8.2 Infinite Series

Given the sequence $\{a_n\} = \{1/2^n\} = 1/2,\ 1/4,\ 1/8,\ \ldots$, consider the following sums:

$$\begin{array}{rcccl} a_1 & = & 1/2 & = & 1/2 \\ a_1 + a_2 & = & 1/2 + 1/4 & = & 3/4 \\ a_1 + a_2 + a_3 & = & 1/2 + 1/4 + 1/8 & = & 7/8 \\ a_1 + a_2 + a_3 + a_4 & = & 1/2 + 1/4 + 1/8 + 1/16 & = & 15/16 \end{array}$$

In general, we can show that

$$a_1 + a_2 + a_3 + \cdots + a_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}.$$

Let S_n be the sum of the first n terms of the sequence $\{1/2^n\}$. From the above, we see that $S_1 = 1/2$, $S_2 = 3/4$, etc. Our formula at the end shows that $S_n = 1 - 1/2^n$.

Now consider the following limit: $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \left(1 - 1/2^n\right) = 1$. This limit can be interpreted as saying something amazing: *the sum of* all *the terms of the sequence* $\{1/2^n\}$ *is 1.*

This example illustrates some interesting concepts that we explore in this section. We begin this exploration with some definitions.

Definition 34 Infinite Series, n^{th} Partial Sums, Convergence, Divergence

Let $\{a_n\}$ be a sequence.

1. The sum $\displaystyle\sum_{n=1}^{\infty} a_n$ is an **infinite series** (or simply **series**).
2. Let $S_n = \displaystyle\sum_{i=1}^{n} a_i$; the sequence $\{S_n\}$ is the sequence of n^{th} **partial sums** of $\{a_n\}$.
3. If the sequence $\{S_n\}$ converges to L, we say the series $\displaystyle\sum_{n=1}^{\infty} a_n$ **converges** to L, and we write $\displaystyle\sum_{n=1}^{\infty} a_n = L$.
4. If the sequence $\{S_n\}$ diverges, the series $\displaystyle\sum_{n=1}^{\infty} a_n$ **diverges**.

Notes:

Using our new terminology, we can state that the series $\sum_{n=1}^{\infty} 1/2^n$ converges, and $\sum_{n=1}^{\infty} 1/2^n = 1$.

We will explore a variety of series in this section. We start with two series that diverge, showing how we might discern divergence.

Example 8.8 Showing series diverge

1. Let $\{a_n\} = \{n^2\}$. Show $\sum_{n=1}^{\infty} a_n$ diverges.
2. Let $\{b_n\} = \{(-1)^{n+1}\}$. Show $\sum_{n=1}^{\infty} b_n$ diverges.

Solution

1. Consider S_n, the n^{th} partial sum.

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \cdots + a_n \\ &= 1^2 + 2^2 + 3^2 \cdots + n^2. \end{aligned}$$

   By Theorem 39, this is

$$= \frac{n(n+1)(2n+1)}{6}.$$

   Since $\lim_{n\to\infty} S_n = \infty$, we conclude that the series $\sum_{n=1}^{\infty} n^2$ diverges. It is instructive to write $\sum_{n=1}^{\infty} n^2 = \infty$ for this tells us *how* the series diverges: it grows without bound.

   A scatter plot of the sequences $\{a_n\}$ and $\{S_n\}$ is given in Figure 8.7(a). The terms of $\{a_n\}$ are growing, so the terms of the partial sums $\{S_n\}$ are growing even faster, illustrating that the series diverges.

2. The sequence $\{b_n\}$ starts with 1, -1, 1, -1, ... Consider some of the

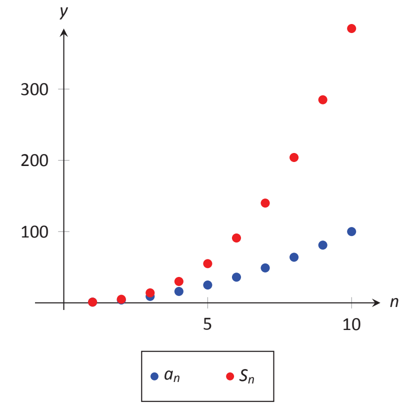
Notes:

partial sums S_n of $\{b_n\}$:

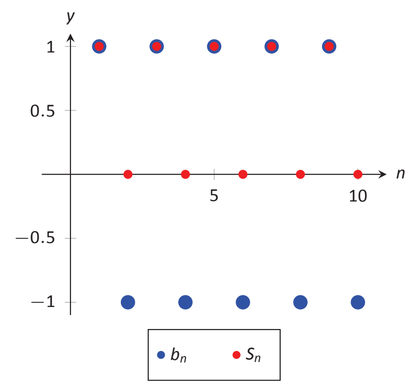
$$S_1 = 1$$
$$S_2 = 0$$
$$S_3 = 1$$
$$S_4 = 0$$

This pattern repeats; we find that $S_n = \begin{cases} 1 & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$. As $\{S_n\}$ oscillates, repeating 1, 0, 1, 0, ..., we conclude that $\lim_{n\to\infty} S_n$ does not exist, hence $\sum_{n=1}^{\infty}(-1)^{n+1}$ diverges.

A scatter plot of the sequence $\{b_n\}$ and the partial sums $\{S_n\}$ is given in Figure 8.7(b). When n is odd, $b_n = S_n$ so the marks for b_n are drawn oversized to show they coincide.



(a)



(b)

Figure 8.7: Scatter plots relating to Example 8.8.

While it is important to recognize when a series diverges, we are generally more interested in the series that converge. In this section we will demonstrate a few general techniques for determining convergence; later sections will delve deeper into this topic.

Geometric Series

One important type of series is a *geometric series*.

Definition 35 Geometric Series

A **geometric series** is a series of the form

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots + r^n + \cdots$$

Note that the index starts at $n = 0$, not $n = 1$.

We started this section with a geometric series, although we dropped the first term of 1. One reason geometric series are important is that they have nice convergence properties.

Notes:

Theorem 62 Convergence of Geometric Series

Consider the geometric series $\displaystyle\sum_{n=0}^{\infty} r^n$.

1. The n^{th} partial sum is: $S_n = \dfrac{1-r^{n+1}}{1-r}$.
2. The series converges if, and only if, $|r| < 1$. When $|r| < 1$,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

According to Theorem 62, the series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$$

converges as $r = 1/2$, and $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-1/2} = 2$. This concurs with our introductory example; while there we got a sum of 1, we skipped the first term of 1.

Example 8.9 Exploring geometric series

Check the convergence of the following series. If the series converges, find its sum.

1. $\displaystyle\sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n$
2. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n$
3. $\displaystyle\sum_{n=0}^{\infty} 3^n$

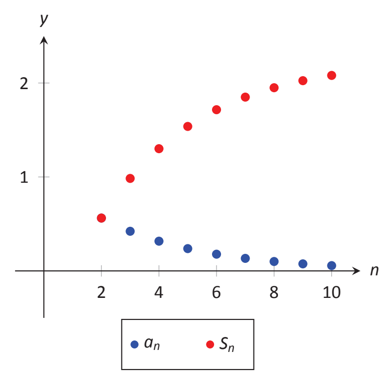


Figure 8.8: Scatter plots relating to the series in Example 8.9.

SOLUTION

1. Since $r = 3/4 < 1$, this series converges. By Theorem 62, we have that

$$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = \frac{1}{1-3/4} = 4.$$

However, note the subscript of the summation in the given series: we are to start with $n = 2$. Therefore we subtract off the first two terms, giving:

$$\sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n = 4 - 1 - \frac{3}{4} = \frac{9}{4}.$$

This is illustrated in Figure 8.8.

Notes:

2. Since $|r| = 1/2 < 1$, this series converges, and by Theorem 62,

$$\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n = \frac{1}{1-(-1/2)} = \frac{2}{3}.$$

   The partial sums of this series are plotted in Figure 8.9(a). Note how the partial sums are not purely increasing as some of the terms of the sequence $\{(-1/2)^n\}$ are negative.

3. Since $r > 1$, the series diverges. (This makes "common sense"; we expect the sum

$$1 + 3 + 9 + 27 + 81 + 243 + \cdots$$

   to diverge.) This is illustrated in Figure 8.9(b).

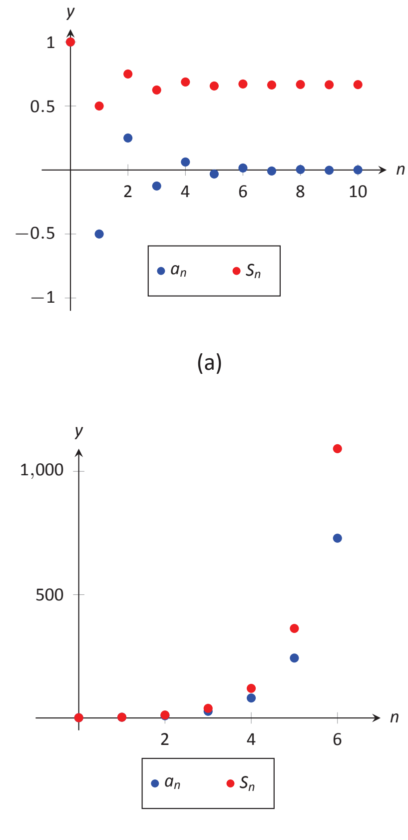


Figure 8.9: Scatter plots relating to the series in Example 8.9.

The Harmonic Series

The series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots$$

is called the *Harmonic Series* because of its relationship to *harmonics* in the study of music and sound. Even though the terms being added are approaching 0, this series diverges, as the sequence of partial sums approaches ∞, as we prove in the following example.

Example 8.10 The Harmonic Series diverges

Show that the Harmonic Series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Solution We begin by realizing the following:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots \\ &\geq 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{=1/2} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{=1/2} + \cdots\end{aligned}$$

We continue by noticing that the terms from $1/9$ through $1/16$ added in the Harmonic series are eight terms all $\geq 1/16$, so they add to 1/2. Then the terms

Notes:

$1/17$ through $1/32$ are sixteen terms all $\geq 1/32$, so they add to 1/2. It should be clear that this pattern continues when we group terms up to a power of 2. Therefore,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n} &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots\\ &\geq 1+\frac{1}{2}+\underbrace{\frac{1}{4}+\frac{1}{4}}_{=1/2}+\underbrace{\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}}_{=1/2}+\cdots\\ &= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\end{aligned}$$

It should be clear that adding infinitely many $1/2$ terms gives ∞. Then the infinite series $\sum\limits_{n=1}^{\infty}$ is greater than or equal to a series that diverges to ∞. Therefore, the Harmonic Series must diverge to ∞.

On the other hand, a related series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\cdots$, called the *Alternating Harmonic Series* actually converges to $\ln 2$. We cannot explain why at the moment. In Section 8.5, we will be able to explain why it converges. However, it won't be until Taylor Series are discussed in Section 8.8 that we can explain why the limit is, in fact, $\ln 2$.

Later sections will provide tests by which we can determine whether or not a given series converges. This, in general, is much easier than determining *what* a given series converges to. There are many cases, though, where the sum can be determined.

Example 8.11 Telescoping series

Evaluate the sum $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$.

Solution It will help to write down some of the first few partial sums

Notes:

of this series.

$$
\begin{aligned}
S_1 &= \frac{1}{1} - \frac{1}{2} && = 1 - \frac{1}{2} \\
S_2 &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) && = 1 - \frac{1}{3} \\
S_3 &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) && = 1 - \frac{1}{4} \\
S_4 &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) && = 1 - \frac{1}{5}
\end{aligned}
$$

Note how most of the terms in each partial sum are canceled out! In general, we see that $S_n = 1 - \dfrac{1}{n+1}$. The sequence $\{S_n\}$ converges, as $\displaystyle\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$, and so we conclude that $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 1$. Partial sums of the series are plotted in Figure 8.10.

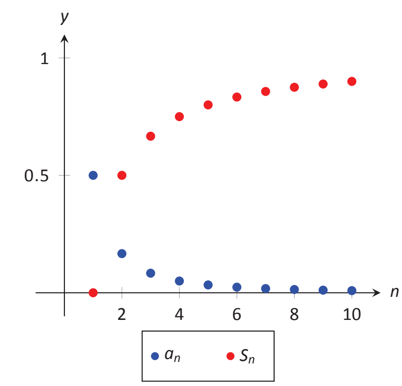


Figure 8.10: Scatter plots relating to the series of Example 8.11.

The series in Example 8.11 is an example of a **telescoping series**. Informally, a telescoping series is one in which the partial sums reduce to just a finite number of terms. The partial sum S_n did not contain n terms, but rather just two: 1 and $1/(n+1)$.

When possible, seek a way to write an explicit formula for the n^{th} partial sum S_n. This makes evaluating the limit $\displaystyle\lim_{n\to\infty} S_n$ much more approachable. We do so in the next example.

Example 8.12 Evaluating series

Evaluate each of the following infinite series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+2n}$ 2. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

Solution

1. We can decompose the fraction $2/(n^2+2n)$ as

$$\frac{2}{n^2+2n} = \frac{1}{n} - \frac{1}{n+2}.$$

(See Section 6.4, Partial Fraction Decomposition, to recall how this is done, if necessary.)

Notes:

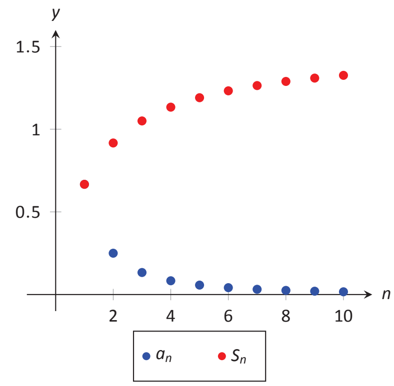
Expressing the terms of $\{S_n\}$ is now more instructive:

$$
\begin{aligned}
S_1 &= 1-\frac{1}{3} && = 1-\frac{1}{3}\\
S_2 &= \left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right) && = 1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}\\
S_3 &= \left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right) && = 1+\frac{1}{2}-\frac{1}{4}-\frac{1}{5}\\
S_4 &= \left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right) && = 1+\frac{1}{2}-\frac{1}{5}-\frac{1}{6}\\
S_5 &= \left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\left(\frac{1}{5}-\frac{1}{7}\right) && = 1+\frac{1}{2}-\frac{1}{6}-\frac{1}{7}
\end{aligned}
$$

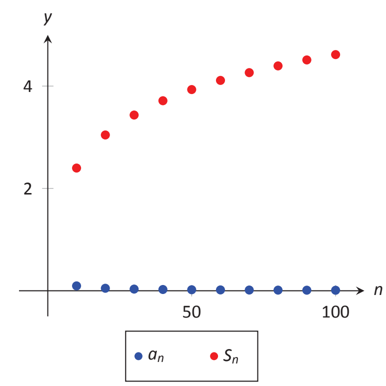
We again have a telescoping series. In each partial sum, most of the terms cancel and we obtain the formula $S_n = 1+\dfrac{1}{2}-\dfrac{1}{n+1}-\dfrac{1}{n+2}$. Taking limits allows us to determine the convergence of the series:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right)=\frac{3}{2}, \quad \text{so} \sum_{n=1}^{\infty}\frac{1}{n^2+2n}=\frac{3}{2}.$$

This is illustrated in Figure 8.11(a).



(a)



(b)

Figure 8.11: Scatter plots relating to the series in Example 8.12.

2. We begin by writing the first few partial sums of the series:

$$
\begin{aligned}
S_1 &= \ln(2)\\
S_2 &= \ln(2)+\ln\left(\frac{3}{2}\right)\\
S_3 &= \ln(2)+\ln\left(\frac{3}{2}\right)+\ln\left(\frac{4}{3}\right)\\
S_4 &= \ln(2)+\ln\left(\frac{3}{2}\right)+\ln\left(\frac{4}{3}\right)+\ln\left(\frac{5}{4}\right)
\end{aligned}
$$

At first, this does not seem helpful, but recall the logarithmic identity: $\ln x+\ln y=\ln(xy)$. Applying this to S_4 gives:

$$S_4=\ln(2)+\ln\left(\frac{3}{2}\right)+\ln\left(\frac{4}{3}\right)+\ln\left(\frac{5}{4}\right)=\ln\left(\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\right)=\ln(5).$$

We can conclude that $\{S_n\}=\big\{\ln(n+1)\big\}$. This sequence does not converge, as $\lim\limits_{n\to\infty} S_n=\infty$. Therefore $\displaystyle\sum_{n=1}^{\infty}\ln\left(\frac{n+1}{n}\right)=\infty$; the series diverges. Note in Figure 8.11(b) how the sequence of partial sums grows

Notes:

slowly; after 100 terms, it is not yet over 5. (Only some terms of a_n and S_n are shown in the plot.) Graphically we may be fooled into thinking the series converges, but our analysis above shows that it does not.

We are learning about a new mathematical object, the series. As done before, we apply "old" mathematics to this new topic.

Theorem 63 Properties of Infinite Series

Let $\displaystyle\sum_{n=1}^{\infty} a_n = L$, $\displaystyle\sum_{n=1}^{\infty} b_n = K$, and let c be a constant.

1. Constant Multiple Rule: $\displaystyle\sum_{n=1}^{\infty} c \cdot a_n = c \cdot \sum_{n=1}^{\infty} a_n = c \cdot L$.

2. Sum/Difference Rule: $\displaystyle\sum_{n=1}^{\infty} \left(a_n \pm b_n\right) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n = L \pm K$.

Before using this theorem, we provide a few "famous" series.

Key Idea 33 Important Series

1. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} = e$. (Note that the index starts with $n = 0$.)

2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$.

4. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. (*Harmonic Series*.)

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$. (*Alternating Harmonic Series*.)

Notes:

Example 8.13 **Evaluating series**

Evaluate the given series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n^2-n)}{n^3}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{1000}{n!}$
3. $\displaystyle\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\frac{1}{49}+\cdots$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n!}$

Solution

1. We start by using algebra to break the series apart:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n^2-n)}{n^3} &= \sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}n^2}{n^3}-\frac{(-1)^{n+1}n}{n^3}\right)\\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}-\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\\ &= \ln(2)-\frac{\pi^2}{12}\approx -0.1293.\end{aligned}$$

This is illustrated in Figure 8.12(a).

2. This looks very similar to the series that involves e in Key Idea 33. Note, however, that the series given in this example starts with $n=1$ and not $n=0$. The first term of the series in the Key Idea is $1/0!=1$, so we will subtract this from our result below:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1000}{n!} &= 1000\cdot\sum_{n=1}^{\infty}\frac{1}{n!}\\ &= 1000\cdot(e-1)\approx 1718.28.\end{aligned}$$

This is illustrated in Figure 8.12(b). The graph shows how this particular series converges very rapidly.

3. The denominators in each term are perfect squares; we are adding $\displaystyle\sum_{n=4}^{\infty}\frac{1}{n^2}$ (note we start with $n=4$, not $n=1$). This series will converge. Using the

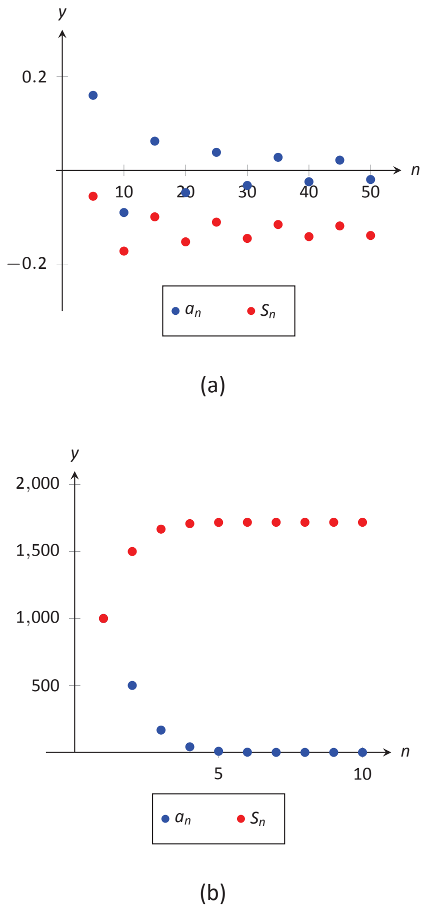


Figure 8.12: Scatter plots relating to the series in Example 8.13.

Notes:

formula from Key Idea 33, we have the following:

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n^2} &= \sum_{n=1}^{3}\frac{1}{n^2}+\sum_{n=4}^{\infty}\frac{1}{n^2}\\
\sum_{n=1}^{\infty}\frac{1}{n^2}-\sum_{n=1}^{3}\frac{1}{n^2} &= \sum_{n=4}^{\infty}\frac{1}{n^2}\\
\frac{\pi^2}{6}-\left(\frac{1}{1}+\frac{1}{4}+\frac{1}{9}\right) &= \sum_{n=4}^{\infty}\frac{1}{n^2}\\
\frac{\pi^2}{6}-\frac{49}{36} &= \sum_{n=4}^{\infty}\frac{1}{n^2}\\
0.2838 &\approx \sum_{n=4}^{\infty}\frac{1}{n^2}
\end{aligned}$$

4. We manipulate this series to look like the series that involves e in Key Idea 33. In the steps below, we twice relabel n as $n+1$ to write it in terms of the series in the table. Note that $n! = n(n-1)!$.

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{n^2}{n!} &= \sum_{n=1}^{\infty}\frac{n}{(n-1)!}\\
&= \sum_{n=0}^{\infty}\frac{n+1}{n!}\\
&= \sum_{n=0}^{\infty}\frac{n}{n!}+\sum_{n=0}^{\infty}\frac{1}{n!}\\
&= \sum_{n=1}^{\infty}\frac{1}{(n-1)!}+e\\
&= \sum_{n=0}^{\infty}\frac{1}{n!}+e\\
&= e+e\\
&= 2e.
\end{aligned}$$

It may take a while before one is comfortable with this statement, whose truth lies at the heart of the study of infinite series: *it is possible that the sum of an infinite list of nonzero numbers is finite.* We have seen this repeatedly in this section, yet it still may "take some getting used to."

As one contemplates the behavior of series, a few facts become clear.

Notes:

1. In order to add an infinite list of nonzero numbers and get a finite result, "most" of those numbers must be "very near" 0.

2. If a series diverges, it means that the sum of an infinite list of numbers is not finite (it may approach $\pm\infty$ or it may oscillate), and:

   (a) The series will still diverge if the first term is removed.

   (b) The series will still diverge if the first 10 terms are removed.

   (c) The series will still diverge if the first $1,000,000$ terms are removed.

   (d) The series will still diverge if any finite number of terms from anywhere in the series are removed.

These concepts are very important and lie at the heart of the next two theorems.

Theorem 64 $\quad$ **n^{th}–Term Test for Convergence/Divergence**

Consider the series $\displaystyle\sum_{n=1}^{\infty} a_n$.

1. If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\lim_{n\to\infty} a_n = 0$.

2. If $\displaystyle\lim_{n\to\infty} a_n \neq 0$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

Note that the two statements in Theorem 64 are really the same. In order to converge, the limit of the terms of the sequence must approach 0; if they do not, the series will not converge.

Looking back, we can apply this theorem to the series in Example 8.8. In that example, the n^{th} terms of both sequences do not converge to 0, therefore we can quickly conclude that each series diverges.

Important! This theorem *does not state* that if $\displaystyle\lim_{n\to\infty} a_n = 0$ then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. The standard example of this is the Harmonic Series. The Harmonic Sequence, $\{1/n\}$, converges to 0; the Harmonic Series, $\displaystyle\sum_{n=1}^{\infty} 1/n$, diverges.

Notes:

Theorem 65 Infinite Nature of Series

The convergence or divergence remains unchanged by the addition or subtraction of any finite number of terms. That is:

1. A divergent series will remain divergent with the addition or subtraction of any finite number of terms.

2. A convergent series will remain convergent with the addition or subtraction of any finite number of terms. (Of course, the *sum* will likely change.)

Consider once more the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges; that is, the sequence of partial sums $\{S_n\}$ grows (very, very slowly) without bound. One might think that by removing the "large" terms of the sequence that perhaps the series will converge. This is simply not the case. For instance, the sum of the first 10 million terms of the Harmonic Series is about 16.7. Removing the first 10 million terms from the Harmonic Series changes the n^{th} partial sums, effectively subtracting 16.7 from the sum. However, a sequence that is growing without bound will still grow without bound when 16.7 is subtracted from it.

The equations below illustrate this. The first line shows the infinite sum of the Harmonic Series split into the sum of the first 10 million terms plus the sum of "everything else." The next equation shows us subtracting these first 10 million terms from both sides. The final equation employs a bit of "psuedo–math": subtracting 16.7 from "infinity" still leaves one with "infinity."

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} &= \sum_{n=1}^{10{,}000{,}000} \frac{1}{n} + \sum_{n=10{,}000{,}001}^{\infty} \frac{1}{n} \\
\sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{10{,}000{,}000} \frac{1}{n} &= \sum_{n=10{,}000{,}001}^{\infty} \frac{1}{n} \\
\infty - 16.7 &= \infty.
\end{aligned}$$

This section introduced us to series and defined a few special types of series whose convergence properties are well known, such as geometric and telescoping series. Most series that we encounter are not one of these types, but we are still interested in knowing whether or not they converge. The next three sections introduce tests that help us determine whether or not a given series converges.

Notes:

Exercises 8.2

Terms and Concepts

1. Use your own words to describe how sequences and series are related.

2. Use your own words to define a *partial sum*.

3. Given a series $\sum_{n=1}^{\infty} a_n$, describe the two sequences related to the series that are important.

4. Use your own words to explain what a geometric series is.

5. T/F: If $\{a_n\}$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

Problems

In Exercises 6 – 13, a series $\sum_{n=1}^{\infty} a_n$ is given.

(a) Give the first 5 partial sums of the series.

(b) Give a graph of the first 5 terms of a_n and S_n on the same axes.

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

8. $\displaystyle\sum_{n=1}^{\infty} \cos(\pi n)$

9. $\displaystyle\sum_{n=1}^{\infty} n$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$

12. $\displaystyle\sum_{n=1}^{\infty} \left(-\frac{9}{10}\right)^n$

13. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$

In Exercises 14 – 19, use Theorem 64 to show the given series diverges.

14. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2}{n(n+2)}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n - n^5}{5^n + n^5}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 1}{2^{n+1}}$

19. $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$

In Exercises 20 – 24, state whether the given series converges or diverges.

20. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{5^n}$

21. $\displaystyle\sum_{n=0}^{\infty} \frac{6^n}{5^n}$

22. $\displaystyle\sum_{n=1}^{\infty} \sqrt{n}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{10}{n!}$

24. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n!} + \frac{1}{n}\right)$

In Exercises 25 – 39, a series is given.

(a) Find a formula for S_n, the n^{th} partial sum of the series.

(b) Determine whether the series converges or diverges. If it converges, state what it converges to.

25. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{4^n}$

26. $1^3 + 2^3 + 3^3 + 4^3 + \cdots$

27. $\displaystyle\sum_{n=1}^{\infty} (-1)^n n$

28. $\displaystyle\sum_{n=0}^{\infty} \frac{5}{2^n}$

29. $\displaystyle\sum_{n=1}^{\infty} e^{-n}$

30. $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} + \cdots$

31. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+2)}$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$

34. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

36. $\frac{1}{1\cdot 4} + \frac{1}{2\cdot 5} + \frac{1}{3\cdot 6} + \frac{1}{4\cdot 7} + \cdots$

37. $2 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{9}\right) + \left(\frac{1}{8} + \frac{1}{27}\right) + \cdots$

38. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$

39. $\displaystyle\sum_{n=0}^{\infty} \left(\sin 1\right)^n$

40. Break the Harmonic Series into the sum of the odd and even terms:

$$\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{2n-1} + \sum_{n=1}^{\infty} \frac{1}{2n}.$$

The goal is to show that each of the series on the right diverge.

(a) Show why $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1} > \sum_{n=1}^{\infty} \frac{1}{2n}$.

(Compare each n^{th} partial sum.)

(b) Show why $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1} < 1 + \sum_{n=1}^{\infty} \frac{1}{2n}$

(c) Explain why (a) and (b) demonstrate that the series of odd terms is convergent, if, and only if, the series of even terms is also convergent. (That is, show both converge or both diverge.)

(d) Explain why knowing the Harmonic Series is divergent determines that the even and odd series are also divergent.

41. Show the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(2n-1)(2n+1)}$ diverges.

42. Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n!}$ using the series in Key Idea 33.