Emitter Location Accuracy Using TDOA and Differential Doppler

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Two methods for locating a radio or sonar transmitter are briefly described: time difference of arrival (TDOA) and differential Doppler. Two formulas are derived which relate the accuracy of the time and frequency measurements to the "one-sigma widths" of the lines of constant TDOA and constant differential Doppler on the surface of the Earth.

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I. INTRODUCTION

Estimating the location of radio transmitters is becoming a very important problem in a variety of applications. In recent years, two methods have been shown to be capable of providing highly accurate estimates of the location of transmitters: time difference of arrival (TDOA) [1, 2] and frequency difference of arrival (FDOA), also called differential Doppler [3]. This paper presents some relationships between the accuracy of these measurements and the accuracy of the location estimates obtained from the measurements. The objective is to develop, under simplifying assumptions, relationships that can be used by the systems engineer to evaluate proposed systems and to determine system specifications in order to satisfy given requirements on emitter location accuracy.

The tool used here for characterizing emitter location accuracy is the formula for the "one-sigma width" of the lines of constant TDOA and FDOA on the surface of the Earth. That is, one measurement of either FDOA or TDOA provides a curve on the surface of the Earth on which the emitter is known to lie. Two such measurements provide an estimate of the location of the transmitter. Since the measurements are subject to various types of errors, the line on the surface of the Earth may be considered to be a random variable whose statistics are induced by statistics on the measured quantities. The variances of these curves can be used to derive the error ellipse associated with the estimate of emitter location [4]. Therefore, when a system for estimation of emitter location is designed, the specifications of the accuracy of the measurement equipment can be related to the desired accuracy of emitter location.

II. TDOA METHOD

Suppose that a signal propagates from a transmitter at point A to known receiver locations at points B and C, and that it is desired to estimate the location of point A. If the spatially separated receivers at B and C are time synchronized so that the TDOA of the signal can be measured, then the range difference between the two paths can be estimated:

$$\tau = (r_2 - r_1)/c \tag{1}$$

where τ is the TDOA, r_2 and r_1 are the distances from the transmitter to points B and C, and c is the speed of propagation. For a given value of τ, this equation represents a hyperboloid of revolution on which the emitter lies. If the transmitter is known to lie on the surface of the Earth, then the intersection of this hyperboloid and the Earth is a curve on which the emitter lies.

To estimate the location of the transmitter, at least one more measurement is required.

Lines of constant circular error probability (CEP) for two TDOA measurements from three receiving platforms have been given by Wegner [5]. (The CEP is the radius of the circle, centered at the emitter location, which contains 50 percent of the volume under the two-dimensional probability density function.)

III. FDOA METHOD

A method for estimating transmitter locations that has been developed over the past decade is based upon the measurement of the frequency of the received signal at two spatially separated moving receivers. The method was originally developed by radio astronomers who called it the very long baseline interferometer (VLBI). The same basic concept has been called differential Doppler, frequency difference of arrival (FDOA), integrated Doppler, and TDOA-DOT (for time derivative of TDOA).

All of the above methods share the following basic characteristics:

1) The physical quantity that is ultimately obtained and allows estimation of transmitter coordinates is change in range difference (from transmitter to receivers) as a function of time.
2) The methods exploit receiver motion to obtain information. Therefore, if there is no motion, then no measurement is possible; the greater the motion, the more accurate the method.
3) The periodicity of the signal is utilized to effect the measurements. The smaller the period (the higher the RF frequency), the more accurate the estimate of emitter location. The more this periodicity is disrupted (for example, by frequency modulation of the waveform), the less accurate the estimation.
4) The estimate of emitter location is extremely sensitive to transmitter motion.

If a signal from a stationary transmitter is received by a moving receiver, the frequency observed is different from the frequency transmitted because of the Doppler effect. If the signal is monitored for a time T, then the average frequency over this interval is

$$f_{av} = f_c - (r_{12} - r_{11})/\lambda T \qquad (2)$$

where f_c is the transmitted frequency (a constant[1]), λ is the wavelength, and r_{12}, r_{11} are the distances from transmitter to receiver at the beginning and end of the transmission interval. A measure of this average frequency therefore provides a measure of the change in range during the measurement interval. Thus a single frequency measurement is analogous to a TDOA measurement wherein the two receiver locations are the points corresponding to the beginning and ending of the transmission. However, the transmitted frequency is not known at the receiver, and the frequency should not be considered a constant. Therefore, to realize the potential accuracy of this method, it is necessary to employ a second receiver which measures the average frequency over precisely the same time interval. The difference between the two frequency measurements provides a single FDOA measurement. In this difference frequency, lack of knowledge of the precise transmitted frequency is no longer critical; the deviations of the transmitted signal from a pure CW signal are the same at the two receivers, and therefore most of the effects of this type will cancel out. The resulting measurement is a measure of the difference between changes in range of two receivers. This information provides a surface in space on which the receiver must lie, similar to but quite distinct from the hyperboloid associated with a TDOA measurement. By obtaining two or more such measurements, the transmitter location can be estimated. Alternatively, two receivers can obtain one TDOA measurement and one FDOA measurement, and the transmitter location can be estimated without the need for a third receiver.

The technique can be applied to pulsed signals provided that the duty cycle is not too low and the differential Doppler is not too large, so that all cycles of the difference signal can be accounted for without any ambiguity.

The FDOA method has several practical complications. The receivers must be precisely synchronized in time, and each receiver must have extreme stability. The locations of the receivers and, more importantly, the change in position during the measurement interval must be estimated with high accuracy.

The geometrical interpretation of this method is obtained as follows. Suppose a signal of the form $A \cos(2\pi f_c t)$ is received at two spatially separated receivers. The average frequency over an interval of time from t_1 to t_2 as observed at the two receivers is, in the absence of noise,

$$f_1 = f_c - (r_{12} - r_{11})/\lambda T \qquad (3)$$

$$f_2 = f_c - (r_{22} - r_{21})/\lambda T \qquad (4)$$

where r_{ij} is the distance of receiver i from the transmitter at time $t = t_j$, $T = t_2 - t_1$ and λ is the wavelength. The FDOA is defined by

[1]We have assumed that the transmitted frequency is constant for convenience. If this is not the case, it is necessary to delay one of the signals before measuring the frequency difference so that the same interval of transmitted energy is used at the two locations.

$$f_d = f_1 - f_2$$
$$= (1/\lambda T)[r_{22} - r_{21} - r_{12} + r_{11}]. \qquad (5)$$

This equation defines a surface in three-dimensional space on which the transmitter must lie. If the transmitter is known to lie on the surface of the Earth, then the intersection of the two surfaces defines a curve on the Earth on which the emitter lies. If a second measurement of FDOA can be obtained or if another type of measurement such as TDOA or angle of arrival can be obtained, two curves can be defined, and the intersection of the two curves provides an estimate of the transmitter location.

IV. EMITTER LOCATION ACCURACY AS A FUNCTION OF THE ACCURACY OF FDOA, TDOA, POSITION, AND VELOCITY MEASUREMENTS

In this section, we derive formulas which relate the accuracy of the curves of constant TDOA and FDOA to the accuracies of the measurements of TDOA and FDOA. The influence of errors in the measurements of position and velocity of the receivers is also included.

Let the equation which relates a measured quantity q to the emitter coordinates x, y, z and other parameters $p_1, p_2, \ldots, p_n$ (which will be taken to be the coordinates of position and velocity of the receivers), in a rectangular coordinate system centered at the center of the Earth, be

$$q = h(x, y, z, p_1, \ldots, p_n). \qquad (6)$$

Taking differentials

$$\Delta q = (\partial h/\partial x)\, \Delta x + (\partial h/\partial y)\, \Delta y + (\partial h/\partial z)\, \Delta z + \Sigma_i\, (\partial h/\partial p_i)\, \Delta p_i$$
$$= \nabla h \cdot \Delta u + \Sigma_i\, (\partial h/\partial p_i)\, \Delta p_i \qquad (7)$$

where $\Delta u = (\Delta x, \Delta y, \Delta z)$ and ∇h is the gradient vector of h with respect to x, y, and z. Let Δn be the component of Δu which is in the direction of the normal to the surface defined by (6). The magnitude of Δn is a measure of the deviation of the surface which is incurred by a change Δq in the value of q. Since the gradient vector ∇h is in the direction of the normal to the surface, we have $\nabla h \cdot \Delta u = \nabla h \cdot \Delta n$, and therefore

$$\Delta q = \|\nabla h\| \cdot \|\Delta n\| + \Sigma_i\, (\partial h/\partial p_i)\, \Delta p_i. \qquad (8)$$

Now suppose that measurements of q and $p_1, \ldots, p_n$ are made. Errors in these measurements will incur an error in the estimate of the surface on which the emitter at (x, y, z) lies. Let σ_n be the standard deviation of this error. Then

$$\|\nabla h\|^2\, \sigma_n^2 = \sigma_q^2 + \sum_i\, (\partial h/\partial p_i)^2\, \sigma_{p_i}^2 \qquad (9)$$

where σ_q and σ_{p_i} are the standard deviations of the measurements and where it is assumed that all the measurements are independent.

σ_n is the standard deviation of the estimate of the surface of constant q. It characterizes the random nature of the location of the surface in the direction normal to the surface. If the transmitter is known to be on the surface of the Earth, then the intersection of the surface of constant q and the surface of the Earth is a curve on which the transmitter lies. The location of this curve is also random, and its statistics are induced by the statistics of the random surface of constant q. To determine the standard deviation of this curve, suppose that the surface of constant q makes an angle ψ with the surface of the Earth. The standard deviation of the random location of the curve of intersection of these two surfaces is equal to the standard deviation σ_n of the location of the surface of constant q, divided by $\sin\psi$, which is given by

$$\sin\psi = \sqrt{1 - \langle r, \nabla h\rangle^2 / \|\nabla h\|^2} \qquad (10)$$

where r is the unit vector from the center of the Earth in the direction of the transmitter. Therefore, the desired standard deviation is obtained from (9) upon division of the right-hand side by $\sin\psi$; it is seen to be the product of two factors, one of which consists of statistical measurement parameters, and one of which depends only on the geometrical configuration. Thus the standard deviation of the random location of the line of constant q on the surface of the Earth is

$$\sigma = MG \qquad (11)$$

where M is the measurement factor

$$M = (\sigma_q^2 + \sum_i (\partial h/\partial p_i)^2\, \sigma_{p_i}^2)^{1/2} \qquad (12)$$

and G is the geometrical factor

$$G = (\|\nabla h\|^2 - \langle \nabla h, r\rangle^2)^{-1/2}. \qquad (13)$$

We now specialize these formulas to the case of a TDOA measurement. According to (1),

$$h(x, y, z, p) = (r_2 - r_1)/c \qquad (14)$$

where p is the vector of six receiver coordinates. In this case

$$\nabla h = (1/c)\, (\varrho_2 - \varrho_1) \qquad (15)$$

where ϱ_i is the unit vector from the point (x, y, z) to the ith receiver, and

$$\|\nabla h\| = (2/c)\sin(\theta/2) \tag{16}$$

where θ is the angle subtended at the transmitter by the two receivers. Since

$$\partial h/\partial x_1 = -(x_1 - x)/c\,r_1 \tag{17}$$

$$\partial h/\partial y_1 = (y_1 - y)/c\,r_1 \tag{18}$$

etc., we have

$$\sum_i (\partial h/\partial p_i)^2 \sigma_{p_i}^2 = (2/c^2)\,\sigma_p^2 \tag{19}$$

where it is assumed that the p_i are equidistributed with standard deviation σ_p.

The result for the one-sigma width of the line of constant TDOA is

$$\sigma_1 = MG \tag{20}$$

where M is the measurement factor

$$M = \sqrt{\sigma_{\mathrm{TDOA}}^2 + 2\sigma_p^2/c^2} \tag{21}$$

and G is the geometrical factor

$$G = c/\sqrt{4\sin^2(\theta/2) - (\cos\phi_2 - \cos\phi_1)^2} \tag{22}$$

where ϕ_i is the angle between $\boldsymbol{r}$ and ϱ_i.

Finally, we specialize the formulas for M and G for the case of an FDOA measurement. In this case

$$h(x, y, z, p) = (1/\lambda T)(r_{22} - r_{21} - r_{12} + r_{11}) \tag{23}$$

and p is the vector of six initial receiver coordinates and six velocity coordinates. The coordinates of receiver 1 at time t_2 are given by

$$x_{12} = x_{11} + v_{x1}T \tag{24}$$

$$y_{12} = y_{11} + v_{y1}T$$

$$z_{12} = z_{11} + v_{z1}T$$

where $T = t_2 - t_1$, and similar equations hold for receiver 2 at time t_2. Therefore, the derivatives of h with respect to the elements of the coordinate vector are

$$\partial h/\partial x_{11} = (x_{11} - x)/\lambda T\,r_{11} - (x_{12} - x)/\lambda T\,r_{12} \tag{25}$$

$$\partial h/\partial x_{21} = (x_{21} - y)/\lambda T\,r_{21} - (x_{22} - x)/\lambda T\,r_{22}$$

$$\partial h/\partial v_{x1} = -(x_{12} - x)/\lambda\,r_{12}$$

$$\partial h/\partial v_{x2} = (x_{22} - x)/\lambda r_{22}$$

and similarly for the other components of p. Therefore,

$$\begin{aligned}\Sigma(\partial h/\partial p_i)^2\,\sigma_{p_i}^2 &= (\sigma_p^2/\lambda^2T^2)\\ &\quad\{[(x_{11} - x)/r_{11} - (x_{12} - x)/r_{12}]^2\\ &\quad + [(y_{11} - y)/r_{11} - (y_{12} - y)/r_{12}]^2 + \cdots\}\\ &\quad + (\sigma_v^2/\lambda^2)\{[(x_{12} - x)/r_{12}]^2\\ &\quad + [(y_{12} - y)/r_{12}]^2 + \cdots\}\\ &= (\sigma_p^2/\lambda^2T^2)[4 - 2\cos\alpha_1 - 2\cos\alpha_2]\\ &\quad + 2(\sigma_v^2/\lambda^2)\end{aligned} \tag{26}$$

where α_1 is the angle subtended at the transmitter between receiver 1 at times t_1 and t_2, and α_2 is the corresponding angle for receiver 2. It has been assumed that the coordinates of receiver positions have a common standard deviation σ_p, and that the coordinates of receiver velocities have a common standard deviation σ_v. For small α_1 and α_2, this equation simplifies to

$$\begin{aligned}\Sigma(\partial h/\partial p_i)^2\,\sigma_{p_i}^2 &= [(\dot\alpha_1^2 + \dot\alpha_2^2)/2\lambda^2]\,\sigma_p^2\\ &\quad + (2/\lambda^2)\,\sigma_v^2\end{aligned} \tag{27}$$

where $\alpha_1 = \dot\alpha_1 T$ and $\alpha_2 = \dot\alpha_2 T$.

The gradient (with respect to x, y, and z) is

$$\nabla h = (1/\lambda T)(\varrho_{22} - \varrho_{21} - \varrho_{12} + \varrho_{11}) \tag{28}$$

where ϱ_{ij} in the *unit* vector from the transmitter in the direction of receiver i at time t_j.

The calculation can be carried further by observing that

$$(\varrho_{i2} - \varrho_{i1})/T = \dot\alpha_i \boldsymbol{e}_i, \qquad i = 1, 2 \tag{29}$$

where $\boldsymbol{e}_i$ is the unit vector in the direction of the component of velocity of receiver i which is normal to the line of sight, and α_i is the angle swept out by the receiver i as observed from the transmitter; the gradient becomes

$$\nabla h = (1/\lambda)(\dot\alpha_2\boldsymbol{e}_2 - \dot\alpha_1\boldsymbol{e}_1). \tag{30}$$

The one-sigma width of the line of constant FDOA is obtained upon substitution of (27) and (30) into (12) and (13):

$$\sigma_2 = MG \tag{31}$$

where M is the factor containing the measurement uncertainties:

$$M = \sqrt{\sigma^2_{\text{FDOA}} + 2\sigma_v^2/\lambda^2 + (\dot{\alpha}_1^2 + \dot{\alpha}_2^2)\sigma_p^2/(2\lambda^2)} \tag{32}$$

and G is the factor containing geometrical parameters:

$$G = \lambda(\dot{\alpha}_2^2 - 2\dot{\alpha}_1\dot{\alpha}_2 \cos\gamma + \dot{\alpha}_1^2 - (\dot{\alpha}_2 \cos\beta_2 - \dot{\alpha}_1 \cos\beta_1)^2)^{-1/2} \tag{33}$$

where γ is the angle between $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, and β_i is the angle between $\boldsymbol{r}$ and $\boldsymbol{e}_i$.

If $\dot{\alpha}_1 = \dot{\alpha}_2$, then G simplifies to

$$G = (\lambda/\dot{\alpha})(4\sin^2\gamma/2 - (\cos\beta_2 - \cos\beta_1)^2)^{-1/2} \tag{34}$$

V. CONCLUSION

The objective of this paper has been to derive a relationship between the accuracy of certain types of measurements and the accuracy of emitter location estimates. This relationship in a general form is given by (11) through (13). and in two special cases by (20) through (22) (for TDOA) and (31) through (33) (for FDOA).

The usefulness of equations of this kind is not to allow explicit evaluation of system accuracy, but rather to provide guidelines for the system designer who is required to meet a specification of emitter location accuracy. This specification can be translated into a specification of measurement accuracy for TDOA and FDOA measurements using (22) and (34). These equations relate the one-sigma width of the line of constant TDOA or FDOA to the standard deviations of the measurements of TDOA, FDOA, and receiver position and velocity. The effect of the geometry is explicitly displayed as a multiplicative factor.

REFERENCES

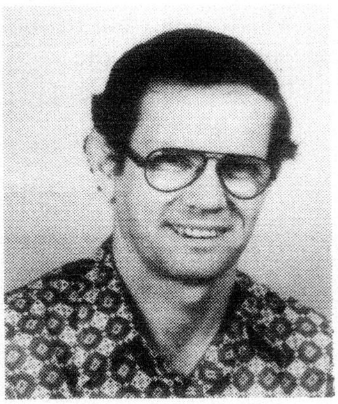
[1] Knapp, C.H., and Carter, G.C. (1976)
The generalized correlation method for estimation of time delay.
IEEE Transactions on Acoustics, Speech, and Signal Processing, Aug. 1976, *ASSP-24,* 320-327.

[2] Chan, Y.T., Hattin, R.V., and Plant, J.B. (1978)
The least squares estimation of time delay and its use in signal detection.
IEEE Transactions on Acoustics, Speech, and Signal Processing, June 1978, *ASSP-26,* 217-222.

[3] Schultheis, P.M., and Weinstein, E. (1979)
Estimation of differential Doppler shifts.
Journal of the Acoustical Society of America, Nov. 1979, *66,* 1412-1419.

[4] Clark, G.P. (1974-1975)
Simplified determination of ellipse of uncertainty.
Journal of the Institute of Navigation, Winter 1974-1975, *21,* 343-350.

[5] Wegner, L.H. (1971)
On the accuracy analysis of airborne techniques for passively locating electromagnetic emitters.
Rand Corp., Report R-722-PR, June 1971.



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