4: Applications of the Derivative

In Chapter 3, we learned how the first and second derivatives of a function influence its graph. In this chapter we explore other applications of the derivative.

4.1 Related Rates

When two quantities are related by an equation, knowing the value of one quantity can determine the value of the other. For instance, the circumference and radius of a circle are related by $C = 2\pi r$; knowing that $C = 6\pi$ in determines the radius must be 3 in.

The topic of **related rates** takes this one step further: knowing the *rate* at which one quantity is changing can determine the *rate* at which the other changes.

We demonstrate the concepts of related rates through examples.

Example 4.1 **Understanding related rates**

The radius of a circle is growing at a rate of 5 in/hr. At what rate is the circumference growing?

Solution The circumference and radius of a circle are related by $C = 2\pi r$. We are given information about how the length of r changes with respect to time; that is, we are told $\frac{dr}{dt} = 5$ in/hr. We want to know how the length of C changes with respect to time, i.e., we want to know $\frac{dC}{dt}$.

Implicitly differentiate both sides of $C = 2\pi r$ with respect to t:

$$\begin{aligned} C &= 2\pi r \\ \frac{d}{dt}\big(C\big) &= \frac{d}{dt}\big(2\pi r\big) \\ \frac{dC}{dt} &= 2\pi\frac{dr}{dt}. \end{aligned}$$

As we know $\frac{dr}{dt} = 5$in/hr, we know

$$\frac{dC}{dt} = 2\pi 5 = 10\pi \approx 31.4\text{in/hr}.$$

Consider another, similar example.

Note: This section relies heavily on implicit differentiation, so referring back to Section 2.6 may help.

Example 4.2 Finding related rates

Water streams out of a faucet at a rate of 2 in^3/s onto a flat surface at a constant rate, forming a circular puddle that is $1/8$ in deep.

1. At what rate is the area of the puddle growing?
2. At what rate is the radius of the circle growing?

Solution

1. We can answer this question two ways: using "common sense" or related rates. The common sense method states that the volume of the puddle is growing by 2 in^3/s, where

   $$\text{volume of puddle} = \text{area of circle} \times \text{depth}.$$

   Since the depth is constant at $1/8$ in, the area must be growing by 16 in^2/s.

   This approach reveals the underlying related rates principle. Let V and A represent the volume and area of the puddle. We know $V = A \times \frac{1}{8}$. Take the derivative of both sides with respect to t, employing implicit differentiation.

   $$\begin{aligned} V &= \frac{1}{8}A \\ \frac{d}{dt}\big(V\big) &= \frac{d}{dt}\left(\frac{1}{8}A\right) \\ \frac{dV}{dt} &= \frac{1}{8}\frac{dA}{dt} \end{aligned}$$

   As $\frac{dV}{dt} = 2$, we know $2 = \frac{1}{8}\frac{dA}{dt}$, and hence $\frac{dA}{dt} = 16$. Thus the area is growing by 16in^2/s.

2. To start, we need an equation that relates what we know to the radius. We just learned something about the surface area of the circular puddle, and we know $A = \pi r^2$. We should be able to learn about the rate at which the radius is growing with this information.

   Implicitly differentiate both sides of $A = \pi r^2$ with respect to t:

   $$\begin{aligned} A &= \pi r^2 \\ \frac{d}{dt}\big(A\big) &= \frac{d}{dt}\big(\pi r^2\big) \\ \frac{dA}{dt} &= 2\pi r\frac{dr}{dt} \end{aligned}$$

Notes:

Our work above told us that $\frac{dA}{dt} = 16\text{in}^2/\text{s}$. Solving for $\frac{dr}{dt}$, we have

$$\frac{dr}{dt} = \frac{8}{\pi r}\text{in/s}.$$

Note how our answer is not a number, but rather a function of r. In other words, *the rate at which the radius is growing depends on how big the circle already is.* If the circle is very large, adding 2 in^3 of water will not make the circle much bigger at all. If the circle is dime sized, adding the same amount of water will make a radical change in the radius of the circle.

In some ways, our problem was (intentionally) ill–posed. We need to specify a current radius in order to know a rate of change. When the puddle has a radius of 10 in, the radius is growing at a rate of

$$\frac{dr}{dt} = \frac{8}{10\pi} = \frac{4}{5\pi} \approx 0.25\text{in/s}.$$

Example 4.3 Studying related rates

Radar guns measure the rate of distance change between the gun and the object it is measuring. For instance, a reading of "55 mph" means the object is moving away from the gun at a rate of 55 miles per hour, whereas a measurement of "-25 mph" would mean that the object is approaching the gun at a rate of 25 miles per hour.

If the radar gun is moving (say, attached to a police car) then radar readouts are only immediately understandable if the gun and the object are moving along the same line. If a police officer is traveling 60mph and gets a readout of 15mph, he knows that the car ahead of him is moving away at a rate of 15 miles an hour, meaning the car is traveling 75mph. (This straight–line principle is one reason officers park on the side of the highway and try to shoot straight back down the road. It gives the most accurate reading.)

Suppose an officer is driving due north at 30 mph and sees a car moving due east, as shown in Figure 4.1. Using his radar gun, he measures a reading of 20 mph. By using landmarks, he believes both he and the other car are about 1/2 mile from the intersection of their two roads.

If the speed limit on the other road is 55 mph, is the other driver speeding?

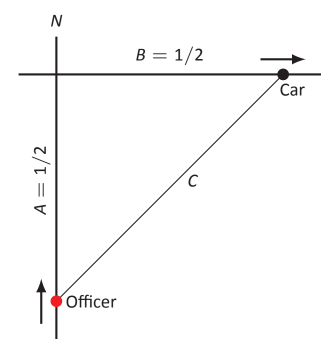


Figure 4.1: A sketch of a police car (at bottom) attempting to measure the speed of a car (at right) in Example 4.3.

Solution Using the diagram in Figure 4.1, let's label what we know about the situation. As both the police officer and other driver are 1/2 mile from the intersection, we have $A = 1/2$, $B = 1/2$, and through the Pythagorean Theorem, $C = 1/\sqrt{2} \approx 0.707$.

We know the police officer is traveling at 30mph; that is, $\frac{dA}{dt} = -30$. The reason this rate of change is negative is that A is getting smaller; the distance

Notes:

Note: Example 4.3 is both interesting and impractical. It highlights the difficulty in using radar in a non–linear fashion, and explains why "in real life" the police officer would follow the other driver to determine their speed, and not pull out pencil and paper.

The principles here are important, though. Many automated vehicles make judgments about other moving objects based on perceived distances, radar–like measurements and the concepts of related rates.

between the officer and the intersection is shrinking. The radar measurement is $\frac{dC}{dt} = 20$. We want to find $\frac{dB}{dt}$.

We need an equation that relates B to A and/or C. The Pythagorean Theorem is a good choice: $A^2 + B^2 = C^2$. Differentiate both sides with respect to t:

$$\begin{aligned} A^2 + B^2 &= C^2 \\ \frac{d}{dt}\left(A^2 + B^2\right) &= \frac{d}{dt}\left(C^2\right) \\ 2A\frac{dA}{dt} + 2B\frac{dB}{dt} &= 2C\frac{dC}{dt} \end{aligned}$$

We have values for everything except $\frac{dB}{dt}$. Solving for this we have

$$\frac{dB}{dt} = \frac{C\frac{dC}{dt} - A\frac{dA}{dt}}{B} = \frac{\frac{1}{\sqrt{2}}(20) - \frac{1}{2}(-30)}{\left(\frac{1}{2}\right)} \approx 58.28 \text{ mph}.$$

The other driver appears to be speeding slightly.

Example 4.4 Studying related rates

A camera is placed on a tripod 10 ft from the side of a road. The camera is to turn to track a car that is to drive by at 100 mph for a promotional video. The video's planners want to know what kind of motor the tripod should be equipped with in order to properly track the car as it passes by. Figure 4.2 shows the proposed setup.

How fast must the camera be able to turn to track the car?

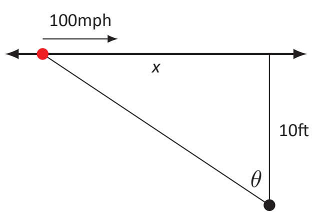


Figure 4.2: Tracking a speeding car (at left) with a rotating camera.

Solution We seek information about how fast the camera is to *turn*; therefore, we need an equation that will relate an angle θ to the position of the camera and the speed and position of the car.

Figure 4.2 suggests we use a trigonometric equation. Letting x represent the distance the car is from the point on the road directly in front of the camera, we have

$$\tan\theta = \frac{x}{10}. \tag{4.1}$$

As the car is moving at 100 mph, we have $\frac{dx}{dt} = -100$ mph (as in the last example, since x is getting smaller as the car travels, $\frac{dx}{dt}$ is negative). We need to convert the measurements so they use the same units; rewrite -100 mph in terms of ft/s:

$$\frac{dx}{dt} = -100\frac{\text{m}}{\text{hr}} = -100\frac{\text{m}}{\text{hr}} \cdot 5280\frac{\text{ft}}{\text{m}} \cdot \frac{1}{3600}\frac{\text{hr}}{\text{s}} = -146.\overline{6}\text{ft/s}.$$

Now take the derivative of both sides of Equation (4.1) using implicit differenti-

Notes:

ation:

$$\begin{aligned}
\tan\theta &= \frac{x}{10} \\
\frac{d}{dt}\big(\tan\theta\big) &= \frac{d}{dt}\left(\frac{x}{10}\right) \\
\sec^2\theta\,\frac{d\theta}{dt} &= \frac{1}{10}\frac{dx}{dt} \\
\frac{d\theta}{dt} &= \frac{\cos^2\theta}{10}\frac{dx}{dt} \qquad (4.2)
\end{aligned}$$

We want to know the fastest the camera has to turn. Common sense tells us this is when the car is directly in front of the camera (i.e., when $\theta = 0$). Our mathematics bears this out. In Equation (4.2) we see this is when $\cos^2\theta$ is largest; this is when $\cos\theta = 1$, or when $\theta = 0$.

With $\frac{dx}{dt} \approx -146.67$ ft/s, we have

$$\frac{d\theta}{dt} = -\frac{1\text{rad}}{10\text{ft}}146.67\text{ft/s} = -14.667\text{radians/s}.$$

We find that $\frac{d\theta}{dt}$ is negative; this matches our diagram in Figure 4.2 for θ is getting smaller as the car approaches the camera.

What is the practical meaning of -14.667 radians/s? Recall that 1 circular revolution goes through 2π radians, thus 14.667 rad/s means $14.667/(2\pi) \approx$ 2.33 revolutions per second. The negative sign indicates the camera is rotating in a clockwise fashion.

We introduced the derivative as a function that gives the slopes of tangent lines of functions. This chapter emphasizes using the derivative in other ways. Newton's Method uses the derivative to approximate roots of functions; this section stresses the "rate of change" aspect of the derivative to find a relationship between the rates of change of two related quantities.

Notes:

Exercises 4.1

Terms and Concepts

1. T/F: Implicit differentiation is often used when solving "related rates" type problems.

2. T/F: A study of related rates is part of the standard police officer training.

Problems

3. Water flows onto a flat surface at a rate of $5\text{cm}^3/\text{s}$ forming a circular puddle 10mm deep. How fast is the radius growing when the radius is:

   (a) 1 cm?

   (b) 10 cm?

   (c) 100 cm?

4. A spherical balloon is inflated with air flowing at a rate of $10\text{cm}^3/\text{s}$. How fast is the radius of the balloon increasing when the radius is:

   (a) 1 cm?

   (b) 10 cm?

   (c) 100 cm?

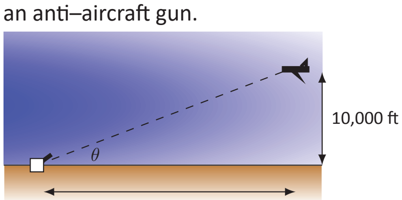
5. Consider the traffic situation introduced in Example 4.3. How fast is the "other car" traveling if the officer and the other car are each 1/2 mile from the intersection, the other car is traveling *due west*, the officer is traveling north at 50mph, and the radar reading is -80mph?

6. Consider the traffic situation introduced in Example 4.3. Calculate how fast the "other car" is traveling in each of the following situations.

   (a) The officer is traveling due north at 50mph and is 1/2 mile from the intersection, while the other car is 1 mile from the intersection traveling west and the radar reading is -80mph?

   (b) The officer is traveling due north at 50mph and is 1 mile from the intersection, while the other car is 1/2 mile from the intersection traveling west and the radar reading is -80mph?

7. An F-22 aircraft is flying at 500mph with an elevation of 10,000ft on a straight–line path that will take it directly over an anti–aircraft gun.



   How fast must the gun be able to turn to accurately track the aircraft when the plane is:

   (a) 1 mile away?

   (b) 1/5 mile away?

   (c) Directly overhead?

8. An F-22 aircraft is flying at 500mph with an elevation of 100ft on a straight–line path that will take it directly over an anti–aircraft gun as in Exercise 7 (note the lower elevation here).

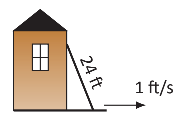
   How fast must the gun be able to turn to accurately track the aircraft when the plane is:

   (a) 1000 feet away?

   (b) 100 feet away?

   (c) Directly overhead?

9. A 24ft. ladder is leaning against a house while the base is pulled away at a constant rate of 1ft/s.



   At what rate is the top of the ladder sliding down the side of the house when the base is:

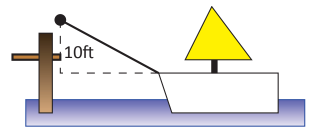
   (a) 1 foot from the house?

   (b) 10 feet from the house?

   (c) 23 feet from the house?

   (d) 24 feet from the house?

10. A rope attached to a boat is being reeled in at a constant rate of 30ft/min by a winch located 10ft above the deck of the boat.



    At what rate is the boat approaching the dock when the boat is:

    (a) 50 feet out?

    (b) 15 feet out?

    (c) 1 foot from the dock?

    (d) What happens when the length of rope pulling in the boat is less than 10 feet long?

11. An inverted circular cone, 20ft deep and 10ft across at the top, is being filled with water at a rate of $10\text{ft}^3/\text{min}$. At what rate is the water rising in the tank when the depth of the water is:

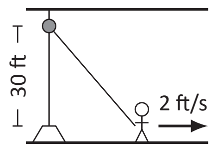
    (a) 1 foot?

    (b) 10 feet?

    (c) 19 feet?

    How long will the tank take to fill when starting at empty?

12. A rope, attached to a weight, goes up through a pulley at the ceiling and back down to a worker. The man holds the rope at the same height as the connection point between rope and weight.



Suppose the man stands directly next to the weight (i.e., a total rope length of 60 ft) and begins to walk away at a rate of 2ft/s. How fast is the weight rising when the man has walked:

(a) 10 feet?

(b) 40 feet?

How far must the man walk to raise the weight all the way to the pulley?

13. Consider the situation described in Exercise 12. Suppose the man starts 40ft from the weight and begins to walk away at a rate of 2ft/s.

(a) How long is the rope?

(b) How fast is the weight rising after the man has walked 10 feet?

(c) How fast is the weight rising after the man has walked 40 feet?

(d) How far must the man walk to raise the weight all the way to the pulley?

14. A hot air balloon lifts off from ground rising vertically. From 100 feet away, a 5' woman tracks the path of the balloon. When her sightline with the balloon makes a 45° angle with the horizontal, she notes the angle is increasing at about 5°/min.

(a) What is the elevation of the balloon?

(b) How fast is it rising?

15. A company that produces landscaping materials is dumping sand into a conical pile. The sand is being poured at a rate of 5ft^3/sec; the physical properties of the sand, in conjunction with gravity, ensure that the cone's height is roughly 2/3 the length of the diameter of the circular base.

How fast is the cone rising when it has a height of 30 feet?

16. A man starts walking north at 4 ft/s, beginning at the Dalton State College Bell Tower. Five minutes later, a woman starts walking at 6 ft/s south, beginning at a point on College Dr. 400 ft due east of the Bell Tower. At what rate are the people moving apart ten minutes after the woman starts walking?