

10.5 Lines

To find the equation of a line in the x - y plane, we need two pieces of information: a point and the slope. The slope conveys *direction* information. As vertical lines have an undefined slope, the following statement is more accurate:

To define a line, one needs a point on the line and the direction of the line.

This holds true for lines in space.

Let P be a point in space, let \vec{p} be the vector with initial point at the origin and terminal point at P (i.e., \vec{p} “points” to P), and let \vec{d} be a vector. Consider the points on the line through P in the direction of \vec{d} .

Clearly one point on the line is P ; we can say that the vector \vec{p} lies at this point on the line. To find another point on the line, we can start at \vec{p} and move in a direction parallel to \vec{d} . For instance, starting at \vec{p} and traveling one length of \vec{d} places one at another point on the line. Consider Figure 10.47 where certain points along the line are indicated.

The figure illustrates how every point on the line can be obtained by starting with \vec{p} and moving a certain distance in the direction of \vec{d} . That is, we can define the line as a function of t :

$$\vec{\ell}(t) = \vec{p} + t\vec{d}. \quad (10.7)$$

In many ways, this is *not* a new concept. Compare Equation (10.7) to the familiar “ $y = mx + b$ ” equation of a line:

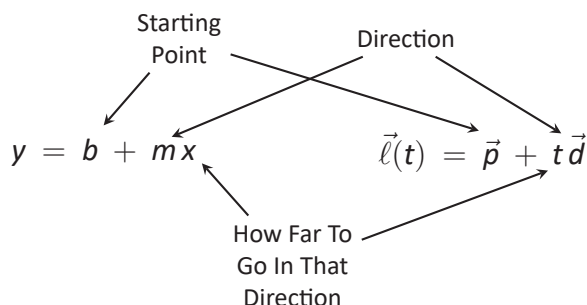


Figure 10.46: Understanding the vector equation of a line.

The equations exhibit the same structure: they give a starting point, define a direction, and state how far in that direction to travel.

Equation (10.7) is an example of a **vector-valued function**; the input of the function is a real number and the output is a vector. We will cover vector-valued functions extensively in the next chapter.

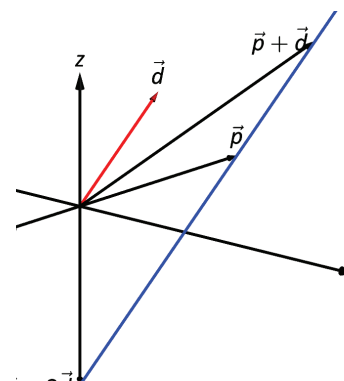


Figure 10.47: Defining a line in space.

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There are other ways to represent a line. Let $\vec{p} = \langle x_0, y_0, z_0 \rangle$ and let $\vec{d} = \langle a, b, c \rangle$. Then the equation of the line through \vec{p} in the direction of \vec{d} is:

$$\begin{aligned}\vec{\ell}(t) &= \vec{p} + t\vec{d} \\ &= \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle \\ &= \langle x_0 + at, y_0 + bt, z_0 + ct \rangle.\end{aligned}$$

The last line states that the x values of the line are given by $x = x_0 + at$, the y values are given by $y = y_0 + bt$, and the z values are given by $z = z_0 + ct$. These three equations, taken together, are the **parametric equations of the line** through \vec{p} in the direction of \vec{d} .

Finally, each of the equations for x , y and z above contain the variable t . We can solve for t in each equation:

$$\begin{aligned}x = x_0 + at &\Rightarrow t = \frac{x - x_0}{a}, \\ y = y_0 + bt &\Rightarrow t = \frac{y - y_0}{b}, \\ z = z_0 + ct &\Rightarrow t = \frac{z - z_0}{c},\end{aligned}$$

assuming $a, b, c \neq 0$. Since t is equal to each expression on the right, we can set these equal to each other, forming the **symmetric equations of the line** through \vec{p} in the direction of \vec{d} :

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Each representation has its own advantages, depending on the context. We summarize these three forms in the following definition, then give examples of their use.

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Definition 66 Equations of Lines in Space

Consider the line in space that passes through $\vec{p} = \langle x_0, y_0, z_0 \rangle$ in the direction of $\vec{d} = \langle a, b, c \rangle$.

1. The **vector equation** of the line is

$$\vec{\ell}(t) = \vec{p} + t\vec{d}.$$

2. The **parametric equations** of the line are

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

3. The **symmetric equations** of the line are

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

when $a, b, c \neq 0$.

Example 10.32 Finding the equation of a line

Give all three equations, as given in Definition 66, of the line through $P = (2, 3, 1)$ in the direction of $\vec{d} = \langle -1, 1, 2 \rangle$. Does the point $Q = (-1, 6, 6)$ lie on this line?

SOLUTION We identify the point $P = (2, 3, 1)$ with the vector $\vec{p} = \langle 2, 3, 1 \rangle$. Following the definition, we have

- the vector equation of the line is $\vec{\ell}(t) = \langle 2, 3, 1 \rangle + t \langle -1, 1, 2 \rangle$;
- the parametric equations of the line are

$$x = 2 - t, \quad y = 3 + t, \quad z = 1 + 2t; \text{ and}$$

- the symmetric equations of the line are

$$\frac{x - 2}{-1} = \frac{y - 3}{1} = \frac{z - 1}{2}.$$

The first two equations of the line are useful when a t value is given: one can immediately find the corresponding point on the line. These forms are good when calculating with a computer; most software programs easily handle equations in these formats. (For instance, to make Figure 10.48, a certain graphics program was given the input $(2-x, 3+x, 1+2*x)$. This particular program requires the variable always be “ x ” instead of “ t ”).

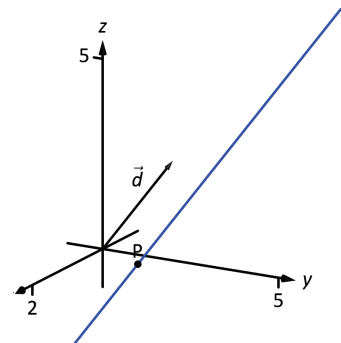


Figure 10.48: Graphing a line in Example 10.32.

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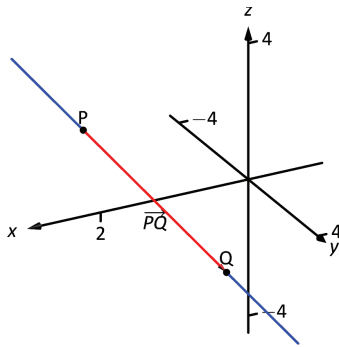


Figure 10.49: A graph of the line in Example 10.33.

Does the point $Q = (-1, 6, 6)$ lie on the line? The graph in Figure 10.48 makes it clear that it does not. We can answer this question without the graph using any of the three equation forms. Of the three, the symmetric equations are probably best suited for this task. Simply plug in the values of x , y and z and see if equality is maintained:

$$\frac{-1-2}{-1} \stackrel{?}{=} \frac{6-3}{1} \stackrel{?}{=} \frac{6-1}{2} \Rightarrow 3 = 3 \neq 2.5.$$

We see that Q does not lie on the line as it did not satisfy the symmetric equations.

Note that the symmetric equations given in Definition 66 only allow for the case where a, b, c are all nonzero. This is because of the denominators. However, we can modify this a bit when one or more of a, b, c are zero. For example, consider the line whose parametric equations are

$$x = 5 + 4t, \quad y = 3, \quad z = -2 + t.$$

This line lies in the plane $y = 3$. So we can define this line using the equations

$$y = 3 \text{ and } \frac{x-5}{4} = \frac{z+2}{1}$$

together. The second equation comes from the usual symmetric equations formula for x and z . This pair is called the symmetric equations for this particular line.

Example 10.33 Finding the equation of a line through two points

Find the parametric equations of the line through the points $P = (2, -1, 2)$ and $Q = (1, 3, -1)$.

SOLUTION Recall the statement made at the beginning of this section: to find the equation of a line, we need a point and a direction. We have *two* points; either one will suffice. The direction of the line can be found by the vector with initial point P and terminal point Q : $\vec{PQ} = \langle -1, 4, -3 \rangle$.

The parametric equations of the line ℓ through P in the direction of \vec{PQ} are:

$$\ell: \quad x = 2 - t \quad y = -1 + 4t \quad z = 2 - 3t.$$

A graph of the points and line are given in Figure 10.49. Note how in the given parameterization of the line, $t = 0$ corresponds to the point P , and $t = 1$ corresponds to the point Q . This relates to the understanding of the vector equation of a line described in Figure 10.46. The parametric equations “start” at the

Notes:

point P , and t determines how far in the direction of \vec{PQ} to travel. When $t = 0$, we travel 0 lengths of \vec{PQ} ; when $t = 1$, we travel one length of \vec{PQ} , resulting in the point Q .

Parallel, Intersecting and Skew Lines

In the plane, two *distinct* lines can either be parallel or they will intersect at exactly one point. In space, given equations of two lines, it can sometimes be difficult to tell whether the lines are distinct or not (i.e., the same line can be represented in different ways). Given lines $\ell_1(t) = \vec{p}_1 + t\vec{d}_1$ and $\ell_2(t) = \vec{p}_2 + t\vec{d}_2$, we have four possibilities: ℓ_1 and ℓ_2 are

the same line	they share all points;
intersecting lines	share only 1 point;
parallel lines	$\vec{d}_1 \parallel \vec{d}_2$, no points in common; or
skew lines	$\vec{d}_1 \nparallel \vec{d}_2$, no points in common.

The next two examples investigate these possibilities.

Example 10.34 Comparing lines

Consider lines ℓ_1 and ℓ_2 , given in parametric equation form:

$$\begin{array}{rcl} x & = & 1 + 3t \\ \ell_1 : y & = & 2 - t \\ z & = & t \end{array} \qquad \begin{array}{rcl} x & = & -2 + 4s \\ \ell_2 : y & = & 3 + s \\ z & = & 5 + 2s. \end{array}$$

Determine whether ℓ_1 and ℓ_2 are the same line, intersect, are parallel, or skew.

SOLUTION We start by looking at the directions of each line. Line ℓ_1 has the direction given by $\vec{d}_1 = \langle 3, -1, 1 \rangle$ and line ℓ_2 has the direction given by $\vec{d}_2 = \langle 4, 1, 2 \rangle$. It should be clear that \vec{d}_1 and \vec{d}_2 are not parallel, hence ℓ_1 and ℓ_2 are not the same line, nor are they parallel. Figure 10.50 verifies this fact (where the points and directions indicated by the equations of each line are identified).

We next check to see if they intersect (if they do not, they are skew lines). To find if they intersect, we look for t and s values such that the respective x , y and z values are the same. That is, we want s and t such that:

$$\begin{array}{rcl} 1 + 3t & = & -2 + 4s \\ 2 - t & = & 3 + s \\ t & = & 5 + 2s. \end{array}$$

This is a relatively simple system of linear equations. Since the last equation is already solved for t , substitute that value of t into the equation above it:

$$2 - (5 + 2s) = 3 + s \Rightarrow s = -2, t = 1.$$

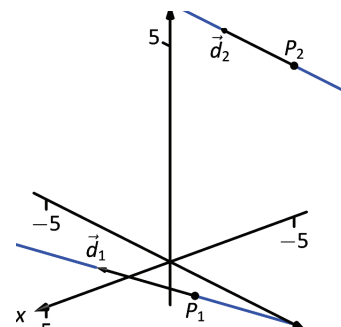


Figure 10.50: Sketching the lines from Example 10.34.

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A key to remember is that we have *three* equations; we need to check if $s = -2$, $t = 1$ satisfies the first equation as well:

$$1 + 3(1) \neq -2 + 4(-2).$$

It does not. Therefore, we conclude that the lines ℓ_1 and ℓ_2 are skew.

Example 10.35 Comparing lines

Consider lines ℓ_1 and ℓ_2 , given in parametric equation form:

$$\begin{array}{ll} \ell_1 : \begin{array}{l} x = -0.7 + 1.6t \\ y = 4.2 + 2.72t \\ z = 2.3 - 3.36t \end{array} & \ell_2 : \begin{array}{l} x = 2.8 - 2.9s \\ y = 10.15 - 4.93s \\ z = -5.05 + 6.09s \end{array} \end{array}$$

Determine whether ℓ_1 and ℓ_2 are the same line, intersect, are parallel, or skew.

SOLUTION It is obviously very difficult to simply look at these equations and discern anything. This is done intentionally. In the “real world,” most equations that are used do not have nice, integer coefficients. Rather, there are lots of digits after the decimal and the equations can look “messy.”

We again start by deciding whether or not each line has the same direction. The direction of ℓ_1 is given by $\vec{d}_1 = \langle 1.6, 2.72, -3.36 \rangle$ and the direction of ℓ_2 is given by $\vec{d}_2 = \langle -2.9, -4.93, 6.09 \rangle$. When it is not clear through observation whether two vectors are parallel or not, the standard way of determining this is by comparing their respective unit vectors. Using a calculator, we find:

$$\begin{aligned} \vec{u}_1 &= \frac{\vec{d}_1}{\|\vec{d}_1\|} = \langle 0.3471, 0.5901, -0.7289 \rangle \\ \vec{u}_2 &= \frac{\vec{d}_2}{\|\vec{d}_2\|} = \langle -0.3471, -0.5901, 0.7289 \rangle. \end{aligned}$$

The two vectors seem to be parallel (at least, their components are equal to 4 decimal places). In most situations, it would suffice to conclude that the lines are at least parallel, if not the same. One way to be sure is to rewrite \vec{d}_1 and \vec{d}_2 in terms of fractions, not decimals. We have

$$\vec{d}_1 = \left\langle \frac{16}{10}, \frac{272}{100}, -\frac{336}{100} \right\rangle \quad \vec{d}_2 = \left\langle -\frac{29}{10}, -\frac{493}{100}, \frac{609}{100} \right\rangle.$$

One can then find the magnitudes of each vector in terms of fractions, then compute the unit vectors likewise. After a lot of manual arithmetic (or after briefly using a computer algebra system), one finds that

$$\vec{u}_1 = \left\langle \sqrt{\frac{10}{83}}, \frac{17}{\sqrt{830}}, -\frac{21}{\sqrt{830}} \right\rangle \quad \vec{u}_2 = \left\langle -\sqrt{\frac{10}{83}}, -\frac{17}{\sqrt{830}}, \frac{21}{\sqrt{830}} \right\rangle.$$

Notes:

We can now say without equivocation that these lines are parallel.

Are they the same line? The parametric equations for a line describe one point that lies on the line, so we know that the point $P_1 = (-0.7, 4.2, 2.3)$ lies on ℓ_1 . To determine if this point also lies on ℓ_2 , plug in the x , y and z values of P_1 into the symmetric equations for ℓ_2 :

$$\frac{(-0.7) - 2.8}{-2.9} \stackrel{?}{=} \frac{(4.2) - 10.15}{-4.93} \stackrel{?}{=} \frac{(2.3) - (-5.05)}{6.09} \Rightarrow 1.2069 = 1.2069 = 1.2069.$$

The point P_1 lies on both lines, so we conclude they are the same line, just parameterized differently. Figure 10.51 graphs this line along with the points and vectors described by the parametric equations. Note how \vec{d}_1 and \vec{d}_2 are parallel, though point in opposite directions (as indicated by their unit vectors above).

Distances

Given a point Q and a line $\vec{\ell}(t) = \vec{p} + t\vec{d}$ in space, it is often useful to know the distance from the point to the line. (Here we use the standard definition of “distance,” i.e., the length of the shortest line segment from the point to the line.) Identifying \vec{p} with the point P , Figure 10.52 will help establish a general method of computing this distance h .

From trigonometry, we know $h = \|\vec{PQ}\| \sin \theta$. We have a similar identity involving the cross product: $\|\vec{PQ} \times \vec{d}\| = \|\vec{PQ}\| \|\vec{d}\| \sin \theta$. Divide both sides of this latter equation by $\|\vec{d}\|$ to obtain h :

$$h = \frac{\|\vec{PQ} \times \vec{d}\|}{\|\vec{d}\|}. \quad (10.8)$$

It is also useful to determine the distance between lines, which we define as the length of the shortest line segment that connects the two lines (an argument from geometry shows that this line segment is perpendicular to both lines). Let lines $\vec{\ell}_1(t) = \vec{p}_1 + t\vec{d}_1$ and $\vec{\ell}_2(t) = \vec{p}_2 + t\vec{d}_2$ be given, as shown in Figure 10.53. To find the direction orthogonal to both \vec{d}_1 and \vec{d}_2 , we take the cross product: $\vec{c} = \vec{d}_1 \times \vec{d}_2$. The magnitude of the orthogonal projection of $\vec{P_1P_2}$ onto \vec{c} is the

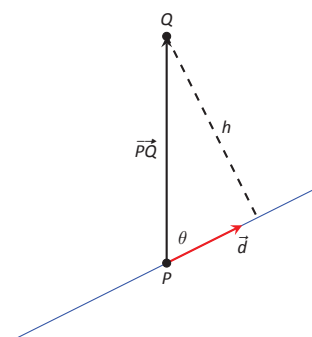


Figure 10.52: Establishing the distance from a point to a line.

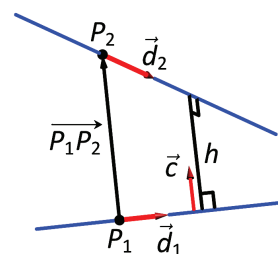


Figure 10.53: Establishing the distance between lines.

Figure 10.51: Graphing the lines in Example 10.35.

Notes:

distance h we seek:

$$\begin{aligned}
 h &= \left\| \text{proj}_{\vec{c}} \overrightarrow{P_1 P_2} \right\| \\
 &= \left\| \frac{\overrightarrow{P_1 P_2} \cdot \vec{c}}{\vec{c} \cdot \vec{c}} \vec{c} \right\| \\
 &= \frac{|\overrightarrow{P_1 P_2} \cdot \vec{c}|}{\|\vec{c}\|^2} \|\vec{c}\| \\
 &= \frac{|\overrightarrow{P_1 P_2} \cdot \vec{c}|}{\|\vec{c}\|}.
 \end{aligned}$$

A problem in the Exercise section is to show that this distance is 0 when the lines intersect. Note the use of the Triple Scalar Product: $\overrightarrow{P_1 P_2} \cdot \vec{c} = \overrightarrow{P_1 P_2} \cdot (\vec{d}_1 \times \vec{d}_2)$.

The following Key Idea restates these two distance formulas.

Key Idea 52 Distances to Lines

1. Let P be a point on a line ℓ that is parallel to \vec{d} . The distance h from a point Q to the line ℓ is:

$$h = \frac{\|\vec{PQ} \times \vec{d}\|}{\|\vec{d}\|}.$$

2. Let P_1 be a point on line ℓ_1 that is parallel to \vec{d}_1 , and let P_2 be a point on line ℓ_2 parallel to \vec{d}_2 . Also, let $\vec{c} = \vec{d}_1 \times \vec{d}_2$, where lines ℓ_1 and ℓ_2 are not parallel. The distance h between the two lines is:

$$h = \frac{|\overrightarrow{P_1 P_2} \cdot \vec{c}|}{\|\vec{c}\|}.$$

Example 10.36 Finding the distance from a point to a line

Find the distance from the point $Q = (1, 1, 3)$ to the line $\vec{\ell}(t) = \langle 1, -1, 1 \rangle + t \langle 2, 3, 1 \rangle$.

SOLUTION The equation of the line gives us the point $P = (1, -1, 1)$ that lies on the line, hence $\vec{PQ} = \langle 0, 2, 2 \rangle$. The equation also gives $\vec{d} = \langle 2, 3, 1 \rangle$.

Notes:

Following Key Idea 52, we have the distance as

$$\begin{aligned} h &= \frac{\|\vec{PQ} \times \vec{d}\|}{\|\vec{d}\|} \\ &= \frac{\|\langle -4, 4, -4 \rangle\|}{\sqrt{14}} \\ &= \frac{4\sqrt{3}}{\sqrt{14}} \approx 1.852. \end{aligned}$$

The point Q is approximately 1.852 units from the line $\vec{\ell}(t)$.

Example 10.37 Finding the distance between lines

Find the distance between the lines

$$\begin{array}{ll} x &= 1 + 3t \\ \ell_1 : y &= 2 - t \\ z &= t \end{array} \qquad \begin{array}{ll} x &= -2 + 4s \\ \ell_2 : y &= 3 + s \\ z &= 5 + 2s. \end{array}$$

SOLUTION These are the same lines as given in Example 10.34, where we showed them to be skew. The equations allow us to identify the following points and vectors:

$$\begin{aligned} P_1 &= (1, 2, 0) \quad P_2 = (-2, 3, 5) \quad \Rightarrow \quad \vec{P_1P_2} = \langle -3, 1, 5 \rangle. \\ \vec{d}_1 &= \langle 3, -1, 1 \rangle \quad \vec{d}_2 = \langle 4, 1, 2 \rangle \quad \Rightarrow \quad \vec{c} = \vec{d}_1 \times \vec{d}_2 = \langle -3, -2, 7 \rangle. \end{aligned}$$

From Key Idea 52 we have the distance h between the two lines is

$$\begin{aligned} h &= \frac{|\vec{P_1P_2} \cdot \vec{c}|}{\|\vec{c}\|} \\ &= \frac{42}{\sqrt{62}} \approx 5.334. \end{aligned}$$

The lines are approximately 5.334 units apart.

One of the key points to understand from this section is this: to describe a line, we need a point and a direction. Whenever a problem is posed concerning a line, one needs to take whatever information is offered and glean point and direction information. Many questions can be asked (and *are* asked in the Exercise section) whose answer immediately follows from this understanding.

Lines are one of two fundamental objects of study in space. The other fundamental object is the *plane*, which we study in detail in the next section. Many complex three dimensional objects are studied by approximating their surfaces with lines and planes.

Notes:

Exercises 10.5

Terms and Concepts

1. To find an equation of a line, what two pieces of information are needed?
2. Two distinct lines in the plane can intersect or be _____.
3. Two distinct lines in space can intersect, be _____ or be _____.
4. Use your own words to describe what it means for two lines in space to be skew.

Problems

In Exercises 5 – 15, write the vector, parametric and symmetric equations of the lines described.

5. Passes through $P = (2, -4, 1)$, parallel to $\vec{d} = \langle 9, 2, 5 \rangle$.
6. Passes through $P = (6, 1, 7)$, parallel to $\vec{d} = \langle -3, 2, 5 \rangle$.
7. Passes through $P = (-2, 5, 4)$, parallel to $\vec{d} = \langle 0, 1, 3 \rangle$.
8. Passes through $P = (1, 1)$, parallel to $\vec{d} = \langle 2, 3 \rangle$.
9. Passes through $P = (2, 1, 5)$ and $Q = (7, -2, 4)$.
10. Passes through $P = (1, -2, 3)$ and $Q = (5, 5, 5)$.
11. Passes through $P = (1, 5, 5)$ and $Q = (2, 2, 5)$.
12. Passes through $P = (0, 1, 2)$ and orthogonal to both $\vec{d}_1 = \langle 2, -1, 7 \rangle$ and $\vec{d}_2 = \langle 7, 1, 3 \rangle$.
13. Passes through $P = (5, 1, 9)$ and orthogonal to both $\vec{d}_1 = \langle 1, 0, 1 \rangle$ and $\vec{d}_2 = \langle 2, 0, 3 \rangle$.
14. Passes through the point of intersection of $\vec{\ell}_1(t)$ and $\vec{\ell}_2(t)$ and orthogonal to both lines, where $\vec{\ell}_1(t) = \langle 2, 1, 1 \rangle + t \langle 5, 1, -2 \rangle$ and $\vec{\ell}_2(t) = \langle -2, -1, 2 \rangle + t \langle 3, 1, -1 \rangle$.
15. Passes through the point of intersection of $\ell_1(t)$ and $\ell_2(t)$ and orthogonal to both lines, where
$$\ell_1 = \begin{cases} x = t \\ y = -2 + 2t \\ z = 1 + t \end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} x = 2 + t \\ y = 2 - t \\ z = 3 + 2t \end{cases}.$$

In Exercises 16 – 23, determine if the described lines are the same line, parallel lines, intersecting or skew lines. If intersecting, give the point of intersection.

16. $\vec{\ell}_1(t) = \langle 1, 2, 1 \rangle + t \langle 2, -1, 1 \rangle$,
 $\vec{\ell}_2(t) = \langle 3, 3, 3 \rangle + t \langle -4, 2, -2 \rangle$.

$$17. \vec{\ell}_1(t) = \langle 2, 1, 1 \rangle + t \langle 5, 1, 3 \rangle, \\ \vec{\ell}_2(t) = \langle 14, 5, 9 \rangle + t \langle 1, 1, 1 \rangle.$$

$$18. \vec{\ell}_1(t) = \langle 3, 4, 1 \rangle + t \langle 2, -3, 4 \rangle, \\ \vec{\ell}_2(t) = \langle -3, 3, -3 \rangle + t \langle 3, -2, 4 \rangle.$$

$$19. \vec{\ell}_1(t) = \langle 1, 1, 1 \rangle + t \langle 3, 1, 3 \rangle, \\ \vec{\ell}_2(t) = \langle 7, 3, 7 \rangle + t \langle 6, 2, 6 \rangle.$$

$$20. \ell_1 = \begin{cases} x = 1 + 2t \\ y = 3 - 2t \\ z = t \end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} x = 3 - t \\ y = 3 + 5t \\ z = 2 + 7t \end{cases}$$

$$21. \ell_1 = \begin{cases} x = 1.1 + 0.6t \\ y = 3.77 + 0.9t \\ z = -2.3 + 1.5t \end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} x = 3.11 + 3.4t \\ y = 2 + 5.1t \\ z = 2.5 + 8.5t \end{cases}$$

$$22. \ell_1 = \begin{cases} x = 0.2 + 0.6t \\ y = 1.33 - 0.45t \\ z = -4.2 + 1.05t \end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} x = 0.86 + 9.2t \\ y = 0.835 - 6.9t \\ z = -3.045 + 16.1t \end{cases}$$

$$23. \ell_1 = \begin{cases} x = 0.1 + 1.1t \\ y = 2.9 - 1.5t \\ z = 3.2 + 1.6t \end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} x = 4 - 2.1t \\ y = 1.8 + 7.2t \\ z = 3.1 + 1.1t \end{cases}$$

In Exercises 24 – 27, find the distance from the point to the line.

$$24. P = (1, 1, 1), \quad \vec{\ell}(t) = \langle 2, 1, 3 \rangle + t \langle 2, 1, -2 \rangle$$

$$25. P = (2, 5, 6), \quad \vec{\ell}(t) = \langle -1, 1, 1 \rangle + t \langle 1, 0, 1 \rangle$$

$$26. P = (0, 3), \quad \vec{\ell}(t) = \langle 2, 0 \rangle + t \langle 1, 1 \rangle$$

$$27. P = (1, 1), \quad \vec{\ell}(t) = \langle 4, 5 \rangle + t \langle -4, 3 \rangle$$

In Exercises 28 – 29, find the distance between the two lines.

$$28. \vec{\ell}_1(t) = \langle 1, 2, 1 \rangle + t \langle 2, -1, 1 \rangle, \\ \vec{\ell}_2(t) = \langle 3, 3, 3 \rangle + t \langle 4, 2, -2 \rangle.$$

$$29. \vec{\ell}_1(t) = \langle 0, 0, 1 \rangle + t \langle 1, 0, 0 \rangle, \\ \vec{\ell}_2(t) = \langle 0, 0, 3 \rangle + t \langle 0, 1, 0 \rangle.$$

Exercises 30 – 32 explore special cases of the distance formulas found in Key Idea 52.

30. Let Q be a point on the line $\ell(t)$. Show why the distance formula correctly gives the distance from the point to the line as 0.

31. Let lines $\ell_1(t)$ and $\ell_2(t)$ be intersecting lines. Show why the distance formula correctly gives the distance between these lines as 0.

32. Let lines $\ell_1(t)$ and $\ell_2(t)$ be parallel.

(a) Show why the distance formula for distance between lines cannot be used as stated to find the distance between the lines.

(b) Show why letting $c = (\overrightarrow{P_1P_2} \times \vec{d}_2) \times \vec{d}_2$ allows one to use the formula.

(c) Show how one can use the formula for the distance between a point and a line to find the distance between parallel lines.