

6.6 Improper Integrals

We begin this section by considering the following definite integrals:

- $\int_0^{100} \frac{1}{1+x^2} dx \approx 1.5608,$
- $\int_0^{1000} \frac{1}{1+x^2} dx \approx 1.5698,$
- $\int_0^{10,000} \frac{1}{1+x^2} dx \approx 1.5707.$

Notice how the integrand is $1/(1+x^2)$ in each integral (which is sketched in Figure 6.15). As the upper bound gets larger, one would expect the “area under the curve” would also grow. While the definite integrals do increase in value as the upper bound grows, they are not increasing by much. In fact, consider:

$$\int_0^b \frac{1}{1+x^2} dx = \tan^{-1} x \Big|_0^b = \tan^{-1} b - \tan^{-1} 0 = \tan^{-1} b.$$

As $b \rightarrow \infty$, $\tan^{-1} b \rightarrow \pi/2$. Therefore it seems that as the upper bound b grows, the value of the definite integral $\int_0^b \frac{1}{1+x^2} dx$ approaches $\pi/2 \approx 1.5708$. This should strike the reader as being a bit amazing: even though the curve extends “to infinity,” it has a finite amount of area underneath it.

When we defined the definite integral $\int_a^b f(x) dx$, we made two stipulations:

1. The interval over which we integrated, $[a, b]$, was a finite interval, and
2. The function $f(x)$ was continuous on $[a, b]$ (ensuring that the range of f was finite).

In this section we consider integrals where one or both of the above conditions do not hold. Such integrals are called **improper integrals**.

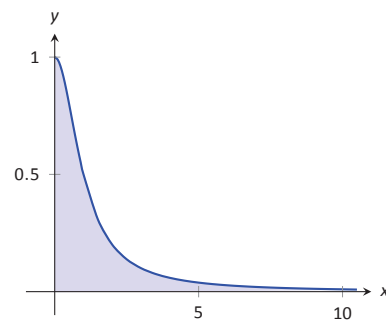


Figure 6.15: Graphing $f(x) = \frac{1}{1+x^2}$.

Notes:

Improper Integrals with Infinite Bounds

Definition 24 Improper Integrals with Infinite Bounds; Converge, Diverge

1. Let f be a continuous function on $[a, \infty)$. Define

$$\int_a^\infty f(x) \, dx \quad \text{to be} \quad \lim_{b \rightarrow \infty} \int_a^b f(x) \, dx.$$

2. Let f be a continuous function on $(-\infty, b]$. Define

$$\int_{-\infty}^b f(x) \, dx \quad \text{to be} \quad \lim_{a \rightarrow -\infty} \int_a^b f(x) \, dx.$$

3. Let f be a continuous function on $(-\infty, \infty)$. Let c be any real number; define

$$\int_{-\infty}^\infty f(x) \, dx \quad \text{to be} \quad \lim_{a \rightarrow -\infty} \int_a^c f(x) \, dx + \lim_{b \rightarrow \infty} \int_c^b f(x) \, dx.$$

An improper integral is said to **converge** if its corresponding limit exists; otherwise, it **diverges**. The improper integral in part 3 converges if and only if both of its limits exist.

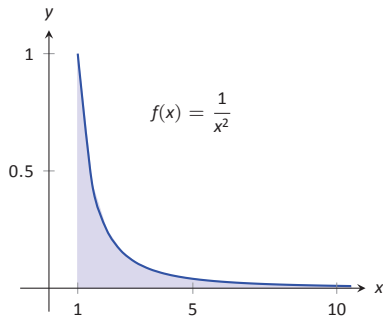


Figure 6.16: A graph of $f(x) = \frac{1}{x^2}$ in Example 6.32.

Example 6.32 Evaluating improper integrals

Evaluate the following improper integrals.

1. $\int_1^\infty \frac{1}{x^2} \, dx$

3. $\int_{-\infty}^0 e^x \, dx$

2. $\int_1^\infty \frac{1}{x} \, dx$

4. $\int_{-\infty}^\infty \frac{1}{1+x^2} \, dx$

SOLUTION

$$\begin{aligned} 1. \quad \int_1^\infty \frac{1}{x^2} \, dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^2} \, dx = \lim_{b \rightarrow \infty} \left. \frac{-1}{x} \right|_1^b \\ &= \lim_{b \rightarrow \infty} \frac{-1}{b} + 1 \\ &= 1. \end{aligned}$$

A graph of the area defined by this integral is given in Figure 6.16.

Notes:

$$\begin{aligned}
 2. \quad \int_1^{\infty} \frac{1}{x} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x} dx \\
 &= \lim_{b \rightarrow \infty} \ln |x| \Big|_1^b \\
 &= \lim_{b \rightarrow \infty} \ln(b) \\
 &= \infty.
 \end{aligned}$$

The limit does not exist, hence the improper integral $\int_1^{\infty} \frac{1}{x} dx$ diverges. Compare the graphs in Figures 6.16 and 6.17; notice how the graph of $f(x) = 1/x$ is noticeably larger. This difference is enough to cause the improper integral to diverge.

$$\begin{aligned}
 3. \quad \int_{-\infty}^0 e^x dx &= \lim_{a \rightarrow -\infty} \int_a^0 e^x dx \\
 &= \lim_{a \rightarrow -\infty} e^x \Big|_a^0 \\
 &= \lim_{a \rightarrow -\infty} e^0 - e^a \\
 &= 1.
 \end{aligned}$$

A graph of the area defined by this integral is given in Figure 6.18.

4. We will need to break this into two improper integrals and choose a value of c as in part 3 of Definition 24. Any value of c is fine; we choose $c = 0$.

$$\begin{aligned}
 \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx &= \lim_{a \rightarrow -\infty} \int_a^0 \frac{1}{1+x^2} dx + \lim_{b \rightarrow \infty} \int_0^b \frac{1}{1+x^2} dx \\
 &= \lim_{a \rightarrow -\infty} \tan^{-1} x \Big|_a^0 + \lim_{b \rightarrow \infty} \tan^{-1} x \Big|_0^b \\
 &= \lim_{a \rightarrow -\infty} (\tan^{-1} 0 - \tan^{-1} a) + \lim_{b \rightarrow \infty} (\tan^{-1} b - \tan^{-1} 0) \\
 &= \left(0 - \frac{-\pi}{2}\right) + \left(\frac{\pi}{2} - 0\right).
 \end{aligned}$$

Each limit exists, hence the original integral converges and has value:

$$= \pi.$$

A graph of the area defined by this integral is given in Figure 6.19.

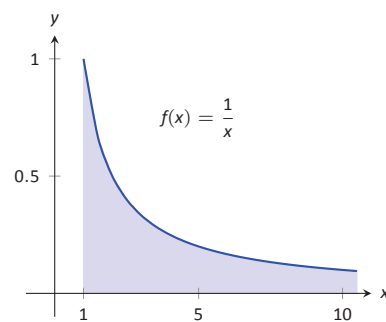


Figure 6.17: A graph of $f(x) = \frac{1}{x}$ in Example 6.32.

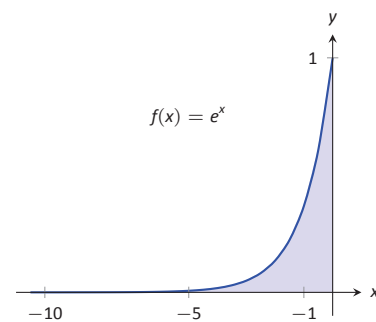


Figure 6.18: A graph of $f(x) = e^x$ in Example 6.32.

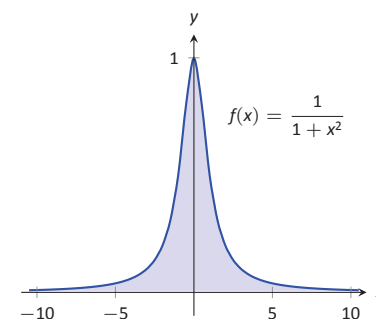


Figure 6.19: A graph of $f(x) = \frac{1}{1+x^2}$ in Example 6.32.

Notes:

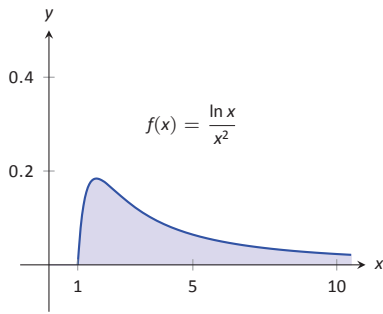


Figure 6.20: A graph of $f(x) = \frac{\ln x}{x^2}$ in Example 6.33.

The previous section introduced l'Hôpital's Rule, a method of evaluating limits that return indeterminate forms. It is not uncommon for the limits resulting from improper integrals to need this rule as demonstrated next.

Example 6.33 Improper integrals and l'Hôpital's Rule

Evaluate the improper integral $\int_1^{\infty} \frac{\ln x}{x^2} dx$.

SOLUTION This integral will require the use of Integration by Parts. Let $u = \ln x$ and $dv = 1/x^2 dx$. Then

$$\begin{aligned} \int_1^{\infty} \frac{\ln x}{x^2} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{\ln x}{x^2} dx \\ &= \lim_{b \rightarrow \infty} \left(-\frac{\ln x}{x} \Big|_1^b + \int_1^b \frac{1}{x^2} dx \right) \\ &= \lim_{b \rightarrow \infty} \left(-\frac{\ln x}{x} - \frac{1}{x} \right) \Big|_1^b \\ &= \lim_{b \rightarrow \infty} \left(-\frac{\ln b}{b} - \frac{1}{b} - (-\ln 1 - 1) \right). \end{aligned}$$

The $1/b$ and $\ln 1$ terms go to 0, leaving $\lim_{b \rightarrow \infty} -\frac{\ln b}{b} + 1$. We need to evaluate $\lim_{b \rightarrow \infty} \frac{\ln b}{b}$ with l'Hôpital's Rule. We have:

$$\begin{aligned} \lim_{b \rightarrow \infty} \frac{\ln b}{b} &\stackrel{\text{by LHR}}{=} \lim_{b \rightarrow \infty} \frac{1/b}{1} \\ &= 0. \end{aligned}$$

Thus the improper integral evaluates as:

$$\int_1^{\infty} \frac{\ln x}{x^2} dx = 1.$$

Improper Integrals with Infinite Range

We have just considered definite integrals where the interval of integration was infinite. We now consider another type of improper integral, where the range of the integrand is infinite.

Notes:

Definition 25 Improper Integrals with Infinite Range

Let $f(x)$ be a continuous function on $[a, b]$ except at c , $a \leq c \leq b$, where $x = c$ is a vertical asymptote of f . Define

$$\int_a^b f(x) dx = \lim_{t \rightarrow c^-} \int_a^t f(x) dx + \lim_{t \rightarrow c^+} \int_t^b f(x) dx.$$

Note: In Definition 25, c can be one of the endpoints (a or b). In that case, there is only one limit to consider as part of the definition.

Example 6.34 Improper integration of functions with infinite range

Evaluate the following improper integrals:

$$1. \int_0^1 \frac{1}{\sqrt{x}} dx \quad 2. \int_{-1}^1 \frac{1}{x^2} dx.$$

SOLUTION

1. A graph of $f(x) = 1/\sqrt{x}$ is given in Figure 6.21. Notice that f has a vertical asymptote at $x = 0$; in some sense, we are trying to compute the area of a region that has no “top.” Could this have a finite value?

$$\begin{aligned} \int_0^1 \frac{1}{\sqrt{x}} dx &= \lim_{a \rightarrow 0^+} \int_a^1 \frac{1}{\sqrt{x}} dx \\ &= \lim_{a \rightarrow 0^+} 2\sqrt{x} \Big|_a^1 \\ &= \lim_{a \rightarrow 0^+} 2(\sqrt{1} - \sqrt{a}) \\ &= 2. \end{aligned}$$

It turns out that the region does have a finite area even though it has no upper bound (strange things can occur in mathematics when considering the infinite).

2. The function $f(x) = 1/x^2$ has a vertical asymptote at $x = 0$, as shown in Figure 6.22, so this integral is an improper integral. Let’s eschew using limits for a moment and proceed without recognizing the improper nature of the integral. This leads to:

$$\begin{aligned} \int_{-1}^1 \frac{1}{x^2} dx &= -\frac{1}{x} \Big|_{-1}^1 \\ &= -1 - (1) \\ &= -2! \end{aligned}$$

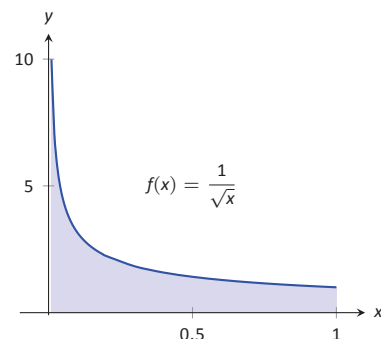


Figure 6.21: A graph of $f(x) = \frac{1}{\sqrt{x}}$ in Example 6.34.

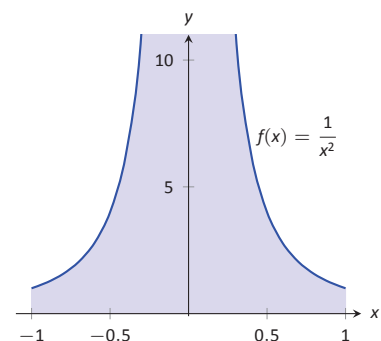


Figure 6.22: A graph of $f(x) = \frac{1}{x^2}$ in Example 6.34.

Notes:

Clearly the area in question is above the x -axis, yet the area is supposedly negative! Why does our answer not match our intuition? To answer this, evaluate the integral using Definition 25.

$$\begin{aligned}
 \int_{-1}^1 \frac{1}{x^2} dx &= \lim_{t \rightarrow 0^-} \int_{-1}^t \frac{1}{x^2} dx + \lim_{t \rightarrow 0^+} \int_t^1 \frac{1}{x^2} dx \\
 &= \lim_{t \rightarrow 0^-} -\frac{1}{x} \Big|_{-1}^t + \lim_{t \rightarrow 0^+} -\frac{1}{x} \Big|_t^1 \\
 &= \lim_{t \rightarrow 0^-} -\frac{1}{t} - 1 + \lim_{t \rightarrow 0^+} -1 + \frac{1}{t} \\
 &\Rightarrow (\infty - 1) + (-1 + \infty).
 \end{aligned}$$

Neither limit converges hence the original improper integral diverges. The nonsensical answer we obtained by ignoring the improper nature of the integral is just that: nonsensical.

Understanding Convergence and Divergence

Oftentimes we are interested in knowing simply whether or not an improper integral converges, and not necessarily the value of a convergent integral. We provide here several tools that help determine the convergence or divergence of improper integrals without integrating.

Our first tool is to understand the behavior of functions of the form $\frac{1}{x^p}$.

Example 6.35 Improper integrals of $1/x^p$

Determine the values of p for which $\int_1^\infty \frac{1}{x^p} dx$ converges.

SOLUTION We begin by integrating and then evaluating the limit.

$$\begin{aligned}
 \int_1^\infty \frac{1}{x^p} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^p} dx \\
 &= \lim_{b \rightarrow \infty} \int_1^b x^{-p} dx \quad (\text{assume } p \neq 1) \\
 &= \lim_{b \rightarrow \infty} \frac{1}{-p+1} x^{-p+1} \Big|_1^b \\
 &= \lim_{b \rightarrow \infty} \frac{1}{1-p} (b^{1-p} - 1^{1-p}).
 \end{aligned}$$

When does this limit converge – i.e., when is this limit *not* ∞ ? This limit converges precisely when the power of b is less than 0: when $1 - p < 0 \Rightarrow 1 < p$.

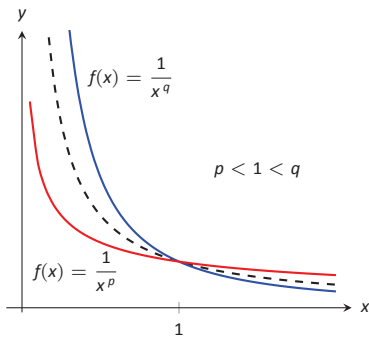


Figure 6.23: Plotting functions of the form $1/x^p$ in Example 6.35.

Notes:

Our analysis shows that if $p > 1$, then $\int_1^\infty \frac{1}{x^p} dx$ converges. When $p < 1$ the improper integral diverges; we showed in Example 6.32 that when $p = 1$ the integral also diverges.

Figure 6.23 graphs $y = 1/x$ with a dashed line, along with graphs of $y = 1/x^p$, $p < 1$, and $y = 1/x^q$, $q > 1$. Somehow the dashed line forms a dividing line between convergence and divergence.

The result of Example 6.35 provides an important tool in determining the convergence of other integrals. A similar result is proved in the exercises about improper integrals of the form $\int_0^1 \frac{1}{x^p} dx$. It can be similarly shown that replacing the bound 1 with any positive real number in these integrals does **not** change if it converges or diverges (though it can change what the integral converges to). These results are summarized in the following Key Idea.

Key Idea 22 **Convergence of Improper Integrals** $\int_a^\infty \frac{1}{x^p} dx$ and $\int_0^b \frac{1}{x^p} dx$

1. For $a > 0$, the improper integral $\int_a^\infty \frac{1}{x^p} dx$ converges when $p > 1$ and diverges when $p \leq 1$.
2. For $b > 0$, the improper integral $\int_0^b \frac{1}{x^p} dx$ converges when $p < 1$ and diverges when $p \geq 1$.

A basic technique in determining convergence of improper integrals is to compare an integrand whose convergence is unknown to an integrand whose convergence is known. We often use integrands of the form $1/x^p$ to compare to as their convergence on certain intervals is known. This is described in the following theorem.

Notes:

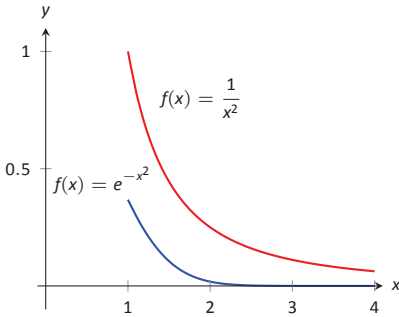


Figure 6.24: Graphs of $f(x) = e^{-x^2}$ and $f(x) = 1/x^2$ in Example 6.36.

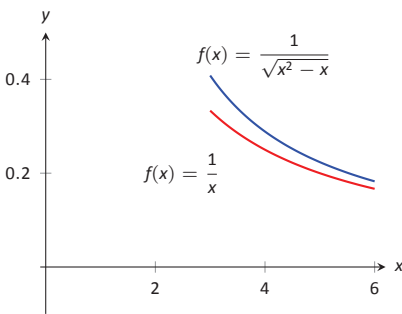


Figure 6.25: Graphs of $f(x) = 1/\sqrt{x^2 - x}$ and $f(x) = 1/x$ in Example 6.36.

Theorem 51 Direct Comparison Test for Improper Integrals

Let f and g be continuous on $[a, \infty)$ where $0 \leq f(x) \leq g(x)$ for all x in $[a, \infty)$.

1. If $\int_a^\infty g(x) dx$ converges, then $\int_a^\infty f(x) dx$ converges.
2. If $\int_a^\infty f(x) dx$ diverges, then $\int_a^\infty g(x) dx$ diverges.

Example 6.36 Determining convergence of improper integrals

Determine the convergence of the following improper integrals.

1. $\int_1^\infty e^{-x^2} dx$
2. $\int_3^\infty \frac{1}{\sqrt{x^2 - x}} dx$

SOLUTION

1. The function $f(x) = e^{-x^2}$ does not have an antiderivative expressible in terms of elementary functions, so we cannot integrate directly. It is comparable to $g(x) = 1/x^2$, and as demonstrated in Figure 6.24, $e^{-x^2} < 1/x^2$ on $[1, \infty)$. We know from Key Idea 22 that $\int_1^\infty \frac{1}{x^2} dx$ converges, hence $\int_1^\infty e^{-x^2} dx$ also converges.

2. Note that for large values of x , $\frac{1}{\sqrt{x^2 - x}} \approx \frac{1}{\sqrt{x^2}} = \frac{1}{x}$. We know from Key Idea 22 and the subsequent note that $\int_3^\infty \frac{1}{x} dx$ diverges, so we seek to compare the original integrand to $1/x$.

It is easy to see that when $x > 0$, we have $x = \sqrt{x^2} > \sqrt{x^2 - x}$. Taking reciprocals reverses the inequality, giving

$$\frac{1}{x} < \frac{1}{\sqrt{x^2 - x}}.$$

Using Theorem 51, we conclude that since $\int_3^\infty \frac{1}{x} dx$ diverges, $\int_3^\infty \frac{1}{\sqrt{x^2 - x}} dx$ diverges as well. Figure 6.25 illustrates this.

Notes:

Being able to compare “unknown” integrals to “known” integrals is very useful in determining convergence. However, some of our examples were a little “too nice.” For instance, it was convenient that $\frac{1}{x} < \frac{1}{\sqrt{x^2 - x}}$, but what if the “ $-x$ ” were replaced with a “ $+2x + 5$ ”? That is, what can we say about the convergence of $\int_3^\infty \frac{1}{\sqrt{x^2 + 2x + 5}} dx$? We have $\frac{1}{x} > \frac{1}{\sqrt{x^2 + 2x + 5}}$, so we cannot use Theorem 51.

In cases like this (and many more) it is useful to employ the following theorem.

Theorem 52 Limit Comparison Test for Improper Integrals

Let f and g be continuous functions on $[a, \infty)$ where $f(x) > 0$ and $g(x) > 0$ for all x . If

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty,$$

then

$$\int_a^\infty f(x) dx \quad \text{and} \quad \int_a^\infty g(x) dx$$

either both converge or both diverge.

Example 6.37 Determining convergence of improper integrals

Determine the convergence of $\int_3^\infty \frac{1}{\sqrt{x^2 + 2x + 5}} dx$.

SOLUTION As x gets large, the quadratic inside the square root function will begin to behave much like $y = x$. So we compare $\frac{1}{\sqrt{x^2 + 2x + 5}}$ to $\frac{1}{x}$ with the Limit Comparison Test:

$$\lim_{x \rightarrow \infty} \frac{1/\sqrt{x^2 + 2x + 5}}{1/x} = \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 2x + 5}}.$$

The immediate evaluation of this limit returns ∞/∞ , an indeterminate form. Using l'Hôpital's Rule seems appropriate, but in this situation, it does not lead to useful results. (We encourage the reader to employ l'Hôpital's Rule at least once to verify this.)

The trouble is the square root function. To get rid of it, we employ the following fact: If $\lim_{x \rightarrow c} f(x) = L$, then $\lim_{x \rightarrow c} f(x)^2 = L^2$. (This is true when either c or L

Notes:

is ∞ .) So we consider now the limit

$$\lim_{x \rightarrow \infty} \frac{x^2}{x^2 + 2x + 5}.$$

This converges to 1, meaning the original limit also converged to 1. As x gets very large, the function $\frac{1}{\sqrt{x^2 + 2x + 5}}$ looks very much like $\frac{1}{x}$. Since we know that

$\int_3^{\infty} \frac{1}{x} dx$ diverges, by the Limit Comparison Test we know that $\int_3^{\infty} \frac{1}{\sqrt{x^2 + 2x + 5}} dx$ also diverges. Figure 6.26 graphs $f(x) = 1/\sqrt{x^2 + 2x + 5}$ and $f(x) = 1/x$, illustrating that as x gets large, the functions become indistinguishable.

Both the Direct and Limit Comparison Tests were given in terms of integrals over an infinite interval. There are versions that apply to improper integrals with an infinite range, but as they are a bit wordy and a little more difficult to employ, they are omitted from this text.

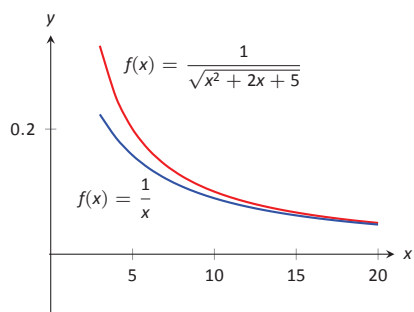


Figure 6.26: Graphing $f(x) = \frac{1}{\sqrt{x^2 + 2x + 5}}$ and $f(x) = \frac{1}{x}$ in Example 6.37.

Notes:

Exercises 6.6

Terms and Concepts

1. The definite integral was defined with what two stipulations?
2. If $\lim_{b \rightarrow \infty} \int_0^b f(x) dx$ exists, then the integral $\int_0^\infty f(x) dx$ is said to _____.
3. If $\int_1^\infty f(x) dx = 10$, and $0 \leq g(x) \leq f(x)$ for all x , then we know that $\int_1^\infty g(x) dx$ _____.
4. For what values of p will $\int_1^\infty \frac{1}{x^p} dx$ converge?
5. For what values of p will $\int_{10}^\infty \frac{1}{x^p} dx$ converge?
6. For what values of p will $\int_0^1 \frac{1}{x^p} dx$ converge?

Problems

In Exercises 7 – 33, evaluate the given improper integral.

7. $\int_0^\infty e^{5-2x} dx$
8. $\int_1^\infty \frac{1}{x^3} dx$
9. $\int_1^\infty x^{-4} dx$
10. $\int_{-\infty}^\infty \frac{1}{x^2 + 9} dx$
11. $\int_{-\infty}^0 2^x dx$
12. $\int_{-\infty}^0 \left(\frac{1}{2}\right)^x dx$
13. $\int_{-\infty}^\infty \frac{x}{x^2 + 1} dx$
14. $\int_{-\infty}^\infty \frac{x}{x^2 + 4} dx$
15. $\int_2^\infty \frac{1}{(x-1)^2} dx$
16. $\int_1^2 \frac{1}{(x-1)^2} dx$
17. $\int_2^\infty \frac{1}{x-1} dx$
18. $\int_1^2 \frac{1}{x-1} dx$
19. $\int_{-1}^1 \frac{1}{x} dx$
20. $\int_1^3 \frac{1}{x-2} dx$
21. $\int_0^\pi \sec^2 x dx$
22. $\int_{-2}^1 \frac{1}{\sqrt{|x|}} dx$
23. $\int_0^\infty xe^{-x} dx$
24. $\int_0^\infty xe^{-x^2} dx$
25. $\int_{-\infty}^\infty xe^{-x^2} dx$
26. $\int_{-\infty}^\infty \frac{1}{e^x + e^{-x}} dx$
27. $\int_0^1 x \ln x dx$
28. $\int_1^\infty \frac{\ln x}{x} dx$
29. $\int_0^1 \ln x dx$
30. $\int_1^\infty \frac{\ln x}{x^2} dx$
31. $\int_1^\infty \frac{\ln x}{\sqrt{x}} dx$
32. $\int_0^\infty e^{-x} \sin x dx$
33. $\int_0^\infty e^{-x} \cos x dx$

In Exercises 34 – 43, use the Direct Comparison Test or the Limit Comparison Test to determine whether the given definite integral converges or diverges. Clearly state what test is being used and what function the integrand is being compared to.

$$34. \int_{10}^{\infty} \frac{3}{\sqrt{3x^2 + 2x - 5}} dx$$

$$35. \int_2^{\infty} \frac{4}{\sqrt{7x^3 - x}} dx$$

$$36. \int_0^{\infty} \frac{\sqrt{x+3}}{\sqrt{x^3 - x^2 + x + 1}} dx$$

$$37. \int_1^{\infty} e^{-x} \ln x dx$$

$$38. \int_5^{\infty} e^{-x^2 + 3x + 1} dx$$

$$39. \int_0^{\infty} \frac{\sqrt{x}}{e^x} dx$$

$$40. \int_2^{\infty} \frac{1}{x^2 + \sin x} dx$$

$$41. \int_0^{\infty} \frac{x}{x^2 + \cos x} dx$$

$$42. \int_0^{\infty} \frac{1}{x + e^x} dx$$

$$43. \int_0^{\infty} \frac{1}{e^x - x} dx$$