13.6 Triple Integration

We learned in Section 13.2 how to compute the signed volume V under a surface $z = f(x, y)$ over a region R: $V = \iint_R f(x, y)\, dA$. It follows naturally that if $f(x, y) \geq g(x, y)$ on R, then the **volume between** $f(x, y)$ **and** $g(x, y)$ **on** R is

$$V = \iint_R f(x, y)\, dA - \iint_R g(x, y)\, dA = \iint_R \big(f(x, y) - g(x, y)\big)\, dA.$$

Theorem 129 Volume Between Surfaces

Let f and g be continuous functions on a closed, bounded region R, where $f(x, y) \geq g(x, y)$ for all (x, y) in R. The volume V between f and g over R is

$$V = \iint_R \big(f(x, y) - g(x, y)\big)\, dA.$$

Example 13.35 Finding volume between surfaces

Find the volume of the space region bounded by the planes $z = 3x + y - 4$ and $z = 8 - 3x - 2y$ in the 1$^{\text{st}}$ octant. In Figure 13.36(a) the planes are drawn; in (b), only the defined region is given.

Solution We need to determine the region R over which we will integrate. To do so, we need to determine where the planes intersect. They have common z-values when $3x + y - 4 = 8 - 3x - 2y$. Applying a little algebra, we have:

$$\begin{aligned} 3x + y - 4 &= 8 - 3x - 2y \\ 6x + 3y &= 12 \\ 2x + y &= 4 \end{aligned}$$

The planes intersect along the line $2x + y = 4$. Therefore the region R is bounded by $x = 0$, $y = 0$, and $y = 4 - 2x$; we can convert these bounds to integration bounds of $0 \leq x \leq 2$, $0 \leq y \leq 4 - 2x$. Thus

$$\begin{aligned} V &= \iint_R \big(8 - 3x - 2y - (3x + y - 4)\big)\, dA \\ &= \int_0^2 \int_0^{4-2x} \big(12 - 6x - 3y\big)\, dy\, dx \\ &= 16. \end{aligned}$$

The volume between the surfaces is 16 cubic units.

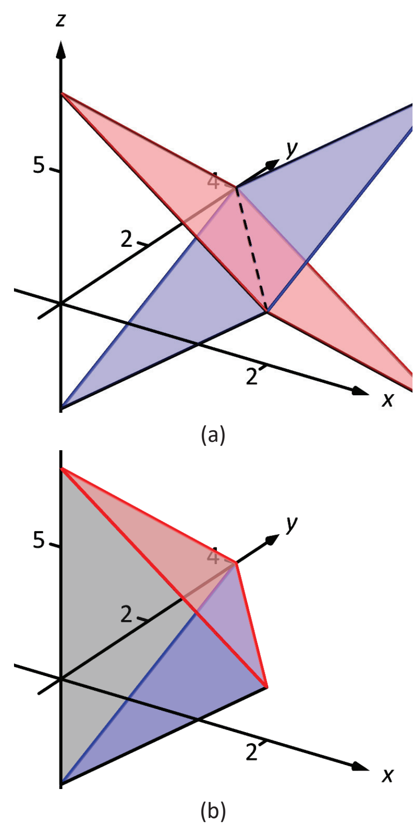


Figure 13.36: Finding the volume between the planes given in Example 13.35.

Notes:

In the preceding example, we found the volume by evaluating the integral

$$\int_0^2 \int_0^{4-2x} \big(8 - 3x - 2y - (3x + y - 4)\big)\, dy\, dx.$$

Note how we can rewrite the integrand as an integral, much as we did in Section 13.1:

$$8 - 3x - 2y - (3x + y - 4) = \int_{3x+y-4}^{8-3x-2y} dz.$$

Thus we can rewrite the double integral that finds volume as

$$\int_0^2 \int_0^{4-2x} \big(8-3x-2y-(3x+y-4)\big)\, dy\, dx = \int_0^2 \int_0^{4-2x} \left(\int_{3x+y-4}^{8-3x-2y} dz\right) dy\, dx.$$

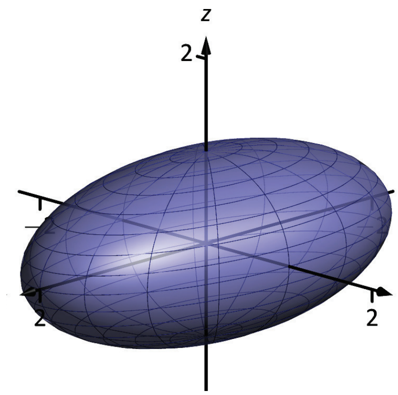
This no longer looks like a "double integral," but more like a "triple integral." Just as our first introduction to double integrals was in the context of finding the area of a plane region, our introduction into triple integrals will be in the context of finding the volume of a space region.

To formally find the volume of a closed, bounded region D in space, such as the one shown in Figure 13.37(a), we start with an approximation. Break D into n rectangular solids; the solids near the boundary of D may possibly not include portions of D and/or include extra space. In Figure 13.37(b), we zoom in on a portion of the boundary of D to show a rectangular solid that contains space not in D; as this is an approximation of the volume, this is acceptable and this error will be reduced as we shrink the size of our solids.

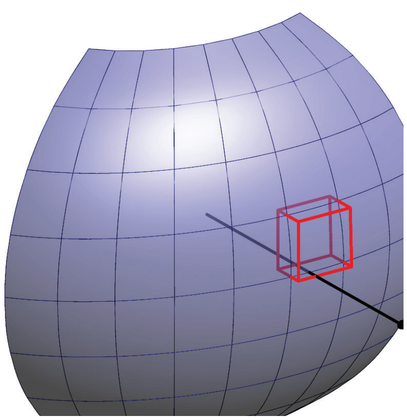
The volume ΔV_i of the i^{th} solid D_i is $\Delta V_i = \Delta x_i \Delta y_i \Delta z_i$, where Δx_i, Δy_i and Δz_i give the dimensions of the rectangular solid in the x, y and z directions, respectively. By summing up the volumes of all n solids, we get an approximation of the volume V of D:

$$V \approx \sum_{i=1}^{n} \Delta V_i = \sum_{i=1}^{n} \Delta x_i \Delta y_i \Delta z_i.$$

Let $||\Delta D||$ represent the length of the longest diagonal of rectangular solids in the subdivision of D. As $||\Delta D|| \to 0$, the volume of each solid goes to 0, as do each of Δx_i, Δy_i and Δz_i, for all i. Our calculus experience tells us that taking a limit as $||\Delta D|| \to 0$ turns our approximation of V into an exact calculation of V. Before we state this result in a theorem, we use a definition to define some terms.



(a)



(b)

Figure 13.37: Approximating the volume of a region D in space.

Notes:

Definition 110 Triple Integrals, Iterated Integration (Part I)

Let D be a closed, bounded region in space. Let a and b be real numbers, let $g_1(x)$ and $g_2(x)$ be continuous functions of x, and let $f_1(x,y)$ and $f_2(x,y)$ be continuous functions of x and y.

1. The volume V of D is denoted by a **triple integral**,
$$V = \iiint_D dV.$$
2. The iterated integral $\displaystyle\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} dz\, dy\, dx$ is evaluated as
$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} dz\, dy\, dx = \int_a^b \int_{g_1(x)}^{g_2(x)} \left(\int_{f_1(x,y)}^{f_2(x,y)} dz \right) dy\, dx.$$
Evaluating the above iterated integral is **triple integration.**

Our informal understanding of the notation $\iiint_D dV$ is "sum up lots of little volumes over D," analogous to our understanding of $\iint_R dA$ and $\iint_R dm$.

We now state the major theorem of this section.

Theorem 130 Triple Integration (Part I)

Let D be a closed, bounded region in space and let ΔD be any subdivision of D into n rectangular solids, where the i^{th} subregion D_i has dimensions $\Delta x_i \times \Delta y_i \times \Delta z_i$ and volume ΔV_i.

1. The volume V of D is
$$V = \iiint_D dV = \lim_{||\Delta D|| \to 0} \sum_{i=1}^{n} \Delta V_i = \lim_{||\Delta D|| \to 0} \sum_{i=1}^{n} \Delta x_i \Delta y_i \Delta z_i.$$
2. If D is defined as the region bounded by the planes $x = a$ and $x = b$, the cylinders $y = g_(x)$ and $y = g_2(x)$, and the surfaces $z = f_1(x,y)$ and $z = f_2(x,y)$, where $a < b$, $g_1(x) \leq g_2(x)$ and $f_1(x,y) \leq f_2(x,y)$ on D, then
$$\iiint_D dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} dz\, dy\, dx.$$
3. V can be determined using iterated integration with other orders of integration (there are 6 total), as long as D is defined by the region enclosed by a pair of planes, a pair of cylinders, and a pair of surfaces.

Notes:

We evaluated the area of a plane region R by iterated integration, where the bounds were "from curve to curve, then from point to point." Theorem 130 allows us to find the volume of a space region with an iterated integral with bounds "from surface to surface, then from curve to curve, then from point to point." In the iterated integral

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} dz\, dy\, dx,$$

the bounds $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$ define a region R in the x-y plane over which the region D exists in space. However, these bounds are also defining surfaces in space; $x = a$ is a plane and $y = g_1(x)$ is a cylinder. The combination of these 6 surfaces enclose, and define, D.

Examples will help us understand triple integration, including integrating with various orders of integration.

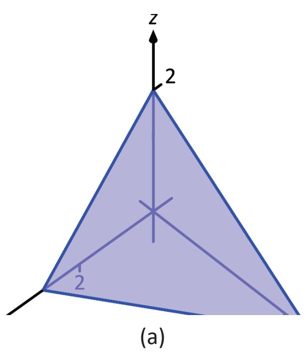


Figure 13.38: The region D used in Example 13.36 in (a); in (b), the region found by collapsing D onto the x-y plane.

Example 13.36 Finding the volume of a space region with triple integration

Find the volume of the space region in the 1^{st} octant bounded by the plane $z = 2 - y/3 - 2x/3$, shown in Figure 13.38(a), using the order of integration $dz\, dy\, dx$. Set up the triple integrals that give the volume in the other 5 orders of integration.

Solution Starting with the order of integration $dz\, dy\, dx$, we need to first find bounds on z. The region D is bounded below by the plane $z = 0$ (because we are restricted to the first octant) and above by $z = 2 - y/3 - 2x/3$; $0 \leq z \leq 2 - y/3 - 2x/3$.

To find the bounds on y and x, we "collapse" the region onto the x-y plane, giving the triangle shown in Figure 13.38(b). (We know the equation of the line $y = 6 - 2x$ in two ways. First, by setting $z = 0$, we have $0 = 2 - y/3 - 2x/3 \Rightarrow y = 6 - 2x$. Secondly, we know this is going to be a straight line between the points $(3, 0)$ and $(0, 6)$ in the x-y plane.)

We define that region R, in the integration order of $dy\, dx$, with bounds $0 \leq$

Notes:

$y \leq 6-2x$ and $0 \leq x \leq 3$. Thus the volume V of the region D is:

$$\begin{aligned}
V &= \iiint_D dV \\
&= \int_0^3 \int_0^{6-2x} \int_0^{2-\frac{1}{3}y-\frac{2}{3}x} dz\, dy\, dz \\
&= \int_0^3 \int_0^{6-2x} \left(\int_0^{2-\frac{1}{3}y-\frac{2}{3}x} dz \right) dy\, dz \\
&= \int_0^3 \int_0^{6-2x} z\Big|_0^{2-\frac{1}{3}y-\frac{2}{3}x} dy\, dz \\
&= \int_0^3 \int_0^{6-2x} \left(2-\frac{1}{3}y-\frac{2}{3}x \right) dy\, dz.
\end{aligned}$$

From this step on, we are evaluating a double integral as done many times before. We skip these steps and give the final volume,

$$= 6\text{units}^3.$$

The order $dz\, dx\, dy$:

Now consider the volume using the order of integration $dz\, dx\, dy$. The bounds on z are the same as before, $0 \leq z \leq 2-y/3-2x/3$. Collapsing the space region on the x-y plane as shown in Figure 13.38(b), we now describe this triangle with the order of integration $dx\, dy$. This gives bounds $0 \leq x \leq 3-y/2$ and $0 \leq y \leq 6$. Thus the volume is given by the triple integral

$$V = \int_0^6 \int_0^{3-\frac{1}{2}y} \int_0^{2-\frac{1}{3}y-\frac{2}{3}x} dz\, dx\, dy.$$

The order $dx\, dy\, dz$:

Following our "surface to surface..." strategy, we need to determine the *x-surfaces* that bound our space region. To do so, approach the region "from behind," in the direction of increasing x. The first surface we hit as we enter the region is the y-z plane, defined by $x=0$. We come out of the region at the plane $z=2-y/3-2x/3$; solving for x, we have $x=3-y/2-3z/2$. Thus the bounds on x are: $0 \leq x \leq 3-y/2-3z/2$.

Now collapse the space region onto the y-z plane, as shown in Figure 13.39(a). (Again, we find the equation of the line $z=2-y/3$ by setting $x=0$ in the equation $x=3-y/2-3z/2$.) We need to find bounds on this region with the order

Notes:

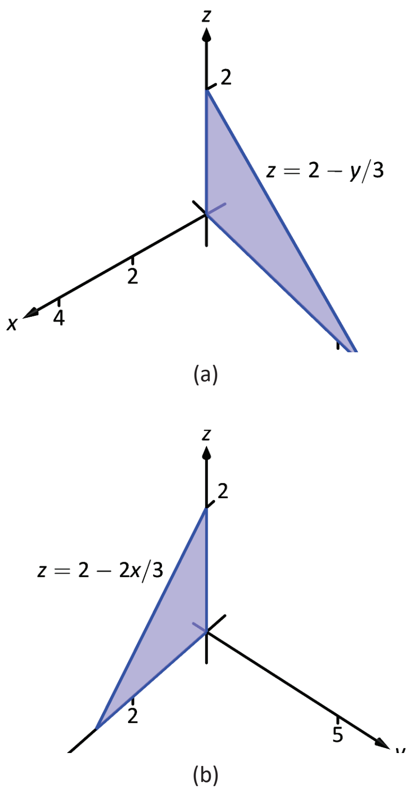


Figure 13.39: The region D in Example 13.36 is collapsed onto the y-z plane in (a); in (b), the region is collapsed onto the x-z plane.

dy dz. The *curves* that bound y are $y = 0$ and $y = 6 - 3z$; the *points* that bound z are 0 and 2. Thus the triple integral giving volume is:

$$\begin{array}{c} 0 \le x \le 3 - y/2 - 3z/2 \\ 0 \le y \le 6 - 3z \\ 0 \le z \le 2 \end{array} \quad \Rightarrow \quad \int_0^2 \int_0^{6-3z} \int_0^{3-y/2-3z/2} dx\, dy\, dz.$$

The order *dx dz dy*:

The x-bounds are the same as the order above. We now consider the triangle in Figure 13.39(a) and describe it with the order *dz dy*: $0 \le z \le 2 - y/3$ and $0 \le y \le 6$. Thus the volume is given by:

$$\begin{array}{c} 0 \le x \le 3 - y/2 - 3z/2 \\ 0 \le z \le 2 - y/3 \\ 0 \le y \le 6 \end{array} \quad \Rightarrow \quad \int_0^6 \int_0^{2-y/3} \int_0^{3-y/2-3z/2} dx\, dz\, dy.$$

The order *dy dz dx*:

We now need to determine the y-surfaces that determine our region. Approaching the space region from "behind" and moving in the direction of increasing y, we first enter the region at $y = 0$, and exit along the plane $z = 2 - y/3 - 2x/3$. Solving for y, this plane has equation $y = 6 - 2x - 3z$. Thus y has bounds $0 \le y \le 6 - 2x - 3z$.

Now collapse the region onto the x-z plane, as shown in Figure 13.39(b). The curves bounding this triangle are $z = 0$ and $z = 2 - 2x/3$; x is bounded by the points $x = 0$ to $x = 3$. Thus the triple integral giving volume is:

$$\begin{array}{c} 0 \le y \le 6 - 2x - 3z \\ 0 \le z \le 2 - 2x/3 \\ 0 \le x \le 3 \end{array} \quad \Rightarrow \quad \int_0^3 \int_0^{2-2x/3} \int_0^{6-2x-3z} dy\, dz\, dx.$$

The order *dy dx dz*:

The y-bounds are the same as in the order above. We now determine the bounds of the triangle in Figure 13.39(b) using the order *dy dx dz*. x is bounded by $x = 0$ and $x = 3 - 3z/2$; z is bounded between $z = 0$ and $z = 2$. This leads to the triple integral:

$$\begin{array}{c} 0 \le y \le 6 - 2x - 3z \\ 0 \le x \le 3 - 3z/2 \\ 0 \le z \le 2 \end{array} \quad \Rightarrow \quad \int_0^2 \int_0^{3-3z/2} \int_0^{6-2x-3z} dy\, dx\, dz.$$

Notes:

This problem was long, but hopefully useful, demonstrating how to determine bounds with every order of integration to describe the region D. In practice, we only need 1, but being able to do them all gives us flexibility to choose the order that suits us best.

In the previous example, we collapsed the surface into the x-y, x-z, and y-z planes as we determined the "curve to curve, point to point" bounds of integration. Since the surface was a triangular portion of a plane, this collapsing, or *projecting*, was simple: the *projection* of a straight line in space onto a coordinate plane is a line.

The following example shows us how to do this when dealing with more complicated surfaces and curves.

Example 13.37 Finding the projection of a curve in space onto the coordinate planes

Consider the surfaces $z = 3 - x^2 - y^2$ and $z = 2y$, as shown in Figure 13.40(a). The curve of their intersection is shown, along with the projection of this curve into the coordinate planes, shown dashed. Find the equations of the projections into the coordinate planes.

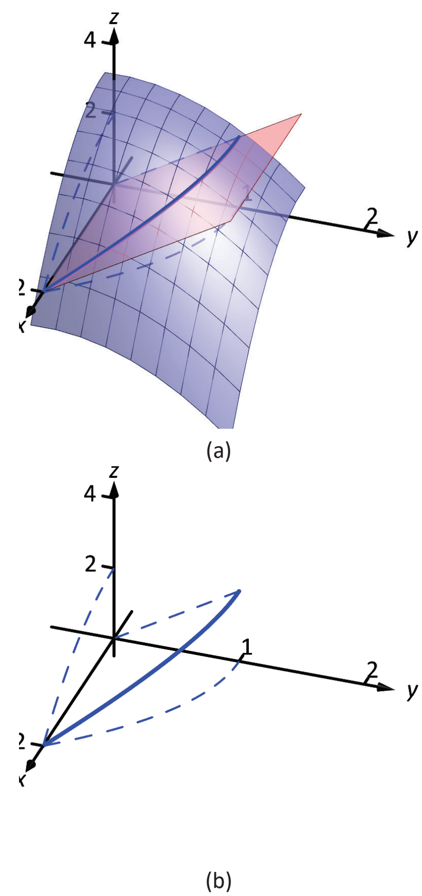


Figure 13.40: Finding the projections of the curve of intersection in Example 13.37.

Solution The two surfaces are $z = 3 - x^2 - y^2$ and $z = 2y$. To find where they intersect, it is natural to set them equal to each other: $3 - x^2 - y^2 = 2y$. This is an implicit relation of x and y that gives all points (x, y) in the x-y plane where the z values of the two surfaces are equal.

We can rewrite this implicit relation by completing the square:

$$3 - x^2 - y^2 = 2y \quad \Rightarrow \quad y^2 + 2y + x^2 = 3 \quad \Rightarrow \quad (y+1)^2 + x^2 = 4.$$

Thus in the x-y plane the projection of the intersection is a circle with radius 2, centered at $(0, -1)$.

To project onto the x-z plane, we do a similar procedure: find the x and z values where the y values on the surface are the same. We start by solving the equation of each surface for y. In this particular case, it works well to actually solve for y^2:

$z = 3 - x^2 - y^2 \quad \Rightarrow \quad y^2 = 3 - x^2 - z$

$z = 2y \quad \Rightarrow \quad y^2 = z^2/4.$

Thus we have (after again completing the square):

$$3 - x^2 - z = z^2/4 \quad \Rightarrow \quad \frac{(z+2)^2}{16} + \frac{x^2}{4} = 1,$$

and ellipse centered at $(0, -2)$ in the x-z plane with a major axis of length 8 and a minor axis of length 4.

Notes:

Finally, to project the curve of intersection into the y-z plane, we solve equation for x. Since $z = 2y$ is a cylinder that lacks the variable x, it becomes our equation of the projection in the y-z plane.

All three projections are shown in Figure 13.40(b).

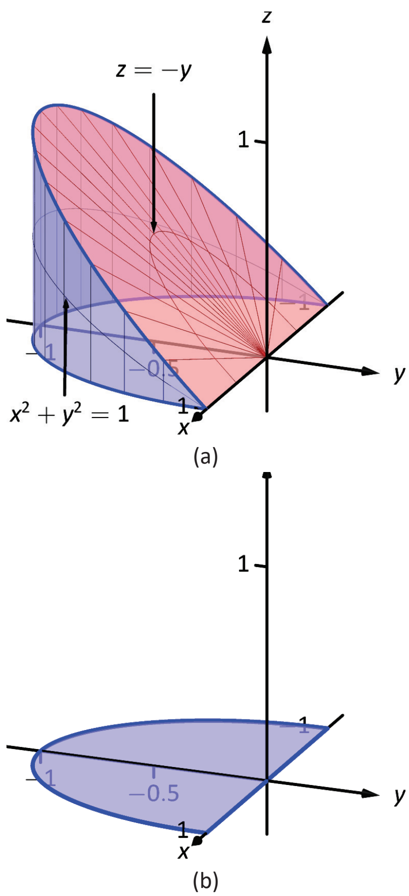


Figure 13.41: The region D in Example 13.38 is shown in (a); in (b), it is collapsed onto the x-y plane.

Example 13.38 Finding the volume of a space region with triple integration

Set up the triple integrals that find the volume of the space region D bounded by the surfaces $x^2+y^2=1$, $z=0$ and $z=-y$, as shown in Figure 13.41(a), with the orders of integration $dz\,dy\,dx$, $dy\,dx\,dz$ and $dx\,dz\,dy$.

Solution The order $dz\,dy\,dx$:

The region D is bounded below by the plane $z=0$ and above by the plane $z=-y$. The cylinder $x^2+y^2=1$ does not offer any bounds in the z-direction, as that surface is parallel to the z-axis. Thus $0\le z\le -y$.

Collapsing the region into the x-y plane, we get part of the circle with equation $x^2+y^2=1$ as shown in Figure 13.41(b). As a function of x, this half circle has equation $y=-\sqrt{1-x^2}$. Thus y is bounded below by $-\sqrt{1-x^2}$ and above by $y=0$: $-\sqrt{1-x^2}\le y\le 0$. The x bounds of the half circle are $-1\le x\le 1$. All together, the bounds of integration and triple integral are as follows:

$$\begin{array}{c} 0\le z\le -y \\ -\sqrt{1-x^2}\le y\le 0 \\ -1\le x\le 1 \end{array} \quad\Rightarrow\quad \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0}\int_{0}^{-y} dz\,dy\,dx.$$

We evaluate this triple integral:

$$\begin{aligned}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0}\int_{0}^{-y} dz\,dy\,dx &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0}\big(-y\big)\,dy\,dx \\ &= \int_{-1}^{1}\Big(-\frac{1}{2}y^2\Big)\Big|_{-\sqrt{1-x^2}}^{0}\,dx \\ &= \int_{-1}^{1}\frac{1}{2}\big(1-x^2\big)\,dx \\ &= \left(\frac{1}{2}\left(x-\frac{1}{3}x^3\right)\right)\Bigg|_{-1}^{1} \\ &= \frac{2}{3}\text{units}^3.\end{aligned}$$

Notes:

With the order $dy\,dx\,dz$:

The region is bounded "below" in the y-direction by the surface $x^2+y^2=1 \Rightarrow y=-\sqrt{1-x^2}$ and "above" by the surface $y=-z$. Thus the y bounds are $-\sqrt{1-x^2}\leq y\leq -z$.

Collapsing the region onto the x-z plane gives the region shown in Figure 13.42(a); this half circle has equation $x^2+z^2=1$. (We find this curve by solving each surface for y^2, then setting them equal to each other. We have $y^2=1-x^2$ and $y=-z \Rightarrow y^2=z^2$. Thus $x^2+z^2=1$.) It is bounded below by $x=-\sqrt{1-z^2}$ and above by $x=\sqrt{1-z^2}$, where z is bounded by $0\leq z\leq 1$. All together, we have:

$$\begin{array}{c} -\sqrt{1-x^2}\leq y\leq -z \\ -\sqrt{1-z^2}\leq x\leq \sqrt{1-z^2} \\ 0\leq z\leq 1 \end{array} \quad\Rightarrow\quad \int_0^1\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{-\sqrt{1-x^2}}^{-z} dy\,dx\,dz.$$

With the order $dx\,dz\,dy$:

D is bounded below by the surface $x=-\sqrt{1-y^2}$ and above by $\sqrt{1-y^2}$. We then collapse the region onto the y-z plane and get the triangle shown in Figure 13.42(b). (The hypotenuse is the line $z=-y$, just as the plane.) Thus z is bounded by $0\leq z\leq -y$ and y is bounded by $-1\leq y\leq 0$. This gives:

$$\begin{array}{c} -\sqrt{1-y^2}\leq x\leq \sqrt{1-y^2} \\ 0\leq z\leq -y \\ -1\leq y\leq 0 \end{array} \quad\Rightarrow\quad \int_{-1}^0\int_0^{-y}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\,dz\,dy.$$

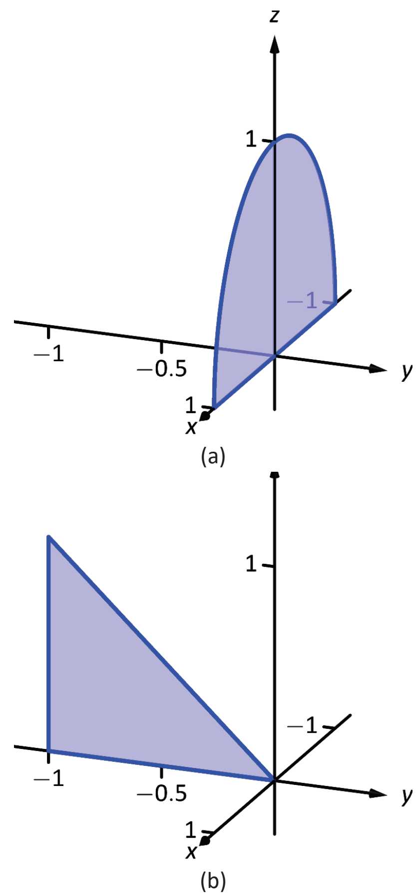


Figure 13.42: The region D in Example 13.38 is shown collapsed onto the x-z plane in (a); in (b), it is collapsed onto the y-z plane.

The following theorem states two things that should make "common sense" to us. First, using the triple integral to find volume of a region D should always return a positive number; we are computing *volume* here, not *signed volume*. Secondly, to compute the volume of a "complicated" region, we could break it up into subregions and compute the volumes of each subregion separately, summing them later to find the total volume.

Notes:

Theorem 131 Properties of Triple Integrals

Let D be a closed, bounded region in space, and let D_1 and D_2 be non-overlapping regions such that $D = D_1 \bigcup D_2$.

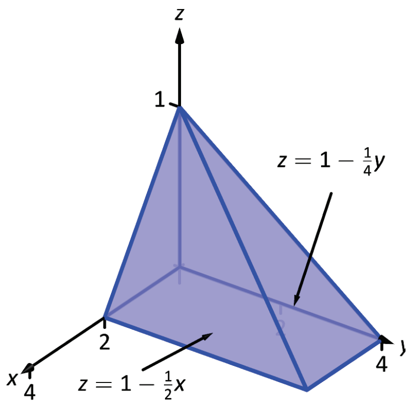
1. $\displaystyle\iiint_D dV \geq 0$

2. $\displaystyle\iiint_D dV = \iiint_{D_1} dV + \iiint_{D_2} dV.$

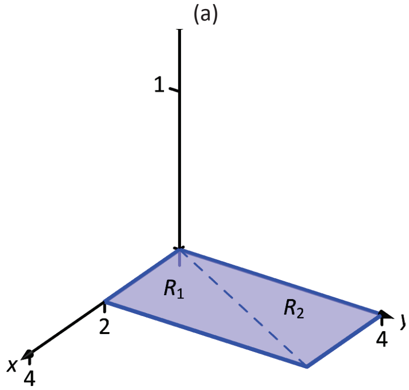
We use this latter property in the next example.

Example 13.39 Finding the volume of a space region with triple integration

Find the volume of the space region D bounded by the coordinate planes, $z = 1 - x/2$ and $z = 1 - y/4$, as shown in Figure 13.43(a). Set up the triple integrals that find the volume of D in all 6 orders of integration.



(a)



(b)

Figure 13.43: The region D in Example 13.39 is shown in (a); in (b), it is collapsed onto the x-y plane.

Solution Following the bounds–determining strategy of "surface to surface, curve to curve, and point to point," we can see that the most difficult orders of integration are the two in which we integrate with respect to z first, for there are two "upper" surfaces that bound D in the z-direction. So we start by noting that we have

$$0 \leq z \leq 1 - \frac{1}{2}x \quad \text{and} \quad 0 \leq z \leq 1 - \frac{1}{4}y.$$

We now collapse the region D onto the x-y axis, as shown in Figure 13.43(b). The boundary of D, the line from $(0,0,1)$ to $(2,4,0)$, is shown in part (b) of the figure as a dashed line; it has equation $y = 2x$. (We can recognize this in two ways: one, in collapsing the line from $(0,0,1)$ to $(2,4,0)$ onto the x-y plane, we simply ignore the z-values, meaning the line now goes from $(0,0)$ to $(2,4)$. Secondly, the two surfaces meet where $z = 1 - x/2$ is equal to $z = 1 - y/4$: thus $1 - x/2 = 1 - y/4 \Rightarrow y = 2x$.)

We use the second property of Theorem 131 to state that

$$\iiint_D dV = \iiint_{D_1} dV + \iiint_{D_2} dV,$$

where D_1 and D_2 are the space regions above the plane regions R_1 and R_2, respectively. Thus we can say

$$\iiint_D dV = \iint_{R_1} \left(\int_0^{1-x/2} dz\right) dA + \iint_{R_2} \left(\int_0^{1-y/4} dz\right) dA.$$

Notes:

All that is left is to determine bounds of R_1 and R_2, depending on whether we are integrating with order *dx dy* or *dy dx*. We give the final integrals here, leaving it to the reader to confirm these results.

dz dy dx:

$$\begin{array}{c} 0 \le z \le 1 - x/2 \\ 0 \le y \le 2x \\ 0 \le x \le 2 \end{array} \qquad\qquad \begin{array}{c} 0 \le z \le 1 - y/4 \\ 2x \le y \le 4 \\ 0 \le x \le 2 \end{array}$$

$$\iiint_D dV = \int_0^2 \int_0^{2x} \int_0^{1-x/2} dz\, dy\, dx + \int_0^2 \int_{2x}^4 \int_0^{1-y/4} dz\, dy\, dx$$

dz dx dy:

$$\begin{array}{c} 0 \le z \le 1 - x/2 \\ y/2 \le x \le 2 \\ 0 \le y \le 4 \end{array} \qquad\qquad \begin{array}{c} 0 \le z \le 1 - y/4 \\ 0 \le x \le y/2 \\ 0 \le y \le 4 \end{array}$$

$$\iiint_D dV = \int_0^4 \int_{y/2}^2 \int_0^{1-x/2} dz\, dx\, dy + \int_0^4 \int_0^{y/2} \int_0^{1-y/4} dz\, dx\, dy$$

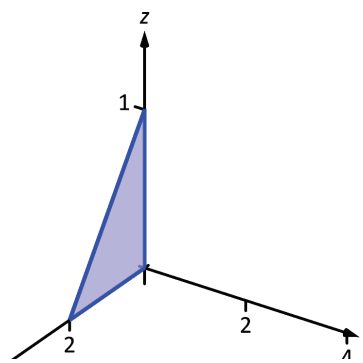
The remaining four orders of integration do not require a sum of triple integrals. In Figure 13.44 we show D collapsed onto the other two coordinate planes. Using these graphs, we give the final orders of integration here, again leaving it to the reader to confirm these results.

dy dx dz:

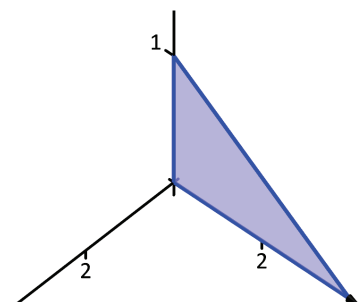
$$\begin{array}{c} 0 \le y \le 4 - 4z \\ 0 \le x \le 2 - 2z \\ 0 \le z \le 1 \end{array} \quad \Rightarrow \int_0^1 \int_0^{2-2z} \int_0^{4-4z} dy\, dx\, dz$$

dy dz dx:

$$\begin{array}{c} 0 \le y \le 4 - 4z \\ 0 \le z \le 1 - x/2 \\ 0 \le x \le 2 \end{array} \quad \Rightarrow \int_0^2 \int_0^{1-x/2} \int_0^{4-4z} dy\, dx\, dz$$



(a)



(b)

Figure 13.44: The region D in Example 13.39 is shown collapsed onto the x-z plane in (a); in (b), it is collapsed onto the y-z plane.

Notes:

dx dy dz:

$$\begin{array}{c} 0 \leq x \leq 2-2z \\ 0 \leq y \leq 4-4z \\ 0 \leq z \leq 1 \end{array} \quad \Rightarrow \int_0^1 \int_0^{4-4z} \int_0^{2-2z} dx\, dy\, dz$$

dx dz dy:

$$\begin{array}{c} 0 \leq x \leq 2-2z \\ 0 \leq z \leq 1-y/4 \\ 0 \leq y \leq 4 \end{array} \quad \Rightarrow \int_0^4 \int_0^{1-y/4} \int_0^{2-2z} dx\, dz\, dy$$

We give one more example of finding the volume of a space region.

Example 13.40 Finding the volume of a space region

Set up a triple integral that gives the volume of the space region D bounded by $z = 2x^2+2$ and $z = 6-2x^2-y^2$. These surfaces are plotted in Figure 13.45(a) and (b), respectively; the region D is shown in part (c) of the figure.

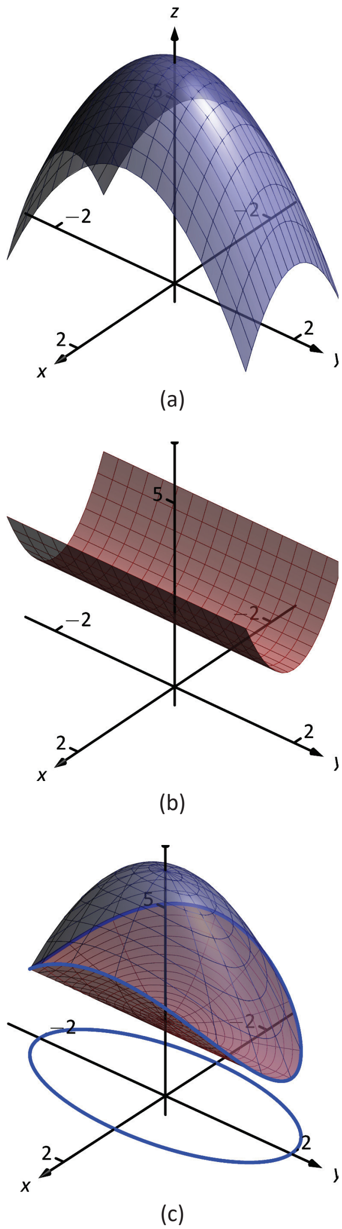


Figure 13.45: The region D is bounded by the surfaces shown in (a) and (b); D is shown in (c).

Solution The main point of this example is this: integrating with respect to z first is rather straightforward; integrating with respect to x first is not.

The order *dz dy dx*:

The bounds on z are clearly $2x^2+2 \leq z \leq 6-2x^2-y^2$. Collapsing D onto the x-y plane gives the ellipse shown in Figure 13.45(c). The equation of this ellipse is found by setting the two surfaces equal to each other:

$$2x^2+2 = 6-2x^2-y^2 \quad \Rightarrow \quad 4x^2+y^2 = 4 \quad \Rightarrow \quad x^2+\frac{y^2}{4} = 1.$$

We can describe this ellipse with the bounds

$$-\sqrt{4-4x^2} \leq y \leq \sqrt{4-4x^2} \quad \text{and} \quad -1 \leq x \leq 1.$$

Thus we find volume as

$$\begin{array}{c} 2x^2+2 \leq z \leq 6-2x^2-y^2 \\ -\sqrt{4-4x^2} \leq y \leq \sqrt{4-4x^2} \\ -1 \leq x \leq 1 \end{array} \quad \Rightarrow \int_{-1}^1 \int_{-\sqrt{4-4x^2}}^{\sqrt{4-4x^2}} \int_{2x^2+2}^{6-2x^2-y^2} dz\, dy\, dx\ .$$

The order *dy dz dx*:

Notes:

Integrating with respect to y is not too difficult. Since the surface $z = 2x^2 + 2$ is a cylinder whose directrix is the y-axis, it does not create a border for y. The paraboloid $z = 6 - 2x^2 - y^2$ does; solving for y, we get the bounds

$$-\sqrt{6 - 2x^2 - z} \leq y \leq \sqrt{6 - 2x^2 - z}.$$

Collapsing D onto the x-z axes gives the region shown in Figure 13.46(a); the lower curve is from the cylinder, with equation $z = 2x^2 + 2$. The upper curve is from the paraboloid; with $y = 0$, the curve is $z = 6 - 2x^2$. Thus bounds on z are $2x^2 + 2 \leq z \leq 6 - 2x^2$; the bounds on x are $-1 \leq x \leq 1$. Thus we have:

$$\begin{array}{c} -\sqrt{6 - 2x^2 - z} \leq y \leq \sqrt{6 - 2x^2 - z} \\ 2x^2 + 2 \leq z \leq 6 - 2x^2 \\ -1 \leq x \leq 1 \end{array} \quad \Rightarrow \quad \int_{-1}^{1} \int_{2x^2+2}^{6-2x^2} \int_{-\sqrt{6-2x^2-z}}^{\sqrt{6-2x^2-z}} dy\, dz\, dx.$$

The order $dx\, dz\, dy$:

This order takes more effort as D must be split into two subregions. The two surfaces create two sets of upper/lower bounds in terms of x; the cylinder creates bounds

$$-\sqrt{z/2 - 1} \leq x \leq \sqrt{z/2 - 1}$$

for region D_1 and the paraboloid creates bounds

$$-\sqrt{3 - y^2/2 - z^2/2} \leq x \leq \sqrt{3 - y^2/2 - z^2/2}$$

for region D_2.

Collapsing D onto the y-z axes gives the regions shown in Figure 13.46(b). We find the equation of the curve $z = 4 - y^2/2$ by noting that the equation of the ellipse seen in Figure 13.45(c) has equation

$$x^2 + y^2/4 = 1 \quad \Rightarrow \quad x = \sqrt{1 - y^2/4}.$$

Substitute this expression for x in either surface equation, $z = 6 - 2x^2 - y^2$ or $z = 2x^2 + 2$. In both cases, we find

$$z = 4 - \frac{1}{2}y^2.$$

Region R_1, corresponding to D_1, has bounds

$$2 \leq z \leq 4 - y^2/2, \quad -2 \leq y \leq 2$$

and region R_2, corresponding to D_2, has bounds

$$4 - y^2/2 \leq z \leq 6 - y^2, \quad -2 \leq y \leq 2.$$

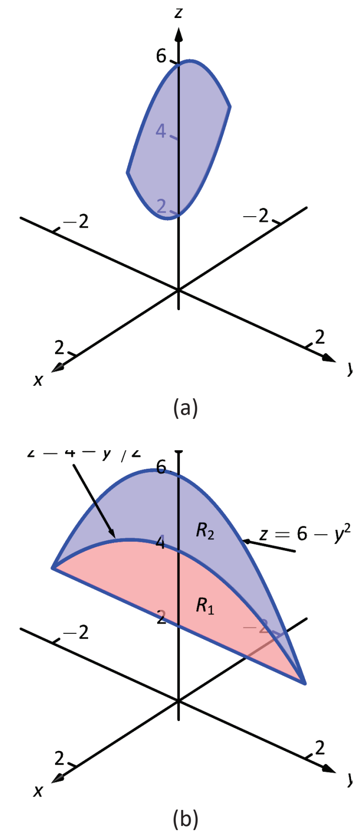


Figure 13.46: The region D in Example 13.40 is collapsed onto the x-z plane in (a); in (b), it is collapsed onto the y-z plane.

Notes:

Thus the volume of D is given by:

$$\int_{-2}^{2}\int_{2}^{4-y^2/2}\int_{-\sqrt{z/2-1}}^{\sqrt{z/2-1}} dx\,dz\,dy \;+\; \int_{-2}^{2}\int_{4-y^2/2}^{6-y^2}\int_{-\sqrt{3-y^2/2-z^2/2}}^{\sqrt{3-y^2/2-z^2/2}} dx\,dz\,dy.$$

If all one wanted to do in Example 13.40 was find the volume of the region D, one would have likely stopped at the first integration setup (with order $dz\,dy\,dx$) and computed the volume from there. However, we included the other two methods 1) to show that it could be done, "messy" or not, and 2) because sometimes we "have" to use a less desirable order of integration in order to actually integrate.

Triple Integration and Functions of Three Variables

There are uses for triple integration beyond merely finding volume, just as there are uses for integration beyond "area under the curve." These uses start with understanding how to integrate functions of three variables, which is effectively no different than integrating functions of two variables. This leads us to a definition, followed by an example.

Definition 111 Iterated Integration, (Part II)

Let D be a closed, bounded region in space, over which $g_1(x)$, $g_2(x)$, $f_1(x,y)$, $f_2(x,y)$ and $h(x,y,z)$ are all continuous, and let a and b be real numbers.

The **iterated integral** $\displaystyle\int_a^b\int_{g_1(x)}^{g_2(x)}\int_{f_1(x,y)}^{f_2(x,y)} h(x,y,z)\,dz\,dy\,dx$ is evaluated as

$$\int_a^b\int_{g_1(x)}^{g_2(x)}\int_{f_1(x,y)}^{f_2(x,y)} h(x,y,z)\,dz\,dy\,dx = \int_a^b\int_{g_1(x)}^{g_2(x)}\left(\int_{f_1(x,y)}^{f_2(x,y)} h(x,y,z)\,dz\right)dy\,dx.$$

Example 13.41 Evaluating a triple integral of a function of three variables

Evaluate $\displaystyle\int_0^1\int_{x^2}^{x}\int_{x^2-y}^{2x+3y}\big(xy+2xz\big)\,dz\,dy\,dx.$

Solution We evaluate this integral according to Definition 111.

Notes:

$$\int_0^1 \int_{x^2}^{x} \int_{x^2-y}^{2x+3y} \big(xy+2xz\big)\, dz\, dy\, dx$$

$$= \int_0^1 \int_{x^2}^{x} \left(\int_{x^2-y}^{2x+3y} \big(xy+2xz\big)\, dz\right) dy\, dx$$

$$= \int_0^1 \int_{x^2}^{x} \left(\big(xyz+xz^2\big)\Big|_{x^2-y}^{2x+3y}\right) dy\, dx$$

$$= \int_0^1 \int_{x^2}^{x} \left(xy(2x+3y)+x(2x+3y)^2-\Big(xy(x^2-y)+x(x^2-y)^2\Big)\right) dy\, dx$$

$$= \int_0^1 \int_{x^2}^{x} \Big(-x^5+x^3y+4x^3+14x^2y+12xy^2\Big)\, dy\, dx.$$

We continue as we have in the past, showing fewer steps.

$$= \int_0^1 \left(-\frac{7}{2}x^7-8x^6-\frac{7}{2}x^5+15x^4\right) dx$$

$$= \frac{281}{336} \approx 0.836.$$

We now know *how* to evaluate a triple integral of a function of three variables; we do not yet understand what it *means*. We build up this understanding in a way very similar to how we have understood integration and double integration.

Let $h(x,y,z)$ a continuous function of three variables, defined over some space region D. We can partition D into n rectangular–solid subregions, each with dimensions $\Delta x_i \times \Delta y_i \times \Delta z_i$. Let (x_i, y_i, z_i) be some point in the i^{th} subregion, and consider the product $h(x_i,y_i,z_i)\Delta x_i\Delta y_i\Delta z_i$. It is the product of a function value (that's the $h(x_i,y_i,z_i)$ part) and a small volume ΔV_i (that's the $\Delta x_i\Delta y_i\Delta z_i$ part). One of the simplest understanding of this type of product is when h describes the density of an object, for then $h \times$ volume $=$ mass.

We can sum up all n products over D. Again letting $||\Delta D||$ represent the length of the longest diagonal of the n rectangular solids in the partition, we can take the limit of the sums of products as $||\Delta D|| \to 0$. That is, we can find

$$S = \lim_{||\Delta D||\to 0} \sum_{i=1}^{n} h(x_i,y_i,z_i)\Delta V_i = \lim_{||\Delta D||\to 0} \sum_{i=1}^{n} h(x_i,y_i,z_i)\Delta x_i\Delta y_i\Delta z_i.$$

While this limit has lots of interpretations depending on the function h, in the case where h describes density, S is the total mass of the object described by the region D.

Notes:

We now use the above limit to define the **triple integral**, give a theorem that relates triple integrals to iterated iteration, followed by the application of triple integrals to find the centers of mass of solid objects.

Definition 112 Triple Integral

Let $w = h(x, y, z)$ be a continuous function over a closed, bounded space region D, and let ΔD be any partition of D into n rectangular solids with volume ΔV_i. The **triple integral of** h **over** D is

$$\iiint_D h(x, y, z)\, dV = \lim_{||\Delta D||\to 0} \sum_{i=1}^{n} h(x_i, y_i, z_i)\Delta V_i.$$

Note: In the marginal note on page 829, we showed how the summation of rectangles over a region R in the plane could be viewed as a double sum, leading to the double integral. Likewise, we can view the sum $\sum_{i=1}^{n} h(x_i, y_i, z_i)\Delta x_i \Delta y_i \Delta z_i$ as a triple sum,

$$\sum_{k=1}^{p}\sum_{j=1}^{n}\sum_{i=1}^{m} h(x_i, y_j, z_k)\Delta x_i \Delta y_j \Delta z_k,$$

which we evaluate as

$$\sum_{k=1}^{p}\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m} h(x_i, y_j, z_k)\Delta x_i\right)\Delta y_j\right)\Delta z_k.$$

Here we fix a k value, which establishes the z-height of the rectangular solids on one "level" of all the rectangular solids in the space region D. The inner double summation adds up all the volumes of the rectangular solids on this level, while the outer summation adds up the volumes of each level.

This triple summation understanding leads to the $\iiint_D$ notation of the triple integral, as well as the method of evaluation shown in Theorem 132.

The following theorem assures us that the above limit exists for continuous functions h and gives us a method of evaluating the limit.

Theorem 132 Triple Integration (Part II)

Let $w = h(x, y, z)$ be a continuous function over a closed, bounded space region D, and let ΔD be any partition of D into n rectangular solids with volume V_i.

1. The limit $\displaystyle\lim_{||\Delta D||\to 0} \sum_{i=1}^{n} h(x_i, y_i, z_i)\Delta V_i$ exists.

2. If D is defined as the region bounded by the planes $x = a$ and $x = b$, the cylinders $y = g_1(x)$ and $y = g_2(x)$, and the surfaces $z = f_1(x, y)$ and $z = f_2(x, y)$, where $a < b$, $g_1(x) \le g_2(x)$ and $f_1(x, y) \le f_2(x, y)$ on D, then

$$\iiint_D h(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} h(x, y, z)\, dz\, dy\, dx.$$

We now apply triple integration to find the centers of mass of solid objects.

Mass and Center of Mass

One may wish to review Section 13.4 for a reminder of the relevant terms and concepts.

Notes:

Definition 113 Mass, Center of Mass of Solids

Let a solid be represented by a region D in space with variable density function $\delta(x,y,z)$.

1. The **mass** of the object is $M = \iiint_D dm = \iiint_D \delta(x,y,z)\,dV$.
2. The **moment about the x-y plane** is $M_{xy} = \iiint_D z\delta(x,y,z)\,dV$.
3. The **moment about the x-z plane** is $M_{xz} = \iiint_D y\delta(x,y,z)\,dV$.
4. The **moment about the y-z plane** is $M_{yz} = \iiint_D x\delta(x,y,z)\,dV$.
5. The **center of mass** of the object is

$$(\overline{x},\overline{y},\overline{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right).$$

Example 13.42 Finding the center of mass of a solid

Find the mass and center of mass of the solid represented by the space region bounded by the coordinate planes and $z = 2 - y/3 - 2x/3$, shown in Figure 13.47, with constant density $\delta(x,y,z) = 3\text{gm/cm}^3$. (Note: this space region was used in Example 13.36.)

Solution We apply Definition 113. In Example 13.36, we found bounds for the order of integration $dz\,dy\,dx$ to be $0 \le z \le 2-y/3-2x/3$, $0 \le y \le 6-2x$ and $0 \le x \le 3$. We find the mass of the object:

$$\begin{aligned}
M &= \iiint_D \delta(x,y,z)\,dV \\
&= \int_0^3 \int_0^{6-2x} \int_0^{2-y/3-2x/3} (3)\,dz\,dy\,dx \\
&= 3\int_0^3 \int_0^{6-2x} \int_0^{2-y/3-2x/3} dz\,dy\,dx \\
&= 3(6) = 18\text{gm}.
\end{aligned}$$

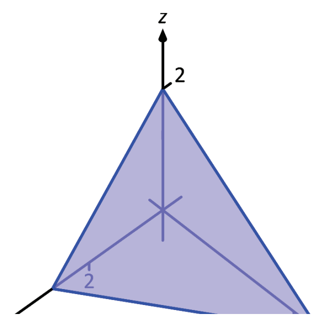


Figure 13.47: Finding the center of mass of this solid in Example 13.42.

The evaluation of the triple integral is done in Example 13.36, so we skipped those steps above. Note how the mass of an object with constant density is simply "density×volume."

Notes:

We now find the moments about the planes.

$$\begin{aligned}
M_{xy} &= \iiint_D 3z\,dV \\
&= \int_0^3 \int_0^{6-2x} \int_0^{2-y/3-2x/3} \big(3z\big)\,dz\,dy\,dx \\
&= \int_0^3 \int_0^{6-2x} \frac{3}{2}\big(2-y/3-2x/3\big)^2\,dy\,dx \\
&= \int_0^3 -\frac{4}{9}\big(x-3\big)^3\,dx \\
&= 9.
\end{aligned}$$

We omit the steps of integrating to find the other moments.

$$\begin{aligned}
M_{yz} &= \iiint_D 3x\,dV \\
&= \frac{27}{2}. \\
M_{xz} &= \iiint_D 3y\,dV \\
&= 27.
\end{aligned}$$

The center of mass is

$$\big(\overline{x},\overline{y},\overline{z}\big) = \left(\frac{27/2}{18},\frac{27}{18},\frac{9}{18}\right) = \big(0.75, 1.5, 0.5\big).$$

Example 13.43 Finding the center of mass of a solid

Find the center of mass of the solid represented by the region bounded by the planes $z = 0$ and $z = -y$ and the cylinder $x^2 + y^2 = 1$, shown in Figure 13.48, with density function $\delta(x, y, z) = 10 + x^2 + 5y - 5z$. (Note: this space region was used in Example 13.38.)

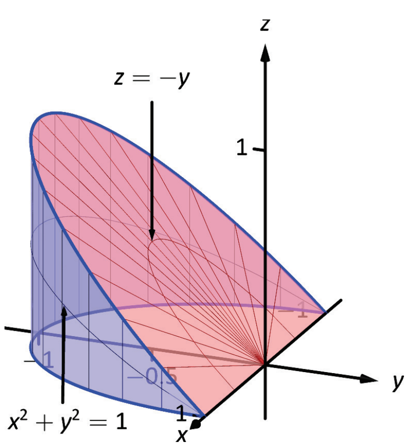


Figure 13.48: Finding the center of mass of this solid in Example 13.43.

Solution As we start, consider the density function. It is symmetric about the y-z plane, and the farther one moves from this plane, the denser the object is. The symmetry indicates that $\overline{x}$ should be 0.

As one moves away from the origin in the y or z directions, the object becomes less dense, though there is more volume in these regions.

Though none of the integrals needed to compute the center of mass are particularly hard, they do require a number of steps. We emphasize here the

Notes:

importance of knowing how to set up the proper integrals; in complex situations we can appeal to technology for a good approximation, if not the exact answer. We use the order of integration $dz\ dy\ dx$, using the bounds found in Example 13.38. (As these are the same for all four triple integrals, we explicitly show the bounds only for M.)

$$\begin{aligned}
M &= \iiint_D \big(10+x^2+5y-5z\big)\ dV \\
&= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0}\int_{0}^{-y} \big(10+x^2+5y-5z\big)\ dV \\
&= \frac{64}{5}-\frac{15\pi}{16} \approx 3.855. \\
M_{yz} &= \iiint_D x\big(10+x^2+5y-5z\big)\ dV \\
&= 0. \\
M_{xz} &= \iiint_D y\big(10+x^2+5y-5z\big)\ dV \\
&= 2-\frac{61\pi}{48} \approx -1.99. \\
M_{xy} &= \iiint_D z\big(10+x^2+5y-5z\big)\ dV \\
&= \frac{61\pi}{96}-\frac{10}{9} \approx 0.885.
\end{aligned}$$

Note how $M_{yz}=0$, as expected. The center of mass is

$$\big(\overline{x},\overline{y},\overline{z}\big) = \left(0,\frac{-1.99}{3.855},\frac{0.885}{3.855}\right) \approx \big(0,-0.516,0.230\big).$$

As stated before, there are many uses for triple integration beyond finding volume. When $h(x,y,z)$ describes a rate of change function over some space region D, then $\displaystyle\iiint_D h(x,y,z)\ dV$ gives the total change over D. Our one specific example of this was computing mass; a density function is simply a "rate of mass change per volume" function. Integrating density gives total mass.

While knowing *how to integrate* is important, it is arguably much more important to know *how to set up* integrals. It takes skill to create a formula that describes a desired quantity; modern technology is very useful in evaluating these formulas quickly and accurately.

Notes:

Exercises 13.6

Terms and Concepts

1. The strategy for establishing bounds for triple integrals is "________ to ________, ________ to ________ and ________ to ________."

2. Give an informal interpretation of what "$\iiint_D dV$" means.

3. Give two uses of triple integration.

4. If an object has a constant density δ and a volume V, what is its mass?

Problems

In Exercises 5 – 8, two surfaces $f_1(x,y)$ and $f_2(x,y)$ and a region R in the x,y plane are given. Set up and evaluate the double integral that finds the volume between these surfaces over R.

5. $f_1(x,y) = 8 - x^2 - y^2$, $f_2(x,y) = 2x + y$;
R is the square with corners $(-1,-1)$ and $(1,1)$.

6. $f_1(x,y) = x^2 + y^2$, $f_2(x,y) = -x^2 - y^2$;
R is the square with corners $(0,0)$ and $(2,3)$.

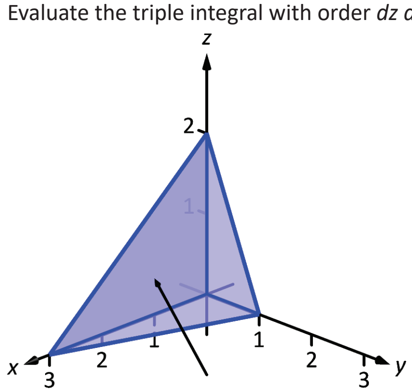
7. $f_1(x,y) = \sin x \cos y$, $f_2(x,y) = \cos x \sin y + 2$;
R is the triangle with corners $(0,0)$, $(\pi,0)$ and (π,π).

8. $f_1(x,y) = 2x^2 + 2y^2 + 3$, $f_2(x,y) = 6 - x^2 - y^2$;
R is the circle $x^2 + y^2 = 1$.

In Exercises 9 – 16, a domain D is described by its bounding surfaces, along with a graph. Set up the triple integrals that give the volume of D in all 6 orders of integration, and find the volume of D by evaluating the indicated triple integral.

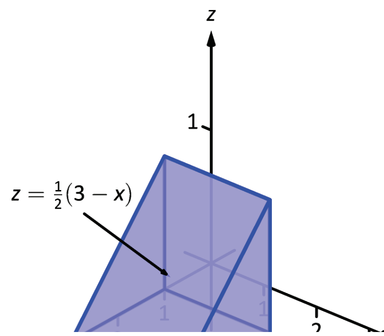
9. D is bounded by the coordinate planes and $z = 2 - 2x/3 - 2y$.

   Evaluate the triple integral with order $dz\,dy\,dx$.



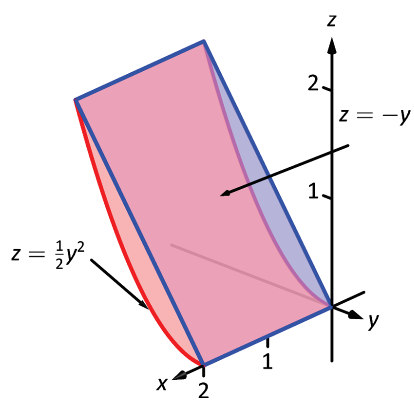
10. D is bounded by the planes $y=0$, $y=2$, $x=1$, $z=0$ and $z=(3-x)/2$.

    Evaluate the triple integral with order $dx\,dy\,dz$.



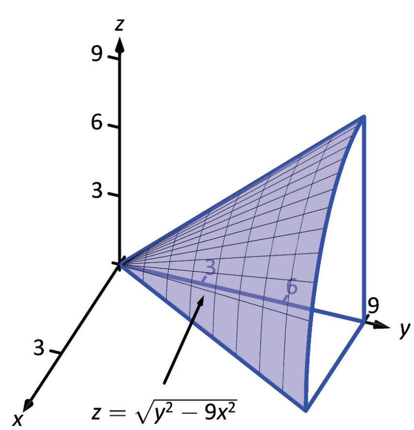
11. D is bounded by the planes $x=0$, $x=2$, $z=-y$ and by $z = y^2/2$.

    Evaluate the triple integral with the order $dy\,dz\,dx$.



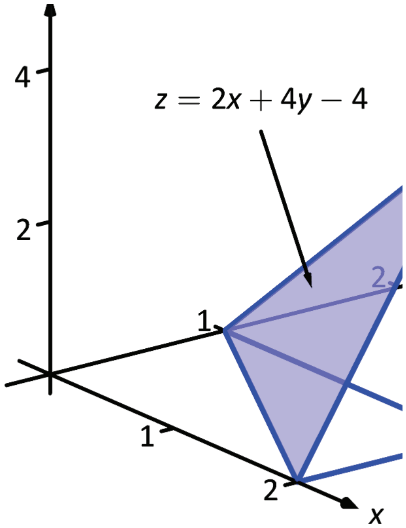
12. D is bounded by the planes $z=0$, $y=9$, $x=0$ and by $z = \sqrt{y^2 - 9x^2}$.

    Do not evaluate any triple integral.



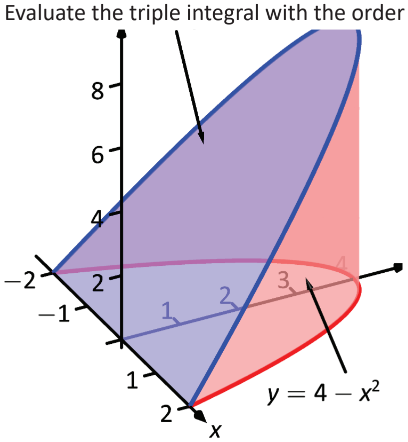
13. D is bounded by the planes $x = 2$, $y = 1$, $z = 0$ and $z = 2x + 4y - 4$.

    Evaluate the triple integral with the order $dx\,dy\,dz$.

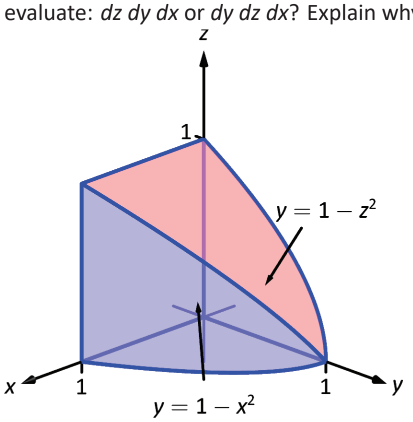


14. D is bounded by the plane $z = 2y$ and by $y = 4 - x^2$.

    Evaluate the triple integral with the order $dz\,dy\,dx$.

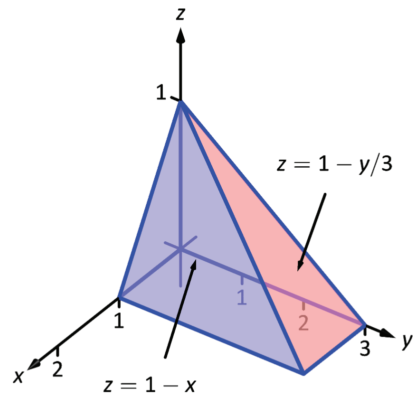


15. D is bounded by the coordinate planes and by $y = 1 - x^2$ and $y = 1 - z^2$.
    Do not evaluate any triple integral. Which order is easier to evaluate: $dz\,dy\,dx$ or $dy\,dz\,dx$? Explain why.



16. D is bounded by the coordinate planes and by $z = 1 - y/3$ and $z = 1 - x$.

    Evaluate the triple integral with order $dx\,dy\,dz$.



In Exercises 17 – 20, evaluate the triple integral.

17. $\displaystyle\int_{-\pi/2}^{\pi/2}\int_0^{\pi}\int_0^{\pi}\big(\cos x\sin y\sin z\big)\,dz\,dy\,dx$

18. $\displaystyle\int_0^1\int_0^x\int_0^{x+y}\big(x + y + z\big)\,dz\,dy\,dx$

19. $\displaystyle\int_0^{\pi}\int_0^1\int_0^z\big(\sin(yz)\big)\,dx\,dy\,dz$

20. $\displaystyle\int_{\pi}^{\pi^2}\int_x^{x^3}\int_{-y^2}^{y^2}\left(z\frac{x^2y + y^2x}{e^{x^2+y^2}}\right)\,dz\,dy\,dx$

In Exercises 21 – 24, find the center of mass of the solid represented by the indicated space region D with density function $\delta(x, y, z)$.

21. D is bounded by the coordinate planes and $z = 2 - 2x/3 - 2y$; $\delta(x, y, z) = 10\text{gm/cm}^3$.
    (Note: this is the same region as used in Exercise 9.)

22. D is bounded by the planes $y = 0$, $y = 2$, $x = 1$, $z = 0$ and $z = (3 - x)/2$; $\delta(x, y, z) = 2\text{gm/cm}^3$.
    (Note: this is the same region as used in Exercise 10.)

23. D is bounded by the planes $x = 2$, $y = 1$, $z = 0$ and $z = 2x + 4y - 4$; $\delta(x, y, z) = x^2\text{lb/in}^3$.
    (Note: this is the same region as used in Exercise 13.)

24. D is bounded by the plane $z = 2y$ and by $y = 4 - x^2$. $\delta(x, y, z) = y^2\text{lb/in}^3$.
    (Note: this is the same region as used in Exercise 14.)