13: Multiple Integration

The previous chapter introduced multivariable functions and we applied concepts of differential calculus to these functions. We learned how we can view a function of two variables as a surface in space, and learned how partial derivatives convey information about how the surface is changing in any direction.

In this chapter we apply techniques of integral calculus to multivariable functions. In Chapter 5 we learned how the definite integral of a single variable function gave us "area under the curve." In this chapter we will see that integration applied to a multivariable function gives us "volume under a surface." And just as we learned applications of integration beyond finding areas, we will find applications of integration in this chapter beyond finding volume.

13.1 Iterated Integrals and Area

In Chapter 12 we found that it was useful to differentiate functions of several variables with respect to one variable, while treating all the other variables as constants or coefficients. We can integrate functions of several variables in a similar way. For instance, if we are told that $f_x(x,y) = 2xy$, we can treat y as staying constant and integrate to obtain $f(x,y)$:

$$\begin{aligned} f(x,y) &= \int f_x(x,y)\, dx \\ &= \int 2xy\, dx \\ &= x^2y + C. \end{aligned}$$

Make a careful note about the constant of integration, C. This "constant" is something with a derivative of 0 with respect to x, so it could be any expression that contains only constants and functions of y. For instance, if $f(x,y) = x^2y + \sin y + y^3 + 17$, then $f_x(x,y) = 2xy$. To signify that C is actually a function of y, we write:

$$f(x,y) = \int f_x(x,y)\, dx = x^2y + C(y).$$

Using this process we can even evaluate definite integrals.

Example 13.1 Integrating functions of more than one variable

Evaluate the integral $\displaystyle\int_1^{2y} 2xy\, dx$.

Solution We find the indefinite integral as before, then apply the Fundamental Theorem of Calculus to evaluate the definite integral:

$$\begin{aligned} \int_1^{2y} 2xy\, dx &= x^2y\Big|_1^{2y} \\ &= (2y)^2y - (1)^2y \\ &= 4y^3 - y. \end{aligned}$$

We can also integrate with respect to y. In general,

$$\int_{h_1(y)}^{h_2(y)} f_x(x,y)\,dx = f(x,y)\Big|_{h_1(y)}^{h_2(y)} = f\big(h_2(y),y\big) - f\big(h_1(y),y\big),$$

and

$$\int_{g_1(x)}^{g_2(x)} f_y(x,y)\,dy = f(x,y)\Big|_{g_1(x)}^{g_2(x)} = f\big(x,g_2(x)\big) - f\big(x,g_1(x)\big).$$

Note that when integrating with respect to x, the bounds are functions of y (of the form $x = h_1(y)$ and $x = h_2(y)$) and the final result is also a function of y. When integrating with respect to y, the bounds are functions of x (of the form $y = g_1(x)$ and $y = g_2(x)$) and the final result is a function of x. Another example will help us understand this.

Example 13.2 Integrating functions of more than one variable

Evaluate $\displaystyle\int_1^x \left(5x^3y^{-3} + 6y^2\right)\,dy$.

Solution We consider x as staying constant and integrate with respect to y:

$$\begin{aligned}\int_1^x \left(5x^3y^{-3} + 6y^2\right)\,dy &= \left(\frac{5x^3y^{-2}}{-2} + \frac{6y^3}{3}\right)\Bigg|_1^x \\ &= \left(-\frac{5}{2}x^3x^{-2} + 2x^3\right) - \left(-\frac{5}{2}x^3 + 2\right) \\ &= \frac{9}{2}x^3 - \frac{5}{2}x - 2.\end{aligned}$$

Note how the bounds of the integral are from $y = 1$ to $y = x$ and that the final answer is a function of x.

In the previous example, we integrated a function with respect to y and ended up with a function of x. We can integrate this as well. This process is known as **iterated integration**, or **multiple integration.**

Example 13.3 Integrating an integral

Evaluate $\displaystyle\int_1^2 \left(\int_1^x \left(5x^3y^{-3} + 6y^2\right)\,dy\right)\,dx$.

Solution We follow a standard "order of operations" and perform the operations inside parentheses first (which is the integral evaluated in Example

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13.2.)

$$\begin{aligned}\int_1^2 \left(\int_1^x \left(5x^3y^{-3}+6y^2\right)\,dy\right)\,dx &= \int_1^2 \left(\left[\frac{5x^3y^{-2}}{-2}+\frac{6y^3}{3}\right]\Bigg|_1^x\right)\,dx\\ &= \int_1^2 \left(\frac{9}{2}x^3-\frac{5}{2}x-2\right)\,dx\\ &= \left(\frac{9}{8}x^4-\frac{5}{4}x^2-2x\right)\Bigg|_1^2\\ &= \frac{89}{8}.\end{aligned}$$

Note how the bounds of x were $x=1$ to $x=2$ and the final result was a number.

The previous example showed how we could perform something called an iterated integral; we do not yet know *why* we would be interested in doing so nor what the result, such as the number 89/8, *means*. Before we investigate these questions, we offer some definitions.

Definition 104 Iterated Integration

Iterated integration is the process of repeatedly integrating the results of previous integrations. Integrating one integral is denoted as follows.

Let a, b, c and d be numbers and let $g_1(x)$, $g_2(x)$, $h_1(y)$ and $h_2(y)$ be functions of x and y, respectively. Then:

1. $\displaystyle\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right)\,dy.$
2. $\displaystyle\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right)\,dx.$

Again make note of the bounds of these iterated integrals.

With $\displaystyle\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$, x varies from $h_1(y)$ to $h_2(y)$, whereas y varies from c to d. That is, the bounds of x are *curves*, the curves $x=h_1(y)$ and $x=h_2(y)$, whereas the bounds of y are *constants*, $y=c$ and $y=d$. It is useful to remember that when setting up and evaluating such iterated integrals, we integrate "from

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curve to curve, then from point to point."

We now begin to investigate *why* we are interested in iterated integrals and *what* they mean.

Area of a plane region

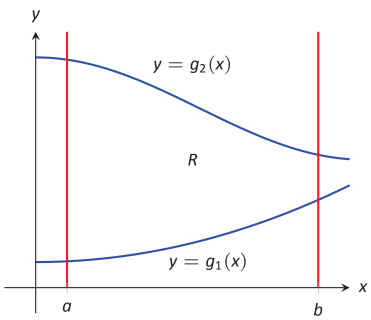

Figure 13.1: Calculating the area of a plane region R with an iterated integral.

Consider the plane region R bounded by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, shown in Figure 13.1. We learned in Section 7.1 that the area of R is given by

$$\int_a^b \big(g_2(x) - g_1(x)\big)\, dx.$$

We can view the expression $\big(g_2(x) - g_1(x)\big)$ as

$$\big(g_2(x) - g_1(x)\big) = \int_{g_1(x)}^{g_2(x)} 1\, dy = \int_{g_1(x)}^{g_2(x)} dy,$$

meaning we can express the area of R as an iterated integral:

$$\text{area of } R = \int_a^b \big(g_2(x) - g_1(x)\big)\, dx = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} dy\right) dx = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx.$$

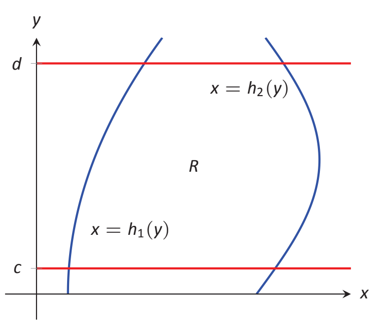

Figure 13.2: Calculating the area of a plane region R with an iterated integral.

In short: a certain iterated integral can be viewed as giving the area of a plane region.

A region R could also be defined by $c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(y)$, as shown in Figure 13.2. Using a process similar to that above, we have

$$\text{the area of } R = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\, dy.$$

We state this formally in a theorem.

Notes:

Theorem 121 Area of a plane region

1. Let R be a plane region bounded by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, where g_1 and g_2 are continuous functions on $[a, b]$. The area A of R is
$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx.$$

2. Let R be a plane region bounded by $c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(y)$, where h_1 and h_2 are continuous functions on $[c, d]$. The area A of R is
$$A = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\, dy.$$

The following examples should help us understand this theorem.

Example 13.4 Area of a rectangle

Find the area A of the rectangle with corners $(-1, 1)$ and $(3, 3)$, as shown in Figure 13.3.

Solution Multiple integration is obviously overkill in this situation, but we proceed to establish its use.

The region R is bounded by $x = -1$, $x = 3$, $y = 1$ and $y = 3$. Choosing to integrate with respect to y first, we have

$$A = \int_{-1}^{3} \int_{1}^{3} 1\, dy\, dx = \int_{-1}^{3} \left(y \Big|_1^3 \right) dx = \int_{-1}^{3} 2\, dx = 2x \Big|_{-1}^{3} = 8.$$

We could also integrate with respect to x first, giving:

$$A = \int_{1}^{3} \int_{-1}^{3} 1\, dx\, dy = \int_{1}^{3} \left(x \Big|_{-1}^3 \right) dy = \int_{1}^{3} 4\, dy = 4y \Big|_{1}^{3} = 8.$$

Clearly there are simpler ways to find this area, but it is interesting to note that this method works.

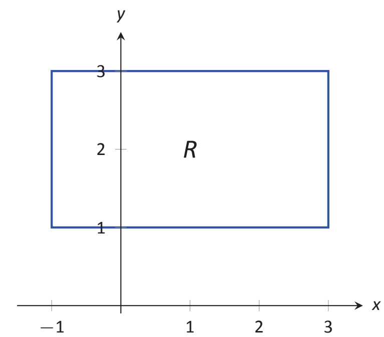

Figure 13.3: Calculating the area of a rectangle with an iterated integral in Example 13.4.

Example 13.5 Area of a triangle

Find the area A of the triangle with vertices at $(1, 1)$, $(3, 1)$ and $(5, 5)$, as shown in Figure 13.4.

Solution The triangle is bounded by the lines as shown in the figure. Choosing to integrate with respect to x first gives that x is bounded by $x = y$

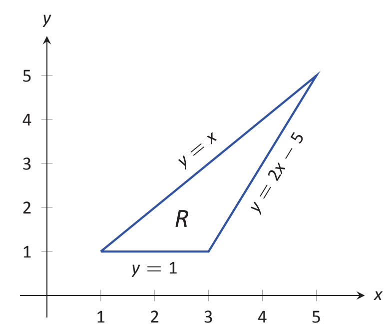

Figure 13.4: Calculating the area of a triangle with iterated integrals in Example 13.5.

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to $x = \frac{y+5}{2}$, while y is bounded by $y = 1$ to $y = 5$. (Recall that since x-values increase from left to right, the leftmost curve, $x = y$, is the lower bound and the rightmost curve, $x = (y + 5)/2$, is the upper bound.) The area is

$$\begin{aligned} A &= \int_1^5 \int_y^{\frac{y+5}{2}} dx\,dy \\ &= \int_1^5 \left(x \Big|_y^{\frac{y+5}{2}} \right) dy \\ &= \int_1^5 \left(-\frac{1}{2}y + \frac{5}{2} \right) dy \\ &= \left(-\frac{1}{4}y^2 + \frac{5}{2}y \right) \Big|_1^5 \\ &= 4. \end{aligned}$$

We can also find the area by integrating with respect to y first. In this situation, though, we have two functions that act as the lower bound for the region R, $y = 1$ and $y = 2x - 5$. This requires us to use two iterated integrals. Note how the x-bounds are different for each integral:

$$\begin{aligned} A &= \int_1^3 \int_1^x 1\,dy\,dx &&+ \int_3^5 \int_{2x-5}^x 1\,dy\,dx \\ &= \int_1^3 \big(y\big)\Big|_1^x dx &&+ \int_3^5 \big(y\big)\Big|_{2x-5}^x dx \\ &= \int_1^3 \big(x-1\big)\,dx &&+ \int_3^5 \big(-x+5\big)\,dx \\ &= 2 &&+ 2 \\ &= 4. \end{aligned}$$

As expected, we get the same answer both ways.

Example 13.6 **Area of a plane region**

Find the area of the region enclosed by $y = 2x$ and $y = x^2$, as shown in Figure 13.5.

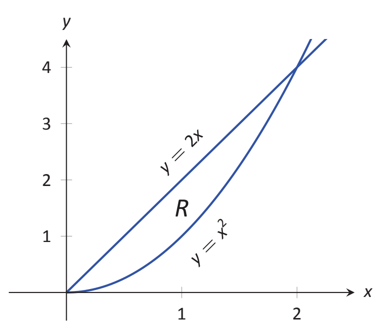

Figure 13.5: Calculating the area of a plane region with iterated integrals in Example 13.6.

SOLUTION Once again we'll find the area of the region using both orders of integration.

Using $dy\,dx$:

$$\int_0^2 \int_{x^2}^{2x} 1\,dy\,dx = \int_0^2 (2x - x^2)\,dx = \left(x^2 - \frac{1}{3}x^3\right)\Big|_0^2 = \frac{4}{3}.$$

Notes:

Using $dx\,dy$:

$$\int_0^4\int_{y/2}^{\sqrt{y}} 1\,dx\,dy = \int_0^4 \left(\sqrt{y}-y/2\right)dy = \left(\frac{2}{3}y^{3/2}-\frac{1}{4}y^2\right)\Big|_0^4 = \frac{4}{3}.$$

Changing Order of Integration

In each of the previous examples, we have been given a region R and found the bounds needed to find the area of R using both orders of integration. We integrated using both orders of integration to demonstrate their equality.

We now approach the skill of describing a region using both orders of integration from a different perspective. Instead of starting with a region and creating iterated integrals, we will start with an iterated integral and rewrite it in the other integration order. To do so, we'll need to understand the region over which we are integrating.

The simplest of all cases is when both integrals are bound by constants. The region described by these bounds is a rectangle (see Example 13.4), and so:

$$\int_a^b\int_c^d 1\,dy\,dx = \int_c^d\int_a^b 1\,dx\,dy.$$

When the inner integral's bounds are not constants, it is generally very useful to sketch the bounds to determine what the region we are integrating over looks like. From the sketch we can then rewrite the integral with the other order of integration.

Examples will help us develop this skill.

Example 13.7 Changing the order of integration

Rewrite the iterated integral $\displaystyle\int_0^6\int_0^{x/3} 1\,dy\,dx$ with the order of integration $dx\,dy$.

Solution We need to use the bounds of integration to determine the region we are integrating over.

The bounds tell us that y is bounded by 0 and $x/3$; x is bounded by 0 and 6. We plot these four curves: $y=0$, $y=x/3$, $x=0$ and $x=6$ to find the region described by the bounds. Figure 13.6 shows these curves, indicating that R is a triangle.

To change the order of integration, we need to consider the curves that bound the x-values. We see that the lower bound is $x=3y$ and the upper bound is $x=6$. The bounds on y are 0 to 2. Thus we can rewrite the integral as $\displaystyle\int_0^2\int_{3y}^6 1\,dx\,dy.$

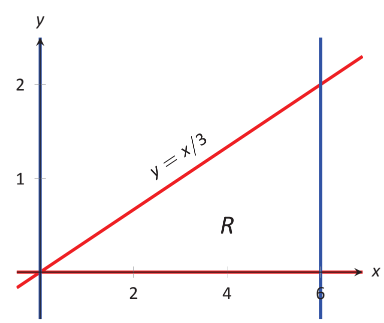

Figure 13.6: Sketching the region R described by the iterated integral in Example 13.7.

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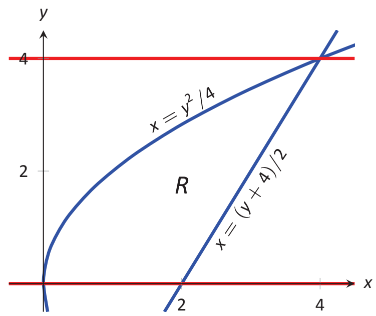

Figure 13.7: Drawing the region determined by the bounds of integration in Example 13.8.

Example 13.8 Changing the order of integration

Change the order of integration of $\displaystyle\int_0^4 \int_{y^2/4}^{(y+4)/2} 1\, dx\, dy$.

Solution We sketch the region described by the bounds to help us change the integration order. x is bounded below and above (i.e., to the left and right) by $x = y^2/4$ and $x = (y+4)/2$ respectively, and y is bounded between 0 and 4. Graphing the previous curves, we find the region R to be that shown in Figure 13.7.

To change the order of integration, we need to establish curves that bound y. The figure makes it clear that there are two lower bounds for y: $y = 0$ on $0 \le x \le 2$, and $y = 2x - 4$ on $2 \le x \le 4$. Thus we need two double integrals. The upper bound for each is $y = 2\sqrt{x}$. Thus we have

$$\int_0^4 \int_{y^2/4}^{(y+4)/2} 1\, dx\, dy = \int_0^2 \int_0^{2\sqrt{x}} 1\, dy\, dx + \int_2^4 \int_{2x-4}^{2\sqrt{x}} 1\, dy\, dx.$$

There is a useful shortcut that applies to a handful of iterated integrals. It can be used the integrand has the form $f(x)g(y)$ of a function $f(x)$ of x alone and a function $g(y)$ of y alone (for example $\sin(x)y^2$ but not $\sin x + y^2$ nor $\sin\left(xy^2\right)$) and the bounds contain only numbers. In this case the double integral can be broken into the product of two single integrals which are faster to compute. This is the next theorem.

Theorem 122

Let $f(x)$ and $g(y)$ be continuous functions on $[a, b]$ and $[c, d]$, respectively. Then

$$\int_c^d \int_a^b f(x)g(y)dxdy = \left(\int_a^b f(x)dx\right)\left(\int_c^d g(y)dy\right).$$

To explain this shortcut, we begin by realizing that $g(y)$ is considered a "constant" multiple when integrating with respect to x and can pulled outside of the first iterated integral:

$$\int_c^d \int_a^b f(x)g(y)dxdy = \int_c^d \left(\int_a^b f(x)g(y)dx\right) dy = \int_c^d g(y)\left(\int_a^b f(x)dx\right) dy.$$

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Then the entire integral $\int_a^b f(x)dx$ is a constant that can be pulled outside the y-integral. It s very important that the bounds do not contain y. So

$$\int_c^d g(y)\left(\int_a^b f(x)dx\right)dy = \left(\int_a^b f(x)dx\right)\left(\int_c^d g(y)dy\right).$$

We use this shortcut in the next example.

Example 13.9 Using Theorem 122

Compute $\displaystyle\int_2^5 \int_0^\pi \sin(x)y^2\,dx\,dy$.

SOLUTION

$$\begin{aligned}\int_2^5 \int_0^\pi \sin(x)y^2\,dx\,dy &= \left(\int_0^\pi \sin(x)\,dx\right)\left(\int_2^5 y^2\,dy\right)\\ &= \left(-\cos x\Big|_0^\pi\right)\left(\frac{y^3}{3}\Big|_2^5\right)\\ &= \big(1-(-1)\big)\left(\frac{5^3}{3}-\frac{2^3}{3}\right)\\ &= 78.\end{aligned}$$

This section has introduced a new concept, the iterated integral. We developed one application for iterated integration: area between curves. However, this is not new, for we already know how to find areas bounded by curves. In the next section we apply iterated integration to solve problems we currently do not know how to handle. The "real" goal of this section was not to learn a new way of computing area. Rather, our goal was to learn how to define a region in the plane using the bounds of an iterated integral. That skill is very important in the following sections.

Notes:

Exercises 13.1

Terms and Concepts

1. When integrating $f_x(x, y)$ with respect to x, the constant of integration C is really which: $C(x)$ or $C(y)$? What does this mean?

2. Integrating an integral is called _________ _________.

3. When evaluating an iterated integral, we integrate from _________ to _________, then from _________ to _________.

4. One understanding of an iterated integral is that $\int_a^b \int_{g_1(x)}^{g_2(x)} dy\,dx$ gives the _________ of a plane region.

Problems

In Exercises 5 – 10, evaluate the integral and subsequent iterated integral.

5. (a) $\displaystyle\int_2^5 \left(6x^2 + 4xy - 3y^2\right) dy$

   (b) $\displaystyle\int_{-3}^{-2} \int_2^5 \left(6x^2 + 4xy - 3y^2\right) dy\,dx$

6. (a) $\displaystyle\int_0^{\pi} \left(2x\cos y + \sin x\right) dx$

   (b) $\displaystyle\int_0^{\pi/2} \int_0^{\pi} \left(2x\cos y + \sin x\right) dx\,dy$

7. (a) $\displaystyle\int_1^x \left(x^2 y - y + 2\right) dy$

   (b) $\displaystyle\int_0^2 \int_1^x \left(x^2 y - y + 2\right) dy\,dx$

8. (a) $\displaystyle\int_y^{y^2} \left(x - y\right) dx$

   (b) $\displaystyle\int_{-1}^1 \int_y^{y^2} \left(x - y\right) dx\,dy$

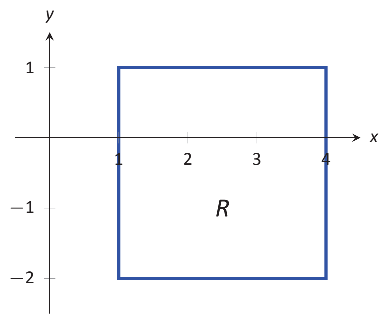
9. (a) $\displaystyle\int_0^y \left(\cos x \sin y\right) dx$

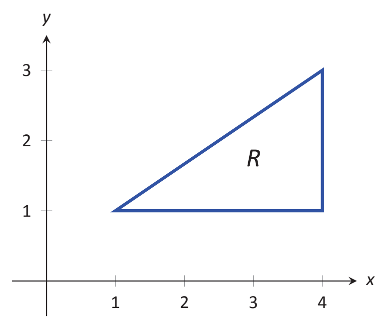
   (b) $\displaystyle\int_0^{\pi} \int_0^y \left(\cos x \sin y\right) dx\,dy$

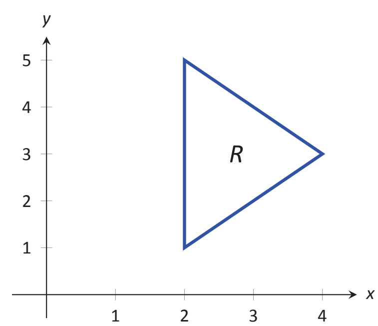
10. (a) $\displaystyle\int_0^x \left(\frac{1}{1+x^2}\right) dy$

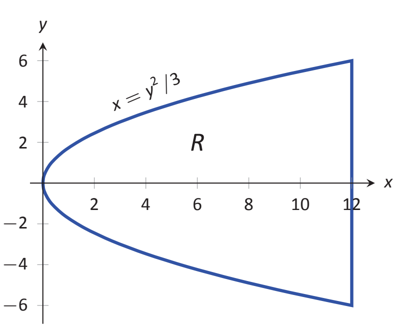
    (b) $\displaystyle\int_1^2 \int_0^x \left(\frac{1}{1+x^2}\right) dy\,dx$

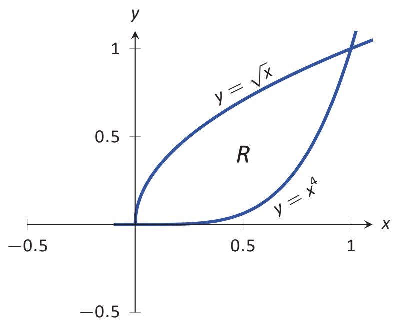
In Exercises 11 – 16, a graph of a planar region R is given. Give the iterated integrals, with both orders of integration $dy\,dx$ and $dx\,dy$, that give the area of R. Evaluate one of the iterated integrals to find the area.

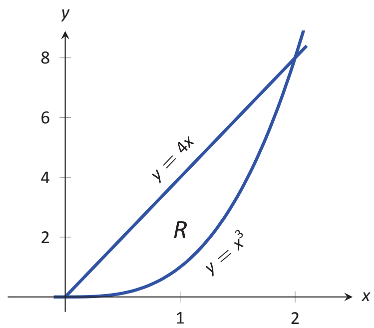
11. 

12. 

13. 

14. 

15. 

16. 

In Exercises 17 – 22, iterated integrals are given that compute the area of a region *R* in the *x*-*y* plane. Sketch the region *R*, and give the iterated integral(s) that give the area of *R* with the opposite order of integration.

17. $\displaystyle\int_{-2}^{2}\int_{0}^{4-x^2} dy\,dx$

18. $\displaystyle\int_{0}^{1}\int_{5-5x}^{5-5x^2} dy\,dx$

19. $\displaystyle\int_{-2}^{2}\int_{0}^{2\sqrt{4-y^2}} dx\,dy$

20. $\displaystyle\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dy\,dx$

21. $\displaystyle\int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy + \int_{1}^{4}\int_{y-2}^{\sqrt{y}} dx\,dy$

22. $\displaystyle\int_{-1}^{1}\int_{(x-1)/2}^{(1-x)/2} dy\,dx$

In Exercises 23 – 27, use the shortcut of Theorem 122 to compute the iterated integral.

23. $\displaystyle\int_{2}^{4}\int_{0}^{1} x^2y^3\,dx\,dy$

24. $\displaystyle\int_{1}^{e}\int_{-1}^{3} \frac{x}{y}\,dx\,dy$

25. $\displaystyle\int_{0}^{\pi/3}\int_{0}^{\pi/4} 4\sin x\cos y\,dx\,dy$

26. $\displaystyle\int_{0}^{2}\int_{0}^{\sqrt{3}} \frac{3^y}{1+x^2}\,dx\,dy$

27. $\displaystyle\int_{1}^{4}\int_{0}^{6} \left(x^2+x^2y\right)\,dx\,dy$

28. Why can't Theorem 122 be used to compute $\displaystyle\int_{0}^{\pi}\int_{0}^{\sin y} e^x\cos y\,dx\,dy$?

29. Why can't Theorem 122 be used to compute $\displaystyle\int_{2}^{4}\int_{0}^{2} \sqrt{x+y}\,dx\,dy$?