APEX Calculus II, Version 3.0
Dalton State College Edition

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6: Techniques of Antidifferentiation

6.1 Integration by Parts

Here's a simple integral that we can't yet evaluate:

$$\int x \cos x \, dx.$$

It's a simple matter to take the derivative of the integrand using the Product Rule, but there is no Product Rule for integrals. However, this section introduces *Integration by Parts*, a method of integration that is based on the Product Rule for derivatives. It will enable us to evaluate this integral.

The Product Rule says that if u and v are functions of x, then $(uv)' = u'v+uv'$. For simplicity, we've written u for $u(x)$ and v for $v(x)$. Suppose we integrate both sides with respect to x. This gives

$$\int (uv)' \, dx = \int (u'v + uv') \, dx.$$

By the Fundamental Theorem of Calculus, the left side integrates to uv. The right side can be broken up into two integrals, and we have

$$uv = \int u'v \, dx + \int uv' \, dx.$$

Solving for the second integral we have

$$\int uv' \, dx = uv - \int u'v \, dx.$$

Using differential notation, we can write $du = u'(x)dx$ and $dv = v'(x)dx$ and the expression above can be written as follows:

$$\int u \, dv = uv - \int v \, du.$$

This is the Integration by Parts formula. For reference purposes, we state this in a theorem.

Theorem 50 Integration by Parts

Let u and v be differentiable functions of x on an interval I containing a and b. Then

$$\int u\,dv = uv - \int v\,du,$$

and

$$\int_{x=a}^{x=b} u\,dv = uv\Big|_a^b - \int_{x=a}^{x=b} v\,du.$$

Let's try an example to understand our new technique.

Example 6.1 Integrating using Integration by Parts

Evaluate $\displaystyle\int x\cos x\,dx$.

Solution The key to Integration by Parts is to identify part of the integrand as "u" and part as "dv." Regular practice will help one make good identifications, and later we will introduce some principles that help. For now, let $u = x$ and $dv = \cos x\,dx$.

It is generally useful to make a small table of these values as done below. Right now we only know u and dv as shown on the left of Figure 6.1; on the right we fill in the rest of what we need. If $u = x$, then $du = dx$. Since $dv = \cos x\,dx$, v is an antiderivative of $\cos x$. We choose $v = \sin x$.

$$\begin{array}{cc} u = x & v = ? \\ du = ? & dv = \cos x\,dx \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = x & v = \sin x \\ du = dx & dv = \cos x\,dx \end{array}$$

Figure 6.1: Setting up Integration by Parts.

Now substitute all of this into the Integration by Parts formula, giving

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx.$$

We can then integrate $\sin x$ to get $-\cos x + C$ and overall our answer is

$$\int x\cos x\,dx = x\sin x + \cos x + C.$$

Note how the antiderivative contains a product, $x\sin x$. This product is what makes Integration by Parts necessary.

Notes:

The example above demonstrates how Integration by Parts works in general. We try to identify u and dv in the integral we are given, and the key is that we usually want to choose u and dv so that du is simpler than u and v is hopefully not too much more complicated than dv. This will mean that the integral on the right side of the Integration by Parts formula, $\int v\,du$ will be simpler to integrate than the original integral $\int u\,dv$.

In the example above, we chose $u = x$ and $dv = \cos x\,dx$. Then $du = dx$ was simpler than u and $v = \sin x$ is no more complicated than dv. Therefore, instead of integrating $x\cos x\,dx$, we could integrate $\sin x\,dx$, which we knew how to do.

A useful mnemonic for helping to determine u is "LIATE," where

L = **L**ogarithmic, I = **I**nverse Trig.,
A = **A**lgebraic (polynomials, roots, power functions),
T = **T**rigonometric, and E = **E**xponential.

If the integrand contains both a logarithmic and an algebraic term, in general letting u be the logarithmic term works best, as indicated by L coming before A in LIATE.

We now consider another example.

Example 6.2 **Integrating using Integration by Parts**

Evaluate $\displaystyle\int xe^x\,dx$.

Solution The integrand contains an **A**lgebraic term (x) and an **E**xponential term (e^x). Our mnemonic suggests letting u be the algebraic term, so we choose $u = x$ and $dv = e^x\,dx$. Then $du = dx$ and $v = e^x$ as indicated by the tables below.

$$\begin{array}{cc} u = x & v = ? \\ du = ? & dv = e^x\,dx \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = x & v = e^x \\ du = dx & dv = e^x\,dx \end{array}$$

Figure 6.2: Setting up Integration by Parts.

We see du is simpler than u, while there is no change in going from dv to v. This is good. The Integration by Parts formula gives

$$\int xe^x\,dx = xe^x - \int e^x\,dx.$$

The integral on the right is simple; our final answer is

$$\int xe^x\,dx = xe^x - e^x + C.$$

Notes:

Note again how the antiderivatives contain a product term.

Example 6.3 **Integrating using Integration by Parts**

Evaluate $\displaystyle\int x^2 \cos x\, dx$.

Solution The mnemonic suggests letting $u = x^2$ instead of the trigonometric function, hence $dv = \cos x\, dx$. Then $du = 2x\, dx$ and $v = \sin x$ as shown below.

$$\begin{array}{cc} u = x^2 & v = ? \\ du = ? & dv = \cos x\, dx \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = x^2 & v = \sin x \\ du = 2x\, dx & dv = \cos x\, dx \end{array}$$

Figure 6.3: Setting up Integration by Parts.

The Integration by Parts formula gives

$$\int x^2 \cos x\, dx = x^2 \sin x - \int 2x \sin x\, dx.$$

At this point, the integral on the right is indeed simpler than the one we started with, but to evaluate it, we need to do Integration by Parts again. Here we choose $u = 2x$ and $dv = \sin x$ and fill in the rest below.

$$\begin{array}{cc} u = 2x & v = ? \\ du = ? & dv = \sin x\, dx \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = 2x & v = -\cos x \\ du = 2\, dx & dv = \sin x\, dx \end{array}$$

Figure 6.4: Setting up Integration by Parts (again).

$$\int x^2 \cos x\, dx = x^2 \sin x - \left(-2x \cos x - \int -2 \cos x\, dx \right).$$

The integral all the way on the right is now something we can evaluate. It evaluates to $-2 \sin x$. Then going through and simplifying, being careful to keep all the signs straight, our answer is

$$\int x^2 \cos x\, dx = x^2 \sin x + 2x \cos x - 2 \sin x + C.$$

Example 6.4 **Integrating using Integration by Parts**

Evaluate $\displaystyle\int e^x \cos x\, dx$.

Notes:

Solution This is a classic problem. Our mnemonic suggests letting u be the trigonometric function instead of the exponential. In this particular example, one can let u be either $\cos x$ or e^x; to demonstrate that we do not have to follow LIATE, we choose $u = e^x$ and hence $dv = \cos x\,dx$. Then $du = e^x\,dx$ and $v = \sin x$ as shown below.

$$\begin{array}{cc} u = e^x & v = ? \\ du = ? & dv = \cos x\,dx \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = e^x & v = \sin x \\ du = e^x\,dx & dv = \cos x\,dx \end{array}$$

Figure 6.5: Setting up Integration by Parts.

Notice that du is no simpler than u, going against our general rule (but bear with us). The Integration by Parts formula yields

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx.$$

The integral on the right is not much different than the one we started with, so it seems like we have gotten nowhere. Let's keep working and apply Integration by Parts to the new integral, using $u = e^x$ and $dv = \sin x\,dx$. This leads us to the following:

$$\begin{array}{cc} u = e^x & v = ? \\ du = ? & dv = \sin x\,dx \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = e^x & v = -\cos x \\ du = e^x\,dx & dv = \sin x\,dx \end{array}$$

Figure 6.6: Setting up Integration by Parts (again).

The Integration by Parts formula then gives:

$$\begin{aligned} \int e^x \cos x\,dx &= e^x \sin x - \left(-e^x \cos x - \int -e^x \cos x\,dx\right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx. \end{aligned}$$

It seems we are back right where we started, as the right hand side contains $\int e^x \cos x\,dx$. But this is actually a good thing.

Add $\displaystyle\int e^x \cos x\,dx$ to both sides. This gives

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x$$

Notes:

Now divide both sides by 2:

$$\int e^x \cos x \, dx = \frac{1}{2}\left(e^x \sin x + e^x \cos x\right).$$

Simplifying a little and adding the constant of integration, our answer is thus

$$\int e^x \cos x \, dx = \frac{1}{2}e^x \left(\sin x + \cos x\right) + C.$$

Example 6.5 **Integrating using Integration by Parts: antiderivative of** $\ln x$

Evaluate $\displaystyle\int \ln x \, dx$.

Solution One may have noticed that we have rules for integrating the familiar trigonometric functions and e^x, but we have not yet given a rule for integrating $\ln x$. That is because $\ln x$ can't easily be integrated with any of the rules we have learned up to this point. But we can find its antiderivative by a clever application of Integration by Parts. Set $u = \ln x$ and $dv = dx$. This is a good, sneaky trick to learn as it can help in other situations. This determines $du = (1/x)\, dx$ and $v = x$ as shown below.

$$\begin{array}{ll} u = \ln x & v = ? \\ du = ? & dv = dx \end{array} \quad \Rightarrow \quad \begin{array}{ll} u = \ln x & v = x \\ du = 1/x \, dx & dv = dx \end{array}$$

Figure 6.7: Setting up Integration by Parts.

Putting this all together in the Integration by Parts formula, things work out very nicely:

$$\int \ln x \, dx = x \ln x - \int x \, \frac{1}{x} \, dx.$$

The new integral simplifies to $\int 1 \, dx$, which is about as simple as things get. Its integral is $x + C$ and our answer is

$$\int \ln x \, dx = x \ln x - x + C.$$

Example 6.6 **Integrating using Int. by Parts: antiderivative of** $\arctan x$

Evaluate $\displaystyle\int \arctan x \, dx$.

Solution The same sneaky trick we used above works here. Let $u = \arctan x$ and $dv = dx$. Then $du = 1/(1 + x^2)\, dx$ and $v = x$. The Integration by

Notes:

Parts formula gives

$$\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}\,dx.$$

The integral on the right can be solved by substitution. Taking $u = 1 + x^2$, we get $du = 2x\,dx$. The integral then becomes

$$\int \arctan x\,dx = x\arctan x - \frac{1}{2}\int \frac{1}{u}\,du.$$

The integral on the right evaluates to $\ln|u| + C$, which becomes $\ln(1+x^2)+C$. (We can drop the absolute value signs because $1 + x^2$ is always positive.) Therefore, the answer is

$$\int \arctan x\,dx = x\arctan x - \ln(1+x^2) + C.$$

Substitution Before Integration

When taking derivatives, it was common to employ multiple rules (such as using both the Quotient and the Chain Rules). It should then come as no surprise that some integrals are best evaluated by combining integration techniques. In particular, here we illustrate making an "unusual" substitution first before using Integration by Parts.

Example 6.7 Integration by Parts after substitution

Evaluate $\displaystyle\int \cos(\ln x)\,dx$.

Solution The integrand contains a composition of functions, leading us to think Substitution would be beneficial. Letting $u = \ln x$, we have $du = 1/x\,dx$. This seems problematic, as we do not have a $1/x$ in the integrand. But consider:

$$du = \frac{1}{x}\,dx \Rightarrow x \cdot du = dx.$$

Since $u = \ln x$, we can use inverse functions and conclude that $x = e^u$. Therefore we have that

$$\begin{aligned} dx &= x \cdot du \\ &= e^u\,du. \end{aligned}$$

We can thus replace $\ln x$ with u and dx with $e^u\,du$. Thus we rewrite our integral as

$$\int \cos(\ln x)\,dx = \int e^u \cos u\,du.$$

Notes:

We evaluated this integral in Example 6.4. Using the result there, we have:

$$\begin{aligned}\int \cos(\ln x)\,dx &= \int e^u \cos u\,du \\ &= \frac{1}{2}e^u\big(\sin u + \cos u\big) + C \\ &= \frac{1}{2}e^{\ln x}\big(\sin(\ln x) + \cos(\ln x)\big) + C \\ &= \frac{1}{2}x\big(\sin(\ln x) + \cos(\ln x)\big) + C.\end{aligned}$$

Definite Integrals and Integration By Parts

So far we have focused only on evaluating indefinite integrals. Of course, we can use Integration by Parts to evaluate definite integrals as well, as Theorem 50 states. We do so in the next example.

Example 6.8 Definite integration using Integration by Parts

Evaluate $\displaystyle\int_1^2 x^2 \ln x\,dx$.

Solution Our mnemonic suggests letting $u = \ln x$, hence $dv = x^2\,dx$. We then get $du = (1/x)\,dx$ and $v = x^3/3$ as shown below.

$$\begin{array}{cc} u = \ln x & v = ? \\ du = ? & dv = x^2\,dx \end{array} \quad\Rightarrow\quad \begin{array}{cc} u = \ln x & v = x^3/3 \\ du = 1/x\,dx & dv = x^2\,dx \end{array}$$

Figure 6.8: Setting up Integration by Parts.

Notes:

The Integration by Parts formula then gives

$$\begin{aligned}
\int_1^2 x^2 \ln x\, dx &= \frac{x^3}{3}\ln x\Big|_1^2 - \int_1^2 \frac{x^3}{3}\frac{1}{x}\, dx \\
&= \frac{x^3}{3}\ln x\Big|_1^2 - \int_1^2 \frac{x^2}{3}\, dx \\
&= \frac{x^3}{3}\ln x\Big|_1^2 - \frac{x^3}{9}\Big|_1^2 \\
&= \left(\frac{x^3}{3}\ln x - \frac{x^3}{9}\right)\Big|_1^2 \\
&= \left(\frac{8}{3}\ln 2 - \frac{8}{9}\right) - \left(\frac{1}{3}\ln 1 - \frac{1}{9}\right) \\
&= \frac{8}{3}\ln 2 - \frac{7}{9} \\
&\approx 1.07.
\end{aligned}$$

In general, Integration by Parts is useful for integrating certain products of functions, like $\int xe^x\, dx$ or $\int x^3 \sin x\, dx$. It is also useful for integrals involving logarithms and inverse trigonometric functions.

As stated before, integration is generally more difficult than derivation. We are developing tools for handling a large array of integrals, and experience will tell us when one tool is preferable/necessary over another. For instance, consider the three similar–looking integrals

$$\int xe^x\, dx, \qquad \int xe^{x^2}\, dx \qquad \text{and} \qquad \int xe^{x^3}\, dx.$$

While the first is calculated easily with Integration by Parts, the second is best approached with Substitution. Taking things one step further, the third integral has no answer in terms of elementary functions, so none of the methods we learn in calculus will get us the exact answer.

Integration by Parts is a very useful method, second only to substitution. In the following sections of this chapter, we continue to learn other integration techniques. The next section focuses on handling integrals containing trigonometric functions.

Notes:

Exercises 6.1

Terms and Concepts

1. T/F: Integration by Parts is useful in evaluating integrands that contain products of functions.

2. T/F: Integration by Parts can be thought of as the "opposite of the Chain Rule."

3. For what is "LIATE" useful?

Problems

In Exercises 4 – 37, evaluate the given indefinite integral.

4. $\int x \sin x \, dx$

5. $\int x e^{-x} \, dx$

6. $\int x^2 \sin x \, dx$

7. $\int x^3 \sin x \, dx$

8. $\int x e^{x^2} \, dx$

9. $\int x^3 e^x \, dx$

10. $\int x 7^x \, dx$

11. $\int x e^{-2x} \, dx$

12. $\int e^x \sin x \, dx$

13. $\int e^{2x} \cos x \, dx$

14. $\int e^{5x} \cos(5x) \, dx$

15. $\int e^{2x} \sin(3x) \, dx$

16. $\int e^{ax} \sin(bx) \, dx$

17. $\int \sin x \cos x \, dx$

18. $\int \sin^{-1} x \, dx$

19. $\int \tan^{-1}(2x) \, dx$

20. $\int x \tan^{-1} x \, dx$

21. $\int \cos^{-1} x \, dx$

22. $\int x \ln x \, dx$

23. $\int (x-2) \ln x \, dx$

24. $\int x \ln(x-1) \, dx$

25. $\int x \ln\left(x^2\right) \, dx$

26. $\int x^2 \ln x \, dx$

27. $\int \frac{\ln x}{\sqrt{x}} \, dx$

28. $\int (\ln x)^2 \, dx$

29. $\int (\ln(x+1))^2 \, dx$

30. $\int x \sec^2 x \, dx$

31. $\int x \csc^2 x \, dx$

32. $\int x \sqrt{x-2} \, dx$

33. $\int x \sqrt{x^2-2} \, dx$

34. $\int \sec x \tan x \, dx$

35. $\int x \sec x \tan x \, dx$

36. $\int x \csc x \cot x \, dx$

37. $\int x^n \ln x \, dx$ for $n \neq -1$

In Exercises 38 – 42, evaluate the indefinite integral after first making a substitution.

38. $\displaystyle\int \sin(\ln x)\,dx$

39. $\displaystyle\int \sin(\sqrt{x})\,dx$

40. $\displaystyle\int \ln(\sqrt{x})\,dx$

41. $\displaystyle\int e^{\sqrt{x}}\,dx$

42. $\displaystyle\int e^{\ln x}\,dx$

In Exercises 43 – 52, evaluate the definite integral. Note: the corresponding indefinite integrals appear in Exercises 4 – 13.

43. $\displaystyle\int_0^{\pi} x\sin x\,dx$

44. $\displaystyle\int_{-1}^{1} xe^{-x}\,dx$

45. $\displaystyle\int_{-\pi/4}^{\pi/4} x^2\sin x\,dx$

46. $\displaystyle\int_{-\pi/2}^{\pi/2} x^3\sin x\,dx$

47. $\displaystyle\int_0^{\sqrt{\ln 2}} xe^{x^2}\,dx$

48. $\displaystyle\int_0^{1} x^3e^{x}\,dx$

49. $\displaystyle\int_0^{2} x7^{x}\,dx$

50. $\displaystyle\int_1^{2} xe^{-2x}\,dx$

51. $\displaystyle\int_0^{\pi} e^{x}\sin x\,dx$

52. $\displaystyle\int_{-\pi/2}^{\pi/2} e^{2x}\cos x\,dx$

In Exercises 53 – 54, evaluate the indefinite integral. These require first using integration by parts to determine v from dv.

53. $\displaystyle\int (x+1)e^{x}\ln x\,dx$

54. $\displaystyle\int xe^{x}\cos x\,dx$

In Exercises 55 – 58, find $f(x)$ described by the given initial value problem.

55. $f'(x) = x\sin x$ and $f(\pi/2) = 10$

56. $f'(x) = xe^{-x}$ and $f(-1) = 10$

57. $f'(x) = x^2\ln x$ and $f(e) = e^3$

58. $f''(x) = \dfrac{1}{x}$ and $f'(1) = 4, f(1) = -11$ for $x > 0$

6.2 Trigonometric Integrals

Functions involving trigonometric functions are useful as they are good at describing periodic behavior. This section describes several techniques for finding antiderivatives of certain combinations of trigonometric functions.

Integrals of the form $\int \sin^m x \cos^n x \, dx$

In learning the technique of Substitution, we saw the integral $\int \sin x \cos x \, dx$ in Example 5.29. The integration was not difficult, and one could easily evaluate the indefinite integral by letting $u = \sin x$ or by letting $u = \cos x$. This integral is easy since the power of both sine and cosine is 1.

We generalize this integral and consider integrals of the form $\int \sin^m x \cos^n x \, dx$, where m, n are nonnegative integers. Our strategy for evaluating these integrals is to use the identity $\cos^2 x + \sin^2 x = 1$ to convert high powers of one trigonometric function into the other, leaving a single sine or cosine term in the integrand. We summarize the general technique in the following Key Idea.

Key Idea 13 Integrals Involving Powers of Sine and Cosine

Consider $\displaystyle\int \sin^m x \cos^n x \, dx$, where m, n are nonnegative integers.

1. If m is odd, then $m = 2k + 1$ for some integer k. Rewrite

$$\sin^m x = \sin^{2k+1} x = \sin^{2k} x \sin x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x.$$

   Then

$$\int \sin^m x \cos^n x \, dx = \int (1 - \cos^2 x)^k \sin x \cos^n x \, dx = -\int (1 - u^2)^k u^n \, du,$$

   where $u = \cos x$ and $du = -\sin x \, dx$.

2. If n is odd, then using substitutions similar to that outlined above we have

$$\int \sin^m x \cos^n x \, dx = \int u^m (1 - u^2)^k \, du,$$

   where $u = \sin x$ and $du = \cos x \, dx$.

3. If both m and n are even, use the power–reducing identities

$$\cos^2 x = \frac{1 + \cos(2x)}{2} \quad \text{and} \quad \sin^2 x = \frac{1 - \cos(2x)}{2}$$

   to reduce the degree of the integrand. Expand the result and apply the principles of this Key Idea again.

Notes:

We practice applying Key Idea 13 in the next examples.

Example 6.9 **Integrating powers of sine and cosine**

Evaluate $\displaystyle\int \sin^5 x \cos^8 x\, dx$.

Solution The power of the sine term is odd, so we rewrite $\sin^5 x$ as

$$\sin^5 x = \sin^4 x \sin x = (\sin^2 x)^2 \sin x = (1-\cos^2 x)^2 \sin x.$$

Our integral is now $\displaystyle\int (1-\cos^2 x)^2 \cos^8 x \sin x\, dx$. Let $u = \cos x$, hence $du = -\sin x\, dx$. Making the substitution and expanding the integrand gives

$$\begin{aligned}\int (1-\cos^2)^2 \cos^8 x \sin x\, dx &= -\int (1-u^2)^2 u^8\, du = -\int \big(1-2u^2+u^4\big)u^8\, du \\ &= -\int \big(u^8 - 2u^{10} + u^{12}\big)\, du.\end{aligned}$$

This final integral is not difficult to evaluate, giving

$$\begin{aligned}-\int \big(u^8 - 2u^{10} + u^{12}\big)\, du &= -\frac{1}{9}u^9 + \frac{2}{11}u^{11} - \frac{1}{13}u^{13} + C \\ &= -\frac{1}{9}\cos^9 x + \frac{2}{11}\cos^{11} x - \frac{1}{13}\cos^{13} x + C.\end{aligned}$$

Example 6.10 **Integrating powers of sine and cosine**

Evaluate $\displaystyle\int \sin^5 x \cos^9 x\, dx$.

Solution The powers of both the sine and cosine terms are odd, therefore we can apply the techniques of Key Idea 13 to either power. We choose to work with the power of the cosine term since the previous example used the sine term's power.

We rewrite $\cos^9 x$ as

$$\begin{aligned}\cos^9 x &= \cos^8 x \cos x \\ &= (\cos^2 x)^4 \cos x \\ &= (1-\sin^2 x)^4 \cos x \\ &= (1 - 4\sin^2 x + 6\sin^4 x - 4\sin^6 x + \sin^8 x)\cos x.\end{aligned}$$

We rewrite the integral as

$$\int \sin^5 x \cos^9 x\, dx = \int \sin^5 x\big(1 - 4\sin^2 x + 6\sin^4 x - 4\sin^6 x + \sin^8 x\big)\cos x\, dx.$$

Notes:

Now substitute and integrate, using $u = \sin x$ and $du = \cos x\,dx$.

$$\int \sin^5 x\left(1 - 4\sin^2 x + 6\sin^4 x - 4\sin^6 x + \sin^8 x\right)\cos x\,dx =$$

$$\int u^5(1 - 4u^2 + 6u^4 - 4u^6 + u^8)\,du = \int \left(u^5 - 4u^7 + 6u^9 - 4u^{11} + u^{13}\right)du$$

$$= \frac{1}{6}u^6 - \frac{1}{2}u^8 + \frac{3}{5}u^{10} - \frac{1}{3}u^{12} + \frac{1}{14}u^{14} + C$$

$$= \frac{1}{6}\sin^6 x - \frac{1}{2}\sin^8 x + \frac{3}{5}\sin^{10} x + \ldots$$

$$- \frac{1}{3}\sin^{12} x + \frac{1}{14}\sin^{14} x + C.$$

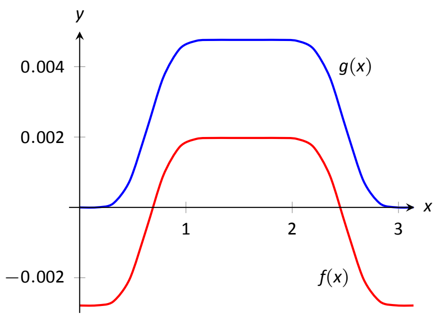


Figure 6.9: A plot of $f(x)$ and $g(x)$ from Example 6.10 and the Technology Note.

Technology Note: The work we are doing here can be a bit tedious, but the skills developed (problem solving, algebraic manipulation, etc.) are important. Nowadays problems of this sort are often solved using a computer algebra system. The powerful program *Mathematica*® integrates $\int \sin^5 x \cos^9 x\,dx$ as

$$f(x) = -\frac{45\cos(2x)}{16384} - \frac{5\cos(4x)}{8192} + \frac{19\cos(6x)}{49152} + \frac{\cos(8x)}{4096} - \frac{\cos(10x)}{81920} - \frac{\cos(12x)}{24576} - \frac{\cos(14x)}{114688},$$

which clearly has a different form than our answer in Example 6.10, which is

$$g(x) = \frac{1}{6}\sin^6 x - \frac{1}{2}\sin^8 x + \frac{3}{5}\sin^{10} x - \frac{1}{3}\sin^{12} x + \frac{1}{14}\sin^{14} x.$$

Figure 6.9 shows a graph of f and g; they are clearly not equal, but they differ *only by a constant*. That is $g(x) = f(x) + C$ for some constant C. So we have two different antiderivatives of the same function, meaning both answers are correct.

Example 6.11 **Integrating powers of sine and cosine**

Evaluate $\displaystyle\int \cos^4 x \sin^2 x\,dx$.

Solution The powers of sine and cosine are both even, so we employ the power–reducing formulas and algebra as follows.

$$\int \cos^4 x\sin^2 x\,dx = \int \left(\frac{1+\cos(2x)}{2}\right)^2\left(\frac{1-\cos(2x)}{2}\right)dx$$

$$= \int \frac{1 + 2\cos(2x) + \cos^2(2x)}{4}\cdot\frac{1-\cos(2x)}{2}\,dx$$

$$= \int \frac{1}{8}\left(1 + \cos(2x) - \cos^2(2x) - \cos^3(2x)\right)dx$$

The $\cos(2x)$ term is easy to integrate, especially with Key Idea 11. The $\cos^2(2x)$ term is another trigonometric integral with an even power, requiring the power–reducing formula again. The $\cos^3(2x)$ term is a cosine function with an odd power, requiring a substitution as done before. We integrate each in turn below.

Notes:

$$\int \cos(2x)\,dx = \frac{1}{2}\sin(2x) + C.$$

$$\int \cos^2(2x)\,dx = \int \frac{1+\cos(4x)}{2}\,dx = \frac{1}{2}\big(x + \frac{1}{4}\sin(4x)\big) + C.$$

Finally, we rewrite $\cos^3(2x)$ as

$$\cos^3(2x) = \cos^2(2x)\cos(2x) = \big(1-\sin^2(2x)\big)\cos(2x).$$

Letting $u = \sin(2x)$, we have $du = 2\cos(2x)\,dx$, hence

$$\begin{aligned}\int \cos^3(2x)\,dx &= \int \big(1-\sin^2(2x)\big)\cos(2x)\,dx\\ &= \int \frac{1}{2}(1-u^2)\,du\\ &= \frac{1}{2}\Big(u - \frac{1}{3}u^3\Big) + C\\ &= \frac{1}{2}\Big(\sin(2x) - \frac{1}{3}\sin^3(2x)\Big) + C\end{aligned}$$

Putting all the pieces together, we have

$$\begin{aligned}\int \cos^4 x\sin^2 x\,dx &= \int \frac{1}{8}\big(1+\cos(2x)-\cos^2(2x)-\cos^3(2x)\big)\,dx\\ &= \frac{1}{8}\Big[x + \frac{1}{2}\sin(2x) - \frac{1}{2}\Big(x+\frac{1}{4}\sin(4x)\Big) - \frac{1}{2}\Big(\sin(2x) - \frac{1}{3}\sin^3(2x)\Big)\Big] + C\\ &= \frac{1}{8}\Big[\frac{1}{2}x - \frac{1}{8}\sin(4x) + \frac{1}{6}\sin^3(2x)\Big] + C\end{aligned}$$

The process above was a bit long and tedious, but being able to work a problem such as this from start to finish is important.

Integrals of the form $\int \sin(mx)\sin(nx)\,dx$, $\int \cos(mx)\cos(nx)\,dx$, and $\int \sin(mx)\cos(nx)\,dx$.

Functions that contain products of sines and cosines of differing periods are important in many applications including the analysis of sound waves. Integrals of the form

$$\int \sin(mx)\sin(nx)\,dx,\quad \int \cos(mx)\cos(nx)\,dx \quad\text{and}\quad \int \sin(mx)\cos(nx)\,dx$$

Notes:

are best approached by first applying the Product to Sum Formulas found in the back cover of this text, namely

$$\sin(x)\sin(y) = \frac{1}{2}\Big[\cos(x-y) - \cos\big(x+y\big)\Big]$$
$$\cos(x)\cos(y) = \frac{1}{2}\Big[\cos\big(x-y\big) + \cos\big(x+y\big)\Big]$$
$$\sin(x)\cos(y) = \frac{1}{2}\Big[\sin\big(x-y\big) + \sin\big(x+y\big)\Big]$$

Example 6.12 Integrating products of $\sin(mx)$ and $\cos(nx)$

Evaluate $\displaystyle\int \sin(5x)\cos(2x)\,dx$.

Solution The application of the formula and subsequent integration are straightforward:

$$\int \sin(5x)\cos(2x)\,dx = \int \frac{1}{2}\Big[\sin(3x) + \sin(7x)\Big]\,dx$$
$$= -\frac{1}{6}\cos(3x) - \frac{1}{14}\cos(7x) + C$$

Integrating more general products of sines and cosines such as $\int \sin\left(3x+\frac{\pi}{4}\right)\sin(3x)dx$ are also best approached by converting to a sum of trigonometric functions.

Integrals of the form $\displaystyle\int \tan^m x \sec^n x\,dx$.

When evaluating integrals of the form $\int \sin^m x\cos^n x\,dx$, the Pythagorean Identity allowed us to convert even powers of sine into even powers of cosine, and vice versa. If, for instance, the power of sine was odd, we pulled out one $\sin x$ and converted the remaining even power of $\sin x$ into a function using powers of $\cos x$, leading to an easy substitution.

The same basic strategy applies to integrals of the form $\int \tan^m x\sec^n x\,dx$, albeit a bit more nuanced. The following three facts will prove useful:

- $\frac{d}{dx}(\tan x) = \sec^2 x$,
- $\frac{d}{dx}(\sec x) = \sec x\tan x$, and
- $1 + \tan^2 x = \sec^2 x$ (from the Pythagorean Identity).

If the integrand can be manipulated to separate a $\sec^2 x$ term with the remaining secant power even, or if a $\sec x\tan x$ term can be separated with the

Notes:

remaining $\tan x$ power even, the Pythagorean Theorem can be employed, leading to a simple substitution. This strategy is outlined in the following Key Idea.

Key Idea 14 Integrals Involving Powers of Tangent and Secant

Consider $\displaystyle\int \tan^m x \sec^n x\, dx$, where m, n are nonnegative integers.

1. If n is even, then $n = 2k$ for some integer k. Rewrite $\sec^n x$ as

$$\sec^n x = \sec^{2k} x = \sec^{2k-2} x \sec^2 x = (1+\tan^2 x)^{k-1} \sec^2 x.$$

   Then

$$\int \tan^m x \sec^n x\, dx = \int \tan^m x(1+\tan^2 x)^{k-1} \sec^2 x\, dx = \int u^m (1+u^2)^{k-1}\, du,$$

   where $u = \tan x$ and $du = \sec^2 x\, dx$.

2. If m is odd, then $m = 2k+1$ for some integer k. Rewrite $\tan^m x \sec^n x$ as

$$\tan^m x \sec^n x = \tan^{2k+1} x \sec^n x = \tan^{2k} x \sec^{n-1} x \sec x \tan x = (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x.$$

   Then

$$\int \tan^m x \sec^n x\, dx = \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x\, dx = \int (u^2-1)^k u^{n-1}\, du,$$

   where $u = \sec x$ and $du = \sec x \tan x\, dx$.

3. If n is odd and m is even, then $m = 2k$ for some integer k. Convert $\tan^m x$ to $(\sec^2 x - 1)^k$. Expand the new integrand and use Integration By Parts, with $dv = \sec^2 x\, dx$.

4. If $m \geq 2$ and $n = 0$, rewrite $\tan^m x$ as

$$\tan^m x = \tan^{m-2} x \tan^2 x = \tan^{m-2} x(\sec^2 x - 1) = \tan^{m-2} x \sec^2 x - \tan^{m-2} x.$$

   So

$$\int \tan^m x\, dx = \underbrace{\int \tan^{m-2} x \sec^2 x\, dx}_{\text{apply rule \#1}} - \underbrace{\int \tan^{m-2} x\, dx}_{\text{apply rule \#4 again}}.$$

The techniques described in items 1 and 2 of Key Idea 14 are relatively straightforward, but the techniques in items 3 and 4 can be rather tedious. A few examples will help with these methods.

Notes:

Example 6.13 **Integrating powers of tangent and secant**

Evaluate $\int \tan^2 x \sec^6 x \, dx$.

Solution Since the power of secant is even, we use rule #1 from Key Idea 14 and pull out a $\sec^2 x$ in the integrand. We convert the remaining powers of secant into powers of tangent.

$$\begin{aligned}\int \tan^2 x \sec^6 x \, dx &= \int \tan^2 x \sec^4 x \sec^2 x \, dx \\ &= \int \tan^2 x \big(1+\tan^2 x\big)^2 \sec^2 x \, dx\end{aligned}$$

Now substitute, with $u = \tan x$, with $du = \sec^2 x \, dx$.

$$= \int u^2 \big(1+u^2\big)^2 \, du$$

We leave the integration and subsequent substitution to the reader. The final answer is

$$= \frac{1}{3}\tan^3 x + \frac{2}{5}\tan^5 x + \frac{1}{7}\tan^7 x + C.$$

Example 6.14 **Integrating powers of tangent and secant**

Evaluate $\int \sec^3 x \, dx$.

Solution We apply rule #3 from Key Idea 14 as the power of secant is odd and the power of tangent is even (0 is an even number). We use Integration by Parts; the rule suggests letting $dv = \sec^2 x \, dx$, meaning that $u = \sec x$.

$$\begin{array}{ll} u = \sec x & v = \,? \\ du = \,? & dv = \sec^2 x \, dx \end{array} \quad \Rightarrow \quad \begin{array}{ll} u = \sec x & v = \tan x \\ du = \sec x \tan x \, dx & dv = \sec^2 x \, dx \end{array}$$

Figure 6.10: Setting up Integration by Parts.

Employing Integration by Parts, we have

$$\begin{aligned}\int \sec^3 x \, dx &= \int \underbrace{\sec x}_{u} \cdot \underbrace{\sec^2 x \, dx}_{dv} \\ &= \sec x \tan x - \int \sec x \tan^2 x \, dx.\end{aligned}$$

Notes:

This new integral also requires applying rule #3 of Key Idea 14:

$$\begin{aligned}
&= \sec x \tan x - \int \sec x\big(\sec^2 x - 1\big)\,dx \\
&= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx \\
&= \sec x \tan x - \int \sec^3 x\,dx + \ln|\sec x + \tan x|
\end{aligned}$$

In previous applications of Integration by Parts, we have seen where the original integral has reappeared in our work. We resolve this by adding $\int \sec^3 x\,dx$ to both sides, giving:

$$\begin{aligned}
2\int \sec^3 x\,dx &= \sec x \tan x + \ln|\sec x + \tan x| \\
\int \sec^3 x\,dx &= \frac{1}{2}\Big(\sec x \tan x + \ln|\sec x + \tan x|\Big) + C
\end{aligned}$$

We give one more example.

Example 6.15 **Integrating powers of tangent and secant**

Evaluate $\displaystyle\int \tan^6 x\,dx$.

Solution We employ rule #4 of Key Idea 14.

$$\begin{aligned}
\int \tan^6 x\,dx &= \int \tan^4 x \tan^2 x\,dx \\
&= \int \tan^4 x\big(\sec^2 x - 1\big)\,dx \\
&= \int \tan^4 x \sec^2 x\,dx - \int \tan^4 x\,dx
\end{aligned}$$

Integrate the first integral with substitution, $u = \tan x$; integrate the second by employing rule #4 again.

$$\begin{aligned}
&= \frac{1}{5}\tan^5 x - \int \tan^2 x \tan^2 x\,dx \\
&= \frac{1}{5}\tan^5 x - \int \tan^2 x\big(\sec^2 x - 1\big)\,dx \\
&= \frac{1}{5}\tan^5 x - \int \tan^2 x \sec^2 x\,dx + \int \tan^2 x\,dx
\end{aligned}$$

Notes:

Again, use substitution for the first integral and rule #4 for the second.

$$= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \int \left(\sec^2 x - 1\right)\,dx$$
$$= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + C.$$

These latter examples were admittedly long, with repeated applications of the same rule. Try to not be overwhelmed by the length of the problem, but rather admire how robust this solution method is. A trigonometric function of a high power can be systematically reduced to trigonometric functions of lower powers until all antiderivatives can be computed.

The next section introduces an integration technique known as Trigonometric Substitution, a clever combination of Substitution and the Pythagorean Theorem.

Notes:

Exercises 6.2

Terms and Concepts

1. T/F: $\int \sin^2 x \cos^2 x \, dx$ cannot be evaluated using the techniques described in this section since both powers of $\sin x$ and $\cos x$ are even.

2. T/F: $\int \sin^3 x \cos^3 x \, dx$ cannot be evaluated using the techniques described in this section since both powers of $\sin x$ and $\cos x$ are odd.

3. T/F: This section addresses how to evaluate indefinite integrals such as $\int \sin^5 x \tan^3 x \, dx$.

Problems

In Exercises 4 – 35, evaluate the indefinite integral.

4. $\int \sin x \cos^4 x \, dx$

5. $\int \sin^3 x \cos x \, dx$

6. $\int \sin^3 x \, dx$

7. $\int \sin^4 x \, dx$

8. $\int \cos^3 x \, dx$

9. $\int \cos^4 x \, dx$

10. $\int \sin^3 x \cos^2 x \, dx$

11. $\int \sin^3 x \cos^3 x \, dx$

12. $\int \sin^6 x \cos^5 x \, dx$

13. $\int \sin^2 x \cos^7 x \, dx$

14. $\int \sin^2 x \cos^2 x \, dx$

15. $\int \frac{1}{1+\sin x} \, dx$ (Hint: Tricky. First multiply the numerator and denominator by some expression that makes the denominator $1 - \sin^2 x$.)

16. $\int \sin(5x) \cos(3x) \, dx$

17. $\int \sin(x) \cos(2x) \, dx$

18. $\int \sin(3x) \sin(7x) \, dx$

19. $\int \sin(\pi x) \sin(2\pi x) \, dx$

20. $\int \cos(x) \cos(2x) \, dx$

21. $\int \cos\left(\frac{\pi}{2}x\right) \cos(\pi x) \, dx$

22. $\int 4\cos(12x) \cos(7x) \cos(4x) \, dx$

23. $\int \sin\left(3x + \frac{\pi}{4}\right) \sin(3x) \, dx$

24. $\int \tan^2(x) dx$

25. $\int \tan^4 x \sec^2 x \, dx$

26. $\int \tan^2 x \sec^4 x \, dx$

27. $\int \tan^3 x \sec^4 x \, dx$

28. $\int \tan^3 x \sec^2 x \, dx$

29. $\int \tan^3 x \sec^3 x \, dx$

30. $\int \tan^5 x \sec^5 x \, dx$

31. $\int \tan^4 x \, dx$

32. $\int \tan^5 x \, dx$

33. $\int \sec^5 x \, dx$

34. $\int \tan^2 x \sec x \, dx$

35. $\int \tan^2 x \sec^3 x \, dx$

In Exercises 36 – 45, evaluate the definite integral. Note: the corresponding indefinite integrals appear in the previous set.

36. $\displaystyle\int_0^{\pi} \sin x \cos^4 x \, dx$

37. $\displaystyle\int_{-\pi}^{\pi} \sin^3 x \cos x \, dx$

38. $\displaystyle\int_0^{\pi} \sin^3 x \, dx$

39. $\displaystyle\int_{-\pi}^{\pi} \sin^4 x \, dx$

40. $\displaystyle\int_{-\pi/2}^{\pi/2} \sin^2 x \cos^7 x \, dx$

41. $\displaystyle\int_0^{\pi/2} \sin(5x) \cos(3x) \, dx$

42. $\displaystyle\int_{-\pi/2}^{\pi/2} \cos(x) \cos(2x) \, dx$

43. $\displaystyle\int_0^{\pi/4} \tan^4 x \sec^2 x \, dx$

44. $\displaystyle\int_{-\pi/4}^{\pi/4} \tan^2 x \sec^4 x \, dx$

45. $\displaystyle\int_0^{\pi/3} \tan^5 x \, dx$

In Exercises 46 – 49, find $f(x)$ described by the given initial value problem.

46. $f'(x) = \sin^3 x$ and $f(0) = 4$

47. $f'(x) = \tan^3 x$ and $f(0) = 4$

48. $f'(x) = \sin^2 x \cos^2 x$ and $f(\pi) = 0$

49. $f'(x) = \sin(5x) \cos(3x)$ and $f(0) = -1$

6.3 Trigonometric Substitution

In Section 5.2 we defined the definite integral as the "signed area under the curve." In that section we had not yet learned the Fundamental Theorem of Calculus, so we evaluated special definite integrals which described nice, geometric shapes. For instance, we were able to evaluate

$$\int_{-3}^{3} \sqrt{9-x^2}\,dx = \frac{9\pi}{2} \tag{6.1}$$

as we recognized that $f(x) = \sqrt{9-x^2}$ described the upper half of a circle with radius 3.

We have since learned a number of integration techniques, including Substitution and Integration by Parts, yet we are still unable to evaluate the above integral without resorting to a geometric interpretation. This section introduces Trigonometric Substitution, a method of integration that fills this gap in our integration skill. This technique works on the same principle as Substitution as found in Section 5.5, though it can feel "backward." In Section 5.5, we set $u = f(x)$, for some function f, and replaced $f(x)$ with u. In this section, we will set $x = f(\theta)$, where f is a trigonometric function, then replace x with $f(\theta)$.

We start by demonstrating this method in evaluating the integral in (6.1). After the example, we will generalize the method and give more examples.

Example 6.16 Using Trigonometric Substitution

Evaluate $\displaystyle\int_{-3}^{3} \sqrt{9-x^2}\,dx$.

Solution We begin by noting that $9\sin^2\theta + 9\cos^2\theta = 9$, and hence $9\cos^2\theta = 9 - 9\sin^2\theta$. If we let $x = 3\sin\theta$, then $9 - x^2 = 9 - 9\sin^2\theta = 9\cos^2\theta$.

Setting $x = 3\sin\theta$ gives $dx = 3\cos\theta\,d\theta$. We are almost ready to substitute. We also wish to change our bounds of integration. The bound $x = -3$ corresponds to $\theta = -\pi/2$ (for when $\theta = -\pi/2$, $x = 3\sin\theta = -3$). Likewise, the bound of $x = 3$ is replaced by the bound $\theta = \pi/2$. Thus

$$\begin{aligned}
\int_{-3}^{3} \sqrt{9-x^2}\,dx &= \int_{-\pi/2}^{\pi/2} \sqrt{9-9\sin^2\theta}(3\cos\theta)\,d\theta \\
&= \int_{-\pi/2}^{\pi/2} 3\sqrt{9\cos^2\theta}\cos\theta\,d\theta \\
&= \int_{-\pi/2}^{\pi/2} 3|3\cos\theta|\cos\theta\,d\theta.
\end{aligned}$$

On $[-\pi/2, \pi/2]$, $\cos\theta$ is always positive, so we can drop the absolute value bars, then employ a power–reducing formula:

Notes:

$$= \int_{-\pi/2}^{\pi/2} 9\cos^2\theta\, d\theta$$
$$= \int_{-\pi/2}^{\pi/2} \frac{9}{2}\big(1+\cos(2\theta)\big)\, d\theta$$
$$= \frac{9}{2}\big(\theta + \frac{1}{2}\sin(2\theta)\big)\Big|_{-\pi/2}^{\pi/2} = \frac{9}{2}\pi.$$

This matches our answer from before.

We now describe in detail Trigonometric Substitution. This method excels when dealing with integrands that contain $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$ and $\sqrt{x^2+a^2}$. The following Key Idea outlines the procedure for each case, followed by more examples. Each right triangle acts as a reference to help us understand the relationships between x and θ.

Key Idea 15 Trigonometric Substitution

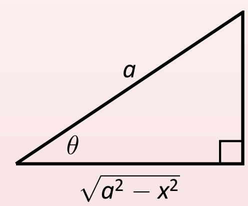
(a) For integrands containing $\sqrt{a^2-x^2}$:

Let $x = a\sin\theta$, $dx = a\cos\theta\, d\theta$

Thus $\theta = \sin^{-1}(x/a)$, for $-\pi/2 \le \theta \le \pi/2$.

On this interval, $\cos\theta \ge 0$, so

$\sqrt{a^2-x^2} = a\cos\theta$



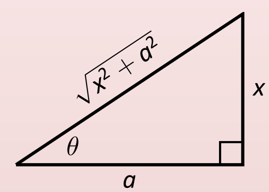
(b) For integrands containing $\sqrt{x^2+a^2}$:

Let $x = a\tan\theta$, $dx = a\sec^2\theta\, d\theta$

Thus $\theta = \tan^{-1}(x/a)$, for $-\pi/2 < \theta < \pi/2$.

On this interval, $\sec\theta > 0$, so

$\sqrt{x^2+a^2} = a\sec\theta$



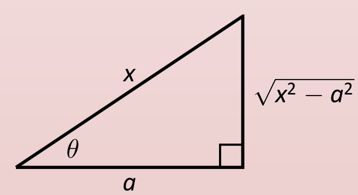
(c) For integrands containing $\sqrt{x^2-a^2}$:

Let $x = a\sec\theta$, $dx = a\sec\theta\tan\theta\, d\theta$

Thus $\theta = \sec^{-1}(x/a)$. If $x/a \ge 1$, then $0 \le \theta < \pi/2$; if $x/a \le -1$, then $\pi/2 < \theta \le \pi$.

We restrict our work to where $x \ge a$, so $x/a \ge 1$, and $0 \le \theta < \pi/2$. On this interval, $\tan\theta \ge 0$, so

$\sqrt{x^2-a^2} = a\tan\theta$



Notes:

Example 6.17 Using Trigonometric Substitution

Evaluate $\displaystyle\int \frac{1}{\sqrt{5+x^2}}\,dx$.

Solution Using Key Idea 15(b), we recognize $a=\sqrt{5}$ and set $x=\sqrt{5}\tan\theta$. This makes $dx=\sqrt{5}\sec^2\theta\,d\theta$. We will use the fact that $\sqrt{5+x^2}=\sqrt{5+5\tan^2\theta}=\sqrt{5\sec^2\theta}=\sqrt{5}\sec\theta$. Substituting, we have:

$$\begin{aligned}\int \frac{1}{\sqrt{5+x^2}}\,dx &= \int \frac{1}{\sqrt{5+5\tan^2\theta}}\sqrt{5}\sec^2\theta\,d\theta\\ &= \int \frac{\sqrt{5}\sec^2\theta}{\sqrt{5}\sec\theta}\,d\theta\\ &= \int \sec\theta\,d\theta\\ &= \ln\big|\sec\theta+\tan\theta\big|+C.\end{aligned}$$

While the integration steps are over, we are not yet done. The original problem was stated in terms of x, whereas our answer is given in terms of θ. We must convert back to x.

The reference triangle given in Key Idea 15(b) helps. With $x=\sqrt{5}\tan\theta$, we have

$$\tan\theta=\frac{x}{\sqrt{5}} \quad\text{and}\quad \sec\theta=\frac{\sqrt{x^2+5}}{\sqrt{5}}.$$

This gives

$$\begin{aligned}\int \frac{1}{\sqrt{5+x^2}}\,dx &= \ln\big|\sec\theta+\tan\theta\big|+C\\ &= \ln\left|\frac{\sqrt{x^2+5}}{\sqrt{5}}+\frac{x}{\sqrt{5}}\right|+C.\end{aligned}$$

We can leave this answer as is, or we can use a logarithmic identity to simplify it. Note:

$$\begin{aligned}\ln\left|\frac{\sqrt{x^2+5}}{\sqrt{5}}+\frac{x}{\sqrt{5}}\right|+C &= \ln\left|\frac{1}{\sqrt{5}}\left(\sqrt{x^2+5}+x\right)\right|+C\\ &= \ln\left|\frac{1}{\sqrt{5}}\right|+\ln\left|\sqrt{x^2+5}+x\right|+C\\ &= \ln\left|\sqrt{x^2+5}+x\right|+C,\end{aligned}$$

where the $\ln\left(1/\sqrt{5}\right)$ term is absorbed into the constant C. (In Section 6.5 we will learn another way of approaching this problem.)

Notes:

Example 6.18 Using Trigonometric Substitution

Evaluate $\displaystyle\int \sqrt{4x^2-1}\,dx$.

Solution We start by rewriting the integrand so that it looks like $\sqrt{x^2-a^2}$ for some value of a:

$$\begin{aligned}\sqrt{4x^2-1} &= \sqrt{4\left(x^2-\frac{1}{4}\right)}\\ &= 2\sqrt{x^2-\left(\frac{1}{2}\right)^2}.\end{aligned}$$

So we have $a=1/2$, and following Key Idea 15(c), we set $x=\frac{1}{2}\sec\theta$, and hence $dx=\frac{1}{2}\sec\theta\tan\theta\,d\theta$. We now rewrite the integral with these substitutions:

$$\begin{aligned}\int\sqrt{4x^2-1}\,dx &= \int 2\sqrt{x^2-\left(\frac{1}{2}\right)^2}\,dx\\ &= \int 2\sqrt{\frac{1}{4}\sec^2\theta-\frac{1}{4}}\left(\frac{1}{2}\sec\theta\tan\theta\right)\,d\theta\\ &= \int\sqrt{\frac{1}{4}(\sec^2\theta-1)}\Big(\sec\theta\tan\theta\Big)\,d\theta\\ &= \int\sqrt{\frac{1}{4}\tan^2\theta}\Big(\sec\theta\tan\theta\Big)\,d\theta\\ &= \int\frac{1}{2}\tan^2\theta\sec\theta\,d\theta\\ &= \frac{1}{2}\int\Big(\sec^2\theta-1\Big)\sec\theta\,d\theta\\ &= \frac{1}{2}\int\big(\sec^3\theta-\sec\theta\big)\,d\theta.\end{aligned}$$

We integrated $\sec^3\theta$ in Example 6.14, finding its antiderivatives to be

$$\int\sec^3\theta\,d\theta=\frac{1}{2}\Big(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\Big)+C.$$

Thus

$$\begin{aligned}\int\sqrt{4x^2-1}\,dx &= \frac{1}{2}\int\big(\sec^3\theta-\sec\theta\big)\,d\theta\\ &= \frac{1}{2}\left(\frac{1}{2}\Big(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\Big)-\ln|\sec\theta+\tan\theta|\right)+C\\ &= \frac{1}{4}\left(\sec\theta\tan\theta-\ln|\sec\theta+\tan\theta|\right)+C.\end{aligned}$$

Notes:

We are not yet done. Our original integral is given in terms of x, whereas our final answer, as given, is in terms of θ. We need to rewrite our answer in terms of x. With $a = 1/2$, and $x = \frac{1}{2}\sec\theta$, the reference triangle in Key Idea 15(c) shows that

$$\tan\theta = \sqrt{x^2 - 1/4}\Big/(1/2) = 2\sqrt{x^2-1/4} \quad \text{and} \quad \sec\theta = 2x.$$

Thus

$$\begin{aligned}\frac{1}{4}\Big(\sec\theta\tan\theta - \ln\big|\sec\theta+\tan\theta\big|\Big) + C &= \frac{1}{4}\Big(2x\cdot 2\sqrt{x^2-1/4} - \ln\big|2x+2\sqrt{x^2-1/4}\big|\Big) + C\\ &= \frac{1}{4}\Big(4x\sqrt{x^2-1/4} - \ln\big|2x+2\sqrt{x^2-1/4}\big|\Big) + C.\end{aligned}$$

The final answer is given in the last line above, repeated here:

$$\int \sqrt{4x^2-1}\,dx = \frac{1}{4}\Big(4x\sqrt{x^2-1/4} - \ln\big|2x+2\sqrt{x^2-1/4}\big|\Big) + C.$$

Example 6.19 Using Trigonometric Substitution

Evaluate $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2}\,dx$.

Solution We use Key Idea 15(a) with $a = 2$, $x = 2\sin\theta$, $dx = 2\cos\theta$ and hence $\sqrt{4-x^2} = 2\cos\theta$. This gives

$$\begin{aligned}\int \frac{\sqrt{4-x^2}}{x^2}\,dx &= \int \frac{2\cos\theta}{4\sin^2\theta}(2\cos\theta)\,d\theta\\ &= \int \cot^2\theta\,d\theta\\ &= \int (\csc^2\theta - 1)\,d\theta\\ &= -\cot\theta - \theta + C.\end{aligned}$$

We need to rewrite our answer in terms of x. Using the reference triangle found in Key Idea 15(a), we have $\cot\theta = \sqrt{4-x^2}/x$ and $\theta = \sin^{-1}(x/2)$. Thus

$$\int \frac{\sqrt{4-x^2}}{x^2}\,dx = -\frac{\sqrt{4-x^2}}{x} - \sin^{-1}\left(\frac{x}{2}\right) + C.$$

Trigonometric Substitution can be applied in many situations, even those not of the form $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$ or $\sqrt{x^2+a^2}$. In the following example, we apply it to an integral we already know how to handle.

Notes:

Example 6.20 Using Trigonometric Substitution

Evaluate $\displaystyle\int \frac{1}{x^2+1}\,dx$.

Solution We know the answer already as $\tan^{-1} x + C$. We apply Trigonometric Substitution here to show that we get the same answer without inherently relying on knowledge of the derivative of the arctangent function.

Using Key Idea 15(b), let $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$ and note that $x^2+1 = \tan^2\theta + 1 = \sec^2\theta$. Thus

$$\begin{aligned}\int \frac{1}{x^2+1}\,dx &= \int \frac{1}{\sec^2\theta}\sec^2\theta\,d\theta\\ &= \int 1\,d\theta\\ &= \theta + C.\end{aligned}$$

Since $x = \tan\theta$, $\theta = \tan^{-1} x$, and we conclude that $\displaystyle\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C$.

The next example is similar to the previous one in that it does not involve a square–root. It shows how several techniques and identities can be combined to obtain a solution.

Example 6.21 Using Trigonometric Substitution

Evaluate $\displaystyle\int \frac{1}{(x^2+6x+10)^2}\,dx$.

Solution We start by completing the square, then make the substitution $u = x+3$, followed by the trigonometric substitution of $u = \tan\theta$:

$$\int \frac{1}{(x^2+6x+10)^2}\,dx = \int \frac{1}{\big((x+3)^2+1\big)^2}\,dx = \int \frac{1}{(u^2+1)^2}\,du.$$

Now make the substitution $u = \tan\theta$, $du = \sec^2\theta\,d\theta$:

$$\begin{aligned}&= \int \frac{1}{(\tan^2\theta+1)^2}\sec^2\theta\,d\theta\\ &= \int \frac{1}{(\sec^2\theta)^2}\sec^2\theta\,d\theta\\ &= \int \cos^2\theta\,d\theta.\end{aligned}$$

Notes:

Applying a power reducing formula, we have

$$\begin{aligned} &= \int \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right)\,d\theta \\ &= \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C. \quad (6.2) \end{aligned}$$

We need to return to the variable x. As $u = \tan\theta$, $\theta = \tan^{-1} u$. Using the identity $\sin(2\theta) = 2\sin\theta\cos\theta$ and using the reference triangle found in Key Idea 15(b), we have

$$\frac{1}{4}\sin(2\theta) = \frac{1}{2}\frac{u}{\sqrt{u^2+1}} \cdot \frac{1}{\sqrt{u^2+1}} = \frac{1}{2}\frac{u}{u^2+1}.$$

Finally, we return to x with the substitution $u = x+3$. We start with the expression in Equation (6.2):

$$\begin{aligned} \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C &= \frac{1}{2}\tan^{-1} u + \frac{1}{2}\frac{u}{u^2+1} + C \\ &= \frac{1}{2}\tan^{-1}(x+3) + \frac{x+3}{2(x^2+6x+10)} + C. \end{aligned}$$

Stating our final result in one line,

$$\int \frac{1}{(x^2+6x+10)^2}\,dx = \frac{1}{2}\tan^{-1}(x+3) + \frac{x+3}{2(x^2+6x+10)} + C.$$

Our last example returns us to definite integrals, as seen in our first example. Given a definite integral that can be evaluated using Trigonometric Substitution, we could first evaluate the corresponding indefinite integral (by changing from an integral in terms of x to one in terms of θ, then converting back to x) and then evaluate using the original bounds. It is much more straightforward, though, to change the bounds as we substitute.

Example 6.22 Definite integration and Trigonometric Substitution

Evaluate $\displaystyle\int_0^5 \frac{x^2}{\sqrt{x^2+25}}\,dx$.

Solution Using Key Idea 15(b), we set $x = 5\tan\theta$, $dx = 5\sec^2\theta\,d\theta$, and note that $\sqrt{x^2+25} = 5\sec\theta$. As we substitute, we can also change the bounds of integration.

The lower bound of the original integral is $x = 0$. As $x = 5\tan\theta$, we solve for θ and find $\theta = \tan^{-1}(x/5)$. Thus the new lower bound is $\theta = \tan^{-1}(0) = 0$. The

Notes:

original upper bound is $x = 5$, thus the new upper bound is $\theta = \tan^{-1}(5/5) = \pi/4$.

Thus we have

$$\int_0^5 \frac{x^2}{\sqrt{x^2+25}}\,dx = \int_0^{\pi/4} \frac{25\tan^2\theta}{5\sec\theta}5\sec^2\theta\,d\theta$$
$$= 25\int_0^{\pi/4} \tan^2\theta\sec\theta\,d\theta.$$

We encountered this indefinite integral in Example 6.18 where we found

$$\int \tan^2\theta\sec\theta\,d\theta = \frac{1}{2}\big(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\big).$$

So

$$25\int_0^{\pi/4}\tan^2\theta\sec\theta\,d\theta = \frac{25}{2}\big(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\big)\Big|_0^{\pi/4}$$
$$= \frac{25}{2}\big(\sqrt{2} - \ln(\sqrt{2}+1)\big)$$
$$\approx 6.661.$$

The following equalities are very useful when evaluating integrals using Trigonometric Substitution.

Key Idea 16 Useful Equalities with Trigonometric Substitution

1. $\sin(2\theta) = 2\sin\theta\cos\theta$
2. $\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
3. $\displaystyle\int \sec^3\theta\,d\theta = \frac{1}{2}\Big(\sec\theta\tan\theta + \ln\big|\sec\theta+\tan\theta\big|\Big) + C$
4. $\displaystyle\int \cos^2\theta\,d\theta = \int \frac{1}{2}\big(1+\cos(2\theta)\big)\,d\theta = \frac{1}{2}\big(\theta + \sin\theta\cos\theta\big) + C.$

The next section introduces Partial Fraction Decomposition, which is an algebraic technique that turns "complicated" fractions into sums of "simpler" fractions, making integration easier.

Notes:

Exercises 6.3

Terms and Concepts

1. Trigonometric Substitution works on the same principles as Integration by Substitution, though it can feel "________".

2. If one uses Trigonometric Substitution on an integrand containing $\sqrt{25-x^2}$, then one should set $x =$________.

3. Consider the Pythagorean Identity $\sin^2\theta + \cos^2\theta = 1$.
   (a) What identity is obtained when both sides are divided by $\cos^2\theta$?
   (b) Use the new identity to simplify $9\tan^2\theta + 9$.

4. Why does Key Idea 15(a) state that $\sqrt{a^2-x^2} = a\cos\theta$, and not $|a\cos\theta|$?

Problems

In Exercises 5 – 18, apply Trigonometric Substitution to evaluate the indefinite integrals.

5. $\displaystyle\int \sqrt{x^2+1}\,dx$

6. $\displaystyle\int \sqrt{x^2+4}\,dx$

7. $\displaystyle\int \sqrt{1-x^2}\,dx$

8. $\displaystyle\int \sqrt{9-x^2}\,dx$

9. $\displaystyle\int \sqrt{x^2-1}\,dx$

10. $\displaystyle\int \sqrt{x^2-16}\,dx$

11. $\displaystyle\int \sqrt{4x^2+1}\,dx$

12. $\displaystyle\int \sqrt{1-9x^2}\,dx$

13. $\displaystyle\int \sqrt{16x^2-1}\,dx$

14. $\displaystyle\int \sqrt{a^2-x^2}\,dx$ for $a > 0$

15. $\displaystyle\int \sqrt{x^2+a^2}\,dx$ for $a > 0$

16. $\displaystyle\int \frac{8}{\sqrt{x^2+2}}\,dx$

17. $\displaystyle\int \frac{3}{\sqrt{7-x^2}}\,dx$

18. $\displaystyle\int \frac{5}{\sqrt{x^2-8}}\,dx$

In Exercises 19 – 29, evaluate the indefinite integrals. Some may be evaluated without Trigonometric Substitution.

19. $\displaystyle\int \frac{\sqrt{x^2-11}}{x}\,dx$

20. $\displaystyle\int \frac{1}{(x^2+1)^2}\,dx$

21. $\displaystyle\int \frac{x}{\sqrt{x^2-3}}\,dx$

22. $\displaystyle\int x^2\sqrt{1-x^2}\,dx$

23. $\displaystyle\int \frac{x}{(x^2+9)^{3/2}}\,dx$

24. $\displaystyle\int \frac{5x^2}{\sqrt{x^2-10}}\,dx$

25. $\displaystyle\int \frac{1}{(x^2+4x+13)^2}\,dx$

26. $\displaystyle\int x^2(1-x^2)^{-3/2}\,dx$

27. $\displaystyle\int \frac{\sqrt{5-x^2}}{7x^2}\,dx$

28. $\displaystyle\int \frac{x^2}{\sqrt{x^2+3}}\,dx$

29. $e^{\sin^{-1}x}\,dx$

30. Consider the integral $\displaystyle\int_{-r}^{r}\sqrt{r^2-x^2}\,dx$ where $r > 0$ is some constant.
    (a) This integral represents the area of what type of region?
    (b) Compute the integral. Check that your answer matches the formula you know for the area of the region in (a).

In Exercises 31 – 36, evaluate the definite integrals by making the proper trigonometric substitution *and* changing the bounds of integration. (Note: each of the corresponding indefinite integrals has appeared previously in this Exercise set.)

31. $\displaystyle\int_{-1}^{1}\sqrt{1-x^2}\,dx$

32. $\int_4^8 \sqrt{x^2-16}\,dx$

33. $\int_0^2 \sqrt{x^2+4}\,dx$

34. $\int_{-1}^1 \frac{1}{(x^2+1)^2}\,dx$

35. $\int_{-1}^1 \sqrt{9-x^2}\,dx$

36. $\int_{-1}^1 x^2\sqrt{1-x^2}\,dx$

In Exercises 37 – 40, find $f(x)$ described by the given initial value problem. (Note: each of the corresponding indefinite integrals has appeared previously in this Exercise set.)

37. $f'(x) = \sqrt{x^2+1}$ and $f(0) = 6$

38. $f'(x) = \sqrt{9-x^2}$ and $f(3/2) = 3\pi/2$

39. $f'(x) = \frac{5}{\sqrt{x^2-8}}$ and $f\left(\sqrt{8}\right) = 4$

40. $f'(x) = \frac{x}{(x^2+9)^{3/2}}$ and $f(4) = 4/5$

6.4 Partial Fraction Decomposition

In this section we investigate the antiderivatives of rational functions. Recall that rational functions are functions of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$. Such functions arise in many contexts, one of which is the solving of certain fundamental differential equations.

We begin with an example that demonstrates the motivation behind this section. Consider the integral $\displaystyle\int \frac{1}{x^2-1}\,dx$. We do not have a simple formula for this (if the denominator were x^2+1, we would recognize the antiderivative as being the arctangent function). It can be solved using Trigonometric Substitution, but note how the integral is easy to evaluate once we realize:

$$\frac{1}{x^2-1} = \frac{1/2}{x-1} - \frac{1/2}{x+1}.$$

Thus

$$\begin{aligned}\int \frac{1}{x^2-1}\,dx &= \int \frac{1/2}{x-1}\,dx - \int \frac{1/2}{x+1}\,dx\\ &= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.\end{aligned}$$

This section teaches how to *decompose*

$$\frac{1}{x^2-1} \quad \text{into} \quad \frac{1/2}{x-1} - \frac{1/2}{x+1}.$$

We start with a rational function $f(x) = \frac{p(x)}{q(x)}$, where p and q do not have any common factors and the degree of p is less than the degree of q. It can be shown that any polynomial, and hence q, can be factored into a product of linear and irreducible quadratic terms. The following Key Idea states how to decompose a rational function into a sum of rational functions whose denominators are all of lower degree than q.

Notes:

Key Idea 17 Partial Fraction Decomposition

Let $\dfrac{p(x)}{q(x)}$ be a rational function, where the degree of p is less than the degree of q.

1. **Linear Terms:** Let $(x-a)$ divide $q(x)$, where $(x-a)^n$ is the highest power of $(x-a)$ that divides $q(x)$. Then the decomposition of $\frac{p(x)}{q(x)}$ will contain the sum

$$\frac{A_1}{(x-a)}+\frac{A_2}{(x-a)^2}+\cdots+\frac{A_n}{(x-a)^n}.$$

2. **Quadratic Terms:** Let x^2+bx+c divide $q(x)$, where $(x^2+bx+c)^n$ is the highest power of x^2+bx+c that divides $q(x)$. Then the decomposition of $\frac{p(x)}{q(x)}$ will contain the sum

$$\frac{B_1x+C_1}{x^2+bx+c}+\frac{B_2x+C_2}{(x^2+bx+c)^2}+\cdots+\frac{B_nx+C_n}{(x^2+bx+c)^n}.$$

To find the coefficients A_i, B_i and C_i:

1. Multiply all fractions by $q(x)$, clearing the denominators. Collect like terms.
2. Equate the resulting coefficients of the powers of x and solve the resulting system of linear equations.

The following examples will demonstrate how to put this Key Idea into practice. Example 6.23 stresses the decomposition aspect of the Key Idea.

Example 6.23 Decomposing into partial fractions

Decompose $f(x)=\dfrac{1}{(x+5)(x-2)^3(x^2+x+2)(x^2+x+7)^2}$ without solving for the resulting coefficients.

Solution The denominator is already factored, as both x^2+x+2 and x^2+x+7 cannot be factored further. We need to decompose $f(x)$ properly. Since $(x+5)$ is a linear term that divides the denominator, there will be a

$$\frac{A}{x+5}$$

Notes:

term in the decomposition.

As $(x-2)^3$ divides the denominator, we will have the following terms in the decomposition:

$$\frac{B}{x-2}, \quad \frac{C}{(x-2)^2} \quad \text{and} \quad \frac{D}{(x-2)^3}.$$

The x^2+x+2 term in the denominator results in a $\dfrac{Ex+F}{x^2+x+2}$ term.

Finally, the $(x^2+x+7)^2$ term results in the terms

$$\frac{Gx+H}{x^2+x+7} \quad \text{and} \quad \frac{Ix+J}{(x^2+x+7)^2}.$$

All together, we have

$$\frac{1}{(x+5)(x-2)^3(x^2+x+2)(x^2+x+7)^2} = \frac{A}{x+5}+\frac{B}{x-2}+\frac{C}{(x-2)^2}+\frac{D}{(x-2)^3}+ \\ \frac{Ex+F}{x^2+x+2}+\frac{Gx+H}{x^2+x+7}+\frac{Ix+J}{(x^2+x+7)^2}$$

Solving for the coefficients A, $B\ldots J$ would be a bit tedious but not "hard." By this, we mean it follows a systematic process, but would take too long to do by hand. It would be easy, however, to program a computer to solve this integral.

Example 6.24 Decomposing into partial fractions

Perform the partial fraction decomposition of $\dfrac{1}{x^2-1}$.

Solution The denominator factors into two linear terms: $x^2-1=(x-1)(x+1)$. Thus

$$\frac{1}{x^2-1}=\frac{A}{x-1}+\frac{B}{x+1}.$$

To solve for A and B, first multiply through by $x^2-1=(x-1)(x+1)$:

$$\begin{aligned}1&=\frac{A(x-1)(x+1)}{x-1}+\frac{B(x-1)(x+1)}{x+1}\\&=A(x+1)+B(x-1)\\&=Ax+A+Bx-B\end{aligned}$$

Now collect like terms.

$$=(A+B)x+(A-B).$$

The next step is key. Note the equality we have:

$$1=(A+B)x+(A-B).$$

Notes:

For clarity's sake, rewrite the left hand side as

$$0x + 1 = (A + B)x + (A - B).$$

On the left, the coefficient of the x term is 0; on the right, it is $(A + B)$. Since both sides are equal, we must have that $0 = A + B$.

Likewise, on the left, we have a constant term of 1; on the right, the constant term is $(A - B)$. Therefore we have $1 = A - B$.

We have two linear equations with two unknowns. This one is easy to solve by hand, leading to

$$\begin{array}{c} A + B = 0 \\ A - B = 1 \end{array} \Rightarrow \begin{array}{c} A = 1/2 \\ B = -1/2 \end{array}.$$

Thus

$$\frac{1}{x^2 - 1} = \frac{1/2}{x - 1} - \frac{1/2}{x + 1}.$$

Example 6.25 Integrating using partial fractions

Use partial fraction decomposition to integrate $\displaystyle\int \frac{1}{(x-1)(x+2)^2}\, dx$.

Solution We decompose the integrand as follows, as described by Key Idea 17:

$$\frac{1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}.$$

To solve for A, B and C, we multiply both sides by $(x - 1)(x + 2)^2$ and collect like terms:

$$\begin{aligned} 1 &= A(x+2)^2 + B(x-1)(x+2) + C(x-1) \qquad (6.3) \\ &= Ax^2 + 4Ax + 4A + Bx^2 + Bx - 2B + Cx - C \\ &= (A+B)x^2 + (4A+B+C)x + (4A-2B-C) \end{aligned}$$

We have

$$0x^2 + 0x + 1 = (A + B)x^2 + (4A + B + C)x + (4A - 2B - C)$$

leading to the equations

$$A + B = 0, \quad 4A + B + C = 0 \quad \text{and} \quad 4A - 2B - C = 1.$$

These three equations of three unknowns lead to a unique solution:

$$A = 1/9, \quad B = -1/9 \quad \text{and} \quad C = -1/3.$$

Note: Equation 6.3 offers a direct route to finding the values of A, B and C. Since the equation holds for all values of x, it holds in particular when $x = 1$. However, when $x = 1$, the right hand side simplifies to $A(1 + 2)^2 = 9A$. Since the left hand side is still 1, we have $1 = 9A$. Hence $A = 1/9$. Likewise, the equality holds when $x = -2$; this leads to the equation $1 = -3C$. Thus $C = -1/3$.
Knowing A and C, we can find the value of B by choosing yet another value of x, such as $x = 0$, and solving for B.

Notes:

Thus

$$\int \frac{1}{(x-1)(x+2)^2}\,dx = \int \frac{1/9}{x-1}\,dx + \int \frac{-1/9}{x+2}\,dx + \int \frac{-1/3}{(x+2)^2}\,dx.$$

Each can be integrated with a simple substitution with $u = x-1$ or $u = x+2$ (or by directly applying Key Idea 11 as the denominators are linear functions). The end result is

$$\int \frac{1}{(x-1)(x+2)^2}\,dx = \frac{1}{9}\ln|x-1| - \frac{1}{9}\ln|x+2| + \frac{1}{3(x+2)} + C.$$

Example 6.26 Integrating using partial fractions

Use partial fraction decomposition to integrate $\displaystyle\int \frac{x^3}{(x-5)(x+3)}\,dx.$

SOLUTION Key Idea 17 presumes that the degree of the numerator is less than the degree of the denominator. Since this is not the case here, we begin by using polynomial division to reduce the degree of the numerator. We divide x^3 by $(x-5)(x+3) = x^2-2x-15$. At each step, divide the leading terms. We finish when the degree of the remainder is less than the degree of the divisor $x^2-2x-15$.

$$\begin{array}{r|l}
 & \quad x + 2 \\ \hline
x^2-2x-15 & \quad x^3 \\
 & -(x^3 - 2x^2 - 15x) \\ \hline
 & \qquad 2x^2 + 15x \\
 & -(2x^2 - 4x - 30) \\ \hline
 & \qquad\qquad 19x + 30
\end{array}$$

Therefore

$$\frac{x^3}{(x-5)(x+3)} = x+2+\frac{19x+30}{(x-5)(x+3)}.$$

Using Key Idea 17, we can rewrite the new rational function as:

$$\frac{19x+30}{(x-5)(x+3)} = \frac{A}{x-5} + \frac{B}{x+3}$$

for appropriate values of A and B. Clearing denominators, we have

$$\begin{aligned} 19x+30 &= A(x+3)+B(x-5) \\ &= (A+B)x + (3A-5B). \end{aligned}$$

Note: The values of A and B in Example 6.26 can be quickly found using the technique described in the margin of Example 6.25.

Notes:

This implies that:

$$\begin{aligned} 19 &= A + B \\ 30 &= 3A - 5B. \end{aligned}$$

Solving this system of linear equations gives

$$\begin{aligned} 125/8 &= A \\ 27/8 &= B. \end{aligned}$$

We can now integrate.

$$\begin{aligned} \int \frac{x^3}{(x-5)(x+3)}\,dx &= \int \left(x + 2 + \frac{125/8}{x-5} + \frac{27/8}{x+3} \right)\,dx \\ &= \frac{x^2}{2} + 2x + \frac{125}{8}\ln|x-5| + \frac{27}{8}\ln|x+3| + C. \end{aligned}$$

Example 6.27 Integrating using partial fractions

Use partial fraction decomposition to evaluate $\displaystyle\int \frac{7x^2+31x+54}{(x+1)(x^2+6x+11)}\,dx$.

Solution The degree of the numerator is less than the degree of the denominator so we begin by applying Key Idea 17. We have:

$$\frac{7x^2+31x+54}{(x+1)(x^2+6x+11)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+6x+11}.$$

Now clear the denominators.

$$\begin{aligned} 7x^2+31x+54 &= A(x^2+6x+11) + (Bx+C)(x+1) \\ &= (A+B)x^2 + (6A+B+C)x + (11A+C). \end{aligned}$$

This implies that:

$$\begin{aligned} 7 &= A + B \\ 31 &= 6A + B + C \\ 54 &= 11A + C. \end{aligned}$$

Solving this system of linear equations gives the nice result of $A = 5$, $B = 2$ and $C = -1$. Thus

$$\int \frac{7x^2+31x+54}{(x+1)(x^2+6x+11)}\,dx = \int \left(\frac{5}{x+1} + \frac{2x-1}{x^2+6x+11} \right)\,dx.$$

Notes:

The first term of this new integrand is easy to evaluate; it leads to a $5\ln|x+1|$ term. The second term is not hard, but takes several steps and uses substitution techniques.

The integrand $\dfrac{2x-1}{x^2+6x+11}$ has a quadratic in the denominator and a linear term in the numerator. This leads us to try substitution. Let $u=x^2+6x+11$, so $du=(2x+6)\,dx$. The numerator is $2x-1$, not $2x+6$, but we can get a $2x+6$ term in the numerator by adding 0 in the form of "$7-7$."

$$\begin{aligned}\frac{2x-1}{x^2+6x+11} &= \frac{2x-1+7-7}{x^2+6x+11}\\ &= \frac{2x+6}{x^2+6x+11}-\frac{7}{x^2+6x+11}.\end{aligned}$$

We can now integrate the first term with substitution, leading to a $\ln|x^2+6x+11|$ term. The final term can be integrated using arctangent. First, complete the square in the denominator:

$$\frac{7}{x^2+6x+11}=\frac{7}{(x+3)^2+2}.$$

An antiderivative of the latter term can be found using Theorem 47 and substitution:

$$\int\frac{7}{x^2+6x+11}\,dx=\frac{7}{\sqrt{2}}\tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right)+C.$$

Let's start at the beginning and put all of the steps together.

$$\begin{aligned}\int\frac{7x^2+31x+54}{(x+1)(x^2+6x+11)}\,dx &= \int\left(\frac{5}{x+1}+\frac{2x-1}{x^2+6x+11}\right)dx\\ &= \int\frac{5}{x+1}\,dx+\int\frac{2x+6}{x^2+6x+11}\,dx-\int\frac{7}{x^2+6x+11}\,dx\\ &= 5\ln|x+1|+\ln|x^2+6x+11|-\frac{7}{\sqrt{2}}\tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right)+C.\end{aligned}$$

As with many other problems in calculus, it is important to remember that one is not expected to "see" the final answer immediately after seeing the problem. Rather, given the initial problem, we break it down into smaller problems that are easier to solve. The final answer is a combination of the answers of the smaller problems.

Substitution Before Partial Fraction Decomposition

It should come as no surprise that some integrals are solved by first using another technique such as Substitution or Integration by Parts before Partial Fraction Decomposition.

Notes:

Example 6.28 Partial fractions after substitution

Evaluate $\displaystyle\int \frac{1}{2-e^x}\,dx$.

Solution In this integral, we begin with the substitution $u = e^x$. Then

$$du = e^x dx \Rightarrow dx = \frac{1}{e^x}du = \frac{1}{u}du.$$

Thus we can rewrite the integral as

$$\int \frac{1}{2-e^x}\,dx = \int \frac{1}{u(2-u)}\,du = \int \frac{-1}{u(u-2)}\,du.$$

Now we decompose the integrand as

$$\frac{-1}{u(u-2)} = \frac{A}{u} + \frac{B}{u-2}.$$

Now clear the denominators.

$$\begin{aligned} -1 &= A(u-2) + Bu \\ &= (A+B)u - 2A. \end{aligned}$$

This implies that:

$$\begin{aligned} 0 &= A + B \\ -1 &= -2A \end{aligned}$$

Then $A = 1/2$ and $B = -1/2$, so the integral becomes

$$\begin{aligned} \int \frac{1}{2-e^x}\,dx &= \int \frac{-1}{u(u-2)}\,du \\ &= \int \left(\frac{1/2}{u} - \frac{1/2}{u-2}\right) du \\ &= \frac{1}{2}\ln|u| - \frac{1}{2}\ln|u-2| + C \\ &= \frac{1}{2}\ln\left|e^x\right| - \frac{1}{2}\ln\left|e^x - 2\right| + C \\ &= \frac{x}{2} - \frac{1}{2}\ln\left|e^x - 2\right| + C. \end{aligned}$$

Notes:

Partial Fraction Decomposition is an important tool when dealing with rational functions. Note that at its heart, it is a technique of algebra, not calculus, as we are rewriting a fraction in a new form. Regardless, it is very useful in the realm of calculus as it lets us evaluate a certain set of "complicated" integrals.

The next section introduces new functions, called the Hyperbolic Functions. They will allow us to make substitutions similar to those found when studying Trigonometric Substitution, allowing us to approach even more integration problems.

Notes:

Exercises 6.4

Terms and Concepts

1. Fill in the blank: Partial Fraction Decomposition is a method of rewriting ________ functions.

2. T/F: It is sometimes necessary to use polynomial division before using Partial Fraction Decomposition.

3. Decompose $\dfrac{1}{x^2-3x}$ without solving for the coefficients, as done in Example 6.23.

4. Decompose $\dfrac{7-x}{x^2-9}$ without solving for the coefficients, as done in Example 6.23.

5. Decompose $\dfrac{x-3}{x^2-7}$ without solving for the coefficients, as done in Example 6.23.

6. Decompose $\dfrac{2x+5}{x^3+7x}$ without solving for the coefficients, as done in Example 6.23.

Problems

In Exercises 7 – 32, evaluate the indefinite integral.

7. $\displaystyle\int \frac{7x+7}{x^2+3x-10}\,dx$

8. $\displaystyle\int \frac{7x-2}{x^2+x}\,dx$

9. $\displaystyle\int \frac{-4}{3x^2-12}\,dx$

10. $\displaystyle\int \frac{x+7}{(x+5)^2}\,dx$

11. $\displaystyle\int \frac{-3x-20}{(x+8)^2}\,dx$

12. $\displaystyle\int \frac{9x^2+11x+7}{x(x+1)^2}\,dx$

13. $\displaystyle\int \frac{-12x^2-x+33}{(x-1)(x+3)(3-2x)}\,dx$

14. $\displaystyle\int \frac{94x^2-10x}{(7x+3)(5x-1)(3x-1)}\,dx$

15. $\displaystyle\int \frac{x^2+x+1}{x^2+x-2}\,dx$

16. $\displaystyle\int \frac{1}{x^3+x}\,dx$

17. $\displaystyle\int \frac{6x+1}{(x+5)(x^2+4)}\,dx$

18. $\displaystyle\int \frac{18x^2+12x+14}{(2x+1)(9x^2+4)}\,dx$

19. $\displaystyle\int \frac{x^3}{x^2-x-20}\,dx$

20. $\displaystyle\int \frac{2x^2-4x+6}{x^2-2x+3}\,dx$

21. $\displaystyle\int \frac{1}{x^3+2x^2+3x}\,dx$

22. $\displaystyle\int \frac{x^2+x+5}{x^2+4x+10}\,dx$

23. $\displaystyle\int \frac{12x^2+21x+3}{(x+1)(3x^2+5x-1)}\,dx$

24. $\displaystyle\int \frac{6x^2+8x-4}{(x-3)(x^2+6x+10)}\,dx$

25. $\displaystyle\int \frac{2x^2+x+1}{(x+1)(x^2+9)}\,dx$

26. $\displaystyle\int \frac{x^2-20x-69}{(x-7)(x^2+2x+17)}\,dx$

27. $\displaystyle\int \frac{1}{x^5+x^3}\,dx$

28. $\displaystyle\int \frac{9x^2-60x+33}{(x-9)(x^2-2x+11)}\,dx$

29. $\displaystyle\int \frac{6x^2+45x+121}{(x+2)(x^2+10x+27)}\,dx$

30. $\displaystyle\int \frac{1}{e^x+3}\,dx$

31. $\displaystyle\int \frac{4\sin x}{3\cos^2 x-12}\,dx$

32. $\displaystyle\int \frac{7e^x-2}{e^x+1}\,dx$

In Exercises 33 – 38, evaluate the definite integral.

33. $\displaystyle\int_1^2 \frac{8x+21}{(x+2)(x+3)}\,dx$

34. $\displaystyle\int_0^5 \frac{14x+6}{(3x+2)(x+4)}\,dx$

35. $\displaystyle\int_1^3 \frac{2}{x^3+x}\,dx$

36. $\displaystyle\int_{-1}^{1} \frac{x^2 + 5x - 5}{(x-10)(x^2+4x+5)}\, dx$

37. $\displaystyle\int_{0}^{1} \frac{x}{(x+1)(x^2+2x+1)}\, dx$

38. $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\cos x}{\sin^2 x - \sin x}\, dx$

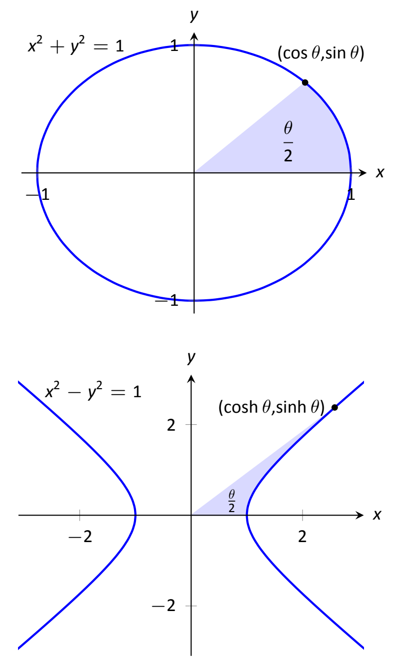


Figure 6.11: Using trigonometric functions to define points on a circle and hyperbolic functions to define points on a hyperbola. The area of the shaded regions are included in them.

Pronunciation Note:
"cosh" rhymes with "gosh,"
"sinh" rhymes with "pinch," and
"tanh" rhymes with "ranch."

6.5 Hyperbolic Functions (Optional)

The **hyperbolic functions** are a set of functions that have many applications to mathematics, physics, and engineering. Among many other applications, they are used to describe the formation of satellite rings around planets, to describe the shape of a rope hanging from two points, and have application to the theory of special relativity. This section defines the hyperbolic functions and describes many of their properties, especially their usefulness to calculus.

These functions are sometimes referred to as the "hyperbolic trigonometric functions" as there are many, many connections between them and the standard trigonometric functions. Figure 6.11 demonstrates one such connection. Just as cosine and sine are used to define points on the circle defined by $x^2+y^2 = 1$, the functions **hyperbolic cosine** and **hyperbolic sine** are used to define points on the hyperbola $x^2 - y^2 = 1$.

We begin with their definition.

Definition 23 Hyperbolic Functions

1. $\sinh x = \dfrac{e^x - e^{-x}}{2}$
2. $\cosh x = \dfrac{e^x + e^{-x}}{2}$
3. $\tanh x = \dfrac{\sinh x}{\cosh x}$
4. $\coth x = \dfrac{\cosh x}{\sinh x}$
5. $\operatorname{sech} x = \dfrac{1}{\cosh x}$
6. $\operatorname{csch} x = \dfrac{1}{\sinh x}$

These hyperbolic functions are graphed in Figure 6.12. In the graphs of $\cosh x$ and $\sinh x$, graphs of $e^x/2$ and $e^{-x}/2$ are included with dashed lines. As x gets "large," $\cosh x$ and $\sinh x$ each act like $e^x/2$; when x is a large negative number, $\cosh x$ acts like $e^{-x}/2$ whereas $\sinh x$ acts like $-e^{-x}/2$.

Notice the domains of $\tanh x$ and $\operatorname{sech} x$ are $(-\infty, \infty)$, whereas both $\coth x$ and $\operatorname{csch} x$ have vertical asymptotes at $x = 0$. Also note the ranges of these functions, especially $\tanh x$: as $x \to \infty$, both $\sinh x$ and $\cosh x$ approach $e^{-x}/2$, hence $\tanh x$ approaches 1.

The following example explores some of the properties of these functions that bear remarkable resemblance to the properties of their trigonometric counterparts.

Notes:

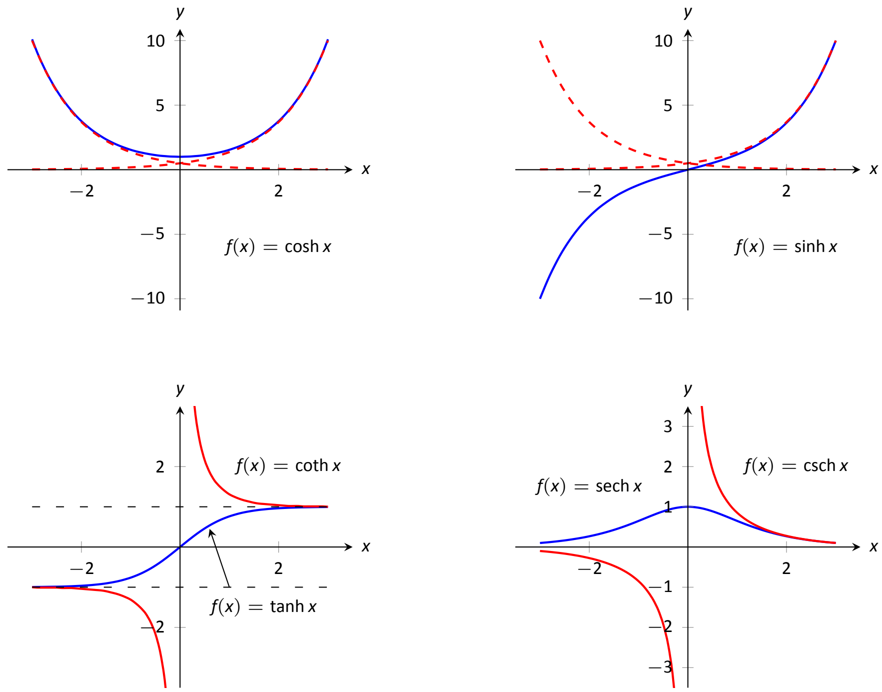


Figure 6.12: Graphs of the hyperbolic functions.

Example 6.29 Exploring properties of hyperbolic functions

Use Definition 23 to rewrite the following expressions.

1. $\cosh^2 x - \sinh^2 x$
2. $\tanh^2 x + \operatorname{sech}^2 x$
3. $2\cosh x \sinh x$
4. $\frac{d}{dx}\big(\cosh x\big)$
5. $\frac{d}{dx}\big(\sinh x\big)$
6. $\frac{d}{dx}\big(\tanh x\big)$

SOLUTION

Notes:

1.
$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x+e^{-x}}{2}\right)^2 - \left(\frac{e^x-e^{-x}}{2}\right)^2\\ &= \frac{e^{2x}+2e^xe^{-x}+e^{-2x}}{4} - \frac{e^{2x}-2e^xe^{-x}+e^{-2x}}{4}\\ &= \frac{4}{4} = 1.\end{aligned}$$

So $\cosh^2 x - \sinh^2 x = 1$.

2.
$$\begin{aligned}\tanh^2 x + \operatorname{sech}^2 x &= \frac{\sinh^2 x}{\cosh^2 x} + \frac{1}{\cosh^2 x}\\ &= \frac{\sinh^2 x + 1}{\cosh^2 x} && \text{Now use identity from \#1.}\\ &= \frac{\cosh^2 x}{\cosh^2 x} = 1.\end{aligned}$$

So $\tanh^2 x + \operatorname{sech}^2 x = 1$.

3.
$$\begin{aligned}2\cosh x \sinh x &= 2\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^x-e^{-x}}{2}\right)\\ &= 2\cdot\frac{e^{2x}-e^{-2x}}{4}\\ &= \frac{e^{2x}-e^{-2x}}{2} = \sinh(2x).\end{aligned}$$

Thus $2\cosh x \sinh x = \sinh(2x)$.

4.
$$\begin{aligned}\frac{d}{dx}\big(\cosh x\big) &= \frac{d}{dx}\left(\frac{e^x+e^{-x}}{2}\right)\\ &= \frac{e^x-e^{-x}}{2}\\ &= \sinh x.\end{aligned}$$

So $\frac{d}{dx}\big(\cosh x\big) = \sinh x$.

5.
$$\begin{aligned}\frac{d}{dx}\big(\sinh x\big) &= \frac{d}{dx}\left(\frac{e^x-e^{-x}}{2}\right)\\ &= \frac{e^x+e^{-x}}{2}\\ &= \cosh x.\end{aligned}$$

So $\frac{d}{dx}\big(\sinh x\big) = \cosh x$.

Notes:

6.
$$
\begin{aligned}
\frac{d}{dx}\big(\tanh x\big) &= \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right)\\
&= \frac{\cosh x\cosh x - \sinh x\sinh x}{\cosh^2 x}\\
&= \frac{1}{\cosh^2 x}\\
&= \operatorname{sech}^2 x.
\end{aligned}
$$

So $\frac{d}{dx}\big(\tanh x\big) = \operatorname{sech}^2 x$.

The following Key Idea summarizes many of the important identities relating to hyperbolic functions. Each can be verified by referring back to Definition 23.

Key Idea 18 Useful Hyperbolic Function Properties

Basic Identities

1. $\cosh^2 x - \sinh^2 x = 1$
2. $\tanh^2 x + \operatorname{sech}^2 x = 1$
3. $\coth^2 x - \operatorname{csch}^2 x = 1$
4. $\cosh 2x = \cosh^2 x + \sinh^2 x$
5. $\sinh 2x = 2\sinh x\cosh x$
6. $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$
7. $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

Derivatives

1. $\frac{d}{dx}\big(\sinh x\big) = \cosh x$
2. $\frac{d}{dx}\big(\cosh x\big) = \sinh x$
3. $\frac{d}{dx}\big(\tanh x\big) = \operatorname{sech}^2 x$
4. $\frac{d}{dx}\big(\coth x\big) = -\operatorname{csch}^2 x$
5. $\frac{d}{dx}\big(\operatorname{sech} x\big) = -\operatorname{sech} x\tanh x$
6. $\frac{d}{dx}\big(\operatorname{csch} x\big) = -\operatorname{csch} x\coth x$

Integrals

1. $\displaystyle\int \sinh x\,dx = \cosh x + C$
2. $\displaystyle\int \cosh x\,dx = \sinh x + C$
3. $\displaystyle\int \tanh x\,dx = \ln(\cosh x) + C$
4. $\displaystyle\int \coth x\,dx = \ln|\sinh x| + C$

We practice using Key Idea 18.

Example 6.30 Derivatives and integrals of hyperbolic functions

Evaluate the following derivatives and integrals.

1. $\dfrac{d}{dx}\big(\cosh 2x\big)$
2. $\displaystyle\int \operatorname{sech}^2(7t-3)\,dt$
3. $\displaystyle\int_0^{\ln 2} \cosh x\,dx$

Notes:

Solution

1. Using the Chain Rule directly, we have $\frac{d}{dx}\big(\cosh 2x\big) = 2\sinh 2x$.

   Just to demonstrate that it works, let's also use the Basic Identity found in Key Idea 18: $\cosh 2x = \cosh^2 x + \sinh^2 x$.

   $$\begin{aligned}\frac{d}{dx}\big(\cosh 2x\big) = \frac{d}{dx}\big(\cosh^2 x + \sinh^2 x\big) &= 2\cosh x\sinh x + 2\sinh x\cosh x \\ &= 4\cosh x\sinh x.\end{aligned}$$

   Using another Basic Identity, we can see that $4\cosh x\sinh x = 2\sinh 2x$. We get the same answer either way.

2. We employ substitution, with $u = 7t - 3$ and $du = 7dt$. Applying Key Ideas 11 and 18 we have:

   $$\int \operatorname{sech}^2(7t-3)\,dt = \frac{1}{7}\tanh(7t-3) + C.$$

3. $$\int_0^{\ln 2}\cosh x\,dx = \sinh x\Big|_0^{\ln 2} = \sinh(\ln 2) - \sinh 0 = \sinh(\ln 2).$$

   We can simplify this last expression as $\sinh x$ is based on exponentials:

   $$\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2 - 1/2}{2} = \frac{3}{4}.$$

Inverse Hyperbolic Functions

Just as the inverse trigonometric functions are useful in certain integrations, the inverse hyperbolic functions are useful with others. Figure 6.13 shows the restrictions on the domains to make each function one-to-one and the resulting domains and ranges of their inverse functions. Their graphs are shown in Figure 6.14.

Because the hyperbolic functions are defined in terms of exponential functions, their inverses can be expressed in terms of logarithms as shown in Key Idea 19. It is often more convenient to refer to $\sinh^{-1} x$ than to $\ln\big(x + \sqrt{x^2+1}\big)$, especially when one is working on theory and does not need to compute actual values. On the other hand, when computations are needed, technology is often helpful but many hand-held calculators lack a *convenient* $\sinh^{-1} x$ button. (Often it can be accessed under a menu system, but not conveniently.) In such a situation, the logarithmic representation is useful. The reader is not encouraged

Notes:

to memorize these, but rather know they exist and know how to use them when needed.

Function	Domain	Range
$\sinh x$	$(-\infty, \infty)$	$(-\infty, \infty)$
$\cosh x$	$[0, \infty)$	$[1, \infty)$
$\tanh x$	$(-\infty, \infty)$	$(-1, 1)$
$\coth x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, -1) \cup (1, \infty)$
$\operatorname{sech} x$	$[0, \infty)$	$(0, 1]$
$\operatorname{csch} x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$

Function	Domain	Range
$\sinh^{-1} x$	$(-\infty, \infty)$	$(-\infty, \infty)$
$\cosh^{-1} x$	$[1, \infty)$	$[0, \infty)$
$\tanh^{-1} x$	$(-1, 1)$	$(-\infty, \infty)$
$\coth^{-1} x$	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$\operatorname{sech}^{-1} x$	$(0, 1]$	$[0, \infty)$
$\operatorname{csch}^{-1} x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$

Figure 6.13: Domains and ranges of the hyperbolic and inverse hyperbolic functions.

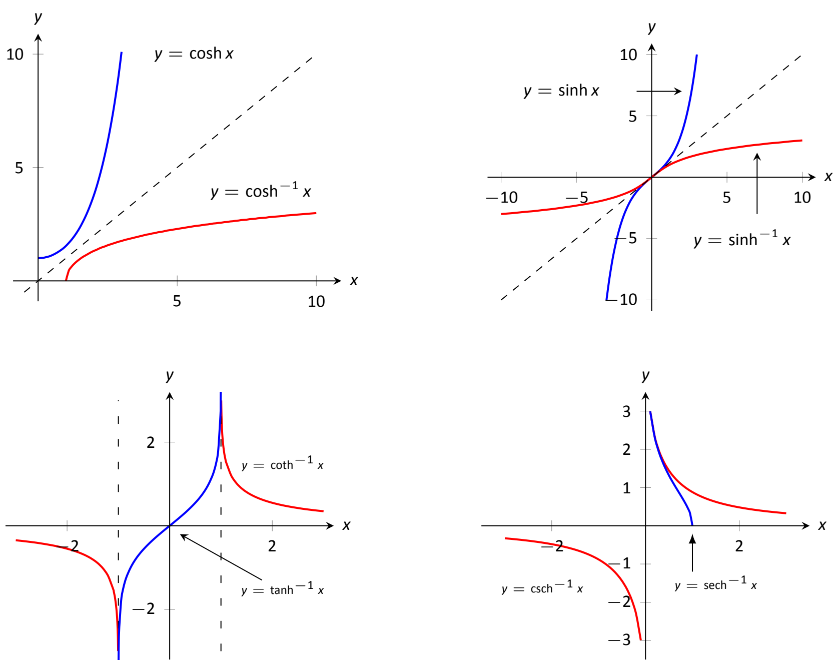


Figure 6.14: Graphs of the hyperbolic functions and their inverses.

Notes:

Key Idea 19 Logarithmic definitions of Inverse Hyperbolic Functions

1. $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$
2. $\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right);\ x \geq 1$
3. $\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right);\ |x| < 1$
4. $\coth^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{x+1}{x-1}\right);\ |x| > 1$
5. $\operatorname{sech}^{-1} x = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right);\ 0 < x \leq 1$
6. $\operatorname{csch}^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{1+x^2}}{|x|}\right);\ x \neq 0$

The following Key Ideas give the derivatives and integrals relating to the inverse hyperbolic functions. In Key Idea 21, both the inverse hyperbolic and logarithmic function representations of the antiderivative are given, based on Key Idea 19. Again, these latter functions are often more useful than the former. Note how inverse hyperbolic functions can be used to solve integrals we used Trigonometric Substitution to solve in Section 6.3.

Key Idea 20 Derivatives Involving Inverse Hyperbolic Functions

1. $\dfrac{d}{dx}\left(\sinh^{-1} x\right) = \dfrac{1}{\sqrt{x^2+1}}$
2. $\dfrac{d}{dx}\left(\cosh^{-1} x\right) = \dfrac{1}{\sqrt{x^2-1}};\ x > 1$
3. $\dfrac{d}{dx}\left(\tanh^{-1} x\right) = \dfrac{1}{1-x^2};\ |x| < 1$
4. $\dfrac{d}{dx}\left(\coth^{-1} x\right) = \dfrac{1}{1-x^2};\ |x| > 1$
5. $\dfrac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = \dfrac{-1}{x\sqrt{1-x^2}}; 0 < x < 1$
6. $\dfrac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = \dfrac{-1}{|x|\sqrt{1+x^2}};\ x \neq 0$

Notes:

Key Idea 21 Integrals Involving Inverse Hyperbolic Functions

1. $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right)+C;\ a>0 \qquad = \ln\left|x+\sqrt{x^2+a^2}\right|+C$

2. $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\left(\frac{x}{a}\right)+C;\ 0<a<x \qquad = \ln\left|x+\sqrt{x^2-a^2}\right|+C$

3. $\displaystyle\int \frac{1}{a^2-x^2}\,dx = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)+C & x^2<a^2 \\ \frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right)+C & a^2<x^2 \end{cases} \qquad = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+C$

4. $\displaystyle\int \frac{1}{x\sqrt{a^2-x^2}}\,dx = -\frac{1}{a}\,\mathrm{sech}^{-1}\left(\frac{x}{a}\right)+C;\ 0<x<a \qquad = \frac{1}{a}\ln\left(\frac{x}{a+\sqrt{a^2-x^2}}\right)+C$

5. $\displaystyle\int \frac{1}{x\sqrt{x^2+a^2}}\,dx = -\frac{1}{a}\,\mathrm{csch}^{-1}\left|\frac{x}{a}\right|+C;\ x\neq 0,\ a>0 \qquad = \frac{1}{a}\ln\left|\frac{x}{a+\sqrt{a^2+x^2}}\right|+C$

We practice using the derivative and integral formulas in the following example.

Example 6.31 Derivatives and integrals involving inverse hyperbolic functions

Evaluate the following.

1. $\displaystyle\frac{d}{dx}\left[\cosh^{-1}\left(\frac{3x-2}{5}\right)\right]$

2. $\displaystyle\int \frac{1}{x^2-1}\,dx$

3. $\displaystyle\int \frac{1}{\sqrt{9x^2+10}}\,dx$

Solution

1. Applying Key Idea 20 with the Chain Rule gives:
$$\frac{d}{dx}\left[\cosh^{-1}\left(\frac{3x-2}{5}\right)\right] = \frac{1}{\sqrt{\left(\frac{3x-2}{5}\right)^2-1}}\cdot\frac{3}{5}.$$

2. Multiplying the numerator and denominator by (-1) gives: $\displaystyle\int \frac{1}{x^2-1}\,dx = \int \frac{-1}{1-x^2}\,dx$. The second integral can be solved with a direct application

Notes:

of item #3 from Key Idea 21, with $a = 1$. Thus

$$\begin{aligned}\int \frac{1}{x^2-1}\,dx &= -\int \frac{1}{1-x^2}\,dx \\ &= \begin{cases} -\tanh^{-1}(x) + C & x^2 < 1 \\ -\coth^{-1}(x) + C & 1 < x^2 \end{cases} \\ &= -\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + C \\ &= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C. \end{aligned} \tag{6.4}$$

We should note that this exact problem was solved at the beginning of Section 6.4. In that example the answer was given as $\frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$. Note that this is equivalent to the answer given in Equation 6.4, as $\ln(a/b) = \ln a - \ln b$.

3. This requires a substitution, then item #1 of Key Idea 21 can be applied.

   Let $u = 3x$, hence $du = 3dx$. We have

$$\int \frac{1}{\sqrt{9x^2+10}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{u^2+10}}\,du.$$

   Note $a^2 = 10$, hence $a = \sqrt{10}$. Now apply the integral rule.

$$\begin{aligned} &= \frac{1}{3}\sinh^{-1}\left(\frac{3x}{\sqrt{10}}\right) + C \\ &= \frac{1}{3}\ln\left|3x + \sqrt{9x^2+10}\right| + C. \end{aligned}$$

This section covers a lot of ground. New functions were introduced, along with some of their fundamental identities, their derivatives and antiderivatives, their inverses, and the derivatives and antiderivatives of these inverses. Four Key Ideas were presented, each including quite a bit of information.

Do not view this section as containing a source of information to be memorized, but rather as a reference for future problem solving. Key Idea 21 contains perhaps the most useful information. Know the integration forms it helps evaluate and understand how to use the inverse hyperbolic answer and the logarithmic answer.

Notes:

Exercises 6.5

Terms and Concepts

1. In Key Idea 18, the equation $\int \tanh x\,dx = \ln(\cosh x) + C$ is given. Why is "$\ln|\cosh x|$" not used – i.e., why are absolute values not necessary?

2. The hyperbolic functions are used to define points on the right hand portion of the hyperbola $x^2 - y^2 = 1$, as shown in Figure 6.11. How can we use the hyperbolic functions to define points on the left hand portion of the hyperbola?

Problems

In Exercises 3 – 10, verify the given identity using Definition 23, as done in Example 6.29.

3. $\coth^2 x - \operatorname{csch}^2 x = 1$

4. $\cosh 2x = \cosh^2 x + \sinh^2 x$

5. $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$

6. $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

7. $\dfrac{d}{dx}[\operatorname{sech} x] = -\operatorname{sech} x \tanh x$

8. $\dfrac{d}{dx}[\coth x] = -\operatorname{csch}^2 x$

9. $\displaystyle\int \tanh x\,dx = \ln(\cosh x) + C$

10. $\displaystyle\int \coth x\,dx = \ln|\sinh x| + C$

In Exercises 11 – 21, find the derivative of the given function.

11. $f(x) = \cosh 2x$

12. $f(x) = \tanh(x^2)$

13. $f(x) = \ln(\sinh x)$

14. $f(x) = \sinh x \cosh x$

15. $f(x) = x\sinh x - \cosh x$

16. $f(x) = \operatorname{sech}^{-1}(x^2)$

17. $f(x) = \sinh^{-1}(3x)$

18. $f(x) = \cosh^{-1}(2x^2)$

19. $f(x) = \tanh^{-1}(x+5)$

20. $f(x) = \tanh^{-1}(\cos x)$

21. $f(x) = \cosh^{-1}(\sec x)$

In Exercises 22 – 26, find the equation of the line tangent to the function at the given x-value.

22. $f(x) = \sinh x$ at $x = 0$

23. $f(x) = \cosh x$ at $x = \ln 2$

24. $f(x) = \operatorname{sech}^2 x$ at $x = \ln 3$

25. $f(x) = \sinh^{-1} x$ at $x = 0$

26. $f(x) = \cosh^{-1} x$ at $x = \sqrt{2}$

In Exercises 27 – 40, evaluate the given indefinite integral.

27. $\displaystyle\int \tanh(2x)\,dx$

28. $\displaystyle\int \cosh(3x-7)\,dx$

29. $\displaystyle\int \sinh x \cosh x\,dx$

30. $\displaystyle\int x\cosh x\,dx$

31. $\displaystyle\int x\sinh x\,dx$

32. $\displaystyle\int \frac{1}{9-x^2}\,dx$

33. $\displaystyle\int \frac{2x}{\sqrt{x^4-4}}\,dx$

34. $\displaystyle\int \frac{\sqrt{x}}{\sqrt{1+x^3}}\,dx$

35. $\displaystyle\int \frac{1}{x^4-16}\,dx$

36. $\displaystyle\int \frac{1}{x^2+x}\,dx$

37. $\displaystyle\int \frac{e^x}{e^{2x}+1}\,dx$

38. $\displaystyle\int \sinh^{-1} x\,dx$

39. $\displaystyle\int \tanh^{-1} x\,dx$

40. $\int \operatorname{sech} x \, dx$ (Hint: mutiply by $\frac{\cosh x}{\cosh x}$; set $u = \sinh x$.)

In Exercises 41 – 43, evaluate the given definite integral.

41. $\int_{-1}^{1} \sinh x \, dx$

42. $\int_{-\ln 2}^{\ln 2} \cosh x \, dx$

43. $\int_{0}^{1} \tanh^{-1} x \, dx$

6.6 Improper Integrals

We begin this section by considering the following definite integrals:

- $\displaystyle\int_0^{100} \frac{1}{1+x^2}\,dx \approx 1.5608,$
- $\displaystyle\int_0^{1000} \frac{1}{1+x^2}\,dx \approx 1.5698,$
- $\displaystyle\int_0^{10,000} \frac{1}{1+x^2}\,dx \approx 1.5707.$

Notice how the integrand is $1/(1+x^2)$ in each integral (which is sketched in Figure 6.15). As the upper bound gets larger, one would expect the "area under the curve" would also grow. While the definite integrals do increase in value as the upper bound grows, they are not increasing by much. In fact, consider:

$$\int_0^b \frac{1}{1+x^2}\,dx = \tan^{-1} x\Big|_0^b = \tan^{-1} b - \tan^{-1} 0 = \tan^{-1} b.$$

As $b \to \infty$, $\tan^{-1} b \to \pi/2$. Therefore it seems that as the upper bound b grows, the value of the definite integral $\displaystyle\int_0^b \frac{1}{1+x^2}\,dx$ approaches $\pi/2 \approx 1.5708$. This should strike the reader as being a bit amazing: even though the curve extends "to infinity," it has a finite amount of area underneath it.

When we defined the definite integral $\displaystyle\int_a^b f(x)\,dx$, we made two stipulations:

1. The interval over which we integrated, $[a,b]$, was a finite interval, and
2. The function $f(x)$ was continuous on $[a,b]$ (ensuring that the range of f was finite).

In this section we consider integrals where one or both of the above conditions do not hold. Such integrals are called **improper integrals.**

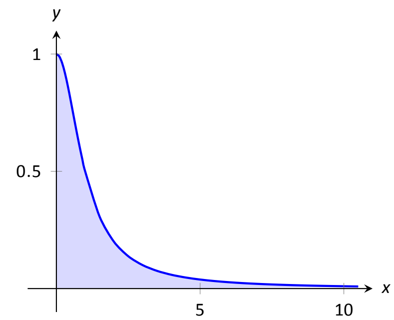


Figure 6.15: Graphing $f(x) = \dfrac{1}{1+x^2}$.

Notes:

Improper Integrals with Infinite Bounds

Definition 24 Improper Integrals with Infinite Bounds; Converge, Diverge

1. Let f be a continuous function on $[a, \infty)$. Define

$$\int_a^\infty f(x)\,dx \quad \text{to be} \quad \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

2. Let f be a continuous function on $(-\infty, b]$. Define

$$\int_{-\infty}^b f(x)\,dx \quad \text{to be} \quad \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

3. Let f be a continuous function on $(-\infty, \infty)$. Let c be any real number; define

$$\int_{-\infty}^\infty f(x)\,dx \quad \text{to be} \quad \lim_{a\to-\infty}\int_a^c f(x)\,dx + \lim_{b\to\infty}\int_c^b f(x)\,dx.$$

An improper integral is said to **converge** if its corresponding limit exists; otherwise, it **diverges**. The improper integral in part 3 converges if and only if both of its limits exist.

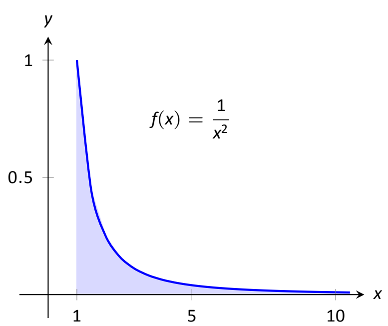


Figure 6.16: A graph of $f(x) = \frac{1}{x^2}$ in Example 6.32.

Example 6.32 Evaluating improper integrals

Evaluate the following improper integrals.

1. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$
2. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$
3. $\displaystyle\int_{-\infty}^0 e^x\,dx$
4. $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$

SOLUTION

1.
$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left.\frac{-1}{x}\right|_1^b$$
$$= \lim_{b\to\infty} \frac{-1}{b} + 1$$
$$= 1.$$

A graph of the area defined by this integral is given in Figure 6.16.

Notes:

2.
$$\begin{aligned}\int_1^\infty \frac{1}{x}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx\\ &= \lim_{b\to\infty} \ln|x|\Big|_1^b\\ &= \lim_{b\to\infty} \ln(b)\\ &= \infty.\end{aligned}$$

The limit does not exist, hence the improper integral $\int_1^\infty \frac{1}{x}\,dx$ diverges.

Compare the graphs in Figures 6.16 and 6.17; notice how the graph of $f(x) = 1/x$ is noticeably larger. This difference is enough to cause the improper integral to diverge.

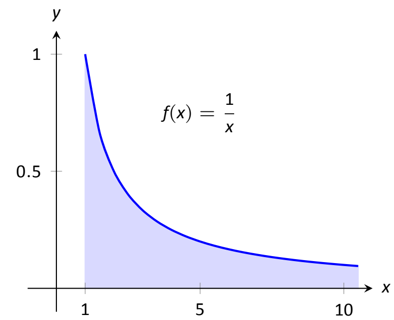


Figure 6.17: A graph of $f(x) = \frac{1}{x}$ in Example 6.32.

3.
$$\begin{aligned}\int_{-\infty}^0 e^x\,dx &= \lim_{a\to-\infty}\int_a^0 e^x\,dx\\ &= \lim_{a\to-\infty} e^x\Big|_a^0\\ &= \lim_{a\to-\infty} e^0 - e^a\\ &= 1.\end{aligned}$$

A graph of the area defined by this integral is given in Figure 6.18.

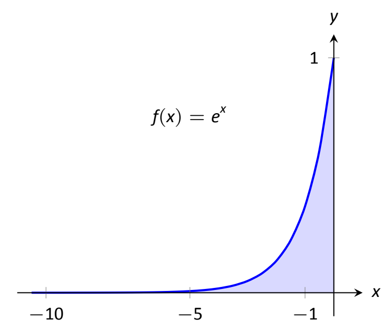


Figure 6.18: A graph of $f(x) = e^x$ in Example 6.32.

4. We will need to break this into two improper integrals and choose a value of c as in part 3 of Definition 24. Any value of c is fine; we choose $c = 0$.

$$\begin{aligned}\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx &= \lim_{a\to-\infty}\int_a^0 \frac{1}{1+x^2}\,dx + \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx\\ &= \lim_{a\to-\infty} \tan^{-1}x\Big|_a^0 + \lim_{b\to\infty} \tan^{-1}x\Big|_0^b\\ &= \lim_{a\to-\infty}\left(\tan^{-1}0 - \tan^{-1}a\right) + \lim_{b\to\infty}\left(\tan^{-1}b - \tan^{-1}0\right)\\ &= \left(0 - \frac{-\pi}{2}\right) + \left(\frac{\pi}{2} - 0\right).\end{aligned}$$

Each limit exists, hence the original integral converges and has value:

$$= \pi.$$

A graph of the area defined by this integral is given in Figure 6.19.

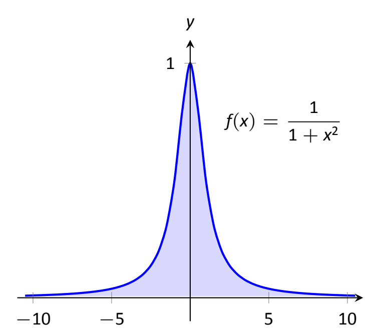


Figure 6.19: A graph of $f(x) = \frac{1}{1+x^2}$ in Example 6.32.

Notes:

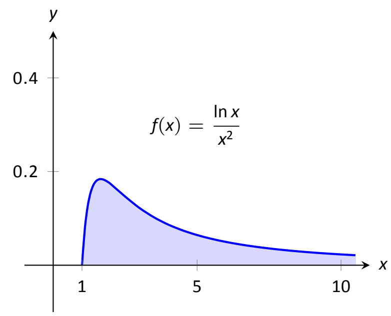


Figure 6.20: A graph of $f(x) = \frac{\ln x}{x^2}$ in Example 6.33.

The previous section introduced l'Hôpital's Rule, a method of evaluating limits that return indeterminate forms. It is not uncommon for the limits resulting from improper integrals to need this rule as demonstrated next.

Example 6.33 Improper integrals and l'Hôpital's Rule

Evaluate the improper integral $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$.

Solution This integral will require the use of Integration by Parts. Let $u = \ln x$ and $dv = 1/x^2\,dx$. Then

$$\begin{aligned}\int_1^\infty \frac{\ln x}{x^2}\,dx &= \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx\\ &= \lim_{b\to\infty}\left(-\frac{\ln x}{x}\Big|_1^b + \int_1^b \frac{1}{x^2}\,dx\right)\\ &= \lim_{b\to\infty}\left(-\frac{\ln x}{x} - \frac{1}{x}\right)\Big|_1^b\\ &= \lim_{b\to\infty}\left(-\frac{\ln b}{b} - \frac{1}{b} - (-\ln 1 - 1)\right).\end{aligned}$$

The $1/b$ and $\ln 1$ terms go to 0, leaving $\displaystyle\lim_{b\to\infty} -\frac{\ln b}{b} + 1$. We need to evaluate $\displaystyle\lim_{b\to\infty}\frac{\ln b}{b}$ with l'Hôpital's Rule. We have:

$$\begin{aligned}\lim_{b\to\infty}\frac{\ln b}{b} &\overset{\text{by LHR}}{=} \lim_{b\to\infty}\frac{1/b}{1}\\ &= 0.\end{aligned}$$

Thus the improper integral evaluates as:

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = 1.$$

Improper Integrals with Infinite Range

We have just considered definite integrals where the interval of integration was infinite. We now consider another type of improper integral, where the range of the integrand is infinite.

Notes:

Definition 25 Improper Integrals with Infinite Range

Let $f(x)$ be a continuous function on $[a,b]$ except at c, $a \leq c \leq b$, where $x = c$ is a vertical asymptote of f. Define

$$\int_a^b f(x)\,dx = \lim_{t\to c^-}\int_a^t f(x)\,dx + \lim_{t\to c^+}\int_t^b f(x)\,dx.$$

Note: In Definition 25, c can be one of the endpoints (a or b). In that case, there is only one limit to consider as part of the definition.

Example 6.34 Improper integration of functions with infinite range

Evaluate the following improper integrals:

1. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$
2. $\displaystyle\int_{-1}^1 \frac{1}{x^2}\,dx.$

Solution

1. A graph of $f(x) = 1/\sqrt{x}$ is given in Figure 6.21. Notice that f has a vertical asymptote at $x = 0$; in some sense, we are trying to compute the area of a region that has no "top." Could this have a finite value?

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}}\,dx &= \lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}\,dx\\ &= \lim_{a\to 0^+} 2\sqrt{x}\Big|_a^1\\ &= \lim_{a\to 0^+} 2\left(\sqrt{1}-\sqrt{a}\right)\\ &= 2.\end{aligned}$$

   It turns out that the region does have a finite area even though it has no upper bound (strange things can occur in mathematics when considering the infinite).

2. The function $f(x) = 1/x^2$ has a vertical asymptote at $x = 0$, as shown in Figure 6.22, so this integral is an improper integral. Let's eschew using limits for a moment and proceed without recognizing the improper nature of the integral. This leads to:

$$\begin{aligned}\int_{-1}^1 \frac{1}{x^2}\,dx &= -\frac{1}{x}\Big|_{-1}^1\\ &= -1-(1)\\ &= -2!\end{aligned}$$

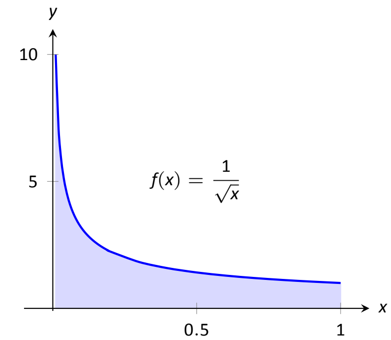


Figure 6.21: A graph of $f(x) = \frac{1}{\sqrt{x}}$ in Example 6.34.

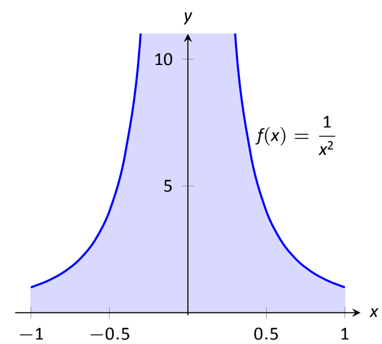


Figure 6.22: A graph of $f(x) = \frac{1}{x^2}$ in Example 6.34.

Notes:

Clearly the area in question is above the x-axis, yet the area is supposedly negative! Why does our answer not match our intuition? To answer this, evaluate the integral using Definition 25.

$$\begin{aligned}\int_{-1}^{1}\frac{1}{x^2}\,dx &= \lim_{t\to0^-}\int_{-1}^{t}\frac{1}{x^2}\,dx + \lim_{t\to0^+}\int_{t}^{1}\frac{1}{x^2}\,dx\\ &= \lim_{t\to0^-} -\frac{1}{x}\Big|_{-1}^{t} + \lim_{t\to0^+} -\frac{1}{x}\Big|_{t}^{1}\\ &= \lim_{t\to0^-} -\frac{1}{t} - 1 + \lim_{t\to0^+} -1 + \frac{1}{t}\\ &\Rightarrow \Big(\infty - 1\Big) + \Big(-1 + \infty\Big).\end{aligned}$$

Neither limit converges hence the original improper integral diverges. The nonsensical answer we obtained by ignoring the improper nature of the integral is just that: nonsensical.

Understanding Convergence and Divergence

Oftentimes we are interested in knowing simply whether or not an improper integral converges, and not necessarily the value of a convergent integral. We provide here several tools that help determine the convergence or divergence of improper integrals without integrating.

Our first tool is to understand the behavior of functions of the form $\dfrac{1}{x^p}$.

Example 6.35 Improper integrals of $1/x^p$

Determine the values of p for which $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges.

Solution We begin by integrating and then evaluating the limit.

$$\begin{aligned}\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx\\ &= \lim_{b\to\infty}\int_1^b x^{-p}\,dx \qquad \text{(assume } p\neq 1\text{)}\\ &= \lim_{b\to\infty}\frac{1}{-p+1}x^{-p+1}\Big|_1^b\\ &= \lim_{b\to\infty}\frac{1}{1-p}\big(b^{1-p}-1^{1-p}\big).\end{aligned}$$

When does this limit converge – i.e., when is this limit *not* ∞? This limit converges precisely when the power of b is less than 0: when $1-p<0 \Rightarrow 1<p$.

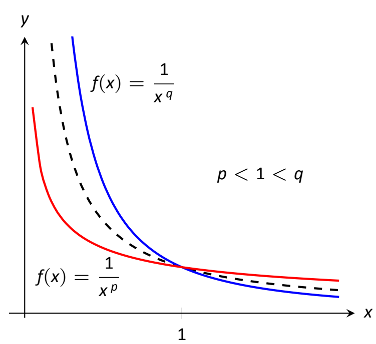


Figure 6.23: Plotting functions of the form $1/x^p$ in Example 6.35.

Notes:

Our analysis shows that if $p > 1$, then $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges. When $p < 1$ the improper integral diverges; we showed in Example 6.32 that when $p = 1$ the integral also diverges.

Figure 6.23 graphs $y = 1/x$ with a dashed line, along with graphs of $y = 1/x^p$, $p < 1$, and $y = 1/x^q$, $q > 1$. Somehow the dashed line forms a dividing line between convergence and divergence.

The result of Example 6.35 provides an important tool in determining the convergence of other integrals. A similar result is proved in the exercises about improper integrals of the form $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$. It can be similarly shown that replacing the bound 1 with any positive real number in these integrals does **not** change if it converges or diverges (though it can change what the integral converges to). These results are summarized in the following Key Idea.

Key Idea 22 Convergence of Improper Integrals $\displaystyle\int_a^\infty \frac{1}{x^p}\,dx$ and $\displaystyle\int_0^b \frac{1}{x^p}\,dx$

1. For $a > 0$, the improper integral $\displaystyle\int_a^\infty \frac{1}{x^p}\,dx$ converges when $p > 1$ and diverges when $p \leq 1$.
2. For $b > 0$, the improper integral $\displaystyle\int_0^b \frac{1}{x^p}\,dx$ converges when $p < 1$ and diverges when $p \geq 1$.

A basic technique in determining convergence of improper integrals is to compare an integrand whose convergence is unknown to an integrand whose convergence is known. We often use integrands of the form $1/x^p$ to compare to as their convergence on certain intervals is known. This is described in the following theorem.

Notes:

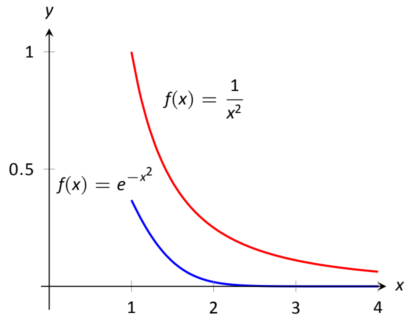


Figure 6.24: Graphs of $f(x) = e^{-x^2}$ and $f(x) = 1/x^2$ in Example 6.36.

Theorem 51 Direct Comparison Test for Improper Integrals

Let f and g be continuous on $[a,\infty)$ where $0 \leq f(x) \leq g(x)$ for all x in $[a,\infty)$.

1. If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then $\displaystyle\int_a^\infty f(x)\,dx$ converges.
2. If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then $\displaystyle\int_a^\infty g(x)\,dx$ diverges.

Example 6.36 Determining convergence of improper integrals

Determine the convergence of the following improper integrals.

1. $\displaystyle\int_1^\infty e^{-x^2}\,dx$ 2. $\displaystyle\int_3^\infty \frac{1}{\sqrt{x^2-x}}\,dx$

Solution

1. The function $f(x) = e^{-x^2}$ does not have an antiderivative expressible in terms of elementary functions, so we cannot integrate directly. It is comparable to $g(x) = 1/x^2$, and as demonstrated in Figure 6.24, $e^{-x^2} < 1/x^2$ on $[1,\infty)$. We know from Key Idea 22 that $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges, hence $\displaystyle\int_1^\infty e^{-x^2}\,dx$ also converges.

2. Note that for large values of x, $\dfrac{1}{\sqrt{x^2-x}} \approx \dfrac{1}{\sqrt{x^2}} = \dfrac{1}{x}$. We know from Key Idea 22 and the subsequent note that $\displaystyle\int_3^\infty \frac{1}{x}\,dx$ diverges, so we seek to compare the original integrand to $1/x$.

   It is easy to see that when $x > 0$, we have $x = \sqrt{x^2} > \sqrt{x^2-x}$. Taking reciprocals reverses the inequality, giving
   $$\frac{1}{x} < \frac{1}{\sqrt{x^2-x}}.$$
   Using Theorem 51, we conclude that since $\displaystyle\int_3^\infty \frac{1}{x}\,dx$ diverges, $\displaystyle\int_3^\infty \frac{1}{\sqrt{x^2-x}}\,dx$ diverges as well. Figure 6.25 illustrates this.

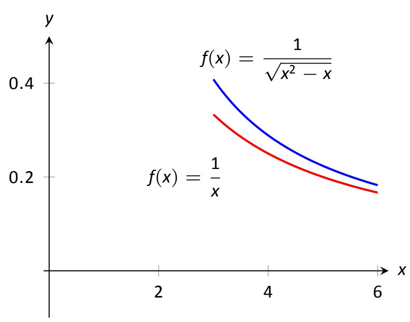


Figure 6.25: Graphs of $f(x) = 1/\sqrt{x^2-x}$ and $f(x) = 1/x$ in Example 6.36.

Notes:

Being able to compare "unknown" integrals to "known" integrals is very useful in determining convergence. However, some of our examples were a little "too nice." For instance, it was convenient that $\frac{1}{x} < \frac{1}{\sqrt{x^2-x}}$, but what if the "$-x$" were replaced with a "$+2x+5$"? That is, what can we say about the convergence of $\int_3^\infty \frac{1}{\sqrt{x^2+2x+5}}\,dx$? We have $\frac{1}{x} > \frac{1}{\sqrt{x^2+2x+5}}$, so we cannot use Theorem 51.

In cases like this (and many more) it is useful to employ the following theorem.

Theorem 52 Limit Comparison Test for Improper Integrals

Let f and g be continuous functions on $[a,\infty)$ where $f(x) > 0$ and $g(x) > 0$ for all x. If

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

either both converge or both diverge.

Example 6.37 Determining convergence of improper integrals

Determine the convergence of $\int_3^\infty \frac{1}{\sqrt{x^2+2x+5}}\,dx$.

Solution As x gets large, the quadratic inside the square root function will begin to behave much like $y = x$. So we compare $\frac{1}{\sqrt{x^2+2x+5}}$ to $\frac{1}{x}$ with the Limit Comparison Test:

$$\lim_{x\to\infty} \frac{1/\sqrt{x^2+2x+5}}{1/x} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+2x+5}}.$$

The immediate evaluation of this limit returns ∞/∞, an indeterminate form. Using l'Hôpital's Rule seems appropriate, but in this situation, it does not lead to useful results. (We encourage the reader to employ l'Hôpital's Rule at least once to verify this.)

The trouble is the square root function. To get rid of it, we employ the following fact: If $\lim_{x\to c} f(x) = L$, then $\lim_{x\to c} f(x)^2 = L^2$. (This is true when either c or L

Notes:

is ∞.) So we consider now the limit

$$\lim_{x\to\infty} \frac{x^2}{x^2+2x+5}.$$

This converges to 1, meaning the original limit also converged to 1. As x gets very large, the function $\dfrac{1}{\sqrt{x^2+2x+5}}$ looks very much like $\dfrac{1}{x}$. Since we know that $\displaystyle\int_3^\infty \frac{1}{x}\,dx$ diverges, by the Limit Comparison Test we know that $\displaystyle\int_3^\infty \frac{1}{\sqrt{x^2+2x+5}}\,dx$ also diverges. Figure 6.26 graphs $f(x) = 1/\sqrt{x^2+2x+5}$ and $f(x) = 1/x$, illustrating that as x gets large, the functions become indistinguishable.

Both the Direct and Limit Comparison Tests were given in terms of integrals over an infinite interval. There are versions that apply to improper integrals with an infinite range, but as they are a bit wordy and a little more difficult to employ, they are omitted from this text.

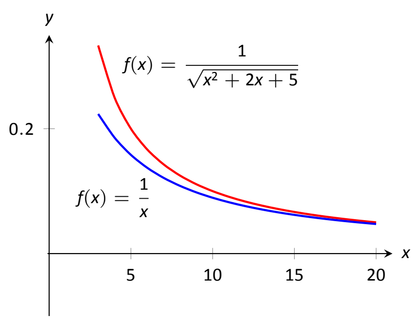


Figure 6.26: Graphing $f(x) = \frac{1}{\sqrt{x^2+2x+5}}$ and $f(x) = \frac{1}{x}$ in Example 6.37.

Notes:

Exercises 6.6

Terms and Concepts

1. The definite integral was defined with what two stipulations?

2. If $\lim_{b\to\infty} \int_0^b f(x)\ dx$ exists, then the integral $\int_0^\infty f(x)\ dx$ is said to ________________.

3. If $\int_1^\infty f(x)\ dx = 10$, and $0 \le g(x) \le f(x)$ for all x, then we know that $\int_1^\infty g(x)\ dx$ ________________.

4. For what values of p will $\int_1^\infty \frac{1}{x^p}\ dx$ converge?

5. For what values of p will $\int_{10}^\infty \frac{1}{x^p}\ dx$ converge?

6. For what values of p will $\int_0^1 \frac{1}{x^p}\ dx$ converge?

Problems

In Exercises 7 – 33, evaluate the given improper integral.

7. $\int_0^\infty e^{5-2x}\ dx$

8. $\int_1^\infty \frac{1}{x^3}\ dx$

9. $\int_1^\infty x^{-4}\ dx$

10. $\int_{-\infty}^\infty \frac{1}{x^2+9}\ dx$

11. $\int_{-\infty}^0 2^x\ dx$

12. $\int_{-\infty}^0 \left(\frac{1}{2}\right)^x\ dx$

13. $\int_{-\infty}^\infty \frac{x}{x^2+1}\ dx$

14. $\int_{-\infty}^\infty \frac{x}{x^2+4}\ dx$

15. $\int_2^\infty \frac{1}{(x-1)^2}\ dx$

16. $\int_1^2 \frac{1}{(x-1)^2}\ dx$

17. $\int_2^\infty \frac{1}{x-1}\ dx$

18. $\int_1^2 \frac{1}{x-1}\ dx$

19. $\int_{-1}^1 \frac{1}{x}\ dx$

20. $\int_1^3 \frac{1}{x-2}\ dx$

21. $\int_0^\pi \sec^2 x\ dx$

22. $\int_{-2}^1 \frac{1}{\sqrt{|x|}}\ dx$

23. $\int_0^\infty xe^{-x}\ dx$

24. $\int_0^\infty xe^{-x^2}\ dx$

25. $\int_{-\infty}^\infty xe^{-x^2}\ dx$

26. $\int_{-\infty}^\infty \frac{1}{e^x+e^{-x}}\ dx$

27. $\int_0^1 x\ln x\ dx$

28. $\int_1^\infty \frac{\ln x}{x}\ dx$

29. $\int_0^1 \ln x\ dx$

30. $\int_1^\infty \frac{\ln x}{x^2}\ dx$

31. $\int_1^\infty \frac{\ln x}{\sqrt{x}}\ dx$

32. $\int_0^\infty e^{-x}\sin x\ dx$

33. $\int_0^\infty e^{-x}\cos x\ dx$

In Exercises 34 – 43, use the Direct Comparison Test or the Limit Comparison Test to determine whether the given definite integral converges or diverges. Clearly state what test is being used and what function the integrand is being compared to.

34. $\displaystyle\int_{10}^{\infty} \frac{3}{\sqrt{3x^2+2x-5}}\,dx$

35. $\displaystyle\int_{2}^{\infty} \frac{4}{\sqrt{7x^3-x}}\,dx$

36. $\displaystyle\int_{0}^{\infty} \frac{\sqrt{x+3}}{\sqrt{x^3-x^2+x+1}}\,dx$

37. $\displaystyle\int_{1}^{\infty} e^{-x}\ln x\,dx$

38. $\displaystyle\int_{5}^{\infty} e^{-x^2+3x+1}\,dx$

39. $\displaystyle\int_{0}^{\infty} \frac{\sqrt{x}}{e^x}\,dx$

40. $\displaystyle\int_{2}^{\infty} \frac{1}{x^2+\sin x}\,dx$

41. $\displaystyle\int_{0}^{\infty} \frac{x}{x^2+\cos x}\,dx$

42. $\displaystyle\int_{0}^{\infty} \frac{1}{x+e^x}\,dx$

43. $\displaystyle\int_{0}^{\infty} \frac{1}{e^x-x}\,dx$

6.7 Exploring Functions Defined as Integrals (Optional)

We have explored many integration techniques in this chapter and the previous one. We learned Substitution, which "undoes" the Chain Rule of differentiation, as well as Integration by Parts, which "undoes" the Product Rule. We learned specialized techniques for handling trigonometric functions as well as the method of Partial Fraction Decomposition. Additionally, we have needed several algebra techniques for integrating certain functions, such as long polynomial division and completing the square. All techniques effectively have this goal in common: rewrite the integrand in a new way so that the integration step is easier to see and implement.

As stated before, integration is, in general, hard. When computing derivatives, there can always be a systematic process used. When one encounters division of functions, the Quotient Rule may be used. Sometimes it is faster and easier to simplify the function, such as $f(x) = \dfrac{x^2+3}{x}$, by distributing the division. With this method, the Quotient Rule may be avoided to take the derivative, but the point is that we *can* use the Quotient Rule whenever we encounter function division. To integrate this function, on the other hand, we absolutely need to distribute the division first. For integration, we have many "techniques" instead of "rules". Finding an antiderivative of a function is a puzzle, not a systematic process.

Consider a not-so-complicated integral such as $\int \sqrt{\tan x}\, dx$. It turns out that

$$\int \sqrt{\tan x}\, dx = \frac{\sqrt{2}}{4}\ln\left(\frac{\tan x - \sqrt{2\tan x}+1}{\tan x + \sqrt{2\tan x}+1}\right) + \frac{\sqrt{2}}{2}\tan^{-1}\left(1+2\sqrt{\tan x}\right) - \frac{\sqrt{2}}{2}\tan^{-1}\left(1-2\sqrt{\tan x}\right) + C.$$

This answer is surprisingly complicated for integrating a simple function. To arrive at this, one first uses the substitution $u = \tan x$ and a trigonometric identity to rewrite the integrand as a rational function. Then, one uses partial fraction decomposition, by first factoring u^4+1 as $\left(u^2+\sqrt{2}u+1\right)\left(u^2-\sqrt{2}u+1\right)$. Not only is it far from straightforward to factor u^4+1, but $\sqrt{2}$ also appears in the coefficients, leading to an ugly answer.

Since we do not have any "rules" that always work for integrating products, it may not be surprising that there are some functions, in fact many, that do **not** have antiderivatives expressible in terms of *elementary functions*. "Elementary functions" is a term used to describe the functions most often encountered in mathematics. These include polynomials, exponential, logarithmic, trigonometric inverse trigonometric functions, as well as sums, differences, products, quotients, and compositions of these. Some basic antiderivatives that cannot be expressed in terms of elementary functions include the following.

Notes:

$$\int e^{-x^2}\,dx \qquad \int \frac{\sin x}{x}\,dx \qquad \int \sin(\sin x)\,dx$$

One might wonder whether it is truly impossible to express these antiderivatives in terms of elementary functions, or if they may be unsolved puzzles where no one has found the correct technique yet. This is natural to think, as one might think $\sqrt{\tan x}$ does not have an elementary antiderivative if no one had thought to combine all the steps. Also $\int \sec x\,dx$ involved a strange substitution we do not expect students to come up with on their own. What if no one had thought of a trigonometric substitution? Could there be another technique no one has come up with yet to tackle other hard integrals?

Possibly surprisingly, it actually has been proven that antiderivatives of the above functions, and many more, cannot be expressed in terms of elementary functions. These proofs come from a branch of mathematics called Differential Galois Theory, which is far beyond the scope of this text. The point is that it is easy to write a function whose antiderivative is impossible to write in terms of elementary functions, and even when a function does have an antiderivative expressible by elementary functions, it may be really hard to discover what it is. The powerful computer algebra system *Mathematica*® has approximately 1,000 pages of code dedicated to integration, and also has the functions described in this section programmed in.

Recall that from the Fundamental Theorem of Calculus, Part 1, given a continuous function $f(x)$, the function $\int_a^x f(t)\,dt$ is an antiderivative of $f(x)$. Then in Section 5.6, we discussed techniques for approximating integrals, which is often necessary for functions without elementary antiderivatives. Now that we have learned integration techniques, we emphasize the need to define new functions as integrals. We will explore a small handful of these functions to gain an understanding as to how such functions operate. Such functions appear in formulas in the sciences. Despite being more difficult to grasp, they act just like other functions we have studied.

Error Functions

The **error function** $\operatorname{erf} x$ and **imaginary error function** $\operatorname{erfi} x$ are important in statistical theory. Calculators and computer algebra systems vary between having neither, one, or both of these functions programmed.

Notes:

Definition 26 Error Function and Imaginary Error Function

The **error function** is

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

The **imaginary error function** is

$$\operatorname{erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{t^2}\, dt.$$

The variable t in the definitions is just a dummy variable used to express the integral. For example, $\operatorname{erf} 1 = \frac{2}{\sqrt{\pi}} \int_0^1 e^{-t^2}\, dt$ is the area between the t-axis and the graph of $y = \frac{2}{\sqrt{\pi}} e^{-t^2}$ from $t = 0$ to $t = 1$. We use the variable t to draw this area, but the calculation is just a number. This area is drawn in Figure 6.27. It is approximately 0.84270079, but it is expressed in exact form as erf 1.

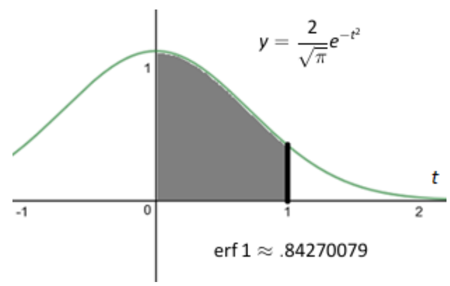


Figure 6.27: Showing erf 1 as an area.

From the Fundamental Theorem of Calculus, Part 1 (and the Constant Multiple Rule),

$$\frac{d}{dx}(\operatorname{erf} x) = \frac{2}{\sqrt{\pi}} e^{-x^2} \text{ and } \frac{d}{dx}(\operatorname{erfi} x) = \frac{2}{\sqrt{\pi}} e^{x^2}.$$

A couple things may seem strange. Why do the definitions contain $\dfrac{2}{\sqrt{\pi}}$? It turns out that the improper integral $\displaystyle\int_0^\infty e^{-x^2}$ converges to $\sqrt{\pi}/2$. (To prove this, one has to use a certain double integral over two variables. We will just accept this on faith.) So the function $\operatorname{erf} x$ is defined so that $\displaystyle\lim_{x\to\infty} \operatorname{erf} x = 1$. This function is vary closely tied to the **normal distribution**

$$\frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$$

in probability and statistics (where μ and σ represent the mean and standard deviation, respectively). Probability theory involves integration, though we wil not go into more details. In a statistics, one may learn that in a normal distribution, 68% of data points lie within 1 standard deviation of the mean, 95% within 2 standard deviations, and 99.7% within 3 standard deviations. These values are actually the following.

$$\operatorname{erf}\left(\frac{1}{\sqrt{2}}\right) \approx 0.6827 \qquad \operatorname{erf}\left(\frac{2}{\sqrt{2}}\right) \approx 0.9545 \qquad \operatorname{erf}\left(\frac{3}{\sqrt{2}}\right) \approx 0.9973$$

Notes:

The name "error function" comes from this connection with statistical error. The name "imaginary error function" is from the identity $\operatorname{erf}(xi) = \operatorname{erfi}(x)i$, though we do not know how to make sense of this fact since we have not discussed calculus in complex numbers.

Below are the graphs of $\operatorname{erf} x$, $\operatorname{erfi} x$, and their derivatives for comparison.

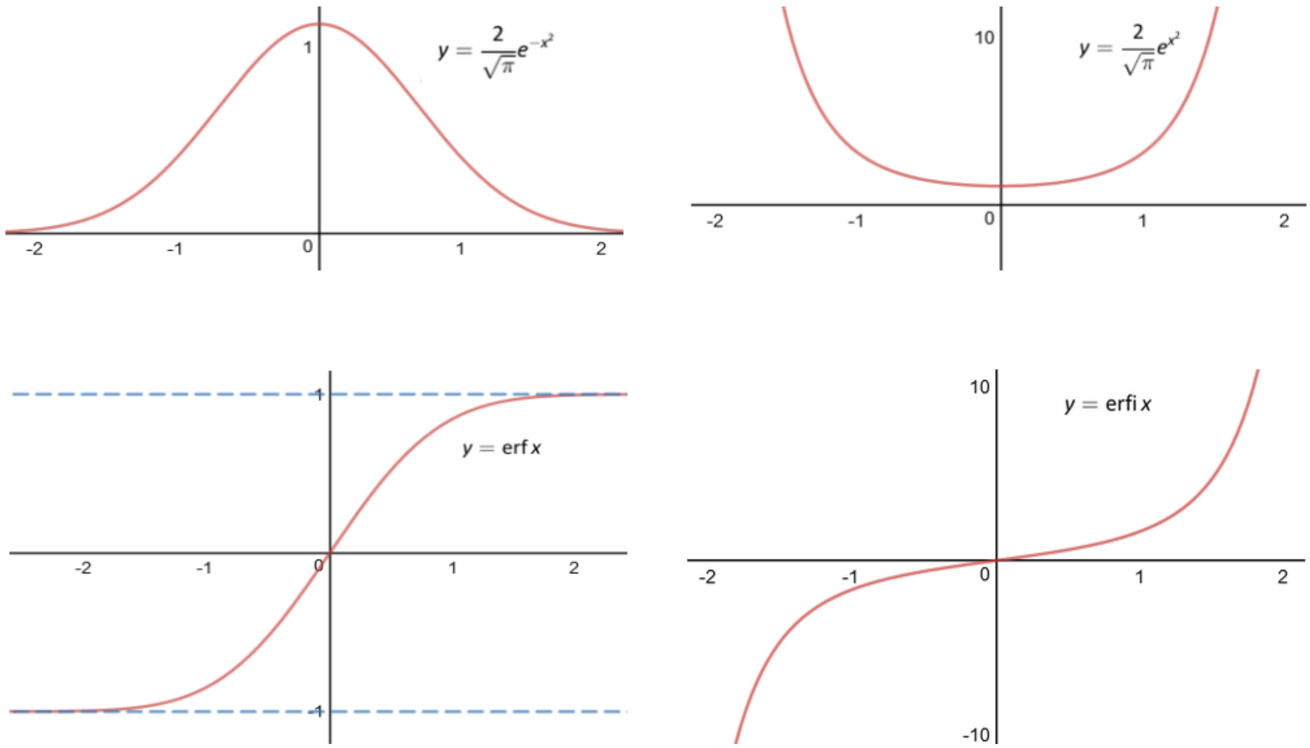


Figure 6.28: The error functions and their derivatives.

There are a few key characteristics of $\operatorname{erf} x$ and $\operatorname{erfi} x$ we can quickly observe.

- They are both increasing functions on $(-\infty, \infty)$. This is easy to explain since their derivatives are always positive.
- Both graphs go through the origin. This is also easy to explain since

$$\operatorname{erf} 0 = \frac{2}{\sqrt{\pi}} \int_0^0 e^{-x^2}\,dx = \frac{2}{\sqrt{\pi}} \cdot 0 = 0 \text{ and } \operatorname{erfi} 0 = \frac{2}{\sqrt{\pi}} \int_0^0 e^{x^2}\,dx = 0.$$

Notes:

- The functions are both odd. That is, $\operatorname{erf}(-x) = -\operatorname{erf} x$ and $\operatorname{erfi}(-x) = -\operatorname{erfi} x$ for any real number x. This can be explained by the symmetry of their derivatives.

So $\dfrac{d}{dx}(\operatorname{erf} x) = \dfrac{2}{\sqrt{\pi}} e^{-x^2}$. How do we come up with an antiderivative of e^{-x^2} without the $\dfrac{2}{\sqrt{\pi}}$ coefficient? We just multiply by a constant! Note that

$$\frac{d}{dx}\left(\frac{\sqrt{\pi}}{2}\operatorname{erf} x\right) = \frac{\sqrt{\pi}}{2}\left(\frac{2}{\sqrt{\pi}} e^{-x^2}\right) = e^{-x^2}$$

and thus

$$\int e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}\operatorname{erf} x + C.$$

We can do something similar for integrals of e^{x^2}. We will soon see that by defining these two functions, we can express many more antiderivatives in terms of them. We summarize the key properties of the error functions in the theorem below.

Theorem 53 Basic Properties of Error Functions

$\dfrac{d}{dx}(\operatorname{erf} x) = \dfrac{2}{\sqrt{\pi}} e^{-x^2}$	$\dfrac{d}{dx}(\operatorname{erfi} x) = \dfrac{2}{\sqrt{\pi}} e^{x^2}$
$\displaystyle\int e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}\operatorname{erf} x + C$	$\displaystyle\int e^{x^2}\,dx = \frac{\sqrt{\pi}}{2}\operatorname{erfi} x + C$
$\operatorname{erf} 0 = 0$	$\operatorname{erfi} 0 = 0$
$\displaystyle\lim_{x\to-\infty} \operatorname{erf} x = -1$	$\displaystyle\lim_{x\to-\infty} \operatorname{erfi} x = -\infty$
$\displaystyle\lim_{x\to\infty} \operatorname{erf} x = 1$	$\displaystyle\lim_{x\to\infty} \operatorname{erfi} x = \infty$
$\operatorname{erf}(-x) = -\operatorname{erf} x$	$\operatorname{erfi}(-x) = -\operatorname{erfi} x$

$\operatorname{erf} x$ has a domain of $(-\infty, \infty)$ and range of $(-1, 1)$

$\operatorname{erfi} x$ has a domain of $(-\infty, \infty)$ and range of $(-\infty, \infty)$

Notes:

In the next example, the integrand has a function that looks similar to e^{x^2} so the answer can be written in exact form in terms of the imaginary error function.

Example 6.38 Integral whose solution can be expressed in terms of an error function

Evaluate $\displaystyle\int_1^5 e^{9x^2}\,dx$

SOLUTION We recognize the integrand as being like e^{x^2} except with a 9 in it. If we rewrite the integrand as $e^{(3x)^2}$, we see we can use $u = 3x$, $du = 3dx$. So $du/3 = dx$. Remembering to change the bounds, we obtain:

$$\begin{aligned}\int_1^5 e^{9x^2}\,dx &= \int_1^5 e^{(3x)^2}\,dx\\ &= \frac{1}{3}\int_3^{15} e^{u^2}\,du\\ &= \frac{1}{3}\cdot\frac{\sqrt{\pi}}{2}\operatorname{erfi} u\,\Big|_3^{15}\\ &= \frac{\sqrt{\pi}}{6}\big(\operatorname{erfi} 15 - \operatorname{erfi} 3\big)\end{aligned}$$

With technology, we can approximate this answer as 5.7941×10^{95}, a huge number because the integrand grows extremely quickly.

One might naively expect that we would have to define yet another function as an antiderivative of $\operatorname{erf} x$. This is not the case, however. Using integration by parts, we can express an antiderivative of $\operatorname{erf} x$ in terms of itself.

Example 6.39 Integrating $\operatorname{erf} x$

Evaluate $\displaystyle\int \operatorname{erf} x\,dx$.

SOLUTION We use Integration by Parts. The LIATE mnemonic is just for elementary functions. However, as with logarithms and inverse trigonometric functions, the derivative of $\operatorname{erf} x$ is much simpler than itself.

$$\begin{array}{ll} u = \operatorname{erf} x & v = ? \\ du = ? & dv = dx \end{array} \quad\Rightarrow\quad \begin{array}{ll} u = \operatorname{erf} x & v = x \\ du = \dfrac{2}{\sqrt{\pi}} e^{-x^2}\,dx & dv = dx \end{array}$$

Figure 6.29: Setting up Integration by Parts.

Notes:

Putting this all together in the Integration by Parts formula, we obtain:

$$\begin{aligned}\int \operatorname{erf} x\,dx &= x\operatorname{erf} x - \int x\left(\frac{2}{\sqrt{\pi}}e^{-x^2}\right)\,dx\\ &= x\operatorname{erf} x - \frac{2}{\sqrt{\pi}}\int xe^{-x^2}\,dx\\ &= x\operatorname{erf} x + \frac{e^{-x^2}}{\sqrt{\pi}} + C\end{aligned}$$

where we used Substitution in the final step. We can check that this is indeed an antiderivative by taking the derivative. Everything is normal, we of course need to use the Product Rule on $x\operatorname{erf} x$.

$$\begin{aligned}\frac{d}{dx}\left(x\operatorname{erf} x + \frac{e^{-x^2}}{\sqrt{\pi}}\right) &= 1\operatorname{erf} x + x\cdot\frac{2}{\sqrt{\pi}}e^{-x^2} + \frac{-2xe^{-x^2}}{\sqrt{\pi}}\\ &= \operatorname{erf} x.\end{aligned}$$

We do three more examples of antiderivatives that cannot be expressed in terms of elementary functions, but can be expressed in terms of error functions.

Example 6.40 Integrating a function
Evaluate $\int e^{x^2+2x}\,dx$

Solution If not for the $2x$ in the exponent, we would be able to state the answer immediately from Theorem 53. This suggests completing the square in the exponent, which is how we will proceed.

$$\begin{aligned}\int e^{x^2+2x}\,dx &= \int e^{x^2+2x+1-1}\,dx\\ &= \int e^{(x+1)^2-1}\,dx\\ &= \int \frac{e^{(x+1)^2}}{e}\,dx\\ &= \frac{\sqrt{\pi}}{2e}\operatorname{erfi}(x+1) + C.\end{aligned}$$

Notes:

Example 6.41 Integrating a function

Evaluate $\int e^{2x}\,\mathrm{erfi}\,x\,dx$

Solution This is a product of functions, suggesting integration by parts. Again, while it makes no sense to use "LIATE", it is still ideal to avoid the exponential when picking u, so that du is simpler than u. So we will pick $u = \mathrm{erfi}\,x$.

$$\begin{array}{ll} u = \mathrm{erfi}\,x & v = ? \\ du = ? & dv = e^{2x}\,dx \end{array} \quad \Rightarrow \quad \begin{array}{ll} u = \mathrm{erfi}\,x & v = \dfrac{1}{2}e^{2x} \\ du = \dfrac{2}{\sqrt{\pi}}e^{x^2}\,dx & dv = e^{2x}\,dx \end{array}$$

Figure 6.30: Setting up Integration by Parts.

Integrating by Parts,

$$\begin{aligned} \int e^{2x}\,\mathrm{erfi}\,x\,dx &= \frac{1}{2}e^{2x}\,\mathrm{erfi}\,x - \int \frac{1}{2}e^{2x}\cdot\frac{2}{\sqrt{\pi}}e^{x^2}\,dx \\ &= \frac{1}{2}e^{2x}\,\mathrm{erfi}\,x - \frac{1}{\sqrt{\pi}}\int e^{x^2+2x}\,dx \end{aligned}$$

by rules of exponents. Now we recognize the last integral as being the one in Example 6.40, from which we obtain our final answer.

$$\begin{aligned} \int e^{2x}\,\mathrm{erfi}\,x\,dx &= \frac{1}{2}e^{2x}\,\mathrm{erfi}\,x - \int \frac{1}{\sqrt{\pi}}e^{x^2+2x}\,dx \\ &= \frac{1}{2}e^{2x}\,\mathrm{erfi}\,x - \frac{1}{2e}\,\mathrm{erfi}(x+1) + C. \end{aligned}$$

Example 6.42 Integrating a function

Evaluate $\displaystyle\int \frac{e^{-x}}{\sqrt{\pi x}}\,dx$

Solution If we expect to use the functions in this section, we expect to have e raised to $\pm$ something squared. The existence of a square root in the integrand also suggests using the substitution $u = \sqrt{x}$. Then we have

$$du = \frac{1}{2\sqrt{x}}dx \Rightarrow \frac{2}{\sqrt{\pi}}du = \frac{1}{\sqrt{\pi x}}dx.$$

Thus,

$$\int \frac{e^{-x}}{\sqrt{\pi x}}\,dx = \frac{2}{\sqrt{\pi}}\int e^{-u^2}\,du = \mathrm{erf}\,u + C = \mathrm{erf}\,\sqrt{x} + C.$$

Notes:

Before we move to exploring other functions, we compute a limit. We remind ourselves that we can use L'Hôpital's Rule on several new limits, if they yield the appropriate indeterminate form. The one below is type $0 \cdot \infty$.

Example 6.43 **A limit involving** $\operatorname{erf} x$

Evaluate $\displaystyle\lim_{x\to\infty} (1 - \operatorname{erf} x)e^x$

Solution We note from Theorem 53 that $\lim_{x\to\infty} 1 - \operatorname{erf} x = 1 - 1 = 0$, so this limit is of the form $0 \cdot \infty$. The easiest way to rewrite this as a quotient to use L'Hôpital's Rule is below.

$$\begin{aligned}
\lim_{x\to\infty} (1 - \operatorname{erf} x)e^x &= \lim_{x\to\infty} \frac{1 - \operatorname{erf} x}{1/e^x} \\
&= \lim_{x\to\infty} \frac{1 - \operatorname{erf} x}{e^{-x}} \\
&\overset{\text{by LHR}}{=} \lim_{x\to\infty} \frac{-\frac{2}{\sqrt{\pi}} e^{-x^2}}{-e^{-x}} \\
&= \lim_{x\to\infty} \frac{2}{\sqrt{\pi}} e^{x - x^2} \\
&= 0.
\end{aligned}$$

Inverse Error Functions

Both $\operatorname{erf} x$ and $\operatorname{erfi} x$ are one-to-one functions, as they are increasing. Therefore, we can define their inverse functions $\operatorname{erf}^{-1} x$ and $\operatorname{erfi}^{-1} x$.

Definition 27 **Inverse Error Function and Inverse Imaginary Error Function**

1. For $-1 < x < 1$, let $\operatorname{erf}^{-1} x$ denote the unique number y for which $\operatorname{erf} y = x$. The function $\operatorname{erf}^{-1} x$ is called the **inverse error function**.
2. For any real number x, let $\operatorname{erfi}^{-1} x$ denote the unique number y for which $\operatorname{erfi} y = x$. The function $\operatorname{erfi}^{-1} x$ is called the **inverse imaginary error function**.

We said before that $\operatorname{erf} 1 \approx 0.8427$. So $\operatorname{erf}^{-1}(0.8427) \approx 1$. Suppose we wish to determine, for a normal distribution, how many standard deviations

Notes:

from the mean we expect half of the data to be within. This involves solving the equation $\operatorname{erf}\left(x/\sqrt{2}\right) = 0.5$. The solution is $x = \sqrt{2}\operatorname{erf}^{-1}(0.5) \approx 0.6745$. This approximation came from a computer algebra system in which erf^{-1} is programmed. This means that roughly half of the data points are within 0.6745 standard deviations of the mean.

We can use Theorem 22 from Section 2.7 to determine that

$$\frac{d}{dx}\left(\operatorname{erf}^{-1} x\right) = \frac{1}{\frac{2}{\sqrt{\pi}}e^{-\operatorname{erf}^{-1}(x)^2}} = \frac{\sqrt{\pi}}{2}e^{\operatorname{erf}^{-1}(x)^2}.$$

However, what about the integral of $\operatorname{erf}^{-1} x$? As typical, that is more difficult to compute than the derivative. We tackle this in the following example.

Example 6.44 **Antiderivatives of** $\operatorname{erf}^{-1} x$

Evaluate $\displaystyle\int \operatorname{erf}^{-1} x\, dx$.

Solution There is not really a big hint as to how to approach this, as is often typical of integration. One might take the approach that we did for integrating $\operatorname{erf} x$ and use Integration by Parts. However, the fastest way is to use the Substitution $u = \operatorname{erf}^{-1} x$. Then

$$du = \frac{\sqrt{\pi}}{2}e^{\operatorname{erf}^{-1}(x)^2}dx = \frac{\sqrt{\pi}}{2}e^{u^2}dx.$$

This implies $dx = \frac{2}{\sqrt{\pi}}e^{-u^2}du$, so

$$\begin{aligned}\int \operatorname{erf}^{-1} x\, dx &= \frac{2}{\sqrt{\pi}}\int ue^{-u^2}du\\ &= -\frac{1}{\sqrt{\pi}}e^{-u^2} + C\\ &= -\frac{1}{\sqrt{\pi}}e^{-\operatorname{erf}^{-1}(x)^2} + C.\end{aligned}$$

We can do the same for $\operatorname{erfi}^{-1} x$. The derivatives and antiderivatives of inverse error functions are summarized in the following theorem.

Notes:

Theorem 54 Derivatives and Antiderivatives of $\text{erf}^{-1}x$ and $\text{erfi}^{-1}x$

The functions $\text{erf}^{-1}x$ and $\text{erfi}^{-1}x$ are differentiable on their domains. Their derivatives and antiderivatives are below.

1. $\dfrac{d}{dx}\left(\text{erf}^{-1}x\right) = \dfrac{\sqrt{\pi}}{2}e^{\text{erf}^{-1}(x)^2}$
2. $\dfrac{d}{dx}\left(\text{erfi}^{-1}x\right) = \dfrac{\sqrt{\pi}}{2}e^{-\,\text{erfi}^{-1}(x)^2}$
3. $\displaystyle\int \text{erf}^{-1}x\,dx = -\frac{1}{\sqrt{\pi}}e^{-\,\text{erf}^{-1}(x)^2} + C$
4. $\displaystyle\int \text{erfi}^{-1}x\,dx = \frac{1}{\sqrt{\pi}}e^{\text{erfi}^{-1}(x)^2} + C$

Sine Integral Function

We consider one more function, defined as an antiderivative of $\dfrac{\sin x}{x}$.

Definition 28 Sine Integral

The **sine integral** is the function

$$\text{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt.$$

Something may seem a bit strange with this definition. The integrand $\frac{\sin t}{t}$ is not defined at $t = 0$, which is one of the bounds. This may suggest that, no matter what x-value is plugged in, $\text{Si}(x)$ refers to an improper integral. However, this is not the case. Recall that $\displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1$. Since this limit exists, the discontinuity $\frac{\sin x}{x}$ has at $x = 0$ is removable. When computing area under the curve, we can ignore the hole. The graph of $y = \dfrac{\sin x}{x}$ is shown in Figure 6.31.

Notes:

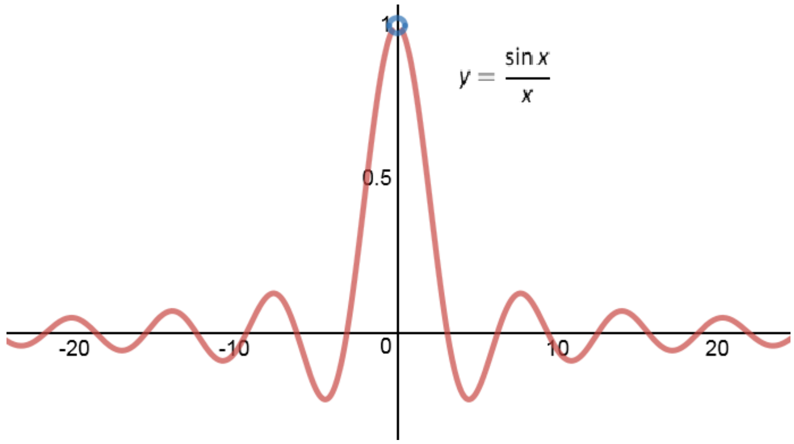


Figure 6.31: The function $y = \sin(x)/x$.

And now in Figure 6.32, we show the graph of $\mathrm{Si}(x)$. It has a rather unusual shape compared to other functions we have seen.

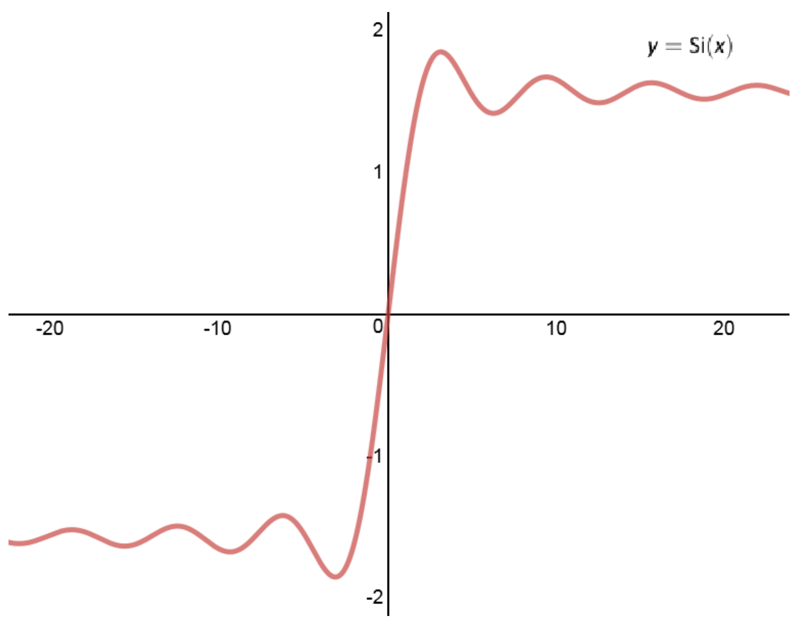


Figure 6.32: The function $y = \mathrm{Si}(x)$.

Notes:

One noticeable characteristic it has many relative extrema. By putting together previous concepts in this text, and an understanding of what the $\text{Si}(x)$ function means, one can easily describe the x-values of the relative extrema. This is done in one of the exercises.

Another interesting characteristic is that it appears as $x \to \infty$ and $x \to -\infty$, the graph settles more and more on a single value. But what value? It turns out that

$$\lim_{x\to\infty} \text{Si}(x) = \int_0^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2} \text{ and } \lim_{x\to-\infty} \text{Si}(x) = \int_0^{-\infty} \frac{\sin x}{x}\,dx = -\frac{\pi}{2}.$$

However, we have not yet learned any technique that can be used to prove this. It involves rewriting the integral as a double integral first. In fact, it is far from straightforward to even show those integrals converge at all.

It is not the case that the particular functions explored in this section are more important than other functions defined as integrals. Rather, the significance of this section is to gain an understanding for how such functions operate, and also to realize that (despite the abstract definitions) they operate just like every other differentiable function. There are many more functions defined as integrals. The computer algebra system *Mathematica*® includes many more, such as the logarithmic integral, cosine integral, dilogarithm, trilogarithm, elliptic integrals, and Fresnel integrals. In Section 8.8, there will be a series of challenging exercises in which one can explore the dilogarithm.

While elementary functions are defined based solely on being easier to describe and appear most often, other mathematical functions arise in numerous various applications. One of the most famous unsolved problems in mathematics is the Riemann Hypothesis, which involves the Riemann Zeta Function. That function is not defined in terms of integral, and cannot be defined with the material in this text. However, the Riemann Hypothesis has applications to number theory, specifically the distribution of prime numbers, which in turn is very important in cryptography (theory of codes). Due to its significance, the Riemann Hypothesis was chosen by the Clay Mathematical Institute to be one of the Millennium Prize Problems. Anyone that proves (or disproves) it can earn an award of 1,000,000 US dollars!

Do not let the complexity of integrals discourage you. There is great value in learning integration techniques, as they allow one to manipulate an integral in ways that can illuminate a concept for greater understanding. The next chapter stresses the uses of integration. We generally do not find antiderivatives for antiderivative's sake, but rather because they provide the solution to some type of problem. The following chapter introduces us to a number of different problems whose solution is provided by integration.

Notes:

Exercises 6.7

Terms and Concepts

1. T/F: The calculus concepts, theorems, and methods only work on elementary functions.

2. What are the domains and ranges of $\operatorname{erf} x$, $\operatorname{erfi} x$, $\operatorname{erf}^{-1} x$, and $\operatorname{erfi}^{-1} x$?

Problems

In Exercises 3 – 12, compute the derivative of the given function.

3. $f(x) = \sqrt{\operatorname{erfi} x}$

4. $f(x) = \operatorname{erfi}\left(\sqrt{x}\right)$

5. $f(x) = \operatorname{erf}(2x+3)$

6. $f(x) = x^2 + 4\operatorname{erfi}(3x)$

7. $f(x) = x^2 \operatorname{erfi}(3x)$

8. $g(x) = \ln\left(3 + \sqrt{\pi}\operatorname{erfi} x\right)$

9. $h(x) = \operatorname{erf} x \operatorname{erfi} x$

10. $f(x) = (\operatorname{erf} x)^2$

11. $k(x) = \operatorname{erf}^{-1}(4x) + 5x + 3$

12. $f(x) = \operatorname{Si}(x)\operatorname{erfi} x$

In Exercises 13 – 31, evaluate the indefinite integral. Write your answer in terms of error functions $\operatorname{erf} x$ **and/or** $\operatorname{erfi} x$ **if necessary.**

13. $\displaystyle\int e^{-4x^2}\,dx$

14. $\displaystyle\int e^{5x^2}\,dx$

15. $\displaystyle\int e^{-ax^2}\,dx$

16. $\displaystyle\int e^{ax^2}\,dx$

17. $\displaystyle\int \operatorname{erfi} x\,dx$

18. $\displaystyle\int xe^{-x^4}\,dx$

19. $\displaystyle\int e^{-4x^2-20x-25}\,dx$

20. $\displaystyle\int e^{x^2} + e^{6x}\,dx$

21. $\displaystyle\int e^{x^2}e^{6x}\,dx$

22. $\displaystyle\int xe^{x^2}\,dx$

23. $\displaystyle\int x^2e^{x^2}\,dx$ (Hint: Use integration by parts with $u = x$, $dv = xe^{x^2}\,dx$.)

24. $\displaystyle\int \frac{e^{x^2}}{x^2}\,dx$

25. $\displaystyle\int \frac{e^{x^2}}{\operatorname{erfi} x}\,dx$

26. $\displaystyle\int \frac{\operatorname{erf} x}{e^{x^2}}\,dx$

27. $\displaystyle\int xe^{-x^2+4x}\,dx$

28. $\displaystyle\int 3^{x^2}\,dx$ (Hint: Rewrite the integrand in terms of base e.)

29. $\displaystyle\int x^{\ln x}\,dx$

30. $\displaystyle\int \frac{1}{\sqrt{\pi}\ln x}\,dx$

31. $\displaystyle\int \sqrt{x}e^{-x^3}\,dx$

In Exercises 32 – 38,evaluate the definite integral. Write your answer in terms of error functions $\operatorname{erf} x$ **and/or** $\operatorname{erfi} x$ **if necessary.**

32. $\displaystyle\int_1^{16} \sqrt{x}e^{-x^3}\,dx$

33. $\displaystyle\int_1^{10} x^2e^{-x^6}\,dx$

34. $\displaystyle\int_0^{1} e^{x^2-2x}\,dx$

35. $\displaystyle\int_0^{\pi/2} e^{-\sin^2 x}\cos x\,dx$

36. $\displaystyle\int_0^{2} \frac{e^{-x^2}}{\sqrt{\pi}\operatorname{erf} x}\,dx$

37. $\displaystyle\int_0^{3} 8\sinh\left(4x^2\right)\,dx$

38. $\displaystyle\int_0^{\infty} \frac{e^{-x}}{\sqrt{x}}\,dx$

In Exercises 39 – 43, evaluate the given limit, using l'Hôpital's Rule if necessary.

39. $\displaystyle\lim_{x\to 0} \frac{\operatorname{erf} x}{x}$

40. $\displaystyle\lim_{x\to 0} \frac{\operatorname{erf} x}{x+3}$

41. $\displaystyle\lim_{x\to 0^+} \frac{\operatorname{erf} x}{\ln(x-1)}$

42. $\displaystyle\lim_{x\to\infty} \frac{\operatorname{erfi}(2x)}{\operatorname{erfi}(x)}$

43. $\displaystyle\lim_{x\to 1^-} \operatorname{erf}^{-1} x$

In Exercises 44 – 45, evaluate the integral involving inverse error functions.

44. $\displaystyle\int e^{3\operatorname{erfi}^{-1}(x)^2}\,dx$

45. $\displaystyle\int_0^1 \operatorname{erf}^{-1} x\,dx$

46. Describe the x-values at which $\operatorname{Si}(x)$ has relative maxima. Describe the x-values at which $\operatorname{Si}(x)$ has relative minima.

In Exercises 47 – 51, evaluate the integral, whose solution is in terms of the Sine Integral function $\operatorname{Si}(x)$.

47. $\displaystyle\int \frac{\sin(7x)}{x}\,dx$

48. $\displaystyle\int \sin(e^x)\,dx$

49. $\displaystyle\int \operatorname{Si}(x)\,dx$

50. $\displaystyle\int \cos\left(\frac{1}{x}\right)\,dx$

51. $\displaystyle\int_0^1 \frac{\sin(ax)}{x}\,dx$

7: Applications of Integration

We begin this chapter with a reminder of a few key concepts from Chapter 5. Let f be a continuous function on $[a, b]$ which is partitioned into n equally spaced subintervals as

$$a < x_1 < x_2 < \cdots < x_n < x_{n+1} = b.$$

Let $\Delta x = (b-a)/n$ denote the length of the subintervals, and let c_i be any x-value in the i^{th} subinterval. Definition 21 states that the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x$$

is a *Riemann Sum.* Riemann Sums are often used to approximate some quantity (area, volume, work, pressure, etc.). The *approximation* becomes *exact* by taking the limit

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x.$$

Theorem 40 connects limits of Riemann Sums to definite integrals:

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)\,dx.$$

Finally, the Fundamental Theorem of Calculus states how definite integrals can be evaluated using antiderivatives.

This chapter employs the following technique to a variety of applications. Suppose the value Q of a quantity is to be calculated. We first approximate the value of Q using a Riemann Sum, then find the exact value via a definite integral. We spell out this technique in the following Key Idea.

Key Idea 23 Application of Definite Integrals Strategy

Let a quantity be given whose value Q is to be computed.

1. Divide the quantity into n smaller "subquantities" of value Q_i.
2. Identify a variable x and function $f(x)$ such that each subquantity can be approximated with the product $f(c_i)\Delta x$, where Δx represents a small change in x. Thus $Q_i \approx f(c_i)\Delta x$. A sample approximation $f(c_i)\Delta x$ of Q_i is called a *differential element*.
3. Recognize that $Q = \sum_{i=1}^{n} Q_i \approx \sum_{i=1}^{n} f(c_i)\Delta x$, which is a Riemann Sum.
4. Taking the appropriate limit gives $Q = \int_a^b f(x)\,dx$

This Key Idea will make more sense after we have had a chance to use it several times. We begin with Area Between Curves, which we addressed briefly in Section 5.5.4.

7.1 Area Between Curves

We are often interested in knowing the area of a region. Forget momentarily that we addressed this already in Section 5.4 and approach it instead using the technique described in Key Idea 23.

Let Q be the area of a region bounded by continuous functions f and g. If we break the region into many subregions, we have an obvious equation:

Total Area = sum of the areas of the subregions.

The issue to address next is how to systematically break a region into subregions. A graph will help. Consider Figure 7.1 (a) where a region between two curves is shaded. While there are many ways to break this into subregions, one particularly efficient way is to "slice" it vertically, as shown in Figure 7.1 (b), into n equally spaced slices.

We now approximate the area of a slice. Again, we have many options, but using a rectangle seems simplest. Picking any x-value c_i in the i^{th} slice, we set the height of the rectangle to be $f(c_i) - g(c_i)$, the difference of the corresponding y-values. The width of the rectangle is a small difference in x-values, which we represent with Δx. Figure 7.1 (c) shows sample points c_i chosen in each subinterval and appropriate rectangles drawn. (Each of these rectangles represents a differential element.) Each slice has an area approximately equal to $\big(f(c_i) - g(c_i)\big)\Delta x$; hence, the total area is approximately the Riemann Sum

$$Q = \sum_{i=1}^{n} \big(f(c_i) - g(c_i)\big)\Delta x.$$

Taking the limit as $n \to \infty$ gives the exact area as $\int_a^b \big(f(x) - g(x)\big)\,dx$.

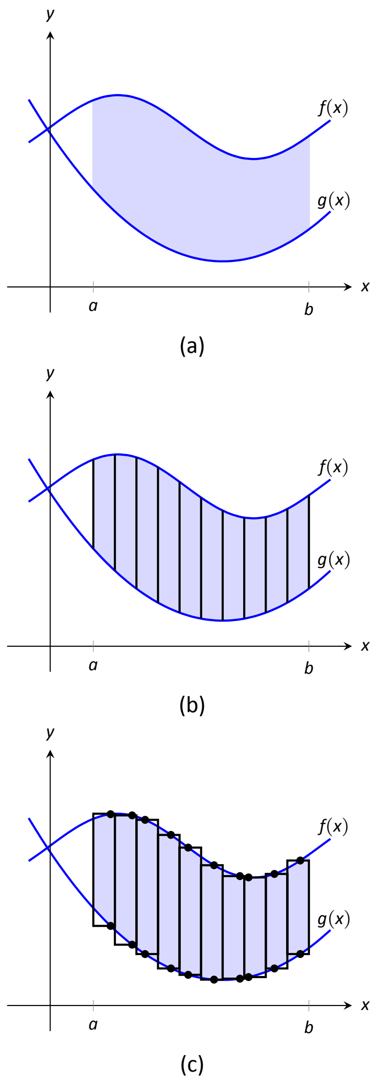


Figure 7.1: Subdividing a region into vertical slices and approximating the areas with rectangles.

Theorem 55 Area Between Curves (restatement of Theorem 43)

Let $f(x)$ and $g(x)$ be continuous functions defined on $[a, b]$ where $f(x) \geq g(x)$ for all x in $[a, b]$. The area of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ is

$$\int_a^b \big(f(x) - g(x)\big)\,dx.$$

Example 7.1 Finding area enclosed by curves

Find the area of the region bounded by $f(x) = \sin x + 2$, $g(x) = \frac{1}{2}\cos(2x) - 1$, $x = 0$ and $x = 4\pi$, as shown in Figure 7.2.

Solution The graph verifies that the upper boundary of the region is

Notes:

given by f and the lower bound is given by g. Therefore the area of the region is the value of the integral

$$\begin{aligned}\int_0^{4\pi}\big(f(x)-g(x)\big)\,dx &= \int_0^{4\pi}\left(\sin x + 2 - \left(\frac{1}{2}\cos(2x)-1\right)\right)\,dx\\ &= -\cos x - \frac{1}{4}\sin(2x) + 3x\Big|_0^{4\pi}\\ &= 12\pi \approx 37.7 \text{ units}^2.\end{aligned}$$

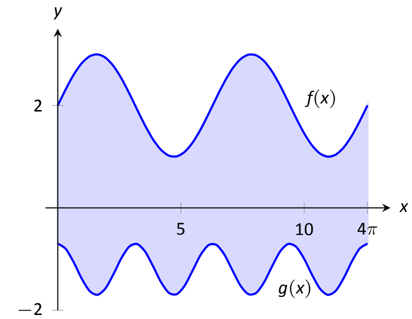


Figure 7.2: Graphing an enclosed region in Example 7.1.

Example 7.2 Finding total area enclosed by curves

Find the total area of the region enclosed by the functions $f(x) = -2x+5$ and $g(x) = x^3 - 7x^2 + 12x - 3$ as shown in Figure 7.3.

SOLUTION A quick calculation shows that $f = g$ at $x = 1, 2$ and 4. One can proceed thoughtlessly by computing $\int_1^4 \big(f(x)-g(x)\big)\,dx$, but this ignores the fact that on $[1,2]$, $g(x) > f(x)$. (In fact, the thoughtless integration returns $-9/4$, hardly the expected value of an *area*.) Thus we compute the total area by breaking the interval $[1,4]$ into two subintervals, $[1,2]$ and $[2,4]$ and using the proper integrand in each.

$$\begin{aligned}\text{Total Area} &= \int_1^2\big(g(x)-f(x)\big)\,dx + \int_2^4\big(f(x)-g(x)\big)\,dx\\ &= \int_1^2\big(x^3-7x^2+14x-8\big)\,dx + \int_2^4\big(-x^3+7x^2-14x+8\big)\,dx\\ &= 5/12 + 8/3\\ &= 37/12 = 3.083 \text{ units}^2.\end{aligned}$$

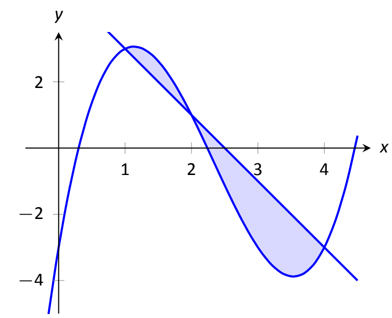


Figure 7.3: Graphing a region enclosed by two functions in Example 7.2.

The previous example makes note that we are expecting area to be *positive*. When first learning about the definite integral, we interpreted it as "signed area under the curve," allowing for "negative area." That doesn't apply here; area is to be positive.

The previous example also demonstrates that we often have to break a given region into subregions before applying Theorem 55. The following example shows another situation where this is applicable, along with an alternate view of applying the Theorem.

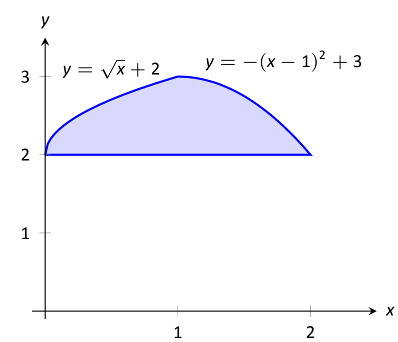


Figure 7.4: Graphing a region for Example 7.3.

Example 7.3 Finding area: integrating with respect to y

Find the area of the region enclosed by the functions $y = \sqrt{x}+2$, $y = -(x-1)^2+3$ and $y = 2$, as shown in Figure 7.4.

Notes:

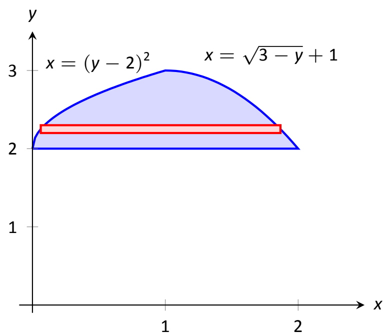


Figure 7.5: The region used in Example 7.3 with boundaries relabeled as functions of y.

Solution We give two approaches to this problem. In the first approach, we notice that the region's "top" is defined by two different curves. On $[0,1]$, the top function is $y = \sqrt{x}+2$; on $[1,2]$, the top function is $y = -(x-1)^2+3$. Thus we compute the area as the sum of two integrals:

$$\begin{aligned}\text{Total Area} &= \int_0^1 \Big(\big(\sqrt{x}+2\big)-2\Big)\,dx + \int_1^2 \Big(\big(-(x-1)^2+3\big)-2\Big)\,dx\\ &= 2/3+2/3\\ &= 4/3.\end{aligned}$$

The second approach is clever and very useful in certain situations. We are used to viewing curves as functions of x; we input an x-value and a y-value is returned. Some curves can also be described as functions of y: input a y-value and an x-value is returned. We can rewrite the equations describing the boundary by solving for x:

$$\begin{aligned} y &= \sqrt{x}+2 &\Rightarrow\quad x &= (y-2)^2\\ y &= -(x-1)^2+3 &\Rightarrow\quad x &= \sqrt{3-y}+1.\end{aligned}$$

Figure 7.5 shows the region with the boundaries relabeled. A differential element, a horizontal rectangle, is also pictured. The width of the rectangle is a small change in y: Δy. The height of the rectangle is a difference in x-values. The "top" x-value is the largest value, i.e., the rightmost. The "bottom" x-value is the smaller, i.e., the leftmost. Therefore the height of the rectangle is

$$\big(\sqrt{3-y}+1\big)-(y-2)^2.$$

The area is found by integrating the above function with respect to y with the appropriate bounds. We determine these by considering the y-values the region occupies. It is bounded below by $y=2$, and bounded above by $y=3$. That is, both the "top" and "bottom" functions exist on the y interval $[2,3]$. Thus

$$\begin{aligned}\text{Total Area} &= \int_2^3 \big(\sqrt{3-y}+1-(y-2)^2\big)\,dy\\ &= \left(-\frac{2}{3}(3-y)^{3/2}+y-\frac{1}{3}(y-2)^3\right)\Bigg|_2^3\\ &= 4/3.\end{aligned}$$

This calculus–based technique of finding area can be useful even with shapes that we normally think of as "easy." Example 7.4 computes the area of a triangle. While the formula "$\frac{1}{2}\times$ base $\times$ height" is well known, in arbitrary triangles it can be nontrivial to compute the height. Calculus makes the problem simple.

Notes:

Example 7.4 Finding the area of a triangle

Compute the area of the regions bounded by the lines $y = x + 1$, $y = -2x + 7$ and $y = -\frac{1}{2}x + \frac{5}{2}$, as shown in Figure 7.6.

Solution Recognize that there are two "top" functions to this region, causing us to use two definite integrals.

$$\begin{aligned}\text{Total Area} &= \int_1^2 \left((x+1) - \left(-\frac{1}{2}x + \frac{5}{2}\right)\right)\,dx + \int_2^3 \left((-2x+7) - \left(-\frac{1}{2}x + \frac{5}{2}\right)\right)\,dx\\ &= 3/4 + 3/4\\ &= 3/2.\end{aligned}$$

We can also approach this by converting each function into a function of y. This also requires 2 integrals, so there isn't really any advantage to doing so. We do it here for demonstration purposes.

The "top" function is always $x = \frac{7-y}{2}$ while there are two "bottom" functions. Being mindful of the proper integration bounds, we have

$$\begin{aligned}\text{Total Area} &= \int_1^2 \left(\frac{7-y}{2} - (5-2y)\right)\,dy + \int_2^3 \left(\frac{7-y}{2} - (y-1)\right)\,dy\\ &= 3/4 + 3/4\\ &= 3/2.\end{aligned}$$

Of course, the final answer is the same. (It is interesting to note that the area of all 4 subregions used is 3/4. This is coincidental.)

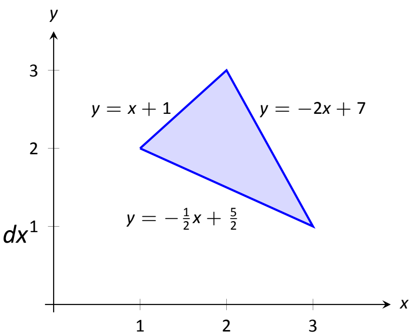


Figure 7.6: Graphing a triangular region in Example 7.4.

While we have focused on producing exact answers, we are also able to make approximations using the principle of Theorem 55. The integrand in the theorem is a distance ("top minus bottom"); integrating this distance function gives an area. By taking discrete measurements of distance, we can approximate an area using numerical integration techniques developed in Section 5.6. The following example demonstrates this.

Example 7.5 Numerically approximating area

To approximate the area of a lake, shown in Figure 7.7 (a), the "length" of the lake is measured at 200-foot increments as shown in Figure 7.7 (b), where the lengths are given in hundreds of feet. Approximate the area of the lake.

Solution The measurements of length can be viewed as measuring "top minus bottom" of two functions. The exact answer is found by integrating $\int_0^{12} \big(f(x) - g(x)\big)\,dx$, but of course we don't know the functions f and g. Our discrete measurements instead allow us to approximate.

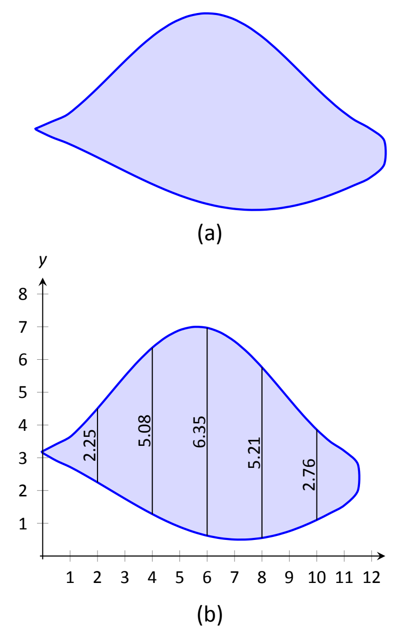


Figure 7.7: (a) A sketch of a lake, and (b) the lake with length measurements.

Notes:

We have the following data points:

$$(0,0),\ (2,2.25),\ (4,5.08),\ (6,6.35),\ (8,5.21),\ (10,2.76),\ (12,0).$$

We also have that $\Delta x = \frac{b-a}{n} = 2$, so Simpson's Rule gives

$$\begin{aligned}\text{Area} &\approx \frac{2}{3}\Big(1\cdot 0 + 4\cdot 2.25 + 2\cdot 5.08 + 4\cdot 6.35 + 2\cdot 5.21 + 4\cdot 2.76 + 1\cdot 0\Big)\\ &= 44.01\overline{3}\ \text{units}^2.\end{aligned}$$

Since the measurements are in hundreds of feet, $\text{units}^2 = (100\ \text{ft})^2 = 10,000\ \text{ft}^2$, giving a total area of $440,133\ \text{ft}^2$. (Since we are approximating, we'd likely say the area was about $440,000\ \text{ft}^2$, which is a little more than 10 acres.)

In the next section we apply our applications–of–integration techniques to finding the volumes of certain solids.

Notes:

Exercises 7.1

Terms and Concepts

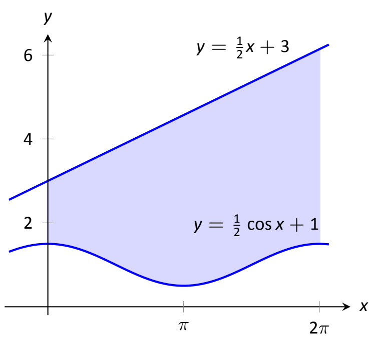
1. T/F: The area between curves is always positive.

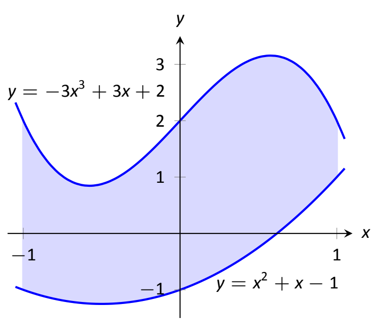
2. T/F: Calculus can be used to find the area of basic geometric shapes.

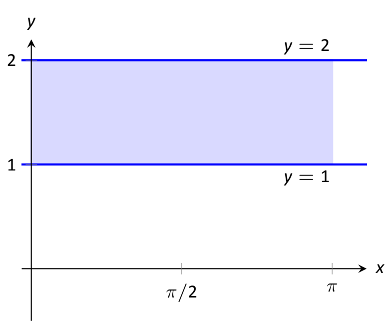
3. In your own words, describe how to find the total area enclosed by $y = f(x)$ and $y = g(x)$.

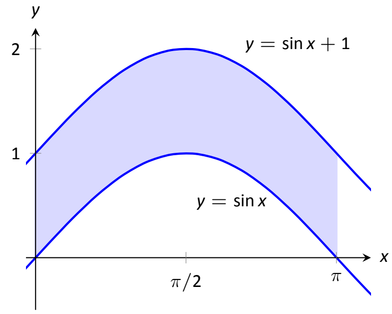
Problems

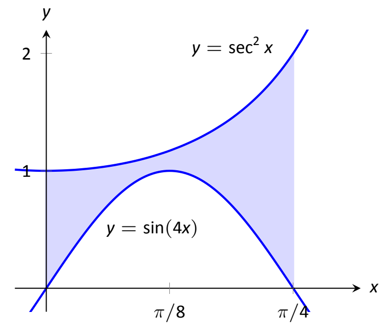
In Exercises 4 – 10, find the area of the shaded region in the given graph.

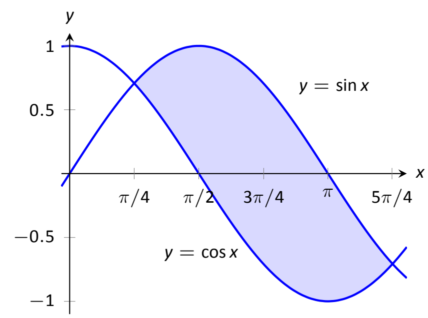
4. 

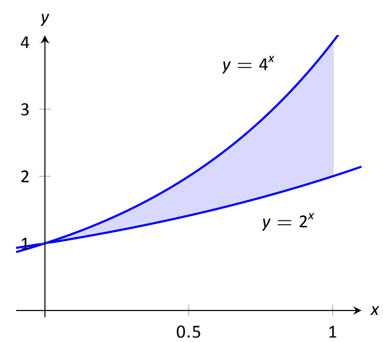
5. 

6. 

7. 

8. 

9. 

10. 

In Exercises 11 – 16, find the total area enclosed by the functions f and g.

11. $f(x) = 2x^2 + 5x - 3$, $g(x) = x^2 + 4x - 1$

12. $f(x) = x^2 - 3x + 2$, $g(x) = -3x + 3$

13. $f(x) = \sin x$, $g(x) = 2x/\pi$

14. $f(x) = x^3 - 4x^2 + x - 1$, $g(x) = -x^2 + 2x - 4$

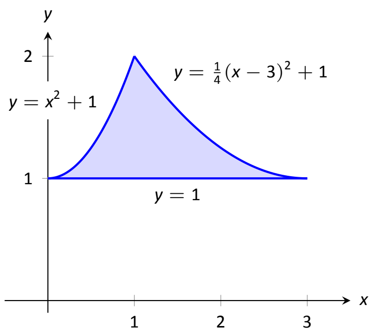
15. $f(x) = x$, $g(x) = \sqrt{x}$

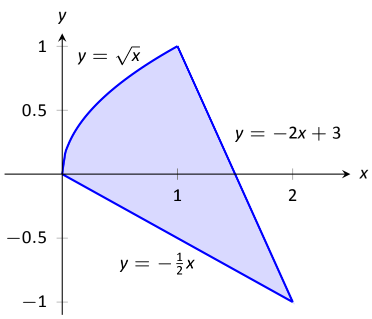
16. $f(x) = -x^3 + 5x^2 + 2x + 1$, $g(x) = 3x^2 + x + 3$

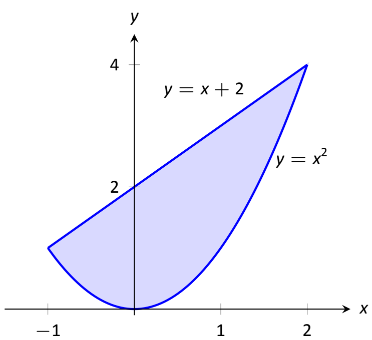
17. The functions $f(x) = \cos(2x)$ and $g(x) = \sin x$ intersect infinitely many times, forming an infinite number of repeated, enclosed regions. Find the areas of these regions.

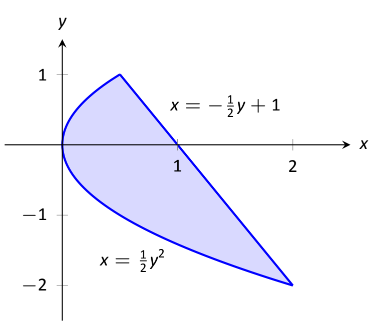
In Exercises 18 – 22, find the area of the enclosed region in two ways:

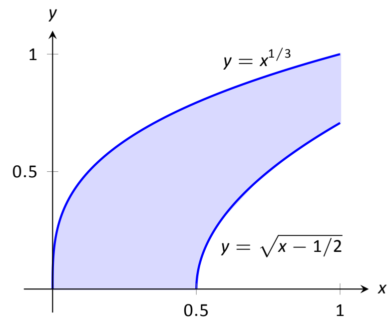
1. **by treating the boundaries as functions of x, and**
2. **by treating the boundaries as functions of y.**

18. 

19. 

20. 

21. 

22. 

In Exercises 23 – 26, find the area of the triangle formed by the given three points.

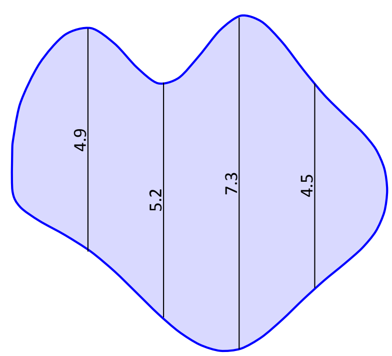
23. $(1, 1)$, $(2, 3)$, and $(3, 3)$

24. $(-1, 1)$, $(1, 3)$, and $(2, -1)$

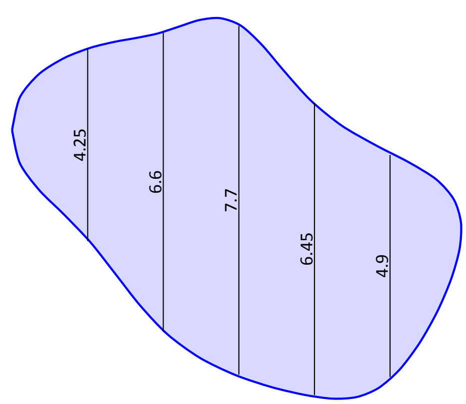
25. $(1, 1)$, $(3, 3)$, and $(3, 3)$

26. $(0, 0)$, $(2, 5)$, and $(5, 2)$

27. Use the Trapezoidal Rule to approximate the area of the pictured lake whose lengths, in hundreds of feet, are measured in 100-foot increments.



28. Use Simpson's Rule to approximate the area of the pictured lake whose lengths, in hundreds of feet, are measured in 200-foot increments.



7.2 Volume by Cross-Sectional Area; Disk and Washer Methods

The volume of a general right cylinder, as shown in Figure 7.8, is

Area of the base $\times$ height.

We can use this fact as the building block in finding volumes of a variety of shapes.

Given an arbitrary solid, we can *approximate* its volume by cutting it into n thin slices. When the slices are thin, each slice can be approximated well by a general right cylinder. Thus the volume of each slice is approximately its cross-sectional area $\times$ thickness. (These slices are the differential elements.)

By orienting a solid along the x-axis, we can let $A(x_i)$ represent the cross-sectional area of the i^{th} slice, and let Δx_i represent the thickness of this slice (the thickness is a small change in x). The total volume of the solid is approximately:

$$\begin{aligned}\text{Volume} &\approx \sum_{i=1}^{n} \Big[\text{Area } \times \text{ thickness}\Big] \\ &= \sum_{i=1}^{n} A(x_i)\Delta x_i.\end{aligned}$$

Recognize that this is a Riemann Sum. By taking a limit (as the thickness of the slices goes to 0) we can find the volume exactly.

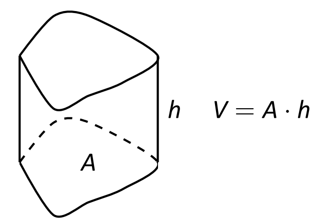


Figure 7.8: The volume of a general right cylinder

Theorem 56 Volume By Cross-Sectional Area

The volume V of a solid, oriented along the x-axis with cross-sectional area $A(x)$ from $x = a$ to $x = b$, is

$$V = \int_a^b A(x)\, dx.$$

Example 7.6 Finding the volume of a solid

Find the volume of a pyramid with a square base of side length 10 in and a height of 5 in.

Solution There are many ways to "orient" the pyramid along the x-axis; Figure 7.9 gives one such way, with the pointed top of the pyramid at the origin and the x-axis going through the center of the base.

Each cross section of the pyramid is a square; this is a sample differential element. To determine its area $A(x)$, we need to determine the side lengths of

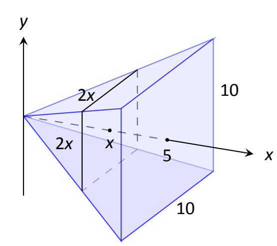


Figure 7.9: Orienting a pyramid along the x-axis in Example 7.6.

Notes:

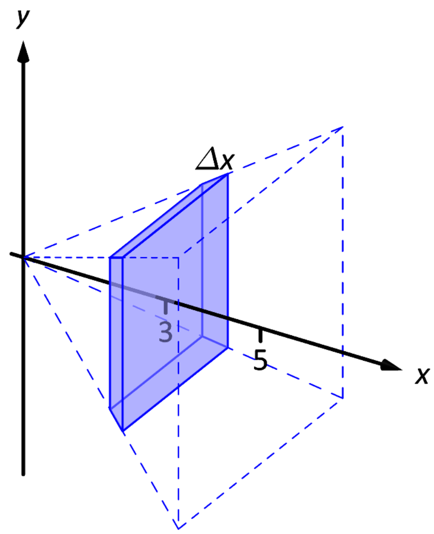


Figure 7.10: Cutting a slice in the pyramid in Example 7.6 at $x = 3$.

the square.

When $x = 5$, the square has side length 10; when $x = 0$, the square has side length 0. Since the edges of the pyramid are lines, it is easy to figure that each cross-sectional square has side length $2x$, giving $A(x) = (2x)^2 = 4x^2$.

If one were to cut a slice out of the pyramid at $x = 3$, as shown in Figure 7.10, one would have a shape with square bottom and top with sloped sides. If the slice were thin, both the bottom and top squares would have sides lengths of about 6, and thus the cross–sectional area of the bottom and top would be about 36in^2. Letting Δx_i represent the thickness of the slice, the volume of this slice would then be about $36\Delta x_i$in^3.

Cutting the pyramid into n slices divides the total volume into n equally–spaced smaller pieces, each with volume $(2x_i)^2\Delta x$, where x_i is the approximate location of the slice along the x-axis and Δx represents the thickness of each slice. One can approximate total volume of the pyramid by summing up the volumes of these slices:

$$\text{Approximate volume } = \sum_{i=1}^{n} (2x_i)^2 \Delta x.$$

Taking the limit as $n \to \infty$ gives the actual volume of the pyramid; recoginizing this sum as a Riemann Sum allows us to find the exact answer using a definite integral, matching the definite integral given by Theorem 56.

We have

$$\begin{aligned} V &= \lim_{n\to\infty} \sum_{i=1}^{n} (2x_i)^2 \Delta x \\ &= \int_0^5 4x^2\, dx \\ &= \frac{4}{3}x^3\Big|_0^5 \\ &= \frac{500}{3}\text{ in}^3 \approx 166.67\text{ in}^3. \end{aligned}$$

We can check our work by consulting the general equation for the volume of a pyramid (see the back cover under "Volume of A General Cone"):

$$\tfrac{1}{3} \times \text{area of base} \times \text{height}.$$

Certainly, using this formula from geometry is faster than our new method, but the calculus–based method can be applied to much more than just cones.

An important special case of Theorem 56 is when the solid is a **solid of revolution**, that is, when the solid is formed by rotating a shape around an axis.

Start with a function $y = f(x)$ from $x = a$ to $x = b$. Revolving this curve about a horizontal axis creates a three-dimensional solid whose cross sections

Notes:

are disks (thin circles). Let $R(x)$ represent the radius of the cross-sectional disk at x; the area of this disk is $\pi R(x)^2$. Applying Theorem 56 gives the Disk Method.

Key Idea 24 The Disk Method

Let a solid be formed by revolving the curve $y = f(x)$ from $x = a$ to $x = b$ around a horizontal axis, and let $R(x)$ be the radius of the cross-sectional disk at x. The volume of the solid is

$$V = \pi \int_a^b R(x)^2\, dx.$$

Example 7.7 Finding volume using the Disk Method

Find the volume of the solid formed by revolving the curve $y = 1/x$, from $x = 1$ to $x = 2$, around the x-axis.

Solution A sketch can help us understand this problem. In Figure 7.11(a) the curve $y = 1/x$ is sketched along with the differential element – a disk – at x with radius $R(x) = 1/x$. In Figure 7.11 (b) the whole solid is pictured, along with the differential element.

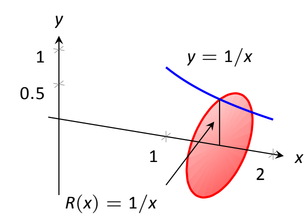
The volume of the differential element shown in part (a) of the figure is approximately $\pi R(x_i)^2 \Delta x$, where $R(x_i)$ is the radius of the disk shown and Δx is the thickness of that slice. The radius $R(x_i)$ is the distance from the x-axis to the curve, hence $R(x_i) = 1/x_i$.

Slicing the solid into n equally–spaced slices, we can approximate the total volume by adding up the approximate volume of each slice:

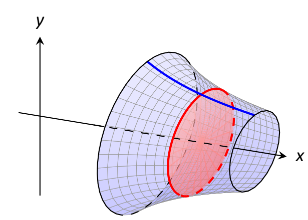
$$\text{Approximate volume } = \sum_{i=1}^{n} \pi \left(\frac{1}{x_i}\right)^2 \Delta x.$$

Taking the limit of the above sum as $n \to \infty$ gives the actual volume; recognizing this sum as a Riemann sum allows us to evaluate the limit with a definite integral, which matches the formula given in Key Idea 24:

$$\begin{aligned} V &= \lim_{n\to\infty} \sum_{i=1}^{n} \pi \left(\frac{1}{x_i}\right)^2 \Delta x \\ &= \pi \int_1^2 \left(\frac{1}{x}\right)^2 dx \\ &= \pi \int_1^2 \frac{1}{x^2}\, dx \end{aligned}$$



(a)



(b)

Figure 7.11: Sketching a solid in Example 7.7.

Notes:

$$
\begin{aligned}
&= \pi \left[-\frac{1}{x} \right] \Big|_1^2 \\
&= \pi \left[-\frac{1}{2} - (-1) \right] \\
&= \frac{\pi}{2} \text{ units}^3.
\end{aligned}
$$

While Key Idea 24 is given in terms of functions of x, the principle involved can be applied to functions of y when the axis of rotation is vertical, not horizontal. We demonstrate this in the next example.

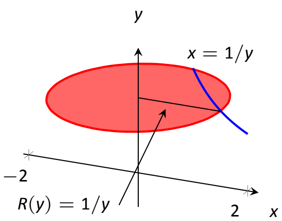
Example 7.8 Finding volume using the Disk Method

Find the volume of the solid formed by revolving the curve $y = 1/x$, from $x = 1$ to $x = 2$, about the y-axis.

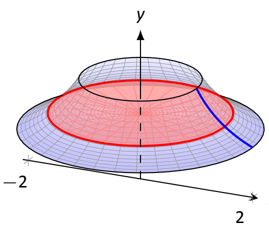
SOLUTION Since the axis of rotation is vertical, we need to convert the function into a function of y and convert the x-bounds to y-bounds. Since $y = 1/x$ defines the curve, we rewrite it as $x = 1/y$. The bound $x = 1$ corresponds to the y-bound $y = 1$, and the bound $x = 2$ corresponds to the y-bound $y = 1/2$.

Thus we are rotating the curve $x = 1/y$, from $y = 1/2$ to $y = 1$ about the y-axis to form a solid. The curve and sample differential element are sketched in Figure 7.12 (a), with a full sketch of the solid in Figure 7.12 (b). We integrate to find the volume:

$$
\begin{aligned}
V &= \pi \int_{1/2}^{1} \frac{1}{y^2}\, dy \\
&= -\frac{\pi}{y}\Big|_{1/2}^{1} \\
&= \pi \text{ units}^3.
\end{aligned}
$$



(a)



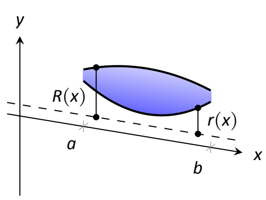
(b)

Figure 7.12: Sketching a solid in Example 7.8.

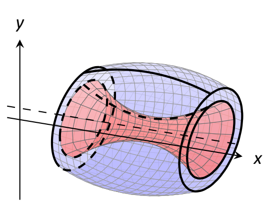
We can also compute the volume of solids of revolution that have a hole in the center. The general principle is simple: compute the volume of the solid irrespective of the hole, then subtract the volume of the hole. If the outside radius of the solid is $R(x)$ and the inside radius (defining the hole) is $r(x)$, then the volume is

$$
V = \pi \int_a^b R(x)^2\, dx - \pi \int_a^b r(x)^2\, dx = \pi \int_a^b \left(R(x)^2 - r(x)^2 \right)\, dx.
$$

One can generate a solid of revolution with a hole in the middle by revolving a region about an axis. Consider Figure 7.13(a), where a region is sketched along



(a)



(b)

Figure 7.13: Establishing the Washer Method; see also Figure 7.14.

Notes:

with a dashed, horizontal axis of rotation. By rotating the region about the axis, a solid is formed as sketched in Figure 7.13(b). The outside of the solid has radius $R(x)$, whereas the inside has radius $r(x)$. Each cross section of this solid will be a washer (a disk with a hole in the center) as sketched in Figure 7.14(c). This leads us to the Washer Method.

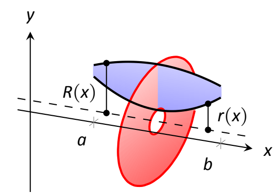


Figure 7.14: Establishing the Washer Method; see also Figure 7.13.

Key Idea 25 The Washer Method

Let a region bounded by $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$ be rotated about a horizontal axis that does not intersect the region, forming a solid. Each cross section at x will be a washer with outside radius $R(x)$ and inside radius $r(x)$. The volume of the solid is

$$V = \pi \int_a^b \left(R(x)^2 - r(x)^2 \right)\, dx.$$

Even though we introduced it first, the Disk Method is just a special case of the Washer Method with an inside radius of $r(x) = 0$.

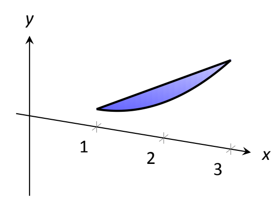
Example 7.9 Finding volume with the Washer Method

Find the volume of the solid formed by rotating the region bounded by $y = x^2 - 2x + 2$ and $y = 2x - 1$ about the x-axis.

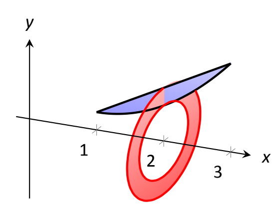
Solution A sketch of the region will help, as given in Figure 7.15(a). Rotating about the x-axis will produce cross sections in the shape of washers, as shown in Figure 7.15(b); the complete solid is shown in part (c). The outside radius of this washer is $R(x) = 2x + 1$; the inside radius is $r(x) = x^2 - 2x + 2$. As the region is bounded from $x = 1$ to $x = 3$, we integrate as follows to compute the volume.

$$\begin{aligned} V &= \pi \int_1^3 \left((2x-1)^2 - (x^2 - 2x + 2)^2 \right) dx \\ &= \pi \int_1^3 \left(-x^4 + 4x^3 - 4x^2 + 4x - 3 \right) dx \\ &= \pi \left[-\frac{1}{5}x^5 + x^4 - \frac{4}{3}x^3 + 2x^2 - 3x \right]\Bigg|_1^3 \\ &= \frac{104}{15}\pi \approx 21.78 \text{ units}^3. \end{aligned}$$

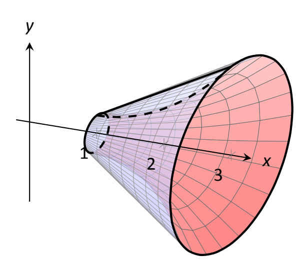
When rotating about a vertical axis, the outside and inside radius functions must be functions of y.



(a)



(b)



(c)

Figure 7.15: Sketching the differential element and solid in Example 7.9.

Notes:

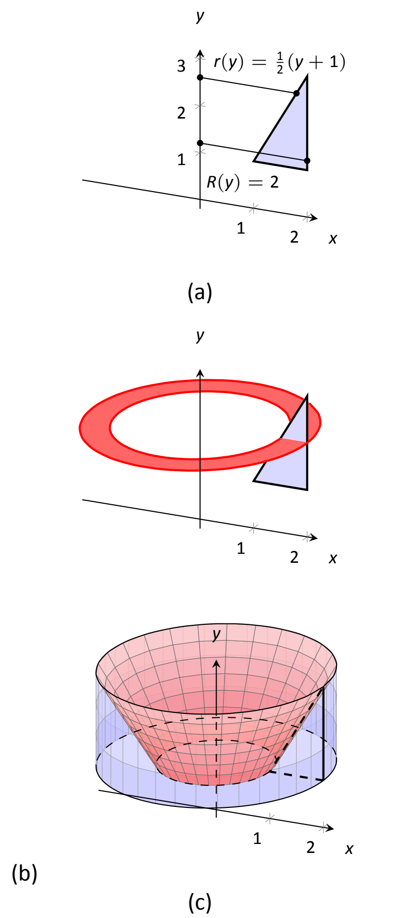


Figure 7.16: Sketching the solid in Example 7.10.

Example 7.10 Finding volume with the Washer Method

Find the volume of the solid formed by rotating the triangular region with vertices at $(1, 1)$, $(2, 1)$ and $(2, 3)$ about the y-axis.

Solution The triangular region is sketched in Figure 7.16(a); the differential element is sketched in (b) and the full solid is drawn in (c). They help us establish the outside and inside radii. Since the axis of rotation is vertical, each radius is a function of y.

The outside radius $R(y)$ is formed by the line connecting $(2, 1)$ and $(2, 3)$; it is a constant function, as regardless of the y-value the distance from the line to the axis of rotation is 2. Thus $R(y) = 2$.

The inside radius is formed by the line connecting $(1, 1)$ and $(2, 3)$. The equation of this line is $y = 2x - 1$, but we need to refer to it as a function of y. Solving for x gives $r(y) = \frac{1}{2}(y + 1)$.

We integrate over the y-bounds of $y = 1$ to $y = 3$. Thus the volume is

$$\begin{aligned}
V &= \pi \int_1^3 \left(2^2 - \left(\frac{1}{2}(y+1)\right)^2\right) dy \\
&= \pi \int_1^3 \left(-\frac{1}{4}y^2 - \frac{1}{2}y + \frac{15}{4}\right) dy \\
&= \pi \left[-\frac{1}{12}y^3 - \frac{1}{4}y^2 + \frac{15}{4}y\right]\Bigg|_1^3 \\
&= \frac{10}{3}\pi \approx 10.47 \text{ units}^3.
\end{aligned}$$

This section introduced a new application of the definite integral. Our default view of the definite integral is that it gives "the area under the curve." However, we can establish definite integrals that represent other quantities; in this section, we computed volume.

The ultimate goal of this section is not to compute volumes of solids. That can be useful, but what is more useful is the understanding of this basic principle of integral calculus, outlined in Key Idea 23: to find the exact value of some quantity,

- we start with an approximation (in this section, slice the solid and approximate the volume of each slice),
- then make the approximation better by refining our original approximation (i.e., use more slices),
- then use limits to establish a definite integral which gives the exact value.

We practice this principle in the next section where we find volumes by slicing solids in a different way.

Notes:

Exercises 7.2

Terms and Concepts

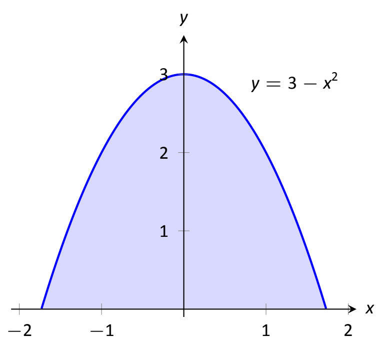
1. T/F: A solid of revolution is formed by revolving a shape around an axis.

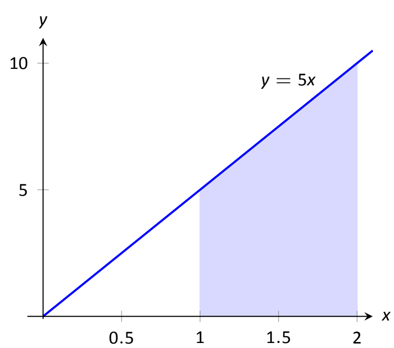
2. In your own words, explain how the Disk and Washer Methods are related.

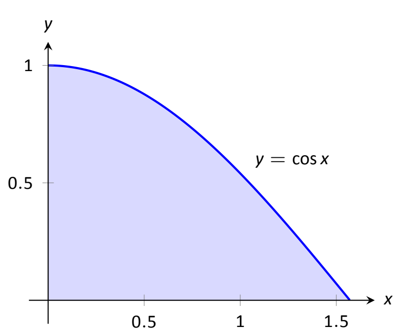
3. Explain the how the units of volume are found in the integral of Theorem 56: if $A(x)$ has units of in^2, how does $\int A(x)\,dx$ have units of in^3?

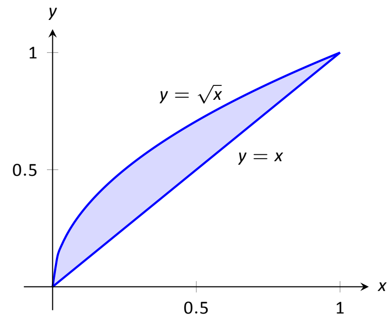
Problems

In Exercises 4 – 7, a region of the Cartesian plane is shaded. Use the Disk/Washer Method to find the volume of the solid of revolution formed by revolving the region about the x-axis.

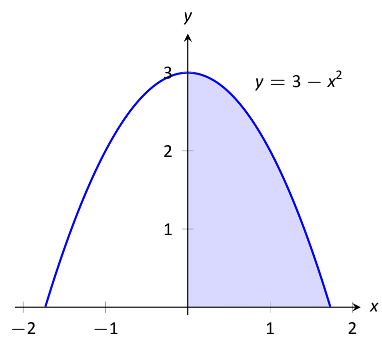
4. 

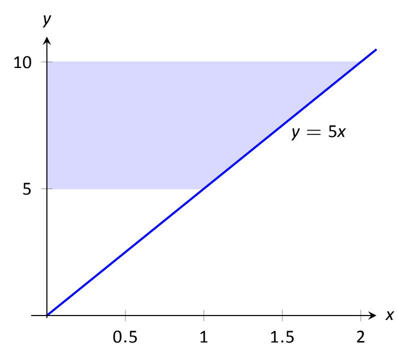
5. 

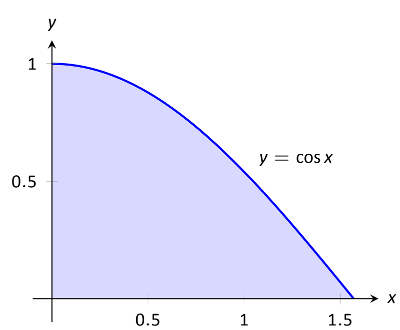
6. 

7. 

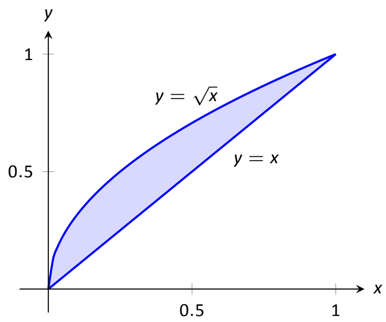
In Exercises 8 – 11, a region of the Cartesian plane is shaded. Use the Disk/Washer Method to find the volume of the solid of revolution formed by revolving the region about the y-axis.

8. 

9. 

10. 

(Hint: Integration By Parts will be necessary, twice. First let $u = \arccos^2 x$, then let $u = \arccos x$.)

11. 

In Exercises 12 – 17, a region of the Cartesian plane is described. Use the Disk/Washer Method to find the volume of the solid of revolution formed by rotating the region about each of the given axes.

12. Region bounded by: $y = \sqrt{x}$, $y = 0$ and $x = 1$.

    Rotate about:

    (a) the x-axis (c) the y-axis
    (b) $y = 1$ (d) $x = 1$

13. Region bounded by: $y = 4 - x^2$ and $y = 0$.

    Rotate about:

    (a) the x-axis (c) $y = -1$
    (b) $y = 4$ (d) $x = 2$

14. The triangle with vertices $(1, 1)$, $(1, 2)$ and $(2, 1)$.

    Rotate about:

    (a) the x-axis (c) the y-axis
    (b) $y = 2$ (d) $x = 1$

15. Region bounded by $y = x^2 - 2x + 2$ and $y = 2x - 1$.

    Rotate about:

    (a) the x-axis (c) $y = 5$
    (b) $y = 1$

16. Region bounded by $y = 1/\sqrt{x^2 + 1}$, $x = -1$, $x = 1$ and the x-axis.

    Rotate about:

    (a) the x-axis (c) $y = -1$
    (b) $y = 1$

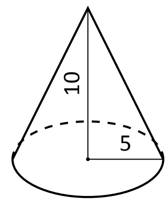
17. Region bounded by $y = 2x$, $y = x$ and $x = 2$.

    Rotate about:

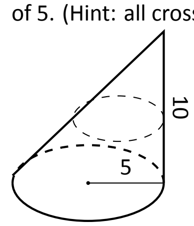
    (a) the x-axis (c) the y-axis
    (b) $y = 4$ (d) $x = 2$

In Exercises 18 – 21, a solid is described. Orient the solid along the x-axis such that a cross-sectional area function $A(x)$ can be obtained, then apply Theorem 56 to find the volume of the solid.

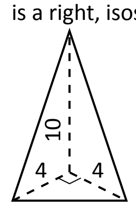
18. A right circular cone with height of 10 and base radius of 5.



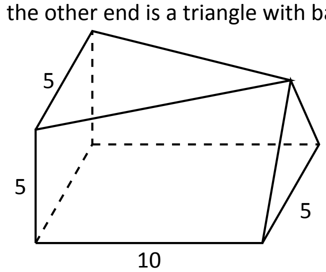
19. A skew right circular cone with height of 10 and base radius of 5. (Hint: all cross-sections are circles.)



20. A right triangular pyramid with height of 10 and whose base is a right, isosceles triangle with side length 4.



21. A solid with length 10 with a rectangular base and triangular top, wherein one end is a square with side length 5 and the other end is a triangle with base and height of 5.



22. The goal is to derive the formula for the volume V of a sphere with radius r.

    (a) A sphere with radius r can be formed by revolving what curve about which line?

    (b) Use your answer to (a) and the disk method to determine a formula for the volume of a sphere.

23. The goal is to derive the formula for the volume V of a right circular cone with height h and base radius r.

    (a) Such a cone can be formed by revolving what curve about which line?

    (b) Use your answer to (a) and the disk method to determine a formula for the volume.

7.3 The Shell Method

Often a given problem can be solved in more than one way. A particular method may be chosen out of convenience, personal preference, or perhaps necessity. Ultimately, it is good to have options.

The previous section introduced the Disk and Washer Methods, which computed the volume of solids of revolution by integrating the cross–sectional area of the solid. This section develops another method of computing volume, the **Shell Method.** Instead of slicing the solid perpendicular to the axis of rotation creating cross-sections, we now slice it parallel to the axis of rotation, creating "shells."

Consider Figure 7.17, where the region shown in (a) is rotated around the y-axis forming the solid shown in (b). A small slice of the region is drawn in (a), parallel to the axis of rotation. When the region is rotated, this thin slice forms a **cylindrical shell**, as pictured in part (c) of the figure. The previous section approximated a solid with lots of thin disks (or washers); we now approximate a solid with many thin cylindrical shells.

To compute the volume of one shell, first consider the paper label on a soup can with radius r and height h. What is the area of this label? A simple way of determining this is to cut the label and lay it out flat, forming a rectangle with height h and length $2\pi r$. Thus the area is $A = 2\pi rh$; see Figure 7.18 (a).

Do a similar process with a cylindrical shell, with height h, thickness Δx, and approximate radius r. Cutting the shell and laying it flat forms a rectangular solid with length $2\pi r$, height h and depth Δx. Thus the volume is $V \approx 2\pi rh\Delta x$; see Figure 7.18 (b). (We say "approximately" since our radius was an approximation.)

By breaking the solid into n cylindrical shells, we can approximate the volume of the solid as

$$V = \sum_{i=1}^{n} 2\pi r_i h_i \Delta x_i,$$

where r_i, h_i and Δx_i are the radius, height and thickness of the i^{th} shell, respectively.

This is a Riemann Sum. Taking a limit as the thickness of the shells approaches 0 leads to a definite integral.

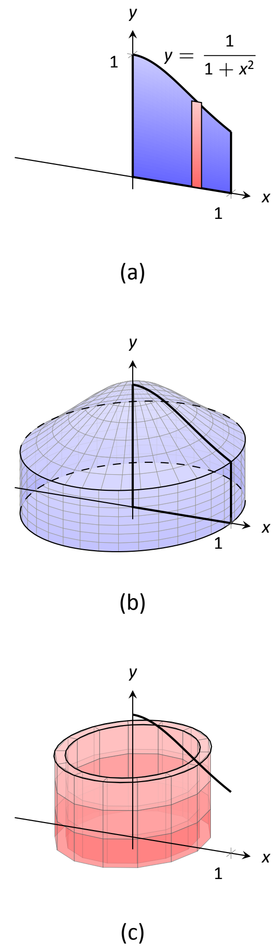


Figure 7.17: Introducing the Shell Method.

Notes:

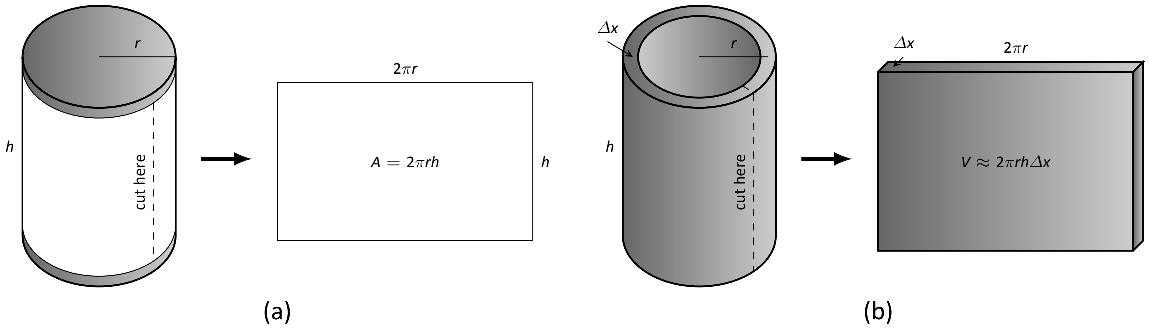


Figure 7.18: Determining the volume of a thin cylindrical shell.

Key Idea 26 The Shell Method

Let a solid be formed by revolving a region R, bounded by $x = a$ and $x = b$, around a vertical axis. Let $r(x)$ represent the distance from the axis of rotation to x (i.e., the radius of a sample shell) and let $h(x)$ represent the height of the solid at x (i.e., the height of the shell). The volume of the solid is

$$V = 2\pi \int_a^b r(x)h(x)\,dx.$$

Special Cases:

1. When the region R is bounded above by $y = f(x)$ and below by $y = g(x)$, then $h(x) = f(x) - g(x)$.
2. When the axis of rotation is the y-axis (i.e., $x = 0$) then $r(x) = x$.

Let's practice using the Shell Method.

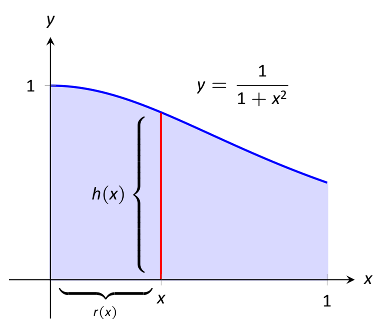


Figure 7.19: Graphing a region in Example 7.11.

Example 7.11 Finding volume using the Shell Method

Find the volume of the solid formed by rotating the region bounded by $y = 0$, $y = 1/(1 + x^2)$, $x = 0$ and $x = 1$ about the y-axis.

Solution This is the region used to introduce the Shell Method in Figure 7.17, but is sketched again in Figure 7.19 for closer reference. A line is drawn in the region parallel to the axis of rotation representing a shell that will be

Notes:

carved out as the region is rotated about the y-axis. (This is the differential element.)

The distance this line is from the axis of rotation determines $r(x)$; as the distance from x to the y-axis is x, we have $r(x) = x$. The height of this line determines $h(x)$; the top of the line is at $y = 1/(1+x^2)$, whereas the bottom of the line is at $y = 0$. Thus $h(x) = 1/(1+x^2) - 0 = 1/(1+x^2)$. The region is bounded from $x = 0$ to $x = 1$, so the volume is

$$V = 2\pi\int_0^1 \frac{x}{1+x^2}\,dx.$$

This requires substitution. Let $u = 1 + x^2$, so $du = 2x\,dx$. We also change the bounds: $u(0) = 1$ and $u(1) = 2$. Thus we have:

$$\begin{aligned} &= \pi\int_1^2 \frac{1}{u}\,du \\ &= \pi \ln u\Big|_1^2 \\ &= \pi \ln 2 \approx 2.178 \text{ units}^3. \end{aligned}$$

Note: in order to find this volume using the Disk Method, two integrals would be needed to account for the regions above and below $y = 1/2$.

With the Shell Method, nothing special needs to be accounted for to compute the volume of a solid that has a hole in the middle, as demonstrated next.

Example 7.12 Finding volume using the Shell Method

Find the volume of the solid formed by rotating the triangular region determined by the points $(0,1)$, $(1,1)$ and $(1,3)$ about the line $x = 3$.

Solution The region is sketched in Figure 7.20(a) along with the differential element, a line within the region parallel to the axis of rotation. In part (b) of the figure, we see the shell traced out by the differential element, and in part (c) the whole solid is shown.

The height of the differential element is the distance from $y = 1$ to $y = 2x + 1$, the line that connects the points $(0,1)$ and $(1,3)$. Thus $h(x) = 2x+1-1 = 2x$. The radius of the shell formed by the differential element is the distance from x to $x = 3$; that is, it is $r(x) = 3 - x$. The x-bounds of the region are $x = 0$ to

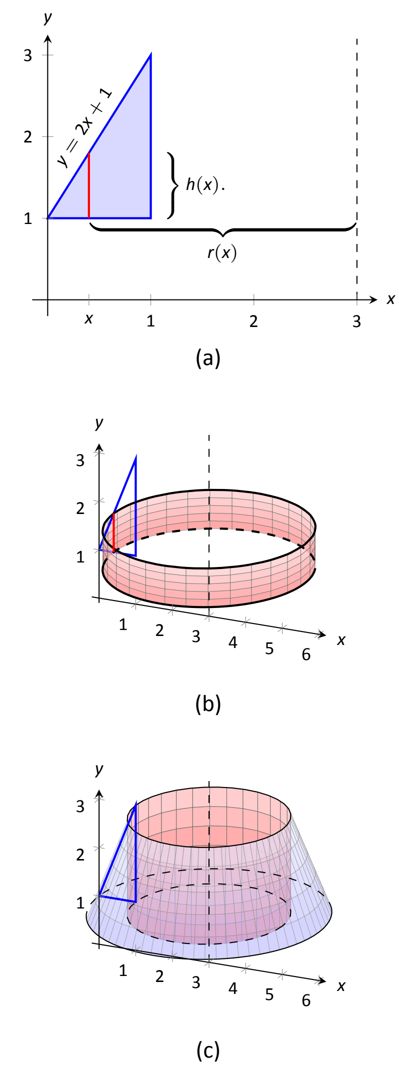


Figure 7.20: Graphing a region in Example 7.12.

Notes:

$x = 1$, giving

$$\begin{aligned} V &= 2\pi \int_0^1 (3-x)(2x)\,dx \\ &= 2\pi \int_0^1 \left(6x - 2x^2\right)dx \\ &= 2\pi\left(3x^2 - \frac{2}{3}x^3\right)\Big|_0^1 \\ &= \frac{14}{3}\pi \approx 14.66 \text{ units}^3. \end{aligned}$$

When revolving a region around a horizontal axis, we must consider the radius and height functions in terms of y, not x.

Example 7.13 Finding volume using the Shell Method

Find the volume of the solid formed by rotating the region given in Example 7.12 about the x-axis.

Solution The region is sketched in Figure 7.21(a) with a sample differential element. In part (b) of the figure the shell formed by the differential element is drawn, and the solid is sketched in (c). (Note that the triangular region looks "short and wide" here, whereas in the previous example the same region looked "tall and narrow." This is because the bounds on the graphs are different.)

The height of the differential element is an x-distance, between $x = \frac{1}{2}y - \frac{1}{2}$ and $x = 1$. Thus $h(y) = 1 - (\frac{1}{2}y - \frac{1}{2}) = -\frac{1}{2}y + \frac{3}{2}$. The radius is the distance from y to the x-axis, so $r(y) = y$. The y bounds of the region are $y = 1$ and $y = 3$, leading to the integral

$$\begin{aligned} V &= 2\pi \int_1^3 \left[y\left(-\frac{1}{2}y + \frac{3}{2}\right)\right] dy \\ &= 2\pi \int_1^3 \left[-\frac{1}{2}y^2 + \frac{3}{2}y\right] dy \\ &= 2\pi \left[-\frac{1}{6}y^3 + \frac{3}{4}y^2\right]\Big|_1^3 \\ &= 2\pi\left[\frac{9}{4} - \frac{7}{12}\right] \\ &= \frac{10}{3}\pi \approx 10.472 \text{ units}^3. \end{aligned}$$

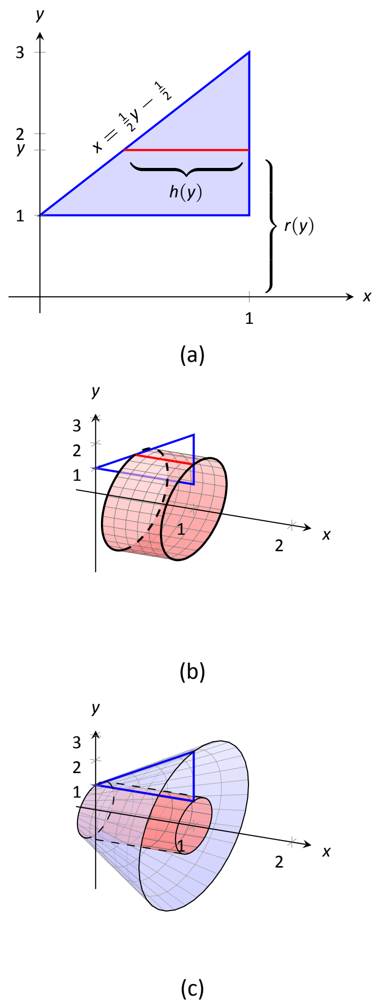


Figure 7.21: Graphing a region in Example 7.13.

Notes:

At the beginning of this section it was stated that "it is good to have options." The next example finds the volume of a solid rather easily with the Shell Method, but using the Washer Method would be quite a chore.

Example 7.14 Finding volume using the Shell Method

Find the volume of the solid formed by revolving the region bounded by $y = \sin x$ and the x-axis from $x = 0$ to $x = \pi$ about the y-axis.

Solution The region and a differential element, the shell formed by this differential element, and the resulting solid are given in Figure 7.22. The radius of a sample shell is $r(x) = x$; the height of a sample shell is $h(x) = \sin x$, each from $x = 0$ to $x = \pi$. Thus the volume of the solid is

$$V = 2\pi \int_0^{\pi} x \sin x \, dx.$$

This requires Integration By Parts. Set $u = x$ and $dv = \sin x \, dx$; we leave it to the reader to fill in the rest. We have:

$$\begin{aligned} &= 2\pi \Big[-x\cos x \Big|_0^{\pi} + \int_0^{\pi} \cos x \, dx \Big] \\ &= 2\pi \Big[\pi + \sin x \Big|_0^{\pi} \Big] \\ &= 2\pi \Big[\pi + 0 \Big] \\ &= 2\pi^2 \approx 19.74 \text{ units}^3. \end{aligned}$$

Note that in order to use the Washer Method, we would need to solve $y = \sin x$ for x, requiring the use of the arcsine function. We leave it to the reader to verify that the outside radius function is $R(y) = \pi - \arcsin y$ and the inside radius function is $r(y) = \arcsin y$. Thus the volume can be computed as

$$\pi \int_0^1 \Big[(\pi - \arcsin y)^2 - (\arcsin y)^2 \Big] \, dy.$$

This integral isn't terrible given that the $\arcsin^2 y$ terms cancel, but it is more onerous than the integral created by the Shell Method.

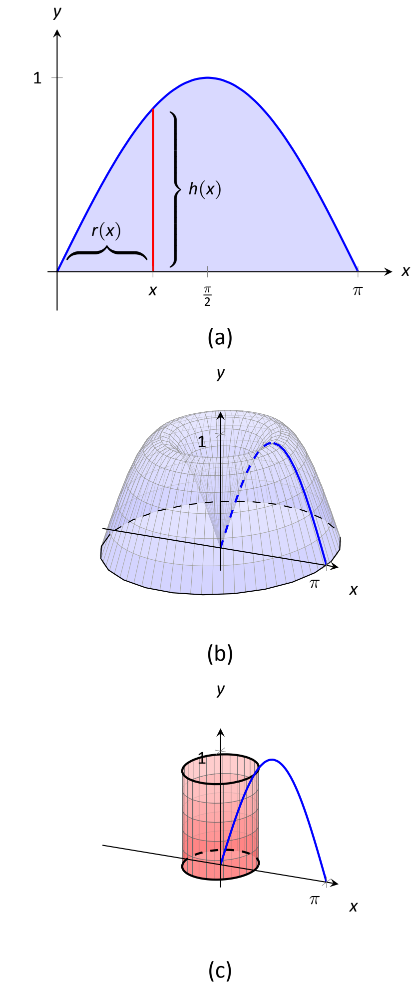


Figure 7.22: Graphing a region in Example 7.14.

We end this section with a table summarizing the usage of the Washer and Shell Methods.

Notes:

Key Idea 27 Summary of the Washer and Shell Methods

Let a region R be given with x-bounds $x = a$ and $x = b$ and y-bounds $y = c$ and $y = d$.

	Washer Method	Shell Method
Horizontal Axis	$\pi \int_a^b \big(R(x)^2 - r(x)^2\big)\, dx$	$2\pi \int_c^d r(y)h(y)\, dy$
Vertical Axis	$\pi \int_c^d \big(R(y)^2 - r(y)^2\big)\, dy$	$2\pi \int_a^b r(x)h(x)\, dx$

As in the previous section, the real goal of this section is not to be able to compute volumes of certain solids. Rather, it is to be able to solve a problem by first approximating, then using limits to refine the approximation to give the exact value. In this section, we approximate the volume of a solid by cutting it into thin cylindrical shells. By summing up the volumes of each shell, we get an approximation of the volume. By taking a limit as the number of equally spaced shells goes to infinity, our summation can be evaluated as a definite integral, giving the exact value.

We use this same principle again in the next section, where we find the length of curves in the plane.

Notes:

Exercises 7.3

Terms and Concepts

1. T/F: A solid of revolution is formed by revolving a shape around an axis.

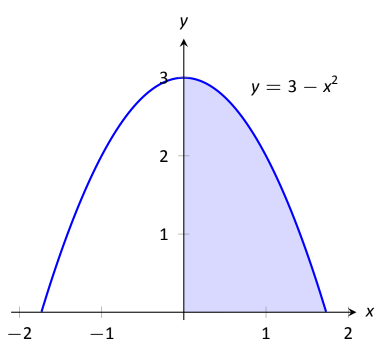
2. T/F: The Shell Method can only be used when the Washer Method fails.

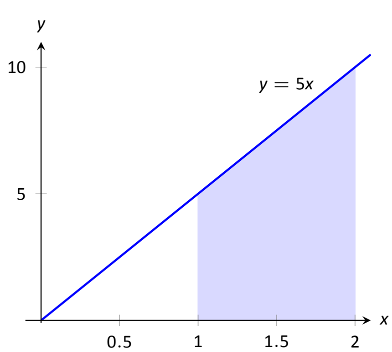
3. T/F: The Shell Method works by integrating cross–sectional areas of a solid.

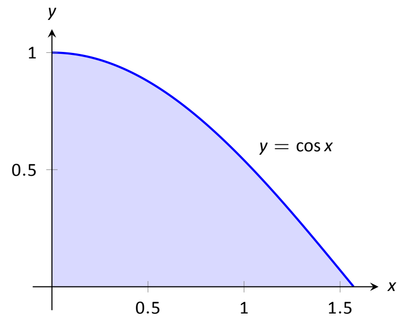
4. T/F: When finding the volume of a solid of revolution that was revolved around a vertical axis, the Shell Method integrates with respect to *x*.

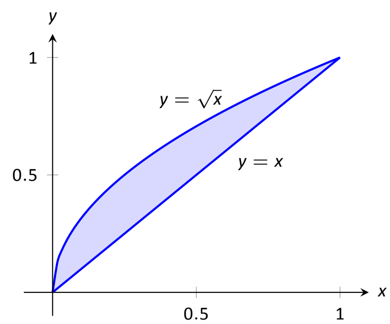
Problems

In Exercises 5 – 8, a region of the Cartesian plane is shaded. Use the Shell Method to find the volume of the solid of revolution formed by revolving the region about the *y*-axis.

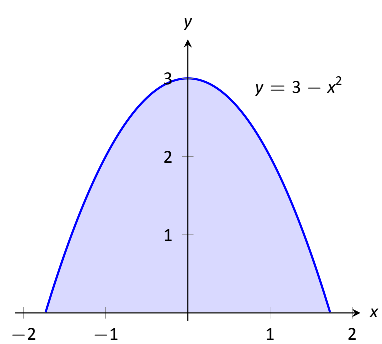
5. 

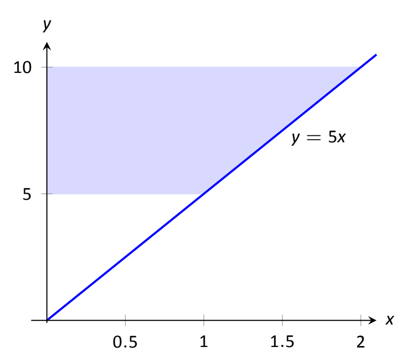
6. 

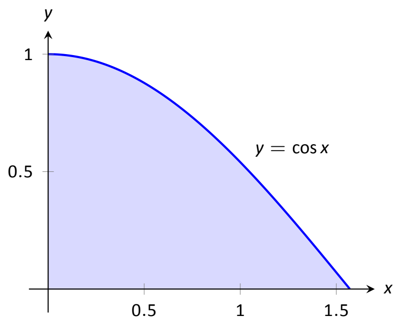
7. 

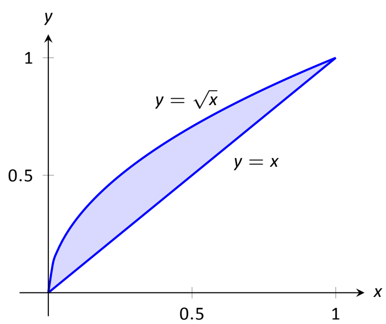
8. 

In Exercises 9 – 12, a region of the Cartesian plane is shaded. Use the Shell Method to find the volume of the solid of revolution formed by revolving the region about the *x*-axis.

9. 

10. 

11. 

12. 

In Exercises 13 – 18, a region of the Cartesian plane is described. Use the Shell Method to find the volume of the solid of revolution formed by rotating the region about each of the given axes.

13. Region bounded by: $y = \sqrt{x}$, $y = 0$ and $x = 1$.
Rotate about:
   (a) the y-axis
   (b) $x = 1$
   (c) the x-axis
   (d) $y = 1$

14. Region bounded by: $y = 4 - x^2$ and $y = 0$.
Rotate about:
   (a) $x = 2$
   (b) $x = -2$
   (c) the x-axis
   (d) $y = 4$

15. The triangle with vertices $(1,1)$, $(1,2)$ and $(2,1)$.
Rotate about:
   (a) the y-axis
   (b) $x = 1$
   (c) the x-axis
   (d) $y = 2$

16. Region bounded by $y = x^2 - 2x + 2$ and $y = 2x - 1$.
Rotate about:
   (a) the y-axis
   (b) $x = 1$
   (c) $x = -1$

17. Region bounded by $y = 1/\sqrt{x^2 + 1}$, $x = 1$ and the x and y-axes.
Rotate about:
   (a) the y-axis
   (b) $x = 1$

18. Region bounded by $y = 2x$, $y = x$ and $x = 2$.
Rotate about:
   (a) the y-axis
   (b) $x = 2$
   (c) the x-axis
   (d) $y = 4$

7.4 Arc Length and Surface Area

In previous sections we have used integration to answer the following questions:

1. Given a region, what is its area?

2. Given a solid, what is its volume?

In this section, we address a related question: Given a curve, what is its length? This is often referred to as **arc length**.

Consider the graph of $y = \sin x$ on $[0, \pi]$ given in Figure 7.23 (a). How long is this curve? That is, if we were to use a piece of string to exactly match the shape of this curve, how long would the string be?

As we have done in the past, we start by approximating; later, we will refine our answer using limits to get an exact solution.

The length of straight–line segments is easy to compute using the Distance Formula. We can approximate the length of the given curve by approximating the curve with straight lines and measuring their lengths.

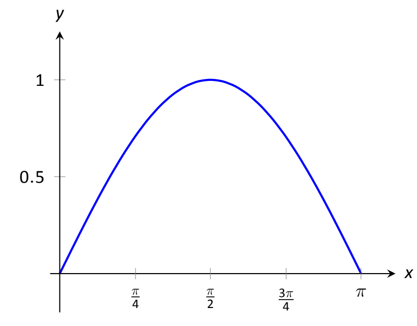
In Figure 7.23 (b), the curve $y = \sin x$ has been approximated with 4 line segments (the interval $[0, \pi]$ has been divided into 4 equally–lengthed subintervals). It is clear that these four line segments approximate $y = \sin x$ very well on the first and last subinterval, though not so well in the middle. Regardless, the sum of the lengths of the line segments is 3.79, so we approximate the arc length of $y = \sin x$ on $[0, \pi]$ to be 3.79.

In general, we can approximate the arc length of $y = f(x)$ on $[a, b]$ in the following manner. Let $a = x_1 < x_2 < \ldots < x_n < x_{n+1} = b$ be a partition of $[a, b]$ into n subintervals. Let Δx_i represent the length of the i^{th} subinterval $[x_i, x_{i+1}]$.

Figure 7.24 zooms in on the i^{th} subinterval where $y = f(x)$ is approximated by a straight line segment. The dashed lines show that we can view this line segment as they hypotenuse of a right triangle whose sides have length Δx_i and Δy_i. Using the Pythagorean Theorem, the length of this line segment is $\sqrt{\Delta x_i^2 + \Delta y_i^2}$. Summing over all subintervals gives an arc length approximation

$$L \approx \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2}.$$

As shown here, this is *not* a Riemann Sum. While we could conclude that taking a limit as the subinterval length goes to zero gives the exact arc length, we would not be able to compute the answer with a definite integral. We need first to do a little algebra.



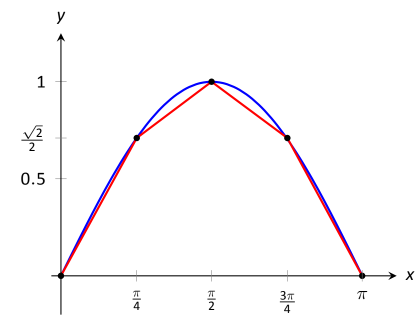


Figure 7.23: Graphing $y = \sin x$ on $[0, \pi]$ and approximating the curve with line segments.

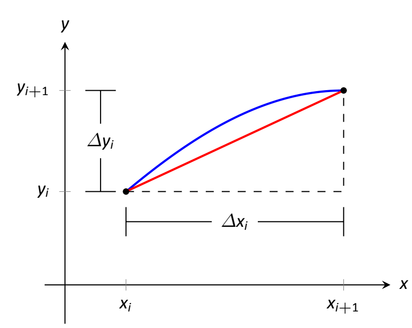


Figure 7.24: Zooming in on the i^{th} subinterval $[x_i, x_{i+1}]$ of a partition of $[a, b]$.

Notes:

In the above expression factor out a Δx_i^2 term:

$$\sum_{i=1}^{n}\sqrt{\Delta x_i^2+\Delta y_i^2}=\sum_{i=1}^{n}\sqrt{\Delta x_i^2\left(1+\frac{\Delta y_i^2}{\Delta x_i^2}\right)}.$$

Now pull the Δx_i^2 term out of the square root:

$$=\sum_{i=1}^{n}\sqrt{1+\frac{\Delta y_i^2}{\Delta x_i^2}}\ \Delta x_i.$$

This is nearly a Riemann Sum. Consider the $\Delta y_i^2/\Delta x_i^2$ term. The expression $\Delta y_i/\Delta x_i$ measures the "change in y/change in x," that is, the "rise over run" of f on the i^{th} subinterval. The Mean Value Theorem of Differentiation (Theorem 27) states that there is a c_i in the i^{th} subinterval where $f'(c_i)=\Delta y_i/\Delta x_i$. Thus we can rewrite our above expression as:

$$=\sum_{i=1}^{n}\sqrt{1+f'(c_i)^2}\ \Delta x_i.$$

This *is* a Riemann Sum. As long as f' is continuous, we can invoke Theorem 40 and conclude

$$=\int_a^b\sqrt{1+f'(x)^2}\ dx.$$

Key Idea 28 Arc Length

Let f be differentiable on an open interval containing $[a,b]$, where f' is also continuous on $[a,b]$. Then the arc length of f from $x=a$ to $x=b$ is

$$L=\int_a^b\sqrt{1+f'(x)^2}\ dx.$$

As the integrand contains a square root, it is often difficult to use the formula in Key Idea 28 to find the length exactly. When exact answers are difficult to come by, we resort to using numerical methods of approximating definite integrals. The following examples will demonstrate this.

Notes:

Example 7.15 **Finding arc length**

Find the arc length of $f(x) = x^{3/2}$ from $x = 0$ to $x = 4$.

Solution We begin by finding $f'(x) = \frac{3}{2}x^{1/2}$. Using the formula, we find the arc length L as

$$\begin{aligned} L &= \int_0^4 \sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}\,dx \\ &= \int_0^4 \sqrt{1+\frac{9}{4}x}\,dx \\ &= \int_0^4 \left(1+\frac{9}{4}x\right)^{1/2}\,dx \\ &= \frac{2}{3}\cdot\frac{4}{9}\left(1+\frac{9}{4}x\right)^{3/2}\Big|_0^4 \\ &= \frac{8}{27}\left(10^{3/2}-1\right) \approx 9.07\text{units}. \end{aligned}$$

A graph of f is given in Figure 7.25.

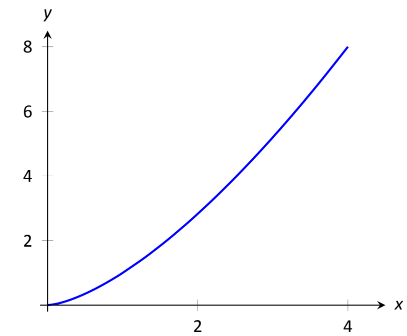


Figure 7.25: A graph of $f(x) = x^{3/2}$ from Example 7.15.

Example 7.16 **Finding arc length**

Find the arc length of $f(x) = \dfrac{1}{8}x^2 - \ln x$ from $x = 1$ to $x = 2$.

Solution This function was chosen specifically because the resulting integral can be evaluated exactly. We begin by finding $f'(x) = x/4 - 1/x$. The arc length is

$$\begin{aligned} L &= \int_1^2 \sqrt{1+\left(\frac{x}{4}-\frac{1}{x}\right)^2}\,dx \\ &= \int_1^2 \sqrt{1+\frac{x^2}{16}-\frac{1}{2}+\frac{1}{x^2}}\,dx \\ &= \int_1^2 \sqrt{\frac{x^2}{16}+\frac{1}{2}+\frac{1}{x^2}}\,dx \\ &= \int_1^2 \sqrt{\left(\frac{x}{4}+\frac{1}{x}\right)^2}\,dx \end{aligned}$$

Notes:

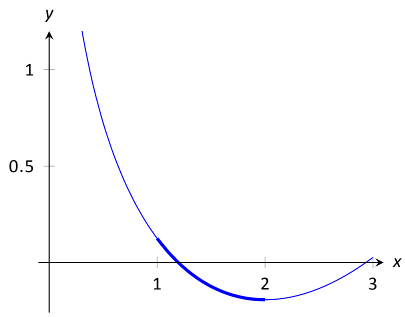


Figure 7.26: A graph of $f(x) = \frac{1}{8}x^2 - \ln x$ from Example 7.16.

$$= \int_1^2 \left(\frac{x}{4} + \frac{1}{x}\right)\,dx$$
$$= \left(\frac{x^2}{8} + \ln x\right)\Bigg|_1^2$$
$$= \frac{3}{8} + \ln 2 \approx 1.07 \text{ units.}$$

A graph of f is given in Figure 7.26; the portion of the curve measured in this problem is in bold.

The previous examples found the arc length exactly through careful choice of the functions. In general, exact answers are much more difficult to come by and numerical approximations are necessary.

Example 7.17 Approximating arc length numerically

Find the length of the sine curve from $x = 0$ to $x = \pi$.

Solution This is somewhat of a mathematical curiosity; in Example 5.19 we found the area under one "hump" of the sine curve is 2 square units; now we are measuring its arc length.

The setup is straightforward: $f(x) = \sin x$ and $f'(x) = \cos x$. Thus

$$L = \int_0^\pi \sqrt{1+\cos^2 x}\,dx.$$

This integral *cannot* be evaluated in terms of elementary functions so we will approximate it with Simpson's Method with $n = 4$. Figure 7.27 gives $\sqrt{1+\cos^2 x}$ evaluated at 5 evenly spaced points in $[0, \pi]$. Simpson's Rule then states that

x	$\sqrt{1+\cos^2 x}$
0	$\sqrt{2}$
$\pi/4$	$\sqrt{3/2}$
$\pi/2$	1
$3\pi/4$	$\sqrt{3/2}$
π	$\sqrt{2}$

Figure 7.27: A table of values of $y = \sqrt{1+\cos^2 x}$ to evaluate a definite integral in Example 7.17.

$$\int_0^\pi \sqrt{1+\cos^2 x}\,dx \approx \frac{\pi - 0}{4\cdot 3}\left(\sqrt{2} + 4\sqrt{3/2} + 2(1) + 4\sqrt{3/2} + \sqrt{2}\right)$$
$$= 3.82918.$$

Using a computer with $n = 100$ the approximation is $L \approx 3.8202$; our approximation with $n = 4$ is quite good.

Notes:

Surface Area of Solids of Revolution

We have already seen how a curve $y = f(x)$ on $[a, b]$ can be revolved around an axis to form a solid. Instead of computing its volume, we now consider its surface area.

We begin as we have in the previous sections: we partition the interval $[a, b]$ with n subintervals, where the i^{th} subinterval is $[x_i, x_{i+1}]$. On each subinterval, we can approximate the curve $y = f(x)$ with a straight line that connects $f(x_i)$ and $f(x_{i+1})$ as shown in Figure 7.28(a). Revolving this line segment about the x-axis creates part of a cone (called a *frustum* of a cone) as shown in Figure 7.28(b). The surface area of a frustum of a cone is

$$2\pi \cdot \text{ length } \cdot \text{average of the two radii } R \text{ and } r.$$

The length is given by L; we use the material just covered by arc length to state that

$$L \approx \sqrt{1 + f'(c_i)}\Delta x_i$$

for some c_i in the i^{th} subinterval. The radii are just the function evaluated at the endpoints of the interval. That is,

$$R = f(x_{i+1}) \quad \text{and} \quad r = f(x_i).$$

Thus the surface area of this sample frustum of the cone is approximately

$$2\pi \frac{f(x_i) + f(x_{i+1})}{2}\sqrt{1 + f'(c_i)^2}\Delta x_i.$$

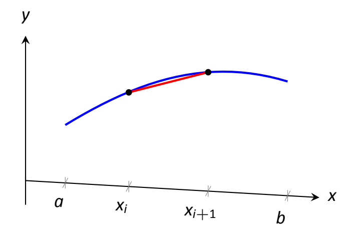
Since f is a continuous function, the Intermediate Value Theorem states there is some d_i in $[x_i, x_{i+1}]$ such that $f(d_i) = \dfrac{f(x_i) + f(x_{i+1})}{2}$; we can use this to rewrite the above equation as

$$2\pi f(d_i)\sqrt{1 + f'(c_i)^2}\Delta x_i.$$

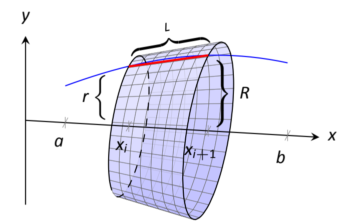
Summing over all the subintervals we get the total surface area to be approximately

$$\text{Surface Area} \approx \sum_{i=1}^{n} 2\pi f(d_i)\sqrt{1 + f'(c_i)^2}\Delta x_i,$$

which is a Riemann Sum. Taking the limit as the subinterval lengths go to zero gives us the exact surface area, given in the following Key Idea.



(a)



(b)

Figure 7.28: Establishing the formula for surface area.

Notes:

Key Idea 29 Surface Area of a Solid of Revolution

Let f be differentiable on an open interval containing $[a,b]$ where f' is also continuous on $[a,b]$.

1. The surface area of the solid formed by revolving the graph of $y = f(x)$, where $f(x) \geq 0$, about the x-axis is

$$\text{Surface Area} = 2\pi \int_a^b f(x)\sqrt{1+f'(x)^2}\,dx.$$

2. The surface area of the solid formed by revolving the graph of $y = f(x)$ about the y-axis, where $a, b \geq 0$, is

$$\text{Surface Area} = 2\pi \int_a^b x\sqrt{1+f'(x)^2}\,dx.$$

(When revolving $y = f(x)$ about the y-axis, the radii of the resulting frustum are x_i and x_{i+1}; their average value is simply the midpoint of the interval. In the limit, this midpoint is just x. This gives the second part of Key Idea 29.)

Example 7.18 Finding surface area of a solid of revolution

Find the surface area of the solid formed by revolving $y = \sin x$ on $[0,\pi]$ around the x-axis, as shown in Figure 7.29.

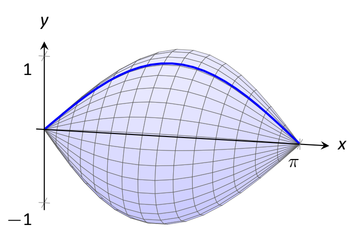


Figure 7.29: Revolving $y = \sin x$ on $[0,\pi]$ about the x-axis.

Solution The setup is relatively straightforward. Using Key Idea 29, we have the surface area SA is:

$$\begin{aligned} SA &= 2\pi \int_0^\pi \sin x\sqrt{1+\cos^2 x}\,dx \\ &= -2\pi\frac{1}{2}\left(\ln\left(\cos x + \sqrt{1+\cos^2 x}\right) + \cos x\sqrt{1+\cos^2 x}\right)\Big|_0^\pi \\ &= 2\pi\left(\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right) \\ &\approx 14.42 \text{ units}^2. \end{aligned}$$

The integration step above is nontrivial, utilizing an integration method called Trigonometric Substitution.

It is interesting to see that the surface area of a solid, whose shape is defined by a trigonometric function, involves both a square root and a logarithm.

Example 7.19 Finding surface area of a solid of revolution

Notes:

Find the surface area of the solid formed by revolving the curve $y = x^2$ on $[0, 1]$ about:

1. the x-axis
2. the y-axis.

Solution

1. The integral is straightforward to setup:

$$SA = 2\pi \int_0^1 x^2 \sqrt{1 + (2x)^2}\, dx.$$

Like the integral in Example 7.18, this requires Trigonometric Substitution.

$$\begin{aligned} &= \frac{\pi}{32}\left(2(8x^3 + x)\sqrt{1 + 4x^2} - \ln\left(2x + \sqrt{1 + 4x^2}\right)\right)\Big|_0^1 \\ &= \frac{\pi}{32}\left(18\sqrt{5} - \ln\left(2 + \sqrt{5}\right)\right) \\ &\approx 3.81 \text{ units}^2. \end{aligned}$$

The solid formed by revolving $y = x^2$ around the x-axis is graphed in Figure 7.30 (a).

2. Since we are revolving around the y-axis, the "radius" of the solid is not $f(x)$ but rather x. Thus the integral to compute the surface area is:

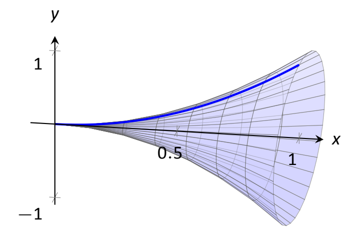
$$SA = 2\pi \int_0^1 x\sqrt{1 + (2x)^2}\, dx.$$

This integral can be solved using substitution. Set $u = 1 + 4x^2$; the new bounds are $u = 1$ to $u = 5$. We then have

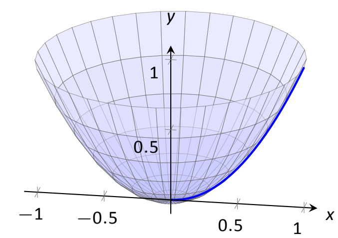
$$\begin{aligned} &= \frac{\pi}{4}\int_1^5 \sqrt{u}\, du \\ &= \frac{\pi}{4} \cdot \frac{2}{3} u^{3/2}\Big|_1^5 \\ &= \frac{\pi}{6}\left(5\sqrt{5} - 1\right) \\ &\approx 5.33 \text{ units}^2. \end{aligned}$$

The solid formed by revolving $y = x^2$ about the y-axis is graphed in Figure 7.30 (b).

Our final example is a famous mathematical "paradox."



(a)



(b)

Figure 7.30: The solids used in Example 7.19.

Notes:

Example 7.20 The surface area and volume of Gabriel's Horn

Consider the solid formed by revolving $y = 1/x$ about the x-axis on $[1, \infty)$. Find the volume and surface area of this solid. (This shape, as graphed in Figure 7.31, is known as "Gabriel's Horn" since it looks like a very long horn that only a supernatural person, such as an angel, could play.)

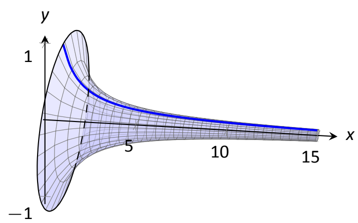


Figure 7.31: A graph of Gabriel's Horn.

SOLUTION To compute the volume it is natural to use the Disk Method. We have:

$$\begin{aligned} V &= \pi \int_1^{\infty} \frac{1}{x^2}\,dx \\ &= \lim_{b\to\infty} \pi \int_1^{b} \frac{1}{x^2}\,dx \\ &= \lim_{b\to\infty} \pi \left(\frac{-1}{x}\right)\Big|_1^b \\ &= \lim_{b\to\infty} \pi \left(1 - \frac{1}{b}\right) \\ &= \pi \text{ units}^3. \end{aligned}$$

Gabriel's Horn has a finite volume of π cubic units. Since we have already seen that regions with infinite length can have a finite area, this is not too difficult to accept.

We now consider its surface area. The integral is straightforward to setup:

$$SA = 2\pi \int_1^{\infty} \frac{1}{x}\sqrt{1 + 1/x^4}\,dx.$$

Integrating this expression is not trivial. We can, however, compare it to other improper integrals. Since $1 < \sqrt{1 + 1/x^4}$ on $[1, \infty)$, we can state that

$$2\pi \int_1^{\infty} \frac{1}{x}\,dx < 2\pi \int_1^{\infty} \frac{1}{x}\sqrt{1 + 1/x^4}\,dx.$$

By Key Idea 22, the improper integral on the left diverges. Since the integral on the right is larger, we conclude it also diverges, meaning Gabriel's Horn has infinite surface area.

Hence the "paradox": we can fill Gabriel's Horn with a finite amount of paint, but since it has infinite surface area, we can never paint it.

Somehow this paradox is striking when we think about it in terms of volume and area. However, we have seen a similar paradox before, as referenced above. We know that the area under the curve $y = 1/x^2$ on $[1, \infty)$ is finite, yet the shape has an infinite perimeter. Strange things can occur when we deal with the infinite.

A standard equation from physics is "Work = force $\times$ distance", when the force applied is constant. In the next section we learn how to compute work when the force applied is variable.

Notes:

Exercises 7.4

Terms and Concepts

1. T/F: The integral formula for computing Arc Length was found by first approximating arc length with straight line segments.

2. T/F: The integral formula for computing Arc Length includes a square root, meaning the integration is probably easy.

Problems

In Exercises 3 – 12, find the arc length of the function on the given interval.

3. $f(x) = x$ on $[0, 1]$.

4. $f(x) = \sqrt{8}x$ on $[-1, 1]$.

5. $f(x) = \frac{1}{3}x^{3/2} - x^{1/2}$ on $[0, 1]$.

6. $f(x) = \frac{1}{12}x^3 + \frac{1}{x}$ on $[1, 4]$.

7. $f(x) = 2x^{3/2} - \frac{1}{6}\sqrt{x}$ on $[0, 9]$.

8. $f(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$ on $[-\ln 2, \ln 2]$.

9. $f(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$ on $[0, \ln 5]$.

10. $f(x) = \frac{1}{12}x^5 + \frac{1}{5x^3}$ on $[.1, 1]$.

11. $f(x) = \ln\left(\sin x\right)$ on $[\pi/6, \pi/2]$.

12. $f(x) = \ln\left(\cos x\right)$ on $[0, \pi/4]$.

In Exercises 13 – 20, set up the integral to compute the arc length of the function on the given interval. Do not evaluate the integral.

13. $f(x) = x^2$ on $[0, 1]$.

14. $f(x) = x^{10}$ on $[0, 1]$.

15. $f(x) = \sqrt{x}$ on $[0, 1]$.

16. $f(x) = \ln x$ on $[1, e]$.

17. $f(x) = \sqrt{1 - x^2}$ on $[-1, 1]$. (Note: this describes the top half of a circle with radius 1.)

18. $f(x) = \sqrt{1 - x^2/9}$ on $[-3, 3]$. (Note: this describes the top half of an ellipse with a major axis of length 6 and a minor axis of length 2.)

19. $f(x) = \frac{1}{x}$ on $[1, 2]$.

20. $f(x) = \sec x$ on $[-\pi/4, \pi/4]$.

In Exercises 21 – 28, use Simpson's Rule, with $n = 4$, to approximate the arc length of the function on the given interval. Note: these are the same problems as in Exercises 13–20.

21. $f(x) = x^2$ on $[0, 1]$.

22. $f(x) = x^{10}$ on $[0, 1]$.

23. $f(x) = \sqrt{x}$ on $[0, 1]$. (Note: $f'(x)$ is not defined at $x = 0$.)

24. $f(x) = \ln x$ on $[1, e]$.

25. $f(x) = \sqrt{1 - x^2}$ on $[-1, 1]$. (Note: $f'(x)$ is not defined at the endpoints.)

26. $f(x) = \sqrt{1 - x^2/9}$ on $[-3, 3]$. (Note: $f'(x)$ is not defined at the endpoints.)

27. $f(x) = \frac{1}{x}$ on $[1, 2]$.

28. $f(x) = \sec x$ on $[-\pi/4, \pi/4]$.

In Exercises 29 – 33, find the surface area of the described solid of revolution.

29. The solid formed by revolving $y = 2x$ on $[0, 1]$ about the x-axis.

30. The solid formed by revolving $y = x^2$ on $[0, 1]$ about the y-axis.

31. The solid formed by revolving $y = x^3$ on $[0, 1]$ about the x-axis.

32. The solid formed by revolving $y = \sqrt{x}$ on $[0, 1]$ about the x-axis.

33. The sphere formed by revolving $y = \sqrt{1 - x^2}$ on $[-1, 1]$ about the x-axis.

7.5 Work

Work is the scientific term used to describe the action of a force which moves an object. When a constant force F is applied to move an object a distance d, the amount of work performed is $W = F \cdot d$.

The SI unit of force is the Newton, (kg·m/s^2), and the SI unit of distance is a meter (m). The fundamental unit of work is one Newton–meter, or a joule (J). That is, applying a force of one Newton for one meter performs one joule of work. In Imperial units (as used in the United States), force is measured in pounds (lb) and distance is measured in feet (ft), hence work is measured in ft–lb.

When force is constant, the measurement of work is straightforward. For instance, lifting a 200 lb object 5 ft performs $200 \cdot 5 = 1000$ ft–lb of work.

What if the force applied is variable? For instance, imagine a climber pulling a 200 ft rope up a vertical face. The rope becomes lighter as more is pulled in, requiring less force and hence the climber performs less work.

In general, let $F(x)$ be a force function on an interval $[a,b]$. We want to measure the amount of work done applying the force F from $x = a$ to $x = b$. We can approximate the amount of work being done by partitioning $[a,b]$ into subintervals $a = x_1 < x_2 < \cdots < x_{n+1} = b$ and assuming that F is constant on each subinterval. Let c_i be a value in the i^{th} subinterval $[x_i, x_{i+1}]$. Then the work done on this interval is approximately $W_i \approx F(c_i) \cdot (x_{i+1} - x_i) = F(c_i)\Delta x_i$, a constant force $\times$ the distance over which it is applied. The total work is

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} F(c_i)\Delta x_i.$$

This, of course, is a Riemann sum. Taking a limit as the subinterval lengths go to zero give an exact value of work which can be evaluated through a definite integral.

Key Idea 30 Work

Let $F(x)$ be a continuous function on $[a,b]$ describing the amount of force being applied to an object in the direction of travel from distance $x = a$ to distance $x = b$. The total work W done on $[a,b]$ is

$$W = \int_a^b F(x)\,dx.$$

Note: *Mass* and *weight* are closely related, yet different, concepts. The mass m of an object is a quantitative measure of that object's resistance to acceleration. The weight w of an object is a measurement of the force applied to the object by the acceleration of gravity g.

Since the two measurements are proportional, $w = m \cdot g$, they are often used interchangeably in everyday conversation. When computing work, one must be careful to note which is being referred to. When mass is given, it must be multiplied by the acceleration of gravity to reference the related force.

Notes:

Example 7.21 Computing work performed: applying variable force
A 60m climbing rope is hanging over the side of a tall cliff. How much work is performed in pulling the rope up to the top, where the rope has a mass of 66g/m?

Solution We need to create a force function $F(x)$ on the interval $[0, 60]$. To do so, we must first decide what x is measuring: is it the length of the rope still hanging or is it the amount of rope pulled in? As long as we are consistent, either approach is fine. We adopt for this example the convention that x is the amount of rope pulled in. This seems to match intuition better; pulling up the first 10 meters of rope involves $x = 0$ to $x = 10$ instead of $x = 60$ to $x = 50$.

As x is the amount of rope pulled in, the amount of rope still hanging is $60-x$. This length of rope has a mass of 66 g/m, or 0.066 kg/m. Then the mass of the rope still hanging is $0.066(60 - x)$ kg; multiplying this mass by the acceleration of gravity, 9.8 m/s^2, gives our variable force function

$$F(x) = (9.8)(0.066)(60 - x) = 0.6468(60 - x).$$

Thus the total work performed in pulling up the rope is

$$W = \int_0^{60} 0.6468(60 - x)\, dx = 1,164.24 \text{ J}.$$

By comparison, consider the work done in lifting the entire rope 60 meters. The rope weights $60 \times 0.066 \times 9.8 = 38.808$ N, so the work applying this force for 60 meters is $60 \times 38.808 = 2,328.48$ J. This is exactly twice the work calculated before (and we leave it to the reader to understand why.)

Example 7.22 Computing work performed: applying variable force
Consider again pulling a 60 m rope up a cliff face, where the rope has a mass of 66 g/m. At what point is exactly half the work performed?

Solution From Example 7.21 we know the total work performed is $1,164.24$ J. We want to find a height h such that the work in pulling the rope from a height of $x = 0$ to a height of $x = h$ is 582.12, half the total work. Thus we want to solve the equation

$$\int_0^{h} 0.6468(60 - x)\, dx = 582.12$$

for h.

Notes:

Note: In Example 7.22, we find that half of the work performed in pulling up a 60 m rope is done in the last 42.43 m. Why is it not coincidental that $60/\sqrt{2} = 42.43$?

$$\begin{aligned}\int_0^h 0.6468(60-x)\,dx &= 582.12\\ \left(38.808x - 0.3234x^2\right)\Big|_0^h &= 582.12\\ 38.808h - 0.3234h^2 &= 582.12\\ -0.3234h^2 + 38.808h - 582.12 &= 0.\end{aligned}$$

Apply the Quadratic Formula.

$$h = 17.57 \text{ and } 102.43$$

As the rope is only 60m long, the only sensible answer is $h = 17.57$. Thus about half the work is done pulling up the first 17.5m the other half of the work is done pulling up the remaining 42.43m.

Example 7.23 Computing work performed: applying variable force

A box of 100 lb of sand is being pulled up at a uniform rate a distance of 50 ft over 1 minute. The sand is leaking from the box at a rate of 1 lb/s. The box itself weighs 5 lb and is pulled by a rope weighing .2 lb/ft.

1. How much work is done lifting just the rope?
2. How much work is done lifting just the box and sand?
3. What is the total amount of work performed?

Solution

1. We start by forming the force function $F_r(x)$ for the rope (where the subscript denotes we are considering the rope). As in the previous example, let x denote the amount of rope, in feet, pulled in. (This is the same as saying x denotes the height of the box.) The weight of the rope with x feet pulled in is $F_r(x) = 0.2(50-x) = 10 - 0.2x$. (Note that we do not have to include the acceleration of gravity here, for the *weight* of the rope per foot is given, not its *mass* per meter as before.) The work performed lifting the rope is

$$W_r = \int_0^{50} (10 - 0.2x)\,dx = 250 \text{ ft–lb}.$$

Notes:

2. The sand is leaving the box at a rate of 1 lb/s. As the vertical trip is to take one minute, we know that 60 lb will have left when the box reaches its final height of 50 ft. Again letting x represent the height of the box, we have two points on the line that describes the weight of the sand: when $x = 0$, the sand weight is 100 lb, producing the point $(0, 100)$; when $x = 50$, the sand in the box weighs 40 lb, producing the point $(50, 40)$. The slope of this line is $\frac{100-40}{0-50} = -1.2$, giving the equation of the weight of the sand at height x as $w(x) = -1.2x + 100$. The box itself weighs a constant 5 lb, so the total force function is $F_b(x) = -1.2x + 105$. Integrating from $x = 0$ to $x = 50$ gives the work performed in lifting box and sand:

$$W_b = \int_0^{50} (-1.2x + 105)\, dx = 3750 \text{ ft–lb.}$$

3. The total work is the sum of W_r and W_b: $250 + 3750 = 4000$ ft–lb. We can also arrive at this via integration:

$$\begin{aligned} W &= \int_0^{50} (F_r(x) + F_b(x))\, dx \\ &= \int_0^{50} (10 - 0.2x - 1.2x + 105)\, dx \\ &= \int_0^{50} (-1.4x + 115)\, dx \\ &= 4000 \text{ ft–lb.} \end{aligned}$$

Hooke's Law and Springs

Hooke's Law states that the force required to compress or stretch a spring x units from its natural length is proportional to x; that is, this force is $F(x) = kx$ for some constant k. For example, if a force of 1 N stretches a given spring 2 cm, then a force of 5 N will stretch the spring 10 cm. Converting the distances to meters, we have that stretching this spring 0.02 m requires a force of $F(0.02) = k(0.02) = 1$ N, hence $k = 1/0.02 = 50$ N/m.

Example 7.24 Computing work performed: stretching a spring
A force of 20 lb stretches a spring from a natural length of 7 inches to a length of 12 inches. How much work was performed in stretching the spring to this length?

Solution In many ways, we are not at all concerned with the actual length of the spring, only with the amount of its change. Hence, we do not care

---

Notes:

that 20 lb of force stretches the spring to a length of 12 inches, but rather that a force of 20 lb stretches the spring by 5 in. This is illustrated in Figure 7.32; we only measure the change in the spring's length, not the overall length of the spring.

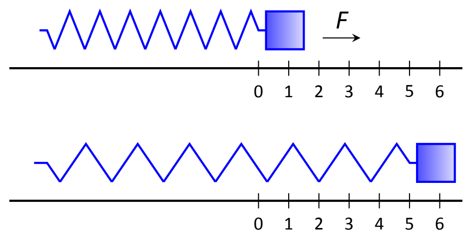


Figure 7.32: Illustrating the important aspects of stretching a spring in computing work in Example 7.24.

Converting the units of length to feet, we have

$$F(5/12) = 5/12k = 20 \text{ lb}.$$

Thus $k = 48$ lb/ft and $F(x) = 48x$.

We compute the total work performed by integrating $F(x)$ from $x = 0$ to $x = 5/12$:

$$\begin{aligned} W &= \int_0^{5/12} 48x\,dx \\ &= 24x^2\Big|_0^{5/12} \\ &= 25/6 \approx 4.1667 \text{ ft–lb}. \end{aligned}$$

Pumping Fluids

Fluid	lb/ft^3	kg/m^3
Concrete	150	2400
Fuel Oil	55.46	890.13
Gasoline	45.93	737.22
Iodine	307	4927
Methanol	49.3	791.3
Mercury	844	13546
Milk	63.6–65.4	1020 – 1050
Water	62.4	1000

Figure 7.33: Weight and Mass densities

Another useful example of the application of integration to compute work comes in the pumping of fluids, often illustrated in the context of emptying a storage tank by pumping the fluid out the top. This situation is different than our previous examples for the forces involved are constant. After all, the force required to move one cubic foot of water (about 62.4 lb) is the same regardless of its location in the tank. What is variable is the distance that cubic foot of water has to travel; water closer to the top travels less distance than water at the bottom, producing less work.

We demonstrate how to compute the total work done in pumping a fluid out of the top of a tank in the next two examples.

Notes:

Example 7.25 Computing work performed: pumping fluids

A cylindrical storage tank with a radius of 10 ft and a height of 30 ft is filled with water, which weighs approximately 62.4 lb/ft^3. Compute the amount of work performed by pumping the water up to a point 5 feet above the top of the tank.

Solution We will refer often to Figure 7.34 which illustrates the salient aspects of this problem.

We start as we often do: we partition an interval into subintervals. We orient our tank vertically since this makes intuitive sense with the base of the tank at $y = 0$. Hence the top of the water is at $y = 30$, meaning we are interested in subdividing the y-interval $[0, 30]$ into n subintervals as

$$0 = y_1 < y_2 < \cdots < y_{n+1} = 30.$$

Consider the work W_i of pumping only the water residing in the i^{th} subinterval, illustrated in Figure 7.34. The force required to move this water is equal to its weight which we calculate as volume $\times$ density. The volume of water in this subinterval is $V_i = 10^2\pi\Delta y_i$; its density is 62.4 lb/ft^3. Thus the required force is $6240\pi\Delta y_i$ lb.

We approximate the distance the force is applied by using any y-value contained in the i^{th} subinterval; for simplicity, we arbitrarily use y_i for now (it will not matter later on). The water will be pumped to a point 5 feet above the top of the tank, that is, to the height of $y = 35$ ft. Thus the distance the water at height y_i travels is $35 - y_i$ ft.

In all, the approximate work W_i peformed in moving the water in the i^{th} subinterval to a point 5 feet above the tank is

$$W_i \approx 6240\pi\Delta y_i(35 - y_i).$$

To approximate the total work performed in pumping out all the water from the tank, we sum all the work W_i performed in pumping the water from each of the n subintervals of $[0, 30]$:

$$W \approx \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} 6240\pi\Delta y_i(35 - y_i).$$

This is a Riemann sum. Taking the limit as the subinterval length goes to 0 gives

$$\begin{aligned} W &= \int_0^{30} 6240\pi(35 - y)\,dy \\ &= 6240\pi\left(35y - \frac{1}{2}y^2\right)\Bigg|_0^{30} \\ &= 11,762,123 \text{ ft–lb} \\ &\approx 1.176 \times 10^7 \text{ ft–lb}. \end{aligned}$$

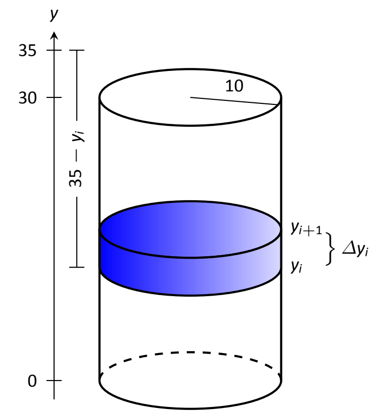


Figure 7.34: Illustrating a water tank in order to compute the work required to empty it in Example 7.25.

Notes:

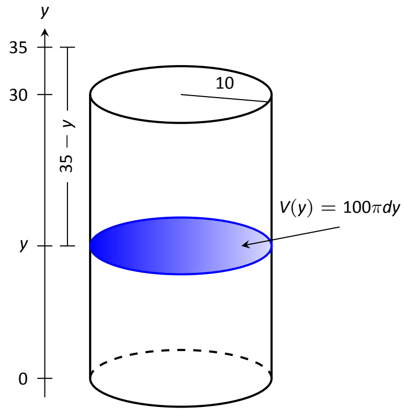


Figure 7.35: A simplified illustration for computing work.

We can "streamline" the above process a bit as we may now recognize what the important features of the problem are. Figure 7.35 shows the tank from Example 7.25 without the i^{th} subinterval identified. Instead, we just draw one differential element. This helps establish the height a small amount of water must travel along with the force required to move it (where the force is volume $\times$ density).

We demonstrate the concepts again in the next examples.

Example 7.26 Computing work performed: pumping fluids

A conical water tank has its top at ground level and its base 10 feet below ground. The radius of the cone at ground level is 2 ft. It is filled with water weighing 62.4 lb/ft^3 and is to be emptied by pumping the water to a spigot 3 feet above ground level. Find the total amount of work performed in emptying the tank.

Solution The conical tank is sketched in Figure 7.36. We can orient the tank in a variety of ways; we could let $y = 0$ represent the base of the tank and $y = 10$ represent the top of the tank, but we choose to keep the convention of the wording given in the problem and let $y = 0$ represent ground level and hence $y = -10$ represents the bottom of the tank. The actual "height" of the water does not matter; rather, we are concerned with the distance the water travels.

The figure also sketches a differential element, a cross–sectional circle. The radius of this circle is variable, depending on y. When $y = -10$, the circle has radius 0; when $y = 0$, the circle has radius 2. These two points, $(-10, 0)$ and $(0, 2)$, allow us to find the equation of the line that gives the radius of the cross–sectional circle, which is $r(y) = 1/5y + 2$. Hence the volume of water at this height is $V(y) = \pi(1/5y + 2)^2 dy$, where dy represents a very small height of the differential element. The force required to move the water at height y is $F(y) = 62.4 \times V(y)$.

The distance the water at height y travels is given by $h(y) = 3 - y$. Thus the total work done in pumping the water from the tank is

$$\begin{aligned} W &= \int_{-10}^{0} 62.4\pi(1/5y + 2)^2(3 - y)\, dy \\ &= 62.4\pi \int_{-10}^{0} \left(-\frac{1}{25}y^3 - \frac{17}{25}y^2 - \frac{8}{5}y + 12\right) dy \\ &= 62.4\pi \cdot \frac{220}{3} \approx 14,376 \text{ ft–lb}. \end{aligned}$$

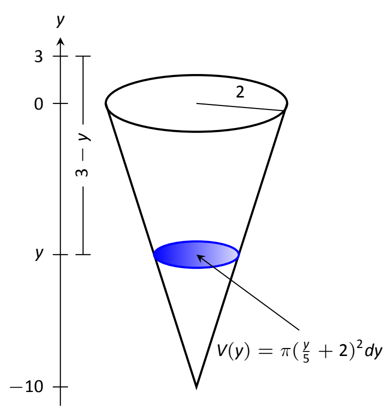


Figure 7.36: A graph of the conical water tank in Example 7.26.

Notes:

Example 7.27 Computing work performed: pumping fluids

A rectangular swimming pool is 20 ft wide and has a 3 ft "shallow end" and a 6 ft "deep end." It is to have its water pumped out to a point 2 ft above the current top of the water. The cross–sectional dimensions of the water in the pool are given in Figure 7.37; note that the dimensions are for the water, not the pool itself. Compute the amount of work performed in draining the pool.

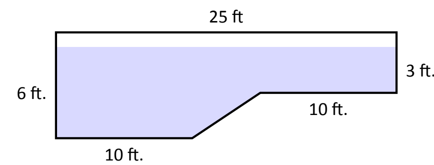


Figure 7.37: The cross–section of a swimming pool filled with water in Example 7.27.

Solution For the purposes of this problem we choose to set $y = 0$ to represent the bottom of the pool, meaning the top of the water is at $y = 6$. Figure 7.38 shows the pool oriented with this y-axis, along with 2 differential elements as the pool must be split into two different regions.

The top region lies in the y-interval of $[3, 6]$, where the length of the differential element is 25 ft as shown. As the pool is 20 ft wide, this differential element represents a this slice of water with volume $V(y) = 20 \cdot 25 \cdot dy$. The water is to be pumped to a height of $y = 8$, so the height function is $h(y) = 8 - y$. The work done in pumping this top region of water is

$$W_t = 62.4 \int_3^6 500(8 - y)\,dy = 327,600 \text{ ft–lb}.$$

The bottom region lies in the y-interval of $[0, 3]$; we need to compute the length of the differential element in this interval.

One end of the differential element is at $x = 0$ and the other is along the line segment joining the points $(10, 0)$ and $(15, 3)$. The equation of this line is $y = \frac{3}{5}(x - 10)$; as we will be integrating with respect to y, we rewrite this equation as $x = \frac{5}{3}y + 10$. So the length of the differential element is a difference of x-values: $x = 0$ and $x = \frac{5}{3}y + 10$, giving a length of $x = \frac{5}{3}y + 10$.

Again, as the pool is 20 ft wide, this differential element represents a thin slice of water with volume $V(y) = 20 \cdot \left(\frac{5}{3}y + 10\right) \cdot dy$; the height function is the same as before at $h(y) = 8 - y$. The work performed in emptying this part of the pool is

$$W_b = 62.4 \int_0^3 20\left(\frac{5}{3}y + 10\right)(8 - y)\,dy = 299,520 \text{ ft–lb}.$$

The total work in empyting the pool is

$$W = W_b + W_t = 327,600 + 299,520 = 627,120 \text{ ft–lb}.$$

Notice how the emptying of the bottom of the pool performs almost as much work as emptying the top. The top portion travels a shorter distance but has more water. In the end, this extra water produces more work.

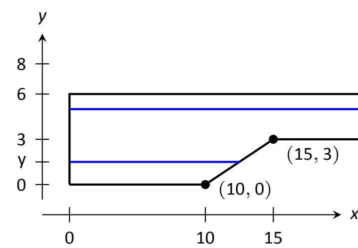


Figure 7.38: Orienting the pool and showing differential elements for Example 7.27.

The next section introduces one final application of the definite integral, the calculation of fluid force on a plate.

Notes:

Exercises 7.5

Terms and Concepts

1. What are the typical units of work?

2. If a man has a mass of 80 kg on Earth, will his mass on the moon be bigger, smaller, or the same?

3. If a woman weighs 130 lb on Earth, will her weight on the moon be bigger, smaller, or the same?

Problems

4. A 100 ft rope, weighing 0.1 lb/ft, hangs over the edge of a tall building.

   (a) How much work is done pulling the entire rope to the top of the building?

   (b) How much rope is pulled in when half of the total work is done?

5. A 50 m rope, with a mass density of 0.2 kg/m, hangs over the edge of a tall building.

   (a) How much work is done pulling the entire rope to the top of the building?

   (b) How much work is done pulling in the first 20 m?

6. A rope of length ℓ ft hangs over the edge of tall cliff. (Assume the cliff is taller than the length of the rope.) The rope has a weight density of *d* lb/ft.

   (a) How much work is done pulling the entire rope to the top of the cliff?

   (b) What percentage of the total work is done pulling in the first half of the rope?

   (c) How much rope is pulled in when half of the total work is done?

7. A 20 m rope with mass density of 0.5 kg/m hangs over the edge of a 10 m building. How much work is done pulling the rope to the top?

8. A crane lifts a 2,000 lb load vertically 30 ft with a 1" cable weighing 1.68 lb/ft.

   (a) How much work is done lifting the cable alone?

   (b) How much work is done lifting the load alone?

   (c) Could one conclude that the work done lifting the cable is negligible compared to the work done lifting the load?

9. A 100 lb bag of sand is lifted uniformly 120 ft in one minute. Sand leaks from the bag at a rate of 1/4 lb/s. What is the total work done in lifting the bag?

10. A box weighing 2 lb lifts 10 lb of sand vertically 50 ft. A crack in the box allows the sand to leak out such that 9 lb of sand is in the box at the end of the trip. Assume the sand leaked out at a uniform rate. What is the total work done in lifting the box and sand?

11. A force of 1000 lb compresses a spring 3 in. How much work is performed in compressing the spring?

12. A force of 2 N stretches a spring 5 cm. How much work is performed in stretching the spring?

13. A force of 50 lb compresses a spring from a natural length of 18 in to 12 in. How much work is performed in compressing the spring?

14. A force of 20 lb stretches a spring from a natural length of 6 in to 8 in. How much work is performed in stretching the spring?

15. 100 ft–lb of work is performed when a 60-lb force stretches a spring from its natural length. How far is the spring being stretched?

16. A force of 7 N stretches a spring from a natural length of 11 cm to 21 cm. How much work is performed in stretching the spring from a length of 16 cm to 21 cm?

17. A force of *f* N stretches a spring *d* m from its natural length. How much work is performed in stretching the spring?

18. A 20 lb weight is attached to a spring. The weight rests on the spring, compressing the spring from a natural length of 1 ft to 6 in.

    How much work is done in lifting the box 1.5 ft (i.e, the spring will be stretched 1 ft beyond its natural length)?

19. A 20 lb weight is attached to a spring. The weight rests on the spring, compressing the spring from a natural length of 1 ft to 6 in.

    How much work is done in lifting the box 6 in (i.e, bringing the spring back to its natural length)?

20. A 5 m tall cylindrical tank with radius of 2 m is filled with 3 m of gasoline, with a mass density of 737.22 kg/m^3. Compute the total work performed in pumping all the gasoline to the top of the tank.

21. A 6 ft cylindrical tank with a radius of 3 ft is filled with water, which has a weight density of 62.4 lb/ft^3. The water is to be pumped to a point 2 ft above the top of the tank.

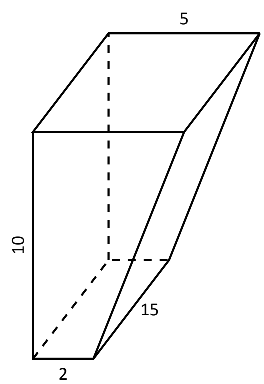
    (a) How much work is performed in pumping all the water from the tank?

    (b) How much work is performed in pumping 3 ft of water from the tank?

    (c) At what point is 1/2 of the total work done?

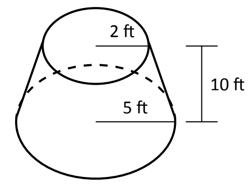
22. A gasoline tanker is filled with gasoline with a weight density of 45.93 lb/ft^3. The dispensing valve at the base is jammed shut, forcing the operator to empty the tank via pumping the gas to a point 1 ft above the top of the tank. Assume the tank is a perfect cylinder, 20 ft long with a diameter of 7.5 ft. How much work is performed in pumping all the gasoline from the tank?

23. A fuel oil storage tank is 10 ft deep with trapezoidal sides, 5 ft at the top and 2 ft at the bottom, and is 15 ft wide (see diagram below). Given that fuel oil weighs 55.46 lb/ft^3, find the work performed in pumping all the oil from the tank to a point 3 ft above the top of the tank.

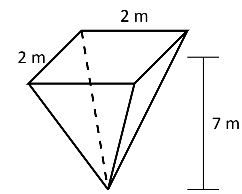


24. A conical water tank is 5 m deep with a top radius of 3 m. (This is similar to Example 7.26.) The tank is filled with pure water, with a mass density of 1000 kg/m^3.
    (a) Find the work performed in pumping all the water to the top of the tank.
    (b) Find the work performed in pumping the top 2.5 m of water to the top of the tank.
    (c) Find the work performed in pumping the top half of the water, by volume, to the top of the tank.

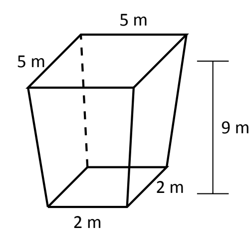
25. A water tank has the shape of a truncated cone, with dimensions given below, and is filled with water with a weight density of 62.4 lb/ft^3. Find the work performed in pumping all water to a point 1 ft above the top of the tank.



26. A water tank has the shape of an inverted pyramid, with dimensions given below, and is filled with water with a mass density of 1000 kg/m^3. Find the work performed in pumping all water to a point 5 m above the top of the tank.



27. A water tank has the shape of an truncated, inverted pyramid, with dimensions given below, and is filled with water with a mass density of 1000 kg/m^3. Find the work performed in pumping all water to a point 1 m above the top of the tank.



7.6 Fluid Forces (Optional)

In the unfortunate situation of a car driving into a body of water, the conventional wisdom is that the water pressure on the doors will quickly be so great that they will be effectively unopenable. (Survival techniques suggest immediately opening the door, rolling down or breaking the window, or waiting until the water fills up the interior at which point the pressure is equalized and the door will open. See Mythbusters episode #72 to watch Adam Savage test these options.)

How can this be true? How much force does it take to open the door of a submerged car? In this section we will find the answer to this question by examining the forces exerted by fluids.

We start with **pressure**, which is related to **force** by the following equations:

$$\text{Pressure} = \frac{\text{Force}}{\text{Area}} \quad \Leftrightarrow \quad \text{Force} = \text{Pressure} \times \text{Area}.$$

In the context of fluids, we have the following definition.

Definition 29 Fluid Pressure

Let w be the weight–density of a fluid. The **pressure** p exerted on an object at depth d in the fluid is $p = w \cdot d$.

We use this definition to find the **force** exerted on a horizontal sheet by considering the sheet's area.

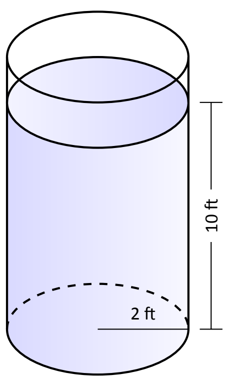


Figure 7.39: A cylindrical tank in Example 7.28.

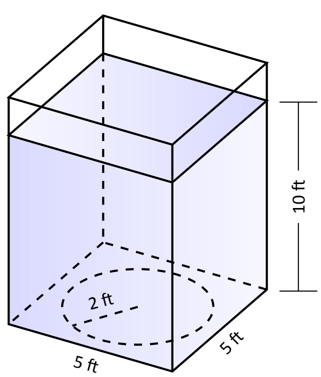


Figure 7.40: A rectangular tank in Example 7.28.

Example 7.28 Computing fluid force

1. A cylindrical storage tank has a radius of 2 ft and holds 10 ft of a fluid with a weight–density of 50 lb/ft^3. (See Figure 7.39.) What is the force exerted on the base of the cylinder by the fluid?
2. A rectangular tank whose base is a 5 ft square has a circular hatch at the bottom with a radius of 2 ft. The tank holds 10 ft of a fluid with a weight–density of 50 lb/ft^3. (See Figure 7.40.) What is the force exerted on the hatch by the fluid?

Solution

1. Using Definition 29, we calculate that the pressure exerted on the cylinder's base is $w \cdot d = 50\text{ lb/ft}^3 \times 10\text{ ft} = 500\text{ lb/ft}^2$. The area of the base is

Notes:

$\pi \cdot 2^2 = 4\pi$ ft^2. So the force exerted by the fluid is

$$F = 500 \times 4\pi = 6283 \text{ lb}.$$

Note that we effectively just computed the *weight* of the fluid in the tank.

2. The dimensions of the tank in this problem are irrelevant. All we are concerned with are the dimensions of the hatch and the depth of the fluid. Since the dimensions of the hatch are the same as the base of the tank in the previous part of this example, as is the depth, we see that the fluid force is the same. That is, $F = 6283$ lb.

   A key concept to understand here is that we are effectively measuring the weight of a 10 ft column of water above the hatch. The size of the tank holding the fluid does not matter.

The previous example demonstrates that computing the force exerted on a horizontally oriented plate is relatively easy to compute. What about a vertically oriented plate? For instance, suppose we have a circular porthole located on the side of a submarine. How do we compute the fluid force exerted on it?

Pascal's Principle states that the pressure exerted by a fluid at a depth is equal in all directions. Thus the pressure on any portion of a plate that is 1 ft below the surface of water is the same no matter how the plate is oriented. (Thus a hollow cube submerged at a great depth will not simply be "crushed" from above, but the sides will also crumple in. The fluid will exert force on *all* sides of the cube.)

So consider a vertically oriented plate as shown in Figure 7.41 submerged in a fluid with weight–density w. What is the total fluid force exerted on this plate? We find this force by first approximating the force on small horizontal strips.

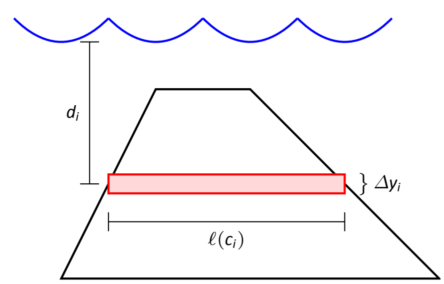


Figure 7.41: A thin, vertically oriented plate submerged in a fluid with weight–density w.

Let the top of the plate be at depth b and let the bottom be at depth a. (For now we assume that surface of the fluid is at depth 0, so if the bottom of the plate is 3 ft under the surface, we have $a = -3$. We will come back to this later.) We partition the interval $[a, b]$ into n subintervals

$$a = y_1 < y_2 < \cdots < y_{n+1} = b,$$

with the i^{th} subinterval having length Δy_i. The force F_i exerted on the plate in the i^{th} subinterval is $F_i = \text{Pressure} \times \text{Area}$.

The pressure is depth $\times w$. We approximate the depth of this thin strip by choosing any value d_i in $[y_i, y_{i+1}]$; the depth is approximately $-d_i$. (Our convention has d_i being a negative number, so $-d_i$ is positive.) For convenience, we let d_i be an endpoint of the subinterval; we let $d_i = y_i$.

The area of the thin strip is approximately length $\times$ width. The width is Δy_i. The length is a function of some y-value c_i in the i^{th} subinterval. We state the

Notes:

length is $\ell(c_i)$. Thus

$$\begin{aligned} F_i &= \text{Pressure} \times \text{Area} \\ &= -y_i \cdot w \times \ell(c_i) \cdot \Delta y_i. \end{aligned}$$

To approximate the total force, we add up the approximate forces on each of the n thin strips:

$$F = \sum_{i=1}^{n} F_i \approx \sum_{i=1}^{n} -w \cdot y_i \cdot \ell(c_i) \cdot \Delta y_i.$$

This is, of course, another Riemann Sum. We can find the exact force by taking a limit as the subinterval lengths go to 0; we evaluate this limit with a definite integral.

Key Idea 31 Fluid Force on a Vertically Oriented Plate

Let a vertically oriented plate be submerged in a fluid with weight–density w where the top of the plate is at $y = b$ and the bottom is at $y = a$. Let $\ell(y)$ be the length of the plate at y.

1. If $y = 0$ corresponds to the surface of the fluid, then the force exerted on the plate by the fluid is
$$F = \int_a^b w \cdot (-y) \cdot \ell(y)\, dy.$$
2. In general, let $d(y)$ represent the distance between the surface of the fluid and the plate at y. Then the force exerted on the plate by the fluid is
$$F = \int_a^b w \cdot d(y) \cdot \ell(y)\, dy.$$

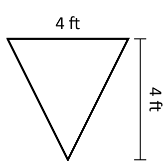


Figure 7.42: A thin plate in the shape of an isosceles triangle in Example 7.29.

Example 7.29 Finding fluid force

Consider a thin plate in the shape of an isosceles triangle as shown in Figure 7.42 submerged in water with a weight–density of 62.4 lb/ft^3. If the bottom of the plate is 10 ft below the surface of the water, what is the total fluid force exerted on this plate?

Solution We approach this problem in two different ways to illustrate the different ways Key Idea 31 can be implemented. First we will let $y = 0$ represent the surface of the water, then we will consider an alternate convention.

Notes:

1. We let $y = 0$ represent the surface of the water; therefore the bottom of the plate is at $y = -10$. We center the triangle on the y-axis as shown in Figure 7.43. The depth of the plate at y is $-y$ as indicated by the Key Idea. We now consider the length of the plate at y.

   We need to find equations of the left and right edges of the plate. The right hand side is a line that connects the points $(0, -10)$ and $(2, -6)$: that line has equation $x = \frac{1}{2}(y + 10)$. (Find the equation in the familiar $y = mx + b$ format and solve for x.) Likewise, the left hand side is described by the line $x = -\frac{1}{2}(y + 10)$. The total length is the distance between these two lines: $\ell(y) = \frac{1}{2}(y + 10) - \left(-\frac{1}{2}(y + 10)\right) = y + 10$.

   The total fluid force is then:

$$F = \int_{-10}^{-6} 62.4(-y)(y + 10)\, dy$$
$$= 62.4 \cdot \frac{176}{3} \approx 3660.8 \text{ lb.}$$

2. Sometimes it seems easier to orient the thin plate nearer the origin. For instance, consider the convention that the bottom of the triangular plate is at $(0, 0)$, as shown in Figure 7.44. The equations of the left and right hand sides are easy to find. They are $y = 2x$ and $y = -2x$, respectively, which we rewrite as $x = \frac{1}{2}y$ and $x = -\frac{1}{2}y$. Thus the length function is $\ell(y) = \frac{1}{2}y - \left(-\frac{1}{2}y\right) = y$.

   As the surface of the water is 10 ft above the base of the plate, we have that the surface of the water is at $y = 10$. Thus the depth function is the distance between $y = 10$ and y; $d(y) = 10 - y$. We compute the total fluid force as:

$$F = \int_{0}^{4} 62.4(10 - y)(y)\, dy$$
$$\approx 3660.8 \text{ lb.}$$

The correct answer is, of course, independent of the placement of the plate in the coordinate plane as long as we are consistent.

Example 7.30 Finding fluid force

Find the total fluid force on a car door submerged up to the bottom of its window in water, where the car door is a rectangle 40” long and 27” high (based on the dimensions of a 2005 Fiat Grande Punto.)

Solution The car door, as a rectangle, is drawn in Figure 7.45. Its length is $10/3$ ft and its height is 2.25 ft. We adopt the convention that the

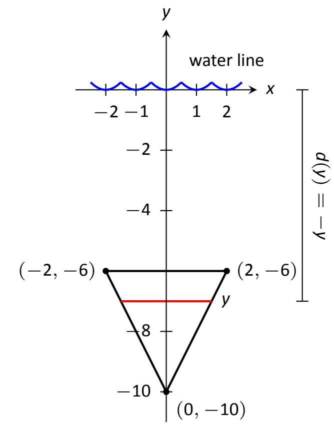


Figure 7.43: Sketching the triangular plate in Example 7.29 with the convention that the water level is at $y = 0$.

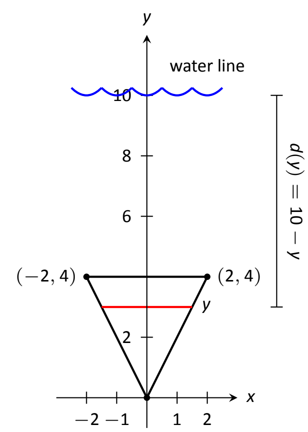


Figure 7.44: Sketching the triangular plate in Example 7.29 with the convention that the base of the triangle is at $(0, 0)$.

Notes:

top of the door is at the surface of the water, both of which are at $y = 0$. Using the weight–density of water of 62.4 lb/ft^3, we have the total force as

$$\begin{aligned} F &= \int_{-2.25}^{0} 62.4(-y)10/3\, dy \\ &= \int_{-2.25}^{0} -208y\, dy \\ &= -104y^2\Big|_{-2.25}^{0} \\ &= 526.5 \text{ lb.} \end{aligned}$$

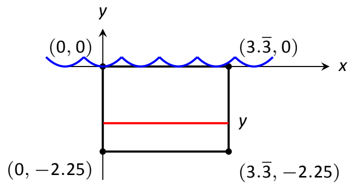


Figure 7.45: Sketching a submerged car door in Example 7.30.

Most adults would find it very difficult to apply over 500 lb of force to a car door while seated inside, making the door effectively impossible to open. This is counter–intuitive as most assume that the door would be relatively easy to open. The truth is that it is not, hence the survival tips mentioned at the beginning of this section.

Example 7.31 Finding fluid force

An underwater observation tower is being built with circular viewing portholes enabling visitors to see underwater life. Each vertically oriented porthole is to have a 3 ft diameter whose center is to be located 50 ft underwater. Find the total fluid force exerted on each porthole. Also, compute the fluid force on a horizontally oriented porthole that is under 50 ft of water.

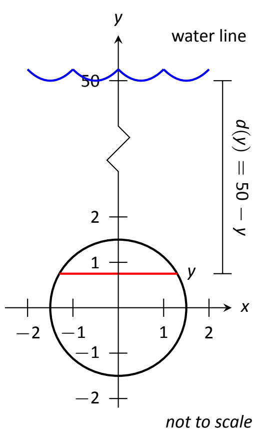


Figure 7.46: Measuring the fluid force on an underwater porthole in Example 7.31.

Solution We place the center of the porthole at the origin, meaning the surface of the water is at $y = 50$ and the depth function will be $d(y) = 50 - y$; see Figure 7.46

The equation of a circle with a radius of 1.5 is $x^2 + y^2 = 2.25$; solving for x we have $x = \pm\sqrt{2.25 - y^2}$, where the positive square root corresponds to the right side of the circle and the negative square root corresponds to the left side of the circle. Thus the length function at depth y is $\ell(y) = 2\sqrt{2.25 - y^2}$. Integrating on $[-1.5, 1.5]$ we have:

$$\begin{aligned} F &= 62.4\int_{-1.5}^{1.5} 2(50 - y)\sqrt{2.25 - y^2}\, dy \\ &= 62.4\int_{-1.5}^{1.5} \left(100\sqrt{2.25 - y^2} - 2y\sqrt{2.25 - y^2}\right) dy \\ &= 6240\int_{-1.5}^{1.5} \left(\sqrt{2.25 - y^2}\right) dy - 62.4\int_{-1.5}^{1.5} \left(2y\sqrt{2.25 - y^2}\right) dy. \end{aligned}$$

Notes:

The second integral above can be evaluated using Substitution. Let $u = 2.25 - y^2$ with $du = -2y\,dy$. The new bounds are: $u(-1.5) = 0$ and $u(1.5) = 0$; the new integral will integrate from $u = 0$ to $u = 0$, hence the integral is 0.

The first integral above finds the area of half a circle of radius 1.5, thus the first integral evaluates to $6240 \cdot \pi \cdot 1.5^2/2 = 22,054$. Thus the total fluid force on a vertically oriented porthole is $22,054$ lb.

Finding the force on a horizontally oriented porthole is more straightforward:

$$F = \text{Pressure} \times \text{Area} = 62.4 \cdot 50 \times \pi \cdot 1.5^2 = 22,054 \text{ lb.}$$

That these two forces are equal is not coincidental; it turns out that the fluid force applied to a vertically oriented circle whose center is at depth d is the same as force applied to a horizontally oriented circle at depth d.

We end this chapter with a reminder of the true skills meant to be developed here. We are not truly concerned with an ability to find fluid forces or the volumes of solids of revolution. Work done by a variable force is important, though measuring the work done in pulling a rope up a cliff is probably not.

What we are actually concerned with is the ability to solve certain problems by first approximating the solution, then refining the approximation, then recognizing if/when this refining process results in a definite integral through a limit. Knowing the formulas found inside the special boxes within this chapter is beneficial as it helps solve problems found in the exercises, and other mathematical skills are strengthened by properly applying these formulas. However, more importantly, understand how each of these formulas was constructed. Each is the result of a summation of approximations; each summation was a Riemann sum, allowing us to take a limit and find the exact answer through a definite integral.

The next chapter addresses an entirely different topic: sequences and series. In short, a sequence is a list of numbers, where a series is the summation of a list of numbers. These seemingly–simple ideas lead to very powerful mathematics.

Notes:

Exercises 7.6

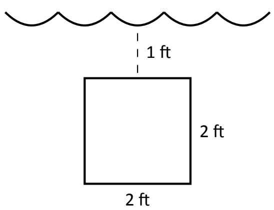
Terms and Concepts

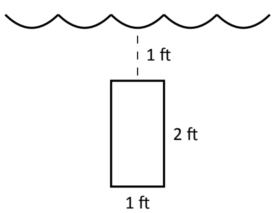
1. State in your own words Pascal's Principle.

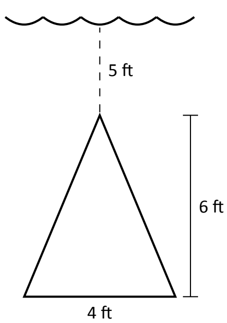
2. State in your own words how pressure is different from force.

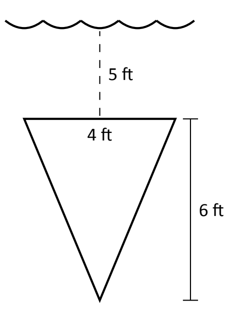
Problems

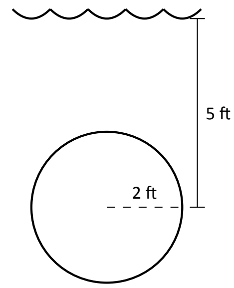
In Exercises 3 – 12, find the fluid force exerted on the given plate, submerged in water with a weight density of 62.4 lb/ft^3.

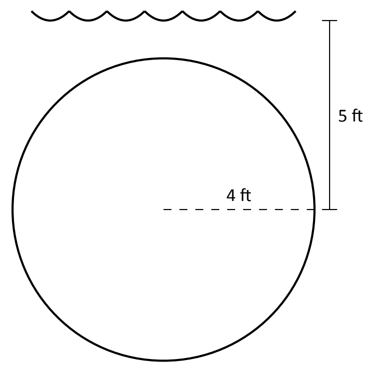
3. 

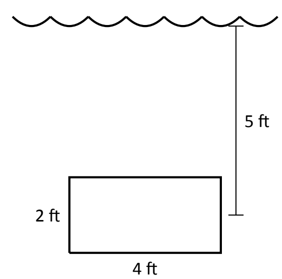
4. 

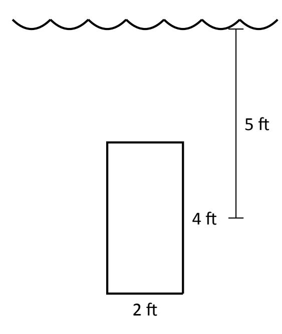
5. 

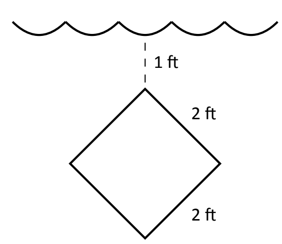
6. 

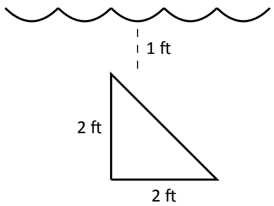
7. 

8. 

9. 

10. 

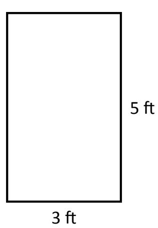
11. 

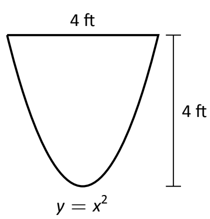
12. 

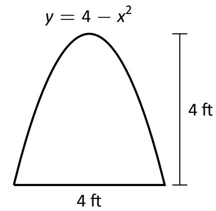
In Exercises 13 – 18, the side of a container is pictured. Find the fluid force exerted on this plate when the container is full of:

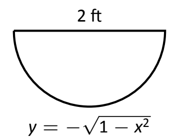
1. **water, with a weight density of 62.4 lb/ft^3, and**

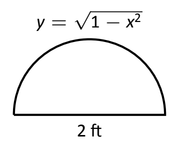
2. **concrete, with a weight density of 150 lb/ft^3.**

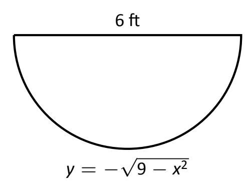
13. 

14. 

15. 

16. 

17. 

18. 

19. How deep must the center of a vertically oriented circular plate with a radius of 1 ft be submerged in water, with a weight density of 62.4 lb/ft^3, for the fluid force on the plate to reach 1,000 lb?

20. How deep must the center of a vertically oriented square plate with a side length of 2 ft be submerged in water, with a weight density of 62.4 lb/ft^3, for the fluid force on the plate to reach 1,000 lb?

8: Sequences and Series

This chapter introduces **sequences** and **series**, important mathematical constructions that are useful when solving a large variety of mathematical problems. The content of this chapter is considerably different from the content of the chapters before it. While the material we learn here definitely falls under the scope of "calculus," we will make very little use of derivatives or integrals. Limits are extremely important, though, especially limits that involve infinity.

One of the problems addressed by this chapter is this: suppose we know information about a function and its derivatives at a point, such as $f(1) = 3$, $f'(1) = 1$, $f''(1) = -2$, $f'''(1) = 7$, and so on. What can I say about $f(x)$ itself? Is there any reasonable approximation of the value of $f(2)$? The topic of Taylor Series addresses this problem, and allows us to make excellent approximations of functions when limited knowledge of the function is available.

Notation: We use $\mathbb{N}$ to describe the set of natural numbers, that is, the integers 1, 2, 3, ...

Factorial: The expression 3! refers to the number $3 \cdot 2 \cdot 1 = 6$.

In general, $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$, where n is a natural number.

We define $0! = 1$. This is so that the property $(n+1)! = (n+1)n!$ that holds for $n \geq 1$ also holds for $n = 0$. This is similar to how the definition of an exponent is extended from positive integers to zero in the natural way.

8.1 Sequences

We commonly refer to a set of events that occur one after the other as a *sequence* of events. In mathematics, we use the word *sequence* to refer to an ordered set of numbers, i.e., a set of numbers that "occur one after the other."

For instance, the numbers 2, 4, 6, 8, ..., form a sequence. The order is important; the first number is 2, the second is 4, etc. It seems natural to seek a formula that describes a given sequence, and often this can be done. For instance, the sequence above could be described by the function $a(n) = 2n$, for the values of $n = 1, 2, \ldots$ To find the 10$^{\text{th}}$ term in the sequence, we would compute $a(10)$. This leads us to the following, formal definition of a sequence.

Definition 30 Sequence

A **sequence** is a function $a(n)$ whose domain is $\mathbb{N}$ (the set $\{1, 2, 3, 4, \ldots\}$ of natural numbers). The **range** of a sequence is the set of all distinct values of $a(n)$.

The **terms** of a sequence are the values $a(1)$, $a(2)$, ..., which are usually denoted with subscripts as a_1, a_2, ...

A sequence $a(n)$ is often denoted as $\{a_n\}$.

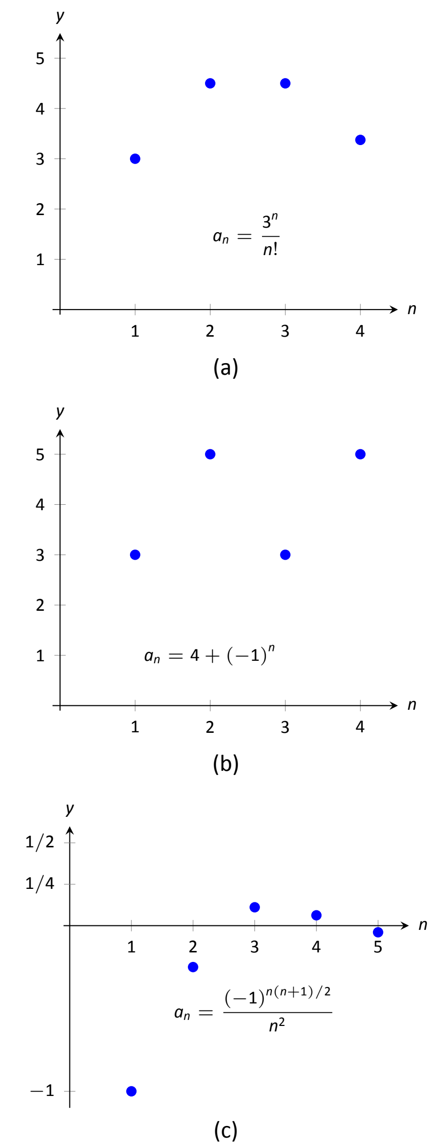


Figure 8.1: Plotting sequences in Example 8.1.

Example 8.1 Listing terms of a sequence

List the first four terms of the following sequences.

1. $\{a_n\} = \left\{\dfrac{3^n}{n!}\right\}$ 2. $\{a_n\} = \{4+(-1)^n\}$ 3. $\{a_n\} = \left\{\dfrac{(-1)^{n(n+1)/2}}{n^2}\right\}$

Solution

1. $a_1 = \dfrac{3^1}{1!} = 3;$ $a_2 = \dfrac{3^2}{2!} = \dfrac{9}{2};$ $a_3 = \dfrac{3^3}{3!} = \dfrac{9}{2};$ $a_4 = \dfrac{3^4}{4!} = \dfrac{27}{8}$

   We can plot the terms of a sequence with a scatter plot. The "x"-axis is used for the values of n, and the values of the terms are plotted on the y-axis. To visualize this sequence, see Figure 8.1(a).

2. $a_1 = 4 + (-1)^1 = 3;$ $a_2 = 4 + (-1)^2 = 5;$

   $a_3 = 4+(-1)^3 = 3;$ $a_4 = 4+(-1)^4 = 5.$ Note that the range of this sequence is finite, consisting of only the values 3 and 5. This sequence is plotted in Figure 8.1(b).

3. $a_1 = \dfrac{(-1)^{1(2)/2}}{1^2} = -1;$ $a_2 = \dfrac{(-1)^{2(3)/2}}{2^2} = -\dfrac{1}{4}$

   $a_3 = \dfrac{(-1)^{3(4)/2}}{3^2} = \dfrac{1}{9}$ $a_4 = \dfrac{(-1)^{4(5)/2}}{4^2} = \dfrac{1}{16};$

   $a_5 = \dfrac{(-1)^{5(6)/2}}{5^2} = -\dfrac{1}{25}.$

   We gave one extra term to begin to show the pattern of signs is "$-, -, +, +, -, -, \ldots$, due to the fact that the exponent of -1 is a special quadratic. Note that the sign is $+$ exactly when $n(n+1)$ is divisible by 4, which is exactly when either n or $n+1$ is divisible by 4. This sequence is plotted in Figure 8.1(c).

Example 8.2 Determining a formula for a sequence

Find the n^{th} term of the following sequences, i.e., find a function that describes each of the given sequences.

1. $2, 5, 8, 11, 14, \ldots$
2. $2, -5, 10, -17, 26, -37, \ldots$
3. $1, 1, 2, 6, 24, 120, 720, \ldots$
4. $\dfrac{5}{2}, \dfrac{5}{2}, \dfrac{15}{8}, \dfrac{5}{4}, \dfrac{25}{32}, \ldots$

Notes:

Solution We should first note that there is never exactly one function that describes a finite set of numbers as a sequence. There are many sequences that start with 2, then 5, as our first example does. We are looking for a simple formula that describes the terms given, knowing there is possibly more than one answer.

1. Note how each term is 3 more than the previous one. This implies a linear function would be appropriate: $a(n) = a_n = 3n + b$ for some appropriate value of b. As we want $a_1 = 2$, we set $b = -1$. Thus $a_n = 3n - 1$.

2. First notice how the sign changes from term to term. This is most commonly accomplished by multiplying the terms by either $(-1)^n$ or $(-1)^{n+1}$. Using $(-1)^n$ multiplies the odd terms by (-1); using $(-1)^{n+1}$ multiplies the even terms by (-1). As this sequence has negative even terms, we will multiply by $(-1)^{n+1}$.

   After this, we might feel a bit stuck as to how to proceed. At this point, we are just looking for a pattern of some sort: what do the numbers 2, 5, 10, 17, etc., have in common? There are many correct answers, but the one that we'll use here is that each is one more than a perfect square. That is, $2 = 1^1 + 1$, $5 = 2^2 + 1$, $10 = 3^2 + 1$, etc. Thus our formula is $a_n = (-1)^{n+1}(n^2 + 1)$.

3. One who is familiar with the factorial function will readily recognize these numbers. They are 0!, 1!, 2!, 3!, etc. Since our sequences start with $n = 1$, we cannot write $a_n = n!$, for this misses the 0! term. Instead, we shift by 1, and write $a_n = (n - 1)!$.

4. This one may appear difficult, especially as the first two terms are the same, but a little "sleuthing" will help. Notice how the terms in the numerator are always multiples of 5, and the terms in the denominator are always powers of 2. Does something as simple as $a_n = \frac{5n}{2^n}$ work?

   When $n = 1$, we see that we indeed get $5/2$ as desired. When $n = 2$, we get $10/4 = 5/2$. Further checking shows that this formula indeed matches the other terms of the sequence.

A common mathematical endeavor is to create a new mathematical object (for instance, a sequence) and then apply previously known mathematics to the new object. We do so here. The fundamental concept of calculus is the limit, so we will investigate what it means to find the limit of a sequence.

Notes:

Definition 31 Limit of a Sequence, Convergent, Divergent

Let $\{a_n\}$ be a sequence and let L be a real number. Given any $\varepsilon > 0$, if an m can be found such that $|a_n - L| < \varepsilon$ for all $n > m$, then we say the **limit of $\{a_n\}$, as n approaches infinity, is L**, denoted

$$\lim_{n\to\infty} a_n = L.$$

If $\lim\limits_{n\to\infty} a_n$ exists, we say the sequence **converges**; otherwise, the sequence **diverges**.

This definition states, informally, that if the limit of a sequence is L, then if you go far enough out along the sequence, all subsequent terms will be *really close* to L. Of course, the terms "far enough" and "really close" are subjective terms, but hopefully the intent is clear.

This definition is reminiscent of the ε–δ proofs of Chapter 1. In that chapter we developed other tools to evaluate limits apart from the formal definition; we do so here as well.

Theorem 57 Limit of a Sequence

Let $\{a_n\}$ be a sequence and let $f(x)$ be a function whose domain contains the positive real numbers where $f(n) = a_n$ for all n in $\mathbb{N}$.

If $\lim\limits_{x\to\infty} f(x) = L$, then $\lim\limits_{n\to\infty} a_n = L$.

Theorem 57 allows us, in certain cases, to apply the tools developed in Chapter 1 to limits of sequences. Note two things *not* stated by the theorem:

1. If $\lim\limits_{x\to\infty} f(x)$ does not exist, we cannot conclude that $\lim\limits_{n\to\infty} a_n$ does not exist. It may, or may not, exist. For instance, we can define a sequence $\{a_n\} = \{\cos(2\pi n)\}$. Let $f(x) = \cos(2\pi x)$. Since the cosine function oscillates over the real numbers, the limit $\lim\limits_{x\to\infty} f(x)$ does not exist.

   However, for every positive integer n, $\cos(2\pi n) = 1$, so $\lim\limits_{n\to\infty} a_n = 1$.

2. If we cannot find a function $f(x)$ whose domain contains the positive real numbers where $f(n) = a_n$ for all n in $\mathbb{N}$, we cannot conclude $\lim\limits_{n\to\infty} a_n$ does not exist. It may, or may not, exist.

Notes:

Example 8.3 Determining convergence/divergence of a sequence

Determine the convergence or divergence of the following sequences.

1. $\{a_n\} = \left\{\dfrac{3n^2-2n+1}{n^2-1000}\right\}$ 2. $\{a_n\} = \{\cos n\}$ 3. $\{a_n\} = \left\{\dfrac{(-1)^n}{n}\right\}$

SOLUTION

1. Using Theorem 11, we can state that $\lim\limits_{x\to\infty} \dfrac{3x^2-2x+1}{x^2-1000} = 3$. (We could have also directly applied l'Hôpital's Rule.) Thus the sequence $\{a_n\}$ converges, and its limit is 3. A scatter plot of every 5 values of a_n is given in Figure 8.2 (a). The values of a_n vary widely near $n = 30$, ranging from about -73 to 125, but as n grows, the values approach 3.

2. The limit $\lim\limits_{x\to\infty} \cos x$ does not exist, as $\cos x$ oscillates (and takes on every value in $[-1, 1]$ infinitely many times). Thus we cannot apply Theorem 57.

   The fact that the cosine function oscillates strongly hints that $\cos n$, when n is restricted to $\mathbb{N}$, will also oscillate. Figure 8.2 (b), where the sequence is plotted, shows that this is true. Because only discrete values of cosine are plotted, it does not bear strong resemblance to the familiar cosine wave.

   We conclude that $\lim\limits_{n\to\infty} a_n$ does not exist.

3. We cannot actually apply Theorem 57 here, as the function $f(x) = (-1)^x/x$ is not well defined. (What does $(-1)^{\sqrt{2}}$ mean? In actuality, there is an answer, but it involves *complex analysis*, beyond the scope of this text.) So for now we say that we cannot determine the limit. (But we will be able to very soon.) By looking at the plot in Figure 8.2 (c), we would like to conclude that the sequence converges to 0. That is true, but at this point we are unable to decisively say so.

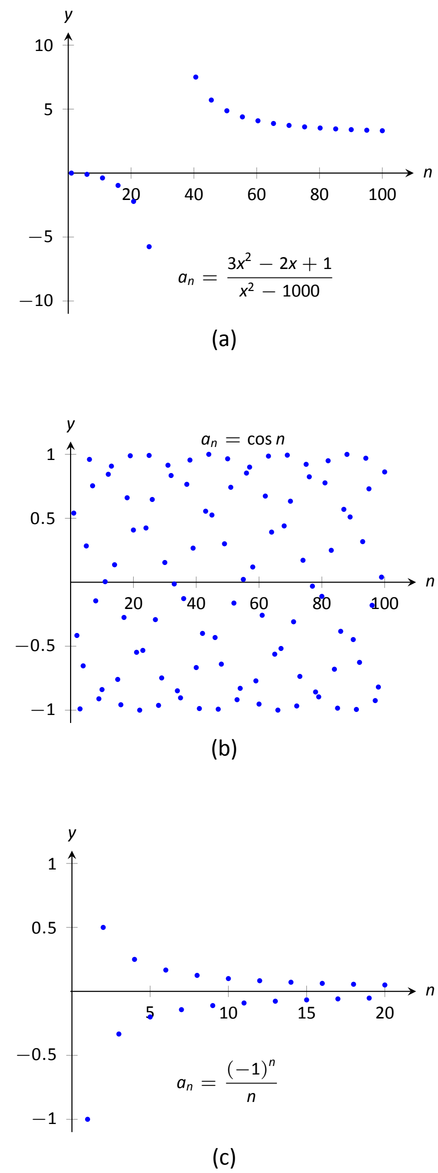


Figure 8.2: Scatter plots of the sequences in Example 8.3.

It seems that $\{(-1)^n/n\}$ converges to 0 but we lack the formal tool to prove it. The following theorem gives us that tool.

Theorem 58 Absolute Value Theorem

Let $\{a_n\}$ be a sequence. Then $\lim\limits_{n\to\infty} |a_n| = 0$ if and only if $\lim\limits_{n\to\infty} a_n = 0$

Example 8.4 Determining the convergence/divergence of a sequence

Determine the convergence or divergence of the following sequences.

Notes:

1. $\{a_n\} = \left\{\dfrac{(-1)^n}{n}\right\}$ 2. $\{a_n\} = \left\{\dfrac{(-1)^n(n+1)}{n}\right\}$

SOLUTION

1. This appeared in Example 8.3. We want to apply Theorem 58, so consider the limit of $\{|a_n|\}$:

$$\begin{aligned}\lim_{n\to\infty} |a_n| &= \lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| \\ &= \lim_{n\to\infty} \frac{1}{n} \\ &= 0.\end{aligned}$$

Since this limit is 0, we can apply Theorem 58 and state that $\lim\limits_{n\to\infty} a_n = 0$.

2. Because of the alternating nature of this sequence (i.e., every other term is multiplied by -1), we cannot simply look at the limit $\lim\limits_{x\to\infty} \dfrac{(-1)^x(x+1)}{x}$. We can try to apply the techniques of Theorem 58:

$$\begin{aligned}\lim_{n\to\infty} |a_n| &= \lim_{n\to\infty} \left|\frac{(-1)^n(n+1)}{n}\right| \\ &= \lim_{n\to\infty} \frac{n+1}{n} \\ &= 1.\end{aligned}$$

We have concluded that when we ignore the alternating sign, the sequence approaches 1. This means we cannot apply Theorem 58; it states the the limit must be 0 in order to conclude anything.

Since we know that the signs of the terms alternate *and* we know that the limit of $|a_n|$ is 1, we know that as n approaches infinity, the terms will alternate between values close to 1 and -1, meaning the sequence diverges. A plot of this sequence is given in Figure 8.3.

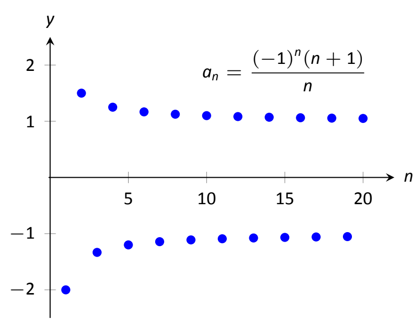


Figure 8.3: A plot of a sequence in Example 8.4, part 2.

We continue our study of the limits of sequences by considering some of the properties of these limits.

Notes:

Theorem 59 Properties of the Limits of Sequences

Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $\lim\limits_{n\to\infty} a_n = L$, $\lim\limits_{n\to\infty} b_n = K$, and let c be a real number.

1. $\lim\limits_{n\to\infty} (a_n \pm b_n) = L \pm K$
2. $\lim\limits_{n\to\infty} (a_n \cdot b_n) = L \cdot K$
3. $\lim\limits_{n\to\infty} (a_n/b_n) = L/K$ when $K \neq 0$
4. $\lim\limits_{n\to\infty} c \cdot a_n = c \cdot L$

Example 8.5 Applying properties of limits of sequences

Let the following sequences, and their limits, be given:

- $\{a_n\} = \left\{\dfrac{n+1}{n^2}\right\}$, and $\lim\limits_{n\to\infty} a_n = 0$;
- $\{b_n\} = \left\{\left(1+\dfrac{1}{n}\right)^n\right\}$, and $\lim\limits_{n\to\infty} b_n = e$; and
- $\{c_n\} = \{n \cdot \sin(5/n)\}$, and $\lim\limits_{n\to\infty} c_n = 5$.

Evaluate the following limits.

1. $\lim\limits_{n\to\infty} (a_n + b_n)$ 2. $\lim\limits_{n\to\infty} (b_n \cdot c_n)$ 3. $\lim\limits_{n\to\infty} (1000 \cdot a_n)$

Solution We will use Theorem 59 to answer each of these.

1. Since $\lim\limits_{n\to\infty} a_n = 0$ and $\lim\limits_{n\to\infty} b_n = e$, we conclude that $\lim\limits_{n\to\infty} (a_n + b_n) = 0 + e = e$. So even though we are adding something to each term of the sequence b_n, we are adding something so small that the final limit is the same as before.

2. Since $\lim\limits_{n\to\infty} b_n = e$ and $\lim\limits_{n\to\infty} c_n = 5$, we conclude that $\lim\limits_{n\to\infty} (b_n \cdot c_n) = e \cdot 5 = 5e$.

3. Since $\lim\limits_{n\to\infty} a_n = 0$, we have $\lim\limits_{n\to\infty} 1000a_n = 1000 \cdot 0 = 0$. It does not matter that we multiply each term by 1000; the sequence still approaches 0. (It just takes longer to get close to 0.)

There is more to learn about sequences than just their limits. We will also study their range and the relationships terms have with the terms that follow. We start with some definitions describing properties of the range.

Notes:

Definition 32 Bounded and Unbounded Sequences

A sequence $\{a_n\}$ is said to be **bounded** if there exists real numbers m and M such that $m \leq a_n \leq M$ for all n in $\mathbb{N}$.

A sequence $\{a_n\}$ is said to be **unbounded** if it is not bounded.

A sequence $\{a_n\}$ is said to be **bounded above** if there exists an M such that $a_n \leq M$ for all n in $\mathbb{N}$; it is **bounded below** if there exists an m such that $m \leq a_n$ for all n in $\mathbb{N}$.

It follows from this definition that an unbounded sequence may be bounded above or bounded below; a sequence that is both bounded above and below is simply a bounded sequence.

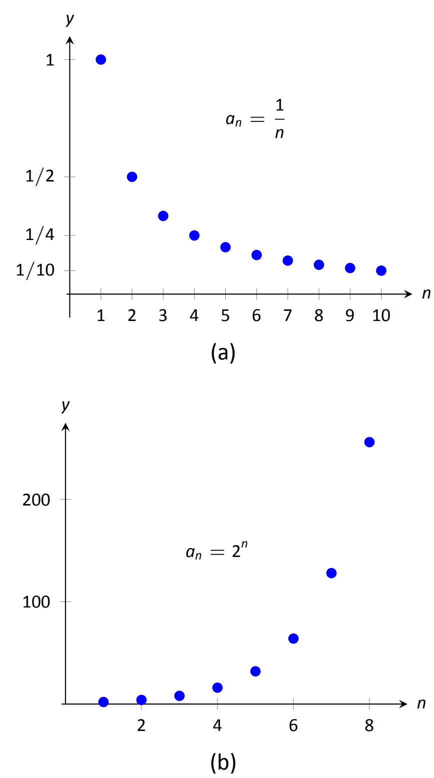


Figure 8.4: A plot of $\{a_n\} = \{1/n\}$ and $\{a_n\} = \{2^n\}$ from Example 8.6.

Example 8.6 Determining boundedness of sequences

Determine the boundedness of the following sequences.

1. $\{a_n\} = \left\{\dfrac{1}{n}\right\}$ 2. $\{a_n\} = \{2^n\}$

SOLUTION

1. The terms of this sequence are always positive but are decreasing, so we have $0 \leq a_n \leq 1$ for all n. Thus this sequence is bounded. Figure 8.4(a) illustrates this.

2. The terms of this sequence obviously grow without bound. However, it is also true that these terms are all positive, meaning $0 < a_n$. Thus we can say the sequence is unbounded, but also bounded below. Figure 8.4(b) illustrates this.

The previous example produces some interesting concepts. First, we can recognize that the sequence $\{1/n\}$ converges to 0. This says, informally, that "most" of the terms of the sequence are "really close" to 0. This implies that the sequence is bounded, using the following logic. First, "most" terms are near 0, so we could find some sort of bound on these terms (using Definition 31, the bound is ε). That leaves a "few" terms that are not near 0 (i.e., a *finite* number of terms). A finite list of numbers is always bounded.

This logic implies that if a sequence converges, it must be bounded. This is indeed true, as stated by the following theorem.

Notes:

Theorem 60 Convergent Sequences are Bounded

Let $\{a_n\}$ be a convergent sequence. Then $\{a_n\}$ is bounded.

Note: Keep in mind what Theorem 60 does *not* say. It does not say that bounded sequences must converge, nor does it say that if a sequence does not converge, it is not bounded.

In Example 8.5 we saw the sequence $\{b_n\} = \left\{(1+1/n)^n\right\}$, where it was stated that $\lim_{n\to\infty} b_n = e$. (Note that this is simply restating part of Theorem 5.) Even though it may be difficult to intuitively grasp the behavior of this sequence, we know immediately that it is bounded.

Another interesting concept to come out of Example 8.6 again involves the sequence $\{1/n\}$. We stated, without proof, that the terms of the sequence were decreasing. That is, that $a_{n+1} < a_n$ for all n. (This is easy to show. Clearly $n < n+1$. Taking reciprocals flips the inequality: $1/n > 1/(n+1)$. This is the same as $a_n > a_{n+1}$.) Sequences that either steadily increase or decrease are important, so we give this property a name.

Definition 33 Monotonic Sequences

1. A sequence $\{a_n\}$ is **monotonically increasing** if $a_n \leq a_{n+1}$ for all n, i.e.,
$$a_1 \leq a_2 \leq a_3 \leq \cdots a_n \leq a_{n+1} \cdots$$
2. A sequence $\{a_n\}$ is **monotonically decreasing** if $a_n \geq a_{n+1}$ for all n, i.e.,
$$a_1 \geq a_2 \geq a_3 \geq \cdots a_n \geq a_{n+1} \cdots$$
3. A sequence is **monotonic** if it is monotonically increasing or monotonically decreasing.

Note: It is sometimes useful to call a monotonically increasing sequence *strictly increasing* if $a_n < a_{n+1}$ for all n; i.e, we remove the possibility that subsequent terms are equal.
A similar statement holds for *strictly decreasing.*

Example 8.7 Determining monotonicity
Determine the monotonicity of the following sequences.

1. $\{a_n\} = \left\{\dfrac{n+1}{n}\right\}$
2. $\{a_n\} = \left\{\dfrac{n^2+1}{n+1}\right\}$
3. $\{a_n\} = \left\{\dfrac{n^2-9}{n^2-10n+26}\right\}$
4. $\{a_n\} = \left\{\dfrac{n^2}{n!}\right\}$

Solution In each of the following, we will examine $a_{n+1}-a_n$. If $a_{n+1}-a_n > 0$, we conclude that $a_n < a_{n+1}$ and hence the sequence is increasing. If

Notes:

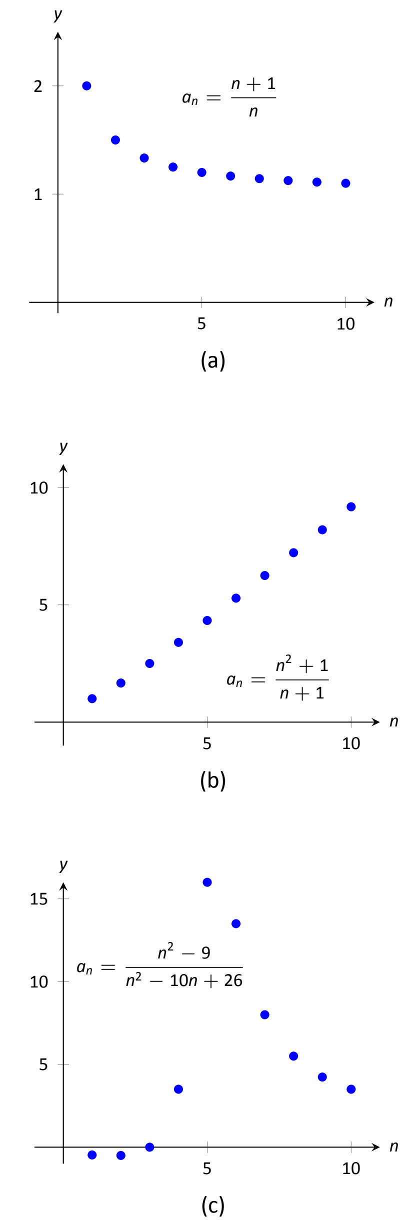


Figure 8.5: Plots of sequences in Example 8.7.

$a_{n+1} - a_n < 0$, we conclude that $a_n > a_{n+1}$ and the sequence is decreasing. Of course, a sequence need not be monotonic and perhaps neither of the above will apply.

We also give a scatter plot of each sequence. These are useful as they suggest a pattern of monotonicity, but analytic work should be done to confirm a graphical trend.

1. $$\begin{aligned} a_{n+1} - a_n &= \frac{n+2}{n+1} - \frac{n+1}{n} \\ &= \frac{(n+2)(n) - (n+1)^2}{(n+1)n} \\ &= \frac{-1}{n(n+1)} \\ &< 0 \quad \text{for all } n. \end{aligned}$$

   Since $a_{n+1} - a_n < 0$ for all n, we conclude that the sequence is decreasing.

2. $$\begin{aligned} a_{n+1} - a_n &= \frac{(n+1)^2+1}{n+2} - \frac{n^2+1}{n+1} \\ &= \frac{\big((n+1)^2+1\big)(n+1) - (n^2+1)(n+2)}{(n+1)(n+2)} \\ &= \frac{n^2+4n+1}{(n+1)(n+2)} \\ &> 0 \quad \text{for all } n. \end{aligned}$$

   Since $a_{n+1} - a_n > 0$ for all n, we conclude the sequence is increasing.

3. We can clearly see in Figure 8.5 (c), where the sequence is plotted, that it is not monotonic. However, it does seem that after the first 4 terms it is decreasing. To understand why, perform the same analysis as done before:

   $$\begin{aligned} a_{n+1} - a_n &= \frac{(n+1)^2 - 9}{(n+1)^2 - 10(n+1) + 26} - \frac{n^2-9}{n^2-10n+26} \\ &= \frac{n^2+2n-8}{n^2-8n+17} - \frac{n^2-9}{n^2-10n+26} \\ &= \frac{(n^2+2n-8)(n^2-10n+26) - (n^2-9)(n^2-8n+17)}{(n^2-8n+17)(n^2-10n+26)} \\ &= \frac{-10n^2+60n-55}{(n^2-8n+17)(n^2-10n+26)}. \end{aligned}$$

   We want to know when this is greater than, or less than, 0. The denominator is always positive, therefore we are only concerned with the numerator. Using the quadratic formula, we can determine that $-10n^2 + 60n -$

Notes:

$55 = 0$ when $n \approx 1.13, 4.87$. So for $n < 1.13$, the sequence is decreasing. Since we are only dealing with the natural numbers, this means that $a_1 > a_2$.

Between 1.13 and 4.87, i.e., for $n = 2$, 3 and 4, we have that $a_{n+1} > a_n$ and the sequence is increasing. (That is, when $n = 2$, 3 and 4, the numerator $-10n^2 + 60n + 55$ from the fraction above is > 0.)

When $n > 4.87$, i.e, for $n \geq 5$, we have that $-10n^2 + 60n + 55 < 0$, hence $a_{n+1} - a_n < 0$, so the sequence is decreasing.

In short, the sequence is simply not monotonic. However, it is useful to note that for $n \geq 5$, the sequence is monotonically decreasing.

4. Again, the plot in Figure 8.6 shows that the sequence is not monotonic, but it suggests that it is monotonically decreasing after the first term. We perform the usual analysis to confirm this.

$$\begin{aligned} a_{n+1} - a_n &= \frac{(n+1)^2}{(n+1)!} - \frac{n^2}{n!} \\ &= \frac{(n+1)^2 - n^2(n+1)}{(n+1)!} \\ &= \frac{-n^3 + 2n + 1}{(n+1)!} \end{aligned}$$

When $n = 1$, the above expression is > 0; for $n \geq 2$, the above expression is < 0. Thus this sequence is not monotonic, but it is monotonically decreasing after the first term.

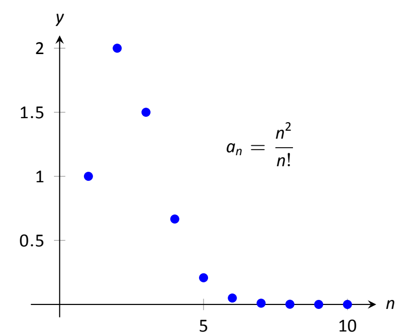


Figure 8.6: A plot of $\{a_n\} = \{n^2/n!\}$ in Example 8.7.

Knowing that a sequence is monotonic can be useful. In particular, if we know that a sequence is bounded and monotonic, we can conclude it converges! Consider, for example, a sequence that is monotonically decreasing and is bounded below. We know the sequence is always getting smaller, but that there is a bound to how small it can become. This is enough to prove that the sequence will converge, as stated in the following theorem.

Theorem 61 Bounded Monotonic Sequences are Convergent

1. Let $\{a_n\}$ be a bounded, monotonic sequence. Then $\{a_n\}$ converges; i.e., $\lim_{n\to\infty} a_n$ exists.
2. Let $\{a_n\}$ be a monotonically increasing sequence that is bounded above. Then $\{a_n\}$ converges.
3. Let $\{a_n\}$ be a monotonically decreasing sequence that is bounded below. Then $\{a_n\}$ converges.

Notes:

Consider once again the sequence $\{a_n\} = \{1/n\}$. It is easy to show it is monotonically decreasing and that it is always positive (i.e., bounded below by 0). Therefore we can conclude by Theorem 61 that the sequence converges. We already knew this by other means, but in the following section this theorem will become very useful.

Sequences are a great source of mathematical inquiry. The On-Line Encyclopedia of Integer Sequences (`http://oeis.org`) contains thousands of sequences with formulas when known. (As of this writing, there are 305,558 sequences in the database. Not all sequences have a known formula.) Perusing this database quickly demonstrates that a single sequence can represent several different "real life" phenomena.

Interesting as this is, our interest actually lies elsewhere. We are more interested in the *sum* of a sequence. That is, given a sequence $\{a_n\}$, we are very interested in $a_1 + a_2 + a_3 + \cdots$. Of course, one might immediately counter with "Doesn't this just add up to 'infinity'?" Many times, yes, but there are many important cases where the answer is no. This is the topic of *series*, which we begin to investigate in the next section.

Notes:

Exercises 8.1

Terms and Concepts

1. Use your own words to define a *sequence.*

2. The domain of a sequence is the ________ numbers.

3. Use your own words to describe the *range* of a sequence.

4. Describe what it means for a sequence to be *bounded*.

Problems

In Exercises 5 – 8, give the first five terms of the given sequence.

5. $\{a_n\} = \left\{ \dfrac{4^n}{(n+1)!} \right\}$

6. $\{b_n\} = \left\{ \left(-\dfrac{3}{2}\right)^n \right\}$

7. $\{c_n\} = \left\{ -\dfrac{n^{n+1}}{n+2} \right\}$

8. $\{d_n\} = \left\{ \dfrac{1}{\sqrt{5}} \left(\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n \right) \right\}$

In Exercises 9 – 12, determine the n^{th} term of the given sequence.

9. 4, 7, 10, 13, 16, ...

10. $3, -\dfrac{3}{2}, \dfrac{3}{4}, -\dfrac{3}{8}, \ldots$

11. 10, 20, 40, 80, 160, ...

12. $1, 1, \dfrac{1}{2}, \dfrac{1}{6}, \dfrac{1}{24}, \dfrac{1}{120}, \ldots$

In Exercises 13 – 16, use the following information to determine the limit of the given sequences.

- $\{a_n\} = \left\{ \dfrac{2^n - 20}{2^n} \right\}; \qquad \lim\limits_{n\to\infty} a_n = 1$
- $\{b_n\} = \left\{ \left(1 + \dfrac{2}{n}\right)^n \right\}; \qquad \lim\limits_{n\to\infty} b_n = e^2$
- $\{c_n\} = \{\sin(3/n)\}; \qquad \lim\limits_{n\to\infty} c_n = 0$

13. $\{a_n\} = \left\{ \dfrac{2^n - 20}{7 \cdot 2^n} \right\}$

14. $\{a_n\} = \{3b_n - a_n\}$

15. $\{a_n\} = \left\{ \sin(3/n) \left(1 + \dfrac{2}{n}\right)^n \right\}$

16. $\{a_n\} = \left\{ \left(1 + \dfrac{2}{n}\right)^{2n} \right\}$

In Exercises 17 – 28, determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.

17. $\{a_n\} = \left\{ (-1)^n \dfrac{n}{n+1} \right\}$

18. $\{a_n\} = \left\{ \dfrac{4n^2 - n + 5}{3n^2 + 1} \right\}$

19. $\{a_n\} = \left\{ \dfrac{4^n}{5^n} \right\}$

20. $\{a_n\} = \left\{ \dfrac{n-1}{n} - \dfrac{n}{n-1} \right\}, n \geq 2$

21. $\{a_n\} = \{\ln(n)\}$

22. $\{a_n\} = \left\{ \dfrac{3n}{\sqrt{n^2+1}} \right\}$

23. $\{a_n\} = \left\{ \left(1 + \dfrac{1}{n}\right)^n \right\}$

24. $\{a_n\} = \left\{ 5 - \dfrac{1}{n} \right\}$

25. $\{a_n\} = \left\{ \dfrac{(-1)^{n+1}}{n} \right\}$

26. $\{a_n\} = \left\{ \dfrac{1.1^n}{n} \right\}$

27. $\{a_n\} = \left\{ \dfrac{2n}{n+1} \right\}$

28. $\{a_n\} = \left\{ (-1)^n \dfrac{n^2}{2^n - 1} \right\}$

In Exercises 29 – 34, determine whether the sequence is bounded, bounded above, bounded below, or none of the above.

29. $\{a_n\} = \{\sin n\}$

30. $\{a_n\} = \{\tan n\}$

31. $\{a_n\} = \left\{ (-1)^n \dfrac{3n-1}{n} \right\}$

32. $\{a_n\} = \left\{ \dfrac{3n^2 - 1}{n} \right\}$

33. $\{a_n\} = \{n \cos n\}$

34. $\{a_n\} = \{2^n - n!\}$

In Exercises 35 – 38, determine whether the sequence is monotonically increasing or decreasing. If it is not, determine if there is an m such that it is monotonic for all $n \geq m$.

35. $\{a_n\} = \left\{\dfrac{n}{n+2}\right\}$

36. $\{a_n\} = \left\{\dfrac{n^2 - 6n + 9}{n}\right\}$

37. $\{a_n\} = \left\{(-1)^n \dfrac{1}{n^3}\right\}$

38. $\{a_n\} = \left\{\dfrac{n^2}{2^n}\right\}$

39. Prove Theorem 58; that is, use the definition of the limit of a sequence to show that if $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

40. Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = K$.

    (a) Show that if $a_n < b_n$ for all n, then $L \leq K$.

    (b) Give an example where $L = K$.

41. Prove the Squeeze Theorem for sequences: Let $\{a_n\}$ and $\{b_n\}$ be such that $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = L$, and let $\{c_n\}$ be such that $a_n \leq c_n \leq b_n$ for all n. Then $\lim\limits_{n\to\infty} c_n = L$

8.2 Infinite Series

Given the sequence $\{a_n\} = \{1/2^n\} = 1/2,\ 1/4,\ 1/8,\ \ldots$, consider the following sums:

$$\begin{array}{ccccc} a_1 & = & 1/2 & = & 1/2 \\ a_1 + a_2 & = & 1/2 + 1/4 & = & 3/4 \\ a_1 + a_2 + a_3 & = & 1/2 + 1/4 + 1/8 & = & 7/8 \\ a_1 + a_2 + a_3 + a_4 & = & 1/2 + 1/4 + 1/8 + 1/16 & = & 15/16 \end{array}$$

In general, we can show that

$$a_1 + a_2 + a_3 + \cdots + a_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}.$$

Let S_n be the sum of the first n terms of the sequence $\{1/2^n\}$. From the above, we see that $S_1 = 1/2$, $S_2 = 3/4$, etc. Our formula at the end shows that $S_n = 1 - 1/2^n$.

Now consider the following limit: $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \left(1 - 1/2^n\right) = 1$. This limit can be interpreted as saying something amazing: *the sum of* all *the terms of the sequence* $\{1/2^n\}$ *is 1.*

This example illustrates some interesting concepts that we explore in this section. We begin this exploration with some definitions.

Definition 34 Infinite Series, n^{th} Partial Sums, Convergence, Divergence

Let $\{a_n\}$ be a sequence.

1. The sum $\displaystyle\sum_{n=1}^{\infty} a_n$ is an **infinite series** (or simply **series**).
2. Let $S_n = \displaystyle\sum_{i=1}^{n} a_i$; the sequence $\{S_n\}$ is the sequence of n^{th} **partial sums** of $\{a_n\}$.
3. If the sequence $\{S_n\}$ converges to L, we say the series $\displaystyle\sum_{n=1}^{\infty} a_n$ **converges** to L, and we write $\displaystyle\sum_{n=1}^{\infty} a_n = L$.
4. If the sequence $\{S_n\}$ diverges, the series $\displaystyle\sum_{n=1}^{\infty} a_n$ **diverges**.

Notes:

Using our new terminology, we can state that the series $\displaystyle\sum_{n=1}^{\infty} 1/2^n$ converges, and $\displaystyle\sum_{n=1}^{\infty} 1/2^n = 1$.

We will explore a variety of series in this section. We start with two series that diverge, showing how we might discern divergence.

Example 8.8 **Showing series diverge**

1. Let $\{a_n\} = \{n^2\}$. Show $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

2. Let $\{b_n\} = \{(-1)^{n+1}\}$. Show $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges.

Solution

1. Consider S_n, the n^{th} partial sum.

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \cdots + a_n \\ &= 1^2 + 2^2 + 3^2 \cdots + n^2. \end{aligned}$$

By Theorem 39, this is

$$= \frac{n(n+1)(2n+1)}{6}.$$

Since $\displaystyle\lim_{n\to\infty} S_n = \infty$, we conclude that the series $\displaystyle\sum_{n=1}^{\infty} n^2$ diverges. It is instructive to write $\displaystyle\sum_{n=1}^{\infty} n^2 = \infty$ for this tells us *how* the series diverges: it grows without bound.

A scatter plot of the sequences $\{a_n\}$ and $\{S_n\}$ is given in Figure 8.7(a). The terms of $\{a_n\}$ are growing, so the terms of the partial sums $\{S_n\}$ are growing even faster, illustrating that the series diverges.

2. The sequence $\{b_n\}$ starts with 1, -1, 1, -1, ... Consider some of the

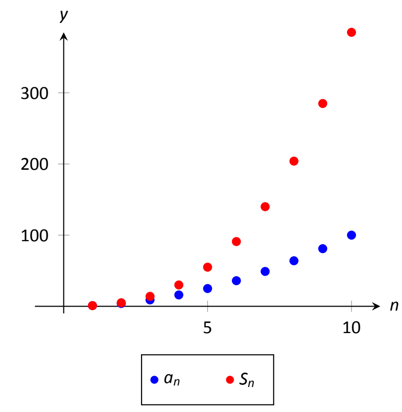
Notes:

partial sums S_n of $\{b_n\}$:

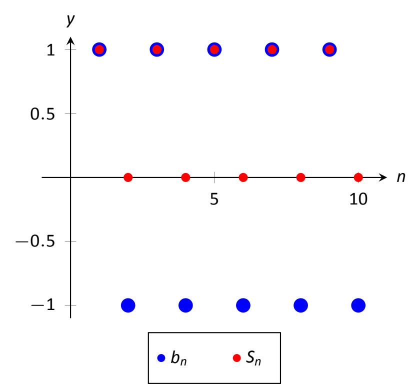
$$S_1 = 1$$
$$S_2 = 0$$
$$S_3 = 1$$
$$S_4 = 0$$

This pattern repeats; we find that $S_n = \begin{cases} 1 & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$. As $\{S_n\}$ oscillates, repeating 1, 0, 1, 0, ..., we conclude that $\lim_{n\to\infty} S_n$ does not exist, hence $\sum_{n=1}^{\infty}(-1)^{n+1}$ diverges.

A scatter plot of the sequence $\{b_n\}$ and the partial sums $\{S_n\}$ is given in Figure 8.7(b). When n is odd, $b_n = S_n$ so the marks for b_n are drawn oversized to show they coincide.



(a)



(b)

Figure 8.7: Scatter plots relating to Example 8.8.

While it is important to recognize when a series diverges, we are generally more interested in the series that converge. In this section we will demonstrate a few general techniques for determining convergence; later sections will delve deeper into this topic.

Geometric Series

One important type of series is a *geometric series*.

Definition 35 Geometric Series

A **geometric series** is a series of the form

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots + r^n + \cdots$$

Note that the index starts at $n = 0$, not $n = 1$.

We started this section with a geometric series, although we dropped the first term of 1. One reason geometric series are important is that they have nice convergence properties.

Notes:

Theorem 62 Convergence of Geometric Series

Consider the geometric series $\displaystyle\sum_{n=0}^{\infty} r^n$.

1. The n^{th} partial sum is: $S_n = \dfrac{1-r^{n+1}}{1-r}$.
2. The series converges if, and only if, $|r| < 1$. When $|r| < 1$,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

According to Theorem 62, the series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$$

converges as $r = 1/2$, and $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-1/2} = 2$. This concurs with our introductory example; while there we got a sum of 1, we skipped the first term of 1.

Example 8.9 Exploring geometric series

Check the convergence of the following series. If the series converges, find its sum.

1. $\displaystyle\sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n$ 2. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n$ 3. $\displaystyle\sum_{n=0}^{\infty} 3^n$

Solution

1. Since $r = 3/4 < 1$, this series converges. By Theorem 62, we have that

$$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = \frac{1}{1-3/4} = 4.$$

However, note the subscript of the summation in the given series: we are to start with $n = 2$. Therefore we subtract off the first two terms, giving:

$$\sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n = 4 - 1 - \frac{3}{4} = \frac{9}{4}.$$

This is illustrated in Figure 8.8.

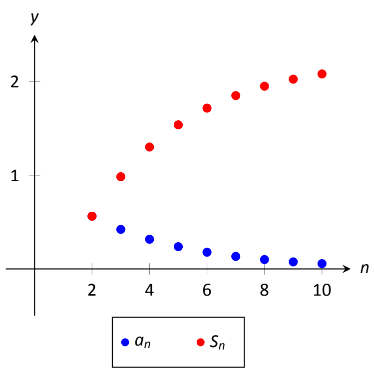


Figure 8.8: Scatter plots relating to the series in Example 8.9.

Notes:

2. Since $|r| = 1/2 < 1$, this series converges, and by Theorem 62,

$$\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n = \frac{1}{1-(-1/2)} = \frac{2}{3}.$$

The partial sums of this series are plotted in Figure 8.9(a). Note how the partial sums are not purely increasing as some of the terms of the sequence $\{(-1/2)^n\}$ are negative.

3. Since $r > 1$, the series diverges. (This makes "common sense"; we expect the sum

$$1+3+9+27+81+243+\cdots$$

to diverge.) This is illustrated in Figure 8.9(b).

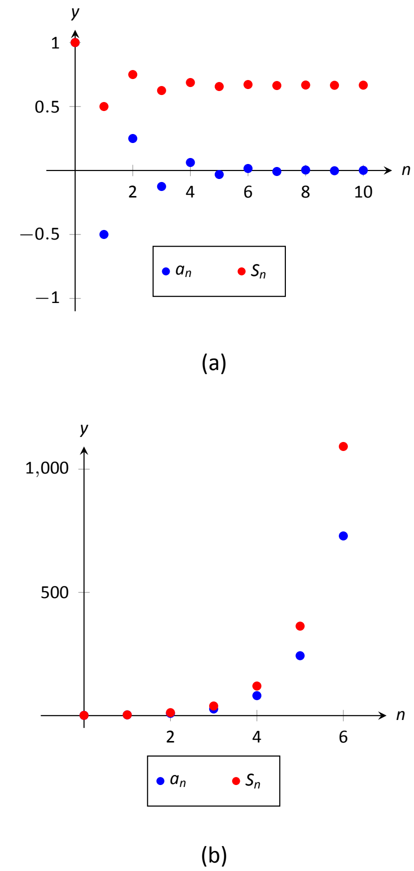


Figure 8.9: Scatter plots relating to the series in Example 8.9.

The Harmonic Series

The series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots$$

is called the *Harmonic Series* because of its relationship to *harmonics* in the study of music and sound. Even though the terms being added are approaching 0, this series diverges, as the sequence of partial sums approaches ∞, as we prove in the following example.

Example 8.10 The Harmonic Series diverges

Show that the Harmonic Series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

SOLUTION We begin by realizing the following:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots \\ &\geq 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{=1/2} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{=1/2} + \cdots\end{aligned}$$

We continue by noticing that the terms from $1/9$ through $1/16$ added in the Harmonic series are eight terms all $\geq 1/16$, so they add to $1/2$. Then the terms

Notes:

$1/17$ through $1/32$ are sixteen terms all $\geq 1/32$, so they add to 1/2. It should be clear that this pattern continues when we group terms up to a power of 2. Therefore,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n} &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots\\ &\geq 1+\frac{1}{2}+\underbrace{\frac{1}{4}+\frac{1}{4}}_{=1/2}+\underbrace{\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}}_{=1/2}+\cdots\\ &= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\end{aligned}$$

It should be clear that adding infinitely many $1/2$ terms gives ∞. Then the infinite series $\sum\limits_{n=1}^{\infty}$ is greater than or equal to a series that diverges to ∞. Therefore, the Harmonic Series must diverge to ∞.

On the other hand, a related series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\cdots$, called the *Alternating Harmonic Series* actually converges to $\ln 2$. We cannot explain why at the moment. In Section 8.5, we will be able to explain why it converges. However, it won't be until Taylor Series are discussed in Section 8.8 that we can explain why the limit is, in fact, $\ln 2$.

Later sections will provide tests by which we can determine whether or not a given series converges. This, in general, is much easier than determining *what* a given series converges to. There are many cases, though, where the sum can be determined.

Example 8.11 Telescoping series

Evaluate the sum $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$.

Solution It will help to write down some of the first few partial sums

Notes:

of this series.

$$\begin{aligned}
S_1 &= \frac{1}{1} - \frac{1}{2} && = 1 - \frac{1}{2}\\
S_2 &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) && = 1 - \frac{1}{3}\\
S_3 &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) && = 1 - \frac{1}{4}\\
S_4 &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) && = 1 - \frac{1}{5}
\end{aligned}$$

Note how most of the terms in each partial sum are canceled out! In general, we see that $S_n = 1 - \dfrac{1}{n+1}$. The sequence $\{S_n\}$ converges, as $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty}\left(1 - \dfrac{1}{n+1}\right) = 1$, and so we conclude that $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 1$. Partial sums of the series are plotted in Figure 8.10.

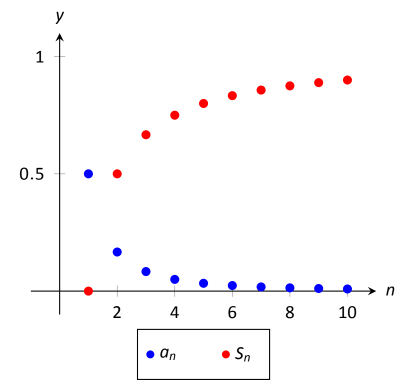


Figure 8.10: Scatter plots relating to the series of Example 8.11.

The series in Example 8.11 is an example of a **telescoping series**. Informally, a telescoping series is one in which the partial sums reduce to just a finite number of terms. The partial sum S_n did not contain n terms, but rather just two: 1 and $1/(n+1)$.

When possible, seek a way to write an explicit formula for the n^{th} partial sum S_n. This makes evaluating the limit $\lim\limits_{n\to\infty} S_n$ much more approachable. We do so in the next example.

Example 8.12 Evaluating series

Evaluate each of the following infinite series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+2n}$ 2. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

Solution

1. We can decompose the fraction $2/(n^2+2n)$ as

$$\frac{2}{n^2+2n} = \frac{1}{n} - \frac{1}{n+2}.$$

(See Section 6.4, Partial Fraction Decomposition, to recall how this is done, if necessary.)

Notes:

Expressing the terms of $\{S_n\}$ is now more instructive:

$$\begin{aligned}
S_1 &= 1-\frac{1}{3} && = 1-\frac{1}{3}\\
S_2 &= \left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right) && = 1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}\\
S_3 &= \left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right) && = 1+\frac{1}{2}-\frac{1}{4}-\frac{1}{5}\\
S_4 &= \left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right) && = 1+\frac{1}{2}-\frac{1}{5}-\frac{1}{6}\\
S_5 &= \left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\left(\frac{1}{5}-\frac{1}{7}\right) && = 1+\frac{1}{2}-\frac{1}{6}-\frac{1}{7}
\end{aligned}$$

We again have a telescoping series. In each partial sum, most of the terms cancel and we obtain the formula $S_n = 1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}$. Taking limits allows us to determine the convergence of the series:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right)=\frac{3}{2}, \quad \text{so } \sum_{n=1}^{\infty}\frac{1}{n^2+2n}=\frac{3}{2}.$$

This is illustrated in Figure 8.11(a).

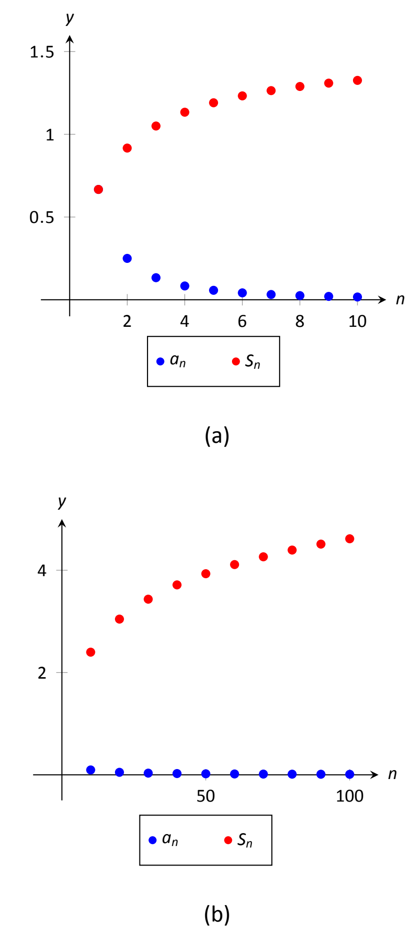


Figure 8.11: Scatter plots relating to the series in Example 8.12.

2. We begin by writing the first few partial sums of the series:

$$\begin{aligned}
S_1 &= \ln(2)\\
S_2 &= \ln(2)+\ln\left(\frac{3}{2}\right)\\
S_3 &= \ln(2)+\ln\left(\frac{3}{2}\right)+\ln\left(\frac{4}{3}\right)\\
S_4 &= \ln(2)+\ln\left(\frac{3}{2}\right)+\ln\left(\frac{4}{3}\right)+\ln\left(\frac{5}{4}\right)
\end{aligned}$$

At first, this does not seem helpful, but recall the logarithmic identity: $\ln x+\ln y=\ln(xy)$. Applying this to S_4 gives:

$$S_4 = \ln(2)+\ln\left(\frac{3}{2}\right)+\ln\left(\frac{4}{3}\right)+\ln\left(\frac{5}{4}\right)=\ln\left(\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\right)=\ln(5).$$

We can conclude that $\{S_n\}=\big\{\ln(n+1)\big\}$. This sequence does not converge, as $\lim_{n\to\infty} S_n=\infty$. Therefore $\sum_{n=1}^{\infty}\ln\left(\frac{n+1}{n}\right)=\infty$; the series diverges. Note in Figure 8.11(b) how the sequence of partial sums grows

Notes:

slowly; after 100 terms, it is not yet over 5. (Only some terms of a_n and S_n are shown in the plot.) Graphically we may be fooled into thinking the series converges, but our analysis above shows that it does not.

We are learning about a new mathematical object, the series. As done before, we apply "old" mathematics to this new topic.

Theorem 63 Properties of Infinite Series

Let $\displaystyle\sum_{n=1}^{\infty} a_n = L$, $\displaystyle\sum_{n=1}^{\infty} b_n = K$, and let c be a constant.

1. Constant Multiple Rule: $\displaystyle\sum_{n=1}^{\infty} c \cdot a_n = c \cdot \sum_{n=1}^{\infty} a_n = c \cdot L.$

2. Sum/Difference Rule: $\displaystyle\sum_{n=1}^{\infty} \big(a_n \pm b_n\big) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n = L \pm K.$

Before using this theorem, we provide a few "famous" series.

Key Idea 32 Important Series

1. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} = e.$ (Note that the index starts with $n = 0$.)

2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$

4. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}.$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. (*Harmonic Series*.)

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2.$ (*Alternating Harmonic Series*.)

Notes:

Example 8.13 Evaluating series

Evaluate the given series.

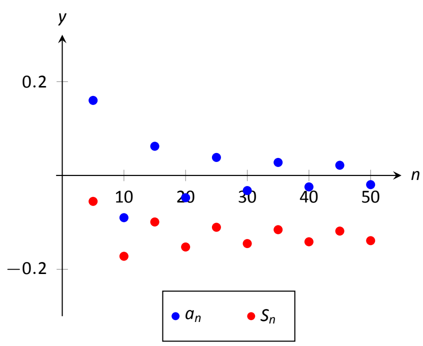
1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n^2-n)}{n^3}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{1000}{n!}$
3. $\displaystyle\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\frac{1}{49}+\cdots$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n!}$

Solution

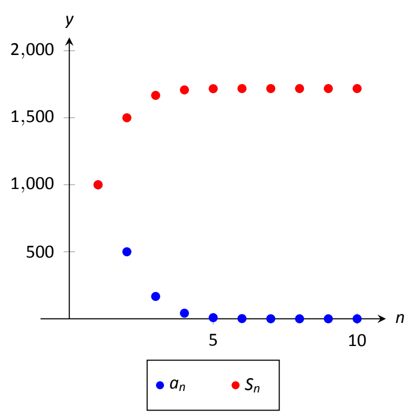
1. We start by using algebra to break the series apart:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n^2-n)}{n^3} &= \sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}n^2}{n^3}-\frac{(-1)^{n+1}n}{n^3}\right)\\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}-\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\\ &= \ln(2)-\frac{\pi^2}{12}\approx -0.1293.\end{aligned}$$

This is illustrated in Figure 8.12(a).



(a)



(b)

Figure 8.12: Scatter plots relating to the series in Example 8.13.

2. This looks very similar to the series that involves e in Key Idea 32. Note, however, that the series given in this example starts with $n = 1$ and not $n = 0$. The first term of the series in the Key Idea is $1/0! = 1$, so we will subtract this from our result below:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1000}{n!} &= 1000\cdot\sum_{n=1}^{\infty}\frac{1}{n!}\\ &= 1000\cdot(e-1)\approx 1718.28.\end{aligned}$$

This is illustrated in Figure 8.12(b). The graph shows how this particular series converges very rapidly.

3. The denominators in each term are perfect squares; we are adding $\displaystyle\sum_{n=4}^{\infty}\frac{1}{n^2}$ (note we start with $n = 4$, not $n = 1$). This series will converge. Using the

Notes:

formula from Key Idea 32, we have the following:

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n^2} &= \sum_{n=1}^{3}\frac{1}{n^2}+\sum_{n=4}^{\infty}\frac{1}{n^2}\\
\sum_{n=1}^{\infty}\frac{1}{n^2}-\sum_{n=1}^{3}\frac{1}{n^2} &= \sum_{n=4}^{\infty}\frac{1}{n^2}\\
\frac{\pi^2}{6}-\left(\frac{1}{1}+\frac{1}{4}+\frac{1}{9}\right) &= \sum_{n=4}^{\infty}\frac{1}{n^2}\\
\frac{\pi^2}{6}-\frac{49}{36} &= \sum_{n=4}^{\infty}\frac{1}{n^2}\\
0.2838 &\approx \sum_{n=4}^{\infty}\frac{1}{n^2}
\end{aligned}$$

4. We manipulate this series to look like the series that involves e in Key Idea 32. In the steps below, we twice relabel n as $n+1$ to write it in terms of the series in the table. Note that $n! = n(n-1)!$.

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{n^2}{n!} &= \sum_{n=1}^{\infty}\frac{n}{(n-1)!}\\
&= \sum_{n=0}^{\infty}\frac{n+1}{n!}\\
&= \sum_{n=0}^{\infty}\frac{n}{n!}+\sum_{n=0}^{\infty}\frac{1}{n!}\\
&= \sum_{n=1}^{\infty}\frac{1}{(n-1)!}+e\\
&= \sum_{n=0}^{\infty}\frac{1}{n!}+e\\
&= e+e\\
&= 2e.
\end{aligned}$$

It may take a while before one is comfortable with this statement, whose truth lies at the heart of the study of infinite series: *it is possible that the sum of an infinite list of nonzero numbers is finite.* We have seen this repeatedly in this section, yet it still may "take some getting used to."

As one contemplates the behavior of series, a few facts become clear.

Notes:

1. In order to add an infinite list of nonzero numbers and get a finite result, "most" of those numbers must be "very near" 0.

2. If a series diverges, it means that the sum of an infinite list of numbers is not finite (it may approach $\pm\infty$ or it may oscillate), and:

   (a) The series will still diverge if the first term is removed.

   (b) The series will still diverge if the first 10 terms are removed.

   (c) The series will still diverge if the first $1,000,000$ terms are removed.

   (d) The series will still diverge if any finite number of terms from anywhere in the series are removed.

These concepts are very important and lie at the heart of the next two theorems.

Theorem 64 $\quad$ ***n*^th^–Term Test for Convergence/Divergence**

Consider the series $\displaystyle\sum_{n=1}^{\infty} a_n$.

1. If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\lim_{n\to\infty} a_n = 0$.

2. If $\displaystyle\lim_{n\to\infty} a_n \neq 0$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

Note that the two statements in Theorem 64 are really the same. In order to converge, the limit of the terms of the sequence must approach 0; if they do not, the series will not converge.

Looking back, we can apply this theorem to the series in Example 8.8. In that example, the n^{th} terms of both sequences do not converge to 0, therefore we can quickly conclude that each series diverges.

Important! This theorem *does not state* that if $\displaystyle\lim_{n\to\infty} a_n = 0$ then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. The standard example of this is the Harmonic Series. The Harmonic Sequence, $\{1/n\}$, converges to 0; the Harmonic Series, $\displaystyle\sum_{n=1}^{\infty} 1/n$, diverges.

Notes:

Theorem 65 Infinite Nature of Series

The convergence or divergence remains unchanged by the addition or subtraction of any finite number of terms. That is:

1. A divergent series will remain divergent with the addition or subtraction of any finite number of terms.

2. A convergent series will remain convergent with the addition or subtraction of any finite number of terms. (Of course, the *sum* will likely change.)

Consider once more the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges; that is, the sequence of partial sums $\{S_n\}$ grows (very, very slowly) without bound. One might think that by removing the "large" terms of the sequence that perhaps the series will converge. This is simply not the case. For instance, the sum of the first 10 million terms of the Harmonic Series is about 16.7. Removing the first 10 million terms from the Harmonic Series changes the n^{th} partial sums, effectively subtracting 16.7 from the sum. However, a sequence that is growing without bound will still grow without bound when 16.7 is subtracted from it.

The equations below illustrate this. The first line shows the infinite sum of the Harmonic Series split into the sum of the first 10 million terms plus the sum of "everything else." The next equation shows us subtracting these first 10 million terms from both sides. The final equation employs a bit of "psuedo–math": subtracting 16.7 from "infinity" still leaves one with "infinity."

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} &= \sum_{n=1}^{10{,}000{,}000} \frac{1}{n} + \sum_{n=10{,}000{,}001}^{\infty} \frac{1}{n} \\
\sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{10{,}000{,}000} \frac{1}{n} &= \sum_{n=10{,}000{,}001}^{\infty} \frac{1}{n} \\
\infty - 16.7 &= \infty.
\end{aligned}$$

This section introduced us to series and defined a few special types of series whose convergence properties are well known, such as geometric and telescoping series. Most series that we encounter are not one of these types, but we are still interested in knowing whether or not they converge. The next three sections introduce tests that help us determine whether or not a given series converges.

Notes:

Exercises 8.2

Terms and Concepts

1. Use your own words to describe how sequences and series are related.

2. Use your own words to define a *partial sum*.

3. Given a series $\sum_{n=1}^{\infty} a_n$, describe the two sequences related to the series that are important.

4. Use your own words to explain what a geometric series is.

5. T/F: If $\{a_n\}$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

Problems

In Exercises 6 – 13, a series $\sum_{n=1}^{\infty} a_n$ is given.

(a) Give the first 5 partial sums of the series.

(b) Give a graph of the first 5 terms of a_n and S_n on the same axes.

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

8. $\displaystyle\sum_{n=1}^{\infty} \cos(\pi n)$

9. $\displaystyle\sum_{n=1}^{\infty} n$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$

12. $\displaystyle\sum_{n=1}^{\infty} \left(-\frac{9}{10}\right)^n$

13. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$

In Exercises 14 – 19, use Theorem 64 to show the given series diverges.

14. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2}{n(n+2)}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n - n^5}{5^n + n^5}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 1}{2^{n+1}}$

19. $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$

In Exercises 20 – 24, state whether the given series converges or diverges.

20. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{5^n}$

21. $\displaystyle\sum_{n=0}^{\infty} \frac{6^n}{5^n}$

22. $\displaystyle\sum_{n=1}^{\infty} \sqrt{n}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{10}{n!}$

24. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n!} + \frac{1}{n}\right)$

In Exercises 25 – 39, a series is given.

(a) Find a formula for S_n, the n^{th} partial sum of the series.

(b) Determine whether the series converges or diverges. If it converges, state what it converges to.

25. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{4^n}$

26. $1^3 + 2^3 + 3^3 + 4^3 + \cdots$

27. $\displaystyle\sum_{n=1}^{\infty} (-1)^n n$

28. $\displaystyle\sum_{n=0}^{\infty} \frac{5}{2^n}$

29. $\displaystyle\sum_{n=1}^{\infty} e^{-n}$

30. $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} + \cdots$

31. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+2)}$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$

34. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

36. $\frac{1}{1\cdot 4} + \frac{1}{2\cdot 5} + \frac{1}{3\cdot 6} + \frac{1}{4\cdot 7} + \cdots$

37. $2 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{9}\right) + \left(\frac{1}{8} + \frac{1}{27}\right) + \cdots$

38. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$

39. $\displaystyle\sum_{n=0}^{\infty} \big(\sin 1\big)^n$

40. Break the Harmonic Series into the sum of the odd and even terms:

$$\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{2n-1} + \sum_{n=1}^{\infty} \frac{1}{2n}.$$

The goal is to show that each of the series on the right diverge.

(a) Show why $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1} > \sum_{n=1}^{\infty} \frac{1}{2n}$.

(Compare each n^{th} partial sum.)

(b) Show why $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1} < 1 + \sum_{n=1}^{\infty} \frac{1}{2n}$

(c) Explain why (a) and (b) demonstrate that the series of odd terms is convergent, if, and only if, the series of even terms is also convergent. (That is, show both converge or both diverge.)

(d) Explain why knowing the Harmonic Series is divergent determines that the even and odd series are also divergent.

41. Show the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(2n-1)(2n+1)}$ diverges.

42. Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n!}$ using the series in Key Idea 32.

8.3 Integral and Comparison Tests

Knowing whether or not a series converges is very important, especially when we discuss Power Series in Section 8.6. Theorem 62 gives a criterion for when geometric series converge, and Theorem 64 gives a quick test to determine if a series diverges. There are many important series whose convergence cannot be determined by these theorems, though, so we introduce a set of tests that allow us to handle a broad range of series. We start with the Integral Test.

Integral Test

We stated in Section 8.1 that a sequence $\{a_n\}$ is a function $a(n)$ whose domain is $\mathbb{N}$, the set of natural numbers. If we can extend $a(n)$ to $\mathbb{R}$, the real numbers, and it is both positive and decreasing on $[1,\infty)$, then the convergence of $\displaystyle\sum_{n=1}^{\infty} a_n$ is the same as $\displaystyle\int_1^{\infty} a(x)\,dx$.

Note: Theorem 66 does not state that the integral and the summation have the same value.

Theorem 66 Integral Test

Let a sequence $\{a_n\}$ be defined by $a_n = a(n)$, where $a(n)$ is continuous, positive and decreasing on $[1,\infty)$. Then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, if, and only if, $\displaystyle\int_1^{\infty} a(x)\,dx$ converges.

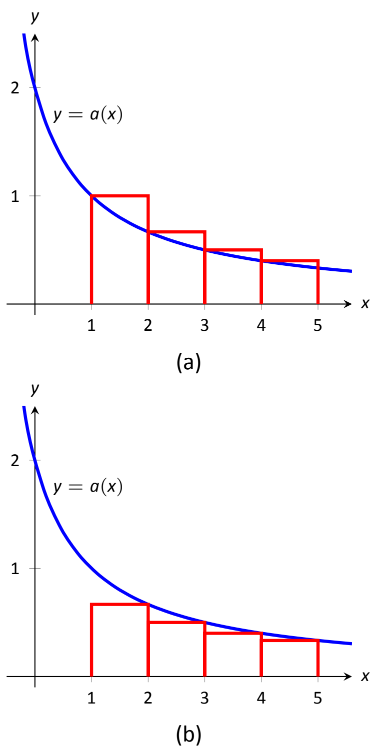


Figure 8.13: Illustrating the truth of the Integral Test.

We can demonstrate the truth of the Integral Test with two simple graphs. In Figure 8.13(a), the height of each rectangle is $a(n) = a_n$ for $n = 1, 2, \ldots$, and clearly the rectangles enclose more area than the area under $y = a(x)$. Therefore we can conclude that

$$\int_1^{\infty} a(x)\,dx < \sum_{n=1}^{\infty} a_n. \tag{8.1}$$

In Figure 8.13(b), we draw rectangles under $y = a(x)$ with the Right-Hand rule, starting with $n = 2$. This time, the area of the rectangles is less than the area under $y = a(x)$, so $\displaystyle\sum_{n=2}^{\infty} a_n < \int_1^{\infty} a(x)\,dx$. Note how this summation starts with $n = 2$; adding a_1 to both sides lets us rewrite the summation starting with

Notes:

$n = 1$:

$$\sum_{n=1}^{\infty} a_n < a_1 + \int_1^{\infty} a(x)\, dx. \tag{8.2}$$

Combining Equations (8.1) and (8.2), we have

$$\sum_{n=1}^{\infty} a_n < a_1 + \int_1^{\infty} a(x)\, dx < a_1 + \sum_{n=1}^{\infty} a_n. \tag{8.3}$$

From Equation (8.3) we can make the following two statements:

1. If $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, so does $\displaystyle\int_1^{\infty} a(x)\, dx$ (because $\displaystyle\sum_{n=1}^{\infty} a_n < a_1 + \int_1^{\infty} a(x)\, dx$)
2. If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, so does $\displaystyle\int_1^{\infty} a(x)\, dx$ (because $\displaystyle\int_1^{\infty} a(x)\, dx < \sum_{n=1}^{\infty} a_n$.)

Therefore the series and integral either both converge or both diverge. Theorem 65 allows us to extend this theorem to series where $a(n)$ is positive and decreasing on $[b, \infty)$ for some $b > 1$.

Example 8.14 **Using the Integral Test**

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$. (The terms of the sequence $\{a_n\} = \{\ln n/n^2\}$ and the n$^{\text{th}}$ partial sums are given in Figure 8.14.)

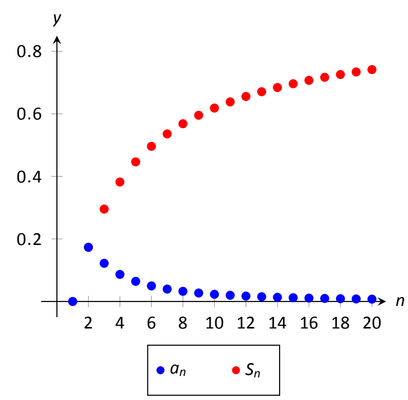


Figure 8.14: Plotting the sequence and series in Example 8.14.

Solution Figure 8.14 implies that $a(n) = (\ln n)/n^2$ is positive and decreasing on $[2, \infty)$. We can determine this analytically, too. We know $a(n)$ is positive as both $\ln n$ and n^2 are positive on $[2, \infty)$. To determine that $a(n)$ is decreasing, consider $a'(n) = (1 - 2\ln n)/n^3$, which is negative for $n \geq 2$. Since $a'(n)$ is negative, $a(n)$ is decreasing.

Applying the Integral Test, we test the convergence of $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}\, dx$. Integrating this improper integral requires the use of Integration by Parts, with $u = \ln x$ and $dv = 1/x^2\, dx$.

$$\begin{aligned}\int_1^{\infty} \frac{\ln x}{x^2}\, dx &= \lim_{b\to\infty} \int_1^{b} \frac{\ln x}{x^2}\, dx \\ &= \lim_{b\to\infty} -\frac{1}{x}\ln x\Big|_1^b + \int_1^b \frac{1}{x^2}\, dx\end{aligned}$$

Notes:

$$\begin{aligned}&= \lim_{b\to\infty} -\frac{1}{x}\ln x - \frac{1}{x}\Big|_1^b\\ &= \lim_{b\to\infty} 1 - \frac{1}{b} - \frac{\ln b}{b}. \quad \text{Apply L'Hôpital's Rule:}\\ &= 1.\end{aligned}$$

Since $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$ converges, so does $\displaystyle\sum_{n=1}^\infty \frac{\ln n}{n^2}$.

p–Series

In Section 6.6 about improper integrals, we determined for which p the integral $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges, and for which p it diverges. Combining this with the Integral Test, we can easily determine convergence for an important class of series called p–series.

Definition 36 p**–Series, General** p**–Series**

1. A p**–series** is a series of the form

$$\sum_{n=1}^\infty \frac{1}{n^p}, \qquad \text{where } p > 0.$$

2. A **general** p**–series** is a series of the form

$$\sum_{n=1}^\infty \frac{1}{(an+b)^p}, \qquad \text{where } p > 0 \text{ and } a, b \text{ are real numbers.}$$

Theorem 67 **Convergence of General** p**–Series**

A general p–series $\displaystyle\sum_{n=1}^\infty \frac{1}{(an+b)^p}$ converges if and only if $p > 1$.

Note: Theorem 67 assumes that $an+b \neq 0$ for all n. If $an+b=0$ for some n, then of course the series does not converge regardless of p as not all of the terms of the sequence are defined.

We prove the convergence of general p–series in the next example.

Notes:

Example 8.15 Proving Theorem 67.

Use the Integral Test to prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$ converges if, and only if, $p > 1$.

Solution Consider the integral $\displaystyle\int_1^{\infty} \frac{1}{(ax+b)^p}\,dx$. Assume $ax+b \neq 0$ for $x \geq 1$. (If $ax+b=0$ for some non-integer $x > 1$, then there exists N for which $ax+b=0$ for $x \geq N$, so integrate on $[N, \infty)$ instead.) Assuming $p \neq 1$,

$$\begin{aligned}\int_1^{\infty} \frac{1}{(ax+b)^p}\,dx &= \lim_{c\to\infty}\int_1^{c} \frac{1}{(ax+b)^p}\,dx \\ &= \lim_{c\to\infty} \frac{1}{a(1-p)}(ax+b)^{1-p}\Big|_1^c \\ &= \lim_{c\to\infty} \frac{1}{a(1-p)}\big((ac+b)^{1-p}-(a+b)^{1-p}\big).\end{aligned}$$

This limit converges if, and only if, $p > 1$. It is easy to show that the integral also diverges in the case of $p = 1$. (This result is similar to the work preceding Key Idea 22.)

Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$ converges if, and only if, $p > 1$.

Example 8.16 Determining convergence of series

Determine the convergence of the following series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$
5. $\displaystyle\sum_{n=11}^{\infty} \frac{1}{\left(\frac{1}{2}n-5\right)^3}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$

Solution

1. This is the Harmonic Series, which we proved to be divergent in the previous section. Note, however, that it is also a p–series with $p = 1$, another way to explain that it diverges.

2. This is a p–series with $p = 2$. By Theorem 67, it converges. Note that the theorem does not give a formula by which we can determine *what* the series converges to; we just know it converges. A famous, unexpected result is that this series converges to $\pi^2/6$.

Notes:

3. This is a p–series with $p = 1/2$; the theorem states that it diverges.

4. This is not a p–series; the definition does not allow for alternating signs. Therefore we cannot apply Theorem 67. Also, we cannot use the Integral Test at all, because that can only be used when the terms are positive. As stated in the previous section, this Alternating Harmonic Series converges to $\ln 2$ though we must wait for later sections to explain why.

5. This is a general p–series with $p = 3$, therefore it converges.

6. This is not a p–series, but a geometric series with $r = 1/2$. It converges.

We consider two more convergence tests in this section, both *comparison* tests. That is, we determine the convergence of one series by comparing it to another series with known convergence.

Direct Comparison Test

Note: A sequence $\{a_n\}$ is a **positive sequence** if $a_n > 0$ for all n.

Because of Theorem 65, any theorem that relies on a positive sequence still holds true when $a_n > 0$ for all but a finite number of values of n.

Theorem 68 Direct Comparison Test

Let $\{a_n\}$ and $\{b_n\}$ be positive sequences where $a_n \leq b_n$ for all $n \geq N$, for some $N \geq 1$.

1. If $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

2. If $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, then $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges.

Example 8.17 Applying the Direct Comparison Test

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n + n^2}$.

Solution This series is neither a geometric or p-series, but seems related. We predict it will converge, so we look for a series with larger terms that converges. (Note too that the Integral Test seems difficult to apply here.)

Since $3^n < 3^n + n^2$, it follows that $\dfrac{1}{3^n} > \dfrac{1}{3^n + n^2}$ for all $n \geq 1$. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ is a convergent geometric series; by Theorem 68, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n + n^2}$ converges.

Notes:

Example 8.18 Applying the Direct Comparison Test

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n-\ln n}$.

Solution We know the Harmonic Series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, and it seems that the given series is closely related to it, hence we predict it will diverge.

Since $n \geq n - \ln n$ for all $n \geq 1$, $\dfrac{1}{n} \leq \dfrac{1}{n-\ln n}$ for all $n \geq 1$.

The Harmonic Series diverges, so we conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n-\ln n}$ diverges as well.

The concept of direct comparison is powerful and often relatively easy to apply. Practice helps one develop the necessary intuition to quickly pick a proper series with which to compare. However, it is easy to construct a series for which it is difficult to apply the Direct Comparison Test.

Consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\ln n}$. It is very similar to the divergent series given in Example 8.18. We suspect that it also diverges, as $\dfrac{1}{n} \approx \dfrac{1}{n+\ln n}$ for large n. However, the inequality that we naturally want to use "goes the wrong way": since $n \leq n + \ln n$ for all $n \geq 1$, $\dfrac{1}{n} \geq \dfrac{1}{n+\ln n}$ for all $n \geq 1$. The given series has terms *less than* the terms of a divergent series, and we cannot conclude anything from this.

Fortunately, we can apply another test to the given series to determine its convergence.

Notes:

Limit Comparison Test

Theorem 69 Limit Comparison Test

Let $\{a_n\}$ and $\{b_n\}$ be positive sequences.

1. If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = L$, where L is a positive real number, then $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ either both converge or both diverge.

2. If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$, then if $\sum\limits_{n=1}^{\infty} b_n$ converges, then so does $\sum\limits_{n=1}^{\infty} a_n$.

3. If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \infty$, then if $\sum\limits_{n=1}^{\infty} b_n$ diverges, then so does $\sum\limits_{n=1}^{\infty} a_n$.

Theorem 69 is most useful when the convergence of the series from $\{b_n\}$ is known and we are trying to determine the convergence of the series from $\{a_n\}$.

We use the Limit Comparison Test in the next example to examine the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n+\ln n}$ which motivated this new test.

Example 8.19 Applying the Limit Comparison Test

Determine the convergence of $\sum\limits_{n=1}^{\infty} \dfrac{1}{n+\ln n}$ using the Limit Comparison Test.

Solution We compare the terms of $\sum\limits_{n=1}^{\infty} \dfrac{1}{n+\ln n}$ to the terms of the Harmonic Sequence $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$:

$$\begin{aligned}\lim_{n\to\infty} \frac{1/(n+\ln n)}{1/n} &= \lim_{n\to\infty} \frac{n}{n+\ln n}\\ &= 1 \quad \text{(after applying L'Hôpital's Rule).}\end{aligned}$$

Since the Harmonic Series diverges, we conclude that $\sum\limits_{n=1}^{\infty} \dfrac{1}{n+\ln n}$ diverges as

Notes:

well.

Example 8.20 **Applying the Limit Comparison Test**

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n - n^2}$

Solution This series is similar to the one in Example 8.17, but now we are considering "$3^n - n^2$" instead of "$3^n + n^2$." This difference makes applying the Direct Comparison Test difficult.

Instead, we use the Limit Comparison Test and compare with the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$:

$$\begin{aligned}\lim_{n\to\infty} \frac{1/(3^n - n^2)}{1/3^n} &= \lim_{n\to\infty} \frac{3^n}{3^n - n^2}\\ &= 1 \quad \text{(after applying L'Hôpital's Rule twice).}\end{aligned}$$

We know $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ is a convergent geometric series, hence $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n - n^2}$ converges as well.

As mentioned before, practice helps one develop the intuition to quickly choose a series with which to compare. A general rule of thumb is to pick a series based on the dominant term in the expression of $\{a_n\}$. It is also helpful to note that among sequences that approach ∞ as $n \to \infty$, factorials dominate exponentials, which dominate algebraic functions (e.g., polynomials), which dominate logarithms. In the previous example, the dominant term of $\dfrac{1}{3^n - n^2}$ was 3^n, so we compared the series to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$. It is hard to apply the Limit Comparison Test to series containing factorials, though, as we have not learned how to apply L'Hôpital's Rule to $n!$.

Example 8.21 **Applying the Limit Comparison Test**

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+3}{n^2 - n + 1}$.

Solution We naïvely attempt to apply the rule of thumb given above and note that the dominant term in the expression of the series is $1/n^2$. Knowing

Notes:

that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, we attempt to apply the Limit Comparison Test:

$$\begin{aligned}\lim_{n\to\infty} \frac{(\sqrt{n}+3)/(n^2-n+1)}{1/n^2} &= \lim_{n\to\infty} \frac{n^2(\sqrt{n}+3)}{n^2-n+1} \\ &= \infty \quad \text{(Apply L'Hôpital's Rule).}\end{aligned}$$

Theorem 69 part (3) only applies when $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges; in our case, it converges. Ultimately, our test has not revealed anything about the convergence of our series.

The problem is that we chose a poor series with which to compare. Since the numerator and denominator of the terms of the series are both algebraic functions, we should have compared our series to the dominant term of the numerator divided by the dominant term of the denominator.

The dominant term of the numerator is $n^{1/2}$ and the dominant term of the denominator is n^2. Thus we should compare the terms of the given series to $n^{1/2}/n^2 = 1/n^{3/2}$:

$$\begin{aligned}\lim_{n\to\infty} \frac{(\sqrt{n}+3)/(n^2-n+1)}{1/n^{3/2}} &= \lim_{n\to\infty} \frac{n^{3/2}(\sqrt{n}+3)}{n^2-n+1} \\ &= 1 \quad \text{(Apply L'Hôpital's Rule).}\end{aligned}$$

Since the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, we conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+3}{n^2-n+1}$ converges as well.

We mentioned earlier that the Integral Test did not work well with series containing factorial terms. The next section introduces the Ratio Test, which does handle such series well. We also introduce the Root Test, which is good for series where each term is raised to a power.

Notes:

Exercises 8.3

Terms and Concepts

1. In order to apply the Integral Test to a sequence $\{a_n\}$, the function $a(n) = a_n$ must be ______, ______ and ______.

2. T/F: The Integral Test can be used to determine the sum of a convergent series.

3. What test(s) in this section do not work well with factorials?

4. Suppose $\displaystyle\sum_{n=0}^{\infty} a_n$ is convergent, and there are sequences $\{b_n\}$ and $\{c_n\}$ such that $b_n \leq a_n \leq c_n$ for all n. What can be said about the series $\displaystyle\sum_{n=0}^{\infty} b_n$ and $\displaystyle\sum_{n=0}^{\infty} c_n$?

Problems

In Exercises 5 – 20, use the Integral Test or General p–Series Test to determine the convergence of the given series.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

10. $\displaystyle\sum_{n=1}^{\infty} n^{-1/3}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{0.999}}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.001}}$

13. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{(2n+8)^2}$

17. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$

19. $\displaystyle\sum_{n=23}^{\infty} \frac{1}{\sqrt{7n-13}}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

In Exercises 21 – 30, use the Direct Comparison Test to determine the convergence of the given series; state what series is used for comparison.

21. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3n-5}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n+n^2-n}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!+n}$

25. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$

26. $\displaystyle\sum_{n=5}^{\infty} \frac{1}{\sqrt{n}-2}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^3-5}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{5^n+10}$

29. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{n^2-1}$

30. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$

In Exercises 31 – 40, use the Limit Comparison Test to determine the convergence of the given series; state what series is used for comparison.

31. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2-3n+5}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n-n^2}$

33. $\displaystyle\sum_{n=4}^{\infty} \frac{\ln n}{n-3}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}}$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{n-10}{n^2+10n+10}$

37. $\displaystyle\sum_{n=1}^{\infty} \sin\left(1/n\right)$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{n+5}{n^3-5}$

39. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+3}{n^2+17}$

40. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+100}$

In Exercises 41 – 48, determine the convergence of the given series. State the test used; more than one test may be appropriate.

41. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

42. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n+5)^3}$

43. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$

44. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n!}$

45. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n+n}$

46. $\displaystyle\sum_{n=1}^{\infty} \frac{n-2}{10n+5}$

47. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

48. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(1/n)}{\sqrt{n}}$

49. Given that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, state which of the following series converges, may converge, or does not converge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n}$

(b) $\displaystyle\sum_{n=1}^{\infty} a_n a_{n+1}$

(c) $\displaystyle\sum_{n=1}^{\infty} (a_n)^2$

(d) $\displaystyle\sum_{n=1}^{\infty} n a_n$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n}$

8.4 Ratio and Root Tests

The n^{th}–Term Test of Theorem 64 states that in order for a series $\sum_{n=1}^{\infty} a_n$ to converge, $\lim_{n\to\infty} a_n = 0$. That is, the terms of $\{a_n\}$ must get very small. Not only must the terms approach 0, they must approach 0 "fast enough": while $\lim_{n\to\infty} 1/n = 0$, the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges as the terms of $\{1/n\}$ do not approach 0 "fast enough."

The comparison tests of the previous section determine convergence by comparing terms of a series to terms of another series whose convergence is known. This section introduces the Ratio and Root Tests, which determine convergence by analyzing the terms of a series to see if they approach 0 "fast enough."

Ratio Test

Theorem 70 Ratio Test

Let $\{a_n\}$ be a positive sequence where $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$.

1. If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $L > 1$ or $L = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $L = 1$, the Ratio Test is inconclusive.

Note: Theorem 65 allows us to apply the Ratio Test to series where $\{a_n\}$ is positive for all but a finite number of terms.

The principle of the Ratio Test is this: if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$, then for large n, each term of $\{a_n\}$ is significantly smaller than its previous term which is enough to ensure convergence.

Example 8.22 Applying the Ratio Test

Use the Ratio Test to determine the convergence of the following series:

1. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ 2. $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$ 3. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

Notes:

Solution

1. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$:

$$\begin{aligned} \lim_{n\to\infty} \frac{2^{n+1}/(n+1)!}{2^n/n!} &= \lim_{n\to\infty} \frac{2^{n+1}n!}{2^n(n+1)!} \\ &= \lim_{n\to\infty} \frac{2}{n+1} \\ &= 0. \end{aligned}$$

Since the limit is $0 < 1$, by the Ratio Test $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3}$:

$$\begin{aligned} \lim_{n\to\infty} \frac{3^{n+1}/(n+1)^3}{3^n/n^3} &= \lim_{n\to\infty} \frac{3^{n+1}n^3}{3^n(n+1)^3} \\ &= \lim_{n\to\infty} \frac{3n^3}{(n+1)^3} \\ &= 3. \end{aligned}$$

Since the limit is $3 > 1$, by the Ratio Test $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3}$ diverges.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$:

$$\begin{aligned} \lim_{n\to\infty} \frac{1/\big((n+1)^2+1\big)}{1/(n^2+1)} &= \lim_{n\to\infty} \frac{n^2+1}{(n+1)^2+1} \\ &= 1. \end{aligned}$$

Since the limit is 1, the Ratio Test is inconclusive. We can easily show this series converges using the Direct or Limit Comparison Tests, with each comparing to the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Notes:

The Ratio Test is not effective when the terms of a series *only* contain algebraic functions (e.g., polynomials). It is most effective when the terms contain some factorials or exponentials. The previous example also reinforces our developing intuition: factorials dominate exponentials, which dominate algebraic functions, which dominate logarithmic functions. In Part 1 of the example, the factorial in the denominator dominated the exponential in the numerator, causing the series to converge. In Part 2, the exponential in the numerator dominated the algebraic function in the denominator, causing the series to diverge.

While we have used factorials in previous sections, we have not explored them closely and one is likely to not yet have a strong intuitive sense for how they behave. The following example gives more practice with factorials.

Example 8.23 **Applying the Ratio Test**

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{n!n!}{(2n)!}$.

Solution Before we begin, be sure to note the difference between $(2n)!$ and $2n!$. When $n = 4$, the former is $8! = 8 \cdot 7 \cdot \ldots \cdot 2 \cdot 1 = 40,320$, whereas the latter is $2(4 \cdot 3 \cdot 2 \cdot 1) = 48$.

Applying the Ratio Test:

$$\lim_{n\to\infty} \frac{(n+1)!(n+1)!/\big(2(n+1)\big)!}{n!n!/(2n)!} = \lim_{n\to\infty} \frac{(n+1)!(n+1)!(2n)!}{n!n!(2n+2)!}$$

Noting that $(2n+2)! = (2n+2)\cdot(2n+1)\cdot(2n)!$, we have

$$\begin{aligned} &= \lim_{n\to\infty} \frac{(n+1)(n+1)}{(2n+2)(2n+1)} \\ &= 1/4. \end{aligned}$$

Since the limit is $1/4 < 1$, by the Ratio Test we conclude $\displaystyle\sum_{n=1}^{\infty} \frac{n!n!}{(2n)!}$ converges.

Root Test

The final test we introduce is the Root Test, which works particularly well on series where each term is raised to a power, and does not work well with terms containing factorials.

Notes:

Note: Theorem 65 allows us to apply the Root Test to series where $\{a_n\}$ is positive for all but a finite number of terms.

Theorem 71 Root Test

Let $\{a_n\}$ be a positive sequence, and let $\lim\limits_{n\to\infty} (a_n)^{1/n} = L$.

1. If $L < 1$, then $\sum\limits_{n=1}^{\infty} a_n$ converges.
2. If $L > 1$ or $L = \infty$, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.
3. If $L = 1$, the Root Test is inconclusive.

Example 8.24 Applying the Root Test

Determine the convergence of the following series using the Root Test:

1. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3n+1}{5n-2}\right)^n$
2. $\displaystyle\sum_{n=1}^{\infty}\frac{n^4}{(\ln n)^n}$
3. $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{n^2}$.

Solution

1. $\displaystyle\lim_{n\to\infty}\left(\left(\frac{3n+1}{5n-2}\right)^n\right)^{1/n} = \lim_{n\to\infty}\frac{3n+1}{5n-2} = \frac{3}{5}.$

   Since the limit is less than 1, we conclude the series converges. Note: it is difficult to apply the Ratio Test to this series.

2. $\displaystyle\lim_{n\to\infty}\left(\frac{n^4}{(\ln n)^n}\right)^{1/n} = \lim_{n\to\infty}\frac{\left(n^{1/n}\right)^4}{\ln n}.$

   As n grows, the numerator approaches 1 (apply L'Hôpital's Rule) and the denominator grows to infinity. Thus

   $$\lim_{n\to\infty}\frac{\left(n^{1/n}\right)^4}{\ln n} = 0.$$

   Since the limit is less than 1, we conclude the series converges.

3. $\displaystyle\lim_{n\to\infty}\left(\frac{2^n}{n^2}\right)^{1/n} = \lim_{n\to\infty}\frac{2}{\left(n^{1/n}\right)^2} = 2.$

   Since this is greater than 1, we conclude the series diverges.

Each of the tests we have encountered so far has required that we analyze series from *positive* sequences. The next section relaxes this restriction by considering *alternating series*, where the underlying sequence has terms that alternate between being positive and negative.

Notes:

Exercises 8.4

Terms and Concepts

1. The Ratio Test is not effective when the terms of a sequence only contain ______ functions.

2. The Ratio Test is most effective when the terms of a sequence contains ______ and/or ______ functions.

3. What three convergence tests do not work well with terms containing factorials?

4. The Root Test works particularly well on series where each term is ______ to a ______.

Problems

In Exercises 5 – 14, determine the convergence of the given series using the Ratio Test. If the Ratio Test is inconclusive, state so and determine convergence with another test.

5. $\displaystyle\sum_{n=0}^{\infty} \frac{2n}{n!}$

6. $\displaystyle\sum_{n=0}^{\infty} \frac{5^n - 3n}{4^n}$

7. $\displaystyle\sum_{n=0}^{\infty} \frac{n!10^n}{(2n)!}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n + n^4}{7^n + n^2}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^3 + 7}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{10 \cdot 5^n}{7^n - 3}$

12. $\displaystyle\sum_{n=1}^{\infty} n \cdot \left(\frac{3}{5}\right)^n$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdot 8 \cdots 2n}{3 \cdot 6 \cdot 9 \cdot 12 \cdots 3n}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{5 \cdot 10 \cdot 15 \cdots (5n)}$

In Exercises 15 – 24, determine the convergence of the given series using the Root Test. If the Root Test is inconclusive, state so and determine convergence with another test.

15. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+5}{3n+11}\right)^n$

16. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{0.9n^2 - n - 3}{n^2 + n + 3}\right)^n$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n^2}{3^n}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^2 2^{n+1}}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{4^{n+7}}{7^n}$

21. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2 - n}{n^2 + n}\right)^n$

22. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$

23. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$

24. $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{(\ln n)^n}$

In Exercises 25 – 34, determine the convergence of the given series. State the test used; more than one test may be appropriate.

25. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + 4n - 2}{n^3 + 4n^2 - 3n + 7}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{n^4 4^n}{n!}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{3^n + n}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^n}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2 + 4n + 1}}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{n!n!n!}{(3n)!}$

31. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$

32. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n+2}{n+1}\right)^n$

33. $\displaystyle\sum_{n=2}^{\infty} \frac{n^3}{\left(\ln n\right)^n}$

34. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$

8.5 Alternating Series and Absolute Convergence

All of the series convergence tests we have used require that the underlying sequence $\{a_n\}$ be a positive sequence. (We can relax this with Theorem 65 and state that there must be an $N > 0$ such that $a_n > 0$ for all $n > N$; that is, $\{a_n\}$ is positive for all but a finite number of values of n.)

In this section we explore series whose summation includes negative terms. We start with a very specific form of series, where the terms of the summation alternate between being positive and negative.

Definition 37 Alternating Series

Let $\{a_n\}$ be a positive sequence. An **alternating series** is a series of either the form

$$\sum_{n=1}^{\infty}(-1)^n a_n \qquad \text{or} \qquad \sum_{n=1}^{\infty}(-1)^{n+1}a_n.$$

Recall the terms of Harmonic Series come from the Harmonic Sequence $\{a_n\} = \{1/n\}$. An important alternating series is the **Alternating Harmonic Series**:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

Geometric Series can also be alternating series when $r < 0$. For instance, if $r = -1/2$, the geometric series is

$$\sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots$$

Theorem 62 states that geometric series converge when $|r| < 1$ and gives the sum: $\displaystyle\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$. When $r = -1/2$ as above, we find

$$\sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n = \frac{1}{1-(-1/2)} = \frac{1}{3/2} = \frac{2}{3}.$$

A powerful convergence theorem exists for other alternating series that meet a few conditions.

Notes:

Theorem 72 Alternating Series Test

Let $\{a_n\}$ be a positive, decreasing sequence where $\lim_{n\to\infty} a_n = 0$. Then

$$\sum_{n=1}^{\infty}(-1)^n a_n \quad \text{and} \quad \sum_{n=1}^{\infty}(-1)^{n+1} a_n$$

converge.

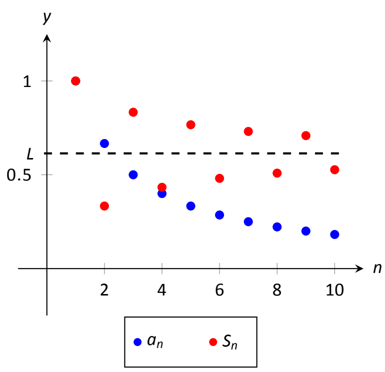


Figure 8.15: Illustrating convergence with the Alternating Series Test.

The basic idea behind Theorem 72 is illustrated in Figure 8.15. A positive, decreasing sequence $\{a_n\}$ is shown along with the partial sums

$$S_n = \sum_{i=1}^{n}(-1)^{i+1}a_i = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n+1}a_n.$$

Because $\{a_n\}$ is decreasing, the amount by which S_n bounces up/down decreases. Moreover, the odd terms of S_n form a decreasing, bounded sequence, while the even terms of S_n form an increasing, bounded sequence. Since bounded, monotonic sequences converge (see Theorem 61) and the terms of $\{a_n\}$ approach 0, one can show the odd and even terms of S_n converge to the same common limit L, the sum of the series.

Example 8.25 Applying the Alternating Series Test

Determine if the Alternating Series Test applies to each of the following series.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ 2. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{\ln n}{n}$ 3. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{|\sin n|}{n^2}$

Solution

1. This is the Alternating Harmonic Series as seen previously. The underlying sequence is $\{a_n\} = \{1/n\}$, which is positive, decreasing, and approaches 0 as $n \to \infty$. Therefore we can apply the Alternating Series Test and conclude this series converges.

   While the test does not state what the series converges to, we will see later that $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = \ln 2$.

2. The underlying sequence is $\{a_n\} = \{\ln n/n\}$. This is positive and approaches 0 as $n \to \infty$ (use L'Hôpital's Rule). However, the sequence is not decreasing for all n. It is straightforward to compute $a_1 = 0$, $a_2 \approx 0.347$,

Notes:

$a_3 \approx 0.366$, and $a_4 \approx 0.347$: the sequence is increasing for at least the first 3 terms.

We do not immediately conclude that we cannot apply the Alternating Series Test. Rather, consider the long–term behavior of $\{a_n\}$. Treating $a_n = a(n)$ as a continuous function of n defined on $[1, \infty)$, we can take its derivative:

$$a'(n) = \frac{1 - \ln n}{n^2}.$$

The derivative is negative for all $n \geq 3$ (actually, for all $n > e$), meaning $a(n) = a_n$ is decreasing on $[3, \infty)$. We can apply the Alternating Series Test to the series when we start with $n = 3$ and conclude that $\displaystyle\sum_{n=3}^{\infty}(-1)^n\frac{\ln n}{n}$ converges; adding the terms with $n = 1$ and $n = 2$ do not change the convergence (i.e., we apply Theorem 65).

The important lesson here is that as before, if a series fails to meet the criteria of the Alternating Series Test on only a finite number of terms, we can still apply the test.

3. The underlying sequence is $\{a_n\} = |\sin n|/n^2$. This sequence is positive and approaches 0 as $n \to \infty$. However, it is not a decreasing sequence; the value of $|\sin n|$ oscillates between 0 and 1 as $n \to \infty$. We cannot remove a finite number of terms to make $\{a_n\}$ decreasing, therefore we cannot apply the Alternating Series Test.

   Keep in mind that this does not mean we conclude the series diverges; in fact, it does converge. We are just unable to conclude this based on Theorem 72.

Key Idea 32 gives the sum of some important series. Two of these are

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.64493 \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} \approx 0.82247.$$

These two series converge to their sums at different rates. To be accurate to two places after the decimal, we need 202 terms of the first series though only 13 of the second. To get 3 places of accuracy, we need 1069 terms of the first series though only 33 of the second. Why is it that the second series converges so much faster than the first?

While there are many factors involved when studying rates of convergence, the alternating structure of an alternating series gives us a powerful tool when approximating the sum of a convergent series.

Notes:

Theorem 73 The Alternating Series Approximation Theorem

Let $\{a_n\}$ be a sequence that satisfies the hypotheses of the Alternating Series Test, and let S_n and L be the n^{th} partial sums and sum, respectively, of either $\displaystyle\sum_{n=1}^{\infty}(-1)^n a_n$ or $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}a_n$. Then

1. $|S_n - L| < a_{n+1}$, and
2. L is between S_n and S_{n+1}.

Part 1 of Theorem 73 states that the n^{th} partial sum of a convergent alternating series will be within a_{n+1} of its total sum. Consider the alternating series we looked at before the statement of the theorem, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$. Since $a_{14} = 1/14^2 \approx 0.0051$, we know that S_{13} is within 0.0051 of the total sum.

Moreover, Part 2 of the theorem states that since $S_{13} \approx 0.8252$ and $S_{14} \approx 0.8201$, we know the sum L lies between 0.8201 and 0.8252. One use of this is the knowledge that S_{14} is accurate to two places after the decimal.

Some alternating series converge slowly. In Example 8.25 we determined the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$ converged. With $n = 1001$, we find $\ln n/n \approx 0.0069$, meaning that $S_{1000} \approx 0.1633$ is accurate to one, maybe two, places after the decimal. Since $S_{1001} \approx 0.1564$, we know the sum L is $0.1564 \leq L \leq 0.1633$.

Example 8.26 Approximating the sum of convergent alternating series

Approximate the sum of the following series, accurate to within 0.001.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^3}$
2. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$.

Solution

Notes:

1. Using Theorem 73 with $a_n = \frac{1}{n^3}$, find n where $a_{n+1} = \frac{1}{(n+1)^3} < 0.001$:

$$\begin{aligned}\frac{1}{(n+1)^3} &\leq 0.001 = \frac{1}{1000}\\ (n+1)^3 &\geq 1000\\ (n+1) &\geq \sqrt[3]{1000}\\ (n+1) &\geq 10\\ n &\geq 9.\end{aligned}$$

   Let L be the sum of this series. By Part 1 of the theorem, $|S_9 - L| < a_{10} = 1/1000$. We can compute $S_9 = 0.902116$, which our theorem states is within 0.001 of the total sum.

   We can use Part 2 of the theorem to obtain an even more accurate result. As we know the 10$^{\text{th}}$ term of the series is $-1/1000$, we can easily compute $S_{10} = 0.901116$. Part 2 of the theorem states that L is between S_9 and S_{10}, so $0.901116 < L < 0.902116$.

2. We want to find n where $\ln(n+1)/(n+1) < 0.001$. We start by solving $\ln(n+1)/(n+1) = 0.001$ for n. This cannot be solved algebraically (in terms of elementary functions), so we will use Newton's Method to approximate a solution.

   Let $f(x) = \ln(x)/x - 0.001$; we want to know where $f(x) = 0$. We make a guess that x must be "large," so our initial guess will be $x_1 = 1000$. Recall how Newton's Method works: given an approximate solution x_n, our next approximation x_{n+1} is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

   We find $f'(x) = \big(1 - \ln(x)\big)/x^2$. This gives

$$\begin{aligned}x_2 &= 1000 - \frac{\ln(1000)/1000 - 0.001}{\big(1 - \ln(1000)\big)/1000^2}\\ &= 2000.\end{aligned}$$

   Using a computer, we find that Newton's Method seems to converge to a solution $x = 9118.01$ after 8 iterations. Thus, we need $n + 1 \geq 9118.01$ to get the guaranteed accuracy. Taking the next integer higher, we have $n + 1 = 9119$, where $\ln(9119)/9119 = 0.000999903 < 0.001$, so use $n = 9118$.

Notes:

Again using a computer, we find $S_{9118} = -0.160369$. Part 1 of the theorem states that this is within 0.001 of the actual sum L. Already knowing the 9,119$^{\text{th}}$ term, we can compute $S_{9119} = -0.159369$, meaning $-0.159369 < L < -0.160369$.

Notice how the first series converged quite quickly, where we needed only 10 terms to reach the desired accuracy, whereas the second series took over 9,000 terms.

One of the famous results of mathematics is that the Harmonic Series, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, yet the Alternating Harmonic Series, $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$, converges. The notion that alternating the signs of the terms in a series can make a series converge leads us to the following definitions.

Definition 38 Absolute and Conditional Convergence

1. A series $\displaystyle\sum_{n=1}^{\infty} a_n$ **converges absolutely** if $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges.
2. A series $\displaystyle\sum_{n=1}^{\infty} a_n$ **converges conditionally** if $\displaystyle\sum_{n=1}^{\infty} a_n$ converges but $\displaystyle\sum_{n=1}^{\infty} |a_n|$ diverges.

Note: In Definition 38, $\displaystyle\sum_{n=1}^{\infty} a_n$ is not necessarily an alternating series; it just may have some negative terms.

Thus we say the Alternating Harmonic Series converges conditionally.

Example 8.27 Determining absolute and conditional convergence.

Determine if the following series converge absolutely, conditionally, or diverge.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n+3}{n^2+2n+5}$
2. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^2+2n+5}{2^n}$
3. $\displaystyle\sum_{n=3}^{\infty}(-1)^n\frac{3n-3}{5n-10}$

SOLUTION

1. We can show the series

$$\sum_{n=1}^{\infty}\left|(-1)^n\frac{n+3}{n^2+2n+5}\right| = \sum_{n=1}^{\infty}\frac{n+3}{n^2+2n+5}$$

Notes:

diverges using the Limit Comparison Test, comparing with $1/n$.

The series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n+3}{n^2+2n+5}$ converges using the Alternating Series Test; we conclude it converges conditionally.

2. We can show the series

$$\sum_{n=1}^{\infty}\left|(-1)^n\frac{n^2+2n+5}{2^n}\right| = \sum_{n=1}^{\infty}\frac{n^2+2n+5}{2^n}$$

converges using the Ratio Test.

Therefore we conclude $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^2+2n+5}{2^n}$ converges absolutely.

3. The series

$$\sum_{n=3}^{\infty}\left|(-1)^n\frac{3n-3}{5n-10}\right| = \sum_{n=3}^{\infty}\frac{3n-3}{5n-10}$$

diverges using the n^{th} Term Test, so it does not converge absolutely.

The series $\displaystyle\sum_{n=3}^{\infty}(-1)^n\frac{3n-3}{5n-10}$ fails the conditions of the Alternating Series Test as $(3n-3)/(5n-10)$ does not approach 0 as $n\to\infty$. We can state further that this series diverges; as $n\to\infty$, the series effectively adds and subtracts $3/5$ over and over. This causes the sequence of partial sums to oscillate and not converge.

Therefore the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{3n-3}{5n-10}$ diverges.

Knowing that a series converges absolutely allows us to make two important statements, given in the following theorem. The first is that absolute convergence is "stronger" than regular convergence. That is, just because $\displaystyle\sum_{n=1}^{\infty}a_n$ converges, we cannot conclude that $\displaystyle\sum_{n=1}^{\infty}|a_n|$ will converge, but knowing a series converges absolutely tells us that $\displaystyle\sum_{n=1}^{\infty}a_n$ will converge.

One reason this is important is that our convergence tests all require that the underlying sequence of terms be positive. By taking the absolute value of the terms of a series where not all terms are positive, we are often able to apply an appropriate test and determine absolute convergence. This, in turn, determines that the series we are given also converges.

Notes:

The second statement relates to **rearrangements** of series. When dealing with a finite set of numbers, the sum of the numbers does not depend on the order which they are added. (So $1+2+3=3+1+2$.) One may be surprised to find out that when dealing with an infinite set of numbers, the same statement does not always hold true: some infinite lists of numbers may be rearranged in different orders to achieve different sums. The theorem states that the terms of an absolutely convergent series can be rearranged in any way without affecting the sum.

Theorem 74 Absolute Convergence Theorem

Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be a series that converges absolutely.

1. $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.
2. Let $\{b_n\}$ be any rearrangement of the sequence $\{a_n\}$. Then
$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n.$$

In Example 8.27, we determined the series in part 2 converges absolutely. Theorem 74 tells us the series converges (which we could also determine using the Alternating Series Test).

The theorem states that rearranging the terms of an absolutely convergent series does not affect its sum. This implies that perhaps the sum of a conditionally convergent series can change based on the arrangement of terms. Indeed, it can. The Riemann Rearrangement Theorem (named after Bernhard Riemann) states that any conditionally convergent series can have its terms rearranged so that the sum is any desired value, including ∞!

As an example, consider the Alternating Harmonic Series once more. We have stated that
$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}\cdots = \ln 2,$$
(see Key Idea 32 or Example 8.25).

Consider the rearrangement where every positive term is followed by two negative terms:
$$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}\cdots$$

Notes:

(Convince yourself that these are exactly the same numbers as appear in the Alternating Harmonic Series, just in a different order.) Now group some terms and simplify:

$$\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\cdots=$$

$$\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\cdots=$$

$$\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots\right)=\frac{1}{2}\ln 2.$$

By rearranging the terms of the series, we have arrived at a different sum! (One could *try* to argue that the Alternating Harmonic Series does not actually converge to $\ln 2$, because rearranging the terms of the series *shouldn't* change the sum. However, the Alternating Series Test proves this series converges to L, for some number L, and if the rearrangement does not change the sum, then $L = L/2$, implying $L = 0$. But the Alternating Series Approximation Theorem quickly shows that $L > 0$. The only conclusion is that the rearrangement *did* change the sum.) This is an incredible result. In fact, if the Harmonic Series $\sum_{n=1}^{\infty}\frac{1}{n}$ were to converge, then $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ would be absolutely convergent, so this would violate Theorem 74. So we have given another proof for the divergence of the Harmonic Series.

We end here our study of tests to determine convergence. The back cover of this text contains a table summarizing the tests that one may find useful.

While series are worthy of study in and of themselves, our ultimate goal within calculus is the study of Power Series, which we will consider in the next section. We will use power series to create functions where the output is the result of an infinite summation.

Notes:

Exercises 8.5

Terms and Concepts

1. Why is $\sum_{n=1}^{\infty} \sin n$ not an alternating series?

2. A series $\sum_{n=1}^{\infty} (-1)^n a_n$ converges when $\{a_n\}$ is ______, ______ and $\lim_{n\to\infty} a_n =$______.

3. Give an example of a series where $\sum_{n=0}^{\infty} a_n$ converges but $\sum_{n=0}^{\infty} |a_n|$ does not.

4. The sum of a ______ convergent series can be changed by rearranging the order of its terms.

Problems

In Exercises 5 – 20, an alternating series $\sum_{n=i}^{\infty} a_n$ is given.

(a) Determine if the series converges or diverges.

(b) Determine if $\sum_{n=0}^{\infty} |a_n|$ converges or diverges.

(c) If $\sum_{n=0}^{\infty} a_n$ converges, determine if the convergence is conditional or absolute.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n!}}$

7. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{n+5}{3n-5}$

8. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n^2}$

9. $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1} \frac{3n+5}{n^2-3n+1}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n + 1}$

11. $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{n}{\ln n}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{1+3+5+\cdots+(2n-1)}$

13. $\displaystyle\sum_{n=1}^{\infty} \cos(\pi n)$

14. $\displaystyle\sum_{n=2}^{\infty} \frac{\sin((2n+1/2)\pi)}{n \ln n}$

15. $\displaystyle\sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)^n$

16. $\displaystyle\sum_{n=0}^{\infty} (-e)^{-n}$

17. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n^2}{n!}$

18. $\displaystyle\sum_{n=0}^{\infty} (-1)^n 2^{-n^2}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1000)^n}{n!}$

Let S_n be the n^{th} partial sum of a series. In Exercises 21 – 24, a convergent alternating series is given and a value of n. Compute S_n and S_{n+1} and use these values to find bounds on the sum of the series.

21. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}, \quad n=5$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}, \quad n=4$

23. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}, \quad n=6$

24. $\displaystyle\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n, \quad n=9$

In Exercises 25 – 29, a convergent alternating series is given along with its sum and a value of ε. Use Theorem 73 to find n such that the n^{th} partial sum of the series is within ε of the sum of the series.

25. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720}, \quad \varepsilon = 0.001$

26. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}, \quad \varepsilon = 0.00001$

27. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}, \quad \varepsilon = 0.0001$

28. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}, \quad \varepsilon = 0.001$

29. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} = \cos 1, \quad \varepsilon = 10^{-8}$

8.6 Power Series

So far, our study of series has examined the question of "Is the sum of these infinite terms finite?," i.e., "Does the series converge?" We now approach series from a different perspective: as a function. Given a value of x, we evaluate $f(x)$ by finding the sum of a particular series that depends on x (assuming the series converges). We start this new approach to series with a definition.

In working with power series, we tweak the definition of a sequence $\{a_n\}$ so that the zeroth term a_0 is defined. In the past, sequences were only defined for $n \geq 1$.

Definition 39 Power Series

Let $\{a_n\}$ be a sequence, let x be a variable, and let c be a real number.

1. The **power series in** x is the series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

2. The **power series in** x **centered at** c is the series

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \ldots$$

Example 8.28 Examples of power series

Write out the first five terms of the following power series:

1. $\displaystyle\sum_{n=0}^{\infty} x^n$ 2. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{(x+1)^n}{n}$ 3. $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1}\frac{(x-\pi)^{2n}}{(2n)!}$.

Solution

1. One of the conventions we adopt is that $x^0 = 1$ regardless of the value of x. Therefore
$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots$$
This is a geometric series in x.

2. This series is centered at $c = -1$. Note how this series starts with $n = 1$. We could rewrite this series starting at $n = 0$ with the understanding that

Notes:

$a_0 = 0$, and hence the first term is 0.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x+1)^n}{n} = (x+1) - \frac{(x+1)^2}{2} + \frac{(x+1)^3}{3} - \frac{(x+1)^4}{4} + \frac{(x+1)^5}{5} \dots$$

3. This series is centered at $c = \pi$. Recall that $0! = 1$.

$$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-\pi)^{2n}}{(2n)!} = -1 + \frac{(x-\pi)^2}{2} - \frac{(x-\pi)^4}{24} + \frac{(x-\pi)^6}{6!} - \frac{(x-\pi)^8}{8!} \dots$$

We introduced power series as a type of function, where a value of x is given and the sum of a series is returned. Of course, not every series converges. For instance, in part 1 of Example 8.28, we recognized the series $\sum_{n=0}^{\infty} x^n$ as a geometric series in x. Theorem 62 states that this series converges only when $|x| < 1$.

This raises the question: "For what values of x will a given power series converge?," which leads us to a theorem and definition.

Theorem 75 Convergence of Power Series

Let a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ be given. Then one of the following is true:

1. The series converges only at $x = c$.
2. There is an $R > 0$ such that the series converges for all x in $(c-R, c+R)$ and diverges for all $x < c-R$ and $x > c+R$.
3. The series converges for all x.

The value of R is important when understanding a power series, hence it is given a name in the following definition. Also, note that part 2 of Theorem 75 makes a statement about the interval $(c-R, c+R)$, but not the endpoints of that interval. A series may/may not converge at these endpoints.

Notes:

Definition 40 Radius and Interval of Convergence

1. The number R given in Theorem 75 is the **radius of convergence** of a given series. When a series converges for only $x = c$, we say the radius of convergence is 0, i.e., $R = 0$. When a series converges for all x, we say the series has an infinite radius of convergence, i.e., $R = \infty$.

2. The **interval of convergence** is the set of all values of x for which the series converges.

To find the values of x for which a given series converges, we will use the convergence tests we studied previously (especially the Ratio Test). However, the tests all required that the terms of a series be positive. The following theorem gives us a work–around to this problem.

Theorem 76 The Radius of Convergence of a Series and Absolute Convergence

The series $\displaystyle\sum_{n=0}^{\infty} a_n(x-c)^n$ and $\displaystyle\sum_{n=0}^{\infty} \left|a_n(x-c)^n\right|$ have the same radius of convergence R.

Theorem 76 allows us to find the radius of convergence R of a series by applying the Ratio Test (or any applicable test) to the absolute value of the terms of the series. We practice this in the following example.

Example 8.29 Determining the radius and interval of convergence.
Find the radius and interval of convergence for each of the following series:

1. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$
2. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n}$
3. $\displaystyle\sum_{n=0}^{\infty} 2^n(x-3)^n$
4. $\displaystyle\sum_{n=0}^{\infty} n!x^n$

SOLUTION

Notes:

1. We apply the Ratio Test to the series $\displaystyle\sum_{n=0}^{\infty}\left|\frac{x^n}{n!}\right|$:

$$\lim_{n\to\infty}\frac{\left|x^{n+1}/(n+1)!\right|}{\left|x^n/n!\right|} = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\cdot\frac{n!}{(n+1)!}\right|$$
$$= \lim_{n\to\infty}\left|\frac{x}{n+1}\right|$$
$$= 0 \text{ for all } x.$$

The Ratio Test shows us that regardless of the choice of x, the series converges. Therefore the radius of convergence is $R = \infty$, and the interval of convergence is $(-\infty, \infty)$.

2. We apply the Ratio Test to the series $\displaystyle\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{x^n}{n}\right| = \sum_{n=1}^{\infty}\left|\frac{x^n}{n}\right|$:

$$\lim_{n\to\infty}\frac{\left|x^{n+1}/(n+1)\right|}{\left|x^n/n\right|} = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\cdot\frac{n}{n+1}\right|$$
$$= \lim_{n\to\infty}|x|\frac{n}{n+1}$$
$$= |x|.$$

The Ratio Test states a series converges if the limit of $|a_{n+1}/a_n| = L < 1$. We found the limit above to be $|x|$; therefore, the power series converges when $|x| < 1$, or when x is in $(-1, 1)$. Thus the radius of convergence is $R = 1$.

To determine the interval of convergence, we need to check the endpoints of $(-1, 1)$. When $x = -1$, we have the opposite of the Harmonic Series:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-1)^n}{n} = \sum_{n=1}^{\infty}\frac{-1}{n}$$
$$= -\infty.$$

The series diverges when $x = -1$.

When $x = 1$, we have the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(1)^n}{n}$, which is the Alternating Harmonic Series, which converges. Therefore the interval of convergence is $(-1, 1]$.

Notes:

3. We apply the Ratio Test to the series $\displaystyle\sum_{n=0}^{\infty}\left|2^n(x-3)^n\right|$:

$$\begin{aligned}\lim_{n\to\infty}\frac{\left|2^{n+1}(x-3)^{n+1}\right|}{\left|2^n(x-3)^n\right|} &= \lim_{n\to\infty}\left|\frac{2^{n+1}}{2^n}\cdot\frac{(x-3)^{n+1}}{(x-3)^n}\right|\\ &= \lim_{n\to\infty}\left|2(x-3)\right|.\end{aligned}$$

   According to the Ratio Test, the series converges when $\left|2(x-3)\right|<1 \implies \left|x-3\right|<1/2$. The series is centered at 3, and x must be within $1/2$ of 3 in order for the series to converge. Therefore the radius of convergence is $R=1/2$, and we know that the series converges absolutely for all x in $(3-1/2, 3+1/2)=(2.5, 3.5)$.

   We check for convergence at the endpoints to find the interval of convergence. When $x=2.5$, we have:

$$\begin{aligned}\sum_{n=0}^{\infty}2^n(2.5-3)^n &= \sum_{n=0}^{\infty}2^n(-1/2)^n\\ &= \sum_{n=0}^{\infty}(-1)^n,\end{aligned}$$

   which diverges. A similar process shows that the series also diverges at $x=3.5$. Therefore the interval of convergence is $(2.5, 3.5)$.

4. We apply the Ratio Test to $\displaystyle\sum_{n=0}^{\infty}\left|n!x^n\right|$:

$$\begin{aligned}\lim_{n\to\infty}\frac{\left|(n+1)!x^{n+1}\right|}{\left|n!x^n\right|} &= \lim_{n\to\infty}\left|(n+1)x\right|\\ &= \infty \quad \text{for all } x\text{, except } x=0.\end{aligned}$$

   The Ratio Test shows that the series diverges for all x except $x=0$. Therefore the radius of convergence is $R=0$.

We can use a power series to define a function:

$$f(x)=\sum_{n=0}^{\infty}a_nx^n$$

where the domain of f is a subset of the interval of convergence of the power series. One can apply calculus techniques to such functions; in particular, we can find derivatives and antiderivatives.

Notes:

Theorem 77 **Derivatives and Indefinite Integrals of Power Series Functions**

Let $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ be a function defined by a power series, with radius of convergence R.

1. $f(x)$ is continuous and differentiable on $(c-R, c+R)$.

2. $f'(x) = \sum_{n=1}^{\infty} a_n \cdot n \cdot (x-c)^{n-1}$, with radius of convergence R.

3. $\int f(x)\,dx = C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1}$, with radius of convergence R.

A few notes about Theorem 77:

1. The theorem states that differentiation and integration do not change the radius of convergence. It does not state anything about the *interval* of convergence. They are not always the same.
2. Notice how the summation for $f'(x)$ starts with $n = 1$. This is because the constant term a_0 of $f(x)$ goes to 0.
3. Differentiation and integration are simply calculated term–by–term using the Power Rules.

Example 8.30 **Derivatives and indefinite integrals of power series**

Let $f(x) = \sum_{n=0}^{\infty} x^n$. Find $f'(x)$ and $F(x) = \int f(x)\,dx$, along with their respective intervals of convergence.

Solution We find the derivative and indefinite integral of $f(x)$, following Theorem 77.

1. $f'(x) = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots$.

   In Example 8.28, we recognized that $\sum_{n=0}^{\infty} x^n$ is a geometric series in x. We know that such a geometric series converges when $|x| < 1$; that is, the interval of convergence is $(-1, 1)$.

Notes:

To determine the interval of convergence of $f'(x)$, we consider the endpoints of $(-1, 1)$:

$$f'(-1) = 1 - 2 + 3 - 4 + \cdots, \quad \text{which diverges.}$$

$$f'(1) = 1 + 2 + 3 + 4 + \cdots, \quad \text{which diverges.}$$

Therefore, the interval of convergence of $f'(x)$ is $(-1, 1)$.

2. $F(x) = \displaystyle\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

To find the interval of convergence of $F(x)$, we again consider the endpoints of $(-1, 1)$:

$$F(-1) = C - 1 + 1/2 - 1/3 + 1/4 + \cdots$$

The value of C is irrelevant; notice that the rest of the series is an Alternating Series that whose terms converge to 0. By the Alternating Series Test, this series converges. (In fact, we can recognize that the terms of the series after C are the opposite of the Alternating Harmonic Series. We can thus say that $F(-1) = C - \ln 2$.)

$$F(1) = C + 1 + 1/2 + 1/3 + 1/4 + \cdots$$

Notice that this summation is $C +$ the Harmonic Series, which diverges. Since F converges for $x = -1$ and diverges for $x = 1$, the interval of convergence of $F(x)$ is $[-1, 1)$.

The previous example showed how to take the derivative and indefinite integral of a power series without motivation for why we care about such operations. We may care for the sheer mathematical enjoyment "that we can", which is motivation enough for many. However, we would be remiss to not recognize that we can learn a great deal from taking derivatives and indefinite integrals.

Recall that $f(x) = \displaystyle\sum_{n=0}^{\infty} x^n$ in Example 8.30 is a geometric series. According to Theorem 62, this series converges to $1/(1-x)$ when $|x| < 1$. Thus we can say

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \text{on} \quad (-1, 1).$$

Integrating the power series, (as done in Example 8.30,) we find

$$F(x) = C_1 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}, \tag{8.4}$$

Notes:

while integrating the function $f(x) = 1/(1-x)$ gives

$$F(x) = -\ln|1-x| + C_2. \tag{8.5}$$

Equating Equations (8.4) and (8.5), we have

$$F(x) = C_1 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\ln|1-x| + C_2.$$

Letting $x = 0$, we have $F(0) = C_1 = C_2$. This implies that we can drop the constants and conclude

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\ln|1-x|.$$

We established in Example 8.30 that the series on the left converges at $x = -1$; substituting $x = -1$ on both sides of the above equality gives

$$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots = -\ln 2.$$

On the left we have the opposite of the Alternating Harmonic Series; on the right, we have $-\ln 2$. We conclude that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$$

Important: We stated in Key Idea 32 (in Section 8.2) that the Alternating Harmonic Series converges to $\ln 2$, and referred to this fact again in Example 8.25 of Section 8.5. However, we never gave an argument for why this was the case. The work above finally shows how we conclude that the Alternating Harmonic Series converges to $\ln 2$.

We use this type of analysis in the next example.

Example 8.31 **Analyzing power series functions**

Let $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$. Find $f'(x)$ and $\displaystyle\int f(x)\,dx$, and use these to analyze the behavior of $f(x)$.

Solution We start by making two notes: first, in Example 8.29, we found the interval of convergence of this power series is $(-\infty, \infty)$. Second, we will find it useful later to have a few terms of the series written out:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \tag{8.6}$$

Notes:

We now find the derivative:

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} n\frac{x^{n-1}}{n!} \\ &= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = 1 + x + \frac{x^2}{2!} + \cdots . \end{aligned}$$

Since the series starts at $n = 1$ and each term refers to $(n-1)$, we can re-index the series starting with $n = 0$:

$$\begin{aligned} &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= f(x). \end{aligned}$$

We found the derivative of $f(x)$ is $f(x)$. The only functions for which this is true are of the form $y = ce^x$ for some constant c. As $f(0) = 1$ (see Equation (8.6)), c must be 1. Therefore we conclude that

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

for all x.

We can also find $\displaystyle\int f(x)\,dx$:

$$\begin{aligned} \int f(x)\,dx &= C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!(n+1)} \\ &= C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \end{aligned}$$

We write out a few terms of this last series:

$$C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} = C + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

The integral of $f(x)$ differs from $f(x)$ only by a constant, again indicating that $f(x) = e^x$.

Example 8.31 and the work following Example 8.30 established relationships between a power series function and "regular" functions that we have dealt with in the past. In general, given a power series function, it is difficult (if not

Notes:

impossible) to express the function in terms of elementary functions. We chose examples where things worked out nicely.

In this section's last example, we show how to solve a simple differential equation with a power series.

Example 8.32 **Solving a differential equation with a power series.**
Give the first 4 terms of the power series solution to $y' = 2y$, where $y(0) = 1$.

Solution The differential equation $y' = 2y$ describes a function $y = f(x)$ where the derivative of y is twice y and $y(0) = 1$. This is a rather simple differential equation; with a bit of thought one should realize that if $y = Ce^{2x}$, then $y' = 2Ce^{2x}$, and hence $y' = 2y$. By letting $C = 1$ we satisfy the initial condition of $y(0) = 1$.

Let's ignore the fact that we already know the solution and find a power series function that satisfies the equation. The solution we seek will have the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

for unknown coefficients a_n. We can find $f'(x)$ using Theorem 77:

$$f'(x) = \sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 \cdots .$$

Since $f'(x) = 2f(x)$, we have

$$\begin{aligned} a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 \cdots &= 2\big(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\big) \\ &= 2a_0 + 2a_1 x + 2a_2 x^2 + 2a_3 x^3 + \cdots \end{aligned}$$

The coefficients of like powers of x must be equal, so we find that

$$a_1 = 2a_0, \quad 2a_2 = 2a_1, \quad 3a_3 = 2a_2, \quad 4a_4 = 2a_3, \quad \text{etc.}$$

The initial condition $y(0) = f(0) = 1$ indicates that $a_0 = 1$; with this, we can find the values of the other coefficients:

$$\begin{aligned} a_0 = 1 \text{ and } a_1 = 2a_0 &\Rightarrow a_1 = 2; \\ a_1 = 2 \text{ and } 2a_2 = 2a_1 &\Rightarrow a_2 = 4/2 = 2; \\ a_2 = 2 \text{ and } 3a_3 = 2a_2 &\Rightarrow a_3 = 8/(2 \cdot 3) = 4/3; \\ a_3 = 4/3 \text{ and } 4a_4 = 2a_3 &\Rightarrow a_4 = 16/(2 \cdot 3 \cdot 4) = 2/3. \end{aligned}$$

Thus the first 5 terms of the power series solution to the differential equation $y' = 2y$ is

$$f(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \cdots$$

Notes:

In Section 8.8, as we study Taylor Series, we will learn how to recognize this series as describing $y = e^{2x}$.

This example illustrates that it can be difficult to recognize an elementary function by its power series expansion. It is far easier to start with a known function, expressed in terms of elementary functions, and represent it as a power series function. One may wonder why we would bother doing so, as the latter function probably seems more complicated. In the next two sections, we show both *how* to do this and *why* such a process can be beneficial.

Notes:

Exercises 8.6

Terms and Concepts

1. We adopt the convenction that $x^0 =$______, regardless of the value of x.

2. What is the difference between the radius of convergence and the interval of convergence?

3. If the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is 5, what is the radius of convergence of $\sum_{n=1}^{\infty} n \cdot a_n x^{n-1}$?

4. If the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is 5, what is the radius of convergence of $\sum_{n=0}^{\infty} (-1)^n a_n x^n$?

Problems

In Exercises 5 – 8, write out the sum of the first 5 terms of the given power series.

5. $\sum_{n=0}^{\infty} 2^n x^n$

6. $\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$

7. $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$

8. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

In Exercises 9 – 25, a power series is given.

(a) Find the radius of convergence.

(b) Find the interval of convergence.

9. $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!} x^n$

10. $\sum_{n=0}^{\infty} n x^n$

11. $\sum_{n=1}^{\infty} \frac{(-1)^n (x-3)^n}{n}$

12. $\sum_{n=0}^{\infty} \frac{(x+4)^n}{n!}$

13. $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$

14. $\sum_{n=0}^{\infty} \frac{(-1)^n (x-5)^n}{10^n}$

15. $\sum_{n=0}^{\infty} 5^n (x-1)^n$

16. $\sum_{n=0}^{\infty} (-2)^n x^n$

17. $\sum_{n=0}^{\infty} \sqrt{n} x^n$

18. $\sum_{n=0}^{\infty} \frac{n}{3^n} x^n$

19. $\sum_{n=0}^{\infty} \frac{3^n}{n!} (x-5)^n$

20. $\sum_{n=0}^{\infty} (-1)^n n! (x-10)^n$

21. $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

22. $\sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n$

23. $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n^3}$

24. $\sum_{n=0}^{\infty} n! \left(\frac{x}{10}\right)^n$

25. $\sum_{n=0}^{\infty} n^2 \left(\frac{x+4}{4}\right)^n$

In Exercises 26 – 31, a function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is given.

(a) Give a power series for $f'(x)$ and its interval of convergence.

(b) Give a power series for $\int f(x)\,dx$ and its interval of convergence.

26. $\sum_{n=0}^{\infty} n x^n$

27. $\sum_{n=1}^{\infty} \frac{x^n}{n}$

28. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$

29. $\displaystyle\sum_{n=0}^{\infty}(-3x)^n$

30. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$

31. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}$

In Exercises 32 – 37, give the first 5 terms of the series that is a solution to the given differential equation.

32. $y' = 3y, \quad y(0) = 1$

33. $y' = 5y, \quad y(0) = 5$

34. $y' = y^2, \quad y(0) = 1$

35. $y' = y + 1, \quad y(0) = 1$

36. $y'' = -y, \quad y(0) = 0, y'(0) = 1$

37. $y'' = 2y, \quad y(0) = 1, y'(0) = 1$

8.7 Taylor Polynomials

Consider a function $y = f(x)$ and a point $\big(c, f(c)\big)$. The derivative, $f'(c)$, gives the instantaneous rate of change of f at $x = c$. Of all lines that pass through the point $\big(c, f(c)\big)$, the line that best approximates f at this point is the tangent line; that is, the line whose slope (rate of change) is $f'(c)$.

In Figure 8.16, we see a function $y = f(x)$ graphed. The table below the graph shows that $f(0) = 2$ and $f'(0) = 1$; therefore, the tangent line to f at $x = 0$ is $p_1(x) = 1(x-0)+2 = x+2$. The tangent line is also given in the figure. Note that "near" $x = 0$, $p_1(x) \approx f(x)$; that is, the tangent line approximates f well.

One shortcoming of this approximation is that the tangent line only matches the slope of f; it does not, for instance, match the concavity of f. We can find a polynomial, $p_2(x)$, that does match the concavity without much difficulty, though. The table in Figure 8.16 gives the following information:

$$f(0) = 2 \qquad f'(0) = 1 \qquad f''(0) = 2.$$

Therefore, we want our polynomial $p_2(x)$ to have these same properties. That is, we need

$$p_2(0) = 2 \qquad p_2'(0) = 1 \qquad p_2''(0) = 2.$$

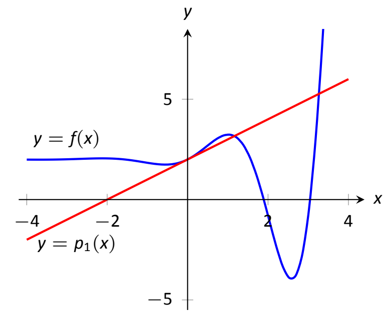
This is simply an initial–value problem. We can solve this using the techniques first described in Section 5.1. To keep $p_2(x)$ as simple as possible, we'll assume that not only $p_2''(0) = 2$, but that $p_2''(x) = 2$. That is, the second derivative of p_2 is constant.

If $p_2''(x) = 2$, then $p_2'(x) = 2x + C$ for some constant C. Since we have determined that $p_2'(0) = 1$, we find that $C = 1$ and so $p_2'(x) = 2x + 1$. Finally, we can compute $p_2(x) = x^2 + x + C$. Using our initial values, we know $p_2(0) = 2$ so $C = 2$. We conclude that $p_2(x) = x^2 + x + 2$. This function is plotted with f in Figure 8.17.

We can repeat this approximation process by creating polynomials of higher degree that match more of the derivatives of f at $x = 0$. In general, a polynomial of degree n can be created to match the first n derivatives of f. Figure 8.17 also shows $p_4(x) = -x^4/2 - x^3/6 + x^2 + x + 2$, whose first four derivatives at 0 match those of f. (Using the table in Figure 8.16, start with $p_4^{(4)}(x) = -12$ and solve the related initial–value problem.)

As we use more and more derivatives, our polynomial approximation to f gets better and better. In this example, the interval on which the approximation is "good" gets bigger and bigger. Figure 8.18 shows $p_{13}(x)$; we can visually affirm that this polynomial approximates f very well on $[-2, 3]$. (The polynomial $p_{13}(x)$ is not particularly "nice". It is

$$\frac{16901x^{13}}{6227020800}+\frac{13x^{12}}{1209600}-\frac{1321x^{11}}{39916800}-\frac{779x^{10}}{1814400}-\frac{359x^9}{362880}+\frac{x^8}{240}+\frac{139x^7}{5040}+\frac{11x^6}{360}-\frac{19x^5}{120}-\frac{x^4}{2}-\frac{x^3}{6}+x^2+x+2.)$$



$f(0) = 2$	$f'''(0) = -1$
$f'(0) = 1$	$f^{(4)}(0) = -12$
$f''(0) = 2$	$f^{(5)}(0) = -19$

Figure 8.16: Plotting $y = f(x)$ and a table of derivatives of f evaluated at 0.

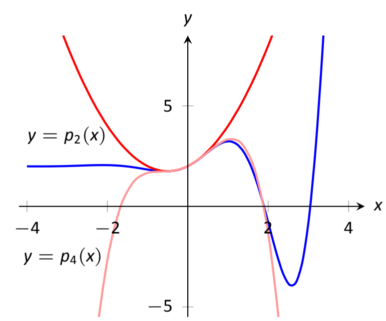


Figure 8.17: Plotting f, p_2 and p_4.

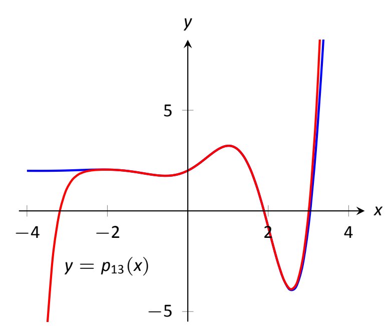


Figure 8.18: Plotting f and p_{13}.

Notes:

The polynomials we have created are examples of *Taylor polynomials*, named after the British mathematician Brook Taylor who made important discoveries about such functions. While we created the above Taylor polynomials by solving initial–value problems, it can be shown that Taylor polynomials follow a general pattern that make their formation much more direct. This is described in the following definition.

Definition 41 Taylor Polynomial, Maclaurin Polynomial

Let f be a function whose first n derivatives exist at $x = c$.

1. The **Taylor polynomial of degree** n **of** f **at** $x = c$ is

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

2. A special case of the Taylor polynomial is the Maclaurin polynomial, where $c = 0$. That is, the **Maclaurin polynomial of degree** n **of** f is

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

We will practice creating Taylor and Maclaurin polynomials in the following examples.

Example 8.33 Finding and using Maclaurin polynomials

1. Find the n^{th} Maclaurin polynomial for $f(x) = e^x$.

2. Use $p_5(x)$ to approximate the value of e.

SOLUTION

1. We start with creating a table of the derivatives of e^x evaluated at $x = 0$. In this particular case, this is relatively simple, as shown in Figure 8.19. By the definition of the Maclaurin series, we have

$$\begin{array}{lll} f(x) = e^x & \Rightarrow & f(0) = 1 \\ f'(x) = e^x & \Rightarrow & f'(0) = 1 \\ f''(x) = e^x & \Rightarrow & f''(0) = 1 \\ \vdots & & \vdots \\ f^{(n)}(x) = e^x & \Rightarrow & f^{(n)}(0) = 1 \end{array}$$

Figure 8.19: The derivatives of $f(x) = e^x$ evaluated at $x = 0$.

Notes:

$$\begin{aligned}p_n(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^n(0)}{n!}x^n\\ &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots + \frac{1}{n!}x^n.\end{aligned}$$

2. Using our answer from part 1, we have

$$p_5 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5.$$

To approximate the value of e, note that $e = e^1 = f(1) \approx p_5(1)$. It is very straightforward to evaluate $p_5(1)$:

$$p_5(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = \frac{163}{60} \approx 2.71667.$$

A plot of $f(x) = e^x$ and $p_5(x)$ is given in Figure 8.20.

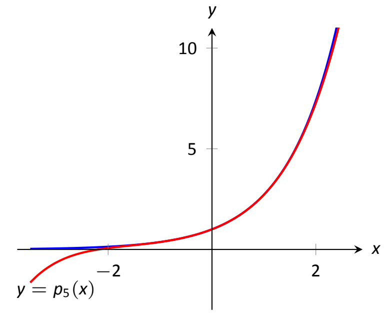


Figure 8.20: A plot of $f(x) = e^x$ and its 5th degree Maclaurin polynomial $p_5(x)$.

Example 8.34 Finding and using Taylor polynomials

1. Find the n^{th} Taylor polynomial of $y = \ln x$ at $x = 1$.
2. Use $p_6(x)$ to approximate the value of $\ln 1.5$.
3. Use $p_6(x)$ to approximate the value of $\ln 2$.

SOLUTION

1. We begin by creating a table of derivatives of $\ln x$ evaluated at $x = 1$. While this is not as straightforward as it was in the previous example, a pattern does emerge, as shown in Figure 8.21.

   Using Definition 41, we have

$$\begin{aligned}p_n(x) &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^n(c)}{n!}(x-c)^n\\ &= 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots + \frac{(-1)^{n+1}}{n}(x-1)^n.\end{aligned}$$

   Note how the coefficients of the $(x-1)$ terms turn out to be "nice."

2. We can compute $p_6(x)$ using our work above:

$$p_6(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6.$$

$$\begin{array}{lcl} f(x) = \ln x & \Rightarrow & f(1) = 0\\ f'(x) = 1/x & \Rightarrow & f'(1) = 1\\ f''(x) = -1/x^2 & \Rightarrow & f''(1) = -1\\ f'''(x) = 2/x^3 & \Rightarrow & f'''(1) = 2\\ f^{(4)}(x) = -6/x^4 & \Rightarrow & f^{(4)}(1) = -6\\ \vdots & & \vdots\\ f^{(n)}(x) = \dfrac{(-1)^{n+1}(n-1)!}{x^n} & \Rightarrow & f^{(n)}(1) = (-1)^{n+1}(n-1)! \end{array}$$

Figure 8.21: Derivatives of $\ln x$ evaluated at $x = 1$.

Notes:

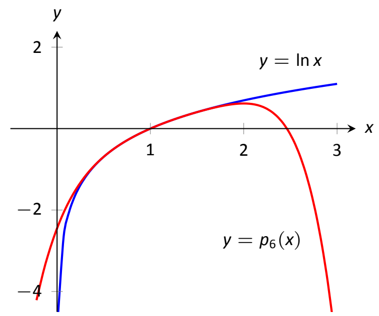


Figure 8.22: A plot of $y = \ln x$ and its 6th degree Taylor polynomial at $x = 1$.

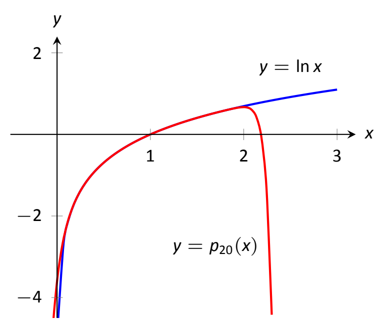


Figure 8.23: A plot of $y = \ln x$ and its 20th degree Taylor polynomial at $x = 1$.

Since $p_6(x)$ approximates $\ln x$ well near $x = 1$, we approximate $\ln 1.5 \approx p_6(1.5)$:

$$\begin{aligned} p_6(1.5) &= (1.5-1) - \frac{1}{2}(1.5-1)^2 + \frac{1}{3}(1.5-1)^3 - \frac{1}{4}(1.5-1)^4 + \cdots \\ &\cdots + \frac{1}{5}(1.5-1)^5 - \frac{1}{6}(1.5-1)^6 \\ &= \frac{259}{640} \\ &\approx 0.404688. \end{aligned}$$

This is a good approximation as a calculator shows that $\ln 1.5 \approx 0.4055$. Figure 8.22 plots $y = \ln x$ with $y = p_6(x)$. We can see that $\ln 1.5 \approx p_6(1.5)$.

3. We approximate $\ln 2$ with $p_6(2)$:

$$\begin{aligned} p_6(2) &= (2-1) - \frac{1}{2}(2-1)^2 + \frac{1}{3}(2-1)^3 - \frac{1}{4}(2-1)^4 + \cdots \\ &\cdots + \frac{1}{5}(2-1)^5 - \frac{1}{6}(2-1)^6 \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \\ &= \frac{37}{60} \\ &\approx 0.616667. \end{aligned}$$

This approximation is not terribly impressive: a hand held calculator shows that $\ln 2 \approx 0.693147$. The graph in Figure 8.22 shows that $p_6(x)$ provides less accurate approximations of $\ln x$ as x gets close to 0 or 2.

Surprisingly enough, even the 20th degree Taylor polynomial fails to approximate $\ln x$ for $x > 2$, as shown in Figure 8.23. We'll soon discuss why this is.

Taylor polynomials are used to approximate functions $f(x)$ in mainly two situations:

1. When $f(x)$ is known, but perhaps "hard" to compute directly. For instance, we can define $y = \cos x$ as either the ratio of sides of a right triangle ("adjacent over hypotenuse") or with the unit circle. However, neither of these provides a convenient way of computing $\cos 2$. A Taylor polynomial of sufficiently high degree can provide a reasonable method of computing such values using only operations usually hard–wired into a computer ($+$, $-$, $\times$ and $\div$).

Notes:

2. When $f(x)$ is not known, but information about its derivatives is known. This occurs more often than one might think, especially in the study of differential equations.

In both situations, a critical piece of information to have is "How good is my approximation?" If we use a Taylor polynomial to compute $\cos 2$, how do we know how accurate the approximation is?

We had the same problem when studying Numerical Integration. Theorem 49 provided bounds on the error when using, say, Simpson's Rule to approximate a definite integral. These bounds allowed us to determine that, for instance, using 10 subintervals provided an approximation within $\pm .01$ of the exact value. The following theorem gives similar bounds for Taylor (and hence Maclaurin) polynomials.

Note: Even though Taylor polynomials *could* be used in calculators and computers to calculate values of trigonometric functions, in practice they generally aren't. Other more efficient and accurate methods have been developed, such as the CORDIC algorithm.

Theorem 78 Taylor's Theorem

1. Let f be a function whose $n+1^{\text{th}}$ derivative exists on an interval I and let c be in I. Then, for each x in I, there exists z_x between x and c such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x),$$

where $R_n(x) = \dfrac{f^{(n+1)}(z_x)}{(n+1)!}(x-c)^{(n+1)}$.

2. $\big|R_n(x)\big| \leq \dfrac{\max\big|f^{(n+1)}(z)\big|}{(n+1)!}\big|(x-c)^{(n+1)}\big|$

The first part of Taylor's Theorem states that $f(x) = p_n(x) + R_n(x)$, where $p_n(x)$ is the n^{th} order Taylor polynomial and $R_n(x)$ is the remainder, or error, in the Taylor approximation. The second part gives bounds on how big that error can be. If the $(n+1)^{\text{th}}$ derivative is large, the error may be large; if x is far from c, the error may also be large. However, the $(n+1)!$ term in the denominator tends to ensure that the error gets smaller as n increases.

The following example computes error estimates for the approximations of $\ln 1.5$ and $\ln 2$ made in Example 8.34.

Example 8.35 Finding error bounds of a Taylor polynomial

Use Theorem 78 to find error bounds when approximating $\ln 1.5$ and $\ln 2$ with $p_6(x)$, the Taylor polynomial of degree 6 of $f(x) = \ln x$ at $x = 1$, as calculated in Example 8.34.

Notes:

Solution

1. We start with the approximation of $\ln 1.5$ with $p_6(1.5)$. The theorem references an open interval I that contains both x and c. The smaller the interval we use the better; it will give us a more accurate (and smaller!) approximation of the error. We let $I = (0.9, 1.6)$, as this interval contains both $c = 1$ and $x = 1.5$.

   The theorem references $\max \left|f^{(n+1)}(z)\right|$. In our situation, this is asking "How big can the 7th derivative of $y = \ln x$ be on the interval $(0.9, 1.6)$?" The seventh derivative is $y = -6!/x^7$. The largest value it attains on I is about 1506. Thus we can bound the error as:

   $$\begin{aligned} \left|R_6(1.5)\right| &\leq \frac{\max \left|f^{(7)}(z)\right|}{7!}\left|(1.5-1)^7\right| \\ &\leq \frac{1506}{5040}\cdot\frac{1}{2^7} \\ &\approx 0.0023. \end{aligned}$$

   We computed $p_6(1.5) = 0.404688$; using a calculator, we find $\ln 1.5 \approx 0.405465$, so the actual error is about 0.000778, which is less than our bound of 0.0023. This affirms Taylor's Theorem; the theorem states that our approximation would be within about 2 thousandths of the actual value, whereas the approximation was actually closer.

2. We again find an interval I that contains both $c = 1$ and $x = 2$; we choose $I = (0.9, 2.1)$. The maximum value of the seventh derivative of f on this interval is again about 1506 (as the largest values come near $x = 0.9$). Thus

   $$\begin{aligned} \left|R_6(2)\right| &\leq \frac{\max \left|f^{(7)}(z)\right|}{7!}\left|(2-1)^7\right| \\ &\leq \frac{1506}{5040}\cdot 1^7 \\ &\approx 0.30. \end{aligned}$$

   This bound is not as nearly as good as before. Using the degree 6 Taylor polynomial at $x = 1$ will bring us within 0.3 of the correct answer. As $p_6(2) \approx 0.61667$, our error estimate guarantees that the actual value of $\ln 2$ is somewhere between 0.31667 and 0.91667. These bounds are not particularly useful.

   In reality, our approximation was only off by about 0.07. However, we are approximating ostensibly because we do not know the real answer. In order to be assured that we have a good approximation, we would have to resort to using a polynomial of higher degree.

Notes:

We practice again. This time, we use Taylor's theorem to find n that guarantees our approximation is within a certain amount.

Example 8.36 Finding sufficiently accurate Taylor polynomials

Find n such that the n^{th} Taylor polynomial of $f(x) = \cos x$ at $x = 0$ approximates $\cos 2$ to within 0.001 of the actual answer. What is $p_n(2)$?

Solution Following Taylor's theorem, we need bounds on the size of the derivatives of $f(x) = \cos x$. In the case of this trigonometric function, this is easy. All derivatives of cosine are $\pm \sin x$ or $\pm \cos x$. In all cases, these functions are never greater than 1 in absolute value. We want the error to be less than 0.001. To find the appropriate n, consider the following inequalities:

$$\frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|(2-0)^{(n+1)}\right| \leq 0.001$$

$$\frac{1}{(n+1)!}\cdot 2^{(n+1)} \leq 0.001$$

We find an n that satisfies this last inequality with trial–and–error. When $n = 8$, we have $\dfrac{2^{8+1}}{(8+1)!} \approx 0.0014$; when $n = 9$, we have $\dfrac{2^{9+1}}{(9+1)!} \approx 0.000282 < 0.001$. Thus we want to approximate $\cos 2$ with $p_9(2)$.

We now set out to compute $p_9(x)$. We again need a table of the derivatives of $f(x) = \cos x$ evaluated at $x = 0$. A table of these values is given in Figure 8.24. Notice how the derivatives, evaluated at $x = 0$, follow a certain pattern. All the odd powers of x in the Taylor polynomial will disappear as their coefficient is 0. While our error bounds state that we need $p_9(x)$, our work shows that this will be the same as $p_8(x)$.

Since we are forming our polynomial at $x = 0$, we are creating a Maclaurin polynomial, and:

$$\begin{aligned} p_8(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(8)}}{8!}x^8 \\ &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 \end{aligned}$$

We finally approximate $\cos 2$:

$$\cos 2 \approx p_8(2) = -\frac{131}{315} \approx -0.41587.$$

Our error bound guarantee that this approximation is within 0.001 of the correct answer. Technology shows us that our approximation is actually within about 0.0003 of the correct answer.

Figure 8.25 shows a graph of $y = p_8(x)$ and $y = \cos x$. Note how well the two functions agree on about $(-\pi, \pi)$.

$$\begin{array}{lcl}
f(x) = \cos x & \Rightarrow & f(0) = 1 \\
f'(x) = -\sin x & \Rightarrow & f'(0) = 0 \\
f''(x) = -\cos x & \Rightarrow & f''(0) = -1 \\
f'''(x) = \sin x & \Rightarrow & f'''(0) = 0 \\
f^{(4)}(x) = \cos x & \Rightarrow & f^{(4)}(0) = 1 \\
f^{(5)}(x) = -\sin x & \Rightarrow & f^{(5)}(0) = 0 \\
f^{(6)}(x) = -\cos x & \Rightarrow & f^{(6)}(0) = -1 \\
f^{(7)}(x) = \sin x & \Rightarrow & f^{(7)}(0) = 0 \\
f^{(8)}(x) = \cos x & \Rightarrow & f^{(8)}(0) = 1 \\
f^{(9)}(x) = -\sin x & \Rightarrow & f^{(9)}(0) = 0
\end{array}$$

Figure 8.24: A table of the derivatives of $f(x) = \cos x$ evaluated at $x = 0$.

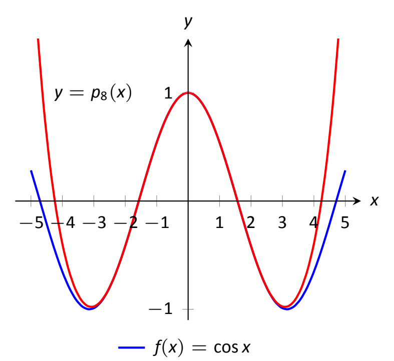


Figure 8.25: A graph of $f(x) = \cos x$ and its degree 8 Maclaurin polynomial.

Notes:

Example 8.37 Finding and using Taylor polynomials

1. Find the degree 4 Taylor polynomial, $p_4(x)$, for $f(x) = \sqrt{x}$ at $x = 4$.
2. Use $p_4(x)$ to approximate $\sqrt{3}$.
3. Find bounds on the error when approximating $\sqrt{3}$ with $p_4(3)$.

Solution

1. We begin by evaluating the derivatives of f at $x = 4$. This is done in Figure 8.26. These values allow us to form the Taylor polynomial $p_4(x)$:

$$p_4(x) = 2+\frac{1}{4}(x-4)+\frac{-1/32}{2!}(x-4)^2+\frac{3/256}{3!}(x-4)^3+\frac{-15/2048}{4!}(x-4)^4.$$

2. As $p_4(x) \approx \sqrt{x}$ near $x = 4$, we approximate $\sqrt{3}$ with $p_4(3) = 1.73212$.
3. To find a bound on the error, we need an open interval that contains $x = 3$ and $x = 4$. We set $I = (2.9, 4.1)$. The largest value the fifth derivative of $f(x) = \sqrt{x}$ takes on this interval is near $x = 2.9$, at about 0.0273. Thus

$$\big|R_4(3)\big| \leq \frac{0.0273}{5!}\big|(3-4)^5\big| \approx 0.00023.$$

This shows our approximation is accurate to at least the first 2 places after the decimal. (It turns out that our approximation is actually accurate to 4 places after the decimal.) A graph of $f(x) = \sqrt{x}$ and $p_4(x)$ is given in Figure 8.27. Note how the two functions are nearly indistinguishable on $(2, 7)$.

$$\begin{array}{lll}
f(x) = \sqrt{x} & \Rightarrow & f(4) = 2 \\
f'(x) = \dfrac{1}{2\sqrt{x}} & \Rightarrow & f'(4) = \dfrac{1}{4} \\
f''(x) = \dfrac{-1}{4x^{3/2}} & \Rightarrow & f''(4) = \dfrac{-1}{32} \\
f'''(x) = \dfrac{3}{8x^{5/2}} & \Rightarrow & f'''(4) = \dfrac{3}{256} \\
f^{(4)}(x) = \dfrac{-15}{16x^{7/2}} & \Rightarrow & f^{(4)}(4) = \dfrac{-15}{2048}
\end{array}$$

Figure 8.26: A table of the derivatives of $f(x) = \sqrt{x}$ evaluated at $x = 4$.

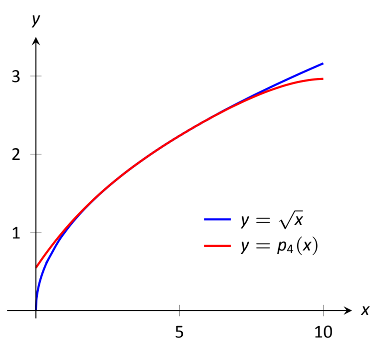


Figure 8.27: A graph of $f(x) = \sqrt{x}$ and its degree 4 Taylor polynomial at $x = 4$.

Our final example gives a brief introduction to using Taylor polynomials to solve differential equations.

Example 8.38 Approximating an unknown function

A function $y = f(x)$ is unknown save for the following two facts.

1. $y(0) = f(0) = 1$, and
2. $y' = y^2$

(This second fact says that amazingly, the derivative of the function is actually the function squared!)

Find the degree 3 Maclaurin polynomial $p_3(x)$ of $y = f(x)$.

Notes:

Solution One might initially think that not enough information is given to find $p_3(x)$. However, note how the second fact above actually lets us know what $y'(0)$ is:

$$y' = y^2 \Rightarrow y'(0) = y^2(0).$$

Since $y(0) = 1$, we conclude that $y'(0) = 1$.

Now we find information about y''. Starting with $y' = y^2$, take derivatives of both sides, *with respect to* x. That means we must use implicit differentiation.

$$\begin{aligned} y' &= y^2 \\ \frac{d}{dx}\big(y'\big) &= \frac{d}{dx}\big(y^2\big) \\ y'' &= 2y \cdot y'. \end{aligned}$$

Now evaluate both sides at $x = 0$:

$$\begin{aligned} y''(0) &= 2y(0) \cdot y'(0) \\ y''(0) &= 2 \end{aligned}$$

We repeat this once more to find $y'''(0)$. We again use implicit differentiation; this time the Product Rule is also required.

$$\begin{aligned} \frac{d}{dx}\big(y''\big) &= \frac{d}{dx}\big(2yy'\big) \\ y''' &= 2y' \cdot y' + 2y \cdot y''. \end{aligned}$$

Now evaluate both sides at $x = 0$:

$$\begin{aligned} y'''(0) &= 2y'(0)^2 + 2y(0)y''(0) \\ y'''(0) &= 2 + 4 = 6 \end{aligned}$$

In summary, we have:

$$y(0) = 1 \qquad y'(0) = 1 \qquad y''(0) = 2 \qquad y'''(0) = 6.$$

We can now form $p_3(x)$:

$$\begin{aligned} p_3(x) &= 1 + x + \frac{2}{2!}x^2 + \frac{6}{3!}x^3 \\ &= 1 + x + x^2 + x^3. \end{aligned}$$

It turns out that the differential equation we started with, $y' = y^2$, where $y(0) = 1$, can be solved without too much difficulty: $y = \dfrac{1}{1-x}$. Figure 8.28 shows this function plotted with $p_3(x)$. Note how similar they are near $x = 0$.

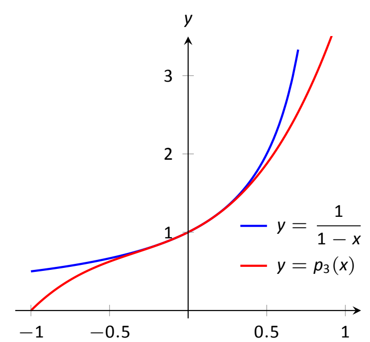


Figure 8.28: A graph of $y = -1/(x-1)$ and $y = p_3(x)$ from Example 8.38.

Notes:

It is beyond the scope of this text to pursue error analysis when using Taylor polynomials to approximate solutions to differential equations. This topic is often broached in introductory Differential Equations courses and usually covered in depth in Numerical Analysis courses. Such an analysis is very important; one needs to know how good their approximation is. We explored this example simply to demonstrate the usefulness of Taylor polynomials.

Most of this chapter has been devoted to the study of infinite series. This section has taken a step back from this study, focusing instead on finite summation of terms. In the next section, we explore **Taylor Series**, where we represent a function with an infinite series.

Notes:

Exercises 8.7

Terms and Concepts

1. What is the difference between a Taylor polynomial and a Maclaurin polynomial?

2. T/F: In general, $p_n(x)$ approximates $f(x)$ better and better as n gets larger.

3. For some function $f(x)$, the Maclaurin polynomial of degree 4 is $p_4(x) = 6 + 3x - 4x^2 + 5x^3 - 7x^4$. What is $p_2(x)$?

4. For some function $f(x)$, the Maclaurin polynomial of degree 4 is $p_4(x) = 6 + 3x - 4x^2 + 5x^3 - 7x^4$. What is $f'''(0)$?

Problems

In Exercises 5 – 12, find the Maclaurin polynomial of degree n for the given function.

5. $f(x) = e^{-x}, \quad n = 3$

6. $f(x) = \sin x, \quad n = 8$

7. $f(x) = x \cdot e^x, \quad n = 5$

8. $f(x) = \tan x, \quad n = 6$

9. $f(x) = e^{2x}, \quad n = 4$

10. $f(x) = \dfrac{1}{1-x}, \quad n = 4$

11. $f(x) = \dfrac{1}{1+x}, \quad n = 4$

12. $f(x) = \dfrac{1}{1+x}, \quad n = 7$

In Exercises 13 – 20, find the Taylor polynomial of degree n, at $x = c$, for the given function.

13. $f(x) = \sqrt{x}, \quad n = 4, \quad c = 1$

14. $f(x) = \ln(x+1), \quad n = 4, \quad c = 1$

15. $f(x) = \cos x, \quad n = 6, \quad c = \pi/4$

16. $f(x) = \sin x, \quad n = 5, \quad c = \pi/6$

17. $f(x) = \dfrac{1}{x}, \quad n = 5, \quad c = 2$

18. $f(x) = \dfrac{1}{x^2}, \quad n = 8, \quad c = 1$

19. $f(x) = \dfrac{1}{x^2+1}, \quad n = 3, \quad c = -1$

20. $f(x) = x^2 \cos x, \quad n = 2, \quad c = \pi$

In Exercises 21 – 24, approximate the function value with the indicated Taylor polynomial and give approximate bounds on the error.

21. Approximate $\sin 0.1$ with the Maclaurin polynomial of degree 3.

22. Approximate $\cos 1$ with the Maclaurin polynomial of degree 4.

23. Approximate $\sqrt{10}$ with the Taylor polynomial of degree 2 centered at $x = 9$.

24. Approximate $\ln 1.5$ with the Taylor polynomial of degree 3 centered at $x = 1$.

Exercises 25 – 28 ask for an n to be found such that $p_n(x)$ approximates $f(x)$ within a certain bound of accuracy.

25. Find n such that the Maclaurin polynomial of degree n of $f(x) = e^x$ approximates e within 0.0001 of the actual value.

26. Find n such that the Taylor polynomial of degree n of $f(x) = \sqrt{x}$, centered at $x = 4$, approximates $\sqrt{3}$ within 0.0001 of the actual value.

27. Find n such that the Maclaurin polynomial of degree n of $f(x) = \cos x$ approximates $\cos \pi/3$ within 0.0001 of the actual value.

28. Find n such that the Maclaurin polynomial of degree n of $f(x) = \sin x$ approximates $\cos \pi$ within 0.0001 of the actual value.

In Exercises 29 – 33, find the n^{th} term of the indicated Taylor polynomial.

29. Find a formula for the n^{th} term of the Maclaurin polynomial for $f(x) = e^x$.

30. Find a formula for the n^{th} term of the Maclaurin polynomial for $f(x) = \cos x$.

31. Find a formula for the n^{th} term of the Maclaurin polynomial for $f(x) = \dfrac{1}{1-x}$.

32. Find a formula for the n^{th} term of the Maclaurin polynomial for $f(x) = \dfrac{1}{1+x}$.

33. Find a formula for the n^{th} term of the Taylor polynomial for $f(x) = \ln x$ centered at $x = 1$.

In Exercises 34 – 36, approximate the solution to the given differential equation with a degree 4 Maclaurin polynomial.

34. $y' = y, \qquad y(0) = 1$

35. $y' = 5y, \qquad y(0) = 3$

36. $y' = \dfrac{2}{y}, \qquad y(0) = 1$

8.8 Taylor Series

In Section 8.6, we showed how certain functions can be represented by a power series function. In 8.7, we showed how we can approximate functions with polynomials, given that enough derivative information is available. In this section we combine these concepts: if a function $f(x)$ is infinitely differentiable, we show how to represent it with a power series function.

Definition 42 Taylor and Maclaurin Series

Let $f(x)$ have derivatives of all orders at $x = c$.

1. The **Taylor Series of** $f(x)$**, centered at** c is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

2. Setting $c = 0$ gives the **Maclaurin Series of** $f(x)$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

The difference between a Taylor polynomial and a Taylor series is the former is a polynomial, containing only a finite number of terms, whereas the latter is a series, a summation of an infinite set of terms. When creating the Taylor polynomial of degree n for a function $f(x)$ at $x = c$, we needed to evaluate f, and the first n derivatives of f, at $x = c$. When creating the Taylor series of f, it helps to find a pattern that describes the n^{th} derivative of f at $x = c$. We demonstrate this in the next two examples.

Example 8.39 The Maclaurin series of $f(x) = \cos x$

Find the Maclaurin series of $f(x) = \cos x$.

SOLUTION In Example 8.36 we found the 8^{th} degree Maclaurin polynomial of $\cos x$. In doing so, we created the table shown in Figure 8.29. Notice how $f^{(n)}(0) = 0$ when n is odd, $f^{(n)}(0) = 1$ when n is divisible by 4, and $f^{(n)}(0) = -1$ when n is even but not divisible by 4. Thus the Maclaurin series of $\cos x$ is

$$1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

We can go further and write this as a summation. Since we only need the terms

$$\begin{array}{lcl}
f(x) = \cos x & \Rightarrow & f(0) = 1 \\
f'(x) = -\sin x & \Rightarrow & f'(0) = 0 \\
f''(x) = -\cos x & \Rightarrow & f''(0) = -1 \\
f'''(x) = \sin x & \Rightarrow & f'''(0) = 0 \\
f^{(4)}(x) = \cos x & \Rightarrow & f^{(4)}(0) = 1 \\
f^{(5)}(x) = -\sin x & \Rightarrow & f^{(5)}(0) = 0 \\
f^{(6)}(x) = -\cos x & \Rightarrow & f^{(6)}(0) = -1 \\
f^{(7)}(x) = \sin x & \Rightarrow & f^{(7)}(0) = 0 \\
f^{(8)}(x) = \cos x & \Rightarrow & f^{(8)}(0) = 1 \\
f^{(9)}(x) = -\sin x & \Rightarrow & f^{(9)}(0) = 0
\end{array}$$

Figure 8.29: A table of the derivatives of $f(x) = \cos x$ evaluated at $x = 0$.

Notes:

where the power of x is even, we write the power series in terms of x^{2n}:

$$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}.$$

Example 8.40 The Taylor series of $f(x) = \ln x$ at $x = 1$

Find the Taylor series of $f(x) = \ln x$ centered at $x = 1$.

SOLUTION Figure 8.30 shows the n^{th} derivative of $\ln x$ evaluated at $x = 1$ for $n = 0, \ldots, 5$, along with an expression for the n^{th} term:

$$f^{(n)}(1) = (-1)^{n+1}(n-1)! \quad \text{for } n \geq 1.$$

Remember that this is what distinguishes Taylor series from Taylor polynomials; we are very interested in finding a pattern for the n^{th} term, not just finding a finite set of coefficients for a polynomial. Since $f(1) = \ln 1 = 0$, we skip the first term and start the summation with $n = 1$, giving the Taylor series for $\ln x$, centered at $x = 1$, as

$$\sum_{n=1}^{\infty}(-1)^{n+1}(n-1)!\frac{1}{n!}(x-1)^n = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}.$$

$$\begin{array}{lcl}
f(x) = \ln x & \Rightarrow & f(1) = 0 \\
f'(x) = 1/x & \Rightarrow & f'(1) = 1 \\
f''(x) = -1/x^2 & \Rightarrow & f''(1) = -1 \\
f'''(x) = 2/x^3 & \Rightarrow & f'''(1) = 2 \\
f^{(4)}(x) = -6/x^4 & \Rightarrow & f^{(4)}(1) = -6 \\
f^{(5)}(x) = 24/x^5 & \Rightarrow & f^{(5)}(1) = 24 \\
\vdots & & \vdots \\
f^{(n)}(x) = & \Rightarrow & f^{(n)}(1) = \\
\dfrac{(-1)^{n+1}(n-1)!}{x^n} & & (-1)^{n+1}(n-1)!
\end{array}$$

Figure 8.30: Derivatives of $\ln x$ evaluated at $x = 1$.

It is important to note that Definition 42 defines a Taylor series given a function $f(x)$; however, we *cannot* yet state that $f(x)$ *is equal* to its Taylor series. We will find that "most of the time" they are equal, but we need to consider the conditions that allow us to conclude this.

Theorem 78 states that the error between a function $f(x)$ and its n^{th}–degree Taylor polynomial $p_n(x)$ is $R_n(x)$, where

$$\left|R_n(x)\right| \leq \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|(x-c)^{(n+1)}\right|.$$

If $R_n(x)$ goes to 0 for each x in an interval I as n approaches infinity, we conclude that the function is equal to its Taylor series expansion.

Theorem 79 Function and Taylor Series Equality

Let $f(x)$ have derivatives of all orders at $x = c$, let $R_n(x)$ be as stated in Theorem 78, and let I be an interval on which the Taylor series of $f(x)$ converges. If $\lim_{n\to\infty} R_n(x) = 0$ for all x in I, then

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n \quad \text{on } I.$$

Notes:

We demonstrate the use of this theorem in an example.

Example 8.41 Establishing equality of a function and its Taylor series
Show that $f(x) = \cos x$ is equal to its Maclaurin series, as found in Example 8.39, for all x.

Solution Given a value x, the magnitude of the error term $R_n(x)$ is bounded by

$$\left|R_n(x)\right| \leq \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|x^{n+1}\right|.$$

Since all derivatives of $\cos x$ are $\pm\sin x$ or $\pm\cos x$, whose magnitudes are bounded by 1, we can state

$$\left|R_n(x)\right| \leq \frac{1}{(n+1)!}\left|x^{n+1}\right|$$

which implies

$$-\frac{|x^{n+1}|}{(n+1)!} \leq R_n(x) \leq \frac{|x^{n+1}|}{(n+1)!}. \tag{8.7}$$

For any x, $\lim\limits_{n\to\infty} \dfrac{x^{n+1}}{(n+1)!} = 0$. Applying the Squeeze Theorem to Equation (8.7), we conclude that $\lim\limits_{n\to\infty} R_n(x) = 0$ for all x, and hence

$$\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \quad \text{for all } x.$$

It is natural to assume that a function is equal to its Taylor series on the series' interval of convergence, but this is not the case. In order to properly establish equality, one must use Theorem 79. This is a bit disappointing, as we developed beautiful techniques for determining the interval of convergence of a power series, and proving that $R_n(x) \to 0$ can be cumbersome as it deals with high order derivatives of the function.

There is good news. A function $f(x)$ that is equal to its Taylor series, centered at any point the domain of $f(x)$, is said to be an **analytic function**, and most, if not all, functions that we encounter within this course are analytic functions. Generally speaking, any function that one creates with elementary functions (polynomials, exponentials, trigonometric functions, etc.) that is not piecewise defined is probably analytic. For most functions, we assume the function is equal to its Taylor series on the series' interval of convergence and only use Theorem 79 when we suspect something may not work as expected.

Notes:

We develop the Taylor series for one more important function, then give a table of the Taylor series for a number of common functions.

Example 8.42 The Binomial Series

Find the Maclaurin series of $f(x) = (1+x)^k$, $k \neq 0$.

Solution When k is a positive integer, the Maclaurin series is finite. For instance, when $k = 4$, we have

$$f(x) = (1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4.$$

The coefficients of x when k is a positive integer are known as the *binomial coefficients*, giving the series we are developing its name.

When $k = 1/2$, we have $f(x) = \sqrt{1+x}$. Knowing a series representation of this function would give a useful way of approximating $\sqrt{1.3}$, for instance.

To develop the Maclaurin series for $f(x) = (1+x)^k$ for any value of $k \neq 0$, we consider the derivatives of f evaluated at $x = 0$:

$$\begin{aligned}
f(x) &= (1+x)^k & f(0) &= 1\\
f'(x) &= k(1+x)^{k-1} & f'(0) &= k\\
f''(x) &= k(k-1)(1+x)^{k-2} & f''(0) &= k(k-1)\\
f'''(x) &= k(k-1)(k-2)(1+x)^{k-3} & f'''(0) &= k(k-1)(k-2)\\
&\vdots & &\vdots\\
f^{(n)}(x) &= k(k-1)\cdots\big(k-(n-1)\big)(1+x)^{k-n} & f^{(n)}(0) &= k(k-1)\cdots\big(k-(n-1)\big)
\end{aligned}$$

Thus the Maclaurin series for $f(x) = (1+x)^k$ is

$$1 + k + \frac{k(k-1)}{2!} + \frac{k(k-1)(k-2)}{3!} + \ldots + \frac{k(k-1)\cdots\big(k-(n-1)\big)}{n!} + \ldots$$

It is important to determine the interval of convergence of this series. With

$$a_n = \frac{k(k-1)\cdots\big(k-(n-1)\big)}{n!}x^n,$$

we apply the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty}\left|\frac{k(k-1)\cdots(k-n)}{(n+1)!}x^{n+1}\right| \Big/ \left|\frac{k(k-1)\cdots\big(k-(n-1)\big)}{n!}x^n\right|\\
&= \lim_{n\to\infty}\left|\frac{k-n}{n}x\right|\\
&= |x|.
\end{aligned}$$

Notes:

The series converges absolutely when the limit of the Ratio Test is less than 1; therefore, we have absolute convergence when $|x| < 1$.

While outside the scope of this text, the interval of convergence depends on the value of k. When $k > 0$, the interval of convergence is $[-1, 1]$. When $-1 < k < 0$, the interval of convergence is $[-1, 1)$. If $k \leq -1$, the interval of convergence is $(-1, 1)$.

We learned that Taylor polynomials offer a way of approximating a "difficult to compute" function with a polynomial. Taylor series offer a way of exactly representing a function with a series. One probably can see the use of a good approximation; is there any use of representing a function exactly as a series?

While we should not overlook the mathematical beauty of Taylor series (which is reason enough to study them), there are practical uses as well. They provide a valuable tool for solving a variety of problems, including problems relating to integration and differential equations.

In Key Idea 33 (on the following page) we give a table of the Taylor series of a number of common functions. We then give a theorem about the "algebra of power series," that is, how we can combine power series to create power series of new functions. This allows us to find the Taylor series of functions like $f(x) = e^x \cos x$ by knowing the Taylor series of e^x and $\cos x$.

Before we investigate combining functions, consider the Taylor series for the arctangent function (see Key Idea 33). Knowing that $\tan^{-1}(1) = \pi/4$, we can use this series to approximate the value of π:

$$\begin{aligned} \frac{\pi}{4} &= \tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots \\ \pi &= 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right) \end{aligned}$$

Unfortunately, this particular expansion of π converges very slowly. The first 100 terms approximate π as 3.13159, which is not particularly good.

Notes:

Key Idea 33 Important Taylor Series Expansions

Function and Series	First Few Terms	Interval of Convergence
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$(-\infty, \infty)$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$(-\infty, \infty)$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$(-\infty, \infty)$
$\ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$	$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$	$(0, 2]$
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$	$1 + x + x^2 + x^3 + \cdots$	$(-1, 1)$
$(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots\big(k-(n-1)\big)}{n!} x^n$	$1 + kx + \frac{k(k-1)}{2!} x^2 + \cdots$	$(-1, 1)$[a]
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$[-1, 1]$

[a] Convergence at $x = \pm 1$ depends on the value of k.

Theorem 80 Algebra of Power Series

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and let $h(x)$ be continuous.

1. $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$ for $|x| < R$.
2. $f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} \big(a_0 b_n + a_1 b_{n-1} + \ldots a_n b_0\big) x^n$ for $|x| < R$.
3. $f\big(h(x)\big) = \sum_{n=0}^{\infty} a_n \big(h(x)\big)^n$ for $|h(x)| < R$.

Notes:

Example 8.43 Combining Taylor series

Write out the first 3 terms of the Taylor Series for $f(x) = e^x \cos x$ using Key Idea 33 and Theorem 80.

Solution Key Idea 33 informs us that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{and} \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots.$$

Applying Theorem 80, we find that

$$e^x \cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right).$$

Distribute the right hand expression across the left:

$$\begin{aligned} &= 1\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \frac{x^2}{2!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) \\ &\quad + \frac{x^3}{3!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \frac{x^4}{4!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \cdots \end{aligned}$$

Distribute again and collect like terms.

$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30} + \frac{x^7}{630} + \cdots$$

While this process is a bit tedious, it is much faster than evaluating all the necessary derivatives of $e^x \cos x$ and computing the Taylor series directly.

Because the series for e^x and $\cos x$ both converge on $(-\infty, \infty)$, so does the series expansion for $e^x \cos x$.

Example 8.44 Creating new Taylor series

Use Theorem 80 to create series for $y = \sin(x^2)$ and $y = \ln(\sqrt{x})$.

Solution Given that

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

we simply substitute x^2 for x in the series, giving

$$\sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} \cdots.$$

Notes:

Since the Taylor series for $\sin x$ has an infinite radius of convergence, so does the Taylor series for $\sin(x^2)$.

The Taylor expansion for $\ln x$ given in Key Idea 33 is centered at $x = 1$, so we will center the series for $\ln(\sqrt{x})$ at $x = 1$ as well. With

$$\ln x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots,$$

we substitute $\sqrt{x}$ for x to obtain

$$\ln(\sqrt{x}) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(\sqrt{x}-1)^n}{n} = (\sqrt{x}-1) - \frac{(\sqrt{x}-1)^2}{2} + \frac{(\sqrt{x}-1)^3}{3} - \cdots.$$

While this is not strictly a power series, it is a series that allows us to study the function $\ln(\sqrt{x})$. Since the interval of convergence of $\ln x$ is $(0, 2]$, and the range of $\sqrt{x}$ on $(0, 4]$ is $(0, 2]$, the interval of convergence of this series expansion of $\ln(\sqrt{x})$ is $(0, 4]$.

Note: In Example 8.44, one could create a series for $\ln(\sqrt{x})$ by simply recognizing that $\ln(\sqrt{x}) = \ln(x^{1/2}) = 1/2 \ln x$, and hence multiplying the Taylor series for $\ln x$ by $1/2$. This example was chosen to demonstrate other aspects of series, such as the fact that the interval of convergence changes.

Example 8.45 Using Taylor series to evaluate definite integrals

Use the Taylor series of e^{-x^2} to evaluate $\displaystyle\int_0^1 e^{-x^2}\,dx$.

Solution We learned, when studying Numerical Integration, that e^{-x^2} does not have an antiderivative expressible in terms of elementary functions. This means any definite integral of this function must have its value approximated, and not computed exactly. (In Section 6.7, we defined the error function $\operatorname{erf} x$ so we can express this integral's value exactly in terms of that function. However, that does not help us approximate this integral.)

We can quickly write out the Taylor series for e^{-x^2} using the Taylor series of e^x:

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

and so

$$\begin{aligned} e^{-x^2} &= \sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!} \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{n!} \\ &= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots. \end{aligned}$$

Notes:

We use Theorem 77 to integrate:

$$\int e^{-x^2}\,dx = C + x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)n!} + \cdots$$

This *is* the antiderivative of e^{-x^2}; while we can write it out as a series, we cannot write it out in terms of elementary functions. We can evaluate the definite integral $\int_0^1 e^{-x^2}\,dx$ using this antiderivative; substituting 1 and 0 for x and subtracting gives

$$\int_0^1 e^{-x^2}\,dx = 1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!}\cdots.$$

Summing the 5 terms shown above give the approximation of 0.74749. Since this is an alternating series, we can use the Alternating Series Approximation Theorem, (Theorem 73), to determine how accurate this approximation is. The next term of the series is $1/(11\cdot 5!) \approx 0.00075758$. Thus we know our approximation is within 0.00075758 of the actual value of the integral. This is arguably much less work than using Simpson's Rule to approximate the value of the integral.

Example 8.46 Using Taylor series to solve differential equations

Solve the differential equation $y' = 2y$ in terms of a power series, and use the theory of Taylor series to recognize the solution in terms of an elementary function.

Solution We found the first 5 terms of the power series solution to this differential equation in Example 8.32 in Section 8.6. These are:

$$a_0 = 1,\quad a_1 = 2,\quad a_2 = \frac{4}{2} = 2,\quad a_3 = \frac{8}{2\cdot 3} = \frac{4}{3},\quad a_4 = \frac{16}{2\cdot 3\cdot 4} = \frac{2}{3}.$$

We include the "unsimplified" expressions for the coefficients found in Example 8.32 as we are looking for a pattern. It can be shown that $a_n = 2^n/n!$. Thus the solution, written as a power series, is

$$y = \sum_{n=0}^{\infty}\frac{2^n}{n!}x^n = \sum_{n=0}^{\infty}\frac{(2x)^n}{n!}.$$

Using Key Idea 33 and Theorem 80, we recognize $f(x) = e^{2x}$:

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \qquad \Rightarrow \qquad e^{2x} = \sum_{n=0}^{\infty}\frac{(2x)^n}{n!}.$$

Notes:

Finding a pattern in the coefficients that match the series expansion of a known function, such as those shown in Key Idea 33, can be difficult. What if the coefficients in the previous example were given in their reduced form; how could we still recover the function $y = e^{2x}$?

Suppose that all we know is that

$$a_0 = 1, \quad a_1 = 2, \quad a_2 = 2, \quad a_3 = \frac{4}{3}, \quad a_4 = \frac{2}{3}.$$

Definition 42 states that each term of the Taylor expansion of a function includes an $n!$. This allows us to say that

$$a_2 = 2 = \frac{b_2}{2!}, \quad a_3 = \frac{4}{3} = \frac{b_3}{3!}, \quad \text{and} \quad a_4 = \frac{2}{3} = \frac{b_4}{4!}$$

for some values b_2, b_3 and b_4. Solving for these values, we see that $b_2 = 4$, $b_3 = 8$ and $b_4 = 16$. That is, we are recovering the pattern we had previously seen, allowing us to write

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n \\ &= 1 + 2x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 + \frac{16}{4!}x^4 + \cdots \end{aligned}$$

From here it is easier to recognize that the series is describing an exponential function.

There are simpler, more direct ways of solving the differential equation $y' = 2y$. We applied power series techniques to this equation to demonstrate its utility, and went on to show how *sometimes* we are able to recover the solution in terms of elementary functions using the theory of Taylor series. Most differential equations faced in real scientific and engineering situations are much more complicated than this one, but power series can offer a valuable tool in finding, or at least approximating, the solution.

This chapter introduced sequences, which are ordered lists of numbers, followed by series, wherein we add up the terms of a sequence. We quickly saw that such sums do not always add up to “infinity,” but rather converge. We studied tests for convergence, then ended the chapter with a formal way of defining functions based on series. Such “series–defined functions” are a valuable tool in solving a number of different problems throughout science and engineering.

Coming in the next chapters are new ways of defining curves in the plane apart from using functions of the form $y = f(x)$. Curves created by these new methods can be beautiful, useful, and important.

Notes:

Exercises 8.8

Terms and Concepts

1. What is the difference between a Taylor polynomial and a Taylor series?

2. What theorem must we use to show that a function is equal to its Taylor series?

Problems

Key Idea 33 gives the n^{th} term of the Taylor series of common functions. In Exercises 3 – 6, verify the formula given in the Key Idea by finding the first few terms of the Taylor series of the given function and identifying a pattern.

3. $f(x) = e^x; \quad c = 0$

4. $f(x) = \sin x; \quad c = 0$

5. $f(x) = 1/(1-x); \quad c = 0$

6. $f(x) = \tan^{-1} x; \quad c = 0$

In Exercises 7 – 12, find a formula for the n^{th} term of the Taylor series of $f(x)$, centered at c, by finding the coefficients of the first few powers of x and looking for a pattern. (The formulas for several of these are found in Key Idea 33; show work verifying these formula.)

7. $f(x) = \cos x; \quad c = \pi/2$

8. $f(x) = 1/x; \quad c = 1$

9. $f(x) = e^{-x}; \quad c = 0$

10. $f(x) = \ln(1+x); \quad c = 0$

11. $f(x) = x/(x+1); \quad c = 1$

12. $f(x) = \sin x; \quad c = \pi/4$

In Exercises 13 – 16, show that the Taylor series for $f(x)$, as given in Key Idea 33, is equal to $f(x)$ by applying Theorem 79; that is, show $\lim\limits_{n\to\infty} R_n(x) = 0$.

13. $f(x) = e^x$

14. $f(x) = \sin x$

15. $f(x) = \ln x$

16. $f(x) = 1/(1-x)$ (show equality only on $(-1, 0)$)

In Exercises 17 – 20, use the Taylor series given in Key Idea 33 to verify the given identity.

17. $\cos(-x) = \cos x$

18. $\sin(-x) = -\sin x$

19. $\frac{d}{dx}\big(\sin x\big) = \cos x$

20. $\frac{d}{dx}\big(\cos x\big) = -\sin x$

In Exercises 21 – 24, write out the first 5 terms of the Binomial series with the given k-value.

21. $k = 1/2$

22. $k = -1/2$

23. $k = 1/3$

24. $k = 4$

In Exercises 25 – 34, use the Taylor series given in Key Idea 33 to create the Taylor series of the given functions.

25. $f(x) = \cos\left(x^2\right)$

26. $f(x) = e^{-x}$

27. $f(x) = \sin\left(2x+3\right)$

28. $f(x) = \tan^{-1}\left(x/2\right)$

29. $f(x) = \int e^{-x^2}\,dx$

30. $f(x) = \sin^{-1}(x)$ (Only find the first four terms. Hint: Plug in $-x^2$ into the appropriate series from Key Idea 33 and then integrate.)

31. $f(x) = e^x \sin x$ (only find the first 4 terms)

32. $f(x) = (1+x)^{1/2}\cos x$ (only find the first 4 terms)

33. $f(x) = \dfrac{x}{(1-x)^2}$

34. Use the result of Exercise 33 to determine the limit of the series $\frac{1}{10^2} + \frac{2}{10^3} + \frac{3}{10^4} + \frac{4}{10^5} + \frac{5}{10^6} + \cdots$. (This will explain why the decimal expansion of the fraction you obtain is 0.012345679 . . . .)

In Exercises 35 – 36, approximate the value of the given definite integral by using the first 4 nonzero terms of the integrand's Taylor series.

35. $\displaystyle\int_0^{\sqrt{\pi}} \sin\left(x^2\right)\,dx$

36. $\displaystyle\int_0^{\pi^2/4} \cos\left(\sqrt{x}\right)\,dx$

In Exercises 37 – 43, we explore the dilogarithm function $\text{dilog}(x) = \int_0^x -\frac{\ln(1-t)}{t}\,dt$**, a non-elementary function which is an antiderivative of** $-\ln(1-x)/x$**. (Since** $-\ln(1-x)/x$ **has a removable discontinuity at** $x=0$**, there is no problem with this integration.)**

37. Using the series in Key Idea 33, come up with a Taylor series for $\text{dilog}(x)$.

38. Determine the interval of convergence of the series in Exercise 37.

39. Evaluate $\text{dilog}(1)$ using the series in Exercise 37.

40. Evaluate $\text{dilog}(-1)$ using the series in Exercise 37.

41. Verify the identity $\text{dilog}(x) + \text{dilog}(-x) = \frac{1}{2}\,\text{dilog}\left(x^2\right)$ by verifying equality for a particular value of x and verifying the derivatives of each side are equal.

42. Verify the identity $\text{dilog}(1-x)+\text{dilog}\left(1-\frac{1}{x}\right) = -\frac{1}{2}(\ln x)^2$ for $x > 0$ by verifying equality for a particular value of x and verifying the derivatives of each side are equal.

43. Use the identity from Exercise 42 and the value computed in Exercise 40 to evaluate $\text{dilog}\left(\frac{1}{2}\right)$.

So far we have only made sense of calculus with real numbers. However, plugging xi **into the series for** e^x**, we can define complex exponents. In Exercises 44 – 46, we explore this.**

44. Write e^{xi} in the form $f(x) + g(x)i$ where $f(x)$ and $g(x)$ are functions of real numbers.

45. Evaluate $e^{\pi i}$.

46. Evaluate $e^{\ln 2+\pi i/3}$. (Use the fact that the properties of real exponents also hold for complex numbers.)

9: Curves in the Plane

We have explored functions of the form $y = f(x)$ closely throughout this text. We have explored their limits, their derivatives and their antiderivatives; we have learned to identify key features of their graphs, such as relative maxima and minima, inflection points and asymptotes; we have found equations of their tangent lines, the areas between portions of their graphs and the x-axis, and the volumes of solids generated by revolving portions of their graphs about a horizontal or vertical axis.

Despite all this, the graphs created by functions of the form $y = f(x)$ are limited. Since each x-value can correspond to only 1 y-value, common shapes like circles cannot be fully described by a function in this form. Fittingly, the "vertical line test" excludes vertical lines from being functions of x, even though these lines are important in mathematics.

In this chapter we'll explore new ways of drawing curves in the plane. We'll still work within the framework of functions, as an input will still only correspond to one output. However, our new techniques of drawing curves will render the vertical line test pointless, and allow us to create important – and beautiful – new curves. Once these curves are defined, we'll apply the concepts of calculus to them, continuing to find equations of tangent lines and the areas of enclosed regions.

9.1 Conic Sections (Optional)

The ancient Greeks recognized that interesting shapes can be formed by intersecting a plane with a *double napped* cone (i.e., two identical cones placed tip–to–tip as shown in the following figures). As these shapes are formed as sections of conics, they have earned the official name "conic sections."

The three "most interesting" conic sections are given in the top row of Figure 9.1. They are the parabola, the ellipse (which includes circles) and the hyperbola. In each of these cases, the plane does not intersect the tips of the cones (usually taken to be the origin).

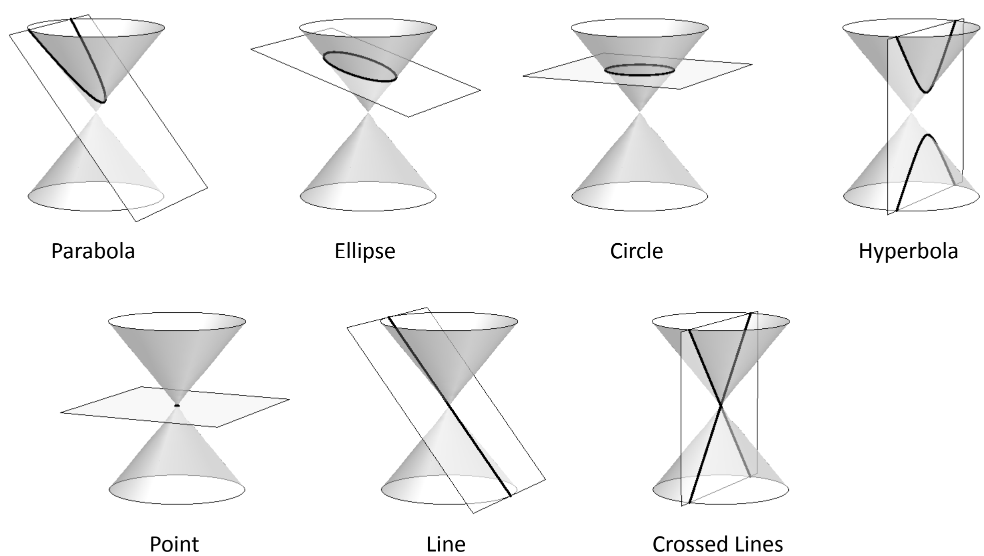


Figure 9.1: Conic Sections

When the plane does contain the origin, three **degenerate** cones can be formed as shown the bottom row of Figure 9.1: a point, a line, and crossed lines. We focus here on the nondegenerate cases.

While the above geometric constructs define the conics in an intuitive, visual way, these constructs are not very helpful when trying to analyze the shapes algebraically or consider them as the graph of a function. It can be shown that all conics can be defined by the general second–degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

While this algebraic definition has its uses, most find another geometric perspective of the conics more beneficial.

Each nondegenerate conic can be defined as the **locus**, or set, of points that satisfy a certain distance property. These distance properties can be used to generate an algebraic formula, allowing us to study each conic as the graph of a function.

Parabolas

Definition 43 Parabola

A **parabola** is the locus of all points equidistant from a point (called a **focus**) and a line (called the **directrix**) that does not contain the focus.

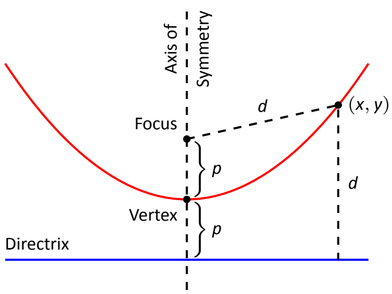


Figure 9.2: Illustrating the definition of the parabola and establishing an algebraic formula.

Figure 9.2 illustrates this definition. The point halfway between the focus and the directrix is the **vertex**. The line through the focus, perpendicular to the directrix, is the **axis of symmetry**, as the portion of the parabola on one side of this line is the mirror–image of the portion on the opposite side.

The definition leads us to an algebraic formula for the parabola. Let $P = (x, y)$ be a point on a parabola whose focus is at $F = (0, p)$ and whose directrix is at $y = -p$. (We'll assume for now that the focus lies on the y-axis; by placing the focus p units above the x-axis and the directrix p units below this axis, the vertex will be at $(0, 0)$.)

We use the Distance Formula to find the distance d_1 between F and P:

$$d_1 = \sqrt{(x-0)^2 + (y-p)^2}.$$

The distance d_2 from P to the directrix is more straightforward:

$$d_2 = y - (-p) = y + p.$$

Notes:

These two distances are equal. Setting $d_1 = d_2$, we can solve for y in terms of x:

$$\begin{aligned} d_1 &= d_2 \\ \sqrt{x^2 + (y-p)^2} &= y + p \end{aligned}$$

Now square both sides.

$$\begin{aligned} x^2 + (y-p)^2 &= (y+p)^2 \\ x^2 + y^2 - 2yp + p^2 &= y^2 + 2yp + p^2 \\ x^2 &= 4yp \\ y &= \frac{1}{4p}x^2. \end{aligned}$$

The geometric definition of the parabola has led us to the familiar quadratic function whose graph is a parabola with vertex at the origin. When we allow the vertex to not be at $(0,0)$, we get the following standard form of the parabola.

Key Idea 34 General Equation of a Parabola

1. **Vertical Axis of Symmetry:** The equation of the parabola with vertex at (h,k) and directrix $y = k - p$ in standard form is

$$y = \frac{1}{4p}(x-h)^2 + k.$$

The focus is at $(h, k+p)$.

2. **Horizontal Axis of Symmetry:** The equation of the parabola with vertex at (h,k) and directrix $x = h - p$ in standard form is

$$x = \frac{1}{4p}(y-k)^2 + h.$$

The focus is at $(h+p, k)$.

Note: p is not necessarily a positive number.

Example 9.1 Finding the equation of a parabola

Give the equation of the parabola with focus at $(1,2)$ and directrix at $y = 3$.

Solution The vertex is located halfway between the focus and directrix, so $(h,k) = (1, 2.5)$. This gives $p = -0.5$. Using Key Idea 34 we have the

Notes:

equation of the parabola as

$$y = \frac{1}{4(-0.5)}(x-1)^2 + 2.5 = -\frac{1}{2}(x-1)^2 + 2.5.$$

The parabola is sketched in Figure 9.3.

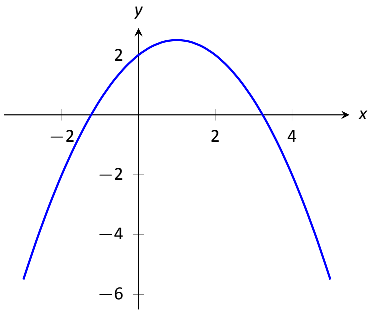


Figure 9.3: The parabola described in Example 9.1.

Example 9.2 Finding the focus and directrix of a parabola

Find the focus and directrix of the parabola $x = \frac{1}{8}y^2 - y + 1$. The point $(7, 12)$ lies on the graph of this parabola; verify that it is equidistant from the focus and directrix.

Solution We need to put the equation of the parabola in its general form. This requires us to complete the square:

$$\begin{aligned} x &= \frac{1}{8}y^2 - y + 1 \\ &= \frac{1}{8}\left(y^2 - 8y + 8\right) \\ &= \frac{1}{8}\left(y^2 - 8y + 16 - 16 + 8\right) \\ &= \frac{1}{8}\left((y-4)^2 - 8\right) \\ &= \frac{1}{8}(y-4)^2 - 1. \end{aligned}$$

Hence the vertex is located at $(-1, 4)$. We have $\frac{1}{8} = \frac{1}{4p}$, so $p = 2$. We conclude that the focus is located at $(1, 4)$ and the directrix is $x = -3$. The parabola is graphed in Figure 9.4, along with its focus and directrix.

The point $(7, 12)$ lies on the graph and is $7 - (-3) = 10$ units from the directrix. The distance from $(7, 12)$ to the focus is:

$$\sqrt{(7-1)^2 + (12-4)^2} = \sqrt{100} = 10.$$

Indeed, the point on the parabola is equidistant from the focus and directrix.

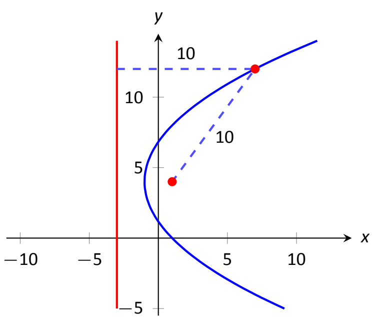


Figure 9.4: The parabola described in Example 9.2. The distances from a point on the parabola to the focus and directrix is given.

Reflective Property

One of the fascinating things about the nondegenerate conic sections is their reflective properties. Parabolas have the following reflective property:

> Any ray emanating from the focus that intersects the parabola reflects off along a line perpendicular to the directrix.

This is illustrated in Figure 9.5. The following theorem states this more rigorously.

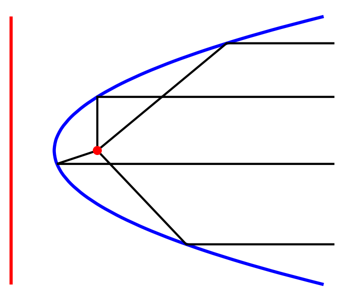


Figure 9.5: Illustrating the parabola's reflective property.

Notes:

Theorem 81 Reflective Property of the Parabola

Let P be a point on a parabola. The tangent line to the parabola at P makes equal angles with the following two lines:

1. The line containing P and the focus F, and
2. The line perpendicular to the directrix through P.

Because of this reflective property, paraboloids (the 3D analogue of parabolas) make for useful flashlight reflectors as the light from the bulb, ideally located at the focus, is reflected along parallel rays. Satellite dishes also have paraboloid shapes. Signals coming from satellites effectively approach the dish along parallel rays. The dish then *focuses* these rays at the focus, where the sensor is located.

Ellipses

Definition 44 Ellipse

An **ellipse** is the locus of all points whose sum of distances from two fixed points, each a **focus** of the ellipse, is constant.

An easy way to visualize this construction of an ellipse is to pin both ends of a string to a board. The pins become the foci. Holding a pencil tight against the string places the pencil on the ellipse; the sum of distances from the pencil to the pins is constant: the length of the string. See Figure 9.6.

We can again find an algebraic equation for an ellipse using this geometric definition. Let the foci be located along the x-axis, c units from the origin. Let these foci be labeled as $F_1 = (-c, 0)$ and $F_2 = (c, 0)$. Let $P = (x, y)$ be a point on the ellipse. The sum of distances from F_1 to P (d_1) and from F_2 to P (d_2) is a constant d. That is, $d_1 + d_2 = d$. Using the Distance Formula, we have

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=d.$$

Using a fair amount of algebra can produce the following equation of an ellipse (note that the equation is an implicitly defined relation, and not a function; it has to be, as an ellipse fails the Vertical Line Test):

$$\frac{x^2}{\left(\frac{d}{2}\right)^2}+\frac{y^2}{\left(\frac{d}{2}\right)^2-c^2}=1.$$

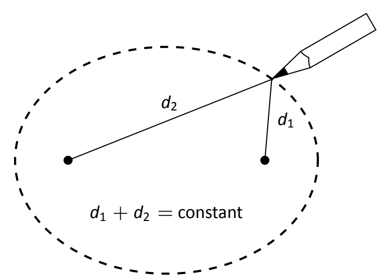


Figure 9.6: Illustrating the construction of an ellipse with pins, pencil and string.

Notes:

This is not particularly illuminating, but by making the substitution $a = d/2$ and $b = \sqrt{a^2 - c^2}$, we can rewrite the above equation as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

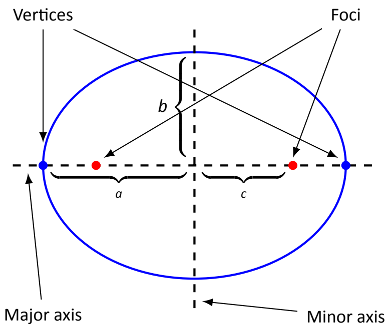


Figure 9.7: Labeling the significant features of an ellipse.

This choice of a and b is not without reason; as shown in Figure 9.7, the values of a and b have geometric meaning in the graph of the ellipse.

In general, the two foci of an ellipse lie on the **major axis** of the ellipse, and the midpoint of the segment joining the two foci is the **center**. The major axis intersects the ellipse at two points, each of which is a **vertex**. The line segment through the center and perpendicular to the major axis is the **minor axis**. The "constant sum of distances" that defines the ellipse is the length of the major axis, i.e., $2a$.

Allowing for the shifting of the ellipse gives the following standard equations.

Key Idea 35 Standard Equation of the Ellipse

The equation of an ellipse centered at (h, k) with major axis of length $2a$ and minor axis of length $2b$ in standard form is:

1. **Horizontal major axis:** $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1.$
2. **Vertical major axis:** $\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1.$

The foci lie along the major axis, c units from the center, where $c^2 = a^2 - b^2$.

Example 9.3 Finding the equation of an ellipse

Find the general equation of the ellipse graphed in Figure 9.8.

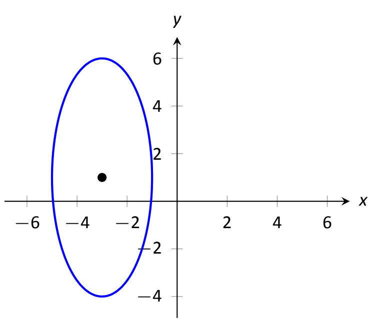


Figure 9.8: The ellipse used in Example 9.3.

Solution The center is located at $(-3, 1)$. The distance from the center to a vertex is 5 units, hence $a = 5$. The minor axis seems to have length 4, so $b = 2$. Thus the equation of the ellipse is

$$\frac{(x+3)^2}{4} + \frac{(y-1)^2}{25} = 1.$$

Example 9.4 Graphing an ellipse

Graph the ellipse defined by $4x^2 + 9y^2 - 8x - 36y = -4$.

Notes:

Solution It is simple to graph an ellipse once it is in standard form. In order to put the given equation in standard form, we must complete the square with both the x and y terms. We first rewrite the equation by regrouping:

$$4x^2+9y^2-8x-36y=-4 \quad\Rightarrow\quad (4x^2-8x)+(9y^2-36y)=-4.$$

Now we complete the squares.

$$\begin{aligned}
(4x^2-8x)+(9y^2-36y)&=-4\\
4(x^2-2x)+9(y^2-4y)&=-4\\
4(x^2-2x+1-1)+9(y^2-4y+4-4)&=-4\\
4\big((x-1)^2-1\big)+9\big((y-2)^2-4\big)&=-4\\
4(x-1)^2-4+9(y-2)^2-36&=-4\\
4(x-1)^2+9(y-2)^2&=36\\
\frac{(x-1)^2}{9}+\frac{(y-2)^2}{4}&=1.
\end{aligned}$$

We see the center of the ellipse is at $(1,2)$. We have $a=3$ and $b=2$; the major axis is horizontal, so the vertices are located at $(-2,2)$ and $(4,2)$. We find $c=\sqrt{9-4}=\sqrt{5}\approx 2.24$. The foci are located along the major axis, approximately 2.24 units from the center, at $(1\pm 2.24,2)$. This is all graphed in Figure 9.9 .

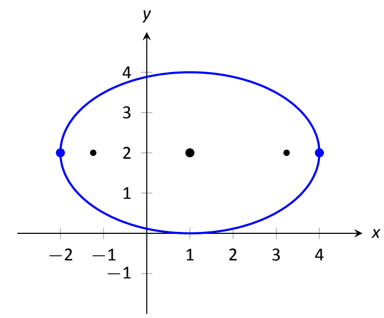


Figure 9.9: Graphing the ellipse in Example 9.4.

Eccentricity

When $a=b$, we have a circle. The general equation becomes

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{a^2}=1 \quad\Rightarrow (x-h)^2+(y-k)^2=a^2,$$

the familiar equation of the circle centered at (h,k) with radius a. Since $a=b$, $c=\sqrt{a^2-b^2}=0$. The circle has "two" foci, but they lie on the same point, the center of the circle.

Consider Figure 9.10, where several ellipses are graphed with $a=1$. In (a), we have $c=0$ and the ellipse is a circle. As c grows, the resulting ellipses look less and less circular. A measure of this "noncircularness" is *eccentricity*.

> **Definition 45 Eccentricity of an Ellipse**
>
> The eccentricity e of an ellipse is $e=\dfrac{c}{a}$.

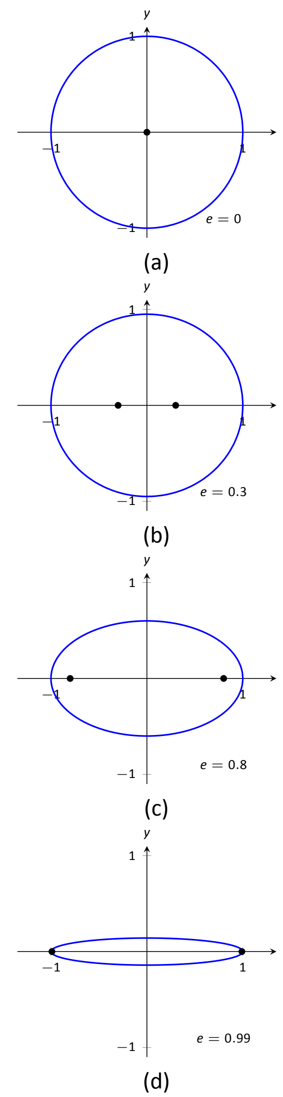


Figure 9.10: Understanding the eccentricity of an ellipse.

Notes:

The eccentricity of a circle is 0; that is, a circle has no "noncircularness." As c approaches a, e approaches 1, giving rise to a very noncircular ellipse, as seen in Figure 9.10 (d).

It was long assumed that planets had circular orbits. This is known to be incorrect; the orbits are elliptical. Earth has an eccentricity of 0.0167 – it has a nearly circular orbit. Mercury's orbit is the most eccentric, with $e = 0.2056$. (Pluto's eccentricity is greater, at $e = 0.248$, the greatest of all the currently known dwarf planets.) The planet with the most circular orbit is Venus, with $e = 0.0068$. The Earth's moon has an eccentricity of $e = 0.0549$, also very circular.

Reflective Property

The ellipse also possesses an interesting reflective property. Any ray emanating from one focus of an ellipse reflects off the ellipse along a line through the other focus, as illustrated in Figure 9.11. This property is given formally in the following theorem.

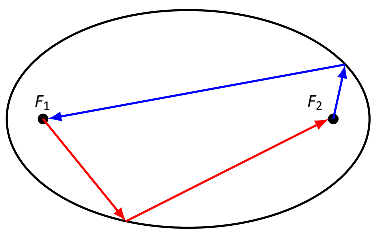


Figure 9.11: Illustrating the reflective property of an ellipse.

Theorem 82 Reflective Property of an Ellipse

Let P be a point on a ellipse with foci F_1 and F_2. The tangent line to the ellipse at P makes equal angles with the following two lines:

1. The line through F_1 and P, and
2. The line through F_2 and P.

This reflective property is useful in optics and is the basis of the phenomena experienced in whispering halls.

Hyperbolas

The definition of a hyperbola is very similar to the definition of an ellipse; we essentially just change the word "sum" to "difference."

Definition 46 Hyperbola

A **hyperbola** is the locus of all points where the absolute value of difference of distances from two fixed points, each a focus of the hyperbola, is constant.

Notes:

We do not have a convenient way of visualizing the construction of a hyperbola as we did for the ellipse. The geometric definition does allow us to find an algebraic expression that describes it. It will be useful to define some terms first.

The two foci lie on the **transverse axis** of the hyperbola; the midpoint of the line segment joining the foci is the **center** of the hyperbola. The transverse axis intersects the hyperbola at two points, each a **vertex** of the hyperbola. The line through the center and perpendicular to the transverse axis is the **conjugate axis.** This is illustrated in Figure 9.12. It is easy to show that the constant difference of distances used in the definition of the hyperbola is the distance between the vertices, i.e., $2a$.

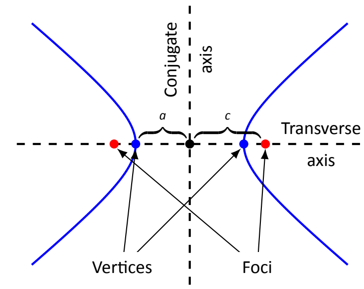


Figure 9.12: Labeling the significant features of a hyperbola.

Key Idea 36 Standard Equation of a Hyperbola

The equation of a hyperbola centered at (h, k) in standard form is:

1. **Horizontal Transverse Axis:** $$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$$
2. **Vertical Transverse Axis:** $$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1.$$

The vertices are located a units from the center and the foci are located c units from the center, where $c^2 = a^2 + b^2$.

Graphing Hyperbolas

Consider the hyperbola $\frac{x^2}{9} - \frac{y^2}{1} = 1$. Solving for y, we find $y = \pm\sqrt{x^2/9 - 1}$. As x grows large, the "-1" part of the equation for y becomes less significant and $y \approx \pm\sqrt{x^2/9} = \pm x/3$. That is, as x gets large, the graph of the hyperbola looks very much like the lines $y = \pm x/3$. These lines are asymptotes of the hyperbola, as shown in Figure 9.13.

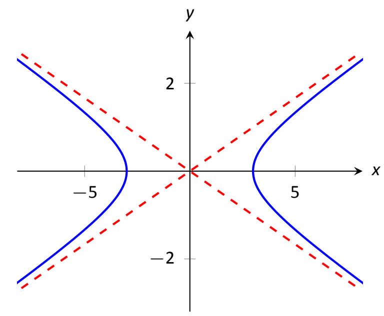


Figure 9.13: Graphing the hyperbola $\frac{x^2}{9} - \frac{y^2}{1} = 1$ along with its asymptotes, $y = \pm x/3$.

This is a valuable tool in sketching. Given the equation of a hyperbola in general form, draw a rectangle centered at (h, k) with sides of length $2a$ parallel to the transverse axis and sides of length $2b$ parallel to the conjugate axis. (See Figure 9.14 for an example with a horizontal transverse axis.) The diagonals of the rectangle lie on the asymptotes.

These lines pass through (h, k). When the transverse axis is horizontal, the slopes are $\pm b/a$; when the transverse axis is vertical, their slopes are $\pm a/b$. This gives equations:

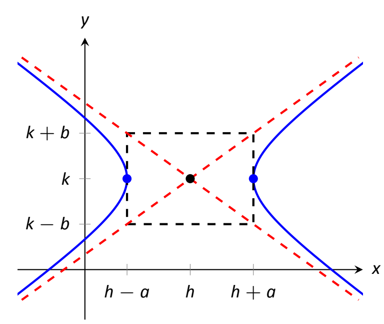


Figure 9.14: Using the asymptotes of a hyperbola as a graphing aid.

Notes:

Horizontal Transverse Axis	Vertical Transverse Axis
$y = \pm\frac{b}{a}(x-h)+k$	$y = \pm\frac{a}{b}(x-h)+k.$

Example 9.5 Graphing a hyperbola

Sketch the hyperbola given by $\dfrac{(y-2)^2}{25} - \dfrac{(x-1)^2}{4} = 1.$

Solution The hyperbola is centered at $(1,2)$; $a = 5$ and $b = 2$. In Figure 9.15 we draw the prescribed rectangle centered at $(1,2)$ along with the asymptotes defined by its diagonals. The hyperbola has a vertical transverse axis, so the vertices are located at $(1,7)$ and $(1,-3)$. This is enough to make a good sketch.

We also find the location of the foci: as $c^2 = a^2 + b^2$, we have $c = \sqrt{29} \approx 5.4$. Thus the foci are located at $(1, 2 \pm 5.4)$ as shown in the figure.

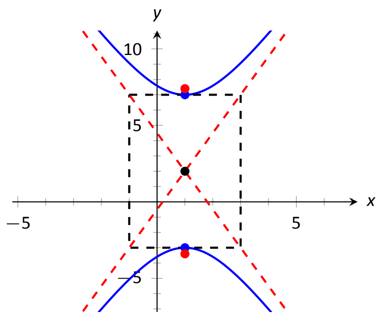


Figure 9.15: Graphing the hyperbola in Example 9.5.

Example 9.6 Graphing a hyperbola

Sketch the hyperbola given by $9x^2 - y^2 + 2y = 10$.

Solution We must complete the square to put the equation in general form. (We recognize this as a hyperbola since it is a general quadratic equation and the x^2 and y^2 terms have opposite signs.)

$$\begin{aligned}
9x^2 - y^2 + 2y &= 10\\
9x^2 - (y^2 - 2y) &= 10\\
9x^2 - (y^2 - 2y + 1 - 1) &= 10\\
9x^2 - \big((y-1)^2 - 1\big) &= 10\\
9x^2 - (y-1)^2 &= 9\\
x^2 - \frac{(y-1)^2}{9} &= 1
\end{aligned}$$

We see the hyperbola is centered at $(0,1)$, with a horizontal transverse axis, where $a = 1$ and $b = 3$. The appropriate rectangle is sketched in Figure 9.16 along with the asymptotes of the hyperbola. The vertices are located at $(\pm 1, 1)$. We have $c = \sqrt{10} \approx 3.2$, so the foci are located at $(\pm 3.2, 1)$ as shown in the figure.

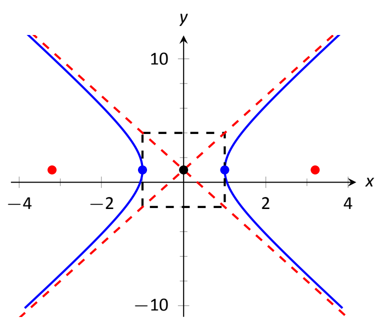


Figure 9.16: Graphing the hyperbola in Example 9.6.

Notes:

Eccentricity

Definition 47 Eccentricity of a Hyperbola

The eccentricity of a hyperbola is $e = \dfrac{c}{a}$.

Note that this is the definition of eccentricity as used for the ellipse. When c is close in value to a (i.e., $e \approx 1$), the hyperbola is very narrow (looking almost like crossed lines). Figure 9.17 shows hyperbolas centered at the origin with $a = 1$. The graph in (a) has $c = 1.05$, giving an eccentricity of $e = 1.05$, which is close to 1. As c grows larger, the hyperbola widens and begins to look like parallel lines, as shown in part (d) of the figure.

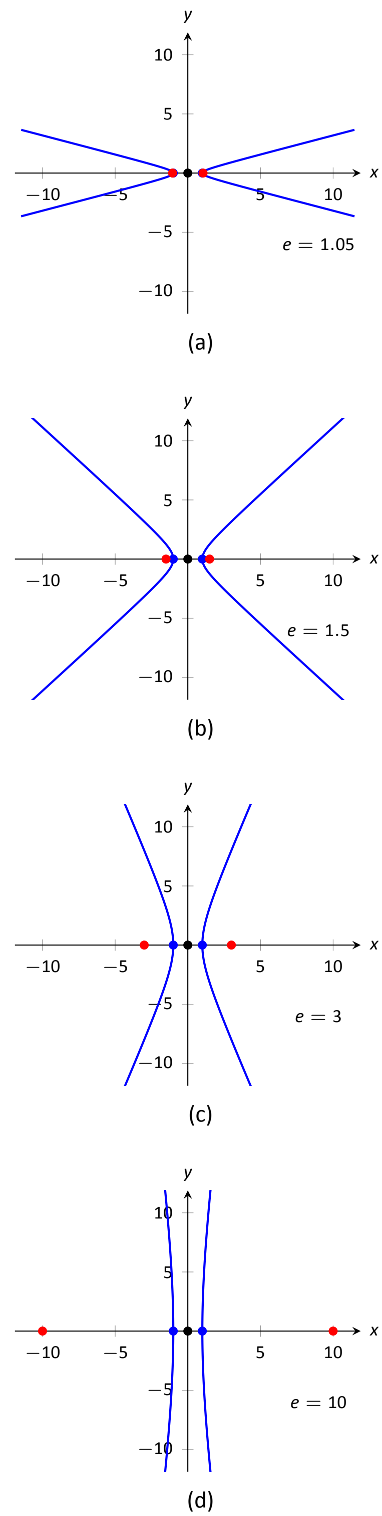


Figure 9.17: Understanding the eccentricity of a hyperbola.

Reflective Property

Hyperbolas share a similar reflective property with ellipses. However, in the case of a hyperbola, a ray emanating from a focus that intersects the hyperbola reflects along a line containing the other focus, but moving *away* from that focus. This is illustrated in Figure 9.19 (on the next page). Hyperbolic mirrors are commonly used in telescopes because of this reflective property. It is stated formally in the following theorem.

Theorem 83 Reflective Property of Hyperbolas

Let P be a point on a hyperbola with foci F_1 and F_2. The tangent line to the hyperbola at P makes equal angles with the following two lines:

1. The line through F_1 and P, and
2. The line through F_2 and P.

Location Determination

Determining the location of a known event has many practical uses (locating the epicenter of an earthquake, an airplane crash site, the position of the person speaking in a large room, etc.).

To determine the location of an earthquake's epicenter, seismologists use *trilateration* (not to be confused with *triangulation*). A seismograph allows one

Notes:

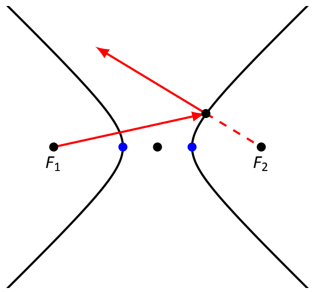


Figure 9.19: Illustrating the reflective property of a hyperbola.

to determine how far away the epicenter was; using three separate readings, the location of the epicenter can be approximated.

A key to this method is knowing distances. What if this information is not available? Consider three microphones at positions A, B and C which all record a noise (a person's voice, an explosion, etc.) created at unknown location D. The microphone does not "know" when the sound was *created*, only when the sound was *detected*. How can the location be determined in such a situation?

If each location has a clock set to the same time, hyperbolas can be used to determine the location. Suppose the microphone at position A records the sound at exactly 12:00, location B records the time exactly 1 second later, and location C records the noise exactly 2 seconds after that. We are interested in the *difference* of times. Since the speed of sound is approximately 340 m/s, we can conclude quickly that the sound was created 340 meters closer to position A than position B. If A and B are a known distance apart (as shown in Figure 9.18 (a)), then we can determine a hyperbola on which D must lie.

The "difference of distances" is 340; this is also the distance between vertices of the hyperbola. So we know $2a = 340$. Positions A and B lie on the foci, so $2c = 1000$. From this we can find $b \approx 470$ and can sketch the hyperbola, given in part (b) of the figure. We only care about the side closest to A. (Why?)

We can also find the hyperbola defined by positions B and C. In this case, $2a = 680$ as the sound traveled an extra 2 seconds to get to C. We still have $2c = 1000$, centering this hyperbola at $(-500, 500)$. We find $b \approx 367$. This hyperbola is sketched in part (c) of the figure. The intersection point of the two graphs is the location of the sound, at approximately $(188, -222.5)$.

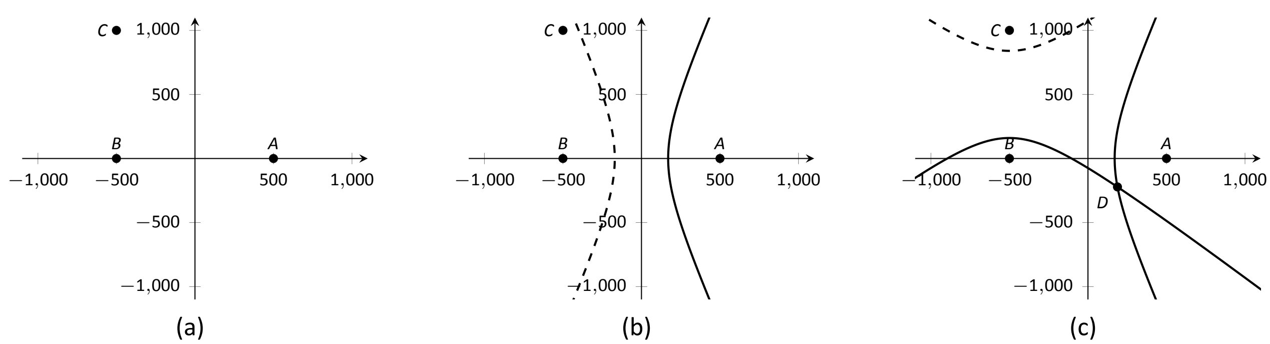


Figure 9.18: Using hyperbolas in location detection.

This chapter explores curves in the plane, in particular curves that cannot be described by functions of the form $y = f(x)$. In this section, we learned of ellipses and hyperbolas that are defined implicitly, not explicitly. In the following sections, we will learn completely new ways of describing curves in the plane, using *parametric equations* and *polar coordinates*, then study these curves using calculus techniques.

Notes:

Exercises 9.1

Terms and Concepts

1. What is the difference between degenerate and nondegenerate conics?

2. Use your own words to explain what the eccentricity of an ellipse measures.

3. What has the largest eccentricity: an ellipse or a hyperbola?

4. Explain why the following is true: "If the coefficient of the x^2 term in the equation of an ellipse in standard form is smaller than the coefficient of the y^2 term, then the ellipse has a horizontal major axis."

5. Explain how one can quickly look at the equation of a hyperbola in standard form and determine whether the transverse axis is horizontal or vertical.

Problems

In Exercises 6 – 13, find the equation of the parabola defined by the given information. Sketch the parabola.

6. Focus: $(3, 2)$; directrix: $y = 1$

7. Focus: $(-1, -4)$; directrix: $y = 2$

8. Focus: $(1, 5)$; directrix: $x = 3$

9. Focus: $(1/4, 0)$; directrix: $x = -1/4$

10. Focus: $(1, 1)$; vertex: $(1, 2)$

11. Focus: $(-3, 0)$; vertex: $(0, 0)$

12. Vertex: $(0, 0)$; directrix: $y = -1/16$

13. Vertex: $(2, 3)$; directrix: $x = 4$

In Exercises 14 – 15, the equation of a parabola and a point on its graph are given. Find the focus and directrix of the parabola, and verify that the given point is equidistant from the focus and directrix.

14. $y = \frac{1}{4}x^2$, $P = (2, 1)$

15. $x = \frac{1}{8}(y-2)^2 + 3$, $P = (11, 10)$

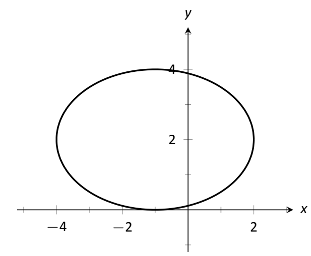
In Exercises 16 – 17, sketch the ellipse defined by the given equation. Label the center, foci and vertices.

16. $\dfrac{(x-1)^2}{3} + \dfrac{(y-2)^2}{5} = 1$

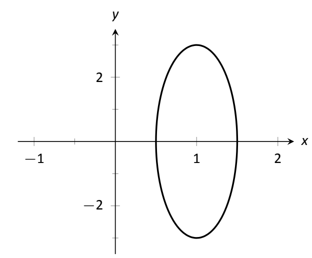
17. $\dfrac{1}{25}x^2 + \dfrac{1}{9}(y+3)^2 = 1$

In Exercises 18 – 19, find the equation of the ellipse shown in the graph. Give the location of the foci and the eccentricity of the ellipse.

18.



19.



In Exercises 20 – 23, find the equation of the ellipse defined by the given information. Sketch the elllipse.

20. Foci: $(\pm 2, 0)$; vertices: $(\pm 3, 0)$

21. Foci: $(-1, 3)$ and $(5, 3)$; vertices: $(-3, 3)$ and $(7, 3)$

22. Foci: $(2, \pm 2)$; vertices: $(2, \pm 7)$

23. Focus: $(-1, 5)$; vertex: $(-1, -4)$; center: $(-1, 1)$

In Exercises 24 – 27, write the equation of the given ellipse in standard form.

24. $x^2 - 2x + 2y^2 - 8y = -7$

25. $5x^2 + 3y^2 = 15$

26. $3x^2 + 2y^2 - 12y + 6 = 0$

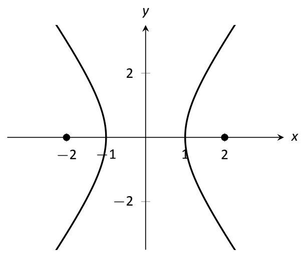
27. $x^2 + y^2 - 4x - 4y + 4 = 0$

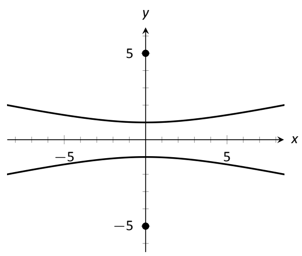
28. Consider the ellipse given by $\dfrac{(x-1)^2}{4} + \dfrac{(y-3)^2}{12} = 1.$

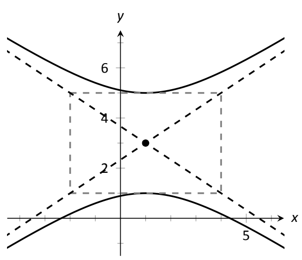
    (a) Verify that the foci are located at $(1, 3 \pm 2\sqrt{2})$.

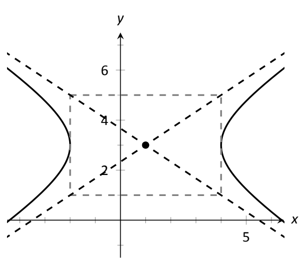
    (b) The points $P_1 = (2, 6)$ and $P_2 = (1+\sqrt{2}, 3+\sqrt{6}) \approx (2.414, 5.449)$ lie on the ellipse. Verify that the sum of distances from each point to the foci is the same.

In Exercises 29 – 32, find the equation of the hyperbola shown in the graph.

29. 

30. 

31. 

32. 

In Exercises 33 – 34, sketch the hyperbola defined by the given equation. Label the center and foci.

33. $\dfrac{(x-1)^2}{16} - \dfrac{(y+2)^2}{9} = 1$

34. $(y-4)^2 - \dfrac{(x+1)^2}{25} = 1$

In Exercises 35 – 38, find the equation of the hyperbola defined by the given information. Sketch the hyperbola.

35. Foci: $(\pm 3, 0)$; vertices: $(\pm 2, 0)$

36. Foci: $(0, \pm 3)$; vertices: $(0, \pm 2)$

37. Foci: $(-2, 3)$ and $(8, 3)$; vertices: $(-1, 3)$ and $(7, 3)$

38. Foci: $(3, -2)$ and $(3, 8)$; vertices: $(3, 0)$ and $(3, 6)$

In Exercises 39 – 42, write the equation of the hyperbola in standard form.

39. $3x^2 - 4y^2 = 12$

40. $3x^2 - y^2 + 2y = 10$

41. $x^2 - 10y^2 + 40y = 30$

42. $(4y - x)(4y + x) = 4$

43. Johannes Kepler discovered that the planets of our solar system have elliptical orbits with the Sun at one focus. The Earth's elliptical orbit is used as a standard unit of distance; the distance from the center of Earth's elliptical orbit to one vertex is 1 Astronomical Unit, or A.U.

    The following table gives information about the orbits of three planets.

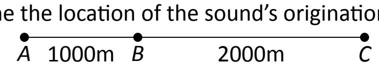
	Distance from center to vertex	eccentricity
Mercury	0.387 A.U.	0.2056
Earth	1 A.U.	0.0167
Mars	1.524 A.U.	0.0934

   (a) In an ellipse, knowing $c^2 = a^2 - b^2$ and $e = c/a$ allows us to find b in terms of a and e. Show $b = a\sqrt{1-e^2}$.

   (b) For each planet, find equations of their elliptical orbit of the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. (This places the center at $(0, 0)$, but the Sun is in a different location for each planet.)

   (c) Shift the equations so that the Sun lies at the origin. Plot the three elliptical orbits.

44. A loud sound is recorded at three stations that lie on a line as shown in the figure below. Station A recorded the sound 1 second after Station B, and Station C recorded the sound 3 seconds after B. Using the speed of sound as 340m/s, determine the location of the sound's origination.



9.2 Parametric Equations

We are familiar with sketching shapes, such as parabolas, by following this basic procedure:

Choose x → Use a function f to find y $\big(y = f(x)\big)$ → Plot point (x, y)

The **rectangular equation** $y = f(x)$ works well for some shapes like a parabola with a vertical axis of symmetry, but in the previous section we encountered several shapes that could not be sketched in this manner. (To plot an ellipse using the above procedure, we need to plot the "top" and "bottom" separately.)

In this section we introduce a new sketching procedure:

Choose t → Use a function f to find x $\big(x = f(t)\big)$ → Plot point (x, y)

Choose t → Use a function g to find y $\big(y = g(t)\big)$ → Plot point (x, y)

Here, x and y are found separately but then plotted together. This leads us to a definition.

Definition 48 Parametric Equations and Curves

Let f and g be continuous functions on an interval I. The set of all points $(x, y) = \big(f(t), g(t)\big)$ in the Cartesian plane, as t varies over I, is the **graph** of the **parametric equations** $x = f(t)$ and $y = g(t)$, where t is the **parameter**. A **curve** is a graph along with the parametric equations that define it.

This is a formal definition of the word *curve*. When a curve lies in a plane (such as the Cartesian plane), it is often referred to as a **plane curve**. Examples will help us understand the concepts introduced in the definition.

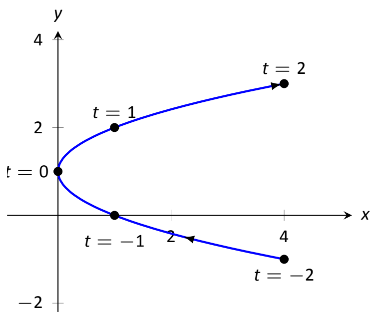
Example 9.7 Plotting parametric functions

Plot the graph of the parametric equations $x = t^2$, $y = t + 1$ for t in $[-2, 2]$.

Notes:

t	x	y
-2	4	-1
-1	1	0
0	0	1
1	1	2
2	4	3

(a)



(b)

Figure 9.20: A table of values of the parametric equations in Example 9.7 along with a sketch of their graph.

Solution We plot the graphs of parametric equations in much the same manner as we plotted graphs of functions like $y = f(x)$: we make a table of values, plot points, then connect these points with a "reasonable" looking curve. Figure 9.20(a) shows such a table of values; note how we have 3 columns.

The points (x, y) from the table are plotted in Figure 9.20(b). The points have been connected with a smooth curve. Each point has been labeled with its corresponding t-value. These values, along with the two arrows along the curve, are used to indicate the **orientation** of the graph. This information helps us determine the direction in which the graph is "moving."

We often use the letter t as the parameter as we often regard t as representing *time*. Certainly there are many contexts in which the parameter is not time, but it can be helpful to think in terms of time as one makes sense of parametric plots and their orientation (for instance, "At time $t = 0$ the position is $(1, 2)$ and at time $t = 3$ the position is $(5, 1)$.").

Example 9.8 Plotting parametric functions

Sketch the graph of the parametric equations $x = \cos^2 t$, $y = \cos t + 1$ for t in $[0, \pi]$.

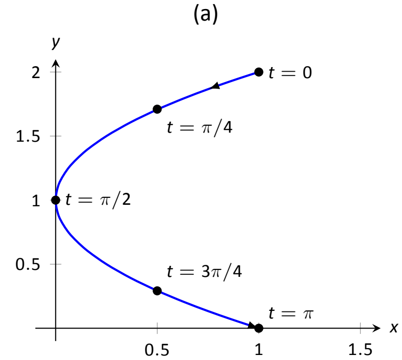
Solution We again start by making a table of values in Figure 9.21(a), then plot the points (x, y) on the Cartesian plane in Figure 9.21(b).

It is not difficult to show that the curves in Examples 9.7 and 9.8 are portions of the same parabola. While the *parabola* is the same, the *curves* are different. In Example 9.7, if we let t vary over all real numbers, we'd obtain the entire parabola. In this example, letting t vary over all real numbers would still produce the same graph; this portion of the parabola would be traced, and re–traced, infinitely. The orientation shown in Figure 9.21 shows the orientation on $[0, \pi]$, but this orientation is reversed on $[\pi, 2\pi]$.

These examples begin to illustrate the powerful nature of parametric equations. Their graphs are far more diverse than the graphs of functions produced by "$y = f(x)$" functions.

t	x	y
0	1	2
$\pi/4$	$1/2$	$1 + \sqrt{2}/2$
$\pi/2$	0	1
$3\pi/4$	$1/2$	$1 - \sqrt{2}/2$
π	1	0

(a)



(b)

Figure 9.21: A table of values of the parametric equations in Example 9.8 along with a sketch of their graph.

Technology Note: Most graphing utilities can graph functions given in parametric form. Often the word "parametric" is abbreviated as "PAR" or "PARAM" in the options. The user usually needs to determine the graphing window (i.e, the minimum and maximum x- and y-values), along with the values of t that are to be plotted. The user is often prompted to give a t minimum, a t maximum, and a "t-step" or "Δt." Graphing utilities effectively plot parametric functions just as we've shown here: they plot lots of points. A smaller t-step plots more points, making for a smoother graph (but may take longer). In Figure 9.20, the t-step is

Notes:

1; in Figure 9.21, the t-step is $\pi/4$.

One nice feature of parametric equations is that their graphs are easy to shift. While this is not too difficult in the "$y = f(x)$" context, the resulting function can look rather messy. (Plus, to shift to the right by two, we replace x with $x - 2$, which is counter–intuitive.) The following example demonstrates this.

Example 9.9 Shifting the graph of parametric functions

Sketch the graph of the parametric equations $x = t^2 + t$, $y = t^2 - t$. Find new parametric equations that shift this graph to the right 3 places and down 2.

Solution The graph of the parametric equations is given in Figure 9.22 (a). It is a parabola with a axis of symmetry along the line $y = x$; the vertex is at $(0, 0)$.

In order to shift the graph to the right 3 units, we need to increase the x-value by 3 for every point. The straightforward way to accomplish this is simply to add 3 to the function defining x: $x = t^2 + t + 3$. To shift the graph down by 2 units, we wish to decrease each y-value by 2, so we subtract 2 from the function defining y: $y = t^2 - t - 2$. Thus our parametric equations for the shifted graph are $x = t^2 + t + 3$, $y = t^2 - t - 2$. This is graphed in Figure 9.22 (b). Notice how the vertex is now at $(3, -2)$.

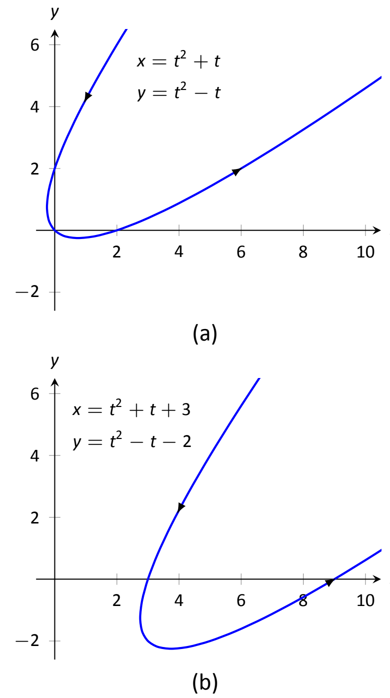


Figure 9.22: Illustrating how to shift graphs in Example 9.9.

Because the x- and y-values of a graph are determined independently, the graphs of parametric functions often possess features not seen on "$y = f(x)$" type graphs. The next example demonstrates how such graphs can arrive at the same point more than once.

Example 9.10 Graphs that cross themselves

Plot the parametric functions $x = t^3 - 5t^2 + 3t + 11$ and $y = t^2 - 2t + 3$ and determine the t-values where the graph crosses itself.

Solution Using the methods developed in this section, we again plot points and graph the parametric equations as shown in Figure 9.23. It appears that the graph crosses itself at the point $(2, 6)$, but we'll need to analytically determine this.

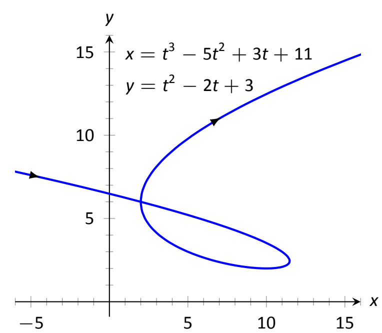


Figure 9.23: A graph of the parametric equations from Example 9.10.

We are looking for two different values, say, s and t, where $x(s) = x(t)$ and $y(s) = y(t)$. That is, the x-values are the same precisely when the y-values are the same. This gives us a system of 2 equations with 2 unknowns:

$$s^3 - 5s^2 + 3s + 11 = t^3 - 5t^2 + 3t + 11$$
$$s^2 - 2s + 3 = t^2 - 2t + 3$$

Solving this system is not trivial but involves only algebra. Using the quadratic

Notes:

formula, one can solve for t in the second equation and find that $t = 1 \pm \sqrt{s^2 - 2s + 1}$. This can be substituted into the first equation, revealing that the graph crosses itself at $t = -1$ and $t = 3$. We confirm our result by computing $x(-1) = x(3) = 2$ and $y(-1) = y(3) = 6$.

Converting between rectangular and parametric equations

It is sometimes useful to rewrite equations in rectangular form (i.e., $y = f(x)$) into parametric form, and vice–versa. Converting from rectangular to parametric can be very simple: given $y = f(x)$, the parametric equations $x = t$, $y = f(t)$ produce the same graph. As an example, given $y = x^2$, the parametric equations $x = t$, $y = t^2$ produce the familiar parabola. However, other parameterizations can be used. The following example demonstrates one possible alternative.

Example 9.11 Converting from rectangular to parametric

Consider $y = x^2$. Find parametric equations $x = f(t)$, $y = g(t)$ for the parabola where $t = \frac{dy}{dx}$. That is, $t = a$ corresponds to the point on the graph whose tangent line has slope a.

Solution We start by computing $\frac{dy}{dx}$: $y' = 2x$. Thus we set $t = 2x$. We can solve for x and find $x = t/2$. Knowing that $y = x^2$, we have $y = t^2/4$. Thus parametric equations for the parabola $y = x^2$ are

$$x = t/2 \quad y = t^2/4.$$

To find the point where the tangent line has a slope of -2, we set $t = -2$. This gives the point $(-1, 1)$. We can verify that the slope of the line tangent to the curve at this point indeed has a slope of -2.

We sometimes chose the parameter to accurately model physical behavior.

Example 9.12 Motion of a projectile

An object is fired from a height of 0ft and lands 6 seconds later, 192ft away. Find parametric equations $x = f(t)$, $y = g(t)$ for the path of the projectile where x is the horizontal distance the object has traveled at time t (in seconds) and y is the height at time t.

Solution Physics tells us that the horizontal motion of the projectile is linear; that is, the horizontal speed of the projectile is constant. Since the object travels 192ft in 6s, we deduce that the object is moving horizontally at a rate of $\frac{192\text{ft}}{6\text{s}} = 32\text{ft/s}$, giving the equation $x = 32t$. For the vertical motion, $y'' = -32$ ft/s^2, which is Earth's gravitational constant. Therefore, $y' = -32t + v_0$. Since the projectile begins on the ground and reaches the ground 6 seconds later, the

Notes:

maximum height is reached after 3 seconds. At this point $t = 3$, the vertical velocity is 0 as it changes direction. Plugging into the y' equation, we obtain $0 = -32(3) + v_0$, so $v_0 = 96$ ft/s. Since $y' = -32t + 96$, it follows that $y = -16t^2 + 96t + h_0$. Since the initial height $h_0 = 0$, $y = -16t^2 + 96t$.

These parametric equations make certain determinations about the object's location easy: 2 seconds into the flight the object is at the point $\big(x(2), y(2)\big) = \big(64, 128\big)$. That is, it has traveled horizontally 64ft and is at a height of 128ft, as shown in Figure 9.24.

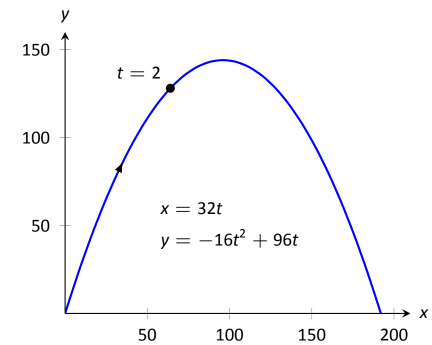


Figure 9.24: Graphing projectile motion in Example 9.12.

To determine if a point (x, y) lies on a given parametric curve, we solve the two equations to find a corresponding t-value that satisfies both equations. The number of such t-value(s) determines how many times the curve goes through the given point. This is illustrated in the next examples.

Example 9.13 Determine if a point lies on a parametric curve

Determine whether the following points lie on the parametric curve described by $x = t^2 + 1$ and $y = t^3 + 3t^2 - 4t$. If so, determine the corresponding t-values.

1. $(4, 3)$ 2. $(2, 0)$ 3. $(5, 12)$

Solution

1. For this point $(4, 3)$, we solve the equations $x = t^2 + 1 = 4$ and $y = t^3 + 3t^2 - 4t = 3$ to determine if there is a single t-value that satisfies both equations simultaneously. The quadratic equation $t^2 + 1 = 4$ is much easier to solve than the cubic equation. It has solutions $t = \pm\sqrt{3}$. Now rather than solve the cubic equation, we plug $t = \pm\sqrt{3}$ into $t^3 + 3t^2 - 4t$ to see if either value gives $y = 3$.

$$t = -\sqrt{3} \quad \Rightarrow \quad \left(-\sqrt{3}\right)^3 + 3\left(-\sqrt{3}\right)^2 - 4\left(-\sqrt{3}\right) = 9 + \sqrt{3} \neq 3$$
$$t = \sqrt{3} \quad \Rightarrow \quad \left(\sqrt{3}\right)^3 + 3\left(\sqrt{3}\right)^2 - 4\left(\sqrt{3}\right) = 9 - \sqrt{3} \neq 3$$

Since no t-value satisfies both equations $x = t^2 + 1 = 4$ and $y = t^3 + 3t^2 - 4t = 3$, the point $(4, 3)$ does not lie on the parametric curve.

2. For this point $(2, 0)$, we solve the equations $x = t^2 + 1 = 2$ and $y = t^3 + 3t^2 - 4t = 0$. The equation $t^2 + 1 = 2$ has the solutions $t = \pm 1$, while $t^3 + 3t^2 - 4t = 0$ is easily solved by factoring: $t(t + 4)(t - 1) = 0$ gives $t = -4, 0, 1$. As $t = 1$ is a solution to both equations, the point $(2, 0)$ is on the curve, at $t = 1$.

Notes:

3. For this point $(5, 12)$, we solve the equations $x = t^2 + 1 = 5$ and $y = t^3 + 3t^2 - 4t = 12$. The equation $t^2 + 1 = 5$ has the solutions $t = \pm 2$. Now we plug $t = \pm 2$ into $t^3 + 3t^2 - 4t$ to see if either value gives $y = 12$.

$$\begin{aligned} t = -2 \quad &\Rightarrow \quad (-2)^3 + 3(-2)^2 - 4(-2) = 12 \\ t = 2 \quad &\Rightarrow \quad (2)^3 + 3(2)^2 - 4(2) = 12 \end{aligned}$$

Both $t = -2$ and $t = 2$ satisfy both equations. Therefore, the curve goes through the point $(5, 12)$ twice: once at $t = -2$ and once at $t = 2$.

It is sometimes necessary to convert given parametric equations into rectangular form. This can be decidedly more difficult, as some "simple" looking parametric equations can have very "complicated" rectangular equations. This conversion is often referred to as "eliminating the parameter," as we are looking for a relationship between x and y that does not involve the parameter t.

Example 9.14 Eliminating the parameter

Find a rectangular equation for the curve described by

$$x = \frac{1}{t^2 + 1} \quad \text{and} \quad y = \frac{t^2}{t^2 + 1}.$$

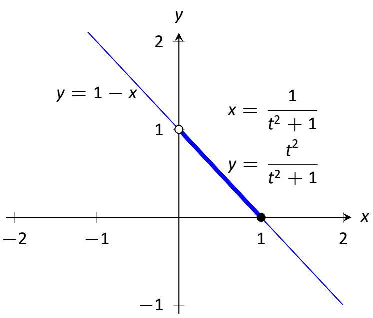


Figure 9.25: Graphing parametric and rectangular equations for a graph in Example 9.14.

Solution There is not a set way to eliminate a parameter. One method is to solve for t in one equation and then substitute that value in the second. We use that technique here, then show a second, simpler method.

Starting with $x = 1/(t^2 + 1)$, solve for t: $t = \pm\sqrt{1/x - 1}$. Substitute this value for t in the equation for y:

$$\begin{aligned} y &= \frac{t^2}{t^2 + 1} \\ &= \frac{1/x - 1}{1/x - 1 + 1} \\ &= \frac{1/x - 1}{1/x} \\ &= \left(\frac{1}{x} - 1\right) \cdot x \\ &= 1 - x. \end{aligned}$$

Notes:

Thus $y = 1 - x$. One may have recognized this earlier by manipulating the equation for y:

$$y = \frac{t^2}{t^2+1} = 1 - \frac{1}{t^2+1} = 1 - x.$$

This is a shortcut that is very specific to this problem; sometimes shortcuts exist and are worth looking for.

We should be careful to limit the domain of the function $y = 1 - x$. The parametric equations limit x to values in $(0, 1]$, thus to produce the same graph we should limit the domain of $y = 1 - x$ to the same.

The graphs of these functions is given in Figure 9.25. The portion of the graph defined by the parametric equations is given in a thick line; the graph defined by $y = 1 - x$ with unrestricted domain is given in a thin line.

Example 9.15 Eliminating the parameter

Eliminate the parameter in $x = 4\cos t + 3$, $y = 2\sin t + 1$

Solution We should not try to solve for t in this situation as the resulting algebra/trig would be messy. Rather, we solve for $\cos t$ and $\sin t$ in each equation, respectively. This gives

$$\cos t = \frac{x-3}{4} \quad \text{and} \quad \sin t = \frac{y-1}{2}.$$

The Pythagorean Theorem gives $\cos^2 t + \sin^2 t = 1$, so:

$$\begin{aligned} \cos^2 t + \sin^2 t &= 1 \\ \left(\frac{x-3}{4}\right)^2 + \left(\frac{y-1}{2}\right)^2 &= 1 \\ \frac{(x-3)^2}{16} + \frac{(y-1)^2}{4} &= 1 \end{aligned}$$

This final equation should look familiar – it is the equation of an ellipse! Figure 9.26 plots the parametric equations, demonstrating that the graph is indeed of an ellipse with a horizontal major axis and center at $(3, 1)$.

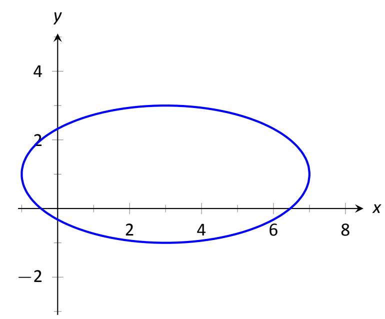
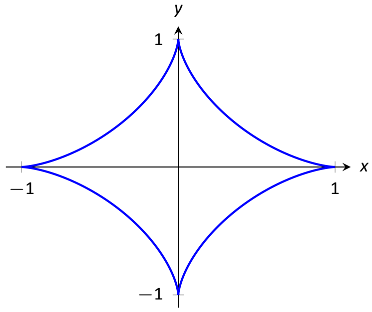


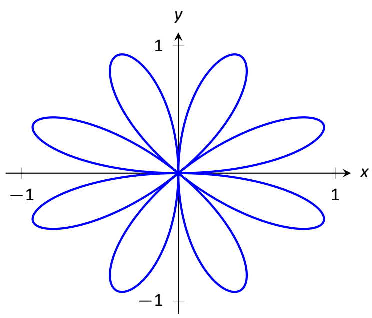
Figure 9.26: Graphing the parametric equations $x = 4\cos t + 3$, $y = 2\sin t + 1$ in Example 9.15.

The Pythagorean Theorem can also be used to identify parametric equations for hyperbolas. We give the parametric equations for ellipses and hyperbolas in the following Key Idea.

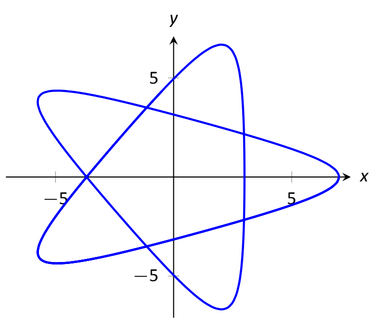
Notes:



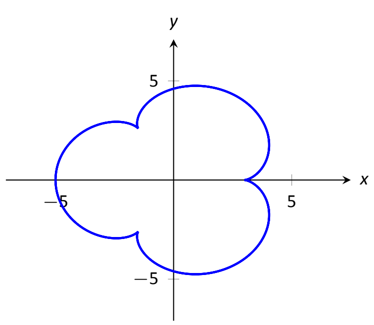
Astroid
$x = \cos^3 t$
$y = \sin^3 t$



Rose Curve
$x = \cos(t)\sin(4t)$
$y = \sin(t)\sin(4t)$



Hypotrochoid
$x = 2\cos(t) + 5\cos(2t/3)$
$y = 2\sin(t) - 5\sin(2t/3)$



Epicycloid
$x = 4\cos(t) - \cos(4t)$
$y = 4\sin(t) - \sin(4t)$

Figure 9.27: A gallery of interesting planar curves.

Key Idea 37 Parametric Equations of Ellipses and Hyperbolas

- The parametric equations

$$x = a\cos t + h, \quad y = b\sin t + k$$

define an ellipse with horizontal axis of length $2a$ and vertical axis of length $2b$, centered at (h, k).

- The parametric equations

$$x = a\tan t + h, \quad y = \pm b\sec t + k$$

define a hyperbola with vertical transverse axis centered at (h, k), and

$$x = \pm a\sec t + h, \quad y = b\tan t + k$$

defines a hyperbola with horizontal transverse axis. Each has asymptotes at $y = \pm b/a(x - h) + k$.

Special Curves

Figure 9.27 gives a small gallery of "interesting" and "famous" curves along with parametric equations that produce them. Interested readers can begin learning more about these curves through internet searches.

One might note a feature shared by two of these graphs: "sharp corners," or **cusps**. We have seen graphs with cusps before and determined that such functions are not differentiable at these points. This leads us to a definition.

Definition 49 Smooth

A curve C defined by $x = f(t)$, $y = g(t)$ is **smooth** on an interval I if f' and g' are continuous on I and not simultaneously 0 (except possibly at the endpoints of I). A curve is **piecewise smooth** on I if I can be partitioned into subintervals where C is smooth on each subinterval.

Consider the astroid, given by $x = \cos^3 t$, $y = \sin^3 t$. Taking derivatives, we have:

$$x' = -3\cos^2 t\sin t \quad \text{and} \quad y' = 3\sin^2 t\cos t.$$

It is clear that each is 0 when $t = 0,\ \pi/2,\ \pi, \ldots$. Thus the astroid is not smooth

Notes:

at these points, corresponding to the cusps seen in the figure.

We demonstrate this once more.

Example 9.16 Determine where a curve is not smooth

Let a curve C be defined by the parametric equations $x = t^3 - 12t + 17$ and $y = t^2 - 4t + 8$. Determine the points, if any, where it is not smooth.

Solution We begin by taking derivatives.

$$x' = 3t^2 - 12, \quad y' = 2t - 4.$$

We set each equal to 0:

$$\begin{aligned} x' &= 0 \Rightarrow 3t^2 - 12 = 0 \Rightarrow t = \pm 2 \\ y' &= 0 \Rightarrow 2t - 4 = 0 \Rightarrow t = 2 \end{aligned}$$

We see at $t = 2$ both x' and y' are 0; thus C is not smooth at $t = 2$, corresponding to the point $(1,4)$. The curve is graphed in Figure 9.28, illustrating the cusp at $(1,4)$.

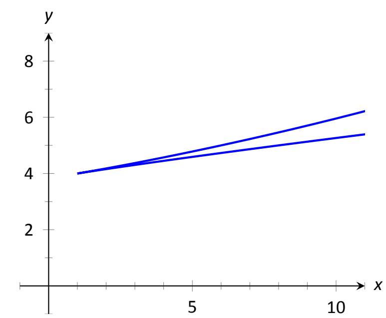


Figure 9.28: Graphing the curve in Example 9.16; note it is not smooth at $(1,4)$.

If a curve is not smooth at $t = t_0$, it means that $x'(t_0) = y'(t_0) = 0$ as defined. This, in turn, means that rate of change of x (and y) is 0; that is, at that instant, neither x nor y is changing. If the parametric equations describe the path of some object, this means the object is at rest at t_0. An object at rest can make a "sharp" change in direction, whereas moving objects tend to change direction in a "smooth" fashion.

One should be careful to note that a "sharp corner" does not have to occur when a curve is not smooth. For instance, one can verify that $x = t^3$, $y = t^6$ produce the familiar $y = x^2$ parabola. However, in this parameterization, the curve is not smooth. A particle traveling along the parabola according to the given parametric equations comes to rest at $t = 0$, though no sharp point is created.

Our previous experience with cusps taught us that a function was not differentiable at a cusp. This can lead us to wonder about derivatives in the context of parametric equations and the application of other calculus concepts. Given a curve defined parametrically, how do we find the slopes of tangent lines? Can we determine concavity? We explore these concepts and more in the next section.

Notes:

Exercises 9.2

Terms and Concepts

1. T/F: When sketching the graph of parametric equations, the x and y values are found separately, then plotted together.

2. The direction in which a graph is "moving" is called the ____ of the graph.

3. An equation written as $y = f(x)$ is written in ____ form.

4. Create parametric equations $x = f(t)$, $y = g(t)$ and sketch their graph. Explain any interesting features of your graph based on the functions f and g.

Problems

In Exercises 5 – 8, sketch the graph of the given parametric equations by hand, making a table of points to plot. Be sure to indicate the orientation of the graph.

5. $x = t^2 + t, \quad y = 1 - t^2, \quad -3 \leq t \leq 3$

6. $x = 1, \quad y = 5\sin t, \quad -\pi/2 \leq t \leq \pi/2$

7. $x = t^2, \quad y = 2, \quad -2 \leq t \leq 2$

8. $x = t^3 - t + 3, \quad y = t^2 + 1, \quad -2 \leq t \leq 2$

In Exercises 9 – 17, sketch the graph of the given parametric equations; using a graphing utility is advisable. Be sure to indicate the orientation of the graph.

9. $x = t^3 - 2t^2, \quad y = t^2, \quad -2 \leq t \leq 3$

10. $x = 1/t, \quad y = \sin t, \quad 0 < t \leq 10$

11. $x = 3\cos t, \quad y = 5\sin t, \quad 0 \leq t \leq 2\pi$

12. $x = 3\cos t + 2, \quad y = 5\sin t + 3, \quad 0 \leq t \leq 2\pi$

13. $x = \cos t, \quad y = \cos(2t), \quad 0 \leq t \leq \pi$

14. $x = \cos t, \quad y = \sin(2t), \quad 0 \leq t \leq 2\pi$

15. $x = 2\sec t, \quad y = 3\tan t, \quad -\pi/2 < t < \pi/2$

16. $x = \cos t + \frac{1}{4}\cos(8t), \quad y = \sin t + \frac{1}{4}\sin(8t), \quad 0 \leq t \leq 2\pi$

17. $x = \cos t + \frac{1}{4}\sin(8t), \quad y = \sin t + \frac{1}{4}\cos(8t), \quad 0 \leq t \leq 2\pi$

In Exercises 18 – 19, four sets of parametric equations are given. Describe how their graphs are similar and different. Be sure to discuss orientation and ranges.

18. (a) $x = t \quad y = t^2, \quad -\infty < t < \infty$
    (b) $x = \sin t \quad y = \sin^2 t, \quad -\infty < t < \infty$
    (c) $x = e^t \quad y = e^{2t}, \quad -\infty < t < \infty$
    (d) $x = -t \quad y = t^2, \quad -\infty < t < \infty$

19. (a) $x = \cos t \quad y = \sin t, \quad 0 \leq t \leq 2\pi$
    (b) $x = \cos(t^2) \quad y = \sin(t^2), \quad 0 \leq t \leq 2\pi$
    (c) $x = \cos(1/t) \quad y = \sin(1/t), \quad 0 < t < 1$
    (d) $x = \cos(\cos t) \quad y = \sin(\cos t), \quad 0 \leq t \leq 2\pi$

In Exercises 20 – 23, determine if the given point is on the parametric curve. If so, find the t-value(s) corresponding to that point.

20. $x = 4t - 5, \quad y = 3t + 10, \quad$ point: $(-9, 2)$

21. $x = t^2, \quad y = t^3 - t, \quad$ point: $(4, -6)$

22. $x = t + 2^t, \quad y = t^2, \quad$ point: $(8, -1)$

23. $x = \cos(3t) - \sin t, \quad y = 2\cos t, \quad$ point: $\left(-1 - \sqrt{3}/2, 1\right)$

In Exercises 24 – 35, eliminate the parameter in the given parametric equations.

24. $x = 2t + 5, \quad y = -3t + 1$

25. $x = \sec t, \quad y = \tan t$

26. $x = 4\sin t + 1, \quad y = 3\cos t - 2$

27. $x = t^2, \quad y = t^3$

28. $x = \dfrac{1}{t+1}, \quad y = \dfrac{3t+5}{t+1}$

29. $x = e^t, \quad y = e^{3t} - 3$

30. $x = \cos^3 t, \quad y = \sin^3 t$

31. $x = \ln t, \quad y = t^2 - 1$

32. $x = \cot t, \quad y = \csc t$

33. $x = \cosh t, \quad y = \sinh t$

34. $x = \cos(2t), \quad y = \sin t$

35. $x = \cos t, \quad y = \sin^2 t$

In Exercises 36 – 39, eliminate the parameter in the given parametric equations. Describe the curve defined by the parametric equations based on its rectangular form.

36. $x = at + x_0, \quad y = bt + y_0$

37. $x = r\cos t, \quad y = r\sin t$

38. $x = a\cos t + h, \quad y = b\sin t + k$

39. $x = a\sec t + h, \quad y = b\tan t + k$

In Exercises 40 – 43, find parametric equations for the given rectangular equation using the parameter $t = \dfrac{dy}{dx}$. Verify that at $t = 1$, the point on the graph has a tangent line with slope of 1.

40. $y = 3x^2 - 11x + 2$

41. $y = e^x$

42. $y = \sin x$ on $[0, \pi]$

43. $y = \sqrt{x}$ on $[0, \infty)$

In Exercises 44 – 47, find the values of t where the graph of the parametric equations crosses itself.

44. $x = t^3 - t + 3, \quad y = t^2 - 3$

45. $x = t^3 - 4t^2 + t + 7, \quad y = t^2 - t$

46. $x = \cos t, \quad y = \sin(2t)$ on $[0, 2\pi]$

47. $x = \cos t\cos(3t), \quad y = \sin t\cos(3t)$ on $[0, \pi]$

In Exercises 48 – 51, find the value(s) of t where the curve defined by the parametric equations is not smooth.

48. $x = t^3 + t^2 - t, \quad y = t^2 + 2t + 3$

49. $x = t^2 - 4t, \quad y = t^3 - 2t^2 - 4t$

50. $x = \cos t, \quad y = 2\cos t$

51. $x = 2\cos t - \cos(2t), \quad y = 2\sin t - \sin(2t)$

In Exercises 52 – 60, find parametric equations that describe the given situation.

52. A projectile is fired from a height of 0ft, landing 16ft away in 4s.

53. A projectile is fired from a height of 0ft, landing 200ft away in 4s.

54. A projectile is fired from a height of 0ft, landing 200ft away in 20s.

55. A circle of radius 2, centered at the origin, that is traced clockwise once on $[0, 2\pi]$.

56. A circle of radius 3, centered at $(1, 1)$, that is traced once counter–clockwise on $[0, 1]$.

57. An ellipse centered at $(1, 3)$ with vertical major axis of length 6 and minor axis of length 2.

58. An ellipse with foci at $(\pm 1, 0)$ and vertices at $(\pm 5, 0)$.

59. A hyperbola with foci at $(5, -3)$ and $(-1, -3)$, and with vertices at $(1, -3)$ and $(3, -3)$.

60. A hyperbola with vertices at $(0, \pm 6)$ and asymptotes $y = \pm 3x$.

9.3 Calculus and Parametric Equations

The previous section defined curves based on parametric equations. In this section we'll employ the techniques of calculus to study these curves.

We are still interested in lines tangent to points on a curve. They describe how the y-values are changing with respect to the x-values, they are useful in making approximations, and they indicate instantaneous direction of travel.

The slope of the tangent line is still $\frac{dy}{dx}$, and the Chain Rule allows us to calculate this in the context of parametric equations. If $x = f(t)$ and $y = g(t)$, the Chain Rule states that

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

Solving for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = \frac{dy}{dt} \bigg/ \frac{dx}{dt} = \frac{g'(t)}{f'(t)},$$

provided that $f'(t) \neq 0$. This is important so we label it a Key Idea.

Key Idea 38 Finding $\frac{dy}{dx}$ with Parametric Equations.

Let $x = f(t)$ and $y = g(t)$, where f and g are differentiable on some open interval I and $f'(t) \neq 0$ on I. Then

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)}.$$

We use this to define the tangent line.

Definition 50 Tangent and Normal Lines

Let a curve C be parameterized by $x = f(t)$ and $y = g(t)$, where f and g are differentiable functions on some interval I containing $t = t_0$. The **tangent line** to C at $t = t_0$ is the line through $\big(f(t_0), g(t_0)\big)$ with slope $m = g'(t_0)/f'(t_0)$, provided $f'(t_0) \neq 0$.

The **normal line** to C at $t = t_0$ is the line through $\big(f(t_0), g(t_0)\big)$ with slope $m = -f'(t_0)/g'(t_0)$, provided $g'(t_0) \neq 0$.

The definition leaves two special cases to consider. When the tangent line is horizontal, the normal line is undefined by the above definition as $g'(t_0) = 0$.

Notes:

Likewise, when the normal line is horizontal, the tangent line is undefined. It seems reasonable that these lines be defined (one can draw a line tangent to the "right side" of a circle, for instance), so we add the following to the above definition.

1. If the tangent line at $t = t_0$ has a slope of 0, the normal line to C at $t = t_0$ is the vertical line $x = f(t_0)$.
2. If the normal line at $t = t_0$ has a slope of 0, the tangent line to C at $t = t_0$ is the vertical line $x = f(t_0)$.

Example 9.17 Tangent and Normal Lines to Curves

Let $x = 5t^2 - 6t + 4$ and $y = t^2 + 6t - 1$, and let C be the curve defined by these equations.

1. Find the equations of the tangent and normal lines to C at $t = 3$.
2. Find where C has vertical and horizontal tangent lines.

Solution

1. We start by computing $f'(t) = 10t - 6$ and $g'(t) = 2t + 6$. Thus

$$\frac{dy}{dx} = \frac{2t+6}{10t-6}.$$

Make note of something that might seem unusual: $\frac{dy}{dx}$ is a function of t, not x. Just as points on the curve are found in terms of t, so are the slopes of the tangent lines.

The point on C at $t = 3$ is $(31, 26)$. The slope of the tangent line is $m = 1/2$ and the slope of the normal line is $m = -2$. Thus,

- the equation of the tangent line is $y = \frac{1}{2}(x - 31) + 26$, and
- the equation of the normal line is $y = -2(x - 31) + 26$.

This is illustrated in Figure 9.29.

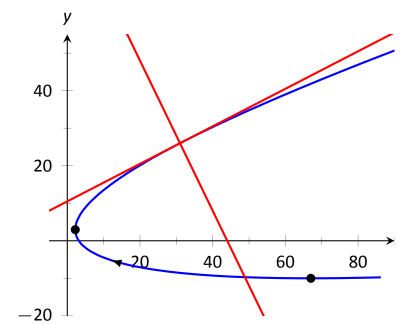


Figure 9.29: Graphing tangent and normal lines in Example 9.17.

2. To find where C has a horizontal tangent line, we set $\frac{dy}{dx} = 0$ and solve for t. In this case, this amounts to setting $g'(t) = 0$ and solving for t (and making sure that $f'(t) \neq 0$).

$$g'(t) = 0 \quad \Rightarrow \quad 2t + 6 = 0 \quad \Rightarrow \quad t = -3.$$

The point on C corresponding to $t = -3$ is $(67, -10)$; the tangent line at that point is horizontal (hence with equation $y = -10$).

Notes:

To find where C has a vertical tangent line, we find where it has a horizontal normal line, and set $-\frac{f'(t)}{g'(t)} = 0$. This amounts to setting $f'(t) = 0$ and solving for t (and making sure that $g'(t) \neq 0$).

$$f'(t) = 0 \quad \Rightarrow \quad 10t - 6 = 0 \quad \Rightarrow \quad t = 0.6.$$

The point on C corresponding to $t = 0.6$ is $(2.2, 2.96)$. The tangent line at that point is $x = 2.2$.

The points where the tangent lines are vertical and horizontal are indicated on the graph in Figure 9.29.

Example 9.18 Tangent and Normal Lines to a Circle

1. Find where the unit circle, defined by $x = \cos t$ and $y = \sin t$ on $[0, 2\pi]$, has vertical and horizontal tangent lines.
2. Find the equation of the normal line at $t = t_0$.

SOLUTION

1. We compute the derivative following Key Idea 38:

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)} = -\frac{\cos t}{\sin t}.$$

The derivative is 0 when $\cos t = 0$; that is, when $t = \pi/2,\ 3\pi/2$. These are the points $(0, 1)$ and $(0, -1)$ on the circle.

The normal line is horizontal (and hence, the tangent line is vertical) when $\sin t = 0$; that is, when $t = 0,\ \pi,\ 2\pi$, corresponding to the points $(-1, 0)$ and $(0, 1)$ on the circle. These results should make intuitive sense.

2. The slope of the normal line at $t = t_0$ is $m = \dfrac{\sin t_0}{\cos t_0} = \tan t_0$. This normal line goes through the point $(\cos t_0, \sin t_0)$, giving the line

$$\begin{aligned} y &= \frac{\sin t_0}{\cos t_0}(x - \cos t_0) + \sin t_0 \\ &= (\tan t_0)x, \end{aligned}$$

as long as $\cos t_0 \neq 0$. It is an important fact to recognize that the normal lines to a circle pass through its center, as illustrated in Figure 9.30. Stated in another way, any line that passes through the center of a circle intersects the circle at right angles.

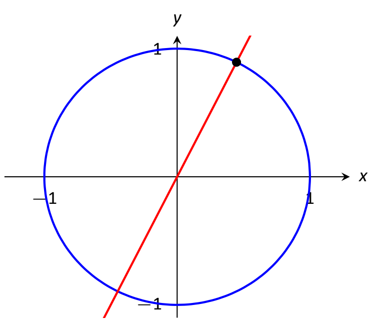


Figure 9.30: Illustrating how a circle's normal lines pass through its center.

Notes:

Example 9.19 Tangent lines when $\frac{dy}{dx}$ is not defined

Find the equation of the tangent line to the astroid $x = \cos^3 t$, $y = \sin^3 t$ at $t = 0$, shown in Figure 9.31.

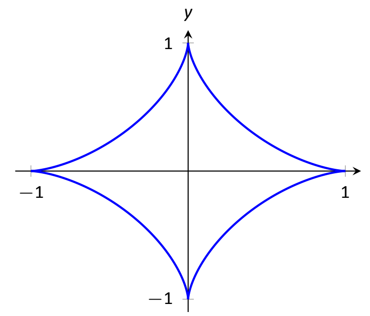


Figure 9.31: A graph of an astroid.

SOLUTION We start by finding $x'(t)$ and $y'(t)$:

$$x'(t) = -3\sin t\cos^2 t, \qquad y'(t) = 3\cos t\sin^2 t.$$

Note that both of these are 0 at $t = 0$; the curve is not smooth at $t = 0$ forming a cusp on the graph. Evaluating $\frac{dy}{dx}$ at this point returns the indeterminate form of "0/0".

We can, however, examine the slopes of tangent lines near $t = 0$, and take the limit as $t \to 0$.

$$\begin{aligned}\lim_{t\to 0}\frac{y'(t)}{x'(t)} &= \lim_{t\to 0}\frac{3\cos t\sin^2 t}{-3\sin t\cos^2 t} \quad \text{(We can cancel as } t \neq 0.\text{)}\\ &= \lim_{t\to 0} -\frac{\sin t}{\cos t}\\ &= 0.\end{aligned}$$

We have accomplished something significant. When the derivative $\frac{dy}{dx}$ returns an indeterminate form at $t = t_0$, we can define its value by setting it to be $\lim\limits_{t\to t_0}\frac{dy}{dx}$, if that limit exists. This allows us to find slopes of tangent lines at cusps, which can be very beneficial.

We found the slope of the tangent line at $t = 0$ to be 0; therefore the tangent line is $y = 0$, the x-axis.

Concavity

We continue to analyze curves in the plane by considering their concavity; that is, we are interested in $\frac{d^2y}{dx^2}$, "the second derivative of y with respect to x." To find this, we need to find the derivative of $\frac{dy}{dx}$ with respect to x; that is,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right],$$

but recall that $\frac{dy}{dx}$ is a function of t, not x, making this computation not straightforward.

To make the upcoming notation a bit simpler, let $h(t) = \frac{dy}{dx}$. We want $\frac{d}{dx}[h(t)]$; that is, we want $\frac{dh}{dx}$. We again appeal to the Chain Rule. Note:

$$\frac{dh}{dt} = \frac{dh}{dx}\cdot\frac{dx}{dt} \quad \Rightarrow \quad \frac{dh}{dx} = \frac{dh}{dt}\bigg/\frac{dx}{dt}.$$

Notes:

In words, to find $\frac{d^2y}{dx^2}$, we first take the derivative of $\frac{dy}{dx}$ *with respect to t*, then divide by $x'(t)$. We restate this as a Key Idea.

Key Idea 39 Finding $\frac{d^2y}{dx^2}$ with Parametric Equations

Let $x = f(t)$ and $y = g(t)$ be twice differentiable functions on an open interval I, where $f'(t) \neq 0$ on I. Then

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left[\frac{dy}{dx}\right] \bigg/ \frac{dx}{dt} = \frac{d}{dt}\left[\frac{dy}{dx}\right] \bigg/ f'(t).$$

Examples will help us understand this Key Idea.

Example 9.20 Concavity of Plane Curves

Let $x = 5t^2 - 6t + 4$ and $y = t^2 + 6t - 1$ as in Example 9.17. Determine the t-intervals on which the graph is concave up/down.

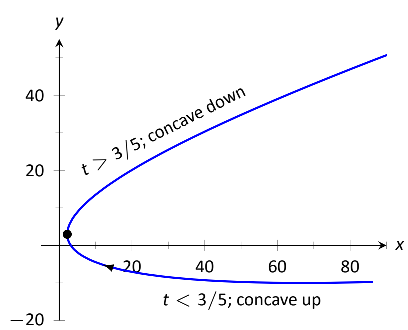


Figure 9.32: Graphing the parametric equations in Example 9.20 to demonstrate concavity.

Solution Concavity is determined by the second derivative of y with respect to x, $\frac{d^2y}{dx^2}$, so we compute that here following Key Idea 39.

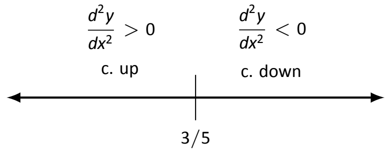
In Example 9.17, we found $\frac{dy}{dx} = \frac{2t+6}{10t-6}$ and $f'(t) = 10t - 6$. So:

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dt}\left[\frac{2t+6}{10t-6}\right] \bigg/ (10t-6)\\ &= -\frac{72}{(10t-6)^2} \bigg/ (10t-6)\\ &= -\frac{72}{(10t-6)^3}\\ &= -\frac{9}{(5t-3)^3}\end{aligned}$$

The graph of the parametric functions is concave up when $\frac{d^2y}{dx^2} > 0$ and concave down when $\frac{d^2y}{dx^2} < 0$. We determine the intervals when the second derivative is greater/less than 0 by first finding when it is 0 or undefined.

As the numerator of $-\frac{9}{(5t-3)^3}$ is never 0, $\frac{d^2y}{dx^2} \neq 0$ for all t. It is undefined when $5t - 3 = 0$; that is, when $t = 3/5$. Following the work established in Section 3.4, we look at values of t greater/less than $3/5$ on a number line:

Notes:



Reviewing Example 9.17, we see that when $t = 3/5 = 0.6$, the graph of the parametric equations has a vertical tangent line. This point is also a point of inflection for the graph, illustrated in Figure 9.32.

Example 9.21 **Concavity of Plane Curves**

Find the points of inflection of the graph of the parametric equations $x = \sqrt{t}$, $y = \sin t$, for $0 \leq t \leq 16$.

Solution We need to compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\cos t}{1/(2\sqrt{t})} = 2\sqrt{t}\cos t.$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{x'(t)} = \frac{\cos t/\sqrt{t} - 2\sqrt{t}\sin t}{1/(2\sqrt{t})} = 2\cos t - 4t\sin t.$$

The points of inflection are found by setting $\frac{d^2y}{dx^2} = 0$. This is not trivial, as equations that mix polynomials and trigonometric functions generally do not have "nice" solutions.

In Figure 9.33(a) we see a plot of the second derivative. It shows that it has zeros at approximately $t = 0.5,\ 3.5,\ 6.5,\ 9.5,\ 12.5$ and 16. These approximations are not very good, made only by looking at the graph. Newton's Method provides more accurate approximations. Accurate to 2 decimal places, we have:

$$t = 0.65,\ 3.29,\ 6.36,\ 9.48,\ 12.61 \text{ and } 15.74.$$

The corresponding points have been plotted on the graph of the parametric equations in Figure 9.33(b). Note how most occur near the x-axis, but not exactly on the axis.

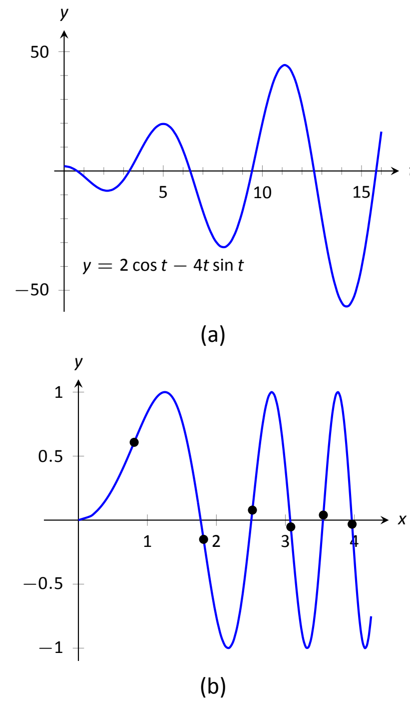


Figure 9.33: In (a), a graph of $\frac{d^2y}{dx^2}$, showing where it is approximately 0. In (b), graph of the parametric equations in Example 9.21 along with the points of inflection.

Arc Length

We continue our study of the features of the graphs of parametric equations by computing their arc length.

Recall in Section 7.4 we found the arc length of the graph of a function, from $x = a$ to $x = b$, to be

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Notes:

We can use this equation and convert it to the parametric equation context. Letting $x = f(t)$ and $y = g(t)$, we know that $\frac{dy}{dx} = g'(t)/f'(t)$. It will also be useful to calculate the differential of x:

$$dx = f'(t)dt \qquad \Rightarrow \qquad dt = \frac{1}{f'(t)} \cdot dx.$$

Starting with the arc length formula above, consider:

$$\begin{aligned} L &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_a^b \sqrt{1 + \frac{g'(t)^2}{f'(t)^2}}\, dx. \end{aligned}$$

Factor out the $f'(t)^2$:

$$\begin{aligned} &= \int_a^b \sqrt{f'(t)^2 + g'(t)^2} \cdot \underbrace{\frac{1}{f'(t)}\, dx}_{=dt} \\ &= \int_{t_1}^{t_2} \sqrt{f'(t)^2 + g'(t)^2}\, dt. \end{aligned}$$

Note the new bounds (no longer "x" bounds, but "t" bounds). They are found by finding t_1 and t_2 such that $a = f(t_1)$ and $b = f(t_2)$. This formula is important, so we restate it as a theorem.

Theorem 84 Arc Length of Parametric Curves

Let $x = f(t)$ and $y = g(t)$ be parametric equations with f' and g' continuous on some open interval I containing t_1 and t_2 on which the graph traces itself only once. The arc length of the graph, from $t = t_1$ to $t = t_2$, is

$$L = \int_{t_1}^{t_2} \sqrt{f'(t)^2 + g'(t)^2}\, dt.$$

As before, these integrals are often not easy to compute. We start with a simple example, then give another where we approximate the solution.

Notes:

Example 9.22 Arc Length of a Circle

Find the arc length of the circle parameterized by $x = 3\cos t$, $y = 3\sin t$ on $[0, 3\pi/2]$.

SOLUTION By direct application of Theorem 84, we have

$$L = \int_0^{3\pi/2} \sqrt{(-3\sin t)^2 + (3\cos t)^2}\, dt.$$

Apply the Pythagorean Theorem.

$$= \int_0^{3\pi/2} 3\, dt$$
$$= 3t\Big|_0^{3\pi/2} = 9\pi/2.$$

This should make sense; we know from geometry that the circumference of a circle with radius 3 is 6π; since we are finding the arc length of $3/4$ of a circle, the arc length is $3/4 \cdot 6\pi = 9\pi/2$.

Example 9.23 Arc Length of a Parametric Curve

The graph of the parametric equations $x = t(t^2 - 1)$, $y = t^2 - 1$ crosses itself as shown in Figure 9.34, forming a "teardrop." Find the arc length of the teardrop.

SOLUTION We can see by the parameterizations of x and y that when $t = \pm 1$, $x = 0$ and $y = 0$. This means we'll integrate from $t = -1$ to $t = 1$. Applying Theorem 84, we have

$$L = \int_{-1}^{1} \sqrt{(3t^2 - 1)^2 + (2t)^2}\, dt$$
$$= \int_{-1}^{1} \sqrt{9t^4 - 2t^2 + 1}\, dt.$$

Unfortunately, the integrand does not have an antiderivative expressible by elementary functions. We turn to numerical integration to approximate its value. Using 4 subintervals, Simpson's Rule approximates the value of the integral as 2.65051. Using a computer, more subintervals are easy to employ, and $n = 20$ gives a value of 2.71559. Increasing n shows that this value is stable and a good approximation of the actual value.

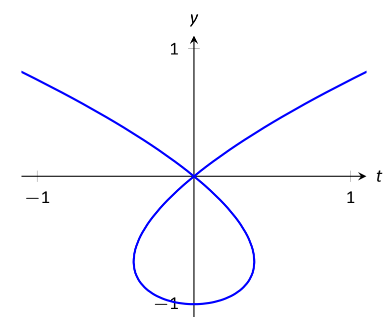


Figure 9.34: A graph of the parametric equations in Example 9.23, where the arc length of the teardrop is calculated.

Notes:

Example 9.24 Arc Length of a Parametric Curve

Determine the length of the graph of the parametric equations $x = 20/t$, $y = t^{3/2}$ from the point $(20, 1)$ to the point $(5, 8)$.

SOLUTION In this problem, we need to determine the t-value(s) for the given points, as in the previous section. For the point $(20, 1)$, solving $x = 20/t = 20$ gives $t = 1$. We plug this into $y = t^{3/2} = 1^{3/2} = 1$, confirming the point is on the curve at $t = 1$. Similarly, we can determine the point $(5, 8)$ corresponds to $t = 4$. So we'll integrate from $t = 1$ to $t = 4$. Applying Theorem 84, we have

$$\begin{aligned} L &= \int_1^4 \sqrt{\left(-\frac{20}{t^2}\right)^2 + \left(\frac{3}{2}t^{1/2}\right)^2}\, dt \\ &= \int_1^4 \sqrt{\frac{400}{t^4} + \frac{9}{4}t}\, dt. \end{aligned}$$

This integrand does not have an antiderivative expressible by elementary functions, but a computer approximates this integral as 17.6087.

Surface Area of a Solid of Revolution

Related to the formula for finding arc length is the formula for finding surface area. We can adapt the formula found in Key Idea 29 from Section 7.4 in a similar way as done to produce the formula for arc length done before.

Key Idea 40 Surface Area of a Solid of Revolution

Consider the graph of the parametric equations $x = f(t)$ and $y = g(t)$, where f' and g' are continuous on an open interval I containing t_1 and t_2 on which the graph does not cross itself.

1. The surface area of the solid formed by revolving the graph about the x-axis is (where $g(t) \geq 0$ on $[t_1, t_2]$):

$$\text{Surface Area} = 2\pi \int_{t_1}^{t_2} g(t)\sqrt{f'(t)^2 + g'(t)^2}\, dt.$$

2. The surface area of the solid formed by revolving the graph about the y-axis is (where $f(t) \geq 0$ on $[t_1, t_2]$):

$$\text{Surface Area} = 2\pi \int_{t_1}^{t_2} f(t)\sqrt{f'(t)^2 + g'(t)^2}\, dt.$$

Notes:

Example 9.25 Surface Area of a Solid of Revolution

Consider the teardrop shape formed by the parametric equations $x = t(t^2 - 1)$, $y = t^2 - 1$ as seen in Example 9.23. Find the surface area if this shape is rotated about the x-axis, as shown in Figure 9.35.

Solution The teardrop shape is formed between $t = -1$ and $t = 1$. Using Key Idea 40, we see we need for $g(t) \geq 0$ on $[-1, 1]$, and this is not the case. To fix this, we simplify replace $g(t)$ with $-g(t)$, which flips the whole graph about the x-axis (and does not change the surface area of the resulting solid). The surface area is:

$$\begin{aligned}\text{Area } S &= 2\pi \int_{-1}^{1} (1 - t^2)\sqrt{(3t^2 - 1)^2 + (2t)^2}\, dt \\ &= 2\pi \int_{-1}^{1} (1 - t^2)\sqrt{9t^4 - 2t^2 + 1}\, dt.\end{aligned}$$

Once again we arrive at an integral that we cannot compute in terms of elementary functions. Using Simpson's Rule with $n = 20$, we find the area to be $S = 9.44$. Using larger values of n shows this is accurate to 2 places after the decimal.

After defining a new way of creating curves in the plane, in this section we have applied calculus techniques to the parametric equation defining these curves to study their properties. In the next section, we define another way of forming curves in the plane. To do so, we create a new coordinate system, called *polar coordinates*, that identifies points in the plane in a manner different than from measuring distances from the y- and x- axes.

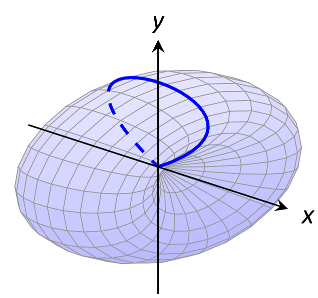


Figure 9.35: Rotating a teardrop shape about the x-axis in Example 9.25.

Notes:

Exercises 9.3

Terms and Concepts

1. T/F: Given parametric equations $x = f(t)$ and $y = g(t)$, $\frac{dy}{dx} = f'(t)/g'(t)$, as long as $g'(t) \neq 0$.

2. Given parametric equations $x = f(t)$ and $y = g(t)$, the derivative $\frac{dy}{dx}$ as given in Key Idea 38 is a function of ________?

3. T/F: Given parametric equations $x = f(t)$ and $y = g(t)$, to find $\frac{d^2y}{dx^2}$, one simply computes $\frac{d}{dt}\left(\frac{dy}{dx}\right)$.

4. T/F: If $\frac{dy}{dx} = 0$ at $t = t_0$, then the normal line to the curve at $t = t_0$ is a vertical line.

Problems

In Exercises 5 – 12, parametric equations for a curve are given.

(a) Find $\dfrac{dy}{dx}$.

(b) Find the equations of the tangent and normal line(s) at the point(s) given.

(c) Sketch the graph of the parametric functions along with the found tangent and normal lines.

5. $x = t, y = t^2; \quad t = 1$

6. $x = \sqrt{t}, y = 5t + 2; \quad t = 4$

7. $x = t^2 - t, y = t^2 + t; \quad t = 1$

8. $x = t^2 - 1, y = t^3 - t; \quad t = 0$ and $t = 1$

9. $x = \sec t, y = \tan t$ on $(-\pi/2, \pi/2); \quad t = \pi/4$

10. $x = \cos t, y = \sin(2t)$ on $[0, 2\pi]; \quad t = \pi/4$

11. $x = \cos t \sin(2t), y = \sin t \sin(2t)$ on $[0, 2\pi]; \quad t = 3\pi/4$

12. $x = e^{t/10} \cos t, y = e^{t/10} \sin t; \quad t = \pi/2$

In Exercises 13 – 20, find t-values where the curve defined by the given parametric equations has a horizontal tangent line. Note: these are the same equations as in Exercises 5 – 12.

13. $x = t, y = t^2$

14. $x = \sqrt{t}, y = 5t + 2$

15. $x = t^2 - t, y = t^2 + t$

16. $x = t^2 - 1, y = t^3 - t$

17. $x = \sec t, y = \tan t$ on $(-\pi/2, \pi/2)$

18. $x = \cos t, y = \sin(2t)$ on $[0, 2\pi]$

19. $x = \cos t \sin(2t), y = \sin t \sin(2t)$ on $[0, 2\pi]$

20. $x = e^{t/10} \cos t, y = e^{t/10} \sin t$

In Exercises 21 – 24, find $t = t_0$ where the graph of the given parametric equations is not smooth, then find $\displaystyle\lim_{t \to t_0} \frac{dy}{dx}$.

21. $x = \dfrac{1}{t^2 + 1}, \quad y = t^3$

22. $x = -t^3 + 7t^2 - 16t + 13, \quad y = t^3 - 5t^2 + 8t - 2$

23. $x = t^3 - 3t^2 + 3t - 1, \quad y = t^2 - 2t + 1$

24. $x = \cos^2 t, \quad y = 1 - \sin^2 t$

In Exercises 25 – 32, parametric equations for a curve are given. Find $\frac{d^2y}{dx^2}$, then determine the intervals on which the graph of the curve is concave up/down. Note: these are the same equations as in Exercises 5 – 12.

25. $x = t, \quad y = t^2$

26. $x = \sqrt{t}, \quad y = 5t + 2$

27. $x = t^2 - t, \quad y = t^2 + t$

28. $x = t^2 - 1, \quad y = t^3 - t$

29. $x = \sec t, \quad y = \tan t$ on $(-\pi/2, \pi/2)$

30. $x = \cos t, \quad y = \sin(2t)$ on $[0, 2\pi]$

31. $x = \cos t \sin(2t), \quad y = \sin t \sin(2t)$ on $[-\pi/2, \pi/2]$

32. $x = e^{t/10} \cos t, \quad y = e^{t/10} \sin t$

In Exercises 33 – 36, find the arc length of the graph of the parametric equations on the given interval(s).

33. $x = -3\sin(2t), \quad y = 3\cos(2t)$ on $[0, \pi]$

34. $x = e^{t/10} \cos t, \quad y = e^{t/10} \sin t$ on $[0, 2\pi]$ and $[2\pi, 4\pi]$

35. $x = 5t + 2, \quad y = 1 - 3t$ on $[-1, 1]$

36. $x = 2t^{3/2}, \quad y = 3t$ on $[0, 1]$

In Exercises 37 – 38, set up an integral expression for the arc length of the curve between the specified points. Do not evaluate the integral.

37. $x = t^3 - t, \quad y = t^3 + 2t$ from $(0, -3)$ to $(6, 12)$

38. $x = \cos(3t) - \sin t, \quad y = 2\cos t$ from $(1, 2)$ to $(-1, 0)$

39. A projectile is fired from a height of 0ft, landing 16ft away in 4s. Determine the total distance the projectile traveled.

40. A projectile is fired from a height of 0ft, landing 200ft away in 20s. Determine the total distance the projectile traveled.

In Exercises 41 – 44, numerically approximate the given arc length.

41. Approximate the arc length of one petal of the rose curve $x = \cos t\cos(2t), \quad y = \sin t\cos(2t)$ using Simpson's Rule and $n = 4$.

42. Approximate the arc length of the "bow tie curve" $x = \cos t, \quad y = \sin(2t)$ using Simpson's Rule and $n = 6$.

43. Approximate the arc length of the parabola $x = t^2 - t$, $y = t^2 + t$ on $[-1, 1]$ using Simpson's Rule and $n = 4$.

44. A common approximate of the circumference of an ellipse given by $x = a\cos t, \quad y = b\sin t$ is $C \approx 2\pi\sqrt{\dfrac{a^2+b^2}{2}}$. Use this formula to approximate the circumference of $x = 5\cos t, \quad y = 3\sin t$ and compare this to the approximation given by Simpson's Rule and $n = 6$.

In Exercises 45 – 48, a solid of revolution is described. Find or approximate its surface area as specified.

45. Find the surface area of the sphere formed by rotating the circle $x = 2\cos t, \quad y = 2\sin t$ about:
    (a) the x-axis and
    (b) the y-axis.

46. Find the surface area of the torus (or "donut") formed by rotating the circle $x = \cos t + 2, \quad y = \sin t$ about the y-axis.

47. Approximate the surface area of the solid formed by rotating the "upper right half" of the bow tie curve $x = \cos t$, $y = \sin(2t)$ on $[0, \pi/2]$ about the x-axis, using Simpson's Rule and $n = 4$.

48. Approximate the surface area of the solid formed by rotating the one petal of the rose curve $x = \cos t\cos(2t)$, $y = \sin t\cos(2t)$ on $[0, \pi/4]$ about the x-axis, using Simpson's Rule and $n = 4$.

9.4 Introduction to Polar Coordinates

We are generally introduced to the idea of graphing curves by relating x-values to y-values through a function f. That is, we set $y = f(x)$, and plot lots of point pairs (x, y) to get a good notion of how the curve looks. This method is useful but has limitations, not least of which is that curves that "fail the vertical line test" cannot be graphed without using multiple functions.

The previous two sections introduced and studied a new way of plotting points in the x, y-plane. Using parametric equations, x and y values are computed independently and then plotted together. This method allows us to graph an extraordinary range of curves. This section introduces yet another way to plot points in the plane: using **polar coordinates**.

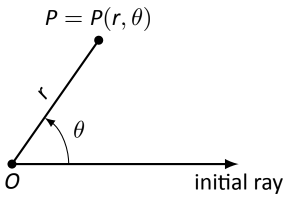


Figure 9.36: Illustrating polar coordinates.

Polar Coordinates

Start with a point O in the plane called the **pole** (we will always identify this point with the origin). From the pole, draw a ray, called the **initial ray** (we will always draw this ray horizontally, identifying it with the positive x-axis). A point P in the plane is determined by the distance r that P is from O, and the angle θ formed between the initial ray and the segment $\overline{OP}$ (measured counter-clockwise). We record the distance and angle as an ordered pair (r, θ). To avoid confusion with rectangular coordinates, we will denote polar coordinates with the letter P, as in $P(r, \theta)$. This is illustrated in Figure 9.36

Practice will make this process more clear.

Example 9.26 Plotting Polar Coordinates

Plot the following polar coordinates:

$$A = P(1, \pi/4) \quad B = P(1.5, \pi) \quad C = P(2, -\pi/3) \quad D = P(-1, \pi/4)$$

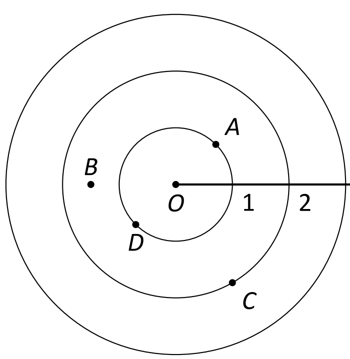


Figure 9.37: Plotting polar points in Example 9.26.

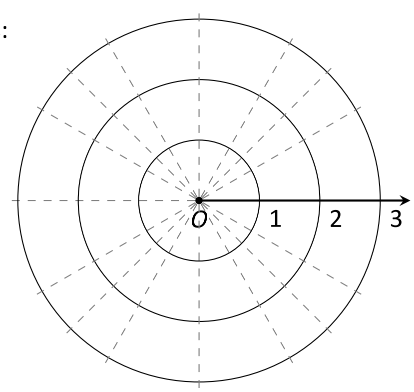
Solution To aid in the drawing, a polar grid is provided at the bottom of this page. To place the point A, go out 1 unit along the initial ray (putting you on the inner circle shown on the grid), then rotate counter-clockwise $\pi/4$ radians (or 45°). Alternately, one can consider the rotation first: think about the ray from O that forms an angle of $\pi/4$ with the initial ray, then move out 1 unit along this ray (again placing you on the inner circle of the grid).

To plot B, go out 1.5 units along the initial ray and rotate π radians (180°).

To plot C, go out 2 units along the initial ray then rotate *clockwise* $\pi/3$ radians, as the angle given is negative.

To plot D, move along the initial ray "-1" units – in other words, "back up" 1 unit, then rotate counter-clockwise by $\pi/4$. The results are given in Figure 9.37.

Notes:



Consider the following two points: $A = P(1, \pi)$ and $B = P(-1, 0)$. To locate A, go out 1 unit on the initial ray then rotate π radians; to locate B, go out -1 units on the initial ray and don't rotate. One should see that A and B are located at the same point in the plane. We can also consider $C = P(1, 3\pi)$, or $D = P(1, -\pi)$; all four of these points share the same location.

This ability to identify a point in the plane with multiple polar coordinates is both a "blessing" and a "curse." We will see that it is beneficial as we can plot beautiful functions that intersect themselves (much like we saw with parametric functions). The unfortunate part of this is that it can be difficult to determine when this happens. We'll explore this more later in this section.

Polar to Rectangular Conversion

It is useful to recognize both the rectangular (or, Cartesian) coordinates of a point in the plane and its polar coordinates. Figure 9.38 shows a point P in the plane with rectangular coordinates (x, y) and polar coordinates $P(r, \theta)$. Using trigonometry, we can make the identities given in the following Key Idea.

Key Idea 41 Converting Between Rectangular and Polar Coordinates

Given the polar point $P(r, \theta)$, the rectangular coordinates are determined by

$$x = r\cos\theta \qquad y = r\sin\theta.$$

Given the rectangular coordinates (x, y), the polar coordinates are determined by

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \text{ (for } x \neq 0).$$

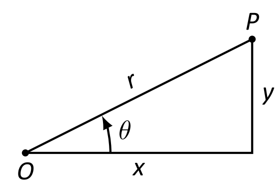


Figure 9.38: Converting between rectangular and polar coordinates.

Example 9.27 Converting Between Polar and Rectangular Coordinates

1. Convert the polar coordinates $P(2, 2\pi/3)$ and $P(-1, 5\pi/4)$ to rectangular coordinates.

2. Convert the rectangular coordinates $(1, 2)$ and $(-1, 1)$ to polar coordinates.

Solution

Notes:

1. (a) We start with $P(2, 2\pi/3)$. Using Key Idea 41, we have

$$x = 2\cos(2\pi/3) = -1 \qquad y = 2\sin(2\pi/3) = \sqrt{3}.$$

So the rectangular coordinates are $(-1, \sqrt{3}) \approx (-1, 1.732)$.

(b) The polar point $P(-1, 5\pi/4)$ is converted to rectangular with:

$$x = -1\cos(5\pi/4) = \sqrt{2}/2 \qquad y = -1\sin(5\pi/4) = \sqrt{2}/2.$$

So the rectangular coordinates are $(\sqrt{2}/2, \sqrt{2}/2) \approx (0.707, 0.707)$.

These points are plotted in Figure 9.39 (a). The rectangular coordinate system is drawn lightly under the polar coordinate system so that the relationship between the two can be seen.

2. (a) To convert the rectangular point $(1, 2)$ to polar coordinates, we use the Key Idea to form the following two equations:

$$1^2 + 2^2 = r^2 \qquad \tan\theta = \frac{2}{1}.$$

The first equation tells us that $r = \sqrt{5}$. Using the inverse tangent function, we find

$$\tan\theta = 2 \quad \Rightarrow \quad \theta = \tan^{-1} 2 \approx 1.11 \approx 63.43^\circ.$$

Thus polar coordinates of $(1, 2)$ are $P\left(\sqrt{5}, \tan^{-1} 2\right)$.

(b) To convert $(-1, 1)$ to polar coordinates, we form the equations

$$(-1)^2 + 1^2 = r^2 \qquad \tan\theta = \frac{1}{-1}.$$

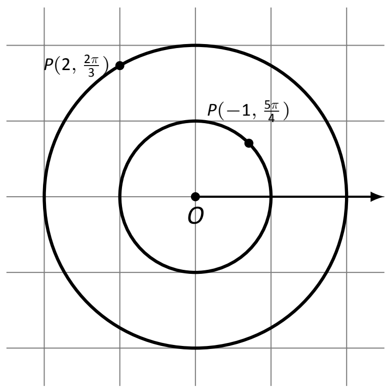
Thus $r = \sqrt{2}$. We need to be careful in computing θ: using the inverse tangent function, we have

$$\tan\theta = -1 \quad \Rightarrow \quad \theta = \tan^{-1}(-1) = -\pi/4 = -45^\circ.$$

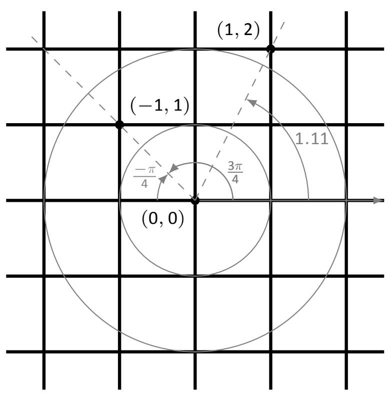
This is not the angle we desire. The range of $\tan^{-1} x$ is $(-\pi/2, \pi/2)$; that is, it returns angles that lie in the 1^{st} and 4^{th} quadrants. To find locations in the 2^{nd} and 3^{rd} quadrants, add π to the result of $\tan^{-1} x$. So $\pi + (-\pi/4)$ puts the angle at $3\pi/4$. Thus the polar point is $P\left(\sqrt{2}, 3\pi/4\right)$.

An alternate method is to use the angle θ given by arctangent, but change the sign of r. Thus we could also refer to $(-1, 1)$ as $P\left(-\sqrt{2}, -\pi/4\right)$.

These points are plotted in Figure 9.39 (b). The polar system is drawn lightly under the rectangular grid with rays to demonstrate the angles used.



(a)



(b)

Figure 9.39: Plotting rectangular and polar points in Example 9.27.

Notes:

Polar Functions and Polar Graphs

Defining a new coordinate system allows us to create a new kind of function, a **polar function.** Rectangular coordinates lent themselves well to creating functions that related x and y, such as $y = x^2$. Polar coordinates allow us to create functions that relate r and θ. Normally these functions look like $r = f(\theta)$, although we can create functions of the form $\theta = f(r)$. The following examples introduce us to this concept.

Example 9.28 Introduction to Graphing Polar Functions
Describe the graphs of the following polar functions.

1. $r = 1.5$
2. $\theta = \pi/4$

Solution

1. The equation $r = 1.5$ describes all points that are 1.5 units from the pole; as the angle is not specified, any θ is allowable. All points 1.5 units from the pole describes a circle of radius 1.5.

   We can consider the rectangular equivalent of this equation; using $r^2 = x^2 + y^2$, we see that $1.5^2 = x^2 + y^2$, which we recognize as the equation of a circle centered at $(0, 0)$ with radius 1.5. This is sketched in Figure 9.40.

2. The equation $\theta = \pi/4$ describes all points such that the line through them and the pole make an angle of $\pi/4$ with the initial ray. As the radius r is not specified, it can be any value (even negative). Thus $\theta = \pi/4$ describes the line through the pole that makes an angle of $\pi/4 = 45^\circ$ with the initial ray.

   We can again consider the rectangular equivalent of this equation. Combine $\tan\theta = y/x$ and $\theta = \pi/4$:

   $$\tan \pi/4 = y/x \quad \Rightarrow x \tan \pi/4 = y \quad \Rightarrow y = x.$$

   This graph is also plotted in Figure 9.40.

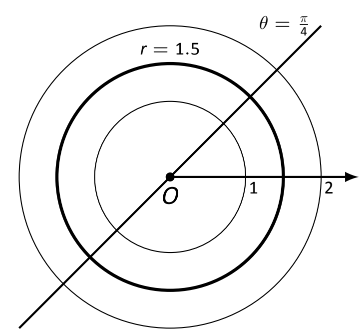


Figure 9.40: Plotting standard polar plots.

The basic rectangular equations of the form $x = h$ and $y = k$ create vertical and horizontal lines, respectively; the basic polar equations $r = h$ and $\theta = \alpha$ create circles and lines through the pole, respectively. With this as a foundation, we can create more complicated polar functions of the form $r = f(\theta)$. The input is an angle; the output is a length, how far in the direction of the angle to go out.

Notes:

We sketch these functions much like we sketch rectangular and parametric functions: we plot lots of points and "connect the dots" with curves. We demonstrate this in the following example.

Example 9.29 Sketching Polar Functions
Sketch the polar function $r = 1 + \cos\theta$ on $[0, 2\pi]$ by plotting points.

Solution A common question when sketching curves by plotting points is "Which points should I plot?" With rectangular equations, we often choose "easy" values – integers, then added more if needed. When plotting polar equations, start with the "common" angles – multiples of $\pi/6$ and $\pi/4$. Figure 9.41 gives a table of just a few values of θ in $[0, \pi]$.

θ	$r = 1 + \cos\theta$
0	2
$\pi/6$	1.86603
$\pi/2$	1
$4\pi/3$	0.5
$7\pi/4$	1.70711

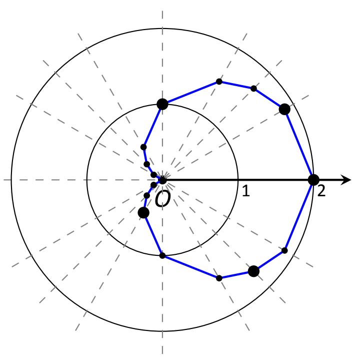


Figure 9.41: Graphing a polar function in Example 9.29 by plotting points.

Consider the point $P(0, 2)$ determined by the first line of the table. The angle is 0 radians – we do not rotate from the initial ray – then we go out 2 units from the pole. When $\theta = \pi/6$, $r = 1.866$ (actually, it is $1 + \sqrt{3}/2$); so rotate by $\pi/6$ radians and go out 1.866 units.

The graph shown uses more points, connected with straight lines. (The points on the graph that correspond to points in the table are signified with larger dots.) Such a sketch is likely good enough to give one an idea of what the graph looks like.

Technology Note: Plotting functions in this way can be tedious, just as it was with rectangular functions. To obtain very accurate graphs, technology is a great aid. Most graphing calculators can plot polar functions; in the menu, set the plotting mode to something like `polar` or `POL`, depending on one's calculator. As with plotting parametric functions, the viewing "window" no longer determines the *x*-values that are plotted, so additional information needs to be provided. Often with the "window" settings are the settings for the beginning and ending θ values (often called $\theta_{\texttt{min}}$ and $\theta_{\texttt{max}}$) as well as the $\theta_{\texttt{step}}$ – that is, how far apart the θ values are spaced. The smaller the $\theta_{\texttt{step}}$ value, the more accurate the graph (which also increases plotting time). Using technology, we graphed the polar function $r = 1 + \cos\theta$ from Example 9.29 in Figure 9.42.

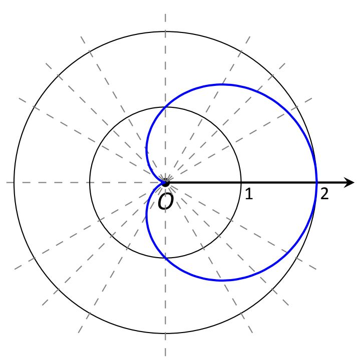


Figure 9.42: Using technology to graph a polar function.

Example 9.30 Sketching Polar Functions
Sketch the polar function $r = \cos(2\theta)$ on $[0, 2\pi]$ by plotting points.

Solution We start by making a table of $\cos(2\theta)$ evaluated at common angles θ, as shown in Figure 9.43. These points are then plotted in Figure 9.44 (a). This particular graph "moves" around quite a bit and one can easily forget which points should be connected to each other. To help us with this, we numbered each point in the table and on the graph.

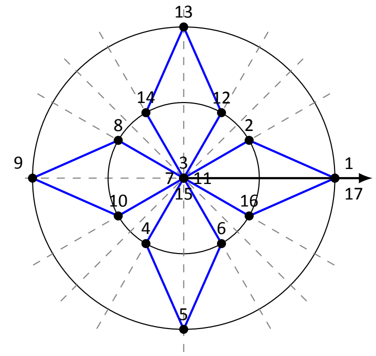
Notes:

Pt.	θ	$\cos(2\theta)$
1	0	1.
2	$\pi/6$	0.5
3	$\pi/4$	0.
4	$\pi/3$	-0.5
5	$\pi/2$	$-1.$
6	$2\pi/3$	-0.5
7	$3\pi/4$	0.
8	$5\pi/6$	0.5
9	π	1.

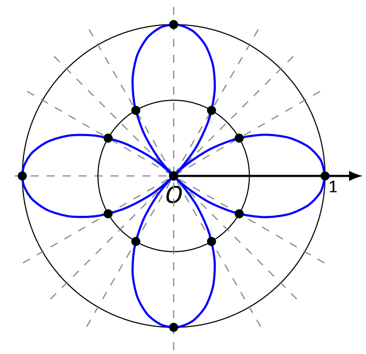
Pt.	θ	$\cos(2\theta)$
10	$7\pi/6$	0.5
11	$5\pi/4$	0.
12	$4\pi/3$	-0.5
13	$3\pi/2$	$-1.$
14	$5\pi/3$	-0.5
15	$7\pi/4$	0.
16	$11\pi/6$	0.5
17	2π	1.

Figure 9.43: Tables of points for plotting a polar curve.

Using more points (and the aid of technology) a smoother plot can be made as shown in Figure 9.44 (b). This plot is an example of a *rose curve*.



(a)



(b)

Figure 9.44: Polar plots from Example 9.30.

It is sometimes desirable to refer to a graph via a polar equation, and other times by a rectangular equation. Therefore it is necessary to be able to convert between polar and rectangular functions, which we practice in the following example. We will make frequent use of the identities found in Key Idea 41.

Example 9.31 Converting between rectangular and polar equations.

Convert from rectangular to polar.

1. $y = x^2$
2. $xy = 1$

Convert from polar to rectangular.

3. $r = \dfrac{2}{\sin\theta - \cos\theta}$
4. $r = 2\cos\theta$

Solution

1. Replace y with $r\sin\theta$ and replace x with $r\cos\theta$, giving:

$$\begin{aligned} y &= x^2 \\ r\sin\theta &= r^2\cos^2\theta \\ \frac{\sin\theta}{\cos^2\theta} &= r \end{aligned}$$

We have found that $r = \sin\theta/\cos^2\theta = \tan\theta\sec\theta$. The domain of this polar function is $(-\pi/2, \pi/2)$; plot a few points to see how the familiar parabola is traced out by the polar equation.

Notes:

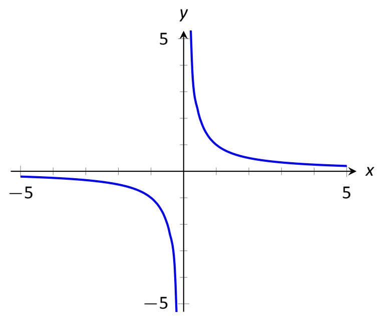


Figure 9.45: Graphing $xy = 1$ from Example 9.31.

2. We again replace x and y using the standard identities and work to solve for r:

$$\begin{aligned} xy &= 1 \\ r\cos\theta \cdot r\sin\theta &= 1 \\ r^2 &= \frac{1}{\cos\theta\sin\theta} \\ r &= \frac{1}{\sqrt{\cos\theta\sin\theta}} \end{aligned}$$

This function is valid only when the product of $\cos\theta\sin\theta$ is positive. This occurs in the first and third quadrants, meaning the domain of this polar function is $(0, \pi/2) \cup (\pi, 3\pi/2)$.

We can rewrite the original rectangular equation $xy = 1$ as $y = 1/x$. This is graphed in Figure 9.45; note how it only exists in the first and third quadrants.

3. There is no set way to convert from polar to rectangular; in general, we look to form the products $r\cos\theta$ and $r\sin\theta$, and then replace these with x and y, respectively. We start in this problem by multiplying both sides by $\sin\theta - \cos\theta$:

$$\begin{aligned} r &= \frac{2}{\sin\theta - \cos\theta} \\ r(\sin\theta - \cos\theta) &= 2 \\ r\sin\theta - r\cos\theta &= 2. \qquad \text{Now replace with } y \text{ and } x\text{:} \\ y - x &= 2 \\ y &= x + 2. \end{aligned}$$

The original polar equation, $r = 2/(\sin\theta - \cos\theta)$ does not easily reveal that its graph is simply a line. However, our conversion shows that it is. The upcoming gallery of polar curves gives the general equations of lines in polar form.

4. By multiplying both sides by r, we obtain both an r^2 term and an $r\cos\theta$ term, which we replace with $x^2 + y^2$ and x, respectively.

$$\begin{aligned} r &= 2\cos\theta \\ r^2 &= 2r\cos\theta \\ x^2 + y^2 &= 2x. \end{aligned}$$

Notes:

We recognize this as a circle; by completing the square we can find its radius and center.

$$x^2 - 2x + y^2 = 0$$
$$(x-1)^2 + y^2 = 1.$$

The circle is centered at $(1, 0)$ and has radius 1. The upcoming gallery of polar curves gives the equations of *some* circles in polar form; circles with arbitrary centers have a complicated polar equation that we do not consider here.

Some curves have very simple polar equations but rather complicated rectangular ones. For instance, the equation $r = 1 + \cos\theta$ describes a *cardioid* (a shape important the sensitivity of microphones, among other things; one is graphed in the gallery in the Limaçon section). It's rectangular form is not nearly as simple; it is the implicit equation $x^4 + y^4 + 2x^2y^2 - 2xy^2 - 2x^3 - y^2 = 0$. The conversion is not "hard," but takes several steps, and is left as a problem in the Exercise section.

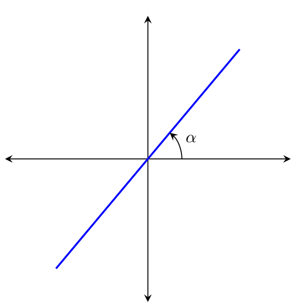
Gallery of Polar Curves

There are a number of basic and "classic" polar curves, famous for their beauty and/or applicability to the sciences. This section ends with a small gallery of some of these graphs. We encourage the reader to understand how these graphs are formed, and to investigate with technology other types of polar functions.

Lines

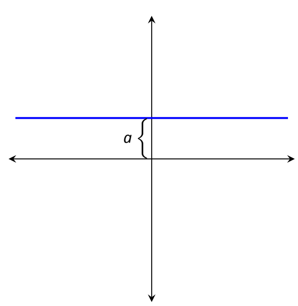
Through the origin:

$\theta = \alpha$



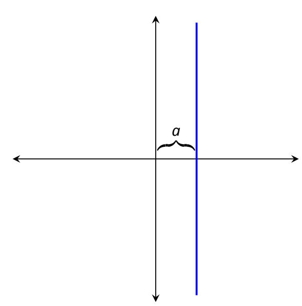
Horizontal line:

$r = a\csc\theta$



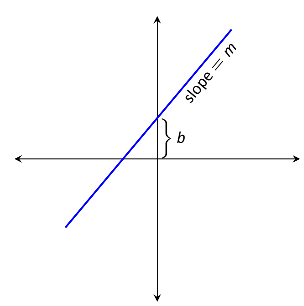
Vertical line:

$r = a\sec\theta$



Not through origin:

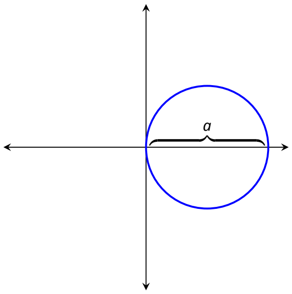
$$r = \frac{b}{\sin\theta - m\cos\theta}$$



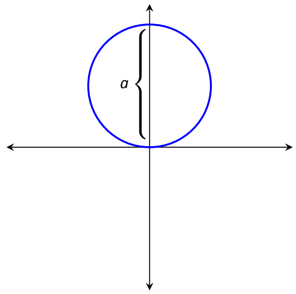
Notes:

Circles

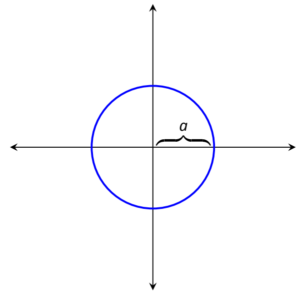
Centered on x-axis:
$r = a\cos\theta$



Centered on y-axis:
$r = a\sin\theta$

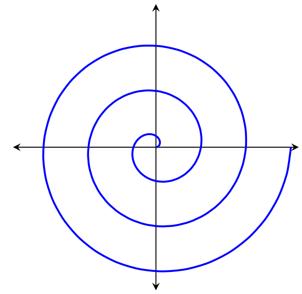


Centered on origin:
$r = a$



Spiral

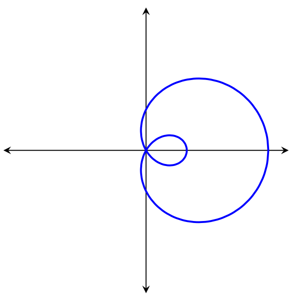
Archimedean spiral
$r = \theta$



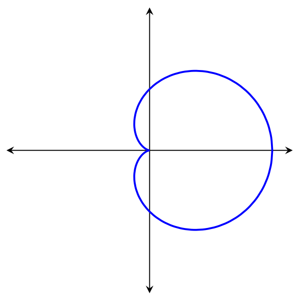
Limaçons

Symmetric about x-axis: $r = a \pm b\cos\theta$; Symmetric about y-axis: $r = a \pm b\sin\theta$; $a, b > 0$

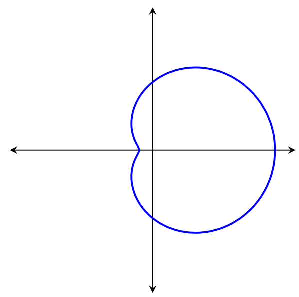
With inner loop:
$\dfrac{a}{b} < 1$



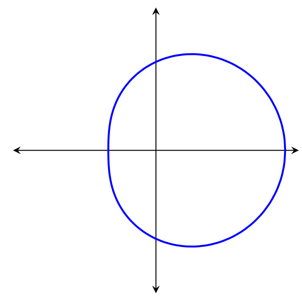
Cardioid:
$\dfrac{a}{b} = 1$



Dimpled:
$1 < \dfrac{a}{b} < 2$



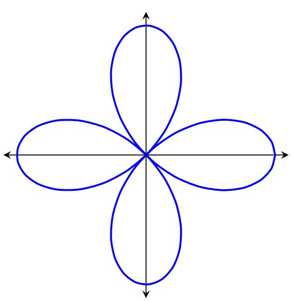
Convex:
$\dfrac{a}{b} > 2$



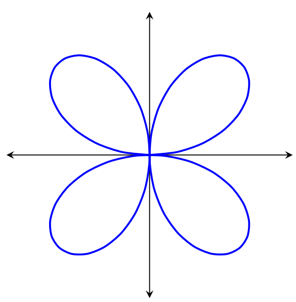
Rose Curves

Symmetric about x-axis: $r = a\cos(n\theta)$; Symmetric about y-axis: $r = a\sin(n\theta)$
Curve contains $2n$ petals when n is even and n petals when n is odd.

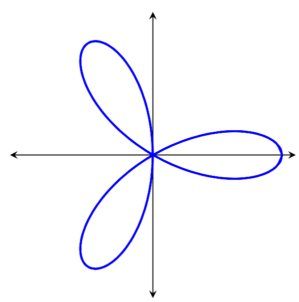
$r = a\cos(2\theta)$



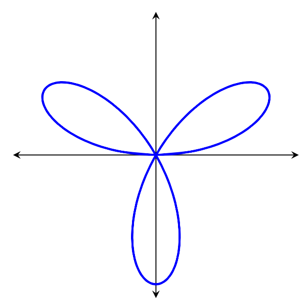
$r = a\sin(2\theta)$



$r = a\cos(3\theta)$



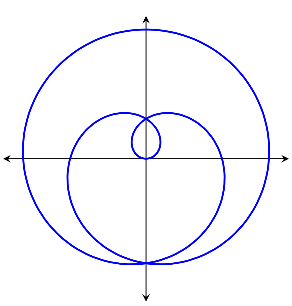
$r = a\sin(3\theta)$



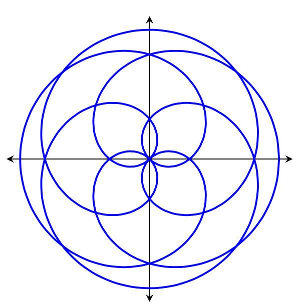
Special Curves

Rose curves

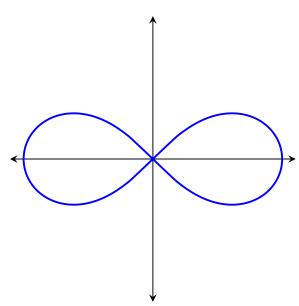
$r = a\sin(\theta/5)$



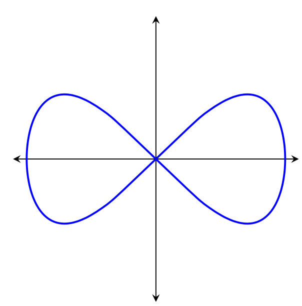
$r = a\sin(2\theta/5)$



Lemniscate:
$r^2 = a^2\cos(2\theta)$



Eight Curve:
$r^2 = a^2\sec^4\theta\cos(2\theta)$



Earlier we discussed how each point in the plane does not have a unique representation in polar form. This can be a "good" thing, as it allows for the beautiful and interesting curves seen in the preceding gallery. However, it can also be a "bad" thing, as it can be difficult to determine where two curves intersect.

Example 9.32 Finding points of intersection with polar curves

Determine where the graphs of the polar equations $r = 1+3\cos\theta$ and $r = \cos\theta$ intersect.

Solution As technology is generally readily available, it is usually a good idea to start with a graph. We have graphed the two functions in Figure 9.46(a); to better discern the intersection points, part (b) of the figure zooms in around the origin. We start by setting the two functions equal to each other and solving for θ:

$$\begin{aligned} 1+3\cos\theta &= \cos\theta \\ 2\cos\theta &= -1 \\ \cos\theta &= -\frac{1}{2} \\ \theta &= \frac{2\pi}{3}, \frac{4\pi}{3}. \end{aligned}$$

(There are, of course, infinite solutions to the equation $\cos\theta = -1/2$; as the limaçon is traced out once on $[0, 2\pi]$, we restrict our solutions to this interval.)

We need to analyze this solution. When $\theta = 2\pi/3$ we obtain the point of intersection that lies in the 4$^{\text{th}}$ quadrant. When $\theta = 4\pi/3$, we get the point of intersection that lies in the 2$^{\text{nd}}$ quadrant. There is more to say about this second intersection point, however. The circle defined by $r = \cos\theta$ is traced out once on $[0, \pi]$, meaning that this point of intersection occurs while tracing out the circle a second time. It seems strange to pass by the point once and then recognize it as a point of intersection only when arriving there a "second time." The first time the circle arrives at this point is when $\theta = \pi/3$. It is key to understand that these two points are the same: $(\cos\pi/3, \pi/3)$ and $(\cos 4\pi/3, 4\pi/3)$.

To summarize what we have done so far, we have found two points of intersection: when $\theta = 2\pi/3$ and when $\theta = 4\pi/3$. When referencing the circle $r = \cos\theta$, the latter point is better referenced as when $\theta = \pi/3$.

There is yet another point of intersection: the pole (or, the origin). We did not recognize this intersection point using our work above as each graph arrives at the pole at a different θ value.

A graph intersects the pole when $r = 0$. Considering the circle $r = \cos\theta$, $r = 0$ when $\theta = \pi/2$ (and odd multiples thereof, as the circle is repeatedly

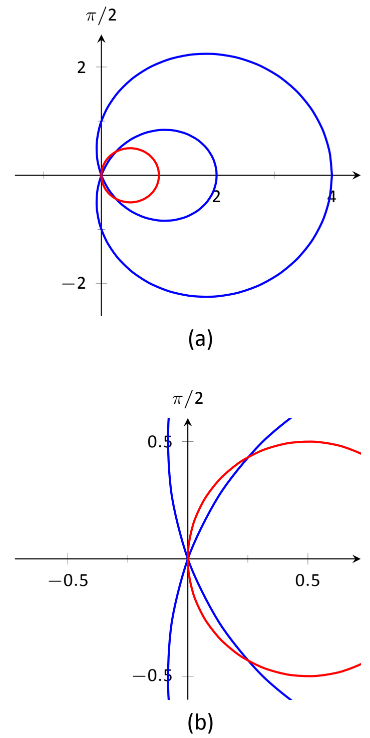


Figure 9.46: Graphs to help determine the points of intersection of the polar functions given in Example 9.32.

Notes:

traced). The limaçon intersects the pole when $1+3\cos\theta = 0$; this occurs when $\cos\theta = -1/3$, or for $\theta = \cos^{-1}(-1/3)$. This is a nonstandard angle, approximately $\theta = 1.9106 = 109.47^\circ$. The limaçon intersects the pole twice in $[0, 2\pi]$; the other angle at which the limaçon is at the pole is the reflection of the first angle across the x-axis. That is, $\theta = 4.3726 = 250.53^\circ$.

If all one is concerned with is the (x, y) coordinates at which the graphs intersect, much of the above work is extraneous. We know they intersect at $(0, 0)$; we might not care at what θ value. Likewise, using $\theta = 2\pi/3$ and $\theta = 4\pi/3$ can give us the needed rectangular coordinates. However, in the next section we apply calculus concepts to polar functions. When computing the area of a region bounded by polar curves, understanding the nuances of the points of intersection becomes important.

Notes:

Exercises 9.4

Terms and Concepts

1. In your own words, describe how to plot the polar point $P(r, \theta)$.

2. T/F: When plotting a point with polar coordinate $P(r, \theta)$, r must be positive.

3. T/F: Every point in the Cartesian plane can be represented by a polar coordinate.

4. T/F: Every point in the Cartesian plane can be represented uniquely by a polar coordinate.

Problems

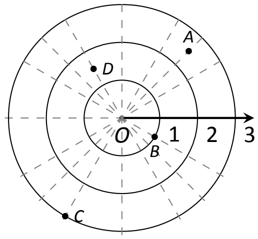
5. Plot the points with the given polar coordinates.

   (a) $A = P(2, 0)$ (b) $B = P(1, \pi)$ (c) $C = P(-2, \pi/2)$ (d) $D = P(1, \pi/4)$

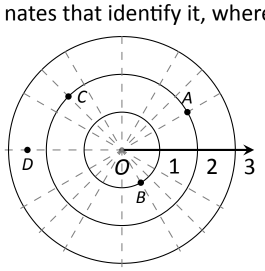
6. Plot the points with the given polar coordinates.

   (a) $A = P(2, 3\pi)$ (b) $B = P(1, -\pi)$ (c) $C = P(1, 2)$ (d) $D = P(1/2, 5\pi/6)$

7. For each of the given points give two sets of polar coordinates that identify it, where $0 \le \theta \le 2\pi$.



8. For each of the given points give two sets of polar coordinates that identify it, where $-\pi \le \theta \le \pi$.



9. Convert each of the following polar coordinates to rectangular, and each of the following rectangular coordinates to polar.

   (a) $A = P(2, \pi/4)$ (b) $B = P(2, -\pi/4)$ (c) $C = (2, -1)$ (d) $D = (-2, 1)$

10. Convert each of the following polar coordinates to rectangular, and each of the following rectangular coordinates to polar.

   (a) $A = P(3, \pi)$ (b) $B = P(1, 2\pi/3)$ (c) $C = (0, 4)$ (d) $D = (1, -\sqrt{3})$

In Exercises 11 – 29, graph the polar function on the given interval.

11. $r = 2, \quad 0 \le \theta \le \pi/2$

12. $\theta = \pi/6, \quad -1 \le r \le 2$

13. $r = 1 - \cos\theta, \quad [0, 2\pi]$

14. $r = 2 + \sin\theta, \quad [0, 2\pi]$

15. $r = 2 - \sin\theta, \quad [0, 2\pi]$

16. $r = 1 - 2\sin\theta, \quad [0, 2\pi]$

17. $r = 1 + 2\sin\theta, \quad [0, 2\pi]$

18. $r = \cos(2\theta), \quad [0, 2\pi]$

19. $r = \sin(3\theta), \quad [0, \pi]$

20. $r = \cos(\theta/3), \quad [0, 3\pi]$

21. $r = \cos(2\theta/3), \quad [0, 6\pi]$

22. $r = \theta/2, \quad [0, 4\pi]$

23. $r = 3\sin(\theta), \quad [0, \pi]$

24. $r = \cos\theta \sin\theta, \quad [0, 2\pi]$

25. $r = \theta^2 - (\pi/2)^2, \quad [-\pi, \pi]$

26. $r = \dfrac{3}{5\sin\theta - \cos\theta}, \quad [0, 2\pi]$

27. $r = \dfrac{-2}{3\cos\theta - 2\sin\theta}, \quad [0, 2\pi]$

28. $r = 3\sec\theta, \quad (-\pi/2, \pi/2)$

29. $r = 3\csc\theta, \quad (0, \pi)$

In Exercises 30 – 38, convert the polar equation to a rectangular equation.

30. $r = 2\cos\theta$

31. $r = -4\sin\theta$

32. $r = \cos\theta + \sin\theta$

33. $r = \dfrac{7}{5\sin\theta - 2\cos\theta}$

34. $r = \dfrac{3}{\cos\theta}$

35. $r = \dfrac{4}{\sin\theta}$

36. $r = \tan\theta$

37. $r = 2$

38. $\theta = \pi/6$

In Exercises 39 – 46, convert the rectangular equation to a polar equation.

39. $y = x$

40. $y = 4x + 7$

41. $x = 5$

42. $y = 5$

43. $x = y^2$

44. $x^2y = 1$

45. $x^2 + y^2 = 7$

46. $(x+1)^2 + y^2 = 1$

In Exercises 47 – 54, find the points of intersection of the polar graphs.

47. $r = \sin(2\theta)$ and $r = \cos\theta$ on $[0, \pi]$

48. $r = \cos(2\theta)$ and $r = \cos\theta$ on $[0, \pi]$

49. $r = 2\cos\theta$ and $r = 2\sin\theta$ on $[0, \pi]$

50. $r = \sin\theta$ and $r = \sqrt{3} + 3\sin\theta$ on $[0, 2\pi]$

51. $r = \sin(3\theta)$ and $r = \cos(3\theta)$ on $[0, \pi]$

52. $r = 3\cos\theta$ and $r = 1 + \cos\theta$ on $[-\pi, \pi]$

53. $r = 1$ and $r = 2\sin(2\theta)$ on $[0, 2\pi]$

54. $r = 1 - \cos\theta$ and $r = 1 + \sin\theta$ on $[0, 2\pi]$

55. Pick a integer value for n, where $n \neq 2, 3$, and use technology to plot $r = \sin\left(\frac{m}{n}\theta\right)$ for three different integer values of m. Sketch these and determine a minimal interval on which the entire graph is shown.

56. Create your own polar function, $r = f(\theta)$ and sketch it. Describe why the graph looks as it does.

9.5 Calculus and Polar Functions

The previous section defined polar coordinates, leading to polar functions. We investigated plotting these functions and solving a fundamental question about their graphs, namely, where do two polar graphs intersect?

We now turn our attention to answering other questions, whose solutions require the use of calculus. A basis for much of what is done in this section is the ability to turn a polar function $r = f(\theta)$ into a set of parametric equations. Using the identities $x = r\cos\theta$ and $y = r\sin\theta$, we can create the parametric equations $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$ and apply the concepts of Section 9.3.

Polar Functions and $\dfrac{dy}{dx}$

We are interested in the lines tangent a given graph, regardless of whether that graph is produced by rectangular, parametric, or polar equations. In each of these contexts, the slope of the tangent line is $\frac{dy}{dx}$. Given $r = f(\theta)$, we are generally *not* concerned with $r' = f'(\theta)$; that describes how fast r changes with respect to θ. Instead, we will use $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$ to compute $\frac{dy}{dx}$.

Using Key Idea 38 we have

$$\frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta}.$$

Each of the two derivatives on the right hand side of the equality requires the use of the Product Rule. We state the important result as a Key Idea.

Key Idea 42 Finding $\frac{dy}{dx}$ with Polar Functions

Let $r = f(\theta)$ be a polar function. With $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$,

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$$

Example 9.33 Finding $\frac{dy}{dx}$ with polar functions.

Consider the limaçon $r = 1 + 2\sin\theta$ on $[0, 2\pi]$.

1. Find the equations of the tangent and normal lines to the graph at $\theta = \pi/4$.

2. Find where the graph has vertical and horizontal tangent lines.

Notes:

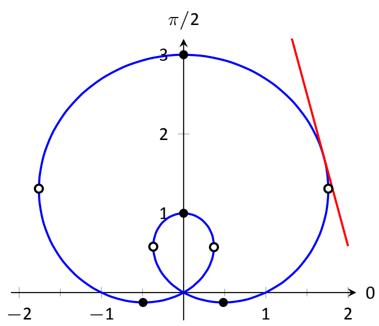


Figure 9.47: The limaçon in Example 9.33 with its tangent line at $\theta = \pi/4$ and points of vertical and horizontal tangency.

Solution

1. We start by computing $\frac{dy}{dx}$. With $f'(\theta) = 2\cos\theta$, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{2\cos\theta\sin\theta + \cos\theta(1+2\sin\theta)}{2\cos^2\theta - \sin\theta(1+2\sin\theta)}\\ &= \frac{\cos\theta(4\sin\theta+1)}{2(\cos^2\theta - \sin^2\theta) - \sin\theta}.\end{aligned}$$

When $\theta = \pi/4$, $\frac{dy}{dx} = -2\sqrt{2} - 1$ (this requires a bit of simplification). In rectangular coordinates, the point on the graph at $\theta = \pi/4$ is $(1+\sqrt{2}/2, 1+\sqrt{2}/2)$. Thus the rectangular equation of the line tangent to the limaçon at $\theta = \pi/4$ is

$$y = (-2\sqrt{2}-1)\big(x - (1+\sqrt{2}/2)\big) + 1 + \sqrt{2}/2 \approx -3.83x + 8.24.$$

The limaçon and the tangent line are graphed in Figure 9.47.

The normal line has the opposite–reciprocal slope as the tangent line, so its equation is

$$y \approx \frac{1}{3.83}x + 1.26.$$

2. To find the horizontal lines of tangency, we find where $\frac{dy}{dx} = 0$; thus we find where the numerator of our equation for $\frac{dy}{dx}$ is 0.

$$\cos\theta(4\sin\theta + 1) = 0 \quad \Rightarrow \quad \cos\theta = 0 \quad \text{or} \quad 4\sin\theta + 1 = 0.$$

On $[0, 2\pi]$, $\cos\theta = 0$ when $\theta = \pi/2,\ 3\pi/2$.

Setting $4\sin\theta + 1 = 0$ gives $\theta = \sin^{-1}(-1/4) \approx -0.2527 = -14.48^\circ$. We want the results in $[0, 2\pi]$; we also recognize there are two solutions, one in the 3rd quadrant and one in the 4th. Using reference angles, we have our two solutions as $\theta = 3.39$ and 6.03 radians. The four points we obtained where the limaçon has a horizontal tangent line are given in Figure 9.47 with black–filled dots.

To find the vertical lines of tangency, we set the denominator of $\frac{dy}{dx} = 0$.

$$2(\cos^2\theta - \sin^2\theta) - \sin\theta = 0.$$

Convert the $\cos^2\theta$ term to $1 - \sin^2\theta$:

$$\begin{aligned}2(1 - \sin^2\theta - \sin^2\theta) - \sin\theta &= 0\\ 4\sin^2\theta + \sin\theta - 1 &= 0.\end{aligned}$$

Notes:

Recognize this as a quadratic in the variable $\sin\theta$. Using the quadratic formula, we have

$$\sin\theta = \frac{-1 \pm \sqrt{33}}{8}.$$

We solve $\sin\theta = \frac{-1+\sqrt{33}}{8}$ and $\sin\theta = \frac{-1-\sqrt{33}}{8}$:

$$\begin{aligned} \sin\theta &= \frac{-1+\sqrt{33}}{8} \\ \theta &= \sin^{-1}\left(\frac{-1+\sqrt{33}}{8}\right) \\ \theta &= 0.6399 \end{aligned} \qquad \begin{aligned} \sin\theta &= \frac{-1-\sqrt{33}}{8} \\ \theta &= \sin^{-1}\left(\frac{-1-\sqrt{33}}{8}\right) \\ \theta &= -1.0030 \end{aligned}$$

In each of the solutions above, we only get one of the possible two solutions as $\sin^{-1} x$ only returns solutions in $[-\pi/2, \pi/2]$, the 4th and 1st quadrants. Again using reference angles, we have:

$$\sin\theta = \frac{-1+\sqrt{33}}{8} \quad \Rightarrow \quad \theta = 0.6399,\ 3.7815 \text{ radians}$$

and

$$\sin\theta = \frac{-1-\sqrt{33}}{8} \quad \Rightarrow \quad \theta = 4.1446,\ 5.2802 \text{ radians.}$$

These points are also shown in Figure 9.47 with white–filled dots.

When the graph of the polar function $r = f(\theta)$ intersects the pole, it means that $f(\alpha) = 0$ for some angle α. Thus the formula for $\frac{dy}{dx}$ in such instances is very simple, reducing simply to

$$\frac{dy}{dx} = \tan\alpha.$$

This equation makes an interesting point. It tells us the slope of the tangent line at the pole is $\tan\alpha$; some of our previous work (see, for instance, Example 9.28) shows us that the line through the pole with slope $\tan\alpha$ has polar equation $\theta = \alpha$. Thus when a polar graph touches the pole at $\theta = \alpha$, the equation of the tangent line at the pole is $\theta = \alpha$.

Example 9.34 Finding tangent lines at the pole.

Let $r = 1 + 2\sin\theta$, a limaçon. Find the equations of the lines tangent to the graph at the pole.

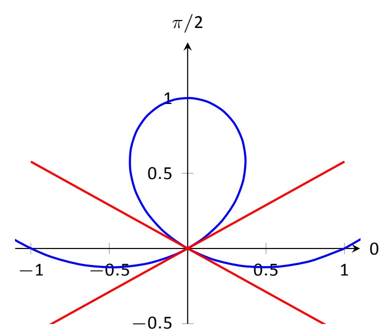
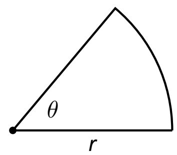


Figure 9.48: Graphing the tangent lines at the pole in Example 9.34.

Notes:

Note: Recall that the area of a sector of a circle with radius r subtended by an angle θ is $A = \frac{1}{2}\theta r^2$.



SOLUTION We need to know when $r = 0$.

$$\begin{aligned} 1 + 2\sin\theta &= 0 \\ \sin\theta &= -1/2 \\ \theta &= \frac{7\pi}{6},\ \frac{11\pi}{6}. \end{aligned}$$

Thus the equations of the tangent lines, in polar, are $\theta = 7\pi/6$ and $\theta = 11\pi/6$. In rectangular form, the tangent lines are $y = \tan(7\pi/6)x$ and $y = \tan(11\pi/6)x$. The full limaçon can be seen in Figure 9.47; we zoom in on the tangent lines in Figure 9.48.

Area

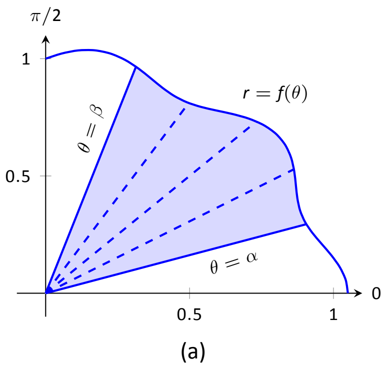
When using rectangular coordinates, the equations $x = h$ and $y = k$ defined vertical and horzontal lines, respectively, and combinations of these lines create rectangles (hence the name "rectangular coordinates"). It is then somewhat natural to use rectangles to approximate area as we did when learning about the definite integral.

When using polar coordinates, the equations $\theta = \alpha$ and $r = c$ form lines through the origin and circles centered at the origin, respectively, and combinations of these curves form sectors of circles. It is then somewhat natural to calculate the area of regions defined by polar functions by first approximating with sectors of circles.

Consider Figure 9.49 (a) where a region defined by $r = f(\theta)$ on $[\alpha, \beta]$ is given. (Note how the "sides" of the region are the lines $\theta = \alpha$ and $\theta = \beta$, whereas in rectangular coordinates the "sides" of regions were often the vertical lines $x = a$ and $x = b$.)

Partition the interval $[\alpha, \beta]$ into n equally spaced subintervals as $\alpha = \theta_1 < \theta_2 < \cdots < \theta_{n+1} = \beta$. The length of each subinterval is $\Delta\theta = (\beta - \alpha)/n$, representing a small change in angle. The area of the region defined by the i^{th} subinterval $[\theta_i, \theta_{i+1}]$ can be approximated with a sector of a circle with radius $f(c_i)$, for some c_i in $[\theta_i, \theta_{i+1}]$. The area of this sector is $\frac{1}{2}f(c_i)^2\Delta\theta$. This is shown in part (b) of the figure, where $[\alpha, \beta]$ has been divided into 4 subintervals. We approximate the area of the whole region by summing the areas of all sectors:

$$\text{Area} \approx \sum_{i=1}^{n} \frac{1}{2} f(c_i)^2 \Delta\theta.$$



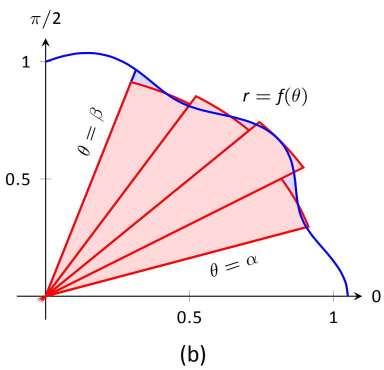


Figure 9.49: Computing the area of a polar region.

This is a Riemann sum. By taking the limit of the sum as $n \to \infty$, we find the

Notes:

exact area of the region in the form of a definite integral.

Theorem 85 Area of a Polar Region

Let f be continuous on $[\alpha, \beta]$, for which the region bounded by $r = f(\theta)$ and the lines $\theta = \alpha$ and $\theta = \beta$ does not overlap itself. The area A of the region bounded by the curve $r = f(\theta)$ and the lines $\theta = \alpha$ and $\theta = \beta$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} f(\theta)^2\,d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta$$

Note: Example 9.35 requires the use of the integral $\int \cos^2\theta\,d\theta$. This is handled well by using the power reducing formula as found in the back of this text. Due to the nature of the area formula, integrating $\cos^2\theta$ and $\sin^2\theta$ is required often. We offer here these indefinite integrals as a time–saving measure.

$$\int \cos^2\theta\,d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C$$
$$\int \sin^2\theta\,d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin(2\theta) + C$$

If the region overlapped itself, the area would be counted multiple times, which would not give the true area.

Example 9.35 Area of a polar region

Find the area of the circle defined by $r = \cos\theta$. (Recall this circle has radius $1/2$.)

SOLUTION This is a direct application of Theorem 85. The circle is traced out exactly once on $[0, \pi]$, leading to the integral

$$\begin{aligned}\text{Area} &= \frac{1}{2}\int_0^{\pi} \cos^2\theta\,d\theta\\ &= \frac{1}{2}\int_0^{\pi} \frac{1+\cos(2\theta)}{2}\,d\theta\\ &= \frac{1}{4}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Bigg|_0^{\pi}\\ &= \frac{1}{4}\pi.\end{aligned}$$

Of course, we already knew the area of a circle with radius $1/2$. We did this example to demonstrate that the area formula is correct. The interval $[0, \pi]$ is very important, as choosing $[0, 2\pi]$ would give the circle traced out twice (thus double the area).

Example 9.36 Area of a polar region

Find the area of the cardioid $r = 1+\cos\theta$ bound between $\theta = \pi/6$ and $\theta = \pi/3$, as shown in Figure 9.50.

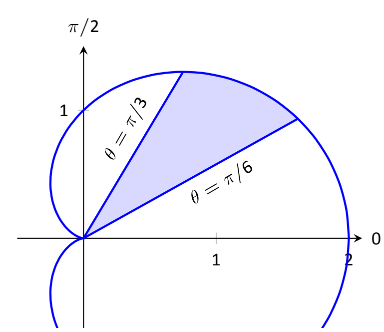


Figure 9.50: Finding the area of the shaded region of a cardioid in Example 9.36.

Notes:

Solution This is again a direct application of Theorem 85.

$$\begin{aligned}\text{Area} &= \frac{1}{2}\int_{\pi/6}^{\pi/3} (1+\cos\theta)^2\, d\theta\\ &= \frac{1}{2}\int_{\pi/6}^{\pi/3} (1+2\cos\theta+\cos^2\theta)\, d\theta\\ &= \frac{1}{2}\left(\theta+2\sin\theta+\frac{1}{2}\theta+\frac{1}{4}\sin(2\theta)\right)\Bigg|_{\pi/6}^{\pi/3}\\ &= \frac{1}{8}\big(\pi+4\sqrt{3}-4\big)\approx 0.7587.\end{aligned}$$

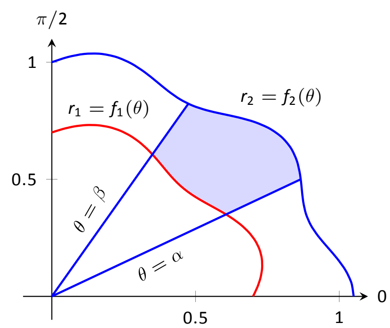


Figure 9.51: Illustrating area bound between two polar curves.

Area Between Curves

Our study of area in the context of rectangular functions led naturally to finding area bounded between curves. We consider the same in the context of polar functions.

Consider the shaded region shown in Figure 9.51. We can find the area of this region by computing the area bounded by $r_2 = f_2(\theta)$ and subtracting the area bounded by $r_1 = f_1(\theta)$ on $[\alpha,\beta]$. Thus

$$\text{Area} = \frac{1}{2}\int_\alpha^\beta r_2^2\, d\theta - \frac{1}{2}\int_\alpha^\beta r_1^2\, d\theta = \frac{1}{2}\int_\alpha^\beta \left(r_2^2 - r_1^2\right) d\theta.$$

Key Idea 43 Area Between Polar Curves

The area A of the non-overlapping region bounded by $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$, $\theta=\alpha$ and $\theta=\beta$, where $f_1(\theta)\le f_2(\theta)$ on $[\alpha,\beta]$, is

$$A = \frac{1}{2}\int_\alpha^\beta \left(r_2^2 - r_1^2\right) d\theta.$$

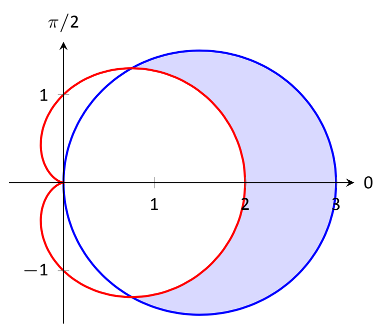


Figure 9.52: Finding the area between polar curves in Example 9.37.

Example 9.37 Area between polar curves

Find the area bounded between the curves $r = 1+\cos\theta$ and $r = 3\cos\theta$, as shown in Figure 9.52.

Solution We need to find the points of intersection between these

Notes:

two functions. Setting them equal to each other, we find:

$$\begin{aligned} 1 + \cos\theta &= 3\cos\theta \\ \cos\theta &= 1/2 \\ \theta &= \pm\pi/3 \end{aligned}$$

Thus we integrate $\frac{1}{2}\left((3\cos\theta)^2 - (1+\cos\theta)^2\right)$ on $[-\pi/3, \pi/3]$.

$$\begin{aligned} \text{Area} &= \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left((3\cos\theta)^2 - (1+\cos\theta)^2\right) d\theta \\ &= \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left(8\cos^2\theta - 2\cos\theta - 1\right) d\theta \\ &= \frac{1}{2}\big(2\sin(2\theta) - 2\sin\theta + 3\theta\big)\Big|_{-\pi/3}^{\pi/3} \\ &= \pi. \end{aligned}$$

Amazingly enough, the area between these curves has a “nice” value.

Example 9.38 Area defined by polar curves

Find the area bounded between the polar curves $r = 1$ and $r = 2\cos(2\theta)$, as shown in Figure 9.53 (a).

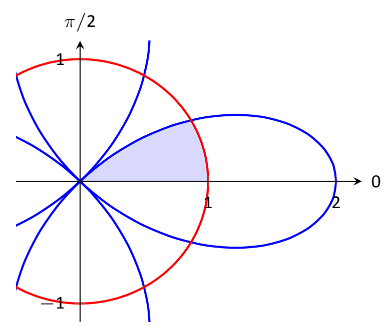
Solution We need to find the point of intersection between the two curves. Setting the two functions equal to each other, we have

$$2\cos(2\theta) = 1 \quad\Rightarrow\quad \cos(2\theta) = \frac{1}{2} \quad\Rightarrow\quad 2\theta = \pi/3 \quad\Rightarrow\quad \theta = \pi/6.$$

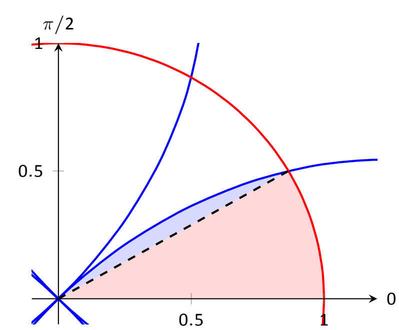
In part (b) of the figure, we zoom in on the region and note that it is not really bounded *between* two polar curves, but rather *by* two polar curves, along with $\theta = 0$. The dashed line breaks the region into its component parts. Below the dashed line, the region is defined by $r = 1$, $\theta = 0$ and $\theta = \pi/6$. (Note: the dashed line lies on the line $\theta = \pi/6$.) Above the dashed line the region is bounded by $r = 2\cos(2\theta)$ and $\theta = \pi/6$. Since we have two separate regions, we find the area using two separate integrals.

Call the area below the dashed line A_1 and the area above the dashed line A_2. They are determined by the following integrals:

$$A_1 = \frac{1}{2}\int_0^{\pi/6} (1)^2\, d\theta \qquad A_2 = \frac{1}{2}\int_{\pi/6}^{\pi/4} \big(2\cos(2\theta)\big)^2\, d\theta.$$



(a)



(b)

Figure 9.53: Graphing the region bounded by the functions in Example 9.38.

Notes:

(The upper bound of the integral computing A_2 is $\pi/4$ as $r = 2\cos(2\theta)$ is at the pole when $\theta = \pi/4$.)

We omit the integration details and let the reader verify that $A_1 = \pi/12$ and $A_2 = \pi/12 - \sqrt{3}/8$; the total area is $A = \pi/6 - \sqrt{3}/8$.

Arc Length

As we have already considered the arc length of curves defined by rectangular and parametric equations, we now consider it in the context of polar equations. Recall that the arc length L of the graph defined by the parametric equations $x = f(t)$, $y = g(t)$ on $[a, b]$ is

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\,dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt. \tag{9.1}$$

Now consider the polar function $r = f(\theta)$. We again use the identities $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$ to create parametric equations based on the polar function. We compute $x'(\theta)$ and $y'(\theta)$ as done before when computing $\frac{dy}{dx}$, then apply Equation (9.1).

The expression $x'(\theta)^2 + y'(\theta)^2$ can be simplified a great deal; we leave this as an exercise and state that

$$x'(\theta)^2 + y'(\theta)^2 = f'(\theta)^2 + f(\theta)^2.$$

This leads us to the arc length formula.

Key Idea 44 Arc Length of Polar Curves

Let $r = f(\theta)$ be a polar function with f' continuous on an open interval I containing $[\alpha, \beta]$, on which the graph traces itself only once. The arc length L of the graph on $[\alpha, \beta]$ is

$$L = \int_\alpha^\beta \sqrt{f'(\theta)^2 + f(\theta)^2}\,d\theta = \int_\alpha^\beta \sqrt{(r')^2 + r^2}\,d\theta.$$

Example 9.39 Arc length of a limaçon

Find the arc length of the limaçon $r = 1 + 2\sin\theta$.

Solution With $r = 1 + 2\sin\theta$, we have $r' = 2\cos\theta$. The limaçon is traced out once on $[0, 2\pi]$, giving us our bounds of integration. Applying Key

Notes:

Idea 44, we have

$$
\begin{aligned}
L &= \int_0^{2\pi} \sqrt{(2\cos\theta)^2 + (1+2\sin\theta)^2}\,d\theta \\
&= \int_0^{2\pi} \sqrt{4\cos^2\theta + 4\sin^2\theta + 4\sin\theta + 1}\,d\theta \\
&= \int_0^{2\pi} \sqrt{4\sin\theta + 5}\,d\theta \\
&\approx 13.3649.
\end{aligned}
$$

The final integral cannot be solved in terms of elementary functions, so we resorted to a numerical approximation. (Simpson's Rule, with $n=4$, approximates the value with 13.0608. Using $n=22$ gives the value above, which is accurate to 4 places after the decimal.)

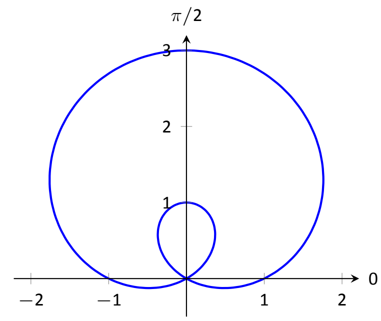


Figure 9.54: The limaçon in Example 9.39 whose arc length is measured.

Surface Area

The formula for arc length leads us to a formula for surface area. The following Key Idea is based on Key Idea 40.

Key Idea 45 Surface Area of a Solid of Revolution

Consider the graph of the polar equation $r = f(\theta)$, where f' is continuous on an open interval containing $[\alpha, \beta]$ on which the graph does not cross itself.

1. The surface area of the solid formed by revolving the graph about the initial ray ($\theta = 0$) is:

$$\text{Surface Area} = 2\pi\int_\alpha^\beta f(\theta)\sin\theta\sqrt{f'(\theta)^2 + f(\theta)^2}\,d\theta.$$

2. The surface area of the solid formed by revolving the graph about the line $\theta = \pi/2$ is:

$$\text{Surface Area} = 2\pi\int_\alpha^\beta f(\theta)\cos\theta\sqrt{f'(\theta)^2 + f(\theta)^2}\,d\theta.$$

Notes:

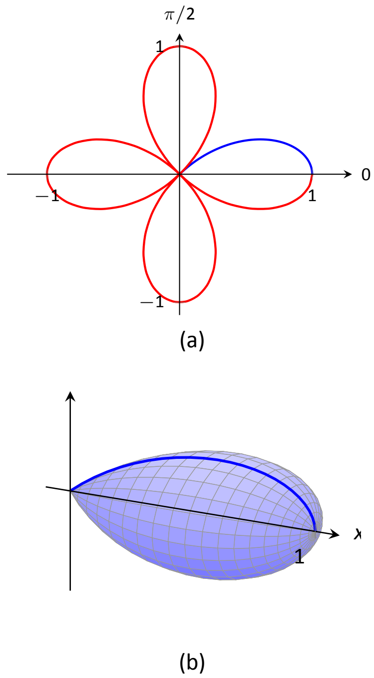


Figure 9.55: Finding the surface area of a rose–curve petal that is revolved around its central axis.

Example 9.40 Surface area determined by a polar curve

Find the surface area formed by revolving one petal of the rose curve $r = \cos(2\theta)$ about its central axis (see Figure 9.55).

Solution We choose, as implied by the figure, to revolve the portion of the curve that lies on $[0, \pi/4]$ about the initial ray. Using Key Idea 45 and the fact that $f'(\theta) = -2\sin(2\theta)$, we have

$$\begin{aligned}\text{Surface Area} &= 2\pi\int_0^{\pi/4}\cos(2\theta)\sin(\theta)\sqrt{\big(-2\sin(2\theta)\big)^2 + \big(\cos(2\theta)\big)^2}\,d\theta\\ &\approx 1.36707.\end{aligned}$$

The integral is another that cannot be evaluated in terms of elementary functions. Simpson's Rule, with $n = 4$, approximates the value at 1.36751.

This chapter has been about curves in the plane. While there is great mathematics to be discovered in the two dimensions of a plane, we live in a three dimensional world and hence we should also look to do mathematics in 3D – that is, in *space*. The next chapter begins our exploration into space by introducing the topic of *vectors*, which are incredibly useful and powerful mathematical objects.

Notes:

Exercises 9.5

Terms and Concepts

1. Given polar equation $r = f(\theta)$, how can one create parametric equations of the same curve?

2. With rectangular coordinates, it is natural to approximate area with ________; with polar coordinates, it is natural to approximate area with ________.

Problems

In Exercises 3 – 10, find:

(a) $\dfrac{dy}{dx}$

(b) the equation of the tangent and normal lines to the curve at the indicated θ–value.

3. $r = 1; \quad \theta = \pi/4$

4. $r = \cos\theta; \quad \theta = \pi/4$

5. $r = 1 + \sin\theta; \quad \theta = \pi/6$

6. $r = 1 - 3\cos\theta; \quad \theta = 3\pi/4$

7. $r = \theta; \quad \theta = \pi/2$

8. $r = \cos(3\theta); \quad \theta = \pi/6$

9. $r = \sin(4\theta); \quad \theta = \pi/3$

10. $r = \dfrac{1}{\sin\theta - \cos\theta}; \quad \theta = \pi$

In Exercises 11 – 14, find the values of θ in the given interval where the graph of the polar function has horizontal and vertical tangent lines.

11. $r = 3; \quad [0, 2\pi]$

12. $r = 2\sin\theta; \quad [0, \pi]$

13. $r = \cos(2\theta); \quad [0, 2\pi]$

14. $r = 1 + \cos\theta; \quad [0, 2\pi]$

In Exercises 15 – 16, find the equation of the lines tangent to the graph at the pole.

15. $r = \sin\theta; \quad [0, \pi]$

16. $r = \sin(3\theta); \quad [0, \pi]$

In Exercises 17 – 27, find the area of the described region.

17. Enclosed by the circle: $r = 4\sin\theta$

18. Enclosed by the circle $r = 5$

19. Enclosed by one petal of $r = \sin(3\theta)$

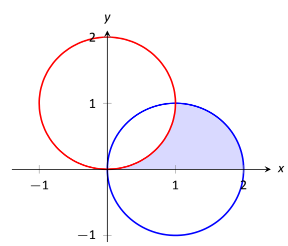
20. Enclosed by the cardioid $r = 1 - \sin\theta$

21. Enclosed by the inner loop of the limaçon $r = 1 + 2\cos\theta$

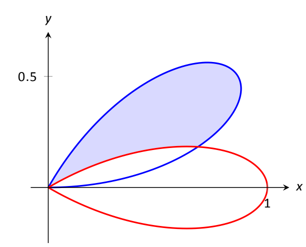
22. Enclosed by the outer loop of the limaçon $r = 1 + 2\cos\theta$ (including area enclosed by the inner loop)

23. Enclosed between the inner and outer loop of the limaçon $r = 1 + 2\cos\theta$

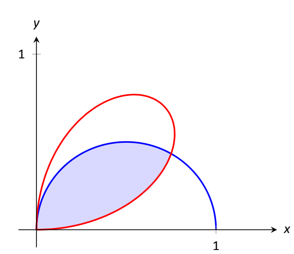
24. Enclosed by $r = 2\cos\theta$ and $r = 2\sin\theta$, as shown:



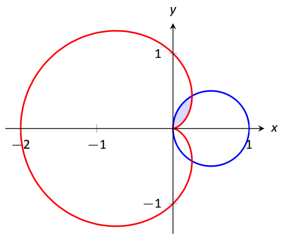
25. Enclosed by $r = \cos(3\theta)$ and $r = \sin(3\theta)$, as shown:



26. Enclosed by $r = \cos\theta$ and $r = \sin(2\theta)$, as shown:



27. Enclosed by $r = \cos\theta$ and $r = 1 - \cos\theta$, as shown:



In Exercises 28 – 32, answer the questions involving arc length.

28. Let $x(\theta) = f(\theta)\cos\theta$ and $y(\theta) = f(\theta)\sin\theta$. Show, as suggested by the text, that

$$x'(\theta)^2 + y'(\theta)^2 = f'(\theta)^2 + f(\theta)^2.$$

29. Use the arc length formula to compute the arc length of the circle $r = 2$.

30. Use the arc length formula to compute the arc length of the circle $r = 4\sin\theta$.

31. Approximate the arc length of one petal of the rose curve $r = \sin(3\theta)$ with Simpson's Rule and $n = 4$.

32. Approximate the arc length of the cardiod $r = 1 + \cos\theta$ with Simpson's Rule and $n = 6$.

In Exercises 33 – 37, answer the questions involving surface area.

33. Use Key Idea 45 to find the surface area of the sphere formed by revolving the circle $r = 2$ about the initial ray.

34. Use Key Idea 45 to find the surface area of the sphere formed by revolving the circle $r = 2\cos\theta$ about the initial ray.

35. Find the surface area of the solid formed by revolving the cardiod $r = 1 + \cos\theta$ about the initial ray.

36. Find the surface area of the solid formed by revolving the circle $r = 2\cos\theta$ about the line $\theta = \pi/2$.

37. Find the surface area of the solid formed by revolving the line $r = 3\sec\theta$, $-\pi/4 \le \theta \le \pi/4$, about the line $\theta = \pi/2$.

A: Solutions To Selected Problems

Chapter 6

Section 6.1

1. T
3. Determining which functions in the integrand to set equal to "u" and which to set equal to "dv".
5. $-e^{-x} - xe^{-x} + C$
7. $-x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$
9. $x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + C$
11. $-\frac{1}{2}xe^{-2x} - \frac{e^{-2x}}{4} + C$
13. $\frac{1}{5}e^{2x}(\sin x + 2\cos x) + C$
15. $\frac{1}{13}e^{2x}(2\sin(3x) - 3\cos(3x)) + C$
17. $-\frac{1}{2}\cos^2 x + C$
19. $x\tan^{-1}(2x) - \frac{1}{4}\ln\left(4x^2+1\right) + C$
21. $x\cos^{-1}x - \sqrt{1-x^2} + C$
23. $-\frac{x^2}{4} + \frac{1}{2}x^2 \ln x + 2x - 2x\ln x + C$
25. $\frac{1}{2}x^2 \ln\left(x^2\right) - \frac{x^2}{2} + C$
27. $2\sqrt{x}\ln x - 4\sqrt{x} + C$
29. $2x + (x+1)\left(\ln(x+1)\right)^2 - (2x+2)\ln(x+1) + C$
31. $\ln|\sin(x)| - x\cot(x) + C$
33. $\frac{1}{3}(x^2-2)^{3/2} + C$
35. $x\sec x - \ln|\sec x + \tan x| + C$
37. $\dfrac{x^{n+1}\ln x}{n+1} - \dfrac{x^{n+1}}{(n+1)^2} + C$
39. $2\sin\left(\sqrt{x}\right) - 2\sqrt{x}\cos\left(\sqrt{x}\right) + C$
41. $2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$
43. π
45. 0
47. $1/2$
49. $\dfrac{98}{\ln 7} - \dfrac{48}{(\ln 7)^2}$
51. $\frac{1}{2} + \frac{e^\pi}{2}$
53. $xe^x \ln x - e^x + C$
55. $\sin x - x\cos x + 9$
57. $\frac{1}{3}x^3 \ln x - \frac{x^3}{9} + \frac{7e^3}{9}$

Section 6.2

1. F
3. F
5. $\frac{1}{4}\sin^4(x) + C$
7. $\frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}sin(4x) + C$
9. $\frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}sin(4x) + C$
11. $\frac{1}{6}\cos^6 x - \frac{1}{4}\cos^4 x + C$
13. $-\frac{1}{9}\sin^9(x) + \frac{3\sin^7(x)}{7} - \frac{3\sin^5(x)}{5} + \frac{\sin^3(x)}{3} + C$
15. $\tan x - \sec x + C$
17. $\frac{1}{2}\left(-\frac{1}{3}\cos(3x) + \cos(-x)\right) + C$
19. $\frac{1}{2}\left(\frac{1}{\pi}\sin(\pi x) - \frac{1}{3\pi}\sin(3\pi x)\right) + C$
21. $\frac{1}{\pi}\sin(\frac{\pi}{2}x) + \frac{1}{3\pi}\sin(\pi x) + C$
23. $\frac{\sqrt{2}}{4}x - \frac{1}{12}\sin\left(6x + \frac{\pi}{4}\right) + C$
25. $\frac{\tan^5(x)}{5} + C$
27. $\frac{\tan^6(x)}{6} + \frac{\tan^4(x)}{4} + C$
29. $\frac{\sec^5(x)}{5} - \frac{\sec^3(x)}{3} + C$
31. $\frac{1}{3}\tan^3 x - \tan x + x + C$
33. $\frac{1}{4}\tan x\sec^3 x + \frac{3}{8}\left(\sec x\tan x + \ln|\sec x + \tan x|\right) + C$
35. $\frac{1}{4}\tan x\sec^3 x - \frac{1}{8}\left(\sec x\tan x + \ln|\sec x + \tan x|\right) + C$
37. 0
39. $3\pi/4$
41. $1/2$
43. $1/5$
45. $\frac{3}{4} + \ln 2$
47. $\frac{1}{2}\tan^2 x + \ln|\cos x| + 4$
49. $\frac{1}{2}\left(-\frac{1}{8}\cos(8x) - \frac{1}{2}\cos(2x)\right) - \frac{11}{16}$

Section 6.3

1. backwards
3. (a) $\tan^2\theta + 1 = \sec^2\theta$
   (b) $9\sec^2\theta$.
5. $\frac{1}{2}\left(x\sqrt{x^2+1} + \ln\left|\sqrt{x^2+1} + x\right|\right) + C$
7. $\frac{1}{2}\left(\sin^{-1}x + x\sqrt{1-x^2}\right) + C$
9. $\frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln|x + \sqrt{x^2-1}| + C$
11. $x\sqrt{x^2+1/4} + \frac{1}{4}\ln|2\sqrt{x^2+1/4} + 2x| + C =$
    $\frac{1}{2}x\sqrt{4x^2+1} + \frac{1}{4}\ln|\sqrt{4x^2+1} + 2x| + C$
13. $4\left(\frac{1}{2}x\sqrt{x^2-1/16} - \frac{1}{32}\ln|4x + 4\sqrt{x^2-1/16}|\right) + C =$
    $\frac{1}{2}x\sqrt{16x^2-1} - \frac{1}{8}\ln|4x + \sqrt{16x^2-1}| + C$
15. $\frac{1}{2}a^2\ln\left(\sqrt{x^2+a^2} + x\right) + \frac{1}{2}x\sqrt{x^2+a^2} + C$
17. $3\sin^{-1}\left(\frac{x}{\sqrt{7}}\right) + C$ (Trig. Subst. is not needed)
19. $\sqrt{x^2-11} - \sqrt{11}\sec^{-1}(x/\sqrt{11}) + C$
21. $\sqrt{x^2-3} + C$ (Trig. Subst. is not needed)
23. $-\frac{1}{\sqrt{x^2+9}} + C$ (Trig. Subst. is not needed)
25. $\frac{1}{18}\frac{x+2}{x^2+4x+13} + \frac{1}{54}\tan^{-1}\left(\frac{x+2}{2}\right) + C$
27. $\frac{1}{7}\left(-\frac{\sqrt{5-x^2}}{x} - \sin^{-1}(x/\sqrt{5})\right) + C$
29. $\frac{1}{2}e^{\sin^{-1}x}\left(x + \sqrt{1-x^2}\right) + C$
31. $\pi/2$
33. $2\sqrt{2} + 2\ln(1+\sqrt{2})$

35. $9\sin^{-1}(1/3)+\sqrt{8}$ Note: the new lower bound is $\theta=\sin^{-1}(-1/3)$ and the new upper bound is $\theta=\sin^{-1}(1/3)$. The final answer comes with recognizing that $\sin^{-1}(-1/3)=-\sin^{-1}(1/3)$ and that $\cos\left(\sin^{-1}(1/3)\right)=\cos\left(\sin^{-1}(-1/3)\right)=\sqrt{8}/3$.

37. $\frac{1}{2}\left(x\sqrt{x^2+1}+\ln|\sqrt{x^2+1}+x|\right)+6$

39. $5\ln\left|\frac{x}{\sqrt{8}}+\frac{\sqrt{x^2-8}}{\sqrt{8}}\right|+4$

Section 6.4

1. rational
3. $\frac{A}{x}+\frac{B}{x-3}$
5. $\frac{A}{x-\sqrt{7}}+\frac{B}{x+\sqrt{7}}$
7. $3\ln|x-2|+4\ln|x+5|+C$
9. $\frac{1}{3}(\ln|x+2|-\ln|x-2|)+C$
11. $-\frac{4}{x+8}-3\ln|x+8|+C$
13. $-\ln|2x-3|+5\ln|x-1|+2\ln|x+3|+C$
15. $x+\ln|x-1|-\ln|x+2|+C$
17. $-\ln|x+5|+\dfrac{1}{2}\ln\left(x^2+4\right)+\dfrac{1}{2}\tan^{-1}\left(\dfrac{x}{2}\right)+C$
19. $\frac{x^2}{2}+x+\frac{125}{9}\ln|x-5|+\frac{64}{9}\ln|x+4|-\frac{35}{2}+C$
21. $\frac{1}{6}\left(-\ln\left|x^2+2x+3\right|+2\ln|x|-\sqrt{2}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)\right)+C$
23. $\ln\left|3x^2+5x-1\right|+2\ln|x+1|+C$
25. $\frac{9}{10}\ln\left|x^2+9\right|+\frac{1}{5}\ln|x+1|-\frac{4}{15}\tan^{-1}\left(\frac{x}{3}\right)+C$
27. $-\ln|x|-\dfrac{1}{2x^2}+\dfrac{1}{2}\ln\left(x^2+1\right)+C$
29. $\frac{1}{2}\ln\left|x^2+10x+27\right|+5\ln|x+2|-6\sqrt{2}\tan^{-1}\left(\frac{x+5}{\sqrt{2}}\right)+C$
31. $\frac{1}{3}(\ln|\cos x+2|-\ln|\cos x-2|)+C$
33. $\ln(2000/243)\approx 2.108$
35. $\ln(9/5)$
37. $1/8$

Section 6.5

1. Because $\cosh x$ is always positive.

3.
$$\begin{aligned}\coth^2 x-\operatorname{csch}^2 x&=\left(\frac{e^x+e^{-x}}{e^x-e^{-x}}\right)^2-\left(\frac{2}{e^x-e^{-x}}\right)^2\\&=\frac{\left(e^{2x}+2+e^{-2x}\right)-(4)}{e^{2x}-2+e^{-2x}}\\&=\frac{e^{2x}-2+e^{-2x}}{e^{2x}-2+e^{-2x}}\\&=1\end{aligned}$$

5.
$$\begin{aligned}\cosh^2 x&=\left(\frac{e^x+e^{-x}}{2}\right)^2\\&=\frac{e^{2x}+2+e^{-2x}}{4}\\&=\frac{1}{2}\,\frac{\left(e^{2x}+e^{-2x}\right)+2}{2}\\&=\frac{1}{2}\left(\frac{e^{2x}+e^{-2x}}{2}+1\right)\\&=\frac{\cosh 2x+1}{2}.\end{aligned}$$

7.
$$\begin{aligned}\frac{d}{dx}\left[\operatorname{sech}x\right]&=\frac{d}{dx}\left[\frac{2}{e^x+e^{-x}}\right]\\&=\frac{-2\left(e^x-e^{-x}\right)}{\left(e^x+e^{-x}\right)^2}\\&=-\frac{2\left(e^x-e^{-x}\right)}{\left(e^x+e^{-x}\right)\left(e^x+e^{-x}\right)}\\&=-\frac{2}{e^x+e^{-x}}\cdot\frac{e^x-e^{-x}}{e^x+e^{-x}}\\&=-\operatorname{sech}x\tanh x\end{aligned}$$

9. $\displaystyle\int\tanh x\,dx=\int\frac{\sinh x}{\cosh x}\,dx$
Let $u=\cosh x$; $du=(\sinh x)dx$
$$\begin{aligned}&=\int\frac{1}{u}\,du\\&=\ln|u|+C\\&=\ln(\cosh x)+C.\end{aligned}$$

11. $2\sinh 2x$
13. $\coth x$
15. $x\cosh x$
17. $\frac{3}{\sqrt{9x^2+1}}$
19. $\frac{1}{1-(x+5)^2}$
21. $\sec x$
23. $y=3/4(x-\ln 2)+5/4$
25. $y=x$
27. $1/2\ln(\cosh(2x))+C$
29. $1/2\sinh^2 x+C$ or $1/2\cosh^2 x+C$
31. $x\cosh(x)-\sinh(x)+C$
33. $\cosh^{-1}(x^2/2)+C=\ln(x^2+\sqrt{x^4-4})+C$
35. $\frac{1}{16}\tan^{-1}(x/2)+\frac{1}{32}\ln|x-2|+\frac{1}{32}\ln|x+2|+C$
37. $\tan^{-1}(e^x)+C$
39. $x\tanh^{-1}x+1/2\ln|x^2-1|+C$
41. 0
43. 2

Section 6.6

1. The interval of integration is finite, and the integrand is continuous on that interval.
3. converges; could also state <10.
5. $p>1$
7. $e^5/2$
9. $1/3$
11. $1/\ln 2$
13. diverges
15. 1
17. diverges
19. diverges
21. diverges
23. 1
25. 0
27. $-1/4$
29. -1

31. diverges
33. $1/2$
35. converges; Limit Comparison Test with $1/x^{3/2}$.
37. converges; Direct Comparison Test with xe^{-x}.
39. converges; Direct Comparison Test with xe^{-x}.
41. diverges; Direct Comparison Test with $x/(x^2+\cos x)$.
43. converges; Limit Comparison Test with $1/e^x$.

Section 6.7

1. F
3. $f'(x) = \frac{e^{x^2}}{\sqrt{\pi}\operatorname{erfi} x}$
5. $f'(x) = \frac{4}{\sqrt{\pi}}e^{-(2x+3)^2}$
7. $f'(x) = 2x\operatorname{erfi} x + \frac{6}{\sqrt{\pi}}x^2e^{9x^2}$
9. $h'(x) = \frac{2}{\sqrt{\pi}}\left(e^{-x^2}\operatorname{erfi} x + e^{x^2}\operatorname{erf} x\right)$
11. $k'(x) = 2\sqrt{\pi}e^{\operatorname{erf}^{-1}(4x)^2} + 5$
13. $\frac{\sqrt{\pi}}{4}\operatorname{erf}(2x) + C$
15. $\frac{\sqrt{\pi}}{2\sqrt{a}}\operatorname{erf}\left(\sqrt{a}x\right) + C$
17. $x\operatorname{erfi} x - \frac{1}{\sqrt{\pi}}e^{x^2} + C$
19. $\frac{\sqrt{\pi}}{4}\operatorname{erf}(2x+5) + C$
21. $\frac{\sqrt{\pi}}{2e^9}\operatorname{erfi}(x+3) + C$
23. $\frac{1}{2}xe^{x^2} - \frac{\sqrt{\pi}}{4}\operatorname{erfi} x + C$
25. $\frac{\sqrt{\pi}}{2}\ln|\operatorname{erfi} x| + C$
27. $e^4\sqrt{\pi}\operatorname{erf}(x-2) - \frac{1}{2}e^{-x^2+4x} + C$
29. $\frac{\sqrt{\pi}}{2e^{1/4}}\operatorname{erfi}\left(\ln x + \frac{1}{2}\right) + C$
31. $\frac{\sqrt{\pi}}{3}\operatorname{erf}\left(x^{3/2}\right) + C$
33. $\frac{\sqrt{\pi}}{6}(\operatorname{erf} 1000 - \operatorname{erf} 1)$
35. $\frac{\sqrt{\pi}}{2}\operatorname{erf} 1$
37. $\sqrt{\pi}(\operatorname{erfi} 6 - \operatorname{erf} 6)$
39. $2/\sqrt{\pi}$
41. 0
43. ∞
45. $1/\sqrt{\pi}$
47. $\operatorname{Si}(7x) + C$
49. $x\operatorname{Si}(x) + \cos x + C$
51. $\operatorname{Si}(a)$

Chapter 7

Section 7.1

1. T
3. Answers will vary.
5. $16/3$
7. π
9. $2\sqrt{2}$
11. 4.5
13. $2-\pi/2$
15. $1/6$
17. On regions such as $[\pi/6, 5\pi/6]$, the area is $3\sqrt{3}/2$. On regions such as $[-\pi/2, \pi/6]$, the area is $3\sqrt{3}/4$.
19. $5/3$
21. $9/4$
23. 1
25. 4
27. 219,000 ft^2

Section 7.2

1. T
3. Recall that "dx" does not just "sit there;" it is multiplied by $A(x)$ and represents the thickness of a small slice of the solid. Therefore dx has units of in, giving $A(x)\,dx$ the units of in^3.
5. $175\pi/3$ units^3
7. $\pi/6$ units^3
9. $35\pi/3$ units^3
11. $2\pi/15$ units^3
13. (a) $512\pi/15$
    (b) $256\pi/5$
    (c) $832\pi/15$
    (d) $128\pi/3$
15. (a) $104\pi/15$
    (b) $64\pi/15$
    (c) $32\pi/5$
17. (a) 8π
    (b) 8π
    (c) $16\pi/3$
    (d) $8\pi/3$
19. The cross–sections of this cone are the same as the cone in Exercise 18. Thus they have the same volume of $250\pi/3$ units^3.
21. Orient the solid so that the x-axis is parallel to long side of the base. All cross–sections are trapezoids (at the far left, the trapezoid is a square; at the far right, the trapezoid has a top length of 0, making it a triangle). The area of the trapezoid at x is $A(x) = 1/2(-1/2x+5+5)(5) = -5/4x+25$. The volume is 187.5 units^3.
23. (a) Answers may vary.
    (b) $V = \frac{1}{3}\pi hr^2$

Section 7.3

1. T
3. F
5. $9\pi/2$ units^3
7. $\pi^2 - 2\pi$ units^3
9. $48\pi\sqrt{3}/5$ units^3
11. $\pi^2/4$ units^3
13. (a) $4\pi/5$
    (b) $8\pi/15$
    (c) $\pi/2$

(d) $5\pi/6$

15. (a) $4\pi/3$
    (b) $\pi/3$
    (c) $4\pi/3$
    (d) $2\pi/3$

17. (a) $2\pi\left(\sqrt{2}-1\right)$
    (b) $2\pi\left(1-\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right)$

Section 7.4

1. T
3. $\sqrt{2}$
5. $4/3$
7. $109/2$
9. $12/5$
11. $-\ln(2-\sqrt{3}) \approx 1.31696$
13. $\int_0^1 \sqrt{1+4x^2}\,dx$
15. $\int_0^1 \sqrt{1+\frac{1}{4x}}\,dx$
17. $\int_{-1}^1 \sqrt{1+\frac{x^2}{1-x^2}}\,dx$
19. $\int_1^2 \sqrt{1+\frac{1}{x^4}}\,dx$
21. 1.4790
23. Simpson's Rule fails, as it requires one to divide by 0. However, recognize the answer should be the same as for $y=x^2$; why?
25. Simpson's Rule fails.
27. 1.4058
29. $2\pi\int_0^1 2x\sqrt{5}\,dx = 2\pi\sqrt{5}$
31. $2\pi\int_0^1 x^3\sqrt{1+9x^4}\,dx = \pi/27(10\sqrt{10}-1)$
33. $2\pi\int_0^1 \sqrt{1-x^2}\sqrt{1+x/(1-x^2)}\,dx = 4\pi$

Section 7.5

1. In SI units, it is one joule, i.e., one Newton–meter, or kg·m/s^2·m. In Imperial Units, it is ft–lb.
3. Smaller.
5. (a) 2450 J
   (b) 1568 J
7. 735 J
9. 11,100 ft–lb
11. 125 ft–lb
13. 12.5 ft–lb
15. $10/3$ ft $=40$ in
17. $f\cdot d/2$ Joules
19. 5 ft–lb
21. (a) 52,929.6 ft–lb
    (b) 18,525.3 ft–lb
    (c) When 3.83 ft of water have been pumped from the tank, leaving about 2.17 ft in the tank.
23. 212,135 ft–lb
25. 187,214 ft–lb
27. 4,917,150 J

Section 7.6

1. Answers will vary.
3. 499.2 lb
5. 6739.2 lb
7. 3920.7 lb
9. 2496 lb
11. 602.59 lb
13. (a) 2340 lb
    (b) 5625 lb
15. (a) 1597.44 lb
    (b) 3840 lb
17. (a) 56.42 lb
    (b) 135.62 lb
19. 5.1 ft

Chapter 8

Section 8.1

1. Answers will vary.
3. Answers will vary.
5. $2, \frac{8}{3}, \frac{8}{3}, \frac{32}{15}, \frac{64}{45}$
7. $\frac{1}{3}, 2, \frac{81}{5}, \frac{512}{3}, \frac{15625}{7}$
9. $a_n = 3n+1$
11. $a_n = 10\cdot 2^{n-1}$
13. $1/7$
15. 0
17. diverges
19. converges to 0
21. diverges
23. converges to e
25. converges to 0
27. converges to 2
29. bounded
31. bounded
33. neither bounded above or below
35. monotonically increasing
37. never monotonic
39. Let $\{a_n\}$ be given such that $\lim\limits_{n\to\infty}|a_n| = 0$. By the definition of the limit of a sequence, given any $\varepsilon > 0$, there is a m such that for all $n > m$, $|\,|a_n| - 0| < \varepsilon$. Since $|\,|a_n| - 0| = |a_n - 0|$, this directly implies that for all $n > m$, $|a_n - 0| < \varepsilon$, meaning that $\lim\limits_{n\to\infty} a_n = 0$.
41. Left to reader

Section 8.2

1. Answers will vary.
3. One sequence is the sequence of terms $\{a_n\}$. The other is the sequence of n^{th} partial sums, $\{S_n\} = \{\sum_{i=1}^n a_i\}$.
5. F
7. (a) $1, \frac{5}{4}, \frac{49}{36}, \frac{205}{144}, \frac{5269}{3600}$

(b) Plot omitted

9. (a) $1, 3, 6, 10, 15$
   (b) Plot omitted

11. (a) $\frac{1}{3}, \frac{4}{9}, \frac{13}{27}, \frac{40}{81}, \frac{121}{243}$
    (b) Plot omitted

13. (a) $0.1, 0.11, 0.111, 0.1111, 0.11111$
    (b) Plot omitted

15. $\lim_{n\to\infty} a_n = \infty$; by Theorem 64 the series diverges.

17. $\lim_{n\to\infty} a_n = 1$; by Theorem 64 the series diverges.

19. $\lim_{n\to\infty} a_n = e$; by Theorem 64 the series diverges.

21. Diverges

23. Converges

25. (a) $S_n = \frac{1-(1/4)^n}{3/4}$
    (b) Converges to $4/3$.

27. (a) $S_n = \begin{cases} \frac{n+1}{2} & \text{n is odd} \\ -\frac{n}{2} & \text{n is even} \end{cases}$
    (b) Diverges

29. (a) $S_n = \frac{1-(1/e)^{n+1}}{1-1/e}$.
    (b) Converges to $1/(1-1/e) = e/(e-1)$.

31. (a) With partial fractions, $a_n = \frac{1}{n} - \frac{1}{n+1}$. Thus $S_n = 1 - \frac{1}{n+1}$.
    (b) Converges to 1.

33. (a) Use partial fraction decomposition to recognize the telescoping series: $a_n = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$. Then $S_n = \frac{n}{2n+1}$.
    (b) Converges to $1/2$.

35. (a) $S_n = 1 - \frac{1}{(n+1)^2}$
    (b) Converges to 1.

37. (a) $a_n = 1/2^n + 1/3^n$ for $n \geq 0$. Thus $S_n = \frac{1-1/2^2}{1/2} + \frac{1-1/3^n}{2/3}$.
    (b) Converges to $2 + 3/2 = 7/2$.

39. (a) $S_n = \frac{1-(\sin 1)^{n+1}}{1-\sin 1}$
    (b) Converges to $\frac{1}{1-\sin 1}$..

41. Using partial fractions, we can show that $a_n = \frac{1}{4}\left(\frac{1}{2n-1} + \frac{1}{2n+1}\right)$. The series is effectively twice the sum of the odd terms of the Harmonic Series which was shown to diverge in Exercise 40. Thus this series diverges.

Section 8.3

1. continuous, positive and decreasing

3. The Integral Test (we do not have a continuous definition of $n!$ yet) and the Limit Comparison Test (same as above, hence we cannot take its derivative).

5. Converges

7. Diverges

9. Diverges

11. Diverges

13. Diverges

15. Converges

17. Converges

19. Diverges

21. Converges; compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, as $1/(n^2+3n-5) \leq 1/n^2$ for all $n > 1$.

23. Diverges; compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, as $1/n \leq \ln n/n$ for all $n \geq 2$.

25. Diverges; compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. Since $n = \sqrt{n^2} > \sqrt{n^2-1}$, $1/n \leq 1/\sqrt{n^2-1}$ for all $n \geq 2$.

27. Diverges; compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$:
$$\frac{1}{n} = \frac{n^2}{n^3} < \frac{n^2+n+1}{n^3} < \frac{n^2+n+1}{n^3-5},$$
for all $n \geq 1$.

29. Diverges; compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. Note that
$$\frac{n}{n^2-1} = \frac{n^2}{n^2-1} \cdot \frac{1}{n} > \frac{1}{n},$$
as $\frac{n^2}{n^2-1} > 1$, for all $n \geq 2$.

31. Converges; compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

33. Diverges; compare to $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$.

35. Diverges; compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

37. Diverges; compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. Just as $\displaystyle\lim_{n\to 0} \frac{\sin n}{n} = 1$, $\displaystyle\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1$.

39. Converges; compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$.

41. Converges; Integral Test

43. Diverges; the n^{th} Term Test and Direct Comparison Test can be used.

45. Converges; the Direct Comparison Test can be used with sequence $1/3^n$.

47. Diverges; the n^{th} Term Test can be used, along with the Integral Test.

49. (a) Converges; use Direct Comparison Test as $\frac{a_n}{n} < n$.
    (b) Converges; since original series converges, we know $\lim_{n\to\infty} a_n = 0$. Thus for large n, $a_n a_{n+1} < a_n$.
    (c) Converges; similar logic to part (b) so $(a_n)^2 < a_n$.
    (d) May converge; certainly $na_n > a_n$ but that does not mean it does not converge.
    (e) Does not converge, using logic from (b) and n^{th} Term Test.

Section 8.4

1. algebraic, or polynomial.

3. Integral Test, Limit Comparison Test, and Root Test

5. Converges

7. Converges

9. The Ratio Test is inconclusive; the p-Series Test states it diverges.
11. Converges
13. Converges; note the summation can be rewritten as $\displaystyle\sum_{n=1}^{\infty}\frac{2^n n!}{3^n n!}$, from which the Ratio Test can be applied.
15. Converges
17. Converges
19. Diverges
21. Diverges. The Root Test is inconclusive, but the n^{th}-Term Test shows divergence. (The terms of the sequence approach e^2, not 0, as $n \to \infty$.)
23. Converges
25. Diverges; Limit Comparison Test
27. Converges; Ratio Test or Limit Comparison Test with $1/3^n$.
29. Diverges; n^{th}-Term Test or Limit Comparison Test with 1.
31. Diverges; Direct Comparison Test with $1/n$
33. Converges; Root Test

Section 8.5

1. The signs of the terms do not alternate; in the given series, some terms are negative and the others positive, but they do not necessarily alternate.
3. Many examples exist; one common example is $a_n = (-1)^n/n$.
5. (a) converges
   (b) converges (p-Series)
   (c) absolute
7. (a) diverges (limit of terms is not 0)
   (b) diverges
   (c) n/a; diverges
9. (a) converges
   (b) diverges (Limit Comparison Test with $1/n$)
   (c) conditional
11. (a) diverges (limit of terms is not 0)
    (b) diverges
    (c) n/a; diverges
13. (a) diverges (terms oscillate between ± 1)
    (b) diverges
    (c) n/a; diverges
15. (a) converges
    (b) converges (Geometric Series with $r = 2/3$)
    (c) absolute
17. (a) converges
    (b) converges (Ratio Test)
    (c) absolute
19. (a) converges
    (b) diverges (p-Series Test with $p = 1/2$)
    (c) conditional
21. $S_5 = -1.1906$; $S_6 = -0.6767$;
    $$-1.1906 \leq \sum_{n=1}^{\infty}\frac{(-1)^n}{\ln(n+1)} \leq -0.6767$$
23. $S_6 = 0.3681$; $S_7 = 0.3679$;
    $$0.3681 \leq \sum_{n=0}^{\infty}\frac{(-1)^n}{n!} \leq 0.3679$$
25. $n = 5$
27. $n = 7$
29. $n = 5$ ($(2n)! > 10^8$ when $n \geq 6$)

Section 8.6

1. 1
3. 5
5. $1 + 2x + 4x^2 + 8x^3 + 16x^4$
7. $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$
9. (a) $R = \infty$
   (b) $(-\infty, \infty)$
11. (a) $R = 1$
    (b) $(2, 4]$
13. (a) $R = 2$
    (b) $(-2, 2)$
15. (a) $R = 1/5$
    (b) $(4/5, 6/5)$
17. (a) $R = 1$
    (b) $(-1, 1)$
19. (a) $R = \infty$
    (b) $(-\infty, \infty)$
21. (a) $R = 1$
    (b) $[-1, 1]$
23. (a) $R = 1$
    (b) $[-3, -1]$
25. (a) $R = 4$
    (b) $x = (-8, 0)$
27. (a) $f'(x) = \displaystyle\sum_{n=1}^{\infty} x^{n-1}; \quad (-1, 1)$
    (b) $\displaystyle\int f(x)\,dx = C + \sum_{n=1}^{\infty}\frac{1}{n(n+1)}x^{n+1}; \quad [-1, 1]$
29. (a) $f'(x) = \displaystyle\sum_{n=1}^{\infty} n(-3)^n x^{n-1}; \quad (-1/3, 1/3)$
    (b) $\displaystyle\int f(x)\,dx = C + \sum_{n=0}^{\infty}\frac{(-3)^n}{n+1}x^{n+1}; \quad (-1/3, 1/3]$
31. (a) $f'(x) = \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}x^n}{n!}; \quad (-\infty, \infty)$
    (b) $\displaystyle\int f(x)\,dx = C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{(n+1)!}; \quad (-\infty, \infty)$
33. $5 + 25x + \frac{125}{2}x^2 + \frac{625}{6}x^3 + \frac{3125}{24}x^4$
35. $1 + 2x + x^2 + \frac{1}{3}x^3 + \frac{1}{12}x^4$
37. $1 + x + x^2 + \frac{1}{3}x^3 + \frac{1}{6}x^4$

Section 8.7

1. The Maclaurin polynomial is a special case of Taylor polynomials. Taylor polynomials are centered at a specific x-value; when that x-value is 0, it is a Maclauring polynomial.
3. $p_2(x) = 6 + 3x - 4x^2$.

5. $p_3(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3$

7. $p_8(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5$

9. $p_4(x) = \frac{2x^4}{3} + \frac{4x^3}{3} + 2x^2 + 2x + 1$

11. $p_4(x) = x^4 - x^3 + x^2 - x + 1$

13. $p_4(x) = 1 + \frac{1}{2}(-1+x) - \frac{1}{8}(-1+x)^2 + \frac{1}{16}(-1+x)^3 - \frac{5}{128}(-1+x)^4$

15. $p_6(x) = \frac{1}{\sqrt{2}} - \frac{-\frac{\pi}{4}+x}{\sqrt{2}} - \frac{(-\frac{\pi}{4}+x)^2}{2\sqrt{2}} + \frac{(-\frac{\pi}{4}+x)^3}{6\sqrt{2}} + \frac{(-\frac{\pi}{4}+x)^4}{24\sqrt{2}} - \frac{(-\frac{\pi}{4}+x)^5}{120\sqrt{2}} - \frac{(-\frac{\pi}{4}+x)^6}{720\sqrt{2}}$

17. $p_5(x) = \frac{1}{2} - \frac{x-2}{4} + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3 + \frac{1}{32}(x-2)^4 - \frac{1}{64}(x-2)^5$

19. $p_3(x) = \frac{1}{2} + \frac{1+x}{2} + \frac{1}{4}(1+x)^2$

21. $p_3(x) = x - \frac{x^3}{6}$; $p_3(0.1) = 0.09983$. Error is bounded by $\pm\frac{1}{4!} \cdot 0.1^4 \approx \pm 0.000004167$.

23. $p_2(x) = 3 + \frac{1}{6}(-9+x) - \frac{1}{216}(-9+x)^2$; $p_2(10) = 3.16204$. The third derivative of $f(x) = \sqrt{x}$ is bounded on $(8, 11)$ by 0.003. Error is bounded by $\pm\frac{0.003}{3!} \cdot 1^3 = \pm 0.0005$.

25. The n^{th} derivative of $f(x) = e^x$ is bounded by 3 on intervals containing 0 and 1. Thus $|R_n(1)| \leq \frac{3}{(n+1)!}1^{(n+1)}$. When $n = 7$, this is less than 0.0001.

27. The n^{th} derivative of $f(x) = \cos x$ is bounded by 1 on intervals containing 0 and $\pi/3$. Thus $|R_n(\pi/3)| \leq \frac{1}{(n+1)!}(\pi/3)^{(n+1)}$. When $n = 7$, this is less than 0.0001. Since the Maclaurin polynomial of $\cos x$ only uses even powers, we can actually just use $n = 6$.

29. The n^{th} term is $\frac{1}{n!}x^n$.

31. The n^{th} term is x^n.

33. The n^{th} term is $(-1)^n\frac{(x-1)^n}{n}$.

35. $3 + 15x + \dfrac{75}{2}x^2 + \dfrac{375}{6}x^3 + \dfrac{1875}{24}x^4$

Section 8.8

1. A Taylor polynomial is a **polynomial**, containing a finite number of terms. A Taylor series is a **series**, the summation of an infinite number of terms.

3. All derivatives of e^x are e^x which evaluate to 1 at $x = 0$.
The Taylor series starts $1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$;
the Taylor series is $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$

5. The n^{th} derivative of $1/(1-x)$ is $f^{(n)}(x) = (n)!/(1-x)^{n+1}$, which evaluates to $n!$ at $x = 0$.
The Taylor series starts $1 + x + x^2 + x^3 + \cdots$;
the Taylor series is $\displaystyle\sum_{n=0}^{\infty}x^n$

7. The Taylor series starts
$0 - (x - \pi/2) + 0x^2 + \frac{1}{6}(x-\pi/2)^3 + 0x^4 - \frac{1}{120}(x-\pi/2)^5$;
the Taylor series is $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-\pi/2)^{2n+1}}{(2n+1)!}$

9. $f^{(n)}(x) = (-1)^n e^{-x}$; at $x = 0$, $f^{(n)}(0) = -1$ when n is odd and $f^{(n)}(0) = 1$ when n is even.
The Taylor series starts $1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \cdots$;
the Taylor series is $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}$.

11. $f^{(n)}(x) = (-1)^{n+1}\frac{n!}{(x+1)^{n+1}}$; at $x = 1$, $f^{(n)}(1) = (-1)^{n+1}\frac{n!}{2^{n+1}}$
The Taylor series starts
$\frac{1}{2} + \frac{1}{4}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3\cdots$;
the Taylor series is $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{2^{n+1}}$.

13. Given a value x, the magnitude of the error term $R_n(x)$ is bounded by
$$\left|R_n(x)\right| \leq \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|x^{(n+1)}\right|,$$
where z is between 0 and x.
If $x > 0$, then $z < x$ and $f^{(n+1)}(z) = e^z < e^x$. If $x < 0$, then $x < z < 0$ and $f^{(n+1)}(z) = e^z < 1$. So given a fixed x value, let $M = \max\{e^x, 1\}$; $f^{(n)}(z) < M$. This allows us to state
$$\left|R_n(x)\right| \leq \frac{M}{(n+1)!}\left|x^{(n+1)}\right|.$$
For any x, $\displaystyle\lim_{n\to\infty}\frac{M}{(n+1)!}\left|x^{(n+1)}\right| = 0$. Thus by the Squeeze Theorem, we conclude that $\displaystyle\lim_{n\to\infty}R_n(x) = 0$ for all x, and hence
$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \quad \text{for all } x.$$

15. Given a value x, the magnitude of the error term $R_n(x)$ is bounded by
$$\left|R_n(x)\right| \leq \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|(x-1)^{(n+1)}\right|,$$
where z is between 1 and x.
Note that $\left|f^{(n+1)}(x)\right| = \frac{n!}{x^{n+1}}$.
We consider the cases when $x > 1$ and when $x < 1$ separately.
If $x > 1$, then $1 < z < x$ and $f^{(n+1)}(z) = \frac{n!}{z^{n+1}} < n!$. Thus
$$\left|R_n(x)\right| \leq \frac{n!}{(n+1)!}\left|(x-1)^{(n+1)}\right| = \frac{(x-1)^{n+1}}{n+1}.$$
For a fixed x,
$$\lim_{n\to\infty}\frac{(x-1)^{n+1}}{n+1} = 0.$$
If $0 < x < 1$, then $x < z < 1$ and $f^{(n+1)}(z) = \frac{n!}{z^{n+1}} < \frac{n!}{x^{n+1}}$. Thus
$$\left|R_n(x)\right| \leq \frac{n!/x^{n+1}}{(n+1)!}\left|(x-1)^{(n+1)}\right| = \frac{x^{n+1}}{n+1}(1-x)^{n+1}.$$
Since $0 < x < 1$, $x^{n+1} < 1$ and $(1-x)^{n+1} < 1$. We can then extend the inequality from above to state
$$\left|R_n(x)\right| \leq \frac{x^{n+1}}{n+1}(1-x)^{n+1} < \frac{1}{n+1}.$$
As $n \to \infty$, $1/(n+1) \to 0$. Thus by the Squeeze Theorem, we conclude that $\displaystyle\lim_{n\to\infty}R_n(x) = 0$ for all x, and hence
$$\ln x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n} \quad \text{for all } 0 < x \leq 2.$$

17. Given $\cos x = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$,
$\cos(-x) = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(-x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = \cos x$, as all powers in the series are even.

19. Given $\sin x = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$,
$\dfrac{d}{dx}\left(\sin x\right) = \dfrac{d}{dx}\left(\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right) =$
$\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = \cos x$. (The summation still starts at $n = 0$ as there was no constant term in the expansion of $\sin x$).

21. $1+\dfrac{x}{2}-\dfrac{x^2}{8}+\dfrac{x^3}{16}-\dfrac{5x^4}{128}$

23. $1+\dfrac{x}{3}-\dfrac{x^2}{9}+\dfrac{5x^3}{81}-\dfrac{10x^4}{243}$

25. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(x^2)^{2n}}{(2n)!}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{(2n)!}$

27. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(2x+3)^{2n+1}}{(2n+1)!}$

29. $\displaystyle C+\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{n!(2n+1)}$

31. $x+x^2+\dfrac{x^3}{3}-\dfrac{x^5}{30}$

33. $\displaystyle\sum_{n=1}^{\infty}nx^n$

35. $\displaystyle\int_0^{\sqrt{\pi}}\sin\left(x^2\right)dx\approx\int_0^{\sqrt{\pi}}\left(x^2-\frac{x^6}{6}+\frac{x^{10}}{120}-\frac{x^{14}}{5040}\right)dx=$ 0.8877

37. $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n^2}$

39. $\pi^2/6$

41. At $x=0$, both sides are 0. The derivative of the left side is $-\frac{\ln(1-x)}{x}-\frac{\ln(1+x)}{x}$. By algebra and rules of logarithms, this can be shown equivalent to the derivative of the right side, which is $-\frac{\ln\left(1-x^2\right)}{x}$.

43. $\frac{\pi^2}{12}-\frac{(\ln 2)^2}{2}$

45. -1

Chapter 9

Section 9.1

1. When defining the conics as the intersections of a plane and a double napped cone, degenerate conics are created when the plane intersects the tips of the cones (usually taken as the origin). Nondegenerate conics are formed when this plane does not contain the origin.

3. Hyperbola

5. With a horizontal transverse axis, the x^2 term has a positive coefficient; with a vertical transverse axis, the y^2 term has a positive coefficient.

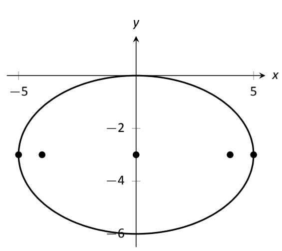
7. $y=\frac{-1}{12}(x+1)^2-1$

9. $x=y^2$

11. $x=-\frac{1}{12}y^2$

13. $x=-\frac{1}{8}(y-3)^2+2$

15. focus: $(5,2)$; directrix: $x=1$. The point P is 10 units from each.

17. 

19. $\frac{(x-1)^2}{1/4}+\frac{y^2}{9}=1$; foci at $(1,\pm\sqrt{8.75})$; $e=\sqrt{8.75}/3\approx 0.99$

21. $\frac{(x-2)^2}{25}+\frac{(y-3)^2}{16}=1$

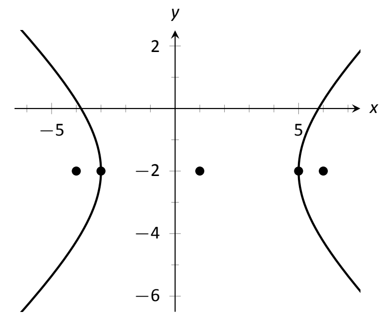
23. $\frac{(x+1)^2}{9}+\frac{(y-1)^2}{25}=1$

25. $\frac{x^2}{3}+\frac{y^2}{5}=1$

27. $\frac{(x-2)^2}{4}+\frac{(y-2)^2}{4}=1$

29. $x^2-\frac{y^2}{3}=1$

31. $\frac{(y-3)^2}{4}-\frac{(x-1)^2}{9}=1$

33. 

35. $\frac{x^2}{4}-\frac{y^2}{5}=1$

37. $\frac{(x-3)^2}{16}-\frac{(y-3)^2}{9}=1$

39. $\frac{x^2}{4}-\frac{y^2}{3}=1$

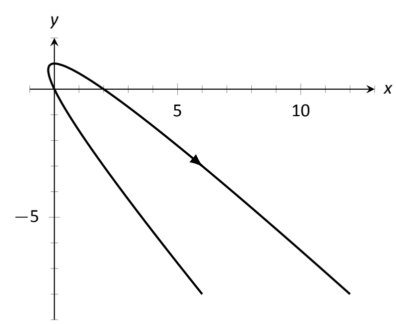
41. $(y-2)^2-\frac{x^2}{10}=1$

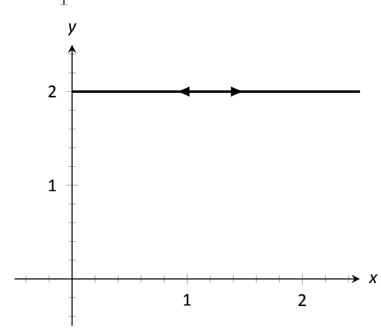
43. (a) Solve for c in $e=c/a$: $c=ae$. Thus $a^2e^2=a^2-b^2$, and $b^2=a^2-a^2e^2$. The result follows.
 (b) Mercury: $x^2/(0.387)^2+y^2/(0.3787)^2=1$
 Earth: $x^2+y^2/(0.99986)^2=1$
 Mars: $x^2/(1.524)^2+y^2/(1.517)^2=1$
 (c) Mercury: $(x-0.08)^2/(0.387)^2+y^2/(0.3787)^2=1$
 Earth: $(x-0.0167)^2+y^2/(0.99986)^2=1$
 Mars: $(x-0.1423)^2/(1.524)^2+y^2/(1.517)^2=1$

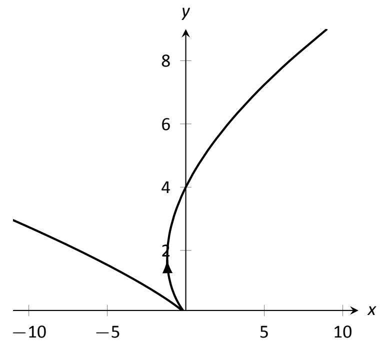
Section 9.2

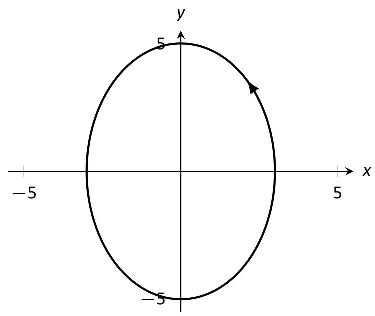
1. T

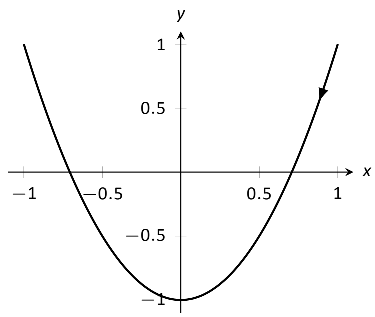
3. rectangular

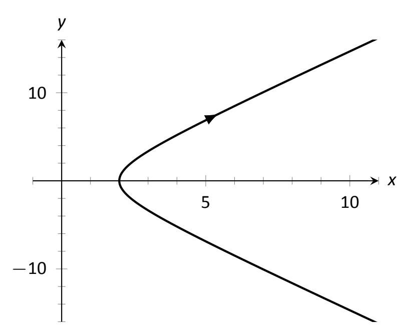
5. 

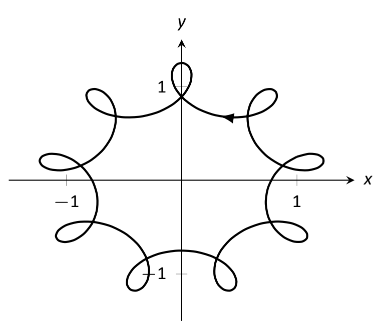
7. 

9. 

11. 

13. 

15. 

17. 

19. (a) Traces a circle of radius 1 counterclockwise once.
(b) Traces a circle of radius 1 counterclockwise over 6 times.
(c) Traces a circle of radius 1 clockwise infinite times.
(d) Traces an arc of a circle of radius 1, from an angle of -1 radians to 1 radian, twice.

21. The point is on the curve, at $t = -2$.

23. The point is on the curve, at $t = \pi/3 + 2k\pi$ for any integer k.

25. $x^2 - y^2 = 1$

27. $y = x^{3/2}$

29. $y = x^3 - 3$

31. $y = e^{2x} - 1$

33. $x^2 - y^2 = 1$

35. $y = 1 - x^2$

37. $x^2 + y^2 = r^2$; circle centered at $(0, 0)$ with radius r.

39. $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$; hyperbola centered at (h, k) with horizontal transverse axis and asymptotes with slope b/a. The parametric equations only give half of the hyperbola. When $a > 0$, the right half; when $a < 0$, the left half.

41. $x = \ln t$, $y = t$. At $t = 1$, $x = 0$, $y = 1$.
$y' = e^x$; when $x = 0$, $y' = 1$.

43. $x = 1/(4t^2)$, $y = 1/(2t)$. At $t = 1$, $x = 1/4$, $y = 1/2$.
$y' = 1/(2\sqrt{x})$; when $x = 1/4$, $y' = 1$.

45. $t = -1,\ 2$

47. $t = \pi/6, \pi/2, 5\pi/6$

49. $t = 2$

51. $t = \ldots 0,\ 2\pi,\ 4\pi,\ \ldots$

53. $x = 50t$, $y = -16t^2 + 64t$

55. $x = 2\cos t$, $y = -2\sin t$; other answers possible

57. $x = \cos t + 1$, $y = 3\sin t + 3$; other answers possible

59. $x = \pm\sec t + 2$, $y = \sqrt{8}\tan t - 3$; other answers possible

Section 9.3

1. F

3. F

5. (a) $\frac{dy}{dx} = 2t$
(b) Tangent line: $y = 2(x-1) + 1$; normal line: $y = -1/2(x-1) + 1$

7. (a) $\frac{dy}{dx} = \frac{2t+1}{2t-1}$
(b) Tangent line: $y = 3x + 2$; normal line: $y = -1/3x + 2$

9. (a) $\frac{dy}{dx} = \csc t$
(b) $t = \pi/4$: Tangent line: $y = \sqrt{2}(x - \sqrt{2}) + 1$; normal line: $y = -1/\sqrt{2}(x - \sqrt{2}) + 1$

11. (a) $\frac{dy}{dx} = \frac{\cos t\sin(2t) + \sin t\cos(2t)}{-\sin t\sin(2t) + 2\cos t\cos(2t)}$
(b) Tangent line: $y = x - \sqrt{2}$; normal line: $y = -x - \sqrt{2}$

13. $t = 0$

15. $t = -1/2$

17. The graph does not have a horizontal tangent line.

19. The solution is non-trivial; use identities $\sin(2t) = 2\sin t\cos t$ and $\cos(2t) = \cos^2 t - \sin^2 t$ to rewrite $g'(t) = 2\sin t(2\cos^2 t - \sin^2 t)$. On $[0, 2\pi]$, $\sin t = 0$ when $t = 0, \pi, 2\pi$, and $2\cos^2 t - \sin^2 t = 0$ when $t = \tan^{-1}(\sqrt{2}),\ \pi \pm \tan^{-1}(\sqrt{2}),\ 2\pi - \tan^{-1}(\sqrt{2})$.

21. $t_0 = 0$; $\lim_{t\to 0}\frac{dy}{dx} = 0$.

23. $t_0 = 1$; $\lim_{t\to 1}\frac{dy}{dx} = \infty$.

25. $\frac{d^2y}{dx^2} = 2$; always concave up

27. $\frac{d^2y}{dx^2} = -\frac{4}{(2t-1)^3}$; concave up on $(-\infty, 1/2)$; concave down on $(1/2, \infty)$.

29. $\frac{d^2y}{dx^2} = -\cot^3 t$; concave up on $(-\infty, 0)$; concave down on $(0, \infty)$.

31. $\frac{d^2y}{dx^2} = \frac{4(13+3\cos(4t))}{(\cos t+3\cos(3t))^3}$, obtained with a computer algebra system; concave up on $\big(-\tan^{-1}(\sqrt{2}/2), \tan^{-1}(\sqrt{2}/2)\big)$, concave down on $\big(-\pi/2, -\tan^{-1}(\sqrt{2}/2)\big) \cup \big(\tan^{-1}(\sqrt{2}/2), \pi/2\big)$

33. $L = 6\pi$

35. $L = 2\sqrt{34}$

37. $L = \int_0^6 \sqrt{(3t^2-1)^2 + (3t^2+2)^2}\, dt$

39. $8\sqrt{257} + \frac{1}{2}\ln\big(16 + \sqrt{257}\big)$

41. $L \approx 2.4416$ (actual value: $L = 2.42211$)

43. $L \approx 4.19216$ (actual value: $L = 4.18308$)

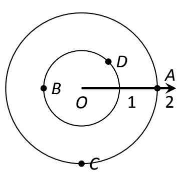
45. The answer is 16π for both (of course), but the integrals are different.

47. $SA \approx 8.50101$ (actual value $SA = 8.02851$

Section 9.4

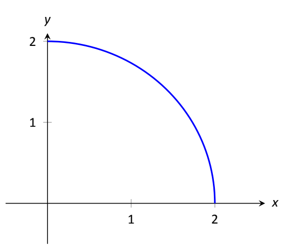
1. Answers will vary.

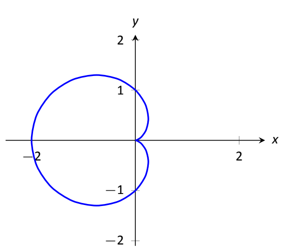
3. T

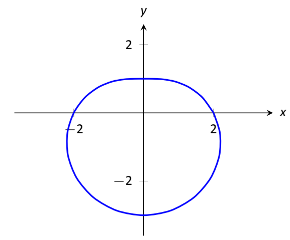
5. 

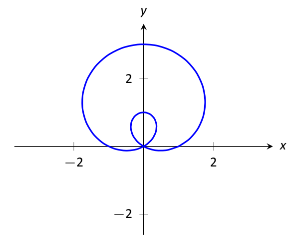
7. $A = P(2.5, \pi/4)$ and $P(-2.5, 5\pi/4)$;
$B = P(-1, 5\pi/6)$ and $P(1, 11\pi/6)$;
$C = P(3, 4\pi/3)$ and $P(-3, \pi/3)$;
$D = P(1.5, 2\pi/3)$ and $P(-1.5, 5\pi/3)$;

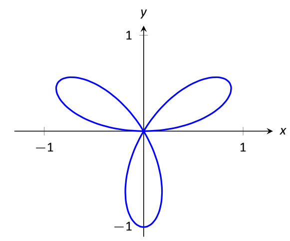
9. $A = (\sqrt{2}, \sqrt{2})$
$B = (\sqrt{2}, -\sqrt{2})$
$C = P(\sqrt{5}, -0.46)$
$D = P(\sqrt{5}, 2.68)$

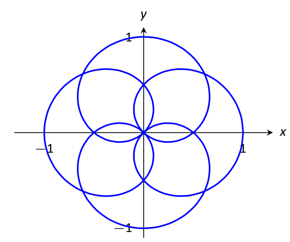
11. 

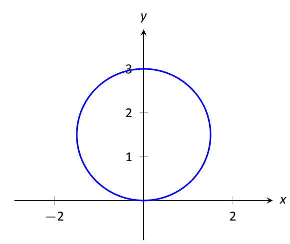
13. 

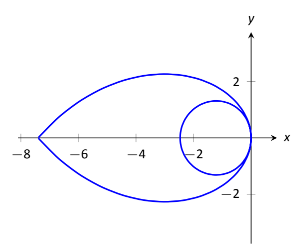
15. 

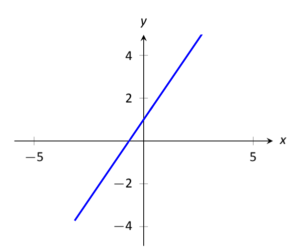
17. 

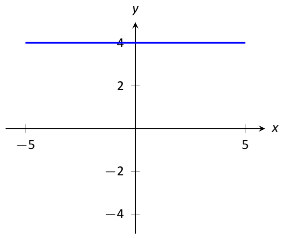
19. 

21. 

23. 

25. 

27. 

29. 

31. $x^2 + (y+2)^2 = 4$

33. $y = 2/5x + 7/5$

35. $y = 4$

37. $x^2 + y^2 = 4$

39. $\theta = \pi/4$

41. $r = 5\sec\theta$

43. $r = \cos\theta / \sin^2\theta$

45. $r = \sqrt{7}$

47. $P(\sqrt{3}/2, \pi/6)$, $P(0, \pi/2)$, $P(-\sqrt{3}/2, 5\pi/6)$

49. $P(0,0) = P(0,\pi/2)$, $P(\sqrt{2}, \pi/4)$

51. $P(\sqrt{2}/2, \pi/12)$, $P(-\sqrt{2}/2, 5\pi/12)$, $P(\sqrt{2}/2, 3\pi/4)$

53. For all points, $r = 1$; $\theta = \pi/12,\ 5\pi/12,\ 7\pi/12,\ 11\pi/12,\ 13\pi/12,\ 17\pi/12,\ 19\pi/12,\ 23\pi/12.$

55. Answers will vary. If m and n do not have any common factors, then an interval of $2n\pi$ is needed to sketch the entire graph.

Section 9.5

1. Using $x = r\cos\theta$ and $y = r\sin\theta$, we can write $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$.

3. (a) $\frac{dy}{dx} = -\cot\theta$

   (b) tangent line: $y = -(x - \sqrt{2}/2) + \sqrt{2}/2$; normal line: $y = x$

5. (a) $\frac{dy}{dx} = \frac{\cos\theta(1+2\sin\theta)}{\cos^2\theta - \sin\theta(1+\sin\theta)}$

   (b) tangent line: $x = 3\sqrt{3}/4$; normal line: $y = 3/4$

7. (a) $\frac{dy}{dx} = \frac{\theta\cos\theta + \sin\theta}{\cos\theta - \theta\sin\theta}$

   (b) tangent line: $y = -2/\pi x + \pi/2$; normal line: $y = \pi/2x + \pi/2$

9. (a) $\frac{dy}{dx} = \frac{4\sin(t)\cos(4t) + \sin(4t)\cos(t)}{4\cos(t)\cos(4t) - \sin(t)\sin(4t)}$

   (b) tangent line: $y = 5\sqrt{3}(x + \sqrt{3}/4) - 3/4$; normal line: $y = -1/5\sqrt{3}(x + \sqrt{3}/4) - 3/4$

11. horizontal: $\theta = \pi/2, 3\pi/2$;
    vertical: $\theta = 0, \pi, 2\pi$

13. horizontal: $\theta = \tan^{-1}(1/\sqrt{5}),\ \pi/2,\ \pi - \tan^{-1}(1/\sqrt{5}),\ \pi + \tan^{-1}(1/\sqrt{5}),\ 3\pi/2,\ 2\pi - \tan^{-1}(1/\sqrt{5})$;
    vertical: $\theta = 0,\ \tan^{-1}(\sqrt{5}),\ \pi - \tan^{-1}(\sqrt{5}),\ \pi,\ \pi + \tan^{-1}(\sqrt{5}),\ 2\pi - \tan^{-1}(\sqrt{5})$

15. In polar: $\theta = 0 \cong \theta = \pi$
    In rectangular: $y = 0$

17. area = 4π

19. area = $\pi/12$

21. area = $\pi - 3\sqrt{3}/2$

23. area = $\pi + 3\sqrt{3}$

25. area = $\displaystyle\int_{\pi/12}^{\pi/3} \frac{1}{2}\sin^2(3\theta)\,d\theta - \int_{\pi/12}^{\pi/6} \frac{1}{2}\cos^2(3\theta)\,d\theta = \frac{1}{12} + \frac{\pi}{24}$

27. area = $\displaystyle\int_{0}^{\pi/3} \frac{1}{2}(1-\cos\theta)^2\,d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2}(\cos\theta)^2\,d\theta = \frac{7\pi}{24} - \frac{\sqrt{3}}{2} \approx 0.0503$

29. 4π

31. $L \approx 2.2592$; (actual value $L = 2.22748$)

33. $SA = 16\pi$

35. $SA = 32\pi/5$

37. $SA = 36\pi$

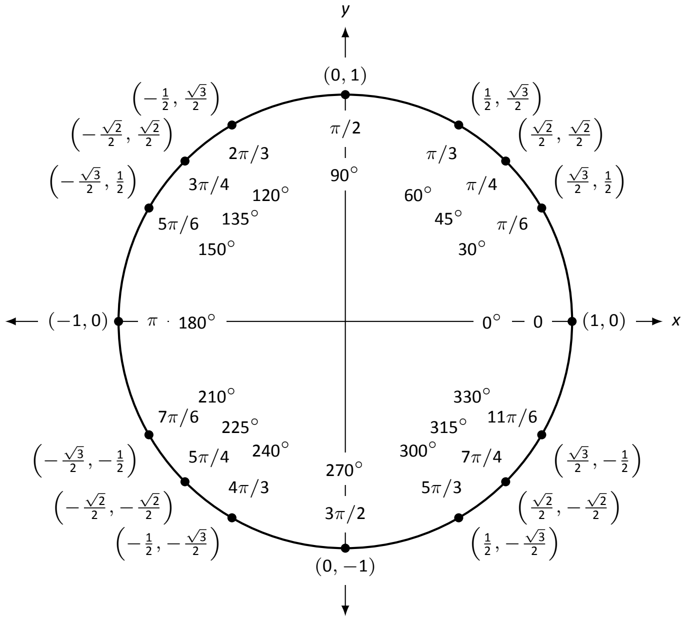
Differentiation Rules

1. $\dfrac{d}{dx}(cx) = c$
2. $\dfrac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$
3. $\dfrac{d}{dx}(f(x) \cdot g(x)) = f'(x)g(x) + f(x)g'(x)$
4. $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$
5. $\dfrac{d}{dx}(f(g(x))) = f'(g(x)g'(x)$
6. $\dfrac{d}{dx}(c) = 0$
7. $\dfrac{d}{dx}(x) = 1$
8. $\dfrac{d}{dx}(x^n) = nx^{n-1}$
9. $\dfrac{d}{dx}(e^x) = e^x$
10. $\dfrac{d}{dx}(a^x) = \ln a \cdot a^x$
11. $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$
12. $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{\ln(a)x}$
13. $\dfrac{d}{dx}(\sin x) = \cos x$
14. $\dfrac{d}{dx}(\cos x) = -\sin x$
15. $\dfrac{d}{dx}(\tan x) = \sec^2 x$
16. $\dfrac{d}{dx}(\cot x) = -\csc^2 x$
17. $\dfrac{d}{dx}(\sec x) = \sec x \tan x$
18. $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$
19. $\dfrac{d}{dx}\left(\sin^{-1} x\right) = \dfrac{1}{\sqrt{1-x^2}}$
20. $\dfrac{d}{dx}\left(\cos^{-1} x\right) = \dfrac{-1}{\sqrt{1-x^2}}$
21. $\dfrac{d}{dx}\left(\tan^{-1} x\right) = \dfrac{1}{1+x^2}$
22. $\dfrac{d}{dx}\left(\cot^{-1} x\right) = \dfrac{-1}{1+x^2}$
23. $\dfrac{d}{dx}\left(\sec^{-1} x\right) = \dfrac{1}{|x|\sqrt{x^2-1}}$
24. $\dfrac{d}{dx}\left(\csc^{-1} x\right) = \dfrac{-1}{|x|\sqrt{x^2-1}}$
25. $\dfrac{d}{dx}(\sinh x) = \cosh x$
26. $\dfrac{d}{dx}(\cosh x) = \sinh x$
27. $\dfrac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
28. $\dfrac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$
29. $\dfrac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
30. $\dfrac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$
31. $\dfrac{d}{dx}\left(\sinh^{-1} x\right) = \dfrac{1}{\sqrt{x^2+1}}$
32. $\dfrac{d}{dx}\left(\cosh^{-1} x\right) = \dfrac{1}{\sqrt{x^2-1}}$
33. $\dfrac{d}{dx}\left(\tanh^{-1} x\right) = \dfrac{1}{1-x^2}$
34. $\dfrac{d}{dx}\left(\coth^{-1} x\right) = \dfrac{1}{1-x^2}$
35. $\dfrac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = \dfrac{-1}{x\sqrt{1-x^2}}$
36. $\dfrac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = \dfrac{-1}{|x|\sqrt{1+x^2}}$

Integration Rules

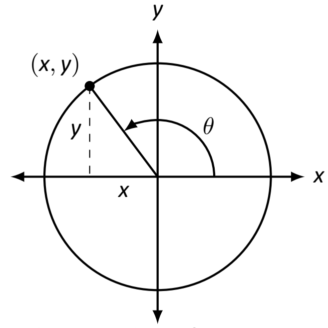
1. $\int c \cdot f(x)\,dx = c\int f(x)\,dx$
2. $\int f(x) \pm g(x)\,dx =$ $\int f(x)\,dx \pm \int g(x)\,dx$
3. $\int 0\,dx = C$
4. $\int 1\,dx = x + C$
5. $\int x^n\,dx = \dfrac{1}{n+1}x^{n+1} + C,\ n \neq -1$ $n \neq -1$
6. $\int e^x\,dx = e^x + C$
7. $\int a^x\,dx = \dfrac{1}{\ln a} \cdot a^x + C$
8. $\int \dfrac{1}{x}\,dx = \ln|x| + C$
9. $\int \sin x\,dx = -\cos x + C$
10. $\int \cos x\,dx = \sin x + C$
11. $\int \tan x\,dx = -\ln|\cos x| + C$
12. $\int \cot x\,dx = \ln|\sin x| + C$
13. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
14. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
15. $\int \sec^2 x\,dx = \tan x + C$
16. $\int \csc^2 x\,dx = -\cot x + C$
17. $\int \sec x \tan x\,dx = \sec x + C$
18. $\int \csc x \cot x\,dx = -\csc x + C$
19. $\int \sin^2 x\,dx = \dfrac{1}{2}x - \dfrac{1}{4}\sin(2x) + C$
20. $\int \cos^2 x\,dx = \dfrac{1}{2}x + \dfrac{1}{4}\sin(2x) + C$
21. $\int \dfrac{1}{x^2+a^2}\,dx = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + C$
22. $\int \dfrac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\dfrac{x}{a}\right) + C$
23. $\int \dfrac{1}{x\sqrt{x^2-a^2}}\,dx = \dfrac{1}{a}\sec^{-1}\left(\dfrac{|x|}{a}\right) + C$
24. $\int \sinh x\,dx = \cosh x + C$
25. $\int \cosh x\,dx = \sinh x + C$
26. $\int \tanh x\,dx = \ln(\cosh x) + C$
27. $\int \coth x\,dx = \ln|\sinh x| + C$
28. $\int \dfrac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left|x + \sqrt{x^2-a^2}\right| + C$
29. $\int \dfrac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left|x + \sqrt{x^2+a^2}\right| + C$
30. $\int \dfrac{1}{a^2-x^2}\,dx = \dfrac{1}{2a}\ln\left|\dfrac{a+x}{a-x}\right| + C$
31. $\int \dfrac{1}{x\sqrt{a^2-x^2}}\,dx = \dfrac{1}{a}\ln\left(\dfrac{x}{a+\sqrt{a^2-x^2}}\right) + C$
32. $\int \dfrac{1}{x\sqrt{x^2+a^2}}\,dx = \dfrac{1}{a}\ln\left|\dfrac{x}{a+\sqrt{x^2+a^2}}\right| + C$

The Unit Circle



Definitions of the Trigonometric Functions

Unit Circle Definition

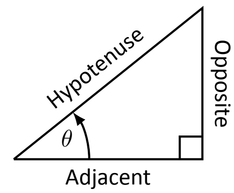


$$\sin\theta = y \qquad \cos\theta = x$$

$$\csc\theta = \frac{1}{y} \qquad \sec\theta = \frac{1}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Right Triangle Definition



$$\sin\theta = \frac{\mathrm{O}}{\mathrm{H}} \qquad \csc\theta = \frac{\mathrm{H}}{\mathrm{O}}$$

$$\cos\theta = \frac{\mathrm{A}}{\mathrm{H}} \qquad \sec\theta = \frac{\mathrm{H}}{\mathrm{A}}$$

$$\tan\theta = \frac{\mathrm{O}}{\mathrm{A}} \qquad \cot\theta = \frac{\mathrm{A}}{\mathrm{O}}$$

Common Trigonometric Identities

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x \qquad \csc\left(\frac{\pi}{2} - x\right) = \sec x$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x \qquad \sec\left(\frac{\pi}{2} - x\right) = \csc x$$

$$\tan\left(\frac{\pi}{2} - x\right) = \cot x \qquad \cot\left(\frac{\pi}{2} - x\right) = \tan x$$

Double Angle Formulas

$$\sin 2x = 2\sin x\cos x$$

$$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1 \\ &= 1 - 2\sin^2 x\end{aligned}$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Sum to Product Formulas

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Power–Reducing Formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

Even/Odd Identities

$$\sin(-x) = -\sin x$$

$$\cos(-x) = \cos x$$

$$\tan(-x) = -\tan x$$

$$\cot(-x) = -\cot x$$

$$\sec(-x) = \sec x$$

$$\csc(-x) = -\csc x$$

Product to Sum Formulas

$$\sin x\sin y = \frac{1}{2}\big(\cos(x-y) - \cos(x+y)\big)$$

$$\cos x\cos y = \frac{1}{2}\big(\cos(x-y) + \cos(x+y)\big)$$

$$\sin x\cos y = \frac{1}{2}\big(\sin(x+y) + \sin(x-y)\big)$$

Angle Sum/Difference Formulas

$$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$$

$$\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x\tan y}$$

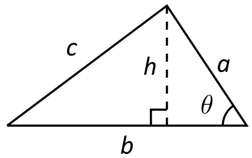
Areas and Volumes

Triangles

$h = a \sin \theta$

Area = $\frac{1}{2}bh$

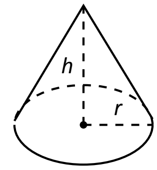
Law of Cosines:
$c^2 = a^2 + b^2 - 2ab \cos \theta$



Right Circular Cone

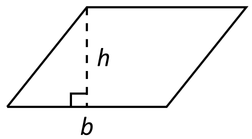
Volume = $\frac{1}{3}\pi r^2 h$

Surface Area =
$\pi r \sqrt{r^2 + h^2} + \pi r^2$



Parallelograms

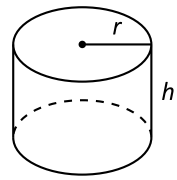
Area = bh



Right Circular Cylinder

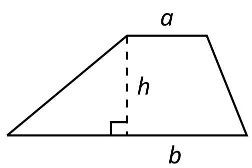
Volume = $\pi r^2 h$

Surface Area =
$2\pi rh + 2\pi r^2$



Trapezoids

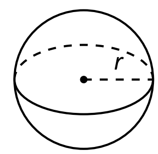
Area = $\frac{1}{2}(a + b)h$



Sphere

Volume = $\frac{4}{3}\pi r^3$

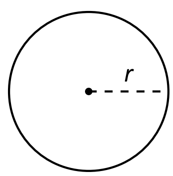
Surface Area =$4\pi r^2$



Circles

Area = πr^2

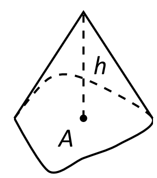
Circumference = $2\pi r$



General Cone

Area of Base = A

Volume = $\frac{1}{3}Ah$

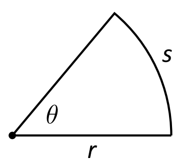


Sectors of Circles

θ in radians

Area = $\frac{1}{2}\theta r^2$

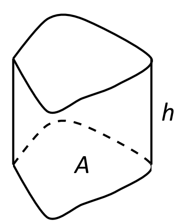
$s = r\theta$



General Right Cylinder

Area of Base = A

Volume = Ah



Algebra

Factors and Zeros of Polynomials

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial. If $p(a) = 0$, then a is a *zero* of the polynomial and a solution of the equation $p(x) = 0$. Furthermore, $(x - a)$ is a *factor* of the polynomial.

Fundamental Theorem of Algebra

An nth degree polynomial has n (not necessarily distinct) zeros. Although all of these zeros may be imaginary, a real polynomial of odd degree must have at least one real zero.

Quadratic Formula

If $p(x) = ax^2 + bx + c$, and $0 \le b^2 - 4ac$, then the real zeros of p are $x = (-b \pm \sqrt{b^2 - 4ac})/2a$

Special Factors

$x^2 - a^2 = (x - a)(x + a)$ $\qquad$ $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$

$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$ $\qquad$ $x^4 - a^4 = (x^2 - a^2)(x^2 + a^2)$

$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdots + nxy^{n-1} + y^n$

$(x - y)^n = x^n - nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 - \cdots \pm nxy^{n-1} \mp y^n$

Binomial Theorem

$(x + y)^2 = x^2 + 2xy + y^2$ $\qquad$ $(x - y)^2 = x^2 - 2xy + y^2$

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ $\qquad$ $(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ $\qquad$ $(x - y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

Rational Zero Theorem

If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients, then every *rational zero* of p is of the form $x = r/s$, where r is a factor of a_0 and s is a factor of a_n.

Factoring by Grouping

$acx^3 + adx^2 + bcx + bd = ax^2(cx + d) + b(cx + d) = (ax^2 + b)(cx + d)$

Arithmetic Operations

$$ab + ac = a(b + c) \qquad \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right) = \frac{ad}{bc} \qquad \frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc} \qquad \frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b}$$

$$a\left(\frac{b}{c}\right) = \frac{ab}{c} \qquad \frac{a - b}{c - d} = \frac{b - a}{d - c} \qquad \frac{ab + ac}{a} = b + c$$

Exponents and Radicals

$$a^0 = 1,\ \ a \ne 0 \qquad (ab)^x = a^x b^x \qquad a^x a^y = a^{x+y} \qquad \sqrt{a} = a^{1/2} \qquad \frac{a^x}{a^y} = a^{x-y} \qquad \sqrt[n]{a} = a^{1/n}$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \qquad \sqrt[n]{a^m} = a^{m/n} \qquad a^{-x} = \frac{1}{a^x} \qquad \sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b} \qquad (a^x)^y = a^{xy} \qquad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

Additional Formulas

Summation Formulas:

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left[f(x_1) + 2f(x_2) + 2f(x_3) + \dots + 2f(x_n) + f(x_{n+1})\right]$$

with Error $\leq \dfrac{(b-a)^3}{12n^2}\left[\max\left|f''(x)\right|\right]$

Simpson's Rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + \dots + 2f(x_{n-1}) + 4f(x_n) + f(x_{n+1})\right]$$

with Error $\leq \dfrac{(b-a)^5}{180n^4}\left[\max\left|f^{(4)}(x)\right|\right]$

Arc Length:

$$L = \int_a^b \sqrt{1 + f'(x)^2}\;dx$$

Surface of Revolution:

$$S = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\;dx$$

(where $f(x) \geq 0$)

$$S = 2\pi \int_a^b x\sqrt{1 + f'(x)^2}\;dx$$

(where $a, b \geq 0$)

Work Done by a Variable Force:

$$W = \int_a^b F(x)\;dx$$

Force Exerted by a Fluid:

$$F = \int_a^b w\,d(y)\,\ell(y)\;dy$$

Taylor Series Expansion for $f(x)$:

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Maclaurin Series Expansion for $f(x)$, where $c = 0$:

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

Summary of Tests for Series:

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} r^n$	$\|r\| < 1$	$\|r\| \geq 1$	Sum $= \dfrac{1}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+a})$	$\lim_{n\to\infty} b_n = L$		Sum $= \left(\sum_{n=1}^{a} b_n\right) - L$
p-Series	$\sum_{n=1}^{\infty} \dfrac{1}{(an+b)^p}$	$p > 1$	$p \leq 1$	
Integral Test	$\sum_{n=0}^{\infty} a_n$	$\int_1^{\infty} a(n)\, dn$ is convergent	$\int_1^{\infty} a(n)\, dn$ is divergent	$a_n = a(n)$ must be positive, continuous, and decreasing.
Direct Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$	$\sum_{n=0}^{\infty} b_n$ diverges and $0 \leq b_n \leq a_n$	$\{a_n\}$ must be positive
Limit Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $\lim_{n\to\infty} a_n/b_n \geq 0$ (but not ∞)	$\sum_{n=0}^{\infty} b_n$ diverges and $\lim_{n\to\infty} a_n/b_n > 0$ (or ∞)	$\{a_n\}$ must be positive
Alternating Series	$\sum_{n=0}^{\infty} (-1)^n a_n$ or $\sum_{n=0}^{\infty} (-1)^{n+1} a_n$	a_n positive, decreasing, and $\lim_{n\to\infty} a_n = 0$		Can't be used to show divergence, though if $\lim_{n\to\infty} a_n \neq 0$, it diverges by the nth-Term test.
Ratio Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} < 1$	$\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n\to\infty} a_{n+1}/a_n = \infty$
Root Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} (a_n)^{1/n} < 1$	$\lim_{n\to\infty} (a_n)^{1/n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n\to\infty} (a_n)^{1/n} = \infty$