APEX CALCULUS III

Version 3.0

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Preface

A Note on Using this Text

Thank you for reading this short preface. Allow us to share a few key points about the text so that you may better understand what you will find beyond this page.

This text is Part III of a three–text series on Calculus. The first part covers material taught in many "Calc 1" courses: limits, derivatives, and the basics of integration, found in Chapters 1 through 6.1. The second text covers material often taught in "Calc 2:" integration and its applications, along with an introduction to sequences, series and Taylor Polynomials, found in Chapters 5 through 8. The third text covers topics common in "Calc 3" or "multivariable calc:" parametric equations, polar coordinates, vector–valued functions, and functions of more than one variable, found in Chapters 9 through 13. All three are available separately for free at `www.vmi.edu/APEX`. These three texts are intended to work together and make one cohesive text, *APEX Calculus*, which can also be downloaded from the website.

Printing the entire text as one volume makes for a large, heavy, cumbersome book. One can certainly only print the pages they currently need, but some prefer to have a nice, bound copy of the text. Therefore this text has been split into these three manageable parts, each of which can be purchased for under $15 at Amazon.com.

A result of this splitting is that sometimes a concept is said to be explored in an "earlier section," though that section does not actually appear in this particular text. Also, the index makes reference to topics, and page numbers, that do not appear in this text. This is done intentionally to show the reader what topics are available for study. Downloading the .pdf of *APEX Calculus* will ensure that you have all the content.

For Students: How to Read this Text

Mathematics textbooks have a reputation for being hard to read. High–level mathematical writing often seeks to say much with few words, and this style often seeps into texts of lower–level topics. This book was written with the goal of being easier to read than many other calculus textbooks, without becoming too verbose.

Each chapter and section starts with an introduction of the coming material, hopefully setting the stage for "why you should care," and ends with a look ahead to see how the just–learned material helps address future problems.

Please read the text; it is written to explain the concepts of Calculus. There are numerous examples to demonstrate the meaning of definitions, the truth of theorems, and the application of mathematical techniques. When you encounter a sentence you don't understand, read it again. If it still doesn't make sense, read on anyway, as sometimes confusing sentences are explained by later sentences.

You don't have to read every equation. The examples generally show "all" the steps needed to solve a problem. Sometimes reading through each step is helpful; sometimes it is confusing. When the steps are illustrating a new technique, one probably should follow each step closely to learn the new technique. When the steps are showing the mathematics needed to find a number to be used later, one can usually skip ahead and see how that number is being used, instead of getting bogged down in reading how the number was found.

Most proofs have been omitted. In mathematics, *proving* something is always true is extremely important, and entails much more than testing to see if it works twice. However, students often are confused by the details of a proof, or become concerned that they should have been able to construct this proof on their own. To alleviate this potential problem, we do not include the proofs to most theorems in the text. The interested reader is highly encouraged to find proofs online or from their instructor. In most cases, one is very capable of understanding what a theorem *means* and *how to apply it* without knowing fully *why* it is true.

Interactive, 3D Graphics

New to Version 3.0 is the addition of interactive, 3D graphics in the .pdf version. Nearly all graphs of objects in space can be rotated, shifted, and zoomed in/out so the reader can better understand the object illustrated.

As of this writing, the only pdf viewers that support these 3D graphics are Adobe Reader & Acrobat (and only the versions for PC/Mac/Unix/Linux computers, not tablets or smartphones). To activate the interactive mode, click on the image. Once activated, one can click/drag to rotate the object and use the scroll wheel on a mouse to zoom in/out. (A great way to investigate an image is to first zoom in on the page of the pdf viewer so the graphic itself takes up much of the screen, then zoom inside the graphic itself.) A CTRL-click/drag pans the object left/right or up/down. By right-clicking on the graph one can access a menu of other options, such as changing the lighting scheme or perspective. One can also revert the graph back to its default view. If you wish to deactive the interactivity, one can right-click and choose the "Disable Content" option.

Thanks

There are many people who deserve recognition for the important role they have played in the development of this text. First, I thank Michelle for her support and encouragement, even as this "project from work" occupied my time and attention at home. Many thanks to Troy Siemers, whose most important contributions extend far beyond the sections he wrote or the 227 figures he coded in Asymptote for 3D interaction. He provided incredible support, advice and encouragement for which I am very grateful. My thanks to Brian Heinold and Dimplekumar Chalishajar for their contributions and to Jennifer Bowen for reading through so much material and providing great feedback early on. Thanks to Troy, Lee Dewald, Dan Joseph, Meagan Herald, Bill Lowe, John David, Vonda Walsh, Geoff Cox, Jessica Libertini and other faculty of VMI who have given me numerous suggestions and corrections based on their experience with teaching from the text. (Special thanks to Troy, Lee & Dan for their patience in teaching Calc III while I was still writing the Calc III material.) Thanks to Randy Cone for encouraging his tutors of VMI's Open Math Lab to read through the text and check the solutions, and thanks to the tutors for spending their time doing so. A very special thanks to Kristi Brown and Paul Janiczek who took this opportunity far above & beyond what I expected, meticulously checking every solution and carefully reading every example. Their comments have been extraordinarily helpful. I am also thankful for the support provided by Wane Schneiter, who as my Dean provided me with extra time to work on this project. I am blessed to have so many people give of their time to make this book better.

APEX – Affordable Print and Electronic teXts

APEX is a consortium of authors who collaborate to produce high–quality, low–cost textbooks. The current textbook–writing paradigm is facing a potential revolution as desktop publishing and electronic formats increase in popularity. However, writing a good textbook is no easy task, as the time requirements alone are substantial. It takes countless hours of work to produce text, write examples and exercises, edit and publish. Through collaboration, however, the cost to any individual can be lessened, allowing us to create texts that we freely distribute electronically and sell in printed form for an incredibly low cost. Having said that, nothing is entirely free; someone always bears some cost. This text "cost" the authors of this book their time, and that was not enough. *APEX Calculus* would not exist had not the Virginia Military Institute, through a generous Jackson–Hope grant, given the lead author significant time away from teaching so he could focus on this text.

Each text is available as a free .pdf, protected by a Creative Commons Attribution - Noncommercial 4.0 copyright. That means you can give the .pdf to anyone you like, print it in any form you like, and even edit the original content and redistribute it. If you do the latter, you must clearly reference this work and you cannot sell your edited work for money.

We encourage others to adapt this work to fit their own needs. One might add sections that are "missing" or remove sections that your students won't need. The source files can be found at `github.com/APEXCalculus`.

You can learn more at `www.vmi.edu/APEX`.

9: VECTORS

This chapter introduces a new mathematical object, the **vector**. Defined in Section 9.2, we will see that vectors provide a powerful language for describing quantities that have magnitude and direction aspects. A simple example of such a quantity is force: when applying a force, one is generally interested in how much force is applied (i.e., the magnitude of the force) and the direction in which the force was applied. Vectors will play an important role in many of the subsequent chapters in this text.

This chapter begins with moving our mathematics out of the plane and into "space." That is, we begin to think mathematically not only in two dimensions, but in three. With this foundation, we can explore vectors both in the plane and in space.

9.1 Introduction to Cartesian Coordinates in Space

Up to this point in this text we have considered mathematics in a 2–dimensional world. We have plotted graphs on the x-y plane using rectangular and polar coordinates and found the area of regions in the plane. We have considered properties of *solid* objects, such as volume and surface area, but only by first defining a curve in the plane and then rotating it out of the plane.

While there is wonderful mathematics to explore in "2D," we live in a "3D" world and eventually we will want to apply mathematics involving this third dimension. In this section we introduce Cartesian coordinates in space and explore basic surfaces. This will lay a foundation for much of what we do in the remainder of the text.

Each point P in space can be represented with an ordered triple, $P = (a, b, c)$, where a, b and c represent the relative position of P along the x-, y- and z-axes, respectively. Each axis is perpendicular to the other two.

Visualizing points in space on paper can be problematic, as we are trying to represent a 3-dimensional concept on a 2–dimensional medium. We cannot draw three lines representing the three axes in which each line is perpendicular to the other two. Despite this issue, standard conventions exist for plotting shapes in space that we will discuss that are more than adequate.

One convention is that the axes must conform to the **right hand rule**. This rule states that when the index finger of the right hand is extended in the direction of the positive x-axis, and the middle finger (bent "inward" so it is perpendicular to the palm) points along the positive y-axis, then the extended thumb will point in the direction of the positive z-axis. (It may take some thought to ve-

Figure 9.1: Plotting the point $P = (2, 1, 3)$ in space.

Figure 9.2: Plotting the point $P = (2, 1, 3)$ in space with a perspective used in this text.

rify this, but this system is inherently different from the one created by using the "left hand rule.")

As long as the coordinate axes are positioned so that they follow this rule, it does not matter how the axes are drawn on paper. There are two popular methods that we briefly discuss.

In Figure 9.1 we see the point $P = (2, 1, 3)$ plotted on a set of axes. The basic convention here is that the x-y plane is drawn in its standard way, with the z-axis down to the left. The perspective is that the paper represents the x-y plane and the positive z axis is coming up, off the page. This method is preferred by many engineers. Because it can be hard to tell where a single point lies in relation to all the axes, dashed lines have been added to let one see how far along each axis the point lies.

One can also consider the x-y plane as being a horizontal plane in, say, a room, where the positive z-axis is pointing up. When one steps back and looks at this room, one might draw the axes as shown in Figure 9.2. The same point P is drawn, again with dashed lines. This point of view is preferred by most mathematicians, and is the convention adopted by this text.

Measuring Distances

It is of critical importance to know how to measure distances between points in space. The formula for doing so is based on measuring distance in the plane, and is known (in both contexts) as the Euclidean measure of distance.

Definition 0 Distance In Space

Let $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ be points in space. The distance D between P and Q is

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

We refer to the line segment that connects points P and Q in space as $\overline{PQ}$, and refer to the length of this segment as $||\overline{PQ}||$. The above distance formula allows us to compute the length of this segment.

Example 0 Length of a line segment

Let $P = (1, 4, -1)$ and let $Q = (2, 1, 1)$. Draw the line segment $\overline{PQ}$ and find its length.

Solution The points P and Q are plotted in Figure 9.3; no special consideration need be made to draw the line segment connecting these two points;

Notes:

simply connect them with a straight line. One *cannot* actually measure this line on the page and deduce anything meaningful; its true length must be measured analytically. Applying Definition 0, we have

$$||\overline{PQ}|| = \sqrt{(2-1)^2 + (1-4)^2 + (1-(-1))^2} = \sqrt{14} \approx 3.74.$$

Figure 9.3: Plotting points P and Q in Example 0.

Spheres

Just as a circle is the set of all points in the *plane* equidistant from a given point (its center), a sphere is the set of all points in *space* that are equidistant from a given point. Definition 0 allows us to write an equation of the sphere.

We start with a point $C = (a, b, c)$ which is to be the center of a sphere with radius r. If a point $P = (x, y, z)$ lies on the sphere, then P is r units from C; that is,

$$||\overline{PC}|| = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = r.$$

Squaring both sides, we get the standard equation of a sphere in space with center at $C = (a, b, c)$ with radius r, as given in the following Key Idea.

Key Idea 0 Standard Equation of a Sphere in Space

The standard equation of the sphere with radius r, centered at $C = (a, b, c)$, is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

Example 1 Equation of a sphere

Find the center and radius of the sphere defined by $x^2+2x+y^2-4y+z^2-6z = 2$.

Solution To determine the center and radius, we must put the equation in standard form. This requires us to complete the square (three times).

$$\begin{aligned} x^2 + 2x + y^2 - 4y + z^2 - 6z &= 2 \\ (x^2 + 2x + 1) + (y^2 - 4y + 4) + (z^2 - 6z + 9) - 14 &= 2 \\ (x+1)^2 + (y-2)^2 + (z-3)^2 &= 16 \end{aligned}$$

The sphere is centered at $(-1, 2, 3)$ and has a radius of 4.

The equation of a sphere is an example of an implicit function defining a surface in space. In the case of a sphere, the variables x, y and z are all used. We now consider situations where surfaces are defined where one or two of these

Notes:

variables are absent.

Introduction to Planes in Space

The coordinate axes naturally define three planes (shown in Figure 9.4), the **coordinate planes**: the x-y plane, the y-z plane and the x-z plane. The x-y plane is characterized as the set of all points in space where the z-value is 0. This, in fact, gives us an equation that describes this plane: $z = 0$. Likewise, the x-z plane is all points where the y-value is 0, characterized by $y = 0$.

Figure 9.4: The coordinate planes.

The equation $x = 2$ describes all points in space where the x-value is 2. This is a plane, parallel to the y-z coordinate plane, shown in Figure 9.5.

Figure 9.5: The plane $x = 2$.

Example 2 Regions defined by planes

Sketch the region defined by the inequalities $-1 \leq y \leq 2$.

Solution The region is all points between the planes $y = -1$ and $y = 2$. These planes are sketched in Figure 9.6, which are parallel to the x-z plane. Thus the region extends infinitely in the x and z directions, and is bounded by planes in the y direction.

Figure 9.6: Sketching the boundaries of a region in Example 2.

Cylinders

The equation $x = 1$ obviously lacks the y and z variables, meaning it defines points where the y and z coordinates can take on any value. Now consider the equation $x^2 + y^2 = 1$ *in space.* In *the plane*, this equation describes a circle of radius 1, centered at the origin. In space, the z coordinate is not specified, meaning it can take on any value. In Figure 9.8 (a), we show part of the graph of the equation $x^2 + y^2 = 1$ by sketching 3 circles: the bottom one has a constant z-value of -1.5, the middle one has a z-value of 0 and the top circle has a z-value

Notes:

of 1. By plotting *all* possible z-values, we get the surface shown in Figure 9.8 (b). This surface looks like a "tube," or a "cylinder"; mathematicians call this surface a **cylinder** for an entirely different reason.

Figure 9.8: Sketching $x^2 + y^2 = 1$.

Definition 1 Cylinder

Let C be a curve in a plane and let L be a line not parallel to C. A **cylinder** is the set of all lines parallel to L that pass through C. The curve C is the **directrix** of the cylinder, and the lines are the **rulings**.

In this text, we consider curves C that lie in planes parallel to one of the coordinate planes, and lines L that are perpendicular to these planes, forming **right cylinders**. Thus the directrix can be defined using equations involving 2 variables, and the rulings will be parallel to the axis of the 3rd variable.

In the example preceding the definition, the curve $x^2 + y^2 = 1$ in the x-y plane is the directrix and the rulings are lines parallel to the z-axis. (Any circle shown in Figure 9.8 can be considered a directrix; we simply choose the one where $z = 0$.) Sample rulings can also be viewed in part (b) of the figure. More examples will help us understand this definition.

Example 3 Graphing cylinders

Graph the cylinder following cylinders.

1. $z = y^2$ 2. $x = \sin z$

Solution

1. We can view the equation $z = y^2$ as a parabola in the y-z plane, as illustrated in Figure 9.7 (a). As x does not appear in the equation, the rulings are lines through this parabola parallel to the x-axis, shown in (b). These rulings give a general idea as to what the surface looks like, drawn in (c).

Figure 9.7: Sketching the cylinder defined by $z = y^2$.

Notes:

2. We can view the equation $x = \sin z$ as a sine curve that exists in the x-z plane, as shown in Figure 9.9 (a). The rules are parallel to the y axis as the variable y does not appear in the equation $x = \sin z$; some of these are shown in part (b). The surface is shown in part (c) of the figure.

Figure 9.9: Sketching the cylinder defined by $x = \sin z$.

Figure 9.10: Introducing surfaces of revolution.

Surfaces of Revolution

One of the applications of integration we learned previously was to find the volume of solids of revolution – solids formed by revolving a curve about a horizontal or vertical axis. We now consider how to find the equation of the surface of such a solid.

Consider the surface formed by revolving $y = \sqrt{x}$ about the x-axis. Cross–sections of this surface parallel to the y-z plane are circles, as shown in Figure 9.10(a). Each circle has equation of the form $y^2 + z^2 = r^2$ for some radius r. The radius is a function of x; in fact, it is $r(x) = \sqrt{x}$. Thus the equation of the surface shown in Figure 9.10b is $y^2 + z^2 = (\sqrt{x})^2$.

We generalize the above principles to give the equations of surfaces formed by revolving curves about the coordinate axes.

Key Idea 1 Surfaces of Revolution, Part 1

Let r be a radius function.

1. The equation of the surface formed by revolving $y = r(x)$ or $z = r(x)$ about the x-axis is $y^2 + z^2 = r(x)^2$.
2. The equation of the surface formed by revolving $x = r(y)$ or $z = r(y)$ about the y-axis is $x^2 + z^2 = r(y)^2$.
3. The equation of the surface formed by revolving $x = r(z)$ or $y = r(z)$ about the z-axis is $x^2 + y^2 = r(z)^2$.

Notes:

Example 4 **Finding equation of a surface of revolution**

Let $y = \sin z$ on $[0, \pi]$. Find the equation of the surface of revolution formed by revolving $y = \sin z$ about the z-axis.

Solution Using Key Idea 1, we find the surface has equation $x^2 + y^2 = \sin^2 z$. The curve is sketched in Figure 9.11(a) and the surface is drawn in Figure 9.11(b).

Note how the surface (and hence the resulting equation) is the same if we began with the curve $x = \sin z$, which is also drawn in Figure 9.11(a).

Figure 9.11: Revolving $y = \sin z$ about the z-axis in Example 4.

This particular method of creating surfaces of revolution is limited. For instance, in Example **??** of Section **??** we found the volume of the solid formed by revolving $y = \sin x$ about the y-axis. Our current method of forming surfaces can only rotate $y = \sin x$ about the x-axis. Trying to rewrite $y = \sin x$ as a function of y is not trivial, as simply writing $x = \sin^{-1} y$ only gives part of the region we desire.

What we desire is a way of writing the surface of revolution formed by rotating $y = f(x)$ about the y-axis. We start by first recognizing this surface is the same as revolving $z = f(x)$ about the z-axis. This will give us a more natural way of viewing the surface.

A value of x is a measurement of distance from the z-axis. At the distance r, we plot a z-height of $f(r)$. When rotating $f(x)$ about the z-axis, we want all points a distance of r from the z-axis in the x-y plane to have a z-height of $f(r)$. All such points satisfy the equation $r^2 = x^2 + y^2$; hence $r = \sqrt{x^2 + y^2}$. Replacing r with $\sqrt{x^2 + y^2}$ in $f(r)$ gives $z = f(\sqrt{x^2 + y^2})$. This is the equation of the surface.

Key Idea 2 **Surfaces of Revolution, Part 2**

Let $z = f(x)$, $x \geq 0$, be a curve in the x-z plane. The surface formed by revolving this curve about the z-axis has equation $z = f\left(\sqrt{x^2 + y^2}\right)$.

Example 5 **Finding equation of surface of revolution**

Find the equation of the surface found by revolving $z = \sin x$ about the z-axis.

Solution Using Key Idea 2, the surface has equation $z = \sin\left(\sqrt{x^2 + y^2}\right)$. The curve and surface are graphed in Figure 9.12.

Figure 9.12: Revolving $z = \sin x$ about the z-axis in Example 5.

Notes:

Quadric Surfaces

Spheres, planes and cylinders are important surfaces to understand. We now consider one last type of surface, a **quadric surface**. The definition may look intimidating, but we will show how to analyze these surfaces in an illuminating way.

Definition 2 Quadric Surface

A **quadric surface** is the graph of the general second–degree equation in three variables:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

When the coefficients D, E or F are not zero, the basic shapes of the quadric surfaces are rotated in space. We will focus on quadric surfaces where these coeffiecients are 0; we will not consider rotations. There are six basic quadric surfaces: the elliptic paraboloid, elliptic cone, ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, and the hyperbolic paraboloid.

We study each shape by considering **traces**, that is, intersections of each surface with a plane parallel to a coordinate plane. For instance, consider the elliptic paraboloid $z = x^2/4 + y^2$, shown in Figure 9.13. If we intersect this shape with the plane $z = d$ (i.e., replace z with d), we have the equation:

$$d = \frac{x^2}{4} + y^2.$$

Divide both sides by d:

$$1 = \frac{x^2}{4d} + \frac{y^2}{d}.$$

This describes an ellipse – so cross sections parallel to the x-y coordinate plane are ellipses. This ellipse is drawn in the figure.

Now consider cross sections parallel to the x-z plane. For instance, letting $y = 0$ gives the equation $z = x^2/4$, clearly a parabola. Intersecting with the plane $x = 0$ gives a cross section defined by $z = y^2$, another parabola. These parabolas are also sketched in the figure.

Thus we see where the elliptic paraboloid gets its name: some cross sections are ellipses, and others are parabolas.

Such an analysis can be made with each of the quadric surfaces. We give a sample equation of each, provide a sketch with representative traces, and describe these traces.

Figure 9.13: The elliptic paraboloid $z = x^2/4 + y^2$.

Notes:

Elliptic Paraboloid, $z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$

Plane	Trace
$x = d$	Parabola
$y = d$	Parabola
$z = d$	Ellipse

One variable in the equation of the elliptic paraboloid will be raised to the first power; above, this is the z variable. The paraboloid will "open" in the direction of this variable's axis. Thus $x = y^2/a^2 + z^2/b^2$ is an elliptic paraboloid that opens along the x-axis.

Multiplying the right hand side by (-1) defines an elliptic paraboloid that "opens" in the opposite direction.

Elliptic Cone, $z^2 = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$

Plane	Trace
$x = 0$	Crossed Lines
$y = 0$	Crossed Lines
$x = d$	Hyperbola
$y = d$	Hyperbola
$z = d$	Ellipse

One can rewrite the equation as $z^2 - x^2/a^2 - y^2/b^2 = 0$. The one variable with a positive coefficient corresponds to the axis that the cones "open" along.

Ellipsoid, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

Plane	Trace
$x = d$	Ellipse
$y = d$	Ellipse
$z = d$	Ellipse

If $a = b = c \neq 0$, the ellipsoid is a sphere with radius a; compare to Key Idea 0.

Hyperboloid of One Sheet, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

Plane	Trace
$x = d$	Hyperbola
$y = d$	Hyperbola
$z = d$	Ellipse

The one variable with a negative coefficient corresponds to the axis that the hyperboloid "opens" along.

Hyperboloid of Two Sheets, $\dfrac{z^2}{c^2} - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

Plane	Trace
$x = d$	Hyperbola
$y = d$	Hyperbola
$z = d$	Ellipse

The one variable with a positive coefficient corresponds to the axis that the hyperboloid "opens" along. In the case illustrated, when $|d| < |c|$, there is no trace.

Hyperbolic Paraboloid, $z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$

Plane	Trace
$x = d$	Parabola
$y = d$	Parabola
$z = d$	Hyperbola

The parabolic traces will open along the axis of the one variable that is raised to the first power.

Figure 9.14: Sketching an elliptic paraboloid.

Figure 9.15: Sketching an ellipsoid.

Example 6 Sketching quadric surfaces

Sketch the quadric surface defined by the given equation.

1. $y = \dfrac{x^2}{4} + \dfrac{z^2}{16}$ 2. $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1.$ 3. $z = y^2 - x^2.$

Solution

1. $y = \dfrac{x^2}{4} + \dfrac{z^2}{16}$:

   We first identify the quadric by pattern–matching with the equations given previously. Only two surfaces have equations where one variable is raised to the first power, the elliptic paraboloid and the hyperbolic paraboloid. In the latter case, the other variables have different signs, so we conclude that this describes a hyperbolic paraboloid. As the variable with the first power is y, we note the paraboloid opens along the y-axis.

   To make a decent sketch by hand, we need only draw a few traces. In this case, the traces $x = 0$ and $z = 0$ form parabolas that outline the shape.

   $x = 0$: The trace is the parabola $y = z^2/16$

   $z = 0$: The trace is the parabola $y = x^2/4$.

   Graphing each trace in the respective plane creates a sketch as shown in Figure 9.14(a). This is enough to give an idea of what the paraboloid looks like. The surface is filled in in (b).

2. $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$:

   This is an ellipsoid. We can get a good idea of its shape by drawing the traces in the coordinate planes.

   $x = 0$: The trace is the ellipse $\dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$. The major axis is along the y–axis with length 6 (as $b = 3$, the length of the axis is 6); the minor axis is along the z-axis with length 4.

   $y = 0$: The trace is the ellipse $x^2 + \dfrac{z^2}{4} = 1$. The major axis is along the z-axis, and the minor axis has length 2 along the x-axis.

   $z = 0$: The trace is the ellipse $x^2 + \dfrac{y^2}{9} = 1$, with major axis along the y-axis.

   Graphing each trace in the respective plane creates a sketch as shown in Figure 9.15(a). Filling in the surface gives Figure 9.15(b).

3. $z = y^2 - x^2$:

Notes:

This defines a hyperbolic paraboloid, very similar to the one shown in the gallery of quadric sections. Consider the traces in the $y-z$ and $x-z$ planes:

$x = 0$: The trace is $z = y^2$, a parabola opening up in the $y - z$ plane.

$y = 0$: The trace is $z = -x^2$, a parabola opening down in the $x - z$ plane.

Sketching these two parabolas gives a sketch like that in Figure 9.16 (a), and filling in the surface gives a sketch like (b).

Figure 9.16: Sketching a hyperbolic paraboloid.

Example 7 Identifying quadric surfaces

Consider the quadric surface shown in Figure 9.17. Which of the following equations best fits this surface?

(a) $x^2 - y^2 - \dfrac{z^2}{9} = 0$ (c) $z^2 - x^2 - y^2 = 1$

(b) $x^2 - y^2 - z^2 = 1$ (d) $4x^2 - y^2 - \dfrac{z^2}{9} = 1$

Solution The image clearly displays a hyperboloid of two sheets. The gallery informs us that the equation will have a form similar to $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

We can immediately eliminate option (a), as the constant in that equation is not 1.

The hyperboloid "opens" along the x-axis, meaning x must be the only variable with a positive coefficient, eliminating (c).

The hyperboloid is wider in the z-direction than in the y-direction, so we need an equation where $c > b$. This eliminates (b), leaving us with (d). We should verify that the equation given in (d), $4x^2 - y^2 - \frac{z^2}{9} = 1$, fits.

We already established that this equation describes a hyperboloid of two sheets that opens in the x-direction and is wider in the z-direction than in the y. Now note the coefficient of the x-term. Rewriting $4x^2$ in standard form, we have: $4x^2 = \dfrac{x^2}{(1/2)^2}$. Thus when $y = 0$ and $z = 0$, x must be $1/2$; i.e., each hyperboloid "starts" at $x = 1/2$. This matches our figure.

We conclude that $4x^2 - y^2 - \dfrac{z^2}{9} = 1$ best fits the graph.

Figure 9.17: A possible equation of this quadric surface is found in Example 7.

This section has introduced points in space and shown how equations can describe surfaces. The next sections explore *vectors*, an important mathematical object that we'll use to explore curves in space.

Notes:

Exercises 9.1

Terms and Concepts

1. Axes drawn in space must conform to the _____ _____ rule.

2. In the plane, the equation $x = 2$ defines a _____; in space, $x = 2$ defines a _____.

3. In the plane, the equation $y = x^2$ defines a _____; in space, $y = x^2$ defines a _____.

4. Which quadric surface looks like a Pringles® chip?

5. Consider the hyperbola $x^2 - y^2 = 1$ in the plane. If this hyperbola is rotated about the x-axis, what quadric surface is formed?

6. Consider the hyperbola $x^2 - y^2 = 1$ in the plane. If this hyperbola is rotated about the y-axis, what quadric surface is formed?

Problems

7. The points $A = (1, 4, 2)$, $B = (2, 6, 3)$ and $C = (4, 3, 1)$ form a triangle in space. Find the distances between each pair of points and determine if the triangle is a right triangle.

8. The points $A = (1, 1, 3)$, $B = (3, 2, 7)$, $C = (2, 0, 8)$ and $D = (0, -1, 4)$ form a quadrilateral $ABCD$ in space. Is this a parallelogram?

9. Find the center and radius of the sphere defined by $x^2 - 8x + y^2 + 2y + z^2 + 8 = 0$.

10. Find the center and radius of the sphere defined by $x^2 + y^2 + z^2 + 4x - 2y - 4z + 4 = 0$.

In Exercises 11 – 14, describe the region in space defined by the inequalities.

11. $x^2 + y^2 + z^2 < 1$

12. $0 \le x \le 3$

13. $x \ge 0,\ y \ge 0,\ z \ge 0$

14. $y \ge 3$

In Exercises 15 – 18, sketch the cylinder in space.

15. $z = x^3$

16. $y = \cos z$

17. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

18. $y = \dfrac{1}{x}$

In Exercises 19 – 22, give the equation of the surface of revolution described.

19. Revolve $z = \dfrac{1}{1 + y^2}$ about the y-axis.

20. Revolve $y = x^2$ about the x-axis.

21. Revolve $z = x^2$ about the z-axis.

22. Revolve $z = 1/x$ about the z-axis.

In Exercises 23 – 26, a quadric surface is sketched. Determine which of the given equations best fits the graph.

23.

(a) $x = y^2 + \dfrac{z^2}{9}$ (b) $x = y^2 + \dfrac{z^2}{3}$

24.

(a) $x^2 - y^2 - z^2 = 0$ (b) $x^2 - y^2 + z^2 = 0$

25.

(a) $x^2 + \dfrac{y^2}{3} + \dfrac{z^2}{2} = 1$ (b) $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$

26.

(a) $y^2 - x^2 - z^2 = 1$ (b) $y^2 + x^2 - z^2 = 1$

In Exercises 27 – 32, sketch the quadric surface.

27. $z - y^2 + x^2 = 0$

28. $z^2 = x^2 + \dfrac{y^2}{4}$

29. $x = -y^2 - z^2$

30. $16x^2 - 16y^2 - 16z^2 = 1$

31. $\dfrac{x^2}{9} - y^2 + \dfrac{z^2}{25} = 1$

32. $4x^2 + 2y^2 + z^2 = 4$

9.2 An Introduction to Vectors

Many quantities we think about daily can be described by a single number: temperature, speed, cost, weight and height. There are also many other concepts we encounter daily that cannot be described with just one number. For instance, a weather forecaster often describes wind with its speed and its direction ("... with winds from the southeast gusting up to 30 mph ..."). When applying a force, we are concerned with both the magnitude and direction of that force. In both of these examples, *direction* is important. Because of this, we study *vectors*, mathematical objects that convey both magnitude and direction information.

One "bare–bones" definition of a vector is based on what we wrote above: "a vector is a mathematical object with magnitude and direction parameters." This definition leaves much to be desired, as it gives no indication as to how such an object is to be used. Several other definitions exist; we choose here a definition rooted in a geometric visualization of vectors. It is very simplistic but readily permits further investigation.

Figure 9.18: Drawing the same vector with different initial points.

Definition 3 Vector

A **vector** is a directed line segment.

Given points P and Q (either in the plane or in space), we denote with $\overrightarrow{PQ}$ the vector *from P to Q*. The point P is said to be the **initial point** of the vector, and the point Q is the **terminal point**.

The **magnitude**, **length** or **norm** of $\overrightarrow{PQ}$ is the length of the line segment $\overline{PQ}$: $\|\overrightarrow{PQ}\| = \|\overline{PQ}\|$.

Two vectors are **equal** if they have the same magnitude and direction.

Figure 9.19: Illustrating how equal vectors have the same displacement.

Figure 9.18 shows multiple instances of the same vector. Each directed line segment has the same direction and length (magnitude), hence each is the same vector.

We use $\mathbb{R}^2$ (pronounced "r two") to represent all the vectors in the plane, and use $\mathbb{R}^3$ (pronounced "r three") to represent all the vectors in space.

Consider the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ as shown in Figure 9.19. The vectors look to be equal; that is, they seem to have the same length and direction. Indeed, they are. Both vectors move 2 units to the right and 1 unit up from the initial point to reach the terminal point. One can analyze this movement to measure the

Notes:

magnitude of the vector, and the movement itself gives direction information (one could also measure the slope of the line passing through P and Q or R and S). Since they have the same length and direction, these two vectors are equal.

This demonstrates that inherently all we care about is *displacement*; that is, how far in the x, y and possibly z directions the terminal point is from the initial point. Both the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ in Figure 9.19 have an x-displacement of 2 and a y-displacement of 1. This suggests a standard way of describing vectors in the plane. A vector whose x-displacement is a and whose y-displacement is b will have terminal point (a, b) when the initial point is the origin, $(0, 0)$. This leads us to a definition of a standard and concise way of referring to vectors.

Definition 4 Component Form of a Vector

1. The **component form** of a vector $\vec{v}$ in $\mathbb{R}^2$, whose terminal point is (a, b) when its initial point is $(0, 0)$, is $\langle a, b\rangle$.
2. The **component form** of a vector $\vec{v}$ in $\mathbb{R}^3$, whose terminal point is (a, b, c) when its initial point is $(0, 0, 0)$, is $\langle a, b, c\rangle$.

The numbers a, b (and c, respectively) are the **components** of $\vec{v}$.

It follows from the definition that the component form of the vector $\overrightarrow{PQ}$, where $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ is

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1\rangle ;$$

in space, where $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$, the component form of $\overrightarrow{PQ}$ is

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle .$$

We practice using this notation in the following example.

Example 8 Using component form notation for vectors

1. Sketch the vector $\vec{v} = \langle 2, -1\rangle$ starting at $P = (3, 2)$ and find its magnitude.
2. Find the component form of the vector $\vec{w}$ whose initial point is $R = (-3, -2)$ and whose terminal point is $S = (-1, 2)$.
3. Sketch the vector $\vec{u} = \langle 2, -1, 3\rangle$ starting at the point $Q = (1, 1, 1)$ and find its magnitude.

Notes:

(a)

(b)

Figure 9.20: Graphing vectors in Example 8.

Solution

1. Using P as the initial point, we move 2 units in the positive x-direction and -1 units in the positive y-direction to arrive at the terminal point $P' = (5, 1)$, as drawn in Figure 9.20(a).

   The magnitude of $\vec{v}$ is determined directly from the component form:

$$||\,\vec{v}\,|| = \sqrt{2^2 + (-1)^2} = \sqrt{5}.$$

2. Using the note following Definition 4, we have

$$\overrightarrow{RS} = \langle -1 - (-3), 2 - (-2) \rangle = \langle 2, 4 \rangle\,.$$

   One can readily see from Figure 9.20(a) that the x- and y-displacement of $\overrightarrow{RS}$ is 2 and 4, respectively, as the component form suggests.

3. Using Q as the initial point, we move 2 units in the positive x-direction, -1 unit in the positive y-direction, and 3 units in the positive z-direction to arrive at the terminal point $Q' = (3, 0, 4)$, illustrated in Figure 9.20(b).

   The magnitude of $\vec{u}$ is:

$$||\,\vec{u}\,|| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}.$$

Now that we have defined vectors, and have created a nice notation by which to describe them, we start considering how vectors interact with each other. That is, we define an *algebra* on vectors.

Notes:

Definition 5 Vector Algebra

1. Let $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$ be vectors in $\mathbb{R}^2$, and let c be a scalar.

   (a) The addition, or sum, of the vectors $\vec{u}$ and $\vec{v}$ is the vector

   $$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2 \rangle .$$

   (b) The scalar product of c and $\vec{v}$ is the vector

   $$c\vec{v} = c \langle v_1, v_2 \rangle = \langle cv_1, cv_2 \rangle .$$

2. Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ be vectors in $\mathbb{R}^3$, and let c be a scalar.

   (a) The addition, or sum, of the vectors $\vec{u}$ and $\vec{v}$ is the vector

   $$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle .$$

   (b) The scalar product of c and $\vec{v}$ is the vector

   $$c\vec{v} = c \langle v_1, v_2, v_3 \rangle = \langle cv_1, cv_2, cv_3 \rangle .$$

Figure 9.22: Illustrating how to add vectors using the Head to Tail Rule and Parallelogram Law.

In short, we say addition and scalar multiplication are computed "component–wise."

Example 9 Adding vectors

Sketch the vectors $\vec{u} = \langle 1, 3 \rangle$, $\vec{v} = \langle 2, 1 \rangle$ and $\vec{u} + \vec{v}$ all with initial point at the origin.

Solution We first compute $\vec{u} + \vec{v}$.

$$\begin{aligned} \vec{u} + \vec{v} &= \langle 1, 3 \rangle + \langle 2, 1 \rangle \\ &= \langle 3, 4 \rangle . \end{aligned}$$

These are all sketched in Figure 9.21.

Figure 9.21: Graphing the sum of vectors in Example 9.

As vectors convey magnitude and direction information, the sum of vectors also convey length and magnitude information. Adding $\vec{u} + \vec{v}$ suggests the following idea:

"Starting at an initial point, go out $\vec{u}$, then go out $\vec{v}$."

Notes:

This idea is sketched in Figure 9.22, where the initial point of $\vec{v}$ is the terminal point of $\vec{u}$. This is known as the "Head to Tail Rule" of adding vectors. Vector addition is very important. For instance, if the vectors $\vec{u}$ and $\vec{v}$ represent forces acting on a body, the sum $\vec{u}+\vec{v}$ gives the resulting force. Because of various physical applications of vector addition, the sum $\vec{u}+\vec{v}$ is often referred to as the **resultant vector**, or just the "resultant."

Analytically, it is easy to see that $\vec{u}+\vec{v}=\vec{v}+\vec{u}$. Figure 9.22 also gives a graphical representation of this, using gray vectors. Note that the vectors $\vec{u}$ and $\vec{v}$, when arranged as in the figure, form a parallelogram. Because of this, the Head to Tail Rule is also known as the Parallelogram Law: the vector $\vec{u}+\vec{v}$ is defined by forming the parallelogram defined by the vectors $\vec{u}$ and $\vec{v}$; the initial point of $\vec{u}+\vec{v}$ is the common initial point of parallelogram, and the terminal point of the sum is the common terminal point of the parallelogram.

While not illustrated here, the Head to Tail Rule and Parallelogram Law hold for vectors in $\mathbb{R}^3$ as well.

It follows from the properties of the real numbers and Definition 5 that

$$\vec{u}-\vec{v}=\vec{u}+(-1)\vec{v}.$$

The Parallelogram Law gives us a good way to visualize this subtraction. We demonstrate this in the following example.

Example 10 Vector Subtraction

Let $\vec{u}=\langle 3,1\rangle$ and $\vec{v}=\langle 1,2\rangle$. Compute and sketch $\vec{u}-\vec{v}$.

Solution The computation of $\vec{u}-\vec{v}$ is straightforward, and we show all steps below. Usually the formal step of multiplying by (-1) is omitted and we "just subtract."

$$\begin{aligned}\vec{u}-\vec{v}&=\vec{u}+(-1)\vec{v}\\&=\langle 3,1\rangle+\langle -1,-2\rangle\\&=\langle 2,-1\rangle.\end{aligned}$$

Figure 9.23 illustrates, using the Head to Tail Rule, how the subtraction can be viewed as the sum $\vec{u}+(-\vec{v})$. The figure also illustrates how $\vec{u}-\vec{v}$ can be obtained by looking only at the terminal points of $\vec{u}$ and $\vec{v}$ (when their initial points are the same).

Figure 9.23: Illustrating how to subtract vectors graphically.

Example 11 Scaling vectors

1. Sketch the vectors $\vec{v}=\langle 2,1\rangle$ and $2\vec{v}$ with initial point at the origin.
2. Compute the magnitudes of $\vec{v}$ and $2\vec{v}$.

Notes:

Solution

1. We compute $2\vec{v}$:

$$\begin{aligned} 2\vec{v} &= 2\langle 2, 1\rangle \\ &= \langle 4, 2\rangle . \end{aligned}$$

   Both $\vec{v}$ and $2\vec{v}$ are sketched in Figure 9.24. Make note that $2\vec{v}$ does not start at the terminal point of $\vec{v}$; rather, its initial point is also the origin.

2. The figure suggests that $2\vec{v}$ is twice as long as $\vec{v}$. We compute their magnitudes to confirm this.

$$\begin{aligned} ||\,\vec{v}\,|| &= \sqrt{2^2 + 1^2} \\ &= \sqrt{5}. \\ ||\,2\vec{v}\,|| &= \sqrt{4^2 + 2^2} \\ &= \sqrt{20} \\ &= \sqrt{4 \cdot 5} = 2\sqrt{5}. \end{aligned}$$

   As we suspected, $2\vec{v}$ is twice as long as $\vec{v}$.

Figure 9.24: Graphing vectors $\vec{v}$ and $2\vec{v}$ in Example 11.

The **zero vector** is the vector whose initial point is also its terminal point. It is denoted by $\vec{0}$. Its component form, in $\mathbb{R}^2$, is $\langle 0, 0\rangle$; in $\mathbb{R}^3$, it is $\langle 0, 0, 0\rangle$. Usually the context makes is clear whether $\vec{0}$ is referring to a vector in the plane or in space.

Our examples have illustrated key principles in vector algebra: how to add and subtract vectors and how to multiply vectors by a scalar. The following theorem states formally the properties of these operations.

Notes:

Theorem 0 Properties of Vector Operations

The following are true for all scalars c and d, and for all vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, where $\vec{u}$, $\vec{v}$ and $\vec{w}$ are all in $\mathbb{R}^2$ or where $\vec{u}$, $\vec{v}$ and $\vec{w}$ are all in $\mathbb{R}^3$:

1. $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ — Commutative Property
2. $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$ — Associative Property
3. $\vec{v}+\vec{0}=\vec{v}$ — Additive Identity
4. $(cd)\vec{v}=c(d\vec{v})$
5. $c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$ — Distributive Property
6. $(c+d)\vec{v}=c\vec{v}+d\vec{v}$ — Distributive Property
7. $0\vec{v}=\vec{0}$
8. $||\,c\vec{v}\,||=|c|\cdot||\,\vec{v}\,||$
9. $||\,\vec{u}\,||=0$ if, and only if, $\vec{u}=\vec{0}$.

As stated before, each vector $\vec{v}$ conveys magnitude and direction information. We have a method of extracting the magnitude, which we write as $||\,\vec{v}\,||$. *Unit vectors* are a way of extracting just the direction information from a vector.

Definition 6 Unit Vector

A **unit vector** is a vector $\vec{v}$ with a magnitude of 1; that is,

$$||\,\vec{v}\,||=1.$$

Consider this scenario: you are given a vector $\vec{v}$ and are told to create a vector of length 10 in the direction of $\vec{v}$. How does one do that? If we knew that $\vec{u}$ was the unit vector in the direction of $\vec{v}$, the answer would be easy: $10\vec{u}$. So how do we find $\vec{u}$?

Property 8 of Theorem 0 holds the key. If we divide $\vec{v}$ by its magnitude, it becomes a vector of length 1. Consider:

$$\begin{aligned}\left|\left|\frac{1}{||\,\vec{v}\,||}\vec{v}\right|\right| &= \frac{1}{||\,\vec{v}\,||}||\,\vec{v}\,|| \qquad \text{(we can pull out } \frac{1}{||\,\vec{v}\,||} \text{ as it is a scalar)}\\ &= 1.\end{aligned}$$

Notes:

So the vector of length 10 in the direction of $\vec{v}$ is $10\dfrac{1}{||\,\vec{v}\,||}\vec{v}$. An example will make this more clear.

Example 12 Using Unit Vectors

Let $\vec{v} = \langle 3, 1\rangle$ and let $\vec{w} = \langle 1, 2, 2\rangle$.

1. Find the unit vector in the direction of $\vec{v}$.
2. Find the unit vector in the direction of $\vec{w}$.
3. Find the vector in the direction of $\vec{v}$ with magnitude 5.

Solution

1. We find $||\,\vec{v}\,|| = \sqrt{10}$. So the unit vector $\vec{u}$ in the direction of $\vec{v}$ is
$$\vec{u} = \frac{1}{\sqrt{10}}\vec{v} = \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle.$$
2. We find $||\,\vec{w}\,|| = 3$, so the unit vector $\vec{z}$ in the direction of $\vec{w}$ is
$$\vec{u} = \frac{1}{3}\vec{w} = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle.$$
3. To create a vector with magnitude 5 in the direction of $\vec{v}$, we multiply the unit vector $\vec{u}$ by 5. Thus $5\vec{u} = \langle 15/\sqrt{10}, 5/\sqrt{10}\rangle$ is the vector we seek. This is sketched in Figure 9.25.

Figure 9.25: Graphing vectors in Example 12. All vectors shown have their initial point at the origin.

The basic formation of the unit vector $\vec{u}$ in the direction of a vector $\vec{v}$ leads to a interesting equation. It is:

$$\vec{v} = ||\,\vec{v}\,||\frac{1}{||\,\vec{v}\,||}\vec{v}.$$

We rewrite the equation with parentheses to make a point:

$$\vec{v} = \underbrace{||\,\vec{v}\,||}_{\text{magnitude}} \cdot \underbrace{\left(\frac{1}{||\,\vec{v}\,||}\vec{v}\right)}_{\text{direction}}.$$

This equation illustrates the fact that a vector has both magnitude and direction, where we view a unit vector as supplying *only* direction information. Identifying unit vectors with direction allows us to define **parallel vectors**.

Notes:

Note: $\vec{0}$ is directionless; because $||\vec{0}|| = 0$, there is no unit vector in the "direction" of $\vec{0}$.
Some texts define two vectors as being parallel if one is a scalar multiple of the other. By this definition, $\vec{0}$ is parallel to all vectors as $\vec{0} = 0\vec{v}$ for all $\vec{v}$.
We prefer the given definition of parallel as it is grounded in the fact that unit vectors provide direction information. One may adopt the convention that $\vec{0}$ is parallel to all vectors if they desire. (See also the marginal note on page 534.)

Definition 7 Parallel Vectors

1. Unit vectors $\vec{u}_1$ and $\vec{u}_2$ are **parallel** if $\vec{u}_1 = \pm\vec{u}_2$.
2. Nonzero vectors $\vec{v}_1$ and $\vec{v}_2$ are **parallel** if their respective unit vectors are parallel.

It is equivalent to say that vectors $\vec{v}_1$ and $\vec{v}_2$ are parallel if there is a scalar $c \neq 0$ such that $\vec{v}_1 = c\vec{v}_2$ (see marginal note).

If one graphed all unit vectors in $\mathbb{R}^2$ with the initial point at the origin, then the terminal points would all lie on the unit circle. Based on what we know from trigonometry, we can then say that the component form of all unit vectors in $\mathbb{R}^2$ is $\langle \cos\theta, \sin\theta \rangle$ for some angle θ.

A similar construction in $\mathbb{R}^3$ shows that the terminal points all lie on the unit sphere. These vectors also have a particular component form, but its derivation is not as straightforward as the one for unit vectors in $\mathbb{R}^2$. Important concepts about unit vectors are given in the following Key Idea.

Key Idea 3 Unit Vectors

1. The unit vector in the direction of $\vec{v}$ is
$$\vec{u} = \frac{1}{||\vec{v}||}\vec{v}.$$
2. A vector $\vec{u}$ in $\mathbb{R}^2$ is a unit vector if, and only if, its component form is $\langle \cos\theta, \sin\theta \rangle$ for some angle θ.
3. A vector $\vec{u}$ in $\mathbb{R}^3$ is a unit vector if, and only if, its component form is $\langle \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta \rangle$ for some angles θ and φ.

These formulas can come in handy in a variety of situations, especially the formula for unit vectors in the plane.

Example 13 Finding Component Forces

Consider a weight of 50lb hanging from two chains, as shown in Figure 9.26. One chain makes an angle of 30° with the vertical, and the other an angle of 45°. Find the force applied to each chain.

Figure 9.26: A diagram of a weight hanging from 2 chains in Example 13.

Solution Knowing that gravity is pulling the 50lb weight straight down,

Notes:

we can create a vector $\vec{F}$ to represent this force.

$$\vec{F} = 50\,\langle 0, -1\rangle = \langle 0, -50\rangle\,.$$

We can view each chain as "pulling" the weight up, preventing it from falling. We can represent the force from each chain with a vector. Let $\vec{F}_1$ represent the force from the chain making an angle of 30° with the vertical, and let $\vec{F}_2$ represent the force form the other chain. Convert all angles to be measured from the horizontal (as shown in Figure 9.27), and apply Key Idea 3. As we do not yet know the magnitudes of these vectors, (that is the problem at hand), we use m_1 and m_2 to represent them.

Figure 9.27: A diagram of the force vectors from Example 13.

$$\vec{F}_1 = m_1\,\langle \cos 120^\circ, \sin 120^\circ\rangle$$

$$\vec{F}_2 = m_2\,\langle \cos 45^\circ, \sin 45^\circ\rangle$$

As the weight is not moving, we know the sum of the forces is $\vec{0}$. This gives:

$$\begin{aligned} \vec{F} + \vec{F}_1 + \vec{F}_2 &= \vec{0} \\ \langle 0, -50\rangle + m_1\,\langle \cos 120^\circ, \sin 120^\circ\rangle + m_2\,\langle \cos 45^\circ, \sin 45^\circ\rangle &= \vec{0} \end{aligned}$$

The sum of the entries in the first component is 0, and the sum of the entries in the second component is also 0. This leads us to the following two equations:

$$\begin{aligned} m_1 \cos 120^\circ + m_2 \cos 45^\circ &= 0 \\ m_1 \sin 120^\circ + m_2 \sin 45^\circ &= 50 \end{aligned}$$

This is a simple 2-equation, 2-unkown system of linear equations. We leave it to the reader to verify that the solution is

$$m_1 = 50(\sqrt{3} - 1) \approx 36.6; \qquad m_2 = \frac{50\sqrt{2}}{1+\sqrt{3}} \approx 25.88.$$

It might seem odd that the sum of the forces applied to the chains is more than 50lb. We leave it to a physics class to discuss the full details, but offer this short explanation. Our equations were established so that the *vertical* components of each force sums to 50lb, thus supporting the weight. Since the chains are at an angle, they also pull against each other, creating an "additional" horizontal force while holding the weight in place.

Unit vectors were very important in the previous calculation; they allowed us to define a vector in the proper direction but with an unknown magnitude. Our computations were then computed component–wise. Because such calculations are often necessary, the *standard unit vectors* can be useful.

Notes:

Definition 8 Standard Unit Vectors

1. In $\mathbb{R}^2$, the standard unit vectors are

$$\vec{i} = \langle 1, 0 \rangle \quad \text{and} \quad \vec{j} = \langle 0, 1 \rangle .$$

2. In $\mathbb{R}^3$, the standard unit vectors are

$$\vec{i} = \langle 1, 0, 0 \rangle \quad \text{and} \quad \vec{j} = \langle 0, 1, 0 \rangle \quad \text{and} \quad \vec{k} = \langle 0, 0, 1 \rangle .$$

Example 14 Using standard unit vectors

1. Rewrite $\vec{v} = \langle 2, -3 \rangle$ using the standard unit vectors.

2. Rewrite $\vec{w} = 4\vec{i} - 5\vec{j} + 2\vec{k}$ in component form.

Solution

1.
$$\begin{aligned} \vec{v} &= \langle 2, -3 \rangle \\ &= \langle 2, 0 \rangle + \langle 0, -3 \rangle \\ &= 2 \langle 1, 0 \rangle - 3 \langle 0, 1 \rangle \\ &= 2\vec{i} - 3\vec{j} \end{aligned}$$

2.
$$\begin{aligned} \vec{w} &= 4\vec{i} - 5\vec{j} + 2\vec{k} \\ &= \langle 4, 0, 0 \rangle + \langle 0, -5, 0 \rangle + \langle 0, 0, 2 \rangle \\ &= \langle 4, -5, 2 \rangle \end{aligned}$$

These two examples demonstrate that converting between component form and the standard unit vectors is rather straightforward. Many mathematicians prefer component form, and it is the preferred notation in this text. Many engineers prefer using the standard unit vectors, and many engineering text use that notation.

Example 15 Finding Component Force

A weight of 25lb is suspended from a chain of length 2ft while a wind pushes the weight to the right with constant force of 5lb as shown in Figure 9.28. What angle will the chain make with the vertical as a result of the wind's pushing? How much higher will the weight be?

Figure 9.28: A figure of a weight being pushed by the wind in Example 15.

Notes:

Solution The force of the wind is represented by the vector $\vec{F}_w = 5\vec{i}$. The force of gravity on the weight is represented by $\vec{F}_g = -25\vec{j}$. The direction and magnitude of the vector representing the force on the chain are both unknown. We represent this force with

$$\vec{F}_c = m\langle \cos\varphi, \sin\varphi\rangle = m\cos\varphi\,\vec{i} + m\sin\varphi\,\vec{j}$$

for some magnitude m and some angle with the horizontal φ. (Note: θ is the angle the chain makes with the *vertical*; φ is the angle with the *horizontal*.)

As the weight is at equilibrium, the sum of the forces is $\vec{0}$:

$$\begin{aligned} \vec{F}_c + \vec{F}_w + \vec{F}_g &= \vec{0} \\ m\cos\varphi\,\vec{i} + m\sin\varphi\,\vec{j} + 5\vec{i} - 25\vec{j} &= \vec{0} \end{aligned}$$

Thus the sum of the $\vec{i}$ and $\vec{j}$ components are 0, leading us to the following system of equations:

$$\begin{aligned} 5 + m\cos\varphi &= 0 \\ -25 + m\sin\varphi &= 0 \end{aligned} \tag{9.1}$$

This is enough to determine $\vec{F}_c$ already, as we know $m\cos\varphi = -5$ and $m\sin\varphi = 25$. Thus $F_c = \langle -5, 25\rangle$. We can use this to find the magnitude m:

$$m = \sqrt{(-5)^2 + 25^2} = 5\sqrt{26} \approx 25.5\text{lb}.$$

We can then use either equality from Equation (9.1) to solve for φ. We choose the first equality as using arccosine will return an angle in the 2nd quadrant:

$$5 + 5\sqrt{26}\cos\varphi = 0 \quad \Rightarrow \quad \varphi = \cos^{-1}\left(\frac{-5}{5\sqrt{26}}\right) \approx 1.7682 \approx 101.31^\circ.$$

Subtracting 90° from this angle gives us an angle of 11.31° with the vertical.

We can now use trigonometry to find out how high the weight is lifted. The diagram shows that a right triangle is formed with the 2ft chain as the hypotenuse with an interior angle of 11.31°. The length of the adjacent side (in the diagram, the dashed vertical line) is $2\cos 11.31^\circ \approx 1.96$ft. Thus the weight is lifted by about 0.04ft, almost 1/2in.

The algebra we have applied to vectors is already demonstrating itself to be very useful. There are two more fundamental operations we can perform with vectors, the *dot product* and the *cross product*. The next two sections explore each in turn.

Notes:

Exercises 9.2

Terms and Concepts

1. Name two different things that cannot be described with just one number, but rather need 2 or more numbers to fully describe them.

2. What is the difference between $(1, 2)$ and $\langle 1, 2\rangle$?

3. What is a unit vector?

4. What does it mean for two vectors to be parallel?

5. What effect does multiplying a vector by -2 have?

Problems

In Exercises 6 – 9, points P and Q are given. Write the vector $\overrightarrow{PQ}$ in component form and using the standard unit vectors.

6. $P = (2, -1), \quad Q = (3, 5)$

7. $P = (3, 2), \quad Q = (7, -2)$

8. $P = (0, 3, -1), \quad Q = (6, 2, 5)$

9. $P = (2, 1, 2), \quad Q = (4, 3, 2)$

10. Let $\vec{u} = \langle 1, -2\rangle$ and $\vec{v} = \langle 1, 1\rangle$.
    (a) Find $\vec{u} + \vec{v}$, $\vec{u} - \vec{v}$, $2\vec{u} - 3\vec{v}$.
    (b) Sketch the above vectors on the same axes, along with $\vec{u}$ and $\vec{v}$.
    (c) Find $\vec{x}$ where $\vec{u} + \vec{x} = 2\vec{v} - \vec{x}$.

11. Let $\vec{u} = \langle 1, 1, -1\rangle$ and $\vec{v} = \langle 2, 1, 2\rangle$.
    (a) Find $\vec{u} + \vec{v}$, $\vec{u} - \vec{v}$, $\pi\vec{u} - \sqrt{2}\vec{v}$.
    (b) Sketch the above vectors on the same axes, along with $\vec{u}$ and $\vec{v}$.
    (c) Find $\vec{x}$ where $\vec{u} + \vec{x} = \vec{v} + 2\vec{x}$.

In Exercises 12 – 15, sketch $\vec{u}$, $\vec{v}$, $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$ on the same axes.

12.

13.

14.

15.

In Exercises 16 – 19, find $||\vec{u}||$, $||\vec{v}||$, $||\vec{u} + \vec{v}||$ and $||\vec{u} - \vec{v}||$.

16. $\vec{u} = \langle 2, 1\rangle, \quad \vec{v} = \langle 3, -2\rangle$

17. $\vec{u} = \langle -3, 2, 2\rangle, \quad \vec{v} = \langle 1, -1, 1\rangle$

18. $\vec{u} = \langle 1, 2\rangle, \quad \vec{v} = \langle -3, -6\rangle$

19. $\vec{u} = \langle 2, -3, 6\rangle, \quad \vec{v} = \langle 10, -15, 30\rangle$

20. Under what conditions is $||\vec{u}|| + ||\vec{v}|| = ||\vec{u} + \vec{v}||$?

In Exercises 21 – 24, find the unit vector $\vec{u}$ in the direction of $\vec{v}$.

21. $\vec{v} = \langle 3, 7\rangle$

22. $\vec{v} = \langle 6, 8\rangle$

23. $\vec{v} = \langle 1, -2, 2\rangle$

24. $\vec{v} = \langle 2, -2, 2\rangle$

25. Find the unit vector in the first quadrant of $\mathbb{R}^2$ that makes a 50° angle with the x-axis.

26. Find the unit vector in the second quadrant of $\mathbb{R}^2$ that makes a 30° angle with the y-axis.

27. Verify, from Key Idea 3, that $\vec{u} = \langle \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta\rangle$ is a unit vector for all angles θ and φ.

A weight of 100lb is suspended from two chains, making angles with the vertical of θ and φ as shown in the figure below.

In Exercises 28 – 31, angles θ and φ are given. Find the force applied to each chain.

28. $\theta = 30^\circ, \quad \varphi = 30^\circ$

29. $\theta = 60^\circ, \quad \varphi = 60^\circ$

30. $\theta = 20^\circ, \quad \varphi = 15^\circ$

31. $\theta = 0^\circ, \quad \varphi = 0^\circ$

A weight of $p$lb is suspended from a chain of length ℓ while a constant force of $\vec{F}_w$ pushes the weight to the right, making an angle of θ with the vertical, as shown in the figure below.

In Exercises 32 – 35, a force $\vec{F}_w$ and length ℓ are given. Find the angle θ and the height the weight is lifted as it moves to the right.

32. $\vec{F}_w = 1\text{lb}, \quad \ell = 1\text{ft}, \quad p = 1\text{lb}$

33. $\vec{F}_w = 1\text{lb}, \quad \ell = 1\text{ft}, \quad p = 10\text{lb}$

34. $\vec{F}_w = 1\text{lb}, \quad \ell = 10\text{ft}, \quad p = 1\text{lb}$

35. $\vec{F}_w = 10\text{lb}, \quad \ell = 10\text{ft}, \quad p = 1\text{lb}$

9.3 The Dot Product

The previous section introduced vectors and described how to add them together and how to multiply them by scalars. This section introduces *a* multiplication on vectors called the **dot product**.

Definition 9 Dot Product

1. Let $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$ in $\mathbb{R}^2$. The **dot product** of $\vec{u}$ and $\vec{v}$, denoted $\vec{u} \cdot \vec{v}$, is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2.$$

2. Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ in $\mathbb{R}^3$. The **dot product** of $\vec{u}$ and $\vec{v}$, denoted $\vec{u} \cdot \vec{v}$, is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Note how this product of vectors returns a *scalar*, not another vector. We practice evaluating a dot product in the following example, then we will discuss why this product is useful.

Example 16 Evaluating dot products

1. Let $\vec{u} = \langle 1, 2 \rangle$, $\vec{v} = \langle 3, -1 \rangle$ in $\mathbb{R}^2$. Find $\vec{u} \cdot \vec{v}$.
2. Let $\vec{x} = \langle 2, -2, 5 \rangle$ and $\vec{y} = \langle -1, 0, 3 \rangle$ in $\mathbb{R}^3$. Find $\vec{x} \cdot \vec{y}$.

Solution

1. Using Definition 9, we have

$$\vec{u} \cdot \vec{v} = 1(3) + 2(-1) = 1.$$

2. Using the definition, we have

$$\vec{x} \cdot \vec{y} = 2(-1) - 2(0) + 5(3) = 13.$$

The dot product, as shown by the preceding example, is very simple to evaluate. It is only the sum of products. While the definition gives no hint as to why we

Notes:

would care about this operation, there is an amazing connection between the dot product and angles formed by the vectors. Before stating this connection, we give a theorem stating some of the properties of the dot product.

Theorem 1 Properties of the Dot Product

Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ and let c be a scalar.

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ Commutative Property
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ Distributive Property
3. $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$
4. $\vec{0} \cdot \vec{v} = 0$
5. $\vec{v} \cdot \vec{v} = ||\,\vec{v}\,||^2$

The last statement of the theorem makes a handy connection between the magnitude of a vector and the dot product with itself. Our definition and theorem give properties of the dot product, but we are still likely wondering "What does the dot product *mean*?" It is helpful to understand that the dot product of a vector with itself is connected to its magnitude.

The next theorem extends this understanding by connecting the dot product to magnitudes and angles. Given vectors $\vec{u}$ and $\vec{v}$ in the plane, an angle θ is clearly formed when $\vec{u}$ and $\vec{v}$ are drawn with the same initial point as illustrated in Figure 9.29(a). (We always take θ to be the angle in $[0, \pi]$ as two angles are actually created.)

The same is also true of 2 vectors in space: given $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ with the same initial point, there is a plane that contains both $\vec{u}$ and $\vec{v}$. (When $\vec{u}$ and $\vec{v}$ are colinear, there are infinite planes that contain both vectors.) In that plane, we can again find an angle θ between them (and again, $0 \leq \theta \leq \pi$). This is illustrated in Figure 9.29(b).

The following theorem connects this angle θ to the dot product of $\vec{u}$ and $\vec{v}$.

Figure 9.29: Illustrating the angle formed by two vectors with the same initial point.

Notes:

Theorem 2 The Dot Product and Angles

Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then

$$\vec{u} \cdot \vec{v} = || \vec{u} || || \vec{v} || \cos\theta,$$

where θ, $0 \leq \theta \leq \pi$, is the angle between $\vec{u}$ and $\vec{v}$.

When θ is an acute angle (i.e., $0 \leq \theta < \pi/2$), $\cos\theta$ is positive; when $\theta = \pi/2$, $\cos\theta = 0$; when θ is an obtuse angle ($\pi/2 < \theta \leq \pi$), $\cos\theta$ is negative. Thus the sign of the dot product gives a general indication of the angle between the vectors, illustrated in Figure 9.30.

Figure 9.30: Illustrating the relationship between the angle between vectors and the sign of their dot product.

We *can* use Theorem 2 to compute the dot product, but generally this theorem is used to find the angle between known vectors (since the dot product is generally easy to compute). To this end, we rewrite the theorem's equation as

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|| \vec{u} || || \vec{v} ||} \quad \Leftrightarrow \quad \theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{|| \vec{u} || || \vec{v} ||}\right).$$

We practice using this theorem in the following example.

Figure 9.31: Vectors used in Example 17.

Example 17 Using the dot product to find angles

Let $\vec{u} = \langle 3, 1 \rangle$, $\vec{v} = \langle -2, 6 \rangle$ and $\vec{w} = \langle -4, 3 \rangle$, as shown in Figure 9.31. Find the angles α, β and θ.

Solution We start by computing the magnitude of each vector.

$$|| \vec{u} || = \sqrt{10}; \quad || \vec{v} || = 2\sqrt{10}; \quad || \vec{w} || = 5.$$

Notes:

We now apply Theorem 2 to find the angles.

$$\begin{aligned}\alpha &= \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{(\sqrt{10})(2\sqrt{10})}\right)\\ &= \cos^{-1}(0) = \frac{\pi}{2} = 90^\circ.\end{aligned}$$

$$\begin{aligned}\beta &= \cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{(2\sqrt{10})(5)}\right)\\ &= \cos^{-1}\left(\frac{26}{10\sqrt{10}}\right)\\ &\approx 0.6055 \approx 34.7^\circ.\end{aligned}$$

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{u}\cdot\vec{w}}{(\sqrt{10})(5)}\right)\\ &= \cos^{-1}\left(\frac{-9}{5\sqrt{10}}\right)\\ &\approx 2.1763 \approx 124.7^\circ\end{aligned}$$

We see from our computation that $\alpha + \beta = \theta$, as indicated by Figure 9.31. While we knew this should be the case, it is nice to see that this non-intuitive formula indeed returns the results we expected.

We do a similar example next in the context of vectors in space.

Example 18 Using the dot product to find angles

Let $\vec{u} = \langle 1, 1, 1\rangle$, $\vec{v} = \langle -1, 3, -2\rangle$ and $\vec{w} = \langle -5, 1, 4\rangle$, as illustrated in Figure 9.32. Find the angle between each pair of vectors.

Solution

1. Between $\vec{u}$ and $\vec{v}$:

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{||\,\vec{u}\,||\,||\,\vec{v}\,||}\right)\\ &= \cos^{-1}\left(\frac{0}{\sqrt{3}\sqrt{14}}\right)\\ &= \frac{\pi}{2}.\end{aligned}$$

Figure 9.32: Vectors used in Example 18.

Notes:

2. Between $\vec{u}$ and $\vec{w}$:

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{u}\cdot\vec{w}}{||\,\vec{u}\,||\,||\,\vec{w}\,||}\right)\\ &= \cos^{-1}\left(\frac{0}{\sqrt{3}\sqrt{42}}\right)\\ &= \frac{\pi}{2}.\end{aligned}$$

3. Between $\vec{v}$ and $\vec{w}$:

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{||\,\vec{v}\,||\,||\,\vec{w}\,||}\right)\\ &= \cos^{-1}\left(\frac{0}{\sqrt{14}\sqrt{42}}\right)\\ &= \frac{\pi}{2}.\end{aligned}$$

While our work shows that each angle is $\pi/2$, i.e., 90°, none of these angles looks to be a right angle in Figure 9.32. Such is the case when drawing three–dimensional objects on the page.

All three angles between these vectors was $\pi/2$, or 90°. We know from geometry and everyday life that 90° angles are "nice" for a variety of reasons, so it should seem significant that these angles are all $\pi/2$. Notice the common feature in each calculation (and also the calculation of α in Example 17): the dot products of each pair of angles was 0. We use this as a basis for a definition of the term **orthogonal**, which is essentially synonymous to *perpendicular*.

Note: The term *perpendicular* originally referred to lines. As mathematics progressed, the concept of "being at right angles to" was applied to other objects, such as vectors and planes, and the term *orthogonal* was introduced. It is especially used when discussing objects that are hard, or impossible, to visualize: two vectors in 5-dimensional space are orthogonal if their dot product is 0. It is not wrong to say they are *perpendicular*, but common convention gives preference to the word *orthogonal*.

Definition 10 Orthogonal

Vectors $\vec{u}$ and $\vec{v}$ are **orthogonal** if their dot product is 0.

Example 19 Finding orthogonal vectors

Let $\vec{u} = \langle 3, 5\rangle$ and $\vec{v} = \langle 1, 2, 3\rangle$.

1. Find two vectors in $\mathbb{R}^2$ that are orthogonal to $\vec{u}$.
2. Find two non–parallel vectors in $\mathbb{R}^3$ that are orthogonal to $\vec{v}$.

Solution

Notes:

1. Recall that a line perpendicular to a line with slope m has slope $-1/m$, the "opposite reciprocal slope." We can think of the slope of $\vec{u}$ as $5/3$, its "rise over run." A vector orthogonal to $\vec{u}$ will have slope $-3/5$. There are many such choices, though all parallel:

$$\langle -5, 3\rangle \quad \text{or} \quad \langle 5, -3\rangle \quad \text{or} \quad \langle -10, 6\rangle \quad \text{or} \quad \langle 15, -9\rangle\text{, etc.}$$

2. There are infinite directions in space orthogonal to any given direction, so there are an infinite number of non–parallel vectors orthogonal to $\vec{v}$. Since there are so many, we have great leeway in finding some.

   One way is to arbitrarily pick values for the first two components, leaving the third unknown. For instance, let $\vec{v}_1 = \langle 2, 7, z\rangle$. If $\vec{v}_1$ is to be orthogonal to $\vec{v}$, then $\vec{v}_1 \cdot \vec{v} = 0$, so

$$2 + 14 + 3z = 0 \quad \Rightarrow z = \frac{-16}{3}.$$

   So $\vec{v}_1 = \langle 2, 7, -16/3\rangle$ is orthogonal to $\vec{v}$. We can apply a similar technique by leaving the first or second component unknown.

   Another method of finding a vector orthogonal to $\vec{v}$ mirrors what we did in part 1. Let $\vec{v}_2 = \langle -2, 1, 0\rangle$. Here we switched the first two components of $\vec{v}$, changing the sign of one of them (similar to the "opposite reciprocal" concept before). Letting the third component be 0 effectively ignores the third component of $\vec{v}$, and it is easy to see that

$$\vec{v}_2 \cdot \vec{v} = \langle -2, 1, 0\rangle \cdot \langle 1, 2, 3\rangle = 0.$$

   Clearly $\vec{v}_1$ and $\vec{v}_2$ are not parallel.

An important construction is illustrated in Figure 9.33, where vectors $\vec{u}$ and $\vec{v}$ are sketched. In part (a), a dotted line is drawn from the tip of $\vec{u}$ to the line containing $\vec{v}$, where the dotted line is orthogonal to $\vec{v}$. In part (b), the dotted line is replaced with the vector $\vec{z}$ and $\vec{w}$ is formed, parallel to $\vec{v}$. It is clear by the diagram that $\vec{u} = \vec{w} + \vec{z}$. What is important about this construction is this: $\vec{u}$ is *decomposed* as the sum of two vectors, one of which is parallel to $\vec{v}$ and one that is perpendicular to $\vec{v}$. It is hard to overstate the importance of this construction (as we'll see in upcoming examples).

The vectors $\vec{w}$, $\vec{z}$ and $\vec{u}$ as shown in Figure 9.33 (b) form a right triangle, where the angle between $\vec{v}$ and $\vec{u}$ is labeled θ. We can find $\vec{w}$ in terms of $\vec{v}$ and $\vec{u}$.

Using trigonometry, we can state that

$$||\,\vec{w}\,|| = ||\,\vec{u}\,||\cos\theta. \tag{9.2}$$

Figure 9.33: Developing the construction of the *orthogonal projection*.

Notes:

We also know that $\vec{w}$ is parallel to to $\vec{v}$; that is, the direction of $\vec{w}$ is the direction of $\vec{v}$, described by the unit vector $\frac{1}{||\vec{v}||}\vec{v}$. The vector $\vec{w}$ is the vector in the direction $\frac{1}{||\vec{v}||}\vec{v}$ with magnitude $||\vec{u}||\cos\theta$:

$$\vec{w} = \left(||\vec{u}||\cos\theta\right)\frac{1}{||\vec{v}||}\vec{v}.$$

Replace $\cos\theta$ using Theorem 2:

$$\begin{aligned} &= \left(||\vec{u}||\frac{\vec{u}\cdot\vec{v}}{||\vec{u}||||\vec{v}||}\right)\frac{1}{||\vec{v}||}\vec{v} \\ &= \frac{\vec{u}\cdot\vec{v}}{||\vec{v}||^2}\vec{v}. \end{aligned}$$

Now apply Theorem 1.

$$= \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}.$$

Since this construction is so important, it is given a special name.

Definition 11 Orthogonal Projection

Let $\vec{u}$ and $\vec{v}$ be given. The **orthogonal projection of $\vec{u}$ onto $\vec{v}$**, denoted $\text{proj}_{\vec{v}}\vec{u}$, is

$$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}.$$

Example 20 Computing the orthogonal projection

1. Let $\vec{u} = \langle -2, 1\rangle$ and $\vec{v} = \langle 3, 1\rangle$. Find $\text{proj}_{\vec{v}}\vec{u}$, and sketch all three vectors with initial points at the origin.

2. Let $\vec{w} = \langle 2, 1, 3\rangle$ and $\vec{x} = \langle 1, 1, 1\rangle$. Find $\text{proj}_{\vec{x}}\vec{w}$, and sketch all three vectors with initial points at the origin.

Solution

Notes:

1. Applying Definition 11, we have

$$\begin{aligned}\text{proj}_{\vec{v}}\,\vec{u} &= \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}\\ &= \frac{-5}{10}\langle 3,1\rangle\\ &= \left\langle -\frac{3}{2}, -\frac{1}{2}\right\rangle.\end{aligned}$$

   Vectors $\vec{u}$, $\vec{v}$ and $\text{proj}_{\vec{v}}\,\vec{u}$ are sketched in Figure 9.34(a). Note how the projection is parallel to $\vec{v}$; that is, it lies on the same line through the origin as $\vec{v}$, although it points in the opposite direction. That is because the angle between $\vec{u}$ and $\vec{v}$ is obtuse (i.e., greater than 90°).

2. Apply the definition:

$$\begin{aligned}\text{proj}_{\vec{x}}\,\vec{w} &= \frac{\vec{w}\cdot\vec{x}}{\vec{x}\cdot\vec{x}}\vec{x}\\ &= \frac{6}{3}\langle 1,1,1\rangle\\ &= \langle 2,2,2\rangle.\end{aligned}$$

   These vectors are sketched in Figure 9.34(b), and again in part (c) from a different perspective. Because of the nature of graphing these vectors, the sketch in part (b) makes it difficult to recognize that the drawn projection has the geometric properties it should. The graph shown in part (c) illustrates these properties better.

Figure 9.34: Graphing the vectors used in Example 20.

Consider Figure 9.35 where the concept of the orthogonal projection is again illustrated. It is clear that

$$\vec{u} = \text{proj}_{\vec{v}}\,\vec{u} + \vec{z}. \tag{9.3}$$

As we know what $\vec{u}$ and $\text{proj}_{\vec{v}}\,\vec{u}$ are, we can solve for $\vec{z}$ and state that

$$\vec{z} = \vec{u} - \text{proj}_{\vec{v}}\,\vec{u}.$$

This leads us to rewrite Equation (9.3) in a seemingly silly way:

$$\vec{u} = \text{proj}_{\vec{v}}\,\vec{u} + (\vec{u} - \text{proj}_{\vec{v}}\,\vec{u}).$$

This is not nonsense, as pointed out in the following Key Idea. (Notation note: the expression "$\parallel \vec{y}$" means "is parallel to $\vec{y}$." We can use this notation to state "$\vec{x} \parallel \vec{y}$" which means "$\vec{x}$ is parallel to $\vec{y}$." The expression "$\perp \vec{y}$" means "is orthogonal to $\vec{y}$," and is used similarly.)

Figure 9.35: Illustrating the orthogonal projection.

Notes:

Key Idea 4 Orthogonal Decomposition of Vectors

Let $\vec{u}$ and $\vec{v}$ be given. Then $\vec{u}$ can be written as the sum of two vectors, one of which is parallel to $\vec{v}$, and one of which is orthogonal to $\vec{v}$:

$$\vec{u} = \underbrace{\text{proj}_{\vec{v}}\,\vec{u}}_{\parallel\,\vec{v}} + \underbrace{(\vec{u} - \text{proj}_{\vec{v}}\,\vec{u})}_{\perp\,\vec{v}}.$$

We illustrate the use of this equality in the following example.

Example 21 Orthogonal decomposition of vectors

1. Let $\vec{u} = \langle -2, 1\rangle$ and $\vec{v} = \langle 3, 1\rangle$ as in Example 20. Decompose $\vec{u}$ as the sum of a vector parallel to $\vec{v}$ and a vector orthogonal to $\vec{v}$.
2. Let $\vec{w} = \langle 2, 1, 3\rangle$ and $\vec{x} = \langle 1, 1, 1\rangle$ as in Example 20. Decompose $\vec{w}$ as the sum of a vector parallel to $\vec{x}$ and a vector orthogonal to $\vec{x}$.

Solution

1. In Example 20, we found that $\text{proj}_{\vec{v}}\,\vec{u} = \langle -1.5, -0.5\rangle$. Let

$$\vec{z} = \vec{u} - \text{proj}_{\vec{v}}\,\vec{u} = \langle -2, 1\rangle - \langle -1.5, -0.5\rangle = \langle -0.5, 1.5\rangle\,.$$

Is $\vec{z}$ orthogonal to $\vec{v}$? (I.e, is $\vec{z} \perp \vec{v}$?) We check for orthogonality with the dot product:

$$\vec{z}\cdot\vec{v} = \langle -0.5, 1.5\rangle \cdot \langle 3, 1\rangle = 0.$$

Since the dot product is 0, we know $\vec{z} \perp \vec{v}$. Thus:

$$\begin{aligned} \vec{u} &= \text{proj}_{\vec{v}}\,\vec{u} + (\vec{u} - \text{proj}_{\vec{v}}\,\vec{u}) \\ \langle -2, 1\rangle &= \underbrace{\langle -1.5, -0.5\rangle}_{\parallel\,\vec{v}} + \underbrace{\langle -0.5, 1.5\rangle}_{\perp\,\vec{v}}. \end{aligned}$$

2. We found in Example 20 that $\text{proj}_{\vec{x}}\,\vec{w} = \langle 2, 2, 2\rangle$. Applying the Key Idea, we have:

$$\vec{z} = \vec{w} - \text{proj}_{\vec{x}}\,\vec{w} = \langle 2, 1, 3\rangle - \langle 2, 2, 2\rangle = \langle 0, -1, 1\rangle\,.$$

We check to see if $\vec{z} \perp \vec{x}$:

$$\vec{z}\cdot\vec{x} = \langle 0, -1, 1\rangle \cdot \langle 1, 1, 1\rangle = 0.$$

Notes:

Since the dot product is 0, we know the two vectors are orthogonal. We now write $\vec{w}$ as the sum of two vectors, one parallel and one orthogonal to $\vec{x}$:

$$\begin{aligned}\vec{w} &= \text{proj}_{\vec{x}}\,\vec{w} \;+\; (\vec{w} - \text{proj}_{\vec{x}}\,\vec{w})\\ \langle 2,1,3\rangle &= \underbrace{\langle 2,2,2\rangle}_{\parallel\,\vec{x}} \;+\; \underbrace{\langle 0,-1,1\rangle}_{\perp\,\vec{x}}\end{aligned}$$

We give an example of where this decomposition is useful.

Example 22 Orthogonally decomposing a force vector

Consider Figure 9.36(a), showing a box weighing 50lb on a ramp that rises 5ft over a span of 20ft. Find the components of force, and their magnitudes, acting on the box (as sketched in part (b) of the figure):

1. in the direction of the ramp, and
2. orthogonal to the ramp.

Solution As the ramp rises 5ft over a horizontal distance of 20ft, we can represent the direction of the ramp with the vector $\vec{r} = \langle 20, 5\rangle$. Gravity pulls down with a force of 50lb, which we represent with $\vec{g} = \langle 0, -50\rangle$.

1. To find the force of gravity in the direction of the ramp, we compute $\text{proj}_{\vec{r}}\,\vec{g}$:

$$\begin{aligned}\text{proj}_{\vec{r}}\,\vec{g} &= \frac{\vec{g}\cdot\vec{r}}{\vec{r}\cdot\vec{r}}\vec{r}\\ &= \frac{-250}{425}\langle 20,5\rangle\\ &= \left\langle -\frac{200}{17}, -\frac{50}{17}\right\rangle \approx \langle -11.76, -2.94\rangle\,.\end{aligned}$$

   The magnitude of $\text{proj}_{\vec{r}}\,\vec{g}$ is $\|\,\text{proj}_{\vec{r}}\,\vec{g}\,\| = 50/\sqrt{17} \approx 12.13$lb. Though the box weighs 50lb, a force of about 12lb is enough to keep the box from sliding down the ramp.

2. To find the component $\vec{z}$ of gravity orthogonal to the ramp, we use Key Idea 4.

$$\begin{aligned}\vec{z} &= \vec{g} - \text{proj}_{\vec{r}}\,\vec{g}\\ &= \left\langle \frac{200}{17}, -\frac{800}{17}\right\rangle \approx \langle 11.76, -47.06\rangle\,.\end{aligned}$$

Figure 9.36: Sketching the ramp and box in Example 22. Note: *The vectors are not drawn to scale.*

Notes:

The magnitude of this force is $||\vec{z}|| \approx 48.51$lb. In physics and engineering, knowing this force is important when computing things like static frictional force. (For instance, we could easily compute if the static frictional force alone was enough to keep the box from sliding down the ramp.)

Application to Work

In physics, the application of a force F to move an object in a straight line a distance d produces *work*; the amount of work W is $W = Fd$, (where F is in the direction of travel). The orthogonal projection allows us to compute work when the force is not in the direction of travel.

Consider Figure 9.37, where a force $\vec{F}$ is being applied to an object moving in the direction of $\vec{d}$. (The distance the object travels is the magnitude of $\vec{d}$.) The work done is the amount of force in the direction of $\vec{d}$, $||\text{proj}_{\vec{d}}\vec{F}||$, times $||\vec{d}||$:

Figure 9.37: Finding work when the force and direction of travel are given as vectors.

$$\begin{aligned}
||\text{proj}_{\vec{d}}\vec{F}|| \cdot ||\vec{d}|| &= \left|\left|\frac{\vec{F}\cdot\vec{d}}{\vec{d}\cdot\vec{d}}\vec{d}\right|\right| \cdot ||\vec{d}|| \\
&= \left|\frac{\vec{F}\cdot\vec{d}}{||\vec{d}||^2}\right| \cdot ||\vec{d}|| \cdot ||\vec{d}|| \\
&= \frac{\left|\vec{F}\cdot\vec{d}\right|}{||\vec{d}||^2}||\vec{d}||^2 \\
&= \left|\vec{F}\cdot\vec{d}\right|.
\end{aligned}$$

The expression $\vec{F}\cdot\vec{d}$ will be positive if the angle between $\vec{F}$ and $\vec{d}$ is acute; when the angle is obtuse (hence $\vec{F}\cdot\vec{d}$ is negative), the force is causing motion in the opposite direction of $\vec{d}$, resulting in "negative work." We want to capture this sign, so we drop the absolute value and find that $W = \vec{F}\cdot\vec{d}$.

Definition 12 Work

Let $\vec{F}$ be a constant force that moves an object in a straight line from point P to point Q. Let $\vec{d} = \overrightarrow{PQ}$. The **work** W done by $\vec{F}$ along $\vec{d}$ is $W = \vec{F}\cdot\vec{d}$.

Example 23 Computing work

A man slides a box along a ramp that rises 3ft over a distance of 15ft by applying 50lb of force as shown in Figure 9.38. Compute the work done.

Notes:

Solution The figure indicates that the force applied makes a 30° angle with the horizontal, so $\vec{F} = 50\,\langle \cos 30^\circ, \sin 30^\circ \rangle \approx \langle 43.3, 25 \rangle$. The ramp is represented by $\vec{d} = \langle 15, 3 \rangle$. The work done is simply

$$\vec{F} \cdot \vec{d} = 50\,\langle \cos 30^\circ, \sin 30^\circ \rangle \cdot \langle 15, 3 \rangle \approx 724.5\text{ft–lb}.$$

Note how we did not actually compute the distance the object traveled, nor the magnitude of the force in the direction of travel; this is all inherently computed by the dot product!

Figure 9.38: Computing work when sliding a box up a ramp in Example 23.

The dot product is a powerful way of evaluating computations that depend on angles without actually using angles. The next section explores another "product" on vectors, the *cross product.* Once again, angles play an important role, though in a much different way.

Notes:

Exercises 9.3

Terms and Concepts

1. The dot product of two vectors is a _____, not a vector.

2. How are the concepts of the dot product and vector magnitude related?

3. How can one quickly tell if the angle between two vectors is acute or obtuse?

4. Give a synonym for "orthogonal."

Problems

In Exercises 5 – 10, find the dot product of the given vectors.

5. $\vec{u} = \langle 2, -4\rangle$, $\vec{v} = \langle 3, 7\rangle$

6. $\vec{u} = \langle 5, 3\rangle$, $\vec{v} = \langle 6, 1\rangle$

7. $\vec{u} = \langle 1, -1, 2\rangle$, $\vec{v} = \langle 2, 5, 3\rangle$

8. $\vec{u} = \langle 3, 5, -1\rangle$, $\vec{v} = \langle 4, -1, 7\rangle$

9. $\vec{u} = \langle 1, 1\rangle$, $\vec{v} = \langle 1, 2, 3\rangle$

10. $\vec{u} = \langle 1, 2, 3\rangle$, $\vec{v} = \langle 0, 0, 0\rangle$

11. Create your own vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^2$ and show that $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$.

12. Create your own vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ and scalar c and show that $c(\vec{u}\cdot\vec{v}) = \vec{u}\cdot(c\vec{v})$.

In Exercises 13 – 16, find the measure of the angle between the two vectors in both radians and degrees.

13. $\vec{u} = \langle 1, 1\rangle$, $\vec{v} = \langle 1, 2\rangle$

14. $\vec{u} = \langle -2, 1\rangle$, $\vec{v} = \langle 3, 5\rangle$

15. $\vec{u} = \langle 8, 1, -4\rangle$, $\vec{v} = \langle 2, 2, 0\rangle$

16. $\vec{u} = \langle 1, 7, 2\rangle$, $\vec{v} = \langle 4, -2, 5\rangle$

In Exercises 17 – 20, a vector $\vec{v}$ is given. Give two vectors that are orthogonal to $\vec{v}$.

17. $\vec{v} = \langle 4, 7\rangle$

18. $\vec{v} = \langle -3, 5\rangle$

19. $\vec{v} = \langle 1, 1, 1\rangle$

20. $\vec{v} = \langle 1, -2, 3\rangle$

In Exercises 21 – 26, vectors $\vec{u}$ and $\vec{v}$ are given. Find $\text{proj}_{\vec{v}}\,\vec{u}$, the orthogonal projection of $\vec{u}$ onto $\vec{v}$, and sketch all three vectors on the same axes.

21. $\vec{u} = \langle 1, 2\rangle$, $\vec{v} = \langle -1, 3\rangle$

22. $\vec{u} = \langle 5, 5\rangle$, $\vec{v} = \langle 1, 3\rangle$

23. $\vec{u} = \langle -3, 2\rangle$, $\vec{v} = \langle 1, 1\rangle$

24. $\vec{u} = \langle -3, 2\rangle$, $\vec{v} = \langle 2, 3\rangle$

25. $\vec{u} = \langle 1, 5, 1\rangle$, $\vec{v} = \langle 1, 2, 3\rangle$

26. $\vec{u} = \langle 3, -1, 2\rangle$, $\vec{v} = \langle 2, 2, 1\rangle$

In Exercises 27 – 32, vectors $\vec{u}$ and $\vec{v}$ are given. Write $\vec{u}$ as the sum of two vectors, one of which is parallel to $\vec{v}$ and one of which is perpendicular to $\vec{v}$. Note: these are the same pairs of vectors as found in Exercises 21 – 26.

27. $\vec{u} = \langle 1, 2\rangle$, $\vec{v} = \langle -1, 3\rangle$

28. $\vec{u} = \langle 5, 5\rangle$, $\vec{v} = \langle 1, 3\rangle$

29. $\vec{u} = \langle -3, 2\rangle$, $\vec{v} = \langle 1, 1\rangle$

30. $\vec{u} = \langle -3, 2\rangle$, $\vec{v} = \langle 2, 3\rangle$

31. $\vec{u} = \langle 1, 5, 1\rangle$, $\vec{v} = \langle 1, 2, 3\rangle$

32. $\vec{u} = \langle 3, -1, 2\rangle$, $\vec{v} = \langle 2, 2, 1\rangle$

33. A 10lb box sits on a ramp that rises 4ft over a distance of 20ft. How much force is required to keep the box from sliding down the ramp?

34. A 10lb box sits on a 15ft ramp that makes a 30° angle with the horizontal. How much force is required to keep the box from sliding down the ramp?

35. How much work is performed in moving a box horizontally 10ft with a force of 20lb applied at an angle of 45° to the horizontal?

36. How much work is performed in moving a box horizontally 10ft with a force of 20lb applied at an angle of 10° to the horizontal?

37. How much work is performed in moving a box up the length of a ramp that rises 2ft over a distance of 10ft, with a force of 50lb applied horizontally?

38. How much work is performed in moving a box up the length of a ramp that rises 2ft over a distance of 10ft, with a force of 50lb applied at an angle of 45° to the horizontal?

39. How much work is performed in moving a box up the length of a 10ft ramp that makes a 5° angle with the horizontal, with 50lb of force applied in the direction of the ramp?

9.4 The Cross Product

"Orthogonality" is immensely important. A quick scan of your current environment will undoubtedly reveal numerous surfaces and edges that are perpendicular to each other (including the edges of this page). The dot product provides a quick test for orthogonality: vectors $\vec{u}$ and $\vec{v}$ are perpendicular if, and only if, $\vec{u} \cdot \vec{v} = 0$.

Given two non–parallel, nonzero vectors $\vec{u}$ and $\vec{v}$ in space, it is very useful to find a vector $\vec{w}$ that is perpendicular to both $\vec{u}$ and $\vec{v}$. There is a operation, called the **cross product**, that creates such a vector. This section defines the cross product, then explores its properties and applications.

Definition 13 Cross Product

Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ be vectors in $\mathbb{R}^3$. The **cross product of $\vec{u}$ and $\vec{v}$**, denoted $\vec{u} \times \vec{v}$, is the vector

$$\vec{u} \times \vec{v} = \langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1 \rangle .$$

This definition can be a bit cumbersome to remember. After an example we will give a convenient method for computing the cross product. For now, careful examination of the products and differences given in the definition should reveal a pattern that is not too difficult to remember. (For instance, in the first component only 2 and 3 appear as subscripts; in the second component, only 1 and 3 appear as subscripts. Further study reveals the order in which they appear.)

Let's practice using this definition by computing a cross product.

Example 24 Computing a cross product

Let $\vec{u} = \langle 2, -1, 4 \rangle$ and $\vec{v} = \langle 3, 2, 5 \rangle$. Find $\vec{u} \times \vec{v}$, and verify that it is orthogonal to both $\vec{u}$ and $\vec{v}$.

Solution Using Definition 13, we have

$$\vec{u} \times \vec{v} = \big\langle (-1)5 - (4)2, -\big((2)5 - (4)3\big), (2)2 - (-1)3 \big\rangle = \langle -13, 2, 7 \rangle .$$

(We encourage the reader to compute this product on their own, then verify their result.)

We test whether or not $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$ using the dot product:

$$\big(\vec{u} \times \vec{v}\big) \cdot \vec{u} = \langle -13, 2, 7 \rangle \cdot \langle 2, -1, 4 \rangle = 0,$$

$$\big(\vec{u} \times \vec{v}\big) \cdot \vec{v} = \langle -13, 2, 7 \rangle \cdot \langle 3, 2, 5 \rangle = 0.$$

Since both dot products are zero, $\vec{u} \times \vec{v}$ is indeed orthogonal to both $\vec{u}$ and $\vec{v}$.

Notes:

A convenient method of computing the cross product starts with forming a particular 3×3 *matrix*, or rectangular array. The first row comprises the standard unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$. The second and third rows are the vectors $\vec{u}$ and $\vec{v}$, respectively. Using $\vec{u}$ and $\vec{v}$ from Example 24, we begin with:

$$\begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 4 \\ 3 & 2 & 5 \end{array}$$

Now repeat the first two columns after the original three:

$$\begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 2 & -1 & 4 & 2 & -1 \\ 3 & 2 & 5 & 3 & 2 \end{array}$$

This gives three full "upper left to lower right" diagonals, and three full "upper right to lower left" diagonals, as shown. Compute the products along each diagonal, then add the products on the right and subtract the products on the left:

$$\begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 2 & -1 & 4 & 2 & -1 \\ 3 & 2 & 5 & 3 & 2 \\ -3\vec{k} & 8\vec{i} & 10\vec{j} & -5\vec{i} & 12\vec{j} \quad 4\vec{k} \end{array}$$

$$\vec{u} \times \vec{v} = \left(-5\vec{i} + 12\vec{j} + 4\vec{k}\right) - \left(-3\vec{k} + 8\vec{i} + 10\vec{j}\right) = -13\vec{i} + 2\vec{j} + 7\vec{k} = \langle -13, 2, 7 \rangle .$$

We practice using this method.

Example 25 Computing a cross product

Let $\vec{u} = \langle 1, 3, 6 \rangle$ and $\vec{v} = \langle -1, 2, 1 \rangle$. Compute both $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$.

Solution To compute $\vec{u} \times \vec{v}$, we form the matrix as prescribed above, complete with repeated first columns:

$$\begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 1 & 3 & 6 & 1 & 3 \\ -1 & 2 & 1 & -1 & 2 \end{array}$$

We let the reader compute the products of the diagonals; we give the result:

$$\vec{u} \times \vec{v} = \left(3\vec{i} - 6\vec{j} + 2\vec{k}\right) - \left(-3\vec{k} + 12\vec{i} + \vec{j}\right) = \langle -9, -7, 5 \rangle .$$

Notes:

To compute $\vec{v} \times \vec{u}$, we switch the second and third rows of the above matrix, then multiply along diagonals and subtract:

$$\begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ -1 & 2 & 1 & -1 & 2 \\ 1 & 3 & 6 & 1 & 3 \end{array}$$

Note how with the rows being switched, the products that once appeared on the right now appear on the left, and vice–versa. Thus the result is:

$$\vec{v} \times \vec{u} = \left(12\vec{i} + \vec{j} - 3\vec{k}\right) - \left(2\vec{k} + 3\vec{i} - 6\vec{j}\right) = \langle 9, 7, -5 \rangle\,,$$

which is the opposite of $\vec{u} \times \vec{v}$. We leave it to the reader to verify that each of these vectors is orthogonal to $\vec{u}$ and $\vec{v}$.

Properties of the Cross Product

It is not coincidence that $\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$ in the preceding example; one can show using Definition 13 that this will always be the case. The following theorem states several useful properties of the cross product, each of which can be verified by referring to the definition.

Theorem 3 Properties of the Cross Product

Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^3$ and let c be a scalar. The following identities hold:

1. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$ — Anticommutative Property
2. (a) $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$ — Distributive Properties
   (b) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
3. $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$
4. (a) $(\vec{u} \times \vec{v}) \cdot \vec{u} = 0$ — Orthogonality Properties
   (b) $(\vec{u} \times \vec{v}) \cdot \vec{v} = 0$
5. $\vec{u} \times \vec{u} = \vec{0}$
6. $\vec{u} \times \vec{0} = \vec{0}$
7. $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$ — Triple Scalar Product

Notes:

We introduced the cross product as a way to find a vector orthogonal to two given vectors, but we did not give a proof that the construction given in Definition 13 satisfies this property. Theorem 3 asserts this property holds; we leave it as a problem in the Exercise section to verify this.

Property 5 from the theorem is also left to the reader to prove in the Exercise section, but it reveals something more interesting than "the cross product of a vector with itself is $\vec{0}$." Let $\vec{u}$ and $\vec{v}$ be parallel vectors; that is, let there be a scalar c such that $\vec{v} = c\vec{u}$. Consider their cross product:

$$\begin{aligned} \vec{u} \times \vec{v} &= \vec{u} \times (c\vec{u}) \\ &= c(\vec{u} \times \vec{u}) && \text{(by Property 3 of Theorem 3)} \\ &= \vec{0}. && \text{(by Property 5 of Theorem 3)} \end{aligned}$$

We have just shown that the cross product of parallel vectors is $\vec{0}$. This hints at something deeper. Theorem 2 related the angle between two vectors and their dot product; there is a similar relationship relating the cross product of two vectors and the angle between them, given by the following theorem.

Note: Definition 10 (through Theorem 2) defines $\vec{u}$ and $\vec{v}$ to be orthogonal if $\vec{u} \cdot \vec{v} = 0$. We could use Theorem 4 to define $\vec{u}$ and $\vec{v}$ are parallel if $\vec{u} \times \vec{v} = 0$. By such a definition, $\vec{0}$ would be both orthogonal and parallel to every vector. Apparent paradoxes such as this are not uncommon in mathematics and can be very useful. (See also the marginal note on page 512.)

Theorem 4 The Cross Product and Angles

Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^3$. Then

$$|| \vec{u} \times \vec{v} || = || \vec{u} || \, || \vec{v} || \sin\theta,$$

where θ, $0 \leq \theta \leq \pi$, is the angle between $\vec{u}$ and $\vec{v}$.

Note that this theorem makes a statement about the *magnitude* of the cross product. When the angle between $\vec{u}$ and $\vec{v}$ is 0 or π (i.e., the vectors are parallel), the magnitude of the cross product is 0. The only vector with a magnitude of 0 is $\vec{0}$ (see Property 9 of Theorem 0), hence the cross product of parallel vectors is $\vec{0}$.

We demonstrate the truth of this theorem in the following example.

Example 26 The cross product and angles

Let $\vec{u} = \langle 1, 3, 6 \rangle$ and $\vec{v} = \langle -1, 2, 1 \rangle$ as in Example 25. Verify Theorem 4 by finding θ, the angle between $\vec{u}$ and $\vec{v}$, and the magnitude of $\vec{u} \times \vec{v}$.

Notes:

Solution We use Theorem 2 to find the angle between $\vec{u}$ and $\vec{v}$.

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{||\,\vec{u}\,||\,||\,\vec{v}\,||}\right)\\ &= \cos^{-1}\left(\frac{11}{\sqrt{46}\sqrt{6}}\right)\\ &\approx 0.8471 = 48.54^\circ.\end{aligned}$$

Our work in Example 25 showed that $\vec{u}\times\vec{v} = \langle -9, -7, 5\rangle$, hence $||\,\vec{u}\times\vec{v}\,|| = \sqrt{155}$. Is $||\,\vec{u}\times\vec{v}\,|| = ||\,\vec{u}\,||\,||\,\vec{v}\,||\sin\theta$? Using numerical approximations, we find:

$$\begin{aligned}||\,\vec{u}\times\vec{v}\,|| &= \sqrt{155} & ||\,\vec{u}\,||\,||\,\vec{v}\,||\sin\theta &= \sqrt{46}\sqrt{6}\sin 0.8471\\ &\approx 12.45. & &\approx 12.45.\end{aligned}$$

Numerically, they seem equal. Using a right triangle, one can show that

$$\sin\left(\cos^{-1}\left(\frac{11}{\sqrt{46}\sqrt{6}}\right)\right) = \frac{\sqrt{155}}{\sqrt{46}\sqrt{6}},$$

which allows us to verify the theorem exactly.

Right Hand Rule

The anticommutative property of the cross product demonstrates that $\vec{u}\times\vec{v}$ and $\vec{v}\times\vec{u}$ differ only by a sign – these vectors have the same magnitude but point in the opposite direction. When seeking a vector perpendicular to $\vec{u}$ and $\vec{v}$, we essentially have two directions to choose from, one in the direction of $\vec{u}\times\vec{v}$ and one in the direction of $\vec{v}\times\vec{u}$. Does it matter which we choose? How can we tell which one we will get without graphing, etc.?

Another wonderful property of the cross product, as defined, is that it follows the **right hand rule.** Given $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ with the same initial point, point the index finger of your right hand in the direction of $\vec{u}$ and let your middle finger point in the direction of $\vec{v}$ (much as we did when establishing the right hand rule for the 3-dimensional coordinate system). Your thumb will naturally extend in the direction of $\vec{u}\times\vec{v}$. One can "practice" this using Figure 9.39. If you switch, and point the index finder in the direction of $\vec{v}$ and the middle finger in the direction of $\vec{u}$, your thumb will now point in the opposite direction, allowing you to "visualize" the anticommutative property of the cross product.

Figure 9.39: Illustrating the Right Hand Rule of the cross product.

Applications of the Cross Product

There are a number of ways in which the cross product is useful in mathematics, physics and other areas of science beyond "just" finding a vector perpendicular to two others. We highlight a few here.

Notes:

(a)

(b)

Figure 9.40: Using the cross product to find the area of a parallelogram.

(a)

(b)

Figure 9.41: Sketching the parallelograms in Example 27.

Area of a Parallelogram

It is a standard geometry fact that the area of a parallelogram is $A = bh$, where b is the length of the base and h is the height of the parallelogram, as illustrated in Figure 9.40(a). As shown when defining the Parallelogram Law of vector addition, two vectors $\vec{u}$ and $\vec{v}$ define a parallelogram when drawn from the same initial point, as illustrated in Figure 9.40(b). Trigonometry tells us that $h = ||\,\vec{u}\,||\sin\theta$, hence the area of the parallelogram is

$$A = ||\,\vec{u}\,||\,||\,\vec{v}\,||\sin\theta = ||\,\vec{u}\times\vec{v}\,||, \tag{9.4}$$

where the second equality comes from Theorem 4. We illustrate using Equation (9.4) in the following example.

Example 27 Finding the area of a parallelogram

1. Find the area of the parallelogram defined by the vectors $\vec{u} = \langle 2, 1\rangle$ and $\vec{v} = \langle 1, 3\rangle$.
2. Verify that the points $A = (1, 1, 1)$, $B = (2, 3, 2)$, $C = (4, 5, 3)$ and $D = (3, 3, 2)$ are the vertices of a parallelogram. Find the area of the parallelogram.

Solution

1. Figure 9.41(a) sketches the parallelogram defined by the vectors $\vec{u}$ and $\vec{v}$. We have a slight problem in that our vectors exist in $\mathbb{R}^2$, not $\mathbb{R}^3$, and the cross product is only defined on vectors in $\mathbb{R}^3$. We skirt this issue by viewing $\vec{u}$ and $\vec{v}$ as vectors in the $x-y$ plane of $\mathbb{R}^3$, and rewrite them as $\vec{u} = \langle 2, 1, 0\rangle$ and $\vec{v} = \langle 1, 3, 0\rangle$. We can now compute the cross product. It is easy to show that $\vec{u}\times\vec{v} = \langle 0, 0, 5\rangle$; therefore the area of the parallelogram is $A = ||\,\vec{u}\times\vec{v}\,|| = 5$.
2. To show that the quadrilateral $ABCD$ is a parallelogram (shown in Figure 9.41(b)), we need to show that the opposite sides are parallel. We can quickly show that $\overrightarrow{AB} = \overrightarrow{DC} = \langle 1, 2, 1\rangle$ and $\overrightarrow{BC} = \overrightarrow{AD} = \langle 2, 2, 1\rangle$. We find the area by computing the magnitude of the cross product of $\overrightarrow{AB}$ and $\overrightarrow{BC}$:

$$\overrightarrow{AB}\times\overrightarrow{BC} = \langle 0, 1, -2\rangle \quad\Rightarrow\quad ||\,\overrightarrow{AB}\times\overrightarrow{BC}\,|| = \sqrt{5} \approx 2.236.$$

This application is perhaps more useful in finding the area of a triangle (in short, triangles are used more often than parallelograms). We illustrate this in the following example.

Notes:

Example 28 Area of a triangle

Find the area of the triangle with vertices $A = (1, 2)$, $B = (2, 3)$ and $C = (3, 1)$, as pictured in Figure 9.42.

Figure 9.42: Finding the area of a triangle in Example 28.

Solution We found the area of this triangle in Example **??** to be 1.5 using integration. There we discussed the fact that finding the area of a triangle can be inconvenient using the "$\frac{1}{2}bh$" formula as one has to compute the height, which generally involves finding angles, etc. Using a cross product is much more direct.

We can choose any two sides of the triangle to use to form vectors; we choose $\overrightarrow{AB} = \langle 1, 1\rangle$ and $\overrightarrow{AC} = \langle 2, -1\rangle$. As in the previous example, we will rewrite these vectors with a third component of 0 so that we can apply the cross product. The area of the triangle is

$$\frac{1}{2}||\overrightarrow{AB} \times \overrightarrow{AC}|| = \frac{1}{2}||\ \langle 1, 1, 0\rangle \times \langle 2, -1, 0\rangle\ || = \frac{1}{2}||\ \langle 0, 0, -3\rangle\ || = \frac{3}{2}.$$

We arrive at the same answer as before with less work.

Note: The word "parallelepiped" is pronounced "parallel–eh–pipe–ed."

Volume of a Parallelepiped

The three dimensional analogue to the parallelogram is the **parallelepiped**. Each face is parallel to the face opposite face, as illustrated in Figure 9.43. By crossing $\vec{v}$ and $\vec{w}$, one gets a vector whose magnitude is the area of the base. Dotting this vector with $\vec{u}$ computes the volume of parallelepiped! (Up to a sign; take the absolute value.)

Figure 9.43: A parallelepiped is the three dimensional analogue to the parallelogram.

Thus the volume of a parallelepiped defined by vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ is

$$V = |\vec{u} \cdot (\vec{v} \times \vec{w})|. \tag{9.5}$$

Note how this is the Triple Scalar Product, first seen in Theorem 3. Applying the identities given in the theorem shows that we can apply the Triple Scalar Product in any "order" we choose to find the volume. That is,

$$V = |\vec{u} \cdot (\vec{v} \times \vec{w})| = |\vec{u} \cdot (\vec{w} \times \vec{v})| = |(\vec{u} \times \vec{v}) \cdot \vec{w}|, \quad \text{etc.}$$

Example 29 Finding the volume of parallelepiped

Find the volume of the parallepiped defined by the vectors $\vec{u} = \langle 1, 1, 0\rangle$, $\vec{v} = \langle -1, 1, 0\rangle$ and $\vec{w} = \langle 0, 1, 1\rangle$.

Figure 9.44: A parallelepiped in Example 29.

Solution We apply Equation (9.5). We first find $\vec{v} \times \vec{w} = \langle 1, 1, -1\rangle$. Then

$$|\vec{u} \cdot (\vec{v} \times \vec{w})| = |\ \langle 1, 1, 0\rangle \cdot \langle 1, 1, -1\rangle\ | = 2.$$

So the volume of the parallelepiped is 2 cubic units.

Notes:

While this application of the Triple Scalar Product is interesting, it is not used all that often: parallelepipeds are not a common shape in physics and engineering. The last application of the cross product is very applicable in engineering.

Torque

Torque is a measure of the turning force applied to an object. A classic scenario involving torque is the application of a wrench to a bolt. When a force is applied to the wrench, the bolt turns. When we represent the force and wrench with vectors $\vec{F}$ and $\vec{\ell}$, we see that the bolt moves (because of the threads) in a direction orthogonal to $\vec{F}$ and $\vec{\ell}$. Torque is usually represented by the Greek letter τ, or tau, and has units of N·m, a Newton–meter, or ft·lb, a foot–pound.

While a full understanding of torque is beyond the purposes of this book, when a force $\vec{F}$ is applied to a lever arm $\vec{\ell}$, the resulting torque is

$$\vec{\tau} = \vec{\ell} \times \vec{F}. \tag{9.6}$$

Figure 9.45: Showing a force being applied to a lever in Example 30.

Example 30 Computing torque

A lever of length 2ft makes an angle with the horizontal of 45°. Find the resulting torque when a force of 10lb is applied to the end of the level where:

1. the force is perpendicular to the lever, and
2. the force makes an angle of 60° with the lever, as shown in Figure 9.45.

Solution

1. We start by determining vectors for the force and lever arm. Since the lever arm makes a 45° angle with the horizontal and is 2ft long, we can state that $\vec{\ell} = 2\left\langle \cos 45^\circ, \sin 45^\circ \right\rangle = \left\langle \sqrt{2}, \sqrt{2} \right\rangle$.

   Since the force vector is perpendicular to the lever arm (as seen in the left hand side of Figure 9.45), we can conclude it is making an angle of -45° with the horizontal. As it has a magnitude of 10lb, we can state $\vec{F} = 10\left\langle \cos(-45^\circ), \sin(-45^\circ) \right\rangle = \left\langle 5\sqrt{2}, -5\sqrt{2} \right\rangle$.

   Using Equation (9.6) to find the torque requires a cross product. We again let the third component of each vector be 0 and compute the cross product:

$$\begin{aligned}\vec{\tau} &= \vec{\ell} \times \vec{F} \\ &= \left\langle \sqrt{2}, \sqrt{2}, 0 \right\rangle \times \left\langle 5\sqrt{2}, -5\sqrt{2}, 0 \right\rangle \\ &= \left\langle 0, 0, -20 \right\rangle\end{aligned}$$

Notes:

This clearly has a magnitude of 20 ft-lb.

We can view the force and lever arm vectors as lying "on the page"; our computation of $\vec{\tau}$ shows that the torque goes "into the page." This follows the Right Hand Rule of the cross product, and it also matches well with the example of the wrench turning the bolt. Turning a bolt clockwise moves it in.

2. Our lever arm can still be represented by $\vec{\ell} = \left\langle \sqrt{2}, \sqrt{2} \right\rangle$. As our force vector makes a 60° angle with $\vec{\ell}$, we can see (referencing the right hand side of the figure) that $\vec{F}$ makes a -15° angle with the horizontal. Thus

$$\begin{aligned}\vec{F} = 10 \left\langle \cos -15^\circ, \sin -15^\circ \right\rangle &= \left\langle \frac{5(1+\sqrt{3})}{\sqrt{2}}, -\frac{5(1+\sqrt{3})}{\sqrt{2}} \right\rangle \\ &\approx \left\langle 9.659, -2.588 \right\rangle .\end{aligned}$$

We again make the third component 0 and take the cross product to find the torque:

$$\begin{aligned}\vec{\tau} &= \vec{\ell} \times \vec{F} \\ &= \left\langle \sqrt{2}, \sqrt{2}, 0 \right\rangle \times \left\langle \frac{5(1+\sqrt{3})}{\sqrt{2}}, -\frac{5(1+\sqrt{3})}{\sqrt{2}}, 0 \right\rangle \\ &= \left\langle 0, 0, -10\sqrt{3} \right\rangle \\ &\approx \left\langle 0, 0, -17.321 \right\rangle .\end{aligned}$$

As one might expect, when the force and lever arm vectors *are* orthogonal, the magnitude of force is greater than when the vectors *are not* orthogonal.

While the cross product has a variety of applications (as noted in this chapter), its fundamental use is finding a vector perpendicular to two others. Knowing a vector is orthogonal to two others is of incredible importance, as it allows us to find the equations of lines and planes in a variety of contexts. The importance of the cross product, in some sense, relies on the importance of lines and planes, which see widespread use throughout engineering, physics and mathematics. We study lines and planes in the next two sections.

Notes:

Exercises 9.4

Terms and Concepts

1. The cross product of two vectors is a _____, not a scalar.

2. One can visualize the direction of $\vec{u} \times \vec{v}$ using the _____ _____ _____.

3. Give a synonym for "orthogonal."

4. T/F: A fundamental principle of the cross product is that $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$.

5. _____ is a measure of the turning force applied to an object.

Problems

In Exercises 6 – 14, vectors $\vec{u}$ and $\vec{v}$ are given. Compute $\vec{u} \times \vec{v}$ and show this is orthogonal to both $\vec{u}$ and $\vec{v}$.

6. $\vec{u} = \langle 3, 2, -2 \rangle, \quad \vec{v} = \langle 0, 1, 5 \rangle$

7. $\vec{u} = \langle 5, -4, 3 \rangle, \quad \vec{v} = \langle 2, -5, 1 \rangle$

8. $\vec{u} = \langle 4, -5, -5 \rangle, \quad \vec{v} = \langle 3, 3, 4 \rangle$

9. $\vec{u} = \langle -4, 7, -10 \rangle, \quad \vec{v} = \langle 4, 4, 1 \rangle$

10. $\vec{u} = \langle 1, 0, 1 \rangle, \quad \vec{v} = \langle 5, 0, 7 \rangle$

11. $\vec{u} = \langle 1, 5, -4 \rangle, \quad \vec{v} = \langle -2, -10, 8 \rangle$

12. $\vec{u} = \vec{i}, \quad \vec{v} = \vec{j}$

13. $\vec{u} = \vec{i}, \quad \vec{v} = \vec{k}$

14. $\vec{u} = \vec{j}, \quad \vec{v} = \vec{k}$

15. Pick any vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ and show that $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.

16. Pick any vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ and show that $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$.

In Exercises 17 – 20, the magnitudes of vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ are given, along with the angle θ between them. Use this information to find the magnitude of $\vec{u} \times \vec{v}$.

17. $||\vec{u}|| = 2, \quad ||\vec{v}|| = 5, \quad \theta = 30^\circ$

18. $||\vec{u}|| = 3, \quad ||\vec{v}|| = 7, \quad \theta = \pi/2$

19. $||\vec{u}|| = 3, \quad ||\vec{v}|| = 4, \quad \theta = \pi$

20. $||\vec{u}|| = 2, \quad ||\vec{v}|| = 5, \quad \theta = 5\pi/6$

In Exercises 21 – 24, find the area of the parallelogram defined by the given vectors.

21. $\vec{u} = \langle 1, 1, 2 \rangle, \quad \vec{v} = \langle 2, 0, 3 \rangle$

22. $\vec{u} = \langle -2, 1, 5 \rangle, \quad \vec{v} = \langle -1, 3, 1 \rangle$

23. $\vec{u} = \langle 1, 2 \rangle, \quad \vec{v} = \langle 2, 1 \rangle$

24. $\vec{u} = \langle 2, 0 \rangle, \quad \vec{v} = \langle 0, 3 \rangle$

In Exercises 25 – 28, find the area of the triangle with the given vertices.

25. Vertices: $(0, 0, 0)$, $(1, 3, -1)$ and $(2, 1, 1)$.

26. Vertices: $(5, 2, -1)$, $(3, 6, 2)$ and $(1, 0, 4)$.

27. Vertices: $(1, 1)$, $(1, 3)$ and $(2, 2)$.

28. Vertices: $(3, 1)$, $(1, 2)$ and $(4, 3)$.

In Exercises 29 – 30, find the area of the quadrilateral with the given vertices. (Hint: break the quadrilateral into 2 triangles.)

29. Vertices: $(0, 0)$, $(1, 2)$, $(3, 0)$ and $(4, 3)$.

30. Vertices: $(0, 0, 0)$, $(2, 1, 1)$, $(-1, 2, -8)$ and $(1, -1, 5)$.

In Exercises 31 – 32, find the volume of the parallelepiped defined by the given vectors.

31. $\vec{u} = \langle 1, 1, 1 \rangle, \quad \vec{v} = \langle 1, 2, 3 \rangle, \quad \vec{w} = \langle 1, 0, 1 \rangle$

32. $\vec{u} = \langle -1, 2, 1 \rangle, \quad \vec{v} = \langle 2, 2, 1 \rangle, \quad \vec{w} = \langle 3, 1, 3 \rangle$

In Exercises 33 – 36, find a unit vector orthogonal to both $\vec{u}$ and $\vec{v}$.

33. $\vec{u} = \langle 1, 1, 1 \rangle, \quad \vec{v} = \langle 2, 0, 1 \rangle$

34. $\vec{u} = \langle 1, -2, 1 \rangle, \quad \vec{v} = \langle 3, 2, 1 \rangle$

35. $\vec{u} = \langle 5, 0, 2 \rangle, \quad \vec{v} = \langle -3, 0, 7 \rangle$

36. $\vec{u} = \langle 1, -2, 1 \rangle, \quad \vec{v} = \langle -2, 4, -2 \rangle$

37. A bicycle rider applies 150lb of force, straight down, onto a pedal that extends 7in horizontally from the crankshaft. Find the magnitude of the torque applied to the crankshaft.

38. A bicycle rider applies 150lb of force, straight down, onto a pedal that extends 7in from the crankshaft, making a 30° angle with the horizontal. Find the magnitude of the torque applied to the crankshaft.

39. To turn a stubborn bolt, 80lb of force is applied to a 10in wrench. What is the maximum amount of torque that can be applied to the bolt?

40. To turn a stubborn bolt, 80lb of force is applied to a 10in wrench in a confined space, where the direction of applied force makes a 10° angle with the wrench. How much torque is subsequently applied to the wrench?

41. Show, using the definition of the Cross Product, that $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$; that is, that $\vec{u}$ is orthogonal to the cross product of $\vec{u}$ and $\vec{v}$.

42. Show, using the definition of the Cross Product, that $\vec{u} \times \vec{u} = \vec{0}$.

9.5 Lines

To find the equation of a line in the x-y plane, we need two pieces of information: a point and the slope. The slope conveys *direction* information. As vertical lines have an undefined slope, the following statement is more accurate:

> To define a line, one needs a point on the line and the direction of the line.

This holds true for lines in space.

Figure 9.47: Defining a line in space.

Let P be a point in space, let $\vec{p}$ be the vector with initial point at the origin and terminal point at P (i.e., $\vec{p}$ "points" to P), and let $\vec{d}$ be a vector. Consider the points on the line through P in the direction of $\vec{d}$.

Clearly one point on the line is P; we can say that the *vector* $\vec{p}$ lies at this point on the line. To find another point on the line, we can start at $\vec{p}$ and move in a direction parallel to $\vec{d}$. For instance, starting at $\vec{p}$ and traveling one length of $\vec{d}$ places one at another point on the line. Consider Figure 9.47 where certain points along the line are indicated.

The figure illustrates how every point on the line can be obtained by starting with $\vec{p}$ and moving a certain distance in the direction of $\vec{d}$. That is, we can define the line as a function of t:

$$\vec{\ell}(t) = \vec{p} + t\,\vec{d}. \tag{9.7}$$

In many ways, this is *not* a new concept. Compare Equation (9.7) to the familiar "$y = mx + b$" equation of a line:

Figure 9.46: Understanding the vector equation of a line.

The equations exhibit the same structure: they give a starting point, define a direction, and state how far in that direction to travel.

Equation (9.7) is an example of a **vector–valued function**; the input of the function is a real number and the output is a vector. We will cover vector–valued functions extensively in the next chapter.

Notes:

There are other ways to represent a line. Let $\vec{p} = \langle x_0, y_0, z_0 \rangle$ and let $\vec{d} = \langle a, b, c \rangle$. Then the equation of the line through $\vec{p}$ in the direction of $\vec{d}$ is:

$$\begin{aligned}\vec{\ell}(t) &= \vec{p} + t\vec{d} \\ &= \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle \\ &= \langle x_0 + at, y_0 + bt, z_0 + ct \rangle .\end{aligned}$$

The last line states that the x values of the line are given by $x = x_0 + at$, the y values are given by $y = y_0 + bt$, and the z values are given by $z = z_0 + ct$. These three equations, taken together, are the **parametric equations of the line** through $\vec{p}$ in the direction of $\vec{d}$.

Finally, each of the equations for x, y and z above contain the variable t. We can solve for t in each equation:

$$\begin{aligned} x = x_0 + at \quad &\Rightarrow \quad t = \frac{x - x_0}{a}, \\ y = y_0 + bt \quad &\Rightarrow \quad t = \frac{y - y_0}{b}, \\ z = z_0 + ct \quad &\Rightarrow \quad t = \frac{z - z_0}{c}, \end{aligned}$$

assuming $a, b, c \neq 0$. Since t is equal to each expression on the right, we can set these equal to each other, forming the **symmetric equations of the line** through $\vec{p}$ in the direction of $\vec{d}$:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Each representation has its own advantages, depending on the context. We summarize these three forms in the following definition, then give examples of their use.

Notes:

Definition 14 Equations of Lines in Space

Consider the line in space that passes through $\vec{p} = \langle x_0, y_0, z_0 \rangle$ in the direction of $\vec{d} = \langle a, b, c \rangle$.

1. The **vector equation** of the line is
$$\vec{\ell}(t) = \vec{p} + t\vec{d}.$$
2. The **parametric equations** of the line are
$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$
3. The **symmetric equations** of the line are
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Example 31 Finding the equation of a line

Give all three equations, as given in Definition 14, of the line through $P = (2, 3, 1)$ in the direction of $\vec{d} = \langle -1, 1, 2 \rangle$. Does the point $Q = (-1, 6, 6)$ lie on this line?

Figure 9.48: Graphing a line in Example 31.

Solution We identify the point $P = (2, 3, 1)$ with the vector $\vec{p} = \langle 2, 3, 1 \rangle$. Following the definition, we have

- the vector equation of the line is $\vec{\ell}(t) = \langle 2, 3, 1 \rangle + t\langle -1, 1, 2 \rangle$;
- the parametric equations of the line are
$$x = 2 - t, \quad y = 3 + t, \quad z = 1 + 2t; \text{ and}$$
- the symmetric equations of the line are
$$\frac{x - 2}{-1} = \frac{y - 3}{1} = \frac{z - 1}{2}.$$

The first two equations of the line are useful when a t value is given: one can immediately find the corresponding point on the line. These forms are good when calculating with a computer; most software programs easily handle equations in these formats. (For instance, to make Figure 9.48, a certain graphics program was given the input `(2-x,3+x,1+2*x)`. This particular program requires the variable always be "x" instead of "t").

Does the point $Q = (-1, 6, 6)$ lie on the line? The graph in Figure 9.48 makes it clear that it does not. We can answer this question without the graph

Notes:

using any of the three equation forms. Of the three, the symmetric equations are probably best suited for this task. Simply plug in the values of x, y and z and see if equality is maintained:

$$\frac{-1-2}{-1} \stackrel{?}{=} \frac{6-3}{1} \stackrel{?}{=} \frac{6-1}{2} \quad \Rightarrow \quad 3 = 3 \neq 2.5.$$

We see that Q does not lie on the line as it did not satisfy the symmetric equations.

Example 32 Finding the equation of a line through two points

Find the parametric equations of the line through the points $P = (2, -1, 2)$ and $Q = (1, 3, -1)$.

Solution Recall the statement made at the beginning of this section: to find the equation of a line, we need a point and a direction. We have *two* points; either one will suffice. The direction of the line can be found by the vector with initial point P and terminal point Q: $\overrightarrow{PQ} = \langle -1, 4, -3 \rangle$.

The parametric equations of the line ℓ through P in the direction of $\overrightarrow{PQ}$ are:

$$\ell: \quad x = 2 - t \quad y = -1 + 4t \quad z = 2 - 3t.$$

A graph of the points and line are given in Figure 9.49. Note how in the given parametrization of the line, $t = 0$ corresponds to the point P, and $t = 1$ corresponds to the point Q. This relates to the understanding of the vector equation of a line described in Figure 9.46. The parametric equations "start" at the point P, and t determines how far in the direction of $\overrightarrow{PQ}$ to travel. When $t = 0$, we travel 0 lengths of $\overrightarrow{PQ}$; when $t = 1$, we travel one length of $\overrightarrow{PQ}$, resulting in the point Q.

Figure 9.49: A graph of the line in Example 32.

Parallel, Intersecting and Skew Lines

In the plane, two *distinct* lines can either be parallel or they will intersect at exactly one point. In space, given equations of two lines, it can sometimes be difficult to tell whether the lines are distinct or not (i.e., the same line can be represented in different ways). Given lines $\vec{\ell}_1(t) = \vec{p}_1 + t\vec{d}_1$ and $\vec{\ell}_2(t) = \vec{p}_2 + t\vec{d}_2$, we have four possibilities: $\vec{\ell}_1$ and $\vec{\ell}_2$ are

the same line	they share all points;
intersecting lines	share only 1 point;
parallel lines	$\vec{d}_1 \parallel \vec{d}_2$, no points in common; or
skew lines	$\vec{d}_1 \nparallel \vec{d}_2$, no points in common.

Notes:

The next two examples investigate these possibilities.

Example 33 Comparing lines

Consider lines ℓ_1 and ℓ_2, given in parametric equation form:

$$\ell_1: \begin{array}{rcl} x &=& 1+3t \\ y &=& 2-t \\ z &=& t \end{array} \qquad \ell_2: \begin{array}{rcl} x &=& -2+4s \\ y &=& 3+s \\ z &=& 5+2s. \end{array}$$

Determine whether ℓ_1 and ℓ_2 are the same line, intersect, are parallel, or skew.

Solution We start by looking at the directions of each line. Line ℓ_1 has the direction given by $\vec{d}_1 = \langle 3, -1, 1\rangle$ and line ℓ_2 has the direction given by $\vec{d}_2 = \langle 4, 1, 2\rangle$. It should be clear that $\vec{d}_1$ and $\vec{d}_2$ are not parallel, hence ℓ_1 and ℓ_2 are not the same line, nor are they parallel. Figure 9.50 verifies this fact (where the points and directions indicated by the equations of each line are identified).

We next check to see if they intersect (if they do not, they are skew lines). To find if they intersect, we look for t and s values such that the respective x, y and z values are the same. That is, we want s and t such that:

$$\begin{array}{rcl} 1+3t &=& -2+4s \\ 2-t &=& 3+s \\ t &=& 5+2s. \end{array}$$

This is a relatively simple system of linear equations. Since the last equation is already solved for t, substitute that value of t into the equation above it:

$$2-(5+2s) = 3+s \quad \Rightarrow \quad s=-2,\ t=1.$$

A key to remember is that we have *three* equations; we need to check if $s = -2,\ t = 1$ satisfies the first equation as well:

$$1+3(1) \neq -2+4(-2).$$

It does not. Therefore, we conclude that the lines ℓ_1 and ℓ_2 are skew.

Figure 9.50: Sketching the lines from Example 33.

Example 34 Comparing lines

Consider lines ℓ_1 and ℓ_2, given in parametric equation form:

$$\ell_1: \begin{array}{rcl} x &=& -0.7+1.6t \\ y &=& 4.2+2.72t \\ z &=& 2.3-3.36t \end{array} \qquad \ell_2: \begin{array}{rcl} x &=& 2.8-2.9s \\ y &=& 10.15-4.93s \\ z &=& -5.05+6.09s. \end{array}$$

Determine whether ℓ_1 and ℓ_2 are the same line, intersect, are parallel, or skew.

Notes:

Solution It is obviously very difficult to simply look at these equations and discern anything. This is done intentionally. In the "real world," most equations that are used do not have nice, integer coefficients. Rather, there are lots of digits after the decimal and the equations can look "messy."

We again start by deciding whether or not each line has the same direction. The direction of ℓ_1 is given by $\vec{d}_1 = \langle 1.6, 2.72, -3.36\rangle$ and the direction of ℓ_2 is given by $\vec{d}_2 = \langle -2.9, -4.93, 6.09\rangle$. When it is not clear through observation whether two vectors are parallel or not, the standard way of determining this is by comparing their respective unit vectors. Using a calculator, we find:

$$\vec{u}_1 = \frac{\vec{d}_1}{||\,\vec{d}_1\,||} = \langle 0.3471, 0.5901, -0.7289\rangle$$
$$\vec{u}_2 = \frac{\vec{d}_2}{||\,\vec{d}_2\,||} = \langle -0.3471, -0.5901, 0.7289\rangle\,.$$

The two vectors seem to be parallel (at least, their components are equal to 4 decimal places). In most situations, it would suffice to conclude that the lines are at least parallel, if not the same. One way to be sure is to rewrite $\vec{d}_1$ and $\vec{d}_2$ in terms of fractions, not decimals. We have

$$\vec{d}_1 = \left\langle \frac{16}{10}, \frac{272}{100}, -\frac{336}{100}\right\rangle \qquad \vec{d}_2 = \left\langle -\frac{29}{10}, -\frac{493}{100}, \frac{609}{100}\right\rangle.$$

One can then find the magnitudes of each vector in terms of fractions, then compute the unit vectors likewise. After a lot of manual arithmetic (or after briefly using a computer algebra system), one finds that

$$\vec{u}_1 = \left\langle \sqrt{\frac{10}{83}}, \frac{17}{\sqrt{830}}, -\frac{21}{\sqrt{830}}\right\rangle \qquad \vec{u}_2 = \left\langle -\sqrt{\frac{10}{83}}, -\frac{17}{\sqrt{830}}, \frac{21}{\sqrt{830}}\right\rangle.$$

We can now say without equivocation that these lines are parallel.

Are they the same line? The parametric equations for a line describe one point that lies on the line, so we know that the point $P_1 = (-0.7, 4.2, 2.3)$ lies on ℓ_1. To determine if this point also lies on ℓ_2, plug in the x, y and z values of P_1 into the symmetric equations for ℓ_2:

$$\frac{(-0.7) - 2.8}{-2.9} \stackrel{?}{=} \frac{(4.2) - 10.15}{-4.93} \stackrel{?}{=} \frac{(2.3) - (-5.05)}{6.09} \quad \Rightarrow \quad 1.2069 = 1.2069 = 1.2069.$$

The point P_1 lies on both lines, so we conclude they are the same line, just parametrized differently. Figure 9.51 graphs this line along with the points and vectors described by the parametric equations. Note how $\vec{d}_1$ and $\vec{d}_2$ are parallel, though point in opposite directions (as indicated by their unit vectors above).

Figure 9.51: Graphing the lines in Example 34.

Notes:

Distances

Figure 9.52: Establishing the distance from a point to a line.

Given a point Q and a line $\vec{\ell}(t) = \vec{p} + t\vec{d}$ in space, it is often useful to know the distance from the point to the line. (Here we use the standard definition of "distance," i.e., the length of the shortest line segment from the point to the line.) Identifying $\vec{p}$ with the point P, Figure 9.52 will help establish a general method of computing this distance h.

From trigonometry, we know $h = \|\overrightarrow{PQ}\| \sin\theta$. We have a similar identity involving the cross product: $\|\overrightarrow{PQ} \times \vec{d}\| = \|\overrightarrow{PQ}\|\,\|\vec{d}\| \sin\theta$. Divide both sides of this latter equation by $\|\vec{d}\|$ to obtain h:

$$h = \frac{\|\overrightarrow{PQ} \times \vec{d}\|}{\|\vec{d}\|}. \tag{9.8}$$

Figure 9.53: Establishing the distance between lines.

It is also useful to determine the distance between lines, which we define as the length of the shortest line segment that connects the two lines (an argument from geometry shows that this line segments is perpendicular to both lines). Let lines $\vec{\ell}_1(t) = \vec{p}_1 + t\vec{d}_1$ and $\vec{\ell}_2(t) = \vec{p}_2 + t\vec{d}_2$ be given, as shown in Figure 9.53. To find the direction orthogonal to both $\vec{d}_1$ and $\vec{d}_2$, we take the cross product: $\vec{c} = \vec{d}_1 \times \vec{d}_2$. The magnitude of the orthogonal projection of $\overrightarrow{P_1P_2}$ onto $\vec{c}$ is the distance h we seek:

$$\begin{aligned} h &= \left\| \operatorname{proj}_{\vec{c}} \overrightarrow{P_1P_2} \right\| \\ &= \left\| \frac{\overrightarrow{P_1P_2} \cdot \vec{c}}{\vec{c} \cdot \vec{c}} \vec{c} \right\| \\ &= \frac{|\overrightarrow{P_1P_2} \cdot \vec{c}|}{\|\vec{c}\|^2} \|\vec{c}\| \\ &= \frac{|\overrightarrow{P_1P_2} \cdot \vec{c}|}{\|\vec{c}\|}. \end{aligned}$$

A problem in the Exercise section is to show that this distance is 0 when the lines intersect. Note the use of the Triple Scalar Product: $\overrightarrow{P_1P_2} \cdot c = \overrightarrow{P_1P_2} \cdot (\vec{d}_1 \times \vec{d}_2)$.

The following Key Idea restates these two distance formulas.

Notes:

Key Idea 5 Distances to Lines

1. Let P be a point on a line ℓ that is parallel to $\vec{d}$. The distance h from a point Q to the line ℓ is:

$$h = \frac{||\,\overrightarrow{PQ} \times \vec{d}\,||}{||\,\vec{d}\,||}.$$

2. Let P_1 be a point on line ℓ_1 that is parallel to $\vec{d}_1$, and let P_2 be a point on line ℓ_2 parallel to $\vec{d}_2$, and let $\vec{c} = \vec{d}_1 \times \vec{d}_2$, where lines ℓ_1 and ℓ_2 are not parallel. The distance h between the two lines is:

$$h = \frac{|\overrightarrow{P_1P_2} \cdot \vec{c}|}{||\,\vec{c}\,||}.$$

Example 35 Finding the distance from a point to a line

Find the distance from the point $Q = (1, 1, 3)$ to the line $\vec{\ell}(t) = \langle 1, -1, 1\rangle + t\,\langle 2, 3, 1\rangle$.

Solution The equation of the line gives us the point $P = (1, -1, 1)$ that lies on the line, hence $\overrightarrow{PQ} = \langle 0, 2, 2\rangle$. The equation also gives $\vec{d} = \langle 2, 3, 1\rangle$. Following Key Idea 5, we have the distance as

$$\begin{aligned} h &= \frac{||\,\overrightarrow{PQ} \times \vec{d}\,||}{||\,\vec{d}\,||} \\ &= \frac{||\ \langle -4, 4, -4\rangle\ ||}{\sqrt{14}} \\ &= \frac{4\sqrt{3}}{\sqrt{14}} \approx 1.852. \end{aligned}$$

The point Q is approximately 1.852 units from the line $\vec{\ell}(t)$.

Example 36 Finding the distance between lines

Find the distance between the lines

$$\ell_1: \begin{array}{rcl} x &=& 1 + 3t \\ y &=& 2 - t \\ z &=& t \end{array} \qquad \ell_2: \begin{array}{rcl} x &=& -2 + 4s \\ y &=& 3 + s \\ z &=& 5 + 2s. \end{array}$$

Solution These are the sames lines as given in Example 33, where we

Notes:

showed them to be skew. The equations allow us to identify the following points and vectors:

$$P_1 = (1, 2, 0) \quad P_2 = (-2, 3, 5) \quad \Rightarrow \quad \overrightarrow{P_1P_2} = \langle -3, 1, 5 \rangle .$$

$$\vec{d}_1 = \langle 3, -1, 1 \rangle \quad \vec{d}_2 = \langle 4, 1, 2 \rangle \quad \Rightarrow \quad \vec{c} = \vec{d}_1 \times \vec{d}_2 = \langle -3, -2, 7 \rangle .$$

From Key Idea 5 we have the distance h between the two lines is

$$\begin{aligned} h &= \frac{|\overrightarrow{P_1P_2} \cdot \vec{c}|}{||\,\vec{c}\,||} \\ &= \frac{42}{\sqrt{62}} \approx 5.334. \end{aligned}$$

The lines are approximately 5.334 units apart.

One of the key points to understand from this section is this: to describe a line, we need a point and a direction. Whenever a problem is posed concerning a line, one needs to take whatever information is offered and glean point and direction information. Many questions can be asked (and *are* asked in the Exercise section) whose answer immediately follows from this understanding.

Lines are one of two fundamental objects of study in space. The other fundamental object is the *plane*, which we study in detail in the next section. Many complex three dimensional objects are studied by approximating their surfaces with lines and planes.

Notes:

Exercises 9.5

Terms and Concepts

1. To find an equation of a line, what two pieces of information are needed?

2. Two distinct lines in the plane can intersect or be _____.

3. Two distinct lines in space can intersect, be _____ or be _____.

4. Use your own words to describe what it means for two lines in space to be skew.

Problems

In Exercises 5 – 14, write the vector, parametric and symmetric equations of the lines described.

5. Passes through $P = (2, -4, 1)$, parallel to $\vec{d} = \langle 9, 2, 5\rangle$.

6. Passes through $P = (6, 1, 7)$, parallel to $\vec{d} = \langle -3, 2, 5\rangle$.

7. Passes through $P = (2, 1, 5)$ and $Q = (7, -2, 4)$.

8. Passes through $P = (1, -2, 3)$ and $Q = (5, 5, 5)$.

9. Passes through $P = (0, 1, 2)$ and orthogonal to both $\vec{d}_1 = \langle 2, -1, 7\rangle$ and $\vec{d}_2 = \langle 7, 1, 3\rangle$.

10. Passes through $P = (5, 1, 9)$ and orthogonal to both $\vec{d}_1 = \langle 1, 0, 1\rangle$ and $\vec{d}_2 = \langle 2, 0, 3\rangle$.

11. Passes through the point of intersection of $\vec{\ell}_1(t)$ and $\vec{\ell}_2(t)$ and orthogonal to both lines, where $\vec{\ell}_1(t) = \langle 2, 1, 1\rangle + t\langle 5, 1, -2\rangle$ and $\vec{\ell}_2(t) = \langle -2, -1, 2\rangle + t\langle 3, 1, -1\rangle$.

12. Passes through the point of intersection of $\ell_1(t)$ and $\ell_2(t)$ and orthogonal to both lines, where
$$\ell_1 = \begin{cases} x = t \\ y = -2 + 2t \\ z = 1 + t \end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} x = 2 + t \\ y = 2 - t \\ z = 3 + 2t \end{cases}.$$

13. Passes through $P = (1, 1)$, parallel to $\vec{d} = \langle 2, 3\rangle$.

14. Passes through $P = (-2, 5)$, parallel to $\vec{d} = \langle 0, 1\rangle$.

In Exercises 15 – 22, determine if the described lines are the same line, parallel lines, intersecting or skew lines. If intersecting, give the point of intersection.

15. $\vec{\ell}_1(t) = \langle 1, 2, 1\rangle + t\langle 2, -1, 1\rangle$,
$\vec{\ell}_2(t) = \langle 3, 3, 3\rangle + t\langle -4, 2, -2\rangle$.

16. $\vec{\ell}_1(t) = \langle 2, 1, 1\rangle + t\langle 5, 1, 3\rangle$,
$\vec{\ell}_2(t) = \langle 14, 5, 9\rangle + t\langle 1, 1, 1\rangle$.

17. $\vec{\ell}_1(t) = \langle 3, 4, 1\rangle + t\langle 2, -3, 4\rangle$,
$\vec{\ell}_2(t) = \langle -3, 3, -3\rangle + t\langle 3, -2, 4\rangle$.

18. $\vec{\ell}_1(t) = \langle 1, 1, 1\rangle + t\langle 3, 1, 3\rangle$,
$\vec{\ell}_2(t) = \langle 7, 3, 7\rangle + t\langle 6, 2, 6\rangle$.

19. $\ell_1 = \begin{cases} x = 1 + 2t \\ y = 3 - 2t \\ z = t \end{cases}$ and $\ell_2 = \begin{cases} x = 3 - t \\ y = 3 + 5t \\ z = 2 + 7t \end{cases}$

20. $\ell_1 = \begin{cases} x = 1.1 + 0.6t \\ y = 3.77 + 0.9t \\ z = -2.3 + 1.5t \end{cases}$ and $\ell_2 = \begin{cases} x = 3.11 + 3.4t \\ y = 2 + 5.1t \\ z = 2.5 + 8.5t \end{cases}$

21. $\ell_1 = \begin{cases} x = 0.2 + 0.6t \\ y = 1.33 - 0.45t \\ z = -4.2 + 1.05t \end{cases}$ and $\ell_2 = \begin{cases} x = 0.86 + 9.2t \\ y = 0.835 - 6.9t \\ z = -3.045 + 16.1t \end{cases}$

22. $\ell_1 = \begin{cases} x = 0.1 + 1.1t \\ y = 2.9 - 1.5t \\ z = 3.2 + 1.6t \end{cases}$ and $\ell_2 = \begin{cases} x = 4 - 2.1t \\ y = 1.8 + 7.2t \\ z = 3.1 + 1.1t \end{cases}$

In Exercises 23 – 26, find the distance from the point to the line.

23. $P = (1, 1, 1)$, $\vec{\ell}(t) = \langle 2, 1, 3\rangle + t\langle 2, 1, -2\rangle$

24. $P = (2, 5, 6)$, $\vec{\ell}(t) = \langle -1, 1, 1\rangle + t\langle 1, 0, 1\rangle$

25. $P = (0, 3)$, $\vec{\ell}(t) = \langle 2, 0\rangle + t\langle 1, 1\rangle$

26. $P = (1, 1)$, $\vec{\ell}(t) = \langle 4, 5\rangle + t\langle -4, 3\rangle$

In Exercises 27 – 28, find the distance between the two lines.

27. $\vec{\ell}_1(t) = \langle 1, 2, 1\rangle + t\langle 2, -1, 1\rangle$,
$\vec{\ell}_2(t) = \langle 3, 3, 3\rangle + t\langle 4, 2, -2\rangle$.

28. $\vec{\ell}_1(t) = \langle 0, 0, 1\rangle + t\langle 1, 0, 0\rangle$,
$\vec{\ell}_2(t) = \langle 0, 0, 3\rangle + t\langle 0, 1, 0\rangle$.

Exercises 29 – 31 explore special cases of the distance formulas found in Key Idea 5.

29. Let Q be a point on the line $\ell(t)$. Show why the distance formula correctly gives the distance from the point to the line as 0.

30. Let lines $\ell_1(t)$ and $\ell_2(t)$ be intersecting lines. Show why the distance formula correctly gives the distance between these lines as 0.

31. Let lines $\ell_1(t)$ and $\ell_2(t)$ be parallel.

(a) Show why the distance formula for distance between lines cannot be used as stated to find the distance between the lines.

(b) Show why letting $c = (\overrightarrow{P_1P_2} \times \vec{d}_2) \times \vec{d}_2$ allows one to the use the formula.

(c) Show how one can use the formula for the distance between a point and a line to find the distance between parallel lines.

9.6 Planes

Any flat surface, such as a wall, table top or stiff piece of cardboard can be thought of as representing part of a plane. Consider a piece of cardboard with a point P marked on it. One can take a nail and stick it into the cardboard at P such that the nail is perpendicular to the cardboard; see Figure 9.54

Figure 9.54: Illustrating defining a plane with a sheet of cardboard and a nail.

This nail provides a "handle" for the cardboard. Moving the cardboard around moves P to different locations in space. Tilting the nail (but keeping P fixed) tilts the cardboard. Both moving and tilting the cardboard defines a different plane in space. In fact, we can define a plane by: 1) the location of P in space, and 2) the direction of the nail.

The previous section showed that one can define a line given a point on the line and the direction of the line (usually given by a vector). One can make a similar statement about planes: we can define a plane in space given a point on the plane and the direction the plane "faces" (using the description above, the direction of the nail). Once again, the direction information will be supplied by a vector, called a **normal vector**, that is orthogonal to the plane.

What exactly does "orthogonal to the plane" mean? Choose any two points P and Q in the plane, and consider the vector $\overrightarrow{PQ}$. We say a vector $\vec{n}$ is orthogonal to the plane if $\vec{n}$ is perpendicular to $\overrightarrow{PQ}$ for all choices of P and Q; that is, if $\vec{n} \cdot \overrightarrow{PQ} = 0$ for all P and Q.

This gives us way of writing an equation describing the plane. Let $P = (x_0, y_0, z_0)$ be a point in the plane and let $\vec{n} = \langle a, b, c\rangle$ be a normal vector to the plane. A point $Q = (x, y, z)$ lies in the plane defined by P and $\vec{n}$ if, and only if, $\overrightarrow{PQ}$ is orthogonal to $\vec{n}$. Knowing $\overrightarrow{PQ} = \langle x - x_0, y - y_0, z - z_0\rangle$, consider:

$$\begin{aligned} \overrightarrow{PQ} \cdot \vec{n} &= 0 \\ \langle x - x_0, y - y_0, z - z_0\rangle \cdot \langle a, b, c\rangle &= 0 \\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 \end{aligned} \tag{9.9}$$

Equation (9.9) defines an *implicit* function describing the plane. More algebra produces:

$$ax + by + cz = ax_0 + by_0 + cz_0.$$

The right hand side is just a number, so we replace it with d:

$$ax + by + cz = d. \tag{9.10}$$

As long as $c \neq 0$, we can solve for z:

$$z = \frac{1}{c}(d - ax - by). \tag{9.11}$$

Notes:

Equation (9.11) is especially useful as many computer programs can graph functions in this form. Equations (9.9) and (9.10) have specific names, given next.

Definition 15 Equations of a Plane in Standard and General Forms

The plane passing through the point $P = (x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c\rangle$ can be described by an equation with **standard form**

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0;$$

the equation's **general form** is

$$ax + by + cz = d.$$

A key to remember throughout this section is this: to find the equation of a plane, we need a point and a normal vector. We will give several examples of finding the equation of a plane, and in each one different types of information are given. In each case, we need to use the given information to find a point on the plane and a normal vector.

Figure 9.55: Sketching the plane in Example 37.

Example 37 Finding the equation of a plane.
Write the equation of the plane that passes through the points $P = (1, 1, 0)$, $Q = (1, 2, -1)$ and $R = (0, 1, 2)$ in standard form.

Solution We need a vector $\vec{n}$ that is orthogonal to the plane. Since P, Q and R are in the plane, so are the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$; $\overrightarrow{PQ} \times \overrightarrow{PR}$ is orthogonal to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and hence the plane itself.

It is straightforward to compute $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \langle 2, 1, 1\rangle$. We can use any point we wish in the plane (any of P, Q or R will do) and we arbitrarily choose P. Following Definition 15, the equation of the plane in standard form is

$$2(x - 1) + (y - 1) + z = 0.$$

The plane is sketched in Figure 9.55.

We have just demonstrated the fact that any three non-collinear points define a plane. (This is why a three-legged stool does not "rock;" it's three feet always lie in a plane. A four-legged stool will rock unless all four feet lie in the same plane.)

Example 38 Finding the equation of a plane.
Verify that lines ℓ_1 and ℓ_2, whose parametric equations are given below, inter-

Notes:

sect, then give the equation of the plane that contains these two lines in general form.

$$\ell_1: \begin{array}{rcl} x &=& -5+2s \\ y &=& 1+s \\ z &=& -4+2s \end{array} \qquad \ell_2: \begin{array}{rcl} x &=& 2+3t \\ y &=& 1-2t \\ z &=& 1+t \end{array}$$

Solution The lines clearly are not parallel. If they do not intersect, they are skew, meaning there is not a plane that contains them both. If they do intersect, there is such a plane.

To find their point of intersection, we set the x, y and z equations equal to each other and solve for s and t:

$$\begin{array}{rcl} -5+2s &=& 2+3t \\ 1+s &=& 1-2t \\ -4+2s &=& 1+t \end{array} \quad\Rightarrow\quad s=2, \quad t=-1.$$

When $s=2$ and $t=-1$, the lines intersect at the point $P=(-1,3,0)$.

Let $\vec{d}_1=\langle 2,1,2\rangle$ and $\vec{d}_2=\langle 3,-2,1\rangle$ be the directions of lines ℓ_1 and ℓ_2, respectively. A normal vector to the plane containing these the two lines will also be orthogonal to $\vec{d}_1$ and $\vec{d}_2$. Thus we find a normal vector $\vec{n}$ by computing $\vec{n}=\vec{d}_1\times\vec{d}_2=\langle 5,4-7\rangle$.

We can pick any point in the plane with which to write our equation; each line gives us infinite choices of points. We choose P, the point of intersection. We follow Definition 15 to write the plane's equation in general form:

$$\begin{aligned} 5(x+1)+4(y-3)-7z &= 0 \\ 5x+5+4y-12-7z &= 0 \\ 5x+4y-7z &= 7. \end{aligned}$$

The plane's equation in general form is $5x+4y-7z=7$; it is sketched in Figure 9.56.

Figure 9.56: Sketching the plane in Example 38.

Example 39 **Finding the equation of a plane**

Give the equation, in standard form, of the plane that passes through the point $P=(-1,0,1)$ and is orthogonal to the line with vector equation $\vec{\ell}(t)=\langle -1,0,1\rangle + t\langle 1,2,2\rangle$.

Solution As the plane is to be orthogonal to the line, the plane must be orthogonal to the direction of the line given by $\vec{d}=\langle 1,2,2\rangle$. We use this as our normal vector. Thus the plane's equation, in standard form, is

$$(x+1)+2y+2(z-1)=0.$$

The line and plane are sketched in Figure 9.57.

Figure 9.57: The line and plane in Example 39.

Notes:

Example 40 Finding the intersection of two planes

Give the parametric equations of the line that is the intersection of the planes p_1 and p_2, where:

$$p_1 : x - (y-2) + (z-1) = 0$$
$$p_2 : -2(x-2) + (y+1) + (z-3) = 0$$

Solution To find an equation of a line, we need a point on the line and the direction of the line.

We can find a point on the line by solving each equation of the planes for z:

$$p_1 : z = -x + y - 1$$
$$p_2 : z = 2x - y - 2$$

We can now set these two equations equal to each other (i.e., we are finding values of x and y where the planes have the same z value):

$$\begin{aligned} -x + y - 1 &= 2x - y - 2 \\ 2y &= 3x - 1 \\ y &= \frac{1}{2}(3x-1) \end{aligned}$$

We can choose any value for x; we choose $x = 1$. This determines that $y = 1$. We can now use the equations of either plane to find z: when $x = 1$ and $y = 1$, $z = -1$ on both planes. We have found a point P on the line: $P = (1, 1, -1)$.

We now need the direction of the line. Since the line lies in each plane, its direction is orthogonal to a normal vector for each plane. Considering the equations for p_1 and p_2, we can quickly determine their normal vectors. For p_1, $\vec{n}_1 = \langle 1, -1, 1\rangle$ and for p_2, $\vec{n}_2 = \langle -2, 1, 1\rangle$. A direction orthogonal to both of these directions is their cross product: $\vec{d} = \vec{n}_1 \times \vec{n}_2 = \langle -2, -3, -1\rangle$.

The parametric equations of the line through $P = (1, 1, -1)$ in the direction of $d = \langle -2, -3, -1\rangle$ is:

$$\ell : \quad x = -2t + 1 \quad y = -3t + 1 \quad z = -t - 1.$$

The planes and line are graphed in Figure 9.58.

Figure 9.58: Graphing the planes and their line of intersection in Example 40.

Example 41 Finding the intersection of a plane and a line

Find the point of intersection, if any, of the line $\ell(t) = \langle 3, -3, -1\rangle + t\langle -1, 2, 1\rangle$ and the plane with equation in general form $2x + y + z = 4$.

Solution The equation of the plane shows that the vector $\vec{n} = \langle 2, 1, 1\rangle$ is a normal vector to the plane, and the equation of the line shows that the line

Notes:

moves parallel to $\vec{d} = \langle -1, 2, 1 \rangle$. Since these are not orthogonal, we know there is a point of intersection. (If there were orthogonal, it would mean that the plane and line were parallel to each other, either never intersecting or the line was in the plane itself.)

To find the point of intersection, we need to find a t value such that $\ell(t)$ satisfies the equation of the plane. Rewriting the equation of the line with parametric equations will help:

$$\ell(t) = \begin{cases} x = 3 - t \\ y = -3 + 2t \\ z = -1 + t \end{cases}.$$

Replacing x, y and z in the equation of the plane with the expressions containing t found in the equation of the line allows us to determine a t value that indicates the point of intersection:

$$\begin{aligned} 2x + y + z &= 4 \\ 2(3 - t) + (-3 + 2t) + (-1 + t) &= 4 \\ t &= 2. \end{aligned}$$

When $t = 2$, the point on the line satisfies the equation of the plane; that point is $\ell(2) = \langle 1, 1, 1 \rangle$. Thus the point $(1, 1, 1)$ is the point of intersection between the plane and the line, illustrated in Figure 9.59.

Figure 9.59: Illustrating the intersection of a line and a plane in Example 41.

Distances

Just as it was useful to find distances between points and lines in the previous section, it is also often necessary to find the distance from a point to a plane.

Consider Figure 9.60, where a plane with normal vector $\vec{n}$ is sketched containing a point P and a point Q, not on the plane, is given. We measure the distance from Q to the plane by measuring the length of the projection of $\overrightarrow{PQ}$ onto $\vec{n}$. That is, we want:

$$\left|\left| \text{proj}_{\vec{n}} \overrightarrow{PQ} \right|\right| = \left|\left| \frac{\vec{n} \cdot \overrightarrow{PQ}}{||\vec{n}||^2} \vec{n} \right|\right| = \frac{|\vec{n} \cdot \overrightarrow{PQ}|}{||\vec{n}||} \tag{9.12}$$

Figure 9.60: Illustrating finding the distance from a point to a plane.

Equation (9.12) is important as it does more than just give the distance between a point and a plane. We will see how it allows us to find several other distances as well: the distance between parallel planes and the distance from a line and a plane. Because Equation (9.12) is important, we restate it as a Key Idea.

Notes:

Key Idea 6 Distance from a Point to a Plane

Let a plane with normal vector $\vec{n}$ be given, and let Q be a point. The distance h from Q to the plane is

$$h = \frac{|\vec{n} \cdot \overrightarrow{PQ}|}{||\,\vec{n}\,||},$$

where P is any point in the plane.

Example 42 Distance between a point and a plane

Find the distance bewteen the point $Q = (2, 1, 4)$ and the plane with equation $2x - 5y + 6z = 9$.

Solution Using the equation of the plane, we find the normal vector $\vec{n} = \langle 2, -5, 6\rangle$. To find a point on the plane, we can let x and y be anything we choose, then let z be whatever satisfies the equation. Letting x and y be 0 seems simple; this makes $z = 1.5$. Thus we let $P = \langle 0, 0, 1.5\rangle$, and $\overrightarrow{PQ} = \langle 2, 1, 2.5\rangle$.

The distance h from Q to the plane is given by Key Idea 6:

$$\begin{aligned} h &= \frac{|\vec{n} \cdot \overrightarrow{PQ}|}{||\,\vec{n}\,||} \\ &= \frac{|\,\langle 2, -5, 6\rangle \cdot \langle 2, 1, 2.5\rangle\,|}{||\,\langle 2, -5, 6\rangle\,||} \\ &= \frac{|14|}{\sqrt{65}} \\ &\approx 1.74. \end{aligned}$$

We can use Key Idea 6 to find other distances. Given two parallel planes, we can find the distance between these planes by letting P be a point on one plane and Q a point on the other. If ℓ is a line parallel to a plane, we can use the Key Idea to find the distance between them as well: again, let P be a point in the plane and let Q be any point on the line. (One can also use Key Idea 5.) The Exercise section contains problems of these types.

These past two sections have not explored lines and planes in space as an exercise of mathematical curiosity. However, there are many, many applications of these fundamental concepts. Complex shapes can be modeled (or, *approximated*) using planes. For instance, part of the exterior of an aircraft may have a complex, yet smooth, shape, and engineers will want to know how air flows across this piece as well as how heat might build up due to air friction. Many equations that help determine air flow and heat dissipation are difficult to apply to arbitrary surfaces, but simple to apply to planes. By approximating

Notes:

a surface with millions of small planes one can more readily model the needed behavior.

Notes:

Exercises 9.6

Terms and Concepts

1. In order to find the equation of a plane, what two pieces of information must one have?

2. What is the relationship between a plane and one of its normal vectors?

Problems

In Exercises 3 – 6, give any two points in the given plane.

3. $2x - 4y + 7z = 2$

4. $3(x+2) + 5(y-9) - 4z = 0$

5. $x = 2$

6. $4(y+2) - (z-6) = 0$

In Exercises 7 – 20, give the equation of the described plane in standard and general forms.

7. Passes through $(2,3,4)$ and has normal vector $\vec{n} = \langle 3, -1, 7\rangle$.

8. Passes through $(1,3,5)$ and has normal vector $\vec{n} = \langle 0, 2, 4\rangle$.

9. Passes through the points $(1,2,3)$, $(3,-1,4)$ and $(1,0,1)$.

10. Passes through the points $(5,3,8)$, $(6,4,9)$ and $(3,3,3)$.

11. Contains the intersecting lines $\ell_1(t) = \langle 2,1,2\rangle + t\langle 1,2,3\rangle$ and $\ell_2(t) = \langle 2,1,2\rangle + t\langle 2,5,4\rangle$.

12. Contains the intersecting lines $\ell_1(t) = \langle 5,0,3\rangle + t\langle -1,1,1\rangle$ and $\ell_2(t) = \langle 1,4,7\rangle + t\langle 3,0,-3\rangle$.

13. Contains the parallel lines $\ell_1(t) = \langle 1,1,1\rangle + t\langle 1,2,3\rangle$ and $\ell_2(t) = \langle 1,1,2\rangle + t\langle 1,2,3\rangle$.

14. Contains the parallel lines $\ell_1(t) = \langle 1,1,1\rangle + t\langle 4,1,3\rangle$ and $\ell_2(t) = \langle 2,2,2\rangle + t\langle 4,1,3\rangle$.

15. Contains the point $(2,-6,1)$ and the line
$$\ell(t) = \begin{cases} x = 2 + 5t \\ y = 2 + 2t \\ z = -1 + 2t \end{cases}$$

16. Contains the point $(5,7,3)$ and the line
$$\ell(t) = \begin{cases} x = t \\ y = t \\ z = t \end{cases}$$

17. Contains the point $(5,7,3)$ and is orthogonal to the line $\ell(t) = \langle 4,5,6\rangle + t\langle 1,1,1\rangle$.

18. Contains the point $(4,1,1)$ and is orthogonal to the line
$$\ell(t) = \begin{cases} x = 4 + 4t \\ y = 1 + 1t \\ z = 1 + 1t \end{cases}$$

19. Contains the point $(-4,7,2)$ and is parallel to the plane $3(x-2) + 8(y+1) - 10z = 0$.

20. Contains the point $(1,2,3)$ and is parallel to the plane $x = 5$.

In Exercises 21 – 22, give the equation of the line that is the intersection of the given planes.

21. $p1:\ 3(x-2) + (y-1) + 4z = 0$, and
$p2:\ 2(x-1) - 2(y+3) + 6(z-1) = 0$.

22. $p1:\ 5(x-5) + 2(y+2) + 4(z-1) = 0$, and
$p2:\ 3x - 4(y-1) + 2(z-1) = 0$.

In Exercises 23 – 26, find the point of intersection between the line and the plane.

23. line: $\langle 5,1,-1\rangle + t\langle 2,2,1\rangle$,
plane: $5x - y - z = -3$

24. line: $\langle 4,1,0\rangle + t\langle 1,0,-1\rangle$,
plane: $3x + y - 2z = 8$

25. line: $\langle 1,2,3\rangle + t\langle 3,5,-1\rangle$,
plane: $3x - 2y - z = 4$

26. line: $\langle 1,2,3\rangle + t\langle 3,5,-1\rangle$,
plane: $3x - 2y - z = -4$

In Exercises 27 – 30, find the given distances.

27. The distance from the point $(1,2,3)$ to the plane $3(x-1) + (y-2) + 5(z-2) = 0$.

28. The distance from the point $(2,6,2)$ to the plane $2(x-1) - y + 4(z+1) = 0$.

29. The distance between the parallel planes $x + y + z = 0$ and $(x-2) + (y-3) + (z+4) = 0$

30. The distance between the parallel planes
$2(x-1)+2(y+1)+(z-2)=0$ and
$2(x-3)+2(y-1)+(z-3)=0$

31. Show why if the point Q lies in a plane, then the distance formula correctly gives the distance from the point to the plane as 0.

32. How is Exercise 30 in Section 9.5 easier to answer once we have an understanding of planes?

10: Vector Valued Functions

In the previous chapter, we learned about vectors and were introduced to the power of vectors within mathematics. In this chapter, we'll build on this foundation to define functions whose input is a real number and whose output is a vector. We'll see how to graph these functions and apply calculus techniques to analyze their behavior. Most importantly, we'll see *why* we are interested in doing this: we'll see beautiful applications to the study of moving objects.

10.1 Vector–Valued Functions

We are very familiar with **real valued functions**, that is, functions whose output is a real number. This section introduces **vector–valued functions** – functions whose output is a vector.

Definition 16 Vector–Valued Functions

A **vector–valued function** is a function of the form

$$\vec{r}(t) = \langle f(t), g(t) \rangle \quad \text{or} \quad \vec{r}(t) = \langle f(t), g(t), h(t) \rangle,$$

where f, g and h are real valued functions.

The **domain** of $\vec{r}$ is the set of all values of t for which $\vec{r}(t)$ is defined. The **range** of $\vec{r}$ is the set of all possible output vectors $\vec{r}(t)$.

Figure 10.1: Sketching the graph of a vector–valued function.

Evaluating and Graphing Vector–Valued Functions

Evaluating a vector–valued function at a specific value of t is straightforward; simply evaluate each component function at that value of t. For instance, if $\vec{r}(t) = \langle t^2, t^2 + t - 1 \rangle$, then $\vec{r}(-2) = \langle 4, 1 \rangle$. We can sketch this vector, as is done in Figure 10.1(a). Plotting lots of vectors is cumbersome, though, so generally we do not sketch the whole vector but just the terminal point. The **graph** of a vector–valued function is the set of all terminal points of $\vec{r}(t)$, where the initial point of each vector is always the origin. In Figure 10.1(b) we sketch the graph of $\vec{r}$; we can indicate individual points on the graph with their respective vector, as shown.

Vector–valued functions are closely related to parametric equations of graphs. While in both methods we plot points $(x(t), y(t))$ or $(x(t), y(t), z(t))$ to produce a graph, in the context of vector–valued functions each such point represents a vector. The implications of this will be more fully realized in the next section as we apply calculus ideas to these functions.

t	$t^3 - t$	$\dfrac{1}{t^2+1}$
-2	-6	$1/5$
-1	0	$1/2$
0	0	1
1	0	$1/2$
2	6	$1/5$

(a)

(b)

Figure 10.2: Sketching the vector–valued function of Example 43.

Figure 10.3: The graph of $\vec{r}(t)$ in Example 44.

Example 43 Graphing vector–valued functions

Graph $\vec{r}(t) = \left\langle t^3 - t, \dfrac{1}{t^2+1} \right\rangle$, for $-2 \leq t \leq 2$. Sketch $\vec{r}(-1)$ and $\vec{r}(2)$.

Solution We start by making a table of t, x and y values as shown in Figure 10.2(a). Plotting these points gives an indication of what the graph looks like. In Figure 10.2(b), we indicate these points and sketch the full graph. We also highlight $\vec{r}(-1)$ and $\vec{r}(2)$ on the graph.

Example 44 Graphing vector–valued functions.

Graph $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ for $0 \leq t \leq 4\pi$.

Solution We can again plot points, but careful consideration of this function is very revealing. Momentarily ignoring the third component, we see the x and y components trace out a circle of radius 1 centered at the origin. Noticing that the z component is t, we see that as the graph winds around the z-axis, it is also increasing at a constant rate in the positive z direction, forming a spiral. This is graphed in Figure 10.3. In the graph $\vec{r}(7\pi/4) \approx (0.707, -0.707, 5.498)$ is highlighted to help us understand the graph.

Algebra of Vector–Valued Functions

Definition 17 Operations on Vector–Valued Functions

Let $\vec{r}_1(t) = \langle f_1(t), g_1(t) \rangle$ and $\vec{r}_2(t) = \langle f_2(t), g_2(t) \rangle$ be vector–valued functions in $\mathbb{R}^2$ and let c be a scalar. Then:

1. $\vec{r}_1(t) \pm \vec{r}_2(t) = \langle f_1(t) \pm f_2(t), g_1(t) \pm g_2(t) \rangle$.
2. $c\vec{r}_1(t) = \langle cf_1(t), cg_1(t) \rangle$.

A similar definition holds for vector–valued functions in $\mathbb{R}^3$.

This definition states that we add, subtract and scale vector-valued functions component–wise. Combining vector–valued functions in this way can be very useful (as well as create interesting graphs).

Example 45 Adding and scaling vector–valued functions.

Let $\vec{r}_1(t) = \langle 0.2t, 0.3t \rangle$, $\vec{r}_2(t) = \langle \cos t, \sin t \rangle$ and $\vec{r}(t) = \vec{r}_1(t) + \vec{r}_2(t)$. Graph $\vec{r}_1(t)$, $\vec{r}_2(t)$, $\vec{r}(t)$ and $5\vec{r}(t)$ on $-10 \leq t \leq 10$.

Notes:

Solution We can graph $\vec{r}_1$ and $\vec{r}_2$ easily by plotting points (or just using technology). Let's think about each for a moment to better understand how vector–valued functions work.

We can rewrite $\vec{r}_1(t) = \langle 0.2t, 0.3t \rangle$ as $\vec{r}_1(t) = t\langle 0.2, 0.3\rangle$. That is, the function $\vec{r}_1$ scales the vector $\langle 0.2, 0.3\rangle$ by t. This scaling of a vector produces a line in the direction of $\langle 0.2, 0.3\rangle$.

We are familiar with $\vec{r}_2(t) = \langle \cos t, \sin t \rangle$; it traces out a circle, centered at the origin, of radius 1. Figure 10.4(a) graphs $\vec{r}_1(t)$ and $\vec{r}_2(t)$.

Adding $\vec{r}_1(t)$ to $\vec{r}_2(t)$ produces $\vec{r}(t) = \langle \cos t + 0.2t, \sin t + 0.3t \rangle$, graphed in Figure 10.4(b). The linear movement of the line combines with the circle to create loops that move in the direction of $\langle 0.2, 0.3\rangle$. (We encourage the reader to experiment by changing $\vec{r}_1(t)$ to $\langle 2t, 3t\rangle$, etc., and observe the effects on the loops.)

Multiplying $\vec{r}(t)$ by 5 scales the function by 5, producing $5\vec{r}(t) = \langle 5\cos t + 1, 5\sin t + 1.5\rangle$, which is graphed in Figure 10.4(c) along with $\vec{r}(t)$. The new function is "5 times bigger" than $\vec{r}(t)$. Note how the graph of $5\vec{r}(t)$ in (c) looks identical to the graph of $\vec{r}(t)$ in (b). This is due to the fact that the x and y bounds of the plot in (c) are exactly 5 times larger than the bounds in (b).

Figure 10.4: Graphing the functions in Example 45.

Example 46 **Adding and scaling vector–valued functions.**

A **cycloid** is a graph traced by a point p on a rolling circle, as shown in Figure 10.5. Find an equation describing the cycloid, where the circle has radius 1.

Figure 10.5: Tracing a cycloid.

Solution This problem is not very difficult if we approach it in a clever way. We start by letting $\vec{p}(t)$ describe the position of the point p on the circle, where the circle is centered at the origin and only rotates clockwise (i.e., it does not roll). This is relatively simple given our previous experiences with parametric equations; $\vec{p}(t) = \langle \cos t, -\sin t\rangle$.

We now want the circle to roll. We represent this by letting $\vec{c}(t)$ represent the location of the center of the circle. It should be clear that the y component of $\vec{c}(t)$ should be 1; the center of the circle is always going to be 1 if it rolls on a horizontal surface.

The x component of $\vec{c}(t)$ is a linear function of t: $f(t) = mt$ for some scalar m. When $t = 0$, $f(t) = 0$ (the circle starts centered on the y-axis). When $t = 2\pi$, the circle has made one complete revolution, traveling a distance equal to its

Notes:

Figure 10.6: The cycloid in Example 46.

circumference, which is also 2π. This gives us a point on our line $f(t) = mt$, the point $(2\pi, 2\pi)$. It should be clear that $m = 1$ and $f(t) = t$. So $\vec{c}(t) = \langle t, 1\rangle$.

We now combine $\vec{p}$ and $\vec{c}$ together to form the equation of the cycloid: $\vec{r}(t) = \vec{p}(t) + \vec{c}(t) = \langle \cos t + t, -\sin t + 1\rangle$, which is graphed in Figure 10.6.

Displacement

A vector–valued function $\vec{r}(t)$ is often used to describe the position of a moving object at time t. At $t = t_0$, the object is at $\vec{r}(t_0)$; at $t = t_1$, the object is at $\vec{r}(t_1)$. Knowing the locations $\vec{r}(t_0)$ and $\vec{r}(t_1)$ give no indication of the path taken between them, but often we only care about the difference of the locations, $\vec{r}(t_1) - \vec{r}(t_0)$, the **displacement**.

Definition 18 Displacement

Let $\vec{r}(t)$ be a vector–valued function and let $t_0 < t_1$ be values in the domain. The **displacement** $\vec{d}$ of $\vec{r}$, from $t = t_0$ to $t = t_1$, is

$$\vec{d} = \vec{r}(t_1) - \vec{r}(t_0).$$

When the displacement vector is drawn with initial point at $\vec{r}(t_0)$, its terminal point is $\vec{r}(t_1)$. We think of it as the vector which points from a starting position to an ending position.

Figure 10.7: Graphing the displacement of a position function in Example 47.

Example 47 Finding and graphing displacement vectors

Let $\vec{r}(t) = \left\langle \cos(\frac{\pi}{2}t), \sin(\frac{\pi}{2}t)\right\rangle$. Graph $\vec{r}(t)$ on $-1 \leq t \leq 1$, and find the displacement of $\vec{r}(t)$ on this interval.

Solution The function $\vec{r}(t)$ traces out the unit circle, though at a different rate than the "usual" $\langle \cos t, \sin t\rangle$ parametrization. At $t_0 = -1$, we have $\vec{r}(t_0) = \langle 0, -1\rangle$; at $t_1 = 1$, we have $\vec{r}(t_1) = \langle 0, 1\rangle$. The displacement of $\vec{r}(t)$ on $[-1, 1]$ is thus $\vec{d} = \langle 0, 1\rangle - \langle 0, -1\rangle = \langle 0, 2\rangle$.

A graph of $\vec{r}(t)$ on $[-1, 1]$ is given in Figure 10.7, along with the displacement vector $\vec{d}$ on this interval.

Measuring displacement makes us contemplate related, yet very different, concepts. Considering the semi–circular path the object in Example 47 took, we can quickly verify that the object ended up a distance of 2 units from its initial location. That is, we can compute $||\vec{d}|| = 2$. However, measuring *distance from the starting point* is different from measuring *distance traveled*. Being a semi–

Notes:

circle, we can measure the distance traveled by this object as $\pi \approx 3.14$ units. Knowing *distance from the starting point* allows us to compute **average rate of change.**

Definition 19 Average Rate of Change

Let $\vec{r}(t)$ be a vector–valued function, where each of its component functions is continuous on its domain, and let $t_0 < t_1$. The **average rate of change** of $\vec{r}(t)$ on $[t_0, t_1]$ is

$$\text{average rate of change} = \frac{\vec{r}(t_1) - \vec{r}(t_0)}{t_1 - t_0}.$$

Example 48 Average rate of change

Let $\vec{r}(t) = \left\langle \cos(\frac{\pi}{2}t), \sin(\frac{\pi}{2}t) \right\rangle$ as in Example 47. Find the average rate of change of $\vec{r}(t)$ on $[-1, 1]$ and on $[-1, 5]$.

Solution We computed in Example 47 that the displacement of $\vec{r}(t)$ on $[-1, 1]$ was $\vec{d} = \langle 0, 2 \rangle$. Thus the average rate of change of $\vec{r}(t)$ on $[-1, 1]$ is:

$$\frac{\vec{r}(1) - \vec{r}(-1)}{1 - (-1)} = \frac{\langle 0, 2 \rangle}{2} = \langle 0, 1 \rangle .$$

We interpret this as follows: the object followed a semi–circular path, meaning it moved towards the right then moved back to the left, while climbing slowly, then quickly, then slowly again. *On average*, however, it progressed straight up at a constant rate of $\langle 0, 1 \rangle$ per unit of time.

We can quickly see that the displacement on $[-1, 5]$ is the same as on $[-1, 1]$, so $\vec{d} = \langle 0, 2 \rangle$. The average rate of change is different, though:

$$\frac{\vec{r}(5) - \vec{r}(-1)}{5 - (-1)} = \frac{\langle 0, 2 \rangle}{6} = \langle 0, 1/3 \rangle .$$

As it took "3 times as long" to arrive at the same place, this average rate of change on $[-1, 5]$ is $1/3$ the average rate of change on $[-1, 1]$.

We considered average rates of change in Sections **??** and **??** as we studied limits and derivatives. The same is true here; in the following section we apply calculus concepts to vector–valued functions as we find limits, derivatives, and integrals. Understanding the average rate of change will give us an understanding of the derivative; displacement gives us one application of integration.

Notes:

Exercises 10.1

Terms and Concepts

1. Vector–valued functions are closely related to _____ _____ of graphs.

2. When sketching vector–valued functions, technically one isn't graphing points, but rather _____.

3. It can be useful to think of _____ as a vector that points from a starting position to an ending position.

Problems

In Exercises 4 – 11, sketch the vector–valued function on the given interval.

4. $\vec{r}(t) = \left\langle t^2, t^2 - 1\right\rangle$, for $-2 \leq t \leq 2$.

5. $\vec{r}(t) = \left\langle t^2, t^3\right\rangle$, for $-2 \leq t \leq 2$.

6. $\vec{r}(t) = \left\langle 1/t, 1/t^2\right\rangle$, for $-2 \leq t \leq 2$.

7. $\vec{r}(t) = \left\langle \frac{1}{10}t^2, \sin t\right\rangle$, for $-2\pi \leq t \leq 2\pi$.

8. $\vec{r}(t) = \left\langle \frac{1}{10}t^2, \sin t\right\rangle$, for $-2\pi \leq t \leq 2\pi$.

9. $\vec{r}(t) = \langle 3\sin(\pi t), 2\cos(\pi t)\rangle$, on $[0, 2]$.

10. $\vec{r}(t) = \langle 3\cos t, 2\sin(2t)\rangle$, on $[0, 2\pi]$.

11. $\vec{r}(t) = \langle 2\sec t, \tan t\rangle$, on $[-\pi, \pi]$.

In Exercises 12 – 15, sketch the vector–valued function on the given interval in $\mathbb{R}^3$. Technology may be useful in creating the sketch.

12. $\vec{r}(t) = \langle 2\cos t, t, 2\sin t\rangle$, on $[0, 2\pi]$.

13. $\vec{r}(t) = \langle 3\cos t, \sin t, t/\pi\rangle$ on $[0, 2\pi]$.

14. $\vec{r}(t) = \langle \cos t, \sin t, \sin t\rangle$ on $[0, 2\pi]$.

15. $\vec{r}(t) = \langle \cos t, \sin t, \sin(2t)\rangle$ on $[0, 2\pi]$.

In Exercises 16 – 19, find $||\vec{r}(t)||$.

16. $\vec{r}(t) = \left\langle t, t^2\right\rangle$.

17. $\vec{r}(t) = \langle 5\cos t, 3\sin t\rangle$.

18. $\vec{r}(t) = \langle 2\cos t, 2\sin t, t\rangle$.

19. $\vec{r}(t) = \left\langle \cos t, t, t^2\right\rangle$.

In Exercises 20 – 27, create a vector–valued function whose graph matches the given description.

20. A circle of radius 2, centered at $(1, 2)$, traced counter–clockwise once on $[0, 2\pi]$.

21. A circle of radius 3, centered at $(5, 5)$, traced clockwise once on $[0, 2\pi]$.

22. An ellipse, centered at $(0, 0)$ with vertical major axis of length 10 and minor axis of length 3, traced once counter–clockwise on $[0, 2\pi]$.

23. An ellipse, centered at $(3, -2)$ with horizontal major axis of length 6 and minor axis of length 4, traced once clockwise on $[0, 2\pi]$.

24. A line through $(2, 3)$ with a slope of 5.

25. A line through $(1, 5)$ with a slope of $-1/2$.

26. A vertically oriented helix with radius of 2 that starts at $(2, 0, 0)$ and ends at $(2, 0, 4\pi)$ after 1 revolution on $[0, 2\pi]$.

27. A vertically oriented helix with radius of 3 that starts at $(3, 0, 0)$ and ends at $(3, 0, 3)$ after 2 revolutions on $[0, 1]$.

In Exercises 28 – 31, find the average rate of change of $\vec{r}(t)$ on the given interval.

28. $\vec{r}(t) = \left\langle t, t^2\right\rangle$ on $[-2, 2]$.

29. $\vec{r}(t) = \langle t, t + \sin t\rangle$ on $[0, 2\pi]$.

30. $\vec{r}(t) = \langle 3\cos t, 2\sin t, t\rangle$ on $[0, 2\pi]$.

31. $\vec{r}(t) = \left\langle t, t^2, t^3\right\rangle$ on $[-1, 3]$.

10.2 Calculus and Vector–Valued Functions

The previous section introduced us to a new mathematical object, the vector–valued function. We now apply calculus concepts to these functions. We start with the limit, then work our way through derivatives to integrals.

Limits of Vector–Valued Functions

The initial definition of the limit of a vector–valued function is a bit intimidating, as was the definition of the limit in Definition **??**. The theorem following the definition shows that in practice, taking limits of vector–valued functions is no more difficult than taking limits of real–valued functions.

Definition 20 Limits of Vector–Valued Functions

Let I be an open interval containing c, and let $\vec{r}(t)$ be a vector–valued function defined on I, except possibly at c. The **limit of $\vec{r}(t)$, as t approaches c, is $\vec{L}$**, expressed as

$$\lim_{t\to c} \vec{r}(t) = \vec{L},$$

means that given any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $t \neq c$, if $|t - c| < \delta$, we have $||\,\vec{r}(t) - \vec{L}\,|| < \varepsilon$.

Note how the measurement of distance between real numbers is the absolute value of their difference; the measure of distance between vectors is the vector norm, or magnitude, of their difference.

Theorem 5 Limits of Vector–Valued Functions

1. Let $\vec{r}(t) = \langle f(t), g(t) \rangle$ be a vector–valued function in $\mathbb{R}^2$ defined on an open interval I containing c. Then

$$\lim_{t\to c} \vec{r}(t) = \left\langle \lim_{t\to c} f(t)\,,\ \lim_{t\to c} g(t) \right\rangle.$$

2. Let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ be a vector–valued function in $\mathbb{R}^3$ defined on an open interval I containing c. Then

$$\lim_{t\to c} \vec{r}(t) = \left\langle \lim_{t\to c} f(t)\,,\ \lim_{t\to c} g(t)\,,\ \lim_{t\to c} h(t) \right\rangle$$

Theorem 5 states that we compute limits component–wise.

Example 49 Finding limits of vector–valued functions

Notes:

Let $\vec{r}(t) = \left\langle \frac{\sin t}{t},\ t^2 - 3t + 3,\ \cos t \right\rangle$. Find $\lim_{t\to 0} \vec{r}(t)$.

Solution We apply the theorem and compute limits component–wise.

$$\begin{aligned}\lim_{t\to 0} \vec{r}(t) &= \left\langle \lim_{t\to 0} \frac{\sin t}{t},\ \lim_{t\to 0} t^2 - 3t + 3,\ \lim_{t\to 0} \cos t \right\rangle \\ &= \langle 1, 3, 1 \rangle .\end{aligned}$$

Continuity

Definition 21 Continuity of Vector–Valued Functions

Let $\vec{r}(t)$ be a vector–valued function defined on an open interval I containing c.

1. $\vec{r}(t)$ is **continuous at** c if $\lim_{t\to c} \vec{r}(t) = r(c)$.
2. If $\vec{r}(t)$ is continuous at all c in I, then $\vec{r}(t)$ is **continuous on** I**.**

We again have a theorem that lets us evaluate continuity component–wise.

Theorem 6 Continuity of Vector–Valued Functions

Let $\vec{r}(t)$ be a vector–valued function defined on an open interval I containing c. $\vec{r}(t)$ is continuous at c if, and only if, each of its component functions is continuous at c.

Example 50 Evaluating continuity of vector–valued functions

Let $\vec{r}(t) = \left\langle \frac{\sin t}{t},\ t^2 - 3t + 3,\ \cos t \right\rangle$. Determine whether $\vec{r}$ is continuous at $t = 0$ and $t = 1$.

Solution While the second and third components of $\vec{r}(t)$ are defined at $t = 0$, the first component, $(\sin t)/t$, is not. Since the first component is not even defined at $t = 0$, $\vec{r}(t)$ is not defined at $t = 0$, and hence it is not continuous at $t = 0$.

At $t = 1$ each of the component functions is continuous. Therefore $\vec{r}(t)$ is continuous at $t = 1$.

Notes:

Derivatives

Consider a vector–valued function $\vec{r}$ defined on an open interval I containing t_0 and t_1. We can compute the displacement of $\vec{r}$ on $[t_0, t_1]$, as shown in Figure 10.8(a). Recall that dividing the displacement vector by $t_1 - t_0$ gives the average rate of change on $[t_0, t_1]$, as shown in (b).

Figure 10.8: Illustrating displacement, leading to an understanding of the derivative of vector–valued functions.

The **derivative** of a vector–valued function is a measure of the *instantaneous* rate of change, measured by taking the limit as the length of $[t_0, t_1]$ goes to 0. Instead of thinking of an interval as $[t_0, t_1]$, we think of it as $[c, c+h]$ for some value of h (hence the interval has length h). The *average* rate of change is

$$\frac{\vec{r}(c+h) - \vec{r}(c)}{h}$$

for any value of $h \neq 0$. We take the limit as $h \to 0$ to measure the instantaneous rate of change; this is the derivative of $\vec{r}$.

Definition 22 Derivative of a Vector–Valued Function

Let $\vec{r}(t)$ be continuous on an open interval I containing c.

1. The derivative of $\vec{r}$ at $t = c$ is

$$\vec{r}\,'(c) = \lim_{h \to 0} \frac{\vec{r}(c+h) - \vec{r}(c)}{h}.$$

2. The derivative of $\vec{r}$ is

$$\vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}.$$

Alternate notations for the derivative of $\vec{r}$ include:

$$\vec{r}\,'(t) = \frac{d}{dt}\big(\vec{r}(t)\big) = \frac{d\vec{r}}{dt}.$$

If a vector–valued function has a derivative for all c in an open interval I, we say that $\vec{r}(t)$ is **differentiable** on I.

Once again we might view this definition as intimidating, but recall that we

Notes:

can evaluate limits component–wise. The following theorem verifies that this means we can compute derivatives component–wise as well, making the task not too difficult.

Theorem 7 Derivatives of Vector–Valued Functions

1. Let $\vec{r}(t) = \langle f(t), g(t) \rangle$. Then
$$\vec{r}\,'(t) = \langle f'(t), g'(t) \rangle .$$
2. Let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$. Then
$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle .$$

Example 51 Derivatives of vector–valued functions

Let $\vec{r}(t) = \langle t^2, t \rangle$.

1. Sketch $\vec{r}(t)$ and $\vec{r}\,'(t)$ on the same axes.
2. Compute $\vec{r}\,'(1)$ and sketch this vector with its initial point at the origin and at $\vec{r}(1)$.

Solution

1. Theorem 7 allows us to compute derivatives component–wise, so
$$\vec{r}\,'(t) = \langle 2t, 1 \rangle .$$
$\vec{r}(t)$ and $\vec{r}\,'(t)$ are graphed together in Figure 10.9(a). Note how plotting the two of these together, in this way, is not very illuminating. When dealing with real–valued functions, plotting $f(x)$ with $f'(x)$ gave us useful information as we were able to compare f and f' at the same x-values. When dealing with vector–valued functions, it is hard to tell which points on the graph of $\vec{r}\,'$ correspond to which points on the graph of $\vec{r}$.
2. We easily compute $\vec{r}\,'(1) = \langle 2, 1 \rangle$, which is drawn in Figure 10.9 with its initial point at the origin, as well as at $\vec{r}(1) = \langle 1, 1 \rangle$. These are sketched in Figure 10.9(b).

Figure 10.9: Graphing the derivative of a vector–valued function in Example 51.

Notes:

Example 52 Derivatives of vector–valued functions

Let $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$. Compute $\vec{r}\,'(t)$ and $\vec{r}\,'(\pi/2)$. Sketch $\vec{r}\,'(\pi/2)$ with its initial point at the origin and at $\vec{r}(\pi/2)$.

Solution We compute $\vec{r}\,'$ as $\vec{r}\,'(t) = \langle -\sin t, \cos t, 1\rangle$. At $t = \pi/2$, we have $\vec{r}\,'(\pi/2) = \langle -1, 0, 1\rangle$. Figure 10.10 shows a graph of $\vec{r}(t)$, with $\vec{r}\,'(\pi/2)$ plotted with its initial point at the origin and at $\vec{r}(\pi/2)$.

Figure 10.10: Viewing a vector–valued function and its derivative at one point.

In Examples 51 and 52, sketching a particular derivative with its initial point at the origin did not seem to reveal anything significant. However, when we sketched the vector with its initial point on the corresponding point on the graph, we did see something significant: the vector appeared to be *tangent* to the graph. We have not yet defined what "tangent" means in terms of curves in space; in fact, we use the derivative to define this term.

Definition 23 Tangent Vector, Tangent Line

Let $\vec{r}(t)$ be a differentiable vector–valued function on an open interval I containing c, where $\vec{r}\,'(c) \neq \vec{0}$.

1. A vector $\vec{v}$ is **tangent to the graph of** $\vec{r}(t)$ **at** $t = c$ if $\vec{v}$ is parallel to $\vec{r}\,'(c)$.
2. The **tangent line** to the graph of $\vec{r}(t)$ at $t = c$ is the line through $\vec{r}(c)$ with direction parallel to $\vec{r}\,'(c)$. An equation of the tangent line is
$$\vec{\ell}(t) = \vec{r}(c) + t\,\vec{r}\,'(c).$$

Example 53 Finding tangent lines to curves in space

Let $\vec{r}(t) = \left\langle t, t^2, t^3\right\rangle$ on $[-1.5, 1.5]$. Find the vector equation of the line tangent to the graph of $\vec{r}$ at $t = -1$.

Solution To find the equation of a line, we need a point on the line and the line's direction. The point is given by $\vec{r}(-1) = \langle -1, 1, -1\rangle$. (To be clear, $\langle -1, 1, -1\rangle$ is a *vector*, not a point, but we use the point "pointed to" by this vector.)

The direction comes from $\vec{r}\,'(-1)$. We compute, component–wise, $\vec{r}\,'(t) = \left\langle 1, 2t, 3t^2\right\rangle$. Thus $\vec{r}\,'(-1) = \langle 1, -2, 3\rangle$.

The vector equation of the line is $\ell(t) = \langle -1, 1, -1\rangle + t\,\langle 1, -2, 3\rangle$. This line and $\vec{r}(t)$ are sketched in Figure 10.11.

Figure 10.11: Graphing a curve in space with its tangent line.

Example 54 Finding tangent lines to curves

Notes:

Find the equations of the lines tangent to $\vec{r}(t) = \left\langle t^3, t^2 \right\rangle$ at $t = -1$ and $t = 0$.

Solution We find that $\vec{r}\,'(t) = \left\langle 3t^2, 2t \right\rangle$. At $t = -1$, we have

$$\vec{r}(-1) = \langle -1, 1 \rangle \quad \text{and} \quad \vec{r}\,'(-1) = \langle 3, -2 \rangle,$$

so the equation of the line tangent to the graph of $\vec{r}(t)$ at $t = -1$ is

$$\ell(t) = \langle -1, 1 \rangle + t \langle 3, -2 \rangle.$$

This line is graphed with $\vec{r}(t)$ in Figure 10.12.

At $t = 0$, we have $\vec{r}\,'(0) = \langle 0, 0 \rangle = \vec{0}$! This implies that the tangent line "has no direction." We cannot apply Definition 23, hence cannot find the equation of the tangent line.

Figure 10.12: Graphing $\vec{r}(t)$ and its tangent line in Example 54.

We were unable to compute the equation of the tangent line to $\vec{r}(t) = \left\langle t^3, t^2 \right\rangle$ at $t = 0$ because $\vec{r}\,'(0) = \vec{0}$. The graph in Figure 10.12 shows that there is a cusp at this point. This leads us to another definition of **smooth**, previously defined by Definition **??** in Section **??**.

Definition 24 Smooth Vector–Valued Functions

Let $\vec{r}(t)$ be a differentiable vector–valued function on an open interval I. $\vec{r}(t)$ is **smooth** on I if $\vec{r}\,'(t) \neq \vec{0}$ on I.

Having established derivatives of vector–valued functions, we now explore the relationships between the derivative and other vector operations. The following theorem states how the derivative interacts with vector addition and the various vector products.

Notes:

Theorem 8 Properies of Derivatives of Vector–Valued Functions

Let $\vec{r}$ and $\vec{s}$ be differentiable vector–valued functions, let f be a differentiable real–valued function, and let c be a real number.

1. $\dfrac{d}{dt}\Big(\vec{r}(t) \pm \vec{s}(t)\Big) = \vec{r}\,'(t) \pm \vec{s}\,'(t)$
2. $\dfrac{d}{dt}\Big(c\vec{r}(t)\Big) = c\vec{r}\,'(t)$
3. $\dfrac{d}{dt}\Big(f(t)\vec{r}(t)\Big) = f'(t)\vec{r}(t) + f(t)\vec{r}\,'(t)$ **Product Rule**
4. $\dfrac{d}{dt}\Big(\vec{r}(t) \cdot \vec{s}(t)\Big) = \vec{r}\,'(t) \cdot \vec{s}(t) + \vec{r}(t) \cdot \vec{s}\,'(t)$ **Product Rule**
5. $\dfrac{d}{dt}\Big(\vec{r}(t) \times \vec{s}(t)\Big) = \vec{r}\,'(t) \times \vec{s}(t) + \vec{r}(t) \times \vec{s}\,'(t)$ **Product Rule**
6. $\dfrac{d}{dt}\Big(\vec{r}\big(f(t)\big)\Big) = \vec{r}\,'\big(f(t)\big)f'(t)$ **Chain Rule**

Example 55 Using derivative properties of vector–valued functions

Let $\vec{r}(t) = \langle t, t^2 - 1\rangle$ and let $\vec{u}(t)$ be the unit vector that points in the direction of $\vec{r}(t)$.

1. Graph $\vec{r}(t)$ and $\vec{u}(t)$ on the same axes, on $[-2, 2]$.
2. Find $\vec{u}\,'(t)$ and sketch $\vec{u}\,'(-2)$, $\vec{u}\,'(-1)$ and $\vec{u}\,'(0)$. Sketch each with initial point the corresponding point on the graph of $\vec{u}$.

Figure 10.13: Graphing $\vec{r}(t)$ and $\vec{u}(t)$ in Example 55.

Solution

1. To form the unit vector that points in the direction of $\vec{r}$, we need to divide $\vec{r}(t)$ by its magnitude.

$$||\,\vec{r}(t)\,|| = \sqrt{t^2 + (t^2-1)^2} \quad \Rightarrow \quad \vec{u}(t) = \frac{1}{\sqrt{t^2 + (t^2-1)^2}}\,\langle t, t^2 - 1\rangle\,.$$

$\vec{r}(t)$ and $\vec{u}(t)$ are graphed in Figure 10.13. Note how the graph of $\vec{u}(t)$ forms part of a circle; this must be the case, as the length of $\vec{u}(t)$ is 1 for all t.

Notes:

Figure 10.14: Graphing some of the derivatives of $\vec{u}(t)$ in Example 55.

2. To compute $\vec{u}\,'(t)$, we use Theorem 8, writing

$$\vec{u}(t) = f(t)\vec{r}(t), \quad \text{where} \quad f(t) = \frac{1}{\sqrt{t^2 + (t^2-1)^2}} = \left(t^2 + (t^2-1)^2\right)^{-1/2}.$$

(We *could* write

$$\vec{u}(t) = \left\langle \frac{t}{\sqrt{t^2 + (t^2-1)^2}}, \frac{t^2-1}{\sqrt{t^2 + (t^2-1)^2}} \right\rangle$$

and then take the derivative. It is a matter of preference; this latter method requires two applications of the Quotient Rule where our method uses the Product and Chain Rules.)

We find $f'(t)$ using the Chain Rule:

$$\begin{aligned} f'(t) &= -\frac{1}{2}\left(t^2 + (t^2-1)^2\right)^{-3/2}\left(2t + 2(t^2-1)(2t)\right) \\ &= -\frac{2t(2t^2-1)}{2\left(\sqrt{t^2 + (t^2-1)^2}\,\right)^3} \end{aligned}$$

We now find $\vec{u}\,'(t)$ using part 3 of Theorem 8:

$$\begin{aligned} \vec{u}\,'(t) &= f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t) \\ &= -\frac{2t(2t^2-1)}{2\left(\sqrt{t^2 + (t^2-1)^2}\,\right)^3}\left\langle t, t^2-1\right\rangle + \frac{1}{\sqrt{t^2 + (t^2-1)^2}}\left\langle 1, 2t\right\rangle. \end{aligned}$$

This is admittedly very "messy;" such is usually the case when we deal with unit vectors. We can use this formula to compute $\vec{u}\,(-2)$, $\vec{u}\,(-1)$ and $\vec{u}\,(0)$:

$$\begin{aligned} \vec{u}\,(-2) &= \left\langle -\frac{15}{13\sqrt{13}}, -\frac{10}{13\sqrt{13}} \right\rangle \approx \langle -0.320, -0.213\rangle \\ \vec{u}\,(-1) &= \langle 0, -2\rangle \\ \vec{u}\,(0) &= \langle 1, 0\rangle \end{aligned}$$

Each of these is sketched in Figure 10.14. Note how the length of the vector gives an indication of how quickly the circle is being traced at that point. When $t = -2$, the circle is being drawn relatively slow; when $t = -1$, the circle is being traced much more quickly.

It is a basic geometric fact that a line tangent to a circle at a point P is perpendicular to the line passing through the center of the circle and P. This is

Notes:

illustrated in Figure 10.14; each tangent vector is perpendicular to the line that passes through its initial point and the center of the circle. Since the center of the circle is the origin, we can state this another way: $\vec{u}\,'(t)$ is orthogonal to $\vec{u}(t)$.

Recall that the dot product serves as a test for orthogonality: if $\vec{u} \cdot \vec{v} = 0$, then $\vec{u}$ is orthogonal to $\vec{v}$. Thus in the above example, $\vec{u}(t) \cdot \vec{u}\,'(t) = 0$.

This is true of any vector–valued function that has a constant length, that is, that traces out part of a circle. It has important implications later on, so we state it as a theorem (and leave its formal proof as an Exercise.)

Theorem 9 Vector–Valued Functions of Constant Length

Let $\vec{r}(t)$ be a differentiable vector–valued function on an open interval I of constant length. That is, $||\,\vec{r}(t)\,|| = c$ for all t in I (equivalently, $\vec{r}(t) \cdot \vec{r}(t) = c^2$ for all t in I). Then $\vec{r}(t) \cdot \vec{r}\,'(t) = 0$ for all t in I.

Integration

Indefinite and definite integrals of vector–valued functions are also evaluated component–wise.

Theorem 10 Indefinite and Definite Integrals of Vector–Valued Functions

Let $\vec{r}(t) = \langle f(t), g(t)\rangle$ be a vector–valued function in $\mathbb{R}^2$.

1. $\displaystyle\int \vec{r}(t)\,dt = \left\langle \int f(t)\,dt, \int g(t)\,dt \right\rangle$
2. $\displaystyle\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt \right\rangle$

A similar statement holds for vector–valued functions in $\mathbb{R}^3$.

Example 56 Evaluating a definite integral of a vector–valued function

Let $\vec{r}(t) = \left\langle e^{2t}, \sin t\right\rangle$. Evaluate $\displaystyle\int_0^1 \vec{r}(t)\,dt$.

Notes:

Solution We follow Theorem 10.

$$\begin{aligned}\int_0^1 \vec{r}(t)\,dt &= \int_0^1 \left\langle e^{2t}, \sin t\right\rangle\,dt\\ &= \left\langle \int_0^1 e^{2t}\,dt\,, \int_0^1 \sin t\,dt\right\rangle\\ &= \left\langle \frac{1}{2}e^{2t}\Big|_0^1\,, -\cos t\Big|_0^1\right\rangle\\ &= \left\langle \frac{1}{2}(e^2-1)\,, -\cos(1)+1\right\rangle\\ &\approx \langle 3.19, 0.460\rangle\,.\end{aligned}$$

Example 57 Solving an initial value problem

Let $\vec{r}''(t) = \langle 2, \cos t, 12t\rangle$. Find $\vec{r}(t)$ where:

- $\vec{r}(0) = \langle -7, -1, 2\rangle$ and
- $\vec{r}'(0) = \langle 5, 3, 0\rangle$.

Solution Knowing $\vec{r}''(t) = \langle 2, \cos t, 12t\rangle$, we find $\vec{r}'(t)$ by evaluating the indefinite integral.

$$\begin{aligned}\int \vec{r}''(t)\,dt &= \left\langle \int 2\,dt\,, \int \cos t\,dt\,, \int 12t\,dt\right\rangle\\ &= \left\langle 2t + C_1, \sin t + C_2, 6t^2 + C_3\right\rangle\\ &= \left\langle 2t, \sin t, 6t^2\right\rangle + \langle C_1, C_2, C_3\rangle\\ &= \left\langle 2t, \sin t, 6t^2\right\rangle + \vec{C}.\end{aligned}$$

Note how each indefinite integral creates its own constant which we collect as one constant vector $\vec{C}$. Knowing $\vec{r}'(0) = \langle 5, 3, 0\rangle$ allows us to solve for $\vec{C}$:

$$\begin{aligned}\vec{r}'(t) &= \left\langle 2t, \sin t, 6t^2\right\rangle + \vec{C}\\ \vec{r}'(0) &= \langle 0, 0, 0\rangle + \vec{C}\\ \langle 5, 3, 0\rangle &= \vec{C}.\end{aligned}$$

So $\vec{r}'(t) = \left\langle 2t, \sin t, 6t^2\right\rangle + \langle 5, 3, 0\rangle = \left\langle 2t+5, \sin t + 3, 6t^2\right\rangle$. To find $\vec{r}(t)$, we integrate once more.

$$\begin{aligned}\int \vec{r}'(t)\,dt &= \left\langle \int 2t+5\,dt, \int \sin t + 3\,dt, \int 6t^2\,dt\right\rangle\\ &= \left\langle t^2+5t, -\cos t + 3t, 2t^3\right\rangle + \vec{C}.\end{aligned}$$

With $\vec{r}(0) = \langle -7, -1, 2\rangle$, we solve for $\vec{C}$:

Notes:

$$\begin{aligned}\vec{r}(t) &= \left\langle t^2+5t, -\cos t+3t, 2t^3\right\rangle + \vec{C}\\ \vec{r}(0) &= \langle 0,-1,0\rangle + \vec{C}\\ \langle -7,-1,2\rangle &= \langle 0,-1,0\rangle + \vec{C}\\ \langle -7,0,2\rangle &= \vec{C}.\end{aligned}$$

So $\vec{r}(t) = \left\langle t^2+5t, -\cos t+3t, 2t^3\right\rangle + \langle -7,0,2\rangle = \left\langle t^2+5t-7, -\cos t+3t, 2t^3+2\right\rangle$.

What does the integration of a vector–valued function *mean*? There are many applications, but none as direct as "the area under the curve" that we used in understanding the integral of a real–valued function.

A key understanding for us comes from considering the integral of a derivative:

$$\int_a^b \vec{r}\,'(t)\,dt = \vec{r}(t)\Big|_a^b = \vec{r}(b) - \vec{r}(a).$$

Integrating a *rate of change* function gives *displacement*.

Noting that vector–valued functions are closely related to parametric equations, we can describe the arc length of the graph of a vector–valued function as an integral. Given parametric equations $x = f(t)$, $y = g(t)$, the arc length on $[a,b]$ of the graph is

$$\text{Arc Length} = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\,dt,$$

as stated in Theorem **??** in Section **??**. If $\vec{r}(t) = \langle f(t), g(t)\rangle$, note that $\sqrt{f'(t)^2+g'(t)^2} = ||\,\vec{r}\,'(t)\,||$. Therefore we can express the arc length of the graph of a vector–valued function as an integral of the magnitude of its derivative.

Theorem 11 Arc Length of a Vector–Valued Function

Let $\vec{r}(t)$ be a vector–valued function where $\vec{r}\,'(t)$ is continuous on $[a,b]$. The arc length L of the graph of $\vec{r}(t)$ is

$$L = \int_a^b ||\,\vec{r}\,'(t)\,||\,dt.$$

Note that we are actually integrating a scalar–function here, not a vector–valued function.

The next section takes what we have established thus far and applies it to objects in motion. We will let $\vec{r}(t)$ describe the path of an object in the plane or in space and will discover the information provided by $\vec{r}\,'(t)$ and $\vec{r}\,''(t)$.

Notes:

Exercises 10.2

Terms and Concepts

1. Limits, derivatives and integrals of vector–valued functions are all evaluated _____–wise.

2. The definite integral of a rate of change function gives _____.

3. Why is it generally not useful to graph both $\vec{r}(t)$ and $\vec{r}\,'(t)$ on the same axes?

Problems

In Exercises 4 – 7, evaluate the given limit.

4. $\lim\limits_{t\to 5} \left\langle 2t+1, 3t^2-1, \sin t\right\rangle$

5. $\lim\limits_{t\to 3} \left\langle e^t, \dfrac{t^2-9}{t+3}\right\rangle$

6. $\lim\limits_{t\to 0} \left\langle \dfrac{t}{\sin t}, (1+t)^{\frac{1}{t}}\right\rangle$

7. $\lim\limits_{h\to 0} \dfrac{\vec{r}(t+h)-\vec{r}(t)}{h}$, where $\vec{r}(t) = \left\langle t^2, t, 1\right\rangle$.

In Exercises 8 – 9, identify the interval(s) on which $\vec{r}(t)$ is continuous.

8. $\vec{r}(t) = \left\langle t^2, 1/t\right\rangle$

9. $\vec{r}(t) = \left\langle \cos t, e^t, \ln t\right\rangle$

In Exercises 10 – 14, find the derivative of the given function.

10. $\vec{r}(t) = \left\langle \cos t, e^t, \ln t\right\rangle$

11. $\vec{r}(t) = \left\langle \dfrac{1}{t}, \dfrac{2t-1}{3t+1}, \tan t\right\rangle$

12. $\vec{r}(t) = (t^2)\left\langle \sin t, 2t+5\right\rangle$

13. $\vec{r}(t) = \left\langle t^2+1, t-1\right\rangle \cdot \left\langle \sin t, 2t+5\right\rangle$

14. $\vec{r}(t) = \left\langle t^2+1, t-1, 1\right\rangle \times \left\langle \sin t, 2t+5, 1\right\rangle$

In Exercises 15 – 18, find $\vec{r}\,'(t)$. Sketch $\vec{r}(t)$ and $\vec{r}\,'(1)$, with the initial point of $\vec{r}\,'(1)$ at $\vec{r}(1)$.

15. $\vec{r}(t) = \left\langle t^2+t, t^2-t\right\rangle$

16. $\vec{r}(t) = \left\langle t^2-2t+2, t^3-3t^2+2t\right\rangle$

17. $\vec{r}(t) = \left\langle t^2+1, t^3-t\right\rangle$

18. $\vec{r}(t) = \left\langle t^2-4t+5, , t^3-6t^2+11t-6\right\rangle$

In Exercises 19 – 22, give the equation of the line tangent to the graph of $\vec{r}(t)$ at the given t value.

19. $\vec{r}(t) = \left\langle t^2+t, t^2-t\right\rangle$ at $t=1$.

20. $\vec{r}(t) = \left\langle 3\cos t, \sin t\right\rangle$ at $t=\pi/4$.

21. $\vec{r}(t) = \left\langle 3\cos t, 3\sin t, t\right\rangle$ at $t=\pi$.

22. $\vec{r}(t) = \left\langle e^t, \tan t, t\right\rangle$ at $t=0$.

In Exercises 23 – 26, find the value(s) of t for which $\vec{r}(t)$ is not smooth.

23. $\vec{r}(t) = \left\langle \cos t, \sin t - t\right\rangle$

24. $\vec{r}(t) = \left\langle t^2-2t+1, t^3+t^2-5t+3\right\rangle$

25. $\vec{r}(t) = \left\langle \cos t - \sin t, \sin t - \cos t, \cos(4t)\right\rangle$

26. $\vec{r}(t) = \left\langle t^3-3t+2, -\cos(\pi t), \sin^2(\pi t)\right\rangle$

Exercises 27 – 29 ask you to verify parts of Theorem 8. In each let $f(t) = t^3$, $\vec{r}(t) = \left\langle t^2, t-1, 1\right\rangle$ and $\vec{s}(t) = \left\langle \sin t, e^t, t\right\rangle$. Compute the various derivatives as indicated.

27. Simplify $f(t)\vec{r}(t)$, then find its derivative; show this is the same as $f'(t)\vec{r}(t) + f(t)\vec{r}\,'(t)$.

28. Simplify $\vec{r}(t)\cdot\vec{s}(t)$, then find its derivative; show this is the same as $\vec{r}\,'(t)\cdot\vec{s}(t) + \vec{r}(t)\cdot\vec{s}\,'(t)$.

29. Simplify $\vec{r}(t)\times\vec{s}(t)$, then find its derivative; show this is the same as $\vec{r}\,'(t)\times\vec{s}(t) + \vec{r}(t)\times\vec{s}\,'(t)$.

In Exercises 30 – 33 , evaluate the given definite or indefinite integral.

30. $\displaystyle\int \left\langle t^3, \cos t, te^t\right\rangle\, dt$

31. $\displaystyle\int \left\langle \frac{1}{1+t^2}, \sec^2 t\right\rangle\, dt$

32. $\displaystyle\int_0^{\pi} \left\langle -\sin t, \cos t\right\rangle\, dt$

33. $\displaystyle\int_{-2}^{2} \left\langle 2t+1, 2t-1\right\rangle\, dt$

In Exercises 34 – 37 , solve the given initial value problems.

34. Find $\vec{r}(t)$, given that $\vec{r}\,'(t) = \left\langle t, \sin t\right\rangle$ and $\vec{r}(0) = \left\langle 2, 2\right\rangle$.

35. Find $\vec{r}(t)$, given that $\vec{r}\,'(t) = \left\langle 1/(t+1), \tan t\right\rangle$ and $\vec{r}(0) = \left\langle 1, 2\right\rangle$.

36. Find $\vec{r}(t)$, given that $\vec{r}''(t) = \left\langle t^2, t, 1 \right\rangle$, $\vec{r}'(0) = \langle 1, 2, 3 \rangle$ and $\vec{r}(0) = \langle 4, 5, 6 \rangle$.

37. Find $\vec{r}(t)$, given that $\vec{r}''(t) = \left\langle \cos t, \sin t, e^t \right\rangle$, $\vec{r}'(0) = \langle 0, 0, 0 \rangle$ and $\vec{r}(0) = \langle 0, 0, 0 \rangle$.

In Exercises 38 – 41 , find the arc length of $\vec{r}(t)$ on the indicated interval.

38. $\vec{r}(t) = \langle 2\cos t, 2\sin t, 3t \rangle$ on $[0, 2\pi]$.

39. $\vec{r}(t) = \langle 5\cos t, 3\sin t, 4\sin t \rangle$ on $[0, 2\pi]$.

40. $\vec{r}(t) = \left\langle t^3, t^2, t^3 \right\rangle$ on $[0, 1]$.

41. $\vec{r}(t) = \left\langle e^{-t}\cos t, e^{-t}\sin t \right\rangle$ on $[0, 1]$.

42. Prove Theorem 9; that is, show if $\vec{r}(t)$ has constant length and is differentiable, then $\vec{r}(t) \cdot \vec{r}'(t) = 0$. (Hint: use the Product Rule to compute $\frac{d}{dt}\left(\vec{r}(t) \cdot \vec{r}(t)\right)$.)

10.3 The Calculus of Motion

A common use of vector–valued functions is to describe the motion of an object in the plane or in space. A **position function** $\vec{r}(t)$ gives the position of an object at **time** t. This section explores how derivatives and integrals are used to study the motion described by such a function.

Definition 25 Velocity, Speed and Acceleration

Let $\vec{r}(t)$ be a position function in $\mathbb{R}^2$ or $\mathbb{R}^3$.

1. **Velocity**, denoted $\vec{v}(t)$, is the instantaneous rate of position change; that is, $\vec{v}(t) = \vec{r}\,'(t)$.
2. **Speed** is the magnitude of velocity, $||\,\vec{v}(t)\,||$.
3. **Acceleration**, denoted $\vec{a}(t)$, is the instantaneous rate of velocity change; that is, $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$.

Example 58 Finding velocity and acceleration

An object is moving with position function $\vec{r}(t) = \left\langle t^2 - t, t^2 + t\right\rangle$, $-3 \leq t \leq 3$, where distances are measured in feet and time is measured in seconds.

1. Find $\vec{v}(t)$ and $\vec{a}(t)$.
2. Sketch $\vec{r}(t)$; plot $\vec{v}(-1)$, $\vec{a}(-1)$, $\vec{v}(1)$ and $\vec{a}(1)$, each with their initial point at their corresponding point on the graph of $\vec{r}(t)$.
3. When is the object's speed minimized?

Solution

1. Taking derivatives, we find

$$\vec{v}(t) = \vec{r}\,'(t) = \langle 2t - 1, 2t + 1\rangle \quad \text{and} \quad \vec{a}(t) = \vec{r}\,''(t) = \langle 2, 2\rangle\,.$$

   Note that acceleration is constant.

2. $\vec{v}(-1) = \langle -3, -1\rangle$, $\vec{a}(-1) = \langle 2, 2\rangle$; $\vec{v}(1) = \langle 1, 3\rangle$, $\vec{a}(1) = \langle 2, 2\rangle$. These are plotted with $\vec{r}(t)$ in Figure 10.15(a).

   We can think of acceleration as "pulling" the velocity vector in a certain direction. At $t = -1$, the velocity vector points down and to the left; at $t = 1$, the velocity vector has been pulled in the $\langle 2, 2\rangle$ direction and is

Figure 10.15: Graphing the position, velocity and acceleration of an object in Example 58.

Notes:

now pointing up and to the right. In Figure 10.15(b) we plot more velocity/acceleration vectors, making more clear the effect acceleration has on velocity.

Since $\vec{a}(t)$ is constant in this example, as t grows large $\vec{v}(t)$ becomes almost parallel to $\vec{a}(t)$. For instance, when $t = 10$, $\vec{v}(10) = \langle 19, 21\rangle$, which is nearly parallel to $\langle 2, 2\rangle$.

3. The object's speed is given by

$$||\,\vec{v}(t)\,|| = \sqrt{(2t-1)^2 + (2t+1)^2} = \sqrt{8t^2+2}.$$

To find the minimal speed, we could apply calculus techniques (such as set the derivative equal to 0 and solve for t, etc.) but we can find it by inspection. Inside the square root we have a quadratic which is minimized when $t = 0$. Thus the speed is minimized at $t = 0$, with a speed of $\sqrt{2}$ ft/s.

The graph in Figure 10.15(b) also implies speed is minimized here. The filled dots on the graph are located at integer values of t between -3 and 3. Dots that are far apart imply the object traveled a far distance in 1 second, indicating high speed; dots that are close together imply the object did not travel far in 1 second, indicating a low speed. The dots are closest together near $t = 0$, implying the speed is minimized near that value.

Example 59 Analyzing Motion

Two objects follow an identical path at different rates on $[-1, 1]$. The position function for Object 1 is $\vec{r}_1(t) = \langle t, t^2\rangle$; the position function for Object 2 is $\vec{r}_2(t) = \langle t^3, t^6\rangle$, where distances are measured in feet and time is measured in seconds. Compare the velocity, speed and acceleration of the two objects on the path.

Solution We begin by computing the velocity and acceleration function for each object:

$$\vec{v}_1(t) = \langle 1, 2t\rangle \qquad \vec{v}_2(t) = \langle 3t^2, 6t^5\rangle$$
$$\vec{a}_1(t) = \langle 0, 2\rangle \qquad \vec{a}_2(t) = \langle 6t, 30t^4\rangle$$

We immediately see that Object 1 has constant acceleration, whereas Object 2 does not.

At $t = -1$, we have $\vec{v}_1(-1) = \langle 1, -2\rangle$ and $\vec{v}_2(-1) = \langle 3, -6\rangle$; the velocity of Object 2 is three times that of Object 1 and so it follows that the speed of Object 2 is three times that of Object 1 ($3\sqrt{5}$ ft/s compared to $\sqrt{5}$ ft/s.)

Figure 10.16: Plotting velocity and acceleration vectors for Object 1 in Example 59.

Notes:

Figure 10.17: Comparing the positions of Objects 1 and 2 in Example 59.

At $t = 0$, the velocity of Object 1 is $\vec{v}(1) = \langle 1, 0\rangle$ and the velocity of Object 2 is $\vec{0}$! This tells us that Object 2 comes to a complete stop at $t = 0$.

In Figure 10.16, we see the velocity and acceleration vectors for Object 1 plotted for $t = -1, -1/2, 0, 1/2$ and $t = 1$. Note again how the constant acceleration vector seems to "pull" the velocity vector from pointing down, right to up, right. We could plot the analogous picture for Object 2, but the velocity and acceleration vectors are rather large ($\vec{a}_2(-1) = \langle -6, 30\rangle$!)

Instead, we simply plot the locations of Object 1 and 2 on intervals of $1/10^{\text{th}}$ of a second, shown in Figure 10.17(a) and (b). Note how the x-values of Object 1 increase at a steady rate. This is because the x-component of $\vec{a}(t)$ is 0; there is no acceleration in the x-component. The dots are not evenly spaced; the object is moving faster near $t = -1$ and $t = 1$ than near $t = 0$.

In part (b) of the Figure, we see the points plotted for Object 2. Note the large change in position from $t = -1$ to $t = -0.9$; the object starts moving very quickly. However, it slows considerably at it approaches the origin, and comes to a complete stop at $t = 0$. While it looks like there are 3 points near the origin, there are in reality 5 points there.

Since the objects begin and end at the same location, the have the same displacement. Since they begin and end at the same time, with the same displacement, they have they have the same average rate of change (i.e, they have the same average velocity). Since they follow the same path, they have the same distance traveled. Even though these three measurements are the same, the objects obviously travel the path in very different ways.

Example 60 Analyzing the motion of a whirling ball on a string

A young boy whirls a ball, attached to a string, above his head in a counterclockwise circle. The ball follows a circular path and makes 2 revolutions per second. The string has length 2ft.

1. Find the position function $\vec{r}(t)$ that describes this situation.
2. Find the acceleration of the ball and derive a physical interpretation of it.
3. A tree stands 10ft in front of the boy. At what t-values should the boy release the string so that the ball hits the tree?

Solution

1. The ball whirls in a circle. Since the string is 2ft long, the radius of the circle is 2. The position function $\vec{r}(t) = \langle 2\cos t, 2\sin t\rangle$ describes a circle with radius 2, centered at the origin, but makes a full revolution every 2π seconds, not two revolutions per second. We modify the period of the

Notes:

trigonometric functions to be 1/2 by multiplying t by 4π. The final position function is thus

$$\vec{r}(t) = \langle 2\cos(4\pi t), 2\sin(4\pi t)\rangle .$$

(Plot this for $0 \leq t \leq 1/2$ to verify that one revolution is made in 1/2 a second.)

2. To find $\vec{a}(t)$, we derive $\vec{r}(t)$ twice.

$$\begin{aligned}\vec{v}(t) = \vec{r}\,'(t) &= \langle -8\pi\sin(4\pi t), 8\pi\cos(4\pi t)\rangle \\ \vec{a}(t) = \vec{r}\,''(t) &= \left\langle -32\pi^2\cos(4\pi t), -32\pi^2\sin(4\pi t)\right\rangle \\ &= -32\pi^2\langle\cos(4\pi t), \sin(4\pi t)\rangle .\end{aligned}$$

Note how $\vec{a}(t)$ is parallel to $\vec{r}(t)$, but has a different magnitude and points in the opposite direction. Why is this?

Recall the classic physics equation, "Force = mass $\times$ acceleration." A force acting on a mass induces acceleration (i.e., the mass moves); acceleration acting on a mass induces a force (gravity gives our mass a *weight*). Thus force and acceleration are closely related. A moving ball "wants" to travel in a straight line. Why does the ball in our example move in a circle? It is attached to the boy's hand by a string. The string applies a force to the ball, affecting it's motion: the string *accelerates* the ball. This is not acceleration in the sense of "it travels faster;" rather, this acceleration is changing the velocity of the ball. In what direction is this force/acceleration being applied? In the direction of the string, towards the boy's hand.

The magnitude of the acceleration is related to the speed at which the ball is traveling. A ball whirling quickly is rapidly changing direction/velocity. When velocity is changing rapidly, the acceleration must be "large."

3. When the boy releases the string, the string no longer applies a force to the ball, meaning acceleration is $\vec{0}$ and the ball can now move in a straight line in the direction of $\vec{v}(t)$.

Let $t = t_0$ be the time when the boy lets go of the string. The ball will be at $\vec{r}(t_0)$, traveling in the direction of $\vec{v}(t_0)$. We want to find t_0 so that this line contains the point $(0, 10)$ (since the tree is 10ft directly in front of the boy).

There are many ways to find this time value. We choose one that is relatively simple computationally. As shown in Figure 10.18, the vector from the release point to the tree is $\langle 0, 10\rangle - \vec{r}(t_0)$. This line segment is tangent to the circle, which means it is also perpendicular to $\vec{r}(t_0)$ itself, so their dot product is 0.

Figure 10.18: Modeling the flight of a ball in Example 60.

Notes:

$$\begin{aligned}
\vec{r}(t_0)\cdot\big(\langle 0,10\rangle-\vec{r}(t_0)\big)&=0\\
\langle 2\cos(4\pi t_0),2\sin(4\pi t_0)\rangle\cdot\langle -2\cos(4\pi t_0),10-2\sin(4\pi t_0)\rangle&=0\\
-4\cos^2(4\pi t_0)+20\sin(4\pi t_0)-4\sin^2(4\pi t_0)&=0\\
20\sin(4\pi t_0)-4&=0\\
\sin(4\pi t_0)&=1/5\\
4\pi t_0&=\sin^{-1}(1/5)\\
4\pi t_0&\approx 0.2+2\pi n,
\end{aligned}$$

where n is an integer. Solving for t_0 we have:

$$t_0\approx 0.016+n/2$$

This is a wonderful formula. Every 1/2 second after $t=0.016$s the boy can release the string (since the ball makes 2 revolutions per second, he has two chances each second to release the ball).

Example 61 Analyzing motion in space

An object moves in a spiral with position function $\vec{r}(t)=\langle\cos t,\sin t,t\rangle$, where distances are measured in meters and time is in minutes. Describe the object's speed and acceleration at time t.

Solution With $\vec{r}(t)=\langle\cos t,\sin t,t\rangle$, we have:

$$\begin{aligned}
\vec{v}(t)&=\langle-\sin t,\cos t,1\rangle \quad\text{and}\\
\vec{a}(t)&=\langle-\cos t,-\sin t,0\rangle\,.
\end{aligned}$$

The speed of the object is $||\,\vec{v}(t)\,||=\sqrt{(-\sin t)^2+\cos^2 t+1}=\sqrt{2}$m/min; it moves at a constant speed. Note that the object does not accelerate in the z-direction, but rather moves up at a constant rate of 1m/min.

The objects in Examples 60 and 61 traveled at a constant speed. That is, $||\,\vec{v}(t)\,||=c$ for some constant c. Recall Theorem 9, which states that if a vector–valued function $\vec{r}(t)$ has constant length, then $\vec{r}(t)$ is perpendicular to its derivative: $\vec{r}(t)\cdot\vec{r}\,'(t)=0$. In these examples, the velocity function has constant length, therefore we can conclude that the velocity is perpendicular to the acceleration: $\vec{v}(t)\cdot\vec{a}(t)=0$. A quick check verifies this.

There is an intuitive understanding of this. If acceleration is parallel to velocity, then it is only affecting the object's speed; it does not change the direction of travel. (For example, consider a dropped stone. Acceleration and velocity are

Notes:

parallel – straight down – and the direction of velocity never changes, though speed does increase.) If acceleration is not perpendicular to velocity, then there is some acceleration in the direction of travel, influencing the speed. If speed is constant, then acceleration must be orthogonal to velocity, as it then only affects direction, and not speed.

Key Idea 7 Objects With Constant Speed

If an object moves with constant speed, then its velocity and acceleration vectors are orthogonal. That is, $\vec{v}(t) \cdot \vec{a}(t) = 0$.

Projectile Motion

An important application of vector–valued position functions is *projectile motion*: the motion of objects under only the influence of gravity. We will measure time in seconds, and distances will either be in meters or feet. We will show that we can completely describe the path of such an object knowing its initial position and initial velocity (i.e., where it *is* and where it *is going.*)

Suppose an object has initial position $\vec{r}(0) = \langle x_0, y_0 \rangle$ and initial velocity $\vec{v}(0) = \langle v_x, v_y \rangle$. It is customary to rewrite $\vec{v}(0)$ in terms of its speed v_0 and direction $\vec{u}$, where $\vec{u}$ is a unit vector. Recall all unit vectors in $\mathbb{R}^2$ can be written as $\langle \cos\theta, \sin\theta \rangle$, where θ is an angle measure counter–clockwise from the x-axis. (We refer to θ as the **angle of elevation.**) Thus $\vec{v}(0) = v_0 \langle \cos\theta, \sin\theta \rangle$.

Note: In this text we use $g = 32$ft/s when using Imperial units, and $g = 9.8$m/s when using SI units.

Since the acceleration of the object is known, namely $\vec{a}(t) = \langle 0, -g \rangle$, where g is the gravitational constant, we can find $\vec{r}(t)$ knowing our two initial conditions. We first find $\vec{v}(t)$:

$$\begin{aligned}
\vec{v}(t) &= \int \vec{a}(t)\, dt \\
\vec{v}(t) &= \int \langle 0, -g \rangle \; dt \\
\vec{v}(t) &= \langle 0, -gt \rangle + \vec{C}.
\end{aligned}$$

Knowing $\vec{v}(0) = v_0 \langle \cos\theta, \sin\theta \rangle$, we have $\vec{C} = v_0 \langle \cos t, \sin t \rangle$ and so

$$\vec{v}(t) = \left\langle v_0 \cos\theta, -gt + v_0 \sin\theta \right\rangle .$$

Notes:

We integrate once more to find $\vec{r}(t)$:

$$\begin{aligned}\vec{r}(t) &= \int \vec{v}(t)\,dt\\ \vec{r}(t) &= \int \left\langle v_0\cos\theta, -gt + v_0\sin\theta\right\rangle\,dt\\ \vec{r}(t) &= \left\langle \big(v_0\cos\theta\big)t, -\frac{1}{2}gt^2 + \big(v_0\sin\theta\big)t\right\rangle + \vec{C}.\end{aligned}$$

Knowing $\vec{r}(0) = \langle x_0, y_0\rangle$, we conclude $\vec{C} = \langle x_0, y_0\rangle$ and

$$\vec{r}(t) = \left\langle \big(v_0\cos\theta\big)t + x_0\,, -\frac{1}{2}gt^2 + \big(v_0\sin\theta\big)t + y_0\right\rangle.$$

Key Idea 8 Projectile Motion

The position function of a projectile propelled from an initial position of $\vec{r}_0 = \langle x_0, y_0\rangle$, with initial speed v_0, with angle of elevation θ and neglecting all accelerations but gravity is

$$\vec{r}(t) = \left\langle \big(v_0\cos\theta\big)t + x_0\,, -\frac{1}{2}gt^2 + \big(v_0\sin\theta\big)t + y_0\right\rangle.$$

Letting $\vec{v}_0 = v_0\langle\cos\theta, \sin\theta\rangle$, $\vec{r}(t)$ can be written as

$$\vec{r}(t) = \left\langle 0, -\frac{1}{2}gt^2\right\rangle + \vec{v}_0 t + \vec{r}_0.$$

We demonstrate how to use this position function in the next two examples.

Example 62 Projectile Motion

Sydney shoots her Red Ryder® bb gun across level ground from an elevation of 4ft, where the barrel of the gun makes a 5° angle with the horizontal. Find how far the bb travels before landing, assuming the bb is fired at the advertised rate of 350ft/s and ignoring air resistance.

Solution A direct application of Key Idea 8 gives

$$\begin{aligned}\vec{r}(t) &= \left\langle (350\cos 5^\circ)t, -16t^2 + (350\sin 5^\circ)t + 4\right\rangle\\ &\approx \left\langle 346.67t, -16t^2 + 30.50t + 4\right\rangle,\end{aligned}$$

Notes:

where we set her initial position to be $\langle 0, 4\rangle$. We need to find *when* the bb lands, then we can find *where*. We accomplish this by setting the y-component equal to 0 and solving for t:

$$\begin{aligned} -16t^2 + 30.50t + 4 &= 0 \\ t &= \frac{-30.50 \pm \sqrt{30.50^2 - 4(-16)(4)}}{-32} \\ t &\approx 2.03s. \end{aligned}$$

(We discarded a negative solution that resulted from our quadratic equation.)

We have found that the bb lands 2.03s after firing; with $t = 2.03$, we find the x-component of our position function is $346.67(2.03) = 703.74$ft. The bb lands about 704 feet away.

Example 63 Projectile Motion

Alex holds his sister's bb gun at a height of 3ft and wants to shoot a target that is 6ft above the ground, 25ft away. At what angle should he hold the gun to hit his target? (We still assume the muzzle velocity is 350ft/s.)

Solution The position function for the path of Alex's bb is

$$\vec{r}(t) = \left\langle (350\cos\theta)t, -16t^2 + (350\sin\theta)t + 3\right\rangle.$$

We need to find θ so that $\vec{r}(t) = \langle 25, 6\rangle$ for some value of t. That is, we want to find θ and t such that

$$(350\cos\theta)t = 25 \quad \text{and} \quad -16t^2 + (350\sin\theta)t + 3 = 6.$$

This is not trivial (though not "hard"). We start by solving each equation for $\cos\theta$ and $\sin\theta$, respectively.

$$\cos\theta = \frac{25}{350t} \quad \text{and} \quad \sin\theta = \frac{3 + 16t^2}{350t}.$$

Using the Pythagorean Identity $\cos^2\theta + \sin^2\theta = 1$, we have

$$\left(\frac{25}{350t}\right)^2 + \left(\frac{3 + 16t^2}{350t}\right)^2 = 1$$

Multiply both sides by $(350t)^2$:

$$\begin{aligned} 25^2 + (3 + 16t^2)^2 &= 350^2t^2 \\ 256t^4 - 122,404t^2 + 634 &= 0. \end{aligned}$$

Notes:

This is a quadratic *in* t^2. That is, we can apply the quadratic formula to find t^2, then solve for t itself.

$$\begin{aligned} t^2 &= \frac{122,404 \pm \sqrt{122,404^2 - 4(256)(634)}}{512} \\ t^2 &= 0.0052,\ 478.135 \\ t &= \pm 0.072,\ \pm 21.866 \end{aligned}$$

Clearly the negative t values do not fit our context, so we have $t = 0.072$ and $t = 21.866$. Using $\cos\theta = 25/(350t)$, we can solve for θ:

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{25}{350 \cdot 0.072}\right) \quad \text{and} \quad \cos^{-1}\left(\frac{25}{350 \cdot 21.866}\right) \\ \theta &= 7.03^\circ \quad \text{and} \quad 89.8^\circ. \end{aligned}$$

Alex has two choices of angle. He can hold the rifle at an angle of about 7° with the horizontal and hit his target 0.07s after firing, or he can hold his rifle almost straight up, with an angle of 89.8°, where he'll hit his target about 22s later. The first option is clearly the option he should choose.

Distance Traveled

Consider a driver who sets her cruise–control to 60mph, and travels at this speed for an hour. We can ask:

1. How far did the driver travel?
2. How far from her starting position is the driver?

The first is easy to answer: she traveled 60 miles. The second is impossible to answer with the given information. We do not know if she traveled in a straight line, on an oval racetrack, or along a slowly–winding highway.

This highlights an important fact: to compute distance traveled, we need only to know the speed, given by $||\ \vec{v}(t)\ ||$.

Theorem 12 Distance Traveled

Let $\vec{v}(t)$ be a velocity function for a moving object. The distance traveled by the object on $[a, b]$ is:

$$\text{distance traveled} = \int_a^b ||\ \vec{v}(t)\ ||\ dt.$$

Note that this is just a restatement of Theorem 11: arc length is the same as distance traveled, just viewed in a different context.

Notes:

Example 64 Distance Traveled, Displacement, and Average Speed

A particle moves in space with position function $\vec{r}(t) = \langle t, t^2, \sin(\pi t)\rangle$ on $[-2, 2]$, where t is measured in seconds and distances are in meters. Find:

1. The distance traveled by the particle on $[-2, 2]$.
2. The displacement of the particle on $[-2, 2]$.
3. The particle's average speed.

Figure 10.19: The path of the particle in Example 64.

Solution

1. We use Theorem 12 to establish the integral:

$$\begin{aligned}\text{distance traveled} &= \int_{-2}^{2} \|\,\vec{v}(t)\,\|\,dt \\ &= \int_{-2}^{2} \sqrt{1 + (2t)^2 + \pi^2\cos^2(\pi t)}\,dt.\end{aligned}$$

   This cannot be solved in terms of elementary functions so we turn to numerical integration, finding the distance to be 12.88m.

2. The displacement is the vector

$$\vec{r}(2) - \vec{r}(-2) = \langle 2, 4, 0\rangle - \langle -2, 4, 0\rangle = \langle 4, 0, 0\rangle\,.$$

   That is, the particle ends with an x-value increased by 4 and with y- and z-values the same (see Figure 10.19).

3. We found above that the particle traveled 12.88m over 4 seconds. We can compute average speed by dividing: 12.88/4 = 3.22m/s.

   We should also consider Definition **??** of Section **??**, which says that the average value of a function f on $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$. In our context, the average value of the speed is

$$\text{average speed} = \frac{1}{2-(-2)}\int_{-2}^{2} \|\,\vec{v}(t)\,\|\,dt \approx \frac{1}{4}12.88 = 3.22\text{m/s}.$$

   Note how the physical context of a particle traveling gives meaning to a more abstract concept learned earlier.

In Definition **??** of Chapter **??** we defined the average value of a function $f(x)$ on $[a, b]$ to be

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Notes:

Note how in Example 64 we computed the average speed as

$$\frac{\text{distance traveled}}{\text{travel time}} = \frac{1}{2-(-2)}\int_{-2}^{2} \|\,\vec{v}(t)\,\|\,dt;$$

that is, we just found the average value of $\|\,\vec{v}(t)\,\|$ on $[-2, 2]$.

Likewise, given position function $\vec{r}(t)$, the average velocity on $[a, b]$ is

$$\frac{\text{displacement}}{\text{travel time}} = \frac{1}{b-a}\int_a^b \vec{r}\,'(t)\,dt = \frac{\vec{r}(b)-\vec{r}(a)}{b-a};$$

that is, it is the average value of $\vec{r}\,'(t)$, or $\vec{v}(t)$, on $[a, b]$.

Key Idea 9 Average Speed, Average Velocity

Let $\vec{r}(t)$ be a continuous position function on an open interval I containing $a < b$.

The **average speed** is:

$$\frac{\text{distance traveled}}{\text{travel time}} = \frac{\int_a^b \|\,\vec{v}(t)\,\|\,dt}{b-a} = \frac{1}{b-a}\int_a^b \|\,\vec{v}(t)\,\|\,dt.$$

The **average velocity** is:

$$\frac{\text{displacement}}{\text{travel time}} = \frac{\int_a^b \vec{r}\,'(t)\,dt}{b-a} = \frac{1}{b-a}\int_a^b \vec{r}\,'(t)\,dt.$$

The next two sections investigate more properties of the graphs of vector–valued functions and we'll apply these new ideas to what we just learned about motion.

Notes:

Exercises 10.3

Terms and Concepts

1. How is *velocity* different from *speed*?

2. What is the difference between *displacement* and *distance traveled*?

3. What is the difference between *average velocity* and *average speed*?

4. *Distance traveled* is the same as _____ _____, just viewed in a different context.

5. Describe a scenario where an object's average speed is a large number, but the magnitude of the average velocity is not a large number.

6. Explain why it is not possible to have an average velocity with a large magnitude but a small average speed.

Problems

In Exercises 7 – 10 , a position function $\vec{r}(t)$ is given. Find $\vec{v}(t)$ and $\vec{a}(t)$.

7. $\vec{r}(t) = \langle 2t+1, 5t-2, 7\rangle$

8. $\vec{r}(t) = \left\langle 3t^2-2t+1, -t^2+t+14\right\rangle$

9. $\vec{r}(t) = \langle \cos t, \sin t\rangle$

10. $\vec{r}(t) = \langle t/10, -\cos t, \sin t\rangle$

In Exercises 11 – 14 , a position function $\vec{r}(t)$ is given. Sketch $\vec{r}(t)$ on the indicated interval. Find $\vec{v}(t)$ and $\vec{a}(t)$, then add $\vec{v}(t_0)$ and $\vec{a}(t_0)$ to your sketch, with their initial points at $\vec{r}(t_0)$, for the given value of t_0.

11. $\vec{r}(t) = \langle t, \sin t\rangle$ on $[0, \pi/2]$; $t_0 = \pi/4$

12. $\vec{r}(t) = \left\langle t^2, \sin t^2\right\rangle$ on $[0, \pi/2]$; $t_0 = \sqrt{\pi/4}$

13. $\vec{r}(t) = \left\langle t^2+t, -t^2+2t\right\rangle$ on $[-2, 2]$; $t_0 = 1$

14. $\vec{r}(t) = \left\langle \dfrac{2t+3}{t^2+1}, t^2\right\rangle$ on $[-1, 1]$; $t_0 = 0$

In Exercises 15 – 24 , a position function $\vec{r}(t)$ of an object is given. Find the speed of the object in terms of t, and find where the speed is minimized/maximized on the indicated interval.

15. $\vec{r}(t) = \left\langle t^2, t\right\rangle$ on $[-1, 1]$

16. $\vec{r}(t) = \left\langle t^2, t^2-t^3\right\rangle$ on $[-1, 1]$

17. $\vec{r}(t) = \langle 5\cos t, 5\sin t\rangle$ on $[0, 2\pi]$

18. $\vec{r}(t) = \langle 2\cos t, 5\sin t\rangle$ on $[0, 2\pi]$

19. $\vec{r}(t) = \langle \sec t, \tan t\rangle$ on $[0, \pi/4]$

20. $\vec{r}(t) = \langle t+\cos t, 1-\sin t\rangle$ on $[0, 2\pi]$

21. $\vec{r}(t) = \langle 12t, 5\cos t, 5\sin t\rangle$ on $[0, 4\pi]$

22. $\vec{r}(t) = \left\langle t^2-t, t^2+t, t\right\rangle$ on $[0, 1]$

23. $\vec{r}(t) = \left\langle t, t^2, \sqrt{1-t^2}\right\rangle$ on $[-1, 1]$

24. **Projectile Motion:** $\vec{r}(t) = \left\langle (v_0\cos\theta)t, -\dfrac{1}{2}gt^2+(v_0\sin\theta)t\right\rangle$ on $\left[0, \dfrac{2v_0\sin\theta}{g}\right]$

In Exercises 25 – 28 , position functions $\vec{r}_1(t)$ and $\vec{r}_2(s)$ for two objects are given that follow the same path on the respective intervals.

(a) Show that the positions are the same at the indicated t_0 and s_0 values; i.e., show $\vec{r}_1(t_0) = \vec{r}_2(s_0)$.

(b) Find the velocity, speed and acceleration of the two objects at t_0 and s_0, respectively.

25. $\vec{r}_1(t) = \left\langle t, t^2\right\rangle$ on $[0, 1]$; $t_0 = 1$
$\vec{r}_2(s) = \left\langle s^2, s^4\right\rangle$ on $[0, 1]$; $s_0 = 1$

26. $\vec{r}_1(t) = \langle 3\cos t, 3\sin t\rangle$ on $[0, 2\pi]$; $t_0 = \pi/2$
$\vec{r}_2(s) = \langle 3\cos(4s), 3\sin(4s)\rangle$ on $[0, \pi/2]$; $s_0 = \pi/8$

27. $\vec{r}_1(t) = \langle 3t, 2t\rangle$ on $[0, 2]$; $t_0 = 2$
$\vec{r}_2(s) = \langle 6t-6, 4t-4\rangle$ on $[1, 2]$; $s_0 = 2$

28. $\vec{r}_1(t) = \left\langle t, \sqrt{t}\right\rangle$ on $[0, 1]$; $t_0 = 1$
$\vec{r}_2(s) = \left\langle \sin t, \sqrt{\sin t}\right\rangle$ on $[0, \pi/2]$; $s_0 = \pi/2$

In Exercises 29 – 32 , find the position function of an object given its acceleration and initial velocity and position.

29. $\vec{a}(t) = \langle 2, 3\rangle$; $\vec{v}(0) = \langle 1, 2\rangle$, $\vec{r}(0) = \langle 5, -2\rangle$

30. $\vec{a}(t) = \langle 2, 3\rangle$; $\vec{v}(1) = \langle 1, 2\rangle$, $\vec{r}(1) = \langle 5, -2\rangle$

31. $\vec{a}(t) = \langle \cos t, -\sin t\rangle$; $\vec{v}(0) = \langle 0, 1\rangle$, $\vec{r}(0) = \langle 0, 0\rangle$

32. $\vec{a}(t) = \langle 0, -32\rangle$; $\vec{v}(0) = \langle 10, 50\rangle$, $\vec{r}(0) = \langle 0, 0\rangle$

In Exercises 33 – 36 , find the displacement, distance traveled, average velocity and average speed of the described object on the given interval.

33. An object with position function $\vec{r}(t) = \langle 2\cos t, 2\sin t, 3t\rangle$, where distances are measured in feet and time is in seconds, on $[0, 2\pi]$.

34. An object with position function $\vec{r}(t) = \langle 5\cos t, -5\sin t\rangle$, where distances are measured in feet and time is in seconds, on $[0, \pi]$.

35. An object with velocity function $\vec{v}(t) = \langle \cos t, \sin t\rangle$, where distances are measured in feet and time is in seconds, on $[0, 2\pi]$.

36. An object with velocity function $\vec{v}(t) = \langle 1, 2, -1\rangle$, where distances are measured in feet and time is in seconds, on $[0, 10]$.

Exercises 37 – 42 ask you to solve a variety of problems based on the principles of projectile motion.

37. A boy whirls a ball, attached to a 3ft string, above his head in a counter–clockwise circle. The ball makes 2 revolutions per second.

    At what t-values should the boy release the string so that the ball heads directly for a tree standing 10ft in front of him?

38. David faces Goliath with only a stone in a 3ft sling, which he whirls above his head at 4 revolutions per second. They stand 20ft apart.

    (a) At what t-values must David release the stone in his sling in order to hit Goliath?

    (b) What is the speed at which the stone is traveling when released?

    (c) Assume David releases the stone from a height of 6ft and Goliath's forehead is 9ft above the ground. What angle of elevation must David apply to the stone to hit Goliath's head?

39. A hunter aims at a deer which is 40 yards away. Her crossbow is at a height of 5ft, and she aims for a spot on the deer 4ft above the ground. The crossbow fires her arrows at 300ft/s.

    (a) At what angle of elevation should she hold the crossbow to hit her target?

    (b) If the deer is moving perpendicularly to her line of sight at a rate of 20mph, by approximately how much should she lead the deer in order to hit it in the desired location?

40. A baseball player hits a ball at 100mph, with an initial height of 3ft and an angle of elevation of 20°, at Boston's Fenway Park. The ball flies towards the famed "Green Monster," a wall 37ft high located 310ft from home plate.

    (a) Show that as hit, the ball hits the wall.

    (b) Show that if the angle of elevation is 21°, the ball clears the Green Monster.

41. A Cessna flies at 1000ft at 150mph and drops a box of supplies to the professor (and his wife) on an island. Ignoring wind resistance, how far horizontally will the supplies travel before they land?

42. A football quarterback throws a pass from a height of 6ft, intending to hit his receiver 20yds away at a height of 5ft.

    (a) If the ball is thrown at a rate of 50mph, what angle of elevation is needed to hit his intended target?

    (b) If the ball is thrown at with an angle of elevation of 8°, what initial ball speed is needed to hit his target?

10.4 Unit Tangent and Normal Vectors

Unit Tangent Vector

Given a smooth vector–valued function $\vec{r}(t)$, we defined in Definition 23 that any vector parallel to $\vec{r}\,'(t_0)$ is *tangent* to the graph of $\vec{r}(t)$ at $t = t_0$. It is often useful to consider just the *direction* of $\vec{r}\,'(t)$ and not its magnitude. Therefore we are interested in the unit vector in the direction of $\vec{r}\,'(t)$. This leads to a definition.

Definition 26 Unit Tangent Vector

Let $\vec{r}(t)$ be a smooth function on an open interval I. The unit tangent vector $\vec{T}(t)$ is

$$\vec{T}(t) = \frac{1}{||\,\vec{r}\,'(t)\,||}\vec{r}\,'(t).$$

Example 65 Computing the unit tangent vector

Let $\vec{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$. Find $\vec{T}(t)$ and compute $\vec{T}(0)$ and $\vec{T}(1)$.

Solution We apply Definition 26 to find $\vec{T}(t)$.

$$\begin{aligned}\vec{T}(t) &= \frac{1}{||\,\vec{r}\,'(t)\,||}\vec{r}\,'(t)\\ &= \frac{1}{\sqrt{\big(-3\sin t\big)^2 + \big(3\cos t\big)^2 + 4^2}}\langle -3\sin t, 3\cos t, 4\rangle\\ &= \left\langle -\frac{3}{5}\sin t, \frac{3}{5}\cos t, \frac{4}{5}\right\rangle.\end{aligned}$$

We can now easily compute $\vec{T}(0)$ and $\vec{T}(1)$:

$$\vec{T}(0) = \left\langle 0, \frac{3}{5}, \frac{4}{5}\right\rangle; \quad \vec{T}(1) = \left\langle -\frac{3}{5}\sin 1, \frac{3}{5}\cos 1, \frac{4}{5}\right\rangle \approx \langle -0.505, 0.324, 0.8\rangle.$$

These are plotted in Figure 10.20 with their initial points at $\vec{r}(0)$ and $\vec{r}(1)$, respectively. (They look rather "short" since they are only length 1.)

The unit tangent vector $\vec{T}(t)$ always has a magnitude of 1, though it is sometimes easy to doubt that is true. We can help solidify this thought in our minds by computing $||\,\vec{T}(1)\,||$:

$$||\,\vec{T}(1)\,|| \approx \sqrt{(-0.505)^2 + 0.324^2 + 0.8^2} = 1.000001.$$

Figure 10.20: Plotting unit tangent vectors in Example 65.

Notes:

We have rounded in our computation of $\vec{T}(1)$, so we don't get 1 exactly. We leave it to the reader to use the exact representation of $\vec{T}(1)$ to verify it has length 1.

In many ways, the previous example was "too nice." It turned out that $\vec{r}\,'(t)$ was always of length 5. In the next example the length of $\vec{r}\,'(t)$ is variable, leaving us with a formula that is not as clean.

Figure 10.21: Plotting unit tangent vectors in Example 66.

Example 66 Computing the unit tangent vector

Let $\vec{r}(t) = \langle t^2 - t, t^2 + t\rangle$. Find $\vec{T}(t)$ and compute $\vec{T}(0)$ and $\vec{T}(1)$.

Solution We find $\vec{r}\,'(t) = \langle 2t - 1, 2t + 1\rangle$, and

$$||\,\vec{r}\,'(t)\,|| = \sqrt{(2t-1)^2 + (2t+1)^2} = \sqrt{8t^2 + 2}.$$

Therefore

$$\vec{T}(t) = \frac{1}{\sqrt{8t^2+2}}\langle 2t-1, 2t+1\rangle = \left\langle \frac{2t-1}{\sqrt{8t^2+2}}, \frac{2t+1}{\sqrt{8t^2+2}}\right\rangle.$$

When $t = 0$, we have $\vec{T}(0) = \langle -1/\sqrt{2}, 1/\sqrt{2}\rangle$; when $t = 1$, we have $\vec{T}(1) = \langle 1/\sqrt{10}, 3/\sqrt{10}\rangle$. We leave it to the reader to verify each of these is a unit vector. They are plotted in Figure 10.21

Unit Normal Vector

Figure 10.22: Given a direction in the plane, there are always two directions orthogonal to it.

Just as knowing the direction tangent to a path is important, knowing a direction orthogonal to a path is important. When dealing with real-valued functions, we defined the normal line at a point to the be the line through the point that was perpendicular to the tangent line at that point. We can do a similar thing with vector–valued functions. Given $\vec{r}(t)$ in $\mathbb{R}^2$, we have 2 directions perpendicular to the tangent vector, as shown in Figure 10.22. It is good to wonder "Is one of these two directions preferable over the other?"

Given $\vec{r}(t)$ in $\mathbb{R}^3$, there are infinite vectors orthogonal to the tangent vector at a given point. Again, we might wonder "Is one of these infinite choices preferable over the others? Is one of these the 'right' choice?"

The answer in both $\mathbb{R}^2$ and $\mathbb{R}^3$ is "Yes, there is one vector that is not only preferable, it is the 'right' one to choose." Recall Theorem 9, which states that if $\vec{r}(t)$ has constant length, then $\vec{r}(t)$ is orthogonal to $\vec{r}\,'(t)$ for all t. We know $\vec{T}(t)$, the unit tangent vector, has constant length. Therefore $\vec{T}(t)$ is orthogonal to $\vec{T}\,'(t)$.

Note: $\vec{T}(t)$ is a unit vector, by definition. This *does not* imply that $\vec{T}\,'(t)$ is also a unit vector.

We'll see that $\vec{T}\,'(t)$ is more than just a convenient choice of vector that is orthogonal to $\vec{r}\,'(t)$; rather, it is the "right" choice. Since all we care about is the direction, we define this newly found vector to be a unit vector.

Notes:

Definition 27 Unit Normal Vector

Let $\vec{r}(t)$ be a vector–valued function where the unit tangent vector, $\vec{T}(t)$, is smooth on an open interval I. The **unit normal vector** $\vec{N}(t)$ is

$$\vec{N}(t) = \frac{1}{\|\vec{T}'(t)\|}\vec{T}'(t).$$

Example 67 Computing the unit normal vector

Let $\vec{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$ as in Example 65. Sketch both $\vec{T}(\pi/2)$ and $\vec{N}(\pi/2)$ with initial points at $\vec{r}(\pi/2)$.

Solution In Example 65, we found $\vec{T}(t) = \left\langle (-3/5)\sin t, (3/5)\cos t, 4/5\right\rangle$. Therefore

$$\vec{T}'(t) = \left\langle -\frac{3}{5}\cos t, -\frac{3}{5}\sin t, 0\right\rangle \quad \text{and} \quad \|\vec{T}'(t)\| = \frac{3}{5}.$$

Thus

$$\vec{N}(t) = \frac{\vec{T}'(t)}{3/5} = \langle -\cos t, -\sin t, 0\rangle.$$

We compute $\vec{T}(\pi/2) = \langle -3/5, 0, 4/5\rangle$ and $\vec{N}(\pi/2) = \langle 0, -1, 0\rangle$. These are sketched in Figure 10.23.

Figure 10.23: Plotting unit tangent and normal vectors in Example 10.23.

The previous example was once again "too nice." In general, the expression for $\vec{T}(t)$ contains fractions of square–roots, hence the expression of $\vec{T}'(t)$ is very messy. We demonstrate this in the next example.

Example 68 Computing the unit normal vector

Let $\vec{r}(t) = \left\langle t^2 - t, t^2 + t\right\rangle$ as in Example 66. Find $\vec{N}(t)$ and sketch $\vec{r}(t)$ with the unit tangent and normal vectors at $t = -1, 0$ and 1.

Solution In Example 66, we found

$$\vec{T}(t) = \left\langle \frac{2t-1}{\sqrt{8t^2+2}}, \frac{2t+1}{\sqrt{8t^2+2}}\right\rangle.$$

Finding $\vec{T}'(t)$ requires two applications of the Quotient Rule:

Notes:

$$
\begin{aligned}
T'(t) &= \left\langle \frac{\sqrt{8t^2+2}(2) - (2t-1)\left(\frac{1}{2}(8t^2+2)^{-1/2}(16t)\right)}{8t^2+2}, \right. \\
&\qquad \left. \frac{\sqrt{8t^2+2}(2) - (2t+1)\left(\frac{1}{2}(8t^2+2)^{-1/2}(16t)\right)}{8t^2+2} \right\rangle \\
&= \left\langle \frac{4(2t+1)}{(8t^2+2)^{3/2}}, \frac{4(1-2t)}{(8t^2+2)^{3/2}} \right\rangle
\end{aligned}
$$

This is not a unit vector; to find $\vec{N}(t)$, we need to divide $\vec{T}'(t)$ by it's magnitude.

$$
\begin{aligned}
\|\,\vec{T}'(t)\,\| &= \sqrt{\frac{16(2t+1)^2}{(8t^2+2)^3} + \frac{16(1-2t)^2}{(8t^2+2)^3}} \\
&= \sqrt{\frac{16(8t^2+2)}{(8t^2+2)^3}} \\
&= \frac{4}{8t^2+2}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\vec{N}(t) &= \frac{1}{4/(8t^2+2)} \left\langle \frac{4(2t+1)}{(8t^2+2)^{3/2}}, \frac{4(1-2t)}{(8t^2+2)^{3/2}} \right\rangle \\
&= \left\langle \frac{2t+1}{\sqrt{8t^2+2}}, -\frac{2t-1}{\sqrt{8t^2+2}} \right\rangle.
\end{aligned}
$$

Using this formula for $\vec{N}(t)$, we compute the unit tangent and normal vectors for $t = -1, 0$ and 1 and sketch them in Figure 10.24.

Figure 10.24: Plotting unit tangent and normal vectors in Example 68.

The final result for $\vec{N}(t)$ in Example 68 is suspiciously similar to $\vec{T}(t)$. There is a clear reason for this. If $\vec{u} = \langle u_1, u_2 \rangle$ is a unit vector in $\mathbb{R}^2$, then the *only* unit vectors orthogonal to $\vec{u}$ are $\langle -u_2, u_1 \rangle$ and $\langle u_2, -u_1 \rangle$. Given $\vec{T}(t)$, we can quickly determine $\vec{N}(t)$ if we know which term to multiply by (-1).

Consider again Figure 10.24, where we have plotted some unit tangent and normal vectors. Note how $\vec{N}(t)$ always points "inside" the curve, or to the concave side of the curve. This is not a coincidence; this is true in general. Knowing the direction that $\vec{r}(t)$ "turns" allows us to quickly find $\vec{N}(t)$.

Notes:

Theorem 13 Unit Normal Vectors in $\mathbb{R}^2$

Let $\vec{r}(t)$ be a vector–valued function in $\mathbb{R}^2$ where $\vec{T}'(t)$ is smooth on an open interval I. Let t_0 be in I and $\vec{T}(t_0) = \langle t_1, t_2 \rangle$ Then $\vec{N}(t_0)$ is either

$$\vec{N}(t_0) = \langle -t_2, t_1 \rangle \quad \text{or} \quad \vec{N}(t_0) = \langle t_2, -t_1 \rangle\,,$$

whichever is the vector that points to the concave side of the graph of $\vec{r}$.

Application to Acceleration

Let $\vec{r}(t)$ be a position function. It is a fact (stated later in Theorem 14) that acceleration, $\vec{a}(t)$, lies in the plane defined by $\vec{T}$ and $\vec{N}$. That is, there are scalars a_T and a_N such that

$$\vec{a}(t) = a_\mathrm{T}\vec{T}(t) + a_\mathrm{N}\vec{N}(t).$$

The scalar a_T measures "how much" acceleration is in the direction of travel, that is, it measures the component of acceleration that affects the speed. The scalar a_N measures "how much" acceleration is perpendicular to the direction of travel, that is, it measures the component of acceleration that affects the direction of travel.

We can find a_T using the orthogonal projection of $\vec{a}(t)$ onto $\vec{T}(t)$ (review Definition 11 in Section 9.3 if needed). Recalling that since $\vec{T}(t)$ is a unit vector, $\vec{T}(t) \cdot \vec{T}(t) = 1$, so we have

$$\text{proj}_{\vec{T}(t)}\,\vec{a}(t) = \frac{\vec{a}(t)\cdot\vec{T}(t)}{\vec{T}(t)\cdot\vec{T}(t)}\vec{T}(t) = \underbrace{\left(\vec{a}(t)\cdot\vec{T}(t)\right)}_{a_\mathrm{T}}\vec{T}(t).$$

Note: Keep in mind that both a_T and a_N are functions of t; that is, the scalar changes depending on t. It is convention to drop the "(t)" notation from $a_\mathrm{T}(t)$ and simply write a_T.

Thus the amount of $\vec{a}(t)$ in the direction of $\vec{T}(t)$ is $a_\mathrm{T} = \vec{a}(t)\cdot\vec{T}(t)$. The same logic gives $a_\mathrm{N} = \vec{a}(t)\cdot\vec{N}(t)$.

While this is a fine way of computing a_T, there are simpler ways of finding a_N (as finding $\vec{N}$ itself can be complicated). The following theorem gives alternate formulas for a_T and a_N.

Notes:

Theorem 14 Acceleration in the Plane Defined by $\vec{T}$ and $\vec{N}$

Let $\vec{r}(t)$ be a position function with acceleration $\vec{a}(t)$ and unit tangent and normal vectors $\vec{T}(t)$ and $\vec{N}(t)$. Then $\vec{a}(t)$ lies in the plane defined by $\vec{T}(t)$ and $\vec{N}(t)$; that is, there exists scalars a_{T} and a_{N} such that

$$\vec{a}(t) = a_{\mathrm{T}}\vec{T}(t) + a_{\mathrm{N}}\vec{N}(t).$$

Moreover,

$$a_{\mathrm{T}} = \vec{a}(t)\cdot\vec{T}(t) = \frac{d}{dt}\Big(||\,\vec{v}(t)\,||\Big)$$
$$a_{\mathrm{N}} = \vec{a}(t)\cdot\vec{N}(t) = \sqrt{||\,\vec{a}(t)\,||^2 - a_{\mathrm{T}}^2} = \frac{||\,\vec{a}(t)\times\vec{v}(t)\,||}{||\,\vec{v}(t)\,||} = ||\,\vec{v}(t)\,||\,||\,\vec{T}'(t)\,||$$

Note the second formula for a_{T}: $\dfrac{d}{dt}\Big(||\,\vec{v}(t)\,||\Big)$. This measures the rate of change of speed, which again is the amount of acceleration in the direction of travel.

Example 69 Computing a_{T} and a_{N}

Let $\vec{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$ as in Examples 65 and 67. Find a_{T} and a_{N}.

Solution The previous examples give $\vec{a}(t) = \langle -3\cos t, -3\sin t, 0\rangle$ and

$$\vec{T}(t) = \left\langle -\frac{3}{5}\sin t, \frac{3}{5}\cos t, \frac{4}{5}\right\rangle \quad\text{and}\quad \vec{N}(t) = \langle -\cos t, -\sin t, 0\rangle\,.$$

We can find a_{T} and a_{N} directly with dot products:

$$a_{\mathrm{T}} = \vec{a}(t)\cdot\vec{T}(t) = \frac{9}{5}\cos t\sin t - \frac{9}{5}\cos t\sin t + 0 = 0.$$
$$a_{\mathrm{N}} = \vec{a}(t)\cdot\vec{N}(t) = 3\cos^2 t + 3\sin^2 t + 0 = 3.$$

Thus $\vec{a}(t) = 0\vec{T}(t) + 3\vec{N}(t) = 3\vec{N}(t)$, which is clearly the case.

What is the practical interpretation of these numbers? $a_{\mathrm{T}} = 0$ means the object is moving at a constant speed, and hence all acceleration comes in the form of direction change.

Example 70 Computing a_{T} and a_{N}

Let $\vec{r}(t) = \left\langle t^2 - t, t^2 + t\right\rangle$ as in Examples 66 and 68. Find a_{T} and a_{N}.

Notes:

Solution The previous examples give $\vec{a}(t) = \langle 2, 2\rangle$ and

$$\vec{T}(t) = \left\langle \frac{2t-1}{\sqrt{8t^2+2}}, \frac{2t+1}{\sqrt{8t^2+2}} \right\rangle \quad \text{and} \quad \vec{N}(t) = \left\langle \frac{2t+1}{\sqrt{8t^2+2}}, -\frac{2t-1}{\sqrt{8t^2+2}} \right\rangle.$$

While we can compute a_{N} using $\vec{N}(t)$, we instead demonstrate using another formula from Theorem 14.

$$a_{\mathrm{T}} = \vec{a}(t) \cdot \vec{T}(t) = \frac{4t-2}{\sqrt{8t^2+2}} + \frac{4t+2}{\sqrt{8t^2+2}} = \frac{8t}{\sqrt{8t^2+2}}.$$

$$a_{\mathrm{N}} = \sqrt{\|\,\vec{a}(t)\,\|^2 - a_{\mathrm{T}}^2} = \sqrt{8 - \left(\frac{8t}{\sqrt{8t^2+2}}\right)^2} = \frac{4}{\sqrt{8t^2+2}}.$$

When $t = 2$, $a_{\mathrm{T}} = \dfrac{16}{\sqrt{34}} \approx 2.74$ and $a_{\mathrm{N}} = \dfrac{4}{\sqrt{34}} \approx 0.69$. We interpret this to mean that at $t = 2$, the particle is acclerating mostly by increasing speed, not by changing direction. As the path near $t = 2$ is relatively straight, this should make intuitive sense. Figure 10.25 gives a graph of the path for reference.

Contrast this with $t = 0$, where $a_{\mathrm{T}} = 0$ and $a_{\mathrm{N}} = 4/\sqrt{2} \approx 2.82$. Here the particle's speed is not changing and all acceleration is in the form of direction change.

Figure 10.25: Graphing $\vec{r}(t)$ in Example 70.

Example 71 **Analyzing projectile motion**

A ball is thrown from a height of 240ft with an initial speed of 64ft/s and an angle of elevation of 30°. Find the position function $\vec{r}(t)$ of the ball and analyze a_{T} and a_{N}.

Solution Using Key Idea 8 of Section 10.3 we form the position function of the ball:

$$\vec{r}(t) = \left\langle \left(64\cos 30^\circ\right)t, -16t^2 + \left(64\sin 30^\circ\right)t + 240 \right\rangle,$$

which we plot in Figure 10.26.

From this we find $\vec{v}(t) = \langle 64\cos 30^\circ, -32t + 64\sin 30^\circ\rangle$ and $\vec{a}(t) = \langle 0, -32\rangle$. Computing $\vec{T}(t)$ is not difficult, and with some simplification we find

$$\vec{T}(t) = \left\langle \frac{\sqrt{3}}{\sqrt{t^2-2t+4}}, \frac{1-t}{\sqrt{t^2-2t+4}} \right\rangle.$$

With $\vec{a}(t)$ as simple as it is, finding a_{T} is also simple:

$$a_{\mathrm{T}} = \vec{a}(t) \cdot \vec{T}(t) = \frac{32t-32}{\sqrt{t^2-2t+4}}.$$

Figure 10.26: Plotting the position of a thrown ball, with 1s increments shown.

Notes:

We choose to not find $\vec{N}(t)$ and find a_N through the formula $a_N = \sqrt{||\,\vec{a}(t)\,||^2 - a_T^2}$:

$$a_N = \sqrt{32^2 - \left(\frac{32t - 32}{\sqrt{t^2 - 2t + 4}}\right)^2} = \frac{32\sqrt{3}}{\sqrt{t^2 - 2t + 4}}.$$

Figure 10.27 gives a table of values of a_T and a_N. When $t = 0$, we see the ball's speed is decreasing; when $t = 1$ the speed of the ball is unchanged. This corresponds to the fact that at $t = 1$ the ball reaches its highest point.

After $t = 1$ we see that a_N is decreasing in value. This is because as the ball falls, it's path becomes straighter and most of the acceleration is in the form of speeding up the ball, and not in changing its direction.

t	a_T	a_N
0	-16	27.7
1	0	32
2	16	27.7
3	24.2	20.9
4	27.7	16
5	29.4	12.7

Figure 10.27: A table of values of a_T and a_N in Example 71.

Our understanding of the unit tangent and normal vectors is aiding our understanding of motion. The work in Example 71 gave quantitative analysis of what we intuitively knew.

The next section provides two more important steps towards this analysis. We currently describe position only in terms of time. In everyday life, though, we often describe position in terms of distance ("The gas station is about 2 miles ahead, on the left."). The *arc length parameter* allows us to reference position in terms of distance traveled.

We also intuitively know that some paths are straighter than others – and some are curvier than others, but we lack a measurement of "curviness." The arc length parameter provides a way for us to compute *curvature*, a quantitative measurement of how curvy a curve is.

Notes:

Exercises 10.4

Terms and Concepts

1. If $\vec{T}(t)$ is a unit tangent vector, what is $\|\vec{T}(t)\|$?

2. If $\vec{N}(t)$ is a unit normal vector, what is $\vec{N}(t)\cdot\vec{r}\,'(t)$?

3. The acceleration vector $\vec{a}(t)$ lies in the plane defined by what two vectors?

4. a_{T} measures how much the acceleration is affecting the _____ of an object.

Problems

In Exercises 5 – 8 , given $\vec{r}(t)$, find $\vec{T}(t)$ and evaluate it at the indicated value of t.

5. $\vec{r}(t) = \left\langle 2t^2, t^2 - t\right\rangle, \quad t = 1$

6. $\vec{r}(t) = \langle t, \cos t\rangle, \quad t = \pi/4$

7. $\vec{r}(t) = \left\langle \cos^3 t, \sin^3 t\right\rangle, \quad t = \pi/4$

8. $\vec{r}(t) = \langle \cos t, \sin t\rangle, \quad t = \pi$

In Exercises 9 – 12 , find the equation of the line tangent to the curve at the indicated t-value using the unit tangent vector. Note: these are the same problems as in Exercises 5 – 8.

9. $\vec{r}(t) = \left\langle 2t^2, t^2 - t\right\rangle, \quad t = 1$

10. $\vec{r}(t) = \langle t, \cos t\rangle, \quad t = \pi/4$

11. $\vec{r}(t) = \left\langle \cos^3 t, \sin^3 t\right\rangle, \quad t = \pi/4$

12. $\vec{r}(t) = \langle \cos t, \sin t\rangle, \quad t = \pi$

In Exercises 13 – 16 , find $\vec{N}(t)$ using Definition 27. Confirm the result using Theorem 13.

13. $\vec{r}(t) = \langle 3\cos t, 3\sin t\rangle$

14. $\vec{r}(t) = \left\langle t, t^2\right\rangle$

15. $\vec{r}(t) = \langle \cos t, 2\sin t\rangle$

16. $\vec{r}(t) = \left\langle e^t, e^{-t}\right\rangle$

In Exercises 17 – 20 , a position function $\vec{r}(t)$ is given along with its unit tangent vector $\vec{T}(t)$ evaluated at $t = a$, for some value of a.

(a) Confirm that $\vec{T}(a)$ is as stated.

(b) Using a graph of $\vec{r}(t)$ and Theorem 13, find $\vec{N}(a)$.

17. $\vec{r}(t) = \langle 3\cos t, 5\sin t\rangle; \quad \vec{T}(\pi/4) = \left\langle -\dfrac{3}{\sqrt{34}}, \dfrac{5}{\sqrt{34}}\right\rangle.$

18. $\vec{r}(t) = \left\langle t, \dfrac{1}{t^2+1}\right\rangle; \quad \vec{T}(1) = \left\langle \dfrac{2}{\sqrt{5}}, -\dfrac{1}{\sqrt{5}}\right\rangle.$

19. $\vec{r}(t) = (1 + 2\sin t)\,\langle \cos t, \sin t\rangle; \quad \vec{T}(0) = \left\langle \dfrac{2}{\sqrt{5}}, \dfrac{1}{\sqrt{5}}\right\rangle.$

20. $\vec{r}(t) = \left\langle \cos^3 t, \sin^3 t\right\rangle; \quad \vec{T}(\pi/4) = \left\langle -\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right\rangle.$

In Exercises 21 – 24 , find $\vec{N}(t)$.

21. $\vec{r}(t) = \langle 4t, 2\sin t, 2\cos t\rangle$

22. $\vec{r}(t) = \langle 5\cos t, 3\sin t, 4\sin t\rangle$

23. $\vec{r}(t) = \langle a\cos t, a\sin t, bt\rangle; \quad a > 0$

24. $\vec{r}(t) = \langle \cos(at), \sin(at), t\rangle$

In Exercises 25 – 30 , find a_{T} and a_{N} given $\vec{r}(t)$. Sketch $\vec{r}(t)$ on the indicated interval, and comment on the relative sizes of a_{T} and a_{N} at the indicated t values.

25. $\vec{r}(t) = \left\langle t, t^2\right\rangle$ on $[-1, 1]$; consider $t = 0$ and $t = 1$.

26. $\vec{r}(t) = \langle t, 1/t\rangle$ on $(0, 4]$; consider $t = 1$ and $t = 2$.

27. $\vec{r}(t) = \langle 2\cos t, 2\sin t\rangle$ on $[0, 2\pi]$; consider $t = 0$ and $t = \pi/2$.

28. $\vec{r}(t) = \left\langle \cos(t^2), \sin(t^2)\right\rangle$ on $(0, 2\pi]$; consider $t = \sqrt{\pi/2}$ and $t = \sqrt{\pi}$.

29. $\vec{r}(t) = \langle a\cos t, a\sin t, bt\rangle$ on $[0, 2\pi]$, where $a, b > 0$; consider $t = 0$ and $t = \pi/2$.

30. $\vec{r}(t) = \langle 5\cos t, 4\sin t, 3\sin t\rangle$ on $[0, 2\pi]$; consider $t = 0$ and $t = \pi/2$.

10.5 The Arc Length Parameter and Curvature

Figure 10.28: Introducing the arc length parameter.

In normal conversation we describe position in terms of both *time* and *distance*. For instance, imagine driving to visit a friend. If she calls and asks where you are, you might answer "I am 20 minutes from your house," or you might say "I am 10 miles from your house." Both answers provide your friend with a general idea of where you are.

Currently, our vector–valued functions have defined points with a parameter t, which we often take to represent time. Consider Figure 10.28(a), where $\vec{r}(t) = \langle t^2 - t, t^2 + t\rangle$ is graphed and the points corresponding to $t = 0,\ 1$ and 2 are shown. Note how the arc length between $t = 0$ and $t = 1$ is smaller than the arc length between $t = 1$ and $t = 2$; if the parameter t is time and $\vec{r}$ is position, we can say that the particle traveled faster on $[1, 2]$ than on $[0, 1]$.

Now consider Figure 10.28(b), where the same graph is parametrized by a different variable s. Points corresponding to $s = 0$ through $s = 6$ are plotted. The arc length of the graph between each adjacent pair of points is 1. We can view this parameter s as distance; that is, the arc length of the graph from $s = 0$ to $s = 3$ is 3, the arc length from $s = 2$ to $s = 6$ is 4, etc. If one wants to find the point 2.5 units from an initial location (i.e., $s = 0$), one would compute $\vec{r}(2.5)$. This parameter s is very useful, and is called the **arc length parameter**.

How do we find the arc length parameter?

Start with any parametrization of $\vec{r}$. We can compute the arc length of the graph of $\vec{r}$ on the interval $[0, t]$ with

$$\text{arc length } = \int_0^t ||\,\vec{r}\,'(u)\,||\,du.$$

We can turn this into a function: as t varies, we find the arc length s from 0 to t. This function is

$$s(t) = \int_0^t ||\,\vec{r}\,'(u)\,||\,du. \tag{10.1}$$

This establishes a relationship between s and t. Knowing this relationship explicitly, we can rewrite $\vec{r}(t)$ as a function of s: $\vec{r}(s)$. We demonstrate this in an example.

Example 72 Finding the arc length parameter

Let $\vec{r}(t) = \langle 3t - 1, 4t + 2\rangle$. Parametrize $\vec{r}$ with the arc length parameter s.

Solution Using Equation (10.1), we write

$$s(t) = \int_0^t ||\,\vec{r}\,'(u)\,||\,du.$$

Notes:

We can integrate this, explicitly finding a relationship between s and t:

$$\begin{aligned} s(t) &= \int_0^t \,||\,\vec{r}\,'(u)\,||\,du \\ &= \int_0^t \sqrt{3^2+4^2}\,du \\ &= \int_0^t 5\,du \\ &= 5t. \end{aligned}$$

Since $s = 5t$, we can write $t = s/5$ and replace t in $\vec{r}(t)$ with $s/5$:

$$\vec{r}(s) = \langle 3(s/5)-1, 4(s/5)+2\rangle = \left\langle \frac{3}{5}s - 1, \frac{4}{5}s + 2\right\rangle.$$

Clearly, as shown in Figure 10.29, the graph of $\vec{r}$ is a line, where $t = 0$ corresponds to the point $(-1, 2)$. What point on the line is 2 units away from this initial point? We find it with $s(2) = \langle 1/5, 18/5\rangle$.

Is the point $(1/5, 18/5)$ really 2 units away from $(-1, 2)$? We use the Distance Formula to check:

$$d = \sqrt{\left(\frac{1}{5}-(-1)\right)^2 + \left(\frac{18}{5}-2\right)^2} = \sqrt{\frac{36}{25}+\frac{64}{25}} = \sqrt{4} = 2.$$

Yes, $s(2)$ is indeed 2 units away, in the direction of travel, from the initial point.

Figure 10.29: Graphing $\vec{r}$ in Example 72 with parameters t and s.

Things worked out very nicely in Example 72; we were able to establish directly that $s = 5t$. Usually, the arc length parameter is much more difficult to describe in terms of t, a result of integrating a square–root. There are a number of things that we can learn about the arc length parameter from Equation (10.1), though, that are incredibly useful.

First, take the derivative of s with respect to t. The Fundamental Theorem of Calculus (see Theorem **??**) states that

$$\frac{ds}{dt} = s\,'(t) = ||\,\vec{r}\,'(t)\,||. \tag{10.2}$$

Letting t represent time and $\vec{r}(t)$ represent position, we see that the rate of change of s with respect to t is speed; that is, the rate of change of "distance traveled" is speed, which should match our intuition.

The Chain Rule states that

$$\begin{aligned} \frac{d\vec{r}}{dt} &= \frac{d\vec{r}}{ds}\cdot\frac{ds}{dt} \\ \vec{r}\,'(t) &= \vec{r}\,'(s)\cdot ||\,\vec{r}\,'(t)\,||. \end{aligned}$$

Notes:

Solving for $\vec{r}\,'(s)$, we have

$$\vec{r}\,'(s) = \frac{\vec{r}\,'(t)}{||\,\vec{r}\,'(t)\,||} = \vec{T}(t), \tag{10.3}$$

where $\vec{T}(t)$ is the unit tangent vector. Equation 10.3 is often misinterpreted, as one is tempted to think it states $\vec{r}\,'(t) = \vec{T}(t)$, but there is a big difference between $\vec{r}\,'(s)$ and $\vec{r}\,'(t)$. The key to take from it is that $\vec{r}\,'(s)$ is a unit vector. In fact, the following theorem states that this characterizes the arc length parameter.

Theorem 15 Arc Length Parameter

Let $\vec{r}(s)$ be a vector–valued function. The parameter s is the arc length parameter if, and only if, $||\,\vec{r}\,'(s)\,|| = 1$.

(a)

(b)

Figure 10.30: Establishing the concept of curvature.

Curvature

Consider points A and B on the curve graphed in Figure 10.30(a). One can readily argue that the curve curves more sharply at A than at B. It is useful to use a number to describe how sharply the curve bends; that number is the **curvature** of the curve.

We derive this number in the following way. Consider Figure 10.30(b), where unit tangent vectors are graphed around points A and B. Notice how the direction of the unit tangent vector changes quite a bit near A, whereas it does not change as much around B. This leads to an important concept: measuring the rate of change of the unit tangent vector with respect to arc length gives us a measurement of curvature.

Definition 28 Curvature

Let $\vec{r}(s)$ be a vector–valued function where s is the arc length parameter. The curvature κ of the graph of $\vec{r}(s)$ is

$$\kappa = \left|\left|\frac{d\vec{T}}{ds}\right|\right| = ||\,\vec{T}\,'(s)\,||\,.$$

If $\vec{r}(s)$ is parametrized by the arc length parameter, then

$$\vec{T}(s) = \frac{\vec{r}\,'(s)}{||\,\vec{r}\,'(s)\,||} \quad \text{and} \quad \vec{N}(s) = \frac{\vec{T}\,'(s)}{||\,\vec{T}\,'(s)\,||}.$$

Notes:

Having defined $|| \vec{T}'(s) || = \kappa$, we can rewrite the second equation as

$$\vec{T}'(s) = \kappa \vec{N}(s). \tag{10.4}$$

We already knew that $\vec{T}'(s)$ is in the same direction as $\vec{N}(s)$; that is, we can think of $\vec{T}(s)$ as being "pulled" in the direction of $\vec{N}(s)$. How "hard" is it being pulled? By a factor of κ. When the curvature is large, $\vec{T}(s)$ is being "pulled hard" and the direction of $\vec{T}(s)$ changes rapidly. When κ is small, $T(s)$ is not being pulled hard and hence its direction is not changing rapidly.

We use Definition 28 to find the curvature of the line in Example 72.

Example 73 Finding the curvature of a line

Use Definition 28 to find the curvature of $\vec{r}(t) = \langle 3t - 1, 4t + 2 \rangle$.

Solution In Example 72, we found that the arc length parameter was defined by $s = 5t$, so $\vec{r}(s) = \langle 3t/5 - 1, 4t/5 + 2 \rangle$ parametrized $\vec{r}$ with the arc length parameter. To find κ, we need to find $\vec{T}'(s)$.

$$\begin{aligned} \vec{T}(s) &= \vec{r}'(s) \quad \text{(recall this is a unit vector)} \\ &= \langle 3/5, 4/5 \rangle . \end{aligned}$$

Therefore

$$\vec{T}'(s) = \langle 0, 0 \rangle$$

and

$$\kappa = \left|\left| \vec{T}'(s) \right|\right| = 0.$$

It probably comes as no surprise that the curvature of a line is 0. (How "curvy" is a line? It is not curvy at all.)

While the definition of curvature is a beautiful mathematical concept, it is nearly impossible to use most of the time; writing $\vec{r}$ in terms of the arc length parameter is generally very hard. Fortunately, there are other methods of calculating this value that are much easier. There is a tradeoff: the definition is "easy" to understand though hard to compute, whereas these other formulas are easy to compute though it may be hard to understand why they work.

Notes:

Theorem 16 Formulas for Curvature

Let C be a smooth curve on an open interval I in the plane or in space.

1. If C is defined by $y = f(x)$, then
$$\kappa = \frac{|f''(x)|}{\left(1 + \left(f'(x)\right)^2\right)^{3/2}}.$$
2. If C is defined as a vector–valued function in the plane, $\vec{r}(t) = \langle x(t), y(t)\rangle$, then
$$\kappa = \frac{|x'y'' - x''y'|}{\left((x')^2 + (y')^2\right)^{3/2}}.$$
3. If C is defined in space by a vector–valued function $\vec{r}(t)$, then
$$\kappa = \frac{||\,\vec{T}'(t)\,||}{||\,\vec{r}'(t)\,||} = \frac{||\,\vec{r}'(t) \times \vec{r}''(t)\,||}{||\,\vec{r}'(t)\,||^3} = \frac{\vec{a}(t) \cdot \vec{N}(t)}{||\,\vec{v}(t)\,||^2}.$$

We practice using these formulas.

Example 74 Finding the curvature of a circle

Find the curvature of a circle with radius r, defined by $\vec{c}(t) = \langle r\cos t, r\sin t\rangle$.

Solution Before we start, we should expect the curvature of a circle to be constant, and not dependent on t. (Why?)

We compute κ using the second part of Theorem 16.

$$\begin{aligned}\kappa &= \frac{|(-r\sin t)(-r\sin t) - (-r\cos t)(r\cos t)|}{\left((-r\sin t)^2 + (r\cos t)^2\right)^{3/2}}\\ &= \frac{r^2(\sin^2 t + \cos^2 t)}{\left(r^2(\sin^2 t + \cos^2 t)\right)^{3/2}}\\ &= \frac{r^2}{r^3} = \frac{1}{r}.\end{aligned}$$

We have found that a circle with radius r has curvature $\kappa = 1/r$.

Example 74 gives a great result. Before this example, if we were told "The

Notes:

curve has a curvature of 5 at point A," we would have no idea what this really meant. Is 5 "big" – does is correspond to a really sharp turn, or a not-so-sharp turn? Now we can think of 5 in terms of a circle with radius 1/5. Knowing the units (inches vs. miles, for instance) allows us to determine how sharply the curve is curving.

Let a point P on a smooth curve C be given, and let κ be the curvature of the curve at P. A circle that:

- passes through P,
- lies on the concave side of C,
- has a common tangent line as C at P and
- has radius $r = 1/\kappa$ (hence has curvature κ)

is the **osculating circle**, or **circle of curvature**, to C at P, and r is the **radius of curvature**. Figure 10.31 shows the graph of the curve seen earlier in Figure 10.30 and its osculating circles at A and B. A sharp turn corresponds to a circle with a small radius; a gradual turn corresponds to a circle with a large radius. Being able to think of curvature in terms of the radius of a circle is very useful. (The word "osculating" comes from a Latin word related to kissing; an osculating circle "kisses" the graph at a particular point. Many beautiful ideas in mathematics have come from studying the osculating circles to a curve.)

Figure 10.31: Illustrating the osculating circles for the curve seen in Figure 10.30.

Example 75 Finding curvature

Find the curvature of the parabola defined by $y = x^2$ at the vertex and at $x = 1$.

Solution We use the first formula found in Theorem 16.

$$\begin{aligned}\kappa(x) &= \frac{|2|}{\left(1+(2x)^2\right)^{3/2}}\\ &= \frac{2}{\left(1+4x^2\right)^{3/2}}.\end{aligned}$$

At the vertex ($x = 0$), the curvature is $\kappa = 2$. At $x = 1$, the curvature is $\kappa = 2/(5)^{3/2} \approx 0.179$. So at $x = 0$, the curvature of $y = x^2$ is that of a circle of radius $1/2$; at $x = 1$, the curvature is that of a circle with radius $\approx 1/0.179 \approx 5.59$. This is illustrated in Figure 10.32. At $x = 3$, the curvature is 0.009; the graph is nearly straight as the curvature is very close to 0.

Figure 10.32: Examining the curvature of $y = x^2$.

Example 76 Finding curvature

Find where the curvature of $\vec{r}(t) = \langle t, t^2, 2t^3\rangle$ is maximized.

Notes:

(a)

(b)

Figure 10.33: Understanding the curvature of a curve in space.

Solution We use the third formula in Theorem 16 as $\vec{r}(t)$ is defined in space. We leave it to the reader to verify that

$$\vec{r}\,'(t) = \left\langle 1, 2t, 6t^2 \right\rangle, \quad \vec{r}\,''(t) = \langle 0, 2, 12t \rangle, \quad \text{and} \quad \vec{r}\,'(t) \times \vec{r}\,''(t) = \left\langle 12t^2, -12t, 2 \right\rangle.$$

Thus

$$\begin{aligned}\kappa(t) &= \frac{||\,\vec{r}\,'(t) \times \vec{r}\,''(t)\,||}{||\,\vec{r}\,'(t)\,||^3} \\ &= \frac{||\,\left\langle 12t^2, -12t, 2 \right\rangle\,||}{||\,\langle 1, 2t, 6t^2 \rangle\,||^3} \\ &= \frac{\sqrt{144t^4 + 144t^2 + 4}}{\left(\sqrt{1 + 4t^2 + 36t^4}\right)^3}\end{aligned}$$

While this is not a particularly "nice" formula, it does explictly tell us what the curvature is at a given t value. To maximize $\kappa(t)$, we should solve $\kappa'(t) = 0$ for t. This is doable, but *very* time consuming. Instead, consider the graph of $\kappa(t)$ as given in Figure 10.33(a). We see that κ is maximized at two t values; using a numerical solver, we find these values are $t \approx \pm 0.189$. In part (b) of the figure we graph $\vec{r}(t)$ and indicate the points where curvature is maximized.

Curvature and Motion

Let $\vec{r}(t)$ be a position function of an object, with velocity $\vec{v}(t) = \vec{r}\,'(t)$ and acceleration $\vec{a}(t) = \vec{r}\,''(t)$. In Section 10.4 we established that acceleration is in the plane formed by $\vec{T}(t)$ and $\vec{N}(t)$, and that we can find scalars a_{T} and a_{N} such that

$$\vec{a}(t) = a_{\mathrm{T}}\vec{T}(t) + a_{\mathrm{N}}\vec{N}(t).$$

Theorem 14 gives formulas for a_{T} and a_{N}:

$$a_{\mathrm{T}} = \frac{d}{dt}\Big(||\,\vec{v}(t)\,||\Big) \quad \text{and} \quad a_{\mathrm{N}} = \frac{||\,\vec{v}(t) \times \vec{a}(t)\,||}{||\,\vec{v}(t)\,||}.$$

We understood that the amount of acceleration in the direction of $\vec{T}$ relates only to how the speed of the object is changing, and that the amount of acceleration in the direction of $\vec{N}$ relates to how the direction of travel of the object is changing. (That is, if the object travels at constant speed, $a_{\mathrm{T}} = 0$; if the object travels in a constant direction, $a_{\mathrm{N}} = 0$.)

In Equation (10.2) at the beginning of this section, we found $s'(t) = ||\,\vec{v}(t)\,||$. We can combine this fact with the above formula for a_{T} to write

$$a_{\mathrm{T}} = \frac{d}{dt}\Big(||\,\vec{v}(t)\,||\Big) = \frac{d}{dt}\big(s'(t)\big) = s''(t).$$

Notes:

Since $s'(t)$ is speed, $s''(t)$ is the rate at which speed is changing with respect to time. We see once more that the component of acceleration in the direction of travel relates only to speed, not to a change in direction.

Now compare the formula for a_{N} above to the formula for curvature in Theorem 16:

$$a_{\mathrm{N}} = \frac{||\,\vec{v}(t) \times \vec{a}(t)\,||}{||\,\vec{v}(t)\,||} \quad \text{and} \quad \kappa = \frac{||\,\vec{r}\,'(t) \times \vec{r}\,''(t)\,||}{||\,\vec{r}\,'(t)\,||^3} = \frac{||\,\vec{v}(t) \times \vec{a}(t)\,||}{||\,\vec{v}(t)\,||^3}.$$

Thus

$$\begin{aligned} a_{\mathrm{N}} &= \kappa ||\,\vec{v}(t)\,||^2 \\ &= \kappa \Big(s'(t)\Big)^2 \end{aligned} \tag{10.5}$$

This last equation shows that the component of acceleration that changes the object's direction is dependent on two things: the curvature of the path and the speed of the object.

Imagine driving a car in a clockwise circle. You will naturally feel a force pushing you towards the door (more accurately, the door is pushing you as the car is turning and you want to travel in a straight line). If you keep the radius of the circle constant but speed up (i.e., increasing $s'(t)$), the door pushes harder against you (a_{N} has increased). If you keep your speed constant but tighten the turn (i.e., increase κ), once again the door will push harder against you.

Putting our new formulas for a_{T} and a_{N} together, we have

$$\vec{a}(t) = s''(t)\vec{T}(t) + \kappa ||\,\vec{v}(t)\,||^2 \vec{N}(t).$$

This is not a particularly practical way of finding a_{T} and a_{N}, but it reveals some great concepts about how acceleration interacts with speed and the shape of a curve.

Example 77 Curvature and road design

The minimum radius of the curve in a highway cloverleaf is determined by the operating speed, as given in the table in Figure 10.34. For each curve and speed, compute a_{N}.

Operating Speed (mph)	Minimum Radius (ft)
35	310
40	430
45	540

Figure 10.34: Operating speed and minimum radius in highway cloverleaf design.

Solution Using Equation (10.5), we can compute the acceleration normal to the curve in each case. We start by converting each speed from "miles per hour" to "feet per second" by multiplying by $5280/3600$.

Notes:

$$\begin{aligned}
\text{35mph, 310ft} &\Rightarrow 51.33\text{ft/s}, \quad \kappa = 1/310 \\
a_{\mathrm{N}} &= \kappa \,||\, \vec{v}(t) \,||^2 \\
&= \frac{1}{310}\big(51.33\big)^2 \\
&= 8.50\text{ft/s}^2.
\end{aligned}$$

$$\begin{aligned}
\text{40mph, 430ft} &\Rightarrow 58.67\text{ft/s}, \quad \kappa = 1/430 \\
a_{\mathrm{N}} &= \frac{1}{430}\big(58.67\big)^2 \\
&= 8.00\text{ft/s}^2.
\end{aligned}$$

$$\begin{aligned}
\text{45mph,540ft} &\Rightarrow 66\text{ft/s}, \quad \kappa = 1/540 \\
a_{\mathrm{N}} &= \frac{1}{540}\big(66\big)^2 \\
&= 8.07\text{ft/s}^2.
\end{aligned}$$

Note that each acceleration is similar; this is by design. Considering the classic "Force $=$ mass $\times$ acceleration" formula, this acceleration must be kept small in order for the tires of a vehicle to keep a "grip" on the road. If one travels on a turn of radius 310ft at a rate of 50mph, the acceleration is double, at 17.35ft/s^2. If the acceleration is too high, the frictional force created by the tires may not be enough to keep the car from sliding. Civil engineers routinely compute a "safe" design speed, then subtract 5-10mph to create the posted speed limit for additional safety.

We end this chapter with a reflection on what we've covered. We started with vector–valued functions, which may have seemed at the time to be just another way of writing parametric equations. However, we have seen that the vector perspective has given us great insight into the behavior of functions and the study of motion. Vector–valued position functions convey displacement, distance traveled, speed, velocity, acceleration and curvature information, each of which has great importance in science and engineering.

Notes:

Exercises 10.5

Terms and Concepts

1. It is common to describe position in terms of both _____ and/or _____.

2. A measure of the "curviness" of a curve is _____.

3. Give two shapes with constant curvature.

4. Describe in your own words what an "osculating circle" is.

5. Complete the identity: $\vec{T}'(s) =$ _____$\vec{N}(s)$.

6. Given a position function $\vec{r}(t)$, how are a_{T} and a_{N} affected by the curvature?

Problems

In Exercises 7 – 10 , a position function $\vec{r}(t)$ is given, where $t = 0$ corresponds to the initial position. Find the arc length parameter s, and rewrite $\vec{r}(t)$ in terms of s; that is, find $\vec{r}(s)$.

7. $\vec{r}(t) = \langle 2t, t, -2t\rangle$

8. $\vec{r}(t) = \langle 7\cos t, 7\sin t\rangle$

9. $\vec{r}(t) = \langle 3\cos t, 3\sin t, 2t\rangle$

10. $\vec{r}(t) = \langle 5\cos t, 13\sin t, 12\cos t\rangle$

In Exercises 11 – 22 , a curve C is described along with 2 points on C.

(a) Using a sketch, determine at which of these points the curvature is greater.

(b) Find the curvature κ of C, and evaluate κ at each of the 2 given points.

11. C is defined by $y = x^3 - x$; points given at $x = 0$ and $x = 1/2$.

12. C is defined by $y = \dfrac{1}{x^2+1}$; points given at $x = 0$ and $x = 2$.

13. C is defined by $y = \cos x$; points given at $x = 0$ and $x = \pi/2$.

14. C is defined by $y = \sqrt{1-x^2}$ on $(-1, 1)$; points given at $x = 0$ and $x = 1/2$.

15. C is defined by $\vec{r}(t) = \langle \cos t, \sin(2t)\rangle$; points given at $t = 0$ and $t = \pi/4$.

16. C is defined by $\vec{r}(t) = \left\langle \cos^2 t, \sin t\cos t\right\rangle$; points given at $t = 0$ and $t = \pi/3$.

17. C is defined by $\vec{r}(t) = \left\langle t^2 - 1, t^3 - t\right\rangle$; points given at $t = 0$ and $t = 5$.

18. C is defined by $\vec{r}(t) = \langle \tan t, \sec t\rangle$; points given at $t = 0$ and $t = \pi/6$.

19. C is defined by $\vec{r}(t) = \langle 4t + 2, 3t - 1, 2t + 5\rangle$; points given at $t = 0$ and $t = 1$.

20. C is defined by $\vec{r}(t) = \left\langle t^3 - t, t^3 - 4, t^2 - 1\right\rangle$; points given at $t = 0$ and $t = 1$.

21. C is defined by $\vec{r}(t) = \langle 3\cos t, 3\sin t, 2t\rangle$; points given at $t = 0$ and $t = \pi/2$.

22. C is defined by $\vec{r}(t) = \langle 5\cos t, 13\sin t, 12\cos t\rangle$; points given at $t = 0$ and $t = \pi/2$.

In Exercises 23 – 26 , find the value of x or t where curvature is maximized.

23. $y = \dfrac{1}{6}x^3$

24. $y = \sin x$

25. $\vec{r}(t) = \left\langle t^2 + 2t, 3t - t^2\right\rangle$

26. $\vec{r}(t) = \langle t, 4/t, 3/t\rangle$

In Exercises 27 – 30 , find the radius of curvature at the indicated value.

27. $y = \tan x$, at $x = \pi/4$

28. $y = x^2 + x - 3$, at $x = \pi/4$

29. $\vec{r}(t) = \langle \cos t, \sin(3t)\rangle$, at $t = 0$

30. $\vec{r}(t) = \langle 5\cos(3t), t\rangle$, at $t = 0$

In Exercises 31 – 34 , find the equation of the osculating circle to the curve at the indicated t-value.

31. $\vec{r}(t) = \left\langle t, t^2\right\rangle$, at $t = 0$

32. $\vec{r}(t) = \langle 3\cos t, \sin t\rangle$, at $t = 0$

33. $\vec{r}(t) = \langle 3\cos t, \sin t\rangle$, at $t = \pi/2$

34. $\vec{r}(t) = \left\langle t^2 - t, t^2 + t\right\rangle$, at $t = 0$

11: Functions of Several Variables

A function of the form $y = f(x)$ is a function of a single variable; given a value of x, we can find a value y. Even the vector–valued functions of Chapter 10 are single–variable functions; the input is a single variable though the output is a vector.

There are many situations where a desired quantity is a function of two or more variables. For instance, wind chill is measured by knowing the temperature and wind speed; the volume of a gas can be computed knowing the pressure and temperature of the gas; to compute a baseball player's batting average, one needs to know the number of hits and the number of at–bats.

This chapter studies **multivariable** functions, that is, functions with more than one input.

11.1 Introduction to Multivariable Functions

Definition 29 Function of Two Variables

Let D be a subset of $\mathbb{R}^2$. A **function f of two variables** is a rule that assigns each pair (x, y) in D a value $z = f(x, y)$ in $\mathbb{R}$. D is the **domain** of f; the set of all outputs of f is the **range**.

Example 78 Understanding a function of two variables

Let $z = f(x, y) = x^2 - y$. Evaluate $f(1, 2)$, $f(2, 1)$, and $f(-2, 4)$; find the domain and range of f.

Solution Using the definition $f(x, y) = x^2 - y$, we have:

$$\begin{aligned} f(1, 2) &= 1^2 - 2 = -1 \\ f(2, 1) &= 2^2 - 1 = 3 \\ f(-2, 4) &= (-2)^2 - 4 = 0 \end{aligned}$$

The domain is not specified, so we take it to be all possible pairs in $\mathbb{R}^2$ for which f is defined. In this example, f is defined for *all* pairs (x, y), so the domain D of f is $\mathbb{R}^2$.

The output of f can be made as large or small as possible; any real number r can be the output. (In fact, given any real number r, $f(0, -r) = r$.) So the range R of f is $\mathbb{R}$.

Figure 11.1: Illustrating the domain of $f(x, y)$ in Example 79.

Figure 11.2: Graphing a function of two variables.

Example 79 Understanding a function of two variables

Let $f(x, y) = \sqrt{1 - \dfrac{x^2}{9} - \dfrac{y^2}{4}}$. Find the domain and range of f.

Solution The domain is all pairs (x, y) allowable as input in f. Because of the square–root, we need (x, y) such that $0 \leq 1 - \frac{x^2}{9} - \frac{y^2}{4}$:

$$0 \leq 1 - \frac{x^2}{9} - \frac{y^2}{4}$$
$$\frac{x^2}{9} + \frac{y^2}{4} \leq 1$$

The above equation describes the interior of an ellipse as shown in Figure 11.1. We can represent the domain D graphically with the figure; in set notation, we can write $D = \{(x, y) | \frac{x^2}{9} + \frac{y^2}{4} \leq 1\}$.

The range is the set of all possible output values. The square–root ensures that all output is ≥ 0. Since the x and y terms are squared, then subtracted, inside the square–root, the largest output value comes at $x = 0$, $y = 0$: $f(0, 0) = 1$. Thus the range R is the interval $[0, 1]$.

Graphing Functions of Two Variables

The **graph** of a function f of two variables is the set of all points $\big(x, y, f(x, y)\big)$ where (x, y) is in the domain of f. This creates a **surface** in space.

One can begin sketching a graph by plotting points, but this has limitations. Consider Figure 11.2(a) where 25 points have been plotted of $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$. More points have been plotted than one would reasonably want to do by hand, yet it is not clear at all what the graph of the function looks like. Technology allows us to plot lots of points, connect adjacent points with lines and add shading to create a graph like Figure 11.2b which does a far better job of illustrating the behavior of f.

While technology is readily available to help us graph functions of two variables, there is still a paper–and–pencil approach that is useful to understand and master as it, combined with high–quality graphics, gives one great insight into the behavior of a function. This technique is known as sketching **level curves**.

Level Curves

It may be surprising to find that the problem of representing a three dimensional surface on paper is familiar to most people (they just don't realize it). Topographical maps, like the one shown in Figure 11.3, represent the surface of Earth by indicating points with the same elevation with **contour lines**. The

Notes:

elevations marked are equally spaced; in this example, each thin line indicates an elevation change in 50ft increments and each thick line indicates a change of 200ft. When lines are drawn close together, elevation changes rapidly (as one does not have to travel far to rise 50ft). When lines are far apart, such as near "Aspen Campground," elevation changes more gradually as one has to walk farther to rise 50ft.

Given a function $z = f(x, y)$, we can draw a "topographical map" of f by drawing **level curves** (or, contour lines). A level curve at $z = c$ is a curve in the x-y plane such that for all points (x, y) on the curve, $f(x, y) = c$.

When drawing level curves, it is important that the c values are spaced equally apart as that gives the best insight to how quickly the "elevation" is changing. Examples will help one understand this concept.

Figure 11.3: A topographical map displays elevation by drawing contour lines, along with the elevation is constant.
Sample taken from the public domain USGS Digital Raster Graphics, http://topmaps.usgs.gove/drg/.

Example 80 Drawing Level Curves

Let $f(x, y) = \sqrt{1 - \dfrac{x^2}{9} - \dfrac{y^2}{4}}$. Find the level curves of f for $c = 0$, 0.2, 0.4, 0.6, 0.8 and 1.

Solution Consider first $c = 0$. The level curve for $c = 0$ is the set of all points (x, y) such that $0 = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}$. Squaring both sides gives us

$$\frac{x^2}{9} + \frac{y^2}{4} = 1,$$

an ellipse centered at $(0, 0)$ with horizontal major axis of length 6 and minor axis of length 4. Thus for any point (x, y) on this curve, $f(x, y) = 0$.

Now consider the level curve for $c = 0.2$

$$\begin{aligned} 0.2 &= \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}} \\ 0.04 &= 1 - \frac{x^2}{9} - \frac{y^2}{4} \\ \frac{x^2}{9} + \frac{y^2}{4} &= 0.96 \\ \frac{x^2}{8.64} + \frac{y^2}{3.84} &= 1. \end{aligned}$$

This is also an ellipse, where $a = \sqrt{8.64} \approx 2.94$ and $b = \sqrt{3.84} \approx 1.96$.

Notes:

(a)

(b)

Figure 11.4: Graphing the level curves in Example 80.

In general, for $z = c$, the level curve is:

$$\begin{aligned} c &= \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}} \\ c^2 &= 1 - \frac{x^2}{9} - \frac{y^2}{4} \\ \frac{x^2}{9} + \frac{y^2}{4} &= 1 - c^2 \\ \frac{x^2}{9(1-c^2)} + \frac{y^2}{4(1-c^2)} &= 1, \end{aligned}$$

ellipses that are decreasing in size as c increases. A special case is when $c = 1$; there the ellipse is just the point $(0, 0)$.

The level curves are shown in Figure 11.4(a). Note how the level curves for $c = 0$ and $c = 0.2$ are very, very close together: this indicates that f is growing rapidly along those curves.

In Figure 11.4(b), the curves are drawn on a graph of f in space. Note how the elevations are evenly spaced. Near the level curves of $c = 0$ and $c = 0.2$ we can see that f indeed is growing quickly.

Example 81 Analyzing Level Curves

Let $f(x, y) = \dfrac{x + y}{x^2 + y^2 + 1}$. Find the level curves for $z = c$.

Solution We begin by setting $f(x, y) = c$ for an arbitrary c and seeing if algebraic manipulation of the equation reveals anything significant.

$$\begin{aligned} \frac{x + y}{x^2 + y^2 + 1} &= c \\ x + y &= c(x^2 + y^2 + 1). \end{aligned}$$

We recognize this as a circle, though the center and radius are not yet clear. By completing the square, we can obtain:

$$\left(x - \frac{1}{2c}\right)^2 + \left(y - \frac{1}{2c}\right)^2 = \frac{1}{2c^2} - 1,$$

a circle centered at $\big(1/(2c), 1/(2c)\big)$ with radius $\sqrt{1/(2c^2) - 1}$, where $|c| < 1/\sqrt{2}$. The level curves for $c = \pm 0.2$, ± 0.4 and ± 0.6 are sketched in Figure 11.5(a). To help illustrate "elevation," we use thicker lines for c values near 0, and dashed lines indicate where $c < 0$.

There is one special level curve, when $c = 0$. The level curve in this situation is $x + y = 0$, the line $y = -x$.

Notes:

In Figure 11.5(b) we see a graph of the surface. Note how the y-axis is pointing away from the viewer to more closely resemble the orientation of the level curves in (a).

Seeing the level curves helps us understand the graph. For instance, the graph does not make it clear that one can "walk" along the line $y = -x$ without elevation change, though the level curve does.

Figure 11.5: Graphing the level curves in Example 81.

Functions of Three Variables

We extend our study of multivariable functions to functions of three variables. (One can make a function of as many variables as one likes; we limit our study to three variables.)

Definition 30 Function of Three Variables

Let D be a subset of $\mathbb{R}^3$. A **function f of three variables** is a rule that assigns each triple (x, y, z) in D a value $w = f(x, y, z)$ in $\mathbb{R}$. D is the **domain** of f; the set of all outputs of f is the **range**.

Note how this definition closely resembles that of Definition 29.

Example 82 Understanding a function of three variables

Let $f(x, y, z) = \dfrac{x^2 + z + 3\sin y}{x + 2y - z}$. Evaluate f at the point $(3, 0, 2)$ and find the domain and range of f.

Solution $$f(3, 0, 2) = \frac{3^2 + 2 + 3\sin 0}{3 + 2(0) - 2} = 11.$$

As the domain of f is not specified, we take it to be the set of all triples (x, y, z) for which $f(x, y, z)$ is defined. As we cannot divide by 0, we find the domain D is

$$D = \{(x, y, z) \mid x + 2y - z \neq 0\}.$$

We recognize that the set of all points in $\mathbb{R}^3$ that *are not* in D form a plane in space that passes through the origin (with normal vector $\langle 1, 2, -1\rangle$).

We determine the range R is $\mathbb{R}$; that is, all real numbers are possible outputs of f. There is no set way of establishing this. Rather, to get numbers near 0 we can let $y = 0$ and choose $z \approx -x^2$. To get numbers of arbitrarily large magnitude, we can let $z \approx x + 2y$.

Notes:

Level Surfaces

It is very difficult to produce a meaningful graph of a function of three variables. A function of *one* variable is a *curve* drawn in *2* dimensions; a function of *two* variables is a *surface* drawn in *3* dimensions; a function of *three* variables is a *hypersurface* drawn in *4* dimensions.

There are a few techniques one can employ to try to "picture" a graph of three variables. One is an analogue of level curves: **level surfaces**. Given $w = f(x, y, z)$, the level surface at $w = c$ is the surface in space formed by all points (x, y, z) where $f(x, y, z) = c$.

Example 83 Finding level surfaces

If a point source S is radiating energy, the intensity I at a given point P in space is inversely proportional to the square of the distance between S and P. That is, when $S = (0, 0, 0)$, $I(x, y, z) = \dfrac{k}{x^2 + y^2 + z^2}$ for some constant k.

Let $k = 1$; find the level surfaces of I.

Solution We can (mostly) answer this question using "common sense." If energy (say, in the form of light) is emanating from the origin, its intensity will be the same at all points equidistant from the origin. That is, at any point on the surface of a sphere centered at the origin, the intensity should be the same. Therefore, the level surfaces are spheres.

We now find this mathematically. The level surface at $I = c$ is defined by

$$c = \frac{1}{x^2 + y^2 + z^2}.$$

A small amount of algebra reveals

$$x^2 + y^2 + z^2 = \frac{1}{c}.$$

Given an intensity c, the level surface $I = c$ is a sphere of radius $1/\sqrt{c}$, centered at the origin.

Figure 11.6 gives a table of the radii of the spheres for given c values. Normally one would use equally spaced c values, but these values have been chosen purposefully. At a distance of 0.25 from the point source, the intensity is 16; to move to a point of half that intensity, one just moves out 0.1 to 0.35 – not much at all. To again halve the intensity, one moves 0.15, a little more than before.

Note how each time the intensity if halved, the distance required to move away grows. We conclude that the closer one is to the source, the more rapidly the intensity changes.

c	r
16.	0.25
8.	0.35
4.	0.5
2.	0.71
1.	1.
0.5	1.41
0.25	2.
0.125	2.83
0.0625	4.

Figure 11.6: A table of c values and the corresponding radius r of the spheres of constant value in Example 83.

In the next section we apply the concepts of limits to functions of two or more variables.

Notes:

Exercises 11.1

Terms and Concepts

1. Give two examples (other than those given in the text) of "real world" functions that require more than one input.

2. The graph of a function of two variables is a _____.

3. Most people are familiar with the concept of level curves in the context of _____ maps.

4. T/F: Along a level curve, the output of a function does not change.

5. The analogue of a level curve for functions of three variables is a level _____.

6. What does it mean when level curves are close together? Far apart?

Problems

In Exercises 7 – 14, give the domain and range of the multivariable function.

7. $f(x, y) = x^2 + y^2 + 2$

8. $f(x, y) = x + 2y$

9. $f(x, y) = x - 2y$

10. $f(x, y) = \dfrac{1}{x + 2y}$

11. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$

12. $f(x, y) = \sin x \cos y$

13. $f(x, y) = \sqrt{9 - x^2 - y^2}$

14. $f(x, y) = \dfrac{1}{\sqrt{x^2 + y^2 - 9}}$

In Exercises 15 – 22, describe in words and sketch the level curves for the function and given c values.

15. $f(x, y) = 3x - 2y$; $c = -2, 0, 2$

16. $f(x, y) = x^2 - y^2$; $c = -1, 0, 1$

17. $f(x, y) = x - y^2$; $c = -2, 0, 2$

18. $f(x, y) = \dfrac{1 - x^2 - y^2}{2y - 2x}$; $c = -2, 0, 2$

19. $f(x, y) = \dfrac{2x - 2y}{x^2 + y^2 + 1}$; $c = -1, 0, 1$

20. $f(x, y) = \dfrac{y - x^3 - 1}{x}$; $c = -3, -1, 0, 1, 3$

21. $f(x, y) = \sqrt{x^2 + 4y^2}$; $c = 1, 2, 3, 4$

22. $f(x, y) = x^2 + 4y^2$; $c = 1, 2, 3, 4$

In Exercises 23 – 26, give the domain and range of the functions of three variables.

23. $f(x, y, z) = \dfrac{x}{x + 2y - 4z}$

24. $f(x, y, z) = \dfrac{1}{1 - x^2 - y^2 - z^2}$

25. $f(x, y, z) = \sqrt{z - x^2 + y^2}$

26. $f(x, y, z) = z^2 \sin x \cos y$

In Exercises 27 – 30, describe the level surfaces of the given functions of three variables.

27. $f(x, y, z) = x^2 + y^2 + z^2$

28. $f(x, y, z) = z - x^2 + y^2$

29. $f(x, y, z) = \dfrac{x^2 + y^2}{z}$

30. $f(x, y, z) = \dfrac{z}{x - y}$

31. Compare the level curves of Exercises 21 and 22. How are they similar, and how are they different? Each surface is a quadric surface; describe how the level curves are consistent with what we know about each surface.

Figure 11.7: Illustrating open and closed sets in the *x*-*y* plane.

11.2 Limits and Continuity of Multivariable Functions

We continue with the pattern we have established in this text: after defining a new kind of function, we apply calculus ideas to it. The previous section defined functions of two and three variables; this section investigates what it means for these functions to be "continuous."

We begin with a series of definitions. We are used to "open intervals" such as $(1, 3)$, which represents the set of all x such that $1 < x < 3$, and "closed intervals" such as $[1, 3]$, which represents the set of all x such that $1 \leq x \leq 3$. We need analogous definitions for open and closed sets in the *x*-*y* plane.

Definition 31 Open Disk, Boundary and Interior Points, Open and Closed Sets, Bounded Sets

An **open disk** B in $\mathbb{R}^2$ centered at (x_0, y_0) with radius r is the set of all points (x, y) such that $\sqrt{(x - x_0)^2 + (y - y_0)^2} < r$.

Let S be a set of points in $\mathbb{R}^2$. A point P in $\mathbb{R}^2$ is a **boundary point** of S if all open disks centered at P contain both points in S and points not in S.

A point P in S is an **interior point** of S if there is an open disk centered at P that contains only points in S.

A set S is **open** if every point in S is an interior point.

A set S is **closed** if it contains all of its boundary points.

A set S is **bounded** if there is an $M > 0$ such that the open disk, centered at the origin with radius M, contains S. A set that is not bounded is **unbounded**.

Figure 11.7 shows several sets in the *x*-*y* plane. In each set, point P_1 lies on the boundary of the set as all open disks centered there contain both points in, and not in, the set. In contrast, point P_2 is an interior point for there is an open disk centered there that lies entirely within the set.

The set depicted in Figure 11.7(a) is a closed set as it contains all of its boundary points. The set in (b) is open, for all of its points are interior points (or, equivalently, it does not contain any of its boundary points). The set in (c) is neither open nor closed as it contains some of its boundary points.

Notes:

Notes:

Example 84 Determining open/closed, bounded/unbounded

Determine if the domain of the function $f(x,y) = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}$ is open, closed, or neither, and if it is bounded.

Solution This domain of this function was found in Example 79 to be $D = \{(x,y) \mid \frac{x^2}{9} + \frac{y^2}{4} \leq 1\}$, the region *bounded* by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$. Since the region includes the boundary (indicated by the use of "$\leq$"), the set contains all of its boundary points and hence is closed. The region is bounded as a disk of radius 4, centered at the origin, contains D.

Figure 11.8: Sketching the domain of the function in Example 85.

Example 85 Determining open/closed, bounded/unbounded

Determine if the domain of $f(x,y) = \frac{1}{x-y}$ is open, closed, or neither.

Solution As we cannot divide by 0, we find the domain to be $D = \{(x,y) \mid x - y \neq 0\}$. In other words, the domain is the set of all points (x,y) *not* on the line $y = x$.

The domain is sketched in Figure 11.8. Note how we can draw an open disk around any point in the domain that lies entirely inside the domain, and also note how the only boundary points of the domain are the points on the line $y = x$. We conclude the domain is an open set. The set is unbounded.

Limits

Recall a pseudo–definition of the limit of a function of one variable: "$\lim_{x \to c} f(x) = L$" means that if x is "really close" to c, then $f(x)$ is "really close" to L. A similar pseudo–definition holds for functions of two variables. We'll say that

$$\text{“}\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L\text{”}$$

means "if the point (x,y) is really close to the point (x_0,y_0), then $f(x,y)$ is really close to L." The formal definition is given below.

Definition 32 Limit of a Function of Two Variables

Let S be an open set containing (x_0,y_0), and let f be a function of two variables defined on S, except possibly at (x_0,y_0). The **limit** of $f(x,y)$ as (x,y) approaches (x_0,y_0) is L, denoted

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L,$$

means that given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $(x,y) \neq (x_0,y_0)$, if (x,y) is in the open disk centered at (x_0,y_0) with radius δ, then $|f(x,y) - L| < \varepsilon$.

Notes:

The concept behind Definition 32 is sketched in Figure 11.9. Given $\varepsilon > 0$, find $\delta > 0$ such that if (x, y) is any point in the open disk centered at (x_0, y_0) in the x-y plane with radius δ, then $f(x, y)$ should be within ε of L.

Computing limits using this definition is rather cumbersome. The following theorem allows us to evaluate limits much more easily.

Figure 11.9: **Illustrating the definition of a limit.** The open disk in the x-y plane has radius δ. Let (x, y) be any point in this disk; $f(x, y)$ is within ε of L.

Theorem 17 Basic Limit Properties of Functions of Two Variables

Let b, x_0, y_0, L and K be real numbers, let n be a positive integer, and let f and g be functions with the following limits:

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x, y) = K.$$

The following limits hold.

1. Constants: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} b = b$
2. Identity $\displaystyle\lim_{(x,y)\to(x_0,y_0)} x = x_0; \qquad \lim_{(x,y)\to(x_0,y_0)} y = y_0$
3. Sums/Differences: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \big(f(x, y) \pm g(x, y)\big) = L \pm K$
4. Scalar Multiples: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} b \cdot f(x, y) = bL$
5. Products: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x, y) \cdot g(x, y) = LK$
6. Quotients: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x, y)/g(x, y) = L/K, (K \neq 0)$
7. Powers: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x, y)^n = L^n$

This theorem, combined with Theorems **??** and **??** of Section **??**, allows us to evaluate many limits.

Example 86 Evaluating a limit

Evaluate the following limits:

1. $\displaystyle\lim_{(x,y)\to(1,\pi)} \frac{y}{x} + \cos(xy)$
2. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2 + y^2}$

Notes:

Solution

1. The aforementioned theorems allow us to simply evaluate $y/x + \cos(xy)$ when $x = 1$ and $y = \pi$. If an indeterminate form is returned, we must do more work to evaluate the limit; otherwise, the result is the limit. Therefore

$$\begin{aligned}\lim_{(x,y)\to(1,\pi)} \frac{y}{x} + \cos(xy) &= \frac{\pi}{1} + \cos\pi \\ &= \pi - 1.\end{aligned}$$

2. We attempt to evaluate the limit by substituting 0 in for x and y, but the result is the indeterminate form "$0/0$." To evaluate this limit, we must "do more work," but we have not yet learned what "kind" of work to do. Therefore we cannot yet evaluate this limit.

When dealing with functions of a single variable we also considered one–sided limits and stated

$$\lim_{x\to c} f(x) = L \quad \text{if, and only if,} \quad \lim_{x\to c^+} f(x) = L \quad \textbf{and} \quad \lim_{x\to c^-} f(x) = L.$$

That is, the limit is L if and only if $f(x)$ approaches L when x approaches c from **either** direction, the left or the right.

In the plane, there are infinite directions from which (x, y) might approach (x_0, y_0). In fact, we do not have to restrict ourselves to approaching (x_0, y_0) from a particular direction, but rather we can approach that point along a path that is not a straight line. It is possible to arrive at different limiting values by approaching (x_0, y_0) along different paths. If this happens, we say that $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist (this is analogous to the left and right hand limits of single variable functions not being equal).

Our theorems tell us that we can evaluate most limits quite simply, without worrying about paths. When indeterminate forms arise, the limit may or may not exist. If it does exist, it can be difficult to prove this as we need to show the same limiting value is obtained regardless of the path chosen. The case where the limit does not exist is often easier to deal with, for we can often pick two paths along which the limit is different.

Example 87 Showing limits do not exist

1. Show $\lim\limits_{(x,y)\to(0,0)} \dfrac{3xy}{x^2 + y^2}$ does not exist by finding the limits along the lines $y = mx$.

Notes:

2. Show $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x+y}$ does not exist by finding the limit along the path $y = -\sin x$.

SOLUTION

1. Evaluating $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^2}$ along the lines $y = mx$ means replace all y's with mx and evaluating the resulting limit:

$$\begin{aligned}\lim_{(x,mx)\to(0,0)} \frac{3x(mx)}{x^2+(mx)^2} &= \lim_{x\to 0} \frac{3mx^2}{x^2(m^2+1)}\\ &= \lim_{x\to 0} \frac{3m}{m^2+1}\\ &= \frac{3m}{m^2+1}.\end{aligned}$$

While the limit exists for each choice of m, we get a *different* limit for each choice of m. That is, along different lines we get differing limiting values, meaning *the* limit does not exist.

2. Let $f(x,y) = \frac{\sin(xy)}{x+y}$. We are to show that $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist by finding the limit along the path $y = -\sin x$. First, however, consider the limits found along the lines $y = mx$ as done above.

$$\begin{aligned}\lim_{(x,mx)\to(0,0)} \frac{\sin\big(x(mx)\big)}{x+mx} &= \lim_{x\to 0} \frac{\sin(mx^2)}{x(m+1)}\\ &= \lim_{x\to 0} \frac{\sin(mx^2)}{x}\cdot\frac{1}{m+1}.\end{aligned}$$

By applying L'Hôpital's Rule, we can show this limit is 0 *except* when $m = -1$, that is, along the line $y = -x$. This line is not in the domain of f, so we have found the following fact: along every line $y = mx$ in the domain of f, $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Now consider the limit along the path $y = -\sin x$:

$$\lim_{(x,-\sin x)\to(0,0)} \frac{\sin\big(-x\sin x\big)}{x-\sin x} = \lim_{x\to 0} \frac{\sin\big(-x\sin x\big)}{x-\sin x}$$

Now apply L'Hôpital's Rule twice:

$$\begin{aligned}&= \lim_{x\to 0} \frac{\cos\big(-x\sin x\big)(-\sin x - x\cos x)}{1-\cos x} \qquad (\text{“}=0/0\text{”})\\ &= \lim_{x\to 0} \frac{-\sin\big(-x\sin x\big)(-\sin x - x\cos x)^2 + \cos\big(-x\sin x\big)(-2\cos x + x\sin x)}{\sin x}\\ &= \text{“2/0”} \Rightarrow \text{the limit does not exist.}\end{aligned}$$

Notes:

Step back and consider what we have just discovered. Along any line $y = mx$ in the domain of the $f(x, y)$, the limit is 0. However, along the path $y = -\sin x$, which lies in the domain of $f(x, y)$ for all $x \neq 0$, the limit does not exist. Since the limit is not the same along every path to $(0, 0)$, we say $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x+y}$ does not exist.

Example 88 Finding a limit

Let $f(x, y) = \dfrac{5x^2y^2}{x^2+y^2}$. Find $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y)$.

Solution It is relatively easy to show that along any line $y = mx$, the limit is 0. This is not enough to prove that the limit exists, as demonstrated in the previous example, but it tells us that if the limit does exist then it must be 0.

To prove the limit is 0, we apply Definition 32. Let $\varepsilon > 0$ be given. We want to find $\delta > 0$ such that if $\sqrt{(x-0)^2 + (y-0)^2} < \delta$, then $|f(x, y) - 0| < \varepsilon$.

Set $\delta < \sqrt{\varepsilon/5}$. Note that $\left|\dfrac{5y^2}{x^2+y^2}\right| < 5$ for all $(x, y) \neq (0, 0)$, and that if $\sqrt{x^2+y^2} < \delta$, then $x^2 < \delta^2$.

Let $\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2+y^2} < \delta$. Consider $|f(x, y) - 0|$:

$$\begin{aligned}
|f(x, y) - 0| &= \left|\frac{5x^2y^2}{x^2+y^2} - 0\right| \\
&= \left|x^2 \cdot \frac{5y^2}{x^2+y^2}\right| \\
&< \delta^2 \cdot 5 \\
&< \frac{\varepsilon}{5} \cdot 5 \\
&= \varepsilon.
\end{aligned}$$

Thus if $\sqrt{(x-0)^2 + (y-0)^2} < \delta$ then $|f(x, y) - 0| < \varepsilon$, which is what we wanted to show. Thus $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5x^2y^2}{x^2+y^2} = 0$.

Continuity

Definition **??** defines what it means for a function of one variable to be continuous. In brief, it meant that the graph of the function did not have breaks, holes, jumps, etc. We define continuity for functions of two variables in a similar way as we did for functions of one variable.

Notes:

Definition 33 Continuous

Let a function $f(x, y)$ be defined on an open disk B containing the point (x_0, y_0).

1. f is **continuous** at (x_0, y_0) if $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.
2. f is **continuous on** B if f is continuous at all points in B. If f is continuous at all points in $\mathbb{R}^2$, we say that f is **continuous everywhere**.

Example 89 Continuity of a function of two variables

Let $f(x, y) = \begin{cases} \frac{\cos y \sin x}{x} & x \neq 0 \\ \cos y & x = 0 \end{cases}$. Is f continuous at $(0, 0)$? Is f continuous everywhere?

Solution To determine if f is continuous at $(0, 0)$, we need to compare $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ to $f(0, 0)$.

Applying the definition of f, we see that $f(0, 0) = \cos 0 = 1$.

We now consider the limit $\lim\limits_{(x,y)\to(0,0)} f(x, y)$. Substituting 0 for x and y in $(\cos y \sin x)/x$ returns the indeterminate form "0/0", so we need to do more work to evaluate this limit.

Consider two related limits: $\lim\limits_{(x,y)\to(0,0)} \cos y$ and $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin x}{x}$. The first limit does not contain x, and since $\cos y$ is continuous,

$$\lim_{(x,y)\to(0,0)} \cos y = \lim_{y\to 0} \cos y = \cos 0 = 1.$$

The second limit does not contain y. By Theorem **??** we can say

$$\lim_{(x,y)\to(0,0)} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Finally, Theorem 17 of this section states that we can combine these two limits as follows:

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{\cos y \sin x}{x} &= \lim_{(x,y)\to(0,0)} (\cos y)\left(\frac{\sin x}{x}\right) \\ &= \left(\lim_{(x,y)\to(0,0)} \cos y\right)\left(\lim_{(x,y)\to(0,0)} \frac{\sin x}{x}\right) \\ &= (1)(1) \\ &= 1. \end{aligned}$$

Notes:

Figure 11.10: A graph of $f(x, y)$ in Example 89.

We have found that $\lim\limits_{(x,y)\to(0,0)} \dfrac{\cos y \sin x}{x} = f(0,0)$, so f is continuous at $(0,0)$.

A similar analysis shows that f is continuous at all points in $\mathbb{R}^2$. As long as $x \neq 0$, we can evaluate the limit directly; when $x = 0$, a similar analysis shows that the limit is $\cos y$. Thus we can say that f is continuous everywhere. A graph of f is given in Figure 11.10. Notice how it has no breaks, jumps, etc.

The following theorem is very similar to Theorem **??**, giving us ways to combine continuous functions to create other continuous functions.

Theorem 18 Properties of Continuous Functions

Let f and g be continuous on an open disk B, let c be a real number, and let n be a positive integer. The following functions are continuous on B.

1. Sums/Differences: $f \pm g$
2. Constant Multiples: $c \cdot f$
3. Products: $f \cdot g$
4. Quotients: f/g (as longs as $g \neq 0$ on B)
5. Powers: f^n
6. Roots: $\sqrt[n]{f}$ (if n is even then $f \geq 0$ on B; if n is odd, then true for all values of f on B.)
7. Compositions: Adjust the definitions of f and g to: Let f be continuous on B, where the range of f on B is J, and let g be a single variable function that is continuous on J. Then $g \circ f$, i.e., $g(f(x, y))$, is continuous on B.

Example 90 Establishing continuity of a function

Let $f(x, y) = \sin(x^2 \cos y)$. Show f is continuous everywhere.

Solution We will apply both Theorems **??** and 18. Let $f_1(x, y) = x^2$. Since y is not actually used in the function, and polynomials are continuous (by Theorem **??**), we conclude f_1 is continuous everywhere. A similar statement can be made about $f_2(x, y) = \cos y$. Part 3 of Theorem 18 states that $f_3 = f_1 \cdot f_2$ is continuous everywhere, and Part 7 of the theorem states the composition of sine with f_3 is continuous: that is, $\sin(f_3) = \sin(x^2 \cos y)$ is continuous everywhere.

Notes:

Functions of Three Variables

The definitions and theorems given in this section can be extended in a natural way to definitions and theorems about functions of three (or more) variables. We cover the key concepts here; some terms from Definitions 31 and 33 are not redefined but their analogous meanings should be clear to the reader.

Definition 34 Open Balls, Limit, Continuous

1. An **open ball** in $\mathbb{R}^3$ centered at (x_0, y_0, z_0) with radius r is the set of all points (x, y, z) such that $\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} = r$.

2. Let D be an open set in $\mathbb{R}^3$ containing (x_0, y_0, z_0), and let $f(x, y, z)$ be a function of three variables defined on D, except possibly at (x_0, y_0, z_0). The **limit** of $f(x, y, z)$ as (x, y, z) approaches (x_0, y_0, z_0) is L, denoted

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z) = L,$$

means that given any $\varepsilon > 0$, there is a $\delta > 0$ such that for all $(x, y, z) \neq (x_0, y_0, z_0)$, if (x, y, z) is in the open ball centered at (x_0, y_0, z_0) with radius δ, then $|f(x, y, z) - L| < \varepsilon$.

3. Let $f(x, y, z)$ be defined on an open ball B containing (x_0, y_0, z_0). f is **continuous** at (x_0, y_0, z_0) if $\lim\limits_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z) = f(x_0, y_0, z_0)$.

These definitions can also be extended naturally to apply to functions of four or more variables. Theorem 18 also applies to function of three or more variables, allowing us to say that the function

$$f(x, y, z) = \frac{e^{x^2+y}\sqrt{y^2+z^2+3}}{\sin(xyz)+5}$$

is continuous everywhere.

When considering single variable functions, we studied limits, then continuity, then the derivative. In our current study of multivariable functions, we have studied limits and continuity. In the next section we study derivation, which takes on a slight twist as we are in a multivarible context.

Notes:

Exercises 11.2

Terms and Concepts

1. Describe in your own words the difference between boundary and interior point of a set.

2. Use your own words to describe (informally) what $\lim\limits_{(x,y)\to(1,2)} f(x,y) = 17$ means.

3. Give an example of a closed, bounded set.

4. Give an example of a closed, unbounded set.

5. Give an example of a open, bounded set.

6. Give an example of a open, unbounded set.

Problems

In Exercises 7 – 10, a set S is given.

(a) Give one boundary point and one interior point, when possible, of S.

(b) State whether S is open, closed, or neither.

(c) State whether S is bounded or unbounded.

7. $S = \left\{(x,y) \;\middle|\; \dfrac{(x-1)^2}{4} + \dfrac{(y-3)^2}{9} \leq 1\right\}$

8. $S = \left\{(x,y) \mid y \neq x^2\right\}$

9. $S = \left\{(x,y) \mid x^2 + y^2 = 1\right\}$

10. $S = \{(x,y) | y > \sin x\}$

In Exercises 11 – 14:

(a) Find the domain D of the given function.

(b) State whether D is an open or closed set.

(c) State whether D is bounded or unbounded.

11. $f(x,y) = \sqrt{9 - x^2 - y^2}$

12. $f(x,y) = \sqrt{y - x^2}$

13. $f(x,y) = \dfrac{1}{\sqrt{y - x^2}}$

14. $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

In Exercises 15 – 20, a limit is given. Evaluate the limit along the paths given, then state why these results show the given limit does not exist.

15. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$

    (a) Along the path $y = 0$.

    (b) Along the path $x = 0$.

16. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x + y}{x - y}$

    (a) Along the path $y = mx$.

17. $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy - y^2}{y^2 + x}$

    (a) Along the path $y = mx$.

    (b) Along the path $x = 0$.

18. $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2)}{y}$

    (a) Along the path $y = mx$.

    (b) Along the path $y = x^2$.

19. $\lim\limits_{(x,y)\to(1,2)} \dfrac{x + y - 3}{x^2 - 1}$

    (a) Along the path $y = 2$.

    (b) Along the path $y = x + 1$.

20. $\lim\limits_{(x,y)\to(\pi,\pi/2)} \dfrac{\sin x}{\cos y}$

    (a) Along the path $x = \pi$.

    (b) Along the path $y = x - \pi/2$.

11.3 Partial Derivatives

Let y be a function of x. We have studied in great detail the derivative of y with respect to x, that is, $\frac{dy}{dx}$, which measures the rate at which y changes with respect to x. Consider now $z = f(x, y)$. It makes sense to want to know how z changes with respect to x and/or y. This section begins our investigation into these rates of change.

Consider the function $z = f(x, y) = x^2 + 2y^2$, as graphed in Figure 11.11(a). By fixing $y = 2$, we focus our attention to all points on the surface where the y-value is 2, shown in both parts (a) and (b) of the figure. These points form a curve in space: $z = f(x, 2) = x^2 + 8$ which is a function of just one variable. We can take the derivative of z with respect to x along this curve and find equations of tangent lines, etc.

The key notion to extract from this example is: by treating y as constant (it does not vary) we can consider how z changes with respect to x. In a similar fashion, we can hold x constant and consider how z changes with respect to y. This is the underlying principle of **partial derivatives**. We state the formal, limit–based definition first, then show how to compute these partial derivatives without directly taking limits.

Figure 11.11: By fixing $y = 2$, the surface $f(x, y) = x^2 + 2y^2$ is a curve in space.

Definition 35 Partial Derivative

Let $z = f(x, y)$ be a continuous function on an open set S in $\mathbb{R}^2$.

1. The **partial derivative of f with respect to x** is:

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}.$$

2. The **partial derivative of f with respect to y** is:

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

Alternate notations for $f_x(x, y)$ include:

$$\frac{\partial}{\partial x} f(x, y), \quad \frac{\partial f}{\partial x}, \quad \frac{\partial z}{\partial x}, \quad \text{and } z_x,$$

with similar notations for $f_y(x, y)$. For ease of notation, $f_x(x, y)$ is often abbreviated f_x.

Example 91 Computing partial derivatives with the limit definition

Let $f(x, y) = x^2y + 2x + y^3$. Find $f_x(x, y)$ using the limit definition.

Notes:

Solution Using Definition 35, we have:

$$\begin{aligned}
f_x(x,y) &= \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}\\
&= \lim_{h\to 0}\frac{(x+h)^2y+2(x+h)+y^3-(x^2y+2x+y^3)}{h}\\
&= \lim_{h\to 0}\frac{(x^2y+2xhy+h^2y+2x+2h+y^3-(x^2y+2x+y^3)}{h}\\
&= \lim_{h\to 0}\frac{2xhy+h^2y+2h}{h}\\
&= \lim_{h\to 0}2xy+hy+2\\
&= 2xy+2.
\end{aligned}$$

We have found $f_x(x,y)=2xy+2$.

Example 91 found a partial derivative using the formal, limit–based definition. Using limits is not necessary, though, as we can rely on our previous knowledge of derivatives to compute partial derivatives easily. When computing $f_x(x,y)$, we hold y fixed – it does not vary. Therefore we can compute the derivative with respect to x by treating y as a constant or coefficient.

Just as $\frac{d}{dx}\left(5x^2\right)=10x$, we compute $\frac{\partial}{\partial x}\left(x^2y\right)=2xy$. Here we are treating y as a coefficient.

Just as $\frac{d}{dx}\left(5^3\right)=0$, we compute $\frac{\partial}{\partial x}\left(y^3\right)=0$. Here we are treating y as a constant. More examples will help make this clear.

Example 92 Finding partial derivatives

Find $f_x(x,y)$ and $f_y(x,y)$ in each of the following.

1. $f(x,y)=x^3y^2+5y^2-x+7$
2. $f(x,y)=\cos(xy^2)+\sin x$
3. $f(x,y)=e^{x^2y^3}\sqrt{x^2+1}$

Solution

1. We have $f(x,y)=x^3y^2+5y^2-x+7$.
   Begin with $f_x(x,y)$. Keep y fixed, treating it as a constant or coefficient, as appropriate:
   $$f_x(x,y)=3x^2y^2-1.$$
   Note how the $5y^2$ and 7 terms go to zero.

Notes:

To compute $f_y(x, y)$, we hold x fixed:

$$f_y(x, y) = 2x^3y + 10y.$$

Note how the $-x$ and 7 terms go to zero.

2. We have $f(x, y) = \cos(xy^2) + \sin x$.
Begin with $f_x(x, y)$. We need to apply the Chain Rule with the cosine term; y^2 is the coefficient of the x-term inside the cosine function.

$$f_x(x, y) = -\sin(xy^2)(y^2) + \cos x = -y^2 \sin(xy^2) + \cos x.$$

To find $f_y(x, y)$, note that x is the coefficient of the y^2 term inside of the cosine term; also note that since x is fixed, $\sin x$ is also fixed, and we treat it as a constant.

$$f_y(x, y) = -\sin(xy^2)(2xy) = -2xy \sin(xy^2).$$

3. We have $f(x, y) = e^{x^2y^3}\sqrt{x^2 + 1}$.
Beginning with $f_x(x, y)$, note how we need to apply the Product Rule.

$$\begin{aligned} f_x(x, y) &= e^{x^2y^3}(2xy^3)\sqrt{x^2 + 1} + e^{x^2y^3}\frac{1}{2}\left(x^2 + 1\right)^{-1/2}(2x) \\ &= 2xy^3e^{x^2y^3}\sqrt{x^2 + 1} + \frac{xe^{x^2y^3}}{\sqrt{x^2 + 1}}. \end{aligned}$$

Note that when finding $f_y(x, y)$ we do not have to apply the Product Rule; since $\sqrt{x^2 + 1}$ does not contain y, we treat it as fixed and hence becomes a coefficient of the $e^{x^2y^3}$ term.

$$f_y(x, y) = e^{x^2y^3}(3x^2y^2)\sqrt{x^2 + 1} = 3x^2y^2e^{x^2y^3}\sqrt{x^2 + 1}.$$

We have shown *how* to compute a partial derivative, but it may still not be clear what a partial derivative *means*. Given $z = f(x, y)$, $f_x(x, y)$ measures the rate at which z changes as only x varies: y is held constant.

Imagine standing in a rolling meadow, then beginning to walk due east. Depending on your location, you might walk up, sharply down, or perhaps not change elevation at all. This is similar to measuring z_x: you are moving only east (in the "x"-direction) and not north/south at all. Going back to your original location, imagine now walking due north (in the "y"-direction). Perhaps walking due north does not change your elevation at all. This is analogous to $z_y = 0$: z does not change with respect to y. We can see that z_x and z_y do not have to be the same, or even similar, as it is easy to imagine circumstances where walking east means you walk downhill, though walking north makes you walk uphill.

Notes:

(a)

(b)

Figure 11.12: Illustrating the meaning of partial derivatives.

The following example helps us visualize this more.

Example 93 Evaluating partial derivatives

Let $z = f(x, y) = -x^2 - \frac{1}{2}y^2 + xy + 10$. Find $f_x(2, 1)$ and $f_y(2, 1)$ and interpret their meaning.

Solution We begin by computing $f_x(x, y) = -2x + y$ and $f_y(x, y) = -y + x$. Thus

$$f_x(2, 1) = -3 \quad \text{and} \quad f_y(2, 1) = 1.$$

It is also useful to note that $f(2, 1) = 7.5$. What does each of these numbers mean?

Consider $f_x(2, 1) = -3$, along with Figure 11.12(a). If one "stands" on the surface at the point $(2, 1, 7.5)$ and moves parallel to the x-axis (i.e., only the x-value changes, not the y-value), then the instantaneous rate of change is -3. Increasing the x-value will decrease the z-value; decreasing the x-value will increase the z-value.

Now consider $f_y(2, 1) = 1$, illustrated in Figure 11.12(b). Moving along the curve drawn on the surface, i.e., parallel to the y-axis and not changing the x-values, increases the z-value instantaneously at a rate of 1. Increasing the y-value by 1 would increase the z-value by approximately 1.

Since the magnitude of f_x is greater than the magnitude of f_y at $(2, 1)$, it is "steeper" in the x-direction than in the y-direction.

Second Partial Derivatives

Let $z = f(x, y)$. We have learned to find the partial derivatives $f_x(x, y)$ and $f_y(x, y)$, which are each functions of x and y. Therefore we can take partial derivatives of them, each with respect to x and y. We define these "second partials" along with the notation, give examples, then discuss their meaning.

Notes:

Definition 36 Second Partial Derivative, Mixed Partial Derivative

Let $z = f(x, y)$ be continuous on an open set S.

1. The **second partial derivative of** f **with respect to** x **then** x is

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \left(f_x\right)_x = f_{xx}$$

2. The **second partial derivative of** f **with respect to** x **then** y is

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \left(f_x\right)_y = f_{xy}$$

Similar definitions hold for $\dfrac{\partial^2 f}{\partial y^2} = f_{yy}$ and $\dfrac{\partial^2 f}{\partial x \partial y} = f_{yx}$.

The second partial derivatives f_{xy} and f_{yx} are **mixed partial derivatives**.

Note: The terms in Definition 36 all depend on limits, so each definition comes with the caveat "where the limit exists."

The notation of second partial derivatives gives some insight into the notation of the second derivative of a function of a single variable. If $y = f(x)$, then $f''(x) = \dfrac{d^2y}{dx^2}$. The "$d^2y$" portion means "take the derivative of y twice," while "dx^2" means "with respect to x both times." When we only know of functions of a single variable, this latter phrase seems silly: there is only one variable to take the derivative with respect to. Now that we understand functions of multiple variables, we see the importance of specifying which variables we are referring to.

Example 94 Second partial derivatives

For each of the following, find all six first and second partial derivatives. That is, find

$$f_x, \quad f_y, \quad f_{xx}, \quad f_{yy}, \quad f_{xy} \quad \text{and} \quad f_{yx}.$$

1. $f(x, y) = x^3y^2 + 2xy^3 + \cos x$
2. $f(x, y) = \dfrac{x^3}{y^2}$
3. $f(x, y) = e^x \sin(x^2y)$

Solution In each, we give f_x and f_y immediately and then spend time deriving the second partial derivatives.

Notes:

1. $f(x,y) = x^3y^2 + 2xy^3 + \cos x$

   $f_x(x,y) = 3x^2y^2 + 2y^3 - \sin x$

   $f_y(x,y) = 2x^3y + 6xy^2$

   $f_{xx}(x,y) = \dfrac{\partial}{\partial x}(f_x) = \dfrac{\partial}{\partial x}\left(3x^2y^2 + 2y^3 - \sin x\right) = 6xy^2 - \cos x$

   $f_{yy}(x,y) = \dfrac{\partial}{\partial y}(f_y) = \dfrac{\partial}{\partial y}\left(2x^3y + 6xy^2\right) = 2x^3 + 12xy$

   $f_{xy}(x,y) = \dfrac{\partial}{\partial y}(f_x) = \dfrac{\partial}{\partial y}\left(3x^2y^2 + 2y^3 - \sin x\right) = 6x^2y + 6y^2$

   $f_{yx}(x,y) = \dfrac{\partial}{\partial x}(f_x) = \dfrac{\partial}{\partial x}\left(2x^3y + 6xy^2\right) = 6x^2y + 6y^2$

2. $f(x,y) = \dfrac{x^3}{y^2} = x^3y^{-2}$

   $f_x(x,y) = \dfrac{3x^2}{y^2}$

   $f_y(x,y) = -\dfrac{2x^3}{y^3}$

   $f_{xx}(x,y) = \dfrac{\partial}{\partial x}(f_x) = \dfrac{\partial}{\partial x}\left(\dfrac{3x^2}{y^2}\right) = \dfrac{6x}{y^2}$

   $f_{yy}(x,y) = \dfrac{\partial}{\partial y}(f_y) = \dfrac{\partial}{\partial y}\left(-\dfrac{2x^3}{y^3}\right) = \dfrac{6x^3}{y^4}$

   $f_{xy}(x,y) = \dfrac{\partial}{\partial y}(f_x) = \dfrac{\partial}{\partial y}\left(\dfrac{3x^2}{y^2}\right) = -\dfrac{6x^2}{y^3}$

   $f_{yx}(x,y) = \dfrac{\partial}{\partial x}(f_x) = \dfrac{\partial}{\partial x}\left(-\dfrac{2x^3}{y^3}\right) = -\dfrac{6x^2}{y^3}$

3. $f(x,y) = e^x \sin(x^2y)$

   Because the following partial derivatives get rather long, we omit the extra notation and just give the results. In several cases, multiple applications of the Product and Chain Rules will be necessary, followed by some basic combination of like terms.

   $f_x(x,y) = e^x \sin(x^2y) + 2xye^x \cos(x^2y)$

   $f_y(x,y) = x^2e^x \cos(x^2y)$

   $f_{xx}(x,y) = e^x \sin(x^2y) + 4xye^x \cos(x^2y) + 2ye^x \cos(x^2y) - 4x^2y^2e^x \sin(x^2y)$

   $f_{yy}(x,y) = -x^4e^x \sin(x^2y)$

   $f_{xy}(x,y) = x^2e^x \cos(x^2y) + 2xe^x \cos(x^2y) - 2x^3ye^x \sin(x^2y)$

   $f_{yx}(x,y) = x^2e^x \cos(x^2y) + 2xe^x \cos(x^2y) - 2x^3ye^x \sin(x^2y)$

Notes:

Notice how in each of the three functions in Example 94, $f_{xy} = f_{yx}$. Due to the complexity of the examples, this likely is not a coincidence. The following theorem states that it is not.

Theorem 19 Mixed Partial Derivatives

Let f be defined such that f_{xy} and f_{yx} are continuous on an open set S. Then for each point (x, y) in S, $f_{xy}(x, y) = f_{yx}(x, y)$.

Finding f_{xy} and f_{yx} independently and comparing the results provides a convenient way of checking our work.

Understanding Second Partial Derivatives

Now that we know *how* to find second partials, we investigate *what* they tell us.

Again we refer back to a function $y = f(x)$ of a single variable. The second derivative of f is "the derivative of the derivative," or "the rate of change of the rate of change." The second derivative measures how much the derivative is changing. If $f''(x) < 0$, then the derivative is getting smaller (so the graph of f is concave down); if $f''(x) > 0$, then the derivative is growing, making the graph of f concave up.

Now consider $z = f(x, y)$. Similar statements can be made about f_{xx} and f_{yy} as could be made about $f''(x)$ above. When taking derivatives with respect to x twice, we measure how much f_x changes with respect to x. If $f_{xx}(x, y) < 0$, it means that as x increases, f_x decreases, and the graph of f will be concave down *in the x-direction*. Using the analogy of standing in the rolling meadow used earlier in this section, f_{xx} measures whether one's path is concave up/down when walking due east.

Similarly, f_{yy} measures the concavity in the y-direction. If $f_{yy}(x, y) > 0$, then f_y is increasing with respect to y and the graph of f will be concave up in the y-direction. Appealing to the rolling meadow analogy again, f_{yy} measures whether one's path is concave up/down when walking due north.

We now consider the mixed partials f_{xy} and f_{yx}. The mixed partial f_{xy} measures how much f_x changes with respect to y. Once again using the rolling meadow analogy, f_x measures the slope if one walks due east. Looking east, begin walking *north* (side–stepping). Is the path towards the east getting steeper? If so, $f_{xy} > 0$. Is the path towards the east not changing in steepness? If so, then $f_{xy} = 0$. A similar thing can be said about f_{yx}: consider the steepness of paths heading north while side–stepping to the east.

The following example examines these ideas with concrete numbers and

Notes:

graphs.

Example 95 Understanding second partial derivatives

Let $z = x^2 - y^2 + xy$. Evaluate the 6 first and second partial derivatives at $(-1/2, 1/2)$ and interpret what each of these numbers mean.

Solution We find that:

$f_x(x, y) = 2x + y$, $f_y(x, y) = -2y + x$, $f_{xx}(x, y) = 2$, $f_{yy}(x, y) = -2$ and $f_{xy}(x, y) = f_{yx}(x, y) = 1$. Thus at $(-1/2, 1/2)$ we have

$$f_x(-1/2, 1/2) = -1/2, \qquad f_y(-1/2, 1/2) = -3/2.$$

The slope of the tangent line at $(-1/2, 1/2, -1/4)$ in the direction of x is $-1/2$: if one moves from that point parallel to the x-axis, the instantaneous rate of change will be $-1/2$. The slope of the tangent line at this point in the direction of y is $-3/2$: if one moves from this point parallel to the y-axis, the instantaneous rate of change will be $-3/2$. These tangents lines are graphed in Figure 11.13(a) and (b), respectively, where the tangent lines are drawn in a solid line.

Now consider only Figure 11.13(a). Three directed tangent lines are drawn (two are dashed), each in the direction of x; that is, each has a slope determined by f_x. Note how as y increases, the slope of these lines get closer to 0. Since the slopes are all negative, getting closer to 0 means the *slopes are increasing.* The slopes given by f_x are increasing as y increases, meaning f_{xy} must be positive.

Since $f_{xy} = f_{yx}$, we also expect f_y to increase as x increases. Consider Figure 11.13(b) where again three directed tangent lines are drawn, this time each in the direction of y with slopes determined by f_y. As x increases, the slopes become less steep (closer to 0). Since these are negative slopes, this means the slopes are increasing.

Thus far we have a visual understanding of f_x, f_y, and $f_{xy} = f_{yx}$. We now interpret f_{xx} and f_{yy}. In Figure 11.13(a), we see a curve drawn where x is held constant at $x = -1/2$: only y varies. This curve is clearly concave down, corresponding to the fact that $f_{yy} < 0$. In part (b) of the figure, we see a similar curve where y is constant and only x varies. This curve is concave up, corresponding to the fact that $f_{xx} > 0$.

Figure 11.13: Understanding the second partial derivatives in Example 95.

Partial Derivatives and Functions of Three Variables

The concepts underlying partial derivatives can be easily extend to more than two variables. We give some definitions and examples in the case of three variables and trust the reader can extend these definitions to more variables if needed.

Notes:

Definition 37 Partial Derivatives with Three Variables

Let $w = f(x, y, z)$ be a continuous function on an open set S in $\mathbb{R}^3$.
The **partial derivative of f with respect to x** is:

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}.$$

Similar definitions hold for $f_y(x, y, z)$ and $f_z(x, y, z)$.

By taking partial derivatives of partial derivatives, we can find second partial derivatives of f with respect to z then y, for instance, just as before.

Example 96 Partial derivatives of functions of three variables

For each of the following, find f_x, f_y, f_z, f_{xz}, f_{yz}, and f_{zz}.

1. $f(x, y, z) = x^2y^3z^4 + x^2y^2 + x^3z^3 + y^4z^4$
2. $f(x, y, z) = x\sin(yz)$

Solution

1. $f_x = 2xy^3z^4 + 2xy^2 + 3x^2z^3$; $f_y = 3x^2y^2z^4 + 2x^2y + 4y^3z^4$;
   $f_z = 4x^2y^3z^3 + 3x^3z^2 + 4y^4z^3$; $f_{xz} = 8xy^3z^3 + 9x^2z^2$;
   $f_{yz} = 12x^2y^2z^3 + 16y^3z^3$; $f_{zz} = 12x^2y^3z^2 + 6x^3z + 12y^4z^2$
2. $f_x = \sin(yz)$; $f_y = xz\cos(yz)$; $f_z = xy\cos(yz)$;
   $f_{xz} = y\cos(yz)$; $f_{yz} = x\cos(yz) - xyz\sin(yz)$; $f_{zz} = -xy^2\sin(xy)$

Higher Order Partial Derivatives

We can continue taking partial derivatives of partial derivatives of partial derivatives of ...; we do not have to stop with second partial derivatives. These higher order partial derivatives do not have a tidy graphical interpretation; nevertheless they are not hard to compute and worthy of some practice.

We do not formally define each higher order derivative, but rather give just a few examples of the notation.

$$f_{xyx}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right) \quad \text{and}$$
$$f_{xyz}(x, y, z) = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right).$$

Notes:

Example 97 Higher order partial derivatives

1. Let $f(x, y) = x^2y^2 + \sin(xy)$. Find f_{xxy} and f_{yxx}.
2. Let $f(x, y, z) = x^3e^{xy} + \cos(z)$. Find f_{xyz}.

Solution

1. To find f_{xxy}, we first find f_x, then f_{xx}, then f_{xxy}:

$$\begin{aligned} f_x &= 2xy^2 + y\cos(xy) \qquad f_{xx} = 2y^2 - y^2\sin(xy) \\ f_{xxy} &= 4y - 2y\sin(xy) - xy^2\cos(xy). \end{aligned}$$

   To find f_{yxx}, we first find f_y, then f_{yx}, then f_{yxx}:

$$\begin{aligned} f_y &= 2x^2y + x\cos(xy) \qquad f_{yx} = 4xy + \cos(xy) - xy\sin(xy) \\ f_{yxx} &= 4y - y\sin(xy) - \big(y\sin(xy) + xy^2\cos(xy)\big) \\ &= 4y - 2y\sin(xy) - xy^2\cos(xy). \end{aligned}$$

   Note how $f_{xxy} = f_{yxx}$.

2. To find f_{xyz}, we find f_x, then f_{xy}, then f_{xyz}:

$$\begin{aligned} f_x &= 3x^2e^{xy} + x^3ye^{xy} \qquad f_{xy} = 3x^3e^{xy} + x^3e^{xy} + x^4ye^{xy} = 4x^3e^{xy} + x^4ye^{xy} \\ f_{xyz} &= 0. \end{aligned}$$

In the previous example we saw that $f_{xxy} = f_{yxx}$; this is not a coincidence. While we do not state this as a formal theorem, as long as each partial derivative is continuous, it does not matter the order in which the partial derivatives are taken. For instance, $f_{xxy} = f_{xyx} = f_{yxx}$.

This can be useful at times. Had we known this, the second part of Example 97 would have been much simpler to compute. Instead of computing f_{xyz} in the x, y then z orders, we could have applied the z, then x then y order (as $f_{xyz} = f_{zxy}$). It is easy to see that $f_z = -\sin z$; then f_{zx} and f_{zxy} are clearly 0 as f_z does not contain an x or y.

Notes:

A brief review of this section: partial derivatives measure the instantaneous rate of change of a multivariable function with respect to one variable. With $z = f(x, y)$, the partial derivatives f_x and f_y measure the instantaneous rate of change of z when moving parallel to the x- and y-axes, respectively. How do we measure the rate of change at a point when we do not move parallel to one of these axes? What if we move in the direction given by the vector $\langle 2, 1\rangle$? Can we measure that rate of change? The answer is, of course, yes, we can. This is the topic of Section 11.6. First, we need to define what it means for a function of two variables to be *differentiable.*

Notes:

Exercises 11.3

Terms and Concepts

1. What is the difference between a constant and a coefficient?

2. Given a function $z = f(x, y)$, explain in your own words how to compute f_x.

3. In the mixed partial fraction f_{xy}, which is computed first, f_x or f_y?

4. In the mixed partial fraction $\dfrac{\partial^2 f}{\partial x \partial y}$, which is computed first, f_x or f_y?

Problems

In Exercises 5 – 8, evaluate $f_x(x, y)$ and $f_y(x, y)$ at the indicated point.

5. $f(x, y) = x^2y - x + 2y + 3$ at $(1, 2)$

6. $f(x, y) = x^3 - 3x + y^2 - 6y$ at $(-1, 3)$

7. $f(x, y) = \sin y \cos x$ at $(\pi/3, \pi/3)$

8. $f(x, y) = \ln(xy)$ at $(-2, -3)$

In Exercises 9 – 26, find f_x, f_y, f_{xx}, f_{yy}, f_{xy} and f_{yx}.

9. $f(x, y) = x^2y + 3x^2 + 4y - 5$

10. $f(x, y) = y^3 + 3xy^2 + 3x^2y + x^3$

11. $f(x, y) = \dfrac{x}{y}$

12. $f(x, y) = \dfrac{4}{xy}$

13. $f(x, y) = e^{x^2+y^2}$

14. $f(x, y) = e^{x+2y}$

15. $f(x, y) = \sin x \cos y$

16. $f(x, y) = (x + y)^3$

17. $f(x, y) = \cos(5xy^3)$

18. $f(x, y) = \sin(5x^2 + 2y^3)$

19. $f(x, y) = \sqrt{4xy^2 + 1}$

20. $f(x, y) = (2x + 5y)\sqrt{y}$

21. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$

22. $f(x, y) = 5x - 17y$

23. $f(x, y) = 3x^2 + 1$

24. $f(x, y) = \ln(x^2 + y)$

25. $f(x, y) = \dfrac{\ln x}{4y}$

26. $f(x, y) = 5e^x \sin y + 9$

In Exercises 27 – 30, form a function $z = f(x, y)$ such that f_x and f_y match those given.

27. $f_x = \sin y + 1, \quad f_y = x \cos y$

28. $f_x = x + y, \quad f_y = x + y$

29. $f_x = 6xy - 4y^2, \quad f_y = 3x^2 - 8xy + 2$

30. $f_x = \dfrac{2x}{x^2 + y^2}, \quad f_y = \dfrac{2y}{x^2 + y^2}$

In Exercises 31 – 34, find f_x, f_y, f_z, f_{yz} and f_{zy}.

31. $f(x, y, z) = x^2e^{2y-3z}$

32. $f(x, y, z) = x^3y^2 + x^3z + y^2z$

33. $f(x, y, z) = \dfrac{3x}{7y^2z}$

34. $f(x, y, z) = \ln(xyz)$

11.4 Differentiability and the Total Differential

We studied **differentials** in Section **??**, where Definition **??** states that if $y = f(x)$ and f is differentiable, then $dy = f'(x)dx$. One important use of this differential is in Integration by Substitution. Another important application is approximation. Let $\Delta x = dx$ represent a change in x. When dx is small, $dy \approx \Delta y$, the change in y resulting from the change in x. Fundamental in this understanding is this: as dx gets small, the difference between Δy and dy goes to 0. Another way of stating this: as dx goes to 0, the *error* in approximating Δy with dy goes to 0.

We extend this idea to functions of two variables. Let $z = f(x, y)$, and let $\Delta x = dx$ and $\Delta y = dy$ represent changes in x and y, respectively. Let $\Delta z = f(x+dx, y+dy) - f(x, y)$ be the change in z over the change in x and y. Recalling that f_x and f_y give the instantaneous rates of z-change in the x- and y-directions, respectively, we can approximate Δz with $dz = f_x dx + f_y dy$; in words, the total change in z is approximately the change caused by changing x plus the change caused by changing y. In a moment we give an indication of whether or not this approximation is any good. First we give a name to dz.

Definition 38 Total Differential

Let $z = f(x, y)$ be continuous on an open set S. Let dx and dy represent changes in x and y, respectively. Where the partial derivatives f_x and f_y exist, the **total differential of** z is

$$dz = f_x(x, y)dx + f_y(x, y)dy.$$

Example 98 Finding the total differential

Let $z = x^4 e^{3y}$. Find dz.

Solution We compute the partial derivatives: $f_x = 4x^3 e^{3y}$ and $f_y = 3x^4 e^{3y}$. Following Definition 38, we have

$$dz = 4x^3 e^{3y} dx + 3x^4 e^{3y} dy.$$

We *can* approximate Δz with dz, but as with all approximations, there is error involved. A good approximation is one in which the error is small. At a given point (x_0, y_0), let E_x and E_y be functions of dx and dy such that $E_x dx + E_y dy$ describes this error. Then

$$\begin{aligned}\Delta z &= dz + E_x dx + E_y dy \\ &= f_x(x_0, y_0)dx + f_y(x_0, y_0)dy + E_x dx + E_y dy.\end{aligned}$$

Notes:

If the approximation of Δz by dz is good, then as dx and dy get small, so does $E_x dx + E_y dy$. The approximation of Δz by dz is even better if, as dx and dy go to 0, so do E_x and E_y. This leads us to our definition of differentiability.

Definition 39 Multivariable Differentiability

Let $z = f(x, y)$ be defined on an open set S containing (x_0, y_0) where $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist. Let dz be the total differential of z at (x_0, y_0), let $\Delta z = f(x_0 + dx, y_0 + dy) - f(x_0, y_0)$, and let E_x and E_y be functions of dx and dy such that

$$\Delta z = dz + E_x dx + E_y dy.$$

1. f is **differentiable at** (x_0, y_0) if, given $\varepsilon > 0$, there is a $\delta > 0$ such that if $|| \langle dx, dy \rangle || < \delta$, then $|| \langle E_x, E_y \rangle || < \varepsilon$. That is, as dx and dy go to 0, so do E_x and E_y.
2. f is **differentiable on** S if f is differentiable at every point in S. If f is differentiable on $\mathbb{R}^2$, we say that f is **differentiable everywhere**.

Example 99 Showing a function is differentiable

Show $f(x, y) = xy + 3y^2$ is differentiable using Definition 39.

Solution We begin by finding $f(x + dx, y + dy)$, Δz, f_x and f_y.

$$\begin{aligned} f(x + dx, y + dy) &= (x + dx)(y + dy) + 3(y + dy)^2 \\ &= xy + xdy + ydx + dxdy + 3y^2 + 6ydy + 3dy^2. \end{aligned}$$

$\Delta z = f(x + dx, y + dy) - f(x, y)$, so

$$\Delta z = xdy + ydx + dxdy + 6ydy + 3dy^2.$$

It is straightforward to compute $f_x = y$ and $f_y = x + 6y$. Consider once more Δz:

$$\begin{aligned} \Delta z &= xdy + ydx + dxdy + 6ydy + 3dy^2 \qquad \text{(now reorder)} \\ &= ydx + xdy + 6ydy + dxdy + 3dy^2 \\ &= \underbrace{(y)}_{f_x} dx + \underbrace{(x + 6y)}_{f_y} dy + \underbrace{(dy)}_{E_x} dx + \underbrace{(3dy)}_{E_y} dy \\ &= f_x dx + f_y dy + E_x dx + E_y dy. \end{aligned}$$

With $E_x = dy$ and $E_y = 3dy$, it is clear that as dx and dy go to 0, E_x and E_y also go to 0. Since this did not depend on a specific point (x_0, y_0), we can say that $f(x, y)$

Notes:

is differentiable for all pairs (x, y) in $\mathbb{R}^2$, or, equivalently, that f is differentiable everywhere.

Our intuitive understanding of differentiability of functions $y = f(x)$ of one variable was that the graph of f was "smooth." A similar intuitive understanding of functions $z = f(x, y)$ of two variables is that the surface defined by f is also "smooth," not containing cusps, edges, breaks, etc. The following theorem states that differentiable functions are continuous, followed by another theorem that provides a more tangible way of determining whether a great number of functions are differentiable or not.

Theorem 20 Continuity and Differentiability of Multivariable Functions

Let $z = f(x, y)$ be defined on an open set S containing (x_0, y_0). If f is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

Theorem 21 Differentiability of Multivariable Functions

Let $z = f(x, y)$ be defined on an open set S containing (x_0, y_0). If f_x and f_y are both continuous on S, then f is differentiable on S.

The theorems assure us that essentially all functions that we see in the course of our studies here are differentiable (and hence continuous) on their natural domains. There is a difference between Definition 39 and Theorem 21, though: it is possible for a function f to be differentiable yet f_x and/or f_y is *not* continuous. Such strange behavior of functions is a source of delight for many mathematicians.

When f_x and f_y exist at a point but are not continuous at that point, we need to use other methods to determine whether or not f is differentiable at that point.

For instance, consider the function

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Notes:

We can find $f_x(0,0)$ and $f_y(0,0)$ using Definition 35:

$$\begin{aligned} f_x(0,0) &= \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{0}{h^2} = 0; \\ f_y(0,0) &= \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{0}{h^2} = 0. \end{aligned}$$

Both f_x and f_y *exist* at $(0,0)$, but they are not continuous at $(0,0)$, as

$$f_x(x,y) = \frac{y(y^2-x^2)}{(x^2+y^2)^2} \qquad \text{and} \qquad f_y(x,y) = \frac{x(x^2-y^2)}{(x^2+y^2)^2}$$

are not continuous at $(0,0)$. (Take the limit of f_x as $(x,y)\to(0,0)$ along the x- and y-axes; they give different results.) So even though f_x and f_y *exist* at every point in the x-y plane, they are not continuous. Therefore it is possible, by Theorem 21, for f to not be differentiable.

Indeed, it is not. One can show that f is not continuous at $(0,0)$ (see Example 87), and by Theorem 20, this means f is not differentiable at $(0,0)$.

Approximating with the Total Differential

By the definition, when f is differentiable dz is a good approximation for Δz when dx and dy are small. We give some simple examples of how this is used here.

Example 100 Approximating with the total differential

Let $z = \sqrt{x}\sin y$. Approximate $f(4.1, 0.8)$.

Solution Recognizing that $\pi/4 \approx 0.785 \approx 0.8$, we can approximate $f(4.1,0.8)$ using $f(4,\pi/4)$. We can easily compute $f(4,\pi/4) = \sqrt{4}\sin(\pi/4) = 2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2} \approx 1.414$. Without calculus, this is the best approximation we could reasonably come up with. The total differential gives us a way of adjusting this initial approximation to hopefully get a more accurate answer.

We let $\Delta z = f(4.1,0.8) - f(4,\pi/4)$. The total differential dz is approximately equal to Δz, so

$$f(4.1,0.8) - f(4,\pi/4) \approx dz \quad\Rightarrow\quad f(4.1,0.8) \approx dz + f(4,\pi/4). \tag{11.1}$$

To find dz, we need f_x and f_y.

Notes:

$$f_x(x,y) = \frac{\sin y}{2\sqrt{x}} \quad \Rightarrow \quad f_x(4,\pi/4) = \frac{\sin \pi/4}{2\sqrt{4}} = \frac{\sqrt{2}/2}{4} = \sqrt{2}/8.$$

$$f_y(x,y) = \sqrt{x}\cos y \quad \Rightarrow \quad f_y(4,\pi/4) = \sqrt{4}\frac{\sqrt{2}}{2} = \sqrt{2}.$$

Approximating 4.1 with 4 gives $dx = 0.1$; approximating 0.8 with $\pi/4$ gives $dy \approx 0.015$. Thus

$$\begin{aligned} dz(4,\pi/4) &= f_x(4,\pi/4)(0.1) + f_y(4,\pi/4)(0.015) \\ &= \frac{\sqrt{2}}{8}(0.1) + \sqrt{2}(0.015) \\ &\approx 0.039. \end{aligned}$$

Returning to Equation (11.1), we have

$$f(4.1, 0.8) \approx 0.039 + 1.414 = 1.4531.$$

We, of course, can compute the actual value of $f(4.1, 0.8)$ with a calculator; the actual value, accurate to 5 places after the decimal, is 1.45254. Obviously our approximation is quite good.

The point of the previous example was *not* to develop an approximation method for known functions. After all, we can very easily compute $f(4.1, 0.8)$ using readily available technology. Rather, it serves to illustrate how well this method of approximation works, and to reinforce the following concept:

"New position = old position + amount of change," so
"New position $\approx$ old position + approximate amount of change."

In the previous example, we could easily compute $f(4, \pi/4)$ and could approximate the amount of z-change when computing $f(4.1, 0.8)$, letting us approximate the new z-value.

It may be surprising to learn that it is not uncommon to know the values of f, f_x and f_y at a particular point without actually knowing the function f. The total differential gives a good method of approximating f at nearby points.

Example 101 Approximating an unknown function

Given that $f(2,-3) = 6$, $f_x(2,-3) = 1.3$ and $f_y(2,-3) = -0.6$, approximate $f(2.1, -3.03)$.

Notes:

Solution The total differential approximates how much f changes from the point $(2, -3)$ to the point $(2.1, -3.03)$. With $dx = 0.1$ and $dy = -0.03$, we have

$$\begin{aligned} dz &= f_x(2,-3)dx + f_y(2,-3)dy \\ &= 1.3(0.1) + (-0.6)(-0.03) \\ &= 0.148. \end{aligned}$$

The change in z is approximately 0.148, so we approximate $f(2.1, -3.03) \approx 6.148$.

Error/Sensitivity Analysis

The total differential gives an approximation of the change in z given small changes in x and y. We can use this to approximate error propagation; that is, if the input is a little off from what it should be, how far from correct will the output be? We demonstrate this in an example.

Example 102 Sensitivity analysis

A cylindrical steel storage tank is to be built that is 10ft tall and 4ft across in diameter. It is known that the steel will expand/contract with temperature changes; is the overall volume of the tank more sensitive to changes in the diameter or in the height of the tank?

Solution A cylindrical solid with height h and radius r has volume $V = \pi r^2 h$. We can view V as a function of two variables, r and h. We can compute partial derivatives of V:

$$\frac{\partial V}{\partial r} = V_r(r,h) = 2\pi rh \qquad \text{and} \qquad \frac{\partial V}{\partial h} = V_h(r,h) = \pi r^2.$$

The total differential is $dV = (2\pi rh)dr + (\pi r^2)dh$. When $h = 10$ and $r = 2$, we have $dV = 40\pi dr + 4\pi dh$. Note that the coefficient of dr is $40\pi \approx 125.7$; the coefficient of dh is a tenth of that, approximately 12.57. A small change in radius will be multiplied by 125.7, whereas a small change in height will be multiplied by 12.57. Thus the volume of the tank is more sensitive to changes in radius than in height.

The previous example showed that the volume of a particular tank was more sensitive to changes in radius than in height. Keep in mind that this analysis only applies to a tank of those dimensions. A tank with a height of 1ft and radius of 5ft would be more sensitive to changes in height than in radius.

Notes:

One could make a chart of small changes in radius and height and find exact changes in volume given specific changes. While this provides exact numbers, it does not give as much insight as the error analysis using the total differential.

Differentiability of Functions of Three Variables

The definition of differentiability for functions of three variables is very similar to that of functions of two variables. We again start with the total differential.

Definition 40 Total Differential

Let $w = f(x, y, z)$ be continuous on an open set S. Let dx, dy and dz represent changes in x, y and z, respectively. Where the partial derivatives f_x, f_y and f_z exist, the **total differential of** w is

$$dz = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz.$$

This differential can be a good approximation of the change in w when $w = f(x, y, z)$ is **differentiable**.

Definition 41 Multivariable Differentiability

Let $w = f(x, y, z)$ be defined on an open ball B containing (x_0, y_0, z_0) where $f_x(x_0, y_0, z_0)$, $f_y(x_0, y_0, z_0)$ and $f_z(x_0, y_0, z_0)$ exist. Let dw be the total differential of w at (x_0, y_0, z_0), let $\Delta w = f(x_0 + dx, y_0 + dy, z_0 + dz) - f(x_0, y_0, z_0)$, and let E_x, E_y and E_z be functions of dx, dy and dz such that

$$\Delta w = dw + E_x dx + E_y dy + E_z dz.$$

1. f is **differentiable at** (x_0, y_0, z_0) if, given $\varepsilon > 0$, there is a $\delta > 0$ such that if $|| \langle dx, dy, dz \rangle || < \delta$, then $|| \langle E_x, E_y, E_z \rangle || < \varepsilon$.
2. f is **differentiable on** B if f is differentiable at every point in B. If f is differentiable on $\mathbb{R}^3$, we say that f is **differentiable everywhere**.

Just as before, this definition gives a rigorous statement about what it means to be differentiable that is not very intuitive. We follow it with a theorem similar to Theorem 21.

Notes:

Theorem 22 Continuity and Differentiability of Functions of Three Variables

Let $w = f(x, y, z)$ be defined on an open ball B containing (x_0, y_0, z_0).

1. If f is differentiable at (x_0, y_0, z_0), then f is continuous at (x_0, y_0, z_0).
2. If f_x, f_y and f_z are continuous on B, then f is differentiable on B.

This set of definition and theorem extends to functions of any number of variables. The theorem again gives us a simple way of verifying that most functions that we enounter are differentiable on their natural domains.

This section has given us a formal definition of what it means for a functions to be "differentiable," along with a theorem that gives a more accessible understanding. The following sections return to notions prompted by our study of partial derivatives that make use of the fact that most functions we encounter are differentiable.

Notes:

Exercises 11.4

Terms and Concepts

1. T/F: If $f(x, y)$ is differentiable on S, the f is continuous on S.

2. T/F: If f_x and f_y are continuous on S, then f is differentiable on S.

3. T/F: If $z = f(x, y)$ is differentiable, then the change in z over small changes dx and dy in x and y is approximately dz.

4. Finish the sentence: "The new z-value is approximately the old z-value plus the approximate _____."

Problems

In Exercises 5 – 8, find the total differential dz.

5. $z = x \sin y + x^2$

6. $z = (2x^2 + 3y)^2$

7. $z = 5x - 7y$

8. $z = xe^{x+y}$

In Exercises 9 – 12, a function $z = f(x, y)$ is given. Give the indicated approximation using the total differential.

9. $f(x, y) = \sqrt{x^2 + y}$. Approximate $f(2.95, 7.1)$ knowing $f(3, 7) = 4$.

10. $f(x, y) = \sin x \cos y$. Approximate $f(0.1, -0.1)$ knowing $f(0, 0) = 0$.

11. $f(x, y) = x^2y - xy^2$. Approximate $f(2.04, 3.06)$ knowing $f(2, 3) = -6$.

12. $f(x, y) = \ln(x - y)$. Approximate $f(5.1, 3.98)$ knowing $f(5, 4) = 0$.

Exercises 13 – 16 ask a variety of questions dealing with approximating error and sensitivity analysis.

13. A cylindrical storage tank is to be 2ft tall with a radius of 1ft. Is the volume of the tank more sensitive to changes in the radius or the height?

14. **Projectile Motion:** The x-value of an object moving under the principles of projectile motion is $x(\theta, v_0, t) = (v_0 \cos\theta)t$. A particular projectile is fired with an initial velocity of $v_0 = 250$ft/s and an angle of elevation of $\theta = 60^\circ$. It travels a distance of 375ft in 3 seconds.

    Is the projectile more sensitive to errors in initial speed or angle of elevation?

15. The length ℓ of a long wall is to be approximated. The angle θ, as shown in the diagram (not to scale), is measured to be 85°, and the distance x is measured to be 30'. Assume that the triangle formed is a right triangle.

    Is the measurement of the length of ℓ more sensitive to errors in the measurement of x or in θ?

16. It is "common sense" that it is far better to measure a long distance with a long measuring tape rather than a short one. A measured distance D can be viewed as the product of the length ℓ of a measuring tape times the number n of times it was used. For instance, using a 3' tape 10 times gives a length of 30'. To measure the same distance with a 12' tape, we would use the tape 2.5 times. (I.e., $30 = 12 \times 2.5$.) Thus $D = n\ell$.

    Suppose each time a measurement is taken with the tape, the recorded distance is within 1/16" of the actual distance. (I.e., $d\ell = 1/16'' \approx 0.005$ft). Using differentials, show why common sense proves correct in that it is better to use a long tape to measure long distances.

In Exercises 17 – 18, find the total differential dw.

17. $w = x^2yz^3$

18. $w = e^x \sin y \ln z$

In Exercises 19 – 22, use the information provided and the total differential to make the given approximation.

19. $f(3, 1) = 7$, $f_x(3, 1) = 9$, $f_y(3, 1) = -2$. Approximate $f(3.05, 0.9)$.

20. $f(-4, 2) = 13$, $f_x(-4, 2) = 2.6$, $f_y(-4, 2) = 5.1$. Approximate $f(-4.12, 2.07)$.

21. $f(2, 4, 5) = -1$, $f_x(2, 4, 5) = 2$, $f_y(2, 4, 5) = -3$, $f_z(2, 4, 5) = 3.7$. Approximate $f(2.5, 4.1, 4.8)$.

22. $f(3, 3, 3) = 5$, $f_x(3, 3, 3) = 2$, $f_y(3, 3, 3) = 0$, $f_z(3, 3, 3) = -2$. Approximate $f(3.1, 3.1, 3.1)$.

11.5 The Multivariable Chain Rule

The Chain Rule, as learned in Section **??**, states that $\dfrac{d}{dx}\Big(f\big(g(x)\big)\Big) = f'\big(g(x)\big)g'(x)$. If $t = g(x)$, we can express the Chain Rule as

$$\frac{df}{dx} = \frac{df}{dt}\frac{dt}{dx}.$$

In this section we extend the Chain Rule to functions of more than one variable.

Theorem 23 Multivariable Chain Rule, Part I

Let $z = f(x, y)$, $x = g(t)$ and $y = h(t)$, where f, g and h are differentiable functions. Then $z = f(x, y) = f\big(g(t), h(t)\big)$ is a function of t, and

$$\begin{aligned}\frac{dz}{dt} = \frac{df}{dt} &= f_x(x, y)\frac{dx}{dt} + f_y(x, y)\frac{dy}{dt}\\ &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.\end{aligned}$$

Figure 11.14: Understanding the application of the Multivariable Chain Rule.

It is good to understand what the situation of $z = f(x, y)$, $x = g(t)$ and $y = h(t)$ describes. We know that $z = f(x, y)$ describes a surface; we also recognize that $x = g(t)$ and $y = h(t)$ are parametric equations for a curve in the x-y plane. Combining these together, we are describing a curve that lies on the surface described by f. The parametric equations for this curve are $x = g(t)$, $y = h(t)$ and $z = f\big(g(t), h(t)\big)$.

Consider Figure 11.14 in which a surface is drawn, along with a dashed curve in the x-y plane. Restricting f to just the points on this circle gives the curve shown on the surface. The derivative $\frac{df}{dt}$ gives the instantaneous rate of change of f with respect to t. If we consider an object traveling along this path, $\frac{df}{dt}$ gives the rate at which the object rises/falls.

We now practice applying the Multivariable Chain Rule.

Example 103 Using the Multivariable Chain Rule

Let $z = x^2y + x$, where $x = \sin t$ and $y = e^{5t}$. Find $\dfrac{dz}{dt}$ using the Chain Rule.

Solution Following Theorem 23, we find

$$f_x(x, y) = 2xy + 1, \qquad f_y(x, y) = x^2, \qquad \frac{dx}{dt} = \cos t, \qquad \frac{dy}{dt} = 5e^{5t}.$$

Notes:

Applying the theorem, we have

$$\frac{dz}{dt} = (2xy + 1)\cos t + 5x^2e^{5t}.$$

This may look odd, as it seems that $\frac{dz}{dt}$ is a function of x, y and t. Since x and y are functions of t, $\frac{dz}{dt}$ is really just a function of t, and we can replace x with $\sin t$ and y with e^{5t}:

$$\frac{dz}{dt} = (2xy + 1)\cos t + 5x^2e^{5t} = (2\sin(t)e^{5t} + 1)\cos t + 5e^{5t}\sin^2 t.$$

The previous example can make us wonder: if we substituted for x and y at the end to show that $\frac{dz}{dt}$ is really just a function of t, why not substitute before differentiating, showing clearly that z is a function of t?

That is, $z = x^2y + x = (\sin t)^2e^{5t} + \sin t$. Applying the Chain and Product Rules, we have

$$\frac{dz}{dt} = 2\sin t\cos t\, e^{5t} + 5\sin^2 t\, e^{5t} + \cos t,$$

which matches the result from the example.

This may now make one wonder "What's the point? If we could already find the derivative, why learn another way of finding it?" In some cases, applying this rule makes deriving simpler, but this is hardly the power of the Chain Rule. Rather, in the case where $z = f(x, y)$, $x = g(t)$ and $y = h(t)$, the Chain Rule is extremely powerful when *we do not know what f, g and/or h are*. It may be hard to believe, but often in "the real world" we know rate–of–change information (i.e., information about derivatives) without explicitly knowing the underlying functions. The Chain Rule allows us to combine several rates of change to find another rate of change. The Chain Rule also has theoretic use, giving us insight into the behavior of certain constructions (as we'll see in the next section).

We demonstrate this in the next example.

Example 104 Applying the Multivarible Chain Rule

An object travels along a path on a surface. The exact path and surface are not known, but at time $t = t_0$ it is known that :

$$\frac{\partial z}{\partial x} = 5, \qquad \frac{\partial z}{\partial y} = -2, \qquad \frac{dx}{dt} = 3 \qquad \text{and} \qquad \frac{dy}{dt} = 7.$$

Find $\frac{dz}{dt}$ at time t_0.

Notes:

Solution The Multivariable Chain Rule states that

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\\ &= 5(3) + (-2)(7)\\ &= 1.\end{aligned}$$

By knowing certain rates–of–change information about the surface and about the path of the particle in the x-y plane, we can determine how quickly the object is rising/falling.

We next apply the Chain Rule to solve a max/min problem.

Example 105 Applying the Multivariable Chain Rule

Consider the surface $z = x^2 + y^2 - xy$, a paraboloid, on which a particle moves with x and y coordinates given by $x = \cos t$ and $y = \sin t$. Find $\frac{dz}{dt}$ when $t = 0$, and find where the particle reaches its maximum/minimum z-values.

Solution It is straightforward to compute

$$f_x(x,y) = 2x - y, \qquad f_y(x,y) = 2y - x, \qquad \frac{dx}{dt} = -\sin t, \qquad \frac{dy}{dt} = \cos t.$$

Combining these according to the Chain Rule gives:

$$\frac{dz}{dt} = -(2x - y)\sin t + (2y - x)\cos t.$$

When $t = 0$, $x = 1$ and $y = 0$. Thus $\dfrac{dz}{dt} = -(2)(0) + (-1)(1) = -1$. When $t = 0$, the particle is moving down, as shown in Figure 11.15.

To find where z-value is maximized/minimized on the particle's path, we set $\frac{dz}{dt} = 0$ and solve for t:

$$\begin{aligned}\frac{dz}{dt} = 0 &= -(2x - y)\sin t + (2y - x)\cos t\\ 0 &= -(2\cos t - \sin t)\sin t + (2\sin t - \cos t)\cos t\\ 0 &= \sin^2 t - \cos^2 t\\ \cos^2 t &= \sin^2 t\\ t &= n\frac{\pi}{4} \quad \text{(for odd } n\text{)}\end{aligned}$$

We can use the First Derivative Test to find that on $[0, 2\pi]$, z has reaches its absolute minimum at $t = \pi/4$ and $5\pi/4$; it reaches its absolute maximum at

Figure 11.15: Plotting the path of a particle on a surface in Example 105.

Notes:

$t = 3\pi/4$ and $7\pi/4$, as shown in Figure 11.15.

We can extend the Chain Rule to include the situation where z is a function of more than one variable, and each of these variables is also a function of more than one variable. The basic case of this is where $z = f(x, y)$, and x and y are functions of two variables, say s and t.

Theorem 24 Multivariable Chain Rule, Part II

1. Let $z = f(x, y)$, $x = g(s, t)$ and $y = h(s, t)$, where f, g and h are differentiable functions. Then z is a function of s and t, and
   - $\dfrac{\partial z}{\partial s} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial s}$, and
   - $\dfrac{\partial z}{\partial t} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial t}$.
2. Let $z = f(x_1, x_2, \ldots, x_m)$ be a differentiable function of m variables, where each of the x_i is a differentiable function of the variables $t_1, t_2, \ldots, t_n$. Then z is a function of the t_i, and

$$\frac{\partial z}{\partial t_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial f}{\partial x_m}\frac{\partial x_m}{\partial t_i}.$$

Example 106 Using the Multivarible Chain Rule, Part II

Let $z = x^2y + x$, $x = s^2 + 3t$ and $y = 2s - t$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$, and evaluate each when $s = 1$ and $t = 2$.

Solution Following Theorem 24, we compute the following partial derivatives:

$$\frac{\partial f}{\partial x} = 2xy + 1 \qquad \frac{\partial f}{\partial y} = x^2,$$

$$\frac{\partial x}{\partial s} = 2s \qquad \frac{\partial x}{\partial t} = 3 \qquad \frac{\partial y}{\partial s} = 2 \qquad \frac{\partial y}{\partial t} = -1.$$

Thus

$$\frac{\partial z}{\partial s} = (2xy + 1)(2s) + (x^2)(2) = 4xys + 2s + 2x^2, \quad \text{and}$$

$$\frac{\partial z}{\partial t} = (2xy + 1)(3) + (x^2)(-1) = 6xy - x^2 + 3.$$

Notes:

When $s = 1$ and $t = 2$, $x = 7$ and $y = 0$, so

$$\frac{\partial z}{\partial s} = 100 \qquad \text{and} \qquad \frac{\partial z}{\partial t} = -46.$$

Example 107 Using the Multivarible Chain Rule, Part II

Let $w = xy + z^2$, where $x = t^2 e^s$, $y = t\cos s$, and $z = s\sin t$. Find $\frac{\partial w}{\partial t}$ when $s = 0$ and $t = \pi$.

Solution Following Theorem 24, we compute the following partial derivatives:

$$\frac{\partial f}{\partial x} = y \qquad \frac{\partial f}{\partial y} = x \qquad \frac{\partial f}{\partial z} = 2z,$$

$$\frac{\partial x}{\partial t} = 2te^s \qquad \frac{\partial y}{\partial t} = \cos s \qquad \frac{\partial z}{\partial t} = s\cos t.$$

Thus

$$\frac{\partial w}{\partial t} = y(2te^s) + x(\cos s) + 2z(s\cos t).$$

When $s = 0$ and $t = \pi$, we have $x = \pi^2$, $y = \pi$ and $z = 0$. Thus

$$\frac{\partial w}{\partial t} = \pi(2\pi) + \pi^2 = 3\pi^2.$$

Implicit Differentiation

We studied finding $\frac{dy}{dx}$ when y is given as an implicit function of x in detail in Section **??**. We find here that the Multivariable Chain Rule gives a simpler method of finding $\frac{dy}{dx}$.

For instance, consider the implicit function $x^2y - xy^3 = 3$. We learned to use the following steps to find $\frac{dy}{dx}$:

$$\begin{aligned} \frac{d}{dx}\left(x^2y - xy^3\right) &= \frac{d}{dx}\left(3\right) \\ 2xy + x^2\frac{dy}{dx} - y^3 - 3xy^2\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{2xy - y^3}{x^2 - 3xy^2}. \end{aligned} \tag{11.2}$$

Instead of using this method, consider $z = x^2y - xy^3$. The implicit function above describes the level curve $z = 3$. Considering x and y as functions of x, the Multivariable Chain Rule states that

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\frac{dx}{dx} + \frac{\partial z}{\partial y}\frac{dy}{dx}. \tag{11.3}$$

Notes:

Since z is constant (in our example, $z = 3$), $\frac{dz}{dx} = 0$. We also know $\frac{dx}{dx} = 1$. Equation (11.3) becomes

$$\begin{aligned} 0 &= \frac{\partial z}{\partial x}(1) + \frac{\partial z}{\partial y}\frac{dy}{dx} \quad \Rightarrow \\ \frac{dy}{dx} &= -\frac{\partial z}{\partial x}\Big/\frac{\partial z}{\partial y} \\ &= -\frac{f_x}{f_y}. \end{aligned}$$

Note how our solution for $\frac{dy}{dx}$ in Equation (11.2) is just the partial derivative of z with respect to x, divided by the partial derivative of z with respect to y.

We state the above as a theorem.

Theorem 25 Implicit Differentiation

Let f be a differentiable function of x and y, where $f(x, y) = c$ defines y as an implicit function of x, for some constant c. Then

$$\frac{dy}{dx} = -\frac{f_x(x, y)}{f_y(x, y)}.$$

We practice using Theorem 25 by applying it to a problem from Section **??**.

Example 108 Implicit Differentiation

Given the implicitly defined function $\sin(x^2y^2) + y^3 = x + y$, find y'. Note: this is the same problem as given in Example **??** of Section **??**, where the solution took about a full page to find.

Solution Let $f(x, y) = \sin(x^2y^2) + y^3 - x - y$; the implicitly defined function above is equivalent to $f(x, y) = 0$. We find $\frac{dy}{dx}$ by applying Theorem 25. We find

$$f_x(x, y) = 2xy^2\cos(x^2y^2) - 1 \qquad \text{and} \qquad f_y(x, y) = 2x^2y\cos(x^2y^2) + 3y^2 - 1,$$

so

$$\frac{dy}{dx} = -\frac{2xy^2\cos(x^2y^2) - 1}{2x^2y\cos(x^2y^2) + 3y^2 - 1},$$

which matches our solution from Example **??**.

Notes:

Exercises 11.5

Terms and Concepts

1. Let a level curve of $z = f(x, y)$ be described by $x = g(t)$, $y = h(t)$. Explain why $\frac{dz}{dt} = 0$.

2. Fill in the blank: The single variable Chain Rule states $\dfrac{d}{dx}\Big(f\big(g(x)\big)\Big) = f'\big(g(x)\big)\cdot$_____.

3. Fill in the blank: The Multivariable Chain Rule states $\dfrac{df}{dt} = \dfrac{\partial f}{\partial x}\cdot$ _____ $+$ _____ $\cdot\dfrac{dy}{dt}$.

4. If $z = f(x, y)$, where $x = g(t)$ and $y = h(t)$, we can substitute and write z as an explicit function of t.
T/F: Using the Multivariable Chain Rule to find $\frac{dz}{dt}$ is sometimes easier than first substituting and then taking the derivative.

5. T/F: The Multivariable Chain Rule is only useful when all the related functions are known explicitly.

6. The Multivariable Chain Rule allows us to compute implicit derivatives easily by just computing two _____ derivatives.

Problems

In Exercises 7 – 12, functions $z = f(x, y)$, $x = g(t)$ and $y = h(t)$ are given.

(a) Use the Multivariable Chain Rule to compute $\dfrac{dz}{dt}$.

(b) Evaluate $\dfrac{dz}{dt}$ at the indicated t-value.

7. $z = 3x + 4y, \quad x = t^2, \quad y = 2t; \quad t = 1$

8. $z = x^2 - y^2, \quad x = t, \quad y = t^2 - 1; \quad t = 1$

9. $z = 5x + 2y, \quad x = 2\cos t + 1, \quad y = \sin t - 3;$ $t = \pi/4$

10. $z = \dfrac{x}{y^2 + 1}, \quad x = \cos t, \quad y = \sin t; \quad t = \pi/2$

11. $z = x^2 + 2y^2, \quad x = \sin t, \quad y = 3\sin t; \quad t = \pi/4$

12. $z = \cos x \sin y, \quad x = \pi t, \quad y = 2\pi t + \pi/2; \quad t = 3$

In Exercises 13 – 18, functions $z = f(x, y)$, $x = g(t)$ and $y = h(t)$ are given. Find the values of t where $\frac{dz}{dt} = 0$. Note: these are the same surfaces/curves as found in Exercises 7 – 12.

13. $z = 3x + 4y, \quad x = t^2, \quad y = 2t$

14. $z = x^2 - y^2, \quad x = t, \quad y = t^2 - 1$

15. $z = 5x + 2y, \quad x = 2\cos t + 1, \quad y = \sin t - 3$

16. $z = \dfrac{x}{y^2 + 1}, \quad x = \cos t, \quad y = \sin t$

17. $z = x^2 + 2y^2, \quad x = \sin t, \quad y = 3\sin t$

18. $z = \cos x \sin y, \quad x = \pi t, \quad y = 2\pi t + \pi/2$

In Exercises 19 – 22, functions $z = f(x, y)$, $x = g(s, t)$ and $y = h(s, t)$ are given.

(a) Use the Multivariable Chain Rule to compute $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

(b) Evaluate $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ at the indicated s and t values.

19. $z = x^2 y, \quad x = s - t, \quad y = 2s + 4t; \quad s = 1, t = 0$

20. $z = \cos\left(\pi x + \dfrac{\pi}{2}y\right), \quad x = st^2, \quad y = s^2 t; \quad s = 1, t = 1$

21. $z = x^2 + y^2, \quad x = s\cos t, \quad y = s\sin t; \quad s = 2, t = \pi/4$

22. $z = e^{-(x^2+y^2)}, \quad x = t, \quad y = st^2; \quad s = 1, t = 1$

In Exercises 23 – 26, find $\dfrac{dy}{dx}$ using Implicit Differentiation and Theorem 25.

23. $x^2 \tan y = 50$

24. $(3x^2 + 2y^3)^4 = 2$

25. $\dfrac{x^2 + y}{x + y^2} = 17$

26. $\ln(x^2 + xy + y^2) = 1$

In Exercises 27 – 30, find $\dfrac{dz}{dt}$, or $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$, using the supplied information.

27. $\dfrac{\partial z}{\partial x} = 2, \quad \dfrac{\partial z}{\partial y} = 1, \quad \dfrac{dx}{dt} = 4, \quad \dfrac{dy}{dt} = -5$

28. $\dfrac{\partial z}{\partial x} = 1, \quad \dfrac{\partial z}{\partial y} = -3, \quad \dfrac{dx}{dt} = 6, \quad \dfrac{dy}{dt} = 2$

29. $\dfrac{\partial z}{\partial x} = -4, \quad \dfrac{\partial z}{\partial y} = 9,$
$\dfrac{\partial x}{\partial s} = 5, \quad \dfrac{\partial x}{\partial t} = 7, \quad \dfrac{\partial y}{\partial s} = -2, \quad \dfrac{\partial y}{\partial t} = 6$

30. $\dfrac{\partial z}{\partial x} = 2, \quad \dfrac{\partial z}{\partial y} = 1,$
$\dfrac{\partial x}{\partial s} = -2, \quad \dfrac{\partial x}{\partial t} = 3, \quad \dfrac{\partial y}{\partial s} = 2, \quad \dfrac{\partial y}{\partial t} = -1$

11.6 Directional Derivatives

Partial derivatives give us an understanding of how a surface changes when we move in the x and y directions. We made the comparison to standing in a rolling meadow and heading due east: the amount of rise/fall in doing so is comparable to f_x. Likewise, the rise/fall in moving due north is comparable to f_y. The steeper the slope, the greater in magnitude f_y.

But what if we didn't move due north or east? What if we needed to move northeast and wanted to measure the amount of rise/fall? Partial derivatives alone cannot measure this. This section investigates **directional derivatives**, which do measure this rate of change.

We begin with a definition.

Definition 42 Directional Derivatives

Let $z = f(x, y)$ be continuous on an open set S and let $\vec{u} = \langle u_1, u_2 \rangle$ be a unit vector. For all points (x, y), the **directional derivative of f at (x, y) in the direction of $\vec{u}$** is

$$D_{\vec{u}} f(x, y) = \lim_{h \to 0} \frac{f(x + hu_1, y + hu_2) - f(x, y)}{h}.$$

The partial derivatives f_x and f_y are defined with similar limits, but only x or y varies with h, not both. Here both x and y vary with a weighted h, determined by a particular unit vector $\vec{u}$. This may look a bit intimidating but in reality it is not too difficult to deal with; it often just requires extra algebra. However, the following theorem reduces this algebraic load.

Theorem 26 Directional Derivatives

Let $z = f(x, y)$ be differentiable on an open set S containing (x_0, y_0), and let $\vec{u} = \langle u_1, u_2 \rangle$ be a unit vector. The directional derivative of f at (x_0, y_0) in the direction of $\vec{u}$ is

$$D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2.$$

Example 109 Computing directional derivatives

Let $z = 14 - x^2 - y^2$ and let $P = (1, 2)$. Find the directional derivative of f, at P, in the following directions:

1. toward the point $Q = (3, 4)$,
2. in the direction of $\langle 2, -1 \rangle$, and

Notes:

Figure 11.16: Understanding the directional derivative in Example 109.

3. toward the origin.

Solution The surface is plotted in Figure 11.16, where the point $P = (1, 2)$ is indicated in the x, y-plane as well as the point $(1, 2, 9)$ which lies on the surface of f. We find that $f_x(x, y) = -2x$ and $f_x(1, 2) = -2$; $f_y(x, y) = -2y$ and $f_y(1, 2) = -4$.

1. Let $\vec{u}_1$ be the unit vector that points from the point $(1, 2)$ to the point $Q = (3, 4)$, as shown in the figure. The vector $\overrightarrow{PQ} = \langle 2, 2\rangle$; the unit vector in this direction is $\vec{u}_1 = \left\langle 1/\sqrt{2}, 1/\sqrt{2}\right\rangle$. Thus the directional derivative of f at $(1, 2)$ in the direction of $\vec{u}_1$ is

$$D_{\vec{u}_1}f(1, 2) = -2(1/\sqrt{2}) + (-4)(1/\sqrt{2}) = -6/\sqrt{2} \approx -4.24.$$

   Thus the instantaneous rate of change in moving from the point $(1, 2, 9)$ on the surface in the direction of $\vec{u}_1$ (which points toward the point Q) is about -4.24. Moving in this direction moves one steeply downward.

2. We seek the directional derivative in the direction of $\langle 2, -1\rangle$. The unit vector in this direction is $\vec{u}_2 = \left\langle 2/\sqrt{5}, -1/\sqrt{5}\right\rangle$. Thus the directional derivative of f at $(1, 2)$ in the direction of $\vec{u}_2$ is

$$D_{\vec{u}_2}f(1, 2) = -2(2/\sqrt{5}) + (-4)(-1/\sqrt{5}) = 0.$$

   Starting on the surface of f at $(1, 2)$ and moving in the direction of $\langle 2, -1\rangle$ (or $\vec{u}_2$) results in no instantaneous change in z-value. This is analogous to standing on the side of a hill and choosing a direction to walk that does not change the elevation. One neither walks up nor down, rather just "along the side" of the hill.

   Finding these directions of "no elevation change" is important.

3. At $P = (1, 2)$, the direction towards the origin is given by the vector $\langle -1, -2\rangle$; the unit vector in this direction is $\vec{u}_3 = \left\langle -1/\sqrt{5}, -2/\sqrt{5}\right\rangle$. The directional derivative of f at P in the direction of the origin is

$$D_{\vec{u}_3}f(1, 2) = -2(-1/\sqrt{5}) + (-4)(-2/\sqrt{5}) = 10/\sqrt{5} \approx 4.47.$$

   Moving towards the origin means "walking uphill" quite steeply, with an initial slope of about 4.47.

As we study directional derivatives, it will help to make an important connection between the unit vector $\vec{u} = \langle u_1, u_2\rangle$ that describes the direction and the partial derivatives f_x and f_y. We start with a definition and follow this with a Key Idea.

Notes:

Definition 43 Gradient

Let $z = f(x, y)$ be differentiable on an open set S that contains the point (x_0, y_0).

1. The **gradient of** f is $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$.
2. The **gradient of** f **at** (x_0, y_0) **is** $\nabla f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle$.

To simplify notation, we often express the gradient as $\nabla f = \langle f_x, f_y\rangle$. The gradient allows us to compute directional derivatives in terms of a dot product.

Note: The symbol "∇" is named "nabla," derived from the Greek name of a Jewish harp. Oddly enough, in mathematics the expression ∇f is pronounced "del f."

Key Idea 10 The Gradient and Directional Derivatives

The directional derivative of $z = f(x, y)$ in the direction of $\vec{u}$ is

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}.$$

The properties of the dot product previously studied allow us to investigate the properties of the directional derivative. Given that the directional derivative gives the instantaneous rate of change of z when moving in the direction of $\vec{u}$, three questions naturally arise:

1. In what direction(s) is the change in z the greatest (i.e., the "steepest uphill")?
2. In what direction(s) is the change in z the least (i.e., the "steepest downhill")?
3. In what direction(s) is there no change in z?

Using the key property of the dot product, we have

$$\nabla f \cdot \vec{u} = ||\nabla f|| \, ||\vec{u}|| \cos\theta = ||\nabla f|| \cos\theta, \tag{11.4}$$

where θ is the angle between the gradient and $\vec{u}$. (Since $\vec{u}$ is a unit vector, $||\vec{u}|| = 1$.) This equation allows us to answer the three questions stated previously.

1. Equation 11.4 is maximized when $\cos\theta = 1$, i.e., when the gradient and $\vec{u}$ have the same direction. We conclude the gradient points in the direction of greatest z change.

Notes:

2. Equation 11.4 is minimized when $\cos\theta = -1$, i.e., when the gradient and $\vec{u}$ have opposite directions. We conclude the gradient points in the opposite direction of the least z change.

3. Equation 11.4 is 0 when $\cos\theta = 0$, i.e., when the gradient and $\vec{u}$ are orthogonal to each other. We conclude the gradient is orthogonal to directions of no z change.

This result is rather amazing. Once again imagine standing in a rolling meadow and face the direction that leads you steepest uphill. Then the direction that leads steepest downhill is directly behind you, and side–stepping either left or right (i.e., moving perpendicularly to the direction you face) does not change your elevation at all.

Recall that a level curve is defined as a curve in the x-y plane along which the z-values of a function do not change. Let a surface $z = f(x, y)$ be given, and let's represent one such level curve as a vector–valued function, $\vec{r}(t) = \langle x(t), y(t)\rangle$. As the output of f does not change along this curve, $f\big(x(t), y(t)\big) = c$ for all t, for some constant c.

Since f is constant for all t, $\frac{df}{dt} = 0$. By the Multivariable Chain Rule, we also know

$$\begin{aligned}\frac{df}{dt} &= f_x(x,y)x'(t) + f_y(x,y)y'(t)\\ &= \langle f_x(x,y), f_y(x,y)\rangle \cdot \langle x'(t), y'(t)\rangle\\ &= \nabla f \cdot \vec{r}\,'(t)\\ &= 0.\end{aligned}$$

This last equality states $\nabla f \cdot \vec{r}\,'(t) = 0$: the gradient is orthogonal to the derivative of $\vec{r}$, meaning the gradient is orthogonal to $\vec{r}$ itself. Our conclusion: at any point on a surface, the gradient at that point is orthogonal to the level curve that passes through that point.

We restate these ideas in a theorem, then use them in an example.

Theorem 27 The Gradient and Directional Derivatives

Let $z = f(x, y)$ be differentiable on an open set S with gradient ∇f, let $P = (x_0, y_0)$ be a point in S and let $\vec{u}$ be a unit vector.

1. The maximum value of $D_{\vec{u}}f(x_0, y_0)$ is $||\ \nabla f(x_0, y_0)\ ||$; the direction of maximal z increase is $\nabla f(x_0, y_0)$.

2. The minimum value of $D_{\vec{u}}f(x_0, y_0)$ is $-||\ \nabla f(x_0, y_0)\ ||$; the direction of minimal z increase is $-\nabla f(x_0, y_0)$.

3. At P, $\nabla f(x_0, y_0)$ is orthogonal to the level curve passing through $\big(x_0, y_0, f(x_0, y_0)\big)$.

Notes:

Example 110 Finding directions of maximal and minimal increase

Let $f(x, y) = \sin x \cos y$ and let $P = (\pi/3, \pi/3)$. Find the directions of maximal/minimal increase, and find a direction where the instantaneous rate of z change is 0.

Solution We begin by finding the gradient. $f_x = \cos x \cos y$ and $f_y = -\sin x \sin y$, thus

$$\nabla f = \langle \cos x \cos y, -\sin x \sin y \rangle \quad \text{and, at } P, \quad \nabla f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \left\langle \frac{1}{4}, -\frac{3}{4} \right\rangle.$$

Thus the direction of maximal increase is $\langle 1/4, -3/4 \rangle$. In this direction, the instantaneous rate of z change is $|| \langle 1/4, -3/4 \rangle || = \sqrt{10}/4 \approx 0.79$.

Figure 11.17 shows the surface plotted from two different perspectives. In each, the gradient is drawn at P with a dashed line (because of the nature of this surface, the gradient points "into" the surface). Let $\vec{u} = \langle u_1, u_2 \rangle$ be the unit vector in the direction of ∇f at P. Each graph of the figure also contains the vector $\langle u_1, u_2, ||\nabla f|| \rangle$. This vector has a "run" of 1 (because in the x-y plane it moves 1 unit) and a "rise" of $||\nabla f||$, hence we can think of it as a vector with slope of $||\nabla f||$ in the direction of ∇f, helping us visualize how "steep" the surface is in its steepest direction.

The direction of minimal increase is $\langle -1/4, 3/4 \rangle$; in this direction the instantaneous rate of z change is $-\sqrt{10}/4 \approx -0.79$.

Any direction orthogonal to ∇f is a direction of no z change. We have two choices: the direction of $\langle 3, 1 \rangle$ and the direction of $\langle -3, -1 \rangle$. The unit vector in the direction of $\langle 3, 1 \rangle$ is shown in each graph of the figure as well. The level curve at $z = \sqrt{3}/4$ is drawn: recall that along this curve the z-values do not change. Since $\langle 3, 1 \rangle$ is a direction of no z-change, this vector is tangent to the level curve at P.

Figure 11.17: Graphing the surface and important directions in Example 110.

Example 111 Understanding when $\nabla f = \vec{0}$

Let $f(x, y) = -x^2 + 2x - y^2 + 2y + 1$. Find the directional derivative of f in any direction at $P = (1, 1)$.

Solution We find $\nabla f = \langle -2x + 2, -2y + 2 \rangle$. At P, we have $\nabla f(1, 1) = \langle 0, 0 \rangle$. According to Theorem 27, this is the direction of maximal increase. However, $\langle 0, 0 \rangle$ is directionless; it has no displacement. And regardless of the unit vector $\vec{u}$ chosen, $D_{\vec{u}} f = 0$.

Figure 11.18 helps us understand what this means. We can see that P lies at the top of a paraboloid. In all directions, the instantaneous rate of change is 0.

So what is the direction of maximal increase? It is fine to give an answer of $\vec{0} = \langle 0, 0 \rangle$, as this indicates that all directional derivatives are 0.

Figure 11.18: At the top of a paraboloid, all directional derivatives are 0.

Notes:

The fact that the gradient of a surface always points in the direction of steepest increase/decrease is very useful, as illustrated in the following example.

Example 112 The flow of water downhill

Consider the surface given by $f(x, y) = 20 - x^2 - 2y^2$. Water is poured on the surface at $(1, 1/4)$. What path does it take as it flows downhill?

Solution Let $\vec{r}(t) = \langle x(t), y(t)\rangle$ be the vector–valued function describing the path of the water in the x-y plane; we seek $x(t)$ and $y(t)$. We know that water will always flow downhill in the steepest direction; therefore, at any point on its path, it will be moving in the direction of $-\nabla f$. (We ignore the physical effects of momentum on the water.) Thus $\vec{r}\,'(t)$ will be parallel to ∇f, and there is some constant c such that $c\nabla f = \vec{r}\,'(t) = \langle x'(t), y'(t)\rangle$.

We find $\nabla f = \langle -2x, -4y\rangle$ and write $x'(t)$ as $\frac{dx}{dt}$ and $y'(t)$ as $\frac{dy}{dt}$. Then

$$\begin{aligned} c\nabla f &= \langle x'(t), y'(t)\rangle \\ \langle -2cx, -4cy\rangle &= \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle. \end{aligned}$$

This implies

$$-2cx = \frac{dx}{dt} \quad \text{and} \quad -4cy = \frac{dy}{dt}, \text{ i.e.,}$$

$$c = -\frac{1}{2x}\frac{dx}{dt} \quad \text{and} \quad c = -\frac{1}{4y}\frac{dy}{dt}.$$

As c equals both expressions, we have

$$\frac{1}{2x}\frac{dx}{dt} = \frac{1}{4y}\frac{dy}{dt}.$$

To find an explicit relationship between x and y, we can integrate both sides with respect to t. Recall from our study of differentials that $\dfrac{dx}{dt}dt = dx$. Thus:

$$\begin{aligned} \int \frac{1}{2x}\frac{dx}{dt}\,dt &= \int \frac{1}{4y}\frac{dy}{dt}\,dt \\ \int \frac{1}{2x}\,dx &= \int \frac{1}{4y}\,dy \\ \frac{1}{2}\ln|x| &= \frac{1}{4}\ln|y| + C_1 \\ 2\ln|x| &= \ln|y| + C_1 \\ \ln|x^2| &= \ln|y| + C_1 \end{aligned}$$

Notes:

Now raise both sides as a power of e:

$$\begin{aligned} x^2 &= e^{\ln|y|+C_1} \\ x^2 &= e^{\ln|y|}e^{C_1} \qquad \text{(Note that } e^{C_1} \text{ is just a constant.)} \\ x^2 &= yC_2 \\ \frac{1}{C_2}x^2 &= y \qquad \text{(Note that } 1/C_2 \text{ is just a constant.)} \\ Cx^2 &= y. \end{aligned}$$

As the water started at the point $(1, 1/4)$, we can solve for C:

$$C(1)^2 = \frac{1}{4} \quad \Rightarrow \quad C = \frac{1}{4}.$$

Thus the water follows the curve $y = x^2/4$ in the x-y plane. The surface and the path of the water is graphed in Figure 11.19(a). In part (b) of the figure, the level curves of the surface are plotted in the x-y plane, along with the curve $y = x^2/4$. Notice how the path intersects the level curves at right angles. As the path follows the gradient downhill, this reinforces the fact that the gradient is orthogonal to level curves.

Figure 11.19: A graph of the surface described in Example 112 along with the path in the x-y plane with the level curves.

Functions of Three Variables

The concepts of directional derivatives and the gradient are easily extended to three (and more) variables. We combine the concepts behind Definitions 42 and 43 and Theorem 26 into one set of definitions.

Definition 44 Directional Derivatives and Gradient with Three Variables

Let $w = F(x, y, z)$ be differentiable on an open ball B and let $\vec{u}$ be a unit vector in $\mathbb{R}^3$.

1. The **gradient** of F is $\nabla F = \langle F_x, F_y, F_z \rangle$.
2. The **directional derivative of F in the direction of $\vec{u}$** is

$$D_{\vec{u}}F = \nabla F \cdot \vec{u}.$$

The same properties of the gradient given in Theorem 27, when f is a function

Notes:

of two variables, hold for F, a function of three variables.

Theorem 28 The Gradient and Directional Derivatives with Three Variables

Let $w = F(x, y, z)$ be differentiable on an open ball B, let ∇F be the gradient of F, and let $\vec{u}$ be a unit vector.

1. The maximum value of $D_{\vec{u}}F$ is $|| \nabla F ||$, obtained when the angle between ∇F and $\vec{u}$ is 0, i.e., the direction of maximal increase is ∇F.
2. The minimum value of $D_{\vec{u}}F$ is $-|| \nabla F ||$, obtained when the angle between ∇F and $\vec{u}$ is π, i.e., the direction of minimal increase is $-\nabla F$.
3. $D_{\vec{u}}F = 0$ when ∇F and $\vec{u}$ are orthogonal.

We interpret the third statement of the theorem as "the gradient is orthogonal to level surfaces," the three–variable analogue to level curves.

Example 113 Finding directional derivatives with functions of three variables

If a point source S is radiating energy, the intensity I at a given point P in space is inversely proportional to the square of the distance between S and P. That is, when $S = (0, 0, 0)$, $I(x, y, z) = \dfrac{k}{x^2 + y^2 + z^2}$ for some constant k.

Let $k = 1$, let $\vec{u} = \langle 2/3, 2/3, 1/3 \rangle$ be a unit vector, and let $P = (2, 5, 3)$. Measure distances in inches. Find the directional derivative of I at P in the direction of $\vec{u}$, and find the direction of greatest intensity increase at P.

Solution We need the gradient ∇I, meaning we need I_x, I_y and I_z. Each partial derivative requires a simple application of the Quotient Rule, giving

$$\begin{aligned}
\nabla I &= \left\langle \frac{-2x}{(x^2+y^2+z^2)^2}, \frac{-2y}{(x^2+y^2+z^2)^2}, \frac{-2z}{(x^2+y^2+z^2)^2} \right\rangle \\
\nabla I(2,5,3) &= \left\langle \frac{-4}{1444}, \frac{-10}{1444}, \frac{-6}{1444} \right\rangle \approx \langle -0.003, -0.007, -0.004 \rangle \\
D_{\vec{u}} I &= \nabla I(2,5,3) \cdot \vec{u} \\
&= -\frac{17}{2166} \approx -0.0078.
\end{aligned}$$

The directional derivative tells us that moving in the direction of $\vec{u}$ from P results in a decrease in intensity of about -0.008 units per inch. (The intensity is decreasing as $\vec{u}$ moves one farther from the origin than P.)

Notes:

The gradient gives the direction of greatest intensity increase. Notice that

$$\begin{aligned}\nabla I(2,5,3) &= \left\langle \frac{-4}{1444}, \frac{-10}{1444}, \frac{-6}{1444} \right\rangle \\ &= \frac{2}{1444} \langle -2, -5, -3 \rangle .\end{aligned}$$

That is, the gradient at $(2,5,3)$ is pointing in the direction of $\langle -2, -5, -3 \rangle$, that is, towards the origin. That should make intuitive sense: the greatest increase in intensity is found by moving towards to source of the energy.

The directional derivative allows us to find the instantaneous rate of z change in any direction at a point. We can use these instantaneous rates of change to define lines and planes that are *tangent* to a surface at a point, which is the topic of the next section.

Notes:

Exercises 11.6

Terms and Concepts

1. What is the difference between a directional derivative and a partial derivative?

2. For what $\vec{u}$ is $D_{\vec{u}}f = f_x$?

3. For what $\vec{u}$ is $D_{\vec{u}}f = f_y$?

4. The gradient is _____ to level curves.

5. The gradient points in the direction of _____ increase.

6. It is generally more informative to view the directional derivative not as the result of a limit, but rather as the result of a _____ product.

Problems

In Exercises 7 – 12, a function $z = f(x, y)$ is given. Find ∇f.

7. $f(x, y) = -x^2y + xy^2 + xy$

8. $f(x, y) = \sin x \cos y$

9. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$

10. $f(x, y) = -4x + 3y$

11. $f(x, y) = x^2 + 2y^2 - xy - 7x$

12. $f(x, y) = x^2y^3 - 2x$

In Exercises 13 – 18, a function $z = f(x, y)$ and a point P are given. Find the directional derivative of f in the indicated directions. Note: these are the same functions as in Exercises 7 through 12.

13. $f(x, y) = -x^2y + xy^2 + xy$, $P = (2, 1)$
    (a) In the direction of $\vec{v} = \langle 3, 4\rangle$
    (b) In the direction toward the point $Q = (1, -1)$.

14. $f(x, y) = \sin x \cos y$, $P = \left(\dfrac{\pi}{4}, \dfrac{\pi}{3}\right)$
    (a) In the direction of $\vec{v} = \langle 1, 1\rangle$.
    (b) In the direction toward the point $Q = (0, 0)$.

15. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$, $P = (1, 1)$.
    (a) In the direction of $\vec{v} = \langle 1, -1\rangle$.
    (b) In the direction toward the point $Q = (-2, -2)$.

16. $f(x, y) = -4x + 3y$, $P = (5, 2)$
    (a) In the direction of $\vec{v} = \langle 3, 1\rangle$.
    (b) In the direction toward the point $Q = (2, 7)$.

17. $f(x, y) = x^2 + 2y^2 - xy - 7x$, $P = (4, 1)$
    (a) In the direction of $\vec{v} = \langle -2, 5\rangle$
    (b) In the direction toward the point $Q = (4, 0)$.

18. $f(x, y) = x^2y^3 - 2x$, $P = (1, 1)$
    (a) In the direction of $\vec{v} = \langle 3, 3\rangle$
    (b) In the direction toward the point $Q = (1, 2)$.

In Exercises 19 – 24, a function $z = f(x, y)$ and a point P are given.

(a) Find the direction of maximal increase of f at P.
(b) What is the maximal value of $D_{\vec{u}}f$ at P?
(c) Find the direction of minimal increase of f at P.
(d) Give a direction $\vec{u}$ such that $D_{\vec{u}}f = 0$ at P.

Note: these are the same functions and points as in Exercises 13 through 18.

19. $f(x, y) = -x^2y + xy^2 + xy$, $P = (2, 1)$

20. $f(x, y) = \sin x \cos y$, $P = \left(\dfrac{\pi}{4}, \dfrac{\pi}{3}\right)$

21. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$, $P = (1, 1)$.

22. $f(x, y) = -4x + 3y$, $P = (5, 4)$.

23. $f(x, y) = x^2 + 2y^2 - xy - 7x$, $P = (4, 1)$

24. $f(x, y) = x^2y^3 - 2x$, $P = (1, 1)$

In Exercises 25 – 28, a function $w = F(x, y, z)$, a vector $\vec{v}$ and a point P are given.

(a) Find $\nabla F(x, y, z)$.
(b) Find $D_{\vec{u}}F$ at P.

25. $F(x, y, z) = 3x^2z^3 + 4xy - 3z^2$, $\vec{v} = \langle 1, 1, 1\rangle$, $P = (3, 2, 1)$

26. $F(x, y, z) = \sin(x)\cos(y)e^z$, $\vec{v} = \langle 2, 2, 1\rangle$, $P = (0, 0, 0)$

27. $F(x, y, z) = x^2y^2 - y^2z^2$, $\vec{v} = \langle -1, 7, 3\rangle$, $P = (1, 0, -1)$

28. $F(x, y, z) = \dfrac{2}{x^2 + y^2 + z^2}$, $\vec{v} = \langle 1, 1, -2\rangle$, $P = (1, 1, 1)$

11.7 Tangent Lines, Normal Lines, and Tangent Planes

Figure 11.20: Showing various lines tangent to a surface.

Derivatives and tangent lines go hand–in–hand. Given $y = f(x)$, the line tangent to the graph of f at $x = x_0$ is the line through $\big(x_0, f(x_0)\big)$ with slope $f'(x_0)$; that is, the slope of the tangent line is the instantaneous rate of change of f at x_0.

When dealing with functions of two variables, the graph is no longer a curve but a surface. At a given point on the surface, it seems there are many lines that fit our intuition of being "tangent" to the surface.

In Figures 11.20 we see lines that are tangent to curves in space. Since each curve lies on a surface, it makes sense to say that the lines are also tangent to the surface. The next definition formally defines what it means to be "tangent to a surface."

Definition 45 Directional Tangent Line

Let $z = f(x, y)$ be differentiable on an open set S containing (x_0, y_0) and let $\vec{u} = \langle u_1, u_2\rangle$ be a unit vector.

1. The line ℓ_x through $\big(x_0, y_0, f(x_0, y_0)\big)$ parallel to $\langle 1, 0, f_x(x_0, y_0)\rangle$ is the **tangent line to f in the direction of x at** (x_0, y_0).
2. The line ℓ_y through $\big(x_0, y_0, f(x_0, y_0)\big)$ parallel to $\langle 0, 1, f_y(x_0, y_0)\rangle$ is the **tangent line to f in the direction of y at** (x_0, y_0).
3. The line $\ell_{\vec{u}}$ through $\big(x_0, y_0, f(x_0, y_0)\big)$ parallel to $\langle u_1, u_2, D_{\vec{u}}f(x_0, y_0)\rangle$ is the **tangent line to f in the direction of $\vec{u}$ at** (x_0, y_0).

It is instructive to consider each of three directions given in the definition in terms of "slope." The direction of ℓ_x is $\langle 1, 0, f_x(x_0, y_0)\rangle$; that is, the "run" is one unit in the x-direction and the "rise" is $f_x(x_0, y_0)$ units in the z-direction. Note how the slope is just the partial derivative with respect to x. A similar statement can be made for ℓ_y. The direction of $\ell_{\vec{u}}$ is $\langle u_1, u_2, D_{\vec{u}}f(x_0, y_0)\rangle$; the "run" is one unit in the $\vec{u}$ direction (where $\vec{u}$ is a unit vector) and the "rise" is the directional derivative of z in that direction.

Definition 45 leads to the following parametric equations of directional tangent lines:

$$\begin{aligned} x &= x_0 + t \\ y &= y_0 \\ z &= z_0 + f_x(x_0, y_0)t \end{aligned}, \quad \ell_y(t) = \begin{cases} x = x_0 \\ y = y_0 + t \\ z = z_0 + f_y(x_0, y_0)t \end{cases} \quad \text{and} \quad \ell_{\vec{u}}(t) = \begin{cases} x = x_0 + u_1 t \\ y = y_0 + u_2 t \\ z = z_0 + D_{\vec{u}}f(x_0, y_0)t \end{cases}.$$

Notes:

Figure 11.21: A surface and directional tangent lines in Example 114.

Example 114 Finding directional tangent lines

Find the lines tangent to the surface $z = \sin x \cos y$ at $(\pi/2, \pi/2)$ in the x and y directions and also in the direction of $\vec{v} = \langle -1, 1\rangle$.

Solution The partial derivatives with respect to x and y are:

$$\begin{aligned} f_x(x,y) &= \cos x \cos y \quad \Rightarrow \quad f_x(\pi/2,\pi/2) = 0 \\ f_y(x,y) &= -\sin x \sin y \quad \Rightarrow \quad f_y(\pi/2,\pi/2) = -1. \end{aligned}$$

At $(\pi/2, \pi/2)$, the z-value is 0.

Thus the parametric equations of the line tangent to f at $(\pi/2, \pi/2)$ in the directions of x and y are:

$$\ell_x(t) = \begin{cases} x = \pi/2 + t \\ y = \pi/2 \\ z = 0 \end{cases} \quad \text{and} \quad \ell_y(t) = \begin{cases} x = \pi/2 \\ y = \pi/2 + t \\ z = -t \end{cases}.$$

The two lines are shown with the surface in Figure 11.21(a). To find the equation of the tangent line in the direction of $\vec{v}$, we first find the unit vector in the direction of $\vec{v}$: $\vec{u} = \left\langle -1/\sqrt{2}, 1/\sqrt{2}\right\rangle$. The directional derivative at $(\pi/2, \pi, 2)$ in the direction of $\vec{u}$ is

$$D_{\vec{u}} f(\pi/2, \pi, 2) = \langle 0, -1\rangle \cdot \left\langle -1/\sqrt{2}, 1/\sqrt{2}\right\rangle = -1/\sqrt{2}.$$

Thus the directional tangent line is

$$\ell_{\vec{u}}(t) = \begin{cases} x = \pi/2 - t/\sqrt{2} \\ y = \pi/2 + t/\sqrt{2} \\ z = -t/\sqrt{2} \end{cases}.$$

The curve through $(\pi/2, \pi/2, 0)$ in the direction of $\vec{v}$ is shown in Figure 11.21(b) along with $\ell_{\vec{u}}(t)$.

Example 115 Finding directional tangent lines

Let $f(x,y) = 4xy - x^4 - y^4$. Find the equations of *all* directional tangent lines to f at $(1,1)$.

Solution First note that $f(1,1) = 2$. We need to compute directional derivatives, so we need ∇f. We begin by computing partial derivatives.

$$f_x = 4y - 4x^3 \Rightarrow f_x(1,1) = 0; \quad f_y = 4x - 4y^3 \Rightarrow f_y(1,1) = 0.$$

Thus $\nabla f(1,1) = \langle 0, 0\rangle$. Let $\vec{u} = \langle u_1, u_2\rangle$ be any unit vector. The directional derivative of f at $(1,1)$ will be $D_{\vec{u}} f(1,1) = \langle 0,0\rangle \cdot \langle u_1, u_2\rangle = 0$. It does not matter

Notes:

what direction we choose; the directional derivative is always 0. Therefore

$$\ell_{\vec{u}}(t) = \begin{cases} x = 1 + u_1 t \\ y = 1 + u_2 t \\ z = 2 \end{cases} .$$

Figure 11.22 shows a graph of f and the point $(1, 1, 2)$. Note that this point comes at the top of a "hill," and therefore every tangent line through this point will have a "slope" of 0.

That is, consider any curve on the surface that goes through this point. Each curve will have a relative maximum at this point, hence its tangent line will have a slope of 0. The following section investigates the points on surfaces where all tangent lines have a slope of 0.

Figure 11.22: Graphing f in Example 115.

Normal Lines

When dealing with a function $y = f(x)$ of one variable, we stated that a line through $(c, f(c))$ was *tangent* to f if the line had a slope of $f'(c)$ and was *normal* (or, *perpendicular, orthogonal*) to f if it had a slope of $-1/f'(c)$. We extend the concept of normal, or orthogonal, to functions of two variables.

Let $z = f(x, y)$ be a differentiable function of two variables. By Definition 45, at (x_0, y_0), $\ell_x(t)$ is a line parallel to the vector $\vec{d}_x = \langle 1, 0, f_x(x_0, y_0)\rangle$ and $\ell_y(t)$ is a line parallel to $\vec{d}_y = \langle 0, 1, f_y(x_0, y_0)\rangle$. Since lines in these directions through $\big(x_0, y_0, f(x_0, y_0)\big)$ are *tangent* to the surface, a line through this point and orthogonal to these directions would be *orthogonal*, or *normal*, to the surface. We can use this direction to create a normal line.

The direction of the normal line is orthogonal to $\vec{d}_x$ and $\vec{d}_y$, hence the direction is parallel to $\vec{d}_n = \vec{d}_x \times \vec{d}_y$. It turns out this cross product has a very simple form:

$$\vec{d}_x \times \vec{d}_y = \langle 1, 0, f_x\rangle \times \langle 0, 1, f_y\rangle = \langle -f_x, -f_y, 1\rangle .$$

It is often more convenient to refer to the opposite of this direction, namely $\langle f_x, f_y, -1\rangle$. This leads to a definition.

Notes:

Definition 46 Normal Line

Let $z = f(x, y)$ be differentiable on an open set S containing (x_0, y_0) where

$$a = f_x(x_0, y_0) \quad \text{and} \quad b = f_y(x_0, y_0)$$

are defined.

1. A nonzero vector parallel to $\vec{n} = \langle a, b, -1\rangle$ is **orthogonal to f at** $P = \big(x_0, y_0, f(x_0, y_0)\big)$.
2. The line ℓ_n through P with direction parallel to $\vec{n}$ is the **normal line to f at** P.

Thus the parametric equations of the normal line to a surface f at $\big(x_0, y_0, f(x_0, y_0)\big)$ is:

$$\ell_n(t) = \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = f(x_0, y_0) - t \end{cases}.$$

Figure 11.23: Graphing a surface with a normal line from Example 116.

Example 116 Finding a normal line

Find the equation of the normal line to $z = -x^2 - y^2 + 2$ at $(0, 1)$.

Solution We find $z_x(x, y) = -2x$ and $z_y(x, y) = -2y$; at $(0, 1)$, we have $z_x = 0$ and $z_y = -2$. We take the direction of the normal line, following Definition 46, to be $\vec{n} = \langle 0, -2, -1\rangle$. The line with this direction going through the point $(0, 1, 1)$ is

$$\ell_n(t) = \begin{cases} x = 0 \\ y = -2t + 1 \\ z = -t + 1 \end{cases} \quad \text{or} \quad \ell_n(t) = \langle 0, -2, -1\rangle\, t + \langle 0, 1, 1\rangle\,.$$

The surface $z = -x^2 + y^2$, along with the found normal line, is graphed in Figure 11.23.

The direction of the normal line has many uses, one of which is the definition of the **tangent plane** which we define shortly. Another use is in measuring distances from the surface to a point. Given a point Q in space, it is general geometric concept to define the distance from Q to the surface as being the length of the shortest line segment $\overline{PQ}$ over all points P on the surface. This, in turn, implies that $\overrightarrow{PQ}$ will be orthogonal to the surface at P. Therefore we can measure the distance from Q to the surface f by finding a point P on the surface such

Notes:

that $\overrightarrow{PQ}$ is parallel to the normal line to f at P.

Example 117 Finding the distance from a point to a surface

Let $f(x, y) = 2 - x^2 - y^2$ and let $Q = (2, 2, 2)$. Find the distance from Q to the surface defined by f.

Solution This surface is used in Example 115, so we know that at (x, y), the direction of the normal line will be $\vec{d}_n = \langle -2x, -2y, -1\rangle$. A point P on the surface will have coordinates $(x, y, 2-x^2-y^2)$, so $\overrightarrow{PQ} = \left\langle 2 - x, 2 - y, x^2 + y^2\right\rangle$. To find where $\overrightarrow{PQ}$ is parallel to $\vec{d}_n$, we need to find x, y and c such that $c\overrightarrow{PQ} = \vec{d}_n$.

$$\begin{aligned} c\overrightarrow{PQ} &= \vec{d}_n \\ c\left\langle 2 - x, 2 - y, x^2 + y^2\right\rangle &= \langle -2x, -2y, -1\rangle . \end{aligned}$$

This implies

$$\begin{aligned} c(2 - x) &= -2x \\ c(2 - y) &= -2y \\ c(x^2 + y^2) &= -1 \end{aligned}$$

In each equation, we can solve for c:

$$c = \frac{-2x}{2 - x} = \frac{-2y}{2 - y} = \frac{-1}{x^2 + y^2}.$$

The first two fractions imply $x = y$, and so the last fraction can be rewritten as $c = -1/(2x^2)$. Then

$$\begin{aligned} \frac{-2x}{2 - x} &= \frac{-1}{2x^2} \\ -2x(2x^2) &= -1(2 - x) \\ 4x^3 &= 2 - x \\ 4x^3 + x - 2 &= 0. \end{aligned}$$

This last equation is a cubic, which is not difficult to solve with a numeric solver. We find that $x = 0.689$, hence $P = (0.689, 0.689, 1.051)$. We find the distance from Q to the surface of f is

$$||\overrightarrow{PQ}|| = \sqrt{(2 - 0.689)^2 + (2 - 0.689)^2 + (2 - 1.051)^2} = 2.083.$$

We can take the concept of measuring the distance from a point to a surface to find a point Q a particular distance from a surface at a given point P on the

Notes:

surface.

Example 118 Finding a point a set distance from a surface

Let $f(x,y) = x - y^2 + 3$. Let $P = \big(2, 1, f(2,1)\big) = (2,1,4)$. Find points Q in space that are 4 units from the surface of f at P. That is, find Q such that $\|\overrightarrow{PQ}\| = 4$ and $\overrightarrow{PQ}$ is orthogonal to f at P.

Solution We begin by finding partial derivatives:

$$\begin{aligned} f_x(x,y) &= 1 \quad &\Rightarrow \quad f_x(2,1) &= 1 \\ f_y(x,y) &= -2y \quad &\Rightarrow \quad f_y(2,1) &= -2 \end{aligned}$$

The vector $\vec{n} = \langle 1, -2, -1\rangle$ is orthogonal to f at P. For reasons that will become more clear in a moment, we find the unit vector in the direction of $\vec{n}$:

$$\vec{u} = \frac{\vec{n}}{\|\vec{n}\|} = \left\langle 1/\sqrt{6}, -2/\sqrt{6}, -1/\sqrt{6}\right\rangle \approx \langle 0.408, -0.816, -0.408\rangle .$$

Thus a the normal line to f at P can be written as

$$\ell_n(t) = \langle 2, 1, 4\rangle + t\,\langle 0.408, -0.816, -0.408\rangle .$$

An advantage of this parametrization of the line is that letting $t = t_0$ gives a point on the line that is $|t_0|$ units from P. (This is because the direction of the line is given in terms of a unit vector.) There are thus two points in space 4 units from P:

$$\begin{aligned} Q_1 &= \ell_n(4) & Q_2 &= \ell_n(-4) \\ &\approx \langle 3.63, -2.27, 2.37\rangle & &\approx \langle 0.37, 4.27, 5.63\rangle \end{aligned}$$

The surface is graphed along with points P, Q_1, Q_2 and a portion of the normal line to f at P.

Figure 11.24: Graphing the surface in Example 118 along with points 4 units from the surface.

Tangent Planes

We can use the direction of the normal line to define a plane. With $a = f_x(x_0, y_0)$, $b = f_y(x_0, y_0)$ and $P = \big(x_0, y_0, f(x_0, y_0)\big)$, the vector $\vec{n} = \langle a, b, -1\rangle$ is orthogonal to f at P. The plane through P with normal vector $\vec{n}$ is therefore **tangent** to f at P.

Notes:

Definition 47 Tangent Plane

Let $z = f(x, y)$ be differentiable on an open set S containing (x_0, y_0), where $a = f_x(x_0, y_0)$, $b = f_y(x_0, y_0)$, $\vec{n} = \langle a, b, -1\rangle$ and $P = \big(x_0, y_0, f(x_0, y_0)\big)$.

The plane through P with normal vector $\vec{n}$ is the **tangent plane to f at P**. The standard form of this plane is

$$a(x - x_0) + b(y - y_0) - \big(z - f(x_0, y_0)\big) = 0.$$

Figure 11.25: Graphing a surface with tangent plane from Example 119.

Example 119 Finding tangent planes

Find the equation of the tangent plane to $z = -x^2 - y^2 + 2$ at $(0, 1)$.

Solution Note that this is the same surface and point used in Example 116. There we found $\vec{n} = \langle 0, -2, -1\rangle$ and $P = (0, 1, 1)$. Therefore the equation of the tangent plane is

$$-2(y - 1) - (z - 1) = 0.$$

The surface $z = -x^2 + y^2$ and tangent plane are graphed in Figure 11.25.

Example 120 Using the tangent plane to approximate function values

The point $(3, -1, 4)$ lies on the surface of an unknown differentiable function f where $f_x(3, -1) = 2$ and $f_y(3, -1) = -1/2$. Find the equation of the tangent plane to f at P, and use this to approximate the value of $f(2.9, -0.8)$.

Solution Knowing the partial derivatives at $(3, -1)$ allows us to form the normal vector to the tangent plane, $\vec{n} = \langle 2, -1/2, -1\rangle$. Thus the equation of the tangent line to f at P is:

$$2(x-3)-1/2(y+1)-(z-4) = 0 \quad \Rightarrow \quad z = 2(x-3)-1/2(y+1)+4. \quad (11.5)$$

Just as tangent lines provide excellent approximations of curves near their point of intersection, tangent planes provide excellent approximations of surfaces near their point of intersection. So $f(2.9, -0.8) \approx z(2.9, -0.8) = 3.7$.

This is not a new method of approximation. Compare the right hand expression for z in Equation (11.5) to the total differential:

$$dz = f_x dx + f_y dy \quad \text{and} \quad z = \underbrace{\underbrace{2}_{f_x}\underbrace{(x-3)}_{dx} + \underbrace{-1/2}_{f_y}\underbrace{(y+1)}_{dy}}_{dz} + 4.$$

Notes:

Thus the "new z-value" is the sum of the change in z (i.e., dz) and the old z-value (4). As mentioned when studying the total differential, it is not uncommon to know partial derivative information about a unknown function, and tangent planes are used to give accurate approximations of the function.

The Gradient and Normal Lines, Tangent Planes

The methods developed in this section so far give a straightforward method of finding equations of normal lines and tangent planes for surfaces with explicit equations of the form $z = f(x, y)$. However, they do not handle implicit equations well, such as $x^2 + y^2 + z^2 = 1$. There is a technique that allows us to find vectors orthogonal to these surfaces based on the **gradient**.

Definition 48 Gradient

Let $w = F(x, y, z)$ be differentiable on an open ball B that contains the point (x_0, y_0, z_0).

1. The **gradient of** F is $\nabla F(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle$.
2. The **gradient of** F **at** (x_0, y_0, z_0) **is**

$$\nabla F(x_0, y_0, z_0) = \langle f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0)\rangle .$$

Recall that when $z = f(x, y)$, the gradient $\nabla f = \langle f_x, f_y\rangle$ is orthogonal to level curves of f. An analogous statement can be made about the gradient ∇F, where $w = F(x, y, z)$. Given a point (x_0, y_0, z_0), let $c = F(x_0, y_0, z_0)$. Then $F(x, y, z) = c$ is a **level surface** that contains the point (x_0, y_0, z_0). The following theorem states that $\nabla F(x_0, y_0, z_0)$ is orthogonal to this level surface.

Theorem 29 The Gradient and Level Surfaces

Let $w = F(x, y, z)$ be differentiable on an open ball B containing (x_0, y_0, z_0) with gradient ∇F, where $F(x_0, y_0, z_0) = c$.

The vector $\nabla F(x_0, y_0, z_0)$ is orthogonal to the level surface $F(x, y, z) = c$ at (x_0, y_0, z_0).

The gradient at a point gives a vector orthogonal to the surface at that point. This direction can be used to find tangent planes and normal lines.

Notes:

Example 121 Using the gradient to find a tangent plane

Find the equation of the plane tangent to the ellipsoid $\dfrac{x^2}{12} + \dfrac{y^2}{6} + \dfrac{z^2}{4} = 1$ at $P = (1, 2, 1)$.

Solution We consider the equation of the ellipsoid as a level surface of a function F of three variables, where $F(x, y, z) = \frac{x^2}{12} + \frac{y^2}{6} + \frac{z^2}{4}$. The gradient is:

$$\begin{aligned}\nabla F(x, y, z) &= \langle F_x, F_y, F_z\rangle \\ &= \left\langle \frac{x}{6}, \frac{y}{3}, \frac{z}{2}\right\rangle.\end{aligned}$$

At P, the gradient is $\nabla F(1, 2, 1) = \langle 1/6, 2/3, 1/2\rangle$. Thus the equation of the plane tangent to the ellipsoid at P is

$$\frac{1}{6}(x-1) + \frac{2}{3}(y-2) + \frac{1}{2}(z-1) = 0.$$

The ellipsoid and tangent plane are graphed in Figure 11.26.

Figure 11.26: An ellipsoid and its tangent plane at a point.

Tangent lines and planes to surfaces have many uses, including the study of instantaneous rates of changes and making approximations. Normal lines also have many uses. In this section we focused on using them to measure distances from a surface. Another interesting application is in computer graphics, where the effects of light on a surface are determined using normal vectors.

The next section investigates another use of partial derivatives: determining relative extrema. When dealing with functions of the form $y = f(x)$, we found relative extrema by finding x where $f'(x) = 0$. We can *start* finding relative extrema of $z = f(x, y)$ by setting f_x and f_y to 0, but it turns out that there is more to consider.

Notes:

Exercises 11.7

Terms and Concepts

1. Explain how the vector $\vec{v} = \langle 1, 0, 3\rangle$ can be thought of as having a "slope" of 3.

2. Explain how the vector $\vec{v} = \langle 0.6, 0.8, -2\rangle$ can be thought of as having a "slope" of -2.

3. T/F: Let $z = f(x, y)$ be differentiable at P. If $\vec{n}$ is a normal vector to the tangent plane of f at P, then $\vec{n}$ is orthogonal to f_x and f_y at P.

4. Explain in your own words why we do not refer to *the* tangent line to a surface at a point, but rather to *directional* tangent lines to a surface at a point.

Problems

In Exercises 5 – 8, a function $z = f(x, y)$, a vector $\vec{v}$ and a point P are given. Give the parametric equations of the following directional tangent lines to f at P:

(a) $\ell_x(t)$

(b) $\ell_y(t)$

(c) $\ell_{\vec{u}}(t)$**, where $\vec{u}$ is the unit vector in the direction of $\vec{v}$.**

5. $f(x, y) = 2x^2y - 4xy^2$, $\vec{v} = \langle 1, 3\rangle$, $P = (2, 3)$.

6. $f(x, y) = 3\cos x \sin y$, $\vec{v} = \langle 1, 2\rangle$, $P = (\pi/3, \pi/6)$.

7. $f(x, y) = 3x - 5y$, $\vec{v} = \langle 1, 1\rangle$, $P = (4, 2)$.

8. $f(x, y) = x^2 - 2x - y^2 + 4y$, $\vec{v} = \langle 1, 1\rangle$, $P = (1, 2)$.

In Exercises 9 – 12, a function $z = f(x, y)$ and a point P are given. Find the equation of the normal line to f at P. Note: these are the same functions as in Exercises 5 – 8.

9. $f(x, y) = 2x^2y - 4xy^2$, $P = (2, 3)$.

10. $f(x, y) = 3\cos x \sin y$, $P = (\pi/3, \pi/6)$.

11. $f(x, y) = 3x - 5y$, $P = (4, 2)$.

12. $f(x, y) = x^2 - 2x - y^2 + 4y$, $P = (1, 2)$.

In Exercises 13 – 16, a function $z = f(x, y)$ and a point P are given. Find the two points that are 2 units from the surface f at P. Note: these are the same functions as in Exercises 5 – 8.

13. $f(x, y) = 2x^2y - 4xy^2$, $P = (2, 3)$.

14. $f(x, y) = 3\cos x \sin y$, $P = (\pi/3, \pi/6)$.

15. $f(x, y) = 3x - 5y$, $P = (4, 2)$.

16. $f(x, y) = x^2 - 2x - y^2 + 4y$, $P = (1, 2)$.

In Exercises 17 – 20, a function $z = f(x, y)$ and a point P are given. Find the equation of the tangent plane to f at P. Note: these are the same functions as in Exercises 5 – 8.

17. $f(x, y) = 2x^2y - 4xy^2$, $P = (2, 3)$.

18. $f(x, y) = 3\cos x \sin y$, $P = (\pi/3, \pi/6)$.

19. $f(x, y) = 3x - 5y$, $P = (4, 2)$.

20. $f(x, y) = x^2 - 2x - y^2 + 4y$, $P = (1, 2)$.

In Exercises 21 – 24, an implicitly defined function of x, y and z is given along with a point P that lies on the surface. Use the gradient ∇F to:

(a) find the equation of the normal line to the surface at P, and

(b) find the equation of the plane tangent to the surface at P.

21. $\dfrac{x^2}{8} + \dfrac{y^2}{4} + \dfrac{z^2}{16} = 1$, at $P = (1, \sqrt{2}, \sqrt{6})$

22. $z^2 - \dfrac{x^2}{4} - \dfrac{y^2}{9} = 0$, at $P = (4, -3, \sqrt{5})$

23. $xy^2 - xz^2 = 0$, at $P = (2, 1, -1)$

24. $\sin(xy) + \cos(yz) = 0$, at $P = (2, \pi/12, 4)$

11.8 Extreme Values

Given a function $z = f(x, y)$, we are often interested in points where z takes on the largest or smallest values. For instance, if z represents a cost function, we would likely want to know what (x, y) values minimize the cost. If z represents the ratio of a volume to surface area, we would likely want to know where z is greatest. This leads to the following definition.

Definition 49 Relative and Absolute Extrema

Let $z = f(x, y)$ be defined on a set S containing the point $P = (x_0, y_0)$.

1. If there is an open disk D containing P such that $f(x_0, y_0) \geq f(x, y)$ for all (x, y) in D, then f has a **relative maximum** at P; if $f(x_0, y_0) \leq f(x, y)$ for all (x, y) in D, then f has a **relative minimum** at P.
2. If $f(x_0, y_0) \geq f(x, y)$ for all (x, y) in S, then f has an **absolute maximum** at P; if $f(x_0, y_0) \leq f(x, y)$ for all (x, y) in S, then f has an **absolute minimum** at P.
3. If f has a relative maximum or minimum at P, then f has a **relative extrema** at P; if f has an absolute maximum or minimum at P, then f has a **absolute extrema** at P.

If f has a relative or absolute maximum at $P = (x_0, y_0)$, it means every curve on the surface of f through P will also have a relative or absolute maximum at P. Recalling what we learned in Section **??**, the slopes of the tangent lines to these curves at P must be 0 or undefined. Since directional derivatives are computed using f_x and f_y, we are led to the following definition and theorem.

Definition 50 Critical Point

Let $z = f(x, y)$ be continuous on an open set S. A **critical point** $P = (x_0, y_0)$ of f is a point in S such that

- $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, or
- $f_x(x_0, y_0)$ and/or $f_y(x_0, y_0)$ is undefined.

Notes:

Theorem 30 Critical Points and Relative Extrema

Let $z = f(x, y)$ be defined on an open set S containing $P = (x_0, y_0)$. If f has a relative extrema at P, then P is a critical point of f.

Therefore, to find relative extrema, we find the critical points of f and determine which correspond to relative maxima, relative minima, or neither. The following examples demonstrate this process.

Figure 11.27: The surface in Example 122 with its absolute minimum indicated.

Example 122 Finding critical points and relative extrema

Let $f(x, y) = x^2 + y^2 - xy - x - 2$. Find the relative extrema of f.

Solution We start by computing the partial derivatives of f:

$$f_x(x, y) = 2x - y - 1 \qquad \text{and} \qquad f_y(x, y) = 2y - x.$$

Each is never undefined. A critical point occurs when f_x and f_y are simultaneously 0, leading us to solve the following system of linear equations:

$$2x - y - 1 = 0 \qquad \text{and} \qquad -x + 2y = 0.$$

This solution to this system is $x = 2/3$, $y = 1/3$. (Check that at $(2/3, 1/3)$, both f_x and f_y are 0.)

The graph in Figure 11.27 shows f along with this critical point. It is clear from the graph that this is a relative minimum; further consideration of the function shows that this is actually the absolute minimum.

Figure 11.28: The surface in Example 123 with its absolute maximum indicated.

Example 123 Finding critical points and relative extrema

Let $f(x, y) = -\sqrt{x^2 + y^2} + 2$. Find the relative extrema of f.

Solution We start by computing the partial derivatives of f:

$$f_x(x, y) = \frac{-x}{\sqrt{x^2 + y^2}} \qquad \text{and} \qquad f_y(x, y) = \frac{-y}{\sqrt{x^2 + y^2}}.$$

It is clear that $f_x = 0$ when $x = 0$ & $y \neq 0$, and that $f_y = 0$ when $y = 0$ & $x \neq 0$. At $(0, 0)$, both f_x and f_y are *not* 0, but rather undefined. The point $(0, 0)$ is still a critical point, though, because the partial derivatives are undefined. This is the only critical point of f.

The surface of f is graphed in Figure 11.28 along with the point $(0, 0, 2)$. The graph shows that this point is the absolute maximum of f.

Notes:

In each of the previous two examples, we found a critical point of f and then determined whether or not it was a relative (or absolute) maximum or minimum by graphing. It would be nice to be able to determine whether a critical point corresponded to a max or a min without a graph. Before we develop such a test, we do one more example that sheds more light on the issues our test needs to consider.

Example 124 Finding critical points and relative extrema

Let $f(x, y) = x^3 - 3x - y^2 + 4y$. Find the relative extrema of f.

Solution Once again we start by finding the partial derivatives of f:

$$f_x(x, y) = 3x^2 - 3 \qquad \text{and} \qquad f_y(x, y) = -2y + 4.$$

Each is always defined. Setting each equal to 0 and solving for x and y, we find

$$\begin{aligned} f_x(x, y) = 0 \quad &\Rightarrow x = \pm 1 \\ f_y(x, y) = 0 \quad &\Rightarrow y = 2. \end{aligned}$$

We have two critical points: $(-1, 2)$ and $(1, 2)$. To determine if they correspond to a relative maximum or minimum, we consider the graph of f in Figure 11.29.

Figure 11.29: The surface in Example 124 with both critical points marked.

The critical point $(-1, 2)$ clearly corresponds to a relative maximum. However, the critical point at $(1, 2)$ is neither a maximum nor a minimum, displaying a different, interesting characteristic.

If one walks parallel to the y-axis towards this critical point, then this point becomes a relative maximum along this path. But if one walks towards this point parallel to the x-axis, this point becomes a relative minimum along this path. A point that seems to act as both a max and a min is a **saddle point**. A formal definition follows.

Definition 51 Saddle Point

Let $P = (x_0, y_0)$ be in the domain of f where $f_x = 0$ and $f_y = 0$ at P. P is a **saddle point** of f if, for every open disk D containing P, there are points (x_1, y_1) and (x_2, y_2) in D such that $f(x_0, y_0) > f(x_1, y_1)$ and $f(x_0, y_0) < f(x_2, y_2)$.

At a saddle point, the instantaneous rate of change in all directions is 0 and there are points nearby with z-values both less than and greater than the z-value of the saddle point.

Notes:

Before Example 124 we mentioned the need for a test to differentiate between relative maxima and minima. We now recognize that our test also needs to account for saddle points. To do so, we consider the second partial derivatives of f.

Recall that with single variable functions, such as $y = f(x)$, if $f''(c) > 0$, then f is concave up at c, and if $f'(c) = 0$, then f has a relative minimum at $x = c$. (We called this the Second Derivative Test.) Note that at a saddle point, it seems the graph is "both" concave up and concave down, depending on which direction you are considering.

It would be nice if the following were true:

$$\begin{array}{rcl} f_{xx} \text{ and } f_{yy} > 0 & \Rightarrow & \text{relative minimum} \\ f_{xx} \text{ and } f_{yy} < 0 & \Rightarrow & \text{relative maximum} \\ f_{xx} \text{ and } f_{yy} \text{ have opposite signs} & \Rightarrow & \text{saddle point.} \end{array}$$

However, this is not the case. Functions f exist where f_{xx} and f_{yy} are both positive but a saddle point still exists. In such a case, while the concavity in the x-direction is up (i.e., $f_{xx} > 0$) and the concavity in the y-direction is also up (i.e., $f_{yy} > 0$), the concavity switches somewhere in between the x- and y-directions.

To account for this, consider $D = f_{xx}f_{yy} - f_{xy}f_{yx}$. Since f_{xy} and f_{yx} are equal when continuous (refer back to Theorem 19), we can rewrite this as $D = f_{xx}f_{yy} - f_{xy}^2$. D can be used to test whether the concavity at a point changes depending on direction. If $D > 0$, the concavity does not switch (i.e., at that point, the graph is concave up or down in all directions). If $D < 0$, the concavity does switch. If $D = 0$, our test fails to determine whether concavity switches or not. We state the use of D in the following theorem.

Theorem 31 Second Derivative Test

Let $z = f(x, y)$ be differentiable on an open set containing $P = (x_0, y_0)$, and let

$$D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0).$$

1. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then P is a relative minimum of f.
2. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then P is a relative maximum of f.
3. If $D < 0$, then P is a saddle point of f.
4. If $D = 0$, the test is inconclusive.

We first practice using this test with the function in the previous example, where we visually determined we had a relative maximum and a saddle point.

Notes:

Example 125 Using the Second Derivative Test

Let $f(x, y) = x^3 - 3x - y^2 + 4y$ as in Example 124. Determine whether the function has a relative minimum, maximum, or saddle point at each critical point.

Solution We determined previously that the critical points of f are $(-1, 2)$ and $(1, 2)$. To use the Second Derivative Test, we must find the second partial derivatives of f:

$$f_{xx} = 6x; \qquad f_{yy} = -2; \qquad f_{xy} = 0.$$

Thus $D(x, y) = -12x$.

At $(-1, 2)$: $D(-1, 2) = 12 > 0$, and $f_{xx}(-1, 2) = -6$. By the Second Derivative Test, f has a relative maximum at $(-1, 2)$.

At $(1, 2)$: $D(1, 2) = -12 < 0$. The Second Derivative Test states that f has a saddle point at $(1, 2)$.

The Second Derivative Test confirmed what we determined visually.

Example 126 Using the Second Derivative Test

Find the relative extrema of $f(x, y) = x^2y + y^2 + xy$.

Solution We start by finding the first and second partial derivatives of f:

$$\begin{aligned} f_x &= 2xy + y & f_y &= x^2 + 2y + x \\ f_{xx} &= 2y & f_{yy} &= 2 \\ f_{xy} &= 2x + 1 & f_{yx} &= 2x + 1. \end{aligned}$$

We find the critical points by finding where f_x and f_y are simultaneously 0 (they are both never undefined). Setting $f_x = 0$, we have:

$$f_x = 0 \quad \Rightarrow \quad 2xy + y = 0 \quad \Rightarrow \quad y(2x + 1) = 0.$$

This implies that for $f_x = 0$, either $y = 0$ or $2x + 1 = 0$.

Assume $y = 0$ then consider $f_y = 0$:

$$\begin{aligned} f_y &= 0 \\ x^2 + 2y + x &= 0, \qquad \text{and since } y = 0\text{, we have} \\ x^2 + x &= 0 \\ x(x + 1) &= 0. \end{aligned}$$

Thus if $y = 0$, we have either $x = 0$ or $x = -1$, giving two critical points: $(-1, 0)$ and $(0, 0)$.

Notes:

Figure 11.30: Graphing f from Example 126 and its relative extrema.

Going back to f_x, now assume $2x+1=0$, i.e., that $x=-1/2$, then consider $f_y=0$:

$$\begin{aligned} f_y &= 0 \\ x^2+2y+x &= 0, \qquad \text{and since } x=-1/2\text{, we have} \\ 1/4+2y-1/2 &= 0 \\ y &= 1/8. \end{aligned}$$

Thus if $x=-1/2$, $y=1/8$ giving the critical point $(-1/2, 1/8)$.

With $D=4y-(2x+1)^2$, we apply the Second Derivative Test to each critical point.

At $(-1,0)$, $D<0$, so $(-1,0)$ is a saddle point.

At $(0,0)$, $D<0$, so $(0,0)$ is also a saddle point.

At $(-1/2,1/8)$, $D>0$ and $f_{xx}>0$, so $(-1/2,1/8)$ is a relative minimum.

Figure 11.30 shows a graph of f and the three critical points. Note how this function does not vary much near the critical points – that is, visually it is difficult to determine whether a point is a saddle point or relative minimum (or even a critical point at all!). This is one reason why the Second Derivative Test is so important to have.

Constrained Optimization

When optimizing functions of one variable such as $y=f(x)$, we made use of Theorem **??**, the Extreme Value Theorem, that said that over a closed interval I, a continuous function has both a maximum and minimum value. To find these maximum and minimum values, we evaluated f at all critical points in the interval, as well as at the endpoints (the "boundary") of the interval.

A similar theorem and procedure applies to functions of two variables. A continuous function over a closed set also attains a maximum and minimum value (see the following theorem). We can find these values by evaluating the function at the critical values in the set and over the boundary of the set. After formally stating this extreme value theorem, we give examples.

Theorem 32 Extreme Value Theorem

Let $z=f(x,y)$ be a continuous function on a closed, bounded set S. Then f has a maximum and minimum value on S.

Example 127 Finding extrema on a closed set

Let $f(x,y)=x^2-y^2+5$ and let S be the triangle with vertices $(-1,-2)$, $(0,1)$ and $(2,-2)$. Find the maximum and minimum values of f on S.

Notes:

Solution It can help to see a graph of f along with the set S. In Figure 11.31(a) the triangle defining S is shown in the x-y plane in a dashed line. Above it is the surface of f; we are only concerned with the portion of f enclosed by the "triangle" on its surface.

We begin by finding the critical points of f. With $f_x = 2x$ and $f_y = -2y$, we find only one critical point, at $(0, 0)$.

We now find the maximum and minimum values that f attains along the boundary of S, that is, along the edges of the triangle. In Figure 11.31(b) we see the triangle sketched in the plane with the equations of the lines forming its edges labeled.

Start with the bottom edge, along the line $y = -2$. If y is -2, then on the surface, we are considering points $f(x, -2)$; that is, our function reduces to $f(x, -2) = x^2 - (-2)^2 + 5 = x^2 + 1 = f_1(x)$. We want to maximize/minimize $f_1(x) = x^2 + 1$ on the interval $[-1, 2]$. To do so, we evaluate $f_1(x)$ at its critical points and at the endpoints.

The critical points of f_1 are found by setting its derivative equal to 0:

$$f_1'(x) = 0 \qquad \Rightarrow x = 0.$$

Evaluating f_1 at this critical point, and at the endpoints of $[-1, 1]$ gives:

$$\begin{aligned} f_1(-1) &= 2 & \Rightarrow \quad & f(-1, -2) = 2 \\ f_1(0) &= 1 & \Rightarrow \quad & f(0, -2) = 1 \\ f_1(2) &= 5 & \Rightarrow \quad & f(2, -2) = 5. \end{aligned}$$

Notice how evaluating f_1 at a point is the same as evaluating f at its corresponding point.

We need to do this process twice more, for the other two edges of the triangle.

Along the left edge, along the line $y = 3x + 1$, we substitute $3x + 1$ in for y in $f(x, y)$:

$$f(x, y) = f(x, 3x + 1) = x^2 - (3x + 1)^2 + 5 = -8x^2 - 6x + 4 = f_2(x).$$

We want the maximum and minimum values of f_2 on the interval $[-1, 0]$, so we evaluate f_2 at its critical points and the endpoints of the interval. We find the critical points:

$$f_2'(x) = -16x - 6 = 0 \qquad \Rightarrow \qquad x = -3/8.$$

Evaluate f_2 at its critical point and the endpoints of $[-1, 0]$:

$$\begin{aligned} f_2(-1) &= 2 & \Rightarrow \quad & f(-1, -2) = 2 \\ f_2(-3/8) &= 41/8 = 5.125 & \Rightarrow \quad & f(-3/8, -0.125) = 5.125 \\ f_2(0) &= 1 & \Rightarrow \quad & f(0, 1) = 4. \end{aligned}$$

Figure 11.31: Plotting the surface of f along with the restricted domain S.

Notes:

Figure 11.32: The surface of f along with important points along the boundary of S and the interior.

Finally, we evaluate f along the right edge of the triangle, where $y = -3/2x + 1$.

$$f(x, y) = f(x, -3/2x + 1) = x^2 - (-3/2x + 1)^2 + 5 = -\frac{5}{4}x^2 + 3x + 4 = f_3(x).$$

The critical points of $f_3(x)$ are:

$$f_3'(x) = 0 \qquad \Rightarrow \qquad x = 6/5 = 1.2.$$

We evaluate f_3 at this critical point and at the endpoints of the interval $[0, 2]$:

$$\begin{aligned} f_3(0) = 4 \quad &\Rightarrow \quad f(0, 1) = 4 \\ f_3(1.2) = 5.8 \quad &\Rightarrow \quad f(1.2, -0.8) = 5.8 \\ f_3(2) = 5 \quad &\Rightarrow \quad f(2, -2) = 5. \end{aligned}$$

One last point to test: the critical point of f, $(0, 0)$. We find $f(0, 0) = 5$.

We have evaluated f at a total of 7 different places, all shown in Figure 11.32. We checked each vertex of the triangle twice, as each showed up as the endpoint of an interval twice. Of all the z-values found, the maximum is 5.8, found at $(1.2, -0.8)$; the minimum is 1, found at $(0, -2)$.

This portion of the text is entitled "Constrained Optimization" because we want to optimize a function (i.e., find its maximum and/or minimum values) subject to a *constraint* – some limit to what values the function can attain. In the previous example, we constrained ourselves by considering a function only within the boundary of a triangle. This was largely arbitrary; the function and the boundary were chosen just as an example, with no real "meaning" behind the function or the chosen constraint.

However, solving constrained optimization problems is a very important topic in applied mathematics. The techniques developed here are the basis for solving larger problems, where more than two variables are involved.

We illustrate the technique once more with a classic problem.

Example 128 Constrained Optimization

The U.S. Postal Service states that the girth+length of Standard Post Package must not exceed 130". Given a rectangular box, the "length" is the longest side, and the "girth" is twice the width+height.

Given a rectangular box where the width and height are equal, what are the dimensions of the box that give the maximum volume subject to the constraint of the size of a Standard Post Package?

Solution Let w, h and ℓ denote the width, height and length of a rectangular box; we assume here that $w = h$. The girth is then $2(w + h) = 4w$. The

Notes:

volume of the box is $V(w,\ell) = wh\ell = w^2\ell$. We wish to maximize this volume subject to the constraint $4w + \ell \leq 130$, or $\ell \leq 130 - 4w$. (Common sense also indicates that $\ell > 0, w > 0$.)

We begin by finding the critical values of V. We find that $V_w = 2w\ell$ and $V_\ell = w^2$; these are simultaneously 0 only at $(0,0)$. This gives a volume of 0, so we can ignore this critical point.

We now consider the volume along the constraint $\ell = 130 - 4w$. Along this line, we have:

$$V(w\ell) = V(w, 130 - 4w) = w^2(130 - 4w) = 130w^2 - 4w^3 = V_1(w).$$

The constraint is applicable on the w-interval $[0, 32.5]$ as indicated in the figure. Thus we want to maximize V_1 on $[0, 32.5]$.

Finding the critical values of V_1, we take the derivative and set it equal to 0:

$$V_1'(w) = 260w - 12w^2 = 0 \quad \Rightarrow \quad w(260 - 12w) = 0 \quad \Rightarrow \quad w = 0, \frac{260}{12} \approx 21.67.$$

We found two critical values: when $w = 0$ and when $w = 21.67$. We again ignore the $w = 0$ solution; the maximum volume, subject to the constraint, comes at $w = h = 21.67$, $\ell = 130 - 4(21.6) = 43.33$. This gives a volume of $V(21.67, 43.33) \approx 19,408\text{in}^3$.

The volume function $V(w,\ell)$ is shown in Figure 11.33 along with the constraint $\ell = 130 - 4w$. As done previously, the constraint is drawn dashed in the x-y plane and also along the surface of the function. The point where the volume is maximized is indicated.

Figure 11.33: Graphing the volume of a box with girth $4w$ and length ℓ, subject to a size constraint.

It is hard to overemphasize the importance of optimization. In "the real world," we routinely seek to make *something* better. By expressing the *something* as a mathematical function, "making *something* better" means "optimize *some function*."

The techniques shown here are only the beginning of an incredibly important field. Many functions that we seek to optimize are incredibly complex, making the step of "find the gradient and set it equal to $\vec{0}$" highly nontrivial. Mastery of the principles here are key to being able to tackle these more complicated problems.

Notes:

Exercises 11.8

Terms and Concepts

1. T/F: Theorem 30 states that if f has a critical point at P, then f has a relative extrema at P.

2. T/F: A point P is a critical point of f if f_x and f_y are both 0 at P.

3. T/F: A point P is a critical point of f if f_x or f_y are undefined at P.

4. Explain what it means to "solve a constrained optimization" problem.

Problems

In Exercises 5 – 14, find the critical points of the given function. Use the Second Derivative Test to determine if each critical point corresponds to a relative maximum, minimum, or saddle point.

5. $f(x, y) = \frac{1}{2}x^2 + 2y^2 - 8y + 4x$

6. $f(x, y) = x^2 + 4x + y^2 - 9y + 3xy$

7. $f(x, y) = x^2 + 3y^2 - 6y + 4xy$

8. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$

9. $f(x, y) = x^2 + y^3 - 3y + 1$

10. $f(x, y) = \dfrac{1}{3}x^3 - x + \dfrac{1}{3}y^3 - 4y$

11. $f(x, y) = x^2y^2$

12. $f(x, y) = x^4 - 2x^2 + y^3 - 27y - 15$

13. $f(x, y) = \sqrt{16 - (x - 3)^2 - y^2}$

14. $f(x, y) = \sqrt{x^2 + y^2}$

In Exercises 15 – 18, find the absolute maximum and minimum of the function subject to the given constraint.

15. $f(x, y) = x^2 + y^2 + y + 1$, constrained to the triangle with vertices $(0, 1)$, $(-1, -1)$ and $(1, -1)$.

16. $f(x, y) = 5x - 7y$, constrained to the region bounded by $y = x^2$ and $y = 1$.

17. $f(x, y) = x^2 + 2x + y^2 + 2y$, constrained to the region bounded by the circle $x^2 + y^2 = 4$.

18. $f(x, y) = 3y - 2x^2$, constrained to the region bounded by the parabola $y = x^2 + x - 1$ and the line $y = x$.

12: Multiple Integration

The previous chapter introduced multivariable functions and we applied concepts of differential calculus to these functions. We learned how we can view a function of two variables as a surface in space, and learned how partial derivatives convey information about how the surface is changing in any direction.

In this chapter we apply techniques of integral calculus to multivariable functions. In Chapter **??** we learned how the definite integral of a single variable function gave us "area under the curve." In this chapter we will see that integration applied to a multivariable function gives us "volume under a surface." And just as we learned applications of integration beyond finding areas, we will find applications of integration in this chapter beyond finding volume.

12.1 Iterated Integrals and Area

In Chapter 11 we found that it was useful to differentiate functions of several variables with respect to one variable, while treating all the other variables as constants or coefficients. We can integrate functions of several variables in a similar way. For instance, if we are told that $f_x(x, y) = 2xy$, we can treat y as staying constant and integrate to obtain $f(x, y)$:

$$\begin{aligned} f(x, y) &= \int f_x(x, y)\, dx \\ &= \int 2xy\, dx \\ &= x^2y + C. \end{aligned}$$

Make a careful note about the constant of integration, C. This "constant" is something with a derivative of 0 with respect to x, so it could be any expression that contains only constants and functions of y. For instance, if $f(x, y) = x^2y + \sin y + y^3 + 17$, then $f_x(x, y) = 2xy$. To signify that C is actually a function of y, we write:

$$f(x, y) = \int f_x(x, y)\, dx = x^2y + C(y).$$

Using this process we can even evaluate definite integrals.

Example 129 **Integrating functions of more than one variable**

Evaluate the integral $\displaystyle\int_1^{2y} 2xy\, dx$.

Solution We find the indefinite integral as before, then apply the Fundamental Theorem of Calculus to evaluate the definite integral:

$$\begin{aligned} \int_1^{2y} 2xy\, dx &= x^2y\Big|_1^{2y} \\ &= (2y)^2y - (1)^2y \\ &= 4y^3 - y. \end{aligned}$$

We can also integrate with respect to y. In general,

$$\int_{h_1(y)}^{h_2(y)} f_x(x,y)\,dx = f(x,y)\Big|_{h_1(y)}^{h_2(y)} = f\big(h_2(y),y\big) - f\big(h_1(y),y\big),$$

and

$$\int_{g_1(x)}^{g_2(x)} f_y(x,y)\,dy = f(x,y)\Big|_{g_1(x)}^{g_2(x)} = f\big(x,g_2(x)\big) - f\big(x,g_1(x)\big).$$

Note that when integrating with respect to x, the bounds are functions of y (of the form $x = h_1(y)$ and $x = h_2(y)$) and the final result is also a function of y. When integrating with respect to y, the bounds are functions of x (of the form $y = g_1(x)$ and $y = g_2(x)$) and the final result is a function of x. Another example will help us understand this.

Example 130 Integrating functions of more than one variable

Evaluate $\displaystyle\int_1^x \left(5x^3y^{-3} + 6y^2\right)\,dy$.

Solution We consider x as staying constant and integrate with respect to y:

$$\begin{aligned}\int_1^x \left(5x^3y^{-3} + 6y^2\right)\,dy &= \left(\frac{5x^3y^{-2}}{-2} + \frac{6y^3}{3}\right)\Bigg|_1^x \\ &= \left(-\frac{5}{2}x^3x^{-2} + 2x^3\right) - \left(-\frac{5}{2}x^3 + 2\right) \\ &= \frac{9}{2}x^3 - \frac{5}{2}x - 2.\end{aligned}$$

Note how the bounds of the integral are from $y = 1$ to $y = x$ and that the final answer is a function of x.

In the previous example, we integrated a function with respect to y and ended up with a function of x. We can integrate this as well. This process is known as **iterated integration**, or **multiple integration.**

Example 131 Integrating an integral

Evaluate $\displaystyle\int_1^2 \left(\int_1^x \left(5x^3y^{-3} + 6y^2\right)\,dy\right)\,dx$.

Solution We follow a standard "order of operations" and perform the operations inside parentheses first (which is the integral evaluated in Example

Notes:

130.)

$$\begin{aligned}\int_1^2 \left(\int_1^x \left(5x^3y^{-3}+6y^2\right)\,dy\right)\,dx &= \int_1^2 \left(\left[\frac{5x^3y^{-2}}{-2}+\frac{6y^3}{3}\right]\Bigg|_1^x\right)\,dx\\ &= \int_1^2 \left(\frac{9}{2}x^3-\frac{5}{2}x-2\right)\,dx\\ &= \left(\frac{9}{8}x^4-\frac{5}{4}x^2-2x\right)\Bigg|_1^2\\ &= \frac{89}{8}.\end{aligned}$$

Note how the bounds of x were $x=1$ to $x=2$ and the final result was a number.

The previous example showed how we could perform something called an iterated integral; we do not yet know *why* we would be interested in doing so nor what the result, such as the number $89/8$, *means*. Before we investigate these questions, we offer some definitions.

Definition 52 Iterated Integration

Iterated integration is the process of repeatedly integrating the results of previous integrations. Integrating one integral is denoted as follows.

Let a, b, c and d be numbers and let $g_1(x)$, $g_2(x)$, $h_1(y)$ and $h_2(y)$ be functions of x and y, respectively. Then:

1. $\displaystyle\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right)\,dy.$
2. $\displaystyle\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right)\,dx.$

Again make note of the bounds of these iterated integrals.
With $\displaystyle\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$, x varies from $h_1(y)$ to $h_2(y)$, whereas y varies from c to d. That is, the bounds of x are *curves*, the curves $x=h_1(y)$ and $x=h_2(y)$, whereas the bounds of y are *constants*, $y=c$ and $y=d$. It is useful to remember that when setting up and evaluating such iterated integrals, we integrate "from

Notes:

curve to curve, then from point to point."

We now begin to investigate *why* we are interested in iterated integrals and *what* they mean.

Area of a plane region

Figure 12.1: Calculating the area of a plane region R with an iterated integral.

Consider the plane region R bounded by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, shown in Figure 12.1. We learned in Section **??** that the area of R is given by

$$\int_a^b \big(g_2(x) - g_1(x)\big)\, dx.$$

We can view the expression $\big(g_2(x) - g_1(x)\big)$ as

$$\big(g_2(x) - g_1(x)\big) = \int_{g_1(x)}^{g_2(x)} 1\, dy = \int_{g_1(x)}^{g_2(x)} dy,$$

meaning we can express the area of R as an iterated integral:

$$\text{area of } R = \int_a^b \big(g_2(x) - g_1(x)\big)\, dx = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} dy\right) dx = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx.$$

Figure 12.2: Calculating the area of a plane region R with an iterated integral.

In short: a certain iterated integral can be viewed as giving the area of a plane region.

A region R could also be defined by $c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(y)$, as shown in Figure 12.2. Using a process similar to that above, we have

$$\text{the area of } R = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\, dy.$$

We state this formally in a theorem.

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Theorem 33 Area of a plane region

1. Let R be a plane region bounded by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, where g_1 and g_2 are continuous functions on $[a, b]$. The area A of R is
$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx.$$
2. Let R be a plane region bounded by $c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(y)$, where h_1 and h_2 are continuous functions on $[c, d]$. The area A of R is
$$A = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\, dy.$$

The following examples should help us understand this theorem.

Example 132 Area of a rectangle

Find the area A of the rectangle with corners $(-1, 1)$ and $(3, 3)$, as shown in Figure 12.3.

Solution Multiple integration is obviously overkill in this situation, but we proceed to establish its use.

The region R is bounded by $x = -1$, $x = 3$, $y = 1$ and $y = 3$. Choosing to integrate with respect to y first, we have

$$A = \int_{-1}^{3} \int_{1}^{3} 1\, dy\, dx = \int_{-1}^{3} \left(y \Big|_1^3 \right) dx = \int_{-1}^{3} 2\, dx = 2x \Big|_{-1}^{3} = 8.$$

We could also integrate with respect to x first, giving:

$$A = \int_{1}^{3} \int_{-1}^{3} 1\, dx\, dy = \int_{1}^{3} \left(x \Big|_{-1}^{3} \right) dy = \int_{1}^{3} 4\, dy = 4y \Big|_{1}^{3} = 8.$$

Clearly there are simpler ways to find this area, but it is interesting to note that this method works.

Figure 12.3: Calculating the area of a rectangle with an iterated integral in Example 132.

Example 133 Area of a triangle

Find the area A of the triangle with vertices at $(1, 1)$, $(3, 1)$ and $(5, 5)$, as shown in Figure 12.4.

Solution The triangle is bounded by the lines as shown in the figure. Choosing to integrate with respect to x first gives that x is bounded by $x = y$

Figure 12.4: Calculating the area of a triangle with iterated integrals in Example 133.

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to $x = \frac{y+5}{2}$, while y is bounded by $y = 1$ to $y = 5$. (Recall that since x-values increase from left to right, the leftmost curve, $x = y$, is the lower bound and the rightmost curve, $x = (y+5)/2$, is the upper bound.) The area is

$$\begin{aligned} A &= \int_1^5 \int_y^{\frac{y+5}{2}} dx\, dy \\ &= \int_1^5 \left(x \Big|_y^{\frac{y+5}{2}} \right) dy \\ &= \int_1^5 \left(-\frac{1}{2}y + \frac{5}{2} \right) dy \\ &= \left(-\frac{1}{4}y^2 + \frac{5}{2}y \right) \Big|_1^5 \\ &= 4. \end{aligned}$$

We can also find the area by integrating with respect to y first. In this situation, though, we have two functions that act as the lower bound for the region R, $y = 1$ and $y = 2x - 5$. This requires us to use two iterated integrals. Note how the x-bounds are different for each integral:

$$\begin{aligned} A &= \int_1^3 \int_1^x 1\, dy\, dx &&+ \int_3^5 \int_{2x-5}^x 1\, dy\, dx \\ &= \int_1^3 (y)\Big|_1^x dx &&+ \int_3^5 (y)\Big|_{2x-5}^x dx \\ &= \int_1^3 (x-1)\, dx &&+ \int_3^5 (-x+5)\, dx \\ &= 2 &&+ 2 \\ &= 4. \end{aligned}$$

As expected, we get the same answer both ways.

Example 134 Area of a plane region

Find the area of the region enclosed by $y = 2x$ and $y = x^2$, as shown in Figure 12.5.

Figure 12.5: Calculating the area of a plane region with iterated integrals in Example 134.

Solution Once again we'll find the area of the region using both orders of integration.

Using $dy\, dx$:

$$\int_0^2 \int_{x^2}^{2x} 1\, dy\, dx = \int_0^2 (2x - x^2)\, dx = \left(x^2 - \frac{1}{3}x^3\right)\Big|_0^2 = \frac{4}{3}.$$

Notes:

Using $dx\,dy$:

$$\int_0^4 \int_{y/2}^{\sqrt{y}} 1\,dx\,dy = \int_0^4 (\sqrt{y} - y/2)\,dy = \left(\frac{2}{3}y^{3/2} - \frac{1}{4}y^2\right)\Big|_0^4 = \frac{4}{3}.$$

Changing Order of Integration

In each of the previous examples, we have been given a region R and found the bounds needed to find the area of R using both orders of integration. We integrated using both orders of integration to demonstrate their equality.

We now approach the skill of describing a region using both orders of integration from a different perspective. Instead of starting with a region and creating iterated integrals, we will start with an iterated integral and rewrite it in the other integration order. To do so, we'll need to understand the region over which we are integrating.

The simplest of all cases is when both integrals are bound by constants. The region described by these bounds is a rectangle (see Example 132), and so:

$$\int_a^b \int_c^d 1\,dy\,dx = \int_c^d \int_a^b 1\,dx\,dy.$$

When the inner integral's bounds are not constants, it is generally very useful to sketch the bounds to determine what the region we are integrating over looks like. From the sketch we can then rewrite the integral with the other order of integration.

Examples will help us develop this skill.

Example 135 Changing the order of integration

Rewrite the iterated integral $\displaystyle\int_0^6 \int_0^{x/3} 1\,dy\,dx$ with the order of integration $dx\,dy$.

Solution We need to use the bounds of integration to determine the region we are integrating over.

The bounds tell us that y is bounded by 0 and $x/3$; x is bounded by 0 and 6. We plot these four curves: $y = 0$, $y = x/3$, $x = 0$ and $x = 6$ to find the region described by the bounds. Figure 12.6 shows these curves, indicating that R is a triangle.

To change the order of integration, we need to consider the curves that bound the x-values. We see that the lower bound is $x = 3y$ and the upper bound is $x = 6$. The bounds on y are 0 to 2. Thus we can rewrite the integral as $\displaystyle\int_0^2 \int_{3y}^6 1\,dx\,dy$.

Figure 12.6: Sketching the region R described by the iterated integral in Example 135.

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Figure 12.7: Drawing the region determined by the bounds of integration in Example 136.

Example 136 Changing the order of integration

Change the order of integration of $\displaystyle\int_0^4 \int_{y^2/4}^{(y+4)/2} 1\,dx\,dy$.

Solution We sketch the region described by the bounds to help us change the integration order. x is bounded below and above (i.e., to the left and right) by $x = y^2/4$ and $x = (y+4)/2$ respectively, and y is bounded between 0 and 4. Graphing the previous curves, we find the region R to be that shown in Figure 12.7.

To change the order of integration, we need to establish curves that bound y. The figure makes it clear that there are two lower bounds for y: $y = 0$ on $0 \le x \le 2$, and $y = 2x - 4$ on $2 \le x \le 4$. Thus we need two double integrals. The upper bound for each is $y = 2\sqrt{x}$. Thus we have

$$\int_0^4 \int_{y^2/4}^{(y+4)/2} 1\,dx\,dy = \int_0^2 \int_0^{2\sqrt{x}} 1\,dy\,dx + \int_2^4 \int_{2x-4}^{2\sqrt{x}} 1\,dy\,dx.$$

This section has introduced a new concept, the iterated integral. We developed one application for iterated integration: area between curves. However, this is not new, for we already know how to find areas bounded by curves.

In the next section we apply iterated integration to solve problems we currently do not know how to handle. The "real" goal of this section was not to learn a new way of computing area. Rather, our goal was to learn how to define a region in the plane using the bounds of an iterated integral. That skill is very important in the following sections.

Notes:

Exercises 12.1

Terms and Concepts

1. When integrating $f_x(x, y)$ with respect to x, the constant of integration C is really which: $C(x)$ or $C(y)$? What does this mean?

2. Integrating an integral is called _____ _____.

3. When evaluating an iterated integral, we integrate from _____ to _____, then from _____ to _____.

4. One understanding of an iterated integral is that $\displaystyle\int_a^b \int_{g_1(x)}^{g_2(x)} dy\,dx$ gives the _____ of a plane region.

Problems

In Exercises 5 – 10, evaluate the integral and subsequent iterated integral.

5. (a) $\displaystyle\int_2^5 \left(6x^2 + 4xy - 3y^2\right) dy$

   (b) $\displaystyle\int_{-3}^{-2} \int_2^5 \left(6x^2 + 4xy - 3y^2\right) dy\,dx$

6. (a) $\displaystyle\int_0^{\pi} \left(2x\cos y + \sin x\right) dx$

   (b) $\displaystyle\int_0^{\pi/2} \int_0^{\pi} \left(2x\cos y + \sin x\right) dx\,dy$

7. (a) $\displaystyle\int_1^x \left(x^2y - y + 2\right) dy$

   (b) $\displaystyle\int_0^2 \int_1^x \left(x^2y - y + 2\right) dy\,dx$

8. (a) $\displaystyle\int_y^{y^2} \left(x - y\right) dx$

   (b) $\displaystyle\int_{-1}^1 \int_y^{y^2} \left(x - y\right) dx\,dy$

9. (a) $\displaystyle\int_0^y \left(\cos x \sin y\right) dx$

   (b) $\displaystyle\int_0^{\pi} \int_0^y \left(\cos x \sin y\right) dx\,dy$

10. (a) $\displaystyle\int_0^x \left(\frac{1}{1+x^2}\right) dy$

    (b) $\displaystyle\int_1^2 \int_0^x \left(\frac{1}{1+x^2}\right) dy\,dx$

In Exercises 11 – 16, a graph of a planar region R is given. Give the iterated integrals, with both orders of integration $dy\,dx$ and $dx\,dy$, that give the area of R. Evaluate one of the iterated integrals to find the area.

11.

12.

13.

14.

15.

16.

In Exercises 17 – 22, iterated integrals are given that compute the area of a region R in the x-y plane. Sketch the region R, and give the iterated integral(s) that give the area of R with the opposite order of integration.

17. $\displaystyle\int_{-2}^{2}\int_{0}^{4-x^2} dy\,dx$

18. $\displaystyle\int_{0}^{1}\int_{5-5x}^{5-5x^2} dy\,dx$

19. $\displaystyle\int_{-2}^{2}\int_{0}^{2\sqrt{4-y^2}} dx\,dy$

20. $\displaystyle\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dy\,dx$

21. $\displaystyle\int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy + \int_{1}^{4}\int_{y-2}^{\sqrt{y}} dx\,dy$

22. $\displaystyle\int_{-1}^{1}\int_{(x-1)/2}^{(1-x)/2} dy\,dx$

12.2 Double Integration and Volume

The definite integral of f over $[a, b]$, $\int_a^b f(x)\,dx$, was introduced as "the signed area under the curve." We approximated the value of this area by first subdividing $[a, b]$ into n subintervals, where the ith subinterval has length Δx_i, and letting c_i be any value in the ith subinterval. We formed rectangles that approximated part of the region under the curve with width Δx_i, height $f(c_i)$, and hence with area $f(c_i)\Delta x_i$. Summing all the rectangle's areas gave an approximation of the definite integral, and Theorem **??** stated that

$$\int_a^b f(x)\,dx = \lim_{\|\Delta x\| \to 0} \sum f(c_i)\Delta x_i,$$

connecting the area under the curve with sums of the areas of rectangles.

We use a similar approach in this section to find volume under a surface.

Let R be a closed, bounded region in the x-y plane and let $z = f(x, y)$ be a continuous function defined on R. We wish to find the signed volume under the surface of f over R. (We use the term "signed volume" to denote that space above the x-y plane, under f, will have a positive volume; space above f and under the x-y plane will have a "negative" volume, similar to the notion of signed area used before.)

We start by partitioning R into n rectangular subregions as shown in Figure 12.8(a). For simplicity's sake, we let all widths be Δx and all heights be Δy. Note that the sum of the areas of the rectangles is not equal to the area of R, but rather is a close approximation. Arbitrarily number the rectangles 1 through n, and pick a point (x_i, y_i) in the ith subregion.

The volume of the rectangular solid whose base is the ith subregion and whose height is $f(x_i, y_i)$ is $V_i = f(x_i, y_i)\Delta x\Delta y$. Such a solid is shown in Figure 12.8(b). Note how this rectangular solid only approximates the true volume under the surface; part of the solid is above the surface and part is below.

For each subregion R_i used to approximate R, create the rectangular solid with base area $\Delta x\Delta y$ and height $f(x_i, y_i)$. The sum of all rectangular solids is

$$\sum_{i=1}^{n} f(x_i, y_i)\Delta x\Delta y.$$

This approximates the signed volume under f over R. As we have done before, to get a better approximation we can use more rectangles to approximate the region R.

In general, each rectangle could have a different width Δx_j and height Δy_k, giving the ith rectangle an area $\Delta A_i = \Delta x_j \Delta y_k$ and the ith rectangular solid a

Figure 12.8: Developing a method for finding signed volume under a surface.

Notes:

volume of $f(x_i, y_i)\Delta A_i$. Let $||\Delta A||$ denote the length of the longest diagonal of all rectangles in the subdivision of R; $||\Delta A|| \to 0$ means each rectangle's width and height are both approaching 0. If f is a continuous function, as $||\Delta A||$ shrinks (and hence $n \to \infty$) the summation $\sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$ approximates the signed volume better and better. This leads to a definition.

Note: Recall that the integration symbol "$\int$" is an "elongated S," representing the word "sum." We interpreted $\int_a^b f(x)\,dx$ as "take the *sum* of the areas of rectangles over the interval $[a, b]$." The double integral uses two integration symbols to represent a "double sum." When adding up the volumes of rectangular solids over a partition of a region R, as done in Figure 12.8, one could first add up the volumes across each row (one type of sum), then add these totals together (another sum), as in

$$\sum_{j=1}^{n}\sum_{i=1}^{m} f(x_i, y_j)\Delta x_i \Delta y_j.$$

One can rewrite this as

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{m} f(x_i, y_j)\Delta x_i\right)\Delta y_j.$$

The summation inside the parenthesis indicates the sum of heights $\times$ widths, which gives an area; multiplying these areas by the thickness Δy_j gives a volume. The illustration in Figure 12.9 relates to this understanding.

Definition 53 Double Integral, Signed Volume

Let $z = f(x, y)$ be a continuous function defined over a closed region R in the x-y plane. The **signed volume** V under f over R is denoted by the **double integral**

$$V = \iint_R f(x, y)\,dA.$$

Alternate notations for the double integral are

$$\iint_R f(x, y)\,dA = \iint_R f(x, y)\,dx\,dy = \iint_R f(x, y)\,dy\,dx.$$

The definition above does not state how to find the signed volume, though the notation offers a hint. We need the next two theorems to evaluate double integrals to find volume.

Theorem 34 Double Integrals and Signed Volume

Let $z = f(x, y)$ be a continuous function defined over a closed region R in the x-y plane. Then the signed volume V under f over R is

$$V = \iint_R f(x, y)\,dA = \lim_{||\Delta A||\to 0}\sum_{i=1}^{n} f(x_i, y_i)\Delta A_i.$$

This theorem states that we can find the exact signed volume using a limit of sums. The partition of the region R is not specified, so any partitioning where the diagonal of each rectangle shrinks to 0 results in the same answer.

This does not offer a very satisfying way of computing area, though. Our experience has shown that evaluating the limits of sums can be tedious. We seek a more direct method.

Recall Theorem **??** in Section **??**. This stated that if $A(x)$ gives the cross-sectional area of a solid at x, then $\int_a^b A(x)\,dx$ gave the volume of that solid over

Notes:

$[a, b]$.

Consider Figure 12.9, where a surface $z = f(x, y)$ is drawn over a region R. Fixing a particular x value, we can consider the area under f over R where x has that fixed value. That area can be found with a definite integral, namely

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy.$$

Remember that though the integrand contains x, we are viewing x as fixed. Also note that the bounds of integration are functions of x: the bounds depend on the value of x.

As $A(x)$ is a cross-sectional area function, we can find the signed volume V under f by integrating it:

$$V = \int_a^b A(x)\, dx = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x, y)\, dy \right) dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

Figure 12.9: Finding volume under a surface by sweeping out a cross–sectional area.

This gives a concrete method for finding signed volume under a surface. We could do a similar procedure where we started with y fixed, resulting in a iterated integral with the order of integration $dx\, dy$. The following theorem states that both methods give the same result, which is the value of the double integral. It is such an important theorem it has a name associated with it.

Theorem 35 Fubini's Theorem

Let R be a closed, bounded region in the x-y plane and let $z = f(x, y)$ be a continuous function on R.

1. If R is bounded by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, where g_1 and g_2 are continuous functions on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is bounded by $c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(y)$, where h_1 and h_2 are continuous functions on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

Note that once again the bounds of integration follow the "curve to curve, point to point" pattern discussed in the previous section. In fact, one of the

Notes:

main points of the previous section is developing the skill of describing a region R with the bounds of an iterated integral. Once this skill is developed, we can use double integrals to compute many quantities, not just signed volume under a surface.

Figure 12.10: Finding the signed volume under a surface in Example 137.

Example 137 Evaluating a double integral

Let $f(x, y) = xy + e^y$. Find the signed volume under f on the region R, which is the rectangle with corners $(3, 1)$ and $(4, 2)$ pictured in Figure 12.10, using Fubini's Theorem and both orders of integration.

Solution We wish to evaluate $\iint_R \left(xy + e^y\right)\, dA$. As R is a rectangle, the bounds are easily described as $3 \leq x \leq 4$ and $1 \leq y \leq 2$.

Using the order $dy\, dx$:

$$\begin{aligned}
\iint_R \left(xy + e^y\right)\, dA &= \int_3^4 \int_1^2 \left(xy + e^y\right)\, dy\, dx \\
&= \int_3^4 \left(\left[\frac{1}{2}xy^2 + e^y\right]\Big|_1^2 \right) dx \\
&= \int_3^4 \left(\frac{3}{2}x + e^2 - e\right) dx \\
&= \left(\frac{3}{4}x^2 + \left(e^2 - e\right)x\right)\Big|_3^4 \\
&= \frac{21}{4} + e^2 - e \approx 9.92.
\end{aligned}$$

Now we check the validity of Fubini's Theorem by using the order $dx\, dy$:

$$\begin{aligned}
\iint_R \left(xy + e^y\right)\, dA &= \int_1^2 \int_3^4 \left(xy + e^y\right)\, dx\, dy \\
&= \int_1^2 \left(\left[\frac{1}{2}x^2y + xe^y\right]\Big|_3^4 \right) dy \\
&= \int_1^2 \left(\frac{7}{2}y + e^y\right)\, dy \\
&= \left(\frac{7}{4}y^2 + e^y\right)\Big|_1^2 \\
&= \frac{21}{4} + e^2 - e \approx 9.92.
\end{aligned}$$

Both orders of integration return the same result, as expected.

Notes:

Example 138 Evaluating a double integral

Evaluate $\iint_R \left(3xy - x^2 - y^2 + 6\right)\, dA$, where R is the triangle bounded by $x = 0$, $y = 0$ and $x/2 + y = 1$, as shown in Figure 12.11.

Figure 12.11: Finding the signed volume under the surface in Example 138.

Solution While it is not specified which order we are to use, we will evaluate the double integral using both orders to help drive home the point that it does not matter which order we use.

Using the order $dy\,dx$: The bounds on y go from "curve to curve," i.e., $0 \leq y \leq 1 - x/2$, and the bounds on x go from "point to point," i.e., $0 \leq x \leq 2$.

$$\begin{aligned}
\iint_R \left(3xy - x^2 - y^2 + 6\right)\, dA &= \int_0^2 \int_0^{-\frac{x}{2}+1} \left(3xy - x^2 - y^2 + 6\right)\, dy\, dx \\
&= \int_0^2 \left(\frac{3}{2}xy^2 - x^2y - \frac{1}{3}y^3 + 6y\right)\Bigg|_0^{-\frac{x}{2}+1} dx \\
&= \int_0^2 \left(\frac{11}{12}x^3 - \frac{11}{4}x^2 - x - \frac{17}{3}\right) dx \\
&= \left(\frac{11}{48}x^4 - \frac{11}{12}x^3 - \frac{1}{2}x^2 - \frac{17}{3}x\right)\Bigg|_0^2 \\
&= \frac{17}{3} = 5.\overline{6}.
\end{aligned}$$

Now lets consider the order $dx\,dy$. Here x goes from "curve to curve," $0 \leq x \leq 2 - 2y$, and y goes from "point to point," $0 \leq y \leq 1$:

$$\begin{aligned}
\iint_R \left(3xy - x^2 - y^2 + 6\right)\, dA &= \int_0^1 \int_0^{2-2y} \left(3xy - x^2 - y^2 + 6\right)\, dx\, dy \\
&= \int_0^1 \left(\frac{3}{2}x^2y - \frac{1}{3}x^3 - xy^2 + 6x\right)\Bigg|_0^{2-2y} dy \\
&= \int_0^1 \left(\frac{32}{3}y^3 - 22y^2 + 2y + \frac{28}{3}\right) dy \\
&= \left(\frac{8}{3}y^4 - \frac{22}{3}y^3 + y^2 + \frac{28}{3}y\right)\Bigg|_0^1 \\
&= \frac{17}{3} = 5.\overline{6}.
\end{aligned}$$

We obtained the same result using both orders of integration.

Note how in these two examples that the bounds of integration depend only on R; the bounds of integration have nothing to do with $f(x, y)$. This is an important concept, so we include it as a Key Idea.

Notes:

Key Idea 11 Double Integration Bounds

When evaluating $\int\int_R f(x,y)\,dA$ using an iterated integral, the bounds of integration depend only on R. The surface f does not determine the bounds of integration.

Before doing another example, we give some properties of double integrals. Each should make sense if we view them in the context of finding signed volume under a surface, over a region.

Figure 12.12: R is the union of two nonoverlapping regions, R_1 and R_2.

Theorem 36 Properties of Double Integrals

Let f and g be continuous functions over a closed, bounded plane region R, and let c be a constant.

1. $\displaystyle\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA.$
2. $\displaystyle\iint_R \big(f(x,y) \pm g(x,y)\big)\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$
3. If $f(x,y) \geq 0$ on R, then $\displaystyle\iint_R f(x,y)\,dA \geq 0.$
4. If $f(x,y) \geq g(x,y)$ on R, then $\displaystyle\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA.$
5. Let R be the union of two nonoverlapping regions, $R = R_1 \bigcup R_2$ (see Figure 12.12). Then
$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA.$$

Figure 12.13: Finding the signed volume under a surface in Example 139.

Example 139 Evaluating a double integral

Let $f(x,y) = \sin x \cos y$ and R be the triangle with vertices $(-1,0)$, $(1,0)$ and $(0,1)$ (see Figure 12.13). Evaluate the double integral $\int\int_R f(x,y)\,dA$.

Solution If we attempt to integrate using an iterated integral with the order $dy\,dx$, note how there are two upper bounds on R meaning we'll need to use two iterated integrals. We would need to split the triangle into two regions along the y-axis, then use Theorem 36, part 5.

Instead, let's use the order $dx\,dy$. The curves bounding x are $y - 1 \leq x \leq$

Notes:

$1-y$; the bounds on y are $0 \le y \le 1$. This gives us:

$$\begin{aligned}\iint_R f(x,y)\,dA &= \int_0^1 \int_{y-1}^{1-y} \sin x \cos y\,dx\,dy \\ &= \int_0^1 \Big(-\cos x \cos y\Big)\Big|_{y-1}^{1-y} dy \\ &= \int_0^1 \cos y\Big(-\cos(1-y)+\cos(y-1)\Big)dy.\end{aligned}$$

Recall that the cosine function is an even function; that is, $\cos x = \cos(-x)$. Therefore, from the last integral above, we have $\cos(y-1) = \cos(1-y)$. Thus the integrand simplifies to 0, and we have

$$\begin{aligned}\iint_R f(x,y)\,dA &= \int_0^1 0\,dy \\ &= 0.\end{aligned}$$

It turns out that over R, there is just as much volume above the x-y plane as below (look again at Figure 12.13), giving a final signed volume of 0.

Example 140 Evaluating a double integral

Evaluate $\iint_R (4-y)\,dA$, where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$, graphed in Figure 12.14.

Solution Graphing each curve can help us find their points of intersection. Solving analytically, the second equation tells us that $y = x^2/4$. Substituting this value in for y in the first equation gives us $x^4/16 = 4x$. Solving for x:

$$\begin{aligned}\frac{x^4}{16} &= 4x \\ x^4 - 64x &= 0 \\ x(x^3-64) &= 0 \\ x &= 0,\ 4.\end{aligned}$$

Thus we've found analytically what was easy to approximate graphically: the regions intersect at $(0,0)$ and $(4,4)$, as shown in Figure 12.14.

We now choose an order of integration: $dy\,dx$ or $dx\,dy$? Either order works; since the integrand does not contain x, choosing $dx\,dy$ might be simpler – at least, the first integral is very simple.

Figure 12.14: Finding the volume under the surface in Example 140.

Notes:

Thus we have the following "curve to curve, point to point" bounds: $y^2/4 \leq x \leq 2\sqrt{y}$, and $0 \leq y \leq 4$.

$$\begin{aligned}
\iint_R (4-y)\,dA &= \int_0^4 \int_{y^2/4}^{2\sqrt{y}} (4-y)\,dx\,dy \\
&= \int_0^4 \big(x(4-y)\big)\Big|_{y^2/4}^{2\sqrt{y}} dy \\
&= \int_0^4 \left(\left(2\sqrt{y} - \frac{y^2}{4}\right)(4-y)\right) dy = \int_0^4 \left(\frac{y^3}{4} - y^2 - 2y^{3/2} + 8y^{1/2}\right) dy \\
&= \left(\frac{y^4}{16} - \frac{y^3}{3} - \frac{4y^{5/2}}{5} + \frac{16y^{3/2}}{3}\right)\Bigg|_0^4 \\
&= \frac{176}{15} = 11.7\overline{3}.
\end{aligned}$$

The signed volume under the surface f is about 11.7 cubic units.

In the previous section we practiced changing the order of integration of a given iterated integral, where the region R was not explicitly given. Changing the bounds of an integral is more than just an test of understanding. Rather, there are cases where integrating in one order is really hard, if not impossible, whereas integrating with the other order is feasible.

Example 141 Changing the order of integration

Rewrite the iterated integral $\displaystyle\int_0^3 \int_y^3 e^{-x^2}\,dx\,dy$ with the order $dy\,dx$. Comment on the feasibility to evaluate each integral.

Figure 12.15: Determining the region R determined by the bounds of integration in Example 141.

Solution Once again we make a sketch of the region over which we are integrating to facilitate changing the order. The bounds on x are from $x = y$ to $x = 3$; the bounds on y are from $y = 0$ to $y = 3$. These curves are sketched in Figure 12.15, enclosing the region R.

To change the bounds, note that the curves bounding y are $y = 0$ up to $y = x$; the triangle is enclosed between $x = 0$ and $x = 3$. Thus the new bounds of integration are $0 \leq y \leq x$ and $0 \leq x \leq 3$, giving the iterated integral $\displaystyle\int_0^3 \int_0^x e^{-x^2}\,dy\,dx$.

How easy is it to evaluate each iterated integral? Consider the order of integrating $dx\,dy$, as given in the original problem. The first indefinite integral we need to evaluate is $\int e^{-x^2}\,dx$; we have stated before (see Section **??**) that this integral cannot be evaluated in terms of elementary functions. We are stuck.

Changing the order of integration makes a big difference here. In the second

Notes:

iterated integral, we are faced with $\int e^{-x^2}\,dy$; integrating with respect to y gives us $ye^{-x^2}+C$, and the first definite integral evaluates to

$$\int_0^x e^{-x^2}\,dy = xe^{-x^2}.$$

Thus

$$\int_0^3\int_0^x e^{-x^2}\,dy\,dx = \int_0^3 \left(xe^{-x^2}\right)dx.$$

This last integral is easy to evaluate with substitution, giving a final answer of $\frac{1}{2}(1-e^{-9}) \approx 0.5$. Figure 12.16 shows the surface over R.

In short, evaluating one iterated integral is impossible; the other iterated integral is relatively simple.

Figure 12.16: Showing the surface f defined in Example 141 over its region R.

Definition **??** defines the average value of a single–variable function $f(x)$ on the interval $[a,b]$ as

$$\text{average value of } f(x) \text{ on } [a,b] = \frac{1}{b-a}\int_a^b f(x)\,dx;$$

that is, it is the "area under f over an interval divided by the length of the interval." We make an analogous statement here: the average value of $z=f(x,y)$ over a region R is the volume under f over R divided by the area of R.

Definition 54 The Average Value of f on R

Let $z=f(x,y)$ be a continuous function defined over a closed region R in the x-y plane. The **average value of f on R** is

$$\text{average value of } f \text{ on } R = \frac{\displaystyle\iint_R f(x,y)\,dA}{\displaystyle\iint_R dA}.$$

Example 142 Finding average value of a function over a region R

Find the average value of $f(x,y)=4-y$ over the region R, which is bounded by the parabolas $y^2=4x$ and $x^2=4y$. Note: this is the same function and region as used in Example 140.

Solution In Example 140 we found

$$\iint_R f(x,y)\,dA = \int_0^4\int_{y^2/4}^{2\sqrt{y}} (4-y)\,dx\,dy = \frac{176}{15}.$$

Notes:

Figure 12.17: Finding the average value of f in Example 142.

We find the area of R by computing $\iint_R dA$:

$$\iint_R dA = \int_0^4 \int_{y^2/4}^{2\sqrt{y}} dx\, dy = \frac{16}{3}.$$

Dividing the volume under the surface by the area gives the average value:

$$\text{average value of } f \text{ on } R = \frac{176/15}{16/3} = \frac{11}{5} = 2.2.$$

While the surface, as shown in Figure 12.17, covers z-values from $z = 0$ to $z = 4$, the "average" z-value on R is 2.2.

The previous section introduced the iterated integral in the context of finding the area of plane regions. This section has extended our understanding of iterated integrals; now we see they can be used to find the signed volume under a surface.

This new understanding allows us to revisit what we did in the previous section. Given a region R in the plane, we computed $\iint_R 1\, dA$; again, our understanding at the time was that we were finding the area of R. However, we can now view the function $z = 1$ as a surface, a flat surface with constant z-value of 1. The double integral $\iint_R 1\, dA$ finds the volume, under $z = 1$, over R, as shown in Figure 12.18. Basic geometry tells us that if the base of a general right cylinder has area A, its volume is $A \cdot h$, where h is the height. In our case, the height is 1. We were "actually" computing the volume of a solid, though we interpreted the number as an area.

The next section extends our abilities to find "volumes under surfaces." Currently, some integrals are hard to compute because either the region R we are integrating over is hard to define with rectangular curves, or the integrand itself is hard to deal with. Some of these problems can be solved by converting everything into polar coordinates.

Figure 12.18: Showing how an iterated integral used to find area also finds a certain volume.

Notes:

Exercises 12.2

Terms and Concepts

1. An integral can be interpreted as giving the signed area over an interval; a double integral can be interpreted as giving the signed _____ over a region.

2. Explain why the following statement is false: "Fubini's Theorem states that $\displaystyle\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ dy\ dx = \int_a^b \int_{g_1(y)}^{g_2(y)} f(x,y)\ dx\ dy$."

3. Explain why if $f(x,y) > 0$ over a region R, then $\iint_R f(x,y)\ dA > 0$.

4. If $\iint_R f(x,y)\ dA = \iint_R g(x,y)\ dA$, does this imply $f(x,y) = g(x,y)$?

Problems

In Exercises 5 – 10,

(a) Evaluate the given iterated integral, and

(b) rewrite the integral using the other order of integration.

5. $\displaystyle\int_1^2 \int_{-1}^1 \left(\frac{x}{y} + 3\right)\ dx\ dy$

6. $\displaystyle\int_{-\pi/2}^{\pi/2} \int_0^{\pi} (\sin x \cos y)\ dx\ dy$

7. $\displaystyle\int_0^4 \int_0^{-x/2+2} \left(3x^2 - y + 2\right)\ dy\ dx$

8. $\displaystyle\int_1^3 \int_y^3 \left(x^2y - xy^2\right)\ dx\ dy$

9. $\displaystyle\int_0^1 \int_{-\sqrt{1-y}}^{\sqrt{1-y}} (x + y + 2)\ dx\ dy$

10. $\displaystyle\int_0^9 \int_{y/3}^{\sqrt{y}} \left(xy^2\right)\ dx\ dy$

In Exercises 11 – 18:

(a) Sketch the region R given by the problem.

(b) Set up the iterated integrals, in both orders, that evaluate the given double integral for the described region R.

(c) Evaluate one of the iterated integrals to find the signed volume under the surface $z = f(x,y)$ over the region R.

11. $\displaystyle\iint_R x^2y\ dA$, where R is bounded by $y = \sqrt{x}$ and $y = x^2$.

12. $\displaystyle\iint_R x^2y\ dA$, where R is bounded by $y = \sqrt[3]{x}$ and $y = x^3$.

13. $\displaystyle\iint_R x^2 - y^2\ dA$, where R is the rectangle with corners $(-1,-1)$, $(1,-1)$, $(1,1)$ and $(-1,1)$.

14. $\displaystyle\iint_R ye^x\ dA$, where R is bounded by $x = 0$, $x = y^2$ and $y = 1$.

15. $\displaystyle\iint_R \left(6 - 3x - 2y\right)\ dA$, where R is bounded by $x = 0$, $y = 0$ and $3x + 2y = 6$.

16. $\displaystyle\iint_R e^y\ dA$, where R is bounded by $y = \ln x$ and $y = \dfrac{1}{e-1}(x-1)$.

17. $\displaystyle\iint_R \left(x^3y - x\right)\ dA$, where R is the half of the circle $x^2 + y^2 = 9$ in the first and second quadrants.

18. $\displaystyle\iint_R \left(4 - 3y\right)\ dA$, where R is bounded by $y = 0$, $y = x/e$ and $y = \ln x$.

In Exercises 19 – 22, state why it is difficult/impossible to integrate the iterated integral in the given order of integration. Change the order of integration and evaluate the new iterated integral.

19. $\displaystyle\int_0^4 \int_{y/2}^2 e^{x^2}\ dx\ dy$

20. $\displaystyle\int_0^{\sqrt{\pi/2}} \int_x^{\sqrt{\pi/2}} \cos\left(y^2\right)\ dy\ dx$

21. $\displaystyle\int_0^1 \int_y^1 \frac{2y}{x^2 + y^2}\ dx\ dy$

22. $\displaystyle\int_{-1}^1 \int_1^2 \frac{x \tan^2 y}{1 + \ln y}\ dy\ dx$

In Exercises 23 – 26, find the average value of f over the region R. Notice how these functions and regions are related to the iterated integrals given in Exercises 5 – 8.

23. $f(x,y) = \dfrac{x}{y} + 3$; R is the rectangle with opposite corners $(-1,1)$ and $(1,2)$.

24. $f(x,y) = \sin x \cos y$; R is bounded by $x = 0$, $x = \pi$, $y = -\pi/2$ and $y = \pi/2$.

25. $f(x,y) = 3x^2 - y + 2$; R is bounded by the lines $y = 0$, $y = 2 - x/2$ and $x = 0$.

26. $f(x,y) = x^2y - xy^2$; R is bounded by $y = x$, $y = 1$ and $x = 3$.

Figure 12.19: Approximating a region R with portions of sectors of circles.

12.3 Double Integration with Polar Coordinates

We have used iterated integrals to evaluate double integrals, which give the signed volume under a surface, $z = f(x, y)$, over a region R of the x-y plane. The integrand is simply $f(x, y)$, and the bounds of the integrals are determined by the region R.

Some regions R are easy to describe using rectangular coordinates – that is, with equations of the form $y = f(x)$, $x = a$, etc. However, some regions are easier to handle if we represent their boundaries with polar equations of the form $r = f(\theta)$, $\theta = \alpha$, etc.

The basic form of the double integral is $\int\int_R f(x, y)\, dA$. We interpret this integral as follows: over the region R, sum up lots of products of heights (given by $f(x_i, y_i)$) and areas (given by ΔA_i). That is, dA represents "a little bit of area." In rectangular coordinates, we can describe a small rectangle as having area $dx\, dy$ or $dy\, dx$ – the area of a rectangle is simply length$\times$width – a small change in x times a small change in y. Thus we replace dA in the double integral with $dx\, dy$ or $dy\, dx$.

Now consider representing a region R with polar coordinates. Consider Figure 12.19(a). Let R be the region in the first quadrant bounded by the curve. We can approximate this region using the natural shape of polar coordinates: portions of sectors of circles. In the figure, one such region is shaded, shown again in part (b) of the figure.

As the area of a sector of a circle with radius r, subtended by an angle θ, is $A = \frac{1}{2}r^2\theta$, we can find the area of the shaded region. The whole sector has area $\frac{1}{2}r_2^2\Delta\theta$, whereas the smaller, unshaded sector has area $\frac{1}{2}r_1^2\Delta\theta$. The area of the shaded region is the difference of these areas:

$$\Delta A_i = \frac{1}{2}r_2^2\Delta\theta - \frac{1}{2}r_1^2\Delta\theta = \frac{1}{2}\left(r_2^2 - r_1^2\right)\left(\Delta\theta\right) = \frac{r_2 + r_1}{2}\left(r_2 - r_1\right)\Delta\theta.$$

Note that $(r_2 + r_1)/2$ is just the average of the two radii.

To approximate the region R, we use many such subregions; doing so shrinks the difference $r_2 - r_1$ between radii to 0 and shrinks the change in angle $\Delta\theta$ also to 0. We represent these infinitesimal changes in radius and angle as dr and $d\theta$, respectively. Finally, as dr is small, $r_2 \approx r_1$, and so $(r_2 + r_1)/2 \approx r_1$. Thus, when dr and $d\theta$ are small,

$$\Delta A_i \approx r_i\, dr\, d\theta.$$

Taking a limit, where the number of subregions goes to infinity and both $r_2 - r_1$ and $\Delta\theta$ go to 0, we get

$$dA = r\, dr\, d\theta.$$

So to evaluate $\int\int_R f(x, y)\, dA$, replace dA with $r\, dr\, d\theta$. Convert the function $z = f(x, y)$ to a function with polar coordinates with the substitutions $x = r\cos\theta$,

Notes:

$y = r\sin\theta$. Finally, find bounds $g_1(\theta) \le r \le g_2(\theta)$ and $\alpha \le \theta \le \beta$ that describe R. This is the key principle of this section, so we restate it here as a Key Idea.

Key Idea 12 Evaluating Double Integrals with Polar Coordinates

Let R be a plane region bounded by the polar equations $\alpha \le \theta \le \beta$ and $g_1(\theta) \le r \le g_2(\theta)$. Then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f\big(r\cos\theta, r\sin\theta\big)\,r\,dr\,d\theta.$$

Examples will help us understand this Key Idea.

Example 143 Evaluating a double integral with polar coordinates

Find the signed volume under the plane $z = 4 - x - 2y$ over the circle with equation $x^2 + y^2 = 1$.

Solution The bounds of the integral are determined solely by the region R over which we are integrating. In this case, it is a circle with equation $x^2 + y^2 = 1$. We need to find polar bounds for this region. It may help to review Section **??**; bounds for this circle are $0 \le r \le 1$ and $0 \le \theta \le 2\pi$.

We replace $f(x,y)$ with $f(r\cos\theta, r\sin\theta)$. That means we make the following substitutions:

$$4 - x - 2y \quad\Rightarrow\quad 4 - r\cos\theta - 2r\sin\theta.$$

Finally, we replace dA in the double integral with $r\,dr\,d\theta$. This gives the final iterated integral, which we evaluate:

$$\begin{aligned}
\iint_R f(x,y)\,dA &= \int_0^{2\pi}\int_0^1 \big(4 - r\cos\theta - 2r\sin\theta\big)r\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 \big(4r - r^2(\cos\theta - 2\sin\theta)\big)\,dr\,d\theta \\
&= \int_0^{2\pi} \left(2r^2 - \frac{1}{3}r^3(\cos\theta - 2\sin\theta)\right)\Big|_0^1\,d\theta \\
&= \int_0^{2\pi} \left(2 - \frac{1}{3}\big(\cos\theta - 2\sin\theta\big)\right)\,d\theta \\
&= \left(2\theta - \frac{1}{3}\big(\sin\theta + 2\cos\theta\big)\right)\Big|_0^{2\pi} \\
&= 4\pi \approx 12.566.
\end{aligned}$$

The surface and region R are shown in Figure 12.20.

Figure 12.20: Evaluating a double integral with polar coordinates in Example 143.

Notes:

(a)

(b)

Figure 12.21: Showing the region R and surface used in Example 144.

Example 144 Evaluating a double integral with polar coordinates

Find the volume under the paraboloid $z = 4 - (x-2)^2 - y^2$ over the region bounded by the circles $(x-1)^2 + y^2 = 1$ and $(x-2)^2 + y^2 = 4$.

Solution At first glance, this seems like a very hard volume to compute as the region R (shown in Figure 12.21(a)) has a hole in it, cutting out a strange portion of the surface, as shown in part (b) of the figure. However, by describing R in terms of polar equations, the volume is not very difficult to compute. It is straightforward to show that the circle $(x-1)^2 + y^2 = 1$ has polar equation $r = 2\cos\theta$, and that the circle $(x-2)^2 + y^2 = 4$ has polar equation $r = 4\cos\theta$. Each of these circles is traced out on the interval $0 \le \theta \le \pi$. The bounds on r are $2\cos\theta \le r \le 4\cos\theta$.

Replacing x with $r\cos\theta$ in the integrand, along with replacing y with $r\sin\theta$, prepares us to evaluate the double integral $\iint_R f(x,y)\,dA$:

$$\begin{aligned}
\iint_R f(x,y)\,dA &= \int_0^\pi \int_{2\cos\theta}^{4\cos\theta} \Big(4 - \big(r\cos\theta - 2\big)^2 - \big(r\sin\theta\big)^2\Big) r\,dr\,d\theta \\
&= \int_0^\pi \int_{2\cos\theta}^{4\cos\theta} \big(-r^3 + 4r^2\cos\theta\big)\,dr\,d\theta \\
&= \int_0^\pi \left(-\frac{1}{4}r^4 + \frac{4}{3}r^3\cos\theta\right)\Bigg|_{2\cos\theta}^{4\cos\theta} d\theta \\
&= \int_0^\pi \left(\left[-\frac{1}{4}(256\cos^4\theta) + \frac{4}{3}(64\cos^4\theta)\right] - \right. \\
&\quad \left. \left[-\frac{1}{4}(16\cos^4\theta) + \frac{4}{3}(8\cos^4\theta)\right]\right) d\theta \\
&= \int_0^\pi \frac{44}{3}\cos^4\theta\,d\theta.
\end{aligned}$$

To integrate $\cos^4\theta$, rewrite it as $\cos^2\theta\cos^2\theta$ and employ the power-reducing formula twice:

$$\begin{aligned}
\cos^4\theta &= \cos^2\theta\cos^2\theta \\
&= \frac{1}{2}\big(1+\cos(2\theta)\big)\frac{1}{2}\big(1+\cos(2\theta)\big) \\
&= \frac{1}{4}\big(1 + 2\cos(2\theta) + \cos^2(2\theta)\big) \\
&= \frac{1}{4}\left(1 + 2\cos(2\theta) + \frac{1}{2}\big(1+\cos(4\theta)\big)\right) \\
&= \frac{3}{8} + \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta).
\end{aligned}$$

Notes:

Picking up from where we left off above, we have

$$\begin{aligned}
&= \int_0^{\pi} \frac{44}{3}\cos^4\theta\, d\theta \\
&= \int_0^{\pi} \frac{44}{3}\left(\frac{3}{8} + \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta)\right) d\theta \\
&= \frac{44}{3}\left(\frac{3}{8}\theta + \frac{1}{4}\sin(2\theta) + \frac{1}{32}\sin(4\theta)\right)\Bigg|_0^{\pi} \\
&= \frac{11}{2}\pi \approx 17.279.
\end{aligned}$$

While this example was not trivial, the double integral would have been *much* harder to evaluate had we used rectangular coordinates.

Example 145 Evaluating a double integral with polar coordinates

Find the volume under the surface $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$ over the sector of the circle with radius a centered at the origin in the first quadrant, as shown in Figure 12.22.

Figure 12.22: The surface and region R used in Example 145.

Solution The region R we are integrating over is a circle with radius a, restricted to the first quadrant. Thus, in polar, the bounds on R are $0 \leq r \leq a$, $0 \leq \theta \leq \pi/2$. The integrand is rewritten in polar as

$$\frac{1}{x^2 + y^2 + 1} \Rightarrow \frac{1}{r^2\cos^2\theta + r^2\sin^2\theta + 1} = \frac{1}{r^2 + 1}.$$

We find the volume as follows:

$$\begin{aligned}
\iint_R f(x, y)\, dA &= \int_0^{\pi/2}\int_0^{a} \frac{r}{r^2 + 1}\, dr\, d\theta \\
&= \int_0^{\pi/2} \frac{1}{2}\left(\ln|r^2 + 1|\right)\Big|_0^{a} d\theta \\
&= \int_0^{\pi/2} \frac{1}{2}\ln(a^2 + 1)\, d\theta \\
&= \left(\frac{1}{2}\ln(a^2 + 1)\theta\right)\Bigg|_0^{\pi/2} \\
&= \frac{\pi}{4}\ln(a^2 + 1).
\end{aligned}$$

Figure 12.22 shows that f shrinks to near 0 very quickly. Regardless, as a grows, so does the volume, without bound.

Note: Previous work has shown that there is finite *area* under $\frac{1}{x^2+1}$ over the entire x-axis. However, Example 145 shows that there is infinite *volume* under $\frac{1}{x^2+y^2+1}$ over the entire x-y plane.

Notes:

Example 146 Finding the volume of a sphere

Find the volume of a sphere with radius a.

Solution The sphere of radius a, centered at the origin, has equation $x^2+y^2+z^2=a^2$; solving for z, we have $z=\sqrt{a^2-x^2-y^2}$. This gives the upper half of a sphere. We wish to find the volume under this top half, then double it to find the total volume.

The region we need to integrate over is the circle of radius a, centered at the origin. Polar bounds for this equation are $0\leq r\leq a$, $0\leq\theta\leq 2\pi$.

All together, the volume of a sphere with radius a is:

$$\begin{aligned}2\iint_R\sqrt{a^2-x^2-y^2}\,dA &= 2\int_0^{2\pi}\int_0^a\sqrt{a^2-(r\cos\theta)^2-(r\sin\theta)^2}r\,dr\,d\theta\\ &= 2\int_0^{2\pi}\int_0^a r\sqrt{a^2-r^2}\,dr\,d\theta.\end{aligned}$$

We can evaluate this inner integral with substitution. With $u=a^2-r^2$, $du=-2r\,dr$. The new bounds of integration are $u(0)=a^2$ to $u(a)=0$. Thus we have:

$$\begin{aligned}&= \int_0^{2\pi}\int_{a^2}^0\left(-u^{1/2}\right)du\,d\theta\\ &= \int_0^{2\pi}\left(-\frac{2}{3}u^{3/2}\right)\Big|_{a^2}^0 d\theta\\ &= \int_0^{2\pi}\left(\frac{2}{3}a^3\right)d\theta\\ &= \left(\frac{2}{3}a^3\theta\right)\Big|_0^{2\pi}\\ &= \frac{4}{3}\pi a^3.\end{aligned}$$

Generally, the formula for the volume of a sphere with radius r is given as $4/3\pi r^3$; we have justified this formula with our calculation.

Figure 12.23: Visualizing the solid used in Example 147.

Example 147 Finding the volume of a solid

A sculptor wants to make a solid bronze cast of the solid shown in Figure 12.23, where the base of the solid has boundary, in polar coordinates, $r=\cos(3\theta)$, and the top is defined by the plane $z=1-x+0.1y$. Find the volume of the solid.

Solution From the outset, we should recognize that knowing *how to set up* this problem is probably more important than knowing *how to compute*

Notes:

the integrals. The iterated integral to come is not "hard" to evaluate, though it is long, requiring lots of algebra. Once the proper iterated integral is determined, one can use readily–available technology to help compute the final answer.

The region R that we are integrating over is bound by $0 \leq r \leq \cos(3\theta)$, for $0 \leq \theta \leq \pi$ (note that this rose curve is traced out on the interval $[0, \pi]$, not $[0, 2\pi]$). This gives us our bounds of integration. The integrand is $z = 1-x+0.1y$; converting to polar, we have that the volume V is:

$$V = \iint_R f(x, y)\, dA = \int_0^{\pi} \int_0^{\cos(3\theta)} \big(1 - r\cos\theta + 0.1r\sin\theta\big) r\, dr\, d\theta.$$

Distributing the r, the inner integral is easy to evaluate, leading to

$$\int_0^{\pi} \left(\frac{1}{2}\cos^2(3\theta) - \frac{1}{3}\cos^3(3\theta)\cos\theta + \frac{0.1}{3}\cos^3(3\theta)\sin\theta \right)\, d\theta.$$

This integral takes time to compute by hand; it is rather long and cumbersome. The powers of cosine need to be reduced, and products like $\cos(3\theta)\cos\theta$ need to be turned to sums using the Product To Sum formulas in the back cover of this text.

We rewrite $\frac{1}{2}\cos^2(3\theta)$ as $\frac{1}{4}(1+\cos(6\theta))$. We can also rewrite $\frac{1}{3}\cos^3(3\theta)\cos\theta$ as:

$$\frac{1}{3}\cos^3(3\theta)\cos\theta = \frac{1}{3}\cos^2(3\theta)\cos(3\theta)\cos\theta = \frac{1}{3}\frac{1+\cos(6\theta)}{2}\big(\cos(4\theta)+\cos(2\theta)\big).$$

This last expression still needs simplification, but eventually all terms can be reduced to the form $a\cos(m\theta)$ or $a\sin(m\theta)$ for various values of a and m.

We forgo the algebra and recommend the reader employ technology, such as WolframAlpha®, to compute the numeric answer. Such technology gives:

$$\int_0^{\pi} \int_0^{\cos(3\theta)} \big(1 - r\cos\theta + 0.1r\sin\theta\big) r\, dr\, d\theta = \frac{\pi}{4} \approx 0.785u^3.$$

Since the units were not specified, we leave the result as almost 0.8 cubic units (meters, feet, etc.) Should the artist want to scale the piece uniformly, so that each rose petal had a length other than 1, she should keep in mind that scaling by a factor of k scales the volume by a factor of k^3.

We have used iterated integrals to find areas of plane regions and volumes under surfaces. Just as a single integral can be used to compute much more than "area under the curve," iterated integrals can be used to compute much more than we have thus far seen. The next two sections show two, among many, applications of iterated integrals.

Notes:

Exercises 12.3

Terms and Concepts

1. When evaluating $\iint_R f(x,y)\ dA$ using polar coordinates, $f(x,y)$ is replaced with _____ and dA is replaced with _____.

2. Why would one be interested in evaluating a double integral with polar coordinates?

Problems

In Exercises 3 – 10, a function $f(x,y)$ is given and a region R of the x-y plane is described. Set up and evaluate $\iint_R f(x,y)\ dA$ using polar coordinates.

3. $f(x,y) = 3x - y + 4$; R is the region enclosed by the circle $x^2 + y^2 = 1$.

4. $f(x,y) = 4x + 4y$; R is the region enclosed by the circle $x^2 + y^2 = 4$.

5. $f(x,y) = 8 - y$; R is the region enclosed by the circles with polar equations $r = \cos\theta$ and $r = 3\cos\theta$.

6. $f(x,y) = 4$; R is the region enclosed by the petal of the rose curve $r = \sin(2\theta)$ in the first quadrant.

7. $f(x,y) = \ln\left(x^2 + y^2\right)$; R is the annulus enclosed by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

8. $f(x,y) = 1 - x^2 - y^2$; R is the region enclosed by the circle $x^2 + y^2 = 1$.

9. $f(x,y) = x^2 - y^2$; R is the region enclosed by the circle $x^2 + y^2 = 36$ in the first and fourth quadrants.

10. $f(x,y) = (x-y)/(x+y)$; R is the region enclosed by the lines $y = x$, $y = 0$ and the circle $x^2 + y^2 = 1$ in the first quadrant.

In Exercises 11 – 14, an iterated integral in rectangular coordinates is given. Rewrite the integral using polar coordinates and evaluate the new double integral.

11. $\displaystyle\int_0^5 \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} \sqrt{x^2+y^2}\ dy\ dx$

12. $\displaystyle\int_{-4}^4 \int_{-\sqrt{16-y^2}}^{0} \left(2y - x\right)\ dx\ dy$

13. $\displaystyle\int_0^2 \int_y^{\sqrt{8-y^2}} \left(x + y\right)\ dx\ dy$

14. $\displaystyle\int_{-2}^{-1} \int_0^{\sqrt{4-x^2}} \left(x+5\right)\ dy\ dx + \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} \left(x+5\right)\ dy\ dx + \int_1^2 \int_0^{\sqrt{4-x^2}} \left(x+5\right)\ dy\ dx$

    Hint: draw the region of each integral carefully and see how they all connect.

In Exercises 15 – 16, special double integrals are presented that are especially well suited for evaluation in polar coordinates.

15. Consider $\displaystyle\iint_R e^{-(x^2+y^2)}\ dA$.

    (a) Why is this integral difficult to evaluate in rectangular coordinates, regardless of the region R?

    (b) Let R be the region bounded by the circle of radius a centered at the origin. Evaluate the double integral using polar coordinates.

    (c) Take the limit of your answer from (b), as $a \to \infty$. What does this imply about the volume under the surface of $e^{-(x^2+y^2)}$ over the entire x-y plane?

16. The surface of a right circular cone with height h and base radius a can be described by the equation $f(x,y) = h - h\sqrt{\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2}}$, where the tip of the cone lies at $(0,0,h)$ and the circular base lies in the x-y plane, centered at the origin.

    Confirm that the volume of a right circular cone with height h and base radius a is $V = \dfrac{1}{3}\pi a^2 h$ by evaluating $\displaystyle\iint_R f(x,y)\ dA$ in polar coordinates.

12.4 Center of Mass

We have used iterated integrals to find areas of plane regions and signed volumes under surfaces. A brief recap of these uses will be useful in this section as we apply iterated integrals to compute the **mass** and **center of mass** of planar regions.

To find the area of a planar region, we evaluated the double integral $\int\int_R\ dA$. That is, summing up the areas of lots of little subregions of R gave us the total area. Informally, we think of $\int\int_R\ dA$ as meaning "sum up lots of little areas over R."

To find the signed volume under a surface, we evaluated the double integral $\int\int_R f(x,y)\ dA$. Recall that the "dA" is not just a "bookend" at the end of an integral; rather, it is multiplied by $f(x,y)$. We regard $f(x,y)$ as giving a height, and dA still giving an area: $f(x,y)\ dA$ gives a volume. Thus, informally, $\int\int_R f(x,y)\ dA$ means "sum up lots of little volumes over R."

We now extend these ideas to other contexts.

Figure 12.24: Illustrating the concept of a lamina.

Mass and Weight

Consider a thin sheet of material with constant thickness and finite area. Mathematicians (and physicists and engineers) call such a sheet a **lamina**. So consider a lamina, as shown in Figure 12.24(a), with the shape of some planar region R, as shown in part (b).

We can write a simple double integral that represents the mass of the lamina: $\int\int_R\ dm$, where "dm" means "a little mass." That is, the double integral states the total mass of the lamina can be found by "summing up lots of little masses over R."

To evaluate this double integral, partition R into n subregions as we have done in the past. The ith subregion has area ΔA_i. A fundamental property of mass is that "mass=density$\times$area." If the lamina has a constant density δ, then the mass of this i^{th} subregion is $\Delta m_i = \delta \Delta A_i$. That is, we can compute a small amount of mass by multiplying a small amount of area by the density.

If density is variable, with density function $\delta = \delta(x,y)$, then we can approximate the mass of the ith subregion of R by multiplying ΔA_i by $\delta(x_i,y_i)$, where (x_i,y_i) is a point in that subregion. That is, for a small enough subregion of R, the density across that region is almost constant.

The total mass M of the lamina is approximately the sum of approximate masses of subregions:

$$M \approx \sum_{i=1}^{n} \Delta m_i = \sum_{i=1}^{n} \delta(x_i,y_i)\Delta A_i.$$

Note: *Mass* and *weight* are different measures. Since they are scalar multiples of each other, it is often easy to treat them as the same measure. In this section we effectively treat them as the same, as our technique for finding mass is the same as for finding weight. The density functions used will simply have different units.

Notes:

Taking the limit as the size of the subregions shrinks to 0 gives us the actual mass; that is, integrating $\delta(x, y)$ over R gives the mass of the lamina.

Definition 55 Mass of a Lamina with Vairable Density

Let $\delta(x, y)$ be a continuous density function of a lamina corresponding to a plane region R. The mass M of the lamina is

$$\text{mass } M = \iint_R dm = \iint_R \delta(x, y)\, dA.$$

Example 148 Finding the mass of a lamina with constant density

Find the mass of a square lamina, with side length 1, with a density of $\delta = 3\text{gm/cm}^2$.

Figure 12.25: A region R representing a lamina in Example 148.

Solution We represent the lamina with a square region in the plane as shown in Figure 12.25. As the density is constant, it does not matter where we place the square.

Following Definition 55, the mass M of the lamina is

$$M = \iint_R 3\, dA = \int_0^1 \int_0^1 3\, dx\, dy = 3 \int_0^1 \int_0^1 dx\, dy = 3\text{gm}.$$

This is all very straightforward; note that all we really did was find the area of the lamina and multiply it by the constant density of 3gm/cm^2.

Example 149 Finding the mass of a lamina with variable density

Find the mass of a square lamina, represented by the unit square with lower lefthand corner at the origin (see Figure 12.25), with variable density $\delta(x, y) = (x + y + 2)\text{gm/cm}^2$.

Solution The variable density δ, in this example, is very uniform, giving a density of 3 in the center of the square and changing linearly. A graph of $\delta(x, y)$ can be seen in Figure 12.26; notice how "same amount" of density is above $z = 3$ as below. We'll comment on the significance of this momentarily.

The mass M is found by integrating $\delta(x, y)$ over R. The order of integration

Notes:

is not important; we choose $dx\,dy$ arbitrarily. Thus:

$$\begin{aligned} M = \iint_R (x+y+2)\,dA &= \int_0^1 \int_0^1 (x+y+2)\,dx\,dy \\ &= \int_0^1 \left(\frac{1}{2}x^2 + x(y+2)\right)\Big|_0^1 dy \\ &= \int_0^1 \left(\frac{5}{2} + y\right)\,dy \\ &= \left(\frac{5}{2}y + \frac{1}{2}y^2\right)\Big|_0^1 \\ &= 3\text{gm}. \end{aligned}$$

It turns out that since since the density of the lamina is so uniformly distributed "above and below" $z = 3$ that the mass of the lamina is the same as if it had a constant density of 3. The density functions in Examples 148 and 149 are graphed in Figure 12.26, which illustrates this concept.

Figure 12.26: Graphing the density functions in Examples 148 and 149.

Example 150 Finding the weight of a lamina with variable density

Find the weight of the lamina represented by the circle with radius 2ft, centered at the origin, with density function $\delta(x,y) = (x^2+y^2+1)\text{lb/ft}^2$. Compare this to the weight of the same lamina with density $\delta(x,y) = (2\sqrt{x^2+y^2}+1)\text{lb/ft}^2$.

Solution A direct application of Definition 55 states that the weight of the lamina is $\iint_R \delta(x,y)\,dA$. Since our lamina is in the shape of a circle, it makes sense to approach the double integral using polar coordinates.

The density function $\delta(x,y) = x^2+y^2+1$ becomes $\delta(r,\theta) = (r\cos\theta)^2 + (r\sin\theta)^2 + 1 = r^2+1$. The circle is bounded by $0 \le r \le 2$ and $0 \le \theta \le 2\pi$. Thus the weight W is:

$$\begin{aligned} W &= \int_0^{2\pi} \int_0^2 (r^2+1)r\,dr\,d\theta \\ &= \int_0^{2\pi} \left(\frac{1}{4}r^4 + \frac{1}{2}r^2\right)\Big|_0^2 d\theta \\ &= \int_0^{2\pi} (6)\,d\theta \\ &= 12\pi \approx 37.70\text{lb}. \end{aligned}$$

Now compare this with the density function $\delta(x,y) = 2\sqrt{x^2+y^2}+1$. Converting this to polar coordinates gives $\delta(r,\theta) = 2\sqrt{(r\cos\theta)^2+(r\sin\theta)^2}+1 =$

Notes:

$2r+1$. Thus the weight W is:

$$\begin{aligned} W &= \int_0^{2\pi}\int_0^2 (2r+1)r\,dr\,d\theta \\ &= \int_0^{2\pi} \left(\frac{2}{3}r^3+\frac{1}{2}r^2\right)\Big|_0^2 d\theta \\ &= \int_0^{2\pi} \left(\frac{22}{3}\right)\,d\theta \\ &= \frac{44}{3}\pi \approx 46.08\text{lb}. \end{aligned}$$

One would expect different density functions to return different weights, as we have here. The density functions were chosen, though, to be similar: each gives a density of 1 at the origin and a density of 5 at the outside edge of the circle, as seen in Figure 12.27.

Figure 12.27: Graphing the density functions in Example 150. In (a) is the density function $\delta(x,y)=x^2+y^2+1$; in (b) is $\delta(x,y)=2\sqrt{x^2+y^2}+1$.

Notice how $x^2+y^2+1 \le 2\sqrt{x^2+y^2}+1$ over the circle; this results in less weight.

Plotting the density functions can be useful as our understanding of mass can be related to our understanding of "volume under a surface." We interpreted $\int\int_R f(x,y)\,dA$ as giving the volume under f over R; we can understand $\int\int_R \delta(x,y)\,dA$ in the same way. The "volume" under δ over R is actually mass;

Notes:

by compressing the "volume" under δ onto the x-y plane, we get "more mass" in some areas than others – i.e., areas of greater density.

Knowing the mass of a lamina is one of several important measures. Another is the **center of mass**, which we discuss next.

Center of Mass

Consider a disk of radius 1 with uniform density. It is common knowledge that the disk will balance on a point if the point is placed at the center of the disk. What if the disk does not have a uniform density? Through trial-and-error, we should still be able to find a spot on the disk at which the disk will balance on a point. This balance point is referred to as the **center of mass**, or **center of gravity**. It is though all the mass is "centered" there. In fact, if the disk has a mass of 3kg, the disk will behave physically as though it were a point-mass of 3kg located at its center of mass. For instance, the disk will naturally spin with an axis through its center of mass (which is why it is important to "balance" the tires of your car: if they are "out of balance", their center of mass will be outside of the axle and it will shake terribly).

We find the center of mass based on the principle of a **weighted average**. Consider a college class in which your homework average is 90%, your test average is 73%, and your final exam grade is an 85%. Experience tells us that our final grade *is not* the *average* of these three grades: that is, it is not:

$$\frac{0.9 + 0.73 + 0.85}{3} \approx 0.837 = 83.7\%.$$

That is, you are probably not pulling a B in the course. Rather, your grades are *weighted*. Let's say the homework is worth 10% of the grade, tests are 60% and the exam is 30%. Then your final grade is:

$$(0.1)(0.9) + (0.6)(0.73) + (0.3)(0.85) = 0.783 = 78.3\%.$$

Each grade is multiplied by a **weight**.

In general, given values $x_1, x_2, \ldots, x_n$ and weights $w_1, w_2, \ldots, w_n$, the weighted average of the n values is

$$\sum_{i=1}^{n} w_i x_i \Bigg/ \sum_{i=1}^{n} w_i.$$

In the grading example above, the sum of the weights 0.1, 0.6 and 0.3 is 1, so we don't see the division by the sum of weights in that instance.

How this relates to center of mass is given in the following theorem.

Notes:

Theorem 37 Center of Mass of Discrete Linear System

Let point masses $m_1, m_2, \ldots, m_n$ be distributed along the x-axis at locations $x_1, x_2, \ldots, x_n$, respectively. The center of mass $\bar{x}$ of the system is located at

$$\bar{x} = \sum_{i=1}^{n} m_i x_i \Big/ \sum_{i=1}^{n} m_i.$$

Example 151 Finding the center of mass of a discrete linear system

1. Point masses of 2gm are located at $x = -1$, $x = 2$ and $x = 3$ are connected by a thin rod of negligible weight. Find the center of mass of the system.

2. Point masses of 10gm, 2gm and 1gm are located at $x = -1$, $x = 2$ and $x = 3$, respectively, are connected by a thin rod of negligible weight. Find the center of mass of the system.

Figure 12.28: Illustrating point masses along a thin rod and the center of mass.

Solution

1. Following Theorem 37, we compute the center of mass as:

$$\bar{x} = \frac{2(-1) + 2(2) + 2(3)}{2 + 2 + 2} = \frac{4}{3} = 1.\overline{3}.$$

So the system would balance on a point placed at $x = 4/3$, as illustrated in Figure 12.28(a).

2. Again following Theorem 37, we find:

$$\bar{x} = \frac{10(-1) + 2(2) + 1(3)}{10 + 2 + 1} = \frac{-3}{13} \approx -0.23.$$

Placing a large weight at the left hand side of the system moves the center of mass left, as shown in Figure 12.28(b).

In a discrete system (i.e., mass is located at individual points, not along a continuum) we find the center of mass by dividing the mass into a **moment** of the system. In general, a moment is a weighted measure of distance from a particular point or line. In the case described by Theorem 37, we are finding a weighted measure of distances from the y-axis, so we refer to this as **the moment about**

Notes:

the y-axis, represented by M_y. Letting M be the total mass of the system, we have $\bar{x} = M_y/M$.

We can extend the concept of the center of mass of discrete points along a line to the center of mass of discrete points in the plane rather easily. To do so, we define some terms then give a theorem.

Definition 56 Moments about the x- and y- Axes.

Let point masses m_1, $m_2, \ldots, m_n$ be located at points (x_1, y_1), $(x_2, y_2) \ldots, (x_n, y_n)$, respectively, in the x-y plane.

1. The **moment about the y-axis**, M_y, is $M_y = \displaystyle\sum_{i=1}^{n} m_i x_i$.
2. The **moment about the x-axis**, M_x, is $M_x = \displaystyle\sum_{i=1}^{n} m_i y_i$.

One can think that these definitions are "backwards" as M_y sums up "x" distances. But remember, "x" distances are measurements of distance from the y-axis, hence defining the moment about the y-axis.

We now define the center of mass of discrete points in the plane.

Theorem 38 Center of Mass of Discrete Planar System

Let point masses m_1, $m_2, \ldots, m_n$ be located at points (x_1, y_1), $(x_2, y_2) \ldots, (x_n, y_n)$, respectively, in the x-y plane, and let $M = \displaystyle\sum_{i=1}^{n} m_i$.

The center of mass of the system is at $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{M_y}{M} \quad \text{and} \quad \bar{y} = \frac{M_x}{M}.$$

Example 152 Finding the center of mass of a discrete planar system

Let point masses of 1kg, 2kg and 5kg be located at points $(2, 0)$, $(1, 1)$ and $(3, 1)$, respectively, and are connected by thin rods of negligible weight. Find the center of mass of the system.

Solution We follow Theorem 38 and Definition 56 to find M, M_x and M_y:

Notes:

$M = 1 + 2 + 5 = 8$kg.

$$\begin{aligned} M_x &= \sum_{i=1}^{n} m_i y_i \\ &= 1(0) + 2(1) + 5(1) \\ &= 7. \end{aligned} \qquad \begin{aligned} M_y &= \sum_{i=1}^{n} m_i x_i \\ &= 1(2) + 2(1) + 5(3) \\ &= 19. \end{aligned}$$

Thus the center of mass is $(\overline{x}, \overline{y}) = \left(\dfrac{M_y}{M}, \dfrac{M_x}{M}\right) = \left(\dfrac{19}{8}, \dfrac{7}{8}\right) = (2.375, 0.875)$, illustrated in Figure 12.29.

We finally arrive at our true goal of this section: finding the center of mass of a lamina with variable density. While the above measurement of center of mass is interesting, it does not directly answer more realistic situations where we need to find the center of mass of a contiguous region. However, understanding the discrete case allows us to approximate the center of mass of a planar lamina; using calculus, we can refine the approximation to an exact value.

We begin by representing a planar lamina with a region R in the x-y plane with density function $\delta(x, y)$. Partition R into n subdivisions, each with area ΔA_i. As done before, we can approximate the mass of the ith subregion with $\delta(x_i, y_i)\Delta A_i$, where (x_i, y_i) is a point inside the ith subregion. We can approximate the moment of this subregion about the y-axis with $x_i\delta(x_i, y_i)\Delta A_i$ – that is, by multiplying the approximate mass of the region by its approximate distance from the y-axis. Similarly, we can approximate the moment about the x-axis with $y_i\delta(x_i, y_i)\Delta A_i$. By summing over all subregions, we have:

$$\text{mass: } M \approx \sum_{i=1}^{n} \delta(x_i, y_i)\Delta A_i \quad \text{(as seen before)}$$

$$\text{moment about the } x\text{-axis: } M_x \approx \sum_{i=1}^{n} y_i\delta(x_i, y_i)\Delta A_i$$

$$\text{moment about the } y\text{-axis: } M_y \approx \sum_{i=1}^{n} x_i\delta(x_i, y_i)\Delta A_i$$

Figure 12.29: Illustrating the center of mass of a discrete planar system in Example 152.

By taking limits, where size of each subregion shrinks to 0 in both the x and y directions, we arrive at the double integrals given in the following theorem.

Notes:

Theorem 39 Center of Mass of a Planar Lamina, Moments

Let a planar lamina be represented by a region R in the x-y plane with density function $\delta(x, y)$.

1. mass: $M = \iint_R \delta(x, y)\, dA$
2. moment about the x-axis: $M_x = \iint_R y\delta(x, y)\, dA$
3. moment about the y-axis: $M_y = \iint_R x\delta(x, y)\, dA$
4. The center of mass of the lamina is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right).$$

We start our practice of finding centers of mass by revisiting some of the lamina used previously in this section when finding mass. We will just set up the integrals needed to compute M, M_x and M_y and leave the details of the integration to the reader.

Example 153 Finding the center of mass of a lamina

Find the center mass of a square lamina, with side length 1, with a density of $\delta = 3\text{gm/cm}^2$. (Note: this is the lamina from Example 148.)

Figure 12.30: A region R representing a lamina in Example 148.

Solution We represent the lamina with a square region in the plane as shown in Figure 12.30 as done previously.

Following Theorem 39, we find M, M_x and M_y:

$$M = \iint_R 3\, dA = \int_0^1 \int_0^1 3\, dx\, dy = 3\text{gm}.$$
$$M_x = \iint_R 3y\, dA = \int_0^1 \int_0^1 3y\, dx\, dy = 3/2 = 1.5.$$
$$M_y = \iint_R 3x\, dA = \int_0^1 \int_0^1 3x\, dx\, dy = 3/2 = 1.5.$$

Thus the center of mass is $(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = (1.5/3, 1.5/3) = (0.5, 0.5)$. This is what we should have expected: the center of mass of a square with constant density is the center of the square.

Notes:

Example 154 Finding the center of mass of a lamina

Find the center of mass of a square lamina, represented by the unit square with lower lefthand corner at the origin (see Figure 12.30), with variable density $\delta(x, y) = (x + y + 2)$gm/cm^2. (Note: this is the lamina from Example 149.)

Solution We follow Theorem 39, to find M, M_x and M_y:

$$M = \iint_R (x + y + 2)\, dA = \int_0^1 \int_0^1 (x + y + 2)\, dx\, dy = 3\text{gm}.$$
$$M_x = \iint_R y(x + y + 2)\, dA = \int_0^1 \int_0^1 y(x + y + 2)\, dx\, dy = \frac{19}{12}.$$
$$M_y = \iint_R x(x + y + 2)\, dA = \int_0^1 \int_0^1 x(x + y + 2)\, dx\, dy = \frac{19}{12}.$$

Thus the center of mass is $(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{19}{36}, \frac{19}{36}\right) \approx (0.528, 0.528)$. While the mass of this lamina is the same as the lamina in the previous example, the greater density found with greater x and y values pulls the center of mass from the center slightly towards the upper righthand corner.

Example 155 Finding the center of mass of a lamina

Find the center of mass of the lamina represented by the circle with radius 2ft, centered at the origin, with density function $\delta(x, y) = (x^2 + y^2 + 1)$lb/ft^2. (Note: this is one of the lamina used in Example 150.)

Solution As done in Example 150, it is best to describe R using polar coordinates. Thus when we compute M_y, we will integrate not $x\delta(x, y) = x(x^2 + y^2 + 1)$, but rather $\big(r\cos\theta\big)\delta(r\cos\theta, r\sin\theta) = \big(r\cos\theta\big)\big(r^2 + 1\big)$. We compute M, M_x and M_y:

$$M = \int_0^{2\pi} \int_0^2 (r^2 + 1) r\, dr\, d\theta = 12\pi \approx 37.7\text{lb}.$$
$$M_x = \int_0^{2\pi} \int_0^2 (r\sin\theta)(r^2 + 1) r\, dr\, d\theta = 0.$$
$$M_y = \int_0^{2\pi} \int_0^2 (r\cos\theta)(r^2 + 1) r\, dr\, d\theta = 0.$$

Since R and the density of R are both symmetric about the x and y axes, it should come as no big surprise that the moments about each axis is 0. Thus the center

Notes:

of mass is $(\bar{x}, \bar{y}) = (0, 0)$.

Example 156 Finding the center of mass of a lamina

Find the center of mass of the lamina represented by the region R shown in Figure 12.31, half an annulus with outer radius 6 and inner radius 5, with constant density 2lb/ft^2.

Figure 12.31: Illustrating the region R in Example 156.

Solution Once again it will be useful to represent R in polar coordinates. Using the description of R and/or the illustration, we see that R is bounded by $5 \leq r \leq 6$ and $0 \leq \theta \leq \pi$. As the lamina is symmetric about the y-axis, we should expect $M_y = 0$. We compute M, M_x and M_y:

$$M = \int_0^\pi \int_5^6 (2) r \, dr \, d\theta = 11\pi \text{lb}.$$

$$M_x = \int_0^\pi \int_5^6 (r \sin\theta)(2) r \, dr \, d\theta = \frac{364}{3} \approx 121.33.$$

$$M_y = \int_0^\pi \int_5^6 (r \cos\theta)(2) r \, dr \, d\theta = 0.$$

Thus the center of mass is $(\bar{x}, \bar{y}) = \left(0, \frac{364}{33\pi}\right) \approx (0, 3.51)$. The center of mass is indicated in Figure 12.31; note how it lies outside of R!

This section has shown us another use for iterated integrals beyond finding area or signed volume under the curve. While there are many uses for iterated integrals, we give one more application in the following section: computing surface area.

Notes:

Exercises 12.4

Terms and Concepts

1. Why is it easy to use "mass" and "weight" interchangeably, even though they are different measures?

2. Given a point (x, y), the value of x is a measure of distance from the _____-axis.

3. We can think of $\iint_R dm$ as meaning "sum up lots of _____"

4. What is a "discrete planar system?"

5. Why does M_x use $\iint_R y\delta(x, y)\, dA$ instead of $\iint_R x\delta(x, y)\, dA$; that is, why do we use "y" and not "x"?

6. Describe a situation where the center of mass of a lamina does not lie within the region of the lamina itself.

Problems

In Exercises 7 – 10, point masses are given along a line or in the plane. Find the center of mass $\bar{x}$ or $(\bar{x}, \bar{y})$, as appropriate. (All masses are in grams and distances are in cm.)

7. $m_1 = 4$ at $x = 1$; $m_2 = 3$ at $x = 3$; $m_3 = 5$ at $x = 10$

8. $m_1 = 2$ at $x = -3$; $m_2 = 2$ at $x = -1$; $m_3 = 3$ at $x = 0$; $m_4 = 3$ at $x = 7$

9. $m_1 = 2$ at $(-2, -2)$; $m_2 = 2$ at $(2, -2)$; $m_3 = 20$ at $(0, 4)$

10. $m_1 = 1$ at $(-1, -1)$; $m_2 = 2$ at $(-1, 1)$; $m_3 = 2$ at $(1, 1)$; $m_4 = 1$ at $(1, -1)$

In Exercises 11 – 18, find the mass/weight of the lamina described by the region R in the plane and its density function $\delta(x, y)$.

11. R is the rectangle with corners $(1, -3)$, $(1, 2)$, $(7, 2)$ and $(7, -3)$; $\delta(x, y) = 5\text{gm/cm}^2$

12. R is the rectangle with corners $(1, -3)$, $(1, 2)$, $(7, 2)$ and $(7, -3)$; $\delta(x, y) = (x + y^2)\text{gm/cm}^2$

13. R is the triangle with corners $(-1, 0)$, $(1, 0)$, and $(0, 1)$; $\delta(x, y) = 2\text{lb/in}^2$

14. R is the triangle with corners $(0, 0)$, $(1, 0)$, and $(0, 1)$; $\delta(x, y) = (x^2 + y^2 + 1)\text{lb/in}^2$

15. R is the circle centered at the origin with radius 2; $\delta(x, y) = (x + y + 4)\text{kg/m}^2$

16. R is the circle sector bounded by $x^2 + y^2 = 25$ in the first quadrant; $\delta(x, y) = (\sqrt{x^2 + y^2} + 1)\text{kg/m}^2$

17. R is the annulus in the first and second quadrants bounded by $x^2 + y^2 = 9$ and $x^2 + y^2 = 36$; $\delta(x, y) = 4\text{lb/ft}^2$

18. R is the annulus in the first and second quadrants bounded by $x^2 + y^2 = 9$ and $x^2 + y^2 = 36$; $\delta(x, y) = \sqrt{x^2 + y^2}\text{lb/ft}^2$

In Exercises 19 – 26, find the center of mass of the lamina described by the region R in the plane and its density function $\delta(x, y)$.
Note: these are the same lamina as in Exercises 11 – 18.

19. R is the rectangle with corners $(1, -3)$, $(1, 2)$, $(7, 2)$ and $(7, -3)$; $\delta(x, y) = 5\text{gm/cm}^2$

20. R is the rectangle with corners $(1, -3)$, $(1, 2)$, $(7, 2)$ and $(7, -3)$; $\delta(x, y) = (x + y^2)\text{gm/cm}^2$

21. R is the triangle with corners $(-1, 0)$, $(1, 0)$, and $(0, 1)$; $\delta(x, y) = 2\text{lb/in}^2$

22. R is the triangle with corners $(0, 0)$, $(1, 0)$, and $(0, 1)$; $\delta(x, y) = (x^2 + y^2 + 1)\text{lb/in}^2$

23. R is the circle centered at the origin with radius 2; $\delta(x, y) = (x + y + 4)\text{kg/m}^2$

24. R is the circle sector bounded by $x^2 + y^2 = 25$ in the first quadrant; $\delta(x, y) = (\sqrt{x^2 + y^2} + 1)\text{kg/m}^2$

25. R is the annulus in the first and second quadrants bounded by $x^2 + y^2 = 9$ and $x^2 + y^2 = 36$; $\delta(x, y) = 4\text{lb/ft}^2$

26. R is the annulus in the first and second quadrants bounded by $x^2 + y^2 = 9$ and $x^2 + y^2 = 36$; $\delta(x, y) = \sqrt{x^2 + y^2}\text{lb/ft}^2$

The *moment of inertia* I is a measure of the tendency of a lamina to resist rotating about an axis or continue to rotate about an axis. I_x is the moment of inertia about the x-axis, I_x is the moment of inertia about the x-axis, and I_O is the moment of inertia about the origin. These are computed as follows:

- $I_x = \iint_R y^2\, dm$
- $I_y = \iint_R x^2\, dm$
- $I_O = \iint_R \left(x^2 + y^2\right) dm$

In Exercises 27 – 30, a lamina corresponding to a planar region R is given with a mass of 16 units. For each, compute I_x, I_y and I_O.

27. R is the 4×4 square with corners at $(-2, -2)$ and $(2, 2)$ with density $\delta(x, y) = 1$.

28. R is the 8×2 rectangle with corners at $(-4, -1)$ and $(4, 1)$ with density $\delta(x, y) = 1$.

29. R is the 4×2 rectangle with corners at $(-2, -1)$ and $(2, 1)$ with density $\delta(x, y) = 2$.

30. R is the circle with radius 2 centered at the origin with density $\delta(x, y) = 4/\pi$.

12.5 Surface Area

In Section **??** we used definite integrals to compute the arc length of plane curves of the form $y = f(x)$. We later extended these ideas to compute the arc length of plane curves defined by parametric or polar equations.

The natural extension of the concept of "arc length over an interval" to surfaces is "surface area over a region."

Consider the surface $z = f(x, y)$ over a region R in the x-y plane, shown in Figure 12.32(a). Because of the domed shape of the surface, the surface area will be greater than that of the area of the region R. We can find this area using the same basic technique we have used over and over: we'll make an approximation, then using limits, we'll refine the approximation to the exact value.

As done to find the volume under a surface or the mass of a lamina, we subdivide R into n subregions. Here we subdivide R into rectangles, as shown in the figure. One such subregion is outlined in the figure, where the rectangle has dimensions Δx_i and Δy_i, along with its corresponding region on the surface.

In part (b) of the figure, we zoom in on this portion of the surface. When Δx_i and Δy_i are small, the function is approximated well by the tangent plane at any point (x_i, y_i) in this subregion, which is graphed in part (b). In fact, the tangent plane approximates the function so well that in this figure, it is virtually indistinguishable from the surface itself! Therefore we can approximate the surface area S_i of this region of the surface with the area T_i of the corresponding portion of the tangent plane.

This portion of the tangent plane is a parallelogram, defined by sides $\vec{u}$ and $\vec{v}$, as shown. One of the applications of the cross product from Section 9.4 is that the area of this parallelogram is $||\,\vec{u} \times \vec{v}\,||$. Once we can determine $\vec{u}$ and $\vec{v}$, we can determine the area.

$\vec{u}$ is tangent to the surface in the direction of x, therefore, from Section 11.7, $\vec{u}$ is parallel to $\langle 1, 0, f_x(x_i, y_i)\rangle$. The x-displacement of $\vec{u}$ is Δx_i, so we know that $\vec{u} = \Delta x_i \langle 1, 0, f_x(x_i, y_i)\rangle$. Similar logic shows that $\vec{v} = \Delta y_i \langle 0, 1, f_y(x_i, y_i)\rangle$. Thus:

$$\begin{aligned}
\text{surface area } S_i &\approx \text{area of } T_i \\
&= ||\,\vec{u} \times \vec{v}\,|| \\
&= \big|\big|\Delta x_i \langle 1, 0, f_x(x_i, y_i)\rangle \times \Delta y_i \langle 0, 1, f_y(x_i, y_i)\rangle\big|\big| \\
&= \sqrt{1 + f_x(x_i, y_i)^2 + f_y(x_i, y_i)^2}\,\Delta x_i \Delta y_i.
\end{aligned}$$

Note that $\Delta x_i \Delta y_i = \Delta A_i$, the area of the ith subregion.

Summing up all n of the approximations to the surface area gives

$$\text{surface area over } R \approx \sum_{i=1}^{n} \sqrt{1 + f_x(x_i, y_i)^2 + f_y(x_i, y_i)^2}\,\Delta A_i.$$

Figure 12.32: Developing a method of computing surface area.

Notes:

Once again take a limit as all of the Δx_i and Δy_i shrink to 0; this leads to a double integral.

Note: as done before, we think of "$\iint_R dS$" as meaning "sum up lots of little surface areas over R."

The concept of surface area is *defined* here, for while we already have a notion of the area of a region in the *plane*, we did not yet have a solid grasp of what "the area of a surface in *space*" means.

Definition 57 Surface Area

Let $z = f(x, y)$ where f_x and f_y are continuous over a closed, bounded region R. The surface area S over R is

$$\begin{aligned} S &= \iint_R dS \\ &= \iint_R \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\, dA. \end{aligned}$$

We test this definition by using it to compute surface areas of known surfaces. We start with a triangle.

Example 157 Finding the surface area of a plane over a triangle

Let $f(x, y) = 4 - x - 2y$, and let R be the region in the plane bounded by $x = 0$, $y = 0$ and $y = 2 - x/2$, as shown in Figure 12.33. Find the surface area of f over R.

Figure 12.33: Finding the area of a triangle in space in Example 157.

Solution We follow Definition 57. We start by noting that $f_x(x, y) = -1$ and $f_y(x, y) = -2$. To define R, we use bounds $0 \le y \le 2 - x/2$ and $0 \le x \le 4$. Therefore

$$\begin{aligned} S &= \iint_R dS \\ &= \int_0^4 \int_0^{2-x/2} \sqrt{1 + (-1)^2 + (-2)^2}\, dy\, dx \\ &= \int_0^4 \sqrt{6}\left(2 - \frac{x}{2}\right) dx \\ &= 4\sqrt{6}. \end{aligned}$$

Because the surface is a triangle, we can figure out the area using geometry. Considering the base of the triangle to be the side in the x-y plane, we find the length of the base to be $\sqrt{20}$. We can find the height using our knowledge of vectors: let $\vec{u}$ be the side in the x-z plane and let $\vec{v}$ be the side in the x-y plane. The height is then $\| \vec{u} - \text{proj}_{\vec{v}}\, \vec{u} \| = 4\sqrt{6/5}$. Geometry states that the area is thus

$$\frac{1}{2} \cdot 4\sqrt{6/5} \cdot \sqrt{20} = 4\sqrt{6}.$$

We affirm the validity of our formula.

Notes:

It is "common knowledge" that the surface area of a sphere of radius r is $4\pi r^2$. We confirm this in the following example, which involves using our formula with polar coordinates.

Example 158 The surface area of a sphere.
Find the surface area of the sphere with radius a centered at the origin, whose top hemisphere has equation $f(x,y) = \sqrt{a^2 - x^2 - y^2}$.

Solution We start by computing partial derivatives and find

$$f_x(x,y) = \frac{-x}{\sqrt{a^2 - x^2 - y^2}} \quad \text{and} \quad f_y(x,y) = \frac{-y}{\sqrt{a^2 - x^2 - y^2}}.$$

As our function f only defines the top upper hemisphere of the sphere, we double our surface area result to get the total area:

$$\begin{aligned} S &= 2\iint_R \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\, dA \\ &= 2\iint_R \sqrt{1 + \frac{x^2+y^2}{a^2 - x^2 - y^2}}\, dA. \end{aligned}$$

The region R that we are integrating over is the circle, centered at the origin, with radius a: $x^2 + y^2 = a^2$. Because of this region, we are likely to have greater success with our integration by converting to polar coordinates. Using the substitutions $x = r\cos\theta$, $y = r\sin\theta$, $dA = r\,dr\,d\theta$ and bounds $0 \le \theta \le 2\pi$ and $0 \le r \le a$, we have:

$$\begin{aligned} S &= 2\int_0^{2\pi}\int_0^a \sqrt{1 + \frac{r^2\cos^2\theta + r^2\sin^2\theta}{a^2 - r^2\cos^2\theta - r^2\sin^2\theta}}\, r\,dr\,d\theta \\ &= 2\int_0^{2\pi}\int_0^a r\sqrt{1 + \frac{r^2}{a^2 - r^2}}\, dr\,d\theta \\ &= 2\int_0^{2\pi}\int_0^a r\sqrt{\frac{a^2}{a^2 - r^2}}\, dr\,d\theta. \end{aligned} \tag{12.1}$$

Apply substitution $u = a^2 - r^2$ and integrate the inner integral, giving

$$\begin{aligned} &= 2\int_0^{2\pi} a^2\, d\theta \\ &= 4\pi a^2. \end{aligned}$$

Our work confirms our previous formula.

Note: The inner integral in Equation (12.1) is an improper integral, as the integrand of $\int_0^a r\sqrt{\frac{a^2}{a^2 - r^2}}\, dr$ is not defined at $r = a$. To properly evaluate this integral, one must use the techniques of Section **??**.

The reason this need arises is that the function $f(x,y) = \sqrt{a^2 - x^2 - y^2}$ fails the requirements of Definition 57, as f_x and f_y are not continuous on the boundary of the circle $x^2 + y^2 = a^2$.

The computation of the surface area is still valid. The definition makes stronger requirements than necessary in part to avoid the use of improper integration, as when f_x and/or f_y are not continuous, the resulting improper integral may not converge. Since the improper integral does converge in this example, the surface area is accurately computed.

Notes:

Figure 12.34: Finding the surface area of a cone in Example 159.

Example 159 Finding the surface area of a cone

The general formula for a right cone with height h and base radius a is

$$f(x,y) = h - \frac{h}{a}\sqrt{x^2+y^2},$$

shown in Figure 12.34. Find the surface area of this cone.

Solution We begin by computing partial derivatives.

$$f_x(x,y) = -\frac{xh}{a\sqrt{x^2+y^2}} \quad \text{and} \quad -\frac{yh}{a\sqrt{x^2+y^2}}.$$

Since we are integrating over the circle $x^2+y^2=a^2$, we again use polar coordinates. Using the standard substitutions, our integrand becomes

$$\sqrt{1+\left(\frac{hr\cos\theta}{a\sqrt{r^2}}\right)^2+\left(\frac{hr\sin\theta}{a\sqrt{r^2}}\right)^2}.$$

This may look intimidating at first, but there are lots of simple simplifications to be done. It amazingly reduces to just

$$\sqrt{1+\frac{h^2}{a^2}} = \frac{1}{a}\sqrt{a^2+h^2}.$$

Our polar bounds are $0 \le \theta \le 2\pi$ and $0 \le r \le a$. Thus

$$\begin{aligned} S &= \int_0^{2\pi}\int_0^a r\frac{1}{a}\sqrt{a^2+h^2}\,dr\,d\theta \\ &= \int_0^{2\pi}\left(\frac{1}{2}r^2\frac{1}{a}\sqrt{a^2+h^2}\right)\Bigg|_0^a d\theta \\ &= \int_0^{2\pi}\frac{1}{2}a\sqrt{a^2+h^2}\,d\theta \\ &= \pi a\sqrt{a^2+h^2}. \end{aligned}$$

This matches the formula found in the back of this text.

Note: Note that once again f_x and f_y are not continuous on the domain of f, as both are undefined at $(0,0)$. (A similar problem occurred in the previous example.) Once again the resulting improper integral converges and the computation of the surface area is valid.

Figure 12.35: Graphing the surface in Example 160.

Example 160 Finding surface area over a region

Find the area of the surface $f(x,y) = x^2 - 3y + 3$ over the region R bounded by $-x \le y \le x$, $0 \le x \le 4$, as pictured in Figure 12.35.

Solution It is straightforward to compute $f_x(x,y) = 2x$ and $f_y(x,y) = -3$. Thus the surface area is described by the double integral

$$\iint_R \sqrt{1+(2x)^2+(-3)^2}\,dA = \iint_R \sqrt{10+4x^2}\,dA.$$

Notes:

As with integrals describing arc length, double integrals describing surface area are in general hard to evaluate directly because of the square–root. This particular integral can be easily evaluated, though, with judicious choice of our order of integration.

Integrating with order $dx\,dy$ requires us to evaluate $\int \sqrt{10+4x^2}\,dx$. This can be done, though it involves Integration By Parts and $\sinh^{-1} x$. Integrating with order $dy\,dx$ has as its first integral $\int \sqrt{10+4x^2}\,dy$, which is easy to evaluate: it is simply $y\sqrt{10+4x^2}+C$. So we proceed with the order $dy\,dx$; the bounds are already given in the statement of the problem.

$$\begin{aligned}\iint_R \sqrt{10+4x^2}\,dA &= \int_0^4 \int_{-x}^{x} \sqrt{10+4x^2}\,dy\,dx \\ &= \int_0^4 \left(y\sqrt{10+4x^2}\right)\Big|_{-x}^{x} dx \\ &= \int_0^4 \left(2x\sqrt{10+4x^2}\right) dx.\end{aligned}$$

Apply substitution with $u = 10+4x^2$:

$$\begin{aligned}&= \left(\frac{1}{6}\left(10+4x^2\right)^{3/2}\right)\Bigg|_0^4 \\ &= \frac{1}{3}\left(37\sqrt{74}-5\sqrt{10}\right) \approx 100.825\text{u}^2.\end{aligned}$$

So while the region R over which we integrate has an area of 16u^2, the surface has a much greater area as its z-values change dramatically over R.

In practice, technology helps greatly in the evaluation of such integrals. High powered computer algebra systems can compute integrals that are difficult, or at least time consuming, by hand, and can at the least produce very accurate approximations with numerical methods. In general, just knowing *how* to set up the proper integrals brings one very close to being able to compute the needed value. Most of the work is actually done in just describing the region R in terms of polar or rectangular coordinates. Once this is done, technology can usually provide a good answer.

We have learned how to integrate integrals; that is, we have learned to evaluate double integrals. In the next section, we learn how to integrate double integrals – that is, we learn to evaluate *triple integrals*, along with learning some uses for this operation.

Notes:

Exercises 12.5

Terms and Concepts

1. "Surface area" is analogous to what previously studied concept?

2. To approximate the area of a small portion of a surface, we computed the area of its _____ plane.

3. We interpret $\iint_R dS$ as "sum up lots of little _____ _____."

4. Why is it important to know how to set up a double integral to compute surface area, even if the resulting integral is hard to evaluate?

5. Why do $z = f(x, y)$ and $z = g(x, y) = f(x, y) + h$, for some real number h, have the same surface area over a region R?

6. Let $z = f(x, y)$ and $z = g(x, y) = 2f(x, y)$. Why is the surface area of g over a region R not twice the surface area of f over R?

Problems

In Exercises 7 – 10, *set up* the iterated integral that computes the surface area of the given surface over the region R.

7. $f(x, y) = \sin x \cos y$; R is the rectangle with bounds $0 \le x \le 2\pi$, $0 \le y \le 2\pi$.

8. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$; R is the circle $x^2 + y^2 = 9$.

9. $f(x, y) = x^2 - y^2$; R is the rectangle with opposite corners $(-1, -1)$ and $(1, 1)$.

10. $f(x, y) = \dfrac{1}{e^{x^2} + 1}$; R is the rectangle bounded by $-5 \le x \le 5$ and $0 \le y \le 1$.

In Exercises 11 – 19, find the area of the given surface over the region R.

11. $f(x, y) = 3x - 7y + 2$; R is the rectangle with opposite corners $(-1, 0)$ and $(1, 3)$.

12. $f(x, y) = 2x + 2y + 2$; R is the triangle with corners $(0, 0)$, $(1, 0)$ and $(0, 1)$.

13. $f(x, y) = x^2 + y^2 + 10$; R is the circle $x^2 + y^2 = 16$.

14. $f(x, y) = -2x + 4y^2 + 7$ over R, the triangle bounded by $y = -x$, $y = x$, $0 \leq y \leq 1$.

15. $f(x, y) = x^2 + y$ over R, the triangle bounded by $y = 2x$, $y = 0$ and $x = 2$.

16. $f(x, y) = \frac{2}{3}x^{3/2} + 2y^{3/2}$ over R, the rectangle with opposite corners $(0, 0)$ and $(1, 1)$.

17. $f(x, y) = 10 - 2\sqrt{x^2 + y^2}$ over R, the circle $x^2 + y^2 = 25$. (This is the cone with height 10 and base radius 5; be sure to compare you result with the known formula.)

18. Find the surface area of the sphere with radius 5 by doubling the surface area of $f(x, y) = \sqrt{25 - x^2 - y^2}$ over R, the circle $x^2 + y^2 = 25$. (Be sure to compare you result with the known formula.)

19. Find the surface area of the ellipse formed by restricting the plane $f(x, y) = cx + dy + h$ to the region R, the circle $x^2 + y^2 = 1$, where c, d and h are some constants. Your answer should be given in terms of c and d; why does the value of h not matter?

12.6 Volume Between Surfaces and Triple Integration

We learned in Section 12.2 how to compute the signed volume V under a surface $z = f(x, y)$ over a region R: $V = \iint_R f(x, y)\, dA$. It follows naturally that if $f(x, y) \geq g(x, y)$ on R, then the **volume between** $f(x, y)$ **and** $g(x, y)$ **on** R is

$$V = \iint_R f(x, y)\, dA - \iint_R g(x, y)\, dA = \iint_R \big(f(x, y) - g(x, y)\big)\, dA.$$

Theorem 40 Volume Between Surfaces

Let f and g be continuous functions on a closed, bounded region R, where $f(x, y) \geq g(x, y)$ for all (x, y) in R. The volume V between f and g over R is

$$V = \iint_R \big(f(x, y) - g(x, y)\big)\, dA.$$

Figure 12.36: Finding the volume between the planes given in Example 161.

Example 161 Finding volume between surfaces

Find the volume of the space region bounded by the planes $z = 3x + y - 4$ and $z = 8 - 3x - 2y$ in the 1st octant. In Figure 12.36(a) the planes are drawn; in (b), only the defined region is given.

Solution We need to determine the region R over which we will integrate. To do so, we need to determine where the planes intersect. They have common z-values when $3x + y - 4 = 8 - 3x - 2y$. Applying a little algebra, we have:

$$\begin{aligned} 3x + y - 4 &= 8 - 3x - 2y \\ 6x + 3y &= 12 \\ 2x + y &= 4 \end{aligned}$$

The planes intersect along the line $2x + y = 4$. Therefore the region R is bounded by $x = 0$, $y = 0$, and $y = 4 - 2x$; we can convert these bounds to integration bounds of $0 \leq x \leq 2$, $0 \leq y \leq 4 - 2x$. Thus

$$\begin{aligned} V &= \iint_R \big(8 - 3x - 2y - (3x + y - 4)\big)\, dA \\ &= \int_0^2 \int_0^{4-2x} \big(12 - 6x - 3y\big)\, dy\, dx \\ &= 16\text{u}^3. \end{aligned}$$

The volume between the surfaces is 16 cubic units.

Notes:

In the preceding example, we found the volume by evaluating the integral

$$\int_0^2 \int_0^{4-2x} \big(8 - 3x - 2y - (3x + y - 4)\big)\, dy\, dx.$$

Note how we can rewrite the integrand as an integral, much as we did in Section 12.1:

$$8 - 3x - 2y - (3x + y - 4) = \int_{3x+y-4}^{8-3x-2y} dz.$$

Thus we can rewrite the double integral that finds volume as

$$\int_0^2 \int_0^{4-2x} \big(8-3x-2y-(3x+y-4)\big)\, dy\, dx = \int_0^2 \int_0^{4-2x} \left(\int_{3x+y-4}^{8-3x-2y} dz \right) dy\, dx.$$

This no longer looks like a "double integral," but more like a "triple integral." Just as our first introduction to double integrals was in the context of finding the area of a plane region, our introduction into triple integrals will be in the context of finding the volume of a space region.

To formally find the volume of a closed, bounded region D in space, such as the one shown in Figure 12.37(a), we start with an approximation. Break D into n rectangular solids; the solids near the boundary of D may possibly not include portions of D and/or include extra space. In Figure 12.37(b), we zoom in on a portion of the boundary of D to show a rectangular solid that contains space not in D; as this is an approximation of the volume, this is acceptable and this error will be reduced as we shrink the size of our solids.

The volume ΔV_i of the ith solid D_i is $\Delta V_i = \Delta x_i \Delta y_i \Delta z_i$, where Δx_i, Δy_i and Δz_i give the dimensions of the rectangular solid in the x, y and z directions, respectively. By summing up the volumes of all n solids, we get an approximation of the volume V of D:

$$V \approx \sum_{i=1}^{n} \Delta V_i = \sum_{i=1}^{n} \Delta x_i \Delta y_i \Delta z_i.$$

Let $||\Delta D||$ represent the length of the longest diagonal of rectangular solids in the subdivision of D. As $||\Delta D|| \to 0$, the volume of each solid goes to 0, as do each of Δx_i, Δy_i and Δz_i, for all i. Our calculus experience tells us that taking a limit as $||\Delta D|| \to 0$ turns our approximation of V into an exact calculation of V. Before we state this result in a theorem, we use a definition to define some terms.

Figure 12.37: Approximating the volume of a region D in space.

Notes:

Definition 58 Triple Integrals, Iterated Integration (Part I)

Let D be a closed, bounded region in space. Let a and b be real numbers, let $g_1(x)$ and $g_2(x)$ be continuous functions of x, and let $f_1(x,y)$ and $f_2(x,y)$ be continuous functions of x and y.

1. The volume V of D is denoted by a **triple integral**,

$$V = \iiint_D dV.$$

2. The iterated integral $\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} dz\, dy\, dx$ is evaluated as

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} dz\, dy\, dx = \int_a^b \int_{g_1(x)}^{g_2(x)} \left(\int_{f_1(x,y)}^{f_2(x,y)} dz \right) dy\, dx.$$

Evaluating the above iterated integral is **triple integration.**

Our informal understanding of the notation $\iiint_D dV$ is "sum up lots of little volumes over D," analogous to our understanding of $\iint_R dA$ and $\iint_R dm$.

We now state the major theorem of this section.

Theorem 41 Triple Integration (Part I)

Let D be a closed, bounded region in space and let ΔD be any subdivision of D into n rectangular solids, where the ith subregion D_i has dimensions $\Delta x_i \times \Delta y_i \times \Delta z_i$ and volume ΔV_i.

1. The volume V of D is

$$V = \iiint_D dV = \lim_{||\Delta D|| \to 0} \sum_{i=1}^{n} \Delta V_i = \lim_{||\Delta D|| \to 0} \sum_{i=1}^{n} \Delta x_i \Delta y_i \Delta z_i.$$

2. If D is defined as the region bounded by the planes $x = a$ and $x = b$, the cylinders $y = g_(x)$ and $y = g_2(x)$, and the surfaces $z = f_1(x,y)$ and $z = f_2(x,y)$, where $a < b$, $g_1(x) \leq g_2(x)$ and $f_1(x,y) \leq f_2(x,y)$ on D, then

$$\iiint_D dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} dz\, dy\, dx.$$

3. V can be determined using iterated integration with other orders of integration (there are 6 total), as long as D is defined by the region enclosed by a pair of planes, a pair of cylinders, and a pair of surfaces.

Notes:

We evaluated the area of a plane region R by iterated integration, where the bounds were "from curve to curve, then from point to point." Theorem 41 allows us to find the volume of a space region with an iterated integral with bounds "from surface to surface, then from curve to curve, then from point to point." In the iterated integral

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} dz\, dy\, dx,$$

the bounds $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$ define a region R in the x-y plane over which the region D exists in space. However, these bounds are also defining surfaces in space; $x = a$ is a plane and $y = g_1(x)$ is a cylinder. The combination of these 6 surfaces enclose, and define, D.

Examples will help us understand triple integration, including integrating with various orders of integration.

Example 162 Finding the volume of a space region with triple integration
Find the volume of the space region in the 1st octant bounded by the plane $z = 2 - y/3 - 2x/3$, shown in Figure 12.38(a), using the order of integration $dz\, dy\, dx$. Set up the triple integrals that give the volume in the other 5 orders of integration.

Solution Starting with the order of integration $dz\, dy\, dx$, we need to first find bounds on z. The region D is bounded below by the plane $z = 0$ (because we are restricted to the first octant) and above by $z = 2 - y/3 - 2x/3$; $0 \leq z \leq 2 - y/3 - 2x/3$.

To find the bounds on y and x, we "collapse" the region onto the x-y plane, giving the triangle shown in Figure 12.38(b). (We know the equation of the line $y = 6 - 2x$ in two ways. First, by setting $z = 0$, we have $0 = 2 - y/3 - 2x/3 \Rightarrow y = 6 - 2x$. Secondly, we know this is going to be a straight line between the points $(3, 0)$ and $(0, 6)$ in the x-y plane.)

We define that region R, in the integration order of $dy\, dx$, with bounds $0 \leq$

Figure 12.38: The region D used in Example 162 in (a); in (b), the region found by collapsing D onto the x-y plane.

Notes:

$y \leq 6 - 2x$ and $0 \leq x \leq 3$. Thus the volume V of the region D is:

$$\begin{aligned} V &= \iiint_D dV \\ &= \int_0^3 \int_0^{6-2x} \int_0^{2-\frac{1}{3}y-\frac{2}{3}x} dz\, dy\, dz \\ &= \int_0^3 \int_0^{6-2x} \left(\int_0^{2-\frac{1}{3}y-\frac{2}{3}x} dz \right) dy\, dz \\ &= \int_0^3 \int_0^{6-2x} z\Big|_0^{2-\frac{1}{3}y-\frac{2}{3}x} dy\, dz \\ &= \int_0^3 \int_0^{6-2x} \left(2 - \frac{1}{3}y - \frac{2}{3}x \right) dy\, dz. \end{aligned}$$

From this step on, we are evaluating a double integral as done many times before. We skip these steps and give the final volume,

$$= 6\text{u}^3.$$

The order $dz\, dx\, dy$:

Now consider the volume using the order of integration $dz\, dx\, dy$. The bounds on z are the same as before, $0 \leq z \leq 2 - y/3 - 2x/3$. Collapsing the space region on the x-y plane as shown in Figure 12.38(b), we now describe this triangle with the order of integration $dx\, dy$. This gives bounds $0 \leq x \leq 3 - y/2$ and $0 \leq y \leq 6$. Thus the volume is given by the triple integral

$$V = \int_0^6 \int_0^{3-\frac{1}{2}y} \int_0^{2-\frac{1}{3}y-\frac{2}{3}x} dz\, dx\, dy.$$

The order $dx\, dy\, dz$:

Following our "surface to surface..." strategy, we need to determine the *x-surfaces* that bound our space region. To do so, approach the region "from behind," in the direction of increasing x. The first surface we hit as we enter the region is the y-z plane, defined by $x = 0$. We come out of the region at the plane $z = 2 - y/3 - 2x/3$; solving for x, we have $x = 3 - y/2 - 3z/2$. Thus the bounds on x are: $0 \leq x \leq 3 - y/2 - 3z/2$.

Now collapse the space region onto the y-z plane, as shown in Figure 12.39(a). (Again, we find the equation of the line $z = 2 - y/3$ by setting $x = 0$ in the equation $x = 3 - y/2 - 3z/2$.) We need to find bounds on this region with the order

Notes:

dy dz. The *curves* that bound y are $y = 0$ and $y = 6 - 3z$; the *points* that bound z are 0 and 2. Thus the triple integral giving volume is:

$$\begin{array}{c} 0 \le x \le 3 - y/2 - 3z/2 \\ 0 \le y \le 6 - 3z \\ 0 \le z \le 2 \end{array} \quad \Rightarrow \quad \int_0^2 \int_0^{6-3z} \int_0^{3-y/2-3z/2} dx\, dy\, dz.$$

The order *dx dz dy*:

The x-bounds are the same as the order above. We now consider the triangle in Figure 12.39(a) and describe it with the order *dz dy*: $0 \le z \le 2 - y/3$ and $0 \le y \le 6$. Thus the volume is given by:

$$\begin{array}{c} 0 \le x \le 3 - y/2 - 3z/2 \\ 0 \le z \le 2 - y/3 \\ 0 \le y \le 6 \end{array} \quad \Rightarrow \quad \int_0^6 \int_0^{2-y/3} \int_0^{3-y/2-3z/2} dx\, dz\, dy.$$

The order *dy dz dx*:

We now need to determine the y-surfaces that determine our region. Approaching the space region from "behind" and moving in the direction of increasing y, we first enter the region at $y = 0$, and exit along the plane $z = 2 - y/3 - 2x/3$. Solving for y, this plane has equation $y = 6 - 2x - 3z$. Thus y has bounds $0 \le y \le 6 - 2x - 3z$.

Now collapse the region onto the x-z plane, as shown in Figure 12.39(b). The curves bounding this triangle are $z = 0$ and $z = 2 - 2x/3$; x is bounded by the points $x = 0$ to $x = 3$. Thus the triple integral giving volume is:

$$\begin{array}{c} 0 \le y \le 6 - 2x - 3z \\ 0 \le z \le 2 - 2x/3 \\ 0 \le x \le 3 \end{array} \quad \Rightarrow \quad \int_0^3 \int_0^{2-2x/3} \int_0^{6-2x-3z} dy\, dz\, dx.$$

The order *dy dx dz*:

The y-bounds are the same as in the order above. We now determine the bounds of the triangle in Figure 12.39(b) using the order *dy dx dz*. x is bounded by $x = 0$ and $x = 3 - 3z/2$; z is bounded between $z = 0$ and $z = 2$. This leads to the triple integral:

$$\begin{array}{c} 0 \le y \le 6 - 2x - 3z \\ 0 \le x \le 3 - 3z/2 \\ 0 \le z \le 2 \end{array} \quad \Rightarrow \quad \int_0^2 \int_0^{3-3z/2} \int_0^{6-2x-3z} dy\, dx\, dz.$$

(a)

(b)

Figure 12.39: The region D in Example 162 is collapsed onto the y-z plane in (a); in (b), the region is collapsed onto the x-z plane.

Notes:

This problem was long, but hopefully useful, demonstrating how to determine bounds with every order of integration to describe the region D. In practice, we only need 1, but being able to do them all gives us flexibility to choose the order that suits us best.

In the previous example, we collapsed the surface into the x-y, x-z, and y-z planes as we determined the "curve to curve, point to point" bounds of integration. Since the surface was a triangular portion of a plane, this collapsing, or *projecting*, was simple: the *projection* of a straight line in space onto a coordinate plane is a line.

The following example shows us how to do this when dealing with more complicated surfaces and curves.

(a)

(b)

Figure 12.40: Finding the projections of the curve of intersection in Example 163.

Example 163 Finding the projection of a curve in space onto the coordinate planes

Consider the surfaces $z = 3 - x^2 - y^2$ and $z = 2y$, as shown in Figure 12.40(a). The curve of their intersection is shown, along with the projection of this curve into the coordinate planes, shown dashed. Find the equations of the projections into the coordinate planes.

Solution The two surfaces are $z = 3 - x^2 - y^2$ and $z = 2y$. To find where they intersect, it is natural to set them equal to each other: $3 - x^2 - y^2 = 2y$. This is an implicit function of x and y that gives all points (x, y) in the x-y plane where the z values of the two surfaces are equal.

We can rewrite this implicit function by completing the square:

$$3 - x^2 - y^2 = 2y \quad \Rightarrow \quad y^2 + 2y + x^2 = 3 \quad \Rightarrow \quad (y+1)^2 + x^2 = 4.$$

Thus in the x-y plane the projection of the intersection is a circle with radius 2, centered at $(0, -1)$.

To project onto the x-z plane, we do a similar procedure: find the x and z values where the y values on the surface are the same. We start by solving the equation of each surface for y. In this particular case, it works well to actually solve for y^2:

$z = 3 - x^2 - y^2 \quad \Rightarrow \quad y^2 = 3 - x^2 - z$

$z = 2y \quad \Rightarrow \quad y^2 = z^2/4.$

Thus we have (after again completing the square):

$$3 - x^2 - z = z^2/4 \quad \Rightarrow \quad \frac{(z+2)^2}{16} + \frac{x^2}{4} = 1,$$

and ellipse centered at $(0, -2)$ in the x-z plane with a major axis of length 8 and a minor axis of length 4.

Notes:

Finally, to project the curve of intersection into the y-z plane, we solve equation for x. Since $z = 2y$ is a cylinder that lacks the variable x, it becomes our equation of the projection in the y-z plane.

All three projections are shown in Figure 12.40(b).

Example 164 Finding the volume of a space region with triple integration

Set up the triple integrals that find the volume of the space region D bounded by the surfaces $x^2 + y^2 = 1$, $z = 0$ and $z = -y$, as shown in Figure 12.41(a), with the orders of integration $dz\,dy\,dx$, $dy\,dx\,dz$ and $dx\,dz\,dy$.

Solution The order $dz\,dy\,dx$:

The region D is bounded below by the plane $z = 0$ and above by the plane $z = -y$. The cylinder $x^2 + y^2 = 1$ does not offer any bounds in the z-direction, as that surface is parallel to the z-axis. Thus $0 \leq z \leq -y$.

Collapsing the region into the x-y plane, we get part of the circle with equation $x^2 + y^2 = 1$ as shown in Figure 12.41(b). As a function of x, this half circle has equation $y = -\sqrt{1 - x^2}$. Thus y is bounded below by $-\sqrt{1 - x^2}$ and above by $y = 0$: $-\sqrt{1 - x^2} \leq y \leq 0$. The x bounds of the half circle are $-1 \leq x \leq 1$. All together, the bounds of integration and triple integral are as follows:

$$\begin{array}{c} 0 \leq z \leq -y \\ -\sqrt{1-x^2} \leq y \leq 0 \\ -1 \leq x \leq 1 \end{array} \quad \Rightarrow \quad \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{0} \int_{0}^{-y} dz\,dy\,dx.$$

We evaluate this triple integral:

$$\begin{aligned} \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{0} \int_{0}^{-y} dz\,dy\,dx &= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{0} \big(-y\big)\,dy\,dx \\ &= \int_{-1}^{1} \Big(-\frac{1}{2}y^2\Big)\Big|_{-\sqrt{1-x^2}}^{0} dx \\ &= \int_{-1}^{1} \frac{1}{2}\big(1 - x^2\big)\,dx \\ &= \left(\frac{1}{2}\left(x - \frac{1}{3}x^3\right)\right)\Big|_{-1}^{1} \\ &= \frac{2}{3}\text{units}^3. \end{aligned}$$

Figure 12.41: The region D in Example 164 is shown in (a); in (b), it is collapsed onto the x-y plane.

Notes:

Figure 12.42: The region D in Example 164 is shown collapsed onto the x-z plane in (a); in (b), it is collapsed onto the y-z plane.

With the order $dy\,dx\,dz$:

The region is bounded "below" in the y-direction by the surface $x^2 + y^2 = 1 \Rightarrow y = -\sqrt{1-x^2}$ and "above" by the surface $y = -z$. Thus the y bounds are $-\sqrt{1-x^2} \leq y \leq -z$.

Collapsing the region onto the x-z plane gives the region shown in Figure 12.42(a); this half circle has equation $x^2 + z^2 = 1$. (We find this curve by solving each surface for y^2, then setting them equal to each other. We have $y^2 = 1 - x^2$ and $y = -z \Rightarrow y^2 = z^2$. Thus $x^2 + z^2 = 1$.) It is bounded below by $x = -\sqrt{1-z^2}$ and above by $x = \sqrt{1-z^2}$, where z is bounded by $0 \leq z \leq 1$. All together, we have:

$$\begin{array}{c} -\sqrt{1-x^2} \leq y \leq -z \\ -\sqrt{1-z^2} \leq x \leq \sqrt{1-z^2} \\ 0 \leq z \leq 1 \end{array} \quad \Rightarrow \quad \int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-x^2}}^{-z} dy\,dx\,dz.$$

With the order $dx\,dz\,dy$:

D is bounded below by the surface $x = -\sqrt{1-y^2}$ and above by $\sqrt{1-y^2}$. We then collapse the region onto the y-z plane and get the triangle shown in Figure 12.42(b). (The hypotenuse is the line $z = -y$, just as the plane.) Thus z is bounded by $0 \leq z \leq -y$ and y is bounded by $-1 \leq y \leq 0$. This gives:

$$\begin{array}{c} -\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2} \\ 0 \leq z \leq -y \\ -1 \leq y \leq 0 \end{array} \quad \Rightarrow \quad \int_{-1}^{0} \int_0^{-y} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\,dz\,dy.$$

The following theorem states two things that should make "common sense" to us. First, using the triple integral to find volume of a region D should always return a positive number; we are computing *volume* here, not *signed volume*. Secondly, to compute the volume of a "complicated" region, we could break it up into subregions and compute the volumes of each subregion separately, summing them later to find the total volume.

Notes:

Theorem 42 Properties of Triple Integrals

Let D be a closed, bounded region in space, and let D_1 and D_2 be non-overlapping regions such that $D = D_1 \bigcup D_2$.

1. $\displaystyle\iiint_D dV \geq 0$
2. $\displaystyle\iiint_D dV = \iiint_{D_1} dV + \iiint_{D_2} dV.$

We use this latter property in the next example.

Example 165 Finding the volume of a space region with triple integration
Find the volume of the space region D bounded by the coordinate planes, $z = 1 - x/2$ and $z = 1 - y/4$, as shown in Figure 12.43(a). Set up the triple integrals that find the volume of D in all 6 orders of integration.

Solution Following the bounds–determining strategy of "surface to surface, curve to curve, and point to point," we can see that the most difficult orders of integration are the two in which we integrate with respect to z first, for there are two "upper" surfaces that bound D in the z-direction. So we start by noting that we have

$$0 \leq z \leq 1 - \frac{1}{2}x \quad \text{and} \quad 0 \leq z \leq 1 - \frac{1}{4}y.$$

We now collapse the region D onto the x-y axis, as shown in Figure 12.43(b). The boundary of D, the line from $(0, 0, 1)$ to $(2, 4, 0)$, is shown in part (b) of the figure as a dashed line; it has equation $y = 2x$. (We can recognize this in two ways: one, in collapsing the line from $(0, 0, 1)$ to $(2, 4, 0)$ onto the x-y plane, we simply ignore the z-values, meaning the line now goes from $(0, 0)$ to $(2, 4)$. Secondly, the two surfaces meet where $z = 1 - x/2$ is equal to $z = 1 - y/4$: thus $1 - x/2 = 1 - y/4 \Rightarrow y = 2x$.)

We use the second property of Theorem 42 to state that

$$\iiint_D dV = \iiint_{D_1} dV + \iiint_{D_2} dV,$$

where D_1 and D_2 are the space regions above the plane regions R_1 and R_2, respectively. Thus we can say

$$\iiint_D dV = \iint_{R_1} \left(\int_0^{1-x/2} dz \right) dA + \iint_{R_2} \left(\int_0^{1-y/4} dz \right) dA.$$

Figure 12.43: The region D in Example 165 is shown in (a); in (b), it is collapsed onto the x-y plane.

Notes:

Figure 12.44: The region D in Example 165 is shown collapsed onto the x-z plane in (a); in (b), it is collapsed onto the y-z plane.

All that is left is to determine bounds of R_1 and R_2, depending on whether we are integrating with order $dx\,dy$ or $dy\,dx$. We give the final integrals here, leaving it to the reader to confirm these results.

$dz\,dy\,dx$:

$$\begin{array}{c} 0 \le z \le 1 - x/2 \\ 0 \le y \le 2x \\ 0 \le x \le 2 \end{array} \qquad \begin{array}{c} 0 \le z \le 1 - y/4 \\ 2x \le y \le 4 \\ 0 \le x \le 2 \end{array}$$

$$\iiint_D dV = \int_0^2 \int_0^{2x} \int_0^{1-x/2} dz\,dy\,dx + \int_0^2 \int_{2x}^4 \int_0^{1-y/4} dz\,dy\,dx$$

$dz\,dx\,dy$:

$$\begin{array}{c} 0 \le z \le 1 - x/2 \\ y/2 \le x \le 2 \\ 0 \le y \le 4 \end{array} \qquad \begin{array}{c} 0 \le z \le 1 - y/4 \\ 0 \le x \le y/2 \\ 0 \le y \le 4 \end{array}$$

$$\iiint_D dV = \int_0^4 \int_{y/2}^2 \int_0^{1-x/2} dz\,dx\,dy + \int_0^4 \int_0^{y/2} \int_0^{1-y/4} dz\,dx\,dy$$

The remaining four orders of integration do not require a sum of triple integrals. In Figure 12.44 we show D collapsed onto the other two coordinate planes. Using these graphs, we give the final orders of integration here, again leaving it to the reader to confirm these results.

$dy\,dx\,dz$:

$$\begin{array}{c} 0 \le y \le 4 - 4z \\ 0 \le x \le 2 - 2z \\ 0 \le z \le 1 \end{array} \quad \Rightarrow \int_0^1 \int_0^{2-2z} \int_0^{4-4z} dy\,dx\,dz$$

$dy\,dz\,dx$:

$$\begin{array}{c} 0 \le y \le 4 - 4z \\ 0 \le z \le 1 - x/2 \\ 0 \le x \le 2 \end{array} \quad \Rightarrow \int_0^2 \int_0^{1-x/2} \int_0^{4-4z} dy\,dx\,dz$$

Notes:

dx dy dz:

$$\begin{array}{c} 0 \le x \le 2-2z \\ 0 \le y \le 4-4z \\ 0 \le z \le 1 \end{array} \quad \Rightarrow \int_0^1 \int_0^{4-4z} \int_0^{2-2z} dx\, dy\, dz$$

dx dz dy:

$$\begin{array}{c} 0 \le x \le 2-2z \\ 0 \le z \le 1-y/4 \\ 0 \le y \le 4 \end{array} \quad \Rightarrow \int_0^4 \int_0^{1-y/4} \int_0^{2-2z} dx\, dz\, dy$$

We give one more example of finding the volume of a space region.

Example 166 Finding the volume of a space region

Set up a triple integral that gives the volume of the space region D bounded by $z = 2x^2 + 2$ and $z = 6 - 2x^2 - y^2$. These surfaces are plotted in Figure 12.45(a) and (b), respectively; the region D is shown in part (c) of the figure.

Solution The main point of this example is this: integrating with respect to z first is rather straightforward; integrating with respect to x first is not.

The order *dz dy dx*:

The bounds on z are clearly $2x^2+2 \le z \le 6-2x^2-y^2$. Collapsing D onto the x-y plane gives the ellipse shown in Figure 12.45(c). The equation of this ellipse is found by setting the two surfaces equal to each other:

$$2x^2 + 2 = 6 - 2x^2 - y^2 \quad \Rightarrow \quad 4x^2 + y^2 = 4 \quad \Rightarrow \quad x^2 + \frac{y^2}{4} = 1.$$

We can describe this ellipse with the bounds

$$-\sqrt{4-4x^2} \le y \le \sqrt{4-4x^2} \quad \text{and} \quad -1 \le x \le 1.$$

Thus we find volume as

$$\begin{array}{c} 2x^2+2 \le z \le 6-2x^2-y^2 \\ -\sqrt{4-4x^2} \le y \le \sqrt{4-4x^2} \\ -1 \le x \le 1 \end{array} \quad \Rightarrow \int_{-1}^{1} \int_{-\sqrt{4-4x^2}}^{\sqrt{4-4x^2}} \int_{2x^2+2}^{6-2x^2-y^2} dz\, dy\, dx \ .$$

The order *dy dz dx*:

Figure 12.45: The region D is bounded by the surfaces shown in (a) and (b); D is shown in (c).

Notes:

Figure 12.46: The region D in Example 166 is collapsed onto the x-z plane in (a); in (b), it is collapsed onto the y-z plane.

Integrating with respect to y is not too difficult. Since the surface $z = 2x^2 + 2$ is a cylinder whose directrix is the y-axis, it does not create a border for y. The paraboloid $z = 6 - 2x^2 - y^2$ does; solving for y, we get the bounds

$$-\sqrt{6 - 2x^2 - z} \leq y \leq \sqrt{6 - 2x^2 - z}.$$

Collapsing D onto the x-z axes gives the region shown in Figure 12.46(a); the lower curve is from the cylinder, with equation $z = 2x^2 + 2$. The upper curve is from the paraboloid; with $y = 0$, the curve is $z = 6 - 2x^2$. Thus bounds on z are $2x^2 + 2 \leq z \leq 6 - 2x^2$; the bounds on x are $-1 \leq x \leq 1$. Thus we have:

$$\begin{aligned} -\sqrt{6 - 2x^2 - z} \leq y \leq \sqrt{6 - 2x^2 - z} \\ 2x^2 + 2 \leq z \leq 6 - 2x^2 \\ -1 \leq x \leq 1 \end{aligned} \qquad \Rightarrow \int_{-1}^{1} \int_{2x^2+2}^{6-2x^2} \int_{-\sqrt{6-2x^2-z}}^{\sqrt{6-2x^2-z}} dy\, dz\, dx.$$

The order $dx\, dz\, dy$:

This order takes more effort as D must be split into two subregions. The two surfaces create two sets of upper/lower bounds in terms of x; the cylinder creates bounds

$$-\sqrt{z/2 - 1} \leq x \leq \sqrt{z/2 - 1}$$

for region D_1 and the paraboloid creates bounds

$$-\sqrt{3 - y^2/2 - z^2/2} \leq x \leq \sqrt{3 - y^2/2 - z^2/2}$$

for region D_2.

Collapsing D onto the y-z axes gives the regions shown in Figure 12.46(b). We find the equation of the curve $z = 4 - y^2/2$ by noting that the equation of the ellipse seen in Figure 12.45(c) has equation

$$x^2 + y^2/4 = 1 \quad \Rightarrow \quad x = \sqrt{1 - y^2/4}.$$

Substitute this expression for x in either surface equation, $z = 6 - 2x^2 - y^2$ or $z = 2x^2 + 2$. In both cases, we find

$$z = 4 - \frac{1}{2}y^2.$$

Region R_1, corresponding to D_1, has bounds

$$2 \leq z \leq 4 - y^2/2, \quad -2 \leq y \leq 2$$

and region R_2, corresponding to D_2, has bounds

$$4 - y^2/2 \leq z \leq 6 - y^2, \quad -2 \leq y \leq 2.$$

Notes:

Thus the volume of D is given by:

$$\int_{-2}^{2}\int_{2}^{4-y^2/2}\int_{-\sqrt{z/2-1}}^{\sqrt{z/2-1}} dx\,dz\,dy \;+\; \int_{-2}^{2}\int_{4-y^2/2}^{6-y^2}\int_{-\sqrt{3-y^2/2-z^2/2}}^{\sqrt{3-y^2/2-z^2/2}} dx\,dz\,dy.$$

If all one wanted to do in Example 166 was find the volume of the region D, one would have likely stopped at the first integration setup (with order $dz\,dy\,dx$) and computed the volume from there. However, we included the other two methods 1) to show that it could be done, "messy" or not, and 2) because sometimes we "have" to use a less desirable order of integration in order to actually integrate.

Triple Integration and Functions of Three Variables

There are uses for triple integration beyond merely finding volume, just as there are uses for integration beyond "area under the curve." These uses start with understanding how to integrate functions of three variables, which is effectively no different than integrating functions of two variables. This leads us to a definition, followed by an example.

Definition 59 Iterated Integration, (Part II)

Let D be a closed, bounded region in space, over which $g_1(x)$, $g_2(x)$, $f_1(x,y)$, $f_2(x,y)$ and $h(x,y,z)$ are all continuous, and let a and b be real numbers.

The **iterated integral** $\displaystyle\int_a^b\int_{g_1(x)}^{g_2(x)}\int_{f_1(x,y)}^{f_2(x,y)} h(x,y,z)\,dz\,dy\,dx$ is evaluated as

$$\int_a^b\int_{g_1(x)}^{g_2(x)}\int_{f_1(x,y)}^{f_2(x,y)} h(x,y,z)\,dz\,dy\,dx = \int_a^b\int_{g_1(x)}^{g_2(x)}\left(\int_{f_1(x,y)}^{f_2(x,y)} h(x,y,z)\,dz\right)dy\,dx.$$

Example 167 Evaluating a triple integral of a function of three variables

Evaluate $\displaystyle\int_0^1\int_{x^2}^{x}\int_{x^2-y}^{2x+3y}\left(xy+2xz\right)dz\,dy\,dx$.

Solution We evaluate this integral according to Definition 59.

Notes:

$$\begin{aligned}
&\int_0^1 \int_{x^2}^{x} \int_{x^2-y}^{2x+3y} \big(xy + 2xz\big)\, dz\, dy\, dx \\
&= \int_0^1 \int_{x^2}^{x} \left(\int_{x^2-y}^{2x+3y} \big(xy + 2xz\big)\, dz \right) dy\, dx \\
&= \int_0^1 \int_{x^2}^{x} \left(\big(xyz + xz^2\big)\Big|_{x^2-y}^{2x+3y} \right) dy\, dx \\
&= \int_0^1 \int_{x^2}^{x} \left(xy(2x+3y) + x(2x+3y)^2 - \Big(xy(x^2-y) + x(x^2-y)^2\Big) \right) dy\, dx \\
&= \int_0^1 \int_{x^2}^{x} \Big(-x^5 + x^3y + 4x^3 + 14x^2y + 12xy^2 \Big)\, dy\, dx.
\end{aligned}$$

We continue as we have in the past, showing fewer steps.

$$\begin{aligned}
&= \int_0^1 \left(-\frac{7}{2}x^7 - 8x^6 - \frac{7}{2}x^5 + 15x^4 \right) dx \\
&= \frac{281}{336} \approx 0.836.
\end{aligned}$$

We now know *how* to evaluate a triple integral of a function of three variables; we do not yet understand what it *means*. We build up this understanding in a way very similar to how we have understood integration and double integration.

Let $h(x, y, z)$ a continuous function of three variables, defined over some space region D. We can partition D into n rectangular–solid subregions, each with dimensions $\Delta x_i \times \Delta y_i \times \Delta z_i$. Let (x_i, y_i, z_i) be some point in the ith subregion, and consider the product $h(x_i, y_i, z_i)\Delta x_i \Delta y_i \Delta z_i$. It is the product of a function value (that's the $h(x_i, y_i, z_i)$ part) and a small volume ΔV_i (that's the $\Delta x_i \Delta y_i \Delta z_i$ part). One of the simplest understanding of this type of product is when h describes the density of an object, for then $h \times$ volume $=$ mass.

We can sum up all n products over D. Again letting $||\Delta D||$ represent the length of the longest diagonal of the n rectangular solids in the partition, we can take the limit of the sums of products as $||\Delta D|| \to 0$. That is, we can find

$$S = \lim_{||\Delta D|| \to 0} \sum_{i=1}^{n} h(x_i, y_i, z_i)\Delta V_i = \lim_{||\Delta D|| \to 0} \sum_{i=1}^{n} h(x_i, y_i, z_i)\Delta x_i \Delta y_i \Delta z_i.$$

While this limit has lots of interpretations depending on the function h, in the case where h describes density, S is the total mass of the object described by the region D.

Notes:

We now use the above limit to define the **triple integral**, give a theorem that relates triple integrals to iterated iteration, followed by the application of triple integrals to find the centers of mass of solid objects.

Definition 60 Triple Integral

Let $w = h(x, y, z)$ be a continuous function over a closed, bounded space region D, and let ΔD be any partition of D into n rectangular solids with volume ΔV_i. The **triple integral of h over D** is

$$\iiint_D h(x, y, z)\, dV = \lim_{||\Delta D|| \to 0} \sum_{i=1}^{n} h(x_i, y_i, z_i)\Delta V_i.$$

The following theorem assures us that the above limit exists for continuous functions h and gives us a method of evaluating the limit.

Theorem 43 Triple Integration (Part II)

Let $w = h(x, y, z)$ be a continuous function over a closed, bounded space region D, and let ΔD be any partition of D into n rectangular solids with volume V_i.

1. The limit $\lim_{||\Delta D|| \to 0} \sum_{i=1}^{n} h(x_i, y_i, z_i)\Delta V_i$ exists.
2. If D is defined as the region bounded by the planes $x = a$ and $x = b$, the cylinders $y = g_1(x)$ and $y = g_2(x)$, and the surfaces $z = f_1(x, y)$ and $z = f_2(x, y)$, where $a < b$, $g_1(x) \leq g_2(x)$ and $f_1(x, y) \leq f_2(x, y)$ on D, then

$$\iiint_D h(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} h(x, y, z)\, dz\, dy\, dx.$$

We now apply triple integration to find the centers of mass of solid objects.

Mass and Center of Mass

One may wish to review Section 12.4 for a reminder of the relevant terms and concepts.

Note: In the marginal note on page 702, we showed how the summation of rectangles over a region R in the plane could be viewed as a double sum, leading to the double integral. Likewise, we can view the sum $\sum_{i=1}^{n} h(x_i, y_i, z_i)\Delta x_i \Delta y_i \Delta z_i$ as a triple sum,

$$\sum_{k=1}^{p} \sum_{j=1}^{n} \sum_{i=1}^{m} h(x_i, y_j, z_k)\Delta x_i \Delta y_j \Delta z_k,$$

which we evaluate as

$$\sum_{k=1}^{p} \left(\sum_{j=1}^{n} \left(\sum_{i=1}^{m} h(x_i, y_j, z_k)\Delta x_i \right) \Delta y_j \right) \Delta z_k.$$

Here we fix a k value, which establishes the z-height of the rectangular solids on one "level" of all the rectangular solids in the space region D. The inner double summation adds up all the volumes of the rectangular solids on this level, while the outer summation adds up the volumes of each level.
This triple summation understanding leads to the $\iint_D$ notation of the triple integral, as well as the method of evaluation shown in Theorem 43.

Notes:

Definition 61 Mass, Center of Mass of Solids

Let a solid be represented by a region D in space with variable density function $\delta(x, y, z)$.

1. The **mass** of the object is $M = \iiint_D dm = \iiint_D \delta(x, y, z)\, dV$.
2. The **moment about the x-y plane** is $M_{xy} = \iiint_D z\delta(x, y, z)\, dV$.
3. The **moment about the x-z plane** is $M_{xz} = \iiint_D y\delta(x, y, z)\, dV$.
4. The **moment about the y-z plane** is $M_{yz} = \iiint_D x\delta(x, y, z)\, dV$.
5. The **center of mass** of the object is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right).$$

Example 168 Finding the center of mass of a solid

Find the mass and center of mass of the solid represented by the space region bounded by the coordinate planes and $z = 2 - y/3 - 2x/3$, shown in Figure 12.47, with constant density $\delta(x, y, z) = 3\text{gm/cm}^3$. (Note: this space region was used in Example 162.)

Figure 12.47: Finding the center of mass of this solid in Example 168.

Solution We apply Definition 61. In Example 162, we found bounds for the order of integration $dz\, dy\, dx$ to be $0 \le z \le 2 - y/3 - 2x/3$, $0 \le y \le 6 - 2x$ and $0 \le x \le 3$. We find the mass of the object:

$$\begin{aligned} M &= \iiint_D \delta(x, y, z)\, dV \\ &= \int_0^3 \int_0^{6-2x} \int_0^{2-y/3-2x/3} (3)\, dz\, dy\, dx \\ &= 3 \int_0^3 \int_0^{6-2x} \int_0^{2-y/3-2x/3} dz\, dy\, dx \\ &= 3(6) = 18\text{gm}. \end{aligned}$$

The evaluation of the triple integral is done in Example 162, so we skipped those steps above. Note how the mass of an object with constant density is simply "density$\times$volume."

Notes:

We now find the moments about the planes.

$$\begin{aligned} M_{xy} &= \iiint_D 3z\,dV \\ &= \int_0^3 \int_0^{6-2x} \int_0^{2-y/3-2x/3} (3z)\,dz\,dy\,dx \\ &= \int_0^3 \int_0^{6-2x} \frac{3}{2}\big(2 - y/3 - 2x/3\big)^2\,dy\,dx \\ &= \int_0^3 -\frac{4}{9}(x-3)^3\,dx \\ &= 9. \end{aligned}$$

We omit the steps of integrating to find the other moments.

$$\begin{aligned} M_{yz} &= \iiint_D 3x\,dV \\ &= \frac{27}{2}. \\ M_{xz} &= \iiint_D 3y\,dV \\ &= 27. \end{aligned}$$

The center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{27/2}{18}, \frac{27}{18}, \frac{9}{18}\right) = (0.75, 1.5, 0.5).$$

Example 169 Finding the center of mass of a solid

Find the center of mass of the solid represented by the region bounded by the planes $z = 0$ and $z = -y$ and the cylinder $x^2 + y^2 = 1$, shown in Figure 12.48, with density function $\delta(x, y, z) = 10 + x^2 + 5y - 5z$. (Note: this space region was used in Example 164.)

Figure 12.48: Finding the center of mass of this solid in Example 169.

Solution As we start, consider the density function. It is symmetric about the y-z plane, and the farther one moves from this plane, the denser the object is. The symmetry indicates that $\bar{x}$ should be 0.

As one moves away from the origin in the y or z directions, the object becomes less dense, though there is more volume in these regions.

Though none of the integrals needed to compute the center of mass are particularly hard, they do require a number of steps. We emphasize here the

Notes:

importance of knowing how to set up the proper integrals; in complex situations we can appeal to technology for a good approximation, if not the exact answer. We use the order of integration *dz dy dx*, using the bounds found in Example 164. (As these are the same for all four triple integrals, we explicitly show the bounds only for M.)

$$\begin{aligned}
M &= \iiint_D \left(10 + x^2 + 5y - 5z\right)\, dV \\
&= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0}\int_{0}^{-y} \left(10 + x^2 + 5y - 5z\right)\, dV \\
&= \frac{64}{5} - \frac{15\pi}{16} \approx 3.855. \\
M_{yz} &= \iiint_D x\left(10 + x^2 + 5y - 5z\right)\, dV \\
&= 0. \\
M_{xz} &= \iiint_D y\left(10 + x^2 + 5y - 5z\right)\, dV \\
&= 2 - \frac{61\pi}{48} \approx -1.99. \\
M_{xy} &= \iiint_D z\left(10 + x^2 + 5y - 5z\right)\, dV \\
&= \frac{61\pi}{96} - \frac{10}{9} \approx 0.885.
\end{aligned}$$

Note how $M_{yz} = 0$, as expected. The center of mass is

$$\left(\bar{x}, \bar{y}, \bar{z}\right) = \left(0, \frac{-1.99}{3.855}, \frac{0.885}{3.855}\right) \approx \left(0, -0.516, 0.230\right).$$

As stated before, there are many uses for triple integration beyond finding volume. When $h(x, y, z)$ describes a rate of change function over some space region D, then $\iiint_D h(x, y, z)\, dV$ gives the total change over D. Our one specific example of this was computing mass; a density function is simply a "rate of mass change per volume" function. Integrating density gives total mass.

While knowing *how to integrate* is important, it is arguably much more important to know *how to set up* integrals. It takes skill to create a formula that describes a desired quantity; modern technology is very useful in evaluating these formulas quickly and accurately.

Notes:

This chapter investigated the natural follow–on to partial derivatives: iterated integration. We learned how to use the bounds of a double integral to describe a region in the plane using both rectangular and polar coordinates, then later expanded to use the bounds of a triple integral to describe a region in space. We used double integrals to find volumes under surfaces, surface area, and the center of mass of lamina; we used triple integrals as an alternate method of finding volumes of space regions and also to find the center of mass of a region in space.

Integration does not stop here. We could continue to iterate our integrals, next investigating "quadruple integrals" whose bounds describe a region in 4–dimensional space (which are very hard to visualize). We can also look back to "regular" integration where we found the area under a curve in the plane. A natural analogue to this is finding the "area under a curve," where the curve is in space, not in a plane. These are just two of many avenues to explore under the heading of "integration."

Notes:

Exercises 12.6

Terms and Concepts

1. The strategy for establishing bounds for triple integrals is "_____ to _____, _____ to _____ and _____ to _____."

2. Give an informal interpretation of what "$\iiint_D dV$" means.

3. Give two uses of triple integration.

4. If an object has a constant density δ and a volume V, what is its mass?

Problems

In Exercises 5 – 8, two surfaces $f_1(x, y)$ and $f_2(x, y)$ and a region R in the x, y plane are given. Set up and evaluate the double integral that finds the volume between these surfaces over R.

5. $f_1(x, y) = 8 - x^2 - y^2$, $f_2(x, y) = 2x + y$;
   R is the square with corners $(-1, -1)$ and $(1, 1)$.

6. $f_1(x, y) = x^2 + y^2$, $f_2(x, y) = -x^2 - y^2$;
   R is the square with corners $(0, 0)$ and $(2, 3)$.

7. $f_1(x, y) = \sin x \cos y$, $f_2(x, y) = \cos x \sin y + 2$;
   R is the triangle with corners $(0, 0)$, $(\pi, 0)$ and (π, π).

8. $f_1(x, y) = 2x^2 + 2y^2 + 3$, $f_2(x, y) = 6 - x^2 - y^2$;
   R is the circle $x^2 + y^2 = 1$.

In Exercises 9 – 16, a domain D is described by its bounding surfaces, along with a graph. Set up the triple integrals that give the volume of D in all 6 orders of integration, and find the volume of D by evaluating the indicated triple integral.

9. D is bounded by the coordinate planes and $z = 2 - 2x/3 - 2y$.

   Evaluate the triple integral with order $dz\,dy\,dx$.

10. D is bounded by the planes $y = 0$, $y = 2$, $x = 1$, $z = 0$ and $z = (3 - x)/2$.

    Evaluate the triple integral with order $dx\,dy\,dz$.

11. D is bounded by the planes $x = 0$, $x = 2$, $z = -y$ and by $z = y^2/2$.

    Evaluate the triple integral with the order $dy\,dz\,dx$.

12. D is bounded by the planes $z = 0$, $y = 9$, $x = 0$ and by $z = \sqrt{y^2 - 9x^2}$.

    Do not evaluate any triple integral.

13. D is bounded by the planes $x = 2$, $y = 1$, $z = 0$ and $z = 2x + 4y - 4$.

Evaluate the triple integral with the order $dx\,dy\,dz$.

14. D is bounded by the plane $z = 2y$ and by $y = 4 - x^2$.

Evaluate the triple integral with the order $dz\,dy\,dx$.

15. D is bounded by the coordinate planes and by $y = 1 - x^2$ and $y = 1 - z^2$.
Do not evaluate any triple integral. Which order is easier to evaluate: $dz\,dy\,dx$ or $dy\,dz\,dx$? Explain why.

16. D is bounded by the coordinate planes and by $z = 1 - y/3$ and $z = 1 - x$.

Evaluate the triple integral with order $dx\,dy\,dz$.

In Exercises 17 – 20, evaluate the triple integral.

17. $\displaystyle\int_{-\pi/2}^{\pi/2}\int_0^{\pi}\int_0^{\pi} \big(\cos x \sin y \sin z\big)\, dz\, dy\, dx$

18. $\displaystyle\int_0^1\int_0^x\int_0^{x+y} \big(x + y + z\big)\, dz\, dy\, dx$

19. $\displaystyle\int_0^{\pi}\int_0^1\int_0^z \big(\sin(yz)\big)\, dx\, dy\, dz$

20. $\displaystyle\int_{\pi}^{\pi^2}\int_x^{x^3}\int_{-y^2}^{y^2} \left(z\frac{x^2y + y^2x}{e^{x^2+y^2}}\right) dz\, dy\, dx$

In Exercises 21 – 24, find the center of mass of the solid represented by the indicated space region D with density function $\delta(x, y, z)$.

21. D is bounded by the coordinate planes and $z = 2 - 2x/3 - 2y$; $\delta(x, y, z) = 10\text{gm/cm}^3$.
(Note: this is the same region as used in Exercise 9.)

22. D is bounded by the planes $y = 0$, $y = 2$, $x = 1$, $z = 0$ and $z = (3 - x)/2$; $\delta(x, y, z) = 2\text{gm/cm}^3$.
(Note: this is the same region as used in Exercise 10.)

23. D is bounded by the planes $x = 2$, $y = 1$, $z = 0$ and $z = 2x + 4y - 4$; $\delta(x, y, z) = x^2\text{lb/in}^3$.
(Note: this is the same region as used in Exercise 13.)

24. D is bounded by the plane $z = 2y$ and by $y = 4 - x^2$. $\delta(x, y, z) = y^2\text{lb/in}^3$.
(Note: this is the same region as used in Exercise 14.)

13: Vector Calculus

13.1 Vector Fields

Chapter 14introduced double and triple integrals. We went from $\int dx$ to $\int\int dxdy$ and $\int\int\int dxdydz$. All those integrals add up small pieces, and the limit gives area or volume or mass. What could be more natural than that? I regret to say, after the success of those multiple integrals, that something is missing. It is even more regrettable that we didn't notice it. The missing piece is nothing less than the Fundamental Theorem of Calculus.

The double integral $\int\int dxdy$ equals the area. To compute it, we did not use an antiderivative of 1. At least not consciously. The method was almost trial and error, and the hard part was to find the limits of integration. This chapter goes deeper, to show how the step from a double integral to a single integral is really a new form of the Fundamental Theorem-when it is done right.

Two new ideas are needed early, one pleasant and one not. You will like vector fields. You may not think so highly of line integrals. Those are ordinary single integrals like $\int v(x)dx$, but they go along curves instead of straight lines. The nice step dx becomes the confusing step ds. Where $\int dx$ equals the length of the interval, $\int ds$ is the length of the curve. The point is that regions are enclosed by curves, and we have to integrate along them. The Fundamental Theorem in its two-dimensional form (Green's Theorem) connects a **double integral over the region to a single integral along its boundary curve**.

The great applications are in science and engineering, where vector fields are so natural. But there are changes in the language. Instead of an antiderivative, we speak about a **potential function**. Instead of the derivative, we take the "divergence" and "curl." Instead of area, we compute **flux** and **circulation** and **work**. Examples come first.

15.1 Vector Fields

For an ordinary scalar function, the input is a number x and the output is a number $f(x)$. For a vector field (or vector function), the input is a point (x, y) and the output is a two-dimensional vector $F(x, y)$. There is a "field" of vectors, one at every point. In three dimensions the input point is (x, y, z) and the output vector F has three components.

DEFINITION Let R be a region in the xy plane. A **vector field** F assigns to every point (x, y) in R a vector $F(x, y)$ with two components: $F(x, y) = M(x, y)i + N(x, y)j.(1)$ This plane vector field involves two functions of two variables. They are the components M and N, which vary from point to point. A vector has fixed components, a vector field has varying components. A three-dimensional vector field has components M(x, y, z) and N(x, y, z) and P(x, y, 2). Then the vectors are F = Mi + Nj + Pk. EXAMPLE 1 The position vector at (x, y) is R = xi

+ yj. Its components are M = x and N = y. The vectors grow larger as we leave the origin (Figure 15.la). Their direction is outward and their length is IRI = J;i?;i = r, The vector R is boldface, the number r is lightface. EXAMPLE 2 The vector field R/r consists of unit vectors u„ pointing outward. We divide R = xi + yj by its length, at every point except the origin. The components of Rlr are M = xlr and N = y/r. Figure 15.1 shows a third field /r , whose length is 1 /r. Fig. 15.1 The vector fields R and R/r and R/r2 are radial. Lengths r and 1 and l/r EXAMPLE 3 The spin field or rotation field or turning field goes around the origin instead of away from it. The field is S. Its components are M = - y and N = x: S = - yi + xj also has length IS1 = J(-y)2 + x2 = r. (2) S is perpendicular to R-their dot product is zero: S R = (- y)(x) + (x)(y) = 0. The spin fields S/r and S/r2 have lengths 1 and llr: The unit vector S/r is u,. Notice the blank at (O,O), where this field is not defined. Fig. 15.2 The spin fields S and S/r and S/r2 go around the origin. Lengths r and 1 and l/r. 15.1 Vector Fields EXAMPLE 4 A gradientfield starts with an ordinary function f(x, y). The components M and PJ are the partial derivatives df/dx and dfldy. Then the field F is the gradient off: F = grad f = Vf= dfldx i + dfldy j. (3) This vector field grad f is everywhere perpendicular to the level curves f(x, y) = c. The length lgrad f 1 tells how fast f is changing (in the direction it changes fastest). Invent a function like f = x2y, and you immediately have its gradient field F = 2xyi + x2j. To repealt, M is df/dx and N is dfldy. For every vector field you should ask two questions: Is it a gradient field? If so, what is f? Here are answers for the radial fields and spin fields: MA The radial fields R and R/r and /r are a11 gradient fields. The spin fields S and S/r are not gradients of any f(x, y), The spin field S/r2 is the gradient of the polar angle 0 = tan- ’(ylx). The derivatives off = f(x2+ y2) are x and y. Thus R is a gradient field. The gradient off = r is the unit vector R/r pointing outwards. Both fields are perpendicular to circles around the origin. Those are the level curves off = f r2 and f = r. Question Is every R/rn a gradient field? Answer Yes. But among the spin fields, the only gradient is S/r2. A ma-jor goal of this chapter is to recognize gradient fields by a simple test. The rejection of S and S/r will be interesting. For some reason -yi + xj is rejected and yi + xj is accepted. (It is the gradient of .) The acceptance of S/r2 as the gradient off = 0 contains a surprise at the origin (Section 15.3). Gradient fields are called conservative. The function f is the potential function. These words, and the next examples, come from physics and engineering. EXAMPLE5 The velocity field is V and the flow field is pV. Suppose: fluid moves steadily down a pipe. Or a river flows smoothly (no waterfall). Or the air circulates in a fixed pattern. The velocity can be different at different points, but there is no change with time. The velocity vector V gives the direction offlow and speed of Jow at every point. In reality the velocity field is V(x, y, z), with three components M, N, P. Those are the velocities v„ v2, v, in the x, y, z directions. The speed (VI is the length: IVI2 = v: + v: -t v:. In a ”plane flow” the k component is zero, and the velocity field is v,i+v2j= Mi+ Nj. gravity F = -R//.” Fig. 15.3 A steady velocity field V and two force

Notes:

fields F. 15 Vector Calculus For a compact disc or a turning wheel, V is a spin field (V =US, co =angular velocity). A tornado might be closer to V =S/r2 (except for a dead spot at the center). An explosion could have V =R/r2. A quieter example is flow in and out of a lake with steady rain as a source term. TheJlowJield pV is the density p times the velocity field. While V gives the rate of movement, pV gives the rate of movement of mass. A greater density means a greater rate IpVJof "mass transport." It is like the number of passengers on a bus times the speed of the bus. EXAMPLE 6 Force fields from gravity: F is downward in the classroom, F is radial in space. When gravity pulls downward, it has only one nonzero component: F = -mgk. This assumes that vectors to the center of the Earth are parallel-almost true in a classroom. Then F is the gradient of -mgz (note dfldz = -mg). In physics the usual potential is not -mgz but +mgz. The force field is minus grad f also in electrical engineering. Electrons flow from high potential to low potential. The mathematics is the same, but the sign is reversed. In space, the force is radial inwards: F = -mMGR/r3. Its magnitude is proportional to l/r2 (Newton's inverse square law). The masses are m and M, and the gravitational constant is G =6.672 x 10-"–with distance in meters, mass in kilograms, and time in seconds. The dimensions of G are (force)(di tance) /(mass) . This is different from the acceleration g =9.8m/sec2, which already accounts for the mass and radius of the Earth. Like all radial fields, gravity is a gradient field. It comes from a potential f: EXAMPLE 7 (a short example) Current in a wire produces a magnetic field B. It is the spin field S/r2 around the wire, times the strength of the current. STREAMLINES AND LINES OF FORCE Drawing a vector field is not always easy. Even the spin field looks messy when the vectors are too long (they go in circles and fall across each other). The circles give a clearer picture than the vectors. In any field, the vectors are tangent to "jield lineswwhich in the spin case are circles. DEFINITION C is afield line or integral curve if the vectors F(x, y) are tangent to C. The slope dyldx of the curve C equals the slope N/M of the vector F =Mi + Nj: We are still drawing the field of vectors, but now they are infinitesimally short. They are connected into curves! What is lost is their length, because S and S/r and S/r2 all have the same field lines (circles). For the position field R and gravity field R/r3, the field lines are rays from the origin. In this case the "curves" are actually straight. EXAMPLE 8 Show that the field lines for the velocity field V =yi +xj are hyperbolas. dy N x – — * y dy =x dx *y2 -x2 =constant. X-M- 15.1 Vector Fields reamlines x2 - y2 = C Fig. 15.4 Velocity fields are tangent to streamlines. Gradient fields also have equipotentials. At every point these hyperbolas line up with the velocity V. Each particle of fluid travels on afield line. In fluid flow those hyperbolas are called streamlines. Drop a leaf into a river, and it follows a streamline. Figure 15.4 shows the streamlines for a river going around a bend. Don't forget the essential question about each vector field. Is it a gradient field? For V = yi + xj the answer is yes, and the potential is f = xy: the gradient of xy is (8flax)i + (8flay)j = yi + xj. (7) When there is a potential, it

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has level curves. They connect points of equal potential, so the curves f (x, y) = c are called equipotentials. Here they are the curves xy = calso hyperbolas. Since gradients are perpendicular to level curves, the streamlines are perpendicular to the equipotentials. Figure 15.4 is sliced one way by streamlines and the other way by equipotentials. A gradient field F = afldx i + afldy j is tangent to the field lines (streamlines) and perpendicular to the equipotentials (level curves off). In the gradient direction f changes fastest. In the level direction f doesn't change at all. The chain rule along f (x, y) = c proves these directions to be perpendicular: af dx af dy – + - = 0 or (grad f) (tangent to level curve) = 0. ax dt oy dt EXAMPLE 9 The streamlines of S/r2 are circles around (0,O). The equipotentials are rays 0 = c. Add rays to Figure 15.2 for the gradient field S/r2. For the gravity field those are reversed. A body is pulled in along the field lines (rays). The equipotentials are the circles where f = llr is constant. The plane is crisscrossed by "orthogonal trajectories9'-curves that meet everywhere at right angles. If you bring a magnet near a pile of iron filings, a little shake will display the field lines. In a force field, they are "lines of force." Here are the other new words. Vector hid F, y, z) = Mi + Nj + Pk Plane field F = M(x, y)i + N(x, y)j Radial field: multiple of R = xi + yj + zk Spifl field: multiple of = - yi + xj Gradient ktd = conservative field: A4 = wax, N = af, P = 18 Potmtialf(x, yf (not a vector) Equipotential curves f(x, y) = c Streamline = field line = integral curve: a curve that has F(x, y) as its tangent vectors. 554 15 Vector Calculus 15.1 EXERCISES Read-through questions A vector field assigns a a to each point (x, y) or (x, y, z). In two dimensions F(x,y) = b i + c j. An example is the position field R = d . Its magnitude is IRI = e and its direction is f . It is the gradient field for f = g . The level curves are h , and they are i to the vectors R. Reversing this picture, the spin field is S = i . Its magnitude is IS1 = k and its direction is I . It is not a gradient field, because no function has af/ax = m and af/ay = n . S is the velocity field for flow going 0 . The streamlines or P lines or integral s are r . The flow field pV gives the rate at which s is moved by the flow. A gravity field from the origin is proportional to F = t which has IF1 = u . This is Newton's v square law. It is a gradient field, with potential f = w .The equipotential curves f(x, y) = c are x . They are Y to the field lines which are . This illustrates that the A of a function f(x, y) is B to its level curves. The velocity field yi + xj is the gradient off = c . Its streamlines are D .The slope dyldx of a streamline equals the ratio E of velocity components. The field is F to the streamlines. Drop a leaf onto the flow, and it goes along G . Find a potential f(x, y) for the gradient fields 1-8. Draw the streamlines perpendicular to the equipotentials f(x, y) = c. 1 F = i + 2j (constant field) 2 F = xi +j 7 F=xyi+ j 8 F=i+ j 9 Draw the shear field F =xj. Check that it is not a gradient field: If af/ax =0 then af/ay =x is impossible. What are the streamlines (field lines) in the direction of F? 10 Find all functions that satisfy af/ax = -y and show that none of them satisfy af/ay = x. Then the spin field S = -yi + xj is not a gradient field. Compute af/ax and af/ay in 11-18. Draw the gradient field F =padf and the equipotentials f(x, y) = c: 15f=x2-y2 16

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f = ex cos y Find equations for the streamlines in 19-24 by solving dyldx = N/M (including a constant C). Draw the streamlines. 21 F =S (spin field) 22 F =S/r (spin field) 23 F =grad (xly) 24 F =grad (2x + y). 25 The Earth's gravity field is radial, but in a room the field lines seem to go straight down into the floor. This is because nearby field lines always look . 26 A line of charges produces the electrostatic force field F = R/r2=(xi + yj)/(x2+ y2). Find the potential f(x, y). (F is also the gravity field for a line of =asses.) In 27-32 write down the vector fields Mi + Nj. 27 F points radially away from the origin with magnitude 5. 28 The velocity is perpendicular to the curves x3 + y3 =c and the speed is 1. 29 The gravitational force F comes from two unit masses at (0,O) and (1,O). 30 The streamlines are in the 45" direction and the speed is 4. 31 The streamlines are circles clockwise around the origin and the speed is 1. 32 The equipotentials are the parabolas y = x2+ c and F is a gradient field. 33 Show directly that the hyperbolas xy = 2 and x2 -y2 = 3 are perpendicular at the point (2, l), by computing both slopes dyldx and multiplying to get -1. 34 The derivative off (x, y) = c isf, +f,(dy/dx) =0. Show that the slope of this level curve is dyldx = -MIN. It is perpendicular to streamlines because (- M/N)(N/M)= . 35 The x and y derivatives of f(r) are dfldx = and dflay =-by the chain rule. (Test f =r2.) The equipotentials are . 36 F =(ax + by)i + (bx + cy)j is a gradient field. Find the potential f and describe the equipotentials. 37 True or false: I. The constant field i + 2k is a gradient field. 2. For non-gradient fields, equipotentials meet streamlines at non-right angles. 3. In three dimensions the equipotentials are surfaces instead of curves. 4. F = x2i+ y2j+ z2k points outward from (0,0,0)- a radial field. 38 Create and draw f and F and your own equipotentials and streamlines. 15.2 Line Integrals 555 39 How can different vector fields have the same streamlines? 40 Draw arrows at six or eight points to show the direction Can they have the same equipotentials? Can they have the and magnitude of each field: same f? (a) R +S (b) Rlr -S/r (c) x2i+x2j (d)yi

13.2 Line Integrals

15.2 along a Line curve. It Integrals can equal an area, but .- that is a special case and not typical. Instead of area, here are two important line integrals in physics and engineering: Work along a curve = F T ds Flow across a curve = In the first integral, F is a force field. In the second integral, F is a flow field. Work is done in the direction of movement, so we integrate F T. Flow is measured through the curve C, so we integrate F n. Here T is the unit tangent vector, and F T is the force cornponent along the curve. Similarly n is the unit normal vector, at right angles with T. Then F n is the component of flow perpendicular to the curve. We will write those integrals in several forms. They may never be as comfortable as J y(x) dx, but eventually we get them under control. I mention these applications

Notes:

early, so you can see where we are going. This section concentrates on work, and flow comes later. (It is also calledflux-the Latin word for flow.) You recognize ds as the step along the curve, corresponding to dx on the x axis. Where f dx gives the length of an interval (it equals b - a), 5 ds is the length of the curve. EXAMPLE 1 Flight from Atlanta to Los Angeles on a straight line and a semicircle. According to Delta Airlines, the distance straight west is 2000 miles. Atlanta is at (1000,O) and Los Angeles is at (-1000, O), with the origin halfway between. The semicircle route C has radius 1000. This is not a great circle route. It is more of a "flat circle," which goes north past Chicago. No plane could fly it (it probably goes into space). The equation for the semicircle is x2 + y2 = 10002. Parametrically this path is x = 1000 cos t, y = 1000 sin t. For a line integral the parameter is better. The plane leaves Atlanta at t = 0 and reaches L.A. at t = n, more than three hours later. On the straight 2000-mile path, Delta could almost do it. Around the semicircle C, the distance is lOOOn miles and the speed has to be 1000 miles per hour. Remember that speed is distance ds divided by time dt: dsldt = ,/(dx dt) + (dyldt)' = l000,/(- sin t)2 + (cos t)2 = 1000. (1) The tangent vector to C is proportional to (dxldt, dyldt) = (-1000 sin t, 1000 cos t). But T is a unit vector, so we divide by 1000-which is the speed. Suppose the wind blows due east with force F = Mi. The components are M and zero. Foir M =constant, compute the dot product F *T and the work -2000 M: FwT= Mi*(-sin t i+cos tj)= M(-sin t)+O(cos t)= - M sin t 15 Vector Calculus Work is force times distance moved. It is negative, because the wind acts against the movement. You may point out that the work could have been found more simplygo 2000 miles and multiply by - M. I would object that this straight route is a dzrerent path. But you claim that the path doesn't matter-the work of the wind is -2000M on every path. I concede that this time you are right (but not always). Most line integrals depend on the path. Those that don't are crucially important. For a gradient field, we only need to know the starting point P and the finish Q. 158 When F is the gradient of a potential function f (x, y), the work J, F T ds depends only on the endpoints P and Q. The work is the change in if F = afpx i + af/ay j then F T ds = f (Q) -f(P). When F = Mi, its components M and zero are the partial derivatives off = Mx. To compute the line integral, just evaluate f at the endpoints. Atlanta has x = 1000, Los Angeles has x = - 1000, and the potential function f = Mx is like an antiderivative: work = f (Q) - f (P) = M(- 1000) - M(1000) = - 2000M. (3) LAX LAX - 1000 , - 1000 1000 J F . Tdr = - 2000M depends on path Fig. 15.5 Force Mi, work -2000M on all paths. Force Myi, no work on straight path. May I give a rough explanation of the work integral 5 F T ds? It becomes clearer when the small movement Tds is written as dx i + dy j. The work is the dot product with F: The infinitesimal work is df: The total work is 5 df= f(Q) - f (P). This is the Fundamental Theorem for a line integral. Only one warning: When F is not the gradient of any f (Example 2), the Theorem does not apply. EXAMPLE 2 Fly these paths against the non-constant force field F = Myi. Compute the work. There is no force on the straight path

Notes:

where y = 0. Along the x axis the wind does no work. But the semicircle goes up where y = 1000 sin t and the wind is strong: F*T=(Myi)*(-sinti+costj)= -My sin t= - lOOOM sin2t This work is enormous (and unrealistic). But the calculations make an important point-everything is converted to the parameter t. The second point is that F = Myi is not a gradient field. First reason: The work was zero on the straight path and 15.2 Line Integrals nonzero on the semicircle. Second reason: No function has df/ dx = My and df /dy = 0. (The first makes f depend on y and the second forbids it. This F is called a shear force.) Without a potential we cannot substitute P and Q-and the work depends on the path. THE DEFINITION OF LINE INTEGRALS We go back to the start, to define F T ds. We can think of F T as a function g(x, y) along the path, and define its integral as a limit of sums: N IC g(, y)ds = limit of xi, yi)Asi as (As),,,., -i 0. i= 1 (5) The points (xi, y,) lie on the curve C. The last point Q is (x,, y,); the first point P is (xo, yo). The step Asi is the distance to (xi, yi) from the previous point. As the steps get small (As -, 0) the straight pieces follow the curve. Exactly as in Section 8.2, the special case g = 1 gives the arc length. As long as g(x, y) is piecewise continuous (jumps allowed) and the path is piecewise smooth (corners allowed), the limit exists and defines the line integral. When g is the density of a wire, the line integral is the total mass. When g is F T, the integral is the work. But nobody does the calculation by formula (5). We now introduce a parameter t-which could be the time, or the arc length s, or the distance x along the base. The diflerential ds becomes (ds/dt)dt. Everything changes over to t: The curve starts when t = a, runs through the points (x(t), y(t)), and ends when t = b. The square root in the integral is the speed dsldt. In three dimensions the points on C are (x(t), y(t), z(t)) and (dz/dt)l is in the square root. EXAMPLE 3 The points on a coil spring are (x, y, z) = (cos t, sin t, t). Find the mass of two complete turns (from t = 0 to t = 44 if the density is p = 4. Solution The key is (d /dt) + (d /dt) + (d ldt) = sin2t + cos2t + 1 = 2. Thus dsldt = fi. To find the mass, integrate the mass per unit length which is g = p = 4: That is a line integral in three-dimensional space. It shows how to introduce t. But it misses the main point of this section, because it contains no vector field F. This section is about work, not just mass. DIFFERENT FORMS OF THE WORK INTEGRAL The work integral IF T ds can be written in a better way. The force is F = Mi + Nj. A small step along the curve is dx i + dy j. Work is force times distance, but it is only the force component along the path that counts. The dot product F -Tds finds that component automatically. 15 Vector Calculus I 15C The vector to a point on C is R = xi + yj. Then dR = Tds = dx i + dy j: I Along a space curve the work is jF* ds=f *d =j dx+ d + dz. The product M dx is (force in x direction)(movement in x direction). This is zero if either factor is zero. When the only force is gravity, pushing a piano takes no work. It is friction that hurts. Carrying the piano up the stairs brings in Pdz, and the total work is the piano weight P times the change in z. To connect the new I F dR with the old IF *T ds, remember the tangent vector T. It is dRlds. herefoie Tds is dR. The best for computations is dR, because the

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unit vector T has a division by dsldt = ,/(d /dt) + (d ldt) . Later we multiply by this square root, in converting ds to (dsldtjdt. It makes no sense to compute the square root, divide by it, and then multiply by it. That is avoided in the improved form M x+N Y. EXAMPLE 4 Vector field F = - yi + xj, path from (1,O) to (0, 1): Find the work. Note 1 This F is the spin field S. It goes around the origin, while R = xi + yj goes outward. Their dot product is F R = - yx + xy = 0. This does not mean that F dR = 0. The force is perpendicular to R, but not to the change in R. The work to move from (I, 0) to (0, I), x axis to y axis, is not zero. Note 2 We have not described the path C. That must be done. The spin field is not a gradient field, and the work along a straight line does not equal the work on a quarter-circle: straight line x = 1 - t, y = t quarter-circle x = cos t, y = sin t. Calculation of work Change F dR = M dx + N dy to the parameter t: Straight line: - y dx + x dy = - t(-dt) + (1 - t)dt = 1 l lo1 S 7T Quarter-circle: - y dx + x dy = -sin t(- sin t dt) + cos t(cos t dt) = -. 2 General method The path is given by x(t) and y(t). Substitute those into M(x, y) and N(x, y)-then F is a function of t. Also find dxldt and dyldt. Integrate M dxldt + N dyldt from the starting time t to the finish. I work 7[: / 2 no work ' work F.dR = 1 Fig.15.6 Threepath for F dR= - dx+.udy=l,n/2,0. 15.2 Llne Integrals For practice, take the path down the x axis to the origin (x = 1 - t, y = 0). Then go up the y axis (x = 0, y = t - 1). The starting time at (1,O) is t = 0. The turning time at the origin is t = 1. The finishing time at (0, 1) is t = 2. The integral has two parts because this new path has two parts: Bent path: J-ydx+xdy=O+O (y=O on one part, then x=O). Note 3 The answer depended on the path, for this spin field F = S. The answer did not depend on the choice of parameter. If we follow the same path at a different speed, the work is the same. We can choose another parameter 2, since (ds/dt)dt and (ds/dz)dz both equal ds. Traveling twice as fast on the straight path (x = 1 - 22, y = 22) we finish at t = 4 instead of t = 1. The work is still 1: CONSERVNION OF TOTAL ENERGY (KINETIC + POTENTIAL) When a force field does work on a mass m, it normally gives that mass a new velocity. Newton's Law is F =ma = mdvldt. (It is a vector law. Why write out three components?) The work F dR is J (m dvldt) (v dt) = *mv v]: = mv(Q)12 - mlv(P)12. The work equals the change in the kinetic energy 4mlv12. But for a gradient field the work is also the change in potential-with a minus sign from physics: Comparing (8) with (9), the combination m1vl2 + f is the same at P and Q. The total energy, kinetic plus potential, is conserved. INDEPENDENCE OF PATH: GRADIENT FIELDS The work of the spin field S depends on the path. Example 4 took three pathsstraight line, quarter-circle, bent line. The work was 1, 42, and 0, different on each path. This happens for more than 99.99the most important fields. Mathematics and physics concentrate on very special fields-for which the work depends only on the endpoints. We now explain what happens, when the integral is independent of the path. Suppose you integrate from P to Q on one path, and back to P on another path. Combined, that is a closed path from P to P (Figure 15.7). But a backward integral is the negative of a forward

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integral, since dR switches sign. If the integrals from P to Q are equal, the integral around the closed path is zero: closed path 1 back path 2 path 1 path 2 The circle on the first integral indicates a closed path. Later we will drop the P's. Not all closed path integrals are zero! For most fields F, different paths yield different work. For "conservative" fields, all paths yield the same work. Then zero 15 Vector Calculus work around a closed path conserves energy. The big question is: How to decide which fields are conservative, without trying all paths? Here is the crucial information about conservative fields, in a plane region R with no holes: 15D F = M(x, y)i + N(x, y)j is a conservative field if it has these properties: A. The work J F dR around every closed path is zero. B. The work F d R depends only on P and Q, not on the path. C. F is a gradient field: M = df/ax and N = df/dy for some potential f(x, y). D. The components satisfy dM/ay = (3Nldx. A field with one of these properties has them all. D is the quick test. These statements A-D bring everything together for conservative fields (alias gradient fields). A closed path goes one way to Q and back the other way to P. The work cancels, and statements A and B are equivalent. We now connect them to C. Note: Test D says that the "curl" of F is zero. That can wait for Green's Theorem in the next section-the full discussion of the curl comes in 15.6. First, a gradient field F = grad f is conservative. The work is f (Q) - f (P), by the fundamental theorem for line integrals. It depends only on the endpoints and not the path. Therefore statement C leads back to B. Our job is in the other direction, to show that conservative fields Mi + Nj are gradients. Assume that the work integral depends only on the endpoints. We must construct a potentialf, so that F is its gradient. In other words, dfldx must be M and dfldy must be N. Fix the point P. Define f (Q) as the work to reach Q. Then F equals grad Check the reasoning. At the starting point P, f is zero. At every other point Q, f is the work J M dx + N dy to reach that point. Allpathshsfom P to Q give the same f(Q), because the field is assumed conservative. After two examples we prove that grad f agrees with F-the construction succeeds. back path 2 - Fig. 15.7 Conservative fields: F dR = 0 and j@ F d R = f (Q) - f (P). Here f (P) = 0. EXAMPLE 5 Find f (x, y) when F = Mi + Nj = 2xyi + x2j. We want (: f /ax = 2xy and dfldy = x2. Solution 1 Choose P = (0,O). Integrate M dx + N dy along to (x, 0) and up to (x, y): (x. 0) 0, Y) 2xy dx = 0 (since y = 0) x2dY = x2y (which is f). S (0.0) S (x, 0) Certainly f = x2y meets the requirements: f, = 2xy and f, = x2. Thus F = gradf Note that dy = 0 in the first integral (on the x axis). Then dx = 0 in the second integral (X is fixed). The integrals add to f = x2y. 15.2 Line Integrals Solution 2 Integrate 2xydx + x2dy on the straight line (xt, yt) from t = 0 to t = 1: Iol 2(xt)(yt)(x dt) + (t) (y dt)= So13x2yt2dt= x2yt3]: = x2y. Most authors use Solution 1. I use Solution 2. Most students use Solution 3: Solution 3 Directly solve df/dx = M = 2xy and then fix up dfldy = N = x2: af/dx = 2xy gives f = x2y (plus any function of y). In this example x2y already has the correct derivative dfldy = x2. No additional function of y is necessary. When we integrate with respect to x, the constant of

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integration (usually C) becomes a function C(y). You will get practice in finding f. This is only possible for conservative fields! I tested M = 2xy and N = x2 in advance (using D) to be sure that dM/dy = dN/dx. EXAMPLE 6 Look for f (x, y) when Mi + Nj is the spin field -yi + xj. Attempted solution 1 Integrate -y dx + x dy from (0,O)to (x, 0) to (x, y): I(x, 0) -ydx=O and I (x. Y) x dy = xy (which seems like f). (0,O) (x. 0) Attempted solution 2 Integrate -y dx + x dy on the line (xt, yt) from t = 0 to 1 : So1-(yt)(x dt) + (xt)(y dt) = 0 (a different f, also wrong). Aitempted solution 3 Directly solve dfldx = -y and try to fix up af/dy = x: af/dx = -y gives f = -xy (plus any function C(y)). The y derivative of this f is -x + dC/dy. That does not agree with the required dfldy = x. Conclusion: The spin field -yi + xj is not conservative. There is no f. Test D gives dM/dy = -1 and dN/dx = + 1. To finish this section, we move from examples to a proof. The potential f(Q) is defined as the work to reach Q. We must show that its partial derivatives are M and N. This seems reasonable from the formula f (Q) = I M dx + N dy, but we have to think it through. Remember statement A, that all paths give the same f(Q). Take a path that goes from P to the left of Q. It comes in to Q on a line y = constant (so dy = 0). As the path reaches Q, we are only integrating M dx. The derivative of this integral, at Q, is df/ax = M. That is the Fundamental Theorem of Calculus. To show that af/ay= N, take a different path. Go from P to a point below Q. The path comes up to Q on a vertical line (so dx = 0). Near Q we are only integrating N dy, so i?f/dy= N. The requirement that the region must have no holes will be critical for test D. EXAMPLE 7 Find f (x, y) = x dx + y dy. Test D is passed: aN/ax= 0= dM/dy. Solution 1 j:",: x dx = +x2is added to j:;:; y dy = fy2. Solution 2 1; (xt)(x dt) + (yt)(y dt) = 1; (x2+ y2)t dt =f (x2+ y2). Solution 3 afjax = x gives f = +x2+ C(y).Then af/dy = y needs C(y)= :y2. 562 15 Vector Calculus 15.2 EXERCISES Read-through questions Work is the a of F dR. Here F is the b and R is the c . The d product finds the component of in the direction of movement dR = dxi + dyj. The straight path (x, y) = (t, 2t) goes from f at t = 0 to g at t = 1 withdR=dti+ h .TheworkofF=3i+jisjF=dR= j i dt= i . Another form of d R is T ds, where T is the k vector to the path and ds = ,/T. For the path (t, 2t), the unit vector Tis m andds= n dt.ForF=3i+j,F*Tdsisstill 0 dt. This F is the gradient off = P . The change in f= 3x +y from (0,O) to (1,2) is q . When F = gradf, the dot product F dR is (af/dx)dx + r = df: The work integral from P to Q is j df = s . In this case the work depends on the t but not on the u . Around a closed path the work is v . The field is called w . F = (1 + y)i + xj is the gradient off = x . The work from (0,O) to (1,2) is Y , the change in potential. For the spin field S = 2 , the work (does)(does not) depend on the path. The path (x, y) = (3 cos t, 3 sin t) is a circle with SgdR = A . The work is B around the complete circle. Formally jg(x, y)ds is the limit of the sum c. The four equivalent properties of a conservative field F = Mi+ Nj are A: D , B: E , C: F , and D: G . Test D is (passed)(not passed) by F = (y + 1)i + xj. The work IF dR around the circle (cos t, sin t) is H . The work on the upper semicircle equals the work on I . This field is the gradient off = J , so the work to (- 1,0) is K . Compute the line

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integrals in 1-6. jcds and jcdy: x = t, y = 2t, 0 6 t < 1. fcxds and jcxyds: x=cost, y=sint, O<t<n/2. S, xy ds: bent line from (0,O) to (1, 1) to (1,O). 1, y dx - x dy: any square path, sides of length 3. fc dx and jc y dx: any closed circle of radius 3. Jc (dsldt) dt: any path of length 5. Does if xy dy equal f xy2]:? Does jfx dx equal fx2]:? Does (jc d) = (IC d) + (fC dy)l? Does jc (d) make sense? 11-16 find the work in moving from (1,O) to (0,l). When F is conservative, construct f: choose your own path when F is not conservative. 11 F=i+yj 12 F=yi+j 17 For which powers n is S/rn a gradient by test D? 18 For which powers n is R/rn a gradient by test D? 19 A wire hoop around a vertical circle x2 + z2 = a2 has density p = a + z. Find its mass M = pds. 20 A wire of constant density p lies on the semicircle x2 + Y2 = a2, y 3 0. Find its mass M and also its moment Mx = 1 py ds. Where is its center of mass 2 = My/M, j = Mx/ M? 21 If the density around the circle x2 + y2 = a2 is p = x2, what is the mass and where is the center of mass? 22 Find F dR along the space curve x = t, y = t2, z = t3, O<t<l. (a) F = grad (xy + xz) (b) F = yi - xj + zk 23 (a) Find the unit tangent vector T and the speed dsldt along the path R = 2t i + t2 j. (b) For F = 3xi + 4j, find F T ds using (a) and F dR directly. (c) What is the work from (2, 1) to (4,4)? 24 If M(x, y, z)i + N(x, y, z)j is the gradient of f(x, y, z), show that none of these functions can depend on z. 25 Find all gradient fields of the form M(y)i + N(x)j. 26 Compute the work W(x, y) = j M dx + N dy on the straight line path (xt, yt) from t = 0 to t = 1. Test to see if aW/ ax = M and aWpy = N. (a) M = y3, N = 3xy2 (b) M = x3, N = 3yx2 (c)M=x/y,N=y/x (d)M=ex+Y,N=e”+Y 27 Find a field F whose work around the unit square (y = 0 then x = 1 then y = 1 then x = 0) equals 4. 28 Find a nonconservative F whose work around the unit circle x2 + y2 = 1 is zero. In 29-34 compute 1 F dR along the straight line R = ti + tj and the parabola R = ti + t2j, from (0,O) to (1,l). When F is a gradient field, use its potential f (x, y). 29 F=i-2j 30 F = x2j 33 F=yi-xj 34 F = (xi + yj)/(x2 + y2 + 1) 35 For which numbers a and b is F = axyi + (x2 + by)j a gradient field? 36 Compute j - y dx + x dy from (1,O) to (0,l) on the line x = 1 - t2, y = t2 and the quarter-circle x = cos 2t, y = sin 2t. Example 4 found 1 and n/2 with different parameters.

Apply the test Nx = My to 37-42. Find f when test D is passed. 43 Around the unit circle find 4 ds and dx and 8 xds. 44 True or false, with reason: (a) When F = yi the line integral lFedR along a curve xi + yj grad xy from P to Q equals the usual area-under the curve. 39 $F = \frac{xi+yj}{|xi+yj|}$ 40 $F = \frac{gradxy}{|gradxy|}$ (b) That line integral depends only on P and Q, not on the curve. 41 F=R+S 42 F =(ax + by)i + (cx + dy)j (c) That line integral around the unit circle equals n.

In previous chapters we have explored a relationship between vectors and integration. Our most tangible result: if $\vec{v}(t)$ is a vector–valued velocity function of a moving object, then integrating $\vec{v}(t)$ from $t = a$ to $t = b$ gives the displacement of that object over that time interval.

This chapter explores completely different relationships between vectors and integration. These relationships will enable us to compute the work done by a magnetic field in moving an object along a path and how much air moves

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through an oddly–shaped screen in space, among other things.

Our upcoming work in integration will benefit from a review. We are not concerned here with techniques of integration, but rather what an integral "does" and how that relates to the notation we use to describe it.

Integration Review

Recall from Section 12.1 that when R is a region in the x-y plane, $\int\int_R dA$ gives the area of the region R. The integral symbols are "elongated esses" meaning "sum" and dA represents "a small amount of area." Taken together, $\int\int_R dA$ means "sum up, over R, small amounts of area." This sum then gives the total area of R. We use two integral symbols since R is a two–dimensional region.

Now let $z = f(x, y)$ represent a surface. The double integral $\int\int_R f(x, y)\ dA$ means "sum up, over R, function values (heights) given by f times small amounts of area." Since "height $\times$ area = volume," we are summing small amounts of volume over R, giving the total signed volume under the surface $z = f(x, y)$ and above the x-y plane.

This notation does not directly inform us *how* to evaluate the double integrals to find an area or a volume. With additional work, we recognize that a small amount of area dA can be measured as the area of a small rectangle, with one side length a small change in x and the other side length a small change in y. That is, $dA = dx\,dy$ or $dA = dy\,dx$. We could also compute a small amount of area by thinking in terms of polar coordinates, where $dA = r\,dr\,d\theta$. These understandings lead us to the iterated integrals we used in Chapter 12.

Let us back our review up farther. Note that $\int_1^3\ dx = x\big|_1^3 = 3 - 1 = 2$. We have simply measured the length of the interval $[1, 3]$. We could rewrite the above integral using syntax similar to the double integral syntax above:

$$\int_1^3\ dx = \int_I dx, \qquad \text{where } I = [1, 3].$$

We interpret "$\int_I dx$" as meaning "sum up, over the interval I, small changes in x." A change in x is a length along the x-axis, so we are adding up along I small lengths, giving the total length of I.

We could also write $\int_1^3 f(x)\ dx$ as $\int_I f(x)\ dx$, interpreted as "sum up, over I, heights given by $y = f(x)$ times small changes in x." Since "height$\times$length = area," we are summing up areas and finding the total signed area between $y = f(x)$ and the x-axis.

This method of referring to the process of integration can be very powerful. It is the core of our notion of the Riemann Sum. When faced with a quantity to compute, if one can think of a way to approximate its value through a sum, the one is well on their way to constructing an integral (or, double or triple in-

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tegral) that computes the desired quantity. We will demonstrate this process throughout this chapter, starting with the next section.

13.3 Introduction to Line Integrals

We first used integration to find "area under a curve." In this section, we learn to do this (again), but in a different context.

Consider the surface and curve shown in Figure 13.10(a). The surface is given by $f(x, y) = 1 - \cos(x)\sin(y)$. The dashed curve lies in the x-y plane and is the familiar $y = x^2$ parabola from $-1 \leq x \leq 1$; we'll call this curve C. The curve drawn with a solid line in the graph is the curve in space that lies on our surface with x and y values that lie on C.

The question we want to answer is this: what is the area that lies below curve drawn with the solid line? In other words, what is the area above C and under the the surface f? This area is shown in Figure 13.10(b).

We suspect the answer can be found using an integral, but before trying to figure out what that integral is, let us first try to approximate its value.

In Figure 13.10(c), 4 rectangles have been drawn over the curve C. The bottom corners of each rectangle lies on C, and each rectangle has a height given by the function $f(x, y)$ for some (x, y) pair along C between the rectangle's bottom corners.

As we know how to find the area of each rectangle, we are able to approximate the area above C and under f. Clearly, our approximation will be *an approximation*. The heights of the rectangles do not match exactly with the surface f, nor does the base of each rectangle follow perfectly the path of C.

In typical calculus fashion, our approximation can be improved by using more rectangles. The sum of the areas of these rectangles gives an approximate value of the true area above C and under f. As the area of each rectangle is "height$\times$width", we assert that the

$$\text{area above } C \approx \sum(\text{heights} \times \text{widths}).$$

When first learning of the integral, and approximating areas with "heights$\times$ widths", the width was a small change in x: dx. That will not suffice in this context. Rather, each width of a rectangle is actually approximating the arc length of a small portion of C. In Section 10.5, we used s to represent the arc–length parameter of a curve. A small amount of arc length will thus be represented by ds.

The height of each rectangle will be determined in some way by the surface f. If we parametrize C by s, an s-value corresponds to an (x, y) pair that lies on the parabola C. Since f is a function of x and y, and x and y are functions of s, we

Figure 13.1: Finding area under a curve in space.

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can say that f is a function of s. Given a value s, we can compute $f(s)$ and find a height. Thus

$$\begin{aligned}\text{area under } f \text{ and above } C &\approx \sum(\text{heights} \times \text{widths});\\ \text{area under } f \text{ and above } C &= \lim_{||\Delta s|| \to 0} \sum f(c_i)\Delta s_i\\ &= \int_C f(s)\, ds. \qquad (13.1)\end{aligned}$$

Here we have introduce a new notation, the integral symbol with a subscript of C. It is reminiscent of our usage of $\int\int_R$. Using the train of thought found in the Integration Review preceding this section, we interpret "$\int_C f(s)\, ds$" as meaning "sum up, along a curve C, function values $f(s) \times$small arc lengths." It is understood here that s represents the arc–length parameter.

All this leads us to a definition. The integral found in Equation 13.4 is called a **line integral**. We formally define it below, but note that the definition is very abstract. On one hand, one is apt to say "the defintion makes sense," while on the other, one is equally apt to say "but I don't know what I'm supposed to do with this definition." We'll address that after the definition, and actually find an answer to the area problem we posed at the beginning of this section.

Definition 62 Line Integral

Let C be a curve continuously parametrized by s, the arc–length parameter, and let f be a continuous function of s. A **line integral** is an integral of the form

$$\int_C f(s)\, ds = \lim_{||\Delta s|| \to 0} \sum_{i=1}^{n} f(c_i)\Delta s_i,$$

where $s_1 < s_2 < \ldots < s_n$ is any partition of the s-interval over which C is defined, c_i is any value in the ith subinterval, Δ_i is the width of the ith subinterval, and $||\Delta s||$ is the length of the longest subinterval in the partition.

The definition of the line integral does not specify whether C is a curve in the plane or space (or hyperspace), as the definition holds regardless. For now, we'll assume C lies in the x-y plane.

This definition of the line integral doesn't really say anything new. If C is a curve and s is the arc–length parameter of C on $a \leq s \leq b$, then

$$\int_C f(s)\, ds = \int_a^b f(s)\, ds.$$

Notes:

The real difference with this integral from the standard "$\int_a^b f(x)\,dx$" we used in the past is that of context. Our previous integrals naturally summed up values over an interval on the x-axis, whereas now we are summing up values over a curve. If we can parametrize the curve with the arc–length parameter, we can evaluate the line integral just as before.

The trouble here is that we have generally avoided direct use of the arc–length parameter s in the past as it is usually difficult to use. We continue that methodology here.

Given a curve C, find a parametrization of C: $x = g(t)$ and $y = h(t)$, for continuous functions g and h, where $a \leq t \leq b$. We can represent this parametrization with a vector–valued function, $\vec{r}(t) = \langle g(t), h(t)\rangle$.

In Section 10.5, we defined the arc–length parameter in Equation 10.1 as

$$s(t) = \int_0^t \|\,\vec{r}\,'(u)\,\|\,du.$$

By the Fundamental Theorem of Calculus, $ds = \|\,\vec{r}\,'(t)\,\|\,dt$. We can substitute the right hand side of this equation for ds in the line integral definition.

We can view f as being a function of x and y since it is a function of s. Thus $f(s) = f(x, y) = f\big(g(t), h(t)\big)$. This gives us a concrete way to evaluate a line integral:

$$\int_C f(s)\,ds = \int_a^b f\big(g(t), h(t)\big)\|\,\vec{r}\,'(t)\,\|\,dt.$$

We write this as a Key Idea, along with its three–dimensional analogue, followed by an example where we finally evaluate an integral and find an area.

Key Idea 13 Evaluating a Line Integral, Part I

- Let C be a curve parametrized by $\vec{r}(t) = \langle g(t), h(t)\rangle$, $a \leq t \leq b$, where g and h are continuously differentiable, and let $z = f(x, y)$. Then
$$\int_C f(s)\,ds = \int_a^b f\big(g(t), h(t)\big)\|\,\vec{r}\,'(t)\,\|\,dt.$$
- Let C be a curve parametrized by $\vec{r}(t) = \langle g(t), h(t), k(t)\rangle$, $a \leq t \leq b$, where g, h and k are continuously differentiable, and let $w = f(x, y, z)$. Then
$$\int_C f(s)\,ds = \int_a^b f\big(g(t), h(t), k(t)\big)\|\,\vec{r}\,'(t)\,\|\,dt.$$

Notes:

To be clear, the first point of Key Idea 16 can be used to find the area under a surface $z = f(x, y)$ and above a curve C.

Let's finally do an example where we actually compute an area.

Example 170 Evaluating a line integral.

Find the area under the surface $f(x, y) = \cos(x) + \sin(y) + 2$ over the curve C, which is the segment of the line $y = 2x + 1$ on $-1 \leq x \leq 1$, as shown in Figure 11.

Solution Our first step is to represent C with a vector–valued function. Since C is a simple line, and we have a explicit relationship between y and x (namely, that y is $2x+1$), we can let $x = t$, $y = 2t+1$, and write $\vec{r}(t) = \langle t, 2t+1\rangle$ for $-1 \leq t \leq 1$.

We find the values of f over C as $f(x, y) = f(t, 2t+1) = \cos(t)+\sin(2t+1)+2$. We also need $\| \vec{r}\,'(t) \|$; with $\vec{r}\,'(t) = \langle 1, 2\rangle$, we have $\| \vec{r}\,'(t) \| = \sqrt{5}$. Thus $ds = \sqrt{5}\, dt$.

The area we seek is

$$\begin{aligned}\int_C f(s)\, ds &= \int_{-1}^{1} \big(\cos(t) + \sin(2t+1) + 2\big)\sqrt{5}\, dt \\ &= \sqrt{5}\Big(\sin(t) - \frac{1}{2}\cos(2t+1) + 2t\Big)\Big|_{-1}^{1} \\ &\approx 14.418 \text{ units}^2.\end{aligned}$$

filler text

filler

filler

In previous chapters we have explored a relationship between vectors and integration. Our most tangible result: if $\vec{v}(t)$ is a vector–valued velocity function of a moving object, then integrating $\vec{v}(t)$ from $t = a$ to $t = b$ gives the displacement of that object over that time interval.

This chapter explores completely different relationships between vectors and integration. These relationships will enable us to compute the work done by a magnetic field in moving an object along a path and how much air moves through an oddly–shaped screen in space, among other things.

Our upcoming work in integration will benefit from a review. We are not concerned here with techniques of integration, but rather what an integral "does" and how that relates to the notation we use to describe it.

(a)

(b)

Figure 13.3: Finding area under a curve in Example 173.

Notes:

Integration Review

Recall from Section 12.1 that when R is a region in the x-y plane, $\int\int_R dA$ gives the area of the region R. The integral symbols are "elongated esses" meaning "sum" and dA represents "a small amount of area." Taken together, $\int\int_R dA$ means "sum up, over R, small amounts of area." This sum then gives the total area of R. We use two integral symbols since R is a two–dimensional region.

Now let $z = f(x, y)$ represent a surface. The double integral $\int\int_R f(x, y)\ dA$ means "sum up, over R, function values (heights) given by f times small amounts of area." Since "height $\times$ area = volume," we are summing small amounts of volume over R, giving the total signed volume under the surface $z = f(x, y)$ and above the x-y plane.

This notation does not directly inform us *how* to evaluate the double integrals to find an area or a volume. With additional work, we recognize that a small amount of area dA can be measured as the area of a small rectangle, with one side length a small change in x and the other side length a small change in y. That is, $dA = dx\,dy$ or $dA = dy\,dx$. We could also compute a small amount of area by thinking in terms of polar coordinates, where $dA = r\,dr\,d\theta$. These understandings lead us to the iterated integrals we used in Chapter 12.

Let us back our review up farther. Note that $\int_1^3\ dx = x\big|_1^3 = 3 - 1 = 2$. We have simply measured the length of the interval $[1, 3]$. We could rewrite the above integral using syntax similar to the double integral syntax above:

$$\int_1^3\ dx = \int_I dx, \qquad \text{where } I = [1, 3].$$

We interpret "$\int_I dx$" as meaning "sum up, over the interval I, small changes in x." A change in x is a length along the x-axis, so we are adding up along I small lengths, giving the total length of I.

We could also write $\int_1^3 f(x)\ dx$ as $\int_I f(x)\ dx$, interpreted as "sum up, over I, heights given by $y = f(x)$ times small changes in x." Since "height$\times$length = area," we are summing up areas and finding the total signed area between $y = f(x)$ and the x-axis.

This method of referring to the process of integration can be very powerful. It is the core of our notion of the Riemann Sum. When faced with a quantity to compute, if one can think of a way to approximate its value through a sum, the one is well on their way to constructing an integral (or, double or triple integral) that computes the desired quantity. We will demonstrate this process throughout this chapter, starting with the next section.

Notes:

13.4 Introduction to Line Integrals

We first used integration to find "area under a curve." In this section, we learn to do this (again), but in a different context.

Consider the surface and curve shown in Figure 13.10(a). The surface is given by $f(x, y) = 1 - \cos(x)\sin(y)$. The dashed curve lies in the x-y plane and is the familiar $y = x^2$ parabola from $-1 \leq x \leq 1$; we'll call this curve C. The curve drawn with a solid line in the graph is the curve in space that lies on our surface with x and y values that lie on C.

The question we want to answer is this: what is the area that lies below curve drawn with the solid line? In other words, what is the area above C and under the the surface f? This area is shown in Figure 13.10(b).

We suspect the answer can be found using an integral, but before trying to figure out what that integral is, let us first try to approximate its value.

In Figure 13.10(c), 4 rectangles have been drawn over the curve C. The bottom corners of each rectangle lies on C, and each rectangle has a height given by the function $f(x, y)$ for some (x, y) pair along C between the rectangle's bottom corners.

As we know how to find the area of each rectangle, we are able to approximate the area above C and under f. Clearly, our approximation will be *an approximation*. The heights of the rectangles do not match exactly with the surface f, nor does the base of each rectangle follow perfectly the path of C.

In typical calculus fashion, our approximation can be improved by using more rectangles. The sum of the areas of these rectangles gives an approximate value of the true area above C and under f. As the area of each rectangle is "height$\times$width", we assert that the

$$\text{area above } C \approx \sum(\text{heights} \times \text{widths}).$$

When first learning of the integral, and approximating areas with "heights$\times$ widths", the width was a small change in x: dx. That will not suffice in this context. Rather, each width of a rectangle is actually approximating the arc length of a small portion of C. In Section 10.5, we used s to represent the arc–length parameter of a curve. A small amount of arc length will thus be represented by ds.

The height of each rectangle will be determined in some way by the surface f. If we parametrize C by s, an s-value corresponds to an (x, y) pair that lies on the parabola C. Since f is a function of x and y, and x and y are functions of s, we can say that f is a function of s. Given a value s, we can compute $f(s)$ and find a

Figure 13.4: Finding area under a curve in space.

Notes:

height. Thus

$$\begin{aligned}\text{area under } f \text{ and above } C &\approx \sum(\text{heights} \times \text{widths});\\ \text{area under } f \text{ and above } C &= \lim_{||\Delta s||\to 0} \sum f(c_i)\Delta s_i\\ &= \int_C f(s)\ ds. \qquad (13.2)\end{aligned}$$

Here we have introduce a new notation, the integral symbol with a subscript of C. It is reminiscent of our usage of $\int\int_R$. Using the train of thought found in the Integration Review preceding this section, we interpret "$\int_C f(s)\ ds$" as meaning "sum up, along a curve C, function values $f(s)\times$small arc lengths." It is understood here that s represents the arc–length parameter.

All this leads us to a definition. The integral found in Equation 13.4 is called a **line integral**. We formally define it below, but note that the definition is very abstract. On one hand, one is apt to say "the defintion makes sense," while on the other, one is equally apt to say "but I don't know what I'm supposed to do with this definition." We'll address that after the definition, and actually find an answer to the area problem we posed at the beginning of this section.

Definition 63 Line Integral

Let C be a curve continuously parametrized by s, the arc–length parameter, and let f be a continuous function of s. A **line integral** is an integral of the form

$$\int_C f(s)\ ds = \lim_{||\Delta s||\to 0} \sum_{i=1}^{n} f(c_i)\Delta s_i,$$

where $s_1 < s_2 < \ldots < s_n$ is any partition of the s-interval over which C is defined, c_i is any value in the ith subinterval, Δ_i is the width of the ith subinterval, and $||\Delta s||$ is the length of the longest subinterval in the partition.

The definition of the line integral does not specify whether C is a curve in the plane or space (or hyperspace), as the definition holds regardless. For now, we'll assume C lies in the x-y plane.

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$$\int_C f(s)\ ds = \int_a^b f(s)\ ds.$$

Notes:

The real difference with this integral from the standard "$\int_a^b f(x)\,dx$" we used in the past is that of context. Our previous integrals naturally summed up values over an interval on the x-axis, whereas now we are summing up values over a curve. If we can parametrize the curve with the arc–length parameter, we can evaluate the line integral just as before.

The trouble here is that we have generally avoided direct use of the arc–length parameter s in the past as it is usually difficult to use. We continue that methodology here.

Given a curve C, find a parametrization of C: $x = g(t)$ and $y = h(t)$, for continuous functions g and h, where $a \leq t \leq b$. We can represent this parametrization with a vector–valued function, $\vec{r}(t) = \langle g(t), h(t)\rangle$.

In Section 10.5, we defined the arc–length parameter in Equation 10.1 as

$$s(t) = \int_0^t \|\,\vec{r}\,'(u)\,\|\,du.$$

By the Fundamental Theorem of Calculus, $ds = \|\,\vec{r}\,'(t)\,\|\,dt$. We can substitute the right hand side of this equation for ds in the line integral definition.

We can view f as being a function of x and y since it is a function of s. Thus $f(s) = f(x, y) = f\big(g(t), h(t)\big)$. This gives us a concrete way to evaluate a line integral:

$$\int_C f(s)\,ds = \int_a^b f\big(g(t), h(t)\big)\|\,\vec{r}\,'(t)\,\|\,dt.$$

We write this as a Key Idea, along with its three–dimensional analogue, followed by an example where we finally evaluate an integral and find an area.

Key Idea 14 Evaluating a Line Integral, Part I

- Let C be a curve parametrized by $\vec{r}(t) = \langle g(t), h(t)\rangle$, $a \leq t \leq b$, where g and h are continuously differentiable, and let $z = f(x, y)$. Then
$$\int_C f(s)\,ds = \int_a^b f\big(g(t), h(t)\big)\|\,\vec{r}\,'(t)\,\|\,dt.$$
- Let C be a curve parametrized by $\vec{r}(t) = \langle g(t), h(t), k(t)\rangle$, $a \leq t \leq b$, where g, h and k are continuously differentiable, and let $w = f(x, y, z)$. Then
$$\int_C f(s)\,ds = \int_a^b f\big(g(t), h(t), k(t)\big)\|\,\vec{r}\,'(t)\,\|\,dt.$$

Notes:

To be clear, the first point of Key Idea 16 can be used to find the area under a surface $z = f(x, y)$ and above a curve C.

Let's finally do an example where we actually compute an area.

Example 171 Evaluating a line integral.

Find the area under the surface $f(x, y) = \cos(x) + \sin(y) + 2$ over the curve C, which is the segment of the line $y = 2x + 1$ on $-1 \leq x \leq 1$, as shown in Figure 11.

Solution Our first step is to represent C with a vector–valued function. Since C is a simple line, and we have a explicit relationship between y and x (namely, that y is $2x+1$), we can let $x = t$, $y = 2t+1$, and write $\vec{r}(t) = \langle t, 2t+1\rangle$ for $-1 \leq t \leq 1$.

We find the values of f over C as $f(x, y) = f(t, 2t+1) = \cos(t)+\sin(2t+1)+2$. We also need $\| \vec{r}\,'(t) \|$; with $\vec{r}\,'(t) = \langle 1, 2\rangle$, we have $\| \vec{r}\,'(t) \| = \sqrt{5}$. Thus $ds = \sqrt{5}\, dt$.

The area we seek is

$$\begin{aligned}\int_C f(s)\, ds &= \int_{-1}^{1} \big(\cos(t) + \sin(2t+1) + 2\big)\sqrt{5}\, dt \\ &= \sqrt{5}\Big(\sin(t) - \frac{1}{2}\cos(2t+1) + 2t\Big)\Big|_{-1}^{1} \\ &\approx 14.418 \text{ units}^2.\end{aligned}$$

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In previous chapters we have explored a relationship between vectors and integration. Our most tangible result: if $\vec{v}(t)$ is a vector–valued velocity function of a moving object, then integrating $\vec{v}(t)$ from $t = a$ to $t = b$ gives the displacement of that object over that time interval.

This chapter explores completely different relationships between vectors and integration. These relationships will enable us to compute the work done by a magnetic field in moving an object along a path and how much air moves through an oddly–shaped screen in space, among other things.

Our upcoming work in integration will benefit from a review. We are not concerned here with techniques of integration, but rather what an integral "does" and how that relates to the notation we use to describe it.

Figure 13.6: Finding area under a curve in Example 173.

Notes:

Integration Review

Recall from Section 12.1 that when R is a region in the x-y plane, $\iint_R dA$ gives the area of the region R. The integral symbols are "elongated esses" meaning "sum" and dA represents "a small amount of area." Taken together, $\iint_R dA$ means "sum up, over R, small amounts of area." This sum then gives the total area of R. We use two integral symbols since R is a two–dimensional region.

Now let $z = f(x, y)$ represent a surface. The double integral $\iint_R f(x, y)\ dA$ means "sum up, over R, function values (heights) given by f times small amounts of area." Since "height $\times$ area = volume," we are summing small amounts of volume over R, giving the total signed volume under the surface $z = f(x, y)$ and above the x-y plane.

This notation does not directly inform us *how* to evaluate the double integrals to find an area or a volume. With additional work, we recognize that a small amount of area dA can be measured as the area of a small rectangle, with one side length a small change in x and the other side length a small change in y. That is, $dA = dx\,dy$ or $dA = dy\,dx$. We could also compute a small amount of area by thinking in terms of polar coordinates, where $dA = r\,dr\,d\theta$. These understandings lead us to the iterated integrals we used in Chapter 12.

Let us back our review up farther. Note that $\int_1^3\ dx = x\big|_1^3 = 3 - 1 = 2$. We have simply measured the length of the interval $[1, 3]$. We could rewrite the above integral using syntax similar to the double integral syntax above:

$$\int_1^3\ dx = \int_I dx, \qquad \text{where } I = [1, 3].$$

We interpret "$\int_I dx$" as meaning "sum up, over the interval I, small changes in x." A change in x is a length along the x-axis, so we are adding up along I small lengths, giving the total length of I.

We could also write $\int_1^3 f(x)\ dx$ as $\int_I f(x)\ dx$, interpreted as "sum up, over I, heights given by $y = f(x)$ times small changes in x." Since "height$\times$length = area," we are summing up areas and finding the total signed area between $y = f(x)$ and the x-axis.

This method of referring to the process of integration can be very powerful. It is the core of our notion of the Riemann Sum. When faced with a quantity to compute, if one can think of a way to approximate its value through a sum, the one is well on their way to constructing an integral (or, double or triple integral) that computes the desired quantity. We will demonstrate this process throughout this chapter, starting with the next section.

Notes:

13.5 Introduction to Line Integrals

We first used integration to find "area under a curve." In this section, we learn to do this (again), but in a different context.

Consider the surface and curve shown in Figure 13.10(a). The surface is given by $f(x, y) = 1 - \cos(x)\sin(y)$. The dashed curve lies in the x-y plane and is the familiar $y = x^2$ parabola from $-1 \leq x \leq 1$; we'll call this curve C. The curve drawn with a solid line in the graph is the curve in space that lies on our surface with x and y values that lie on C.

The question we want to answer is this: what is the area that lies below curve drawn with the solid line? In other words, what is the area above C and under the the surface f? This area is shown in Figure 13.10(b).

We suspect the answer can be found using an integral, but before trying to figure out what that integral is, let us first try to approximate its value.

In Figure 13.10(c), 4 rectangles have been drawn over the curve C. The bottom corners of each rectangle lies on C, and each rectangle has a height given by the function $f(x, y)$ for some (x, y) pair along C between the rectangle's bottom corners.

As we know how to find the area of each rectangle, we are able to approximate the area above C and under f. Clearly, our approximation will be *an approximation*. The heights of the rectangles do not match exactly with the surface f, nor does the base of each rectangle follow perfectly the path of C.

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$$\text{area above } C \approx \sum(\text{heights} \times \text{widths}).$$

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The height of each rectangle will be determined in some way by the surface f. If we parametrize C by s, an s-value corresponds to an (x, y) pair that lies on the parabola C. Since f is a function of x and y, and x and y are functions of s, we can say that f is a function of s. Given a value s, we can compute $f(s)$ and find a

Figure 13.7: Finding area under a curve in space.

Notes:

height. Thus

$$\begin{aligned}\text{area under } f \text{ and above } C &\approx \sum(\text{heights} \times \text{widths});\\ \text{area under } f \text{ and above } C &= \lim_{||\Delta s|| \to 0} \sum f(c_i)\Delta s_i\\ &= \int_C f(s)\, ds. \qquad (13.3)\end{aligned}$$

Here we have introduce a new notation, the integral symbol with a subscript of C. It is reminiscent of our usage of $\int\int_R$. Using the train of thought found in the Integration Review preceding this section, we interpret "$\int_C f(s)\, ds$" as meaning "sum up, along a curve C, function values $f(s) \times$small arc lengths." It is understood here that s represents the arc–length parameter.

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Definition 64 Line Integral

Let C be a curve continuously parametrized by s, the arc–length parameter, and let f be a continuous function of s. A **line integral** is an integral of the form

$$\int_C f(s)\, ds = \lim_{||\Delta s|| \to 0} \sum_{i=1}^{n} f(c_i)\Delta s_i,$$

where $s_1 < s_2 < \ldots < s_n$ is any partition of the s-interval over which C is defined, c_i is any value in the ith subinterval, Δ_i is the width of the ith subinterval, and $||\Delta s||$ is the length of the longest subinterval in the partition.

The definition of the line integral does not specify whether C is a curve in the plane or space (or hyperspace), as the definition holds regardless. For now, we'll assume C lies in the x-y plane.

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$$\int_C f(s)\, ds = \int_a^b f(s)\, ds.$$

Notes:

The real difference with this integral from the standard "$\int_a^b f(x)\,dx$" we used in the past is that of context. Our previous integrals naturally summed up values over an interval on the x-axis, whereas now we are summing up values over a curve. If we can parametrize the curve with the arc–length parameter, we can evaluate the line integral just as before.

The trouble here is that we have generally avoided direct use of the arc–length parameter s in the past as it is usually difficult to use. We continue that methodology here.

Given a curve C, find a parametrization of C: $x = g(t)$ and $y = h(t)$, for continuous functions g and h, where $a \leq t \leq b$. We can represent this parametrization with a vector–valued function, $\vec{r}(t) = \langle g(t), h(t)\rangle$.

In Section 10.5, we defined the arc–length parameter in Equation 10.1 as

$$s(t) = \int_0^t \|\,\vec{r}\,'(u)\,\|\,du.$$

By the Fundamental Theorem of Calculus, $ds = \|\,\vec{r}\,'(t)\,\|\,dt$. We can substitute the right hand side of this equation for ds in the line integral definition.

We can view f as being a function of x and y since it is a function of s. Thus $f(s) = f(x, y) = f\big(g(t), h(t)\big)$. This gives us a concrete way to evaluate a line integral:

$$\int_C f(s)\,ds = \int_a^b f\big(g(t), h(t)\big)\|\,\vec{r}\,'(t)\,\|\,dt.$$

We write this as a Key Idea, along with its three–dimensional analogue, followed by an example where we finally evaluate an integral and find an area.

Key Idea 15 Evaluating a Line Integral, Part I

- Let C be a curve parametrized by $\vec{r}(t) = \langle g(t), h(t)\rangle$, $a \leq t \leq b$, where g and h are continuously differentiable, and let $z = f(x, y)$. Then
$$\int_C f(s)\,ds = \int_a^b f\big(g(t), h(t)\big)\|\,\vec{r}\,'(t)\,\|\,dt.$$
- Let C be a curve parametrized by $\vec{r}(t) = \langle g(t), h(t), k(t)\rangle$, $a \leq t \leq b$, where g, h and k are continuously differentiable, and let $w = f(x, y, z)$. Then
$$\int_C f(s)\,ds = \int_a^b f\big(g(t), h(t), k(t)\big)\|\,\vec{r}\,'(t)\,\|\,dt.$$

Notes:

To be clear, the first point of Key Idea 16 can be used to find the area under a surface $z = f(x, y)$ and above a curve C.

Let's finally do an example where we actually compute an area.

Example 172 Evaluating a line integral.

Find the area under the surface $f(x, y) = \cos(x) + \sin(y) + 2$ over the curve C, which is the segment of the line $y = 2x + 1$ on $-1 \leq x \leq 1$, as shown in Figure 11.

Solution Our first step is to represent C with a vector–valued function. Since C is a simple line, and we have a explicit relationship between y and x (namely, that y is $2x+1$), we can let $x = t$, $y = 2t+1$, and write $\vec{r}(t) = \langle t, 2t+1\rangle$ for $-1 \leq t \leq 1$.

We find the values of f over C as $f(x, y) = f(t, 2t+1) = \cos(t)+\sin(2t+1)+2$. We also need $\| \vec{r}\,'(t) \|$; with $\vec{r}\,'(t) = \langle 1, 2\rangle$, we have $\| \vec{r}\,'(t) \| = \sqrt{5}$. Thus $ds = \sqrt{5}\, dt$.

The area we seek is

$$\begin{aligned}\int_C f(s)\, ds &= \int_{-1}^{1} \big(\cos(t) + \sin(2t+1) + 2\big)\sqrt{5}\, dt \\ &= \sqrt{5}\Big(\sin(t) - \frac{1}{2}\cos(2t+1) + 2t\Big)\Big|_{-1}^{1} \\ &\approx 14.418 \text{ units}^2.\end{aligned}$$

filler text

filler

filler

In previous chapters we have explored a relationship between vectors and integration. Our most tangible result: if $\vec{v}(t)$ is a vector–valued velocity function of a moving object, then integrating $\vec{v}(t)$ from $t = a$ to $t = b$ gives the displacement of that object over that time interval.

This chapter explores completely different relationships between vectors and integration. These relationships will enable us to compute the work done by a magnetic field in moving an object along a path and how much air moves through an oddly–shaped screen in space, among other things.

Our upcoming work in integration will benefit from a review. We are not concerned here with techniques of integration, but rather what an integral "does" and how that relates to the notation we use to describe it.

(a)

(b)

Figure 13.8: Finding area under a curve in Example 173.

Notes:

Integration Review

Recall from Section 12.1 that when R is a region in the x-y plane, $\int\int_R dA$ gives the area of the region R. The integral symbols are "elongated esses" meaning "sum" and dA represents "a small amount of area." Taken together, $\int\int_R dA$ means "sum up, over R, small amounts of area." This sum then gives the total area of R. We use two integral symbols since R is a two–dimensional region.

Now let $z = f(x, y)$ represent a surface. The double integral $\int\int_R f(x, y)\ dA$ means "sum up, over R, function values (heights) given by f times small amounts of area." Since "height $\times$ area = volume," we are summing small amounts of volume over R, giving the total signed volume under the surface $z = f(x, y)$ and above the x-y plane.

This notation does not directly inform us *how* to evaluate the double integrals to find an area or a volume. With additional work, we recognize that a small amount of area dA can be measured as the area of a small rectangle, with one side length a small change in x and the other side length a small change in y. That is, $dA = dx\,dy$ or $dA = dy\,dx$. We could also compute a small amount of area by thinking in terms of polar coordinates, where $dA = r\,dr\,d\theta$. These understandings lead us to the iterated integrals we used in Chapter 12.

Let us back our review up farther. Note that $\int_1^3\ dx = x\big|_1^3 = 3 - 1 = 2$. We have simply measured the length of the interval $[1, 3]$. We could rewrite the above integral using syntax similar to the double integral syntax above:

$$\int_1^3\ dx = \int_I dx, \qquad \text{where } I = [1, 3].$$

We interpret "$\int_I dx$" as meaning "sum up, over the interval I, small changes in x." A change in x is a length along the x-axis, so we are adding up along I small lengths, giving the total length of I.

We could also write $\int_1^3 f(x)\ dx$ as $\int_I f(x)\ dx$, interpreted as "sum up, over I, heights given by $y = f(x)$ times small changes in x." Since "height$\times$length = area," we are summing up areas and finding the total signed area between $y = f(x)$ and the x-axis.

This method of referring to the process of integration can be very powerful. It is the core of our notion of the Riemann Sum. When faced with a quantity to compute, if one can think of a way to approximate its value through a sum, the one is well on their way to constructing an integral (or, double or triple integral) that computes the desired quantity. We will demonstrate this process throughout this chapter, starting with the next section.

Notes:

13.6 Introduction to Line Integrals

We first used integration to find "area under a curve." In this section, we learn to do this (again), but in a different context.

Consider the surface and curve shown in Figure 13.10(a). The surface is given by $f(x, y) = 1 - \cos(x)\sin(y)$. The dashed curve lies in the x-y plane and is the familiar $y = x^2$ parabola from $-1 \leq x \leq 1$; we'll call this curve C. The curve drawn with a solid line in the graph is the curve in space that lies on our surface with x and y values that lie on C.

The question we want to answer is this: what is the area that lies below curve drawn with the solid line? In other words, what is the area above C and under the the surface f? This area is shown in Figure 13.10(b).

We suspect the answer can be found using an integral, but before trying to figure out what that integral is, let us first try to approximate its value.

In Figure 13.10(c), 4 rectangles have been drawn over the curve C. The bottom corners of each rectangle lies on C, and each rectangle has a height given by the function $f(x, y)$ for some (x, y) pair along C between the rectangle's bottom corners.

As we know how to find the area of each rectangle, we are able to approximate the area above C and under f. Clearly, our approximation will be *an approximation*. The heights of the rectangles do not match exactly with the surface f, nor does the base of each rectangle follow perfectly the path of C.

In typical calculus fashion, our approximation can be improved by using more rectangles. The sum of the areas of these rectangles gives an approximate value of the true area above C and under f. As the area of each rectangle is "height$\times$width", we assert that the

$$\text{area above } C \approx \sum(\text{heights} \times \text{widths}).$$

When first learning of the integral, and approximating areas with "heights$\times$widths", the width was a small change in x: dx. That will not suffice in this context. Rather, each width of a rectangle is actually approximating the arc length of a small portion of C. In Section 10.5, we used s to represent the arc–length parameter of a curve. A small amount of arc length will thus be represented by ds.

The height of each rectangle will be determined in some way by the surface f. If we parametrize C by s, an s-value corresponds to an (x, y) pair that lies on the parabola C. Since f is a function of x and y, and x and y are functions of s, we can say that f is a function of s. Given a value s, we can compute $f(s)$ and find a

Figure 13.10: Finding area under a curve in space.

Notes:

height. Thus

$$\begin{aligned}\text{area under } f \text{ and above } C &\approx \sum(\text{heights} \times \text{widths});\\ \text{area under } f \text{ and above } C &= \lim_{||\Delta s||\to 0} \sum f(c_i)\Delta s_i\\ &= \int_C f(s)\ ds. \qquad (13.4)\end{aligned}$$

Here we have introduce a new notation, the integral symbol with a subscript of C. It is reminiscent of our usage of $\int\int_R$. Using the train of thought found in the Integration Review preceding this section, we interpret "$\int_C f(s)\ ds$" as meaning "sum up, along a curve C, function values $f(s)\times$small arc lengths." It is understood here that s represents the arc–length parameter.

All this leads us to a definition. The integral found in Equation 13.4 is called a **line integral**. We formally define it below, but note that the definition is very abstract. On one hand, one is apt to say "the defintion makes sense," while on the other, one is equally apt to say "but I don't know what I'm supposed to do with this definition." We'll address that after the definition, and actually find an answer to the area problem we posed at the beginning of this section.

Definition 65 Line Integral

Let C be a curve continuously parametrized by s, the arc–length parameter, and let f be a continuous function of s. A **line integral** is an integral of the form

$$\int_C f(s)\ ds = \lim_{||\Delta s||\to 0} \sum_{i=1}^{n} f(c_i)\Delta s_i,$$

where $s_1 < s_2 < \ldots < s_n$ is any partition of the s-interval over which C is defined, c_i is any value in the ith subinterval, Δ_i is the width of the ith subinterval, and $||\Delta s||$ is the length of the longest subinterval in the partition.

The definition of the line integral does not specify whether C is a curve in the plane or space (or hyperspace), as the definition holds regardless. For now, we'll assume C lies in the x-y plane.

This definition of the line integral doesn't really say anything new. If C is a curve and s is the arc–length parameter of C on $a \le s \le b$, then

$$\int_C f(s)\ ds = \int_a^b f(s)\ ds.$$

Notes:

The real difference with this integral from the standard "$\int_a^b f(x)\,dx$" we used in the past is that of context. Our previous integrals naturally summed up values over an interval on the x-axis, whereas now we are summing up values over a curve. If we can parametrize the curve with the arc–length parameter, we can evaluate the line integral just as before.

The trouble here is that we have generally avoided direct use of the arc–length parameter s in the past as it is usually difficult to use. We continue that methodology here.

Given a curve C, find a parametrization of C: $x = g(t)$ and $y = h(t)$, for continuous functions g and h, where $a \leq t \leq b$. We can represent this parametrization with a vector–valued function, $\vec{r}(t) = \langle g(t), h(t)\rangle$.

In Section 10.5, we defined the arc–length parameter in Equation 10.1 as

$$s(t) = \int_0^t \|\,\vec{r}\,'(u)\,\|\,du.$$

By the Fundamental Theorem of Calculus, $ds = \|\,\vec{r}\,'(t)\,\|\,dt$. We can substitute the right hand side of this equation for ds in the line integral definition.

We can view f as being a function of x and y since it is a function of s. Thus $f(s) = f(x, y) = f\big(g(t), h(t)\big)$. This gives us a concrete way to evaluate a line integral:

$$\int_C f(s)\,ds = \int_a^b f\big(g(t), h(t)\big)\|\,\vec{r}\,'(t)\,\|\,dt.$$

We write this as a Key Idea, along with its three–dimensional analogue, followed by an example where we finally evaluate an integral and find an area.

Key Idea 16 Evaluating a Line Integral, Part I

- Let C be a curve parametrized by $\vec{r}(t) = \langle g(t), h(t)\rangle$, $a \leq t \leq b$, where g and h are continuously differentiable, and let $z = f(x, y)$. Then
$$\int_C f(s)\,ds = \int_a^b f\big(g(t), h(t)\big)\|\,\vec{r}\,'(t)\,\|\,dt.$$
- Let C be a curve parametrized by $\vec{r}(t) = \langle g(t), h(t), k(t)\rangle$, $a \leq t \leq b$, where g, h and k are continuously differentiable, and let $w = f(x, y, z)$. Then
$$\int_C f(s)\,ds = \int_a^b f\big(g(t), h(t), k(t)\big)\|\,\vec{r}\,'(t)\,\|\,dt.$$

Notes:

To be clear, the first point of Key Idea 16 can be used to find the area under a surface $z = f(x, y)$ and above a curve C.

Let's finally do an example where we actually compute an area.

Example 173 Evaluating a line integral.

Find the area under the surface $f(x, y) = \cos(x) + \sin(y) + 2$ over the curve C, which is the segment of the line $y = 2x + 1$ on $-1 \leq x \leq 1$, as shown in Figure 11.

Solution Our first step is to represent C with a vector–valued function. Since C is a simple line, and we have a explicit relationship between y and x (namely, that y is $2x+1$), we can let $x = t$, $y = 2t+1$, and write $\vec{r}(t) = \langle t, 2t+1\rangle$ for $-1 \leq t \leq 1$.

We find the values of f over C as $f(x, y) = f(t, 2t+1) = \cos(t)+\sin(2t+1)+2$. We also need $\| \vec{r}\,'(t) \|$; with $\vec{r}\,'(t) = \langle 1, 2\rangle$, we have $\| \vec{r}\,'(t) \| = \sqrt{5}$. Thus $ds = \sqrt{5}\, dt$.

The area we seek is

$$\begin{aligned}\int_C f(s)\, ds &= \int_{-1}^{1} \big(\cos(t) + \sin(2t+1) + 2\big)\sqrt{5}\, dt \\ &= \sqrt{5}\Big(\sin(t) - \frac{1}{2}\cos(2t+1) + 2t\Big)\Big|_{-1}^{1} \\ &\approx 14.418 \text{ units}^2.\end{aligned}$$

filler text

filler

filler

Figure 12: Finding area under a curve in Example 173.

Notes:

A: Solutions To Selected Problems

Chapter 9

Section 9.1

1. right hand
3. curve (a parabola); surface (a cylinder)
5. a hyperboloid of two sheets
7. $\|\overline{AB}\| = \sqrt{6}$; $\|\overline{BC}\| = \sqrt{17}$; $\|\overline{AC}\| = \sqrt{11}$. Yes, it is a right triangle as $\|\overline{AB}\|^2 + \|\overline{AC}\|^2 = \|\overline{BC}\|^2$.
9. Center at $(4, -1, 0)$; radius = 3
11. Interior of a sphere with radius 1 centered at the origin.
13. The first octant of space; all points (x, y, z) where each of x, y and z are positive. (Analogous to the first quadrant in the plane.)

15.

17.

19. $y^2 + z^2 = x^4$
21. $z = (\sqrt{x^2 + y^2})^2 = x^2 + y^2$
23. (a) $x = y^2 + \dfrac{z^2}{9}$
25. (b) $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$

27.

29.

31.

Section 9.2

1. Answers will vary.
3. A vector with magnitude 1.
5. It stretches the vector by a factor of 2, and points it in the opposite direction.
7. $\overrightarrow{PQ} = \langle -4, 4\rangle = -4\vec{i} + 4\vec{j}$
9. $\overrightarrow{PQ} = \langle 2, 2, 0\rangle = 2\vec{i} + 2\vec{j}$
11. (a) $\vec{u} + \vec{v} = \langle 3, 2, 1\rangle$; $\vec{u} - \vec{v} = \langle -1, 0, -3\rangle$; $\pi\vec{u} - \sqrt{2}\vec{v} = \langle \pi - 2\sqrt{2}, \pi - \sqrt{2}, -\pi - 2\sqrt{2}\rangle$.

    (c) $\vec{x} = \langle -1, 0, -3\rangle$.

13.

Sketch of $\vec{u} - \vec{v}$ shifted for clarity.

15.

17. $||\vec{u}|| = \sqrt{17}$, $||\vec{v}|| = \sqrt{3}$, $||\vec{u}+\vec{v}|| = \sqrt{14}$, $||\vec{u}-\vec{v}|| = \sqrt{26}$
19. $||\vec{u}|| = 7$, $||\vec{v}|| = 35$, $||\vec{u}+\vec{v}|| = 42$, $||\vec{u}-\vec{v}|| = 28$
21. $\vec{u} = \langle 3/\sqrt{58}, 7/\sqrt{58}\rangle$
23. $\vec{u} = \langle 1/3, -2/3, 2/3\rangle$
25. $\vec{u} = \langle \cos 50^\circ, \sin 50^\circ\rangle \approx \langle 0.643, 0.766\rangle$.
27.
$$\begin{aligned} ||\vec{u}|| &= \sqrt{\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta} \\ &= \sqrt{\sin^2\theta(\cos^2\varphi + \sin^2\varphi) + \cos^2\theta} \\ &= \sqrt{\sin^2\theta + \cos^2\theta} \\ &= 1. \end{aligned}$$
29. The force on each chain is 100lb.
31. The force on each chain is 50lb.
33. $\theta = 5.71^\circ$; the weight is lifted 0.005 ft (about 1/16th of an inch).
35. $\theta = 84.29^\circ$; the weight is lifted 9 ft.

Section 9.3

1. Scalar
3. By considering the sign of the dot product of the two vectors. If the dot product is positive, the angle is acute; if the dot product is negative, the angle is obtuse.
5. -22
7. 3
9. not defined
11. Answers will vary.
13. $\theta = 0.3218 \approx 18.43^\circ$
15. $\theta = \pi/4 = 45^\circ$
17. Answers will vary; two possible answers are $\langle -7, 4\rangle$ and $\langle 14, -8\rangle$.
19. Answers will vary; two possible answers are $\langle 1, 0, -1\rangle$ and $\langle 4, 5, -9\rangle$.
21. $\text{proj}_{\vec{v}}\,\vec{u} = \langle -1/2, 3/2\rangle$.
23. $\text{proj}_{\vec{v}}\,\vec{u} = \langle -1/2, -1/2\rangle$.
25. $\text{proj}_{\vec{v}}\,\vec{u} = \langle 1, 2, 3\rangle$.
27. $\vec{u} = \langle -1/2, 3/2\rangle + \langle 3/2, 1/2\rangle$.
29. $\vec{u} = \langle -1/2, -1/2\rangle + \langle -5/2, 5/2\rangle$.
31. $\vec{u} = \langle 1, 2, 3\rangle + \langle 0, 3, -2\rangle$.
33. 1.96lb
35. 141.42ft–lb
37. 500ft–lb
39. 500ft–lb

Section 9.4

1. vector
3. "Perpendicular" is one answer.
5. Torque
7. $\vec{u}\times\vec{v} = \langle 11, 1, -17\rangle$
9. $\vec{u}\times\vec{v} = \langle 47, -36, -44\rangle$
11. $\vec{u}\times\vec{v} = \langle 0, 0, 0\rangle$
13. $\vec{i}\times\vec{k} = -\vec{j}$
15. Answers will vary.
17. 5
19. 0
21. $\sqrt{14}$
23. 3
25. $5\sqrt{2}/2$
27. 1
29. 7
31. 2
33. $\pm\frac{1}{\sqrt{6}}\langle 1, 1, -2\rangle$
35. $\langle 0, \pm 1, 0\rangle$
37. 87.5ft–lb
39. $200/3 \approx 66.67$ft–lb
41. With $\vec{u} = \langle u_1, u_2, u_3\rangle$ and $\vec{v} = \langle v_1, v_2, v_3\rangle$, we have
$$\begin{aligned} \vec{u}\cdot(\vec{u}\times\vec{v}) &= \langle u_1, u_2, u_3\rangle \cdot (\langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1\rangle) \\ &= u_1(u_2v_3 - u_3v_2) - u2(u_1v_3 - u_3v_1) + u3(u_1v_2 - u_2v_1) \\ &= 0. \end{aligned}$$

Section 9.5

1. A point on the line and the direction of the line.
3. parallel, skew
5. vector: $\ell(t) = \langle 2, -4, 1\rangle + t\langle 9, 2, 5\rangle$
parametric: $x = 2 + 9t$, $y = -4 + 2t$, $z = 1 + 5t$
symmetric: $(x-2)/9 = (y+4)/2 = (z-1)/5$
7. Answers can vary: vector: $\ell(t) = \langle 2, 1, 5\rangle + t\langle 5, -3, -1\rangle$
parametric: $x = 2 + 5t$, $y = 1 - 3t$, $z = 5 - t$
symmetric: $(x-2)/5 = -(y-1)/3 = -(z-5)$
9. Answers can vary; here the direction is given by $\vec{d}_1\times\vec{d}_2$: vector:
$\ell(t) = \langle 0, 1, 2\rangle + t\langle -10, 43, 9\rangle$
parametric: $x = -10t$, $y = 1 + 43t$, $z = 2 + 9t$
symmetric: $-x/10 = (y-1)/43 = (z-2)/9$
11. Answers can vary; here the direction is given by $\vec{d}_1\times\vec{d}_2$: vector:
$\ell(t) = \langle 7, 2, -1\rangle + t\langle 1, -1, 2\rangle$
parametric: $x = 7 + t$, $y = 2 - t$, $z = -1 + 2t$
symmetric: $x - 7 = 2 - y = (z+1)/2$
13. vector: $\ell(t) = \langle 1, 1\rangle + t\langle 2, 3\rangle$
parametric: $x = 1 + 2t$, $y = 1 + 3t$
symmetric: $(x-1)/2 = (y-1)/3$
15. parallel
17. intersecting; $\vec{\ell}_1(3) = \vec{\ell}_2(4) = \langle 9, -5, 13\rangle$
19. skew
21. same
23. $\sqrt{41}/3$
25. $5\sqrt{2}/2$
27. $3/\sqrt{2}$
29. Since both P and Q are on the line, $\overrightarrow{PQ}$ is parallel to $\vec{d}$. Thus $\overrightarrow{PQ}\times\vec{d} = \vec{0}$, giving a distance of 0.
31. (a) The distance formula cannot be used because since $\vec{d}_1$ and $\vec{d}_2$ are parallel, $\vec{c}$ is $\vec{0}$ and we cannot divide by $||\vec{0}||$.
    (b) Since $\vec{d}_1$ and $\vec{d}_2$ are parallel, $\overrightarrow{P_1P_2}$ lies in the plane formed by the two lines. Thus $\overrightarrow{P_1P_2}\times\vec{d}_2$ is orthogonal to this plane, and $\vec{c} = (\overrightarrow{P_1P_2}\times\vec{d}_2)\times\vec{d}_2$ is parallel to the plane, but still orthogonal to both $\vec{d}_1$ and $\vec{d}_2$. We desire the length of the projection of $\overrightarrow{P_1P_2}$ onto $\vec{c}$, which is what the formula provides.

(c) Since the lines are parallel, one can measure the distance between the lines at any location on either line (just as to find the distance between straight railroad tracks, one can use a measuring tape anywhere along the track, not just at one specific place.) Let $P = P_1$ and $Q = P_2$ as given by the equations of the lines, and apply the formula for distance between a point and a line.

Section 9.6

1. A point in the plane and a normal vector (i.e., a direction orthogonal to the plane).
3. Answers will vary.
5. Answers will vary.
7. Standard form: $3(x-2)-(y-3)+7(z-4)=0$
   general form: $3x-y+7z=31$
9. Answers may vary;
   Standard form: $8(x-1)+4(y-2)-4(z-3)=0$
   general form: $8x+4y-4z=4$
11. Answers may vary;
    Standard form: $-7(x-2)+2(y-1)+(z-2)=0$
    general form: $-7x+2y+z=-10$
13. Answers may vary;
    Standard form: $2(x-1)-(y-1)=0$
    general form: $2x-y=1$
15. Answers may vary;
    Standard form: $2(x-2)-(y+6)-4(z-1)=0$
    general form: $2x-y-4z=6$
17. Answers may vary;
    Standard form: $(x-5)+(y-7)+(z-3)=0$
    general form: $x+y+z=15$
19. Answers may vary;
    Standard form: $3(x+4)+8(y-7)-10(z-2)=0$
    general form: $3x+8y-10z=24$
21. Answers may vary:
    $$\ell = \begin{cases} x = 14t \\ y = -1-10t \\ z = 2-8t \end{cases}$$
23. $(-3,-7,-5)$
25. No point of intersection; the plane and line are parallel.
27. $\sqrt{5/7}$
29. $1/\sqrt{3}$
31. If P is any point in the plane, and Q is also in the plane, then $\overrightarrow{PQ}$ lies parallel to the plane and is orthogonal to $\vec{n}$, the normal vector. Thus $\vec{n}\cdot\overrightarrow{PQ}=0$, giving the distance as 0.

Chapter 10

Section 10.1

1. parametric equations
3. displacement

5.

7.

9.

11.

13.

15.

17. $||\vec{r}(t)|| = \sqrt{25\cos^2 t + 9\sin^2 t}$.
19. $||\vec{r}(t)|| = \sqrt{\cos^2 t + t^2 + t^4}$.
21. Answers may vary; three solutions are
    $\vec{r}(t) = \langle 3\sin t+5, 3\cos t+5\rangle$,
    $\vec{r}(t) = \langle -3\cos t+5, 3\sin t+5\rangle$ and
    $\vec{r}(t) = \langle 3\cos t+5, -3\sin t+5\rangle$.

23. Answers may vary, though most direct solutions are
$\vec{r}(t) = \langle -3\cos t + 3, 2\sin t - 2\rangle$,
$\vec{r}(t) = \langle 3\cos t + 3, -2\sin t - 2\rangle$ and
$\vec{r}(t) = \langle 3\sin t + 3, 2\cos t - 2\rangle$.

25. Answers may vary, though most direct solutions are
$\vec{r}(t) = \langle t, -1/2(t-1) + 5\rangle$,
$\vec{r}(t) = \langle t+1, -1/2t + 5\rangle$,
$\vec{r}(t) = \langle -2t+1, t+5\rangle$ and
$\vec{r}(t) = \langle 2t+1, -t+5\rangle$.

27. Answers may vary, though most direct solution is
$\vec{r}(t) = \langle 3\cos(4\pi t), 3\sin(4\pi t), 3t\rangle$.

29. $\langle 1, 1\rangle$

31. $\langle 1, 2, 7\rangle$

Section 10.2

1. component

3. It is difficult to identify the points on the graphs of $\vec{r}(t)$ and $\vec{r}\,'(t)$ that correspond to each other.

5. $\langle e^3, 0\rangle$

7. $\langle 2t, 1, 0\rangle$

9. $(0, \infty)$

11. $\vec{r}\,'(t) = \langle -1/t^2, 5/(3t+1)^2, \sec^2 t\rangle$

13. $\vec{r}\,'(t) = \langle 2t, 1\rangle \cdot \langle \sin t, 2t+5\rangle + \langle t^2+1, t-1\rangle \cdot \langle \cos t, 2\rangle = (t^2+1)\cos t + 2t\sin t + 4t + 3$

15.

$\vec{r}\,'(t) = \langle 2t+1, 2t-1\rangle$

17.

$\vec{r}\,'(t) = \langle 2t, 3t^2 - 1\rangle$

19. $\ell(t) = \langle 2, 0\rangle + t\langle 3, 1\rangle$

21. $\ell(t) = \langle -3, 0, \pi\rangle + t\langle 0, -3, 1\rangle$

23. $t = 2n\pi$, where n is an integer; so
$t = \ldots -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots$

25. $\vec{r}(t)$ is not smooth at $t = 3\pi/4 + n\pi$, where n is an integer

27. Both derivatives return $\langle 5t^4, 4t^3 - 3t^2, 3t^2\rangle$.

29. Both derivatives return
$\langle 2t - e^t - 1, \cos t - 3t^2, (t^2+2t)e^t - (t-1)\cos t - \sin t\rangle$.

31. $\langle \tan^{-1} t, \tan t\rangle + \vec{C}$

33. $\langle 4, -4\rangle$

35. $\vec{r}(t) = \langle \ln|t+1| + 1, -\ln|\cos t| + 2\rangle$

37. $\vec{r}(t) = \langle -\cos t + 1, t - \sin t, e^t - t - 1\rangle$

39. 10π

41. $\sqrt{2}(1 - e^{-1})$

Section 10.3

1. Velocity is a vector, indicating an objects direction of travel and its rate of distance change (i.e., its speed). Speed is a scalar.

3. The average velocity is found by dividing the displacement by the time traveled – it is a vector. The average speed is found by dividing the distance traveled by the time traveled – it is a scalar.

5. One example is traveling at a constant speed s in a circle, ending at the starting position. Since the displacement is $\vec{0}$, the average velocity is $\vec{0}$, hence $||\,\vec{0}\,|| = 0$. But traveling at constant speed s means the average speed is also $s > 0$.

7. $\vec{v}(t) = \langle 2, 5, 0\rangle$, $\vec{a}(t) = \langle 0, 0, 0\rangle$

9. $\vec{v}(t) = \langle -\sin t, \cos t\rangle$, $\vec{a}(t) = \langle -\cos t, -\sin t\rangle$

11. $\vec{v}(t) = \langle 1, \cos t\rangle$, $\vec{a}(t) = \langle 0, -\sin t\rangle$

13. $\vec{v}(t) = \langle 2t+1, -2t+2\rangle$, $\vec{a}(t) = \langle 2, -2\rangle$

15. $||\,\vec{v}(t)\,|| = \sqrt{4t^2+1}$.
Min at $t = 0$; Max at $t = \pm 1$.

17. $||\,\vec{v}(t)\,|| = 5$.
Speed is constant, so there is no difference between min/max

19. $||\,\vec{v}(t)\,|| = |\sec t|\sqrt{\tan^2 t + \sec^2 t}$.
min: $t = 0$; max: $t = \pi/4$

21. $||\,\vec{v}(t)\,|| = 13$.
speed is constant, so there is no difference between min/max

23. $||\,\vec{v}(t)\,|| = \sqrt{4t^2 + 1 + t^2/(1-t^2)}$.
min: $t = 0$; max: there is no max; speed approaches ∞ as $t \to \pm 1$

25. (a) $\vec{r}_1(1) = \langle 1, 1\rangle$; $\vec{r}_2(1) = \langle 1, 1\rangle$
(b) $\vec{v}_1(1) = \langle 1, 2\rangle$; $||\,\vec{v}_1(1)\,|| = \sqrt{5}$; $\vec{a}_1(1) = \langle 0, 2\rangle$
$\vec{v}_2(1) = \langle 2, 4\rangle$; $||\,\vec{v}_2(1)\,|| = 2\sqrt{5}$; $\vec{a}_2(1) = \langle 2, 12\rangle$

27. (a) $\vec{r}_1(2) = \langle 6, 4\rangle$; $\vec{r}_2(2) = \langle 6, 4\rangle$

(b) $\vec{v}_1(2) = \langle 3, 2\rangle$; $||\vec{v}_1(2)|| = \sqrt{13}$; $\vec{a}_1(2) = \langle 0, 0\rangle$
$\vec{v}_2(2) = \langle 6, 4\rangle$; $||\vec{v}_2(2)|| = 2\sqrt{13}$; $\vec{a}_2(2) = \langle 0, 0\rangle$

29. $\vec{v}(t) = \langle 2t+1, 3t+2\rangle$, $\vec{r}(t) = \left\langle t^2+t+5, 3t^2/2+2t-2\right\rangle$

31. $\vec{v}(t) = \langle \sin t, \cos t\rangle$, $\vec{r}(t) = \langle 1-\cos t, \sin t\rangle$

33. Displacement: $\langle 0, 0, 6\pi\rangle$; distance traveled: $2\sqrt{13}\pi \approx 22.65$ft; average velocity: $\langle 0, 0, 3\rangle$; average speed: $\sqrt{13} \approx 3.61$ft/s

35. Displacement: $\langle 0, 0\rangle$; distance traveled: $2\pi \approx 6.28$ft; average velocity: $\langle 0, 0\rangle$; average speed: 1ft/s

37. At t-values of $\sin^{-1}(9/30)/(4\pi) + n/2 \approx 0.024 + n/2$ seconds, where n is an integer.

39. (a) Holding the crossbow at an angle of 0.013 radians, $\approx 0.745^\circ$ will hit the target 0.4s later. (Another solution exists, with an angle of 89°, landing 18.75s later, but this is impractical.)
(b) In the .4 seconds the arrow travels, a deer, traveling at 20mph or 29.33ft/s, can travel 11.7ft. So she needs to lead the deer by 11.7ft.

41. The position function is $\vec{r}(t) = \left\langle 220t, -16t^2+1000\right\rangle$. The y-component is 0 when $t = 7.9$; $\vec{r}(7.9) = \langle 1739.25, 0\rangle$, meaning the box will travel about 1740ft horizontally before it lands.

Section 10.4

1. 1

3. $\vec{T}(t)$ and $\vec{N}(t)$.

5. $\vec{T}(t) = \left\langle \frac{4t}{\sqrt{20t^2-4t+1}}, \frac{2t-1}{\sqrt{20t^2-4t+1}}\right\rangle$; $\vec{T}(1) = \left\langle 4/\sqrt{17}, 1/\sqrt{17}\right\rangle$

7. $\vec{T}(t) = \frac{\cos t \sin t}{\sqrt{\cos^2 t \sin^2 t}}\langle -\cos t, \sin t\rangle$. (Be careful; this cannot be simplified as just $\langle -\cos t, \sin t\rangle$ as $\sqrt{\cos^2 t \sin^2 t} \neq \cos t \sin t$, but rather $|\cos t \sin t|$.) $\vec{T}(\pi/4) = \left\langle -\sqrt{2}/2, \sqrt{2}/2\right\rangle$

9. $\ell(t) = \langle 2, 0\rangle + t\left\langle 4/\sqrt{17}, 1/\sqrt{17}\right\rangle$; in parametric form,
$$\ell(t) = \begin{cases} x &= 2+4t/\sqrt{17} \\ y &= t/\sqrt{17} \end{cases}$$

11. $\ell(t) = \left\langle \sqrt{2}/4, \sqrt{2}/4\right\rangle + t\left\langle -\sqrt{2}/2, \sqrt{2}/2\right\rangle$; in parametric form,
$$\ell(t) = \begin{cases} x &= \sqrt{2}/4 - \sqrt{2}t/2 \\ y &= \sqrt{2}/4 + \sqrt{2}t/2 \end{cases}$$

13. $\vec{T}(t) = \langle -\sin t, \cos t\rangle$; $\vec{N}(t) = \langle -\cos t, -\sin t\rangle$

15. $\vec{T}(t) = \left\langle -\frac{\sin t}{\sqrt{4\cos^2 t+\sin^2 t}}, \frac{2\cos t}{\sqrt{4\cos^2 t+\sin^2 t}}\right\rangle$;
$\vec{N}(t) = \left\langle -\frac{2\cos t}{\sqrt{4\cos^2 t+\sin^2 t}}, -\frac{\sin t}{\sqrt{4\cos^2 t+\sin^2 t}}\right\rangle$

17. (a) Be sure to show work
(b) $\vec{N}(\pi/4) = \left\langle -5/\sqrt{34}, -3/\sqrt{34}\right\rangle$

19. (a) Be sure to show work
(b) $\vec{N}(0) = \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle$

21. $\vec{T}(t) = \frac{1}{\sqrt{5}}\langle 2, \cos t, -\sin t\rangle$; $\vec{N}(t) = \langle 0, -\sin t, -\cos t\rangle$

23. $\vec{T}(t) = \frac{1}{\sqrt{a^2+b^2}}\langle -a\sin t, a\cos t, b\rangle$; $\vec{N}(t) = \langle -\cos t, -\sin t, 0\rangle$

25. $a_T = \frac{4t}{\sqrt{1+4t^2}}$ and $a_N = \sqrt{4 - \frac{16t^2}{1+4t^2}}$
At $t = 0$, $a_T = 0$ and $a_N = 2$;
At $t = 1$, $a_T = 4/\sqrt{5}$ and $a_N = 2/\sqrt{5}$.
At $t = 0$, all acceleration comes in the form of changing the direction of velocity and not the speed; at $t = 1$, more acceleration comes in changing the speed than in changing direction.

27. $a_T = 0$ and $a_N = 2$
At $t = 0$, $a_T = 0$ and $a_N = 2$;
At $t = \pi/2$, $a_T = 0$ and $a_N = 2$.
The object moves at constant speed, so all acceleration comes from changing direction, hence $a_T = 0$. $\vec{a}(t)$ is always parallel to $\vec{N}(t)$, but twice as long, hence $a_N = 2$.

29. $a_T = 0$ and $a_N = a$
At $t = 0$, $a_T = 0$ and $a_N = a$;
At $t = \pi/2$, $a_T = 0$ and $a_N = a$.
The object moves at constant speed, meaning that a_T is always 0. The object "rises" along the z-axis at a constant rate, so all acceleration comes in the form of changing direction circling the z-axis. The greater the radius of this circle the greater the acceleration, hence $a_N = a$.

Section 10.5

1. time and/or distance

3. Answers may include lines, circles, helixes

5. κ

7. $s = 3t$, so $\vec{r}(s) = \langle 2s/3, s/3, -2s/3\rangle$

9. $s = \sqrt{13}t$, so $\vec{r}(s) = \left\langle 3\cos(s/\sqrt{13}), 3\sin(s/\sqrt{13}), 2s/\sqrt{13}\right\rangle$

11. $\kappa = \frac{|6x|}{\left(1+(3x^2-1)^2\right)^{3/2}}$;
$\kappa(0) = 0$, $\kappa(1/2) = \frac{192}{17\sqrt{17}} \approx 2.74$.

13. $\kappa = \frac{|\cos x|}{\left(1+\sin^2 x\right)^{3/2}}$;
$\kappa(0) = 1$, $\kappa(\pi/2) = 0$

15. $\kappa = \frac{|2\cos t\cos(2t)+4\sin t\sin(2t)|}{\left(4\cos^2(2t)+\sin^2 t\right)^{3/2}}$;
$\kappa(0) = 1/4$, $\kappa(\pi/4) = 8$

17. $\kappa = \frac{|6t^2+2|}{\left(4t^2+(3t^2-1)^2\right)^{3/2}}$;
$\kappa(0) = 2$, $\kappa(5) = \frac{19}{1394\sqrt{1394}} \approx 0.0004$

19. $\kappa = 0$;
$\kappa(0) = 0$, $\kappa(1) = 0$

21. $\kappa = \frac{3}{13}$;
$\kappa(0) = 3/13$, $\kappa(\pi/2) = 3/13$

23. maximized at $x = \pm\frac{\sqrt{2}}{\sqrt[4]{5}}$

25. maximized at $t = 1/4$

27. radius of curvature is $5\sqrt{5}/4$.

29. radius of curvature is 9.

31. $x^2 + (y-1/2)^2 = 1/4$, or $\vec{c}(t) = \langle 1/2\cos t, 1/2\sin t + 1/2\rangle$

33. $x^2 + (y+8)^2 = 81$, or $\vec{c}(t) = \langle 9\cos t, 9\sin t - 8\rangle$

Chapter 11

Section 11.1

1. Answers will vary.

3. topographical

5. surface

7. domain: $\mathbb{R}^2$
range: $z \geq 2$

9. domain: $\mathbb{R}^2$
range: $\mathbb{R}$

11. domain: $\mathbb{R}^2$
range: $0 < z \leq 1$

13. domain: $\{(x, y) \mid x^2 + y^2 \leq 9\}$, i.e., the domain is the circle and interior of a circle centered at the origin with radius 3.
range: $0 \leq z \leq 3$

15. Level curves are lines $y = (3/2)x - c/2$.

17. Level curves are parabolas $x = y^2 + c$.

19. When $c \neq 0$, the level curves are circles, centered at $(1/c, -1/c)$ with radius $\sqrt{2/c^2 - 1}$. When $c = 0$, the level curve is the line $y = x$.

21. Level curves are ellipses of the form $\frac{x^2}{c^2} + \frac{y^2}{c^2/4} = 1$, i.e., $a = c$ and $b = c/2$.

23. domain: $x + 2y - 4z \neq 0$; the set of points in $\mathbb{R}^3$ NOT in the domain form a plane through the origin.
range: $\mathbb{R}$

25. domain: $z \geq x^2 - y^2$; the set of points in $\mathbb{R}^3$ above (and including) the hyperbolic paraboloid $z = x^2 - y^2$.
range: $[0, \infty)$

27. The level surfaces are spheres, centered at the origin, with radius $\sqrt{c}$.

29. The level surfaces are paraboloids of the form $z = \frac{x^2}{c} + \frac{y^2}{c}$; the larger c, the "wider" the paraboloid.

31. The level curves for each surface are similar; for $z = \sqrt{x^2 + 4y^2}$ the level curves are ellipses of the form $\frac{x^2}{c^2} + \frac{y^2}{c^2/4} = 1$, i.e., $a = c$ and $b = c/2$; whereas for $z = x^2 + 4y^2$ the level curves are ellipses of the form $\frac{x^2}{c} + \frac{y^2}{c/4} = 1$, i.e., $a = \sqrt{c}$ and $b = \sqrt{c}/2$. The first set of ellipses are spaced evenly apart, meaning the function grows at a constant rate; the second set of ellipses are more closely spaced together as c grows, meaning the function grows faster and faster as c increases.
The function $z = \sqrt{x^2 + 4y^2}$ can be rewritten as $z^2 = x^2 + 4y^2$, an elliptic cone; the function $z = x^2 + 4y^2$ is a paraboloid, each matching the description above.

Section 11.2

1. Answers will vary.

3. Answers will vary.
One possible answer: $\{(x, y) | x^2 + y^2 \leq 1\}$

5. Answers will vary.
One possible answer: $\{(x, y) | x^2 + y^2 < 1\}$

7. (a) Answers will vary.
interior point: $(1, 3)$
boundary point: $(3, 3)$
   (b) S is a closed set
   (c) S is bounded

9. (a) Answers will vary.
interior point: none
boundary point: $(0, -1)$
   (b) S is a closed set, consisting only of boundary points
   (c) S is bounded

11. (a) $D = \{(x, y) \mid 9 - x^2 - y^2 \geq 0\}$.
   (b) D is a closed set.
   (c) D is bounded.

13. (a) $D = \{(x, y) \mid y > x^2\}$.
   (b) D is an open set.
   (c) D is unbounded.

15. (a) Along $y = 0$, the limit is 1.
   (b) Along $x = 0$, the limit is -1.
Since the above limits are not equal, the limit does not exist.

17. (a) Along $y = mx$, the limit is $\dfrac{mx(1 - m)}{m^2x + 1}$.
   (b) Along $x = 0$, the limit is -1.
Since the above limits are not equal, the limit does not exist.

19. (a) Along $y = 2$, the limit is:

$$\begin{aligned}\lim_{(x,y)\to(1,2)} \frac{x + y - 3}{x^2 - 1} &= \lim_{x\to 1} \frac{x - 1}{x^2 - 1} \\ &= \lim_{x\to 1} \frac{1}{x + 1} \\ &= 1/2.\end{aligned}$$

(b) Along $y = x + 1$, the limit is:

$$\lim_{(x,y)\to(1,2)} \frac{x+y-3}{x^2-1} = \lim_{x\to 1} \frac{2(x-1)}{x^2-1} = \lim_{x\to 1} \frac{2}{x+1} = 1.$$

Since the limits along the lines $y = 2$ and $y = x + 1$ differ, the overall limit does not exist.

Section 11.3

1. A constant is a number that is added or subtracted in an expression; a coefficient is a number that is being multiplied by a nonconstant function.

3. f_x

5. $f_x = 2xy - 1, f_y = x^2 + 2$
$f_x(1,2) = 3, f_y(1,2) = 3$

7. $f_x = -\sin x \sin y, f_y = \cos x \cos y$
$f_x(\pi/3, \pi/3) = -3/4, f_y(\pi/3, \pi/3) = 1/4$

9. $f_x = 2xy + 6x, f_y = x^2 + 4$
$f_{xx} = 2y + 6, f_{yy} = 0$
$f_{xy} = 2x, f_{yx} = 2x$

11. $f_x = 1/y, f_y = -x/y^2$
$f_{xx} = 0, f_{yy} = 2x/y^3$
$f_{xy} = -1/y^2, f_{yx} = -1/y^2$

13. $f_x = 2xe^{x^2+y^2}, f_y = 2ye^{x^2+y^2}$
$f_{xx} = 2e^{x^2+y^2} + 4x^2e^{x^2+y^2}, f_{yy} = 2e^{x^2+y^2} + 4y^2e^{x^2+y^2}$
$f_{xy} = 4xye^{x^2+y^2}, f_{yx} = 4xye^{x^2+y^2}$

15. $f_x = \cos x \cos y, f_y = -\sin x \sin y$
$f_{xx} = -\sin x \cos y, f_{yy} = -\sin x \cos y$
$f_{xy} = -\sin y \cos x, f_{yx} = -\sin y \cos x$

17. $f_x = -5y^3 \sin(5xy^3), f_y = -15xy^2 \sin(5xy^3)$
$f_{xx} = -25y^6 \cos(5xy^3),$
$f_{yy} = -225x^2y^4 \cos(5xy^3) - 30xy \sin(5xy^3)$
$f_{xy} = -75xy^5 \cos(5xy^3) - 15y^2 \sin(5xy^3),$
$f_{yx} = -75xy^5 \cos(5xy^3) - 15y^2 \sin(5xy^3)$

19. $f_x = \frac{2y^2}{\sqrt{4xy^2+1}}, f_y = \frac{4xy}{\sqrt{4xy^2+1}}$
$f_{xx} = -\frac{4y^4}{\sqrt{4xy^2+1}^3}, f_{yy} = -\frac{16x^2y^2}{\sqrt{4xy^2+1}^3} + \frac{4x}{\sqrt{4xy^2+1}}$
$f_{xy} = -\frac{8xy^3}{\sqrt{4xy^2+1}^3} + \frac{4y}{\sqrt{4xy^2+1}}, f_{yx} = -\frac{8xy^3}{\sqrt{4xy^2+1}^3} + \frac{4y}{\sqrt{4xy^2+1}}$

21. $f_x = -\frac{2x}{(x^2+y^2+1)^2}, f_y = -\frac{2y}{(x^2+y^2+1)^2}$
$f_{xx} = \frac{8x^2}{(x^2+y^2+1)^3} - \frac{2}{(x^2+y^2+1)^2}, f_{yy} = \frac{8y^2}{(x^2+y^2+1)^3} - \frac{2}{(x^2+y^2+1)^2}$
$f_{xy} = \frac{8xy}{(x^2+y^2+1)^3}, f_{yx} = \frac{8xy}{(x^2+y^2+1)^3}$

23. $f_x = 6x, f_y = 0$
$f_{xx} = 6, f_{yy} = 0$
$f_{xy} = 0, f_{yx} = 0$

25. $f_x = \frac{1}{4xy}, f_y = -\frac{\ln x}{4y^2}$
$f_{xx} = -\frac{1}{4x^2y}, f_{yy} = \frac{\ln x}{2y^3}$
$f_{xy} = -\frac{1}{4xy^2}, f_{yx} = -\frac{1}{4xy^2}$

27. $f(x, y) = x \sin y + x + C$, where C is any constant.

29. $f(x, y) = 3x^2y - 4xy^2 + 2y + C$, where C is any constant.

31. $f_x = 2xe^{2y-3z}, f_y = 2x^2e^{2y-3z}, f_z = -3x^2e^{2y-3z}$
$f_{yz} = -6x^2e^{2y-3z}, f_{zy} = -6x^2e^{2y-3z}$

33. $f_x = \frac{3}{7y^2z}, f_y = -\frac{6x}{7y^3z}, f_z = -\frac{3x}{7y^2z^2}$
$f_{yz} = \frac{6x}{7y^3z^2}, f_{zy} = \frac{6x}{7y^3z^2}$

Section 11.4

1. T

3. T

5. $dz = (\sin y + 2x)dx + (x \cos y)dy$

7. $dz = 5dx - 7dy$

9. $dz = \frac{x}{\sqrt{x^2+y}}dx + \frac{1}{2\sqrt{x^2+y}}dy$, with $dx = -0.05$ and $dy = .1$. At $(3, 7)$, $dz = 3/4(-0.05) + 1/8(.1) = -0.025$, so $f(2.95, 7.1) \approx -0.025 + 4 = 3.975$.

11. $dz = (2xy - y^2)dx + (x^2 - 2xy)dy$, with $dx = 0.04$ and $dy = 0.06$. At $(2, 3)$, $dz = 3(0.04) + (-8)(0.06) = -0.36$, so $f(2.04, 3.06) \approx -0.36 - 6 = -6.36$.

13. The total differential of volume is $dV = 4\pi dr + \pi dh$. The coefficient of dr is greater than the coefficient of dh, so the volume is more sensitive to changes in the radius.

15. Using trigonometry, $\ell = x \tan\theta$, so $d\ell = \tan\theta dx + x \sec^2\theta d\theta$. With $\theta = 85°$ and $x = 30$, we have $d\ell = 11.43dx + 3949.38d\theta$. The measured length of the wall is much more sensitive to errors in θ than in x. While it can be difficult to compare sensitivities between measuring feet and measuring degrees (it is somewhat like "comparing apples to oranges"), here the coefficients are so different that the result is clear: a small error in degree has a much greater impact than a small error in distance.

17. $dw = 2xyz^3\,dx + x^2z^3\,dy + 3x^2yz^2\,dz$

19. $dx = 0.05$, $dy = -0.1$. $dz = 9(.05) + (-2)(-0.1) = 0.65$. So $f(3.05, 0.9) \approx 7 + 0.65 = 7.65$.

21. $dx = 0.5$, $dy = 0.1$, $dz = -0.2$.
$dw = 2(0.5) + (-3)(0.1) + 3.7(-0.2) = -0.04$, so
$f(2.5, 4.1, 4.8) \approx -1 - 0.04 = -1.04$.

Section 11.5

1. Because the parametric equations describe a level curve, z is constant for all t. Therefore $\frac{dz}{dt} = 0$.

3. $\frac{dx}{dt}$, and $\frac{\partial f}{\partial y}$

5. F

7. (a) $\frac{dz}{dt} = 3(2t) + 4(2) = 6t + 8$.
(b) At $t = 1$, $\frac{dz}{dt} = 14$.

9. (a) $\frac{dz}{dt} = 5(-2\sin t) + 2(\cos t) = -10\sin t + 2\cos t$
(b) At $t = \pi/4$, $\frac{dz}{dt} = -4\sqrt{2}$.

11. (a) $\dfrac{dz}{dt} = 2x(\cos t) + 4y(3\cos t)$.
(b) At $t = \pi/4$, $x = \sqrt{2}/2$, $y = 3\sqrt{2}/2$, and $\frac{dz}{dt} = 19$.

13. $t = -4/3$; this corresponds to a minimum

15. $t = \tan^{-1}(1/5) + n\pi$, where n is an integer

17. We find that
$$\frac{dz}{dt} = 38\cos t \sin t.$$
Thus $\frac{dz}{dt} = 0$ when $t = \pi n$ or $\pi n + \pi/2$, where n is any integer.

19. (a) $\frac{\partial z}{\partial s} = 2xy(1) + x^2(2) = 2xy + 2x^2$;
$\frac{\partial z}{\partial t} = 2xy(-1) + x^2(4) = -2xy + 4x^2$
(b) With $s = 1$, $t = 0$, $x = 1$ and $y = 2$. Thus $\frac{\partial z}{\partial s} = 6$ and $\frac{\partial z}{\partial t} = 0$

21. (a) $\frac{\partial z}{\partial s} = 2x(\cos t) + 2y(\sin t) = 2x\cos t + 2y\sin t$;
$\frac{\partial z}{\partial t} = 2x(-s\sin t) + 2y(s\cos t) = -2xs\sin t + 2ys\cos t$
(b) With $s = 2$, $t = \pi/4$, $x = \sqrt{2}$ and $y = \sqrt{2}$. Thus $\frac{\partial z}{\partial s} = 4$ and $\frac{\partial z}{\partial t} = 0$

23. $f_x = 2x\tan y, f_y = x^2\sec^2 y$;
$\dfrac{dy}{dx} = -\dfrac{2\tan y}{x\sec^2 y}$

25. $f_x = \dfrac{(x+y^2)(2x)-(x^2+y)(1)}{(x+y^2)^2}$,
$f_y = \dfrac{(x+y^2)(1)-(x^2+y)(2y)}{(x+y^2)^2}$;
$\dfrac{dy}{dx} = -\dfrac{2x(x+y^2)-(x^2+y)}{x+y^2-2y(x^2+y)}$

27. $\frac{dz}{dt} = 2(4)+1(-5) = 3.$

29. $\frac{\partial z}{\partial s} = -4(5)+9(-2) = -38,$
$\frac{\partial z}{\partial t} = -4(7)+9(6) = 26.$

Section 11.6

1. A partial derivative is essentially a special case of a directional derivative; it is the directional derivative in the direction of x or y, i.e., $\langle 1,0\rangle$ or $\langle 0,1\rangle$.

3. $\vec{u} = \langle 0,1\rangle$

5. maximal, or greatest

7. $\nabla f = \left\langle -2xy+y^2+y, -x^2+2xy+x\right\rangle$

9. $\nabla f = \left\langle \frac{-2x}{(x^2+y^2+1)^2}, \frac{-2y}{(x^2+y^2+1)^2}\right\rangle$

11. $\nabla f = \langle 2x-y-7, 4y-x\rangle$

13. $\nabla f = \left\langle -2xy+y^2+y, -x^2+2xy+x\right\rangle$; $\nabla f(2,1) = \langle -2,2\rangle$. Be sure to change all directions to unit vectors.
    (a) $2/5$ ($\vec{u} = \langle 3/5, 4/5\rangle$)
    (b) $-2\sqrt{5}$ ($\vec{u} = \left\langle -1/\sqrt{5}, -2\sqrt{5}\right\rangle$)

15. $\nabla f = \left\langle \frac{-2x}{(x^2+y^2+1)^2}, \frac{-2y}{(x^2+y^2+1)^2}\right\rangle$; $\nabla f(1,1) = \langle -2/9, -2/9\rangle$. Be sure to change all directions to unit vectors.
    (a) 0 ($\vec{u} = \left\langle 1/\sqrt{2}, -1/\sqrt{2}\right\rangle$)
    (b) $2\sqrt{2}/9$ ($\vec{u} = \left\langle -1/\sqrt{2}, -1/\sqrt{2}\right\rangle$)

17. $\nabla f = \langle 2x-y-7, 4y-x\rangle$; $\nabla f(4,1) = \langle 0,0\rangle$.
    (a) 0
    (b) 0

19. $\nabla f = \left\langle -2xy+y^2+y, -x^2+2xy+x\right\rangle$
    (a) $\nabla f(2,1) = \langle -2,2\rangle$
    (b) $||\nabla f(2,1)|| = ||\langle -2,2\rangle|| = \sqrt{8}$
    (c) $\langle 2,-2\rangle$
    (d) $\left\langle 1/\sqrt{2}, 1/\sqrt{2}\right\rangle$

21. $\nabla f = \left\langle \frac{-2x}{(x^2+y^2+1)^2}, \frac{-2y}{(x^2+y^2+1)^2}\right\rangle$
    (a) $\nabla f(1,1) = \langle -2/9, -2/9\rangle$.
    (b) $||\nabla f(1,1)|| = ||\langle -2/9, -2/9\rangle|| = 2\sqrt{2}/9$
    (c) $\langle 2/9, 2/9\rangle$
    (d) $\left\langle 1/\sqrt{2}, -1/\sqrt{2}\right\rangle$

23. $\nabla f = \langle 2x-y-7, 4y-x\rangle$
    (a) $\nabla f(4,1) = \langle 0,0\rangle$
    (b) 0
    (c) $\langle 0,0\rangle$
    (d) All directions give a directional derivative of 0.

25. (a) $\nabla F(x,y,z) = \left\langle 6xz^3+4y, 4x, 9x^2z^2-6z\right\rangle$
    (b) $113/\sqrt{3}$

27. (a) $\nabla F(x,y,z) = \left\langle 2xy^2, 2y(x^2-z^2), -2y^2z\right\rangle$
    (b) 0

Section 11.7

1. Answers will vary. The displacement of the vector is one unit in the x-direction and 3 units in the z-direction, with no change in y. Thus along a line parallel to $\vec{v}$, the change in z is 3 times the change in x – i.e., a "slope" of 3. Specifically, the line in the x-z plane parallel to z has a slope of 3.

3. T

5. (a) $\ell_x(t) = \begin{cases} x = 2+t \\ y = 3 \\ z = -48-12t \end{cases}$
   (b) $\ell_y(t) = \begin{cases} x = 2 \\ y = 3+t \\ z = -48-40t \end{cases}$
   (c) $\ell_{\vec{u}}(t) = \begin{cases} x = 2+t/\sqrt{10} \\ y = 3+3t/\sqrt{10} \\ z = -48-66\sqrt{2/5}t \end{cases}$

7. (a) $\ell_x(t) = \begin{cases} x = 4+t \\ y = 2 \\ z = 2+3t \end{cases}$
   (b) $\ell_y(t) = \begin{cases} x = 4 \\ y = 2+t \\ z = 2-5t \end{cases}$
   (c) $\ell_{\vec{u}}(t) = \begin{cases} x = 4+t/\sqrt{2} \\ y = 2+t/\sqrt{2} \\ z = 2-\sqrt{2}t \end{cases}$

9. $\ell_{\vec{n}}(t) = \begin{cases} x = 2-12t \\ y = 3-40t \\ z = -48-t \end{cases}$

11. $\ell_{\vec{n}}(t) = \begin{cases} x = 4+3t \\ y = 2-5t \\ z = 2-t \end{cases}$

13. $(1.425, 1.085, -48.078)$, $(2.575, 4.915, -47.952)$

15. $(5.014, 0.31, 1.662)$ and $(2.986, 3.690, 2.338)$

17. $-12(x-2)-40(y-3)-(z+48) = 0$

19. $3(x-4)-5(y-2)-(z-2) = 0$ (Note that this tangent plane is the same as the original function, a plane.)

21. $\nabla F = \langle x/4, y/2, z/8\rangle$; at P, $\nabla F = \left\langle 1/4, \sqrt{2}/2, \sqrt{6}/8\right\rangle$
    (a) $\ell_{\vec{n}}(t) = \begin{cases} x = 1+t/4 \\ y = \sqrt{2}+\sqrt{2}t/2 \\ z = \sqrt{6}+\sqrt{6}t/8 \end{cases}$
    (b) $\frac{1}{4}(x-1)+\frac{\sqrt{2}}{2}(y-\sqrt{2})+\frac{\sqrt{6}}{8}(z-\sqrt{6}) = 0.$

23. $\nabla F = \left\langle y^2-z^2, 2xy, -2xz\right\rangle$; at P, $\nabla F = \langle 0,4,4\rangle$
    (a) $\ell_{\vec{n}}(t) = \begin{cases} x = 2 \\ y = 1+4t \\ z = -1+4t \end{cases}$
    (b) $4(y-1)+4(z+1) = 0.$

Section 11.8

1. F; it is the "other way around."

3. T

5. One critical point at $(-4,2)$; $f_{xx} = 1$ and $D = 4$, so this point corresponds to a relative minimum.

7. One critical point at $(6, -3)$; $D = -4$, so this point corresponds to a saddle point.

9. Two critical points: at $(0, -1)$; $f_{xx} = 2$ and $D = -12$, so this point corresponds to a saddle point;
at $(0, 1)$, $f_{xx} = 2$ and $D = 12$, so this corresponds to a relative minimum.

11. One critical point at $(0, 0)$. $D = -12x^2y^2$, so at $(0, 0)$, $D = 0$ and the test is inconclusive. (Some elementary thought shows that it is the absolute minimum.)

13. One critical point: $f_x = 0$ when $x = 3$; $f_y = 0$ when $y = 0$, so one critical point at $(3, 0)$, which is a relative maximum, where
$f_{xx} = \frac{y^2-16}{(16-(x-3)^2-y^2)^{3/2}}$ and $D = \frac{16}{(16-(x-3)^2-y^2)^2}$.
Both f_x and f_y are undefined along the circle $(x-3)^2 + y^2 = 16$; at any point along this curve, $f(x, y) = 0$, the absolute minimum of the function.

15. The triangle is bound by the lines $y = -1$, $y = 2x + 1$ and $y = -2x + 1$.
Along $y = -1$, there is a critical point at $(0, -1)$.
Along $y = 2x + 1$, there is a critical point at $(-3/5, -1/5)$.
Along $y = -2x + 1$, there is a critical point at $(3/5, -1/5)$.
The function f has one critical point, irrespective of the constraint, at $(0, -1/2)$.
Checking the value of f at these four points, along with the three vertices of the triangle, we find the absolute maximum is at $(0, 1, 3)$ and the absolute minimum is at $(0, -1/2, 3/4)$.

17. The region has no "corners" or "vertices," just a smooth edge.
To find critical points along the circle $x^2 + y^2 = 4$, we solve for y^2: $y^2 = 4 - x^2$. We can go further and state $y = \pm\sqrt{4 - x^2}$.
We can rewrite f as
$f(x) = x^2 + 2x + (4 - x^2) + \sqrt{4 - x^2} = 2x + 4 + \sqrt{4 - x^2}$. (We will return and use $-\sqrt{4 - x^2}$ later.) Solving $f'(x) = 0$, we get $x = \sqrt{2} \Rightarrow y = \sqrt{2}$. $f'(x)$ is also undefined at $x = \pm 2$, where $y = 0$.
Using $y = -\sqrt{4 - x^2}$, we rewrite $f(x, y)$ as
$f(x) = 2x + 4 - \sqrt{4 - x^2}$. Solving $f'(x) = 0$, we get $x = -\sqrt{2},\ y = -\sqrt{2}$.
The function f itself has a critical point at $(-1, -1)$.
Checking the value of f at $(-1, -1)$, $(\sqrt{2}, \sqrt{2})$, $(-\sqrt{2}, -\sqrt{2})$, $(2, 0)$ and $(-2, 0)$, we find the absolute maximum is at $(2, 0, 8)$ and the absolute minimum is at $(-1, -1, -2)$.

Chapter 12

Section 12.1

1. $C(y)$, meaning that instead of being just a constant, like the number 5, it is a function of y, which acts like a constant when taking derivatives with respect to x.

3. curve to curve, then from point to point

5. (a) $18x^2 + 42x - 117$
   (b) -108

7. (a) $x^4/2 - x^2 + 2x - 3/2$
   (b) $23/15$

9. (a) $\sin^2 y$
   (b) $\pi/2$

11. $\int_1^4 \int_{-2}^1 dy\,dx$ and $\int_{-2}^1 \int_1^4 dx\,dy$.
area of $R = 9u^2$

13. $\int_2^4 \int_{x-1}^{7-x} dy\,dx$. The order $dx\,dy$ needs two iterated integrals as x is bounded above by two different functions. This gives:
$$\int_1^3 \int_2^{y+1} dx\,dy + \int_3^5 \int_2^{7-y} dx\,dy.$$
area of $R = 4u^2$

15. $\int_0^1 \int_{x^4}^{\sqrt{x}} dy\,dx$ and $\int_0^1 \int_{y^2}^{\sqrt[4]{y}} dx\,dy$
area of $R = 7/15u^2$

17.

area of $R = \int_0^4 \int_{-\sqrt{4-y}}^{\sqrt{4-y}} dx\,dy$

19.

area of $R = \int_0^4 \int_{-\sqrt{4-x^2/4}}^{\sqrt{4-x^2/4}} dy\,dx$

21.

area of $R = \int_{-1}^2 \int_{x^2}^{x+2} dy\,dx$

Section 12.2

1. volume

3. The double integral gives the signed volume under the surface. Since the surface is always positive, it is always above the x-y plane and hence produces only "positive" volume.

5. 6; $\int_{-1}^1 \int_1^2 \left(\frac{x}{y} + 3\right) dy\,dx$

7. 112/3; $\int_0^2 \int_0^{4-2y} \left(3x^2 - y + 2\right) dx\,dy$

9. 16/5; $\int_{-1}^1 \int_0^{1-x^2} (x + y + 2)\ dy\,dx$

11. (a)

(b) $\int_0^1 \int_{x^2}^{\sqrt{x}} x^2 y \, dy \, dx = \int_0^1 \int_{y^2}^{\sqrt{y}} x^2 y \, dx \, dy.$

(c) $\frac{3}{56}$

13. (a)

(b) $\int_{-1}^1 \int_{-1}^1 x^2 - y^2 \, dy \, dx = \int_{-1}^1 \int_{-1}^1 x^2 - y^2 \, dx \, dy.$

(c) 0

15. (a)

(b)

(c) $\int_0^2 \int_0^{3-3/2x} (6 - 3x - 2y) \, dy \, dx =$
$\int_0^3 \int_0^{2-2/3y} (6 - 3x - 2y) \, dx \, dy.$

(d) 6

17. (a)

(b) $\int_{-3}^3 \int_0^{\sqrt{9-x^2}} (x^3 y - x) \, dy \, dx =$
$\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} (x^3 y - x) \, dx \, dy.$

(c) 0

19. Integrating e^{x^2} with respect to x is not possible in terms of elementary functions. $\int_0^2 \int_0^{2x} e^{x^2} \, dy \, dx = e^4 - 1.$

21. Integrating $\int_y^1 \frac{2y}{x^2 + y^2} \, dx$ gives $\tan^{-1}(1/y) - \pi/4$; integrating $\tan^{-1}(1/y)$ is hard.
$\int_0^1 \int_0^x \frac{2y}{x^2 + y^2} \, dy \, dx = \ln 2.$

23. average value of $f = 6/2 = 3$

25. average value of $f = \frac{112/3}{4} = 28/3$

Section 12.3

1. $f(r\cos\theta, r\sin\theta), r \, dr \, d\theta$

3. $\int_0^{2\pi} \int_0^1 (3r\cos\theta - r\sin\theta + 4) r \, dr \, d\theta = 4\pi$

5. $\int_0^{\pi} \int_{\cos\theta}^{3\cos\theta} (8 - r\sin\theta) r \, dr \, d\theta = 16\pi$

7. $\int_0^{2\pi} \int_1^2 (\ln(r^2)) r \, dr \, d\theta = 2\pi(\ln 16 - 3/2)$

9. $\int_{-\pi/2}^{\pi/2} \int_0^6 (r^2 \cos^2\theta - r^2 \sin^2\theta) r \, dr \, d\theta =$
$\int_{-\pi/2}^{\pi/2} \int_0^6 (r^2 \cos(2\theta)) r \, dr \, d\theta = 0$

11. $\int_{-\pi/2}^{\pi/2} \int_0^5 (r^2) \, dr \, d\theta = 125\pi/3$

13. $\int_0^{\pi/4} \int_0^{\sqrt{8}} (r\cos\theta + r\sin\theta) r \, dr \, d\theta = 16\sqrt{2}/3$

15. (a) This is impossible to integrate with rectangular coordinates as $e^{-(x^2+y^2)}$ does not have an antiderivative in terms of elementary functions.

(b) $\int_0^{2\pi} \int_0^a r e^{r^2} \, dr \, d\theta = \pi(1 - e^{-a^2}).$

(c) $\lim_{a\to\infty} \pi(1 - e^{-a^2}) = \pi$. This implies that there is a finite volume under the surface $e^{-(x^2+y^2)}$ over the entire x-y plane.

Section 12.4

1. Because they are scalar multiples of each other.

3. "little masses"

5. M_x measures the moment about the x-axis, meaning we need to measure distance from the x-axis. Such measurements are measures in the y-direction.

7. $\bar{x} = 5.25$

9. $(\bar{x}, \bar{y}) = (0, 3)$

11. $M = 150$gm;

13. $M = 2$lb

15. $M = 16\pi \approx 50.27$kg

17. $M = 54\pi \approx 169.65$lb

19. $M = 150$gm; $M_y = 600$; $M_x = -75$; $(\bar{x}, \bar{y}) = (4, -0.5)$

21. $M = 2$lb; $M_y = 0$; $M_x = 2/3$; $(\bar{x}, \bar{y}) = (0, 1/3)$

23. $M = 16\pi \approx 50.27$kg; $M_y = 4\pi$; $M_x = 4\pi$; $(\bar{x}, \bar{y}) = (1/4, 1/4)$

25. $M = 54\pi \approx 169.65$lb; $M_y = 0$; $M_x = 504$; $(\bar{x}, \bar{y}) = (0, 2.97)$

27. $I_x = 64/3$; $I_y = 64/3$; $I_O = 128/3$
29. $I_x = 16/3$; $I_y = 64/3$; $I_O = 80/3$

Section 12.5

1. arc length
3. surface areas
5. Intuitively, adding h to f only shifts f up (i.e., parallel to the z-axis) and does not change its shape. Therefore it will not change the surface area over R.
   Analytically, $f_x = g_x$ and $f_y = g_y$; therefore, the surface area of each is computed with identical double integrals.
7. $SA = \int_0^{2\pi}\int_0^{2\pi} \sqrt{1+\cos^2 x\cos^2 y + \sin^2 x \sin^2 y}\, dx\, dy$
9. $SA = \int_{-1}^{1}\int_{-1}^{1} \sqrt{1+4x^2+4y^2}\, dx\, dy$
11. $SA = \int_0^3\int_{-1}^{1} \sqrt{1+9+49}\, dx\, dy = 6\sqrt{59} \approx 46.09$
13. This is easier in polar:
$$\begin{aligned} SA &= \int_0^{2\pi}\int_0^4 r\sqrt{1+4r^2\cos^2 t + 4r^2\sin^2 t}\, dr\, d\theta \\ &= \int_0^{2\pi}\int_0^4 r\sqrt{1+4r^2}\, dr\, d\theta \\ &= \frac{\pi}{6}\left(65\sqrt{65}-1\right) \approx 273.87 \end{aligned}$$
15.
$$\begin{aligned} SA &= \int_0^2\int_0^{2x} \sqrt{1+1+4x^2}\, dy\, dx \\ &= \int_0^2 \left(2x\sqrt{2+4x^2}\right) dx \\ &= \frac{26}{3}\sqrt{2} \approx 12.26 \end{aligned}$$
17. This is easier in polar:
$$\begin{aligned} SA &= \int_0^{2\pi}\int_0^5 r\sqrt{1+\frac{4r^2\cos^2 t + 4r^2\sin^2 t}{r^2\sin^2 t + r^2\cos^2 t}}\, dr\, d\theta \\ &= \int_0^{2\pi}\int_0^5 r\sqrt{5}\, dr\, d\theta \\ &= 25\pi\sqrt{5} \approx 175.62 \end{aligned}$$
19. Integrating in polar is easiest considering R:
$$\begin{aligned} SA &= \int_0^{2\pi}\int_0^1 r\sqrt{1+c^2+d^2}\, dr\, d\theta \\ &= \int_0^{2\pi} \frac{1}{2}\left(\sqrt{1+c^2+d^2}\right) dy \\ &= \pi\sqrt{1+c^2+d^2}. \end{aligned}$$
   The value of h does not matter as it only shifts the plane vertically (i.e., parallel to the z-axis). Different values of h do not create different ellipses in the plane.

Section 12.6

1. surface to surface, curve to curve and point to point
3. Answers can vary. From this section we used triple integration to find the volume of a solid region, the mass of a solid, and the center of mass of a solid.
5. $V = \int_{-1}^{1}\int_{-1}^{1} \left(8 - x^2 - y^2 - (2x+y)\right) dx\, dy = 88/3$
7. $V = \int_0^{\pi}\int_0^{x} \left(\cos x \sin y + 2 - \sin x \cos y\right) dy\, dx = \pi^2 - \pi \approx 6.728$
9. $dz\, dy\, dx$: $\int_0^3\int_0^{1-x/3}\int_0^{2-2x/3-2y} dz\, dy\, dx$
   $dz\, dx\, dy$: $\int_0^1\int_0^{3-3y}\int_0^{2-2x/3-2y} dz\, dx\, dy$
   $dy\, dz\, dx$: $\int_0^3\int_0^{2-2x/3}\int_0^{1-x/3-z/2} dy\, dz\, dx$
   $dy\, dx\, dz$: $\int_0^2\int_0^{3-3z/2}\int_0^{1-x/3-z/2} dy\, dx\, dz$
   $dx\, dz\, dy$: $\int_0^1\int_0^{2-2y}\int_0^{3-3y-3z/2} dx\, dz\, dy$
   $dx\, dy\, dz$: $\int_0^2\int_0^{1-z/2}\int_0^{3-3y-3z/2} dx\, dy\, dz$
   $V = \int_0^3\int_0^{1-x/3}\int_0^{2-2x/3-2y} dz\, dy\, dx = 1.$
11. $dz\, dy\, dx$: $\int_0^2\int_{-2}^{0}\int_{y^2/2}^{-y} dz\, dy\, dx$
   $dz\, dx\, dy$: $\int_{-2}^{0}\int_0^2\int_{y^2/2}^{-y} dz\, dx\, dy$
   $dy\, dz\, dx$: $\int_0^2\int_0^2\int_{-\sqrt{2z}}^{-z} dy\, dz\, dx$
   $dy\, dx\, dz$: $\int_0^2\int_0^2\int_{-\sqrt{2z}}^{-z} dy\, dx\, dz$
   $dx\, dz\, dy$: $\int_{-2}^{0}\int_{y^2/2}^{-y}\int_0^2 dx\, dz\, dy$
   $dx\, dy\, dz$: $\int_0^2\int_{-\sqrt{2z}}^{-z}\int_0^2 dx\, dy\, dz$
   $V = \int_0^2\int_0^2\int_{-\sqrt{2z}}^{-z} dy\, dz\, dx = 4/3.$
13. $dz\, dy\, dx$: $\int_0^2\int_{1-x/2}^{1}\int_0^{2x+4y-4} dz\, dy\, dx$
   $dz\, dx\, dy$: $\int_0^1\int_{2-2y}^{2}\int_0^{2x+4y-4} dz\, dx\, dy$
   $dy\, dz\, dx$: $\int_0^2\int_0^{2x}\int_{z/4-x/2+1}^{1} dy\, dz\, dx$
   $dy\, dx\, dz$: $\int_0^4\int_{z/2}^{2}\int_{z/4-x/2+1}^{1} dy\, dx\, dz$
   $dx\, dz\, dy$: $\int_0^1\int_0^{4y}\int_{z/2-2y+2}^{2} dx\, dz\, dy$
   $dx\, dy\, dz$: $\int_0^4\int_{z/4}^{1}\int_{z/2-2y+2}^{2} dx\, dy\, dz$
   $V = \int_0^4\int_{z/4}^{1}\int_{2y-z/2-2}^{2} dx\, dy\, dz = 4/3.$
15. $dz\, dy\, dx$: $\int_0^1\int_0^{1-x^2}\int_0^{\sqrt{1-y}} dz\, dy\, dx$
   $dz\, dx\, dy$: $\int_0^1\int_0^{\sqrt{1-y}}\int_0^{\sqrt{1-y}} dz\, dx\, dy$
   $dy\, dz\, dx$: $\int_0^1\int_0^{x}\int_0^{1-x^2} dy\, dz\, dx + \int_0^1\int_x^1\int_0^{1-z^2} dy\, dz\, dx$
   $dy\, dx\, dz$: $\int_0^1\int_0^{z}\int_0^{1-z^2} dy\, dx\, dz + \int_0^1\int_z^1\int_0^{1-x^2} dy\, dx\, dz$
   $dx\, dz\, dy$: $\int_0^1\int_0^{\sqrt{1-y}}\int_0^{\sqrt{1-y}} dx\, dz\, dy$
   $dx\, dy\, dz$: $\int_0^1\int_0^{1-z^2}\int_0^{\sqrt{1-y}} dx\, dy\, dz$
   Answers will vary. Neither order is particularly "hard." The order $dz\, dy\, dx$ requires integrating a square root, so powers can be messy; the order $dy\, dz\, dx$ requires two triple integrals, but each uses only polynomials.
17. 8
19. π
21. $M = 10$, $M_{yz} = 15/2$, $M_{xz} = 5/2$, $M_{xy} = 5$; $(\bar{x}, \bar{y}, \bar{z}) = (3/4, 1/4, 1/2)$
23. $M = 16/5$, $M_{yz} = 16/3$, $M_{xz} = 104/45$, $M_{xy} = 32/9$; $(\bar{x}, \bar{y}, \bar{z}) = (5/3, 13/18, 10/9) \approx (1.67, 0.72, 1.11)$

22. Converges; compare to $\sum_{n=1}^{\infty} \frac{1}{n^2}$, as $1/(n^2 \ln n) \leq 1/n^2$ for all $n \geq 2$.

23. Converges; compare to $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

24. Converges; compare to $\sum_{n=1}^{\infty} \frac{1}{4^n}$.

25. Diverges; compare to $\sum_{n=1}^{\infty} \frac{\ln n}{n}$.

26. Diverges; compare to $\sum_{n=1}^{\infty} \frac{1}{n}$.

27. Diverges; compare to $\sum_{n=1}^{\infty} \frac{1}{n}$.

28. Diverges; compare to $\sum_{n=1}^{\infty} \frac{1}{n}$.

29. Diverges; compare to $\sum_{n=1}^{\infty} \frac{1}{n}$. Just as $\lim_{n\to 0} \frac{\sin n}{n} = 1$, $\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1$.

30. Converges; compare to $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

31. Converges; compare to $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$.

32. Diverges; compare to $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$.

33. Converges; Integral Test

34. Converges; Integral Test, p-Series Test, Direct & Limit Comparison Tests can all be used.

35. Diverges; the nth Term Test and Direct Comparison Test can be used.

36. Converges; the Direct Comparison Test can be used with sequence $1/(n-1)!$.

37. Converges; the Direct Comparison Test can be used with sequence $1/3^n$.

38. Diverges; the nth Term Test can be used, along with the Limit Comparison Test (compare with 1/10).

39. Diverges; the nth Term Test can be used, along with the Integral Test.

40. Converges; the Direct Comparison Test can be used with sequence $1/\sqrt{n}$.

41. (a) Converges; use Direct Comparison Test as $\frac{a_n}{n} < n$.
    (b) Converges; since original series converges, we know $\lim_{n\to\infty} a_n = 0$. Thus for large n, $a_n a_{n+1} < a_n$.
    (c) Converges; similar logic to part (b) so $(a_n)^2 < a_n$.
    (d) May converge; certainly $na_n > a_n$ but that does not mean it does not converge.
    (e) Does not converge, using logic from (b) and n^{th} Term Test.

Section 8.4

1. algebraic, or polynomial.
2. factorial and/or exponential
3. Integral Test, Limit Comparison Test, and Root Test
4. raised to a power
5. Converges
6. Diverges
7. Converges
8. Converges
9. The Ratio Test is inconclusive; the p-Series Test states it diverges.
10. The Ratio Test is inconclusive; the Direct Comparison Test with $1/n^3$ shows it converges.
11. Converges
12. Converges
13. Converges; note the summation can be rewritten as $\sum_{n=1}^{\infty} \frac{2^n n!}{3^n n!}$, from which the Ratio Test can be applied.
14. Converges; rewrite the summation as $\sum_{n=1}^{\infty} \frac{n!}{5^n n!}$ then apply the Ratio Test.
15. Converges
16. Converges
17. Converges
18. Converges
19. Diverges
20. Converges
21. Diverges. The Root Test is inconclusive, but the nth-Term Test shows divergence. (The terms of the sequence approach e^2, not 0, as $n \to \infty$.)
22. Converges
23. Converges
24. Converges
25. Diverges; Limit Comparison Test
26. Converges; Ratio Test
27. Converges; Ratio Test or Limit Comparison Test with $1/3^n$.
28. Converges; Root Test
29. Diverges; n^{th}-Term Test or Limit Comparison Test with 1.
30. Converges; Ratio Test
31. Diverges; Direct Comparison Test with $1/n$
32. Diverges; n^{th}-Term Test (n^{th} term approaches e.)
33. Converges; Root Test
34. Converges; Limit Comparison Test with $1/n^2$ (get common denominator first). It is also a Telescoping Series.

Section 8.5

1. The signs of the terms do not alternate; in the given series, some terms are negative and the others positive, but they do not necessarily alternate.
2. postive, decreasing, 0
3. Many examples exist; one common example is $a_n = (-1)^n/n$.
4. conditionally
5. (a) converges
   (b) converges (p-Series)
   (c) absolute
6. (a) converges

(b) converges (use Ratio Test)
(c) absolute

7. (a) diverges (limit of terms is not 0)
(b) diverges
(c) n/a; diverges

8. (a) diverges (limit of terms is not 0)
(b) diverges
(c) n/a; diverges

9. (a) converges
(b) diverges (Limit Comparison Test with $1/n$)
(c) conditional

10. (a) converges
(b) diverges (Direct Comparison Test with $1/n$)
(c) conditional

11. (a) diverges (limit of terms is not 0)
(b) diverges
(c) n/a; diverges

12. (a) converges
(b) converges (the sum in the denominator is n^2)
(c) absolute

13. (a) diverges (terms oscillate between ± 1)
(b) diverges
(c) n/a; diverges

14. (a) converges
(b) diverges (Integral Test)
(c) conditional

15. (a) converges
(b) converges (Geometric Series with $r = 2/3$)
(c) absolute

16. (a) converges
(b) converges (Geometric Series with $r = 1/e$)
(c) absolute

17. (a) converges
(b) converges (Ratio Test)
(c) absolute

18. (a) converges
(b) converges (Ratio Test)
(c) absolute

19. (a) converges
(b) diverges (p-Series Test with $p = 1/2$)
(c) conditional

20. (a) converges
(b) converges (Ratio Test)
(c) absolute

21. $S_5 = -1.1906$; $S_6 = -0.6767$;
$$-1.1906 \le \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)} \le -0.6767$$

22. $S_4 = 0.9459$; $S_5 = 0.9475$;
$$0.9459 \le \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \le 0.9475$$

23. $S_6 = 0.3681$; $S_7 = 0.3679$;
$$0.3681 \le \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \le 0.3679$$

24. $S_9 = 0.666016$; $S_{10} = 0.666992$;
$$0.666016 \le \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n \le 0.666992$$

25. $n = 5$

26. $n = 7$

27. Using the theorem, we find $n = 499$ guarantees the sum is within 0.001 of $\pi/4$. (Convergence is actually faster, as the sum is within ε of $\pi/24$ when $n \ge 249$.)

28. $n = 5$ ($(2n)! > 10^8$ when $n \ge 6$)

Section 8.6

1. 1

2. The radius of convergence is a **value** R such that a power series, centered at $x = c$, converges for all values of x in $(c - R, c + R)$. The interval of convergence is an **interval** on which the power series converges; it may differ from $(c - R, c + R)$ only at the endpoints.

3. 5

4. 5

5. $1 + 2x + 4x^2 + 8x^3 + 16x^4$

6. $x + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \frac{x^5}{25}$

7. $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$

8. $1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320}$

9. (a) $R = \infty$
(b) $(-\infty, \infty)$

10. (a) $R = 1$
(b) $(-1, 1)$

11. (a) $R = 1$
(b) $(2, 4]$

12. (a) $R = \infty$
(b) $(-\infty, \infty)$

13. (a) $R = 2$
(b) $(-2, 2)$

14. (a) $R = 10$
(b) $(-5, 15)$

15. (a) $R = 1/5$
(b) $(4/5, 6/5)$

16. (a) $R = 1/2$
(b) $(-1/2, 1/2)$

17. (a) $R = 1$
(b) $(-1, 1)$

18. (a) $R = 3$
(b) $(-3, 3)$

19. (a) $R = \infty$
(b) $(-\infty, \infty)$

20. (a) $R = 0$
(b) $x = 10$

21. (a) $R = 1$

(b) $[-1, 1]$

22. (a) $R = 1$

(b) $[-3, -1]$

23. (a) $R = 0$

(b) $x = 0$

24. (a) $R = 4$

(b) $x = (-8, 0)$

25. (a) $f'(x) = \sum_{n=1}^{\infty} n^2 x^{n-1}; \quad (-1, 1)$

(b) $\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{n}{n+1} x^{n+1}; \quad (-1, 1)$

26. (a) $f'(x) = \sum_{n=1}^{\infty} x^{n-1}; \quad (-1, 1)$

(b) $\int f(x)\,dx = C + \sum_{n=1}^{\infty} \frac{1}{n(n+1)} x^{n+1}; \quad [-1, 1]$

27. (a) $f'(x) = \sum_{n=1}^{\infty} \frac{n}{2^n} x^{n-1}; \quad (-2, 2)$

(b) $\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{1}{(n+1)2^n} x^{n+1}; \quad [-2, 2)$

28. (a) $f'(x) = \sum_{n=1}^{\infty} n(-3)^n x^{n-1}; \quad (-1/3, 1/3)$

(b) $\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{(-3)^n}{n+1} x^{n+1}; \quad (-1/3, 1/3]$

29. (a) $f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!}$; $(-\infty, \infty)$

(b) $\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}; \quad (-\infty, \infty)$

30. (a) $f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^n}{n!}; \quad (-\infty, \infty)$

(b) $\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)!}; \quad (-\infty, \infty)$

31. $1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4$

32. $5 + 25x + \frac{125}{2}x^2 + \frac{625}{6}x^3 + \frac{3125}{24}x^4$

33. $1 + x + x^2 + x^3 + x^4$

34. $1 + 2x + x^2 + \frac{1}{3}x^3 + \frac{1}{12}x^4$

35. $0 + x + 0x^2 - \frac{1}{6}x^3 + 0x^4$

36. $1 + x + x^2 + \frac{1}{3}x^3 + \frac{1}{6}x^4$

Section 8.7

1. The Maclaurin polynomial is a special case of Taylor polynomials. Taylor polynomials are centered at a specific x-value; when that x-value is 0, it is a Maclauring polynomial.

2. T

3. $p_2(x) = 6 + 3x - 4x^2$.

4. $f'''(0) = 30$

5. $p_3(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3$

6. $p_8(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7$

7. $p_8(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5$

8. $p_6(x) = \frac{2x^5}{15} + \frac{x^3}{3} + x$

9. $p_4(x) = \frac{2x^4}{3} + \frac{4x^3}{3} + 2x^2 + 2x + 1$

10. $p_4(x) = x^4 + x^3 + x^2 + x + 1$

11. $p_4(x) = x^4 - x^3 + x^2 - x + 1$

12. $p_7(x) = -\frac{x^7}{7} + \frac{x^5}{5} - \frac{x^3}{3} + x$

13. $p_4(x) = 1 + \frac{1}{2}(-1+x) - \frac{1}{8}(-1+x)^2 + \frac{1}{16}(-1+x)^3 - \frac{5}{128}(-1+x)^4$

14. $p_4(x) =$ $\ln(2) + \frac{1}{2}(-1+x) - \frac{1}{8}(-1+x)^2 + \frac{1}{24}(-1+x)^3 - \frac{1}{64}(-1+x)^4$

15. $p_6(x) = \frac{1}{\sqrt{2}} - \frac{-\frac{\pi}{4}+x}{\sqrt{2}} - \frac{\left(-\frac{\pi}{4}+x\right)^2}{2\sqrt{2}} + \frac{\left(-\frac{\pi}{4}+x\right)^3}{6\sqrt{2}} + \frac{\left(-\frac{\pi}{4}+x\right)^4}{24\sqrt{2}} - \frac{\left(-\frac{\pi}{4}+x\right)^5}{120\sqrt{2}} - \frac{\left(-\frac{\pi}{4}+x\right)^6}{720\sqrt{2}}$

16. $p_5(x) = \frac{1}{2} + \frac{1}{2}\sqrt{3}\left(-\frac{\pi}{6}+x\right) - \frac{1}{4}\left(-\frac{\pi}{6}+x\right)^2 - \frac{\left(-\frac{\pi}{6}+x\right)^3}{4\sqrt{3}} + \frac{1}{48}\left(-\frac{\pi}{6}+x\right)^4 + \frac{\left(-\frac{\pi}{6}+x\right)^5}{80\sqrt{3}}$

17. $p_5(x) = \frac{1}{2} - \frac{x-2}{4} + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3 + \frac{1}{32}(x-2)^4 - \frac{1}{64}(x-2)^5$

18. $p_8(x) = 1 - 2(-1+x) + 3(-1+x)^2 - 4(-1+x)^3 + 5(-1+x)^4 - 6(-1+x)^5 + 7(-1+x)^6 - 8(-1+x)^7 + 9(-1+x)^8$

19. $p_3(x) = \frac{1}{2} + \frac{1+x}{2} + \frac{1}{4}(1+x)^2$

20. $p_2(x) = -\pi^2 - 2\pi(x-\pi) + \frac{1}{2}\left(\pi^2 - 2\right)(x-\pi)^2$

21. $p_3(x) = x - \frac{x^3}{6}$; $p_3(0.1) = 0.09983$. Error is bounded by $\pm\frac{1}{4!} \cdot 0.1^4 \approx \pm 0.000004167$.

22. $p_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$; $p_4(1) = 13/24 \approx 0.54167$. Error is bounded by $\pm\frac{1}{5!} \cdot 1^5 \approx \pm 0.00833$

23. $p_2(x) = 3 + \frac{1}{6}(-9+x) - \frac{1}{216}(-9+x)^2$; $p_2(10) = 3.16204$. The third derivative of $f(x) = \sqrt{x}$ is bounded on $(8, 11)$ by 0.003. Error is bounded by $\pm\frac{0.003}{3!} \cdot 1^3 = \pm 0.0005$.

24. $p_3(x) = -1 + x - \frac{1}{2}(-1+x)^2 + \frac{1}{3}(-1+x)^3$; $p_3(1.5) = 0.41667$. The fourth derivative of $f(x) = \ln x$ is bounded on $(.9, 2)$ by 10. Error is bounded by $\pm\frac{10}{4!} \cdot .5^4 = \pm 0.026$.

25. The nth derivative of $f(x) = e^x$ is bounded by 3 on intervals containing 0 and 1. Thus $|R_n(1)| \le \frac{3}{(n+1)!} 1^{(n+1)}$. When $n = 7$, this is less than 0.0001.

26. The nth derivative of $f(x) = \sqrt{x}$ is bounded by 0.1 on intervals containing 3 and 4. Thus $|R_n(\pi)| \le \frac{0.1}{(n+1)!}(1)^{(n+1)}$. When $n = 4$, this is less than 0.0001.

27. The nth derivative of $f(x) = \cos x$ is bounded by 1 on intervals containing 0 and $\pi/3$. Thus $|R_n(\pi/3)| \le \frac{1}{(n+1)!}(\pi/3)^{(n+1)}$. When $n = 7$, this is less than 0.0001. Since the Maclaurin polynomial of $\cos x$ only uses even powers, we can actually just use $n = 6$.

28. The nth derivative of $f(x) = \sin x$ is bounded by 1 on intervals containing 0 and π. Thus $|R_n(\pi)| \le \frac{1}{(n+1)!}(\pi)^{(n+1)}$. When $n = 12$, this is less than 0.0001. Since the Maclaurin polynomial of $\sin x$ only uses odd powers, we can actually just use $n = 11$.

29. The nth term is $\frac{1}{n!}x^n$.

30. The nth term is: when n is even, $\frac{(-1)^{n/2}}{n!}x^n$; when n is odd, 0.

31. The nth term is x^n.

32. The nth term is $(-1)^n x^n$.

33. The nth term is $(-1)^n \frac{(x-1)^n}{n}$.

34. $1 + x + \dfrac{1}{2}x^2 + \dfrac{1}{6}x^3 + \dfrac{1}{24}x^4$

35. $3 + 15x + \dfrac{75}{2}x^2 + \dfrac{375}{6}x^3 + \dfrac{1875}{24}x^4$

36. $1 + 2x - 2x^2 + 4x^3 - 10x^4$

Section 8.8

1. A Taylor polynomial is a **polynomial**, containing a finite number of terms. A Taylor series is a **series**, the summation of an infinite number of terms.

2. Theorem **??**, entitled "Function and Taylor Series Equality"

3. All derivatives of e^x are e^x which evaluate to 1 at $x = 0$.
   The Taylor series starts $1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$;
   the Taylor series is $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

4. All derivatives of $\sin x$ are either $\pm\cos x$ or $\pm\sin x$, which evaluate to ± 1 or 0 at $x = 0$. The Taylor series starts
   $0 + x + 0x^2 - \frac{1}{6}x^3 + 0x^4 + \frac{1}{120}x^5$;
   the Taylor series is $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

5. The nth derivative of $1/(1-x)$ is $f^{(n)}(x) = (n)!/(1-x)^{n+1}$, which evaluates to $n!$ at $x = 0$.
   The Taylor series starts $1 + x + x^2 + x^3 + \cdots$;
   the Taylor series is $\displaystyle\sum_{n=0}^{\infty} x^n$

6. The derivative of $\tan^{-1} x$ is $1/(1+x^2)$. Taking successive derivatives using the Quotient Rule, the derivatives of $\tan^{-1} x$ fall into two categories in terms of their evaluation at $x = 0$.
   When n is even, $f^{(n)}(x) = (-1)^{(n-1)/2} \dfrac{p(x)}{(1+x^2)^n}$, where $p(x)$ is a polynomial such that $p(0) = 0$. Hence $f^{(n)}(0) = 0$ when n is even.
   When n is odd, $f^{(n)}(x) = (-1)^{(n-1)/2} \dfrac{p(x)}{(1+x^2)^n}$, where $p(x)$ is a polynomial such that $p(0) = (n-1)!$. Hence $f^{(n)}(0) = (-1)^{(n-1)/2}(n-1)!$ when n is odd. (The unusual power of (-1) is such that every other odd term is negative.)
   The Taylor series starts $x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots$; by reindexing to only obtain odd powers of x, we get
   the Taylor series is $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

7. The Taylor series starts
   $0 - (x - \pi/2) + 0x^2 + \frac{1}{6}(x - \pi/2)^3 + 0x^4 - \frac{1}{120}(x - \pi/2)^5$;
   the Taylor series is $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-\pi/2)^{2n+1}}{(2n+1)!}$

8. The Taylor series starts
   $1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - (x-1)^5$;
   the Taylor series is $\displaystyle\sum_{n=0}^{\infty} (-1)^n (x-1)^n$

9. $f^{(n)}(x) = (-1)^n e^{-x}$; at $x = 0$, $f^{(n)}(0) = -1$ when n is odd and $f^{(n)}(0) = 1$ when n is even.
   The Taylor series starts $1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \cdots$;
   the Taylor series is $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$.

10. $f^{(n)}(x) = (-1)^{n+1} \frac{(n-1)!}{(1+x)^n}$; at $x = 0$, $f^{(n)}(0) = (-1)^{n+1}(n-1)!$
    The Taylor series starts $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$;
    the Taylor series is $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$.

11. $f^{(n)}(x) = (-1)^{n+1} \frac{n!}{(x+1)^{n+1}}$; at $x = 1$, $f^{(n)}(1) = (-1)^{n+1} \frac{n!}{2^{n+1}}$
    The Taylor series starts
    $\frac{1}{2} + \frac{1}{4}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 \cdots$;
    the Taylor series is $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{2^{n+1}}$.

12. The derivatives of $\sin x$ are $\pm\cos x$ and $\pm\sin x$; at $x = \pi/4$, these derivatives evaluate to $\pm\sqrt{2}/2$.
    The Taylor series starts $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - \pi/4) - \frac{\sqrt{2}}{2}\frac{(x-\pi/4)^2}{2} - \frac{\sqrt{2}}{2}\frac{(x-\pi/4)^3}{3!} + \frac{\sqrt{2}}{2}\frac{(x-\pi/4)^4}{4!} + \frac{\sqrt{2}}{2}\frac{(x-\pi/4)^5}{5!} \cdots$. Note how the signs are "even, even, odd, odd, even, even, odd, odd,. . . We saw signs like these in Example **??** of Section **??**; one way of producing such signs is to raise (-1) to a special quadratic power. While many possibilities exist, one such quadratic is $(n+3)(n+4)/2$.
    Thus the Taylor series is $\displaystyle\sum_{n=0}^{\infty} (-1)^{\frac{(n+3)(n+4)}{2}} \frac{\sqrt{2}}{2} \frac{(x-\pi/4)^n}{n!}$.

13. Given a value x, the magnitude of the error term $R_n(x)$ is bounded by
    $$\left|R_n(x)\right| \le \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|x^{(n+1)}\right|,$$
    where z is between 0 and x.
    If $x > 0$, then $z < x$ and $f^{(n+1)}(z) = e^z < e^x$. If $x < 0$, then $x < z < 0$ and $f^{(n+1)}(z) = e^z < 1$. So given a fixed x value, let $M = \max\{e^x, 1\}$; $f^{(n)}(z) < M$. This allows us to state
    $$\left|R_n(x)\right| \le \frac{M}{(n+1)!}\left|x^{(n+1)}\right|.$$
    For any x, $\displaystyle\lim_{n\to\infty} \frac{M}{(n+1)!}\left|x^{(n+1)}\right| = 0$. Thus by the Squeeze Theorem, we conclude that $\displaystyle\lim_{n\to\infty} R_n(x) = 0$ for all x, and hence
    $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x.$$

14. The following argument is essentially the same as that given for $f(x) = \cos x$ in Example **??**.
    Given a value x, the magnitude of the error term $R_n(x)$ is bounded by
    $$\left|R_n(x)\right| \le \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|x^{(n+1)}\right|.$$
    Since all derivatives of $\sin x$ are $\pm\cos x$ or $\pm\sin x$, whose magnitudes are bounded by 1, we can state
    $$\left|R_n(x)\right| \le \frac{1}{(n+1)!}\left|x^{(n+1)}\right|.$$
    For any x, $\displaystyle\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$. Thus by the Squeeze Theorem, we conclude that $\displaystyle\lim_{n\to\infty} R_n(x) = 0$ for all x, and hence
    $$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{for all } x.$$

15. Given a value x, the magnitude of the error term $R_n(x)$ is bounded by
    $$\left|R_n(x)\right| \le \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|(x-1)^{(n+1)}\right|,$$
    where z is between 1 and x.
    Note that $\left|f^{(n+1)}(x)\right| = \frac{n!}{x^{n+1}}$.
    We consider the cases when $x > 1$ and when $x < 1$ separately.

If $x > 1$, then $1 < z < x$ and $f^{(n+1)}(z) = \frac{n!}{z^{n+1}} < n!$. Thus

$$\left|R_n(x)\right| \leq \frac{n!}{(n+1)!}\left|(x-1)^{(n+1)}\right| = \frac{(x-1)^{n+1}}{n+1}.$$

For a fixed x,

$$\lim_{n\to\infty} \frac{(x-1)^{n+1}}{n+1} = 0.$$

If $0 < x < 1$, then $x < z < 1$ and $f^{(n+1)}(z) = \frac{n!}{z^{n+1}} < \frac{n!}{x^{n+1}}$. Thus

$$\left|R_n(x)\right| \leq \frac{n!/x^{n+1}}{(n+1)!}\left|(x-1)^{(n+1)}\right| = \frac{x^{n+1}}{n+1}(1-x)^{n+1}.$$

Since $0 < x < 1$, $x^{n+1} < 1$ and $(1-x)^{n+1} < 1$. We can then extend the inequality from above to state

$$\left|R_n(x)\right| \leq \frac{x^{n+1}}{n+1}(1-x)^{n+1} < \frac{1}{n+1}.$$

As $n \to \infty$, $1/(n+1) \to 0$. Thus by the Squeeze Theorem, we conclude that $\lim_{n\to\infty} R_n(x) = 0$ for all x, and hence

$$\ln x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n} \quad \text{for all } 0 < x \leq 2.$$

16. Given a value x, the magnitude of the error term $R_n(x)$ is bounded by

$$\left|R_n(x)\right| \leq \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|x^{(n+1)}\right|,$$

where z is between 0 and x.
Note that $\left|f^{(n+1)}(x)\right| = \frac{(n+1)!}{(1-x)^{n+2}}$.
If $0 < x < 1$, then $0 < z < x$ and $f^{(n+1)}(z) = \frac{(n+1)!}{(1-z)^{n+2}} < \frac{(n+1)!}{(1-x)^{n+2}}$. Thus

$$\left|R_n(x)\right| \leq \frac{(n+1)!}{(1-x)^{n+2}}\frac{1}{(n+1)!}\left|x^{n+1}\right| = \frac{(x-1)^{n+1}}{n+1}.$$

For a fixed x,

$$\lim_{n\to\infty} \frac{(x-1)^{n+1}}{n+1} = 0,$$

hence

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ on } (-1, 0).$$

17. Given $\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$,
$\cos(-x) = \sum_{n=0}^{\infty}(-1)^n \frac{(-x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} = \cos x$, as all powers in the series are even.

18. Given $\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$,
$\sin(-x) = \sum_{n=0}^{\infty}(-1)^n \frac{(-x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \frac{-x^{2n+1}}{(2n+1)!} = -\sin x$, as all powers in the series are odd.

19. Given $\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$,
$\frac{d}{dx}\left(\sin x\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}\right) = \sum_{n=0}^{\infty}(-1)^n \frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} = \cos x$. (The summation still starts at $n = 0$ as there was no constant term in the expansion of $\sin x$).

20. Given $\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$,
$\frac{d}{dx}\left(\cos x\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}\right) = \sum_{n=1}^{\infty}(-1)^n \frac{(2n)x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty}(-1)^n \frac{x^{2n-1}}{(2n-1)!}$. We can re-index this summation to start at $n = 0$ by replacing n with $n+1$ in the summation:

$$\sum_{n=1}^{\infty}(-1)^n \frac{x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty}(-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!}.$$

Note that this series has the opposite sign of the Taylor series for $\sin x$; thus $\frac{d}{dx}(\cos x) = -\sin x$.

21. $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128}$

22. $1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + \frac{35x^4}{128}$

23. $1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243}$

24. $1 + 4x + 6x^2 + 4x^3 + x^4$ (note the series is finite, and the formula still applies)

25. $\sum_{n=0}^{\infty}(-1)^n \frac{(x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n}}{(2n)!}$.

26. $\sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$.

27. $\sum_{n=0}^{\infty}(-1)^n \frac{(2x+3)^{2n+1}}{(2n+1)!}$.

28. $\sum_{n=0}^{\infty}(-1)^n \frac{(x/2)^{2n+1}}{(2n+1)}$.

29. $x + x^2 + \frac{x^3}{3} - \frac{x^5}{30}$

30. $1 + \frac{x}{2} - \frac{5x^2}{8} - \frac{3x^3}{16}$

31. $\int_0^{\sqrt{\pi}} \sin\left(x^2\right)\,dx \approx \int_0^{\sqrt{\pi}} \left(x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \frac{x^{14}}{5040}\right)\,dx = 0.8877$

32. $\int_0^{\pi^2/4} \cos\left(\sqrt{x}\right)\,dx \approx \int_0^{\pi^2/4} \left(1 - \frac{x}{2} + \frac{x^2}{24} - \frac{x^3}{720}\right)\,dx = 1.1412$. (Actual answer: $\pi - 2$)

Chapter 9

Section 9.1

1. When defining the conics as the intersections of a plane and a double napped cone, degenerate conics are created when the plane intersects the tips of the cones (usually taken as the origin). Nondegenerate conics are formed when this plane does not contain the origin.
2. Answers will vary.
3. Hyperbola
4. With the equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, the ellipse has a horizontal major axis if $a > b$. But the coefficient of the x^2 term is $1/a^2$ (not a^2), so if $1/a^2 < 1/b^2$, then $a > b$ and the major axis is horizontal.
5. With a horizontal transverse axis, the x^2 term has a positive coefficient; with a vertical transverse axis, the y^2 term has a positive coefficient.
6. $y = \frac{1}{2}(x-3)^2 + \frac{3}{2}$

7. $y = \frac{-1}{12}(x+1)^2 - 1$

8. $x = -\frac{1}{4}(y-5)^2 + 2$

9. $x = y^2$

10. $y = -\frac{1}{4}(x-1)^2 + 2$

11. $x = -\frac{1}{12}y^2$

12. $y = 4x^2$

13. $x = -\frac{1}{8}(y-3)^2 + 2$

14. focus: $(0,1)$; directrix: $y = -1$. The point P is 2 units from each.

15. focus: $(5,2)$; directrix: $x = 1$. The point P is 10 units from each.

16.

17.

18. $\frac{(x+1)^2}{9} + \frac{(y-2)^2}{4} = 1$; foci at $(-1 \pm \sqrt{5}, 2)$; $e = \sqrt{5}/3$

19. $\frac{(x-1)^2}{1/4} + \frac{y^2}{9} = 1$; foci at $(1, \pm\sqrt{8.75})$; $e = \sqrt{8.75}/3 \approx 0.99$

20. $\frac{x^2}{9} + \frac{y^2}{5} = 1$

21. $\frac{(x-2)^2}{25} + \frac{(y-3)^2}{16} = 1$

22. $\frac{(x-2)^2}{45} + \frac{y^2}{49} = 1$

23. $\frac{(x+1)^2}{9} + \frac{(y-1)^2}{25} = 1$

24. $\frac{(x-1)^2}{2} + (y-2)^2 = 1$

25. $\frac{x^2}{3} + \frac{y^2}{5} = 1$

26. $\frac{x^2}{4} + \frac{(y-3)^2}{6} = 1$

27. $\frac{(x-2)^2}{4} + \frac{(y-2)^2}{4} = 1$

28. (a) $c = \sqrt{12-4} = 2\sqrt{2}$.

    (b) The sum of distances for each point is $2\sqrt{12} \approx 6.9282$.

29. $x^2 - \frac{y^2}{3} = 1$

30. $y^2 - \frac{x^2}{24} = 1$

31. $\frac{(y-3)^2}{4} - \frac{(x-1)^2}{9} = 1$

32. $\frac{(x-1)^2}{9} - \frac{(y-3)^2}{4} = 1$

33.

34.

35. $\frac{x^2}{4} - \frac{y^2}{5} = 1$

36. $\frac{y^2}{4} - \frac{x^2}{5} = 1$

37. $\frac{(x-3)^2}{16} - \frac{(y-3)^2}{9} = 1$

38. $\frac{(y-3)^2}{9} - \frac{(x-3)^2}{16} = 1$

39. $\frac{x^2}{4} - \frac{y^2}{3} = 1$

40. $\frac{x^2}{3} - \frac{(y-1)^2}{9} = 1$

41. $(y-2)^2 - \frac{x^2}{10} = 1$

42. $4y^2 - \frac{x^2}{4} = 1$

43. (a) Solve for c in $e = c/a$: $c = ae$. Thus $a^2e^2 = a^2 - b^2$, and $b^2 = a^2 - a^2e^2$. The result follows.

    (b) Mercury: $x^2/(0.387)^2 + y^2/(0.3787)^2 = 1$
    Earth: $x^2 + y^2/(0.99986)^2 = 1$
    Mars: $x^2/(1.524)^2 + y^2/(1.517)^2 = 1$

    (c) Mercury: $(x-0.08)^2/(0.387)^2 + y^2/(0.3787)^2 = 1$
    Earth: $(x-0.0167)^2 + y^2/(0.99986)^2 = 1$
    Mars: $(x-0.1423)^2/(1.524)^2 + y^2/(1.517)^2 = 1$

44. The sound originated from a point approximately 31m to the left of B and 1340m above it.

Section 9.2

1. T
2. orientation
3. rectangular
4. Answers will vary.
5.

Differentiation Rules

1. $\frac{d}{dx}(cx) = c$
2. $\frac{d}{dx}(u \pm v) = u' \pm v'$
3. $\frac{d}{dx}(u \cdot v) = uv' + u'v$
4. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$
5. $\frac{d}{dx}(u(v)) = u'(v)v'$
6. $\frac{d}{dx}(c) = 0$
7. $\frac{d}{dx}(x) = 1$
8. $\frac{d}{dx}(x^n) = nx^{n-1}$
9. $\frac{d}{dx}(e^x) = e^x$
10. $\frac{d}{dx}(a^x) = \ln a \cdot a^x$
11. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
12. $\frac{d}{dx}(\log_a x) = \frac{1}{\ln a} \cdot \frac{1}{x}$
13. $\frac{d}{dx}(\sin x) = \cos x$
14. $\frac{d}{dx}(\cos x) = -\sin x$
15. $\frac{d}{dx}(\csc x) = -\csc x \cot x$
16. $\frac{d}{dx}(\sec x) = \sec x \tan x$
17. $\frac{d}{dx}(\tan x) = \sec^2 x$
18. $\frac{d}{dx}(\cot x) = -\csc^2 x$
19. $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$
20. $\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$
21. $\frac{d}{dx}\left(\csc^{-1} x\right) = \frac{-1}{|x|\sqrt{x^2-1}}$
22. $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}$
23. $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$
24. $\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$
25. $\frac{d}{dx}(\cosh x) = \sinh x$
26. $\frac{d}{dx}(\sinh x) = \cosh x$
27. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
28. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
29. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$
30. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$
31. $\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}}$
32. $\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{x^2+1}}$
33. $\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = \frac{-1}{x\sqrt{1-x^2}}$
34. $\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = \frac{-1}{|x|\sqrt{1+x^2}}$
35. $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2}$
36. $\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1-x^2}$

Integration Rules

1. $\int c \cdot f(x)\, dx = c \int f(x)\, dx$
2. $\int f(x) \pm g(x)\, dx = \int f(x)\, dx \pm \int g(x)\, dx$
3. $\int 0\, dx = C$
4. $\int 1\, dx = x + C$
5. $\int x^n\, dx = \frac{1}{n+1} x^{n+1} + C,\ n \neq -1$
   $n \neq -1$
6. $\int e^x\, dx = e^x + C$
7. $\int a^x\, dx = \frac{1}{\ln a} \cdot a^x + C$
8. $\int \frac{1}{x}\, dx = \ln|x| + C$
9. $\int \cos x\, dx = \sin x + C$
10. $\int \sin x\, dx = -\cos x + C$
11. $\int \tan x\, dx = -\ln|\cos x| + C$
12. $\int \sec x\, dx = \ln|\sec x + \tan x| + C$
13. $\int \csc x\, dx = -\ln|\csc x + \cot x| + C$
14. $\int \cot x\, dx = \ln|\sin x| + C$
15. $\int \sec^2 x\, dx = \tan x + C$
16. $\int \csc^2 x\, dx = -\cot x + C$
17. $\int \sec x \tan x\, dx = \sec x + C$
18. $\int \csc x \cot x\, dx = -\csc x + C$
19. $\int \cos^2 x\, dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$
20. $\int \sin^2 x\, dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$
21. $\int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
22. $\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
23. $\int \frac{1}{x\sqrt{x^2-a^2}}\, dx = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right) + C$
24. $\int \cosh x\, dx = \sinh x + C$
25. $\int \sinh x\, dx = \cosh x + C$
26. $\int \tanh x\, dx = \ln(\cosh x) + C$
27. $\int \coth x\, dx = \ln|\sinh x| + C$
28. $\int \frac{1}{\sqrt{x^2-a^2}}\, dx = \ln\left|x + \sqrt{x^2-a^2}\right| + C$
29. $\int \frac{1}{\sqrt{x^2+a^2}}\, dx = \ln\left|x + \sqrt{x^2+a^2}\right| + C$
30. $\int \frac{1}{a^2-x^2}\, dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$
31. $\int \frac{1}{x\sqrt{a^2-x^2}}\, dx = \frac{1}{a}\ln\left(\frac{x}{a+\sqrt{a^2-x^2}}\right) + C$
32. $\int \frac{1}{x\sqrt{x^2+a^2}}\, dx = \frac{1}{a}\ln\left|\frac{x}{a+\sqrt{x^2+a^2}}\right| + C$

The Unit Circle

Definitions of the Trigonometric Functions

Unit Circle Definition

$$\sin\theta = y \qquad \cos\theta = x$$

$$\csc\theta = \frac{1}{y} \qquad \sec\theta = \frac{1}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Right Triangle Definition

$$\sin\theta = \frac{\text{O}}{\text{H}} \qquad \csc\theta = \frac{\text{H}}{\text{O}}$$

$$\cos\theta = \frac{\text{A}}{\text{H}} \qquad \sec\theta = \frac{\text{H}}{\text{A}}$$

$$\tan\theta = \frac{\text{O}}{\text{A}} \qquad \cot\theta = \frac{\text{A}}{\text{O}}$$

Common Trigonometric Identities

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x \qquad \csc\left(\frac{\pi}{2} - x\right) = \sec x$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x \qquad \sec\left(\frac{\pi}{2} - x\right) = \csc x$$

$$\tan\left(\frac{\pi}{2} - x\right) = \cot x \qquad \cot\left(\frac{\pi}{2} - x\right) = \tan x$$

Double Angle Formulas

$$\sin 2x = 2\sin x\cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$= 2\cos^2 x - 1$$

$$= 1 - 2\sin^2 x$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Sum to Product Formulas

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Power–Reducing Formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

Even/Odd Identities

$$\sin(-x) = -\sin x$$

$$\cos(-x) = \cos x$$

$$\tan(-x) = -\tan x$$

$$\csc(-x) = -\csc x$$

$$\sec(-x) = \sec x$$

$$\cot(-x) = -\cot x$$

Product to Sum Formulas

$$\sin x\sin y = \frac{1}{2}\big(\cos(x-y) - \cos(x+y)\big)$$

$$\cos x\cos y = \frac{1}{2}\big(\cos(x-y) + \cos(x+y)\big)$$

$$\sin x\cos y = \frac{1}{2}\big(\sin(x+y) + \sin(x-y)\big)$$

Angle Sum/Difference Formulas

$$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$$

$$\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x\tan y}$$

Areas and Volumes

Triangles

$h = a \sin\theta$

Area = $\frac{1}{2}bh$

Law of Cosines:
$c^2 = a^2 + b^2 - 2ab\cos\theta$

Right Circular Cone

Volume = $\frac{1}{3}\pi r^2 h$

Surface Area =
$\pi r\sqrt{r^2 + h^2} + \pi r^2$

Parallelograms

Area = bh

Right Circular Cylinder

Volume = $\pi r^2 h$

Surface Area =
$2\pi rh + 2\pi r^2$

Trapezoids

Area = $\frac{1}{2}(a + b)h$

Sphere

Volume = $\frac{4}{3}\pi r^3$

Surface Area =$4\pi r^2$

Circles

Area = πr^2

Circumference = $2\pi r$

General Cone

Area of Base = A

Volume = $\frac{1}{3}Ah$

Sectors of Circles

θ in radians

Area = $\frac{1}{2}\theta r^2$

$s = r\theta$

General Right Cylinder

Area of Base = A

Volume = Ah

Algebra

Factors and Zeros of Polynomials

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial. If $p(a) = 0$, then a is a *zero* of the polynomial and a solution of the equation $p(x) = 0$. Furthermore, $(x - a)$ is a *factor* of the polynomial.

Fundamental Theorem of Algebra

An nth degree polynomial has n (not necessarily distinct) zeros. Although all of these zeros may be imaginary, a real polynomial of odd degree must have at least one real zero.

Quadratic Formula

If $p(x) = ax^2 + bx + c$, and $0 \leq b^2 - 4ac$, then the real zeros of p are $x = (-b \pm \sqrt{b^2 - 4ac})/2a$

Special Factors

$x^2 - a^2 = (x - a)(x + a)$ $\qquad$ $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$

$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$ $\qquad$ $x^4 - a^4 = (x^2 - a^2)(x^2 + a^2)$

$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdots + nxy^{n-1} + y^n$

$(x - y)^n = x^n - nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 - \cdots \pm nxy^{n-1} \mp y^n$

Binomial Theorem

$(x + y)^2 = x^2 + 2xy + y^2$ $\qquad$ $(x - y)^2 = x^2 - 2xy + y^2$

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ $\qquad$ $(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ $\qquad$ $(x - y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

Rational Zero Theorem

If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients, then every *rational zero* of p is of the form $x = r/s$, where r is a factor of a_0 and s is a factor of a_n.

Factoring by Grouping

$acx^3 + adx^2 + bcx + bd = ax^2(cs + d) + b(cx + d) = (ax^2 + b)(cx + d)$

Arithmetic Operations

$ab + ac = a(b + c)$ $\qquad$ $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd}$ $\qquad$ $\dfrac{a + b}{c} = \dfrac{a}{c} + \dfrac{b}{c}$

$\dfrac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \left(\dfrac{a}{b}\right)\left(\dfrac{d}{c}\right) = \dfrac{ad}{bc}$ $\qquad$ $\dfrac{\left(\frac{a}{b}\right)}{c} = \dfrac{a}{bc}$ $\qquad$ $\dfrac{a}{\left(\frac{b}{c}\right)} = \dfrac{ac}{b}$

$a\left(\dfrac{b}{c}\right) = \dfrac{ab}{c}$ $\qquad$ $\dfrac{a - b}{c - d} = \dfrac{b - a}{d - c}$ $\qquad$ $\dfrac{ab + ac}{a} = b + c$

Exponents and Radicals

$a^0 = 1, \quad a \neq 0$ $\qquad$ $(ab)^x = a^x b^x$ $\qquad$ $a^x a^y = a^{x+y}$ $\qquad$ $\sqrt{a} = a^{1/2}$ $\qquad$ $\dfrac{a^x}{a^y} = a^{x-y}$ $\qquad$ $\sqrt[n]{a} = a^{1/n}$

$\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$ $\qquad$ $\sqrt[n]{a^m} = a^{m/n}$ $\qquad$ $a^{-x} = \dfrac{1}{a^x}$ $\qquad$ $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ $\qquad$ $(a^x)^y = a^{xy}$ $\qquad$ $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Additional Formulas

Summation Formulas:

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left[f(x_1) + 2f(x_2) + 2f(x_3) + \ldots + 2f(x_n) + f(x_{n+1})\right]$$

with Error $\leq \dfrac{(b-a)^3}{12n^2}\left[\max\left|f''(x)\right|\right]$

Simpson's Rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + \ldots + 2f(x_{n-1}) + 4f(x_n) + f(x_{n+1})\right]$$

with Error $\leq \dfrac{(b-a)^5}{180n^4}\left[\max\left|f^{(4)}(x)\right|\right]$

Arc Length:

$$L = \int_a^b \sqrt{1 + f'(x)^2}\;dx$$

Surface of Revolution:

$$S = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\;dx$$

(where $f(x) \geq 0$)

$$S = 2\pi \int_a^b x\sqrt{1 + f'(x)^2}\;dx$$

(where $a, b \geq 0$)

Work Done by a Variable Force:

$$W = \int_a^b F(x)\;dx$$

Force Exerted by a Fluid:

$$F = \int_a^b w\,d(y)\,\ell(y)\;dy$$

Taylor Series Expansion for $f(x)$:

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Maclaurin Series Expansion for $f(x)$, where $c = 0$:

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n$$

Summary of Tests for Series:

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} r^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	Sum $= \dfrac{1}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+a})$	$\lim_{n\to\infty} b_n = L$		Sum $= \left(\sum_{n=1}^{a} b_n\right) - L$
p-Series	$\sum_{n=1}^{\infty} \dfrac{1}{(an+b)^p}$	$p > 1$	$p \leq 1$	
Integral Test	$\sum_{n=0}^{\infty} a_n$	$\int_1^{\infty} a(n)\, dn$ is convergent	$\int_1^{\infty} a(n)\, dn$ is divergent	$a_n = a(n)$ must be continuous
Direct Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$	$\sum_{n=0}^{\infty} b_n$ diverges and $0 \leq b_n \leq a_n$	
Limit Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $\lim_{n\to\infty} a_n/b_n \geq 0$	$\sum_{n=0}^{\infty} b_n$ diverges and $\lim_{n\to\infty} a_n/b_n > 0$	Also diverges if $\lim_{n\to\infty} a_n/b_n = \infty$
Ratio Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} < 1$	$\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n\to\infty} a_{n+1}/a_n = \infty$
Root Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} (a_n)^{1/n} < 1$	$\lim_{n\to\infty} (a_n)^{1/n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n\to\infty} (a_n)^{1/n} = \infty$