APEX CALCULUS III, VERSION 3.0

DALTON STATE COLLEGE EDITION

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10: VECTORS

This chapter introduces a new mathematical object, the **vector**. Defined in Section 10.2, we will see that vectors provide a powerful language for describing quantities that have magnitude and direction aspects. A simple example of such a quantity is force: when applying a force, one is generally interested in how much force is applied (i.e., the magnitude of the force) and the direction in which the force was applied. Vectors will play an important role in many of the subsequent chapters in this text.

This chapter begins with moving our mathematics out of the plane and into "space." That is, we begin to think mathematically not only in two dimensions, but in three. With this foundation, we can explore vectors both in the plane and in space.

10.1 Introduction to Cartesian Coordinates in Space

Up to this point in this text we have considered mathematics in a 2–dimensional world. We have plotted graphs on the x-y plane using rectangular and polar coordinates and found the area of regions in the plane. We have considered properties of *solid* objects, such as volume and surface area, but only by first defining a curve in the plane and then rotating it out of the plane.

While there is wonderful mathematics to explore in "2D," we live in a "3D" world and eventually we will want to apply mathematics involving this third dimension. In this section we introduce Cartesian coordinates in space and explore basic surfaces. This will lay a foundation for much of what we do in the remainder of the text.

Each point P in space can be represented with an ordered triple, $P = (a, b, c)$, where a, b and c represent the relative position of P along the x-, y- and z-axes, respectively. Each axis is perpendicular to the other two.

Visualizing points in space on paper can be problematic, as we are trying to represent a 3-dimensional concept on a 2–dimensional medium. We cannot draw three lines representing the three axes in which each line is perpendicular to the other two. Despite this issue, standard conventions exist for plotting shapes in space that we will discuss that are more than adequate.

One convention is that the axes must conform to the **right hand rule**. This rule states that when the index finger of the right hand is extended in the direction of the positive x-axis, and the middle finger (bent "inward" so it is perpendicular to the palm) points along the positive y-axis, then the extended thumb will point in the direction of the positive z-axis. (It may take some thought to

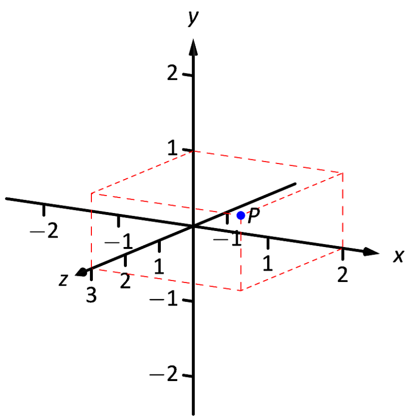


Figure 10.1: Plotting the point $P = (2, 1, 3)$ in space.

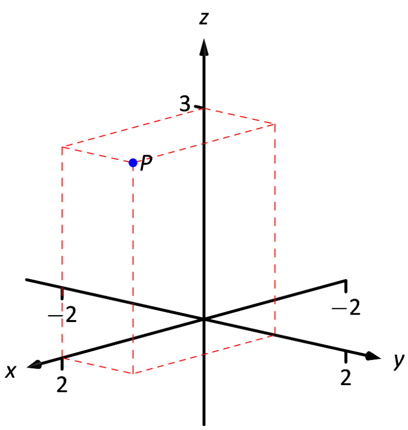


Figure 10.2: Plotting the point $P = (2, 1, 3)$ in space with a perspective used in this text.

verify this, but this system is inherently different from the one created by using the "left hand rule.")

As long as the coordinate axes are positioned so that they follow this rule, it does not matter how the axes are drawn on paper. There are two popular methods that we briefly discuss.

In Figure 10.1 we see the point $P = (2, 1, 3)$ plotted on a set of axes. The basic convention here is that the x-y plane is drawn in its standard way, with the z-axis down to the left. The perspective is that the paper represents the x-y plane and the positive z axis is coming up, off the page. This method is preferred by many engineers. Because it can be hard to tell where a single point lies in relation to all the axes, dashed lines have been added to let one see how far along each axis the point lies.

One can also consider the x-y plane as being a horizontal plane in, say, a room, where the positive z-axis is pointing up. When one steps back and looks at this room, one might draw the axes as shown in Figure 10.2. The same point P is drawn, again with dashed lines. This point of view is preferred by most mathematicians, and is the convention adopted by this text.

As how the x and y axes divide the plane into four quadrants, the x, y, and z axes divide space into eight **octants**. Only the "first octant" (analogous to the first quadrant) will be referred to by name.

Definition 51 First Octant

The **first octant** is the set of points in space for which $x > 0$, $y > 0$, and $z > 0$.

Measuring Distances

It is of critical importance to know how to measure distances between points in space. The formula for doing so is based on measuring distance in the plane, and is known (in both contexts) as the Euclidean measure of distance.

Definition 52 Distance In Space

Let $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ be points in space. The distance D between P and Q is

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Notes:

We refer to the line segment that connects points P and Q in space as $\overline{PQ}$, and refer to the length of this segment as $||\overline{PQ}||$. The above distance formula allows us to compute the length of this segment.

Example 10.1 Length of a line segment

Let $P = (1, 4, -1)$ and let $Q = (2, 1, 1)$. Draw the line segment $\overline{PQ}$ and find its length.

Solution The points P and Q are plotted in Figure 10.3; no special consideration need be made to draw the line segment connecting these two points; simply connect them with a straight line. One *cannot* actually measure this line on the page and deduce anything meaningful; its true length must be measured analytically. Applying Definition 52, we have

$$||\overline{PQ}|| = \sqrt{(2-1)^2 + (1-4)^2 + (1-(-1))^2} = \sqrt{14} \approx 3.74.$$

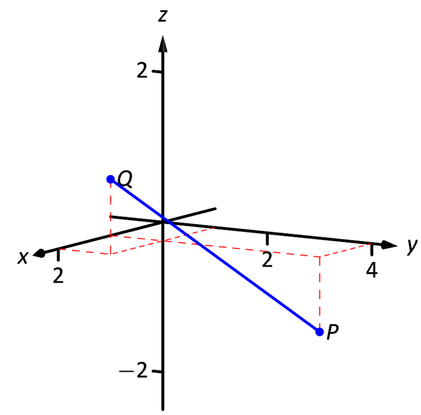


Figure 10.3: Plotting points P and Q in Example 10.1.

Spheres

Just as a circle is the set of all points in the *plane* equidistant from a given point (its center), a sphere is the set of all points in *space* that are equidistant from a given point. Definition 52 allows us to write an equation of the sphere.

We start with a point $C = (a, b, c)$ which is to be the center of a sphere with radius r. If a point $P = (x, y, z)$ lies on the sphere, then P is r units from C; that is,

$$||\overline{PC}|| = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = r.$$

Squaring both sides, we get the standard equation of a sphere in space with center at $C = (a, b, c)$ with radius r, as given in the following Key Idea.

Key Idea 46 Standard Equation of a Sphere in Space

The standard equation of the sphere with radius r, centered at $C = (a, b, c)$, is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

Example 10.2 Equation of a sphere

Find the center and radius of the sphere defined by $x^2+2x+y^2-4y+z^2-6z = 2$.

Solution To determine the center and radius, we must put the equa-

Notes:

tion in standard form. This requires us to complete the square (three times).

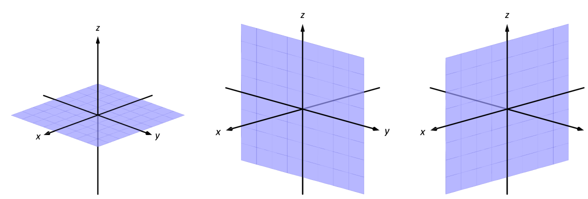
$$\begin{aligned} x^2+2x+y^2-4y+z^2-6z &= 2 \\ (x^2+2x+1)+(y^2-4y+4)+(z^2-6z+9)-14 &= 2 \\ (x+1)^2+(y-2)^2+(z-3)^2 &= 16 \end{aligned}$$

The sphere is centered at $(-1, 2, 3)$ and has a radius of 4.

The equation of a sphere is an example of an implicit relation defining a surface in space. In the case of a sphere, the variables x, y and z are all used. We now consider situations where surfaces are defined where one or two of these variables are absent.

Introduction to Planes in Space

The coordinate axes naturally define three planes (shown in Figure 10.4), the **coordinate planes**: the x-y plane, the y-z plane and the x-z plane. The x-y plane is characterized as the set of all points in space where the z-value is 0. This, in fact, gives us an equation that describes this plane: $z = 0$. Likewise, the x-z plane is all points where the y-value is 0, characterized by $y = 0$.



the x-y plane　the y-z plane　the x-z plane

Figure 10.4: The coordinate planes.

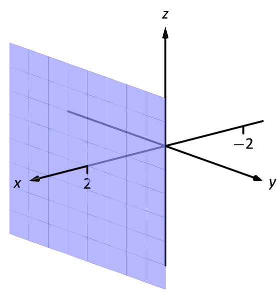


Figure 10.5: The plane $x = 2$.

The equation $x = 2$ describes all points in space where the x-value is 2. This is a plane, parallel to the y-z coordinate plane, shown in Figure 10.5.

Example 10.3　Regions defined by planes

Sketch the region defined by the inequalities $-1 \leq y \leq 2$.

Solution The region is all points between the planes $y = -1$ and $y = 2$. These planes are sketched in Figure 10.6, which are parallel to the x-z plane. Thus the region extends infinitely in the x and z directions, and is bounded by planes in the y direction.

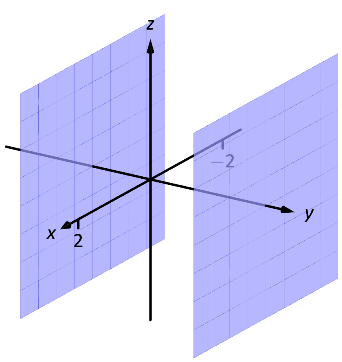


Figure 10.6: Sketching the boundaries of a region in Example 10.3.

Notes:

Cylinders

The equation $x = 1$ obviously lacks the y and z variables, meaning it defines points where the y and z coordinates can take on any value. Now consider the equation $x^2 + y^2 = 1$ *in space.* In *the plane*, this equation describes a circle of radius 1, centered at the origin. In space, the z coordinate is not specified, meaning it can take on any value. In Figure 10.8 (a), we show part of the graph of the equation $x^2 + y^2 = 1$ by sketching 3 circles: the bottom one has a constant z-value of -1.5, the middle one has a z-value of 0 and the top circle has a z-value of 1. By plotting *all* possible z-values, we get the surface shown in Figure 10.8 (b). This surface looks like a "tube," or a "cylinder"; mathematicians call this surface a **cylinder** for an entirely different reason.

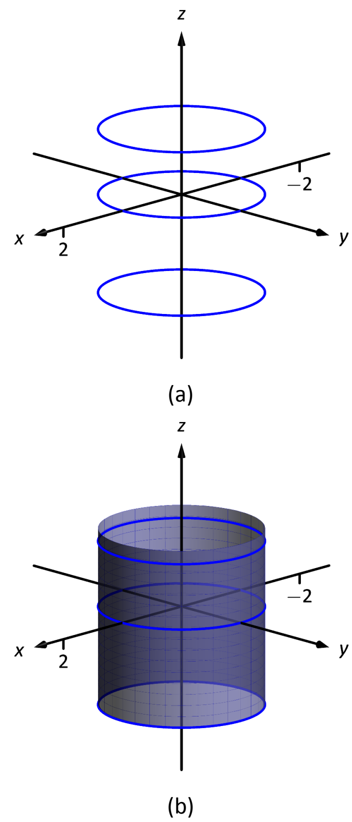


Figure 10.8: Sketching $x^2 + y^2 = 1$.

Definition 53 Cylinder

Let C be a curve in a plane and let L be a line not parallel to C. A **cylinder** is the set of all lines parallel to L that pass through C. The curve C is the **directrix** of the cylinder, and the lines are the **rulings**.

In this text, we consider curves C that lie in planes parallel to one of the coordinate planes, and lines L that are perpendicular to these planes, forming **right cylinders**. Thus the directrix can be defined using equations involving 2 variables, and the rulings will be parallel to the axis of the 3$^{\text{rd}}$ variable.

In the example preceding the definition, the curve $x^2 + y^2 = 1$ in the x-y plane is the directrix and the rulings are lines parallel to the z-axis. (Any circle shown in Figure 10.8 can be considered a directrix; we simply choose the one where $z = 0$.) Sample rulings can also be viewed in part (b) of the figure. More examples will help us understand this definition.

Example 10.4 Graphing cylinders

Graph the cylinder following cylinders.

1. $z = y^2$ 2. $x = \sin z$

Solution

1. We can view the equation $z = y^2$ as a parabola in the y-z plane, as illustrated in Figure 10.7 (a). As x does not appear in the equation, the rulings are lines through this parabola parallel to the x-axis, shown in (b). These

Notes:

rulings give a general idea as to what the surface looks like, drawn in (c).

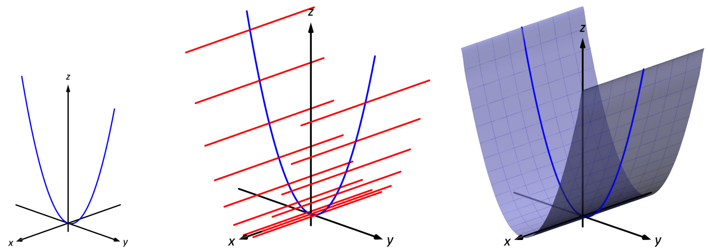


Figure 10.8: Sketching the cylinder defined by $z = y^2$.

2. We can view the equation $x = \sin z$ as a sine curve that exists in the x-z plane, as shown in Figure 10.9 (a). The rules are parallel to the y axis as the variable y does not appear in the equation $x = \sin z$; some of these are shown in part (b). The surface is shown in part (c) of the figure.

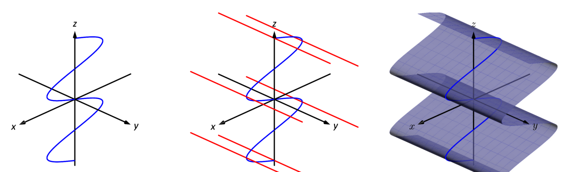


Figure 10.9: Sketching the cylinder defined by $x = \sin z$.

Surfaces of Revolution

One of the applications of integration we learned previously was to find the volume of solids of revolution – solids formed by revolving a curve about a horizontal or vertical axis. We now consider how to find the equation of the surface of such a solid.

Consider the surface formed by revolving $y = \sqrt{x}$ about the x-axis. Cross–sections of this surface parallel to the y-z plane are circles, as shown in Figure 10.10(a). Each circle has equation of the form $y^2 + z^2 = r^2$ for some radius r. The radius is a function of x; in fact, it is $r(x) = \sqrt{x}$. Thus the equation of the surface shown in Figure 10.10b is $y^2 + z^2 = (\sqrt{x})^2$.

We generalize the above principles to give the equations of surfaces formed by revolving curves about the coordinate axes.

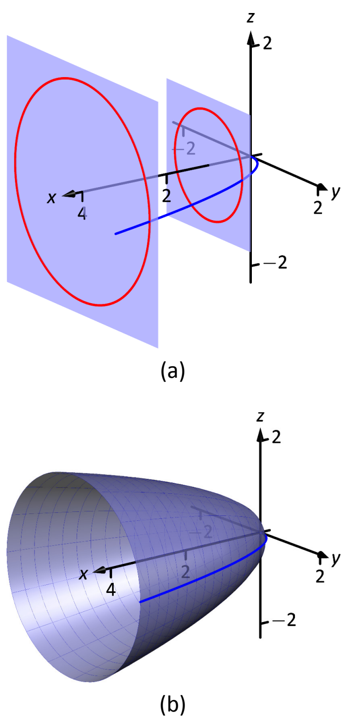


Figure 10.10: Introducing surfaces of revolution.

Notes:

Key Idea 47 Surfaces of Revolution, Part 1

Let r be a radius function.

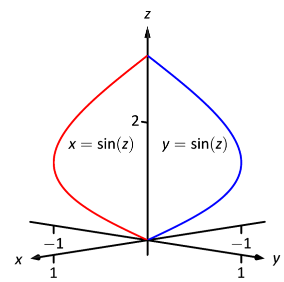
1. The equation of the surface formed by revolving $y = r(x)$ or $z = r(x)$ about the x-axis is $y^2 + z^2 = r(x)^2$.
2. The equation of the surface formed by revolving $x = r(y)$ or $z = r(y)$ about the y-axis is $x^2 + z^2 = r(y)^2$.
3. The equation of the surface formed by revolving $x = r(z)$ or $y = r(z)$ about the z-axis is $x^2 + y^2 = r(z)^2$.

Example 10.5 Finding equation of a surface of revolution

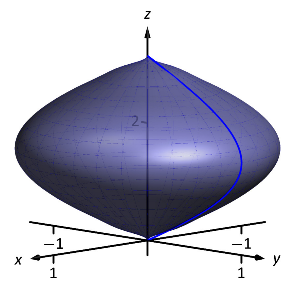
Let $y = \sin z$ on $[0, \pi]$. Find the equation of the surface of revolution formed by revolving $y = \sin z$ about the z-axis.

Solution Using Key Idea 47, we find the surface has equation $x^2+y^2 = \sin^2 z$. The curve is sketched in Figure 10.11(a) and the surface is drawn in Figure 10.11(b).

Note how the surface (and hence the resulting equation) is the same if we began with the curve $x = \sin z$, which is also drawn in Figure 10.11(a).



(a)



(b)

Figure 10.11: Revolving $y = \sin z$ about the z-axis in Example 10.5.

This particular method of creating surfaces of revolution is limited. For instance, in Example 7.14 of Section 7.3 we found the volume of the solid formed by revolving $y = \sin x$ about the y-axis. Our current method of forming surfaces can only rotate $y = \sin x$ about the x-axis. Trying to rewrite $y = \sin x$ as a function of y is not trivial, as simply writing $x = \sin^{-1} y$ only gives part of the region we desire.

What we desire is a way of writing the surface of revolution formed by rotating $y = f(x)$ about the y-axis. We start by first recognizing this surface is the same as revolving $z = f(x)$ about the z-axis. This will give us a more natural way of viewing the surface.

A value of x is a measurement of distance from the z-axis. At the distance r, we plot a z-height of $f(r)$. When rotating $f(x)$ about the z-axis, we want all points a distance of r from the z-axis in the x-y plane to have a z-height of $f(r)$. All such points satisfy the equation $r^2 = x^2 + y^2$; hence $r = \sqrt{x^2 + y^2}$. Replacing r with $\sqrt{x^2 + y^2}$ in $f(r)$ gives $z = f(\sqrt{x^2 + y^2})$. This is the equation of the surface.

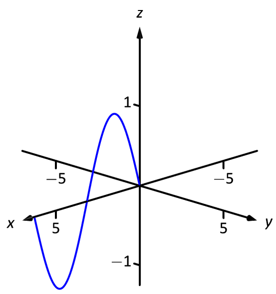
Notes:

> **Key Idea 48 Surfaces of Revolution, Part 2**
>
> Let $z = f(x)$, $x \geq 0$, be a curve in the x-z plane. The surface formed by revolving this curve about the z-axis has equation $z = f\big(\sqrt{x^2+y^2}\big)$.

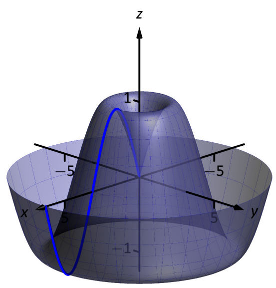
Example 10.6 Finding equation of surface of revolution

Find the equation of the surface found by revolving $z = \sin x$ about the z-axis.

Solution Using Key Idea 48, the surface has equation $z = \sin\left(\sqrt{x^2+y^2}\right)$. The curve and surface are graphed in Figure 10.12.



(a)



(b)

Figure 10.12: Revolving $z = \sin x$ about the z-axis in Example 10.6.

Quadric Surfaces

Spheres, planes and cylinders are important surfaces to understand. We now consider one last type of surface, a **quadric surface**. The definition may look intimidating, but we will show how to analyze these surfaces in an illuminating way.

> **Definition 54 Quadric Surface**
>
> A **quadric surface** is the graph of the general second–degree equation in three variables:
>
> $$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

When the coefficients D, E or F are not zero, the basic shapes of the quadric surfaces are rotated in space. We will focus on quadric surfaces where these coeffiecients are 0; we will not consider rotations. There are six basic quadric surfaces: the elliptic paraboloid, elliptic cone, ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, and the hyperbolic paraboloid.

We study each shape by considering **traces**, that is, intersections of each surface with a plane parallel to a coordinate plane. For instance, consider the elliptic paraboloid $z = x^2/4 + y^2$, shown in Figure 10.13. If we intersect this shape with the plane $z = d$ (i.e., replace z with d), we have the equation:

$$d = \frac{x^2}{4} + y^2.$$

Divide both sides by d:

$$1 = \frac{x^2}{4d} + \frac{y^2}{d}.$$

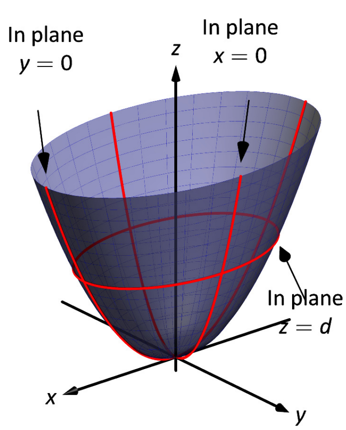


Figure 10.13: The elliptic paraboloid $z = x^2/4 + y^2$.

Notes:

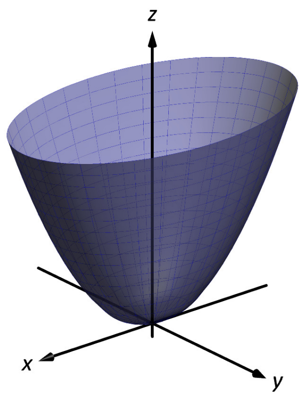
This describes an ellipse – so cross sections parallel to the x-y coordinate plane are ellipses. This ellipse is drawn in the figure.

Now consider cross sections parallel to the x-z plane. For instance, letting $y = 0$ gives the equation $z = x^2/4$, clearly a parabola. Intersecting with the plane $x = 0$ gives a cross section defined by $z = y^2$, another parabola. These parabolas are also sketched in the figure.

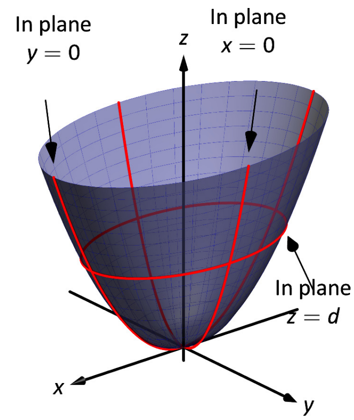
Thus we see where the elliptic paraboloid gets its name: some cross sections are ellipses, and others are parabolas.

Notes:

Elliptic Paraboloid, $z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$



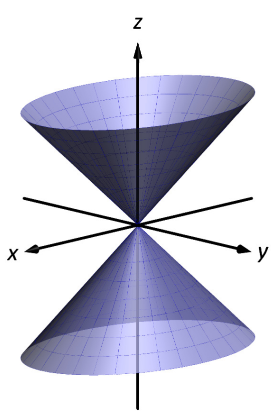
Plane	Trace
$x = d$	Parabola
$y = d$	Parabola
$z = d$	Ellipse



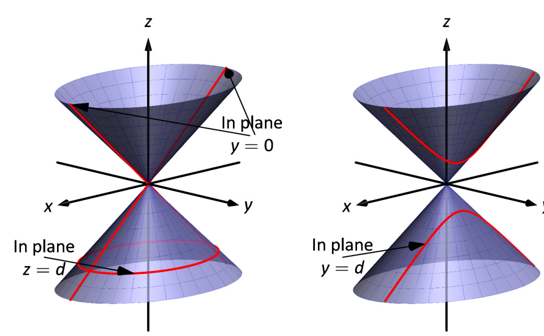
One variable in the equation of the elliptic paraboloid will be raised to the first power; above, this is the z variable. The paraboloid will "open" in the direction of this variable's axis. Thus $x = y^2/a^2 + z^2/b^2$ is an elliptic paraboloid that opens along the x-axis.

Multiplying the right hand side by (-1) defines an elliptic paraboloid that "opens" in the opposite direction.

Elliptic Cone, $z^2 = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$

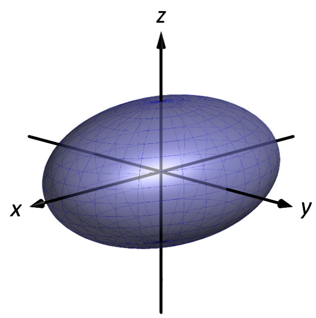


Plane	Trace
$x = 0$	Crossed Lines
$y = 0$	Crossed Lines
$x = d$	Hyperbola
$y = d$	Hyperbola
$z = d$	Ellipse

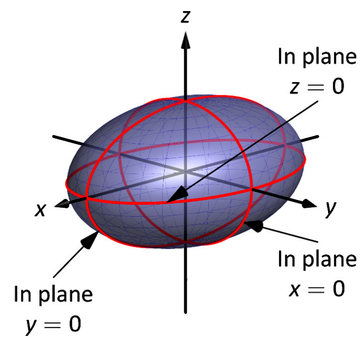


One can rewrite the equation as $z^2 - x^2/a^2 - y^2/b^2 = 0$. The one variable with a positive coefficient corresponds to the axis that the cones "open" along.

Ellipsoid, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

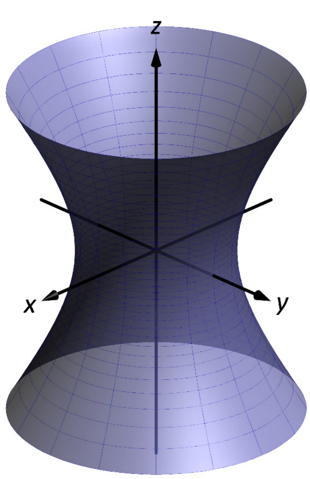


Plane	Trace
$x = d$	Ellipse
$y = d$	Ellipse
$z = d$	Ellipse

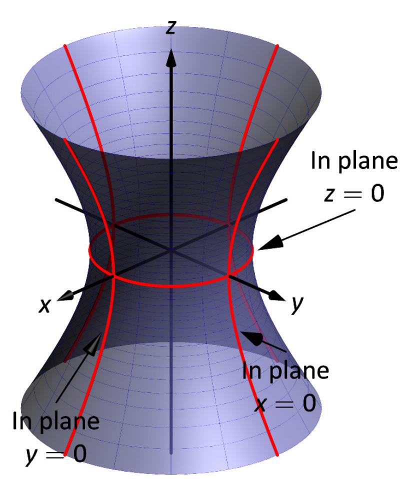


If $a = b = c \neq 0$, the ellipsoid is a sphere with radius a; compare to Key Idea 46.

Hyperboloid of One Sheet, $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

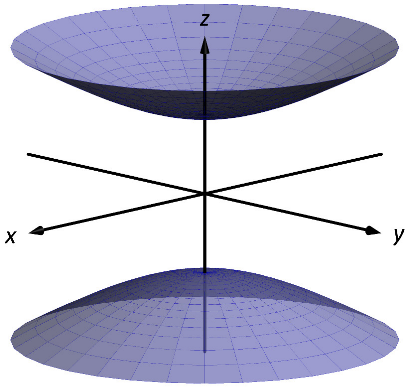


Plane	Trace
$x = d$	Hyperbola
$y = d$	Hyperbola
$z = d$	Ellipse

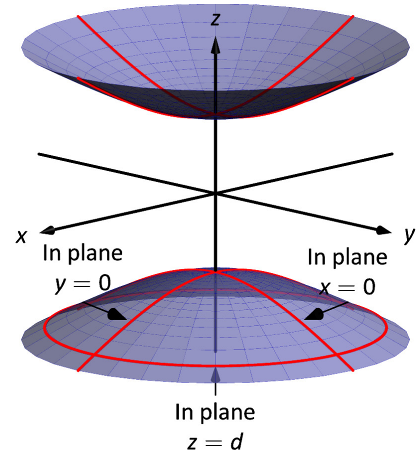


The one variable with a negative coefficient corresponds to the axis that the hyperboloid "opens" along.

Hyperboloid of Two Sheets, $\dfrac{z^2}{c^2} - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

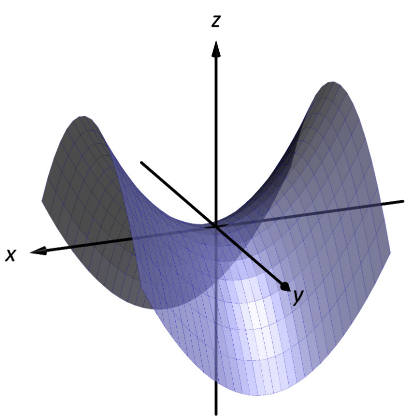


Plane	Trace
$x = d$	Hyperbola
$y = d$	Hyperbola
$z = d$	Ellipse

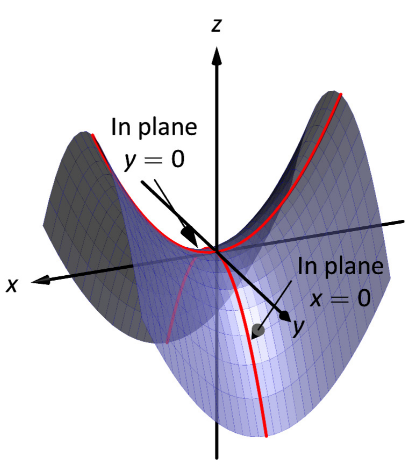


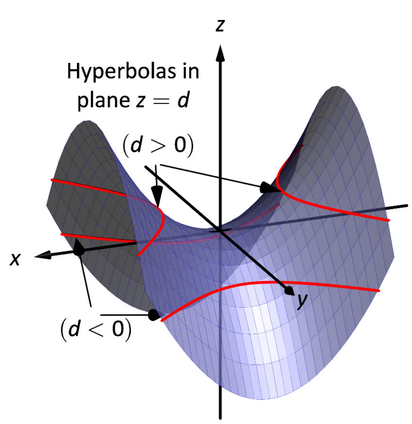
The one variable with a positive coefficient corresponds to the axis that the hyperboloid "opens" along. In the case illustrated, when $|d| < |c|$, there is no trace.

Hyperbolic Paraboloid, $z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$



Plane	Trace
$x = d$	Parabola
$y = d$	Parabola
$z = d$	Hyperbola





The parabolic traces will open along the axis of the one variable that is raised to the first power.

Example 10.7 Sketching quadric surfaces

Sketch the quadric surface defined by the given equation.

1. $y = \dfrac{x^2}{4} + \dfrac{z^2}{16}$ 2. $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$. 3. $z = y^2 - x^2$.

Solution

1. $y = \dfrac{x^2}{4} + \dfrac{z^2}{16}$:

   We first identify the quadric by pattern–matching with the equations given previously. Only two surfaces have equations where one variable is raised to the first power, the elliptic paraboloid and the hyperbolic paraboloid. In the latter case, the other variables have different signs, so we conclude that this describes an elliptic paraboloid. As the variable with the first power is y, we note the paraboloid opens along the y-axis.

   To make a decent sketch by hand, we need only draw a few traces. In this case, the traces $x = 0$ and $z = 0$ form parabolas that outline the shape.

   $x = 0$: The trace is the parabola $y = z^2/16$

   $z = 0$: The trace is the parabola $y = x^2/4$.

   Graphing each trace in the respective plane creates a sketch as shown in Figure 10.14(a). This is enough to give an idea of what the paraboloid looks like. The surface is filled in in (b).

2. $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$:

   This is an ellipsoid. We can get a good idea of its shape by drawing the traces in the coordinate planes.

   $x = 0$: The trace is the ellipse $\dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$. The major axis is along the y–axis with length 6 (as $b = 3$, the length of the axis is 6); the minor axis is along the z-axis with length 4.

   $y = 0$: The trace is the ellipse $x^2 + \dfrac{z^2}{4} = 1$. The major axis is along the z-axis, and the minor axis has length 2 along the x-axis.

   $z = 0$: The trace is the ellipse $x^2 + \dfrac{y^2}{9} = 1$, with major axis along the y-axis.

   Graphing each trace in the respective plane creates a sketch as shown in Figure 10.15(a). Filling in the surface gives Figure 10.15(b).

3. $z = y^2 - x^2$:

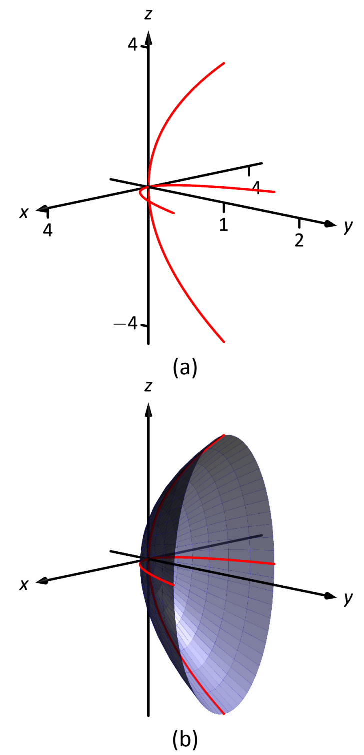


Figure 10.14: Sketching an elliptic paraboloid.

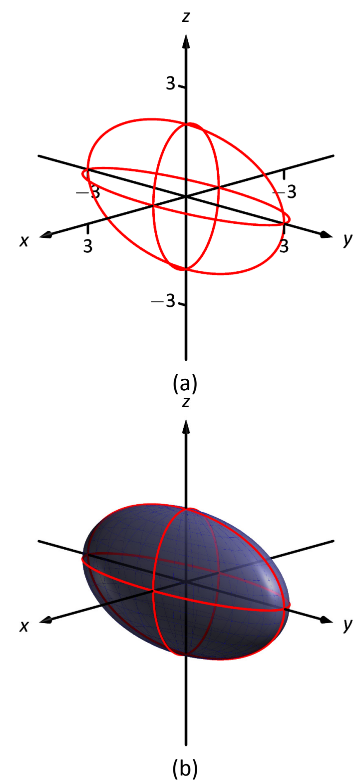


Figure 10.15: Sketching an ellipsoid.

Notes:

This defines a hyperbolic paraboloid, very similar to the one shown in the gallery of quadric sections. Consider the traces in the $y-z$ and $x-z$ planes:

$x = 0$: The trace is $z = y^2$, a parabola opening up in the $y - z$ plane.

$y = 0$: The trace is $z = -x^2$, a parabola opening down in the $x - z$ plane.

Sketching these two parabolas gives a sketch like that in Figure 10.16 (a), and filling in the surface gives a sketch like (b).

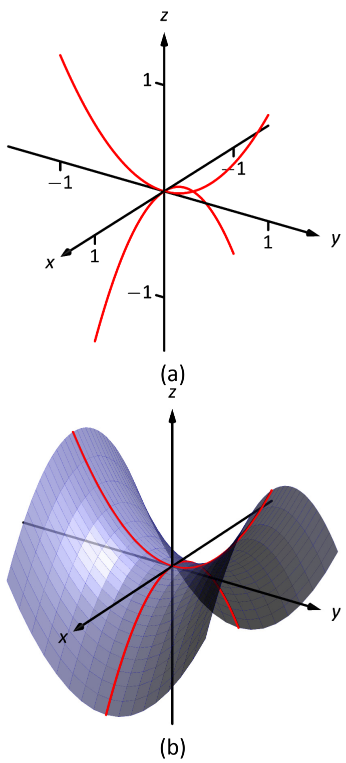


Figure 10.16: Sketching a hyperbolic paraboloid.

Example 10.8 Identifying quadric surfaces

Consider the quadric surface shown in Figure 10.17. Which of the following equations best fits this surface?

(a) $x^2 - y^2 - \dfrac{z^2}{9} = 0$ (c) $z^2 - x^2 - y^2 = 1$

(b) $x^2 - y^2 - z^2 = 1$ (d) $4x^2 - y^2 - \dfrac{z^2}{9} = 1$

Solution The image clearly displays a hyperboloid of two sheets. The gallery informs us that the equation will have a form similar to $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

We can immediately eliminate option (a), as the constant in that equation is not 1.

The hyperboloid "opens" along the x-axis, meaning x must be the only variable with a positive coefficient, eliminating (c).

The hyperboloid is wider in the z-direction than in the y-direction, so we need an equation where $c > b$. This eliminates (b), leaving us with (d). We should verify that the equation given in (d), $4x^2 - y^2 - \frac{z^2}{9} = 1$, fits.

We already established that this equation describes a hyperboloid of two sheets that opens in the x-direction and is wider in the z-direction than in the y. Now note the coefficient of the x-term. Rewriting $4x^2$ in standard form, we have: $4x^2 = \dfrac{x^2}{(1/2)^2}$. Thus when $y = 0$ and $z = 0$, x must be $1/2$; i.e., each hyperboloid "starts" at $x = 1/2$. This matches our figure.

We conclude that $4x^2 - y^2 - \dfrac{z^2}{9} = 1$ best fits the graph.

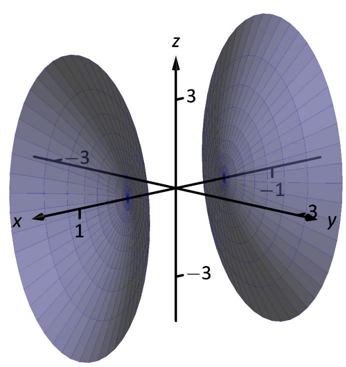


Figure 10.17: A possible equation of this quadric surface is found in Example 10.8.

This section has introduced points in space and shown how equations can describe surfaces. The next sections explore *vectors*, an important mathematical object that we'll use to explore curves in space.

Notes:

Exercises 10.1

Terms and Concepts

1. Axes drawn in space must conform to the ________ ________ rule.

2. In the plane, the equation $x = 2$ defines a ________; in space, $x = 2$ defines a ________.

3. In the plane, the equation $y = x^2$ defines a ________; in space, $y = x^2$ defines a ________.

4. Which quadric surface looks like a Pringles® chip?

5. Consider the hyperbola $x^2 - y^2 = 1$ in the plane. If this hyperbola is rotated about the x-axis, what quadric surface is formed?

6. Consider the hyperbola $x^2 - y^2 = 1$ in the plane. If this hyperbola is rotated about the y-axis, what quadric surface is formed?

Problems

7. The points $A = (1, 4, 2)$, $B = (2, 6, 3)$ and $C = (4, 3, 1)$ form a triangle in space. Find the distances between each pair of points and determine if the triangle is a right triangle.

8. The points $A = (1, 1, 3)$, $B = (3, 2, 7)$, $C = (2, 0, 8)$ and $D = (0, -1, 4)$ form a quadrilateral $ABCD$ in space. Is this a parallelogram?

9. Find the center and radius of the sphere defined by $(x-4)^2 + y^2 + (z+7)^2 = 118$.

10. Write an equation for the sphere centered at $(2, 4, -9)$ with radius 12.

11. Write an equation for the sphere centered at $(6, 0, 0)$ with radius 5.

12. Find the center and radius of the sphere defined by $x^2 - 8x + y^2 + 2y + z^2 + 8 = 0$.

13. Find the center and radius of the sphere defined by $x^2 + y^2 + z^2 + 4x - 2y - 4z + 4 = 0$.

14. Write an equation of the elliptic paraboloid formed by taking the elliptic paraboloid $x = y^2 + 2z^2$ and shifting it 3 units in the positive z-direction and 5 units in the negative x-direction.

15. Write an equation for the ellipsoid centered at $(3, -2, 0)$ that extends 4 units from the center in the x-direction, 5 units from the center in the y-direction, and 1 unit from the center in the z-direction.

16. What type of curves are the traces of the surface $z = x^2+y^2$ in planes of the form $x = d$?

17. What type of curves are the traces of the surface $z = x^2+y^2$ in planes of the form $z = d$, $d > 0$?

18. What type of curves are the traces of the surface $z = x^2+y^2$ in planes of the form $z = d$, $d < 0$?

19. What type of curve is the trace of the surface $z = x^2 + y^2$ in the plane $z = 0$?

In Exercises 20 – 23, describe the region in space defined by the inequalities.

20. $x^2 + y^2 + z^2 < 1$

21. $0 \leq x \leq 3$

22. $x \geq 0,\ y \geq 0,\ z \geq 0$

23. $y \geq 3$

In Exercises 24 – 27, sketch the cylinder in space.

24. $z = x^3$

25. $y = \cos z$

26. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

27. $y = \dfrac{1}{x}$

In Exercises 28 – 31, give the equation of the surface of revolution described.

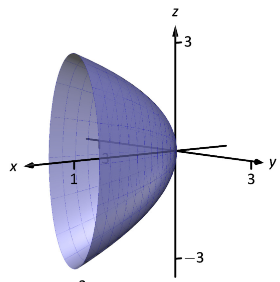
28. Revolve $z = \dfrac{1}{1+y^2}$ about the y-axis.

29. Revolve $y = x^2$ about the x-axis.

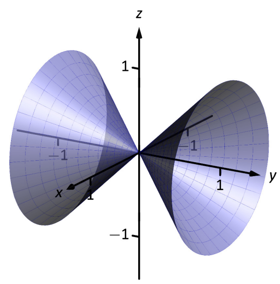
30. Revolve $z = x^2$ about the z-axis.

31. Revolve $z = 1/x$ about the z-axis.

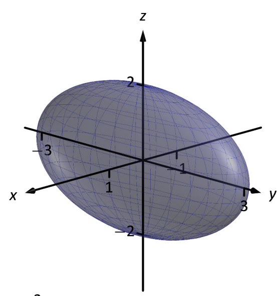
In Exercises 32 – 35, a quadric surface is sketched. Determine which of the given equations best fits the graph.

32. 

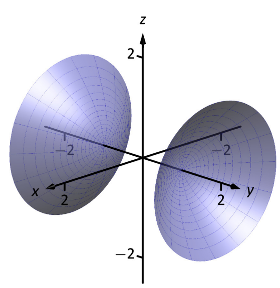
(a) $x = y^2 + \dfrac{z^2}{9}$ (b) $x = y^2 + \dfrac{z^2}{3}$

33. 

(a) $x^2 - y^2 - z^2 = 0$ (b) $x^2 - y^2 + z^2 = 0$

34. 

(a) $x^2 + \dfrac{y^2}{3} + \dfrac{z^2}{2} = 1$ (b) $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$

35. 

(a) $y^2 - x^2 - z^2 = 1$ (b) $y^2 + x^2 - z^2 = 1$

In Exercises 36 – 41, sketch the quadric surface.

36. $z - y^2 + x^2 = 0$

37. $z^2 = x^2 + \dfrac{y^2}{4}$

38. $x = -y^2 - z^2$

39. $16x^2 - 16y^2 - 16z^2 = 1$

40. $\dfrac{x^2}{9} - y^2 + \dfrac{z^2}{25} = 1$

41. $4x^2 + 2y^2 + z^2 = 4$

10.2 An Introduction to Vectors

Many quantities we think about daily can be described by a single number: temperature, speed, cost, weight and height. There are also many other concepts we encounter daily that cannot be described with just one number. For instance, a weather forecaster often describes wind with its speed and its direction (". . . with winds from the southeast gusting up to 30 mph . . ."). When applying a force, we are concerned with both the magnitude and direction of that force. In both of these examples, *direction* is important. Because of this, we study *vectors*, mathematical objects that convey both magnitude and direction information.

One "bare–bones" definition of a vector is based on what we wrote above: "a vector is a mathematical object with magnitude and direction parameters." This definition leaves much to be desired, as it gives no indication as to how such an object is to be used. Several other definitions exist; we choose here a definition rooted in a geometric visualization of vectors. It is very simplistic but readily permits further investigation.

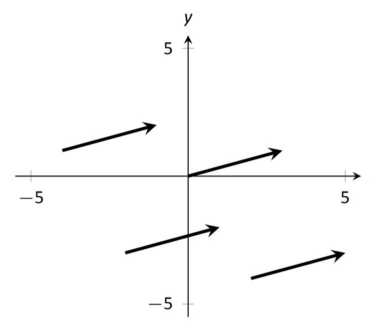


Figure 10.18: Drawing the same vector with different initial points.

Definition 55 Vector

A **vector** is a directed line segment.

Given points P and Q (either in the plane or in space), we denote with $\overrightarrow{PQ}$ the vector *from P to Q*. The point P is said to be the **initial point** of the vector, and the point Q is the **terminal point**.

The **magnitude**, **length** or **norm** of $\overrightarrow{PQ}$ is the length of the line segment $\overline{PQ}$: $|| \overrightarrow{PQ} || = || \overline{PQ} ||$.

Two vectors are **equal** if they have the same magnitude and direction.

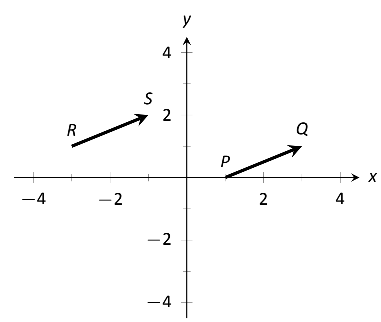


Figure 10.19: Illustrating how equal vectors have the same displacement.

Figure 10.18 shows multiple instances of the same vector. Each directed line segment has the same direction and length (magnitude), hence each is the same vector.

We use $\mathbb{R}^2$ (pronounced "r two") to represent all the vectors in the plane, and use $\mathbb{R}^3$ (pronounced "r three") to represent all the vectors in space.

Consider the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ as shown in Figure 10.19. The vectors look to be equal; that is, they seem to have the same length and direction. Indeed, they are. Both vectors move 2 units to the right and 1 unit up from the initial point to reach the terminal point. One can analyze this movement to measure the

Notes:

magnitude of the vector, and the movement itself gives direction information (one could also measure the slope of the line passing through P and Q or R and S). Since they have the same length and direction, these two vectors are equal.

This demonstrates that inherently all we care about is *displacement*; that is, how far in the x, y and possibly z directions the terminal point is from the initial point. Both the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ in Figure 10.19 have an x-displacement of 2 and a y-displacement of 1. This suggests a standard way of describing vectors in the plane. A vector whose x-displacement is a and whose y-displacement is b will have terminal point (a, b) when the initial point is the origin, $(0, 0)$. This leads us to a definition of a standard and concise way of referring to vectors.

Definition 56 Component Form of a Vector

1. The **component form** of a vector $\vec{v}$ in $\mathbb{R}^2$, whose terminal point is (a, b) when its initial point is $(0, 0)$, is $\langle a, b\rangle$.
2. The **component form** of a vector $\vec{v}$ in $\mathbb{R}^3$, whose terminal point is (a, b, c) when its initial point is $(0, 0, 0)$, is $\langle a, b, c\rangle$.

The numbers a, b (and c, respectively) are the **components** of $\vec{v}$.

It follows from the definition that the component form of the vector $\overrightarrow{PQ}$, where $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ is

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1\rangle ;$$

in space, where $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$, the component form of $\overrightarrow{PQ}$ is

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle .$$

We practice using this notation in the following example.

Example 10.9 Using component form notation for vectors

1. Sketch the vector $\vec{v} = \langle 2, -1\rangle$ starting at $P = (3, 2)$ and find its magnitude.
2. Find the component form of the vector $\vec{w}$ whose initial point is $R = (-3, -2)$ and whose terminal point is $S = (-1, 2)$.
3. Sketch the vector $\vec{u} = \langle 2, -1, 3\rangle$ starting at the point $Q = (1, 1, 1)$ and find its magnitude.

Notes:

Solution

1. Using P as the initial point, we move 2 units in the positive x-direction and -1 units in the positive y-direction to arrive at the terminal point $P' = (5,1)$, as drawn in Figure 10.20(a).

   The magnitude of $\vec{v}$ is determined directly from the component form:

   $$|| \vec{v} \,|| = \sqrt{2^2 + (-1)^2} = \sqrt{5}.$$

2. Using the note following Definition 56, we have

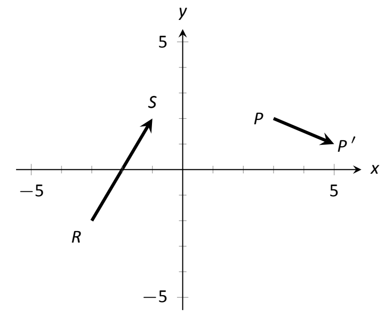
   $$\overrightarrow{RS} = \langle -1 - (-3), 2 - (-2) \rangle = \langle 2, 4 \rangle .$$

   One can readily see from Figure 10.20(a) that the x- and y-displacement of $\overrightarrow{RS}$ is 2 and 4, respectively, as the component form suggests.

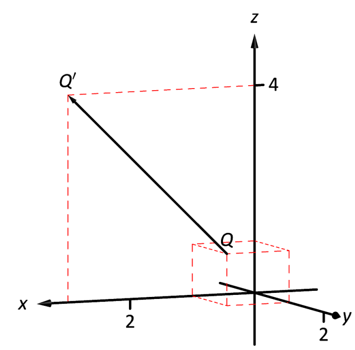
3. Using Q as the initial point, we move 2 units in the positive x-direction, -1 unit in the positive y-direction, and 3 units in the positive z-direction to arrive at the terminal point $Q' = (3,0,4)$, illustrated in Figure 10.20(b).

   The magnitude of $\vec{u}$ is:

   $$|| \vec{u} \,|| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}.$$



(a)



(b)

Figure 10.20: Graphing vectors in Example 10.9.

Now that we have defined vectors, and have created a nice notation by which to describe them, we start considering how vectors interact with each other. That is, we define an *algebra* on vectors.

Notes:

Definition 57 Vector Algebra

1. Let $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$ be vectors in $\mathbb{R}^2$, and let c be a **scalar** (a real number or a variable expression that represents a real number).

   (a) The addition, or sum, of the vectors $\vec{u}$ and $\vec{v}$ is the vector
   $$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2 \rangle .$$

   (b) The scalar product of c and $\vec{v}$ is the vector
   $$c\vec{v} = c \langle v_1, v_2 \rangle = \langle cv_1, cv_2 \rangle .$$

2. Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ be vectors in $\mathbb{R}^3$, and let c be a scalar.

   (a) The addition, or sum, of the vectors $\vec{u}$ and $\vec{v}$ is the vector
   $$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle .$$

   (b) The scalar product of c and $\vec{v}$ is the vector
   $$c\vec{v} = c \langle v_1, v_2, v_3 \rangle = \langle cv_1, cv_2, cv_3 \rangle .$$

In short, we say addition and scalar multiplication are computed "component–wise."

Example 10.10 Adding vectors

Sketch the vectors $\vec{u} = \langle 1, 3 \rangle$, $\vec{v} = \langle 2, 1 \rangle$ and $\vec{u} + \vec{v}$ all with initial point at the origin.

SOLUTION We first compute $\vec{u} + \vec{v}$.

$$\begin{aligned} \vec{u} + \vec{v} &= \langle 1, 3 \rangle + \langle 2, 1 \rangle \\ &= \langle 3, 4 \rangle . \end{aligned}$$

These are all sketched in Figure 10.21.

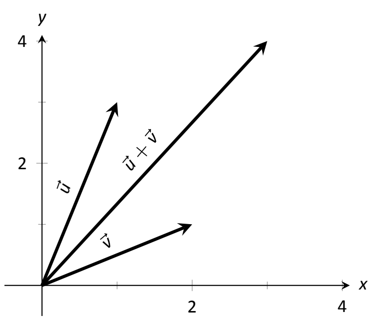


Figure 10.21: Graphing the sum of vectors in Example 10.10.

As vectors convey magnitude and direction information, the sum of vectors also convey length and magnitude information. Adding $\vec{u} + \vec{v}$ suggests the following idea:

Notes:

"Starting at an initial point, go out $\vec{u}$, then go out $\vec{v}$."

This idea is sketched in Figure 10.22, where the initial point of $\vec{v}$ is the terminal point of $\vec{u}$. This is known as the "Head to Tail Rule" of adding vectors. Vector addition is very important. For instance, if the vectors $\vec{u}$ and $\vec{v}$ represent forces acting on a body, the sum $\vec{u}+\vec{v}$ gives the resulting force. Because of various physical applications of vector addition, the sum $\vec{u}+\vec{v}$ is often referred to as the **resultant vector**, or just the "resultant."

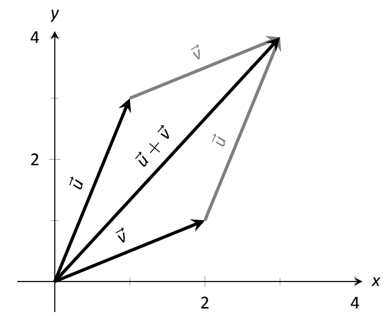


Figure 10.22: Illustrating how to add vectors using the Head to Tail Rule and Parallelogram Law.

Analytically, it is easy to see that $\vec{u}+\vec{v}=\vec{v}+\vec{u}$. Figure 10.22 also gives a graphical representation of this, using gray vectors. Note that the vectors $\vec{u}$ and $\vec{v}$, when arranged as in the figure, form a parallelogram. Because of this, the Head to Tail Rule is also known as the Parallelogram Law: the vector $\vec{u}+\vec{v}$ is defined by forming the parallelogram defined by the vectors $\vec{u}$ and $\vec{v}$; the initial point of $\vec{u}+\vec{v}$ is the common initial point of parallelogram, and the terminal point of the sum is the common terminal point of the parallelogram.

While not illustrated here, the Head to Tail Rule and Parallelogram Law hold for vectors in $\mathbb{R}^3$ as well.

It follows from the properties of the real numbers and Definition 57 that

$$\vec{u}-\vec{v}=\vec{u}+(-1)\vec{v}.$$

The Parallelogram Law gives us a good way to visualize this subtraction. We demonstrate this in the following example.

Example 10.11 Vector Subtraction

Let $\vec{u}=\langle 3,1\rangle$ and $\vec{v}=\langle 1,2\rangle$. Compute and sketch $\vec{u}-\vec{v}$.

Solution The computation of $\vec{u}-\vec{v}$ is straightforward, and we show all steps below. Usually the formal step of multiplying by (-1) is omitted and we "just subtract."

$$\begin{aligned}\vec{u}-\vec{v}&=\vec{u}+(-1)\vec{v}\\&=\langle 3,1\rangle+\langle -1,-2\rangle\\&=\langle 2,-1\rangle.\end{aligned}$$

Figure 10.23 illustrates, using the Head to Tail Rule, how the subtraction can be viewed as the sum $\vec{u}+(-\vec{v})$. The figure also illustrates how $\vec{u}-\vec{v}$ can be obtained by looking only at the terminal points of $\vec{u}$ and $\vec{v}$ (when their initial points are the same).

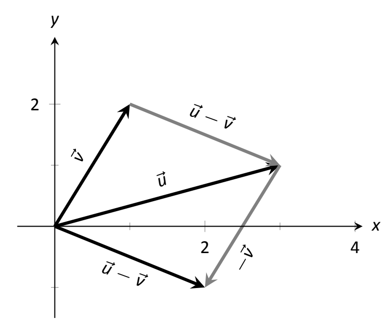


Figure 10.23: Illustrating how to subtract vectors graphically.

Example 10.12 Scaling vectors

1. Sketch the vectors $\vec{v}=\langle 2,1\rangle$ and $2\vec{v}$ with initial point at the origin.

Notes:

2. Compute the magnitudes of $\vec{v}$ and $2\vec{v}$.

Solution

1. We compute $2\vec{v}$:

$$\begin{aligned} 2\vec{v} &= 2\left\langle 2, 1 \right\rangle \\ &= \left\langle 4, 2 \right\rangle . \end{aligned}$$

Both $\vec{v}$ and $2\vec{v}$ are sketched in Figure 10.24. Make note that $2\vec{v}$ does not start at the terminal point of $\vec{v}$; rather, its initial point is also the origin.

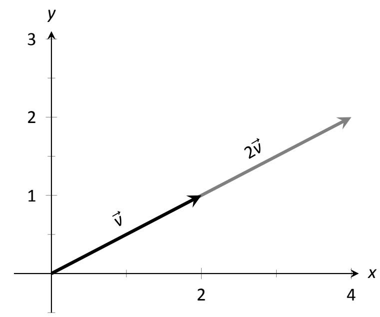


Figure 10.24: Graphing vectors $\vec{v}$ and $2\vec{v}$ in Example 10.12.

2. The figure suggests that $2\vec{v}$ is twice as long as $\vec{v}$. We compute their magnitudes to confirm this.

$$\begin{aligned} ||\,\vec{v}\,|| &= \sqrt{2^2+1^2} \\ &= \sqrt{5}. \\ ||\,2\vec{v}\,|| &= \sqrt{4^2+2^2} \\ &= \sqrt{20} \\ &= \sqrt{4\cdot 5} = 2\sqrt{5}. \end{aligned}$$

As we suspected, $2\vec{v}$ is twice as long as $\vec{v}$.

The **zero vector** is the vector whose initial point is also its terminal point. It is denoted by $\vec{0}$. Its component form, in $\mathbb{R}^2$, is $\langle 0, 0\rangle$; in $\mathbb{R}^3$, it is $\langle 0, 0, 0\rangle$. Usually the context makes is clear whether $\vec{0}$ is referring to a vector in the plane or in space.

For any vector $\vec{v}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$, the vector $(-1)\vec{v}$ is denoted by $-\vec{v}$. The vector $-\vec{v}$ has the same magnitude as $\vec{v}$ but points in the *opposite* direction.

Our examples have illustrated key principles in vector algebra: how to add and subtract vectors and how to multiply vectors by a scalar. The following theorem states formally the properties of these operations.

Notes:

Theorem 86 Properties of Vector Operations

The following are true for all scalars c and d, and for all vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, where $\vec{u}$, $\vec{v}$ and $\vec{w}$ are all in $\mathbb{R}^2$ or where $\vec{u}$, $\vec{v}$ and $\vec{w}$ are all in $\mathbb{R}^3$:

1. $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ — Commutative Property
2. $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$ — Associative Property
3. $\vec{v}+\vec{0}=\vec{v}$ — Additive Identity
4. $(cd)\vec{v}=c(d\vec{v})$
5. $c(\vec{u}\pm\vec{v})=c\vec{u}\pm c\vec{v}$ — Distributive Property
6. $(c\pm d)\vec{v}=c\vec{v}\pm d\vec{v}$ — Distributive Property
7. $0\vec{v}=\vec{0}$
8. $||\, c\vec{v}\,|| = |c|\cdot||\,\vec{v}\,||$
9. $||\,\vec{u}\,|| = 0$ if and only if $\vec{u}=\vec{0}$.

As stated before, each vector $\vec{v}$ conveys magnitude and direction information. We have a method of extracting the magnitude, which we write as $||\,\vec{v}\,||$. *Unit vectors* are a way of extracting just the direction information from a vector.

Definition 58 Unit Vector

A **unit vector** is a vector $\vec{v}$ with a magnitude of 1; that is,

$$||\,\vec{v}\,|| = 1.$$

Consider this scenario: you are given a vector $\vec{v}$ and are told to create a vector of length 10 in the direction of $\vec{v}$. How does one do that? If we knew that $\vec{u}$ was the unit vector in the direction of $\vec{v}$, the answer would be easy: $10\vec{u}$. So how do we find $\vec{u}$?

Property 8 of Theorem 86 holds the key. If we divide $\vec{v}$ by its magnitude, it becomes a vector of length 1. Consider:

$$\begin{aligned}\left|\left|\frac{1}{||\,\vec{v}\,||}\vec{v}\right|\right| &= \frac{1}{||\,\vec{v}\,||}||\,\vec{v}\,|| \qquad \text{(we can pull out } \tfrac{1}{||\,\vec{v}\,||} \text{ as it is a scalar)}\\ &= 1.\end{aligned}$$

Notes:

So the vector of length 10 in the direction of $\vec{v}$ is $10\dfrac{1}{\|\,\vec{v}\,\|}\vec{v}$. An example will make this more clear.

Example 10.13 Using Unit Vectors

Let $\vec{v} = \langle 3, 1\rangle$ and let $\vec{w} = \langle 1, 2, 2\rangle$.

1. Find the unit vector in the direction of $\vec{v}$.
2. Find the unit vector in the direction of $\vec{w}$.
3. Find the vector in the direction of $\vec{v}$ with magnitude 5.

Solution

1. We find $\|\,\vec{v}\,\| = \sqrt{10}$. So the unit vector $\vec{u}$ in the direction of $\vec{v}$ is
$$\vec{u} = \frac{1}{\sqrt{10}}\vec{v} = \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle.$$
2. We find $\|\,\vec{w}\,\| = 3$, so the unit vector $\vec{z}$ in the direction of $\vec{w}$ is
$$\vec{u} = \frac{1}{3}\vec{w} = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle.$$
3. To create a vector with magnitude 5 in the direction of $\vec{v}$, we multiply the unit vector $\vec{u}$ by 5. Thus $5\vec{u} = \langle 15/\sqrt{10}, 5/\sqrt{10}\rangle$ is the vector we seek. This is sketched in Figure 10.25.

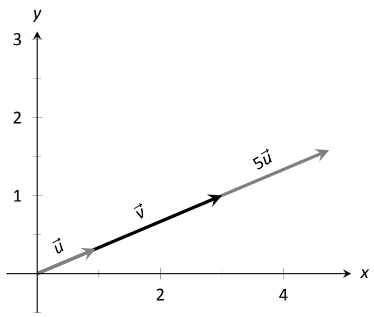


Figure 10.25: Graphing vectors in Example 10.13. All vectors shown have their initial point at the origin.

The basic formation of the unit vector $\vec{u}$ in the direction of a vector $\vec{v}$ leads to a interesting equation. It is:

$$\vec{v} = \|\,\vec{v}\,\|\frac{1}{\|\,\vec{v}\,\|}\vec{v}.$$

We rewrite the equation with parentheses to make a point:

$$\vec{v} = \underbrace{\|\,\vec{v}\,\|}_{\text{magnitude}} \cdot \underbrace{\left(\frac{1}{\|\,\vec{v}\,\|}\vec{v}\right)}_{\text{direction}}.$$

This equation illustrates the fact that a vector has both magnitude and direction, where we view a unit vector as supplying *only* direction information. Identifying unit vectors with direction allows us to define **parallel vectors**.

Notes:

Definition 59 Parallel Vectors

1. Unit vectors $\vec{u}_1$ and $\vec{u}_2$ are **parallel** if $\vec{u}_1 = \pm\vec{u}_2$.
2. Nonzero vectors $\vec{v}_1$ and $\vec{v}_2$ are **parallel** if their respective unit vectors are parallel.

It is equivalent to say that vectors $\vec{v}_1$ and $\vec{v}_2$ are parallel if there is a scalar $c \neq 0$ such that $\vec{v}_1 = c\vec{v}_2$ (see marginal note).

If one graphed all unit vectors in $\mathbb{R}^2$ with the initial point at the origin, then the terminal points would all lie on the unit circle. Based on what we know from trigonometry, we can then say that the component form of all unit vectors in $\mathbb{R}^2$ is $\langle \cos\theta, \sin\theta\rangle$ for some angle θ.

A similar construction in $\mathbb{R}^3$ shows that the terminal points all lie on the unit sphere. These vectors also have a particular component form, but its derivation is not as straightforward as the one for unit vectors in $\mathbb{R}^2$. Important concepts about unit vectors are given in the following Key Idea.

Note: $\vec{0}$ is directionless; because $||\,\vec{0}\,|| = 0$, there is no unit vector in the "direction" of $\vec{0}$.
Some texts define two vectors as being parallel if one is a scalar multiple of the other. By this definition, $\vec{0}$ is parallel to all vectors as $\vec{0} = 0\vec{v}$ for all $\vec{v}$.
We prefer the given definition of parallel as it is grounded in the fact that unit vectors provide direction information. One may adopt the convention that $\vec{0}$ is parallel to all vectors if they desire. (See also the marginal note on page 641.)

Key Idea 49 Unit Vectors

1. The unit vector in the direction of $\vec{v}$ is
$$\vec{u} = \frac{1}{||\,\vec{v}\,||}\vec{v}.$$
2. A vector $\vec{u}$ in $\mathbb{R}^2$ is a unit vector if, and only if, its component form is $\langle \cos\theta, \sin\theta\rangle$ for some angle θ.
3. A vector $\vec{u}$ in $\mathbb{R}^3$ is a unit vector if, and only if, its component form is $\langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi\rangle$ for some angles θ and φ.

These formulas can come in handy in a variety of situations, especially the formula for unit vectors in the plane.

Example 10.14 Finding Component Forces

Consider a weight of 50lb hanging from two chains, as shown in Figure 10.26. One chain makes an angle of 30° with the vertical, and the other an angle of 45°. Find the force applied to each chain.

Solution Knowing that gravity is pulling the 50lb weight straight down,

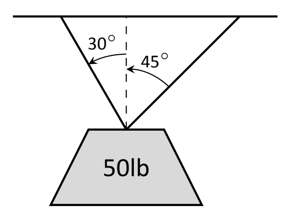


Figure 10.26: A diagram of a weight hanging from 2 chains in Example 10.14.

Notes:

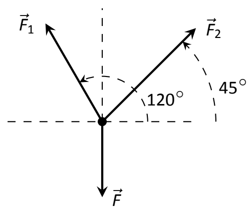


Figure 10.27: A diagram of the force vectors from Example 10.14.

we can create a vector $\vec{F}$ to represent this force.

$$\vec{F} = 50\langle 0, -1\rangle = \langle 0, -50\rangle .$$

We can view each chain as "pulling" the weight up, preventing it from falling. We can represent the force from each chain with a vector. Let $\vec{F_1}$ represent the force from the chain making an angle of 30° with the vertical, and let $\vec{F_2}$ represent the force form the other chain. Convert all angles to be measured from the horizontal (as shown in Figure 10.27), and apply Key Idea 49. As we do not yet know the magnitudes of these vectors, (that is the problem at hand), we use m_1 and m_2 to represent them.

$$\vec{F_1} = m_1\langle\cos 120°, \sin 120°\rangle$$
$$\vec{F_2} = m_2\langle\cos 45°, \sin 45°\rangle$$

As the weight is not moving, we know the sum of the forces is $\vec{0}$. This gives:

$$\begin{aligned} \vec{F} + \vec{F_1} + \vec{F_2} &= \vec{0} \\ \langle 0, -50\rangle + m_1\langle\cos 120°, \sin 120°\rangle + m_2\langle\cos 45°, \sin 45°\rangle &= \vec{0} \end{aligned}$$

The sum of the entries in the first component is 0, and the sum of the entries in the second component is also 0. This leads us to the following two equations:

$$\begin{aligned} m_1\cos 120° + m_2\cos 45° &= 0 \\ m_1\sin 120° + m_2\sin 45° &= 50 \end{aligned}$$

This is a simple 2-equation, 2-unknown system of linear equations. We leave it to the reader to verify that the solution is

$$m_1 = 50(\sqrt{3} - 1) \approx 36.6; \qquad m_2 = \frac{50\sqrt{2}}{1+\sqrt{3}} \approx 25.88.$$

It might seem odd that the sum of the forces applied to the chains is more than 50lb. We leave it to a physics class to discuss the full details, but offer this short explanation. Our equations were established so that the *vertical* components of each force sums to 50lb, thus supporting the weight. Since the chains are at an angle, they also pull against each other, creating an "additional" horizontal force while holding the weight in place.

Unit vectors were very important in the previous calculation; they allowed us to define a vector in the proper direction but with an unknown magnitude. Our computations were then computed component–wise. Because such calculations are often necessary, the *standard unit vectors* can be useful.

Notes:

Definition 60 Standard Unit Vectors

1. In $\mathbb{R}^2$, the standard unit vectors are
$$\vec{i} = \langle 1, 0 \rangle \quad \text{and} \quad \vec{j} = \langle 0, 1 \rangle .$$
2. In $\mathbb{R}^3$, the standard unit vectors are
$$\vec{i} = \langle 1, 0, 0 \rangle \quad \text{and} \quad \vec{j} = \langle 0, 1, 0 \rangle \quad \text{and} \quad \vec{k} = \langle 0, 0, 1 \rangle .$$

Example 10.15 Using standard unit vectors

1. Rewrite $\vec{v} = \langle 2, -3 \rangle$ using the standard unit vectors.
2. Rewrite $\vec{w} = 4\vec{i} - 5\vec{j} + 2\vec{k}$ in component form.

Solution

1.
$$\begin{aligned} \vec{v} &= \langle 2, -3 \rangle \\ &= \langle 2, 0 \rangle + \langle 0, -3 \rangle \\ &= 2\langle 1, 0 \rangle - 3\langle 0, 1 \rangle \\ &= 2\vec{i} - 3\vec{j} \end{aligned}$$

2.
$$\begin{aligned} \vec{w} &= 4\vec{i} - 5\vec{j} + 2\vec{k} \\ &= \langle 4, 0, 0 \rangle + \langle 0, -5, 0 \rangle + \langle 0, 0, 2 \rangle \\ &= \langle 4, -5, 2 \rangle \end{aligned}$$

These two examples demonstrate that converting between component form and the standard unit vectors is rather straightforward. Many mathematicians prefer component form, and it is the preferred notation in this text. Many engineers prefer using the standard unit vectors, and many engineering texts use that notation.

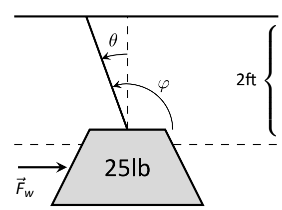


Figure 10.28: A figure of a weight being pushed by the wind in Example 10.16.

Example 10.16 Finding Component Force

A weight of 25lb is suspended from a chain of length 2ft while a wind pushes the weight to the right with constant force of 5lb as shown in Figure 10.28. What angle will the chain make with the vertical as a result of the wind's pushing? How much higher will the weight be?

Notes:

Solution The force of the wind is represented by the vector $\vec{F}_w = 5\vec{i}$. The force of gravity on the weight is represented by $\vec{F}_g = -25\vec{j}$. The direction and magnitude of the vector representing the force on the chain are both unknown. We represent this force with

$$\vec{F}_c = m\left\langle \cos\varphi, \sin\varphi \right\rangle = m\cos\varphi\,\vec{i} + m\sin\varphi\,\vec{j}$$

for some magnitude m and some angle with the horizontal φ. (Note: θ is the angle the chain makes with the *vertical*; φ is the angle with the *horizontal*.)

As the weight is at equilibrium, the sum of the forces is $\vec{0}$:

$$\begin{aligned} \vec{F}_c + \vec{F}_w + \vec{F}_g &= \vec{0} \\ m\cos\varphi\,\vec{i} + m\sin\varphi\,\vec{j} + 5\vec{i} - 25\vec{j} &= \vec{0} \end{aligned}$$

Thus the sum of the $\vec{i}$ and $\vec{j}$ components are 0, leading us to the following system of equations:

$$\begin{aligned} 5 + m\cos\varphi &= 0 \\ -25 + m\sin\varphi &= 0 \end{aligned} \tag{10.1}$$

This is enough to determine $\vec{F}_c$ already, as we know $m\cos\varphi = -5$ and $m\sin\varphi = 25$. Thus $F_c = \langle -5, 25 \rangle$. We can use this to find the magnitude m:

$$m = \sqrt{(-5)^2 + 25^2} = 5\sqrt{26} \approx 25.5\text{lb}.$$

We can then use either equality from Equation (10.1) to solve for φ. We choose the first equality as using arccosine will return an angle in the 2nd quadrant:

$$5 + 5\sqrt{26}\cos\varphi = 0 \quad \Rightarrow \quad \varphi = \cos^{-1}\left(\frac{-5}{5\sqrt{26}}\right) \approx 1.7682 \approx 101.31^\circ.$$

Subtracting 90° from this angle gives us an angle of 11.31° with the vertical.

We can now use trigonometry to find out how high the weight is lifted. The diagram shows that a right triangle is formed with the 2ft chain as the hypotenuse with an interior angle of 11.31°. The length of the adjacent side (in the diagram, the dashed vertical line) is $2\cos 11.31^\circ \approx 1.96$ft. Thus the weight is lifted by about 0.04ft, almost 1/2in.

The algebra we have applied to vectors is already demonstrating itself to be very useful. There are two more fundamental operations we can perform with vectors, the *dot product* and the *cross product*. The next two sections explore each in turn.

Notes:

Exercises 10.2

Terms and Concepts

1. Name two different things that cannot be described with just one number, but rather need 2 or more numbers to fully describe them.

2. What is the difference between $(1, 2)$ and $\langle 1, 2 \rangle$?

3. What is a unit vector?

4. What does it mean for two vectors to be parallel?

5. What effect does multiplying a vector by -2 have?

Problems

In Exercises 6 – 9, points P and Q are given. Write the vector $\overrightarrow{PQ}$ in component form and using the standard unit vectors.

6. $P = (2, -1), \quad Q = (3, 5)$

7. $P = (3, 2), \quad Q = (7, -2)$

8. $P = (0, 3, -1), \quad Q = (6, 2, 5)$

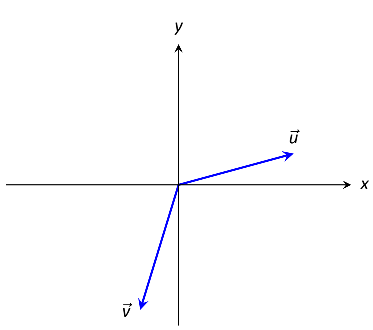
9. $P = (2, 1, 2), \quad Q = (4, 3, 2)$

10. Let $\vec{u} = \langle 1, -2 \rangle$ and $\vec{v} = \langle 1, 1 \rangle$.
    (a) Find $\vec{u} + \vec{v}$, $\vec{u} - \vec{v}$, $2\vec{u} - 3\vec{v}$.
    (b) Sketch the above vectors on the same axes, along with $\vec{u}$ and $\vec{v}$.
    (c) Find $\vec{x}$ where $\vec{u} + \vec{x} = 2\vec{v} - \vec{x}$.

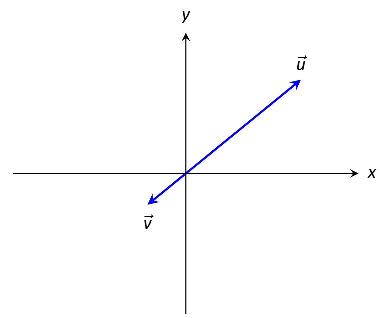
11. Let $\vec{u} = \langle 1, 1, -1 \rangle$ and $\vec{v} = \langle 2, 1, 2 \rangle$.
    (a) Find $\vec{u} + \vec{v}$, $\vec{u} - \vec{v}$, $\pi\vec{u} - \sqrt{2}\vec{v}$.
    (b) Sketch the above vectors on the same axes, along with $\vec{u}$ and $\vec{v}$.
    (c) Find $\vec{x}$ where $\vec{u} + \vec{x} = \vec{v} + 2\vec{x}$.

In Exercises 12 – 15, sketch $\vec{u}$, $\vec{v}$, $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$ on the same axes.

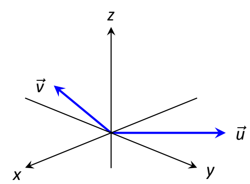
12.



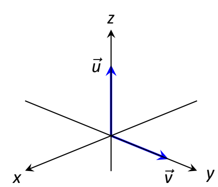
13.



14.



15.



In Exercises 16 – 19, find $||\,\vec{u}\,||$, $||\,\vec{v}\,||$, $||\,\vec{u}+\vec{v}\,||$ and $||\,\vec{u}-\vec{v}\,||$.

16. $\vec{u} = \langle 2, 1 \rangle, \quad \vec{v} = \langle 3, -2 \rangle$

17. $\vec{u} = \langle -3, 2, 2 \rangle, \quad \vec{v} = \langle 1, -1, 1 \rangle$

18. $\vec{u} = \langle 1, 2 \rangle, \quad \vec{v} = \langle -3, -6 \rangle$

19. $\vec{u} = \langle 2, -3, 6 \rangle, \quad \vec{v} = \langle 10, -15, 30 \rangle$

20. Under what conditions is $||\,\vec{u}\,|| + ||\,\vec{v}\,|| = ||\,\vec{u}+\vec{v}\,||$?

In Exercises 21 – 24, find the unit vector $\vec{u}$ in the direction of $\vec{v}$.

21. $\vec{v} = \langle 3, 7 \rangle$

22. $\vec{v} = \langle 6, 8 \rangle$

23. $\vec{v} = \langle 1, -2, 2 \rangle$

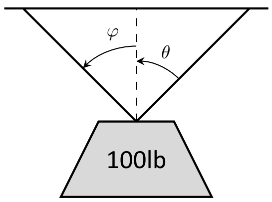
24. $\vec{v} = \langle 2, -2, 2 \rangle$

25. Find the unit vector in the first quadrant of $\mathbb{R}^2$ that makes a 50° angle with the x-axis.

26. Find the unit vector in the second quadrant of $\mathbb{R}^2$ that makes a 30° angle with the y-axis.

27. Verify, from Key Idea 49, that $\vec{u} = \langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi\rangle$ is a unit vector for all angles φ and θ.

A weight of 100lb is suspended from two chains, making angles with the vertical of θ and φ as shown in the figure below.



In Exercises 28 – 31, angles θ and φ are given. Find the force applied to each chain.

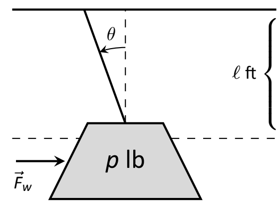
28. $\theta = 30^\circ, \quad \varphi = 30^\circ$

29. $\theta = 60^\circ, \quad \varphi = 60^\circ$

30. $\theta = 20^\circ, \quad \varphi = 15^\circ$

31. $\theta = 0^\circ, \quad \varphi = 0^\circ$

A weight of p lb is suspended from a chain of length ℓ while a constant force of $\vec{F}_w$ pushes the weight to the right, making an angle of θ with the vertical, as shown in the figure below.



In Exercises 32 – 35, a force $\vec{F}_w$ and length ℓ are given. Find the angle θ and the height the weight is lifted as it moves to the right.

32. $\vec{F}_w = 1\text{lb}, \quad \ell = 1\text{ft}, \quad p = 1\text{lb}$

33. $\vec{F}_w = 1\text{lb}, \quad \ell = 1\text{ft}, \quad p = 10\text{lb}$

34. $\vec{F}_w = 1\text{lb}, \quad \ell = 10\text{ft}, \quad p = 1\text{lb}$

35. $\vec{F}_w = 10\text{lb}, \quad \ell = 10\text{ft}, \quad p = 1\text{lb}$

10.3 The Dot Product

The previous section introduced vectors and described how to add them together and how to multiply them by scalars. This section introduces *a* multiplication on vectors called the **dot product**.

Definition 61 Dot Product

1. Let $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$ in $\mathbb{R}^2$. The **dot product** of $\vec{u}$ and $\vec{v}$, denoted $\vec{u} \cdot \vec{v}$, is
$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2.$$
2. Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ in $\mathbb{R}^3$. The **dot product** of $\vec{u}$ and $\vec{v}$, denoted $\vec{u} \cdot \vec{v}$, is
$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

Note how this product of vectors returns a *scalar*, not another vector. We practice evaluating a dot product in the following example, then we will discuss why this product is useful.

Example 10.17 Evaluating dot products

1. Let $\vec{u} = \langle 1, 2 \rangle$, $\vec{v} = \langle 3, -1 \rangle$ in $\mathbb{R}^2$. Find $\vec{u} \cdot \vec{v}$.
2. Let $\vec{x} = \langle 2, -2, 5 \rangle$ and $\vec{y} = \langle -1, 0, 3 \rangle$ in $\mathbb{R}^3$. Find $\vec{x} \cdot \vec{y}$.

Solution

1. Using Definition 61, we have
$$\vec{u} \cdot \vec{v} = 1(3) + 2(-1) = 1.$$
2. Using the definition, we have
$$\vec{x} \cdot \vec{y} = 2(-1) - 2(0) + 5(3) = 13.$$

The dot product, as shown by the preceding example, is very simple to evaluate. It is only the sum of products. While the definition gives no hint as to why

Notes:

we would care about this operation, there is an amazing connection between the dot product and angles formed by the vectors. Before stating this connection, we give a theorem stating some of the properties of the dot product.

Theorem 87 Properties of the Dot Product

Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ and let c be a scalar.

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ Commutative Property
2. $\vec{u} \cdot (\vec{v} \pm \vec{w}) = \vec{u} \cdot \vec{v} \pm \vec{u} \cdot \vec{w}$ Distributive Property
3. $(\vec{u} \pm \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} \pm \vec{v} \cdot \vec{w}$ Distributive Property
4. $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$
5. $\vec{0} \cdot \vec{v} = \vec{v} \cdot \vec{0} = 0$
6. $\vec{v} \cdot \vec{v} = ||\,\vec{v}\,||^2$

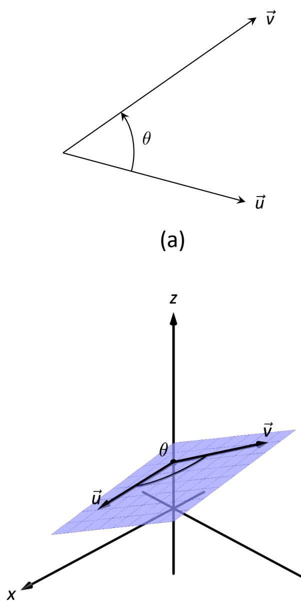


Figure 10.29: Illustrating the angle formed by two vectors with the same initial point.

The last statement of the theorem makes a handy connection between the magnitude of a vector and the dot product with itself. Our definition and theorem give properties of the dot product, but we are still likely wondering "What does the dot product *mean*?" It is helpful to understand that the dot product of a vector with itself is connected to its magnitude.

The next theorem extends this understanding by connecting the dot product to magnitudes and angles. Given vectors $\vec{u}$ and $\vec{v}$ in the plane, an angle θ is clearly formed when $\vec{u}$ and $\vec{v}$ are drawn with the same initial point as illustrated in Figure 10.29(a). (We always take θ to be the angle in $[0, \pi]$ as two angles are actually created.)

The same is also true of 2 vectors in space: given $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ with the same initial point, there is a plane that contains both $\vec{u}$ and $\vec{v}$. (When $\vec{u}$ and $\vec{v}$ are colinear, there are infinite planes that contain both vectors.) In that plane, we can again find an angle θ between them (and again, $0 \leq \theta \leq \pi$). This is illustrated in Figure 10.29(b).

The following theorem connects this angle θ to the dot product of $\vec{u}$ and $\vec{v}$.

Notes:

Theorem 88 The Dot Product and Angles

Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then

$$\vec{u} \cdot \vec{v} = \|\, \vec{u} \,\| \, \|\, \vec{v} \,\| \cos\theta,$$

where θ, $0 \le \theta \le \pi$, is the angle between $\vec{u}$ and $\vec{v}$.

When θ is an acute angle (i.e., $0 \le \theta < \pi/2$), $\cos\theta$ is positive; when $\theta = \pi/2$, $\cos\theta = 0$; when θ is an obtuse angle ($\pi/2 < \theta \le \pi$), $\cos\theta$ is negative. Thus the sign of the dot product gives a general indication of the angle between the vectors, illustrated in Figure 10.30.

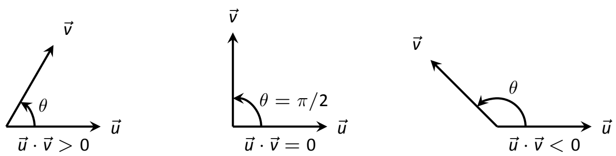


Figure 10.30: Illustrating the relationship between the angle between vectors and the sign of their dot product.

We *can* use Theorem 88 to compute the dot product, but generally this theorem is used to find the angle between known vectors (since the dot product is generally easy to compute). To this end, we rewrite the theorem's equation as

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\, \vec{u} \,\| \|\, \vec{v} \,\|} \quad \Leftrightarrow \quad \theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{\|\, \vec{u} \,\| \|\, \vec{v} \,\|}\right).$$

We practice using this theorem in the following example.

Example 10.18 Using the dot product to find angles

Let $\vec{u} = \langle 3, 1\rangle$, $\vec{v} = \langle -2, 6\rangle$ and $\vec{w} = \langle -4, 3\rangle$, as shown in Figure 10.31. Find the angles α, β and θ.

Solution We start by computing the magnitude of each vector.

$$\|\, \vec{u} \,\| = \sqrt{10}; \quad \|\, \vec{v} \,\| = 2\sqrt{10}; \quad \|\, \vec{w} \,\| = 5.$$

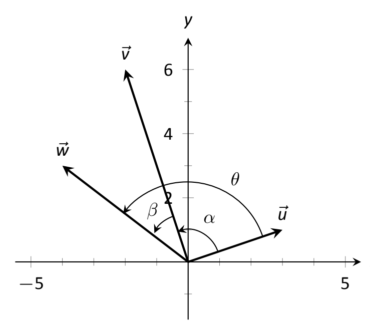


Figure 10.31: Vectors used in Example 10.18.

Notes:

We now apply Theorem 88 to find the angles.

$$\begin{aligned}\alpha &= \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{(\sqrt{10})(2\sqrt{10})}\right)\\ &= \cos^{-1}(0) = \frac{\pi}{2} = 90^\circ.\end{aligned}$$

$$\begin{aligned}\beta &= \cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{(2\sqrt{10})(5)}\right)\\ &= \cos^{-1}\left(\frac{26}{10\sqrt{10}}\right)\\ &\approx 0.6055 \approx 34.7^\circ.\end{aligned}$$

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{u}\cdot\vec{w}}{(\sqrt{10})(5)}\right)\\ &= \cos^{-1}\left(\frac{-9}{5\sqrt{10}}\right)\\ &\approx 2.1763 \approx 124.7^\circ\end{aligned}$$

We see from our computation that $\alpha+\beta=\theta$, as indicated by Figure 10.31. While we knew this should be the case, it is nice to see that this non-intuitive formula indeed returns the results we expected.

We do a similar example next in the context of vectors in space.

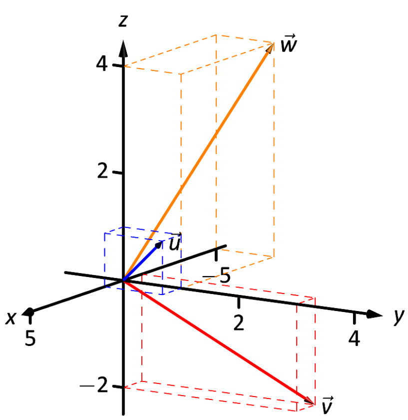


Figure 10.32: Vectors used in Example 10.19.

Example 10.19 Using the dot product to find angles

Let $\vec{u}=\langle 1,1,1\rangle$, $\vec{v}=\langle -1,3,-2\rangle$ and $\vec{w}=\langle -5,1,4\rangle$, as illustrated in Figure 10.32. Find the angle between each pair of vectors.

Solution

1. Between $\vec{u}$ and $\vec{v}$:

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{||\,\vec{u}\,||||\,\vec{v}\,||}\right)\\ &= \cos^{-1}\left(\frac{0}{\sqrt{3}\sqrt{14}}\right)\\ &= \frac{\pi}{2}.\end{aligned}$$

Notes:

2. Between $\vec{u}$ and $\vec{w}$:

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{u}\cdot\vec{w}}{\|\vec{u}\|\|\vec{w}\|}\right)\\ &= \cos^{-1}\left(\frac{0}{\sqrt{3}\sqrt{42}}\right)\\ &= \frac{\pi}{2}.\end{aligned}$$

3. Between $\vec{v}$ and $\vec{w}$:

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right)\\ &= \cos^{-1}\left(\frac{0}{\sqrt{14}\sqrt{42}}\right)\\ &= \frac{\pi}{2}.\end{aligned}$$

While our work shows that each angle is $\pi/2$, i.e., 90°, none of these angles looks to be a right angle in Figure 10.32. Such is the case when drawing three–dimensional objects on the page.

All three angles between these vectors was $\pi/2$, or 90°. We know from geometry and everyday life that 90° angles are "nice" for a variety of reasons, so it should seem significant that these angles are all $\pi/2$. Notice the common feature in each calculation (and also the calculation of α in Example 10.18): the dot products of each pair of angles was 0. We use this as a basis for a definition of the term **orthogonal**, which is essentially synonymous to *perpendicular*.

Note: The term *perpendicular* originally referred to lines. As mathematics progressed, the concept of "being at right angles to" was applied to other objects, such as vectors and planes, and the term *orthogonal* was introduced. It is especially used when discussing objects that are hard, or impossible, to visualize: two vectors in 5-dimensional space are orthogonal if their dot product is 0. It is not wrong to say they are *perpendicular*, but common convention gives preference to the word *orthogonal*.

Definition 62 Orthogonal

Vectors $\vec{u}$ and $\vec{v}$ are **orthogonal** if their dot product is 0.

Example 10.20 Finding orthogonal vectors

Let $\vec{u} = \langle 3, 5\rangle$ and $\vec{v} = \langle 1, 2, 3\rangle$.

1. Find two vectors in $\mathbb{R}^2$ that are orthogonal to $\vec{u}$.
2. Find two non–parallel vectors in $\mathbb{R}^3$ that are orthogonal to $\vec{v}$.

Solution

Notes:

1. Recall that a line perpendicular to a line with slope m has slope $-1/m$, the "opposite reciprocal slope." We can think of the slope of $\vec{u}$ as $5/3$, its "rise over run." A vector orthogonal to $\vec{u}$ will have slope $-3/5$. There are many such choices, though all parallel:

$$\langle -5, 3 \rangle \quad \text{or} \quad \langle 5, -3 \rangle \quad \text{or} \quad \langle -10, 6 \rangle \quad \text{or} \quad \langle 15, -9 \rangle \text{, etc.}$$

2. There are infinite directions in space orthogonal to any given direction, so there are an infinite number of non–parallel vectors orthogonal to $\vec{v}$. Since there are so many, we have great leeway in finding some.

   One way is to arbitrarily pick values for the first two components, leaving the third unknown. For instance, let $\vec{v}_1 = \langle 2, 7, z \rangle$. If $\vec{v}_1$ is to be orthogonal to $\vec{v}$, then $\vec{v}_1 \cdot \vec{v} = 0$, so

$$2 + 14 + 3z = 0 \qquad \Rightarrow z = \frac{-16}{3}.$$

   So $\vec{v}_1 = \langle 2, 7, -16/3 \rangle$ is orthogonal to $\vec{v}$. We can apply a similar technique by leaving the first or second component unknown.

   Another method of finding a vector orthogonal to $\vec{v}$ mirrors what we did in part 1. Let $\vec{v}_2 = \langle -2, 1, 0 \rangle$. Here we switched the first two components of $\vec{v}$, changing the sign of one of them (similar to the "opposite reciprocal" concept before). Letting the third component be 0 effectively ignores the third component of $\vec{v}$, and it is easy to see that

$$\vec{v}_2 \cdot \vec{v} = \langle -2, 1, 0 \rangle \cdot \langle 1, 2, 3 \rangle = 0.$$

   Clearly $\vec{v}_1$ and $\vec{v}_2$ are not parallel.

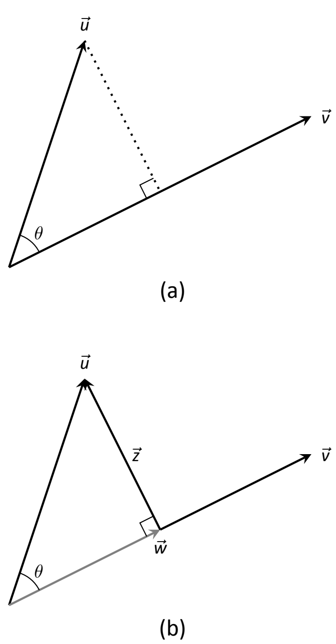


Figure 10.33: Developing the construction of the *orthogonal projection*.

An important construction is illustrated in Figure 10.33, where vectors $\vec{u}$ and $\vec{v}$ are sketched. In part (a), a dotted line is drawn from the tip of $\vec{u}$ to the line containing $\vec{v}$, where the dotted line is orthogonal to $\vec{v}$. In part (b), the dotted line is replaced with the vector $\vec{z}$ and $\vec{w}$ is formed, parallel to $\vec{v}$. It is clear by the diagram that $\vec{u} = \vec{w} + \vec{z}$. What is important about this construction is this: $\vec{u}$ is *decomposed* as the sum of two vectors, one of which is parallel to $\vec{v}$ and one that is perpendicular to $\vec{v}$. It is hard to overstate the importance of this construction (as we'll see in upcoming examples).

The vectors $\vec{w}$, $\vec{z}$ and $\vec{u}$ as shown in Figure 10.33 (b) form a right triangle, where the angle between $\vec{v}$ and $\vec{u}$ is labeled θ. We can find $\vec{w}$ in terms of $\vec{v}$ and $\vec{u}$.

Using trigonometry, we can state that

$$||\,\vec{w}\,|| = ||\,\vec{u}\,|| \cos\theta. \tag{10.2}$$

Notes:

We also know that $\vec{w}$ is parallel to to $\vec{v}$; that is, the direction of $\vec{w}$ is the direction of $\vec{v}$, described by the unit vector $\frac{1}{||\,\vec{v}\,||}\vec{v}$. The vector $\vec{w}$ is the vector in the direction $\frac{1}{||\,\vec{v}\,||}\vec{v}$ with magnitude $||\,\vec{u}\,||\cos\theta$:

$$\vec{w} = \Big(||\,\vec{u}\,||\cos\theta\Big)\frac{1}{||\,\vec{v}\,||}\vec{v}.$$

Replace $\cos\theta$ using Theorem 88:

$$\begin{aligned} &= \left(||\,\vec{u}\,||\frac{\vec{u}\cdot\vec{v}}{||\,\vec{u}\,||||\,\vec{v}\,||}\right)\frac{1}{||\,\vec{v}\,||}\vec{v} \\ &= \frac{\vec{u}\cdot\vec{v}}{||\,\vec{v}\,||^2}\vec{v}. \end{aligned}$$

Now apply Theorem 87.

$$= \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}.$$

Since this construction is so important, it is given a special name.

Definition 63 Orthogonal Projection

Let $\vec{u}$ and $\vec{v}$ be given. The **orthogonal projection of $\vec{u}$ onto $\vec{v}$**, denoted $\text{proj}_{\,\vec{v}}\,\vec{u}$, is

$$\text{proj}_{\,\vec{v}}\,\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}.$$

Example 10.21 Computing the orthogonal projection

1. Let $\vec{u} = \langle -2, 1\rangle$ and $\vec{v} = \langle 3, 1\rangle$. Find $\text{proj}_{\,\vec{v}}\,\vec{u}$, and sketch all three vectors with initial points at the origin.

2. Let $\vec{w} = \langle 2, 1, 3\rangle$ and $\vec{x} = \langle 1, 1, 1\rangle$. Find $\text{proj}_{\,\vec{x}}\,\vec{w}$, and sketch all three vectors with initial points at the origin.

Solution

Notes:

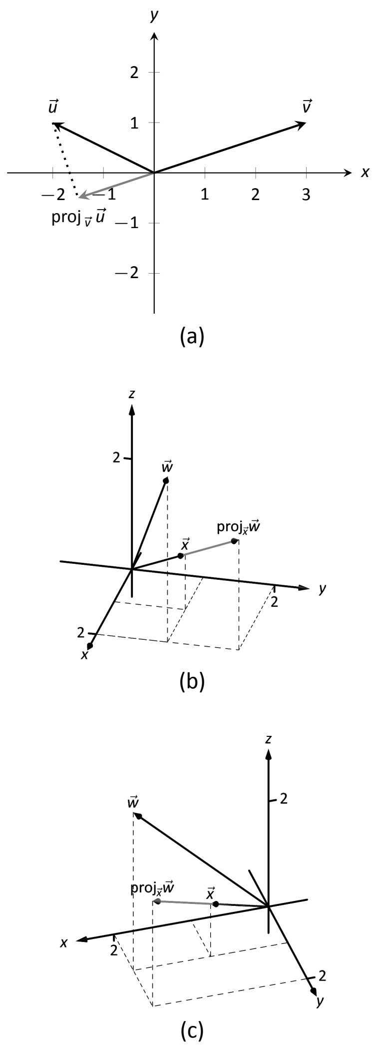


Figure 10.34: Graphing the vectors used in Example 10.21.

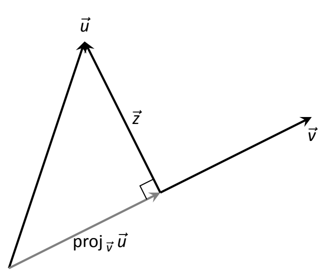


Figure 10.35: Illustrating the orthogonal projection.

1. Applying Definition 63, we have

$$\begin{aligned}\text{proj}_{\vec{v}}\,\vec{u} &= \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}\\ &= \frac{-5}{10}\langle 3,1\rangle\\ &= \left\langle -\frac{3}{2}, -\frac{1}{2}\right\rangle.\end{aligned}$$

Vectors $\vec{u}$, $\vec{v}$ and $\text{proj}_{\vec{v}}\,\vec{u}$ are sketched in Figure 10.34(a). Note how the projection is parallel to $\vec{v}$; that is, it lies on the same line through the origin as $\vec{v}$, although it points in the opposite direction. That is because the angle between $\vec{u}$ and $\vec{v}$ is obtuse (i.e., greater than 90°).

2. Apply the definition:

$$\begin{aligned}\text{proj}_{\vec{x}}\,\vec{w} &= \frac{\vec{w}\cdot\vec{x}}{\vec{x}\cdot\vec{x}}\vec{x}\\ &= \frac{6}{3}\langle 1,1,1\rangle\\ &= \langle 2,2,2\rangle\,.\end{aligned}$$

These vectors are sketched in Figure 10.34(b), and again in part (c) from a different perspective. Because of the nature of graphing these vectors, the sketch in part (b) makes it difficult to recognize that the drawn projection has the geometric properties it should. The graph shown in part (c) illustrates these properties better.

Consider Figure 10.35 where the concept of the orthogonal projection is again illustrated. It is clear that

$$\vec{u} = \text{proj}_{\vec{v}}\,\vec{u} + \vec{z}. \tag{10.3}$$

As we know what $\vec{u}$ and $\text{proj}_{\vec{v}}\,\vec{u}$ are, we can solve for $\vec{z}$ and state that

$$\vec{z} = \vec{u} - \text{proj}_{\vec{v}}\,\vec{u}.$$

This leads us to rewrite Equation (10.3) in a seemingly silly way:

$$\vec{u} = \text{proj}_{\vec{v}}\,\vec{u} + (\vec{u} - \text{proj}_{\vec{v}}\,\vec{u}).$$

This is not nonsense, as pointed out in the following Key Idea. (Notation note: the expression "$\parallel\vec{y}$" means "is parallel to $\vec{y}$." We can use this notation to state

Notes:

"$\vec{x} \parallel \vec{y}$" which means "$\vec{x}$ is parallel to $\vec{y}$." The expression "$\perp\ \vec{y}$" means "is orthogonal to $\vec{y}$," and is used similarly.)

Key Idea 50 Orthogonal Decomposition of Vectors

Let $\vec{u}$ and $\vec{v}$ be given. Then $\vec{u}$ can be written as the sum of two vectors, one of which is parallel to $\vec{v}$, and one of which is orthogonal to $\vec{v}$:

$$\vec{u} = \underbrace{\text{proj}_{\vec{v}}\,\vec{u}}_{\parallel\ \vec{v}} + \underbrace{(\vec{u} - \text{proj}_{\vec{v}}\,\vec{u})}_{\perp\ \vec{v}}.$$

We illustrate the use of this equality in the following example.

Example 10.22 Orthogonal decomposition of vectors

1. Let $\vec{u} = \langle -2, 1\rangle$ and $\vec{v} = \langle 3, 1\rangle$ as in Example 10.21. Decompose $\vec{u}$ as the sum of a vector parallel to $\vec{v}$ and a vector orthogonal to $\vec{v}$.
2. Let $\vec{w} = \langle 2, 1, 3\rangle$ and $\vec{x} = \langle 1, 1, 1\rangle$ as in Example 10.21. Decompose $\vec{w}$ as the sum of a vector parallel to $\vec{x}$ and a vector orthogonal to $\vec{x}$.

Solution

1. In Example 10.21, we found that $\text{proj}_{\vec{v}}\,\vec{u} = \langle -1.5, -0.5\rangle$. Let

   $$\vec{z} = \vec{u} - \text{proj}_{\vec{v}}\,\vec{u} = \langle -2, 1\rangle - \langle -1.5, -0.5\rangle = \langle -0.5, 1.5\rangle\,.$$

   Is $\vec{z}$ orthogonal to $\vec{v}$? (I.e, is $\vec{z} \perp \vec{v}$?) We check for orthogonality with the dot product:

   $$\vec{z}\cdot\vec{v} = \langle -0.5, 1.5\rangle\cdot\langle 3, 1\rangle = 0.$$

   Since the dot product is 0, we know $\vec{z} \perp \vec{v}$. Thus:

   $$\begin{aligned} \vec{u} &= \text{proj}_{\vec{v}}\,\vec{u} \ + \ (\vec{u} - \text{proj}_{\vec{v}}\,\vec{u}) \\ \langle -2, 1\rangle &= \underbrace{\langle -1.5, -0.5\rangle}_{\parallel\ \vec{v}} + \underbrace{\langle -0.5, 1.5\rangle}_{\perp\ \vec{v}}. \end{aligned}$$

2. We found in Example 10.21 that $\text{proj}_{\vec{x}}\,\vec{w} = \langle 2, 2, 2\rangle$. Applying the Key Idea, we have:

   $$\vec{z} = \vec{w} - \text{proj}_{\vec{x}}\,\vec{w} = \langle 2, 1, 3\rangle - \langle 2, 2, 2\rangle = \langle 0, -1, 1\rangle\,.$$

Notes:

We check to see if $\vec{z} \perp \vec{x}$:

$$\vec{z} \cdot \vec{x} = \langle 0, -1, 1 \rangle \cdot \langle 1, 1, 1 \rangle = 0.$$

Since the dot product is 0, we know the two vectors are orthogonal. We now write $\vec{w}$ as the sum of two vectors, one parallel and one orthogonal to $\vec{x}$:

$$\begin{aligned} \vec{w} &= \text{proj}_{\vec{x}}\,\vec{w} \;+\; (\vec{w} - \text{proj}_{\vec{x}}\,\vec{w}) \\ \langle 2, 1, 3 \rangle &= \underbrace{\langle 2, 2, 2 \rangle}_{\parallel\, \vec{x}} \;+\; \underbrace{\langle 0, -1, 1 \rangle}_{\perp\, \vec{x}} \end{aligned}$$

We give an example of where this decomposition is useful.

Example 10.23 Orthogonally decomposing a force vector

Consider Figure 10.36(a), showing a box weighing 50lb on a ramp that rises 5ft over a span of 20ft. Find the components of force, and their magnitudes, acting on the box (as sketched in part (b) of the figure):

1. in the direction of the ramp, and
2. orthogonal to the ramp.

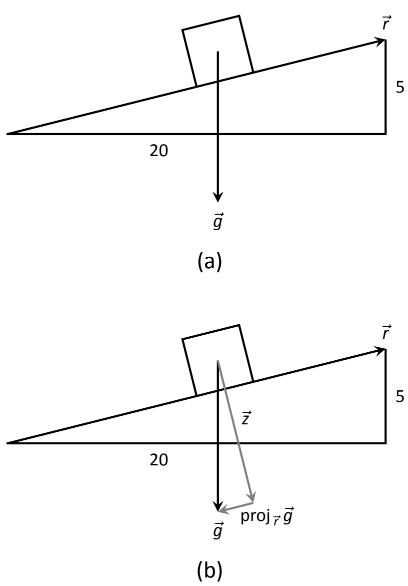


Figure 10.36: Sketching the ramp and box in Example 10.23. Note: *The vectors are not drawn to scale.*

Solution As the ramp rises 5ft over a horizontal distance of 20ft, we can represent the direction of the ramp with the vector $\vec{r} = \langle 20, 5 \rangle$. Gravity pulls down with a force of 50lb, which we represent with $\vec{g} = \langle 0, -50 \rangle$.

1. To find the force of gravity in the direction of the ramp, we compute $\text{proj}_{\vec{r}}\,\vec{g}$:

$$\begin{aligned} \text{proj}_{\vec{r}}\,\vec{g} &= \frac{\vec{g} \cdot \vec{r}}{\vec{r} \cdot \vec{r}}\vec{r} \\ &= \frac{-250}{425}\langle 20, 5 \rangle \\ &= \left\langle -\frac{200}{17}, -\frac{50}{17} \right\rangle \approx \langle -11.76, -2.94 \rangle . \end{aligned}$$

The magnitude of $\text{proj}_{\vec{r}}\,\vec{g}$ is $||\ \text{proj}_{\vec{r}}\,\vec{g}\ || = 50/\sqrt{17} \approx 12.13$lb. Though the box weighs 50lb, a force of about 12lb is enough to keep the box from sliding down the ramp.

2. To find the component $\vec{z}$ of gravity orthogonal to the ramp, we use Key Idea 50.

$$\begin{aligned} \vec{z} &= \vec{g} - \text{proj}_{\vec{r}}\,\vec{g} \\ &= \left\langle \frac{200}{17}, -\frac{800}{17} \right\rangle \approx \langle 11.76, -47.06 \rangle . \end{aligned}$$

Notes:

The magnitude of this force is $|| \vec{z} || \approx 48.51$lb. In physics and engineering, knowing this force is important when computing things like static frictional force. (For instance, we could easily compute if the static frictional force alone was enough to keep the box from sliding down the ramp.)

Application to Work

In physics, the application of a force F to move an object in a straight line a distance d produces *work*; the amount of work W is $W = Fd$, (where F is in the direction of travel). The orthogonal projection allows us to compute work when the force is not in the direction of travel.

Consider Figure 10.37, where a force $\vec{F}$ is being applied to an object moving in the direction of $\vec{d}$. (The distance the object travels is the magnitude of $\vec{d}$.) The work done is the amount of force in the direction of $\vec{d}$, $|| \operatorname{proj}_{\vec{d}} \vec{F} ||$, times $|| \vec{d} ||$:

$$\begin{aligned} || \operatorname{proj}_{\vec{d}} \vec{F} || \cdot || \vec{d} || &= \left|\left| \frac{\vec{F} \cdot \vec{d}}{\vec{d} \cdot \vec{d}} \vec{d} \right|\right| \cdot || \vec{d} || \\ &= \left| \frac{\vec{F} \cdot \vec{d}}{|| \vec{d} ||^2} \right| \cdot || \vec{d} || \cdot || \vec{d} || \\ &= \frac{\left| \vec{F} \cdot \vec{d} \right|}{|| \vec{d} ||^2} || \vec{d} ||^2 \\ &= \left| \vec{F} \cdot \vec{d} \right|. \end{aligned}$$

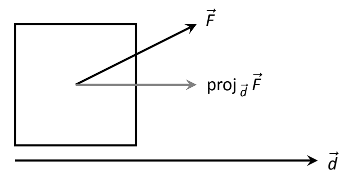


Figure 10.37: Finding work when the force and direction of travel are given as vectors.

The expression $\vec{F} \cdot \vec{d}$ will be positive if the angle between $\vec{F}$ and $\vec{d}$ is acute; when the angle is obtuse (hence $\vec{F} \cdot \vec{d}$ is negative), the force is causing motion in the opposite direction of $\vec{d}$, resulting in "negative work." We want to capture this sign, so we drop the absolute value and find that $W = \vec{F} \cdot \vec{d}$.

Definition 64 Work

Let $\vec{F}$ be a constant force that moves an object in a straight line from point P to point Q. Let $\vec{d} = \overrightarrow{PQ}$. The **work** W done by $\vec{F}$ along $\vec{d}$ is $W = \vec{F} \cdot \vec{d}$.

Example 10.24 Computing work

A man slides a box along a ramp that rises 3ft over a distance of 15ft by applying 50lb of force as shown in Figure 10.38. Compute the work done.

Notes:

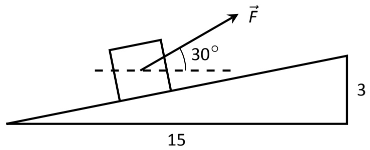


Figure 10.38: Computing work when sliding a box up a ramp in Example 10.24.

Solution The figure indicates that the force applied makes a 30° angle with the horizontal, so $\vec{F} = 50\langle\cos 30^\circ, \sin 30^\circ\rangle \approx \langle 43.3, 25\rangle$. The ramp is represented by $\vec{d} = \langle 15, 3\rangle$. The work done is simply

$$\vec{F}\cdot\vec{d} = 50\langle\cos 30^\circ, \sin 30^\circ\rangle \cdot \langle 15, 3\rangle \approx 724.5\text{ft–lb}.$$

Note how we did not actually compute the distance the object traveled, nor the magnitude of the force in the direction of travel; this is all inherently computed by the dot product!

The dot product is a powerful way of evaluating computations that depend on angles without actually using angles. The next section explores another "product" on vectors, the *cross product.* Once again, angles play an important role, though in a much different way.

Notes:

Exercises 10.3

Terms and Concepts

1. The dot product of two vectors is a ________, not a vector.

2. How are the concepts of the dot product and vector magnitude related?

3. How can one quickly tell if the angle between two vectors is acute or obtuse?

4. Give a synonym for "orthogonal."

Problems

In Exercises 5 – 10, find the dot product of the given vectors.

5. $\vec{u} = \langle 2, -4\rangle$, $\vec{v} = \langle 3, 7\rangle$

6. $\vec{u} = \langle 5, 3\rangle$, $\vec{v} = \langle 6, 1\rangle$

7. $\vec{u} = \langle 1, -1, 2\rangle$, $\vec{v} = \langle 2, 5, 3\rangle$

8. $\vec{u} = \langle 3, 5, -1\rangle$, $\vec{v} = \langle 4, -1, 7\rangle$

9. $\vec{u} = \langle 1, 1\rangle$, $\vec{v} = \langle 1, 2, 3\rangle$

10. $\vec{u} = \langle 1, 2, 3\rangle$, $\vec{v} = \langle 0, 0, 0\rangle$

11. Create your own vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^2$ and show that $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$.

12. Create your own vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ and scalar c and show that $c(\vec{u}\cdot\vec{v}) = \vec{u}\cdot(c\vec{v})$.

In Exercises 13 – 16, find the measure of the angle between the two vectors in both radians and degrees.

13. $\vec{u} = \langle 1, 1\rangle$, $\vec{v} = \langle 1, 2\rangle$

14. $\vec{u} = \langle -2, 1\rangle$, $\vec{v} = \langle 3, 5\rangle$

15. $\vec{u} = \langle 8, 1, -4\rangle$, $\vec{v} = \langle 2, 2, 0\rangle$

16. $\vec{u} = \langle 1, 7, 2\rangle$, $\vec{v} = \langle 4, -2, 5\rangle$

In Exercises 17 – 20, a vector $\vec{v}$ is given. Give two vectors that are orthogonal to $\vec{v}$.

17. $\vec{v} = \langle 4, 7\rangle$

18. $\vec{v} = \langle -3, 5\rangle$

19. $\vec{v} = \langle 1, 1, 1\rangle$

20. $\vec{v} = \langle 1, -2, 3\rangle$

In Exercises 21 – 26, vectors $\vec{u}$ and $\vec{v}$ are given. Find $\text{proj}_{\vec{v}}\,\vec{u}$, the orthogonal projection of $\vec{u}$ onto $\vec{v}$, and sketch all three vectors on the same axes.

21. $\vec{u} = \langle 1, 2\rangle$, $\vec{v} = \langle -1, 3\rangle$

22. $\vec{u} = \langle 5, 5\rangle$, $\vec{v} = \langle 1, 3\rangle$

23. $\vec{u} = \langle -3, 2\rangle$, $\vec{v} = \langle 1, 1\rangle$

24. $\vec{u} = \langle -3, 2\rangle$, $\vec{v} = \langle 2, 3\rangle$

25. $\vec{u} = \langle 1, 5, 1\rangle$, $\vec{v} = \langle 1, 2, 3\rangle$

26. $\vec{u} = \langle 3, -1, 2\rangle$, $\vec{v} = \langle 2, 2, 1\rangle$

In Exercises 27 – 32, vectors $\vec{u}$ and $\vec{v}$ are given. Write $\vec{u}$ as the sum of two vectors, one of which is parallel to $\vec{v}$ and one of which is perpendicular to $\vec{v}$. Note: these are the same pairs of vectors as found in Exercises 21 – 26.

27. $\vec{u} = \langle 1, 2\rangle$, $\vec{v} = \langle -1, 3\rangle$

28. $\vec{u} = \langle 5, 5\rangle$, $\vec{v} = \langle 1, 3\rangle$

29. $\vec{u} = \langle -3, 2\rangle$, $\vec{v} = \langle 1, 1\rangle$

30. $\vec{u} = \langle -3, 2\rangle$, $\vec{v} = \langle 2, 3\rangle$

31. $\vec{u} = \langle 1, 5, 1\rangle$, $\vec{v} = \langle 1, 2, 3\rangle$

32. $\vec{u} = \langle 3, -1, 2\rangle$, $\vec{v} = \langle 2, 2, 1\rangle$

33. A 10lb box sits on a ramp that rises 4ft over a distance of 20ft. How much force is required to keep the box from sliding down the ramp?

34. A 10lb box sits on a 15ft ramp that makes a 30° angle with the horizontal. How much force is required to keep the box from sliding down the ramp?

35. How much work is performed in moving a box horizontally 10ft with a force of 20lb applied at an angle of 45° to the horizontal?

36. How much work is performed in moving a box horizontally 10ft with a force of 20lb applied at an angle of 10° to the horizontal?

37. How much work is performed in moving a box up the length of a ramp that rises 2ft over a distance of 10ft, with a force of 50lb applied horizontally?

38. How much work is performed in moving a box up the length of a ramp that rises 2ft over a distance of 10ft, with a force of 50lb applied at an angle of 45° to the horizontal?

39. How much work is performed in moving a box up the length of a 10ft ramp that makes a 5° angle with the horizontal, with 50lb of force applied in the direction of the ramp?

10.4 The Cross Product

"Orthogonality" is immensely important. A quick scan of your current environment will undoubtedly reveal numerous surfaces and edges that are perpendicular to each other (including the edges of this page). The dot product provides a quick test for orthogonality: vectors $\vec{u}$ and $\vec{v}$ are perpendicular if, and only if, $\vec{u} \cdot \vec{v} = 0$.

Given two non–parallel, nonzero vectors $\vec{u}$ and $\vec{v}$ in space, it is very useful to find a vector $\vec{w}$ that is perpendicular to both $\vec{u}$ and $\vec{v}$. There is a operation, called the **cross product**, that creates such a vector. This section defines the cross product, then explores its properties and applications.

Definition 65 Cross Product

Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ be vectors in $\mathbb{R}^3$. The **cross product of $\vec{u}$ and $\vec{v}$**, denoted $\vec{u} \times \vec{v}$, is the vector

$$\vec{u} \times \vec{v} = \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle .$$

This definition can be a bit cumbersome to remember. After an example we will give a convenient method for computing the cross product. For now, careful examination of the products and differences given in the definition should reveal a pattern that is not too difficult to remember. (For instance, in the first component only 2 and 3 appear as subscripts; in the second component, only 1 and 3 appear as subscripts. Further study reveals the order in which they appear.)

Let's practice using this definition by computing a cross product.

Example 10.25 Computing a cross product

Let $\vec{u} = \langle 2, -1, 4 \rangle$ and $\vec{v} = \langle 3, 2, 5 \rangle$. Find $\vec{u} \times \vec{v}$, and verify that it is orthogonal to both $\vec{u}$ and $\vec{v}$.

Solution Using Definition 65, we have

$$\vec{u} \times \vec{v} = \langle (-1)5 - (4)2, (4)3 - (2)5, (2)2 - (-1)3 \rangle = \langle -13, 2, 7 \rangle .$$

(We encourage the reader to compute this product on their own, then verify their result.)

We test whether or not $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$ using the dot product:

$$\big(\vec{u} \times \vec{v}\big) \cdot \vec{u} = \langle -13, 2, 7 \rangle \cdot \langle 2, -1, 4 \rangle = 0,$$

$$\big(\vec{u} \times \vec{v}\big) \cdot \vec{v} = \langle -13, 2, 7 \rangle \cdot \langle 3, 2, 5 \rangle = 0.$$

Since both dot products are zero, $\vec{u} \times \vec{v}$ is indeed orthogonal to both $\vec{u}$ and $\vec{v}$.

Notes:

A convenient method of computing the cross product starts with forming a particular 3×3 *matrix*, or rectangular array. The first row comprises the standard unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$. The second and third rows are the vectors $\vec{u}$ and $\vec{v}$, respectively. Using $\vec{u}$ and $\vec{v}$ from Example 10.25, we begin with:

$$\begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 4 \\ 3 & 2 & 5 \end{array}$$

Now repeat the first two columns after the original three:

$$\begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 2 & -1 & 4 & 2 & -1 \\ 3 & 2 & 5 & 3 & 2 \end{array}$$

This gives three full "upper left to lower right" diagonals, and three full "upper right to lower left" diagonals, as shown. Compute the products along each diagonal, then add the products on the right and subtract the products on the left:

$$\begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 2 & -1 & 4 & 2 & -1 \\ 3 & 2 & 5 & 3 & 2 \end{array}$$

$$-3\vec{k} \quad 8\vec{i} \quad 10\vec{j} \qquad\qquad -5\vec{i} \quad 12\vec{j} \quad 4\vec{k}$$

$$\vec{u} \times \vec{v} = \big(-5\vec{i}+12\vec{j}+4\vec{k}\big) - \big(-3\vec{k}+8\vec{i}+10\vec{j}\big) = -13\vec{i}+2\vec{j}+7\vec{k} = \langle -13, 2, 7\rangle .$$

We practice using this method.

Example 10.26 Computing a cross product

Let $\vec{u} = \langle 1, 3, 6\rangle$ and $\vec{v} = \langle -1, 2, 1\rangle$. Compute both $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$.

Solution To compute $\vec{u} \times \vec{v}$, we form the matrix as prescribed above, complete with repeated first columns:

$$\begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 1 & 3 & 6 & 1 & 3 \\ -1 & 2 & 1 & -1 & 2 \end{array}$$

We let the reader compute the products of the diagonals; we give the result:

$$\vec{u} \times \vec{v} = \big(3\vec{i} - 6\vec{j} + 2\vec{k}\big) - \big(-3\vec{k} + 12\vec{i} + \vec{j}\big) = \langle -9, -7, 5\rangle .$$

Notes:

To compute $\vec{v} \times \vec{u}$, we switch the second and third rows of the above matrix, then multiply along diagonals and subtract:

$$\begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ -1 & 2 & 1 & -1 & 2 \\ 1 & 3 & 6 & 1 & 3 \end{array}$$

Note how with the rows being switched, the products that once appeared on the right now appear on the left, and vice–versa. Thus the result is:

$$\vec{v} \times \vec{u} = \left(12\vec{i} + \vec{j} - 3\vec{k}\right) - \left(2\vec{k} + 3\vec{i} - 6\vec{j}\right) = \langle 9, 7, -5 \rangle ,$$

which is the opposite of $\vec{u} \times \vec{v}$. We leave it to the reader to verify that each of these vectors is orthogonal to $\vec{u}$ and $\vec{v}$.

Properties of the Cross Product

It is not coincidence that $\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$ in the preceding example; one can show using Definition 65 that this will always be the case. The following theorem states several useful properties of the cross product, each of which can be verified by referring to the definition.

Theorem 89 Properties of the Cross Product

Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^3$ and let c be a scalar. The following identities hold:

1. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$ — Anticommutative Property
2. (a) $(\vec{u} \pm \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} \pm \vec{v} \times \vec{w}$ — Distributive Properties
   (b) $\vec{u} \times (\vec{v} \pm \vec{w}) = \vec{u} \times \vec{v} \pm \vec{u} \times \vec{w}$
3. $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$
4. (a) $(\vec{u} \times \vec{v}) \cdot \vec{u} = 0$ — Orthogonality Properties
   (b) $(\vec{u} \times \vec{v}) \cdot \vec{v} = 0$
5. $\vec{u} \times \vec{u} = \vec{0}$
6. $\vec{u} \times \vec{0} = \vec{0} \times \vec{u} = \vec{0}$
7. $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$ — Triple Scalar Product

Notes:

We introduced the cross product as a way to find a vector orthogonal to two given vectors, but we did not give a proof that the construction given in Definition 65 satisfies this property. Theorem 89 asserts this property holds; we leave it as a problem in the Exercise section to verify this.

Property 5 from the theorem is also left to the reader to prove in the Exercise section, but it reveals something more interesting than "the cross product of a vector with itself is $\vec{0}$." Let $\vec{u}$ and $\vec{v}$ be parallel vectors; that is, let there be a scalar c such that $\vec{v} = c\vec{u}$. Consider their cross product:

$$\begin{aligned} \vec{u} \times \vec{v} &= \vec{u} \times (c\vec{u}) \\ &= c(\vec{u} \times \vec{u}) && \text{(by Property 3 of Theorem 89)} \\ &= \vec{0}. && \text{(by Property 5 of Theorem 89)} \end{aligned}$$

We have just shown that the cross product of parallel vectors is $\vec{0}$. This hints at something deeper. Theorem 88 related the angle between two vectors and their dot product; there is a similar relationship relating the cross product of two vectors and the angle between them, given by the following theorem.

Theorem 90 The Cross Product and Angles

Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^3$. Then

$$|| \vec{u} \times \vec{v} || = || \vec{u} || \, || \vec{v} || \sin\theta,$$

where θ, $0 \leq \theta \leq \pi$, is the angle between $\vec{u}$ and $\vec{v}$.

Note: Definition 62 (through Theorem 88) defines $\vec{u}$ and $\vec{v}$ to be orthogonal if $\vec{u} \cdot \vec{v} = 0$. We could use Theorem 90 to define $\vec{u}$ and $\vec{v}$ are parallel if $\vec{u} \times \vec{v} = 0$. By such a definition, $\vec{0}$ would be both orthogonal and parallel to every vector. Apparent paradoxes such as this are not uncommon in mathematics and can be very useful. (See also the marginal note on page 619.)

Note that this theorem makes a statement about the *magnitude* of the cross product. When the angle between $\vec{u}$ and $\vec{v}$ is 0 or π (i.e., the vectors are parallel), the magnitude of the cross product is 0. The only vector with a magnitude of 0 is $\vec{0}$ (see Property 9 of Theorem 86), hence the cross product of parallel vectors is $\vec{0}$.

We demonstrate the truth of this theorem in the following example.

Example 10.27 The cross product and angles

Let $\vec{u} = \langle 1, 3, 6 \rangle$ and $\vec{v} = \langle -1, 2, 1 \rangle$ as in Example 10.26. Verify Theorem 90 by finding θ, the angle between $\vec{u}$ and $\vec{v}$, and the magnitude of $\vec{u} \times \vec{v}$.

Notes:

Solution We use Theorem 88 to find the angle between $\vec{u}$ and $\vec{v}$.

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{||\,\vec{u}\,||\,||\,\vec{v}\,||}\right)\\ &= \cos^{-1}\left(\frac{11}{\sqrt{46}\sqrt{6}}\right)\\ &\approx 0.8471 = 48.54^\circ.\end{aligned}$$

Our work in Example 10.26 showed that $\vec{u}\times\vec{v} = \langle -9, -7, 5\rangle$, hence $||\,\vec{u}\times\vec{v}\,|| = \sqrt{155}$. Is $||\,\vec{u}\times\vec{v}\,|| = ||\,\vec{u}\,||\,||\,\vec{v}\,||\sin\theta$? Using numerical approximations, we find:

$$\begin{aligned}||\,\vec{u}\times\vec{v}\,|| &= \sqrt{155} & \qquad ||\,\vec{u}\,||\,||\,\vec{v}\,||\sin\theta &= \sqrt{46}\sqrt{6}\sin 0.8471\\ &\approx 12.45. & &\approx 12.45.\end{aligned}$$

Numerically, they seem equal. Using a right triangle, one can show that

$$\sin\left(\cos^{-1}\left(\frac{11}{\sqrt{46}\sqrt{6}}\right)\right) = \frac{\sqrt{155}}{\sqrt{46}\sqrt{6}},$$

which allows us to verify the theorem exactly.

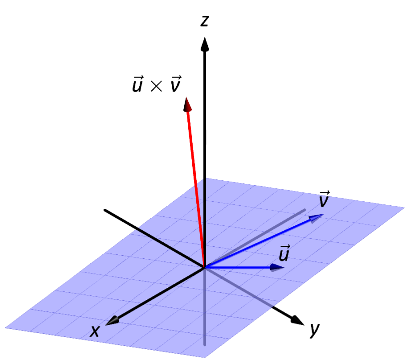


Figure 10.39: Illustrating the Right Hand Rule of the cross product.

Right Hand Rule

The anticommutative property of the cross product demonstrates that $\vec{u}\times\vec{v}$ and $\vec{v}\times\vec{u}$ differ only by a sign – these vectors have the same magnitude but point in the opposite direction. When seeking a vector perpendicular to $\vec{u}$ and $\vec{v}$, we essentially have two directions to choose from, one in the direction of $\vec{u}\times\vec{v}$ and one in the direction of $\vec{v}\times\vec{u}$. Does it matter which we choose? How can we tell which one we will get without graphing, etc.?

Another wonderful property of the cross product, as defined, is that it follows the **right hand rule.** Given $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ with the same initial point, point the index finger of your right hand in the direction of $\vec{u}$ and let your middle finger point in the direction of $\vec{v}$ (much as we did when establishing the right hand rule for the 3-dimensional coordinate system). Your thumb will naturally extend in the direction of $\vec{u}\times\vec{v}$. One can "practice" this using Figure 10.39. If you switch, and point the index finder in the direction of $\vec{v}$ and the middle finger in the direction of $\vec{u}$, your thumb will now point in the opposite direction, allowing you to "visualize" the anticommutative property of the cross product.

There is another equivalent way to use the right hand rule that some may find easier. If you point your right thumb in the direction of $\vec{u}$ and the remaining fingers of your right hand in the direction of $\vec{v}$, then the palm side of your right hand will face in the direction of $\vec{u}\times\vec{v}$.

Notes:

Applications of the Cross Product

There are a number of ways in which the cross product is useful in mathematics, physics and other areas of science beyond "just" finding a vector perpendicular to two others. We highlight a few here.

Area of a Parallelogram

It is a standard geometry fact that the area of a parallelogram is $A = bh$, where b is the length of the base and h is the height of the parallelogram, as illustrated in Figure 10.40(a). As shown when defining the Parallelogram Law of vector addition, two vectors $\vec{u}$ and $\vec{v}$ define a parallelogram when drawn from the same initial point, as illustrated in Figure 10.40(b). Trigonometry tells us that $h = ||\,\vec{u}\,||\sin\theta$, hence the area of the parallelogram is

$$A = ||\,\vec{u}\,||\,||\,\vec{v}\,||\sin\theta = ||\,\vec{u}\times\vec{v}\,||, \tag{10.4}$$

where the second equality comes from Theorem 90. We illustrate using Equation (10.4) in the following example.

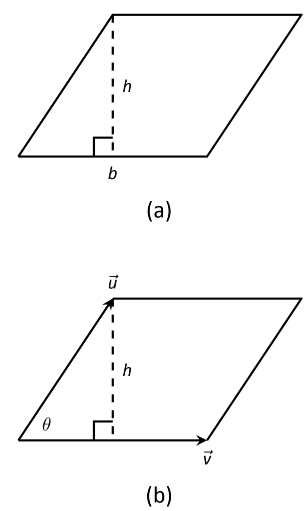


Figure 10.40: Using the cross product to find the area of a parallelogram.

Example 10.28 Finding the area of a parallelogram

1. Find the area of the parallelogram defined by the vectors $\vec{u} = \langle 2, 1\rangle$ and $\vec{v} = \langle 1, 3\rangle$.
2. Verify that the points $A = (1,1,1)$, $B = (2,3,2)$, $C = (4,5,3)$ and $D = (3,3,2)$ are the vertices of a parallelogram. Find the area of the parallelogram.

SOLUTION

1. Figure 10.41(a) sketches the parallelogram defined by the vectors $\vec{u}$ and $\vec{v}$. We have a slight problem in that our vectors exist in $\mathbb{R}^2$, not $\mathbb{R}^3$, and the cross product is only defined on vectors in $\mathbb{R}^3$. We skirt this issue by viewing $\vec{u}$ and $\vec{v}$ as vectors in the $x-y$ plane of $\mathbb{R}^3$, and rewrite them as $\vec{u} = \langle 2,1,0\rangle$ and $\vec{v} = \langle 1,3,0\rangle$. We can now compute the cross product. It is easy to show that $\vec{u}\times\vec{v} = \langle 0,0,5\rangle$; therefore the area of the parallelogram is $A = ||\,\vec{u}\times\vec{v}\,|| = 5$.
2. To show that the quadrilateral $ABCD$ is a parallelogram (shown in Figure 10.41(b)), we need to show that the opposite sides are parallel. We can quickly show that $\overrightarrow{AB} = \overrightarrow{DC} = \langle 1,2,1\rangle$ and $\overrightarrow{BC} = \overrightarrow{AD} = \langle 2,2,1\rangle$. We find the area by computing the magnitude of the cross product of $\overrightarrow{AB}$ and $\overrightarrow{BC}$:
$$\overrightarrow{AB}\times\overrightarrow{BC} = \langle 0,1,-2\rangle \quad\Rightarrow\quad ||\,\overrightarrow{AB}\times\overrightarrow{BC}\,|| = \sqrt{5}\approx 2.236.$$

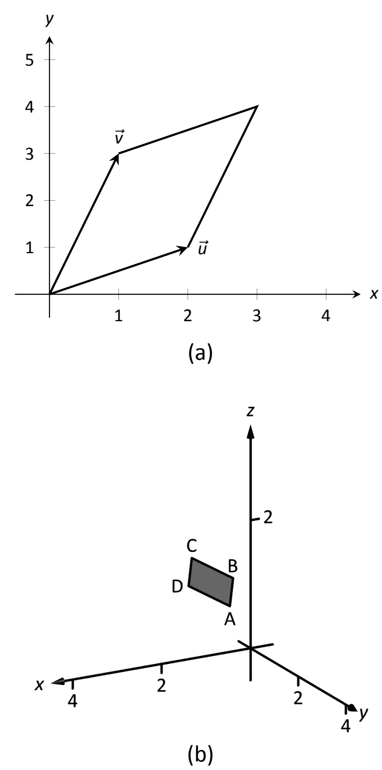


Figure 10.41: Sketching the parallelograms in Example 10.28.

Notes:

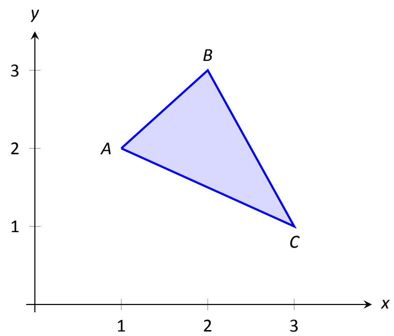


Figure 10.42: Finding the area of a triangle in Example 10.29.

Note: The word "parallelepiped" is pronounced "parallel–eh–pipe–ed."

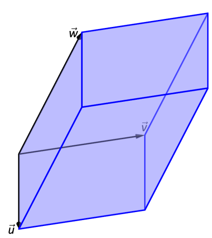


Figure 10.43: A parallelepiped is the three dimensional analogue to the parallelogram.

This application is perhaps more useful in finding the area of a triangle (in short, triangles are used more often than parallelograms). We illustrate this in the following example.

Example 10.29 Area of a triangle

Find the area of the triangle with vertices $A = (1, 2)$, $B = (2, 3)$ and $C = (3, 1)$, as pictured in Figure 10.42.

Solution We found the area of this triangle in Example 7.4 to be 1.5 using integration. There we discussed the fact that finding the area of a triangle can be inconvenient using the "$\frac{1}{2}bh$" formula as one has to compute the height, which generally involves finding angles, etc. Using a cross product is much more direct.

We can choose any two sides of the triangle to use to form vectors; we choose $\overrightarrow{AB} = \langle 1, 1\rangle$ and $\overrightarrow{AC} = \langle 2, -1\rangle$. As in the previous example, we will rewrite these vectors with a third component of 0 so that we can apply the cross product. The area of the triangle is

$$\frac{1}{2}||\,\overrightarrow{AB}\times\overrightarrow{AC}\,|| = \frac{1}{2}||\ \langle 1,1,0\rangle \times \langle 2,-1,0\rangle\ || = \frac{1}{2}||\ \langle 0,0,-3\rangle\ || = \frac{3}{2}.$$

We arrive at the same answer as before with less work.

Volume of a Parallelepiped

The three dimensional analogue to the parallelogram is the **parallelepiped**. Each face is parallel to the face opposite face, as illustrated in Figure 10.43. By crossing $\vec{v}$ and $\vec{w}$, one gets a vector whose magnitude is the area of the base. Dotting this vector with $\vec{u}$ computes the volume of parallelepiped! (Up to a sign; take the absolute value.)

Thus the volume of a parallelepiped defined by vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ is

$$V = |\vec{u}\cdot(\vec{v}\times\vec{w})|. \tag{10.5}$$

Note how this is the Triple Scalar Product, first seen in Theorem 89. Applying the identities given in the theorem shows that we can apply the Triple Scalar Product in any "order" we choose to find the volume. That is,

$$V = |\vec{u}\cdot(\vec{v}\times\vec{w})| = |\vec{u}\cdot(\vec{w}\times\vec{v})| = |(\vec{u}\times\vec{v})\cdot\vec{w}|,\quad \text{etc.}$$

Example 10.30 Finding the volume of parallelepiped

Find the volume of the parallepiped defined by the vectors $\vec{u} = \langle 1, 1, 0\rangle$, $\vec{v} = \langle -1, 1, 0\rangle$ and $\vec{w} = \langle 0, 1, 1\rangle$.

Notes:

Solution We apply Equation (10.5). We first find $\vec{v} \times \vec{w} = \langle 1, 1, -1 \rangle$. Then

$$|\vec{u} \cdot (\vec{v} \times \vec{w})| = |\, \langle 1, 1, 0 \rangle \cdot \langle 1, 1, -1 \rangle \,| = 2.$$

So the volume of the parallelepiped is 2 cubic units.

While this application of the Triple Scalar Product is interesting, it is not used all that often: parallelepipeds are not a common shape in physics and engineering. The last application of the cross product is very applicable in engineering.

Torque

Torque is a measure of the turning force applied to an object. A classic scenario involving torque is the application of a wrench to a bolt. When a force is applied to the wrench, the bolt turns. When we represent the force and wrench with vectors $\vec{F}$ and $\vec{\ell}$, we see that the bolt moves (because of the threads) in a direction orthogonal to $\vec{F}$ and $\vec{\ell}$. Torque is usually represented by the Greek letter τ, or tau, and has units of N·m, a Newton–meter, or ft·lb, a foot–pound.

While a full understanding of torque is beyond the purposes of this book, when a force $\vec{F}$ is applied to a lever arm $\vec{\ell}$, the resulting torque is

$$\vec{\tau} = \vec{\ell} \times \vec{F}. \tag{10.6}$$

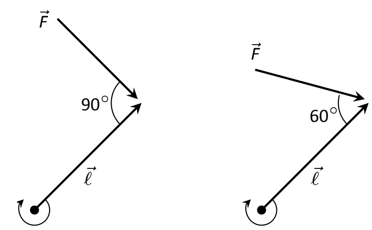


Figure 10.45: Showing a force being applied to a lever in Example 10.31.

Example 10.31 Computing torque

A lever of length 2ft makes an angle with the horizontal of 45°. Find the resulting torque when a force of 10lb is applied to the end of the level where:

1. the force is perpendicular to the lever, and
2. the force makes an angle of 60° with the lever, as shown in Figure 10.45.

Solution

1. We start by determining vectors for the force and lever arm. Since the lever arm makes a 45° angle with the horizontal and is 2ft long, we can state that $\vec{\ell} = 2\,\langle \cos 45^\circ, \sin 45^\circ \rangle = \left\langle \sqrt{2}, \sqrt{2} \right\rangle$.

   Since the force vector is perpendicular to the lever arm (as seen in the left hand side of Figure 10.45), we can conclude it is making an angle of -45° with the horizontal. As it has a magnitude of 10lb, we can state $\vec{F} = 10\,\langle \cos(-45^\circ), \sin(-45^\circ) \rangle = \left\langle 5\sqrt{2}, -5\sqrt{2} \right\rangle$.

   Using Equation (10.6) to find the torque requires a cross product. We again let the third component of each vector be 0 and compute the cross

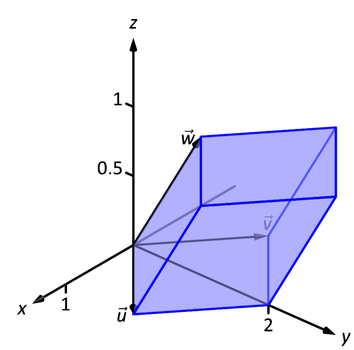


Figure 10.44: A parallelepiped in Example 10.30.

Notes:

product:

$$\begin{aligned}\vec{\tau} &= \vec{\ell} \times \vec{F}\\ &= \left\langle \sqrt{2}, \sqrt{2}, 0\right\rangle \times \left\langle 5\sqrt{2}, -5\sqrt{2}, 0\right\rangle\\ &= \langle 0, 0, -20\rangle\end{aligned}$$

This clearly has a magnitude of 20 ft-lb.

We can view the force and lever arm vectors as lying "on the page"; our computation of $\vec{\tau}$ shows that the torque goes "into the page." This follows the Right Hand Rule of the cross product, and it also matches well with the example of the wrench turning the bolt. Turning a bolt clockwise moves it in.

2. Our lever arm can still be represented by $\vec{\ell} = \left\langle \sqrt{2}, \sqrt{2}\right\rangle$. As our force vector makes a 60° angle with $\vec{\ell}$, we can see (referencing the right hand side of the figure) that $\vec{F}$ makes a $-15°$ angle with the horizontal. Thus

$$\begin{aligned}\vec{F} = 10\left\langle \cos -15°, \sin -15°\right\rangle &= \left\langle \frac{5\left(\sqrt{6}+\sqrt{2}\right)}{2}, -\frac{5\left(\sqrt{6}-\sqrt{2}\right)}{2}, 0\right\rangle\\ &\approx \langle 9.659, -2.588\rangle .\end{aligned}$$

We again make the third component 0 and take the cross product to find the torque:

$$\begin{aligned}\vec{\tau} &= \vec{\ell} \times \vec{F}\\ &= \left\langle \sqrt{2}, \sqrt{2}, 0\right\rangle \times \left\langle \frac{5\left(\sqrt{6}+\sqrt{2}\right)}{2}, -\frac{5\left(\sqrt{6}-\sqrt{2}\right)}{2}, 0\right\rangle\\ &= \left\langle 0, 0, -10\sqrt{3}\right\rangle\\ &\approx \langle 0, 0, -17.321\rangle .\end{aligned}$$

As one might expect, when the force and lever arm vectors *are* orthogonal, the magnitude of force is greater than when the vectors *are not* orthogonal.

While the cross product has a variety of applications (as noted in this chapter), its fundamental use is finding a vector perpendicular to two others. Knowing a vector is orthogonal to two others is of incredible importance, as it allows us to find the equations of lines and planes in a variety of contexts. The importance of the cross product, in some sense, relies on the importance of lines and planes, which see widespread use throughout engineering, physics and mathematics. We study lines and planes in the next two sections.

Notes:

Exercises 10.4

Terms and Concepts

1. The cross product of two vectors is a ________, not a scalar.

2. One can visualize the direction of $\vec{u} \times \vec{v}$ using the ________ ________ ________.

3. Give a synonym for "orthogonal."

4. T/F: A fundamental principle of the cross product is that $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$.

5. ________ is a measure of the turning force applied to an object.

Problems

In Exercises 6 – 14, vectors $\vec{u}$ and $\vec{v}$ are given. Compute $\vec{u} \times \vec{v}$ and show this is orthogonal to both $\vec{u}$ and $\vec{v}$.

6. $\vec{u} = \langle 3, 2, -2 \rangle, \quad \vec{v} = \langle 0, 1, 5 \rangle$

7. $\vec{u} = \langle 5, -4, 3 \rangle, \quad \vec{v} = \langle 2, -5, 1 \rangle$

8. $\vec{u} = \langle 4, -5, -5 \rangle, \quad \vec{v} = \langle 3, 3, 4 \rangle$

9. $\vec{u} = \langle -4, 7, -10 \rangle, \quad \vec{v} = \langle 4, 4, 1 \rangle$

10. $\vec{u} = \langle 1, 0, 1 \rangle, \quad \vec{v} = \langle 5, 0, 7 \rangle$

11. $\vec{u} = \langle 1, 5, -4 \rangle, \quad \vec{v} = \langle -2, -10, 8 \rangle$

12. $\vec{u} = \vec{i}, \quad \vec{v} = \vec{j}$

13. $\vec{u} = \vec{i}, \quad \vec{v} = \vec{k}$

14. $\vec{u} = \vec{j}, \quad \vec{v} = \vec{k}$

15. Pick any vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ and show that $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.

16. Pick any vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ and show that $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$.

In Exercises 17 – 20, the magnitudes of vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ are given, along with the angle θ between them. Use this information to find the magnitude of $\vec{u} \times \vec{v}$.

17. $|| \vec{u} || = 2, \quad || \vec{v} || = 5, \quad \theta = 30^\circ$

18. $|| \vec{u} || = 3, \quad || \vec{v} || = 7, \quad \theta = \pi/2$

19. $|| \vec{u} || = 3, \quad || \vec{v} || = 4, \quad \theta = \pi$

20. $|| \vec{u} || = 2, \quad || \vec{v} || = 5, \quad \theta = 5\pi/6$

In Exercises 21 – 24, find the area of the parallelogram defined by the given vectors.

21. $\vec{u} = \langle 1, 1, 2 \rangle, \quad \vec{v} = \langle 2, 0, 3 \rangle$

22. $\vec{u} = \langle -2, 1, 5 \rangle, \quad \vec{v} = \langle -1, 3, 1 \rangle$

23. $\vec{u} = \langle 1, 2 \rangle, \quad \vec{v} = \langle 2, 1 \rangle$

24. $\vec{u} = \langle 2, 0 \rangle, \quad \vec{v} = \langle 0, 3 \rangle$

In Exercises 25 – 28, find the area of the triangle with the given vertices.

25. Vertices: $(0, 0, 0)$, $(1, 3, -1)$ and $(2, 1, 1)$.

26. Vertices: $(5, 2, -1)$, $(3, 6, 2)$ and $(1, 0, 4)$.

27. Vertices: $(1, 1)$, $(1, 3)$ and $(2, 2)$.

28. Vertices: $(3, 1)$, $(1, 2)$ and $(4, 3)$.

In Exercises 29 – 30, find the area of the quadrilateral with the given vertices. (Hint: break the quadrilateral into 2 triangles.)

29. Vertices: $(0, 0)$, $(1, 2)$, $(3, 0)$ and $(4, 3)$.

30. Vertices: $(0, 0, 0)$, $(2, 1, 1)$, $(-1, 2, -8)$ and $(1, -1, 5)$.

In Exercises 31 – 32, find the volume of the parallelepiped defined by the given vectors.

31. $\vec{u} = \langle 1, 1, 1 \rangle, \quad \vec{v} = \langle 1, 2, 3 \rangle, \quad \vec{w} = \langle 1, 0, 1 \rangle$

32. $\vec{u} = \langle -1, 2, 1 \rangle, \quad \vec{v} = \langle 2, 2, 1 \rangle, \quad \vec{w} = \langle 3, 1, 3 \rangle$

In Exercises 33 – 36, find a unit vector orthogonal to both $\vec{u}$ and $\vec{v}$.

33. $\vec{u} = \langle 1, 1, 1 \rangle, \quad \vec{v} = \langle 2, 0, 1 \rangle$

34. $\vec{u} = \langle 1, -2, 1 \rangle, \quad \vec{v} = \langle 3, 2, 1 \rangle$

35. $\vec{u} = \langle 5, 0, 2 \rangle, \quad \vec{v} = \langle -3, 0, 7 \rangle$

36. $\vec{u} = \langle 1, -2, 1 \rangle, \quad \vec{v} = \langle -2, 4, -2 \rangle$

37. A bicycle rider applies 150lb of force, straight down, onto a pedal that extends 7in horizontally from the crankshaft. Find the magnitude of the torque applied to the crankshaft.

38. A bicycle rider applies 150lb of force, straight down, onto a pedal that extends 7in from the crankshaft, making a 30° angle with the horizontal. Find the magnitude of the torque applied to the crankshaft.

39. To turn a stubborn bolt, 80lb of force is applied to a 10in wrench. What is the maximum amount of torque that can be applied to the bolt?

40. To turn a stubborn bolt, 80lb of force is applied to a 10in wrench in a confined space, where the direction of applied force makes a 10° angle with the wrench. How much torque is subsequently applied to the wrench?

41. Show, using the definition of the Cross Product, that $\vec{u}\cdot(\vec{u}\times\vec{v}) = 0$; that is, that $\vec{u}$ is orthogonal to the cross product of $\vec{u}$ and $\vec{v}$.

42. Show, using the definition of the Cross Product, that $\vec{u}\times\vec{u} = \vec{0}$.

43. Let $\vec{u}$ and $\vec{v}$ be vectors, and θ be the angle between them. If $\vec{u}\times\vec{v} = \langle 8, 4, 1\rangle$ and $\tan\theta = 1/6$, what is $\vec{u}\cdot\vec{v}$?

10.5 Lines

To find the equation of a line in the *x*-*y* plane, we need two pieces of information: a point and the slope. The slope conveys *direction* information. As vertical lines have an undefined slope, the following statement is more accurate:

> To define a line, one needs a point on the line and the direction of the line.

This holds true for lines in space.

Let P be a point in space, let $\vec{p}$ be the vector with initial point at the origin and terminal point at P (i.e., $\vec{p}$ "points" to P), and let $\vec{d}$ be a vector. Consider the points on the line through P in the direction of $\vec{d}$.

Clearly one point on the line is P; we can say that the *vector* $\vec{p}$ lies at this point on the line. To find another point on the line, we can start at $\vec{p}$ and move in a direction parallel to $\vec{d}$. For instance, starting at $\vec{p}$ and traveling one length of $\vec{d}$ places one at another point on the line. Consider Figure 10.47 where certain points along the line are indicated.

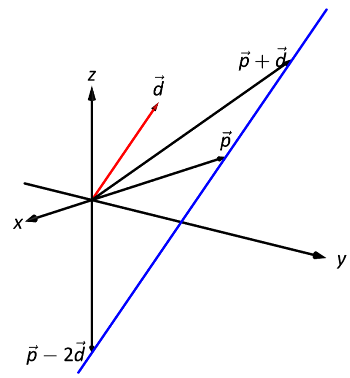


Figure 10.47: Defining a line in space.

The figure illustrates how every point on the line can be obtained by starting with $\vec{p}$ and moving a certain distance in the direction of $\vec{d}$. That is, we can define the line as a function of t:

$$\vec{\ell}(t) = \vec{p} + t\,\vec{d}. \tag{10.7}$$

In many ways, this is *not* a new concept. Compare Equation (10.7) to the familiar "$y = mx + b$" equation of a line:

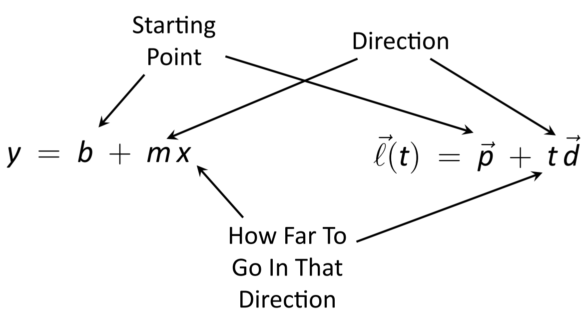


Figure 10.46: Understanding the vector equation of a line.

The equations exhibit the same structure: they give a starting point, define a direction, and state how far in that direction to travel.

Equation (10.7) is an example of a **vector–valued function**; the input of the function is a real number and the output is a vector. We will cover vector–valued functions extensively in the next chapter.

Notes:

There are other ways to represent a line. Let $\vec{p} = \langle x_0, y_0, z_0 \rangle$ and let $\vec{d} = \langle a, b, c \rangle$. Then the equation of the line through $\vec{p}$ in the direction of $\vec{d}$ is:

$$\begin{aligned}\vec{\ell}(t) &= \vec{p} + t\vec{d} \\ &= \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle \\ &= \langle x_0 + at, y_0 + bt, z_0 + ct \rangle .\end{aligned}$$

The last line states that the x values of the line are given by $x = x_0 + at$, the y values are given by $y = y_0 + bt$, and the z values are given by $z = z_0 + ct$. These three equations, taken together, are the **parametric equations of the line** through $\vec{p}$ in the direction of $\vec{d}$.

Finally, each of the equations for x, y and z above contain the variable t. We can solve for t in each equation:

$$\begin{aligned} x = x_0 + at \quad &\Rightarrow \quad t = \frac{x - x_0}{a}, \\ y = y_0 + bt \quad &\Rightarrow \quad t = \frac{y - y_0}{b}, \\ z = z_0 + ct \quad &\Rightarrow \quad t = \frac{z - z_0}{c}, \end{aligned}$$

assuming $a, b, c \neq 0$. Since t is equal to each expression on the right, we can set these equal to each other, forming the **symmetric equations of the line** through $\vec{p}$ in the direction of $\vec{d}$:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Each representation has its own advantages, depending on the context. We summarize these three forms in the following definition, then give examples of their use.

Notes:

Definition 66 Equations of Lines in Space

Consider the line in space that passes through $\vec{p} = \langle x_0, y_0, z_0\rangle$ in the direction of $\vec{d} = \langle a, b, c\rangle$.

1. The **vector equation** of the line is
$$\vec{\ell}(t) = \vec{p} + t\vec{d}.$$
2. The **parametric equations** of the line are
$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$
3. The **symmetric equations** of the line are
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$
when $a, b, c \neq 0$.

Example 10.32 Finding the equation of a line

Give all three equations, as given in Definition 66, of the line through $P = (2, 3, 1)$ in the direction of $\vec{d} = \langle -1, 1, 2\rangle$. Does the point $Q = (-1, 6, 6)$ lie on this line?

Solution We identify the point $P = (2, 3, 1)$ with the vector $\vec{p} = \langle 2, 3, 1\rangle$. Following the definition, we have

- the vector equation of the line is $\vec{\ell}(t) = \langle 2, 3, 1\rangle + t\langle -1, 1, 2\rangle$;
- the parametric equations of the line are
$$x = 2 - t, \quad y = 3 + t, \quad z = 1 + 2t; \text{ and}$$
- the symmetric equations of the line are
$$\frac{x - 2}{-1} = \frac{y - 3}{1} = \frac{z - 1}{2}.$$

The first two equations of the line are useful when a t value is given: one can immediately find the corresponding point on the line. These forms are good when calculating with a computer; most software programs easily handle equations in these formats. (For instance, to make Figure 10.48, a certain graphics program was given the input `(2-x,3+x,1+2*x)`. This particular program requires the variable always be "x" instead of "t").

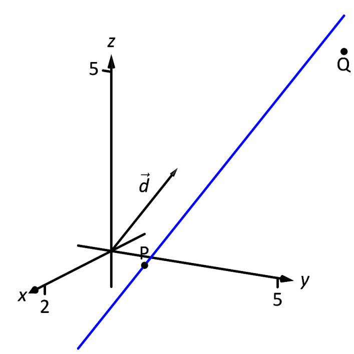


Figure 10.48: Graphing a line in Example 10.32.

Notes:

Does the point $Q = (-1, 6, 6)$ lie on the line? The graph in Figure 10.48 makes it clear that it does not. We can answer this question without the graph using any of the three equation forms. Of the three, the symmetric equations are probably best suited for this task. Simply plug in the values of x, y and z and see if equality is maintained:

$$\frac{-1-2}{-1} \stackrel{?}{=} \frac{6-3}{1} \stackrel{?}{=} \frac{6-1}{2} \quad \Rightarrow \quad 3 = 3 \neq 2.5.$$

We see that Q does not lie on the line as it did not satisfy the symmetric equations.

Note that the symmetric equations given in Definition 66 only allow for the case where a, b, c are all nonzero. This is because of the denominators. However, we can modify this a bit when one or more of a, b, c are zero. For example, consider the line whose parametric equations are

$$x = 5 + 4t, \quad y = 3, \quad z = -2 + t.$$

This line lies in the plane $y = 3$. So we can define this line using the equations

$$y = 3\text{and } \frac{x-5}{4} = \frac{z+2}{1}$$

together. The second equation comes from the usual symmetric equations formula for x and z. This pair is called the symmetric equations for this particular line.

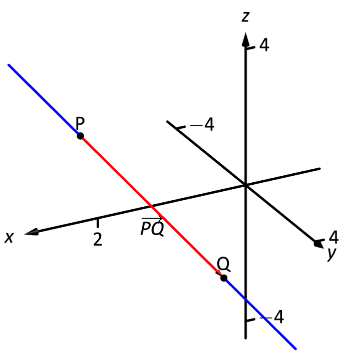


Figure 10.49: A graph of the line in Example 10.33.

Example 10.33 Finding the equation of a line through two points

Find the parametric equations of the line through the points $P = (2, -1, 2)$ and $Q = (1, 3, -1)$.

Solution Recall the statement made at the beginning of this section: to find the equation of a line, we need a point and a direction. We have *two* points; either one will suffice. The direction of the line can be found by the vector with initial point P and terminal point Q: $\overrightarrow{PQ} = \langle -1, 4, -3 \rangle$.

The parametric equations of the line ℓ through P in the direction of $\overrightarrow{PQ}$ are:

$$\ell : \quad x = 2 - t \quad y = -1 + 4t \quad z = 2 - 3t.$$

A graph of the points and line are given in Figure 10.49. Note how in the given parameterization of the line, $t = 0$ corresponds to the point P, and $t = 1$ corresponds to the point Q. This relates to the understanding of the vector equation of a line described in Figure 10.46. The parametric equations "start" at the

Notes:

point P, and t determines how far in the direction of $\overrightarrow{PQ}$ to travel. When $t = 0$, we travel 0 lengths of $\overrightarrow{PQ}$; when $t = 1$, we travel one length of $\overrightarrow{PQ}$, resulting in the point Q.

Parallel, Intersecting and Skew Lines

In the plane, two *distinct* lines can either be parallel or they will intersect at exactly one point. In space, given equations of two lines, it can sometimes be difficult to tell whether the lines are distinct or not (i.e., the same line can be represented in different ways). Given lines $\vec{\ell}_1(t) = \vec{p}_1 + t\vec{d}_1$ and $\vec{\ell}_2(t) = \vec{p}_2 + t\vec{d}_2$, we have four possibilities: $\vec{\ell}_1$ and $\vec{\ell}_2$ are

the same line	they share all points;
intersecting lines	share only 1 point;
parallel lines	$\vec{d}_1 \parallel \vec{d}_2$, no points in common; or
skew lines	$\vec{d}_1 \nparallel \vec{d}_2$, no points in common.

The next two examples investigate these possibilities.

Example 10.34 Comparing lines

Consider lines ℓ_1 and ℓ_2, given in parametric equation form:

$$\ell_1: \begin{array}{rcl} x &=& 1+3t \\ y &=& 2-t \\ z &=& t \end{array} \qquad \ell_2: \begin{array}{rcl} x &=& -2+4s \\ y &=& 3+s \\ z &=& 5+2s. \end{array}$$

Determine whether ℓ_1 and ℓ_2 are the same line, intersect, are parallel, or skew.

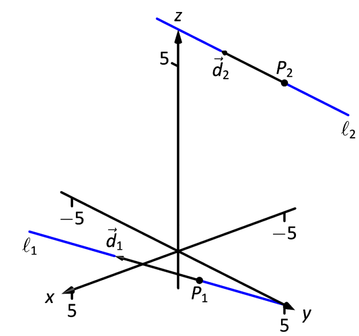


Figure 10.50: Sketching the lines from Example 10.34.

Solution We start by looking at the directions of each line. Line ℓ_1 has the direction given by $\vec{d}_1 = \langle 3, -1, 1\rangle$ and line ℓ_2 has the direction given by $\vec{d}_2 = \langle 4, 1, 2\rangle$. It should be clear that $\vec{d}_1$ and $\vec{d}_2$ are not parallel, hence ℓ_1 and ℓ_2 are not the same line, nor are they parallel. Figure 10.50 verifies this fact (where the points and directions indicated by the equations of each line are identified).

We next check to see if they intersect (if they do not, they are skew lines). To find if they intersect, we look for t and s values such that the respective x, y and z values are the same. That is, we want s and t such that:

$$\begin{array}{rcl} 1+3t &=& -2+4s \\ 2-t &=& 3+s \\ t &=& 5+2s. \end{array}$$

This is a relatively simple system of linear equations. Since the last equation is already solved for t, substitute that value of t into the equation above it:

$$2-(5+2s) = 3+s \quad \Rightarrow \quad s=-2,\ t=1.$$

Notes:

A key to remember is that we have *three* equations; we need to check if $s = -2,\ t = 1$ satisfies the first equation as well:

$$1 + 3(1) \neq -2 + 4(-2).$$

It does not. Therefore, we conclude that the lines ℓ_1 and ℓ_2 are skew.

Example 10.35 Comparing lines

Consider lines ℓ_1 and ℓ_2, given in parametric equation form:

$$\ell_1 : \begin{array}{rcl} x &=& -0.7 + 1.6t \\ y &=& 4.2 + 2.72t \\ z &=& 2.3 - 3.36t \end{array} \qquad \ell_2 : \begin{array}{rcl} x &=& 2.8 - 2.9s \\ y &=& 10.15 - 4.93s \\ z &=& -5.05 + 6.09s. \end{array}$$

Determine whether ℓ_1 and ℓ_2 are the same line, intersect, are parallel, or skew.

Solution It is obviously very difficult to simply look at these equations and discern anything. This is done intentionally. In the "real world," most equations that are used do not have nice, integer coefficients. Rather, there are lots of digits after the decimal and the equations can look "messy."

We again start by deciding whether or not each line has the same direction. The direction of ℓ_1 is given by $\vec{d}_1 = \langle 1.6, 2.72, -3.36\rangle$ and the direction of ℓ_2 is given by $\vec{d}_2 = \langle -2.9, -4.93, 6.09\rangle$. When it is not clear through observation whether two vectors are parallel or not, the standard way of determining this is by comparing their respective unit vectors. Using a calculator, we find:

$$\vec{u}_1 = \frac{\vec{d}_1}{||\,\vec{d}_1\,||} = \langle 0.3471, 0.5901, -0.7289\rangle$$
$$\vec{u}_2 = \frac{\vec{d}_2}{||\,\vec{d}_2\,||} = \langle -0.3471, -0.5901, 0.7289\rangle\,.$$

The two vectors seem to be parallel (at least, their components are equal to 4 decimal places). In most situations, it would suffice to conclude that the lines are at least parallel, if not the same. One way to be sure is to rewrite $\vec{d}_1$ and $\vec{d}_2$ in terms of fractions, not decimals. We have

$$\vec{d}_1 = \left\langle \frac{16}{10}, \frac{272}{100}, -\frac{336}{100}\right\rangle \qquad \vec{d}_2 = \left\langle -\frac{29}{10}, -\frac{493}{100}, \frac{609}{100}\right\rangle.$$

One can then find the magnitudes of each vector in terms of fractions, then compute the unit vectors likewise. After a lot of manual arithmetic (or after briefly using a computer algebra system), one finds that

$$\vec{u}_1 = \left\langle \sqrt{\frac{10}{83}}, \frac{17}{\sqrt{830}}, -\frac{21}{\sqrt{830}}\right\rangle \qquad \vec{u}_2 = \left\langle -\sqrt{\frac{10}{83}}, -\frac{17}{\sqrt{830}}, \frac{21}{\sqrt{830}}\right\rangle.$$

Notes:

We can now say without equivocation that these lines are parallel.

Are they the same line? The parametric equations for a line describe one point that lies on the line, so we know that the point $P_1 = (-0.7, 4.2, 2.3)$ lies on ℓ_1. To determine if this point also lies on ℓ_2, plug in the x, y and z values of P_1 into the symmetric equations for ℓ_2:

$$\frac{(-0.7) - 2.8}{-2.9} \stackrel{?}{=} \frac{(4.2) - 10.15}{-4.93} \stackrel{?}{=} \frac{(2.3) - (-5.05)}{6.09} \quad \Rightarrow \quad 1.2069 = 1.2069 = 1.2069.$$

The point P_1 lies on both lines, so we conclude they are the same line, just parameterized differently. Figure 10.51 graphs this line along with the points and vectors described by the parametric equations. Note how $\vec{d}_1$ and $\vec{d}_2$ are parallel, though point in opposite directions (as indicated by their unit vectors above).

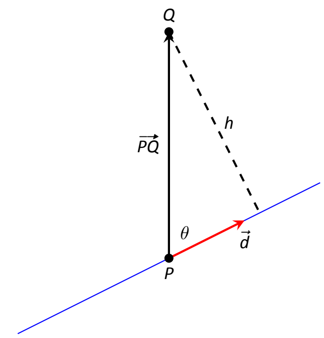


Figure 10.52: Establishing the distance from a point to a line.

Distances

Given a point Q and a line $\vec{\ell}(t) = \vec{p} + t\vec{d}$ in space, it is often useful to know the distance from the point to the line. (Here we use the standard definition of "distance," i.e., the length of the shortest line segment from the point to the line.) Identifying $\vec{p}$ with the point P, Figure 10.52 will help establish a general method of computing this distance h.

From trigonometry, we know $h = \parallel \vec{PQ} \parallel \sin\theta$. We have a similar identity involving the cross product: $\parallel \vec{PQ} \times \vec{d} \parallel = \parallel \vec{PQ} \parallel \parallel \vec{d} \parallel \sin\theta$. Divide both sides of this latter equation by $\parallel \vec{d} \parallel$ to obtain h:

$$h = \frac{\parallel \vec{PQ} \times \vec{d} \parallel}{\parallel \vec{d} \parallel}. \tag{10.8}$$

It is also useful to determine the distance between lines, which we define as the length of the shortest line segment that connects the two lines (an argument from geometry shows that this line segments is perpendicular to both lines). Let lines $\vec{\ell_1}(t) = \vec{p}_1 + t\vec{d}_1$ and $\vec{\ell_2}(t) = \vec{p}_2 + t\vec{d}_2$ be given, as shown in Figure 10.53. To find the direction orthogonal to both $\vec{d}_1$ and $\vec{d}_2$, we take the cross product: $\vec{c} = \vec{d}_1 \times \vec{d}_2$. The magnitude of the orthogonal projection of $\overrightarrow{P_1P_2}$ onto $\vec{c}$ is the

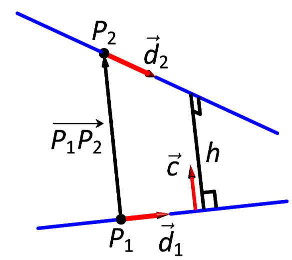


Figure 10.53: Establishing the distance between lines.

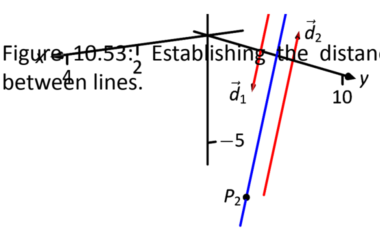


Figure 10.51: Graphing the lines in Example 10.35.

Notes:

distance h we seek:

$$\begin{aligned} h &= \left|\left|\ \text{proj}_{\vec{c}}\, \overrightarrow{P_1P_2}\ \right|\right| \\ &= \left|\left| \frac{\overrightarrow{P_1P_2}\cdot\vec{c}}{\vec{c}\cdot\vec{c}}\vec{c} \right|\right| \\ &= \frac{|\overrightarrow{P_1P_2}\cdot\vec{c}|}{||\,\vec{c}\,||^2}||\,\vec{c}\,|| \\ &= \frac{|\overrightarrow{P_1P_2}\cdot\vec{c}|}{||\,\vec{c}\,||}. \end{aligned}$$

A problem in the Exercise section is to show that this distance is 0 when the lines intersect. Note the use of the Triple Scalar Product: $\overrightarrow{P_1P_2}\cdot c = \overrightarrow{P_1P_2}\cdot(\vec{d_1}\times\vec{d_2})$.

The following Key Idea restates these two distance formulas.

Key Idea 51 Distances to Lines

1. Let P be a point on a line ℓ that is parallel to $\vec{d}$. The distance h from a point Q to the line ℓ is:

$$h = \frac{||\,\overrightarrow{PQ}\times\vec{d}\,||}{||\,\vec{d}\,||}.$$

2. Let P_1 be a point on line ℓ_1 that is parallel to $\vec{d_1}$, and let P_2 be a point on line ℓ_2 parallel to $\vec{d_2}$. Also, let $\vec{c}=\vec{d_1}\times\vec{d_2}$, where lines ℓ_1 and ℓ_2 are not parallel. The distance h between the two lines is:

$$h = \frac{|\overrightarrow{P_1P_2}\cdot\vec{c}|}{||\,\vec{c}\,||}.$$

Example 10.36 Finding the distance from a point to a line

Find the distance from the point $Q=(1,1,3)$ to the line $\vec{\ell}(t)=\langle 1,-1,1\rangle + t\,\langle 2,3,1\rangle$.

SOLUTION The equation of the line gives us the point $P=(1,-1,1)$ that lies on the line, hence $\overrightarrow{PQ}=\langle 0,2,2\rangle$. The equation also gives $\vec{d}=\langle 2,3,1\rangle$.

Notes:

Following Key Idea 51, we have the distance as

$$\begin{aligned} h &= \frac{\|\,\overrightarrow{PQ} \times \vec{d}\,\|}{\|\,\vec{d}\,\|} \\ &= \frac{\|\ \langle -4, 4, -4\rangle\ \|}{\sqrt{14}} \\ &= \frac{4\sqrt{3}}{\sqrt{14}} \approx 1.852. \end{aligned}$$

The point Q is approximately 1.852 units from the line $\vec{\ell}(t)$.

Example 10.37 Finding the distance between lines

Find the distance between the lines

$$\ell_1: \begin{array}{rcl} x &=& 1+3t \\ y &=& 2-t \\ z &=& t \end{array} \qquad \ell_2: \begin{array}{rcl} x &=& -2+4s \\ y &=& 3+s \\ z &=& 5+2s. \end{array}$$

Solution These are the same lines as given in Example 10.34, where we showed them to be skew. The equations allow us to identify the following points and vectors:

$$P_1 = (1,2,0) \quad P_2 = (-2,3,5) \quad \Rightarrow \quad \overrightarrow{P_1P_2} = \langle -3, 1, 5\rangle\,.$$

$$\vec{d}_1 = \langle 3, -1, 1\rangle \quad \vec{d}_2 = \langle 4, 1, 2\rangle \quad \Rightarrow \quad \vec{c} = \vec{d}_1 \times \vec{d}_2 = \langle -3, -2, 7\rangle\,.$$

From Key Idea 51 we have the distance h between the two lines is

$$\begin{aligned} h &= \frac{|\overrightarrow{P_1P_2} \cdot \vec{c}|}{\|\,\vec{c}\,\|} \\ &= \frac{42}{\sqrt{62}} \approx 5.334. \end{aligned}$$

The lines are approximately 5.334 units apart.

One of the key points to understand from this section is this: to describe a line, we need a point and a direction. Whenever a problem is posed concerning a line, one needs to take whatever information is offered and glean point and direction information. Many questions can be asked (and *are* asked in the Exercise section) whose answer immediately follows from this understanding.

Lines are one of two fundamental objects of study in space. The other fundamental object is the *plane*, which we study in detail in the next section. Many complex three dimensional objects are studied by approximating their surfaces with lines and planes.

Notes:

Exercises 10.5

Terms and Concepts

1. To find an equation of a line, what two pieces of information are needed?

2. Two distinct lines in the plane can intersect or be ________.

3. Two distinct lines in space can intersect, be ________ or be ________.

4. Use your own words to describe what it means for two lines in space to be skew.

Problems

In Exercises 5 – 16, write the vector, parametric and symmetric equations of the lines described.

5. Passes through $P = (2, -4, 1)$, parallel to $\vec{d} = \langle 9, 2, 5\rangle$.

6. Passes through $P = (6, 1, 7)$, parallel to $\vec{d} = \langle -3, 2, 5\rangle$.

7. Passes through $P = (-2, 5, 4)$, parallel to $\vec{d} = \langle 0, 1, 3\rangle$.

8. Passes through $P = (1, 1)$, parallel to $\vec{d} = \langle 2, 3\rangle$.

9. Passes through $P = (2, 1, 5)$ and $Q = (7, -2, 4)$.

10. Passes through $P = (1, -2, 3)$ and $Q = (5, 5, 5)$.

11. Passes through $P = (1, 5, 5)$ and $Q = (2, 2, 5)$.

12. Passes through $P = (0, 1, 2)$ and orthogonal to both $\vec{d_1} = \langle 2, -1, 7\rangle$ and $\vec{d_2} = \langle 7, 1, 3\rangle$.

13. Passes through $P = (5, 1, 9)$ and orthogonal to both $\vec{d_1} = \langle 1, 0, 1\rangle$ and $\vec{d_2} = \langle 2, 0, 3\rangle$.

14. Passes through the point of intersection of $\vec{\ell_1}(t)$ and $\vec{\ell_2}(t)$ and orthogonal to both lines, where
$\vec{\ell_1}(t) = \langle 2, 1, 1\rangle + t\langle 5, 1, -2\rangle$ and
$\vec{\ell_2}(t) = \langle -2, -1, 2\rangle + t\langle 3, 1, -1\rangle$.

15. Passes through the point of intersection of $\ell_1(t)$ and $\ell_2(t)$ and orthogonal to both lines, where
$$\ell_1 = \begin{cases} x = t \\ y = -2 + 2t \\ z = 1 + t \end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} x = 2 + t \\ y = 2 - t \\ z = 3 + 2t \end{cases}.$$

In Exercises 16 – 23, determine if the described lines are the same line, parallel lines, intersecting or skew lines. If intersecting, give the point of intersection.

16. $\vec{\ell_1}(t) = \langle 1, 2, 1\rangle + t\langle 2, -1, 1\rangle$,
$\vec{\ell_2}(t) = \langle 3, 3, 3\rangle + t\langle -4, 2, -2\rangle$.

17. $\vec{\ell_1}(t) = \langle 2, 1, 1\rangle + t\langle 5, 1, 3\rangle$,
$\vec{\ell_2}(t) = \langle 14, 5, 9\rangle + t\langle 1, 1, 1\rangle$.

18. $\vec{\ell_1}(t) = \langle 3, 4, 1\rangle + t\langle 2, -3, 4\rangle$,
$\vec{\ell_2}(t) = \langle -3, 3, -3\rangle + t\langle 3, -2, 4\rangle$.

19. $\vec{\ell_1}(t) = \langle 1, 1, 1\rangle + t\langle 3, 1, 3\rangle$,
$\vec{\ell_2}(t) = \langle 7, 3, 7\rangle + t\langle 6, 2, 6\rangle$.

20. $$\ell_1 = \begin{cases} x = 1 + 2t \\ y = 3 - 2t \\ z = t \end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} x = 3 - t \\ y = 3 + 5t \\ z = 2 + 7t \end{cases}$$

21. $$\ell_1 = \begin{cases} x = 1.1 + 0.6t \\ y = 3.77 + 0.9t \\ z = -2.3 + 1.5t \end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} x = 3.11 + 3.4t \\ y = 2 + 5.1t \\ z = 2.5 + 8.5t \end{cases}$$

22. $$\ell_1 = \begin{cases} x = 0.2 + 0.6t \\ y = 1.33 - 0.45t \\ z = -4.2 + 1.05t \end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} x = 0.86 + 9.2t \\ y = 0.835 - 6.9t \\ z = -3.045 + 16.1t \end{cases}$$

23. $$\ell_1 = \begin{cases} x = 0.1 + 1.1t \\ y = 2.9 - 1.5t \\ z = 3.2 + 1.6t \end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} x = 4 - 2.1t \\ y = 1.8 + 7.2t \\ z = 3.1 + 1.1t \end{cases}$$

In Exercises 24 – 27, find the distance from the point to the line.

24. $P = (1, 1, 1)$, $\quad \vec{\ell}(t) = \langle 2, 1, 3\rangle + t\langle 2, 1, -2\rangle$

25. $P = (2, 5, 6)$, $\quad \vec{\ell}(t) = \langle -1, 1, 1\rangle + t\langle 1, 0, 1\rangle$

26. $P = (0, 3)$, $\quad \vec{\ell}(t) = \langle 2, 0\rangle + t\langle 1, 1\rangle$

27. $P = (1, 1)$, $\quad \vec{\ell}(t) = \langle 4, 5\rangle + t\langle -4, 3\rangle$

In Exercises 28 – 29, find the distance between the two lines.

28. $\vec{\ell_1}(t) = \langle 1, 2, 1\rangle + t\langle 2, -1, 1\rangle$,
$\vec{\ell_2}(t) = \langle 3, 3, 3\rangle + t\langle 4, 2, -2\rangle$.

29. $\vec{\ell_1}(t) = \langle 0, 0, 1\rangle + t\langle 1, 0, 0\rangle$,
$\vec{\ell_2}(t) = \langle 0, 0, 3\rangle + t\langle 0, 1, 0\rangle$.

Exercises 30 – 32 explore special cases of the distance formulas found in Key Idea 51.

30. Let Q be a point on the line $\ell(t)$. Show why the distance formula correctly gives the distance from the point to the line as 0.

31. Let lines $\ell_1(t)$ and $\ell_2(t)$ be intersecting lines. Show why the distance formula correctly gives the distance between these lines as 0.

32. Let lines $\ell_1(t)$ and $\ell_2(t)$ be parallel.

(a) Show why the distance formula for distance between lines cannot be used as stated to find the distance between the lines.

(b) Show why letting $c = (\overrightarrow{P_1P_2} \times \vec{d}_2) \times \vec{d}_2$ allows one to the use the formula.

(c) Show how one can use the formula for the distance between a point and a line to find the distance between parallel lines.

10.6 Planes

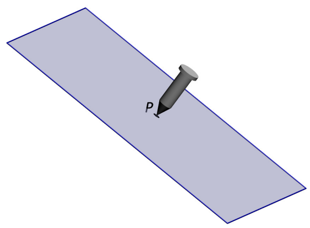


Figure 10.54: Illustrating defining a plane with a sheet of cardboard and a nail.

Any flat surface, such as a wall, table top or stiff piece of cardboard can be thought of as representing part of a plane. Consider a piece of cardboard with a point P marked on it. One can take a nail and stick it into the cardboard at P such that the nail is perpendicular to the cardboard; see Figure 10.54

This nail provides a "handle" for the cardboard. Moving the cardboard around moves P to different locations in space. Tilting the nail (but keeping P fixed) tilts the cardboard. Both moving and tilting the cardboard defines a different plane in space. In fact, we can define a plane by: 1) the location of P in space, and 2) the direction of the nail.

The previous section showed that one can define a line given a point on the line and the direction of the line (usually given by a vector). One can make a similar statement about planes: we can define a plane in space given a point on the plane and the direction the plane "faces" (using the description above, the direction of the nail). Once again, the direction information will be supplied by a vector, called a **normal vector**, that is orthogonal to the plane.

What exactly does "orthogonal to the plane" mean? Choose any two points P and Q in the plane, and consider the vector $\overrightarrow{PQ}$. We say a vector $\vec{n}$ is orthogonal to the plane if $\vec{n}$ is perpendicular to $\overrightarrow{PQ}$ for all choices of P and Q; that is, if $\vec{n}\cdot\overrightarrow{PQ}=0$ for all P and Q.

This gives us way of writing an equation describing the plane. Let $P=(x_0,y_0,z_0)$ be a point in the plane and let $\vec{n}=\langle a,b,c\rangle$ be a normal vector to the plane. A point $Q=(x,y,z)$ lies in the plane defined by P and $\vec{n}$ if, and only if, $\overrightarrow{PQ}$ is orthogonal to $\vec{n}$. Knowing $\overrightarrow{PQ}=\langle x-x_0,y-y_0,z-z_0\rangle$, consider:

$$\begin{aligned}\overrightarrow{PQ}\cdot\vec{n}&=0\\ \langle x-x_0,y-y_0,z-z_0\rangle\cdot\langle a,b,c\rangle&=0\\ a(x-x_0)+b(y-y_0)+c(z-z_0)&=0 \qquad (10.9)\end{aligned}$$

Equation (10.9) defines an *implicit* relation describing the plane. More algebra produces:

$$ax+by+cz=ax_0+by_0+cz_0.$$

The right hand side is just a number, so we replace it with d:

$$ax+by+cz=d. \qquad (10.10)$$

As long as $c\neq 0$, we can solve for z:

$$z=\frac{1}{c}(d-ax-by). \qquad (10.11)$$

Notes:

Equation (10.11) is especially useful as many computer programs can graph functions in this form. Equations (10.9) and (10.10) have specific names, given next.

Definition 67 Equations of a Plane in Standard and General Forms

The plane passing through the point $P = (x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c\rangle$ can be described by an equation with **standard form**

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0;$$

the equation's **general form** is

$$ax + by + cz = d.$$

A key to remember throughout this section is this: to find the equation of a plane, we need a point and a normal vector. We will give several examples of finding the equation of a plane, and in each one different types of information are given. In each case, we need to use the given information to find a point on the plane and a normal vector.

Example 10.38 Finding the equation of a plane.

Write the equation of the plane that passes through the points $P = (1, 1, 0)$, $Q = (1, 2, -1)$ and $R = (0, 1, 2)$ in standard form.

Solution We need a vector $\vec{n}$ that is orthogonal to the plane. Since P, Q and R are in the plane, so are the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$; $\overrightarrow{PQ} \times \overrightarrow{PR}$ is orthogonal to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and hence the plane itself.

It is straightforward to compute $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \langle 2, 1, 1\rangle$. We can use any point we wish in the plane (any of P, Q or R will do) and we arbitrarily choose P. Following Definition 67, the equation of the plane in standard form is

$$2(x - 1) + (y - 1) + z = 0.$$

The plane is sketched in Figure 10.55.

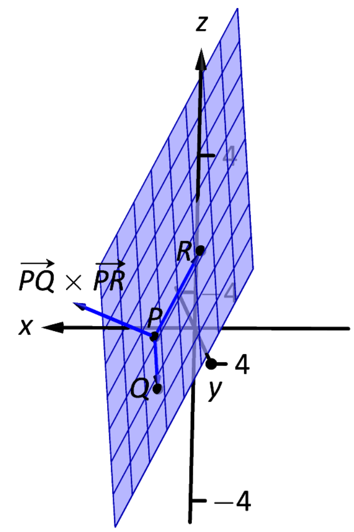


Figure 10.55: Sketching the plane in Example 10.38.

We have just demonstrated the fact that any three non-collinear points define a plane. (This is why a three-legged stool does not "rock;" it's three feet always lie in a plane. A four-legged stool will rock unless all four feet lie in the same plane.)

Example 10.39 Finding the equation of a plane.

Verify that lines ℓ_1 and ℓ_2, whose parametric equations are given below, inter-

Notes:

sect, then give the equation of the plane that contains these two lines in general form.

$$\ell_1: \begin{array}{rcl} x &=& -5+2s \\ y &=& 1+s \\ z &=& -4+2s \end{array} \qquad \ell_2: \begin{array}{rcl} x &=& 2+3t \\ y &=& 1-2t \\ z &=& 1+t \end{array}$$

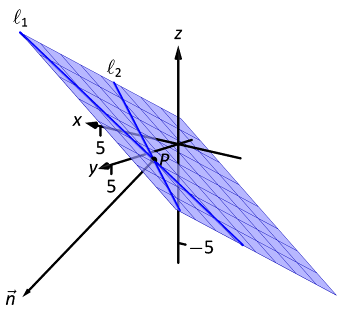


Figure 10.56: Sketching the plane in Example 10.39.

Solution The lines clearly are not parallel. If they do not intersect, they are skew, meaning there is not a plane that contains them both. If they do intersect, there is such a plane.

To find their point of intersection, we set the x, y and z equations equal to each other and solve for s and t:

$$\begin{array}{rcl} -5+2s &=& 2+3t \\ 1+s &=& 1-2t \\ -4+2s &=& 1+t \end{array} \quad \Rightarrow \quad s=2, \quad t=-1.$$

When $s=2$ and $t=-1$, the lines intersect at the point $P=(-1,3,0)$.

Let $\vec{d_1}=\langle 2,1,2\rangle$ and $\vec{d_2}=\langle 3,-2,1\rangle$ be the directions of lines ℓ_1 and ℓ_2, respectively. A normal vector to the plane containing these the two lines will also be orthogonal to $\vec{d_1}$ and $\vec{d_2}$. Thus we find a normal vector $\vec{n}$ by computing $\vec{n}=\vec{d_1}\times\vec{d_2}=\langle 5,4-7\rangle$.

We can pick any point in the plane with which to write our equation; each line gives us infinite choices of points. We choose P, the point of intersection. We follow Definition 67 to write the plane's equation in general form:

$$\begin{aligned} 5(x+1)+4(y-3)-7z&=0 \\ 5x+5+4y-12-7z&=0 \\ 5x+4y-7z&=7. \end{aligned}$$

The plane's equation in general form is $5x+4y-7z=7$; it is sketched in Figure 10.56.

Example 10.40 Finding the equation of a plane

Give the equation, in standard form, of the plane that passes through the point $P=(-1,0,1)$ and is orthogonal to the line with vector equation $\vec{\ell}(t)=\langle -1,0,1\rangle+t\langle 1,2,2\rangle$.

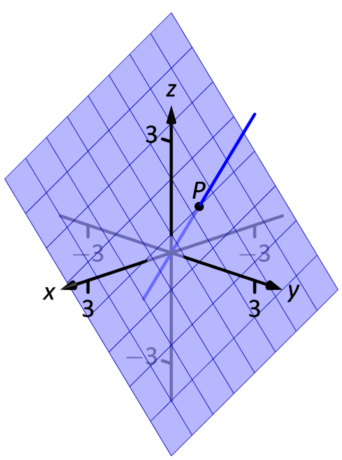


Figure 10.57: The line and plane in Example 10.40.

Solution As the plane is to be orthogonal to the line, the plane must be orthogonal to the direction of the line given by $\vec{d}=\langle 1,2,2\rangle$. We use this as our normal vector. Thus the plane's equation, in standard form, is

$$(x+1)+2y+2(z-1)=0.$$

The line and plane are sketched in Figure 10.57.

Notes:

Example 10.41 Finding the intersection of two planes

Give the parametric equations of the line that is the intersection of the planes p_1 and p_2, where:

$$\begin{aligned} p_1 &: x-(y-2)+(z-1)=0 \\ p_2 &: -2(x-2)+(y+1)+(z-3)=0 \end{aligned}$$

Solution To find an equation of a line, we need a point on the line and the direction of the line.

We can find a point on the line by solving each equation of the planes for z:

$$\begin{aligned} p_1 &: z=-x+y-1 \\ p_2 &: z=2x-y-2 \end{aligned}$$

We can now set these two equations equal to each other (i.e., we are finding values of x and y where the planes have the same z value):

$$\begin{aligned} -x+y-1 &= 2x-y-2 \\ 2y &= 3x-1 \\ y &= \frac{1}{2}(3x-1) \end{aligned}$$

We can choose any value for x; we choose $x=1$. This determines that $y=1$. We can now use the equations of either plane to find z: when $x=1$ and $y=1$, $z=-1$ on both planes. We have found a point P on the line: $P=(1,1,-1)$.

We now need the direction of the line. Since the line lies in each plane, its direction is orthogonal to a normal vector for each plane. Considering the equations for p_1 and p_2, we can quickly determine their normal vectors. For p_1, $\vec{n}_1=\langle 1,-1,1\rangle$ and for p_2, $\vec{n}_2=\langle -2,1,1\rangle$. A direction orthogonal to both of these directions is their cross product: $\vec{d}=\vec{n}_1\times\vec{n}_2=\langle -2,-3,-1\rangle$.

The parametric equations of the line through $P=(1,1,-1)$ in the direction of $d=\langle -2,-3,-1\rangle$ is:

$$\ell: \quad x=-2t+1 \quad y=-3t+1 \quad z=-t-1.$$

The planes and line are graphed in Figure 10.58.

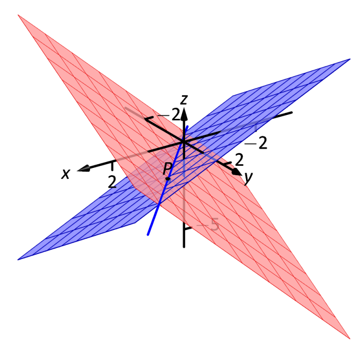


Figure 10.58: Graphing the planes and their line of intersection in Example 10.41.

Example 10.42 Finding the intersection of a plane and a line

Find the point of intersection, if any, of the line $\ell(t)=\langle 3,-3,-1\rangle+t\langle -1,2,1\rangle$ and the plane with equation in general form $2x+y+z=4$.

Solution The equation of the plane shows that the vector $\vec{n}=\langle 2,1,1\rangle$ is a normal vector to the plane, and the equation of the line shows that the line

Notes:

moves parallel to $\vec{d} = \langle -1, 2, 1 \rangle$. Since these are not orthogonal, we know there is a point of intersection. (If they were orthogonal, it would mean that the plane and line were parallel to each other, either never intersecting or the line was in the plane itself.)

To find the point of intersection, we need to find a t value such that $\ell(t)$ satisfies the equation of the plane. Rewriting the equation of the line with parametric equations will help:

$$\ell(t) = \begin{cases} x = 3 - t \\ y = -3 + 2t \\ z = -1 + t \end{cases}.$$

Replacing x, y and z in the equation of the plane with the expressions containing t found in the equation of the line allows us to determine a t value that indicates the point of intersection:

$$\begin{aligned} 2x + y + z &= 4 \\ 2(3 - t) + (-3 + 2t) + (-1 + t) &= 4 \\ t &= 2. \end{aligned}$$

When $t = 2$, the point on the line satisfies the equation of the plane; that point is $\ell(2) = \langle 1, 1, 1 \rangle$. Thus the point $(1, 1, 1)$ is the point of intersection between the plane and the line, illustrated in Figure 10.59.

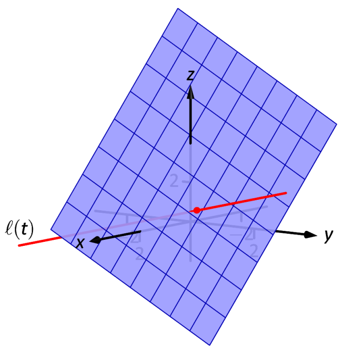


Figure 10.59: Illustrating the intersection of a line and a plane in Example 10.42.

Distances

Just as it was useful to find distances between points and lines in the previous section, it is also often necessary to find the distance from a point to a plane.

Consider Figure 10.60, where a plane with normal vector $\vec{n}$ is sketched containing a point P and a point Q, not on the plane, is given. We measure the distance from Q to the plane by measuring the length of the projection of $\overrightarrow{PQ}$ onto $\vec{n}$. That is, we want:

$$\left|\left| \operatorname{proj}_{\vec{n}} \overrightarrow{PQ} \right|\right| = \left|\left| \frac{\vec{n} \cdot \overrightarrow{PQ}}{|| \vec{n} ||^2} \vec{n} \right|\right| = \frac{|\vec{n} \cdot \overrightarrow{PQ}|}{|| \vec{n} ||} \qquad (10.12)$$

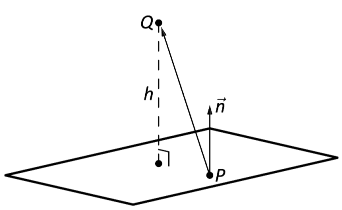


Figure 10.60: Illustrating finding the distance from a point to a plane.

Equation (10.12) is important as it does more than just give the distance between a point and a plane. We will see how it allows us to find several other distances as well: the distance between parallel planes and the distance from a line and a plane. Because Equation (10.12) is important, we restate it as a Key Idea.

Notes:

Key Idea 52 Distance from a Point to a Plane

Let a plane with normal vector $\vec{n}$ be given, and let Q be a point. The distance h from Q to the plane is

$$h = \frac{|\vec{n} \cdot \overrightarrow{PQ}|}{||\,\vec{n}\,||},$$

where P is any point in the plane.

Example 10.43 Distance between a point and a plane

Find the distance bewteen the point $Q = (2, 1, 4)$ and the plane with equation $2x - 5y + 6z = 9$.

Solution Using the equation of the plane, we find the normal vector $\vec{n} = \langle 2, -5, 6\rangle$. To find a point on the plane, we can let x and y be anything we choose, then let z be whatever satisfies the equation. Letting x and y be 0 seems simple; this makes $z = 1.5$. Thus we let $P = \langle 0, 0, 1.5\rangle$, and $\overrightarrow{PQ} = \langle 2, 1, 2.5\rangle$.

The distance h from Q to the plane is given by Key Idea 52:

$$\begin{aligned} h &= \frac{|\vec{n} \cdot \overrightarrow{PQ}|}{||\,\vec{n}\,||} \\ &= \frac{|\,\langle 2, -5, 6\rangle \cdot \langle 2, 1, 2.5\rangle\,|}{||\,\langle 2, -5, 6\rangle\,||} \\ &= \frac{|14|}{\sqrt{65}} \\ &\approx 1.74. \end{aligned}$$

We can use Key Idea 52 to find other distances. Given two parallel planes, we can find the distance between these planes by letting P be a point on one plane and Q a point on the other. If ℓ is a line parallel to a plane, we can use the Key Idea to find the distance between them as well: again, let P be a point in the plane and let Q be any point on the line. (One can also use Key Idea 51.) The Exercise section contains problems of these types.

These past two sections have not explored lines and planes in space as an exercise of mathematical curiosity. However, there are many, many applications of these fundamental concepts. Complex shapes can be modeled (or, *approximated*) using planes. For instance, part of the exterior of an aircraft may have a complex, yet smooth, shape, and engineers will want to know how air flows across this piece as well as how heat might build up due to air friction. Many equations that help determine air flow and heat dissipation are difficult to apply to arbitrary surfaces, but simple to apply to planes. By approximating a surface with millions of small planes one can more readily model the needed behavior.

Notes:

Exercises 10.6

Terms and Concepts

1. In order to find the equation of a plane, what two pieces of information must one have?

2. What is the relationship between a plane and one of its normal vectors?

Problems

In Exercises 3 – 6, give any two points in the given plane.

3. $2x - 4y + 7z = 2$

4. $3(x + 2) + 5(y - 9) - 4z = 0$

5. $x = 2$

6. $4(y + 2) - (z - 6) = 0$

In Exercises 7 – 20, give the equation of the described plane in standard and general forms.

7. Passes through $(2, 3, 4)$ and has normal vector $\vec{n} = \langle 3, -1, 7\rangle$.

8. Passes through $(1, 3, 5)$ and has normal vector $\vec{n} = \langle 0, 2, 4\rangle$.

9. Passes through the points $(1, 2, 3)$, $(3, -1, 4)$ and $(1, 0, 1)$.

10. Passes through the points $(5, 3, 8)$, $(6, 4, 9)$ and $(3, 3, 3)$.

11. Contains the intersecting lines $\ell_1(t) = \langle 2, 1, 2\rangle + t\,\langle 1, 2, 3\rangle$ and $\ell_2(t) = \langle 2, 1, 2\rangle + t\,\langle 2, 5, 4\rangle$.

12. Contains the intersecting lines $\ell_1(t) = \langle 5, 0, 3\rangle + t\,\langle -1, 1, 1\rangle$ and $\ell_2(t) = \langle 1, 4, 7\rangle + t\,\langle 3, 0, -3\rangle$.

13. Contains the parallel lines $\ell_1(t) = \langle 1, 1, 1\rangle + t\,\langle 1, 2, 3\rangle$ and $\ell_2(t) = \langle 1, 1, 2\rangle + t\,\langle 1, 2, 3\rangle$.

14. Contains the parallel lines $\ell_1(t) = \langle 1, 1, 1\rangle + t\,\langle 4, 1, 3\rangle$ and $\ell_2(t) = \langle 2, 2, 2\rangle + t\,\langle 4, 1, 3\rangle$.

15. Contains the point $(2, -6, 1)$ and the line
$$\ell(t) = \begin{cases} x = 2 + 5t \\ y = 2 + 2t \\ z = -1 + 2t \end{cases}$$

16. Contains the point $(5, 7, 3)$ and the line
$$\ell(t) = \begin{cases} x = t \\ y = t \\ z = t \end{cases}$$

17. Contains the point $(5, 7, 3)$ and is orthogonal to the line $\ell(t) = \langle 4, 5, 6\rangle + t\,\langle 1, 1, 1\rangle$.

18. Contains the point $(4, 1, 1)$ and is orthogonal to the line
$$\ell(t) = \begin{cases} x = 4 + 4t \\ y = 1 + 1t \\ z = 1 + 1t \end{cases}$$

19. Contains the point $(-4, 7, 2)$ and is parallel to the plane $3(x - 2) + 8(y + 1) - 10z = 0$.

20. Contains the point $(1, 2, 3)$ and is parallel to the plane $x = 5$.

In Exercises 21 – 22, give the equation of the line that is the intersection of the given planes.

21. $p1:\ 3(x - 2) + (y - 1) + 4z = 0$, and
$p2:\ 2(x - 1) - 2(y + 3) + 6(z - 1) = 0$.

22. $p1:\ 5(x - 5) + 2(y + 2) + 4(z - 1) = 0$, and
$p2:\ 3x - 4(y - 1) + 2(z - 1) = 0$.

In Exercises 23 – 26, find the point of intersection between the line and the plane.

23. line: $\langle 5, 1, -1\rangle + t\,\langle 2, 2, 1\rangle$,
plane: $5x - y - z = -3$

24. line: $\langle 4, 1, 0\rangle + t\,\langle 1, 0, -1\rangle$,
plane: $3x + y - 2z = 8$

25. line: $\langle 1, 2, 3\rangle + t\,\langle 3, 5, -1\rangle$,
plane: $3x - 2y - z = 4$

26. line: $\langle 1, 2, 3\rangle + t\,\langle 3, 5, -1\rangle$,
plane: $3x - 2y - z = -4$

In Exercises 27 – 30, find the given distances.

27. The distance from the point $(1, 2, 3)$ to the plane $3(x - 1) + (y - 2) + 5(z - 2) = 0$.

28. The distance from the point $(2, 6, 2)$ to the plane $2(x - 1) - y + 4(z + 1) = 0$.

29. The distance between the parallel planes
$x + y + z = 0$ and
$(x - 2) + (y - 3) + (z + 4) = 0$

30. The distance between the parallel planes $2(x-1)+2(y+1)+(z-2)=0$ and $2(x-3)+2(y-1)+(z-3)=0$

31. Show why if the point Q lies in a plane, then the distance formula correctly gives the distance from the point to the plane as 0.

32. How is Exercise 31 in Section 10.5 easier to answer once we have an understanding of planes?

11: Vector Valued Functions

In the previous chapter, we learned about vectors and were introduced to the power of vectors within mathematics. In this chapter, we'll build on this foundation to define functions whose input is a real number and whose output is a vector. We'll see how to graph these functions and apply calculus techniques to analyze their behavior. Most importantly, we'll see *why* we are interested in doing this: we'll see beautiful applications to the study of moving objects.

11.1 Vector–Valued Functions

We are very familiar with **real valued functions**, that is, functions whose output is a real number. This section introduces **vector–valued functions** – functions whose output is a vector.

Definition 68 Vector–Valued Functions

A **vector–valued function** is a function of the form

$$\vec{r}(t) = \langle f(t), g(t) \rangle \quad \text{or} \quad \vec{r}(t) = \langle f(t), g(t), h(t) \rangle,$$

where f, g and h are real valued functions.

The **domain** of $\vec{r}$ is the set of all values of t for which $\vec{r}(t)$ is defined. The **range** of $\vec{r}$ is the set of all possible output vectors $\vec{r}(t)$.

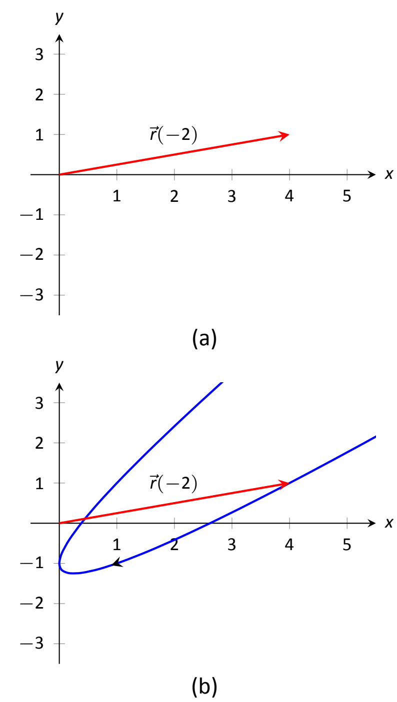


Figure 11.1: Sketching the graph of a vector–valued function.

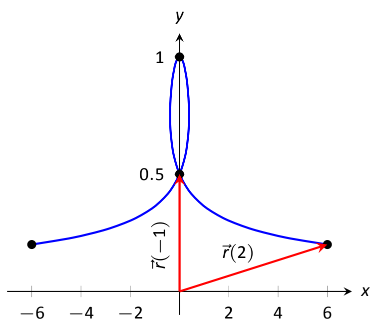
Evaluating and Graphing Vector–Valued Functions

Evaluating a vector–valued function at a specific value of t is straightforward; simply evaluate each component function at that value of t. For instance, if $\vec{r}(t) = \langle t^2, t^2 + t - 1 \rangle$, then $\vec{r}(-2) = \langle 4, 1 \rangle$. We can sketch this vector, as is done in Figure 11.1(a). Plotting lots of vectors is cumbersome, though, so generally we do not sketch the whole vector but just the terminal point. The **graph** of a vector–valued function is the set of all terminal points of $\vec{r}(t)$, where the initial point of each vector is always the origin. In Figure 11.1(b) we sketch the graph of $\vec{r}$; we can indicate individual points on the graph with their respective vector, as shown.

Vector–valued functions are closely related to parametric equations of graphs. While in both methods we plot points $\big(x(t), y(t)\big)$ or $\big(x(t), y(t), z(t)\big)$ to produce a graph, in the context of vector–valued functions each such point represents a vector. The implications of this will be more fully realized in the next section as we apply calculus ideas to these functions.

t	t^3-t	$\dfrac{1}{t^2+1}$
-2	-6	1/5
-1	0	1/2
0	0	1
1	0	1/2
2	6	1/5

(a)



(b)

Figure 11.2: Sketching the vector–valued function of Example 11.1.

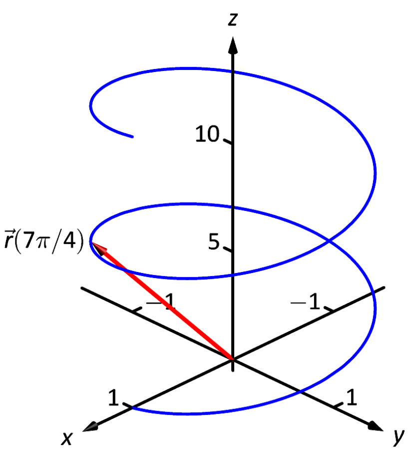


Figure 11.3: The graph of $\vec r(t)$ in Example 11.2.

Example 11.1 Graphing vector–valued functions

Graph $\vec r(t) = \left\langle t^3-t, \dfrac{1}{t^2+1}\right\rangle$, for $-2\le t\le 2$. Sketch $\vec r(-1)$ and $\vec r(2)$.

Solution We start by making a table of t, x and y values as shown in Figure 11.2(a). Plotting these points gives an indication of what the graph looks like. In Figure 11.2(b), we indicate these points and sketch the full graph. We also highlight $\vec r(-1)$ and $\vec r(2)$ on the graph.

Example 11.2 Graphing vector–valued functions.

Graph $\vec r(t) = \langle \cos t, \sin t, t\rangle$ for $0\le t\le 4\pi$.

Solution We can again plot points, but careful consideration of this function is very revealing. Momentarily ignoring the third component, we see the x and y components trace out a circle of radius 1 centered at the origin. Noticing that the z component is t, we see that as the graph winds around the z-axis, it is also increasing at a constant rate in the positive z direction, forming a spiral. This is graphed in Figure 11.3. In the graph $\vec r(7\pi/4)\approx(0.707, -0.707, 5.498)$ is highlighted to help us understand the graph.

Algebra of Vector–Valued Functions

Definition 69 Operations on Vector–Valued Functions

Let $\vec r_1(t) = \langle f_1(t), g_1(t)\rangle$ and $\vec r_2(t) = \langle f_2(t), g_2(t)\rangle$ be vector–valued functions in $\mathbb{R}^2$ and let c be a scalar. Then:

1. $\vec r_1(t)\pm\vec r_2(t) = \langle f_1(t)\pm f_2(t), g_1(t)\pm g_2(t)\rangle$.
2. $c\vec r_1(t) = \langle cf_1(t), cg_1(t)\rangle$.

A similar definition holds for vector–valued functions in $\mathbb{R}^3$.

This definition states that we add, subtract and scale vector-valued functions component–wise. Combining vector–valued functions in this way can be very useful (as well as create interesting graphs).

Example 11.3 Adding and scaling vector–valued functions.

Let $\vec r_1(t) = \langle 0.2t, 0.3t\rangle$, $\vec r_2(t) = \langle \cos t, \sin t\rangle$ and $\vec r(t) = \vec r_1(t)+\vec r_2(t)$. Graph $\vec r_1(t)$, $\vec r_2(t)$, $\vec r(t)$ and $5\vec r(t)$ on $-10\le t\le 10$.

Notes:

Solution We can graph $\vec{r}_1$ and $\vec{r}_2$ easily by plotting points (or just using technology). Let's think about each for a moment to better understand how vector–valued functions work.

We can rewrite $\vec{r}_1(t) = \langle 0.2t, 0.3t \rangle$ as $\vec{r}_1(t) = t\langle 0.2, 0.3\rangle$. That is, the function $\vec{r}_1$ scales the vector $\langle 0.2, 0.3\rangle$ by t. This scaling of a vector produces a line in the direction of $\langle 0.2, 0.3\rangle$.

We are familiar with $\vec{r}_2(t) = \langle \cos t, \sin t \rangle$; it traces out a circle, centered at the origin, of radius 1. Figure 11.4(a) graphs $\vec{r}_1(t)$ and $\vec{r}_2(t)$.

Adding $\vec{r}_1(t)$ to $\vec{r}_2(t)$ produces $\vec{r}(t) = \langle \cos t + 0.2t, \sin t + 0.3t \rangle$, graphed in Figure 11.4(b). The linear movement of the line combines with the circle to create loops that move in the direction of $\langle 0.2, 0.3\rangle$. (We encourage the reader to experiment by changing $\vec{r}_1(t)$ to $\langle 2t, 3t\rangle$, etc., and observe the effects on the loops.)

Multiplying $\vec{r}(t)$ by 5 scales the function by 5, producing $5\vec{r}(t) = \langle 5\cos t + 1, 5\sin t + 1.5\rangle$, which is graphed in Figure 11.4(c) along with $\vec{r}(t)$. The new function is "5 times bigger" than $\vec{r}(t)$. Note how the graph of $5\vec{r}(t)$ in (c) looks identical to the graph of $\vec{r}(t)$ in (b). This is due to the fact that the x and y bounds of the plot in (c) are exactly 5 times larger than the bounds in (b).

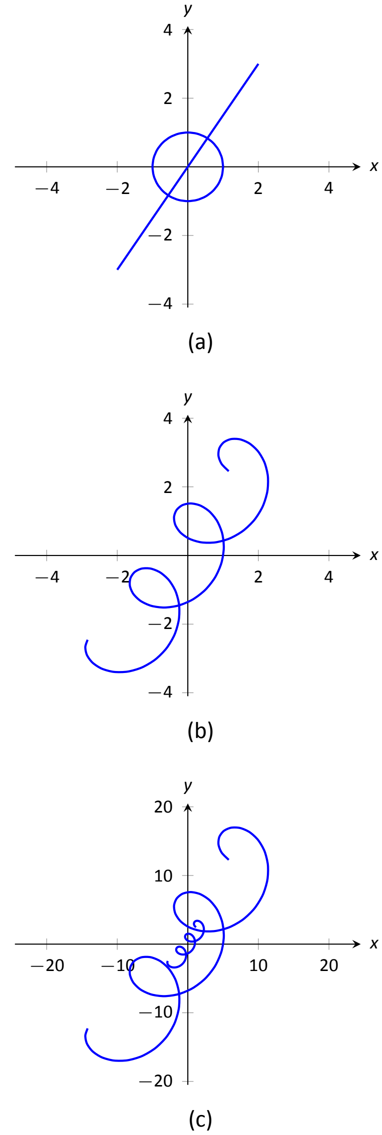


Figure 11.4: Graphing the functions in Example 11.3.

Example 11.4 **Adding and scaling vector–valued functions.**

A **cycloid** is a graph traced by a point p on a rolling circle, as shown in Figure 11.5. Find an equation describing the cycloid, where the circle has radius 1.

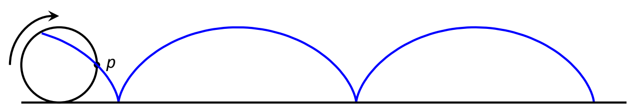


Figure 11.5: Tracing a cycloid.

Solution This problem is not very difficult if we approach it in a clever way. We start by letting $\vec{p}(t)$ describe the position of the point p on the circle, where the circle is centered at the origin and only rotates clockwise (i.e., it does not roll). This is relatively simple given our previous experiences with parametric equations; $\vec{p}(t) = \langle \cos t, -\sin t\rangle$.

We now want the circle to roll. We represent this by letting $\vec{c}(t)$ represent the location of the center of the circle. It should be clear that the y component of $\vec{c}(t)$ should be 1; the center of the circle is always going to be 1 if it rolls on a horizontal surface.

The x component of $\vec{c}(t)$ is a linear function of t: $f(t) = mt$ for some scalar m. When $t = 0$, $f(t) = 0$ (the circle starts centered on the y-axis). When $t = 2\pi$, the circle has made one complete revolution, traveling a distance equal to its

Notes:

circumference, which is also 2π. This gives us a point on our line $f(t) = mt$, the point $(2\pi, 2\pi)$. It should be clear that $m = 1$ and $f(t) = t$. So $\vec{c}(t) = \langle t, 1 \rangle$.

We now combine $\vec{p}$ and $\vec{c}$ together to form the equation of the cycloid: $\vec{r}(t) = \vec{p}(t) + \vec{c}(t) = \langle \cos t + t, -\sin t + 1 \rangle$, which is graphed in Figure 11.6.

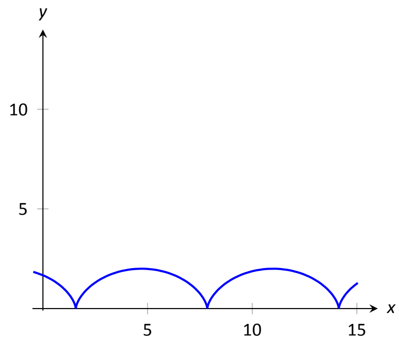


Figure 11.6: The cycloid in Example 11.4.

Displacement

A vector–valued function $\vec{r}(t)$ is often used to describe the position of a moving object at time t. At $t = t_0$, the object is at $\vec{r}(t_0)$; at $t = t_1$, the object is at $\vec{r}(t_1)$. Knowing the locations $\vec{r}(t_0)$ and $\vec{r}(t_1)$ give no indication of the path taken between them, but often we only care about the difference of the locations, $\vec{r}(t_1) - \vec{r}(t_0)$, the **displacement**.

Definition 70 Displacement

Let $\vec{r}(t)$ be a vector–valued function and let $t_0 < t_1$ be values in the domain. The **displacement** $\vec{d}$ of $\vec{r}$, from $t = t_0$ to $t = t_1$, is

$$\vec{d} = \vec{r}(t_1) - \vec{r}(t_0).$$

When the displacement vector is drawn with initial point at $\vec{r}(t_0)$, its terminal point is $\vec{r}(t_1)$. We think of it as the vector which points from a starting position to an ending position.

Example 11.5 Finding and graphing displacement vectors

Let $\vec{r}(t) = \left\langle \cos(\frac{\pi}{2}t), \sin(\frac{\pi}{2}t) \right\rangle$. Graph $\vec{r}(t)$ on $-1 \le t \le 1$, and find the displacement of $\vec{r}(t)$ on this interval.

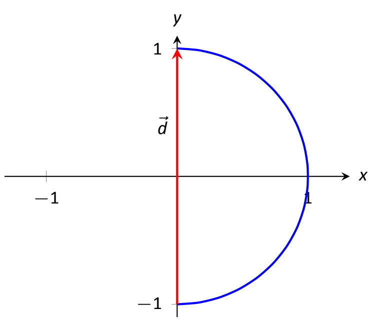


Figure 11.7: Graphing the displacement of a position function in Example 11.5.

Solution The function $\vec{r}(t)$ traces out the unit circle, though at a different rate than the "usual" $\langle \cos t, \sin t \rangle$ parametrization. At $t_0 = -1$, we have $\vec{r}(t_0) = \langle 0, -1 \rangle$; at $t_1 = 1$, we have $\vec{r}(t_1) = \langle 0, 1 \rangle$. The displacement of $\vec{r}(t)$ on $[-1, 1]$ is thus $\vec{d} = \langle 0, 1 \rangle - \langle 0, -1 \rangle = \langle 0, 2 \rangle$.

A graph of $\vec{r}(t)$ on $[-1, 1]$ is given in Figure 11.7, along with the displacement vector $\vec{d}$ on this interval.

Measuring displacement makes us contemplate related, yet very different, concepts. Considering the semi–circular path the object in Example 11.5 took, we can quickly verify that the object ended up a distance of 2 units from its initial location. That is, we can compute $\| \vec{d} \| = 2$. However, measuring *distance from the starting point* is different from measuring *distance traveled*. Being a semi–

Notes:

circle, we can measure the distance traveled by this object as $\pi \approx 3.14$ units. Knowing *distance from the starting point* allows us to compute **average rate of change.**

Definition 71 Average Rate of Change

Let $\vec{r}(t)$ be a vector–valued function, where each of its component functions is continuous on its domain, and let $t_0 < t_1$. The **average rate of change** of $\vec{r}(t)$ on $[t_0, t_1]$ is

$$\text{average rate of change} = \frac{\vec{r}(t_1) - \vec{r}(t_0)}{t_1 - t_0}.$$

Example 11.6 Average rate of change

Let $\vec{r}(t) = \left\langle \cos(\frac{\pi}{2}t), \sin(\frac{\pi}{2}t) \right\rangle$ as in Example 11.5. Find the average rate of change of $\vec{r}(t)$ on $[-1, 1]$ and on $[-1, 5]$.

Solution We computed in Example 11.5 that the displacement of $\vec{r}(t)$ on $[-1, 1]$ was $\vec{d} = \langle 0, 2 \rangle$. Thus the average rate of change of $\vec{r}(t)$ on $[-1, 1]$ is:

$$\frac{\vec{r}(1) - \vec{r}(-1)}{1 - (-1)} = \frac{\langle 0, 2 \rangle}{2} = \langle 0, 1 \rangle\,.$$

We interpret this as follows: the object followed a semi–circular path, meaning it moved towards the right then moved back to the left, while climbing slowly, then quickly, then slowly again. *On average*, however, it progressed straight up at a constant rate of $\langle 0, 1 \rangle$ per unit of time.

We can quickly see that the displacement on $[-1, 5]$ is the same as on $[-1, 1]$, so $\vec{d} = \langle 0, 2 \rangle$. The average rate of change is different, though:

$$\frac{\vec{r}(5) - \vec{r}(-1)}{5 - (-1)} = \frac{\langle 0, 2 \rangle}{6} = \langle 0, 1/3 \rangle\,.$$

As it took "3 times as long" to arrive at the same place, this average rate of change on $[-1, 5]$ is $1/3$ the average rate of change on $[-1, 1]$.

We considered average rates of change in Sections 1.1 and 2.1 as we studied limits and derivatives. The same is true here; in the following section we apply calculus concepts to vector–valued functions as we find limits, derivatives, and integrals. Understanding the average rate of change will give us an understanding of the derivative; displacement gives us one application of integration.

Notes:

Exercises 11.1

Terms and Concepts

1. Vector–valued functions are closely related to ________ ________ of graphs.

2. When sketching vector–valued functions, technically one isn't graphing points, but rather ________.

3. It can be useful to think of ________ as a vector that points from a starting position to an ending position.

Problems

In Exercises 4 – 11, sketch the vector–valued function on the given interval.

4. $\vec{r}(t) = \left\langle t^2, t^2 - 1 \right\rangle$, for $-2 \leq t \leq 2$.

5. $\vec{r}(t) = \left\langle t^2, t^3 \right\rangle$, for $-2 \leq t \leq 2$.

6. $\vec{r}(t) = \left\langle 1/t, 1/t^2 \right\rangle$, for $-2 \leq t \leq 2$.

7. $\vec{r}(t) = \left\langle \frac{1}{10}t^2, \sin t \right\rangle$, for $-2\pi \leq t \leq 2\pi$.

8. $\vec{r}(t) = \left\langle \frac{1}{10}t^2, \sin t \right\rangle$, for $-2\pi \leq t \leq 2\pi$.

9. $\vec{r}(t) = \langle 3\sin(\pi t), 2\cos(\pi t) \rangle$, on $[0, 2]$.

10. $\vec{r}(t) = \langle 3\cos t, 2\sin(2t) \rangle$, on $[0, 2\pi]$.

11. $\vec{r}(t) = \langle 2\sec t, \tan t \rangle$, on $[-\pi, \pi]$.

In Exercises 12 – 15, sketch the vector–valued function on the given interval in $\mathbb{R}^3$. Technology may be useful in creating the sketch.

12. $\vec{r}(t) = \langle 2\cos t, t, 2\sin t \rangle$, on $[0, 2\pi]$.

13. $\vec{r}(t) = \langle 3\cos t, \sin t, t/\pi \rangle$ on $[0, 2\pi]$.

14. $\vec{r}(t) = \langle \cos t, \sin t, \sin t \rangle$ on $[0, 2\pi]$.

15. $\vec{r}(t) = \langle \cos t, \sin t, \sin(2t) \rangle$ on $[0, 2\pi]$.

In Exercises 16 – 19, find $||\,\vec{r}(t)\,||$.

16. $\vec{r}(t) = \left\langle t, t^2 \right\rangle$.

17. $\vec{r}(t) = \langle 5\cos t, 3\sin t \rangle$.

18. $\vec{r}(t) = \langle 2\cos t, 2\sin t, t \rangle$.

19. $\vec{r}(t) = \left\langle \cos t, t, t^2 \right\rangle$.

In Exercises 20 – 27, create a vector–valued function whose graph matches the given description.

20. A circle of radius 2, centered at $(1, 2)$, traced counter–clockwise once on $[0, 2\pi]$.

21. A circle of radius 3, centered at $(5, 5)$, traced clockwise once on $[0, 2\pi]$.

22. An ellipse, centered at $(0, 0)$ with vertical major axis of length 10 and minor axis of length 3, traced once counter–clockwise on $[0, 2\pi]$.

23. An ellipse, centered at $(3, -2)$ with horizontal major axis of length 6 and minor axis of length 4, traced once clockwise on $[0, 2\pi]$.

24. A line through $(2, 3)$ with a slope of 5.

25. A line through $(1, 5)$ with a slope of $-1/2$.

26. A vertically oriented helix with radius of 2 that starts at $(2, 0, 0)$ and ends at $(2, 0, 4\pi)$ after 1 revolution on $[0, 2\pi]$.

27. A vertically oriented helix with radius of 3 that starts at $(3, 0, 0)$ and ends at $(3, 0, 3)$ after 2 revolutions on $[0, 1]$.

In Exercises 28 – 31, find the average rate of change of $\vec{r}(t)$ on the given interval.

28. $\vec{r}(t) = \left\langle t, t^2 \right\rangle$ on $[-2, 2]$.

29. $\vec{r}(t) = \langle t, t + \sin t \rangle$ on $[0, 2\pi]$.

30. $\vec{r}(t) = \langle 3\cos t, 2\sin t, t \rangle$ on $[0, 2\pi]$.

31. $\vec{r}(t) = \left\langle t, t^2, t^3 \right\rangle$ on $[-1, 3]$.

11.2 Calculus and Vector–Valued Functions

The previous section introduced us to a new mathematical object, the vector–valued function. We now apply calculus concepts to these functions. We start with the limit, then work our way through derivatives to integrals.

Limits of Vector–Valued Functions

The initial definition of the limit of a vector–valued function is a bit intimidating, as was the definition of the limit in Definition 1. The theorem following the definition shows that in practice, taking limits of vector–valued functions is no more difficult than taking limits of real–valued functions.

Definition 72 Limits of Vector–Valued Functions

Let I be an open interval containing c, and let $\vec{r}(t)$ be a vector–valued function defined on I, except possibly at c. The **limit of $\vec{r}(t)$, as t approaches c, is $\vec{L}$**, expressed as

$$\lim_{t\to c} \vec{r}(t) = \vec{L},$$

means that given any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $t \neq c$, if $|t - c| < \delta$, we have $|| \, \vec{r}(t) - \vec{L} \, || < \varepsilon$.

Note how the measurement of distance between real numbers is the absolute value of their difference; the measure of distance between vectors is the vector norm, or magnitude, of their difference.

Theorem 91 Limits of Vector–Valued Functions

1. Let $\vec{r}(t) = \langle f(t), g(t) \rangle$ be a vector–valued function in $\mathbb{R}^2$ defined on an open interval I containing c. Then
$$\lim_{t\to c} \vec{r}(t) = \left\langle \lim_{t\to c} f(t) \, , \, \lim_{t\to c} g(t) \right\rangle .$$

2. Let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ be a vector–valued function in $\mathbb{R}^3$ defined on an open interval I containing c. Then
$$\lim_{t\to c} \vec{r}(t) = \left\langle \lim_{t\to c} f(t) \, , \, \lim_{t\to c} g(t) \, , \, \lim_{t\to c} h(t) \right\rangle$$

Theorem 91 states that we compute limits component–wise.

Notes:

Example 11.7 Finding limits of vector–valued functions

Let $\vec{r}(t) = \left\langle \frac{\sin t}{t}, \ t^2 - 3t + 3, \ \cos t \right\rangle$. Find $\lim\limits_{t\to 0} \vec{r}(t)$.

Solution We apply the theorem and compute limits component–wise.

$$\begin{aligned}\lim_{t\to 0} \vec{r}(t) &= \left\langle \lim_{t\to 0} \frac{\sin t}{t}, \ \lim_{t\to 0} t^2 - 3t + 3, \ \lim_{t\to 0} \cos t \right\rangle \\ &= \langle 1, 3, 1 \rangle .\end{aligned}$$

Continuity

Definition 73 Continuity of Vector–Valued Functions

Let $\vec{r}(t)$ be a vector–valued function defined on an open interval I containing c.

1. $\vec{r}(t)$ is **continuous at** c if $\lim\limits_{t\to c} \vec{r}(t) = r(c)$.
2. If $\vec{r}(t)$ is continuous at all c in I, then $\vec{r}(t)$ is **continuous on** I.

We again have a theorem that lets us evaluate continuity component–wise.

Theorem 92 Continuity of Vector–Valued Functions

Let $\vec{r}(t)$ be a vector–valued function defined on an open interval I containing c. $\vec{r}(t)$ is continuous at c if, and only if, each of its component functions is continuous at c.

Example 11.8 Evaluating continuity of vector–valued functions

Let $\vec{r}(t) = \left\langle \frac{\sin t}{t}, \ t^2 - 3t + 3, \ \cos t \right\rangle$. Determine whether $\vec{r}$ is continuous at $t = 0$ and $t = 1$.

Solution While the second and third components of $\vec{r}(t)$ are defined at $t = 0$, the first component, $(\sin t)/t$, is not. Since the first component is not even defined at $t = 0$, $\vec{r}(t)$ is not defined at $t = 0$, and hence it is not continuous at $t = 0$.

At $t = 1$ each of the component functions is continuous. Therefore $\vec{r}(t)$ is continuous at $t = 1$.

Notes:

Derivatives

Consider a vector–valued function $\vec{r}$ defined on an open interval I containing t_0 and t_1. We can compute the displacement of $\vec{r}$ on $[t_0, t_1]$, as shown in Figure 11.8(a). Recall that dividing the displacement vector by $t_1 - t_0$ gives the average rate of change on $[t_0, t_1]$, as shown in (b).

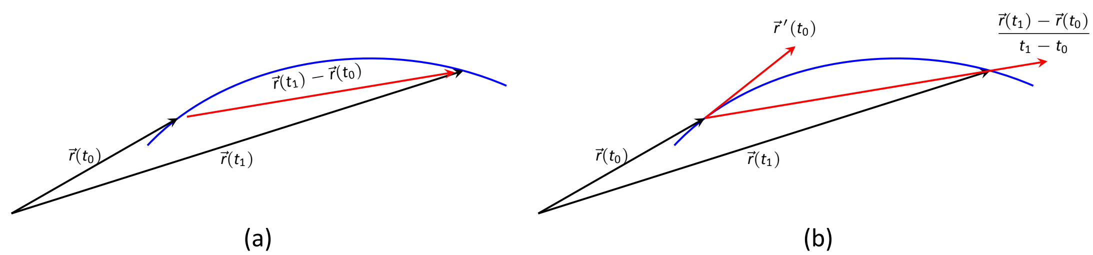


Figure 11.8: Illustrating displacement, leading to an understanding of the derivative of vector–valued functions.

The **derivative** of a vector–valued function is a measure of the *instantaneous* rate of change, measured by taking the limit as the length of $[t_0, t_1]$ goes to 0. Instead of thinking of an interval as $[t_0, t_1]$, we think of it as $[c, c+h]$ for some value of h (hence the interval has length h). The *average* rate of change is

$$\frac{\vec{r}(c+h) - \vec{r}(c)}{h}$$

for any value of $h \neq 0$. We take the limit as $h \to 0$ to measure the instantaneous rate of change; this is the derivative of $\vec{r}$.

Definition 74　　Derivative of a Vector–Valued Function

Let $\vec{r}(t)$ be continuous on an open interval I containing c.

1. The derivative of $\vec{r}$ at $t = c$ is
$$\vec{r}\,'(c) = \lim_{h\to 0} \frac{\vec{r}(c+h) - \vec{r}(c)}{h}.$$

2. The derivative of $\vec{r}$ is
$$\vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}.$$

Alternate notations for the derivative of $\vec{r}$ include:

$$\vec{r}\,'(t) = \frac{d}{dt}\big(\vec{r}(t)\big) = \frac{d\vec{r}}{dt}.$$

If a vector–valued function has a derivative for all c in an open interval I, we say that $\vec{r}(t)$ is **differentiable** on I.

Once again we might view this definition as intimidating, but recall that we

Notes:

can evaluate limits component–wise. The following theorem verifies that this means we can compute derivatives component–wise as well, making the task not too difficult.

Theorem 93 Derivatives of Vector–Valued Functions

1. Let $\vec{r}(t) = \langle f(t), g(t) \rangle$. Then
$$\vec{r}\,'(t) = \langle f'(t), g'(t) \rangle .$$

2. Let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$. Then
$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle .$$

Example 11.9 Derivatives of vector–valued functions

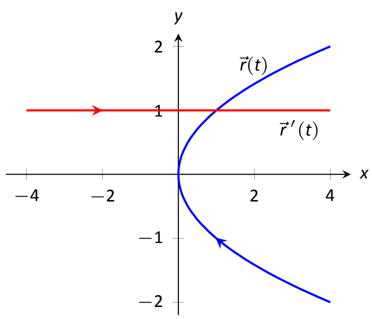
Let $\vec{r}(t) = \left\langle t^2, t \right\rangle$.

1. Sketch $\vec{r}(t)$ and $\vec{r}\,'(t)$ on the same axes.
2. Compute $\vec{r}\,'(1)$ and sketch this vector with its initial point at the origin and at $\vec{r}(1)$.

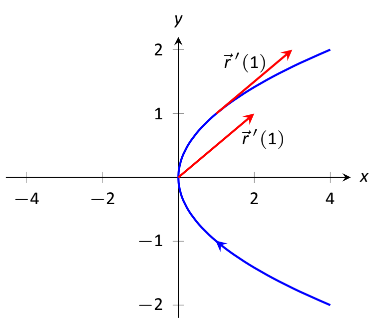
Solution

1. Theorem 93 allows us to compute derivatives component–wise, so
$$\vec{r}\,'(t) = \langle 2t, 1 \rangle .$$
$\vec{r}(t)$ and $\vec{r}\,'(t)$ are graphed together in Figure 11.9(a). Note how plotting the two of these together, in this way, is not very illuminating. When dealing with real–valued functions, plotting $f(x)$ with $f'(x)$ gave us useful information as we were able to compare f and f' at the same x-values. When dealing with vector–valued functions, it is hard to tell which points on the graph of $\vec{r}\,'$ correspond to which points on the graph of $\vec{r}$.

2. We easily compute $\vec{r}\,'(1) = \langle 2, 1 \rangle$, which is drawn in Figure 11.9 with its initial point at the origin, as well as at $\vec{r}(1) = \langle 1, 1 \rangle$. These are sketched in Figure 11.9(b).



(a)



(b)

Figure 11.9: Graphing the derivative of a vector–valued function in Example 11.9.

Notes:

Example 11.10 Derivatives of vector–valued functions

Let $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$. Compute $\vec{r}\,'(t)$ and $\vec{r}\,'(\pi/2)$. Sketch $\vec{r}\,'(\pi/2)$ with its initial point at the origin and at $\vec{r}(\pi/2)$.

Solution We compute $\vec{r}\,'$ as $\vec{r}\,'(t) = \langle -\sin t, \cos t, 1\rangle$. At $t = \pi/2$, we have $\vec{r}\,'(\pi/2) = \langle -1, 0, 1\rangle$. Figure 11.10 shows a graph of $\vec{r}(t)$, with $\vec{r}\,'(\pi/2)$ plotted with its initial point at the origin and at $\vec{r}(\pi/2)$.

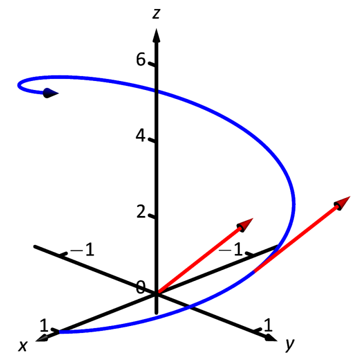


Figure 11.10: Viewing a vector–valued function and its derivative at one point.

In Examples 11.9 and 11.10, sketching a particular derivative with its initial point at the origin did not seem to reveal anything significant. However, when we sketched the vector with its initial point on the corresponding point on the graph, we did see something significant: the vector appeared to be *tangent* to the graph. We have not yet defined what "tangent" means in terms of curves in space; in fact, we use the derivative to define this term.

Definition 75 Tangent Vector, Tangent Line

Let $\vec{r}(t)$ be a differentiable vector–valued function on an open interval I containing c, where $\vec{r}\,'(c) \neq \vec{0}$.

1. A vector $\vec{v}$ is **tangent to the graph of** $\vec{r}(t)$ **at** $t = c$ if $\vec{v}$ is parallel to $\vec{r}\,'(c)$.
2. The **tangent line** to the graph of $\vec{r}(t)$ at $t = c$ is the line through $\vec{r}(c)$ with direction parallel to $\vec{r}\,'(c)$. An equation of the tangent line is
$$\vec{\ell}(t) = \vec{r}(c) + t\,\vec{r}\,'(c).$$

Example 11.11 Finding tangent lines to curves in space

Let $\vec{r}(t) = \left\langle t, t^2, t^3\right\rangle$ on $[-1.5, 1.5]$. Find the vector equation of the line tangent to the graph of $\vec{r}$ at $t = -1$.

Solution To find the equation of a line, we need a point on the line and the line's direction. The point is given by $\vec{r}(-1) = \langle -1, 1, -1\rangle$. (To be clear, $\langle -1, 1, -1\rangle$ is a *vector*, not a point, but we use the point "pointed to" by this vector.)

The direction comes from $\vec{r}\,'(-1)$. We compute, component–wise, $\vec{r}\,'(t) = \left\langle 1, 2t, 3t^2\right\rangle$. Thus $\vec{r}\,'(-1) = \langle 1, -2, 3\rangle$.

The vector equation of the line is $\ell(t) = \langle -1, 1, -1\rangle + t\,\langle 1, -2, 3\rangle$. This line and $\vec{r}(t)$ are sketched in Figure 11.11.

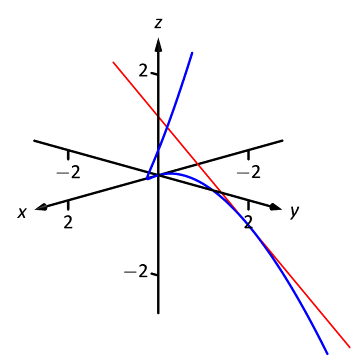


Figure 11.11: Graphing a curve in space with its tangent line.

Example 11.12 Finding tangent lines to curves

Notes:

Find the equations of the lines tangent to $\vec{r}(t) = \langle t^3, t^2 \rangle$ at $t = -1$ and $t = 0$.

Solution We find that $\vec{r}\,'(t) = \langle 3t^2, 2t \rangle$. At $t = -1$, we have

$$\vec{r}(-1) = \langle -1, 1 \rangle \quad \text{and} \quad \vec{r}\,'(-1) = \langle 3, -2 \rangle,$$

so the equation of the line tangent to the graph of $\vec{r}(t)$ at $t = -1$ is

$$\ell(t) = \langle -1, 1 \rangle + t \langle 3, -2 \rangle.$$

This line is graphed with $\vec{r}(t)$ in Figure 11.12.

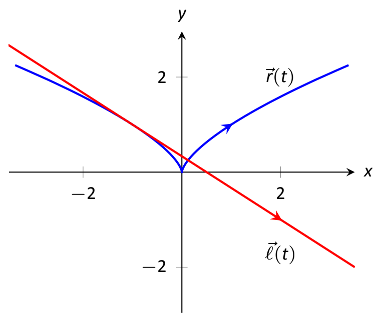


Figure 11.12: Graphing $\vec{r}(t)$ and its tangent line in Example 11.12.

At $t = 0$, we have $\vec{r}\,'(0) = \langle 0, 0 \rangle = \vec{0}$! This implies that the tangent line "has no direction." We cannot apply Definition 75, hence cannot find the equation of the tangent line.

We were unable to compute the equation of the tangent line to $\vec{r}(t) = \langle t^3, t^2 \rangle$ at $t = 0$ because $\vec{r}\,'(0) = \vec{0}$. The graph in Figure 11.12 shows that there is a cusp at this point. This leads us to another definition of **smooth**, previously defined by Definition 49 in Section 9.2.

Definition 76 Smooth Vector–Valued Functions

Let $\vec{r}(t)$ be a differentiable vector–valued function on an open interval I. $\vec{r}(t)$ is **smooth** on I if $\vec{r}\,'(t) \neq \vec{0}$ on I.

Having established derivatives of vector–valued functions, we now explore the relationships between the derivative and other vector operations. The following theorem states how the derivative interacts with vector addition and the various vector products.

Notes:

Theorem 94 Properies of Derivatives of Vector–Valued Functions

Let $\vec{r}$ and $\vec{s}$ be differentiable vector–valued functions, let f be a differentiable real–valued function, and let c be a real number.

1. $\dfrac{d}{dt}\Big(\vec{r}(t) \pm \vec{s}(t)\Big) = \vec{r}\,'(t) \pm \vec{s}\,'(t)$
2. $\dfrac{d}{dt}\Big(c\vec{r}(t)\Big) = c\vec{r}\,'(t)$
3. $\dfrac{d}{dt}\Big(f(t)\vec{r}(t)\Big) = f\,'(t)\vec{r}(t) + f(t)\vec{r}\,'(t)$ **Product Rule**
4. $\dfrac{d}{dt}\Big(\vec{r}(t)\cdot\vec{s}(t)\Big) = \vec{r}\,'(t)\cdot\vec{s}(t) + \vec{r}(t)\cdot\vec{s}\,'(t)$ **Product Rule**
5. $\dfrac{d}{dt}\Big(\vec{r}(t)\times\vec{s}(t)\Big) = \vec{r}\,'(t)\times\vec{s}(t) + \vec{r}(t)\times\vec{s}\,'(t)$ **Product Rule**
6. $\dfrac{d}{dt}\Big(\vec{r}\big(f(t)\big)\Big) = \vec{r}\,'\big(f(t)\big)f\,'(t)$ **Chain Rule**

Example 11.13 Using derivative properties of vector–valued functions

Let $\vec{r}(t) = \langle t, t^2 - 1\rangle$ and let $\vec{u}(t)$ be the unit vector that points in the direction of $\vec{r}(t)$.

1. Graph $\vec{r}(t)$ and $\vec{u}(t)$ on the same axes, on $[-2, 2]$.
2. Find $\vec{u}\,'(t)$ and sketch $\vec{u}\,'(-2)$, $\vec{u}\,'(-1)$ and $\vec{u}\,'(0)$. Sketch each with initial point the corresponding point on the graph of $\vec{u}$.

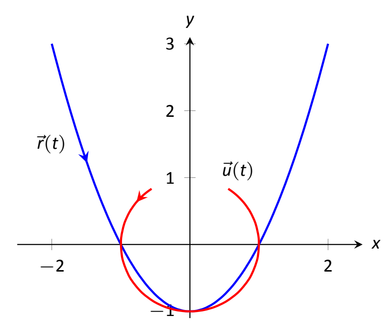


Figure 11.13: Graphing $\vec{r}(t)$ and $\vec{u}(t)$ in Example 11.13.

Solution

1. To form the unit vector that points in the direction of $\vec{r}$, we need to divide $\vec{r}(t)$ by its magnitude.

$$||\,\vec{r}(t)\,|| = \sqrt{t^2 + (t^2-1)^2} \quad \Rightarrow \quad \vec{u}(t) = \frac{1}{\sqrt{t^2+(t^2-1)^2}}\left\langle t, t^2-1\right\rangle.$$

$\vec{r}(t)$ and $\vec{u}(t)$ are graphed in Figure 11.13. Note how the graph of $\vec{u}(t)$ forms part of a circle; this must be the case, as the length of $\vec{u}(t)$ is 1 for all t.

Notes:

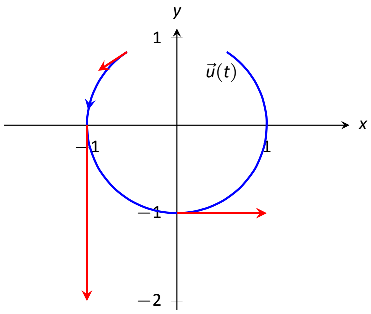


Figure 11.14: Graphing some of the derivatives of $\vec{u}(t)$ in Example 11.13.

2. To compute $\vec{u}\,'(t)$, we use Theorem 94, writing

$$\vec{u}(t) = f(t)\vec{r}(t), \quad \text{where} \quad f(t) = \frac{1}{\sqrt{t^2+(t^2-1)^2}} = \left(t^2+(t^2-1)^2\right)^{-1/2}.$$

(We *could* write

$$\vec{u}(t) = \left\langle \frac{t}{\sqrt{t^2+(t^2-1)^2}}, \frac{t^2-1}{\sqrt{t^2+(t^2-1)^2}} \right\rangle$$

and then take the derivative. It is a matter of preference; this latter method requires two applications of the Quotient Rule where our method uses the Product and Chain Rules.)

We find $f'(t)$ using the Chain Rule:

$$\begin{aligned} f'(t) &= -\frac{1}{2}\left(t^2+(t^2-1)^2\right)^{-3/2}\left(2t+2(t^2-1)(2t)\right) \\ &= -\frac{2t(2t^2-1)}{2\left(\sqrt{t^2+(t^2-1)^2}\,\right)^3} \end{aligned}$$

We now find $\vec{u}\,'(t)$ using part 3 of Theorem 94:

$$\begin{aligned} \vec{u}\,'(t) &= f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t) \\ &= -\frac{2t(2t^2-1)}{2\left(\sqrt{t^2+(t^2-1)^2}\,\right)^3}\left\langle t, t^2-1\right\rangle + \frac{1}{\sqrt{t^2+(t^2-1)^2}}\left\langle 1, 2t\right\rangle. \end{aligned}$$

This is admittedly very "messy;" such is usually the case when we deal with unit vectors. We can use this formula to compute $\vec{u}\,(-2)$, $\vec{u}\,(-1)$ and $\vec{u}\,(0)$:

$$\begin{aligned} \vec{u}\,(-2) &= \left\langle -\frac{15}{13\sqrt{13}}, -\frac{10}{13\sqrt{13}} \right\rangle \approx \langle -0.320, -0.213\rangle \\ \vec{u}\,(-1) &= \langle 0, -2\rangle \\ \vec{u}\,(0) &= \langle 1, 0\rangle \end{aligned}$$

Each of these is sketched in Figure 11.14. Note how the length of the vector gives an indication of how quickly the circle is being traced at that point. When $t = -2$, the circle is being drawn relatively slow; when $t = -1$, the circle is being traced much more quickly.

It is a basic geometric fact that a line tangent to a circle at a point P is perpendicular to the line passing through the center of the circle and P. This is

Notes:

illustrated in Figure 11.14; each tangent vector is perpendicular to the line that passes through its initial point and the center of the circle. Since the center of the circle is the origin, we can state this another way: $\vec{u}\,'(t)$ is orthogonal to $\vec{u}(t)$.

Recall that the dot product serves as a test for orthogonality: if $\vec{u} \cdot \vec{v} = 0$, then $\vec{u}$ is orthogonal to $\vec{v}$. Thus in the above example, $\vec{u}(t) \cdot \vec{u}\,'(t) = 0$.

This is true of any vector–valued function that has a constant length, that is, that traces out part of a circle. It has important implications later on, so we state it as a theorem (and leave its formal proof as an Exercise.)

Theorem 95 Vector–Valued Functions of Constant Length

Let $\vec{r}(t)$ be a differentiable vector–valued function on an open interval I of constant length. That is, $||\,\vec{r}(t)\,|| = c$ for all t in I (equivalently, $\vec{r}(t) \cdot \vec{r}(t) = c^2$ for all t in I). Then $\vec{r}(t) \cdot \vec{r}\,'(t) = 0$ for all t in I.

Integration

Indefinite and definite integrals of vector–valued functions are also evaluated component–wise.

Theorem 96 Indefinite and Definite Integrals of Vector–Valued Functions

Let $\vec{r}(t) = \langle f(t), g(t) \rangle$ be a vector–valued function in $\mathbb{R}^2$.

1. $\displaystyle \int \vec{r}(t)\ dt = \left\langle \int f(t)\ dt, \int g(t)\ dt \right\rangle$

2. $\displaystyle \int_a^b \vec{r}(t)\ dt = \left\langle \int_a^b f(t)\ dt, \int_a^b g(t)\ dt \right\rangle$

A similar statement holds for vector–valued functions in $\mathbb{R}^3$.

Example 11.14 Evaluating a definite integral of a vector–valued function

Let $\vec{r}(t) = \left\langle e^{2t}, \sin t \right\rangle$. Evaluate $\displaystyle \int_0^1 \vec{r}(t)\ dt$.

Notes:

SOLUTION We follow Theorem 96.

$$\begin{aligned}\int_0^1 \vec{r}(t)\,dt &= \int_0^1 \left\langle e^{2t}, \sin t\right\rangle\,dt\\ &= \left\langle \int_0^1 e^{2t}\,dt\,, \int_0^1 \sin t\,dt\right\rangle\\ &= \left\langle \frac{1}{2}e^{2t}\Big|_0^1\,, -\cos t\Big|_0^1\right\rangle\\ &= \left\langle \frac{1}{2}(e^2-1)\,, -\cos(1)+1\right\rangle\\ &\approx \langle 3.19, 0.460\rangle\,.\end{aligned}$$

Example 11.15 Solving an initial value problem

Let $\vec{r}\,''(t) = \langle 2, \cos t, 12t\rangle$. Find $\vec{r}(t)$ where:

- $\vec{r}(0) = \langle -7, -1, 2\rangle$ and
- $\vec{r}\,'(0) = \langle 5, 3, 0\rangle$.

SOLUTION Knowing $\vec{r}\,''(t) = \langle 2, \cos t, 12t\rangle$, we find $\vec{r}\,'(t)$ by evaluating the indefinite integral.

$$\begin{aligned}\int \vec{r}\,''(t)\,dt &= \left\langle \int 2\,dt\,, \int \cos t\,dt\,, \int 12t\,dt\right\rangle\\ &= \left\langle 2t + C_1, \sin t + C_2, 6t^2 + C_3\right\rangle\\ &= \left\langle 2t, \sin t, 6t^2\right\rangle + \langle C_1, C_2, C_3\rangle\\ &= \left\langle 2t, \sin t, 6t^2\right\rangle + \vec{C}.\end{aligned}$$

Note how each indefinite integral creates its own constant which we collect as one constant vector $\vec{C}$. Knowing $\vec{r}\,'(0) = \langle 5, 3, 0\rangle$ allows us to solve for $\vec{C}$:

$$\begin{aligned}\vec{r}\,'(t) &= \left\langle 2t, \sin t, 6t^2\right\rangle + \vec{C}\\ \vec{r}\,'(0) &= \langle 0, 0, 0\rangle + \vec{C}\\ \langle 5, 3, 0\rangle &= \vec{C}.\end{aligned}$$

So $\vec{r}\,'(t) = \left\langle 2t, \sin t, 6t^2\right\rangle + \langle 5, 3, 0\rangle = \left\langle 2t+5, \sin t + 3, 6t^2\right\rangle$. To find $\vec{r}(t)$, we integrate once more.

$$\begin{aligned}\int \vec{r}\,'(t)\,dt &= \left\langle \int 2t+5\,dt, \int \sin t + 3\,dt, \int 6t^2\,dt\right\rangle\\ &= \left\langle t^2 + 5t, -\cos t + 3t, 2t^3\right\rangle + \vec{C}.\end{aligned}$$

With $\vec{r}(0) = \langle -7, -1, 2\rangle$, we solve for $\vec{C}$:

Notes:

$$\begin{aligned}\vec{r}(t) &= \left\langle t^2+5t, -\cos t+3t, 2t^3\right\rangle + \vec{C}\\ \vec{r}(0) &= \langle 0,-1,0\rangle + \vec{C}\\ \langle -7,-1,2\rangle &= \langle 0,-1,0\rangle + \vec{C}\\ \langle -7,0,2\rangle &= \vec{C}.\end{aligned}$$

So $\vec{r}(t) = \left\langle t^2+5t, -\cos t+3t, 2t^3\right\rangle + \langle -7,0,2\rangle = \left\langle t^2+5t-7, -\cos t+3t, 2t^3+2\right\rangle$.

What does the integration of a vector–valued function *mean*? There are many applications, but none as direct as "the area under the curve" that we used in understanding the integral of a real–valued function.

A key understanding for us comes from considering the integral of a derivative:

$$\int_a^b \vec{r}\,'(t)\,dt = \vec{r}(t)\Big|_a^b = \vec{r}(b)-\vec{r}(a).$$

Integrating a *rate of change* function gives *displacement*.

Noting that vector–valued functions are closely related to parametric equations, we can describe the arc length of the graph of a vector–valued function as an integral. Given parametric equations $x=f(t)$, $y=g(t)$, the arc length on $[a,b]$ of the graph is

$$\text{Arc Length} = \int_a^b \sqrt{f\,'(t)^2+g\,'(t)^2}\,dt,$$

as stated in Theorem 84 in Section 9.3. If $\vec{r}(t) = \langle f(t), g(t)\rangle$, note that $\sqrt{f\,'(t)^2+g\,'(t)^2} = ||\ \vec{r}\,'(t)\ ||$. Therefore we can express the arc length of the graph of a vector–valued function as an integral of the magnitude of its derivative.

Theorem 97 Arc Length of a Vector–Valued Function

Let $\vec{r}(t)$ be a vector–valued function where $\vec{r}\,'(t)$ is continuous on $[a,b]$. The arc length L of the graph of $\vec{r}(t)$ is

$$L = \int_a^b ||\ \vec{r}\,'(t)\ ||\,dt.$$

Note that we are actually integrating a scalar–function here, not a vector–valued function.

The next section takes what we have established thus far and applies it to objects in motion. We will let $\vec{r}(t)$ describe the path of an object in the plane or in space and will discover the information provided by $\vec{r}\,'(t)$ and $\vec{r}\,''(t)$.

Notes:

Exercises 11.2

Terms and Concepts

1. Limits, derivatives and integrals of vector–valued functions are all evaluated ________–wise.

2. The definite integral of a rate of change function gives ________.

3. Why is it generally not useful to graph both $\vec{r}(t)$ and $\vec{r}\,'(t)$ on the same axes?

Problems

In Exercises 4 – 7, evaluate the given limit.

4. $\displaystyle\lim_{t\to 5} \left\langle 2t+1, 3t^2-1, \sin t\right\rangle$

5. $\displaystyle\lim_{t\to 3} \left\langle e^t, \frac{t^2-9}{t+3}\right\rangle$

6. $\displaystyle\lim_{t\to 0} \left\langle \frac{t}{\sin t}, (1+t)^{\frac{1}{t}}\right\rangle$

7. $\displaystyle\lim_{h\to 0} \frac{\vec{r}(t+h)-\vec{r}(t)}{h}$, where $\vec{r}(t) = \left\langle t^2, t, 1\right\rangle$.

In Exercises 8 – 9, identify the interval(s) on which $\vec{r}(t)$ is continuous.

8. $\vec{r}(t) = \left\langle t^2, 1/t\right\rangle$

9. $\vec{r}(t) = \left\langle \cos t, e^t, \ln t\right\rangle$

In Exercises 10 – 14, find the derivative of the given function.

10. $\vec{r}(t) = \left\langle \cos t, e^t, \ln t\right\rangle$

11. $\vec{r}(t) = \left\langle \frac{1}{t}, \frac{2t-1}{3t+1}, \tan t\right\rangle$

12. $\vec{r}(t) = (t^2)\left\langle \sin t, 2t+5\right\rangle$

13. $\vec{r}(t) = \left\langle t^2+1, t-1\right\rangle \cdot \left\langle \sin t, 2t+5\right\rangle$

14. $\vec{r}(t) = \left\langle t^2+1, t-1, 1\right\rangle \times \left\langle \sin t, 2t+5, 1\right\rangle$

In Exercises 15 – 18, find $\vec{r}\,'(t)$. Sketch $\vec{r}(t)$ and $\vec{r}\,'(1)$, with the initial point of $\vec{r}\,'(1)$ at $\vec{r}(1)$.

15. $\vec{r}(t) = \left\langle t^2+t, t^2-t\right\rangle$

16. $\vec{r}(t) = \left\langle t^2-2t+2, t^3-3t^2+2t\right\rangle$

17. $\vec{r}(t) = \left\langle t^2+1, t^3-t\right\rangle$

18. $\vec{r}(t) = \left\langle t^2-4t+5, , t^3-6t^2+11t-6\right\rangle$

In Exercises 19 – 22, give the equation of the line tangent to the graph of $\vec{r}(t)$ at the given t value.

19. $\vec{r}(t) = \left\langle t^2+t, t^2-t\right\rangle$ at $t=1$.

20. $\vec{r}(t) = \left\langle 3\cos t, \sin t\right\rangle$ at $t=\pi/4$.

21. $\vec{r}(t) = \left\langle 3\cos t, 3\sin t, t\right\rangle$ at $t=\pi$.

22. $\vec{r}(t) = \left\langle e^t, \tan t, t\right\rangle$ at $t=0$.

In Exercises 23 – 26, find the value(s) of t for which $\vec{r}(t)$ is not smooth.

23. $\vec{r}(t) = \left\langle \cos t, \sin t - t\right\rangle$

24. $\vec{r}(t) = \left\langle t^2-2t+1, t^3+t^2-5t+3\right\rangle$

25. $\vec{r}(t) = \left\langle \cos t - \sin t, \sin t - \cos t, \cos(4t)\right\rangle$

26. $\vec{r}(t) = \left\langle t^3-3t+2, -\cos(\pi t), \sin^2(\pi t)\right\rangle$

Exercises 27 – 29 ask you to verify parts of Theorem 94. In each let $f(t) = t^3$, $\vec{r}(t) = \left\langle t^2, t-1, 1\right\rangle$ and $\vec{s}(t) = \left\langle \sin t, e^t, t\right\rangle$. Compute the various derivatives as indicated.

27. Simplify $f(t)\vec{r}(t)$, then find its derivative; show this is the same as $f\,'(t)\vec{r}(t) + f(t)\vec{r}\,'(t)$.

28. Simplify $\vec{r}(t)\cdot\vec{s}(t)$, then find its derivative; show this is the same as $\vec{r}\,'(t)\cdot\vec{s}(t) + \vec{r}(t)\cdot\vec{s}\,'(t)$.

29. Simplify $\vec{r}(t)\times\vec{s}(t)$, then find its derivative; show this is the same as $\vec{r}\,'(t)\times\vec{s}(t) + \vec{r}(t)\times\vec{s}\,'(t)$.

In Exercises 30 – 33 , evaluate the given definite or indefinite integral.

30. $\displaystyle\int \left\langle t^3, \cos t, te^t\right\rangle\,dt$

31. $\displaystyle\int \left\langle \frac{1}{1+t^2}, \sec^2 t\right\rangle\,dt$

32. $\displaystyle\int_0^{\pi} \left\langle -\sin t, \cos t\right\rangle\,dt$

33. $\displaystyle\int_{-2}^{2} \left\langle 2t+1, 2t-1\right\rangle\,dt$

In Exercises 34 – 37 , solve the given initial value problems.

34. Find $\vec{r}(t)$, given that $\vec{r}\,'(t) = \left\langle t, \sin t\right\rangle$ and $\vec{r}(0) = \left\langle 2, 2\right\rangle$.

35. Find $\vec{r}(t)$, given that $\vec{r}\,'(t) = \left\langle 1/(t+1), \tan t\right\rangle$ and $\vec{r}(0) = \left\langle 1, 2\right\rangle$.

36. Find $\vec{r}(t)$, given that $\vec{r}\,''(t) = \left\langle t^2, t, 1 \right\rangle$, $\vec{r}\,'(0) = \langle 1, 2, 3\rangle$ and $\vec{r}(0) = \langle 4, 5, 6\rangle$.

37. Find $\vec{r}(t)$, given that $\vec{r}\,''(t) = \left\langle \cos t, \sin t, e^t \right\rangle$, $\vec{r}\,'(0) = \langle 0, 0, 0\rangle$ and $\vec{r}(0) = \langle 0, 0, 0\rangle$.

In Exercises 38 – 41 , find the arc length of $\vec{r}(t)$ on the indicated interval.

38. $\vec{r}(t) = \langle 2\cos t, 2\sin t, 3t\rangle$ on $[0, 2\pi]$.

39. $\vec{r}(t) = \langle 5\cos t, 3\sin t, 4\sin t\rangle$ on $[0, 2\pi]$.

40. $\vec{r}(t) = \left\langle t^3, t^2, t^3 \right\rangle$ on $[0, 1]$.

41. $\vec{r}(t) = \left\langle e^{-t}\cos t, e^{-t}\sin t \right\rangle$ on $[0, 1]$.

42. Prove Theorem 95; that is, show if $\vec{r}(t)$ has constant length and is differentiable, then $\vec{r}(t)\cdot\vec{r}\,'(t) = 0$. (Hint: use the Product Rule to compute $\frac{d}{dt}\big(\vec{r}(t)\cdot\vec{r}(t)\big)$.)

11.3 The Calculus of Motion

A common use of vector–valued functions is to describe the motion of an object in the plane or in space. A **position function** $\vec{r}(t)$ gives the position of an object at **time** t. This section explores how derivatives and integrals are used to study the motion described by such a function.

Definition 77 Velocity, Speed and Acceleration

Let $\vec{r}(t)$ be a position function in $\mathbb{R}^2$ or $\mathbb{R}^3$.

1. **Velocity**, denoted $\vec{v}(t)$, is the instantaneous rate of position change; that is, $\vec{v}(t) = \vec{r}\,'(t)$.
2. **Speed** is the magnitude of velocity, $\|\,\vec{v}(t)\,\|$.
3. **Acceleration**, denoted $\vec{a}(t)$, is the instantaneous rate of velocity change; that is, $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$.

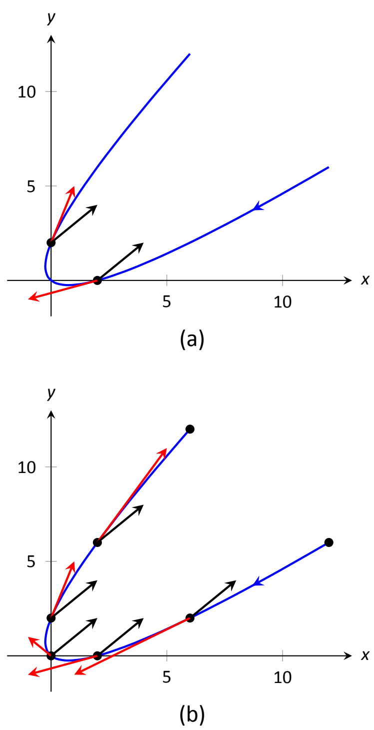


Figure 11.15: Graphing the position, velocity and acceleration of an object in Example 11.16.

Example 11.16 Finding velocity and acceleration

An object is moving with position function $\vec{r}(t) = \left\langle t^2 - t, t^2 + t \right\rangle$, $-3 \le t \le 3$, where distances are measured in feet and time is measured in seconds.

1. Find $\vec{v}(t)$ and $\vec{a}(t)$.
2. Sketch $\vec{r}(t)$; plot $\vec{v}(-1)$, $\vec{a}(-1)$, $\vec{v}(1)$ and $\vec{a}(1)$, each with their initial point at their corresponding point on the graph of $\vec{r}(t)$.
3. When is the object's speed minimized?

Solution

1. Taking derivatives, we find

$$\vec{v}(t) = \vec{r}\,'(t) = \langle 2t-1, 2t+1 \rangle \quad \text{and} \quad \vec{a}(t) = \vec{r}\,''(t) = \langle 2, 2 \rangle .$$

   Note that acceleration is constant.

2. $\vec{v}(-1) = \langle -3, -1 \rangle$, $\vec{a}(-1) = \langle 2, 2 \rangle$; $\vec{v}(1) = \langle 1, 3 \rangle$, $\vec{a}(1) = \langle 2, 2 \rangle$. These are plotted with $\vec{r}(t)$ in Figure 11.15(a).

   We can think of acceleration as "pulling" the velocity vector in a certain direction. At $t = -1$, the velocity vector points down and to the left; at $t = 1$, the velocity vector has been pulled in the $\langle 2, 2 \rangle$ direction and is

Notes:

now pointing up and to the right. In Figure 11.15(b) we plot more velocity/acceleration vectors, making more clear the effect acceleration has on velocity.

Since $\vec{a}(t)$ is constant in this example, as t grows large $\vec{v}(t)$ becomes almost parallel to $\vec{a}(t)$. For instance, when $t = 10$, $\vec{v}(10) = \langle 19, 21\rangle$, which is nearly parallel to $\langle 2, 2\rangle$.

3. The object's speed is given by

$$|| \, \vec{v}(t) \, || = \sqrt{(2t-1)^2 + (2t+1)^2} = \sqrt{8t^2+2}.$$

To find the minimal speed, we could apply calculus techniques (such as set the derivative equal to 0 and solve for t, etc.) but we can find it by inspection. Inside the square root we have a quadratic which is minimized when $t = 0$. Thus the speed is minimized at $t = 0$, with a speed of $\sqrt{2}$ ft/s.

The graph in Figure 11.15(b) also implies speed is minimized here. The filled dots on the graph are located at integer values of t between -3 and 3. Dots that are far apart imply the object traveled a far distance in 1 second, indicating high speed; dots that are close together imply the object did not travel far in 1 second, indicating a low speed. The dots are closest together near $t = 0$, implying the speed is minimized near that value.

Example 11.17 Analyzing Motion

Two objects follow an identical path at different rates on $[-1, 1]$. The position function for Object 1 is $\vec{r}_1(t) = \langle t, t^2\rangle$; the position function for Object 2 is $\vec{r}_2(t) = \langle t^3, t^6\rangle$, where distances are measured in feet and time is measured in seconds. Compare the velocity, speed and acceleration of the two objects on the path.

Solution We begin by computing the velocity and acceleration function for each object:

$$\vec{v}_1(t) = \langle 1, 2t\rangle \qquad \vec{v}_2(t) = \langle 3t^2, 6t^5\rangle$$
$$\vec{a}_1(t) = \langle 0, 2\rangle \qquad \vec{a}_2(t) = \langle 6t, 30t^4\rangle$$

We immediately see that Object 1 has constant acceleration, whereas Object 2 does not.

At $t = -1$, we have $\vec{v}_1(-1) = \langle 1, -2\rangle$ and $\vec{v}_2(-1) = \langle 3, -6\rangle$; the velocity of Object 2 is three times that of Object 1 and so it follows that the speed of Object 2 is three times that of Object 1 ($3\sqrt{5}$ ft/s compared to $\sqrt{5}$ ft/s.)

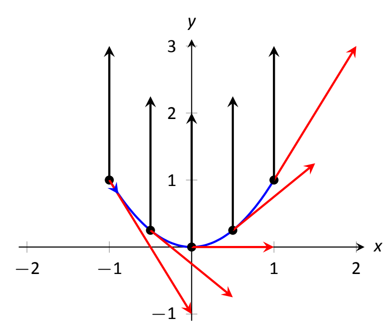


Figure 11.16: Plotting velocity and acceleration vectors for Object 1 in Example 11.17.

Notes:

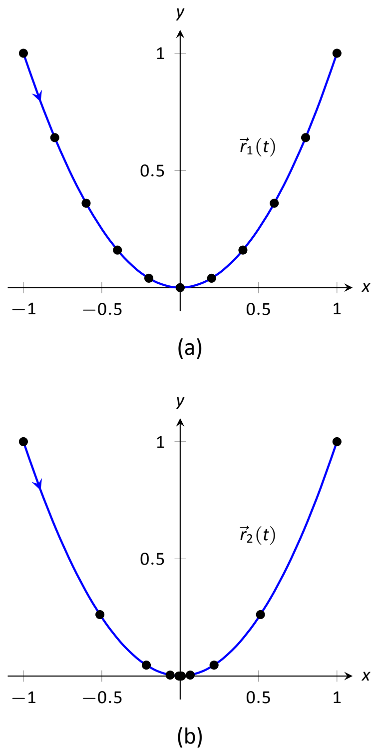


Figure 11.17: Comparing the positions of Objects 1 and 2 in Example 11.17.

At $t = 0$, the velocity of Object 1 is $\vec{v}(1) = \langle 1, 0\rangle$ and the velocity of Object 2 is $\vec{0}$! This tells us that Object 2 comes to a complete stop at $t = 0$.

In Figure 11.16, we see the velocity and acceleration vectors for Object 1 plotted for $t = -1, -1/2, 0, 1/2$ and $t = 1$. Note again how the constant acceleration vector seems to "pull" the velocity vector from pointing down, right to up, right. We could plot the analogous picture for Object 2, but the velocity and acceleration vectors are rather large ($\vec{a}_2(-1) = \langle -6, 30\rangle$!)

Instead, we simply plot the locations of Object 1 and 2 on intervals of $1/10^{\text{th}}$ of a second, shown in Figure 11.17(a) and (b). Note how the x-values of Object 1 increase at a steady rate. This is because the x-component of $\vec{a}(t)$ is 0; there is no acceleration in the x-component. The dots are not evenly spaced; the object is moving faster near $t = -1$ and $t = 1$ than near $t = 0$.

In part (b) of the Figure, we see the points plotted for Object 2. Note the large change in position from $t = -1$ to $t = -0.9$; the object starts moving very quickly. However, it slows considerably at it approaches the origin, and comes to a complete stop at $t = 0$. While it looks like there are 3 points near the origin, there are in reality 5 points there.

Since the objects begin and end at the same location, the have the same displacement. Since they begin and end at the same time, with the same displacement, they have they have the same average rate of change (i.e, they have the same average velocity). Since they follow the same path, they have the same distance traveled. Even though these three measurements are the same, the objects obviously travel the path in very different ways.

Example 11.18 Analyzing the motion of a whirling ball on a string

A young boy whirls a ball, attached to a string, above his head in a counterclockwise circle. The ball follows a circular path and makes 2 revolutions per second. The string has length 2ft.

1. Find the position function $\vec{r}(t)$ that describes this situation.
2. Find the acceleration of the ball and derive a physical interpretation of it.
3. A tree stands 10ft in front of the boy. At what t-values should the boy release the string so that the ball hits the tree?

Solution

1. The ball whirls in a circle. Since the string is 2ft long, the radius of the circle is 2. The position function $\vec{r}(t) = \langle 2\cos t, 2\sin t\rangle$ describes a circle with radius 2, centered at the origin, but makes a full revolution every 2π seconds, not two revolutions per second. We modify the period of the

Notes:

trigonometric functions to be 1/2 by multiplying t by 4π. The final position function is thus

$$\vec{r}(t) = \langle 2\cos(4\pi t), 2\sin(4\pi t)\rangle .$$

(Plot this for $0 \leq t \leq 1/2$ to verify that one revolution is made in 1/2 a second.)

2. To find $\vec{a}(t)$, we derive $\vec{r}(t)$ twice.

$$\begin{aligned}\vec{v}(t) = \vec{r}\,'(t) &= \langle -8\pi\sin(4\pi t), 8\pi\cos(4\pi t)\rangle \\ \vec{a}(t) = \vec{r}\,''(t) &= \left\langle -32\pi^2\cos(4\pi t), -32\pi^2\sin(4\pi t)\right\rangle \\ &= -32\pi^2\langle\cos(4\pi t), \sin(4\pi t)\rangle .\end{aligned}$$

Note how $\vec{a}(t)$ is parallel to $\vec{r}(t)$, but has a different magnitude and points in the opposite direction. Why is this?

Recall the classic physics equation, "Force = mass × acceleration." A force acting on a mass induces acceleration (i.e., the mass moves); acceleration acting on a mass induces a force (gravity gives our mass a *weight*). Thus force and acceleration are closely related. A moving ball "wants" to travel in a straight line. Why does the ball in our example move in a circle? It is attached to the boy's hand by a string. The string applies a force to the ball, affecting it's motion: the string *accelerates* the ball. This is not acceleration in the sense of "it travels faster;" rather, this acceleration is changing the velocity of the ball. In what direction is this force/acceleration being applied? In the direction of the string, towards the boy's hand.

The magnitude of the acceleration is related to the speed at which the ball is traveling. A ball whirling quickly is rapidly changing direction/velocity. When velocity is changing rapidly, the acceleration must be "large."

3. When the boy releases the string, the string no longer applies a force to the ball, meaning acceleration is $\vec{0}$ and the ball can now move in a straight line in the direction of $\vec{v}(t)$.

Let $t = t_0$ be the time when the boy lets go of the string. The ball will be at $\vec{r}(t_0)$, traveling in the direction of $\vec{v}(t_0)$. We want to find t_0 so that this line contains the point $(0, 10)$ (since the tree is 10ft directly in front of the boy).

There are many ways to find this time value. We choose one that is relatively simple computationally. As shown in Figure 11.18, the vector from the release point to the tree is $\langle 0, 10\rangle - \vec{r}(t_0)$. This line segment is tangent to the circle, which means it is also perpendicular to $\vec{r}(t_0)$ itself, so their dot product is 0.

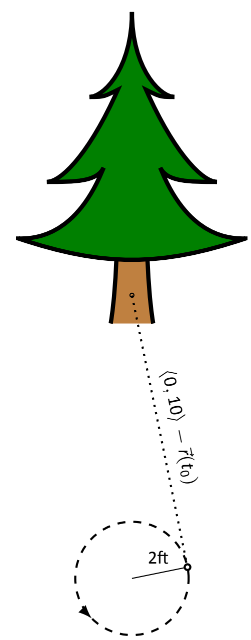


Figure 11.18: Modeling the flight of a ball in Example 11.18.

Notes:

$$\begin{aligned}
\vec{r}(t_0)\cdot\big(\langle 0,10\rangle - \vec{r}(t_0)\big) &= 0\\
\langle 2\cos(4\pi t_0), 2\sin(4\pi t_0)\rangle\cdot\langle -2\cos(4\pi t_0), 10-2\sin(4\pi t_0)\rangle &= 0\\
-4\cos^2(4\pi t_0)+20\sin(4\pi t_0)-4\sin^2(4\pi t_0) &= 0\\
20\sin(4\pi t_0)-4 &= 0\\
\sin(4\pi t_0) &= 1/5\\
4\pi t_0 &= \sin^{-1}(1/5)\\
4\pi t_0 &\approx 0.2+2\pi n,
\end{aligned}$$

where n is an integer. Solving for t_0 we have:

$$t_0 \approx 0.016 + n/2$$

This is a wonderful formula. Every 1/2 second after $t = 0.016$s the boy can release the string (since the ball makes 2 revolutions per second, he has two chances each second to release the ball).

Example 11.19 Analyzing motion in space

An object moves in a spiral with position function $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$, where distances are measured in meters and time is in minutes. Describe the object's speed and acceleration at time t.

Solution With $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$, we have:

$$\begin{aligned}
\vec{v}(t) &= \langle -\sin t, \cos t, 1\rangle \quad \text{and}\\
\vec{a}(t) &= \langle -\cos t, -\sin t, 0\rangle .
\end{aligned}$$

The speed of the object is $||\,\vec{v}(t)\,|| = \sqrt{(-\sin t)^2+\cos^2 t+1} = \sqrt{2}$m/min; it moves at a constant speed. Note that the object does not accelerate in the z-direction, but rather moves up at a constant rate of 1m/min.

The objects in Examples 11.18 and 11.19 traveled at a constant speed. That is, $||\,\vec{v}(t)\,|| = c$ for some constant c. Recall Theorem 95, which states that if a vector–valued function $\vec{r}(t)$ has constant length, then $\vec{r}(t)$ is perpendicular to its derivative: $\vec{r}(t)\cdot\vec{r}\,'(t) = 0$. In these examples, the velocity function has constant length, therefore we can conclude that the velocity is perpendicular to the acceleration: $\vec{v}(t)\cdot\vec{a}(t) = 0$. A quick check verifies this.

There is an intuitive understanding of this. If acceleration is parallel to velocity, then it is only affecting the object's speed; it does not change the direction of travel. (For example, consider a dropped stone. Acceleration and velocity are

Notes:

parallel – straight down – and the direction of velocity never changes, though speed does increase.) If acceleration is not perpendicular to velocity, then there is some acceleration in the direction of travel, influencing the speed. If speed is constant, then acceleration must be orthogonal to velocity, as it then only affects direction, and not speed.

Key Idea 53 Objects With Constant Speed

If an object moves with constant speed, then its velocity and acceleration vectors are orthogonal. That is, $\vec{v}(t) \cdot \vec{a}(t) = 0$.

Projectile Motion

An important application of vector–valued position functions is *projectile motion*: the motion of objects under only the influence of gravity. We will measure time in seconds, and distances will either be in meters or feet. We will show that we can completely describe the path of such an object knowing its initial position and initial velocity (i.e., where it *is* and where it *is going.*)

Suppose an object has initial position $\vec{r}(0) = \langle x_0, y_0 \rangle$ and initial velocity $\vec{v}(0) = \langle v_x, v_y \rangle$. It is customary to rewrite $\vec{v}(0)$ in terms of its speed v_0 and direction $\vec{u}$, where $\vec{u}$ is a unit vector. Recall all unit vectors in $\mathbb{R}^2$ can be written as $\langle \cos\theta, \sin\theta \rangle$, where θ is an angle measure counter–clockwise from the x-axis. (We refer to θ as the **angle of elevation.**) Thus $\vec{v}(0) = v_0 \langle \cos\theta, \sin\theta \rangle$.

Note: In this text we use $g = 32$ft/s when using Imperial units, and $g = 9.8$m/s when using SI units.

Since the acceleration of the object is known, namely $\vec{a}(t) = \langle 0, -g \rangle$, where g is the gravitational constant, we can find $\vec{r}(t)$ knowing our two initial conditions. We first find $\vec{v}(t)$:

$$\begin{aligned}
\vec{v}(t) &= \int \vec{a}(t)\, dt \\
\vec{v}(t) &= \int \langle 0, -g \rangle\ dt \\
\vec{v}(t) &= \langle 0, -gt \rangle + \vec{C}.
\end{aligned}$$

Knowing $\vec{v}(0) = v_0 \langle \cos\theta, \sin\theta \rangle$, we have $\vec{C} = v_0 \langle \cos t, \sin t \rangle$ and so

$$\vec{v}(t) = \left\langle v_0 \cos\theta, -gt + v_0 \sin\theta \right\rangle.$$

Notes:

We integrate once more to find $\vec{r}(t)$:

$$\begin{aligned}
\vec{r}(t) &= \int \vec{v}(t)\,dt \\
\vec{r}(t) &= \int \left\langle v_0\cos\theta, -gt + v_0\sin\theta\right\rangle\,dt \\
\vec{r}(t) &= \left\langle \big(v_0\cos\theta\big)t, -\frac{1}{2}gt^2 + \big(v_0\sin\theta\big)t\right\rangle + \vec{C}.
\end{aligned}$$

Knowing $\vec{r}(0) = \langle x_0, y_0\rangle$, we conclude $\vec{C} = \langle x_0, y_0\rangle$ and

$$\vec{r}(t) = \left\langle \big(v_0\cos\theta\big)t + x_0\,, -\frac{1}{2}gt^2 + \big(v_0\sin\theta\big)t + y_0\,\right\rangle.$$

Key Idea 54 Projectile Motion

The position function of a projectile propelled from an initial position of $\vec{r}_0 = \langle x_0, y_0\rangle$, with initial speed v_0, with angle of elevation θ and neglecting all accelerations but gravity is

$$\vec{r}(t) = \left\langle \big(v_0\cos\theta\big)t + x_0\,, -\frac{1}{2}gt^2 + \big(v_0\sin\theta\big)t + y_0\,\right\rangle.$$

Letting $\vec{v}_0 = v_0\,\langle\cos\theta, \sin\theta\rangle$, $\vec{r}(t)$ can be written as

$$\vec{r}(t) = \left\langle 0, -\frac{1}{2}gt^2\right\rangle + \vec{v}_0 t + \vec{r}_0.$$

We demonstrate how to use this position function in the next two examples.

Example 11.20 Projectile Motion

Sydney shoots her Red Ryder® bb gun across level ground from an elevation of 4ft, where the barrel of the gun makes a 5° angle with the horizontal. Find how far the bb travels before landing, assuming the bb is fired at the advertised rate of 350ft/s and ignoring air resistance.

Solution A direct application of Key Idea 54 gives

$$\begin{aligned}
\vec{r}(t) &= \left\langle (350\cos 5^\circ)t, -16t^2 + (350\sin 5^\circ)t + 4\right\rangle \\
&\approx \left\langle 346.67t, -16t^2 + 30.50t + 4\right\rangle,
\end{aligned}$$

Notes:

where we set her initial position to be $\langle 0, 4\rangle$. We need to find *when* the bb lands, then we can find *where*. We accomplish this by setting the y-component equal to 0 and solving for t:

$$\begin{aligned} -16t^2 + 30.50t + 4 &= 0 \\ t &= \frac{-30.50 \pm \sqrt{30.50^2 - 4(-16)(4)}}{-32} \\ t &\approx 2.03s. \end{aligned}$$

(We discarded a negative solution that resulted from our quadratic equation.)

We have found that the bb lands 2.03s after firing; with $t = 2.03$, we find the x-component of our position function is $346.67(2.03) = 703.74$ft. The bb lands about 704 feet away.

Example 11.21 Projectile Motion

Alex holds his sister's bb gun at a height of 3ft and wants to shoot a target that is 6ft above the ground, 25ft away. At what angle should he hold the gun to hit his target? (We still assume the muzzle velocity is 350ft/s.)

Solution The position function for the path of Alex's bb is

$$\vec{r}(t) = \left\langle (350\cos\theta)t, -16t^2 + (350\sin\theta)t + 3\right\rangle .$$

We need to find θ so that $\vec{r}(t) = \langle 25, 6\rangle$ for some value of t. That is, we want to find θ and t such that

$$(350\cos\theta)t = 25 \quad \text{and} \quad -16t^2 + (350\sin\theta)t + 3 = 6.$$

This is not trivial (though not "hard"). We start by solving each equation for $\cos\theta$ and $\sin\theta$, respectively.

$$\cos\theta = \frac{25}{350t} \quad \text{and} \quad \sin\theta = \frac{3 + 16t^2}{350t}.$$

Using the Pythagorean Identity $\cos^2\theta + \sin^2\theta = 1$, we have

$$\left(\frac{25}{350t}\right)^2 + \left(\frac{3+16t^2}{350t}\right)^2 = 1$$

Multiply both sides by $(350t)^2$:

$$\begin{aligned} 25^2 + (3 + 16t^2)^2 &= 350^2t^2 \\ 256t^4 - 122,404t^2 + 634 &= 0. \end{aligned}$$

Notes:

This is a quadratic *in* t^2. That is, we can apply the quadratic formula to find t^2, then solve for t itself.

$$\begin{aligned} t^2 &= \frac{122,404 \pm \sqrt{122,404^2 - 4(256)(634)}}{512} \\ t^2 &= 0.0052,\ 478.135 \\ t &= \pm 0.072,\ \pm 21.866 \end{aligned}$$

Clearly the negative t values do not fit our context, so we have $t = 0.072$ and $t = 21.866$. Using $\cos\theta = 25/(350t)$, we can solve for θ:

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{25}{350 \cdot 0.072}\right) \quad \text{and} \quad \cos^{-1}\left(\frac{25}{350 \cdot 21.866}\right) \\ \theta &= 7.03^\circ \quad \text{and} \quad 89.8^\circ. \end{aligned}$$

Alex has two choices of angle. He can hold the rifle at an angle of about 7° with the horizontal and hit his target 0.07s after firing, or he can hold his rifle almost straight up, with an angle of 89.8°, where he'll hit his target about 22s later. The first option is clearly the option he should choose.

Distance Traveled

Consider a driver who sets her cruise–control to 60mph, and travels at this speed for an hour. We can ask:

1. How far did the driver travel?
2. How far from her starting position is the driver?

The first is easy to answer: she traveled 60 miles. The second is impossible to answer with the given information. We do not know if she traveled in a straight line, on an oval racetrack, or along a slowly–winding highway.

This highlights an important fact: to compute distance traveled, we need only to know the speed, given by $||\ \vec{v}(t)\ ||$.

Theorem 98 Distance Traveled

Let $\vec{v}(t)$ be a velocity function for a moving object. The distance traveled by the object on $[a, b]$ is:

$$\text{distance traveled} = \int_a^b ||\ \vec{v}(t)\ ||\ dt.$$

Note that this is just a restatement of Theorem 97: arc length is the same as distance traveled, just viewed in a different context.

Notes:

Example 11.22 Distance Traveled, Displacement, and Average Speed

A particle moves in space with position function $\vec{r}(t) = \langle t, t^2, \sin(\pi t)\rangle$ on $[-2, 2]$, where t is measured in seconds and distances are in meters. Find:

1. The distance traveled by the particle on $[-2, 2]$.
2. The displacement of the particle on $[-2, 2]$.
3. The particle's average speed.

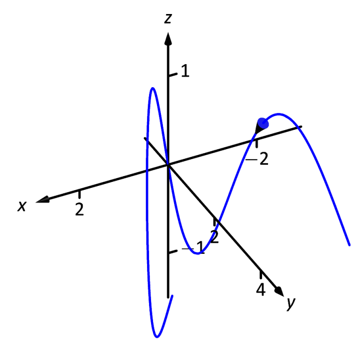


Figure 11.19: The path of the particle in Example 11.22.

SOLUTION

1. We use Theorem 98 to establish the integral:

$$\begin{aligned}\text{distance traveled} &= \int_{-2}^{2} ||\,\vec{v}(t)\,||\,dt\\ &= \int_{-2}^{2}\sqrt{1+(2t)^2+\pi^2\cos^2(\pi t)}\,dt.\end{aligned}$$

   This cannot be solved in terms of elementary functions so we turn to numerical integration, finding the distance to be 12.88m.

2. The displacement is the vector

$$\vec{r}(2) - \vec{r}(-2) = \langle 2, 4, 0\rangle - \langle -2, 4, 0\rangle = \langle 4, 0, 0\rangle\,.$$

   That is, the particle ends with an x-value increased by 4 and with y- and z-values the same (see Figure 11.19).

3. We found above that the particle traveled 12.88m over 4 seconds. We can compute average speed by dividing: 12.88/4 = 3.22m/s.

   We should also consider Definition 22 of Section 5.4, which says that the average value of a function f on $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$. In our context, the average value of the speed is

$$\text{average speed} = \frac{1}{2-(-2)}\int_{-2}^{2} ||\,\vec{v}(t)\,||\,dt \approx \frac{1}{4}12.88 = 3.22\text{m/s}.$$

   Note how the physical context of a particle traveling gives meaning to a more abstract concept learned earlier.

In Definition 22 of Chapter 5 we defined the average value of a function $f(x)$ on $[a, b]$ to be

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Notes:

Note how in Example 11.22 we computed the average speed as

$$\frac{\text{distance traveled}}{\text{travel time}} = \frac{1}{2-(-2)}\int_{-2}^{2} \|\, \vec{v}(t) \,\|\, dt;$$

that is, we just found the average value of $\|\, \vec{v}(t) \,\|$ on $[-2, 2]$.

Likewise, given position function $\vec{r}(t)$, the average velocity on $[a, b]$ is

$$\frac{\text{displacement}}{\text{travel time}} = \frac{1}{b-a}\int_{a}^{b} \vec{r}\,'(t)\, dt = \frac{\vec{r}(b)-\vec{r}(a)}{b-a};$$

that is, it is the average value of $\vec{r}\,'(t)$, or $\vec{v}(t)$, on $[a, b]$.

Key Idea 55 Average Speed, Average Velocity

Let $\vec{r}(t)$ be a continuous position function on an open interval I containing $a < b$.

The **average speed** is:

$$\frac{\text{distance traveled}}{\text{travel time}} = \frac{\int_a^b \|\, \vec{v}(t) \,\|\, dt}{b-a} = \frac{1}{b-a}\int_{a}^{b} \|\, \vec{v}(t) \,\|\, dt.$$

The **average velocity** is:

$$\frac{\text{displacement}}{\text{travel time}} = \frac{\int_a^b \vec{r}\,'(t)\, dt}{b-a} = \frac{1}{b-a}\int_{a}^{b} \vec{r}\,'(t)\, dt.$$

The next two sections investigate more properties of the graphs of vector–valued functions and we'll apply these new ideas to what we just learned about motion.

Notes:

Exercises 11.3

Terms and Concepts

1. How is *velocity* different from *speed*?

2. What is the difference between *displacement* and *distance traveled*?

3. What is the difference between *average velocity* and *average speed*?

4. *Distance traveled* is the same as ________ ________, just viewed in a different context.

5. Describe a scenario where an object's average speed is a large number, but the magnitude of the average velocity is not a large number.

6. Explain why it is not possible to have an average velocity with a large magnitude but a small average speed.

Problems

In Exercises 7 – 13 , a position function $\vec{r}(t)$ is given.

1. **Find $\vec{v}(t)$.**
2. **Find $\vec{a}(t)$.**
3. **Find $\vec{v}(t)\cdot\vec{a}(t)$.**
4. **Is the speed constant? If so, determine the constant speed.**

7. $\vec{r}(t) = \langle 2t+1, 5t-2, 7\rangle$

8. $\vec{r}(t) = \left\langle 3t^2-2t+1, -t^2+t+14\right\rangle$

9. $\vec{r}(t) = \langle \cos t, \sin t\rangle$

10. $\vec{r}(t) = \langle t/10, -\cos t, \sin t\rangle$

11. $\vec{r}(t) = \left\langle t^2, \sin t, \cos t\right\rangle$

12. $\vec{r}(t) = \langle \cos(3t) - \sin t, 2\cos t\rangle$

13. $\vec{r}(t) = \langle \sin t + \cos t, 3t, \sin t - \cos t\rangle$

In Exercises 14 – 17 , a position function $\vec{r}(t)$ is given. Sketch $\vec{r}(t)$ on the indicated interval. Find $\vec{v}(t)$ and $\vec{a}(t)$, then add $\vec{v}(t_0)$ and $\vec{a}(t_0)$ to your sketch, with their initial points at $\vec{r}(t_0)$, for the given value of t_0.

14. $\vec{r}(t) = \langle t, \sin t\rangle$ on $[0, \pi/2]$; $t_0 = \pi/4$

15. $\vec{r}(t) = \left\langle t^2, \sin t^2\right\rangle$ on $[0, \pi/2]$; $t_0 = \sqrt{\pi/4}$

16. $\vec{r}(t) = \left\langle t^2+t, -t^2+2t\right\rangle$ on $[-2, 2]$; $t_0 = 1$

17. $\vec{r}(t) = \left\langle \dfrac{2t+3}{t^2+1}, t^2\right\rangle$ on $[-1, 1]$; $t_0 = 0$

In Exercises 18 – 27 , a position function $\vec{r}(t)$ of an object is given. Find the speed of the object in terms of t, and find where the speed is minimized/maximized on the indicated interval.

18. $\vec{r}(t) = \left\langle t^2, t\right\rangle$ on $[-1, 1]$

19. $\vec{r}(t) = \left\langle t^2, t^2-t^3\right\rangle$ on $[-1, 1]$

20. $\vec{r}(t) = \langle 5\cos t, 5\sin t\rangle$ on $[0, 2\pi]$

21. $\vec{r}(t) = \langle 2\cos t, 5\sin t\rangle$ on $[0, 2\pi]$

22. $\vec{r}(t) = \langle \sec t, \tan t\rangle$ on $[0, \pi/4]$

23. $\vec{r}(t) = \langle t+\cos t, 1-\sin t\rangle$ on $[0, 2\pi]$

24. $\vec{r}(t) = \langle 12t, 5\cos t, 5\sin t\rangle$ on $[0, 4\pi]$

25. $\vec{r}(t) = \left\langle t^2-t, t^2+t, t\right\rangle$ on $[0, 1]$

26. $\vec{r}(t) = \left\langle t, t^2, \sqrt{1-t^2}\right\rangle$ on $[-1, 1]$

27. **Projectile Motion:** $\vec{r}(t) = \left\langle (v_0\cos\theta)t, -\dfrac{1}{2}gt^2 + (v_0\sin\theta)t\right\rangle$ on $\left[0, \dfrac{2v_0\sin\theta}{g}\right]$

In Exercises 28 – 31 , position functions $\vec{r}_1(t)$ and $\vec{r}_2(s)$ for two objects are given that follow the same path on the respective intervals.

(a) Show that the positions are the same at the indicated t_0 and s_0 values; i.e., show $\vec{r}_1(t_0) = \vec{r}_2(s_0)$.

(b) Find the velocity, speed and acceleration of the two objects at t_0 and s_0, respectively.

28. $\vec{r}_1(t) = \left\langle t, t^2\right\rangle$ on $[0, 1]$; $t_0 = 1$
$\vec{r}_2(s) = \left\langle s^2, s^4\right\rangle$ on $[0, 1]$; $s_0 = 1$

29. $\vec{r}_1(t) = \langle 3\cos t, 3\sin t\rangle$ on $[0, 2\pi]$; $t_0 = \pi/2$
$\vec{r}_2(s) = \langle 3\cos(4s), 3\sin(4s)\rangle$ on $[0, \pi/2]$; $s_0 = \pi/8$

30. $\vec{r}_1(t) = \langle 3t, 2t\rangle$ on $[0, 2]$; $t_0 = 2$
$\vec{r}_2(s) = \langle 6t-6, 4t-4\rangle$ on $[1, 2]$; $s_0 = 2$

31. $\vec{r}_1(t) = \left\langle t, \sqrt{t}\right\rangle$ on $[0, 1]$; $t_0 = 1$
$\vec{r}_2(s) = \left\langle \sin t, \sqrt{\sin t}\right\rangle$ on $[0, \pi/2]$; $s_0 = \pi/2$

In Exercises 32 – 35 , find the position function of an object given its acceleration and initial velocity and position.

32. $\vec{a}(t) = \langle 2, 3\rangle$; $\quad\vec{v}(0) = \langle 1, 2\rangle$, $\quad\vec{r}(0) = \langle 5, -2\rangle$

33. $\vec{a}(t) = \langle 2, 3\rangle$; $\quad\vec{v}(1) = \langle 1, 2\rangle$, $\quad\vec{r}(1) = \langle 5, -2\rangle$

34. $\vec{a}(t) = \langle \cos t, -\sin t \rangle; \quad \vec{v}(0) = \langle 0, 1 \rangle, \quad \vec{r}(0) = \langle 0, 0 \rangle$

35. $\vec{a}(t) = \langle 0, -32 \rangle; \quad \vec{v}(0) = \langle 10, 50 \rangle, \quad \vec{r}(0) = \langle 0, 0 \rangle$

In Exercises 36 – 39 , find the displacement, distance traveled, average velocity and average speed of the described object on the given interval.

36. An object with position function $\vec{r}(t) = \langle 2\cos t, 2\sin t, 3t \rangle$, where distances are measured in feet and time is in seconds, on $[0, 2\pi]$.

37. An object with position function $\vec{r}(t) = \langle 5\cos t, -5\sin t \rangle$, where distances are measured in feet and time is in seconds, on $[0, \pi]$.

38. An object with velocity function $\vec{v}(t) = \langle \cos t, \sin t \rangle$, where distances are measured in feet and time is in seconds, on $[0, 2\pi]$.

39. An object with velocity function $\vec{v}(t) = \langle 1, 2, -1 \rangle$, where distances are measured in feet and time is in seconds, on $[0, 10]$.

Exercises 40 – 45 ask you to solve a variety of problems based on the principles of projectile motion.

40. A boy whirls a ball, attached to a 3ft string, above his head in a counter–clockwise circle. The ball makes 2 revolutions per second.

    At what t-values should the boy release the string so that the ball heads directly for a tree standing 10ft in front of him?

41. David faces Goliath with only a stone in a 3ft sling, which he whirls above his head at 4 revolutions per second. They stand 20ft apart.

    (a) At what t-values must David release the stone in his sling in order to hit Goliath?

    (b) What is the speed at which the stone is traveling when released?

    (c) Assume David releases the stone from a height of 6ft and Goliath's forehead is 9ft above the ground. What angle of elevation must David apply to the stone to hit Goliath's head?

42. A hunter aims at a deer which is 40 yards away. Her crossbow is at a height of 5ft, and she aims for a spot on the deer 4ft above the ground. The crossbow fires her arrows at 300ft/s.

    (a) At what angle of elevation should she hold the crossbow to hit her target?

    (b) If the deer is moving perpendicularly to her line of sight at a rate of 20mph, by approximately how much should she lead the deer in order to hit it in the desired location?

43. A baseball player hits a ball at 100mph, with an initial height of 3ft and an angle of elevation of 20°, at Boston's Fenway Park. The ball flies towards the famed "Green Monster," a wall 37ft high located 310ft from home plate.

    (a) Show that as hit, the ball hits the wall.

    (b) Show that if the angle of elevation is 21°, the ball clears the Green Monster.

44. A Cessna flies at 1000ft at 150mph and drops a box of supplies to the professor (and his wife) on an island. Ignoring wind resistance, how far horizontally will the supplies travel before they land?

45. A football quarterback throws a pass from a height of 6ft, intending to hit his receiver 20yds away at a height of 5ft.

    (a) If the ball is thrown at a rate of 50mph, what angle of elevation is needed to hit his intended target?

    (b) If the ball is thrown at with an angle of elevation of 8°, what initial ball speed is needed to hit his target?

11.4 Unit Tangent and Normal Vectors

Unit Tangent Vector

Given a smooth vector–valued function $\vec{r}(t)$, we defined in Definition 75 that any vector parallel to $\vec{r}\,'(t_0)$ is *tangent* to the graph of $\vec{r}(t)$ at $t = t_0$. It is often useful to consider just the *direction* of $\vec{r}\,'(t)$ and not its magnitude. Therefore we are interested in the unit vector in the direction of $\vec{r}\,'(t)$. This leads to a definition.

Definition 78 Unit Tangent Vector

Let $\vec{r}(t)$ be a smooth function on an open interval I. The unit tangent vector $\vec{T}(t)$ is

$$\vec{T}(t) = \frac{1}{\|\,\vec{r}\,'(t)\,\|}\vec{r}\,'(t).$$

Example 11.23 Computing the unit tangent vector

Let $\vec{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$. Find $\vec{T}(t)$ and compute $\vec{T}(0)$ and $\vec{T}(1)$.

Solution We apply Definition 78 to find $\vec{T}(t)$.

$$\begin{aligned}\vec{T}(t) &= \frac{1}{\|\,\vec{r}\,'(t)\,\|}\vec{r}\,'(t)\\ &= \frac{1}{\sqrt{\big(-3\sin t\big)^2 + \big(3\cos t\big)^2 + 4^2}}\langle -3\sin t, 3\cos t, 4\rangle\\ &= \left\langle -\frac{3}{5}\sin t, \frac{3}{5}\cos t, \frac{4}{5}\right\rangle.\end{aligned}$$

We can now easily compute $\vec{T}(0)$ and $\vec{T}(1)$:

$$\vec{T}(0) = \left\langle 0, \frac{3}{5}, \frac{4}{5}\right\rangle; \quad \vec{T}(1) = \left\langle -\frac{3}{5}\sin 1, \frac{3}{5}\cos 1, \frac{4}{5}\right\rangle \approx \langle -0.505, 0.324, 0.8\rangle.$$

These are plotted in Figure 11.20 with their initial points at $\vec{r}(0)$ and $\vec{r}(1)$, respectively. (They look rather "short" since they are only length 1.)

The unit tangent vector $\vec{T}(t)$ always has a magnitude of 1, though it is sometimes easy to doubt that is true. We can help solidify this thought in our minds by computing $\|\,\vec{T}(1)\,\|$:

$$\|\,\vec{T}(1)\,\| \approx \sqrt{(-0.505)^2 + 0.324^2 + 0.8^2} = 1.000001.$$

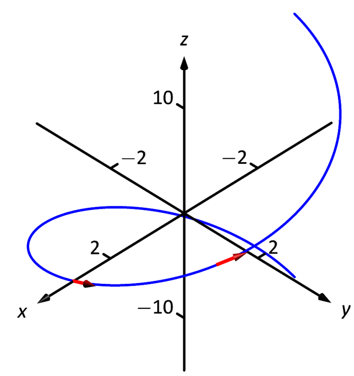


Figure 11.20: Plotting unit tangent vectors in Example 11.23.

Notes:

We have rounded in our computation of $\vec{T}(1)$, so we don't get 1 exactly. We leave it to the reader to use the exact representation of $\vec{T}(1)$ to verify it has length 1.

In many ways, the previous example was "too nice." It turned out that $\vec{r}\,'(t)$ was always of length 5. In the next example the length of $\vec{r}\,'(t)$ is variable, leaving us with a formula that is not as clean.

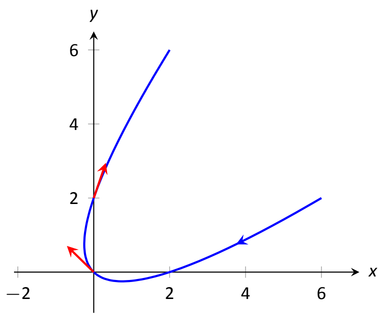


Figure 11.21: Plotting unit tangent vectors in Example 11.24.

Example 11.24 Computing the unit tangent vector

Let $\vec{r}(t) = \langle t^2 - t, t^2 + t\rangle$. Find $\vec{T}(t)$ and compute $\vec{T}(0)$ and $\vec{T}(1)$.

Solution We find $\vec{r}\,'(t) = \langle 2t - 1, 2t + 1\rangle$, and

$$\|\,\vec{r}\,'(t)\,\| = \sqrt{(2t-1)^2 + (2t+1)^2} = \sqrt{8t^2+2}.$$

Therefore

$$\vec{T}(t) = \frac{1}{\sqrt{8t^2+2}}\langle 2t-1, 2t+1\rangle = \left\langle \frac{2t-1}{\sqrt{8t^2+2}}, \frac{2t+1}{\sqrt{8t^2+2}}\right\rangle.$$

When $t = 0$, we have $\vec{T}(0) = \langle -1/\sqrt{2}, 1/\sqrt{2}\rangle$; when $t = 1$, we have $\vec{T}(1) = \langle 1/\sqrt{10}, 3/\sqrt{10}\rangle$. We leave it to the reader to verify each of these is a unit vector. They are plotted in Figure 11.21

Unit Normal Vector

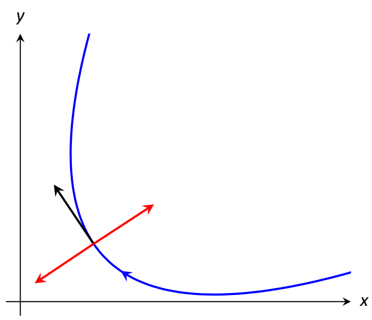


Figure 11.22: Given a direction in the plane, there are always two directions orthogonal to it.

Just as knowing the direction tangent to a path is important, knowing a direction orthogonal to a path is important. When dealing with real-valued functions, we defined the normal line at a point to the be the line through the point that was perpendicular to the tangent line at that point. We can do a similar thing with vector–valued functions. Given $\vec{r}(t)$ in $\mathbb{R}^2$, we have 2 directions perpendicular to the tangent vector, as shown in Figure 11.22. It is good to wonder "Is one of these two directions preferable over the other?"

Given $\vec{r}(t)$ in $\mathbb{R}^3$, there are infinite vectors orthogonal to the tangent vector at a given point. Again, we might wonder "Is one of these infinite choices preferable over the others? Is one of these the "correct" choice?"

The answer in both $\mathbb{R}^2$ and $\mathbb{R}^3$ is "Yes, there is one vector that is not only preferable, it is the "correct" one to choose." Recall Theorem 95, which states that if $\vec{r}(t)$ has constant length, then $\vec{r}(t)$ is orthogonal to $\vec{r}\,'(t)$ for all t. We know $\vec{T}(t)$, the unit tangent vector, has constant length. Therefore $\vec{T}(t)$ is orthogonal to $\vec{T}\,'(t)$.

Note: $\vec{T}(t)$ is a unit vector, by definition. This *does not* imply that $\vec{T}\,'(t)$ is also a unit vector.

We'll see that $\vec{T}\,'(t)$ is more than just a convenient choice of vector that is orthogonal to $\vec{r}\,'(t)$; rather, it is the "correct" choice. Since all we care about is the direction, we define this newly found vector to be a unit vector.

Notes:

Definition 79 Unit Normal Vector

Let $\vec{r}(t)$ be a vector–valued function where the unit tangent vector, $\vec{T}(t)$, is smooth on an open interval I. The **unit normal vector** $\vec{N}(t)$ is

$$\vec{N}(t) = \frac{1}{\|\,\vec{T}\,'(t)\,\|}\vec{T}\,'(t).$$

Example 11.25 Computing the unit normal vector

Let $\vec{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$ as in Example 11.23. Sketch both $\vec{T}(\pi/2)$ and $\vec{N}(\pi/2)$ with initial points at $\vec{r}(\pi/2)$.

Solution In Example 11.23, we found $\vec{T}(t) = \Big\langle (-3/5)\sin t, (3/5)\cos t, 4/5\Big\rangle$. Therefore

$$\vec{T}\,'(t) = \left\langle -\frac{3}{5}\cos t, -\frac{3}{5}\sin t, 0\right\rangle \quad \text{and} \quad \|\,\vec{T}\,'(t)\,\| = \frac{3}{5}.$$

Thus

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{3/5} = \langle -\cos t, -\sin t, 0\rangle\,.$$

We compute $\vec{T}(\pi/2) = \langle -3/5, 0, 4/5\rangle$ and $\vec{N}(\pi/2) = \langle 0, -1, 0\rangle$. These are sketched in Figure 11.23.

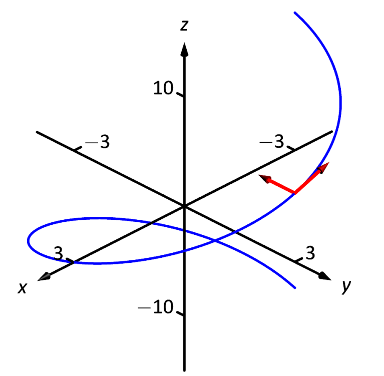


Figure 11.23: Plotting unit tangent and normal vectors in Example 11.23.

The previous example was once again "too nice." In general, the expression for $\vec{T}(t)$ contains fractions of square–roots, hence the expression of $\vec{T}\,'(t)$ is very messy. We demonstrate this in the next example.

Example 11.26 Computing the unit normal vector

Let $\vec{r}(t) = \langle t^2 - t, t^2 + t\rangle$ as in Example 11.24. Find $\vec{N}(t)$ and sketch $\vec{r}(t)$ with the unit tangent and normal vectors at $t = -1, 0$ and 1.

Solution In Example 11.24, we found

$$\vec{T}(t) = \left\langle \frac{2t-1}{\sqrt{8t^2+2}}, \frac{2t+1}{\sqrt{8t^2+2}}\right\rangle.$$

Finding $\vec{T}\,'(t)$ requires two applications of the Quotient Rule:

Notes:

$$T'(t) = \left\langle \frac{\sqrt{8t^2+2}(2) - (2t-1)\left(\frac{1}{2}(8t^2+2)^{-1/2}(16t)\right)}{8t^2+2}, \right.$$

$$\left. \frac{\sqrt{8t^2+2}(2) - (2t+1)\left(\frac{1}{2}(8t^2+2)^{-1/2}(16t)\right)}{8t^2+2} \right\rangle$$

$$= \left\langle \frac{4(2t+1)}{(8t^2+2)^{3/2}}, \frac{4(1-2t)}{(8t^2+2)^{3/2}} \right\rangle$$

This is not a unit vector; to find $\vec{N}(t)$, we need to divide $\vec{T}'(t)$ by it's magnitude.

$$\|\,\vec{T}'(t)\,\| = \sqrt{\frac{16(2t+1)^2}{(8t^2+2)^3} + \frac{16(1-2t)^2}{(8t^2+2)^3}}$$

$$= \sqrt{\frac{16(8t^2+2)}{(8t^2+2)^3}}$$

$$= \frac{4}{8t^2+2}.$$

Finally,

$$\vec{N}(t) = \frac{1}{4/(8t^2+2)} \left\langle \frac{4(2t+1)}{(8t^2+2)^{3/2}}, \frac{4(1-2t)}{(8t^2+2)^{3/2}} \right\rangle$$

$$= \left\langle \frac{2t+1}{\sqrt{8t^2+2}}, -\frac{2t-1}{\sqrt{8t^2+2}} \right\rangle.$$

Using this formula for $\vec{N}(t)$, we compute the unit tangent and normal vectors for $t = -1, 0$ and 1 and sketch them in Figure 11.24.

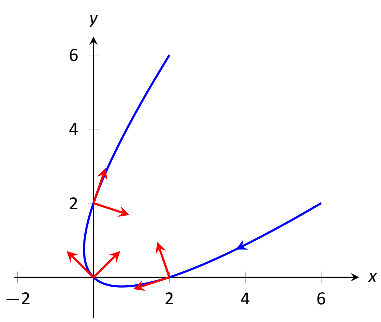


Figure 11.24: Plotting unit tangent and normal vectors in Example 11.26.

The final result for $\vec{N}(t)$ in Example 11.26 is suspiciously similar to $\vec{T}(t)$. There is a clear reason for this. If $\vec{u} = \langle u_1, u_2 \rangle$ is a unit vector in $\mathbb{R}^2$, then the *only* unit vectors orthogonal to $\vec{u}$ are $\langle -u_2, u_1 \rangle$ and $\langle u_2, -u_1 \rangle$. Given $\vec{T}(t)$, we can quickly determine $\vec{N}(t)$ if we know which term to multiply by (-1).

Consider again Figure 11.24, where we have plotted some unit tangent and normal vectors. Note how $\vec{N}(t)$ always points "inside" the curve, or to the concave side of the curve. This is not a coincidence; this is true in general. Knowing the direction that $\vec{r}(t)$ "turns" allows us to quickly find $\vec{N}(t)$.

Notes:

Theorem 99 Unit Normal Vectors in $\mathbb{R}^2$

Let $\vec{r}(t)$ be a vector–valued function in $\mathbb{R}^2$ where $\vec{T}\,'(t)$ is smooth on an open interval I. Let t_0 be in I and $\vec{T}(t_0) = \langle t_1, t_2 \rangle$ Then $\vec{N}(t_0)$ is either

$$\vec{N}(t_0) = \langle -t_2, t_1 \rangle \quad \text{or} \quad \vec{N}(t_0) = \langle t_2, -t_1 \rangle\,,$$

whichever is the vector that points to the concave side of the graph of $\vec{r}$.

Application to Acceleration

Let $\vec{r}(t)$ be a position function. It is a fact (stated later in Theorem 100) that acceleration, $\vec{a}(t)$, lies in the plane defined by $\vec{T}$ and $\vec{N}$. That is, there are scalars a_{T} and a_{N} such that

$$\vec{a}(t) = a_{\mathrm{T}}\vec{T}(t) + a_{\mathrm{N}}\vec{N}(t).$$

The scalar a_{T} measures "how much" acceleration is in the direction of travel, that is, it measures the component of acceleration that affects the speed. The scalar a_{N} measures "how much" acceleration is perpendicular to the direction of travel, that is, it measures the component of acceleration that affects the direction of travel.

We can find a_{T} using the orthogonal projection of $\vec{a}(t)$ onto $\vec{T}(t)$ (review Definition 63 in Section 10.3 if needed). Recalling that since $\vec{T}(t)$ is a unit vector, $\vec{T}(t)\cdot\vec{T}(t) = 1$, so we have

$$\text{proj}_{\,\vec{T}(t)}\,\vec{a}(t) = \frac{\vec{a}(t)\cdot\vec{T}(t)}{\vec{T}(t)\cdot\vec{T}(t)}\vec{T}(t) = \underbrace{\big(\vec{a}(t)\cdot\vec{T}(t)\big)}_{a_{\mathrm{T}}}\,\vec{T}(t).$$

Thus the amount of $\vec{a}(t)$ in the direction of $\vec{T}(t)$ is $a_{\mathrm{T}} = \vec{a}(t)\cdot\vec{T}(t)$. The same logic gives $a_{\mathrm{N}} = \vec{a}(t)\cdot\vec{N}(t)$.

While this is a fine way of computing a_{T}, there are simpler ways of finding a_{N} (as finding $\vec{N}$ itself can be complicated). The following theorem gives alternate formulas for a_{T} and a_{N}.

Note: Keep in mind that both a_{T} and a_{N} are functions of t; that is, the scalar changes depending on t. It is convention to drop the "(t)" notation from $a_{\mathrm{T}}(t)$ and simply write a_{T}.

Notes:

Theorem 100 Acceleration in the Plane Defined by $\vec{T}$ and $\vec{N}$

Let $\vec{r}(t)$ be a position function with acceleration $\vec{a}(t)$ and unit tangent and normal vectors $\vec{T}(t)$ and $\vec{N}(t)$. Then $\vec{a}(t)$ lies in the plane defined by $\vec{T}(t)$ and $\vec{N}(t)$; that is, there exists scalars a_T and a_N such that

$$\vec{a}(t) = a_\text{T}\vec{T}(t) + a_\text{N}\vec{N}(t).$$

Moreover,

$$a_\text{T} = \vec{a}(t)\cdot\vec{T}(t) = \frac{d}{dt}\Big(||\,\vec{v}(t)\,||\Big)$$
$$a_\text{N} = \vec{a}(t)\cdot\vec{N}(t) = \sqrt{||\,\vec{a}(t)\,||^2 - a_\text{T}^2} = \frac{||\,\vec{a}(t)\times\vec{v}(t)\,||}{||\,\vec{v}(t)\,||} = ||\,\vec{v}(t)\,||\,||\,\vec{T}\,'(t)\,||$$

Note the second formula for a_T: $\frac{d}{dt}\Big(||\,\vec{v}(t)\,||\Big)$. This measures the rate of change of speed, which again is the amount of acceleration in the direction of travel.

Example 11.27 Computing a_T and a_N

Let $\vec{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$ as in Examples 11.23 and 11.25. Find a_T and a_N.

Solution The previous examples give $\vec{a}(t) = \langle -3\cos t, -3\sin t, 0\rangle$ and

$$\vec{T}(t) = \left\langle -\frac{3}{5}\sin t, \frac{3}{5}\cos t, \frac{4}{5}\right\rangle \quad \text{and} \quad \vec{N}(t) = \langle -\cos t, -\sin t, 0\rangle\,.$$

We can find a_T and a_N directly with dot products:

$$a_\text{T} = \vec{a}(t)\cdot\vec{T}(t) = \frac{9}{5}\cos t\sin t - \frac{9}{5}\cos t\sin t + 0 = 0.$$
$$a_\text{N} = \vec{a}(t)\cdot\vec{N}(t) = 3\cos^2 t + 3\sin^2 t + 0 = 3.$$

Thus $\vec{a}(t) = 0\vec{T}(t) + 3\vec{N}(t) = 3\vec{N}(t)$, which is clearly the case.

What is the practical interpretation of these numbers? $a_\text{T} = 0$ means the object is moving at a constant speed, and hence all acceleration comes in the form of direction change.

Example 11.28 Computing a_T and a_N

Let $\vec{r}(t) = \langle t^2 - t, t^2 + t\rangle$ as in Examples 11.24 and 11.26. Find a_T and a_N.

Notes:

Solution The previous examples give $\vec{a}(t) = \langle 2, 2\rangle$ and

$$\vec{T}(t) = \left\langle \frac{2t-1}{\sqrt{8t^2+2}}, \frac{2t+1}{\sqrt{8t^2+2}} \right\rangle \quad \text{and} \quad \vec{N}(t) = \left\langle \frac{2t+1}{\sqrt{8t^2+2}}, -\frac{2t-1}{\sqrt{8t^2+2}} \right\rangle.$$

While we can compute a_{N} using $\vec{N}(t)$, we instead demonstrate using another formula from Theorem 100.

$$a_{\text{T}} = \vec{a}(t)\cdot\vec{T}(t) = \frac{4t-2}{\sqrt{8t^2+2}} + \frac{4t+2}{\sqrt{8t^2+2}} = \frac{8t}{\sqrt{8t^2+2}}.$$

$$a_{\text{N}} = \sqrt{\|\,\vec{a}(t)\,\|^2 - a_{\text{T}}^2} = \sqrt{8 - \left(\frac{8t}{\sqrt{8t^2+2}}\right)^2} = \frac{4}{\sqrt{8t^2+2}}.$$

When $t = 2$, $a_{\text{T}} = \dfrac{16}{\sqrt{34}} \approx 2.74$ and $a_{\text{N}} = \dfrac{4}{\sqrt{34}} \approx 0.69$. We interpret this to mean that at $t = 2$, the particle is acclerating mostly by increasing speed, not by changing direction. As the path near $t = 2$ is relatively straight, this should make intuitive sense. Figure 11.25 gives a graph of the path for reference.

Contrast this with $t = 0$, where $a_{\text{T}} = 0$ and $a_{\text{N}} = 4/\sqrt{2} \approx 2.82$. Here the particle's speed is not changing and all acceleration is in the form of direction change.

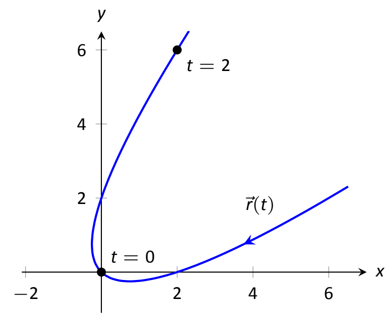


Figure 11.25: Graphing $\vec{r}(t)$ in Example 11.28.

Example 11.29 Analyzing projectile motion

A ball is thrown from a height of 240ft with an initial speed of 64ft/s and an angle of elevation of 30°. Find the position function $\vec{r}(t)$ of the ball and analyze a_{T} and a_{N}.

Solution Using Key Idea 54 of Section 11.3 we form the position function of the ball:

$$\vec{r}(t) = \left\langle \big(64\cos 30^\circ\big)t, -16t^2 + \big(64\sin 30^\circ\big)t + 240 \right\rangle,$$

which we plot in Figure 11.26.

From this we find $\vec{v}(t) = \langle 64\cos 30^\circ, -32t + 64\sin 30^\circ\rangle$ and $\vec{a}(t) = \langle 0, -32\rangle$. Computing $\vec{T}(t)$ is not difficult, and with some simplification we find

$$\vec{T}(t) = \left\langle \frac{\sqrt{3}}{\sqrt{t^2-2t+4}}, \frac{1-t}{\sqrt{t^2-2t+4}} \right\rangle.$$

With $\vec{a}(t)$ as simple as it is, finding a_{T} is also simple:

$$a_{\text{T}} = \vec{a}(t)\cdot\vec{T}(t) = \frac{32t-32}{\sqrt{t^2-2t+4}}.$$

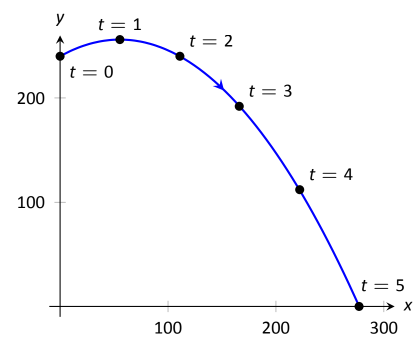


Figure 11.26: Plotting the position of a thrown ball, with 1s increments shown.

Notes:

t	a_T	a_N
0	-16	27.7
1	0	32
2	16	27.7
3	24.2	20.9
4	27.7	16
5	29.4	12.7

Figure 11.27: A table of values of a_T and a_N in Example 11.29.

We choose to not find $\vec{N}(t)$ and find a_N through the formula $a_N = \sqrt{||\ \vec{a}(t)\ ||^2 - a_T^2}$:

$$a_N = \sqrt{32^2 - \left(\frac{32t - 32}{\sqrt{t^2 - 2t + 4}}\right)^2} = \frac{32\sqrt{3}}{\sqrt{t^2 - 2t + 4}}.$$

Figure 11.27 gives a table of values of a_T and a_N. When $t = 0$, we see the ball's speed is decreasing; when $t = 1$ the speed of the ball is unchanged. This corresponds to the fact that at $t = 1$ the ball reaches its highest point.

After $t = 1$ we see that a_N is decreasing in value. This is because as the ball falls, it's path becomes straighter and most of the acceleration is in the form of speeding up the ball, and not in changing its direction.

Our understanding of the unit tangent and normal vectors is aiding our understanding of motion. The work in Example 11.29 gave quantitative analysis of what we intuitively knew.

The next section provides two more important steps towards this analysis. We currently describe position only in terms of time. In everyday life, though, we often describe position in terms of distance ("The gas station is about 2 miles ahead, on the left."). The *arc length parameter* allows us to reference position in terms of distance traveled.

We also intuitively know that some paths are straighter than others – and some are curvier than others, but we lack a measurement of "curviness." The arc length parameter provides a way for us to compute *curvature*, a quantitative measurement of how curvy a curve is.

Notes:

Exercises 11.4

Terms and Concepts

1. If $\vec{T}(t)$ is a unit tangent vector, what is $\| \vec{T}(t) \|$?

2. If $\vec{N}(t)$ is a unit normal vector, what is $\vec{N}(t) \cdot \vec{r}\,'(t)$?

3. The acceleration vector $\vec{a}(t)$ lies in the plane defined by what two vectors?

4. a_{T} measures how much the acceleration is affecting the ________ of an object.

Problems

In Exercises 5 – 8 , given $\vec{r}(t)$, find $\vec{T}(t)$ and evaluate it at the indicated value of t.

5. $\vec{r}(t) = \left\langle 2t^2, t^2 - t \right\rangle, \quad t = 1$

6. $\vec{r}(t) = \langle t, \cos t \rangle, \quad t = \pi/4$

7. $\vec{r}(t) = \left\langle \cos^3 t, \sin^3 t \right\rangle, \quad t = \pi/4$

8. $\vec{r}(t) = \langle \cos t, \sin t \rangle, \quad t = \pi$

In Exercises 9 – 12 , find the equation of the line tangent to the curve at the indicated t-value using the unit tangent vector. Note: these are the same problems as in Exercises 5 – 8.

9. $\vec{r}(t) = \left\langle 2t^2, t^2 - t \right\rangle, \quad t = 1$

10. $\vec{r}(t) = \langle t, \cos t \rangle, \quad t = \pi/4$

11. $\vec{r}(t) = \left\langle \cos^3 t, \sin^3 t \right\rangle, \quad t = \pi/4$

12. $\vec{r}(t) = \langle \cos t, \sin t \rangle, \quad t = \pi$

In Exercises 13 – 16 , find $\vec{N}(t)$ using Definition 79. Confirm the result using Theorem 99.

13. $\vec{r}(t) = \langle 3\cos t, 3\sin t \rangle$

14. $\vec{r}(t) = \left\langle t, t^2 \right\rangle$

15. $\vec{r}(t) = \langle \cos t, 2\sin t \rangle$

16. $\vec{r}(t) = \left\langle e^t, e^{-t} \right\rangle$

In Exercises 17 – 20 , a position function $\vec{r}(t)$ is given along with its unit tangent vector $\vec{T}(t)$ evaluated at $t = a$, for some value of a.

(a) Confirm that $\vec{T}(a)$ is as stated.

(b) Using a graph of $\vec{r}(t)$ and Theorem 99, find $\vec{N}(a)$.

17. $\vec{r}(t) = \langle 3\cos t, 5\sin t \rangle; \quad \vec{T}(\pi/4) = \left\langle -\frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right\rangle.$

18. $\vec{r}(t) = \left\langle t, \frac{1}{t^2+1} \right\rangle; \quad \vec{T}(1) = \left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle.$

19. $\vec{r}(t) = (1 + 2\sin t)\,\langle \cos t, \sin t \rangle; \quad \vec{T}(0) = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle.$

20. $\vec{r}(t) = \left\langle \cos^3 t, \sin^3 t \right\rangle; \quad \vec{T}(\pi/4) = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle.$

In Exercises 21 – 24 , find $\vec{N}(t)$.

21. $\vec{r}(t) = \langle 4t, 2\sin t, 2\cos t \rangle$

22. $\vec{r}(t) = \langle 5\cos t, 3\sin t, 4\sin t \rangle$

23. $\vec{r}(t) = \langle a\cos t, a\sin t, bt \rangle; \quad a > 0$

24. $\vec{r}(t) = \langle \cos(at), \sin(at), t \rangle$

In Exercises 25 – 30 , find a_{T} and a_{N} given $\vec{r}(t)$. Sketch $\vec{r}(t)$ on the indicated interval, and comment on the relative sizes of a_{T} and a_{N} at the indicated t values.

25. $\vec{r}(t) = \left\langle t, t^2 \right\rangle$ on $[-1, 1]$; consider $t = 0$ and $t = 1$.

26. $\vec{r}(t) = \langle t, 1/t \rangle$ on $(0, 4]$; consider $t = 1$ and $t = 2$.

27. $\vec{r}(t) = \langle 2\cos t, 2\sin t \rangle$ on $[0, 2\pi]$; consider $t = 0$ and $t = \pi/2$.

28. $\vec{r}(t) = \left\langle \cos(t^2), \sin(t^2) \right\rangle$ on $(0, 2\pi]$; consider $t = \sqrt{\pi/2}$ and $t = \sqrt{\pi}$.

29. $\vec{r}(t) = \langle a\cos t, a\sin t, bt \rangle$ on $[0, 2\pi]$, where $a, b > 0$; consider $t = 0$ and $t = \pi/2$.

30. $\vec{r}(t) = \langle 5\cos t, 4\sin t, 3\sin t \rangle$ on $[0, 2\pi]$; consider $t = 0$ and $t = \pi/2$.

11.5 The Arc Length Parameter and Curvature

In normal conversation we describe position in terms of both *time* and *distance*. For instance, imagine driving to visit a friend. If she calls and asks where you are, you might answer "I am 20 minutes from your house," or you might say "I am 10 miles from your house." Both answers provide your friend with a general idea of where you are.

Currently, our vector–valued functions have defined points with a parameter t, which we often take to represent time. Consider Figure 11.28(a), where $\vec{r}(t) = \left\langle t^2 - t, t^2 + t \right\rangle$ is graphed and the points corresponding to $t = 0$, 1 and 2 are shown. Note how the arc length between $t = 0$ and $t = 1$ is smaller than the arc length between $t = 1$ and $t = 2$; if the parameter t is time and $\vec{r}$ is position, we can say that the particle traveled faster on $[1, 2]$ than on $[0, 1]$.

Now consider Figure 11.28(b), where the same graph is parameterized by a different variable s. Points corresponding to $s = 0$ through $s = 6$ are plotted. The arc length of the graph between each adjacent pair of points is 1. We can view this parameter s as distance; that is, the arc length of the graph from $s = 0$ to $s = 3$ is 3, the arc length from $s = 2$ to $s = 6$ is 4, etc. If one wants to find the point 2.5 units from an initial location (i.e., $s = 0$), one would compute $\vec{r}(2.5)$. This parameter s is very useful, and is called the **arc length parameter**.

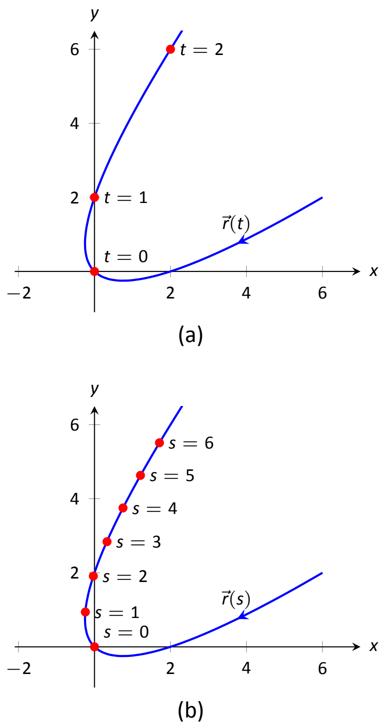


Figure 11.28: Introducing the arc length parameter.

How do we find the arc length parameter?

Start with any parameterization of $\vec{r}$. We can compute the arc length of the graph of $\vec{r}$ on the interval $[0, t]$ with

$$\text{arc length } = \int_0^t ||\,\vec{r}\,'(u)\,||\,du.$$

We can turn this into a function: as t varies, we find the arc length s from 0 to t. This function is

$$s(t) = \int_0^t ||\,\vec{r}\,'(u)\,||\,du. \tag{11.1}$$

This establishes a relationship between s and t. Knowing this relationship explicitly, we can rewrite $\vec{r}(t)$ as a function of s: $\vec{r}(s)$. We demonstrate this in an example.

Example 11.30 Finding the arc length parameter

Let $\vec{r}(t) = \langle 3t - 1, 4t + 2\rangle$. parameterize $\vec{r}$ with the arc length parameter s.

SOLUTION Using Equation (11.1), we write

$$s(t) = \int_0^t ||\,\vec{r}\,'(u)\,||\,du.$$

Notes:

We can integrate this, explicitly finding a relationship between s and t:

$$\begin{aligned} s(t) &= \int_0^t ||\,\vec{r}\,'(u)\,||\,du \\ &= \int_0^t \sqrt{3^2+4^2}\,du \\ &= \int_0^t 5\,du \\ &= 5t. \end{aligned}$$

Since $s = 5t$, we can write $t = s/5$ and replace t in $\vec{r}(t)$ with $s/5$:

$$\vec{r}(s) = \langle 3(s/5)-1, 4(s/5)+2\rangle = \left\langle \frac{3}{5}s-1, \frac{4}{5}s+2\right\rangle.$$

Clearly, as shown in Figure 11.29, the graph of $\vec{r}$ is a line, where $t = 0$ corresponds to the point $(-1, 2)$. What point on the line is 2 units away from this initial point? We find it with $s(2) = \langle 1/5, 18/5\rangle$.

Is the point $(1/5, 18/5)$ really 2 units away from $(-1, 2)$? We use the Distance Formula to check:

$$d = \sqrt{\left(\frac{1}{5}-(-1)\right)^2+\left(\frac{18}{5}-2\right)^2} = \sqrt{\frac{36}{25}+\frac{64}{25}} = \sqrt{4} = 2.$$

Yes, $s(2)$ is indeed 2 units away, in the direction of travel, from the initial point.

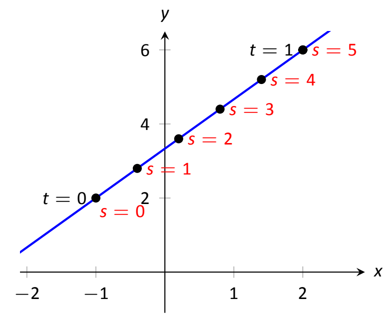


Figure 11.29: Graphing $\vec{r}$ in Example 11.30 with parameters t and s.

Things worked out very nicely in Example 11.30; we were able to establish directly that $s = 5t$. Usually, the arc length parameter is much more difficult to describe in terms of t, a result of integrating a square–root. There are a number of things that we can learn about the arc length parameter from Equation (11.1), though, that are incredibly useful.

First, take the derivative of s with respect to t. The Fundamental Theorem of Calculus (see Theorem 41) states that

$$\frac{ds}{dt} = s\,'(t) = ||\,\vec{r}\,'(t)\,||. \tag{11.2}$$

Letting t represent time and $\vec{r}(t)$ represent position, we see that the rate of change of s with respect to t is speed; that is, the rate of change of "distance traveled" is speed, which should match our intuition.

The Chain Rule states that

$$\begin{aligned} \frac{d\vec{r}}{dt} &= \frac{d\vec{r}}{ds}\cdot\frac{ds}{dt} \\ \vec{r}\,'(t) &= \vec{r}\,'(s)\cdot||\,\vec{r}\,'(t)\,||. \end{aligned}$$

Notes:

Solving for $\vec{r}\,'(s)$, we have

$$\vec{r}\,'(s) = \frac{\vec{r}\,'(t)}{||\,\vec{r}\,'(t)\,||} = \vec{T}(t), \tag{11.3}$$

where $\vec{T}(t)$ is the unit tangent vector. Equation 11.3 is often misinterpreted, as one is tempted to think it states $\vec{r}\,'(t) = \vec{T}(t)$, but there is a big difference between $\vec{r}\,'(s)$ and $\vec{r}\,'(t)$. The key to take from it is that $\vec{r}\,'(s)$ is a unit vector. In fact, the following theorem states that this characterizes the arc length parameter.

Theorem 101 Arc Length Parameter

Let $\vec{r}(s)$ be a vector–valued function. The parameter s is the arc length parameter if, and only if, $||\,\vec{r}\,'(s)\,|| = 1$.

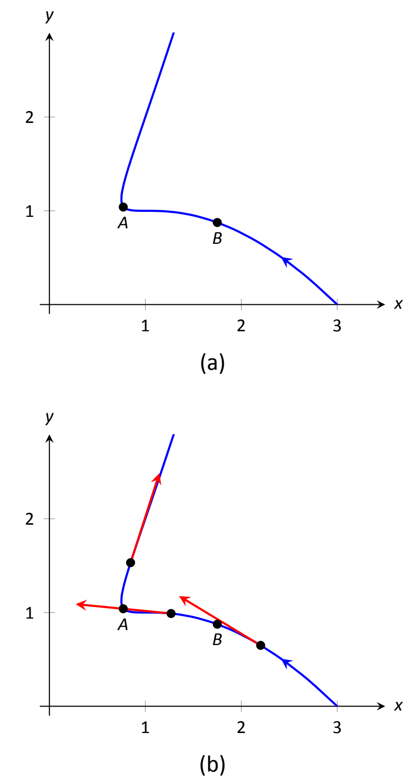


Figure 11.30: Establishing the concept of curvature.

Curvature

Consider points A and B on the curve graphed in Figure 11.30(a). One can readily argue that the curve curves more sharply at A than at B. It is useful to use a number to describe how sharply the curve bends; that number is the **curvature** of the curve.

We derive this number in the following way. Consider Figure 11.30(b), where unit tangent vectors are graphed around points A and B. Notice how the direction of the unit tangent vector changes quite a bit near A, whereas it does not change as much around B. This leads to an important concept: measuring the rate of change of the unit tangent vector with respect to arc length gives us a measurement of curvature.

Definition 80 Curvature

Let $\vec{r}(s)$ be a vector–valued function where s is the arc length parameter. The curvature κ of the graph of $\vec{r}(s)$ is

$$\kappa = \left|\left|\frac{d\vec{T}}{ds}\right|\right| = ||\,\vec{T}\,'(s)\,||\,.$$

If $\vec{r}(s)$ is parameterized by the arc length parameter, then

$$\vec{T}(s) = \frac{\vec{r}\,'(s)}{||\,\vec{r}\,'(s)\,||} \quad \text{and} \quad \vec{N}(s) = \frac{\vec{T}\,'(s)}{||\,\vec{T}\,'(s)\,||}.$$

Notes:

Having defined $|| \vec{T}'(s) || = \kappa$, we can rewrite the second equation as

$$\vec{T}'(s) = \kappa\vec{N}(s). \tag{11.4}$$

We already knew that $\vec{T}'(s)$ is in the same direction as $\vec{N}(s)$; that is, we can think of $\vec{T}(s)$ as being "pulled" in the direction of $\vec{N}(s)$. How "hard" is it being pulled? By a factor of κ. When the curvature is large, $\vec{T}(s)$ is being "pulled hard" and the direction of $\vec{T}(s)$ changes rapidly. When κ is small, $T(s)$ is not being pulled hard and hence its direction is not changing rapidly.

We use Definition 80 to find the curvature of the line in Example 11.30.

Example 11.31 Finding the curvature of a line

Use Definition 80 to find the curvature of $\vec{r}(t) = \langle 3t - 1, 4t + 2\rangle$.

Solution In Example 11.30, we found that the arc length parameter was defined by $s = 5t$, so $\vec{r}(s) = \langle 3t/5 - 1, 4t/5 + 2\rangle$ parameterized $\vec{r}$ with the arc length parameter. To find κ, we need to find $\vec{T}'(s)$.

$$\begin{aligned}\vec{T}(s) &= \vec{r}'(s) \quad \text{(recall this is a unit vector)}\\ &= \langle 3/5, 4/5\rangle .\end{aligned}$$

Therefore

$$\vec{T}'(s) = \langle 0, 0\rangle$$

and

$$\kappa = \left|\left| \vec{T}'(s) \right|\right| = 0.$$

It probably comes as no surprise that the curvature of a line is 0. (How "curvy" is a line? It is not curvy at all.)

While the definition of curvature is a beautiful mathematical concept, it is nearly impossible to use most of the time; writing $\vec{r}$ in terms of the arc length parameter is generally very hard. Fortunately, there are other methods of calculating this value that are much easier. There is a tradeoff: the definition is "easy" to understand though hard to compute, whereas these other formulas are easy to compute though it may be hard to understand why they work.

Notes:

Theorem 102 Formulas for Curvature

Let C be a smooth curve on an open interval I in the plane or in space.

1. If C is defined by $y = f(x)$, then
$$\kappa = \frac{|f''(x)|}{\Big(1+\big(f'(x)\big)^2\Big)^{3/2}}.$$

2. If C is defined as a vector–valued function in the plane, $\vec{r}(t) = \langle x(t), y(t)\rangle$, then
$$\kappa = \frac{|x'y'' - x''y'|}{\big((x')^2+(y')^2\big)^{3/2}}.$$

3. If C is defined in space by a vector–valued function $\vec{r}(t)$, then
$$\kappa = \frac{\|\,\vec{T}\,'(t)\,\|}{\|\,\vec{r}\,'(t)\,\|} = \frac{\|\,\vec{r}\,'(t)\times\vec{r}\,''(t)\,\|}{\|\,\vec{r}\,'(t)\,\|^3} = \frac{\vec{a}(t)\cdot\vec{N}(t)}{\|\,\vec{v}(t)\,\|^2}.$$

We practice using these formulas.

Example 11.32 Finding the curvature of a circle

Find the curvature of a circle with radius r, defined by $\vec{c}(t) = \langle r\cos t, r\sin t\rangle$.

Solution Before we start, we should expect the curvature of a circle to be constant, and not dependent on t. (Why?)

We compute κ using the second part of Theorem 102.

$$\begin{aligned}\kappa &= \frac{|(-r\sin t)(-r\sin t) - (-r\cos t)(r\cos t)|}{\big((-r\sin t)^2+(r\cos t)^2\big)^{3/2}}\\ &= \frac{r^2(\sin^2 t+\cos^2 t)}{\big(r^2(\sin^2 t+\cos^2 t)\big)^{3/2}}\\ &= \frac{r^2}{r^3} = \frac{1}{r}.\end{aligned}$$

We have found that a circle with radius r has curvature $\kappa = 1/r$.

Example 11.32 gives a great result. Before this example, if we were told

Notes:

"The curve has a curvature of 5 at point A," we would have no idea what this really meant. Is 5 "big" – does is correspond to a really sharp turn, or a not-so-sharp turn? Now we can think of 5 in terms of a circle with radius 1/5. Knowing the units (inches vs. miles, for instance) allows us to determine how sharply the curve is curving.

Let a point P on a smooth curve C be given, and let κ be the curvature of the curve at P. A circle that:

- passes through P,
- lies on the concave side of C,
- has a common tangent line as C at P and
- has radius $r = 1/\kappa$ (hence has curvature κ)

is the **osculating circle**, or **circle of curvature**, to C at P, and r is the **radius of curvature**. Figure 11.31 shows the graph of the curve seen earlier in Figure 11.30 and its osculating circles at A and B. A sharp turn corresponds to a circle with a small radius; a gradual turn corresponds to a circle with a large radius. Being able to think of curvature in terms of the radius of a circle is very useful. (The word "osculating" comes from a Latin word related to kissing; an osculating circle "kisses" the graph at a particular point. Many beautiful ideas in mathematics have come from studying the osculating circles to a curve.)

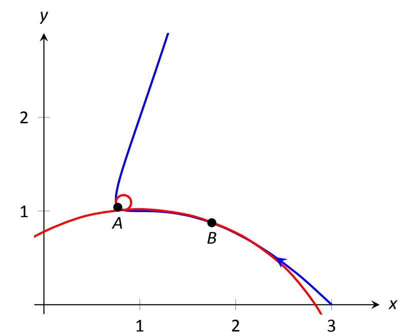


Figure 11.31: Illustrating the osculating circles for the curve seen in Figure 11.30.

Example 11.33 Finding curvature

Find the curvature of the parabola defined by $y = x^2$ at the vertex and at $x = 1$.

Solution We use the first formula found in Theorem 102.

$$\kappa(x) = \frac{|2|}{\left(1+(2x)^2\right)^{3/2}} = \frac{2}{\left(1+4x^2\right)^{3/2}}.$$

At the vertex ($x = 0$), the curvature is $\kappa = 2$. At $x = 1$, the curvature is $\kappa = 2/(5)^{3/2} \approx 0.179$. So at $x = 0$, the curvature of $y = x^2$ is that of a circle of radius $1/2$; at $x = 1$, the curvature is that of a circle with radius $\approx 1/0.179 \approx 5.59$. This is illustrated in Figure 11.32. At $x = 3$, the curvature is 0.009; the graph is nearly straight as the curvature is very close to 0.

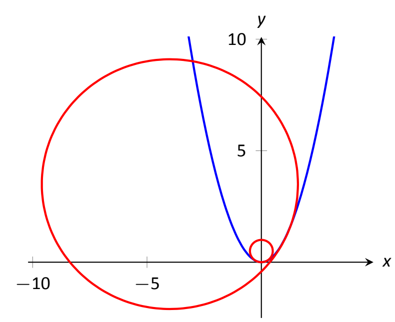


Figure 11.32: Examining the curvature of $y = x^2$.

Example 11.34 Finding curvature

Find where the curvature of $\vec{r}(t) = \langle t, t^2, 2t^3 \rangle$ is maximized.

Notes:

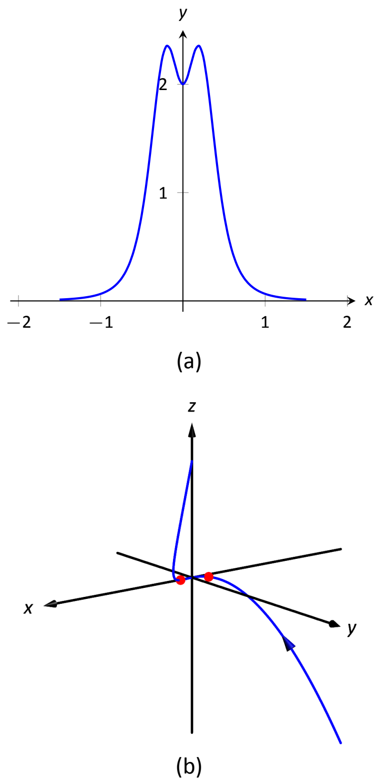


Figure 11.33: Understanding the curvature of a curve in space.

Solution We use the third formula in Theorem 102 as $\vec{r}(t)$ is defined in space. We leave it to the reader to verify that

$$\vec{r}\,'(t) = \left\langle 1, 2t, 6t^2 \right\rangle, \quad \vec{r}\,''(t) = \langle 0, 2, 12t \rangle, \quad \text{and} \quad \vec{r}\,'(t)\times\vec{r}\,''(t) = \left\langle 12t^2, -12t, 2 \right\rangle.$$

Thus

$$\begin{aligned}\kappa(t) &= \frac{||\,\vec{r}\,'(t)\times\vec{r}\,''(t)\,||}{||\,\vec{r}\,'(t)\,||^3}\\ &= \frac{||\,\left\langle 12t^2, -12t, 2\right\rangle\,||}{||\,\left\langle 1, 2t, 6t^2\right\rangle\,||^3}\\ &= \frac{\sqrt{144t^4+144t^2+4}}{\left(\sqrt{1+4t^2+36t^4}\right)^3}\end{aligned}$$

While this is not a particularly "nice" formula, it does explictly tell us what the curvature is at a given t value. To maximize $\kappa(t)$, we should solve $\kappa'(t) = 0$ for t. This is doable, but *very* time consuming. Instead, consider the graph of $\kappa(t)$ as given in Figure 11.33(a). We see that κ is maximized at two t values; using a numerical solver, we find these values are $t \approx \pm 0.189$. In part (b) of the figure we graph $\vec{r}(t)$ and indicate the points where curvature is maximized.

Curvature and Motion

Let $\vec{r}(t)$ be a position function of an object, with velocity $\vec{v}(t) = \vec{r}\,'(t)$ and acceleration $\vec{a}(t) = \vec{r}\,''(t)$. In Section 11.4 we established that acceleration is in the plane formed by $\vec{T}(t)$ and $\vec{N}(t)$, and that we can find scalars a_{T} and a_{N} such that

$$\vec{a}(t) = a_{\text{T}}\vec{T}(t) + a_{\text{N}}\vec{N}(t).$$

Theorem 100 gives formulas for a_{T} and a_{N}:

$$a_{\text{T}} = \frac{d}{dt}\Big(||\,\vec{v}(t)\,||\Big) \quad \text{and} \quad a_{\text{N}} = \frac{||\,\vec{v}(t)\times\vec{a}(t)\,||}{||\,\vec{v}(t)\,||}.$$

We understood that the amount of acceleration in the direction of $\vec{T}$ relates only to how the speed of the object is changing, and that the amount of acceleration in the direction of $\vec{N}$ relates to how the direction of travel of the object is changing. (That is, if the object travels at constant speed, $a_{\text{T}} = 0$; if the object travels in a constant direction, $a_{\text{N}} = 0$.)

In Equation (11.2) at the beginning of this section, we found $s\,'(t) = ||\,\vec{v}(t)\,||$. We can combine this fact with the above formula for a_{T} to write

$$a_{\text{T}} = \frac{d}{dt}\Big(||\,\vec{v}(t)\,||\Big) = \frac{d}{dt}\big(s\,'(t)\big) = s\,''(t).$$

Notes:

Since $s'(t)$ is speed, $s''(t)$ is the rate at which speed is changing with respect to time. We see once more that the component of acceleration in the direction of travel relates only to speed, not to a change in direction.

Now compare the formula for a_{N} above to the formula for curvature in Theorem 102:

$$a_{\text{N}} = \frac{||\,\vec{v}(t) \times \vec{a}(t)\,||}{||\,\vec{v}(t)\,||} \quad \text{and} \quad \kappa = \frac{||\,\vec{r}\,'(t) \times \vec{r}\,''(t)\,||}{||\,\vec{r}\,'(t)\,||^3} = \frac{||\,\vec{v}(t) \times \vec{a}(t)\,||}{||\,\vec{v}(t)\,||^3}.$$

Thus

$$\begin{aligned} a_{\text{N}} &= \kappa||\,\vec{v}(t)\,||^2 \qquad (11.5)\\ &= \kappa\Big(s'(t)\Big)^2 \end{aligned}$$

This last equation shows that the component of acceleration that changes the object's direction is dependent on two things: the curvature of the path and the speed of the object.

Imagine driving a car in a clockwise circle. You will naturally feel a force pushing you towards the door (more accurately, the door is pushing you as the car is turning and you want to travel in a straight line). If you keep the radius of the circle constant but speed up (i.e., increasing $s'(t)$), the door pushes harder against you (a_{N} has increased). If you keep your speed constant but tighten the turn (i.e., increase κ), once again the door will push harder against you.

Putting our new formulas for a_{T} and a_{N} together, we have

$$\vec{a}(t) = s''(t)\vec{T}(t) + \kappa||\,\vec{v}(t)\,||^2\vec{N}(t).$$

This is not a particularly practical way of finding a_{T} and a_{N}, but it reveals some great concepts about how acceleration interacts with speed and the shape of a curve.

Example 11.35 Curvature and road design

The minimum radius of the curve in a highway cloverleaf is determined by the operating speed, as given in the table in Figure 11.34. For each curve and speed, compute a_{N}.

Solution Using Equation (11.5), we can compute the acceleration normal to the curve in each case. We start by converting each speed from "miles per hour" to "feet per second" by multiplying by $5280/3600$.

Operating Speed (mph)	Minimum Radius (ft)
35	310
40	430
45	540

Figure 11.34: Operating speed and minimum radius in highway cloverleaf design.

Notes:

$$\begin{aligned}
\text{35mph, 310ft} &\Rightarrow \text{51.33ft/s}, \quad \kappa = 1/310 \\
a_\text{N} &= \kappa \mid\mid \vec{v}(t) \mid\mid^2 \\
&= \frac{1}{310}\big(51.33\big)^2 \\
&= 8.50\text{ft/s}^2.
\end{aligned}$$

$$\begin{aligned}
\text{40mph, 430ft} &\Rightarrow \text{58.67ft/s}, \quad \kappa = 1/430 \\
a_\text{N} &= \frac{1}{430}\big(58.67\big)^2 \\
&= 8.00\text{ft/s}^2.
\end{aligned}$$

$$\begin{aligned}
\text{45mph,540ft} &\Rightarrow \text{66ft/s}, \quad \kappa = 1/540 \\
a_\text{N} &= \frac{1}{540}\big(66\big)^2 \\
&= 8.07\text{ft/s}^2.
\end{aligned}$$

Note that each acceleration is similar; this is by design. Considering the classic "Force $=$ mass $\times$ acceleration" formula, this acceleration must be kept small in order for the tires of a vehicle to keep a "grip" on the road. If one travels on a turn of radius 310ft at a rate of 50mph, the acceleration is double, at 17.35ft/s^2. If the acceleration is too high, the frictional force created by the tires may not be enough to keep the car from sliding. Civil engineers routinely compute a "safe" design speed, then subtract 5-10mph to create the posted speed limit for additional safety.

We end this chapter with a reflection on what we've covered. We started with vector–valued functions, which may have seemed at the time to be just another way of writing parametric equations. However, we have seen that the vector perspective has given us great insight into the behavior of functions and the study of motion. Vector–valued position functions convey displacement, distance traveled, speed, velocity, acceleration and curvature information, each of which has great importance in science and engineering.

Notes:

Exercises 11.5

Terms and Concepts

1. It is common to describe position in terms of both ________ and/or ________.

2. A measure of the "curviness" of a curve is ________.

3. Give two shapes with constant curvature.

4. Describe in your own words what an "osculating circle" is.

5. Complete the identity: $\vec{T}\,'(s) =$ ________$\vec{N}(s)$.

6. Given a position function $\vec{r}(t)$, how are a_{T} and a_{N} affected by the curvature?

Problems

In Exercises 7 – 10 , a position function $\vec{r}(t)$ is given, where $t = 0$ corresponds to the initial position. Find the arc length parameter s, and rewrite $\vec{r}(t)$ in terms of s; that is, find $\vec{r}(s)$.

7. $\vec{r}(t) = \langle 2t, t, -2t \rangle$

8. $\vec{r}(t) = \langle 7\cos t, 7\sin t \rangle$

9. $\vec{r}(t) = \langle 3\cos t, 3\sin t, 2t \rangle$

10. $\vec{r}(t) = \langle 5\cos t, 13\sin t, 12\cos t \rangle$

In Exercises 11 – 22 , a curve C is described along with 2 points on C.

(a) Using a sketch, determine at which of these points the curvature is greater.

(b) Find the curvature κ of C, and evaluate κ at each of the 2 given points.

11. C is defined by $y = x^3 - x$; points given at $x = 0$ and $x = 1/2$.

12. C is defined by $y = \dfrac{1}{x^2+1}$; points given at $x = 0$ and $x = 2$.

13. C is defined by $y = \cos x$; points given at $x = 0$ and $x = \pi/2$.

14. C is defined by $y = \sqrt{1-x^2}$ on $(-1, 1)$; points given at $x = 0$ and $x = 1/2$.

15. C is defined by $\vec{r}(t) = \langle \cos t, \sin(2t) \rangle$; points given at $t = 0$ and $t = \pi/4$.

16. C is defined by $\vec{r}(t) = \left\langle \cos^2 t, \sin t\cos t \right\rangle$; points given at $t = 0$ and $t = \pi/3$.

17. C is defined by $\vec{r}(t) = \left\langle t^2 - 1, t^3 - t \right\rangle$; points given at $t = 0$ and $t = 5$.

18. C is defined by $\vec{r}(t) = \langle \tan t, \sec t \rangle$; points given at $t = 0$ and $t = \pi/6$.

19. C is defined by $\vec{r}(t) = \langle 4t+2, 3t-1, 2t+5 \rangle$; points given at $t = 0$ and $t = 1$.

20. C is defined by $\vec{r}(t) = \left\langle t^3 - t, t^3 - 4, t^2 - 1 \right\rangle$; points given at $t = 0$ and $t = 1$.

21. C is defined by $\vec{r}(t) = \langle 3\cos t, 3\sin t, 2t \rangle$; points given at $t = 0$ and $t = \pi/2$.

22. C is defined by $\vec{r}(t) = \langle 5\cos t, 13\sin t, 12\cos t \rangle$; points given at $t = 0$ and $t = \pi/2$.

In Exercises 23 – 26 , find the value of x or t where curvature is maximized.

23. $y = \dfrac{1}{6}x^3$

24. $y = \sin x$

25. $\vec{r}(t) = \left\langle t^2 + 2t, 3t - t^2 \right\rangle$

26. $\vec{r}(t) = \langle t, 4/t, 3/t \rangle$

In Exercises 27 – 30 , find the radius of curvature at the indicated value.

27. $y = \tan x$, at $x = \pi/4$

28. $y = x^2 + x - 3$, at $x = \pi/4$

29. $\vec{r}(t) = \langle \cos t, \sin(3t) \rangle$, at $t = 0$

30. $\vec{r}(t) = \langle 5\cos(3t), t \rangle$, at $t = 0$

In Exercises 31 – 34 , find the equation of the osculating circle to the curve at the indicated t-value.

31. $\vec{r}(t) = \left\langle t, t^2 \right\rangle$, at $t = 0$

32. $\vec{r}(t) = \langle 3\cos t, \sin t \rangle$, at $t = 0$

33. $\vec{r}(t) = \langle 3\cos t, \sin t \rangle$, at $t = \pi/2$

34. $\vec{r}(t) = \left\langle t^2 - t, t^2 + t \right\rangle$, at $t = 0$

12: Functions of Several Variables

A function of the form $y = f(x)$ is a function of a single variable; given a value of x, we can find a value y. Even the vector–valued functions of Chapter 11 are single–variable functions; the input is a single variable though the output is a vector.

There are many situations where a desired quantity is a function of two or more variables. For instance, wind chill is measured by knowing the temperature and wind speed; the volume of a gas can be computed knowing the pressure and temperature of the gas; to compute a baseball player's batting average, one needs to know the number of hits and the number of at–bats.

This chapter studies **multivariable** functions, that is, functions with more than one input.

12.1 Introduction to Multivariable Functions

Definition 81 Function of Two Variables

Let D be a subset of $\mathbb{R}^2$. A **function f of two variables** is a rule that assigns each pair (x, y) in D a value $z = f(x, y)$ in $\mathbb{R}$. D is the **domain** of f; the set of all outputs of f is the **range**.

Example 12.1 Understanding a function of two variables

Let $z = f(x, y) = x^2 - y$. Evaluate $f(1, 2)$, $f(2, 1)$, and $f(-2, 4)$; find the domain and range of f.

Solution Using the definition $f(x, y) = x^2 - y$, we have:

$$\begin{aligned} f(1,2) &= 1^2 - 2 = -1 \\ f(2,1) &= 2^2 - 1 = 3 \\ f(-2,4) &= (-2)^2 - 4 = 0 \end{aligned}$$

The domain is not specified, so we take it to be all possible pairs in $\mathbb{R}^2$ for which f is defined. In this example, f is defined for *all* pairs (x, y), so the domain D of f is $\mathbb{R}^2$.

The output of f can be made as large or small as possible; any real number r can be the output. (In fact, given any real number r, $f(0, -r) = r$.) So the range R of f is $\mathbb{R}$.

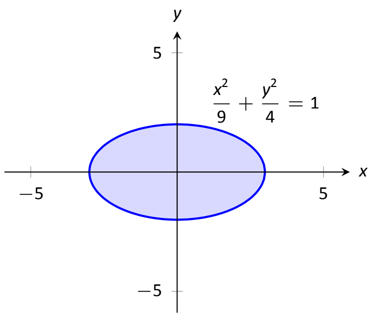


Figure 12.1: Illustrating the domain of $f(x, y)$ in Example 12.2.

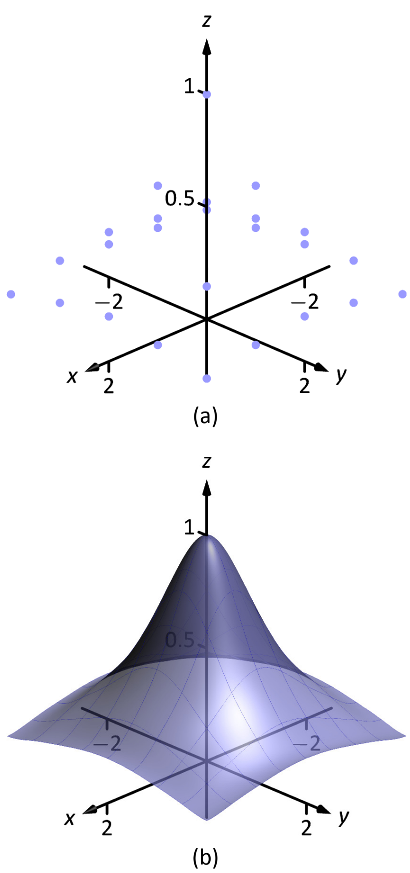


Figure 12.2: Graphing a function of two variables.

Example 12.2 **Understanding a function of two variables**

Let $f(x, y) = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}$. Find the domain and range of f.

Solution The domain is all pairs (x, y) allowable as input in f. Because of the square–root, we need (x, y) such that $0 \leq 1 - \frac{x^2}{9} - \frac{y^2}{4}$:

$$0 \leq 1 - \frac{x^2}{9} - \frac{y^2}{4}$$
$$\frac{x^2}{9} + \frac{y^2}{4} \leq 1$$

The above equation describes the interior of an ellipse as shown in Figure 12.1. We can represent the domain D graphically with the figure; in set notation, we can write $D = \{(x, y) | \frac{x^2}{9} + \frac{y^2}{4} \leq 1\}$.

The range is the set of all possible output values. The square–root ensures that all output is ≥ 0. Since the x and y terms are squared, then subtracted, inside the square–root, the largest output value comes at $x = 0$, $y = 0$: $f(0, 0) = 1$. Thus the range R is the interval $[0, 1]$.

Graphing Functions of Two Variables

The **graph** of a function f of two variables is the set of all points $\big(x, y, f(x, y)\big)$ where (x, y) is in the domain of f. This creates a **surface** in space.

One can begin sketching a graph by plotting points, but this has limitations. Consider Figure 12.2(a) where 25 points have been plotted of $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$. More points have been plotted than one would reasonably want to do by hand, yet it is not clear at all what the graph of the function looks like. Technology allows us to plot lots of points, connect adjacent points with lines and add shading to create a graph like Figure 12.2b which does a far better job of illustrating the behavior of f.

While technology is readily available to help us graph functions of two variables, there is still a paper–and–pencil approach that is useful to understand and master as it, combined with high–quality graphics, gives one great insight into the behavior of a function. This technique is known as sketching **level curves**.

Level Curves

It may be surprising to find that the problem of representing a three dimensional surface on paper is familiar to most people (they just don't realize it). Topographical maps, like the one shown in Figure 12.3, represent the surface of Earth by indicating points with the same elevation with **contour lines**. The

Notes:

elevations marked are equally spaced; in this example, each thin line indicates an elevation change in 50ft increments and each thick line indicates a change of 200ft. When lines are drawn close together, elevation changes rapidly (as one does not have to travel far to rise 50ft). When lines are far apart, such as near "Aspen Campground," elevation changes more gradually as one has to walk farther to rise 50ft.

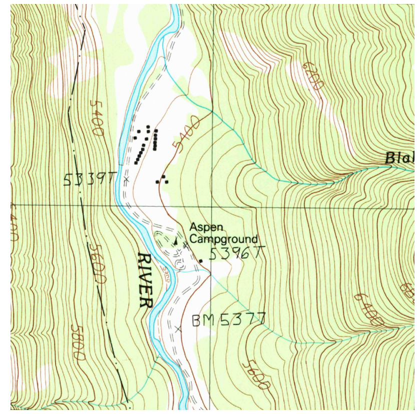


Figure 12.3: A topographical map displays elevation by drawing contour lines, along with the elevation is constant.
Sample taken from the public domain USGS Digital Raster Graphics, http://topmaps.usgs.gove/drg/.

Given a function $z = f(x, y)$, we can draw a "topographical map" of f by drawing **level curves** (or, contour lines). A level curve at $z = c$ is a curve in the x-y plane such that for all points (x, y) on the curve, $f(x, y) = c$.

When drawing level curves, it is important that the c values are spaced equally apart as that gives the best insight to how quickly the "elevation" is changing. Examples will help one understand this concept.

Example 12.3 Drawing Level Curves

Let $f(x, y) = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}$. Find the level curves of f for $c = 0$, 0.2, 0.4, 0.6, 0.8 and 1.

Solution Consider first $c = 0$. The level curve for $c = 0$ is the set of all points (x, y) such that $0 = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}$. Squaring both sides gives us

$$\frac{x^2}{9} + \frac{y^2}{4} = 1,$$

an ellipse centered at $(0, 0)$ with horizontal major axis of length 6 and minor axis of length 4. Thus for any point (x, y) on this curve, $f(x, y) = 0$.

Now consider the level curve for $c = 0.2$

$$\begin{aligned}
0.2 &= \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}} \\
0.04 &= 1 - \frac{x^2}{9} - \frac{y^2}{4} \\
\frac{x^2}{9} + \frac{y^2}{4} &= 0.96 \\
\frac{x^2}{8.64} + \frac{y^2}{3.84} &= 1.
\end{aligned}$$

This is also an ellipse, where $a = \sqrt{8.64} \approx 2.94$ and $b = \sqrt{3.84} \approx 1.96$.

Notes:

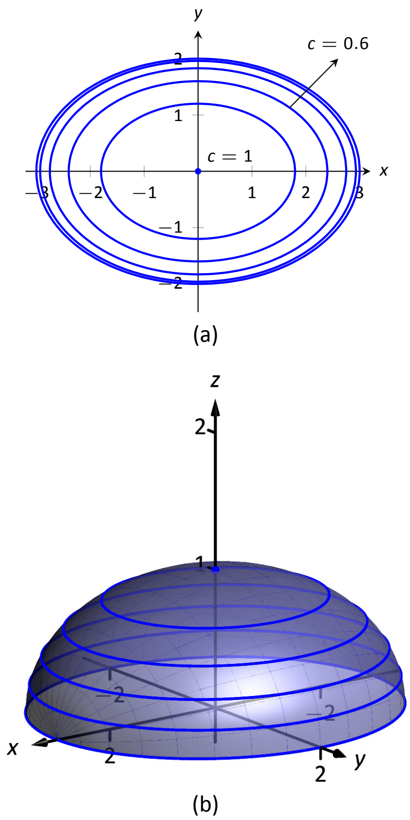


Figure 12.4: Graphing the level curves in Example 12.3.

In general, for $z = c$, the level curve is:

$$\begin{aligned} c &= \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}} \\ c^2 &= 1 - \frac{x^2}{9} - \frac{y^2}{4} \\ \frac{x^2}{9} + \frac{y^2}{4} &= 1 - c^2 \\ \frac{x^2}{9(1-c^2)} + \frac{y^2}{4(1-c^2)} &= 1, \end{aligned}$$

ellipses that are decreasing in size as c increases. A special case is when $c = 1$; there the ellipse is just the point $(0, 0)$.

The level curves are shown in Figure 12.4(a). Note how the level curves for $c = 0$ and $c = 0.2$ are very, very close together: this indicates that f is growing rapidly along those curves.

In Figure 12.4(b), the curves are drawn on a graph of f in space. Note how the elevations are evenly spaced. Near the level curves of $c = 0$ and $c = 0.2$ we can see that f indeed is growing quickly.

Example 12.4 **Analyzing Level Curves**

Let $f(x, y) = \dfrac{x+y}{x^2+y^2+1}$. Find the level curves for $z = c$.

Solution We begin by setting $f(x, y) = c$ for an arbitrary c and seeing if algebraic manipulation of the equation reveals anything significant.

$$\begin{aligned} \frac{x+y}{x^2+y^2+1} &= c \\ x + y &= c(x^2 + y^2 + 1). \end{aligned}$$

We recognize this as a circle, though the center and radius are not yet clear. By completing the square, we can obtain:

$$\left(x - \frac{1}{2c}\right)^2 + \left(y - \frac{1}{2c}\right)^2 = \frac{1}{2c^2} - 1,$$

a circle centered at $\big(1/(2c), 1/(2c)\big)$ with radius $\sqrt{1/(2c^2) - 1}$, where $|c| < 1/\sqrt{2}$. The level curves for $c = \pm 0.2$, ± 0.4 and ± 0.6 are sketched in Figure 12.5(a). To help illustrate "elevation," we use thicker lines for c values near 0, and dashed lines indicate where $c < 0$.

There is one special level curve, when $c = 0$. The level curve in this situation is $x + y = 0$, the line $y = -x$.

Notes:

In Figure 12.5(b) we see a graph of the surface. Note how the y-axis is pointing away from the viewer to more closely resemble the orientation of the level curves in (a).

Seeing the level curves helps us understand the graph. For instance, the graph does not make it clear that one can "walk" along the line $y = -x$ without elevation change, though the level curve does.

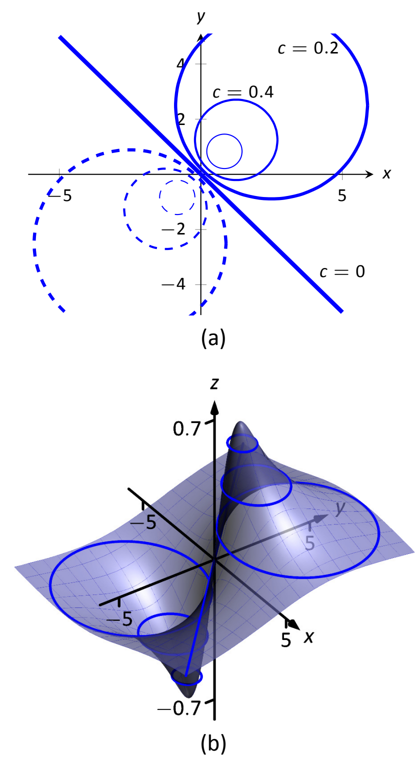


Figure 12.5: Graphing the level curves in Example 12.4.

Functions of Three Variables

We extend our study of multivariable functions to functions of three variables. (One can make a function of as many variables as one likes; we limit our study to three variables.)

Definition 82 Function of Three Variables

Let D be a subset of $\mathbb{R}^3$. A **function f of three variables** is a rule that assigns each triple (x, y, z) in D a value $w = f(x, y, z)$ in $\mathbb{R}$. D is the **domain** of f; the set of all outputs of f is the **range**.

Note how this definition closely resembles that of Definition 81.

Example 12.5 Understanding a function of three variables

Let $f(x, y, z) = \dfrac{x^2 + z + 3\sin y}{x + 2y - z}$. Evaluate f at the point $(3, 0, 2)$ and find the domain and range of f.

Solution $$f(3, 0, 2) = \frac{3^2 + 2 + 3\sin 0}{3 + 2(0) - 2} = 11.$$

As the domain of f is not specified, we take it to be the set of all triples (x, y, z) for which $f(x, y, z)$ is defined. As we cannot divide by 0, we find the domain D is

$$D = \{(x, y, z) \mid x + 2y - z \neq 0\}.$$

We recognize that the set of all points in $\mathbb{R}^3$ that *are not* in D form a plane in space that passes through the origin (with normal vector $\langle 1, 2, -1\rangle$).

We determine the range R is $\mathbb{R}$; that is, all real numbers are possible outputs of f. There is no set way of establishing this. Rather, to get numbers near 0 we can let $y = 0$ and choose $z \approx -x^2$. To get numbers of arbitrarily large magnitude, we can let $z \approx x + 2y$.

Notes:

Level Surfaces

It is very difficult to produce a meaningful graph of a function of three variables. A function of *one* variable is a *curve* drawn in *2* dimensions; a function of *two* variables is a *surface* drawn in *3* dimensions; a function of *three* variables is a *hypersurface* drawn in *4* dimensions.

There are a few techniques one can employ to try to "picture" a graph of three variables. One is an analogue of level curves: **level surfaces**. Given $w = f(x, y, z)$, the level surface at $w = c$ is the surface in space formed by all points (x, y, z) where $f(x, y, z) = c$.

Example 12.6 Finding level surfaces

If a point source S is radiating energy, the intensity I at a given point P in space is inversely proportional to the square of the distance between S and P. That is, when $S = (0, 0, 0)$, $I(x, y, z) = \dfrac{k}{x^2 + y^2 + z^2}$ for some constant k.

Let $k = 1$; find the level surfaces of I.

Solution We can (mostly) answer this question using "common sense." If energy (say, in the form of light) is emanating from the origin, its intensity will be the same at all points equidistant from the origin. That is, at any point on the surface of a sphere centered at the origin, the intensity should be the same. Therefore, the level surfaces are spheres.

We now find this mathematically. The level surface at $I = c$ is defined by

$$c = \frac{1}{x^2 + y^2 + z^2}.$$

A small amount of algebra reveals

$$x^2 + y^2 + z^2 = \frac{1}{c}.$$

Given an intensity c, the level surface $I = c$ is a sphere of radius $1/\sqrt{c}$, centered at the origin.

Figure 12.6 gives a table of the radii of the spheres for given c values. Normally one would use equally spaced c values, but these values have been chosen purposefully. At a distance of 0.25 from the point source, the intensity is 16; to move to a point of half that intensity, one just moves out 0.1 to 0.35 – not much at all. To again halve the intensity, one moves 0.15, a little more than before.

Note how each time the intensity if halved, the distance required to move away grows. We conclude that the closer one is to the source, the more rapidly the intensity changes.

c	r
16.	0.25
8.	0.35
4.	0.5
2.	0.71
1.	1.
0.5	1.41
0.25	2.
0.125	2.83
0.0625	4.

Figure 12.6: A table of c values and the corresponding radius r of the spheres of constant value in Example 12.6.

In the next section we apply the concepts of limits to functions of two or more variables.

Notes:

Exercises 12.1

Terms and Concepts

1. Give two examples (other than those given in the text) of "real world" functions that require more than one input.

2. The graph of a function of two variables is a ________.

3. Most people are familiar with the concept of level curves in the context of ________ maps.

4. T/F: Along a level curve, the output of a function does not change.

5. The analogue of a level curve for functions of three variables is a level ________.

6. What does it mean when level curves are close together? Far apart?

Problems

In Exercises 7 – 14, give the domain and range of the multivariable function.

7. $f(x,y) = x^2 + y^2 + 2$

8. $f(x,y) = x + 2y$

9. $f(x,y) = x - 2y$

10. $f(x,y) = \dfrac{1}{x+2y}$

11. $f(x,y) = \dfrac{1}{x^2+y^2+1}$

12. $f(x,y) = \sin x \cos y$

13. $f(x,y) = \sqrt{9 - x^2 - y^2}$

14. $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2-9}}$

In Exercises 15 – 22, describe in words and sketch the level curves for the function and given c values.

15. $f(x,y) = 3x - 2y$; $c = -2, 0, 2$

16. $f(x,y) = x^2 - y^2$; $c = -1, 0, 1$

17. $f(x,y) = x - y^2$; $c = -2, 0, 2$

18. $f(x,y) = \dfrac{1 - x^2 - y^2}{2y - 2x}$; $c = -2, 0, 2$

19. $f(x,y) = \dfrac{2x - 2y}{x^2 + y^2 + 1}$; $c = -1, 0, 1$

20. $f(x,y) = \dfrac{y - x^3 - 1}{x}$; $c = -3, -1, 0, 1, 3$

21. $f(x,y) = \sqrt{x^2 + 4y^2}$; $c = 1, 2, 3, 4$

22. $f(x,y) = x^2 + 4y^2$; $c = 1, 2, 3, 4$

In Exercises 23 – 26, give the domain and range of the functions of three variables.

23. $f(x,y,z) = \dfrac{x}{x + 2y - 4z}$

24. $f(x,y,z) = \dfrac{1}{1 - x^2 - y^2 - z^2}$

25. $f(x,y,z) = \sqrt{z - x^2 + y^2}$

26. $f(x,y,z) = z^2 \sin x \cos y$

In Exercises 27 – 30, describe the level surfaces of the given functions of three variables.

27. $f(x,y,z) = x^2 + y^2 + z^2$

28. $f(x,y,z) = z - x^2 + y^2$

29. $f(x,y,z) = \dfrac{x^2 + y^2}{z}$

30. $f(x,y,z) = \dfrac{z}{x - y}$

31. Compare the level curves of Exercises 21 and 22. How are they similar, and how are they different? Each surface is a quadric surface; describe how the level curves are consistent with what we know about each surface.

12.2 Limits and Continuity of Multivariable Functions

We continue with the pattern we have established in this text: after defining a new kind of function, we apply calculus ideas to it. The previous section defined functions of two and three variables; this section investigates what it means for these functions to be "continuous."

We begin with a series of definitions. We are used to "open intervals" such as $(1,3)$, which represents the set of all x such that $1 < x < 3$, and "closed intervals" such as $[1,3]$, which represents the set of all x such that $1 \leq x \leq 3$. We need analogous definitions for open and closed sets in the x-y plane.

Definition 83 Open Disk, Boundary and Interior Points, Open and Closed Sets, Bounded Sets

An **open disk** B in $\mathbb{R}^2$ centered at (x_0, y_0) with radius r is the set of all points (x, y) such that $\sqrt{(x-x_0)^2+(y-y_0)^2} < r$.

Let S be a set of points in $\mathbb{R}^2$. A point P in $\mathbb{R}^2$ is a **boundary point** of S if all open disks centered at P contain both points in S and points not in S.

A point P in S is an **interior point** of S if there is an open disk centered at P that contains only points in S.

A set S is **open** if every point in S is an interior point.

A set S is **closed** if it contains all of its boundary points.

A set S is **bounded** if there is an $M > 0$ such that the open disk, centered at the origin with radius M, contains S. A set that is not bounded is **unbounded**.

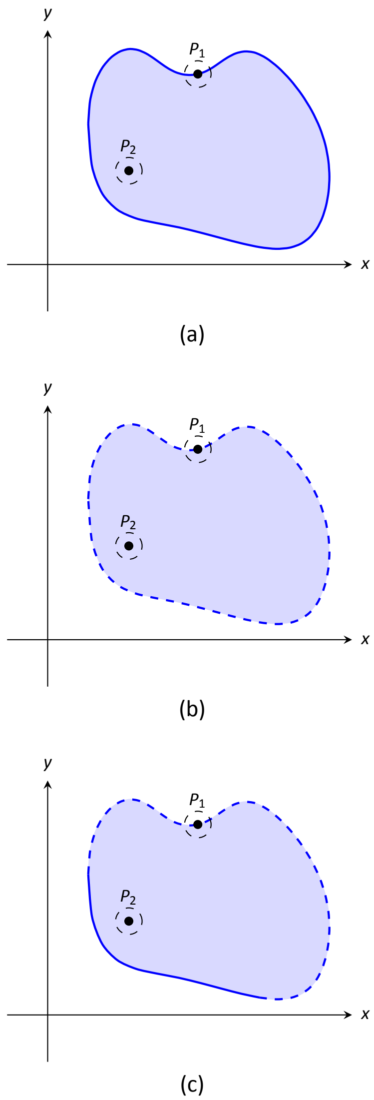


Figure 12.7: Illustrating open and closed sets in the x-y plane.

Figure 12.7 shows several sets in the x-y plane. In each set, point P_1 lies on the boundary of the set as all open disks centered there contain both points in, and not in, the set. In contrast, point P_2 is an interior point for there is an open disk centered there that lies entirely within the set.

The set depicted in Figure 12.7(a) is a closed set as it contains all of its boundary points. The set in (b) is open, for all of its points are interior points (or, equivalently, it does not contain any of its boundary points). The set in (c) is neither open nor closed as it contains some of its boundary points.

Notes:

Example 12.7 Determining open/closed, bounded/unbounded

Determine if the domain of the function $f(x,y) = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}$ is open, closed, or neither, and if it is bounded.

Solution This domain of this function was found in Example 12.2 to be $D = \{(x,y) \mid \frac{x^2}{9} + \frac{y^2}{4} \le 1\}$, the region *bounded* by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$. Since the region includes the boundary (indicated by the use of "$\le$"), the set contains all of its boundary points and hence is closed. The region is bounded as a disk of radius 4, centered at the origin, contains D.

Example 12.8 Determining open/closed, bounded/unbounded

Determine if the domain of $f(x,y) = \frac{1}{x-y}$ is open, closed, or neither.

Solution As we cannot divide by 0, we find the domain to be $D = \{(x,y) \mid x - y \neq 0\}$. In other words, the domain is the set of all points (x,y) *not* on the line $y = x$.

The domain is sketched in Figure 12.8. Note how we can draw an open disk around any point in the domain that lies entirely inside the domain, and also note how the only boundary points of the domain are the points on the line $y = x$. We conclude the domain is an open set. The set is unbounded.

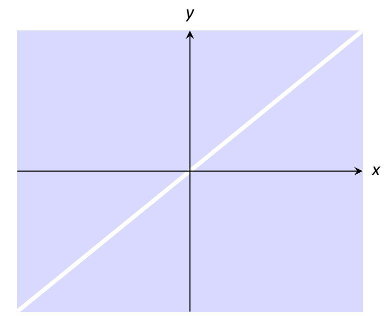


Figure 12.8: Sketching the domain of the function in Example 12.8.

Limits

Recall a pseudo–definition of the limit of a function of one variable: "$\lim_{x \to c} f(x) = L$" means that if x is "really close" to c, then $f(x)$ is "really close" to L. A similar pseudo–definition holds for functions of two variables. We'll say that

$$\text{“}\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L\text{”}$$

means "if the point (x,y) is really close to the point (x_0,y_0), then $f(x,y)$ is really close to L." The formal definition is given below.

Definition 84 Limit of a Function of Two Variables

Let S be an open set containing (x_0,y_0), and let f be a function of two variables defined on S, except possibly at (x_0,y_0). The **limit** of $f(x,y)$ as (x,y) approaches (x_0,y_0) is L, denoted

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L,$$

means that given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $(x,y) \neq (x_0,y_0)$, if (x,y) is in the open disk centered at (x_0,y_0) with radius δ, then $|f(x,y) - L| < \varepsilon$.

Notes:

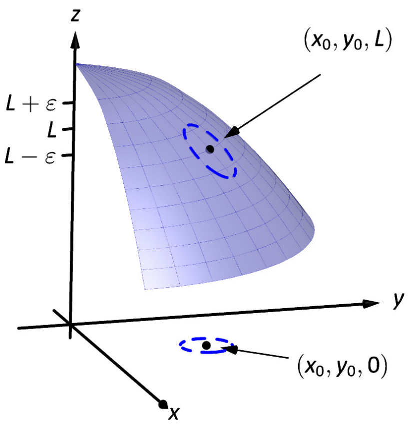


Figure 12.9: **Illustrating the definition of a limit.** The open disk in the x-y plane has radius δ. Let (x, y) be any point in this disk; $f(x, y)$ is within ε of L.

The concept behind Definition 84 is sketched in Figure 12.9. Given $\varepsilon > 0$, find $\delta > 0$ such that if (x, y) is any point in the open disk centered at (x_0, y_0) in the x-y plane with radius δ, then $f(x, y)$ should be within ε of L.

Computing limits using this definition is rather cumbersome. The following theorem allows us to evaluate limits much more easily.

Theorem 103 Basic Limit Properties of Functions of Two Variables

Let b, x_0, y_0, L and K be real numbers, let n be a positive integer, and let f and g be functions with the following limits:

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x, y) = K.$$

The following limits hold.

1. Constants: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} b = b$
2. Identity $\displaystyle\lim_{(x,y)\to(x_0,y_0)} x = x_0; \quad \lim_{(x,y)\to(x_0,y_0)} y = y_0$
3. Sums/Differences: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \big(f(x, y) \pm g(x, y)\big) = L \pm K$
4. Scalar Multiples: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} b \cdot f(x, y) = bL$
5. Products: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x, y) \cdot g(x, y) = LK$
6. Quotients: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x, y)/g(x, y) = L/K, (K \neq 0)$
7. Powers: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x, y)^n = L^n$

This theorem, combined with Theorems 2 and 3 of Section 1.3, allows us to evaluate many limits.

Example 12.9 Evaluating a limit

Evaluate the following limits:

$$1.\ \lim_{(x,y)\to(1,\pi)} \frac{y}{x} + \cos(xy) \qquad 2.\ \lim_{(x,y)\to(0,0)} \frac{3xy}{x^2 + y^2}$$

Notes:

Solution

1. The aforementioned theorems allow us to simply evaluate $y/x + \cos(xy)$ when $x = 1$ and $y = \pi$. If an indeterminate form is returned, we must do more work to evaluate the limit; otherwise, the result is the limit. Therefore

$$\begin{aligned}\lim_{(x,y)\to(1,\pi)} \frac{y}{x} + \cos(xy) &= \frac{\pi}{1} + \cos\pi \\ &= \pi - 1.\end{aligned}$$

2. We attempt to evaluate the limit by substituting 0 in for x and y, but the result is the indeterminate form "0/0." To evaluate this limit, we must "do more work," but we have not yet learned what "kind" of work to do. Therefore we cannot yet evaluate this limit.

When dealing with functions of a single variable we also considered one–sided limits and stated

$$\lim_{x\to c} f(x) = L \quad \text{if, and only if,} \quad \lim_{x\to c^+} f(x) = L \quad \textbf{and} \quad \lim_{x\to c^-} f(x) = L.$$

That is, the limit is L if and only if $f(x)$ approaches L when x approaches c from **either** direction, the left or the right.

In the plane, there are infinite directions from which (x, y) might approach (x_0, y_0). In fact, we do not have to restrict ourselves to approaching (x_0, y_0) from a particular direction, but rather we can approach that point along a path that is not a straight line. It is possible to arrive at different limiting values by approaching (x_0, y_0) along different paths. If this happens, we say that $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist (this is analogous to the left and right hand limits of single variable functions not being equal).

Our theorems tell us that we can evaluate most limits quite simply, without worrying about paths. When indeterminate forms arise, the limit may or may not exist. If it does exist, it can be difficult to prove this as we need to show the same limiting value is obtained regardless of the path chosen. The case where the limit does not exist is often easier to deal with, for we can often pick two paths along which the limit is different.

Example 12.10 Showing limits do not exist

1. Show $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2 + y^2}$ does not exist by finding the limits along the lines $y = mx$.

Notes:

2. Show $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x+y}$ does not exist by finding the limit along the path $y = -\sin x$.

Solution

1. Evaluating $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^2}$ along the lines $y = mx$ means replace all y's with mx and evaluating the resulting limit:

$$\begin{aligned}\lim_{(x,mx)\to(0,0)} \frac{3x(mx)}{x^2+(mx)^2} &= \lim_{x\to0} \frac{3mx^2}{x^2(m^2+1)}\\ &= \lim_{x\to0}\frac{3m}{m^2+1}\\ &= \frac{3m}{m^2+1}.\end{aligned}$$

While the limit exists for each choice of m, we get a *different* limit for each choice of m. That is, along different lines we get differing limiting values, meaning *the* limit does not exist.

2. Let $f(x,y) = \frac{\sin(xy)}{x+y}$. We are to show that $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist by finding the limit along the path $y = -\sin x$. First, however, consider the limits found along the lines $y = mx$ as done above.

$$\begin{aligned}\lim_{(x,mx)\to(0,0)} \frac{\sin\big(x(mx)\big)}{x+mx} &= \lim_{x\to0}\frac{\sin(mx^2)}{x(m+1)}\\ &= \lim_{x\to0}\frac{\sin(mx^2)}{x}\cdot\frac{1}{m+1}.\end{aligned}$$

By applying L'Hôpital's Rule, we can show this limit is 0 *except* when $m = -1$, that is, along the line $y = -x$. This line is not in the domain of f, so we have found the following fact: along every line $y = mx$ in the domain of f, $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Now consider the limit along the path $y = -\sin x$:

$$\lim_{(x,-\sin x)\to(0,0)} \frac{\sin\big(-x\sin x\big)}{x-\sin x} = \lim_{x\to0}\frac{\sin\big(-x\sin x\big)}{x-\sin x}$$

Now apply L'Hôpital's Rule twice:

$$\begin{aligned}&= \lim_{x\to0}\frac{\cos\big(-x\sin x\big)(-\sin x - x\cos x)}{1-\cos x} \qquad (\text{“}=0/0\text{”})\\ &= \lim_{x\to0}\frac{-\sin\big(-x\sin x\big)(-\sin x - x\cos x)^2 + \cos\big(-x\sin x\big)(-2\cos x + x\sin x)}{\sin x}\\ &= \text{“}2/0\text{”} \Rightarrow \text{the limit does not exist.}\end{aligned}$$

Notes:

Step back and consider what we have just discovered. Along any line $y = mx$ in the domain of the $f(x, y)$, the limit is 0. However, along the path $y = -\sin x$, which lies in the domain of $f(x, y)$ for all $x \neq 0$, the limit does not exist. Since the limit is not the same along every path to $(0, 0)$, we say $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x+y}$ does not exist.

Example 12.11 Finding a limit

Let $f(x, y) = \dfrac{5x^2y^2}{x^2+y^2}$. Find $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y)$.

Solution It is relatively easy to show that along any line $y = mx$, the limit is 0. This is not enough to prove that the limit exists, as demonstrated in the previous example, but it tells us that if the limit does exist then it must be 0.

To prove the limit is 0, we apply Definition 84. Let $\varepsilon > 0$ be given. We want to find $\delta > 0$ such that if $\sqrt{(x-0)^2 + (y-0)^2} < \delta$, then $|f(x, y) - 0| < \varepsilon$.

Set $\delta < \sqrt{\varepsilon/5}$. Note that $\left|\dfrac{5y^2}{x^2+y^2}\right| < 5$ for all $(x, y) \neq (0, 0)$, and that if $\sqrt{x^2+y^2} < \delta$, then $x^2 < \delta^2$.

Let $\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2+y^2} < \delta$. Consider $|f(x, y) - 0|$:

$$\begin{aligned}
|f(x, y) - 0| &= \left|\frac{5x^2y^2}{x^2+y^2} - 0\right| \\
&= \left|x^2 \cdot \frac{5y^2}{x^2+y^2}\right| \\
&< \delta^2 \cdot 5 \\
&< \frac{\varepsilon}{5} \cdot 5 \\
&= \varepsilon.
\end{aligned}$$

Thus if $\sqrt{(x-0)^2 + (y-0)^2} < \delta$ then $|f(x, y) - 0| < \varepsilon$, which is what we wanted to show. Thus $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5x^2y^2}{x^2+y^2} = 0$.

Using Polar Coordinates to Determine a Limit as $(x, y) \to (0, 0)$

Recall from Section 9.4 that in polar coordinates, we have $x = r\cos\theta$, $y = r\sin\theta$, where r is a directed distance from the origin. Also recall that the origin $(0, 0)$ (called the *pole* in polar coordinates) is described by $r = 0$ for **every** θ. This gives us another way to compute a limit of a function in two variables *only* for the case $(x, y) \to (0, 0)$ (as points other than $(0, 0)$ depend on more than just r). If we convert to polar, then we can take the limit as r approaches 0. If

Notes:

this limit exists, and does not depend on θ, then it is the limit as $(x, y) \to (0, 0)$. If it depends on θ, then the limit does not exist. This is often easier but not always. In the next two example, we redo two earlier limits with this method, and obtain the same results.

Example 12.12 Finding a limit by conversion to polar coordinates

Let $f(x, y) = \dfrac{5x^2y^2}{x^2+y^2}$. Find $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y)$ by converting to polar coordinates.

Solution Converting to polar coordinates, and replacing "$(x, y) \to (0, 0)$" with "$r \to 0$", we obtain

$$\begin{aligned}\lim_{(x,y)\to(0,0)} f(x, y) &= \lim_{(x,y)\to(0,0)} \frac{5x^2y^2}{x^2+y^2}\\ &= \lim_{r\to0} \frac{5(r\cos\theta)^2(r\sin\theta)^2}{r^2}\\ &= \lim_{r\to0} 5r^2\cos^2\theta\sin^2\theta\\ &= 0.\end{aligned}$$

That is the same result we previously obtained in Example 12.11. Next we will redo the second limit in Example 12.9 by converting to polar coordinates and once again we will see the limit does not exist.

Example 12.13 Finding a limit by conversion to polar coordinates

Let $f(x, y) = \dfrac{3xy}{x^2+y^2}$. Find $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y)$ by converting to polar coordinates.

Solution Converting to polar coordinates, we obtain

$$\begin{aligned}\lim_{(x,y)\to(0,0)} f(x, y) &= \lim_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^2}\\ &= \lim_{r\to0} \frac{3r\cos\theta\cdot r\sin\theta}{r^2}\\ &= \lim_{r\to0} \frac{3r^2\cos\theta\sin\theta}{r^2}\\ &= \lim_{r\to0} 3\cos\theta\sin\theta\end{aligned}$$

which does not exist because it depends on θ. Therefore, we get different results along different paths headed to the origin, so the multivariable limit does

Notes:

not exist.

Continuity

Definition 3 defines what it means for a function of one variable to be continuous. In brief, it meant that the graph of the function did not have breaks, holes, jumps, etc. We define continuity for functions of two variables in a similar way as we did for functions of one variable.

Definition 85 Continuous

Let a function $f(x,y)$ be defined on an open disk B containing the point (x_0,y_0).

1. f is **continuous** at (x_0,y_0) if $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$.
2. f is **continuous on** B if f is continuous at all points in B. If f is continuous at all points in $\mathbb{R}^2$, we say that f is **continuous everywhere**.

Example 12.14 Continuity of a function of two variables

Let $f(x,y) = \begin{cases} \frac{\cos y \sin x}{x} & x \neq 0 \\ \cos y & x = 0 \end{cases}$. Is f continuous at $(0,0)$? Is f continuous everywhere?

Solution To determine if f is continuous at $(0,0)$, we need to compare $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ to $f(0,0)$.

Applying the definition of f, we see that $f(0,0) = \cos 0 = 1$.

We now consider the limit $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$. Substituting 0 for x and y in $(\cos y \sin x)/x$ returns the indeterminate form "0/0", so we need to do more work to evaluate this limit.

Consider two related limits: $\displaystyle\lim_{(x,y)\to(0,0)} \cos y$ and $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin x}{x}$. The first limit does not contain x, and since $\cos y$ is continuous,

$$\lim_{(x,y)\to(0,0)} \cos y = \lim_{y\to 0} \cos y = \cos 0 = 1.$$

The second limit does not contain y. By Theorem 5 (or L'Hôpital's Rule), we can say

$$\lim_{(x,y)\to(0,0)} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\sin x}{x} = 1.$$

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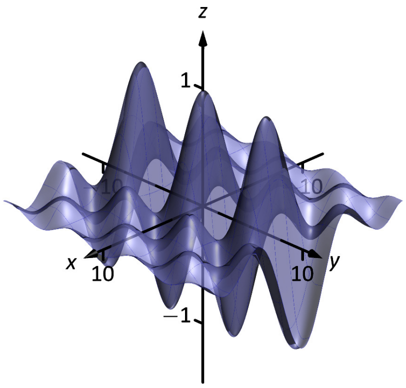


Figure 12.10: A graph of $f(x,y)$ in Example 12.14.

Finally, Theorem 103 of this section states that we can combine these two limits as follows:

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{\cos y \sin x}{x} &= \lim_{(x,y)\to(0,0)} (\cos y)\left(\frac{\sin x}{x}\right) \\
&= \left(\lim_{(x,y)\to(0,0)} \cos y\right)\left(\lim_{(x,y)\to(0,0)} \frac{\sin x}{x}\right) \\
&= (1)(1) \\
&= 1.
\end{aligned}$$

We have found that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\cos y \sin x}{x} = f(0,0)$, so f is continuous at $(0,0)$.

A similar analysis shows that f is continuous at all points in $\mathbb{R}^2$. As long as $x \neq 0$, we can evaluate the limit directly; when $x = 0$, a similar analysis shows that the limit is $\cos y$. Thus we can say that f is continuous everywhere. A graph of f is given in Figure 12.10. Notice how it has no breaks, jumps, etc.

The following theorem is very similar to Theorem 8, giving us ways to combine continuous functions to create other continuous functions.

Theorem 104 Properties of Continuous Functions

Let f and g be continuous on an open disk B, let c be a real number, and let n be a positive integer. The following functions are continuous on B.

1. Sums/Differences: $f \pm g$
2. Constant Multiples: $c \cdot f$
3. Products: $f \cdot g$
4. Quotients: f/g (as longs as $g \neq 0$ on B)
5. Powers: f^n
6. Roots: $\sqrt[n]{f}$ (if n is even then $f \geq 0$ on B; if n is odd, then true for all values of f on B.)
7. Compositions: Adjust the definitions of f and g to: Let f be continuous on B, where the range of f on B is J, and let g be a single variable function that is continuous on J. Then $g \circ f$, i.e., $g(f(x,y))$, is continuous on B.

Notes:

Example 12.15 Establishing continuity of a function

Let $f(x,y) = \sin(x^2\cos y)$. Show f is continuous everywhere.

Solution We will apply both Theorems 8 and 104. Let $f_1(x,y) = x^2$. Since y is not actually used in the function, and polynomials are continuous (by Theorem 8), we conclude f_1 is continuous everywhere. A similar statement can be made about $f_2(x,y) = \cos y$. Part 3 of Theorem 104 states that $f_3 = f_1 \cdot f_2$ is continuous everywhere, and Part 7 of the theorem states the composition of sine with f_3 is continuous: that is, $\sin(f_3) = \sin(x^2\cos y)$ is continuous everywhere.

Functions of Three Variables

The definitions and theorems given in this section can be extended in a natural way to definitions and theorems about functions of three (or more) variables. We cover the key concepts here; some terms from Definitions 83 and 85 are not redefined but their analogous meanings should be clear to the reader.

Definition 86 Open Balls, Limit, Continuous

1. An **open ball** in $\mathbb{R}^3$ centered at (x_0,y_0,z_0) with radius r is the set of all points (x,y,z) such that $\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} = r$.

2. Let D be an open set in $\mathbb{R}^3$ containing (x_0,y_0,z_0), and let $f(x,y,z)$ be a function of three variables defined on D, except possibly at (x_0,y_0,z_0). The **limit** of $f(x,y,z)$ as (x,y,z) approaches (x_0,y_0,z_0) is L, denoted

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = L,$$

means that given any $\varepsilon > 0$, there is a $\delta > 0$ such that for all $(x,y,z) \neq (x_0,y_0,z_0)$, if (x,y,z) is in the open ball centered at (x_0,y_0,z_0) with radius δ, then $|f(x,y,z) - L| < \varepsilon$.

3. Let $f(x,y,z)$ be defined on an open ball B containing (x_0,y_0,z_0). f is **continuous** at (x_0,y_0,z_0) if $\displaystyle\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = f(x_0,y_0,z_0)$.

These definitions can also be extended naturally to apply to functions of four or more variables. Theorem 104 also applies to function of three or more vari-

Notes:

ables, allowing us to say that the function

$$f(x,y,z) = \frac{e^{x^2+y}\sqrt{y^2+z^2+3}}{\sin(xyz)+5}$$

is continuous everywhere.

When considering single variable functions, we studied limits, then continuity, then the derivative. In our current study of multivariable functions, we have studied limits and continuity. In the next section we study derivation, which takes on a slight twist as we are in a multivarible context.

Notes:

Exercises 12.2

Terms and Concepts

1. Describe in your own words the difference between boundary and interior point of a set.

2. Use your own words to describe (informally) what $\lim\limits_{(x,y)\to(1,2)} f(x,y) = 17$ means.

3. Give an example of a closed, bounded set.

4. Give an example of a closed, unbounded set.

5. Give an example of a open, bounded set.

6. Give an example of a open, unbounded set.

Problems

In Exercises 7 – 10, a set S is given.

(a) Give one boundary point and one interior point, when possible, of S.

(b) State whether S is open, closed, or neither.

(c) State whether S is bounded or unbounded.

7. $S = \left\{(x,y) \;\middle|\; \dfrac{(x-1)^2}{4} + \dfrac{(y-3)^2}{9} \leq 1\right\}$

8. $S = \left\{(x,y) \mid y \neq x^2\right\}$

9. $S = \left\{(x,y) \mid x^2 + y^2 = 1\right\}$

10. $S = \{(x,y) | y > \sin x\}$

In Exercises 11 – 14:

(a) Find the domain D of the given function.

(b) State whether D is an open or closed set.

(c) State whether D is bounded or unbounded.

11. $f(x,y) = \sqrt{9 - x^2 - y^2}$

12. $f(x,y) = \sqrt{y - x^2}$

13. $f(x,y) = \dfrac{1}{\sqrt{y - x^2}}$

14. $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

In Exercises 15 – 20, a limit is given. Evaluate the limit along the paths given, then state why these results show the given limit does not exist.

15. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$

 (a) Along the path $y = 0$.

 (b) Along the path $x = 0$.

16. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x + y}{x - y}$

 (a) Along the path $y = mx$.

17. $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy - y^2}{y^2 + x}$

 (a) Along the path $y = mx$.

 (b) Along the path $x = 0$.

18. $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2)}{y}$

 (a) Along the path $y = mx$.

 (b) Along the path $y = x^2$.

19. $\lim\limits_{(x,y)\to(1,2)} \dfrac{x + y - 3}{x^2 - 1}$

 (a) Along the path $y = 2$.

 (b) Along the path $y = x + 1$.

20. $\lim\limits_{(x,y)\to(\pi,\pi/2)} \dfrac{\sin x}{\cos y}$

 (a) Along the path $x = \pi$.

 (b) Along the path $y = x - \pi/2$.

In Exercises 21 – 24, determine the limit by converting to polar coordinates.

21. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3 - 2x^2 - 2y^2}{x^2 + y^2}$

22. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x}{x^2 + xy + y^2}$

23. $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2 + xy + y^2}$

24. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^2 + xy + y^2}$

12.3 Partial Derivatives

Let y be a function of x. We have studied in great detail the derivative of y with respect to x, that is, $\frac{dy}{dx}$, which measures the rate at which y changes with respect to x. Consider now $z = f(x, y)$. It makes sense to want to know how z changes with respect to x and/or y. This section begins our investigation into these rates of change.

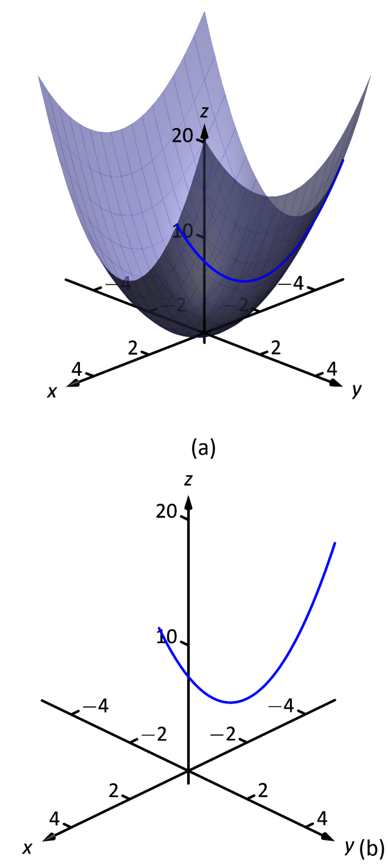


Figure 12.11: By fixing $y = 2$, the surface $f(x, y) = x^2 + 2y^2$ is a curve in space.

Consider the function $z = f(x, y) = x^2 + 2y^2$, as graphed in Figure 12.11(a). By fixing $y = 2$, we focus our attention to all points on the surface where the y-value is 2, shown in both parts (a) and (b) of the figure. These points form a curve in space: $z = f(x, 2) = x^2 + 8$ which is a function of just one variable. We can take the derivative of z with respect to x along this curve and find equations of tangent lines, etc.

The key notion to extract from this example is: by treating y as constant (it does not vary) we can consider how z changes with respect to x. In a similar fashion, we can hold x constant and consider how z changes with respect to y. This is the underlying principle of **partial derivatives**. We state the formal, limit–based definition first, then show how to compute these partial derivatives without directly taking limits.

Definition 87 Partial Derivative

Let $z = f(x, y)$ be a continuous function on an open set S in $\mathbb{R}^2$.

1. The **partial derivative of f with respect to x** is:
$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}.$$
2. The **partial derivative of f with respect to y** is:
$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

Alternate notations for $f_x(x, y)$ include:

$$\frac{\partial}{\partial x} f(x, y), \quad \frac{\partial f}{\partial x}, \quad \frac{\partial z}{\partial x}, \quad \text{and } z_x,$$

with similar notations for $f_y(x, y)$. For ease of notation, $f_x(x, y)$ is often abbreviated f_x.

Example 12.16 Computing partial derivatives with the limit definition

Let $f(x, y) = x^2y + 2x + y^3$. Find $f_x(x, y)$ using the limit definition.

Notes:

Solution Using Definition 87, we have:

$$\begin{aligned}
f_x(x,y) &= \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2y+2(x+h)+y^3-(x^2y+2x+y^3)}{h} \\
&= \lim_{h\to 0} \frac{(x^2y+2xhy+h^2y+2x+2h+y^3-(x^2y+2x+y^3)}{h} \\
&= \lim_{h\to 0} \frac{2xhy+h^2y+2h}{h} \\
&= \lim_{h\to 0} 2xy+hy+2 \\
&= 2xy+2.
\end{aligned}$$

We have found $f_x(x,y) = 2xy+2$.

Example 12.16 found a partial derivative using the formal, limit–based definition. Using limits is not necessary, though, as we can rely on our previous knowledge of derivatives to compute partial derivatives easily. When computing $f_x(x,y)$, we hold y fixed – it does not vary. Therefore we can compute the derivative with respect to x by treating y as a constant or coefficient.

Just as $\frac{d}{dx}\big(5x^2\big) = 10x$, we compute $\frac{\partial}{\partial x}\big(x^2y\big) = 2xy$. Here we are treating y as a coefficient.

Just as $\frac{d}{dx}\big(5^3\big) = 0$, we compute $\frac{\partial}{\partial x}\big(y^3\big) = 0$. Here we are treating y as a constant. More examples will help make this clear.

Example 12.17 Finding partial derivatives

Find $f_x(x,y)$ and $f_y(x,y)$ in each of the following.

1. $f(x,y) = x^3y^2+5y^2-x+7$
2. $f(x,y) = \cos(xy^2)+\sin x$
3. $f(x,y) = e^{x^2y^3}\sqrt{x^2+1}$

Solution

1. We have $f(x,y) = x^3y^2+5y^2-x+7$.
   Begin with $f_x(x,y)$. Keep y fixed, treating it as a constant or coefficient, as appropriate:
   $$f_x(x,y) = 3x^2y^2-1.$$
   Note how the $5y^2$ and 7 terms go to zero.

Notes:

To compute $f_y(x,y)$, we hold x fixed:

$$f_y(x,y) = 2x^3y + 10y.$$

Note how the $-x$ and 7 terms go to zero.

2. We have $f(x,y) = \cos(xy^2) + \sin x$.
   Begin with $f_x(x,y)$. We need to apply the Chain Rule with the cosine term; y^2 is the coefficient of the x-term inside the cosine function.

$$f_x(x,y) = -\sin(xy^2)(y^2) + \cos x = -y^2\sin(xy^2) + \cos x.$$

   To find $f_y(x,y)$, note that x is the coefficient of the y^2 term inside of the cosine term; also note that since x is fixed, $\sin x$ is also fixed, and we treat it as a constant.

$$f_y(x,y) = -\sin(xy^2)(2xy) = -2xy\sin(xy^2).$$

3. We have $f(x,y) = e^{x^2y^3}\sqrt{x^2+1}$.
   Beginning with $f_x(x,y)$, note how we need to apply the Product Rule.

$$\begin{aligned} f_x(x,y) &= e^{x^2y^3}(2xy^3)\sqrt{x^2+1} + e^{x^2y^3}\frac{1}{2}\left(x^2+1\right)^{-1/2}(2x) \\ &= 2xy^3e^{x^2y^3}\sqrt{x^2+1} + \frac{xe^{x^2y^3}}{\sqrt{x^2+1}}. \end{aligned}$$

   Note that when finding $f_y(x,y)$ we do not have to apply the Product Rule; since $\sqrt{x^2+1}$ does not contain y, we treat it as fixed and hence becomes a coefficient of the $e^{x^2y^3}$ term.

$$f_y(x,y) = e^{x^2y^3}(3x^2y^2)\sqrt{x^2+1} = 3x^2y^2e^{x^2y^3}\sqrt{x^2+1}.$$

We have shown *how* to compute a partial derivative, but it may still not be clear what a partial derivative *means*. Given $z = f(x,y)$, $f_x(x,y)$ measures the rate at which z changes as only x varies: y is held constant.

Imagine standing in a rolling meadow, then beginning to walk due east. Depending on your location, you might walk up, sharply down, or perhaps not change elevation at all. This is similar to measuring z_x: you are moving only east (in the "x"-direction) and not north/south at all. Going back to your original location, imagine now walking due north (in the "y"-direction). Perhaps walking due north does not change your elevation at all. This is analogous to $z_y = 0$: z does not change with respect to y. We can see that z_x and z_y do not have to be the same, or even similar, as it is easy to imagine circumstances where walking east means you walk downhill, though walking north makes you walk uphill.

Notes:

The following example helps us visualize this more.

Example 12.18 Evaluating partial derivatives
Let $z = f(x, y) = -x^2 - \frac{1}{2}y^2 + xy + 10$. Find $f_x(2, 1)$ and $f_y(2, 1)$ and interpret their meaning.

Solution We begin by computing $f_x(x, y) = -2x + y$ and $f_y(x, y) = -y + x$. Thus

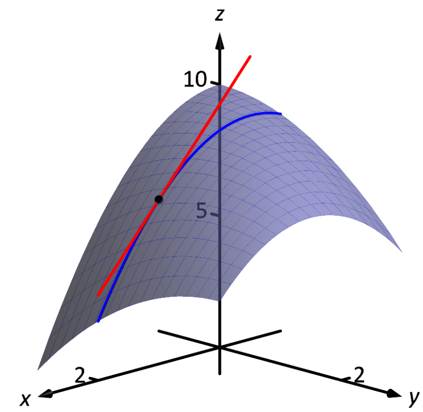
$$f_x(2, 1) = -3 \quad \text{and} \quad f_y(2, 1) = 1.$$

It is also useful to note that $f(2, 1) = 7.5$. What does each of these numbers mean?

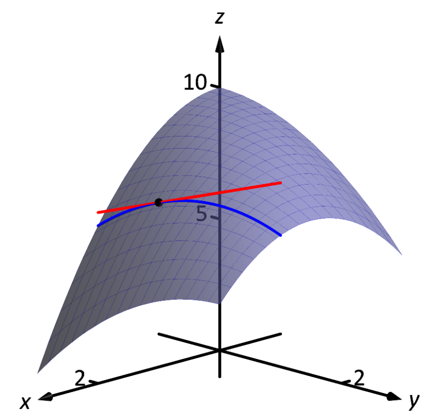
Consider $f_x(2, 1) = -3$, along with Figure 12.12(a). If one "stands" on the surface at the point $(2, 1, 7.5)$ and moves parallel to the x-axis (i.e., only the x-value changes, not the y-value), then the instantaneous rate of change is -3. Increasing the x-value will decrease the z-value; decreasing the x-value will increase the z-value.

Now consider $f_y(2, 1) = 1$, illustrated in Figure 12.12(b). Moving along the curve drawn on the surface, i.e., parallel to the y-axis and not changing the x-values, increases the z-value instantaneously at a rate of 1. Increasing the y-value by 1 would increase the z-value by approximately 1.

Since the magnitude of f_x is greater than the magnitude of f_y at $(2, 1)$, it is "steeper" in the x-direction than in the y-direction.



(a)



(b)

Figure 12.12: Illustrating the meaning of partial derivatives.

Second Partial Derivatives

Let $z = f(x, y)$. We have learned to find the partial derivatives $f_x(x, y)$ and $f_y(x, y)$, which are each functions of x and y. Therefore we can take partial derivatives of them, each with respect to x and y. We define these "second partials" along with the notation, give examples, then discuss their meaning.

Notes:

Definition 88 Second Partial Derivative, Mixed Partial Derivative

Let $z = f(x, y)$ be continuous on an open set S.

1. The **second partial derivative of f with respect to x then x** is

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \left(f_x\right)_x = f_{xx}$$

2. The **second partial derivative of f with respect to x then y** is

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \left(f_x\right)_y = f_{xy}$$

Similar definitions hold for $\dfrac{\partial^2 f}{\partial y^2} = f_{yy}$ and $\dfrac{\partial^2 f}{\partial x \partial y} = f_{yx}$.

The second partial derivatives f_{xy} and f_{yx} are **mixed partial derivatives**.

Note: The terms in Definition 88 all depend on limits, so each definition comes with the caveat "where the limit exists."

The notation of second partial derivatives gives some insight into the notation of the second derivative of a function of a single variable. If $y = f(x)$, then $f''(x) = \dfrac{d^2 y}{dx^2}$. The "$d^2y$" portion means "take the derivative of y twice," while "dx^2" means "with respect to x both times." When we only know of functions of a single variable, this latter phrase seems silly: there is only one variable to take the derivative with respect to. Now that we understand functions of multiple variables, we see the importance of specifying which variables we are referring to.

Example 12.19 Second partial derivatives

For each of the following, find all six first and second partial derivatives. That is, find

$$f_x, \quad f_y, \quad f_{xx}, \quad f_{yy}, \quad f_{xy} \quad \text{and} \quad f_{yx}\,.$$

1. $f(x, y) = x^3y^2 + 2xy^3 + \cos x$
2. $f(x, y) = \dfrac{x^3}{y^2}$
3. $f(x, y) = e^x \sin(x^2 y)$

Solution In each, we give f_x and f_y immediately and then spend time deriving the second partial derivatives.

Notes:

1. $f(x,y) = x^3y^2 + 2xy^3 + \cos x$

   $f_x(x,y) = 3x^2y^2 + 2y^3 - \sin x$

   $f_y(x,y) = 2x^3y + 6xy^2$

   $f_{xx}(x,y) = \dfrac{\partial}{\partial x}\big(f_x\big) = \dfrac{\partial}{\partial x}\big(3x^2y^2 + 2y^3 - \sin x\big) = 6xy^2 - \cos x$

   $f_{yy}(x,y) = \dfrac{\partial}{\partial y}\big(f_y\big) = \dfrac{\partial}{\partial y}\big(2x^3y + 6xy^2\big) = 2x^3 + 12xy$

   $f_{xy}(x,y) = \dfrac{\partial}{\partial y}\big(f_x\big) = \dfrac{\partial}{\partial y}\big(3x^2y^2 + 2y^3 - \sin x\big) = 6x^2y + 6y^2$

   $f_{yx}(x,y) = \dfrac{\partial}{\partial x}\big(f_x\big) = \dfrac{\partial}{\partial x}\big(2x^3y + 6xy^2\big) = 6x^2y + 6y^2$

2. $f(x,y) = \dfrac{x^3}{y^2} = x^3y^{-2}$

   $f_x(x,y) = \dfrac{3x^2}{y^2}$

   $f_y(x,y) = -\dfrac{2x^3}{y^3}$

   $f_{xx}(x,y) = \dfrac{\partial}{\partial x}\big(f_x\big) = \dfrac{\partial}{\partial x}\left(\dfrac{3x^2}{y^2}\right) = \dfrac{6x}{y^2}$

   $f_{yy}(x,y) = \dfrac{\partial}{\partial y}\big(f_y\big) = \dfrac{\partial}{\partial y}\left(-\dfrac{2x^3}{y^3}\right) = \dfrac{6x^3}{y^4}$

   $f_{xy}(x,y) = \dfrac{\partial}{\partial y}\big(f_x\big) = \dfrac{\partial}{\partial y}\left(\dfrac{3x^2}{y^2}\right) = -\dfrac{6x^2}{y^3}$

   $f_{yx}(x,y) = \dfrac{\partial}{\partial x}\big(f_x\big) = \dfrac{\partial}{\partial x}\left(-\dfrac{2x^3}{y^3}\right) = -\dfrac{6x^2}{y^3}$

3. $f(x,y) = e^x\sin(x^2y)$

   Because the following partial derivatives get rather long, we omit the extra notation and just give the results. In several cases, multiple applications of the Product and Chain Rules will be necessary, followed by some basic combination of like terms.

   $f_x(x,y) = e^x\sin(x^2y) + 2xye^x\cos(x^2y)$

   $f_y(x,y) = x^2e^x\cos(x^2y)$

   $f_{xx}(x,y) = e^x\sin(x^2y) + 4xye^x\cos(x^2y) + 2ye^x\cos(x^2y) - 4x^2y^2e^x\sin(x^2y)$

   $f_{yy}(x,y) = -x^4e^x\sin(x^2y)$

   $f_{xy}(x,y) = x^2e^x\cos(x^2y) + 2xe^x\cos(x^2y) - 2x^3ye^x\sin(x^2y)$

   $f_{yx}(x,y) = x^2e^x\cos(x^2y) + 2xe^x\cos(x^2y) - 2x^3ye^x\sin(x^2y)$

Notes:

Notice how in each of the three functions in Example 12.19, $f_{xy} = f_{yx}$. Due to the complexity of the examples, this likely is not a coincidence. The following theorem states that it is not.

Theorem 105 Mixed Partial Derivatives

Let f be defined such that f_{xy} and f_{yx} are continuous on an open set S. Then for each point (x, y) in S, $f_{xy}(x, y) = f_{yx}(x, y)$.

Finding f_{xy} and f_{yx} independently and comparing the results provides a convenient way of checking our work.

Understanding Second Partial Derivatives

Now that we know *how* to find second partials, we investigate *what* they tell us.

Again we refer back to a function $y = f(x)$ of a single variable. The second derivative of f is "the derivative of the derivative," or "the rate of change of the rate of change." The second derivative measures how much the derivative is changing. If $f''(x) < 0$, then the derivative is getting smaller (so the graph of f is concave down); if $f''(x) > 0$, then the derivative is growing, making the graph of f concave up.

Now consider $z = f(x, y)$. Similar statements can be made about f_{xx} and f_{yy} as could be made about $f''(x)$ above. When taking derivatives with respect to x twice, we measure how much f_x changes with respect to x. If $f_{xx}(x, y) < 0$, it means that as x increases, f_x decreases, and the graph of f will be concave down *in the x-direction*. Using the analogy of standing in the rolling meadow used earlier in this section, f_{xx} measures whether one's path is concave up/down when walking due east.

Similarly, f_{yy} measures the concavity in the y-direction. If $f_{yy}(x, y) > 0$, then f_y is increasing with respect to y and the graph of f will be concave up in the y-direction. Appealing to the rolling meadow analogy again, f_{yy} measures whether one's path is concave up/down when walking due north.

We now consider the mixed partials f_{xy} and f_{yx}. The mixed partial f_{xy} measures how much f_x changes with respect to y. Once again using the rolling meadow analogy, f_x measures the slope if one walks due east. Looking east, begin walking *north* (side–stepping). Is the path towards the east getting steeper? If so, $f_{xy} > 0$. Is the path towards the east not changing in steepness? If so, then $f_{xy} = 0$. A similar thing can be said about f_{yx}: consider the steepness of paths heading north while side–stepping to the east.

The following example examines these ideas with concrete numbers and

Notes:

graphs.

Example 12.20 Understanding second partial derivatives

Let $z = x^2 - y^2 + xy$. Evaluate the 6 first and second partial derivatives at $(-1/2, 1/2)$ and interpret what each of these numbers mean.

Solution We find that:

$f_x(x,y) = 2x + y$, $f_y(x,y) = -2y + x$, $f_{xx}(x,y) = 2$, $f_{yy}(x,y) = -2$ and $f_{xy}(x,y) = f_{yx}(x,y) = 1$. Thus at $(-1/2, 1/2)$ we have

$$f_x(-1/2, 1/2) = -1/2, \qquad f_y(-1/2, 1/2) = -3/2.$$

The slope of the tangent line at $(-1/2, 1/2, -1/4)$ in the direction of x is $-1/2$: if one moves from that point parallel to the x-axis, the instantaneous rate of change will be $-1/2$. The slope of the tangent line at this point in the direction of y is $-3/2$: if one moves from this point parallel to the y-axis, the instantaneous rate of change will be $-3/2$. These tangents lines are graphed in Figure 12.13(a) and (b), respectively, where the tangent lines are drawn in a solid line.

Now consider only Figure 12.13(a). Three directed tangent lines are drawn (two are dashed), each in the direction of x; that is, each has a slope determined by f_x. Note how as y increases, the slope of these lines get closer to 0. Since the slopes are all negative, getting closer to 0 means the *slopes are increasing.* The slopes given by f_x are increasing as y increases, meaning f_{xy} must be positive.

Since $f_{xy} = f_{yx}$, we also expect f_y to increase as x increases. Consider Figure 12.13(b) where again three directed tangent lines are drawn, this time each in the direction of y with slopes determined by f_y. As x increases, the slopes become less steep (closer to 0). Since these are negative slopes, this means the slopes are increasing.

Thus far we have a visual understanding of f_x, f_y, and $f_{xy} = f_{yx}$. We now interpret f_{xx} and f_{yy}. In Figure 12.13(a), we see a curve drawn where x is held constant at $x = -1/2$: only y varies. This curve is clearly concave down, corresponding to the fact that $f_{yy} < 0$. In part (b) of the figure, we see a similar curve where y is constant and only x varies. This curve is concave up, corresponding to the fact that $f_{xx} > 0$.

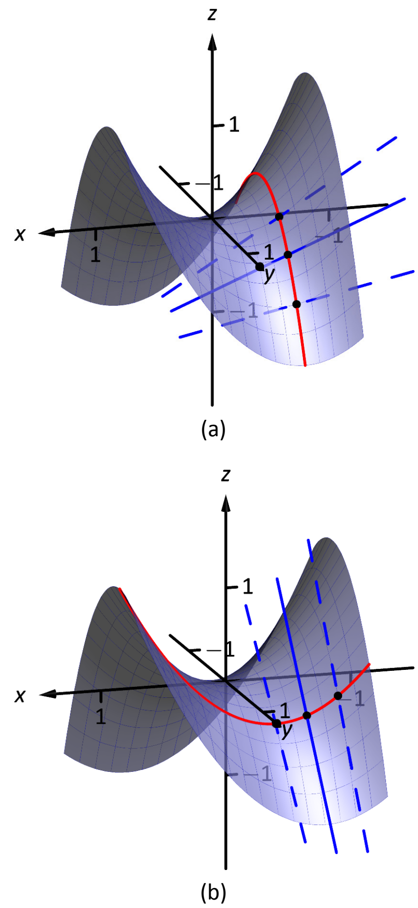


Figure 12.13: Understanding the second partial derivatives in Example 12.20.

Partial Derivatives and Functions of Three Variables

The concepts underlying partial derivatives can be easily extend to more than two variables. We give some definitions and examples in the case of three variables and trust the reader can extend these definitions to more variables if needed.

Notes:

Definition 89 Partial Derivatives with Three Variables

Let $w = f(x, y, z)$ be a continuous function on an open set S in $\mathbb{R}^3$.
The **partial derivative of** f **with respect to** x is:

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}.$$

Similar definitions hold for $f_y(x, y, z)$ and $f_z(x, y, z)$.

By taking partial derivatives of partial derivatives, we can find second partial derivatives of f with respect to z then y, for instance, just as before.

Example 12.21 Partial derivatives of functions of three variables

For each of the following, find f_x, f_y, f_z, f_{xz}, f_{yz}, and f_{zz}.

1. $f(x, y, z) = x^2y^3z^4 + x^2y^2 + x^3z^3 + y^4z^4$
2. $f(x, y, z) = x\sin(yz)$

Solution

1. $f_x = 2xy^3z^4 + 2xy^2 + 3x^2z^3$; $\quad f_y = 3x^2y^2z^4 + 2x^2y + 4y^3z^4$;
   $f_z = 4x^2y^3z^3 + 3x^3z^2 + 4y^4z^3$; $\quad f_{xz} = 8xy^3z^3 + 9x^2z^2$;
   $f_{yz} = 12x^2y^2z^3 + 16y^3z^3$; $\quad f_{zz} = 12x^2y^3z^2 + 6x^3z + 12y^4z^2$
2. $f_x = \sin(yz)$; $\quad f_y = xz\cos(yz)$; $\quad f_z = xy\cos(yz)$;
   $f_{xz} = y\cos(yz)$; $\quad f_{yz} = x\cos(yz) - xyz\sin(yz)$; $\quad f_{zz} = -xy^2\sin(xy)$

Higher Order Partial Derivatives

We can continue taking partial derivatives of partial derivatives of partial derivatives of …; we do not have to stop with second partial derivatives. These higher order partial derivatives do not have a tidy graphical interpretation; nevertheless they are not hard to compute and worthy of some practice.

We do not formally define each higher order derivative, but rather give just a few examples of the notation.

$$f_{xyx}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right) \quad \text{and}$$

$$f_{xyz}(x, y, z) = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right).$$

Notes:

Example 12.22 Higher order partial derivatives

1. Let $f(x, y) = x^2y^2 + \sin(xy)$. Find f_{xxy} and f_{yxx}.
2. Let $f(x, y, z) = x^3e^{xy} + \cos(z)$. Find f_{xyz}.

Solution

1. To find f_{xxy}, we first find f_x, then f_{xx}, then f_{xxy}:

$$f_x = 2xy^2 + y\cos(xy) \qquad f_{xx} = 2y^2 - y^2\sin(xy)$$
$$f_{xxy} = 4y - 2y\sin(xy) - xy^2\cos(xy).$$

   To find f_{yxx}, we first find f_y, then f_{yx}, then f_{yxx}:

$$f_y = 2x^2y + x\cos(xy) \qquad f_{yx} = 4xy + \cos(xy) - xy\sin(xy)$$
$$\begin{aligned} f_{yxx} &= 4y - y\sin(xy) - \big(y\sin(xy) + xy^2\cos(xy)\big) \\ &= 4y - 2y\sin(xy) - xy^2\cos(xy). \end{aligned}$$

   Note how $f_{xxy} = f_{yxx}$.

2. To find f_{xyz}, we find f_x, then f_{xy}, then f_{xyz}:

$$f_x = 3x^2e^{xy} + x^3ye^{xy} \qquad f_{xy} = 3x^3e^{xy} + x^3e^{xy} + x^4ye^{xy} = 4x^3e^{xy} + x^4ye^{xy}$$
$$f_{xyz} = 0.$$

In the previous example we saw that $f_{xxy} = f_{yxx}$; this is not a coincidence. While we do not state this as a formal theorem, as long as each partial derivative is continuous, it does not matter the order in which the partial derivatives are taken. For instance, $f_{xxy} = f_{xyx} = f_{yxx}$.

This can be useful at times. Had we known this, the second part of Example 12.22 would have been much simpler to compute. Instead of computing f_{xyz} in the x, y then z orders, we could have applied the z, then x then y order (as $f_{xyz} = f_{zxy}$). It is easy to see that $f_z = -\sin z$; then f_{zx} and f_{zxy} are clearly 0 as f_z does not contain an x or y.

Notes:

A brief review of this section: partial derivatives measure the instantaneous rate of change of a multivariable function with respect to one variable. With $z = f(x, y)$, the partial derivatives f_x and f_y measure the instantaneous rate of change of z when moving parallel to the x- and y-axes, respectively. How do we measure the rate of change at a point when we do not move parallel to one of these axes? What if we move in the direction given by the vector $\langle 2, 1\rangle$? Can we measure that rate of change? The answer is, of course, yes, we can. This is the topic of Section 12.6. First, we need to define what it means for a function of two variables to be *differentiable.*

Notes:

Exercises 12.3

Terms and Concepts

1. What is the difference between a constant and a coefficient?

2. Given a function $z = f(x, y)$, explain in your own words how to compute f_x.

3. In the mixed partial fraction f_{xy}, which is computed first, f_x or f_y?

4. In the mixed partial fraction $\dfrac{\partial^2 f}{\partial x \partial y}$, which is computed first, f_x or f_y?

Problems

In Exercises 5 – 8, evaluate $f_x(x, y)$ and $f_y(x, y)$ at the indicated point.

5. $f(x, y) = x^2y - x + 2y + 3$ at $(1, 2)$

6. $f(x, y) = x^3 - 3x + y^2 - 6y$ at $(-1, 3)$

7. $f(x, y) = \sin y \cos x$ at $(\pi/3, \pi/3)$

8. $f(x, y) = \ln(xy)$ at $(-2, -3)$

In Exercises 9 – 28, find f_x, f_y, f_{xx}, f_{yy}, f_{xy} and f_{yx}.

9. $f(x, y) = x^2y + 3x^2 + 4y - 5$

10. $f(x, y) = y^3 + 3xy^2 + 3x^2y + x^3$

11. $f(x, y) = \dfrac{x}{y}$

12. $f(x, y) = \dfrac{4}{xy}$

13. $f(x, y) = e^{x^2+y^2}$

14. $f(x, y) = e^{x+2y}$

15. $f(x, y) = \sin x \cos y$

16. $f(x, y) = (x + y)^3$

17. $f(x, y) = \cos(5xy^3)$

18. $f(x, y) = \sin(5x^2 + 2y^3)$

19. $f(x, y) = \sqrt{4xy^2 + 1}$

20. $f(x, y) = (2x + 5y)\sqrt{y}$

21. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$

22. $f(x, y) = 5x - 17y$

23. $f(x, y) = x^y$

24. $f(x, y) = 3x^2 + 1$

25. $f(x, y) = \ln(x^2 + y)$

26. $f(x, y) = \dfrac{\ln x}{4y}$

27. $f(x, y) = 5x + x^{2+\cos y}$

28. $f(x, y) = 5e^x \sin y + 9$

In Exercises 29 – 35, form a function $z = f(x, y)$ such that f_x and f_y match those given.

29. $f_x = \sin y + 1, \quad f_y = x \cos y$

30. $f_x = 2x + y, \quad f_y = 3x + y$

31. $f_x = x + y, \quad f_y = x + y$

32. $f_x = e^y, \quad f_y = e^x$

33. $f_x = x^2 \cos y, \quad f_y = \tan(xy)$

34. $f_x = 6xy - 4y^2, \quad f_y = 3x^2 - 8xy + 2$

35. $f_x = \dfrac{2x}{x^2 + y^2}, \quad f_y = \dfrac{2y}{x^2 + y^2}$

In Exercises 36 – 39, find f_x, f_y, f_z, f_{yz} and f_{zy}.

36. $f(x, y, z) = x^2e^{2y-3z}$

37. $f(x, y, z) = x^3y^2 + x^3z + y^2z$

38. $f(x, y, z) = \dfrac{3x}{7y^2z}$

39. $f(x, y, z) = \ln(xyz)$

12.4 Differentiability and the Total Differential

We studied **differentials** in Section 4.4, where Definition 18 states that if $y = f(x)$ and f is differentiable, then $dy = f'(x)dx$. One important use of this differential is in Integration by Substitution. Another important application is approximation. Let $\Delta x = dx$ represent a change in x. When dx is small, $dy \approx \Delta y$, the change in y resulting from the change in x. Fundamental in this understanding is this: as dx gets small, the difference between Δy and dy goes to 0. Another way of stating this: as dx goes to 0, the *error* in approximating Δy with dy goes to 0.

We extend this idea to functions of two variables. Let $z = f(x, y)$, and let $\Delta x = dx$ and $\Delta y = dy$ represent changes in x and y, respectively. Let $\Delta z = f(x+dx, y+dy) - f(x, y)$ be the change in z over the change in x and y. Recalling that f_x and f_y give the instantaneous rates of z-change in the x- and y-directions, respectively, we can approximate Δz with $dz = f_x dx + f_y dy$; in words, the total change in z is approximately the change caused by changing x plus the change caused by changing y. In a moment we give an indication of whether or not this approximation is any good. First we give a name to dz.

Definition 90 Total Differential

Let $z = f(x, y)$ be continuous on an open set S. Let dx and dy represent changes in x and y, respectively. Where the partial derivatives f_x and f_y exist, the **total differential of** z is

$$dz = f_x(x, y)dx + f_y(x, y)dy.$$

Example 12.23 Finding the total differential

Let $z = x^4e^{3y}$. Find dz.

Solution We compute the partial derivatives: $f_x = 4x^3e^{3y}$ and $f_y = 3x^4e^{3y}$. Following Definition 90, we have

$$dz = 4x^3e^{3y}dx + 3x^4e^{3y}dy.$$

We *can* approximate Δz with dz, but as with all approximations, there is error involved. A good approximation is one in which the error is small. At a given point (x_0, y_0), let E_x and E_y be functions of dx and dy such that $E_x dx + E_y dy$ describes this error. Then

$$\begin{aligned}\Delta z &= dz + E_x dx + E_y dy\\ &= f_x(x_0, y_0)dx + f_y(x_0, y_0)dy + E_x dx + E_y dy.\end{aligned}$$

Notes:

If the approximation of Δz by dz is good, then as dx and dy get small, so does $E_xdx + E_ydy$. The approximation of Δz by dz is even better if, as dx and dy go to 0, so do E_x and E_y. This leads us to our definition of differentiability.

Definition 91 Multivariable Differentiability

Let $z = f(x, y)$ be defined on an open set S containing (x_0, y_0) where $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist. Let dz be the total differential of z at (x_0, y_0), let $\Delta z = f(x_0 + dx, y_0 + dy) - f(x_0, y_0)$, and let E_x and E_y be functions of dx and dy such that

$$\Delta z = dz + E_xdx + E_ydy.$$

1. f is **differentiable at** (x_0, y_0) if, given $\varepsilon > 0$, there is a $\delta > 0$ such that if $|| \langle dx, dy\rangle || < \delta$, then $|| \langle E_x, E_y\rangle || < \varepsilon$. That is, as dx and dy go to 0, so do E_x and E_y.
2. f is **differentiable on** S if f is differentiable at every point in S. If f is differentiable on $\mathbb{R}^2$, we say that f is **differentiable everywhere**.

Example 12.24 Showing a function is differentiable

Show $f(x, y) = xy + 3y^2$ is differentiable using Definition 91.

SOLUTION We begin by finding $f(x + dx, y + dy)$, Δz, f_x and f_y.

$$\begin{aligned} f(x + dx, y + dy) &= (x + dx)(y + dy) + 3(y + dy)^2 \\ &= xy + xdy + ydx + dxdy + 3y^2 + 6ydy + 3dy^2. \end{aligned}$$

$\Delta z = f(x + dx, y + dy) - f(x, y)$, so

$$\Delta z = xdy + ydx + dxdy + 6ydy + 3dy^2.$$

It is straightforward to compute $f_x = y$ and $f_y = x + 6y$. Consider once more Δz:

$$\begin{aligned} \Delta z &= xdy + ydx + dxdy + 6ydy + 3dy^2 \qquad \text{(now reorder)} \\ &= ydx + xdy + 6ydy + dxdy + 3dy^2 \\ &= \underbrace{(y)}_{f_x}\,dx + \underbrace{(x + 6y)}_{f_y}\,dy + \underbrace{(dy)}_{E_x}\,dx + \underbrace{(3dy)}_{E_y}\,dy \\ &= f_xdx + f_ydy + E_xdx + E_ydy. \end{aligned}$$

With $E_x = dy$ and $E_y = 3dy$, it is clear that as dx and dy go to 0, E_x and E_y also go to 0. Since this did not depend on a specific point (x_0, y_0), we can say that $f(x, y)$

Notes:

is differentiable for all pairs (x, y) in $\mathbb{R}^2$, or, equivalently, that f is differentiable everywhere.

Our intuitive understanding of differentiability of functions $y = f(x)$ of one variable was that the graph of f was "smooth." A similar intuitive understanding of functions $z = f(x, y)$ of two variables is that the surface defined by f is also "smooth," not containing cusps, edges, breaks, etc. The following theorem states that differentiable functions are continuous, followed by another theorem that provides a more tangible way of determining whether a great number of functions are differentiable or not.

Theorem 106 Continuity and Differentiability of Multivariable Functions

Let $z = f(x, y)$ be defined on an open set S containing (x_0, y_0). If f is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

Theorem 107 Differentiability of Multivariable Functions

Let $z = f(x, y)$ be defined on an open set S containing (x_0, y_0). If f_x and f_y are both continuous on S, then f is differentiable on S.

The theorems assure us that essentially all functions that we see in the course of our studies here are differentiable (and hence continuous) on their natural domains. There is a difference between Definition 91 and Theorem 107, though: it is possible for a function f to be differentiable yet f_x and/or f_y is *not* continuous. Such strange behavior of functions is a source of delight for many mathematicians.

When f_x and f_y exist at a point but are not continuous at that point, we need to use other methods to determine whether or not f is differentiable at that point.

For instance, consider the function

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Notes:

We can find $f_x(0,0)$ and $f_y(0,0)$ using Definition 87:

$$\begin{aligned} f_x(0,0) &= \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{0}{h^2} = 0; \\ f_y(0,0) &= \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{0}{h^2} = 0. \end{aligned}$$

Both f_x and f_y *exist* at $(0,0)$, but they are not continuous at $(0,0)$, as

$$f_x(x,y) = \frac{y(y^2-x^2)}{(x^2+y^2)^2} \qquad \text{and} \qquad f_y(x,y) = \frac{x(x^2-y^2)}{(x^2+y^2)^2}$$

are not continuous at $(0,0)$. (Take the limit of f_x as $(x,y) \to (0,0)$ along the x- and y-axes; they give different results.) So even though f_x and f_y *exist* at every point in the x-y plane, they are not continuous. Therefore it is possible, by Theorem 107, for f to not be differentiable.

Indeed, it is not. One can show that f is not continuous at $(0,0)$ (see Example 12.10), and by Theorem 106, this means f is not differentiable at $(0,0)$.

Approximating with the Total Differential

By the definition, when f is differentiable dz is a good approximation for Δz when dx and dy are small. We give some simple examples of how this is used here.

Example 12.25 Approximating with the total differential

Let $z = \sqrt{x}\sin y$. Approximate $f(4.1, 0.8)$.

Solution Recognizing that $\pi/4 \approx 0.785 \approx 0.8$, we can approximate $f(4.1,0.8)$ using $f(4,\pi/4)$. We can easily compute $f(4,\pi/4) = \sqrt{4}\sin(\pi/4) = 2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2} \approx 1.414$. Without calculus, this is the best approximation we could reasonably come up with. The total differential gives us a way of adjusting this initial approximation to hopefully get a more accurate answer.

We let $\Delta z = f(4.1,0.8) - f(4,\pi/4)$. The total differential dz is approximately equal to Δz, so

$$f(4.1,0.8) - f(4,\pi/4) \approx dz \quad \Rightarrow \quad f(4.1,0.8) \approx dz + f(4,\pi/4). \qquad (12.1)$$

To find dz, we need f_x and f_y.

Notes:

$$f_x(x,y) = \frac{\sin y}{2\sqrt{x}} \quad \Rightarrow \quad f_x(4,\pi/4) = \frac{\sin \pi/4}{2\sqrt{4}} = \frac{\sqrt{2}/2}{4} = \sqrt{2}/8.$$

$$f_y(x,y) = \sqrt{x}\cos y \quad \Rightarrow \quad f_y(4,\pi/4) = \sqrt{4}\frac{\sqrt{2}}{2} = \sqrt{2}.$$

Approximating 4.1 with 4 gives $dx = 0.1$; approximating 0.8 with $\pi/4$ gives $dy \approx 0.015$. Thus

$$\begin{aligned} dz(4,\pi/4) &= f_x(4,\pi/4)(0.1) + f_y(4,\pi/4)(0.015) \\ &= \frac{\sqrt{2}}{8}(0.1) + \sqrt{2}(0.015) \\ &\approx 0.039. \end{aligned}$$

Returning to Equation (12.1), we have

$$f(4.1, 0.8) \approx 0.039 + 1.414 = 1.4531.$$

We, of course, can compute the actual value of $f(4.1, 0.8)$ with a calculator; the actual value, accurate to 5 places after the decimal, is 1.45254. Obviously our approximation is quite good.

The point of the previous example was *not* to develop an approximation method for known functions. After all, we can very easily compute $f(4.1, 0.8)$ using readily available technology. Rather, it serves to illustrate how well this method of approximation works, and to reinforce the following concept:

"New position = old position $+$ amount of change," so
"New position $\approx$ old position + approximate amount of change."

In the previous example, we could easily compute $f(4, \pi/4)$ and could approximate the amount of z-change when computing $f(4.1, 0.8)$, letting us approximate the new z-value.

It may be surprising to learn that it is not uncommon to know the values of f, f_x and f_y at a particular point without actually knowing the function f. The total differential gives a good method of approximating f at nearby points.

Example 12.26 Approximating an unknown function

Given that $f(2,-3) = 6$, $f_x(2,-3) = 1.3$ and $f_y(2,-3) = -0.6$, approximate $f(2.1, -3.03)$.

Notes:

Solution The total differential approximates how much f changes from the point $(2, -3)$ to the point $(2.1, -3.03)$. With $dx = 0.1$ and $dy = -0.03$, we have

$$\begin{aligned} dz &= f_x(2, -3)dx + f_y(2, -3)dy \\ &= 1.3(0.1) + (-0.6)(-0.03) \\ &= 0.148. \end{aligned}$$

The change in z is approximately 0.148, so we approximate $f(2.1, -3.03) \approx 6.148$.

Error/Sensitivity Analysis

The total differential gives an approximation of the change in z given small changes in x and y. We can use this to approximate error propagation; that is, if the input is a little off from what it should be, how far from correct will the output be? We demonstrate this in an example.

Example 12.27 Sensitivity analysis

A cylindrical steel storage tank is to be built that is 10ft tall and 4ft across in diameter. It is known that the steel will expand/contract with temperature changes; is the overall volume of the tank more sensitive to changes in the diameter or in the height of the tank?

Solution A cylindrical solid with height h and radius r has volume $V = \pi r^2 h$. We can view V as a function of two variables, r and h. We can compute partial derivatives of V:

$$\frac{\partial V}{\partial r} = V_r(r, h) = 2\pi rh \qquad \text{and} \qquad \frac{\partial V}{\partial h} = V_h(r, h) = \pi r^2.$$

The total differential is $dV = (2\pi rh)dr + (\pi r^2)dh$. When $h = 10$ and $r = 2$, we have $dV = 40\pi dr + 4\pi dh$. Note that the coefficient of dr is $40\pi \approx 125.7$; the coefficient of dh is a tenth of that, approximately 12.57. A small change in radius will be multiplied by 125.7, whereas a small change in height will be multiplied by 12.57. Thus the volume of the tank is more sensitive to changes in radius than in height.

The previous example showed that the volume of a particular tank was more sensitive to changes in radius than in height. Keep in mind that this analysis only applies to a tank of those dimensions. A tank with a height of 1ft and radius of 5ft would be more sensitive to changes in height than in radius.

Notes:

One could make a chart of small changes in radius and height and find exact changes in volume given specific changes. While this provides exact numbers, it does not give as much insight as the error analysis using the total differential.

Differentiability of Functions of Three Variables

The definition of differentiability for functions of three variables is very similar to that of functions of two variables. We again start with the total differential.

Definition 92 Total Differential

Let $w = f(x, y, z)$ be continuous on an open set S. Let dx, dy and dz represent changes in x, y and z, respectively. Where the partial derivatives f_x, f_y and f_z exist, the **total differential of** w is

$$dz = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz.$$

This differential can be a good approximation of the change in w when $w = f(x, y, z)$ is **differentiable**.

Definition 93 Multivariable Differentiability

Let $w = f(x, y, z)$ be defined on an open ball B containing (x_0, y_0, z_0) where $f_x(x_0, y_0, z_0)$, $f_y(x_0, y_0, z_0)$ and $f_z(x_0, y_0, z_0)$ exist. Let dw be the total differential of w at (x_0, y_0, z_0), let $\Delta w = f(x_0 + dx, y_0 + dy, z_0 + dz) - f(x_0, y_0, z_0)$, and let E_x, E_y and E_z be functions of dx, dy and dz such that

$$\Delta w = dw + E_x dx + E_y dy + E_z dz.$$

1. f is **differentiable at** (x_0, y_0, z_0) if, given $\varepsilon > 0$, there is a $\delta > 0$ such that if $|| \langle dx, dy, dz \rangle || < \delta$, then $|| \langle E_x, E_y, E_z \rangle || < \varepsilon$.

2. f is **differentiable on** B if f is differentiable at every point in B. If f is differentiable on $\mathbb{R}^3$, we say that f is **differentiable everywhere**.

Just as before, this definition gives a rigorous statement about what it means to be differentiable that is not very intuitive. We follow it with a theorem similar to Theorem 107.

Notes:

Theorem 108 Continuity and Differentiability of Functions of Three Variables

Let $w = f(x, y, z)$ be defined on an open ball B containing (x_0, y_0, z_0).

1. If f is differentiable at (x_0, y_0, z_0), then f is continuous at (x_0, y_0, z_0).
2. If f_x, f_y and f_z are continuous on B, then f is differentiable on B.

This set of definition and theorem extends to functions of any number of variables. The theorem again gives us a simple way of verifying that most functions that we enounter are differentiable on their natural domains.

This section has given us a formal definition of what it means for a functions to be "differentiable," along with a theorem that gives a more accessible understanding. The following sections return to notions prompted by our study of partial derivatives that make use of the fact that most functions we encounter are differentiable.

Notes:

Exercises 12.4

Terms and Concepts

1. T/F: If $f(x,y)$ is differentiable on S, the f is continuous on S.

2. T/F: If f_x and f_y are continuous on S, then f is differentiable on S.

3. T/F: If $z = f(x,y)$ is differentiable, then the change in z over small changes dx and dy in x and y is approximately dz.

4. Finish the sentence: "The new z-value is approximately the old z-value plus the approximate ________."

Problems

In Exercises 5 – 8, find the total differential dz.

5. $z = x\sin y + x^2$

6. $z = (2x^2 + 3y)^2$

7. $z = 5x - 7y$

8. $z = xe^{x+y}$

In Exercises 9 – 12, a function $z = f(x,y)$ is given. Give the indicated approximation using the total differential.

9. $f(x,y) = \sqrt{x^2+y}$. Approximate $f(2.95, 7.1)$ knowing $f(3,7) = 4$.

10. $f(x,y) = \sin x \cos y$. Approximate $f(0.1, -0.1)$ knowing $f(0,0) = 0$.

11. $f(x,y) = x^2y - xy^2$. Approximate $f(2.04, 3.06)$ knowing $f(2,3) = -6$.

12. $f(x,y) = \ln(x-y)$. Approximate $f(5.1, 3.98)$ knowing $f(5,4) = 0$.

Exercises 13 – 16 ask a variety of questions dealing with approximating error and sensitivity analysis.

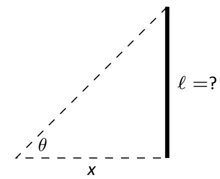
13. A cylindrical storage tank is to be 2ft tall with a radius of 1ft. Is the volume of the tank more sensitive to changes in the radius or the height?

14. **Projectile Motion:** The x-value of an object moving under the principles of projectile motion is $x(\theta, v_0, t) = (v_0\cos\theta)t$. A particular projectile is fired with an initial velocity of $v_0 = 250$ft/s and an angle of elevation of $\theta = 60^\circ$. It travels a distance of 375ft in 3 seconds.

    Is the projectile more sensitive to errors in initial speed or angle of elevation?

15. The length ℓ of a long wall is to be approximated. The angle θ, as shown in the diagram (not to scale), is measured to be 85°, and the distance x is measured to be 30'. Assume that the triangle formed is a right triangle.

    Is the measurement of the length of ℓ more sensitive to errors in the measurement of x or in θ?



16. It is "common sense" that it is far better to measure a long distance with a long measuring tape rather than a short one. A measured distance D can be viewed as the product of the length ℓ of a measuring tape times the number n of times it was used. For instance, using a 3' tape 10 times gives a length of 30'. To measure the same distance with a 12' tape, we would use the tape 2.5 times. (I.e., $30 = 12 \times 2.5$.) Thus $D = n\ell$.

    Suppose each time a measurement is taken with the tape, the recorded distance is within 1/16" of the actual distance. (I.e., $d\ell = 1/16'' \approx 0.005$ft). Using differentials, show why common sense proves correct in that it is better to use a long tape to measure long distances.

In Exercises 17 – 18, find the total differential dw.

17. $w = x^2yz^3$

18. $w = e^x \sin y \ln z$

In Exercises 19 – 22, use the information provided and the total differential to make the given approximation.

19. $f(3,1) = 7$, $f_x(3,1) = 9$, $f_y(3,1) = -2$. Approximate $f(3.05, 0.9)$.

20. $f(-4,2) = 13$, $f_x(-4,2) = 2.6$, $f_y(-4,2) = 5.1$. Approximate $f(-4.12, 2.07)$.

21. $f(2,4,5) = -1$, $f_x(2,4,5) = 2$, $f_y(2,4,5) = -3$, $f_z(2,4,5) = 3.7$. Approximate $f(2.5, 4.1, 4.8)$.

22. $f(3,3,3) = 5$, $f_x(3,3,3) = 2$, $f_y(3,3,3) = 0$, $f_z(3,3,3) = -2$. Approximate $f(3.1, 3.1, 3.1)$.

12.5 The Multivariable Chain Rule

The Chain Rule, as learned in Section 2.5, states that $\dfrac{d}{dx}\Big(f\big(g(x)\big)\Big) = f'\big(g(x)\big)g'(x)$. If $t = g(x)$, we can express the Chain Rule as

$$\frac{df}{dx} = \frac{df}{dt}\frac{dt}{dx}.$$

In this section we extend the Chain Rule to functions of more than one variable.

Theorem 109 Multivariable Chain Rule, Part I

Let $z = f(x,y)$, $x = g(t)$ and $y = h(t)$, where f, g and h are differentiable functions. Then $z = f(x,y) = f\big(g(t), h(t)\big)$ is a function of t, and

$$\begin{aligned}\frac{dz}{dt} = \frac{df}{dt} &= f_x(x,y)\frac{dx}{dt} + f_y(x,y)\frac{dy}{dt}\\ &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.\end{aligned}$$

It is good to understand what the situation of $z = f(x,y)$, $x = g(t)$ and $y = h(t)$ describes. We know that $z = f(x,y)$ describes a surface; we also recognize that $x = g(t)$ and $y = h(t)$ are parametric equations for a curve in the x-y plane. Combining these together, we are describing a curve that lies on the surface described by f. The parametric equations for this curve are $x = g(t)$, $y = h(t)$ and $z = f\big(g(t), h(t)\big)$.

Consider Figure 12.14 in which a surface is drawn, along with a dashed curve in the x-y plane. Restricting f to just the points on this circle gives the curve shown on the surface. The derivative $\frac{df}{dt}$ gives the instantaneous rate of change of f with respect to t. If we consider an object traveling along this path, $\frac{df}{dt}$ gives the rate at which the object rises/falls.

We now practice applying the Multivariable Chain Rule.

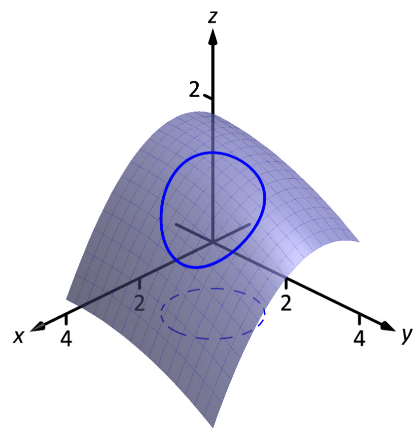


Figure 12.14: Understanding the application of the Multivariable Chain Rule.

Example 12.28 Using the Multivariable Chain Rule

Let $z = x^2y + x$, where $x = \sin t$ and $y = e^{5t}$. Find $\dfrac{dz}{dt}$ using the Chain Rule.

SOLUTION Following Theorem 109, we find

$$f_x(x,y) = 2xy + 1, \qquad f_y(x,y) = x^2, \qquad \frac{dx}{dt} = \cos t, \qquad \frac{dy}{dt} = 5e^{5t}.$$

Notes:

Applying the theorem, we have

$$\frac{dz}{dt} = (2xy + 1)\cos t + 5x^2e^{5t}.$$

This may look odd, as it seems that $\frac{dz}{dt}$ is a function of x, y and t. Since x and y are functions of t, $\frac{dz}{dt}$ is really just a function of t, and we can replace x with $\sin t$ and y with e^{5t}:

$$\frac{dz}{dt} = (2xy + 1)\cos t + 5x^2e^{5t} = (2\sin(t)e^{5t} + 1)\cos t + 5e^{5t}\sin^2 t.$$

The previous example can make us wonder: if we substituted for x and y at the end to show that $\frac{dz}{dt}$ is really just a function of t, why not substitute before differentiating, showing clearly that z is a function of t?

That is, $z = x^2y + x = (\sin t)^2e^{5t} + \sin t$. Applying the Chain and Product Rules, we have

$$\frac{dz}{dt} = 2\sin t\cos t\, e^{5t} + 5\sin^2 t\, e^{5t} + \cos t,$$

which matches the result from the example.

This may now make one wonder "What's the point? If we could already find the derivative, why learn another way of finding it?" In some cases, applying this rule makes deriving simpler, but this is hardly the power of the Chain Rule. Rather, in the case where $z = f(x, y)$, $x = g(t)$ and $y = h(t)$, the Chain Rule is extremely powerful when *we do not know what f, g and/or h are*. It may be hard to believe, but often in "the real world" we know rate–of–change information (i.e., information about derivatives) without explicitly knowing the underlying functions. The Chain Rule allows us to combine several rates of change to find another rate of change. The Chain Rule also has theoretic use, giving us insight into the behavior of certain constructions (as we'll see in the next section).

We demonstrate this in the next example.

Example 12.29 Applying the Multivarible Chain Rule

An object travels along a path on a surface. The exact path and surface are not known, but at time $t = t_0$ it is known that :

$$\frac{\partial z}{\partial x} = 5, \qquad \frac{\partial z}{\partial y} = -2, \qquad \frac{dx}{dt} = 3 \qquad \text{and} \qquad \frac{dy}{dt} = 7.$$

Find $\frac{dz}{dt}$ at time t_0.

Notes:

Solution The Multivariable Chain Rule states that

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\\ &= 5(3) + (-2)(7)\\ &= 1.\end{aligned}$$

By knowing certain rates–of–change information about the surface and about the path of the particle in the x-y plane, we can determine how quickly the object is rising/falling.

We next apply the Chain Rule to solve a max/min problem.

Example 12.30 Applying the Multivariable Chain Rule

Consider the surface $z = x^2 + y^2 - xy$, a paraboloid, on which a particle moves with x and y coordinates given by $x = \cos t$ and $y = \sin t$. Find $\frac{dz}{dt}$ when $t = 0$, and find where the particle reaches its maximum/minimum z-values.

Solution It is straightforward to compute

$$f_x(x,y) = 2x - y, \qquad f_y(x,y) = 2y - x, \qquad \frac{dx}{dt} = -\sin t, \qquad \frac{dy}{dt} = \cos t.$$

Combining these according to the Chain Rule gives:

$$\frac{dz}{dt} = -(2x-y)\sin t + (2y-x)\cos t.$$

When $t = 0$, $x = 1$ and $y = 0$. Thus $\dfrac{dz}{dt} = -(2)(0) + (-1)(1) = -1$. When $t = 0$, the particle is moving down, as shown in Figure 12.15.

To find where z-value is maximized/minimized on the particle's path, we set $\frac{dz}{dt} = 0$ and solve for t:

$$\begin{aligned}\frac{dz}{dt} = 0 &= -(2x-y)\sin t + (2y-x)\cos t\\ 0 &= -(2\cos t - \sin t)\sin t + (2\sin t - \cos t)\cos t\\ 0 &= \sin^2 t - \cos^2 t\\ \cos^2 t &= \sin^2 t\\ t &= n\frac{\pi}{4} \quad \text{(for odd } n\text{)}\end{aligned}$$

We can use the First Derivative Test to find that on $[0, 2\pi]$, z has reaches its absolute minimum at $t = \pi/4$ and $5\pi/4$; it reaches its absolute maximum at

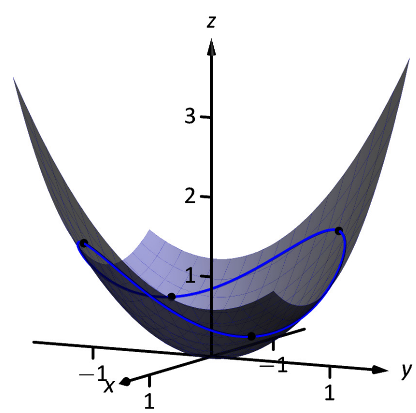


Figure 12.15: Plotting the path of a particle on a surface in Example 12.30.

Notes:

$t = 3\pi/4$ and $7\pi/4$, as shown in Figure 12.15.

We can extend the Chain Rule to include the situation where z is a function of more than one variable, and each of these variables is also a function of more than one variable. The basic case of this is where $z = f(x, y)$, and x and y are functions of two variables, say s and t.

Theorem 110 Multivariable Chain Rule, Part II

1. Let $z = f(x, y)$, $x = g(s, t)$ and $y = h(s, t)$, where f, g and h are differentiable functions. Then z is a function of s and t, and
   - $\dfrac{\partial z}{\partial s} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial s}$, and
   - $\dfrac{\partial z}{\partial t} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial t}$.
2. Let $z = f(x_1, x_2, \ldots, x_m)$ be a differentiable function of m variables, where each of the x_i is a differentiable function of the variables $t_1, t_2, \ldots, t_n$. Then z is a function of the t_i, and

$$\frac{\partial z}{\partial t_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial f}{\partial x_m}\frac{\partial x_m}{\partial t_i}.$$

Example 12.31 Using the Multivarible Chain Rule, Part II

Let $z = x^2y + x$, $x = s^2 + 3t$ and $y = 2s - t$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$, and evaluate each when $s = 1$ and $t = 2$.

Solution Following Theorem 110, we compute the following partial derivatives:

$$\frac{\partial f}{\partial x} = 2xy + 1 \qquad \frac{\partial f}{\partial y} = x^2,$$

$$\frac{\partial x}{\partial s} = 2s \qquad \frac{\partial x}{\partial t} = 3 \qquad \frac{\partial y}{\partial s} = 2 \qquad \frac{\partial y}{\partial t} = -1.$$

Thus

$$\frac{\partial z}{\partial s} = (2xy + 1)(2s) + (x^2)(2) = 4xys + 2s + 2x^2, \quad \text{and}$$

$$\frac{\partial z}{\partial t} = (2xy + 1)(3) + (x^2)(-1) = 6xy - x^2 + 3.$$

Notes:

When $s = 1$ and $t = 2$, $x = 7$ and $y = 0$, so

$$\frac{\partial z}{\partial s} = 100 \qquad \text{and} \qquad \frac{\partial z}{\partial t} = -46.$$

Example 12.32 Using the Multivarible Chain Rule, Part II

Let $w = xy + z^2$, where $x = t^2 e^s$, $y = t\cos s$, and $z = s\sin t$. Find $\frac{\partial w}{\partial t}$ when $s = 0$ and $t = \pi$.

Solution Following Theorem 110, we compute the following partial derivatives:

$$\frac{\partial f}{\partial x} = y \qquad \frac{\partial f}{\partial y} = x \qquad \frac{\partial f}{\partial z} = 2z,$$

$$\frac{\partial x}{\partial t} = 2te^s \qquad \frac{\partial y}{\partial t} = \cos s \qquad \frac{\partial z}{\partial t} = s\cos t.$$

Thus

$$\frac{\partial w}{\partial t} = y(2te^s) + x(\cos s) + 2z(s\cos t).$$

When $s = 0$ and $t = \pi$, we have $x = \pi^2$, $y = \pi$ and $z = 0$. Thus

$$\frac{\partial w}{\partial t} = \pi(2\pi) + \pi^2 = 3\pi^2.$$

Implicit Differentiation

We studied finding $\frac{dy}{dx}$ when y is given by an implicit equation of x in detail in Section 2.6. We find here that the Multivariable Chain Rule gives a simpler method of finding $\frac{dy}{dx}$.

For instance, consider the implicit equation $x^2y - xy^3 = 3$. We learned to use the following steps to find $\frac{dy}{dx}$:

$$\begin{aligned} \frac{d}{dx}\left(x^2y - xy^3\right) &= \frac{d}{dx}\left(3\right) \\ 2xy + x^2\frac{dy}{dx} - y^3 - 3xy^2\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{2xy - y^3}{x^2 - 3xy^2}. \end{aligned} \tag{12.2}$$

Instead of using this method, consider $z = x^2y - xy^3$. The implicit relation above describes the level curve $z = 3$. Considering x and y as functions of x, the Multivariable Chain Rule states that

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\frac{dx}{dx} + \frac{\partial z}{\partial y}\frac{dy}{dx}. \tag{12.3}$$

Notes:

Since z is constant (in our example, $z = 3$), $\frac{dz}{dx} = 0$. We also know $\frac{dx}{dx} = 1$. Equation (12.3) becomes

$$\begin{aligned} 0 &= \frac{\partial z}{\partial x}(1) + \frac{\partial z}{\partial y}\frac{dy}{dx} \quad \Rightarrow \\ \frac{dy}{dx} &= -\frac{\partial z}{\partial x}\Big/\frac{\partial z}{\partial y} \\ &= -\frac{f_x}{f_y}. \end{aligned}$$

Note how our solution for $\frac{dy}{dx}$ in Equation (12.2) is just the partial derivative of z with respect to x, divided by the partial derivative of z with respect to y.

We state the above as a theorem.

Theorem 111 Implicit Differentiation

Let f be a differentiable function of x and y, where $f(x, y) = c$ defines an implicit relation between x and y, for some constant c. Then

$$\frac{dy}{dx} = -\frac{f_x(x, y)}{f_y(x, y)}.$$

We practice using Theorem 111 by applying it to a problem from Section 2.6.

Example 12.33 Implicit Differentiation

Given the implicitly defined relation $\sin(x^2y^2) + y^3 = x + y$, find y'. Note: this is the same problem as given in Example 2.43 of Section 2.6, where the solution took about a full page to find.

Solution Let $f(x, y) = \sin(x^2y^2) + y^3 - x - y$; the implicitly defined relation above is equivalent to $f(x, y) = 0$. We find $\frac{dy}{dx}$ by applying Theorem 111. We find

$$f_x(x, y) = 2xy^2\cos(x^2y^2) - 1 \qquad \text{and} \qquad f_y(x, y) = 2x^2y\cos(x^2y^2) + 3y^2 - 1,$$

so

$$\frac{dy}{dx} = -\frac{2xy^2\cos(x^2y^2) - 1}{2x^2y\cos(x^2y^2) + 3y^2 - 1},$$

which matches our solution from Example 2.43.

Notes:

Exercises 12.5

Terms and Concepts

1. Let a level curve of $z = f(x,y)$ be described by $x = g(t)$, $y = h(t)$. Explain why $\frac{dz}{dt} = 0$.

2. Fill in the blank: The single variable Chain Rule states $\frac{d}{dx}\Big(f\big(g(x)\big)\Big) = f'\big(g(x)\big)\cdot$________.

3. Fill in the blank: The Multivariable Chain Rule states $\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot$ ________ $+$ ________ $\cdot \frac{dy}{dt}$.

4. If $z = f(x,y)$, where $x = g(t)$ and $y = h(t)$, we can substitute and write z as an explicit function of t.
T/F: Using the Multivariable Chain Rule to find $\frac{dz}{dt}$ is sometimes easier than first substituting and then taking the derivative.

5. T/F: The Multivariable Chain Rule is only useful when all the related functions are known explicitly.

6. The Multivariable Chain Rule allows us to compute implicit derivatives easily by just computing two ________ derivatives.

Problems

In Exercises 7 – 12, functions $z = f(x,y)$, $x = g(t)$ and $y = h(t)$ are given.

(a) Use the Multivariable Chain Rule to compute $\frac{dz}{dt}$.

(b) Evaluate $\frac{dz}{dt}$ at the indicated t-value.

7. $z = 3x + 4y, \quad x = t^2, \quad y = 2t; \quad t = 1$

8. $z = x^2 - y^2, \quad x = t, \quad y = t^2 - 1; \quad t = 1$

9. $z = 5x + 2y, \quad x = 2\cos t + 1, \quad y = \sin t - 3; \quad t = \pi/4$

10. $z = \frac{x}{y^2+1}, \quad x = \cos t, \quad y = \sin t; \quad t = \pi/2$

11. $z = x^2 + 2y^2, \quad x = \sin t, \quad y = 3\sin t; \quad t = \pi/4$

12. $z = \cos x \sin y, \quad x = \pi t, \quad y = 2\pi t + \pi/2; \quad t = 3$

In Exercises 13 – 18, functions $z = f(x,y)$, $x = g(t)$ and $y = h(t)$ are given. Find the values of t where $\frac{dz}{dt} = 0$. Note: these are the same surfaces/curves as found in Exercises 7 – 12.

13. $z = 3x + 4y, \quad x = t^2, \quad y = 2t$

14. $z = x^2 - y^2, \quad x = t, \quad y = t^2 - 1$

15. $z = 5x + 2y, \quad x = 2\cos t + 1, \quad y = \sin t - 3$

16. $z = \frac{x}{y^2+1}, \quad x = \cos t, \quad y = \sin t$

17. $z = x^2 + 2y^2, \quad x = \sin t, \quad y = 3\sin t$

18. $z = \cos x \sin y, \quad x = \pi t, \quad y = 2\pi t + \pi/2$

In Exercises 19 – 22, functions $z = f(x,y)$, $x = g(s,t)$ and $y = h(s,t)$ are given.

(a) Use the Multivariable Chain Rule to compute $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

(b) Evaluate $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ at the indicated s and t values.

19. $z = x^2 y, \quad x = s - t, \quad y = 2s + 4t; \quad s = 1, t = 0$

20. $z = \cos\left(\pi x + \frac{\pi}{2}y\right), \quad x = st^2, \quad y = s^2 t; \quad s = 1, t = 1$

21. $z = x^2 + y^2, \quad x = s\cos t, \quad y = s\sin t; \quad s = 2, t = \pi/4$

22. $z = e^{-(x^2+y^2)}, \quad x = t, \quad y = st^2; \quad s = 1, t = 1$

In Exercises 23 – 26, find $\frac{dy}{dx}$ using Implicit Differentiation and Theorem 111.

23. $x^2 \tan y = 50$

24. $(3x^2 + 2y^3)^4 = 2$

25. $\frac{x^2+y}{x+y^2} = 17$

26. $\ln(x^2 + xy + y^2) = 1$

In Exercises 27 – 30, find $\frac{dz}{dt}$, or $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$, using the supplied information.

27. $\frac{\partial z}{\partial x} = 2, \quad \frac{\partial z}{\partial y} = 1, \quad \frac{dx}{dt} = 4, \quad \frac{dy}{dt} = -5$

28. $\frac{\partial z}{\partial x} = 1, \quad \frac{\partial z}{\partial y} = -3, \quad \frac{dx}{dt} = 6, \quad \frac{dy}{dt} = 2$

29. $\frac{\partial z}{\partial x} = -4, \quad \frac{\partial z}{\partial y} = 9,$
$\frac{\partial x}{\partial s} = 5, \quad \frac{\partial x}{\partial t} = 7, \quad \frac{\partial y}{\partial s} = -2, \quad \frac{\partial y}{\partial t} = 6$

30. $\frac{\partial z}{\partial x} = 2, \quad \frac{\partial z}{\partial y} = 1,$
$\frac{\partial x}{\partial s} = -2, \quad \frac{\partial x}{\partial t} = 3, \quad \frac{\partial y}{\partial s} = 2, \quad \frac{\partial y}{\partial t} = -1$

12.6 Directional Derivatives

Partial derivatives give us an understanding of how a surface changes when we move in the x and y directions. We made the comparison to standing in a rolling meadow and heading due east: the amount of rise/fall in doing so is comparable to f_x. Likewise, the rise/fall in moving due north is comparable to f_y. The steeper the slope, the greater in magnitude f_y.

But what if we didn't move due north or east? What if we needed to move northeast and wanted to measure the amount of rise/fall? Partial derivatives alone cannot measure this. This section investigates **directional derivatives**, which do measure this rate of change.

We begin with a definition.

Definition 94 Directional Derivatives

Let $z = f(x, y)$ be continuous on an open set S and let $\vec{u} = \langle u_1, u_2 \rangle$ be a unit vector. For all points (x, y), the **directional derivative of** f **at** (x, y) **in the direction of** $\vec{u}$ is

$$D_{\vec{u}}\, f(x, y) = \lim_{h \to 0} \frac{f(x + hu_1, y + hu_2) - f(x, y)}{h}.$$

The partial derivatives f_x and f_y are defined with similar limits, but only x or y varies with h, not both. Here both x and y vary with a weighted h, determined by a particular unit vector $\vec{u}$. This may look a bit intimidating but in reality it is not too difficult to deal with; it often just requires extra algebra. However, the following theorem reduces this algebraic load.

Theorem 112 Directional Derivatives

Let $z = f(x, y)$ be differentiable on an open set S containing (x_0, y_0), and let $\vec{u} = \langle u_1, u_2 \rangle$ be a unit vector. The directional derivative of f at (x_0, y_0) in the direction of $\vec{u}$ is

$$D_{\vec{u}}\, f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2.$$

Example 12.34 Computing directional derivatives

Let $z = 14 - x^2 - y^2$ and let $P = (1, 2)$. Find the directional derivative of f, at P, in the following directions:

1. toward the point $Q = (3, 4)$,
2. in the direction of $\langle 2, -1 \rangle$, and

Notes:

3. toward the origin.

Solution The surface is plotted in Figure 12.16, where the point $P = (1,2)$ is indicated in the x, y-plane as well as the point $(1,2,9)$ which lies on the surface of f. We find that $f_x(x,y) = -2x$ and $f_x(1,2) = -2$; $f_y(x,y) = -2y$ and $f_y(1,2) = -4$.

1. Let $\vec{u}_1$ be the unit vector that points from the point $(1,2)$ to the point $Q = (3,4)$, as shown in the figure. The vector $\overrightarrow{PQ} = \langle 2,2\rangle$; the unit vector in this direction is $\vec{u}_1 = \left\langle 1/\sqrt{2}, 1/\sqrt{2}\right\rangle$. Thus the directional derivative of f at $(1,2)$ in the direction of $\vec{u}_1$ is

$$D_{\vec{u}_1}f(1,2) = -2(1/\sqrt{2}) + (-4)(1/\sqrt{2}) = -6/\sqrt{2} \approx -4.24.$$

   Thus the instantaneous rate of change in moving from the point $(1,2,9)$ on the surface in the direction of $\vec{u}_1$ (which points toward the point Q) is about -4.24. Moving in this direction moves one steeply downward.

2. We seek the directional derivative in the direction of $\langle 2,-1\rangle$. The unit vector in this direction is $\vec{u}_2 = \left\langle 2/\sqrt{5}, -1/\sqrt{5}\right\rangle$. Thus the directional derivative of f at $(1,2)$ in the direction of $\vec{u}_2$ is

$$D_{\vec{u}_2}f(1,2) = -2(2/\sqrt{5}) + (-4)(-1/\sqrt{5}) = 0.$$

   Starting on the surface of f at $(1,2)$ and moving in the direction of $\langle 2,-1\rangle$ (or $\vec{u}_2$) results in no instantaneous change in z-value. This is analogous to standing on the side of a hill and choosing a direction to walk that does not change the elevation. One neither walks up nor down, rather just "along the side" of the hill.

   Finding these directions of "no elevation change" is important.

3. At $P = (1,2)$, the direction towards the origin is given by the vector $\langle -1,-2\rangle$; the unit vector in this direction is $\vec{u}_3 = \left\langle -1/\sqrt{5}, -2/\sqrt{5}\right\rangle$. The directional derivative of f at P in the direction of the origin is

$$D_{\vec{u}_3}f(1,2) = -2(-1/\sqrt{5}) + (-4)(-2/\sqrt{5}) = 10/\sqrt{5} \approx 4.47.$$

   Moving towards the origin means "walking uphill" quite steeply, with an initial slope of about 4.47.

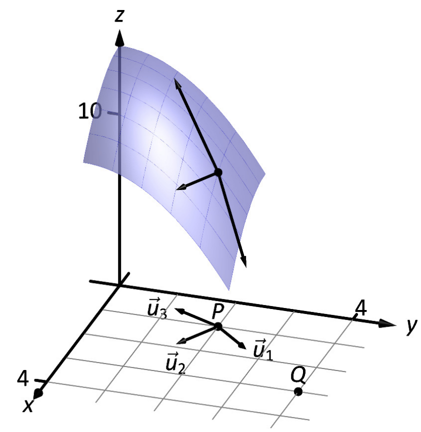


Figure 12.16: Understanding the directional derivative in Example 12.34.

As we study directional derivatives, it will help to make an important connection between the unit vector $\vec{u} = \langle u_1, u_2\rangle$ that describes the direction and the partial derivatives f_x and f_y. We start with a definition and follow this with a Key Idea.

Notes:

Definition 95 Gradient

Let $z = f(x, y)$ be differentiable on an open set S that contains the point (x_0, y_0).

1. The **gradient of** f is $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$.
2. The **gradient of** f **at** (x_0, y_0) **is** $\nabla f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle$.

To simplify notation, we often express the gradient as $\nabla f = \langle f_x, f_y \rangle$. The gradient allows us to compute directional derivatives in terms of a dot product.

Note: The symbol "∇" is named "nabla," derived from the Greek name of a Jewish harp. Oddly enough, in mathematics the expression ∇f is pronounced "del f."

Key Idea 56 The Gradient and Directional Derivatives

The directional derivative of $z = f(x, y)$ in the direction of $\vec{u}$ is

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}.$$

The properties of the dot product previously studied allow us to investigate the properties of the directional derivative. Given that the directional derivative gives the instantaneous rate of change of z when moving in the direction of $\vec{u}$, three questions naturally arise:

1. In what direction(s) is the change in z the greatest (i.e., the "steepest uphill")?
2. In what direction(s) is the change in z the least (i.e., the "steepest downhill")?
3. In what direction(s) is there no change in z?

Using the key property of the dot product, we have

$$\nabla f \cdot \vec{u} = || \nabla f || \, || \vec{u} || \cos\theta = || \nabla f || \cos\theta, \qquad (12.4)$$

where θ is the angle between the gradient and $\vec{u}$. (Since $\vec{u}$ is a unit vector, $|| \vec{u} || = 1$.) This equation allows us to answer the three questions stated previously.

1. Equation 12.4 is maximized when $\cos\theta = 1$, i.e., when the gradient and $\vec{u}$ have the same direction. We conclude the gradient points in the direction of greatest z change.

Notes:

2. Equation 12.4 is minimized when $\cos\theta = -1$, i.e., when the gradient and $\vec{u}$ have opposite directions. We conclude the gradient points in the opposite direction of the least z change.

3. Equation 12.4 is 0 when $\cos\theta = 0$, i.e., when the gradient and $\vec{u}$ are orthogonal to each other. We conclude the gradient is orthogonal to directions of no z change.

This result is rather amazing. Once again imagine standing in a rolling meadow and face the direction that leads you steepest uphill. Then the direction that leads steepest downhill is directly behind you, and side–stepping either left or right (i.e., moving perpendicularly to the direction you face) does not change your elevation at all.

Recall that a level curve is defined as a curve in the x-y plane along which the z-values of a function do not change. Let a surface $z = f(x, y)$ be given, and let's represent one such level curve as a vector–valued function, $\vec{r}(t) = \langle x(t), y(t)\rangle$. As the output of f does not change along this curve, $f\big(x(t), y(t)\big) = c$ for all t, for some constant c.

Since f is constant for all t, $\frac{df}{dt} = 0$. By the Multivariable Chain Rule, we also know

$$\begin{aligned}\frac{df}{dt} &= f_x(x,y)x'(t) + f_y(x,y)y'(t)\\ &= \langle f_x(x,y), f_y(x,y)\rangle \cdot \langle x'(t), y'(t)\rangle\\ &= \nabla f \cdot \vec{r}\,'(t)\\ &= 0.\end{aligned}$$

This last equality states $\nabla f \cdot \vec{r}\,'(t) = 0$: the gradient is orthogonal to the derivative of $\vec{r}$, meaning the gradient is orthogonal to $\vec{r}$ itself. Our conclusion: at any point on a surface, the gradient at that point is orthogonal to the level curve that passes through that point.

We restate these ideas in a theorem, then use them in an example.

Theorem 113 The Gradient and Directional Derivatives

Let $z = f(x, y)$ be differentiable on an open set S with gradient ∇f, let $P = (x_0, y_0)$ be a point in S and let $\vec{u}$ be a unit vector.

1. The maximum value of $D_{\vec{u}}\,f(x_0, y_0)$ is $||\ \nabla f(x_0, y_0)\ ||$; the direction of maximal z increase is $\nabla f(x_0, y_0)$.

2. The minimum value of $D_{\vec{u}}\,f(x_0, y_0)$ is $-||\ \nabla f(x_0, y_0)\ ||$; the direction of minimal z increase is $-\nabla f(x_0, y_0)$.

3. At P, $\nabla f(x_0, y_0)$ is orthogonal to the level curve passing through $\big(x_0, y_0, f(x_0, y_0)\big)$.

Notes:

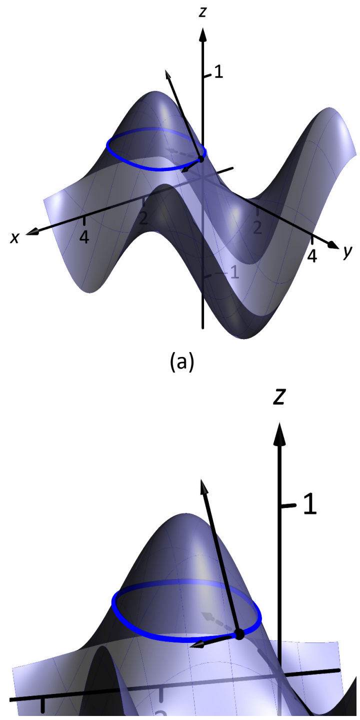


Figure 12.17: Graphing the surface and important directions in Example 12.35.

Example 12.35 Finding directions of maximal and minimal increase

Let $f(x,y) = \sin x\cos y$ and let $P = (\pi/3, \pi/3)$. Find the directions of maximal/minimal increase, and find a direction where the instantaneous rate of z change is 0.

Solution We begin by finding the gradient. $f_x = \cos x\cos y$ and $f_y = -\sin x\sin y$, thus

$$\nabla f = \langle \cos x\cos y, -\sin x\sin y\rangle \quad \text{and, at } P, \quad \nabla f\left(\frac{\pi}{3},\frac{\pi}{3}\right) = \left\langle \frac{1}{4}, -\frac{3}{4}\right\rangle.$$

Thus the direction of maximal increase is $\langle 1/4, -3/4\rangle$. In this direction, the instantaneous rate of z change is $||\langle 1/4, -3/4\rangle|| = \sqrt{10}/4 \approx 0.79$.

Figure 12.17 shows the surface plotted from two different perspectives. In each, the gradient is drawn at P with a dashed line (because of the nature of this surface, the gradient points "into" the surface). Let $\vec{u} = \langle u_1, u_2\rangle$ be the unit vector in the direction of ∇f at P. Each graph of the figure also contains the vector $\langle u_1, u_2, ||\nabla f||\rangle$. This vector has a "run" of 1 (because in the x-y plane it moves 1 unit) and a "rise" of $||\nabla f||$, hence we can think of it as a vector with slope of $||\nabla f||$ in the direction of ∇f, helping us visualize how "steep" the surface is in its steepest direction.

The direction of minimal increase is $\langle -1/4, 3/4\rangle$; in this direction the instantaneous rate of z change is $-\sqrt{10}/4 \approx -0.79$.

Any direction orthogonal to ∇f is a direction of no z change. We have two choices: the direction of $\langle 3, 1\rangle$ and the direction of $\langle -3, -1\rangle$. The unit vector in the direction of $\langle 3, 1\rangle$ is shown in each graph of the figure as well. The level curve at $z = \sqrt{3}/4$ is drawn: recall that along this curve the z-values do not change. Since $\langle 3, 1\rangle$ is a direction of no z-change, this vector is tangent to the level curve at P.

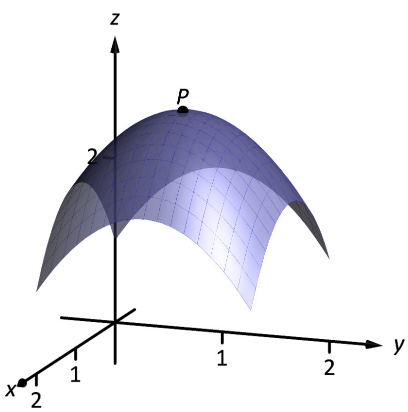


Figure 12.18: At the top of a paraboloid, all directional derivatives are 0.

Example 12.36 Understanding when $\nabla f = \vec{0}$

Let $f(x,y) = -x^2 + 2x - y^2 + 2y + 1$. Find the directional derivative of f in any direction at $P = (1,1)$.

Solution We find $\nabla f = \langle -2x+2, -2y+2\rangle$. At P, we have $\nabla f(1,1) = \langle 0, 0\rangle$. According to Theorem 113, this is the direction of maximal increase. However, $\langle 0, 0\rangle$ is directionless; it has no displacement. And regardless of the unit vector $\vec{u}$ chosen, $D_{\vec{u}}f = 0$.

Figure 12.18 helps us understand what this means. We can see that P lies at the top of a paraboloid. In all directions, the instantaneous rate of change is 0.

So what is the direction of maximal increase? It is fine to give an answer of $\vec{0} = \langle 0, 0\rangle$, as this indicates that all directional derivatives are 0.

Notes:

The fact that the gradient of a surface always points in the direction of steepest increase/decrease is very useful, as illustrated in the following example.

Example 12.37 The flow of water downhill

Consider the surface given by $f(x,y) = 20 - x^2 - 2y^2$. Water is poured on the surface at $(1, 1/4)$. What path does it take as it flows downhill?

SOLUTION Let $\vec{r}(t) = \langle x(t), y(t)\rangle$ be the vector–valued function describing the path of the water in the x-y plane; we seek $x(t)$ and $y(t)$. We know that water will always flow downhill in the steepest direction; therefore, at any point on its path, it will be moving in the direction of $-\nabla f$. (We ignore the physical effects of momentum on the water.) Thus $\vec{r}\,'(t)$ will be parallel to ∇f, and there is some constant c such that $c\nabla f = \vec{r}\,'(t) = \langle x'(t), y'(t)\rangle$.

We find $\nabla f = \langle -2x, -4y\rangle$ and write $x'(t)$ as $\frac{dx}{dt}$ and $y'(t)$ as $\frac{dy}{dt}$. Then

$$\begin{aligned} c\nabla f &= \langle x'(t), y'(t)\rangle \\ \langle -2cx, -4cy\rangle &= \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle. \end{aligned}$$

This implies

$$-2cx = \frac{dx}{dt} \quad \text{and} \quad -4cy = \frac{dy}{dt}, \text{ i.e.,}$$

$$c = -\frac{1}{2x}\frac{dx}{dt} \quad \text{and} \quad c = -\frac{1}{4y}\frac{dy}{dt}.$$

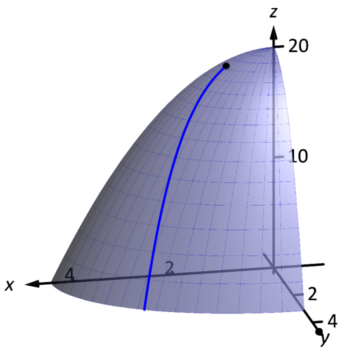
As c equals both expressions, we have

$$\frac{1}{2x}\frac{dx}{dt} = \frac{1}{4y}\frac{dy}{dt}.$$

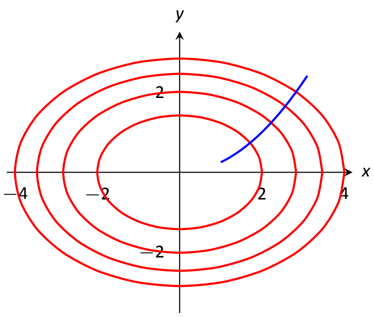
To find an explicit relationship between x and y, we can integrate both sides with respect to t. Recall from our study of differentials that $\dfrac{dx}{dt}dt = dx$. Thus:

$$\begin{aligned} \int \frac{1}{2x}\frac{dx}{dt}\,dt &= \int \frac{1}{4y}\frac{dy}{dt}\,dt \\ \int \frac{1}{2x}\,dx &= \int \frac{1}{4y}\,dy \\ \frac{1}{2}\ln|x| &= \frac{1}{4}\ln|y| + C_1 \\ 2\ln|x| &= \ln|y| + C_1 \\ \ln|x^2| &= \ln|y| + C_1 \end{aligned}$$

Notes:



(a)



(b)

Figure 12.19: A graph of the surface described in Example 12.37 along with the path in the x-y plane with the level curves.

Now raise both sides as a power of e:

$$\begin{aligned} x^2 &= e^{\ln|y|+C_1} \\ x^2 &= e^{\ln|y|}e^{C_1} \qquad \text{(Note that } e^{C_1} \text{ is just a constant.)} \\ x^2 &= yC_2 \\ \frac{1}{C_2}x^2 &= y \qquad \text{(Note that } 1/C_2 \text{ is just a constant.)} \\ Cx^2 &= y. \end{aligned}$$

As the water started at the point $(1, 1/4)$, we can solve for C:

$$C(1)^2 = \frac{1}{4} \quad \Rightarrow \quad C = \frac{1}{4}.$$

Thus the water follows the curve $y = x^2/4$ in the x-y plane. The surface and the path of the water is graphed in Figure 12.19(a). In part (b) of the figure, the level curves of the surface are plotted in the x-y plane, along with the curve $y = x^2/4$. Notice how the path intersects the level curves at right angles. As the path follows the gradient downhill, this reinforces the fact that the gradient is orthogonal to level curves.

Functions of Three Variables

The concepts of directional derivatives and the gradient are easily extended to three (and more) variables. We combine the concepts behind Definitions 94 and 95 and Theorem 112 into one set of definitions.

Definition 96 Directional Derivatives and Gradient with Three Variables

Let $w = F(x, y, z)$ be differentiable on an open ball B and let $\vec{u}$ be a unit vector in $\mathbb{R}^3$.

1. The **gradient** of F is $\nabla F = \langle F_x, F_y, F_z \rangle$.
2. The **directional derivative of F in the direction of $\vec{u}$** is

$$D_{\vec{u}}\, F = \nabla F \cdot \vec{u}.$$

The same properties of the gradient given in Theorem 113, when f is a func-

Notes:

tion of two variables, hold for F, a function of three variables.

Theorem 114 The Gradient and Directional Derivatives with Three Variables

Let $w = F(x, y, z)$ be differentiable on an open ball B, let ∇F be the gradient of F, and let $\vec{u}$ be a unit vector.

1. The maximum value of $D_{\vec{u}}\,F$ is $||\ \nabla F\ ||$, obtained when the angle between ∇F and $\vec{u}$ is 0, i.e., the direction of maximal increase is ∇F.

2. The minimum value of $D_{\vec{u}}\,F$ is $-||\ \nabla F\ ||$, obtained when the angle between ∇F and $\vec{u}$ is π, i.e., the direction of minimal increase is $-\nabla F$.

3. $D_{\vec{u}}\,F = 0$ when ∇F and $\vec{u}$ are orthogonal.

We interpret the third statement of the theorem as "the gradient is orthogonal to level surfaces," the three–variable analogue to level curves.

Example 12.38 Finding directional derivatives with functions of three variables

If a point source S is radiating energy, the intensity I at a given point P in space is inversely proportional to the square of the distance between S and P. That is, when $S = (0, 0, 0)$, $I(x, y, z) = \dfrac{k}{x^2 + y^2 + z^2}$ for some constant k.

Let $k = 1$, let $\vec{u} = \langle 2/3, 2/3, 1/3\rangle$ be a unit vector, and let $P = (2, 5, 3)$. Measure distances in inches. Find the directional derivative of I at P in the direction of $\vec{u}$, and find the direction of greatest intensity increase at P.

SOLUTION We need the gradient ∇I, meaning we need I_x, I_y and I_z. Each partial derivative requires a simple application of the Quotient Rule, giving

$$\begin{aligned}
\nabla I &= \left\langle \frac{-2x}{(x^2+y^2+z^2)^2}, \frac{-2y}{(x^2+y^2+z^2)^2}, \frac{-2z}{(x^2+y^2+z^2)^2} \right\rangle \\
\nabla I(2,5,3) &= \left\langle \frac{-4}{1444}, \frac{-10}{1444}, \frac{-6}{1444} \right\rangle \approx \langle -0.003, -0.007, -0.004\rangle \\
D_{\vec{u}}\,I &= \nabla I(2,5,3) \cdot \vec{u} \\
&= -\frac{17}{2166} \approx -0.0078.
\end{aligned}$$

The directional derivative tells us that moving in the direction of $\vec{u}$ from P results in a decrease in intensity of about -0.008 units per inch. (The intensity is decreasing as $\vec{u}$ moves one farther from the origin than P.)

Notes:

The gradient gives the direction of greatest intensity increase. Notice that

$$\begin{aligned}\nabla I(2,5,3) &= \left\langle \frac{-4}{1444}, \frac{-10}{1444}, \frac{-6}{1444} \right\rangle \\ &= \frac{2}{1444}\left\langle -2,-5,-3\right\rangle.\end{aligned}$$

That is, the gradient at $(2,5,3)$ is pointing in the direction of $\langle -2,-5,-3\rangle$, that is, towards the origin. That should make intuitive sense: the greatest increase in intensity is found by moving towards to source of the energy.

The directional derivative allows us to find the instantaneous rate of z change in any direction at a point. We can use these instantaneous rates of change to define lines and planes that are *tangent* to a surface at a point, which is the topic of the next section.

Notes:

Exercises 12.6

Terms and Concepts

1. What is the difference between a directional derivative and a partial derivative?

2. For what $\vec{u}$ is $D_{\vec{u}}\,f = f_x$?

3. For what $\vec{u}$ is $D_{\vec{u}}\,f = f_y$?

4. The gradient is ________ to level curves.

5. The gradient points in the direction of ________ increase.

6. It is generally more informative to view the directional derivative not as the result of a limit, but rather as the result of a ________ product.

Problems

In Exercises 7 – 12, a function $z = f(x, y)$ is given. Find ∇f.

7. $f(x, y) = -x^2y + xy^2 + xy$

8. $f(x, y) = \sin x \cos y$

9. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$

10. $f(x, y) = -4x + 3y$

11. $f(x, y) = x^2 + 2y^2 - xy - 7x$

12. $f(x, y) = x^2y^3 - 2x$

In Exercises 13 – 18, a function $z = f(x, y)$ and a point P are given. Find the directional derivative of f in the indicated directions. Note: these are the same functions as in Exercises 7 through 12.

13. $f(x, y) = -x^2y + xy^2 + xy$, $P = (2, 1)$
    (a) In the direction of $\vec{v} = \langle 3, 4\rangle$
    (b) In the direction toward the point $Q = (1, -1)$.

14. $f(x, y) = \sin x \cos y$, $P = \left(\dfrac{\pi}{4}, \dfrac{\pi}{3}\right)$
    (a) In the direction of $\vec{v} = \langle 1, 1\rangle$.
    (b) In the direction toward the point $Q = (0, 0)$.

15. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$, $P = (1, 1)$.
    (a) In the direction of $\vec{v} = \langle 1, -1\rangle$.
    (b) In the direction toward the point $Q = (-2, -2)$.

16. $f(x, y) = -4x + 3y$, $P = (5, 2)$
    (a) In the direction of $\vec{v} = \langle 3, 1\rangle$.
    (b) In the direction toward the point $Q = (2, 7)$.

17. $f(x, y) = x^2 + 2y^2 - xy - 7x$, $P = (4, 1)$
    (a) In the direction of $\vec{v} = \langle -2, 5\rangle$
    (b) In the direction toward the point $Q = (4, 0)$.

18. $f(x, y) = x^2y^3 - 2x$, $P = (1, 1)$
    (a) In the direction of $\vec{v} = \langle 3, 3\rangle$
    (b) In the direction toward the point $Q = (1, 2)$.

In Exercises 19 – 24, a function $z = f(x, y)$ and a point P are given.

(a) Find the direction of maximal increase of f at P.

(b) What is the maximal value of $D_{\vec{u}}\,f$ at P?

(c) Find the direction of minimal increase of f at P.

(d) Give a direction $\vec{u}$ such that $D_{\vec{u}}\,f = 0$ at P.

Note: these are the same functions and points as in Exercises 13 through 18.

19. $f(x, y) = -x^2y + xy^2 + xy$, $P = (2, 1)$

20. $f(x, y) = \sin x \cos y$, $P = \left(\dfrac{\pi}{4}, \dfrac{\pi}{3}\right)$

21. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$, $P = (1, 1)$.

22. $f(x, y) = -4x + 3y$, $P = (5, 4)$.

23. $f(x, y) = x^2 + 2y^2 - xy - 7x$, $P = (4, 1)$

24. $f(x, y) = x^2y^3 - 2x$, $P = (1, 1)$

In Exercises 25 – 28, a function $w = F(x, y, z)$, a vector $\vec{v}$ and a point P are given.

(a) Find $\nabla F(x, y, z)$.

(b) Find $D_{\vec{u}}\,F$ at P.

25. $F(x, y, z) = 3x^2z^3 + 4xy - 3z^2$, $\vec{v} = \langle 1, 1, 1\rangle$, $P = (3, 2, 1)$

26. $F(x, y, z) = \sin(x)\cos(y)e^z$, $\vec{v} = \langle 2, 2, 1\rangle$, $P = (0, 0, 0)$

27. $F(x, y, z) = x^2y^2 - y^2z^2$, $\vec{v} = \langle -1, 7, 3\rangle$, $P = (1, 0, -1)$

28. $F(x, y, z) = \dfrac{2}{x^2 + y^2 + z^2}$, $\vec{v} = \langle 1, 1, -2\rangle$, $P = (1, 1, 1)$

12.7 Tangent Lines, Normal Lines, and Tangent Planes

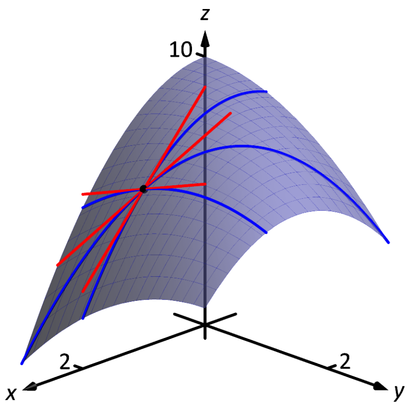


Figure 12.20: Showing various lines tangent to a surface.

Derivatives and tangent lines go hand–in–hand. Given $y = f(x)$, the line tangent to the graph of f at $x = x_0$ is the line through $\big(x_0, f(x_0)\big)$ with slope $f'(x_0)$; that is, the slope of the tangent line is the instantaneous rate of change of f at x_0.

When dealing with functions of two variables, the graph is no longer a curve but a surface. At a given point on the surface, it seems there are many lines that fit our intuition of being "tangent" to the surface.

In Figures 12.20 we see lines that are tangent to curves in space. Since each curve lies on a surface, it makes sense to say that the lines are also tangent to the surface. The next definition formally defines what it means to be "tangent to a surface."

Definition 97 Directional Tangent Line

Let $z = f(x, y)$ be differentiable on an open set S containing (x_0, y_0) and let $\vec{u} = \langle u_1, u_2\rangle$ be a unit vector.

1. The line ℓ_x through $\big(x_0, y_0, f(x_0, y_0)\big)$ parallel to $\langle 1, 0, f_x(x_0, y_0)\rangle$ is the **tangent line to f in the direction of x at** (x_0, y_0).

2. The line ℓ_y through $\big(x_0, y_0, f(x_0, y_0)\big)$ parallel to $\langle 0, 1, f_y(x_0, y_0)\rangle$ is the **tangent line to f in the direction of y at** (x_0, y_0).

3. The line $\ell_{\vec{u}}$ through $\big(x_0, y_0, f(x_0, y_0)\big)$ parallel to $\langle u_1, u_2, D_{\vec{u}}f(x_0, y_0)\rangle$ is the **tangent line to f in the direction of $\vec{u}$ at** (x_0, y_0).

It is instructive to consider each of three directions given in the definition in terms of "slope." The direction of ℓ_x is $\langle 1, 0, f_x(x_0, y_0)\rangle$; that is, the "run" is one unit in the x-direction and the "rise" is $f_x(x_0, y_0)$ units in the z-direction. Note how the slope is just the partial derivative with respect to x. A similar statement can be made for ℓ_y. The direction of $\ell_{\vec{u}}$ is $\langle u_1, u_2, D_{\vec{u}}f(x_0, y_0)\rangle$; the "run" is one unit in the $\vec{u}$ direction (where $\vec{u}$ is a unit vector) and the "rise" is the directional derivative of z in that direction.

Definition 97 leads to the following parametric equations of directional tangent lines:

$$\ell_x(t) = \begin{cases} x = x_0 + t \\ y = y_0 \\ z = z_0 + f_x(x_0, y_0)t \end{cases}, \quad \ell_y(t) = \begin{cases} x = x_0 \\ y = y_0 + t \\ z = z_0 + f_y(x_0, y_0)t \end{cases} \quad \text{and} \quad \ell_{\vec{u}}(t) = \begin{cases} x = x_0 + u_1 t \\ y = y_0 + u_2 t \\ z = z_0 + D_{\vec{u}}f(x_0, y_0)t \end{cases}.$$

Notes:

Example 12.39 Finding directional tangent lines

Find the lines tangent to the surface $z = \sin x \cos y$ at $(\pi/2, \pi/2)$ in the x and y directions and also in the direction of $\vec{v} = \langle -1, 1 \rangle$.

Solution The partial derivatives with respect to x and y are:

$$f_x(x,y) = \cos x \cos y \quad \Rightarrow \quad f_x(\pi/2, \pi/2) = 0$$
$$f_y(x,y) = -\sin x \sin y \quad \Rightarrow \quad f_y(\pi/2, \pi/2) = -1.$$

At $(\pi/2, \pi/2)$, the z-value is 0.

Thus the parametric equations of the line tangent to f at $(\pi/2, \pi/2)$ in the directions of x and y are:

$$\ell_x(t) = \begin{cases} x = \pi/2 + t \\ y = \pi/2 \\ z = 0 \end{cases} \quad \text{and} \quad \ell_y(t) = \begin{cases} x = \pi/2 \\ y = \pi/2 + t \\ z = -t \end{cases} .$$

The two lines are shown with the surface in Figure 12.21(a). To find the equation of the tangent line in the direction of $\vec{v}$, we first find the unit vector in the direction of $\vec{v}$: $\vec{u} = \left\langle -1/\sqrt{2}, 1/\sqrt{2} \right\rangle$. The directional derivative at $(\pi/2, \pi, 2)$ in the direction of $\vec{u}$ is

$$D_{\vec{u}} f(\pi/2, \pi, 2) = \langle 0, -1 \rangle \cdot \left\langle -1/\sqrt{2}, 1/\sqrt{2} \right\rangle = -1/\sqrt{2}.$$

Thus the directional tangent line is

$$\ell_{\vec{u}}(t) = \begin{cases} x = \pi/2 - t/\sqrt{2} \\ y = \pi/2 + t/\sqrt{2} \\ z = -t/\sqrt{2} \end{cases} .$$

The curve through $(\pi/2, \pi/2, 0)$ in the direction of $\vec{v}$ is shown in Figure 12.21(b) along with $\ell_{\vec{u}}(t)$.

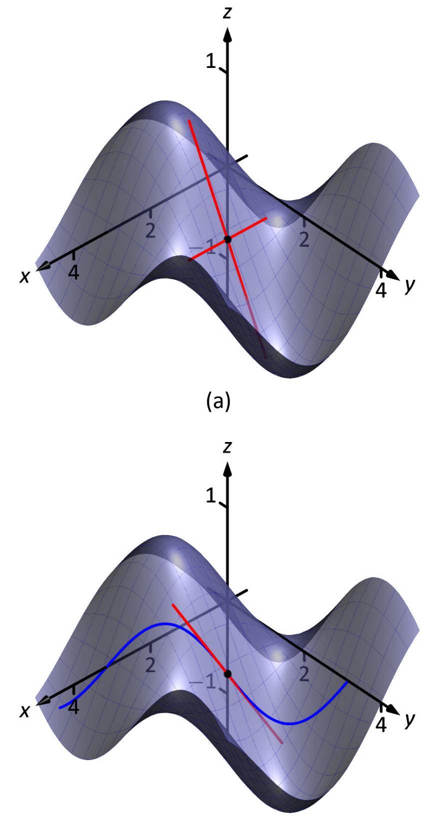


Figure 12.21: A surface and directional tangent lines in Example 12.39.

Example 12.40 Finding directional tangent lines

Let $f(x,y) = 4xy - x^4 - y^4$. Find the equations of *all* directional tangent lines to f at $(1, 1)$.

Solution First note that $f(1,1) = 2$. We need to compute directional derivatives, so we need ∇f. We begin by computing partial derivatives.

$$f_x = 4y - 4x^3 \Rightarrow f_x(1,1) = 0; \quad f_y = 4x - 4y^3 \Rightarrow f_y(1,1) = 0.$$

Thus $\nabla f(1,1) = \langle 0, 0 \rangle$. Let $\vec{u} = \langle u_1, u_2 \rangle$ be any unit vector. The directional derivative of f at $(1,1)$ will be $D_{\vec{u}} f(1,1) = \langle 0, 0 \rangle \cdot \langle u_1, u_2 \rangle = 0$. It does not matter

Notes:

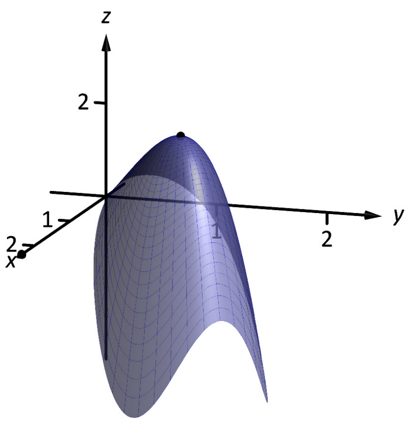


Figure 12.22: Graphing f in Example 12.40.

what direction we choose; the directional derivative is always 0. Therefore

$$\ell_{\vec{u}}(t) = \begin{cases} x = 1 + u_1 t \\ y = 1 + u_2 t \\ z = 2 \end{cases}.$$

Figure 12.22 shows a graph of f and the point $(1, 1, 2)$. Note that this point comes at the top of a "hill," and therefore every tangent line through this point will have a "slope" of 0.

That is, consider any curve on the surface that goes through this point. Each curve will have a relative maximum at this point, hence its tangent line will have a slope of 0. The following section investigates the points on surfaces where all tangent lines have a slope of 0.

Normal Lines

When dealing with a function $y = f(x)$ of one variable, we stated that a line through $(c, f(c))$ was *tangent* to f if the line had a slope of $f'(c)$ and was *normal* (or, *perpendicular, orthogonal*) to f if it had a slope of $-1/f'(c)$. We extend the concept of normal, or orthogonal, to functions of two variables.

Let $z = f(x, y)$ be a differentiable function of two variables. By Definition 97, at (x_0, y_0), $\ell_x(t)$ is a line parallel to the vector $\vec{d}_x = \langle 1, 0, f_x(x_0, y_0)\rangle$ and $\ell_y(t)$ is a line parallel to $\vec{d}_y = \langle 0, 1, f_y(x_0, y_0)\rangle$. Since lines in these directions through $\big(x_0, y_0, f(x_0, y_0)\big)$ are *tangent* to the surface, a line through this point and orthogonal to these directions would be *orthogonal*, or *normal*, to the surface. We can use this direction to create a normal line.

The direction of the normal line is orthogonal to $\vec{d}_x$ and $\vec{d}_y$, hence the direction is parallel to $\vec{d}_n = \vec{d}_x \times \vec{d}_y$. It turns out this cross product has a very simple form:

$$\vec{d}_x \times \vec{d}_y = \langle 1, 0, f_x\rangle \times \langle 0, 1, f_y\rangle = \langle -f_x, -f_y, 1\rangle .$$

It is often more convenient to refer to the opposite of this direction, namely $\langle f_x, f_y, -1\rangle$. This leads to a definition.

Notes:

Definition 98 Normal Line

Let $z = f(x, y)$ be differentiable on an open set S containing (x_0, y_0) where

$$a = f_x(x_0, y_0) \quad \text{and} \quad b = f_y(x_0, y_0)$$

are defined.

1. A nonzero vector parallel to $\vec{n} = \langle a, b, -1\rangle$ is **orthogonal to f at** $P = \big(x_0, y_0, f(x_0, y_0)\big)$.
2. The line ℓ_n through P with direction parallel to $\vec{n}$ is the **normal line to f at** P.

Thus the parametric equations of the normal line to a surface f at $\big(x_0, y_0, f(x_0, y_0)\big)$ is:

$$\ell_n(t) = \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = f(x_0, y_0) - t \end{cases}.$$

Example 12.41 Finding a normal line

Find the equation of the normal line to $z = -x^2 - y^2 + 2$ at $(0, 1)$.

Solution We find $z_x(x, y) = -2x$ and $z_y(x, y) = -2y$; at $(0, 1)$, we have $z_x = 0$ and $z_y = -2$. We take the direction of the normal line, following Definition 98, to be $\vec{n} = \langle 0, -2, -1\rangle$. The line with this direction going through the point $(0, 1, 1)$ is

$$\ell_n(t) = \begin{cases} x = 0 \\ y = -2t + 1 \\ z = -t + 1 \end{cases} \quad \text{or} \quad \ell_n(t) = \langle 0, -2, -1\rangle\, t + \langle 0, 1, 1\rangle\,.$$

The surface $z = -x^2 + y^2$, along with the found normal line, is graphed in Figure 12.23.

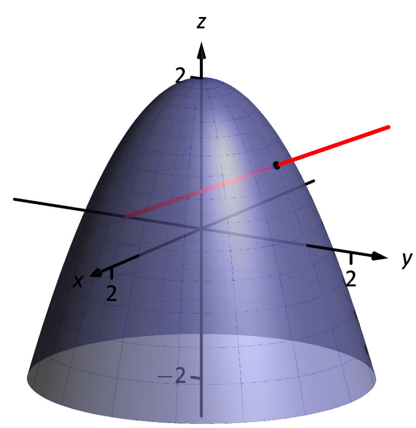


Figure 12.23: Graphing a surface with a normal line from Example 12.41.

The direction of the normal line has many uses, one of which is the definition of the **tangent plane** which we define shortly. Another use is in measuring distances from the surface to a point. Given a point Q in space, it is general geometric concept to define the distance from Q to the surface as being the length of the shortest line segment $\overline{PQ}$ over all points P on the surface. This, in turn, implies that $\overrightarrow{PQ}$ will be orthogonal to the surface at P. Therefore we can measure the distance from Q to the surface f by finding a point P on the surface such

Notes:

that $\overrightarrow{PQ}$ is parallel to the normal line to f at P.

Example 12.42 Finding the distance from a point to a surface
Let $f(x, y) = 2 - x^2 - y^2$ and let $Q = (2, 2, 2)$. Find the distance from Q to the surface defined by f.

Solution This surface is used in Example 12.40, so we know that at (x, y), the direction of the normal line will be $\vec{d}_n = \langle -2x, -2y, -1\rangle$. A point P on the surface will have coordinates $(x, y, 2-x^2-y^2)$, so $\overrightarrow{PQ} = \left\langle 2 - x, 2 - y, x^2 + y^2\right\rangle$. To find where $\overrightarrow{PQ}$ is parallel to $\vec{d}_n$, we need to find x, y and c such that $c\overrightarrow{PQ} = \vec{d}_n$.

$$\begin{aligned} c\overrightarrow{PQ} &= \vec{d}_n \\ c\left\langle 2 - x, 2 - y, x^2 + y^2\right\rangle &= \langle -2x, -2y, -1\rangle . \end{aligned}$$

This implies

$$\begin{aligned} c(2 - x) &= -2x \\ c(2 - y) &= -2y \\ c(x^2 + y^2) &= -1 \end{aligned}$$

In each equation, we can solve for c:

$$c = \frac{-2x}{2 - x} = \frac{-2y}{2 - y} = \frac{-1}{x^2 + y^2}.$$

The first two fractions imply $x = y$, and so the last fraction can be rewritten as $c = -1/(2x^2)$. Then

$$\begin{aligned} \frac{-2x}{2 - x} &= \frac{-1}{2x^2} \\ -2x(2x^2) &= -1(2 - x) \\ 4x^3 &= 2 - x \\ 4x^3 + x - 2 &= 0. \end{aligned}$$

This last equation is a cubic, which is not difficult to solve with a numeric solver. We find that $x = 0.689$, hence $P = (0.689, 0.689, 1.051)$. We find the distance from Q to the surface of f is

$$||\,\overrightarrow{PQ}\,|| = \sqrt{(2 - 0.689)^2 + (2 - 0.689)^2 + (2 - 1.051)^2} = 2.083.$$

We can take the concept of measuring the distance from a point to a surface to find a point Q a particular distance from a surface at a given point P on the

Notes:

surface.

Example 12.43 Finding a point a set distance from a surface
Let $f(x,y) = x - y^2 + 3$. Let $P = \big(2, 1, f(2,1)\big) = (2, 1, 4)$. Find points Q in space that are 4 units from the surface of f at P. That is, find Q such that $\| \overrightarrow{PQ} \| = 4$ and $\overrightarrow{PQ}$ is orthogonal to f at P.

Solution We begin by finding partial derivatives:

$$\begin{aligned} f_x(x,y) &= 1 & \Rightarrow \quad & f_x(2,1) = 1 \\ f_y(x,y) &= -2y & \Rightarrow \quad & f_y(2,1) = -2 \end{aligned}$$

The vector $\vec{n} = \langle 1, -2, -1\rangle$ is orthogonal to f at P. For reasons that will become more clear in a moment, we find the unit vector in the direction of $\vec{n}$:

$$\vec{u} = \frac{\vec{n}}{\| \vec{n} \|} = \left\langle 1/\sqrt{6}, -2/\sqrt{6}, -1/\sqrt{6}\right\rangle \approx \langle 0.408, -0.816, -0.408\rangle .$$

Thus a the normal line to f at P can be written as

$$\ell_n(t) = \langle 2, 1, 4\rangle + t\,\langle 0.408, -0.816, -0.408\rangle .$$

An advantage of this parametrization of the line is that letting $t = t_0$ gives a point on the line that is $|t_0|$ units from P. (This is because the direction of the line is given in terms of a unit vector.) There are thus two points in space 4 units from P:

$$\begin{aligned} Q_1 &= \ell_n(4) & \qquad Q_2 &= \ell_n(-4) \\ &\approx \langle 3.63, -2.27, 2.37\rangle & &\approx \langle 0.37, 4.27, 5.63\rangle \end{aligned}$$

The surface is graphed along with points P, Q_1, Q_2 and a portion of the normal line to f at P.

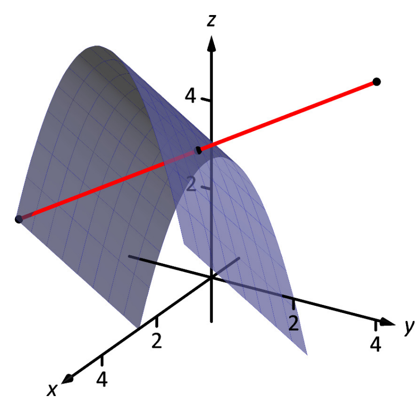


Figure 12.24: Graphing the surface in Example 12.43 along with points 4 units from the surface.

Tangent Planes

We can use the direction of the normal line to define a plane. With $a = f_x(x_0, y_0)$, $b = f_y(x_0, y_0)$ and $P = \big(x_0, y_0, f(x_0, y_0)\big)$, the vector $\vec{n} = \langle a, b, -1\rangle$ is orthogonal to f at P. The plane through P with normal vector $\vec{n}$ is therefore **tangent** to f at P.

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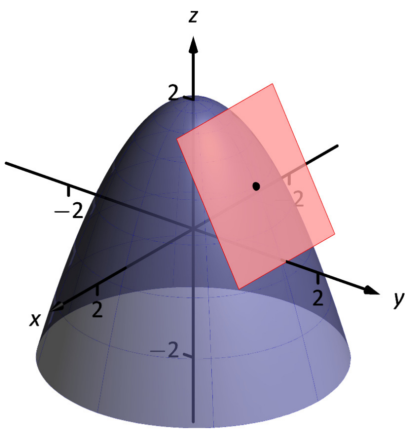

Figure 12.25: Graphing a surface with tangent plane from Example 12.44.

Definition 99 Tangent Plane

Let $z = f(x, y)$ be differentiable on an open set S containing (x_0, y_0), where $a = f_x(x_0, y_0)$, $b = f_y(x_0, y_0)$, $\vec{n} = \langle a, b, -1\rangle$ and $P = \big(x_0, y_0, f(x_0, y_0)\big)$.

The plane through P with normal vector $\vec{n}$ is the **tangent plane to f at P**. The standard form of this plane is

$$a(x - x_0) + b(y - y_0) - \big(z - f(x_0, y_0)\big) = 0.$$

Example 12.44 Finding tangent planes

Find the equation of the tangent plane to $z = -x^2 - y^2 + 2$ at $(0, 1)$.

Solution Note that this is the same surface and point used in Example 12.41. There we found $\vec{n} = \langle 0, -2, -1\rangle$ and $P = (0, 1, 1)$. Therefore the equation of the tangent plane is

$$-2(y - 1) - (z - 1) = 0.$$

The surface $z = -x^2 + y^2$ and tangent plane are graphed in Figure 12.25.

Example 12.45 Using the tangent plane to approximate function values

The point $(3, -1, 4)$ lies on the surface of an unknown differentiable function f where $f_x(3, -1) = 2$ and $f_y(3, -1) = -1/2$. Find the equation of the tangent plane to f at P, and use this to approximate the value of $f(2.9, -0.8)$.

Solution Knowing the partial derivatives at $(3, -1)$ allows us to form the normal vector to the tangent plane, $\vec{n} = \langle 2, -1/2, -1\rangle$. Thus the equation of the tangent line to f at P is:

$$2(x-3) - 1/2(y+1) - (z-4) = 0 \quad \Rightarrow \quad z = 2(x-3) - 1/2(y+1) + 4. \tag{12.5}$$

Just as tangent lines provide excellent approximations of curves near their point of intersection, tangent planes provide excellent approximations of surfaces near their point of intersection. So $f(2.9, -0.8) \approx z(2.9, -0.8) = 3.7$.

This is not a new method of approximation. Compare the right hand expression for z in Equation (12.5) to the total differential:

$$dz = f_x dx + f_y dy \quad \text{and} \quad z = \underbrace{\underbrace{2}_{f_x}\underbrace{(x-3)}_{dx} + \underbrace{-1/2}_{f_y}\underbrace{(y+1)}_{dy}}_{dz} + 4.$$

Notes:

Thus the "new z-value" is the sum of the change in z (i.e., dz) and the old z-value (4). As mentioned when studying the total differential, it is not uncommon to know partial derivative information about a unknown function, and tangent planes are used to give accurate approximations of the function.

The Gradient and Normal Lines, Tangent Planes

The methods developed in this section so far give a straightforward method of finding equations of normal lines and tangent planes for surfaces with explicit equations of the form $z = f(x, y)$. However, they do not handle implicit equations well, such as $x^2 + y^2 + z^2 = 1$. There is a technique that allows us to find vectors orthogonal to these surfaces based on the **gradient**.

Definition 100 Gradient

Let $w = F(x, y, z)$ be differentiable on an open ball B that contains the point (x_0, y_0, z_0).

1. The **gradient of** F is $\nabla F(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle$.
2. The **gradient of** F **at** (x_0, y_0, z_0) **is**

$$\nabla F(x_0, y_0, z_0) = \langle f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0)\rangle .$$

Recall that when $z = f(x, y)$, the gradient $\nabla f = \langle f_x, f_y\rangle$ is orthogonal to level curves of f. An analogous statement can be made about the gradient ∇F, where $w = F(x, y, z)$. Given a point (x_0, y_0, z_0), let $c = F(x_0, y_0, z_0)$. Then $F(x, y, z) = c$ is a **level surface** that contains the point (x_0, y_0, z_0). The following theorem states that $\nabla F(x_0, y_0, z_0)$ is orthogonal to this level surface.

Theorem 115 The Gradient and Level Surfaces

Let $w = F(x, y, z)$ be differentiable on an open ball B containing (x_0, y_0, z_0) with gradient ∇F, where $F(x_0, y_0, z_0) = c$.

The vector $\nabla F(x_0, y_0, z_0)$ is orthogonal to the level surface $F(x, y, z) = c$ at (x_0, y_0, z_0).

The gradient at a point gives a vector orthogonal to the surface at that point. This direction can be used to find tangent planes and normal lines.

Notes:

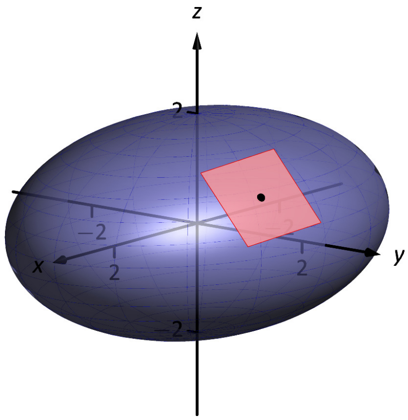

Figure 12.26: An ellipsoid and its tangent plane at a point.

Example 12.46 Using the gradient to find a tangent plane

Find the equation of the plane tangent to the ellipsoid $\dfrac{x^2}{12}+\dfrac{y^2}{6}+\dfrac{z^2}{4}=1$ at $P=(1,2,1)$.

Solution We consider the equation of the ellipsoid as a level surface of a function F of three variables, where $F(x,y,z)=\frac{x^2}{12}+\frac{y^2}{6}+\frac{z^2}{4}$. The gradient is:

$$\begin{aligned}\nabla F(x,y,z) &= \langle F_x, F_y, F_z\rangle \\ &= \left\langle \frac{x}{6}, \frac{y}{3}, \frac{z}{2}\right\rangle.\end{aligned}$$

At P, the gradient is $\nabla F(1,2,1)=\langle 1/6, 2/3, 1/2\rangle$. Thus the equation of the plane tangent to the ellipsoid at P is

$$\frac{1}{6}(x-1)+\frac{2}{3}(y-2)+\frac{1}{2}(z-1)=0.$$

The ellipsoid and tangent plane are graphed in Figure 12.26.

Tangent lines and planes to surfaces have many uses, including the study of instantaneous rates of changes and making approximations. Normal lines also have many uses. In this section we focused on using them to measure distances from a surface. Another interesting application is in computer graphics, where the effects of light on a surface are determined using normal vectors.

The next section investigates another use of partial derivatives: determining relative extrema. When dealing with functions of the form $y=f(x)$, we found relative extrema by finding x where $f'(x)=0$. We can *start* finding relative extrema of $z=f(x,y)$ by setting f_x and f_y to 0, but it turns out that there is more to consider.

Notes:

Exercises 12.7

Terms and Concepts

1. Explain how the vector $\vec{v} = \langle 1, 0, 3 \rangle$ can be thought of as having a "slope" of 3.

2. Explain how the vector $\vec{v} = \langle 0.6, 0.8, -2 \rangle$ can be thought of as having a "slope" of -2.

3. T/F: Let $z = f(x, y)$ be differentiable at P. If $\vec{n}$ is a normal vector to the tangent plane of f at P, then $\vec{n}$ is orthogonal to f_x and f_y at P.

4. Explain in your own words why we do not refer to *the* tangent line to a surface at a point, but rather to *directional* tangent line*s* to a surface at a point.

Problems

In Exercises 5 – 8, a function $z = f(x, y)$, a vector $\vec{v}$ and a point P are given. Give the parametric equations of the following directional tangent lines to f at P:

(a) $\ell_x(t)$

(b) $\ell_y(t)$

(c) $\ell_{\vec{u}}(t)$**, where $\vec{u}$ is the unit vector in the direction of $\vec{v}$.**

5. $f(x, y) = 2x^2y - 4xy^2$, $\vec{v} = \langle 1, 3 \rangle$, $P = (2, 3)$.

6. $f(x, y) = 3\cos x \sin y$, $\vec{v} = \langle 1, 2 \rangle$, $P = (\pi/3, \pi/6)$.

7. $f(x, y) = 3x - 5y$, $\vec{v} = \langle 1, 1 \rangle$, $P = (4, 2)$.

8. $f(x, y) = x^2 - 2x - y^2 + 4y$, $\vec{v} = \langle 1, 1 \rangle$, $P = (1, 2)$.

In Exercises 9 – 12, a function $z = f(x, y)$ and a point P are given. Find the equation of the normal line to f at P. Note: these are the same functions as in Exercises 5 – 8.

9. $f(x, y) = 2x^2y - 4xy^2$, $P = (2, 3)$.

10. $f(x, y) = 3\cos x \sin y$, $P = (\pi/3, \pi/6)$.

11. $f(x, y) = 3x - 5y$, $P = (4, 2)$.

12. $f(x, y) = x^2 - 2x - y^2 + 4y$, $P = (1, 2)$.

In Exercises 13 – 16, a function $z = f(x, y)$ and a point P are given. Find the two points that are 2 units from the surface f at P. Note: these are the same functions as in Exercises 5 – 8.

13. $f(x, y) = 2x^2y - 4xy^2$, $P = (2, 3)$.

14. $f(x, y) = 3\cos x \sin y$, $P = (\pi/3, \pi/6)$.

15. $f(x, y) = 3x - 5y$, $P = (4, 2)$.

16. $f(x, y) = x^2 - 2x - y^2 + 4y$, $P = (1, 2)$.

In Exercises 17 – 20, a function $z = f(x, y)$ and a point P are given. Find the equation of the tangent plane to f at P. Note: these are the same functions as in Exercises 5 – 8.

17. $f(x, y) = 2x^2y - 4xy^2$, $P = (2, 3)$.

18. $f(x, y) = 3\cos x \sin y$, $P = (\pi/3, \pi/6)$.

19. $f(x, y) = 3x - 5y$, $P = (4, 2)$.

20. $f(x, y) = x^2 - 2x - y^2 + 4y$, $P = (1, 2)$.

In Exercises 21 – 24, an implicitly defined relation of x, y and z is given along with a point P that lies on the surface. Use the gradient ∇F to:

(a) find the equation of the normal line to the surface at P, and

(b) find the equation of the plane tangent to the surface at P.

21. $\dfrac{x^2}{8} + \dfrac{y^2}{4} + \dfrac{z^2}{16} = 1$, at $P = (1, \sqrt{2}, \sqrt{6})$

22. $z^2 - \dfrac{x^2}{4} - \dfrac{y^2}{9} = 0$, at $P = (4, -3, \sqrt{5})$

23. $xy^2 - xz^2 = 0$, at $P = (2, 1, -1)$

24. $\sin(xy) + \cos(yz) = 0$, at $P = (2, \pi/12, 4)$

12.8 Extreme Values

Given a function $z = f(x, y)$, we are often interested in points where z takes on the largest or smallest values. For instance, if z represents a cost function, we would likely want to know what (x, y) values minimize the cost. If z represents the ratio of a volume to surface area, we would likely want to know where z is greatest. This leads to the following definition.

Definition 101 Relative and Absolute Extrema

Let $z = f(x, y)$ be defined on a set S containing the point $P = (x_0, y_0)$.

1. If there is an open disk D containing P such that $f(x_0, y_0) \geq f(x, y)$ for all (x, y) in D, then f has a **relative maximum** at P; if $f(x_0, y_0) \leq f(x, y)$ for all (x, y) in D, then f has a **relative minimum** at P.

2. If $f(x_0, y_0) \geq f(x, y)$ for all (x, y) in S, then f has an **absolute maximum** at P; if $f(x_0, y_0) \leq f(x, y)$ for all (x, y) in S, then f has an **absolute minimum** at P.

3. If f has a relative maximum or minimum at P, then f has a **relative extremum** at P; if f has an absolute maximum or minimum at P, then f has na **absolute extremum** at P.

If f has a relative or absolute maximum at $P = (x_0, y_0)$, it means every curve on the surface of f through P will also have a relative or absolute maximum at P. Recalling what we learned in Section 3.1, the slopes of the tangent lines to these curves at P must be 0 or undefined. Since directional derivatives are computed using f_x and f_y, we are led to the following definition and theorem.

Definition 102 Critical Point

Let $z = f(x, y)$ be continuous on an open set S. A **critical point** $P = (x_0, y_0)$ of f is a point in S such that

- $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, or
- $f_x(x_0, y_0)$ and/or $f_y(x_0, y_0)$ is undefined.

Notes:

Theorem 116 Critical Points and Relative Extrema

Let $z = f(x, y)$ be defined on an open set S containing $P = (x_0, y_0)$. If f has a relative extremum at P, then P is a critical point of f.

Therefore, to find relative extrema, we find the critical points of f and determine which correspond to relative maxima, relative minima, or neither. The following examples demonstrate this process.

Example 12.47 Finding critical points and relative extrema

Let $f(x, y) = x^2 + y^2 - xy - x - 2$. Find the relative extrema of f.

SOLUTION We start by computing the partial derivatives of f:

$$f_x(x, y) = 2x - y - 1 \qquad \text{and} \qquad f_y(x, y) = 2y - x.$$

Each is never undefined. A critical point occurs when f_x and f_y are simultaneously 0, leading us to solve the following system of linear equations:

$$2x - y - 1 = 0 \qquad \text{and} \qquad -x + 2y = 0.$$

This solution to this system is $x = 2/3$, $y = 1/3$. (Check that at $(2/3, 1/3)$, both f_x and f_y are 0.)

The graph in Figure 12.27 shows f along with this critical point. It is clear from the graph that this is a relative minimum; further consideration of the function shows that this is actually the absolute minimum.

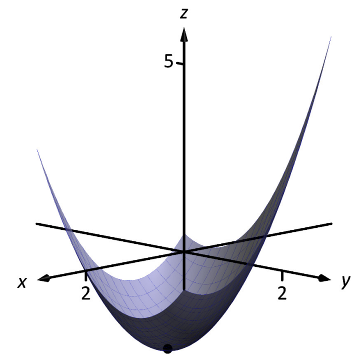

Figure 12.27: The surface in Example 12.47 with its absolute minimum indicated.

Example 12.48 Finding critical points and relative extrema

Let $f(x, y) = -\sqrt{x^2 + y^2} + 2$. Find the relative extrema of f.

SOLUTION We start by computing the partial derivatives of f:

$$f_x(x, y) = \frac{-x}{\sqrt{x^2 + y^2}} \qquad \text{and} \qquad f_y(x, y) = \frac{-y}{\sqrt{x^2 + y^2}}.$$

It is clear that $f_x = 0$ when $x = 0$ & $y \neq 0$, and that $f_y = 0$ when $y = 0$ & $x \neq 0$. At $(0, 0)$, both f_x and f_y are *not* 0, but rather undefined. The point $(0, 0)$ is still a critical point, though, because the partial derivatives are undefined. This is the only critical point of f.

The surface of f is graphed in Figure 12.28 along with the point $(0, 0, 2)$. The graph shows that this point is the absolute maximum of f.

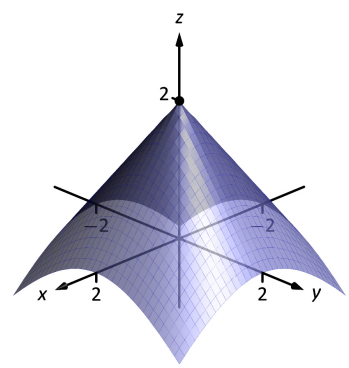

Figure 12.28: The surface in Example 12.48 with its absolute maximum indicated.

Notes:

In each of the previous two examples, we found a critical point of f and then determined whether or not it was a relative (or absolute) maximum or minimum by graphing. It would be nice to be able to determine whether a critical point corresponded to a max or a min without a graph. Before we develop such a test, we do one more example that sheds more light on the issues our test needs to consider.

Example 12.49 Finding critical points and relative extrema

Let $f(x, y) = x^3 - 3x - y^2 + 4y$. Find the relative extrema of f.

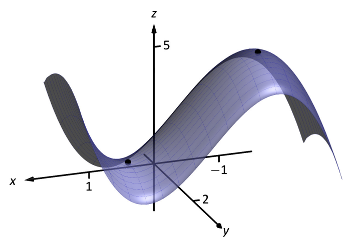

Figure 12.29: The surface in Example 12.49 with both critical points marked.

Solution Once again we start by finding the partial derivatives of f:

$$f_x(x, y) = 3x^2 - 3 \quad \text{and} \quad f_y(x, y) = -2y + 4.$$

Each is always defined. Setting each equal to 0 and solving for x and y, we find

$$\begin{aligned} f_x(x, y) = 0 &\quad \Rightarrow x = \pm 1 \\ f_y(x, y) = 0 &\quad \Rightarrow y = 2. \end{aligned}$$

We have two critical points: $(-1, 2)$ and $(1, 2)$. To determine if they correspond to a relative maximum or minimum, we consider the graph of f in Figure 12.29.

The critical point $(-1, 2)$ clearly corresponds to a relative maximum. However, the critical point at $(1, 2)$ is neither a maximum nor a minimum, displaying a different, interesting characteristic.

If one walks parallel to the y-axis towards this critical point, then this point becomes a relative maximum along this path. But if one walks towards this point parallel to the x-axis, this point becomes a relative minimum along this path. A point that seems to act as both a max and a min is a **saddle point**. A formal definition follows.

Definition 103 Saddle Point

Let $P = (x_0, y_0)$ be in the domain of f where $f_x = 0$ and $f_y = 0$ at P. P is a **saddle point** of f if, for every open disk D containing P, there are points (x_1, y_1) and (x_2, y_2) in D such that $f(x_0, y_0) > f(x_1, y_1)$ and $f(x_0, y_0) < f(x_2, y_2)$.

At a saddle point, the instantaneous rate of change in all directions is 0 and there are points nearby with z-values both less than and greater than the z-value of the saddle point.

Notes:

Before Example 12.49 we mentioned the need for a test to differentiate between relative maxima and minima. We now recognize that our test also needs to account for saddle points. To do so, we consider the second partial derivatives of f.

Recall that with single variable functions, such as $y = f(x)$, if $f''(c) > 0$, then f is concave up at c, and if $f'(c) = 0$, then f has a relative minimum at $x = c$. (We called this the Second Derivative Test.) Note that at a saddle point, it seems the graph is "both" concave up and concave down, depending on which direction you are considering.

It would be nice if the following were true:

$$\begin{array}{rcc} f_{xx} \text{ and } f_{yy} > 0 & \Rightarrow & \text{relative minimum} \\ f_{xx} \text{ and } f_{yy} < 0 & \Rightarrow & \text{relative maximum} \\ f_{xx} \text{ and } f_{yy} \text{ have opposite signs} & \Rightarrow & \text{saddle point.} \end{array}$$

However, this is not the case. Functions f exist where f_{xx} and f_{yy} are both positive but a saddle point still exists. In such a case, while the concavity in the x-direction is up (i.e., $f_{xx} > 0$) and the concavity in the y-direction is also up (i.e., $f_{yy} > 0$), the concavity switches somewhere in between the x- and y-directions.

To account for this, consider $D = f_{xx}f_{yy} - f_{xy}f_{yx}$. Since f_{xy} and f_{yx} are equal when continuous (refer back to Theorem 105), we can rewrite this as $D = f_{xx}f_{yy} - f_{xy}^2$. D can be used to test whether the concavity at a point changes depending on direction. If $D > 0$, the concavity does not switch (i.e., at that point, the graph is concave up or down in all directions). If $D < 0$, the concavity does switch. If $D = 0$, our test fails to determine whether concavity switches or not. We state the use of D in the following theorem.

Theorem 117 Second Derivative Test for Functions of Two Variables

Let $z = f(x, y)$ be differentiable on an open set containing $P = (x_0, y_0)$, and let

$$D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0).$$

1. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then P is a relative minimum of f.
2. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then P is a relative maximum of f.
3. If $D < 0$, then P is a saddle point of f.
4. If $D = 0$, the test is inconclusive.

Note: We can substitute $f_{xx}(x_0, y_0)$ with $f_{yy}(x_0, y_0)$ in (1) and (2) above. If $D > 0$, then $f_{xx}(x_0, y_0)$ and $f_{yy}(x_0, y_0)$ are either both positive or both negative.

Notes:

We first practice using this test with the function in the previous example, where we visually determined we had a relative maximum and a saddle point.

Example 12.50 Using the Second Derivative Test

Let $f(x, y) = x^3 - 3x - y^2 + 4y$ as in Example 12.49. Determine whether the function has a relative minimum, maximum, or saddle point at each critical point.

SOLUTION We determined previously that the critical points of f are $(-1, 2)$ and $(1, 2)$. To use the Second Derivative Test, we must find the second partial derivatives of f:

$$f_{xx} = 6x; \qquad f_{yy} = -2; \qquad f_{xy} = 0.$$

Thus $D(x, y) = -12x$.

At $(-1, 2)$: $D(-1, 2) = 12 > 0$, and $f_{xx}(-1, 2) = -6$. By the Second Derivative Test, f has a relative maximum at $(-1, 2)$.

At $(1, 2)$: $D(1, 2) = -12 < 0$. The Second Derivative Test states that f has a saddle point at $(1, 2)$.

The Second Derivative Test confirmed what we determined visually.

Example 12.51 Using the Second Derivative Test

Find the relative extrema of $f(x, y) = x^2y + y^2 + xy$.

SOLUTION We start by finding the first and second partial derivatives of f:

$$\begin{aligned} f_x &= 2xy + y & f_y &= x^2 + 2y + x \\ f_{xx} &= 2y & f_{yy} &= 2 \\ f_{xy} &= 2x + 1 & f_{yx} &= 2x + 1. \end{aligned}$$

We find the critical points by finding where f_x and f_y are simultaneously 0 (they are both never undefined). Setting $f_x = 0$, we have:

$$f_x = 0 \quad \Rightarrow \quad 2xy + y = 0 \quad \Rightarrow \quad y(2x + 1) = 0.$$

This implies that for $f_x = 0$, either $y = 0$ or $2x + 1 = 0$.

Assume $y = 0$ then consider $f_y = 0$:

$$\begin{aligned} f_y &= 0 \\ x^2 + 2y + x &= 0, \qquad \text{and since } y = 0, \text{ we have} \\ x^2 + x &= 0 \\ x(x + 1) &= 0. \end{aligned}$$

Notes:

Thus if $y = 0$, we have either $x = 0$ or $x = -1$, giving two critical points: $(-1, 0)$ and $(0, 0)$.

Going back to f_x, now assume $2x+1 = 0$, i.e., that $x = -1/2$, then consider $f_y = 0$:

$$\begin{aligned} f_y &= 0 \\ x^2 + 2y + x &= 0, \qquad \text{and since } x = -1/2, \text{ we have} \\ 1/4 + 2y - 1/2 &= 0 \\ y &= 1/8. \end{aligned}$$

Thus if $x = -1/2$, $y = 1/8$ giving the critical point $(-1/2, 1/8)$.

With $D = 4y - (2x+1)^2$, we apply the Second Derivative Test to each critical point.

At $(-1, 0)$, $D < 0$, so $(-1, 0)$ is a saddle point.

At $(0, 0)$, $D < 0$, so $(0, 0)$ is also a saddle point.

At $(-1/2, 1/8)$, $D > 0$ and $f_{xx} > 0$, so $(-1/2, 1/8)$ is a relative minimum.

Figure 12.30 shows a graph of f and the three critical points. Note how this function does not vary much near the critical points – that is, visually it is difficult to determine whether a point is a saddle point or relative minimum (or even a critical point at all!). This is one reason why the Second Derivative Test is so important to have.

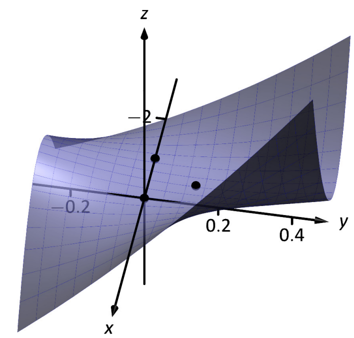

Figure 12.30: Graphing f from Example 12.51 and its relative extrema.

Constrained Optimization

When optimizing functions of one variable such as $y = f(x)$, we made use of Theorem 25, the Extreme Value Theorem, that said that over a closed interval I, a continuous function has both a maximum and minimum value. To find these maximum and minimum values, we evaluated f at all critical points in the interval, as well as at the endpoints (the "boundary") of the interval.

A similar theorem and procedure applies to functions of two variables. A continuous function over a closed set also attains a maximum and minimum value (see the following theorem). We can find these values by evaluating the function at the critical values in the set and over the boundary of the set. After formally stating this extreme value theorem, we give examples.

Theorem 118 Extreme Value Theorem

Let $z = f(x, y)$ be a continuous function on a closed, bounded set S. Then f has a maximum and minimum value on S.

Notes:

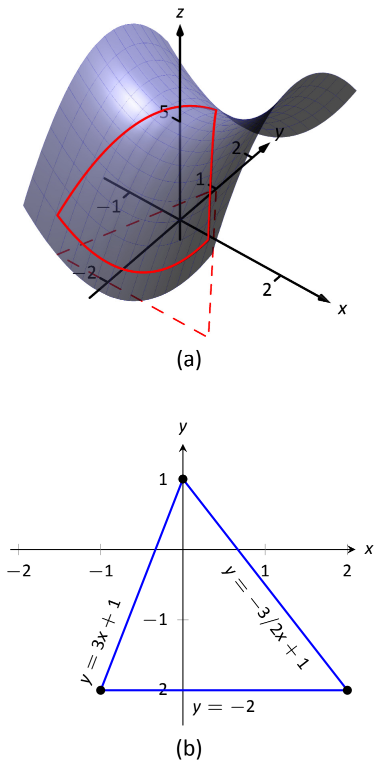

Figure 12.31: Plotting the surface of f along with the restricted domain S.

Example 12.52 Finding extrema on a closed set

Let $f(x, y) = x^2 - y^2 + 5$ and let S be the triangle with vertices $(-1, -2)$, $(0, 1)$ and $(2, -2)$. Find the maximum and minimum values of f on S.

Solution It can help to see a graph of f along with the set S. In Figure 12.31(a) the triangle defining S is shown in the x-y plane in a dashed line. Above it is the surface of f; we are only concerned with the portion of f enclosed by the "triangle" on its surface.

We begin by finding the critical points of f. With $f_x = 2x$ and $f_y = -2y$, we find only one critical point, at $(0, 0)$.

We now find the maximum and minimum values that f attains along the boundary of S, that is, along the edges of the triangle. In Figure 12.31(b) we see the triangle sketched in the plane with the equations of the lines forming its edges labeled.

Start with the bottom edge, along the line $y = -2$. If y is -2, then on the surface, we are considering points $f(x, -2)$; that is, our function reduces to $f(x, -2) = x^2 - (-2)^2 + 5 = x^2 + 1 = f_1(x)$. We want to maximize/minimize $f_1(x) = x^2 + 1$ on the interval $[-1, 2]$. To do so, we evaluate $f_1(x)$ at its critical points and at the endpoints.

The critical points of f_1 are found by setting its derivative equal to 0:

$$f_1'(x) = 0 \qquad \Rightarrow x = 0.$$

Evaluating f_1 at this critical point, and at the endpoints of $[-1, 1]$ gives:

$$\begin{aligned} f_1(-1) = 2 \qquad &\Rightarrow \qquad f(-1, -2) = 2 \\ f_1(0) = 1 \qquad &\Rightarrow \qquad f(0, -2) = 1 \\ f_1(2) = 5 \qquad &\Rightarrow \qquad f(2, -2) = 5. \end{aligned}$$

Notice how evaluating f_1 at a point is the same as evaluating f at its corresponding point.

We need to do this process twice more, for the other two edges of the triangle.

Along the left edge, along the line $y = 3x + 1$, we substitute $3x + 1$ in for y in $f(x, y)$:

$$f(x, y) = f(x, 3x + 1) = x^2 - (3x + 1)^2 + 5 = -8x^2 - 6x + 4 = f_2(x).$$

We want the maximum and minimum values of f_2 on the interval $[-1, 0]$, so we evaluate f_2 at its critical points and the endpoints of the interval. We find the critical points:

$$f_2'(x) = -16x - 6 = 0 \qquad \Rightarrow \qquad x = -3/8.$$

Notes:

Evaluate f_2 at its critical point and the endpoints of $[-1, 0]$:

$$\begin{aligned} f_2(-1) = 2 \quad &\Rightarrow \quad f(-1,-2) = 2 \\ f_2(-3/8) = 41/8 = 5.125 \quad &\Rightarrow \quad f(-3/8,-0.125) = 5.125 \\ f_2(0) = 1 \quad &\Rightarrow \quad f(0,1) = 4. \end{aligned}$$

Finally, we evaluate f along the right edge of the triangle, where $y = -\frac{3}{2}x+1$.

$$f(x,y) = f\left(x, -\frac{3}{2}x+1\right) = x^2 - \left(-\frac{3}{2}x+1\right)^2 + 5 = -\frac{5}{4}x^2 + 3x + 4 = f_3(x).$$

The critical points of $f_3(x)$ are:

$$f_3'(x) = 0 \quad \Rightarrow \quad x = 6/5 = 1.2.$$

We evaluate f_3 at this critical point and at the endpoints of the interval $[0, 2]$:

$$\begin{aligned} f_3(0) = 4 \quad &\Rightarrow \quad f(0,1) = 4 \\ f_3(1.2) = 5.8 \quad &\Rightarrow \quad f(1.2,-0.8) = 5.8 \\ f_3(2) = 5 \quad &\Rightarrow \quad f(2,-2) = 5. \end{aligned}$$

One last point to test: the critical point of f, $(0, 0)$. We find $f(0, 0) = 5$.

We have evaluated f at a total of 7 different places, all shown in Figure 12.32. We checked each vertex of the triangle twice, as each showed up as the endpoint of an interval twice. Of all the z-values found, the maximum is 5.8, found at $(1.2, -0.8)$; the minimum is 1, found at $(0, -2)$.

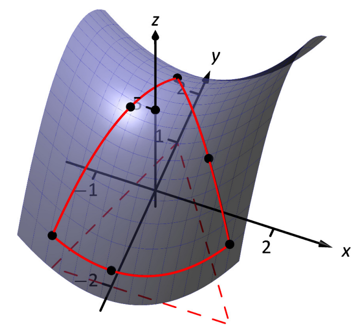

Figure 12.32: The surface of f along with important points along the boundary of S and the interior.

This portion of the text is entitled "Constrained Optimization" because we want to optimize a function (i.e., find its maximum and/or minimum values) subject to a *constraint* – some limit to what values the function can attain. In the previous example, we constrained ourselves by considering a function only within the boundary of a triangle. This was largely arbitrary; the function and the boundary were chosen just as an example, with no real "meaning" behind the function or the chosen constraint.

However, solving constrained optimization problems is a very important topic in applied mathematics. The techniques developed here are the basis for solving larger problems, where more than two variables are involved.

We illustrate the technique once more with a classic problem.

Example 12.53 Constrained Optimization

The U.S. Postal Service states that the girth+length of Standard Post Package must not exceed 130". Given a rectangular box, the "length" is the longest side, and the "girth" is twice the width+height.

Notes:

Given a rectangular box where the width and height are equal, what are the dimensions of the box that give the maximum volume subject to the constraint of the size of a Standard Post Package?

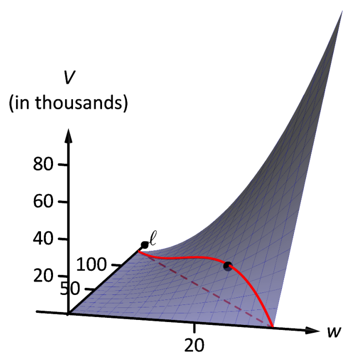

Figure 12.33: Graphing the volume of a box with girth $4w$ and length ℓ, subject to a size constraint.

Solution Let w, h and ℓ denote the width, height and length of a rectangular box; we assume here that $w = h$. The girth is then $2(w + h) = 4w$. The volume of the box is $V(w, \ell) = wh\ell = w^2\ell$. We wish to maximize this volume subject to the constraint $4w + \ell \leq 130$, or $\ell \leq 130 - 4w$. (Common sense also indicates that $\ell > 0, w > 0$.)

We begin by finding the critical values of V. We find that $V_w = 2w\ell$ and $V_\ell = w^2$; these are simultaneously 0 only at $(0, 0)$. This gives a volume of 0, so we can ignore this critical point.

We now consider the volume along the constraint $\ell = 130 - 4w$. Along this line, we have:

$$V(w\ell) = V(w, 130 - 4w) = w^2(130 - 4w) = 130w^2 - 4w^3 = V_1(w).$$

The constraint is applicable on the w-interval $[0, 32.5]$ as indicated in the figure. Thus we want to maximize V_1 on $[0, 32.5]$.

Finding the critical values of V_1, we take the derivative and set it equal to 0:

$$V_1'(w) = 260w - 12w^2 = 0 \quad \Rightarrow \quad w(260 - 12w) = 0 \quad \Rightarrow \quad w = 0, \frac{260}{12} \approx 21.67.$$

We found two critical values: when $w = 0$ and when $w = 21.67$. We again ignore the $w = 0$ solution; the maximum volume, subject to the constraint, comes at $w = h = 21.67$, $\ell = 130 - 4(21.6) = 43.33$. This gives a volume of $V(21.67, 43.33) \approx 19,408\text{in}^3$.

The volume function $V(w, \ell)$ is shown in Figure 12.33 along with the constraint $\ell = 130 - 4w$. As done previously, the constraint is drawn dashed in the x-y plane and also along the surface of the function. The point where the volume is maximized is indicated.

It is hard to overemphasize the importance of optimization. In "the real world," we routinely seek to make *something* better. By expressing the *something* as a mathematical function, "making *something* better" means "optimize *some function.*"

The techniques shown here are only the beginning of an incredibly important field. Many functions that we seek to optimize are incredibly complex, making the step of "find the gradient and set it equal to $\vec{0}$" highly nontrivial. Mastery of the principles here are key to being able to tackle these more complicated problems.

Notes:

Exercises 12.8

Terms and Concepts

1. T/F: Theorem 116 states that if f has a critical point at P, then f has a relative extremum at P.

2. T/F: A point P is a critical point of f if f_x and f_y are both 0 at P.

3. T/F: A point P is a critical point of f if f_x or f_y are undefined at P.

4. Explain what it means to "solve a constrained optimization" problem.

Problems

In Exercises 5 – 15, find the critical points of the given function. Use the Second Derivative Test to determine if each critical point corresponds to a relative maximum, minimum, or saddle point.

5. $f(x, y) = \frac{1}{2}x^2 + 2y^2 - 8y + 4x$

6. $f(x, y) = x^2 + 4x + y^2 - 9y + 3xy$

7. $f(x, y) = x^2 + 3y^2 - 6y + 4xy$

8. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$

9. $f(x, y) = x^2 + y^3 - 3y + 1$

10. $f(x, y) = \dfrac{1}{3}x^3 - x + \dfrac{1}{3}y^3 - 4y$

11. $f(x, y) = x^2y^2$

12. $f(x, y) = \ln x - xy + 9y^3$

13. $f(x, y) = x^4 - 2x^2 + y^3 - 27y - 15$

14. $f(x, y) = \sqrt{16 - (x - 3)^2 - y^2}$

15. $f(x, y) = \sqrt{x^2 + y^2}$

In Exercises 16 – 19, find the absolute maximum and minimum of the function subject to the given constraint.

16. $f(x, y) = x^2 + y^2 + y + 1$, constrained to the triangle with vertices $(0, 1)$, $(-1, -1)$ and $(1, -1)$.

17. $f(x, y) = 5x - 7y$, constrained to the region bounded by $y = x^2$ and $y = 1$.

18. $f(x, y) = x^2 + 2x + y^2 + 2y$, constrained to the region bounded by the circle $x^2 + y^2 = 4$.

19. $f(x, y) = 3y - 2x^2$, constrained to the region bounded by the parabola $y = x^2 + x - 1$ and the line $y = x$.

20. A rectangular box without a lid is to be made from 300 ft^2 of cardboard. Find the maximum possible volume of such a box.

21. An arena is to be made with volume 7680 m^3 and shaped as a rectangular box. The base will be made of slate, which costs $\$20/\text{m}^2$. The roof and all but one of the sides will be made of glass, which costs $\$4/\text{m}^2$. The remaining side will be made of corn, which costs $\$1/\text{m}^2$. Determine the dimensions of the arena at which the cost of materials is minimized.

22. An airline mandates that the total of the length, width, and height of a checked bag must be at most 62 inches. Determine the maximum possible volume of such a bag.

12.9 Method of Lagrange Multipliers

This section faces up to a practical problem we encountered at the end of the last section. We often minimize or maximize one function $f(x, y)$ while another function $g(x, y)$ is fixed. There is a constraint on x and y, given by $g(x, y) = k$ for some constant k. This restricts the material available, or the funds available, or the energy available, for example. With this constraint, the problem is to do the best possible, either maximizing or minimizing the function $f(x, y)$.

At the absolute minimum of $f(x, y)$, if one exists, the requirement $g(x, y) = k$ is probably violated. In that case the minimum point is not allowed. We cannot use just $f_x = 0$ and $f_y = 0$ as those equations don't account for g. We must find equations for the constrained minimum or constrained maximum. They will involve f_x and f_y, and also g_x and g_y, which give local information about f and g. To see the equations, we look at two examples.

Example 12.54

Minimize $f(x, y) = x^2 + y^2$ subject to the restraint $g(x, y) = 2x + y = k$, for a constant k.

Solution Look at the level curves in Figure 12.34. They are circles

$$x^2 + y^2 = c.$$

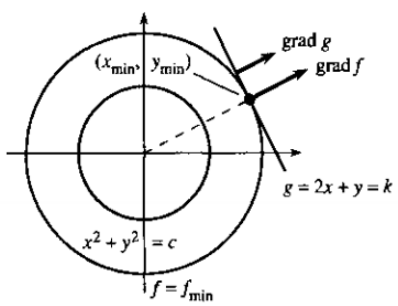

Figure 12.34: Circles $f = c$ tangent to line $g = k$; parallel gradients

When c is small, the circles do not touch the line $2x+y = k$. There are no points that satisfy the constraint, when c is too small. Now increase c. Eventually the growing circles $x^2 + y^2 = c$ will just touch the line $x + 2y = k$. The point where they touch is the winner. It gives the smallest value of c that can be achieved on the line. The touching point is $(x_{\min}, y_{\min})$, and the value of c there is $f_{\min}$.

What equation describes that point? When the circle touches the line, they are tangent. They have the same slope. The perpendiculars to the circle and the line go in the same direction. That is the key fact, which you see in Figure 12.34. The direction perpendicular to $f(x, y) = c$ is given by $\nabla f = (f_x, f_y)$. The direction perpendicular to $g(x, y) = k$ is given by $\nabla g = (g_x, g_y)$. The key idea says that those two vectors are parallel - one gradient vector is a multiple of the other gradient vector, with a multiplier λ that is unknown. That is, $\nabla f = \lambda \nabla g$.

There are now three unknowns x, y, and λ, as well as three equations

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \lambda \frac{\partial g}{\partial x} && \Rightarrow 2x = 2\lambda \\
\frac{\partial f}{\partial y} &= \lambda \frac{\partial g}{\partial y} && \Rightarrow 2y = \lambda \\
g(x, y) &= k && \Rightarrow 2x + y = k.
\end{aligned}$$

Notes:

where the third equation is simply the constraint. Now we solve this system. In the third equation, substitute 2λ for $2x$ and $\frac{1}{2}\lambda$ for y, which follow from the first two equations. This yields $2x + y = \frac{5}{2}\lambda = k$. Knowing $\lambda = \frac{2}{5}k$, the first two of the above equations give

$$x = \lambda = \frac{2}{5}k, \quad y = \frac{1}{2}\lambda = \frac{1}{5}k$$

and so

$$f_{\text{min}} = \left(\frac{2}{5}k\right)^2 + \left(\frac{1}{5}k\right)^2 = \frac{1}{5}k^2.$$

The winning point is $(x_{\text{min}}, y_{\text{min}}) = \left(\frac{2}{5}k, \frac{1}{5}k\right)$. It minimizes the distance squared $f = x^2 + y^2 = \frac{1}{5}k^2$ from the origin to the line. One can reinterpret this problem as finding the point on the line $2x + y = k$ closest to the origin.

Theorem 119 Lagrange Multipliers with One Constraint

At the minimum or maximum of $f(x, y)$ subject to $g(x, y) = k$, the gradient of f is parallel to the gradient of g, with an unknown number λ as the multiplier. That is,

$$\nabla f = \lambda \nabla g, \text{ and so } \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \text{ and } \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \text{ and } g(x, y) = k.$$

To minimize or maximize $f(x, y)$ subject to $g(x, y) = k$, solve this system of three equations. Evaluate $f(x, y)$ at all solutions (x, y) to this system. The largest of these is f_{max}, the maximum of f subject to $g = k$, and the smallest is f_{min}, the minimum of f subject to $g = k$.

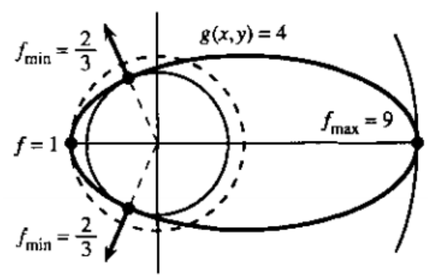

Figure 12.35: Ellipse $g = 4$ tangent to line $g = k$

Example 12.55

Find the maximum and minimum values of $f(x, y) = x^2 + y^2$ on the ellipse $g(x, y) = (x - 1)^2 + 4y^2 = 4$.

Solution As in Figure 12.35, the circles $x^2 + y^2 = c$ grow until they touch the ellipse. The touching point is $(x_{\text{min}}, y_{\text{min}})$ and that smallest value of c is f_{min}. As the circles continue to grow, they cut through the ellipse. Finally there is a point $(x_{\text{max}}, y_{\text{max}})$ where the last circle touches. That largest value of c is f_{max}.

The minimum and maximum are described by the same rule: the circle is tangent to the ellipse. The perpendiculars go in the same direction. Therefore

Notes:

∇f is a multiple of ∇g, and the unknown multiplier is λ.

$$\begin{aligned}
\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \quad &\Rightarrow 2x = 2\lambda(x-1) \\
\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \quad &\Rightarrow 2y = 8\lambda y \\
g(x,y) = k \quad &\Rightarrow (x-1)^2 + 4y^2 = 4.
\end{aligned}$$

The second equation allows two possibilities: $y = 0$ or $\lambda = \frac{1}{4}$. For the case $y = 0$, the last equation gives $(x-1)^2 = 4$ or $x^2 - 2x - 3 = 0$, and thus $x = 3$ or $x = -1$. Then the first equation gives $\lambda = \frac{3}{2}$ or $\lambda = \frac{1}{2}$. The values of f are therefore $x^2 + y^2 = 3^2 + 0^2 = 9$ and $x^2 + y^2 = (-1)^2 + 0^2 = 1$.

In the other case that $\lambda = \frac{1}{4}$, the first equation yields $x = -\frac{1}{3}$. Then the last equation requires $y^2 = \frac{5}{9}$. Since $x^2 = \frac{1}{9}$ we find $x^2 + y^2 = \frac{6}{9} = \frac{2}{3}$.

The equations we started with have four simultaneous solutions, at which the circle and ellipse are tangent. The four points are $(3,0)$, $(-1,0)$, $\left(-\frac{1}{3}, \frac{\sqrt{5}}{3}\right)$, and $\left(-\frac{1}{3}, -\frac{\sqrt{5}}{3}\right)$. The four values of f are, respectively, 9, 1, $\frac{2}{3}$, and $\frac{2}{3}$. Therefore the maximum value of f on the ellipse is 9 and the minimum value is $\frac{2}{3}$.

Using this method, the system we need to solve includes the equations $f_x = \lambda g_x$, $f_y = \lambda g_y$, and $g = k$, with unknowns x, y, and λ. There is no absolute method for solving this system of equations (unless they are all linear equations, then one can use elimination or Cramer's Rule to solve). Often, the first two equations yield x and y in terms of λ, and substituting into $g(x,y) = k$ gives an equation for λ. This next example illustrates this method when one is maximizing or minimizing a function $f(x,y,z)$ of three variables subject to a constraint $g(x,y,z) = k$. The method is the same except that the equation $\nabla f = \lambda \nabla g$ will yield three equations in the variables x, y, and z, with the fourth equation $g(x,y,z) = k$ creating a system of four equations in four unknowns.

Example 12.56

Suppose that 20 square feet of cardboard is to be used to create a rectangular box with an open top. What dimensions will result in a box with the maximum volume?

Solution If we let x, y, and z be the dimensions of the box, then the volume is given by $V(x,y,z) = xyz$ which is the function to be maximized. The constraint in this problem is the amount of cardboard, or surface area, and so

$$g(x,y,z) = 2xz + 2yz + xy = 20$$

since there is no top to this box. The method of Lagrange multipliers yields the

Notes:

equations

$$\begin{aligned} V_x = \lambda g_x &\Rightarrow yz = \lambda(2z + y) \\ V_y = \lambda g_y &\Rightarrow xz = \lambda(2z + x) \\ V_z = \lambda g_z &\Rightarrow xy = \lambda(2x + 2y) \\ g(x, y, z) = 20 &\Rightarrow 2xz + 2yz + xy = 20 \end{aligned}$$

Multiplying each of the first three equations by x, y, and z, respectively, gives the system

$$\begin{aligned} xyz &= \lambda(2xz + xy) \\ xyz &= \lambda(2yz + xy) \\ xyz &= \lambda(2xz + 2yz) \\ 2xz + 2yz + xy &= 20 \end{aligned}$$

Note that the first three equations imply that $\lambda \neq 0$, else one would arrive at the conclusion $xyz = 0$, which does not make sense in this situation. Therefore we can divide by λ and we get from equating the first two equations that

$$2xz + xy = 2yz + xy \text{ or } xz = yz.$$

Since $z \neq 0$, we get that $x = y$. Similarly, the second and third equations imply that

$$2yz + xy = 2xz + 2yz \text{ or } xy = 2xz.$$

Again, since $x \neq 0$, we get that $y = 2z$. Lastly, substitute $x = y = 2z$ into equation (4) and we get

$$4z^2 + 4z^2 + 4z^2 = 20.$$

This gives us $3z^2 = 5$ or $z = \sqrt{\frac{5}{3}}$. Therefore the dimensions of the box with maximum volume must be

$$x = y = 2\sqrt{\frac{5}{3}} \approx 2.58 \text{ feet, and } z = \sqrt{\frac{5}{3}} \approx 1.29 \text{ feet.}$$

Maximum and Minimum with Two Constraints

The whole subject of minimization or maximization is called optimization. Its applications to business decisions make up operations research. The special case of linear functions is always important - in this part of mathematics it is called linear programming. A book about those subjects won't fit inside a calculus book, but we can take one more step - to allow a second constraint.

Notes:

The function to minimize or maximize is now $f(x, y, z)$ and two constraints are given by $g(x, y, z) = k_1$ and $h(x, y, z) = k_2$, for two constants k_1 and k_2. The multipliers are λ_1 and λ_2. We need at least three variables x, y, and z because two constraints would completely determine x and y without a third.

Figure 12.36 shows the geometry behind these equations. For convenience, suppose f is $x^2 + y^2 + z^2$, and so we are minimizing distance squared. The constraints

$$g = x + y + z = 9 \text{ and } h = x + 2y + 3z = 20$$

are linear, and their graphs are planes. The constraints keep the points (x, y, z) on both planes, and therefore on the line where they intersect. We are therefore finding the squared distance from $(0, 0, 0)$ to a line again.

What equations do we solve in this case? The level surfaces $x^2 + y^2 + z^2 = c$ are spheres. They grow as c increases. The first sphere to touch the line is tangent to it, and that touching point gives the minimum solution (the smallest c). All three vectors ∇f, ∇g, ∇h are perpendicular to the line:

$$\begin{aligned} \text{line tangent to sphere} &\Rightarrow \nabla f \text{ perpendicular to line} \\ \text{line in both planes} &\Rightarrow \nabla g \text{ and } \nabla h \text{ perpendicular to line} \end{aligned}$$

Thus ∇f, ∇g, and ∇h are in the same plane, one that is perpendicular to the line. With three vectors in a plane, ∇f must be a combination of ∇g and ∇h:

$$(f_x, f_y, f_z) = \lambda_1(g_x, g_y, g_z) + \lambda_2(h_x, h_y, h_z).$$

This is the key equation. It applies to curved surfaces as well as planes.

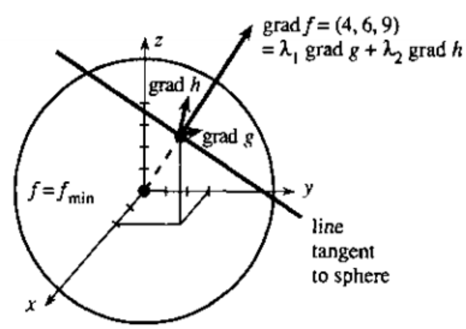

Figure 12.36: Perpendicular vector ∇f is a combination of $\lambda_1 \nabla g + \lambda_2 \nabla h$

Notes:

Theorem 120 Lagrange Multipliers with Two Constraints

At the minimum or maximum of $f(x, y, z)$ subject to $g(x, y, z) = k_1$ and $h(x, y, z) = k_2$, the gradient of f is in the same plane as ∇g and ∇h, or

$$\nabla f = \lambda_1 \nabla g + \lambda_2 \nabla h.$$

That is,

$$\frac{\partial f}{\partial x} = \lambda_1 \frac{\partial g}{\partial x} + \lambda_2 \frac{\partial h}{\partial x}$$
$$\frac{\partial f}{\partial y} = \lambda_1 \frac{\partial g}{\partial y} + \lambda_2 \frac{\partial h}{\partial y}$$
$$\frac{\partial f}{\partial z} = \lambda_1 \frac{\partial g}{\partial z} + \lambda_2 \frac{\partial h}{\partial z}$$

subject to $g(x, y, z) = k_1$ and $h(x, y, z) = k_2$. To minimize or maximize $f(x, y, z)$ subject to $g(x, y, z) = k_1$ and $h(x, y, z) = k_2$, solve this system of five equations in the unknowns x, y, z, λ_1, and λ_2. Evaluate $f(x, y, z)$ at all solutions (x, y, z) to this system. The largest of these is f_{max}, the maximum of f subject to the constraints, and the smallest is f_{min}, the minimum of f subject to the constraints.

Example 12.57

Find the minimum value of $f(x, y, z) = x^2 + y^2 + z^2$ when

$$g = x + y + z = 9 \text{ and } h = x + 2y + 3z = 20.$$

Solution In figure 12.37, the normals to those planes are $\nabla g = (1, 1, 1)$ and $\nabla h = (1, 2, 3)$. The gradient of f is $\nabla f = (2x, 2y, 2z)$. The equation

$$\nabla f = \lambda_1 \nabla g + \lambda_2 \nabla h$$

becomes

$$2x = \lambda_1 + \lambda_2,\ 2y = \lambda_1 + 2\lambda_2,\ 2z = \lambda_1 + 3\lambda_2.$$

Solve for x, y, and z and substitute these into the constraint equations to get

$$\frac{1}{2}\left((\lambda_1 + \lambda_2) + (\lambda_1 + 2\lambda_2) + (\lambda_1 + 3\lambda_2)\right) = 9$$
$$\frac{1}{2}\left((\lambda_1 + \lambda_2) + 2(\lambda_1 + 2\lambda_2) + 3(\lambda_1 + 3\lambda_2) +\right) = 20$$

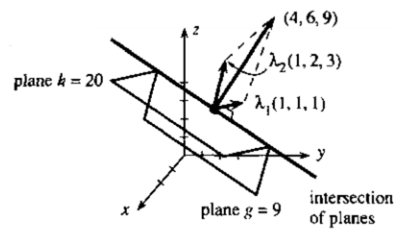

Figure 12.37: Perpendicular vector ∇f is a combination of $\lambda_1 \nabla g + \lambda_2 \nabla h$

Notes:

After multiplying these by 2, they simplify down to

$$3\lambda_1 + 6\lambda_2 = 18 \text{ and } 6\lambda_1 + 14\lambda_2 = 40.$$

This is a linear system with solution $\lambda_1 = 2$ and $\lambda_2 = 2$. The previous equations therefore give $(x, y, z) = (2, 3, 4)$. This point gives $f_{\text{min}} = 29$, the minimum value of f subject to the two constraints.

Notes:

Exercises 12.9

Terms and Concepts

1. Fill in the blank: Geometrically, ∇f is ________ to ∇g at a maximum or minimum of $f(x, y)$ subject to the constraint $g(x, y) = k$.

2. Fill in the blanks: When using the method of Lagrange multipliers to find the maximum or minimum of $f(x, y)$ subject to the constraint $g(x, y) = k$, you must solve a system of ________ equations in the unknowns x, y, and λ.

3. T/F: When finding the maximum of a function $f(x, y)$ subject to the constraint $2x - 3y = 6$, one is finding the largest value of $f(x, y)$ on a line.

4. T/F: When trying to maximize or minimize a function $f(x, y, z)$ subject to two constraints $g(x, y, z) = k_1$ and $h(x, y, z) = k_2$, the additional two equations in the system are $\nabla f = \lambda_1 \nabla g$ and $\nabla f = \lambda_2 \nabla h$.

Problems

In Exercises 5 – 8, find maximum and minimum values of $f(x, y)$ subject to the given restraint, as well as the points (x, y) where these occur.

5. $f(x, y) = x^2 y$ with $g(x, y) = x^2 + y^2 = 1$

6. $f(x, y) = x + y$ with $g(x, y) = \dfrac{1}{x} + \dfrac{1}{y} = 1$

7. $f(x, y) = 3x + y$ with $g(x, y) = x^2 + 9y^2 = 1$

8. $f(x, y) = x^2 + y^2$ with $g(x, y) = x^6 + y^6 = 2$

In Exercises 9 – 12, answer the question using the method of Lagrange multipliers.

9. Find the maximum value of $f(x, y) = xy$ on the circle of radius $\sqrt{2}$ with center at the origin. At what point on the circle does this occur?

10. Find the minimum and maximum values of the function $f(x, y) = 2x - 3y$ on the circle $x^2 + y^2 = 13$.

11. Find the minimum value of $f(x, y, z) = x^2 + 2y^2 + z^2$ if (x, y, z) is restricted to the planes $x + y + z = 0$ and $x - z = 1$. At what point(s) does this occur?

12. Find the maximum value of $f(x, y, z) = x + y + z$ if (x, y, z) is restricted to $x^2 + z^2 = 2$ and $x + y = 1$. At what point(s) does this occur?

13: Multiple Integration

The previous chapter introduced multivariable functions and we applied concepts of differential calculus to these functions. We learned how we can view a function of two variables as a surface in space, and learned how partial derivatives convey information about how the surface is changing in any direction.

In this chapter we apply techniques of integral calculus to multivariable functions. In Chapter 5 we learned how the definite integral of a single variable function gave us "area under the curve." In this chapter we will see that integration applied to a multivariable function gives us "volume under a surface." And just as we learned applications of integration beyond finding areas, we will find applications of integration in this chapter beyond finding volume.

13.1 Iterated Integrals and Area

In Chapter 12 we found that it was useful to differentiate functions of several variables with respect to one variable, while treating all the other variables as constants or coefficients. We can integrate functions of several variables in a similar way. For instance, if we are told that $f_x(x,y) = 2xy$, we can treat y as staying constant and integrate to obtain $f(x,y)$:

$$\begin{aligned} f(x,y) &= \int f_x(x,y)\,dx \\ &= \int 2xy\,dx \\ &= x^2y + C. \end{aligned}$$

Make a careful note about the constant of integration, C. This "constant" is something with a derivative of 0 with respect to x, so it could be any expression that contains only constants and functions of y. For instance, if $f(x,y) = x^2y + \sin y + y^3 + 17$, then $f_x(x,y) = 2xy$. To signify that C is actually a function of y, we write:

$$f(x,y) = \int f_x(x,y)\,dx = x^2y + C(y).$$

Using this process we can even evaluate definite integrals.

Example 13.1 Integrating functions of more than one variable

Evaluate the integral $\displaystyle\int_1^{2y} 2xy\,dx$.

Solution We find the indefinite integral as before, then apply the Fundamental Theorem of Calculus to evaluate the definite integral:

$$\begin{aligned} \int_1^{2y} 2xy\,dx &= x^2y\Big|_1^{2y} \\ &= (2y)^2y - (1)^2y \\ &= 4y^3 - y. \end{aligned}$$

We can also integrate with respect to y. In general,

$$\int_{h_1(y)}^{h_2(y)} f_x(x,y)\,dx = f(x,y)\Big|_{h_1(y)}^{h_2(y)} = f\big(h_2(y),y\big) - f\big(h_1(y),y\big),$$

and

$$\int_{g_1(x)}^{g_2(x)} f_y(x,y)\,dy = f(x,y)\Big|_{g_1(x)}^{g_2(x)} = f\big(x,g_2(x)\big) - f\big(x,g_1(x)\big).$$

Note that when integrating with respect to x, the bounds are functions of y (of the form $x = h_1(y)$ and $x = h_2(y)$) and the final result is also a function of y. When integrating with respect to y, the bounds are functions of x (of the form $y = g_1(x)$ and $y = g_2(x)$) and the final result is a function of x. Another example will help us understand this.

Example 13.2 Integrating functions of more than one variable

Evaluate $\displaystyle\int_1^x \left(5x^3y^{-3} + 6y^2\right)\,dy$.

Solution We consider x as staying constant and integrate with respect to y:

$$\begin{aligned}\int_1^x \left(5x^3y^{-3} + 6y^2\right)\,dy &= \left(\frac{5x^3y^{-2}}{-2} + \frac{6y^3}{3}\right)\Bigg|_1^x \\ &= \left(-\frac{5}{2}x^3x^{-2} + 2x^3\right) - \left(-\frac{5}{2}x^3 + 2\right) \\ &= \frac{9}{2}x^3 - \frac{5}{2}x - 2.\end{aligned}$$

Note how the bounds of the integral are from $y = 1$ to $y = x$ and that the final answer is a function of x.

In the previous example, we integrated a function with respect to y and ended up with a function of x. We can integrate this as well. This process is known as **iterated integration**, or **multiple integration.**

Example 13.3 Integrating an integral

Evaluate $\displaystyle\int_1^2 \left(\int_1^x \left(5x^3y^{-3} + 6y^2\right)\,dy\right)\,dx$.

Solution We follow a standard "order of operations" and perform the operations inside parentheses first (which is the integral evaluated in Example

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Notes:

13.2.)

$$\int_1^2 \left(\int_1^x \left(5x^3y^{-3} + 6y^2\right)\,dy\right)\,dx = \int_1^2 \left(\left[\frac{5x^3y^{-2}}{-2} + \frac{6y^3}{3}\right]\Bigg|_1^x\right)\,dx$$
$$= \int_1^2 \left(\frac{9}{2}x^3 - \frac{5}{2}x - 2\right)\,dx$$
$$= \left(\frac{9}{8}x^4 - \frac{5}{4}x^2 - 2x\right)\Bigg|_1^2$$
$$= \frac{89}{8}.$$

Note how the bounds of x were $x = 1$ to $x = 2$ and the final result was a number.

The previous example showed how we could perform something called an iterated integral; we do not yet know *why* we would be interested in doing so nor what the result, such as the number $89/8$, *means*. Before we investigate these questions, we offer some definitions.

Definition 104 Iterated Integration

Iterated integration is the process of repeatedly integrating the results of previous integrations. Integrating one integral is denoted as follows.

Let a, b, c and d be numbers and let $g_1(x)$, $g_2(x)$, $h_1(y)$ and $h_2(y)$ be functions of x and y, respectively. Then:

1. $\displaystyle\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right)\,dy.$
2. $\displaystyle\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right)\,dx.$

Again make note of the bounds of these iterated integrals.

With $\displaystyle\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$, x varies from $h_1(y)$ to $h_2(y)$, whereas y varies from c to d. That is, the bounds of x are *curves*, the curves $x = h_1(y)$ and $x = h_2(y)$, whereas the bounds of y are *constants*, $y = c$ and $y = d$. It is useful to remember that when setting up and evaluating such iterated integrals, we integrate "from

Notes:

curve to curve, then from point to point."

We now begin to investigate *why* we are interested in iterated integrals and *what* they mean.

Area of a plane region

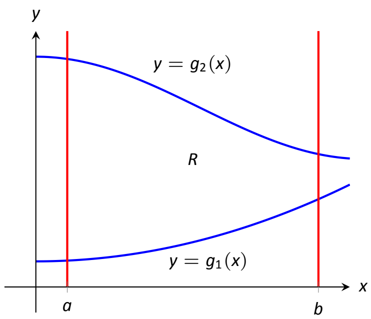

Figure 13.1: Calculating the area of a plane region R with an iterated integral.

Consider the plane region R bounded by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, shown in Figure 13.1. We learned in Section 7.1 that the area of R is given by

$$\int_a^b \big(g_2(x) - g_1(x)\big)\, dx.$$

We can view the expression $\big(g_2(x) - g_1(x)\big)$ as

$$\big(g_2(x) - g_1(x)\big) = \int_{g_1(x)}^{g_2(x)} 1\, dy = \int_{g_1(x)}^{g_2(x)} dy,$$

meaning we can express the area of R as an iterated integral:

$$\text{area of } R = \int_a^b \big(g_2(x) - g_1(x)\big)\, dx = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} dy\right) dx = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx.$$

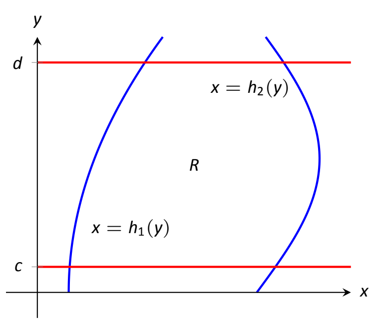

Figure 13.2: Calculating the area of a plane region R with an iterated integral.

In short: a certain iterated integral can be viewed as giving the area of a plane region.

A region R could also be defined by $c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(y)$, as shown in Figure 13.2. Using a process similar to that above, we have

$$\text{the area of } R = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\, dy.$$

We state this formally in a theorem.

Notes:

Theorem 121 Area of a plane region

1. Let R be a plane region bounded by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, where g_1 and g_2 are continuous functions on $[a, b]$. The area A of R is
$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\,dx.$$

2. Let R be a plane region bounded by $c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(y)$, where h_1 and h_2 are continuous functions on $[c, d]$. The area A of R is
$$A = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\,dy.$$

The following examples should help us understand this theorem.

Example 13.4 Area of a rectangle

Find the area A of the rectangle with corners $(-1, 1)$ and $(3, 3)$, as shown in Figure 13.3.

Solution Multiple integration is obviously overkill in this situation, but we proceed to establish its use.

The region R is bounded by $x = -1$, $x = 3$, $y = 1$ and $y = 3$. Choosing to integrate with respect to y first, we have

$$A = \int_{-1}^{3}\int_{1}^{3} 1\,dy\,dx = \int_{-1}^{3}\left(y\Big|_1^3\right)\,dx = \int_{-1}^{3} 2\,dx = 2x\Big|_{-1}^{3} = 8.$$

We could also integrate with respect to x first, giving:

$$A = \int_{1}^{3}\int_{-1}^{3} 1\,dx\,dy = \int_{1}^{3}\left(x\Big|_{-1}^3\right)\,dy = \int_{1}^{3} 4\,dy = 4y\Big|_{1}^{3} = 8.$$

Clearly there are simpler ways to find this area, but it is interesting to note that this method works.

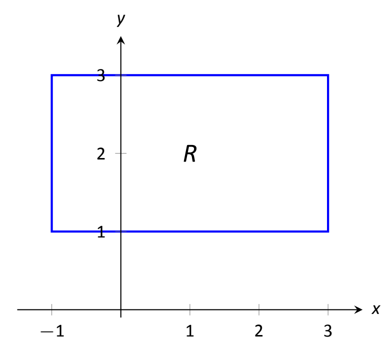

Figure 13.3: Calculating the area of a rectangle with an iterated integral in Example 13.4.

Example 13.5 Area of a triangle

Find the area A of the triangle with vertices at $(1, 1)$, $(3, 1)$ and $(5, 5)$, as shown in Figure 13.4.

Solution The triangle is bounded by the lines as shown in the figure. Choosing to integrate with respect to x first gives that x is bounded by $x = y$

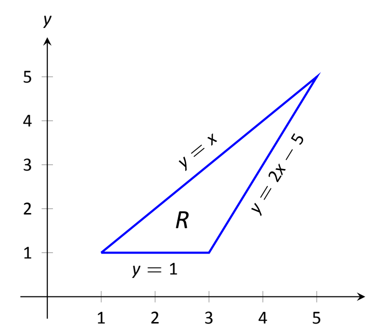

Figure 13.4: Calculating the area of a triangle with iterated integrals in Example 13.5.

Notes:

to $x = \frac{y+5}{2}$, while y is bounded by $y = 1$ to $y = 5$. (Recall that since x-values increase from left to right, the leftmost curve, $x = y$, is the lower bound and the rightmost curve, $x = (y+5)/2$, is the upper bound.) The area is

$$\begin{aligned} A &= \int_1^5 \int_y^{\frac{y+5}{2}} dx\, dy \\ &= \int_1^5 \left(x \Big|_y^{\frac{y+5}{2}} \right) dy \\ &= \int_1^5 \left(-\frac{1}{2}y + \frac{5}{2} \right) dy \\ &= \left(-\frac{1}{4}y^2 + \frac{5}{2}y \right) \Big|_1^5 \\ &= 4. \end{aligned}$$

We can also find the area by integrating with respect to y first. In this situation, though, we have two functions that act as the lower bound for the region R, $y = 1$ and $y = 2x - 5$. This requires us to use two iterated integrals. Note how the x-bounds are different for each integral:

$$\begin{aligned} A &= \int_1^3 \int_1^x 1\, dy\, dx && + && \int_3^5 \int_{2x-5}^x 1\, dy\, dx \\ &= \int_1^3 (y)\Big|_1^x dx && + && \int_3^5 (y)\Big|_{2x-5}^x dx \\ &= \int_1^3 (x-1)\, dx && + && \int_3^5 (-x+5)\, dx \\ &= 2 && + && 2 \\ &= 4. \end{aligned}$$

As expected, we get the same answer both ways.

Example 13.6 Area of a plane region

Find the area of the region enclosed by $y = 2x$ and $y = x^2$, as shown in Figure 13.5.

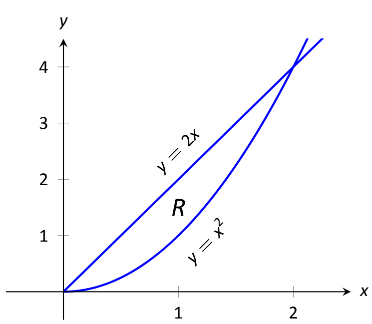

Figure 13.5: Calculating the area of a plane region with iterated integrals in Example 13.6.

Solution Once again we'll find the area of the region using both orders of integration.

Using $dy\,dx$:

$$\int_0^2 \int_{x^2}^{2x} 1\, dy\, dx = \int_0^2 (2x - x^2)\, dx = \left(x^2 - \frac{1}{3}x^3\right)\Big|_0^2 = \frac{4}{3}.$$

Notes:

Using $dx\,dy$:

$$\int_0^4 \int_{y/2}^{\sqrt{y}} 1\,dx\,dy = \int_0^4 (\sqrt{y} - y/2)\,dy = \left(\frac{2}{3}y^{3/2} - \frac{1}{4}y^2\right)\Big|_0^4 = \frac{4}{3}.$$

Changing Order of Integration

In each of the previous examples, we have been given a region R and found the bounds needed to find the area of R using both orders of integration. We integrated using both orders of integration to demonstrate their equality.

We now approach the skill of describing a region using both orders of integration from a different perspective. Instead of starting with a region and creating iterated integrals, we will start with an iterated integral and rewrite it in the other integration order. To do so, we'll need to understand the region over which we are integrating.

The simplest of all cases is when both integrals are bound by constants. The region described by these bounds is a rectangle (see Example 13.4), and so:

$$\int_a^b \int_c^d 1\,dy\,dx = \int_c^d \int_a^b 1\,dx\,dy.$$

When the inner integral's bounds are not constants, it is generally very useful to sketch the bounds to determine what the region we are integrating over looks like. From the sketch we can then rewrite the integral with the other order of integration.

Examples will help us develop this skill.

Example 13.7 Changing the order of integration

Rewrite the iterated integral $\displaystyle\int_0^6 \int_0^{x/3} 1\,dy\,dx$ with the order of integration $dx\,dy$.

Solution We need to use the bounds of integration to determine the region we are integrating over.

The bounds tell us that y is bounded by 0 and $x/3$; x is bounded by 0 and 6. We plot these four curves: $y = 0$, $y = x/3$, $x = 0$ and $x = 6$ to find the region described by the bounds. Figure 13.6 shows these curves, indicating that R is a triangle.

To change the order of integration, we need to consider the curves that bound the x-values. We see that the lower bound is $x = 3y$ and the upper bound is $x = 6$. The bounds on y are 0 to 2. Thus we can rewrite the integral as $\displaystyle\int_0^2 \int_{3y}^6 1\,dx\,dy.$

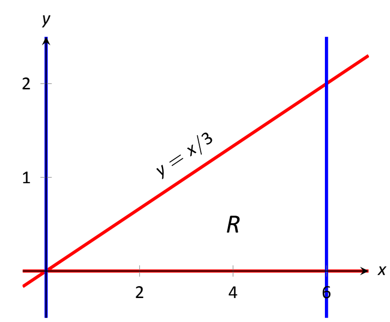

Figure 13.6: Sketching the region R described by the iterated integral in Example 13.7.

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Example 13.8 Changing the order of integration

Change the order of integration of $\int_0^4 \int_{y^2/4}^{(y+4)/2} 1\,dx\,dy$.

Solution We sketch the region described by the bounds to help us change the integration order. x is bounded below and above (i.e., to the left and right) by $x = y^2/4$ and $x = (y+4)/2$ respectively, and y is bounded between 0 and 4. Graphing the previous curves, we find the region R to be that shown in Figure 13.7.

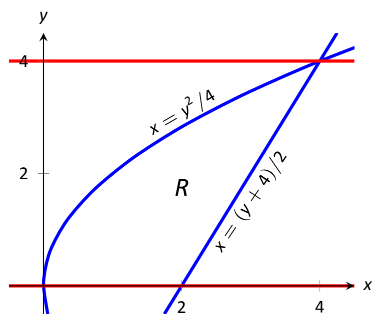


Figure 13.7: Drawing the region determined by the bounds of integration in Example 13.8.

To change the order of integration, we need to establish curves that bound y. The figure makes it clear that there are two lower bounds for y: $y = 0$ on $0 \leq x \leq 2$, and $y = 2x - 4$ on $2 \leq x \leq 4$. Thus we need two double integrals. The upper bound for each is $y = 2\sqrt{x}$. Thus we have

$$\int_0^4 \int_{y^2/4}^{(y+4)/2} 1\,dx\,dy = \int_0^2 \int_0^{2\sqrt{x}} 1\,dy\,dx + \int_2^4 \int_{2x-4}^{2\sqrt{x}} 1\,dy\,dx.$$

There is a useful shortcut that applies to a handful of iterated integrals. It can be used the integrand has the form $f(x)g(y)$ of a function $f(x)$ of x alone and a function $g(y)$ of y alone (for example $\sin(x)y^2$ but not $\sin x + y^2$ nor $\sin\left(xy^2\right)$) and the bounds contain only numbers. In this case the double integral can be broken into the product of two single integrals which are faster to compute. This is the next theorem.

Theorem 122

Let $f(x)$ and $g(y)$ be continuous functions on $[a, b]$ and $[c, d]$, respectively. Then

$$\int_c^d \int_a^b f(x)g(y)dxdy = \left(\int_a^b f(x)dx\right)\left(\int_c^d g(y)dy\right).$$

To explain this shortcut, we begin by realizing that $g(y)$ is considered a "constant" multiple when integrating with respect to x and can pulled outside of the first iterated integral:

$$\int_c^d \int_a^b f(x)g(y)dxdy = \int_c^d \left(\int_a^b f(x)g(y)dx\right) dy = \int_c^d g(y)\left(\int_a^b f(x)dx\right) dy.$$

Notes:

Then the entire integral $\int_a^b f(x)dx$ is a constant that can be pulled outside the y-integral. It s very important that the bounds do not contain y. So

$$\int_c^d g(y)\left(\int_a^b f(x)dx\right)dy = \left(\int_a^b f(x)dx\right)\left(\int_c^d g(y)dy\right).$$

We use this shortcut in the next example.

Example 13.9 Using Theorem 122

Compute $\displaystyle\int_2^5 \int_0^\pi \sin(x)y^2\,dx\,dy$.

Solution

$$\begin{aligned}\int_2^5 \int_0^\pi \sin(x)y^2\,dx\,dy &= \left(\int_0^\pi \sin(x)\,dx\right)\left(\int_2^5 y^2\,dy\right)\\ &= \left(-\cos x\Big|_0^\pi\right)\left(\frac{y^3}{3}\Big|_2^5\right)\\ &= \big(1-(-1)\big)\left(\frac{5^3}{3}-\frac{2^3}{3}\right)\\ &= 78.\end{aligned}$$

This section has introduced a new concept, the iterated integral. We developed one application for iterated integration: area between curves. However, this is not new, for we already know how to find areas bounded by curves. In the next section we apply iterated integration to solve problems we currently do not know how to handle. The “real” goal of this section was not to learn a new way of computing area. Rather, our goal was to learn how to define a region in the plane using the bounds of an iterated integral. That skill is very important in the following sections.

Notes:

Exercises 13.1

Terms and Concepts

1. When integrating $f_x(x,y)$ with respect to x, the constant of integration C is really which: $C(x)$ or $C(y)$? What does this mean?

2. Integrating an integral is called _________ _________.

3. When evaluating an iterated integral, we integrate from _________ to _________, then from _________ to _________.

4. One understanding of an iterated integral is that $\displaystyle\int_a^b \int_{g_1(x)}^{g_2(x)} dy\,dx$ gives the _________ of a plane region.

Problems

In Exercises 5 – 10, evaluate the integral and subsequent iterated integral.

5. (a) $\displaystyle\int_2^5 \left(6x^2+4xy-3y^2\right) dy$

   (b) $\displaystyle\int_{-3}^{-2}\int_2^5 \left(6x^2+4xy-3y^2\right) dy\,dx$

6. (a) $\displaystyle\int_0^{\pi} \left(2x\cos y+\sin x\right) dx$

   (b) $\displaystyle\int_0^{\pi/2}\int_0^{\pi} \left(2x\cos y+\sin x\right) dx\,dy$

7. (a) $\displaystyle\int_1^x \left(x^2y-y+2\right) dy$

   (b) $\displaystyle\int_0^2\int_1^x \left(x^2y-y+2\right) dy\,dx$

8. (a) $\displaystyle\int_y^{y^2} \left(x-y\right) dx$

   (b) $\displaystyle\int_{-1}^1\int_y^{y^2} \left(x-y\right) dx\,dy$

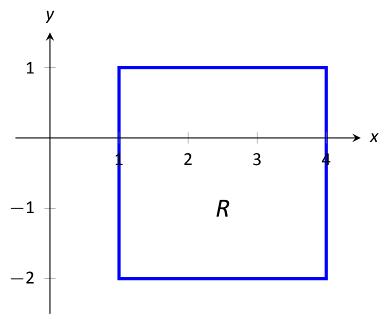
9. (a) $\displaystyle\int_0^y \left(\cos x\sin y\right) dx$

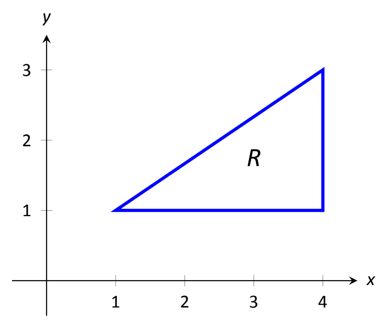
   (b) $\displaystyle\int_0^{\pi}\int_0^y \left(\cos x\sin y\right) dx\,dy$

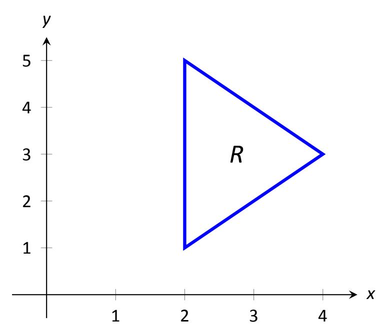
10. (a) $\displaystyle\int_0^x \left(\frac{1}{1+x^2}\right) dy$

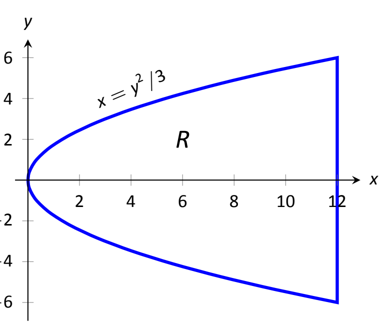
    (b) $\displaystyle\int_1^2\int_0^x \left(\frac{1}{1+x^2}\right) dy\,dx$

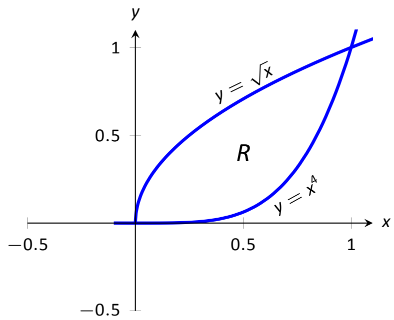
In Exercises 11 – 16, a graph of a planar region R is given. Give the iterated integrals, with both orders of integration $dy\,dx$ and $dx\,dy$, that give the area of R. Evaluate one of the iterated integrals to find the area.

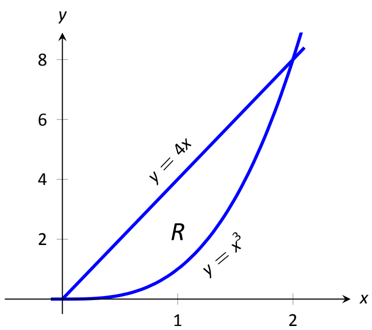
11. 

12. 

13. 

14. 

15. 

16. 

In Exercises 17 – 22, iterated integrals are given that compute the area of a region R in the x-y plane. Sketch the region R, and give the iterated integral(s) that give the area of R with the opposite order of integration.

17. $\displaystyle\int_{-2}^{2}\int_{0}^{4-x^2} dy\,dx$

18. $\displaystyle\int_{0}^{1}\int_{5-5x}^{5-5x^2} dy\,dx$

19. $\displaystyle\int_{-2}^{2}\int_{0}^{2\sqrt{4-y^2}} dx\,dy$

20. $\displaystyle\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dy\,dx$

21. $\displaystyle\int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy+\int_{1}^{4}\int_{y-2}^{\sqrt{y}} dx\,dy$

22. $\displaystyle\int_{-1}^{1}\int_{(x-1)/2}^{(1-x)/2} dy\,dx$

In Exercises 23 – 27, use the shortcut of Theorem 122 to compute the iterated integral.

23. $\displaystyle\int_{2}^{4}\int_{0}^{1} x^2y^3\,dx\,dy$

24. $\displaystyle\int_{1}^{e}\int_{-1}^{3} \frac{x}{y}\,dx\,dy$

25. $\displaystyle\int_{0}^{\pi/3}\int_{0}^{\pi/4} 4\sin x\cos y\,dx\,dy$

26. $\displaystyle\int_{0}^{2}\int_{0}^{\sqrt{3}} \frac{3^y}{1+x^2}\,dx\,dy$

27. $\displaystyle\int_{1}^{4}\int_{0}^{6} \left(x^2+x^2y\right)\,dx\,dy$

28. Why can't Theorem 122 be used to compute $\displaystyle\int_{0}^{\pi}\int_{0}^{\sin y} e^x\cos y\,dx\,dy$?

29. Why can't Theorem 122 be used to compute $\displaystyle\int_{2}^{4}\int_{0}^{2} \sqrt{x+y}\,dx\,dy$?

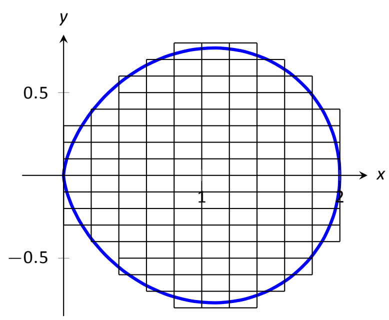
13.2 Double Integration and Volume

The definite integral of f over $[a, b]$, $\int_a^b f(x)\,dx$, was introduced as "the signed area under the curve." We approximated the value of this area by first subdividing $[a, b]$ into n subintervals, where the i^{th} subinterval has length Δx_i, and letting c_i be any value in the i^{th} subinterval. We formed rectangles that approximated part of the region under the curve with width Δx_i, height $f(c_i)$, and hence with area $f(c_i)\Delta x_i$. Summing all the rectangle's areas gave an approximation of the definite integral, and Theorem 40 stated that

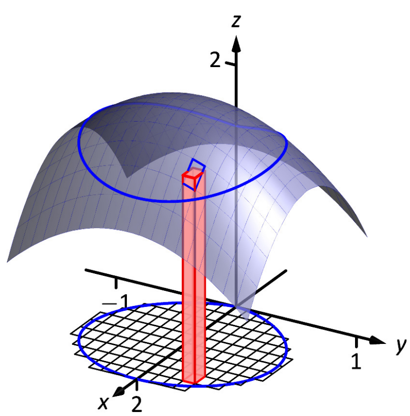
$$\int_a^b f(x)\,dx = \lim_{\|\Delta x\|\to 0} \sum f(c_i)\Delta x_i,$$

connecting the area under the curve with sums of the areas of rectangles.

We use a similar approach in this section to find volume under a surface.



(a)



(b)

Figure 13.8: Developing a method for finding signed volume under a surface.

Let R be a closed, bounded region in the x-y plane and let $z = f(x, y)$ be a continuous function defined on R. We wish to find the signed volume under the surface of f over R. (We use the term "signed volume" to denote that space above the x-y plane, under f, will have a positive volume; space above f and under the x-y plane will have a "negative" volume, similar to the notion of signed area used before.)

We start by partitioning R into n rectangular subregions as shown in Figure 13.8(a). For simplicity's sake, we let all widths be Δx and all heights be Δy. Note that the sum of the areas of the rectangles is not equal to the area of R, but rather is a close approximation. Arbitrarily number the rectangles 1 through n, and pick a point (x_i, y_i) in the i^{th} subregion.

The volume of the rectangular solid whose base is the i^{th} subregion and whose height is $f(x_i, y_i)$ is $V_i = f(x_i, y_i)\Delta x\Delta y$. Such a solid is shown in Figure 13.8(b). Note how this rectangular solid only approximates the true volume under the surface; part of the solid is above the surface and part is below.

For each subregion R_i used to approximate R, create the rectangular solid with base area $\Delta x\Delta y$ and height $f(x_i, y_i)$. The sum of all rectangular solids is

$$\sum_{i=1}^{n} f(x_i, y_i)\Delta x\Delta y.$$

This approximates the signed volume under f over R. As we have done before, to get a better approximation we can use more rectangles to approximate the region R.

In general, each rectangle could have a different width Δx_j and height Δy_k, giving the i^{th} rectangle an area $\Delta A_i = \Delta x_j\Delta y_k$ and the i^{th} rectangular solid a

Notes:

volume of $f(x_i, y_i)\Delta A_i$. Let $||\Delta A||$ denote the length of the longest diagonal of all rectangles in the subdivision of R; $||\Delta A|| \to 0$ means each rectangle's width and height are both approaching 0. If f is a continuous function, as $||\Delta A||$ shrinks (and hence $n \to \infty$) the summation $\displaystyle\sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$ approximates the signed volume better and better. This leads to a definition.

Definition 105 Double Integral, Signed Volume

Let $z = f(x, y)$ be a continuous function defined over a closed region R in the x-y plane. The **signed volume** V under f over R is denoted by the **double integral**

$$V = \iint_R f(x, y)\, dA.$$

Alternate notations for the double integral are

$$\iint_R f(x, y)\, dA = \iint_R f(x, y)\, dx\, dy = \iint_R f(x, y)\, dy\, dx.$$

The definition above does not state how to find the signed volume, though the notation offers a hint. We need the next two theorems to evaluate double integrals to find volume.

Theorem 123 Double Integrals and Signed Volume

Let $z = f(x, y)$ be a continuous function defined over a closed region R in the x-y plane. Then the signed volume V under f over R is

$$V = \iint_R f(x, y)\, dA = \lim_{||\Delta A||\to 0} \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i.$$

This theorem states that we can find the exact signed volume using a limit of sums. The partition of the region R is not specified, so any partitioning where the diagonal of each rectangle shrinks to 0 results in the same answer.

This does not offer a very satisfying way of computing area, though. Our experience has shown that evaluating the limits of sums can be tedious. We seek a more direct method.

Recall Theorem 56 in Section 7.2. This stated that if $A(x)$ gives the cross-sectional area of a solid at x, then $\int_a^b A(x)\, dx$ gave the volume of that solid over

Notes:

Note: Recall that the integration symbol "$\int$" is an "elongated S," representing the word "sum." We interpreted $\int_a^b f(x)\, dx$ as "take the *sum* of the areas of rectangles over the interval $[a, b]$." The double integral uses two integration symbols to represent a "double sum." When adding up the volumes of rectangular solids over a partition of a region R, as done in Figure 13.8, one could first add up the volumes across each row (one type of sum), then add these totals together (another sum), as in

$$\sum_{j=1}^{n}\sum_{i=1}^{m} f(x_i, y_j)\Delta x_i \Delta y_j.$$

One can rewrite this as

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{m} f(x_i, y_j)\Delta x_i\right)\Delta y_j.$$

The summation inside the parenthesis indicates the sum of heights $\times$ widths, which gives an area; multiplying these areas by the thickness Δy_j gives a volume. The illustration in Figure 13.9 relates to this understanding.

$[a, b]$.

Consider Figure 13.9, where a surface $z = f(x, y)$ is drawn over a region R. Fixing a particular x value, we can consider the area under f over R where x has that fixed value. That area can be found with a definite integral, namely

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy.$$

Remember that though the integrand contains x, we are viewing x as fixed. Also note that the bounds of integration are functions of x: the bounds depend on the value of x.

As $A(x)$ is a cross-sectional area function, we can find the signed volume V under f by integrating it:

$$V = \int_a^b A(x)\, dx = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x, y)\, dy \right) dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

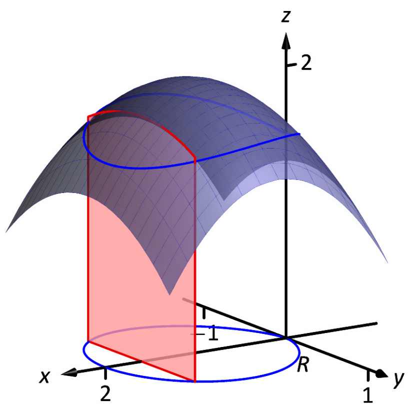


Figure 13.9: Finding volume under a surface by sweeping out a cross–sectional area.

This gives a concrete method for finding signed volume under a surface. We could do a similar procedure where we started with y fixed, resulting in a iterated integral with the order of integration $dx\, dy$. The following theorem states that both methods give the same result, which is the value of the double integral. It is such an important theorem it has a name associated with it.

Theorem 124 Fubini's Theorem

Let R be a closed, bounded region in the x-y plane and let $z = f(x, y)$ be a continuous function on R.

1. If R is bounded by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, where g_1 and g_2 are continuous functions on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is bounded by $c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(y)$, where h_1 and h_2 are continuous functions on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

Note that once again the bounds of integration follow the "curve to curve, point to point" pattern discussed in the previous section. In fact, one of the

Notes:

main points of the previous section is developing the skill of describing a region R with the bounds of an iterated integral. Once this skill is developed, we can use double integrals to compute many quantities, not just signed volume under a surface.

Example 13.10 Evaluating a double integral

Let $f(x,y) = xy + e^y$. Find the signed volume under f on the region R, which is the rectangle with corners $(3,1)$ and $(4,2)$ pictured in Figure 13.10, using Fubini's Theorem and both orders of integration.

Solution We wish to evaluate $\iint_R \left(xy + e^y\right)\, dA$. As R is a rectangle, the bounds are easily described as $3 \le x \le 4$ and $1 \le y \le 2$.

Using the order $dy\,dx$:

$$\begin{aligned}
\iint_R \left(xy + e^y\right)\, dA &= \int_3^4 \int_1^2 \left(xy + e^y\right)\, dy\, dx \\
&= \int_3^4 \left(\left[\frac{1}{2}xy^2 + e^y\right]\Bigg|_1^2 \right) dx \\
&= \int_3^4 \left(\frac{3}{2}x + e^2 - e \right) dx \\
&= \left(\frac{3}{4}x^2 + (e^2 - e)x \right)\Bigg|_3^4 \\
&= \frac{21}{4} + e^2 - e \approx 9.92.
\end{aligned}$$

Now we check the validity of Fubini's Theorem by using the order $dx\,dy$:

$$\begin{aligned}
\iint_R \left(xy + e^y\right)\, dA &= \int_1^2 \int_3^4 \left(xy + e^y\right)\, dx\, dy \\
&= \int_1^2 \left(\left[\frac{1}{2}x^2y + xe^y\right]\Bigg|_3^4 \right) dy \\
&= \int_1^2 \left(\frac{7}{2}y + e^y \right)\, dy \\
&= \left(\frac{7}{4}y^2 + e^y \right)\Bigg|_1^2 \\
&= \frac{21}{4} + e^2 - e \approx 9.92.
\end{aligned}$$

Both orders of integration return the same result, as expected.

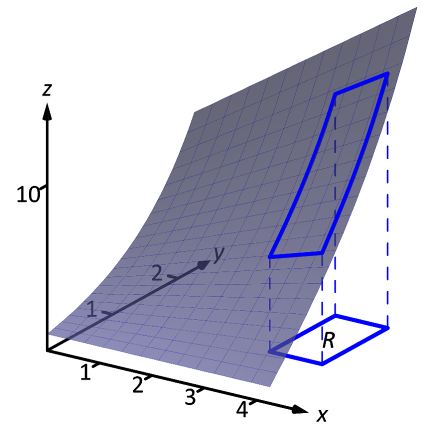


Figure 13.10: Finding the signed volume under a surface in Example 13.10.

Notes:

Example 13.11 Evaluating a double integral

Evaluate $\iint_R \left(3xy - x^2 - y^2 + 6\right)\, dA$, where R is the triangle bounded by $x = 0$, $y = 0$ and $x/2 + y = 1$, as shown in Figure 13.11.

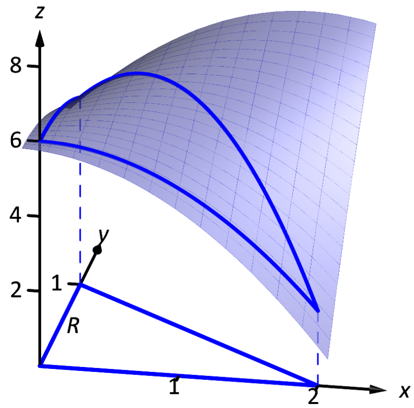


Figure 13.11: Finding the signed volume under the surface in Example 13.11.

Solution While it is not specified which order we are to use, we will evaluate the double integral using both orders to help drive home the point that it does not matter which order we use.

Using the order $dy\, dx$: The bounds on y go from "curve to curve," i.e., $0 \leq y \leq 1 - x/2$, and the bounds on x go from "point to point," i.e., $0 \leq x \leq 2$.

$$\begin{aligned}
\iint_R \left(3xy - x^2 - y^2 + 6\right)\, dA &= \int_0^2 \int_0^{-\frac{x}{2}+1} \left(3xy - x^2 - y^2 + 6\right)\, dy\, dx \\
&= \int_0^2 \left(\frac{3}{2}xy^2 - x^2y - \frac{1}{3}y^3 + 6y\right)\Bigg|_0^{-\frac{x}{2}+1} dx \\
&= \int_0^2 \left(\frac{11}{12}x^3 - \frac{11}{4}x^2 - x - \frac{17}{3}\right) dx \\
&= \left(\frac{11}{48}x^4 - \frac{11}{12}x^3 - \frac{1}{2}x^2 - \frac{17}{3}x\right)\Bigg|_0^2 \\
&= \frac{17}{3} = 5.\overline{6}.
\end{aligned}$$

Now lets consider the order $dx\, dy$. Here x goes from "curve to curve," $0 \leq x \leq 2 - 2y$, and y goes from "point to point," $0 \leq y \leq 1$:

$$\begin{aligned}
\iint_R \left(3xy - x^2 - y^2 + 6\right)\, dA &= \int_0^1 \int_0^{2-2y} \left(3xy - x^2 - y^2 + 6\right)\, dx\, dy \\
&= \int_0^1 \left(\frac{3}{2}x^2y - \frac{1}{3}x^3 - xy^2 + 6x\right)\Bigg|_0^{2-2y} dy \\
&= \int_0^1 \left(\frac{32}{3}y^3 - 22y^2 + 2y + \frac{28}{3}\right) dy \\
&= \left(\frac{8}{3}y^4 - \frac{22}{3}y^3 + y^2 + \frac{28}{3}y\right)\Bigg|_0^1 \\
&= \frac{17}{3} = 5.\overline{6}.
\end{aligned}$$

We obtained the same result using both orders of integration.

Note how in these two examples that the bounds of integration depend only on R; the bounds of integration have nothing to do with $f(x, y)$. This is an important concept, so we include it as a Key Idea.

Notes:

Key Idea 57 Double Integration Bounds

When evaluating $\iint_R f(x,y)\,dA$ using an iterated integral, the bounds of integration depend only on R. The surface f does not determine the bounds of integration.

Before doing another example, we give some properties of double integrals. Each should make sense if we view them in the context of finding signed volume under a surface, over a region.

Theorem 125 Properties of Double Integrals

Let f and g be continuous functions over a closed, bounded plane region R, and let c be a constant.

1. $\displaystyle\iint_R c\,f(x,y)\,dA = c\iint_R f(x,y)\,dA$.
2. $\displaystyle\iint_R \big(f(x,y)\pm g(x,y)\big)\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$
3. If $f(x,y)\geq 0$ on R, then $\displaystyle\iint_R f(x,y)\,dA \geq 0$.
4. If $f(x,y)\geq g(x,y)$ on R, then $\displaystyle\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$.
5. Let R be the union of two nonoverlapping regions, $R = R_1\bigcup R_2$ (see Figure 13.12). Then
$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA.$$

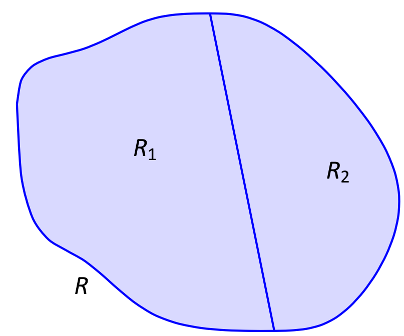


Figure 13.12: R is the union of two nonoverlapping regions, R_1 and R_2.

Example 13.12 Evaluating a double integral

Let $f(x,y) = \sin x\cos y$ and R be the triangle with vertices $(-1,0)$, $(1,0)$ and $(0,1)$ (see Figure 13.13). Evaluate the double integral $\iint_R f(x,y)\,dA$.

Solution If we attempt to integrate using an iterated integral with the order $dy\,dx$, note how there are two upper bounds on R meaning we'll need to use two iterated integrals. We would need to split the triangle into two regions along the y-axis, then use Theorem 125, part 5.

Instead, let's use the order $dx\,dy$. The curves bounding x are $y-1\leq x\leq$

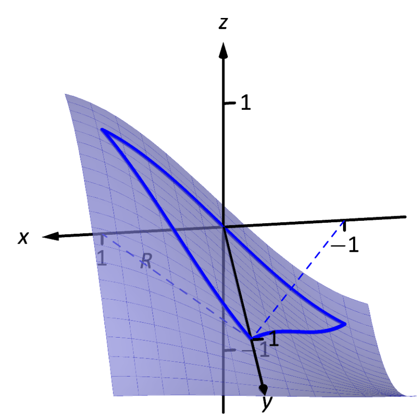


Figure 13.13: Finding the signed volume under a surface in Example 13.12.

Notes:

$1-y$; the bounds on y are $0 \le y \le 1$. This gives us:

$$\begin{aligned}\iint_R f(x,y)\,dA &= \int_0^1 \int_{y-1}^{1-y} \sin x \cos y\,dx\,dy \\ &= \int_0^1 \Big(-\cos x \cos y\Big)\Big|_{y-1}^{1-y}\,dy \\ &= \int_0^1 \cos y\Big(-\cos(1-y)+\cos(y-1)\Big)\,dy.\end{aligned}$$

Recall that the cosine function is an even function; that is, $\cos x = \cos(-x)$. Therefore, from the last integral above, we have $\cos(y-1)=\cos(1-y)$. Thus the integrand simplifies to 0, and we have

$$\begin{aligned}\iint_R f(x,y)\,dA &= \int_0^1 0\,dy \\ &= 0.\end{aligned}$$

It turns out that over R, there is just as much volume above the x-y plane as below (look again at Figure 13.13), giving a final signed volume of 0.

Example 13.13 Evaluating a double integral

Evaluate $\iint_R (4-y)\,dA$, where R is the region bounded by the parabolas $y^2=4x$ and $x^2=4y$, graphed in Figure 13.14.

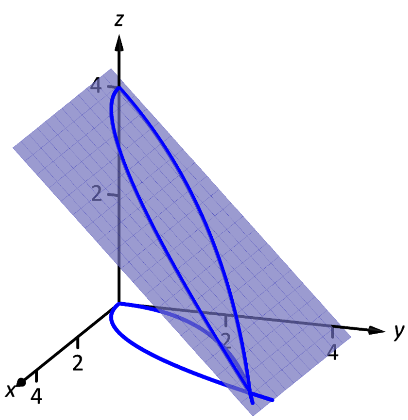


Figure 13.14: Finding the volume under the surface in Example 13.13.

Solution Graphing each curve can help us find their points of intersection. Solving analytically, the second equation tells us that $y=x^2/4$. Substituting this value in for y in the first equation gives us $x^4/16=4x$. Solving for x:

$$\begin{aligned}\frac{x^4}{16} &= 4x \\ x^4-64x &= 0 \\ x(x^3-64) &= 0 \\ x &= 0,\ 4.\end{aligned}$$

Thus we've found analytically what was easy to approximate graphically: the regions intersect at $(0,0)$ and $(4,4)$, as shown in Figure 13.14.

We now choose an order of integration: $dy\,dx$ or $dx\,dy$? Either order works; since the integrand does not contain x, choosing $dx\,dy$ might be simpler – at least, the first integral is very simple.

Notes:

Thus we have the following "curve to curve, point to point" bounds: $y^2/4 \leq x \leq 2\sqrt{y}$, and $0 \leq y \leq 4$.

$$\begin{aligned}
\iint_R (4-y)\,dA &= \int_0^4 \int_{y^2/4}^{2\sqrt{y}} (4-y)\,dx\,dy \\
&= \int_0^4 \Big(x(4-y)\Big)\Big|_{y^2/4}^{2\sqrt{y}} dy \\
&= \int_0^4 \left(\left(2\sqrt{y}-\frac{y^2}{4}\right)(4-y)\right)\,dy = \int_0^4 \left(\frac{y^3}{4}-y^2-2y^{3/2}+8y^{1/2}\right)\,dy \\
&= \left(\frac{y^4}{16}-\frac{y^3}{3}-\frac{4y^{5/2}}{5}+\frac{16y^{3/2}}{3}\right)\Big|_0^4 \\
&= \frac{176}{15} = 11.7\overline{3}.
\end{aligned}$$

The signed volume under the surface f is about 11.7 cubic units.

In the previous section we practiced changing the order of integration of a given iterated integral, where the region R was not explicitly given. Changing the bounds of an integral is more than just an test of understanding. Rather, there are cases where integrating in one order is really hard, if not impossible, whereas integrating with the other order is feasible.

Example 13.14 Changing the order of integration

Rewrite the iterated integral $\displaystyle\int_0^3 \int_y^3 e^{-x^2}\,dx\,dy$ with the order $dy\,dx$. Comment on the feasibility to evaluate each integral.

SOLUTION Once again we make a sketch of the region over which we are integrating to facilitate changing the order. The bounds on x are from $x = y$ to $x = 3$; the bounds on y are from $y = 0$ to $y = 3$. These curves are sketched in Figure 13.15, enclosing the region R.

To change the bounds, note that the curves bounding y are $y = 0$ up to $y = x$; the triangle is enclosed between $x = 0$ and $x = 3$. Thus the new bounds of integration are $0 \leq y \leq x$ and $0 \leq x \leq 3$, giving the iterated integral $\displaystyle\int_0^3 \int_0^x e^{-x^2}\,dy\,dx$.

How easy is it to evaluate each iterated integral? Consider the order of integrating $dx\,dy$, as given in the original problem. The first indefinite integral we need to evaluate is $\int e^{-x^2}\,dx$; we have stated before (see Section 5.6) that this integral cannot be evaluated in terms of elementary functions. We are stuck. (We could actually write it in terms of the nonelementary error function erf x

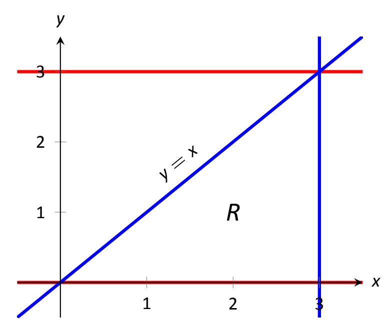


Figure 13.15: Determining the region R determined by the bounds of integration in Example 13.14.

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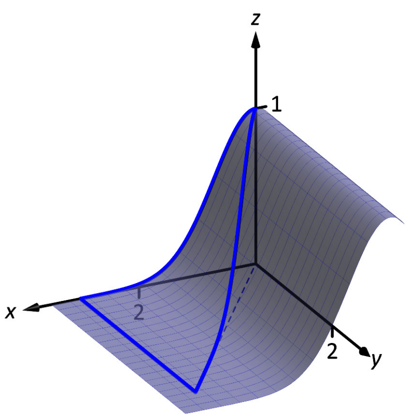


Figure 13.16: Showing the surface f defined in Example 13.14 over its region R.

from Section 6.7 and get cancelation. However, we will do it the simpler way.)

Changing the order of integration makes a big difference here. In the second iterated integral, we are faced with $\int e^{-x^2}\,dy$; integrating with respect to y gives us $ye^{-x^2}+C$, and the first definite integral evaluates to

$$\int_0^x e^{-x^2}\,dy = xe^{-x^2}.$$

Thus

$$\int_0^3 \int_0^x e^{-x^2}\,dy\,dx = \int_0^3 \left(xe^{-x^2}\right)dx.$$

This last integral is easy to evaluate with substitution, giving a final answer of $\frac{1}{2}(1-e^{-9}) \approx 0.5$. Figure 13.16 shows the surface over R.

In short, evaluating one iterated integral is impossible; the other iterated integral is relatively simple.

Definition 22 defines the average value of a single–variable function $f(x)$ on the interval $[a,b]$ as

$$\text{average value of } f(x) \text{ on } [a,b] = \frac{1}{b-a}\int_a^b f(x)\,dx;$$

that is, it is the "area under f over an interval divided by the length of the interval." We make an analogous statement here: the average value of $z=f(x,y)$ over a region R is the volume under f over R divided by the area of R.

Definition 106 The Average Value of f on R

Let $z=f(x,y)$ be a continuous function defined over a closed region R in the x-y plane. The **average value of f on R** is

$$\text{average value of } f \text{ on } R = \frac{\displaystyle\iint_R f(x,y)\,dA}{\displaystyle\iint_R dA}.$$

Example 13.15 Finding average value of a function over a region R

Find the average value of $f(x,y)=4-y$ over the region R, which is bounded by the parabolas $y^2=4x$ and $x^2=4y$. Note: this is the same function and region as used in Example 13.13.

Notes:

Solution In Example 13.13 we found

$$\iint_R f(x,y)\,dA = \int_0^4 \int_{y^2/4}^{2\sqrt{y}} (4-y)\,dx\,dy = \frac{176}{15}.$$

We find the area of R by computing $\iint_R dA$:

$$\iint_R dA = \int_0^4 \int_{y^2/4}^{2\sqrt{y}} dx\,dy = \frac{16}{3}.$$

Dividing the volume under the surface by the area gives the average value:

$$\text{average value of } f \text{ on } R = \frac{176/15}{16/3} = \frac{11}{5} = 2.2.$$

While the surface, as shown in Figure 13.17, covers z-values from $z=0$ to $z=4$, the "average" z-value on R is 2.2.

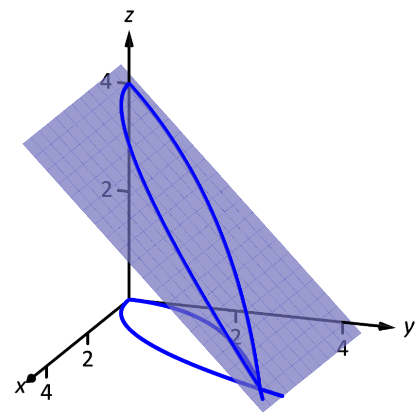


Figure 13.17: Finding the average value of f in Example 13.15.

The previous section introduced the iterated integral in the context of finding the area of plane regions. This section has extended our understanding of iterated integrals; now we see they can be used to find the signed volume under a surface.

This new understanding allows us to revisit what we did in the previous section. Given a region R in the plane, we computed $\iint_R 1\,dA$; again, our understanding at the time was that we were finding the area of R. However, we can now view the function $z=1$ as a surface, a flat surface with constant z-value of 1. The double integral $\iint_R 1\,dA$ finds the volume, under $z=1$, over R, as shown in Figure 13.18. Basic geometry tells us that if the base of a general right cylinder has area A, its volume is $A \cdot h$, where h is the height. In our case, the height is 1. We were "actually" computing the volume of a solid, though we interpreted the number as an area.

The next section extends our abilities to find "volumes under surfaces." Currently, some integrals are hard to compute because either the region R we are integrating over is hard to define with rectangular curves, or the integrand itself is hard to deal with. Some of these problems can be solved by converting everything into polar coordinates.

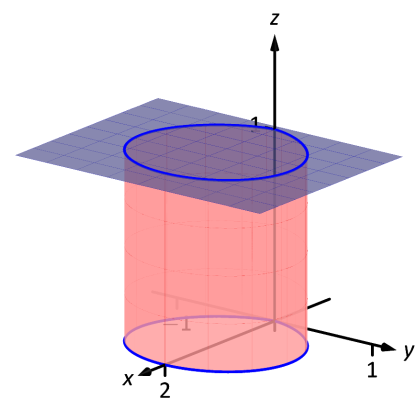


Figure 13.18: Showing how an iterated integral used to find area also finds a certain volume.

Notes:

Exercises 13.2

Terms and Concepts

1. An integral can be interpreted as giving the signed area over an interval; a double integral can be interpreted as giving the signed ________ over a region.

2. Explain why the following statement is false: "Fubini's Theorem states that $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ dy\ dx = \int_a^b \int_{g_1(y)}^{g_2(y)} f(x,y)\ dx\, dy$."

3. Explain why if $f(x,y) > 0$ over a region R, then $\iint_R f(x,y)\ dA > 0$.

4. If $\iint_R f(x,y)\ dA = \iint_R g(x,y)\ dA$, does this imply $f(x,y) = g(x,y)$?

Problems

In Exercises 5 – 10,

(a) Evaluate the given iterated integral, and

(b) rewrite the integral using the other order of integration.

5. $\displaystyle\int_1^2 \int_{-1}^1 \left(\frac{x}{y} + 3\right)\ dx\, dy$

6. $\displaystyle\int_{-\pi/2}^{\pi/2} \int_0^{\pi} (\sin x \cos y)\ dx\, dy$

7. $\displaystyle\int_0^4 \int_0^{-x/2+2} \left(3x^2 - y + 2\right)\ dy\, dx$

8. $\displaystyle\int_1^3 \int_y^3 \left(x^2y - xy^2\right)\ dx\, dy$

9. $\displaystyle\int_0^1 \int_{-\sqrt{1-y}}^{\sqrt{1-y}} (x + y + 2)\ dx\, dy$

10. $\displaystyle\int_0^9 \int_{y/3}^{\sqrt{y}} \left(xy^2\right)\ dx\, dy$

In Exercises 11 – 18:

(a) Sketch the region R given by the problem.

(b) Set up the iterated integrals, in both orders, that evaluate the given double integral for the described region R.

(c) Evaluate one of the iterated integrals to find the signed volume under the surface $z = f(x,y)$ over the region R.

11. $\displaystyle\iint_R x^2y\ dA$, where R is bounded by $y = \sqrt{x}$ and $y = x^2$.

12. $\displaystyle\iint_R x^2y\ dA$, where R is bounded by $y = \sqrt[3]{x}$ and $y = x^3$.

13. $\displaystyle\iint_R x^2 - y^2\ dA$, where R is the rectangle with corners $(-1,-1)$, $(1,-1)$, $(1,1)$ and $(-1,1)$.

14. $\displaystyle\iint_R ye^x\ dA$, where R is bounded by $x = 0$, $x = y^2$ and $y = 1$.

15. $\displaystyle\iint_R \left(6 - 3x - 2y\right)\ dA$, where R is bounded by $x = 0$, $y = 0$ and $3x + 2y = 6$.

16. $\displaystyle\iint_R e^y\ dA$, where R is bounded by $y = \ln x$ and $y = \frac{1}{e-1}(x-1)$.

17. $\displaystyle\iint_R \left(x^3y - x\right)\ dA$, where R is the half of the circle $x^2 + y^2 = 9$ in the first and second quadrants.

18. $\displaystyle\iint_R \left(4 - 3y\right)\ dA$, where R is bounded by $y = 0$, $y = x/e$ and $y = \ln x$.

In Exercises 19 – 22, state why it is difficult/impossible to integrate the iterated integral in the given order of integration. Change the order of integration and evaluate the new iterated integral.

19. $\displaystyle\int_0^4 \int_{y/2}^2 e^{x^2}\ dx\, dy$

20. $\displaystyle\int_0^{\sqrt{\pi/2}} \int_x^{\sqrt{\pi/2}} \cos\left(y^2\right)\ dy\, dx$

21. $\displaystyle\int_0^1 \int_y^1 \frac{2y}{x^2 + y^2}\ dx\, dy$

22. $\displaystyle\int_{-1}^1 \int_1^2 \frac{x \tan^2 y}{1 + \ln y}\ dy\, dx$

In Exercises 23 – 26, find the average value of f over the region R. Notice how these functions and regions are related to the iterated integrals given in Exercises 5 – 8.

23. $f(x,y) = \dfrac{x}{y} + 3$; R is the rectangle with opposite corners $(-1,1)$ and $(1,2)$.

24. $f(x,y) = \sin x \cos y$; R is bounded by $x = 0$, $x = \pi$, $y = -\pi/2$ and $y = \pi/2$.

25. $f(x,y) = 3x^2 - y + 2$; R is bounded by the lines $y = 0$, $y = 2 - x/2$ and $x = 0$.

26. $f(x,y) = x^2y - xy^2$; R is bounded by $y = x$, $y = 1$ and $x = 3$.

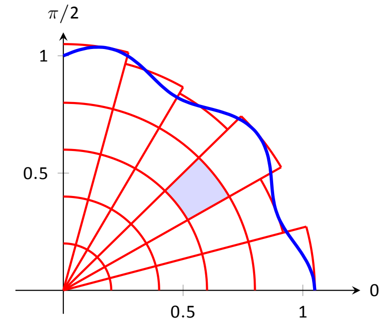
13.3 Double Integration with Polar Coordinates

We have used iterated integrals to evaluate double integrals, which give the signed volume under a surface, $z = f(x, y)$, over a region R of the x-y plane. The integrand is simply $f(x, y)$, and the bounds of the integrals are determined by the region R.

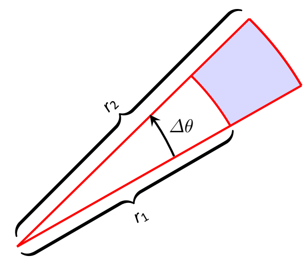
Some regions R are easy to describe using rectangular coordinates – that is, with equations of the form $y = f(x)$, $x = a$, etc. However, some regions are easier to handle if we represent their boundaries with polar equations of the form $r = f(\theta)$, $\theta = \alpha$, etc.

The basic form of the double integral is $\iint_R f(x, y)\, dA$. We interpret this integral as follows: over the region R, sum up lots of products of heights (given by $f(x_i, y_i)$) and areas (given by ΔA_i). That is, dA represents "a little bit of area." In rectangular coordinates, we can describe a small rectangle as having area $dx\,dy$ or $dy\,dx$ – the area of a rectangle is simply length$\times$width – a small change in x times a small change in y. Thus we replace dA in the double integral with $dx\,dy$ or $dy\,dx$.

Now consider representing a region R with polar coordinates. Consider Figure 13.19(a). Let R be the region in the first quadrant bounded by the curve. We can approximate this region using the natural shape of polar coordinates: portions of sectors of circles. In the figure, one such region is shaded, shown again in part (b) of the figure.



(a)



(b)

Figure 13.19: Approximating a region R with portions of sectors of circles.

As the area of a sector of a circle with radius r, subtended by an angle θ, is $A = \frac{1}{2}r^2\theta$, we can find the area of the shaded region. The whole sector has area $\frac{1}{2}r_2^2\Delta\theta$, whereas the smaller, unshaded sector has area $\frac{1}{2}r_1^2\Delta\theta$. The area of the shaded region is the difference of these areas:

$$\Delta A_i = \frac{1}{2}r_2^2\Delta\theta - \frac{1}{2}r_1^2\Delta\theta = \frac{1}{2}\big(r_2^2 - r_1^2\big)\big(\Delta\theta\big) = \frac{r_2 + r_1}{2}\big(r_2 - r_1\big)\Delta\theta.$$

Note that $(r_2 + r_1)/2$ is just the average of the two radii.

To approximate the region R, we use many such subregions; doing so shrinks the difference $r_2 - r_1$ between radii to 0 and shrinks the change in angle $\Delta\theta$ also to 0. We represent these infinitesimal changes in radius and angle as dr and $d\theta$, respectively. Finally, as dr is small, $r_2 \approx r_1$, and so $(r_2 + r_1)/2 \approx r_1$. Thus, when dr and $d\theta$ are small,

$$\Delta A_i \approx r_i\, dr\, d\theta.$$

Taking a limit, where the number of subregions goes to infinity and both $r_2 - r_1$ and $\Delta\theta$ go to 0, we get

$$dA = r\, dr\, d\theta.$$

So to evaluate $\iint_R f(x, y)\, dA$, replace dA with $r\,dr\,d\theta$. Convert the function $z = f(x, y)$ to a function with polar coordinates with the substitutions $x = r\cos\theta$,

Notes:

$y = r\sin\theta$. Finally, find bounds $g_1(\theta) \leq r \leq g_2(\theta)$ and $\alpha \leq \theta \leq \beta$ that describe R. This is the key principle of this section, so we restate it here as a Key Idea.

Key Idea 58 Evaluating Double Integrals with Polar Coordinates

Let R be a plane region bounded by the polar equations $\alpha \leq \theta \leq \beta$ and $g_1(\theta) \leq r \leq g_2(\theta)$. Then

$$\iint_R f(x,y)\, dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f\big(r\cos\theta, r\sin\theta\big)\, r\, dr\, d\theta.$$

Examples will help us understand this Key Idea.

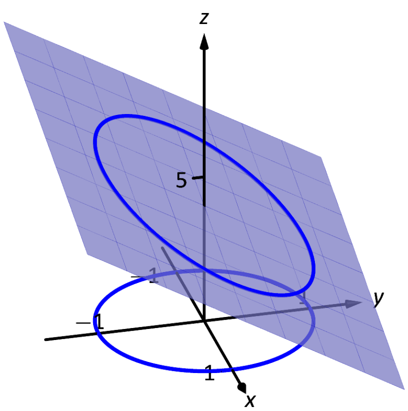


Figure 13.20: Evaluating a double integral with polar coordinates in Example 13.16.

Example 13.16 Evaluating a double integral with polar coordinates

Find the signed volume under the plane $z = 4 - x - 2y$ over the circle with equation $x^2 + y^2 = 1$.

Solution The bounds of the integral are determined solely by the region R over which we are integrating. In this case, it is a circle with equation $x^2 + y^2 = 1$. We need to find polar bounds for this region. It may help to review Section 9.4; bounds for this circle are $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$.

We replace $f(x,y)$ with $f(r\cos\theta, r\sin\theta)$. That means we make the following substitutions:

$$4 - x - 2y \quad \Rightarrow \quad 4 - r\cos\theta - 2r\sin\theta.$$

Finally, we replace dA in the double integral with $r\, dr\, d\theta$. This gives the final iterated integral, which we evaluate:

$$\begin{aligned}
\iint_R f(x,y)\, dA &= \int_0^{2\pi} \int_0^1 \big(4 - r\cos\theta - 2r\sin\theta\big) r\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^1 \big(4r - r^2(\cos\theta - 2\sin\theta)\big)\, dr\, d\theta \\
&= \int_0^{2\pi} \left(2r^2 - \frac{1}{3}r^3(\cos\theta - 2\sin\theta)\right)\Bigg|_0^1 d\theta \\
&= \int_0^{2\pi} \left(2 - \frac{1}{3}\big(\cos\theta - 2\sin\theta\big)\right) d\theta \\
&= \left(2\theta - \frac{1}{3}\big(\sin\theta + 2\cos\theta\big)\right)\Bigg|_0^{2\pi} \\
&= 4\pi \approx 12.566.
\end{aligned}$$

The surface and region R are shown in Figure 13.20.

Notes:

Example 13.17 Evaluating a double integral with polar coordinates

Find the volume under the paraboloid $z = 4 - (x-2)^2 - y^2$ over the region bounded by the circles $(x-1)^2 + y^2 = 1$ and $(x-2)^2 + y^2 = 4$.

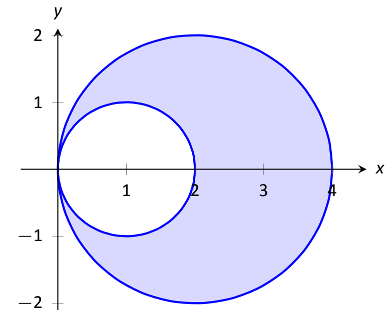
Solution At first glance, this seems like a very hard volume to compute as the region R (shown in Figure 13.21(a)) has a hole in it, cutting out a strange portion of the surface, as shown in part (b) of the figure. However, by describing R in terms of polar equations, the volume is not very difficult to compute. It is straightforward to show that the circle $(x-1)^2 + y^2 = 1$ has polar equation $r = 2\cos\theta$, and that the circle $(x-2)^2 + y^2 = 4$ has polar equation $r = 4\cos\theta$. Each of these circles is traced out on the interval $0 \le \theta \le \pi$. The bounds on r are $2\cos\theta \le r \le 4\cos\theta$.

Replacing x with $r\cos\theta$ in the integrand, along with replacing y with $r\sin\theta$, prepares us to evaluate the double integral $\iint_R f(x,y)\,dA$:

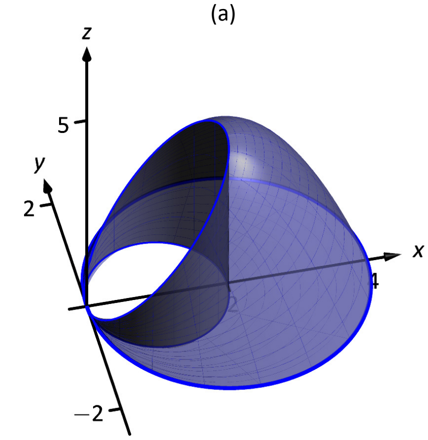
$$\begin{aligned}
\iint_R f(x,y)\,dA &= \int_0^{\pi}\int_{2\cos\theta}^{4\cos\theta} \left(4 - \big(r\cos\theta - 2\big)^2 - \big(r\sin\theta\big)^2\right) r\,dr\,d\theta \\
&= \int_0^{\pi}\int_{2\cos\theta}^{4\cos\theta} \big(-r^3 + 4r^2\cos\theta\big)\,dr\,d\theta \\
&= \int_0^{\pi} \left(-\frac{1}{4}r^4 + \frac{4}{3}r^3\cos\theta\right)\Bigg|_{2\cos\theta}^{4\cos\theta} d\theta \\
&= \int_0^{\pi} \left(\left[-\frac{1}{4}(256\cos^4\theta) + \frac{4}{3}(64\cos^4\theta)\right] - \right. \\
&\qquad \left. \left[-\frac{1}{4}(16\cos^4\theta) + \frac{4}{3}(8\cos^4\theta)\right]\right)\,d\theta \\
&= \int_0^{\pi} \frac{44}{3}\cos^4\theta\,d\theta.
\end{aligned}$$

To integrate $\cos^4\theta$, rewrite it as $\cos^2\theta\cos^2\theta$ and employ the power-reducing formula twice:

$$\begin{aligned}
\cos^4\theta &= \cos^2\theta\cos^2\theta \\
&= \frac{1}{2}\big(1+\cos(2\theta)\big)\frac{1}{2}\big(1+\cos(2\theta)\big) \\
&= \frac{1}{4}\big(1 + 2\cos(2\theta) + \cos^2(2\theta)\big) \\
&= \frac{1}{4}\left(1 + 2\cos(2\theta) + \frac{1}{2}\big(1+\cos(4\theta)\big)\right) \\
&= \frac{3}{8} + \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta).
\end{aligned}$$



(a)



(b)

Figure 13.21: Showing the region R and surface used in Example 13.17.

Notes:

Picking up from where we left off above, we have

$$\begin{aligned}
&= \int_0^{\pi} \frac{44}{3}\cos^4\theta\, d\theta \\
&= \int_0^{\pi} \frac{44}{3}\left(\frac{3}{8}+\frac{1}{2}\cos(2\theta)+\frac{1}{8}\cos(4\theta)\right) d\theta \\
&= \frac{44}{3}\left(\frac{3}{8}\theta+\frac{1}{4}\sin(2\theta)+\frac{1}{32}\sin(4\theta)\right)\Big|_0^{\pi} \\
&= \frac{11}{2}\pi \approx 17.279.
\end{aligned}$$

While this example was not trivial, the double integral would have been *much* harder to evaluate had we used rectangular coordinates.

Example 13.18 Evaluating a double integral with polar coordinates

Find the volume under the surface $f(x,y)=\dfrac{1}{x^2+y^2+1}$ over the sector of the circle with radius a centered at the origin in the first quadrant, as shown in Figure 13.22.

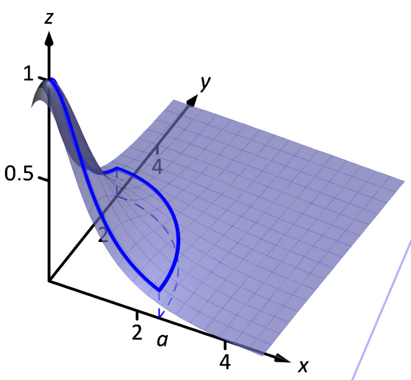


Figure 13.22: The surface and region R used in Example 13.21.

Solution The region R we are integrating over is a circle with radius a, restricted to the first quadrant. Thus, in polar, the bounds on R are $0\le r\le a$, $0\le\theta\le\pi/2$. The integrand is rewritten in polar as

$$\frac{1}{x^2+y^2+1} \Rightarrow \frac{1}{r^2\cos^2\theta+r^2\sin^2\theta+1}=\frac{1}{r^2+1}.$$

We find the volume as follows:

$$\begin{aligned}
\iint_R f(x,y)\, dA &= \int_0^{\pi/2}\int_0^a \frac{r}{r^2+1}\, dr\, d\theta \\
&= \int_0^{\pi/2} \frac{1}{2}\big(\ln|r^2+1|\big)\Big|_0^a\, d\theta \\
&= \int_0^{\pi/2} \frac{1}{2}\ln(a^2+1)\, d\theta \\
&= \left(\frac{1}{2}\ln(a^2+1)\theta\right)\Big|_0^{\pi/2} \\
&= \frac{\pi}{4}\ln(a^2+1).
\end{aligned}$$

Figure 13.22 shows that f shrinks to near 0 very quickly. Regardless, as a grows, so does the volume, without bound.

Note: Previous work has shown that there is finite *area* under $\frac{1}{x^2+1}$ over the entire x-axis. However, Example 13.21 shows that there is infinite *volume* under $\frac{1}{x^2+y^2+1}$ over the entire x-y plane.

Notes:

Example 13.19 Finding the volume of a sphere

Find the volume of a sphere with radius a.

Solution The sphere of radius a, centered at the origin, has equation $x^2+y^2+z^2=a^2$; solving for z, we have $z=\sqrt{a^2-x^2-y^2}$. This gives the upper half of a sphere. We wish to find the volume under this top half, then double it to find the total volume.

The region we need to integrate over is the circle of radius a, centered at the origin. Polar bounds for this equation are $0\leq r\leq a$, $0\leq\theta\leq 2\pi$.

All together, the volume of a sphere with radius a is:

$$\begin{aligned} 2\iint_R \sqrt{a^2-x^2-y^2}\,dA &= 2\int_0^{2\pi}\int_0^a \sqrt{a^2-(r\cos\theta)^2-(r\sin\theta)^2}r\,dr\,d\theta \\ &= 2\int_0^{2\pi}\int_0^a r\sqrt{a^2-r^2}\,dr\,d\theta. \end{aligned}$$

We can evaluate this inner integral with substitution. With $u=a^2-r^2$, $du=-2r\,dr$. The new bounds of integration are $u(0)=a^2$ to $u(a)=0$. Thus we have:

$$\begin{aligned} &= \int_0^{2\pi}\int_{a^2}^0 \left(-u^{1/2}\right)du\,d\theta \\ &= \int_0^{2\pi}\left(-\frac{2}{3}u^{3/2}\right)\Bigg|_{a^2}^0 d\theta \\ &= \int_0^{2\pi}\left(\frac{2}{3}a^3\right)d\theta \\ &= \left(\frac{2}{3}a^3\theta\right)\Bigg|_0^{2\pi} \\ &= \frac{4}{3}\pi a^3. \end{aligned}$$

Generally, the formula for the volume of a sphere with radius r is given as $\frac{4}{3}\pi r^3$; we have justified this formula with our calculation. In Section 13.7, we will be able to use spherical coordinates to write a triple integral that is even easier to compute.

Example 13.20 Finding the volume of a solid

A sculptor wants to make a solid bronze cast of the solid shown in Figure 13.23, where the base of the solid has boundary, in polar coordinates, $r=\cos(3\theta)$, and the top is defined by the plane $z=1-x+0.1y$. Find the volume of the solid.

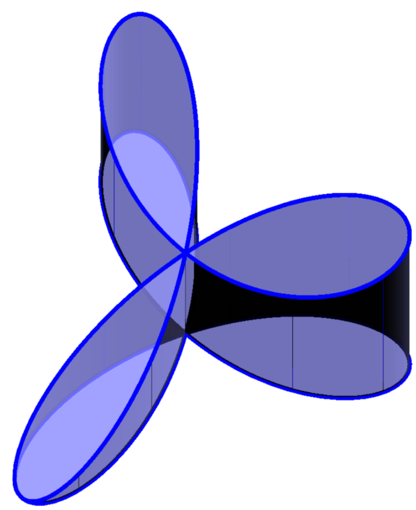


Figure 13.23: Visualizing the solid used in Example 13.20.

Notes:

Solution From the outset, we should recognize that knowing *how to set up* this problem is probably more important than knowing *how to compute the integrals.* The iterated integral to come is not "hard" to evaluate, though it is long, requiring lots of algebra. Once the proper iterated integral is determined, one can use readily–available technology to help compute the final answer.

The region R that we are integrating over is bound by $0 \leq r \leq \cos(3\theta)$, for $0 \leq \theta \leq \pi$ (note that this rose curve is traced out on the interval $[0, \pi]$, not $[0, 2\pi]$). This gives us our bounds of integration. The integrand is $z = 1-x+0.1y$; converting to polar, we have that the volume V is:

$$V = \iint_R f(x,y)\, dA = \int_0^{\pi} \int_0^{\cos(3\theta)} \big(1 - r\cos\theta + 0.1r\sin\theta\big) r\, dr\, d\theta.$$

Distributing the r, the inner integral is easy to evaluate, leading to

$$\int_0^{\pi} \left(\frac{1}{2}\cos^2(3\theta) - \frac{1}{3}\cos^3(3\theta)\cos\theta + \frac{0.1}{3}\cos^3(3\theta)\sin\theta\right)\, d\theta.$$

This integral takes time to compute by hand; it is rather long and cumbersome. The powers of cosine need to be reduced, and products like $\cos(3\theta)\cos\theta$ need to be turned to sums using the Product To Sum formulas in Section 6.2 (and in the back cover of this text).

We rewrite $\frac{1}{2}\cos^2(3\theta)$ as $\frac{1}{4}(1+\cos(6\theta))$. We can also rewrite $\frac{1}{3}\cos^3(3\theta)\cos\theta$ as:

$$\frac{1}{3}\cos^3(3\theta)\cos\theta = \frac{1}{3}\cos^2(3\theta)\cos(3\theta)\cos\theta = \frac{1}{3}\cdot\frac{1+\cos(6\theta)}{2}\big(\cos(4\theta)+\cos(2\theta)\big).$$

This last expression still needs simplification, but eventually all terms can be reduced to the form $a\cos(m\theta)$ or $a\sin(m\theta)$ for various values of a and m.

We forgo the algebra and recommend the reader employ technology, such as WolframAlpha®, to compute the numeric answer. Such technology gives:

$$\int_0^{\pi} \int_0^{\cos(3\theta)} \big(1 - r\cos\theta + 0.1r\sin\theta\big) r\, dr\, d\theta = \frac{\pi}{4} \approx 0.785.$$

Since the units were not specified, we leave the result as 0.785 cubic units (meters, feet, etc.) Should the artist want to scale the piece uniformly, so that each rose petal had a length other than 1, she should keep in mind that scaling by a factor of k scales the volume by a factor of k^3.

We finish up with an example, showing a situation in which we can evaluate a single-variable definite integral (in which the integrand does not have an elementary antiderivative) by rewriting it as a double integral and converting

Notes:

to polar coordinates. Specifically, it is easy with the Direct Comparison Test to show the integral $\displaystyle\int_0^\infty e^{-x^2}\,dx$ converges, but what exactly does it converge to? We mentioned, without proof, in Section 6.7 that it converges to $\sqrt{\pi}/2$, an important result in statistics. Below, we give a proof.

Example 13.21 Proving $\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}/2$ using polar coordinates

Evaluate $\displaystyle\int_0^\infty e^{-x^2}\,dx$.

Solution Let I denote the integral $\displaystyle\int_0^\infty e^{-x^2}\,dx$ we wish to compute. Then we start with I^2 by multiplying I by itself. Since x is just a "dummy variable", we change to y in the second integral, with the corresponding dy. Using two variables allows us to combine the expression into a single integral. We will not discuss improper double integrals, except in this example. The first quadrant $(x > 0, y > 0)$ is expressible in polar coordinates as $(r > 0, 0 \leq \theta \leq \pi/2)$.

$$
\begin{aligned}
I^2 &= \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right) \\
&= \int_0^\infty\int_0^\infty e^{-x^2}e^{-y^2}\,dx\,dy && \text{(use Theorem 122)} \\
&= \int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}\,dx\,dy && \text{(exponent rules)} \\
&= \int_0^{\pi/2}\int_0^\infty e^{-r^2}r\,dr\,d\theta && \text{(convert to polar)} \\
&= \left(\int_0^{\pi/2} d\theta\right)\left(\int_0^\infty e^{-r^2}r\,dr\right) && \text{(use Theorem 122)} \\
&= \frac{\pi}{2}\lim_{b\to\infty}\int_0^b e^{-r^2}r\,dr && \text{(definition of improper integral)} \\
&= \frac{\pi}{2}\lim_{b\to\infty} -\frac{1}{2}\int_0^{-b^2} e^u\,du && \text{(substitution } u = -r^2\text{)} \\
&= -\frac{\pi}{4}\lim_{b\to\infty}\left(e^{-b^2}-1\right) \\
&= \frac{\pi}{4}.
\end{aligned}
$$

Since $I^2 = \pi/4$, it follows that $I = \pm\sqrt{\pi}/2$. It is obvious that $I = \int_0^\infty e^{-x^2}\,dx$, the signed area of a region above the x-axis, must be positive. Therefore,

Notes:

$\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}/2.$

We have used iterated integrals to find areas of plane regions and volumes under surfaces. Just as a single integral can be used to compute much more than "area under the curve," iterated integrals can be used to compute much more than we have thus far seen. The next two sections show two, among many, applications of iterated integrals.

Notes:

Exercises 13.3

Terms and Concepts

1. When evaluating $\iint_R f(x,y)\ dA$ using polar coordinates, $f(x,y)$ is replaced with ________ and dA is replaced with ________.

2. Why would one be interested in evaluating a double integral with polar coordinates?

Problems

In Exercises 3 – 10, a function $f(x,y)$ is given and a region R of the x-y plane is described. Set up and evaluate $\iint_R f(x,y)\ dA$ using polar coordinates.

3. $f(x,y) = 3x - y + 4$; R is the region enclosed by the circle $x^2 + y^2 = 1$.

4. $f(x,y) = 4x + 4y$; R is the region enclosed by the circle $x^2 + y^2 = 4$.

5. $f(x,y) = 8 - y$; R is the region enclosed by the circles with polar equations $r = \cos\theta$ and $r = 3\cos\theta$.

6. $f(x,y) = 4$; R is the region enclosed by the petal of the rose curve $r = \sin(2\theta)$ in the first quadrant.

7. $f(x,y) = \ln\left(x^2 + y^2\right)$; R is the annulus enclosed by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

8. $f(x,y) = 1 - x^2 - y^2$; R is the region enclosed by the circle $x^2 + y^2 = 1$.

9. $f(x,y) = x^2 - y^2$; R is the region enclosed by the circle $x^2 + y^2 = 36$ in the first and fourth quadrants.

10. $f(x,y) = (x-y)/(x+y)$; R is the region enclosed by the lines $y = x$, $y = 0$ and the circle $x^2 + y^2 = 1$ in the first quadrant.

In Exercises 11 – 14, an iterated integral in rectangular coordinates is given. Rewrite the integral using polar coordinates and evaluate the new double integral.

11. $\displaystyle\int_0^5 \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} \sqrt{x^2 + y^2}\ dy\, dx$

12. $\displaystyle\int_{-4}^4 \int_{-\sqrt{16-y^2}}^{0} \left(2y - x\right)\ dx\, dy$

13. $\displaystyle\int_0^2 \int_y^{\sqrt{8-y^2}} \left(x + y\right)\ dx\, dy$

14. $\displaystyle\int_{-2}^{-1} \int_0^{\sqrt{4-x^2}} \left(x+5\right)\ dy\, dx + \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} \left(x+5\right)\ dy\, dx + \int_1^2 \int_0^{\sqrt{4-x^2}} \left(x+5\right)\ dy\, dx$

    Hint: draw the region of each integral carefully and see how they all connect.

In Exercises 15 – 16, special double integrals are presented that are especially well suited for evaluation in polar coordinates.

15. Consider $\displaystyle\iint_R e^{-(x^2+y^2)}\ dA$.

    (a) Why is this integral difficult to evaluate in rectangular coordinates, regardless of the region R?

    (b) Let R be the region bounded by the circle of radius a centered at the origin. Evaluate the double integral using polar coordinates.

    (c) Take the limit of your answer from (b), as $a \to \infty$. What does this imply about the volume under the surface of $e^{-(x^2+y^2)}$ over the entire x-y plane?

16. The surface of a right circular cone with height h and base radius a can be described by the equation $f(x,y) = h - h\sqrt{\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2}}$, where the tip of the cone lies at $(0,0,h)$ and the circular base lies in the x-y plane, centered at the origin.

    Confirm that the volume of a right circular cone with height h and base radius a is $V = \dfrac{1}{3}\pi a^2 h$ by evaluating $\displaystyle\iint_R f(x,y)\ dA$ in polar coordinates.

13.4 Center of Mass

We have used iterated integrals to find areas of plane regions and signed volumes under surfaces. A brief recap of these uses will be useful in this section as we apply iterated integrals to compute the **mass** and **center of mass** of planar regions.

To find the area of a planar region, we evaluated the double integral $\iint_R dA$. That is, summing up the areas of lots of little subregions of R gave us the total area. Informally, we think of $\iint_R dA$ as meaning "sum up lots of little areas over R."

To find the signed volume under a surface, we evaluated the double integral $\iint_R f(x,y)\, dA$. Recall that the "dA" is not just a "bookend" at the end of an integral; rather, it is multiplied by $f(x,y)$. We regard $f(x,y)$ as giving a height, and dA still giving an area: $f(x,y)\, dA$ gives a volume. Thus, informally, $\iint_R f(x,y)\, dA$ means "sum up lots of little volumes over R."

We now extend these ideas to other contexts.

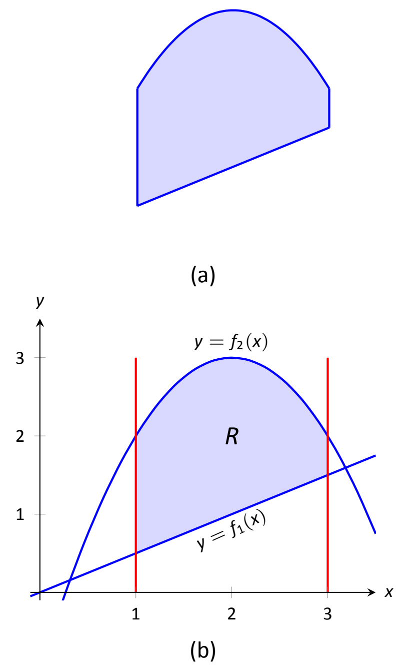


Figure 13.24: Illustrating the concept of a lamina.

Note: *Mass* and *weight* are different measures. Since they are scalar multiples of each other, it is often easy to treat them as the same measure. In this section we effectively treat them as the same, as our technique for finding mass is the same as for finding weight. The density functions used will simply have different units.

Mass and Weight

Consider a thin sheet of material with constant thickness and finite area. Mathematicians (and physicists and engineers) call such a sheet a **lamina**. So consider a lamina, as shown in Figure 13.24(a), with the shape of some planar region R, as shown in part (b).

We can write a simple double integral that represents the mass of the lamina: $\iint_R dm$, where "dm" means "a little mass." That is, the double integral states the total mass of the lamina can be found by "summing up lots of little masses over R."

To evaluate this double integral, partition R into n subregions as we have done in the past. The i^{th} subregion has area ΔA_i. A fundamental property of mass is that "mass=density×area." If the lamina has a constant density δ, then the mass of this i^{th} subregion is $\Delta m_i = \delta \Delta A_i$. That is, we can compute a small amount of mass by multiplying a small amount of area by the density.

If density is variable, with density function $\delta = \delta(x,y)$, then we can approximate the mass of the i^{th} subregion of R by multiplying ΔA_i by $\delta(x_i, y_i)$, where (x_i, y_i) is a point in that subregion. That is, for a small enough subregion of R, the density across that region is almost constant.

The total mass M of the lamina is approximately the sum of approximate masses of subregions:

$$M \approx \sum_{i=1}^{n} \Delta m_i = \sum_{i=1}^{n} \delta(x_i, y_i) \Delta A_i.$$

Notes:

Taking the limit as the size of the subregions shrinks to 0 gives us the actual mass; that is, integrating $\delta(x, y)$ over R gives the mass of the lamina.

Definition 107 Mass of a Lamina with Variable Density

Let $\delta(x, y)$ be a continuous density function of a lamina corresponding to a plane region R. The mass M of the lamina is

$$\text{mass } M = \iint_R dm = \iint_R \delta(x, y)\, dA.$$

Example 13.22 Finding the mass of a lamina with constant density

Find the mass of a square lamina, with side length 1, with a density of $\delta = 3\text{gm/cm}^2$.

Solution We represent the lamina with a square region in the plane as shown in Figure 13.25. As the density is constant, it does not matter where we place the square.

Following Definition 107, the mass M of the lamina is

$$M = \iint_R 3\, dA = \int_0^1 \int_0^1 3\, dx\, dy = 3 \int_0^1 \int_0^1 dx\, dy = 3\text{gm}.$$

This is all very straightforward; note that all we really did was find the area of the lamina and multiply it by the constant density of 3gm/cm^2.

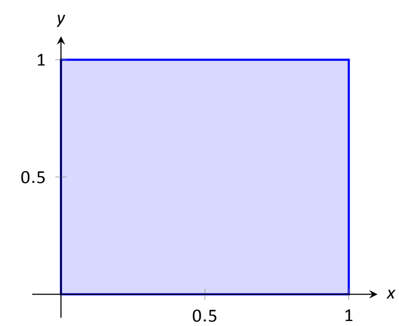


Figure 13.25: A region R representing a lamina in Example 13.22.

Example 13.23 Finding the mass of a lamina with variable density

Find the mass of a square lamina, represented by the unit square with lower lefthand corner at the origin (see Figure 13.25), with variable density $\delta(x, y) = (x + y + 2)\text{gm/cm}^2$.

Solution The variable density δ, in this example, is very uniform, giving a density of 3 in the center of the square and changing linearly. A graph of $\delta(x, y)$ can be seen in Figure 13.26; notice how "same amount" of density is above $z = 3$ as below. We'll comment on the significance of this momentarily.

The mass M is found by integrating $\delta(x, y)$ over R. The order of integration

Notes:

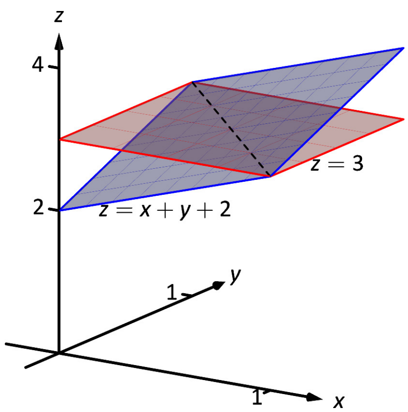


Figure 13.26: Graphing the density functions in Examples 13.22 and 13.23.

is not important; we choose $dx\,dy$ arbitrarily. Thus:

$$\begin{aligned} M &= \iint_R (x+y+2)\,dA = \int_0^1\int_0^1 (x+y+2)\,dx\,dy \\ &= \int_0^1 \left(\frac{1}{2}x^2 + x(y+2)\right)\Big|_0^1\,dy \\ &= \int_0^1 \left(\frac{5}{2}+y\right)\,dy \\ &= \left(\frac{5}{2}y + \frac{1}{2}y^2\right)\Big|_0^1 \\ &= 3\text{gm}. \end{aligned}$$

It turns out that since since the density of the lamina is so uniformly distributed "above and below" $z = 3$ that the mass of the lamina is the same as if it had a constant density of 3. The density functions in Examples 13.22 and 13.23 are graphed in Figure 13.26, which illustrates this concept.

Example 13.24 Finding the weight of a lamina with variable density

Find the weight of the lamina represented by the circle with radius 2ft, centered at the origin, with density function $\delta(x,y) = (x^2+y^2+1)$lb/ft^2. Compare this to the weight of the same lamina with density $\delta(x,y) = (2\sqrt{x^2+y^2}+1)$lb/ft^2.

Solution A direct application of Definition 107 states that the weight of the lamina is $\iint_R \delta(x,y)\,dA$. Since our lamina is in the shape of a circle, it makes sense to approach the double integral using polar coordinates.

The density function $\delta(x,y) = x^2+y^2+1$ becomes $\delta(r,\theta) = (r\cos\theta)^2 + (r\sin\theta)^2 + 1 = r^2+1$. The circle is bounded by $0 \le r \le 2$ and $0 \le \theta \le 2\pi$. Thus the weight W is:

$$\begin{aligned} W &= \int_0^{2\pi}\int_0^2 (r^2+1)r\,dr\,d\theta \\ &= \int_0^{2\pi}\left(\frac{1}{4}r^4 + \frac{1}{2}r^2\right)\Big|_0^2\,d\theta \\ &= \int_0^{2\pi}(6)\,d\theta \\ &= 12\pi \approx 37.70\text{lb}. \end{aligned}$$

Now compare this with the density function $\delta(x,y) = 2\sqrt{x^2+y^2}+1$. Converting this to polar coordinates gives $\delta(r,\theta) = 2\sqrt{(r\cos\theta)^2+(r\sin\theta)^2}+1 =$

Notes:

$2r + 1$. Thus the weight W is:

$$\begin{aligned} W &= \int_0^{2\pi}\int_0^2 (2r+1)r\,dr\,d\theta \\ &= \int_0^{2\pi} \left(\frac{2}{3}r^3 + \frac{1}{2}r^2\right)\Big|_0^2 d\theta \\ &= \int_0^{2\pi} \left(\frac{22}{3}\right)\,d\theta \\ &= \frac{44}{3}\pi \approx 46.08\text{lb}. \end{aligned}$$

One would expect different density functions to return different weights, as we have here. The density functions were chosen, though, to be similar: each gives a density of 1 at the origin and a density of 5 at the outside edge of the circle, as seen in Figure 13.27.

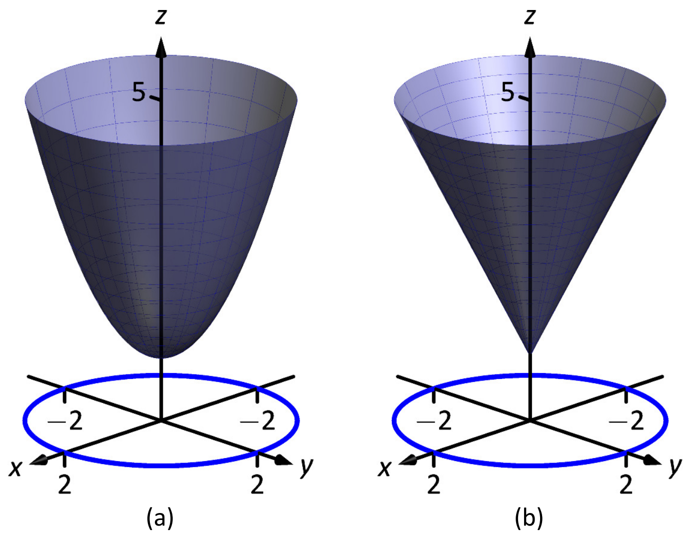


Figure 13.27: Graphing the density functions in Example 13.24. In (a) is the density function $\delta(x, y) = x^2 + y^2 + 1$; in (b) is $\delta(x, y) = 2\sqrt{x^2 + y^2} + 1$.

Notice how $x^2 + y^2 + 1 \leq 2\sqrt{x^2 + y^2} + 1$ over the circle; this results in less weight.

Plotting the density functions can be useful as our understanding of mass can be related to our understanding of "volume under a surface." We interpreted $\iint_R f(x, y)\,dA$ as giving the volume under f over R; we can understand $\iint_R \delta(x, y)\,dA$ in the same way. The "volume" under δ over R is actually mass;

Notes:

by compressing the "volume" under δ onto the *x*-*y* plane, we get "more mass" in some areas than others – i.e., areas of greater density.

Knowing the mass of a lamina is one of several important measures. Another is the **center of mass**, which we discuss next.

Center of Mass

Consider a disk of radius 1 with uniform density. It is common knowledge that the disk will balance on a point if the point is placed at the center of the disk. What if the disk does not have a uniform density? Through trial-and-error, we should still be able to find a spot on the disk at which the disk will balance on a point. This balance point is referred to as the **center of mass**, or **center of gravity**. It is though all the mass is "centered" there. In fact, if the disk has a mass of 3kg, the disk will behave physically as though it were a point-mass of 3kg located at its center of mass. For instance, the disk will naturally spin with an axis through its center of mass (which is why it is important to "balance" the tires of your car: if they are "out of balance", their center of mass will be outside of the axle and it will shake terribly).

We find the center of mass based on the principle of a **weighted average**. Consider a college class in which your homework average is 90%, your test average is 73%, and your final exam grade is an 85%. Experience tells us that our final grade *is not* the *average* of these three grades: that is, it is not:

$$\frac{0.9+0.73+0.85}{3}\approx 0.837 = 83.7\%.$$

That is, you are probably not pulling a B in the course. Rather, your grades are *weighted*. Let's say the homework is worth 10% of the grade, tests are 60% and the exam is 30%. Then your final grade is:

$$(0.1)(0.9)+(0.6)(0.73)+(0.3)(0.85) = 0.783 = 78.3\%.$$

Each grade is multiplied by a **weight**.

In general, given values $x_1, x_2, \ldots, x_n$ and weights $w_1, w_2, \ldots, w_n$, the weighted average of the n values is

$$\sum_{i=1}^{n} w_i x_i \Bigg/ \sum_{i=1}^{n} w_i.$$

In the grading example above, the sum of the weights 0.1, 0.6 and 0.3 is 1, so we don't see the division by the sum of weights in that instance.

How this relates to center of mass is given in the following theorem.

Notes:

Theorem 126 Center of Mass of Discrete Linear System

Let point masses $m_1, m_2, \ldots, m_n$ be distributed along the x-axis at locations $x_1, x_2, \ldots, x_n$, respectively. The center of mass $\overline{x}$ of the system is located at

$$\overline{x} = \sum_{i=1}^{n} m_i x_i \Bigg/ \sum_{i=1}^{n} m_i.$$

Example 13.25 Finding the center of mass of a discrete linear system

1. Point masses of 2gm are located at $x = -1$, $x = 2$ and $x = 3$ are connected by a thin rod of negligible weight. Find the center of mass of the system.
2. Point masses of 10gm, 2gm and 1gm are located at $x = -1$, $x = 2$ and $x = 3$, respectively, are connected by a thin rod of negligible weight. Find the center of mass of the system.

Solution

1. Following Theorem 126, we compute the center of mass as:

$$\overline{x} = \frac{2(-1) + 2(2) + 2(3)}{2 + 2 + 2} = \frac{4}{3} = 1.\overline{3}.$$

   So the system would balance on a point placed at $x = 4/3$, as illustrated in Figure 13.28(a).

2. Again following Theorem 126, we find:

$$\overline{x} = \frac{10(-1) + 2(2) + 1(3)}{10 + 2 + 1} = \frac{-3}{13} \approx -0.23.$$

   Placing a large weight at the left hand side of the system moves the center of mass left, as shown in Figure 13.28(b).

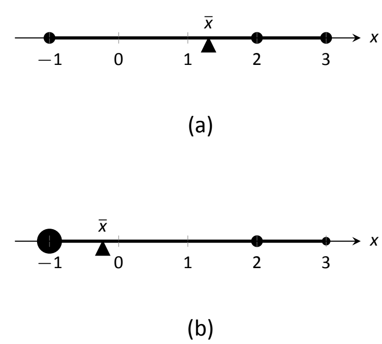


Figure 13.28: Illustrating point masses along a thin rod and the center of mass.

In a discrete system (i.e., mass is located at individual points, not along a continuum) we find the center of mass by dividing the mass into a **moment** of the system. In general, a moment is a weighted measure of distance from a particular point or line. In the case described by Theorem 126, we are finding a weighted measure of distances from the y-axis, so we refer to this as **the moment about the y-axis**, represented by M_y. Letting M be the total mass of the system, we have $\overline{x} = M_y/M$.

Notes:

We can extend the concept of the center of mass of discrete points along a line to the center of mass of discrete points in the plane rather easily. To do so, we define some terms then give a theorem.

Definition 108 Moments about the x- and y- Axes.

Let point masses $m_1, m_2, \ldots, m_n$ be located at points (x_1, y_1), $(x_2, y_2) \ldots, (x_n, y_n)$, respectively, in the x-y plane.

1. The **moment about the y-axis**, M_y, is $M_y = \sum_{i=1}^{n} m_i x_i$.
2. The **moment about the x-axis**, M_x, is $M_x = \sum_{i=1}^{n} m_i y_i$.

One can think that these definitions are "backwards" as M_y sums up "x" distances. But remember, "x" distances are measurements of distance from the y-axis, hence defining the moment about the y-axis.

We now define the center of mass of discrete points in the plane.

Theorem 127 Center of Mass of Discrete Planar System

Let point masses $m_1, m_2, \ldots, m_n$ be located at points (x_1, y_1), $(x_2, y_2) \ldots, (x_n, y_n)$, respectively, in the x-y plane, and let $M = \sum_{i=1}^{n} m_i$.

The center of mass of the system is at $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{M_y}{M} \quad \text{and} \quad \bar{y} = \frac{M_x}{M}.$$

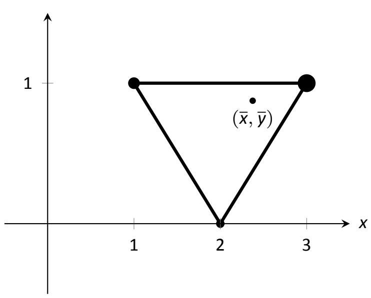


Figure 13.29: Illustrating the center of mass of a discrete planar system in Example 13.26.

Example 13.26 Finding the center of mass of a discrete planar system

Let point masses of 1kg, 2kg and 5kg be located at points $(2, 0)$, $(1, 1)$ and $(3, 1)$, respectively, and are connected by thin rods of negligible weight. Find the center of mass of the system.

SOLUTION We follow Theorem 127 and Definition 108 to find M, M_x and M_y:

$M = 1 + 2 + 5 = 8$kg.

Notes:

$$M_x = \sum_{i=1}^{n} m_i y_i \qquad\qquad M_y = \sum_{i=1}^{n} m_i x_i$$
$$= 1(0) + 2(1) + 5(1) \qquad\qquad = 1(2) + 2(1) + 5(3)$$
$$= 7. \qquad\qquad = 19.$$

Thus the center of mass is $(\overline{x}, \overline{y}) = \left(\dfrac{M_y}{M}, \dfrac{M_x}{M}\right) = \left(\dfrac{19}{8}, \dfrac{7}{8}\right) = (2.375, 0.875)$, illustrated in Figure 13.29.

We finally arrive at our true goal of this section: finding the center of mass of a lamina with variable density. While the above measurement of center of mass is interesting, it does not directly answer more realistic situations where we need to find the center of mass of a contiguous region. However, understanding the discrete case allows us to approximate the center of mass of a planar lamina; using calculus, we can refine the approximation to an exact value.

We begin by representing a planar lamina with a region R in the x-y plane with density function $\delta(x, y)$. Partition R into n subdivisions, each with area ΔA_i. As done before, we can approximate the mass of the i^{th} subregion with $\delta(x_i, y_i)\Delta A_i$, where (x_i, y_i) is a point inside the i^{th} subregion. We can approximate the moment of this subregion about the y-axis with $x_i\delta(x_i, y_i)\Delta A_i$ – that is, by multiplying the approximate mass of the region by its approximate distance from the y-axis. Similarly, we can approximate the moment about the x-axis with $y_i\delta(x_i, y_i)\Delta A_i$. By summing over all subregions, we have:

$$\text{mass: } M \approx \sum_{i=1}^{n} \delta(x_i, y_i)\Delta A_i \quad \text{(as seen before)}$$
$$\text{moment about the } x\text{-axis: } M_x \approx \sum_{i=1}^{n} y_i\delta(x_i, y_i)\Delta A_i$$
$$\text{moment about the } y\text{-axis: } M_y \approx \sum_{i=1}^{n} x_i\delta(x_i, y_i)\Delta A_i$$

By taking limits, where size of each subregion shrinks to 0 in both the x and y directions, we arrive at the double integrals given in the following theorem.

Notes:

Theorem 128 Center of Mass of a Planar Lamina, Moments

Let a planar lamina be represented by a region R in the x-y plane with density function $\delta(x, y)$.

1. mass: $M = \iint_R \delta(x, y)\, dA$
2. moment about the x-axis: $M_x = \iint_R y\delta(x, y)\, dA$
3. moment about the y-axis: $M_y = \iint_R x\delta(x, y)\, dA$
4. The center of mass of the lamina is

$$(\overline{x}, \overline{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right).$$

We start our practice of finding centers of mass by revisiting some of the lamina used previously in this section when finding mass. We will just set up the integrals needed to compute M, M_x and M_y and leave the details of the integration to the reader.

Example 13.27 Finding the center of mass of a lamina

Find the center mass of a square lamina, with side length 1, with a density of $\delta = 3\text{gm/cm}^2$. (Note: this is the lamina from Example 13.22.)

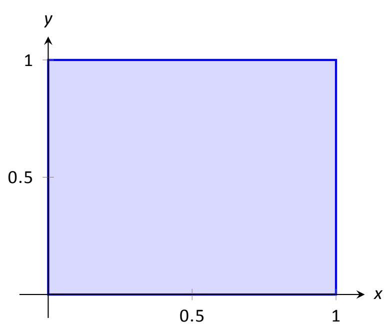


Figure 13.30: A region R representing a lamina in Example 13.22.

Solution We represent the lamina with a square region in the plane as shown in Figure 13.30 as done previously.

Following Theorem 128, we find M, M_x and M_y:

$$M = \iint_R 3\, dA = \int_0^1 \int_0^1 3\, dx\, dy = 3\text{gm}.$$

$$M_x = \iint_R 3y\, dA = \int_0^1 \int_0^1 3y\, dx\, dy = 3/2 = 1.5.$$

$$M_y = \iint_R 3x\, dA = \int_0^1 \int_0^1 3x\, dx\, dy = 3/2 = 1.5.$$

Thus the center of mass is $(\overline{x}, \overline{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = (1.5/3, 1.5/3) = (0.5, 0.5)$.

This is what we should have expected: the center of mass of a square with constant density is the center of the square.

Notes:

Example 13.28 Finding the center of mass of a lamina

Find the center of mass of a square lamina, represented by the unit square with lower lefthand corner at the origin (see Figure 13.30), with variable density $\delta(x,y) = (x+y+2)$gm/cm^2. (Note: this is the lamina from Example 13.23.)

SOLUTION We follow Theorem 128, to find M, M_x and M_y:

$$M = \iint_R (x+y+2)\, dA = \int_0^1 \int_0^1 (x+y+2)\, dx\, dy = 3\text{gm}.$$
$$M_x = \iint_R y(x+y+2)\, dA = \int_0^1 \int_0^1 y(x+y+2)\, dx\, dy = \frac{19}{12}.$$
$$M_y = \iint_R x(x+y+2)\, dA = \int_0^1 \int_0^1 x(x+y+2)\, dx\, dy = \frac{19}{12}.$$

Thus the center of mass is $(\overline{x},\overline{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{19}{36}, \frac{19}{36}\right) \approx (0.528, 0.528)$. While the mass of this lamina is the same as the lamina in the previous example, the greater density found with greater x and y values pulls the center of mass from the center slightly towards the upper righthand corner.

Example 13.29 Finding the center of mass of a lamina

Find the center of mass of the lamina represented by the circle with radius 2ft, centered at the origin, with density function $\delta(x,y) = (x^2+y^2+1)$lb/ft^2. (Note: this is one of the lamina used in Example 13.24.)

SOLUTION As done in Example 13.24, it is best to describe R using polar coordinates. Thus when we compute M_y, we will integrate not $x\delta(x,y) = x(x^2 + y^2+1)$, but rather $\big(r\cos\theta\big)\delta(r\cos\theta, r\sin\theta) = \big(r\cos\theta\big)\big(r^2+1\big)$. We compute M, M_x and M_y:

$$M = \int_0^{2\pi}\int_0^2 (r^2+1)r\, dr\, d\theta = 12\pi \approx 37.7\text{lb}.$$
$$M_x = \int_0^{2\pi}\int_0^2 (r\sin\theta)(r^2+1)r\, dr\, d\theta = 0.$$
$$M_y = \int_0^{2\pi}\int_0^2 (r\cos\theta)(r^2+1)r\, dr\, d\theta = 0.$$

Since R and the density of R are both symmetric about the x and y axes, it should come as no big surprise that the moments about each axis is 0. Thus the center

Notes:

of mass is $(\bar{x}, \bar{y}) = (0, 0)$.

Example 13.30 Finding the center of mass of a lamina

Find the center of mass of the lamina represented by the region R shown in Figure 13.31, half an annulus with outer radius 6 and inner radius 5, with constant density 2lb/ft^2.

Solution Once again it will be useful to represent R in polar coordinates. Using the description of R and/or the illustration, we see that R is bounded by $5 \leq r \leq 6$ and $0 \leq \theta \leq \pi$. As the lamina is symmetric about the y-axis, we should expect $M_y = 0$. We compute M, M_x and M_y:

$$M = \int_0^\pi \int_5^6 (2) r\, dr\, d\theta = 11\pi \text{lb}.$$

$$M_x = \int_0^\pi \int_5^6 (r\sin\theta)(2) r\, dr\, d\theta = \frac{364}{3} \approx 121.33.$$

$$M_y = \int_0^\pi \int_5^6 (r\cos\theta)(2) r\, dr\, d\theta = 0.$$

Thus the center of mass is $(\bar{x}, \bar{y}) = \left(0, \frac{364}{33\pi}\right) \approx (0, 3.51)$. The center of mass is indicated in Figure 13.31; note how it lies outside of R!

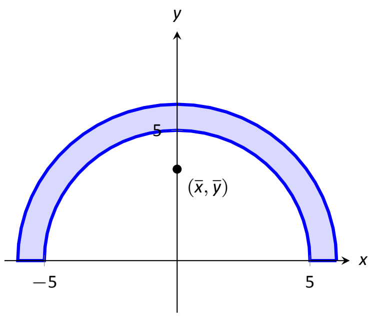


Figure 13.31: Illustrating the region R in Example 13.30.

This section has shown us another use for iterated integrals beyond finding area or signed volume under the curve. While there are many uses for iterated integrals, we give one more application in the following section: computing surface area.

Notes:

Exercises 13.4

Terms and Concepts

1. Why is it easy to use "mass" and "weight" interchangeably, even though they are different measures?

2. Given a point (x, y), the value of x is a measure of distance from the ________-axis.

3. We can think of $\iint_R \, dm$ as meaning "sum up lots of ________"

4. What is a "discrete planar system?"

5. Why does M_x use $\iint_R y\delta(x, y)\, dA$ instead of $\iint_R x\delta(x, y)\, dA$; that is, why do we use "y" and not "x"?

6. Describe a situation where the center of mass of a lamina does not lie within the region of the lamina itself.

Problems

In Exercises 7 – 10, point masses are given along a line or in the plane. Find the center of mass $\bar{x}$ or $(\bar{x}, \bar{y})$, as appropriate. (All masses are in grams and distances are in cm.)

7. $m_1 = 4$ at $x = 1$; $\quad m_2 = 3$ at $x = 3$; $\quad m_3 = 5$ at $x = 10$

8. $m_1 = 2$ at $x = -3$; $\quad m_2 = 2$ at $x = -1$;
   $m_3 = 3$ at $x = 0$; $\quad m_4 = 3$ at $x = 7$

9. $m_1 = 2$ at $(-2, -2)$; $\quad m_2 = 2$ at $(2, -2)$;
   $m_3 = 20$ at $(0, 4)$

10. $m_1 = 1$ at $(-1, -1)$; $\quad m_2 = 2$ at $(-1, 1)$;
    $m_3 = 2$ at $(1, 1)$; $\quad m_4 = 1$ at $(1, -1)$

In Exercises 11 – 18, find the mass/weight of the lamina described by the region R in the plane and its density function $\delta(x, y)$.

11. R is the rectangle with corners $(1, -3)$, $(1, 2)$, $(7, 2)$ and $(7, -3)$; $\delta(x, y) = 5\text{gm/cm}^2$

12. R is the rectangle with corners $(1, -3)$, $(1, 2)$, $(7, 2)$ and $(7, -3)$; $\delta(x, y) = (x + y^2)\text{gm/cm}^2$

13. R is the triangle with corners $(-1, 0)$, $(1, 0)$, and $(0, 1)$; $\delta(x, y) = 2\text{lb/in}^2$

14. R is the triangle with corners $(0, 0)$, $(1, 0)$, and $(0, 1)$; $\delta(x, y) = (x^2 + y^2 + 1)\text{lb/in}^2$

15. R is the circle centered at the origin with radius 2; $\delta(x, y) = (x + y + 4)\text{kg/m}^2$

16. R is the circle sector bounded by $x^2 + y^2 = 25$ in the first quadrant; $\delta(x, y) = (\sqrt{x^2 + y^2} + 1)\text{kg/m}^2$

17. R is the annulus in the first and second quadrants bounded by $x^2 + y^2 = 9$ and $x^2 + y^2 = 36$; $\delta(x, y) = 4\text{lb/ft}^2$

18. R is the annulus in the first and second quadrants bounded by $x^2 + y^2 = 9$ and $x^2 + y^2 = 36$; $\delta(x, y) = \sqrt{x^2 + y^2}\text{lb/ft}^2$

In Exercises 19 – 26, find the center of mass of the lamina described by the region R in the plane and its density function $\delta(x, y)$.
Note: these are the same lamina as in Exercises 11 – 18.

19. R is the rectangle with corners $(1, -3)$, $(1, 2)$, $(7, 2)$ and $(7, -3)$; $\delta(x, y) = 5\text{gm/cm}^2$

20. R is the rectangle with corners $(1, -3)$, $(1, 2)$, $(7, 2)$ and $(7, -3)$; $\delta(x, y) = (x + y^2)\text{gm/cm}^2$

21. R is the triangle with corners $(-1, 0)$, $(1, 0)$, and $(0, 1)$; $\delta(x, y) = 2\text{lb/in}^2$

22. R is the triangle with corners $(0, 0)$, $(1, 0)$, and $(0, 1)$; $\delta(x, y) = (x^2 + y^2 + 1)\text{lb/in}^2$

23. R is the circle centered at the origin with radius 2; $\delta(x, y) = (x + y + 4)\text{kg/m}^2$

24. R is the circle sector bounded by $x^2 + y^2 = 25$ in the first quadrant; $\delta(x, y) = (\sqrt{x^2 + y^2} + 1)\text{kg/m}^2$

25. R is the annulus in the first and second quadrants bounded by $x^2 + y^2 = 9$ and $x^2 + y^2 = 36$; $\delta(x, y) = 4\text{lb/ft}^2$

26. R is the annulus in the first and second quadrants bounded by $x^2 + y^2 = 9$ and $x^2 + y^2 = 36$; $\delta(x, y) = \sqrt{x^2 + y^2}\text{lb/ft}^2$

The *moment of inertia* I is a measure of the tendency of a lamina to resist rotating about an axis or continue to rotate about an axis. I_x is the moment of inertia about the x-axis, I_x is the moment of inertia about the x-axis, and I_O is the moment of inertia about the origin. These are computed as follows:

- $I_x = \displaystyle\iint_R y^2 \, dm$
- $I_y = \displaystyle\iint_R x^2 \, dm$
- $I_O = \displaystyle\iint_R \left(x^2 + y^2\right) dm$

In Exercises 27 – 30, a lamina corresponding to a planar region R is given with a mass of 16 units. For each, compute I_x, I_y and I_O.

27. R is the 4×4 square with corners at $(-2, -2)$ and $(2, 2)$ with density $\delta(x, y) = 1$.

28. R is the 8×2 rectangle with corners at $(-4, -1)$ and $(4, 1)$ with density $\delta(x, y) = 1$.

29. R is the 4×2 rectangle with corners at $(-2, -1)$ and $(2, 1)$ with density $\delta(x, y) = 2$.

30. R is the circle with radius 2 centered at the origin with density $\delta(x, y) = 4/\pi$.

13.5 Surface Area

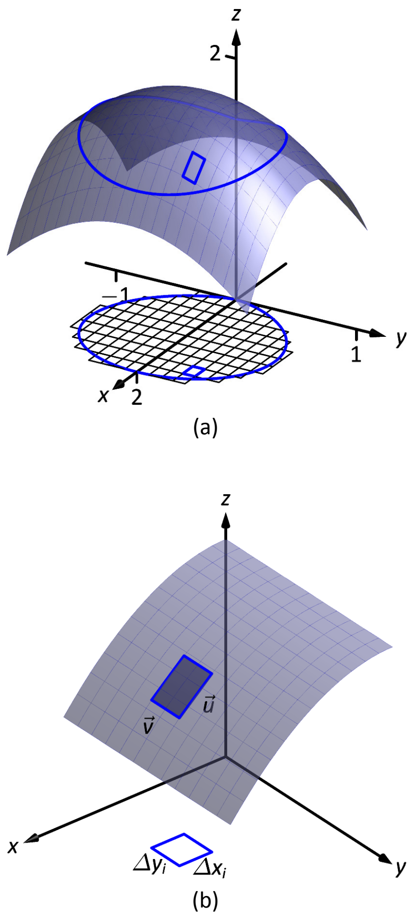


Figure 13.32: Developing a method of computing surface area.

In Section 7.4 we used definite integrals to compute the arc length of plane curves of the form $y = f(x)$. We later extended these ideas to compute the arc length of plane curves defined by parametric or polar equations.

The natural extension of the concept of "arc length over an interval" to surfaces is "surface area over a region."

Consider the surface $z = f(x, y)$ over a region R in the x-y plane, shown in Figure 13.32(a). Because of the domed shape of the surface, the surface area will be greater than that of the area of the region R. We can find this area using the same basic technique we have used over and over: we'll make an approximation, then using limits, we'll refine the approximation to the exact value.

As done to find the volume under a surface or the mass of a lamina, we subdivide R into n subregions. Here we subdivide R into rectangles, as shown in the figure. One such subregion is outlined in the figure, where the rectangle has dimensions Δx_i and Δy_i, along with its corresponding region on the surface.

In part (b) of the figure, we zoom in on this portion of the surface. When Δx_i and Δy_i are small, the function is approximated well by the tangent plane at any point (x_i, y_i) in this subregion, which is graphed in part (b). In fact, the tangent plane approximates the function so well that in this figure, it is virtually indistinguishable from the surface itself! Therefore we can approximate the surface area S_i of this region of the surface with the area T_i of the corresponding portion of the tangent plane.

This portion of the tangent plane is a parallelogram, defined by sides $\vec u$ and $\vec v$, as shown. One of the applications of the cross product from Section 10.4 is that the area of this parallelogram is $|| \vec u \times \vec v ||$. Once we can determine $\vec u$ and $\vec v$, we can determine the area.

$\vec u$ is tangent to the surface in the direction of x, therefore, from Section 12.7, $\vec u$ is parallel to $\langle 1, 0, f_x(x_i, y_i)\rangle$. The x-displacement of $\vec u$ is Δx_i, so we know that $\vec u = \Delta x_i \langle 1, 0, f_x(x_i, y_i)\rangle$. Similar logic shows that $\vec v = \Delta y_i \langle 0, 1, f_y(x_i, y_i)\rangle$. Thus:

$$\begin{aligned}
\text{surface area } S_i &\approx \text{area of } T_i \\
&= || \vec u \times \vec v || \\
&= \left|\left| \Delta x_i \langle 1, 0, f_x(x_i, y_i)\rangle \times \Delta y_i \langle 0, 1, f_y(x_i, y_i)\rangle \right|\right| \\
&= \sqrt{1 + f_x(x_i, y_i)^2 + f_y(x_i, y_i)^2}\,\Delta x_i \Delta y_i.
\end{aligned}$$

Note that $\Delta x_i \Delta y_i = \Delta A_i$, the area of the i^{th} subregion.

Summing up all n of the approximations to the surface area gives

$$\text{surface area over } R \approx \sum_{i=1}^{n} \sqrt{1 + f_x(x_i, y_i)^2 + f_y(x_i, y_i)^2}\,\Delta A_i.$$

Notes:

Once again take a limit as all of the Δx_i and Δy_i shrink to 0; this leads to a double integral.

Definition 109 Surface Area

Let $z = f(x,y)$ where f_x and f_y are continuous over a closed, bounded region R. The surface area S over R is

$$\begin{aligned} S &= \iint_R dS \\ &= \iint_R \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\, dA. \end{aligned}$$

Note: as done before, we think of "$\iint_R dS$" as meaning "sum up lots of little surface areas over R."

The concept of surface area is *defined* here, for while we already have a notion of the area of a region in the *plane*, we did not yet have a solid grasp of what "the area of a surface in *space*" means.

We test this definition by using it to compute surface areas of known surfaces. We start with a triangle.

Example 13.31 Finding the surface area of a plane over a triangle

Let $f(x,y) = 4 - x - 2y$, and let R be the region in the plane bounded by $x = 0$, $y = 0$ and $y = 2 - x/2$, as shown in Figure 13.33. Find the surface area of f over R.

Solution We follow Definition 109. We start by noting that $f_x(x,y) = -1$ and $f_y(x,y) = -2$. To define R, we use bounds $0 \le y \le 2 - x/2$ and $0 \le x \le 4$. Therefore

$$\begin{aligned} S &= \iint_R dS \\ &= \int_0^4 \int_0^{2-x/2} \sqrt{1 + (-1)^2 + (-2)^2}\, dy\, dx \\ &= \int_0^4 \sqrt{6}\left(2 - \frac{x}{2}\right)\, dx \\ &= 4\sqrt{6}. \end{aligned}$$

Because the surface is a triangle, we can figure out the area using geometry. Considering the base of the triangle to be the side in the x-y plane, we find the length of the base to be $\sqrt{20}$. We can find the height using our knowledge of vectors: let $\vec{u}$ be the side in the x-z plane and let $\vec{v}$ be the side in the x-y plane. The height is then $\|\, \vec{u} - \text{proj}_{\vec{v}}\,\vec{u}\, \| = 4\sqrt{6/5}$. Geometry states that the area is thus

$$\frac{1}{2} \cdot 4\sqrt{6/5} \cdot \sqrt{20} = 4\sqrt{6}.$$

We affirm the validity of our formula.

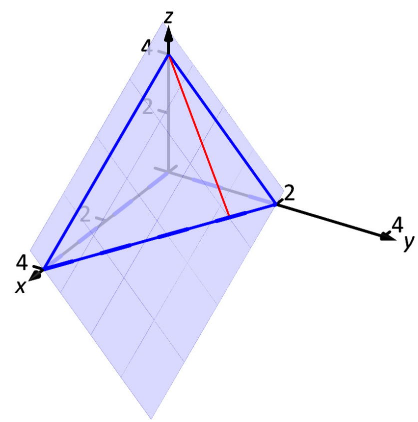


Figure 13.33: Finding the area of a triangle in space in Example 13.31.

Notes:

It is "common knowledge" that the surface area of a sphere of radius r is $4\pi r^2$. We confirm this in the following example, which involves using our formula with polar coordinates.

Example 13.32 The surface area of a sphere.

Find the surface area of the sphere with radius a centered at the origin, whose top hemisphere has equation $f(x,y)=\sqrt{a^2-x^2-y^2}$.

Solution We start by computing partial derivatives and find

$$f_x(x,y)=\frac{-x}{\sqrt{a^2-x^2-y^2}} \quad \text{and} \quad f_y(x,y)=\frac{-y}{\sqrt{a^2-x^2-y^2}}.$$

As our function f only defines the top upper hemisphere of the sphere, we double our surface area result to get the total area:

$$\begin{aligned} S &= 2\iint_R \sqrt{1+f_x(x,y)^2+f_y(x,y)^2}\,dA \\ &= 2\iint_R \sqrt{1+\frac{x^2+y^2}{a^2-x^2-y^2}}\,dA. \end{aligned}$$

The region R that we are integrating over is the circle, centered at the origin, with radius a: $x^2+y^2=a^2$. Because of this region, we are likely to have greater success with our integration by converting to polar coordinates. Using the substitutions $x=r\cos\theta$, $y=r\sin\theta$, $dA=r\,dr\,d\theta$ and bounds $0\leq\theta\leq 2\pi$ and $0\leq r\leq a$, we have:

$$\begin{aligned} S &= 2\int_0^{2\pi}\int_0^a \sqrt{1+\frac{r^2\cos^2\theta+r^2\sin^2\theta}{a^2-r^2\cos^2\theta-r^2\sin^2\theta}}\,r\,dr\,d\theta \\ &= 2\int_0^{2\pi}\int_0^a r\sqrt{1+\frac{r^2}{a^2-r^2}}\,dr\,d\theta \\ &= 2\int_0^{2\pi}\int_0^a r\sqrt{\frac{a^2}{a^2-r^2}}\,dr\,d\theta. \end{aligned} \tag{13.1}$$

Apply substitution $u=a^2-r^2$ and integrate the inner integral, giving

$$\begin{aligned} &= 2\int_0^{2\pi} a^2\,d\theta \\ &= 4\pi a^2. \end{aligned}$$

Our work confirms our previous formula.

Note: The inner integral in Equation (13.1) is an improper integral, as the integrand of $\displaystyle\int_0^a r\sqrt{\frac{a^2}{a^2-r^2}}\,dr$ is not defined at $r=a$. To properly evaluate this integral, one must use the techniques of Section 6.6.

The reason this need arises is that the function $f(x,y)=\sqrt{a^2-x^2-y^2}$ fails the requirements of Definition 109, as f_x and f_y are not continuous on the boundary of the circle $x^2+y^2=a^2$.

The computation of the surface area is still valid. The definition makes stronger requirements than necessary in part to avoid the use of improper integration, as when f_x and/or f_y are not continuous, the resulting improper integral may not converge. Since the improper integral does converge in this example, the surface area is accurately computed.

Notes:

Example 13.33 Finding the surface area of a cone

The general formula for a right cone with height h and base radius a is

$$f(x,y) = h - \frac{h}{a}\sqrt{x^2+y^2},$$

shown in Figure 13.34. Find the surface area of this cone.

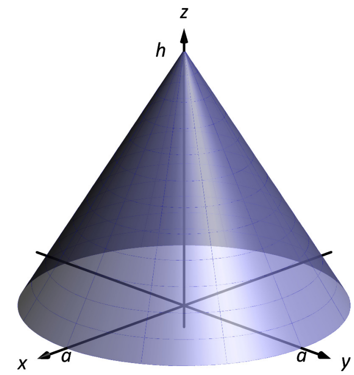


Figure 13.34: Finding the surface area of a cone in Example 13.33.

Solution We begin by computing partial derivatives.

$$f_x(x,y) = -\frac{xh}{a\sqrt{x^2+y^2}} \quad \text{and} \quad -\frac{yh}{a\sqrt{x^2+y^2}}.$$

Since we are integrating over the circle $x^2+y^2 = a^2$, we again use polar coordinates. Using the standard substitutions, our integrand becomes

$$\sqrt{1+\left(\frac{hr\cos\theta}{a\sqrt{r^2}}\right)^2 + \left(\frac{hr\sin\theta}{a\sqrt{r^2}}\right)^2}.$$

This may look intimidating at first, but there are lots of simple simplifications to be done. It amazingly reduces to just

$$\sqrt{1+\frac{h^2}{a^2}} = \frac{1}{a}\sqrt{a^2+h^2}.$$

Our polar bounds are $0 \le \theta \le 2\pi$ and $0 \le r \le a$. Thus

$$\begin{aligned} S &= \int_0^{2\pi}\int_0^a r\frac{1}{a}\sqrt{a^2+h^2}\,dr\,d\theta \\ &= \int_0^{2\pi}\left(\frac{1}{2}r^2\frac{1}{a}\sqrt{a^2+h^2}\right)\Bigg|_0^a\,d\theta \\ &= \int_0^{2\pi}\frac{1}{2}a\sqrt{a^2+h^2}\,d\theta \\ &= \pi a\sqrt{a^2+h^2}. \end{aligned}$$

This matches the formula found in the back of this text.

Note: Note that once again f_x and f_y are not continuous on the domain of f, as both are undefined at $(0,0)$. (A similar problem occurred in the previous example.) Once again the resulting improper integral converges and the computation of the surface area is valid.

Example 13.34 Finding surface area over a region

Find the area of the surface $f(x,y) = x^2-3y+3$ over the region R bounded by $-x \le y \le x$, $0 \le x \le 4$, as pictured in Figure 13.35.

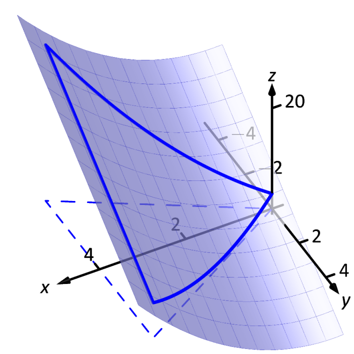


Figure 13.35: Graphing the surface in Example 13.34.

Solution It is straightforward to compute $f_x(x,y) = 2x$ and $f_y(x,y) = -3$. Thus the surface area is described by the double integral

$$\iint_R \sqrt{1+(2x)^2+(-3)^2}\,dA = \iint_R \sqrt{10+4x^2}\,dA.$$

Notes:

As with integrals describing arc length, double integrals describing surface area are in general hard to evaluate directly because of the square–root. This particular integral can be easily evaluated, though, with judicious choice of our order of integration.

Integrating with order $dx\,dy$ requires us to evaluate $\int \sqrt{10+4x^2}\,dx$. This can be done, though it is messier and involves a trigonometric substitution. Integrating with order $dy\,dx$ has as its first integral $\int \sqrt{10+4x^2}\,dy$, which is easy to evaluate: it is simply $y\sqrt{10+4x^2}+C$. So we proceed with the order $dy\,dx$; the bounds are already given in the statement of the problem.

$$\begin{aligned}\iint_R \sqrt{10+4x^2}\,dA &= \int_0^4 \int_{-x}^{x} \sqrt{10+4x^2}\,dy\,dx \\ &= \int_0^4 \left(y\sqrt{10+4x^2}\right)\Big|_{-x}^{x}\,dx \\ &= \int_0^4 \left(2x\sqrt{10+4x^2}\right)\,dx.\end{aligned}$$

Apply substitution with $u = 10+4x^2$:

$$\begin{aligned}&= \left(\frac{1}{6}\left(10+4x^2\right)^{3/2}\right)\Bigg|_0^4 \\ &= \frac{1}{3}\left(37\sqrt{74}-5\sqrt{10}\right) \approx 100.825\text{square units.}\end{aligned}$$

So while the region R over which we integrate has an area of 16 square units, the surface has a much greater area as its z-values change dramatically over R.

In practice, technology helps greatly in the evaluation of such integrals. High powered computer algebra systems can compute integrals that are difficult, or at least time consuming, by hand, and can at the least produce very accurate approximations with numerical methods. In general, just knowing *how* to set up the proper integrals brings one very close to being able to compute the needed value. Most of the work is actually done in just describing the region R in terms of polar or rectangular coordinates. Once this is done, technology can usually provide a good answer.

We have learned how to integrate integrals; that is, we have learned to evaluate double integrals. In the next section, we learn how to integrate double integrals – that is, we learn to evaluate *triple integrals*, along with learning some uses for this operation.

Notes:

Exercises 13.5

Terms and Concepts

1. "Surface area" is analogous to what previously studied concept?

2. To approximate the area of a small portion of a surface, we computed the area of its ________ plane.

3. We interpret $\iint_R dS$ as "sum up lots of little ________ ________."

4. Why is it important to know how to set up a double integral to compute surface area, even if the resulting integral is hard to evaluate?

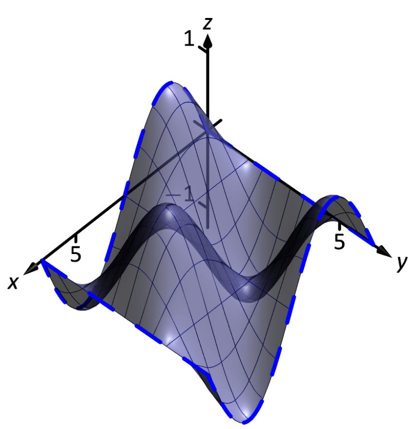
5. Why do $z = f(x, y)$ and $z = g(x, y) = f(x, y) + h$, for some real number h, have the same surface area over a region R?

6. Let $z = f(x, y)$ and $z = g(x, y) = 2f(x, y)$. Why is the surface area of g over a region R not twice the surface area of f over R?

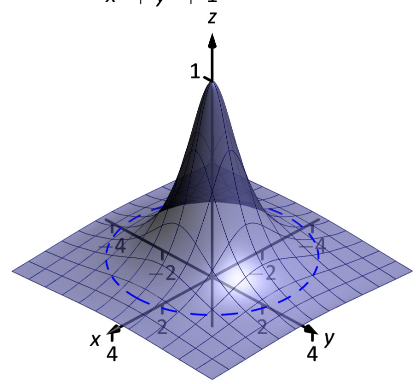
Problems

In Exercises 7 – 10, *set up* the iterated integral that computes the surface area of the given surface over the region R.

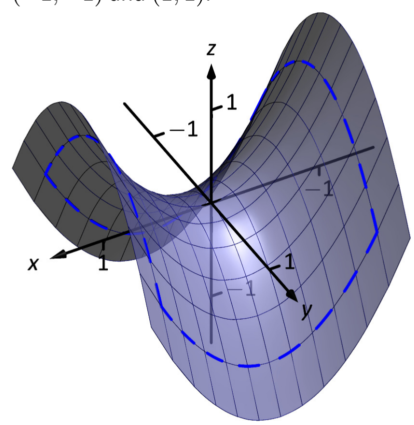
7. $f(x, y) = \sin x \cos y$; R is the rectangle with bounds $0 \leq x \leq 2\pi$, $0 \leq y \leq 2\pi$.



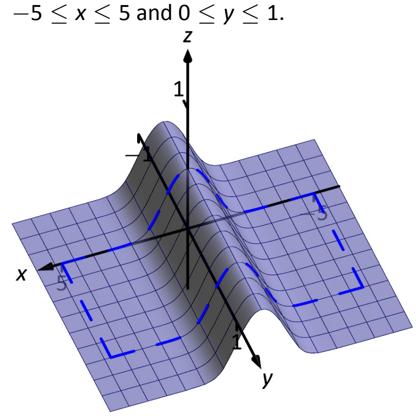
8. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$; R is the circle $x^2 + y^2 = 9$.



9. $f(x, y) = x^2 - y^2$; R is the rectangle with opposite corners $(-1, -1)$ and $(1, 1)$.



10. $f(x, y) = \dfrac{1}{e^{x^2} + 1}$; R is the rectangle bounded by $-5 \leq x \leq 5$ and $0 \leq y \leq 1$.



In Exercises 11 – 19, find the area of the given surface over the region R.

11. $f(x, y) = 3x - 7y + 2$; R is the rectangle with opposite corners $(-1, 0)$ and $(1, 3)$.

12. $f(x, y) = 2x + 2y + 2$; R is the triangle with corners $(0, 0)$, $(1, 0)$ and $(0, 1)$.

13. $f(x,y) = x^2 + y^2 + 10$; R is the circle $x^2 + y^2 = 16$.

14. $f(x,y) = -2x + 4y^2 + 7$ over R, the triangle bounded by $y = -x$, $y = x$, $0 \le y \le 1$.

15. $f(x,y) = x^2 + y$ over R, the triangle bounded by $y = 2x$, $y = 0$ and $x = 2$.

16. $f(x,y) = \frac{2}{3}x^{3/2} + 2y^{3/2}$ over R, the rectangle with opposite corners $(0,0)$ and $(1,1)$.

17. $f(x,y) = 10 - 2\sqrt{x^2 + y^2}$ over R, the circle $x^2 + y^2 = 25$. (This is the cone with height 10 and base radius 5; be sure to compare you result with the known formula.)

18. Find the surface area of the sphere with radius 5 by doubling the surface area of $f(x,y) = \sqrt{25 - x^2 - y^2}$ over R, the circle $x^2 + y^2 = 25$. (Be sure to compare you result with the known formula.)

19. Find the surface area of the ellipse formed by restricting the plane $f(x,y) = cx + dy + h$ to the region R, the circle $x^2 + y^2 = 1$, where c, d and h are some constants. Your answer should be given in terms of c and d; why does the value of h not matter?

13.6 Triple Integration

We learned in Section 13.2 how to compute the signed volume V under a surface $z = f(x,y)$ over a region R: $V = \iint_R f(x,y)\, dA$. It follows naturally that if $f(x,y) \geq g(x,y)$ on R, then the **volume between** $f(x,y)$ **and** $g(x,y)$ **on** R is

$$V = \iint_R f(x,y)\, dA - \iint_R g(x,y)\, dA = \iint_R \big(f(x,y) - g(x,y)\big)\, dA.$$

Theorem 129 Volume Between Surfaces

Let f and g be continuous functions on a closed, bounded region R, where $f(x,y) \geq g(x,y)$ for all (x,y) in R. The volume V between f and g over R is

$$V = \iint_R \big(f(x,y) - g(x,y)\big)\, dA.$$

Example 13.35 Finding volume between surfaces

Find the volume of the space region bounded by the planes $z = 3x + y - 4$ and $z = 8 - 3x - 2y$ in the 1$^{\text{st}}$ octant. In Figure 13.36(a) the planes are drawn; in (b), only the defined region is given.

Solution We need to determine the region R over which we will integrate. To do so, we need to determine where the planes intersect. They have common z-values when $3x + y - 4 = 8 - 3x - 2y$. Applying a little algebra, we have:

$$\begin{aligned} 3x + y - 4 &= 8 - 3x - 2y \\ 6x + 3y &= 12 \\ 2x + y &= 4 \end{aligned}$$

The planes intersect along the line $2x + y = 4$. Therefore the region R is bounded by $x = 0$, $y = 0$, and $y = 4 - 2x$; we can convert these bounds to integration bounds of $0 \leq x \leq 2$, $0 \leq y \leq 4 - 2x$. Thus

$$\begin{aligned} V &= \iint_R \big(8 - 3x - 2y - (3x + y - 4)\big)\, dA \\ &= \int_0^2 \int_0^{4-2x} \big(12 - 6x - 3y\big)\, dy\, dx \\ &= 16. \end{aligned}$$

The volume between the surfaces is 16 cubic units.

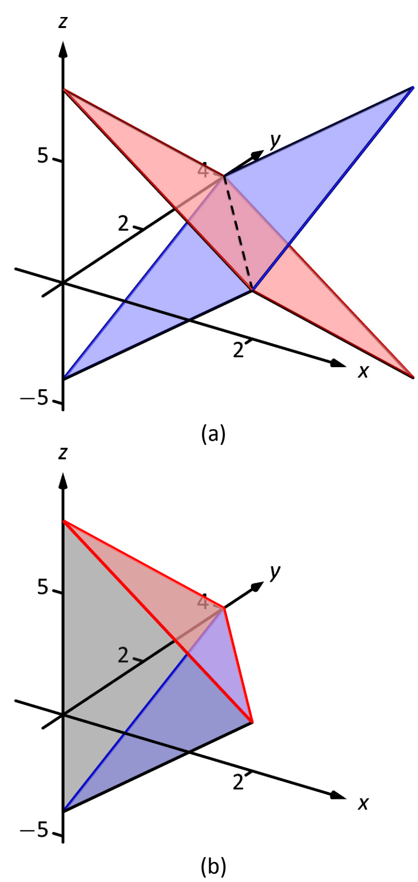


Figure 13.36: Finding the volume between the planes given in Example 13.35.

Notes:

In the preceding example, we found the volume by evaluating the integral

$$\int_0^2 \int_0^{4-2x} \big(8 - 3x - 2y - (3x + y - 4)\big)\, dy\, dx.$$

Note how we can rewrite the integrand as an integral, much as we did in Section 13.1:

$$8 - 3x - 2y - (3x + y - 4) = \int_{3x+y-4}^{8-3x-2y} dz.$$

Thus we can rewrite the double integral that finds volume as

$$\int_0^2 \int_0^{4-2x} \big(8-3x-2y-(3x+y-4)\big)\, dy\, dx = \int_0^2 \int_0^{4-2x} \left(\int_{3x+y-4}^{8-3x-2y} dz\right) dy\, dx.$$

This no longer looks like a "double integral," but more like a "triple integral." Just as our first introduction to double integrals was in the context of finding the area of a plane region, our introduction into triple integrals will be in the context of finding the volume of a space region.

To formally find the volume of a closed, bounded region D in space, such as the one shown in Figure 13.37(a), we start with an approximation. Break D into n rectangular solids; the solids near the boundary of D may possibly not include portions of D and/or include extra space. In Figure 13.37(b), we zoom in on a portion of the boundary of D to show a rectangular solid that contains space not in D; as this is an approximation of the volume, this is acceptable and this error will be reduced as we shrink the size of our solids.

The volume ΔV_i of the i^{th} solid D_i is $\Delta V_i = \Delta x_i \Delta y_i \Delta z_i$, where Δx_i, Δy_i and Δz_i give the dimensions of the rectangular solid in the x, y and z directions, respectively. By summing up the volumes of all n solids, we get an approximation of the volume V of D:

$$V \approx \sum_{i=1}^{n} \Delta V_i = \sum_{i=1}^{n} \Delta x_i \Delta y_i \Delta z_i.$$

Let $||\Delta D||$ represent the length of the longest diagonal of rectangular solids in the subdivision of D. As $||\Delta D|| \to 0$, the volume of each solid goes to 0, as do each of Δx_i, Δy_i and Δz_i, for all i. Our calculus experience tells us that taking a limit as $||\Delta D|| \to 0$ turns our approximation of V into an exact calculation of V. Before we state this result in a theorem, we use a definition to define some terms.

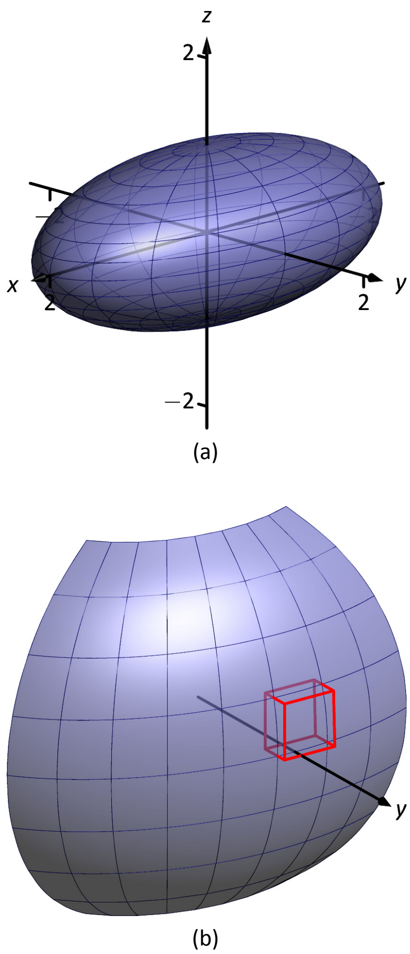


Figure 13.37: Approximating the volume of a region D in space.

Notes:

Definition 110 Triple Integrals, Iterated Integration (Part I)

Let D be a closed, bounded region in space. Let a and b be real numbers, let $g_1(x)$ and $g_2(x)$ be continuous functions of x, and let $f_1(x,y)$ and $f_2(x,y)$ be continuous functions of x and y.

1. The volume V of D is denoted by a **triple integral**,

$$V = \iiint_D dV.$$

2. The iterated integral $\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} dz\,dy\,dx$ is evaluated as

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} dz\,dy\,dx = \int_a^b \int_{g_1(x)}^{g_2(x)} \left(\int_{f_1(x,y)}^{f_2(x,y)} dz \right) dy\,dx.$$

   Evaluating the above iterated integral is **triple integration.**

Our informal understanding of the notation $\iiint_D dV$ is "sum up lots of little volumes over D," analogous to our understanding of $\iint_R dA$ and $\iint_R dm$.

We now state the major theorem of this section.

Theorem 130 Triple Integration (Part I)

Let D be a closed, bounded region in space and let ΔD be any subdivision of D into n rectangular solids, where the i^{th} subregion D_i has dimensions $\Delta x_i \times \Delta y_i \times \Delta z_i$ and volume ΔV_i.

1. The volume V of D is

$$V = \iiint_D dV = \lim_{||\Delta D|| \to 0} \sum_{i=1}^{n} \Delta V_i = \lim_{||\Delta D|| \to 0} \sum_{i=1}^{n} \Delta x_i \Delta y_i \Delta z_i.$$

2. If D is defined as the region bounded by the planes $x = a$ and $x = b$, the cylinders $y = g_(x)$ and $y = g_2(x)$, and the surfaces $z = f_1(x,y)$ and $z = f_2(x,y)$, where $a < b$, $g_1(x) \leq g_2(x)$ and $f_1(x,y) \leq f_2(x,y)$ on D, then

$$\iiint_D dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} dz\,dy\,dx.$$

3. V can be determined using iterated integration with other orders of integration (there are 6 total), as long as D is defined by the region enclosed by a pair of planes, a pair of cylinders, and a pair of surfaces.

Notes:

We evaluated the area of a plane region R by iterated integration, where the bounds were "from curve to curve, then from point to point." Theorem 130 allows us to find the volume of a space region with an iterated integral with bounds "from surface to surface, then from curve to curve, then from point to point." In the iterated integral

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} dz\, dy\, dx,$$

the bounds $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$ define a region R in the x-y plane over which the region D exists in space. However, these bounds are also defining surfaces in space; $x = a$ is a plane and $y = g_1(x)$ is a cylinder. The combination of these 6 surfaces enclose, and define, D.

Examples will help us understand triple integration, including integrating with various orders of integration.

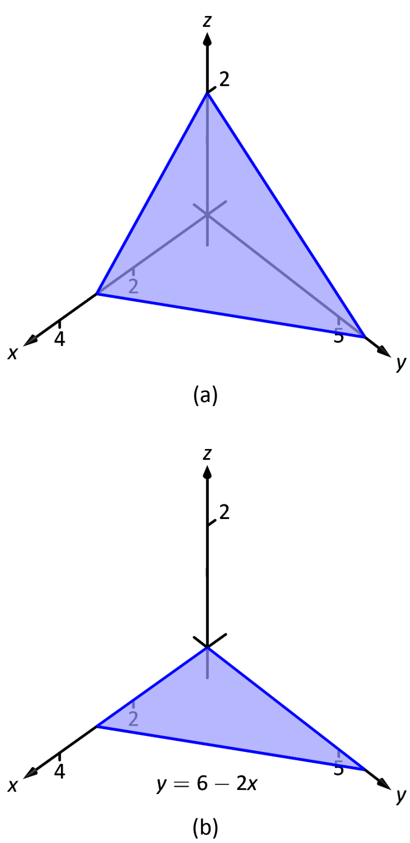


Figure 13.38: The region D used in Example 13.36 in (a); in (b), the region found by collapsing D onto the x-y plane.

Example 13.36 Finding the volume of a space region with triple integration

Find the volume of the space region in the 1^{st} octant bounded by the plane $z = 2 - y/3 - 2x/3$, shown in Figure 13.38(a), using the order of integration $dz\, dy\, dx$. Set up the triple integrals that give the volume in the other 5 orders of integration.

Solution Starting with the order of integration $dz\, dy\, dx$, we need to first find bounds on z. The region D is bounded below by the plane $z = 0$ (because we are restricted to the first octant) and above by $z = 2 - y/3 - 2x/3$; $0 \leq z \leq 2 - y/3 - 2x/3$.

To find the bounds on y and x, we "collapse" the region onto the x-y plane, giving the triangle shown in Figure 13.38(b). (We know the equation of the line $y = 6 - 2x$ in two ways. First, by setting $z = 0$, we have $0 = 2 - y/3 - 2x/3 \Rightarrow y = 6 - 2x$. Secondly, we know this is going to be a straight line between the points $(3, 0)$ and $(0, 6)$ in the x-y plane.)

We define that region R, in the integration order of $dy\, dx$, with bounds $0 \leq$

Notes:

$y \leq 6 - 2x$ and $0 \leq x \leq 3$. Thus the volume V of the region D is:

$$\begin{aligned}
V &= \iiint_D dV \\
&= \int_0^3 \int_0^{6-2x} \int_0^{2-\frac{1}{3}y-\frac{2}{3}x} dz\, dy\, dz \\
&= \int_0^3 \int_0^{6-2x} \left(\int_0^{2-\frac{1}{3}y-\frac{2}{3}x} dz \right) dy\, dz \\
&= \int_0^3 \int_0^{6-2x} z\Big|_0^{2-\frac{1}{3}y-\frac{2}{3}x} dy\, dz \\
&= \int_0^3 \int_0^{6-2x} \left(2 - \frac{1}{3}y - \frac{2}{3}x\right) dy\, dz.
\end{aligned}$$

From this step on, we are evaluating a double integral as done many times before. We skip these steps and give the final volume,

$$= 6\text{units}^3.$$

The order *dz dx dy*:

Now consider the volume using the order of integration *dz dx dy*. The bounds on z are the same as before, $0 \leq z \leq 2-y/3-2x/3$. Collapsing the space region on the x-y plane as shown in Figure 13.38(b), we now describe this triangle with the order of integration *dx dy*. This gives bounds $0 \leq x \leq 3-y/2$ and $0 \leq y \leq 6$. Thus the volume is given by the triple integral

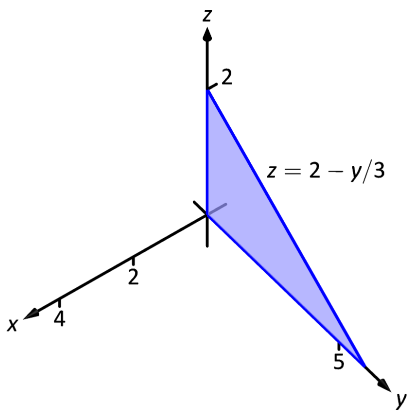
$$V = \int_0^6 \int_0^{3-\frac{1}{2}y} \int_0^{2-\frac{1}{3}y-\frac{2}{3}x} dz\, dx\, dy.$$

The order *dx dy dz*:

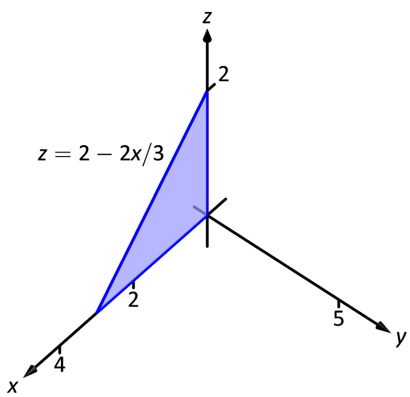
Following our "surface to surface..." strategy, we need to determine the *x-surfaces* that bound our space region. To do so, approach the region "from behind," in the direction of increasing x. The first surface we hit as we enter the region is the y-z plane, defined by $x = 0$. We come out of the region at the plane $z = 2-y/3-2x/3$; solving for x, we have $x = 3-y/2-3z/2$. Thus the bounds on x are: $0 \leq x \leq 3 - y/2 - 3z/2$.

Now collapse the space region onto the y-z plane, as shown in Figure 13.39(a). (Again, we find the equation of the line $z = 2-y/3$ by setting $x = 0$ in the equation $x = 3 - y/2 - 3z/2$.) We need to find bounds on this region with the order

Notes:



(a)



(b)

Figure 13.39: The region D in Example 13.36 is collapsed onto the y-z plane in (a); in (b), the region is collapsed onto the x-z plane.

$dy\,dz$. The *curves* that bound y are $y = 0$ and $y = 6 - 3z$; the *points* that bound z are 0 and 2. Thus the triple integral giving volume is:

$$\begin{array}{c} 0 \le x \le 3 - y/2 - 3z/2 \\ 0 \le y \le 6 - 3z \\ 0 \le z \le 2 \end{array} \quad \Rightarrow \quad \int_0^2 \int_0^{6-3z} \int_0^{3-y/2-3z/2} dx\,dy\,dz.$$

The order $dx\,dz\,dy$:

The x-bounds are the same as the order above. We now consider the triangle in Figure 13.39(a) and describe it with the order $dz\,dy$: $0 \le z \le 2 - y/3$ and $0 \le y \le 6$. Thus the volume is given by:

$$\begin{array}{c} 0 \le x \le 3 - y/2 - 3z/2 \\ 0 \le z \le 2 - y/3 \\ 0 \le y \le 6 \end{array} \quad \Rightarrow \quad \int_0^6 \int_0^{2-y/3} \int_0^{3-y/2-3z/2} dx\,dz\,dy.$$

The order $dy\,dz\,dx$:

We now need to determine the y-surfaces that determine our region. Approaching the space region from "behind" and moving in the direction of increasing y, we first enter the region at $y = 0$, and exit along the plane $z = 2 - y/3 - 2x/3$. Solving for y, this plane has equation $y = 6 - 2x - 3z$. Thus y has bounds $0 \le y \le 6 - 2x - 3z$.

Now collapse the region onto the x-z plane, as shown in Figure 13.39(b). The curves bounding this triangle are $z = 0$ and $z = 2 - 2x/3$; x is bounded by the points $x = 0$ to $x = 3$. Thus the triple integral giving volume is:

$$\begin{array}{c} 0 \le y \le 6 - 2x - 3z \\ 0 \le z \le 2 - 2x/3 \\ 0 \le x \le 3 \end{array} \quad \Rightarrow \quad \int_0^3 \int_0^{2-2x/3} \int_0^{6-2x-3z} dy\,dz\,dx.$$

The order $dy\,dx\,dz$:

The y-bounds are the same as in the order above. We now determine the bounds of the triangle in Figure 13.39(b) using the order $dy\,dx\,dz$. x is bounded by $x = 0$ and $x = 3 - 3z/2$; z is bounded between $z = 0$ and $z = 2$. This leads to the triple integral:

$$\begin{array}{c} 0 \le y \le 6 - 2x - 3z \\ 0 \le x \le 3 - 3z/2 \\ 0 \le z \le 2 \end{array} \quad \Rightarrow \quad \int_0^2 \int_0^{3-3z/2} \int_0^{6-2x-3z} dy\,dx\,dz.$$

Notes:

This problem was long, but hopefully useful, demonstrating how to determine bounds with every order of integration to describe the region D. In practice, we only need 1, but being able to do them all gives us flexibility to choose the order that suits us best.

In the previous example, we collapsed the surface into the x-y, x-z, and y-z planes as we determined the "curve to curve, point to point" bounds of integration. Since the surface was a triangular portion of a plane, this collapsing, or *projecting*, was simple: the *projection* of a straight line in space onto a coordinate plane is a line.

The following example shows us how to do this when dealing with more complicated surfaces and curves.

Example 13.37 Finding the projection of a curve in space onto the coordinate planes

Consider the surfaces $z = 3 - x^2 - y^2$ and $z = 2y$, as shown in Figure 13.40(a). The curve of their intersection is shown, along with the projection of this curve into the coordinate planes, shown dashed. Find the equations of the projections into the coordinate planes.

Solution The two surfaces are $z = 3 - x^2 - y^2$ and $z = 2y$. To find where they intersect, it is natural to set them equal to each other: $3 - x^2 - y^2 = 2y$. This is an implicit relation of x and y that gives all points (x, y) in the x-y plane where the z values of the two surfaces are equal.

We can rewrite this implicit relation by completing the square:

$$3 - x^2 - y^2 = 2y \quad \Rightarrow \quad y^2 + 2y + x^2 = 3 \quad \Rightarrow \quad (y+1)^2 + x^2 = 4.$$

Thus in the x-y plane the projection of the intersection is a circle with radius 2, centered at $(0, -1)$.

To project onto the x-z plane, we do a similar procedure: find the x and z values where the y values on the surface are the same. We start by solving the equation of each surface for y. In this particular case, it works well to actually solve for y^2:

$z = 3 - x^2 - y^2 \quad \Rightarrow \quad y^2 = 3 - x^2 - z$

$z = 2y \quad \Rightarrow \quad y^2 = z^2/4.$

Thus we have (after again completing the square):

$$3 - x^2 - z = z^2/4 \quad \Rightarrow \quad \frac{(z+2)^2}{16} + \frac{x^2}{4} = 1,$$

and ellipse centered at $(0, -2)$ in the x-z plane with a major axis of length 8 and a minor axis of length 4.

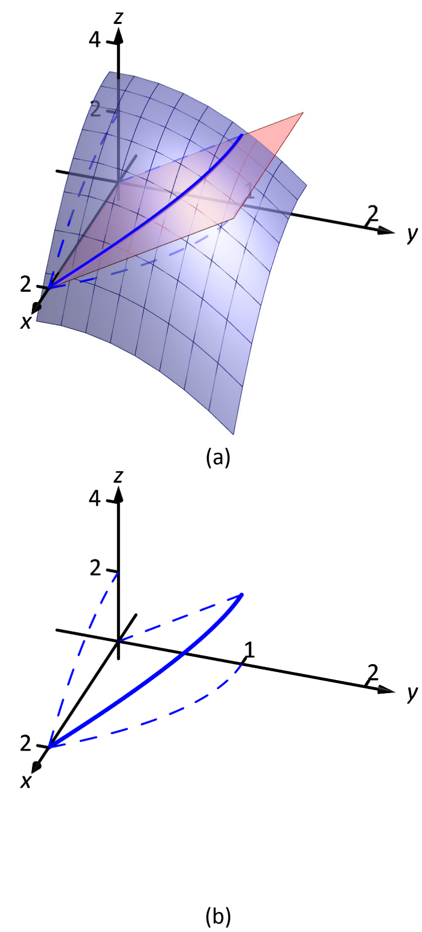


Figure 13.40: Finding the projections of the curve of intersection in Example 13.37.

Notes:

Finally, to project the curve of intersection into the y-z plane, we solve equation for x. Since $z = 2y$ is a cylinder that lacks the variable x, it becomes our equation of the projection in the y-z plane.

All three projections are shown in Figure 13.40(b).

Example 13.38 Finding the volume of a space region with triple integration

Set up the triple integrals that find the volume of the space region D bounded by the surfaces $x^2 + y^2 = 1$, $z = 0$ and $z = -y$, as shown in Figure 13.41(a), with the orders of integration $dz\,dy\,dx$, $dy\,dx\,dz$ and $dx\,dz\,dy$.

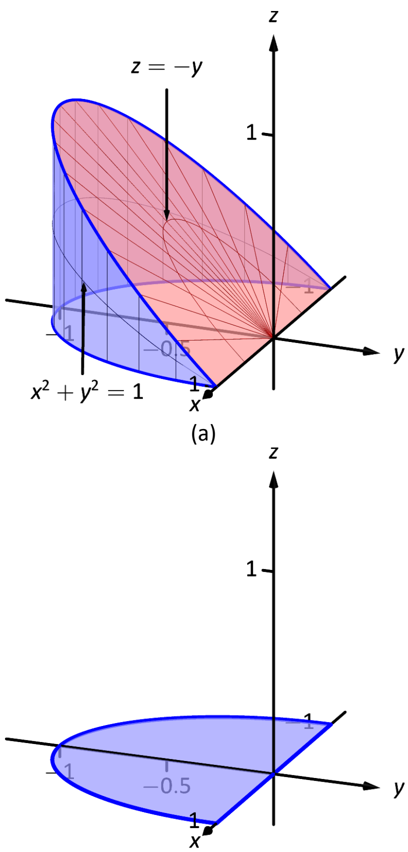


Figure 13.41: The region D in Example 13.38 is shown in (a); in (b), it is collapsed onto the x-y plane.

Solution The order $dz\,dy\,dx$:

The region D is bounded below by the plane $z = 0$ and above by the plane $z = -y$. The cylinder $x^2 + y^2 = 1$ does not offer any bounds in the z-direction, as that surface is parallel to the z-axis. Thus $0 \le z \le -y$.

Collapsing the region into the x-y plane, we get part of the circle with equation $x^2 + y^2 = 1$ as shown in Figure 13.41(b). As a function of x, this half circle has equation $y = -\sqrt{1-x^2}$. Thus y is bounded below by $-\sqrt{1-x^2}$ and above by $y = 0$: $-\sqrt{1-x^2} \le y \le 0$. The x bounds of the half circle are $-1 \le x \le 1$. All together, the bounds of integration and triple integral are as follows:

$$\begin{array}{c} 0 \le z \le -y \\ -\sqrt{1-x^2} \le y \le 0 \\ -1 \le x \le 1 \end{array} \quad \Rightarrow \quad \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0}\int_{0}^{-y} dz\,dy\,dx.$$

We evaluate this triple integral:

$$\begin{aligned} \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0}\int_{0}^{-y} dz\,dy\,dx &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0} \big(-y\big)\,dy\,dx \\ &= \int_{-1}^{1} \Big(-\frac{1}{2}y^2\Big)\Big|_{-\sqrt{1-x^2}}^{0}\,dx \\ &= \int_{-1}^{1} \frac{1}{2}\big(1-x^2\big)\,dx \\ &= \left(\frac{1}{2}\left(x-\frac{1}{3}x^3\right)\right)\Big|_{-1}^{1} \\ &= \frac{2}{3}\text{units}^3. \end{aligned}$$

Notes:

With the order $dy\,dx\,dz$:

The region is bounded "below" in the y-direction by the surface $x^2+y^2=1 \Rightarrow y=-\sqrt{1-x^2}$ and "above" by the surface $y=-z$. Thus the y bounds are $-\sqrt{1-x^2} \leq y \leq -z$.

Collapsing the region onto the x-z plane gives the region shown in Figure 13.42(a); this half circle has equation $x^2+z^2=1$. (We find this curve by solving each surface for y^2, then setting them equal to each other. We have $y^2=1-x^2$ and $y=-z \Rightarrow y^2=z^2$. Thus $x^2+z^2=1$.) It is bounded below by $x=-\sqrt{1-z^2}$ and above by $x=\sqrt{1-z^2}$, where z is bounded by $0 \leq z \leq 1$. All together, we have:

$$\begin{array}{c} -\sqrt{1-x^2} \leq y \leq -z \\ -\sqrt{1-z^2} \leq x \leq \sqrt{1-z^2} \\ 0 \leq z \leq 1 \end{array} \quad \Rightarrow \quad \int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-x^2}}^{-z} dy\,dx\,dz.$$

With the order $dx\,dz\,dy$:

D is bounded below by the surface $x=-\sqrt{1-y^2}$ and above by $\sqrt{1-y^2}$. We then collapse the region onto the y-z plane and get the triangle shown in Figure 13.42(b). (The hypotenuse is the line $z=-y$, just as the plane.) Thus z is bounded by $0 \leq z \leq -y$ and y is bounded by $-1 \leq y \leq 0$. This gives:

$$\begin{array}{c} -\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2} \\ 0 \leq z \leq -y \\ -1 \leq y \leq 0 \end{array} \quad \Rightarrow \quad \int_{-1}^0 \int_0^{-y} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\,dz\,dy.$$

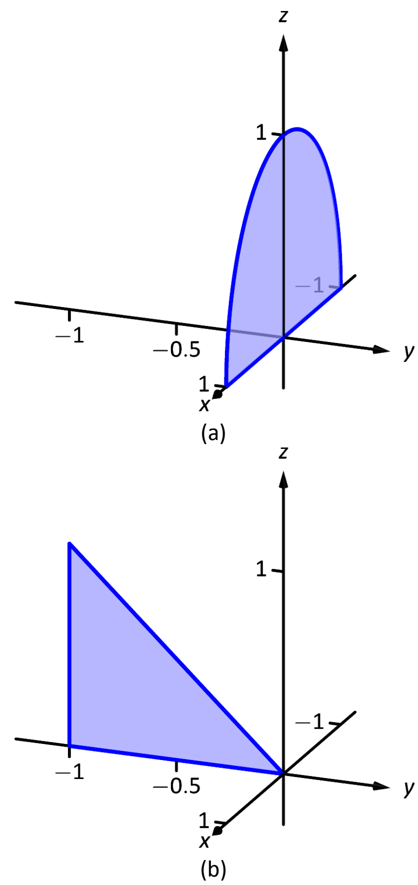


Figure 13.42: The region D in Example 13.38 is shown collapsed onto the x-z plane in (a); in (b), it is collapsed onto the y-z plane.

The following theorem states two things that should make "common sense" to us. First, using the triple integral to find volume of a region D should always return a positive number; we are computing *volume* here, not *signed volume*. Secondly, to compute the volume of a "complicated" region, we could break it up into subregions and compute the volumes of each subregion separately, summing them later to find the total volume.

Notes:

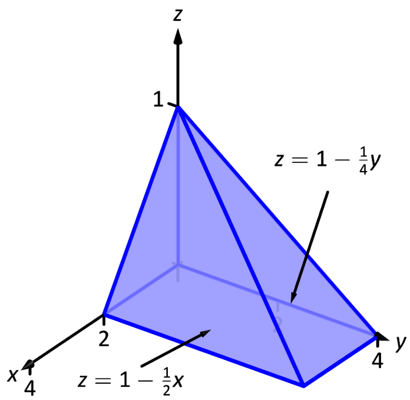
Theorem 131 Properties of Triple Integrals

Let D be a closed, bounded region in space, and let D_1 and D_2 be non-overlapping regions such that $D = D_1 \bigcup D_2$.

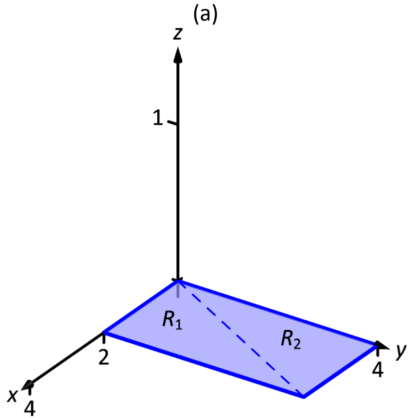
1. $\displaystyle\iiint_D dV \geq 0$

2. $\displaystyle\iiint_D dV = \iiint_{D_1} dV + \iiint_{D_2} dV.$

We use this latter property in the next example.



(a)



(b)

Figure 13.43: The region D in Example 13.39 is shown in (a); in (b), it is collapsed onto the x-y plane.

Example 13.39 Finding the volume of a space region with triple integration

Find the volume of the space region D bounded by the coordinate planes, $z = 1 - x/2$ and $z = 1 - y/4$, as shown in Figure 13.43(a). Set up the triple integrals that find the volume of D in all 6 orders of integration.

Solution Following the bounds–determining strategy of "surface to surface, curve to curve, and point to point," we can see that the most difficult orders of integration are the two in which we integrate with respect to z first, for there are two "upper" surfaces that bound D in the z-direction. So we start by noting that we have

$$0 \leq z \leq 1 - \frac{1}{2}x \quad \text{and} \quad 0 \leq z \leq 1 - \frac{1}{4}y.$$

We now collapse the region D onto the x-y axis, as shown in Figure 13.43(b). The boundary of D, the line from $(0, 0, 1)$ to $(2, 4, 0)$, is shown in part (b) of the figure as a dashed line; it has equation $y = 2x$. (We can recognize this in two ways: one, in collapsing the line from $(0, 0, 1)$ to $(2, 4, 0)$ onto the x-y plane, we simply ignore the z-values, meaning the line now goes from $(0, 0)$ to $(2, 4)$. Secondly, the two surfaces meet where $z = 1 - x/2$ is equal to $z = 1 - y/4$: thus $1 - x/2 = 1 - y/4 \Rightarrow y = 2x$.)

We use the second property of Theorem 131 to state that

$$\iiint_D dV = \iiint_{D_1} dV + \iiint_{D_2} dV,$$

where D_1 and D_2 are the space regions above the plane regions R_1 and R_2, respectively. Thus we can say

$$\iiint_D dV = \iint_{R_1} \left(\int_0^{1-x/2} dz \right) dA + \iint_{R_2} \left(\int_0^{1-y/4} dz \right) dA.$$

Notes:

All that is left is to determine bounds of R_1 and R_2, depending on whether we are integrating with order *dx dy* or *dy dx*. We give the final integrals here, leaving it to the reader to confirm these results.

dz dy dx:

$$\begin{array}{c} 0 \le z \le 1 - x/2 \\ 0 \le y \le 2x \\ 0 \le x \le 2 \end{array} \qquad\qquad \begin{array}{c} 0 \le z \le 1 - y/4 \\ 2x \le y \le 4 \\ 0 \le x \le 2 \end{array}$$

$$\iiint_D dV = \int_0^2 \int_0^{2x} \int_0^{1-x/2} dz\, dy\, dx + \int_0^2 \int_{2x}^4 \int_0^{1-y/4} dz\, dy\, dx$$

dz dx dy:

$$\begin{array}{c} 0 \le z \le 1 - x/2 \\ y/2 \le x \le 2 \\ 0 \le y \le 4 \end{array} \qquad\qquad \begin{array}{c} 0 \le z \le 1 - y/4 \\ 0 \le x \le y/2 \\ 0 \le y \le 4 \end{array}$$

$$\iiint_D dV = \int_0^4 \int_{y/2}^2 \int_0^{1-x/2} dz\, dx\, dy + \int_0^4 \int_0^{y/2} \int_0^{1-y/4} dz\, dx\, dy$$

The remaining four orders of integration do not require a sum of triple integrals. In Figure 13.44 we show D collapsed onto the other two coordinate planes. Using these graphs, we give the final orders of integration here, again leaving it to the reader to confirm these results.

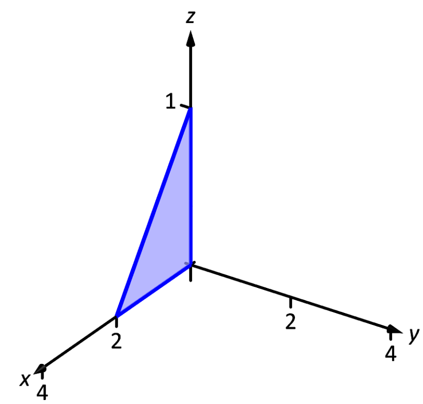
dy dx dz:

$$\begin{array}{c} 0 \le y \le 4 - 4z \\ 0 \le x \le 2 - 2z \\ 0 \le z \le 1 \end{array} \Rightarrow \int_0^1 \int_0^{2-2z} \int_0^{4-4z} dy\, dx\, dz$$

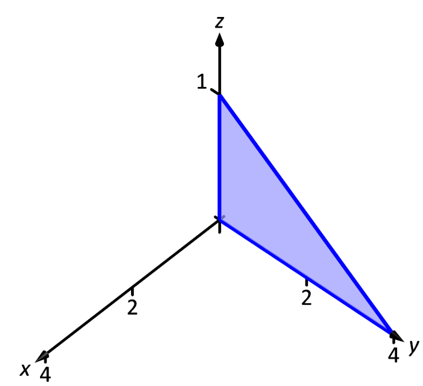
dy dz dx:

$$\begin{array}{c} 0 \le y \le 4 - 4z \\ 0 \le z \le 1 - x/2 \\ 0 \le x \le 2 \end{array} \Rightarrow \int_0^2 \int_0^{1-x/2} \int_0^{4-4z} dy\, dx\, dz$$

Notes:



(a)



(b)

Figure 13.44: The region D in Example 13.39 is shown collapsed onto the x-z plane in (a); in (b), it is collapsed onto the y-z plane.

dx dy dz:

$$\begin{array}{c} 0 \leq x \leq 2-2z \\ 0 \leq y \leq 4-4z \\ 0 \leq z \leq 1 \end{array} \quad \Rightarrow \int_0^1 \int_0^{4-4z} \int_0^{2-2z} dx\, dy\, dz$$

dx dz dy:

$$\begin{array}{c} 0 \leq x \leq 2-2z \\ 0 \leq z \leq 1-y/4 \\ 0 \leq y \leq 4 \end{array} \quad \Rightarrow \int_0^4 \int_0^{1-y/4} \int_0^{2-2z} dx\, dz\, dy$$

We give one more example of finding the volume of a space region.

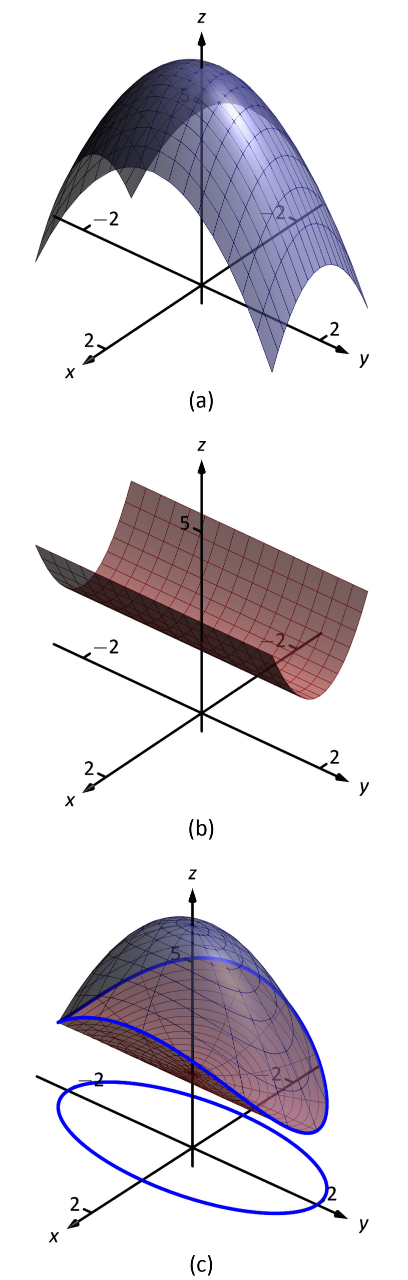


Figure 13.45: The region D is bounded by the surfaces shown in (a) and (b); D is shown in (c).

Example 13.40 Finding the volume of a space region

Set up a triple integral that gives the volume of the space region D bounded by $z = 2x^2+2$ and $z = 6-2x^2-y^2$. These surfaces are plotted in Figure 13.45(a) and (b), respectively; the region D is shown in part (c) of the figure.

Solution The main point of this example is this: integrating with respect to z first is rather straightforward; integrating with respect to x first is not.

The order *dz dy dx*:

The bounds on z are clearly $2x^2+2 \leq z \leq 6-2x^2-y^2$. Collapsing D onto the x-y plane gives the ellipse shown in Figure 13.45(c). The equation of this ellipse is found by setting the two surfaces equal to each other:

$$2x^2+2 = 6-2x^2-y^2 \quad \Rightarrow \quad 4x^2+y^2 = 4 \quad \Rightarrow \quad x^2+\frac{y^2}{4} = 1.$$

We can describe this ellipse with the bounds

$$-\sqrt{4-4x^2} \leq y \leq \sqrt{4-4x^2} \quad \text{and} \quad -1 \leq x \leq 1.$$

Thus we find volume as

$$\begin{array}{c} 2x^2+2 \leq z \leq 6-2x^2-y^2 \\ -\sqrt{4-4x^2} \leq y \leq \sqrt{4-4x^2} \\ -1 \leq x \leq 1 \end{array} \quad \Rightarrow \int_{-1}^1 \int_{-\sqrt{4-4x^2}}^{\sqrt{4-4x^2}} \int_{2x^2+2}^{6-2x^2-y^2} dz\, dy\, dx \ .$$

The order *dy dz dx*:

Notes:

Integrating with respect to y is not too difficult. Since the surface $z = 2x^2 + 2$ is a cylinder whose directrix is the y-axis, it does not create a border for y. The paraboloid $z = 6 - 2x^2 - y^2$ does; solving for y, we get the bounds

$$-\sqrt{6 - 2x^2 - z} \le y \le \sqrt{6 - 2x^2 - z}.$$

Collapsing D onto the x-z axes gives the region shown in Figure 13.46(a); the lower curve is from the cylinder, with equation $z = 2x^2 + 2$. The upper curve is from the paraboloid; with $y = 0$, the curve is $z = 6 - 2x^2$. Thus bounds on z are $2x^2 + 2 \le z \le 6 - 2x^2$; the bounds on x are $-1 \le x \le 1$. Thus we have:

$$\begin{array}{c} -\sqrt{6 - 2x^2 - z} \le y \le \sqrt{6 - 2x^2 - z} \\ 2x^2 + 2 \le z \le 6 - 2x^2 \\ -1 \le x \le 1 \end{array} \quad \Rightarrow \int_{-1}^{1} \int_{2x^2+2}^{6-2x^2} \int_{-\sqrt{6-2x^2-z}}^{\sqrt{6-2x^2-z}} dy\, dz\, dx.$$

The order $dx\, dz\, dy$:

This order takes more effort as D must be split into two subregions. The two surfaces create two sets of upper/lower bounds in terms of x; the cylinder creates bounds

$$-\sqrt{z/2 - 1} \le x \le \sqrt{z/2 - 1}$$

for region D_1 and the paraboloid creates bounds

$$-\sqrt{3 - y^2/2 - z^2/2} \le x \le \sqrt{3 - y^2/2 - z^2/2}$$

for region D_2.

Collapsing D onto the y-z axes gives the regions shown in Figure 13.46(b). We find the equation of the curve $z = 4 - y^2/2$ by noting that the equation of the ellipse seen in Figure 13.45(c) has equation

$$x^2 + y^2/4 = 1 \quad \Rightarrow \quad x = \sqrt{1 - y^2/4}.$$

Substitute this expression for x in either surface equation, $z = 6 - 2x^2 - y^2$ or $z = 2x^2 + 2$. In both cases, we find

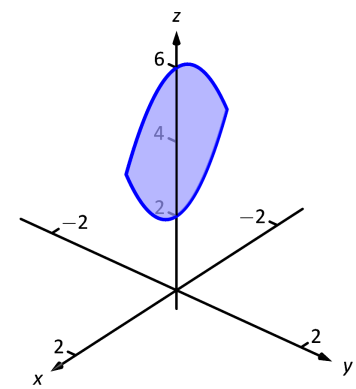
$$z = 4 - \frac{1}{2}y^2.$$

Region R_1, corresponding to D_1, has bounds

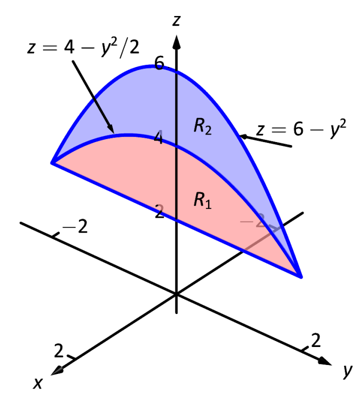
$$2 \le z \le 4 - y^2/2, \quad -2 \le y \le 2$$

and region R_2, corresponding to D_2, has bounds

$$4 - y^2/2 \le z \le 6 - y^2, \quad -2 \le y \le 2.$$



(a)



(b)

Figure 13.46: The region D in Example 13.40 is collapsed onto the x-z plane in (a); in (b), it is collapsed onto the y-z plane.

Notes:

Thus the volume of D is given by:

$$\int_{-2}^{2}\int_{2}^{4-y^2/2}\int_{-\sqrt{z/2-1}}^{\sqrt{z/2-1}} dx\,dz\,dy \;+\; \int_{-2}^{2}\int_{4-y^2/2}^{6-y^2}\int_{-\sqrt{3-y^2/2-z^2/2}}^{\sqrt{3-y^2/2-z^2/2}} dx\,dz\,dy.$$

If all one wanted to do in Example 13.40 was find the volume of the region D, one would have likely stopped at the first integration setup (with order $dz\,dy\,dx$) and computed the volume from there. However, we included the other two methods 1) to show that it could be done, "messy" or not, and 2) because sometimes we "have" to use a less desirable order of integration in order to actually integrate.

Triple Integration and Functions of Three Variables

There are uses for triple integration beyond merely finding volume, just as there are uses for integration beyond "area under the curve." These uses start with understanding how to integrate functions of three variables, which is effectively no different than integrating functions of two variables. This leads us to a definition, followed by an example.

Definition 111 Iterated Integration, (Part II)

Let D be a closed, bounded region in space, over which $g_1(x)$, $g_2(x)$, $f_1(x,y)$, $f_2(x,y)$ and $h(x,y,z)$ are all continuous, and let a and b be real numbers.

The **iterated integral** $\displaystyle\int_a^b\int_{g_1(x)}^{g_2(x)}\int_{f_1(x,y)}^{f_2(x,y)} h(x,y,z)\,dz\,dy\,dx$ is evaluated as

$$\int_a^b\int_{g_1(x)}^{g_2(x)}\int_{f_1(x,y)}^{f_2(x,y)} h(x,y,z)\,dz\,dy\,dx = \int_a^b\int_{g_1(x)}^{g_2(x)}\left(\int_{f_1(x,y)}^{f_2(x,y)} h(x,y,z)\,dz\right)dy\,dx.$$

Example 13.41 Evaluating a triple integral of a function of three variables

Evaluate $\displaystyle\int_0^1\int_{x^2}^{x}\int_{x^2-y}^{2x+3y}\big(xy+2xz\big)\,dz\,dy\,dx.$

Solution We evaluate this integral according to Definition 111.

Notes:

$$\int_0^1 \int_{x^2}^{x} \int_{x^2-y}^{2x+3y} \big(xy+2xz\big)\, dz\, dy\, dx$$

$$= \int_0^1 \int_{x^2}^{x} \left(\int_{x^2-y}^{2x+3y} \big(xy+2xz\big)\, dz\right)\, dy\, dx$$

$$= \int_0^1 \int_{x^2}^{x} \left(\big(xyz+xz^2\big)\Big|_{x^2-y}^{2x+3y}\right)\, dy\, dx$$

$$= \int_0^1 \int_{x^2}^{x} \left(xy(2x+3y)+x(2x+3y)^2 - \Big(xy(x^2-y)+x(x^2-y)^2\Big)\right)\, dy\, dx$$

$$= \int_0^1 \int_{x^2}^{x} \Big(-x^5+x^3y+4x^3+14x^2y+12xy^2\Big)\, dy\, dx.$$

We continue as we have in the past, showing fewer steps.

$$= \int_0^1 \left(-\frac{7}{2}x^7-8x^6-\frac{7}{2}x^5+15x^4\right)\, dx$$

$$= \frac{281}{336} \approx 0.836.$$

We now know *how* to evaluate a triple integral of a function of three variables; we do not yet understand what it *means*. We build up this understanding in a way very similar to how we have understood integration and double integration.

Let $h(x,y,z)$ a continuous function of three variables, defined over some space region D. We can partition D into n rectangular–solid subregions, each with dimensions $\Delta x_i \times \Delta y_i \times \Delta z_i$. Let (x_i, y_i, z_i) be some point in the i^{th} subregion, and consider the product $h(x_i,y_i,z_i)\Delta x_i \Delta y_i \Delta z_i$. It is the product of a function value (that's the $h(x_i,y_i,z_i)$ part) and a small volume ΔV_i (that's the $\Delta x_i \Delta y_i \Delta z_i$ part). One of the simplest understanding of this type of product is when h describes the density of an object, for then $h \times$ volume $=$ mass.

We can sum up all n products over D. Again letting $||\Delta D||$ represent the length of the longest diagonal of the n rectangular solids in the partition, we can take the limit of the sums of products as $||\Delta D|| \to 0$. That is, we can find

$$S = \lim_{||\Delta D|| \to 0} \sum_{i=1}^{n} h(x_i,y_i,z_i)\Delta V_i = \lim_{||\Delta D|| \to 0} \sum_{i=1}^{n} h(x_i,y_i,z_i)\Delta x_i \Delta y_i \Delta z_i.$$

While this limit has lots of interpretations depending on the function h, in the case where h describes density, S is the total mass of the object described by the region D.

Notes:

We now use the above limit to define the **triple integral**, give a theorem that relates triple integrals to iterated iteration, followed by the application of triple integrals to find the centers of mass of solid objects.

Note: In the marginal note on page 819, we showed how the summation of rectangles over a region R in the plane could be viewed as a double sum, leading to the double integral. Likewise, we can view the sum $\sum_{i=1}^{n} h(x_i, y_i, z_i)\Delta x_i \Delta y_i \Delta z_i$ as a triple sum,

$$\sum_{k=1}^{p}\sum_{j=1}^{n}\sum_{i=1}^{m} h(x_i, y_j, z_k)\Delta x_i \Delta y_j \Delta z_k,$$

which we evaluate as

$$\sum_{k=1}^{p}\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m} h(x_i, y_j, z_k)\Delta x_i\right)\Delta y_j\right)\Delta z_k.$$

Here we fix a k value, which establishes the z-height of the rectangular solids on one "level" of all the rectangular solids in the space region D. The inner double summation adds up all the volumes of the rectangular solids on this level, while the outer summation adds up the volumes of each level.
This triple summation understanding leads to the $\iiint_D$ notation of the triple integral, as well as the method of evaluation shown in Theorem 132.

Definition 112 Triple Integral

Let $w = h(x, y, z)$ be a continuous function over a closed, bounded space region D, and let ΔD be any partition of D into n rectangular solids with volume ΔV_i. The **triple integral of** h **over** D is

$$\iiint_D h(x, y, z)\, dV = \lim_{||\Delta D||\to 0}\sum_{i=1}^{n} h(x_i, y_i, z_i)\Delta V_i.$$

The following theorem assures us that the above limit exists for continuous functions h and gives us a method of evaluating the limit.

Theorem 132 Triple Integration (Part II)

Let $w = h(x, y, z)$ be a continuous function over a closed, bounded space region D, and let ΔD be any partition of D into n rectangular solids with volume V_i.

1. The limit $\lim_{||\Delta D||\to 0}\sum_{i=1}^{n} h(x_i, y_i, z_i)\Delta V_i$ exists.
2. If D is defined as the region bounded by the planes $x = a$ and $x = b$, the cylinders $y = g_1(x)$ and $y = g_2(x)$, and the surfaces $z = f_1(x, y)$ and $z = f_2(x, y)$, where $a < b$, $g_1(x) \le g_2(x)$ and $f_1(x, y) \le f_2(x, y)$ on D, then

$$\iiint_D h(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} h(x, y, z)\, dz\, dy\, dx.$$

We now apply triple integration to find the centers of mass of solid objects.

Mass and Center of Mass

One may wish to review Section 13.4 for a reminder of the relevant terms and concepts.

Notes:

Definition 113 Mass, Center of Mass of Solids

Let a solid be represented by a region D in space with variable density function $\delta(x,y,z)$.

1. The **mass** of the object is $M = \iiint_D dm = \iiint_D \delta(x,y,z)\, dV$.
2. The **moment about the x-y plane** is $M_{xy} = \iiint_D z\delta(x,y,z)\, dV$.
3. The **moment about the x-z plane** is $M_{xz} = \iiint_D y\delta(x,y,z)\, dV$.
4. The **moment about the y-z plane** is $M_{yz} = \iiint_D x\delta(x,y,z)\, dV$.
5. The **center of mass** of the object is

$$(\bar{x},\bar{y},\bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right).$$

Example 13.42 Finding the center of mass of a solid

Find the mass and center of mass of the solid represented by the space region bounded by the coordinate planes and $z = 2 - y/3 - 2x/3$, shown in Figure 13.47, with constant density $\delta(x,y,z) = 3\text{gm/cm}^3$. (Note: this space region was used in Example 13.36.)

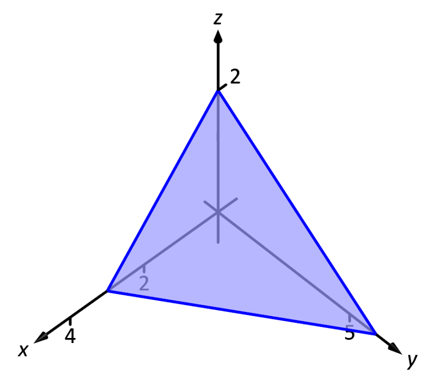


Figure 13.47: Finding the center of mass of this solid in Example 13.42.

Solution We apply Definition 113. In Example 13.36, we found bounds for the order of integration $dz\,dy\,dx$ to be $0 \le z \le 2-y/3-2x/3$, $0 \le y \le 6-2x$ and $0 \le x \le 3$. We find the mass of the object:

$$\begin{aligned} M &= \iiint_D \delta(x,y,z)\, dV \\ &= \int_0^3 \int_0^{6-2x} \int_0^{2-y/3-2x/3} (3)\, dz\, dy\, dx \\ &= 3\int_0^3 \int_0^{6-2x} \int_0^{2-y/3-2x/3} dz\, dy\, dx \\ &= 3(6) = 18\text{gm}. \end{aligned}$$

The evaluation of the triple integral is done in Example 13.36, so we skipped those steps above. Note how the mass of an object with constant density is simply "density$\times$volume."

Notes:

We now find the moments about the planes.

$$\begin{aligned} M_{xy} &= \iiint_D 3z\,dV \\ &= \int_0^3 \int_0^{6-2x} \int_0^{2-y/3-2x/3} \big(3z\big)\,dz\,dy\,dx \\ &= \int_0^3 \int_0^{6-2x} \frac{3}{2}\big(2-y/3-2x/3\big)^2\,dy\,dx \\ &= \int_0^3 -\frac{4}{9}\big(x-3\big)^3\,dx \\ &= 9. \end{aligned}$$

We omit the steps of integrating to find the other moments.

$$\begin{aligned} M_{yz} &= \iiint_D 3x\,dV \\ &= \frac{27}{2}. \\ M_{xz} &= \iiint_D 3y\,dV \\ &= 27. \end{aligned}$$

The center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{27/2}{18}, \frac{27}{18}, \frac{9}{18}\right) = \big(0.75, 1.5, 0.5\big).$$

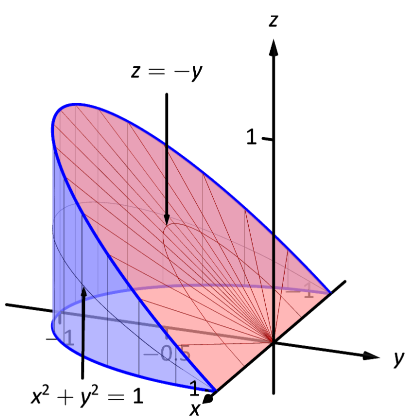


Figure 13.48: Finding the center of mass of this solid in Example 13.43.

Example 13.43 Finding the center of mass of a solid

Find the center of mass of the solid represented by the region bounded by the planes $z = 0$ and $z = -y$ and the cylinder $x^2 + y^2 = 1$, shown in Figure 13.48, with density function $\delta(x, y, z) = 10 + x^2 + 5y - 5z$. (Note: this space region was used in Example 13.38.)

Solution As we start, consider the density function. It is symmetric about the y-z plane, and the farther one moves from this plane, the denser the object is. The symmetry indicates that $\bar{x}$ should be 0.

As one moves away from the origin in the y or z directions, the object becomes less dense, though there is more volume in these regions.

Though none of the integrals needed to compute the center of mass are particularly hard, they do require a number of steps. We emphasize here the

Notes:

importance of knowing how to set up the proper integrals; in complex situations we can appeal to technology for a good approximation, if not the exact answer. We use the order of integration *dz dy dx*, using the bounds found in Example 13.38. (As these are the same for all four triple integrals, we explicitly show the bounds only for *M*.)

$$\begin{aligned}
M &= \iiint_D \big(10 + x^2 + 5y - 5z\big)\, dV \\
&= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{0} \int_{0}^{-y} \big(10 + x^2 + 5y - 5z\big)\, dV \\
&= \frac{64}{5} - \frac{15\pi}{16} \approx 3.855. \\
M_{yz} &= \iiint_D x\big(10 + x^2 + 5y - 5z\big)\, dV \\
&= 0. \\
M_{xz} &= \iiint_D y\big(10 + x^2 + 5y - 5z\big)\, dV \\
&= 2 - \frac{61\pi}{48} \approx -1.99. \\
M_{xy} &= \iiint_D z\big(10 + x^2 + 5y - 5z\big)\, dV \\
&= \frac{61\pi}{96} - \frac{10}{9} \approx 0.885.
\end{aligned}$$

Note how $M_{yz} = 0$, as expected. The center of mass is

$$\big(\overline{x}, \overline{y}, \overline{z}\big) = \left(0, \frac{-1.99}{3.855}, \frac{0.885}{3.855}\right) \approx \big(0, -0.516, 0.230\big).$$

As stated before, there are many uses for triple integration beyond finding volume. When $h(x, y, z)$ describes a rate of change function over some space region *D*, then $\displaystyle\iiint_D h(x, y, z)\, dV$ gives the total change over *D*. Our one specific example of this was computing mass; a density function is simply a "rate of mass change per volume" function. Integrating density gives total mass.

While knowing *how to integrate* is important, it is arguably much more important to know *how to set up* integrals. It takes skill to create a formula that describes a desired quantity; modern technology is very useful in evaluating these formulas quickly and accurately.

Notes:

Exercises 13.6

Terms and Concepts

1. The strategy for establishing bounds for triple integrals is "________ to ________, ________ to ________ and ________ to ________."

2. Give an informal interpretation of what "$\iiint_D \, dV$" means.

3. Give two uses of triple integration.

4. If an object has a constant density δ and a volume V, what is its mass?

Problems

In Exercises 5 – 8, two surfaces $f_1(x,y)$ and $f_2(x,y)$ and a region R in the x,y plane are given. Set up and evaluate the double integral that finds the volume between these surfaces over R.

5. $f_1(x,y) = 8 - x^2 - y^2$, $f_2(x,y) = 2x + y$;
   R is the square with corners $(-1,-1)$ and $(1,1)$.

6. $f_1(x,y) = x^2 + y^2$, $f_2(x,y) = -x^2 - y^2$;
   R is the square with corners $(0,0)$ and $(2,3)$.

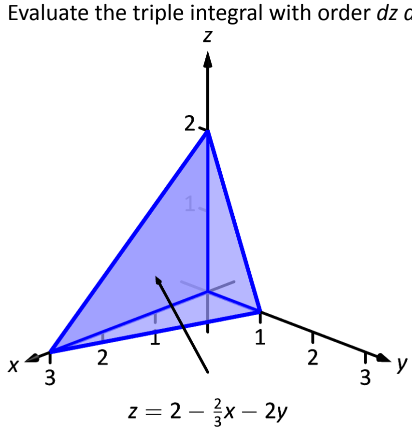
7. $f_1(x,y) = \sin x \cos y$, $f_2(x,y) = \cos x \sin y + 2$;
   R is the triangle with corners $(0,0)$, $(\pi,0)$ and (π,π).

8. $f_1(x,y) = 2x^2 + 2y^2 + 3$, $f_2(x,y) = 6 - x^2 - y^2$;
   R is the circle $x^2 + y^2 = 1$.

In Exercises 9 – 16, a domain D is described by its bounding surfaces, along with a graph. Set up the triple integrals that give the volume of D in all 6 orders of integration, and find the volume of D by evaluating the indicated triple integral.

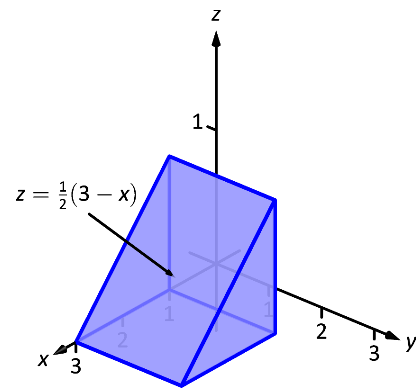
9. D is bounded by the coordinate planes and $z = 2 - 2x/3 - 2y$.

   Evaluate the triple integral with order $dz\,dy\,dx$.



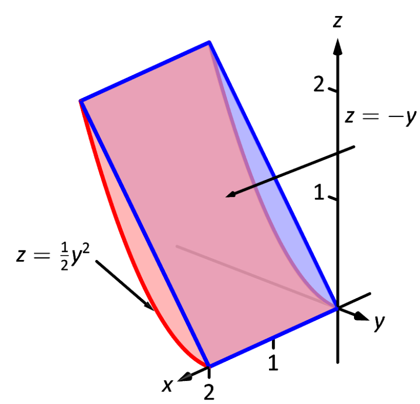
10. D is bounded by the planes $y = 0$, $y = 2$, $x = 1$, $z = 0$ and $z = (3-x)/2$.

    Evaluate the triple integral with order $dx\,dy\,dz$.



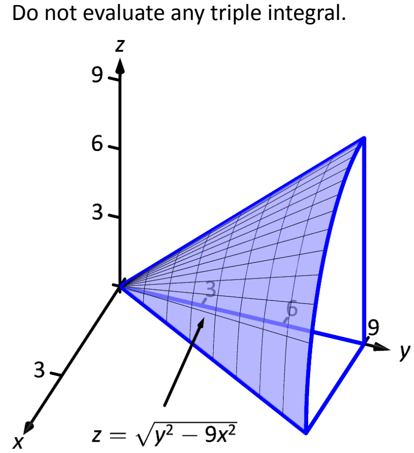
11. D is bounded by the planes $x = 0$, $x = 2$, $z = -y$ and by $z = y^2/2$.

    Evaluate the triple integral with the order $dy\,dz\,dx$.



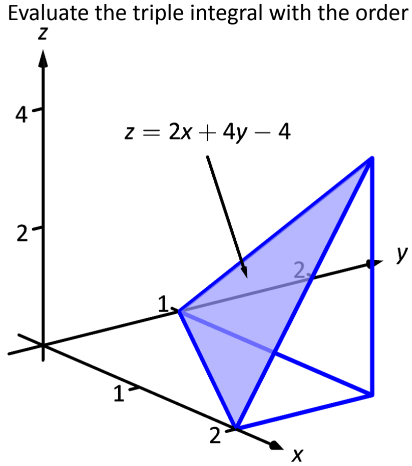
12. D is bounded by the planes $z = 0$, $y = 9$, $x = 0$ and by $z = \sqrt{y^2 - 9x^2}$.

    Do not evaluate any triple integral.



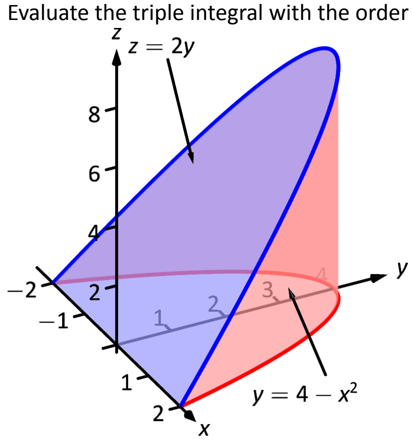
13. D is bounded by the planes $x = 2$, $y = 1$, $z = 0$ and $z = 2x + 4y - 4$.

    Evaluate the triple integral with the order $dx\,dy\,dz$.

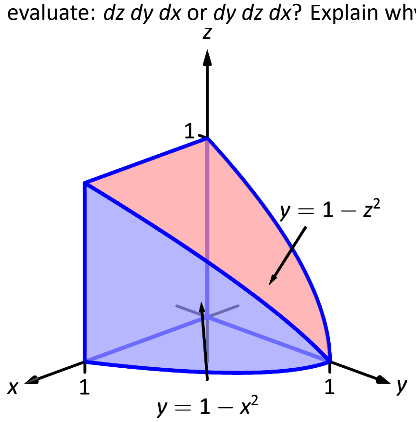


14. D is bounded by the plane $z = 2y$ and by $y = 4 - x^2$.

    Evaluate the triple integral with the order $dz\,dy\,dx$.

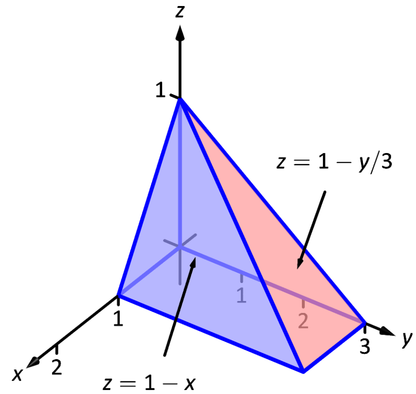


15. D is bounded by the coordinate planes and by $y = 1 - x^2$ and $y = 1 - z^2$.
    Do not evaluate any triple integral. Which order is easier to evaluate: $dz\,dy\,dx$ or $dy\,dz\,dx$? Explain why.



16. D is bounded by the coordinate planes and by $z = 1 - y/3$ and $z = 1 - x$.

    Evaluate the triple integral with order $dx\,dy\,dz$.



In Exercises 17 – 20, evaluate the triple integral.

17. $\displaystyle\int_{-\pi/2}^{\pi/2}\int_0^{\pi}\int_0^{\pi}\big(\cos x\sin y\sin z\big)\,dz\,dy\,dx$

18. $\displaystyle\int_0^1\int_0^x\int_0^{x+y}\big(x+y+z\big)\,dz\,dy\,dx$

19. $\displaystyle\int_0^{\pi}\int_0^1\int_0^z\big(\sin(yz)\big)\,dx\,dy\,dz$

20. $\displaystyle\int_{\pi}^{\pi^2}\int_x^{x^3}\int_{-y^2}^{y^2}\left(z\frac{x^2y+y^2x}{e^{x^2+y^2}}\right)\,dz\,dy\,dx$

In Exercises 21 – 24, find the center of mass of the solid represented by the indicated space region D with density function $\delta(x, y, z)$.

21. D is bounded by the coordinate planes and $z = 2 - 2x/3 - 2y$; $\quad \delta(x, y, z) = 10\text{gm/cm}^3$.
    (Note: this is the same region as used in Exercise 9.)

22. D is bounded by the planes $y = 0$, $y = 2$, $x = 1$, $z = 0$ and $z = (3 - x)/2$; $\quad \delta(x, y, z) = 2\text{gm/cm}^3$.
    (Note: this is the same region as used in Exercise 10.)

23. D is bounded by the planes $x = 2$, $y = 1$, $z = 0$ and $z = 2x + 4y - 4$; $\quad \delta(x, y, z) = x^2\text{lb/in}^3$.
    (Note: this is the same region as used in Exercise 13.)

24. D is bounded by the plane $z = 2y$ and by $y = 4 - x^2$. $\delta(x, y, z) = y^2\text{lb/in}^3$.
    (Note: this is the same region as used in Exercise 14.)

13.7 Cylindrical and Spherical Integration

Triple Integration with Cylindrical Coordinates

Cylindrical coordinates are useful for describing many solids that are symmetric around an axis. The solid is three dimensional, and so we use the coordinates (r, θ, z) for points in this solid. In short, the radius r and angle θ are the same as in polar coordinates, with r being the distance out from axis of symmetry and θ being the angle around the axis, with the positive x-axis being the direction of $\theta = 0$ as usual. The coordinate z is the same as in the Cartesian system - the vertical distance up the axis, with $z = 0$ being the xy-plane. See Figure 13.49.

The conversions between cylindrical and Cartesian coordinates are given as follows, which follows from our previous study of polar coordinates.

Key Idea 59 Converting between cylindrical and Cartesian coordinates

Given a point (r, θ, z) in cylindrical coordinates, its Cartesian coordinates are

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z.$$

Given a point (x, y, z) in Cartesian coordinates, we use

$$r^2 = x^2 + y^2,\ \tan\theta = \frac{y}{x}\ \text{(for } x \neq 0),\ z = z$$

to convert to cylindrical coordinates.

Example 13.44

Write the point $(\sqrt{3}, 1, -1)$ in cylindrical coordinates.

Solution Since we are given the Cartesian coordinates, we use the above conversion formulas to convert this point to cylindrical coordinates. Since $r^2 = x^2 + y^2$, we arrive at

$$r = \sqrt{3+1} = 2.$$

Since $\tan\theta = \dfrac{1}{\sqrt{3}}$, it follows that $\theta = \dfrac{\pi}{6}$ as the point is in the first quadrant of the xy-plane. Therefore the cylindrical coordinates of this point are $\left(2, \frac{\pi}{6}, -1\right)$,

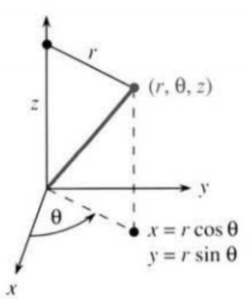


Figure 13.49: Cylindrical coordinates for a point

Notes:

since the z-coordinate remains the same.

As mentioned, certain surfaces and solids are conveniently described using cylindrical coordinates. For example, consider the solid half-cylinder D of radius 1 centered on the z-axis with the bottom on the xy-plane and the top at $z = 3$. If the half-cylinder is above the first two quadrants of the xy-plane, then any point in the cylinder has a θ coordinate between 0 and π. Therefore the coordinates that describe this solid are

$$0 \leq r \leq 1,\ 0 \leq \theta \leq \pi,\ 0 \leq z \leq 3.$$

This half-cylinder is shown in Figure 13.50.

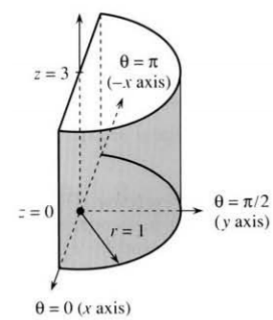


Figure 13.50: Half cylinder

Surfaces and solids described using Cartesian coordinates can also be written using cylindrical coordinates, and vice versa, using our conversion formulas above.

Example 13.45
Consider the surface described by $z = r^2$ in cylindrical coordinates. Write the equation for this surface in Cartesian coordinates and identify the surface.

SOLUTION We know that $r^2 = x^2 + y^2$, and so we can write $z = x^2 + y^2$. This is a paraboloid with vertex at the origin opening up. See Figure 13.51.

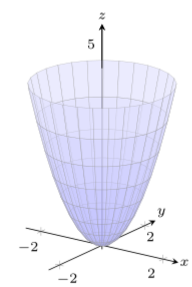


Figure 13.51: $z = r^2$

When integrating over a solid D described with cylindrical coordinates, one would evaluate a triple integral $\iiint_D f(x,y,z)dV$, but using dr, $d\theta$, and dz instead of dx, dy, and dz. The differential of volume dV would be

$$dV = r\,dr\,d\theta\,dz$$

since we are working with the same two coordinates r and θ from polar coordinates and the third coordinate is the same as in the Cartesian system. This is the volume of the curved box in Figure 13.52. In other words, the integral above becomes

$$\iiint_D f(x,y,z)\ dV = \iiint_D f(r\cos\theta, r\sin\theta, z)\ r\,dr\,d\theta\,dz$$

where each occurrence of x and y in the integrand has been converted to cylindrical coordinates. The bounds on each integral would be the corresponding bounds on r, θ, and z that describe the solid D, and we can integrate in any appropriate order.

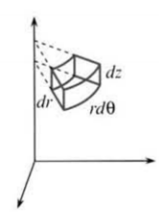


Figure 13.52: Small volume $r\,dr\,d\theta\,dz$

Notes:

Example 13.46

Compute the volume of the half-cylinder D described previously, for $0 \leq r \leq 1$, $0 \leq \theta \leq \pi$, $0 \leq z \leq 3$.

Solution Since we are computing the volume of D, the integrand is $f(x, y, z) = 1$ and so

$$V = \iiint_D 1\, dV = \iiint_D r\, dr\, d\theta\, dz.$$

Using our bounds on r, θ, and z that describe D as our bounds of integration, we get a volume of

$$\int_0^3 \int_0^\pi \int_0^1 r\, dr\, d\theta\, dz = \int_0^3 \int_0^\pi \frac{1}{2}r^2\Big|_0^1\, d\theta\, dz = \int_0^3 \int_0^\pi \frac{1}{2}\, d\theta\, dz$$

which evaluates to a volume of $\frac{3\pi}{2}$ cubic units.

Example 13.47

The surface $z = 1 - r$ encloses a conical solid between itself and the xy-plane. Find the volume of this solid as seen in Figure 13.53.

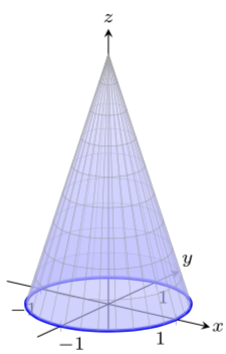


Figure 13.53: $z = 1 - r$

Solution The cone $z = 1 - r$ intersects the xy-plane ($z = 0$) at a radius of $r = 1$. This puts the bounds on r as $0 \leq r \leq 1$, and clearly the bounds on θ would be $0 \leq \theta \leq 2\pi$ since the solid is the full cone. Since the solid is not a cylinder, at least one of the bounds on z must be non-constant. At any radius r, the height of the solid is bounded between the xy-plane and the surface $z = 1 - r$. Therefore the bounds on z are $0 \leq z \leq 1 - r$, and we will integrate with respect to dz first. Therefore the volume is

$$V = \int_0^{2\pi} \int_0^1 \int_0^{1-r} r\, dz\, dr\, d\theta = 2\pi \int_0^1 (1 - r)r\, dr$$

which evaluates to a volume of $\dfrac{\pi}{3}$ cubic units.

Example 13.48

A solid D is the region inside the cylinder $x^2 + y^2 = 1$, below the plane $z = 2$ and above the paraboloid $z = 1 - x^2 - y^2$, with all distances in meters. See Figure 13.55. If the density of this solid is

$$\delta(x, y, z) = \sqrt{x^2 + y^2}$$

Notes:

kilograms per cubic meter, determine the mass of this solid.

Solution As this solid is a cylinder with a paraboloid cut out of the bottom, it makes sense to use cylindrical coordinates here, where the mass is the integral of the density over the solid. The z-coordinates of this solid solid are bounded below by $1 - x^2 - y^2$ or $1 - r^2$, and bounded above by $z = 2$. Therefore the solid D is described by

$$0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi,\ 1 - x^2 - y^2 = 1 - r^2 \leq z \leq 2$$

and the density function, when converted to cylindrical coordinates, is

$$\delta(r, \theta, z) = \sqrt{r^2} = r.$$

Integrating using the order $dz\, d\theta\, dr$ and the additional factor of r for cylindrical coordinates yields a mass of

$$\int_0^1 \int_0^{2\pi} \int_{1-r^2}^2 r^2\, dz\, d\theta\, dr = 2\pi \int_0^1 r^2 + r^4\, dr = 2\pi \left(\frac{1}{12}\right) = \frac{16\pi}{15}$$

kilograms.

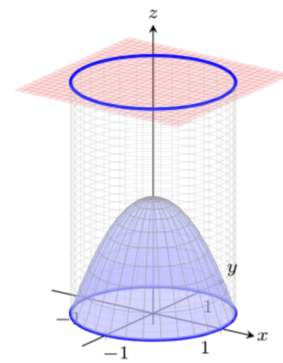


Figure 13.55: Solid D bounded above by $z = 2$, below by $z = 1 - x^2 - y^2$, laterally by $x^2 + y^2 = 1$

Triple Integration with Spherical Coordinates

Objects bounded by spheres or cones are more easily described using a different coordinate system called spherical coordinates. The Earth, for example, is a solid sphere (or near enough). On its surface we use two coordinates - latitude and longitude. To dig inward or fly outward, there is a third coordinate, the distance ρ from the center. This Greek letter *rho* replaces radius r to avoid confusion with cylindrical coordinates. Where r is measured from the z-axis, ρ is measured directly from the origin. Thus for any point (x, y, z),

$$\rho^2 = x^2 + y^2 + z^2$$

which is the square of the distance between the origin and the point. The angle θ is the same as in cylindrical coordinates, and it goes from 0 to 2π on a full sphere with $\theta = 0$ pointing in the direction of the positive x-axis. It is the longitude, which increases as you travel east around the Equator. The angle ϕ is new, however. It equals 0 at the North Pole and π (not 2π) at the South Pole. It is measured down from the z-axis. The Equator, for example, has a latitude of 0 degrees but has angle $\phi = \frac{\pi}{2}$ instead. See Figure 13.54. (The angle ρ is π minus the latitude (in radians) on Earth.)

Notes:

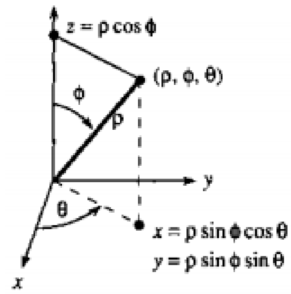


Figure 13.54: Spherical coordinates (ρ, θ, ϕ)

The spherical coordinates of a point (x, y, z) are given by the ordered triple (ρ, θ, ϕ) where ρ, θ, and ϕ can be restricted to $\rho \geq 0$, $0 \leq \theta \leq 2\pi$, and $0 \leq \phi \leq \pi$. The relationship between spherical and Cartesian coordinates is illustrated by Figure 13.54. From the triangles, we have

$$z = \rho \cos \phi \text{ and } r = \rho \sin \phi$$

But we know that $x = r \cos \theta$ and $y = r \sin \theta$ from before, so we end up with the following conversion equations.

Key Idea 60 Converting between spherical and Cartesian coordinates

Given a point (ρ, θ, ϕ) in spherical coordinates, its Cartesian coordinates are

$$x = \rho \sin \phi \cos \theta, \; y = \rho \sin \phi \sin \theta, \; z = \rho \cos \phi.$$

Given a point (x, y, z) in Cartesian coordinates, we use

- $\rho^2 = x^2 + y^2 + z^2$
- $\tan \theta = \dfrac{y}{x}$ for $x \neq 0$
- $\cos \phi = \dfrac{z}{\rho} = \dfrac{z}{\sqrt{x^2 + y^2 + z^2}}$ except at the origin (where ϕ can have any value)

to convert to spherical coordinates.

Notes:

Example 13.49

The point $\left(3, \frac{\pi}{6}, \frac{2\pi}{3}\right)$ is given in spherical coordinates. What are the Cartesian coordinates representing this point?

Solution Using the conversion equations above, we have

$$x = \rho \sin\phi \cos\theta = 3 \sin\left(\frac{2\pi}{3}\right) \cos\left(\frac{\pi}{6}\right) = 3\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{9}{4}$$
$$y = \rho \sin\phi \sin\theta = 3 \sin\left(\frac{2\pi}{3}\right) \sin\left(\frac{\pi}{6}\right) = 3\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = \frac{3\sqrt{3}}{4}$$
$$z = \rho \cos\phi = 3 \cos\left(\frac{2\pi}{3}\right) = 3\left(\frac{1}{2}\right) = \frac{3}{2}$$

which corresponds to the Cartesian coordinates of $\left(\frac{9}{4}, \frac{3\sqrt{3}}{4}, \frac{3}{2}\right)$.

Surfaces like spheres and cones can be easily described in spherical coordinates. For example, $\rho = R$ describes the surface of a sphere of radius R, while $\phi = \frac{\pi}{3}$ describes a conical surface opening up at a 60 degree angle from the z-axis (or a 30 degree angle from the xy-plane). Some solids, such as solid cones or parts of solid spheres, have bounds which are easily described in spherical coordinates. For example, the bounds

$$0 \le \rho \le 1, \ \ 0 \le \theta \le 2\pi, \ \ \frac{\pi}{2} \le \phi \le \pi$$

describes the solid lower hemisphere of radius 1, while the bounds

$$0 \le \rho \le 1, \ \ 0 \le \theta \le 2\pi, \ \ 0 \le \phi \le \frac{\pi}{4}$$

describes a conical solid opening up at a 45 degree angle with a spherical top.

If one is to integrate over a solid D best described in spherical coordinates, such as when finding its volume, one can think of the solid as being divided into spherical boxes or spherical wedges of thickness $d\rho$ and angles $d\theta$ and $d\phi$. The dimensions of this spherical box, as seen in Figure 13.56, are $d\rho$, $\rho\, d\phi$ (arc of a circle of radius ρ and angle $d\phi$), and $\rho \sin\phi\, d\theta$ (arc of a circle with radius $\rho \sin\phi$ and angle $d\theta$). Therefore the approximate volume of the spherical box is

$$dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi.$$

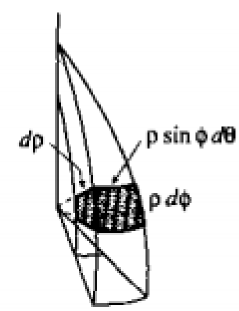


Figure 13.56: Spherical box

Notes:

To integrate a function $f(x,y,z)$ over a solid D described using spherical coordinates, one would therefore compute

$$\iiint_D f(x,y,z)\ dA = \iiint_D f(\rho,\theta,\phi)\ \rho^2\ \sin\phi\ d\rho\ d\theta\ d\phi$$

where the bounds on ρ, θ, and ϕ are as appropriate to describe the solid D. As a first example of this, we will verify the geometric formula for the volume of a sphere. (Note that we have previously done this two other ways before, in Exercise 22 of Section 7.2 using the disk method, and then in Section 13.3 using a polar double integral.)

Example 13.50

Determine the volume of a sphere of radius R.

Solution Let D be the sphere of radius R centered at the origin. Then we can describe D using spherical coordinates as simply all points (ρ,θ,ϕ) where

$$0 \le \rho \le R,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi.$$

Therefore the volume of this ball is equal to

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2 \sin\phi d\rho d\phi d\theta = \left(\frac{1}{3}\rho^3\Big|_0^R\right)\left(-\cos\phi\big|_0^{\pi}\right)\left(\theta\big|_0^{2\pi}\right) = \frac{1}{3}R^3(2)(2\pi)$$

which is equal to the familiar formula for the volume of a sphere being $\dfrac{4}{3}\pi R^3$.

Example 13.51

Evaluate the integral

$$\iiint_D x^2+y^2+z^2\ dV$$

where D is the solid that lies above the cone $z=\sqrt{x^2+y^2}$ but below the sphere of radius $\frac{1}{2}$ centered at $\left(0,0,\frac{1}{2}\right)$.

Solution The cone $z=\sqrt{x^2+y^2}$ is the cone pointing upward from the origin at an angle of $\frac{\pi}{4}$ from the xy-plane. The solid D therefore consists of a half-spherical top with a conical bottom, with the point of the cone at the origin, meeting the spherical top at the horizontal equator of the sphere.

To describe this solid in spherical coordinates, note that the sphere of radius $\frac{1}{2}$ centered at the point $\left(0,0,\frac{1}{2}\right)$ can be written as

$$x^2+y^2+\left(z-\frac{1}{2}\right)^2=\frac{1}{4}$$

Notes:

which is equivalent to $x^2+y^2+z^2=z$. In spherical coordinates, this says $\rho^2=\rho\cos\phi$, or $\rho=\cos\phi$. As for the cone, the equation $\phi=\frac{\pi}{4}$ describes this surface. Therefore D can be described as all points (ρ,θ,ϕ) with $0\leq\rho\leq\cos\phi$, $0\leq\theta\leq 2\pi$, and $0\leq\phi\leq\frac{\pi}{4}$.

Lastly, to integrate we need to write the integrand $x^2+y^2+z^2$ as ρ^2 and include the spherical integration factor $\rho^2\sin\phi$. Therefore

$$\begin{aligned}
\iiint_E x^2+y^2+z^2\,dV &= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi}\rho^4\sin\phi\,d\rho\,d\phi\,d\theta\\
&= 2\pi\int_0^{\pi/4}\sin\phi\left.\frac{1}{5}\rho^5\right|_0^{\cos\phi}d\phi\\
&= \frac{2}{5}\pi\int_0^{\pi/4}\cos^5\phi\sin\phi\,d\phi\\
&= \frac{2}{5}\pi\left(-\frac{1}{6}\cos^6\phi\Big|_0^{\pi/4}\right)\\
&= \frac{2}{5}\pi\left(\frac{1}{8}-1\right)=\frac{7\pi}{120}
\end{aligned}$$

This chapter investigated the natural follow–on to partial derivatives: iterated integration. We learned how to use the bounds of a double integral to describe a region in the plane using both rectangular and polar coordinates, then later expanded to use the bounds of a triple integral to describe a region in space. We used double integrals to find volumes under surfaces, surface area, and the center of mass of lamina; we used triple integrals as an alternate method of finding volumes of space regions and also to find the center of mass of a region in space.

Integration does not stop here. We could continue to iterate our integrals, next investigating "quadruple integrals" whose bounds describe a region in 4–dimensional space (which are very hard to visualize). We can also look back to "regular" integration where we found the area under a curve in the plane. A natural analogue to this is finding the "area under a curve," where the curve is in space, not in a plane. These are just two of many avenues to explore under the heading of "integration."

Notes:

Exercises 13.7

Terms and Concepts

1. Fill in the blank: To describe a full cylinder, the bounds on θ would go from $\theta = 0$ to $\theta =$ ________.

2. Fill in the blank: To describe a full sphere, the bounds on θ would go from $\theta = 0$ to $\theta = 2\pi$ and ϕ would go from $\phi = 0$ to $\phi =$ ________.

3. T/F: In spherical coordinates, the direction of $\phi = 0$ is perpendicular to the z-axis.

4. T/F: In spherical coordinates, the equation $\rho = 4$ describes the surface of a sphere.

5. T/F: In cylindrical or spherical coordinates, the equation $\theta = \dfrac{\pi}{4}$ describes a cone.

Problems

In Exercises 6 – 8, the Cartesian coordinates of a point are given. Write both the cylindrical (r, θ, z) and spherical (ρ, θ, ϕ) coordinates of this point.

6. $(1, 0, 0)$

7. $(0, -2, 0)$

8. $(1, 1, -2)$

In Exercises 9 – 11, the Cartesian coordinates of a point are given. Write both the cylindrical (r, θ, z) and spherical (ρ, θ, ϕ) coordinates of this point.

9. $(1, 0, 0)$

10. $(0, -2, 0)$

11. $(1, 1, -2)$

In Exercises 12 – 14, an integral equal to the volume of a solid is given. Describe the solid.

12. $\int_0^{2\pi}\int_0^1\int_0^4 r\, dz\, dr\, d\theta$

13. $\int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

14. $\int_0^{2\pi}\int_{3\pi/4}^{\pi}\int_0^1 \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

In Exercises 15 – 18, evaluate the integral using the given described solid D using either cylindrical or spherical coordinates as appropriate.

15. $\iiint_D \sqrt{x^2+y^2}\, dV$ where D is the cylinder with axis along the z-axis and radius 4 between $z = 0$ and $z = 1$.

16. $\iiint_D 1\ dV$ where D is the solid trapped between the paraboloid $z = x^2 + y^2$ and the plane $z = 9$.

17. $\iiint_D x^2+y^2+z^2\, dV$ where D is the solid sphere of radius 2 centered at the origin.

18. $\iiint_D \sqrt{x^2+y^2+z^2}\, dV$ where D is the solid between the spheres $x^2+y^2+z^2 = 1$ and $x^2+y^2+z^2 = 4$ and above the cone $z = \sqrt{x^2+y^2}$.

In Exercises 19 – 20, evaluate the integral by first converting to cylindrical or spherical coordinates.

19. $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{\sqrt{x^2+y^2}}^{1} xz\, dz\, dx\, dy$

20. $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} 1\, dz\, dy\, dx$

In Exercises 21 – 24, determine the limit by converting to spherical coordinates, similar to what was done for polar coordinates in Section 12.2.

21. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{z^2}{\sqrt{x^2+y^2}}$

22. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xyz - 3x^2 - 3y^2 - 3z^2}{x^2+y^2+z^2}$

23. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{7xyz}{x^2+y^2+2z^2}$

24. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{e^z - 1}{\sqrt{x^2+y^2+2z^2}}$

14: VECTOR CALCULUS

14.1 Vector Fields

For an ordinary scalar function, the input is a number x and the output is a number $f(x)$. For a vector field (or vector function), the input is a point (x, y) and the output is a two-dimensional vector $F(x, y)$. There is a "field" of vectors, one at every point. In three dimensions the input point is (x, y, z) and the output vector $\vec{F}$ has three components.

Let R be a region in the xy-plane. A **vector field** $\vec{F}$ assigns to every point (x, y) in $\mathbb{R}$ a vector $\vec{F}(x, y)$ with two components:

$$\vec{F}(x, y) = M(x, y)\vec{i} + N(x, y)\vec{j}.$$

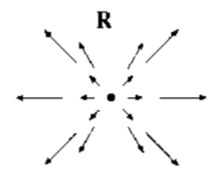


Figure 14.1: The vector field **R**

This plane vector field involves two functions of two variables. They are the components M and N, which vary from point to point. A vector has fixed components; a vector field has varying components. A three-dimensional vector field has components $M(x, y, z)$, $N(x, y, z)$, and $P(x, y, z)$. Then the vectors are

$$\vec{F}(x, y, z) = M(z, y, z)\vec{i} + N(x, y, z)\vec{j} + P(x, y, z)\vec{k}.$$

Example 14.1
Consider the position vector field $\vec{R} = x\vec{i} + y\vec{j}$.

SOLUTION Its components are $M(x, y) = x$ and $N(x, y) = y$. The vectors grow larger as we leave the origin - see Figure 14.1. Their direction is outward and their length is $\mid R \mid = \sqrt{x^2 + y^2} = r$. This type of vector field is often referred to as a radial field.

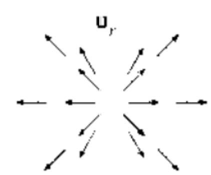


Figure 14.2: The vector field **R**

Example 14.2
Now suppose the vector field $\vec{F}$ is given by $\vec{F} = \dfrac{\vec{R}}{r}$.

SOLUTION Then

$$\vec{F}(x, y) = \frac{x}{\sqrt{x^2 + y^2}}\vec{i} + \frac{y}{\sqrt{x^2 + y^2}}\vec{j}$$

consists of unit vectors pointing outward. We obtain this vector field by dividing $\vec{R} = x\vec{i} + y\vec{j}$ by its length at every point except the origin. See Figure 14.2. Figure 14.3 shows a third vector field $\dfrac{\vec{R}}{r^2}$, where the length of each vector is $1/r$.

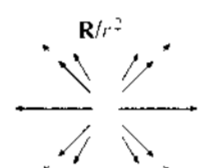


Figure 14.3: The vector field **R**

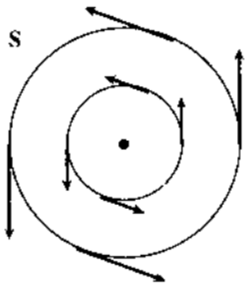


Figure 14.4: The vector field **S**

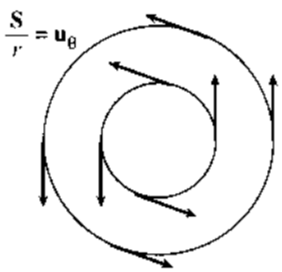


Figure 14.5: The vector field **S**$/r$

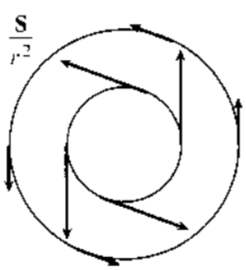


Figure 14.6: The vector field **S**$/r^2$

Example 14.3
A spin field or rotation field goes around the origin instead of away from it.

Solution Such a field in two dimensions is given by

$$\vec{S}(x, y) = -y\vec{i} + x\vec{j}$$

as given in Figure 14.4 where the components are $M(x, y) = -y$ and $N(x, y) = x$. Each vector $\vec{S}(x, y)$ has length

$$|\vec{S}| = \sqrt{(-y)^2 + x^2} = r$$

Note, however, that $\vec{S}$ is perpendicular to $\vec{R}$ as defined above, since their dot product is zero: $\vec{S} \cdot \vec{R} = (-y)(x) + (x)(y) = 0$. The spin fields $\vec{S}/r$ and $\vec{S}/r^2$ have lengths 1 and $1/r$, respectively. See Figure 14.5 and Figure 14.6. Notice the blank at $(0, 0)$ in these second two spin fields, where these fields are not defined.

Gradient Vector Fields

A **gradient field** in two dimensions starts with an ordinary function $f(x, y)$ of two variables. The corresponding vector field $\vec{F}$ is the gradient $\nabla f(x, y)$ given by

$$\vec{F}(x, y) = \nabla f(x, y) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}.$$

This vector field ∇f is everywhere perpendicular to the level curves $f(x, y) = c$. The length $|\nabla f|$ tells how fast f is changing (in the direction it changes fastest). A similar construction of a gradient vector field can be done in three dimensions for a function $f(x, y, z)$ of three variables.

Example 14.4
Consider the function $f(x, y) = x^2 y$. Determine the gradient vector field associated to this function.

Solution The partial derivatives of $f(x, y)$ are

$$\frac{\partial f}{\partial x} = 2xy \text{ and } \frac{\partial f}{\partial y} = x^2$$

and so the gradient vector field corresponding to $f(x, y)$ is given by

$$\vec{F}(x, y) = \nabla f(x, y) = 2xy\vec{i} + x^2\vec{j}$$

Notes:

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This last type of vector field brings up a question. For any given vector field $\vec{F}(x,y)$, is it the gradient field of some function $f(x,y)$? If so, what is the function $f(x,y)$?

Example 14.5
Show that the radial field

$$\vec{R}(x,y) = x\vec{i} + y\vec{j}$$

discussed above is the gradient field of some function $f(x,y)$.

Solution We need a function $f(x,y)$ so that $\dfrac{\partial f}{\partial x} = x$ and $\dfrac{\partial f}{\partial y} = y$. Note that $x^2 + y^2$ has partial derivatives of $2x$ and $2y$, respectively. Therefore $f(x,y) = \dfrac{1}{2}(x^2 + y^2)$ will have a gradient equal to $\vec{R}(x,y)$.

It turns out that the spin fields $\vec{S}(x,y)$ and $\vec{S}(x,y)/r$ discussed above are not the gradients of any function $f(x,y)$, while $\vec{S}(x,y)/r^2$ is. A major goal of this chapter is to recognize gradient fields by a simple test. A vector field $\vec{F}$ which is the gradient field of some function $f(x,y)$ is called **conservative**. The function f is the **potential function** for the gradient field. This terminology, and the next examples, come from physics and engineering.

Example 14.6
Suppose a fluid is moving steadily down a pipe, or that a river flows smoothly with no waterfalls. Let

$$\vec{V}(x,y,z) = v_x(x,y,z)\vec{i} + v_y(x,y,z)\vec{j} + v_z(x,y,z)$$

be the velocity at the point (x,y,z).

Solution This vector field $\vec{V}$ gives the direction of flow and speed of flow at every point, with v_x, v_y, and v_z being the components of the velocity in the direction of x, y and z-axes. The speed of the fluid at any point is the length

$$\mid \vec{V} \mid = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

One could also consider velocity fields for wind or for a moving gas.

Example 14.7
Consider the gravitational force $|\vec{F}|$ between two objects of mass m and M given

---

Notes:

by

$$|\vec{F}| = \frac{mMG}{r^2}$$

where G is the gravitational constant. ($G \approx 6.672 \times 10^{-11}$ when distance is in meters, mass is in kilograms, and time is in seconds.) Assume that the object with mass M is at the origin in $\mathbb{R}^3$.

Solution As a vector field we can write

$$\vec{F} = \frac{mMG}{r^2}\left(-\frac{(x,y,z)}{r}\right) = -\frac{mMG}{r^3}\left(\vec{i}+\vec{j}+\vec{k}\right)$$

since gravitational force is directed inward toward the origin. Like all radial fields, gravity is a gradient field. It comes from the potential function

$$f(x,y,z) = \frac{mMG}{r} = \frac{mMG}{\sqrt{x^2+y^2+z^2}}$$

which one can verify. Note that this shows that the potential function $f(x,y,z)$ gives the gravitational potential energy at a point.

Notes:

Exercises 14.1

Problems

In Exercises 1 – 8, find a potential $f(x,y)$ for the gradient field.

1. $\mathbf{F} = \mathbf{i} + 2\mathbf{j}$

2. $\mathbf{F} = x\mathbf{i} + \mathbf{j}$

3. $\mathbf{F} = \cos(x+y)\mathbf{i} + \cos(x+y)\mathbf{j}$

4. $\mathbf{F} = \frac{1}{y}\mathbf{i} + \frac{x}{y^2}\mathbf{j}$

5. $\mathbf{F} = \frac{2x\mathbf{i}+2y\mathbf{j}}{x^2+y^2}$

6. $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j}$

7. $\mathbf{F} = xy\mathbf{i} + \frac{x^2}{2}\mathbf{j}$

8. $\mathbf{F} = \sqrt{y}\mathbf{i} + \frac{x}{2\sqrt{y}}\mathbf{j}$

In Exercises 9 – 17, find the gradient field.

9. $f(x,y) = 3x + y$

10. $f(x,y) = x - 3y$

11. $f(x,y) = x + y^2$

12. $f(x,y) = (x-1)^2 + y^2$

13. $f(x,y) = x^2 - y^2$

14. $f(x,y) = e^x\cos(y)$

15. $f(x,y) = e^{x-y}$

16. $f(x,y) = \frac{y}{x}$

17. $f(x,y) = y^x$

14.2 Line Integrals

Much as the Riemann integral was defined as integral on an interval of the real number line or on a region of the plane, a *line integral* is an integral along a curve. It can equal an area, but that is a special case and not typical. Instead of area, two motivating examples of line integrals from physics and engineering are work along a two-dimensional or three dimensional curve and fluid flow across a curve.

Key Idea 61 Work and Flow

$$\text{Work along a curve} = \int_C \vec{F}\cdot\vec{T}\,ds$$

$$\text{Flow across a curve} = \int_C \vec{F}\cdot\vec{n}\,ds$$

In the first integral, $\vec{F}$ is a force field. In the second integral, $\vec{F}$ is a flow field. Work is done in the direction of movement, so we integrate $\vec{F}\cdot\vec{T}$, where $\vec{T}$ is a unit vector tangent to the curve. Flow is measured through the curve C, so we integrate $\vec{F}\cdot\vec{n}$, where $\vec{n}$ is the unit normal vector, a vector perpendicular to $\vec{T}$. Then $\vec{F}\cdot\vec{n}$ is the component of flow perpendicular to the curve. We will write those integrals in several forms. This section concentrates on work, and flow comes later. The flow is also called the *flux*. The differential ds represents the step along the curve, corresponding to dx on the x-axis for a one-dimensional integral. Whereas $\int_a^b dx$ gives the length of an interval (it equals $b-a$), the value of $\int_C ds$ is the length of the curve C.

Example 14.8

Consider two possible flight paths from Atlanta to Los Angeles, with Atlanta at $(1000, 0)$ and Los Angeles at $(-1000, 0)$ on what we will assume is a flat plane. One flight path is a straight line while the other is a semicircle from Atlanta to $(0, 1000)$ to Los Angeles. Suppose the wind blows due east with force $\vec{F} = M\vec{i}$, with the component in the direction $\vec{j}$ being zero. Assuming that M is a constant, compute the dot product $\vec{F}\cdot\vec{T}$ and then the work done by the wind over the course of the flight.

Solution First, along the straight line path, the work done will be $-2000M$, given that work is distance multiplied by force, and the force is a constant in the

Notes:

direction opposite the path of travel. Next we consider the semicircular path given by $x^2 + y^2 = 1000^2$. Parametrically we can write this path as

$$x = 1000\cos(t),\ y = 1000\sin(t)$$

for $t = 0$ to $t = \pi$ being the values of the parameter t from Atlanta to Los Angeles, respectively. The plane's speed will be

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 1000\sqrt{\cos(t)^2 + \sin(t)^2} = 1000.$$

Note that the unit tangent vector $\vec{T}$ will be

$$\vec{T} = \frac{1}{1000}\left(\frac{d}{dt}(1000\cos(t))\vec{i} + \frac{d}{dt}(1000\sin(t))\vec{j}\right) = -\sin(t)\vec{i} + \cos(t)\vec{j}$$

and so

$$\vec{F}\cdot\vec{T} = M\vec{i}\cdot(-\sin t\vec{i} + \cos t\vec{j}) = M(-\sin t) + 0(\cos t) = -M\sin t$$

Now to calculate the work done from $t = 0$ to $t = \pi$, we compute

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_0^\pi (-M\sin(t))\left(\frac{ds}{dt}\,dt\right) = -1000M\ \sin(t)\,dt = -2000M$$

which is the same as the straight-line path. It appears that the path of the flight doesn't matter in this example, only the origin and destination.

Work is force times distance moved. In this last example it was negative, because the wind acted against the movement of the plane. Because M was constant, it appears the work along the semicircular path could have been found more simply by simply taking the distance of the straight-line path between the endpoints and multiplying by $-M$. However, in general this will not be the case. Most line integrals depend on the path taken. Those that do not depend on the path are crucially important. For a emphgradient field, we only need to know the starting point P and the end point Q.

Notes:

Key Idea 62

When $\vec{F}$ is the gradient of a potential function $f(x, y)$, the work $\int \vec{F} \cdot \vec{T} ds$ depends only on the endpoints $P = (x_0, y_0)$ and $Q = (x_1, y_1)$. The work is the change in f: if

$$\vec{F} = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

then

$$\int_C \vec{F} \cdot \vec{T} ds = f(Q) - f(P).$$

In the above example, the vector field $\vec{F}$ was the gradient of the potential function $f(x, y) = Mx$, and so the work done would be just $f(-1000, 0) - f(1000, 0) = -2000M$, as we calculated. We move on next to a more formal definition of line integrals.

The Definition of Line Integrals

To define $\vec{F} \cdot \vec{T} ds$. We can think of $\vec{F} \cdot \vec{T}$ as a function $f(x, y)$ along the path C, and define its integral as a limit of sums as we are used to. Suppose we have a smooth plane curve C given parametrically by

$$x = x(t),\ y = y(t),\ \text{for } a \leq t \leq b$$

and defining a vector-valued function $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$. We divide C into n sub-arcs by partitioning $[a, b]$ into n subintervals $[t_{i-1}, t]$. Taking $x_i = x(t_i)$ and $y_i = y(t_i)$, we have divided C into sub-arcs from (x_{i-1}, y_{i-1}) to (x_i, y_i) with corresponding arclengths Δs_i. Choosing any point t_i^* in $[t_{i-1}, t]$ gives a point $(x_i^*, y_i^*) = (x(t_i^*), y(t_i^*))$ in the i^{th} sub-arc.

Now suppose $f(x, y)$ is a function of two variables including the curve C in its domain. We can then form

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

which is a Riemann Sum. Taking the limit we arrive at the definition of the line integral of $f(x, y)$ along the curve C.

Notes:

Definition 114 **Line Integral of $f(x, y)$ along C**

Let C be a smooth curve in the domain of the function $f(x, y)$. Then the line integral of $f(x, y)$ along C is given by

$$\int_C g(x, y)\, ds = \lim_{||\Delta s|| \to 0} \sum_{i=1}^{n} f(x_i, y_i) \Delta s_i$$

The points (x_i, y_i) lie on the curve C. The last point Q is (x_n, y_n) and the first point P is (x_0, y_0). The step Δs_i is the distance to (x_i, y_i) from the previous point. As $\Delta s \to 0$, the straight pieces better and better approximate the path of the curve. Note that the special case $f(x, y) = 1$ gives the arc length of C. As long as $f(x, y)$ is piecewise continuous (a finite number of jumps allowed) and the path is piecewise smooth (a finite number of corners allowed), the limit exists and defines the line integral.

When $f(x, y)$ is the density of a wire C, the line integral is the total mass of the wire. When $f(x, y)$ is $\vec{F} \cdot \vec{T}$, the integral is the work done along the path C. Calculating these values using the above definition is cumbersome. We now parametrize the curve C as above. The parameter could be thought of as the time for a moving object. In doing so, the differential ds becomes $\frac{ds}{dt}\, dt$ and everything in the integral can be changed over the new variable t, where $t = a$ gives the starting point $P = \big(x(a), y(a)\big)$ and $t = b$ gives the ending point $Q = \big(x(b), y(b)\big)$:

$$\int_C f(x, y)\, ds = \int_{t=a}^{t=b} f\big(x(t), y(t)\big) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

We can repeat this process in three dimensions as well. In three dimensions the points on a smooth three-dimensional curve C are given by a parametrization $\big(x(t), y(t), z(t)\big)$, from $t = a$ to $t = b$, and so given a piecewise continuous function $f(x, y, z)$ we obtain

$$\int_C f(x, y, z)\, ds = \int_{t=a}^{t=b} f\big(x(t), y(t), z(t)\big) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$$

Example 14.9

Notes:

Suppose the points on a coil spring C are

$$(x(t), y(t), z(t)) = (\cos t, \sin t, t)$$

for $t = 0$ to $t = 4\pi$. Find the mass of the spring if the density is given by $\rho = 4$ kilograms per meter.

Solution We first calculate

$$(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2 = \sin^2 t + \cos^2 t + 1 = 2.$$

Thus $\dfrac{ds}{dt} = \sqrt{2}$. To find the total mass, integrate the mass per unit length, which is $g = \rho = 4$ from $t = 0$ to $t = 4\pi$.

$$\text{mass} = \int_0^{4\pi} \rho \frac{ds}{dt}\, dt = \int_0^{4\pi} 4\sqrt{2}\, dt = 16\sqrt{2}\pi$$

kilograms.

Example 14.10

Compute the line integral $\displaystyle\int_C f(x, y)\, ds$ where $f(x, y) = 3x$ and C is the portion of the parabola $y = x^2$ from $(0, 0)$ to $(2, 4)$.

Solution First we need to parametrize the curve C. This can be done in any case of a curve $y = g(x)$ by simply letting $t = x$. In this case we obtain $x = x(t) = t$ and $y = y(t) = 3t$, for $t = 0$ to $t = 2$. Note that $f(x, y)$ becomes $f(t) = 3t$. Then

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (1)^2 + (3)^2 = 10$$

and so

$$\int_C f(x, y)\, ds = \int_0^2 3t\,\sqrt{10}\, dt = \frac{3\sqrt{10}}{2}(4 - 0) = 6\sqrt{10}.$$

Different Forms of the Work Integral

The work integral $\int F \cdot \vec{T} ds$ from before can be written in a different way. Suppose the vector field is $\vec{F} = M(x, y)\vec{i} + N(x, y)\vec{j}$. A small step along the curve is $ds = dx\vec{i} + dy\vec{j}$. Work is force multiplied by distance, but it is only the force component along the direction of the path that counts. The dot product $F \cdot Tds$

Notes:

finds that component automatically. The vector to a point on C is $\vec{r} = x\vec{i} + y\vec{j}$. Then $d\vec{r} = \vec{T}ds = dx\vec{i} + dy\vec{j}$. So we arrive at

$$\text{work} = \int_C \vec{F} \cdot d\vec{r} = \int_C M(x,y)dx + N(x,y)dy.$$

If the curve C is in three-dimensions, the work is similarly

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_C \vec{F} \cdot d\vec{r} = \int M(x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz.$$

Consider the expression $M dx$, which is the product of force in the x-direction M and the movement in the x-direction dx. This product is zero if either factor is zero. Ignoring friction, pushing a piano along level ground takes no work. On the other hand, carrying the piano up the stairs brings in a nonzero $P\,dz$ term, and the total work is the piano weight P times the change in z.

To connect the new $\int \vec{F} \cdot d\vec{r}$ line integral with the old $\int \vec{F} \cdot \vec{T}ds$ expression, consider the tangent vector $\vec{T}$. It is $\dfrac{d\vec{r}}{ds}$. Therefore $\vec{T}ds$ is simply $d\vec{r}$. The best form for computations is $\vec{r}$, because the unit vector $\vec{T}$ has a division by $ds/dt = \sqrt{(dx/dt)^2 + (dy/dt)^2}$, which can get messy. Later we multiply by this square root, in converting ds to $(ds/dt)dt$. It makes no sense to compute the square root, divide by it, and then multiply by it. That is avoided in the new form $\int_C M\,dx + N\,dy$.

Example 14.11

Consider the vector field $\vec{F}(x,y) = -y\vec{i} + x\vec{j}$ and two different paths from $(1,0)$ to $(0,1)$. The first path will be a straight line segment path and the second is a quarter-circle path along the unit circle. Compute the work done by $\vec{F}$ in moving along each path.

Solution First we consider the straight-line path. This requires parametrizing a line segment from a point P to a point Q which can always be done using

$$\vec{r}(t) = (1-t)P + tQ$$

for $t = 0$ to $t = 1$. In this example, we would have

$$\vec{r}(t) = (1-t)(1,0) + t(0,1) = (1-t, t)$$

or $x(t) = 1 - t$ and $y(t) = t$ for $0 \le t \le 1$. Therefore $dx = -dt$ and $dy = dt$. Computing the work yields

$$\int_C -y\,dx + x\,dy = \int_0^1 -t(-dt) + (1-t)(dt) = \int_0^1 1\,dt = 1.$$

Notes:

Next we compute the work done along the quarter-circle path. Since this follows the unit circle, it can be parametrized by

$$\vec{r}(t) = \cos(t)\vec{i} + \sin(t)\vec{j}$$

for $t = 0$ to $t = \dfrac{\pi}{2}$. Then

$$dx = -\sin(t)\,dt \text{ and } dy = \cos(t)\,dt$$

giving the work done as

$$\int_C -ydx + xdy = \int_0^{\pi/2} -sin(t)(-\sin(t)dt) + \cos(t)(\cos(t)dt) = \int_0^{\pi/2} 1dt = \frac{\pi}{2}.$$

Note here that the value of the work done does depend on the path taken, not just the starting and ending points.

Key Idea 63 Work done by a force field $\vec{F}$ along a curve C

Suppose a smooth curve C is parametrized by

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$$

and an object is traveling along the curve C from a point at $t = a$ to $t = b$ subject to a force $\vec{F}$. To determine the work done by $\vec{F}$ on the object, one computes $\int_C M\,dx + N\,dy$ by first substituting $x(t)$ and $y(t)$ into $M(x, y)$ and $N(x, y)$ to create a function of the parameter t. Find dx and dy by taking the derivatives of $x(t)$ and $y(t)$ to turn the integral into an integral of t. Then integrate from $t = a$ to $t = b$ to compute the work done. The procedure is similar if C is in three-dimensional space and the object subject to the force $\vec{F}(x, y, z)$.

Example 14.12

Consider a force field $\vec{F}(x, y) = x^2\vec{i} - xy\vec{j}$ and a particle moving along the quarter-circle path from $(1, 0)$ to $(0, 1)$ along the unit circle. Determine the work done by the force field on the particle.

Solution We will again use the parametrization

$$\vec{r}(t) = \cos(t)\vec{i} + \sin(t)\vec{j}$$

Notes:

for $t=0$ to $t=\frac{\pi}{2}$, giving $dx=-\sin(t)\,dt$ and $dy=\sin(t)\,dt$. Then the force field becomes

$$\vec{F}(t)=\cos^2(t)\vec{i}-\cos(t)\sin(t)\vec{j}$$

as a function of the parameter t. Computing the work done yields

$$\int_C Mdx+Ndy=\int_0^{\pi/2}\cos^2(t)\,(-\sin(t)\,dt)+\cos(t)\sin(t)\,(\cos(t)\,dt)=\int_0^{\pi/2}-2\cos^2(t)\sin(t)dt$$

Using a substitution of $u=\cos(t)$ and $du=-\sin(t)\,dt$ gives the work done by $\vec{F}$ is

$$2\left[\frac{\cos^3(t)}{3}\right]_0^{\pi/2}=-\frac{2}{3}.$$

Note that the work is negative because the particle is moving against the force along its path.

Independence of Path and Conservation of Energy

When a force field does work on a mass m, it normally gives that mass a new velocity. Newton's Law is $\vec{F}=m\vec{a}=m\frac{d\vec{v}}{dt}$. The work done $\int_C \vec{F}\cdot d\vec{r}$ along a path C from P to Q is

$$\int\left(m\frac{d\vec{v}}{dt}\right)\cdot(\vec{v}dt)=\left[\frac{1}{2}m\vec{v}\cdot\vec{v}\right]_P^Q=\frac{1}{2}m||\vec{v}(Q)||^2-\frac{1}{2}m||\vec{v}(P)||^2 \qquad (14.1)$$

The work equals the change in the kinetic energy $\frac{1}{2}m||\vec{v}||^2$. However, for a gradient field $\vec{F}=\nabla f$ the work is also the negative change in potential

$$\text{work}=\int\vec{F}\cdot d\vec{r}=-\int df=f(P)-f(Q). \qquad (14.2)$$

Comparing 14.1 with 14.2, the combination $\frac{1}{2}m||\vec{v}||^2+f$ is the same at P and Q. The total energy, kinetic plus potential, is *conserved*.

Most of the examples in physics concentrate on special fields in which the work done depends only on the starting and ending points of the path. We now explain what happens when the integral is independent of the path. Suppose you integrate from P to Q on one path, and then back to P on another path.

Notes:

Combined, that is a closed path from containing P. See Figure 14.7. But a backward integral is the negative of a forward integral, since $d\vec{r}$ switches sign. If the two integrals from P to Q are equal, the integral around the closed path is zero.

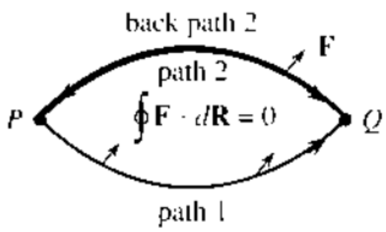


Figure 14.7: The vector field **R**

$$\begin{aligned}\oint_C \vec{F}\cdot d\vec{r} &= \int_P^Q \vec{F}\cdot d\vec{r} + \int_Q^P \vec{F}\cdot d\vec{r}\\ &= \int_P^Q \vec{F}\cdot d\vec{r} - \int_P^Q \vec{F}\cdot d\vec{r} = 0\end{aligned}$$

Notationally, the circle on the first integral indicates a closed path. It is not necessary for closed paths to indicate a particular start and endpoint P. However, not all closed path integrals are zero! For most fields $\vec{F}$, different paths yield different work. For what are called *conservative* fields, all paths with the same starting and ending points yield the same work. Then zero work around a closed path indicates conservation of energy. The crucial question is how to determine which fields are conservative without trying an infinite number of paths.

Key Idea 64 Properties of a Conservative Field

The vector field $\vec{F} = M(x,y)\vec{i} + N(x,y)\vec{j}$ is a conservative field if it has these properties:

1. The work $\oint_C \vec{F}\cdot d\vec{r}$ around every closed path C is zero;
2. The work $\int_C \vec{F}\cdot d\vec{r}$ along a curve C depends only on the starting and ending points of the path, not on actual path itself;
3. $\vec{F}$ is a gradient field. That is, $M = \dfrac{\partial f}{\partial x}$ and $N = \dfrac{\partial f}{\partial y}$ for some potential function $f(x,y)$.

A field with one of these properties has them all.

Consider the case of a smooth path C given by $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$ from $t = a$ to $t = b$. If a particle moving along that path is subject to a conservative vector field $\vec{F} = \nabla f$, then path independence above can be restated as follows, sometimes called the *Fundamental Theorem of Line Integrals*, which is analogous to the second part of the Fundamental Theorem of Calculus for regular integrals.

Notes:

Theorem 133 Fundadmental Theorem for Line Integrals

Given a smooth curve C from P to Q defined parametrically by $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$ for $a \leq t \leq b$ and a **conservative** vector field $\vec{F}(x,y) = \nabla f(x,y)$, then

$$\int_C \vec{F} \cdot d\vec{r} = f\left(\vec{r}(b)\right) - f\left(\vec{r}(a)\right) = f(Q) - f(P)$$

The proof of the above result follows simply from the fact that $\vec{F} = \nabla f$ and that

$$\nabla f(x,y) \cdot d\vec{r}(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{d}{dt}f\left(\vec{r}(t)\right)$$

by the Chain Rule. A similar statement for the Fundamental Theorem holds in three dimensions, as well.

Next we focus on a simple test for determining whether a vector field in two dimensions is conservative. If $\vec{F}(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j}$ is conservative and satisfies $\vec{F} = \nabla f(x,y)$ then

$$\frac{\partial f}{\partial x} = M(x,y) \text{ and } \frac{\partial f}{\partial y} = N(x,y).$$

We know from Chapter 12 that mixed partials are equal under the right circumstances. That is, if $M(x,y)$ and $N(x,y)$ have continuous first-order partial derivatives, then

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

or

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Theorem 134 Two Dimensional Test for a Conservative Vector Field

Let $\vec{F}(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j}$ be a vector field defined on an open simply-connected region D. If $M(x,y)$ and $N(x,y)$ have continuous first-order partial derivatives and $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ then $\vec{F}$ is conservative.

Notes:

The property of region D being open and simply-connected means, essentially, that there are no holes in the region on which the vector field is defined. In such situations, the above test will determine whether or not a given vector field is conservative. In the case that it is, one may be able to employ antiderivatives to find the potential function $f(x,y)$ so that $\vec{F}=\nabla f$. It is necessary to know this potential function if one is to use the Fundamental Theorem to compute line integrals of conservative vector fields.

Example 14.13

Show that $\vec{F}(x,y) = 2xy\vec{i} + x^2\vec{j}$ is conservative. Then find a potential function $f(x,y)$ so that $\nabla f = \vec{F}$.

Solution For $M(x,y) = 2xy$ and $N(x,y) = x^2$, note that $\dfrac{\partial M}{\partial y} = 2x$ and $\dfrac{\partial N}{\partial x} = 2x$, also. Since M and N are defined everywhere and have continuous partial derivatives of every type, it follows that $\vec{F}$ is conservative. Now we find the potential. Start with $M(x,y) = 2xy$. We need $f(x,y)$ so that $\dfrac{\partial f}{\partial x} = 2xy$. This would occur if

$$f(x,y) = \int 2xy\,dx + g(y) = x^2y + g(y)$$

for some function g depending only on y. Take the partial derivative with respect to y now. We should have that $\dfrac{\partial f}{\partial y}$ is $N(x,y) = x^2$, but we also get from above that

$$\frac{\partial f}{\partial y} = x^2 + g'(y).$$

Setting these equal, this implies that $g'(y) = 0$ or the $g(y)$ is a constant. Therefore $f(x,y) = x^2y + C$ for any constant C is a potential function for $\vec{F}$.

Example 14.14

Determine if $\vec{F}(x,y) = ye^{xy}\vec{i} + (xe^{xy} + 2y)\vec{j}$ is conservative. If it is, find a potential function $f(x,y)$ so that $\nabla f = \vec{F}$.

Solution For $M(x,y) = ye^{xy}$ and $N(x,y) = xe^{xy} + 2y$, note that both $\dfrac{\partial M}{\partial y}$ and $\dfrac{\partial N}{\partial x}$ equal $e^{xy} + xye^{xy}$. Since M and N are defined everywhere and have continuous partial derivatives of every type, it follows that $\vec{F}$ is conservative. To find the potential $f(x,y)$, start with $M(x,y) = ye^{xy}$. We need $f(x,y)$ so that

Notes:

$\frac{\partial f}{\partial x} = 2xy$. This would occur if

$$f(x,y) = \int ye^{xy}\,dx + g(y) = y\left(\frac{1}{y}e^{xy}\right) + g(y) = e^{xy} + g(y)$$

for some function g depending only on y. Take the partial derivative with respect to y now. We should have that $\frac{\partial f}{\partial y}$ is $N(x,y) = xe^{xy} + 2y$, but we also get from above that

$$\frac{\partial f}{\partial y} = e^{xy} + g'(y).$$

Setting these equal, this implies that $g'(y) = 2y$ or that $g(y) = y^2 + C$ for any constant C. Therefore $f(x,y) = e^{xy} + y^2 + C$ is a potential function for $\vec{F}$.

Note that this approach of finding $f(x,y)$ should fail if the vector field $\vec{F}$ is not conservative. The next example illustrates this for the spin field from a previous example.

Example 14.15

Attempt to find a potential function $f(x,y)$ for the vector field

$$\vec{F}(x,y) = -y\vec{i} + x\vec{j}.$$

Solution Note that this vector field is not conservative as $\frac{\partial M}{\partial y} = -1$ while $\frac{\partial N}{\partial x} = 1$. To show how finding the potential as in the last example results in a problem, we directly solve $\frac{\partial f}{\partial x} = M(x,y) = -y$ and try to achieve $\frac{\partial f}{\partial y} = x$. First $\frac{\partial f}{\partial x} = -y$ gives

$$f(x,y) = -xy + g(y)$$

for some (possibly constant) function $g(y)$ depending only on y. Taking $\frac{\partial}{\partial y}$ of this yields $-x + g'(y)$. That does not agree with the requirement that $\frac{\partial f}{\partial y} = x$. There is no way to define $g(y)$ to make this work. Therefore we conclude that the spin field $-y\vec{i} + x\vec{j}$ is not conservative as it has no potential function.

Notes:

Exercises 14.2

Problems

In Exercises 1 – 8, compute the line integrals.

1. $\int_C 1\,ds$; C: $x = t$, $y = 2t$, $0 \leq t \leq 1$
2. $\int_C 1\,dy$; C: $x = t$, $y = 2t$, $0 \leq t \leq 1$
3. $\int_C x\,ds$; C: $x = \cos(t)$, $y = \sin(t)$, $0 \leq t \leq \frac{\pi}{2}$
4. $\int_C xy\,ds$; C: $x = \cos(t)$, $y = \sin(t)$, $0 \leq t \leq \frac{\pi}{2}$
5. $\int_C xy\,ds$; C: bent line from $(0,0)$ to $(1,1)$ to $(1,0)$
6. $\int_C y\,dx - x\,dy$ C: any square path, sides of length 3
7. $\int_C 1\,dx$; C: any closed circle of radius 3
8. $\int_C y\,dx$; C: $x = 3\cos(t)$, $y = 3\sin(t)$, $0 \leq t \leq 2\pi$

In Exercises 9 – 14, find the work in moving from $(1,0)$ to $(0,1)$. When F is conservative, construct *f*. Choose your own path when F is not conservative.

9. $\mathbf{F} = \mathbf{i} + y\mathbf{j}$
10. $\mathbf{F} = y\mathbf{i} + \mathbf{j}$
11. $\mathbf{F} = xy^2\mathbf{i} + yx^2\mathbf{j}$
12. $\mathbf{F} = e^y\mathbf{i} + xe^y\mathbf{j}$
13. $\mathbf{F} = \mathbf{i} + y\mathbf{j}$
14. $\mathbf{F} = -y^2\mathbf{i} + x^2\mathbf{j}$

In Exercises 15 – 20, compute $\int F \cdot dR$ along the straight line $\mathbf{R} = t\mathbf{i} + t\mathbf{j}$ and the parabola $\mathbf{R} = t\mathbf{i} + t^2\mathbf{j}$, from $(0,0)$ to $(1,1)$. When F is a gradient field, use its potential $f(x,y)$.

15. $\mathbf{F} = \mathbf{i} - 2\mathbf{j}$
16. $\mathbf{F} = \mathbf{i} - 2\mathbf{j}$
17. $\mathbf{F} = 2xy^2\mathbf{i} - 2yx^2\mathbf{j}$
18. $\mathbf{F} = x^2y\mathbf{i} + xy^2\mathbf{j}$
19. $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$
20. $\mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j}}{x^2 + y^2 + 1}$

In Exercises 21 – 26, determine if $\mathbf{F}(x,y)$ is conservative. If it is conservative, find a potential function.

21. $\mathbf{F} = y^2e^{-x}\mathbf{i} - 2ye^{-x}\mathbf{j}$
22. $\mathbf{F} = y^2e^{x}\mathbf{i} - 2ye^{x}\mathbf{j}$
23. $\mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j}}{x\mathbf{i} + y\mathbf{j}}$
24. $\mathbf{F} = \frac{\nabla f(x,y)}{|\nabla f(x,y)|}$ where $f(x,y) = xy$
25. $\mathbf{F} = (x - y)\mathbf{i} + (y + x)\mathbf{j}$
26. $\mathbf{F} = (-8x + 3y)\mathbf{i} + (3x + 5y)\mathbf{j}$

14.3 Green's Theorem

This section contains the Fundamental Theorem of Calculus extended to two dimensions. The formula was discovered 150 years after Newton and Leibniz, by a self-taught English mathematician named George Green. His theorem connects a double integral over a region R to a line integral along its boundary curve C.

In Section 5.4 you learn that the integral of $\frac{df}{dx}$ from $x = a$ to $x = b$ equals $f(b) - f(a)$. This connects a one-dimensional integral to a zero-dimensional integral - the value $f(b) - f(a)$ is some kind of a point integral. It is this absolutely crucial idea (to integrate a derivative from information at the boundary) that extends Green's Theorem into two dimensions.

There are two important integrals around a curve C. The first is for a force field $\vec{F}$, and the work done is $\int_C \vec{F}\cdot\vec{T}\,ds = \int_C M\,dx + N\,dy$. The second is for a flow field $\vec{F}$, and the flux is $\int_C \vec{F}\cdot\vec{n}\,dx = \int_C M\,dy - N\,dx$, where $\vec{n}$ is the normal vector. Green's Theorem handles both, in two dimensions. In three dimensions they split into the Divergence Theorem (Section 14.5) and Stokes' Theorem (Section 14.6). Green's Theorem applies to "smooth" functions $M(x, y)$ and $N(x, y)$ with continuous first derivatives in a region containing the interior R inside the curve C. $M(x, y)$ and $N(x, y)$ will have a specific meaning in applications (to electricity and magnetism or to fluid flow or to mechanics). We capture the central idea first and the applications follow.

As Green's Theorem is about simple, closed curves, we first define some terminology.

Definition 115 Simple, Closed Curves, and the Counterclockwise Direction

Recall that a curve is **closed** if its starting and ending points are the same. A curve is **simple** if it does not intersect itself, except possibly at the starting and ending points. Any simple, closed curve C encloses a region R. Suppose an ant is crawling along the curve C. The **counterclockwise** direction is the one in which the enclosed region R is always on the ant's left.

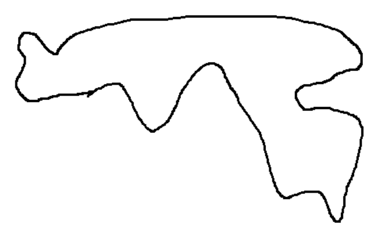


Figure 14.8: A simple closed curve.

We are familiar with the term "counterclockwise" for circles. This allows us to define it for any simple, closed curve, such as the wacky curve in Figure 14.8.

Notes:

Theorem 135 Green's Theorem

Suppose the region R is bounded by a simple, closed, piecewise smooth curve C oriented in the counterclockwise direction. If $M(x, y)$ and $N(x, y)$ have continuous first-order partial derivatives on an open region containing R, then

$$\oint_C M(x, y)\,dx + N(x, y)\,dy = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dA$$

Example 14.16

First consider the case that $\vec{F}(x, y)$ is a conservative force field with a potential function $f(x, y)$ so that $\nabla f = \vec{F}$. Let C be a simple, closed, piecewise smooth curve enclosing a region R. What does Green's Theorem say about the work done around C in this case?

SOLUTION Since $\vec{F} = M\,\vec{i} + N\,\vec{j}$ is conservative, we get that

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

from the last section. Then Green's Theorem implies

$$\oint_C M(x, y)\,dx + N(x, y)\,dy = \iint_R 0\,dA = 0.$$

This verifies the claim in the last section that line integrals of conservative vector fields around closed curves are zero.

Example 14.17

Compute the line integral $\oint_C x^3\,dx + xy^2\,dy$ where C is the triangular curve consisting of line segments connecting $(0, 0)$ to $(1, 0)$, $(1, 0)$ to $(1, 1)$, and $(1, 1)$ to $(0, 0)$.

SOLUTION Computing this directly would require three different line integrals and three different parametrizations for the line segments. Instead we employ Green's Theorem and compute a single double integral. Letting R be the triangle inside the curves bounded by $y = 0$ and $y = x$, we get

$$\oint_C x^3\,dx + xy^2\,dy = \iint_R (y^2 - 0)\,dA = \int_0^1 \int_0^x y^2\,dy\,dx = \frac{1}{12}.$$

Notes:

Proof of Green's Theorem

Green's Theorem is not easy to prove in general. We will, however, verify it is true in the case that R is a simple region - one that is expressible in both the forms

$$R = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

for some curves $y = g_1(x)$ and $y = g_2(x)$, and

$$R = \{(x, y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

for some curves $x = h_1(y)$ and $x = h_2(y)$. Other regions which are the union of simple regions can then be done with this in hand.

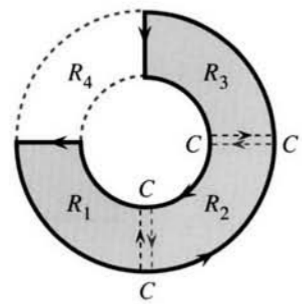


Figure 14.9: Union of simple regions

For a vector field $\vec{F}(x, y) = M(x, y)\,\vec{i} + N(x, y)\vec{j}$ as above, the idea is to prove that

$$\oint_C M(x, y)\,dx = \iint_R -\frac{\partial M}{\partial y}\,dA$$

and

$$\oint_C N(x, y)\,dy = \iint_R \frac{\partial N}{\partial x}\,dA$$

separately. We verify the first and the leave the second, which is very similar, as an exercise.

Suppose that R is the region given by

$$\{(x, y) \mid a \leq x \leq b, g(x) \leq y \leq f(x)\}$$

for top curve $y = f(x)$ and bottom curve $y = g(x)$. In the double integral, first integrate with respect to y to get

$$\int_{g(x)}^{f(x)} -\frac{\partial M}{\partial y}\,dy = \Big(-M(x, y)\Big]_{y=g(x)}^{y=f(x)} = -M(x, f(x)) + M(x, g(x)).$$

Integrate with respect to x to yield

$$-\int_a^b M(x, f(x))\,dx + \int_a^b M(x, g(x))\,dx.$$

The other side of the equation deals with a line integral. For this we get

$$\oint_C M(x, y)dx = \int_{\text{top}} M(x, y)dx + \int_{\text{bottom}} M(x, y)dx = \int_b^a M(x, f(x))dx + \int_a^b M(x, g(x))dx$$

Notes:

which is the same as the other side due to the addition of a minus sign when the bounds of the first integral are switched.

Next consider a region R which is not simple but is instead a union of simple regions, such as in Figure 14.9. As in the figure, we would break the region into three pieces R_1, R_2, and R_3 which are all simple. The three individual double integrals sum to the double integral of R, which would also equal the sum of the three line integrals by the above argument. When we add the line integrals, the line integrals over the cross cuts cancel out as they are going in opposite directions, resulting in only the boundary pieces adding up to the total line integral. This leaves the double integral of the interior equal to the line integral of just the boundary, as Green's Theorem would state. The next example will pertain to a region that is a union of two simple regions.

If the region R contains the piece R_4 in Figure 14.9, then the theorem is still true. The integral around the outside is still counterclockwise, but the integral is clockwise around the inner circle. Keeping the region R to your left as you go around C gives the counterclockwise direction according to our definition. The complete ring is doubly connected, not simply connected. Green's Theorem allows any finite number of regions R_i and cross cuts and holes.

Example 14.18
Compute the line integral $\oint_C y^2\, dx + 3xy\, dy$ where C is the boundary of the semi-annular region in the upper half-plane between $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution The enclosed region R is not simple, but is a union of two simple pieces if you cut down the y-axis. Therefore Green's Theorem applies in this situation. The theorem gives

$$\oint_C y^2\, dx + 3xy\, dy = \iint_R 3y - 2y\, dA = \iint_R y\, dA.$$

Notice that this region can be integrated over easily via polar coordinates. Therefore

$$\oint_C y^2\, dx + 3xy\, dy = \int_0^{\pi} \int_1^2 (r \sin \theta) r\, dr\, d\theta = \frac{14}{3}.$$

Notes:

Usng Green's Theorem to Compute Areas

So far, we have used Green's Theorem to rewrite line integrals as double integrals, making them easier to compute. However, Green's Theorem gives equality between a line integral and a double integral, and therefore can also be used to rewrite a double integral as a line integral. This can be done for any double integral, but we wll only focus on $\iint_R 1dA$, the area of a region R.

Consider the region R in Figure 14.10. This region is inside the simple closed curve given by the parametric equations

$$x = \cos(3t) - \sin t \text{ and } y = 2\cos t$$

traced once on the interval $[0, 2\pi]$. Note that determining bounds for this region in terms of x andor y, as we would typically do to compute rewrite the double integral as an iterated integral, cannot be done in a simple way. However, as we have a parameterization for the boundary curve, it is not hard to compute a line integral along that curve.

Consider any vector field $\vec{F}(x,y) = M\vec{i} + N\vec{j}$ for which $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = 1$. There are infinitely many such vector fields, the most basic of which are $\vec{F}(x,y) = x\vec{j}$ or $\vec{F}(x,y) = -y\,\vec{i}$. What is the line integral of $\vec{F}$ around a simple closed curve C computing? By Green's Theorem

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C M(x,y)\,dx + N(x,y)\,dy = \iint_R 1\,dA$$

which is the area of R, or the area enclosed by C. In this way, one can use a line integral to compute area as long as one knows the boundary curve of the region.

We summarize this using a Key Idea before we continue with examples.

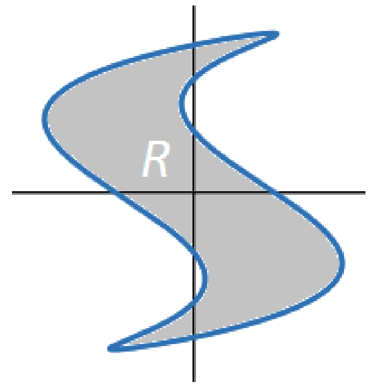


Figure 14.10: The region R is bounded inside the simple closed curve $x = \cos(3t) - \sin t$, $y = 2\cos t$

Key Idea 65 Area inside a simple closed curve C

Let R be the region inside a simple closed curve C, parameterized in the counterclockwise direction. Let $\vec{F}(x,y) = M\vec{i} + N\vec{j}$ be any vector field for which $N_x - M_y = 1$ Then

$$\text{Area of } R = \iint_R 1\,dA = \oint_C M(x,y)\,dx + N(x,y)\,dy.$$

In particular,

$$\text{Area of } R = \iint_R 1\,dA = \oint_C x\,dy = -\oint_C y\,dx.$$

Notes:

Example 14.19 Computing area using Green's Theorem

Compute the area of the region R inside the simple closed parametric curve C given by

$$x = \cos(3t) - \sin t \text{ and } y = 2\cos t$$

traced once on the interval $[0, 2\pi]$.

Solution This is the region shown in Figure 14.10. The given parameterization gives the counterclockwise direction. (If we did not know that, however, and computed a line integral in the opposite direction, it would give the opposite value. Since an area must be positive, we could take the absolute value in which case the direction of the curve would not matter.) Using Key Idea 65,

$$\begin{aligned}
\iint_R 1\, dA &= \oint_C x\, dy \\
&= \int_0^{2\pi} \big(\cos(3t) - \sin t\big)(-2\sin t)\, dt \\
&= -2\int_0^{2\pi} \big(\sin t\cos(3t) - \sin^2 t\big)\, dt.
\end{aligned}$$

Recall now that we must proceed with the Product-to-Sum identities,

$$\begin{aligned}
\iint_R 1\, dA &= -2\int_0^{2\pi} \big(\sin t\cos(3t) - \sin^2 t\big)\, dt \\
&= -2\int_0^{2\pi} \left(\frac{\sin(4t)+\sin(-2t)}{2} - \frac{1-\cos(2t)}{2}\right)\, dt \\
&= \int_0^{2\pi} \big(-\sin(4t) + \sin(2t) + 1 - \cos(2t)\big)\, dt \\
&= \frac{\cos(4t)}{4} - \frac{\cos(2t)}{2} + t - \frac{\sin(2t)}{2}\Bigg|_0^{2\pi} \\
&= 2\pi \text{ square units.}
\end{aligned}$$

The ability to compute an area using Green's Theorem has physical value as well. A **planimeter** is a device used to compute the area of an arbitrary two-dimensional shape. It operates by tracing the boundary of the shape, and computing an appropriate line integral while tracing the boundary.

We finish this section with a more familiar shape.

Notes:

Example 14.20 Area inside an ellipse

Using a line integral, compute the area of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

with semiaxes of length a and b.

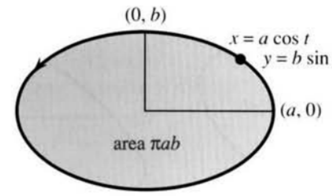


Figure 14.11: Ellipse

Solution See Figure 14.11. By the above equation for the ellipse, we can parameterize the ellipse as

$$x=a\cos(t),\ \ y=b\sin(t)$$

for $0\leq t\leq 2\pi$, which traces the ellipse once in the counterclockwise direction. Using Key Idea 65, we get an area of

$$\begin{aligned}\iint_R 1\,dA &= \oint_C x\,dy\\ &= \int_0^{2\pi} a\cos(t)\,\left(b\cos(t)\right)\,dt\\ &= ab\int_0^{2\pi}\cos^2(t)\,dt\\ &= ab\int_0^{2\pi}\frac{1+\cos(2t)}{2}\,dt\\ &= ab\left(\frac{t}{2}+\frac{\sin(2t)}{4}\right)\Big|_0^{2\pi}\\ &= \pi ab \text{ square units.}\end{aligned}$$

One can consider a region R in the plane as a flat surface in three-dimensional space. Green's Theorem says that we can compute an integral over the surface R by instead computing a line integral around the boundary of the region. This will hold for more general surfaces in space, and the result that generalizes Green's Theorem is called Stokes' Theorem, discussed in Section 14.6.

Notes:

Exercises 14.3

Problems

In Exercises 1 – 6, compute the line integrals and separately compute the double integrals in Green's Theorem. The has parametric equations $x = 2\cos(t)$, $y = 2\sin(t)$, and the triangle has sides $x = 0$, $y = 0$, $x + y = 1$.

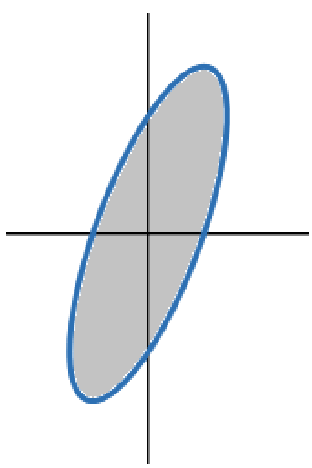
1. $\oint x\,dy$ along the circle
2. $\oint x^2y\,dy$ along the circle
3. $\oint x\,dx$ along the triangle
4. $\oint y\,dx$ along the triangle
5. $\oint x^2y\,dx$ along the circle
6. $\oint x^2y\,dx$ along the triangle

In Exercises 7 – 11, compute the line integrals and separately compute the double integral in Green's Theorem.

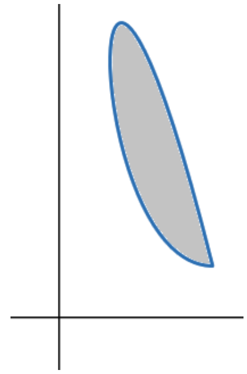
7. $\mathbf{F} = x\mathbf{i} + y\mathbf{j}\,dx$ $R =$ upper half of the disk $x^2 + y^2 \leq 1$.
8. $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j}$, $C =$ square with sides $y = 0$, $x = 1$, $y = 1$, $x = 0$
9. $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$ in the unit circle
10. $\mathbf{F} = xy\mathbf{i}$ in the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$
11. $\mathbf{F} = x^2y\mathbf{j}$ in the triangle with sides $x = 0$, $y = 0$, $x + y = 1$

In Exercises 12 – 15, the given parametric curve is a simple closed curve on the given interval, parameterized in the counterclockwise direction. Use Green's Theorem to compute the area of the region inside the curve.

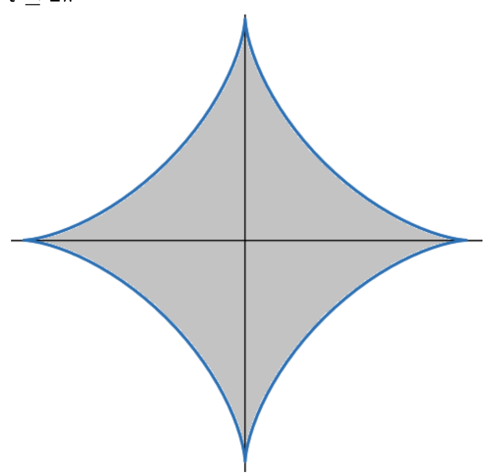
12. $x = \cos t - \sin t$
    $y = 3\cos t$
    $0 \leq t \leq 2\pi$



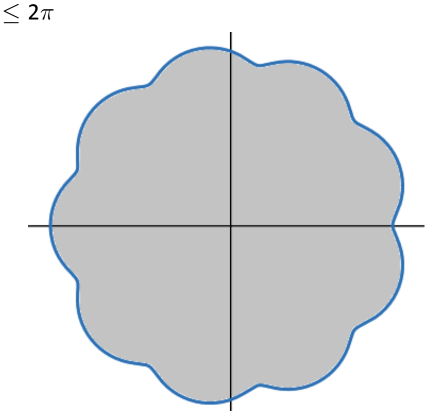
13. $x = t^2 - 2t + 3$
    $y = t^3 - 5t^2 + 3t + 11$
    $-1 \leq t \leq 3$



14. $x = \cos^3 t$
    $y = \sin^3 t$
    $0 \leq t \leq 2\pi$



15. $x = 20\cos t - \cos(10t)$
    $y = 20\sin t - \sin(10t)$
    $0 \leq t \leq 2\pi$



14.4 Surface Integrals

The double integral in Green's Theorem is over a region R in the plane. In this section we consider instead a curved surface T in three-dimensional space, such as part of a sphere or of a cone. When the z is a function of x and y, it can be written $z = f(x, y)$, and the graph is a surface, analogous to how $y = f(x)$ was the graph of a particular type of curve. In more general cases, a curve could be written parametrically, $x = x(t)$ and $y = y(t)$, as functions of a parameter t. Similarly, a surface can be written parametrically as functions of two parameters.

When we compute the length of a curve, we had an integral $\int_C ds$, a line integral. Similarly, when we compute the area of a given surface T, we will evaluate an integral $\iint_T dS$, called a surface integral. This type of integral will be used to compute surface area even when the surface is not given by a function $z = f(x, y)$ of two variables.

We first start with a review of the case of a surface T given by $z = f(x, y)$ for (x, y) in a region R of the xy-plane, as in Section 13.5. When we partitioned the region R into small rectangular pieces, the associated pieces of the surface $z = f(x, y)$ will be almost flat, like small parallelograms. The surface area of such a piece ended up being

$$dS = \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2}\, dx\, dy$$

computed using a cross product of the vectors making up the sides of the parallelogram. The total surface area of T could then be computed as

$$S = \iint_R dS = \iint_R \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2}\, dA.$$

We now generalize this method to the situation that the surface T is given parametrically.

Parametric Surfaces and Surface Area

A **parametric surface** is one defined by three functions of two parameters, which we will denote by u and v. We will write these functions as $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$, where (u, v) come from a region R in the uv-plane. Points in the uv-plane are mapped to three-dimensional points (x, y, z) in three-space.

Notes:

Example 14.21

Determine a parametric representation for a cylindrical surface

$$x^2 + y^2 = 4$$

between $z = 0$ and $z = 8$.

SOLUTION Using cylindrical coordinates, we know that

$$x = 2\cos\theta \;\text{ and }\; y = 2\sin\theta$$

and so we can use $u = \theta$ and $v = z$ as our two parameters. This results in parametric equations

$$x(u, v) = 2\cos u,\; y(u, v) = 2\sin u,\; z(u, v) = v$$

in the region described by $0 \le u \le 2\pi$ and $0 \le v \le 8$. So a rectangle in the uv-plane maps to a cylindrical surface in three-dimensional space.

Example 14.22

Determine a parametric representation for the sphere $x^2 + y^2 + z^2 = a^2$, where a is a constant.

SOLUTION Now using spherical coordinates, we know that

$$x = a\sin\phi\cos\theta,\; y = a\sin\phi\sin\theta,\; z = a\cos\phi$$

and so we can use $u = \phi$ and $v = \theta$ as our two parameters. This results in parametric equations

$$x(u, v) = a\sin u\cos v,\; y(u, v) = a\sin u\sin v,\; z(u, v) = a\cos u$$

for (u, v) in the rectangular region R in the uv-plane given by $0 \le u \le \pi$ and $0 \le v \le 2\pi$.

Example 14.23

Determine a parametric representation for any surface $z = f(x, y)$ given by a function of two variables, for (x, y) in a region R.

SOLUTION In this case, we can simply use $u = x$ and $v = y$ for the two parameters. Then

$$x(u, v) = u,\; y(u, v) = v,\; z = f(u, v)$$

Notes:

gives the parametric equations for the surface. The region R in the xy-plane will be the same as the region R in the uv-plane in this case.

Now we find dS for a surface T given parametrically by equations

$$x = x(u, v),\ y = y(u, v),\ z = z(u, v)$$

for (u, v) in a region R, following the same idea as before. Small increments du and dv in the parameters will result in small parallelogram pieces of the surface of area dS. One side of the parallelogram comes from du while the other comes from dv. The two sides are given by vectors $\vec{a}\ du$ and $\vec{b}\ dv$ where

$$\vec{a} = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k} \text{ and } \vec{a} = \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k}.$$

To find the area of the parallelogram piece, we take the norm (magnitude) of the cross product, so

$$dS = \left|\left|\vec{a} \times \vec{b}\right|\right|\ du\, dv,$$

and we integrate this over R to get the total surface area. To clarify the notation, let us write

$$\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k},$$

a vector function describing the surface. Then

$$\vec{a} = \vec{r}_u(u, v) \text{ and } \vec{b} = \vec{r}_v(u, v)$$

as above. So dS can now be written as

$$dS = ||\vec{r}_u \times \vec{r}_v||\ du\, dv$$

and the total surface area is

$$S = \iint_R ||\vec{r}_u \times \vec{r}_v||\ du\, dv.$$

Example 14.24

Use the above method to compute the surface area of the cone $z = \sqrt{x^2 + y^2}$ up to a height of 4.

Solution Using R for the circular region below the cone, the standard parametrization is

$$x = u,\ y = v,\ z = \sqrt{u^2 + v^2}$$

Notes:

for (u, v) in R, as above for a surface given by a function of two variables. We compute $\vec{r}_u$, $\vec{r}_v$ and $\vec{r}_u \times \vec{r}_v$ first, using

$$\vec{r} = u\,\vec{i} + v\,\vec{j} + \sqrt{u^2 + v^2}\,\vec{k}.$$

We have

$$\vec{r}_u = \vec{i} + \frac{u}{\sqrt{u^2+v^2}}\vec{k} \text{ and } \vec{r}_v = \vec{j} + \frac{v}{\sqrt{u^2+v^2}}\vec{k}$$

and so

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{u}{\sqrt{u^2+v^2}} \\ 0 & 1 & \frac{v}{\sqrt{u^2+v^2}} \end{vmatrix} = -\frac{v}{\sqrt{u^2+v^2}}\vec{i} - \frac{u}{\sqrt{u^2+v^2}}\vec{j} + \vec{k}.$$

The norm $|\vec{r}_u \times \vec{r}_v|$ becomes

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{1 + \frac{u^2}{u^2+v^2} + \frac{v^2}{u^2+v^2}} = \sqrt{2}.$$

Therefore the surface area is

$$S = \iint_R \sqrt{2}\,du\,dv = 16\sqrt{2}\pi$$

square units, since the region R is a circle of radius 4. Notice that $||\vec{r}_u \times \vec{r}_v||$ is the same as $\sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}$ using x and y instead of u and v, as we would have obtained using the method from Section 13.5.

Example 14.25

Determine the surface area of the helicoid (a spiral ramp shown in Figure 14.12) given parametrically by

$$\vec{r}(u, v) = u\cos(v)\,\vec{i} + u\sin(v)\,\vec{j} + v\,\vec{k}$$

for $0 \le u \le 1$ and $0 \le v \le 4\pi$.

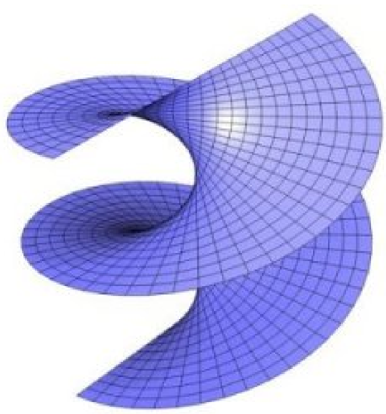


Figure 14.12: Helicoid

Solution Notice that we will be integrating over a rectangle in the uv-plane. We can immediately calculate $\vec{r}_u$ and $\vec{r}_v$ as

$$\vec{r}_u = \cos(v)\,\vec{i} + \sin(v)\,\vec{j} \text{ and } \vec{r}_v = -u\sin(v)\,\vec{i} + u\cos(v)\,\vec{j} + \vec{k}$$

and so

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 0 \\ -u\sin(v) & u\cos(v) & 1 \end{vmatrix} = \sin(v)\,\vec{i} - \cos(v)\,\vec{j} + u\,\vec{k}.$$

Notes:

Therefore the norm of this cross product is

$$||\vec{r}_u \times \vec{r}_v|| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{u^2+1}.$$

Integrating over the rectangle in the uv-plane gives us the surface area

$$S = \int_0^{4\pi}\int_0^1 \sqrt{1+u^2}\,du\,dv = 4\pi\left(\left(\frac{1}{2}\left(u\sqrt{1+u^2} + \ln|\sqrt{1+u^2}+u|\right)\right)\right|_0^1$$

which evaluates to $2\pi\left(\sqrt{2}+\ln(1+\sqrt{2})\right)$ square units.

Vector Fields and Flux

Given a parametrized surface T over a region R in the uv-plane, we have computed the surface area of T as $\int\int_T 1\,dS$. Replacing 1 with any function of three variables, an integral of the form $\int\int_T f(x,y,z)\,dS$ is called the *surface integral* of $f(x,y,z)$ over the surface T. To evaluate such a surface integral, one would translate $f(x,y,z)$ to a function of u and v by using the parametric equation $x = x(u,v)$, $y = y(u,v)$, and $z = z(u,v)$ for the given surface. Then multiply this expression by $dS = |\vec{r}_u \times \vec{r}_v|\ du\,dv$ and integrate over the region R in the uv-plane to obtain the surface integral.

Definition 116 Surface Integral

Let T be a surface defined by

$$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$$

for (u,v) in a region R of the uv-plane. The **surface integral** of a function $f(x,y,z)$ over T is

$$\int\int_T f(x,y,z)\,dS = \int\int_R f(x(u,v),y(u,v),z(u,v))\ ||\vec{r}_u \times \vec{r}_v||\ du\,dv.$$

Example 14.26

Compute the surface integral of $f(x,y,z) = xyz$ over that part of the cylinder $x^2+y^2 = 1$ above the first quadrant in the xy-plane and between $z = 0$ and $z = 2$.

SOLUTION We first need to parametrize the cylinder as we did earlier, to obtain

$$\vec{r}(u,v) = \cos(u)\,\vec{i} + \sin(u)\,\vec{j} + v\,\vec{k}$$

Notes:

for $0 \leq u \leq \frac{\pi}{2}$ and $0 \leq v \leq 2$. Next, we compute dS as before, using

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin(u) & \cos(u) & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos(u)\,\vec{i} + \sin(u)\,\vec{j}$$

This gives

$$||\vec{r}_u \times \vec{r}_v|| = \sqrt{\sin^2(u) + \cos^2(u) + 0} = \sqrt{1} = 1$$

for our surface. Now to integrate $f(x, y, z)$ over T, we change f into a function of u and v by writing

$$f(x, y, z) = xyz = \cos(u)\ \sin(u)\ v$$

as in our parametrization. Therefore the surface integral equals

$$\iint_T f(x, y, z)dS = \int_0^2 \int_0^{\pi/2} \cos(u)\sin(u)vdudv = \left(\frac{1}{2}v^2\Big|_0^2\right)\left(\frac{1}{2}\sin^2(u)\Big|_0^{\pi/2}\right) = 1.$$

A key example of an application of a surface integral is in computing the flow or *flux* of a vector field across a given surface, which we discuss next. As before, we will first address the case when the surface can be written in the form $z = f(x, y)$.

Definition 117 Flux

Let $z = f(x, y)$ define a surface T for (x, y) in a region R in the xy-plane, and let

$$\vec{F}(x, y, z) = M(x, y, z)\vec{i} + N(x, y, z)\vec{j} + P(x, y, z)\vec{k}$$

be a vector field. The **flux** of $\vec{F}$ across T is the surface integral

$$\iint_T \vec{F} \cdot \vec{n}\, dS = \iint_R -M\, f_x(x, y) - N\, f_y(x, y) + P\, dx\, dy$$

where $\vec{n}$ is the unit normal vector to the surface.

Recall that a vector $\vec{N}$ normal to the surface will be

$$\vec{N} = -f_x(x, y)\vec{i} - f_y(x, y)\vec{j} + \vec{k}$$

and so the unit normal vector is found by dividing this by its norm

$$\vec{n} = \frac{\vec{N}}{||\vec{N}||} = \frac{\vec{N}}{\sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2}}.$$

Notes:

Notice that the expression representing $|\vec{N}|$ is exactly the same as in dS, so these will cancel to obtain the right-most definition above. That is,

$$\begin{aligned}\vec{F}\cdot\vec{n}\,dS &= \frac{\vec{F}\cdot\vec{N}}{\sqrt{1+f_x(x,y)^2+f_y(x,y)^2}}\sqrt{1+f_x(x,y)^2+f_y(x,y)^2}\,dx\,dy\\ &= -M\,f_x(x,y)-N\,f_y(x,y)+P\ \ dx\,dy.\end{aligned}$$

It is worth noting that we are only considering *orientable* surfaces here, which are surfaces with two sides. In the case of a sphere, for example, there is a normal vector pointing inward and another pointing outward. For a closed surface such as a sphere it is conventional to take the normal vector pointing outward for $\vec{n}$. That is, we adopt the convention that outward flux is positive. A surface such as a Möbius strip would not be orientable since it has only one side; a pencil traced along one side around the surface will eventually arrive back at its starting position.

Example 14.27

Let $\vec{F}(x,y,z) = x\,\vec{i}+y\,\vec{j}+z\,\vec{k}$ and consider the conical surface T given by $z = f(x,y) = \sqrt{x^2+y^2}$ for (x,y) within the disk R given by $x^2+y^2 \le 4$. Compute the flux of $\vec{F}$ across this surface.

Solution For the above definition, we have $M = x$, $N = y$, and $P = z$. We also have

$$f_x(x,y) = \frac{x}{\sqrt{x^2+y^2}} \text{ and } f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}}$$

as previously. Therefore the flux across T is

$$\iint_T \vec{F}\cdot\vec{n}\,dS = \iint_R -\frac{x^2}{\sqrt{x^2+y^2}} - \frac{y^2}{\sqrt{x^2+y^2}} + \sqrt{x^2+y^2}\,dx\,dy$$

since $P = z = \sqrt{x^2+y^2}$. Notice that the integrand simplifies to zero, and so the flux across T is equal to zero. This makes sense since the vector field $\vec{F}$ goes straight out from the origin, which is along the conical surface, not across it. That is, there is no flow across T because $\vec{F}\cdot\vec{n} = 0$.

Next we address the case when the surface T is given parametrically by a function $\vec{r}(u,v)$ of two parameters u and v from a region R in the uv-plane. For surfaces that fold and twist, the formulas can look more complicated, but often the calculations are simpler. Recall that a small parallelogram piece of the surface T will have area $||\vec{r}_u \times \vec{r}_v||\ \ du\,dv$. The vectors $\vec{r}_u$ and $\vec{r}_v$ are tangent to the

Notes:

surface along the sides.

We now put the cross product to another use, because $\vec{F} \cdot \vec{n}\, dS$ involves not only area but *direction* as well. We need the unit vector $\vec{n}$ to know how much flow goes through, since $\vec{r}_u \times \vec{r}_v$ is perpendicular to the surface. Also since $\vec{n} = \frac{\vec{N}}{||\vec{N}||}$ we get

$$dS = ||\vec{N}||\, du\, dv,$$

as the square root cancels out in $\vec{n}\, dS$. This leave the following formula for the flux of a vector field across the surface T.

Definition 118 Flux

Let $\vec{r}(u, v)$ define a surface T for (u, v) in a region R in the uv-plane, and let

$$\vec{F}(x, y, z) = M(x, y, z)\vec{i} + N(x, y, z)\vec{j} + P(x, y, z)\vec{k}$$

be a vector field. The **flux** of $\vec{F}$ across T is

$$\iint_T \vec{F} \cdot \vec{n}\, dS = \iint_R \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\ du\, dv$$

where $\vec{r}_u \times \vec{r}_v$ is normal to the surface.

Example 14.28

Let $\vec{F}(x, y, z) = \vec{k}$ and consider the spherical surface T given by $x^2 + y^2 + z^2 = 9$. Compute the flux of $\vec{F}$ across the top half of the surface T.

Solution Using spherical coordinates and parameters $u = \phi$ and $v = \theta$, we have

$$\vec{r}(u, v) = 3\sin(u)\cos(v)\,\vec{i} + 3\sin(u)\sin(v)\,\vec{j} + 3\cos(u)\,\vec{k}$$

and so

$$\begin{aligned} \vec{N} = \vec{r}_u \times \vec{r}_v &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3\cos(u)\cos(v) & 3\cos(u)\sin(v) & -3\sin(u) \\ -3\sin(u)\sin(v) & 3\sin(u)\cos(v) & 0 \end{vmatrix} \\ &= 9\left(\sin^2(u)\cos(v)\,\vec{i} + \sin^2(u)\sin(v)\,\vec{j} + \cos(u)\sin(u)\,\vec{k}\right). \end{aligned}$$

Taking the dot product with $\vec{F} = \vec{k}$ we get

$$\vec{F} \cdot \vec{N} = 9\cos(u)\sin(u).$$

Notes:

Lastly, we integrate on $0 \leq u \leq \frac{\pi}{2}$ and $0 \leq v \leq 2\pi$ to obtain a flux of

$$\int_0^{2\pi} \int_0^{\pi/2} 9\cos(u)\sin(u)\, du\, dv = 18\pi \left(\frac{1}{2}\sin^2(u)\right)\Bigg|_0^{\pi/2} = 9\pi$$

across the hemisphere.

We will see another generalization of Green's Theorem in the next section, called the Divergence Theorem, where we look at flux across closed surfaces.

Notes:

Exercises 14.4

Problems

In Exercises 1 – 13, find the surface area.

1. Paraboloid $z = x^2 + y^2$ below the plane $z = 4$
2. Paraboloid $z = x^2 + y^2$ between the plane $z = 4$ and $z = 8$
3. Plane $z = x - y$ inside the cylinder $x^2 + y^2 = 1$
4. Plane $z = 3x + 4y$ above the square $0 \leq x \leq 1, 0 \leq y \leq 1$
5. Spherical cap $x^2 + y^2 + z^2 = 1$ above $z = \frac{1}{\sqrt{2}}$
6. Spherical band $x^2 + y^2 + z^2 = 1$ between $z = 0$ and $z = \frac{1}{\sqrt{2}}$
7. Plane $z = 7y$ above a triangle of area A
8. Cone $z^2 = x^2 + y^2$ between the planes $z = 3$ and $z = 7$
9. The monkey saddle $z = \frac{1}{3}x^3 - xy^2$ inside $x^2 + y^2 = 1$
10. Plane $z = 1 - 2x - 2y$ inside $x \geq 0, y \geq 0, z \geq 0$
11. Plane $z = x - y$ inside the cylinder $x^2 + y^2 = 1$
12. Right circular cone of radius r and height h
13. Gutter $z = x^2$ below $z = 9$ and between $y = \pm 2$

In Exercises 14 – 17, compute the surface integrals $\int\int g(x, y, z)\, dS$.

14. $g = xy$ over the triangle $x + y + z = 1, x, y, z \geq 0$
15. $g = x^2 + y^2$ over the top half of $x^2 + y^2 + z^2 = 1$
16. $g = xyz$ on $x^2 + y^2 + z^2 = 1$ above $z^2 = x^2 + y^2$
17. $g = x$ on the cylinder $x^2 + y^2 = 4$ between $z = 0$ and $z = 3$

In Exercises 18 – 21, find the flux $\int\int$ $\mathbf{F} \cdot \mathbf{n}\, dS$ for $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

18. Paraboloid $z = x^2 + y^2$ below the plane $z = 4$
19. Paraboloid $z = x^2 + y^2$ between $z = 4$ and $z = 8$
20. Plane $z = x - y$ inside the cylinder $x^2 + y^2 = 1$
21. Plane $z = 3x + 4y$ above the square $0 \leq x \leq 1, 0 \leq y \leq 1$

14.5 The Divergence Theorem

Green's Theorem dealt with a closed curve in two dimensions. In this section, we look at a closed surface enclosing a solid region in three dimensions. A surface integral representing the flux through the surface can be written as a triple integral of the enclosed solid. This is the Divergence Theorem, attributed independently to Gauss and Russian mathematician Ostrogradsky. First, we define what we mean by the divergence of a vector field.

Definition 119 Divergence

Let

$$\vec{F}(x,y,z) = M(x,y,z)\,\vec{i} + N(x,y,z)\,\vec{j} + P(x,y,z)\,\vec{k}$$

be a vector field. The divergence div $\vec{F}$ of the field $\vec{F}$ is given by

$$\operatorname{div}\vec{F}(x,y,z) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

which is a real-valued function. One can write this as a dot product of

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

with the vector field $\vec{F}$, or as div $\vec{F} = \nabla \cdot \vec{F}$.

Example 14.29

Determine the divergence div $\vec{F}$ if the field $\vec{F}$ is given by

$$\vec{F}(x,y,z) = xz\,\vec{i} + xyz\,\vec{j} - y^2\,\vec{k}.$$

Solution We compute

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2) = z + xz.$$

The reason for the terminology *divergence* relates again to fluid flow. If $\vec{F}(x,y,z)$ represents the velocity field of a fluid or gas, then div $\vec{F}(x_0,y_0,z_0)$ represents the rate of change of the fluid or gas flowing from the point (x_0,y_0,z_0). That is, div $\vec{F}$ relates to the tendency of the fluid or gas to diverge from the point

Notes:

in question. If div $\vec{F} = 0$ for all points (x, y, z), we say the fluid or gas is **incompressible**. In other words, when div $\vec{F} = 0$, the flow in equals the flow out.

For example, the spin fields

$$\vec{F} = -y\,\vec{i} + x\,\vec{j} + 0\,\vec{k} \quad \text{and} \quad \vec{F} = 0\,\vec{i} - z\,\vec{j} + y\,\vec{k}$$

have zero divergence at all points. The first is a spin around the z-axis while the second is a spin around the x-axis. The flow out across a surface will equal the flow into the surface. However, in a radial flow such as

$$\vec{F} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$$

flow is straight out from the origin. The divergence is div $\vec{F} = 1 + 1 + 1 = 3$ in this case, indicating the tendency of the fluid to diverge from any point. The Divergence Theorem, stated next, will clarify these situations.

Theorem 136 The Divergence Theorem

Consider a vector field

$$\vec{F} = M(x, y, z)\,\vec{i} + N(x, y, z)\,\vec{j} + P(x, y, z)\,\vec{k}$$

whose component functions M, N, and P have continuous partial derivatives within a region containing a simple solid region D whose boundary surface is T. Then the flux of $\vec{F}$ across T equals the triple integral of the divergence div $\vec{F}$ inside D. That is,

$$\iint_T \vec{F} \cdot \vec{n}\, dS = \iiint_D \operatorname{div} \vec{F}\, dV.$$

Example 14.30

Compute the flux of the vector field

$$\vec{F}(x, y, z) = 3xy\,\vec{i} + x^2 e^{xz^2}\,\vec{j} + (y + 2z)\,\vec{k}$$

across the surface of the cylinder $x^2 + y^2 = 1$ between $z = 0$ and $z = 4$.

Solution Let D be the cylindrical solid and let T be its surface. We employ the Divergence Theorem here to compute $\iint_T \vec{F} \cdot \vec{n}\, dS$ via a triple integral over D instead of a surface integral. The divergence of $\vec{F}$ is

$$\operatorname{div} \vec{F}(x, y, z) = 3y + 2$$

Notes:

and so

$$\iint_T \vec{F}\cdot\vec{n}\,dS = \iiint_D 3y+2\,dV.$$

The region D is a cylinder, so cylindrical coordinates make sense here. The triple integral becomes

$$\iint_T \vec{F}\cdot\vec{n}\,dS = \int_0^4\int_0^{2\pi}\int_0^1 (3r\sin\theta+2)r\,dr\,d\theta\,dz = 8\pi.$$

If we think of $\vec{F}$ as a fluid or gas flow, the flux being a positive value here indicates that the net flow across the surface is directed outward from the cylinder.

To see the reasoning behind the Divergence Theorem, consider a small box with center at a point (x, y, z) and edges of length Δx, Δy, Δz. Out of the top and bottom of the box, the normal vectors are $\vec{k}$ and $-\vec{k}$. The dot product with $\vec{F} = M\vec{i}+N\vec{j}+P\vec{k}$ is P or $-P$. So the two fluxes are close to $P(x, y, z+0.5\Delta z)\Delta x\Delta y$ and $-P(x, y, z-0.5\Delta z)\Delta x\,\Delta y$. When the top is combined with the bottom, the difference of those is ΔP:

$$\text{net flux upward} \approx \Delta P\,\Delta x\,\Delta y \approx \frac{\partial P}{\partial z}\Delta V.$$

Similarly, the combined flux on the sides and the front and back will be approximately $\dfrac{\partial P}{\partial y}\Delta V$ and $\dfrac{\partial P}{\partial x}\Delta V$, respectively. Adding the six faces, we reach the key point:

$$\text{flux out of the box} \approx \left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}+\frac{\partial P}{\partial z}\right)\ \Delta V$$

which of course is $\left(\operatorname{div}\vec{F}\right)\ \Delta V$.

Note however that the ratio $\dfrac{\Delta P}{\Delta z}$ is not exactly $\dfrac{\partial P}{\partial z}$ - the difference is of order Δz. So the difference in the net flux upward is of higher order than $\Delta V\,\Delta z$. Added over many boxes, this error disappears as Δz approaches zero.

The sum of $\left(\operatorname{div}\vec{F}\right)\ \Delta V$ over all the boxes approaches $\iiint_D \operatorname{div}\vec{F}\,dV$. On the other side of the equation is a sum of fluxes. There is $\vec{F}\cdot\vec{n}\,\Delta S$ out of the top of one box, plus $\vec{F}\cdot\vec{n}\,\Delta S$ out the bottom of the box above. The first has $\vec{n}=\vec{k}$ and the second has $\vec{n}=-\vec{k}$. They cancel each other out - the flow goes from box to box. This happens every two boxes that meet. The only fluxes to survive are at the outer surface T. The final step, as $\Delta x, \Delta y, \Delta z$ approach zero, gives that those outside terms approach $\iint \vec{F}\cdot\vec{n}\,dS$. This would finish the argument for the Divergence Theorem. A formal proof would be similar to the one done for

Notes:

Green's Theorem. The reasoning above, however, is probably more useful than the detailed proof.

Another example of flux involves *heat flow* and the rate of heat flow across a surface. Suppose that $T(x, y, z)$ is the temperature at the point (x, y, z) in a solid or substance. Then the **heat flow** is defined as the vector field

$$\vec{F} = -K\nabla T$$

where K is a constant called the **conductivity** of the substance. The rate of heat flow across the surface of a simple solid D in the substance is given by the flux integral

$$\iint \vec{F} \cdot \vec{n}\, dS = -K \iint \nabla T \cdot \vec{n}\, dS = -K \iiint_D \text{div } \nabla T\, dS$$

by the Divergence Theorem.

Example 14.31 Heat Flow

Suppose that $T(x, y, z) = x^2 + y^2 + z^2$ is the temperature of a solid sphere centered at the origin of radius 2. Determine the heat flow across the surface of the sphere if the conductivity is the constant K.

Solution The heat flow is the vector field

$$\vec{F} = -K\nabla T = -2K(x\,\vec{i} + y\,\vec{j} + z\,\vec{k})$$

and so the heat flow across the surface of the sphere is

$$\iint \vec{F} \cdot \vec{n}\, dS = -2K \iint (x\,\vec{i} + y\,\vec{j} + z\,\vec{k}) \cdot \vec{n}\, dS$$

which we can evaluate using the Divergence Theorem. This integral becomes

$$\iint \vec{F} \cdot \vec{n}\, dS = -2K \iiint_D (1 + 1 + 1)\, dV$$

where D is the sphere. Noting that the volume of the sphere is $\frac{4}{3}\pi(2)^3$ cubic units, we get a heat flow of $-2K(3)\left(\frac{4}{3}\pi(2)^3\right) = -64K\pi$. The flux is negative, and so heat is flowing inward, which makes sense since the surface of the sphere is warmer than the inside.

In the next final section, we look at the case of a surface which is not closed. In such a case, the boundary of the surface is a curve in three-dimensional space, and the flux across the surface can be evaluated using a line integral around the closed curve. This is yet another generalization of Green's Theorem called Stokes' Theorem.

Notes:

Exercises 14.5

Problems

In Exercises 1 – 6, compute the divergence of each field.

1. $\mathbf{F}(x,y,z) = 2xy\mathbf{i} - y^2\mathbf{j} + 3x^2y\mathbf{k}$

2. $\mathbf{F}(x,y,z) = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$

3. $\mathbf{F}(x,y,z) = 2x^2y\mathbf{i} - (z^3 - 5x)\mathbf{j} + 7x^2\mathbf{k}$

4. $\mathbf{F}(x,y,z) = xe^z\mathbf{i} + (x+y+z)\mathbf{j} + (3y+z^2)\mathbf{k}$

5. $\mathbf{F}(x,y,z) = e^x\sin(y)\mathbf{i} + e^x\cos(y)\mathbf{j} + z\mathbf{k}$

6. $\mathbf{F}(x,y,z) = z^2\sin\left(e^{y^2+2y}\right)\mathbf{i} - x^3\sqrt{z^2+1}\,\mathbf{j} + e^{3y}\cos(4x - 5)\mathbf{k}$

In Exercises 7 – 15, compute the flux $\int\int \mathbf{F}\cdot\mathbf{n}\,dS$ by the Divergence Theorem.

7. $\mathbf{F}(x,y,z) = x\mathbf{i} + x\mathbf{j} + x\mathbf{k}$, S: unit sphere $x^2+y^2+z^2 = 1$

8. $\mathbf{F}(x,y,z) = -y\mathbf{i} + x\mathbf{j}$, V: unit cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$

9. $\mathbf{F}(x,y,z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, S: unit sphere

10. $\mathbf{F}(x,y,z) = x^2\mathbf{i} + 8y^2\mathbf{j} + z^2\mathbf{k}$, V: unit cube

11. $\mathbf{F}(x,y,z) = x\mathbf{i} + 2y\mathbf{j}$, S: sides $x = 0$, $y = 0$, $z = 0$, $x+y+z = 1$

12. $\mathbf{F}(x,y,z) = \rho(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$, S: sphere $\rho = a$

13. $\mathbf{F}(x,y,z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$, S: sphere $\rho = a$

14. $\mathbf{F}(x,y,z) = z^2\mathbf{k}$, V: upper half of ball $\rho \le a$

15. $\mathbf{F}(x,y,z) = \nabla f(x,y,z)$ where $f(x,y,z) = xe^y sin(z)$, S: sphere $\rho = a$

14.6 Stokes' Theorem and Curl

In the last section, the Divergence Theorem gave us a generalization of Green's Theorem to a closed surface. In this section we consider a surface which is not closed, and hence has a boundary. This boundary will be a curve in three-dimensional space. This puts us back in the setting of Green's Theorem - a region bounded by a curve. Stokes' Theorem will be this generalization of Green's Theorem to three-dimensional space. Instead of a region in the plane, the region will be a surface in three-dimensional space.

Before we get to Stokes' Theorem, we will define and discuss the idea of the *curl* of a vector field. In the previous section, we defined divergence as the dot product of $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ with the vector field. We will use another vector operation, the cross product, to define curl.

Curl

Consider a three-dimensional vector field

$$\vec{F}(x, y, z) = M(x, y, z)\,\vec{i} + N(x, y, z)\,\vec{j} + P(x, y, z)\,\vec{k}$$

in which the component functions have continuous partial derivatives. In Green's Theorem, the vector field had only two dimensions and the line integral around the curve turned into a double integral of the scalar function $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$. In three-dimensions, this function changes to something involving what we will call the curl of the vector field.

Notes:

Definition 120 Curl

Let $\vec{F}(x,y,z) = M(x,y,z)\,\vec{i} + N(x,y,z)\,\vec{j} + P(x,y,z)\,\vec{k}$ be a vector field. The **curl** of the field $\vec{F}$, denoted by curl $\vec{F}$, is given by

$$\text{curl}\,\vec{F}(x,y,z) = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\vec{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\vec{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}$$

which is a real-valued function. An easier way to remember this is as the cross product of

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

with the vector field $\vec{F}$, or as

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

Note that if $P(x,y,z) = 0$ above and we have a two-dimensional vector field with $M(x,y,z) = M(x,y)$ and $N(x,y,z) = N(x,y)$ in the plane not depending on z, then $\dfrac{\partial M}{\partial z} = \dfrac{\partial N}{\partial z} = 0$ and

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M(x,y) & N(x,y) & 0 \end{vmatrix} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}$$

as in Green's Theorem. That is, in this case we have the restatement of Green's Theorem as

$$\oint_C M(x,y)\,dx + N(x,y)\,dy = \iint_R \text{curl}\,\vec{F}\cdot\vec{n}\,dA$$

where C is the closed boundary curve of the region R in the plane.

Example 14.32

Determine the curl of the spin field

$$\vec{F}(x,y,z) = (z-y)\,\vec{i} + (x-z)\,\vec{j} + (y-x)\,\vec{k}$$

Notes:

which has an axis of rotation given by the vector $\vec{a} = (1, 1, 1)$.

Solution The curl of $\vec{F}$ is given by the cross product

$$\begin{aligned} \text{curl } F &= \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (z-y) & (x-z) & (y-x) \end{vmatrix} \\ &= (1+1)\vec{i} - (-1-1)\vec{j} + (1+1)\vec{k} = (2, 2, 2) = 2\vec{a} \end{aligned}$$

This result generalizes. The curl of a spin field about an axis of rotation given by the vector $\vec{a}$ is in the direction of the axis of rotation. This gives us our first clue as to what the curl of a vector field represents - it measures the spin.

We have spent a significant amount of time talking about conservative vector fields in this chapter. What is the curl of such a vector field, one which is the gradient of a scalar function? Consider the potential function $f(x, y, z)$ with continuous derivatives with gradient vector field

$$\vec{F} = \nabla f(x, y, z) = \frac{\partial f}{\partial x}\,\vec{i} + \frac{\partial f}{\partial y}\,\vec{j} + \frac{\partial f}{\partial z}\,\vec{k}$$

Then the curl of $\vec{F}$ is

$$\begin{aligned} \text{curl } \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y}\right)\vec{i} + \left(\frac{\partial}{\partial z}\frac{\partial f}{\partial x} - \frac{\partial}{\partial x}\frac{\partial f}{\partial z}\right)\vec{j} + \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\right)\vec{k} \end{aligned}$$

Since $f_{yz} = f_{zy}, f_{xz} = f_{zx}$, and $f_{xy} = f_{yx}$, we arrive at curl $\vec{F} = \vec{0}$. Therefore the curl of any conservative vector field is zero.

Similarly, if $\vec{F}$ is the curl of another vector field, say $\vec{F} = \text{curl } \vec{G}$, then div $\vec{F} =$ div curl $\vec{G} = 0$. That is, the divergence of a curl field is zero. We summarize these last two results in the following theorem.

Notes:

Theorem 137

Consider a vector field $\vec{F}$. If $\vec{F} = \nabla f(x, y, z)$ is a conservative vector field, then curl $\vec{F} = \vec{0}$. If $\vec{G}$ is another vector field and $\vec{F} =$ curl $\vec{G}$, then div $\vec{F} = 0$.

While we may think of curl as measuring the spin of a field, we would be incorrect in thinking that a vector field made up of parallel vectors would always have zero for a curl. The vector field in the next example is such a case, where the parallel vectors have differing lengths.

Example 14.33

Determine the curl of the vector field $\vec{F}(x, y, z) = z\,\vec{i}$, in which every vector is parallel to the x-axis.

Solution We compute

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & 0 & 0 \end{vmatrix} = \vec{j}$$

and so the curl of this vector field is in the direction of the y-axis. To make some sense of this, imagine a wheel placed anywhere in the xz-plane of this velocity vector field. The velocity at the top of the wheel will be greater than that at the bottom, causing the wheel to spin. See Figure 14.13. The axis of rotation is in the direction of the y-axis, and this is the curl vector calculated above.

Conservative Fields and Potential Functions

Recall that a vector field $\vec{F}(x, y) = M(x, y)\,\vec{i} + N(x, y)\,\vec{j}$ in the xy-plane is conservative (is a gradient field) whenever $M_y = N_x$, which is equivalent to saying the curl of this vector field is $\vec{0}$. As we saw above, this extends to vector fields in three dimensions. If

$$\vec{F}(x, y, z) = M(x, y, z)\,\vec{i} + N(x, y, z)\,\vec{j} + P(x, y, z)\,\vec{k}$$

is conservative, then curl $\vec{F}(x, y, z) = \vec{0}$. In particular, curl $\vec{F} = \vec{0}$ means that

$$M_y = N_x, \quad M_z = P_x, \quad \text{and } N_z = P_y.$$

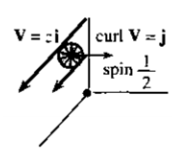


Figure 14.13: Velocity field $\mathbf{V} = z\mathbf{i}$

Notes:

These are three equations we can readily deal with to determine whether there exists a potential function $f(x, y, z)$ so that $\vec{F} = \nabla f$. Recall now the key properties of conservative vector fields, stated here now for three dimensions.

Key Idea 66

A vector field $\vec{F}(x, y, z) = M(x, y, z)\,\vec{i} + N(x, y, z)\,\vec{j} + P(x, y, z)\,\vec{k}$ is conservative if it has the following properties:

1. The work $\oint_C \vec{F} \cdot d\vec{r} = 0$ around every closed path in space;
2. The value of work $\oint_C \vec{F} \cdot d\vec{r}$ depends only on the initial point P and terminal point Q, not on the path in space;
3. $\vec{F}$ is a gradient field: there exists a function $f(x, y, z)$ so that $M = f_x$, $N = f_y$, and $P = f_z$;
4. The components of $\vec{F}$ satisfy $M_y = N_x$, $M_z = P_x$, and $N_z = P_y$.

A field with one of these properties has all of them. The fourth one is a quick check to determine if a given vector field in three dimensions is conservative or not.

Example 14.34
Determine whether or not

$$\vec{F}(x, y, z) = (z - y)\,\vec{i} + (x - z)\,\vec{j} + (y - x)\,\vec{k}$$

is a conservative field.

Solution We compute the appropriate partial derivatives of the components. Notice that $M_z = 1$ while $P_x = -1$. Since these are not equal, the fourth property above fails and so $\vec{F}$ cannot be the gradient field of a potential function.

Example 14.35
Determine whether or not

$$\vec{F}(x, y, z) = e^{yz}\,\vec{i} + xz\,e^{yz}\,\vec{j} + xy\,e^{yz}\,\vec{k}$$

is a conservative field. If so, find a potential function $f(x, y, z)$ for this vector field.

Notes:

Solution We compute the appropriate partial derivatives of the components. Notice that

$$M_y = z\,e^{yz} = N_x,\ M_z = xy^2\,e^{yz} = P_x,\ \text{and } N_z = (x + xyz)\,e^{yz} = P_y$$

and so $\vec{F}$ is conservative. To find its potential function, we use the same technique as in Section 14.2. There is a function $f(x, y, z)$ so that $f_x = e^{yz}$, $f_y = xz\,e^{yz}$, and $f_z = xy\,e^{yz}$. We start with the first and integrate with respect to x, which yields that

$$f(x, y, z) = x\,e^{yz} + g(y, z)$$

where g may depend at most only on y and z. Next take the derivative of this with respect to y and compare with the above f_y. This yields

$$f_y = xz\,e^{yz} + g_y(y, z) = xz\,e^{yz}$$

and so $g_y(y, z) = 0$. This means that $g(y, z) = g(z)$ can only depend on at most z. Similarly, the z-derivative yields

$$f_z = xy\,e^{yz} + g'(z) = xy\,e^{yz}$$

and so $g'(z) = 0$ meaning that g is a constant. Therefore the potential function is $f(x, y, z) = x\,e^{yz} + C$ for any constant C.

Now that we have worked with curl and have see how it relates to conservative fields, we will move on to the last main result in this chapter, Stokes' Theorem.

Stokes' Theorem

Stokes' Theorem will be like Green's Theorem - a line integral equals a surface integral. The line integral is still the work $\oint \vec{F} \cdot d\vec{r}$ around a curve. The surface integral in Green's Theorem is $\iint (N_x - M_y)\,dx\,dy$, with the surface being flat in the xy-plane. The normal vector to this surface is therefore $\vec{k}$, and we recognize $N_x - M_y$ as the $\vec{k}$-component of the curl. Green's Theorem uses only this component because the normal vector direction is always $\vec{k}$. For Stokes' Theorem on a curved surface, we will need all three components of curl as the normal vector is not always as simple.

Notes:

Theorem 138 Stokes' Theorem

Consider a three-dimensional vector field $\vec{F}(x, y, z)$ and a surface T with closed boundary curve C, oriented in the positive direction. Then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_T (\text{curl } \vec{F}) \cdot \vec{n}\, dS$$

where $\vec{n}$ is the unit normal vector of the surface.

While this theorem is not easy to visualize, think of the right-hand side above as the sum of the spins along the surface. The left-hand side is the total circulation (or work) around the boundary C. Notice one simple corollary of this result, however. If $\vec{F}$ is conservative (a gradient field) then curl $\vec{F} = \vec{0}$ and so the work done around the closed curve C is zero, as we know.

We next provide a quick argument as to why this theorem is true, leaving out a formal proof. In Figure 14.14 we see a triangle ABC attached to a triangle ACD, creating a surface. More generally S will be a closed curved surface but two triangles are enough to make the point here. In the plane of each triangle, Green's Theorem is known and gives us

$$\oint_{AB+BC+CA} \vec{F} \cdot d\vec{R} = \iint_{ABC} \text{curl } \vec{F} \cdot \vec{n}_1\, dS$$

and

$$\oint_{AC+CD+DA} \vec{F} \cdot d\vec{R} = \iint_{ACD} \text{curl } \vec{F} \cdot \vec{n}_2\, dS.$$

Now add. The right sides give the integral of curl $\vec{F} \cdot \vec{n}$ over the two-triangle surface. On the left side we get the integral over CA canceling the one over AC - the cross cut disappears. That leaves $AB + BC + CD + DA$, the boundary C of the surface, the left side of Stokes' Theorem. Next we give some examples of Stokes' Theorem in use.

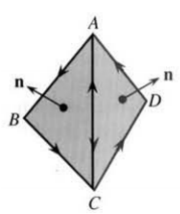


Figure 14.14: Surface formed by two triangles

Example 14.36

Use Stokes' Theorem to compute $\iint_T \text{curl } \vec{F} \cdot \vec{n}\, dS$, if T is the part of the unit sphere $x^2 + y^2 + z^2 = 1$ lying above the xy-plane and

$$\vec{F}(x, y, z) = y\,\vec{i} + x^2\,\vec{j} + z\,\vec{k}$$

Solution Using Stokes' Theorem, we can change this surface integral to a line integral around the boundary of this hemisphere, which is the unit circle.

Notes:

We use the standard parametrization

$$\vec{r}(t) = \cos(t)\,\vec{i} + \sin(t)\,\vec{j} + 0\,\vec{k}$$

for the unit circle C, where $0 \le t \le 2\pi$. By Stokes' Theorem,

$$\begin{aligned}\iint_T \operatorname{curl} \vec{F} \cdot \vec{n}\, dS &= \oint_C \vec{F} \cdot d\vec{r} \\ &= \int_0^{2\pi} \big(\sin(t), \cos^2(t), 0\big) \cdot (-\sin(t), \cos(t), 0)\ dt \\ &= \int_0^{2\pi} -\sin^2(t) + \cos^3(t)\, dt = -\pi\end{aligned}$$

In this way, we have avoided computing the curl of the given vector field as well as avoided computing a surface integral by instead computing a line integral.

Example 14.37

Use Stokes' Theorem to compute $\displaystyle\oint_C \vec{F} \cdot d\vec{r}$ where

$$\vec{F}(x, y, z) = -y^2\,\vec{i} + x\,\vec{j} + z^2\,\vec{k}$$

and C is the curve of intersection of the plane $z = 2-y$ and the cylinder $x^2+y^2 = 1$, with the usual counterclockwise orientation.

Solution Instead of computing a line integral in this case, we will instead compute a surface integral using a surface T that has C as its boundary. The surface T we will choose is the elliptical region in the plane $z = 2 - y$ inside the cylinder $x^2 + y^2 = 1$. The shadow of this surface down on the xy-plane is the unit disc D. And so with $z = 2 - y$ we parametrize

$$\vec{r}(x, y) = x\,\vec{i} + y\,\vec{j} + (2 - y)\,\vec{k}$$

which results in a normal vector

$$\vec{n} = \vec{r}_x \times \vec{r}_y = \vec{j} + \vec{k}$$

For the curl of $\vec{F}$, we compute

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = (1 + 2y)\,\vec{k}$$

Notes:

Therefore by Stokes' Theorem

$$\begin{aligned}
\oint_C \vec{F}\cdot d\vec{r} &= \iint_T \text{curl}\,\vec{F}\cdot\vec{n}\,dS \\
&= \iint_D (0,0,1+2y)\cdot(0,1,1)\,dA \\
&= \iint_D 1+2y\,dA \\
&= \int_0^{2\pi}\int_0^1 (1+2r\sin(\theta))\,r\,dr\,d\theta = \pi
\end{aligned}$$

In this way, we have avoided having to parametrize the boundary curve C and instead only had to do a surface integral over a surface above the unit disk in the xy-plane.

In closing, notice how most of the main results in this chapter (Green's Theorem, The Divergence Theorem, Stokes' Theorem) involve an interchange between one type of integral and another, such as a work line integral being equal to a flux surface integral. Besides being useful theoretically, in certain situations one of these integrals may be easier than the other to compute directly and so we get a computational advantage as well.

Notes:

Exercises 14.6

Problems

In Exercises 1 – 6, find curl F.

1. $\mathbf{F}(x, y, z) = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$

2. $\mathbf{F} = \nabla f(x, y, z)$ where $f(x, y, z) = xe^y sin(z)$

3. $\mathbf{F}(x, y, z) = (x + y + z)(\mathbf{i} + \mathbf{j} + \mathbf{k})$

4. $\mathbf{F}(x, y, z) = (x + y)\mathbf{i} - (x + y)\mathbf{j}$

5. $\mathbf{F}(x, y, z) = \sqrt{x^2 + y^2 + z^2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

6. $\mathbf{F}(x, y, z) = (\mathbf{i} + \mathbf{j}) \times (x\mathbf{i} + y\mathbf{j})$

In Exercises 7 – 10, compute $\oint$ F $\cdot$ *dR* by Stokes' Theorem.

7. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j}$, $C =$ circle $x^2 + z^2 = 1$, $y = 0$

8. $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$

9. $\mathbf{F}(x, y, z) = 2x^2y\mathbf{i} - (z^3 - 5x)\mathbf{j} + 7x^2\mathbf{k}$

10. $\mathbf{F}(x, y, z) = xe^z\mathbf{i} + (x + y + z)\mathbf{j} + (3y + z^2)\mathbf{k}$

In Exercises 11 – 12, compute $\oint$ curl F $\cdot$ n *dR* over the top half of the sphere $x^2+y^2+z^2 = 1$ and separately $\oint$ F *dR* by Stokes' Theorem.

11. $\mathbf{F} = (y\mathbf{i} - x\mathbf{j})$

12. $\mathbf{F} = (z - y)\mathbf{i} + (x - z)\mathbf{j} + (y - x)\mathbf{k}$

A: Solutions To Selected Problems

Chapter 10

Section 10.1

1. right hand

3. curve (a parabola); surface (a cylinder)

5. a hyperboloid of two sheets

7. $\|\overline{AB}\| = \sqrt{6}$; $\|\overline{BC}\| = \sqrt{17}$; $\|\overline{AC}\| = \sqrt{11}$. Yes, it is a right triangle as $\|\overline{AB}\|^2 + \|\overline{AC}\|^2 = \|\overline{BC}\|^2$.

9. Center at $(4, 0, -7)$; radius = $\sqrt{118}$

11. $(x-6)^2 + y^2 + z^2 = 25$

13. Center at $(-2, 1, 2)$; radius = $\sqrt{5}$

15. $\frac{(x-3)^2}{16} + \frac{(y+2)^2}{25} + z^2 = 1$

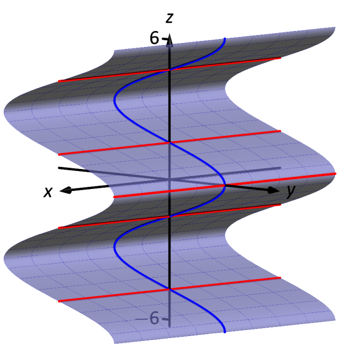
17. Circles

19. A single point.

21. Region bounded between the planes $x = 0$ (the $y-z$ coordinate plane) and $x = 3$.

23. All points in space where the y value is greater than 3; viewing space as often depicted in this text, this is the region "to the right" of the plane $y = 3$ (which is parallel to the $x-z$ coordinate plane.)

25.



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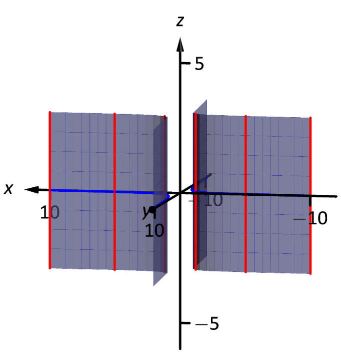


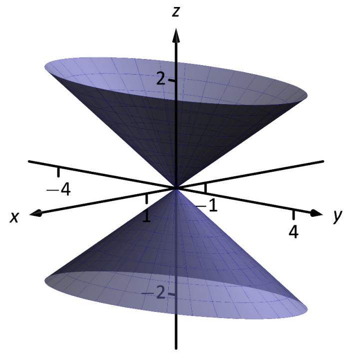
29. $y^2 + z^2 = x^4$

31. $z = \frac{1}{\sqrt{x^2+y^2}}$

33. (b) $x^2 - y^2 + z^2 = 0$

35. (a) $y^2 - x^2 - z^2 = 1$

37.



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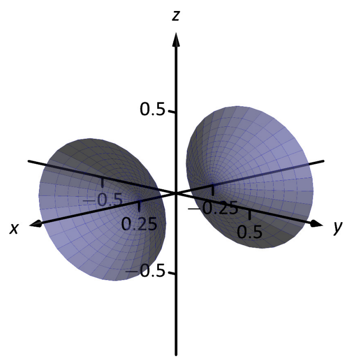

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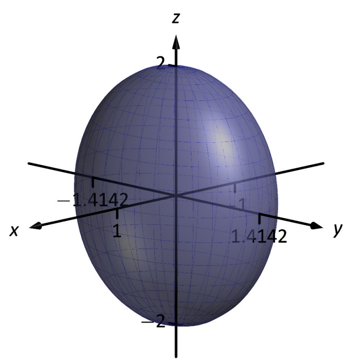


Section 10.2

1. Answers will vary.

3. A vector with magnitude 1.

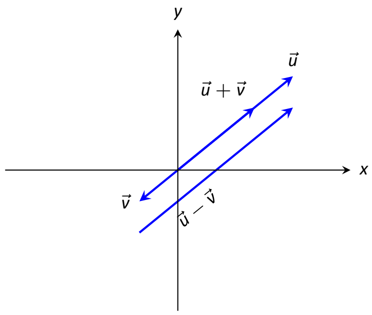
5. It stretches the vector by a factor of 2, and points it in the opposite direction.

7. $\overrightarrow{PQ} = \langle -4, 4\rangle = -4\vec{i} + 4\vec{j}$

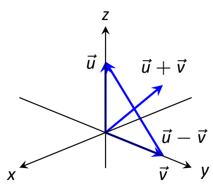
9. $\overrightarrow{PQ} = \langle 2, 2, 0\rangle = 2\vec{i} + 2\vec{j}$

11. (a) $\vec{u} + \vec{v} = \langle 3, 2, 1\rangle$; $\vec{u} - \vec{v} = \langle -1, 0, -3\rangle$; $\pi\vec{u} - \sqrt{2}\vec{v} = \langle \pi - 2\sqrt{2}, \pi - \sqrt{2}, -\pi - 2\sqrt{2}\rangle$.

    (c) $\vec{x} = \langle -1, 0, -3\rangle$.

13. 

Sketch of $\vec u - \vec v$ shifted for clarity.

15. 

17. $\|\,\vec u\,\| = \sqrt{17}$, $\|\,\vec v\,\| = \sqrt{3}$, $\|\,\vec u + \vec v\,\| = \sqrt{14}$, $\|\,\vec u - \vec v\,\| = \sqrt{26}$

19. $\|\,\vec u\,\| = 7$, $\|\,\vec v\,\| = 35$, $\|\,\vec u + \vec v\,\| = 42$, $\|\,\vec u - \vec v\,\| = 28$

21. $\vec u = \left\langle 3/\sqrt{58}, 7/\sqrt{58} \right\rangle$

23. $\vec u = \langle 1/3, -2/3, 2/3 \rangle$

25. $\vec u = \langle \cos 50^\circ, \sin 50^\circ \rangle \approx \langle 0.643, 0.766 \rangle$.

27.
$$\begin{aligned} \|\,\vec u\,\| &= \sqrt{\sin^2\varphi\cos^2\theta + \sin^2\varphi\sin^2\theta + \cos^2\varphi} \\ &= \sqrt{\sin^2\varphi(\cos^2\theta + \sin^2\theta) + \cos^2\varphi} \\ &= \sqrt{\sin^2\varphi + \cos^2\varphi} \\ &= 1. \end{aligned}$$

29. The force on each chain is 100lb.

31. The force on each chain is 50lb.

33. $\theta = 5.71^\circ$; the weight is lifted 0.005 ft (about 1/16th of an inch).

35. $\theta = 84.29^\circ$; the weight is lifted 9 ft.

Section 10.3

1. Scalar

3. By considering the sign of the dot product of the two vectors. If the dot product is positive, the angle is acute; if the dot product is negative, the angle is obtuse.

5. -22

7. 3

9. not defined

11. Answers will vary.

13. $\theta = 0.3218 \approx 18.43^\circ$

15. $\theta = \pi/4 = 45^\circ$

17. Answers will vary; two possible answers are $\langle -7, 4\rangle$ and $\langle 14, -8\rangle$.

19. Answers will vary; two possible answers are $\langle 1, 0, -1\rangle$ and $\langle 4, 5, -9\rangle$.

21. $\text{proj}_{\vec v}\,\vec u = \langle -1/2, 3/2\rangle$.

23. $\text{proj}_{\vec v}\,\vec u = \langle -1/2, -1/2\rangle$.

25. $\text{proj}_{\vec v}\,\vec u = \langle 1, 2, 3\rangle$.

27. $\vec u = \langle -1/2, 3/2\rangle + \langle 3/2, 1/2\rangle$.

29. $\vec u = \langle -1/2, -1/2\rangle + \langle -5/2, 5/2\rangle$.

31. $\vec u = \langle 1, 2, 3\rangle + \langle 0, 3, -2\rangle$.

33. 1.96lb

35. 141.42ft–lb

37. 500ft–lb

39. 500ft–lb

Section 10.4

1. vector

3. “Perpendicular” is one answer.

5. Torque

7. $\vec u \times \vec v = \langle 11, 1, -17\rangle$

9. $\vec u \times \vec v = \langle 47, -36, -44\rangle$

11. $\vec u \times \vec v = \langle 0, 0, 0\rangle$

13. $\vec i \times \vec k = -\vec j$

15. Answers will vary.

17. 5

19. 0

21. $\sqrt{14}$

23. 3

25. $5\sqrt{2}/2$

27. 1

29. 7

31. 2

33. $\pm\frac{1}{\sqrt{6}}\langle 1, 1, -2\rangle$

35. $\langle 0, \pm 1, 0\rangle$

37. 87.5ft–lb

39. $200/3 \approx 66.67$ft–lb

41. With $\vec u = \langle u_1, u_2, u_3\rangle$ and $\vec v = \langle v_1, v_2, v_3\rangle$, we have
$$\begin{aligned} \vec u \cdot (\vec u \times \vec v) &= \langle u_1, u_2, u_3\rangle \cdot (\langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1\rangle) \\ &= u_1(u_2v_3 - u_3v_2) - u2(u_1v_3 - u_3v_1) + u3(u_1v_2 - u_2v_1) \\ &= 0. \end{aligned}$$

43. 54

Section 10.5

1. A point on the line and the direction of the line.

3. parallel, skew

5. vector: $\ell(t) = \langle 2, -4, 1\rangle + t\langle 9, 2, 5\rangle$
parametric: $x = 2 + 9t$, $y = -4 + 2t$, $z = 1 + 5t$
symmetric: $(x - 2)/9 = (y + 4)/2 = (z - 1)/5$

7. vector: $\ell(t) = \langle -2, 5, 4\rangle + t\langle 0, 1, 3\rangle$
parametric: $x = -2$, $y = 5 + t$, $z = 4 + 3t$
symmetric: $x = -2$, $y - 5 = (z - 4)/3$

9. Answers can vary: vector: $\ell(t) = \langle 2, 1, 5\rangle + t\langle 5, -3, -1\rangle$
parametric: $x = 2 + 5t$, $y = 1 - 3t$, $z = 5 - t$
symmetric: $(x - 2)/5 = -(y - 1)/3 = -(z - 5)$

11. vector: $\ell(t) = \langle 1, 5, 5\rangle + t\langle 1, -3, 0\rangle$
parametric: $x = 1 + t$, $y = 5 - 3t$, $z = 5$
symmetric: $x - 1 = (y - 5)/(-3)$, $z = 5$

13. Answers can vary; here the direction is given by $\vec{d}_1 \times \vec{d}_2$: vector: $\ell(t) = \langle 5, 1, 9 \rangle + t\,\langle 0, -1, 0 \rangle$
parametric: $x = 5, y = 1 - t, z = 9$
symmetric: $x = 5, z = 9$

15. Answers can vary; here the direction is given by $\vec{d}_1 \times \vec{d}_2$: vector: $\ell(t) = \langle 2, 2, 3 \rangle + t\,\langle 5, -1, -3 \rangle$
parametric: $x = 2 + 5t, y = 2 - t, z = 3 - 3t$
symmetric: $(x - 2)/5 = -(y - 2) = -(z - 3)/3$

17. intersecting; $\ell_1(2) = \ell_2(-2) = \langle 12, 3, 7 \rangle$

19. same

21. parallel

23. skew

25. $3\sqrt{2}$

27. 5

29. 2

31. (Note: this solution is easier once one has studied Section 10.6.) Since the two lines intersect, we can state $P_2 = P_1 + a\vec{d}_1 + b\vec{d}_2$ for some scalars a and b. (Here we abuse notation slightly and add points to vectors.) Thus $\overrightarrow{P_1P_2} = a\vec{d}_1 + b\vec{d}_2$. Vector $\vec{c}$ is the cross product of $\vec{d}_1$ and $\vec{d}_2$, hence is orthogonal to both, and hence is orthogonal to $\overrightarrow{P_1P_2}$. Thus $\overrightarrow{P_1P_2} \cdot \vec{c} = 0$, and the distance between lines is 0.

Section 10.6

1. A point in the plane and a normal vector (i.e., a direction orthogonal to the plane).

3. Answers will vary.

5. Answers will vary.

7. Standard form: $3(x - 2) - (y - 3) + 7(z - 4) = 0$
general form: $3x - y + 7z = 31$

9. Answers may vary;
Standard form: $8(x - 1) + 4(y - 2) - 4(z - 3) = 0$
general form: $8x + 4y - 4z = 4$

11. Answers may vary;
Standard form: $-7(x - 2) + 2(y - 1) + (z - 2) = 0$
general form: $-7x + 2y + z = -10$

13. Answers may vary;
Standard form: $2(x - 1) - (y - 1) = 0$
general form: $2x - y = 1$

15. Answers may vary;
Standard form: $2(x - 2) - (y + 6) - 4(z - 1) = 0$
general form: $2x - y - 4z = 6$

17. Answers may vary;
Standard form: $(x - 5) + (y - 7) + (z - 3) = 0$
general form: $x + y + z = 15$

19. Answers may vary;
Standard form: $3(x + 4) + 8(y - 7) - 10(z - 2) = 0$
general form: $3x + 8y - 10z = 24$

21. Answers may vary:
$$\ell = \begin{cases} x = 14t \\ y = -1 - 10t \\ z = 2 - 8t \end{cases}$$

23. $(-3, -7, -5)$

25. No point of intersection; the plane and line are parallel.

27. $\sqrt{5/7}$

29. $1/\sqrt{3}$

31. If P is any point in the plane, and Q is also in the plane, then $\overrightarrow{PQ}$ lies parallel to the plane and is orthogonal to $\vec{n}$, the normal vector. Thus $\vec{n} \cdot \overrightarrow{PQ} = 0$, giving the distance as 0.

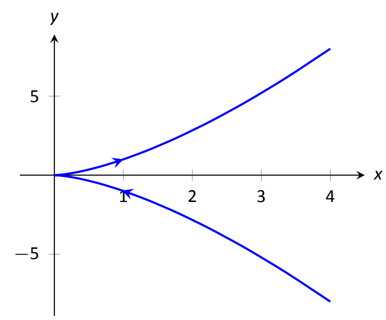
Chapter 11

Section 11.1

1. parametric equations

3. displacement

5.



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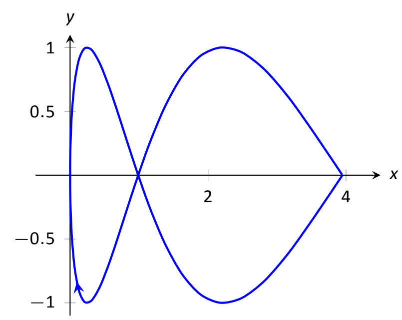

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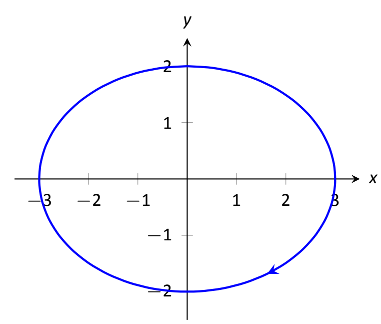

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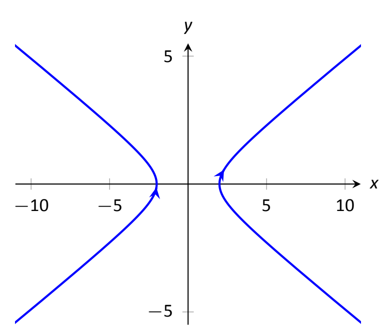
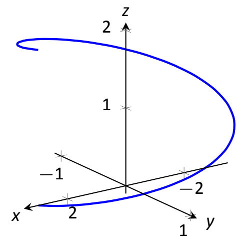
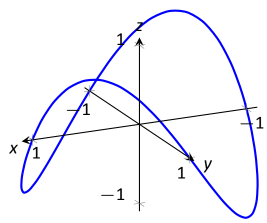


13. 

15. 

17. $\|\vec{r}(t)\| = \sqrt{25\cos^2 t + 9\sin^2 t}$.

19. $\|\vec{r}(t)\| = \sqrt{\cos^2 t + t^2 + t^4}$.

21. Answers may vary; three solutions are
$\vec{r}(t) = \langle 3\sin t + 5, 3\cos t + 5\rangle$,
$\vec{r}(t) = \langle -3\cos t + 5, 3\sin t + 5\rangle$ and
$\vec{r}(t) = \langle 3\cos t + 5, -3\sin t + 5\rangle$.

23. Answers may vary, though most direct solutions are
$\vec{r}(t) = \langle -3\cos t + 3, 2\sin t - 2\rangle$,
$\vec{r}(t) = \langle 3\cos t + 3, -2\sin t - 2\rangle$ and
$\vec{r}(t) = \langle 3\sin t + 3, 2\cos t - 2\rangle$.

25. Answers may vary, though most direct solutions are
$\vec{r}(t) = \langle t, -1/2(t-1) + 5\rangle$,
$\vec{r}(t) = \langle t+1, -1/2t + 5\rangle$,
$\vec{r}(t) = \langle -2t+1, t+5\rangle$ and
$\vec{r}(t) = \langle 2t+1, -t+5\rangle$.

27. Answers may vary, though most direct solution is
$\vec{r}(t) = \langle 3\cos(4\pi t), 3\sin(4\pi t), 3t\rangle$.

29. $\langle 1, 1\rangle$

31. $\langle 1, 2, 7\rangle$

Section 11.2

1. component

3. It is difficult to identify the points on the graphs of $\vec{r}(t)$ and $\vec{r}\,'(t)$ that correspond to each other.

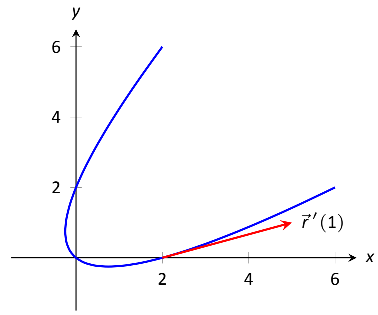
5. $\langle e^3, 0\rangle$

7. $\langle 2t, 1, 0\rangle$

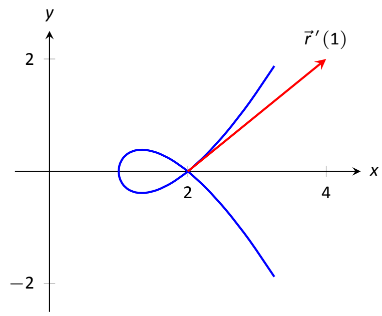
9. $(0, \infty)$

11. $\vec{r}\,'(t) = \langle -1/t^2, 5/(3t+1)^2, \sec^2 t\rangle$

13. $\vec{r}\,'(t) = \langle 2t, 1\rangle \cdot \langle \sin t, 2t+5\rangle + \langle t^2+1, t-1\rangle \cdot \langle \cos t, 2\rangle = (t^2+1)\cos t + 2t\sin t + 4t + 3$

15. 

$\vec{r}\,'(t) = \langle 2t+1, 2t-1\rangle$

17. 

$\vec{r}\,'(t) = \langle 2t, 3t^2 - 1\rangle$

19. $\ell(t) = \langle 2, 0\rangle + t\langle 3, 1\rangle$

21. $\ell(t) = \langle -3, 0, \pi\rangle + t\langle 0, -3, 1\rangle$

23. $t = 2n\pi$, where n is an integer; so
$t = \ldots - 4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots$

25. $\vec{r}(t)$ is not smooth at $t = 3\pi/4 + n\pi$, where n is an integer

27. Both derivatives return $\langle 5t^4, 4t^3 - 3t^2, 3t^2\rangle$.

29. Both derivatives return
$\langle 2t - e^t - 1, \cos t - 3t^2, (t^2+2t)e^t - (t-1)\cos t - \sin t\rangle$.

31. $\langle \tan^{-1} t, \tan t\rangle + \vec{C}$

33. $\langle 4, -4\rangle$

35. $\vec{r}(t) = \langle \ln|t+1| + 1, -\ln|\cos t| + 2\rangle$

37. $\vec{r}(t) = \langle -\cos t + 1, t - \sin t, e^t - t - 1\rangle$

39. 10π

41. $\sqrt{2}(1 - e^{-1})$

Section 11.3

1. Velocity is a vector, indicating an objects direction of travel and its rate of distance change (i.e., its speed). Speed is a scalar.

3. The average velocity is found by dividing the displacement by the time traveled – it is a vector. The average speed is found by dividing the distance traveled by the time traveled – it is a scalar.

5. One example is traveling at a constant speed s in a circle, ending at the starting position. Since the displacement is $\vec{0}$, the average velocity is $\vec{0}$, hence $\|\vec{0}\| = 0$. But traveling at constant speed s means the average speed is also $s > 0$.

7. (a) $\vec{v}(t) = \langle 2, 5, 0\rangle$
   (b) $\vec{a}(t) = \langle 0, 0, 0\rangle$
   (c) $\vec{v}(t) \cdot \vec{a}(t) = 0$
   (d) The speed is constant. The speed is $\sqrt{29}$.

9. (a) $\vec{v}(t) = \langle -\sin t, \cos t\rangle$
   (b) $\vec{a}(t) = \langle -\cos t, -\sin t\rangle$
   (c) $\vec{v}(t) \cdot \vec{a}(t) = 0$

(d) The speed is constant. The speed is 1.

11. (a) $\vec{v}(t) = \langle 2t, \cos t, -\sin t \rangle$

(b) $\vec{a}(t) = \langle 2, -\sin t, -\cos t \rangle$

(c) $\vec{v}(t) \cdot \vec{a}(t) = 4t$

(d) The speed is not constant.

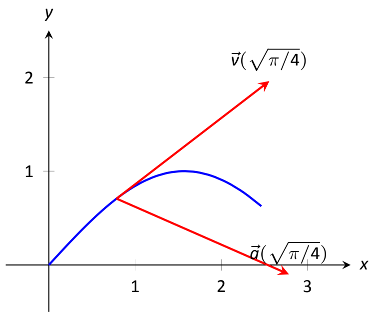
13. (a) $\vec{v}(t) = \langle \cos t - \sin t, 3, \cos t + \sin t \rangle$

(b) $\vec{a}(t) = \langle -\sin t - \cos t, 0, -\sin t + \cos t \rangle$

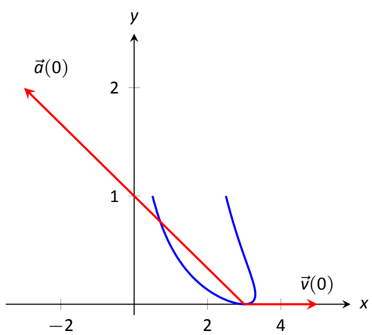
(c) $\vec{v}(t) \cdot \vec{a}(t) = 0$

(d) The speed is constant. The speed is $\sqrt{11}$.

15. $\vec{v}(t) = \left\langle 2t, 2t\cos\left(t^2\right)\right\rangle$, $\vec{a}(t) = \left\langle 2, 2\left(\cos\left(t^2\right) - 2t^2\sin\left(t^2\right)\right)\right\rangle$



17. $\vec{v}(t) = \left\langle -\frac{2(t^2+3t-1)}{(t^2+1)^2}, 2t \right\rangle$, $\vec{a}(t) = \left\langle \frac{2(2t^3+9t^2-6t-3)}{(t^2+1)^3}, 2 \right\rangle$



19. $||\,\vec{v}(t)\,|| = |t|\sqrt{9t^2 - 12t + 8}$.
min: $t = 0$; max: $t = -1$

21. $||\,\vec{v}(t)\,|| = \sqrt{4\sin^2 t + 25\cos^2 t}$.
min: $t = \pi/2,\ 3\pi/2$; max: $t = 0,\ 2\pi$

23. $||\,\vec{v}(t)\,|| = \sqrt{2 - 2\sin t}$.
min: $t = \pi/2$; max: $t = 3\pi/2$

25. $||\,\vec{v}(t)\,|| = \sqrt{8t^2 + 3}$.
min: $t = 0$; max: $t = 1$

27. $||\,\vec{v}(t)\,|| = \sqrt{g^2t^2 - (2gv_0\sin\theta)t + v_0^2}$.
min: $t = (v_0\sin\theta)/g$; max: $t = 0, t = (2v_0\sin\theta)/g$

29. (a) $\vec{r}_1(\pi/2) = \langle 0, 3\rangle$; $\vec{r}_2(\pi/8) = \langle 0, 3\rangle$

(b) $\vec{v}_1(\pi/2) = \langle -3, 0\rangle$; $||\,\vec{v}_1(\pi/2)\,|| = 3$; $\vec{a}_1(\pi/2) = \langle 0, -3\rangle$
$\vec{v}_2(\pi/8) = \langle -12, 0\rangle$; $||\,\vec{v}_2(\pi/8)\,|| = 12$;
$\vec{a}_2(\pi/8) = \langle 0, -48\rangle$

31. (a) $\vec{r}_1(1) = \langle 1, 1\rangle$; $\vec{r}_2(\pi/2) = \langle 1, 1\rangle$

(b) $\vec{v}_1(1) = \langle 1, 1/2\rangle$; $||\,\vec{v}_1(1)\,|| = \sqrt{5}/2$; $\vec{a}_1(1) = \langle 0, -1/4\rangle$
$\vec{v}_2(\pi/2) = \langle 0, 0\rangle$; $||\,\vec{v}_2(\pi/2)\,|| = 0$;
$\vec{a}_2(\pi/2) = \langle -1, -1/2\rangle$

33. $\vec{v}(t) = \langle 2t - 1, 3t - 1\rangle$, $\vec{r}(t) = \left\langle t^2 - t + 5, 3t^2/2 - t - 5/2\right\rangle$

35. $\vec{v}(t) = \langle 10, -32t + 50\rangle$, $\vec{r}(t) = \left\langle 10t, -16t^2 + 50t\right\rangle$

37. Displacement: $\langle -10, 0\rangle$; distance traveled: $5\pi \approx 15.71$ft; average velocity: $\langle -10/\pi, 0\rangle \approx \langle -3.18, 0\rangle$; average speed: 5ft/s

39. Displacement: $\langle 10, 20, -20\rangle$; distance traveled: 30ft; average velocity: $\langle 1, 2, -2\rangle$; average speed: 3ft/s

41. The stone, while whirling, can be modeled by $\vec{r}(t) = \langle 3\cos(8\pi t), 3\sin(8\pi t)\rangle$.

(a) For t-values $t = \sin^{-1}(3/20)/(8\pi) + n/4 \approx 0.006 + n/4$, where n is an integer.

(b) $||\,\vec{r}\,'(t)\,|| = 24\pi \approx 51.4$ft/s

(c) At $t = 0.006$, the stone is approximately 19.77ft from Goliath. Using the formula for projectile motion, we want the angle of elevation that lets a projectile starting at $\langle 0, 6\rangle$ with a initial velocity of 51.4ft/s arrive at $\langle 19.77, 9\rangle$. The desired angle is 0.27 radians, or 15.69°.

43. The position function of the ball is $\vec{r}(t) = \left\langle (146.67\cos\theta)t, -16t^2 + (146.67\sin\theta)t + 3\right\rangle$, where θ is the angle of elevation.

(a) With $\theta = 20^\circ$, the ball reaches 310ft from home plate in 2.25 seconds; at this time, the height of the ball is 34.9ft, not enough to clear the Green Monster.

(b) With $\theta = 21^\circ$, the ball reaches 310ft from home plate in 2.26s, with a height of 40ft, clearing the wall.

45. The position function of the ball is $\vec{r}(t) = \left\langle (v_0\cos\theta)t, -16t^2 + (v_0\sin\theta)t + 6\right\rangle$, where θ is the angle of elevation and v_0 is the initial ball speed.

(a) With $v_0 = 73.33$ft/s, there are two angles of elevation possible. An angle of $\theta = 9.47^\circ$ delivers the ball in 0.83s, while an angle of 79.57° delivers the ball in 4.5s.

(b) With $\theta = 8^\circ$, the initial speed must be 53.8mph$\approx$ 78.9ft/s.

Section 11.4

1. 1

3. $\vec{T}(t)$ and $\vec{N}(t)$.

5. $\vec{T}(t) = \left\langle \frac{4t}{\sqrt{20t^2-4t+1}}, \frac{2t-1}{\sqrt{20t^2-4t+1}} \right\rangle$; $\vec{T}(1) = \langle 4/\sqrt{17}, 1/\sqrt{17}\rangle$

7. $\vec{T}(t) = \frac{\cos t\sin t}{\sqrt{\cos^2 t\sin^2 t}}\langle -\cos t, \sin t\rangle$. (Be careful; this cannot be simplified as just $\langle -\cos t, \sin t\rangle$ as $\sqrt{\cos^2 t\sin^2 t} \neq \cos t\sin t$, but rather $|\cos t\sin t|$.) $\vec{T}(\pi/4) = \langle -\sqrt{2}/2, \sqrt{2}/2\rangle$

9. $\ell(t) = \langle 2, 0\rangle + t\,\langle 4/\sqrt{17}, 1/\sqrt{17}\rangle$; in parametric form,
$$\ell(t) = \begin{cases} x &= 2 + 4t/\sqrt{17} \\ y &= t/\sqrt{17} \end{cases}$$

11. $\ell(t) = \langle \sqrt{2}/4, \sqrt{2}/4\rangle + t\,\langle -\sqrt{2}/2, \sqrt{2}/2\rangle$; in parametric form,
$$\ell(t) = \begin{cases} x &= \sqrt{2}/4 - \sqrt{2}t/2 \\ y &= \sqrt{2}/4 + \sqrt{2}t/2 \end{cases}$$

13. $\vec{T}(t) = \langle -\sin t, \cos t\rangle$; $\vec{N}(t) = \langle -\cos t, -\sin t\rangle$

15. $\vec{T}(t) = \left\langle -\frac{\sin t}{\sqrt{4\cos^2 t+\sin^2 t}}, \frac{2\cos t}{\sqrt{4\cos^2 t+\sin^2 t}} \right\rangle$;
$\vec{N}(t) = \left\langle -\frac{2\cos t}{\sqrt{4\cos^2 t+\sin^2 t}}, -\frac{\sin t}{\sqrt{4\cos^2 t+\sin^2 t}} \right\rangle$

17. (a) Be sure to show work

(b) $\vec{N}(\pi/4) = \langle -5/\sqrt{34}, -3/\sqrt{34}\rangle$

19. (a) Be sure to show work

(b) $\vec{N}(0) = \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$

21. $\vec{T}(t) = \frac{1}{\sqrt{5}}\langle 2, \cos t, -\sin t\rangle$; $\vec{N}(t) = \langle 0, -\sin t, -\cos t\rangle$

23. $\vec{T}(t) = \frac{1}{\sqrt{a^2+b^2}}\langle -a\sin t, a\cos t, b\rangle$; $\vec{N}(t) = \langle -\cos t, -\sin t, 0\rangle$

25. $a_\text{T} = \frac{4t}{\sqrt{1+4t^2}}$ and $a_\text{N} = \sqrt{4 - \frac{16t^2}{1+4t^2}}$
At $t = 0$, $a_\text{T} = 0$ and $a_\text{N} = 2$;
At $t = 1$, $a_\text{T} = 4/\sqrt{5}$ and $a_\text{N} = 2/\sqrt{5}$.
At $t = 0$, all acceleration comes in the form of changing the direction of velocity and not the speed; at $t = 1$, more acceleration comes in changing the speed than in changing direction.

27. $a_\text{T} = 0$ and $a_\text{N} = 2$
At $t = 0$, $a_\text{T} = 0$ and $a_\text{N} = 2$;
At $t = \pi/2$, $a_\text{T} = 0$ and $a_\text{N} = 2$.
The object moves at constant speed, so all acceleration comes from changing direction, hence $a_\text{T} = 0$. $\vec{a}(t)$ is always parallel to $\vec{N}(t)$, but twice as long, hence $a_\text{N} = 2$.

29. $a_\text{T} = 0$ and $a_\text{N} = a$
At $t = 0$, $a_\text{T} = 0$ and $a_\text{N} = a$;
At $t = \pi/2$, $a_\text{T} = 0$ and $a_\text{N} = a$.
The object moves at constant speed, meaning that a_T is always 0. The object "rises" along the z-axis at a constant rate, so all acceleration comes in the form of changing direction circling the z-axis. The greater the radius of this circle the greater the acceleration, hence $a_\text{N} = a$.

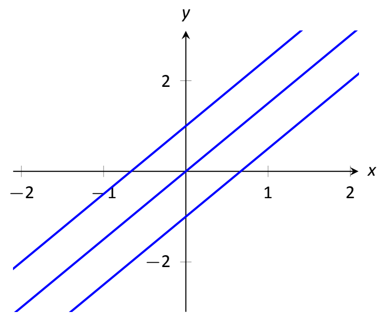
Section 11.5

1. time and/or distance
3. Answers may include lines, circles, helixes
5. κ
7. $s = 3t$, so $\vec{r}(s) = \langle 2s/3, s/3, -2s/3 \rangle$
9. $s = \sqrt{13}t$, so $\vec{r}(s) = \langle 3\cos(s/\sqrt{13}), 3\sin(s/\sqrt{13}), 2s/\sqrt{13} \rangle$
11. $\kappa = \frac{|6x|}{\left(1+(3x^2-1)^2\right)^{3/2}}$;
$\kappa(0) = 0$, $\kappa(1/2) = \frac{192}{17\sqrt{17}} \approx 2.74$.
13. $\kappa = \frac{|\cos x|}{\left(1+\sin^2 x\right)^{3/2}}$;
$\kappa(0) = 1$, $\kappa(\pi/2) = 0$
15. $\kappa = \frac{|2\cos t\cos(2t)+4\sin t\sin(2t)|}{\left(4\cos^2(2t)+\sin^2 t\right)^{3/2}}$;
$\kappa(0) = 1/4$, $\kappa(\pi/4) = 8$
17. $\kappa = \frac{|6t^2+2|}{\left(4t^2+(3t^2-1)^2\right)^{3/2}}$;
$\kappa(0) = 2$, $\kappa(5) = \frac{19}{1394\sqrt{1394}} \approx 0.0004$
19. $\kappa = 0$;
$\kappa(0) = 0$, $\kappa(1) = 0$
21. $\kappa = \frac{3}{13}$;
$\kappa(0) = 3/13$, $\kappa(\pi/2) = 3/13$
23. maximized at $x = \pm\frac{\sqrt{2}}{\sqrt[4]{5}}$
25. maximized at $t = 1/4$
27. radius of curvature is $5\sqrt{5}/4$.
29. radius of curvature is 9.
31. $x^2 + (y - 1/2)^2 = 1/4$, or $\vec{c}(t) = \langle 1/2\cos t, 1/2\sin t + 1/2 \rangle$
33. $x^2 + (y + 8)^2 = 81$, or $\vec{c}(t) = \langle 9\cos t, 9\sin t - 8 \rangle$

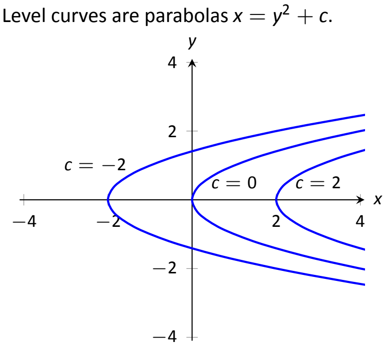
Chapter 12

Section 12.1

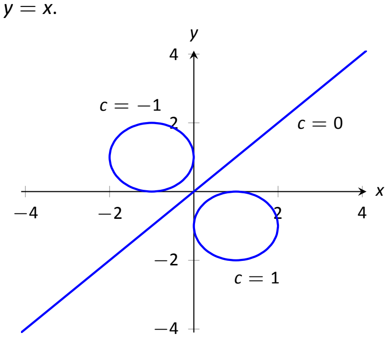
1. Answers will vary.
3. topographical
5. surface
7. domain: $\mathbb{R}^2$
range: $z \geq 2$
9. domain: $\mathbb{R}^2$
range: $\mathbb{R}$
11. domain: $\mathbb{R}^2$
range: $0 < z \leq 1$
13. domain: $\{(x, y) \mid x^2 + y^2 \leq 9\}$, i.e., the domain is the circle and interior of a circle centered at the origin with radius 3.
range: $0 \leq z \leq 3$
15. Level curves are lines $y = (3/2)x - c/2$.



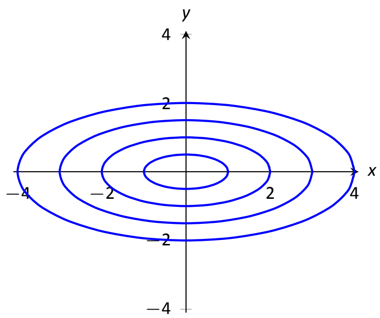
17. Level curves are parabolas $x = y^2 + c$.



19. When $c \neq 0$, the level curves are circles, centered at $(1/c, -1/c)$ with radius $\sqrt{2/c^2 - 1}$. When $c = 0$, the level curve is the line $y = x$.



21. Level curves are ellipses of the form $\frac{x^2}{c^2} + \frac{y^2}{c^2/4} = 1$, i.e., $a = c$ and $b = c/2$.



23. domain: $x + 2y - 4z \neq 0$; the set of points in $\mathbb{R}^3$ NOT in the domain form a plane through the origin.
range: $\mathbb{R}$

25. domain: $z \geq x^2 - y^2$; the set of points in $\mathbb{R}^3$ above (and including) the hyperbolic paraboloid $z = x^2 - y^2$.
range: $[0, \infty)$

27. The level surfaces are spheres, centered at the origin, with radius $\sqrt{c}$.

29. The level surfaces are paraboloids of the form $z = \frac{x^2}{c} + \frac{y^2}{c}$; the larger c, the "wider" the paraboloid.

31. The level curves for each surface are similar; for $z = \sqrt{x^2 + 4y^2}$ the level curves are ellipses of the form $\frac{x^2}{c^2} + \frac{y^2}{c^2/4} = 1$, i.e., $a = c$ and $b = c/2$; whereas for $z = x^2 + 4y^2$ the level curves are ellipses of the form $\frac{x^2}{c} + \frac{y^2}{c/4} = 1$, i.e., $a = \sqrt{c}$ and $b = \sqrt{c}/2$. The first set of ellipses are spaced evenly apart, meaning the function grows at a constant rate; the second set of ellipses are more closely spaced together as c grows, meaning the function grows faster and faster as c increases.
The function $z = \sqrt{x^2 + 4y^2}$ can be rewritten as $z^2 = x^2 + 4y^2$, an elliptic cone; the function $z = x^2 + 4y^2$ is a paraboloid, each matching the description above.

Section 12.2

1. Answers will vary.

3. Answers will vary.
One possible answer: $\{(x, y) | x^2 + y^2 \leq 1\}$

5. Answers will vary.
One possible answer: $\{(x, y) | x^2 + y^2 < 1\}$

7. (a) Answers will vary.
interior point: $(1, 3)$
boundary point: $(3, 3)$
(b) S is a closed set
(c) S is bounded

9. (a) Answers will vary.
interior point: none
boundary point: $(0, -1)$
(b) S is a closed set, consisting only of boundary points
(c) S is bounded

11. (a) $D = \{(x, y) \,|\, 9 - x^2 - y^2 \geq 0\}$.
(b) D is a closed set.
(c) D is bounded.

13. (a) $D = \{(x, y) \,|\, y > x^2\}$.
(b) D is an open set.
(c) D is unbounded.

15. (a) Along $y = 0$, the limit is 1.
(b) Along $x = 0$, the limit is -1.
Since the above limits are not equal, the limit does not exist.

17. (a) Along $y = mx$, the limit is $\dfrac{mx(1 - m)}{m^2x + 1}$.
(b) Along $x = 0$, the limit is -1.
Since the above limits are not equal, the limit does not exist.

19. (a) Along $y = 2$, the limit is:
$$\begin{aligned}\lim_{(x,y)\to(1,2)} \frac{x + y - 3}{x^2 - 1} &= \lim_{x\to 1} \frac{x - 1}{x^2 - 1} \\ &= \lim_{x\to 1} \frac{1}{x + 1} \\ &= 1/2.\end{aligned}$$
(b) Along $y = x + 1$, the limit is:
$$\begin{aligned}\lim_{(x,y)\to(1,2)} \frac{x + y - 3}{x^2 - 1} &= \lim_{x\to 1} \frac{2(x - 1)}{x^2 - 1} \\ &= \lim_{x\to 1} \frac{2}{x + 1} \\ &= 1.\end{aligned}$$
Since the limits along the lines $y = 2$ and $y = x + 1$ differ, the overall limit does not exist.

21. -2

23. The limit does not exist.

Section 12.3

1. A constant is a number that is added or subtracted in an expression; a coefficient is a number that is being multiplied by a nonconstant function.

3. f_x

5. $f_x = 2xy - 1, f_y = x^2 + 2$
$f_x(1, 2) = 3, f_y(1, 2) = 3$

7. $f_x = -\sin x \sin y, f_y = \cos x \cos y$
$f_x(\pi/3, \pi/3) = -3/4, f_y(\pi/3, \pi/3) = 1/4$

9. $f_x = 2xy + 6x, f_y = x^2 + 4$
$f_{xx} = 2y + 6, f_{yy} = 0$
$f_{xy} = 2x, f_{yx} = 2x$

11. $f_x = 1/y, f_y = -x/y^2$
$f_{xx} = 0, f_{yy} = 2x/y^3$
$f_{xy} = -1/y^2, f_{yx} = -1/y^2$

13. $f_x = 2xe^{x^2+y^2}, f_y = 2ye^{x^2+y^2}$
$f_{xx} = 2e^{x^2+y^2} + 4x^2e^{x^2+y^2}, f_{yy} = 2e^{x^2+y^2} + 4y^2e^{x^2+y^2}$
$f_{xy} = 4xye^{x^2+y^2}, f_{yx} = 4xye^{x^2+y^2}$

15. $f_x = \cos x \cos y, f_y = -\sin x \sin y$
$f_{xx} = -\sin x \cos y, f_{yy} = -\sin x \cos y$
$f_{xy} = -\sin y \cos x, f_{yx} = -\sin y \cos x$

17. $f_x = -5y^3 \sin(5xy^3), f_y = -15xy^2 \sin(5xy^3)$
$f_{xx} = -25y^6 \cos(5xy^3)$,
$f_{yy} = -225x^2y^4 \cos(5xy^3) - 30xy \sin(5xy^3)$
$f_{xy} = -75xy^5 \cos(5xy^3) - 15y^2 \sin(5xy^3)$,
$f_{yx} = -75xy^5 \cos(5xy^3) - 15y^2 \sin(5xy^3)$

19. $f_x = \frac{2y^2}{\sqrt{4xy^2+1}}, f_y = \frac{4xy}{\sqrt{4xy^2+1}}$
$f_{xx} = -\frac{4y^4}{\sqrt{4xy^2+1}^3}, f_{yy} = -\frac{16x^2y^2}{\sqrt{4xy^2+1}^3} + \frac{4x}{\sqrt{4xy^2+1}}$
$f_{xy} = -\frac{8xy^3}{\sqrt{4xy^2+1}^3} + \frac{4y}{\sqrt{4xy^2+1}}, f_{yx} = -\frac{8xy^3}{\sqrt{4xy^2+1}^3} + \frac{4y}{\sqrt{4xy^2+1}}$

21. $f_x = -\frac{2x}{(x^2+y^2+1)^2}, f_y = -\frac{2y}{(x^2+y^2+1)^2}$
$f_{xx} = \frac{8x^2}{(x^2+y^2+1)^3} - \frac{2}{(x^2+y^2+1)^2}, f_{yy} = \frac{8y^2}{(x^2+y^2+1)^3} - \frac{2}{(x^2+y^2+1)^2}$
$f_{xy} = \frac{8xy}{(x^2+y^2+1)^3}, f_{yx} = \frac{8xy}{(x^2+y^2+1)^3}$

23. $f_x = yx^{y-1}, f_y = x^y \ln x$
$f_{xx} = y(y-1)x^{y-2}, f_{yy} = x^y(\ln x)^2$
$f_{xy} = x^{y-1}(1 + y\ln x), f_{yx} = x^{y-1}(1 + y\ln x)$

25. $f_x = \frac{2x}{(x^2+y)}, f_y = \frac{1}{(x^2+y)}$
$f_{xx} = -\frac{4x^2}{(x^2+y)^2} + \frac{2}{(x^2+y)}, f_{yy} = -\frac{1}{(x^2+y)^2}$
$f_{xy} = -\frac{2x}{(x^2+y)^2}, f_{yx} = -\frac{2x}{(x^2+y)^2}$

27. $f_x = 5 + (2+\cos y)x^{1+\cos y}, f_y = -x^{2+\cos y}\ln x \sin y$
$f_{xx} = (2+\cos y)(1+\cos y)x^{\cos y}$,
$f_{yy} = x^{2+\cos y}\ln x\left(\ln x \sin^2 y - \cos y\right)$
$f_{xy} = -\sin y x^{1+\cos y}\left(1 + (2+\cos y)\ln x\right)$,
$f_{yx} = -\sin y x^{1+\cos y}\left(1 + (2+\cos y)\ln x\right)$

29. $f(x,y) = x\sin y + x + C$, where C is any constant.

31. $f(x,y) = \frac{1}{2}x^2 + xy + \frac{1}{2}y^2 + C$, where C is any constant.

33. No possible function $f(x,y)$ exists.

35. $f(x,y) = \ln(x^2+y^2) + C$, where C is any constant.

37. $f_x = 3x^2y^2 + 3x^2z, f_y = 2x^3y + 2yz, f_z = x^3 + y^2$
$f_{yz} = 2y, f_{zy} = 2y$

39. $f_x = \frac{1}{x}, f_y = \frac{1}{y}, f_z = \frac{1}{z}$
$f_{yz} = 0, f_{zy} = 0$

Section 12.4

1. T

3. T

5. $dz = (\sin y + 2x)dx + (x\cos y)dy$

7. $dz = 5dx - 7dy$

9. $dz = \frac{x}{\sqrt{x^2+y}}dx + \frac{1}{2\sqrt{x^2+y}}dy$, with $dx = -0.05$ and $dy = .1$. At $(3,7)$, $dz = 3/4(-0.05) + 1/8(.1) = -0.025$, so $f(2.95, 7.1) \approx -0.025 + 4 = 3.975$.

11. $dz = (2xy - y^2)dx + (x^2 - 2xy)dy$, with $dx = 0.04$ and $dy = 0.06$. At $(2,3)$, $dz = 3(0.04) + (-8)(0.06) = -0.36$, so $f(2.04, 3.06) \approx -0.36 - 6 = -6.36$.

13. The total differential of volume is $dV = 4\pi dr + \pi dh$. The coefficient of dr is greater than the coefficient of dh, so the volume is more sensitive to changes in the radius.

15. Using trigonometry, $\ell = x\tan\theta$, so $d\ell = \tan\theta dx + x\sec^2\theta d\theta$. With $\theta = 85^\circ$ and $x = 30$, we have $d\ell = 11.43dx + 3949.38d\theta$. The measured length of the wall is much more sensitive to errors in θ than in x. While it can be difficult to compare sensitivities between measuring feet and measuring degrees (it is somewhat like "comparing apples to oranges"), here the coefficients are so different that the result is clear: a small error in degree has a much greater impact than a small error in distance.

17. $dw = 2xyz^3\,dx + x^2z^3\,dy + 3x^2yz^2\,dz$

19. $dx = 0.05, dy = -0.1$. $dz = 9(.05) + (-2)(-0.1) = 0.65$. So $f(3.05, 0.9) \approx 7 + 0.65 = 7.65$.

21. $dx = 0.5, dy = 0.1, dz = -0.2$.
$dw = 2(0.5) + (-3)(0.1) + 3.7(-0.2) = -0.04$, so $f(2.5, 4.1, 4.8) \approx -1 - 0.04 = -1.04$.

Section 12.5

1. Because the parametric equations describe a level curve, z is constant for all t. Therefore $\frac{dz}{dt} = 0$.

3. $\frac{dx}{dt}$, and $\frac{\partial f}{\partial y}$

5. F

7. (a) $\frac{dz}{dt} = 3(2t) + 4(2) = 6t + 8$.
   (b) At $t = 1$, $\frac{dz}{dt} = 14$.

9. (a) $\frac{dz}{dt} = 5(-2\sin t) + 2(\cos t) = -10\sin t + 2\cos t$
   (b) At $t = \pi/4$, $\frac{dz}{dt} = -4\sqrt{2}$.

11. (a) $\dfrac{dz}{dt} = 2x(\cos t) + 4y(3\cos t)$.
   (b) At $t = \pi/4, x = \sqrt{2}/2, y = 3\sqrt{2}/2$, and $\frac{dz}{dt} = 19$.

13. $t = -4/3$; this corresponds to a minimum

15. $t = \tan^{-1}(1/5) + n\pi$, where n is an integer

17. We find that
$$\frac{dz}{dt} = 38\cos t\sin t.$$
Thus $\frac{dz}{dt} = 0$ when $t = \pi n$ or $\pi n + \pi/2$, where n is any integer.

19. (a) $\frac{\partial z}{\partial s} = 2xy(1) + x^2(2) = 2xy + 2x^2$;
   $\frac{\partial z}{\partial t} = 2xy(-1) + x^2(4) = -2xy + 4x^2$
   (b) With $s = 1, t = 0, x = 1$ and $y = 2$. Thus $\frac{\partial z}{\partial s} = 6$ and $\frac{\partial z}{\partial t} = 0$

21. (a) $\frac{\partial z}{\partial s} = 2x(\cos t) + 2y(\sin t) = 2x\cos t + 2y\sin t$;
   $\frac{\partial z}{\partial t} = 2x(-s\sin t) + 2y(s\cos t) = -2xs\sin t + 2ys\cos t$
   (b) With $s = 2, t = \pi/4, x = \sqrt{2}$ and $y = \sqrt{2}$. Thus $\frac{\partial z}{\partial s} = 4$ and $\frac{\partial z}{\partial t} = 0$

23. $f_x = 2x\tan y, f_y = x^2\sec^2 y$;
$\dfrac{dy}{dx} = -\dfrac{2\tan y}{x\sec^2 y}$

25. $f_x = \dfrac{(x+y^2)(2x) - (x^2+y)(1)}{(x+y^2)^2}$,
$f_y = \dfrac{(x+y^2)(1) - (x^2+y)(2y)}{(x+y^2)^2}$;
$\dfrac{dy}{dx} = -\dfrac{2x(x+y^2) - (x^2+y)}{x + y^2 - 2y(x^2+y)}$

27. $\frac{dz}{dt} = 2(4) + 1(-5) = 3$.

29. $\frac{\partial z}{\partial s} = -4(5) + 9(-2) = -38$,
$\frac{\partial z}{\partial t} = -4(7) + 9(6) = 26$.

Section 12.6

1. A partial derivative is essentially a special case of a directional derivative; it is the directional derivative in the direction of x or y, i.e., $\langle 1,0\rangle$ or $\langle 0,1\rangle$.

3. $\vec{u} = \langle 0,1\rangle$

5. maximal, or greatest

7. $\nabla f = \left\langle -2xy + y^2 + y, -x^2 + 2xy + x\right\rangle$

9. $\nabla f = \left\langle \frac{-2x}{(x^2+y^2+1)^2}, \frac{-2y}{(x^2+y^2+1)^2}\right\rangle$

11. $\nabla f = \langle 2x - y - 7, 4y - x\rangle$

13. $\nabla f = \left\langle -2xy + y^2 + y, -x^2 + 2xy + x\right\rangle$; $\nabla f(2,1) = \langle -2, 2\rangle$. Be sure to change all directions to unit vectors.
   (a) $2/5$ ($\vec{u} = \langle 3/5, 4/5\rangle$)
   (b) $-2\sqrt{5}$ ($\vec{u} = \left\langle -1/\sqrt{5}, -2\sqrt{5}\right\rangle$)

15. $\nabla f = \left\langle \frac{-2x}{(x^2+y^2+1)^2}, \frac{-2y}{(x^2+y^2+1)^2}\right\rangle$; $\nabla f(1,1) = \langle -2/9, -2/9\rangle$. Be sure to change all directions to unit vectors.
   (a) 0 ($\vec{u} = \left\langle 1/\sqrt{2}, -1/\sqrt{2}\right\rangle$)
   (b) $2\sqrt{2}/9$ ($\vec{u} = \left\langle -1/\sqrt{2}, -1/\sqrt{2}\right\rangle$)

17. $\nabla f = \langle 2x - y - 7, 4y - x\rangle$; $\nabla f(4,1) = \langle 0, 0\rangle$.
   (a) 0

(b) 0

19. $\nabla f = \left\langle -2xy + y^2 + y, -x^2 + 2xy + x \right\rangle$

(a) $\nabla f(2,1) = \langle -2, 2 \rangle$

(b) $||\, \nabla f(2,1)\, || = ||\, \langle -2, 2\rangle\, || = \sqrt{8}$

(c) $\langle 2, -2 \rangle$

(d) $\left\langle 1/\sqrt{2}, 1/\sqrt{2} \right\rangle$

21. $\nabla f = \left\langle \frac{-2x}{(x^2+y^2+1)^2}, \frac{-2y}{(x^2+y^2+1)^2} \right\rangle$

(a) $\nabla f(1,1) = \langle -2/9, -2/9 \rangle$.

(b) $||\, \nabla f(1,1)\, || = ||\, \langle -2/9, -2/9\rangle\, || = 2\sqrt{2}/9$

(c) $\langle 2/9, 2/9 \rangle$

(d) $\left\langle 1/\sqrt{2}, -1/\sqrt{2} \right\rangle$

23. $\nabla f = \langle 2x - y - 7, 4y - x \rangle$

(a) $\nabla f(4,1) = \langle 0, 0 \rangle$

(b) 0

(c) $\langle 0, 0 \rangle$

(d) All directions give a directional derivative of 0.

25. (a) $\nabla F(x,y,z) = \left\langle 6xz^3 + 4y, 4x, 9x^2z^2 - 6z \right\rangle$

(b) $113/\sqrt{3}$

27. (a) $\nabla F(x,y,z) = \left\langle 2xy^2, 2y(x^2 - z^2), -2y^2z \right\rangle$

(b) 0

Section 12.7

1. Answers will vary. The displacement of the vector is one unit in the x-direction and 3 units in the z-direction, with no change in y. Thus along a line parallel to $\vec{v}$, the change in z is 3 times the change in x – i.e., a "slope" of 3. Specifically, the line in the x-z plane parallel to z has a slope of 3.

3. T

5. (a) $\ell_x(t) = \begin{cases} x = 2 + t \\ y = 3 \\ z = -48 - 12t \end{cases}$

(b) $\ell_y(t) = \begin{cases} x = 2 \\ y = 3 + t \\ z = -48 - 40t \end{cases}$

(c) $\ell_{\vec{u}}(t) = \begin{cases} x = 2 + t/\sqrt{10} \\ y = 3 + 3t/\sqrt{10} \\ z = -48 - 66\sqrt{2/5}t \end{cases}$

7. (a) $\ell_x(t) = \begin{cases} x = 4 + t \\ y = 2 \\ z = 2 + 3t \end{cases}$

(b) $\ell_y(t) = \begin{cases} x = 4 \\ y = 2 + t \\ z = 2 - 5t \end{cases}$

(c) $\ell_{\vec{u}}(t) = \begin{cases} x = 4 + t/\sqrt{2} \\ y = 2 + t/\sqrt{2} \\ z = 2 - \sqrt{2}t \end{cases}$

9. $\ell_{\vec{n}}(t) = \begin{cases} x = 2 - 12t \\ y = 3 - 40t \\ z = -48 - t \end{cases}$

11. $\ell_{\vec{n}}(t) = \begin{cases} x = 4 + 3t \\ y = 2 - 5t \\ z = 2 - t \end{cases}$

13. $(1.425, 1.085, -48.078)$, $(2.575, 4.915, -47.952)$

15. $(5.014, 0.31, 1.662)$ and $(2.986, 3.690, 2.338)$

17. $-12(x-2) - 40(y-3) - (z+48) = 0$

19. $3(x-4) - 5(y-2) - (z-2) = 0$ (Note that this tangent plane is the same as the original function, a plane.)

21. $\nabla F = \langle x/4, y/2, z/8 \rangle$; at P, $\nabla F = \left\langle 1/4, \sqrt{2}/2, \sqrt{6}/8 \right\rangle$

(a) $\ell_{\vec{n}}(t) = \begin{cases} x = 1 + t/4 \\ y = \sqrt{2} + \sqrt{2}t/2 \\ z = \sqrt{6} + \sqrt{6}t/8 \end{cases}$

(b) $\frac{1}{4}(x-1) + \frac{\sqrt{2}}{2}(y - \sqrt{2}) + \frac{\sqrt{6}}{8}(z - \sqrt{6}) = 0$.

23. $\nabla F = \left\langle y^2 - z^2, 2xy, -2xz \right\rangle$; at P, $\nabla F = \langle 0, 4, 4 \rangle$

(a) $\ell_{\vec{n}}(t) = \begin{cases} x = 2 \\ y = 1 + 4t \\ z = -1 + 4t \end{cases}$

(b) $4(y-1) + 4(z+1) = 0$.

Section 12.8

1. F; it is the "other way around."

3. T

5. One critical point at $(-4, 2)$; $f_{xx} = 1$ and $D = 4$, so this point corresponds to a relative minimum.

7. One critical point at $(6, -3)$; $D = -4$, so this point corresponds to a saddle point.

9. Two critical points: at $(0, -1)$; $f_{xx} = 2$ and $D = -12$, so this point corresponds to a saddle point;
at $(0, 1)$, $f_{xx} = 2$ and $D = 12$, so this corresponds to a relative minimum.

11. One critical point at $(0, 0)$. $D = -12x^2y^2$, so at $(0,0)$, $D = 0$ and the test is inconclusive. (Some elementary thought shows that it is the absolute minimum.)

13. Six critical points: $f_x = 0$ when $x = -1, 0$ and 1; $f_y = 0$ when $y = -3, 3$. Together, we get the points:
$(-1, -3)$ saddle point; $(-1, 3)$ rel. min
$(0, -3)$ rel. max; $(0, 3)$ saddle point
$(1, -3)$ saddle point; $(1, 3)$ relative min
where $f_{xx} = 12x^2 - 4$ and $D = 24y(3x^2 - 1)$.

15. One critical point: $f_x = 0$ when $x = 0$; $f_y = 0$ when $y = 0$, so one critical point at $(0,0)$ (although it should be noted that at $(0,0)$, both f_x and f_y are undefined.) The Second Derivative Test fails at $(0, 0)$, with $D = 0$. A graph, or simple calculation, shows that $(0, 0)$ is the absolute minimum of f.

17. The region has two "corners" at $(1, 1)$ and $(-1, 1)$.
Along $y = 1$, there is no critical point.
Along $y = x^2$, there is a critical point at $(5/14, 25/196) \approx (0.357, 0.128)$.
The function f itself has no critical points. Checking the value of f at the corners $(1,1)$, $(-1,1)$ and the critical point $(5/14, 25/196)$, we find the absolute maximum is at $(5/14, 25/196, 25/28) \approx (0.357, 0.128, 0.893)$ and the absolute minimum is at $(-1, 1, -12)$.

19. The region has two "corners" at $(-1, -1)$ and $(1, 1)$.
Along the line $y = x$, $f(x, y)$ becomes $f(x) = 3x - 2x^2$. Along this line, we have a critical point at $(3/4, 3/4)$.
Along the curve $y = x^2 + x - 1$, $f(x, y)$ becomes $f(x) = x^2 + 3x - 3$. There is a critical point along this curve at $(-3/2, -1/4)$. Since $x = -3/2$ lies outside our bounded region, we ignore this critical point.
The function f itself has no critical points.
Checking the value of f at $(-1, -1)$, $(1, 1)$, $(3/4, 3/4)$, we find the absolute maximum is at $(3/4, 3/4, 9/8)$ and the absolute minimum is at $(-1, -1, -5)$.

21. $10\text{m} \times 16\text{m} \times 48\text{m}$

Section 12.9

1. perpendicular or orthogonal

3. T

5. $f_{\max} = \dfrac{2}{3\sqrt{3}}$ at $\left(\pm\sqrt{\dfrac{2}{3}}, \dfrac{1}{\sqrt{3}}\right)$, and $f_{\min} = -\dfrac{2}{3\sqrt{3}}$ at $\left(\pm\sqrt{\dfrac{2}{3}}, -\dfrac{1}{\sqrt{3}}\right)$

7. $f_{\max} = \dfrac{\sqrt{82}}{3}$ at $\left(\dfrac{9}{\sqrt{82}}, \dfrac{1}{3\sqrt{82}}\right)$, and $f_{\min} = -\dfrac{\sqrt{82}}{3}$ at $\left(-\dfrac{9}{\sqrt{82}}, -\dfrac{1}{3\sqrt{82}}\right)$

9. $f_{\max} = 1$ at $(1, 1)$

11. $f_{\min} = \dfrac{1}{2}$ at the point $\left(\dfrac{1}{2}, 0, -\dfrac{1}{2}\right)$

Chapter 13

Section 13.1

1. $C(y)$, meaning that instead of being just a constant, like the number 5, it is a function of y, which acts like a constant when taking derivatives with respect to x.

3. curve to curve, then from point to point

5. (a) $18x^2 + 42x - 117$

   (b) -108

7. (a) $x^4/2 - x^2 + 2x - 3/2$

   (b) $23/15$

9. (a) $\sin^2 y$

   (b) $\pi/2$

11. $\displaystyle\int_1^4 \int_{-2}^1 dy\,dx$ and $\displaystyle\int_{-2}^1 \int_1^4 dx\,dy$.
area of $R = 9u^2$

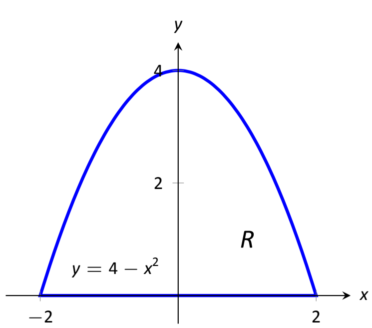
13. $\displaystyle\int_2^4 \int_{x-1}^{7-x} dy\,dx$. The order $dx\,dy$ needs two iterated integrals as x is bounded above by two different functions. This gives:

$$\int_1^3 \int_2^{y+1} dx\,dy + \int_3^5 \int_2^{7-y} dx\,dy.$$

area of $R = 4u^2$

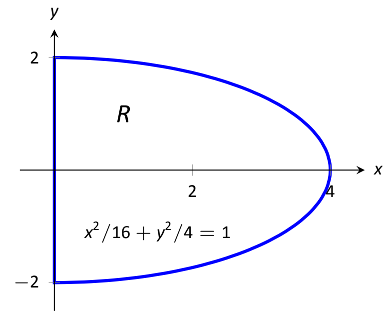
15. $\displaystyle\int_0^1 \int_{x^4}^{\sqrt{x}} dy\,dx$ and $\displaystyle\int_0^1 \int_{y^2}^{\sqrt[4]{y}} dx\,dy$
area of $R = 7/15u^2$

17.



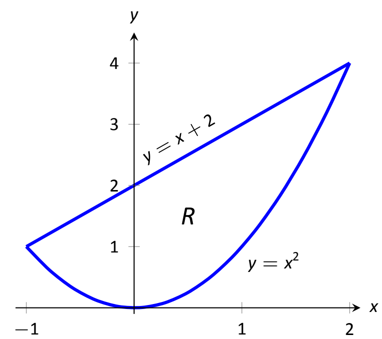
area of $R = \displaystyle\int_0^4 \int_{-\sqrt{4-y}}^{\sqrt{4-y}} dx\,dy$

19.



area of $R = \displaystyle\int_0^4 \int_{-\sqrt{4-x^2/4}}^{\sqrt{4-x^2/4}} dy\,dx$

21.



area of $R = \displaystyle\int_{-1}^2 \int_{x^2}^{x+2} dy\,dx$

23. 20

25. $2\sqrt{3} - \sqrt{6}$

27. 756

29. The integrand $\sqrt{x+y}$ cannot be written as $f(x)g(y)$ where f is a function of only x and g is a function of only y.

Section 13.2

1. volume

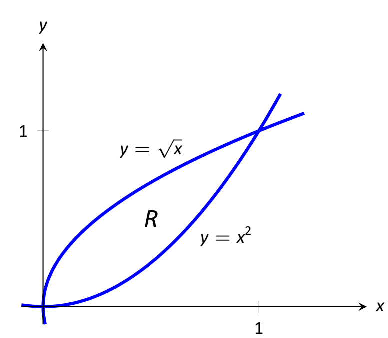
3. The double integral gives the signed volume under the surface. Since the surface is always positive, it is always above the x-y plane and hence produces only "positive" volume.

5. 6; $\displaystyle\int_{-1}^1 \int_1^2 \left(\frac{x}{y} + 3\right)\,dy\,dx$

7. 112/3; $\displaystyle\int_0^2 \int_0^{4-2y} \left(3x^2 - y + 2\right)\,dx\,dy$

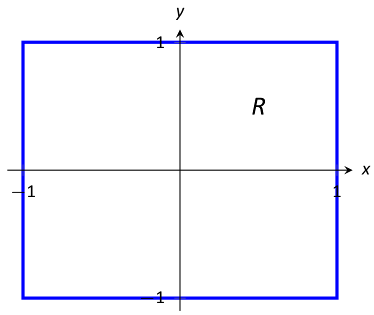
9. 16/5; $\displaystyle\int_{-1}^1 \int_0^{1-x^2} (x + y + 2)\,dy\,dx$

11. (a)



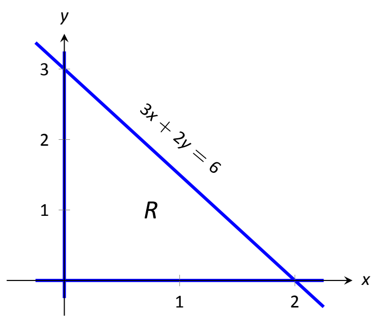
   (b) $\displaystyle\int_0^1 \int_{x^2}^{\sqrt{x}} x^2 y\,dy\,dx = \int_0^1 \int_{y^2}^{\sqrt{y}} x^2 y\,dx\,dy.$

   (c) $\frac{3}{56}$

13. (a) 

(b) $\displaystyle\int_{-1}^{1}\int_{-1}^{1} x^2 - y^2\, dy\, dx = \int_{-1}^{1}\int_{-1}^{1} x^2 - y^2\, dx\, dy.$

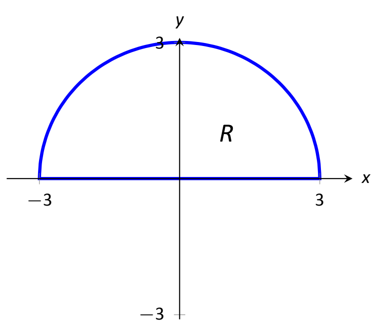
(c) 0

15. (a) 

(b)

(c) $\displaystyle\int_0^2\int_0^{3-3/2x} \big(6 - 3x - 2y\big)\, dy\, dx =$
$\displaystyle\int_0^3\int_0^{2-2/3y} \big(6 - 3x - 2y\big)\, dx\, dy.$

(d) 6

17. (a) 

(b) $\displaystyle\int_{-3}^{3}\int_0^{\sqrt{9-x^2}} \big(x^3y - x\big)\, dy\, dx =$
$\displaystyle\int_0^3\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \big(x^3y - x\big)\, dx\, dy.$

(c) 0

19. Integrating e^{x^2} with respect to x is not possible in terms of elementary functions. $\displaystyle\int_0^2\int_0^{2x} e^{x^2}\, dy\, dx = e^4 - 1.$

21. Integrating $\displaystyle\int_y^1 \frac{2y}{x^2+y^2}\, dx$ gives $\tan^{-1}(1/y) - \pi/4$; integrating $\tan^{-1}(1/y)$ is hard.
$\displaystyle\int_0^1\int_0^x \frac{2y}{x^2+y^2}\, dy\, dx = \ln 2.$

23. average value of $f = 6/2 = 3$

25. average value of $f = \frac{112/3}{4} = 28/3$

Section 13.3

1. $f\big(r\cos\theta, r\sin\theta\big), r\, dr\, d\theta$

3. $\displaystyle\int_0^{2\pi}\int_0^1 \big(3r\cos\theta - r\sin\theta + 4\big) r\, dr\, d\theta = 4\pi$

5. $\displaystyle\int_0^{\pi}\int_{\cos\theta}^{3\cos\theta} \big(8 - r\sin\theta\big) r\, dr\, d\theta = 16\pi$

7. $\displaystyle\int_0^{2\pi}\int_1^2 \big(\ln(r^2)\big) r\, dr\, d\theta = 2\pi\big(\ln 16 - 3/2\big)$

9. $\displaystyle\int_{-\pi/2}^{\pi/2}\int_0^6 \big(r^2\cos^2\theta - r^2\sin^2\theta\big) r\, dr\, d\theta =$
$\displaystyle\int_{-\pi/2}^{\pi/2}\int_0^6 \big(r^2\cos(2\theta)\big) r\, dr\, d\theta = 0$

11. $\displaystyle\int_{-\pi/2}^{\pi/2}\int_0^5 \big(r^2\big)\, dr\, d\theta = 125\pi/3$

13. $\displaystyle\int_0^{\pi/4}\int_0^{\sqrt{8}} \big(r\cos\theta + r\sin\theta\big) r\, dr\, d\theta = 16\sqrt{2}/3$

15. (a) This is impossible to integrate with rectangular coordinates as $e^{-(x^2+y^2)}$ does not have an antiderivative in terms of elementary functions.

(b) $\displaystyle\int_0^{2\pi}\int_0^a re^{r^2}\, dr\, d\theta = \pi\left(1 - e^{-a^2}\right).$

(c) $\displaystyle\lim_{a\to\infty} \pi(1 - e^{-a^2}) = \pi$. This implies that there is a finite volume under the surface $e^{-(x^2+y^2)}$ over the entire x-y plane.

Section 13.4

1. Because they are scalar multiples of each other.

3. "little masses"

5. M_x measures the moment about the x-axis, meaning we need to measure distance from the x-axis. Such measurements are measures in the y-direction.

7. $\bar{x} = 5.25$

9. $(\bar{x}, \bar{y}) = (0, 3)$

11. $M = 150$gm;

13. $M = 2$lb

15. $M = 16\pi \approx 50.27$kg

17. $M = 54\pi \approx 169.65$lb

19. $M = 150$gm; $M_y = 600$; $M_x = -75$; $(\bar{x}, \bar{y}) = (4, -0.5)$

21. $M = 2$lb; $M_y = 0$; $M_x = 2/3$; $(\bar{x}, \bar{y}) = (0, 1/3)$

23. $M = 16\pi \approx 50.27$kg; $M_y = 4\pi$; $M_x = 4\pi$; $(\bar{x}, \bar{y}) = (1/4, 1/4)$

25. $M = 54\pi \approx 169.65$lb; $M_y = 0$; $M_x = 504$; $(\bar{x}, \bar{y}) = (0, 2.97)$

27. $I_x = 64/3$; $I_y = 64/3$; $I_O = 128/3$

29. $I_x = 16/3$; $I_y = 64/3$; $I_O = 80/3$

Section 13.5

1. arc length

3. surface areas

5. Intuitively, adding h to f only shifts f up (i.e., parallel to the z-axis) and does not change its shape. Therefore it will not change the surface area over R.
Analytically, $f_x = g_x$ and $f_y = g_y$; therefore, the surface area of each is computed with identical double integrals.

7. $SA = \displaystyle\int_0^{2\pi}\int_0^{2\pi} \sqrt{1 + \cos^2 x\cos^2 y + \sin^2 x\sin^2 y}\, dx\, dy$

9. $SA = \int_{-1}^{1}\int_{-1}^{1} \sqrt{1+4x^2+4y^2}\,dx\,dy$

11. $SA = \int_{0}^{3}\int_{-1}^{1} \sqrt{1+9+49}\,dx\,dy = 6\sqrt{59} \approx 46.09$

13. This is easier in polar:
$$\begin{aligned} SA &= \int_0^{2\pi}\int_0^4 r\sqrt{1+4r^2\cos^2 t+4r^2\sin^2 t}\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^4 r\sqrt{1+4r^2}\,dr\,d\theta \\ &= \frac{\pi}{6}\left(65\sqrt{65}-1\right) \approx 273.87 \end{aligned}$$

15.
$$\begin{aligned} SA &= \int_0^2\int_0^{2x} \sqrt{1+1+4x^2}\,dy\,dx \\ &= \int_0^2 \left(2x\sqrt{2+4x^2}\right)dx \\ &= \frac{26}{3}\sqrt{2} \approx 12.26 \end{aligned}$$

17. This is easier in polar:
$$\begin{aligned} SA &= \int_0^{2\pi}\int_0^5 r\sqrt{1+\frac{4r^2\cos^2 t+4r^2\sin^2 t}{r^2\sin^2 t+r^2\cos^2 t}}\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^5 r\sqrt{5}\,dr\,d\theta \\ &= 25\pi\sqrt{5} \approx 175.62 \end{aligned}$$

19. Integrating in polar is easiest considering R:
$$\begin{aligned} SA &= \int_0^{2\pi}\int_0^1 r\sqrt{1+c^2+d^2}\,dr\,d\theta \\ &= \int_0^{2\pi} \frac{1}{2}\left(\sqrt{1+c^2+d^2}\right)dy \\ &= \pi\sqrt{1+c^2+d^2}. \end{aligned}$$
The value of h does not matter as it only shifts the plane vertically (i.e., parallel to the z-axis). Different values of h do not create different ellipses in the plane.

Section 13.6

1. surface to surface, curve to curve and point to point
3. Answers can vary. From this section we used triple integration to find the volume of a solid region, the mass of a solid, and the center of mass of a solid.
5. $V = \int_{-1}^{1}\int_{-1}^{1}\left(8-x^2-y^2-(2x+y)\right)dx\,dy = 88/3$
7. $V = \int_0^{\pi}\int_0^{x}\left(\cos x\sin y+2-\sin x\cos y\right)dy\,dx = \pi^2-\pi \approx 6.728$
9. $dz\,dy\,dx$: $\int_0^3\int_0^{1-x/3}\int_0^{2-2x/3-2y} dz\,dy\,dx$

   $dz\,dx\,dy$: $\int_0^1\int_0^{3-3y}\int_0^{2-2x/3-2y} dz\,dx\,dy$

   $dy\,dz\,dx$: $\int_0^3\int_0^{2-2x/3}\int_0^{1-x/3-z/2} dy\,dz\,dx$

   $dy\,dx\,dz$: $\int_0^2\int_0^{3-3z/2}\int_0^{1-x/3-z/2} dy\,dx\,dz$

   $dx\,dz\,dy$: $\int_0^1\int_0^{2-2y}\int_0^{3-3y-3z/2} dx\,dz\,dy$

   $dx\,dy\,dz$: $\int_0^2\int_0^{1-z/2}\int_0^{3-3y-3z/2} dx\,dy\,dz$

   $V = \int_0^3\int_0^{1-x/3}\int_0^{2-2x/3-2y} dz\,dy\,dx = 1.$

11. $dz\,dy\,dx$: $\int_0^2\int_{-2}^{0}\int_{y^2/2}^{-y} dz\,dy\,dx$

   $dz\,dx\,dy$: $\int_{-2}^{0}\int_0^2\int_{y^2/2}^{-y} dz\,dx\,dy$

   $dy\,dz\,dx$: $\int_0^2\int_0^2\int_{-\sqrt{2z}}^{-z} dy\,dz\,dx$

   $dy\,dx\,dz$: $\int_0^2\int_0^2\int_{-\sqrt{2z}}^{-z} dy\,dx\,dz$

   $dx\,dz\,dy$: $\int_{-2}^{0}\int_{y^2/2}^{-y}\int_0^2 dx\,dz\,dy$

   $dx\,dy\,dz$: $\int_0^2\int_{-\sqrt{2z}}^{-z}\int_0^2 dx\,dy\,dz$

   $V = \int_0^2\int_0^2\int_{-\sqrt{2z}}^{-z} dy\,dz\,dx = 4/3.$

13. $dz\,dy\,dx$: $\int_0^2\int_{1-x/2}^{1}\int_0^{2x+4y-4} dz\,dy\,dx$

   $dz\,dx\,dy$: $\int_0^1\int_{2-2y}^{2}\int_0^{2x+4y-4} dz\,dx\,dy$

   $dy\,dz\,dx$: $\int_0^2\int_0^{2x}\int_{z/4-x/2+1}^{1} dy\,dz\,dx$

   $dy\,dx\,dz$: $\int_0^4\int_{z/2}^{2}\int_{z/4-x/2+1}^{1} dy\,dx\,dz$

   $dx\,dz\,dy$: $\int_0^1\int_0^{4y}\int_{z/2-2y+2}^{2} dx\,dz\,dy$

   $dx\,dy\,dz$: $\int_0^4\int_{z/4}^{1}\int_{z/2-2y+2}^{2} dx\,dy\,dz$

   $V = \int_0^4\int_{z/4}^{1}\int_{2y-z/2-2}^{2} dx\,dy\,dz = 4/3.$

15. $dz\,dy\,dx$: $\int_0^1\int_0^{1-x^2}\int_0^{\sqrt{1-y}} dz\,dy\,dx$

   $dz\,dx\,dy$: $\int_0^1\int_0^{\sqrt{1-y}}\int_0^{\sqrt{1-y}} dz\,dx\,dy$

   $dy\,dz\,dx$: $\int_0^1\int_0^{x}\int_0^{1-x^2} dy\,dz\,dx + \int_0^1\int_x^1\int_0^{1-z^2} dy\,dz\,dx$

   $dy\,dx\,dz$: $\int_0^1\int_0^{z}\int_0^{1-z^2} dy\,dx\,dz + \int_0^1\int_z^1\int_0^{1-x^2} dy\,dx\,dz$

   $dx\,dz\,dy$: $\int_0^1\int_0^{\sqrt{1-y}}\int_0^{\sqrt{1-y}} dx\,dz\,dy$

   $dx\,dy\,dz$: $\int_0^1\int_0^{1-z^2}\int_0^{\sqrt{1-y}} dx\,dy\,dz$

   Answers will vary. Neither order is particularly "hard." The order $dz\,dy\,dx$ requires integrating a square root, so powers can be messy; the order $dy\,dz\,dx$ requires two triple integrals, but each uses only polynomials.
17. 8
19. π
21. $M = 10$, $M_{yz} = 15/2$, $M_{xz} = 5/2$, $M_{xy} = 5$; $(\bar{x},\bar{y},\bar{z}) = (3/4, 1/4, 1/2)$
23. $M = 16/5$, $M_{yz} = 16/3$, $M_{xz} = 104/45$, $M_{xy} = 32/9$; $(\bar{x},\bar{y},\bar{z}) = (5/3, 13/18, 10/9) \approx (1.67, 0.72, 1.11)$

Section 13.7

1. 2π
3. F
5. F
7. $(r,\theta,z) = \left(2, \frac{3\pi}{2}, 0\right)$ and $(\rho,\theta,\phi) = \left(2, \frac{3\pi}{2}, \frac{\pi}{2}\right)$
9. $(r,\theta,z) = (1, 0, 0)$ and $(\rho,\theta,\phi) = \left(1, 0, \frac{\pi}{2}\right)$
11. $(r,\theta,z) = \left(\sqrt{2}, \frac{\pi}{4}, -2\right)$ and $(\rho,\theta,\phi) = \left(\sqrt{6}, \frac{\pi}{4}, \cos^{-1}\left(-\frac{2}{\sqrt{6}}\right)\right)$

13. The solid is the upper half hemisphere of a sphere of radius 2
15. $\frac{128\pi}{3}$
17. $\frac{128\pi}{5}$
19. 0
21. The limit does not exist.
23. 0

Chapter 14

Section 14.1

1. $f(x, y) = x + 2y$
3. $f(x, y) = \sin(x + y)$
5. $f(x, y) = \ln(x^2 + y^2)$
7. $f(x, y) = \frac{x^2 y}{2}$
9. $\mathbf{F}(x, y) = 3\mathbf{i} + \mathbf{j}$
11. $\mathbf{F}(x, y) = \mathbf{i} + 2y\mathbf{j}$
13. $\mathbf{F}(x, y) = 2x\mathbf{i} - 2y\mathbf{j}$
15. $\mathbf{F}(x, y) = e^{x-y}\mathbf{i} - e^{x-y}\mathbf{j}$
17. $\mathbf{F}(x, y) = y^x \ln y\mathbf{i} - xy^{x-1}\mathbf{j}$

Section 14.2

1. $\sqrt{5}$
3. 1
5. $\frac{\sqrt{2}}{3} + \frac{1}{2}$
7. 0
9. Conservative. $f(x, y) = x + \frac{y^2}{2}, f(1, 0) - f(0, 1) = -\frac{1}{2}$
11. Conservative. $f(x, y) = \frac{x^2 y^2}{2}, f(1, 0) - f(0, 1) = 0$
13. Conservative. $f(x, y) = x + \frac{y^2}{2}, f(1, 0) - f(0, 1) = -1$
15. Conservative. $f(x, y) = x - 2y, f(1, 1) - f(0, 0) = -1$
17. Conservative. $f(x, y) = x^2 y^2, f(1, 1) - f(0, 0) = 1$
19. Not conservative. $0, -\frac{1}{3}$
21. Conservative. $f(x, y) = -y^2 e^{-x} + C$
23. Conservative. $f(x, y) = \sqrt{x^2 + y^2} + C$
25. Not conservative.

Section 14.3

1. 4π
3. 0
5. -4π
7. 0
9. 0
11. $\frac{1}{12}$
13. $256/15$
15. 410π

Section 14.4

1. $\frac{\pi}{6}\left(17\sqrt{17} - 1\right)$
3. $\pi\sqrt{3}$
5. $\pi\left(2 - \sqrt{2}\right)$
7. $5A\sqrt{2}$
9. $\frac{\pi}{2}\left(\ln\left(1 + \sqrt{2}\right) + \sqrt{2}\right)$
11. $\pi\sqrt{3}$
13. $2\ln(6 + \sqrt{37}) + 12\sqrt{37}$
15. $\frac{4\pi}{3}$
17. 0
19. -24π
21. 0

Section 14.5

1. 0
3. $4xy$
5. 1
7. $\frac{4\pi}{3}$
9. 0
11. $\frac{1}{2}$
13. $\frac{12a^5\pi}{5}$
15. 0

Section 14.6

1. $\mathbf{i} + \mathbf{j} + \mathbf{k}$
3. 0
5. 0
7. 0
9. $4xy$
11. -2π

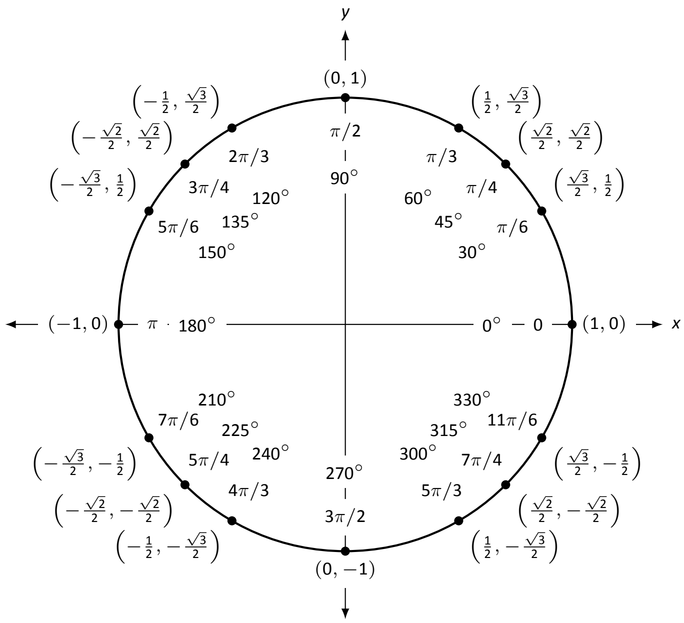
Differentiation Rules

1. $\frac{d}{dx}(cx) = c$
2. $\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$
3. $\frac{d}{dx}(f(x) \cdot g(x)) = f'(x)g(x) + f(x)g'(x)$
4. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$
5. $\frac{d}{dx}(f(g(x))) = f'(g(x)g'(x)$
6. $\frac{d}{dx}(c) = 0$
7. $\frac{d}{dx}(x) = 1$
8. $\frac{d}{dx}(x^n) = nx^{n-1}$
9. $\frac{d}{dx}(e^x) = e^x$
10. $\frac{d}{dx}(a^x) = \ln a \cdot a^x$
11. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
12. $\frac{d}{dx}(\log_a x) = \frac{1}{\ln(a)x}$
13. $\frac{d}{dx}(\sin x) = \cos x$
14. $\frac{d}{dx}(\cos x) = -\sin x$
15. $\frac{d}{dx}(\tan x) = \sec^2 x$
16. $\frac{d}{dx}(\cot x) = -\csc^2 x$
17. $\frac{d}{dx}(\sec x) = \sec x \tan x$
18. $\frac{d}{dx}(\csc x) = -\csc x \cot x$
19. $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$
20. $\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$
21. $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$
22. $\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$
23. $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}$
24. $\frac{d}{dx}\left(\csc^{-1} x\right) = \frac{-1}{|x|\sqrt{x^2-1}}$
25. $\frac{d}{dx}(\sinh x) = \cosh x$
26. $\frac{d}{dx}(\cosh x) = \sinh x$
27. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
28. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$
29. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
30. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$
31. $\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{x^2+1}}$
32. $\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}}$
33. $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2}$
34. $\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1-x^2}$
35. $\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = \frac{-1}{x\sqrt{1-x^2}}$
36. $\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = \frac{-1}{|x|\sqrt{1+x^2}}$

Integration Rules

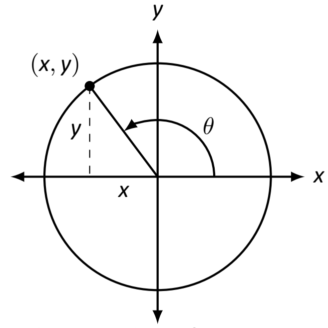
1. $\int c \cdot f(x)\, dx = c \int f(x)\, dx$
2. $\int f(x) \pm g(x)\, dx =$
   $\int f(x)\, dx \pm \int g(x)\, dx$
3. $\int 0\, dx = C$
4. $\int 1\, dx = x + C$
5. $\int x^n\, dx = \frac{1}{n+1}x^{n+1} + C,\ n \neq -1$
   $n \neq -1$
6. $\int e^x\, dx = e^x + C$
7. $\int a^x\, dx = \frac{1}{\ln a} \cdot a^x + C$
8. $\int \frac{1}{x}\, dx = \ln|x| + C$
9. $\int \sin x\, dx = -\cos x + C$
10. $\int \cos x\, dx = \sin x + C$
11. $\int \tan x\, dx = -\ln|\cos x| + C$
12. $\int \cot x\, dx = \ln|\sin x| + C$
13. $\int \sec x\, dx = \ln|\sec x + \tan x| + C$
14. $\int \csc x\, dx = -\ln|\csc x + \cot x| + C$
15. $\int \sec^2 x\, dx = \tan x + C$
16. $\int \csc^2 x\, dx = -\cot x + C$
17. $\int \sec x \tan x\, dx = \sec x + C$
18. $\int \csc x \cot x\, dx = -\csc x + C$
19. $\int \sin^2 x\, dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$
20. $\int \cos^2 x\, dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$
21. $\int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
22. $\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
23. $\int \frac{1}{x\sqrt{x^2-a^2}}\, dx = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right) + C$
24. $\int \sinh x\, dx = \cosh x + C$
25. $\int \cosh x\, dx = \sinh x + C$
26. $\int \tanh x\, dx = \ln(\cosh x) + C$
27. $\int \coth x\, dx = \ln|\sinh x| + C$
28. $\int \frac{1}{\sqrt{x^2-a^2}}\, dx = \ln\left|x + \sqrt{x^2-a^2}\right| + C$
29. $\int \frac{1}{\sqrt{x^2+a^2}}\, dx = \ln\left|x + \sqrt{x^2+a^2}\right| + C$
30. $\int \frac{1}{a^2-x^2}\, dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$
31. $\int \frac{1}{x\sqrt{a^2-x^2}}\, dx = \frac{1}{a}\ln\left(\frac{x}{a+\sqrt{a^2-x^2}}\right) + C$
32. $\int \frac{1}{x\sqrt{x^2+a^2}}\, dx = \frac{1}{a}\ln\left|\frac{x}{a+\sqrt{x^2+a^2}}\right| + C$

The Unit Circle



Definitions of the Trigonometric Functions

Unit Circle Definition

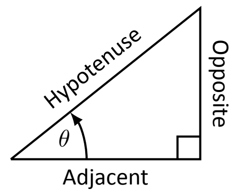


$\sin\theta = y \qquad \cos\theta = x$

$\csc\theta = \dfrac{1}{y} \qquad \sec\theta = \dfrac{1}{x}$

$\tan\theta = \dfrac{y}{x} \qquad \cot\theta = \dfrac{x}{y}$

Right Triangle Definition



$\sin\theta = \dfrac{\text{O}}{\text{H}} \qquad \csc\theta = \dfrac{\text{H}}{\text{O}}$

$\cos\theta = \dfrac{\text{A}}{\text{H}} \qquad \sec\theta = \dfrac{\text{H}}{\text{A}}$

$\tan\theta = \dfrac{\text{O}}{\text{A}} \qquad \cot\theta = \dfrac{\text{A}}{\text{O}}$

Common Trigonometric Identities

Pythagorean Identities

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

$1 + \cot^2 x = \csc^2 x$

Cofunction Identities

$\sin\left(\dfrac{\pi}{2} - x\right) = \cos x \qquad \csc\left(\dfrac{\pi}{2} - x\right) = \sec x$

$\cos\left(\dfrac{\pi}{2} - x\right) = \sin x \qquad \sec\left(\dfrac{\pi}{2} - x\right) = \csc x$

$\tan\left(\dfrac{\pi}{2} - x\right) = \cot x \qquad \cot\left(\dfrac{\pi}{2} - x\right) = \tan x$

Double Angle Formulas

$\sin 2x = 2\sin x\cos x$

$$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x\\ &= 2\cos^2 x - 1\\ &= 1 - 2\sin^2 x\end{aligned}$$

$\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

Sum to Product Formulas

$\sin x + \sin y = 2\sin\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$

$\sin x - \sin y = 2\sin\left(\dfrac{x-y}{2}\right)\cos\left(\dfrac{x+y}{2}\right)$

$\cos x + \cos y = 2\cos\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$

$\cos x - \cos y = -2\sin\left(\dfrac{x+y}{2}\right)\sin\left(\dfrac{x-y}{2}\right)$

Power–Reducing Formulas

$\sin^2 x = \dfrac{1 - \cos 2x}{2}$

$\cos^2 x = \dfrac{1 + \cos 2x}{2}$

$\tan^2 x = \dfrac{1 - \cos 2x}{1 + \cos 2x}$

Even/Odd Identities

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$\tan(-x) = -\tan x$

$\cot(-x) = -\cot x$

$\sec(-x) = \sec x$

$\csc(-x) = -\csc x$

Product to Sum Formulas

$\sin x\sin y = \dfrac{1}{2}\big(\cos(x-y) - \cos(x+y)\big)$

$\cos x\cos y = \dfrac{1}{2}\big(\cos(x-y) + \cos(x+y)\big)$

$\sin x\cos y = \dfrac{1}{2}\big(\sin(x+y) + \sin(x-y)\big)$

Angle Sum/Difference Formulas

$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$

$\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$

$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x\tan y}$

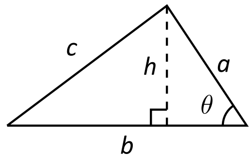
Areas and Volumes

Triangles

$h = a \sin\theta$

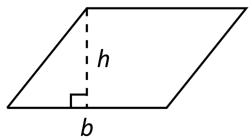
Area = $\frac{1}{2}bh$

Law of Cosines:
$c^2 = a^2 + b^2 - 2ab\cos\theta$



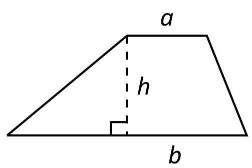
Parallelograms

Area = bh



Trapezoids

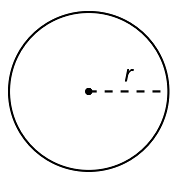
Area = $\frac{1}{2}(a+b)h$



Circles

Area = πr^2

Circumference = $2\pi r$

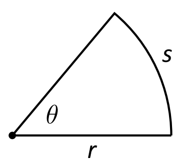


Sectors of Circles

θ in radians

Area = $\frac{1}{2}\theta r^2$

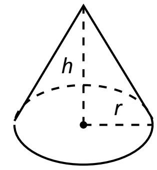
$s = r\theta$



Right Circular Cone

Volume = $\frac{1}{3}\pi r^2 h$

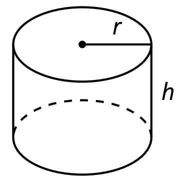
Surface Area =
$\pi r\sqrt{r^2+h^2} + \pi r^2$



Right Circular Cylinder

Volume = $\pi r^2 h$

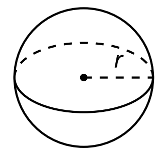
Surface Area =
$2\pi rh + 2\pi r^2$



Sphere

Volume = $\frac{4}{3}\pi r^3$

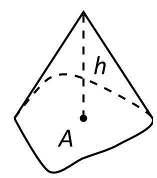
Surface Area =$4\pi r^2$



General Cone

Area of Base = A

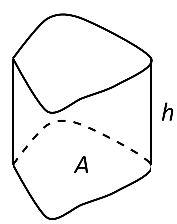
Volume = $\frac{1}{3}Ah$



General Right Cylinder

Area of Base = A

Volume = Ah



Algebra

Factors and Zeros of Polynomials

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial. If $p(a) = 0$, then a is a *zero* of the polynomial and a solution of the equation $p(x) = 0$. Furthermore, $(x - a)$ is a *factor* of the polynomial.

Fundamental Theorem of Algebra

An nth degree polynomial has n (not necessarily distinct) zeros. Although all of these zeros may be imaginary, a real polynomial of odd degree must have at least one real zero.

Quadratic Formula

If $p(x) = ax^2 + bx + c$, and $0 \le b^2 - 4ac$, then the real zeros of p are $x = (-b \pm \sqrt{b^2 - 4ac})/2a$

Special Factors

$$x^2 - a^2 = (x - a)(x + a) \qquad x^3 - a^3 = (x - a)(x^2 + ax + a^2)$$

$$x^3 + a^3 = (x + a)(x^2 - ax + a^2) \qquad x^4 - a^4 = (x^2 - a^2)(x^2 + a^2)$$

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdots + nxy^{n-1} + y^n$$

$$(x - y)^n = x^n - nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 - \cdots \pm nxy^{n-1} \mp y^n$$

Binomial Theorem

$$(x + y)^2 = x^2 + 2xy + y^2 \qquad (x - y)^2 = x^2 - 2xy + y^2$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 \qquad (x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \qquad (x - y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$$

Rational Zero Theorem

If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients, then every *rational zero* of p is of the form $x = r/s$, where r is a factor of a_0 and s is a factor of a_n.

Factoring by Grouping

$acx^3 + adx^2 + bcx + bd = ax^2(cx + d) + b(cx + d) = (ax^2 + b)(cx + d)$

Arithmetic Operations

$$ab + ac = a(b + c) \qquad \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right) = \frac{ad}{bc} \qquad \frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc} \qquad \frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b}$$

$$a\left(\frac{b}{c}\right) = \frac{ab}{c} \qquad \frac{a - b}{c - d} = \frac{b - a}{d - c} \qquad \frac{ab + ac}{a} = b + c$$

Exponents and Radicals

$$a^0 = 1,\ \ a \ne 0 \qquad (ab)^x = a^x b^x \qquad a^x a^y = a^{x+y} \qquad \sqrt{a} = a^{1/2} \qquad \frac{a^x}{a^y} = a^{x-y} \qquad \sqrt[n]{a} = a^{1/n}$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \qquad \sqrt[n]{a^m} = a^{m/n} \qquad a^{-x} = \frac{1}{a^x} \qquad \sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b} \qquad (a^x)^y = a^{xy} \qquad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

Additional Formulas

Summation Formulas:

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left[f(x_1) + 2f(x_2) + 2f(x_3) + \ldots + 2f(x_n) + f(x_{n+1})\right]$$

with Error $\leq \dfrac{(b-a)^3}{12n^2}\left[\max\left|f''(x)\right|\right]$

Simpson's Rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + \ldots + 2f(x_{n-1}) + 4f(x_n) + f(x_{n+1})\right]$$

with Error $\leq \dfrac{(b-a)^5}{180n^4}\left[\max\left|f^{(4)}(x)\right|\right]$

Arc Length:

$$L = \int_a^b \sqrt{1 + f'(x)^2}\ dx$$

Surface of Revolution:

$$S = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\ dx$$

(where $f(x) \geq 0$)

$$S = 2\pi \int_a^b x\sqrt{1 + f'(x)^2}\ dx$$

(where $a, b \geq 0$)

Work Done by a Variable Force:

$$W = \int_a^b F(x)\ dx$$

Force Exerted by a Fluid:

$$F = \int_a^b w\,d(y)\,\ell(y)\ dy$$

Taylor Series Expansion for $f(x)$:

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Maclaurin Series Expansion for $f(x)$, where $c = 0$:

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n$$

Summary of Tests for Series:

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} r^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	Sum $= \frac{1}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+a})$	$\lim_{n\to\infty} b_n = L$		Sum $= \left(\sum_{n=1}^{a} b_n\right) - L$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$	$p > 1$	$p \leq 1$	
Integral Test	$\sum_{n=0}^{\infty} a_n$	$\int_1^{\infty} a(n)\, dn$ is convergent	$\int_1^{\infty} a(n)\, dn$ is divergent	$a_n = a(n)$ must be positive, continuous, and decreasing.
Direct Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$	$\sum_{n=0}^{\infty} b_n$ diverges and $0 \leq b_n \leq a_n$	$\{a_n\}$ must be positive
Limit Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $\lim_{n\to\infty} a_n/b_n \geq 0$ (but not ∞)	$\sum_{n=0}^{\infty} b_n$ diverges and $\lim_{n\to\infty} a_n/b_n > 0$ (or ∞)	$\{a_n\}$ must be positive
Alternating Series	$\sum_{n=0}^{\infty} (-1)^n a_n$ or $\sum_{n=0}^{\infty} (-1)^{n+1} a_n$	a_n positive, decreasing, and $\lim_{n\to\infty} a_n = 0$		Can't be used to show divergence, though if $\lim_{n\to\infty} a_n \neq 0$, it diverges by the nth-Term test.
Ratio Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n\to\infty} a_{n+1}/a_n = \infty$
Root Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} (a_n)^{1/n} < 1$	$\lim_{n\to\infty} (a_n)^{1/n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n\to\infty} (a_n)^{1/n} = \infty$