14.8 Stokes' Theorem

Just as the spatial Divergence Theorem of the previous section is an extension of the planar Divergence Theorem, Stokes' Theorem is the spatial extension of Green's Theorem. Recall that Green's Theorem states that the circulation of a vector field around a closed curve in the plane is equal to the sum of the curl of the field over the region enclosed by the curve. Stokes' Theorem effectively makes the same statement: given a closed curve that lies on a surface $\mathcal{S}$, the circulation of a vector field around that curve is the same as the sum of "the curl of the field" across the enclosed surface. We use quotes around "the curl of the field" to signify that this statement is not quite correct, as we do not sum curl $\vec{F}$, but curl $\vec{F}\cdot\vec{n}$, where $\vec{n}$ is a unit vector normal to $\mathcal{S}$. That is, we sum the portion of curl $\vec{F}$ that is orthogonal to $\mathcal{S}$ at a point.

Green's Theorem dictated that the curve was to be traversed counterclockwise when measuring circulation. Stokes' Theorem will follow a right hand rule: when the thumb of one's right hand points in the direction of $\vec{n}$, the path C will be traversed in the direction of the curling fingers of the hand (this is equivalent to traversing counterclockwise in the plane).

Theorem 145 Stokes' Theorem

Let $\mathcal{S}$ be a piecewise-smooth, orientable surface whose boundary is a piecewise-smooth curve C, let $\vec{n}$ be a unit vector normal to $\mathcal{S}$, let C be traversed with respect to $\vec{n}$ according to the right hand rule, and let the components of $\vec{F}$ have continuous first partial derivatives over $\mathcal{S}$. Then

$$\oint_C \vec{F}\cdot\, d\vec{r} = \iint_{\mathcal{S}} (\text{curl}\,\vec{F})\cdot\vec{n}\,dS.$$

In general, the best approach to evaluating the surface integral in Stokes' Theorem is to parameterize the surface $\mathcal{S}$ with a function $\vec{r}(u,v)$. We can find a unit normal vector $\vec{n}$ as

$$\vec{n} = \frac{\vec{r}_u\times\vec{r}_v}{||\,\vec{r}_u\times\vec{r}_v\,||}.$$

Since $dS = ||\,\vec{r}_u\times\vec{r}_v\,||\;dA$, the surface integral in practice is evaluated as

$$\iint_{\mathcal{S}} (\text{curl}\,\vec{F})\cdot(\vec{r}_u\times\vec{r}_v)\,dA,$$

where $\vec{r}_u\times\vec{r}_v$ may be replaced by $\vec{r}_v\times\vec{r}_u$ to properly match the direction of this vector with the orientation of the parameterization of C.

Notes:

Example 14.40 Verifying Stokes' Theorem

Considering the planar surface $f(x, y) = 7 - 2x - 2y$, let C be the curve in space that lies on this surface above the circle of radius 1 and centered at $(1, 1)$ in the x-y plane, let $\mathcal{S}$ be the planar region enclosed by C, as illustrated in Figure 14.47, and let $\vec{F} = \langle x+y, 2y, y^2\rangle$. Verify Stokes' Theorem by showing $\oint_C \vec{F} \cdot d\vec{r} = \iint_{\mathcal{S}} (\text{curl}\,\vec{F}) \cdot \vec{n}\, dS$.

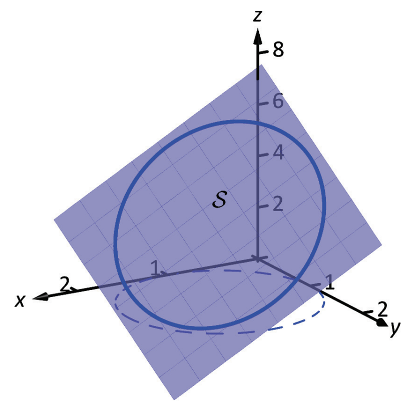


Figure 14.47: As given in Example 14.40, the surface $\mathcal{S}$ is the portion of the plane bounded by the curve.

Solution We begin by parameterizing C and then find the circulation. A unit circle centered at $(1, 1)$ can be parameterized with $x = \cos t + 1$, $y = \sin t + 1$ on $0 \le t \le 2\pi$; to put this curve on the surface f, make the z component equal $f(x, y)$: $z = 7-2(\cos t+1)-2(\sin t+1) = 3-2\cos t-2\sin t$. All together, we parameterize C with $\vec{r}(t) = \langle \cos t + 1, \sin t + 1, 3 - 2\cos t - 2\sin t\rangle$.

The circulation of $\vec{F}$ around C is

$$\begin{aligned}\oint_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \vec{F}\big(\vec{r}(t)\big) \cdot \vec{r}\,'(t)\, dt\\ &= \int_0^{2\pi} \big(2\sin^3 t - 2\cos t\sin^2 t + 3\sin^2 t - 3\cos t\sin t\big)\, dt\\ &= 3\pi.\end{aligned}$$

We now parameterize $\mathcal{S}$. (We reuse the letter "r" for our surface as this is our custom.) Based on the parameterization of C above, we describe $\mathcal{S}$ with $\vec{r}(u, v) = \langle v\cos u + 1, v\sin u + 1, 3 - 2v\cos u - 2v\sin u\rangle$, where $0 \le u \le 2\pi$ and $0 \le v \le 1$.

We leave it to the reader to confirm that $\vec{r}_u \times \vec{r}_v = \langle 2v, 2v, v\rangle$. As $0 \le v \le 1$, this vector always has a non-negative z-component, which the right–hand rule requires given the orientation of C used above. We also leave it to the reader to confirm curl $\vec{F} = \langle 2y, 0, -1\rangle$.

The surface integral of Stokes' Theorem is thus

$$\begin{aligned}\iint_{\mathcal{S}} (\text{curl}\,\vec{F}) \cdot \vec{n}\, dS &= \iint_{\mathcal{S}} (\text{curl}\,\vec{F}) \cdot (\vec{r}_u \times \vec{r}_v)\, dA\\ &= \int_0^1 \int_0^{2\pi} \langle 2v\sin u + 2, 0, -1\rangle \cdot \langle 2v, 2v, v\rangle\, du\, dv\\ &= 3\pi,\end{aligned}$$

which matches our previous result.

One of the interesting results of Stokes' Theorem is that if two surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ share the same boundary, then $\iint_{\mathcal{S}_1} (\text{curl}\,\vec{F}) \cdot \vec{n}\, dS = \iint_{\mathcal{S}_2} (\text{curl}\,\vec{F}) \cdot \vec{n}\, dS$. That is, the value of these two surface integrals is somehow independent of the interior of the surface. We demonstrate this principle in the next example.

Notes:

Example 14.41 Stokes' Theorem and surfaces that share a boundary

Let C be the curve given in Example 14.40 and note that it lies on the surface $z = 6 - x^2 - y^2$. Let $\mathcal{S}$ be the region of this surface bounded by C, and let $\vec{F} = \langle x + y, 2y, y^2\rangle$ as in the previous example. Compute $\iint_{\mathcal{S}} (\text{curl}\,\vec{F}) \cdot \vec{n}\, dS$ to show it equals the result found in the previous example.

Solution We begin by demonstrating that C lies on the surface $z = 6 - x^2 - y^2$. We can parameterize the x and y components of C with $x = \cos t + 1$, $y = \sin t + 1$ as before. Lifting these components to the surface f gives the z component as $z = 6 - x^2 - y^2 = 6 - (\cos t + 1)^2 - (\sin t + 1)^2 = 3 - 2\cos t - 2\sin t$, which is the same z component as found in Example 14.40. Thus the curve C lies on the surface $z = 6 - x^2 - y^2$, as illustrated in Figure 14.48.

Since C and $\vec{F}$ are the same as in the previous example, we already know that $\oint_C \vec{F} \cdot d\vec{r} = 3\pi$. We confirm that this is also the value of $\iint_{\mathcal{S}} (\text{curl}\,\vec{F}) \cdot \vec{n}\, dS$.

We parameterize $\mathcal{S}$ with

$$\vec{r}(u, v) = \langle v\cos u + 1, v\sin u + 1, 6 - (v\cos u + 1)^2 - (v\sin u + 1)^2\rangle,$$

where $0 \le u \le 2\pi$ and $0 \le v \le 1$, and leave it to the reader to confirm that

$$\vec{r}_u \times \vec{r}_v = \big\langle 2v\big(v\cos u + 1\big), 2v\big(v\sin u + 1\big), v\big\rangle\,,$$

which also conforms to the right–hand rule with regard to the orientation of C. With $\text{curl}\,\vec{F} = \langle 2y, 0, -1\rangle$ as before, we have

$$\iint_{\mathcal{S}} (\text{curl}\,\vec{F}) \cdot \vec{n}\, dS =$$

$$\int_0^1 \int_0^{2\pi} \langle 2v\sin u + 2, 0, -1\rangle \cdot \big\langle 2v\big(v\cos u + 1\big), 2v\big(v\sin u + 1\big), v\big\rangle\, du\, dv =$$

$$3\pi.$$

Even though the surfaces used in this example and in Example 14.40 are very different, because they share the same boundary, Stokes' Theorem guarantees they have equal "sum of curls" across their respective surfaces.

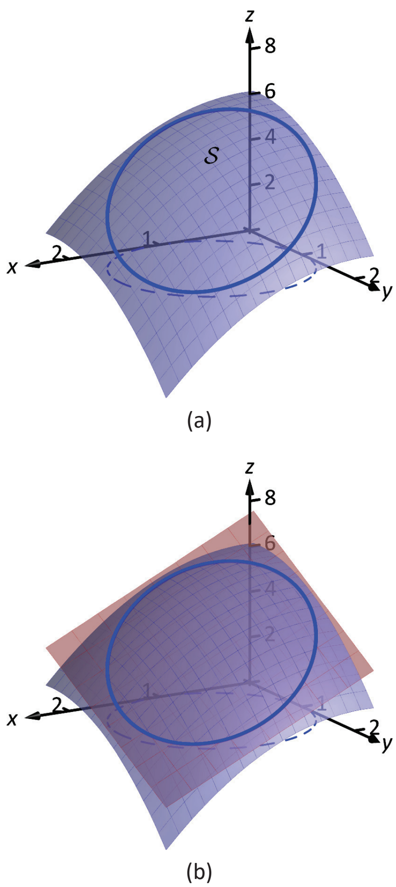


Figure 14.48: As given in Example 14.41, the surface $\mathcal{S}$ is the portion of the plane bounded by the curve.

Notes:

A Common Thread of Calculus

We have threefold interest in each of the major theorems of this chapter: the Fundamental Theorem of Line Integrals, Green's, Stokes' and the Divergence Theorems. First, we find the beauty of their truth interesting. Second, each provides two methods of computing a desired quantity, sometimes offering a simpler method of computation.

There is yet one more reason of interest in the major theorems of this chapter. These important theorems also all share an important principle with the Fundamental Theorem of Calculus, introduced in Chapter 5.

Revisit this fundamental theorem, adopting the notation used heavily in this chapter. Let I be the interval $[a, b]$ and let $y = F(x)$ be differentiable on I, with $F'(x) = f(x)$. The Fundamental Theorem of Calculus states that

$$\int_I f(x)\, dx = F(b) - F(a).$$

That is, the sum of the rates of change of a function F over an interval I can also be calculated with a certain sum of F itself on the boundary of I (in this case, at the points $x = a$ and $x = b$).

Each of the named theorems above can be expressed in similar terms. Consider the Fundamental Theorem of Line Integrals: given a function $z = f(x, y)$, the gradient ∇f is a type of rate of change of f. Given a curve C with initial and terminal points A and B, respectively, this fundamental theorem states that

$$\int_C \nabla f\, ds = f(B) - f(A),$$

where again the sum of a rate of change of f along a curve C can also be evaluated by a certain sum of f at the boundary of C (i.e., the points A and B).

Green's Theorem is essentially a special case of Stokes' Theorem, so we consider just Stokes' Theorem here. Recalling that the curl of a vector field $\vec{F}$ is a measure of a rate of change of $\vec{F}$, Stokes' Theorem states that over a surface $\mathcal{S}$ bounded by a closed curve C,

$$\iint_{\mathcal{S}} \left(\operatorname{curl} \vec{F}\right) \cdot \vec{n}\, dS = \oint_C \vec{F} \cdot d\vec{r},$$

i.e., the sum of a rate of change of $\vec{F}$ can be calculated with a certain sum of $\vec{F}$ itself over the boundary of $\mathcal{S}$. In this case, the latter sum is also an infinite sum, requiring an integral.

Finally, the Divergence Theorems state that the sum of divergences of a vector field (another measure of a rate of change of $\vec{F}$) over a region can also be computed with a certain sum of $\vec{F}$ over the boundary of that region. When the

Notes:

region is planar, the latter sum of $\vec{F}$ is an integral; when the region is spatial, the latter sum of $\vec{F}$ is a double integral.

The common thread among these theorems: the sum of a rate of change of a function over a region can be computed as another sum of the function itself on the boundary of the region. While very general, this is a very powerful and important statement.

Notes:

Exercises 14.8

Terms and Concepts

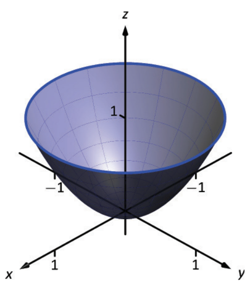
1. What property of a vector field does Stokes' Theorem relate to circulation?

2. Stokes' Theorem is the spatial version of what other theorem?

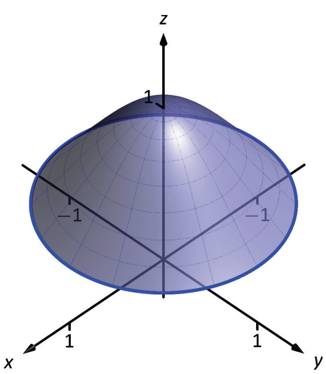
Problems

In Exercises 3 – 6, a closed curve C that is the boundary of a surface $\mathcal{S}$ is given along with a vector field $\vec{F}$. Verify Stokes' Theorem on C; that is, show $\oint_C \vec{F} \cdot d\vec{r} = \iint_{\mathcal{S}} \left(\text{curl}\,\vec{F}\right) \cdot \vec{n}\, dS$.

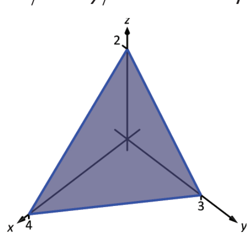
3. C is the curve parameterized by $\vec{r}(t) = \langle \cos t, \sin t, 1\rangle$ and $\mathcal{S}$ is the portion of $z = x^2 + y^2$ enclosed by C; $\vec{F} = \langle z, -x, y\rangle$.



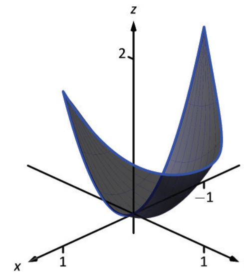
4. C is the curve parameterized by $\vec{r}(t) = \langle \cos t, \sin t, e^{-1}\rangle$ and $\mathcal{S}$ is the portion of $z = e^{-x^2-y^2}$ enclosed by C; $\vec{F} = \langle -y, x, 1\rangle$.



5. C is the curve that follows the triangle with vertices at $(0, 0, 2)$, $(4, 0, 0)$ and $(0, 3, 0)$, traversing the the vertices in that order and returning to $(0, 0, 2)$, and $\mathcal{S}$ is the portion of the plane $z = 2 - x/2 - 2y/3$ enclosed by C; $\vec{F} = \langle y, -z, y\rangle$.

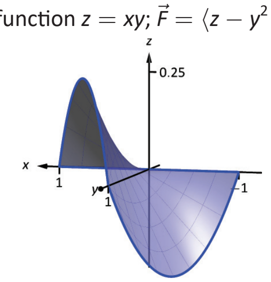


6. C is the curve whose x and y coordinates follow the parabola $y = 1 - x^2$ from $x = 1$ to $x = -1$, then follow the line from $(-1, 0)$ back to $(1, 0)$, where the z coordinates of C are determined by $f(x, y) = 2x^2 + y^2$, and $\mathcal{S}$ is the portion of $z = 2x^2 + y^2$ enclosed by C; $\vec{F} = \langle y^2 + z, x, x^2 - y\rangle$.

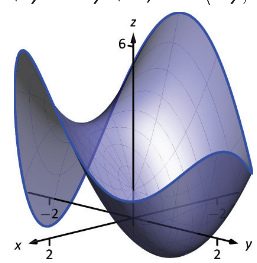


In Exercises 7 – 10, a closed curve C that is the boundary of a surface $\mathcal{S}$ is given along with a vector field $\vec{F}$. Find the circulation of $\vec{F}$ around C either through direct computation or through Stokes' Theorem.

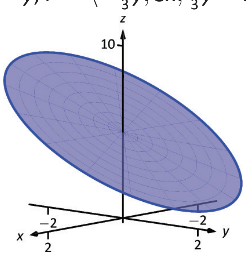
7. C is the curve whose x- and y-values are determined by the three sides of a triangle with vertices at $(-1, 0)$, $(1, 0)$ and $(0, 1)$, traversed in that order, and the z-values are determined by the function $z = xy$; $\vec{F} = \langle z - y^2, x, z\rangle$.



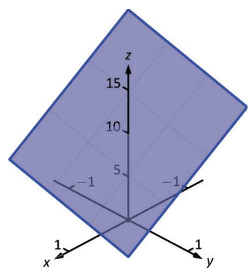
8. C is the curve whose x- and y-values are given by $\vec{r}(t) = \langle 2\cos t, 2\sin t\rangle$ and the z-values are determined by the function $z = x^2 + y^3 - 3y + 1$; $\vec{F} = \langle -y, x, z\rangle$.



9. C is the curve whose x- and y-values are given by $\vec{r}(t) = \langle \cos t, 3\sin t\rangle$ and the z-values are determined by the function $z = 5 - 2x - y$; $\vec{F} = \langle -\frac{1}{3}y, 3x, \frac{2}{3}y - 3x\rangle$.



10. C is the curve whose x- and y-values are sides of the square with vertices at $(1,1)$, $(-1,1)$, $(-1,-1)$ and $(1,-1)$, traversed in that order, and the z-values are determined by the function $z = 10 - 5x - 2y$; $\vec{F} = \langle 5y^2, 2y^2, y^2 \rangle$.



Exercises 11 – 12 are designed to challenge your understanding and require no computation.

11. (a) Green's Theorem can be used to find the area of a region enclosed by a curve by evaluating a line integral with the appropriate choice of vector field $\vec{F}$. What condition on $\vec{F}$ makes this possible?

    (b) Likewise, Stokes' Theorem can be used to find the surface area of a region enclosed by a curve in space by evaluating a line integral with the appropriate choice of vector field $\vec{F}$. What condition on $\vec{F}$ makes this possible?

12. Stokes' Theorem establishes equality between a particular line integral and a particular double integral. What types of circumstances would lead one to choose to evaluate the double integral over the line integral?