Derivative Helper

Note on Notation

There are a few ways that you are likely to see derivatives notated in NSW HSC maths textbooks and exam papers. Just remember that these are not hard rules and you should use the notation that fits with a question or feels most comfortable to you while communicating your intentions clearly.

When a situation involves functions you are likely to see derivatives as $f'(x)$, $g'(x)$ and $f''(x)$, where the derivative is with respect to the input variable—usually x. This apostrophe mark may also be used with variables such as $y = 3x^2$ becomes $y' = 6x$. This mark may be pronounced "prime", so you might say "y prime" to mean the first derivative of y.

The other main form of notation is $\frac{dy}{dx}$, which represents the derivative of y with respect to x. Usually this is seen when equations use y or some other variable such as $u = 3x$ gives $\frac{du}{dx} = 3$. Multiple derivatives are represented with index notation as $\frac{d^2u}{dx^2}$ would be the second derivative of u with respect to x. A rarer use of this notation is $\frac{d}{dx}$ which is an operator that roughly means "take the derivative of [expression] with respect to x". For example, if we wanted the derivative of $2x^4$ we might write $\frac{d}{dx}(2x^4) = 8x$.

Constants and Coefficients

The derivative of a constant expression is 0. For example, if $u = 10$ then $\frac{du}{dx} = 0$. More generally for any constant C,

$$\frac{d}{dx}(C) = 0.$$

Constant coefficients are a bit different, consider a function, f, that is another function, g, multiplied by a constant, C. The derivative of this function, $f(x) = Cg(x)$, would be

$$f'(x) = \frac{d}{dx}(Cg'(x)) = C\frac{d}{dx}(g(x)) = Cg'(x).$$

Notice that the constant coefficient doesn't change, I like to imagine it's just along for the ride. A more concrete example might be the function $f(x) = 4x^3$, in this case 4 is the constant coefficient of the function $g(x) = x^3$. So when we want to find $f'(x)$ we can apply the power rule to x^3 then multiply by 4, which might look like

$$f'(x) = 4 \cdot \frac{d}{dx}(x^3) = 4 \cdot 3x^2 = 12x^2.$$

Sums and Subtraction

When a function is made up of multiple expressions separated by addition or subtraction—which is just funky addition—you can find the derivative of the function, by finding the derivative of each of the smaller parts. For example, if $f(x) = g_1(x) + g_2(x) - g_3(x)$ then $f'(x) = g_1'(x) + g_2'(x) - g_3'(x)$. For a more concrete example consider $f(x) = x^2 + 3x + 9$, to find the derivative we will find the derivative of each part.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^2) + 3 \cdot \frac{d}{dx}(x) - \frac{d}{dx}(9), \\ &= 2x + 3 \cdot 1 - 0, \\ &= 2x + 3. \end{aligned}$$

Formulas

First Principles

The first principles formula is effectively taking the rise over run gradient of a very small section of a function. It looks like

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

To use to to find a derivative, you expand the function applications and simplify until you are no longer dividing by h, at which point you can substitute $h = 0$ and you are done.

Power Rule

The power rule is used for expressions of a variable that have a constant exponent. For simple powers of x use the rule

$$y = x^n \quad \frac{dy}{dx} = nx^{n-1}.$$

The more general formula uses the chain rule, and is

$$y = f(x)^n \quad \frac{dy}{dx} = nf'(x)[f(x)]^{n-1}.$$

Product Rule

The product rule is used when a function is a product of two expressions in terms of x. The common formula is for when $y = uv$,

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Chain Rule

The chain rule is used when finding the derivative of a function that has another function as its input—also called function composition. You might see this written as, for $y = g(u)$ where $u = f(x)$,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

This means to first do the derivative of the outside function with respect to its input function, then multiply by the derivative of the input function with respect to x. Another way of writing this would be, if $y = g(f(x))$ then,

$$y' = f'(x) \times g'(f(x))$$

Quotient Rule

This is used when one function is divided by another. Consider $y = \dfrac{u}{v}$, the derivative of y would be

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

When applying this formula it is important to make sure you keep track of what functions are u and v. Another option is to convert $y = \dfrac{u}{v}$ into $y = uv^{-1}$ and use the product, power, and chain rules.

Exponentials and Logarithms

The derivative formulas for exponentials and logarithms tend to be surprisingly simple. Here are the simple cases,

$$\frac{d}{dx}(e^x) = e^x \text{ and } \frac{d}{dx}(\ln(x)) = \frac{1}{x}.$$

More generally you will apply the chain rule, for $y = e^{f(x)}$, you have

$$\frac{dy}{dx} = f'(x)e^{f(x)},$$

and for $y = \ln(f(x))$ you have,

$$\frac{dy}{dx} = \frac{f'(x)}{f(x)}.$$

These forms are also useful when you have a base that is not e. The helpful rules to remember are,

$$a^x = e^{\ln(a)x} \text{ and } \log_a(x) = \frac{\ln(x)}{\ln(a)}.$$

Trigonometric Functions

The trig functions have slightly strange derivatives. The basic versions are,

$$y = \sin(x) \text{ gives } \frac{dy}{dx} = \cos(x),$$
$$y = \cos(x) \text{ gives } \frac{dy}{dx} = -\sin(x),$$
$$y = \tan(x) \text{ gives } \frac{dy}{dx} = \sec^2(x),$$

Notice that the derivatives of cos and sin eventually make a loop. The chain rule versions of these are,

$$y = \sin(f(x)) \text{ gives } \frac{dy}{dx} = f'(x)\cos(f(x)),$$
$$y = \cos(f(x)) \text{ gives } \frac{dy}{dx} = -f'(x)\sin(f(x)),$$
$$y = \tan(f(x)) \text{ gives } \frac{dy}{dx} = f'(x)\sec^2(f(x)),$$

Inverse Trig Functions

These functions have the strangest derivative formulas. The simple versions are

$$y = \sin^{-1}(x) \text{ gives } \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}},$$
$$y = \cos^{-1}(x) \text{ gives } \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}},$$
$$y = \tan^{-1}(x) \text{ gives } \frac{dy}{dx} = \frac{1}{1+x^2},$$

The chain rule versions are

$$y = \sin^{-1}(f(x)) \text{ gives } \frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-[f(x)]^2}},$$
$$y = \cos^{-1}(f(x)) \text{ gives } \frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1-[f(x)]^2}},$$
$$y = \tan^{-1}(f(x)) \text{ gives } \frac{dy}{dx} = \frac{f'(x)}{1+[f(x)]^2},$$