Analytic Number Theory III

Lecture notes

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Contents

List of lectures

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If you have any corrections, you can write to me at Stud.IP or make a pull request directly at the GitHub repository (which is much more convenient for me than the way via Stud.IP).

glhf,
Alex

1 Number fields

Example (Pell[1] equation)**:** Let $d > 1$ be an integer, which is not a square, and find all integer solutions to Lecture 1, 24.10.2023

$$x^2 - dy^2 = 1. \tag{1.1}$$

Write $\mathbb{Z}[\sqrt{d}] = \{a + \sqrt{d}b \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{Q}[\sqrt{d}]$ with its natural ring structure. If $(x, y) \in \mathbb{Z}^2$ is a solution to (1.1), then

$$(x + \sqrt{d}y)(x - \sqrt{d}y) = x^2 - dy^2 = 1$$

and for every $k \in \mathbb{N}$

$$(x + \sqrt{d}y)^k(x - \sqrt{d}y)^k = x_k^2 - dy_k^2 = 1,$$

with $x_k, y_k \in \mathbb{Z}$. I.e. if $(x, y) \neq (\pm 1, 0)$ we can generate new solutions as above. Define the norm map $N : \mathbb{Z}[\sqrt{d}] \to \mathbb{Z},\ a + \sqrt{d}b \mapsto a^2 - db^2$. Then solutions to (1.1) can be described as units $x + \sqrt{d}y \in \mathbb{Z}[\sqrt{d}]^*$ in the ring $\mathbb{Z}[\sqrt{d}]$ with $N(x + \sqrt{d}y) = 1$.

Example (Gaussian integers)**:** The question is to find all primes p which can be written as a sum of two integer squares

$$p = a^2 + b^2.$$

I.e. we ask for primes p which factor as $p = (a + ib)(a - ib)$ in the ring $\mathbb{Z}[i]$.

1.1 Number fields and number rings, first definitions and examples

Definition (Number field)
A *number field* is a finite field extension of $\mathbb{Q}$.

Example: a) For $d \in \mathbb{Z}$, where d is not a square, the fields $\mathbb{Q}[\sqrt{d}] = \mathbb{Q}[x]/(x^2 - d)$

[1] after John Pell (1611 - 1685), an English mathematician

are number fields (with degree 2 over $\mathbb{Q}$). We call $\mathbb{Q}[\sqrt{d}]$ a *real quadratic field* if $d > 0$ and an *imaginary quadratic field* if $d < 0$.

b) $\mathbb{Q}[\sqrt{d_1}, \sqrt{d_2}]$ are number fields for $d_1, d_2 \in \mathbb{Z}$, usually called *biquadratic fields.*

c) Let $m \in \mathbb{N}$ and $\omega = e^{\frac{2\pi i}{m}}$. Then $\mathbb{Q}[\omega]$ is a number field, called the *m-th cyclotomic field.*

?) What could be an analogue of the integers in a general number field?

$$Z \subset \mathbb{Q} \qquad ? \subset \mathbb{Q}[\sqrt{d}] \qquad ? \subset \mathbb{F}$$

Definition (Algebraic integer)
A complex number $\alpha \in \mathbb{C}$ is called an *algebraic integer*, if there is a monic polynomial $P(x) \in \mathbb{Z}[x]$ with $P(\alpha) = 0$.

Example:

- Every $n \in \mathbb{Z}$ is an algebraic integer.
- $\sqrt{d}$ for $d \in \mathbb{Z}$ is an algebraic integer (take $P(x) = x^2 - d$).
- $e^{\frac{2\pi i}{m}}$ is an algebraic integer for every $m \in \mathbb{N}$ (take $P(x) = x^m - 1$).

Theorem 1.1
Let α be an algebraic integer and $f(x) \in \mathbb{Z}[x]$ a monic polynomial with $f(x) = 0$. If $f(x)$ is of minimal degree with these properties, then f is irreducible.

Remark: Theorem 1.1 shows, that the minimal polynomial of an algebraic integer over $\mathbb{Q}$ has coefficients in $\mathbb{Z}$.

Lemma 1.2
Let $f \in \mathbb{Z}[x]$ be a monic polynomial and $g, k \in \mathbb{Q}[x]$ monic polynomials with $f = gh$. Then, $g, k \in \mathbb{Z}[x]$.

Corollary 1.3
If $\alpha \in \mathbb{Q}$ is an algebraic integer, then $\alpha \in \mathbb{Z}$.

Theorem 1.4 (Characterization of algebraic integers)
Let $\alpha \in \mathbb{C}$. Then the following statements are equivalent:

(i) α *is an algebraic integer.*

(ii) $\mathbb{Z}[\alpha]$ *is a finitely generated group (under addition).*

(iii) There exists a subring $R \subset \mathbb{C}$ *with* $\alpha \in R$ *and such that* $(R, +)$ *is a finitely generated group.*

(iv) There is a non-trivial finitely generated subgroup $(A, +)$ *of* $\mathbb{C}$*, such that* $\alpha A \subseteq A$*.*

Corollary 1.5
The set of algebraic integers in $\mathbb{C}$ *is a ring.*

Lecture 2, 27.10.2023

Definition (Ring of algebraic integers)
Let K be a number field. Then we write $\mathcal{O}_K$ for the set of algebraic integers contained in K and we call $\mathcal{O}_K$ the ring of integers of K.

Example: $\mathcal{O}_{\mathbb{Q}} = \mathbb{Z}$

Proposition 1.6
Let $d \in \mathbb{Z}$ *be a squarefree integer.*

- *If* $d \equiv 2, 3 \bmod 4$ *then* $\mathcal{O}_{\mathbb{Q}[\sqrt{d}]} = \left\{ a + \sqrt{d}b \,\middle|\, a, b \in \mathbb{Z} \right\}$.
- *If* $d \equiv 1 \bmod 4$*, then* $\mathcal{O}_{\mathbb{Q}[\sqrt{d}]} = \left\{ \frac{a+\sqrt{d}b}{2} \,\middle|\, a \equiv b \bmod 2 \right\}$.

1.2 Embeddings, Norm and Trace

Recall: Let L/K be a finite field extension. If $char K = 0$, then L/K is separable. Let $\bar{K}$ be an algebraic closure of K. If L/K is seperable, them $[L : K] = \# \operatorname{Hom}_K(L, \bar{K})$.

Theorem
Let L/K *be a finite separable field extension. Then there exists an element* $\alpha \in L$ *such that* $L = K(\alpha)$*. In particular, for number fields* $Q \subseteq K \subseteq L$ *we obtain the following:*

- *There exists* $\alpha \in L$ *such that* $L = K(\alpha)$
- *If there is an embedding* $\hat{\imath} : K \hookrightarrow \mathbb{C}$*, then there exist* $[L : K]$ *embeddings* $L \hookrightarrow \mathbb{C}$*, which extend* $\hat{\imath}$*. If* $g(x)$ *is a minimal polynomial of* α *over* K *then*

the embeddings are given by $\sigma_i : \alpha \mapsto \beta_i$, where $\beta_1, \ldots, \beta_{[L:K]}$ are the $[L : K]$ distinct conjugates of α.

Example: 1. Let $d \in \mathbb{Z}$ be not a square. Then there are exactly two embeddings of $\mathbb{Q}[\sqrt{d}]$ into $\mathbb{C}$, namely $\sigma_1 : a + \sqrt{d}b \mapsto a + \sqrt{d}b$ and $\sigma_2 : a + \sqrt{d}b \mapsto a - \sqrt{d}b$.

2. We have $[\mathbb{Q}[\sqrt[3]{2} : \mathbb{Q}]] = 3$ and the three embeddings are given by

$$\sigma_1(\sqrt[3]{2}) = \sqrt[3]{2},\ \sigma_2(\sqrt[3]{2}) = e^{\frac{2\pi i}{3}}\sqrt[3]{2},\ \sigma_3(\sqrt[3]{2}) = e^{\frac{4\pi i}{3}}\sqrt[3]{2}.$$

Note that $\sigma_1(\mathbb{Q}[\sqrt[3]{2}]) \subseteq \mathbb{R}$, whereas σ_2 and σ_3 are "complex embeddings". $\mathbb{Q}[\sqrt[3]{2}]/\mathbb{Q}$ is not a normal extension.

Definition (Trace and norm)
Let K be a field and V an n-dimensional K-vector space. For $\varphi : V \to V$ a K-endomorphism, we define the characteristic polynomial

$$\chi_\varphi(x) = \det(xI_n - \varphi) = \sum_{i=0}^{n} c_i x^{n-i}$$

for some $c_0, \ldots, c_n \in K$. We define the determinant and trace of φ by $\det\varphi = (-1)^n c_n$ and $\operatorname{trace}\varphi = -c_1$

Note that if $\varphi, \psi : V \to V$ are both K-endomorphisms of V, then $\det(\varphi \circ \psi) = \det(\varphi)\det(\psi)$ and $\operatorname{trace}(a\varphi + b\psi) = a \operatorname{trace}(\varphi) + b \operatorname{trace}(\psi)\ \forall\, a, b \in K$.

Definition (Trace and norm of a number field)
Let $\mathbb{Q} \subseteq K \subseteq L$ be number fields and $\alpha \in L$. We write $\varphi_\alpha : L \to L,\ x \mapsto \alpha x$ and define the (relative) norm and trace of α by

$$N_{L/K}(\alpha) = \det\varphi_\alpha, \quad \mathrm{Tr}_{L/K}(\alpha) = \operatorname{trace}(\varphi_\alpha).$$

Remark: The map $N_{L/K} : L^* \to K^*$ is a grouphomomorphism as $N_{L/K}(\alpha\beta) = N_{L/K}(\alpha)N_{L/K}(\beta)\ \forall\, \alpha, \beta \in L \setminus \{0\}$. Similarly, $\mathrm{Tr}_{L/K} : L \to K$ is a K-linear map, as

$$\mathrm{Tr}_{L/K}(u\alpha + v\beta) = u\,\mathrm{Tr}_{L/K}(\alpha) + v\,\mathrm{Tr}_{L/K}(\beta)\ \forall\, u, v \in K,\ \alpha, \beta \in L.$$

Example: Let $K = \mathbb{Q},\ L = \mathbb{Q}(i)$ and $\alpha = a+ib \in \mathbb{Q}(i)$. Then φ_α can be represented with respect to the basis 1, i by

$$\varphi_\alpha = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

and hence

$$N_{L/\mathbb{Q}}(a+ib) = a^2 + b^2, \quad \mathrm{Tr}_{L/\mathbb{Q}}(a+ib) = 2a.$$

Lemma 1.7

Let L/K is an extension of number fields with $[L:K] = n$. For $a \in K$ we have

$$N_{L/K}(a) = a^n, \quad \mathrm{Tr}_{L/K} = na.$$

Lemma 1.8

Let L/K be an extension of number fields with $L = K(\alpha)$ and $[L:K] = n$. Let $f(x) = x^n + c_1 x^{n-1} + \cdots + c_n$ be the minimal polynomial of α over K. Then

$$N_{L/K}(\alpha) = (-1)^n c_n, \quad \mathrm{Tr}_{L/K}(\alpha) = -c_1.$$

Lemma 1.9

Let L/K be a number field extension, $\alpha \in L$, $[L : K(\alpha)] = r$. Then we have

$$N_{L/K}(\alpha) = \left(N_{K(\alpha/K)}(\alpha)\right)^r, \quad \mathrm{Tr}_{L/K}(\alpha) = r\,\mathrm{Tr}_{K(\alpha)/K}(\alpha).$$

Corollary 1.10

Let L/K be number fields and $\alpha \in \mathcal{O}_L$. Then $N_{L/K}(\alpha), \mathrm{Tr}_{L/K}(\alpha) \in \mathcal{O}_K$. In particular $N_{L/\mathbb{Q}}(\alpha), \mathrm{Tr}_{L/\mathbb{Q}} \in \mathbb{Z}$.

Theorem 1.11

Let L/K be number fields, $[L:K] = n$ and $\sigma_1, \dots, \sigma_n : L \hookrightarrow \mathbb{C}$ be the n distinct K-linear embeddings of L into $\mathbb{C}$. Then, for $\alpha \in L$, we have

$$N_{L/K}(\alpha) = \prod_{i=1}^n \sigma_i(\alpha), \quad \mathrm{Tr}_{L/K}(\alpha) = \sum_{i=1}^n \sigma_i(\alpha).$$

Corollary 1.12

Let L/K be a Galois extension of number fields. Then, for $\alpha \in L$ and $\sigma \in \mathrm{Gal}(L/K)$, *we have*

$$N_{L/K}(\sigma(\alpha)) = N_{L/K}(\alpha), \quad \mathrm{Tr}_{L/K}(\sigma(\alpha)) = \mathrm{Tr}_{L/K}(\alpha).$$

Theorem 1.13
Let $K \subseteq L \subseteq M$ be a tower of number fields and $\alpha \in M$. Then

$$N_{M/K} = N_{L/K}(N_{M/L}(\alpha)), \quad \mathrm{Tr}_{M/K}(\alpha) = \mathrm{Tr}_{L/K}(\mathrm{Tr}_{M/L}(\alpha)).$$

Lecture 3, 03.11.2023

An application of the norm map

Given a number field K with ring of integers $\mathcal{O}_K$, how can we find $\mathcal{O}_K^*$, i.e. the units in $\mathcal{O}_K$?

- If $\alpha \in \mathcal{O}_K^*$, $\alpha^{-1} \in \mathcal{O}_K$ and $1 = N_{K/\mathbb{Q}}(\alpha\alpha^{-1}) = N_{K/\mathbb{Q}}(\alpha) \cdot N_{K/\mathbb{Q}}(\alpha^{-1})$. By Corollary 1.10, $N_{K/\mathbb{Q}}(\alpha),\ N_{K/\mathbb{Q}}(\alpha^{-1}) \in \mathbb{Z} \implies N_{K/\mathbb{Q}}(\alpha) = \pm 1$.
- If $\alpha \in \mathcal{O}_K$ with $N_{K/\mathbb{Q}}(\alpha) = \pm 1$, then $\alpha \in \mathcal{O}_K^*$.

Example: Let $d \in \mathbb{Z}$, d squarefree. Then, for $a, b \in \mathbb{Q}$, $N_{\mathbb{Q}[\sqrt{d}]/\mathbb{Q}}(a + \sqrt{d}b) = (a + b\sqrt{d})(a - b\sqrt{d}) = a^2 - db^2$. For $d \equiv 2, 3 \bmod 4$, we find that

$$\mathcal{O}_{\mathbb{Q}[\sqrt{d}]} = \left\{ a + b\sqrt{d} \,\middle|\, a, b \in \mathbb{Z},\ a^2 - db^2 = \pm 1 \right\}.$$

The trace as a bilinear form

Let L/K be number fields. Then $\mathrm{Tr}_{L/K}$ induces a bilinear form

$$\mathrm{Tr}_{L/K} : L \times L \to K,\ (x, y) \mapsto \mathrm{Tr}_{L/K}(x \cdot y). \tag{1.2}$$

Write L^* for the dual vector space of L, i.e. the set of all K-linear vector space homomorphisms.

Theorem 1.14
The bilinear form (1.2) induces an isomorphism of K-vector spaces

$$\psi : L \to L^*,\ x \to \mathrm{Tr}_{L/K}(x, \cdot).$$

Corollary 1.15
Let L/K be number fields and $(v_1, \ldots, v_n)$ a K-basis with $n = [L : K]$. Then there exists a unique K-basis $(w_1, \ldots, w_n)$ of L, such that $\mathrm{Tr}_{L/K}(v_i w_j) = \delta_{ij},\ 1 \leq i, j, \leq n$.

1.3 Discriminant

Let $K/\mathbb{Q}$ be a number field of degree $n = [K : \mathbb{Q}]$ and $\sigma_1, \dots, \sigma_n : K \to \mathbb{C}$ its embeddings.

Definition (Discriminant)
For $\alpha_1, \dots, \alpha_n \in K$, we define the *discriminant* as

$$\mathrm{disc}(\alpha_1, \dots, \alpha_n) = \det\Big((\sigma_i(\alpha_j))_{1 \le i,j \le n}\Big)^2.$$

Theorem 1.16
Let $\alpha_1, \dots, \alpha_n \in K$. Then $\alpha_1, \dots, \alpha_n$ are $\mathbb{Q}$-linearly independent if and only if $\mathrm{disc}(\alpha_1, \dots, \alpha_n) \neq 0$.

Lemma 1.17
Let $\alpha_1, \dots, \alpha_n \in K$. Then

$$\mathrm{disc}(\alpha_1, \dots, \alpha_n) = \det\Big(\mathrm{Tr}_{K/\mathbb{Q}}(\alpha_i \alpha_j)\Big)_{1 \le i,j \le n}.$$

Corollary 1.18
Let $\alpha_1, \dots, \alpha_n \in K$. Then $\mathrm{disc}(\alpha_1, \dots, \alpha_n) \in \mathbb{Q}$. *If moreover $\alpha_1, \dots, \alpha_n \in \mathcal{O}_K$, then* $\mathrm{disc}(\alpha_1, \dots, \alpha_n) \in \mathbb{Z}$.

Theorem 1.19
Let α be algebraic over $\mathbb{Q}$ with $\Big[\mathbb{Q}[\alpha] : \mathbb{Q}\Big] = n$, and $\alpha_1, \dots, \alpha_n$ the n different conjugates of α. Then

$$\mathrm{disc}\Big(1, \alpha, \dots, \alpha^{n-1}\Big) = \prod_{1 \le i,j \le n} (a_i - a_j)^2.$$

If moreover $f(x)$ is the minimal polynomial of α over $\mathbb{Q}$, then

$$\mathrm{disc}\Big(1, \alpha, \dots, \alpha^{n-1}\Big) = (-1)^{\frac{n(n-1)}{2}} N_{\mathbb{Q}[\alpha]/\mathbb{Q}}\Big((f'(\alpha)\Big).$$

Question: Let K be a number field with ring of integers $\mathcal{O}_K$ and of degree $n = [K : \mathbb{Q}]$. Then K is an n-dimensional $\mathbb{Q}$-vector space. How can we describe the structure of the group $(\mathcal{O}_K, +)$?

Example: For $d \in \mathbb{Z}$ squarefree and $K = \mathbb{Q}\left[\sqrt{d}\right]$, the ring of integers $\mathcal{O}_K$ is a free abelian group of rank 2, where a $\mathbb{Z}$-basis is given by $(1,\ \omega)$, with

$$\omega = \begin{cases} \sqrt{d} & d \equiv 2, 3 \bmod 4, \\ \frac{1+\sqrt{d}}{2} & d \equiv 1 \bmod 4. \end{cases}$$

Theorem 1.20
Let $K/\mathbb{Q}$ be a number field of degree $n = [K : \mathbb{Q}]$. Then $\mathcal{O}_K$ is a free abelian group of rank n, i.e. there exists $\alpha_1, \dots, \alpha_n \in \mathcal{O}_K$, such that every $\beta \in \mathcal{O}_K$ can be uniquely written in the form

$$\beta = m_1\alpha_1 + \cdots + m_n\alpha_n$$

with $m_1, \dots, m_n \in \mathbb{Z}$.

Remark: In the notation of Theorem 1.20, we call $(\alpha_1, \dots, \alpha_n)$ and integral basis of $\mathcal{O}_K$ (over $\mathbb{Z}$).

Lecture 4, 07.11.2023

Lemma 1.21
Let K be a number field as above. Then there exists a $\mathbb{Q}$-basis of the number field, say $(\alpha_1, \dots, \alpha_n)$, with $\alpha_1, \dots, \alpha_n \in \mathcal{O}_K$.

Proposition 1.22
Let $(\alpha_1, \dots, \alpha_n)$ be a $\mathbb{Q}$-basis of a number field K with $\alpha_1, \dots, \alpha_n \in \mathcal{O}_K$, $d = \operatorname{disc}(\alpha_1, \dots, \alpha_n)$ and $\beta \in \mathcal{O}_K$. Then there exist $m_1, \dots, m_n \in \mathbb{Z}$, such that

$$\beta = \frac{m_1\alpha_1 + \cdots + m_n\alpha_n}{d}$$

and $d \mid m_i^2$ for $1 \leq i \leq n$.

Lemma 1.23
Let K be a number field with integral bases $(\alpha_1, \dots, \alpha_n)$ and $(\beta_1, \dots, \beta_n)$. Then

$$\operatorname{disc}(\alpha_1, \dots, \alpha_n) = \operatorname{disc}(\beta_1, \dots, \beta_n).$$

Definition (Discriminant of K)
Let K be a number field and $(\alpha_1, \dots, \alpha_n)$ a $\mathbb{Z}$-basis for $\mathcal{O}_K$. We define the *discriminant*

disc(K) of K as

$$\operatorname{disc}(K) = \operatorname{disc}(\alpha_1, \dots, \alpha_n).$$

Example: Let $d \in \mathbb{Z}$ be squarefree. Then

$$\operatorname{disc}\left(\left[\sqrt{d}\right]\right) = \begin{cases} 4d & d \equiv 2,3 \mod 4, \\ d & d \equiv 1 \mod 4. \end{cases}$$

1.4 Cyclotomic fields

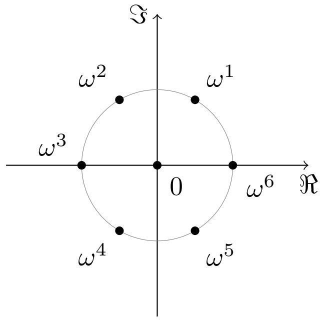
Definition
For $m \in \mathbb{N}$ we call $\mathbb{Q}\left[e^{\frac{2\pi i}{m}}\right]$ the m-th cyclotomic field.

Example:

- The first two cyclotomic fields are equal to $\mathbb{Q}$.
- Let $m = 6$ and write $\omega = e^{\frac{2\pi i}{6}}$. Then $\omega^5 = -\omega^2$, i.e. $\omega = -\omega^4$ and $\mathbb{Q}[\omega] = \mathbb{Q}[\omega^2]$. This means that the third and sixth cyclotomic fields are equal.

In the following let $m \in \mathbb{N}$ and write $\omega = e^{\frac{2\pi i}{m}}$.

Theorem 1.24
The extension $\mathbb{Q}[\omega]$ over $\mathbb{Q}$ is Galois with degree equal to $\varphi(m)$, where φ is Euler's totient function. Moreover, the Galois group is isomorphic to $(\mathbb{Z}/m\mathbb{Z})^* = \{k \in \mathbb{Z}/m\mathbb{Z} \mid \gcd(k,m) = 1\}$.



For $k \in (\mathbb{Z}/m\mathbb{Z})^*$ the corresponding automorphism is given by $\omega \mapsto \omega^k$.

Proposition 1.25
The conjugates of ω are exactly given by ω^k with $\gcd(m,k) = 1$.

Corollary 1.26
Let $m \in \mathbb{N}$ be even. Then the roots of unity contained in $\mathbb{Q}(e^{\frac{2\pi i}{m}})$ are exactly the m-th roots of unity.

Corollary 1.27
The m-th cyclotomic fields, for m even, are all non-isomorphic.

Theorem 1.28
Let $m = p^r$ for some prime p and $\omega = e^{\frac{2\pi i}{m}}$. Then $\mathcal{O}_{Q[\omega]} = \mathbb{Z}[\omega]$.

Remark: More generally, $Z[\omega] = \mathcal{O}_{Q[\omega]}$ for *every* cyclotomic field.

Notation: We write $\operatorname{disc}(\alpha) = \operatorname{disc}(1, \alpha, \ldots, \alpha^{n-1})$.

Lemma 1.29
For $m \in \mathbb{N}$ we have $\operatorname{disc}(\omega) \mid m^{\varphi(m)}$.

Lecture 5, 10.11.2023

Lemma 1.30
For $m \geq 3$ we have $\operatorname{disc}(1-\omega) = \operatorname{disc}(\omega)$.

Lemma 1.31
Let $m = p^r$ be a prime power, $r \in \mathbb{N}$. Then

$$\prod_{\substack{1 \leq k \leq m \\ \gcd(k,m)=1}} \left(1 - \omega^k\right) = p.$$

Remark: In particular for $m = p^r$ we have $\frac{p}{(1-\omega)^{\varphi(m)}} \in \mathbb{Z}[\omega]$.

2 Prime ideal factorisation

2.1 Unique prime ideal factorisation

Motivation: If K is a number field with ring of integers $\mathcal{O}_K$, then we may not have a unique factorisation in $\mathcal{O}_K$ into irreducible elements (up to units and ordering).

Example: Let $K = \mathbb{Q}(\sqrt{-5})$, then $\mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$. In $\mathbb{Z}[\sqrt{-5}]$ we have $2 \cdot 3 = 6 = (1+\sqrt{-5})(1-\sqrt{-5})$, where 2 and 3 are irreducible elements.

Our next goal is to replace factorisation into irreducible elements by prime *ideal* factorisation. Instead of number fields, we consider more generally *Dedekind domains*.

Definition (Integrally closed ring)
Let R be an integral domain and $K = \left\{ \frac{a}{b} \,\middle|\, a, b \in R,\ b \neq 0 \right\}$ its field of fractions. We call R *integrally closed*, if every element $\frac{a}{b} \in K$, which is a zero of a monic polynomial with coefficients in R is contained in R.

Example: Let K be a number field with ring of integers $\mathcal{O}_K$. Then $\mathcal{O}_K$ is integrally closed. Indeed let $\alpha \in K$ satisfy $\alpha^n + b_1\alpha^{n-1} + \cdots + b_n = 0$, with $b_1, \ldots, b_n \in \mathcal{O}_K$. Then $\mathbb{Z}[\alpha, b_1, \ldots, b_n]$ is finitely generated as an additive group and we have $\alpha \in \mathcal{O}_K$.

Definition (Noetherian[1] ring)
We call a commutative ring R *noetherian* if every ideal is finitely generated.

Remark: The following statements about a commutative ring R are equivalent:

1. R is noetherian.
2. Every increasing sequence of ideals is eventually constant, i.e. if $I_1 \subseteq I_2 \subseteq \ldots$, then there is some $n_0 \in \mathbb{N}$, such that $I_n = I_{n_0}$ for every $n > n_0$.
3. Every non-empty set S of ideals has a maximal element, i.e. there is some $M \in S$, such that if $M' \in S$ with $M \subseteq M'$, then $M = M'$.

[1] after Emmy Noether (1882 - 1935), a German mathematician

Example: Principal ideal domains and polynomial rings $\mathbb{Z}[x_1, \ldots, x_n]$ or $K[x_1, \ldots, x_n]$ for any field K are noetherian.

Definition (Dedekind[2] domain)
A *Dedekind domain* is a noetherian integrally closed domain, in which every non-zero prime ideal is maximal.

Theorem 2.1
Let K be a number field. Then its ring of integers $\mathcal{O}_K$ is a Dedekind domain.

Example: Coordinate rings of irreducible smooth curves over an algebraically closed field, e.g. $\mathbb{C}[T]$ is a Dedekind domain.

First properties of Dedekind domains

Lemma 2.2
Let R be a Dedekind domain, which is not a field, and $0 \neq I \subseteq R$ an ideal. Then I contains a product of non-zero prime ideals $P_1 \cdots P_k \subseteq I$.

Lemma 2.3
Let R be a Dedekind domain with field of fractions K and $0 \neq I \subsetneq R$ a ideal. Then there exists $\alpha \in K \setminus R$ with $\alpha I \subseteq R$.

Lecture 6, 17.11.2023

Theorem 2.4
Let R be a Dedekind domain and $0 \neq I \subseteq R$ an ideal. Then there is an ideal $0 \neq J \subseteq R$, such that IJ is principal.

Example: Let $R = \mathbb{Z}\left[\sqrt{-5}\right]$ and $I = \left(2, 1 + \sqrt{-5}\right)$. Then I is not principal, but $\left(2, 1 + \sqrt{-5}\right)\left(2, 1 - \sqrt{-5}\right) = (2)$ is principal.

Observation: Note that $\alpha \in I$ implies that $J \subset A = \frac{1}{\alpha} IJ$. Hence $\gamma JI = \gamma\alpha\left(\frac{1}{\alpha}JI\right) = \alpha\gamma A \subseteq (\alpha)$. As $\gamma J \subseteq \gamma A \subseteq R$, we find that $\gamma J \subseteq J$.

[2] after Richard Dedekind (1831 - 1916), a German mathematician

The ideal class group

Definition (Equivalence of ideals)
Let R be an integral domain. We say that two non-zero ideals I, J are equivalent if and only if there exist $\alpha, \beta \in R \setminus \{0\}$ with $\alpha I = \beta J$.

Remark:

1. This really is an equivalence relation. We call the equivalence classes under this relation *ideal classes.*
2. We can define a multiplication on the set of ideal classes by multiplication of representatives, $[I][J] = [IJ]$, with the neutral element $[R]$.
3. All principal ideals form one ideal class.

Corollary 2.5
Let R be a Dedekind domain. Then the ideal classes form a group under multiplication.

Definition (Ideal class group)
We call the group given by ideal classes under multiplication in the Dedekind domain R the *ideal class group* of R, denoted by $Cl(R)$.

Example: $\mathbb{Z}$ is a principal ideal domain, hence $|Cl(\mathbb{Z})| = 1$.

Remark: There are only finitely many imaginary quadratic fields K with $|Cl(\mathcal{O}_K)| = 1$.

Question (Gauss)**:** Do there exist as many real quadratic number fields K with $|Cl(\mathcal{O}_K)| = 1$?

Corollary 2.6
Let R be a Dedekind domain and A, B, C ideals with $A \neq 0$.

1. *If $AB = AC$ then $B = C$.*
2. *We have $B \mid A$, i.e. $A = BJ$ for some ideal J, if and only if $A \subseteq B$.*

Theorem 2.7 (Unique prime ideal factorisation)
Every ideal $I \neq 0$ in a Dedekind domain R can be written as a product $I = P_1 \cdots P_r$

with non-zero prime ideals $P_1, \dots, P_r$ and this representation is unique up to ordering of $P_1, \dots, P_r$.

Example: In $\mathbb{Z}\left(\sqrt{-5}\right)$ we don't have unique factorisation into reducible elements, e.g. $2 \cdot 3 = \left(1+\sqrt{-5}\right)\left(1-\sqrt{-5}\right)$, but in terms of ideals we have $(2) = \left(2, 1+\sqrt{-5}\right)^2 = P_1^2$, $(3) = \left(3, 1+\sqrt{-5}\right)\left(3, 1-\sqrt{-5}\right) = P_2 \cdot P_3$. Note that P_1, P_2, P_3 are all prime ideals as $\left|\mathbb{Z}\left[\sqrt{-5}\right]/P_i\right| \in \{2, 3\}$ for $1 \leq i \leq 3$. In the ideal class group we find that

$$\begin{aligned}(2) \cdot (3) &= P_1^2 P_2 P_3 \\ &= P_1 P_2 P_1 P_3 \\ &= \left(1+\sqrt{-5}\right)\left(1-\sqrt{-5}\right).\end{aligned}$$

Definition (Greatest common divisor, least common multiple)
Let R be a Dedekind domain and $I, J \neq 0$ ideals with prime factorisation

$$I = \prod_{i=1}^{r} P_1^{a_i}, \;\; J = \prod_{i=1}^{r} P_i^{b_i},$$

where $P_1, \dots, P_r$ are distinct prime ideals and $a_1, \dots, a_r, b_1, \dots, b_r \in \mathbb{Z}_{\geq 0}$. We define the *greatest common divisor* $\gcd(I, J)$ and *least common multiple* $\operatorname{lcm}(I, J)$ by

$$\gcd(I, J) = \prod_{i=1}^{r} P_i^{\min(a_i, b_i)}, \quad \operatorname{lcm}(I, J) = \prod_{i=1}^{r} P_i^{\max(a_i, b_i)}.$$

Exercise
Show that

$$\gcd(I, J) = I + J, \quad \operatorname{lcm}(I, J) = I \cap J.$$

Question: Given the ring of integers $\mathcal{O}_K$ in a number field K, we know that every ideal is finitely generated. Can we say something about the numbers of generators we need? E.g. in $\mathbb{Z}[\sqrt{-5}]$, the prime ideal $(2, 1+\sqrt{-5})$ is not a principal idea, but generated by two elements.

Lecture 7, 21.11.2023

Remark: Chinese Remainder Theorem: Let R be a commutatiove ring with 1 and

$a_1, \dots, a_n$ coprime ideals, i.e. $a_i + a_j = R \; \forall \, i \neq j$. Then there is an isomorphism

$$R/\bigcap_{i=1}^{n} a_i \to R/a_1 \times \cdots \times R/a_n.$$

Theorem 2.8

Let R be a Dedekind domain, $I \subseteq R$ a non-zero ideal and $\alpha \in I \setminus \{0\}$. Then there exists $\beta \in I$ with $I = (\alpha, \beta)$.

Corollary 2.9

A Dedekind domain is a unique factorisation domain (UFD) if and only if is is a principal ideal domain (PID).

Remark: In general, a PID is a UFD but the reverse implication does not hold. For example $\mathbb{Z}[x]$ is a UFD, but not a PID.

2.2 Splitting of primes

Let p be a (rational) prime number. Then (p) is a prime ideal in $\mathbb{Z}$, but the ideal $(p) = p\mathcal{O}_K$ need not be a prime ideal in $\mathcal{O}_K$. For example, let $p \equiv 1 \bmod 4$, then in $\mathbb{Z}[i]$ we have

$$(p) = (a + ib)(a - ib), \tag{2.1}$$

where $a^2 + b^2 = p$ with $a, b \in \mathbb{Z}$. Note that $N_{\mathbb{Q}[i]/\mathbb{Q}}(a + ib) = p$ and hence $a + ib$ is a prime element in the PID $\mathbb{Z}[i]$, and (2.1) is the prime ideal factorisation of (p). Moreover, $a + ib$ and $a - ib$ do not differ by multiplication with one of the units $\pm 1, \pm i$, and hence

$$P_1 = (a + ib) \neq (a - ib) = P_2$$

in $\mathbb{Z}[i]$. The ideal (2) splits in $\mathbb{Z}[i]$ as $2 = (1 + i)^2$, where $(1 + i)$ is a prime ideal. If $p \equiv 3 \bmod 4$ is a rational prime, then (p) remains a prime ideal in $\mathbb{Z}[i]$. (check!)

Question: More generally, let $K \subseteq L$ be number fields with rings of integers $\mathcal{O}_K, \mathcal{O}_L$. Given a non-zero prime ideal P in $\mathcal{O}_K$, how does $P\mathcal{O}_L$ split into prime ideals in $\mathcal{O}_L$?

Notation: In the following, we keep the notation $K \subseteq L$, $\mathcal{O}_K \subseteq \mathcal{O}_L$ as above.

Definition (Primes)

We say that $P \subseteq \mathcal{O}_K$ or $Q \subseteq \mathcal{O}_L$ is a *prime* if P or respectively Q is a non-zero prime ideal in $\mathcal{O}_K$ or respectively $\mathcal{O}_L$. Moreover, we say that Q *lies above* P or P *lies under* Q if $Q \mid P\mathcal{O}_L$.

Lemma 2.10
Let P resp. Q be primes in $\mathcal{O}_K$ resp. $\mathcal{O}_L$. Then Q lies above P if and only if one of the following equivalent conditions holds:

1. $P\mathcal{O}_L \subseteq Q$.
2. $P \subseteq Q$.
3. $Q \cap \mathcal{O}_K = P$.
4. $Q \cap K = P$.

Theorem 2.11
Every prime Q in $\mathcal{O}_L$ lies above a unique prime P in $\mathcal{O}_K$ and for every prime P in $\mathcal{O}_K$ there is some prime Q in $\mathcal{O}_L$, which lies above P.

Lemma 2.12
Let Q be a prime in $\mathcal{O}_L$ lying above P in $\mathcal{O}_K$. Then $\mathcal{O}_L/Q$ and $\mathcal{O}_K/P$ are finite fields with $\mathcal{O}_K/P \hookrightarrow \mathcal{O}_L/Q$.

Let P be a prime in $\mathcal{O}_K$ and consider in $\mathcal{O}_L$ the prime ideal factorisation

$$P\mathcal{O}_L = \prod_{i=1}^{r} Q_i^{e_i}$$

with distinct primes $Q_1, \ldots, Q_r$.

Definition (Ramification index, inertia degree)
We call

$$e_i = e(Q_i \mid P)$$

the *ramification index* of Q_i above P and

$$f_i = f(Q_i \mid P) = \left[\mathcal{O}_L/Q_i : \mathcal{O}_K/P\right]$$

the *inertia degree* of Q_i over P. Moreover, we call $\mathcal{O}_L/Q_i$ and $\mathcal{O}_K/P$ *residue fields* of Q_i or respectively P.

Remark: Let $K \subseteq L \subseteq M$ be number fields with primes $P \subseteq Q \subseteq R$. Then

$$e(R \mid P) = e(R \mid Q)e(Q \mid P), \quad f(R \mid P) = f(R \mid Q)f(Q \mid P).$$

Example: Let $K = \mathbb{Q}$, $L = \mathbb{Q}(i)$. If p is a rational prime with $p \equiv 1 \bmod 4$, then $(p) = P_1 \cdot P_2$, $P_1 = (a + ib)$, $P_2 = (a - ib)$ for some $a, b \in \mathbb{Z}$. We have

$$e\Big(P_i \mid (p)\Big) = 1, \quad f\Big(P_i \mid (p)\Big) = 1.$$

For a rational prime $p \equiv 3 \bmod 4$ we obtain

$$e\Big(\underbrace{(p)}_{\subseteq \mathbb{Z}[i]} \mid \underbrace{(p)}_{\subseteq \mathbb{Z}}\Big) = 1, \quad f\Big((p) \mid (p)\Big) = 2.$$

For $p = 2$ note that $(2) = (1+i)^2$ and $\Big|\mathbb{Z}[i] \mid (1+i)\Big| = 2$, hence

$$e\Big((1+i) \mid (2)\Big) = 2, \quad f\Big((1+i) \mid (2)\Big) = 1.$$

In this example, independent of the rational prime p we find that

$$\sum_{i=1}^{r} e_i f_i = \Big[\mathbb{Q}(i) : \mathbb{Q}\Big].$$

Our goal now is to show the above statement for number fields $K \subseteq L$. Lecture 8, 24.11.2023

Norms of ideals

Definition (Norm of an ideal)
Let K be a number field and $I \subseteq \mathcal{O}_K$ a non-zero ideal. Then we define the *norm* $N(I)$ of the ideal I as

$$N(I) := \Big|\mathcal{O}_K / I\Big|.$$

Lemma 2.13
Let $I, J \subseteq \mathcal{O}_K$ be non-zero ideals. Then

$$N(IJ) = N(I)N(J).$$

Proposition 2.14
Let K be a number field of degree $n = [K : \mathbb{Q}]$ and $p \in \mathbb{Z}$ a prime with prime ideal

factorisation

$$(p) = \prod_{i=1}^{r} P_i^{e_i}$$

in $\mathcal{O}_K$ *and* $f_i = f(P_i \mid p)$ *for* $1 \leq i \leq r$. *Then*

$$\sum_{i=1}^{r} e_i f_i = n.$$

Next, we will look at general number field extensions $L \subseteq K$. We start with some preparations:

Lemma 2.15
Let $0 \neq B \subseteq A \subsetneq R$ *be ideals in a Dedekind domain* R. *Then there exists* $\alpha \in K = Quot(R)$, *such that*

$$\alpha B \subseteq R, \text{ but } \alpha B \subsetneq A.$$

Lemma 2.16
Let $I \neq 0$ *be an ideal in* $\mathcal{O}_K$ *and* $n = [L : K]$. *Then*

$$N(I\mathcal{O}_L) = N(I)^n.$$

Example: For $K = \mathbb{Q}$ we have already used this identity above, in which case it reduces to

$$N\big((p)\big) = p^n,$$

with $(p) \subseteq \mathcal{O}_L$ and p a rational prime.

Theorem 2.17
Let P *be a prime in* $\mathcal{O}_K$ *and* $P\mathcal{O}_L = \prod_{i=1}^{r} Q_i^{e_i}$ *the prime ideal factorisation in* $\mathcal{O}_L$ *with distinct ideals* $Q_1, \ldots, Q_r$ *and inertia degrees* $f_i = f(Q_i \mid P)$. *Then*

$$[L : K] = \sum_{i=1}^{r} e_i f_i.$$

Example: (a) Let p be a rational prime and $\omega = e^{\frac{2\pi i}{p^r}}$ for some $r \in \mathbb{N}$. By Lemma 1.31 we have

$$p = \prod_{\substack{1 \leq k \leq m \\ \gcd(k,m)=1}} \left(1 - \omega^k\right).$$

We show on the exercise sheet that for $p \not| k$

$$(1-\omega^k) = u_k(1-\omega)$$

for some $u_k \in \mathbb{Z}[\omega]$. Hence in $\mathbb{Z}[\omega]$ we have

$$(p) = (1-\omega)^{\varphi(p^r)}.$$

By Theorem 2.17, we deduce that $(1-\omega)$ is a prime ideal in $\mathbb{Z}[\omega]$ and

$$f\Big((1-\omega) \mid (p)\Big) = 1$$

(b) Let α be a root of $\alpha^3 = \alpha + 1$. Then $\mathbb{Q}(\alpha)/\mathbb{Q}$ is an extension of degree 3. One can compute $\operatorname{disc}(1, \alpha, \alpha^2) = -23$. As 23 is square-free, we find that $\mathcal{O}_{\mathbb{Q}(\alpha)} = \mathbb{Z}[\alpha]$ with integral basis $(1, \alpha, \alpha^2)$. Moreover, in $\mathbb{Z}[\alpha]$, we have

$$23 \cdot \mathbb{Z}[\alpha] = (23, \alpha - 10)^2 (23, \alpha - 3), \tag{2.2}$$

where $(23, \alpha - 10)$ and $(23, \alpha - 3)$ are coprime. Hence (2.2) is the prime ideal factorisation of (23) in $\mathbb{Z}[\alpha]$ and

$$f\Big((23, \alpha - 10) \mid 23\Big) = f\Big((23, \alpha - 3) \mid 23\Big) = 1.$$

Remark: In these examples we have found ramification indices $e > 1$, which however is not the "typical" case, as we will see below.

Definition (Ramified prime)
Let P be a prime in $\mathcal{O}_K$. We say that P is *ramified in* $\mathcal{O}_L$, if there is a prime Q in $\mathcal{O}_L$, lying above P, with

$$e(Q \mid P) > 1.$$

Theorem 2.18
Let p be a rational prime (i.e. a prime number in $\mathbb{Z}$), which is ramified in $\mathcal{O}_K$. Then

$$p \mid \operatorname{disc}(\mathcal{O}_K).$$

Remark: One can even show, that $p \mid \operatorname{disc}(\mathcal{O}_K)$ imlies that p is ramified in $\mathcal{O}_K$.

Corollary 2.19
There are only finitely many primes P in $\mathcal{O}_K$ which are ramified in $\mathcal{O}_L$.

Lecture 9, 28.11.2023

Galois extensions

In the proof of Theorem 2.18 we noted that if $L/\mathbb{Q}$ is a Galois extension and Q a prime in $\mathcal{O}_L$ above $p \in \mathbb{Z}$, so is the ideal $\sigma(Q)$ for all $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$.

Theorem 2.20
Let L/K be Galois and Q a prime in $\mathcal{O}_L$ lying above the prime P in $\mathcal{O}_L$. Then $\sigma(Q)$ is a prime above P for every $\sigma \in \mathrm{Gal}(L/K)$. Moreover, if Q' is another prime in $\mathcal{O}_L$ over P, then there exists an automorphism $\sigma \in \mathrm{Gal}(L/K)$ with $\sigma(Q) = Q'$.

Example: $K = \mathbb{Q},\ L = \mathbb{Q}(i),\ p \in \mathbb{Z}$ a prime with $p \equiv 1 \bmod 4$. Write $p = a^2 + b^2$ with $a, b \in \mathbb{Z}$. In $\mathbb{Z}[i]$ we have $(p) = (a + ib)(a - ib)$.

Corollary 2.21
Let L/K be a Galois extension, P a prime in $\mathcal{O}_K$ and Q_1, Q_2 primes in $\mathcal{O}_L$ lying above P. Then

$$e(Q_1 \mid P) = e(Q_2 \mid P), \quad f(Q_1 \mid P) = f(Q_2 \mid P).$$

Remark: In the notation above, we hence obtain

$$P\mathcal{O}_L = (Q_1 \cdots Q_r)^e \text{ with } f(Q_i \mid P) = f(Q_j \mid P).$$

Question: Let L/K be any number fields (not necessarily Galois) and P a prime in $\mathcal{O}_K$. Find explicitly the factorisation

$$P\mathcal{O}_L = \prod_{i=1}^{r} Q_i^{e_i}$$

with $Q_1, \ldots, Q_r$ prime.

Example: Let $m \in \mathbb{Z} \setminus \{1\}$ be odd and square-free and let $K = \mathbb{Q}$, $L = \mathbb{Q}(\sqrt{m})$. Consider an odd prime $p \in \mathbb{Z}$ with $p \nmid m$. By Theorem 2.18, p is not ramified in $\mathcal{O}_K$ as $\mathrm{disc}(K) \in \{m, 4m\}$. Hence we either have $p\mathcal{O}_L = Q_iQ_2$ with distinct primes

Q_1, Q_2 and $f(Q_i \mid p) = 1$ for $i = 1, 2$, *or* $p\mathcal{O}_L$ is prime with $f(p\mathcal{O}_L \mid p) = 2$.

Let Q be a prime above p. Consider the polynomial $g(X) = X^2 - m$. Then $g(X)$ has a zero in $\mathcal{O}_L$ and hence a zero in $\mathcal{O}_L/Q$.

1. If m is not a square modulo p, then $X^2 - m$ has no zero in $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z} \hookrightarrow \mathcal{O}_L/Q$ is a non-trivial field extension, i.e. $f(Q \mid p) = 2$.

2. Let $a \in \mathbb{Z}$ be a solution to $a^2 - m \equiv 0 \bmod p$. Then in $\mathcal{O}_L$ we have the factorisation $(a - \sqrt{m})(a + \sqrt{m}) \in p\mathcal{O}_L$ and in fact

$$(p, a - \sqrt{m})(p, a + \sqrt{m}) = p\mathcal{O}_L. \tag{2.3}$$

As neither of the factors $(p, a - \sqrt{m}), (p, a + \sqrt{m})$ contains 1, and $p\mathcal{O}_L$ factors into a product of at most two primes, we have already found in (2.3) the prime ideal factorisation of $p\mathcal{O}_L$ and

$$f\Big((p, a \pm \sqrt{m}) \mid p\Big) = 1.$$

More generally, let L/K be number fields, say of degree $n = [L : K]$. Fix an element $\alpha \in \mathcal{O}_L$, such that $L = K(\alpha)$. Note, that by Proposition 1.22 the quotient $\mathcal{O}_L/\mathcal{O}_K[\alpha]$ is finite. Let $g(X) \in \mathcal{O}_K[X]$ be the minimal polynomial of α over K.

Theorem 2.22
With notation as above, let P be a prime in $\mathcal{O}_K$ and factor $g(X)$ in $(\mathcal{O}_K/P)[X]$ as

$$g(X) \equiv g_1(X)^{e_1} \cdots g_r(X)^{e_r} \bmod P[X],$$

where $g_1(X), \ldots, g_r(X) \in \mathcal{O}_K[X]$ are monic polynomials, pairwise distinct and irreducible in $(\mathcal{O}_K/P)[X]$. Let $(p) \in P \cap \mathbb{Z}$ and assume $p \nmid \Big|\mathcal{O}_L/\mathcal{O}_K[\alpha]\Big|$. Then we have the factorisation

$$P\mathcal{O}_L = \prod_{i=1}^{r} Q_i^{i_i},$$

where $Q_i = (P, g_i(\alpha))$ is a prime and $f(Q_i \mid P) = \deg g_i$ for $1 \leq i \leq r$.

Example: Let α be a root of $\alpha^3 - \alpha - 1 = 0$. We have from earlier that $\mathcal{O}_{\mathbb{Q}[\alpha]} = \mathbb{Z}[\alpha]$ and $\operatorname{disc}(\mathbb{Q}[\alpha]) = -23$. Modulo 23 we find that Lecture 10, 01.12.2023

$$X^3 - X - 1 \equiv (X - 10)^2(X - 3)$$

and hence by Theorem 2.22

$$23\mathbb{Z}[\alpha] = (23, \alpha - 10)^2(23, \alpha - 3).$$

3 Number fields - Dirichlet's unit theorem, class groups and lattices

3.1 Finiteness of the ideal class group

Let K be a number field with ring of integers $\mathcal{O}_K$. We will keep this notation throughout this chapter.

Recall: We call two non-zero ideals $I, J \subseteq \mathcal{O}_K$ equivalent, if $\exists\, \alpha, \beta \in \mathcal{O}_K \setminus \{0\}$, such that $\alpha I = \beta J$, and we write $Cl(\mathcal{O}_K)$ for the group of equivalence classes under multiplication.

Question: Is $Cl(\mathcal{O}_K)$ finite?

Theorem 3.1
For every number field K there is a constant C_K, such that every non-zero ideal I contains an element $\alpha \in I \setminus \{0\}$ with

$$\left|N_{K/\mathbb{Q}}(\alpha)\right| \leq C_K N(I).$$

Corollary 3.2
Let K and C_K be as in Theorem 3.1. Then every ideal class $C \in Cl(\mathcal{O}_K)$ contains an ideal I with $N(I) \leq C_K$.

Corollary 3.3
For every number field K we have $|Cl(\mathcal{O}_K)| < \infty$.

Example: Let $K = \mathbb{Q}[\sqrt{2}]$, i.e. $\mathcal{O}_K = \mathbb{Z}[\sqrt{2}]$. As in the proof of Theorem 3.1, we can take $C_K = (1+\sqrt{2})^2$ (using the integral basis $(1, \sqrt{2})$). Note that $(1+\sqrt{2})^2 < 6$. We consider the prime ideals in $\mathbb{Z}[\sqrt{2}]$, which lie above 2, 3, 5. Note that $2\mathbb{Z}[\sqrt{2}] = (\sqrt{2})^2$ and that (3), (5) are prime ideals (see Theorem 2.22, noting that $X^2 - 2$ remains irreducible modulo 3, 5). Hence $\left|Cl(\mathbb{Z}[\sqrt{2}])\right| = 1$.

Remark: In the example above and other examples, we would like to take C_K as small as possible.

Our next goal will be to find improvements for the value of C_K using results from the geometry of numbers.

Idea: Let K be a number field of degree n, $\sigma_1, \ldots, \sigma_r : K \hookrightarrow \mathbb{R}$ its real embeddings and $\tau_1, \bar{\tau}_1, \tau_2, \bar{\tau}_2, \ldots, \tau_s, \bar{\tau}_s : K \hookrightarrow \mathbb{C}$ its different complex embedings, where we sort them into pairs $\tau_i, \bar{\tau}_i$, which differ by complex conjugations. Then $n = r + 2s$ and we can define an injective map

$$\varphi : K \to \mathbb{R}^n, \quad \alpha \mapsto \Big(\sigma_1(\alpha), \ldots, \sigma_r(\alpha), \Re\tau_1(\alpha)\Im\tau_1(\alpha), \ldots, \Re\tau_s(\alpha), \Im\tau_s(\alpha)\Big).$$

Let $(\alpha_1, \ldots, \alpha_n)$ be an integral basis of $\mathcal{O}_K$. Then we can view $\varphi(\mathcal{O}_K) = \mathbb{Z}\varphi(\alpha_1) + \cdots + \mathbb{Z}\varphi(\alpha_n) \subseteq \mathbb{R}^n$ as an additive group. Also, if $I \subseteq \mathcal{O}_K$ is a non-zero ideal, then I is a free $\mathbb{Z}$-module of rank n, say with basis $(\beta_1, \ldots, \beta_n)$. Then

$$\varphi(I) = \mathbb{Z}\varphi(\beta_1) + \cdots + \mathbb{Z}\varphi(\beta_n) \subseteq \mathbb{R}^n$$

and we can interpret $\varphi(I)$ as a *lattice* in $\mathbb{R}^n$. In order to improve upon C_K in Theorem 3.1, we would like to find a "small" non-zero element in this lattice.

Lecture 11, 05.12.2023

3.2 Geometry of numbers

Motivation: Consider a lattice L, e.g. $\mathbb{Z}^n \subseteq \mathbb{R}^n$, and a "nice" subset $C \subseteq \mathbb{R}^n$, e.g. a ball of radius r. When does C contain a point in $L \setminus \{0\}$?

Definition (Lattice)
Let $v_1, \ldots, v_n \in \mathbb{R}^n$ be linearly independent vectors (over $\mathbb{R}$). Then we call the group

$$L = \{z_1 v_1 + \cdots + z_n v_n \mid z_1, \ldots, z_n \in \mathbb{Z}\} \subseteq \mathbb{R}^n$$

a (full) *lattice* in $\mathbb{R}^n$ and $v_1, \ldots, v_n$ a basis of L. We define the determinant $d(L)$ of the lattice L as

$$d(L) = |\det(v_1, \ldots, v_n)|.$$

Remark: As additive groups we have $L \cong \mathbb{Z}^n$. If $x \in L$ and $v_1, \ldots, v_n$ as above, then there is exactly one way to write x as $\sum_{i=1}^n x_i v_i$ with $x_1, \ldots, x_n \in \mathbb{Z}$.

Notation: We write $M_{n\times n}(\mathbb{Z})$ for the set of $n \times n$ matrices with coefficients in $\mathbb{Z}$. and $GL(n,\mathbb{Z}) = \{A \in M_{n\times n}(\mathbb{Z}) \mid \det(M) = \pm 1\}$ for the group of invertible matrices in $M_{n\times n}(\mathbb{Z})$.

Lemma 3.4
Let $L \subseteq \mathbb{R}^n$ be a lattice and $\{v_1, \ldots, v_n\}$, $\{w_1, \ldots, w_n\}$ bases of L. Then there exists a matrix $A \in GL(n,\mathbb{Z})$, say $A = (a_{i,j})_{1\leq i,j\leq n}$, such that

$$w_i = \sum_{i=1}^n a_{i,j} v_j, \quad 1 \leq i \leq n.$$

Moreover,

$$|\det(v_1, \ldots, v_n)| = |\det(w_1, \ldots, w_n)|.$$

Remark: In particular, the determinant $d(L)$ of a lattice $L \subseteq \mathbb{R}^n$ is well-defined.

Next, we want to compare the relative "size" of two lattices $M \subseteq L \subseteq \mathbb{R}^n$. Let $L = \{\sum_{i=1}^n z_i v_i \mid z_1, \ldots, z_n \in \mathbb{Z}\}$ and $M = \{\sum_{i=1}^n t_i w_i \mid t_1, \ldots, t_n \in \mathbb{Z}\}$ with $M \subseteq L$. Then $w_i \in L \;\forall\, 1 \leq i \leq n$ and hence there exists an $a_{i,j} \in \mathbb{Z}$ with $w_i = \sum_{j=1}^n a_{i,j} v_j \;\forall\, 1 \leq i \leq n$. Let $A = (a_{i,j})_{1\leq i,j\leq n} \in M_{n\times n}(\mathbb{Z})$.

Definition (Index of a sublattice)
In the notation above, we define the *index* $[L : M]$ of M in L as

$$[L : M] = |\det(A)|.$$

Remark:

1. The index $[L : M]$ does not depend on the choice of bases of L, M. By $w_i = \sum_{j=1}^n a_{i,j} v_j$, we have

$$\underbrace{|\det(w_1, \ldots, w_n)|}_{d(M)} = |\det(A)| \underbrace{|\det(v_1, \ldots, v_n)|}_{d(L)},$$

   and hence $[L : M] = \frac{d(M)}{d(L)}$.

2. One can show that $[L : M] = |L/M|$, where L/M is the quotient group.

Example: Let $e_1, \dots, e_n$ be the unit vectors in $\mathbb{R}^n$, i.e. $e_i = (0, \dots, 0, 1, 0, \dots, 0)$.

1. $\mathbb{Z}^n = \{\sum_{i=1}^n e_i z_i \mid z_1, \dots, z_n \in \mathbb{Z}\}$ is a lattice with $d(\mathbb{Z}^n) = 1$. Let $d_1, \dots, d_n \in \mathbb{N}$ and set $w_i = d_i e_i$ for all $1 \leq i \leq n$. Then $M = \{\sum_{i=1}^n z_i w_i \mid z_1, \dots, z_n \in \mathbb{Z}\} \subseteq \mathbb{Z}^n$ is a sublattice with $d(M) = |\det(d_1 e_1, \dots, d_n e_n)| = d_1 \cdots d_n$ and $[Z^n : M] = d_1 \cdots d_n$. Hence, as abelian groups, $\mathbb{Z}^n / M \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_n\mathbb{Z}$.

2. $L = \left\{ \frac{a_1}{2} e_1 + \cdots + \frac{a_n}{2} e_n \,\middle|\, a_1, \dots, a_n \in \mathbb{Z},\ a_1 \equiv \cdots \equiv a_n \bmod 2 \right\}$ is a lattice in $\mathbb{R}^n$ with basis $e_1, \dots, e_{n-1}, \frac{e_1 + \cdots + e_n}{2}$.

Convex bodies

Definition (Convex set)
We call a subset $C \subseteq \mathbb{R}^n$ *convex* if for all $x, y \in C$ the line segment

$$\{tx + (1-t)y \mid 0 \leq t \leq 1\}$$

is contained in C as well.

Definition (Central symmetric convex body)
A subset $C \subseteq \mathbb{R}^n$ is called a *central symmetric convex body* if it has the following properties:

(a) C is compact (i.e. closed and bounded) and convex. (convex body)

(b) 0 is in the interior of C. (central)

(c) If $x \in C$, then $-x \in C$. (symmetric)

Example:

1. Let $C \subseteq \mathbb{R}^n$ be a central symmetric convex body and $A : \mathbb{R}^n \to \mathbb{R}^n$ an invertible linear map. Then $A(C)$ is a central symmetric convex body.

2. The norm $\|x\|_2 = (\sum_{i=1}^n |x_i|^2)^{\frac{1}{2}}$ leads to the n-dimensional unit ball

$$B_n = \{x \in \mathbb{R}^n \mid \|x\|_2 \leq 1\}.$$

$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$ induces the n-dimensional unit cube

$$K_n = \left\{ x \in \mathbb{R}^n \,\middle|\, \max_{1 \leq i \leq n} |x_i| \leq 1 \right\}.$$

$\|x\|_1 = \sum_i^n |x_i|$ give the n-dimensional unit octahedron

$$O_n = \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^n |x_i| \leq 1 \right\}.$$

Lemma 3.5
Let $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}^n_{\geq 0}$ be a norm. Then $B_{\|\cdot\|} = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ is a central symmetric convex body.

So far we have found that every norm on $\mathbb{R}^n$ "produces" a central symmetric convex body in $\mathbb{R}^n$. Is there a one-to-one correspondence, i.e. are these all the different classes of central symmetric convex bodies?

Remark: Let $C \subseteq \mathbb{R}^n$ be a central symmetric convex body. For $\lambda \geq 0$, set $\lambda C = \{\lambda x \mid x \in C\}$. If $\lambda > 0$, then λC is again a central symmetric body. For $x \in \mathbb{R}^n$, we define $\|x\|_C = \min\{\lambda \in \mathbb{R}_{\geq 0} \mid x \in \lambda C\}$.

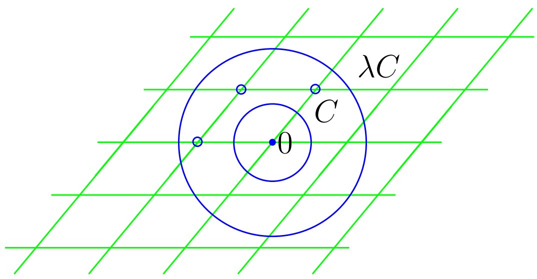
Lemma 3.6
Using the same notation as above, the following statements hold:

1. *$\|\cdot\|_C$ is well-defined.*
2. *$\|\cdot\|_C$ defines a norm on $\mathbb{R}^n$.*
3. *$\lambda C = \{x \in \mathbb{R}^n \mid \|x\|_C \leq \lambda\}$ for $\lambda > 0$.*

In particular, we recover C via $C = \{x \in \mathbb{R}^n \mid \|x\|_C \leq 1\}$.

Minkowski's[1] first convex body theorem

Let $L \subseteq \mathbb{R}^n$ be a lattice and $C \subseteq \mathbb{R}^n$ a central symmetric convex body. When is $C \cap L \neq \{0\}$, i.e. when does C contain more lattice points than just 0?



Theorem 3.7 (Minkowski's first convex body theorem, 1896)
With the same notation as above, let $\mathrm{vol}(C) \geq 2^n d(L)$. Then $C \cap L \neq \{0\}$, i.e. there exists a $x \in L \setminus \{0\}$ with $x \in C$.

[1]after Hermann Minkowski (1864 - 1909), a German mathematician

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Notation: For a lattice $L \subseteq \mathbb{R}^n$ with basis $v_1, \ldots, v_n$, we define

$$F = \left\{ \sum_{i=1}^{n} x_i v_i \,\middle|\, 0 \leq x_i \leq 1 \;\forall\, 1 \leq i > n \right\}$$

as the *fundamental parallelepiped* for L. This is the fundamental domain for $\mathbb{R}^n/L$. (see below)

Example: $[0,1)^n$ is the fundamental parallelepiped for $\mathbb{Z}^n$.

Remark: A fundamental parallelepiped depends on the choice of basis $v_1, \ldots, v_n$, but we have $\mathrm{vol}(F) = |\det(v_1, \ldots, v_n)| = d(L)$.

Lemma 3.8
Using the notation as above and for $u \in \mathbb{R}^n$ we write $u + F = \{u + x \,|\, x \in F\}$. Then

$$\mathbb{R}^n = \bigcup_{u \in L} (u + F)$$

is a disjunction.

Remark: Recall Landau's O-notation: Let $f, g, h : \mathbb{R}_{\geq x_0} \to \mathbb{R}$ for some $x_0 \in \mathbb{R}$. We write $f(x) = g(x) + = (h(x))$ if there exists $x_1 \geq x_0$ and $C \geq 0$, such that

$$|f(x) - g(x)| \leq Ch(x) \quad \forall\, x > x_1.$$

Example: $x^{-1} = O(1)$, $\lfloor x \rfloor = x + O(1)$, $(x+a)^n = x^n + O(x^{n-1})$ for any $a \in \mathbb{R}$, $n \in \mathbb{N}$, $(x+1)^{\frac{1}{2}} = x^{\frac{1}{2}} + O(x^{-\frac{1}{2}})$

Lemma 3.9
Let $L \subseteq \mathbb{R}^n$ be a lattice and $C \subseteq \mathbb{R}^n$ a central symmetric convex body. Then, as $\lambda \to \infty$, we have

$$|\lambda C \cap L| = \frac{\mathrm{vol}(C)}{d(L)} \lambda^n + O\Big(\lambda^{n-1}\Big).$$

Question: Do we need C to be central symmetric or convex in Minkowski's theorem?

Minkowski's second convex body theorem

Let $L \subseteq \mathbb{R}^n$ be a lattice and $C \subseteq \mathbb{R}^n$ a central symmetric convex body. When is $L \cap C \neq \{0\}$?

Definition (Successive minima)
We let

$$\lambda_1 = \min\{\lambda > 0 \mid \lambda C \cap L \neq \{0\}\}$$

and for $2 \leq i \leq n$ we define

$$\lambda_i = \min\{\lambda \in \mathbb{R}_{\geq 0} \mid \lambda C \cap L \text{ contains at least } i \text{ linearly independent points}\}.$$

We call $\lambda_1, \ldots, \lambda_n$ the *successive minima* of L with respect to C.

Lemma 3.10
Let $L, C \subseteq \mathbb{R}^n$ be as above. The successive minima $\lambda_1, \ldots, \lambda_n$ of L with respect to C are well defined and we have $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n < \infty$. Moreover, there exist linearly independent elements $v_1, \ldots, v_n \in L$ with $v_i \in \lambda_i C\ \forall\, 1 \leq i \leq n$.

Caveat: The vectors $v_1, \ldots, v_n$ from Lemma 3.10 may not be a basis of L. Let

$$L = \left\{ \frac{x_1 e_1 + \cdots + x_n e_n}{2} \,\middle|\, x_i \in \mathbb{Z},\ x_1 \equiv \cdots \equiv x_n \bmod 2 \right\}.$$

For $n > 4$ and $C = B_n$ the unit ball, we have

$$\left\| \frac{e_1 + \cdots + e_n}{2} \right\| = \frac{1}{2}\sqrt{n} > 1,$$

but $\|e_1\|_2 = \cdots = \|e_n\|_2 = 1$.

Question: Is there a relation between $d(L)$ and the product $\lambda_1 \cdots \lambda_n$?

Example: The lattice $L = \mathbb{Z}d_1e_1 \oplus \cdots \oplus \mathbb{Z}d_ne_n$ with $0 < d_1 \leq \cdots \leq n_n$ has with respect to $\|\cdot\|_\infty$ the successive minima $d_1 \leq \cdots \leq d_n$ and $d_1 \cdots d_n = d(L)$.

Theorem 3.11 (Minkowski's second convex body theorem, 1910)
Let $L \subseteq \mathbb{R}^n$ be a lattice, $C \subseteq \mathbb{R}^n$ a central symmetric convex body and $\lambda_1, \ldots, \lambda_n$

successive minima of L with respect to C. Then

$$\frac{1}{n!}\frac{2^n d(L)}{\mathrm{vol}(C)} \leq \lambda_1 \cdots \lambda_n \leq \frac{2^n d(L)}{\mathrm{vol}(C)}$$

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Remark: The upper bound is sharp. Take for example $L = \mathbb{Z}^n$ and $C = \{x \in \mathbb{R} \mid \|x\|_\infty \leq 1\}$, then $\mathrm{vol}(C) = 2^n$, $d(L) = 1$, $\lambda_1 = \cdots = \lambda_n = 1$. The following example shows that the lower bound is sharp as well.

Example: Let $0 < \lambda_1 \leq \cdots \leq \lambda_n$, $L = \mathbb{Z}^n$, $C = \{x \in \mathbb{R}^n \mid \sum_{i=1}^n \lambda_i |x_i| \leq 1\}$. Then L has successive minima $\lambda_1, \ldots, \lambda_n$ with respect to C and $\mathrm{vol}(C) = \frac{2^n}{n!}(\lambda_1 \cdots \lambda_n)^{-1}$.

Minkowski's second convex body theorem implies Minkowski's first convex body theorem. Let L, C be as above and assume that $\mathrm{vol}(C) \geq 2^n d(L)$. Then

$$\lambda_1^n \leq \lambda_1 \cdots \lambda_n \leq \frac{2^n d(L)}{\mathrm{vol}(C)} \leq 1\,,$$

i.e. $\lambda_1 \leq 1$ and $C \cap L \neq \{0\}$.

Remark: Theorem 3.11 is invariant under linear transformation. Let $L, C, \lambda_1, \ldots, \lambda_n$ be as above and $\phi : \mathbb{R}^n \to \mathbb{R}^n$ a linear invertible map. Then $\phi(L)$ is a lattice, $\phi(C)$ is a central symmetric convex body and one can show that $\lambda_1, \ldots, \lambda_n$ are the successive minima of $\phi(L)$ with respect to $\phi(C)$ as for $x \in \mathbb{R}^n$ we have $\|x\|_C = \|\phi(x)\|_{\phi(C)}$. We note that

$$\frac{d(\phi(L))}{\mathrm{vol}(\phi(C))} = \frac{|\det \phi| d(L)}{|\det \phi| \,\mathrm{vol}(C)} = \frac{d(L)}{\mathrm{vol}(C)}\,.$$

This means it suffices to prove Theorem 3.11 for $L = \mathbb{Z}^n$.

Lemma 3.12
Let $v_1, \ldots, v_r \in \mathbb{R}^n$. Then $S = \{\sum_{i=1}^r x_i v_i \mid x_i \in \mathbb{R},\ \sum_{i=1}^r |x_i| \leq 1\}$ is the smallest convex subset in $\mathbb{R}^n$ that is symmetric about 0 and contains $v_1, \ldots, v_r$. I.e. S is symmtric about 0 and if $R \subseteq \mathbb{R}^n$ is convex, symmetric about 0 and $v_1, \ldots, v_r \in R$, then $S \subseteq R$.

Theorem 3.13
Let $L \subseteq \mathbb{R}^n$ be a lattice. Then there exist $v_1, \ldots, v_n \in L$, such that $v_1, \ldots, v_n$ are a

basis of L and

$$\|v_1\|_2 \cdots \|v_n\|_2 \le \left(\frac{4}{3}\right)^{\frac{n(n-1)}{4}} d(L)\,.$$

Remark: This is a weaker version of the upper bound in Theorem 3.11. Our constant $\left(\frac{4}{3}\right)^{\frac{n(n-1)}{4}}$ is in general larger than 2^n (and is for large n actually pretty far off, as the exponent grows in n^2), and each successive minimum λ_i is bounded above by $\|v_i\|_2$, so they might be even smaller.

Corollary 3.14
Let $\lambda_1, \ldots, \lambda_n$ be the successive minima of a lattice $L \subseteq \mathbb{R}^n$ with respect to B_n. Then

$$\lambda_1 \cdots \lambda_n \le \left(\frac{4}{3}\right)^{\frac{n(n-1)}{4}} d(L)\,.$$

Corollary 3.15
Let $E \subseteq \mathbb{R}^n$ be an ellipsoid, symmetric about 0 and $L \subseteq \mathbb{R}^n$ a lattice. Let $\lambda_1, \ldots, \lambda_n$ be the successive minima of L ´with respect to E. Then

$$\lambda_1 \cdots \lambda_n \le \left(\frac{4}{3}\right)^{\frac{n(n-1)}{4}} V(n) \frac{d(L)}{\mathrm{vol}(E)}\,,$$

where we write $V(n) = \mathrm{vol}(B_n)$.

Theorem (Jordan's[2] theorem)
Let $C \subseteq \mathbb{R}^n$ be a central symmetric convex body. Then there exists and ellipsoid $E \subseteq \mathbb{R}^n$ with

$$E \subseteq C \subseteq \sqrt{n}E\,.$$

Corollary 3.16
For all $n \in \mathbb{N}$ there exists a constant $c(N) > 0$ with the following property: Let $L \subseteq \mathbb{R}^n$ be a lattice, $C \subseteq \mathbb{R}^n$ a central symmetric convex body, and $\lambda_1, \ldots, \lambda_n$ the successive minima of L with respect to C. Then

$$\lambda_1 \cdots \lambda_n \le c(n) \frac{d(L)}{\mathrm{vol}(C)}\,.$$

[2] after M. E. Camille Jordan (1838 - 1922), a French mathematician

Let $v_1 \in L \setminus \{0\}$ be such that $\|v_1\|_2 = \lambda_1$, where λ_1 is the first successive minimum of L with respect to B_n. Fix an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$, such that $e_1 = \lambda_1^{-1} v_1$. Consider the projection $\rho : \mathbb{R}^n \to \mathbb{R}^{n-1}$, $\sum_{i=1}^n x_i e_i \mapsto (x_2, \ldots, x_n)$. Let $L' = \rho(L)$, e.g. if $L = \mathbb{Z}d_1e_1 \oplus \cdots \oplus \mathbb{Z}d_ne_n$, then $L' = \mathbb{Z}d_2e_2 \oplus \cdots \oplus \mathbb{Z}d_ne_n$.

Lemma 3.17
Using the same notation as above, $L' \subseteq \mathbb{R}^{n-1}$ is a lattice and if $v_1, \ldots, v_n$ is a basis of L then $\rho(v_2), \ldots, \rho(v_n)$ is a bsis of L'.

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Lemma 3.18
Let $\{v_2', \ldots, v_n'\}$ be a basis of L' and $v_2, \ldots, v_n \in L$ with $\rho(v_i) = v_i'$ for $2 \leq i \leq n$. Then $\{v_1, \ldots, v_n\}$ is a basis of L.

Lemma 3.19

$$d(L) = \lambda_1 d(L') .$$

Lemma 3.20
Let $v' \in L'$. Then there exists $v \in L$, such that $\rho(v) = v'$ and

$$\|v\|_2^2 \leq \frac{4}{3}\|v'\|_2^2 .$$

Remark: We always have $\prod_{i=1}^n \|v_i\|_2 \geq d(L)$.

3.3 Bounds for class numbers

For the rest of this section, let K be a number field with ring of integers $\mathcal{O}_K$.

Question: Can we improve upon our earlier upper bounds on $|Cl(\mathcal{O}_K)|$?

Idea: We could interpret the non-zero ideal $I \subseteq \mathcal{O}_K$ as a lattice and apply Minkowski's first convex body theorem to find an element $\alpha \in I \setminus \{0\}$ of small norm.

More concretely, let $\sigma_1, \ldots, \sigma_r : K \hookrightarrow \mathbb{R}$ be the real embeddings and $\tau_1, \bar{\tau}_1, \ldots, \tau_s, \bar{\tau}_s$:

$K \hookrightarrow \mathbb{C}$ be the complex embeddings of K. Note that $r + 2s = n$, where $n = [K : \mathbb{Q}]$. Define the map

$$\varphi : K \to \mathbb{R}^n, \quad \alpha \mapsto \Big(\sigma_1(\alpha), \ldots, \sigma_r(\alpha), \Re\tau_1(\alpha), \Im\tau_1(\alpha), \ldots, \Re\tau_s(\alpha), \Im\tau_s(\alpha)\Big)\,.$$

Lemma 3.21
The image $\varphi(\mathcal{O}_K) =: \Lambda$ is a (full) lattice in $\mathbb{R}^n$ with determinant

$$d(\Lambda) = \frac{1}{2^s}\sqrt{|\operatorname{disc}\mathcal{O}_K|}\,.$$

Remark: If I is a non-zero ideal, then the same argument shows that $\varphi(I)$ is a sublattice of $\mathcal{O}_K$. More precisely, $d\big(\varphi(I)\big) = d\big(\varphi(\mathcal{O}_K)\big)\underbrace{\big|\varphi(\mathcal{O}_K)/\varphi(I)\big|}_{=|\mathcal{O}_K/I|}$, i.e.

$$d\big(\varphi(I)\big) = \frac{1}{2^s}\sqrt{|\operatorname{disc}\mathcal{O}_K|}N(I)\,.$$

Corollary 3.22
$\varphi(K)$ is dense in $\mathbb{R}^n$.

Our next goal is for a non-zero ideal $I \subseteq \mathcal{O}_K$ to find a $\alpha \in I \setminus \{0\}$, such that $|N_{K/\mathbb{Q}}(\alpha)|$ is small. We write $\varphi(\alpha) = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Then

$$N_{K/\mathbb{Q}}(\alpha) = y_1 \cdot y_2 \cdots y_r \cdot \left(y_{r+1}^2 + y_{r+2}^2\right) \cdots \left(y_{n-1}^2 + y_n^2\right).$$

The problem here is that the function $N : \mathbb{R}^n \to \mathbb{R}$ is not a norm on $\mathbb{R}^n$.

Idea: Construct a central symmetric convex body $A \subseteq \mathbb{R}^n$, such that $x \in A$ implies that $|N(x)| \leq 1$.

We define

$$A = \left\{x \in \mathbb{R}^n \,\middle|\, |x_1| + \cdots + |x_r| + 2\left(\sqrt{x_{r+1}^2 + x_{r+2}^2} + \cdots + \sqrt{x_{n-1}^2 + x_n^2}\right) \leq n\right\}$$

Lemma 3.23
A is a central symmetric convex body with the property that $x \in A$ implies $|N(x)| \leq 1$. Moreover,

$$\operatorname{vol}(A) = \frac{n^n}{n!}2^r\left(\frac{\pi}{2}\right)^s\,.$$

Theorem 3.24
Let $I \subseteq \mathcal{O}_K$ be a non-zero ideal. Then there exists an $\alpha \in I \setminus \{0\}$ with

$$|N_{K/\mathbb{Q}}(\alpha)| \leq \frac{n!}{n^n} \left(\frac{4}{\pi}\right)^s \sqrt{|\operatorname{disc} \mathcal{O}_K|} N(I) \,.$$

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Corollary 3.25
Every ideal class $C \in Cl(\mathcal{O}_K)$ contains a representative I with

$$N(I) \leq \frac{n!}{n^n} \left(\frac{4}{\pi}\right)^s \sqrt{|\operatorname{disc} \mathcal{O}_K|} \,.$$

Corollary 3.26
If $K \neq \mathbb{Q}$ (i.e. $n \neq 1$), then

$$|\operatorname{disc} \mathcal{O}_K| > 1 \,.$$

Example: We try to find the class group of $\mathbb{Z}[\sqrt{-5}]$, i.e. we have $K = \mathbb{Q}[\sqrt{-5}]$, $\mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$, $n = 2$, $s = 1$. By Corollary 3.26 it is sufficient to consider ideals $I \subseteq \mathcal{O}_K$ with

$$N(I) \leq \frac{2!}{4} \frac{4}{\pi} \underbrace{\sqrt{|\operatorname{disc}(\mathbb{Z}[\sqrt{-5}])|}}_{=2\sqrt{5}} = \frac{4\sqrt{5}}{\pi} \leq 3 \,,$$

i.e. ideals lying above 2. Recall that

$$2\mathbb{Z}[\sqrt{-5}] = (2, 1 + \sqrt{-5})^2$$

and $(2, 1 + \sqrt{-5})$ is not principal. Hence

$$|Cl(\mathbb{Z}[\sqrt{-5}])| = 2 \,.$$

3.4 Dirichlet's unit theorem

Let K be a number field with ring of integers $\mathcal{O}_K$. What can we say about the group of units $\mathcal{O}_K^*$?

Example:

- For $K = \mathbb{Q}$ we have $\mathbb{Z}^* = \{\pm 1\}$, for $K = \mathbb{Q}(i)$ we have $\mathbb{Z}[i]^* = \{\pm 1, \pm i\}$. In the exercises we have seen that $\mathcal{O}_K^*$ is finite for all imaginary quadratic number fields K.
- If $K = \mathbb{Q}(\sqrt{d})$ with $d \in \mathbb{N}$ square-free, then the Pell equation $x^2 - dy^2 = 1$ has

a non-trivial solution (x_0, y_0) and $x_0 + \sqrt{d}y_0$ generates infinitely many units in $\mathcal{O}_K$

Let $n = [K : \mathbb{Q}]$, $\sigma_1, \ldots, \sigma_r : K \hookrightarrow \mathbb{R}$ and $\tau_1, \bar{\tau}_1, \ldots, \tau_s, \bar{\tau}_s : K \hookrightarrow \mathbb{C}$ be the real and complex embeddings of K. As in Section 3.3, let $\varphi : K \to \mathbb{R}^n$ be defined by

$$\alpha \mapsto \Big(\sigma_1(\alpha), \ldots, \sigma_r(\alpha), \Re\tau_1(\alpha), \Im\tau_1(\alpha), \ldots, \Re\tau_s(\alpha), \Im\tau_s(\alpha)\Big)\,.$$

Definition
In the notation above we define the maps $\log : \varphi(K \setminus \{0\}) \to \mathbb{R}^{r+s}$ as

$$(x_1, \ldots, x_n) \mapsto \Big(\log|x_1|, \ldots, \log|x_r|, \log\left(x_{r+1}^2 + x_{r+2}^2\right), \ldots, \log\left(x_{n-1}^2 + x_n^2\right)\Big)$$

and $\psi : \mathbb{K} \setminus \{0\} \to \mathbb{R}^{r+s}$ as $\psi = \log \circ \varphi$.

First properties of ψ:

(a) For $\alpha, \beta \in K \setminus \{0\}$ we have

$$\psi(\alpha\beta) = \psi(\alpha)\psi(\beta)\,.$$

(b) Let $H \subseteq \mathbb{R}^{r+s}$ be the hyperplane given by $y_1 + \cdots + y_{r+s} = 0$. Then we have $\psi(\mathcal{O}_K^*) \subseteq H$, because every $\alpha \in \mathcal{O}_K^*$ satisfies

$$1 = |N_{K/\mathbb{Q}}(\alpha)| = |\sigma_1(\alpha)| \cdots |\sigma_r(\alpha)||\tau_1(\alpha)|^2 \cdots |\tau_s(\alpha)|^2\,,$$

i.e. $0 = \log|\sigma_1(\alpha)| + \cdots + \log|\tau_s(\alpha)|^2$.

(c) Let $B \subseteq \mathbb{R}^{r+s}$ be a bounded subset. Then $\log^{-1}(B) \cap \varphi(\mathcal{O}_K \setminus \{0\})$ is finite.

Our next goal is to study the image $\psi(\mathcal{O}_K^*) \subseteq H \subseteq \mathbb{R}^{r+s}$. Note that by (a) above, $\psi(\mathcal{O}_K^*)$ is an (additive) subgroup of H.

Lemma 3.27
Let $G \subseteq \mathbb{R}^m$ be a subgroup, such that every bounded subset of G is finite. Then there exist over R linearly independent vectors $v_1, \cdots, v_d \in \mathbb{R}^m$ for some $d \leq m$ such that

$$G = \left\{ \sum_{i=1}^{d} x_i v_i \,\middle|\, x_1, \ldots, x_d \in \mathbb{Z} \right\}\,.$$

Corollary 3.28
$\psi(\mathcal{O}_K^*)$ is a lattice in some linear subspace of H.

Next we will show that $\psi(\mathcal{O}_K^*)$ spans H, i.e. $\psi(\mathcal{O}_K^*)$ is a lattice of full rank in H.

Lemma 3.29
Let $1 \leq k \leq r+s$ and $\alpha \in \mathcal{O}_K \setminus \{0\}$. Write $\psi(\alpha) = (a_1, \ldots, a_{r+s})$. Then there exists $\beta \in \mathcal{O}_K \setminus \{0\}$ with

$$|N_{K/\mathbb{Q}}(\beta)| \leq \left(\frac{2}{\pi}\right)^2 \sqrt{|\operatorname{disc} \mathcal{O}_K|}$$

and with the property that if $\psi(\beta) = (b_1 \ldots, b_{r+s})$, then $b_j < a_j$ for all $1 \leq j \leq r+s,\ j \neq k$

Lemma 3.30
There exist units $u_1, \ldots, u_{r+s} \in \mathcal{O}_K^*$ with the following property: If

$$\psi(u_l) = (u_{l,1}, \ldots, u_{l,r+s}) \ ,$$

then $u_{l,j} < 0$ for all $j \neq l$.

Remark: If we construct a matrix

$$\begin{pmatrix} \psi(u_1) \\ \vdots \\ \psi(u_l) \\ \vdots \\ \psi(u_{r+s}) \end{pmatrix} = \begin{pmatrix} u_{1,1} & \ldots & u_{1,l} & \ldots & u_{1,r+s} \\ \vdots & \ddots & \vdots & & \vdots \\ u_{l,1} & \ldots & u_{l,l} & \ldots & u_{l,r+s} \\ \vdots & & \vdots & \ddots & \vdots \\ u_{r+s,1} & \ldots & u_{r+s,l} & \ldots & u_{r+s,r+s} \end{pmatrix}$$

Lemma 3.30 tells us that the diagonals are positive while all other entries are negative.

Next we will let $u_1, \ldots, u_{r+s}$ be units as in Lemma 3.30. We would lke to show that $\psi(u_1), \ldots, \psi(u_{r+s})$ span H.

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Lemma 3.31
Let $A = (a_{ij})_{1 \leq i,j \leq m} \in Mat_{m \times m}(\mathbb{R})$ and assume the following properties:

(i) $\sum_{j=1}^m a_{ij} = 0$ *for all* $1 \leq i \leq m$

(ii) $a_{ii} > 0$ *for all* $1 \leq i \leq m$

(iii) $a_{ij} < 0$ *for* $i \neq j$, $1 \leq i, j \leq m$

Then $\operatorname{rank}(A) = m - 1$.

Corollary 3.32
The image $\psi(\mathcal{O}_K^*) \subseteq H$ *is a lattice of rank* $r+s-1$.

Theorem 3.33 (Dirichlet's[3] unit theorem)
Let K *be a number field with* r *real and* $2s$ *complex embeddings and* $\mathcal{O}_K$ *its ring of integers. Then there exist units* $u_1, \dots, u_{r+s-1} \in \mathcal{O}_K^*$, *such that every unit* $u \in \mathcal{O}_K^*$ *can be written uniquely in the form*

$$u = \mu \cdot u_1^{e_1} \cdot u_2^{e_2} \cdots u_{r+s-1}^{e_{r+s-1}}$$

with $\mu \in K$ *a root of unity and* $e_1, \dots, e_{r+s-1} \in \mathbb{Z}$.

Remark: We call $u_1, \dots, u_{r+s-1}$ as in Theorem 3.33 a fundamental system of units.

Example:

1. If K is a cubic field with exaclty one real embedding, then the only roots of unity in K are ± 1 (as they are the only roots on unity in $\mathbb{R}$). Hence there exists a fundamental unit $u \in \mathcal{O}_K^*$, such that $\mathcal{O}_K^* = \left\{ \pm u^k \,\middle|\, k \in \mathbb{Z} \right\}$.
2. The only number fields with a finite group of units $\mathcal{O}_K^*$ are $\mathbb{Q}$ and imaginary quadratic number fields.

[3] after Peter Gustav Lejeune Dirichlet (1805 - 1859), a German mathematician

4 Diophantine Approximation

4.1 Introduction

Motivation: Let $\alpha \in \mathbb{R}$, how well can we approximate α with rational numbers of small denominator? Given $\varepsilon > 0$, what is the "smallest" fraction $\frac{x}{y}$ (i.e. y small), such that $\left|\alpha - \frac{x}{y}\right| < \varepsilon$, $x \in \mathbb{Z}, y \in \mathbb{N}$?

Theorem 4.1 (Dirichlet, 1842)
Let $\alpha \in \mathbb{R}$ and $Q \in \mathbb{N}$. Then there exist $x, y \in \mathbb{Z}$, such that $\left|\alpha - \frac{x}{y}\right| \leq \frac{1}{yQ}$, $0 < y \leq Q$ and with $\gcd(x, y) = 1$.

Corollary 4.2
Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then there exist infinitely many pairs $(x, y) \in \mathbb{Z}^2$, such that $y > 0$, $\gcd(x, y) = 1$ and $\left|\alpha - \frac{x}{y}\right| \leq \frac{1}{y^2}$.

Theorem 4.3 (Dirichlet, 1842)

(a) Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ for some $n \in \mathbb{N}$. For all $Q \in \mathbb{N}$ there exists a tuple $x_1, \ldots, x_n, y) \in \mathbb{Z}^{n+1}$ with $0 \leq y \leq Q^n$, such that

$$|\alpha_i y - x_i| \leq \frac{1}{Q} \quad \forall\, 1 \leq i \leq n\,.$$

(b) Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, not all in $\mathbb{Q}$. Then there exist inifinitely many tuples $(x_1, \ldots, x_n, y) \in \mathbb{Z}^{n+1}$ with $\gcd(x_1, \ldots, x_n, y) = 1$, $y > 0$, such that

$$\left|\alpha_i - \frac{x_i}{y}\right| \leq \frac{1}{y^{1+\frac{1}{n}}} \quad \forall\, 1 \leq i \leq n\,.$$

Another application of Minkowski's convex body theorem: Rational points close to hyperplanes.

Theorem 4.4
Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, such that $1, \alpha_1, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}$. Then

there exist infinitely many tuples $(x, y_1, \ldots, y_n) \in \mathbb{Z}^{n+1}$ *with* $y_1, \ldots, y_n) \neq (0, \ldots, 0)$ *and*

$$\left|\alpha_1 y_1 + \cdots + \alpha_n y_n - x\right| \leq \left(\max_{1\leq i\leq n} |y_i|\right)^{-n} .$$

An open problem: Recall the notation $\|y\| = \min_{m\in\mathbb{Z}} |y - m|$ for $y \in \mathbb{R}$. Let $\alpha \in \mathbb{R}$. By Dirichlet's theorem there exist infinitely many $y \in \mathbb{N}$ with $y\|\alpha y\| \leq 1$. Let $\alpha, \beta \in \mathbb{R}$. Then there exist infinitley many $y \in \mathbb{N}$ with $y\|\alpha y\|\|\beta y\| \leq 1$.

Conjecture (Littlewood[1] conjecture)
Let $\alpha, \beta \in \mathbb{R}$. Then

$$\liminf_{y\to\infty} y\|\alpha y\|\|\beta y\| = 0 .$$

Borel[2] showed in 1909 that the exceptional set has Lebesgue measure 0. Einsiedler[3], Katok[4] and Lindenstrauss[5] showed in 2006 that the exceptional set also has Hausdorff dimension 0.

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Question: Can we do better than Corollary 4.2?

Example: Let $A > \sqrt{5}$ and $\alpha = \frac{1+\sqrt{5}}{2}$. Then the inequality $|\alpha - \frac{x}{y}| \leq \frac{1}{Ay^2}$ has only finitely many solutions $x, y \in \mathbb{N}$.

For $\delta > 0$, consider the inequality

$$\left|\alpha - \frac{x}{y}\right| \leq \frac{1}{y^{2+\delta}} \tag{4.1}$$

in $x, y > 0$, $\gcd(x, y) = 1$. For what α does (4.1) have inifinitely many solutions? Khinchin[6] showed in 1927 that the set of such α has Lebesgue measure 0.

Example: Let $a \in \mathbb{N}_{\geq 3}$ and set $\alpha = \sum_{n=1}^{\infty} 10^{-a^{2n}}$. The claim is that there exist infinitely many $(x, y \in \mathbb{Z}^2)$ with $y > 0$ and $\gcd(x, y) = 1$, such that

$$\left|\alpha - \frac{x}{y}\right| \leq \frac{1}{y^a} .$$

[1]after John Edensor Littlewood (1885 - 1977), a British mathematician
[2]Émile Borel (1871 - 1956), a French mathematician and politician
[3]Manfred Einsiedler (*1973), an Austrian mathematician
[4]Anatole Katok (1944-2018), an American mathematician
[5]Elon Lindenstrauss (*1970), an Israeli mathematician
[6]Aleksandr Khinchin (1894 - 1959), a Soviet mathematician

Idea: To construct such well-appropriable numbers we pick α in the decimal expansion (or use any other base) with very few digits 1, which get more and more sparse, and set all other digits equal to zero.

Theorem (Roth[7], 1955)
Let $\alpha \in \mathbb{R}$ be an algebraic number and $\delta > 0$. Then there are only finitely many tuples $(x, y) \in \mathbb{Z}^2$ with $y > 0$, $\gcd(x, y) = 1$ and

$$\left|\alpha - \frac{x}{y}\right| \leq \frac{1}{y^{2+\delta}}\,.$$

Roth's theorem implies that $\alpha = \sum_{n=1}^{\infty} 10^{-a^{2n}}$ for $a \geq 3$ is transcendental.

Definition (Linearly independent complex numbers)
We call a set $\{\alpha_1, \ldots, \alpha_n\} \in \mathbb{C}^n$ linearly independent over $\mathbb{Q}$ if the relation $x_1\alpha_1 + \cdots + x_n\alpha_n = 0$ with $x_1, \ldots, x_n \in \mathbb{Q}$ implies $x_1 = \cdots = x_n = 0$.

Theorem (Schmidt[8], 1971)
Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ algebraic such that $\{1, \alpha_1, \ldots, \alpha_n\}$ is linearly independent over $\mathbb{Q}$. Let $\delta > 0$. Then there exist only finitely many tuples $(x_1, \ldots, x_n, y) \in \mathbb{Z}^{n+1}$ with $y > 0$, $\gcd(x_1, \ldots, x_n, y) = 1$ and

$$\left|\alpha_i - \frac{x_i}{y}\right| \leq y^{-1-\frac{1}{n}} \quad \forall\, 1 \leq i \leq n\,.$$

Theorem (Subspace Theorem, Schmidt, 1972)
Let $n > 2$ and $L_i = \alpha_{i1}x_1 + \cdots + \alpha_{in}x_n$, $1 \leq i \leq n$, be n linearly independent linear forms with coefficients in $\overline{\mathbb{Q}}$. Let $C, \delta > 0$. Then the solution of the inequality

$$|L_1 \cdot L_2 \cdots L_n| \leq C \max\{|x_1|, \ldots, |x_n|\}^{-\delta}$$

with $(x_1, \ldots, x_n) \in \mathbb{Z}^n$ are contained in a finite union of proper linear subspaces of $\mathbb{Q}^n$.

[7] Klaus Roth (1925 - 2015), a British mathematician
[8] Wolfgang M. Schmidt (*1933), an Austrian mathematician

Example: Let α be an algebraic number and consider the linear forms $ax_2 - x_1,\ x_2$.

$$|ax_2 - x_1||x_2| \leq \max\{|x_1|, |x_2|\}^{-\delta}$$

The application of the Subspace Theorem leads us back to Roth's theorem.

4.2 Transcendence

Definition (Algebraic and transcendental numbers)
We call $\alpha \in \mathbb{C}$ *algebraic* (over $\mathbb{Q}$) if there exists a non-zero polynomial $P(x) \in \mathbb{Q}[x]$ such that $P(\alpha) = 0$. If $\alpha \in \mathbb{C}$ is not algebraic, then we call it *transcendental*.

Notation: We write $\overline{\mathbb{Q}} = \{\alpha \in \mathbb{C} \,|\, \alpha \text{ is algebraic}\}$.

Definition (Algebraically independent numbers)
We call $\alpha_1, \dots, \alpha_r \in \mathbb{C}$ *algebraically independent* if there is no non-zero polynomial $P \in \overline{\mathbb{Q}}[x_1, \dots, x_r]$ with $P(\alpha_1, \dots, \alpha_r) = 0$.

Example:

1. $\alpha \in \mathbb{C}$ is transcendental if and only if α is algebraically independent.
2. e is transcendental.
3. $\alpha_1 = e,\ \alpha_2 = e^2$ are linearly independent over $\overline{\mathbb{Q}}$ but not algebraically independent as $\alpha_1^2 - \alpha_2 = 0$.

Definition (Transcendence degree, trancendence basis)
Let $S \subseteq \mathbb{C}$. We define the *transcendence degree* of S as the maximal number $t \in \mathbb{Z}_{\geq 0}$ (or $t = \infty$), such that S contains t algebraically independent elements. We denote it by $\operatorname{trdeg} S$. If $B \subseteq S$ is an algebraically independent subset with $|B| = \operatorname{trdeg} S$, then we call B a *transcendence basis* of S.

Example:

1. $\operatorname{trdeg} \mathbb{Q}(e) = 1$ and $\{e\}$ and $\{e^2\}$ are examples of a transcendence basis for $\mathbb{Q}(e)$.
2. Let $S \subseteq \mathbb{C}$ with transcendence basis $B = \{\alpha_1, \dots, \alpha_r\}$. Then every $x \in S$ is algebraic over $\overline{\mathbb{Q}}(\alpha_1, \dots, \alpha_r)$.

Lemma 4.5
Let $\alpha \in \mathbb{R}$ and assume that there exists a sequence of tuples of integers $(x_{n},_{n}) \in \mathbb{Z}^2$, $n \in \mathbb{N}$, with $y_n > 0$, $\frac{x_n}{y_n} \neq \alpha\ \forall n \in \mathbb{N}$ and

$$|x_n - \alpha y_n| \to 0 \text{ as } n \to \infty .$$

Then $\alpha \notin \mathbb{Q}$.

Theorem 4.6
$e \notin \mathbb{Q}$.

Proof. Write $e = \sum_{k=0}^{\infty} \frac{1}{k!}$. For $n \in \mathbb{N}$ set $x_n = n! \sum_{k=0}^{n} \frac{1}{k!}$ and $y_n = n!$. Then

$$\begin{aligned} 0 < |x_n - ey_n| &= n! \sum_{k=n+1}^{\infty} \frac{1}{k!} = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \dots \\ &< \frac{1}{n+1} + \frac{1}{(n+1)^2} = \frac{1}{n+1} \sum_{q=0}^{\infty} \frac{1}{(n+1)^q} = \frac{1}{n+1} \cdot \frac{1}{1 - \frac{1}{n+1}} \\ &= \frac{1}{n} \to 0 \text{ for } n \to \infty \end{aligned}$$

□

Theorem 4.7
The number $\alpha = \sum_{k=1}^{\infty} 10^{-k!}$ is transcendental.

Transcendence of e

For $z \in \mathbb{C}$ we set $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$.

Theorem 4.8 (Hermite[9], 1873)
e is transcendental.

For a polynomial $f \in \mathbb{C}[x]$ we define the integral transform $F(z) = \int_0^z e^{z-u} f(u) \mathrm{d}u$, where $z \in \mathbb{C}$, and we integrate over the line segment $\{tz \,|\, 0 \leq t \leq 1\}$, i.e.

$$F(z) = \int_0^1 e^{z(1-t)} f(tz) z \mathrm{d}t .$$

[9] Charles Hermite (1822 - 1901), a French mathematician

Example: If $f(u) = u$, then

$$F(z) = \int_0^1 e^{z(1-z)} z^2 t \mathrm{d}t = \left[\frac{1}{z} e^{z(1-t)} z^2 t\right]_0^1 + \int_0^1 \frac{1}{z} e^{z(1-t)} z^2 \mathrm{d}t$$
$$= -z + \left[-e^{z(1-t)}\right]_0^1 = e^z - z - 1$$

Lemma 4.9
Let $f \in \mathbb{C}[x]$ be of degree m. Then

$$F(z) = e^z \left(\sum_{j=0}^{m} f^{(j)}(0)\right) - \sum_{j=0}^{m} f^{(j)}(z)\,.$$

Lemma 4.10
Let $f \in \mathbb{C}[x]$ and $z \in \mathbb{C}$. Then

$$|F(z)| \leq |z| e^{|z|} \sup_{\substack{u \in \mathbb{C} \\ |u| \leq |z|}} |f(u)|\,.$$

Now, assume that e is algebraic. Then there exists $q_0, \ldots, q_n \in \mathbb{Z}$, $n \geq 0$, $q_n \neq 0$, such that

$$q_0 + q_1 e + \cdots + q_n e^n = 0 \tag{4.2}$$

Lemma 4.11
Let $f \in \mathbb{C}[x]$ be of degree n and $q_0, \ldots, q_n$ as in (4.2). Then

$$\sum_{a=0}^{n} q_a F(a) = -\sum_{a=0}^{n} \sum_{j=0}^{m} q_a f^{(j)}(a)\,. \tag{4.3}$$

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Our next step will be to construct a polynomial $f(x) \in \mathbb{C}[x]$, such that $|F(0)|, \ldots, |F(n)|$ are very small and the right-hand side of (4.3) is a non-zero integer.

Let p be a prime number to be chosen later. Define

$$f(X) = \frac{1}{(p-1)!} X^{p-1} \Big((X-1)(X-2)\ldots(X_n)\Big)^p\,.$$

Lemma 4.12
Let f be as above. Then we have

(i) $f^{(p-1)}(0) = \Big((-1)^n n!\Big)^p$

(ii) $f^{(j)}(a)$ *if either* $a \in \{1, \ldots, n\}$ *and* $0 \leq j \leq p-1$ *or* $a = 0$ *and* $0 \leq j \leq p-2$

(iii) Let $0 \le a \le n$ *and* $j \ge p$*. Then* $f^{(j)}(a) \equiv 0 \bmod p$*.*

Lemma 4.13
Let $p > |q_0 n|$*. Then*

$$M := \sum_{a=0}^{n} q_a F(a) \in \mathbb{Z} \setminus \{0\}\,.$$

Lemma 4.14
Let $q_0, \ldots, q_n$ *and* M*,* p *like above. Then* $|M| \to 0$ *for* $p \to \infty$*.*

We summarise: If $q_0 + q_1 e + \cdots + q_n e^n = 0$ for $q_0, \ldots, q_n \in \mathbb{Z}$, $q_0 \neq 0$, and $f(X) = \frac{1}{(p-1)!} X^{p-1}\big((X-1)\cdots(X-n)\big)^p$ for a sufficiently large prime p, then $M = \sum_{a=0}^{n} q_a F(a) \in \mathbb{Z} \setminus \{0\}$ and $|M| < \frac{1}{2}$, which is a contradiction. Hence, e is transcendental.

Remark: In the proof of Theorem 4.8 we showd that for any $n \in \mathbb{N}$, the numbers $1, e, e^2, \ldots, e^n$ are linearly independent over $\mathbb{Q}$ (and hence over $\overline{\mathbb{Q}}$).

Question: Let $\alpha_0, \ldots, \alpha_n \in \overline{\mathbb{Q}}$. Under which assumptions are the numbers $e^{\alpha_0}, \ldots, e^{\alpha_n}$ linearly dependent over $\mathbb{Q}$ or $\overline{\mathbb{Q}}$?

We certainly need the α_i to be distinct, as for example $1 \cdot e^{\alpha} + (-1) \cdot e^{\alpha} = 0$ for all $\alpha \in \overline{\mathbb{Q}}$.

Theorem 4.15 (Baker[10], Lindemann[11]-Weierstraß[12])
Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \overline{\mathbb{Q}}$ *for some* $n \in \mathbb{N}$*. Assume that* $\alpha_1, \ldots, \alpha_n$ *are pairwise distinct and* $\beta_1 \cdots \beta_n \neq 0$*. Then*

$$\beta_1 e^{\alpha_1} \cdots \beta_n^{\alpha_n} \neq 0\,.$$

Remark: This implies that if $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}$ are pairwise distinct, then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are linearly independent over $\overline{\mathbb{Q}}$.

[10] Alan Baker (1939 - 2018), an English mathematician
[11] after Ferdinand von Lindemann (1852-1939), a German mathematician,
[12] and Karl Weierstraß (1815-1879), a German mathematician

Corollary 4.16
Let $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$*. Then* e^{α} *is transcendental.*

Corollary 4.17
π *is transcendental.*

Proof. Assume $\pi \in \overline{\mathbb{Q}}$. Then $i\pi \in \overline{\mathbb{Q}}$, but $e^{i\pi} = -1$ is not transcendental. □

Corollary 4.18
Let $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}$ *be linearly independent over* $\mathbb{Q}$*. Then* $e^{\alpha_1}, \dots, e^{\alpha_n}$ *are algebraically independent.*

Remark: Corollary 4.18 is in fact equivalent to Theorem 4.15.

Example: Imagine we try to show that

$$1 \cdot e^0 + 2 \cdot e^{\sqrt{3}} \neq 0\,.$$

For $\sigma \in \mathrm{Gal}(\mathbb{Q}(\sqrt{3})/\mathbb{Q})$ and $\alpha \in \mathbb{Q}(\sqrt{3})$, set $\sigma(e^{\alpha}) = e^{\sigma(\alpha)}$. Then the non-trivial automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}(\sqrt{3})/\mathbb{Q})$ maps $1 + 2e^{\sqrt{3}}$ to $1 + 2e^{-\sqrt{3}}$. However,

$$\left(1 + e^{\sqrt{3}}\right)\left(1 + 2e^{-\sqrt{3}}\right) = 1 + 4 + 2e^{\sqrt{3}} + 2e^{-\sqrt{3}}$$

is invariant under $\mathrm{Gal}(\mathbb{Q}(\sqrt{3})/\mathbb{Q})$.

We can reduce Theorem 4.15 to the following result:

Theorem 4.19 ("Weak Lindemann-Weierstraß theorem")
Let $\mathbb{Q} \subseteq L \subseteq \mathbb{C}$ *be a normal number field. Let* $\gamma_1, \dots, \gamma_t, \delta_1, \dots, \delta_t \in L$*, such that* $\gamma_1, \dots, \gamma_t$ *are pairwise distinct and* $\delta_1 \cdots \delta_t \neq 0$*. Assume that each* $\tau \in \mathrm{Gal}(L/\mathbb{Q})$ *permutes the pairs* $(\gamma_1, \delta_1), \dots, (\gamma_t, \delta_t)$*. Then*

$$\delta_1 e^{\gamma_1} + \cdots + \delta_t e^{\gamma_t} \neq 0\,.$$

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Let $l \in \mathbb{Z} \setminus \{0\}$, such that $l\gamma_1, \dots, l\gamma_t$ are algebraic integers. Let p be a prime. For

$1 \leq k \leq t$ we define

$$f_K(X) = \frac{1}{(p-1)!} l^{pt} (X - \gamma_k)^{p-1} \prod_{\substack{i=1 \\ i \neq k}}^{t} (X - \gamma_i)^p .$$

Set $F_k(z) = \int_0^1 e^{z-u} f_k(u) \mathrm{d}u$ and $M_k = \delta_1 F_k(\gamma_1) + \cdots + \delta_t F_k(\gamma_t)$.

Lemma 4.20
If $\tau \in \mathrm{Gal}(L/\mathbb{Q})$, then $\tau(M_1) \in \{M_1, \ldots, M_t\}$.

Notation: For $\alpha, \beta \in \overline{\mathbb{Q}}$, $m \in \mathbb{Z} \setminus \{0\}$, we write $\alpha \equiv \beta \bmod m$ if $\frac{\alpha - \beta}{m}$ is an algebraic integer.

Lemma 4.21
Let $1 \leq m \leq t$. Then

(i) $f_1^{(p-1)}(\gamma_1) = l^{pt} \left(\prod_{i=2}^{t} (\gamma_i - \gamma_1) \right)^p$

(ii) If either $2 \leq m \leq t$ and $0 \leq j \leq p-1$ or $m = 1$ and $0 \leq j \leq p-2$, then $f_1^{(j)}(\gamma_m) = 0$.

(iii) $f_1^{(j)}(\gamma_m) \equiv 0 \bmod p$ *if $1 \leq m \leq t$ and $j \geq p$.*

Lemma 4.22
If p is sufficiently large, then $M_1 \neq 0$ is an algebraic integer.

Lemma 4.23
Let $1 \leq k \leq t$. Then $|M_k| \to 0$ for $p \to \infty$.

4.3 More on transcendence results

Recall our definition $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ for $z \in \mathbb{C}$. For $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$, we set $\alpha^b = e^{\beta \log \alpha}$, where $\log \alpha$ is some solution of the equation $\alpha = e^z$. I.e. if we fix one solution $\log \alpha$, then all possibilities for $e^{\beta \log \alpha}$ are given by $e^{\beta(\log \alpha + 2\pi i k)}$, $k \in \mathbb{Z}$. In Section 4.2 we have seen that if $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}$ are pairwise distinct, then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are linearly independent over $\overline{\mathbb{Q}}$. As a corollary: if $\beta \in \overline{\mathbb{Q}} \setminus \{0\}$, then e^{β} is transcendental.

Theorem 4.24 (Gelfond[13], Schneider[14], 1934)
Let $\alpha, \beta \in \overline{\mathbb{Q}}$ *with* $0 \neq \alpha \neq 1$ *and* $\beta \notin \mathbb{Q}$. *Then* $\alpha^\beta = e^{\beta \log \alpha}$ *is transcendental for any solution* $\log \alpha$.

Corollary 4.25
Let $\alpha \in \overline{\mathbb{Q}}$ *with* $\alpha \notin i\mathbb{Q}$. *Then* $e^{\pi\alpha}$ *is transcendental.*

Corollary 4.26
Let $\alpha_1, \alpha_2 \in \overline{\mathbb{Q}} \setminus \{0\}$. *Fix a choice of logarithms of* $\log \alpha_1, \log \alpha_2$ *and assume that* $\log \alpha_1, \log \alpha_2$ *are linearly independent over* $\mathbb{Q}$. *Then if* $\beta_1, \beta_2 \in \overline{\mathbb{Q}} \setminus \{0\}$, *we have* $\beta_1 \log \alpha_1 + \beta_2 \log \alpha_2 \neq 0$.

Example: The real logarithms $\log 2$ and $\log 3$ are linearly independent over $\mathbb{Q}$ and $\overline{\mathbb{Q}}$.

Question: How about elements $\log \alpha_1, \dots, \log \alpha_n$ for $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}$?

Theorem 4.27 (Baker, 1965)
Let $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}} \setminus \{0, 1\}$ *and fix choices for* $\log \alpha_1, \dots, \log \alpha_n$, *such that* $\log \alpha_1, \dots, \log \alpha_n$ *are linearly independent over* $\mathbb{Q}$. *Let* $\beta_1, \dots, \beta_n \in \overline{\mathbb{Q}} \setminus \{0\}$. *Then* $\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$ *is transcendental.*

Remark: Baker's theorem gives us the stronger conclusion that $1, \log \alpha_1, \dots, \log \alpha_n$ are linearly independent over $\overline{\mathbb{Q}}$.

Definition
Let $\alpha \in \overline{\mathbb{Q}}$ with primitive minimal polynomial $f \in \mathbb{Z}[X]$, i.e. a minimal polynomial $f(X) = a_0 + a_1 X + \cdots + a_d X^d$ with $a_0, \dots, a_d$ and $\gcd(a_0, \dots, a_d) = 1$. Then we set $H(\alpha) = \max_{0 \leq i \leq d} |a_i|$.

Theorem 4.28
Let $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}} \setminus \{0, 1\}$, $\gamma \in \overline{\mathbb{Q}}$ *and* $\beta_1, \dots, \beta_n \in \overline{\mathbb{Q}} \setminus \{0\}$. *Pick choices of*

[13] Alexander Gelfond (1906-1968), a Soviet mathematician, who did his Ph. D with Khinchin
[14] Theodor Schneider (1911-1988), a German mathematician, who worked in Göttingen until 1953 and later became the director of the MRI Oberwolfach

$\log\alpha_1,\dots,\log\alpha_n$ *and assume that* $\log\alpha_1,\dots,\log\alpha_n$ *are linearly independent over* $\mathbb{Q}$. *Then*

$$|\gamma+\beta_1\log\alpha_1+\cdots+\beta_n\log\alpha_n|\geq(eB)^{-}c$$

with $B=\max\Big(H(\gamma),H(\beta_1),\dots,H(\beta_n)\Big)$ *and* $c>0$ *an effectively computable constant depending on* $n, H(\alpha_1),\dots,H(\alpha_n)$ *and the choices for* $\log\alpha_1,\dots,\log\alpha_n$.

Question: How can we recognise if $\log\alpha_1,\dots,\log\alpha_n$ are $\mathbb{Q}$-linearly independent.

Assume that $\log\alpha_1,\dots,\log\alpha_n$ are $\mathbb{Q}$-linearly dependent. Then there exist $b_1,\dots,b_n\in\mathbb{Z}$, not all zero, such that

$$b_1\log\alpha_1+\cdots+b_n\log\alpha_n=0\,,$$

i.e.

$$\alpha_1^{b_1}\alpha_2^{b_2}\cdots\alpha_n^{b_n}=0\,.$$

Definition (Multiplicative dependency)
We say that $\alpha_1,\dots,\alpha_n\in\mathbb{C}$ are *multiplicatively dependent* if there exist $b_1,\dots,b_n\in\mathbb{Z}$, not all zero, such that

$$\alpha_1^{b_1}\alpha_2^{b_2}\cdots\alpha_n^{b_n}=0\,.$$

Remark: If $\alpha_1,\dots,\alpha_n\in\overline{\mathbb{Q}}\backslash\{0\}$ are multiplicatively independent, then $\log\alpha_1,\dots,\log\alpha_n$ are $\mathbb{Q}$-linearly independent.

Corollary 4.29
Let $\alpha_1,\dots,\alpha_n\in\overline{\mathbb{Q}}\setminus\{0\},\beta_1,\dots,\beta_n\in\overline{\mathbb{Q}}$, *such that* $\alpha_1,\dots,\alpha_n$ *are multiplicatively independent and* $(\beta_1,\dots,\beta_n)\notin\mathbb{Q}^n$, *then* $\alpha_1^{\beta_1}\cdots\alpha_n^{\beta_n}$ *is transcendental (for any choice of* $\log\alpha_1,\dots,\log\alpha_n$*).*

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Conjecture 4.30 (Schanuel's[15] conjecture)
Let $x_1,\dots,x_n\in\mathbb{C}$ be linearly independent over $\mathbb{Q}$. Then

$$\operatorname{trdeg}\Big(x_1,\dots,x_n,e^{x_1},\dots,e^{x_n}\Big)\geq n\,.$$

Schanuel's conjecture implies the Lindemann-Weierstraß theorem. Let $\alpha_1,\dots,\alpha_n\in$

[15] after Stephen Schanuel (1933 - 2014), an American mathematician

$\overline{\mathbb{Q}}$ be linearly independent over $\mathbb{Q}$. Then $\mathrm{trdeg}(\underbrace{\alpha_1, \ldots, \alpha_n}_{\mathrm{trdeg}(e^{\alpha_1},\ldots,e^{\alpha_n})}, e^{\alpha_1}, \ldots, e^{\alpha_n}) \geq n$, i.e. $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent. It also implies Baker's theorem. Let $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}} \setminus \{0, 1\}$, such that $\log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over $\mathbb{Q}$. Then $n \leq \mathrm{trdeg}(\log \alpha_1, \ldots, \log \alpha_n, \alpha_1, \ldots, \alpha_n) = \mathrm{trdeg}(\log \alpha_1, \ldots, \log \alpha_n)$. I.e. there is a non-trivial algebraic relation over $\overline{\mathbb{Q}}$ of $\log \alpha_1, \ldots, \log \alpha_n$, in particular if $\beta_1, \ldots, \beta_n \in \overline{\mathbb{Q}}$, not all zero, then $\beta \log \alpha_1 + \cdots + \beta_n \log \alpha_n \notin \overline{\mathbb{Q}}$.

What is known towards Schanuel's conjecture?

$n = 1$: If $x \in \mathbb{C}$ is transcendental, then $\mathrm{trdeg}(x, e^x) \geq 1$, if $x \in \overline{\mathbb{Q}} \setminus \{0\}$, then e^x is transcendental by Lindemann-Weierstraß.

$n = 2$: This is still open. For example, we don't know if $(\log 2)(\log 3)$ is transcendental (by Baker's theorem they are $\overline{\mathbb{Q}}$-linearly independent). Schanuel's conjecture would imply that $\mathrm{trdeg}(\log 2, \log 3, 2, 3) = \mathrm{trdeg}(\log 2, \log 3) \geq 2$, i.e. $\log 2, \log 3$ are algebraically independent.

Conjecture 4.31
e and π are algebraically independent.

Conjecture 4.32
Let $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}} \setminus \{0, 1\}$ and assume that $\log \alpha_1, \ldots, \log \alpha_n$ are $\mathbb{Q}$-linearly independent for some choices of $\log \alpha_1, \ldots, \log \alpha_n$. Then $\log \alpha_1, \ldots, \log \alpha_n$ are algebraically independent.

Theorem 4.33
Let $\alpha, \beta \in \overline{\mathbb{Q}} \cap \mathbb{R}$ with $\alpha > 0$, $\alpha \neq 1$ and $\beta \notin \mathbb{Q}$. Let $\log \alpha$ be the real logarithm of α. Then $\alpha^\beta 0 e^{\beta \log \alpha}$ is transcendental.

4.4 Siegel's lemma

Consider m linear equations in n variables, say

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\ &\vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= 0 \end{aligned} \tag{4.4}$$

Assume $m \in n$ and $a_{ij} \in \mathbb{Z}$ for $1 \leq i, j \leq n$. Then there exists a solution $x \in \mathbb{Q}^n \setminus \{0\}$ and hence also a solution $x \in \mathbb{Z}^n \setminus \{0\}$. What can we say about the size of a "smallest" solution?

Example:
- Let $m = 1$, $n = 2$, $a \in \mathbb{Z} \setminus \{0\}$. Then every solution $(x_1, x_2) \in \mathbb{Z}^2 \setminus \{0\}$ to the equations $x_1 + a_x 2 = 0$ satisfies $x_1 \neq 0$ and $a \mid x_1$, i.e. $\|x\|_\infty \geq a$.

- Let $a \in \mathbb{Z} \setminus \{0\}$, m arbitrary and $n = m - 1$. Consider the system1

$$\begin{aligned} x_1 + ax_2 \qquad\qquad\qquad &= 0 \\ x_2 + ax_3 \qquad\quad &= 0 \\ x_{n-1} + ax_n &= 0 \end{aligned}$$

  Every non-trivial solution $x \in \mathbb{Z}^n$ satisfies $x_1 \cdots x_n \neq 0$ and $a^{n-1} \mid x_1$, i.e. $\|x\|_\infty \geq a^{n-1}$

Theorem 4.34 (Siegel's[16] lemma)
Let $n > m > 0$, $A \geq 1$ and $a_{ij} \in \mathbb{Z}$ for $1 \leq i \leq m$, $1 \leq j \leq n$, such that $|a_{ij}| \leq A \ \forall i, j$. Then there exists a solution $x \in \mathbb{Z}^n \setminus \{0\}$ to the system (4.4), such that

$$\max_{1 \leq i \leq n} |x_i| \leq (nA)^{\frac{m}{n-m}}$$

An alternate point of view: Consider $m = 1$. Let $a_1, \ldots, a_n \in \mathbb{Z}$ with $\gcd(a_1, \ldots, a_n) = 1$ and define the hyperplane $H = \{x \in \mathbb{R}^n \mid \sum_{i=1}^n a_i x_i = 0\}$ and the lattice $\Lambda = \{x \in \mathbb{Z}^n \mid \sum_{i=1}^n a_i x_i = 0\}$. Then $\Lambda \subset H$ is a lattice of rank $n - 1$.

Lemma 4.35
Assume that $\gcd(a_1, \ldots, a_n) = 1$. *Then* $d(\Lambda) = \|a\|_2$.

If $\lambda_1, \ldots, \lambda_n$ are successive minima of Λ with respect to $\|\cdot\|_\infty$, then by Minkowski's second convex body theorem, $\lambda_1 \cdots \lambda_{n-1} \ll d(\Lambda) = \|a\|_2$, i.e. $\lambda_1^{n-1} \leq \|a\|_2, \lambda_1 \ll \|a\|_2^{\frac{1}{n-1}}$.

Next, we want to solve (4.4) in $x \in \mathbb{Z}^n$, where the coefficients a_{ij} are elements in some number field.

Example: Let $d \in \mathbb{Z} \setminus \{0, 1\}$ be square-free and $K = \mathbb{Q}(\sqrt{d})$. Set

$$\omega_d = \begin{cases} \sqrt{d}, & d \equiv 2, 3 \bmod 4 \\ \frac{1+\sqrt{d}}{2} & d \equiv 1 \bmod 4 \end{cases}$$

[16]after Carl Ludwig Siegel (1896-1981), a German mathematician

Then $1, \omega_d$ is an integral basis. For $n > 2m > 0$, consider the system

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= 0 \end{aligned} \tag{4.5}$$

with $a_{ij} \in \mathcal{O}_K$. Our goal is to find solutions $(x_1, \ldots, x_n) \in \mathbb{Z}^n \setminus \{0\}$. Our first idea would be to write $a_{ij} = b_{ij} + c_{ij}\omega_d$. Then rewrite (4.5) as

$$\begin{aligned} (b_{11} + c_{ii}\omega_d)x_1 + \ldots + (b_{1n} + c_{1n}\omega_d)x_n &= 0 \\ &\vdots \\ (b_{m1} + c_{m1}\omega_d)x_1 + \ldots + (b_{mn} + c_{mn}\omega_d)x_n &= 0 \end{aligned}$$

This is equivalent to

$$\begin{aligned} b_{11}x_1 + \ldots + b_{1n}x_n &= 0 \\ &\vdots \\ b_{m1}x_1 + \ldots + b_{mn}x_n &= 0 \\ c_{11}x_1 + \ldots + c_{1n}x_n &= 0 \\ &\vdots \\ c_{m1}x_1 + \ldots + x_{mn}x_n &= 0 \end{aligned}$$

If $n > 2m$, then this is a system as in Theorem 4.34 and it has a non-zero solution $x \in \mathbb{Z}^n$ with

$$\max_{1 \le i \le n} |x_i| \le (nA)^{\frac{2m}{n-2m}},$$

where $A = \max i, j\Big\{|b_{ij}|, |c_{ij|}\Big\}$.

Remark: Note that in this construction A depends on the choice of basis $1, \omega_d$. We will use a basis-independent approach below.

Our general set-up will be to let K be a number field of degree d with ring of integers $\mathcal{O}_K$. Let $\sigma_1, \ldots, \sigma_d : K \hookrightarrow \mathbb{C}$ be the d distinct embeddings, such that $\sigma_1, \ldots, \sigma_r : K \hookrightarrow \mathbb{R}$ are the real embeddings and $\sigma_{r+1}, \ldots, \sigma_{r+s} : K \hookrightarrow \mathbb{C}$ complex embeddings with $r + 2s = d$ and $\sigma_{r+s+1} = \overline{\sigma_{r+1}}, \ldots, \sigma_{r+2s} = \overline{\sigma_{r+s}}$. Define the map $\varphi : K \to \mathbb{R}^d$,

$$x \mapsto \Big(\sigma_1(x), \ldots, \sigma_r(x), \Re\sigma_{r+1}(x), \Im\sigma_{r+1}(x), \ldots, \Re\sigma_{r+s}(x), \Im\sigma_{r+s}(x)\Big)$$

Definition (House)
Let $\alpha \in K$. We define the *house of* α as

$$\overline{|\alpha|} = \max\left(|\sigma_1(\alpha)|, \ldots, |\sigma_d(\alpha)|\right).$$

Remark: The definition of $\overline{|\alpha|}$ is independent of the field K with $\alpha \in K$. If $\alpha \in \overline{\mathbb{Q}}$ has a minimal polynomial of degree m over Q and conjugates $\alpha^{(1)}, \ldots, \alpha^{(m)}$, then

$$\overline{|\alpha|} = \max\left(|\alpha^{(1)}|, \ldots, |\alpha^{(m)}|\right).$$

Question: The rational integers $\mathbb{Z} \subset \mathbb{Q}$ are discrete, in particular if $m \in \mathbb{Z}$, $|m| < 1$, then $m = 0$. Is there a similar statement for $\mathcal{O}_K \subset K$?

Observation: Let $\alpha \neq 0$ be an algebraic integer with conjugates $\alpha^{(1)}, \ldots, \alpha^{(m)}$. Then there is at least one index j with

$$|\alpha^{(j)}| \geq 1.$$

Lecture 21, 23.01.2024

Lemma 4.36
Let $\alpha \in \mathcal{O}_K$ with $\|\varphi(\alpha)\|_\infty \leq \frac{2}{3}$, then $\alpha = 0$.

Theorem 4.37
Let $K/\mathbb{Q}$ be as above, i.e. $[K : \mathbb{Q}] = d$. Let $m, n \in \mathbb{N}$ with $n > dm$ and $A \in \mathbb{R}_{\geq 1}$. Suppose $a_{ij} \in \mathcal{O}_K$ with $\overline{|\alpha_{ij}|} \leq A$ for all $1 \leq i \leq m, 1 \leq j \leq n$. Then the system

$$\begin{gathered} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{gathered}$$

has a solution $x \in \mathbb{Z}^n \setminus \{0\}$ with

$$\max_{1 \leq i \leq n} |x_i| \leq (3nA)^{\frac{dm}{n-dm}}.$$

4.5 Approaches towards the Gelfond-Schneider theorem

Theorem 4.38 (Weak version of Gelfond-Schneider theorem)
Let $\alpha, \beta \in \overline{\mathbb{Q}} \cap \mathbb{R}$ with $\alpha > 0$, $\alpha \neq 1$, and $\beta \notin \mathbb{Q}$. Let $\log \alpha$ be the real logarithm of α. Then $\alpha^\beta = e^{\beta \log \alpha}$ is transcendental.

We will give an outline to the proof. Assume that $\gamma = \alpha^\beta \in \overline{\mathbb{Q}}$ and let $K = \mathbb{Q}(\alpha, \beta, \gamma)$, $d = [K : \mathbb{Q}]$. Choose $m \in \mathbb{N}$, such that $m\alpha, m\beta, m\gamma$ are all algebraic integers. Let $D_1, D_2, L \in \mathbb{N}$ be parameters to be chosen later.

Step 1: We construct a function

$$F(z) = \sum_{i=0}^{D_1-1} \sum_{j=0}^{D_2-1} a_{ij} z^i \alpha^{jz}$$

with the following properties:

- $a_{ij} \in \mathbb{Z}$ for all $0 \leq i < D_1$, $0 \leq j < D_2$, not all zero
- $F(a + b\beta) = 0$ for all $1 \leq a, b \leq L$
- there is a constant $c_1 > 0$ only depending on α, β, γ, d, such that

  $$|a_{ij}| \leq e^{c_1(D_1 \log L + D_2 L)}$$

  for all $0 \leq i < D_1$, $0 \leq j < D_2$.

Assume $D_1 D_2 \geq 2dL^2$. Then such a function can be constructed using Siegel's lemma.

Next we choose D_1, D_2, such that $D_1 = D_2 L$ and $D_1 D_2 = 2dL^2$, i.e. $D_1 = \sqrt{2d} L^{\frac{3}{2}}$, $D_2 = \sqrt{2d} L^{\frac{1}{2}}$. If $L = 2dM^2$ for some $M \in \mathbb{N}$, then D_1, D_2 are both integers, and $|a_{ij}| \leq e^{c_3 L^{\frac{3}{2}} \log L}$ for some c_3.

Step 2: We show that $F(z)$ has at most $D_1 D_2 = 2dL^2$ zeros in $\mathbb{R}$. For this we view $F(z)$ as an exponential polynomial, i.e. a function of the form

$$E(z) = \sum_{k=1}^{r} p_k(z) e^{\gamma_k z} \tag{4.6}$$

with $p_k(z) \in \mathbb{R}[z] \setminus \{0\}$ and $\gamma_k \in \mathbb{R}$ distinct.

Lemma 4.39
Let $E(z)$ be as in (4.6). Then $E(z)$ has at most

$$\left(\sum_{k=1}^{r}(1+\deg p_k)\right)-1$$

zeros in $\mathbb{R}$.

Step 3: Let $c = 1 + \lfloor\sqrt{2d}\rfloor$. Using tools from complex analysis we show that $|F(a+b\beta)|$ is very small for $1 \le a, b \le cL$.

Lemma 4.40
Using the same notation as above, let $1 \le a, b \le cL$. Then we have the following estimates for some constants $c_4, c_5 > 0$:

(i) $|F(a+b\beta)| \le e^{c_4 L^{\frac{3}{2}}\log L - L^2}$

(ii) if $\sigma : K \hookrightarrow \mathbb{C}$ is some embedding, then

$$\left|\sigma\Big(F(a+b\beta)\Big)\right| \le e^{c_5 L^{\frac{3}{2}}\log L}$$

From Lemma 4.40 we deduce the following: Let $1 \le a, b \le cL$. Then there is a constant $c_6 > 0$ such that

$$\begin{aligned}\left|N_{K/\mathbb{Q}}\left(m^{D_1+2cLD_2}F(a+b\beta)\right)\right| &= m^{d(D_1+2cLD_2)}\left|\prod_{\sigma:K\hookrightarrow\mathbb{C}}\sigma\Big(F(a+b\beta)\Big)\right| \\ &\le m^{d(D_1+2cLD_2)}e^{c_4L^{\frac{3}{2}}\log L - L^2 + (d-1)c_5L^{\frac{3}{2}}\log L} \\ &\le e^{c_6L^{\frac{3}{2}}\log L - L^2}\end{aligned}$$

Hence, for L sufficiently large, we find

$$\left|N_{K/\mathbb{Q}}\left(\underbrace{m^{D_1+2cLD_2}F(a+b\beta)}_{\in\mathcal{O}_K}\right)\right| < 1$$

and $F(a+b\beta) = 0$ for $1 \le a, b \le cL$. As $1, \beta$ are $\mathbb{Q}$-linearly independent, this implies that F has at least $c^2L^2 > 2dL^2 = D_1D_2$ real zeros, which is a contradiction to step 2.

5 Diophantine Approximations and applications to Diophantine problems

5.1 Linear forms in logarithms

Lecture 22, 26.01.2024

We recall

Theorem 5.1 (Baker, 1975)
Let $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}} \setminus \{0,1\}$, $\gamma, \beta_1, \ldots, \beta_n \in \overline{\mathbb{Q}}$ and pick a choice of logarithms $\log \alpha_1, \ldots, \log \alpha_n$. If $\gamma + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n \neq 0$, then

$$\left|\gamma + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n\right| \geq (eB)^{-C},$$

where $B = \max(H(\gamma), H(\beta_1), \ldots, H(\beta_n))$ and $C = C(\alpha_1, \ldots, \alpha_n)$ is an effectively computable constant.

We move towards applications:

Corollary 5.2
Let $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}} \setminus \{0,1\}$, $b_1, \ldots, b_n \in \mathbb{Z}$ and assume that $\alpha_1^{b_1} \cdots \alpha_n^{b_n} \neq 1$. Then

$$\left|\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1\right| \geq (eB)^{-\tilde{C}},$$

where $B = \max\left(|b_1|, \ldots, |b_n|\right)$ and $\tilde{C} > 0$ is an effectively computable constant depending on n and $\alpha_1, \ldots, \alpha_n$.

Corollary 5.3
Let $a, b \in \mathbb{Z}$ with $a, b \geq 2$. Then there exists an effectively computable constant $C_1 = C_1(a,b) > 0$, such that for every $m, n \in \mathbb{N}$ with $a^m \neq b^n$, we have

$$|a^m - b^n| \geq \frac{\max(a^m, b^n)}{(e \max(m,n))^{C_1}}.$$

Remark: a) Let $a, b \in \mathbb{Z}_{\geq 2}$ and $k \in \mathbb{Z}_{\neq 0}$. Then there is a constant $C_2 > 0$, such that all solutions $m, n \in \mathbb{N}$ to the equation $a^m - b^n = k$ satisfy

$$\max(|m|, |n|) \leq C_2 \,.$$

b) Using techniques from linear forms in logarithms, Tijdeman[1] (1976) even proved that there exists an effectively computable constant $C_3 > 0$, such that if $a^m - b^n = 1$ with $a, b, m, n \in \mathbb{Z}_{\geq 2}$, then

$$a^m, b^n \leq C_3 \,.$$

Catalan's conjecture (1844), that the equation $a^m - b^n = 1$ with $a, b, m, n \in \mathbb{N}_{\geq 2}$ has only one solution, namely $3^2 - 2^3 = 1$, was proven by Mihăilescu[2] in 2002.

Corollary 5.4
Let $p_1, \ldots, p_t$ be prime numbers and let $(a_n)_{n \in \mathbb{N}}$ be the monotonically increasing sequence of natural numbers, which are composed of the primes $p_1, \ldots, p_t$. Then there exist effectively computable constants C_4, C_5 only depending on $t, p_1, \ldots, p_n$, such that

$$a_n - a_{n-1} \geq \frac{a_n}{C_4 (\log a_n)^{C_5}}, \quad n \in \mathbb{N}_{\geq 2} \,.$$

Example: For $t = 1,\ p_1 = 2$, we obtain the sequence $2^n,\ n \geq 0$, and the stronger statement

$$a_n - a_{n-1} = 2^n - 2^{n-1} = \frac{1}{2} 2^n = \frac{1}{2} a_n \,.$$

Unit equations

Let K be a number field with ring of integers $\mathcal{O}_K$. Let $\alpha, \beta \in K^*$. What can we say about solutions to the equation

$$\alpha x + \beta y = 1 \tag{5.1}$$

in $x, y \in \mathcal{O}_K^*$? For example, are there infinitely many units x, y, such that $x = 1 - y$?

Theorem 5.5 (Baker 1960s, Györy 1978)
For every number field K, (5.1) has at most finitely many solutions and they can be effectively determined.

[1] Robert Tijdeman (*1943), a Dutch mathematician
[2] Preda Mihăilescu (*1955), a Romanian mathematician, who currently teaches in Göttingen

Remark: The finiteness statement, though ineffective, has already been proved by Siegel in 1921.

We first make some preparations for the proof of Theorem 5.5: Let $d = [K : \mathbb{Q}]$ and assume that K has r real embeddings $\sigma_1, \dots, \sigma_r : K \hookrightarrow \mathbb{R}$ and $\sigma_{r+1}, \dots, \sigma_{r+s} : K \hookrightarrow \mathbb{C}$ complex embeddings with $r + 2s = d$ and $\sigma_{r+s+1} = \overline{\sigma_{r+1}}, \dots, \sigma_{r+2s} = \overline{\sigma_{r+s}}$. Let $u_1, \dots, u_{r+s-1}$ be a fundamental system of units.

Lemma 5.6
Let $u \in \mathcal{O}_K^*$ and write $u = \xi u_1^{b_1} \cdots u_{r+s-1}^{b_{r+s-1}}$ with $\xi \in K$ a root of unity and $b_1, \dots, b_{r+s-1} \in \mathbb{Z}$. There exists an effectively computable constant $C_6 > 0$ depending only on $K, u_1, \dots, u_{r+s-1}$, such that

$$\max\Big(|b_1|, \dots, |b_{r+s-1}|\Big) \leq C_6 \log \overline{|u|}\,.$$

Lecture 23, 30.01.2024

Thue equations

Consider a binary form $F(X, Y) \in \mathbb{Z}[X, Y]$ of degree d, i.e. an expression of the form $F(X, Y) = a_0 X^d + a_1 X^{d-1} Y + \dots + a_d Y^d$ with $a_0, \dots, a_d \in \mathbb{Z}$. Let $m \in \mathbb{Z}$.

Question: What can we say about solutions to the equation $F(X, Y) = m$ with $x, y \in \mathbb{Z}$? For $d > 3$ we call these Thue[3] equations.

Our next goal will be to prove a finiteness result.

Theorem 5.7
Let $F(X, Y) \in \mathbb{Z}[X, Y]$ be a binary form of degree $d > 3$ with $F(1, 0) \neq 0$ and such that $F(X, 1)$ has at least three distinct roots in $\mathbb{C}$. Let $m \in \mathbb{Z} \setminus \{0\}$. Then there are finitely many solutions $x, y \in \mathbb{Z}$ to the equation $F(X, Y) = m$.

Lemma 5.8
Let K be a number field of degree d. Then there exists an effectively computable constant $C_8 > 1$, such that for every $\alpha \in \mathcal{O}_K$ there exists $u \in \mathcal{O}_K^*$ with

$$C_8^{-1} |N_{K/\mathbb{Q}}(\alpha)|^{\frac{1}{d}} \leq \overline{|u\alpha|} \leq C_8 |N_{K/\mathbb{Q}}(\alpha)|^{\frac{1}{d}}\,.$$

[3] after Axel Thue (1863 - 1922), a Norwegian mathematician

Remark: Note that $\left|\prod_{j=1}^{d}\sigma_j(u\alpha)\right| = |N_{K/\mathbb{Q}}(u\alpha)| = |N_{K/\mathbb{Q}}(\alpha)|$, and hence for every $1 \le j \le d$ we have $|N_{K/\mathbb{Q}}(\alpha)| \le |\sigma_j(u\alpha)| \left(C_8|NK/\mathbb{Q}(\alpha)|^{\frac{1}{d}}\right)^{d-1}$, i.e.

$$|\sigma_j(u\alpha)| \ge C_8^{d-1}|N_{K/\mathbb{Q}}(\alpha)|^{\frac{1}{d}}$$

and we find that all of the conjugates $|\sigma_j(u\alpha)|$ are of computable size.

Lemma 5.9
Let $\alpha \in \mathcal{O}_K \setminus \{0\}$. Then there exist divisors $\beta_1, \dots, \beta_t$ of α in $\mathcal{O}_K$, which can be determined effectively and with the following property: If $\gamma \mid \alpha$ in $\mathcal{O}_K$, then there exists $1 \le j \le t$ and $u \in \mathcal{O}_K^*$, such that $\gamma = u\beta_j$.

An application of Theorem 5.7: Consider the equation

$$y^2 = 2x(x-3) \tag{5.2}$$

in $x, y \in \mathbb{Z}$. For $x \in \mathbb{Z}$ we have $\gcd(2x, x-3) \mid 6$. Hence if $(x,y) \in \mathbb{Z}^2$ is a solution to (5.2), then we can write $2x = au^3$, $x - 3 = bv^3$, $u, v \in \mathbb{Z}$ with $a, b \in \left\{\pm 2^k 3^l \,\middle|\, 0 \le k, l \le 2\right\}$ and such that ab is a cube. Fix such a tuple (a,b). Then we find a solution (u,v) to the equation $au^3 - 2bv^3 = 6$, which is a Thue equation. Using the finiteness result for Thue equations we can show that (5.2) has at most finitely many solutions.

Theorem (Baker, 1968)
Let $f(X) \in \mathbb{Z}[X]$ of degree d, $b \in \mathbb{Z} \setminus \{0\}$ and $n \in \mathbb{N}_{\ge 2}$. Assume that f has no multiple zeros and $d \ge 2$ if $n > 3$ and $d \ge 3$ if $n = 2$. Then the equation

$$by^n = f(x)$$

has at most finitely many integer solutions, which can be determined effectively.

Remark: For $n = 2$ we obtain hyperelliptic equations.

Lecture 24, 02.02.2024

5.2 Roth's theorem

Definition (Approximation exponent)
Let $\alpha \in \mathbb{R}$. We define the *approximation exponent* τ_α of α as the infimum of real

numbers $\tau > 0$, such that for all $\delta > 0$ the inequality

$$|\alpha - \xi| \leq H(\xi)^{-\tau-\delta}$$

has only finitely many solutions $\xi \in \mathbb{Q}$.

Remark: a) For $\xi = \frac{p}{q} \in \mathbb{Q}$ with $p, q \in \mathbb{Z}$, $q \neq 0$, and $\gcd(p, q) = 1$, we have

$$H(\xi) = \max(|p|, |q|)\,.$$

b) For $\alpha \in \mathbb{Q}$ we have $\tau_\alpha = 1$.

c) If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then by Dirichlet's theorem we have $\tau_\alpha \geq 2$.

Theorem (Roth, 1955)
Let $\alpha \in \mathbb{R} \cap \overline{\mathbb{Q}}$ and $\kappa > 2$. Then there exists a constant $c(\alpha, \kappa) > 0$, such that

$$|\xi - \alpha| \geq c(\alpha, \kappa) H(\xi)^{-\kappa}$$

for all $\xi \in \mathbb{Q}$, $\xi \neq \alpha$.

Remark: a) For all $\alpha \in \mathbb{R} \cap \overline{\mathbb{Q}}$ with $\alpha \notin \mathbb{Q}$, we deduce that the approximation exponent $\tau_\alpha = 2$.

b) The constant $c(\alpha, \kappa)$ in Roth's theorem is ineffective, i.e. with the methods of the proof one cannot compute $c(\alpha, \kappa)$ explicitly.

We will prove a weaker result:

Theorem 5.10 (Liouville, 1844)
Let $\alpha \in \mathbb{R} \cap \overline{\mathbb{Q}}$ of degree $d \geq 1$. Then there exits an effectively computable constant $c(\alpha) > 0$, such that

$$|\xi - \alpha| \geq c(\alpha) H(\xi)^{-d}$$

for all $\xi \in \mathbb{Q}$, $\xi \neq \alpha$.

Remark: If $f(X) = a_0 X^d + \cdots + a_d \in \mathbb{Z}[X]$ is a primitive polynomial of α, then we define the *Mahler measure* of α as

$$M(\alpha) = |a_0| \prod_{j=1}^{d} \max(1, |a^{(j)}|)$$

and Liouville's theorem can be formulated as

$$|\alpha - \xi| \geq 2^{1-d} M(\alpha) H(\xi)^{-d}$$

for all $\xi \in \mathbb{Q}$, $\xi \neq \alpha$.

Conjecture (Lehmer, 1930s)
There is a constant $c > 0$, such that for all $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$, which is not a root of unity, one has

$$M(\alpha) \geq 1 + c\,.$$

Roth's theorem and Thue equations

Definition (Square-free binary form)
We call a binary form $F(X, Y) \in \mathbb{Z}[X, Y]$ *binary-free* if it is not divisible by a square $(\alpha X + \beta Y)^2 \in \mathbb{C}[X, Y]$, $\alpha, \beta \in \mathbb{C}$ not both zero.

Theorem 5.11
Let $F(X, Y) \in \mathbb{Z}[X, Y]$ be a square-free form of degree $d \geq 3$ and $\kappa > 2$. Then there exists a constant $c(F, \kappa) > 0$, such that for every $(p, q) \in \mathbb{Z}^2$ with $F(p, q) \neq 0$, we have

$$|F(p, q)| \geq c(F, \kappa) \max(|p|, |q|)^{d-\kappa}\,.$$

Corollary 5.12
Let $F(X, Y) \in \mathbb{Z}[X, Y]$ be a binary form, such that $F(X, 1)$ has at least three distinct roots, and $m \in \mathbb{Z} \setminus \{0\}$. Then the equation $F(X, Y) = m$ has at most finitely many integer solutions.

5.3 Schmidt's subspace theorem

Theorem 5.13 (Schmidt, 1972)
Let $L_i(X_1, \ldots, X_n) = a_{i1}X_1 + \cdots + a_{in}X_n$, $1 \leq i \leq n$, be n linearly independent integer forms with $a_{ij} \in \overline{\mathbb{Q}}$, $1 \leq i, j \leq n$, $n \geq 2$. Let $\delta, C > 0$. Then all integer

solutions $x \in \mathbb{Z}^n$ to the inequality

$$|L_1(X) \cdots L_n(X)| \leq C\|X\|_\infty^{-\delta}$$

are contained in a finite union of proper linear subspaces of $\mathbb{Q}^n$.

Question: Under which conditions does the inequality

$$|L_1(X) \cdots L_n(X)| \leq C\|X\|_\infty^{-\delta} \tag{5.3}$$

in Schmidt's subspace theorem have only *finitely many* solutions?

Example: Let $L_1(X) = a_{11}X_1 + \ldots a_{1n}X_n$ with $a_{11}, \ldots, a_{1n} \in \mathbb{Z}$ and $X^{(0)} = (X_1^{(0)}, \ldots, X_n^{(0)}) \in \mathbb{Z}^n \setminus \{0\}$ a solution to $L_1(X^{(0)}) = 0$. Then (5.3) has infinitely many solutions given for example by $\lambda X^{(0)}$, $\lambda \in \mathbb{Z}$.

Question: How about restricting to the inequality

$$0 < |L_1(X) \cdots L_n(X)| \leq C\|X\|_\infty^{-\delta}\,?$$

Definitions

Important theorems