Polynomials, Powerful Tangents, and Permutations

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Problem 1

Given real numbers, $a, b, c \in \mathbb{R}$, the polynomial

$$g(x) = x^3 + ax^2 + x + 10,$$

is such that all of its roots are also roots of the polynomial

$$f(x) = x^4 + x^3 + bx^2 + 100x + c.$$

What is $f(1)$? (2017 AMC 12A Problem 23)

Claim: $f(1) = -7007$

Proof. Let r_1, r_2, r_3 be the roots of $g(x)$, and r_4 be the additional root of $f(x)$. By Vieta's Formulas, we have that

$$r_1 + r_2 + r_3 = -a$$
$$r_1 + r_2 + r_3 + r_4 = -1$$

so we know that $r_4 = a - 1$.

Vieta's formulas also tell us that

$$r_1 r_2 r_3 = -10$$
$$r_1 r_2 + r_2 r_3 + r_3 r_1 = 1$$
$$r_1 r_2 r_3 + r_2 r_3 r_4 + r_2 r_4 r_1 = -100 \qquad (\star)$$

Substituting $r_1 r_2 r_3 = -10$ into $\star$ gives us that

$$-10 + (r_1 r_2 + r_2 r_3 + r_3 r_1) r_4 = -10 + r_4 = -100.$$

Then $r_4 = 90$, so $a = 89$.

By factoring $f(x)$ in terms of $g(x)$, we get

$$f(x) = (x - r_4) g(x) = (x + 90)\, g(x)$$

Then since $g(1) = -77$, $f(1) = 91 \cdot -77 = -7007$, as desired. $\square$

Problem 2

There is a unique θ between 0° and 90° such that for nonnegative $n \in \mathbb{Z}$, $\tan(2^n\theta)$ is positive if and only if $n \equiv_3 0$. If $\theta = \frac{p}{q}$ for two relatively prime integers, find $p+q$. (2019 AIME II Problem 10)

Proof. Note that if $\tan(\theta)$ is positive, then $0^\circ < \theta < 90^\circ \mod 180$. Furthermore, it must also hold that

$$2^0\theta \equiv 2^3\theta \equiv 2^6\theta \equiv \dots \mod 180$$

as if it did not, the terminal angle would shift out of the first quadrant.

Then, $2^3\theta \equiv 2^0\theta$ so $7\theta \equiv 0^\circ \mod 180$ Thus, θ must be one of

$$\frac{180^\circ}{7}, \quad \frac{360^\circ}{7}, \quad \frac{540^\circ}{7}.$$

The only value of these three that works is $\theta = \frac{540^\circ}{7}$, and these two integers are already coprime, so our answer is 547. $\square$

Problem 3

Prove the Hockey-Stick Identity,

$$\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$$

for all $n, r \in \mathbb{N}, n > r$

Proof. Let $n = r$. Then we have that

$$\sum_{i=r}^{n} \binom{i}{r} = \sum_{i=r}^{r} \binom{i}{r} = \binom{r}{r} = 1 = \binom{r+1}{r+1}.$$

Then let $k \in \mathbb{N}$ such that $k > r$ and

$$\sum_{i=r}^{k} \binom{i}{r} = \binom{k+1}{r+1}.$$

Then we have

$$\begin{aligned} \sum_{i=r}^{k+1} \binom{i}{r} &= \left(\sum_{i=r}^{k} \binom{i}{r} \right) + \binom{k+1}{r} \\ &= \binom{k+1}{r+1} + \binom{k+1}{r} \\ &= \binom{k+2}{r+1} \end{aligned}$$

as desired.

$\square$

Problem 4

Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

over $\mathbb{C}$. Diagonalize if possible.

Proof. The characteristic polynomial of A is $\lambda^2 + 1$. This has roots in $\mathbb{C}$, being $\lambda = \pm i$. We now solve for the eigenvectors.

Case 1: $\lambda = i$

We solve $(A - iI)v = 0$.

$$\left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right] \simeq \left[\begin{array}{cc|c} 1 & -i & 0 \\ -i & -1 & 0 \end{array}\right] \simeq \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, our eigenvector for $\lambda = -i$ is

$$v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}.$$

We solve for $\lambda = i$ similarly to find

$$v_2 = \begin{bmatrix} -i \\ i \end{bmatrix}.$$

Then since we have two linearly independent eigenvectors, A is diagonalizable and $A = P^{-1}DP$ where

$$P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

□