DYNARE SUMMER SCHOOL

Computing optimal policy

Michel Juillard

June 15, 2018

Outline

1. Introduction
2. Optimal policy under commitment (Ramsey policy)
3. Discretionary policy equilibrium
4. Optimal simple rule

Introduction

Two issues:

1. Using welfare to compare policy or institutional arrangements
   - Consistent with objective of households in the model
   - More growth is not always optimal
2. Computing policies that maximize households welfare
   - Tinbergen approach to economic policy
   - Remedies asymmetry with treatment of private agents in most models
   - Provides a benchmark to compare more practical policies
   - Provides a natural framework to discuss how monetary policy should answer different type of structural shocks

Introduction (continued)

- Difficulty of aggregation with social welfare function
- Policy maker maybe only trying to stabilize economy (loss function)
- When agents form expectations about future policy, designing optimal policy becomes a dynamic game
- Issue of time consistency
- Policy rule versus discretion

Time consistency, commitment

- The behavior of private agents and their expectations act as constraints on the choice of the policymaker.
- As private agents anticipate policy, policy announcements (or policy rules) affects the behavior of private agents in previous periods.
- However, this particular constraint for policymaker choices doesn't exist for first period policy decisions (the policymaker has the *benefit of surprise*).
- As a consequence, if the policymaker re–optimize in the future, he will not follow the rule announced at the time of the first optimization. This is *time inconsistency*.
- *Commitment* is the commitment not to re–optimize in the future.

Ramsey policy: General nonlinear case

$$\max_{\{y_\tau\}_{\tau=0}^{\infty}} E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} U(y_\tau)$$

s.t.

$$E_\tau f(y_{\tau+1}, y_\tau, y_{\tau-1}, \varepsilon_\tau) = 0$$

$y_t \in R^n$: endogenous variables

$\varepsilon_t \in R^p$: stochastic shocks

and

$$f : R^{3n+p} \rightarrow R^m$$

There are $n - m$ free policy instruments.

Lagrangian

The Lagrangian is written

$$L\left(y_{t-1},\varepsilon_t,\ldots\right)=E_t\sum_{\tau=t}^{\infty}\beta^{\tau-t}U(y_\tau)-\lambda_\tau' f\left(y_{\tau+1},y_\tau,y_{\tau-1},\varepsilon_\tau\right)$$

where λ_τ is a vector of m Lagrange multipliers.
It turns out that it is the discounted value of the Lagrange multipliers that are stationary and not the multipliers themselves. It is therefore handy to rewrite the Lagrangian as

$$L\left(y_{t-1},\varepsilon_t,\ldots\right)=E_t\sum_{\tau=t}^{\infty}\beta^{\tau-t}\left(U(y_\tau)-\mu_\tau' f\left(y_{\tau+1},y_\tau,y_{\tau-1},\varepsilon_\tau\right)\right)$$

whith $\mu_t=\lambda_\tau/\beta^{\tau-t}$.

Optimization problem reformulated

The optimization problem becomes

$$\max_{\{y_\tau\}_{\tau=t}^{\infty}} \min_{\{\mu_\tau\}_{\tau=t}^{\infty}} E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} \left(U(y_\tau) - \mu_\tau' f\left(y_{\tau+1}, y_\tau, y_{\tau-1}, \varepsilon_\tau\right)\right)$$

for y_{t-1} given.

First order necessary conditions

Derivatives of the Lagrangian with respect to endogenous variables:

$$\begin{aligned}\frac{\partial L}{\partial y_\tau} &= E_t\big\{U_1(y_\tau) - \mu'_\tau f_2(y_{\tau+1}, y_\tau, y_{\tau-1}, \varepsilon_\tau) \\ &\quad - \beta\mu'_{\tau+1} f_3(y_{\tau+2}, y_{\tau+1}, y_\tau, \varepsilon_{\tau+1})\big\} && \tau = t \\ \frac{\partial L}{\partial y_\tau} &= E_t\big\{U_1(y_\tau) - \mu'_\tau f_2(y_{\tau+1}, y_\tau, y_{\tau-1}, \varepsilon_\tau) \\ &\quad - \beta\mu'_{\tau+1} f_3(y_{\tau+2}, y_{\tau+1}, y_\tau, \varepsilon_{\tau+1}) \\ &\quad - \beta^{-1}\mu_{\tau-1} f_1(y_\tau, y_{\tau-1}, y_{\tau-2}, \varepsilon_{\tau-1})\big\} && \tau > t\end{aligned}$$

First order conditions

The first order conditions of this optimization problem are

$$
\begin{aligned}
E_t\{U_1(y_t) - \mu_t' f_2(y_{t+1}, y_t, y_t, \varepsilon_t) \\
-\beta \mu_{t+1}' f_3(y_{t+2}, y_{t+1}, y_t, \varepsilon_{t+1}) \\
-\beta^{-1} \mu_{t-1}' f_1(y_t, y_{t-1}, y_{t-2}, \varepsilon_{t-1})\} &= 0 \\
E_t f\left(y_{t+1}, y_t, y_{t-1}, \varepsilon_t\right) &= 0
\end{aligned}
$$

with $\mu_{t-1} = 0$ and where $U_1()$ is the Jacobian of function $U()$ with respect to y_τ and $f_i()$ is the first order partial derivative of $f()$ with respect to the ith argument.

Cautionary remark

The First Order Conditions for optimality are only necessary conditions for a maximum. Levine, Pearlman and Pierse (2008) propose algorithms to check a sufficient condition.

Nature of the solution

The above system of equations is nothing but a larger system of nonlinear rational expectation equations. As such, it can be approximated either to first order or to second order. The solution takes the form

$$\left[\begin{array}{c} y_t \\ \mu_t \end{array}\right] = \hat{g}\left(y_{t-2}, y_{t-1}, \mu_{t-1}, \varepsilon_{t-1}, \varepsilon_t\right)$$

- The optimal policy is then directly obtained as part of the set of $g()$ functions.
- The optimal policy may depend on previous value of Lagrange multipliers
- This solution doesn't necessarily provide a policy that is implementable.

The steady state problem

The steady state is solution of

$$\begin{aligned} U_1(\bar{y}) - \bar{\mu}' \big[f_2(\bar{y}, \bar{y}, \bar{y}, 0) + \beta f_3(\bar{y}, \bar{y}, \bar{y}, 0) & \\ + \beta^{-1} f_1(\bar{y}, \bar{y}, \bar{y}, 0) \big] &= 0 \\ f(\bar{y}, \bar{y}, \bar{y}, 0) &= 0 \end{aligned}$$

Pitfalls

- A naive approach ot linear-quadratic appoximation that would consider a linear approximation of the dynamics of the system and a second order approximation of the objective function, ignores the second order derivatives f_{ij} that enter in the first order approximation of the dynamics of the model under optimal policy.
- If the initial conditions for the system are far away from the steady state under Ramsey policy, a local approximation may not render accurately the transition.

The approximated solution function

$$\begin{aligned}
y_t &= \bar{y} + g_1 \widehat{y}_{t-2} + g_2 \widehat{y}_{t-1} + g_3 \widehat{\mu}_{t-1} + g_4 \varepsilon_{t-1} + g_5 \varepsilon_t \\
\mu_t &= \bar{\mu} + h_1 \widehat{y}_{t-2} + h_2 \widehat{y}_{t-1} + h_3 \widehat{\mu}_{t-1} + h_4 \varepsilon_{t-1} + h_5 \varepsilon_t
\end{aligned}$$

A New Keynesian model

Sticky prices with price adjustment cost (Rotemberg)

- Household i maximizes discounted utility. Period utility is

$$lnC_{i,t} - \frac{\chi}{2} h_{i,t}^2$$

- Budget constraint:

$$\frac{B_{i,t}}{P_t} = (1 + R_{t-1})\frac{B_{i,t-1}}{P_t} - C_{i,t} + \frac{W_t}{P_t} h_{i,t} + \Pi_{i,t}$$

Firms

Firm j maximizes profits:

$$(1+\tau)\frac{P_{j,t}}{P_t}C_{j,t} - MC_t C_{j,t} - \frac{\phi}{2}\left(\frac{P_{j,t}}{P_{j,t-1}} - 1\right)^2 C_{t,j}$$

Marginal cost is

$$MC_t = \frac{W_t}{P_t exp(Z_t)} = \frac{\chi h_t C_t}{exp(Z_t)}$$

Demand curve (monopolistic competition among producers of imperfectly substituable goods)

$$C_{j,t} = \left(\frac{P_{j,t}}{P_t}\right)^{-\theta}$$

Recursive equilibrium

$$
\begin{aligned}
\frac{1}{C_t} &= \beta \mathbb{E}_t \left\{ \frac{1}{C_{t+1}} \frac{1+r_t}{\pi_{t+1}} \right\} \\
\left[\tau - \frac{1}{\theta - 1} \right] (1-\theta) + \theta \left(\frac{\chi h_t C_t}{\exp Z_t} - 1 \right) &= \phi(\pi_t - 1)\pi_t - \beta \mathbb{E}_t \phi(\pi_{t+1} - 1)\pi_{t-1} \\
\exp Z_t h_t &= C_t \left[1 + \frac{\phi}{2} (\pi_t - 1)^2 \right] \\
Z_t &= \rho Z_{t-1} + u_t
\end{aligned}
$$

Ramsey problem

$$
\begin{aligned}
\max E_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ & \ln C_t - \frac{\chi}{2} h_t^2 \\
& - \mu_{1,t} \left(\frac{1}{C_t} - \beta \frac{1}{C_{t+1}} \frac{1+R_t}{\pi_{t+1}} \right) \\
& - \mu_{2,t} \Bigg(\left[\tau - \frac{1}{\theta - 1} \right] (1-\theta) + \theta \left(\frac{\chi h_t C_t}{\exp Z_t} - 1 \right) \\
& \qquad - \phi(\pi_t - 1)\pi_t + \beta\phi(\pi_{t+1} - 1)\pi_{t-1} \Bigg) \\
& - \mu_{3,t} \left(\exp Z_t h_t - C_t \left[1 + \frac{\phi}{2} (\pi_t - 1)^2 \right] \right) \\
& - \mu_{4,t} \left(z_t - \rho z_{t-1} - u_t \right) \Bigg\}
\end{aligned}
$$

$\mu_{i,t} = \lambda_{i,t}\beta^t$ where $\lambda_{i,t}$ is a Lagrange multiplier.

Dynare code

nk_ramsey_1.mod

```
var pie, c, h, r, z;
varexo u;

parameters beta, rho, chi, theta, phi, tau;

beta=0.99;
chi = 1;
phi=100;
theta=5;
phi=1;
tau=1/(theta-1);
rho=0.9;
```

Dynare code (continued)

```
model;
z=rho*z(-1)+u;
1/c=beta*(1/c(+1))*((1+r)/pie(+1));
(tau-1/(theta-1))*(1-theta)
    +theta*(chi*h*c/exp(z)-1) =
     phi*(pie-1)*pie - beta*phi*(pie(+1)-1)*pie(+1)
exp(z)*h = c*(1+(phi/2)*(pie-1)^2);
end;

write_latex_dynamic_model;
```

Dynare code (continued)

```
initval;
pie=1.1;
r=1/beta;
c=0.9;
h=0.9;
z=0;
end;

shocks;
var u; stderr 0.008;
end;

planner_objective(log(c)-(chi/2)*h^2);

ramsey_policy(planner_discount=0.99,order=1);
```

Outline

1. Introduction
2. Computing welfare in DSGE models
3. Optimal policy under commitment (Ramsey policy)
4. Discretionary policy equilibrium
5. Optimal simple rule

Discretionary policy equilibrium

- Equilibrium when policy maker re-optimizes in every period (resets the Lagrange multipliers to zero) and private agents expect her to do so.
- Solution well known for linear quadratic case. Nonlinear problems are much more difficult to solve.

$$L_t = \sum_{\tau=t}^{\infty} \beta^{\tau-t}(y_t' W y_t + u_t' Q u_t)$$

s.t.

$$A_0 y_\tau = A_1 y_{\tau-1} + A_2 \mathbb{E}_\tau y_{\tau+1} + A_3 u_\tau + A_4 \mathbb{E}_\tau u_{\tau+1} + A_5 \epsilon_t$$

Solution algorithm (I)

- Markov-perfect Stackelberg-Nash equilibrium
- Policy maker optimizing today is Stackelberg leader
- Private sector agents and policy maker optimizing tomorrow are Stackelberg followers

Assume the a solution of the following form:

$$y_t = H_1 y_{t-1} + H_2 \epsilon_t$$
$$u_t = F_1 y_{t-1} + F_2 \epsilon_t$$

$H_1,\ H_2,\ F_1,\ F_2$ are unknown matrices.

Solution algorithm (II)

Step 1 For $i = 0$, initialize $H_1^{(i)}$, $H_2^{(i)}$, $F_1^{(i)}$, $F_2^{(i)}$

Step 2 Compute $D^{(i)} = A_0 - A_2 H_1^{(i)} - A_4 F_1^{(i)}$

Step 3 Solve the Sylvester equation

$$P^{(i)} = W + \beta F_1^{(i)\prime} Q F_1^{(i)} + \beta H_1^{(i)} P^{(i)} H_1^{(i)}$$

Step 4 Set

$$\begin{aligned}
F_1^{(i+1)} &= -(Q + A_3'(D^{(i)\,-1} A_3)^{-1} A_3'(D^{(i)\prime})^{-1} P^{(i)} D^{(i)\,-1} A_1\,, \\
F_2^{(i+1)} &= -(Q + A_3'(D^{(i)\prime})^{-1} P^{(i)} D^{(i)\,-1} A_3)^{-1} A_3'(D^{(i)\prime})^{-1} P^{(i)} D^{(i)\,-1} A_5\,, \\
H_1^{(i+1)} &= D^{(i)\,-1}(A_1 + A_3 F_1^{(i+1)}), \\
H_2^{(i+1)} &= D^{(i)\,-1}(A_5 + A_3 F_2^{(i+1)}).
\end{aligned}$$

Step 5 Go back to Step 2, until convergence.

Example: Clarida, Gali, Gertler

$$
\begin{aligned}
y_t &= \delta y_{t-1} + (1-\delta)E_t y_{t+1} + \sigma(r_t - E_t \pi_{t+1}) + e_{y_t} \\
\pi_t &= \alpha \pi_{-1} + (1-\alpha)E_t \pi_{t+1} + \kappa y_t + e_{\pi_t}
\end{aligned}
$$

Objectif

$$
\arg\min_{\gamma_1,\gamma_2} \mathrm{var}(y) + \mathrm{var}(\pi)
$$

$$
= \arg\min_{\gamma_1,\gamma_2} \lim_{\beta \to 1} E_0 \sum_{t=1}^{\infty} (1-\beta)\beta^t (y_t^2 + \pi_t^2)
$$

Outline

1. Introduction
2. Computing welfare in DSGE models
3. Optimal policy under commitment (Ramsey policy)
4. Discretionary policy equilibrium
5. Optimal simple rule

Optimal simple rule

Example (Clarida, Gali, Gertler)

$$
\begin{aligned}
y_t &= \delta y_{t-1} + (1-\delta) E_t y_{t+1} + \sigma(r_t - E_t \pi_{t+1}) + e_{y_t} \\
\pi_t &= \alpha \pi_{-1} + (1-\alpha) E_t \pi_{t+1} + \kappa y_t + e_{\pi_t} \\
r_t &= \gamma_1 \pi_t + \gamma_2 y_t
\end{aligned}
$$

Objectif

$$
\arg\min_{\gamma_1,\gamma_2} \mathrm{var}(y) + \mathrm{var}(\pi)
$$

$$
= \arg\min_{\gamma_1,\gamma_2} \lim_{\beta \to 1} E_0 \sum_{t=1}^{\infty} (1-\beta)\beta^t (y_t^2 + \pi_t^2)
$$

DYNARE example cgg_osr_1.mod

```
var y pie r;
varexo e_y e_pie;

parameters delta sigma alpha kappa gamma1 gamma2;

delta =  0.44;
kappa =  0.18;
alpha =  0.48;
sigma = -0.06;
```

(continued)

```
model(linear);
y   = delta*y(-1)+(1-delta)*y(+1)+sigma *(r-pie(+1))+e_y;
pie = alpha*pie(-1)+(1-alpha)*pie(+1)+kappa*y+e_pie;
r = gamma1*pie+gamma2*y;
end;

shocks;
var e_y;
stderr 0.63;
var e_pie;
stderr 0.4;
end;
```

(continued)

```
optim_weights;
pie 1;
y 1;
end;

gamma1 = 1.1;
gamma2 = 0;

osr_params gamma1 gamma2;

osr;
```

Controlling for variance of change in interest rate

$$
\begin{aligned}
y_t &= \delta y_{t-1} + (1-\delta)E_t y_{t+1} + \sigma(r_t - E_t pie_{t+1}) + e_{y_t} \\
pie_t &= \alpha pie_{-1} + (1-\alpha)E_t pie_{t+1} + \kappa y_t + e_{pie_t} \\
r_t &= \gamma_1 pie_t + \gamma_2 y_t \\
dr_t &= r_t - r_{t-1}
\end{aligned}
$$

Objectif

$$
\min_{\gamma_1,\gamma_2} \text{var}(y) + \text{var}(pie) + 0.2\text{var}(dr)
$$

See `cgg_osr_2.mod`

A grid search for initial values (I)

```
cgg_osr_3.mod

gamma1_s = [  1.1 2.5 5];
gamma2_s = [  0 1 2];

nres = length(gamma1_s)*length(gamma2_s);
results = zeros(nres,3);

i_y = strmatch('y',M_.endo_names,'exact');
i_inf = strmatch('pie',M_.endo_names,'exact');
i_dr = strmatch('dr',M_.endo_names,'exact');
```