

### 13.3 Double Integration with Polar Coordinates

We have used iterated integrals to evaluate double integrals, which give the signed volume under a surface,  $z = f(x, y)$ , over a region  $R$  of the  $x$ - $y$  plane. The integrand is simply  $f(x, y)$ , and the bounds of the integrals are determined by the region  $R$ .

Some regions  $R$  are easy to describe using rectangular coordinates – that is, with equations of the form  $y = f(x)$ ,  $x = a$ , etc. However, some regions are easier to handle if we represent their boundaries with polar equations of the form  $r = f(\theta)$ ,  $\theta = \alpha$ , etc.

The basic form of the double integral is  $\iint_R f(x, y) dA$ . We interpret this integral as follows: over the region  $R$ , sum up lots of products of heights (given by  $f(x_i, y_i)$ ) and areas (given by  $\Delta A_i$ ). That is,  $dA$  represents “a little bit of area.” In rectangular coordinates, we can describe a small rectangle as having area  $dx dy$  or  $dy dx$  – the area of a rectangle is simply length  $\times$  width – a small change in  $x$  times a small change in  $y$ . Thus we replace  $dA$  in the double integral with  $dx dy$  or  $dy dx$ .

Now consider representing a region  $R$  with polar coordinates. Consider Figure 13.19(a). Let  $R$  be the region in the first quadrant bounded by the curve. We can approximate this region using the natural shape of polar coordinates: portions of sectors of circles. In the figure, one such region is shaded, shown again in part (b) of the figure.

As the area of a sector of a circle with radius  $r$ , subtended by an angle  $\theta$ , is  $A = \frac{1}{2}r^2\theta$ , we can find the area of the shaded region. The whole sector has area  $\frac{1}{2}r_2^2\Delta\theta$ , whereas the smaller, unshaded sector has area  $\frac{1}{2}r_1^2\Delta\theta$ . The area of the shaded region is the difference of these areas:

$$\Delta A_i = \frac{1}{2}r_2^2\Delta\theta - \frac{1}{2}r_1^2\Delta\theta = \frac{1}{2}(r_2^2 - r_1^2)(\Delta\theta) = \frac{r_2 + r_1}{2}(r_2 - r_1)\Delta\theta.$$

Note that  $(r_2 + r_1)/2$  is just the average of the two radii.

To approximate the region  $R$ , we use many such subregions; doing so shrinks the difference  $r_2 - r_1$  between radii to 0 and shrinks the change in angle  $\Delta\theta$  also to 0. We represent these infinitesimal changes in radius and angle as  $dr$  and  $d\theta$ , respectively. Finally, as  $dr$  is small,  $r_2 \approx r_1$ , and so  $(r_2 + r_1)/2 \approx r_1$ . Thus, when  $dr$  and  $d\theta$  are small,

$$\Delta A_i \approx r_i dr d\theta.$$

Taking a limit, where the number of subregions goes to infinity and both  $r_2 - r_1$  and  $\Delta\theta$  go to 0, we get

$$dA = r dr d\theta.$$

So to evaluate  $\iint_R f(x, y) dA$ , replace  $dA$  with  $r dr d\theta$ . Convert the function  $z = f(x, y)$  to a function with polar coordinates with the substitutions  $x =$

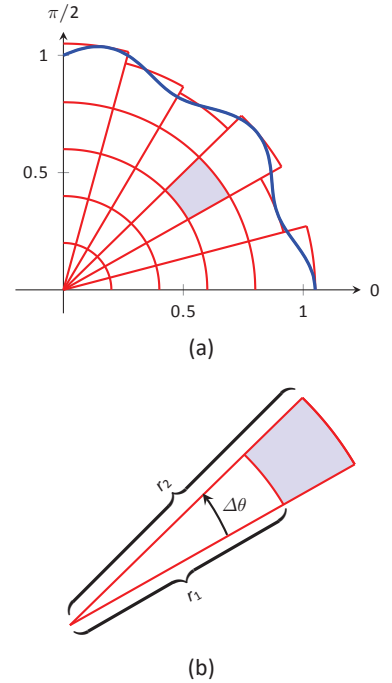


Figure 13.19: Approximating a region  $R$  with portions of sectors of circles.

Notes:

$r \cos \theta$ ,  $y = r \sin \theta$ . Finally, find bounds  $g_1(\theta) \leq r \leq g_2(\theta)$  and  $\alpha \leq \theta \leq \beta$  that describe  $R$ . This is the key principle of this section, so we restate it here as a Key Idea.

**Key Idea 59 Evaluating Double Integrals with Polar Coordinates**

Let  $R$  be a plane region bounded by the polar equations  $\alpha \leq \theta \leq \beta$  and  $g_1(\theta) \leq r \leq g_2(\theta)$ . Then

$$\iint_R f(x, y) \, dA = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta.$$

Examples will help us understand this Key Idea.

**Example 13.16 Evaluating a double integral with polar coordinates**

Find the signed volume under the plane  $z = 4 - x - 2y$  over the circle with equation  $x^2 + y^2 = 1$ .

**SOLUTION** The bounds of the integral are determined solely by the region  $R$  over which we are integrating. In this case, it is a circle with equation  $x^2 + y^2 = 1$ . We need to find polar bounds for this region. It may help to review Section 9.4; bounds for this circle are  $0 \leq r \leq 1$  and  $0 \leq \theta \leq 2\pi$ .

We replace  $f(x, y)$  with  $f(r \cos \theta, r \sin \theta)$ . That means we make the following substitutions:

$$4 - x - 2y \Rightarrow 4 - r \cos \theta - 2r \sin \theta.$$

Finally, we replace  $dA$  in the double integral with  $r \, dr \, d\theta$ . This gives the final iterated integral, which we evaluate:

$$\begin{aligned} \iint_R f(x, y) \, dA &= \int_0^{2\pi} \int_0^1 (4 - r \cos \theta - 2r \sin \theta) r \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^1 (4r - r^2(\cos \theta - 2 \sin \theta)) \, dr \, d\theta \\ &= \int_0^{2\pi} \left( 2r^2 - \frac{1}{3}r^3(\cos \theta - 2 \sin \theta) \right) \Big|_0^1 d\theta \\ &= \int_0^{2\pi} \left( 2 - \frac{1}{3}(\cos \theta - 2 \sin \theta) \right) d\theta \\ &= \left( 2\theta - \frac{1}{3}(\sin \theta + 2 \cos \theta) \right) \Big|_0^{2\pi} \\ &= 4\pi \approx 12.566. \end{aligned}$$

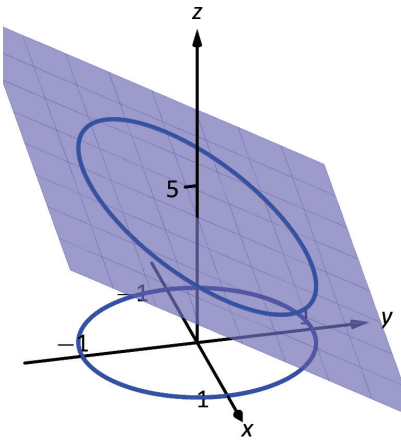


Figure 13.20: Evaluating a double integral with polar coordinates in Example 13.16.

Notes:

The surface and region  $R$  are shown in Figure 13.20.

**Example 13.17 Evaluating a double integral with polar coordinates**

Find the volume under the paraboloid  $z = 4 - (x - 2)^2 - y^2$  over the region bounded by the circles  $(x - 1)^2 + y^2 = 1$  and  $(x - 2)^2 + y^2 = 4$ .

**SOLUTION** At first glance, this seems like a very hard volume to compute as the region  $R$  (shown in Figure 13.21(a)) has a hole in it, cutting out a strange portion of the surface, as shown in part (b) of the figure. However, by describing  $R$  in terms of polar equations, the volume is not very difficult to compute. It is straightforward to show that the circle  $(x - 1)^2 + y^2 = 1$  has polar equation  $r = 2 \cos \theta$ , and that the circle  $(x - 2)^2 + y^2 = 4$  has polar equation  $r = 4 \cos \theta$ . Each of these circles is traced out on the interval  $0 \leq \theta \leq \pi$ . The bounds on  $r$  are  $2 \cos \theta \leq r \leq 4 \cos \theta$ .

Replacing  $x$  with  $r \cos \theta$  in the integrand, along with replacing  $y$  with  $r \sin \theta$ , prepares us to evaluate the double integral  $\iint_R f(x, y) \, dA$ :

$$\begin{aligned} \iint_R f(x, y) \, dA &= \int_0^\pi \int_{2 \cos \theta}^{4 \cos \theta} (4 - (r \cos \theta - 2)^2 - (r \sin \theta)^2) r \, dr \, d\theta \\ &= \int_0^\pi \int_{2 \cos \theta}^{4 \cos \theta} (-r^3 + 4r^2 \cos \theta) \, dr \, d\theta \\ &= \int_0^\pi \left( -\frac{1}{4}r^4 + \frac{4}{3}r^3 \cos \theta \right) \bigg|_{2 \cos \theta}^{4 \cos \theta} d\theta \\ &= \int_0^\pi \left( \left[ -\frac{1}{4}(256 \cos^4 \theta) + \frac{4}{3}(64 \cos^4 \theta) \right] - \left[ -\frac{1}{4}(16 \cos^4 \theta) + \frac{4}{3}(8 \cos^4 \theta) \right] \right) d\theta \\ &= \int_0^\pi \frac{44}{3} \cos^4 \theta \, d\theta. \end{aligned}$$

Notes:

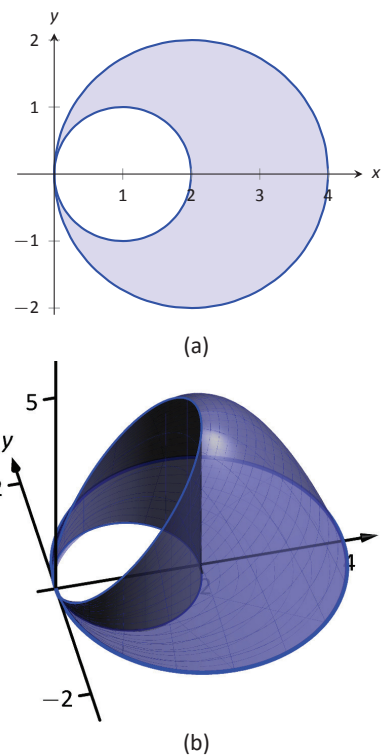


Figure 13.21: Showing the region  $R$  and surface used in Example 13.17.

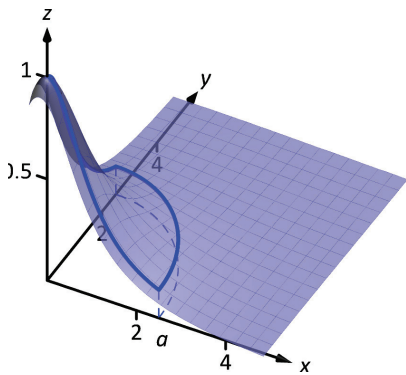


Figure 13.22: The surface and region  $R$  used in Example 13.21.

To integrate  $\cos^4 \theta$ , rewrite it as  $\cos^2 \theta \cos^2 \theta$  and employ the power-reducing formula twice:

$$\begin{aligned}\cos^4 \theta &= \cos^2 \theta \cos^2 \theta \\ &= \frac{1}{2}(1 + \cos(2\theta)) \frac{1}{2}(1 + \cos(2\theta)) \\ &= \frac{1}{4}(1 + 2\cos(2\theta) + \cos^2(2\theta)) \\ &= \frac{1}{4}\left(1 + 2\cos(2\theta) + \frac{1}{2}(1 + \cos(4\theta))\right) \\ &= \frac{3}{8} + \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta).\end{aligned}$$

Picking up from where we left off above, we have

$$\begin{aligned}&= \int_0^\pi \frac{44}{3} \cos^4 \theta \, d\theta \\ &= \int_0^\pi \frac{44}{3} \left( \frac{3}{8} + \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta) \right) d\theta \\ &= \frac{44}{3} \left( \frac{3}{8}\theta + \frac{1}{4}\sin(2\theta) + \frac{1}{32}\sin(4\theta) \right) \Big|_0^\pi \\ &= \frac{11}{2}\pi \approx 17.279.\end{aligned}$$

While this example was not trivial, the double integral would have been *much* harder to evaluate had we used rectangular coordinates.

#### Example 13.18 Evaluating a double integral with polar coordinates

Find the volume under the surface  $f(x, y) = \frac{1}{x^2 + y^2 + 1}$  over the sector of the circle with radius  $a$  centered at the origin in the first quadrant, as shown in Figure 13.22.

**SOLUTION** The region  $R$  we are integrating over is a circle with radius  $a$ , restricted to the first quadrant. Thus, in polar, the bounds on  $R$  are  $0 \leq r \leq a$ ,  $0 \leq \theta \leq \pi/2$ . The integrand is rewritten in polar as

$$\frac{1}{x^2 + y^2 + 1} \Rightarrow \frac{1}{r^2 \cos^2 \theta + r^2 \sin^2 \theta + 1} = \frac{1}{r^2 + 1}.$$

Notes:

We find the volume as follows:

$$\begin{aligned}
 \iint_R f(x, y) \, dA &= \int_0^{\pi/2} \int_0^a \frac{r}{r^2 + 1} \, dr \, d\theta \\
 &= \int_0^{\pi/2} \left. \frac{1}{2} (\ln |r^2 + 1|) \right|_0^a d\theta \\
 &= \int_0^{\pi/2} \frac{1}{2} \ln(a^2 + 1) \, d\theta \\
 &= \left( \frac{1}{2} \ln(a^2 + 1) \theta \right) \Big|_0^{\pi/2} \\
 &= \frac{\pi}{4} \ln(a^2 + 1).
 \end{aligned}$$

Figure 13.22 shows that  $f$  shrinks to near 0 very quickly. Regardless, as  $a$  grows, so does the volume, without bound.

### Example 13.19 Finding the volume of a sphere

Find the volume of a sphere with radius  $a$ .

**SOLUTION** The sphere of radius  $a$ , centered at the origin, has equation  $x^2 + y^2 + z^2 = a^2$ ; solving for  $z$ , we have  $z = \sqrt{a^2 - x^2 - y^2}$ . This gives the upper half of a sphere. We wish to find the volume under this top half, then double it to find the total volume.

The region we need to integrate over is the circle of radius  $a$ , centered at the origin. Polar bounds for this equation are  $0 \leq r \leq a$ ,  $0 \leq \theta \leq 2\pi$ .

All together, the volume of a sphere with radius  $a$  is:

$$\begin{aligned}
 2 \iint_R \sqrt{a^2 - x^2 - y^2} \, dA &= 2 \int_0^{2\pi} \int_0^a \sqrt{a^2 - (r \cos \theta)^2 - (r \sin \theta)^2} r \, dr \, d\theta \\
 &= 2 \int_0^{2\pi} \int_0^a r \sqrt{a^2 - r^2} \, dr \, d\theta.
 \end{aligned}$$

**Note:** Previous work has shown that there is finite *area* under  $\frac{1}{x^2+1}$  over the entire  $x$ -axis. However, Example 13.21 shows that there is infinite *volume* under  $\frac{1}{x^2+y^2+1}$  over the entire  $x$ - $y$  plane.

Notes:

We can evaluate this inner integral with substitution. With  $u = a^2 - r^2$ ,  $du = -2r \, dr$ . The new bounds of integration are  $u(0) = a^2$  to  $u(a) = 0$ . Thus we have:

$$\begin{aligned}
 &= \int_0^{2\pi} \int_{a^2}^0 (-u^{1/2}) \, du \, d\theta \\
 &= \int_0^{2\pi} \left( -\frac{2}{3} u^{3/2} \right) \Big|_{a^2}^0 \, d\theta \\
 &= \int_0^{2\pi} \left( \frac{2}{3} a^3 \right) \, d\theta \\
 &= \left( \frac{2}{3} a^3 \theta \right) \Big|_0^{2\pi} \\
 &= \frac{4}{3} \pi a^3.
 \end{aligned}$$

Generally, the formula for the volume of a sphere with radius  $r$  is given as  $\frac{4}{3}\pi r^3$ ; we have justified this formula with our calculation. In Section 13.8, we will be able to use spherical coordinates to write a triple integral that is even easier to compute.

#### Example 13.20 Finding the volume of a solid

A sculptor wants to make a solid bronze cast of the solid shown in Figure 13.23, where the base of the solid has boundary, in polar coordinates,  $r = \cos(3\theta)$ , and the top is defined by the plane  $z = 1 - x + 0.1y$ . Find the volume of the solid.

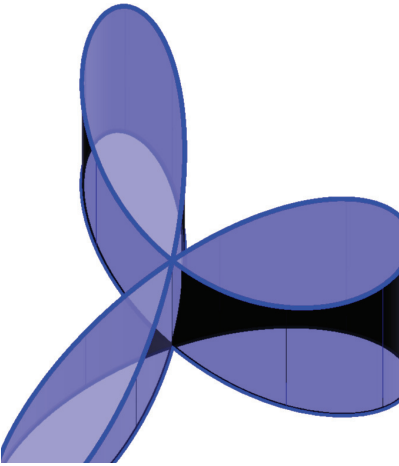


Figure 13.23: Visualizing the solid used in Example 13.20.

**SOLUTION** From the outset, we should recognize that knowing *how to set up* this problem is probably more important than knowing *how to compute the integrals*. The iterated integral to come is not “hard” to evaluate, though it is long, requiring lots of algebra. Once the proper iterated integral is determined, one can use readily-available technology to help compute the final answer.

The region  $R$  that we are integrating over is bound by  $0 \leq r \leq \cos(3\theta)$ , for  $0 \leq \theta \leq \pi$  (note that this rose curve is traced out on the interval  $[0, \pi]$ , not  $[0, 2\pi]$ ). This gives us our bounds of integration. The integrand is  $z = 1 - x + 0.1y$ ; converting to polar, we have that the volume  $V$  is:

$$V = \iint_R f(x, y) \, dA = \int_0^\pi \int_0^{\cos(3\theta)} (1 - r \cos \theta + 0.1r \sin \theta) r \, dr \, d\theta.$$

Distributing the  $r$ , the inner integral is easy to evaluate, leading to

$$\int_0^\pi \left( \frac{1}{2} \cos^2(3\theta) - \frac{1}{3} \cos^3(3\theta) \cos \theta + \frac{0.1}{3} \cos^3(3\theta) \sin \theta \right) d\theta.$$

Notes:

This integral takes time to compute by hand; it is rather long and cumbersome. The powers of cosine need to be reduced, and products like  $\cos(3\theta) \cos \theta$  need to be turned to sums using the Product To Sum formulas in Section 6.2 (and in the back cover of this text).

We rewrite  $\frac{1}{2} \cos^2(3\theta)$  as  $\frac{1}{4}(1 + \cos(6\theta))$ . We can also rewrite  $\frac{1}{3} \cos^3(3\theta) \cos \theta$  as:

$$\frac{1}{3} \cos^3(3\theta) \cos \theta = \frac{1}{3} \cos^2(3\theta) \cos(3\theta) \cos \theta = \frac{1}{3} \cdot \frac{1 + \cos(6\theta)}{2} (\cos(4\theta) + \cos(2\theta)).$$

This last expression still needs simplification, but eventually all terms can be reduced to the form  $a \cos(m\theta)$  or  $a \sin(m\theta)$  for various values of  $a$  and  $m$ .

We forgo the algebra and recommend the reader employ technology, such as WolframAlpha®, to compute the numeric answer. Such technology gives:

$$\int_0^\pi \int_0^{\cos(3\theta)} (1 - r \cos \theta + 0.1r \sin \theta) r \, dr \, d\theta = \frac{\pi}{4} \approx 0.785.$$

Since the units were not specified, we leave the result as 0.785 cubic units (meters, feet, etc.) Should the artist want to scale the piece uniformly, so that each rose petal had a length other than 1, she should keep in mind that scaling by a factor of  $k$  scales the volume by a factor of  $k^3$ .

We finish up with an example, showing a situation in which we can evaluate a single-variable definite integral (in which the integrand does not have an elementary antiderivative) by rewriting it as a double integral and converting to polar coordinates. Specifically, it is easy with the Direct Comparison Test to show the integral  $\int_0^\infty e^{-x^2} dx$  converges, but what exactly does it converge to?

We mentioned, without proof, in Section 6.7 that it converges to  $\sqrt{\pi}/2$ , an important result in statistics. Below, we give a proof.

**Example 13.21** Proving  $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$  using polar coordinates

Evaluate  $\int_0^\infty e^{-x^2} dx$ .

**SOLUTION** Let  $I$  denote the integral  $\int_0^\infty e^{-x^2} dx$  we wish to compute.

Then we start with  $I^2$  by multiplying  $I$  by itself. Since  $x$  is just a “dummy variable”, we change to  $y$  in the second integral, with the corresponding  $dy$ . Using two variables allows us to combine the expression into a single integral. We will not discuss improper double integrals, except in this example. The first quadrant ( $x > 0, y > 0$ ) is expressible in polar coordinates as ( $r > 0, 0 \leq \theta \leq \pi/2$ ).

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Notes:

$$\begin{aligned}
I^2 &= \left( \int_0^\infty e^{-x^2} dx \right) \left( \int_0^\infty e^{-y^2} dy \right) \\
&= \int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2} dx dy && \text{(use Theorem 122)} \\
&= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy && \text{(exponent rules)} \\
&= \int_0^{\pi/2} \int_0^\infty e^{-r^2} r dr d\theta && \text{(convert to polar)} \\
&= \left( \int_0^{\pi/2} d\theta \right) \left( \int_0^\infty e^{-r^2} r dr \right) && \text{(use Theorem 122)} \\
&= \frac{\pi}{2} \lim_{b \rightarrow \infty} \int_0^b e^{-r^2} r dr && \text{(definition of improper integral)} \\
&= \frac{\pi}{2} \lim_{b \rightarrow \infty} -\frac{1}{2} \int_0^{-b^2} e^u du && \text{(substitution } u = -r^2) \\
&= -\frac{\pi}{4} \lim_{b \rightarrow \infty} (e^{-b^2} - 1) \\
&= \frac{\pi}{4}.
\end{aligned}$$

Since  $I^2 = \pi/4$ , it follows that  $I = \pm\sqrt{\pi}/2$ . It is obvious that  $I = \int_0^\infty e^{-x^2} dx$ , the signed area of a region above the  $x$ -axis, must be positive. Therefore,  $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$ .

We have used iterated integrals to find areas of plane regions and volumes under surfaces. Just as a single integral can be used to compute much more than “area under the curve,” iterated integrals can be used to compute much more than we have thus far seen. The next two sections show two, among many, applications of iterated integrals.

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Notes:



## Exercises 13.3

### Terms and Concepts

- When evaluating  $\iint_R f(x, y) \, dA$  using polar coordinates,  $f(x, y)$  is replaced with \_\_\_\_\_ and  $dA$  is replaced with \_\_\_\_\_.
- Why would one be interested in evaluating a double integral with polar coordinates?

### Problems

In Exercises 3 – 10, a function  $f(x, y)$  is given and a region  $R$  of the  $x$ - $y$  plane is described. Set up and evaluate  $\iint_R f(x, y) \, dA$  using polar coordinates.

- $f(x, y) = 3x - y + 4$ ;  $R$  is the region enclosed by the circle  $x^2 + y^2 = 1$ .
- $f(x, y) = 4x + 4y$ ;  $R$  is the region enclosed by the circle  $x^2 + y^2 = 4$ .
- $f(x, y) = 8 - y$ ;  $R$  is the region enclosed by the circles with polar equations  $r = \cos \theta$  and  $r = 3 \cos \theta$ .
- $f(x, y) = 4$ ;  $R$  is the region enclosed by the petal of the rose curve  $r = \sin(2\theta)$  in the first quadrant.
- $f(x, y) = \ln(x^2 + y^2)$ ;  $R$  is the annulus enclosed by the circles  $x^2 + y^2 = 1$  and  $x^2 + y^2 = 4$ .
- $f(x, y) = 1 - x^2 - y^2$ ;  $R$  is the region enclosed by the circle  $x^2 + y^2 = 1$ .
- $f(x, y) = x^2 - y^2$ ;  $R$  is the region enclosed by the circle  $x^2 + y^2 = 36$  in the first and fourth quadrants.
- $f(x, y) = (x - y)/(x + y)$ ;  $R$  is the region enclosed by the lines  $y = x$ ,  $y = 0$  and the circle  $x^2 + y^2 = 1$  in the first quadrant.

In Exercises 11 – 14, an iterated integral in rectangular coordinates is given. Rewrite the integral using polar coordinates and evaluate the new double integral.

- $\int_0^5 \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} \sqrt{x^2 + y^2} \, dy \, dx$
- $\int_{-4}^4 \int_{-\sqrt{16-y^2}}^0 (2y - x) \, dx \, dy$
- $\int_0^2 \int_y^{\sqrt{8-y^2}} (x + y) \, dx \, dy$
- $\int_{-2}^{-1} \int_0^{\sqrt{4-x^2}} (x + 5) \, dy \, dx + \int_{-1}^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} (x + 5) \, dy \, dx + \int_1^2 \int_0^{\sqrt{4-x^2}} (x + 5) \, dy \, dx$

**Hint:** draw the region of each integral carefully and see how they all connect.

In Exercises 15 – 16, special double integrals are presented that are especially well suited for evaluation in polar coordinates.

- Consider  $\iint_R e^{-(x^2+y^2)} \, dA$ .
  - Why is this integral difficult to evaluate in rectangular coordinates, regardless of the region  $R$ ?
  - Let  $R$  be the region bounded by the circle of radius  $a$  centered at the origin. Evaluate the double integral using polar coordinates.
  - Take the limit of your answer from (b), as  $a \rightarrow \infty$ . What does this imply about the volume under the surface of  $e^{-(x^2+y^2)}$  over the entire  $x$ - $y$  plane?
- The surface of a right circular cone with height  $h$  and base radius  $a$  can be described by the equation  $f(x, y) = h - h\sqrt{\frac{x^2}{a^2} + \frac{y^2}{a^2}}$ , where the tip of the cone lies at  $(0, 0, h)$  and the circular base lies in the  $x$ - $y$  plane, centered at the origin.

Confirm that the volume of a right circular cone with height  $h$  and base radius  $a$  is  $V = \frac{1}{3}\pi a^2 h$  by evaluating

$\iint_R f(x, y) \, dA$  in polar coordinates.