References

[1] L. G. Delcaro and G. L. Sicuranza, "Design of two-dimensional recursive digital filters," *IEEE Trans. Acoust., Speech, Signal Processing,* vol. ASSP-25, Dec. 1977.

[2] G. L. Sicuranza, "Linear phase approximation for two-dimensional recursive digital filters," *IEEE Trans. Acoust., Speech, Signal Processing,* vol. ASSP-27, June 1979.

[3] S. A. H. Aly and M. M. Fahmy, "Design of two-dimensional recursive digital filters with specified magnitude and group delay characteristics," *IEEE Trans. Circuits Syst.,* vol. CAS-25, Nov. 1978.

Measurement of the Differential Doppler Shift

EHUD WEINSTEIN

***Abstract*—Doppler shifts between narrow-band signals observed at one or more pairs of receivers and originated from a remote source of radiation are useful for estimating source location and track. This paper deals with an instrumentationally attractive approach of estimating differential Doppler shifts by making center frequency measurements at each receiver output and subtracting them in a pairwise fashion. For low in-band signal-to-noise ratio conditions, center frequency measurements at different receiver outputs are weakly correlated. In this mode of operation, therefore, the mean-square error in the differential Doppler estimate equals the pairwise sum of the mean-square errors in each of the center frequency measurement. For high signal-to-noise ratio conditions, the various center frequency estimates are strongly correlated. In this mode of operation, the accuracy of the resulting differential Doppler estimate improves with the first power of the signal-to-noise ratio, even though the accuracy of each center frequency estimate approaches an absolute bound independent of the noise spectrum.**

The optimal (minimum mean-square error) differential Doppler shift estimate is obtained by simultaneous processing of the receiver outputs jointly. Comparison between the former (indirect) and the latter (direct) estimation techniques yields some interesting insights: the accuracy of the direct estimation procedure is proportional to T^{-3} where T is the observation period. It is basically a coherent procedure. Center frequency measurement and the differential Doppler shift estimate derived from it are basically incoherent procedures and obey the well known T^{-1} dependence. Under high signal-to-noise ratio conditions, the estimation error in both methods decreases with the first power of the signal-to-noise ratio. Under low signal-to-noise ratio conditions, the accuracy of the suboptimal indirect method is inferior to the optimal direct method by a factor proportional to the inverse first power of the individual signal-to-noise ratio.

I. Introduction

The location of a remote source of radiation can be found by observation of its signal at three or more spatially separated receivers. Most systems which have been analyzed depend on measurements of the relative (differential) delay of the signal wavefront between receiver pairs to determine bearing and (at least for relatively nearby sources) range [8]-[11].

Manuscript received September 28, 1979; revised October 9, 1980 and July 20, 1981. This work was supported by the Office of Naval Research under Contract N00014-77-C-0196.

The author is with the Department of Ocean Engineering, Woods Hole Oceanographic Institute, Woods Hole, MA 02543, on leave at the Department of Electronic Systems, School of Engineering, Tel-Aviv University, Tel-Aviv, Israel.

When the source is moving, one can obviously gain important additional information about source location and track by measuring the relative (differential) Doppler shifts. If an adequate number of receivers (at least five) is available, measurements of the differential Doppler shifts are sufficient to determine source trajectory in a plane and can therefore be used for localization when differential delay measurements are unreliable (perhaps because the narrow bandwidth of the signal makes the differential delay observations highly ambiguous).

The general problem may be stated as follows: a moving source radiates a narrow-band Gaussian random process towards a set of spatially separated receivers. Each receiver is contaminated with Gaussian noise so that the actual waveform observed at the ith receiver is given by

$$r_i(t) = s_i(t) + n_i(t). \tag{1}$$

Suppose the source is moving at constant velocity and is sufficiently remote from the receiving array so that the various signal components are frequency shifted by constant amounts relative to each other.

Let ω_0^i denote the center frequency associated with $s_i(t)$. The differential Doppler shift between the (i, j) receiver pair is given by

$$\Delta\omega_0^{ij} = \omega_0^i - \omega_0^j. \tag{2}$$

A minimum mean-square error estimate of the quantity in (2) is achieved by processing the outputs of the (i, j) receiver pair coherently. An instrumentally more attractive alternative is to make separate center frequency estimates at each receiver output and subtract them in pairwise fashion to obtain the various $\Delta\omega_0^{ij}$ estimates.

This paper is concerned with the estimation of differential Doppler shifts using the latter (indirect) technique. We first derive the structure of the optimal center frequency estimator and compute its performance. The accuracy of the resulting differential Doppler estimate is then compared with the optimal estimator which processes the receiver outputs jointly.

II. Center Frequency Estimation

The available data are $r_i(t)$ from which one wishes to estimate ω_0^i. We shall consider observation times T large compared to the inverse bandwidth of the signal ($WT >> 1$). In passive radar/sonar applications, this assumption is likely to be satisfied with two exceptions: the signal may be a pure sinusoid or it may be a sample function from a random process of a bandwidth so narrow that its correlation time exceeds T. In the latter case, the typical sample function looks like a sinusoid of random amplitude and phase. By treating the sinusoidal case in a separate section at the end, we, therefore, accommodate most signals likely to be encountered in practice.

If the bandwidth of the noise is at least equally large, the most convenient form of the data vector uses Fourier coefficients

$$\boldsymbol{r}_i = (R_i(\omega_1), R_i(\omega_2), \cdots, R_i(\omega_N))^T \tag{3}$$

where

$$R_i(\omega_n) = \frac{1}{\sqrt{T}} \int_0^T r_i(t)\, e^{-j2\pi\omega_n t}\, dt, \qquad \omega_n = \frac{n}{T}. \tag{4}$$

Since signal and noise are Gaussian and the components of $\boldsymbol{r}_i$ are generated by linear operations on these time functions, $\boldsymbol{r}_i$ has a multivariate Gaussian distribution

$$P(\mathbf{r}_i/\omega_0^i) = \frac{1}{\det(\pi K_i)}\, e^{-\mathbf{r}_i^* K_i^{-1} \mathbf{r}_i} \tag{5}$$

where $\mathbf{r}_i^*$ denotes the conjugate-transpose of $\mathbf{r}_i$ and K_i is the data covariance matrix

$$K_i = E\{\mathbf{r}_i \mathbf{r}_i^* / \omega_0^i\}. \tag{6}$$

$E\{\cdot\}$ stands for the statistical expectation of the bracketed quantity.

For $WT >> 1$, the Fourier coefficients associated with different frequencies are uncorrelated [3]. Thus, K_i assumes a diagonal form with the diagonal elements

$$(K_i)_{nn} = E\{|R_i(\omega_n)|^2\} = S_i(\omega_{nj}\omega_0^i) + N_i(\omega_n). \tag{7}$$

$S_i(\omega_j\omega_0^i)$ and $N_i(\omega)$ are, respectively, signal and noise spectral densities at the output of the ith receiver.

If one defines an optimal estimator as one which achieves zero bias and minimizes mean-square error, the maximum likelihood method has strong claim to optimality because of its asymptotic efficiency and lack of bias [6]. It works with the likelihood functional

$$z_i(\hat{\omega}_0^i) = \frac{d}{d\omega_0^i} \ln P(\mathbf{r}_i/\omega_0^i)\Big|_{\omega_0^i = \hat{\omega}_0^i}. \tag{8}$$

A value $\hat{\omega}_0^i$ which satisfies $z_i = 0$ is called the maximum likelihood estimate of ω_0^i. To obtain z_i explicitly, one must substitute (5) into (8) and carry out the indicated operations. The result is available in the literature [4]:

$$z_i = \mathbf{r}_i^* \hat{K}_i^{-1} \frac{d\hat{K}_i}{d\omega_0^i} \hat{K}_i^{-1} \mathbf{r}_i - \operatorname{tr}\left(\hat{K}_i^{-1} \frac{d\hat{K}_i}{d\omega_0^i}\right). \tag{9}$$

$\operatorname{tr}(\cdot)$ stands for the trace of the bracketed matrix. $\hat{K}_i$ and $d\hat{K}_i/d\omega_0^i$ are, respectively, the data covariance matrix and its derivative with respect to ω_0^i, evaluated at the estimated value. These are diagonal matrices with the diagonal elements [from (7)]

$$(\hat{K}_i)_{nn} = S_i(\omega_{nj}\hat{\omega}_0^i) + N_i(\omega_n) \tag{10}$$

$$\left(\frac{d\hat{K}_i}{d\omega_0^i}\right)_{nn} = \frac{dS_i(\omega_{nj}\omega_0^i)}{d\omega_0^i}\Big|_{\omega_0^i = \hat{\omega}_0^i} \triangleq \frac{dS_i(\omega_{nj}\hat{\omega}_0^i)}{d\hat{\omega}_0^i}. \tag{11}$$

Using (10) and (11) in (9) yields the simplified form for z_i:

$$z_i = \sum_{n=1}^{N} \frac{dS_i(\omega_{nj}\hat{\omega}_0^i)/d\hat{\omega}_0^i}{[S_i(\omega_{nj}\hat{\omega}_0^i) + N_i(\omega_n)]^2} |R_i(\omega_n)|^2 - \sum_{n=1}^{N} \frac{dS_i(\omega_{nj}\hat{\omega}_0^i)/d\hat{\omega}_0^i}{S_i(\omega_{nj}\omega_0^i) + N_i(\omega_n)}. \tag{12}$$

Substitution of (4) in (12) yields

$$z_i = \int_0^T dt\, r_i(t) \int_0^T d\sigma\, h(t - \sigma_j \hat{\omega}_0^i)\, r_i(\sigma) - c_i \tag{13}$$

where

$$h(\tau_j\hat{\omega}_0^i) = \frac{1}{T}\sum_{n=1}^{N} \frac{dS_i(\omega_{nj}\hat{\omega}_0^i)/d\hat{\omega}_0^i}{[S_i(\omega_{nj}\hat{\omega}_0^i) + N_i(\omega_n)]^2}\, e^{j\omega_n\tau} \tag{14}$$

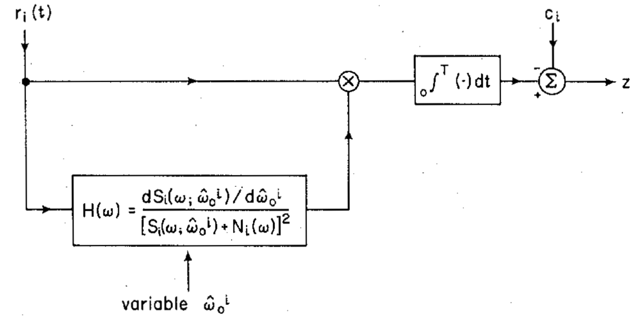

Fig. 1.

and

$$c_i = \sum_{n=1}^{N} \frac{dS_i(\omega_{nj}\hat{\omega}_0^i)/d\hat{\omega}_0^i}{S_i(\omega_{nj}\hat{\omega}_0^i) + N_i(\omega_n)}. \tag{15}$$

For $WT >> 1$ and smoothly varying spectra, the n sum in (14) and (15) can be converted into an integral

$$h(\tau_j\hat{\omega}_0^i) = \int_0^\infty \frac{dS_i(\omega_j\hat{\omega}_0^i)/d\hat{\omega}_0^i)}{[S_i(\omega_j\hat{\omega}_0^i) + N_i(\omega)]^2}\, e^{j\omega\tau}\, d\omega \tag{16}$$

$$c_i = T\int_0^\infty \frac{dS_i(\omega_j\hat{\omega}_0^i)/d\hat{\omega}_0^i}{S_i(\omega_j\hat{\omega}_0^i) + N_i(\omega)}\, d\omega. \tag{17}$$

The realization of z_i is shown in Fig. 1.

The Cramer–Rao formula sets a lower bound on the mean-square error of any unbiased estimator [5]:

$$D^2(\hat{\omega}_0^i) \geqslant \frac{1}{-E\{d^2 \ln P(\mathbf{r}_i/\omega_0^i)/d\omega_0^{i\,2}\}} = \frac{1}{-E\{dz_i(\omega_0^i)/d\omega_0^i\}}. \tag{18}$$

The right-hand side of (18) is obtained by differentiating (9) with respect to $\hat{\omega}_0^i$ and taking the statistical expectation at the true value $\hat{\omega}_0^i = \omega_0^i$. The result is (see [2])

$$-E\left\{\frac{dz_i}{d\omega_0^i}\right\} = \operatorname{tr}\left[\left(K_i^{-1}\frac{dK_i}{d\omega_0^i}\right)^2\right] = \sum_{n=1}^{N}\left[\frac{dS_i(\omega_{nj}\omega_0^i)/d\omega_0^i}{S_i(\omega_{nj}\omega_0^i) + N_i(\omega_n)}\right]^2 \xrightarrow[WT>>1]{} \cdot\, T\int_0^\infty \left[\frac{S_i(\omega_j\omega_0^i)/N_i(\omega)}{1 + S_i(\omega_j\omega_0^i)/N_i(\omega)} \cdot \frac{d}{d\omega_0^i} \ln S_i(\omega_j\omega_0^i)\right]^2 d\omega. \tag{19}$$

For observation time large compared with the inverse bandwidth of signal and noise, the maximum likelihood estimate is asymptotically unbiased and efficient [6]. One can therefore use the Cramer–Rao lower bound with some confidence to characterize the attainable mean-square estimation error. Hence,

$$D^2(\hat{\omega}_0^i) = \frac{1}{T\displaystyle\int_0^\infty \left[\frac{S_i(\omega_j\omega_0^i)/N_i(\omega)}{1 + S_i(\omega_j\omega_0^i)/N_i(\omega)}\, \frac{d}{d\omega_0^i} \ln S_i(\omega_j\omega_0^i)\right]^2 d\omega} \tag{20}$$

where $S_i(\omega_j\omega_0^i)/N_i(\omega)$ is the signal-to-noise ratio at the ith receiver. We shall assume that the noise spectrum is broad compared to the maximum expected Doppler shift (so that source motion does not change the effective signal-to-noise ratio). Thus,

$$S_i(\omega_j\omega_0^i)/N_i(\omega) = \beta_i S(\omega)/N(\omega) \tag{21}$$

where β_i is the relative signal-to-noise level at the ith receiver. $S(\omega)$ is the (unshifted) signal spectra concentrated about ω_0, the radiated center frequency. With this convention, the minimum mean-square estimation error is given by

$$D^2(\hat{\omega}_0^i) = \frac{1}{T\int_0^\infty \left[\frac{\beta_i S(\omega)/N(\omega)}{1+\beta_i S(\omega)/N(\omega)} \frac{d}{d\omega_0}\ln S(\omega)\right]^2 d\omega}. \tag{22}$$

Aside from the output signal-to-noise ratio, (22) depends on the slope of the signal spectrum. This is reasonable since the center frequency estimator compares the average power in a band above and below the center frequency. If there is a frequency band where this power changes rapidly, a more sensitive measurement is performed.

III. Differential Doppler Estimation

An estimate of $\Delta\omega_0^{ij}$ can be obtained by the straightforward substraction $(\hat{\omega}_0^i - \hat{\omega}_0^j)$. If each center frequency estimate is unbiased, so is the resulting differential Doppler estimate. The mean-square error is given by

$$D^2(\hat{\omega}_0^i - \hat{\omega}_0^j) = D^2(\hat{\omega}_0^i) + D^2(\hat{\omega}_0^j) - 2\,\mathrm{cov}\,(\hat{\omega}_0^i, \hat{\omega}_0^j). \tag{23}$$

The covariance between maximum likelihood estimates obtained from disjoint segments of the available data is given by [2, eq. (19)], rewritten here for the pair $(\hat{\omega}_0^i, \hat{\omega}_0^j)$:

$$\mathrm{cov}\,(\hat{\omega}_0^i, \hat{\omega}_0^j) = \frac{\mathrm{tr}\left(K_i^{-1}K_{ij}K_j^{-1}\frac{dK_j}{d\omega_0^j}K_j^{-1}K_{ji}K_i^{-1}\frac{dK_i}{d\omega_0^i}\right)}{\mathrm{tr}\left[\left(K_i^{-1}\frac{dK_i}{d\omega_0^i}\right)^2\right]\mathrm{tr}\left[\left(K_j^{-1}\frac{dK_j}{d\omega_0^j}\right)^2\right]} \tag{24}$$

where $K_{ij} = E\{\mathbf{r}_i\mathbf{r}_j^*\}$. We shall assume that the noise components $n_i(t)$ are uncorrelated (thus independent) from receiver-to-receiver.[1] In that case, K_{ij} assumes a digaonal form with the diagonal elements

$$(K_{ij})_{nn} = E\{R_i(\omega_n)R_j^*(\omega_n)\} = \sqrt{\beta_i\beta_j}\,S(\omega_n). \tag{25}$$

Since all the matrices appearing in (24) are diagonal, the trace operation is simply the n sum. Thus,

$$\mathrm{cov}\,(\hat{\omega}_0^i, \hat{\omega}_0^j) = \frac{\sum_{n=1}^N\left[\frac{\beta_i\beta_j S^2(\omega_n)/N^2(\omega_n)}{(1+\beta_i S(\omega_n)/N(\omega_n))\,(1+\beta_j S(\omega_n)/N(\omega_n))}\frac{d\ln S(\omega_n)}{d\omega_0}\right]^2}{\left(\sum_{n=1}^N\left[\frac{\beta_i S(\omega_n)/N(\omega_n)}{1+\beta_i S(\omega_n)/N(\omega_n)}\frac{d\ln S(\omega_n)}{d\omega_0}\right]^2\right)\left(\sum_{n=1}^N\left[\frac{\beta_j S(\omega_n)/N(\omega_n)}{1+\beta_j S(\omega_n)/N(\omega_n)}\frac{d\ln S(\omega_n)}{d\omega_0}\right]^2\right)}$$

$$\xrightarrow[WT>>1]{} \frac{1}{T}\frac{\int_0^\infty\left[\frac{\beta_i\beta_j S^2(\omega)/N^2(\omega)}{[1+\beta_i S(\omega)/N(\omega)]\,[1+\beta_j S(\omega)/N(\omega)]}\cdot\frac{d\ln S(\omega)}{d\omega_0}\right]^2 d\omega}{\left(\int_0^\infty\left[\frac{\beta_i S(\omega)/N(\omega)}{1+\beta_i S(\omega)/N(\omega)}\cdot\frac{d\ln S(\omega)}{d\omega_0}\right]^2 d\omega\right)\left(\int_0^\infty\left[\frac{\beta_j S(\omega)/N(\omega)}{1+\beta_j S(\omega)/N(\omega)}\frac{d\ln S(\omega)}{d\omega_0}\right]^2 d\omega\right)}. \tag{26}$$

In the low signal-to-noise ratio mode of operation ($S(\omega)/N(\omega) \ll 1$), the third term on the right-hand side of (23) is negligible relative to the first and second terms, and one can write to a very good approximation

$$D^2(\hat{\omega}_0^i - \hat{\omega}_0^j) = D^2(\hat{\omega}_0^i) + D^2(\hat{\omega}_0^j). \tag{27}$$

This should not be surprising. Because of the postulated lack of noise coherence from receiver to receiver, the errors in the two center frequency measurements are essentially uncorrelated so that their mean-square values add.

$\mathrm{cov}\,(\hat{\omega}_0^i, \hat{\omega}_0^j)$ contributes significantly to the sum in (23) only when the signal-to-noise ratio is high throughout the signal frequency band. In this mode of operation the estimation accuracy of the differential Doppler estimate cannot be interpreted from (22) alone. One must substitute (22) and (26) into (23) and carry out the indicated sum. The result is

$$D^2(\hat{\omega}_0^i - \hat{\omega}_0^j) = \frac{1}{T}\frac{\int_0^\infty\left[\frac{\beta_i\beta_j S^2(\omega)/N^2(\omega)}{(1+\beta_i S(\omega)/N(\omega))\,(1+\beta_j S(\omega)/N(\omega))}\frac{d\ln S(\omega)}{d\omega_0}\right]^2\frac{\frac{\beta_i}{\beta_j}+\frac{\beta_j}{\beta_i}+2(\beta_i+\beta_j)\frac{S(\omega)}{N(\omega)}}{\beta_i\beta_j S^2(\omega)/N^2(\omega)}}{\int_0^\infty\left[\frac{\beta_i S(\omega)/N(\omega)}{1+\beta_i S(\omega)/N(\omega)}\frac{d\ln S(\omega)}{d\omega_0}\right]^2 d\omega\cdot\int_0^\infty\left[\frac{\beta_j S(\omega)/N(\omega)}{1+\beta_j S(\omega)/N(\omega)}\frac{d\ln S(\omega)}{d\omega_0}\right]^2 d\omega} \tag{28}$$

[1] We are interested in the differential Doppler shifts. If these are to be significant, the receivers cannot be spaced very closely. The assumption of noise independence from receiver-to-receiver is therefore not very restrictive.

For increasing values of signal-to-noise ratio, (22) approaches a limit independent of the noise spectrum. Equation (28) decreases with the first power of the signal-to-noise ratio even for very high signal-to-noise ratios. The fact that the center frequency estimate has an absolute lower bound greater than zero is another property common to power measurement. Even if there is no noise at all, the randomness of the signal prevents error-free determination of its spectral parameters in a finite time T. However, if $S(\omega)/N(\omega)$ increases without a limit, $\hat{\omega}_0^i$ and $\hat{\omega}_0^j$ become fully correlated thus by taking their difference $(\hat{\omega}_0^i - \hat{\omega}_0^j)$, the estimation errors are subtracted out to yield error-free estimate of the differential Doppler shift.

IV. Comparison with the Direct Estimation Technique

Direct estimation of $\Delta\omega_0^{ij}$ by coherent processing the ith and jth receiver outputs jointly is discussed in [1]. The mean-square estimation is given by

$$D^2(\Delta\hat{\omega}_0^{ij}) = \frac{6}{\beta_i\beta_j T^3 \displaystyle\int_0^{\infty} \frac{S^2(\omega)/N^2(\omega)}{1 + (\beta_i + \beta_j)\, S(\omega)/N(\omega)}\, d\omega}. \tag{29}$$

Perhaps the most striking difference between the direct (optimal) estimation of differential Doppler shift and the present indirect (suboptimal) estimation procedure is the T dependence of the mean-square error. Equation (28) varies as T^{-1}; (29) varies as T^{-3}. Even though the radiated signal is a sample function from a random process, the direct differential Doppler measurement exploits the fact that the waveshapes at the various receivers are deterministically related. It is basically a coherent procedure. Differential Doppler estimation using center frequency measurements is an incoherent procedure. It is a power measurement and obeys the well-known T^{-1} dependence.

The indicated T dependence, however, may appear to be misleading at first glance. It suggests that for sufficiently small T, one could improve the differential Doppler estimate by subtracting center frequency estimates. This is clearly nonsense since we have established the absolute optimality of the direct differential Doppler measurement. It is not difficult to find an explanation for the apparent paradox. Suppose the signal spectrum has the form

$$S(\omega) = S_0\left(\frac{\omega - \omega_0}{W}\right) \tag{30}$$

where $S_0(x)$ does not depend on ω_0 and W. In other words, $S_0(\cdot)$ specifies the basic spectral shape, while the parameters ω_0 and W adjust the center frequency and bandwidth, respectively. Suppose further that the noise spectrum is essentially flat with spectral level N_0 over the signal band. Then (28) can be written in the form

$$D^2(\hat{\omega}_0^i - \hat{\omega}_0^j) = \frac{WT}{T^2} \cdot \frac{\displaystyle\int_{-\infty}^{\infty} \left[\frac{\beta_i\beta_j S_0^2(\omega)/N_0^2}{(1+\beta_i S_0(\omega)/N_0)(1+\beta_j S_0(\omega)/N_0)} \frac{d\ln S_0(\omega)}{d\omega}\right]^2 \frac{\dfrac{\beta_i}{\beta_j} + \dfrac{\beta_j}{\beta_i} + 2(\beta_i+\beta_j)\dfrac{S_0(\omega)}{N_0}}{\beta_i\beta_j S_0^2(\omega)/N_0^2}\, d\omega}{\displaystyle\int_{-\infty}^{\infty} \left[\frac{\beta_i S_0(\omega)/N_0}{1+\beta_i S_0(\omega)/N_0} \frac{d\ln S_0(\omega)}{d\omega}\right]^2 d\omega \int_{-\infty}^{\infty} \left[\frac{\beta_j S_0(\omega)/N_0}{1+\beta_j S_0(\omega)/N_0} \frac{d\ln S_0(\omega)}{d\omega}\right]^2 d\omega}. \tag{31}$$

Similarly, (29) can be written as

$$D^2(\Delta\hat{\omega}_0^{ij}) = \frac{6}{T^2(WT)\beta_i\beta_j} \frac{1}{\displaystyle\int_{-\infty}^{\infty} \frac{S_0^2(\omega)/N_0^2}{1+(\beta_i+\beta_j)\, S_0(\omega)/N_0}\, d\omega}. \tag{32}$$

The integrals appearing in (31) and (32) are independent of ω_0 or W. One therefore obtains immediately

$$\frac{D^2(\Delta\hat{\omega}_0^{ij})}{D_2(\hat{\omega}_0^i - \hat{\omega}_0^j)} \propto \frac{1}{(WT)^2}. \tag{33}$$

Thus, the significant parameter is not the observation time T, but the time-bandwidth product. We have assumed that $WT \gg 1$ so that the advantage rests very clearly with the direct differential Doppler measurement.

To explore further apparent differences between (28) and (29), we consider separately the cases of high and low signal-to-noise ratio. We are concerned with performance degradation in the indirect estimation method relative to the direct (coherent) estimation method. We shall find it convenient to work with (31) and (32).

1) $\beta_i S_0(\omega)/N_0,\ \beta_j S_0(\omega)/N_0 \gg 1$.

If the signal-to-noise ratio at each receiver output is high throughout the signal frequency band, (32) assumes the form

$$D^2(\Delta\hat{\omega}_0^{ij}) = \frac{6(\beta_i+\beta_j)}{\beta_i\beta_j T^2(WT)} \cdot \frac{N_0}{P_s} \tag{34}$$

where $P_s = \int_{-\infty}^{+\infty} S_0(\omega)\, d\omega$ is the average (normalized) signal power. In the high signal-to-noise ratio mode of operation, therefore, the direct differential Doppler estimate does not depend on detailed spectral properties of the signal.

Equation (31) assumes the form

$$D^2(\hat{\omega}_0^i - \hat{\omega}_0^j) = \frac{2(\beta_i+\beta_j)(WT)}{\beta_i\beta_j T^2} \cdot \frac{N_0 \displaystyle\int_{-\infty}^{+\infty} [d\ln S_0(\omega)/d\omega]^2 \frac{d\omega}{S_0(\omega)}}{\left(\displaystyle\int_{-\infty}^{+\infty} [d\ln S_0(\omega)/d\omega]^2\, d\omega\right)^2}. \tag{35}$$

The mean-square error in both estimation techniques improves with the first power of the noise level. Equation (35), however, depends on detailed spectral shape of the signal. Taking the ratio of (35) to (34) and making use of the Cauchy-Schwarz inequality, one obtains

$$\frac{D^2(\hat{\omega}_0^i - \hat{\omega}_0^j)}{D^2(\Delta\hat{\omega}_0^{ij})} = \frac{(WT)^2}{3} \frac{\displaystyle\int_{-\infty}^{+\infty} \left[\frac{d\ln S_0(\omega)}{d\omega}\right]^2 \frac{d\omega}{S_0(\omega)} \int_{-\infty}^{+\infty} S_0(\omega)\, d\omega}{\left(\displaystyle\int_{-\infty}^{+\infty} \left[\frac{d\ln S_0(\omega)}{d\omega}\right]^2 d\omega\right)^2}$$

$$\geqslant \left(\frac{WT}{\sqrt{3}} \frac{\displaystyle\int_{-\infty}^{+\infty} |d\ln S_0(\omega)/d\omega|\, d\omega}{\displaystyle\int_{-\infty}^{+\infty} |d\ln S_0(\omega)/d\omega|^2\, d\omega}\right)^2. \tag{36}$$

The second version of (36) provides an upper bound on the relative efficiency of the indirect estimation method to the direct one. This bound depends explicitly on the slope of the signal spectrum plotted on a logarithmic (dB) scale.

2) $\beta_i S_0(\omega)/N_0 \ll 1, \beta_j S_0(\omega)/N_0 \gg 1$.

Equation (34) remains unchanged whenever the pairwise sum of signal-to-noise ratios is in access of unity (i.e., $(\beta_i + \beta_j) S_0(\omega)/N_0 \gg 1$). Equation (31) assumes the limiting form

$$D^2(\hat{\omega}_0^i - \hat{\omega}_0^j) = \frac{N_0^2}{\beta_i^2 \int_0^{\infty} [dS_0(\omega)/d\omega]^2 \, d\omega}. \tag{37}$$

The right-hand side of (37) describes, to a very good approximation, the right-hand side of (22) under $\beta_i S_0(\omega)/N_0 \ll 1$. Thus, center frequency measurement at the receiver with the low signal-to-noise ratio is the main source of error.

Equation (34) varies with the first power of the noise spectral level. Equation (37) varies with the second power of the noise spectral level. Perhaps a more informative comparison is obtained by considering the following illustration:

$$S_0(\omega) = S_0 e^{-\omega^2/2}. \tag{38}$$

For Gaussian shape signal spectrum, the ratio of (37) to (34) assumes the form

$$\frac{D^2(\hat{\omega}_0^i - \hat{\omega}_0^j)}{D^2(\Delta\hat{\omega}_0^{ij})} = \frac{(WT)^2}{6\beta_i S_0/N_0}. \tag{39}$$

Besides the improvement with the second power of the time-bandwidth product, the mean-square error in the direct method is smaller than the mean-square error in the indirect method by a factor of $(\beta_i S_0/N_0)^{-1}$. Note, however, that the form of (39) depends on the particular choice of $S_0(\omega)$ given by (38). For faster decaying spectral shape, the relative efficiency could vary substantially to the favor of an estimator which uses center frequency measurements.

3) $\beta_i S_0(\omega)/N_0, \beta_j S_0(\omega)/N_0 \ll 1$.

If the pairwise sum of the signal-to-noise ratios does not exceed unity even in the signal frequency band, (32) assumes the form

$$D^2(\Delta\hat{\omega}_0^{ij}) = \frac{6}{T^2(WT)\beta_i\beta_j} \frac{N_0^2}{\int_{-\infty}^{+\infty} S_0^2(\omega)\, d\omega}. \tag{40}$$

Estimation accuracy of the direct procedure now depends to some extent on the form of the signal spectrum. Equation (31) can be approximated by

$$D^2(\hat{\omega}_0^i - \hat{\omega}_0^j) = \frac{WT\left(\frac{1}{\beta_i^2} + \frac{1}{\beta_j^2}\right)}{T^2} \frac{N_0^2}{\int_{-\infty}^{+\infty} [dS_0(\omega)/d\omega]^2 \, d\omega}. \tag{41}$$

As already indicated in (27), (41) simply consists of the sum of the individual mean-square errors in the ith and jth center frequency measurements.

Estimation accuracy of both methods varies with the inverse second power of the noise spectral level. The importance of the ratio β_i/β_j can be observed from (42):

$$\frac{D^2(\hat{\omega}_0^i - \hat{\omega}_0^j)}{D^2(\Delta\hat{\omega}_0^{ij})} = \frac{(WT)^2}{6}\left(\frac{\beta_i}{\beta_j} + \frac{\beta_j}{\beta_i}\right) \frac{\int_{-\infty}^{+\infty} S_0^2(\omega)\, d\omega}{\int_{-\infty}^{+\infty} [dS_0(\omega)/d\omega]^2 \, d\omega}. \tag{42}$$

Since

$$\left(\frac{\beta_i}{\beta_j} + \frac{\beta_j}{\beta_i}\right)\Bigg|_{(\beta_i + \beta_j) = \text{constant}} \geqslant 2. \tag{43}$$

Then the relative efficiency of the indirect estimation method to the direct one reaches its maximum value when a given amount of pairwise signal-to-noise ratio is divided equally between the ith and jth receivers.

V. Discussion

A. Dependence on Spectral Parameters

The discussion in Section IV suggests a possible improvement in estimation accuracy of the indirect method relative to the direct (optimal) one, depending on the spectral shape of the radiated signal. Practically speaking, however, one is unlikely to have accurate prior knowledge of the detailed spectral shape. The indicated improvement is, therefore, not realizable in practice.

The direct (optimal) estimation accuracy is shown to be completely unaffected by lack of exact knowledge of the signal spectral parameters [1]. The resulting mean-square error can, therefore, be used as a benchmark for performance evaluation.

B. M Receivers

Equation (29) presents the minimum mean-square error in the estimation of $\Delta\omega_0^{ij}$ obtainable by processing data from the ith and jth receivers. If the receiving array consists of $M(>2)$ elements, we therefore ignore information available from other receivers. The question to be asked is whether pairwise processing of the receiver outputs is sufficient to ensure optimality of the corresponding pairwise differential Doppler estimate.

Estimation of the various differential Doppler shifts by processing all M receiver outputs jointly is discussed in [1]. It is shown that if the sum of signal-to-noise ratios at the output of receivers i and j is large throughout the signal frequency band, information from receivers other than the (i, j) pair is essentially useless in the estimation of $\Delta\omega_0^{ij}$. If the pairwise signal-to-noise ratio does not exceed unity even in the signal frequency band, the $\Delta\omega_0^{ij}$ estimate obtained from the (i, j) receiver pair only is distinctly suboptimal.

If $(\beta_i + \beta_j) S_0(\omega)/N_0 \ll 1$ but $\sum_{l=1}^{M} \beta_l S_0(\omega)/N_0 \gg 1$, the accuracy of the $\Delta\omega_0^{ij}$ estimate obtainable by processing all M receiver outputs jointly is still characterized by (34). If $\sum_{l=1}^{M} \beta_l S_0(\omega)/N_0$ does not exceed unity even in the signal frequency band, the resulting mean-square error is given by multiplying (40) by the factor $(\beta_i + \beta_j)/\sum_{l=1}^{M} \beta_l$.

C. Sinusoidal Signals

If the source signal is modeled as a sinusoid, the waveshapes at the various receivers are sample functions from uncorrelated Gaussian random processes with mean value provided by the signal component and autocorrelation function provided by the noise component. For spatially incoherent noise, the various receiver outputs are therefore statistically independent. Statistical information from other receivers is not useful in improving the center frequency estimate for a particular receiver. Thus, in the case of sinusoidal signals, one does not suffer any performance degradation by using the instrumentationally more attractive procedure of estimating signal frequency at each receiver separately and then subtracting to generate the various differential Doppler estimates. For detailed analysis of the sinusoidal case, we refer the interested reader to [1] and [7].

ACKNOWLEDGMENT

I wish to thank Dr. E. Hays and Dr. R. Spindel for their helpful comments on a preliminary draft of the paper, to C. Muzzey for her excellent secretarial assistance, and to B. Pratt for her help in the construction of the figure.

REFERENCES

[1] P. M. Schultheiss and E. Weinstein, "Estimation of differential Doppler shifts," *J. Acoust. Soc. Amer.*, vol. 66, Nov. 1979.
[2] E. Weinstein, "Decentralization of the Gaussian maximum likelihood estimator and its applications to passive array processing," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-29, pp. 945-951, Oct. 1981.
[3] A. Papoulis, *Probability, Random Variables and Stochastic Processes*. New York: McGraw-Hill, ch. 13.
[4] W. J. Bangs, "Array processing with generalized beamformers," Ph.D. dissertation, Yale Univ., New Haven, CT, 1971. Also NUSC Tech. Rep.
[5] C. R. Rao, *Linear Statistical Inference and Its Applications*. New York: Wiley, 1965, ch. 5.
[6] H. Cramer, *Mathematical Methods of Statistics*. Princeton, NJ: Princeton Univ., 1951, ch. 22-23.
[7] P. M. Schultheiss, "Frequency estimation using an array," in *Proc. IEEE Electron. Aeros. Syst. Con.*, 1977.
[8] C. H. Knapp and G. C. Carter, "The generalized correlation method for estimation of time delay," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-24, pp. 320-326, Aug. 1976.
[9] W. H. Hahn and S. A. Tretter, "Optimum processing for delay-vector estimation in passive signal arrays," *IEEE Trans. Inform. Theory*, vol. IT-19, no. 5, pp. 608-614.
[10] B. V. Hamon and E. J. Hannan, "Spectral estimation of time delay for dispersive and non-dispersive systems," *Appl. Statist.*, vol. 23, no. 2, p. 134.
[11] V. H. MacDonald and P. M. Schultheiss, "Optimum passive bearing estimation in a spatially incoherent noise environment," *J. Acoust. Soc. Amer.*, vol. 46, pp. 37-43, July 1969.

Comments on the Generalized Ambiguity Function

LAWRENCE J. ZIOMEK

***Abstract*—A straightforward derivation of the form of the generalized (wide-band) ambiguity function is presented. It is obtained by treating the scattered acoustic pressure field from a point target (in relative motion with respect to a bistatic transmit/receive array geometry) as the output of a linear, time-varying, random filter. The results are compared to previous expressions which have appeared in the literature for the generalized ambiguity function, and the differences are discussed.**

I. INTRODUCTION

Since an ambiguity function can be thought of as the inner product between a transmitted signal and its scattered return from a point target in relative motion, the form of the equation used to model the return signal obviously affects the expression for the ambiguity function.

Manuscript received November 7, 1980; revised September 1, 1981.
The author is with the Department of Ocean Technology, Applied Research Laboratory, Pennsylvania State University, State College, PA 16801.

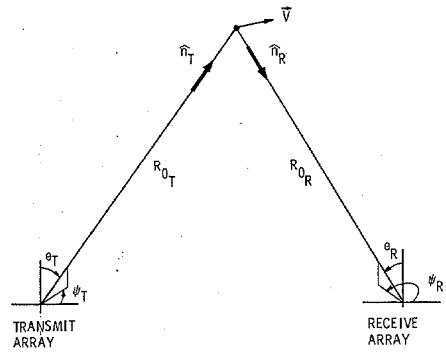

Fig. 1. Bistatic scatter geometry.

The purpose of this correspondence is to derive the equation of the return signal which, in our case, will be considered as the scattered acoustic pressure field from a point target which is in the far field and in relative motion with respect to a bistatic transmit/receive array geometry. The results can then be easily reduced to the more common, monostatic (backscatter) geometry.

The physical situation described above will be treated as a linear, time-varying, random system. The derived equation of the return signal, and hence the resulting generalized or wide-band ambiguity function, will then be compared with other expressions which have appeared in the literature.

II. DERIVATION OF RETURN SIGNAL

The output complex envelope $\tilde{y}(t)$ of a linear, time-varying, random system can be expressed as [1]

$$\tilde{y}(t) = \int_{-\infty}^{\infty} \tilde{X}(f) H(f + f_c, t) e^{+j2\pi f t} \, df \tag{1}$$

where $\tilde{X}(f)$ is the Fourier transform of the input complex envelope $\tilde{x}(t)$; $H(f + f_c, t)$ is the time-varying, random transfer function; f_c is the center or carrier frequency of the input bandpass signal $x(t) = \text{Re}\,\{\tilde{x}(t)e^{+j2\pi f_c t}\}$, where Re means real part; and the frequency variable f represents frequency deviations from f_c. The input $\tilde{x}(t)$ can be thought of as the transmitted signal and the output $\tilde{y}(t)$ as the scattered or return signal.

If one considers a point target to be in relative motion and in the far field with respect to a bistatic transmit/receive array geometry, then it can be shown that the transfer function which corresponds to this physical situation is given by [1] (see Fig. 1)

$$H(f, t) = \frac{F(f)}{R_{O_T} R_{O_R}} \exp\{-jk[R_{O_T} + R_{O_R} + (\hat{n}_T - \hat{n}_R) \cdot \boldsymbol{V} t']\} \tag{2}$$

where

$$F(f) = D_T(ku_T, kv_T) g(\hat{n}_R, \hat{n}_T, f) D_R(ku_R, kv_R) \tag{3}$$

$$u_T = \sin\theta_T \cos\psi_T \tag{4}$$

$$v_T = \sin\theta_T \sin\psi_T \tag{5}$$

$$u_R = \sin\theta_R \cos\psi_R \tag{6}$$

$$v_R = \sin\theta_R \sin\psi_R \tag{7}$$

0096-3518/82/0200-0117$00.75 © 1982 IEEE