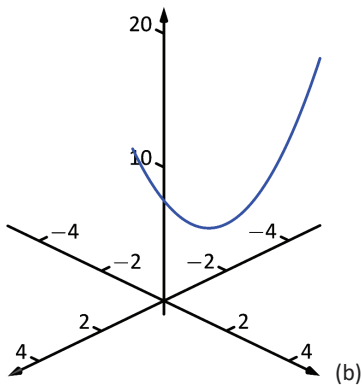


(a)



(b)

Figure 12.11: By fixing $y = 2$, the surface $f(x, y) = x^2 + 2y^2$ is a curve in space.

Alternate notations for $f_x(x, y)$ include:

$$\frac{\partial}{\partial x} f(x, y), \quad \frac{\partial f}{\partial x}, \quad \frac{\partial z}{\partial x}, \quad \text{and } z_x,$$

with similar notations for $f_y(x, y)$. For ease of notation, $f_x(x, y)$ is often abbreviated f_x .

12.3 Partial Derivatives

Let y be a function of x . We have studied in great detail the derivative of y with respect to x , that is, $\frac{dy}{dx}$, which measures the rate at which y changes with respect to x . Consider now $z = f(x, y)$. It makes sense to want to know how z changes with respect to x and/or y . This section begins our investigation into these rates of change.

Consider the function $z = f(x, y) = x^2 + 2y^2$, as graphed in Figure 12.11(a). By fixing $y = 2$, we focus our attention to all points on the surface where the y -value is 2, shown in both parts (a) and (b) of the figure. These points form a curve in space: $z = f(x, 2) = x^2 + 8$ which is a function of just one variable. We can take the derivative of z with respect to x along this curve and find equations of tangent lines, etc.

The key notion to extract from this example is: by treating y as constant (it does not vary) we can consider how z changes with respect to x . In a similar fashion, we can hold x constant and consider how z changes with respect to y . This is the underlying principle of **partial derivatives**. We state the formal, limit-based definition first, then show how to compute these partial derivatives without directly taking limits.

Definition 87 Partial Derivative

Let $z = f(x, y)$ be a continuous function on an open set S in \mathbb{R}^2 .

1. The **partial derivative of f with respect to x** is:

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x + h, y) - f(x, y)}{h}.$$

2. The **partial derivative of f with respect to y** is:

$$f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

Example 12.16 Computing partial derivatives with the limit definition

Let $f(x, y) = x^2y + 2x + y^3$. Find $f_x(x, y)$ using the limit definition.

Notes:

SOLUTION Using Definition 87, we have:

$$\begin{aligned}
 f_x(x, y) &= \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x+h)^2 y + 2(x+h) + y^3 - (x^2 y + 2x + y^3)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x^2 y + 2xhy + h^2 y + 2x + 2h + y^3) - (x^2 y + 2x + y^3)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2xhy + h^2 y + 2h}{h} \\
 &= \lim_{h \rightarrow 0} 2xy + hy + 2 \\
 &= 2xy + 2.
 \end{aligned}$$

We have found $f_x(x, y) = 2xy + 2$.

Example 12.16 found a partial derivative using the formal, limit-based definition. Using limits is not necessary, though, as we can rely on our previous knowledge of derivatives to compute partial derivatives easily. When computing $f_x(x, y)$, we hold y fixed – it does not vary. Therefore we can compute the derivative with respect to x by treating y as a constant or coefficient.

Just as $\frac{d}{dx}(5x^2) = 10x$, we compute $\frac{\partial}{\partial x}(x^2 y) = 2xy$. Here we are treating y as a coefficient.

Just as $\frac{d}{dx}(5^3) = 0$, we compute $\frac{\partial}{\partial x}(y^3) = 0$. Here we are treating y as a constant. More examples will help make this clear.

Example 12.17 Finding partial derivatives

Find $f_x(x, y)$ and $f_y(x, y)$ in each of the following.

1. $f(x, y) = x^3 y^2 + 5y^2 - x + 7$
2. $f(x, y) = \cos(xy^2) + \sin x$
3. $f(x, y) = e^{x^2 y^3} \sqrt{x^2 + 1}$

SOLUTION

1. We have $f(x, y) = x^3 y^2 + 5y^2 - x + 7$.

Begin with $f_x(x, y)$. Keep y fixed, treating it as a constant or coefficient, as appropriate:

$$f_x(x, y) = 3x^2 y^2 - 1.$$

Note how the $5y^2$ and 7 terms go to zero.

Notes:

To compute $f_y(x, y)$, we hold x fixed:

$$f_y(x, y) = 2x^3y + 10y.$$

Note how the $-x$ and 7 terms go to zero.

2. We have $f(x, y) = \cos(xy^2) + \sin x$.

Begin with $f_x(x, y)$. We need to apply the Chain Rule with the cosine term; y^2 is the coefficient of the x -term inside the cosine function.

$$f_x(x, y) = -\sin(xy^2)(y^2) + \cos x = -y^2 \sin(xy^2) + \cos x.$$

To find $f_y(x, y)$, note that x is the coefficient of the y^2 term inside of the cosine term; also note that since x is fixed, $\sin x$ is also fixed, and we treat it as a constant.

$$f_y(x, y) = -\sin(xy^2)(2xy) = -2xy \sin(xy^2).$$

3. We have $f(x, y) = e^{x^2y^3} \sqrt{x^2 + 1}$.

Beginning with $f_x(x, y)$, note how we need to apply the Product Rule.

$$\begin{aligned} f_x(x, y) &= e^{x^2y^3} (2xy^3) \sqrt{x^2 + 1} + e^{x^2y^3} \frac{1}{2} (x^2 + 1)^{-1/2} (2x) \\ &= 2xy^3 e^{x^2y^3} \sqrt{x^2 + 1} + \frac{xe^{x^2y^3}}{\sqrt{x^2 + 1}}. \end{aligned}$$

Note that when finding $f_y(x, y)$ we do not have to apply the Product Rule; since $\sqrt{x^2 + 1}$ does not contain y , we treat it as fixed and hence becomes a coefficient of the $e^{x^2y^3}$ term.

$$f_y(x, y) = e^{x^2y^3} (3x^2y^2) \sqrt{x^2 + 1} = 3x^2y^2 e^{x^2y^3} \sqrt{x^2 + 1}.$$

We have shown *how* to compute a partial derivative, but it may still not be clear what a partial derivative *means*. Given $z = f(x, y)$, $f_x(x, y)$ measures the rate at which z changes as only x varies: y is held constant.

Imagine standing in a rolling meadow, then beginning to walk due east. Depending on your location, you might walk up, sharply down, or perhaps not change elevation at all. This is similar to measuring z_x : you are moving only east (in the “ x ”-direction) and not north/south at all. Going back to your original location, imagine now walking due north (in the “ y ”-direction). Perhaps walking due north does not change your elevation at all. This is analogous to $z_y = 0$: z does not change with respect to y . We can see that z_x and z_y do not have to be the same, or even similar, as it is easy to imagine circumstances where walking east means you walk downhill, though walking north makes you walk uphill.

Notes:

The following example helps us visualize this more.

Example 12.18 Evaluating partial derivatives

Let $z = f(x, y) = -x^2 - \frac{1}{2}y^2 + xy + 10$. Find $f_x(2, 1)$ and $f_y(2, 1)$ and interpret their meaning.

SOLUTION We begin by computing $f_x(x, y) = -2x + y$ and $f_y(x, y) = -y + x$. Thus

$$f_x(2, 1) = -3 \quad \text{and} \quad f_y(2, 1) = 1.$$

It is also useful to note that $f(2, 1) = 7.5$. What does each of these numbers mean?

Consider $f_x(2, 1) = -3$, along with Figure 12.12(a). If one “stands” on the surface at the point $(2, 1, 7.5)$ and moves parallel to the x -axis (i.e., only the x -value changes, not the y -value), then the instantaneous rate of change is -3 . Increasing the x -value will decrease the z -value; decreasing the x -value will increase the z -value.

Now consider $f_y(2, 1) = 1$, illustrated in Figure 12.12(b). Moving along the curve drawn on the surface, i.e., parallel to the y -axis and not changing the x -values, increases the z -value instantaneously at a rate of 1. Increasing the y -value by 1 would increase the z -value by approximately 1.

Since the magnitude of f_x is greater than the magnitude of f_y at $(2, 1)$, it is “steeper” in the x -direction than in the y -direction.

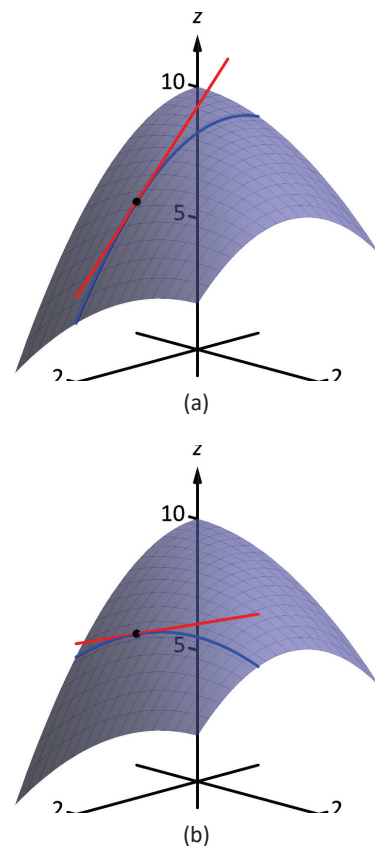


Figure 12.12: Illustrating the meaning of partial derivatives.

Second Partial Derivatives

Let $z = f(x, y)$. We have learned to find the partial derivatives $f_x(x, y)$ and $f_y(x, y)$, which are each functions of x and y . Therefore we can take partial derivatives of them, each with respect to x and y . We define these “second partials” along with the notation, give examples, then discuss their meaning.

Notes:

Definition 88 Second Partial Derivative, Mixed Partial Derivative

Let $z = f(x, y)$ be continuous on an open set S .

1. The **second partial derivative of f with respect to x then x** is

$$\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = (f_x)_x = f_{xx}$$

2. The **second partial derivative of f with respect to x then y** is

$$\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = (f_x)_y = f_{xy}$$

Similar definitions hold for $\frac{\partial^2 f}{\partial y^2} = f_{yy}$ and $\frac{\partial^2 f}{\partial x \partial y} = f_{yx}$.

The second partial derivatives f_{xy} and f_{yx} are **mixed partial derivatives**.

Note: The terms in Definition 88 all depend on limits, so each definition comes with the caveat “where the limit exists.”

The notation of second partial derivatives gives some insight into the notation of the second derivative of a function of a single variable. If $y = f(x)$, then $f''(x) = \frac{d^2 y}{dx^2}$. The “ $d^2 y$ ” portion means “take the derivative of y twice,” while “ dx^2 ” means “with respect to x both times.” When we only know of functions of a single variable, this latter phrase seems silly: there is only one variable to take the derivative with respect to. Now that we understand functions of multiple variables, we see the importance of specifying which variables we are referring to.

Example 12.19 Second partial derivatives

For each of the following, find all six first and second partial derivatives. That is, find

$$f_x, \quad f_y, \quad f_{xx}, \quad f_{yy}, \quad f_{xy} \quad \text{and} \quad f_{yx}.$$

1. $f(x, y) = x^3 y^2 + 2xy^3 + \cos x$

2. $f(x, y) = \frac{x^3}{y^2}$

3. $f(x, y) = e^x \sin(x^2 y)$

SOLUTION In each, we give f_x and f_y immediately and then spend time deriving the second partial derivatives.

Notes:

$$1. f(x, y) = x^3y^2 + 2xy^3 + \cos x$$

$$f_x(x, y) = 3x^2y^2 + 2y^3 - \sin x$$

$$f_y(x, y) = 2x^3y + 6xy^2$$

$$f_{xx}(x, y) = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}(3x^2y^2 + 2y^3 - \sin x) = 6xy^2 - \cos x$$

$$f_{yy}(x, y) = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}(2x^3y + 6xy^2) = 2x^3 + 12xy$$

$$f_{xy}(x, y) = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(3x^2y^2 + 2y^3 - \sin x) = 6x^2y + 6y^2$$

$$f_{yx}(x, y) = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(2x^3y + 6xy^2) = 6x^2y + 6y^2$$

$$2. f(x, y) = \frac{x^3}{y^2} = x^3y^{-2}$$

$$f_x(x, y) = \frac{3x^2}{y^2}$$

$$f_y(x, y) = -\frac{2x^3}{y^3}$$

$$f_{xx}(x, y) = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\left(\frac{3x^2}{y^2}\right) = \frac{6x}{y^2}$$

$$f_{yy}(x, y) = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}\left(-\frac{2x^3}{y^3}\right) = \frac{6x^3}{y^4}$$

$$f_{xy}(x, y) = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left(\frac{3x^2}{y^2}\right) = -\frac{6x^2}{y^3}$$

$$f_{yx}(x, y) = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\left(-\frac{2x^3}{y^3}\right) = -\frac{6x^2}{y^3}$$

$$3. f(x, y) = e^x \sin(x^2y)$$

Because the following partial derivatives get rather long, we omit the extra notation and just give the results. In several cases, multiple applications of the Product and Chain Rules will be necessary, followed by some basic combination of like terms.

$$f_x(x, y) = e^x \sin(x^2y) + 2xye^x \cos(x^2y)$$

$$f_y(x, y) = x^2e^x \cos(x^2y)$$

$$f_{xx}(x, y) = e^x \sin(x^2y) + 4xye^x \cos(x^2y) + 2ye^x \cos(x^2y) - 4x^2y^2e^x \sin(x^2y)$$

$$f_{yy}(x, y) = -x^4e^x \sin(x^2y)$$

$$f_{xy}(x, y) = x^2e^x \cos(x^2y) + 2xe^x \cos(x^2y) - 2x^3ye^x \sin(x^2y)$$

$$f_{yx}(x, y) = x^2e^x \cos(x^2y) + 2xe^x \cos(x^2y) - 2x^3ye^x \sin(x^2y)$$

Notes:

Notice how in each of the three functions in Example 12.19, $f_{xy} = f_{yx}$. Due to the complexity of the examples, this likely is not a coincidence. The following theorem states that it is not.

Theorem 105 Mixed Partial Derivatives

Let f be defined such that f_{xy} and f_{yx} are continuous on an open set S . Then for each point (x, y) in S , $f_{xy}(x, y) = f_{yx}(x, y)$.

Finding f_{xy} and f_{yx} independently and comparing the results provides a convenient way of checking our work.

Understanding Second Partial Derivatives

Now that we know *how* to find second partials, we investigate *what* they tell us.

Again we refer back to a function $y = f(x)$ of a single variable. The second derivative of f is “the derivative of the derivative,” or “the rate of change of the rate of change.” The second derivative measures how much the derivative is changing. If $f''(x) < 0$, then the derivative is getting smaller (so the graph of f is concave down); if $f''(x) > 0$, then the derivative is growing, making the graph of f concave up.

Now consider $z = f(x, y)$. Similar statements can be made about f_{xx} and f_{yy} as could be made about $f''(x)$ above. When taking derivatives with respect to x twice, we measure how much f_x changes with respect to x . If $f_{xx}(x, y) < 0$, it means that as x increases, f_x decreases, and the graph of f will be concave down *in the x -direction*. Using the analogy of standing in the rolling meadow used earlier in this section, f_{xx} measures whether one’s path is concave up/down when walking due east.

Similarly, f_{yy} measures the concavity in the y -direction. If $f_{yy}(x, y) > 0$, then f_y is increasing with respect to y and the graph of f will be concave up in the y -direction. Appealing to the rolling meadow analogy again, f_{yy} measures whether one’s path is concave up/down when walking due north.

We now consider the mixed partials f_{xy} and f_{yx} . The mixed partial f_{xy} measures how much f_x changes with respect to y . Once again using the rolling meadow analogy, f_x measures the slope if one walks due east. Looking east, begin walking *north* (side-stepping). Is the path towards the east getting steeper? If so, $f_{xy} > 0$. Is the path towards the east not changing in steepness? If so, then $f_{xy} = 0$. A similar thing can be said about f_{yx} : consider the steepness of paths heading north while side-stepping to the east.

The following example examines these ideas with concrete numbers and

Notes:

graphs.

Example 12.20 Understanding second partial derivatives

Let $z = x^2 - y^2 + xy$. Evaluate the 6 first and second partial derivatives at $(-1/2, 1/2)$ and interpret what each of these numbers mean.

SOLUTION We find that:

$f_x(x, y) = 2x + y$, $f_y(x, y) = -2y + x$, $f_{xx}(x, y) = 2$, $f_{yy}(x, y) = -2$ and $f_{xy}(x, y) = f_{yx}(x, y) = 1$. Thus at $(-1/2, 1/2)$ we have

$$f_x(-1/2, 1/2) = -1/2, \quad f_y(-1/2, 1/2) = -3/2.$$

The slope of the tangent line at $(-1/2, 1/2, -1/4)$ in the direction of x is $-1/2$: if one moves from that point parallel to the x -axis, the instantaneous rate of change will be $-1/2$. The slope of the tangent line at this point in the direction of y is $-3/2$: if one moves from this point parallel to the y -axis, the instantaneous rate of change will be $-3/2$. These tangent lines are graphed in Figure 12.13(a) and (b), respectively, where the tangent lines are drawn in a solid line.

Now consider only Figure 12.13(a). Three directed tangent lines are drawn (two are dashed), each in the direction of x ; that is, each has a slope determined by f_x . Note how as y increases, the slope of these lines get closer to 0. Since the slopes are all negative, getting closer to 0 means the *slopes are increasing*. The slopes given by f_x are increasing as y increases, meaning f_{xy} must be positive.

Since $f_{xy} = f_{yx}$, we also expect f_y to increase as x increases. Consider Figure 12.13(b) where again three directed tangent lines are drawn, this time each in the direction of y with slopes determined by f_y . As x increases, the slopes become less steep (closer to 0). Since these are negative slopes, this means the slopes are increasing.

Thus far we have a visual understanding of f_x , f_y , and $f_{xy} = f_{yx}$. We now interpret f_{xx} and f_{yy} . In Figure 12.13(a), we see a curve drawn where x is held constant at $x = -1/2$: only y varies. This curve is clearly concave down, corresponding to the fact that $f_{yy} < 0$. In part (b) of the figure, we see a similar curve where y is constant and only x varies. This curve is concave up, corresponding to the fact that $f_{xx} > 0$.

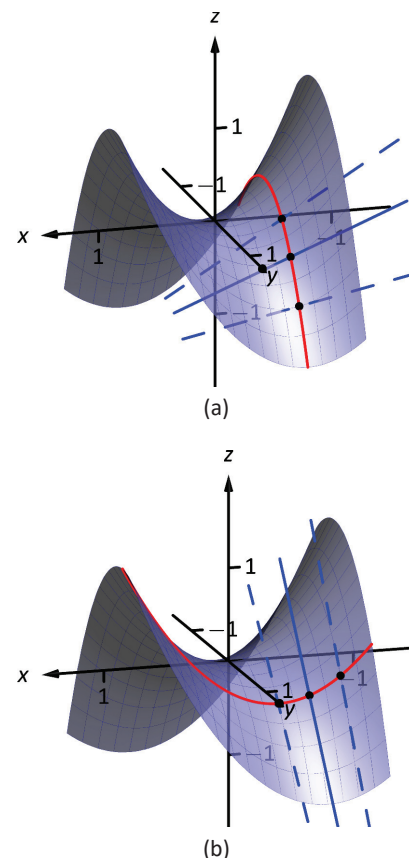


Figure 12.13: Understanding the second partial derivatives in Example 12.20.

Partial Derivatives and Functions of Three Variables

The concepts underlying partial derivatives can be easily extended to more than two variables. We give some definitions and examples in the case of three variables and trust the reader can extend these definitions to more variables if needed.

Notes:

Definition 89 Partial Derivatives with Three Variables

Let $w = f(x, y, z)$ be a continuous function on an open set S in \mathbb{R}^3 . The **partial derivative of f with respect to x** is:

$$f_x(x, y, z) = \lim_{h \rightarrow 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}.$$

Similar definitions hold for $f_y(x, y, z)$ and $f_z(x, y, z)$.

By taking partial derivatives of partial derivatives, we can find second partial derivatives of f with respect to z then y , for instance, just as before.

Example 12.21 Partial derivatives of functions of three variables

For each of the following, find f_x , f_y , f_z , f_{xz} , f_{yz} , and f_{zz} .

1. $f(x, y, z) = x^2y^3z^4 + x^2y^2 + x^3z^3 + y^4z^4$
2. $f(x, y, z) = x \sin(yz)$

SOLUTION

1. $f_x = 2xy^3z^4 + 2xy^2 + 3x^2z^3$; $f_y = 3x^2y^2z^4 + 2x^2y + 4y^3z^4$;
 $f_z = 4x^2y^3z^3 + 3x^3z^2 + 4y^4z^3$; $f_{xz} = 8xy^3z^3 + 9x^2z^2$;
 $f_{yz} = 12x^2y^2z^3 + 16y^3z^3$; $f_{zz} = 12x^2y^3z^2 + 6x^3z + 12y^4z^2$
2. $f_x = \sin(yz)$; $f_y = xz \cos(yz)$; $f_z = xy \cos(yz)$;
 $f_{xz} = y \cos(yz)$; $f_{yz} = x \cos(yz) - xyz \sin(yz)$; $f_{zz} = -xy^2 \sin(xy)$

Higher Order Partial Derivatives

We can continue taking partial derivatives of partial derivatives of partial derivatives of ...; we do not have to stop with second partial derivatives. These higher order partial derivatives do not have a tidy graphical interpretation; nevertheless they are not hard to compute and worthy of some practice.

We do not formally define each higher order derivative, but rather give just a few examples of the notation.

$$f_{xyx}(x, y) = \frac{\partial}{\partial x} \left(\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) \right) \quad \text{and}$$

$$f_{xyz}(x, y, z) = \frac{\partial}{\partial z} \left(\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) \right).$$

Notes:

Example 12.22 Higher order partial derivatives

1. Let $f(x, y) = x^2y^2 + \sin(xy)$. Find f_{xxy} and f_{yxx} .
2. Let $f(x, y, z) = x^3e^{xy} + \cos(z)$. Find f_{xyz} .

SOLUTION

1. To find f_{xxy} , we first find f_x , then f_{xx} , then f_{xxy} :

$$\begin{aligned} f_x &= 2xy^2 + y \cos(xy) & f_{xx} &= 2y^2 - y^2 \sin(xy) \\ f_{xxy} &= 4y - 2y \sin(xy) - xy^2 \cos(xy). \end{aligned}$$

To find f_{yxx} , we first find f_y , then f_{yx} , then f_{yxx} :

$$\begin{aligned} f_y &= 2x^2y + x \cos(xy) & f_{yx} &= 4xy + \cos(xy) - xy \sin(xy) \\ f_{yxx} &= 4y - y \sin(xy) - (y \sin(xy) + xy^2 \cos(xy)) \\ &= 4y - 2y \sin(xy) - xy^2 \cos(xy). \end{aligned}$$

Note how $f_{xxy} = f_{yxx}$.

2. To find f_{xyz} , we find f_x , then f_{xy} , then f_{xyz} :

$$\begin{aligned} f_x &= 3x^2e^{xy} + x^3ye^{xy} & f_{xy} &= 3x^3e^{xy} + x^3e^{xy} + x^4ye^{xy} = 4x^3e^{xy} + x^4ye^{xy} \\ f_{xyz} &= 0. \end{aligned}$$

In the previous example we saw that $f_{xxy} = f_{yxx}$; this is not a coincidence. While we do not state this as a formal theorem, as long as each partial derivative is continuous, it does not matter the order in which the partial derivatives are taken. For instance, $f_{xxy} = f_{xyx} = f_{yxx}$.

This can be useful at times. Had we known this, the second part of Example 12.22 would have been much simpler to compute. Instead of computing f_{xyz} in the x, y then z orders, we could have applied the z , then x then y order (as $f_{xyz} = f_{zxy}$). It is easy to see that $f_z = -\sin z$; then f_{zx} and f_{zxy} are clearly 0 as f_z does not contain an x or y .

Notes:

A brief review of this section: partial derivatives measure the instantaneous rate of change of a multivariable function with respect to one variable. With $z = f(x, y)$, the partial derivatives f_x and f_y measure the instantaneous rate of change of z when moving parallel to the x - and y -axes, respectively. How do we measure the rate of change at a point when we do not move parallel to one of these axes? What if we move in the direction given by the vector $\langle 2, 1 \rangle$? Can we measure that rate of change? The answer is, of course, yes, we can. This is the topic of Section 12.6. First, we need to define what it means for a function of two variables to be *differentiable*.

Notes:

Exercises 12.3

Terms and Concepts

1. What is the difference between a constant and a coefficient?
2. Given a function $z = f(x, y)$, explain in your own words how to compute f_x .
3. In the mixed partial fraction f_{xy} , which is computed first, f_x or f_y ?
4. In the mixed partial fraction $\frac{\partial^2 f}{\partial x \partial y}$, which is computed first, f_x or f_y ?

Problems

In Exercises 5 – 8, evaluate $f_x(x, y)$ and $f_y(x, y)$ at the indicated point.

5. $f(x, y) = x^2y - x + 2y + 3$ at $(1, 2)$
6. $f(x, y) = x^3 - 3x + y^2 - 6y$ at $(-1, 3)$
7. $f(x, y) = \sin y \cos x$ at $(\pi/3, \pi/3)$
8. $f(x, y) = \ln(xy)$ at $(-2, -3)$

In Exercises 9 – 28, find f_x , f_y , f_{xx} , f_{yy} , f_{xy} and f_{yx} .

9. $f(x, y) = x^2y + 3x^2 + 4y - 5$
10. $f(x, y) = y^3 + 3xy^2 + 3x^2y + x^3$
11. $f(x, y) = \frac{x}{y}$
12. $f(x, y) = \frac{4}{xy}$
13. $f(x, y) = e^{x^2+y^2}$
14. $f(x, y) = e^{x+2y}$
15. $f(x, y) = \sin x \cos y$
16. $f(x, y) = (x + y)^3$
17. $f(x, y) = \cos(5xy^3)$
18. $f(x, y) = \sin(5x^2 + 2y^3)$

$$19. f(x, y) = \sqrt{4xy^2 + 1}$$

$$20. f(x, y) = (2x + 5y)\sqrt{y}$$

$$21. f(x, y) = \frac{1}{x^2 + y^2 + 1}$$

$$22. f(x, y) = 5x - 17y$$

$$23. f(x, y) = x^y$$

$$24. f(x, y) = 3x^2 + 1$$

$$25. f(x, y) = \ln(x^2 + y)$$

$$26. f(x, y) = \frac{\ln x}{4y}$$

$$27. f(x, y) = 5x + x^{2+\cos y}$$

$$28. f(x, y) = 5e^x \sin y + 9$$

In Exercises 29 – 35, form a function $z = f(x, y)$ such that f_x and f_y match those given.

29. $f_x = \sin y + 1$, $f_y = x \cos y$
30. $f_x = 2x + y$, $f_y = 3x + y$
31. $f_x = x + y$, $f_y = x + y$
32. $f_x = e^y$, $f_y = e^x$
33. $f_x = x^2 \cos y$, $f_y = \tan(xy)$
34. $f_x = 6xy - 4y^2$, $f_y = 3x^2 - 8xy + 2$
35. $f_x = \frac{2x}{x^2 + y^2}$, $f_y = \frac{2y}{x^2 + y^2}$

In Exercises 36 – 39, find f_x , f_y , f_z , f_{yz} and f_{zy} .

36. $f(x, y, z) = x^2 e^{2y-3z}$
37. $f(x, y, z) = x^3 y^2 + x^3 z + y^2 z$
38. $f(x, y, z) = \frac{3x}{7y^2 z}$
39. $f(x, y, z) = \ln(xyz)$