4.2 L'Hôpital's Rule

Our treatment of limits exposed us to "0/0", an indeterminate form. If $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$, we do not conclude that $\lim_{x\to c} f(x)/g(x)$ is $0/0$; rather, we use $0/0$ as notation to describe the fact that both the numerator and denominator approach 0. The expression 0/0 has no numeric value; other work must be done to evaluate the limit.

Other indeterminate forms exist; they are: ∞/∞, $0\cdot\infty$, $\infty-\infty$, 0^0, 1^∞ and ∞^0. Just as "0/0" does not mean "divide 0 by 0," the expression "∞/∞" does not mean "divide infinity by infinity." Instead, it means "a quantity is growing without bound and is being divided by another quantity that is growing without bound." (Technically, ∞ means $\pm\infty$ in these indeterminate forms, so $\infty/-\infty$, $-\infty/\infty$, and $-\infty/-\infty$ are all indeterminate, and considered a form of "∞/∞".) We cannot determine from such a statement what value results in the limit, or even if the limit exists. Likewise, "$0\cdot\infty$" does not mean "multiply zero by infinity." Instead, it means "one quantity is shrinking to zero, and is being multiplied by a quantity that is growing without bound." We cannot determine from such a description what the result of such a limit will be.

This section introduces l'Hôpital's Rule, a method using derivatives to resolve limits that produce the indeterminate forms 0/0 and ∞/∞. We'll also show how algebraic manipulation can be used to convert other indeterminate expressions into one of these two forms so that our new rule can be applied.

Theorem 34 L'Hôpital's Rule, Part 1

Let $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$, where f and g are differentiable functions on an open interval I containing c, and $g'(x) \neq 0$ on I except possibly at c. Then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}.$$

We demonstrate the use of l'Hôpital's Rule in the following examples; we will often use "LHR" as an abbreviation of "l'Hôpital's Rule."

Notes:

Example 4.5 Using l'Hôpital's Rule

Evaluate the following limits, using l'Hôpital's Rule as needed.

1. $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}$
2. $\displaystyle\lim_{x\to 1}\frac{\sqrt{x+3}-2}{1-x}$
3. $\displaystyle\lim_{x\to 0}\frac{x^2}{1-\cos x}$
4. $\displaystyle\lim_{x\to 2}\frac{x^2+x-6}{x^2-3x+2}$

Solution

1. We proved this limit is 1 in Example 1.13 using the Squeeze Theorem. Here we use l'Hôpital's Rule to show its power.

$$\lim_{x\to 0}\frac{\sin x}{x} \overset{\text{by LHR}}{=} \lim_{x\to 0}\frac{\cos x}{1} = 1.$$

We actually needed to know the value of this limit to prove that the derivative of $\sin x$ is $\cos x$, a fact we used in this computation. Therefore, we did need the geometric argument in Example 1.13 to first prove this. However, since we now know the derivative of $\sin x$ is $\cos x$, we need not remember the value of this limit as we can now determine it with l'Hôpital's Rule.

2. $$\lim_{x\to 1}\frac{\sqrt{x+3}-2}{1-x} \overset{\text{by LHR}}{=} \lim_{x\to 1}\frac{\frac{1}{2}(x+3)^{-1/2}}{-1} = -\frac{1}{4}.$$

3. $$\lim_{x\to 0}\frac{x^2}{1-\cos x} \overset{\text{by LHR}}{=} \lim_{x\to 0}\frac{2x}{\sin x}.$$

This latter limit also evaluates to the 0/0 indeterminate form. To evaluate it, we apply l'Hôpital's Rule again.

$$\lim_{x\to 0}\frac{2x}{\sin x} \overset{\text{by LHR}}{=} \frac{2}{\cos x} = 2.$$

Thus $\displaystyle\lim_{x\to 0}\frac{x^2}{1-\cos x} = 2.$

4. We already know how to evaluate this limit; first factor the numerator and denominator. We then have:

$$\lim_{x\to 2}\frac{x^2+x-6}{x^2-3x+2} = \lim_{x\to 2}\frac{(x-2)(x+3)}{(x-2)(x-1)} = \lim_{x\to 2}\frac{x+3}{x-1} = 5.$$

We now show how to solve this using l'Hôpital's Rule.

$$\lim_{x\to 2}\frac{x^2+x-6}{x^2-3x+2} \overset{\text{by LHR}}{=} \lim_{x\to 2}\frac{2x+1}{2x-3} = 5.$$

Notes:

Note that at each step where l'Hôpital's Rule was applied, it was *needed*: the initial limit returned the indeterminate form of "$0/0$." If the initial limit returns, for example, 1/2, then l'Hôpital's Rule does not apply.

The following theorem extends our initial version of l'Hôpital's Rule in two ways. It allows the technique to be applied to the indeterminate form ∞/∞ and to limits where x approaches $\pm\infty$.

Theorem 35 L'Hôpital's Rule, Part 2

1. Let $\lim\limits_{x\to a} f(x) = \pm\infty$ and $\lim\limits_{x\to a} g(x) = \pm\infty$, where f and g are differentiable on an open interval I containing a. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

2. Let f and g be differentiable functions on the open interval (a,∞) for some value a, where $g'(x) \neq 0$ on (a,∞) and $\lim\limits_{x\to\infty} f(x)/g(x)$ returns either 0/0 or ∞/∞. Then

$$\lim_{x\to \infty} \frac{f(x)}{g(x)} = \lim_{x\to \infty} \frac{f'(x)}{g'(x)}.$$

A similar statement can be made for limits where x approaches $-\infty$.

Example 4.6 Using l'Hôpital's Rule with limits involving ∞

Evaluate the following limits.

1. $\displaystyle\lim_{x\to\infty} \frac{3x^2-100x+2}{4x^2+5x-1000}$ 2. $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^3}$.

Solution

1. We can evaluate this limit already using Theorem 11; the answer is 3/4. We apply l'Hôpital's Rule to demonstrate its applicability.

$$\lim_{x\to\infty} \frac{3x^2-100x+2}{4x^2+5x-1000} \stackrel{\text{by LHR}}{=} \lim_{x\to\infty} \frac{6x-100}{8x+5} \stackrel{\text{by LHR}}{=} \lim_{x\to\infty} \frac{6}{8} = \frac{3}{4}.$$

2. $$\lim_{x\to\infty} \frac{e^x}{x^3} \stackrel{\text{by LHR}}{=} \lim_{x\to\infty} \frac{e^x}{3x^2} \stackrel{\text{by LHR}}{=} \lim_{x\to\infty} \frac{e^x}{6x} \stackrel{\text{by LHR}}{=} \lim_{x\to\infty} \frac{e^x}{6} = \infty.$$

Notes:

Recall that this means that the limit does not exist; as x approaches ∞, the expression e^x/x^3 grows without bound. We can infer from this that e^x grows "faster" than x^3; as x gets large, e^x is far larger than x^3. (This has important implications in computing when considering efficiency of algorithms.)

Indeterminate Forms $0 \cdot \infty$ and $\infty - \infty$

L'Hôpital's Rule can only be applied to ratios of functions. When faced with an indeterminate form such as $0 \cdot \infty$ or $\infty - \infty$, we can sometimes apply algebra to rewrite the limit so that l'Hôpital's Rule can be applied. We demonstrate the general idea in the next example.

Example 4.7 Applying l'Hôpital's Rule to other indeterminate forms
Evaluate the following limits.

1. $\displaystyle\lim_{x\to 0^+} x \cdot e^{1/x}$
2. $\displaystyle\lim_{x\to 0^-} x \cdot e^{1/x}$
3. $\displaystyle\lim_{x\to\infty} \ln(x+1) - \ln x$
4. $\displaystyle\lim_{x\to\infty} x^2 - e^x$

SOLUTION

1. As $x \to 0^+$, $x \to 0$ and $e^{1/x} \to \infty$. Thus we have the indeterminate form $0 \cdot \infty$. We rewrite the expression $x \cdot e^{1/x}$ as $\dfrac{e^{1/x}}{1/x}$; now, as $x \to 0^+$, we get the indeterminate form ∞/∞ to which l'Hôpital's Rule can be applied.

$$\lim_{x\to 0^+} x \cdot e^{1/x} = \lim_{x\to 0^+} \frac{e^{1/x}}{1/x} \overset{\text{by LHR}}{=} \lim_{x\to 0^+} \frac{(-1/x^2)e^{1/x}}{-1/x^2} = \lim_{x\to 0^+} e^{1/x} = \infty.$$

   Interpretation: $e^{1/x}$ grows "faster" than x shrinks to zero, meaning their product grows without bound.

2. As $x \to 0^-$, $x \to 0$ and $e^{1/x} \to e^{-\infty} \to 0$. The the limit evaluates to $0 \cdot 0$ which is not an indeterminate form. We conclude then that

$$\lim_{x\to 0^-} x \cdot e^{1/x} = 0.$$

3. This limit initially evaluates to the indeterminate form $\infty - \infty$. By applying a logarithmic rule, we can rewrite the limit as

$$\lim_{x\to\infty} \ln(x+1) - \ln x = \lim_{x\to\infty} \ln\left(\frac{x+1}{x}\right).$$

Notes:

As $x \to \infty$, the argument of the ln term approaches ∞/∞, to which we can apply l'Hôpital's Rule.

$$\lim_{x\to\infty} \frac{x+1}{x} \overset{\text{by LHR}}{=} \frac{1}{1} = 1.$$

Since $x \to \infty$ implies $\dfrac{x+1}{x} \to 1$, it follows that

$$x \to \infty \quad \text{implies} \quad \ln\left(\frac{x+1}{x}\right) \to \ln 1 = 0.$$

Thus

$$\lim_{x\to\infty} \ln(x+1) - \ln x = \lim_{x\to\infty} \ln\left(\frac{x+1}{x}\right) = 0.$$

Interpretation: since this limit evaluates to 0, it means that for large x, there is essentially no difference between $\ln(x+1)$ and $\ln x$; their difference is essentially 0.

4. The limit $\displaystyle\lim_{x\to\infty} x^2 - e^x$ initially returns the indeterminate form $\infty - \infty$. We can rewrite the expression by factoring out x^2; $x^2 - e^x = x^2\left(1 - \dfrac{e^x}{x^2}\right)$. We need to evaluate how e^x/x^2 behaves as $x \to \infty$:

$$\lim_{x\to\infty} \frac{e^x}{x^2} \overset{\text{by LHR}}{=} \lim_{x\to\infty} \frac{e^x}{2x} \overset{\text{by LHR}}{=} \lim_{x\to\infty} \frac{e^x}{2} = \infty.$$

Thus $\lim_{x\to\infty} x^2(1 - e^x/x^2)$ evaluates to $\infty \cdot (-\infty)$, which is not an indeterminate form; rather, $\infty \cdot (-\infty)$ evaluates to $-\infty$. We conclude that $\displaystyle\lim_{x\to\infty} x^2 - e^x = -\infty$.

Interpretation: as x gets large, the difference between x^2 and e^x grows very large.

Indeterminate Forms 0^0, 1^∞ and ∞^0

When faced with an indeterminate form that involves a power, it often helps to employ the natural logarithmic function. The following Key Idea expresses the concept, which is followed by an example that demonstrates its use.

Notes:

Key Idea 6 Evaluating Limits Involving Indeterminate Forms 0^0, 1^∞ and ∞^0

If $\displaystyle\lim_{x\to c} \ln\big(f(x)\big) = L$, then $\displaystyle\lim_{x\to c} f(x) = \lim_{x\to c} e^{\ln(f(x))} = e^{L}$.

Example 4.8 Using l'Hôpital's Rule with indeterminate forms involving exponents

Evaluate the following limits.

1. $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$
2. $\displaystyle\lim_{x\to 0^+} x^x$.

Solution

1. This equivalent to a special limit given in Theorem 3; these limits have important applications within mathematics and finance. Note that the exponent approaches ∞ while the base approaches 1, leading to the indeterminate form 1^∞. Let $f(x) = (1+1/x)^x$; the problem asks to evaluate $\displaystyle\lim_{x\to\infty} f(x)$. Let's first evaluate $\displaystyle\lim_{x\to\infty} \ln\big(f(x)\big)$.

$$\begin{aligned}\lim_{x\to\infty} \ln\big(f(x)\big) &= \lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^x \\ &= \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) \\ &= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{1/x}\end{aligned}$$

This produces the indeterminate form 0/0, so we apply l'Hôpital's Rule.

$$\begin{aligned}&= \lim_{x\to\infty} \frac{\frac{1}{1+1/x}\cdot(-1/x^2)}{(-1/x^2)} \\ &= \lim_{x\to\infty} \frac{1}{1+1/x} \\ &= 1.\end{aligned}$$

Thus $\displaystyle\lim_{x\to\infty} \ln\big(f(x)\big) = 1$. We return to the original limit and apply Key Idea 6.

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln(f(x))} = e^1 = e.$$

Notes:

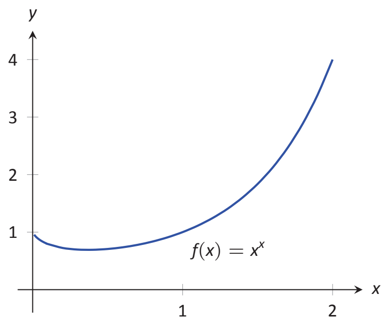

Figure 4.3: A graph of $f(x) = x^x$ supporting the fact that as $x \to 0^+$, $f(x) \to 1$.

2. This limit leads to the indeterminate form 0^0. Let $f(x) = x^x$ and consider first $\lim\limits_{x\to 0^+} \ln\big(f(x)\big)$.

$$\begin{aligned}\lim_{x\to 0^+} \ln\big(f(x)\big) &= \lim_{x\to 0^+} \ln\left(x^x\right)\\ &= \lim_{x\to 0^+} x\ln x\\ &= \lim_{x\to 0^+} \frac{\ln x}{1/x}.\end{aligned}$$

This produces the indeterminate form $-\infty/\infty$ so we apply l'Hôpital's Rule.

$$\begin{aligned}&= \lim_{x\to 0^+} \frac{1/x}{-1/x^2}\\ &= \lim_{x\to 0^+} -x\\ &= 0.\end{aligned}$$

Thus $\lim\limits_{x\to 0^+} \ln\big(f(x)\big) = 0$. We return to the original limit and apply Key Idea 6.

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln(f(x))} = e^0 = 1.$$

This result is supported by the graph of $f(x) = x^x$ given in Figure 4.3.

Notes:

Exercises 4.2

Terms and Concepts

1. List the different indeterminate forms described in this section.

2. T/F: l'Hôpital's Rule provides a faster method of computing derivatives.

3. T/F: l'Hôpital's Rule states that $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)}{g'(x)}$.

4. Explain what the indeterminate form "1^∞" means.

5. Fill in the blanks: The Quotient Rule is applied to $\frac{f(x)}{g(x)}$ when taking ________; l'Hôpital's Rule is applied when taking certain ________.

6. Create (but do not evaluate!) a limit that returns "∞^0".

7. Create a function $f(x)$ such that $\lim\limits_{x\to 1} f(x)$ returns "0^0".

Problems

Exercises 8 – 10 were given back in Section 1.3 as limits that challenge your understanding of the material then. Now compute them much more easily with l'Hôpital's Rule!

8. $\lim\limits_{x\to 0} \frac{\sin(3x)}{x}$

9. $\lim\limits_{x\to 0} \frac{\sin(5x)}{8x}$

10. $\lim\limits_{x\to 0} \frac{\ln(1+x)}{x}$

In Exercises 11 – 74, evaluate the given limit. Note that for some of these, l'Hôpital's Rule is not to be used.

11. $\lim\limits_{x\to 1} \frac{x^2+x-2}{x-1}$

12. $\lim\limits_{x\to 2} \frac{x^3+5}{x+1}$

13. $\lim\limits_{x\to 2} \frac{x^2+x-6}{x^2-7x+10}$

14. $\lim\limits_{x\to 3} \frac{x^2+4x-5}{7x+3}$

15. $\lim\limits_{x\to \pi} \frac{\sin x}{x-\pi}$

16. $\lim\limits_{x\to 0} \frac{\cos x}{x}$

17. $\lim\limits_{x\to \pi/4} \frac{\sin x - \cos x}{\cos(2x)}$

18. $\lim\limits_{x\to 0} \frac{\sin(5x)}{x}$

19. $\lim\limits_{x\to 0} \frac{\sin(2x)}{x+2}$

20. $\lim\limits_{x\to 0} \frac{\sin(2x)}{\sin(3x)}$

21. $\lim\limits_{x\to 0} \frac{\sin(ax)}{\sin(bx)}$

22. $\lim\limits_{x\to 0} \frac{\cos(2x)}{cos(3x)}$

23. $\lim\limits_{x\to 0} \frac{\cos(ax)}{cos(bx)}$

24. $\lim\limits_{x\to 0^+} \frac{e^x-1}{x^2}$

25. $\lim\limits_{x\to 0^+} \frac{e^x-x-1}{x^2}$

26. $\lim\limits_{x\to 0^+} \frac{x-\sin x}{x^3-x^2}$

27. $\lim\limits_{x\to 1} \frac{5^x-1}{x}$

28. $\lim\limits_{x\to 0} \frac{5^x-1}{x}$

29. $\lim\limits_{x\to \infty} \frac{x^4}{e^x}$

30. $\lim\limits_{x\to \infty} \frac{\sqrt{x}}{e^x}$

31. $\lim\limits_{x\to \infty} \frac{e^x}{\sqrt{x}}$

32. $\lim\limits_{x\to \infty} \frac{e^x}{2^x}$

33. $\lim\limits_{x\to \infty} \frac{e^x}{3^x}$

34. $\lim\limits_{x\to 0} \frac{\tan x}{\tan^{-1} x}$

35. $\lim\limits_{x\to 3} \frac{x^3-5x^2+3x+9}{x^3-7x^2+15x-9}$

36. $\lim\limits_{x\to -2} \frac{x^3+4x^2+4x}{x^3+7x^2+16x+12}$

37. $\lim_{x\to 0} \dfrac{3^x - 2^x}{x^2 - x}$

38. $\lim_{x\to\infty} \dfrac{\ln x}{x}$

39. $\lim_{x\to\infty} \dfrac{\ln(x^2)}{x}$

40. $\lim_{x\to\infty} \dfrac{\left(\ln x\right)^2}{x}$

41. $\lim_{x\to 0} \dfrac{x^2}{\ln(\cos x)}$

42. $\lim_{x\to e} \dfrac{\sqrt{\ln x}}{x+4}$

43. $\lim_{x\to\infty} \dfrac{\sqrt{\ln x}}{x+4}$

44. $\lim_{x\to 0^+} x \cdot \ln x$

45. $\lim_{x\to 0^+} \sqrt{x} \cdot \ln x$

46. $\lim_{x\to \pi} x^2 \cos x$

47. $\lim_{x\to 0^+} xe^{1/x}$

48. $\lim_{x\to\infty} x^3 - x^2$

49. $\lim_{x\to\infty} \sqrt{x} - \ln x$

50. $\lim_{x\to -\infty} xe^x$

51. $\lim_{x\to 0^+} \dfrac{1}{x^2} e^{-1/x}$

52. $\lim_{x\to 0^+} (1+x)^{1/x}$

53. $\lim_{x\to\infty} 2x \tan^{-1} x$

54. $\lim_{x\to\infty} 2x \tan^{-1} x - \pi x$

55. $\lim_{x\to 0^+} (2x)^x$

56. $\lim_{x\to 0^+} (2/x)^x$

57. $\lim_{x\to 3^+} (2/x)^x$

58. $\lim_{x\to 0^+} (\sin x)^x$ Hint: use the Squeeze Theorem.

59. $\lim_{x\to 1^+} (1-x)^{1-x}$

60. $\lim_{x\to\infty} (x)^{1/x}$

61. $\lim_{x\to\infty} (1/x)^x$

62. $\lim_{x\to 1^+} (\ln x)^{1-x}$

63. $\lim_{x\to\infty} (1+x)^{1/x}$

64. $\lim_{x\to\infty} (1+x^2)^{1/x}$

65. $\lim_{x\to\infty} x^{-x}$

66. $\lim_{x\to \pi/2} \tan x \cos x$

67. $\lim_{x\to \pi/2} \tan x \sin(2x)$

68. $\lim_{x\to 1^+} \dfrac{1}{\ln x} - \dfrac{1}{x-1}$

69. $\lim_{x\to 1} \dfrac{g(x)}{\ln x}$ where $g(x)$ is the inverse of $f(x) = x + e^x$.

70. $\lim_{x\to 3^+} \dfrac{5}{x^2-9} - \dfrac{x}{x-3}$

71. $\lim_{x\to\infty} x \tan(1/x)$

72. $\lim_{x\to\infty} \dfrac{(\ln x)^3}{x}$

73. $\lim_{x\to 1} \dfrac{x^2+x-2}{\ln x}$

74. $\lim_{x\to\infty} \log_{4x+1}(3x+5)$