9.3   Calculus and Parametric Equations

The previous section defined curves based on parametric equations. In this section we'll employ the techniques of calculus to study these curves.

We are still interested in lines tangent to points on a curve. They describe how the y-values are changing with respect to the x-values, they are useful in making approximations, and they indicate instantaneous direction of travel.

The slope of the tangent line is still $\frac{dy}{dx}$, and the Chain Rule allows us to calculate this in the context of parametric equations. If $x = f(t)$ and $y = g(t)$, the Chain Rule states that

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

Solving for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = \frac{dy}{dt} \bigg/ \frac{dx}{dt} = \frac{g'(t)}{f'(t)},$$

provided that $f'(t) \neq 0$. This is important so we label it a Key Idea.

Key Idea 39   Finding $\frac{dy}{dx}$ with Parametric Equations.

Let $x = f(t)$ and $y = g(t)$, where f and g are differentiable on some open interval I and $f'(t) \neq 0$ on I. Then

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)}.$$

We use this to define the tangent line.

Definition 50   Tangent and Normal Lines

Let a curve C be parameterized by $x = f(t)$ and $y = g(t)$, where f and g are differentiable functions on some interval I containing $t = t_0$. The **tangent line** to C at $t = t_0$ is the line through $\big(f(t_0), g(t_0)\big)$ with slope $m = g'(t_0)/f'(t_0)$, provided $f'(t_0) \neq 0$.

The **normal line** to C at $t = t_0$ is the line through $\big(f(t_0), g(t_0)\big)$ with slope $m = -f'(t_0)/g'(t_0)$, provided $g'(t_0) \neq 0$.

The definition leaves two special cases to consider. When the tangent line is horizontal, the normal line is undefined by the above definition as $g'(t_0) = 0$.

Notes:

Likewise, when the normal line is horizontal, the tangent line is undefined. It seems reasonable that these lines be defined (one can draw a line tangent to the "right side" of a circle, for instance), so we add the following to the above definition.

1. If the tangent line at $t = t_0$ has a slope of 0, the normal line to C at $t = t_0$ is the vertical line $x = f(t_0)$.

2. If the normal line at $t = t_0$ has a slope of 0, the tangent line to C at $t = t_0$ is the vertical line $x = f(t_0)$.

Example 9.17   Tangent and Normal Lines to Curves
Let $x = 5t^2 - 6t + 4$ and $y = t^2 + 6t - 1$, and let C be the curve defined by these equations.

1. Find the equations of the tangent and normal lines to C at $t = 3$.

2. Find where C has vertical and horizontal tangent lines.

SOLUTION

1. We start by computing $f'(t) = 10t - 6$ and $g'(t) = 2t + 6$. Thus

$$\frac{dy}{dx} = \frac{2t + 6}{10t - 6}.$$

   Make note of something that might seem unusual: $\frac{dy}{dx}$ is a function of t, not x. Just as points on the curve are found in terms of t, so are the slopes of the tangent lines.

   The point on C at $t = 3$ is $(31, 26)$. The slope of the tangent line is $m = 1/2$ and the slope of the normal line is $m = -2$. Thus,

   • the equation of the tangent line is $y = \dfrac{1}{2}(x - 31) + 26$, and

   • the equation of the normal line is $y = -2(x - 31) + 26$.

   This is illustrated in Figure 9.29.

Figure 9.29: Graphing tangent and normal lines in Example 9.17.

2. To find where C has a horizontal tangent line, we set $\frac{dy}{dx} = 0$ and solve for t. In this case, this amounts to setting $g'(t) = 0$ and solving for t (and making sure that $f'(t) \neq 0$).

$$g'(t) = 0 \quad \Rightarrow \quad 2t + 6 = 0 \quad \Rightarrow \quad t = -3.$$

   The point on C corresponding to $t = -3$ is $(67, -10)$; the tangent line at that point is horizontal (hence with equation $y = -10$).
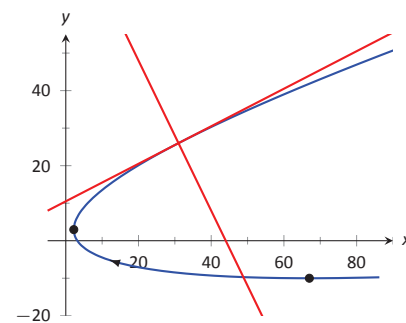
Notes:

To find where C has a vertical tangent line, we find where it has a horizontal normal line, and set $-\frac{f'(t)}{g'(t)} = 0$. This amounts to setting $f'(t) = 0$ and solving for t (and making sure that $g'(t) \neq 0$).

$$f'(t) = 0 \quad \Rightarrow \quad 10t - 6 = 0 \quad \Rightarrow \quad t = 0.6.$$

The point on C corresponding to $t = 0.6$ is $(2.2, 2.96)$. The tangent line at that point is $x = 2.2$.

The points where the tangent lines are vertical and horizontal are indicated on the graph in Figure 9.29.

Example 9.18 Tangent and Normal Lines to a Circle

1. Find where the unit circle, defined by $x = \cos t$ and $y = \sin t$ on $[0, 2\pi]$, has vertical and horizontal tangent lines.

2. Find the equation of the normal line at $t = t_0$.

SOLUTION

1. We compute the derivative following Key Idea 39:

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)} = -\frac{\cos t}{\sin t}.$$

The derivative is 0 when $\cos t = 0$; that is, when $t = \pi/2,\ 3\pi/2$. These are the points $(0, 1)$ and $(0, -1)$ on the circle.

The normal line is horizontal (and hence, the tangent line is vertical) when $\sin t = 0$; that is, when $t = 0,\ \pi,\ 2\pi$, corresponding to the points $(-1, 0)$ and $(0, 1)$ on the circle. These results should make intuitive sense.

2. The slope of the normal line at $t = t_0$ is $m = \dfrac{\sin t_0}{\cos t_0} = \tan t_0$. This normal line goes through the point $(\cos t_0, \sin t_0)$, giving the line

$$\begin{aligned} y &= \frac{\sin t_0}{\cos t_0}(x - \cos t_0) + \sin t_0 \\ &= (\tan t_0)x, \end{aligned}$$

as long as $\cos t_0 \neq 0$. It is an important fact to recognize that the normal lines to a circle pass through its center, as illustrated in Figure 9.30. Stated in another way, any line that passes through the center of a circle intersects the circle at right angles.
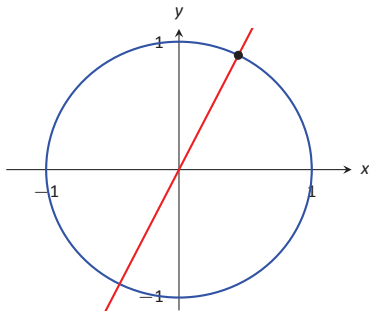

Figure 9.30: Illustrating how a circle's normal lines pass through its center.

Notes:

Example 9.19 **Tangent lines when $\frac{dy}{dx}$ is not defined**

Find the equation of the tangent line to the astroid $x = \cos^3 t$, $y = \sin^3 t$ at $t = 0$, shown in Figure 9.31.

SOLUTION We start by finding $x'(t)$ and $y'(t)$:

$$x'(t) = -3\sin t \cos^2 t, \qquad y'(t) = 3\cos t \sin^2 t.$$

Note that both of these are 0 at $t = 0$; the curve is not smooth at $t = 0$ forming a cusp on the graph. Evaluating $\frac{dy}{dx}$ at this point returns the indeterminate form of "0/0".

We can, however, examine the slopes of tangent lines near $t = 0$, and take the limit as $t \to 0$.

$$\lim_{t \to 0} \frac{y'(t)}{x'(t)} = \lim_{t \to 0} \frac{3\cos t \sin^2 t}{-3\sin t \cos^2 t} \qquad \text{(We can cancel as } t \neq 0.\text{)}$$
$$= \lim_{t \to 0} -\frac{\sin t}{\cos t}$$
$$= 0.$$

We have accomplished something significant. When the derivative $\frac{dy}{dx}$ returns an indeterminate form at $t = t_0$, we can define its value by setting it to be $\lim\limits_{t \to t_0} \frac{dy}{dx}$, if that limit exists. This allows us to find slopes of tangent lines at cusps, which can be very beneficial.

We found the slope of the tangent line at $t = 0$ to be 0; therefore the tangent line is $y = 0$, the x-axis.

Concavity

We continue to analyze curves in the plane by considering their concavity; that is, we are interested in $\frac{d^2y}{dx^2}$, "the second derivative of y with respect to x." To find this, we need to find the derivative of $\frac{dy}{dx}$ with respect to x; that is,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right],$$

but recall that $\frac{dy}{dx}$ is a function of t, not x, making this computation not straightforward.

To make the upcoming notation a bit simpler, let $h(t) = \frac{dy}{dx}$. We want $\frac{d}{dx}[h(t)]$; that is, we want $\frac{dh}{dx}$. We again appeal to the Chain Rule. Note:

$$\frac{dh}{dt} = \frac{dh}{dx} \cdot \frac{dx}{dt} \qquad \Rightarrow \qquad \frac{dh}{dx} = \frac{dh}{dt} \bigg/ \frac{dx}{dt}.$$
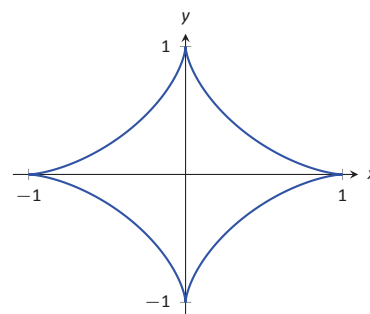

Figure 9.31: A graph of an astroid.

Notes:

In words, to find $\frac{d^2y}{dx^2}$, we first take the derivative of $\frac{dy}{dx}$ *with respect to t*, then divide by $x'(t)$. We restate this as a Key Idea.

Key Idea 40   **Finding $\frac{d^2y}{dx^2}$ with Parametric Equations**

Let $x = f(t)$ and $y = g(t)$ be twice differentiable functions on an open interval I, where $f'(t) \neq 0$ on I. Then

$$\frac{d^2y}{dx^2} \;=\; \frac{d}{dt}\left[\frac{dy}{dx}\right] \Big/ \frac{dx}{dt} \;=\; \frac{d}{dt}\left[\frac{dy}{dx}\right] \Big/ f'(t).$$

Examples will help us understand this Key Idea.

Example 9.20   **Concavity of Plane Curves**

Let $x = 5t^2 - 6t + 4$ and $y = t^2 + 6t - 1$ as in Example 9.17. Determine the t-intervals on which the graph is concave up/down.

SOLUTION   Concavity is determined by the second derivative of y with respect to x, $\frac{d^2y}{dx^2}$, so we compute that here following Key Idea 40.

In Example 9.17, we found $\dfrac{dy}{dx} = \dfrac{2t+6}{10t-6}$ and $f'(t) = 10t - 6$. So:

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left[\frac{2t+6}{10t-6}\right] \Big/ (10t - 6)$$

$$= -\frac{72}{(10t-6)^2} \Big/ (10t - 6)$$

$$= -\frac{72}{(10t-6)^3}$$

$$= -\frac{9}{(5t-3)^3}$$

The graph of the parametric functions is concave up when $\frac{d^2y}{dx^2} > 0$ and concave down when $\frac{d^2y}{dx^2} < 0$. We determine the intervals when the second derivative is greater/less than 0 by first finding when it is 0 or undefined.

As the numerator of $-\dfrac{9}{(5t-3)^3}$ is never 0, $\frac{d^2y}{dx^2} \neq 0$ for all t. It is undefined when $5t - 3 = 0$; that is, when $t = 3/5$. Following the work established in Section 3.4, we look at values of t greater/less than $3/5$ on a number line:
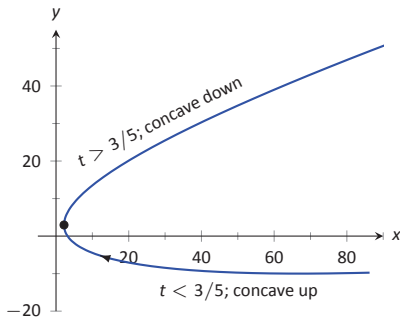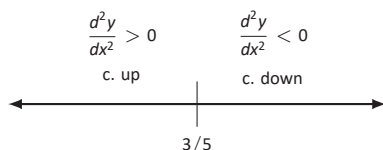

y

40

t > 3/5; concave down

20

20   40   60   80   x

t < 3/5; concave up

−20

Figure 9.32: Graphing the parametric equations in Example 9.20 to demonstrate concavity.

Notes:

$$\frac{d^2y}{dx^2} > 0 \qquad \frac{d^2y}{dx^2} < 0$$

c. up   c. down

3/5

Reviewing Example 9.17, we see that when $t = 3/5 = 0.6$, the graph of the parametric equations has a vertical tangent line. This point is also a point of inflection for the graph, illustrated in Figure 9.32.

Example 9.21   Concavity of Plane Curves

Find the points of inflection of the graph of the parametric equations $x = \sqrt{t}$, $y = \sin t$, for $0 \le t \le 16$.

SOLUTION   We need to compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\cos t}{1/(2\sqrt{t})} = 2\sqrt{t}\cos t.$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{x'(t)} = \frac{\cos t/\sqrt{t} - 2\sqrt{t}\sin t}{1/(2\sqrt{t})} = 2\cos t - 4t\sin t.$$

The points of inflection are found by setting $\frac{d^2y}{dx^2} = 0$. This is not trivial, as equations that mix polynomials and trigonometric functions generally do not have "nice" solutions.

In Figure 9.33(a) we see a plot of the second derivative. It shows that it has zeros at approximately $t = 0.5$, 3.5, 6.5, 9.5, 12.5 and 16. These approximations are not very good, made only by looking at the graph. Newton's Method provides more accurate approximations. Accurate to 2 decimal places, we have:

$$t = 0.65,\ 3.29,\ 6.36,\ 9.48,\ 12.61 \text{ and } 15.74.$$

The corresponding points have been plotted on the graph of the parametric equations in Figure 9.33(b). Note how most occur near the x-axis, but not exactly on the axis.

Arc Length

We continue our study of the features of the graphs of parametric equations by computing their arc length.

Recall in Section 7.4 we found the arc length of the graph of a function, from $x = a$ to $x = b$, to be

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$
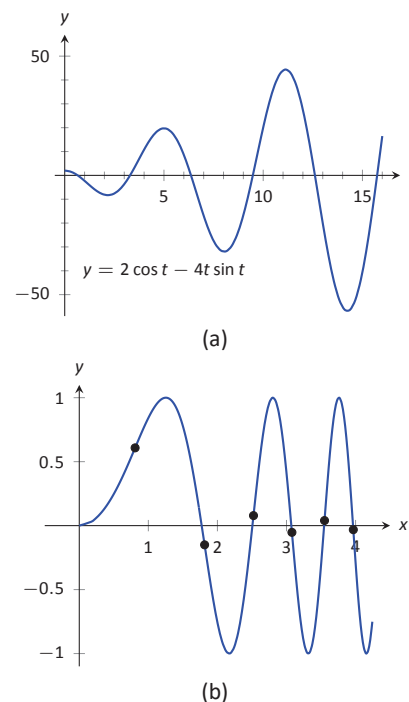

$y = 2\cos t - 4t\sin t$

(a)

(b)

Figure 9.33: In (a), a graph of $\frac{d^2y}{dx^2}$, showing where it is approximately 0. In (b), graph of the parametric equations in Example 9.21 along with the points of inflection.

Notes:

We can use this equation and convert it to the parametric equation context. Letting $x = f(t)$ and $y = g(t)$, we know that $\frac{dy}{dx} = g'(t)/f'(t)$. It will also be useful to calculate the differential of x:

$$dx = f'(t)dt \qquad \Rightarrow \qquad dt = \frac{1}{f'(t)} \cdot dx.$$

Starting with the arc length formula above, consider:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_a^b \sqrt{1 + \frac{g'(t)^2}{f'(t)^2}}\, dx.$$

Factor out the $f'(t)^2$:

$$= \int_a^b \sqrt{f'(t)^2 + g'(t)^2} \cdot \underbrace{\frac{1}{f'(t)}\, dx}_{=dt}$$

$$= \int_{t_1}^{t_2} \sqrt{f'(t)^2 + g'(t)^2}\, dt.$$

Note the new bounds (no longer "x" bounds, but "t" bounds). They are found by finding t_1 and t_2 such that $a = f(t_1)$ and $b = f(t_2)$. This formula is important, so we restate it as a theorem.

Theorem 84   Arc Length of Parametric Curves

Let $x = f(t)$ and $y = g(t)$ be parametric equations with f' and g' continuous on some open interval I containing t_1 and t_2 on which the graph traces itself only once. The arc length of the graph, from $t = t_1$ to $t = t_2$, is

$$L = \int_{t_1}^{t_2} \sqrt{f'(t)^2 + g'(t)^2}\, dt.$$

As before, these integrals are often not easy to compute. We start with a simple example, then give another where we approximate the solution.

Notes:

Example 9.22   **Arc Length of a Circle**

Find the arc length of the circle parameterized by $x = 3\cos t$, $y = 3\sin t$ on $[0, 3\pi/2]$.

> **SOLUTION**   By direct application of Theorem 84, we have
>
> $$L = \int_0^{3\pi/2} \sqrt{(-3\sin t)^2 + (3\cos t)^2}\, dt.$$
>
> Apply the Pythagorean Theorem.
>
> $$= \int_0^{3\pi/2} 3\, dt$$
>
> $$= 3t\Big|_0^{3\pi/2} = 9\pi/2.$$
>
> This should make sense; we know from geometry that the circumference of a circle with radius 3 is 6π; since we are finding the arc length of $3/4$ of a circle, the arc length is $3/4 \cdot 6\pi = 9\pi/2$.

Example 9.23   **Arc Length of a Parametric Curve**

The graph of the parametric equations $x = t(t^2 - 1)$, $y = t^2 - 1$ crosses itself as shown in Figure 9.34, forming a "teardrop." Find the arc length of the teardrop.

> **SOLUTION**   We can see by the parameterizations of x and y that when $t = \pm 1$, $x = 0$ and $y = 0$. This means we'll integrate from $t = -1$ to $t = 1$. Applying Theorem 84, we have
>
> $$L = \int_{-1}^{1} \sqrt{(3t^2 - 1)^2 + (2t)^2}\, dt$$
>
> $$= \int_{-1}^{1} \sqrt{9t^4 - 2t^2 + 1}\, dt.$$

Unfortunately, the integrand does not have an antiderivative expressible by elementary functions. We turn to numerical integration to approximate its value. Using 4 subintervals, Simpson's Rule approximates the value of the integral as 2.65051. Using a computer, more subintervals are easy to employ, and $n = 20$ gives a value of 2.71559. Increasing n shows that this value is stable and a good approximation of the actual value.
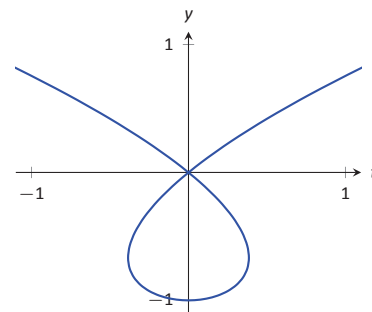

Figure 9.34: A graph of the parametric equations in Example 9.23, where the arc length of the teardrop is calculated.

Notes:

Example 9.24   Arc Length of a Parametric Curve
Determine the length of the graph of the parametric equations $x = 20/t$, $y = t^{3/2}$ from the point $(20, 1)$ to the point $(5, 8)$.

SOLUTION   In this problem, we need to determine the t-value(s) for the given points, as in the previous section. For the point $(20, 1)$, solving $x = 20/t = 20$ gives $t = 1$. We plug this into $y = t^{3/2} = 1^{3/2} = 1$, confirming the point is on the curve at $t = 1$. Similarly, we can determine the point $(5, 8)$ corresponds to $t = 4$. So we'll integrate from $t = 1$ to $t = 4$. Applying Theorem 84, we have

$$L = \int_1^4 \sqrt{\left(-\frac{20}{t^2}\right)^2 + \left(\frac{3}{2}t^{1/2}\right)^2}\, dt$$

$$= \int_1^4 \sqrt{\frac{400}{t^4} + \frac{9}{4}t}\, dt.$$

This integrand does not have an antiderivative expressible by elementary functions, but a computer approximates this integral as 17.6087.

Surface Area of a Solid of Revolution

Related to the formula for finding arc length is the formula for finding surface area. We can adapt the formula found in Key Idea 29 from Section 7.4 in a similar way as done to produce the formula for arc length done before.

Key Idea 41   Surface Area of a Solid of Revolution

Consider the graph of the parametric equations $x = f(t)$ and $y = g(t)$, where f' and g' are continuous on an open interval I containing t_1 and t_2 on which the graph does not cross itself.

1. The surface area of the solid formed by revolving the graph about the x-axis is (where $g(t) \geq 0$ on $[t_1, t_2]$):

$$\text{Surface Area} = 2\pi \int_{t_1}^{t_2} g(t)\sqrt{f'(t)^2 + g'(t)^2}\, dt.$$

2. The surface area of the solid formed by revolving the graph about the y-axis is (where $f(t) \geq 0$ on $[t_1, t_2]$):

$$\text{Surface Area} = 2\pi \int_{t_1}^{t_2} f(t)\sqrt{f'(t)^2 + g'(t)^2}\, dt.$$

Notes:

Example 9.25   Surface Area of a Solid of Revolution
Consider the teardrop shape formed by the parametric equations $x = t(t^2 - 1)$, $y = t^2 - 1$ as seen in Example 9.23. Find the surface area if this shape is rotated about the x-axis, as shown in Figure 9.35.

   Solution   The teardrop shape is formed between $t = -1$ and $t = 1$. Using Key Idea 41, we see we need for $g(t) \geq 0$ on $[-1, 1]$, and this is not the case. To fix this, we simplify replace $g(t)$ with $-g(t)$, which flips the whole graph about the x-axis (and does not change the surface area of the resulting solid). The surface area is:

$$\text{Area } S = 2\pi \int_{-1}^{1} (1 - t^2)\sqrt{(3t^2 - 1)^2 + (2t)^2}\, dt$$

$$= 2\pi \int_{-1}^{1} (1 - t^2)\sqrt{9t^4 - 2t^2 + 1}\, dt.$$

Once again we arrive at an integral that we cannot compute in terms of elementary functions. Using Simpson's Rule with $n = 20$, we find the area to be $S = 9.44$. Using larger values of n shows this is accurate to 2 places after the decimal.

   After defining a new way of creating curves in the plane, in this section we have applied calculus techniques to the parametric equation defining these curves to study their properties. In the next section, we define another way of forming curves in the plane. To do so, we create a new coordinate system, called *polar coordinates*, that identifies points in the plane in a manner different than from measuring distances from the y- and x- axes.
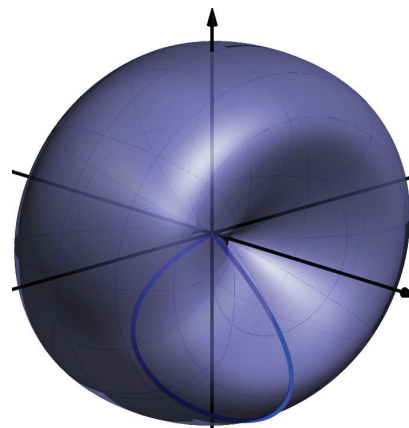

Figure 9.35: Rotating a teardrop shape about the x-axis in Example 9.25.

Notes:

Exercises 9.3

Terms and Concepts

1. T/F: Given parametric equations $x = f(t)$ and $y = g(t)$, $\frac{dy}{dx} = f'(t)/g'(t)$, as long as $g'(t) \neq 0$.

2. Given parametric equations $x = f(t)$ and $y = g(t)$, the derivative $\frac{dy}{dx}$ as given in Key Idea 39 is a function of _____?

3. T/F: Given parametric equations $x = f(t)$ and $y = g(t)$, to find $\frac{d^2y}{dx^2}$, one simply computes $\frac{d}{dt}\left(\frac{dy}{dx}\right)$.

4. T/F: If $\frac{dy}{dx} = 0$ at $t = t_0$, then the normal line to the curve at $t = t_0$ is a vertical line.

Problems

In Exercises 5 – 12, parametric equations for a curve are given.

(a) Find $\dfrac{dy}{dx}$.

(b) Find the equations of the tangent and normal line(s) at the point(s) given.

(c) Sketch the graph of the parametric functions along with the found tangent and normal lines.

5. $x = t, y = t^2; \quad t = 1$

6. $x = \sqrt{t}, y = 5t + 2; \quad t = 4$

7. $x = t^2 - t, y = t^2 + t; \quad t = 1$

8. $x = t^2 - 1, y = t^3 - t; \quad t = 0$ and $t = 1$

9. $x = \sec t, y = \tan t$ on $(-\pi/2, \pi/2); \quad t = \pi/4$

10. $x = \cos t, y = \sin(2t)$ on $[0, 2\pi]; \quad t = \pi/4$

11. $x = \cos t \sin(2t), y = \sin t \sin(2t)$ on $[0, 2\pi]; \quad t = 3\pi/4$

12. $x = e^{t/10} \cos t, y = e^{t/10} \sin t; \quad t = \pi/2$

In Exercises 13 – 20, find t-values where the curve defined by the given parametric equations has a horizontal tangent line. Note: these are the same equations as in Exercises 5 – 12.

13. $x = t, y = t^2$

14. $x = \sqrt{t}, y = 5t + 2$

15. $x = t^2 - t, y = t^2 + t$

16. $x = t^2 - 1, y = t^3 - t$

17. $x = \sec t, y = \tan t$ on $(-\pi/2, \pi/2)$

18. $x = \cos t, y = \sin(2t)$ on $[0, 2\pi]$

19. $x = \cos t \sin(2t), y = \sin t \sin(2t)$ on $[0, 2\pi]$

20. $x = e^{t/10} \cos t, y = e^{t/10} \sin t$

In Exercises 21 – 24, find $t = t_0$ where the graph of the given parametric equations is not smooth, then find $\lim\limits_{t \to t_0} \dfrac{dy}{dx}$.

21. $x = \dfrac{1}{t^2 + 1}, \quad y = t^3$

22. $x = -t^3 + 7t^2 - 16t + 13, \quad y = t^3 - 5t^2 + 8t - 2$

23. $x = t^3 - 3t^2 + 3t - 1, \quad y = t^2 - 2t + 1$

24. $x = \cos^2 t, \quad y = 1 - \sin^2 t$

In Exercises 25 – 32, parametric equations for a curve are given. Find $\frac{d^2y}{dx^2}$, then determine the intervals on which the graph of the curve is concave up/down. Note: these are the same equations as in Exercises 5 – 12.

25. $x = t, \quad y = t^2$

26. $x = \sqrt{t}, \quad y = 5t + 2$

27. $x = t^2 - t, \quad y = t^2 + t$

28. $x = t^2 - 1, \quad y = t^3 - t$

29. $x = \sec t, \quad y = \tan t$ on $(-\pi/2, \pi/2)$

30. $x = \cos t, \quad y = \sin(2t)$ on $[0, 2\pi]$

31. $x = \cos t \sin(2t), \quad y = \sin t \sin(2t)$ on $[-\pi/2, \pi/2]$

32. $x = e^{t/10} \cos t, \quad y = e^{t/10} \sin t$

In Exercises 33 – 36, find the arc length of the graph of the parametric equations on the given interval(s).

33. $x = -3 \sin(2t), \quad y = 3 \cos(2t)$ on $[0, \pi]$

34. $x = e^{t/10} \cos t, \quad y = e^{t/10} \sin t$ on $[0, 2\pi]$ and $[2\pi, 4\pi]$

35. $x = 5t + 2, \quad y = 1 - 3t$ on $[-1, 1]$

36. $x = 2t^{3/2}, \quad y = 3t$ on $[0, 1]$

In Exercises 37 – 38, set up an integral expression for the arc length of the curve between the specified points. Do not evaluate the integral.

37. $x = t^3 - t, \quad y = t^3 + 2t$ from $(0, -3)$ to $(6, 12)$

38. $x = \cos(3t) - \sin t, \quad y = 2 \cos t$ from $(1, 2)$ to $(-1, 0)$

39. A projectile is fired from a height of 0ft, landing 16ft away in 4s. Determine the total distance the projectile traveled.

40. A projectile is fired from a height of 0ft, landing 200ft away in 20s. Determine the total distance the projectile traveled.

In Exercises 41 – 44, numerically approximate the given arc length.

41. Approximate the arc length of one petal of the rose curve $x = \cos t \cos(2t)$, $y = \sin t \cos(2t)$ using Simpson's Rule and $n = 4$.

42. Approximate the arc length of the "bow tie curve" $x = \cos t$, $y = \sin(2t)$ using Simpson's Rule and $n = 6$.

43. Approximate the arc length of the parabola $x = t^2 - t$, $y = t^2 + t$ on $[-1, 1]$ using Simpson's Rule and $n = 4$.

44. A common approximate of the circumference of an ellipse given by $x = a \cos t$, $y = b \sin t$ is $C \approx 2\pi\sqrt{\dfrac{a^2 + b^2}{2}}$. Use this formula to approximate the circumference of $x = 5 \cos t$, $y = 3 \sin t$ and compare this to the approximation given by Simpson's Rule and $n = 6$.

In Exercises 45 – 48, a solid of revolution is described. Find or approximate its surface area as specified.

45. Find the surface area of the sphere formed by rotating the circle $x = 2 \cos t$, $y = 2 \sin t$ about:

   (a) the x-axis and

   (b) the y-axis.

46. Find the surface area of the torus (or "donut") formed by rotating the circle $x = \cos t + 2$, $y = \sin t$ about the y-axis.

47. Approximate the surface area of the solid formed by rotating the "upper right half" of the bow tie curve $x = \cos t$, $y = \sin(2t)$ on $[0, \pi/2]$ about the x-axis, using Simpson's Rule and $n = 4$.

48. Approximate the surface area of the solid formed by rotating the one petal of the rose curve $x = \cos t \cos(2t)$, $y = \sin t \cos(2t)$ on $[0, \pi/4]$ about the x-axis, using Simpson's Rule and $n = 4$.