

13.5 Surface Area

In Section 7.4 we used definite integrals to compute the arc length of plane curves of the form $y = f(x)$. We later extended these ideas to compute the arc length of plane curves defined by parametric or polar equations.

The natural extension of the concept of “arc length over an interval” to surfaces is “surface area over a region.”

Consider the surface $z = f(x, y)$ over a region R in the x - y plane, shown in Figure 13.32(a). Because of the domed shape of the surface, the surface area will be greater than that of the area of the region R . We can find this area using the same basic technique we have used over and over: we’ll make an approximation, then using limits, we’ll refine the approximation to the exact value.

As done to find the volume under a surface or the mass of a lamina, we subdivide R into n subregions. Here we subdivide R into rectangles, as shown in the figure. One such subregion is outlined in the figure, where the rectangle has dimensions Δx_i and Δy_i , along with its corresponding region on the surface.

In part (b) of the figure, we zoom in on this portion of the surface. When Δx_i and Δy_i are small, the function is approximated well by the tangent plane at any point (x_i, y_i) in this subregion, which is graphed in part (b). In fact, the tangent plane approximates the function so well that in this figure, it is virtually indistinguishable from the surface itself! Therefore we can approximate the surface area S_i of this region of the surface with the area T_i of the corresponding portion of the tangent plane.

This portion of the tangent plane is a parallelogram, defined by sides \vec{u} and \vec{v} , as shown. One of the applications of the cross product from Section 10.4 is that the area of this parallelogram is $\|\vec{u} \times \vec{v}\|$. Once we can determine \vec{u} and \vec{v} , we can determine the area.

\vec{u} is tangent to the surface in the direction of x , therefore, from Section 12.7, \vec{u} is parallel to $\langle 1, 0, f_x(x_i, y_i) \rangle$. The x -displacement of \vec{u} is Δx_i , so we know that $\vec{u} = \Delta x_i \langle 1, 0, f_x(x_i, y_i) \rangle$. Similar logic shows that $\vec{v} = \Delta y_i \langle 0, 1, f_y(x_i, y_i) \rangle$. Thus:

$$\begin{aligned} \text{surface area } S_i &\approx \text{area of } T_i \\ &= \|\vec{u} \times \vec{v}\| \\ &= \left\| \Delta x_i \langle 1, 0, f_x(x_i, y_i) \rangle \times \Delta y_i \langle 0, 1, f_y(x_i, y_i) \rangle \right\| \\ &= \sqrt{1 + f_x(x_i, y_i)^2 + f_y(x_i, y_i)^2} \Delta x_i \Delta y_i. \end{aligned}$$

Note that $\Delta x_i \Delta y_i = \Delta A_i$, the area of the i^{th} subregion.

Summing up all n of the approximations to the surface area gives

$$\text{surface area over } R \approx \sum_{i=1}^n \sqrt{1 + f_x(x_i, y_i)^2 + f_y(x_i, y_i)^2} \Delta A_i.$$

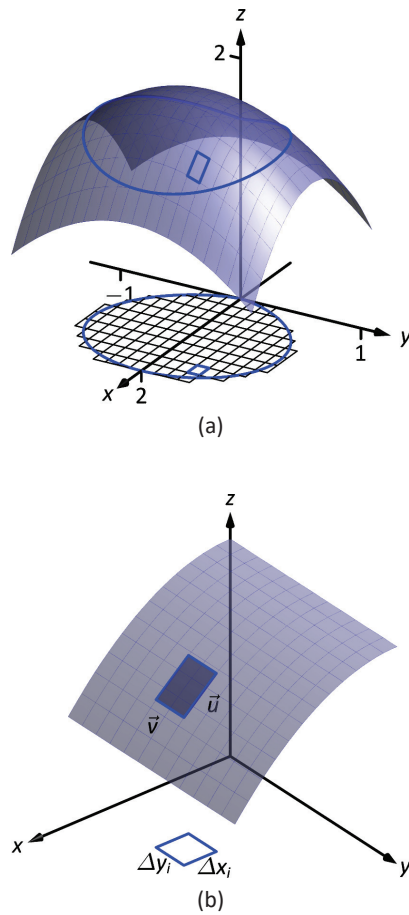


Figure 13.32: Developing a method of computing surface area.

Notes:

Once again take a limit as all of the Δx_i and Δy_i shrink to 0; this leads to a double integral.

Definition 109 Surface Area

Let $z = f(x, y)$ where f_x and f_y are continuous over a closed, bounded region R . The surface area S over R is

$$\begin{aligned} S &= \iint_R dS \\ &= \iint_R \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2} dA. \end{aligned}$$

Note: as done before, we think of “ $\iint_R dS$ ” as meaning “sum up lots of little surface areas over R .”

The concept of surface area is *defined* here, for while we already have a notion of the area of a region in the *plane*, we did not yet have a solid grasp of what “the area of a surface in *space*” means.

We test this definition by using it to compute surface areas of known surfaces. We start with a triangle.

Example 13.31 Finding the surface area of a plane over a triangle

Let $f(x, y) = 4 - x - 2y$, and let R be the region in the plane bounded by $x = 0$, $y = 0$ and $y = 2 - x/2$, as shown in Figure 13.33. Find the surface area of f over R .

SOLUTION We follow Definition 109. We start by noting that $f_x(x, y) = -1$ and $f_y(x, y) = -2$. To define R , we use bounds $0 \leq y \leq 2 - x/2$ and $0 \leq x \leq 4$. Therefore

$$\begin{aligned} S &= \iint_R dS \\ &= \int_0^4 \int_0^{2-x/2} \sqrt{1 + (-1)^2 + (-2)^2} dy dx \\ &= \int_0^4 \sqrt{6} \left(2 - \frac{x}{2}\right) dx \\ &= 4\sqrt{6}. \end{aligned}$$

Because the surface is a triangle, we can figure out the area using geometry. Considering the base of the triangle to be the side in the x - y plane, we find the length of the base to be $\sqrt{20}$. We can find the height using our knowledge of vectors: let \vec{u} be the side in the x - z plane and let \vec{v} be the side in the x - y plane. The height is then $\|\vec{u} - \text{proj}_{\vec{v}} \vec{u}\| = 4\sqrt{6/5}$. Geometry states that the area is thus

$$\frac{1}{2} \cdot 4\sqrt{6/5} \cdot \sqrt{20} = 4\sqrt{6}.$$

We affirm the validity of our formula.

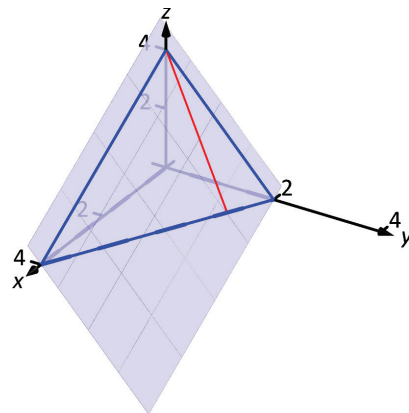


Figure 13.33: Finding the area of a triangle in space in Example 13.31.

Notes:

It is “common knowledge” that the surface area of a sphere of radius r is $4\pi r^2$. We confirm this in the following example, which involves using our formula with polar coordinates.

Example 13.32 The surface area of a sphere.

Find the surface area of the sphere with radius a centered at the origin, whose top hemisphere has equation $f(x, y) = \sqrt{a^2 - x^2 - y^2}$.

SOLUTION We start by computing partial derivatives and find

$$f_x(x, y) = \frac{-x}{\sqrt{a^2 - x^2 - y^2}} \quad \text{and} \quad f_y(x, y) = \frac{-y}{\sqrt{a^2 - x^2 - y^2}}.$$

As our function f only defines the top upper hemisphere of the sphere, we double our surface area result to get the total area:

$$\begin{aligned} S &= 2 \iint_R \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2} \, dA \\ &= 2 \iint_R \sqrt{1 + \frac{x^2 + y^2}{a^2 - x^2 - y^2}} \, dA. \end{aligned}$$

The region R that we are integrating over is the circle, centered at the origin, with radius a : $x^2 + y^2 = a^2$. Because of this region, we are likely to have greater success with our integration by converting to polar coordinates. Using the substitutions $x = r \cos \theta$, $y = r \sin \theta$, $dA = r \, dr \, d\theta$ and bounds $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq a$, we have:

$$\begin{aligned} S &= 2 \int_0^{2\pi} \int_0^a \sqrt{1 + \frac{r^2 \cos^2 \theta + r^2 \sin^2 \theta}{a^2 - r^2 \cos^2 \theta - r^2 \sin^2 \theta}} \, r \, dr \, d\theta \\ &= 2 \int_0^{2\pi} \int_0^a r \sqrt{1 + \frac{r^2}{a^2 - r^2}} \, dr \, d\theta \\ &= 2 \int_0^{2\pi} \int_0^a r \sqrt{\frac{a^2}{a^2 - r^2}} \, dr \, d\theta. \end{aligned} \tag{13.1}$$

Apply substitution $u = a^2 - r^2$ and integrate the inner integral, giving

$$\begin{aligned} &= 2 \int_0^{2\pi} a^2 \, d\theta \\ &= 4\pi a^2. \end{aligned}$$

Our work confirms our previous formula.

Note: The inner integral in Equation (13.1) is an improper integral, as the integrand of $\int_0^a r \sqrt{\frac{a^2}{a^2 - r^2}} \, dr$ is not defined at $r = a$. To properly evaluate this integral, one must use the techniques of Section 6.6.

The reason this need arises is that the function $f(x, y) = \sqrt{a^2 - x^2 - y^2}$ fails the requirements of Definition 109, as f_x and f_y are not continuous on the boundary of the circle $x^2 + y^2 = a^2$.

The computation of the surface area is still valid. The definition makes stronger requirements than necessary in part to avoid the use of improper integration, as when f_x and/or f_y are not continuous, the resulting improper integral may not converge. Since the improper integral does converge in this example, the surface area is accurately computed.

Notes:

Example 13.33 Finding the surface area of a cone

The general formula for a right cone with height h and base radius a is

$$f(x, y) = h - \frac{h}{a} \sqrt{x^2 + y^2},$$

shown in Figure 13.34. Find the surface area of this cone.

SOLUTION We begin by computing partial derivatives.

$$f_x(x, y) = -\frac{xh}{a\sqrt{x^2 + y^2}} \quad \text{and} \quad -\frac{yh}{a\sqrt{x^2 + y^2}}.$$

Since we are integrating over the circle $x^2 + y^2 = a^2$, we again use polar coordinates. Using the standard substitutions, our integrand becomes

$$\sqrt{1 + \left(\frac{hr \cos \theta}{a\sqrt{r^2}}\right)^2 + \left(\frac{hr \sin \theta}{a\sqrt{r^2}}\right)^2}.$$

This may look intimidating at first, but there are lots of simple simplifications to be done. It amazingly reduces to just

$$\sqrt{1 + \frac{h^2}{a^2}} = \frac{1}{a} \sqrt{a^2 + h^2}.$$

Our polar bounds are $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq a$. Thus

$$\begin{aligned} S &= \int_0^{2\pi} \int_0^a r \frac{1}{a} \sqrt{a^2 + h^2} \, dr \, d\theta \\ &= \int_0^{2\pi} \left(\frac{1}{2} r^2 \frac{1}{a} \sqrt{a^2 + h^2} \right) \Big|_0^a d\theta \\ &= \int_0^{2\pi} \frac{1}{2} a \sqrt{a^2 + h^2} \, d\theta \\ &= \pi a \sqrt{a^2 + h^2}. \end{aligned}$$

This matches the formula found in the back of this text.

Example 13.34 Finding surface area over a region

Find the area of the surface $f(x, y) = x^2 - 3y + 3$ over the region R bounded by $-x \leq y \leq x$, $0 \leq x \leq 4$, as pictured in Figure 13.35.

SOLUTION It is straightforward to compute $f_x(x, y) = 2x$ and $f_y(x, y) = -3$. Thus the surface area is described by the double integral

$$\iint_R \sqrt{1 + (2x)^2 + (-3)^2} \, dA = \iint_R \sqrt{10 + 4x^2} \, dA.$$

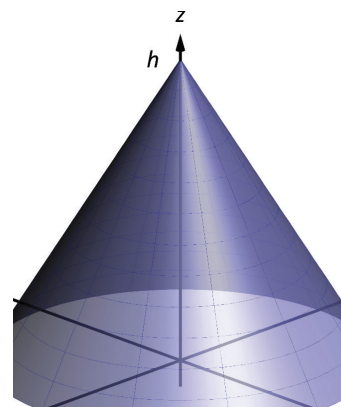


Figure 13.34: Finding the surface area of a cone in Example 13.33.

Note: Note that once again f_x and f_y are not continuous on the domain of f , as both are undefined at $(0, 0)$. (A similar problem occurred in the previous example.) Once again the resulting improper integral converges and the computation of the surface area is valid.

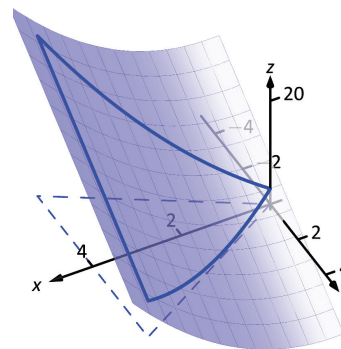


Figure 13.35: Graphing the surface in Example 13.34.

Notes:

As with integrals describing arc length, double integrals describing surface area are in general hard to evaluate directly because of the square-root. This particular integral can be easily evaluated, though, with judicious choice of our order of integration.

Integrating with order $dx\,dy$ requires us to evaluate $\int \sqrt{10 + 4x^2}\,dx$. This can be done, though it is messier and involves a trigonometric substitution. Integrating with order $dy\,dx$ has as its first integral $\int \sqrt{10 + 4x^2}\,dy$, which is easy to evaluate: it is simply $y\sqrt{10 + 4x^2} + C$. So we proceed with the order $dy\,dx$; the bounds are already given in the statement of the problem.

$$\begin{aligned}\iint_R \sqrt{10 + 4x^2}\,dA &= \int_0^4 \int_{-x}^x \sqrt{10 + 4x^2}\,dy\,dx \\ &= \int_0^4 (y\sqrt{10 + 4x^2}) \Big|_{-x}^x dx \\ &= \int_0^4 (2x\sqrt{10 + 4x^2})\,dx.\end{aligned}$$

Apply substitution with $u = 10 + 4x^2$:

$$\begin{aligned}&= \left(\frac{1}{6} (10 + 4x^2)^{3/2} \right) \Big|_0^4 \\ &= \frac{1}{3} (37\sqrt{74} - 5\sqrt{10}) \approx 100.825 \text{ square units.}\end{aligned}$$

So while the region R over which we integrate has an area of 16 square units, the surface has a much greater area as its z -values change dramatically over R .

In practice, technology helps greatly in the evaluation of such integrals. High powered computer algebra systems can compute integrals that are difficult, or at least time consuming, by hand, and can at the least produce very accurate approximations with numerical methods. In general, just knowing *how* to set up the proper integrals brings one very close to being able to compute the needed value. Most of the work is actually done in just describing the region R in terms of polar or rectangular coordinates. Once this is done, technology can usually provide a good answer.

We have learned how to integrate integrals; that is, we have learned to evaluate double integrals. In the next section, we learn how to integrate double integrals – that is, we learn to evaluate *triple integrals*, along with learning some uses for this operation.

Notes:

Exercises 13.5

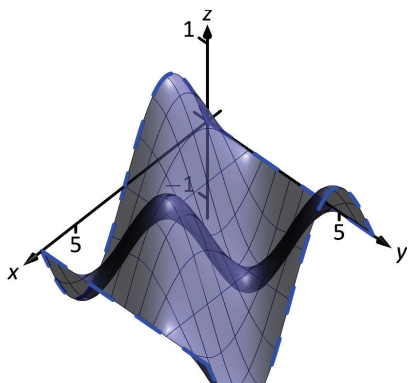
Terms and Concepts

1. "Surface area" is analogous to what previously studied concept?
2. To approximate the area of a small portion of a surface, we computed the area of its _____ plane.
3. We interpret $\iint_R dS$ as "sum up lots of little _____."
4. Why is it important to know how to set up a double integral to compute surface area, even if the resulting integral is hard to evaluate?
5. Why do $z = f(x, y)$ and $z = g(x, y) = f(x, y) + h$, for some real number h , have the same surface area over a region R ?
6. Let $z = f(x, y)$ and $z = g(x, y) = 2f(x, y)$. Why is the surface area of g over a region R not twice the surface area of f over R ?

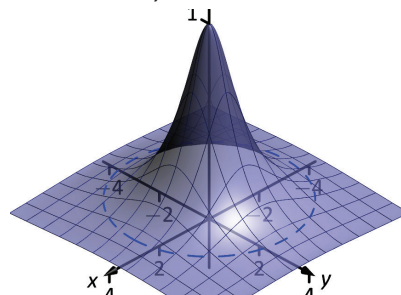
Problems

In Exercises 7 – 10, set up the iterated integral that computes the surface area of the given surface over the region R .

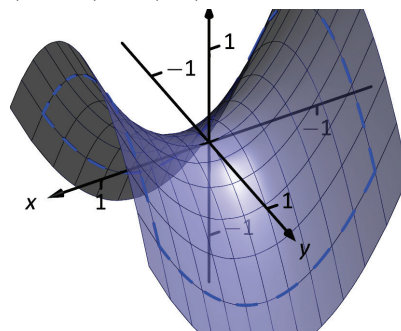
7. $f(x, y) = \sin x \cos y$; R is the rectangle with bounds $0 \leq x \leq 2\pi$, $0 \leq y \leq 2\pi$.



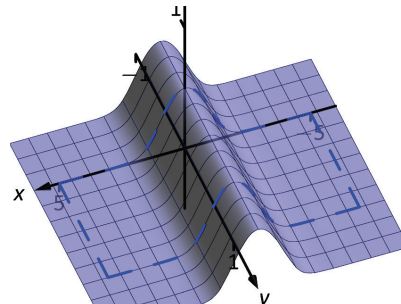
8. $f(x, y) = \frac{1}{x^2 + y^2 + 1}$; R is the circle $x^2 + y^2 = 9$.



9. $f(x, y) = x^2 - y^2$; R is the rectangle with opposite corners $(-1, -1)$ and $(1, 1)$.



10. $f(x, y) = \frac{1}{e^{x^2} + 1}$; R is the rectangle bounded by $-5 \leq x \leq 5$ and $0 \leq y \leq 1$.



In Exercises 11 – 19, find the area of the given surface over the region R .

11. $f(x, y) = 3x - 7y + 2$; R is the rectangle with opposite corners $(-1, 0)$ and $(1, 3)$.
12. $f(x, y) = 2x + 2y + 2$; R is the triangle with corners $(0, 0)$, $(1, 0)$ and $(0, 1)$.
13. $f(x, y) = x^2 + y^2 + 10$; R is the circle $x^2 + y^2 = 16$.
14. $f(x, y) = -2x + 4y^2 + 7$ over R , the triangle bounded by $y = -x$, $y = x$, $0 \leq y \leq 1$.
15. $f(x, y) = x^2 + y$ over R , the triangle bounded by $y = 2x$, $y = 0$ and $x = 2$.

16. $f(x, y) = \frac{2}{3}x^{3/2} + 2y^{3/2}$ over R , the rectangle with opposite corners $(0, 0)$ and $(1, 1)$.
17. $f(x, y) = 10 - 2\sqrt{x^2 + y^2}$ over R , the circle $x^2 + y^2 = 25$. (This is the cone with height 10 and base radius 5; be sure to compare your result with the known formula.)
18. Find the surface area of the sphere with radius 5 by doubling the surface area of $f(x, y) = \sqrt{25 - x^2 - y^2}$ over R , the circle $x^2 + y^2 = 25$. (Be sure to compare your result with the known formula.)
19. Find the surface area of the ellipse formed by restricting the plane $f(x, y) = cx + dy + h$ to the region R , the circle $x^2 + y^2 = 1$, where c , d and h are some constants. Your answer should be given in terms of c and d ; why does the value of h not matter?