A Simple and Efficient Estimator for Hyperbolic Location

Y. T. Chan, *Senior Member, IEEE*, and K. C. Ho, *Member, IEEE*

Abstract—An effective technique in locating a source based on intersections of hyperbolic curves defined by the time differences of arrival of a signal received at a number of sensors is proposed. The approach is noniterative and gives an explicit solution. It is an approximate realization of the maximum-likelihood estimator and is shown to attain the Cramér-Rao lower bound near the small error region. Comparisons of performance with existing techniques of beamformer, spherical-interpolation, divide and conquer, and iterative Taylor-series methods are made. The proposed technique performs significantly better than spherical-interpolation, and has a higher noise threshold than divide and conquer before performance breaks away from the Cramér-Rao lower bound. It provides an explicit solution form that is not available in the beamforming and Taylor-series methods. Computational complexity is comparable to spherical-interpolation but substantially less than the Taylor-series method.

I. INTRODUCTION

IN sonar and radar, it is often of interest to determine the location of an object from its emissions [1]. A number of spatially separated sensors capture the emitted signal and the time differences of arrival (TDOA's) at the sensors are determined. Using the TDOA's, emitter location relative to the sensors can be calculated.

The position fix is simplified when the sensors are arranged in a linear fashion. Many optimum processing techniques have been proposed, with different complexity and restrictions. Carter's focused beamforming [1] requires a search over a set of possible source locations. Hahn's method [2]–[3] assumes a distant source. Abel and Smith [4] provide an explicit solution that can achieve the Cramér-Rao Lower Bound (CRLB) in the small error region.

The situation is more complex when sensors are distributed arbitrarily. In this case, emitter position is determined from the intersection of a set of hyperbolic curves defined by the TDOA estimates. Finding the solution is not easy as the equations are nonlinear. Fang [5] gave an exact solution when the number of TDOA measurements are equal to the number of unknowns (coordinates of transmitter). This solution, however, cannot make use of extra measurements, available when there are extra sensors, to improve position accuracy. The more general situation with extra measurements was considered by Friedlan-

der [6], Schau and Robinson [7], and Smith and Abel [8]–[9]. Although closed-form solutions have been developed, their estimators are not optimum. The divide and conquer (DAC) method [10] from Abel can achieve optimum performance, but it requires that the Fisher information is sufficiently large. To obtain a precise position estimate at reasonable noise levels, the Taylor-series method [11]–[12] is commonly employed. It is an iterative method: it starts with an initial guess and improves the estimate at each step by determining the local linear least-squares (LS) solution. An initial guess close to the true solution is needed to avoid local minima. Selection of such a starting point is not simple in practice. Moreover, convergence of the iterative process is not assured. It is also computationally intensive as LS computation is required in each iteration.

We give an alternative solution for hyperbolic position fix. The solution is in closed-form, valid for both distant and close sources, and is an approximation of the maximum likelihood (ML) estimator when the TDOA estimation errors are small. To illustrate, Section II considers a 2-D localization problem with an arbitrary array manifold. In the special case of a linear array, the solution reduces to the one given by [4]. With three sensors to provide two TDOA estimates, an exact solution is obtained which is equivalent to the result in [7]. With four or more sensors, the original set of TDOA equations are transformed into another set of equations which are linear in source position coordinates and an extra variable. The weighted linear LS gives an initial solution. A second weighted LS, which makes use of the known constraint between source coordinates and the extra variable, gives an improved position estimate. Expression for location variance is derived. Section III compares the estimator's localization accuracy with the CRLB. Performance comparison with those found in the literature is presented in Section IV. Section V is a simulation study on accuracy of the proposed and the other methods. Conclusions are given in Section VI.

II. HYPERBOLIC POSITION FIXING SOLUTION

The development is in a 2-D plane for ease of illustration. Extension to three dimensions is straightforward. Assume that there are M sensors distributed arbitrarily in a 2-D plane as shown in Fig. 1. Let the sampled observations at sensor i be

$$u_i(k) = s(k - d_i) + \eta_i(k), \qquad i = 1, 2, \cdots, M \qquad (1)$$

where $s(k)$ is the signal radiating from the source, d_i the time delay associated with receiver i and $\eta_i(k)$ the additive noise. The signal and noises are assumed to be mutually independent, zero mean stationary Gaussian random processes. To localize the source, we first estimate TDOA of the signal received by sensors i and j by Hahn and Tretter's method [2]–[3]. It is an optimum estimator in the sense that the CRLB can be attained. The technique consists of measuring TDOA's for all possible receiver pairs by generalized cross-correlation [13] and then calculating the Gauss-Markov (weighted) estimate of the TDOA's with respect to the first receiver, $d_{i,1} = d_i - d_1$ for $i = 2, 3, \cdots, M$. TDOA between receivers i and j are computed from

$$d_{i,j} = d_{i,1} - d_{j,1}, \qquad i, j = 2, 3, \cdots, M \qquad (2)$$

Let $\mathbf{d} = [d_{2,1}, d_{3,1}, \cdots, d_{M,1}]^T$ be the estimated TDOA vector. The covariance matrix $\mathbf{Q}$ of $\mathbf{d}$ is given by [2]–[3]

$$\mathbf{Q} = \left\{ \frac{2T}{2\pi} \int_0^\Omega \omega^2 \frac{S(\omega)^2}{1 + S(\omega)\mathrm{tr}(\mathbf{N}(\omega)^{-1})} \right.$$
$$\left. \times \left[\mathrm{tr}\left(\mathbf{N}(\omega)^{-1}\right)\mathbf{N}_p(\omega)^{-1} - \mathbf{N}_p(\omega)^{-1}\mathbf{1}\mathbf{1}^T\mathbf{N}_P(\omega)^{-1} \right] d\omega \right\}^{-1}$$
$$(3)$$

where 0 to Ω is the frequency band processed and T is the observation time. $\mathrm{tr}(*)$ is the trace of the matrix $*$. $S(\omega)$ is the signal power spectrum, $\mathbf{N}(\omega) = \mathrm{diag}\{N_1(\omega), N_2(\omega), \cdots, N_M(\omega)\}$ is the noise power spectral matrix, $\mathbf{N}_p(\omega)$ is the lower right $M-1$ by $M-1$ partition of the matrix $\mathbf{N}(\omega)$ and $\mathbf{1}$ is a vector of unity which has the same size as $\mathbf{N}_p(\omega)$.

Denote the noise free value of $\{*\}$ as $\{*\}^0$. TDOA $d_{i,j}$ will then be

$$d_{i,j} = d_{i,j}^0 + n_{i,j}, \qquad i, j = 1, 2, \cdots, M. \qquad (4)$$

with $n_{i,j}$ representing the noise (delay estimation error) component. Define the noise vector as $\mathbf{n} = [n_{2,1}, n_{3,1}, \cdots, n_{M,1}]^T$. Since the TDOA estimator is unbiased, the mean of $\mathbf{n}$ is zero and its covariance matrix is the same as $\mathbf{Q}$. Notice from (2) that $n_{i,j} = n_{i,1} - n_{j,1}$.

For simplicity, the index i is presumed to run from 2 to M in the sequel unless otherwise specified. Let the source be at
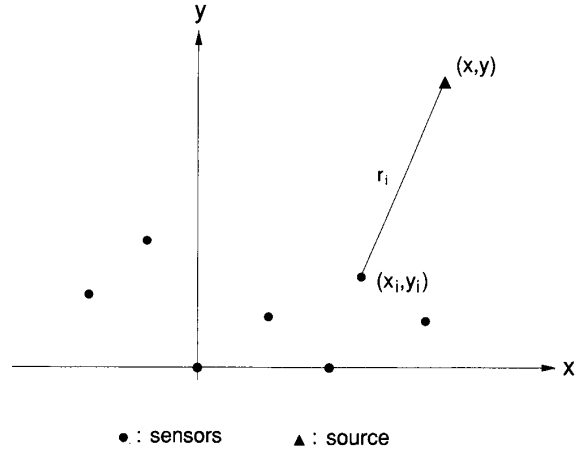

● : sensors   ▲ : source

Fig. 1. Localization in a 2-D plane.

unknown position (x, y) and the sensors at known locations (x_i, y_i). The squared distance between the source and sensor i is

$$r_i^2 = (x_i - x)^2 + (y_i - y)^2$$
$$= K_i - 2x_i x - 2y_i y + x^2 + y^2, \quad i = 1, 2, \cdots, M$$

where

$$K_i = x_i^2 + y_i^2. \qquad (5)$$

If c is the signal propagation speed, then

$$r_{i,1} = cd_{i,1} = r_i - r_1 \qquad (6)$$

define a set of nonlinear equations whose solution gives (x, y).

Solving those nonlinear equations is difficult. Linearizing (6) by Taylor-series expansion and then solving iteratively is one possible way [11]–[12]. With a set of TDOA estimates $d_{i,1}$, the method starts with an initial position guess (x_0, y_0) and computes position deviations

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \left(\mathbf{G}_t^T \mathbf{Q}^{-1} \mathbf{G}_t\right)^{-1} \mathbf{G}_t^T \mathbf{Q}^{-1} \mathbf{h}_t$$

where (see (7), at the bottom of this page). The values $r_i = 1, 2, \cdots, M$ are computed from (5) with $x = x_0$ and

$$\mathbf{h}_t = \begin{bmatrix} r_{2,1} - (r_2 - r_1) \\ r_{3,1} - (r_3 - r_1) \\ r_{M,1} - (r_M - r_1) \end{bmatrix}$$

$$\mathbf{G}_t = \begin{bmatrix} (x_1 - x)/r_1 - (x_2 - x)/r_2 & (y_1 - y)/r_1 - (y_2 - y)/r_2 \\ (x_1 - x)/r_1 - (x_3 - x)/r_3 & (y_1 - y)/r_1 - (y_3 - y)/r_3 \\ (x_1 - x)/r_1 - (x_M - x)/r_M & (y_1 - y)/r_1 - (y_M - y)/r_M \end{bmatrix} \qquad (7)$$

$y = y_0$. In the next iteration, x_0 and y_0 are then set to $x_0 + \Delta x$ and $y_0 + \Delta y$. The whole process is repeated again until Δx and Δy are sufficiently small. This method has the difficulties of requiring a close enough starting point and large computations. Moreover, convergence is not guaranteed.

An alternative [6]–[9] is to first transform (6) into another set of equations. From (6), $r_i^2 = (r_{i,1} + r_1)^2$ so that (5) can be rewritten as

$$r_{i,1}^2 + 2r_{i,1}r_1 + r_1^2 = K_i - 2x_i x - 2y_i y + x^2 + y^2. \quad (8)$$

Subtracting (5) at $i = 1$ from (8), we obtain

$$r_{i,1}^2 + 2r_{i,1}r_1 = -2x_{i,1}x - 2y_{i,1}y + K_i - K_1. \quad (9)$$

The symbols $x_{i,1}$ and $y_{i,1}$ stand for $x_i - x_1$ and $y_i - y_1$ respectively. Note that (9) is a set of linear equations with unknowns x, y and r_1. To solve for x and y, [6] eliminates r_1 from (9) and produces $(M-2)$ linear equations in x and y. The source position is then computed by LS. On the other hand, [8] first solves x and y in terms of r_1. The intermediate result is inserted back to (9) to generate equations in the unknown r_1 only. Substituting the computed r_1 value that minimizes the LS equation error to the intermediate result gives the final solution. This is termed the spherical-interpolation (SI) method [8]. The preceding two solutions are shown to be mathematically equivalent [6]. They are, however, not optimum and the weighting matrices required in LS are not easy to determine. A new estimator for position fixing which is capable of achieving optimum performance is next given.

A. Arbitrary Array

1) Three Sensors $(M = 3)$: With three sensors, x and y can be solved in terms of r_1 from (9). That is

$$\begin{bmatrix} x \\ y \end{bmatrix} = - \begin{bmatrix} x_{2,1} & y_{2,1} \\ x_{3,1} & y_{3,1} \end{bmatrix}^{-1} \\ \times \left\{ \begin{bmatrix} r_{2,1} \\ r_{3,1} \end{bmatrix} r_1 + \frac{1}{2} \begin{bmatrix} r_{2,1}^2 - K_2 + K_1 \\ r_{3,1}^2 - K_3 + K_1 \end{bmatrix} \right\}. \quad (10)$$

Inserting this intermediate result into (5) at $i = 1$ gives a quadratic in r_1. Substitution of the positive root back into (10) produces the solution. On some occasions, there may be two positive roots that produce two different answers. The solution ambiguity can be resolved by restricting the transmitter to lie within the region of interest. This answer is equivalent to the one in [7].

2) Four or More Sensors $(M \geq 4)$: The system is overdetermined as the number of measurements is greater than the number of unknowns. In the presence of noise, the set of equations in (9) will not meet at the same point and the proper answer is the (x, y) that best fit these equations. Let $\mathbf{z}_a = [\mathbf{z}_p^T, r_1]^T$ be the unknown vector, where $\mathbf{z}_p = [x, y]^T$. With TDOA noise, the error vector derived from (9) is

$$\psi = \mathbf{h} - \mathbf{G}_a \mathbf{z}_a^0$$

where

$$\mathbf{h} = \frac{1}{2} \begin{bmatrix} r_{2,1}^2 - K_2 + K_1 \\ r_{3,1}^2 - K_3 + K_1 \\ \cdot \\ r_{M,1}^2 - K_M + K_1 \end{bmatrix}$$

$$\mathbf{G}_a = - \begin{bmatrix} x_{2,1} & y_{2,1} & r_{2,1} \\ x_{3,1} & y_{3,1} & r_{3,1} \\ \cdot & \cdot & \cdot \\ x_{M,1} & y_{M,1} & r_{M,1} \end{bmatrix}. \quad (11)$$

When (4) is used to express $r_{i,1}$ as $r_{i,1}^0 + cn_{i,1}$ and noting from (6) that $r_i^0 = r_{i,1}^0 + r_1^0$, ψ is found to be

$$\psi = c\mathbf{B}\mathbf{n} + 0.5c^2\mathbf{n} \odot \mathbf{n}$$
$$\mathbf{B} = \text{diag}\{r_2^0, r_3^0, \cdots, r_M^0\}. \quad (12)$$

The symbol $\odot$ represents the Schur product (element-by-element product). The TDOA found by generalized cross-correlation with Gaussian data is asymptotically normally distributed when the signal-to-noise ratio (SNR) is high [1]. It follows that the noise vector $\mathbf{n}$ in Hahn and Tretter's [2] estimator is also asymptotically normal. The covariance matrix of ψ can therefore be evaluated. In practice, the condition $cn_{i,1} \ll r_i^0$ is usually satisfied. When ignoring the second term on the right of (12), ψ becomes a Gaussian random vector with covariance matrix given by

$$\Psi = \mathbf{E}[\psi\psi^T] = c^2\mathbf{B}\mathbf{Q}\mathbf{B}. \quad (13)$$

The elements of $\mathbf{z}_a$ are related by (5), which means that (11) is still a set of nonlinear equations in two variables x and y. The approach to solve the nonlinear problem is to first assume that there is no relationship among x, y and r_1. They can then be solved by LS. The final solution is obtained by imposing the known relationship (5) to the computed result via another LS computation. This two step procedure is an approximation of a true ML estimator for emitter location. By considering the elements of $\mathbf{z}_a$ independent, the ML estimate of $\mathbf{z}_a$ is

$$\mathbf{z}_a = \arg \min\{(\mathbf{h} - \mathbf{G}_a\mathbf{z}_a)^T \Psi^{-1}(\mathbf{h} - \mathbf{G}_a\mathbf{z}_a)\}$$
$$= (\mathbf{G}_a^T \Psi^{-1} \mathbf{G}_a)^{-1} \mathbf{G}_a^T \Psi^{-1} \mathbf{h} \quad (14a)$$

which is also readily recognized as the generalized LS solution of (11). Ψ is not known in practice as $\mathbf{B}$ contains the true distances between source and receivers. Further approximation is necessary in order to make the problem solvable. When the source is far from array, each r_i^0 is close to r^0 so that $\mathbf{B} \approx r^0\mathbf{I}$, where r^0 designates the range and $\mathbf{I}$ is an identity matrix of size $M - 1$. Since scaling of Ψ does not affect the answer, an approximation of (14a) is

$$\mathbf{z}_a \approx (\mathbf{G}_a^T \mathbf{Q}^{-1} \mathbf{G}_a)^{-1} \mathbf{G}_a^T \mathbf{Q}^{-1} \mathbf{h}. \quad (14b)$$

If, on the other hand, the source is close, we can first use (14b) to obtain an initial solution to estimate $\mathbf{B}$. The final answer is then computed from (14a). Although (14a) can be iterated to provide an even better answer, simulations show that applying (14a) once is sufficient to give an accurate result.

The covariance matrix of $\mathbf{z}_a$ is obtained by evaluating the expectations of $\mathbf{z}_a$ and $\mathbf{z}_a\mathbf{z}_a^T$ from (14a). The calculation

is quite involved because the matrix $\mathbf{G}_a$ contains random quantities $r_{i,1}$. We compute the covariance matrix by using the perturbation approach. In the presence of noise, $r_{i,1} = r_{i,1}^0 + cn_{i,1}$. $\mathbf{G}_a$ and $\mathbf{h}$ can be expressed as $\mathbf{G}_a = \mathbf{G}_a^0 + \Delta\mathbf{G}_a$ and $\mathbf{h} = \mathbf{h}^0 + \Delta\mathbf{h}$. Since $\mathbf{G}_a^0\mathbf{z}_a^0 = h^0$, (11) implies that

$$\psi = \Delta\mathbf{h} - \Delta\mathbf{G}_a\mathbf{z}_a^0. \tag{15}$$

Let $\mathbf{z}_a = \mathbf{z}_a^0 + \Delta\mathbf{z}_a$. Then from (14a)

$$(\mathbf{G}_a^{0T} + \Delta\mathbf{G}_a^T)\Psi^{-1}(\mathbf{G}_a^0 + \Delta\mathbf{G}_a)(\mathbf{z}_a^0 + \Delta\mathbf{z}_a)$$
$$= (\mathbf{G}_a^{0T} + \Delta\mathbf{G}_a^T)\Psi^{-1}(\mathbf{h} + \Delta\mathbf{h}). \tag{16}$$

Retaining only the linear perturbation terms and then using (12) and (15), $\Delta\mathbf{z}_a$ and its covariance matrix is

$$\Delta\mathbf{z}_a = c(\mathbf{G}_a^{0T}\Psi^{-1}\mathbf{G}_a^0)^{-1}\mathbf{G}_a^{0T}\Psi^{-1}\mathbf{Bn},$$
$$\text{cov}(\mathbf{z}_a) = E[\Delta\mathbf{z}_a\Delta\mathbf{z}_a^T] = (\mathbf{G}_a^{0T}\Psi^{-1}\mathbf{G}_a^0)^{-1} \tag{17}$$

where the square error term in (12) has been ignored and (13) has been used to give $\text{cov}(\mathbf{z}_a)$.

The solution of $\mathbf{z}_a$ assumes that x, y and r_1 are independent. But they are related by (5) at $i = 1$. The remaining question is how to incorporate this relationship to give an improved estimate. When bias is ignored (this is justified when the noises in the TDOA's are small), the vector $\mathbf{z}_a$ is a random vector with its mean centred at the true value and covariance matrix given by (17). Hence the elements of $\mathbf{z}_a$ can be expressed as

$$z_{a,1} = x^0 + e_1, \quad z_{a,2} = y^0 + e_2, \quad z_{a,3} = r_1^0 + e_3 \tag{18}$$

where e_1, e_2 and e_3 are estimation errors of $\mathbf{z}_a$. Subtracting the first two components of $\mathbf{z}_a$ by x_1 and y_1, and then squaring the elements gives another set of equations

$$\psi' = \mathbf{h}' - \mathbf{G}_a'\mathbf{z}_a'^0$$

where

$$\mathbf{h}' = \begin{bmatrix} (z_{a,1} - x_1)^2 \\ (z_{a,2} - y_1)^2 \\ z_{a,3}^2 \end{bmatrix}, \qquad \mathbf{G}_a' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix},$$
$$\mathbf{z}_a' = \begin{bmatrix} (x - x_1)^2 \\ (y - y_1)^2 \end{bmatrix} \tag{19}$$

where ψ' is a vector denoting inaccuracies in $\mathbf{z}_a$. Substituting (18) into (19) gives

$$\psi_1' = 2(x^0 - x_1)e_1 + e_1^2 \approx 2(x^0 - x_1)e_1$$
$$\psi_2' = 2(y^0 - y_1)e_2 + e_2^2 \approx 2(y^0 - y_1)e_2$$
$$\psi_3' = 2r_1^0 e_3 + e_3^2 \approx 2r_1^0 e_3. \tag{20}$$

The approximation is valid as the errors e_i, $i = 1, 2, 3$ are small. This is another approximation to the true maximum likelihood procedure. The covariance matrix of ψ' is therefore

$$\Psi' = [\psi'\psi'^T] = 4\mathbf{B}'\text{cov}(\mathbf{z}_a)\mathbf{B}'$$
$$\mathbf{B}' = \text{diag}\{x^0 - x_1, y^0 - y_1, r_1^0\}. \tag{21}$$

Since ψ is Gaussian, it follows that ψ' is also Gaussian. Thus the ML estimate of $\mathbf{z}_a'$ is

$$\mathbf{z}_a' = (\mathbf{G}_a'^T\Psi'^{-1}\mathbf{G}_a')^{-1}\mathbf{G}_a'^T\Psi'^{-1}\mathbf{h}'. \tag{22a}$$

The matrix Ψ' is not known since it contains the true values. Nevertheless, $\mathbf{B}'$ can be approximated by using the values in $\mathbf{z}_a$, $\mathbf{G}_a^0$ in (17) approximated by $\mathbf{G}_a$, and $\mathbf{B}$ in (13) approximated by the values computed from (14b). If the source is distant, then $\text{cov}(\mathbf{z}_a) \approx c^2 r^{02}(\mathbf{G}_a^{0T}\mathbf{Q}^{-1}\mathbf{G}_a^0)^{-1}$ and (22a) reduces to

$$\mathbf{z}_a' \approx (\mathbf{G}_a'^T\mathbf{B}'^{-1}\mathbf{G}_a\mathbf{Q}^{-1}\mathbf{G}_a\mathbf{B}'^{-1}\mathbf{G}_a')^{-1}$$
$$(\mathbf{G}_a'^T\mathbf{B}'^{-1}\mathbf{G}_a\mathbf{Q}^{-1}\mathbf{G}_a\mathbf{B}'^{-1})\mathbf{h}'. \tag{22b}$$

Notice that the matrix $\mathbf{G}_a'$ is constant. By taking expectations of $\mathbf{z}_a'$ and $\mathbf{z}_a'\mathbf{z}_a'^T$, the covariance matrix of $\mathbf{z}_a'$ is

$$\text{cov}(\mathbf{z}_a') = (\mathbf{G}_a'^T\Psi'^{-1}\mathbf{G}_a')^{-1}. \tag{23}$$

The final position estimate is then obtained from $\mathbf{z}_a'$ as

$$\mathbf{z}_p = \sqrt{\mathbf{z}_a'} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

or

$$\mathbf{z}_p = -\sqrt{\mathbf{z}_a'} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}. \tag{24}$$

The proper solution is selected to be the one which lies in the region of interest. If one of the coordinates of $\mathbf{z}_a'$ is close to zero, the square root in (24) may become imaginary. In such a case, the imaginary component is set to zero. To find the covariance matrix of position estimate, we express our final solution in the form $x = x^0 + e_x$ and $y = y^0 + e_y$. From the definition of $\mathbf{z}_a'$ in (19), it follows that

$$z_{a,1}' - (x^0 - x_1)^2 = 2(x^0 - x_1)e_x + e_x^2,$$
$$z_{a,2}' - (y^0 - y_1)^2 = 2(y^0 - y_1)e_y + e_y^2. \tag{25}$$

The errors e_x and e_y are relatively small compared with x^0 and y^0. Ignoring e_x^2 and e_y^2, and upon using (13), (17), (21) and (23), the covariance matrix of $\mathbf{z}_p$ is found to be

$$\Phi = \text{cov}(\mathbf{z}_p) = \frac{1}{4}\mathbf{B}''^{-1}\text{cov}(\mathbf{z}_a')\mathbf{B}''^{-1}$$
$$= c^2\mathbf{B}''\mathbf{G}_a'^T\mathbf{B}'^{-1}\mathbf{G}_a^{0T}\mathbf{B}^{-1}\mathbf{Q}^{-1}\mathbf{B}^{-1}\mathbf{G}_a^0\mathbf{B}'^{-1}\mathbf{G}_a'\mathbf{B}'')^{-1}$$

where

$$\mathbf{B}'' = \begin{bmatrix} (x^0 - x_1) & 0 \\ 0 & (y^0 - y_1) \end{bmatrix}. \tag{26}$$

In summary, (14a), (22a), and (24) are the solution equations. Since the weighting matrices in (14a) and (22a) are unknown, proper approximation is necessary to find the answer. When the source is far from array, (14b), (22b), and (24) are used. For a near-field source, (14b) is first used to give an approximation of $\mathbf{B}$. Equations (14a), (22a), and (24) then give the solution. Position accuracy is assessed through the covariance matrix in (26).

B. Linear Array

The above formulas (10), (14), and (22) are valid if the matrices involved are full rank. When sensors are arranged linearly, the matrices containing x_i and y_i will be singular because the sensor positions satisfy $y_i = ax_i + b$, $i = 1, 2, \cdots, M$, where a and b are some constants. Rewrite (9) as

$$-2x_{i,1}(x + ay) - 2r_{i,1}r_1 = r_{i,1}^2 - K_i + K_1. \quad (27)$$

Equation (27) is linear in $w = (x + ay)$ and r_1. Using the first stage procedure illustrated in Section II-A gives a solution similar to (14a)

$$\mathbf{z}_l = (\mathbf{G}_l^T \Psi^{-1} \mathbf{G}_l)^{-1} \mathbf{G}_l^T \Psi^{-1} \mathbf{h}:$$

$$\mathbf{z}_l = \begin{bmatrix} x + ay \\ r_1 \end{bmatrix}, \qquad \mathbf{G}_l = - \begin{bmatrix} x_{2,1} & r_{2,1} \\ x_{3,1} & r_{3,1} \\ \cdot & \cdot \\ x_{M,1} & r_{M,1} \end{bmatrix}. \quad (28)$$

The vector $\mathbf{h}$ is defined in (11). A second stage is unnecessary since r_1 and $x + ay$ are two independent variables. To obtain the position estimate, substitute the computed r_1 and $x = w - ay$ into (5) at $i = 1$ to produce a quadratic in y. The location estimate is

$$y = \frac{-E \pm \sqrt{E^2 - 4AC}}{2A}, \qquad x = w - ay$$

where

$$A = 1 + a^2, E = -2(aw + b), C = K_1 - 2x_1w + w^2 - r_1^2. \quad (29)$$

If the array is on the x-axis and if (x_1, y_1) is at the origin, $a = b = x_1 = K_1 = 0$ and (28) reduces to the simple form $y = \pm\sqrt{r_1^2 - w^2}$ and $x = w$. This solution is identical to the one given in [4].

To calculate the covariance matrix, we perturb the random quantities in (28) and proceed as before to obtain

$$E[\Delta\mathbf{z}_l \Delta\mathbf{z}_l^T] = (\mathbf{G}_l^{0T} \Psi^{-1} \mathbf{G}_l^0)^{-1}. \quad (30)$$

According to (5), Δx, Δy and Δr_1 are related by $\Delta r_1 = -\{(x_1 - x^0)\Delta x + (y_1 - y^0)\Delta y\}/r_1^0$. Hence

$$\Delta\mathbf{z}_l = \mathbf{T}\Delta\mathbf{z}_p = \begin{bmatrix} \frac{1}{x_0^0 - x_1} & \frac{a}{y_0^0 - y_1} \\ \frac{r_1^0}{r_1^0} & \frac{r_1^0}{r_1^0} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \quad (31)$$

and the required covariance matrix is

$$\begin{aligned} \Phi &= \mathbf{T}^{-1}(\mathbf{G}_l^{0T} \Psi^{-1} \mathbf{G}_l^0)^{-1} \mathbf{T}^{-T} \\ &= c^2(\mathbf{T}^T \mathbf{G}_l^{0T} \mathbf{B}^{-1} \mathbf{Q}^{-1} \mathbf{B}^{-1} \mathbf{G}_l^0 \mathbf{T})^{-1}. \end{aligned} \quad (32)$$

The proposed solution requires the knowledge of TDOA covariance matrix $\mathbf{Q}$ which may not be known in practice. If the noise power spectral densities are similar at sensors, it

can be replaced by a matrix of diagonal elements 1 and 0.5 for all other elements (see (44) below). If the sensor positions have significant uncertainties, we can also incorporate these uncertainties to the matrix so that more weights are given to the equation containing the more reliable sensor position (x_i, y_i).

The covariance matrix of position estimate contains the uncertainty information in localization. In particular, the position mean-square error (MSE) is equal to the sum of the diagonal elements of Φ. Another commonly used measure of localization accuracy is the circular error probable (CEP) [12]. It is defined as the radius of the circle that has its centre at the mean and contains half the realizations of the random position estimate. Recall that since the noise vector $\mathbf{n}$ is Gaussian, $\Delta\mathbf{z}_p$ is also Gaussian distributed. The CEP is thus related to Φ. Details of its computation from Φ can be found in [11]–[12].

III. COMPARISON WITH THE CRLB

The Cramér-Rao inequality [14] sets a lower bound for the variance of any unbiased parameter estimators. Hence it is of interest to compare the estimator with the optimum.

The CRLB of the localization problem is derived in the Appendix. It is given by

$$\Phi^0 = c^2(\mathbf{G}_t^{0T} \mathbf{Q}^{-1} \mathbf{G}_l^0)^{-1} \quad (33)$$

where the matrix $\mathbf{G}_t^0$ is defined in (7) with $(x, y, r_i) = (x^0, y^0, r_i^0)$.

We shall first consider the arbitrarily distributed array case. The corresponding position covariance matrix is given in (26). Denote the (i, j)th element of a matrix $\mathbf{R}$ as $[\mathbf{R}]_{i,j}$. Then from (11), (12) and (21)

$$[\mathbf{B}^{-1}\mathbf{G}_a^0\mathbf{B}'^{-1}]_{i-1,1} = \frac{-x_{i,1}}{(x^0 - x_1)r_i^0}$$

$$[\mathbf{B}^{-1}\mathbf{G}_a^0\mathbf{B}'^{-1}]_{i-1,2} = \frac{-y_{i,1}}{(y^0 - y_1)r_i^0}$$

$$[\mathbf{B}^{-1}\mathbf{G}_a^0\mathbf{B}'^{-1}]_{i-1,3} = \frac{-r_{i,1}^0}{r_1^0 r_i^0}. \quad (34)$$

Hence from the definition of $\mathbf{G}_a'$ and $\mathbf{B}''$ in (19) and (26), and noting that $x_{i,1} = x_i - x_1$ and $r_{i,1}^0 = r_i^0 - r_1^0$

$$[\mathbf{B}^{-1}\mathbf{G}_a^0\mathbf{B}'^{-1}\mathbf{G}_a'\mathbf{B}'']_{i-1,1}$$

$$= -(x^0 - x_1)\left\{\frac{x_{i,1}}{(x^0 - x_1)r_i^0} + \frac{r_{i,1}^0}{r_1^0 r_i^0}\right\}$$

$$= [\mathbf{G}_t^0]_{i-1,1}.$$

Similarly

$$[\mathbf{B}^{-1}\mathbf{G}_a^0\mathbf{B}'^{-1}\mathbf{G}_a'\mathbf{B}'']_{i-1,2} = [\mathbf{G}_t^0]_{i-1,2}. \quad (35)$$

Thus

$$\mathbf{B}^{-1}\mathbf{G}_a^0\mathbf{B}'^{-1}\mathbf{G}_a'\mathbf{B}'' = \mathbf{G}_t^0. \quad (36)$$

Comparison between (26) and (33) reveals that the position estimate with arbitrary array can achieve the optimum performance and is therefore efficient, when the measurement errors in TDOA's are small.

In the linear array case, using the definitions of $\mathbf{B}$, $\mathbf{G}_l^0$ and $\mathbf{T}$ in (12), (28) and (31), it can be verified that

$$\mathbf{B}^{-1}\mathbf{G}_l^0\mathbf{T} = \mathbf{G}_t^0. \tag{37}$$

It is evident from (32) that the covariance matrix Φ is identical to Φ^0 of (33). As a consequence, our estimator is also efficient for linearly distributed sensors. This is consistent with the results given in [4].

IV. Comparison with Previous Work

We shall next compare our technique with those in the literature.

A. Linear Array

In the special case of a linear array of three sensors, Carter [1] has derived an exact formula for source range and bearing. We next show that the solution of Section II-B will give the same answer.

The localization geometry is shown in Fig. 2. The sensor positions are numbered as $(x_1 = 0, y_1 = 0)$, $(x_2 = -L_1, y_2 = 0)$ and $(x_3 = L_2, y_3 = 0)$ so that r_1 corresponds to the range r of the object. Substituting $a = 0$, $x_{2,1} = -L_1$, $x_{3,1} = L_2$, $K_1 = 0$, $K_2 = L_1^2$ and $K_3 = L_2^2$ into (27) yields

$$\begin{bmatrix} -L_1 & r_{2,1} \\ L_2 & r_{3,1} \end{bmatrix}\begin{bmatrix} x \\ r \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} r_{2,1}^2 - L_1^2 \\ r_{3,1}^2 - L_2^2 \end{bmatrix}. \tag{38}$$

Solving (38) for x and r gives

$$r = \frac{L_1\left[1 - \left(\frac{r_{2,1}}{L_1}\right)^2\right] + L_2\left[1 - \left(\frac{r_{3,1}}{L_2}\right)^2\right]}{2\left(\frac{r_{3,1}}{L_2} + \frac{r_{2,1}}{L_1}\right)} \tag{39}$$

and the bearing angle

$$\theta = \cos^{-1}\left(\frac{x}{r}\right) = \cos^{-1}\left(\frac{L_2^2 - 2rr_{3,1} - r_{3,1}^2}{2rL_2}\right). \tag{40}$$

They are identical to the results given in [1] (where the sensors are numbered as $(x_1 = 0, y_1 = 0)$, $(x_2 = L_1, y_2 = 0)$ and $(x_3 = L_1 + L_2, y_3 = 0)$. Accordingly, our $r_{2,1}$ and $r_{3,1}$ are equal to $-r_{2,1}$ and $r_{3,2}$ in [1]). x is equal to

$$x = \frac{r_{2,1}L_2^2 - r_{3,1}L_1^2 - r_{2,1}r_{3,1}r_{3,2}}{2\{r_{2,1}L_2 + r_{3,1}L_1\}} \tag{41}$$

and y is obtained from $\sqrt{r^2 - x^2}$.

With three or more sensors, a traditional method is the focused beamforming derived by Carter [1], [15]. The sensor outputs are prefiltered and time shifted according to delays chosen by some hypothesized range and bearing. The processed sensor outputs are summed, filtered, squared and averaged to give a final output. The source range and bearing estimate is the pair which maximizes the final output. This technique requires an extensive search in the 2-D range and bearing space. Another approach is by Hahn [2]–[3] in which the TDOA vector $\mathbf{d}$ is first estimated. The source range and bearing are deduced from the weighted sum of ranges and bearings obtained from the associated TDOA's for every triplet $\{(x_1, y_1), (x_i, y_i), (x_j, y_j)\}, i \neq j$. The selection of
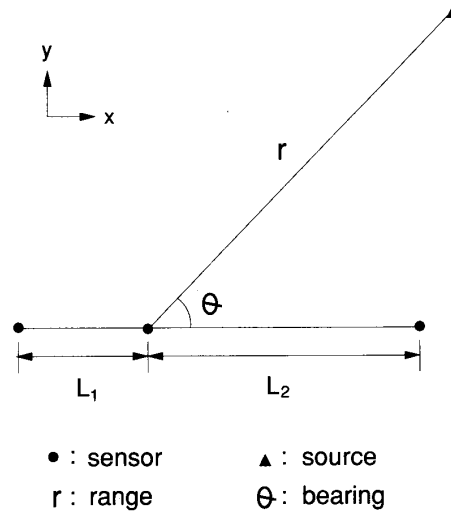
the weights is crucial to achieve optimum performance but expressions for the weights are, in general, very complicated [3]. Moreover, the solution is correct only for distant source. Recently, Abel and Smith [4] deduced an explicit closed-form solution for the problem which is simple to compute and achieves CRLB around small error region. In addition, no distant source assumption is required. Our result is indeed identical to theirs when the coordinate axes are chosen to be the same.

In sonar, the quantities of interest are often range and bearing instead of the x and y coordinates of the emitter. The optimum bounds for range and bearing variances have been derived by Bangs and Schultheiss [16]. They assume spatially incoherent noise fields and a distant source. It is also possible to derive the variances of range and bearing for our estimator. As the estimator has already been shown to be efficient around small error region, the corresponding variances will be the CRLB. It will then be shown that with the two assumptions made in [16], we shall come to Bangs and Schultheiss's solution.

Without loss of generality, let the y-coordinate of all sensors be zero. In terms of range r and bearing θ, x is equal to $r\cos\theta$ and hence $\Delta x = -r^0 \sin\theta^0 \Delta\theta + \cos\theta^0 \Delta r$. We arrange the sensors so that r_1 is the source range. Then

$$\Delta z_l = \begin{bmatrix} \Delta x \\ \Delta r_1 \end{bmatrix} = \mathbf{H}\begin{bmatrix} \Delta\theta \\ \Delta r \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} -r^0 \sin\theta^0 & \cos\theta^0 \\ 0 & 1 \end{bmatrix} \tag{42}$$

From (30), we have

$$\zeta = E\left[\begin{bmatrix} \Delta\theta \\ \Delta r \end{bmatrix}[\Delta\theta \quad \Delta r]\right] = (\mathbf{H}^T\mathbf{G}_l^{0T}\Psi^{-1}\mathbf{G}_l^0\mathbf{H})^{-1} \tag{43}$$

which is a general formula for arbitrary source and noise fields. In spatially incoherent noise fields of identical receiver noise power spectra, the vector TDOA covariance matrix $\mathbf{Q}$ in (3)

Fig. 2. Localization with a linear three sensor array.

can be simplified to

$$Q = \frac{g^{-1}}{M}\begin{bmatrix} 1 & 0.5 & 0.5 & \cdots & 0.5 \\ 0.5 & 1 & 0.5 & \cdots & 0.5 \\ . & . & . & \cdots & . \\ 0.5 & 0.5 & 0.5 & \cdots & 1 \end{bmatrix}$$

$$= \frac{1}{2}g^{-1}\begin{bmatrix} M-1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & M-1 & -1 & \cdots & -1 & -1 \\ . & . & . & \cdots & . & . \\ -1 & . & . & \cdots & -1 & M-1 \end{bmatrix}^{-1}$$

$$g = \left\{ \frac{T}{2\pi}\int_0^\Omega \omega^2 \frac{S(\omega)^2/N(\omega)^2}{1 + MS(\omega)/N(\omega)} d\omega \right\}. \tag{44}$$

When the definitions of H, G_l^0 and Ψ given in (42), (28) and (13) are substituted into (43), we have (45), at the bottom of the next page. If the distant source assumption is imposed, then
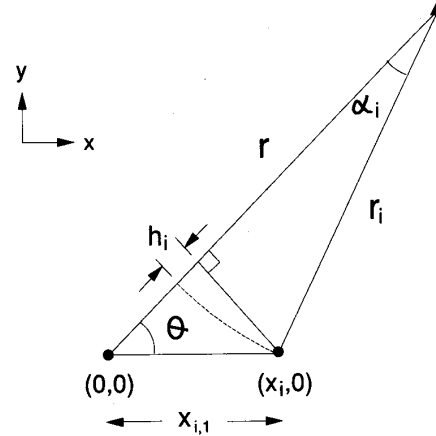
$$r_i^0 \approx r^0$$

and as illustrated in Fig. 3

$$\frac{x_{i,1}\cos\theta^0 + r_{i,1}^0}{r_i^0} = \frac{h_i}{r_i^0} = (1 - \cos\alpha_i)$$

$$= 2\sin^2\frac{\alpha_i}{2} \approx 2\left(\frac{x_{i,1}\sin\theta^0}{2r_i^0}\right)^2$$

$$\approx \frac{1}{2}\frac{\sin^2\theta^0}{r^{02}}x_{i,1}^2. \tag{46}$$

Since $x_{i,1} \ll r^0$, $[\zeta^{-1}]_{1,2} \approx 0$. It follows that

$$\sigma_\theta^2 \approx \left\{ gc^2\sin^2\theta^0 \sum_{i=1}^M\sum_{j=1}^M(x_i - x_j)^2 \right\}^{-1}$$

$$\sigma_r^2 = \left\{ \frac{gc^2\sin^4\theta^0}{4r^{04}}\sum_{i=1}^M\sum_{j=1}^M(x_{i,1}^2 - x_{j,1}^2)^2 \right\}^{-1} \tag{47}$$

where σ_θ^2 and σ_r^2 denote the variances of source bearing and range. They are identical to those given by Bangs and Schultheiss [15]–[16].

● : sensor   ▲ : source

r : range   Θ : bearing

Fig. 3. Sensors and source geometry.

B. Localization with Arbitrary Array

For arbitrarily distributed sensors, spherical-intersection (SX) [7], spherical-interpolation (SI) [6], [8]–[9], penalty function SI [9], DAC [10], and Taylor-series [11]–[12] are commonly used methods. Although the first four have computational advantage over the last, their solution is less accurate.

SX assumes known r_1 and solves x and y in terms of r_1 from (9). (5) is then used to find r_1 and hence x, y. Since r_1 is assumed to be constant in the first step, the degree of freedom to minimize the second norm of ψ is reduced. The solution obtained is therefore nonoptimum as demonstrated in [8]. SI assumes the three variables x, y and r_1 in (9) to be independent, and eliminates r_1 from those equations. The answer obtained is of course nonoptimum as the relationship (5) among x, y and r_1 is ignored. Penalty function SI is an improvement on SI by incorporating (5), but a suitable choice of the parameter α which controls the contributions of (5) to the error criterion is unknown. Our method, on the other hand, incorporates this constraint to the solution by using a

$$[\zeta^{-1}]_{1,1} = gc^2r^{02}\sin^2\theta^0 \sum_{i=1}^M\sum_{j=1}^M\left(\frac{x_{i,1}}{r_i^0} - \frac{x_{j,1}}{r_j^0}\right)^2$$

$$[\zeta^{-1}]_{2,1} = [\zeta^{-1}]_{1,2} = -gc^2r^0\sin\theta^0$$

$$\times \left\{ (M-1)\sum_{i=2}^M \frac{x_{i,1}(x_{i,1}\cos\theta + r_{i,1}^0)}{r_i^{02}} \right.$$

$$\left. - \sum_{i=2}^M\sum_{j=2,j\neq i}^M \frac{x_{i,1}(x_{j,1}\cos\theta^0 + r_{j,1}^e)}{r_i^0 r_j^0} \right\}$$

$$[\zeta^{-1}]_{2,2} = gc^2\sum_{i=1}^M\sum_{j=1}^M\left(\frac{x_{i,1}\cos\theta^0 + r_{i,1}^0}{r_i^0} - \frac{x_{j,1}\cos\theta^0 + r_{j,1}^0}{r_j^0}\right) \tag{45}$$

TABLE I
COMPARISON OF MSE FOR THE SI, TAYLOR-SERIES AND PROPOSED METHODS; ARBITRARY ARRAY AND NEAR SOURCE

MSE	M = 3	M = 4	M = 5	M = 6	M = 7	M = 8	M = 9	M = 10
A	no sol.	1.5768	0.1597	0.1480	0.1229	0.1164	0.1148	0.1103
B	2.1726	0.7061	0.1460	0.1341	0.1144	0.1057	0.1034	0.09462
C	2.1726	0.7282	0.1456	0.1359	0.1159	0.1077	0.1055	0.09697
D	2.1726	0.6986	0.1451	0.1337	0.1141	0.1055	0.1034	0.09480
E	1.9794	0.6884	0.1451	0.1334	0.1143	0.1054	0.1032	0.09432

A: SI method. B: Taylor series method. C: proposed method, { (14b), (22b), (24) }. D: proposed method, { (14b), (14a), (22a), (24) }. E: theoretical MSE of the new method = CRLB.

LS computation with appropriate weighting. It follows that the optimum estimate is obtained.

DAC consists of dividing the sensor measurements (TDOA's in our case) into groups, each having a size equal to the number of unknowns. The unknown parameters are computed from each group and then appropriately combined to give the final answer. The solution uses stochastic approximation and requires the Fisher information in each group to be sufficiently large. As a consequence, optimum performance can be achieved only when the noise is small enough. This implies a low noise threshold before performance deviates from the CRLB. Another difficulty is that if the number of sensors is not equal to an integral multiple of the number of unknowns plus one, the TDOA estimates from those remaining sensors cannot be utilized to improve solution accuracy. Our estimator has no such restriction. Moreover, it does not require stochastic approximation and hence has a larger noise threshold. This is verified by simulations in Section V.

While the Taylor-series method can give an accurate position estimate at reasonable noise levels, the major drawback is that it is iterative, and there is no guarantee for convergence. Our approach does not require iteration. If the source is far from the array, we compute first the intermediate solution z_a by (14b) and the final solution by (22b) followed by (24). It does not require the computation of $1/r_i$ for $i = 1, 2, \cdots, M$, which is most costly for large M as square root and division operations are required. Although computing $1/r_i$ is necessary for a near source, it is needed only once. In the Taylor-series method, calculation of $1/r_i$ is a must for each iteration as can be seen in (7). Finally, no special procedure is required to detect divergence.

Under the condition that the Taylor-series method is properly converged, it is of interest to compare the accuracy of the two techniques. The Taylor-series method assumes that the linearization error is small. It has been illustrated in [17] that the higher order terms are significant in determining the solution in bad geometric dilution of precision (GDOP) situations. Even through the noise power is relatively small, there is no guarantee that the obtained solution is accurate. It is expected that our method works better in cases where GDOP is poor. In fact, our method guarantees optimum performance around small TDOA noise region and thus is always be superior.

V. SIMULATION RESULTS

Simulations are performed to corroborate the theoretical development and to compare the relative localization accuracy

for different methods. For simplicity, we assume that the signal and noises in (1) are white random processes and that the SNR of all sensor inputs are identical. Consequently, the covariance matrix Q is found from (3) to be σ_d^2 for diagonal elements and $0.5\,\sigma_d^2$ for all other elements, where σ_d^2 is the TDOA variance. The TDOA estimates are simulated by adding to the actual TDOA's correlated Gaussian random noises with covariance matrix given by Q.

Table I compares the localization accuracy of the SI, Taylor-series and the new estimator for an arbitrary array with a different number of sensors. The sensor positions are ($x_1 = 0, y_1 = 0$), ($x_2 = -5, y_2 = 8$), ($x_3 = 4, y_3 = 6$), ($x_4 = -2, y_4 = 4$), ($x_5 = 7, y_5 = 3$), ($x_6 = -7, y_6 = 5$), ($x_7 = 2, y_7 = 5$), ($x_8 = -4, y_8 = 2$), ($x_9 = 3, y_9 = 3$) and ($x_{10} = 1, y_{10} = 8$). The source is at ($x^0 = 8, y^0 = 22$). The TDOA noise power is set to $0.001/c^2$ and the MSE $= E[(x - x^0)^2 + (y - y^0)^2]$ are obtained from the average of 100 000 independent runs. The weighting matrices W and V in the SI method [8, eq. (12)] are both set to Q^{-1}. The initial position guess in the Taylor-series method is chosen to be the true solution to allow for convergence. Simulations show that at least three iterations are required for convergence. Two ways are used to compute the solution in the new method. One uses (14b), (22b), and (24). The other first employs (14b) to deduce an estimate of B and then uses (14a) to calculate the intermediate solution z_a. The final result is obtained from (22a) followed by (24). The matrix G_a^0 is replaced by G_a and the true values in B' is substituted by the values in z_a in (22). The differences between the two results are small. This indicates that in most cases, the simplified formulae (14b) and (22b) are sufficient. Among the three methods, SI performs worst and our solution method gives a slightly smaller MSE than the Taylor-series method. Note that the proposed method with the simplified formulae still performs better than the SI method. Additionally, the SI method cannot produce a position estimate for the three sensor situation when the number of independent equations equals the number of unknowns. The error for SI is most significant when there are four sensors. The theoretical MSE given by the sum of the diagonal elements of Φ in (26), which is identical to the CRLB, is also computed. Notice that there is a close match between the predicted and the simulated values and the validity of (26) is confirmed. This also confirms that our estimator achieves the CRLB and that replacing the true values in B, G_a^0 and B' by their noisy versions does not affect the result much.

In the case of a linear array, the results are given in Table II. Only the proposed and Taylor-series method are

TABLE II
COMPARISON OF MSE FOR THE PROPOSED AND TAYLOR-SERIES METHODS; LINEAR ARRAY AND NEAR SOURCE

MSE	M = 3	M = 4	M = 5	M = 6	M = 7	M = 8	M = 9	M = 10
B	8.2574	1.2836	0.3566	0.1222	0.06155	0.02854	0.01746	0.009540
C	8.2574	1.1131	0.3556	0.1225	0.06199	0.02884	0.01772	0.009726
D	8.2574	1.1117	0.3545	0.1219	0.06148	0.02852	0.01746	0.009541
E	7.2819	1.1000	0.3548	0.1219	0.06123	0.02840	0.01750	0.009599

B: Taylor series method. C: proposed method, {(28) with $\Psi = Q$, (29)}. D: proposed method, {(28) with $\Psi = Q$, (28), (29)}. E: theoretical MSE of the new method = CRLB.

compared because SI fails due to singularity problem. The sensor coordinates are $(x_i = -(i - 1), y_i = 0)$ when i is odd and $(x_i = i, y_i = 0)$ when i is even, $i = 1, 2, \cdots, 10$. The source position is the same as before. The TDOA variance is set to $0.0001/c^2$. The solution is computed by (28), (29). The localization MSE decreases as the number of sensor increases. Again, the differences in the results by using the approximation $\Psi = Q$ in (28) and the actual equations by first estimating Ψ are small. The theoretical MSE, which is the same as CRLB, are also evaluated from (32). They are in close agreement with the simulated values. Again, the Taylor-series method gives almost identical results as the new method.

When the source is far away at position $(x^0 = -50, y^0 = 250)$, the MSE for random and linear arrays are given in Tables III and IV respectively. The sensor positions are the same as before and the TDOA noise power is $\sigma_d^2 = 0.00001/c^2$ in both cases. Since the source range is large, the results obtained from the simplified and actual formulae are found to be identical. The source position cannot be estimated if there are only three sensors due to large position variations. As shown in Table III, the proposed method performs much better than SI and slightly better than Taylor-series. Indeed, the new method attains the CRLB for both the random and linear array scenarios. In the case of a linear array, there are some interesting observations. When M is small (4 or 5), the proposed method has significantly less MSE than the Taylor-series method. The larger MSE in the Taylor-series method is due to linearization errors. With a distant source, this error is significant when the array base line is small, which is the situation when M is small. As M increases the base line also increases, reducing the linearization errors. Consequently comparable performance is observed. It must be emphasized that the Taylor-series method has been given a most favorable initial guess, namely, the true position. In practice, this is not possible and solution divergence may occur.

It has been shown in [10] that DAC is an optimum estimator around small error region. The method was tested in an arbitrary array in the same conditions as before. We found that DAC perform equally well with our method in the near-field case. In the far-field situation, DAC gives an MSE in the order of 10^6, even through the noise in the second case is smaller. This is because DAC requires a stochastic approximation which is only valid when the transmitter position variance is small. It can be seen from (33) and the definition of G_t in (7) that transmitter position variance is roughly proportional to the squared range of the object. Hence a far-field emitter

TABLE III
COMPARISON OF MSE FOR THE SI, TAYLOR-SERIES AND PROPOSED METHODS; ARBITRARY ARRAY AND DISTANT SOURCE

MSE	M = 4	M = 5	M = 6	M = 7	M = 8	M = 9	M = 10
A	101881	213.93	48.12	40.40	41.87	40.11	37.23
B	346.86	147.57	44.38	38.64	38.63	36.55	33.80
C	348.74	144.84	44.06	38.41	38.47	36.50	33.87
E	328.82	143.94	44.06	38.54	38.53	36.47	33.73

A: SI method. B: Taylor series method. C: proposed method, {(14b), (22b), (24)}. E: theoretical MSE of the new method = CRLB.

TABLE IV
COMPARISON OF MSE FOR THE PROPOSED AND TAYLOR-SERIES METHODS; LINEAR ARRAY AND DISTANT SOURCE

MSE	M = 4	M = 5	M = 6	M = 7	M = 8	M = 9	M = 10
B	1797.13	432.68	159.50	69.16	34.56	18.56	10.85
C	1588.31	409.66	153.92	68.41	34.44	18.56	10.86
E	1437.25	408.17	154.05	68.06	34.25	18.57	10.90

B: Taylor series method. C: proposed method, {(28) with $\Psi = Q$, (29)}. E: theoretical MSE of the new method = CRLB.

requires an extremely low noise power to make the stochastic approximation valid. The result is that DAC has a low noise threshold.

To give a comparison of the thresholding effect between DAC and our method, the MSE for a fixed array configuration is studied by varying the TDOA noise power. The previous arbitrary array configuration is chosen with $M = 9$. The source is located at $(x^0 = -50, y^0 = 250)$. The result is illustrated in Fig. 4. Both methods perform well at low noise level. Thresholding effect occurs in DAC when $\sigma_d^2 = 0.000016$. On the other hand, our method follows closely with the CRLB until $\sigma_d^2 = 0.0001$ (six times larger then that in DAC). It is also interesting to see that although deviation from CRLB starts at high noise level, the MSE does not jump to a large value as compared to DAC. It is mentioned in [18] that the Bhattachayya bound is tighter than CRLB for nonlinear parameter estimation. Hence it is more realistic to compare our estimator's performance with the Bhattachayya bound at low SNR. It can be seen in Fig. 4 that our method performs only a little worse than the second-order Bhattachayya bound (BHB2). This difference is probably due to the fact that we have neglected second order error terms in our algorithm. It is expected that our estimator
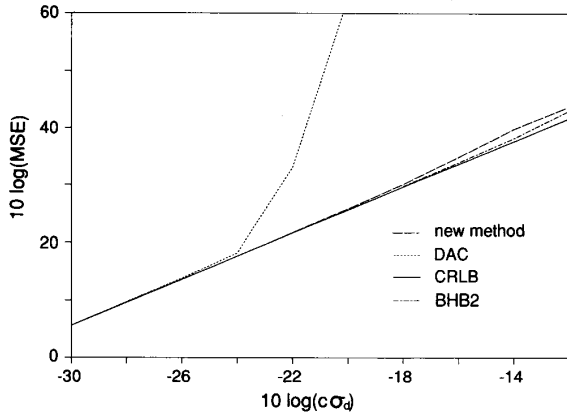
Fig. 4. Comparison of threshold effect between the new method and DAC.

will approach BHB2 if those second order error terms are taken into account. Many simulations has been tried with different array geometry and source locations (near-field and far-field). They all verify that our estimator enters its large error region at a smaller SNR than DAC and follows closely the BHB2.

VI. CONCLUSION

A new approach for localizing a source from a set of hyperbolic curves defined by TDOA measurements is proposed. By introducing an intermediate variable, the nonlinear equations relating TDOA estimates and source position can be transformed into a set of equations which are linear in the unknown parameters and the intermediate variable. A LS then gives their solution. By exploiting the known relation between the intermediate variable and the position coordinates, a second weighted LS gives the final solution for the position coordinates. The covariance matrix of the position estimate was derived and found to meet the CRLB. The estimator is asymptotically efficient for arbitrary or linear array configurations. Indeed, it is an approximation of the ML estimator for hyperbolic position fix when the TDOA error is small. In the case of a linear array, the solution reduces to that of [4]. A comparison with other localization methods was conducted. The proposed method has the same simplicity as the SI method but performs significantly better, particularly when the number of sensors is small. It has a higher noise threshold than the DAC method and provides an explicit solution form which is not available in existing optimum estimators. Finally, the new technique offers a computational advantage over the Taylor-series technique and eliminates the convergence problem.

In this paper, we have only considered TDOA estimation error. In practical localization system, sensor position uncertainty is often encountered [19]. If the variances of the uncertainties of individual sensors are known, it will not be difficult to incorporate the reciprocal of the variances as weights in the weighted LS to give an ML estimator (see discussion after (32)).

APPENDIX

Hahn and Tretter's [2] estimator is an implementation of the ML estimator for a vector of TDOA known to be asymptotically Gaussian with covariance matrix given by $\mathbf{Q}$. Hence the conditional probability density function of $\mathbf{d}$ is

$$p(\mathbf{d} \mid \mathbf{z}_p) = \frac{1}{(2\pi)^{(M-1)/2}|\mathbf{Q}|^{1/2}}$$
$$\times \exp\left\{-\frac{1}{2}\left(\mathbf{d} - \frac{1}{c}\mathbf{r}\right)^T \mathbf{Q}^{-1}\left(\mathbf{d} - \frac{1}{c}\mathbf{r}\right)\right\} \quad (A1)$$

where $\mathbf{r} = [r_{2,1}, r_{3,1}, \cdots, r_{M,1}]^T$ is a function of $\mathbf{z}_p$. The transmitter position can be expressed as a nonlinear function of $\mathbf{d}$, i.e., $x = f_1(\mathbf{d})$ and $y = f_2(\mathbf{d})$. Using Taylor-series expansion of x and y around the true TDOA vector, it can be verified that both the bias and variance of transmitter position are proportional to the TDOA covariance matrix $\mathbf{Q}$. If variations in TDOA's are small so that the bias square is insignificant compared with the variance, the CRLB of $\mathbf{z}_p$ is given by [14]

$$\Phi^0 = \left\{ E\left[\left(\frac{\partial}{\partial \mathbf{z}_p} \ln p(\mathbf{d}|\mathbf{z}_p)\right) \right.\right.$$
$$\left.\left. \left(\frac{\partial}{\partial \mathbf{z}_p} \ln p(\mathbf{d}|\mathbf{z}_p)\right)^T \right]|_{\mathbf{z}_p=\mathbf{z}_p^0} \right\}^{-1}. \quad (A2)$$

The partial derivative of $\ln p(\mathbf{d} \mid \mathbf{z}_p)$ with respect to $\mathbf{z}_p$ is

$$\frac{\partial}{\partial \mathbf{z}_p} \ln p(\mathbf{d} \mid \mathbf{z}_p) = -\frac{1}{c}\frac{\partial \mathbf{r}^T}{\partial \mathbf{z}_p}\mathbf{Q}^{-1}\left(\mathbf{d} - \frac{\mathbf{r}}{c}\right). \quad (A3)$$

Hence

$$\Phi^0 = c^2 \left(\frac{\partial \mathbf{r}^T}{\partial \mathbf{z}_p}\mathbf{Q}^{-1}\frac{\partial \mathbf{r}}{\partial \mathbf{z}_p}\right)^{-1}|_{\mathbf{z}_p=\mathbf{z}_p^0} \quad (A4)$$

where $\partial \mathbf{r}^T/\partial \mathbf{z}_p$ is found from the definition of $\mathbf{r}$ to be $\mathbf{G}_t$.

ACKNOWLEDGMENT

REFERENCES

[1] G. C. Carter, "Time delay estimation for passive sonar signal processing," IEEE Trans. Acoust., Speech, Signal Processing, vol. ASSP-29, pp. 463–470, June 1981.

[2] W. R. Hahn and S. A. Tretter, "Optimum processing for delay-vector estimation in passive signal arrays," IEEE Trans. Inform. Theory, vol. IT-19, pp. 608–614, Sept. 1973.

[3] W. R. Hahn, "Optimum signal processing for passive sonar range and bearing estimation," J. Acoust. Soc. Am., vol. 58, pp. 201–207, July 1975.

[4] J. S. Abel and J. O. Smith, "Source range and depth estimation from multipath range difference measurements," IEEE Trans. Acoust., Speech, Signal Processing, vol. 37, pp. 1157–1165, Aug. 1989.

[5] B. T. Fang, "Simple solutions for hyperbolic and related position fixes," IEEE Trans. Aerosp. Electron. Syst., vol. 26, pp. 748–753, Sept. 1990.

[6] B. Friedlander, "A passive localization algorithm and its accuracy analysis," *IEEE J. Ocean. Eng.*, vol. OE-12, pp. 234–245, Jan. 1987.

[7] H. C. Schau and A. Z. Robinson, "Passive source localization employing intersecting spherical surfaces from time-of-arrival differences," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-35, pp. 1223–1225, Aug. 1987.

[8] J. O. Smith and J. S. Abel, "Closed-form least-squares source location estimation from range-difference measurements," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-35, pp. 1661–1669, Dec. 1987.

[9] J. S. Abel and J. O. Smith, "The spherical interpolation method for closed-form passive source localization using range difference measurements," in *Proc. ICASSP-87* (Dallas, TX), pp. 471–474.

[10] J. S. Abel, "A divide and conquer approach to least-squares estimation," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 26, pp. 423–427, Mar. 1990.

[11] W. H. Foy, "Position-location solutions by Taylor-series estimation," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-12, pp. 187–194, Mar. 1976.

[12] D. J. Torrieri, "Statistical theory of passive location systems," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-20, pp. 183–198, Mar. 1984.

[13] C. H. Knapp and G. C. Carter, "The generalized correlation method for estimation of time delay," *IEEE Trans, Acoust., Speech, Signal Processing*, vol. ASSP-24, pp. 320–327, Apr. 1976.

[14] J. M. Mendel, *Lessons in Digital Estimation Theory.* Englewood Cliffs, NJ: Prentice-Hall, 1987.

[15] G. C. Carter, "Variance.bounds for passively locating an acoustic source with a symmetric line array," *J. Acoust. Soc. Am.*, vol. 62, pp. 922–926, Oct. 1977.

[16] W. J. Bangs and P. M. Schultheiss, "Space-time processing for optimal parameter estimation," in *Signal Processing*, J. W. R. Griffiths, P. L. Stocklin, and C. Van Schooneveld, Eds. New York: Academic, 1973, pp. 577–590.

[17] S. Bancroft, "An algebraic solution of the GPS equations," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-21, pp. 56–59, Jan. 1985.

[18] H. L. Van Trees, *Detection, Estimation and Modulation Theory, Part I.* New York: Wiley, 1968.

[19] G. C. Carter, "Passive ranging errors due to receiving hydrophone position uncertainty," *J. Acoust. Soc. Am.*, vol. 65, pp. 528–530, Feb. 1979.

Y. T. Chan (SM'80) was born in Hong Kong. He received the B.Sc. and M.Sc. degrees from Queen's University, Kingston, Ontario, Canada, in 1963 and 1967, and the Ph.D. degree from the University of New Brunswick, Fredericton, Canada, in 1973, all in electrical engineering.

He has worked with Northern Telecom Ltd. and Bell-Northern Research. Since 1973 he has been at the Royal Military College of Canada, where he is presently a Professor. He has also spent two sabbatical terms at the Chinese University of Hong Kong, during 1985–1986 and 1993. His research interests are in sonar signal processing and passive localization and tracking techniques. He has served as a consultant on sonar systems.

Dr. Chan was an Associate Editor (1980–1982) of the IEEE TRANSACTIONS ON SIGNAL PROCESSING and was the Technical Program Chairman of the 1984 International Conference on Acoustics, Speech, and Signal Processing (ICASSP-84). He directed a NATO Advanced Study Institute on Underwater Acoustic Data Processing in 1988 and was the General Chairman of ICASSP-91 held in Toronto, Canada.

K. C. Ho (S'89–M'91) was born in Hong Kong. He received the B.Sc. degree with First Class Honours in electronics and the Ph.D. degree in electronic engineering from the Chinese University of Hong Kong, in 1988 and 1991, respectively.

From August to November 1991, he was a Research Associate at the Chinese University of Hong Kong. He is currently a Research Fellow in the Department of Electrical and Computer Engineering at the Royal Military College of Canada. His research interests are in digital signal processing, geolocation, and developing efficient adaptive signal processing algorithms for various applications including time delay estimation, noise cancelation, and system identification.

Dr. Ho was the recipient of the Croucher Foundation Studentship from 1988 to 1991.