Machine Intelligence II SoSe 2016 Exercise 5

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5.1.1 Sample Laplace from uniform random variable

Probability density function

$$p_X(x) = \frac{1}{2b} exp\left(-\frac{|x-\mu|}{b}\right)$$

Splitting the pdf into two parts in order to eliminate absolute value

$$p_X(x) = \begin{cases} \frac{1}{2b} exp(\frac{x-\mu}{b}) & \text{if } x < \mu \\ \frac{1}{2b} exp(-\frac{x-\mu}{b}) & \text{if } x \geq \mu \end{cases}$$

Definition cumulative density function

$$F_X(x) = \int_0^x p_X(x) dx$$

Building the antiderivative:

$$\begin{cases} \frac{1}{2} exp(\frac{x-\mu}{b}) & \text{if } x < \mu \\ 1 - \frac{1}{2} exp(-\frac{x-\mu}{b}) & \text{if } x \geq \mu \end{cases}$$

Inverse cumulative density function for $x < \mu$

$$\frac{1}{2} exp(\frac{x-\mu}{b}) = y$$
$$\frac{x-\mu}{b} = ln(2y)$$
$$x = ln(2y) * b + \mu$$

Inverse cumulative density function for $x \geq \mu$

$$1 - \frac{1}{2} exp(-\frac{x-\mu}{b}) = y$$
$$exp(-\frac{x-\mu}{b}) = -2 * (y-1)$$
$$x - \mu = ln(-2 * (y-1)) * b$$
$$x = b * ln(-2y+2) + \mu$$

Complete inverse cumulative density function

$$F_X^{-1}(x) = \begin{cases} b \log(2y) + \mu & \text{if } y < 0.5 \\ \mu - b \log(-2y+2) & \text{if } y \geq 0.5 \end{cases}$$

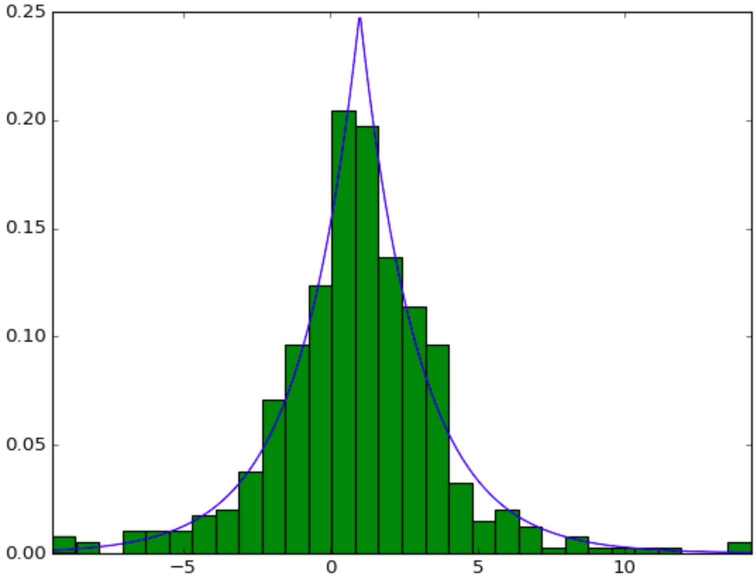
5.1.2 Compute samples and overlay density function

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib notebook
```

In [2]:
```
def p_X(x, μ=1, b=2):
    return 1 / (2 * b) * np.exp(-(np.abs(x - μ) / b))

def F_X_inv(y, μ=1, b=2):
    lower = np.log(2 * y) * b + μ
    upper = μ - np.log(-2 * y + 2) * b
    return (y < 0.5) * lower + (y >= 0.5) * upper

samples = F_X_inv(np.random.random(500))
min_, max_ = samples.min(), samples.max()
steps = np.linspace(min_, max_, 500)
density = p_X(steps)

fig, ax = plt.subplots(ncols=1, nrows=1)
ax.plot(steps, density)
ax.set_xlim(min_, max_)
ax.hist(samples, normed=True, bins=30)
```

```
plt.show()
```

```
<IPython.core.display.Javascript object>
```



5.2.1

Solution on paper on the first page of the three pages appended below.

5.2.2

Solution on paper on the second and third page of the three pages appended below.

Second page: using equations for u1 and u2 to solve for x1 and x2.

Third page: calculating the partial derivatives that comprise the Jacobian matrix, calculating it's determinant. The value of the determinant shows, that u(X) corresponds to two normally distributed random variables (with 0 mean and unit variance).

• pdf of a random var X: $p_X(x) = e^{-x}$, $x \geqslant 0$

• change of vars, u: $u(x) = e^{-x}$

• inverse of the change, x: $x(u) = -\log(u)$

because: $y = e^{-x}$

$\log y = -x$

$-\log y = x$

pdf of transformed rand var $u(X)$: ?

$p_{u(X)}(u) = ?$

1. Since $p_X(x)$ is the pdf of an n-dim random vector X, pdf of the transformed random variable $u(X)$ is given by:

$$p_{u(X)}(u) = p_X(x(u)) \left| \det \frac{\partial x(u)}{\partial u} \right| , \text{ where } \underbrace{p_X(\alpha) = e^{-\alpha}}_{(A)} \text{ and } \underbrace{x(u) = -\log(u)}_{(B)}$$

putting (A) and (B) in:

$$p_{u(X)}(u) = e^{-(-\log(u))} \left| \det \frac{\partial x(u)}{\partial u} \right| = e^{\log(u)} \left| \det \frac{\partial x(u)}{\partial u} \right| = u \left| \det \frac{\partial x(u)}{\partial u} \right|$$

$$u_1 = \sqrt{-2\log x_1}\cos(2\pi x_2) \rightarrow \sqrt{-2\log x_1} = \frac{u_1}{\cos(2\pi x_2)}$$

$$u_2 = \sqrt{-2\log x_1}\sin(2\pi x_2) \rightarrow \sqrt{-2\log x_1} = \frac{u_2}{\sin(2\pi x_2)}$$

$$\frac{u_1}{\cos(2\pi x_2)} = \frac{u_2}{\sin(2\pi x_2)}$$

$$\frac{u_2}{u_1} = \frac{\sin(2\pi x_2)}{\cos(2\pi x_2)}$$

$$\frac{u_2}{u_1} = \tan(2\pi x_2)$$

$$\tan^{-1}\left(\frac{u_2}{u_1}\right) = 2\pi x_2$$

$$\boxed{x_2 = \frac{1}{2\pi}\tan^{-1}\left(\frac{u_2}{u_1}\right)}$$

$$u_1 = \sqrt{-2\log x_1}\cos\left(2\pi\cdot\frac{1}{2\pi}\tan^{-1}\left(\frac{u_2}{u_1}\right)\right) = \sqrt{-2\log x_1}\cos\left(\tan^{-1}\left(\frac{u_2}{u_1}\right)\right) = \sqrt{-2\log x_1}\cdot\frac{1}{\sqrt{\frac{u_2^2}{u_1^2}+1}}$$

$$u_1 = \sqrt{-2\log x_1}\cdot\frac{1}{\sqrt{\frac{u_2^2}{u_1^2}+1}} \quad \Big/ \cdot\sqrt{\frac{u_2^2}{u_1^2}+1}$$

$$u_1\cdot\sqrt{\frac{u_2^2}{u_1^2}+1} = \sqrt{-2\log x_1}$$

$$u_1^2\left(\frac{u_2^2}{u_1^2}+1\right) = -2\log x_1$$

$$u_2^2 + u_1^2 = -2\log x_1$$

$$\boxed{x_1 = e^{-\left(\frac{u_1^2+u_2^2}{2}\right)}}$$

$$\frac{\partial(x_1, x_2)}{\partial(u_1, u_2)} = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} \end{vmatrix}$$

$$\frac{\partial}{\partial u_1} = 2\frac{u_1}{2} = u_1 \qquad \frac{\partial}{\partial u_1} = 0$$

$$\frac{\partial x_1}{\partial u_1} = \frac{\partial}{\partial u_1} e^{-\frac{u_1^2 + u_2^2}{2}} = \frac{\partial}{\partial u_1} e^{-\left(\frac{u_1^2}{2}\right) + \left(\frac{u_2^2}{2}\right)} = u_1 \cdot \left[e^{-\left(\frac{u_1^2 + u_2^2}{2}\right)} \right]$$

$$\frac{\partial x_1}{\partial u_2} = \frac{\partial}{\partial u_2} e^{-\left(\frac{u_1^2 + u_2^2}{2}\right)} = u_2 \left[e^{-\frac{u_1^2 + u_2^2}{2}} \right]$$

$$\frac{\partial x_2}{\partial u_1} = -\frac{u_2}{2\pi u_1^2 + 2\pi u_2^2}$$

$$\frac{\partial x_2}{\partial u_2} = \frac{u_1}{2\pi u_1^2 + 2\pi u_2^2}$$

$$\begin{vmatrix} \frac{\partial x_1}{u_1} & \frac{\partial x_1}{\partial u_2} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} \end{vmatrix} = u_1 \cdot \left[e^{-\left(\frac{u_1^2 + u_2^2}{2}\right)} \right] \cdot \frac{u_1}{2\pi u_1^2 + 2\pi u_2^2} + u_2 \cdot \left[e^{-\left(\frac{u_1^2 + u_2^2}{2}\right)} \right] \cdot \frac{u_2}{2\pi u_1^2 + 2\pi u_2^2} =$$

$$= -\left[\frac{1}{\sqrt{2\pi}} e^{-\frac{u_1^2}{2}} \right] \left[\frac{1}{\sqrt{2\pi}} e^{-\frac{u_2^2}{2}} \right]$$