

Figure 12.7: Illustrating open and closed sets in the x - y plane.

12.2 Limits and Continuity of Multivariable Functions

We continue with the pattern we have established in this text: after defining a new kind of function, we apply calculus ideas to it. The previous section defined functions of two and three variables; this section investigates what it means for these functions to be “continuous.”

We begin with a series of definitions. We are used to “open intervals” such as $(1, 3)$, which represents the set of all x such that $1 < x < 3$, and “closed intervals” such as $[1, 3]$, which represents the set of all x such that $1 \leq x \leq 3$. We need analogous definitions for open and closed sets in the x - y plane.

Definition 83 Open Disk, Boundary and Interior Points, Open and Closed Sets, Bounded Sets

An **open disk** B in \mathbb{R}^2 centered at (x_0, y_0) with radius r is the set of all points (x, y) such that $\sqrt{(x - x_0)^2 + (y - y_0)^2} < r$.

Let S be a set of points in \mathbb{R}^2 . A point P in \mathbb{R}^2 is a **boundary point** of S if all open disks centered at P contain both points in S and points not in S .

A point P in S is an **interior point** of S if there is an open disk centered at P that contains only points in S .

A set S is **open** if every point in S is an interior point.

A set S is **closed** if it contains all of its boundary points.

A set S is **bounded** if there is an $M > 0$ such that the open disk, centered at the origin with radius M , contains S . A set that is not bounded is **unbounded**.

Figure 12.7 shows several sets in the x - y plane. In each set, point P_1 lies on the boundary of the set as all open disks centered there contain both points in, and not in, the set. In contrast, point P_2 is an interior point for there is an open disk centered there that lies entirely within the set.

The set depicted in Figure 12.7(a) is a closed set as it contains all of its boundary points. The set in (b) is open, for all of its points are interior points (or, equivalently, it does not contain any of its boundary points). The set in (c) is neither open nor closed as it contains some of its boundary points.

Notes:

Example 12.7 Determining open/closed, bounded/unbounded

Determine if the domain of the function $f(x, y) = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}$ is open, closed, or neither, and if it is bounded.

SOLUTION This domain of this function was found in Example 12.2 to be $D = \{(x, y) \mid \frac{x^2}{9} + \frac{y^2}{4} \leq 1\}$, the region *bounded* by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$. Since the region includes the boundary (indicated by the use of “ \leq ”), the set contains all of its boundary points and hence is closed. The region is bounded as a disk of radius 4, centered at the origin, contains D .

Example 12.8 Determining open/closed, bounded/unbounded

Determine if the domain of $f(x, y) = \frac{1}{x-y}$ is open, closed, or neither.

SOLUTION As we cannot divide by 0, we find the domain to be $D = \{(x, y) \mid x - y \neq 0\}$. In other words, the domain is the set of all points (x, y) *not* on the line $y = x$.

The domain is sketched in Figure 12.8. Note how we can draw an open disk around any point in the domain that lies entirely inside the domain, and also note how the only boundary points of the domain are the points on the line $y = x$. We conclude the domain is an open set. The set is unbounded.

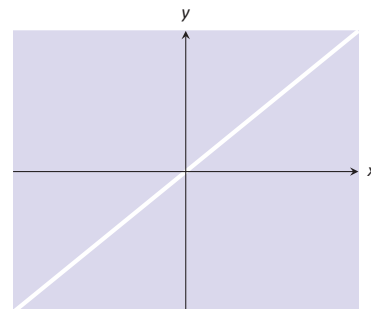


Figure 12.8: Sketching the domain of the function in Example 12.8.

Limits

Recall a pseudo-definition of the limit of a function of one variable: “ $\lim_{x \rightarrow c} f(x) = L$ ” means that if x is “really close” to c , then $f(x)$ is “really close” to L . A similar pseudo-definition holds for functions of two variables. We’ll say that

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = L$$

means “if the point (x, y) is really close to the point (x_0, y_0) , then $f(x, y)$ is really close to L .” The formal definition is given below.

Definition 84 Limit of a Function of Two Variables

Let S be an open set containing (x_0, y_0) , and let f be a function of two variables defined on S , except possibly at (x_0, y_0) . The **limit** of $f(x, y)$ as (x, y) approaches (x_0, y_0) is L , denoted

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = L,$$

means that given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $(x, y) \neq (x_0, y_0)$, if (x, y) is in the open disk centered at (x_0, y_0) with radius δ , then $|f(x, y) - L| < \varepsilon$.

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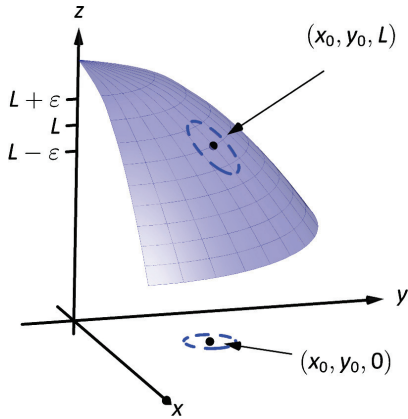


Figure 12.9: **Illustrating the definition of a limit.** The open disk in the x - y plane has radius δ . Let (x, y) be any point in this disk; $f(x, y)$ is within ε of L .

The concept behind Definition 84 is sketched in Figure 12.9. Given $\varepsilon > 0$, find $\delta > 0$ such that if (x, y) is any point in the open disk centered at (x_0, y_0) in the x - y plane with radius δ , then $f(x, y)$ should be within ε of L .

Computing limits using this definition is rather cumbersome. The following theorem allows us to evaluate limits much more easily.

Theorem 103 Basic Limit Properties of Functions of Two Variables

Let b, x_0, y_0, L and K be real numbers, let n be a positive integer, and let f and g be functions with the following limits:

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = L \quad \text{and} \quad \lim_{(x,y) \rightarrow (x_0,y_0)} g(x, y) = K.$$

The following limits hold.

1. Constants: $\lim_{(x,y) \rightarrow (x_0,y_0)} b = b$
2. Identity $\lim_{(x,y) \rightarrow (x_0,y_0)} x = x_0; \quad \lim_{(x,y) \rightarrow (x_0,y_0)} y = y_0$
3. Sums/Differences: $\lim_{(x,y) \rightarrow (x_0,y_0)} (f(x, y) \pm g(x, y)) = L \pm K$
4. Scalar Multiples: $\lim_{(x,y) \rightarrow (x_0,y_0)} b \cdot f(x, y) = bL$
5. Products: $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) \cdot g(x, y) = LK$
6. Quotients: $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y)/g(x, y) = L/K, (K \neq 0)$
7. Powers: $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y)^n = L^n$

This theorem, combined with Theorems 2 and 3 of Section 1.3, allows us to evaluate many limits.

Example 12.9 Evaluating a limit

Evaluate the following limits:

$$1. \lim_{(x,y) \rightarrow (1,\pi)} \frac{y}{x} + \cos(xy)$$

$$2. \lim_{(x,y) \rightarrow (0,0)} \frac{3xy}{x^2 + y^2}$$

Notes:

SOLUTION

1. The aforementioned theorems allow us to simply evaluate $y/x + \cos(xy)$ when $x = 1$ and $y = \pi$. If an indeterminate form is returned, we must do more work to evaluate the limit; otherwise, the result is the limit. Therefore

$$\begin{aligned}\lim_{(x,y) \rightarrow (1,\pi)} \frac{y}{x} + \cos(xy) &= \frac{\pi}{1} + \cos \pi \\ &= \pi - 1.\end{aligned}$$

2. We attempt to evaluate the limit by substituting 0 in for x and y , but the result is the indeterminate form “0/0.” To evaluate this limit, we must “do more work,” but we have not yet learned what “kind” of work to do. Therefore we cannot yet evaluate this limit.

When dealing with functions of a single variable we also considered one-sided limits and stated

$$\lim_{x \rightarrow c} f(x) = L \quad \text{if, and only if,} \quad \lim_{x \rightarrow c^+} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c^-} f(x) = L.$$

That is, the limit is L if and only if $f(x)$ approaches L when x approaches c from **either** direction, the left or the right.

In the plane, there are infinite directions from which (x, y) might approach (x_0, y_0) . In fact, we do not have to restrict ourselves to approaching (x_0, y_0) from a particular direction, but rather we can approach that point along a path that is not a straight line. It is possible to arrive at different limiting values by approaching (x_0, y_0) along different paths. If this happens, we say that $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y)$ does not exist (this is analogous to the left and right hand limits of single variable functions not being equal).

Our theorems tell us that we can evaluate most limits quite simply, without worrying about paths. When indeterminate forms arise, the limit may or may not exist. If it does exist, it can be difficult to prove this as we need to show the same limiting value is obtained regardless of the path chosen. The case where the limit does not exist is often easier to deal with, for we can often pick two paths along which the limit is different.

Example 12.10 **Showing limits do not exist**

1. Show $\lim_{(x,y) \rightarrow (0,0)} \frac{3xy}{x^2 + y^2}$ does not exist by finding the limits along the lines $y = mx$.

Notes:

2. Show $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(xy)}{x+y}$ does not exist by finding the limit along the path $y = -\sin x$.

SOLUTION

1. Evaluating $\lim_{(x,y) \rightarrow (0,0)} \frac{3xy}{x^2 + y^2}$ along the lines $y = mx$ means replace all y 's with mx and evaluating the resulting limit:

$$\begin{aligned} \lim_{(x,mx) \rightarrow (0,0)} \frac{3x(mx)}{x^2 + (mx)^2} &= \lim_{x \rightarrow 0} \frac{3mx^2}{x^2(m^2 + 1)} \\ &= \lim_{x \rightarrow 0} \frac{3m}{m^2 + 1} \\ &= \frac{3m}{m^2 + 1}. \end{aligned}$$

While the limit exists for each choice of m , we get a *different* limit for each choice of m . That is, along different lines we get differing limiting values, meaning *the* limit does not exist.

2. Let $f(x, y) = \frac{\sin(xy)}{x+y}$. We are to show that $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ does not exist by finding the limit along the path $y = -\sin x$. First, however, consider the limits found along the lines $y = mx$ as done above.

$$\begin{aligned} \lim_{(x,mx) \rightarrow (0,0)} \frac{\sin(x(mx))}{x + mx} &= \lim_{x \rightarrow 0} \frac{\sin(mx^2)}{x(m+1)} \\ &= \lim_{x \rightarrow 0} \frac{\sin(mx^2)}{x} \cdot \frac{1}{m+1}. \end{aligned}$$

By applying L'Hôpital's Rule, we can show this limit is 0 *except* when $m = -1$, that is, along the line $y = -x$. This line is not in the domain of f , so we have found the following fact: along every line $y = mx$ in the domain of f , $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$.

Now consider the limit along the path $y = -\sin x$:

$$\lim_{(x, -\sin x) \rightarrow (0,0)} \frac{\sin(-x \sin x)}{x - \sin x} = \lim_{x \rightarrow 0} \frac{\sin(-x \sin x)}{x - \sin x}$$

Now apply L'Hôpital's Rule twice:

$$\begin{aligned} &= \lim_{x \rightarrow 0} \frac{\cos(-x \sin x)(-\sin x - x \cos x)}{1 - \cos x} \quad ("= 0/0") \\ &= \lim_{x \rightarrow 0} \frac{-\sin(-x \sin x)(-\sin x - x \cos x)^2 + \cos(-x \sin x)(-2 \cos x + x \sin x)}{\sin x} \\ &= "2/0" \Rightarrow \text{the limit does not exist.} \end{aligned}$$

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Step back and consider what we have just discovered. Along any line $y = mx$ in the domain of the $f(x, y)$, the limit is 0. However, along the path $y = -\sin x$, which lies in the domain of $f(x, y)$ for all $x \neq 0$, the limit does not exist. Since the limit is not the same along every path to $(0, 0)$, we say

$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(xy)}{x+y}$ does not exist.

Example 12.11 Finding a limit

Let $f(x, y) = \frac{5x^2y^2}{x^2 + y^2}$. Find $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$.

SOLUTION It is relatively easy to show that along any line $y = mx$, the limit is 0. This is not enough to prove that the limit exists, as demonstrated in the previous example, but it tells us that if the limit does exist then it must be 0.

To prove the limit is 0, we apply Definition 84. Let $\varepsilon > 0$ be given. We want to find $\delta > 0$ such that if $\sqrt{(x-0)^2 + (y-0)^2} < \delta$, then $|f(x, y) - 0| < \varepsilon$.

Set $\delta < \sqrt{\varepsilon/5}$. Note that $\left| \frac{5y^2}{x^2 + y^2} \right| < 5$ for all $(x, y) \neq (0, 0)$, and that if $\sqrt{x^2 + y^2} < \delta$, then $x^2 < \delta^2$.

Let $\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2} < \delta$. Consider $|f(x, y) - 0|$:

$$\begin{aligned} |f(x, y) - 0| &= \left| \frac{5x^2y^2}{x^2 + y^2} - 0 \right| \\ &= \left| x^2 \cdot \frac{5y^2}{x^2 + y^2} \right| \\ &< \delta^2 \cdot 5 \\ &< \frac{\varepsilon}{5} \cdot 5 \\ &= \varepsilon. \end{aligned}$$

Thus if $\sqrt{(x-0)^2 + (y-0)^2} < \delta$ then $|f(x, y) - 0| < \varepsilon$, which is what we wanted to show. Thus $\lim_{(x,y) \rightarrow (0,0)} \frac{5x^2y^2}{x^2 + y^2} = 0$.

Using Polar Coordinates to Determine a Limit as $(x, y) \rightarrow (0, 0)$

Recall from Section 9.4 that in polar coordinates, we have $x = r \cos \theta$, $y = r \sin \theta$, where r is a directed distance from the origin. Also recall that the origin $(0, 0)$ (called the *pole* in polar coordinates) is described by $r = 0$ for **every** θ . This gives us another way to compute a limit of a function in two variables *only* for the case $(x, y) \rightarrow (0, 0)$ (as points other than $(0, 0)$ depend on more than just r). If we convert to polar, then we can take the limit as r approaches 0. If

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this limit exists, and does not depend on θ , then it is the limit as $(x, y) \rightarrow (0, 0)$. If it depends on θ , then the limit does not exist. This is often easier but not always. In the next two examples, we redo two earlier limits with this method, and obtain the same results.

Example 12.12 Finding a limit by conversion to polar coordinates

Let $f(x, y) = \frac{5x^2y^2}{x^2 + y^2}$. Find $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ by converting to polar coordinates.

SOLUTION Converting to polar coordinates, and replacing “ $(x, y) \rightarrow (0, 0)$ ” with “ $r \rightarrow 0$ ”, we obtain

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} f(x, y) &= \lim_{(x,y) \rightarrow (0,0)} \frac{5x^2y^2}{x^2 + y^2} \\ &= \lim_{r \rightarrow 0} \frac{5(r \cos \theta)^2(r \sin \theta)^2}{r^2} \\ &= \lim_{r \rightarrow 0} 5r^2 \cos^2 \theta \sin^2 \theta \\ &= 0. \end{aligned}$$

That is the same result we previously obtained in Example 12.11. Next we will redo the second limit in Example 12.9 by converting to polar coordinates and once again we will see the limit does not exist.

Example 12.13 Finding a limit by conversion to polar coordinates

Let $f(x, y) = \frac{3xy}{x^2 + y^2}$. Find $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ by converting to polar coordinates.

SOLUTION Converting to polar coordinates, we obtain

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} f(x, y) &= \lim_{(x,y) \rightarrow (0,0)} \frac{3xy}{x^2 + y^2} \\ &= \lim_{r \rightarrow 0} \frac{3r \cos \theta \cdot r \sin \theta}{r^2} \\ &= \lim_{r \rightarrow 0} \frac{3r^2 \cos \theta \sin \theta}{r^2} \\ &= \lim_{r \rightarrow 0} 3 \cos \theta \sin \theta \end{aligned}$$

which does not exist because it depends on θ . Therefore, we get different results along different paths headed to the origin, so the multivariable limit does

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not exist.

Continuity

Definition 3 defines what it means for a function of one variable to be continuous. In brief, it meant that the graph of the function did not have breaks, holes, jumps, etc. We define continuity for functions of two variables in a similar way as we did for functions of one variable.

Definition 85 Continuous

Let a function $f(x, y)$ be defined on an open disk B containing the point (x_0, y_0) .

1. f is **continuous** at (x_0, y_0) if $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = f(x_0, y_0)$.
2. f is **continuous on B** if f is continuous at all points in B . If f is continuous at all points in \mathbb{R}^2 , we say that f is **continuous everywhere**.

Example 12.14 Continuity of a function of two variables

Let $f(x, y) = \begin{cases} \frac{\cos y \sin x}{x} & x \neq 0 \\ \cos y & x = 0 \end{cases}$. Is f continuous at $(0, 0)$? Is f continuous everywhere?

SOLUTION To determine if f is continuous at $(0, 0)$, we need to compare $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ to $f(0, 0)$.

Applying the definition of f , we see that $f(0, 0) = \cos 0 = 1$.

We now consider the limit $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$. Substituting 0 for x and y in $(\cos y \sin x)/x$ returns the indeterminate form “0/0”, so we need to do more work to evaluate this limit.

Consider two related limits: $\lim_{(x,y) \rightarrow (0,0)} \cos y$ and $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin x}{x}$. The first limit does not contain x , and since $\cos y$ is continuous,

$$\lim_{(x,y) \rightarrow (0,0)} \cos y = \lim_{y \rightarrow 0} \cos y = \cos 0 = 1.$$

The second limit does not contain y . By Theorem 5 (or L'Hôpital's Rule), we can say

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

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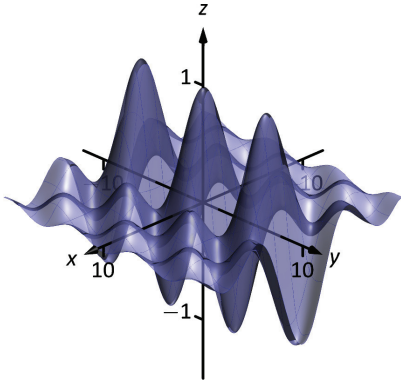


Figure 12.10: A graph of $f(x, y)$ in Example 12.14.

Finally, Theorem 103 of this section states that we can combine these two limits as follows:

$$\begin{aligned}\lim_{(x,y) \rightarrow (0,0)} \frac{\cos y \sin x}{x} &= \lim_{(x,y) \rightarrow (0,0)} (\cos y) \left(\frac{\sin x}{x} \right) \\ &= \left(\lim_{(x,y) \rightarrow (0,0)} \cos y \right) \left(\lim_{(x,y) \rightarrow (0,0)} \frac{\sin x}{x} \right) \\ &= (1)(1) \\ &= 1.\end{aligned}$$

We have found that $\lim_{(x,y) \rightarrow (0,0)} \frac{\cos y \sin x}{x} = f(0, 0)$, so f is continuous at $(0, 0)$.

A similar analysis shows that f is continuous at all points in \mathbb{R}^2 . As long as $x \neq 0$, we can evaluate the limit directly; when $x = 0$, a similar analysis shows that the limit is $\cos y$. Thus we can say that f is continuous everywhere. A graph of f is given in Figure 12.10. Notice how it has no breaks, jumps, etc.

The following theorem is very similar to Theorem 8, giving us ways to combine continuous functions to create other continuous functions.

Theorem 104 Properties of Continuous Functions

Let f and g be continuous on an open disk B , let c be a real number, and let n be a positive integer. The following functions are continuous on B .

1. Sums/Differences: $f \pm g$
2. Constant Multiples: $c \cdot f$
3. Products: $f \cdot g$
4. Quotients: f/g (as long as $g \neq 0$ on B)
5. Powers: f^n
6. Roots: $\sqrt[n]{f}$ (if n is even then $f \geq 0$ on B ; if n is odd, then true for all values of f on B .)
7. Compositions: Adjust the definitions of f and g to: Let f be continuous on B , where the range of f on B is J , and let g be a single variable function that is continuous on J . Then $g \circ f$, i.e., $g(f(x, y))$, is continuous on B .

Notes:

Example 12.15 Establishing continuity of a function

Let $f(x, y) = \sin(x^2 \cos y)$. Show f is continuous everywhere.

SOLUTION We will apply both Theorems 8 and 104. Let $f_1(x, y) = x^2$. Since y is not actually used in the function, and polynomials are continuous (by Theorem 8), we conclude f_1 is continuous everywhere. A similar statement can be made about $f_2(x, y) = \cos y$. Part 3 of Theorem 104 states that $f_3 = f_1 \cdot f_2$ is continuous everywhere, and Part 7 of the theorem states the composition of sine with f_3 is continuous: that is, $\sin(f_3) = \sin(x^2 \cos y)$ is continuous everywhere.

Functions of Three Variables

The definitions and theorems given in this section can be extended in a natural way to definitions and theorems about functions of three (or more) variables. We cover the key concepts here; some terms from Definitions 83 and 85 are not redefined but their analogous meanings should be clear to the reader.

Definition 86 Open Balls, Limit, Continuous

1. An **open ball** in \mathbb{R}^3 centered at (x_0, y_0, z_0) with radius r is the set of all points (x, y, z) such that $\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} = r$.
2. Let D be an open set in \mathbb{R}^3 containing (x_0, y_0, z_0) , and let $f(x, y, z)$ be a function of three variables defined on D , except possibly at (x_0, y_0, z_0) . The **limit** of $f(x, y, z)$ as (x, y, z) approaches (x_0, y_0, z_0) is L , denoted

$$\lim_{(x, y, z) \rightarrow (x_0, y_0, z_0)} f(x, y, z) = L,$$

means that given any $\varepsilon > 0$, there is a $\delta > 0$ such that for all $(x, y, z) \neq (x_0, y_0, z_0)$, if (x, y, z) is in the open ball centered at (x_0, y_0, z_0) with radius δ , then $|f(x, y, z) - L| < \varepsilon$.

3. Let $f(x, y, z)$ be defined on an open ball B containing (x_0, y_0, z_0) . f is **continuous** at (x_0, y_0, z_0) if $\lim_{(x, y, z) \rightarrow (x_0, y_0, z_0)} f(x, y, z) = f(x_0, y_0, z_0)$.

These definitions can also be extended naturally to apply to functions of four or more variables. Theorem 104 also applies to function of three or more vari-

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ables, allowing us to say that the function

$$f(x, y, z) = \frac{e^{x^2+y} \sqrt{y^2 + z^2 + 3}}{\sin(xyz) + 5}$$

is continuous everywhere.

When considering single variable functions, we studied limits, then continuity, then the derivative. In our current study of multivariable functions, we have studied limits and continuity. In the next section we study derivation, which takes on a slight twist as we are in a multivariable context.

Notes:

Exercises 12.2

Terms and Concepts

- Describe in your own words the difference between boundary and interior point of a set.
- Use your own words to describe (informally) what $\lim_{(x,y) \rightarrow (1,2)} f(x,y) = 17$ means.
- Give an example of a closed, bounded set.
- Give an example of a closed, unbounded set.
- Give an example of a open, bounded set.
- Give an example of a open, unbounded set.

Problems

In Exercises 7 – 10, a set S is given.

- Give one boundary point and one interior point, when possible, of S .
- State whether S is open, closed, or neither.
- State whether S is bounded or unbounded.

- $S = \left\{ (x,y) \mid \frac{(x-1)^2}{4} + \frac{(y-3)^2}{9} \leq 1 \right\}$
- $S = \{(x,y) \mid y \neq x^2\}$
- $S = \{(x,y) \mid x^2 + y^2 = 1\}$
- $S = \{(x,y) \mid y > \sin x\}$

In Exercises 11 – 14:

- Find the domain D of the given function.
- State whether D is an open or closed set.
- State whether D is bounded or unbounded.

- $f(x,y) = \sqrt{9 - x^2 - y^2}$
- $f(x,y) = \sqrt{y - x^2}$
- $f(x,y) = \frac{1}{\sqrt{y - x^2}}$
- $f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$

In Exercises 15 – 20, a limit is given. Evaluate the limit along the paths given, then state why these results show the given limit does not exist.

- $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$
 - Along the path $y = 0$.
 - Along the path $x = 0$.
- $\lim_{(x,y) \rightarrow (0,0)} \frac{x + y}{x - y}$
 - Along the path $y = mx$.
- $\lim_{(x,y) \rightarrow (0,0)} \frac{xy - y^2}{y^2 + x}$
 - Along the path $y = mx$.
 - Along the path $x = 0$.
- $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2)}{y}$
 - Along the path $y = mx$.
 - Along the path $y = x^2$.
- $\lim_{(x,y) \rightarrow (1,2)} \frac{x + y - 3}{x^2 - 1}$
 - Along the path $y = 2$.
 - Along the path $y = x + 1$.
- $\lim_{(x,y) \rightarrow (\pi, \pi/2)} \frac{\sin x}{\cos y}$
 - Along the path $x = \pi$.
 - Along the path $y = x - \pi/2$.

In Exercises 21 – 24, determine the limit by converting to polar coordinates.

- $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - 2x^2 - 2y^2}{x^2 + y^2}$
- $\lim_{(x,y) \rightarrow (0,0)} \frac{x}{x^2 + xy + y^2}$
- $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + xy + y^2}$
- $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^2 + xy + y^2}$