14.7   The Divergence Theorem in Space

Theorem 142 gives the Divergence Theorem in the plane, which states that the flux of a vector field across a closed *curve* equals the sum of the divergences over the region enclosed by the curve. Recall that the flux was measured via a line integral, and the sum of the divergences was measured through a double integral.

We now consider the three-dimensional version of the Divergence Theorem. It states, in words, that the flux across a closed *surface* equals the sum of the divergences over the domain enclosed by the surface. Since we are in space (versus the plane), we measure flux via a surface integral, and the sums of divergences will be measured through a triple integral.

Note: the term "outer unit normal vector" used in Theorem 144 means $\vec{n}$ points to the outside of $\mathcal{S}$.

> **Theorem 144   The Divergence Theorem (in space)**
>
> Let D be a closed domain in space whose boundary is an orientable, piecewise-smooth surface $\mathcal{S}$ with outer unit normal vector $\vec{n}$, and let $\vec{F}$ be a vector field whose components are differentiable on D. Then
>
> $$\iint_{\mathcal{S}} \vec{F} \cdot \vec{n} \, dS = \iiint_{D} \operatorname{div} \vec{F} \, dV.$$

Example 14.36   Using the Divergence Theorem in space
Let D be the domain in space bounded by the planes $z = 0$ and $z = 2x$, along with the cylinder $x = 1 - y^2$, as graphed in Figure 14.44, let $\mathcal{S}$ be the boundary of D, and let $\vec{F} = \langle x + y, y^2, 2z \rangle$.

Verify the Divergence Theorem by finding the total outward flux of $\vec{F}$ across $\mathcal{S}$, and show this is equal to $\iiint_{D} \operatorname{div} \vec{F} \, dV$.

SOLUTION   The surface $\mathcal{S}$ is piecewise-smooth, comprising surfaces $\mathcal{S}_1$, which is part of the plane $z = 2x$, surface $\mathcal{S}_2$, which is part of the cylinder $x = 1 - y^2$, and surface $\mathcal{S}_3$, which is part of the plane $z = 0$. To find the total outward flux across $\mathcal{S}$, we need to compute the outward flux across each of these three surfaces.

We leave it to the reader to confirm that surfaces $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ can be parameterized by $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ respectively as

$$\vec{r}_1(u, v) = \langle v(1 - u^2), u, 2v(1 - u^2) \rangle,$$
$$\vec{r}_2(u, v) = \langle (1 - u^2), u, 2v(1 - u^2) \rangle,$$
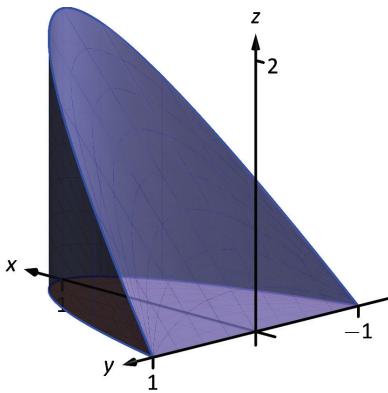$$\vec{r}_3(u, v) = \langle v(1 - u^2), u, 0 \rangle,$$

Figure 14.44: The surfaces used in Example 14.36.

Notes:

where $-1 \leq u \leq 1$ and $0 \leq v \leq 1$ for all three functions.

We compute a unit normal vector $\vec{n}$ for each as $\frac{\vec{r}_u \times \vec{r}_v}{||\vec{r}_u \times \vec{r}_v||}$, though recall that as we are integrating $\vec{F} \cdot \vec{n}\, dS$, we actually only use $\vec{r}_u \times \vec{r}_v$. Finally, in previous flux computations, it did not matter which direction $\vec{n}$ pointed as long as we made note of its direction. When using the Divergence Theorem, we need $\vec{n}$ to point to the outside of the closed surface, so in practice this means we'll either use $\vec{r}_u \times \vec{r}_v$ or $\vec{r}_v \times \vec{r}_u$, depending on which points outside of the closed surface $\mathcal{S}$.

We leave it to the reader to confirm the following cross products and integrations are correct.

For $\mathcal{S}_1$, we need to use $\vec{r}_{1v} \times \vec{r}_{1u} = \langle 2(u^2 - 1), 0, 1 - u^2 \rangle$. (Note the z-component is nonnegative as $u \leq 1$, therefore this vector always points up, meaning *to the outside*, of $\mathcal{S}$.) The flux across $\mathcal{S}_1$ is:

$$
\begin{aligned}
\text{Flux across } \mathcal{S}_1: &= \iint_{\mathcal{S}_1} \vec{F} \cdot \vec{n}_1 \, dS \\
&= \int_0^1 \int_{-1}^1 \vec{F}(\vec{r}_1(u,v)) \cdot (\vec{r}_{1v} \times \vec{r}_{1u}) \, du \, dv \\
&= \int_0^1 \int_{-1}^1 \langle v(1-u^2) + u, u^2, 4v(1-u^2) \rangle \cdot \langle 2(u^2-1), 0, 1-u^2 \rangle \, du \, dv \\
&= \int_0^1 \int_{-1}^1 \left(2u^4 v + 2u^3 - 4u^2 v - 2u + 2v \right) du \, dv \\
&= \frac{16}{15}.
\end{aligned}
$$

For $\mathcal{S}_2$, we use $\vec{r}_{2u} \times \vec{r}_{2v} = \langle 2(1-u^2), 4u(1-u^2), 0 \rangle$. (Note the x-component is always nonnegative, meaning this vector points outside $\mathcal{S}$.) The flux across $\mathcal{S}_2$ is:

$$
\begin{aligned}
\text{Flux across } \mathcal{S}_2: &= \iint_{\mathcal{S}_2} \vec{F} \cdot \vec{n}_2 \, dS \\
&= \int_0^1 \int_{-1}^1 \vec{F}(\vec{r}_2(u,v)) \cdot (\vec{r}_{2u} \times \vec{r}_{2v}) \, du \, dv \\
&= \int_0^1 \int_{-1}^1 \langle 1 - u^2 + u, u^2, 4v(1-u^2) \rangle \cdot \langle 2(1-u^2), 4u(1-u^2), 0 \rangle \, du \, dv \\
&= \int_0^1 \int_{-1}^1 \left(4u^5 - 2u^4 - 2u^3 + 4u^2 - 2u - 2 \right) du \, dv \\
&= \frac{32}{15}.
\end{aligned}
$$

For $\mathcal{S}_3$, we use $\vec{r}_{3u} \times \vec{r}_{3v} = \langle 0, 0, u^2 - 1 \rangle$. (Note the z-component is never

---

Notes:

positive, meaning this vector points down, outside of $\mathcal{S}$.) The flux across $\mathcal{S}_3$ is:

$$
\begin{aligned}
\text{Flux across } \mathcal{S}_3 \colon &= \iint_{\mathcal{S}_3} \vec{F} \cdot \vec{n}_3 \, dS \\
&= \int_0^1 \int_{-1}^1 \vec{F}(\vec{r}_3(u,v)) \cdot (\vec{r}_{3u} \times \vec{r}_{3v}) \, du \, dv \\
&= \int_0^1 \int_{-1}^1 \left\langle v(1-u^2) + u, u^2, 0 \right\rangle \cdot \left\langle 0, 0, u^2 - 1 \right\rangle \, du \, dv \\
&= \int_0^1 \int_{-1}^1 0 \, du \, dv \\
&= 0.
\end{aligned}
$$

Thus the total outward flux, measured by surface integrals across all three component surfaces of $\mathcal{S}$, is $16/15 + 32/15 + 0 = 48/15 = 16/5 = 3.2$. We now find the total outward flux by integrating div $\vec{F}$ over D.

Following the steps outlined in Section 13.6, we see the bounds of x, y and z can be set as (thinking "surface to surface, curve to curve, point to point"):

$$
0 \leq z \leq 2x; \quad 0 \leq x \leq 1 - y^2; \quad -1 \leq y \leq 1.
$$

With div $\vec{F} = 1 + 2y + 2 = 2y + 3$, we find the total outward flux of $\vec{F}$ over $\mathcal{S}$ as:

$$
\text{Flux} = \iiint_D \text{div} \, \vec{F} \, dV = \int_{-1}^1 \int_0^{1-y^2} \int_0^{2x} (2y+3) \, dz \, dx \, dy = 16/5,
$$

the same result we obtained previously.

In Example 14.36 we see that the total outward flux of a vector field across a closed surface can be found two different ways because of the Divergence Theorem. One computation took far less work to obtain. In that particular case, since $\mathcal{S}$ was comprised of three separate surfaces, it was far simpler to compute one triple integral than three surface integrals (each of which required partial derivatives and a cross product). In practice, if outward flux needs to be measured, one would choose only one method. We will use both methods in this section simply to reinforce the truth of the Divergence Theorem.

We practice again in the following example.

Example 14.37   Using the Divergence Theorem in space

Let $\mathcal{S}$ be the surface formed by the paraboloid $z = 1 - x^2 - y^2$, $z \geq 0$, and the unit disk centered at the origin in the x-y plane, graphed in Figure 14.45, and let $\vec{F} = \langle 0, 0, z \rangle$. (This surface and vector field were used in Example 14.35.)

Verify the Divergence Theorem; find the total outward flux across $\mathcal{S}$ and evaluate the triple integral of div $\vec{F}$, showing that these two quantities are equal.

Notes:

SOLUTION   We find the flux across $\mathcal{S}$ first. As $\mathcal{S}$ is piecewise–smooth, we decompose it into smooth components $\mathcal{S}_1$, the disk, and $\mathcal{S}_2$, the paraboloid, and find the flux across each.

In Example 14.35, we found the flux across $\mathcal{S}_1$ is 0. We also found that the flux across $\mathcal{S}_2$ is $\pi/2$. (In that example, the normal vector had a positive z component hence was an outer normal.) Thus the total outward flux is $0 + \pi/2 = \pi/2$.

We now compute $\iiint_D \text{div}\,\vec{F}\,dV$. We can describe D as the domain bounded by (think "surface to surface, curve to curve, point to point"):

$$0 \leq z \leq 1 - x^2 - y^2, \quad -\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}, \quad -1 \leq x \leq 1.$$

This description of D is not very easy to integrate. With polar, we can do better. Let R represent the unit disk, which can be described in polar simply as r, where $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. With $x = r\cos\theta$ and $y = r\sin\theta$, the surface $\mathcal{S}_2$ becomes

$$z = 1 - x^2 - y^2 \Rightarrow 1 - (r\cos\theta)^2 - (r\sin\theta)^2 \Rightarrow 1 - r^2.$$

Thus D can be described as the domain bounded by:

$$0 \leq z \leq 1 - r^2, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.$$

With $\text{div}\,\vec{F} = 1$, we can integrate, recalling that $dV = r\,dz\,dr\,d\theta$:

$$\iiint_D \text{div}\,\vec{F}\,dV = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} r\,dz\,dr\,d\theta = \frac{\pi}{2},$$

which matches our flux computation above.

Figure 14.45: The surfaces used in Example 14.37.

Example 14.38   A "paradox" of the Divergence Theorem and Gauss's Law

The magnitude of many physical quantities (such as light intensity or electromagnetic and gravitational forces) follow an "inverse square law": the magnitude of the quantity at a point is inversely proportional to the square of the distance to the source of the quantity.

Let a point light source be placed at the origin and let $\vec{F}$ be the vector field which describes the intensity and direction of the emanating light. At a point (x, y, z), the unit vector describing the direction of the light passing through that point is $\langle x, y, z \rangle / \sqrt{x^2 + y^2 + z^2}$. As the intensity of light follows the inverse square law, the magnitude of $\vec{F}$ at (x, y, z) is $k/(x^2 + y^2 + z^2)$ for some constant k. Taken together,

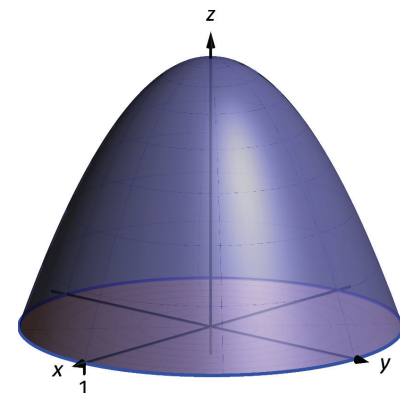$$\vec{F}(x, y, z) = \frac{k}{(x^2 + y^2 + z^2)^{3/2}}\langle x, y, z \rangle.$$

Notes:

Consider the cube, centered at the origin, with sides of length $2a$ for some $a > 0$ (hence corners of the cube lie at (a, a, a), $(-a, -a, -a)$, etc., as shown in Figure 14.46). Find the flux across the six faces of the cube and compare this to $\iint_D \text{div}\, \vec{F}\, dV$.

SOLUTION   Let $\mathcal{S}_1$ be the "top" face of the cube, which can be parameterized by $\vec{r}(u, v) = \langle u, v, a \rangle$ for $-a \leq u \leq a$, $-a \leq v \leq a$. We leave it to the reader to confirm that $\vec{r}_u \times \vec{r}_v = \langle 0, 0, 1 \rangle$, which points outside of the cube.

The flux across this face is:

$$\text{Flux} = \iint_{\mathcal{S}_1} \vec{F} \cdot \vec{n}\, dS$$

$$= \int_{-a}^{a} \int_{-a}^{a} \vec{F}\big(\vec{r}(u, v)\big) \cdot \big(\vec{r}_u \times \vec{r}_v\big)\, du\, dv$$

$$= \int_{-a}^{a} \int_{-a}^{a} \frac{k\, a}{(u^2 + v^2 + a^2)^{3/2}}\, du\, dv.$$

This double integral is *not* trivial to compute, requiring multiple trigonometric substitutions. This example is not meant to stress integration techniques, so we leave it to the reader to confirm the result is

$$= \frac{2k\pi}{3}.$$

Note how the result is independent of a; no matter the size of the cube, the flux through the top surface is always $2k\pi/3$.

An argument of symmetry shows that the flux through each of the six faces is $2k\pi/3$, thus the total flux through the faces of the cube is $6 \times 2k\pi/3 = 4k\pi$.

It takes a bit of algebra, but we can show that $\text{div}\, \vec{F} = 0$. Thus the Divergence Theorem would seem to imply that the total flux through the faces of the cube should be

$$\text{Flux} = \iiint_D \text{div}\, \vec{F}\, dV = \iiint_D 0\, dV = 0,$$

but clearly this does not match the result from above. What went wrong?

Revisit the statement of the Divergence Theorem. One of the conditions is that the components of $\vec{F}$ must be differentiable on the domain enclosed by the surface. In our case, $\vec{F}$ is *not* differentiable at the origin – it is not even defined! As $\vec{F}$ does not satisfy the conditions of the Divergence Theorem, it does not apply, and we cannot expect $\iint_{\mathcal{S}} \vec{F} \cdot \vec{n}\, dA = \iiint_D \text{div}\, \vec{F}\, dV$.

Since $\vec{F}$ is differentiable everywhere except the origin, the Divergence Theorem does apply over any domain that does not include the origin. Let $\mathcal{S}_2$ be any surface that encloses the cube used before, and let $\hat{D}$ be the domain *between*

Figure 14.46: The cube used in Example 14.38.

the cube and S_2; note how $\hat{D}$ does not include the origin and so the Divergence Theorem does apply over this domain. The total outward flux over $\hat{D}$ is thus $\iint_{\hat{D}} \text{div}\,\vec{F}\,dV = 0$, which means the amount of flux coming out of S_2 is the same as the amount of flux coming out of the cube. The conclusion: the flux across *any* surface enclosing the origin will be $4k\pi$.

This has an important consequence in electrodynamics. Let q be a point charge at the origin. The electric field generated by this point charge is

$$\vec{E} = \frac{q\,\langle x, y, z\rangle}{4\pi\varepsilon_0(x^2 + y^2 + z^2)^{3/2}},$$

i.e., it is $\vec{F}$ with $k = q/(4\pi\varepsilon_0)$, where ε_0 is a physical constant (the "permittivity of free space"). Gauss's Law states that the outward flux of $\vec{E}$ across any surface enclosing the origin is q/ε_0.

Our interest in the Divergence Theorem is twofold. First, it's truth alone is interesting: to study the behavior of a vector field across a closed surface, one can examine properties of that field within the surface. Secondly, it offers an alternative way of computing flux. When there are multiple methods of computing a desired quantity, one has power to select the easiest computation as illustrated next.

Example 14.39   Using the Divergence Theorem to compute flux
Let S be the cube bounded by the planes $x = \pm 1$, $y = \pm 1$, $z = \pm 1$, and let $\vec{F} = \langle x^2 y, 2yz, x^2 z^3\rangle$. Compute the outward flux of $\vec{F}$ over S.

SOLUTION   We compute $\text{div}\,\vec{F} = 2xy + 2z + 3x^2 z^2$. By the Divergence Theorem, the outward flux is the triple integral over D enclosed by S:

$$\text{Outward flux:} \quad \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \left(2xy + 2z + 3x^2 z^2\right) dz\,dy\,dx = \frac{8}{3}.$$

The direct flux computation requires six surface integrals, one for each face of the cube. The Divergence Theorem offers a much more simple computation.

Just as the spatial Divergence Theorem of this section is an extension of the planar Divergence Theorem, we will discuss Stokes' Theorem a spatial extension of Green's Theorem, in the next section.

Notes:

Exercises 14.7

Terms and Concepts

1. What are the differences between the Divergence Theorems of Section 14.4 and this section?

2. What property of a vector field does the Divergence Theorem relate to flux?

3. The Divergence Theorem establishes equality between a particular double integral and a particular triple integral. What types of circumstances would lead one to choose to evaluate the triple integral over the double integral?

Problems

In Exercises 4 – 7, a closed surface $\mathcal{S}$ enclosing a domain D and a vector field $\vec{F}$ are given. Verify the Divergence Theorem on $\mathcal{S}$; that is, show $\iint_\mathcal{S} \vec{F} \cdot \vec{n} \, dS = \iiint_D \operatorname{div} \vec{F} \, dV.$
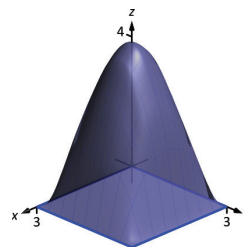
4. $\mathcal{S}$ is the surface bounding the domain D enclosed by the plane $z = 2 - x/2 - 2y/3$ and the coordinate planes in the first octant; $\vec{F} = \langle x^2, y^2, x \rangle$.
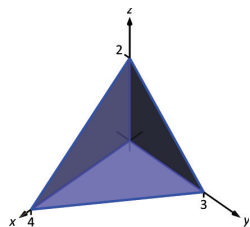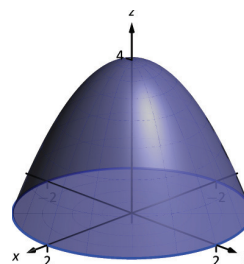

5. $\mathcal{S}$ is the surface bounding the domain D enclosed by the cylinder $x^2 + y^2 = 1$ and the planes $z = -3$ and $z = 3$; $\vec{F} = \langle -x, y, z \rangle$.
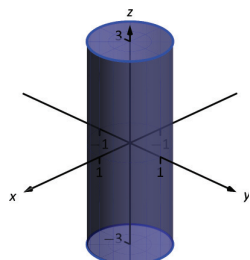
6. $\mathcal{S}$ is the surface bounding the domain D enclosed by $z = xy(3 - x)(3 - y)$ and the plane $z = 0$; $\vec{F} = \langle 3x, 4y, 5z + 1 \rangle$.
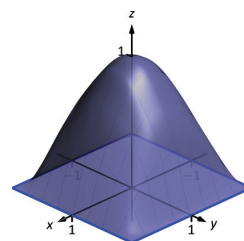

7. $\mathcal{S}$ is the surface composed of $\mathcal{S}_1$, the paraboloid $z = 4 - x^2 - y^2$ for $z \geq 0$, and $\mathcal{S}_2$, the disk of radius 2 centered at the origin; $\vec{F} = \langle x, y, z^2 \rangle$.
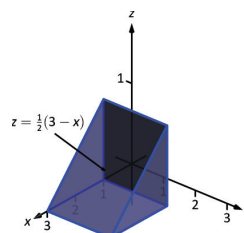

In Exercises 8 – 11, a closed surface $\mathcal{S}$ and a vector field $\vec{F}$ are given. Find the outward flux of $\vec{F}$ over $\mathcal{S}$ either through direct computation or through the Divergence Theorem.
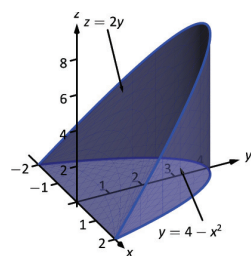
8. $\mathcal{S}$ is the surface formed by the intersections of $z = 0$ and $z = (x^2 - 1)(y^2 - 1)$; $\vec{F} = \langle x^2 + 1, yz, xz^2 \rangle$.

9. $\mathcal{S}$ is the surface formed by the intersections of the planes $z = \frac{1}{2}(3 - x)$, $x = 1$, $y = 0$, $y = 2$ and $z = 0$; $\vec{F} = \langle x, y^2, z \rangle$.

10. $\mathcal{S}$ is the surface formed by the intersections of the planes $z = 2y$, $y = 4 - x^2$ and $z = 0$; $\vec{F} = \langle xz, 0, xz \rangle$.

z = 2y

8

6

4

y

−2
2
−1
1
2
3

1

2

x

$y = 4 - x^2$

11. $\mathcal{S}$ is the surface formed by the intersections of the cylinder $z = 1 - x^2$ and the planes $y = -2$, $y = 2$ and $z = 0$; $\vec{F} = \langle 0, y^3, 0 \rangle$.

z

$z = 1 - x^2$

1

−2

−1

x

1

2

y