Prove or disprove:

$$2024^{2025} < 2025^{2024}$$

Solution by Bram Ogus:

First, note that for all $a, b, n \in \mathbb{N}$ the following holds:

$$a < b \Rightarrow a^n < b^n.$$

Because we know $2024 < 2025$, we also know that $2024^{2024} < 2025^{2024}$. If $2024^{2025} < 2025^{2024}$, we will find that

$$\frac{2025^{2024}}{2024^{2025}} > 1.$$

We know that $\frac{2025^{2024}}{2024^{2024}} > 1$, but we must ask ourselves how much bigger it is. If

$$\frac{2025^{2024}}{2024^{2024}} > 2024,$$

it will hold true that $2024^{2025} < 2025^{2024}$. Otherwise, we will find that $2024^{2025} > 2025^{2024}$.

Observe that we have a fraction of the form

$$\left(\frac{x+1}{x}\right)^x = \left(1 + \frac{1}{x}\right)^x.$$

This equation is well known, specifically for its limit as x approaches ∞,

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e.$$

From this we can see that

$$1 < \frac{2025^{2024}}{2024^{2024}} < e,$$

so it follows that

$$\frac{2025^{2024}}{2024^{2025}} < \frac{e}{2024} < 1,$$

giving us that $2024^{2025} > 2025^{2024}$.

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