

# 14: VECTOR CALCULUS

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In previous chapters we have explored a relationship between vectors and integration. Our most tangible result: if  $\vec{v}(t)$  is the vector-valued velocity function of a moving object, then integrating  $\vec{v}(t)$  from  $t = a$  to  $t = b$  gives the displacement of that object over that time interval.

This chapter explores completely different relationships between vectors and integration. These relationships will enable us to compute the work done by a magnetic field in moving an object along a path and find how much air moves through an oddly-shaped screen in space, among other things.

Our upcoming work with integration will benefit from a review. We are not concerned here with techniques of integration, but rather what an integral “does” and how that relates to the notation we use to describe it.

## Integration Review

Recall from Section 13.1 that when  $R$  is a region in the  $x$ - $y$  plane,  $\iint_R dA$  gives the area of the region  $R$ . The integral symbol is an elongated  $S$  meaning “sum” and  $dA$  represents “a small amount of area.” Taken together,  $\iint_R dA$  means “sum up, over  $R$ , small amounts of area.” This sum then gives the total area of  $R$ . We use two integral symbols since  $R$  is a two-dimensional region.

Now let  $z = f(x, y)$  represent a surface. The double integral  $\iint_R f(x, y) dA$  means “sum up, over  $R$ , function values (heights) given by  $f$  times small amounts of area.” Since “height  $\times$  area = volume,” we are summing small amounts of volume over  $R$ , giving the total signed volume under the surface  $z = f(x, y)$  and above the  $x$ - $y$  plane.

This notation does not directly inform us *how* to evaluate the double integrals to find an area or a volume. With additional work, we recognize that a small amount of area  $dA$  can be measured as the area of a small rectangle, with one side length a small change in  $x$  and the other side length a small change in  $y$ . That is,  $dA = dx dy$  or  $dA = dy dx$ . We could also compute a small amount of area by thinking in terms of polar coordinates, where  $dA = r dr d\theta$ . These understandings lead us to the iterated integrals we used in Chapter 13.

Let us back our review up farther. Note that  $\int_1^3 dx = x|_1^3 = 3 - 1 = 2$ . We have simply measured the length of the interval  $[1, 3]$ . We could rewrite the above integral using syntax similar to the double integral syntax above:

$$\int_1^3 dx = \int_I dx, \quad \text{where } I = [1, 3].$$

We interpret “ $\int_I dx$ ” as meaning “sum up, over the interval  $I$ , small changes in  $x$ .” A change in  $x$  is a length along the  $x$ -axis, so we are adding up along  $I$  small

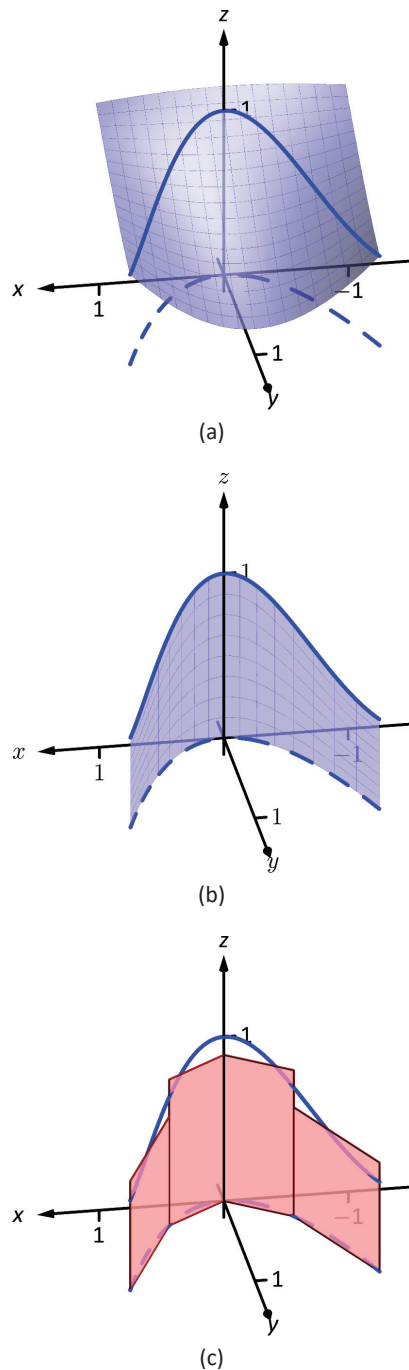


Figure 14.1: Finding area under a curve in space.

lengths, giving the total length of  $I$ .

We could also write  $\int_1^3 f(x) dx$  as  $\int_I f(x) dx$ , interpreted as “sum up, over  $I$ , heights given by  $y = f(x)$  times small changes in  $x$ .” Since “height  $\times$  length = area,” we are summing up areas and finding the total signed area between  $y = f(x)$  and the  $x$ -axis.

This method of referring to the process of integration can be very powerful. It is the core of our notion of the Riemann Sum. When faced with a quantity to compute, if one can think of a way to approximate its value through a sum, the one is well on their way to constructing an integral (or, double or triple integral) that computes the desired quantity. We will demonstrate this process throughout this chapter, starting with the next section.

## 14.1 Introduction to Line Integrals

We first used integration to find “area under a curve.” In this section, we learn to do this (again), but in a different context.

Consider the surface and curve shown in Figure 14.1(a). The surface is given by  $f(x, y) = 1 - \cos(x)\sin(y)$ . The dashed curve lies in the  $x$ - $y$  plane and is the familiar  $y = x^2$  parabola from  $-1 \leq x \leq 1$ ; we’ll call this curve  $C$ . The curve drawn with a solid line in the graph is the curve in space that lies on our surface with  $x$  and  $y$  values that lie on  $C$ .

The question we want to answer is this: what is the area that lies below the curve drawn with the solid line? In other words, what is the area of the region above  $C$  and under the the surface  $f$ ? This region is shown in Figure 14.1(b).

We suspect the answer can be found using an integral, but before trying to figure out what that integral is, let us first try to approximate its value.

In Figure 14.1(c), four rectangles have been drawn over the curve  $C$ . The bottom corners of each rectangle lie on  $C$ , and each rectangle has a height given by the function  $f(x, y)$  for some  $(x, y)$  pair along  $C$  between the rectangle’s bottom corners.

As we know how to find the area of each rectangle, we are able to approximate the area above  $C$  and under  $f$ . Clearly, our approximation will be *an approximation*. The heights of the rectangles do not match exactly with the surface  $f$ , nor does the base of each rectangle follow perfectly the path of  $C$ .

In typical calculus fashion, our approximation can be improved by using more rectangles. The sum of the areas of these rectangles gives an approximate value of the true area above  $C$  and under  $f$ . As the area of each rectangle is “height  $\times$  width”, we assert that the

$$\text{area above } C \approx \sum (\text{heights} \times \text{widths}).$$

When first learning of the integral, and approximating areas with “heights  $\times$

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widths”, the width was a small change in  $x$ :  $dx$ . That will not suffice in this context. Rather, each width of a rectangle is actually approximating the arc length of a small portion of  $C$ . In Section 11.5, we used  $s$  to represent the arc-length parameter of a curve. A small amount of arc length will thus be represented by  $ds$ .

The height of each rectangle will be determined in some way by the surface  $f$ . If we parameterize  $C$  by  $s$ , an  $s$ -value corresponds to an  $(x, y)$  pair that lies on the parabola  $C$ . Since  $f$  is a function of  $x$  and  $y$ , and  $x$  and  $y$  are functions of  $s$ , we can say that  $f$  is a function of  $s$ . Given a value  $s$ , we can compute  $f(s)$  and find a height. Thus

$$\begin{aligned}\text{area under } f \text{ and above } C &\approx \sum (\text{heights} \times \text{widths}); \\ \text{area under } f \text{ and above } C &= \lim_{\|\Delta s\| \rightarrow 0} \sum f(c_i) \Delta s_i \\ &= \int_C f(s) \, ds.\end{aligned}\tag{14.1}$$

Here we have introduced a new notation, the integral symbol with a subscript of  $C$ . It is reminiscent of our usage of  $\iint_R$ . Using the train of thought found in the Integration Review preceding this section, we interpret “ $\int_C f(s) \, ds$ ” as meaning “sum up, along a curve  $C$ , function values  $f(s) \times$  small arc lengths.” It is understood here that  $s$  represents the arc-length parameter.

All this leads us to a definition. The integral found in Equation 14.1 is called a **line integral**. We formally define it below, but note that the definition is very abstract. On one hand, one is apt to say “the definition makes sense,” while on the other, one is equally apt to say “but I don’t know what I’m supposed to do with this definition.” We’ll address that after the definition, and actually find an answer to the area problem we posed at the beginning of this section.

#### Definition 114 Line Integral Over A Scalar Field

Let  $C$  be a smooth curve parameterized by  $s$ , the arc-length parameter, and let  $f$  be a continuous function of  $s$ . A **line integral** is an integral of the form

$$\int_C f(s) \, ds = \lim_{\|\Delta s\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta s_i,$$

where  $s_1 < s_2 < \dots < s_n$  is any partition of the  $s$ -interval over which  $C$  is defined,  $c_i$  is any value in the  $i^{\text{th}}$  subinterval,  $\Delta s_i$  is the width of the  $i^{\text{th}}$  subinterval, and  $\|\Delta s\|$  is the length of the longest subinterval in the partition.

**Note:** Definition 114 uses the term **scalar field** which has not yet been defined. Its meaning is discussed in the paragraph preceding Definition 119 when it is compared to a **vector field**.

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When  $C$  is a **closed** curve, i.e., a curve that ends at the same point at which it starts, we use

$$\oint_C f(s) \, ds \quad \text{instead of} \quad \int_C f(s) \, ds.$$

The definition of the line integral does not specify whether  $C$  is a curve in the plane or space (or hyperspace), as the definition holds regardless. For now, we'll assume  $C$  lies in the  $x$ - $y$  plane.

This definition of the line integral doesn't really say anything new. If  $C$  is a curve and  $s$  is the arc-length parameter of  $C$  on  $a \leq s \leq b$ , then

$$\int_C f(s) \, ds = \int_a^b f(s) \, ds.$$

The real difference with this integral from the standard " $\int_a^b f(x) \, dx$ " we used in the past is that of context. Our previous integrals naturally summed up values over an interval on the  $x$ -axis, whereas now we are summing up values over a curve. *If* we can parameterize the curve with the arc-length parameter, we can evaluate the line integral just as before. Unfortunately, parameterizing a curve in terms of the arc-length parameter is usually very difficult, so we must develop a method of evaluating line integrals using a different parameterization.

Given a curve  $C$ , find any parameterization of  $C$ :  $x = g(t)$  and  $y = h(t)$ , for continuous functions  $g$  and  $h$ , where  $a \leq t \leq b$ . We can represent this parameterization with a vector-valued function,  $\vec{r}(t) = \langle g(t), h(t) \rangle$ .

In Section 11.5, we defined the arc-length parameter in Equation 11.1 as

$$s(t) = \int_0^t \|\vec{r}'(u)\| \, du.$$

By the Fundamental Theorem of Calculus,  $ds = \|\vec{r}'(t)\| \, dt$ . We can substitute the right hand side of this equation for  $ds$  in the line integral definition.

We can view  $f$  as being a function of  $x$  and  $y$  since it is a function of  $s$ . Thus  $f(s) = f(x, y) = f(g(t), h(t))$ . This gives us a concrete way to evaluate a line integral:

$$\int_C f(s) \, ds = \int_a^b f(g(t), h(t)) \|\vec{r}'(t)\| \, dt.$$

We restate this as a theorem, along with its three-dimensional analogue, followed by an example where we finally evaluate an integral and find an area.

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Notes:

**Theorem 135 Evaluating a Line Integral Over A Scalar Field**

- Let  $C$  be a curve parameterized by  $\vec{r}(t) = \langle g(t), h(t) \rangle$ ,  $a \leq t \leq b$ , where  $g$  and  $h$  are continuously differentiable, and let  $z = f(x, y)$ , where  $f$  is continuous over  $C$ . Then

$$\int_C f(s) \, ds = \int_a^b f(g(t), h(t)) \|\vec{r}'(t)\| \, dt.$$

- Let  $C$  be a curve parameterized by  $\vec{r}(t) = \langle g(t), h(t), k(t) \rangle$ ,  $a \leq t \leq b$ , where  $g$ ,  $h$  and  $k$  are continuously differentiable, and let  $w = f(x, y, z)$ , where  $f$  is continuous over  $C$ . Then

$$\int_C f(s) \, ds = \int_a^b f(g(t), h(t), k(t)) \|\vec{r}'(t)\| \, dt.$$

To be clear, the first point of Theorem 135 can be used to find the area under a surface  $z = f(x, y)$  and above a curve  $C$ . We will later give an understanding of the line integral when  $C$  is a curve in space.

Let's do an example where we actually compute an area.

**Example 14.1 Evaluating a line integral: area under a surface over a curve.**

Find the area under the surface  $f(x, y) = \cos(x) + \sin(y) + 2$  over the curve  $C$ , which is the segment of the line  $y = 2x + 1$  on  $-1 \leq x \leq 1$ , as shown in Figure 14.2.

**SOLUTION** Our first step is to represent  $C$  with a vector-valued function. Since  $C$  is a simple line, and we have an explicit relationship between  $y$  and  $x$  (namely, that  $y$  is  $2x + 1$ ), we can let  $x = t$ ,  $y = 2t + 1$ , and write  $\vec{r}(t) = \langle t, 2t + 1 \rangle$  for  $-1 \leq t \leq 1$ .

We find the values of  $f$  over  $C$  as  $f(x, y) = f(t, 2t + 1) = \cos(t) + \sin(2t + 1) + 2$ . We also need  $\|\vec{r}'(t)\|$ ; with  $\vec{r}'(t) = \langle 1, 2 \rangle$ , we have  $\|\vec{r}'(t)\| = \sqrt{5}$ . Thus  $ds = \sqrt{5} \, dt$ .

The area we seek is

$$\begin{aligned} \int_C f(s) \, ds &= \int_{-1}^1 (\cos(t) + \sin(2t + 1) + 2) \sqrt{5} \, dt \\ &= \sqrt{5} \left( \sin(t) - \frac{1}{2} \cos(2t + 1) + 2t \right) \Big|_{-1}^1 \\ &\approx 14.418 \text{ units}^2. \end{aligned}$$

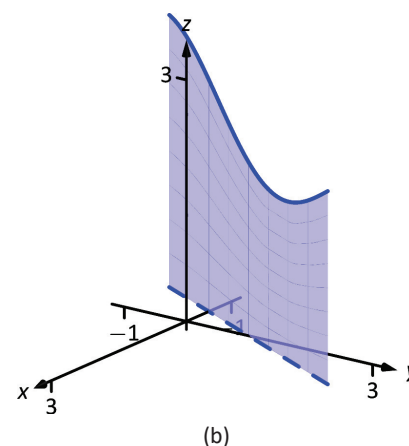
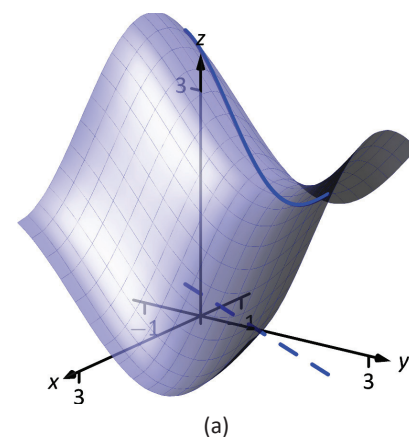


Figure 14.2: Finding area under a curve in Example 14.1.

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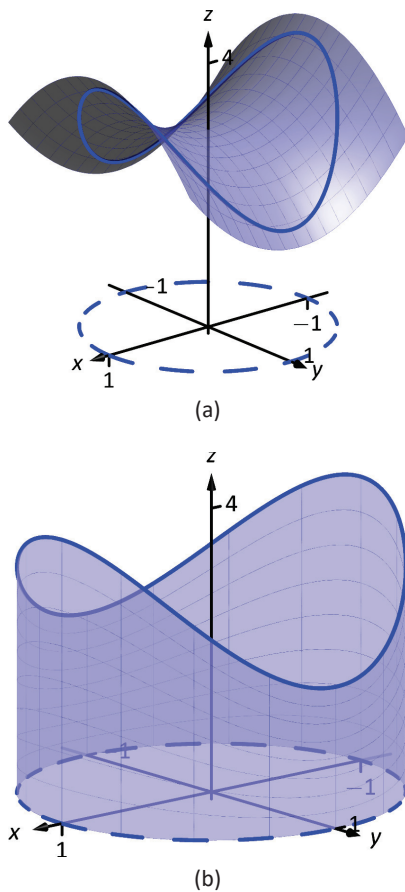


Figure 14.3: Finding area under a curve in Example 14.2.

We will practice setting up and evaluating a line integral in another example, then find the area described at the beginning of this section.

**Example 14.2 Evaluating a line integral: area under a surface over a curve.**

Find the area over the unit circle in the  $x$ - $y$  plane and under the surface  $f(x, y) = x^2 - y^2 + 3$ , shown in Figure 14.3.

**SOLUTION** The curve  $C$  is the unit circle, which we will describe with the parameterization  $\vec{r}(t) = \langle \cos t, \sin t \rangle$  for  $0 \leq t \leq 2\pi$ . We find  $\|\vec{r}'(t)\| = 1$ , so  $ds = 1dt$ .

We find the values of  $f$  over  $C$  as  $f(x, y) = f(\cos t, \sin t) = \cos^2 t - \sin^2 t + 3$ . Thus the area we seek is (note the use of the  $\oint f(s)ds$  notation):

$$\begin{aligned} \oint_C f(s) ds &= \int_0^{2\pi} (\cos^2 t - \sin^2 t + 3) dt \\ &= 6\pi. \end{aligned}$$

(Note: we may have approximated this answer from the start. The unit circle has a circumference of  $2\pi$ , and we may have guessed that due to the apparent symmetry of our surface, the average height of the surface is 3.)

We now consider the example that introduced this section.

**Example 14.3 Evaluating a line integral: area under a surface over a curve.**

Find the area under  $f(x, y) = 1 - \cos(x)\sin(y)$  and over the parabola  $y = x^2$ , from  $-1 \leq x \leq 1$ .

**SOLUTION** We parameterize our curve  $C$  as  $\vec{r}(t) = \langle t, t^2 \rangle$  for  $-1 \leq t \leq 1$ ; we find  $\|\vec{r}'(t)\| = \sqrt{1 + 4t^2}$ , so  $ds = \sqrt{1 + 4t^2} dt$ .

Replacing  $x$  and  $y$  with their respective functions of  $t$ , we have  $f(x, y) = f(t, t^2) = 1 - \cos(t)\sin(t^2)$ . Thus the area under  $f$  and over  $C$  is found to be

$$\int_C f(s) ds = \int_{-1}^1 (1 - \cos(t)\sin(t^2)) \sqrt{1 + 4t^2} dt.$$

This integral is impossible to evaluate using the techniques developed in this text. We resort to a numerical approximation; accurate to two places after the decimal, we find the area is 2.17.

We give one more example of finding area.

**Example 14.4 Evaluating a line integral: area under a curve in space.**

Find the area above the  $x$ - $y$  plane and below the helix parameterized by  $\vec{r}(t) = \langle \cos t, 2 \sin t, t/\pi \rangle$ , for  $0 \leq t \leq 2\pi$ , as shown in Figure 14.4.

Notes:

**SOLUTION** Note how this problem is different than the previous examples: here, the height is not given by a surface, but by the curve itself.

We use the given vector-valued function  $\vec{r}(t)$  to determine the curve  $C$  in the  $x$ - $y$  plane by simply using the first two components of  $\vec{r}(t)$ :  $\vec{c}(t) = \langle \cos t, 2 \sin t \rangle$ . Thus  $ds = \|\vec{c}'(t)\| dt = \sqrt{\sin^2 t + 4 \cos^2 t} dt$ .

The height is not found by evaluating a surface over  $C$ , but rather it is given directly by the third component of  $\vec{r}(t)$ :  $t/\pi$ . Thus

$$\oint_C f(s) ds = \int_0^{2\pi} \frac{t}{\pi} \sqrt{\sin^2 t + 4 \cos^2 t} dt \approx 9.69,$$

where the approximation was obtained using numerical methods.

Note how in each of the previous examples we are effectively finding “area under a curve”, just as we did when first learning of integration. We have used the phrase “area over a curve  $C$  and under a surface,” but that is because of the important role  $C$  plays in the integral. The figures show how the curve  $C$  defines another curve on the surface  $z = f(x, y)$ , and we are finding the area under that curve.

## Properties of Line Integrals

Many properties of line integrals can be inferred from general integration properties. For instance, if  $k$  is a scalar, then  $\int_C k f(s) ds = k \int_C f(s) ds$ .

One property in particular of line integrals is worth noting. If  $C$  is a curve composed of subcurves  $C_1$  and  $C_2$ , where they share only one point in common (see Figure 14.5(a)), then the line integral over  $C$  is the sum of the line integrals over  $C_1$  and  $C_2$ :

$$\int_C f(s) ds = \int_{C_1} f(s) ds + \int_{C_2} f(s) ds.$$

This property allows us to evaluate line integrals over some curves  $C$  that are not smooth. Note how in Figure 14.5(b) the curve is not smooth at  $D$ , so by our definition of the line integral we cannot evaluate  $\int_C f(s) ds$ . However, one can evaluate line integrals over  $C_1$  and  $C_2$  and their sum will be the desired quantity.

A curve  $C$  that is composed of two or more smooth curves is said to be **piecewise smooth**. In this chapter, any statement that is made about smooth curves also holds for piecewise smooth curves.

We state these properties as a theorem.

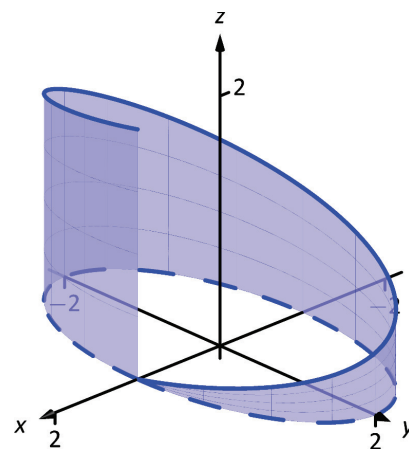


Figure 14.4: Finding area under a curve in Example 14.4.

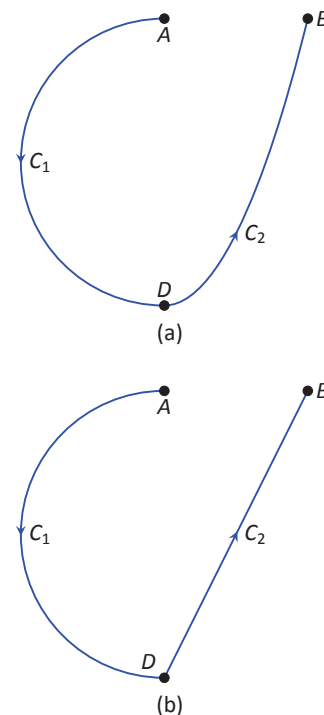


Figure 14.5: Illustrating properties of line integrals.

Notes:

**Theorem 136 Properties of Line Integrals Over Scalar Fields**

1. Let  $C$  be a smooth curve parameterized by the arc-length parameter  $s$ , let  $f$  and  $g$  be continuous functions of  $s$ , and let  $k_1$  and  $k_2$  be scalars. Then

$$\int_C (k_1 f(s) + k_2 g(s)) \, ds = k_1 \int_C f(s) \, ds + k_2 \int_C g(s) \, ds.$$

2. Let  $C$  be piecewise smooth, composed of smooth components  $C_1$  and  $C_2$ . Then

$$\int_C f(s) \, ds = \int_{C_1} f(s) \, ds + \int_{C_2} f(s) \, ds.$$

**Mass and Center of Mass**

We first learned integration as a method to find area under a curve, then later used integration to compute a variety of other quantities, such as arc length, volume, force, etc. In this section, we also introduced line integrals as a method to find area under a curve, and now we explore one more application.

Let a curve  $C$  (either in the plane or in space) represent a thin wire with variable density  $\delta(s)$ . We can approximate the mass of the wire by dividing the wire (i.e., the curve) into small segments of length  $\Delta s_i$  and assume the density is constant across these small segments. The mass of each segment is density of the segment  $\times$  its length; by summing up the approximate mass of each segment we can approximate the total mass:

$$\text{Total Mass of Wire} = \sum \delta(s_i) \Delta s_i.$$

By taking the limit as the length of the segments approaches 0, we have the definition of the line integral as seen in Definition 114. When learning of the line integral, we let  $f(s)$  represent a height; now we let  $f(s) = \delta(s)$  represent a density.

We can extend this understanding of computing mass to also compute the center of mass of a thin wire. (As a reminder, the center of mass can be a useful piece of information as objects rotate about that center.) We give the relevant formulas in the next definition, followed by an example. Note the similarities between this definition and Definition 113, which gives similar properties of solids in space.

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Notes:



**Definition 115** Mass, Center of Mass of Thin Wire

Let a thin wire lie along a smooth curve  $C$  with continuous density function  $\delta(s)$ , where  $s$  is the arc length parameter.

1. The **mass** of the thin wire is  $M = \int_C \delta(s) ds$ .
2. The **moment about the  $y$ - $z$  plane** is  $M_{yz} = \int_C x\delta(s) ds$ .
3. The **moment about the  $x$ - $z$  plane** is  $M_{xz} = \int_C y\delta(s) ds$ .
4. The **moment about the  $x$ - $y$  plane** is  $M_{xy} = \int_C z\delta(s) ds$ .
5. The **center of mass** of the wire is

$$(\bar{x}, \bar{y}, \bar{z}) = \left( \frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M} \right).$$

**Example 14.5** Evaluating a line integral: calculating mass.

A thin wire follows the path  $\vec{r}(t) = \langle 1 + \cos t, 1 + \sin t, 1 + \sin(2t) \rangle$ ,  $0 \leq t \leq 2\pi$ . The density of the wire is determined by its position in space:  $\delta(x, y, z) = y + z$  g/cm. The wire is shown in Figure 14.6, where a light color indicates low density and a dark color represents high density. Find the mass and center of mass of the wire.

**SOLUTION** We compute the density of the wire as

$$\delta(x, y, z) = \delta(1 + \cos t, 1 + \sin t, 1 + \sin(2t)) = 2 + \sin t + \sin(2t).$$

We compute  $ds$  as

$$ds = \|\vec{r}'(t)\| dt = \sqrt{\sin^2 t + \cos^2 t + 4 \cos^2(2t)} dt = \sqrt{1 + 4 \cos^2(2t)} dt.$$

Thus the mass is

$$M = \oint_C \delta(s) ds = \int_0^{2\pi} (2 + \sin t + \sin(2t)) \sqrt{1 + 4 \cos^2(2t)} dt \approx 21.08\text{g}.$$

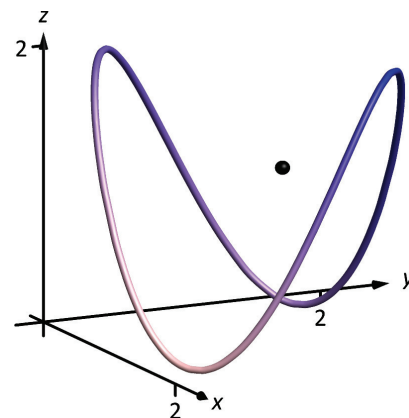


Figure 14.6: Finding the mass of a thin wire in Example 14.5.

Notes:

We compute the moments about the coordinate planes:

$$M_{yz} = \oint_C x\delta(s) ds = \int_0^{2\pi} (1 + \cos t)(2 + \sin t + \sin(2t))\sqrt{1 + 4\cos^2(2t)} dt \approx 21.08.$$

$$M_{xz} = \oint_C y\delta(s) ds = \int_0^{2\pi} (1 + \sin t)(2 + \sin t + \sin(2t))\sqrt{1 + 4\cos^2(2t)} dt \approx 26.35$$

$$M_{xy} = \oint_C z\delta(s) ds = \int_0^{2\pi} (1 + \sin(2t))(2 + \sin t + \sin(2t))\sqrt{1 + 4\cos^2(2t)} dt \approx 25.40$$

Thus the center of mass of the wire is located at

$$(\bar{x}, \bar{y}, \bar{z}) = \left( \frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M} \right) \approx (1, 1.25, 1.20),$$

as indicated by the dot in Figure 14.6. Note how in this example, the curve  $C$  is “centered” about the point  $(1, 1, 1)$ , though the variable density of the wire pulls the center of mass out along the  $y$  and  $z$  axes.

We end this section with a callback to the Integration Review that preceded this section. A line integral looks like:  $\int_C f(s) ds$ . As stated before the definition of the line integral, this means “sum up, along a curve  $C$ , function values  $f(s) \times$  small arc lengths.” When  $f(s)$  represents a height, we have “height  $\times$  length = area.” When  $f(s)$  is a density (and we use  $\delta(s)$  by convention), we have “density (mass per unit length)  $\times$  length = mass.”

In the next section, we investigate a new mathematical object, the *vector field*. The remaining sections of this chapter are devoted to understanding integration in the context of vector fields.

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Notes:

# Exercises 14.1

## Terms and Concepts

1. Explain how a line integral can be used to find the area under a curve.
2. How does the evaluation of a line integral given as  $\int_C f(s) ds$  differ from a line integral given as  $\int_C f(s) ds$ ?
3. Why are most line integrals evaluated using Key Idea 135 instead of “directly” as  $\int_C f(s) ds$ ?
4. Sketch a closed, piecewise smooth curve composed of three subcurves.

## Problems

**In Exercises 5 – 13, a planar curve  $C$  is given along with a surface  $f$  that is defined over  $C$ . Evaluate the line integral  $\int_C f(s) ds$ .**

5.  $C$  is the line segment joining the points  $(-2, -1)$  and  $(1, 2)$ ; the surface is  $f(x, y) = x^2 + y^2 + 2$ .
6.  $C$  is the segment of  $y = 3x + 2$  on  $[1, 2]$ ; the surface is  $f(x, y) = 5x + 2y$ .
7.  $C$  is the curve given by  $\vec{r}(t) = \langle \cos t, \sin t \rangle$  on  $[0, \frac{\pi}{2}]$ ; the surface is  $f(x, y) = x$ .
8.  $C$  is the curve given by  $\vec{r}(t) = \langle \cos t, \sin t \rangle$  on  $[0, \frac{\pi}{2}]$ ; the surface is  $f(x, y) = xy$ .
9.  $C$  is the bent path composed of line segments from  $(0,0)$  to  $(1,1)$  to  $(1,0)$ ; the surface is  $f(x, y) = xy$ .
10.  $C$  is the circle with radius 2 centered at the point  $(4, 2)$ ; the surface is  $f(x, y) = 3x - y$ .
11.  $C$  is the curve given by  $\vec{r}(t) = \langle \cos t + t \sin t, \sin t - t \cos t \rangle$  on  $[0, 2\pi]$ ; the surface is  $f(x, y) = 5$ .
12.  $C$  is the piecewise curve composed of the line segments that connect  $(0, 1)$  to  $(1, 1)$ , then connect  $(1, 1)$  to  $(1, 0)$ ; the surface is  $f(x, y) = x + y^2$ .

13.  $C$  is the piecewise curve composed of the line segment joining the points  $(0, 0)$  and  $(1, 1)$ , along with the quarter-circle parameterized by  $\langle \cos t, -\sin t + 1 \rangle$  on  $[0, \pi/2]$  (which starts at the point  $(1, 1)$  and ends at  $(0, 0)$ ); the surface is  $f(x, y) = x^2 + y^2$ .

**In Exercises 14 – 17, a planar curve  $C$  is given along with a surface  $f$  that is defined over  $C$ . Set up the line integral  $\int_C f(s) ds$ , then approximate its value using technology.**

14.  $C$  is the portion of the parabola  $y = 2x^2 + x + 1$  on  $[0, 1]$ ; the surface is  $f(x, y) = x^2 + 2y$ .
15.  $C$  is the portion of the curve  $y = \sin x$  on  $[0, \pi]$ ; the surface is  $f(x, y) = x$ .
16.  $C$  is the ellipse given by  $\vec{r}(t) = \langle 2 \cos t, \sin t \rangle$  on  $[0, 2\pi]$ ; the surface is  $f(x, y) = 10 - x^2 - y^2$ .
17.  $C$  is the portion of  $y = x^3$  on  $[-1, 1]$ ; the surface is  $f(x, y) = 2x + 3y + 5$ .

**In Exercises 18 – 21, a parameterized curve  $C$  in space is given. Find the area above the  $x$ - $y$  plane that is under  $C$ .**

18.  $C: \vec{r}(t) = \langle 5t, t, t^2 \rangle$  for  $1 \leq t \leq 2$ .
19.  $C: \vec{r}(t) = \langle \cos t, \sin t, \sin(2t) + 1 \rangle$  for  $0 \leq t \leq 2\pi$ .
20.  $C: \vec{r}(t) = \langle 3 \cos t, 3 \sin t, t^2 \rangle$  for  $0 \leq t \leq 2\pi$ .
21.  $C: \vec{r}(t) = \langle 3t, 4t, t \rangle$  for  $0 \leq t \leq 1$ .

**In Exercises 22 – 23, a parameterized curve  $C$  is given that represents a thin wire with density  $\delta$ . Find the mass and center of mass of the thin wire.**

22.  $C: \vec{r}(t) = \langle \cos t, \sin t, t \rangle$  for  $0 \leq t \leq 4\pi$ ;  $\delta(x, y, z) = z$ .
23.  $C: \vec{r}(t) = \langle t - t^2, t^2 - t^3, t^3 - t^4 \rangle$  for  $0 \leq t \leq 1$ ;  $\delta(x, y, z) = x + 2y + 2z$ . Use technology to approximate the value of each integral.