11.4 Unit Tangent and Normal Vectors

Unit Tangent Vector

Given a smooth vector–valued function $\vec{r}(t)$, we defined in Definition 75 that any vector parallel to $\vec{r}\,'(t_0)$ is *tangent* to the graph of $\vec{r}(t)$ at $t = t_0$. It is often useful to consider just the *direction* of $\vec{r}\,'(t)$ and not its magnitude. Therefore we are interested in the unit vector in the direction of $\vec{r}\,'(t)$. This leads to a definition.

Definition 78 Unit Tangent Vector

Let $\vec{r}(t)$ be a smooth function on an open interval I. The unit tangent vector $\vec{T}(t)$ is

$$\vec{T}(t) = \frac{1}{\|\,\vec{r}\,'(t)\,\|}\vec{r}\,'(t).$$

Example 11.23 Computing the unit tangent vector

Let $\vec{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$. Find $\vec{T}(t)$ and compute $\vec{T}(0)$ and $\vec{T}(1)$.

Solution We apply Definition 78 to find $\vec{T}(t)$.

$$\begin{aligned}\vec{T}(t) &= \frac{1}{\|\,\vec{r}\,'(t)\,\|}\vec{r}\,'(t)\\ &= \frac{1}{\sqrt{\big(-3\sin t\big)^2 + \big(3\cos t\big)^2 + 4^2}}\langle -3\sin t, 3\cos t, 4\rangle\\ &= \left\langle -\frac{3}{5}\sin t, \frac{3}{5}\cos t, \frac{4}{5}\right\rangle.\end{aligned}$$

We can now easily compute $\vec{T}(0)$ and $\vec{T}(1)$:

$$\vec{T}(0) = \left\langle 0, \frac{3}{5}, \frac{4}{5}\right\rangle; \quad \vec{T}(1) = \left\langle -\frac{3}{5}\sin 1, \frac{3}{5}\cos 1, \frac{4}{5}\right\rangle \approx \langle -0.505, 0.324, 0.8\rangle.$$

These are plotted in Figure 11.20 with their initial points at $\vec{r}(0)$ and $\vec{r}(1)$, respectively. (They look rather "short" since they are only length 1.)

The unit tangent vector $\vec{T}(t)$ always has a magnitude of 1, though it is sometimes easy to doubt that is true. We can help solidify this thought in our minds by computing $\|\,\vec{T}(1)\,\|$:

$$\|\,\vec{T}(1)\,\| \approx \sqrt{(-0.505)^2 + 0.324^2 + 0.8^2} = 1.000001.$$

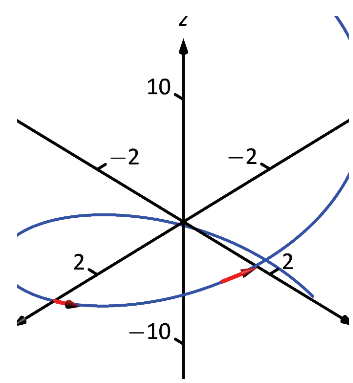

Figure 11.20: Plotting unit tangent vectors in Example 11.23.

Notes:

We have rounded in our computation of $\vec{T}(1)$, so we don't get 1 exactly. We leave it to the reader to use the exact representation of $\vec{T}(1)$ to verify it has length 1.

In many ways, the previous example was "too nice." It turned out that $\vec{r}\,'(t)$ was always of length 5. In the next example the length of $\vec{r}\,'(t)$ is variable, leaving us with a formula that is not as clean.

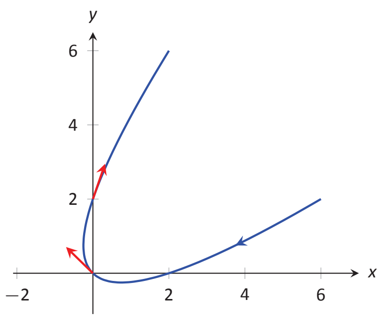

Figure 11.21: Plotting unit tangent vectors in Example 11.24.

Example 11.24 Computing the unit tangent vector

Let $\vec{r}(t) = \langle t^2 - t, t^2 + t\rangle$. Find $\vec{T}(t)$ and compute $\vec{T}(0)$ and $\vec{T}(1)$.

Solution We find $\vec{r}\,'(t) = \langle 2t - 1, 2t + 1\rangle$, and

$$||\,\vec{r}\,'(t)\,|| = \sqrt{(2t-1)^2 + (2t+1)^2} = \sqrt{8t^2 + 2}.$$

Therefore

$$\vec{T}(t) = \frac{1}{\sqrt{8t^2+2}}\langle 2t-1, 2t+1\rangle = \left\langle \frac{2t-1}{\sqrt{8t^2+2}}, \frac{2t+1}{\sqrt{8t^2+2}} \right\rangle.$$

When $t = 0$, we have $\vec{T}(0) = \langle -1/\sqrt{2}, 1/\sqrt{2}\rangle$; when $t = 1$, we have $\vec{T}(1) = \langle 1/\sqrt{10}, 3/\sqrt{10}\rangle$. We leave it to the reader to verify each of these is a unit vector. They are plotted in Figure 11.21

Unit Normal Vector

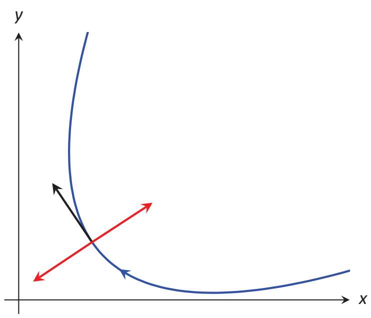

Figure 11.22: Given a direction in the plane, there are always two directions orthogonal to it.

Just as knowing the direction tangent to a path is important, knowing a direction orthogonal to a path is important. When dealing with real-valued functions, we defined the normal line at a point to the be the line through the point that was perpendicular to the tangent line at that point. We can do a similar thing with vector–valued functions. Given $\vec{r}(t)$ in $\mathbb{R}^2$, we have 2 directions perpendicular to the tangent vector, as shown in Figure 11.22. It is good to wonder "Is one of these two directions preferable over the other?"

Given $\vec{r}(t)$ in $\mathbb{R}^3$, there are infinite vectors orthogonal to the tangent vector at a given point. Again, we might wonder "Is one of these infinite choices preferable over the others? Is one of these the "correct" choice?"

The answer in both $\mathbb{R}^2$ and $\mathbb{R}^3$ is "Yes, there is one vector that is not only preferable, it is the "correct" one to choose." Recall Theorem 95, which states that if $\vec{r}(t)$ has constant length, then $\vec{r}(t)$ is orthogonal to $\vec{r}\,'(t)$ for all t. We know $\vec{T}(t)$, the unit tangent vector, has constant length. Therefore $\vec{T}(t)$ is orthogonal to $\vec{T}\,'(t)$.

Note: $\vec{T}(t)$ is a unit vector, by definition. This *does not* imply that $\vec{T}\,'(t)$ is also a unit vector.

We'll see that $\vec{T}\,'(t)$ is more than just a convenient choice of vector that is orthogonal to $\vec{r}\,'(t)$; rather, it is the "correct" choice. Since all we care about is the direction, we define this newly found vector to be a unit vector.

Notes:

Definition 79 Unit Normal Vector

Let $\vec{r}(t)$ be a vector–valued function where the unit tangent vector, $\vec{T}(t)$, is smooth on an open interval I. The **unit normal vector** $\vec{N}(t)$ is

$$\vec{N}(t) = \frac{1}{||\,\vec{T}\,'(t)\,||}\vec{T}\,'(t).$$

Example 11.25 Computing the unit normal vector

Let $\vec{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$ as in Example 11.23. Sketch both $\vec{T}(\pi/2)$ and $\vec{N}(\pi/2)$ with initial points at $\vec{r}(\pi/2)$.

Solution In Example 11.23, we found $\vec{T}(t) = \Big\langle (-3/5)\sin t, (3/5)\cos t, 4/5\Big\rangle$. Therefore

$$\vec{T}\,'(t) = \left\langle -\frac{3}{5}\cos t, -\frac{3}{5}\sin t, 0\right\rangle \quad \text{and} \quad ||\,\vec{T}\,'(t)\,|| = \frac{3}{5}.$$

Thus

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{3/5} = \langle -\cos t, -\sin t, 0\rangle\,.$$

We compute $\vec{T}(\pi/2) = \langle -3/5, 0, 4/5\rangle$ and $\vec{N}(\pi/2) = \langle 0, -1, 0\rangle$. These are sketched in Figure 11.23.

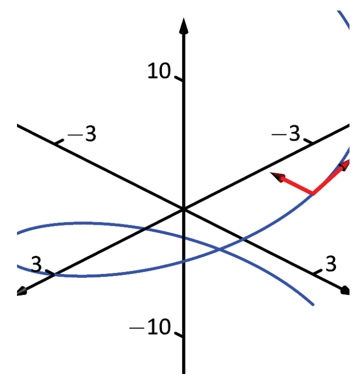

Figure 11.23: Plotting unit tangent and normal vectors in Example 11.23.

The previous example was once again "too nice." In general, the expression for $\vec{T}(t)$ contains fractions of square–roots, hence the expression of $\vec{T}\,'(t)$ is very messy. We demonstrate this in the next example.

Example 11.26 Computing the unit normal vector

Let $\vec{r}(t) = \langle t^2 - t, t^2 + t\rangle$ as in Example 11.24. Find $\vec{N}(t)$ and sketch $\vec{r}(t)$ with the unit tangent and normal vectors at $t = -1, 0$ and 1.

Solution In Example 11.24, we found

$$\vec{T}(t) = \left\langle \frac{2t-1}{\sqrt{8t^2+2}}, \frac{2t+1}{\sqrt{8t^2+2}}\right\rangle.$$

Finding $\vec{T}\,'(t)$ requires two applications of the Quotient Rule:

Notes:

$$T'(t) = \left\langle \frac{\sqrt{8t^2+2}(2) - (2t-1)\left(\frac{1}{2}(8t^2+2)^{-1/2}(16t)\right)}{8t^2+2}, \right.$$
$$\left. \frac{\sqrt{8t^2+2}(2) - (2t+1)\left(\frac{1}{2}(8t^2+2)^{-1/2}(16t)\right)}{8t^2+2} \right\rangle$$
$$= \left\langle \frac{4(2t+1)}{(8t^2+2)^{3/2}}, \frac{4(1-2t)}{(8t^2+2)^{3/2}} \right\rangle$$

This is not a unit vector; to find $\vec{N}(t)$, we need to divide $\vec{T}\,'(t)$ by it's magnitude.

$$|| \vec{T}\,'(t) || = \sqrt{\frac{16(2t+1)^2}{(8t^2+2)^3} + \frac{16(1-2t)^2}{(8t^2+2)^3}}$$
$$= \sqrt{\frac{16(8t^2+2)}{(8t^2+2)^3}}$$
$$= \frac{4}{8t^2+2}.$$

Finally,

$$\vec{N}(t) = \frac{1}{4/(8t^2+2)} \left\langle \frac{4(2t+1)}{(8t^2+2)^{3/2}}, \frac{4(1-2t)}{(8t^2+2)^{3/2}} \right\rangle$$
$$= \left\langle \frac{2t+1}{\sqrt{8t^2+2}}, -\frac{2t-1}{\sqrt{8t^2+2}} \right\rangle.$$

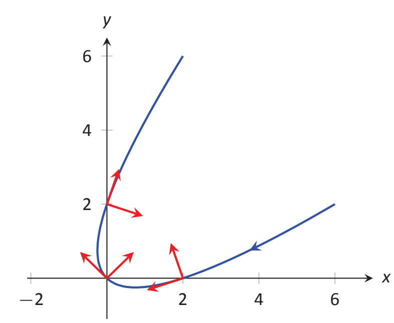

Figure 11.24: Plotting unit tangent and normal vectors in Example 11.26.

Using this formula for $\vec{N}(t)$, we compute the unit tangent and normal vectors for $t = -1, 0$ and 1 and sketch them in Figure 11.24.

The final result for $\vec{N}(t)$ in Example 11.26 is suspiciously similar to $\vec{T}(t)$. There is a clear reason for this. If $\vec{u} = \langle u_1, u_2 \rangle$ is a unit vector in $\mathbb{R}^2$, then the *only* unit vectors orthogonal to $\vec{u}$ are $\langle -u_2, u_1 \rangle$ and $\langle u_2, -u_1 \rangle$. Given $\vec{T}(t)$, we can quickly determine $\vec{N}(t)$ if we know which term to multiply by (-1).

Consider again Figure 11.24, where we have plotted some unit tangent and normal vectors. Note how $\vec{N}(t)$ always points "inside" the curve, or to the concave side of the curve. This is not a coincidence; this is true in general. Knowing the direction that $\vec{r}(t)$ "turns" allows us to quickly find $\vec{N}(t)$.

Notes:

Theorem 99 Unit Normal Vectors in $\mathbb{R}^2$

Let $\vec{r}(t)$ be a vector–valued function in $\mathbb{R}^2$ where $\vec{T}\,'(t)$ is smooth on an open interval I. Let t_0 be in I and $\vec{T}(t_0) = \langle t_1, t_2\rangle$ Then $\vec{N}(t_0)$ is either

$$\vec{N}(t_0) = \langle -t_2, t_1\rangle \quad \text{or} \quad \vec{N}(t_0) = \langle t_2, -t_1\rangle\,,$$

whichever is the vector that points to the concave side of the graph of $\vec{r}$.

Application to Acceleration

Let $\vec{r}(t)$ be a position function. It is a fact (stated later in Theorem 100) that acceleration, $\vec{a}(t)$, lies in the plane defined by $\vec{T}$ and $\vec{N}$. That is, there are scalars a_{T} and a_{N} such that

$$\vec{a}(t) = a_{\mathrm{T}}\vec{T}(t) + a_{\mathrm{N}}\vec{N}(t).$$

The scalar a_{T} measures "how much" acceleration is in the direction of travel, that is, it measures the component of acceleration that affects the speed. The scalar a_{N} measures "how much" acceleration is perpendicular to the direction of travel, that is, it measures the component of acceleration that affects the direction of travel.

We can find a_{T} using the orthogonal projection of $\vec{a}(t)$ onto $\vec{T}(t)$ (review Definition 63 in Section 10.3 if needed). Recalling that since $\vec{T}(t)$ is a unit vector, $\vec{T}(t)\cdot\vec{T}(t) = 1$, so we have

$$\text{proj}_{\,\vec{T}(t)}\,\vec{a}(t) = \frac{\vec{a}(t)\cdot\vec{T}(t)}{\vec{T}(t)\cdot\vec{T}(t)}\vec{T}(t) = \underbrace{\big(\vec{a}(t)\cdot\vec{T}(t)\big)}_{a_{\mathrm{T}}}\vec{T}(t).$$

Thus the amount of $\vec{a}(t)$ in the direction of $\vec{T}(t)$ is $a_{\mathrm{T}} = \vec{a}(t)\cdot\vec{T}(t)$. The same logic gives $a_{\mathrm{N}} = \vec{a}(t)\cdot\vec{N}(t)$.

While this is a fine way of computing a_{T}, there are simpler ways of finding a_{N} (as finding $\vec{N}$ itself can be complicated). The following theorem gives alternate formulas for a_{T} and a_{N}.

Note: Keep in mind that both a_{T} and a_{N} are functions of t; that is, the scalar changes depending on t. It is convention to drop the "(t)" notation from $a_{\mathrm{T}}(t)$ and simply write a_{T}.

Notes:

Theorem 100 Acceleration in the Plane Defined by $\vec{T}$ and $\vec{N}$

Let $\vec{r}(t)$ be a position function with acceleration $\vec{a}(t)$ and unit tangent and normal vectors $\vec{T}(t)$ and $\vec{N}(t)$. Then $\vec{a}(t)$ lies in the plane defined by $\vec{T}(t)$ and $\vec{N}(t)$; that is, there exists scalars a_T and a_N such that

$$\vec{a}(t) = a_\text{T}\vec{T}(t) + a_\text{N}\vec{N}(t).$$

Moreover,

$$a_\text{T} = \vec{a}(t)\cdot\vec{T}(t) = \frac{d}{dt}\Big(\,||\,\vec{v}(t)\,||\,\Big)$$
$$a_\text{N} = \vec{a}(t)\cdot\vec{N}(t) = \sqrt{||\,\vec{a}(t)\,||^2 - a_\text{T}^2} = \frac{||\,\vec{a}(t)\times\vec{v}(t)\,||}{||\,\vec{v}(t)\,||} = ||\,\vec{v}(t)\,||\,||\,\vec{T}\,'(t)\,||$$

Note the second formula for a_T: $\frac{d}{dt}\Big(\,||\,\vec{v}(t)\,||\,\Big)$. This measures the rate of change of speed, which again is the amount of acceleration in the direction of travel.

Example 11.27 Computing a_T and a_N

Let $\vec{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$ as in Examples 11.23 and 11.25. Find a_T and a_N.

Solution The previous examples give $\vec{a}(t) = \langle -3\cos t, -3\sin t, 0\rangle$ and

$$\vec{T}(t) = \left\langle -\frac{3}{5}\sin t, \frac{3}{5}\cos t, \frac{4}{5}\right\rangle \quad\text{and}\quad \vec{N}(t) = \langle -\cos t, -\sin t, 0\rangle\,.$$

We can find a_T and a_N directly with dot products:

$$a_\text{T} = \vec{a}(t)\cdot\vec{T}(t) = \frac{9}{5}\cos t\sin t - \frac{9}{5}\cos t\sin t + 0 = 0.$$
$$a_\text{N} = \vec{a}(t)\cdot\vec{N}(t) = 3\cos^2 t + 3\sin^2 t + 0 = 3.$$

Thus $\vec{a}(t) = 0\vec{T}(t) + 3\vec{N}(t) = 3\vec{N}(t)$, which is clearly the case.

What is the practical interpretation of these numbers? $a_\text{T} = 0$ means the object is moving at a constant speed, and hence all acceleration comes in the form of direction change.

Example 11.28 Computing a_T and a_N

Let $\vec{r}(t) = \big\langle t^2 - t, t^2 + t\big\rangle$ as in Examples 11.24 and 11.26. Find a_T and a_N.

Notes:

Solution The previous examples give $\vec{a}(t) = \langle 2, 2\rangle$ and

$$\vec{T}(t) = \left\langle \frac{2t-1}{\sqrt{8t^2+2}}, \frac{2t+1}{\sqrt{8t^2+2}} \right\rangle \quad \text{and} \quad \vec{N}(t) = \left\langle \frac{2t+1}{\sqrt{8t^2+2}}, -\frac{2t-1}{\sqrt{8t^2+2}} \right\rangle.$$

While we can compute a_N using $\vec{N}(t)$, we instead demonstrate using another formula from Theorem 100.

$$a_\mathrm{T} = \vec{a}(t)\cdot\vec{T}(t) = \frac{4t-2}{\sqrt{8t^2+2}} + \frac{4t+2}{\sqrt{8t^2+2}} = \frac{8t}{\sqrt{8t^2+2}}.$$

$$a_\mathrm{N} = \sqrt{||\,\vec{a}(t)\,||^2 - a_\mathrm{T}^2} = \sqrt{8 - \left(\frac{8t}{\sqrt{8t^2+2}}\right)^2} = \frac{4}{\sqrt{8t^2+2}}.$$

When $t = 2$, $a_\mathrm{T} = \dfrac{16}{\sqrt{34}} \approx 2.74$ and $a_\mathrm{N} = \dfrac{4}{\sqrt{34}} \approx 0.69$. We interpret this to mean that at $t = 2$, the particle is acclerating mostly by increasing speed, not by changing direction. As the path near $t = 2$ is relatively straight, this should make intuitive sense. Figure 11.25 gives a graph of the path for reference.

Contrast this with $t = 0$, where $a_\mathrm{T} = 0$ and $a_\mathrm{N} = 4/\sqrt{2} \approx 2.82$. Here the particle's speed is not changing and all acceleration is in the form of direction change.

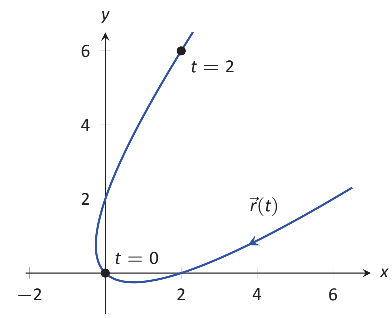

Figure 11.25: Graphing $\vec{r}(t)$ in Example 11.28.

Example 11.29 **Analyzing projectile motion**

A ball is thrown from a height of 240ft with an initial speed of 64ft/s and an angle of elevation of 30°. Find the position function $\vec{r}(t)$ of the ball and analyze a_T and a_N.

Solution Using Key Idea 55 of Section 11.3 we form the position function of the ball:

$$\vec{r}(t) = \left\langle \big(64\cos 30^\circ\big)t, -16t^2 + \big(64\sin 30^\circ\big)t + 240 \right\rangle,$$

which we plot in Figure 11.26.

From this we find $\vec{v}(t) = \langle 64\cos 30^\circ, -32t + 64\sin 30^\circ\rangle$ and $\vec{a}(t) = \langle 0, -32\rangle$. Computing $\vec{T}(t)$ is not difficult, and with some simplification we find

$$\vec{T}(t) = \left\langle \frac{\sqrt{3}}{\sqrt{t^2-2t+4}}, \frac{1-t}{\sqrt{t^2-2t+4}} \right\rangle.$$

With $\vec{a}(t)$ as simple as it is, finding a_T is also simple:

$$a_\mathrm{T} = \vec{a}(t)\cdot\vec{T}(t) = \frac{32t-32}{\sqrt{t^2-2t+4}}.$$

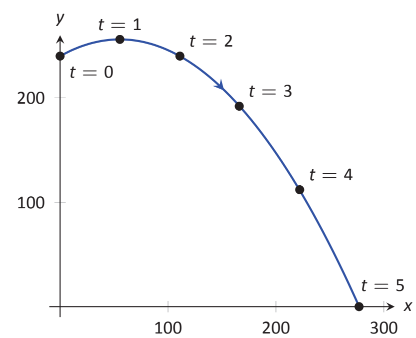

Figure 11.26: Plotting the position of a thrown ball, with 1s increments shown.

Notes:

We choose to not find $\vec{N}(t)$ and find a_{N} through the formula $a_{\text{N}} = \sqrt{||\,\vec{a}(t)\,||^2 - a_{\text{T}}^2}$:

$$a_{\text{N}} = \sqrt{32^2 - \left(\frac{32t-32}{\sqrt{t^2-2t+4}}\right)^2} = \frac{32\sqrt{3}}{\sqrt{t^2-2t+4}}.$$

t	a_{T}	a_{N}
0	-16	27.7
1	0	32
2	16	27.7
3	24.2	20.9
4	27.7	16
5	29.4	12.7

Figure 11.27: A table of values of a_{T} and a_{N} in Example 11.29.

Figure 11.27 gives a table of values of a_{T} and a_{N}. When $t = 0$, we see the ball's speed is decreasing; when $t = 1$ the speed of the ball is unchanged. This corresponds to the fact that at $t = 1$ the ball reaches its highest point.

After $t = 1$ we see that a_{N} is decreasing in value. This is because as the ball falls, it's path becomes straighter and most of the acceleration is in the form of speeding up the ball, and not in changing its direction.

Our understanding of the unit tangent and normal vectors is aiding our understanding of motion. The work in Example 11.29 gave quantitative analysis of what we intuitively knew.

The next section provides two more important steps towards this analysis. We currently describe position only in terms of time. In everyday life, though, we often describe position in terms of distance ("The gas station is about 2 miles ahead, on the left."). The *arc length parameter* allows us to reference position in terms of distance traveled.

We also intuitively know that some paths are straighter than others – and some are curvier than others, but we lack a measurement of "curviness." The arc length parameter provides a way for us to compute *curvature*, a quantitative measurement of how curvy a curve is.

Notes:

Exercises 11.4

Terms and Concepts

1. If $\vec{T}(t)$ is a unit tangent vector, what is $\|\vec{T}(t)\|$?

2. If $\vec{N}(t)$ is a unit normal vector, what is $\vec{N}(t) \cdot \vec{r}\,'(t)$?

3. The acceleration vector $\vec{a}(t)$ lies in the plane defined by what two vectors?

4. a_T measures how much the acceleration is affecting the ________ of an object.

Problems

In Exercises 5 – 8 , given $\vec{r}(t)$, find $\vec{T}(t)$ and evaluate it at the indicated value of t.

5. $\vec{r}(t) = \langle 2t^2, t^2 - t\rangle, \quad t = 1$

6. $\vec{r}(t) = \langle t, \cos t\rangle, \quad t = \pi/4$

7. $\vec{r}(t) = \langle \cos^3 t, \sin^3 t\rangle, \quad t = \pi/4$

8. $\vec{r}(t) = \langle \cos t, \sin t\rangle, \quad t = \pi$

In Exercises 9 – 12 , find the equation of the line tangent to the curve at the indicated t-value using the unit tangent vector. Note: these are the same problems as in Exercises 5 – 8.

9. $\vec{r}(t) = \langle 2t^2, t^2 - t\rangle, \quad t = 1$

10. $\vec{r}(t) = \langle t, \cos t\rangle, \quad t = \pi/4$

11. $\vec{r}(t) = \langle \cos^3 t, \sin^3 t\rangle, \quad t = \pi/4$

12. $\vec{r}(t) = \langle \cos t, \sin t\rangle, \quad t = \pi$

In Exercises 13 – 16 , find $\vec{N}(t)$ using Definition 79. Confirm the result using Theorem 99.

13. $\vec{r}(t) = \langle 3\cos t, 3\sin t\rangle$

14. $\vec{r}(t) = \langle t, t^2\rangle$

15. $\vec{r}(t) = \langle \cos t, 2\sin t\rangle$

16. $\vec{r}(t) = \langle e^t, e^{-t}\rangle$

In Exercises 17 – 20 , a position function $\vec{r}(t)$ is given along with its unit tangent vector $\vec{T}(t)$ evaluated at $t = a$, for some value of a.

(a) Confirm that $\vec{T}(a)$ is as stated.

(b) Using a graph of $\vec{r}(t)$ and Theorem 99, find $\vec{N}(a)$.

17. $\vec{r}(t) = \langle 3\cos t, 5\sin t\rangle; \quad \vec{T}(\pi/4) = \left\langle -\frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}}\right\rangle.$

18. $\vec{r}(t) = \left\langle t, \frac{1}{t^2+1}\right\rangle; \quad \vec{T}(1) = \left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right\rangle.$

19. $\vec{r}(t) = (1 + 2\sin t)\,\langle \cos t, \sin t\rangle; \quad \vec{T}(0) = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right\rangle.$

20. $\vec{r}(t) = \langle \cos^3 t, \sin^3 t\rangle; \quad \vec{T}(\pi/4) = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle.$

In Exercises 21 – 24 , find $\vec{N}(t)$.

21. $\vec{r}(t) = \langle 4t, 2\sin t, 2\cos t\rangle$

22. $\vec{r}(t) = \langle 5\cos t, 3\sin t, 4\sin t\rangle$

23. $\vec{r}(t) = \langle a\cos t, a\sin t, bt\rangle; \quad a > 0$

24. $\vec{r}(t) = \langle \cos(at), \sin(at), t\rangle$

In Exercises 25 – 30 , find a_T and a_N given $\vec{r}(t)$. Sketch $\vec{r}(t)$ on the indicated interval, and comment on the relative sizes of a_T and a_N at the indicated t values.

25. $\vec{r}(t) = \langle t, t^2\rangle$ on $[-1, 1]$; consider $t = 0$ and $t = 1$.

26. $\vec{r}(t) = \langle t, 1/t\rangle$ on $(0, 4]$; consider $t = 1$ and $t = 2$.

27. $\vec{r}(t) = \langle 2\cos t, 2\sin t\rangle$ on $[0, 2\pi]$; consider $t = 0$ and $t = \pi/2$.

28. $\vec{r}(t) = \langle \cos(t^2), \sin(t^2)\rangle$ on $(0, 2\pi]$; consider $t = \sqrt{\pi/2}$ and $t = \sqrt{\pi}$.

29. $\vec{r}(t) = \langle a\cos t, a\sin t, bt\rangle$ on $[0, 2\pi]$, where $a, b > 0$; consider $t = 0$ and $t = \pi/2$.

30. $\vec{r}(t) = \langle 5\cos t, 4\sin t, 3\sin t\rangle$ on $[0, 2\pi]$; consider $t = 0$ and $t = \pi/2$.