4.4 Differentials

In Section 2.2 we explored the meaning and use of the derivative. This section starts by revisiting some of those ideas.

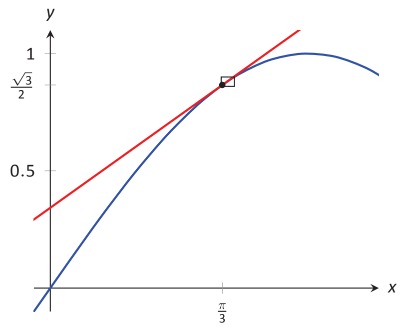
Recall that the derivative of a function f can be used to find the slopes of lines tangent to the graph of f. At $x = c$, the tangent line to the graph of f has equation

$$y = f'(c)(x - c) + f(c).$$

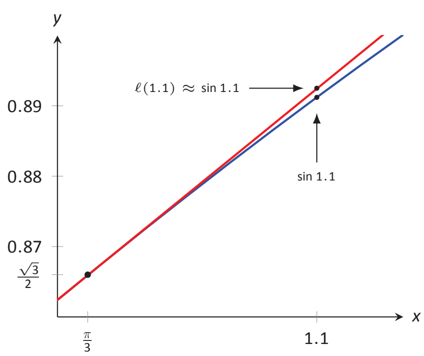
The tangent line can be used to find good approximations of $f(x)$ for values of x near c.

For instance, we can approximate $\sin 1.1$ using the tangent line to the graph of $f(x) = \sin x$ at $x = \pi/3 \approx 1.05$. Recall that $\sin(\pi/3) = \sqrt{3}/2 \approx 0.866$, and $\cos(\pi/3) = 1/2$. Thus the tangent line to $f(x) = \sin x$ at $x = \pi/3$ is:

$$\ell(x) = \frac{1}{2}(x - \pi/3) + 0.866.$$



(a)



(b)

Figure 4.8: Graphing $f(x) = \sin x$ and its tangent line at $x = \pi/3$ in order to estimate $\sin 1.1$.

In Figure 4.8(a), we see a graph of $f(x) = \sin x$ graphed along with its tangent line at $x = \pi/3$. The small rectangle shows the region that is displayed in Figure 4.8(b). In this figure, we see how we are approximating $\sin 1.1$ with the tangent line, evaluated at 1.1. Together, the two figures show how close these values are.

Using this line to approximate $\sin 1.1$, we have:

$$\begin{aligned}\ell(1.1) &= \frac{1}{2}(1.1 - \pi/3) + 0.866\\ &= \frac{1}{2}(0.053) + 0.866 = 0.8925.\end{aligned}$$

(We leave it to the reader to see how good of an approximation this is.)

We now generalize this concept. Given $f(x)$ and an x–value c, the tangent line is $\ell(x) = f'(c)(x - c) + f(c)$. Clearly, $f(c) = \ell(c)$. Let Δx be a small number, representing a small change in x value. We assert that:

$$f(c + \Delta x) \approx \ell(c + \Delta x),$$

since the tangent line to a function approximates well the values of that function near $x = c$.

As the x value changes from c to $c + \Delta x$, the y value of f changes from $f(c)$ to $f(c + \Delta x)$. We call this change of y value Δy. That is:

$$\Delta y = f(c + \Delta x) - f(c).$$

Notes:

Replacing $f(c+\Delta x)$ with its tangent line approximation, we have

$$\begin{aligned}\Delta y &\approx \ell(c+\Delta x) - f(c)\\ &= f'(c)\big((c+\Delta x)-c\big) + f(c) - f(c)\\ &= f'(c)\Delta x \qquad (4.3)\end{aligned}$$

This final equation is important; we'll come back to it in Key Idea 8.

We introduce two new variables, dx and dy in the context of a formal definition.

Definition 18 Differentials of x and y.

Let $y = f(x)$ be differentiable. The **differential of** x, denoted dx, is any nonzero real number (usually taken to be a small number). The **differential of** y, denoted dy, is

$$dy = f'(x)dx.$$

We can solve for $f'(x)$ in the above equation: $f'(x) = dy/dx$. This states that the derivative of f with respect to x is the differential of y divided by the differential of x; this is **not** the alternate notation for the derivative, $\frac{dy}{dx}$. This latter notation was chosen because of the fraction–like qualities of the derivative, but again, it is one symbol and not a fraction.

It is helpful to organize our new concepts and notations in one place.

Key Idea 8 Differential Notation

Let $y = f(x)$ be a differentiable function.

1. Δx represents a small, nonzero change in x value.
2. dx represents a small, nonzero change in x value (i.e., $\Delta x = dx$).
3. Δy is the change in y value as x changes by Δx; hence
$$\Delta y = f(x+\Delta x) - f(x).$$
4. $dy = f'(x)dx$ which, by Equation (4.3), is an *approximation* of the change in y value as x changes by Δx; $dy \approx \Delta y$.

Notes:

What is the value of differentials? Like many mathematical concepts, differentials provide both practical and theoretical benefits. We explore both here.

Example 4.12 Finding and using differentials
Consider $f(x) = x^2$. Knowing $f(3) = 9$, approximate $f(3.1)$.

Solution The x value is changing from $x = 3$ to $x = 3.1$; therefore, we see that $dx = 0.1$. If we know how much the y value changes from $f(3)$ to $f(3.1)$ (i.e., if we know Δy), we will know exactly what $f(3.1)$ is (since we already know $f(3)$). We can approximate Δy with dy.

$$\begin{aligned}\Delta y &\approx dy \\ &= f'(3)dx \\ &= 2 \cdot 3 \cdot 0.1 = 0.6.\end{aligned}$$

We expect the y value to change by about 0.6, so we approximate $f(3.1) \approx 9.6$.

We leave it to the reader to verify this, but the preceding discussion links the differential to the tangent line of $f(x)$ at $x = 3$. One can verify that the tangent line, evaluated at $x = 3.1$, also gives $y = 9.6$.

Of course, it is easy to compute the actual answer (by hand or with a calculator): $3.1^2 = 9.61$. (Before we get too cynical and say "Then why bother?", note our approximation is *really* good!)

So why bother?

In "most" real life situations, we do not know the function that describes a particular behavior. Instead, we can only take measurements of how things change – measurements of the derivative.

Imagine water flowing down a winding channel. It is easy to measure the speed and direction (i.e., the *velocity*) of water at any location. It is very hard to create a function that describes the overall flow, hence it is hard to predict where a floating object placed at the beginning of the channel will end up. However, we can *approximate* the path of an object using differentials. Over small intervals, the path taken by a floating object is essentially linear. Differentials allow us to approximate the true path by piecing together lots of short, linear paths. This technique is called Euler's Method, studied in introductory Differential Equations courses.

We use differentials once more to approximate the value of a function. Even though calculators are very accessible, it is neat to see how these techniques can sometimes be used to easily compute something that looks rather hard.

Notes:

Example 4.13 Using differentials to approximate a function value

Approximate $\sqrt{4.5}$.

Solution We expect $\sqrt{4.5} \approx 2$, yet we can do better. Let $f(x) = \sqrt{x}$, and let $c = 4$. Thus $f(4) = 2$. We can compute $f'(x) = 1/(2\sqrt{x})$, so $f'(4) = 1/4$.

We approximate the difference between $f(4.5)$ and $f(4)$ using differentials, with $dx = 0.5$:

$$f(4.5) - f(4) = \Delta y \approx dy = f'(4) \cdot dx = 1/4 \cdot 1/2 = 1/8 = 0.125.$$

The approximate change in f from $x = 4$ to $x = 4.5$ is 0.125, so we approximate $\sqrt{4.5} \approx 2.125$.

Differentials are important when we discuss *integration*. When we study that topic, we will use notation such as

$$\int f(x)\,dx$$

quite often. While we don't discuss here what all of that notation means, note the existence of the differential dx. Proper handling of *integrals* comes with proper handling of differentials.

In light of that, we practice finding differentials in general.

Example 4.14 Finding differentials

In each of the following, find the differential dy.

1. $y = \sin x$ 2. $y = e^x(x^2 + 2)$ 3. $y = \sqrt{x^2 + 3x - 1}$

Solution

1. $y = \sin x$: As $f(x) = \sin x$, $f'(x) = \cos x$. Thus

$$dy = \cos(x)dx.$$

2. $y = e^x(x^2 + 2)$: Let $f(x) = e^x(x^2 + 2)$. We need $f'(x)$, requiring the Product Rule.

   We have $f'(x) = e^x(x^2 + 2) + 2xe^x$, so

$$dy = \big(e^x(x^2 + 2) + 2xe^x\big)dx.$$

Notes:

3. $y = \sqrt{x^2+3x-1}$: Let $f(x) = \sqrt{x^2+3x-1}$; we need $f'(x)$, requiring the Chain Rule.

   We have $f'(x) = \dfrac{1}{2}(x^2+3x-1)^{-\frac{1}{2}}(2x+3) = \dfrac{2x+3}{2\sqrt{x^2+3x-1}}$. Thus

$$dy = \frac{(2x+3)dx}{2\sqrt{x^2+3x-1}}.$$

Finding the differential dy of $y = f(x)$ is really no harder than finding the derivative of f; we just *multiply* $f'(x)$ by dx. It is important to remember that we are not simply adding the symbol "dx" at the end.

We have seen a practical use of differentials as they offer a good method of making certain approximations. Another use is *error propagation.* Suppose a length is measured to be x, although the actual value is $x+\Delta x$ (where we hope Δx is small). This measurement of x may be used to compute some other value; we can think of this as $f(x)$ for some function f. As the true length is $x+\Delta x$, one really should have computed $f(x+\Delta x)$. The difference between $f(x)$ and $f(x+\Delta x)$ is the propagated error.

How close are $f(x)$ and $f(x+\Delta x)$? This is a difference in "y" values;

$$f(x+\Delta x) - f(x) = \Delta y \approx dy.$$

We can approximate the propagated error using differentials.

Example 4.15 Using differentials to approximate propagated error

A steel ball bearing is to be manufactured with a diameter of 2cm. The manufacturing process has a tolerance of ± 0.1mm in the diameter. Given that the density of steel is about 7.85g/cm^3, estimate the propagated error in the mass of the ball bearing.

Solution The mass of a ball bearing is found using the equation "mass = volume $\times$ density." In this situation the mass function is a product of the radius of the ball bearing, hence it is $m = 7.85\frac{4}{3}\pi r^3$. The differential of the mass is

$$dm = 31.4\pi r^2 dr.$$

The radius is to be 1cm; the manufacturing tolerance in the radius is ± 0.05mm, or ± 0.005cm. The propagated error is approximately:

$$\begin{aligned}\Delta m &\approx dm\\ &= 31.4\pi(1)^2(\pm 0.005)\\ &= \pm 0.493\text{g}\end{aligned}$$

Notes:

Is this error significant? It certainly depends on the application, but we can get an idea by computing the *relative error*. The ratio between amount of error to the total mass is

$$\begin{aligned}\frac{dm}{m} &= \pm\frac{0.493}{7.85\frac{4}{3}\pi}\\ &= \pm\frac{0.493}{32.88}\\ &= \pm 0.015,\end{aligned}$$

or $\pm 1.5\%$.

We leave it to the reader to confirm this, but if the diameter of the ball was supposed to be 10cm, the same manufacturing tolerance would give a propagated error in mass of ± 12.33g, which corresponds to a *percent error* of $\pm 0.188\%$. While the amount of error is much greater ($12.33 > 0.493$), the percent error is much lower.

We first learned of the derivative in the context of instantaneous rates of change and slopes of tangent lines. We furthered our understanding of the power of the derivative by studying how it relates to the graph of a function (leading to ideas of increasing/decreasing and concavity). This chapter has put the derivative to yet more uses:

- Equation solving (Newton's Method)
- Related Rates (furthering our use of the derivative to find instantaneous rates of change)
- Optimization (applied extreme values), and
- Differentials (useful for various approximations and for something called integration).

In the next chapters, we will consider the "reverse" problem to computing the derivative: given a function f, can we find a function whose derivative is f? Be able to do so opens up an incredible world of mathematics and applications.

Notes:

Exercises 4.4

Terms and Concepts

1. T/F: Given a differentiable function $y = f(x)$, we are generally free to choose a value for dx, which then determines the value of dy.

2. T/F: The symbols "dx" and "Δx" represent the same concept.

3. T/F: The symbols "dy" and "Δy" represent the same concept.

4. T/F: Differentials are important in the study of integration.

5. How are differentials and tangent lines related?

Problems

In Exercises 6 – 16, use differentials to approximate the given value by hand.

6. 2.05^2

7. 5.93^2

8. 5.1^3

9. 6.8^3

10. $\sqrt{16.5}$

11. $\sqrt{24}$

12. $\sqrt[3]{63}$

13. $\sqrt[3]{8.5}$

14. $\sin 3$

15. $\cos 1.5$

16. $e^{0.1}$

In Exercises 17 – 29, compute the differential dy.

17. $y = x^2 + 3x - 5$

18. $y = x^7 - x^5$

19. $y = \dfrac{1}{4x^2}$

20. $y = (2x + \sin x)^2$

21. $y = x^2 e^{3x}$

22. $y = \dfrac{4}{x^4}$

23. $y = \dfrac{2x}{\tan x + 1}$

24. $y = \ln(5x)$

25. $y = e^x \sin x$

26. $y = \cos(\sin x)$

27. $y = \dfrac{x+1}{x+2}$

28. $y = 3^x \ln x$

29. $y = x \ln x - x$

30. A set of plastic spheres are to be made with a diameter of 1cm. If the manufacturing process is accurate to 1mm, what is the propagated error in volume of the spheres?

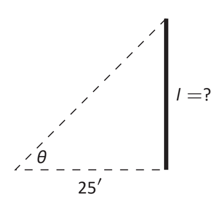
31. The distance, in feet, a stone drops in t seconds is given by $d(t) = 16t^2$. The depth of a hole is to be approximated by dropping a rock and listening for it to hit the bottom. What is the propagated error if the time measurement is accurate to $2/10^{\text{ths}}$ of a second and the measured time is:
    (a) 2 seconds?
    (b) 5 seconds?

32. What is the propagated error in the measurement of the cross sectional area of a circular log if the diameter is measured at $15''$, accurate to $1/4''$?

33. A wall is to be painted that is $8'$ high and is measured to be $10',\ 7''$ long. Find the propagated error in the measurement of the wall's surface area if the measurement is accurate to $1/2''$.

Exercises 34 – 38 explore some issues related to surveying in which distances are approximated using other measured distances and measured angles. *(Hint: Convert all angles to radians before computing.)*

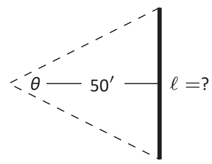
34. The length ℓ of a long wall is to be approximated. The angle θ, as shown in the diagram (not to scale), is measured to be 85.2°, accurate to 1°. Assume that the triangle formed is a right triangle.



    (a) What is the measured length ℓ of the wall?
    (b) What is the propagated error?
    (c) What is the percent error?

35. Answer the questions of Exercise 34, but with a measured angle of 71.5°, accurate to 1°, measured from a point $100'$ from the wall.

36. The length ℓ of a long wall is to be calculated by measuring the angle θ shown in the diagram (not to scale). Assume the formed triangle is an isosceles triangle. The measured angle is 143°, accurate to 1°.



(a) What is the measured length of the wall?

(b) What is the propagated error?

(c) What is the percent error?

37. The length of the walls in Exercises 34 – 36 are essentially the same. Which setup gives the most accurate result?

38. Consider the setup in Exercises 36. This time, assume the angle measurement of 143° is exact but the measured $50'$ from the wall is accurate to $6''$. What is the approximate percent error?