (0,l) \\ 0 & \text{if } x = 0 \\ -f(-x) & \text{if } x \in (-l,0) \end{cases}$$

□

Example 27.2. Consider the function $f\colon (0,l) \to \mathbb{R}$ defined by $f(x) = x$. The sine Fourier coefficients are given by

$$b_n = \frac{2}{l}\int_0^l x \sin\left(\frac{n\pi x}{l}\right)\, dx = \frac{-2l(-1)^n}{n\pi}$$

Thus, we know that the fourier sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) = -\frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi x}{l}\right).$$

Note that this series is conditionally convergent for most $x \in (0,l)$. Moreover, we know that $S_n(l) = S_n(0) = 0$ for all n, and thus it doesn't satisfy the convergence for $x = l$ as $f(l) = l$ at the endpoint.

27.1 Inhomogeneous Boundary Conditions with Source Terms on Semi-Infinite Domains

Now we introduce the **method of subtraction** to solve the inhomogeneous boundary value problems. The steps are

1. Make the boundary condition homogeneous by subtracting the linear function.
2. Use separation of variables to solve the problem with no source term and obtain the series form of the solution.
3. Use the Duhamel's principle to solve the problem with source terms in terms of the ODE formed by the Fourier sine coefficients.
4. Substitute back to get the solution to the original problem.

Example 27.3. Consider the following boundary value problem

$$\begin{cases} v_t = \kappa v_{xx} + f(x,t) & \text{for } t > 0, x \in (0,l) \\ v(0,t) = g(t), v(l,t) = h(t) \\ v(x,0) = \phi(x) \end{cases}$$

We can define a shifted solution

$$w(x,t) = v(x,t) - \frac{x}{l}h(t) - \left(1 - \frac{x}{l}\right)g(t),$$

This satisfies the new set of equations:

$$\begin{cases} w_t = \kappa w_{xx} + f(x,t) - \frac{x}{l}h'(t) - \left(1 - \frac{x}{l}\right)g'(t) \\ w(0,t) = 0, w(l,t) = 0 \\ w(x,0) = \phi(x) - \frac{x}{l}h(0) - \left(1 - \frac{x}{l}\right)g(0) \end{cases}$$

Then, we can use the separation of variables to solve the new problem. We assume the solution for the inhomogeneous boundary value problem can be expressed as a Fourier sine series:

$$u(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin\left(\frac{n\pi x}{l}\right), \quad B_n(t) = \frac{2}{l}\int_0^l \left(\phi(x) - \frac{x}{l}h(0) - \left(1 - \frac{x}{l}\right)g(0)\right) \sin\left(\frac{n\pi x}{l}\right) dx.$$

and we can expand the source term as

$$\tilde{f}(x,t) = \sum_{n=1}^{\infty} \tilde{f}_n(t) \sin\left(\frac{n\pi x}{l}\right), \quad \tilde{f}_n(t) = \frac{2}{l}\int_0^l f(x,t) \sin\left(\frac{n\pi x}{l}\right) dx.$$

Then we know that they satisfy the following ODEs

$$B_n' + \kappa\left(\frac{n\pi}{l}\right)^2 B_n = \tilde{f}_n(t)$$

This can be solved by the integrating factor method, which gives us

$$A_n(t) = e^{-\kappa\left(\frac{n\pi}{l}\right)^2 t}\left(B_n(0) + \int_0^t e^{\kappa\left(\frac{n\pi}{l}\right)^2 s}\tilde{f}_n(s)\, ds\right).$$

for the solution to the inhomogeneous boundary solution

$$w(x,t) = \sum_{n=1}^{\infty} A_n(t) \sin\left(\frac{n\pi x}{l}\right).$$

Substituting back in gives us the solution to the original problem.