Prof. Dr. Johannes Pfeifer
University of Cologne

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Jermann (1998)

1 Firms

1.1 Technology growth

Technology X_t grows deterministically with $X_t/X_{t-1} = \gamma$

1.2 Firm Problem

Firm maximizes the sum of dividends

$$D_t = A_t K_t^{\alpha} (X_t N_t)^{1-\alpha} - W_t N_t - I_t$$

subject to the LOM

$$K_{t+1} = (1-\delta) K_t + \left(\frac{b}{1-a} \left(\frac{I_t}{K_t} \right)^{1-a} + c \right) K_t$$

The parameters a and c need to determined to satisfy steady state relationships. Detrending yields:

$$D_t = A_t K_t^{\alpha} N_t^{1-\alpha} - W_t N_t - I_t$$

and

$$\gamma K_{t+1} = (1-\delta) K_t - \left(\frac{b}{1-a} \left(\frac{I_t}{K_t} \right)^{1-a} + c \right) K_t$$

Using the households SDF, the firm maximizes

$$\max E_t \sum_{t=0}^{\infty} (\beta^*)^k \frac{\Lambda_{t+k}}{\Lambda_t} \left(\begin{matrix} A_{t+k} K_{t+k}^{\alpha} N_{t+k}^{1-\alpha} - W_{t+k} N_{t+k} - I_{t+k} \\ -q_{t+k} \left(\gamma K_{t+k+1} - (1-\delta) K_{t+k} - \left(\frac{b}{1-a} \left(\frac{I_{t+k}}{K_{t+k}} \right)^{1-a} + c \right) K_{t+k} \right) \end{matrix} \right)$$

FOC Capital

$$\begin{aligned} \frac{\partial L}{\partial K_{t+k+1}} =& (\beta^*)^k \frac{\Lambda_{t+k}}{\Lambda_t} (-q_{t+k}\gamma) \\ &+ E_t (\beta^*)^{k+1} \frac{\Lambda_{t+k+1}}{\Lambda_t} \left[\begin{matrix} \alpha A_{t+k+1} K_{t+k+1}^{\alpha-1} N_{t+k+1}^{1-\alpha} \\ -q_{t+k+1} \left(-(1-\delta) - \frac{ba}{1-a} I_{t+k+1}^{1-a} K_{t+k+1}^{a-1} - c \right) \end{matrix} \right] = 0 \end{aligned}$$

Simplified

$$\gamma \Lambda_{t+k} q_{t+k} = \beta^* \Lambda_{t+k+1} \left(\alpha A_{t+k+1} K_{t+k+1}^{\alpha-1} N_{t+k+1}^{1-\alpha} + E_t q_{t+k+1} \left((1-\delta) + \frac{ba}{1-a} I_{t+k+1}^{1-a} K_{t+k+1}^{a-1} + c \right) \right)$$

Investment FOC

$$\frac{\partial L}{\partial I_{t+k}} = \beta^k \frac{\Lambda_{t+k}}{\Lambda_t}\left(1 - q_{t+k}\left(-bI_{t+k}^{-a}K_{t+k}^{a}\right)\right) = 0$$

Simplified

$$1 = q_{t+k} b I_{t+k}^{-a} K_{t+k}^{a}$$

or

$$q_t = \frac{1}{b}\left(\frac{I_t}{K_t}\right)^a$$

Labor FOC

$$W_t = (1-\alpha)\, A_t K_t^{\alpha} N_t^{1-\alpha}$$

Pricing follows

$$1 = \beta^* E_t \frac{\Lambda_{t+1}}{\Lambda_t} x_t$$

where x_t is the payoff of any asset. For capital, from (1.2)

$$1 = \beta^* E_t \frac{\Lambda_{t+1}}{\Lambda_t}\frac{1}{\gamma q_t}\left(\alpha A_{t+1} K_{t+1}^{\alpha-1} N_{t+1}^{1-\alpha} + q_{t+1}\left((1-\delta) + \frac{ba}{1-a} I_{t+1}^{1-a} K_{t+1}^{a-1} + c\right)\right) = 0$$

2 Household

The household maximizes

$$\max E_t \sum_{k=0}^{\infty} \beta^k \left\{\frac{(C_{t+k} - hC_{t+k-1})^{1-\tau}}{1-\tau}\right\} = E_t \sum_{k=0}^{\infty} \underbrace{\left(\tilde{\beta}\gamma^{1-\tau}\right)^k}_{\beta^*} \left\{\frac{\left(C_{t+k} - \frac{h}{\gamma}C_{t+k-1}\right)^{1-\tau}}{1-\tau}\right\}$$

subject to

$$\left[W_{t+k} + a'_{t+k}\left(V_{t+k}^a + D_{t+k}^a\right) = C_{t+k} + a'_{t+k+1} V_{t+k}^a\right]$$

where a is the amount of assets held, V the value of the asset and D its dividend. This equation is already in detrended form. The FOCs are

$$\begin{aligned}
\frac{\partial L}{\partial C_t} &= \left(C_t - \frac{h}{\gamma}C_{t-1}\right)^{-\tau} - \Lambda_t - \beta^* \frac{h}{\gamma}\left(C_{t+1} - \frac{h}{\gamma}C_t\right)^{-\tau} = 0 \\
\Lambda_t &= \left(C_t - \frac{h}{\gamma}C_{t-1}\right)^{-\tau} - \beta^* \frac{h}{\gamma}\left(C_{t+1} - \frac{h}{\gamma}C_t\right)^{-\tau} \\
\frac{\partial L}{\partial a_{t+k}} &= -\beta^k \Lambda_{t+k} V_{t+k}^a + \beta^{*k+1} E_{t+k} \Lambda_{t+k+1}\left(V_{t+k+1}^a + D_{t+k+1}^a\right) = 0 \\
\Lambda_{t+k} V_{t+k}^a &= \beta \Lambda_{t+k+1}\left(V_{t+k+1}^a + D_{t+k+1}^a\right)
\end{aligned}$$

where $N = 1$ has been imposed.

For an one-period risk free bond, this implies

$$1 = \beta^* E_t \frac{\Lambda_{t+1}}{\Lambda_t}\left(1 + r^*\right)$$

For a firm share, this implies

$$\Lambda_t = \beta^* E_t \Lambda_{t+1} \frac{\left(V_{t+1}^k + D_{t+1}\right)}{V_t^k}$$

while for a consol that in perpetuity pays the risk free rate, we have

$$\Lambda_t = \beta^* E_t \Lambda_{t+1} \frac{(V_{t+1}^c + r^*)}{V_t^c}$$

3 Market Clearing

As usual

$$Y = C + I$$

All assets except for firm shares are in zero net supply, i.e. a=0 except for a^k which must be 1 $\forall t$, implying

$$W_t + \left(V_t^k + D_t^k\right) = C_t + V_t^k \Rightarrow W_t + D_t = C_t$$

4 Steady State

In steady state, we have

$$K = \frac{1}{\gamma}(1-\delta) K + \underbrace{\frac{1}{\gamma}\left(\frac{b}{1-a}\left(\frac{I}{K}\right)^{1-a} + c\right) K}_{I}$$

Thus, we need

$$\frac{I}{K} = \left(1 - \frac{1}{\gamma}(1-\delta)\right)$$

i.e. the investment capital ratio can be computed. From the investment FOC with Tobin's $q = 1$ imposed, we have

$$b I_t^{-a} K_t^a = 1 \Rightarrow b = \left(\frac{I}{K}\right)^a$$

Moreover, the LOM implies

$$\frac{1}{\gamma}\left(\frac{b}{1-a}\left(\frac{I}{K}\right)^{1-a} + const\right) = \frac{I}{K}$$

which allows solving for *const*:

$$const = \gamma \frac{I}{K} - \frac{b}{1-a}\left(\frac{I}{K}\right)^{1-a}$$

Finally, the capital FOC implies

$$
\begin{aligned}
&\Lambda_t q_t \gamma = \beta^* \Lambda_{t+1} \left(\alpha A_{t+1} K_{t+1}^{\alpha-1} + q_{t+1} \left((1-\delta) + c + \frac{ba}{1-a} I_{t+1}^{1-a} K_{t+1}^{a-1} \right) \right) \\
&\gamma = \beta^* \left(\alpha K^{\alpha-1} + \left((1-\delta) + c + \frac{ba}{1-a} \left(\frac{I}{K} \right)^{1-a} \right) \right) \\
&\frac{\gamma}{\beta^*} = \left(\alpha K^{\alpha-1} + \left((1-\delta) + c + \frac{ba}{1-a} \left(\frac{I}{K} \right)^{1-a} \right) \right) \\
&\frac{\gamma}{\beta^*} - \left((1-\delta) + c + \frac{ba}{1-a} \left(\frac{I}{K} \right)^{1-a} \right) = \alpha K^{\alpha-1} \\
&\left[\frac{1}{\alpha} \left(\frac{\gamma}{\beta^*} - \left((1-\delta) + c + \frac{ba}{1-a} \left(\frac{I}{K} \right)^{1-a} \right) \right) \right]^{\frac{1}{\alpha-1}} = K
\end{aligned}
$$

From I/K investment immediately follows. Moreover,

$$
\begin{aligned}
W &= (1-\alpha) K^{\alpha} \\
D &= K^{\alpha} - W - I
\end{aligned}
$$

and

$$
\begin{aligned}
&\Lambda_{t+k} V_{t+k}^{a} = \beta^* \Lambda_{t+k+1} \left(V_{t+k+1}^{a} + D_{t+k+1}^{a} \right) \\
&\frac{1}{\beta} = \left(1 + \frac{D}{V} \right) \\
&\left(\frac{1}{\beta^*} - 1 \right) V = D
\end{aligned}
$$

$$
\Lambda = \left(C - \frac{h}{\gamma} C \right)^{-\tau} \left(1 - \beta \frac{h}{\gamma} \right)
$$

$$
\begin{aligned}
&W_t + D_t = C_t \\
&\left(C_t \left(1 - \frac{h}{\gamma} \right) \right)^{-\tau} = \Lambda_t
\end{aligned}
$$

5 Mapping a and ξ

ξ is the elasticity of

$$
\begin{aligned}
&q = \frac{1}{b} \left(\frac{I}{K} \right)^{a} \\
&\log q = \log \left(\frac{1}{b} \right) + a \log \left(\frac{I}{K} \right) \\
&\log \left(\frac{I}{K} \right) = -\frac{1}{a} \log \left(\frac{1}{b} \right) + \frac{1}{a} \log q \\
&\frac{\partial \log \left(\frac{I}{K} \right)}{\partial \log q} = \frac{1}{a} = \xi
\end{aligned}
$$