13.4   Center of Mass

In Section 7.7, we introduced centers of mass, but only computed them for planar laminas with constant density. With the tools of double integrals, we can now compute a broader range of centers of mass.

We have used iterated integrals to find areas of plane regions and signed volumes under surfaces. A brief recap of these uses will be useful in this section as we apply iterated integrals to compute the **mass** and **center of mass** of planar regions.

To find the area of a planar region, we evaluated the double integral $\iint_R dA$. That is, summing up the areas of lots of little subregions of R gave us the total area. Informally, we think of $\iint_R dA$ as meaning "sum up lots of little areas over R."

To find the signed volume under a surface, we evaluated the double integral $\iint_R f(x, y)\, dA$. Recall that the "dA" is not just a "bookend" at the end of an integral; rather, it is multiplied by $f(x, y)$. We regard $f(x, y)$ as giving a height, and dA still giving an area: $f(x, y)\, dA$ gives a volume. Thus, informally, $\iint_R f(x, y)\, dA$ means "sum up lots of little volumes over R."

We now extend these ideas to other contexts.

Mass and Weight

Consider a thin sheet of material with constant thickness and finite area. Mathematicians (and physicists and engineers) call such a sheet a **lamina**. So consider a lamina, as shown in Figure 13.24(a), with the shape of some planar region R, as shown in part (b).

We can write a simple double integral that represents the mass of the lamina: $\iint_R dm$, where "dm" means "a little mass." That is, the double integral states the total mass of the lamina can be found by "summing up lots of little masses over R."

To evaluate this double integral, partition R into n subregions as we have done in the past. The ith subregion has area ΔA_i. A fundamental property of mass is that "mass=density$\times$area." If the lamina has a constant density δ, then the mass of this i^{th} subregion is $\Delta m_i = \delta \Delta A_i$. That is, we can compute a small amount of mass by multiplying a small amount of area by the density.

If density is variable, with density function $\delta = \delta(x, y)$, then we can approximate the mass of the ith subregion of R by multiplying ΔA_i by $\delta(x_i, y_i)$, where (x_i, y_i) is a point in that subregion. That is, for a small enough subregion of R, the density across that region is almost constant.

The total mass M of the lamina is approximately the sum of approximate
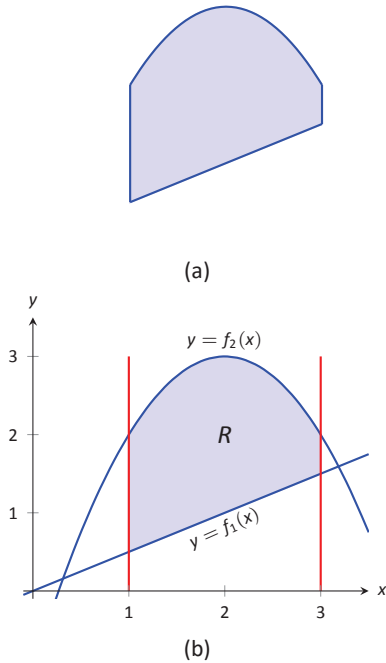

(a)

(b)

Figure 13.24: Illustrating the concept of a lamina.

Note: *Mass* and *weight* are different measures. Since they are scalar multiples of each other, it is often easy to treat them as the same measure. In this section we effectively treat them as the same, as our technique for finding mass is the same as for finding weight. The density functions used will simply have different units.

Notes:

masses of subregions:

$$M \approx \sum_{i=1}^{n} \Delta m_i = \sum_{i=1}^{n} \delta(x_i, y_i) \Delta A_i.$$

Taking the limit as the size of the subregions shrinks to 0 gives us the actual mass; that is, integrating $\delta(x, y)$ over R gives the mass of the lamina.

Definition 107   Mass of a Lamina with Variable Density

Let $\delta(x, y)$ be a continuous density function of a lamina corresponding to a plane region R. The mass M of the lamina is

$$\text{mass } M = \iint_R dm = \iint_R \delta(x, y) \, dA.$$

Example 13.22   Finding the mass of a lamina with constant density
Find the mass of a square lamina, with side length 1, with a density of $\delta = 3\text{g/cm}^2$.

SOLUTION   We represent the lamina with a square region in the plane as shown in Figure 13.25. As the density is constant, it does not matter where we place the square.

Following Definition 107, the mass M of the lamina is

$$M = \iint_R 3 \, dA = \int_0^1 \int_0^1 3 \, dx \, dy = 3 \int_0^1 \int_0^1 dx \, dy = 3\text{g}.$$

This is all very straightforward; note that all we really did was find the area of the lamina and multiply it by the constant density of 3g/cm^2.

Example 13.23   Finding the mass of a lamina with variable density
Find the mass of a square lamina, represented by the unit square with lower lefthand corner at the origin (see Figure 13.25), with variable density $\delta(x, y) = (x + y + 2)\text{g/cm}^2$.

SOLUTION   The variable density δ, in this example, is very uniform, giving a density of 3 in the center of the square and changing linearly. A graph of $\delta(x, y)$ can be seen in Figure 13.26; notice how "same amount" of density is above $z = 3$ as below. We'll comment on the significance of this momentarily.
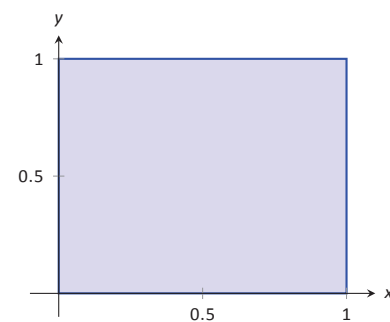

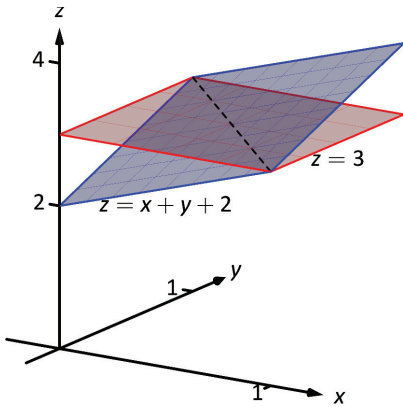Figure 13.25: A region R representing a lamina in Example 13.22.

Notes:

Figure 13.26: Graphing the density functions in Examples 13.22 and 13.23.

The mass M is found by integrating $\delta(x, y)$ over R. The order of integration is not important; we choose $dx\,dy$ arbitrarily. Thus:

$$M = \iint_R (x + y + 2)\, dA = \int_0^1 \int_0^1 (x + y + 2)\, dx\, dy$$

$$= \int_0^1 \left(\frac{1}{2}x^2 + x(y + 2) \right) \Big|_0^1 dy$$

$$= \int_0^1 \left(\frac{5}{2} + y \right) dy$$

$$= \left(\frac{5}{2}y + \frac{1}{2}y^2 \right) \Big|_0^1$$

$$= 3g.$$

It turns out that since since the density of the lamina is so uniformly distributed "above and below" $z = 3$ that the mass of the lamina is the same as if it had a constant density of 3. The density functions in Examples 13.22 and 13.23 are graphed in Figure 13.26, which illustrates this concept.

Example 13.24   Finding the weight of a lamina with variable density
Find the weight of the lamina represented by the circle with radius 2ft, centered at the origin, with density function $\delta(x, y) = (x^2 + y^2 + 1)$lb/ft^2. Compare this to the weight of the same lamina with density $\delta(x, y) = (2\sqrt{x^2 + y^2} + 1)$lb/ft^2.

SOLUTION   A direct application of Definition 107 states that the weight of the lamina is $\iint_R \delta(x, y)\, dA$. Since our lamina is in the shape of a circle, it makes sense to approach the double integral using polar coordinates.

The density function $\delta(x, y) = x^2 + y^2 + 1$ becomes $\delta(r, \theta) = (r\cos\theta)^2 + (r\sin\theta)^2 + 1 = r^2 + 1$. The circle is bounded by $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$. Thus the weight W is:

$$W = \int_0^{2\pi} \int_0^2 (r^2 + 1) r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{4}r^4 + \frac{1}{2}r^2 \right) \Big|_0^2 d\theta$$

$$= \int_0^{2\pi} 6\, d\theta$$

$$= 12\pi \approx 37.70\text{lb.}$$

Now compare this with the density function $\delta(x, y) = 2\sqrt{x^2 + y^2} + 1$. Converting this to polar coordinates gives $\delta(r, \theta) = 2\sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} + 1 =$

Notes:

$2r + 1$. Thus the weight W is:

$$W = \int_0^{2\pi} \int_0^2 (2r + 1)r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left(\frac{2}{3}r^3 + \frac{1}{2}r^2 \right) \Big|_0^2 d\theta$$

$$= \int_0^{2\pi} \frac{22}{3} \, d\theta$$

$$= \frac{44}{3}\pi \approx 46.08\text{lb.}$$

One would expect different density functions to return different weights, as we have here. The density functions were chosen, though, to be similar: each gives a density of 1 at the origin and a density of 5 at the outside edge of the circle, as seen in Figure 13.27.
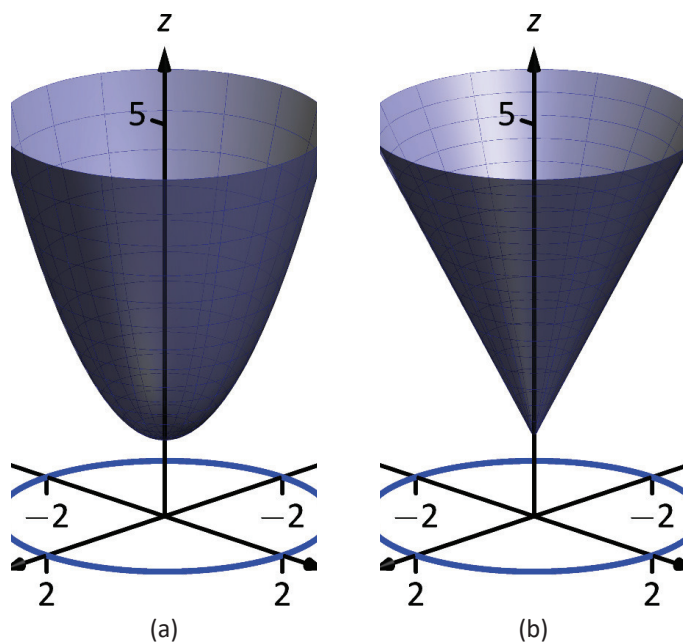

Figure 13.27: Graphing the density functions in Example 13.24. In (a) is the density function $\delta(x, y) = x^2 + y^2 + 1$; in (b) is $\delta(x, y) = 2\sqrt{x^2 + y^2} + 1$.

Notice how $x^2 + y^2 + 1 \leq 2\sqrt{x^2 + y^2} + 1$ over the circle; this results in less weight.

Notes:

Plotting the density functions can be useful as our understanding of mass can be related to our understanding of "volume under a surface." We interpreted $\iint_R f(x, y)\ dA$ as giving the volume under f over R; we can understand $\iint_R \delta(x, y)\ dA$ in the same way. The "volume" under δ over R is actually mass; by compressing the "volume" under δ onto the x-y plane, we get "more mass" in some areas than others – i.e., areas of greater density.

Knowing the mass of a lamina is one of several important measures. Another is the **center of mass**, which we discuss next.

Center of Mass

Consider a disk of radius 1 with uniform density. It is common knowledge that the disk will balance on a point if the point is placed at the center of the disk. What if the disk does not have a uniform density? Through trial-and-error, we should still be able to find a spot on the disk at which the disk will balance on a point. This balance point is referred to as the **center of mass**, or **center of gravity**. It is though all the mass is "centered" there. In fact, if the disk has a mass of 3kg, the disk will behave physically as though it were a point-mass of 3kg located at its center of mass. For instance, the disk will naturally spin with an axis through its center of mass (which is why it is important to "balance" the tires of your car: if they are "out of balance", their center of mass will be outside of the axle and it will shake terribly).

We find the center of mass based on the principle of a **weighted average**. Consider a college class in which your homework average is 90%, your test average is 73%, and your final exam grade is an 85%. Experience tells us that our final grade *is not* the *average* of these three grades: that is, it is not:

$$\frac{0.9 + 0.73 + 0.85}{3} \approx 0.837 = 83.7\%.$$

That is, you are probably not pulling a B in the course. Rather, your grades are *weighted*. Let's say the homework is worth 10% of the grade, tests are 60% and the exam is 30%. Then your final grade is:

$$(0.1)(0.9) + (0.6)(0.73) + (0.3)(0.85) = 0.783 = 78.3\%.$$

Each grade is multiplied by a **weight**.

In general, given values $x_1, x_2, \ldots, x_n$ and weights $w_1, w_2, \ldots, w_n$, the weighted average of the n values is

$$\sum_{i=1}^{n} w_i x_i \Bigg/ \sum_{i=1}^{n} w_i.$$

In the grading example above, the sum of the weights 0.1, 0.6 and 0.3 is 1, so we don't see the division by the sum of weights in that instance.

Notes:

How this relates to center of mass is given in the following theorem.

Theorem 126   Center of Mass of Discrete Linear System

Let point masses $m_1, m_2, \ldots, m_n$ be distributed along the x-axis at locations $x_1, x_2, \ldots, x_n$, respectively. The center of mass $\bar{x}$ of the system is located at

$$\bar{x} = \sum_{i=1}^{n} m_i x_i \left/ \sum_{i=1}^{n} m_i \right. .$$

Example 13.25   Finding the center of mass of a discrete linear system

1. Point masses of 2g are located at $x = -1$, $x = 2$ and $x = 3$ are connected by a thin rod of negligible weight. Find the center of mass of the system.

2. Point masses of 10g, 2g, and 1g are located at $x = -1$, $x = 2$ and $x = 3$, respectively, are connected by a thin rod of negligible weight. Find the center of mass of the system.

SOLUTION

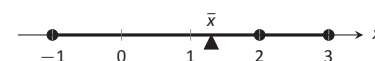1. Following Theorem 126, we compute the center of mass as:

$$\bar{x} = \frac{2(-1) + 2(2) + 2(3)}{2 + 2 + 2} = \frac{4}{3} = 1.\overline{3}.$$

So the system would balance on a point placed at $x = 4/3$, as illustrated in Figure 13.28(a).
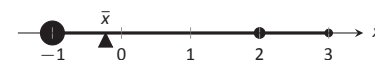
2. Again following Theorem 126, we find:

$$\bar{x} = \frac{10(-1) + 2(2) + 1(3)}{10 + 2 + 1} = \frac{-3}{13} \approx -0.23.$$

Placing a large weight at the left hand side of the system moves the center of mass left, as shown in Figure 13.28(b).

In a discrete system (i.e., mass is located at individual points, not along a continuum) we find the center of mass by dividing the mass into a **moment** of the system. In general, a moment is a weighted measure of distance from a particular point or line. In the case described by Theorem 126, we are finding a

(a)

(b)

Figure 13.28: Illustrating point masses along a thin rod and the center of mass.

Notes:

weighted measure of distances from the y-axis, so we refer to this as **the moment about the y-axis**, represented by M_y. Letting M be the total mass of the system, we have $\bar{x} = M_y/M$.

We can extend the concept of the center of mass of discrete points along a line to the center of mass of discrete points in the plane rather easily. To do so, we define some terms then give a theorem.

Definition 108   Moments About the x- and y- Axes.

Let point masses $m_1,\ m_2,\ldots,m_n$ be located at points (x_1,y_1), $(x_2,y_2),\ldots,(x_n,y_n)$, respectively, in the x-y plane.

1. The **moment about the y-axis**, M_y, is $M_y = \displaystyle\sum_{i=1}^{n} m_i x_i$.

2. The **moment about the x-axis**, M_x, is $M_x = \displaystyle\sum_{i=1}^{n} m_i y_i$.

One can think that these definitions are "backwards" as M_y sums up "x" distances. But remember, "x" distances are measurements of distance from the y-axis, hence defining the moment about the y-axis.

We now define the center of mass of discrete points in the plane.

Theorem 127   Center of Mass of Discrete Planar System

Let point masses $m_1,\ m_2,\ldots,m_n$ be located at points (x_1,y_1), $(x_2,y_2),\ldots,(x_n,y_n)$, respectively, in the x-y plane, and let $M = \displaystyle\sum_{i=1}^{n} m_i$.

The center of mass of the system is at $(\bar{x},\bar{y})$, where

$$\bar{x} = \frac{M_y}{M} \quad\text{and}\quad \bar{y} = \frac{M_x}{M}.$$

Example 13.26   Finding the center of mass of a discrete planar system
Let point masses of 1kg, 2kg, and 5kg be located at points $(2,0)$, $(1,1)$, and $(3,1)$, respectively, and are connected by thin rods of negligible weight. Find the center of mass of the system.

SOLUTION   We follow Theorem 127 and Definition 108 to find M, M_x

---

Notes:

and M_y:

$M = 1 + 2 + 5 = 8\text{kg}.$

$$M_x = \sum_{i=1}^{n} m_i y_i \qquad\qquad M_y = \sum_{i=1}^{n} m_i x_i$$

$$= 1(0) + 2(1) + 5(1) \qquad\qquad\quad = 1(2) + 2(1) + 5(3)$$

$$= 7 \text{ kg.} \qquad\qquad\qquad\qquad\quad = 19 \text{ kg.}$$

Thus the center of mass is $(\bar{x}, \bar{y}) = \left(\dfrac{M_y}{M}, \dfrac{M_x}{M} \right) = \left(\dfrac{19}{8}, \dfrac{7}{8} \right) = (2.375, 0.875),$ illustrated in Figure 13.29.

We finally arrive at our true goal of this section: finding the center of mass of a lamina with variable density. While the above measurement of center of mass is interesting, it does not directly answer more realistic situations where we need to find the center of mass of a contiguous region. However, understanding the discrete case allows us to approximate the center of mass of a planar lamina; using calculus, we can refine the approximation to an exact value.

We begin by representing a planar lamina with a region R in the x-y plane with density function $\delta(x, y)$. Partition R into n subdivisions, each with area ΔA_i. As done before, we can approximate the mass of the ith subregion with $\delta(x_i, y_i)\Delta A_i$, where (x_i, y_i) is a point inside the ith subregion. We can approximate the moment of this subregion about the y-axis with $x_i\delta(x_i, y_i)\Delta A_i$ – that is, by multiplying the approximate mass of the region by its approximate distance from the y-axis. Similarly, we can approximate the moment about the x-axis with $y_i\delta(x_i, y_i)\Delta A_i$. By summing over all subregions, we have:

$$\text{mass: } M \approx \sum_{i=1}^{n} \delta(x_i, y_i)\Delta A_i \quad \text{(as seen before)}$$

$$\text{moment about the } x\text{-axis: } M_x \approx \sum_{i=1}^{n} y_i\delta(x_i, y_i)\Delta A_i$$

$$\text{moment about the } y\text{-axis: } M_y \approx \sum_{i=1}^{n} x_i\delta(x_i, y_i)\Delta A_i$$

By taking limits, where size of each subregion shrinks to 0 in both the x and y directions, we arrive at the double integrals given in the following theorem.
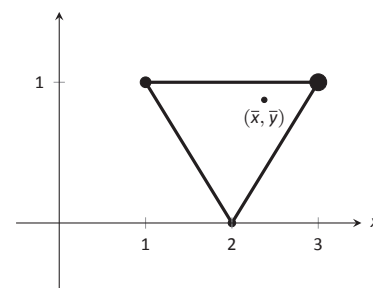

Figure 13.29: Illustrating the center of mass of a discrete planar system in Example 13.26.

Notes:

> **Theorem 128   Center of Mass of a Planar Lamina, Moments**
>
> Let a planar lamina be represented by a region R in the x-y plane with density function $\delta(x, y)$.
>
> 1. mass: $M = \displaystyle\iint_R \delta(x, y) \, dA$
>
> 2. moment about the x-axis: $M_x = \displaystyle\iint_R y\delta(x, y) \, dA$
>
> 3. moment about the y-axis: $M_y = \displaystyle\iint_R x\delta(x, y) \, dA$
>
> 4. The center of mass of the lamina is
>
> $$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right).$$

We start our practice of finding centers of mass by revisiting some of the lamina used previously in this section when finding mass. We will just set up the integrals needed to compute M, M_x and M_y and leave the details of the integration to the reader.
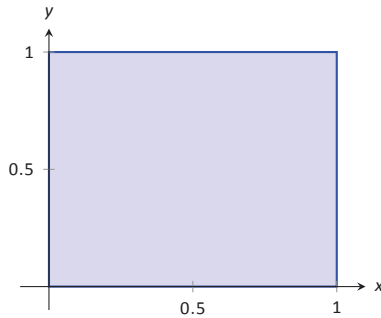
Example 13.27   Finding the center of mass of a lamina
Find the center mass of a square lamina, with side length 1, with a density of $\delta = 3\text{g/cm}^2$. (Note: this is the lamina from Example 13.22.)

SOLUTION   We represent the lamina with a square region in the plane as shown in Figure 13.30 as done previously.
Following Theorem 128, we find M, M_x and M_y:

$$M = \iint_R 3 \, dA = \int_0^1 \int_0^1 3 \, dx \, dy = 3\text{g}.$$

$$M_x = \iint_R 3y \, dA = \int_0^1 \int_0^1 3y \, dx \, dy = 3/2 = 1.5.$$

$$M_y = \iint_R 3x \, dA = \int_0^1 \int_0^1 3x \, dx \, dy = 3/2 = 1.5.$$

Thus the center of mass is $(\bar{x}, \bar{y}) = \left(\dfrac{M_y}{M}, \dfrac{M_x}{M} \right) = (1.5/3, 1.5/3) = (0.5, 0.5)$.
This is what we should have expected: the center of mass of a square with constant density is the center of the square.

Figure 13.30: A region R representing a lamina in Example 13.22.

Notes:

Example 13.28 **Finding the center of mass of a lamina**

Find the center of mass of a square lamina, represented by the unit square with lower lefthand corner at the origin (see Figure 13.30), with variable density $\delta(x, y) = (x + y + 2)$g/cm^2. (Note: this is the lamina from Example 13.23.)

SOLUTION We follow Theorem 128, to find M, M_x and M_y:

$$M = \iint_R (x + y + 2)\, dA = \int_0^1 \int_0^1 (x + y + 2)\, dx\, dy = 3g.$$

$$M_x = \iint_R y(x + y + 2)\, dA = \int_0^1 \int_0^1 y(x + y + 2)\, dx\, dy = \frac{19}{12}.$$

$$M_y = \iint_R x(x + y + 2)\, dA = \int_0^1 \int_0^1 x(x + y + 2)\, dx\, dy = \frac{19}{12}.$$

Thus the center of mass is $(\bar{x}, \bar{y}) = \left(\dfrac{M_y}{M}, \dfrac{M_x}{M}\right) = \left(\dfrac{19}{36}, \dfrac{19}{36}\right) \approx (0.528, 0.528)$.

While the mass of this lamina is the same as the lamina in the previous example, the greater density found with greater x and y values pulls the center of mass from the center slightly towards the upper righthand corner.

Example 13.29 **Finding the center of mass of a lamina**

Find the center of mass of the lamina represented by the circle with radius 2ft, centered at the origin, with density function $\delta(x, y) = (x^2 + y^2 + 1)$lb/ft^2. (Note: this is one of the lamina used in Example 13.24.)

SOLUTION As done in Example 13.24, it is best to describe R using polar coordinates. Thus when we compute M_y, we will integrate not $x\delta(x, y) = x(x^2 + y^2 + 1)$, but rather $(r\cos\theta)\delta(r\cos\theta, r\sin\theta) = (r\cos\theta)(r^2 + 1)$. We compute M, M_x and M_y:

$$M = \int_0^{2\pi} \int_0^2 (r^2 + 1)r\, dr\, d\theta = 12\pi \approx 37.7\text{lb}.$$

$$M_x = \int_0^{2\pi} \int_0^2 (r\sin\theta)(r^2 + 1)r\, dr\, d\theta = 0.$$

$$M_y = \int_0^{2\pi} \int_0^2 (r\cos\theta)(r^2 + 1)r\, dr\, d\theta = 0.$$

Since R and the density of R are both symmetric about the x and y axes, it should come as no big surprise that the moments about each axis is 0. Thus the center
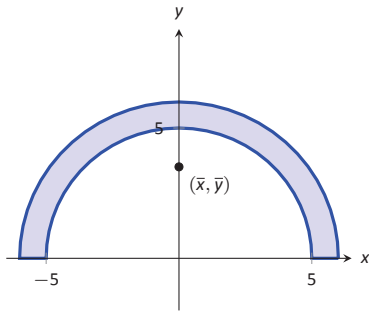
Notes:

of mass is $(\bar{x}, \bar{y}) = (0, 0)$.

Example 13.30   **Finding the center of mass of a lamina**

Find the center of mass of the lamina represented by the region R shown in Figure 13.31, half an annulus with outer radius 6 and inner radius 5, with constant density 2lb/ft^2.

SOLUTION   Once again it will be useful to represent R in polar coordinates. Using the description of R and/or the illustration, we see that R is bounded by $5 \leq r \leq 6$ and $0 \leq \theta \leq \pi$. As the lamina is symmetric about the y-axis, we should expect $M_y = 0$. We compute M, M_x and M_y:

$$M = \int_0^\pi \int_5^6 (2)r \, dr \, d\theta = 11\pi \text{lb}.$$

$$M_x = \int_0^\pi \int_5^6 (r\sin\theta)(2)r \, dr \, d\theta = \frac{364}{3} \approx 121.33.$$

$$M_y = \int_0^\pi \int_5^6 (r\cos\theta)(2)r \, dr \, d\theta = 0.$$

Thus the center of mass is $(\bar{x}, \bar{y}) = \left(0, \frac{364}{33\pi}\right) \approx (0, 3.51)$. The center of mass is indicated in Figure 13.31; note how it lies outside of R!

This section has shown us another use for iterated integrals beyond finding area or signed volume under the curve. While there are many uses for iterated integrals, we give one more application in the following section: computing surface area.

Figure 13.31: Illustrating the region R in Example 13.30.

Notes:

Exercises 13.4

Terms and Concepts

1. Why is it easy to use "mass" and "weight" interchangeably, even though they are different measures?

2. Given a point (x, y), the value of x is a measure of distance from the _____-axis.

3. We can think of $\iint_R dm$ as meaning "sum up lots of _____"

4. What is a "discrete planar system?"

5. Why does M_x use $\iint_R y\delta(x, y)\, dA$ instead of $\iint_R x\delta(x, y)\, dA$; that is, why do we use "y" and not "x"?

6. Describe a situation where the center of mass of a lamina does not lie within the region of the lamina itself.

Problems

In Exercises 7 – 10, point masses are given along a line or in the plane. Find the center of mass $\bar{x}$ or $(\bar{x}, \bar{y})$, as appropriate. (All masses are in grams and distances are in cm.)

7. $m_1 = 4$ at $x = 1$; $m_2 = 3$ at $x = 3$; $m_3 = 5$ at $x = 10$

8. $m_1 = 2$ at $x = -3$; $m_2 = 2$ at $x = -1$; $m_3 = 3$ at $x = 0$; $m_4 = 3$ at $x = 7$

9. $m_1 = 2$ at $(-2, -2)$; $m_2 = 2$ at $(2, -2)$; $m_3 = 20$ at $(0, 4)$

10. $m_1 = 1$ at $(-1, -1)$; $m_2 = 2$ at $(-1, 1)$; $m_3 = 2$ at $(1, 1)$; $m_4 = 1$ at $(1, -1)$

In Exercises 11 – 18, find the mass/weight of the lamina described by the region R in the plane and its density function $\delta(x, y)$.

11. R is the rectangle with corners $(1, -3)$, $(1, 2)$, $(7, 2)$ and $(7, -3)$; $\delta(x, y) = 5$g/cm^2

12. R is the rectangle with corners $(1, -3)$, $(1, 2)$, $(7, 2)$ and $(7, -3)$; $\delta(x, y) = (x + y^2)$g/cm^2

13. R is the triangle with corners $(-1, 0)$, $(1, 0)$, and $(0, 1)$; $\delta(x, y) = 2$lb/in^2

14. R is the triangle with corners $(0, 0)$, $(1, 0)$, and $(0, 1)$; $\delta(x, y) = (x^2 + y^2 + 1)$lb/in^2

15. R is the circle centered at the origin with radius 2; $\delta(x, y) = (x + y + 4)$kg/m^2

16. R is the circle sector bounded by $x^2 + y^2 = 25$ in the first quadrant; $\delta(x, y) = (\sqrt{x^2 + y^2} + 1)$kg/m^2

17. R is the annulus in the first and second quadrants bounded by $x^2 + y^2 = 9$ and $x^2 + y^2 = 36$; $\delta(x, y) = 4$lb/ft^2

18. R is the annulus in the first and second quadrants bounded by $x^2 + y^2 = 9$ and $x^2 + y^2 = 36$; $\delta(x, y) = \sqrt{x^2 + y^2}$lb/ft^2

In Exercises 19 – 26, find the center of mass of the lamina described by the region R in the plane and its density function $\delta(x, y)$.
Note: these are the same lamina as in Exercises 11 – 18.

19. R is the rectangle with corners $(1, -3)$, $(1, 2)$, $(7, 2)$ and $(7, -3)$; $\delta(x, y) = 5$g/cm^2

20. R is the rectangle with corners $(1, -3)$, $(1, 2)$, $(7, 2)$ and $(7, -3)$; $\delta(x, y) = (x + y^2)$g/cm^2

21. R is the triangle with corners $(-1, 0)$, $(1, 0)$, and $(0, 1)$; $\delta(x, y) = 2$lb/in^2

22. R is the triangle with corners $(0, 0)$, $(1, 0)$, and $(0, 1)$; $\delta(x, y) = (x^2 + y^2 + 1)$lb/in^2

23. R is the circle centered at the origin with radius 2; $\delta(x, y) = (x + y + 4)$kg/m^2

24. R is the circle sector bounded by $x^2 + y^2 = 25$ in the first quadrant; $\delta(x, y) = (\sqrt{x^2 + y^2} + 1)$kg/m^2

25. R is the annulus in the first and second quadrants bounded by $x^2 + y^2 = 9$ and $x^2 + y^2 = 36$; $\delta(x, y) = 4$lb/ft^2

26. R is the annulus in the first and second quadrants bounded by $x^2 + y^2 = 9$ and $x^2 + y^2 = 36$; $\delta(x, y) = \sqrt{x^2 + y^2}$lb/ft^2

The *moment of inertia* I is a measure of the tendency of a lamina to resist rotating about an axis or continue to rotate about an axis. I_x is the moment of inertia about the x-axis, I_x is the moment of inertia about the x-axis, and I_O is the moment of inertia about the origin. These are computed as follows:

- $I_x = \iint_R y^2 \, dm$

- $I_y = \iint_R x^2 \, dm$

- $I_O = \iint_R (x^2 + y^2) \, dm$

In Exercises 27 – 30, a lamina corresponding to a planar region R is given with a mass of 16 units. For each, compute I_x, I_y and I_O.

27. R is the 4×4 square with corners at $(-2, -2)$ and $(2, 2)$ with density $\delta(x, y) = 1$.

28. R is the 8×2 rectangle with corners at $(-4, -1)$ and $(4, 1)$ with density $\delta(x, y) = 1$.

29. R is the 4×2 rectangle with corners at $(-2, -1)$ and $(2, 1)$ with density $\delta(x, y) = 2$.

30. R is the circle with radius 2 centered at the origin with density $\delta(x, y) = 4/\pi$.