Let us consider the 2D NSE system with dynamic boundary condition

$$\partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla\pi = f \qquad \operatorname{div} u = 0 \qquad \text{in } \Omega \tag{1}$$
$$\beta\partial_t u + \alpha u + \nu[(Du)n]_\tau = \beta h \qquad u\cdot n = 0 \qquad \text{on } \Gamma = \partial\Omega \tag{2}$$

where $\Omega \subset \mathbb{R}^2$ is a suitable planar domain. It was shown in [2], given that Ω is smooth and *bounded*, the fractal dimension of the global attractor can be estimated as

$$\dim_H(\mathcal{A}) \le c_0 \frac{M_\beta}{m_\alpha^{3/2}} \cdot \frac{\ell^2\|F\|_H}{\nu^2} \tag{3}$$

where $F = (f,h)$, $H = L^2(\Omega\times\Gamma)$, $\ell \sim \operatorname{diam}\Omega$ is the characteristic length and

$$m_\alpha = \min\{1, \alpha\ell/\nu\} \qquad M_\beta = \max\{1, \beta/\ell\} \tag{4}$$

One observes that the last term in (3) corresponds to the so-called Grashof number $G = \|F\|_H/\lambda_1\nu^2$. For $\alpha\to\infty$, $\beta\to\infty$, boundary condition (2) reduces to homogeneous Dirichlet $u=0$, while (3) reduces to $\dim(\mathcal{A}) \le G$, which is consistent with the best available estimate in this setting, see [3].

Question. What about the limit α, $\beta \to 0$? This would reduce (2) to $[(Du)n]_\tau = 0$, where one has an improved estimate

$$\dim(\mathcal{A}) \le c_1 G^{2/3} \tag{5}$$

(up to some logarithmic term); see [1], [4]. Note that (5) also holds in case of periodic boundary conditions, see [3]. Unfortunately, our estimate (3) blows up for for α small, so a different approach is needed.

Some ideas. Let us work with $\omega = \operatorname{curl} u$, which is motivated by two facts. Firstly, $[(Du)n]_\tau = \omega$ on Γ, at least for the flat boundary. Secondly, the estimate (5) is proved while working with $u \in W^{1,2}(\Omega)$, or testing the equation with $Au = -\Delta u$. This is equivalent to working with $\omega \in L^2$, or testing the equation for ω by ω.
So, let us apply curl to (1), to get

$$\partial_t\omega - \nu\Delta\omega + (u\cdot\nabla)\omega = \operatorname{curl} f \qquad \text{in } \Omega \tag{6}$$
$$\beta\partial_t u + \alpha u + \nu\omega = \beta h \qquad \text{on } \Gamma = \partial\Omega \tag{7}$$

Now we have several tasks:

1. Prove the estimate (5) for this system if α, $\beta = 0$. This seems to work, see a sketch (TODO) below.

2. Show that somehow the estimate is robust also for α, β very small, maybe via some asymptotic similarity of both systems?

3. As previous taks is presumably hard, we could also look at some further simplification. For example, we ignore the interior evolution completely, that is to say, to replace (6) by
$$-\nu\Delta u + (u \cdot \nabla)u + \nabla\pi = f \tag{8}$$
or even take $f = 0$. This makes sense since for β small, the boundary evolution is much faster than the interior one.

4. Last, but not least: prove just the existence of global attractor for (1–2) in case of Ω *unbounded*. This is not trivial because of problems with both dissipativity (no Poincaré inequality) and asymptotic compactness (unbouded domain).

References

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