

**A<sub>P</sub>E<sub>X</sub> CALCULUS III**  
Version 3.0

**Gregory Hartman, Ph.D.**

*Department of Applied Mathematics  
Virginia Military Institute*

*Contributing Authors*

**Troy Siemers, Ph.D.**

*Department of Applied Mathematics  
Virginia Military Institute*

**Brian Heinold, Ph.D.**

*Department of Mathematics and Computer Science  
Mount Saint Mary's University*

**Dimplekumar Chalishajar, Ph.D.**

*Department of Applied Mathematics  
Virginia Military Institute*

*Editor*

**Jennifer Bowen, Ph.D.**

*Department of Mathematics and Computer Science  
The College of Wooster*



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# PREFACE

## *A Note on Using this Text*

Thank you for reading this short preface. Allow us to share a few key points about the text so that you may better understand what you will find beyond this page.

This text is Part III of a three-text series on Calculus. The first part covers material taught in many “Calc 1” courses: limits, derivatives, and the basics of integration, found in Chapters 1 through 6.1. The second text covers material often taught in “Calc 2:” integration and its applications, along with an introduction to sequences, series and Taylor Polynomials, found in Chapters 5 through 8. The third text covers topics common in “Calc 3” or “multivariable calc:” parametric equations, polar coordinates, vector-valued functions, and functions of more than one variable, found in Chapters 9 through 13. All three are available separately for free at [www.vmi.edu/APEX](http://www.vmi.edu/APEX). These three texts are intended to work together and make one cohesive text, *APEX Calculus*, which can also be downloaded from the website.

Printing the entire text as one volume makes for a large, heavy, cumbersome book. One can certainly only print the pages they currently need, but some prefer to have a nice, bound copy of the text. Therefore this text has been split into these three manageable parts, each of which can be purchased for under \$15 at Amazon.com.

A result of this splitting is that sometimes a concept is said to be explored in an “earlier section,” though that section does not actually appear in this particular text. Also, the index makes reference to topics, and page numbers, that do not appear in this text. This is done intentionally to show the reader what topics are available for study. Downloading the .pdf of *APEX Calculus* will ensure that you have all the content.

## **For Students: How to Read this Text**

Mathematics textbooks have a reputation for being hard to read. High-level mathematical writing often seeks to say much with few words, and this style often seeps into texts of lower-level topics. This book was written with the goal of being easier to read than many other calculus textbooks, without becoming too verbose.

Each chapter and section starts with an introduction of the coming material, hopefully setting the stage for “why you should care,” and ends with a look ahead to see how the just-learned material helps address future problems.

*Please read the text;* it is written to explain the concepts of Calculus. There are numerous examples to demonstrate the meaning of definitions, the truth of theorems, and the application of mathematical techniques. When you encounter a sentence you don’t understand, read it again. If it still doesn’t make sense, read on anyway, as sometimes confusing sentences are explained by later sentences.

*You don’t have to read every equation.* The examples generally show “all” the steps needed to solve a problem. Sometimes reading through each step is helpful; sometimes it is confusing. When the steps are illustrating a new technique, one probably should follow each step closely to learn the new technique. When the steps are showing the mathematics needed to find a number to be used later, one can usually skip ahead and see how that number is being used, instead of getting bogged down in reading how the number was found.

*Most proofs have been omitted.* In mathematics, *proving* something is always true is extremely important, and entails much more than testing to see if it works twice. However, students often are confused by the details of a proof, or become concerned that they should have been able to construct this proof on their own. To alleviate this potential problem, we do not include the proofs to most theorems in the text. The interested reader is highly encouraged to find proofs online or from their instructor. In most cases, one is very capable of understanding what a theorem *means* and *how to apply it* without knowing fully *why* it is true.

## Interactive, 3D Graphics

New to Version 3.0 is the addition of interactive, 3D graphics in the .pdf version. Nearly all graphs of objects in space can be rotated, shifted, and zoomed in/out so the reader can better understand the object illustrated.

As of this writing, the only pdf viewers that support these 3D graphics are Adobe Reader & Acrobat (and only the versions for PC/Mac/Unix/Linux computers, not tablets or smartphones). To activate the interactive mode, click on the image. Once activated, one can click/drag to rotate the object and use the scroll wheel on a mouse to zoom in/out. (A great way to investigate an image is to first zoom in on the page of the pdf viewer so the graphic itself takes up much of the screen, then zoom inside the graphic itself.) A CTRL-click/drag pans the object left/right or up/down. By right-clicking on the graph one can access a menu of other options, such as changing the lighting scheme or perspective. One can also revert the graph back to its default view. If you wish to deactivate the interactivity, one can right-click and choose the “Disable Content” option.

## Thanks

There are many people who deserve recognition for the important role they have played in the development of this text. First, I thank Michelle for her support and encouragement, even as this “project from work” occupied my time and attention at home. Many thanks to Troy Siemers, whose most important contributions extend far beyond the sections he wrote or the 227 figures he coded in Asymptote for 3D interaction. He provided incredible support, advice and encouragement for which I am very grateful. My thanks to Brian Heinold and Dimplekumar Chalishajar for their contributions and to Jennifer Bowen for reading through so much material and providing great feedback early on. Thanks to Troy, Lee Dewald, Dan Joseph, Meagan Herald, Bill Lowe, John David, Vonda Walsh, Geoff Cox, Jessica Libertini and other faculty of VMI who have given me numerous suggestions and corrections based on their experience with teaching from the text. (Special thanks to Troy, Lee & Dan for their patience in teaching Calc III while I was still writing the Calc III material.) Thanks to Randy Cone for encouraging his tutors of VMI’s Open Math Lab to read through the text and check the solutions, and thanks to the tutors for spending their time doing so. A very special thanks to Kristi Brown and Paul Janiczek who took this opportunity far above & beyond what I expected, meticulously checking every solution and carefully reading every example. Their comments have been extraordinarily helpful. I am also thankful for the support provided by Wane Schneiter, who as my Dean provided me with extra time to work on this project. I am blessed to have so many people give of their time to make this book better.

## **A<sub>P</sub>E<sub>X</sub> – Affordable Print and Electronic teXts**

A<sub>P</sub>E<sub>X</sub> is a consortium of authors who collaborate to produce high-quality, low-cost textbooks. The current textbook-writing paradigm is facing a potential revolution as desktop publishing and electronic formats increase in popularity. However, writing a good textbook is no easy task, as the time requirements alone are substantial. It takes countless hours of work to produce text, write examples and exercises, edit and publish. Through collaboration, however, the cost to any individual can be lessened, allowing us to create texts that we freely distribute electronically and sell in printed form for an incredibly low cost. Having said that, nothing is entirely free; someone always bears some cost. This text “cost” the authors of this book their time, and that was not enough. *APEX Calculus* would not exist had not the Virginia Military Institute, through a generous Jackson-Hope grant, given the lead author significant time away from teaching so he could focus on this text.

Each text is available as a free .pdf, protected by a Creative Commons Attribution - Noncommercial 4.0 copyright. That means you can give the .pdf to anyone you like, print it in any form you like, and even edit the original content and redistribute it. If you do the latter, you must clearly reference this work and you cannot sell your edited work for money.

We encourage others to adapt this work to fit their own needs. One might add sections that are “missing” or remove sections that your students won’t need. The source files can be found at [github.com/APEXCalculus](https://github.com/APEXCalculus).

You can learn more at [www.vmi.edu/APEX](http://www.vmi.edu/APEX).



# 9: VECTORS

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This chapter introduces a new mathematical object, the **vector**. Defined in Section 9.2, we will see that vectors provide a powerful language for describing quantities that have magnitude and direction aspects. A simple example of such a quantity is force: when applying a force, one is generally interested in how much force is applied (i.e., the magnitude of the force) and the direction in which the force was applied. Vectors will play an important role in many of the subsequent chapters in this text.

This chapter begins with moving our mathematics out of the plane and into “space.” That is, we begin to think mathematically not only in two dimensions, but in three. With this foundation, we can explore vectors both in the plane and in space.

## 9.1 Introduction to Cartesian Coordinates in Space

Up to this point in this text we have considered mathematics in a 2-dimensional world. We have plotted graphs on the  $x$ - $y$  plane using rectangular and polar coordinates and found the area of regions in the plane. We have considered properties of *solid* objects, such as volume and surface area, but only by first defining a curve in the plane and then rotating it out of the plane.

While there is wonderful mathematics to explore in “2D,” we live in a “3D” world and eventually we will want to apply mathematics involving this third dimension. In this section we introduce Cartesian coordinates in space and explore basic surfaces. This will lay a foundation for much of what we do in the remainder of the text.

Each point  $P$  in space can be represented with an ordered triple,  $P = (a, b, c)$ , where  $a$ ,  $b$  and  $c$  represent the relative position of  $P$  along the  $x$ -,  $y$ - and  $z$ -axes, respectively. Each axis is perpendicular to the other two.

Visualizing points in space on paper can be problematic, as we are trying to represent a 3-dimensional concept on a 2-dimensional medium. We cannot draw three lines representing the three axes in which each line is perpendicular to the other two. Despite this issue, standard conventions exist for plotting shapes in space that we will discuss that are more than adequate.

One convention is that the axes must conform to the **right hand rule**. This rule states that when the index finger of the right hand is extended in the direction of the positive  $x$ -axis, and the middle finger (bent “inward” so it is perpendicular to the palm) points along the positive  $y$ -axis, then the extended thumb will point in the direction of the positive  $z$ -axis. (It may take some thought to

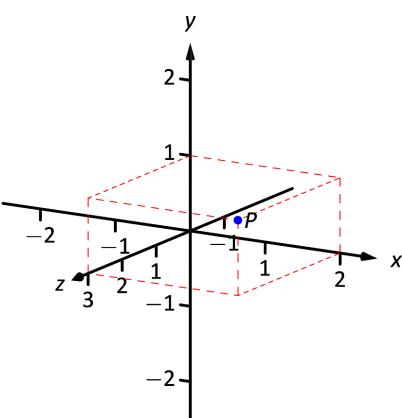


Figure 9.1: Plotting the point  $P = (2, 1, 3)$  in space.

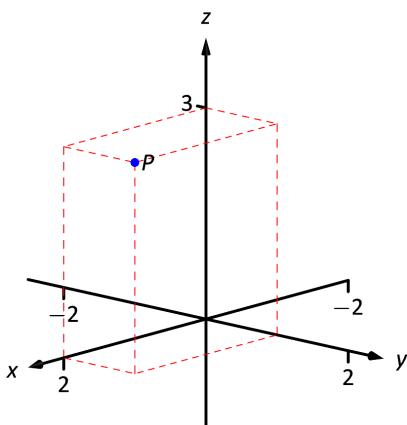


Figure 9.2: Plotting the point  $P = (2, 1, 3)$  in space with a perspective used in this text.

verify this, but this system is inherently different from the one created by using the “left hand rule.”)

As long as the coordinate axes are positioned so that they follow this rule, it does not matter how the axes are drawn on paper. There are two popular methods that we briefly discuss.

In Figure 9.1 we see the point  $P = (2, 1, 3)$  plotted on a set of axes. The basic convention here is that the  $x$ - $y$  plane is drawn in its standard way, with the  $z$ -axis down to the left. The perspective is that the paper represents the  $x$ - $y$  plane and the positive  $z$  axis is coming up, off the page. This method is preferred by many engineers. Because it can be hard to tell where a single point lies in relation to all the axes, dashed lines have been added to let one see how far along each axis the point lies.

One can also consider the  $x$ - $y$  plane as being a horizontal plane in, say, a room, where the positive  $z$ -axis is pointing up. When one steps back and looks at this room, one might draw the axes as shown in Figure 9.2. The same point  $P$  is drawn, again with dashed lines. This point of view is preferred by most mathematicians, and is the convention adopted by this text.

## Measuring Distances

It is of critical importance to know how to measure distances between points in space. The formula for doing so is based on measuring distance in the plane, and is known (in both contexts) as the Euclidean measure of distance.

### Definition 25 Distance In Space

Let  $P = (x_1, y_1, z_1)$  and  $Q = (x_2, y_2, z_2)$  be points in space. The distance  $D$  between  $P$  and  $Q$  is

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

We refer to the line segment that connects points  $P$  and  $Q$  in space as  $\overline{PQ}$ , and refer to the length of this segment as  $\|\overline{PQ}\|$ . The above distance formula allows us to compute the length of this segment.

### Example 150 Length of a line segment

Let  $P = (1, 4, -1)$  and let  $Q = (2, 1, 1)$ . Draw the line segment  $\overline{PQ}$  and find its length.

**SOLUTION** The points  $P$  and  $Q$  are plotted in Figure 9.3; no special consideration need be made to draw the line segment connecting these two points;

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Notes:

simply connect them with a straight line. One *cannot* actually measure this line on the page and deduce anything meaningful; its true length must be measured analytically. Applying Definition 25, we have

$$\|\overline{PQ}\| = \sqrt{(2-1)^2 + (1-4)^2 + (1-(-1))^2} = \sqrt{14} \approx 3.74.$$

## Spheres

Just as a circle is the set of all points in the *plane* equidistant from a given point (its center), a sphere is the set of all points in *space* that are equidistant from a given point. Definition 25 allows us to write an equation of the sphere.

We start with a point  $C = (a, b, c)$  which is to be the center of a sphere with radius  $r$ . If a point  $P = (x, y, z)$  lies on the sphere, then  $P$  is  $r$  units from  $C$ ; that is,

$$\|\overline{PC}\| = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = r.$$

Squaring both sides, we get the standard equation of a sphere in space with center at  $C = (a, b, c)$  with radius  $r$ , as given in the following Key Idea.

### Key Idea 33 Standard Equation of a Sphere in Space

The standard equation of the sphere with radius  $r$ , centered at  $C = (a, b, c)$ , is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

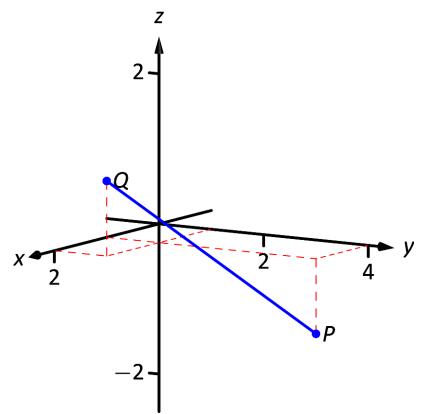


Figure 9.3: Plotting points  $P$  and  $Q$  in Example 150.

### Example 151 Equation of a sphere

Find the center and radius of the sphere defined by  $x^2 + 2x + y^2 - 4y + z^2 - 6z = 2$ .

**SOLUTION** To determine the center and radius, we must put the equation in standard form. This requires us to complete the square (three times).

$$\begin{aligned} x^2 + 2x + y^2 - 4y + z^2 - 6z &= 2 \\ (x^2 + 2x + 1) + (y^2 - 4y + 4) + (z^2 - 6z + 9) - 14 &= 2 \\ (x+1)^2 + (y-2)^2 + (z-3)^2 &= 16 \end{aligned}$$

The sphere is centered at  $(-1, 2, 3)$  and has a radius of 4.

The equation of a sphere is an example of an implicit function defining a surface in space. In the case of a sphere, the variables  $x$ ,  $y$  and  $z$  are all used. We now consider situations where surfaces are defined where one or two of these

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variables are absent.

## Introduction to Planes in Space

The coordinate axes naturally define three planes (shown in Figure 9.4), the **coordinate planes**: the  $x$ - $y$  plane, the  $y$ - $z$  plane and the  $x$ - $z$  plane. The  $x$ - $y$  plane is characterized as the set of all points in space where the  $z$ -value is 0. This, in fact, gives us an equation that describes this plane:  $z = 0$ . Likewise, the  $x$ - $z$  plane is all points where the  $y$ -value is 0, characterized by  $y = 0$ .

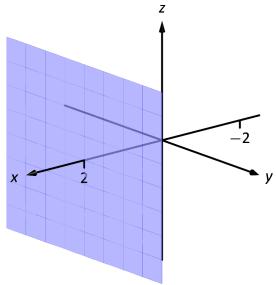


Figure 9.5: The plane  $x = 2$ .

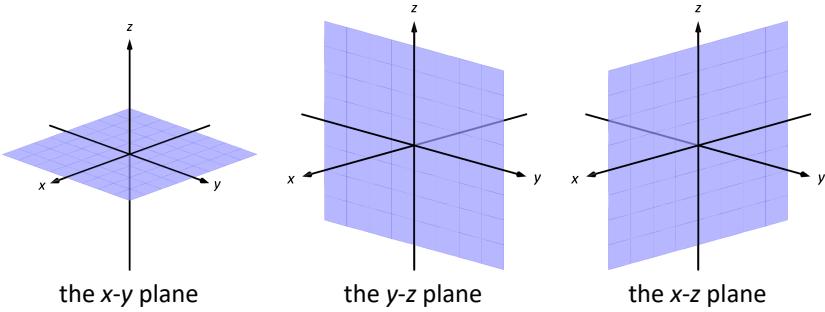


Figure 9.4: The coordinate planes.

The equation  $x = 2$  describes all points in space where the  $x$ -value is 2. This is a plane, parallel to the  $y$ - $z$  coordinate plane, shown in Figure 9.5.

### Example 152 Regions defined by planes

Sketch the region defined by the inequalities  $-1 \leq y \leq 2$ .

**SOLUTION** The region is all points between the planes  $y = -1$  and  $y = 2$ . These planes are sketched in Figure 9.6, which are parallel to the  $x$ - $z$  plane. Thus the region extends infinitely in the  $x$  and  $z$  directions, and is bounded by planes in the  $y$  direction.

## Cylinders

The equation  $x = 1$  obviously lacks the  $y$  and  $z$  variables, meaning it defines points where the  $y$  and  $z$  coordinates can take on any value. Now consider the equation  $x^2 + y^2 = 1$  in space. In the plane, this equation describes a circle of radius 1, centered at the origin. In space, the  $z$  coordinate is not specified, meaning it can take on any value. In Figure 9.8 (a), we show part of the graph of the equation  $x^2 + y^2 = 1$  by sketching 3 circles: the bottom one has a constant  $z$ -value of  $-1.5$ , the middle one has a  $z$ -value of  $0$  and the top circle has a  $z$ -value

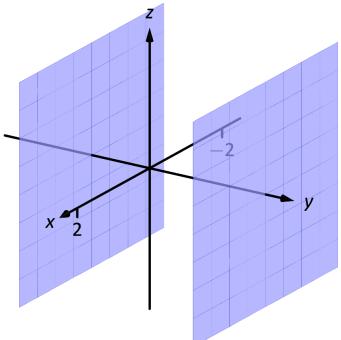


Figure 9.6: Sketching the boundaries of a region in Example 152.

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Notes:

of 1. By plotting *all* possible z-values, we get the surface shown in Figure 9.8 (b). This surface looks like a “tube,” or a “cylinder”; mathematicians call this surface a **cylinder** for an entirely different reason.

### Definition 26 Cylinder

Let  $C$  be a curve in a plane and let  $L$  be a line not parallel to  $C$ . A **cylinder** is the set of all lines parallel to  $L$  that pass through  $C$ . The curve  $C$  is the **directrix** of the cylinder, and the lines are the **rulings**.

In this text, we consider curves  $C$  that lie in planes parallel to one of the coordinate planes, and lines  $L$  that are perpendicular to these planes, forming **right cylinders**. Thus the directrix can be defined using equations involving 2 variables, and the rulings will be parallel to the axis of the 3<sup>rd</sup> variable.

In the example preceding the definition, the curve  $x^2 + y^2 = 1$  in the  $x$ - $y$  plane is the directrix and the rulings are lines parallel to the  $z$ -axis. (Any circle shown in Figure 9.8 can be considered a directrix; we simply choose the one where  $z = 0$ .) Sample rulings can also be viewed in part (b) of the figure. More examples will help us understand this definition.

### Example 153 Graphing cylinders

Graph the cylinder following cylinders.

1.  $z = y^2$
2.  $x = \sin z$

#### SOLUTION

1. We can view the equation  $z = y^2$  as a parabola in the  $y$ - $z$  plane, as illustrated in Figure 9.7 (a). As  $x$  does not appear in the equation, the rulings are lines through this parabola parallel to the  $x$ -axis, shown in (b). These rulings give a general idea as to what the surface looks like, drawn in (c).

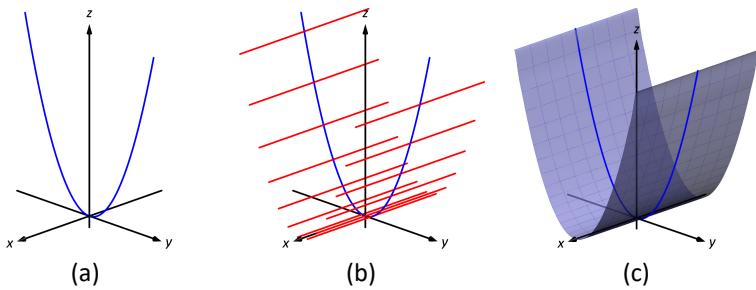


Figure 9.7: Sketching the cylinder defined by  $z = y^2$ .

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Notes:

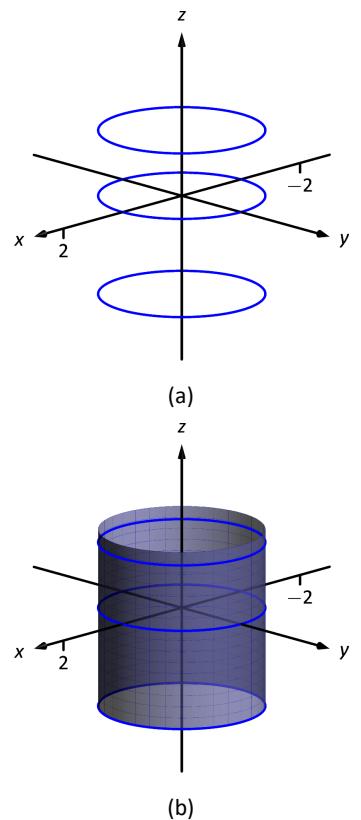


Figure 9.8: Sketching  $x^2 + y^2 = 1$ .

2. We can view the equation  $x = \sin z$  as a sine curve that exists in the  $x$ - $z$  plane, as shown in Figure 9.9 (a). The rules are parallel to the  $y$  axis as the variable  $y$  does not appear in the equation  $x = \sin z$ ; some of these are shown in part (b). The surface is shown in part (c) of the figure.

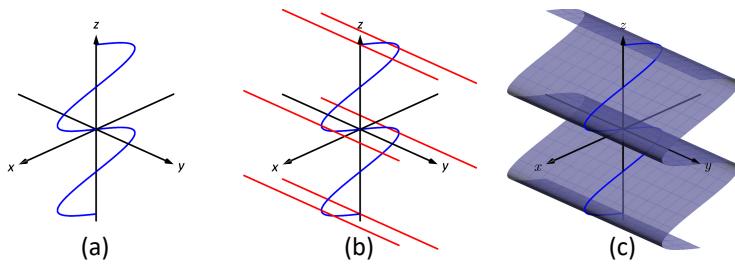
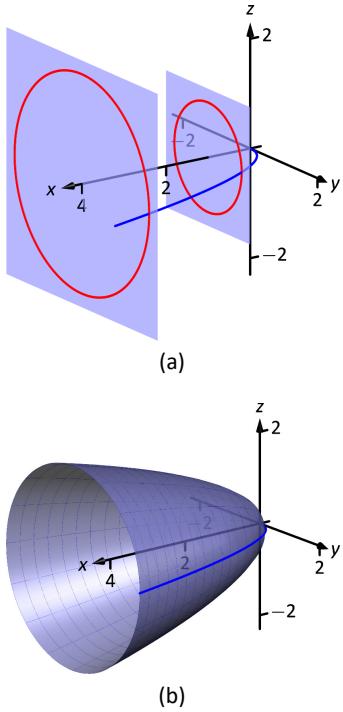
Figure 9.9: Sketching the cylinder defined by  $x = \sin z$ .

Figure 9.10: Introducing surfaces of revolution.

## Surfaces of Revolution

One of the applications of integration we learned previously was to find the volume of solids of revolution – solids formed by revolving a curve about a horizontal or vertical axis. We now consider how to find the equation of the surface of such a solid.

Consider the surface formed by revolving  $y = \sqrt{x}$  about the  $x$ -axis. Cross-sections of this surface parallel to the  $y$ - $z$  plane are circles, as shown in Figure 9.10(a). Each circle has equation of the form  $y^2 + z^2 = r^2$  for some radius  $r$ . The radius is a function of  $x$ ; in fact, it is  $r(x) = \sqrt{x}$ . Thus the equation of the surface shown in Figure 9.10b is  $y^2 + z^2 = (\sqrt{x})^2$ .

We generalize the above principles to give the equations of surfaces formed by revolving curves about the coordinate axes.

### Key Idea 34 Surfaces of Revolution, Part 1

Let  $r$  be a radius function.

1. The equation of the surface formed by revolving  $y = r(x)$  or  $z = r(x)$  about the  $x$ -axis is  $y^2 + z^2 = r(x)^2$ .
2. The equation of the surface formed by revolving  $x = r(y)$  or  $z = r(y)$  about the  $y$ -axis is  $x^2 + z^2 = r(y)^2$ .
3. The equation of the surface formed by revolving  $x = r(z)$  or  $y = r(z)$  about the  $z$ -axis is  $x^2 + y^2 = r(z)^2$ .

Notes:

**Example 154 Finding equation of a surface of revolution**

Let  $y = \sin z$  on  $[0, \pi]$ . Find the equation of the surface of revolution formed by revolving  $y = \sin z$  about the  $z$ -axis.

**SOLUTION** Using Key Idea 34, we find the surface has equation  $x^2 + y^2 = \sin^2 z$ . The curve is sketched in Figure 9.11(a) and the surface is drawn in Figure 9.11(b).

Note how the surface (and hence the resulting equation) is the same if we began with the curve  $x = \sin z$ , which is also drawn in Figure 9.11(a).

This particular method of creating surfaces of revolution is limited. For instance, in Example 49 of Section 7.3 we found the volume of the solid formed by revolving  $y = \sin x$  about the  $y$ -axis. Our current method of forming surfaces can only rotate  $y = \sin x$  about the  $x$ -axis. Trying to rewrite  $y = \sin x$  as a function of  $y$  is not trivial, as simply writing  $x = \sin^{-1} y$  only gives part of the region we desire.

What we desire is a way of writing the surface of revolution formed by rotating  $y = f(x)$  about the  $y$ -axis. We start by first recognizing this surface is the same as revolving  $z = f(x)$  about the  $z$ -axis. This will give us a more natural way of viewing the surface.

A value of  $x$  is a measurement of distance from the  $z$ -axis. At the distance  $r$ , we plot a  $z$ -height of  $f(r)$ . When rotating  $f(x)$  about the  $z$ -axis, we want all points a distance of  $r$  from the  $z$ -axis in the  $x$ - $y$  plane to have a  $z$ -height of  $f(r)$ . All such points satisfy the equation  $r^2 = x^2 + y^2$ ; hence  $r = \sqrt{x^2 + y^2}$ . Replacing  $r$  with  $\sqrt{x^2 + y^2}$  in  $f(r)$  gives  $z = f(\sqrt{x^2 + y^2})$ . This is the equation of the surface.

**Key Idea 35 Surfaces of Revolution, Part 2**

Let  $z = f(x)$ ,  $x \geq 0$ , be a curve in the  $x$ - $z$  plane. The surface formed by revolving this curve about the  $z$ -axis has equation  $z = f(\sqrt{x^2 + y^2})$ .

**Example 155 Finding equation of surface of revolution**

Find the equation of the surface found by revolving  $z = \sin x$  about the  $z$ -axis.

**SOLUTION** Using Key Idea 35, the surface has equation  $z = \sin(\sqrt{x^2 + y^2})$ . The curve and surface are graphed in Figure 9.12.

Notes:

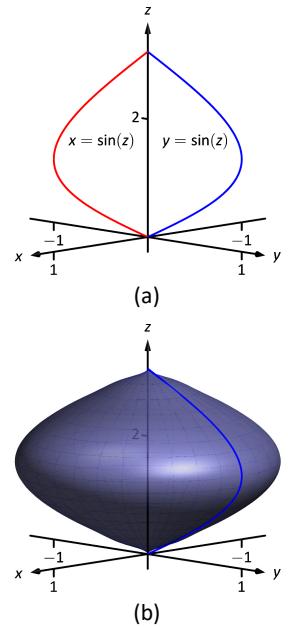


Figure 9.11: Revolving  $y = \sin z$  about the  $z$ -axis in Example 154.

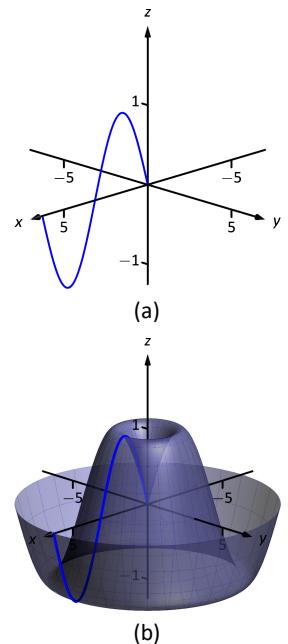


Figure 9.12: Revolving  $z = \sin x$  about the  $z$ -axis in Example 155.

## Quadratic Surfaces

Spheres, planes and cylinders are important surfaces to understand. We now consider one last type of surface, a **quadric surface**. The definition may look intimidating, but we will show how to analyze these surfaces in an illuminating way.

### Definition 27 Quadric Surface

A **quadric surface** is the graph of the general second-degree equation in three variables:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

When the coefficients  $D, E$  or  $F$  are not zero, the basic shapes of the quadric surfaces are rotated in space. We will focus on quadric surfaces where these coefficients are 0; we will not consider rotations. There are six basic quadric surfaces: the elliptic paraboloid, elliptic cone, ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, and the hyperbolic paraboloid.

We study each shape by considering **traces**, that is, intersections of each surface with a plane parallel to a coordinate plane. For instance, consider the elliptic paraboloid  $z = x^2/4 + y^2$ , shown in Figure 9.13. If we intersect this shape with the plane  $z = d$  (i.e., replace  $z$  with  $d$ ), we have the equation:

$$d = \frac{x^2}{4} + y^2.$$

Divide both sides by  $d$ :

$$1 = \frac{x^2}{4d} + \frac{y^2}{d}.$$

This describes an ellipse – so cross sections parallel to the  $x$ - $y$  coordinate plane are ellipses. This ellipse is drawn in the figure.

Now consider cross sections parallel to the  $x$ - $z$  plane. For instance, letting  $y = 0$  gives the equation  $z = x^2/4$ , clearly a parabola. Intersecting with the plane  $x = 0$  gives a cross section defined by  $z = y^2$ , another parabola. These parabolas are also sketched in the figure.

Thus we see where the elliptic paraboloid gets its name: some cross sections are ellipses, and others are parabolas.

Such an analysis can be made with each of the quadric surfaces. We give a sample equation of each, provide a sketch with representative traces, and describe these traces.

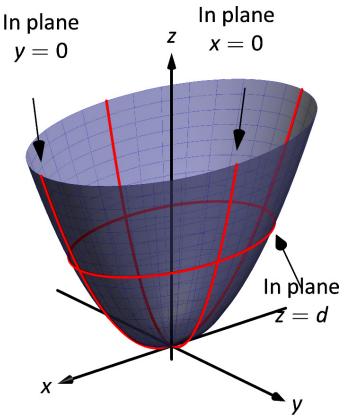
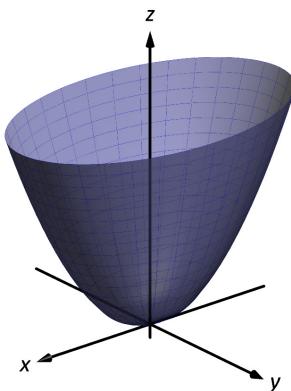


Figure 9.13: The elliptic paraboloid  $z = x^2/4 + y^2$ .

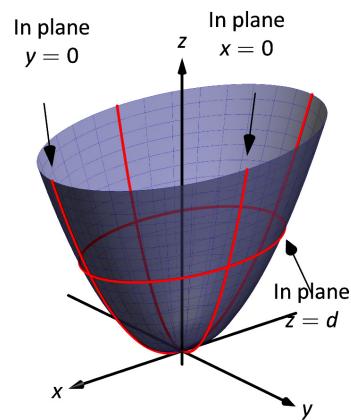
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Notes:

**Elliptic Paraboloid,**  $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$



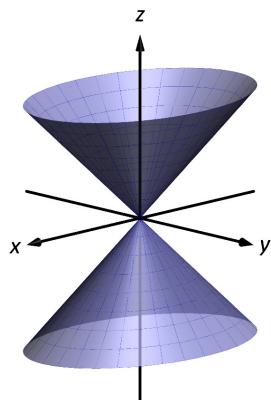
Plane	Trace
$x = d$	Parabola
$y = d$	Parabola
$z = d$	Ellipse



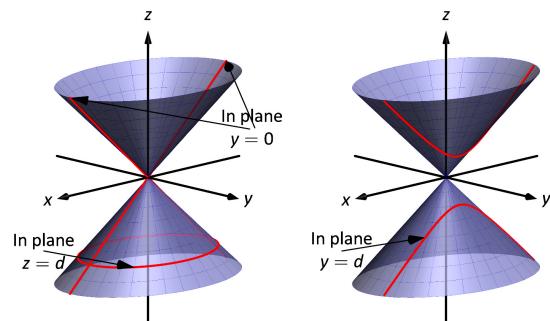
One variable in the equation of the elliptic paraboloid will be raised to the first power; above, this is the  $z$  variable. The paraboloid will “open” in the direction of this variable’s axis. Thus  $x = y^2/a^2 + z^2/b^2$  is an elliptic paraboloid that opens along the  $x$ -axis.

Multiplying the right hand side by  $(-1)$  defines an elliptic paraboloid that “opens” in the opposite direction.

**Elliptic Cone,**  $z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

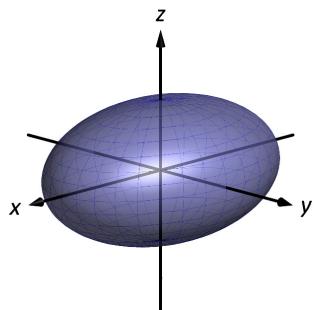


Plane	Trace
$x = 0$	Crossed Lines
$y = 0$	Crossed Lines
$x = d$	Hyperbola
$y = d$	Hyperbola
$z = d$	Ellipse

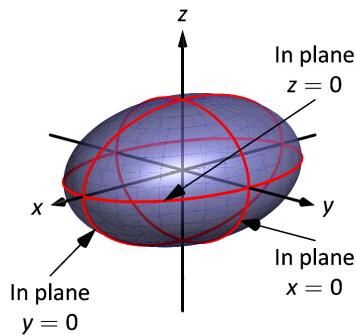


One can rewrite the equation as  $z^2 - x^2/a^2 - y^2/b^2 = 0$ . The one variable with a positive coefficient corresponds to the axis that the cones “open” along.

**Ellipsoid,**  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



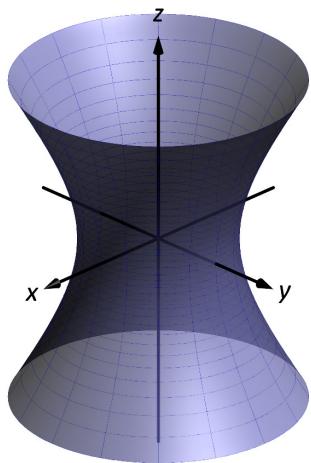
Plane	Trace
$x = d$	Ellipse
$y = d$	Ellipse
$z = d$	Ellipse



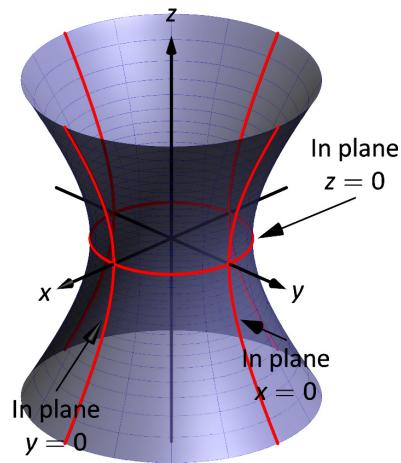
If  $a = b = c \neq 0$ , the ellipsoid is a sphere with radius  $a$ ; compare to Key Idea 33.

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**Hyperboloid of One Sheet,**  $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

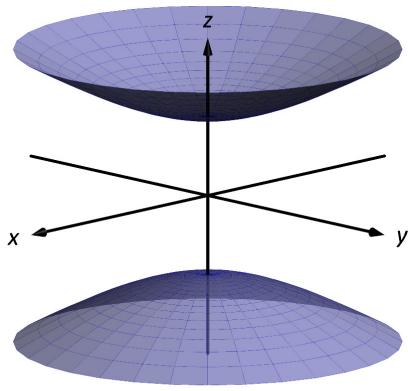


Plane	Trace
$x = d$	Hyperbola
$y = d$	Hyperbola
$z = d$	Ellipse

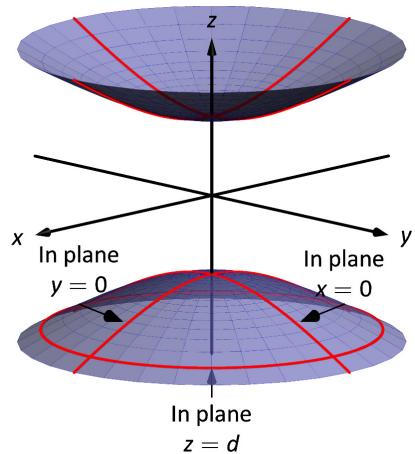


The one variable with a negative coefficient corresponds to the axis that the hyperboloid “opens” along.

**Hyperboloid of Two Sheets,**  $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

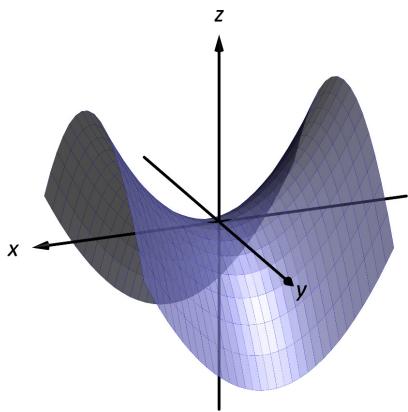


Plane	Trace
$x = d$	Hyperbola
$y = d$	Hyperbola
$z = d$	Ellipse

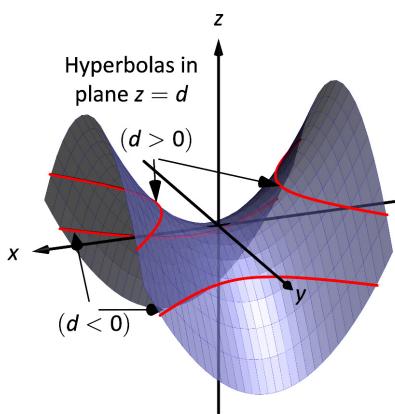
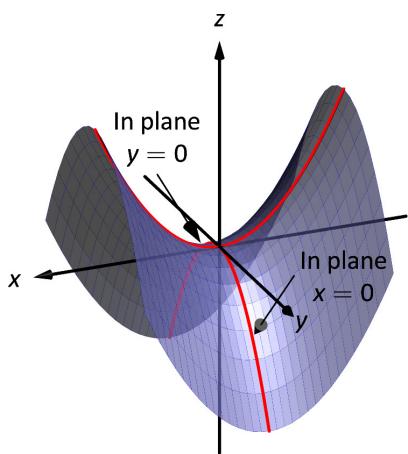


The one variable with a positive coefficient corresponds to the axis that the hyperboloid “opens” along. In the case illustrated, when  $|d| < |c|$ , there is no trace.

**Hyperbolic Paraboloid,**  $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$



Plane	Trace
$x = d$	Parabola
$y = d$	Parabola
$z = d$	Hyperbola



The parabolic traces will open along the axis of the one variable that is raised to the first power.

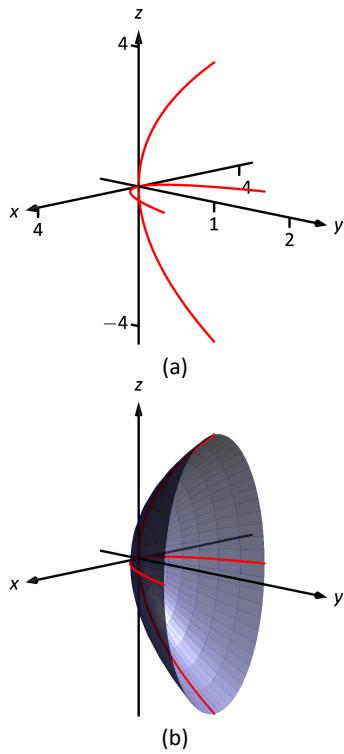


Figure 9.14: Sketching an elliptic paraboloid.

**Example 156 Sketching quadric surfaces**

Sketch the quadric surface defined by the given equation.

$$1. y = \frac{x^2}{4} + \frac{z^2}{16} \quad 2. x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1. \quad 3. z = y^2 - x^2.$$

**SOLUTION**

$$1. y = \frac{x^2}{4} + \frac{z^2}{16}:$$

We first identify the quadric by pattern-matching with the equations given previously. Only two surfaces have equations where one variable is raised to the first power, the elliptic paraboloid and the hyperbolic paraboloid. In the latter case, the other variables have different signs, so we conclude that this describes a hyperbolic paraboloid. As the variable with the first power is  $y$ , we note the paraboloid opens along the  $y$ -axis.

To make a decent sketch by hand, we need only draw a few traces. In this case, the traces  $x = 0$  and  $z = 0$  form parabolas that outline the shape.

$x = 0$ : The trace is the parabola  $y = z^2/16$

$z = 0$ : The trace is the parabola  $y = x^2/4$ .

Graphing each trace in the respective plane creates a sketch as shown in Figure 9.14(a). This is enough to give an idea of what the paraboloid looks like. The surface is filled in in (b).

$$2. x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1 :$$

This is an ellipsoid. We can get a good idea of its shape by drawing the traces in the coordinate planes.

$x = 0$ : The trace is the ellipse  $\frac{y^2}{9} + \frac{z^2}{4} = 1$ . The major axis is along the  $y$ -axis with length 6 (as  $b = 3$ , the length of the axis is 6); the minor axis is along the  $z$ -axis with length 4.

$y = 0$ : The trace is the ellipse  $x^2 + \frac{z^2}{4} = 1$ . The major axis is along the  $z$ -axis, and the minor axis has length 2 along the  $x$ -axis.

$z = 0$ : The trace is the ellipse  $x^2 + \frac{y^2}{9} = 1$ , with major axis along the  $y$ -axis.

Graphing each trace in the respective plane creates a sketch as shown in Figure 9.15(a). Filling in the surface gives Figure 9.15(b).

$$3. z = y^2 - x^2:$$

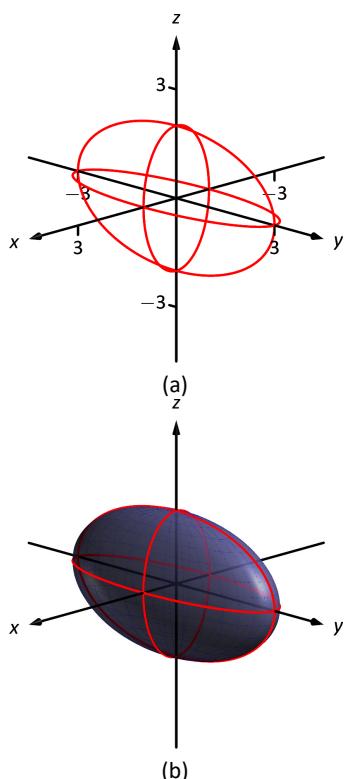


Figure 9.15: Sketching an ellipsoid.

Notes:

This defines a hyperbolic paraboloid, very similar to the one shown in the gallery of quadric sections. Consider the traces in the  $y-z$  and  $x-z$  planes:

$x = 0$ : The trace is  $z = y^2$ , a parabola opening up in the  $y-z$  plane.

$y = 0$ : The trace is  $z = -x^2$ , a parabola opening down in the  $x-z$  plane.

Sketching these two parabolas gives a sketch like that in Figure 9.16 (a), and filling in the surface gives a sketch like (b).

### Example 157 Identifying quadric surfaces

Consider the quadric surface shown in Figure 9.17. Which of the following equations best fits this surface?

- (a)  $x^2 - y^2 - \frac{z^2}{9} = 0$       (c)  $z^2 - x^2 - y^2 = 1$   
 (b)  $x^2 - y^2 - z^2 = 1$       (d)  $4x^2 - y^2 - \frac{z^2}{9} = 1$

**SOLUTION** The image clearly displays a hyperboloid of two sheets. The gallery informs us that the equation will have a form similar to  $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ .

We can immediately eliminate option (a), as the constant in that equation is not 1.

The hyperboloid “opens” along the  $x$ -axis, meaning  $x$  must be the only variable with a positive coefficient, eliminating (c).

The hyperboloid is wider in the  $z$ -direction than in the  $y$ -direction, so we need an equation where  $c > b$ . This eliminates (b), leaving us with (d). We should verify that the equation given in (d),  $4x^2 - y^2 - \frac{z^2}{9} = 1$ , fits.

We already established that this equation describes a hyperboloid of two sheets that opens in the  $x$ -direction and is wider in the  $z$ -direction than in the  $y$ . Now note the coefficient of the  $x$ -term. Rewriting  $4x^2$  in standard form, we have:  $4x^2 = \frac{x^2}{(1/2)^2}$ . Thus when  $y = 0$  and  $z = 0$ ,  $x$  must be  $1/2$ ; i.e., each hyperboloid “starts” at  $x = 1/2$ . This matches our figure.

We conclude that  $4x^2 - y^2 - \frac{z^2}{9} = 1$  best fits the graph.

This section has introduced points in space and shown how equations can describe surfaces. The next sections explore *vectors*, an important mathematical object that we'll use to explore curves in space.

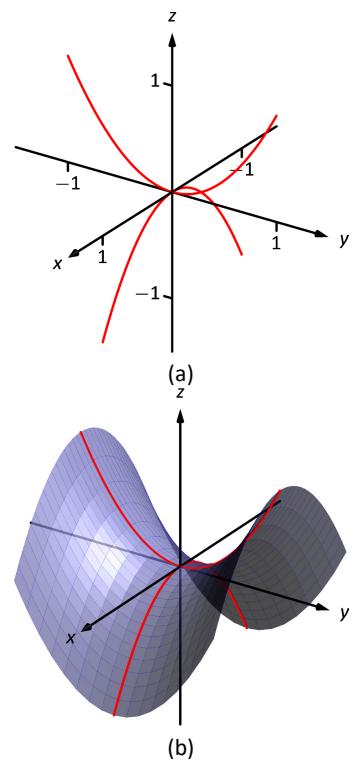


Figure 9.16: Sketching a hyperbolic paraboloid.

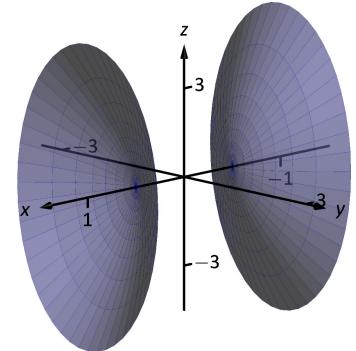


Figure 9.17: A possible equation of this quadric surface is found in Example 157.

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Notes:

# Exercises 9.1

## Terms and Concepts

1. Axes drawn in space must conform to the \_\_\_\_\_ rule.
2. In the plane, the equation  $x = 2$  defines a \_\_\_\_\_; in space,  $x = 2$  defines a \_\_\_\_\_.
3. In the plane, the equation  $y = x^2$  defines a \_\_\_\_\_; in space,  $y = x^2$  defines a \_\_\_\_\_.
4. Which quadric surface looks like a Pringles® chip?
5. Consider the hyperbola  $x^2 - y^2 = 1$  in the plane. If this hyperbola is rotated about the  $x$ -axis, what quadric surface is formed?
6. Consider the hyperbola  $x^2 - y^2 = 1$  in the plane. If this hyperbola is rotated about the  $y$ -axis, what quadric surface is formed?

## Problems

7. The points  $A = (1, 4, 2)$ ,  $B = (2, 6, 3)$  and  $C = (4, 3, 1)$  form a triangle in space. Find the distances between each pair of points and determine if the triangle is a right triangle.
8. The points  $A = (1, 1, 3)$ ,  $B = (3, 2, 7)$ ,  $C = (2, 0, 8)$  and  $D = (0, -1, 4)$  form a quadrilateral  $ABCD$  in space. Is this a parallelogram?
9. Find the center and radius of the sphere defined by  $x^2 - 8x + y^2 + 2y + z^2 + 8 = 0$ .
10. Find the center and radius of the sphere defined by  $x^2 + y^2 + z^2 + 4x - 2y - 4z + 4 = 0$ .

**In Exercises 11 – 14, describe the region in space defined by the inequalities.**

11.  $x^2 + y^2 + z^2 < 1$
12.  $0 \leq x \leq 3$
13.  $x \geq 0, y \geq 0, z \geq 0$
14.  $y \geq 3$

**In Exercises 15 – 18, sketch the cylinder in space.**

15.  $z = x^3$
16.  $y = \cos z$
17.  $\frac{x^2}{4} + \frac{y^2}{9} = 1$

18.  $y = \frac{1}{x}$

**In Exercises 19 – 22, give the equation of the surface of revolution described.**

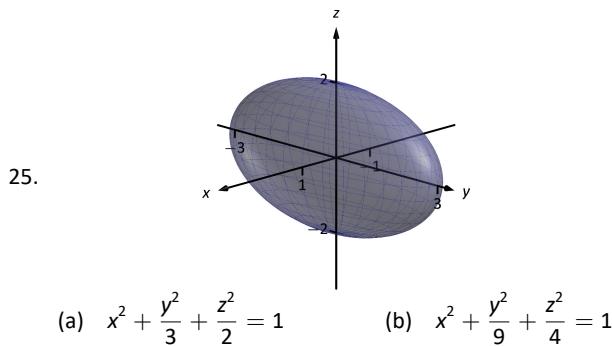
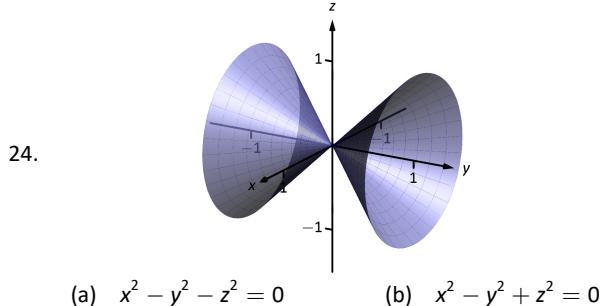
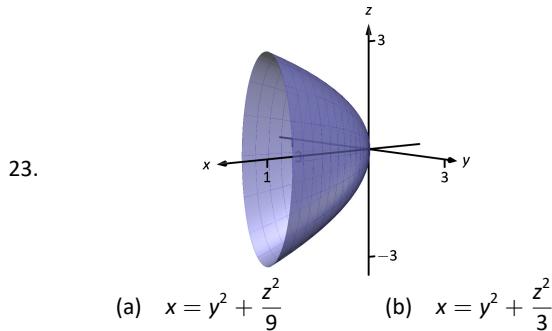
19. Revolve  $z = \frac{1}{1+y^2}$  about the  $y$ -axis.

20. Revolve  $y = x^2$  about the  $x$ -axis.

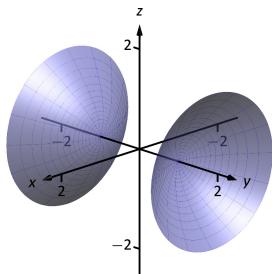
21. Revolve  $z = x^2$  about the  $z$ -axis.

22. Revolve  $z = 1/x$  about the  $z$ -axis.

**In Exercises 23 – 26, a quadric surface is sketched. Determine which of the given equations best fits the graph.**



26.



(a)  $y^2 - x^2 - z^2 = 1$

(b)  $y^2 + x^2 - z^2 = 1$

**In Exercises 27 – 32, sketch the quadric surface.**

27.  $z - y^2 + x^2 = 0$

28.  $z^2 = x^2 + \frac{y^2}{4}$

29.  $x = -y^2 - z^2$

30.  $16x^2 - 16y^2 - 16z^2 = 1$

31.  $\frac{x^2}{9} - y^2 + \frac{z^2}{25} = 1$

32.  $4x^2 + 2y^2 + z^2 = 4$

## 9.2 An Introduction to Vectors

Many quantities we think about daily can be described by a single number: temperature, speed, cost, weight and height. There are also many other concepts we encounter daily that cannot be described with just one number. For instance, a weather forecaster often describes wind with its speed and its direction ("... with winds from the southeast gusting up to 30 mph ..."). When applying a force, we are concerned with both the magnitude and direction of that force. In both of these examples, *direction* is important. Because of this, we study *vectors*, mathematical objects that convey both magnitude and direction information.

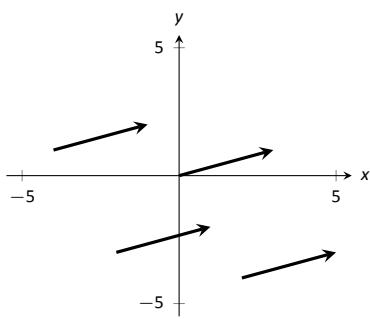


Figure 9.18: Drawing the same vector with different initial points.

### Definition 28 Vector

A **vector** is a directed line segment.

Given points  $P$  and  $Q$  (either in the plane or in space), we denote with  $\vec{PQ}$  the vector from  $P$  to  $Q$ . The point  $P$  is said to be the **initial point** of the vector, and the point  $Q$  is the **terminal point**.

The **magnitude, length** or **norm** of  $\vec{PQ}$  is the length of the line segment  $\overline{PQ}$ :  $\| \vec{PQ} \| = \| \overline{PQ} \|$ .

Two vectors are **equal** if they have the same magnitude and direction.

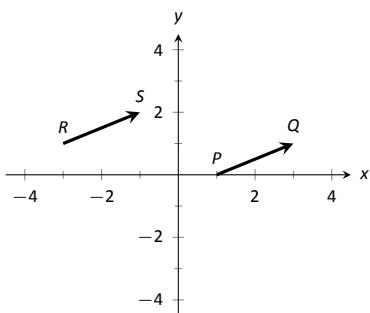


Figure 9.19: Illustrating how equal vectors have the same displacement.

Figure 9.18 shows multiple instances of the same vector. Each directed line segment has the same direction and length (magnitude), hence each is the same vector.

We use  $\mathbb{R}^2$  (pronounced "r two") to represent all the vectors in the plane, and use  $\mathbb{R}^3$  (pronounced "r three") to represent all the vectors in space.

Consider the vectors  $\vec{PQ}$  and  $\vec{RS}$  as shown in Figure 9.19. The vectors look to be equal; that is, they seem to have the same length and direction. Indeed, they are. Both vectors move 2 units to the right and 1 unit up from the initial point to reach the terminal point. One can analyze this movement to measure the

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Notes:

magnitude of the vector, and the movement itself gives direction information (one could also measure the slope of the line passing through  $P$  and  $Q$  or  $R$  and  $S$ ). Since they have the same length and direction, these two vectors are equal.

This demonstrates that inherently all we care about is *displacement*; that is, how far in the  $x$ ,  $y$  and possibly  $z$  directions the terminal point is from the initial point. Both the vectors  $\vec{PQ}$  and  $\vec{RS}$  in Figure 9.19 have an  $x$ -displacement of 2 and a  $y$ -displacement of 1. This suggests a standard way of describing vectors in the plane. A vector whose  $x$ -displacement is  $a$  and whose  $y$ -displacement is  $b$  will have terminal point  $(a, b)$  when the initial point is the origin,  $(0, 0)$ . This leads us to a definition of a standard and concise way of referring to vectors.

### Definition 29 Component Form of a Vector

1. The **component form** of a vector  $\vec{v}$  in  $\mathbb{R}^2$ , whose terminal point is  $(a, b)$  when its initial point is  $(0, 0)$ , is  $\langle a, b \rangle$ .
2. The **component form** of a vector  $\vec{v}$  in  $\mathbb{R}^3$ , whose terminal point is  $(a, b, c)$  when its initial point is  $(0, 0, 0)$ , is  $\langle a, b, c \rangle$ .

The numbers  $a$ ,  $b$  (and  $c$ , respectively) are the **components** of  $\vec{v}$ .

It follows from the definition that the component form of the vector  $\vec{PQ}$ , where  $P = (x_1, y_1)$  and  $Q = (x_2, y_2)$  is

$$\vec{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle;$$

in space, where  $P = (x_1, y_1, z_1)$  and  $Q = (x_2, y_2, z_2)$ , the component form of  $\vec{PQ}$  is

$$\vec{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

We practice using this notation in the following example.

### Example 158 Using component form notation for vectors

1. Sketch the vector  $\vec{v} = \langle 2, -1 \rangle$  starting at  $P = (3, 2)$  and find its magnitude.
2. Find the component form of the vector  $\vec{w}$  whose initial point is  $R = (-3, -2)$  and whose terminal point is  $S = (-1, 2)$ .
3. Sketch the vector  $\vec{u} = \langle 2, -1, 3 \rangle$  starting at the point  $Q = (1, 1, 1)$  and find its magnitude.

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Notes:

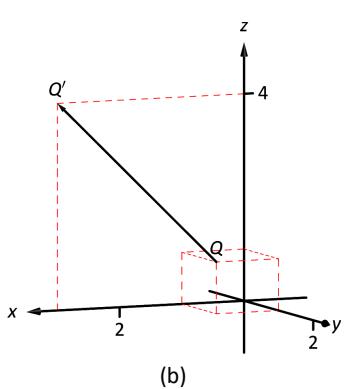
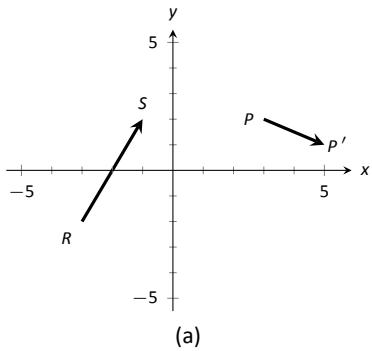


Figure 9.20: Graphing vectors in Example 158.

### SOLUTION

- Using  $P$  as the initial point, we move 2 units in the positive  $x$ -direction and  $-1$  units in the positive  $y$ -direction to arrive at the terminal point  $P' = (5, 1)$ , as drawn in Figure 9.20(a).

The magnitude of  $\vec{v}$  is determined directly from the component form:

$$\|\vec{v}\| = \sqrt{2^2 + (-1)^2} = \sqrt{5}.$$

- Using the note following Definition 29, we have

$$\overrightarrow{RS} = \langle -1 - (-3), 2 - (-2) \rangle = \langle 2, 4 \rangle.$$

One can readily see from Figure 9.20(a) that the  $x$ - and  $y$ -displacement of  $\overrightarrow{RS}$  is 2 and 4, respectively, as the component form suggests.

- Using  $Q$  as the initial point, we move 2 units in the positive  $x$ -direction,  $-1$  unit in the positive  $y$ -direction, and 3 units in the positive  $z$ -direction to arrive at the terminal point  $Q' = (3, 0, 4)$ , illustrated in Figure 9.20(b).

The magnitude of  $\vec{u}$  is:

$$\|\vec{u}\| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}.$$

Now that we have defined vectors, and have created a nice notation by which to describe them, we start considering how vectors interact with each other. That is, we define an *algebra* on vectors.

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Notes:

**Definition 30 Vector Algebra**

1. Let  $\vec{u} = \langle u_1, u_2 \rangle$  and  $\vec{v} = \langle v_1, v_2 \rangle$  be vectors in  $\mathbb{R}^2$ , and let  $c$  be a scalar.

- (a) The addition, or sum, of the vectors  $\vec{u}$  and  $\vec{v}$  is the vector

$$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2 \rangle.$$

- (b) The scalar product of  $c$  and  $\vec{v}$  is the vector

$$c\vec{v} = c \langle v_1, v_2 \rangle = \langle cv_1, cv_2 \rangle.$$

2. Let  $\vec{u} = \langle u_1, u_2, u_3 \rangle$  and  $\vec{v} = \langle v_1, v_2, v_3 \rangle$  be vectors in  $\mathbb{R}^3$ , and let  $c$  be a scalar.

- (a) The addition, or sum, of the vectors  $\vec{u}$  and  $\vec{v}$  is the vector

$$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle.$$

- (b) The scalar product of  $c$  and  $\vec{v}$  is the vector

$$c\vec{v} = c \langle v_1, v_2, v_3 \rangle = \langle cv_1, cv_2, cv_3 \rangle.$$

In short, we say addition and scalar multiplication are computed “component-wise.”

**Example 159 Adding vectors**

Sketch the vectors  $\vec{u} = \langle 1, 3 \rangle$ ,  $\vec{v} = \langle 2, 1 \rangle$  and  $\vec{u} + \vec{v}$  all with initial point at the origin.

**SOLUTION** We first compute  $\vec{u} + \vec{v}$ .

$$\begin{aligned}\vec{u} + \vec{v} &= \langle 1, 3 \rangle + \langle 2, 1 \rangle \\ &= \langle 3, 4 \rangle.\end{aligned}$$

These are all sketched in Figure 9.21.

As vectors convey magnitude and direction information, the sum of vectors also convey length and magnitude information. Adding  $\vec{u} + \vec{v}$  suggests the following idea:

“Starting at an initial point, go out  $\vec{u}$ , then go out  $\vec{v}$ .”

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Notes:

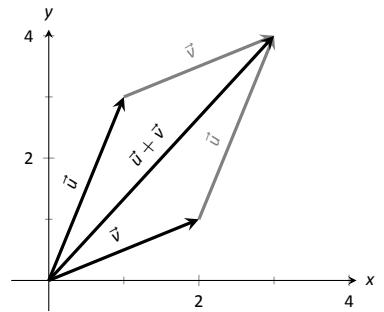


Figure 9.22: Illustrating how to add vectors using the Head to Tail Rule and Parallelogram Law.

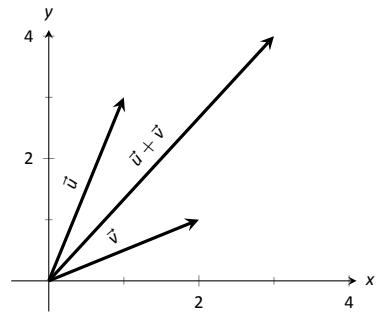


Figure 9.21: Graphing the sum of vectors in Example 159.

This idea is sketched in Figure 9.22, where the initial point of  $\vec{v}$  is the terminal point of  $\vec{u}$ . This is known as the “Head to Tail Rule” of adding vectors. Vector addition is very important. For instance, if the vectors  $\vec{u}$  and  $\vec{v}$  represent forces acting on a body, the sum  $\vec{u} + \vec{v}$  gives the resulting force. Because of various physical applications of vector addition, the sum  $\vec{u} + \vec{v}$  is often referred to as the **resultant vector**, or just the “resultant.”

Analytically, it is easy to see that  $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ . Figure 9.22 also gives a graphical representation of this, using gray vectors. Note that the vectors  $\vec{u}$  and  $\vec{v}$ , when arranged as in the figure, form a parallelogram. Because of this, the Head to Tail Rule is also known as the Parallelogram Law: the vector  $\vec{u} + \vec{v}$  is defined by forming the parallelogram defined by the vectors  $\vec{u}$  and  $\vec{v}$ ; the initial point of  $\vec{u} + \vec{v}$  is the common initial point of parallelogram, and the terminal point of the sum is the common terminal point of the parallelogram.

While not illustrated here, the Head to Tail Rule and Parallelogram Law hold for vectors in  $\mathbb{R}^3$  as well.

It follows from the properties of the real numbers and Definition 30 that

$$\vec{u} - \vec{v} = \vec{u} + (-1)\vec{v}.$$

The Parallelogram Law gives us a good way to visualize this subtraction. We demonstrate this in the following example.

#### Example 160 Vector Subtraction

Let  $\vec{u} = \langle 3, 1 \rangle$  and  $\vec{v} = \langle 1, 2 \rangle$ . Compute and sketch  $\vec{u} - \vec{v}$ .

**SOLUTION** The computation of  $\vec{u} - \vec{v}$  is straightforward, and we show all steps below. Usually the formal step of multiplying by  $(-1)$  is omitted and we “just subtract.”

$$\begin{aligned}\vec{u} - \vec{v} &= \vec{u} + (-1)\vec{v} \\ &= \langle 3, 1 \rangle + \langle -1, -2 \rangle \\ &= \langle 2, -1 \rangle.\end{aligned}$$

Figure 9.23 illustrates, using the Head to Tail Rule, how the subtraction can be viewed as the sum  $\vec{u} + (-\vec{v})$ . The figure also illustrates how  $\vec{u} - \vec{v}$  can be obtained by looking only at the terminal points of  $\vec{u}$  and  $\vec{v}$  (when their initial points are the same).

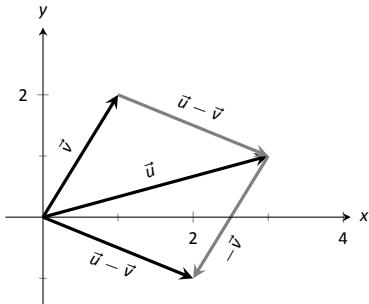


Figure 9.23: Illustrating how to subtract vectors graphically.

#### Example 161 Scaling vectors

1. Sketch the vectors  $\vec{v} = \langle 2, 1 \rangle$  and  $2\vec{v}$  with initial point at the origin.
2. Compute the magnitudes of  $\vec{v}$  and  $2\vec{v}$ .

---

Notes:

**SOLUTION**

1. We compute  $2\vec{v}$ :

$$\begin{aligned} 2\vec{v} &= 2 \langle 2, 1 \rangle \\ &= \langle 4, 2 \rangle. \end{aligned}$$

Both  $\vec{v}$  and  $2\vec{v}$  are sketched in Figure 9.24. Make note that  $2\vec{v}$  does not start at the terminal point of  $\vec{v}$ ; rather, its initial point is also the origin.

2. The figure suggests that  $2\vec{v}$  is twice as long as  $\vec{v}$ . We compute their magnitudes to confirm this.

$$\begin{aligned} \|\vec{v}\| &= \sqrt{2^2 + 1^2} \\ &= \sqrt{5}. \\ \|\mathbf{2}\vec{v}\| &= \sqrt{4^2 + 2^2} \\ &= \sqrt{20} \\ &= \sqrt{4 \cdot 5} = 2\sqrt{5}. \end{aligned}$$

As we suspected,  $2\vec{v}$  is twice as long as  $\vec{v}$ .

The **zero vector** is the vector whose initial point is also its terminal point. It is denoted by  $\vec{0}$ . Its component form, in  $\mathbb{R}^2$ , is  $\langle 0, 0 \rangle$ ; in  $\mathbb{R}^3$ , it is  $\langle 0, 0, 0 \rangle$ . Usually the context makes it clear whether  $\vec{0}$  is referring to a vector in the plane or in space.

Our examples have illustrated key principles in vector algebra: how to add and subtract vectors and how to multiply vectors by a scalar. The following theorem states formally the properties of these operations.

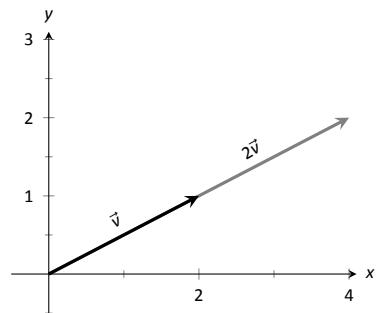


Figure 9.24: Graphing vectors  $\vec{v}$  and  $2\vec{v}$  in Example 161.

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Notes:

**Theorem 34 Properties of Vector Operations**

The following are true for all scalars  $c$  and  $d$ , and for all vectors  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$ , where  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$  are all in  $\mathbb{R}^2$  or where  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$  are all in  $\mathbb{R}^3$ :

1.  $\vec{u} + \vec{v} = \vec{v} + \vec{u}$  Commutative Property
2.  $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$  Associative Property
3.  $\vec{v} + \vec{0} = \vec{v}$  Additive Identity
4.  $(cd)\vec{v} = c(d\vec{v})$
5.  $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$  Distributive Property
6.  $(c + d)\vec{v} = c\vec{v} + d\vec{v}$  Distributive Property
7.  $0\vec{v} = \vec{0}$
8.  $\|c\vec{v}\| = |c| \cdot \|\vec{v}\|$
9.  $\|\vec{u}\| = 0$  if, and only if,  $\vec{u} = \vec{0}$ .

As stated before, each vector  $\vec{v}$  conveys magnitude and direction information. We have a method of extracting the magnitude, which we write as  $\|\vec{v}\|$ . *Unit vectors* are a way of extracting just the direction information from a vector.

**Definition 31 Unit Vector**

A **unit vector** is a vector  $\vec{v}$  with a magnitude of 1; that is,

$$\|\vec{v}\| = 1.$$

Consider this scenario: you are given a vector  $\vec{v}$  and are told to create a vector of length 10 in the direction of  $\vec{v}$ . How does one do that? If we knew that  $\vec{u}$  was the unit vector in the direction of  $\vec{v}$ , the answer would be easy:  $10\vec{u}$ . So how do we find  $\vec{u}$ ?

Property 8 of Theorem 34 holds the key. If we divide  $\vec{v}$  by its magnitude, it becomes a vector of length 1. Consider:

$$\left\| \frac{1}{\|\vec{v}\|} \vec{v} \right\| = \frac{1}{\|\vec{v}\|} \|\vec{v}\| \quad (\text{we can pull out } \frac{1}{\|\vec{v}\|} \text{ as it is a scalar}) \\ = 1.$$

---

Notes:

So the vector of length 10 in the direction of  $\vec{v}$  is  $10 \frac{1}{\|\vec{v}\|} \vec{v}$ . An example will make this more clear.

### Example 162 Using Unit Vectors

Let  $\vec{v} = \langle 3, 1 \rangle$  and let  $\vec{w} = \langle 1, 2, 2 \rangle$ .

1. Find the unit vector in the direction of  $\vec{v}$ .
2. Find the unit vector in the direction of  $\vec{w}$ .
3. Find the vector in the direction of  $\vec{v}$  with magnitude 5.

#### SOLUTION

1. We find  $\|\vec{v}\| = \sqrt{10}$ . So the unit vector  $\vec{u}$  in the direction of  $\vec{v}$  is

$$\vec{u} = \frac{1}{\sqrt{10}} \vec{v} = \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle.$$

2. We find  $\|\vec{w}\| = 3$ , so the unit vector  $\vec{z}$  in the direction of  $\vec{w}$  is

$$\vec{z} = \frac{1}{3} \vec{w} = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle.$$

3. To create a vector with magnitude 5 in the direction of  $\vec{v}$ , we multiply the unit vector  $\vec{u}$  by 5. Thus  $5\vec{u} = \langle 15/\sqrt{10}, 5/\sqrt{10} \rangle$  is the vector we seek. This is sketched in Figure 9.25.

The basic formation of the unit vector  $\vec{u}$  in the direction of a vector  $\vec{v}$  leads to a interesting equation. It is:

$$\vec{v} = \|\vec{v}\| \frac{1}{\|\vec{v}\|} \vec{v}.$$

We rewrite the equation with parentheses to make a point:

$$\vec{v} = \underbrace{\|\vec{v}\|}_{\text{magnitude}} \cdot \underbrace{\left( \frac{1}{\|\vec{v}\|} \vec{v} \right)}_{\text{direction}}.$$

This equation illustrates the fact that a vector has both magnitude and direction, where we view a unit vector as supplying *only* direction information. Identifying unit vectors with direction allows us to define **parallel vectors**.

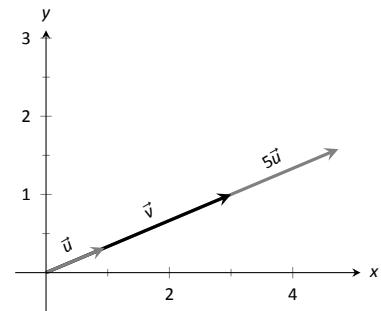


Figure 9.25: Graphing vectors in Example 162. All vectors shown have their initial point at the origin.

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Notes:

**Note:**  $\vec{0}$  is directionless; because  $\|\vec{0}\| = 0$ , there is no unit vector in the “direction” of  $\vec{0}$ .

Some texts define two vectors as being parallel if one is a scalar multiple of the other. By this definition,  $\vec{0}$  is parallel to all vectors as  $\vec{0} = 0\vec{v}$  for all  $\vec{v}$ .

We prefer the given definition of parallel as it is grounded in the fact that unit vectors provide direction information. One may adopt the convention that  $\vec{0}$  is parallel to all vectors if they desire. (See also the marginal note on page 534.)

### Definition 32 Parallel Vectors

- Unit vectors  $\vec{u}_1$  and  $\vec{u}_2$  are **parallel** if  $\vec{u}_1 = \pm\vec{u}_2$ .
- Nonzero vectors  $\vec{v}_1$  and  $\vec{v}_2$  are **parallel** if their respective unit vectors are parallel.

It is equivalent to say that vectors  $\vec{v}_1$  and  $\vec{v}_2$  are parallel if there is a scalar  $c \neq 0$  such that  $\vec{v}_1 = c\vec{v}_2$  (see marginal note).

If one graphed all unit vectors in  $\mathbb{R}^2$  with the initial point at the origin, then the terminal points would all lie on the unit circle. Based on what we know from trigonometry, we can then say that the component form of all unit vectors in  $\mathbb{R}^2$  is  $\langle \cos \theta, \sin \theta \rangle$  for some angle  $\theta$ .

A similar construction in  $\mathbb{R}^3$  shows that the terminal points all lie on the unit sphere. These vectors also have a particular component form, but its derivation is not as straightforward as the one for unit vectors in  $\mathbb{R}^2$ . Important concepts about unit vectors are given in the following Key Idea.

### Key Idea 36 Unit Vectors

- The unit vector in the direction of  $\vec{v}$  is

$$\vec{u} = \frac{1}{\|\vec{v}\|} \vec{v}.$$

- A vector  $\vec{u}$  in  $\mathbb{R}^2$  is a unit vector if, and only if, its component form is  $\langle \cos \theta, \sin \theta \rangle$  for some angle  $\theta$ .
- A vector  $\vec{u}$  in  $\mathbb{R}^3$  is a unit vector if, and only if, its component form is  $\langle \sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta \rangle$  for some angles  $\theta$  and  $\varphi$ .

These formulas can come in handy in a variety of situations, especially the formula for unit vectors in the plane.

### Example 163 Finding Component Forces

Consider a weight of 50lb hanging from two chains, as shown in Figure 9.26. One chain makes an angle of  $30^\circ$  with the vertical, and the other an angle of  $45^\circ$ . Find the force applied to each chain.

**SOLUTION** Knowing that gravity is pulling the 50lb weight straight down,

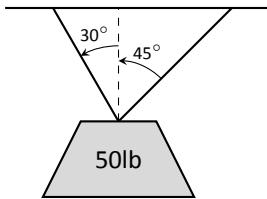


Figure 9.26: A diagram of a weight hanging from 2 chains in Example 163.

Notes:

we can create a vector  $\vec{F}$  to represent this force.

$$\vec{F} = 50 \langle 0, -1 \rangle = \langle 0, -50 \rangle.$$

We can view each chain as “pulling” the weight up, preventing it from falling. We can represent the force from each chain with a vector. Let  $\vec{F}_1$  represent the force from the chain making an angle of  $30^\circ$  with the vertical, and let  $\vec{F}_2$  represent the force from the other chain. Convert all angles to be measured from the horizontal (as shown in Figure 9.27), and apply Key Idea 36. As we do not yet know the magnitudes of these vectors, (that is the problem at hand), we use  $m_1$  and  $m_2$  to represent them.

$$\vec{F}_1 = m_1 \langle \cos 120^\circ, \sin 120^\circ \rangle$$

$$\vec{F}_2 = m_2 \langle \cos 45^\circ, \sin 45^\circ \rangle$$

As the weight is not moving, we know the sum of the forces is  $\vec{0}$ . This gives:

$$\vec{F} + \vec{F}_1 + \vec{F}_2 = \vec{0}$$

$$\langle 0, -50 \rangle + m_1 \langle \cos 120^\circ, \sin 120^\circ \rangle + m_2 \langle \cos 45^\circ, \sin 45^\circ \rangle = \vec{0}$$

The sum of the entries in the first component is 0, and the sum of the entries in the second component is also 0. This leads us to the following two equations:

$$m_1 \cos 120^\circ + m_2 \cos 45^\circ = 0$$

$$m_1 \sin 120^\circ + m_2 \sin 45^\circ = 50$$

This is a simple 2-equation, 2-unknown system of linear equations. We leave it to the reader to verify that the solution is

$$m_1 = 50(\sqrt{3} - 1) \approx 36.6; \quad m_2 = \frac{50\sqrt{2}}{1 + \sqrt{3}} \approx 25.88.$$

It might seem odd that the sum of the forces applied to the chains is more than 50lb. We leave it to a physics class to discuss the full details, but offer this short explanation. Our equations were established so that the *vertical* components of each force sums to 50lb, thus supporting the weight. Since the chains are at an angle, they also pull against each other, creating an “additional” horizontal force while holding the weight in place.

Unit vectors were very important in the previous calculation; they allowed us to define a vector in the proper direction but with an unknown magnitude. Our computations were then computed component-wise. Because such calculations are often necessary, the *standard unit vectors* can be useful.

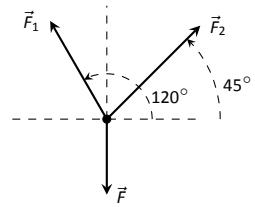


Figure 9.27: A diagram of the force vectors from Example 163.

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Notes:

**Definition 33 Standard Unit Vectors**

1. In  $\mathbb{R}^2$ , the standard unit vectors are

$$\vec{i} = \langle 1, 0 \rangle \quad \text{and} \quad \vec{j} = \langle 0, 1 \rangle.$$

2. In  $\mathbb{R}^3$ , the standard unit vectors are

$$\vec{i} = \langle 1, 0, 0 \rangle \quad \text{and} \quad \vec{j} = \langle 0, 1, 0 \rangle \quad \text{and} \quad \vec{k} = \langle 0, 0, 1 \rangle.$$

**Example 164 Using standard unit vectors**

1. Rewrite  $\vec{v} = \langle 2, -3 \rangle$  using the standard unit vectors.

2. Rewrite  $\vec{w} = 4\vec{i} - 5\vec{j} + 2\vec{k}$  in component form.

**SOLUTION**

$$\begin{aligned} 1. \quad \vec{v} &= \langle 2, -3 \rangle \\ &= \langle 2, 0 \rangle + \langle 0, -3 \rangle \\ &= 2 \langle 1, 0 \rangle - 3 \langle 0, 1 \rangle \\ &= 2\vec{i} - 3\vec{j} \\ 2. \quad \vec{w} &= 4\vec{i} - 5\vec{j} + 2\vec{k} \\ &= \langle 4, 0, 0 \rangle + \langle 0, -5, 0 \rangle + \langle 0, 0, 2 \rangle \\ &= \langle 4, -5, 2 \rangle \end{aligned}$$

These two examples demonstrate that converting between component form and the standard unit vectors is rather straightforward. Many mathematicians prefer component form, and it is the preferred notation in this text. Many engineers prefer using the standard unit vectors, and many engineering text use that notation.

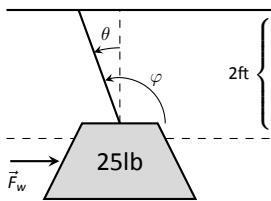


Figure 9.28: A figure of a weight being pushed by the wind in Example 165.

Notes:

**Example 165 Finding Component Force**

A weight of 25lb is suspended from a chain of length 2ft while a wind pushes the weight to the right with constant force of 5lb as shown in Figure 9.28. What angle will the chain make with the vertical as a result of the wind's pushing? How much higher will the weight be?

**SOLUTION** The force of the wind is represented by the vector  $\vec{F}_w = 5\vec{i}$ . The force of gravity on the weight is represented by  $\vec{F}_g = -25\vec{j}$ . The direction and magnitude of the vector representing the force on the chain are both unknown. We represent this force with

$$\vec{F}_c = m \langle \cos \varphi, \sin \varphi \rangle = m \cos \varphi \vec{i} + m \sin \varphi \vec{j}$$

for some magnitude  $m$  and some angle with the horizontal  $\varphi$ . (Note:  $\theta$  is the angle the chain makes with the *vertical*;  $\varphi$  is the angle with the *horizontal*.)

As the weight is at equilibrium, the sum of the forces is  $\vec{0}$ :

$$\begin{aligned}\vec{F}_c + \vec{F}_w + \vec{F}_g &= \vec{0} \\ m \cos \varphi \vec{i} + m \sin \varphi \vec{j} + 5\vec{i} - 25\vec{j} &= \vec{0}\end{aligned}$$

Thus the sum of the  $\vec{i}$  and  $\vec{j}$  components are 0, leading us to the following system of equations:

$$\begin{aligned}5 + m \cos \varphi &= 0 \\ -25 + m \sin \varphi &= 0\end{aligned}\tag{9.1}$$

This is enough to determine  $\vec{F}_c$  already, as we know  $m \cos \varphi = -5$  and  $m \sin \varphi = 25$ . Thus  $F_c = \langle -5, 25 \rangle$ . We can use this to find the magnitude  $m$ :

$$m = \sqrt{(-5)^2 + 25^2} = 5\sqrt{26} \approx 25.5\text{lb}.$$

We can then use either equality from Equation (9.1) to solve for  $\varphi$ . We choose the first equality as using arccosine will return an angle in the 2<sup>nd</sup> quadrant:

$$5 + 5\sqrt{26} \cos \varphi = 0 \Rightarrow \varphi = \cos^{-1} \left( \frac{-5}{5\sqrt{26}} \right) \approx 1.7682 \approx 101.31^\circ.$$

Subtracting  $90^\circ$  from this angle gives us an angle of  $11.31^\circ$  with the vertical.

We can now use trigonometry to find out how high the weight is lifted. The diagram shows that a right triangle is formed with the 2ft chain as the hypotenuse with an interior angle of  $11.31^\circ$ . The length of the adjacent side (in the diagram, the dashed vertical line) is  $2 \cos 11.31^\circ \approx 1.96\text{ft}$ . Thus the weight is lifted by about 0.04ft, almost 1/2in.

The algebra we have applied to vectors is already demonstrating itself to be very useful. There are two more fundamental operations we can perform with vectors, the *dot product* and the *cross product*. The next two sections explore each in turn.

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Notes:

# Exercises 9.2

## Terms and Concepts

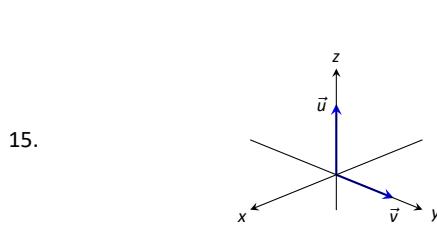
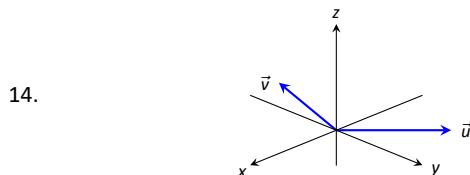
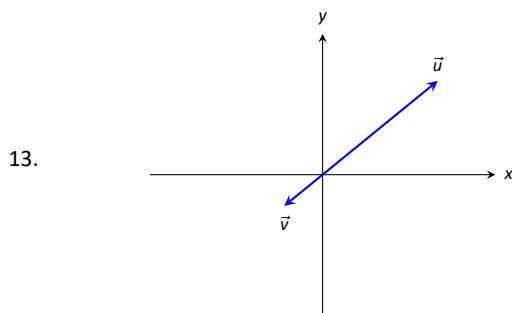
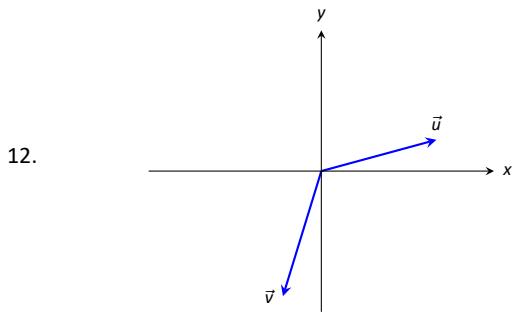
1. Name two different things that cannot be described with just one number, but rather need 2 or more numbers to fully describe them.
2. What is the difference between  $(1, 2)$  and  $\langle 1, 2 \rangle$ ?
3. What is a unit vector?
4. What does it mean for two vectors to be parallel?
5. What effect does multiplying a vector by  $-2$  have?

## Problems

In Exercises 6 – 9, points  $P$  and  $Q$  are given. Write the vector  $\vec{PQ}$  in component form and using the standard unit vectors.

6.  $P = (2, -1)$ ,  $Q = (3, 5)$
7.  $P = (3, 2)$ ,  $Q = (7, -2)$
8.  $P = (0, 3, -1)$ ,  $Q = (6, 2, 5)$
9.  $P = (2, 1, 2)$ ,  $Q = (4, 3, 2)$
10. Let  $\vec{u} = \langle 1, -2 \rangle$  and  $\vec{v} = \langle 1, 1 \rangle$ .
  - (a) Find  $\vec{u} + \vec{v}$ ,  $\vec{u} - \vec{v}$ ,  $2\vec{u} - 3\vec{v}$ .
  - (b) Sketch the above vectors on the same axes, along with  $\vec{u}$  and  $\vec{v}$ .
  - (c) Find  $\vec{x}$  where  $\vec{u} + \vec{x} = 2\vec{v} - \vec{x}$ .
11. Let  $\vec{u} = \langle 1, 1, -1 \rangle$  and  $\vec{v} = \langle 2, 1, 2 \rangle$ .
  - (a) Find  $\vec{u} + \vec{v}$ ,  $\vec{u} - \vec{v}$ ,  $\pi\vec{u} - \sqrt{2}\vec{v}$ .
  - (b) Sketch the above vectors on the same axes, along with  $\vec{u}$  and  $\vec{v}$ .
  - (c) Find  $\vec{x}$  where  $\vec{u} + \vec{x} = \vec{v} + 2\vec{x}$ .

In Exercises 12 – 15, sketch  $\vec{u}$ ,  $\vec{v}$ ,  $\vec{u} + \vec{v}$  and  $\vec{u} - \vec{v}$  on the same axes.



In Exercises 16 – 19, find  $\|\vec{u}\|$ ,  $\|\vec{v}\|$ ,  $\|\vec{u} + \vec{v}\|$  and  $\|\vec{u} - \vec{v}\|$ .

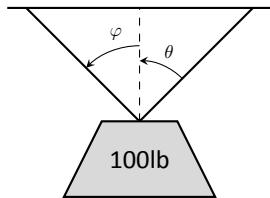
16.  $\vec{u} = \langle 2, 1 \rangle$ ,  $\vec{v} = \langle 3, -2 \rangle$
17.  $\vec{u} = \langle -3, 2, 2 \rangle$ ,  $\vec{v} = \langle 1, -1, 1 \rangle$
18.  $\vec{u} = \langle 1, 2 \rangle$ ,  $\vec{v} = \langle -3, -6 \rangle$
19.  $\vec{u} = \langle 2, -3, 6 \rangle$ ,  $\vec{v} = \langle 10, -15, 30 \rangle$
20. Under what conditions is  $\|\vec{u}\| + \|\vec{v}\| = \|\vec{u} + \vec{v}\|$ ?

In Exercises 21 – 24, find the unit vector  $\vec{u}$  in the direction of  $\vec{v}$ .

21.  $\vec{v} = \langle 3, 7 \rangle$
22.  $\vec{v} = \langle 6, 8 \rangle$
23.  $\vec{v} = \langle 1, -2, 2 \rangle$
24.  $\vec{v} = \langle 2, -2, 2 \rangle$
25. Find the unit vector in the first quadrant of  $\mathbb{R}^2$  that makes a  $50^\circ$  angle with the  $x$ -axis.

26. Find the unit vector in the second quadrant of  $\mathbb{R}^2$  that makes a  $30^\circ$  angle with the  $y$ -axis.
27. Verify, from Key Idea 36, that  $\vec{u} = \langle \sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta \rangle$  is a unit vector for all angles  $\theta$  and  $\varphi$ .
31.  $\theta = 0^\circ, \varphi = 0^\circ$

A weight of 100lb is suspended from two chains, making angles with the vertical of  $\theta$  and  $\varphi$  as shown in the figure below.



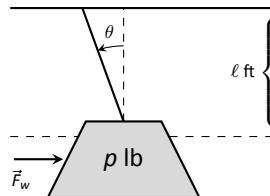
In Exercises 28 – 31, angles  $\theta$  and  $\varphi$  are given. Find the force applied to each chain.

28.  $\theta = 30^\circ, \varphi = 30^\circ$

29.  $\theta = 60^\circ, \varphi = 60^\circ$

30.  $\theta = 20^\circ, \varphi = 15^\circ$

A weight of  $p$ lb is suspended from a chain of length  $\ell$  while a constant force of  $\vec{F}_w$  pushes the weight to the right, making an angle of  $\theta$  with the vertical, as shown in the figure below.



In Exercises 32 – 35, a force  $\vec{F}_w$  and length  $\ell$  are given. Find the angle  $\theta$  and the height the weight is lifted as it moves to the right.

32.  $\vec{F}_w = 1\text{lb}, \ell = 1\text{ft}, p = 1\text{lb}$

33.  $\vec{F}_w = 1\text{lb}, \ell = 1\text{ft}, p = 10\text{lb}$

34.  $\vec{F}_w = 1\text{lb}, \ell = 10\text{ft}, p = 1\text{lb}$

35.  $\vec{F}_w = 10\text{lb}, \ell = 10\text{ft}, p = 1\text{lb}$

### 9.3 The Dot Product

The previous section introduced vectors and described how to add them together and how to multiply them by scalars. This section introduces a multiplication on vectors called the **dot product**.

#### Definition 34 Dot Product

- Let  $\vec{u} = \langle u_1, u_2 \rangle$  and  $\vec{v} = \langle v_1, v_2 \rangle$  in  $\mathbb{R}^2$ . The **dot product** of  $\vec{u}$  and  $\vec{v}$ , denoted  $\vec{u} \cdot \vec{v}$ , is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2.$$

- Let  $\vec{u} = \langle u_1, u_2, u_3 \rangle$  and  $\vec{v} = \langle v_1, v_2, v_3 \rangle$  in  $\mathbb{R}^3$ . The **dot product** of  $\vec{u}$  and  $\vec{v}$ , denoted  $\vec{u} \cdot \vec{v}$ , is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Note how this product of vectors returns a *scalar*, not another vector. We practice evaluating a dot product in the following example, then we will discuss why this product is useful.

#### Example 166 Evaluating dot products

- Let  $\vec{u} = \langle 1, 2 \rangle$ ,  $\vec{v} = \langle 3, -1 \rangle$  in  $\mathbb{R}^2$ . Find  $\vec{u} \cdot \vec{v}$ .
- Let  $\vec{x} = \langle 2, -2, 5 \rangle$  and  $\vec{y} = \langle -1, 0, 3 \rangle$  in  $\mathbb{R}^3$ . Find  $\vec{x} \cdot \vec{y}$ .

#### SOLUTION

- Using Definition 34, we have

$$\vec{u} \cdot \vec{v} = 1(3) + 2(-1) = 1.$$

- Using the definition, we have

$$\vec{x} \cdot \vec{y} = 2(-1) - 2(0) + 5(3) = 13.$$

The dot product, as shown by the preceding example, is very simple to evaluate. It is only the sum of products. While the definition gives no hint as to why

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Notes:

we would care about this operation, there is an amazing connection between the dot product and angles formed by the vectors. Before stating this connection, we give a theorem stating some of the properties of the dot product.

**Theorem 35 Properties of the Dot Product**

Let  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$  be vectors in  $\mathbb{R}^2$  or  $\mathbb{R}^3$  and let  $c$  be a scalar.

1.  $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$  Commutative Property
2.  $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$  Distributive Property
3.  $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$
4.  $\vec{0} \cdot \vec{v} = 0$
5.  $\vec{v} \cdot \vec{v} = ||\vec{v}||^2$

The last statement of the theorem makes a handy connection between the magnitude of a vector and the dot product with itself. Our definition and theorem give properties of the dot product, but we are still likely wondering “What does the dot product *mean*?” It is helpful to understand that the dot product of a vector with itself is connected to its magnitude.

The next theorem extends this understanding by connecting the dot product to magnitudes and angles. Given vectors  $\vec{u}$  and  $\vec{v}$  in the plane, an angle  $\theta$  is clearly formed when  $\vec{u}$  and  $\vec{v}$  are drawn with the same initial point as illustrated in Figure 9.29(a). (We always take  $\theta$  to be the angle in  $[0, \pi]$  as two angles are actually created.)

The same is also true of 2 vectors in space: given  $\vec{u}$  and  $\vec{v}$  in  $\mathbb{R}^3$  with the same initial point, there is a plane that contains both  $\vec{u}$  and  $\vec{v}$ . (When  $\vec{u}$  and  $\vec{v}$  are collinear, there are infinite planes that contain both vectors.) In that plane, we can again find an angle  $\theta$  between them (and again,  $0 \leq \theta \leq \pi$ ). This is illustrated in Figure 9.29(b).

The following theorem connects this angle  $\theta$  to the dot product of  $\vec{u}$  and  $\vec{v}$ .

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Notes:

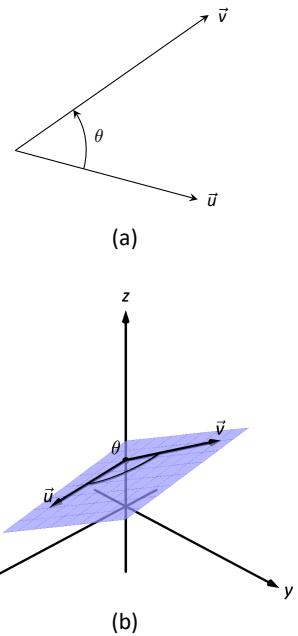


Figure 9.29: Illustrating the angle formed by two vectors with the same initial point.

**Theorem 36 The Dot Product and Angles**

Let  $\vec{u}$  and  $\vec{v}$  be vectors in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ . Then

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta,$$

where  $\theta$ ,  $0 \leq \theta \leq \pi$ , is the angle between  $\vec{u}$  and  $\vec{v}$ .

When  $\theta$  is an acute angle (i.e.,  $0 \leq \theta < \pi/2$ ),  $\cos \theta$  is positive; when  $\theta = \pi/2$ ,  $\cos \theta = 0$ ; when  $\theta$  is an obtuse angle ( $\pi/2 < \theta \leq \pi$ ),  $\cos \theta$  is negative. Thus the sign of the dot product gives a general indication of the angle between the vectors, illustrated in Figure 9.30.

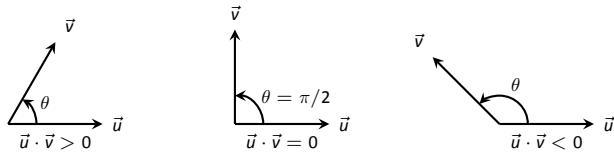


Figure 9.30: Illustrating the relationship between the angle between vectors and the sign of their dot product.

We can use Theorem 36 to compute the dot product, but generally this theorem is used to find the angle between known vectors (since the dot product is generally easy to compute). To this end, we rewrite the theorem's equation as

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \Leftrightarrow \theta = \cos^{-1} \left( \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \right).$$

We practice using this theorem in the following example.

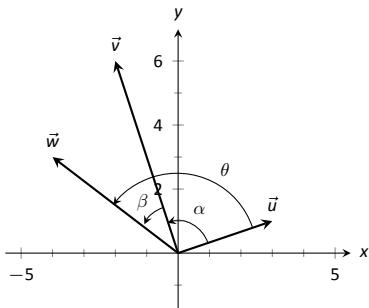


Figure 9.31: Vectors used in Example 167.

**Example 167 Using the dot product to find angles**

Let  $\vec{u} = \langle 3, 1 \rangle$ ,  $\vec{v} = \langle -2, 6 \rangle$  and  $\vec{w} = \langle -4, 3 \rangle$ , as shown in Figure 9.31. Find the angles  $\alpha$ ,  $\beta$  and  $\theta$ .

**SOLUTION**

We start by computing the magnitude of each vector.

$$\|\vec{u}\| = \sqrt{10}; \quad \|\vec{v}\| = 2\sqrt{10}; \quad \|\vec{w}\| = 5.$$

Notes:

We now apply Theorem 36 to find the angles.

$$\begin{aligned}\alpha &= \cos^{-1} \left( \frac{\vec{u} \cdot \vec{v}}{(\sqrt{10})(2\sqrt{10})} \right) \\ &= \cos^{-1}(0) = \frac{\pi}{2} = 90^\circ.\end{aligned}$$

$$\begin{aligned}\beta &= \cos^{-1} \left( \frac{\vec{v} \cdot \vec{w}}{(2\sqrt{10})(5)} \right) \\ &= \cos^{-1} \left( \frac{26}{10\sqrt{10}} \right) \\ &\approx 0.6055 \approx 34.7^\circ.\end{aligned}$$

$$\begin{aligned}\theta &= \cos^{-1} \left( \frac{\vec{u} \cdot \vec{w}}{(\sqrt{10})(5)} \right) \\ &= \cos^{-1} \left( \frac{-9}{5\sqrt{10}} \right) \\ &\approx 2.1763 \approx 124.7^\circ\end{aligned}$$

We see from our computation that  $\alpha + \beta = \theta$ , as indicated by Figure 9.31. While we knew this should be the case, it is nice to see that this non-intuitive formula indeed returns the results we expected.

We do a similar example next in the context of vectors in space.

#### Example 168 Using the dot product to find angles

Let  $\vec{u} = \langle 1, 1, 1 \rangle$ ,  $\vec{v} = \langle -1, 3, -2 \rangle$  and  $\vec{w} = \langle -5, 1, 4 \rangle$ , as illustrated in Figure 9.32. Find the angle between each pair of vectors.

#### SOLUTION

- Between  $\vec{u}$  and  $\vec{v}$ :

$$\begin{aligned}\theta &= \cos^{-1} \left( \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \right) \\ &= \cos^{-1} \left( \frac{0}{\sqrt{3}\sqrt{14}} \right) \\ &= \frac{\pi}{2}.\end{aligned}$$

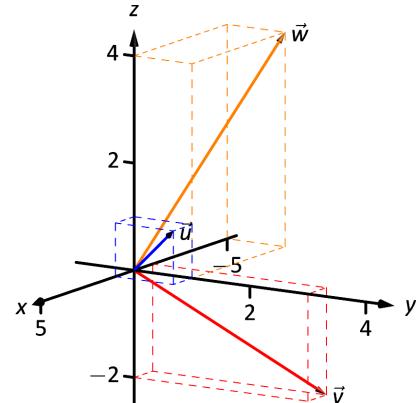


Figure 9.32: Vectors used in Example 168.

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Notes:

2. Between  $\vec{u}$  and  $\vec{w}$ :

$$\begin{aligned}\theta &= \cos^{-1} \left( \frac{\vec{u} \cdot \vec{w}}{\|\vec{u}\| \|\vec{w}\|} \right) \\ &= \cos^{-1} \left( \frac{0}{\sqrt{3}\sqrt{42}} \right) \\ &= \frac{\pi}{2}.\end{aligned}$$

3. Between  $\vec{v}$  and  $\vec{w}$ :

$$\begin{aligned}\theta &= \cos^{-1} \left( \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|} \right) \\ &= \cos^{-1} \left( \frac{0}{\sqrt{14}\sqrt{42}} \right) \\ &= \frac{\pi}{2}.\end{aligned}$$

While our work shows that each angle is  $\pi/2$ , i.e.,  $90^\circ$ , none of these angles looks to be a right angle in Figure 9.32. Such is the case when drawing three-dimensional objects on the page.

All three angles between these vectors was  $\pi/2$ , or  $90^\circ$ . We know from geometry and everyday life that  $90^\circ$  angles are “nice” for a variety of reasons, so it should seem significant that these angles are all  $\pi/2$ . Notice the common feature in each calculation (and also the calculation of  $\alpha$  in Example 167): the dot products of each pair of angles was 0. We use this as a basis for a definition of the term **orthogonal**, which is essentially synonymous to *perpendicular*.

#### Definition 35 Orthogonal

Vectors  $\vec{u}$  and  $\vec{v}$  are **orthogonal** if their dot product is 0.

#### Example 169 Finding orthogonal vectors

Let  $\vec{u} = \langle 3, 5 \rangle$  and  $\vec{v} = \langle 1, 2, 3 \rangle$ .

1. Find two vectors in  $\mathbb{R}^2$  that are orthogonal to  $\vec{u}$ .
2. Find two non-parallel vectors in  $\mathbb{R}^3$  that are orthogonal to  $\vec{v}$ .

#### SOLUTION

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Notes:

1. Recall that a line perpendicular to a line with slope  $m$  has slope  $-1/m$ , the “opposite reciprocal slope.” We can think of the slope of  $\vec{u}$  as  $5/3$ , its “rise over run.” A vector orthogonal to  $\vec{u}$  will have slope  $-3/5$ . There are many such choices, though all parallel:

$$\langle -5, 3 \rangle \quad \text{or} \quad \langle 5, -3 \rangle \quad \text{or} \quad \langle -10, 6 \rangle \quad \text{or} \quad \langle 15, -9 \rangle, \text{ etc.}$$

2. There are infinite directions in space orthogonal to any given direction, so there are an infinite number of non-parallel vectors orthogonal to  $\vec{v}$ . Since there are so many, we have great leeway in finding some.

One way is to arbitrarily pick values for the first two components, leaving the third unknown. For instance, let  $\vec{v}_1 = \langle 2, 7, z \rangle$ . If  $\vec{v}_1$  is to be orthogonal to  $\vec{v}$ , then  $\vec{v}_1 \cdot \vec{v} = 0$ , so

$$2 + 14 + 3z = 0 \Rightarrow z = \frac{-16}{3}.$$

So  $\vec{v}_1 = \langle 2, 7, -16/3 \rangle$  is orthogonal to  $\vec{v}$ . We can apply a similar technique by leaving the first or second component unknown.

Another method of finding a vector orthogonal to  $\vec{v}$  mirrors what we did in part 1. Let  $\vec{v}_2 = \langle -2, 1, 0 \rangle$ . Here we switched the first two components of  $\vec{v}$ , changing the sign of one of them (similar to the “opposite reciprocal” concept before). Letting the third component be 0 effectively ignores the third component of  $\vec{v}$ , and it is easy to see that

$$\vec{v}_2 \cdot \vec{v} = \langle -2, 1, 0 \rangle \cdot \langle 1, 2, 3 \rangle = 0.$$

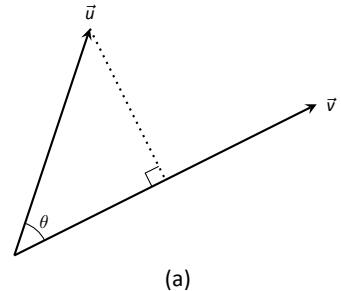
Clearly  $\vec{v}_1$  and  $\vec{v}_2$  are not parallel.

An important construction is illustrated in Figure 9.33, where vectors  $\vec{u}$  and  $\vec{v}$  are sketched. In part (a), a dotted line is drawn from the tip of  $\vec{u}$  to the line containing  $\vec{v}$ , where the dotted line is orthogonal to  $\vec{v}$ . In part (b), the dotted line is replaced with the vector  $\vec{z}$  and  $\vec{w}$  is formed, parallel to  $\vec{v}$ . It is clear by the diagram that  $\vec{u} = \vec{w} + \vec{z}$ . What is important about this construction is this:  $\vec{u}$  is *decomposed* as the sum of two vectors, one of which is parallel to  $\vec{v}$  and one that is perpendicular to  $\vec{v}$ . It is hard to overstate the importance of this construction (as we’ll see in upcoming examples).

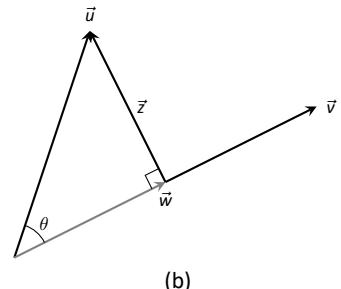
The vectors  $\vec{w}$ ,  $\vec{z}$  and  $\vec{u}$  as shown in Figure 9.33 (b) form a right triangle, where the angle between  $\vec{v}$  and  $\vec{u}$  is labeled  $\theta$ . We can find  $\vec{w}$  in terms of  $\vec{v}$  and  $\vec{u}$ .

Using trigonometry, we can state that

$$\|\vec{w}\| = \|\vec{u}\| \cos \theta. \tag{9.2}$$



(a)



(b)

Figure 9.33: Developing the construction of the *orthogonal projection*.

---

Notes:

We also know that  $\vec{w}$  is parallel to  $\vec{v}$ ; that is, the direction of  $\vec{w}$  is the direction of  $\vec{v}$ , described by the unit vector  $\frac{1}{\|\vec{v}\|}\vec{v}$ . The vector  $\vec{w}$  is the vector in the direction  $\frac{1}{\|\vec{v}\|}\vec{v}$  with magnitude  $\|\vec{u}\| \cos \theta$ :

$$\vec{w} = (\|\vec{u}\| \cos \theta) \frac{1}{\|\vec{v}\|} \vec{v}.$$

Replace  $\cos \theta$  using Theorem 36:

$$\begin{aligned} &= \left( \|\vec{u}\| \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \right) \frac{1}{\|\vec{v}\|} \vec{v} \\ &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v}. \end{aligned}$$

Now apply Theorem 35.

$$= \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}.$$

Since this construction is so important, it is given a special name.

#### Definition 36 Orthogonal Projection

Let  $\vec{u}$  and  $\vec{v}$  be given. The **orthogonal projection of  $\vec{u}$  onto  $\vec{v}$** , denoted  $\text{proj}_{\vec{v}} \vec{u}$ , is

$$\text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}.$$

#### Example 170 Computing the orthogonal projection

1. Let  $\vec{u} = \langle -2, 1 \rangle$  and  $\vec{v} = \langle 3, 1 \rangle$ . Find  $\text{proj}_{\vec{v}} \vec{u}$ , and sketch all three vectors with initial points at the origin.
2. Let  $\vec{w} = \langle 2, 1, 3 \rangle$  and  $\vec{x} = \langle 1, 1, 1 \rangle$ . Find  $\text{proj}_{\vec{x}} \vec{w}$ , and sketch all three vectors with initial points at the origin.

#### SOLUTION

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Notes:

1. Applying Definition 36, we have

$$\begin{aligned}\text{proj}_{\vec{v}} \vec{u} &= \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} \\ &= \frac{-5}{10} \langle 3, 1 \rangle \\ &= \left\langle -\frac{3}{2}, -\frac{1}{2} \right\rangle.\end{aligned}$$

Vectors  $\vec{u}$ ,  $\vec{v}$  and  $\text{proj}_{\vec{v}} \vec{u}$  are sketched in Figure 9.34(a). Note how the projection is parallel to  $\vec{v}$ ; that is, it lies on the same line through the origin as  $\vec{v}$ , although it points in the opposite direction. That is because the angle between  $\vec{u}$  and  $\vec{v}$  is obtuse (i.e., greater than  $90^\circ$ ).

2. Apply the definition:

$$\begin{aligned}\text{proj}_{\vec{x}} \vec{w} &= \frac{\vec{w} \cdot \vec{x}}{\vec{x} \cdot \vec{x}} \vec{x} \\ &= \frac{6}{3} \langle 1, 1, 1 \rangle \\ &= \langle 2, 2, 2 \rangle.\end{aligned}$$

These vectors are sketched in Figure 9.34(b), and again in part (c) from a different perspective. Because of the nature of graphing these vectors, the sketch in part (b) makes it difficult to recognize that the drawn projection has the geometric properties it should. The graph shown in part (c) illustrates these properties better.

Consider Figure 9.35 where the concept of the orthogonal projection is again illustrated. It is clear that

$$\vec{u} = \text{proj}_{\vec{v}} \vec{u} + \vec{z}. \quad (9.3)$$

As we know what  $\vec{u}$  and  $\text{proj}_{\vec{v}} \vec{u}$  are, we can solve for  $\vec{z}$  and state that

$$\vec{z} = \vec{u} - \text{proj}_{\vec{v}} \vec{u}.$$

This leads us to rewrite Equation (9.3) in a seemingly silly way:

$$\vec{u} = \text{proj}_{\vec{v}} \vec{u} + (\vec{u} - \text{proj}_{\vec{v}} \vec{u}).$$

This is not nonsense, as pointed out in the following Key Idea. (Notation note: the expression “ $\parallel \vec{y}$ ” means “is parallel to  $\vec{y}$ .” We can use this notation to state “ $\vec{x} \parallel \vec{y}$ ” which means “ $\vec{x}$  is parallel to  $\vec{y}$ .” The expression “ $\perp \vec{y}$ ” means “is orthogonal to  $\vec{y}$ ,” and is used similarly.)

Notes:

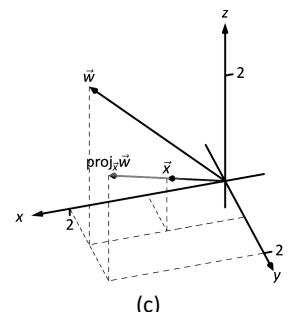
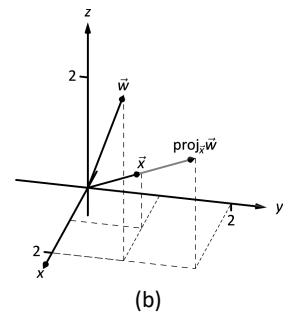
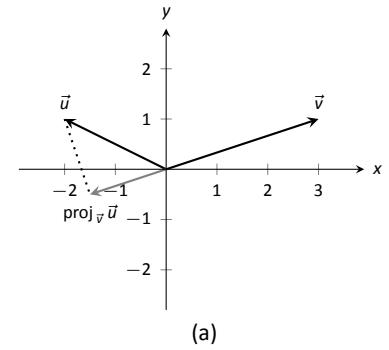


Figure 9.34: Graphing the vectors used in Example 170.

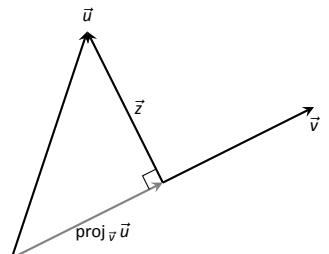


Figure 9.35: Illustrating the orthogonal projection.

**Key Idea 37 Orthogonal Decomposition of Vectors**

Let  $\vec{u}$  and  $\vec{v}$  be given. Then  $\vec{u}$  can be written as the sum of two vectors, one of which is parallel to  $\vec{v}$ , and one of which is orthogonal to  $\vec{v}$ :

$$\vec{u} = \underbrace{\text{proj}_{\vec{v}} \vec{u}}_{\parallel \vec{v}} + \underbrace{(\vec{u} - \text{proj}_{\vec{v}} \vec{u})}_{\perp \vec{v}}.$$

We illustrate the use of this equality in the following example.

**Example 171 Orthogonal decomposition of vectors**

1. Let  $\vec{u} = \langle -2, 1 \rangle$  and  $\vec{v} = \langle 3, 1 \rangle$  as in Example 170. Decompose  $\vec{u}$  as the sum of a vector parallel to  $\vec{v}$  and a vector orthogonal to  $\vec{v}$ .
2. Let  $\vec{w} = \langle 2, 1, 3 \rangle$  and  $\vec{x} = \langle 1, 1, 1 \rangle$  as in Example 170. Decompose  $\vec{w}$  as the sum of a vector parallel to  $\vec{x}$  and a vector orthogonal to  $\vec{x}$ .

**SOLUTION**

1. In Example 170, we found that  $\text{proj}_{\vec{v}} \vec{u} = \langle -1.5, -0.5 \rangle$ . Let

$$\vec{z} = \vec{u} - \text{proj}_{\vec{v}} \vec{u} = \langle -2, 1 \rangle - \langle -1.5, -0.5 \rangle = \langle -0.5, 1.5 \rangle.$$

Is  $\vec{z}$  orthogonal to  $\vec{v}$ ? (I.e., is  $\vec{z} \perp \vec{v}$ ?) We check for orthogonality with the dot product:

$$\vec{z} \cdot \vec{v} = \langle -0.5, 1.5 \rangle \cdot \langle 3, 1 \rangle = 0.$$

Since the dot product is 0, we know  $\vec{z} \perp \vec{v}$ . Thus:

$$\begin{aligned} \vec{u} &= \text{proj}_{\vec{v}} \vec{u} + (\vec{u} - \text{proj}_{\vec{v}} \vec{u}) \\ \langle -2, 1 \rangle &= \underbrace{\langle -1.5, -0.5 \rangle}_{\parallel \vec{v}} + \underbrace{\langle -0.5, 1.5 \rangle}_{\perp \vec{v}}. \end{aligned}$$

2. We found in Example 170 that  $\text{proj}_{\vec{x}} \vec{w} = \langle 2, 2, 2 \rangle$ . Applying the Key Idea, we have:

$$\vec{z} = \vec{w} - \text{proj}_{\vec{x}} \vec{w} = \langle 2, 1, 3 \rangle - \langle 2, 2, 2 \rangle = \langle 0, -1, 1 \rangle.$$

We check to see if  $\vec{z} \perp \vec{x}$ :

$$\vec{z} \cdot \vec{x} = \langle 0, -1, 1 \rangle \cdot \langle 1, 1, 1 \rangle = 0.$$

---

Notes:

Since the dot product is 0, we know the two vectors are orthogonal. We now write  $\vec{w}$  as the sum of two vectors, one parallel and one orthogonal to  $\vec{x}$ :

$$\begin{aligned}\vec{w} &= \text{proj}_{\vec{x}} \vec{w} + (\vec{w} - \text{proj}_{\vec{x}} \vec{w}) \\ \langle 2, 1, 3 \rangle &= \underbrace{\langle 2, 2, 2 \rangle}_{\parallel \vec{x}} + \underbrace{\langle 0, -1, 1 \rangle}_{\perp \vec{x}}\end{aligned}$$

We give an example of where this decomposition is useful.

**Example 172 Orthogonally decomposing a force vector**

Consider Figure 9.36(a), showing a box weighing 50lb on a ramp that rises 5ft over a span of 20ft. Find the components of force, and their magnitudes, acting on the box (as sketched in part (b) of the figure):

1. in the direction of the ramp, and
2. orthogonal to the ramp.

**SOLUTION** As the ramp rises 5ft over a horizontal distance of 20ft, we can represent the direction of the ramp with the vector  $\vec{r} = \langle 20, 5 \rangle$ . Gravity pulls down with a force of 50lb, which we represent with  $\vec{g} = \langle 0, -50 \rangle$ .

1. To find the force of gravity in the direction of the ramp, we compute  $\text{proj}_{\vec{r}} \vec{g}$ :

$$\begin{aligned}\text{proj}_{\vec{r}} \vec{g} &= \frac{\vec{g} \cdot \vec{r}}{\vec{r} \cdot \vec{r}} \vec{r} \\ &= \frac{-250}{425} \langle 20, 5 \rangle \\ &= \left\langle -\frac{200}{17}, -\frac{50}{17} \right\rangle \approx \langle -11.76, -2.94 \rangle.\end{aligned}$$

The magnitude of  $\text{proj}_{\vec{r}} \vec{g}$  is  $\| \text{proj}_{\vec{r}} \vec{g} \| = 50/\sqrt{17} \approx 12.13\text{lb}$ . Though the box weighs 50lb, a force of about 12lb is enough to keep the box from sliding down the ramp.

2. To find the component  $\vec{z}$  of gravity orthogonal to the ramp, we use Key Idea 37.

$$\begin{aligned}\vec{z} &= \vec{g} - \text{proj}_{\vec{r}} \vec{g} \\ &= \left\langle \frac{200}{17}, -\frac{800}{17} \right\rangle \approx \langle 11.76, -47.06 \rangle.\end{aligned}$$

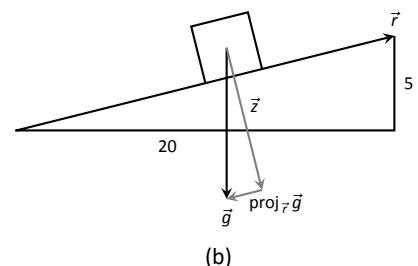
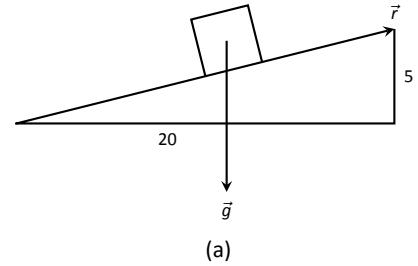


Figure 9.36: Sketching the ramp and box in Example 172. Note: The vectors are not drawn to scale.

---

Notes:

The magnitude of this force is  $\|\vec{z}\| \approx 48.51\text{lb}$ . In physics and engineering, knowing this force is important when computing things like static frictional force. (For instance, we could easily compute if the static frictional force alone was enough to keep the box from sliding down the ramp.)

### Application to Work

In physics, the application of a force  $F$  to move an object in a straight line a distance  $d$  produces *work*; the amount of work  $W$  is  $W = Fd$ , (where  $F$  is in the direction of travel). The orthogonal projection allows us to compute work when the force is not in the direction of travel.

Consider Figure 9.37, where a force  $\vec{F}$  is being applied to an object moving in the direction of  $\vec{d}$ . (The distance the object travels is the magnitude of  $\vec{d}$ .) The work done is the amount of force in the direction of  $\vec{d}$ ,  $\|\text{proj}_{\vec{d}}\vec{F}\|$ , times  $\|\vec{d}\|$ :

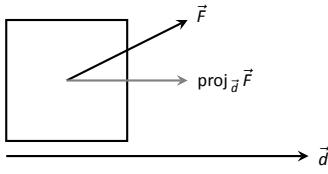


Figure 9.37: Finding work when the force and direction of travel are given as vectors.

$$\begin{aligned} \|\text{proj}_{\vec{d}}\vec{F}\| \cdot \|\vec{d}\| &= \left\| \frac{\vec{F} \cdot \vec{d}}{\vec{d} \cdot \vec{d}} \vec{d} \right\| \cdot \|\vec{d}\| \\ &= \left| \frac{\vec{F} \cdot \vec{d}}{\|\vec{d}\|^2} \right| \cdot \|\vec{d}\| \cdot \|\vec{d}\| \\ &= \frac{|\vec{F} \cdot \vec{d}|}{\|\vec{d}\|^2} \|\vec{d}\|^2 \\ &= |\vec{F} \cdot \vec{d}|. \end{aligned}$$

The expression  $\vec{F} \cdot \vec{d}$  will be positive if the angle between  $\vec{F}$  and  $\vec{d}$  is acute; when the angle is obtuse (hence  $\vec{F} \cdot \vec{d}$  is negative), the force is causing motion in the opposite direction of  $\vec{d}$ , resulting in “negative work.” We want to capture this sign, so we drop the absolute value and find that  $W = \vec{F} \cdot \vec{d}$ .

#### Definition 37 Work

Let  $\vec{F}$  be a constant force that moves an object in a straight line from point  $P$  to point  $Q$ . Let  $\vec{d} = \vec{PQ}$ . The **work**  $W$  done by  $\vec{F}$  along  $\vec{d}$  is  $W = \vec{F} \cdot \vec{d}$ .

#### Example 173 Computing work

A man slides a box along a ramp that rises 3ft over a distance of 15ft by applying 50lb of force as shown in Figure 9.38. Compute the work done.

Notes:

**SOLUTION** The figure indicates that the force applied makes a  $30^\circ$  angle with the horizontal, so  $\vec{F} = 50 \langle \cos 30^\circ, \sin 30^\circ \rangle \approx \langle 43.3, 25 \rangle$ . The ramp is represented by  $\vec{d} = \langle 15, 3 \rangle$ . The work done is simply

$$\vec{F} \cdot \vec{d} = 50 \langle \cos 30^\circ, \sin 30^\circ \rangle \cdot \langle 15, 3 \rangle \approx 724.5 \text{ ft-lb.}$$

Note how we did not actually compute the distance the object traveled, nor the magnitude of the force in the direction of travel; this is all inherently computed by the dot product!

The dot product is a powerful way of evaluating computations that depend on angles without actually using angles. The next section explores another “product” on vectors, the *cross product*. Once again, angles play an important role, though in a much different way.

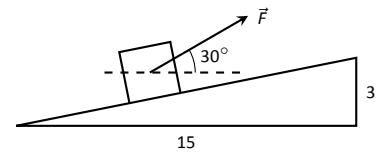


Figure 9.38: Computing work when sliding a box up a ramp in Example 173.

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Notes:

# Exercises 9.3

## Terms and Concepts

1. The dot product of two vectors is a \_\_\_\_\_, not a vector.
2. How are the concepts of the dot product and vector magnitude related?
3. How can one quickly tell if the angle between two vectors is acute or obtuse?
4. Give a synonym for “orthogonal.”

## Problems

In Exercises 5 – 10, find the dot product of the given vectors.

5.  $\vec{u} = \langle 2, -4 \rangle, \vec{v} = \langle 3, 7 \rangle$
6.  $\vec{u} = \langle 5, 3 \rangle, \vec{v} = \langle 6, 1 \rangle$
7.  $\vec{u} = \langle 1, -1, 2 \rangle, \vec{v} = \langle 2, 5, 3 \rangle$
8.  $\vec{u} = \langle 3, 5, -1 \rangle, \vec{v} = \langle 4, -1, 7 \rangle$
9.  $\vec{u} = \langle 1, 1 \rangle, \vec{v} = \langle 1, 2, 3 \rangle$
10.  $\vec{u} = \langle 1, 2, 3 \rangle, \vec{v} = \langle 0, 0, 0 \rangle$
11. Create your own vectors  $\vec{u}, \vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^2$  and show that  $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ .
12. Create your own vectors  $\vec{u}$  and  $\vec{v}$  in  $\mathbb{R}^3$  and scalar  $c$  and show that  $c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$ .

In Exercises 13 – 16, find the measure of the angle between the two vectors in both radians and degrees.

13.  $\vec{u} = \langle 1, 1 \rangle, \vec{v} = \langle 1, 2 \rangle$
14.  $\vec{u} = \langle -2, 1 \rangle, \vec{v} = \langle 3, 5 \rangle$
15.  $\vec{u} = \langle 8, 1, -4 \rangle, \vec{v} = \langle 2, 2, 0 \rangle$
16.  $\vec{u} = \langle 1, 7, 2 \rangle, \vec{v} = \langle 4, -2, 5 \rangle$

In Exercises 17 – 20, a vector  $\vec{v}$  is given. Give two vectors that are orthogonal to  $\vec{v}$ .

17.  $\vec{v} = \langle 4, 7 \rangle$
18.  $\vec{v} = \langle -3, 5 \rangle$
19.  $\vec{v} = \langle 1, 1, 1 \rangle$
20.  $\vec{v} = \langle 1, -2, 3 \rangle$

In Exercises 21 – 26, vectors  $\vec{u}$  and  $\vec{v}$  are given. Find  $\text{proj}_{\vec{v}} \vec{u}$ , the orthogonal projection of  $\vec{u}$  onto  $\vec{v}$ , and sketch all three vectors on the same axes.

21.  $\vec{u} = \langle 1, 2 \rangle, \vec{v} = \langle -1, 3 \rangle$
22.  $\vec{u} = \langle 5, 5 \rangle, \vec{v} = \langle 1, 3 \rangle$
23.  $\vec{u} = \langle -3, 2 \rangle, \vec{v} = \langle 1, 1 \rangle$
24.  $\vec{u} = \langle -3, 2 \rangle, \vec{v} = \langle 2, 3 \rangle$
25.  $\vec{u} = \langle 1, 5, 1 \rangle, \vec{v} = \langle 1, 2, 3 \rangle$
26.  $\vec{u} = \langle 3, -1, 2 \rangle, \vec{v} = \langle 2, 2, 1 \rangle$

In Exercises 27 – 32, vectors  $\vec{u}$  and  $\vec{v}$  are given. Write  $\vec{u}$  as the sum of two vectors, one of which is parallel to  $\vec{v}$  and one of which is perpendicular to  $\vec{v}$ . Note: these are the same pairs of vectors as found in Exercises 21 – 26.

27.  $\vec{u} = \langle 1, 2 \rangle, \vec{v} = \langle -1, 3 \rangle$
28.  $\vec{u} = \langle 5, 5 \rangle, \vec{v} = \langle 1, 3 \rangle$
29.  $\vec{u} = \langle -3, 2 \rangle, \vec{v} = \langle 1, 1 \rangle$
30.  $\vec{u} = \langle -3, 2 \rangle, \vec{v} = \langle 2, 3 \rangle$
31.  $\vec{u} = \langle 1, 5, 1 \rangle, \vec{v} = \langle 1, 2, 3 \rangle$
32.  $\vec{u} = \langle 3, -1, 2 \rangle, \vec{v} = \langle 2, 2, 1 \rangle$
33. A 10lb box sits on a ramp that rises 4ft over a distance of 20ft. How much force is required to keep the box from sliding down the ramp?
34. A 10lb box sits on a 15ft ramp that makes a  $30^\circ$  angle with the horizontal. How much force is required to keep the box from sliding down the ramp?
35. How much work is performed in moving a box horizontally 10ft with a force of 20lb applied at an angle of  $45^\circ$  to the horizontal?
36. How much work is performed in moving a box horizontally 10ft with a force of 20lb applied at an angle of  $10^\circ$  to the horizontal?
37. How much work is performed in moving a box up the length of a ramp that rises 2ft over a distance of 10ft, with a force of 50lb applied horizontally?
38. How much work is performed in moving a box up the length of a ramp that rises 2ft over a distance of 10ft, with a force of 50lb applied at an angle of  $45^\circ$  to the horizontal?
39. How much work is performed in moving a box up the length of a 10ft ramp that makes a  $5^\circ$  angle with the horizontal, with 50lb of force applied in the direction of the ramp?

## 9.4 The Cross Product

“Orthogonality” is immensely important. A quick scan of your current environment will undoubtedly reveal numerous surfaces and edges that are perpendicular to each other (including the edges of this page). The dot product provides a quick test for orthogonality: vectors  $\vec{u}$  and  $\vec{v}$  are perpendicular if, and only if,  $\vec{u} \cdot \vec{v} = 0$ .

Given two non-parallel, nonzero vectors  $\vec{u}$  and  $\vec{v}$  in space, it is very useful to find a vector  $\vec{w}$  that is perpendicular to both  $\vec{u}$  and  $\vec{v}$ . There is a operation, called the **cross product**, that creates such a vector. This section defines the cross product, then explores its properties and applications.

### Definition 38     Cross Product

Let  $\vec{u} = \langle u_1, u_2, u_3 \rangle$  and  $\vec{v} = \langle v_1, v_2, v_3 \rangle$  be vectors in  $\mathbb{R}^3$ . The **cross product of  $\vec{u}$  and  $\vec{v}$** , denoted  $\vec{u} \times \vec{v}$ , is the vector

$$\vec{u} \times \vec{v} = \langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle.$$

This definition can be a bit cumbersome to remember. After an example we will give a convenient method for computing the cross product. For now, careful examination of the products and differences given in the definition should reveal a pattern that is not too difficult to remember. (For instance, in the first component only 2 and 3 appear as subscripts; in the second component, only 1 and 3 appear as subscripts. Further study reveals the order in which they appear.)

Let's practice using this definition by computing a cross product.

### Example 174     Computing a cross product

Let  $\vec{u} = \langle 2, -1, 4 \rangle$  and  $\vec{v} = \langle 3, 2, 5 \rangle$ . Find  $\vec{u} \times \vec{v}$ , and verify that it is orthogonal to both  $\vec{u}$  and  $\vec{v}$ .

**SOLUTION**     Using Definition 38, we have

$$\vec{u} \times \vec{v} = \langle (-1)5 - (4)2, -((2)5 - (4)3), (2)2 - (-1)3 \rangle = \langle -13, 2, 7 \rangle.$$

(We encourage the reader to compute this product on their own, then verify their result.)

We test whether or not  $\vec{u} \times \vec{v}$  is orthogonal to  $\vec{u}$  and  $\vec{v}$  using the dot product:

$$(\vec{u} \times \vec{v}) \cdot \vec{u} = \langle -13, 2, 7 \rangle \cdot \langle 2, -1, 4 \rangle = 0,$$

$$(\vec{u} \times \vec{v}) \cdot \vec{v} = \langle -13, 2, 7 \rangle \cdot \langle 3, 2, 5 \rangle = 0.$$

Since both dot products are zero,  $\vec{u} \times \vec{v}$  is indeed orthogonal to both  $\vec{u}$  and  $\vec{v}$ .

Notes:

A convenient method of computing the cross product starts with forming a particular  $3 \times 3$  *matrix*, or rectangular array. The first row comprises the standard unit vectors  $\vec{i}$ ,  $\vec{j}$ , and  $\vec{k}$ . The second and third rows are the vectors  $\vec{u}$  and  $\vec{v}$ , respectively. Using  $\vec{u}$  and  $\vec{v}$  from Example 174, we begin with:

$$\begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 4 \\ 3 & 2 & 5 \end{array}$$

Now repeat the first two columns after the original three:

$$\begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 2 & -1 & 4 & 2 & -1 \\ 3 & 2 & 5 & 3 & 2 \end{array}$$

This gives three full “upper left to lower right” diagonals, and three full “upper right to lower left” diagonals, as shown. Compute the products along each diagonal, then add the products on the right and subtract the products on the left:

$$\vec{u} \times \vec{v} = (-5\vec{i} + 12\vec{j} + 4\vec{k}) - (-3\vec{k} + 8\vec{i} + 10\vec{j}) = -13\vec{i} + 2\vec{j} + 7\vec{k} = \langle -13, 2, 7 \rangle.$$

We practice using this method.

### Example 175 Computing a cross product

Let  $\vec{u} = \langle 1, 3, 6 \rangle$  and  $\vec{v} = \langle -1, 2, 1 \rangle$ . Compute both  $\vec{u} \times \vec{v}$  and  $\vec{v} \times \vec{u}$ .

**SOLUTION** To compute  $\vec{u} \times \vec{v}$ , we form the matrix as prescribed above, complete with repeated first columns:

$$\begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 1 & 3 & 6 & 1 & 3 \\ -1 & 2 & 1 & -1 & 2 \end{array}$$

We let the reader compute the products of the diagonals; we give the result:

$$\vec{u} \times \vec{v} = (3\vec{i} - 6\vec{j} + 2\vec{k}) - (-3\vec{k} + 12\vec{i} + \vec{j}) = \langle -9, -7, 5 \rangle.$$

---

Notes:

To compute  $\vec{v} \times \vec{u}$ , we switch the second and third rows of the above matrix, then multiply along diagonals and subtract:

$$\begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ -1 & 2 & 1 & -1 & 2 \\ 1 & 3 & 6 & 1 & 3 \end{array}$$

Note how with the rows being switched, the products that once appeared on the right now appear on the left, and vice-versa. Thus the result is:

$$\vec{v} \times \vec{u} = (12\vec{i} + \vec{j} - 3\vec{k}) - (2\vec{k} + 3\vec{i} - 6\vec{j}) = \langle 9, 7, -5 \rangle,$$

which is the opposite of  $\vec{u} \times \vec{v}$ . We leave it to the reader to verify that each of these vectors is orthogonal to  $\vec{u}$  and  $\vec{v}$ .

### Properties of the Cross Product

It is not coincidence that  $\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$  in the preceding example; one can show using Definition 38 that this will always be the case. The following theorem states several useful properties of the cross product, each of which can be verified by referring to the definition.

#### Theorem 37 Properties of the Cross Product

Let  $\vec{u}, \vec{v}$  and  $\vec{w}$  be vectors in  $\mathbb{R}^3$  and let  $c$  be a scalar. The following identities hold:

1.  $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$  Anticommutative Property
2. (a)  $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$  Distributive Properties  
 (b)  $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
3.  $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$
4. (a)  $(\vec{u} \times \vec{v}) \cdot \vec{u} = 0$  Orthogonality Properties  
 (b)  $(\vec{u} \times \vec{v}) \cdot \vec{v} = 0$
5.  $\vec{u} \times \vec{u} = \vec{0}$
6.  $\vec{u} \times \vec{0} = \vec{0}$
7.  $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$  Triple Scalar Product

---

Notes:

We introduced the cross product as a way to find a vector orthogonal to two given vectors, but we did not give a proof that the construction given in Definition 38 satisfies this property. Theorem 37 asserts this property holds; we leave it as a problem in the Exercise section to verify this.

Property 5 from the theorem is also left to the reader to prove in the Exercise section, but it reveals something more interesting than “the cross product of a vector with itself is  $\vec{0}$ .“ Let  $\vec{u}$  and  $\vec{v}$  be parallel vectors; that is, let there be a scalar  $c$  such that  $\vec{v} = c\vec{u}$ . Consider their cross product:

$$\begin{aligned}\vec{u} \times \vec{v} &= \vec{u} \times (c\vec{u}) \\ &= c(\vec{u} \times \vec{u}) \quad (\text{by Property 3 of Theorem 37}) \\ &= \vec{0}. \quad (\text{by Property 5 of Theorem 37})\end{aligned}$$

We have just shown that the cross product of parallel vectors is  $\vec{0}$ . This hints at something deeper. Theorem 36 related the angle between two vectors and their dot product; there is a similar relationship relating the cross product of two vectors and the angle between them, given by the following theorem.

**Note:** Definition 35 (through Theorem 36) defines  $\vec{u}$  and  $\vec{v}$  to be orthogonal if  $\vec{u} \cdot \vec{v} = 0$ . We could use Theorem 38 to define  $\vec{u}$  and  $\vec{v}$  are parallel if  $\vec{u} \times \vec{v} = 0$ . By such a definition,  $\vec{0}$  would be both orthogonal and parallel to every vector. Apparent paradoxes such as this are not uncommon in mathematics and can be very useful. (See also the marginal note on page 512.)

### Theorem 38 The Cross Product and Angles

Let  $\vec{u}$  and  $\vec{v}$  be vectors in  $\mathbb{R}^3$ . Then

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin \theta,$$

where  $\theta$ ,  $0 \leq \theta \leq \pi$ , is the angle between  $\vec{u}$  and  $\vec{v}$ .

Note that this theorem makes a statement about the *magnitude* of the cross product. When the angle between  $\vec{u}$  and  $\vec{v}$  is 0 or  $\pi$  (i.e., the vectors are parallel), the magnitude of the cross product is 0. The only vector with a magnitude of 0 is  $\vec{0}$  (see Property 9 of Theorem 34), hence the cross product of parallel vectors is  $\vec{0}$ .

We demonstrate the truth of this theorem in the following example.

### Example 176 The cross product and angles

Let  $\vec{u} = \langle 1, 3, 6 \rangle$  and  $\vec{v} = \langle -1, 2, 1 \rangle$  as in Example 175. Verify Theorem 38 by finding  $\theta$ , the angle between  $\vec{u}$  and  $\vec{v}$ , and the magnitude of  $\vec{u} \times \vec{v}$ .

---

Notes:

**SOLUTION** We use Theorem 36 to find the angle between  $\vec{u}$  and  $\vec{v}$ .

$$\begin{aligned}\theta &= \cos^{-1} \left( \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \right) \\ &= \cos^{-1} \left( \frac{11}{\sqrt{46}\sqrt{6}} \right) \\ &\approx 0.8471 = 48.54^\circ.\end{aligned}$$

Our work in Example 175 showed that  $\vec{u} \times \vec{v} = \langle -9, -7, 5 \rangle$ , hence  $\|\vec{u} \times \vec{v}\| = \sqrt{155}$ . Is  $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin \theta$ ? Using numerical approximations, we find:

$$\begin{aligned}\|\vec{u} \times \vec{v}\| &= \sqrt{155} \\ &\approx 12.45.\end{aligned}\quad \begin{aligned}\|\vec{u}\| \|\vec{v}\| \sin \theta &= \sqrt{46}\sqrt{6} \sin 0.8471 \\ &\approx 12.45.\end{aligned}$$

Numerically, they seem equal. Using a right triangle, one can show that

$$\sin \left( \cos^{-1} \left( \frac{11}{\sqrt{46}\sqrt{6}} \right) \right) = \frac{\sqrt{155}}{\sqrt{46}\sqrt{6}},$$

which allows us to verify the theorem exactly.

### Right Hand Rule

The anticommutative property of the cross product demonstrates that  $\vec{u} \times \vec{v}$  and  $\vec{v} \times \vec{u}$  differ only by a sign – these vectors have the same magnitude but point in the opposite direction. When seeking a vector perpendicular to  $\vec{u}$  and  $\vec{v}$ , we essentially have two directions to choose from, one in the direction of  $\vec{u} \times \vec{v}$  and one in the direction of  $\vec{v} \times \vec{u}$ . Does it matter which we choose? How can we tell which one we will get without graphing, etc.?

Another wonderful property of the cross product, as defined, is that it follows the **right hand rule**. Given  $\vec{u}$  and  $\vec{v}$  in  $\mathbb{R}^3$  with the same initial point, point the index finger of your right hand in the direction of  $\vec{u}$  and let your middle finger point in the direction of  $\vec{v}$  (much as we did when establishing the right hand rule for the 3-dimensional coordinate system). Your thumb will naturally extend in the direction of  $\vec{u} \times \vec{v}$ . One can “practice” this using Figure 9.39. If you switch, and point the index finger in the direction of  $\vec{v}$  and the middle finger in the direction of  $\vec{u}$ , your thumb will now point in the opposite direction, allowing you to “visualize” the anticommutative property of the cross product.

### Applications of the Cross Product

There are a number of ways in which the cross product is useful in mathematics, physics and other areas of science beyond “just” finding a vector perpendicular to two others. We highlight a few here.

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Notes:

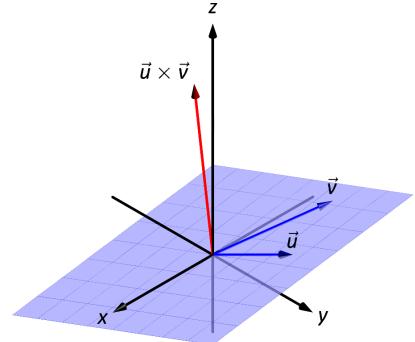


Figure 9.39: Illustrating the Right Hand Rule of the cross product.

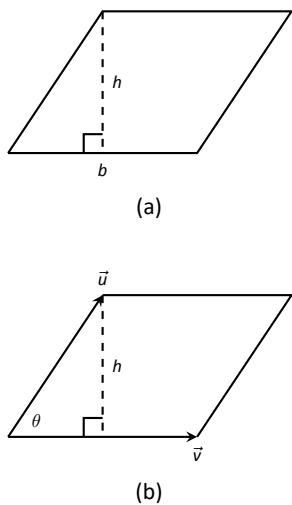


Figure 9.40: Using the cross product to find the area of a parallelogram.

### Area of a Parallelogram

It is a standard geometry fact that the area of a parallelogram is  $A = bh$ , where  $b$  is the length of the base and  $h$  is the height of the parallelogram, as illustrated in Figure 9.40(a). As shown when defining the Parallelogram Law of vector addition, two vectors  $\vec{u}$  and  $\vec{v}$  define a parallelogram when drawn from the same initial point, as illustrated in Figure 9.40(b). Trigonometry tells us that  $h = \|\vec{u}\| \sin \theta$ , hence the area of the parallelogram is

$$A = \|\vec{u}\| \|\vec{v}\| \sin \theta = \|\vec{u} \times \vec{v}\|, \quad (9.4)$$

where the second equality comes from Theorem 38. We illustrate using Equation (9.4) in the following example.

#### Example 177 Finding the area of a parallelogram

- Find the area of the parallelogram defined by the vectors  $\vec{u} = \langle 2, 1 \rangle$  and  $\vec{v} = \langle 1, 3 \rangle$ .
- Verify that the points  $A = (1, 1, 1)$ ,  $B = (2, 3, 2)$ ,  $C = (4, 5, 3)$  and  $D = (3, 3, 2)$  are the vertices of a parallelogram. Find the area of the parallelogram.

#### SOLUTION

- Figure 9.41(a) sketches the parallelogram defined by the vectors  $\vec{u}$  and  $\vec{v}$ . We have a slight problem in that our vectors exist in  $\mathbb{R}^2$ , not  $\mathbb{R}^3$ , and the cross product is only defined on vectors in  $\mathbb{R}^3$ . We skirt this issue by viewing  $\vec{u}$  and  $\vec{v}$  as vectors in the  $x-y$  plane of  $\mathbb{R}^3$ , and rewrite them as  $\vec{u} = \langle 2, 1, 0 \rangle$  and  $\vec{v} = \langle 1, 3, 0 \rangle$ . We can now compute the cross product. It is easy to show that  $\vec{u} \times \vec{v} = \langle 0, 0, 5 \rangle$ ; therefore the area of the parallelogram is  $A = \|\vec{u} \times \vec{v}\| = 5$ .
- To show that the quadrilateral  $ABCD$  is a parallelogram (shown in Figure 9.41(b)), we need to show that the opposite sides are parallel. We can quickly show that  $\vec{AB} = \vec{DC} = \langle 1, 2, 1 \rangle$  and  $\vec{BC} = \vec{AD} = \langle 2, 2, 1 \rangle$ . We find the area by computing the magnitude of the cross product of  $\vec{AB}$  and  $\vec{BC}$ :

$$\vec{AB} \times \vec{BC} = \langle 0, 1, -2 \rangle \Rightarrow \|\vec{AB} \times \vec{BC}\| = \sqrt{5} \approx 2.236.$$

This application is perhaps more useful in finding the area of a triangle (in short, triangles are used more often than parallelograms). We illustrate this in the following example.

Notes:

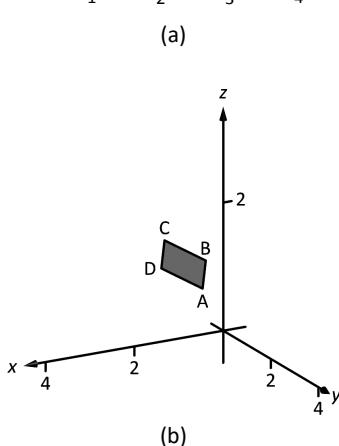


Figure 9.41: Sketching the parallelograms in Example 177.

**Example 178 Area of a triangle**

Find the area of the triangle with vertices  $A = (1, 2)$ ,  $B = (2, 3)$  and  $C = (3, 1)$ , as pictured in Figure 9.42.

**SOLUTION** We found the area of this triangle in Example 39 to be 1.5 using integration. There we discussed the fact that finding the area of a triangle can be inconvenient using the “ $\frac{1}{2}bh$ ” formula as one has to compute the height, which generally involves finding angles, etc. Using a cross product is much more direct.

We can choose any two sides of the triangle to use to form vectors; we choose  $\vec{AB} = \langle 1, 1 \rangle$  and  $\vec{AC} = \langle 2, -1 \rangle$ . As in the previous example, we will rewrite these vectors with a third component of 0 so that we can apply the cross product. The area of the triangle is

$$\frac{1}{2} \|\vec{AB} \times \vec{AC}\| = \frac{1}{2} \| \langle 1, 1, 0 \rangle \times \langle 2, -1, 0 \rangle \| = \frac{1}{2} \| \langle 0, 0, -3 \rangle \| = \frac{3}{2}.$$

We arrive at the same answer as before with less work.

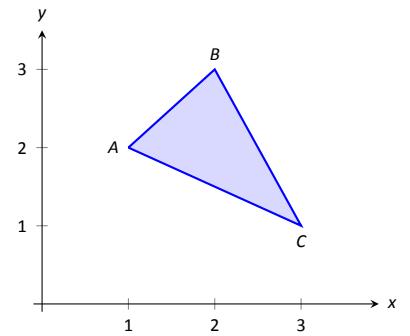


Figure 9.42: Finding the area of a triangle in Example 178.

**Note:** The word “parallelepiped” is pronounced “parallel–eh–pipe–ed.”

**Volume of a Parallelepiped**

The three dimensional analogue to the parallelogram is the **parallelepiped**. Each face is parallel to the face opposite face, as illustrated in Figure 9.43. By crossing  $\vec{v}$  and  $\vec{w}$ , one gets a vector whose magnitude is the area of the base. Dotting this vector with  $\vec{u}$  computes the volume of parallelepiped! (Up to a sign; take the absolute value.)

Thus the volume of a parallelepiped defined by vectors  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$  is

$$V = |\vec{u} \cdot (\vec{v} \times \vec{w})|. \quad (9.5)$$

Note how this is the Triple Scalar Product, first seen in Theorem 37. Applying the identities given in the theorem shows that we can apply the Triple Scalar Product in any “order” we choose to find the volume. That is,

$$V = |\vec{u} \cdot (\vec{v} \times \vec{w})| = |\vec{u} \cdot (\vec{w} \times \vec{v})| = |(\vec{u} \times \vec{v}) \cdot \vec{w}|, \quad \text{etc.}$$

**Example 179 Finding the volume of parallelepiped**

Find the volume of the parallelepiped defined by the vectors  $\vec{u} = \langle 1, 1, 0 \rangle$ ,  $\vec{v} = \langle -1, 1, 0 \rangle$  and  $\vec{w} = \langle 0, 1, 1 \rangle$ .

**SOLUTION** We apply Equation (9.5). We first find  $\vec{v} \times \vec{w} = \langle 1, 1, -1 \rangle$ . Then

$$|\vec{u} \cdot (\vec{v} \times \vec{w})| = | \langle 1, 1, 0 \rangle \cdot \langle 1, 1, -1 \rangle | = 2.$$

So the volume of the parallelepiped is 2 cubic units.

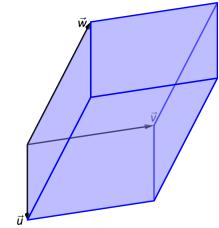


Figure 9.43: A parallelepiped is the three dimensional analogue to the parallelogram.

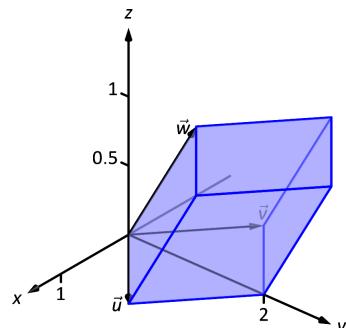


Figure 9.44: A parallelepiped in Example 179.

Notes:

While this application of the Triple Scalar Product is interesting, it is not used all that often: parallelepipeds are not a common shape in physics and engineering. The last application of the cross product is very applicable in engineering.

### Torque

**Torque** is a measure of the turning force applied to an object. A classic scenario involving torque is the application of a wrench to a bolt. When a force is applied to the wrench, the bolt turns. When we represent the force and wrench with vectors  $\vec{F}$  and  $\vec{\ell}$ , we see that the bolt moves (because of the threads) in a direction orthogonal to  $\vec{F}$  and  $\vec{\ell}$ . Torque is usually represented by the Greek letter  $\tau$ , or tau, and has units of N·m, a Newton–meter, or ft·lb, a foot–pound.

While a full understanding of torque is beyond the purposes of this book, when a force  $\vec{F}$  is applied to a lever arm  $\vec{\ell}$ , the resulting torque is

$$\vec{\tau} = \vec{\ell} \times \vec{F}. \quad (9.6)$$

#### Example 180 Computing torque

A lever of length 2ft makes an angle with the horizontal of  $45^\circ$ . Find the resulting torque when a force of 10lb is applied to the end of the level where:

1. the force is perpendicular to the lever, and
2. the force makes an angle of  $60^\circ$  with the lever, as shown in Figure 9.45.

#### SOLUTION

1. We start by determining vectors for the force and lever arm. Since the lever arm makes a  $45^\circ$  angle with the horizontal and is 2ft long, we can state that  $\vec{\ell} = 2 \langle \cos 45^\circ, \sin 45^\circ \rangle = \langle \sqrt{2}, \sqrt{2} \rangle$ .

Since the force vector is perpendicular to the lever arm (as seen in the left hand side of Figure 9.45), we can conclude it is making an angle of  $-45^\circ$  with the horizontal. As it has a magnitude of 10lb, we can state  $\vec{F} = 10 \langle \cos(-45^\circ), \sin(-45^\circ) \rangle = \langle 5\sqrt{2}, -5\sqrt{2} \rangle$ .

Using Equation (9.6) to find the torque requires a cross product. We again let the third component of each vector be 0 and compute the cross product:

$$\begin{aligned}\vec{\tau} &= \vec{\ell} \times \vec{F} \\ &= \langle \sqrt{2}, \sqrt{2}, 0 \rangle \times \langle 5\sqrt{2}, -5\sqrt{2}, 0 \rangle \\ &= \langle 0, 0, -20 \rangle\end{aligned}$$

---

Notes:

This clearly has a magnitude of 20 ft-lb.

We can view the force and lever arm vectors as lying “on the page”; our computation of  $\vec{\tau}$  shows that the torque goes “into the page.” This follows the Right Hand Rule of the cross product, and it also matches well with the example of the wrench turning the bolt. Turning a bolt clockwise moves it in.

- Our lever arm can still be represented by  $\vec{\ell} = \langle \sqrt{2}, \sqrt{2} \rangle$ . As our force vector makes a  $60^\circ$  angle with  $\vec{\ell}$ , we can see (referencing the right hand side of the figure) that  $\vec{F}$  makes a  $-15^\circ$  angle with the horizontal. Thus

$$\begin{aligned}\vec{F} &= 10 \langle \cos -15^\circ, \sin -15^\circ \rangle = \left\langle \frac{5(1 + \sqrt{3})}{\sqrt{2}}, -\frac{5(1 + \sqrt{3})}{\sqrt{2}} \right\rangle \\ &\approx \langle 9.659, -2.588 \rangle.\end{aligned}$$

We again make the third component 0 and take the cross product to find the torque:

$$\begin{aligned}\vec{\tau} &= \vec{\ell} \times \vec{F} \\ &= \langle \sqrt{2}, \sqrt{2}, 0 \rangle \times \left\langle \frac{5(1 + \sqrt{3})}{\sqrt{2}}, -\frac{5(1 + \sqrt{3})}{\sqrt{2}}, 0 \right\rangle \\ &= \langle 0, 0, -10\sqrt{3} \rangle \\ &\approx \langle 0, 0, -17.321 \rangle.\end{aligned}$$

As one might expect, when the force and lever arm vectors *are* orthogonal, the magnitude of force is greater than when the vectors *are not* orthogonal.

While the cross product has a variety of applications (as noted in this chapter), its fundamental use is finding a vector perpendicular to two others. Knowing a vector is orthogonal to two others is of incredible importance, as it allows us to find the equations of lines and planes in a variety of contexts. The importance of the cross product, in some sense, relies on the importance of lines and planes, which see widespread use throughout engineering, physics and mathematics. We study lines and planes in the next two sections.

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Notes:

# Exercises 9.4

## Terms and Concepts

1. The cross product of two vectors is a \_\_\_\_\_, not a scalar.
2. One can visualize the direction of  $\vec{u} \times \vec{v}$  using the \_\_\_\_\_.
3. Give a synonym for “orthogonal.”
4. T/F: A fundamental principle of the cross product is that  $\vec{u} \times \vec{v}$  is orthogonal to  $\vec{u}$  and  $\vec{v}$ .
5. \_\_\_\_\_ is a measure of the turning force applied to an object.

## Problems

In Exercises 6 – 14, vectors  $\vec{u}$  and  $\vec{v}$  are given. Compute  $\vec{u} \times \vec{v}$  and show this is orthogonal to both  $\vec{u}$  and  $\vec{v}$ .

6.  $\vec{u} = \langle 3, 2, -2 \rangle, \quad \vec{v} = \langle 0, 1, 5 \rangle$
7.  $\vec{u} = \langle 5, -4, 3 \rangle, \quad \vec{v} = \langle 2, -5, 1 \rangle$
8.  $\vec{u} = \langle 4, -5, -5 \rangle, \quad \vec{v} = \langle 3, 3, 4 \rangle$
9.  $\vec{u} = \langle -4, 7, -10 \rangle, \quad \vec{v} = \langle 4, 4, 1 \rangle$
10.  $\vec{u} = \langle 1, 0, 1 \rangle, \quad \vec{v} = \langle 5, 0, 7 \rangle$
11.  $\vec{u} = \langle 1, 5, -4 \rangle, \quad \vec{v} = \langle -2, -10, 8 \rangle$
12.  $\vec{u} = \vec{i}, \quad \vec{v} = \vec{j}$
13.  $\vec{u} = \vec{i}, \quad \vec{v} = \vec{k}$
14.  $\vec{u} = \vec{j}, \quad \vec{v} = \vec{k}$

15. Pick any vectors  $\vec{u}, \vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^3$  and show that  $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$ .
16. Pick any vectors  $\vec{u}, \vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^3$  and show that  $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$ .

In Exercises 17 – 20, the magnitudes of vectors  $\vec{u}$  and  $\vec{v}$  in  $\mathbb{R}^3$  are given, along with the angle  $\theta$  between them. Use this information to find the magnitude of  $\vec{u} \times \vec{v}$ .

17.  $\|\vec{u}\| = 2, \quad \|\vec{v}\| = 5, \quad \theta = 30^\circ$
18.  $\|\vec{u}\| = 3, \quad \|\vec{v}\| = 7, \quad \theta = \pi/2$
19.  $\|\vec{u}\| = 3, \quad \|\vec{v}\| = 4, \quad \theta = \pi$
20.  $\|\vec{u}\| = 2, \quad \|\vec{v}\| = 5, \quad \theta = 5\pi/6$

In Exercises 21 – 24, find the area of the parallelogram defined by the given vectors.

21.  $\vec{u} = \langle 1, 1, 2 \rangle, \quad \vec{v} = \langle 2, 0, 3 \rangle$
22.  $\vec{u} = \langle -2, 1, 5 \rangle, \quad \vec{v} = \langle -1, 3, 1 \rangle$
23.  $\vec{u} = \langle 1, 2 \rangle, \quad \vec{v} = \langle 2, 1 \rangle$
24.  $\vec{u} = \langle 2, 0 \rangle, \quad \vec{v} = \langle 0, 3 \rangle$

In Exercises 25 – 28, find the area of the triangle with the given vertices.

25. Vertices:  $(0, 0, 0), (1, 3, -1)$  and  $(2, 1, 1)$ .
26. Vertices:  $(5, 2, -1), (3, 6, 2)$  and  $(1, 0, 4)$ .
27. Vertices:  $(1, 1), (1, 3)$  and  $(2, 2)$ .
28. Vertices:  $(3, 1), (1, 2)$  and  $(4, 3)$ .

In Exercises 29 – 30, find the area of the quadrilateral with the given vertices. (Hint: break the quadrilateral into 2 triangles.)

29. Vertices:  $(0, 0), (1, 2), (3, 0)$  and  $(4, 3)$ .
30. Vertices:  $(0, 0, 0), (2, 1, 1), (-1, 2, -8)$  and  $(1, -1, 5)$ .

In Exercises 31 – 32, find the volume of the parallelepiped defined by the given vectors.

31.  $\vec{u} = \langle 1, 1, 1 \rangle, \quad \vec{v} = \langle 1, 2, 3 \rangle, \quad \vec{w} = \langle 1, 0, 1 \rangle$
32.  $\vec{u} = \langle -1, 2, 1 \rangle, \quad \vec{v} = \langle 2, 2, 1 \rangle, \quad \vec{w} = \langle 3, 1, 3 \rangle$

In Exercises 33 – 36, find a unit vector orthogonal to both  $\vec{u}$  and  $\vec{v}$ .

33.  $\vec{u} = \langle 1, 1, 1 \rangle, \quad \vec{v} = \langle 2, 0, 1 \rangle$
34.  $\vec{u} = \langle 1, -2, 1 \rangle, \quad \vec{v} = \langle 3, 2, 1 \rangle$
35.  $\vec{u} = \langle 5, 0, 2 \rangle, \quad \vec{v} = \langle -3, 0, 7 \rangle$

36.  $\vec{u} = \langle 1, -2, 1 \rangle, \quad \vec{v} = \langle -2, 4, -2 \rangle$
37. A bicycle rider applies 150lb of force, straight down, onto a pedal that extends 7in horizontally from the crankshaft. Find the magnitude of the torque applied to the crankshaft.
38. A bicycle rider applies 150lb of force, straight down, onto a pedal that extends 7in from the crankshaft, making a  $30^\circ$  angle with the horizontal. Find the magnitude of the torque applied to the crankshaft.

39. To turn a stubborn bolt, 80lb of force is applied to a 10in wrench. What is the maximum amount of torque that can be applied to the bolt?
40. To turn a stubborn bolt, 80lb of force is applied to a 10in wrench in a confined space, where the direction of applied force makes a  $10^\circ$  angle with the wrench. How much torque is subsequently applied to the wrench?
41. Show, using the definition of the Cross Product, that  $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$ ; that is, that  $\vec{u}$  is orthogonal to the cross product of  $\vec{u}$  and  $\vec{v}$ .
42. Show, using the definition of the Cross Product, that  $\vec{u} \times \vec{u} = \vec{0}$ .

## 9.5 Lines

To find the equation of a line in the  $x$ - $y$  plane, we need two pieces of information: a point and the slope. The slope conveys *direction* information. As vertical lines have an undefined slope, the following statement is more accurate:

To define a line, one needs a point on the line and the direction of the line.

This holds true for lines in space.

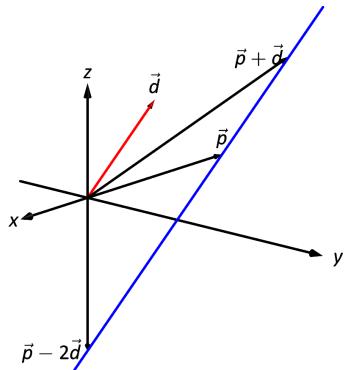


Figure 9.47: Defining a line in space.

Let  $P$  be a point in space, let  $\vec{p}$  be the vector with initial point at the origin and terminal point at  $P$  (i.e.,  $\vec{p}$  “points” to  $P$ ), and let  $\vec{d}$  be a vector. Consider the points on the line through  $P$  in the direction of  $\vec{d}$ .

Clearly one point on the line is  $P$ ; we can say that the vector  $\vec{p}$  lies at this point on the line. To find another point on the line, we can start at  $\vec{p}$  and move in a direction parallel to  $\vec{d}$ . For instance, starting at  $\vec{p}$  and traveling one length of  $\vec{d}$  places one at another point on the line. Consider Figure 9.47 where certain points along the line are indicated.

The figure illustrates how every point on the line can be obtained by starting with  $\vec{p}$  and moving a certain distance in the direction of  $\vec{d}$ . That is, we can define the line as a function of  $t$ :

$$\ell(t) = \vec{p} + t \vec{d}. \quad (9.7)$$

In many ways, this is *not* a new concept. Compare Equation (9.7) to the familiar “ $y = mx + b$ ” equation of a line:

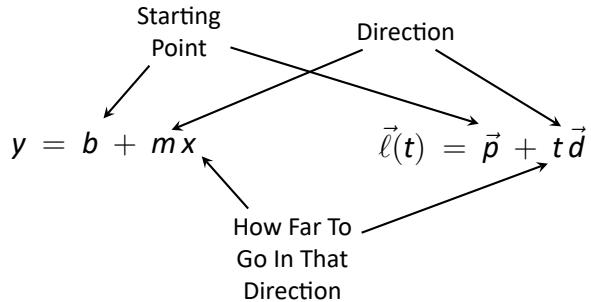


Figure 9.46: Understanding the vector equation of a line.

The equations exhibit the same structure: they give a starting point, define a direction, and state how far in that direction to travel.

Equation (9.7) is an example of a **vector-valued function**; the input of the function is a real number and the output is a vector. We will cover vector-valued functions extensively in the next chapter.

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Notes:

There are other ways to represent a line. Let  $\vec{p} = \langle x_0, y_0, z_0 \rangle$  and let  $\vec{d} = \langle a, b, c \rangle$ . Then the equation of the line through  $\vec{p}$  in the direction of  $\vec{d}$  is:

$$\begin{aligned}\vec{\ell}(t) &= \vec{p} + t\vec{d} \\ &= \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle \\ &= \langle x_0 + at, y_0 + bt, z_0 + ct \rangle.\end{aligned}$$

The last line states that the  $x$  values of the line are given by  $x = x_0 + at$ , the  $y$  values are given by  $y = y_0 + bt$ , and the  $z$  values are given by  $z = z_0 + ct$ . These three equations, taken together, are the **parametric equations of the line** through  $\vec{p}$  in the direction of  $\vec{d}$ .

Finally, each of the equations for  $x$ ,  $y$  and  $z$  above contain the variable  $t$ . We can solve for  $t$  in each equation:

$$\begin{aligned}x = x_0 + at &\Rightarrow t = \frac{x - x_0}{a}, \\ y = y_0 + bt &\Rightarrow t = \frac{y - y_0}{b}, \\ z = z_0 + ct &\Rightarrow t = \frac{z - z_0}{c},\end{aligned}$$

assuming  $a, b, c \neq 0$ . Since  $t$  is equal to each expression on the right, we can set these equal to each other, forming the **symmetric equations of the line** through  $\vec{p}$  in the direction of  $\vec{d}$ :

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Each representation has its own advantages, depending on the context. We summarize these three forms in the following definition, then give examples of their use.

Notes:

**Definition 39 Equations of Lines in Space**

Consider the line in space that passes through  $\vec{p} = \langle x_0, y_0, z_0 \rangle$  in the direction of  $\vec{d} = \langle a, b, c \rangle$ .

1. The **vector equation** of the line is

$$\vec{\ell}(t) = \vec{p} + t\vec{d}.$$

2. The **parametric equations** of the line are

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

3. The **symmetric equations** of the line are

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

**Example 181 Finding the equation of a line**

Give all three equations, as given in Definition 39, of the line through  $P = (2, 3, 1)$  in the direction of  $\vec{d} = \langle -1, 1, 2 \rangle$ . Does the point  $Q = (-1, 6, 6)$  lie on this line?

**SOLUTION** We identify the point  $P = (2, 3, 1)$  with the vector  $\vec{p} = \langle 2, 3, 1 \rangle$ . Following the definition, we have

- the vector equation of the line is  $\vec{\ell}(t) = \langle 2, 3, 1 \rangle + t \langle -1, 1, 2 \rangle$ ;
- the parametric equations of the line are

$$x = 2 - t, \quad y = 3 + t, \quad z = 1 + 2t; \text{ and}$$

- the symmetric equations of the line are

$$\frac{x - 2}{-1} = \frac{y - 3}{1} = \frac{z - 1}{2}.$$

The first two equations of the line are useful when a  $t$  value is given: one can immediately find the corresponding point on the line. These forms are good when calculating with a computer; most software programs easily handle equations in these formats. (For instance, to make Figure 9.48, a certain graphics program was given the input  $(2-x, 3+x, 1+2*x)$ . This particular program requires the variable always be "x" instead of "t").

Does the point  $Q = (-1, 6, 6)$  lie on the line? The graph in Figure 9.48 makes it clear that it does not. We can answer this question without the graph

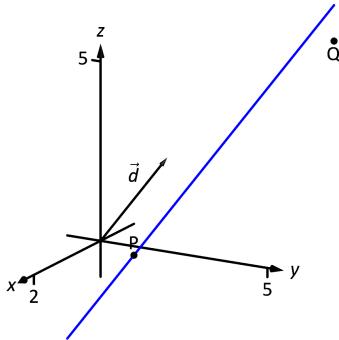


Figure 9.48: Graphing a line in Example 181.

Notes:

using any of the three equation forms. Of the three, the symmetric equations are probably best suited for this task. Simply plug in the values of  $x$ ,  $y$  and  $z$  and see if equality is maintained:

$$\frac{-1 - 2}{-1} \stackrel{?}{=} \frac{6 - 3}{1} \stackrel{?}{=} \frac{6 - 1}{2} \Rightarrow 3 = 3 \neq 2.5.$$

We see that  $Q$  does not lie on the line as it did not satisfy the symmetric equations.

### Example 182 Finding the equation of a line through two points

Find the parametric equations of the line through the points  $P = (2, -1, 2)$  and  $Q = (1, 3, -1)$ .

**SOLUTION** Recall the statement made at the beginning of this section: to find the equation of a line, we need a point and a direction. We have *two* points; either one will suffice. The direction of the line can be found by the vector with initial point  $P$  and terminal point  $Q$ :  $\vec{PQ} = \langle -1, 4, -3 \rangle$ .

The parametric equations of the line  $\ell$  through  $P$  in the direction of  $\vec{PQ}$  are:

$$\ell : x = 2 - t \quad y = -1 + 4t \quad z = 2 - 3t.$$

A graph of the points and line are given in Figure 9.49. Note how in the given parametrization of the line,  $t = 0$  corresponds to the point  $P$ , and  $t = 1$  corresponds to the point  $Q$ . This relates to the understanding of the vector equation of a line described in Figure 9.46. The parametric equations “start” at the point  $P$ , and  $t$  determines how far in the direction of  $\vec{PQ}$  to travel. When  $t = 0$ , we travel 0 lengths of  $\vec{PQ}$ ; when  $t = 1$ , we travel one length of  $\vec{PQ}$ , resulting in the point  $Q$ .

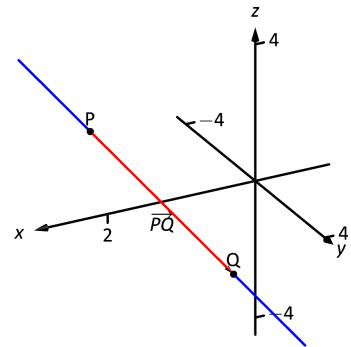


Figure 9.49: A graph of the line in Example 182.

### Parallel, Intersecting and Skew Lines

In the plane, two *distinct* lines can either be parallel or they will intersect at exactly one point. In space, given equations of two lines, it can sometimes be difficult to tell whether the lines are distinct or not (i.e., the same line can be represented in different ways). Given lines  $\vec{\ell}_1(t) = \vec{p}_1 + t\vec{d}_1$  and  $\vec{\ell}_2(t) = \vec{p}_2 + t\vec{d}_2$ , we have four possibilities:  $\vec{\ell}_1$  and  $\vec{\ell}_2$  are

the same line	they share all points;
intersecting lines	share only 1 point;
parallel lines	$\vec{d}_1 \parallel \vec{d}_2$ , no points in common; or
skew lines	$\vec{d}_1 \not\parallel \vec{d}_2$ , no points in common.

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Notes:

The next two examples investigate these possibilities.

### Example 183 Comparing lines

Consider lines  $\ell_1$  and  $\ell_2$ , given in parametric equation form:

$$\ell_1: \begin{aligned} x &= 1 + 3t \\ y &= 2 - t \\ z &= t \end{aligned} \quad \ell_2: \begin{aligned} x &= -2 + 4s \\ y &= 3 + s \\ z &= 5 + 2s. \end{aligned}$$

Determine whether  $\ell_1$  and  $\ell_2$  are the same line, intersect, are parallel, or skew.

**SOLUTION** We start by looking at the directions of each line. Line  $\ell_1$  has the direction given by  $\vec{d}_1 = \langle 3, -1, 1 \rangle$  and line  $\ell_2$  has the direction given by  $\vec{d}_2 = \langle 4, 1, 2 \rangle$ . It should be clear that  $\vec{d}_1$  and  $\vec{d}_2$  are not parallel, hence  $\ell_1$  and  $\ell_2$  are not the same line, nor are they parallel. Figure 9.50 verifies this fact (where the points and directions indicated by the equations of each line are identified).

We next check to see if they intersect (if they do not, they are skew lines). To find if they intersect, we look for  $t$  and  $s$  values such that the respective  $x$ ,  $y$  and  $z$  values are the same. That is, we want  $s$  and  $t$  such that:

$$\begin{aligned} 1 + 3t &= -2 + 4s \\ 2 - t &= 3 + s \\ t &= 5 + 2s. \end{aligned}$$

This is a relatively simple system of linear equations. Since the last equation is already solved for  $t$ , substitute that value of  $t$  into the equation above it:

$$2 - (5 + 2s) = 3 + s \Rightarrow s = -2, t = 1.$$

A key to remember is that we have *three* equations; we need to check if  $s = -2, t = 1$  satisfies the first equation as well:

$$1 + 3(1) \neq -2 + 4(-2).$$

It does not. Therefore, we conclude that the lines  $\ell_1$  and  $\ell_2$  are skew.

### Example 184 Comparing lines

Consider lines  $\ell_1$  and  $\ell_2$ , given in parametric equation form:

$$\ell_1: \begin{aligned} x &= -0.7 + 1.6t \\ y &= 4.2 + 2.72t \\ z &= 2.3 - 3.36t \end{aligned} \quad \ell_2: \begin{aligned} x &= 2.8 - 2.9s \\ y &= 10.15 - 4.93s \\ z &= -5.05 + 6.09s. \end{aligned}$$

Determine whether  $\ell_1$  and  $\ell_2$  are the same line, intersect, are parallel, or skew.

---

Notes:

**SOLUTION** It is obviously very difficult to simply look at these equations and discern anything. This is done intentionally. In the “real world,” most equations that are used do not have nice, integer coefficients. Rather, there are lots of digits after the decimal and the equations can look “messy.”

We again start by deciding whether or not each line has the same direction. The direction of  $\ell_1$  is given by  $\vec{d}_1 = \langle 1.6, 2.72, -3.36 \rangle$  and the direction of  $\ell_2$  is given by  $\vec{d}_2 = \langle -2.9, -4.93, 6.09 \rangle$ . When it is not clear through observation whether two vectors are parallel or not, the standard way of determining this is by comparing their respective unit vectors. Using a calculator, we find:

$$\begin{aligned}\vec{u}_1 &= \frac{\vec{d}_1}{\|\vec{d}_1\|} = \langle 0.3471, 0.5901, -0.7289 \rangle \\ \vec{u}_2 &= \frac{\vec{d}_2}{\|\vec{d}_2\|} = \langle -0.3471, -0.5901, 0.7289 \rangle.\end{aligned}$$

The two vectors seem to be parallel (at least, their components are equal to 4 decimal places). In most situations, it would suffice to conclude that the lines are at least parallel, if not the same. One way to be sure is to rewrite  $\vec{d}_1$  and  $\vec{d}_2$  in terms of fractions, not decimals. We have

$$\vec{d}_1 = \left\langle \frac{16}{10}, \frac{272}{100}, -\frac{336}{100} \right\rangle \quad \vec{d}_2 = \left\langle -\frac{29}{10}, -\frac{493}{100}, \frac{609}{100} \right\rangle.$$

One can then find the magnitudes of each vector in terms of fractions, then compute the unit vectors likewise. After a lot of manual arithmetic (or after briefly using a computer algebra system), one finds that

$$\vec{u}_1 = \left\langle \sqrt{\frac{10}{83}}, \frac{17}{\sqrt{830}}, -\frac{21}{\sqrt{830}} \right\rangle \quad \vec{u}_2 = \left\langle -\sqrt{\frac{10}{83}}, -\frac{17}{\sqrt{830}}, \frac{21}{\sqrt{830}} \right\rangle.$$

We can now say without equivocation that these lines are parallel.

Are they the same line? The parametric equations for a line describe one point that lies on the line, so we know that the point  $P_1 = (-0.7, 4.2, 2.3)$  lies on  $\ell_1$ . To determine if this point also lies on  $\ell_2$ , plug in the  $x$ ,  $y$  and  $z$  values of  $P_1$  into the symmetric equations for  $\ell_2$ :

$$\frac{(-0.7) - 2.8}{-2.9} \stackrel{?}{=} \frac{(4.2) - 10.15}{-4.93} \stackrel{?}{=} \frac{(2.3) - (-5.05)}{6.09} \Rightarrow 1.2069 = 1.2069 = 1.2069.$$

The point  $P_1$  lies on both lines, so we conclude they are the same line, just parametrized differently. Figure 9.51 graphs this line along with the points and vectors described by the parametric equations. Note how  $\vec{d}_1$  and  $\vec{d}_2$  are parallel, though point in opposite directions (as indicated by their unit vectors above).

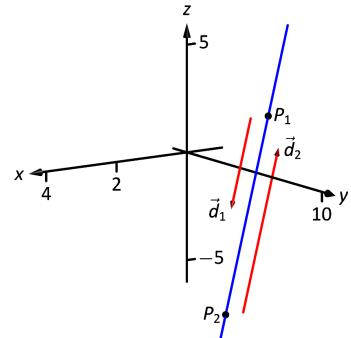


Figure 9.51: Graphing the lines in Example 184.

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Notes:

## Distances

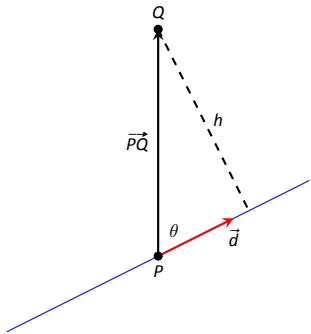


Figure 9.52: Establishing the distance from a point to a line.

Given a point  $Q$  and a line  $\ell(t) = \vec{p} + t\vec{d}$  in space, it is often useful to know the distance from the point to the line. (Here we use the standard definition of “distance,” i.e., the length of the shortest line segment from the point to the line.) Identifying  $\vec{p}$  with the point  $P$ , Figure 9.52 will help establish a general method of computing this distance  $h$ .

From trigonometry, we know  $h = \|\overrightarrow{PQ}\| \sin \theta$ . We have a similar identity involving the cross product:  $\|\overrightarrow{PQ} \times \vec{d}\| = \|\overrightarrow{PQ}\| \|\vec{d}\| \sin \theta$ . Divide both sides of this latter equation by  $\|\vec{d}\|$  to obtain  $h$ :

$$h = \frac{\|\overrightarrow{PQ} \times \vec{d}\|}{\|\vec{d}\|}. \quad (9.8)$$

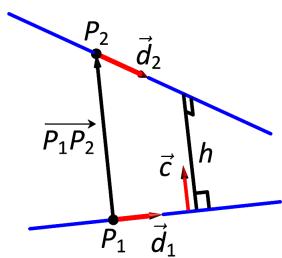


Figure 9.53: Establishing the distance between lines.

It is also useful to determine the distance between lines, which we define as the length of the shortest line segment that connects the two lines (an argument from geometry shows that this line segments is perpendicular to both lines). Let lines  $\ell_1(t) = \vec{p}_1 + t\vec{d}_1$  and  $\ell_2(t) = \vec{p}_2 + t\vec{d}_2$  be given, as shown in Figure 9.53. To find the direction orthogonal to both  $\vec{d}_1$  and  $\vec{d}_2$ , we take the cross product:  $\vec{c} = \vec{d}_1 \times \vec{d}_2$ . The magnitude of the orthogonal projection of  $\overrightarrow{P_1P_2}$  onto  $\vec{c}$  is the distance  $h$  we seek:

$$\begin{aligned} h &= \|\text{proj}_{\vec{c}} \overrightarrow{P_1P_2}\| \\ &= \left\| \frac{\overrightarrow{P_1P_2} \cdot \vec{c}}{\vec{c} \cdot \vec{c}} \vec{c} \right\| \\ &= \frac{|\overrightarrow{P_1P_2} \cdot \vec{c}|}{\|\vec{c}\|^2} \|\vec{c}\| \\ &= \frac{|\overrightarrow{P_1P_2} \cdot \vec{c}|}{\|\vec{c}\|}. \end{aligned}$$

A problem in the Exercise section is to show that this distance is 0 when the lines intersect. Note the use of the Triple Scalar Product:  $\overrightarrow{P_1P_2} \cdot \vec{c} = \overrightarrow{P_1P_2} \cdot (\vec{d}_1 \times \vec{d}_2)$ .

The following Key Idea restates these two distance formulas.

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Notes:

**Key Idea 38 Distances to Lines**

1. Let  $P$  be a point on a line  $\ell$  that is parallel to  $\vec{d}$ . The distance  $h$  from a point  $Q$  to the line  $\ell$  is:

$$h = \frac{\|\overrightarrow{PQ} \times \vec{d}\|}{\|\vec{d}\|}.$$

2. Let  $P_1$  be a point on line  $\ell_1$  that is parallel to  $\vec{d}_1$ , and let  $P_2$  be a point on line  $\ell_2$  parallel to  $\vec{d}_2$ , and let  $\vec{c} = \vec{d}_1 \times \vec{d}_2$ , where lines  $\ell_1$  and  $\ell_2$  are not parallel. The distance  $h$  between the two lines is:

$$h = \frac{|\overrightarrow{P_1P_2} \cdot \vec{c}|}{\|\vec{c}\|}.$$

**Example 185 Finding the distance from a point to a line**

Find the distance from the point  $Q = (1, 1, 3)$  to the line  $\vec{\ell}(t) = \langle 1, -1, 1 \rangle + t \langle 2, 3, 1 \rangle$ .

**SOLUTION** The equation of the line gives us the point  $P = (1, -1, 1)$  that lies on the line, hence  $\overrightarrow{PQ} = \langle 0, 2, 2 \rangle$ . The equation also gives  $\vec{d} = \langle 2, 3, 1 \rangle$ . Following Key Idea 38, we have the distance as

$$\begin{aligned} h &= \frac{\|\overrightarrow{PQ} \times \vec{d}\|}{\|\vec{d}\|} \\ &= \frac{\|\langle -4, 4, -4 \rangle\|}{\sqrt{14}} \\ &= \frac{4\sqrt{3}}{\sqrt{14}} \approx 1.852. \end{aligned}$$

The point  $Q$  is approximately 1.852 units from the line  $\vec{\ell}(t)$ .

**Example 186 Finding the distance between lines**

Find the distance between the lines

$$\begin{array}{ll} \ell_1: \begin{array}{l} x = 1 + 3t \\ y = 2 - t \\ z = t \end{array} & \ell_2: \begin{array}{l} x = -2 + 4s \\ y = 3 + s \\ z = 5 + 2s. \end{array} \end{array}$$

**SOLUTION** These are the same lines as given in Example 183, where

Notes:

we showed them to be skew. The equations allow us to identify the following points and vectors:

$$P_1 = (1, 2, 0) \quad P_2 = (-2, 3, 5) \quad \Rightarrow \quad \overrightarrow{P_1P_2} = \langle -3, 1, 5 \rangle.$$

$$\vec{d}_1 = \langle 3, -1, 1 \rangle \quad \vec{d}_2 = \langle 4, 1, 2 \rangle \quad \Rightarrow \quad \vec{c} = \vec{d}_1 \times \vec{d}_2 = \langle -3, -2, 7 \rangle.$$

From Key Idea 38 we have the distance  $h$  between the two lines is

$$\begin{aligned} h &= \frac{|\overrightarrow{P_1P_2} \cdot \vec{c}|}{\|\vec{c}\|} \\ &= \frac{42}{\sqrt{62}} \approx 5.334. \end{aligned}$$

The lines are approximately 5.334 units apart.

One of the key points to understand from this section is this: to describe a line, we need a point and a direction. Whenever a problem is posed concerning a line, one needs to take whatever information is offered and glean point and direction information. Many questions can be asked (and are asked in the Exercise section) whose answer immediately follows from this understanding.

Lines are one of two fundamental objects of study in space. The other fundamental object is the *plane*, which we study in detail in the next section. Many complex three dimensional objects are studied by approximating their surfaces with lines and planes.

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Notes:

# Exercises 9.5

## Terms and Concepts

- To find an equation of a line, what two pieces of information are needed?
- Two distinct lines in the plane can intersect or be \_\_\_\_\_.
- Two distinct lines in space can intersect, be \_\_\_\_\_ or be \_\_\_\_\_.
- Use your own words to describe what it means for two lines in space to be skew.

## Problems

**In Exercises 5 – 14, write the vector, parametric and symmetric equations of the lines described.**

- Passes through  $P = (2, -4, 1)$ , parallel to  $\vec{d} = \langle 9, 2, 5 \rangle$ .
- Passes through  $P = (6, 1, 7)$ , parallel to  $\vec{d} = \langle -3, 2, 5 \rangle$ .
- Passes through  $P = (2, 1, 5)$  and  $Q = (7, -2, 4)$ .
- Passes through  $P = (1, -2, 3)$  and  $Q = (5, 5, 5)$ .
- Passes through  $P = (0, 1, 2)$  and orthogonal to both  $\vec{d}_1 = \langle 2, -1, 7 \rangle$  and  $\vec{d}_2 = \langle 7, 1, 3 \rangle$ .
- Passes through  $P = (5, 1, 9)$  and orthogonal to both  $\vec{d}_1 = \langle 1, 0, 1 \rangle$  and  $\vec{d}_2 = \langle 2, 0, 3 \rangle$ .
- Passes through the point of intersection of  $\vec{\ell}_1(t)$  and  $\vec{\ell}_2(t)$  and orthogonal to both lines, where  
 $\vec{\ell}_1(t) = \langle 2, 1, 1 \rangle + t \langle 5, 1, -2 \rangle$  and  
 $\vec{\ell}_2(t) = \langle -2, -1, 2 \rangle + t \langle 3, 1, -1 \rangle$ .
- Passes through the point of intersection of  $\ell_1(t)$  and  $\ell_2(t)$  and orthogonal to both lines, where  
 $\ell_1 = \begin{cases} x = t \\ y = -2 + 2t \\ z = 1 + t \end{cases}$  and  $\ell_2 = \begin{cases} x = 2 + t \\ y = 2 - t \\ z = 3 + 2t \end{cases}$
- Passes through  $P = (1, 1)$ , parallel to  $\vec{d} = \langle 2, 3 \rangle$ .
- Passes through  $P = (-2, 5)$ , parallel to  $\vec{d} = \langle 0, 1 \rangle$ .

**In Exercises 15 – 22, determine if the described lines are the same line, parallel lines, intersecting or skew lines. If intersecting, give the point of intersection.**

- $\vec{\ell}_1(t) = \langle 1, 2, 1 \rangle + t \langle 2, -1, 1 \rangle$ ,  
 $\vec{\ell}_2(t) = \langle 3, 3, 3 \rangle + t \langle -4, 2, -2 \rangle$ .

16.  $\vec{\ell}_1(t) = \langle 2, 1, 1 \rangle + t \langle 5, 1, 3 \rangle$ ,  
 $\vec{\ell}_2(t) = \langle 14, 5, 9 \rangle + t \langle 1, 1, 1 \rangle$ .

17.  $\vec{\ell}_1(t) = \langle 3, 4, 1 \rangle + t \langle 2, -3, 4 \rangle$ ,  
 $\vec{\ell}_2(t) = \langle -3, 3, -3 \rangle + t \langle 3, -2, 4 \rangle$ .

18.  $\vec{\ell}_1(t) = \langle 1, 1, 1 \rangle + t \langle 3, 1, 3 \rangle$ ,  
 $\vec{\ell}_2(t) = \langle 7, 3, 7 \rangle + t \langle 6, 2, 6 \rangle$ .

19.  $\ell_1 = \begin{cases} x = 1 + 2t \\ y = 3 - 2t \\ z = t \end{cases}$  and  $\ell_2 = \begin{cases} x = 3 - t \\ y = 3 + 5t \\ z = 2 + 7t \end{cases}$

20.  $\ell_1 = \begin{cases} x = 1.1 + 0.6t \\ y = 3.77 + 0.9t \\ z = -2.3 + 1.5t \end{cases}$  and  $\ell_2 = \begin{cases} x = 3.11 + 3.4t \\ y = 2 + 5.1t \\ z = 2.5 + 8.5t \end{cases}$

21.  $\ell_1 = \begin{cases} x = 0.2 + 0.6t \\ y = 1.33 - 0.45t \\ z = -4.2 + 1.05t \end{cases}$  and  $\ell_2 = \begin{cases} x = 0.86 + 9.2t \\ y = 0.835 - 6.9t \\ z = -3.045 + 16.1t \end{cases}$

22.  $\ell_1 = \begin{cases} x = 0.1 + 1.1t \\ y = 2.9 - 1.5t \\ z = 3.2 + 1.6t \end{cases}$  and  $\ell_2 = \begin{cases} x = 4 - 2.1t \\ y = 1.8 + 7.2t \\ z = 3.1 + 1.1t \end{cases}$

**In Exercises 23 – 26, find the distance from the point to the line.**

23.  $P = (1, 1, 1)$ ,  $\vec{\ell}(t) = \langle 2, 1, 3 \rangle + t \langle 2, 1, -2 \rangle$

24.  $P = (2, 5, 6)$ ,  $\vec{\ell}(t) = \langle -1, 1, 1 \rangle + t \langle 1, 0, 1 \rangle$

25.  $P = (0, 3)$ ,  $\vec{\ell}(t) = \langle 2, 0 \rangle + t \langle 1, 1 \rangle$

26.  $P = (1, 1)$ ,  $\vec{\ell}(t) = \langle 4, 5 \rangle + t \langle -4, 3 \rangle$

**In Exercises 27 – 28, find the distance between the two lines.**

27.  $\vec{\ell}_1(t) = \langle 1, 2, 1 \rangle + t \langle 2, -1, 1 \rangle$ ,  
 $\vec{\ell}_2(t) = \langle 3, 3, 3 \rangle + t \langle 4, 2, -2 \rangle$ .

28.  $\vec{\ell}_1(t) = \langle 0, 0, 1 \rangle + t \langle 1, 0, 0 \rangle$ ,  
 $\vec{\ell}_2(t) = \langle 0, 0, 3 \rangle + t \langle 0, 1, 0 \rangle$ .

**Exercises 29 – 31 explore special cases of the distance formulas found in Key Idea 38.**

29. Let  $Q$  be a point on the line  $\ell(t)$ . Show why the distance formula correctly gives the distance from the point to the line as 0.

30. Let lines  $\ell_1(t)$  and  $\ell_2(t)$  be intersecting lines. Show why the distance formula correctly gives the distance between these lines as 0.

31. Let lines  $\ell_1(t)$  and  $\ell_2(t)$  be parallel.
- (a) Show why the distance formula for distance between lines cannot be used as stated to find the distance between the lines.
  - (b) Show why letting  $c = (\overrightarrow{P_1P_2} \times \vec{d}_2) \times \vec{d}_2$  allows one to use the formula.
  - (c) Show how one can use the formula for the distance between a point and a line to find the distance between parallel lines.

## 9.6 Planes

Any flat surface, such as a wall, table top or stiff piece of cardboard can be thought of as representing part of a plane. Consider a piece of cardboard with a point  $P$  marked on it. One can take a nail and stick it into the cardboard at  $P$  such that the nail is perpendicular to the cardboard; see Figure 9.54

This nail provides a “handle” for the cardboard. Moving the cardboard around moves  $P$  to different locations in space. Tilting the nail (but keeping  $P$  fixed) tilts the cardboard. Both moving and tilting the cardboard defines a different plane in space. In fact, we can define a plane by: 1) the location of  $P$  in space, and 2) the direction of the nail.

The previous section showed that one can define a line given a point on the line and the direction of the line (usually given by a vector). One can make a similar statement about planes: we can define a plane in space given a point on the plane and the direction the plane “faces” (using the description above, the direction of the nail). Once again, the direction information will be supplied by a vector, called a **normal vector**, that is orthogonal to the plane.

What exactly does “orthogonal to the plane” mean? Choose any two points  $P$  and  $Q$  in the plane, and consider the vector  $\vec{PQ}$ . We say a vector  $\vec{n}$  is orthogonal to the plane if  $\vec{n}$  is perpendicular to  $\vec{PQ}$  for all choices of  $P$  and  $Q$ ; that is, if  $\vec{n} \cdot \vec{PQ} = 0$  for all  $P$  and  $Q$ .

This gives us way of writing an equation describing the plane. Let  $P = (x_0, y_0, z_0)$  be a point in the plane and let  $\vec{n} = \langle a, b, c \rangle$  be a normal vector to the plane. A point  $Q = (x, y, z)$  lies in the plane defined by  $P$  and  $\vec{n}$  if, and only if,  $\vec{PQ}$  is orthogonal to  $\vec{n}$ . Knowing  $\vec{PQ} = \langle x - x_0, y - y_0, z - z_0 \rangle$ , consider:

$$\begin{aligned}\vec{PQ} \cdot \vec{n} &= 0 \\ \langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle &= 0 \\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0\end{aligned}\tag{9.9}$$

Equation (9.9) defines an *implicit* function describing the plane. More algebra produces:

$$ax + by + cz = ax_0 + by_0 + cz_0.$$

The right hand side is just a number, so we replace it with  $d$ :

$$ax + by + cz = d.\tag{9.10}$$

As long as  $c \neq 0$ , we can solve for  $z$ :

$$z = \frac{1}{c}(d - ax - by).\tag{9.11}$$

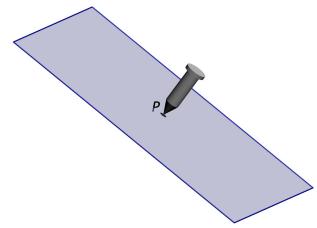


Figure 9.54: Illustrating defining a plane with a sheet of cardboard and a nail.

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Notes:

Equation (9.11) is especially useful as many computer programs can graph functions in this form. Equations (9.9) and (9.10) have specific names, given next.

**Definition 40      Equations of a Plane in Standard and General Forms**

The plane passing through the point  $P = (x_0, y_0, z_0)$  with normal vector  $\vec{n} = \langle a, b, c \rangle$  can be described by an equation with **standard form**

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0;$$

the equation's **general form** is

$$ax + by + cz = d.$$

A key to remember throughout this section is this: to find the equation of a plane, we need a point and a normal vector. We will give several examples of finding the equation of a plane, and in each one different types of information are given. In each case, we need to use the given information to find a point on the plane and a normal vector.

**Example 187      Finding the equation of a plane.**

Write the equation of the plane that passes through the points  $P = (1, 1, 0)$ ,  $Q = (1, 2, -1)$  and  $R = (0, 1, 2)$  in standard form.

**SOLUTION**      We need a vector  $\vec{n}$  that is orthogonal to the plane. Since  $P$ ,  $Q$  and  $R$  are in the plane, so are the vectors  $\vec{PQ}$  and  $\vec{PR}$ ;  $\vec{PQ} \times \vec{PR}$  is orthogonal to  $\vec{PQ}$  and  $\vec{PR}$  and hence the plane itself.

It is straightforward to compute  $\vec{n} = \vec{PQ} \times \vec{PR} = \langle 2, 1, 1 \rangle$ . We can use any point we wish in the plane (any of  $P$ ,  $Q$  or  $R$  will do) and we arbitrarily choose  $P$ . Following Definition 40, the equation of the plane in standard form is

$$2(x - 1) + (y - 1) + z = 0.$$

The plane is sketched in Figure 9.55.

We have just demonstrated the fact that any three non-collinear points define a plane. (This is why a three-legged stool does not “rock;” it’s three feet always lie in a plane. A four-legged stool will rock unless all four feet lie in the same plane.)

**Example 188      Finding the equation of a plane.**

Verify that lines  $\ell_1$  and  $\ell_2$ , whose parametric equations are given below, inter-

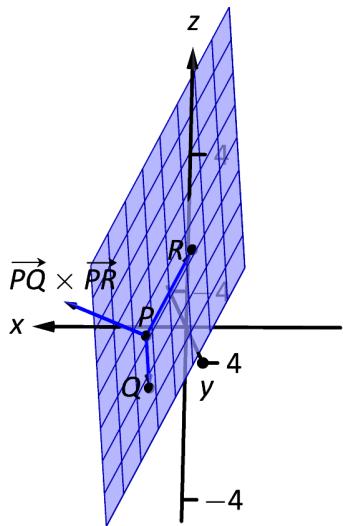


Figure 9.55: Sketching the plane in Example 187.

Notes:

sect, then give the equation of the plane that contains these two lines in general form.

$$\ell_1: \begin{aligned} x &= -5 + 2s \\ y &= 1 + s \\ z &= -4 + 2s \end{aligned} \quad \ell_2: \begin{aligned} x &= 2 + 3t \\ y &= 1 - 2t \\ z &= 1 + t \end{aligned}$$

**SOLUTION** The lines clearly are not parallel. If they do not intersect, they are skew, meaning there is not a plane that contains them both. If they do intersect, there is such a plane.

To find their point of intersection, we set the  $x$ ,  $y$  and  $z$  equations equal to each other and solve for  $s$  and  $t$ :

$$\begin{aligned} -5 + 2s &= 2 + 3t \\ 1 + s &= 1 - 2t \quad \Rightarrow \quad s = 2, \quad t = -1. \\ -4 + 2s &= 1 + t \end{aligned}$$

When  $s = 2$  and  $t = -1$ , the lines intersect at the point  $P = (-1, 3, 0)$ .

Let  $\vec{d}_1 = \langle 2, 1, 2 \rangle$  and  $\vec{d}_2 = \langle 3, -2, 1 \rangle$  be the directions of lines  $\ell_1$  and  $\ell_2$ , respectively. A normal vector to the plane containing these two lines will also be orthogonal to  $\vec{d}_1$  and  $\vec{d}_2$ . Thus we find a normal vector  $\vec{n}$  by computing  $\vec{n} = \vec{d}_1 \times \vec{d}_2 = \langle 5, 4, -7 \rangle$ .

We can pick any point in the plane with which to write our equation; each line gives us infinite choices of points. We choose  $P$ , the point of intersection. We follow Definition 40 to write the plane's equation in general form:

$$\begin{aligned} 5(x + 1) + 4(y - 3) - 7z &= 0 \\ 5x + 5 + 4y - 12 - 7z &= 0 \\ 5x + 4y - 7z &= 7. \end{aligned}$$

The plane's equation in general form is  $5x + 4y - 7z = 7$ ; it is sketched in Figure 9.56.

### Example 189 Finding the equation of a plane

Give the equation, in standard form, of the plane that passes through the point  $P = (-1, 0, 1)$  and is orthogonal to the line with vector equation  $\vec{\ell}(t) = \langle -1, 0, 1 \rangle + t \langle 1, 2, 2 \rangle$ .

**SOLUTION** As the plane is to be orthogonal to the line, the plane must be orthogonal to the direction of the line given by  $\vec{d} = \langle 1, 2, 2 \rangle$ . We use this as our normal vector. Thus the plane's equation, in standard form, is

$$(x + 1) + 2y + 2(z - 1) = 0.$$

The line and plane are sketched in Figure 9.57.

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Notes:

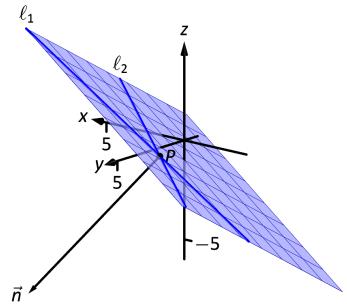


Figure 9.56: Sketching the plane in Example 188.

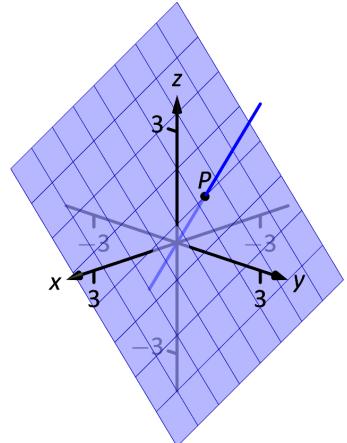


Figure 9.57: The line and plane in Example 189.

**Example 190 Finding the intersection of two planes**

Give the parametric equations of the line that is the intersection of the planes  $p_1$  and  $p_2$ , where:

$$p_1 : x - (y - 2) + (z - 1) = 0$$

$$p_2 : -2(x - 2) + (y + 1) + (z - 3) = 0$$

**SOLUTION** To find an equation of a line, we need a point on the line and the direction of the line.

We can find a point on the line by solving each equation of the planes for  $z$ :

$$p_1 : z = -x + y - 1$$

$$p_2 : z = 2x - y - 2$$

We can now set these two equations equal to each other (i.e., we are finding values of  $x$  and  $y$  where the planes have the same  $z$  value):

$$-x + y - 1 = 2x - y - 2$$

$$2y = 3x - 1$$

$$y = \frac{1}{2}(3x - 1)$$

We can choose any value for  $x$ ; we choose  $x = 1$ . This determines that  $y = 1$ . We can now use the equations of either plane to find  $z$ : when  $x = 1$  and  $y = 1$ ,  $z = -1$  on both planes. We have found a point  $P$  on the line:  $P = (1, 1, -1)$ .

We now need the direction of the line. Since the line lies in each plane, its direction is orthogonal to a normal vector for each plane. Considering the equations for  $p_1$  and  $p_2$ , we can quickly determine their normal vectors. For  $p_1$ ,  $\vec{n}_1 = \langle 1, -1, 1 \rangle$  and for  $p_2$ ,  $\vec{n}_2 = \langle -2, 1, 1 \rangle$ . A direction orthogonal to both of these directions is their cross product:  $\vec{d} = \vec{n}_1 \times \vec{n}_2 = \langle -2, -3, -1 \rangle$ .

The parametric equations of the line through  $P = (1, 1, -1)$  in the direction of  $d = \langle -2, -3, -1 \rangle$  is:

$$\ell : \quad x = -2t + 1 \quad y = -3t + 1 \quad z = -t - 1.$$

The planes and line are graphed in Figure 9.58.

**Example 191 Finding the intersection of a plane and a line**

Find the point of intersection, if any, of the line  $\ell(t) = \langle 3, -3, -1 \rangle + t \langle -1, 2, 1 \rangle$  and the plane with equation in general form  $2x + y + z = 4$ .

**SOLUTION** The equation of the plane shows that the vector  $\vec{n} = \langle 2, 1, 1 \rangle$  is a normal vector to the plane, and the equation of the line shows that the line

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Notes:

moves parallel to  $\vec{d} = \langle -1, 2, 1 \rangle$ . Since these are not orthogonal, we know there is a point of intersection. (If there were orthogonal, it would mean that the plane and line were parallel to each other, either never intersecting or the line was in the plane itself.)

To find the point of intersection, we need to find a  $t$  value such that  $\ell(t)$  satisfies the equation of the plane. Rewriting the equation of the line with parametric equations will help:

$$\ell(t) = \begin{cases} x = 3 - t \\ y = -3 + 2t \\ z = -1 + t \end{cases}$$

Replacing  $x$ ,  $y$  and  $z$  in the equation of the plane with the expressions containing  $t$  found in the equation of the line allows us to determine a  $t$  value that indicates the point of intersection:

$$\begin{aligned} 2x + y + z &= 4 \\ 2(3 - t) + (-3 + 2t) + (-1 + t) &= 4 \\ t &= 2. \end{aligned}$$

When  $t = 2$ , the point on the line satisfies the equation of the plane; that point is  $\ell(2) = \langle 1, 1, 1 \rangle$ . Thus the point  $(1, 1, 1)$  is the point of intersection between the plane and the line, illustrated in Figure 9.59.

## Distances

Just as it was useful to find distances between points and lines in the previous section, it is also often necessary to find the distance from a point to a plane.

Consider Figure 9.60, where a plane with normal vector  $\vec{n}$  is sketched containing a point  $P$  and a point  $Q$ , not on the plane, is given. We measure the distance from  $Q$  to the plane by measuring the length of the projection of  $\overrightarrow{PQ}$  onto  $\vec{n}$ . That is, we want:

$$\left\| \text{proj}_{\vec{n}} \overrightarrow{PQ} \right\| = \left\| \frac{\vec{n} \cdot \overrightarrow{PQ}}{\|\vec{n}\|^2} \vec{n} \right\| = \frac{|\vec{n} \cdot \overrightarrow{PQ}|}{\|\vec{n}\|} \quad (9.12)$$

Equation (9.12) is important as it does more than just give the distance between a point and a plane. We will see how it allows us to find several other distances as well: the distance between parallel planes and the distance from a line and a plane. Because Equation (9.12) is important, we restate it as a Key Idea.

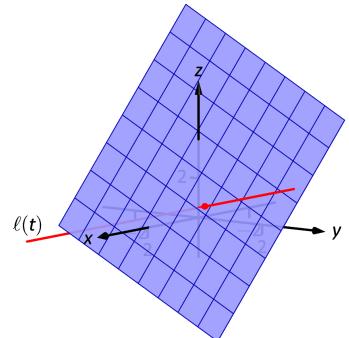


Figure 9.59: Illustrating the intersection of a line and a plane in Example 191.

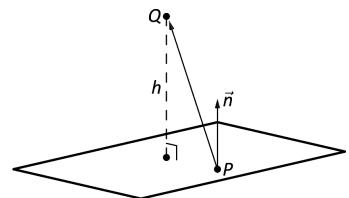


Figure 9.60: Illustrating finding the distance from a point to a plane.

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Notes:

**Key Idea 39 Distance from a Point to a Plane**

Let a plane with normal vector  $\vec{n}$  be given, and let  $Q$  be a point. The distance  $h$  from  $Q$  to the plane is

$$h = \frac{|\vec{n} \cdot \vec{PQ}|}{\|\vec{n}\|},$$

where  $P$  is any point in the plane.

**Example 192 Distance between a point and a plane**

Find the distance between the point  $Q = (2, 1, 4)$  and the plane with equation  $2x - 5y + 6z = 9$ .

**SOLUTION** Using the equation of the plane, we find the normal vector  $\vec{n} = \langle 2, -5, 6 \rangle$ . To find a point on the plane, we can let  $x$  and  $y$  be anything we choose, then let  $z$  be whatever satisfies the equation. Letting  $x$  and  $y$  be 0 seems simple; this makes  $z = 1.5$ . Thus we let  $P = \langle 0, 0, 1.5 \rangle$ , and  $\vec{PQ} = \langle 2, 1, 2.5 \rangle$ .

The distance  $h$  from  $Q$  to the plane is given by Key Idea 39:

$$\begin{aligned} h &= \frac{|\vec{n} \cdot \vec{PQ}|}{\|\vec{n}\|} \\ &= \frac{|\langle 2, -5, 6 \rangle \cdot \langle 2, 1, 2.5 \rangle|}{\|\langle 2, -5, 6 \rangle\|} \\ &= \frac{|14|}{\sqrt{65}} \\ &\approx 1.74. \end{aligned}$$

We can use Key Idea 39 to find other distances. Given two parallel planes, we can find the distance between these planes by letting  $P$  be a point on one plane and  $Q$  a point on the other. If  $\ell$  is a line parallel to a plane, we can use the Key Idea to find the distance between them as well: again, let  $P$  be a point in the plane and let  $Q$  be any point on the line. (One can also use Key Idea 38.) The Exercise section contains problems of these types.

These past two sections have not explored lines and planes in space as an exercise of mathematical curiosity. However, there are many, many applications of these fundamental concepts. Complex shapes can be modeled (or, *approximated*) using planes. For instance, part of the exterior of an aircraft may have a complex, yet smooth, shape, and engineers will want to know how air flows across this piece as well as how heat might build up due to air friction. Many equations that help determine air flow and heat dissipation are difficult to apply to arbitrary surfaces, but simple to apply to planes. By approximating a surface with millions of small planes one can more readily model the needed behavior.

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Notes:

# Exercises 9.6

## Terms and Concepts

- In order to find the equation of a plane, what two pieces of information must one have?
- What is the relationship between a plane and one of its normal vectors?

## Problems

**In Exercises 3 – 6, give any two points in the given plane.**

- $2x - 4y + 7z = 2$
- $3(x + 2) + 5(y - 9) - 4z = 0$
- $x = 2$
- $4(y + 2) - (z - 6) = 0$

**In Exercises 7 – 20, give the equation of the described plane in standard and general forms.**

- Passes through  $(2, 3, 4)$  and has normal vector  $\vec{n} = \langle 3, -1, 7 \rangle$ .
- Passes through  $(1, 3, 5)$  and has normal vector  $\vec{n} = \langle 0, 2, 4 \rangle$ .
- Passes through the points  $(1, 2, 3)$ ,  $(3, -1, 4)$  and  $(1, 0, 1)$ .
- Passes through the points  $(5, 3, 8)$ ,  $(6, 4, 9)$  and  $(3, 3, 3)$ .

- Contains the intersecting lines  
 $\ell_1(t) = \langle 2, 1, 2 \rangle + t \langle 1, 2, 3 \rangle$  and  
 $\ell_2(t) = \langle 2, 1, 2 \rangle + t \langle 2, 5, 4 \rangle$ .

- Contains the intersecting lines  
 $\ell_1(t) = \langle 5, 0, 3 \rangle + t \langle -1, 1, 1 \rangle$  and  
 $\ell_2(t) = \langle 1, 4, 7 \rangle + t \langle 3, 0, -3 \rangle$ .

- Contains the parallel lines  
 $\ell_1(t) = \langle 1, 1, 1 \rangle + t \langle 1, 2, 3 \rangle$  and  
 $\ell_2(t) = \langle 1, 1, 2 \rangle + t \langle 1, 2, 3 \rangle$ .

- Contains the parallel lines  
 $\ell_1(t) = \langle 1, 1, 1 \rangle + t \langle 4, 1, 3 \rangle$  and  
 $\ell_2(t) = \langle 2, 2, 2 \rangle + t \langle 4, 1, 3 \rangle$ .

- Contains the point  $(2, -6, 1)$  and the line  
 $\ell(t) = \begin{cases} x = 2 + 5t \\ y = 2 + 2t \\ z = -1 + 2t \end{cases}$

- Contains the point  $(5, 7, 3)$  and the line

$$\ell(t) = \begin{cases} x = t \\ y = t \\ z = t \end{cases}$$

- Contains the point  $(5, 7, 3)$  and is orthogonal to the line  
 $\ell(t) = \langle 4, 5, 6 \rangle + t \langle 1, 1, 1 \rangle$ .

- Contains the point  $(4, 1, 1)$  and is orthogonal to the line

$$\ell(t) = \begin{cases} x = 4 + 4t \\ y = 1 + 1t \\ z = 1 + 1t \end{cases}$$

- Contains the point  $(-4, 7, 2)$  and is parallel to the plane  
 $3(x - 2) + 8(y + 1) - 10z = 0$ .

- Contains the point  $(1, 2, 3)$  and is parallel to the plane  
 $x = 5$ .

**In Exercises 21 – 22, give the equation of the line that is the intersection of the given planes.**

- $p1 : 3(x - 2) + (y - 1) + 4z = 0$ , and  
 $p2 : 2(x - 1) - 2(y + 3) + 6(z - 1) = 0$ .

- $p1 : 5(x - 5) + 2(y + 2) + 4(z - 1) = 0$ , and  
 $p2 : 3x - 4(y - 1) + 2(z - 1) = 0$ .

**In Exercises 23 – 26, find the point of intersection between the line and the plane.**

- line:  $\langle 5, 1, -1 \rangle + t \langle 2, 2, 1 \rangle$ ,  
plane:  $5x - y - z = -3$

- line:  $\langle 4, 1, 0 \rangle + t \langle 1, 0, -1 \rangle$ ,  
plane:  $3x + y - 2z = 8$

- line:  $\langle 1, 2, 3 \rangle + t \langle 3, 5, -1 \rangle$ ,  
plane:  $3x - 2y - z = 4$

- line:  $\langle 1, 2, 3 \rangle + t \langle 3, 5, -1 \rangle$ ,  
plane:  $3x - 2y - z = -4$

**In Exercises 27 – 30, find the given distances.**

- The distance from the point  $(1, 2, 3)$  to the plane  
 $3(x - 1) + (y - 2) + 5(z - 2) = 0$ .

- The distance from the point  $(2, 6, 2)$  to the plane  
 $2(x - 1) - y + 4(z + 1) = 0$ .

- The distance between the parallel planes  
 $x + y + z = 0$  and  
 $(x - 2) + (y - 3) + (z + 4) = 0$

30. The distance between the parallel planes  
 $2(x - 1) + 2(y + 1) + (z - 2) = 0$  and  
 $2(x - 3) + 2(y - 1) + (z - 3) = 0$
31. Show why if the point  $Q$  lies in a plane, then the distance formula correctly gives the distance from the point to the plane as 0.
32. How is Exercise 30 in Section 9.5 easier to answer once we have an understanding of planes?

# 10: VECTOR VALUED FUNCTIONS

In the previous chapter, we learned about vectors and were introduced to the power of vectors within mathematics. In this chapter, we'll build on this foundation to define functions whose input is a real number and whose output is a vector. We'll see how to graph these functions and apply calculus techniques to analyze their behavior. Most importantly, we'll see *why* we are interested in doing this: we'll see beautiful applications to the study of moving objects.

## 10.1 Vector–Valued Functions

We are very familiar with **real valued functions**, that is, functions whose output is a real number. This section introduces **vector–valued functions** – functions whose output is a vector.

### Definition 41 Vector–Valued Functions

A **vector–valued function** is a function of the form

$$\vec{r}(t) = \langle f(t), g(t) \rangle \quad \text{or} \quad \vec{r}(t) = \langle f(t), g(t), h(t) \rangle,$$

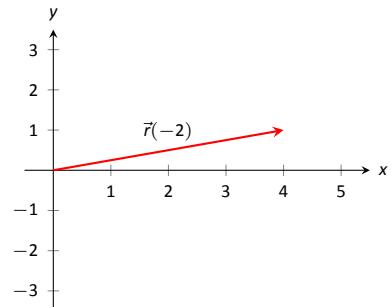
where  $f$ ,  $g$  and  $h$  are real valued functions.

The **domain** of  $\vec{r}$  is the set of all values of  $t$  for which  $\vec{r}(t)$  is defined. The **range** of  $\vec{r}$  is the set of all possible output vectors  $\vec{r}(t)$ .

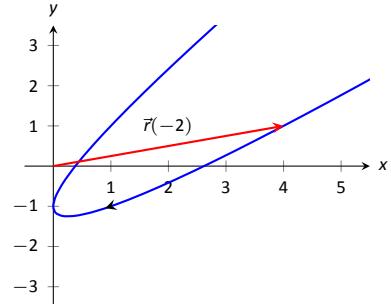
### Evaluating and Graphing Vector–Valued Functions

Evaluating a vector–valued function at a specific value of  $t$  is straightforward; simply evaluate each component function at that value of  $t$ . For instance, if  $\vec{r}(t) = \langle t^2, t^2 + t - 1 \rangle$ , then  $\vec{r}(-2) = \langle 4, 1 \rangle$ . We can sketch this vector, as is done in Figure 10.1(a). Plotting lots of vectors is cumbersome, though, so generally we do not sketch the whole vector but just the terminal point. The **graph** of a vector–valued function is the set of all terminal points of  $\vec{r}(t)$ , where the initial point of each vector is always the origin. In Figure 10.1(b) we sketch the graph of  $\vec{r}$ ; we can indicate individual points on the graph with their respective vector, as shown.

Vector–valued functions are closely related to parametric equations of graphs. While in both methods we plot points  $(x(t), y(t))$  or  $(x(t), y(t), z(t))$  to produce a graph, in the context of vector–valued functions each such point represents a vector. The implications of this will be more fully realized in the next section as we apply calculus ideas to these functions.



(a)

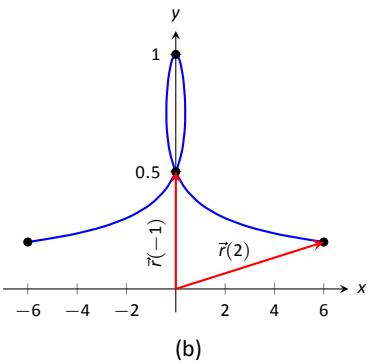


(b)

Figure 10.1: Sketching the graph of a vector–valued function.

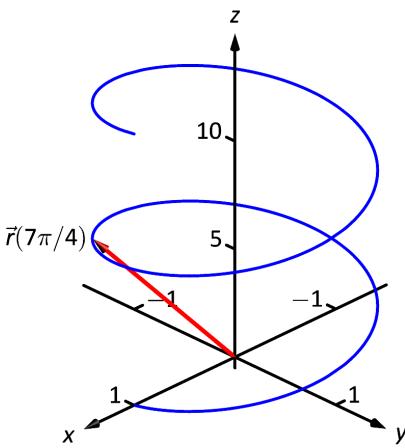
$t$	$t^3 - t$	$\frac{1}{t^2 + 1}$
-2	-6	1/5
-1	0	1/2
0	0	1
1	0	1/2
2	6	1/5

(a)



(b)

Figure 10.2: Sketching the vector-valued function of Example 193.

Figure 10.3: The graph of  $\vec{r}(t)$  in Example 194.**Example 193 Graphing vector-valued functions**

Graph  $\vec{r}(t) = \left\langle t^3 - t, \frac{1}{t^2 + 1} \right\rangle$ , for  $-2 \leq t \leq 2$ . Sketch  $\vec{r}(-1)$  and  $\vec{r}(2)$ .

**SOLUTION** We start by making a table of  $t$ ,  $x$  and  $y$  values as shown in Figure 10.2(a). Plotting these points gives an indication of what the graph looks like. In Figure 10.2(b), we indicate these points and sketch the full graph. We also highlight  $\vec{r}(-1)$  and  $\vec{r}(2)$  on the graph.

**Example 194 Graphing vector-valued functions.**

Graph  $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$  for  $0 \leq t \leq 4\pi$ .

**SOLUTION** We can again plot points, but careful consideration of this function is very revealing. Momentarily ignoring the third component, we see the  $x$  and  $y$  components trace out a circle of radius 1 centered at the origin. Noticing that the  $z$  component is  $t$ , we see that as the graph winds around the  $z$ -axis, it is also increasing at a constant rate in the positive  $z$  direction, forming a spiral. This is graphed in Figure 10.3. In the graph  $\vec{r}(7\pi/4) \approx (0.707, -0.707, 5.498)$  is highlighted to help us understand the graph.

**Algebra of Vector-Valued Functions****Definition 42 Operations on Vector-Valued Functions**

Let  $\vec{r}_1(t) = \langle f_1(t), g_1(t) \rangle$  and  $\vec{r}_2(t) = \langle f_2(t), g_2(t) \rangle$  be vector-valued functions in  $\mathbb{R}^2$  and let  $c$  be a scalar. Then:

1.  $\vec{r}_1(t) \pm \vec{r}_2(t) = \langle f_1(t) \pm f_2(t), g_1(t) \pm g_2(t) \rangle$ .
2.  $c\vec{r}_1(t) = \langle cf_1(t), cg_1(t) \rangle$ .

A similar definition holds for vector-valued functions in  $\mathbb{R}^3$ .

This definition states that we add, subtract and scale vector-valued functions component-wise. Combining vector-valued functions in this way can be very useful (as well as create interesting graphs).

**Example 195 Adding and scaling vector-valued functions.**

Let  $\vec{r}_1(t) = \langle 0.2t, 0.3t \rangle$ ,  $\vec{r}_2(t) = \langle \cos t, \sin t \rangle$  and  $\vec{r}(t) = \vec{r}_1(t) + \vec{r}_2(t)$ . Graph  $\vec{r}_1(t)$ ,  $\vec{r}_2(t)$ ,  $\vec{r}(t)$  and  $5\vec{r}(t)$  on  $-10 \leq t \leq 10$ .

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Notes:

**SOLUTION** We can graph  $\vec{r}_1$  and  $\vec{r}_2$  easily by plotting points (or just using technology). Let's think about each for a moment to better understand how vector-valued functions work.

We can rewrite  $\vec{r}_1(t) = \langle 0.2t, 0.3t \rangle$  as  $\vec{r}_1(t) = t \langle 0.2, 0.3 \rangle$ . That is, the function  $\vec{r}_1$  scales the vector  $\langle 0.2, 0.3 \rangle$  by  $t$ . This scaling of a vector produces a line in the direction of  $\langle 0.2, 0.3 \rangle$ .

We are familiar with  $\vec{r}_2(t) = \langle \cos t, \sin t \rangle$ ; it traces out a circle, centered at the origin, of radius 1. Figure 10.4(a) graphs  $\vec{r}_1(t)$  and  $\vec{r}_2(t)$ .

Adding  $\vec{r}_1(t)$  to  $\vec{r}_2(t)$  produces  $\vec{r}(t) = \langle \cos t + 0.2t, \sin t + 0.3t \rangle$ , graphed in Figure 10.4(b). The linear movement of the line combines with the circle to create loops that move in the direction of  $\langle 0.2, 0.3 \rangle$ . (We encourage the reader to experiment by changing  $\vec{r}_1(t)$  to  $\langle 2t, 3t \rangle$ , etc., and observe the effects on the loops.)

Multiplying  $\vec{r}(t)$  by 5 scales the function by 5, producing  $5\vec{r}(t) = \langle 5 \cos t + 1, 5 \sin t + 1.5 \rangle$ , which is graphed in Figure 10.4(c) along with  $\vec{r}(t)$ . The new function is “5 times bigger” than  $\vec{r}(t)$ . Note how the graph of  $5\vec{r}(t)$  in (c) looks identical to the graph of  $\vec{r}(t)$  in (b). This is due to the fact that the  $x$  and  $y$  bounds of the plot in (c) are exactly 5 times larger than the bounds in (b).

### Example 196 Adding and scaling vector-valued functions.

A **cycloid** is a graph traced by a point  $p$  on a rolling circle, as shown in Figure 10.5. Find an equation describing the cycloid, where the circle has radius 1.

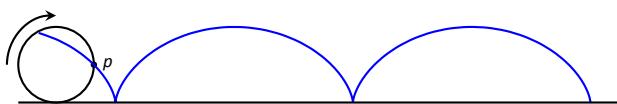


Figure 10.5: Tracing a cycloid.

**SOLUTION** This problem is not very difficult if we approach it in a clever way. We start by letting  $\vec{p}(t)$  describe the position of the point  $p$  on the circle, where the circle is centered at the origin and only rotates clockwise (i.e., it does not roll). This is relatively simple given our previous experiences with parametric equations;  $\vec{p}(t) = \langle \cos t, -\sin t \rangle$ .

We now want the circle to roll. We represent this by letting  $\vec{c}(t)$  represent the location of the center of the circle. It should be clear that the  $y$  component of  $\vec{c}(t)$  should be 1; the center of the circle is always going to be 1 if it rolls on a horizontal surface.

The  $x$  component of  $\vec{c}(t)$  is a linear function of  $t$ :  $f(t) = mt$  for some scalar  $m$ . When  $t = 0$ ,  $f(t) = 0$  (the circle starts centered on the  $y$ -axis). When  $t = 2\pi$ , the circle has made one complete revolution, traveling a distance equal to its

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Notes:

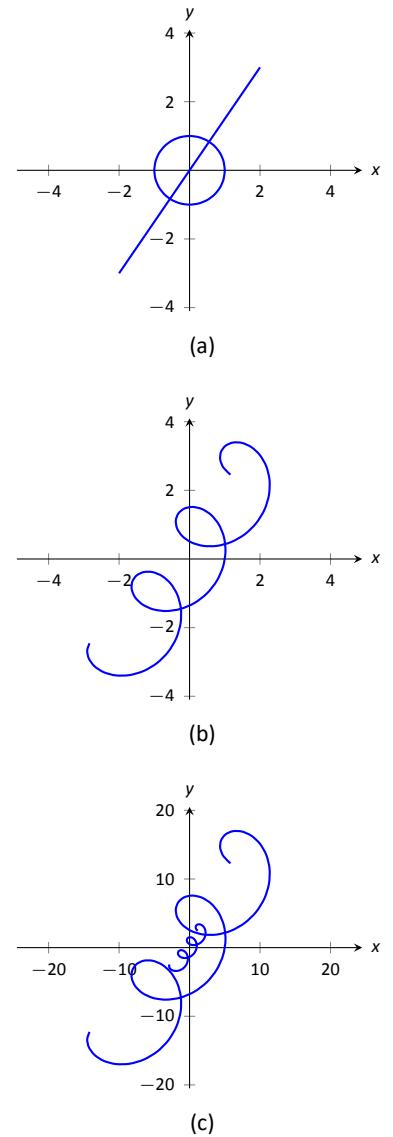


Figure 10.4: Graphing the functions in Example 195.

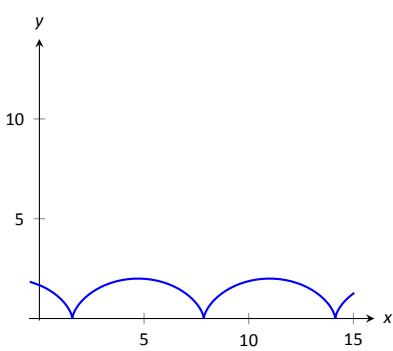


Figure 10.6: The cycloid in Example 196.

circumference, which is also  $2\pi$ . This gives us a point on our line  $f(t) = mt$ , the point  $(2\pi, 2\pi)$ . It should be clear that  $m = 1$  and  $f(t) = t$ . So  $\vec{c}(t) = \langle t, 1 \rangle$ .

We now combine  $\vec{p}$  and  $\vec{c}$  together to form the equation of the cycloid:  $\vec{r}(t) = \vec{p}(t) + \vec{c}(t) = \langle \cos t + t, -\sin t + 1 \rangle$ , which is graphed in Figure 10.6.

### Displacement

A vector-valued function  $\vec{r}(t)$  is often used to describe the position of a moving object at time  $t$ . At  $t = t_0$ , the object is at  $\vec{r}(t_0)$ ; at  $t = t_1$ , the object is at  $\vec{r}(t_1)$ . Knowing the locations  $\vec{r}(t_0)$  and  $\vec{r}(t_1)$  give no indication of the path taken between them, but often we only care about the difference of the locations,  $\vec{r}(t_1) - \vec{r}(t_0)$ , the **displacement**.

#### Definition 43 Displacement

Let  $\vec{r}(t)$  be a vector-valued function and let  $t_0 < t_1$  be values in the domain. The **displacement**  $\vec{d}$  of  $\vec{r}$ , from  $t = t_0$  to  $t = t_1$ , is

$$\vec{d} = \vec{r}(t_1) - \vec{r}(t_0).$$

When the displacement vector is drawn with initial point at  $\vec{r}(t_0)$ , its terminal point is  $\vec{r}(t_1)$ . We think of it as the vector which points from a starting position to an ending position.

#### Example 197 Finding and graphing displacement vectors

Let  $\vec{r}(t) = \langle \cos(\frac{\pi}{2}t), \sin(\frac{\pi}{2}t) \rangle$ . Graph  $\vec{r}(t)$  on  $-1 \leq t \leq 1$ , and find the displacement of  $\vec{r}(t)$  on this interval.

**SOLUTION** The function  $\vec{r}(t)$  traces out the unit circle, though at a different rate than the “usual”  $\langle \cos t, \sin t \rangle$  parametrization. At  $t_0 = -1$ , we have  $\vec{r}(t_0) = \langle 0, -1 \rangle$ ; at  $t_1 = 1$ , we have  $\vec{r}(t_1) = \langle 0, 1 \rangle$ . The displacement of  $\vec{r}(t)$  on  $[-1, 1]$  is thus  $\vec{d} = \langle 0, 1 \rangle - \langle 0, -1 \rangle = \langle 0, 2 \rangle$ .

A graph of  $\vec{r}(t)$  on  $[-1, 1]$  is given in Figure 10.7, along with the displacement vector  $\vec{d}$  on this interval.

Measuring displacement makes us contemplate related, yet very different, concepts. Considering the semi-circular path the object in Example 197 took, we can quickly verify that the object ended up a distance of 2 units from its initial location. That is, we can compute  $\|\vec{d}\| = 2$ . However, measuring *distance from the starting point* is different from measuring *distance traveled*. Being a semi-

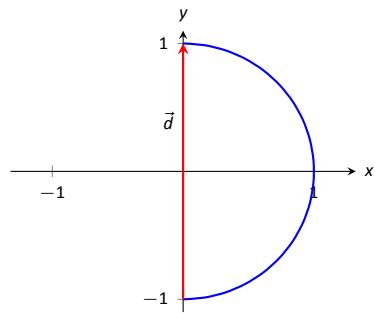


Figure 10.7: Graphing the displacement of a position function in Example 197.

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Notes:

circle, we can measure the distance traveled by this object as  $\pi \approx 3.14$  units. Knowing *distance from the starting point* allows us to compute **average rate of change**.

**Definition 44      Average Rate of Change**

Let  $\vec{r}(t)$  be a vector-valued function, where each of its component functions is continuous on its domain, and let  $t_0 < t_1$ . The **average rate of change** of  $\vec{r}(t)$  on  $[t_0, t_1]$  is

$$\text{average rate of change} = \frac{\vec{r}(t_1) - \vec{r}(t_0)}{t_1 - t_0}.$$

**Example 198      Average rate of change**

Let  $\vec{r}(t) = \langle \cos(\frac{\pi}{2}t), \sin(\frac{\pi}{2}t) \rangle$  as in Example 197. Find the average rate of change of  $\vec{r}(t)$  on  $[-1, 1]$  and on  $[-1, 5]$ .

**SOLUTION** We computed in Example 197 that the displacement of  $\vec{r}(t)$  on  $[-1, 1]$  was  $\vec{d} = \langle 0, 2 \rangle$ . Thus the average rate of change of  $\vec{r}(t)$  on  $[-1, 1]$  is:

$$\frac{\vec{r}(1) - \vec{r}(-1)}{1 - (-1)} = \frac{\langle 0, 2 \rangle}{2} = \langle 0, 1 \rangle.$$

We interpret this as follows: the object followed a semi-circular path, meaning it moved towards the right then moved back to the left, while climbing slowly, then quickly, then slowly again. *On average*, however, it progressed straight up at a constant rate of  $\langle 0, 1 \rangle$  per unit of time.

We can quickly see that the displacement on  $[-1, 5]$  is the same as on  $[-1, 1]$ , so  $\vec{d} = \langle 0, 2 \rangle$ . The average rate of change is different, though:

$$\frac{\vec{r}(5) - \vec{r}(-1)}{5 - (-1)} = \frac{\langle 0, 2 \rangle}{6} = \langle 0, 1/3 \rangle.$$

As it took “3 times as long” to arrive at the same place, this average rate of change on  $[-1, 5]$  is  $1/3$  the average rate of change on  $[-1, 1]$ .

We considered average rates of change in Sections 1.1 and 2.1 as we studied limits and derivatives. The same is true here; in the following section we apply calculus concepts to vector-valued functions as we find limits, derivatives, and integrals. Understanding the average rate of change will give us an understanding of the derivative; displacement gives us one application of integration.

Notes:

# Exercises 10.1

## Terms and Concepts

1. Vector-valued functions are closely related to \_\_\_\_\_ of graphs.
2. When sketching vector-valued functions, technically one isn't graphing points, but rather \_\_\_\_\_.
3. It can be useful to think of \_\_\_\_\_ as a vector that points from a starting position to an ending position.

## Problems

**In Exercises 4 – 11, sketch the vector-valued function on the given interval.**

4.  $\vec{r}(t) = \langle t^2, t^2 - 1 \rangle$ , for  $-2 \leq t \leq 2$ .
5.  $\vec{r}(t) = \langle t^2, t^3 \rangle$ , for  $-2 \leq t \leq 2$ .
6.  $\vec{r}(t) = \langle 1/t, 1/t^2 \rangle$ , for  $-2 \leq t \leq 2$ .
7.  $\vec{r}(t) = \langle \frac{1}{10}t^2, \sin t \rangle$ , for  $-2\pi \leq t \leq 2\pi$ .
8.  $\vec{r}(t) = \langle \frac{1}{10}t^2, \sin t \rangle$ , for  $-2\pi \leq t \leq 2\pi$ .
9.  $\vec{r}(t) = \langle 3 \sin(\pi t), 2 \cos(\pi t) \rangle$ , on  $[0, 2]$ .
10.  $\vec{r}(t) = \langle 3 \cos t, 2 \sin(2t) \rangle$ , on  $[0, 2\pi]$ .
11.  $\vec{r}(t) = \langle 2 \sec t, \tan t \rangle$ , on  $[-\pi, \pi]$ .

**In Exercises 12 – 15, sketch the vector-valued function on the given interval in  $\mathbb{R}^3$ . Technology may be useful in creating the sketch.**

12.  $\vec{r}(t) = \langle 2 \cos t, t, 2 \sin t \rangle$ , on  $[0, 2\pi]$ .
13.  $\vec{r}(t) = \langle 3 \cos t, \sin t, t/\pi \rangle$  on  $[0, 2\pi]$ .
14.  $\vec{r}(t) = \langle \cos t, \sin t, \sin t \rangle$  on  $[0, 2\pi]$ .
15.  $\vec{r}(t) = \langle \cos t, \sin t, \sin(2t) \rangle$  on  $[0, 2\pi]$ .

**In Exercises 16 – 19, find  $\|\vec{r}(t)\|$ .**

16.  $\vec{r}(t) = \langle t, t^2 \rangle$ .
17.  $\vec{r}(t) = \langle 5 \cos t, 3 \sin t \rangle$ .
18.  $\vec{r}(t) = \langle 2 \cos t, 2 \sin t, t \rangle$ .
19.  $\vec{r}(t) = \langle \cos t, t, t^2 \rangle$ .

**In Exercises 20 – 27, create a vector-valued function whose graph matches the given description.**

20. A circle of radius 2, centered at  $(1, 2)$ , traced counter-clockwise once on  $[0, 2\pi]$ .
21. A circle of radius 3, centered at  $(5, 5)$ , traced clockwise once on  $[0, 2\pi]$ .
22. An ellipse, centered at  $(0, 0)$  with vertical major axis of length 10 and minor axis of length 3, traced once counter-clockwise on  $[0, 2\pi]$ .
23. An ellipse, centered at  $(3, -2)$  with horizontal major axis of length 6 and minor axis of length 4, traced once clockwise on  $[0, 2\pi]$ .
24. A line through  $(2, 3)$  with a slope of 5.
25. A line through  $(1, 5)$  with a slope of  $-1/2$ .
26. A vertically oriented helix with radius of 2 that starts at  $(2, 0, 0)$  and ends at  $(2, 0, 4\pi)$  after 1 revolution on  $[0, 2\pi]$ .
27. A vertically oriented helix with radius of 3 that starts at  $(3, 0, 0)$  and ends at  $(3, 0, 3)$  after 2 revolutions on  $[0, 1]$ .

**In Exercises 28 – 31, find the average rate of change of  $\vec{r}(t)$  on the given interval.**

28.  $\vec{r}(t) = \langle t, t^2 \rangle$  on  $[-2, 2]$ .
29.  $\vec{r}(t) = \langle t, t + \sin t \rangle$  on  $[0, 2\pi]$ .
30.  $\vec{r}(t) = \langle 3 \cos t, 2 \sin t, t \rangle$  on  $[0, 2\pi]$ .
31.  $\vec{r}(t) = \langle t, t^2, t^3 \rangle$  on  $[-1, 3]$ .

## 10.2 Calculus and Vector-Valued Functions

The previous section introduced us to a new mathematical object, the vector-valued function. We now apply calculus concepts to these functions. We start with the limit, then work our way through derivatives to integrals.

### Limits of Vector-Valued Functions

The initial definition of the limit of a vector-valued function is a bit intimidating, as was the definition of the limit in Definition 1. The theorem following the definition shows that in practice, taking limits of vector-valued functions is no more difficult than taking limits of real-valued functions.

#### Definition 45 Limits of Vector-Valued Functions

Let  $I$  be an open interval containing  $c$ , and let  $\vec{r}(t)$  be a vector-valued function defined on  $I$ , except possibly at  $c$ . The **limit of  $\vec{r}(t)$ , as  $t$  approaches  $c$ , is  $\vec{L}$** , expressed as

$$\lim_{t \rightarrow c} \vec{r}(t) = \vec{L},$$

means that given any  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for all  $t \neq c$ , if  $|t - c| < \delta$ , we have  $\|\vec{r}(t) - \vec{L}\| < \varepsilon$ .

Note how the measurement of distance between real numbers is the absolute value of their difference; the measure of distance between vectors is the vector norm, or magnitude, of their difference.

#### Theorem 39 Limits of Vector-Valued Functions

- Let  $\vec{r}(t) = \langle f(t), g(t) \rangle$  be a vector-valued function in  $\mathbb{R}^2$  defined on an open interval  $I$  containing  $c$ . Then

$$\lim_{t \rightarrow c} \vec{r}(t) = \left\langle \lim_{t \rightarrow c} f(t), \lim_{t \rightarrow c} g(t) \right\rangle.$$

- Let  $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$  be a vector-valued function in  $\mathbb{R}^3$  defined on an open interval  $I$  containing  $c$ . Then

$$\lim_{t \rightarrow c} \vec{r}(t) = \left\langle \lim_{t \rightarrow c} f(t), \lim_{t \rightarrow c} g(t), \lim_{t \rightarrow c} h(t) \right\rangle$$

Theorem 39 states that we compute limits component-wise.

Notes:

**Example 199      Finding limits of vector-valued functions**

Let  $\vec{r}(t) = \left\langle \frac{\sin t}{t}, t^2 - 3t + 3, \cos t \right\rangle$ . Find  $\lim_{t \rightarrow 0} \vec{r}(t)$ .

**SOLUTION**

We apply the theorem and compute limits component-wise.

$$\begin{aligned}\lim_{t \rightarrow 0} \vec{r}(t) &= \left\langle \lim_{t \rightarrow 0} \frac{\sin t}{t}, \lim_{t \rightarrow 0} t^2 - 3t + 3, \lim_{t \rightarrow 0} \cos t \right\rangle \\ &= \langle 1, 3, 1 \rangle.\end{aligned}$$

**Continuity****Definition 46      Continuity of Vector-Valued Functions**

Let  $\vec{r}(t)$  be a vector-valued function defined on an open interval  $I$  containing  $c$ .

1.  $\vec{r}(t)$  is **continuous at  $c$**  if  $\lim_{t \rightarrow c} \vec{r}(t) = r(c)$ .

2. If  $\vec{r}(t)$  is continuous at all  $c$  in  $I$ , then  $\vec{r}(t)$  is **continuous on  $I$** .

We again have a theorem that lets us evaluate continuity component-wise.

**Theorem 40      Continuity of Vector-Valued Functions**

Let  $\vec{r}(t)$  be a vector-valued function defined on an open interval  $I$  containing  $c$ .  $\vec{r}(t)$  is continuous at  $c$  if, and only if, each of its component functions is continuous at  $c$ .

**Example 200      Evaluating continuity of vector-valued functions**

Let  $\vec{r}(t) = \left\langle \frac{\sin t}{t}, t^2 - 3t + 3, \cos t \right\rangle$ . Determine whether  $\vec{r}$  is continuous at  $t = 0$  and  $t = 1$ .

**SOLUTION** While the second and third components of  $\vec{r}(t)$  are defined at  $t = 0$ , the first component,  $(\sin t)/t$ , is not. Since the first component is not even defined at  $t = 0$ ,  $\vec{r}(t)$  is not defined at  $t = 0$ , and hence it is not continuous at  $t = 0$ .

At  $t = 1$  each of the component functions is continuous. Therefore  $\vec{r}(t)$  is continuous at  $t = 1$ .

---

Notes:

## Derivatives

Consider a vector-valued function  $\vec{r}$  defined on an open interval  $I$  containing  $t_0$  and  $t_1$ . We can compute the displacement of  $\vec{r}$  on  $[t_0, t_1]$ , as shown in Figure 10.8(a). Recall that dividing the displacement vector by  $t_1 - t_0$  gives the average rate of change on  $[t_0, t_1]$ , as shown in (b).

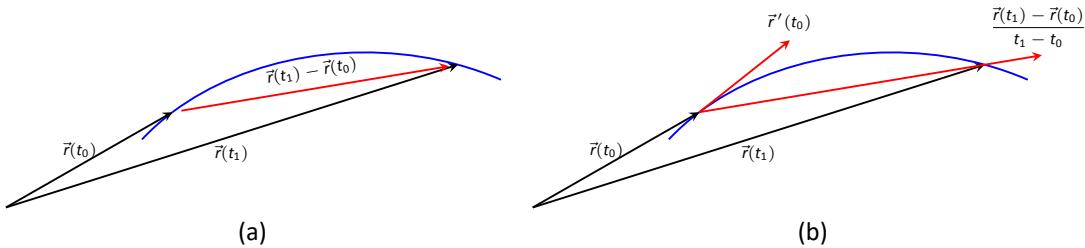


Figure 10.8: Illustrating displacement, leading to an understanding of the derivative of vector-valued functions.

The **derivative** of a vector-valued function is a measure of the *instantaneous* rate of change, measured by taking the limit as the length of  $[t_0, t_1]$  goes to 0. Instead of thinking of an interval as  $[t_0, t_1]$ , we think of it as  $[c, c + h]$  for some value of  $h$  (hence the interval has length  $h$ ). The *average* rate of change is

$$\frac{\vec{r}(c+h) - \vec{r}(c)}{h}$$

for any value of  $h \neq 0$ . We take the limit as  $h \rightarrow 0$  to measure the instantaneous rate of change; this is the derivative of  $\vec{r}$ .

### Definition 47 Derivative of a Vector-Valued Function

Let  $\vec{r}(t)$  be continuous on an open interval  $I$  containing  $c$ .

1. The derivative of  $\vec{r}$  at  $t = c$  is

$$\vec{r}'(c) = \lim_{h \rightarrow 0} \frac{\vec{r}(c+h) - \vec{r}(c)}{h}.$$

2. The derivative of  $\vec{r}$  is

$$\vec{r}'(t) = \lim_{h \rightarrow 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}.$$

Alternate notations for the derivative of  $\vec{r}$  include:

$$\vec{r}'(t) = \frac{d}{dt}(\vec{r}(t)) = \frac{d\vec{r}}{dt}.$$

If a vector-valued function has a derivative for all  $c$  in an open interval  $I$ , we say that  $\vec{r}(t)$  is **differentiable** on  $I$ .

Once again we might view this definition as intimidating, but recall that we

Notes:

can evaluate limits component-wise. The following theorem verifies that this means we can compute derivatives component-wise as well, making the task not too difficult.

**Theorem 41 Derivatives of Vector-Valued Functions**

- Let  $\vec{r}(t) = \langle f(t), g(t) \rangle$ . Then

$$\vec{r}'(t) = \langle f'(t), g'(t) \rangle.$$

- Let  $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ . Then

$$\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle.$$

**Example 201 Derivatives of vector-valued functions**

Let  $\vec{r}(t) = \langle t^2, t \rangle$ .

- Sketch  $\vec{r}(t)$  and  $\vec{r}'(t)$  on the same axes.
- Compute  $\vec{r}'(1)$  and sketch this vector with its initial point at the origin and at  $\vec{r}(1)$ .

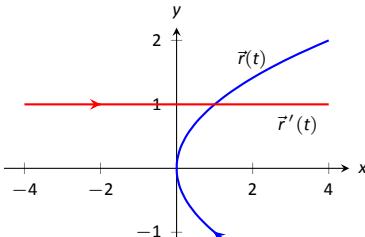
**SOLUTION**

- Theorem 41 allows us to compute derivatives component-wise, so

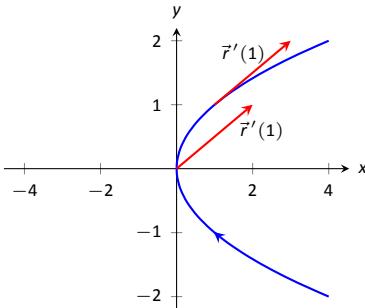
$$\vec{r}'(t) = \langle 2t, 1 \rangle.$$

$\vec{r}(t)$  and  $\vec{r}'(t)$  are graphed together in Figure 10.9(a). Note how plotting the two of these together, in this way, is not very illuminating. When dealing with real-valued functions, plotting  $f(x)$  with  $f'(x)$  gave us useful information as we were able to compare  $f$  and  $f'$  at the same  $x$ -values. When dealing with vector-valued functions, it is hard to tell which points on the graph of  $\vec{r}'$  correspond to which points on the graph of  $\vec{r}$ .

- We easily compute  $\vec{r}'(1) = \langle 2, 1 \rangle$ , which is drawn in Figure 10.9 with its initial point at the origin, as well as at  $\vec{r}(1) = \langle 1, 1 \rangle$ . These are sketched in Figure 10.9(b).



(a)



(b)

Figure 10.9: Graphing the derivative of a vector-valued function in Example 201.

Notes:

**Example 202 Derivatives of vector-valued functions**

Let  $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ . Compute  $\vec{r}'(t)$  and  $\vec{r}'(\pi/2)$ . Sketch  $\vec{r}'(\pi/2)$  with its initial point at the origin and at  $\vec{r}(\pi/2)$ .

**SOLUTION** We compute  $\vec{r}'$  as  $\vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$ . At  $t = \pi/2$ , we have  $\vec{r}'(\pi/2) = \langle -1, 0, 1 \rangle$ . Figure 10.10 shows a graph of  $\vec{r}(t)$ , with  $\vec{r}'(\pi/2)$  plotted with its initial point at the origin and at  $\vec{r}(\pi/2)$ .

In Examples 201 and 202, sketching a particular derivative with its initial point at the origin did not seem to reveal anything significant. However, when we sketched the vector with its initial point on the corresponding point on the graph, we did see something significant: the vector appeared to be *tangent* to the graph. We have not yet defined what “tangent” means in terms of curves in space; in fact, we use the derivative to define this term.

**Definition 48 Tangent Vector, Tangent Line**

Let  $\vec{r}(t)$  be a differentiable vector-valued function on an open interval  $I$  containing  $c$ , where  $\vec{r}'(c) \neq \vec{0}$ .

1. A vector  $\vec{v}$  is **tangent to the graph of  $\vec{r}(t)$  at  $t = c$**  if  $\vec{v}$  is parallel to  $\vec{r}'(c)$ .
2. The **tangent line** to the graph of  $\vec{r}(t)$  at  $t = c$  is the line through  $\vec{r}(c)$  with direction parallel to  $\vec{r}'(c)$ . An equation of the tangent line is

$$\ell(t) = \vec{r}(c) + t\vec{r}'(c).$$

**Example 203 Finding tangent lines to curves in space**

Let  $\vec{r}(t) = \langle t, t^2, t^3 \rangle$  on  $[-1.5, 1.5]$ . Find the vector equation of the line tangent to the graph of  $\vec{r}$  at  $t = -1$ .

**SOLUTION** To find the equation of a line, we need a point on the line and the line's direction. The point is given by  $\vec{r}(-1) = \langle -1, 1, -1 \rangle$ . (To be clear,  $\langle -1, 1, -1 \rangle$  is a *vector*, not a point, but we use the point “pointed to” by this vector.)

The direction comes from  $\vec{r}'(-1)$ . We compute, component-wise,  $\vec{r}'(t) = \langle 1, 2t, 3t^2 \rangle$ . Thus  $\vec{r}'(-1) = \langle 1, -2, 3 \rangle$ .

The vector equation of the line is  $\ell(t) = \langle -1, 1, -1 \rangle + t \langle 1, -2, 3 \rangle$ . This line and  $\vec{r}(t)$  are sketched in Figure 10.11.

**Example 204 Finding tangent lines to curves**

Notes:

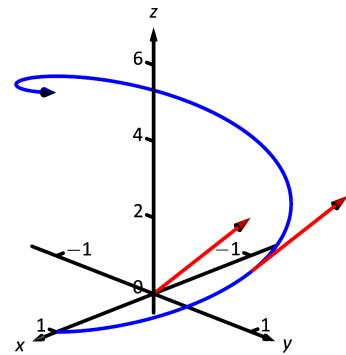


Figure 10.10: Viewing a vector-valued function and its derivative at one point.

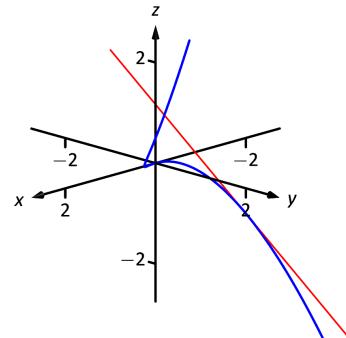


Figure 10.11: Graphing a curve in space with its tangent line.

Find the equations of the lines tangent to  $\vec{r}(t) = \langle t^3, t^2 \rangle$  at  $t = -1$  and  $t = 0$ .

**SOLUTION** We find that  $\vec{r}'(t) = \langle 3t^2, 2t \rangle$ . At  $t = -1$ , we have

$$\vec{r}(-1) = \langle -1, 1 \rangle \quad \text{and} \quad \vec{r}'(-1) = \langle 3, -2 \rangle,$$

so the equation of the line tangent to the graph of  $\vec{r}(t)$  at  $t = -1$  is

$$\ell(t) = \langle -1, 1 \rangle + t \langle 3, -2 \rangle.$$

This line is graphed with  $\vec{r}(t)$  in Figure 10.12.

At  $t = 0$ , we have  $\vec{r}'(0) = \langle 0, 0 \rangle = \vec{0}$ ! This implies that the tangent line “has no direction.” We cannot apply Definition 48, hence cannot find the equation of the tangent line.

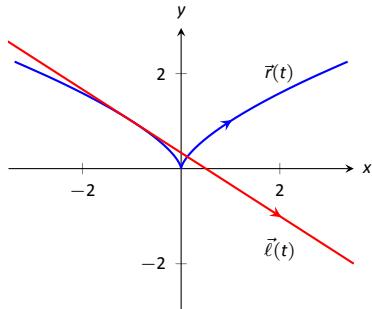


Figure 10.12: Graphing  $\vec{r}(t)$  and its tangent line in Example 204.

We were unable to compute the equation of the tangent line to  $\vec{r}(t) = \langle t^3, t^2 \rangle$  at  $t = 0$  because  $\vec{r}'(0) = \vec{0}$ . The graph in Figure 10.12 shows that there is a cusp at this point. This leads us to another definition of **smooth**, previously defined by Definition 23 in Section 9.2.

#### Definition 49 Smooth Vector-Valued Functions

Let  $\vec{r}(t)$  be a differentiable vector-valued function on an open interval  $I$ .  $\vec{r}(t)$  is **smooth** on  $I$  if  $\vec{r}'(t) \neq \vec{0}$  on  $I$ .

Having established derivatives of vector-valued functions, we now explore the relationships between the derivative and other vector operations. The following theorem states how the derivative interacts with vector addition and the various vector products.

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Notes:

**Theorem 42 Properties of Derivatives of Vector-Valued Functions**

Let  $\vec{r}$  and  $\vec{s}$  be differentiable vector-valued functions, let  $f$  be a differentiable real-valued function, and let  $c$  be a real number.

1.  $\frac{d}{dt}(\vec{r}(t) \pm \vec{s}(t)) = \vec{r}'(t) \pm \vec{s}'(t)$
2.  $\frac{d}{dt}(c\vec{r}(t)) = c\vec{r}'(t)$
3.  $\frac{d}{dt}(f(t)\vec{r}(t)) = f'(t)\vec{r}(t) + f(t)\vec{r}'(t)$  **Product Rule**
4.  $\frac{d}{dt}(\vec{r}(t) \cdot \vec{s}(t)) = \vec{r}'(t) \cdot \vec{s}(t) + \vec{r}(t) \cdot \vec{s}'(t)$  **Product Rule**
5.  $\frac{d}{dt}(\vec{r}(t) \times \vec{s}(t)) = \vec{r}'(t) \times \vec{s}(t) + \vec{r}(t) \times \vec{s}'(t)$  **Product Rule**
6.  $\frac{d}{dt}(\vec{r}(f(t))) = \vec{r}'(f(t))f'(t)$  **Chain Rule**

**Example 205 Using derivative properties of vector-valued functions**

Let  $\vec{r}(t) = \langle t, t^2 - 1 \rangle$  and let  $\vec{u}(t)$  be the unit vector that points in the direction of  $\vec{r}(t)$ .

1. Graph  $\vec{r}(t)$  and  $\vec{u}(t)$  on the same axes, on  $[-2, 2]$ .
2. Find  $\vec{u}'(t)$  and sketch  $\vec{u}'(-2), \vec{u}'(-1)$  and  $\vec{u}'(0)$ . Sketch each with initial point the corresponding point on the graph of  $\vec{u}$ .

**SOLUTION**

1. To form the unit vector that points in the direction of  $\vec{r}$ , we need to divide  $\vec{r}(t)$  by its magnitude.

$$\|\vec{r}(t)\| = \sqrt{t^2 + (t^2 - 1)^2} \Rightarrow \vec{u}(t) = \frac{1}{\sqrt{t^2 + (t^2 - 1)^2}} \langle t, t^2 - 1 \rangle.$$

$\vec{r}(t)$  and  $\vec{u}(t)$  are graphed in Figure 10.13. Note how the graph of  $\vec{u}(t)$  forms part of a circle; this must be the case, as the length of  $\vec{u}(t)$  is 1 for all  $t$ .

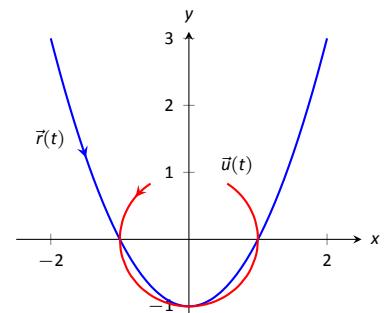


Figure 10.13: Graphing  $\vec{r}(t)$  and  $\vec{u}(t)$  in Example 205.

Notes:

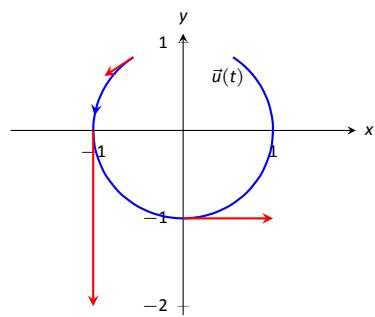


Figure 10.14: Graphing some of the derivatives of  $\vec{u}(t)$  in Example 205.

2. To compute  $\vec{u}'(t)$ , we use Theorem 42, writing

$$\vec{u}(t) = f(t)\vec{r}(t), \quad \text{where } f(t) = \frac{1}{\sqrt{t^2 + (t^2 - 1)^2}} = (t^2 + (t^2 - 1)^2)^{-1/2}.$$

(We could write

$$\vec{u}(t) = \left\langle \frac{t}{\sqrt{t^2 + (t^2 - 1)^2}}, \frac{t^2 - 1}{\sqrt{t^2 + (t^2 - 1)^2}} \right\rangle$$

and then take the derivative. It is a matter of preference; this latter method requires two applications of the Quotient Rule where our method uses the Product and Chain Rules.)

We find  $f'(t)$  using the Chain Rule:

$$\begin{aligned} f'(t) &= -\frac{1}{2}(t^2 + (t^2 - 1)^2)^{-3/2}(2t + 2(t^2 - 1)(2t)) \\ &= -\frac{2t(2t^2 - 1)}{2(\sqrt{t^2 + (t^2 - 1)^2})^3} \end{aligned}$$

We now find  $\vec{u}'(t)$  using part 3 of Theorem 42:

$$\begin{aligned} \vec{u}'(t) &= f'(t)\vec{u}(t) + f(t)\vec{u}'(t) \\ &= -\frac{2t(2t^2 - 1)}{2(\sqrt{t^2 + (t^2 - 1)^2})^3} \langle t, t^2 - 1 \rangle + \frac{1}{\sqrt{t^2 + (t^2 - 1)^2}} \langle 1, 2t \rangle. \end{aligned}$$

This is admittedly very “messy;” such is usually the case when we deal with unit vectors. We can use this formula to compute  $\vec{u}(-2)$ ,  $\vec{u}(-1)$  and  $\vec{u}(0)$ :

$$\begin{aligned} \vec{u}(-2) &= \left\langle -\frac{15}{13\sqrt{13}}, -\frac{10}{13\sqrt{13}} \right\rangle \approx \langle -0.320, -0.213 \rangle \\ \vec{u}(-1) &= \langle 0, -2 \rangle \\ \vec{u}(0) &= \langle 1, 0 \rangle \end{aligned}$$

Each of these is sketched in Figure 10.14. Note how the length of the vector gives an indication of how quickly the circle is being traced at that point. When  $t = -2$ , the circle is being drawn relatively slow; when  $t = -1$ , the circle is being traced much more quickly.

It is a basic geometric fact that a line tangent to a circle at a point  $P$  is perpendicular to the line passing through the center of the circle and  $P$ . This is

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Notes:

illustrated in Figure 10.14; each tangent vector is perpendicular to the line that passes through its initial point and the center of the circle. Since the center of the circle is the origin, we can state this another way:  $\vec{u}'(t)$  is orthogonal to  $\vec{u}(t)$ .

Recall that the dot product serves as a test for orthogonality: if  $\vec{u} \cdot \vec{v} = 0$ , then  $\vec{u}$  is orthogonal to  $\vec{v}$ . Thus in the above example,  $\vec{u}(t) \cdot \vec{u}'(t) = 0$ .

This is true of any vector-valued function that has a constant length, that is, that traces out part of a circle. It has important implications later on, so we state it as a theorem (and leave its formal proof as an Exercise.)

**Theorem 43 Vector-Valued Functions of Constant Length**

Let  $\vec{r}(t)$  be a differentiable vector-valued function on an open interval  $I$  of constant length. That is,  $\|\vec{r}(t)\| = c$  for all  $t$  in  $I$  (equivalently,  $\vec{r}(t) \cdot \vec{r}(t) = c^2$  for all  $t$  in  $I$ ). Then  $\vec{r}(t) \cdot \vec{r}'(t) = 0$  for all  $t$  in  $I$ .

## Integration

Indefinite and definite integrals of vector-valued functions are also evaluated component-wise.

**Theorem 44 Indefinite and Definite Integrals of Vector-Valued Functions**

Let  $\vec{r}(t) = \langle f(t), g(t) \rangle$  be a vector-valued function in  $\mathbb{R}^2$ .

$$1. \int \vec{r}(t) dt = \left\langle \int f(t) dt, \int g(t) dt \right\rangle$$

$$2. \int_a^b \vec{r}(t) dt = \left\langle \int_a^b f(t) dt, \int_a^b g(t) dt \right\rangle$$

A similar statement holds for vector-valued functions in  $\mathbb{R}^3$ .

**Example 206 Evaluating a definite integral of a vector-valued function**

Let  $\vec{r}(t) = \langle e^{2t}, \sin t \rangle$ . Evaluate  $\int_0^1 \vec{r}(t) dt$ .

---

Notes:

**SOLUTION** We follow Theorem 44.

$$\begin{aligned}\int_0^1 \vec{r}(t) dt &= \int_0^1 \langle e^{2t}, \sin t \rangle dt \\ &= \left\langle \int_0^1 e^{2t} dt, \int_0^1 \sin t dt \right\rangle \\ &= \left\langle \frac{1}{2}e^{2t} \Big|_0^1, -\cos t \Big|_0^1 \right\rangle \\ &= \left\langle \frac{1}{2}(e^2 - 1), -\cos(1) + 1 \right\rangle \\ &\approx \langle 3.19, 0.460 \rangle.\end{aligned}$$

**Example 207 Solving an initial value problem**

Let  $\vec{r}''(t) = \langle 2, \cos t, 12t \rangle$ . Find  $\vec{r}(t)$  where:

- $\vec{r}(0) = \langle -7, -1, 2 \rangle$  and
- $\vec{r}'(0) = \langle 5, 3, 0 \rangle$ .

**SOLUTION** Knowing  $\vec{r}''(t) = \langle 2, \cos t, 12t \rangle$ , we find  $\vec{r}'(t)$  by evaluating the indefinite integral.

$$\begin{aligned}\int \vec{r}''(t) dt &= \left\langle \int 2 dt, \int \cos t dt, \int 12t dt \right\rangle \\ &= \langle 2t + C_1, \sin t + C_2, 6t^2 + C_3 \rangle \\ &= \langle 2t, \sin t, 6t^2 \rangle + \langle C_1, C_2, C_3 \rangle \\ &= \langle 2t, \sin t, 6t^2 \rangle + \vec{C}.\end{aligned}$$

Note how each indefinite integral creates its own constant which we collect as one constant vector  $\vec{C}$ . Knowing  $\vec{r}'(0) = \langle 5, 3, 0 \rangle$  allows us to solve for  $\vec{C}$ :

$$\begin{aligned}\vec{r}'(t) &= \langle 2t, \sin t, 6t^2 \rangle + \vec{C} \\ \vec{r}'(0) &= \langle 0, 0, 0 \rangle + \vec{C} \\ \langle 5, 3, 0 \rangle &= \vec{C}.\end{aligned}$$

So  $\vec{r}'(t) = \langle 2t, \sin t, 6t^2 \rangle + \langle 5, 3, 0 \rangle = \langle 2t + 5, \sin t + 3, 6t^2 \rangle$ . To find  $\vec{r}(t)$ , we integrate once more.

$$\begin{aligned}\int \vec{r}'(t) dt &= \left\langle \int 2t + 5 dt, \int \sin t + 3 dt, \int 6t^2 dt \right\rangle \\ &= \langle t^2 + 5t, -\cos t + 3t, 2t^3 \rangle + \vec{C}.\end{aligned}$$

With  $\vec{r}(0) = \langle -7, -1, 2 \rangle$ , we solve for  $\vec{C}$ :

Notes:

$$\begin{aligned}\vec{r}(t) &= \langle t^2 + 5t, -\cos t + 3t, 2t^3 \rangle + \vec{C} \\ \vec{r}(0) &= \langle 0, -1, 0 \rangle + \vec{C} \\ \langle -7, -1, 2 \rangle &= \langle 0, -1, 0 \rangle + \vec{C} \\ \langle -7, 0, 2 \rangle &= \vec{C}.\end{aligned}$$

So  $\vec{r}(t) = \langle t^2 + 5t, -\cos t + 3t, 2t^3 \rangle + \langle -7, 0, 2 \rangle = \langle t^2 + 5t - 7, -\cos t + 3t, 2t^3 + 2 \rangle$ .

What does the integration of a vector-valued function *mean*? There are many applications, but none as direct as “the area under the curve” that we used in understanding the integral of a real-valued function.

A key understanding for us comes from considering the integral of a derivative:

$$\int_a^b \vec{r}'(t) dt = \vec{r}(t) \Big|_a^b = \vec{r}(b) - \vec{r}(a).$$

Integrating a *rate of change* function gives *displacement*.

Noting that vector-valued functions are closely related to parametric equations, we can describe the arc length of the graph of a vector-valued function as an integral. Given parametric equations  $x = f(t)$ ,  $y = g(t)$ , the arc length on  $[a, b]$  of the graph is

$$\text{Arc Length} = \int_a^b \sqrt{f'(t)^2 + g'(t)^2} dt,$$

as stated in Theorem 32 in Section 9.3. If  $\vec{r}(t) = \langle f(t), g(t) \rangle$ , note that  $\sqrt{f'(t)^2 + g'(t)^2} = \|\vec{r}'(t)\|$ . Therefore we can express the arc length of the graph of a vector-valued function as an integral of the magnitude of its derivative.

#### Theorem 45 Arc Length of a Vector-Valued Function

Let  $\vec{r}(t)$  be a vector-valued function where  $\vec{r}'(t)$  is continuous on  $[a, b]$ . The arc length  $L$  of the graph of  $\vec{r}(t)$  is

$$L = \int_a^b \|\vec{r}'(t)\| dt.$$

Note that we are actually integrating a scalar-function here, not a vector-valued function.

The next section takes what we have established thus far and applies it to objects in motion. We will let  $\vec{r}(t)$  describe the path of an object in the plane or in space and will discover the information provided by  $\vec{r}'(t)$  and  $\vec{r}''(t)$ .

Notes:

# Exercises 10.2

## Terms and Concepts

1. Limits, derivatives and integrals of vector-valued functions are all evaluated \_\_\_\_\_-wise.
2. The definite integral of a rate of change function gives \_\_\_\_\_.
3. Why is it generally not useful to graph both  $\vec{r}(t)$  and  $\vec{r}'(t)$  on the same axes?

## Problems

In Exercises 4 – 7, evaluate the given limit.

4.  $\lim_{t \rightarrow 5} \langle 2t + 1, 3t^2 - 1, \sin t \rangle$
5.  $\lim_{t \rightarrow 3} \left\langle e^t, \frac{t^2 - 9}{t + 3} \right\rangle$
6.  $\lim_{t \rightarrow 0} \left\langle \frac{t}{\sin t}, (1 + t)^{\frac{1}{t}} \right\rangle$
7.  $\lim_{h \rightarrow 0} \frac{\vec{r}(t + h) - \vec{r}(t)}{h}$ , where  $\vec{r}(t) = \langle t^2, t, 1 \rangle$ .

In Exercises 8 – 9, identify the interval(s) on which  $\vec{r}(t)$  is continuous.

8.  $\vec{r}(t) = \langle t^2, 1/t \rangle$
9.  $\vec{r}(t) = \langle \cos t, e^t, \ln t \rangle$

In Exercises 10 – 14, find the derivative of the given function.

10.  $\vec{r}(t) = \langle \cos t, e^t, \ln t \rangle$
11.  $\vec{r}(t) = \left\langle \frac{1}{t}, \frac{2t - 1}{3t + 1}, \tan t \right\rangle$
12.  $\vec{r}(t) = (t^2) \langle \sin t, 2t + 5 \rangle$
13.  $\vec{r}(t) = \langle t^2 + 1, t - 1 \rangle \cdot \langle \sin t, 2t + 5 \rangle$
14.  $\vec{r}(t) = \langle t^2 + 1, t - 1, 1 \rangle \times \langle \sin t, 2t + 5, 1 \rangle$

In Exercises 15 – 18, find  $\vec{r}'(t)$ . Sketch  $\vec{r}(t)$  and  $\vec{r}'(1)$ , with the initial point of  $\vec{r}'(1)$  at  $\vec{r}(1)$ .

15.  $\vec{r}(t) = \langle t^2 + t, t^2 - t \rangle$
16.  $\vec{r}(t) = \langle t^2 - 2t + 2, t^3 - 3t^2 + 2t \rangle$
17.  $\vec{r}(t) = \langle t^2 + 1, t^3 - t \rangle$
18.  $\vec{r}(t) = \langle t^2 - 4t + 5, t^3 - 6t^2 + 11t - 6 \rangle$

In Exercises 19 – 22, give the equation of the line tangent to the graph of  $\vec{r}(t)$  at the given  $t$  value.

19.  $\vec{r}(t) = \langle t^2 + t, t^2 - t \rangle$  at  $t = 1$ .
20.  $\vec{r}(t) = \langle 3 \cos t, \sin t \rangle$  at  $t = \pi/4$ .
21.  $\vec{r}(t) = \langle 3 \cos t, 3 \sin t, t \rangle$  at  $t = \pi$ .
22.  $\vec{r}(t) = \langle e^t, \tan t, t \rangle$  at  $t = 0$ .

In Exercises 23 – 26, find the value(s) of  $t$  for which  $\vec{r}(t)$  is not smooth.

23.  $\vec{r}(t) = \langle \cos t, \sin t - t \rangle$
24.  $\vec{r}(t) = \langle t^2 - 2t + 1, t^3 + t^2 - 5t + 3 \rangle$
25.  $\vec{r}(t) = \langle \cos t - \sin t, \sin t - \cos t, \cos(4t) \rangle$
26.  $\vec{r}(t) = \langle t^3 - 3t + 2, -\cos(\pi t), \sin^2(\pi t) \rangle$

Exercises 27 – 29 ask you to verify parts of Theorem 42. In each let  $f(t) = t^3$ ,  $\vec{r}(t) = \langle t^2, t - 1, 1 \rangle$  and  $\vec{s}(t) = \langle \sin t, e^t, t \rangle$ . Compute the various derivatives as indicated.

27. Simplify  $f(t)\vec{r}(t)$ , then find its derivative; show this is the same as  $f'(t)\vec{r}(t) + f(t)\vec{r}'(t)$ .
28. Simplify  $\vec{r}(t) \cdot \vec{s}(t)$ , then find its derivative; show this is the same as  $\vec{r}'(t) \cdot \vec{s}(t) + \vec{r}(t) \cdot \vec{s}'(t)$ .
29. Simplify  $\vec{r}(t) \times \vec{s}(t)$ , then find its derivative; show this is the same as  $\vec{r}'(t) \times \vec{s}(t) + \vec{r}(t) \times \vec{s}'(t)$ .

In Exercises 30 – 33, evaluate the given definite or indefinite integral.

30.  $\int \langle t^3, \cos t, te^t \rangle dt$
31.  $\int \left\langle \frac{1}{1+t^2}, \sec^2 t \right\rangle dt$
32.  $\int_0^\pi \langle -\sin t, \cos t \rangle dt$
33.  $\int_{-2}^2 \langle 2t + 1, 2t - 1 \rangle dt$

In Exercises 34 – 37, solve the given initial value problems.

34. Find  $\vec{r}(t)$ , given that  $\vec{r}'(t) = \langle t, \sin t \rangle$  and  $\vec{r}(0) = \langle 2, 2 \rangle$ .
35. Find  $\vec{r}(t)$ , given that  $\vec{r}'(t) = \langle 1/(t + 1), \tan t \rangle$  and  $\vec{r}(0) = \langle 1, 2 \rangle$ .

36. Find  $\vec{r}(t)$ , given that  $\vec{r}''(t) = \langle t^2, t, 1 \rangle$ ,  
 $\vec{r}'(0) = \langle 1, 2, 3 \rangle$  and  $\vec{r}(0) = \langle 4, 5, 6 \rangle$ .
37. Find  $\vec{r}(t)$ , given that  $\vec{r}''(t) = \langle \cos t, \sin t, e^t \rangle$ ,  
 $\vec{r}'(0) = \langle 0, 0, 0 \rangle$  and  $\vec{r}(0) = \langle 0, 0, 0 \rangle$ .
- In Exercises 38 – 41 , find the arc length of  $\vec{r}(t)$  on the indicated interval.**
38.  $\vec{r}(t) = \langle 2 \cos t, 2 \sin t, 3t \rangle$  on  $[0, 2\pi]$ .
39.  $\vec{r}(t) = \langle 5 \cos t, 3 \sin t, 4 \sin t \rangle$  on  $[0, 2\pi]$ .
40.  $\vec{r}(t) = \langle t^3, t^2, t^3 \rangle$  on  $[0, 1]$ .
41.  $\vec{r}(t) = \langle e^{-t} \cos t, e^{-t} \sin t \rangle$  on  $[0, 1]$ .
42. Prove Theorem 43; that is, show if  $\vec{r}(t)$  has constant length and is differentiable, then  $\vec{r}(t) \cdot \vec{r}'(t) = 0$ . (Hint: use the Product Rule to compute  $\frac{d}{dt}(\vec{r}(t) \cdot \vec{r}(t))$ .)

### 10.3 The Calculus of Motion

A common use of vector-valued functions is to describe the motion of an object in the plane or in space. A **position function**  $\vec{r}(t)$  gives the position of an object at **time**  $t$ . This section explores how derivatives and integrals are used to study the motion described by such a function.

**Definition 50 Velocity, Speed and Acceleration**

Let  $\vec{r}(t)$  be a position function in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ .

1. **Velocity**, denoted  $\vec{v}(t)$ , is the instantaneous rate of position change; that is,  $\vec{v}(t) = \vec{r}'(t)$ .
2. **Speed** is the magnitude of velocity,  $\|\vec{v}(t)\|$ .
3. **Acceleration**, denoted  $\vec{a}(t)$ , is the instantaneous rate of velocity change; that is,  $\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)$ .

**Example 208 Finding velocity and acceleration**

An object is moving with position function  $\vec{r}(t) = \langle t^2 - t, t^2 + t \rangle$ ,  $-3 \leq t \leq 3$ , where distances are measured in feet and time is measured in seconds.

1. Find  $\vec{v}(t)$  and  $\vec{a}(t)$ .
2. Sketch  $\vec{r}(t)$ ; plot  $\vec{v}(-1)$ ,  $\vec{a}(-1)$ ,  $\vec{v}(1)$  and  $\vec{a}(1)$ , each with their initial point at their corresponding point on the graph of  $\vec{r}(t)$ .
3. When is the object's speed minimized?

**SOLUTION**

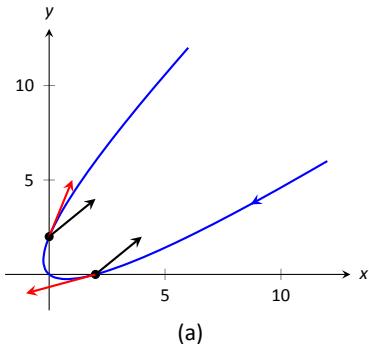
1. Taking derivatives, we find

$$\vec{v}(t) = \vec{r}'(t) = \langle 2t - 1, 2t + 1 \rangle \quad \text{and} \quad \vec{a}(t) = \vec{r}''(t) = \langle 2, 2 \rangle.$$

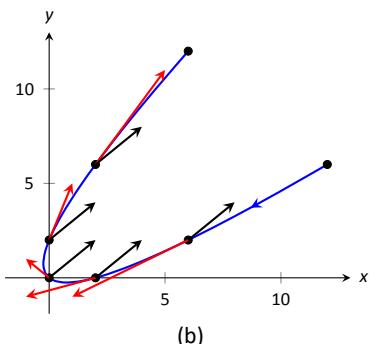
Note that acceleration is constant.

2.  $\vec{v}(-1) = \langle -3, -1 \rangle$ ,  $\vec{a}(-1) = \langle 2, 2 \rangle$ ;  $\vec{v}(1) = \langle 1, 3 \rangle$ ,  $\vec{a}(1) = \langle 2, 2 \rangle$ . These are plotted with  $\vec{r}(t)$  in Figure 10.15(a).

We can think of acceleration as “pulling” the velocity vector in a certain direction. At  $t = -1$ , the velocity vector points down and to the left; at  $t = 1$ , the velocity vector has been pulled in the  $\langle 2, 2 \rangle$  direction and is



(a)



(b)

Figure 10.15: Graphing the position, velocity and acceleration of an object in Example 208.

Notes:

now pointing up and to the right. In Figure 10.15(b) we plot more velocity/acceleration vectors, making more clear the effect acceleration has on velocity.

Since  $\vec{a}(t)$  is constant in this example, as  $t$  grows large  $\vec{v}(t)$  becomes almost parallel to  $\vec{a}(t)$ . For instance, when  $t = 10$ ,  $\vec{v}(10) = \langle 19, 21 \rangle$ , which is nearly parallel to  $\langle 2, 2 \rangle$ .

3. The object's speed is given by

$$\|\vec{v}(t)\| = \sqrt{(2t-1)^2 + (2t+1)^2} = \sqrt{8t^2 + 2}.$$

To find the minimal speed, we could apply calculus techniques (such as set the derivative equal to 0 and solve for  $t$ , etc.) but we can find it by inspection. Inside the square root we have a quadratic which is minimized when  $t = 0$ . Thus the speed is minimized at  $t = 0$ , with a speed of  $\sqrt{2}$  ft/s.

The graph in Figure 10.15(b) also implies speed is minimized here. The filled dots on the graph are located at integer values of  $t$  between  $-3$  and  $3$ . Dots that are far apart imply the object traveled a far distance in 1 second, indicating high speed; dots that are close together imply the object did not travel far in 1 second, indicating a low speed. The dots are closest together near  $t = 0$ , implying the speed is minimized near that value.

#### Example 209 Analyzing Motion

Two objects follow an identical path at different rates on  $[-1, 1]$ . The position function for Object 1 is  $\vec{r}_1(t) = \langle t, t^2 \rangle$ ; the position function for Object 2 is  $\vec{r}_2(t) = \langle t^3, t^6 \rangle$ , where distances are measured in feet and time is measured in seconds. Compare the velocity, speed and acceleration of the two objects on the path.

**SOLUTION** We begin by computing the velocity and acceleration function for each object:

$$\begin{aligned}\vec{v}_1(t) &= \langle 1, 2t \rangle & \vec{v}_2(t) &= \langle 3t^2, 6t^5 \rangle \\ \vec{a}_1(t) &= \langle 0, 2 \rangle & \vec{a}_2(t) &= \langle 6t, 30t^4 \rangle\end{aligned}$$

We immediately see that Object 1 has constant acceleration, whereas Object 2 does not.

At  $t = -1$ , we have  $\vec{v}_1(-1) = \langle 1, -2 \rangle$  and  $\vec{v}_2(-1) = \langle 3, -6 \rangle$ ; the velocity of Object 2 is three times that of Object 1 and so it follows that the speed of Object 2 is three times that of Object 1 ( $3\sqrt{5}$  ft/s compared to  $\sqrt{5}$  ft/s.)

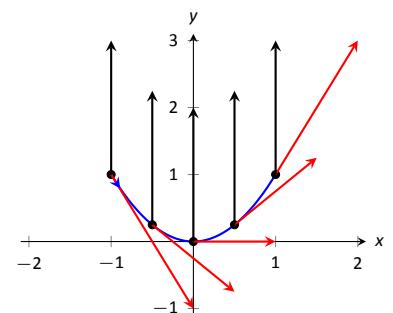


Figure 10.16: Plotting velocity and acceleration vectors for Object 1 in Example 209.

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Notes:

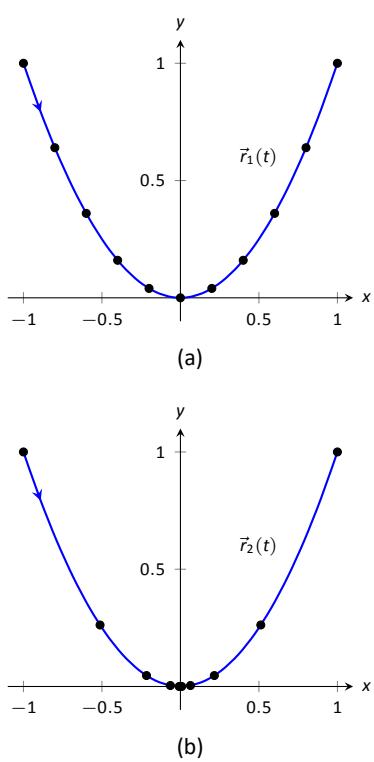


Figure 10.17: Comparing the positions of Objects 1 and 2 in Example 209.

At  $t = 0$ , the velocity of Object 1 is  $\vec{v}(1) = \langle 1, 0 \rangle$  and the velocity of Object 2 is  $\vec{0}$ ! This tells us that Object 2 comes to a complete stop at  $t = 0$ .

In Figure 10.16, we see the velocity and acceleration vectors for Object 1 plotted for  $t = -1, -1/2, 0, 1/2$  and  $t = 1$ . Note again how the constant acceleration vector seems to “pull” the velocity vector from pointing down, right to up, right. We could plot the analogous picture for Object 2, but the velocity and acceleration vectors are rather large ( $\vec{a}_2(-1) = \langle -6, 30 \rangle$ !).

Instead, we simply plot the locations of Object 1 and 2 on intervals of  $1/10^{\text{th}}$  of a second, shown in Figure 10.17(a) and (b). Note how the  $x$ -values of Object 1 increase at a steady rate. This is because the  $x$ -component of  $\vec{a}(t)$  is 0; there is no acceleration in the  $x$ -component. The dots are not evenly spaced; the object is moving faster near  $t = -1$  and  $t = 1$  than near  $t = 0$ .

In part (b) of the Figure, we see the points plotted for Object 2. Note the large change in position from  $t = -1$  to  $t = -0.9$ ; the object starts moving very quickly. However, it slows considerably as it approaches the origin, and comes to a complete stop at  $t = 0$ . While it looks like there are 3 points near the origin, there are in reality 5 points there.

Since the objects begin and end at the same location, they have the same displacement. Since they begin and end at the same time, with the same displacement, they have the same average rate of change (i.e., they have the same average velocity). Since they follow the same path, they have the same distance traveled. Even though these three measurements are the same, the objects obviously travel the path in very different ways.

### Example 210 Analyzing the motion of a whirling ball on a string

A young boy whirls a ball, attached to a string, above his head in a counter-clockwise circle. The ball follows a circular path and makes 2 revolutions per second. The string has length 2ft.

1. Find the position function  $\vec{r}(t)$  that describes this situation.
2. Find the acceleration of the ball and derive a physical interpretation of it.
3. A tree stands 10ft in front of the boy. At what  $t$ -values should the boy release the string so that the ball hits the tree?

### SOLUTION

1. The ball whirls in a circle. Since the string is 2ft long, the radius of the circle is 2. The position function  $\vec{r}(t) = \langle 2 \cos t, 2 \sin t \rangle$  describes a circle with radius 2, centered at the origin, but makes a full revolution every  $2\pi$  seconds, not two revolutions per second. We modify the period of the

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Notes:

trigonometric functions to be  $1/2$  by multiplying  $t$  by  $4\pi$ . The final position function is thus

$$\vec{r}(t) = \langle 2 \cos(4\pi t), 2 \sin(4\pi t) \rangle.$$

(Plot this for  $0 \leq t \leq 1/2$  to verify that one revolution is made in  $1/2$  a second.)

2. To find  $\vec{a}(t)$ , we derive  $\vec{r}(t)$  twice.

$$\begin{aligned}\vec{v}(t) &= \vec{r}'(t) = \langle -8\pi \sin(4\pi t), 8\pi \cos(4\pi t) \rangle \\ \vec{a}(t) &= \vec{r}''(t) = \langle -32\pi^2 \cos(4\pi t), -32\pi^2 \sin(4\pi t) \rangle \\ &= -32\pi^2 \langle \cos(4\pi t), \sin(4\pi t) \rangle.\end{aligned}$$

Note how  $\vec{a}(t)$  is parallel to  $\vec{r}(t)$ , but has a different magnitude and points in the opposite direction. Why is this?

Recall the classic physics equation, “Force = mass  $\times$  acceleration.” A force acting on a mass induces acceleration (i.e., the mass moves); acceleration acting on a mass induces a force (gravity gives our mass a *weight*). Thus force and acceleration are closely related. A moving ball “wants” to travel in a straight line. Why does the ball in our example move in a circle? It is attached to the boy’s hand by a string. The string applies a force to the ball, affecting its motion: the string *accelerates* the ball. This is not acceleration in the sense of “it travels faster;” rather, this acceleration is changing the velocity of the ball. In what direction is this force/acceleration being applied? In the direction of the string, towards the boy’s hand.

The magnitude of the acceleration is related to the speed at which the ball is traveling. A ball whirling quickly is rapidly changing direction/velocity. When velocity is changing rapidly, the acceleration must be “large.”

3. When the boy releases the string, the string no longer applies a force to the ball, meaning acceleration is  $\vec{0}$  and the ball can now move in a straight line in the direction of  $\vec{v}(t)$ .

Let  $t = t_0$  be the time when the boy lets go of the string. The ball will be at  $\vec{r}(t_0)$ , traveling in the direction of  $\vec{v}(t_0)$ . We want to find  $t_0$  so that this line contains the point  $(0, 10)$  (since the tree is 10ft directly in front of the boy).

There are many ways to find this time value. We choose one that is relatively simple computationally. As shown in Figure 10.18, the vector from the release point to the tree is  $\langle 0, 10 \rangle - \vec{r}(t_0)$ . This line segment is tangent to the circle, which means it is also perpendicular to  $\vec{r}(t_0)$  itself, so their dot product is 0.

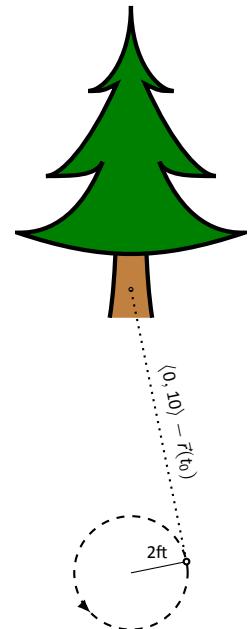


Figure 10.18: Modeling the flight of a ball in Example 210.

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Notes:

$$\begin{aligned}
& \vec{r}(t_0) \cdot (\langle 0, 10 \rangle - \vec{r}(t_0)) = 0 \\
& \langle 2 \cos(4\pi t_0), 2 \sin(4\pi t_0) \rangle \cdot \langle -2 \cos(4\pi t_0), 10 - 2 \sin(4\pi t_0) \rangle = 0 \\
& -4 \cos^2(4\pi t_0) + 20 \sin(4\pi t_0) - 4 \sin^2(4\pi t_0) = 0 \\
& 20 \sin(4\pi t_0) - 4 = 0 \\
& \sin(4\pi t_0) = 1/5 \\
& 4\pi t_0 = \sin^{-1}(1/5) \\
& 4\pi t_0 \approx 0.2 + 2\pi n,
\end{aligned}$$

where  $n$  is an integer. Solving for  $t_0$  we have:

$$t_0 \approx 0.016 + n/2$$

This is a wonderful formula. Every 1/2 second after  $t = 0.016$ s the boy can release the string (since the ball makes 2 revolutions per second, he has two chances each second to release the ball).

### Example 211 Analyzing motion in space

An object moves in a spiral with position function  $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ , where distances are measured in meters and time is in minutes. Describe the object's speed and acceleration at time  $t$ .

**SOLUTION**

With  $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ , we have:

$$\begin{aligned}
\vec{v}(t) &= \langle -\sin t, \cos t, 1 \rangle \quad \text{and} \\
\vec{a}(t) &= \langle -\cos t, -\sin t, 0 \rangle.
\end{aligned}$$

The speed of the object is  $\|\vec{v}(t)\| = \sqrt{(-\sin t)^2 + \cos^2 t + 1} = \sqrt{2}$ m/min; it moves at a constant speed. Note that the object does not accelerate in the  $z$ -direction, but rather moves up at a constant rate of 1m/min.

The objects in Examples 210 and 211 traveled at a constant speed. That is,  $\|\vec{v}(t)\| = c$  for some constant  $c$ . Recall Theorem 43, which states that if a vector-valued function  $\vec{r}(t)$  has constant length, then  $\vec{r}(t)$  is perpendicular to its derivative:  $\vec{r}(t) \cdot \vec{r}'(t) = 0$ . In these examples, the velocity function has constant length, therefore we can conclude that the velocity is perpendicular to the acceleration:  $\vec{v}(t) \cdot \vec{a}(t) = 0$ . A quick check verifies this.

There is an intuitive understanding of this. If acceleration is parallel to velocity, then it is only affecting the object's speed; it does not change the direction of travel. (For example, consider a dropped stone. Acceleration and velocity are

---

Notes:

parallel – straight down – and the direction of velocity never changes, though speed does increase.) If acceleration is not perpendicular to velocity, then there is some acceleration in the direction of travel, influencing the speed. If speed is constant, then acceleration must be orthogonal to velocity, as it then only affects direction, and not speed.

#### Key Idea 40 Objects With Constant Speed

If an object moves with constant speed, then its velocity and acceleration vectors are orthogonal. That is,  $\vec{v}(t) \cdot \vec{a}(t) = 0$ .

## Projectile Motion

An important application of vector-valued position functions is *projectile motion*: the motion of objects under only the influence of gravity. We will measure time in seconds, and distances will either be in meters or feet. We will show that we can completely describe the path of such an object knowing its initial position and initial velocity (i.e., where it *is* and where it *is going*.)

Suppose an object has initial position  $\vec{r}(0) = \langle x_0, y_0 \rangle$  and initial velocity  $\vec{v}(0) = \langle v_x, v_y \rangle$ . It is customary to rewrite  $\vec{v}(0)$  in terms of its speed  $v_0$  and direction  $\vec{u}$ , where  $\vec{u}$  is a unit vector. Recall all unit vectors in  $\mathbb{R}^2$  can be written as  $\langle \cos \theta, \sin \theta \rangle$ , where  $\theta$  is an angle measure counter-clockwise from the  $x$ -axis. (We refer to  $\theta$  as the **angle of elevation**.) Thus  $\vec{v}(0) = v_0 \langle \cos \theta, \sin \theta \rangle$ .

Since the acceleration of the object is known, namely  $\vec{a}(t) = \langle 0, -g \rangle$ , where  $g$  is the gravitational constant, we can find  $\vec{r}(t)$  knowing our two initial conditions. We first find  $\vec{v}(t)$ :

$$\begin{aligned}\vec{v}(t) &= \int \vec{a}(t) \, dt \\ \vec{v}(t) &= \int \langle 0, -g \rangle \, dt \\ \vec{v}(t) &= \langle 0, -gt \rangle + \vec{C}.\end{aligned}$$

Knowing  $\vec{v}(0) = v_0 \langle \cos \theta, \sin \theta \rangle$ , we have  $\vec{C} = v_0 \langle \cos \theta, \sin \theta \rangle$  and so

$$\vec{v}(t) = \langle v_0 \cos \theta, -gt + v_0 \sin \theta \rangle.$$

**Note:** In this text we use  $g = 32 \text{ ft/s}$  when using Imperial units, and  $g = 9.8 \text{ m/s}$  when using SI units.

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Notes:

We integrate once more to find  $\vec{r}(t)$ :

$$\begin{aligned}\vec{r}(t) &= \int \vec{v}(t) \, dt \\ \vec{r}(t) &= \int \langle v_0 \cos \theta, -gt + v_0 \sin \theta \rangle \, dt \\ \vec{r}(t) &= \left\langle (v_0 \cos \theta)t, -\frac{1}{2}gt^2 + (v_0 \sin \theta)t \right\rangle + \vec{C}.\end{aligned}$$

Knowing  $\vec{r}(0) = \langle x_0, y_0 \rangle$ , we conclude  $\vec{C} = \langle x_0, y_0 \rangle$  and

$$\vec{r}(t) = \left\langle (v_0 \cos \theta)t + x_0, -\frac{1}{2}gt^2 + (v_0 \sin \theta)t + y_0 \right\rangle.$$

#### Key Idea 41      Projectile Motion

The position function of a projectile propelled from an initial position of  $\vec{r}_0 = \langle x_0, y_0 \rangle$ , with initial speed  $v_0$ , with angle of elevation  $\theta$  and neglecting all accelerations but gravity is

$$\vec{r}(t) = \left\langle (v_0 \cos \theta)t + x_0, -\frac{1}{2}gt^2 + (v_0 \sin \theta)t + y_0 \right\rangle.$$

Letting  $\vec{v}_0 = v_0 \langle \cos \theta, \sin \theta \rangle$ ,  $\vec{r}(t)$  can be written as

$$\vec{r}(t) = \left\langle 0, -\frac{1}{2}gt^2 \right\rangle + \vec{v}_0 t + \vec{r}_0.$$

We demonstrate how to use this position function in the next two examples.

#### Example 212      Projectile Motion

Sydney shoots her Red Ryder® bb gun across level ground from an elevation of 4ft, where the barrel of the gun makes a  $5^\circ$  angle with the horizontal. Find how far the bb travels before landing, assuming the bb is fired at the advertised rate of 350ft/s and ignoring air resistance.

**SOLUTION**      A direct application of Key Idea 41 gives

$$\begin{aligned}\vec{r}(t) &= \langle (350 \cos 5^\circ)t, -16t^2 + (350 \sin 5^\circ)t + 4 \rangle \\ &\approx \langle 346.67t, -16t^2 + 30.50t + 4 \rangle,\end{aligned}$$

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Notes:

where we set her initial position to be  $\langle 0, 4 \rangle$ . We need to find *when* the bb lands, then we can find *where*. We accomplish this by setting the  $y$ -component equal to 0 and solving for  $t$ :

$$\begin{aligned} -16t^2 + 30.50t + 4 &= 0 \\ t &= \frac{-30.50 \pm \sqrt{30.50^2 - 4(-16)(4)}}{-32} \\ t &\approx 2.03s. \end{aligned}$$

(We discarded a negative solution that resulted from our quadratic equation.)

We have found that the bb lands 2.03s after firing; with  $t = 2.03$ , we find the  $x$ -component of our position function is  $346.67(2.03) = 703.74\text{ft}$ . The bb lands about 704 feet away.

### Example 213 Projectile Motion

Alex holds his sister's bb gun at a height of 3ft and wants to shoot a target that is 6ft above the ground, 25ft away. At what angle should he hold the gun to hit his target? (We still assume the muzzle velocity is 350ft/s.)

**SOLUTION** The position function for the path of Alex's bb is

$$\vec{r}(t) = \langle (350 \cos \theta)t, -16t^2 + (350 \sin \theta)t + 3 \rangle.$$

We need to find  $\theta$  so that  $\vec{r}(t) = \langle 25, 6 \rangle$  for some value of  $t$ . That is, we want to find  $\theta$  and  $t$  such that

$$(350 \cos \theta)t = 25 \quad \text{and} \quad -16t^2 + (350 \sin \theta)t + 3 = 6.$$

This is not trivial (though not "hard"). We start by solving each equation for  $\cos \theta$  and  $\sin \theta$ , respectively.

$$\cos \theta = \frac{25}{350t} \quad \text{and} \quad \sin \theta = \frac{3 + 16t^2}{350t}.$$

Using the Pythagorean Identity  $\cos^2 \theta + \sin^2 \theta = 1$ , we have

$$\left(\frac{25}{350t}\right)^2 + \left(\frac{3 + 16t^2}{350t}\right)^2 = 1$$

Multiply both sides by  $(350t)^2$ :

$$\begin{aligned} 25^2 + (3 + 16t^2)^2 &= 350^2 t^2 \\ 256t^4 - 122,404t^2 + 634 &= 0. \end{aligned}$$

Notes:

This is a quadratic in  $t^2$ . That is, we can apply the quadratic formula to find  $t^2$ , then solve for  $t$  itself.

$$t^2 = \frac{122,404 \pm \sqrt{122,404^2 - 4(256)(634)}}{512}$$

$$t^2 = 0.0052, 478.135$$

$$t = \pm 0.072, \pm 21.866$$

Clearly the negative  $t$  values do not fit our context, so we have  $t = 0.072$  and  $t = 21.866$ . Using  $\cos \theta = 25/(350t)$ , we can solve for  $\theta$ :

$$\theta = \cos^{-1} \left( \frac{25}{350 \cdot 0.072} \right) \quad \text{and} \quad \cos^{-1} \left( \frac{25}{350 \cdot 21.866} \right)$$

$$\theta = 7.03^\circ \quad \text{and} \quad 89.8^\circ.$$

Alex has two choices of angle. He can hold the rifle at an angle of about  $7^\circ$  with the horizontal and hit his target 0.07s after firing, or he can hold his rifle almost straight up, with an angle of  $89.8^\circ$ , where he'll hit his target about 22s later. The first option is clearly the option he should choose.

## Distance Traveled

Consider a driver who sets her cruise-control to 60mph, and travels at this speed for an hour. We can ask:

1. How far did the driver travel?
2. How far from her starting position is the driver?

The first is easy to answer: she traveled 60 miles. The second is impossible to answer with the given information. We do not know if she traveled in a straight line, on an oval racetrack, or along a slowly-winding highway.

This highlights an important fact: to compute distance traveled, we need only to know the speed, given by  $\|\vec{v}(t)\|$ .

### Theorem 46 Distance Traveled

Let  $\vec{v}(t)$  be a velocity function for a moving object. The distance traveled by the object on  $[a, b]$  is:

$$\text{distance traveled} = \int_a^b \|\vec{v}(t)\| dt.$$

Note that this is just a restatement of Theorem 45: arc length is the same as distance traveled, just viewed in a different context.

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Notes:

**Example 214 Distance Traveled, Displacement, and Average Speed**

A particle moves in space with position function  $\vec{r}(t) = \langle t, t^2, \sin(\pi t) \rangle$  on  $[-2, 2]$ , where  $t$  is measured in seconds and distances are in meters. Find:

1. The distance traveled by the particle on  $[-2, 2]$ .
2. The displacement of the particle on  $[-2, 2]$ .
3. The particle's average speed.

**SOLUTION**

1. We use Theorem 46 to establish the integral:

$$\begin{aligned} \text{distance traveled} &= \int_{-2}^2 \|\vec{v}(t)\| dt \\ &= \int_{-2}^2 \sqrt{1 + (2t)^2 + \pi^2 \cos^2(\pi t)} dt. \end{aligned}$$

This cannot be solved in terms of elementary functions so we turn to numerical integration, finding the distance to be 12.88m.

2. The displacement is the vector

$$\vec{r}(2) - \vec{r}(-2) = \langle 2, 4, 0 \rangle - \langle -2, 4, 0 \rangle = \langle 4, 0, 0 \rangle.$$

That is, the particle ends with an  $x$ -value increased by 4 and with  $y$ - and  $z$ -values the same (see Figure 10.19).

3. We found above that the particle traveled 12.88m over 4 seconds. We can compute average speed by dividing:  $12.88/4 = 3.22\text{m/s}$ .

We should also consider Definition 22 of Section 5.4, which says that the average value of a function  $f$  on  $[a, b]$  is  $\frac{1}{b-a} \int_a^b f(x) dx$ . In our context, the average value of the speed is

$$\text{average speed} = \frac{1}{2 - (-2)} \int_{-2}^2 \|\vec{v}(t)\| dt \approx \frac{1}{4} 12.88 = 3.22\text{m/s}.$$

Note how the physical context of a particle traveling gives meaning to a more abstract concept learned earlier.

In Definition 22 of Chapter 5 we defined the average value of a function  $f(x)$  on  $[a, b]$  to be

$$\frac{1}{b-a} \int_a^b f(x) dx.$$

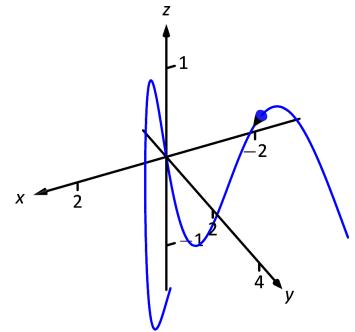


Figure 10.19: The path of the particle in Example 214.

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Notes:

Note how in Example 214 we computed the average speed as

$$\frac{\text{distance traveled}}{\text{travel time}} = \frac{1}{2 - (-2)} \int_{-2}^2 \|\vec{v}(t)\| dt;$$

that is, we just found the average value of  $\|\vec{v}(t)\|$  on  $[-2, 2]$ .

Likewise, given position function  $\vec{r}(t)$ , the average velocity on  $[a, b]$  is

$$\frac{\text{displacement}}{\text{travel time}} = \frac{1}{b - a} \int_a^b \vec{r}'(t) dt = \frac{\vec{r}(b) - \vec{r}(a)}{b - a};$$

that is, it is the average value of  $\vec{r}'(t)$ , or  $\vec{v}(t)$ , on  $[a, b]$ .

#### Key Idea 42      Average Speed, Average Velocity

Let  $\vec{r}(t)$  be a continuous position function on an open interval  $I$  containing  $a < b$ .

The **average speed** is:

$$\frac{\text{distance traveled}}{\text{travel time}} = \frac{\int_a^b \|\vec{v}(t)\| dt}{b - a} = \frac{1}{b - a} \int_a^b \|\vec{v}(t)\| dt.$$

The **average velocity** is:

$$\frac{\text{displacement}}{\text{travel time}} = \frac{\int_a^b \vec{r}'(t) dt}{b - a} = \frac{1}{b - a} \int_a^b \vec{r}'(t) dt.$$

The next two sections investigate more properties of the graphs of vector-valued functions and we'll apply these new ideas to what we just learned about motion.

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Notes:

# Exercises 10.3

## Terms and Concepts

1. How is *velocity* different from *speed*?
2. What is the difference between *displacement* and *distance traveled*?
3. What is the difference between *average velocity* and *average speed*?
4. *Distance traveled* is the same as \_\_\_\_\_, just viewed in a different context.
5. Describe a scenario where an object's average speed is a large number, but the magnitude of the average velocity is not a large number.
6. Explain why it is not possible to have an average velocity with a large magnitude but a small average speed.

## Problems

In Exercises 7 – 10 , a position function  $\vec{r}(t)$  is given. Find  $\vec{v}(t)$  and  $\vec{a}(t)$ .

7.  $\vec{r}(t) = \langle 2t + 1, 5t - 2, 7 \rangle$
8.  $\vec{r}(t) = \langle 3t^2 - 2t + 1, -t^2 + t + 14 \rangle$
9.  $\vec{r}(t) = \langle \cos t, \sin t \rangle$
10.  $\vec{r}(t) = \langle t/10, -\cos t, \sin t \rangle$

In Exercises 11 – 14 , a position function  $\vec{r}(t)$  is given. Sketch  $\vec{r}(t)$  on the indicated interval. Find  $\vec{v}(t)$  and  $\vec{a}(t)$ , then add  $\vec{v}(t_0)$  and  $\vec{a}(t_0)$  to your sketch, with their initial points at  $\vec{r}(t_0)$ , for the given value of  $t_0$ .

11.  $\vec{r}(t) = \langle t, \sin t \rangle$  on  $[0, \pi/2]$ ;  $t_0 = \pi/4$
12.  $\vec{r}(t) = \langle t^2, \sin t^2 \rangle$  on  $[0, \pi/2]$ ;  $t_0 = \sqrt{\pi/4}$
13.  $\vec{r}(t) = \langle t^2 + t, -t^2 + 2t \rangle$  on  $[-2, 2]$ ;  $t_0 = 1$
14.  $\vec{r}(t) = \left\langle \frac{2t+3}{t^2+1}, t^2 \right\rangle$  on  $[-1, 1]$ ;  $t_0 = 0$

In Exercises 15 – 24 , a position function  $\vec{r}(t)$  of an object is given. Find the speed of the object in terms of  $t$ , and find where the speed is minimized/maximized on the indicated interval.

15.  $\vec{r}(t) = \langle t^2, t \rangle$  on  $[-1, 1]$
16.  $\vec{r}(t) = \langle t^2, t^2 - t^3 \rangle$  on  $[-1, 1]$

17.  $\vec{r}(t) = \langle 5 \cos t, 5 \sin t \rangle$  on  $[0, 2\pi]$
18.  $\vec{r}(t) = \langle 2 \cos t, 5 \sin t \rangle$  on  $[0, 2\pi]$
19.  $\vec{r}(t) = \langle \sec t, \tan t \rangle$  on  $[0, \pi/4]$
20.  $\vec{r}(t) = \langle t + \cos t, 1 - \sin t \rangle$  on  $[0, 2\pi]$
21.  $\vec{r}(t) = \langle 12t, 5 \cos t, 5 \sin t \rangle$  on  $[0, 4\pi]$
22.  $\vec{r}(t) = \langle t^2 - t, t^2 + t, t \rangle$  on  $[0, 1]$
23.  $\vec{r}(t) = \left\langle t, t^2, \sqrt{1-t^2} \right\rangle$  on  $[-1, 1]$
24. **Projectile Motion:**  $\vec{r}(t) = \left\langle (v_0 \cos \theta)t, -\frac{1}{2}gt^2 + (v_0 \sin \theta)t \right\rangle$   
on  $\left[0, \frac{2v_0 \sin \theta}{g}\right]$

In Exercises 25 – 28 , position functions  $\vec{r}_1(t)$  and  $\vec{r}_2(s)$  for two objects are given that follow the same path on the respective intervals.

- (a) Show that the positions are the same at the indicated  $t_0$  and  $s_0$  values; i.e., show  $\vec{r}_1(t_0) = \vec{r}_2(s_0)$ .
- (b) Find the velocity, speed and acceleration of the two objects at  $t_0$  and  $s_0$ , respectively.
25.  $\vec{r}_1(t) = \langle t, t^2 \rangle$  on  $[0, 1]$ ;  $t_0 = 1$   
 $\vec{r}_2(s) = \langle s^2, s^4 \rangle$  on  $[0, 1]$ ;  $s_0 = 1$
26.  $\vec{r}_1(t) = \langle 3 \cos t, 3 \sin t \rangle$  on  $[0, 2\pi]$ ;  $t_0 = \pi/2$   
 $\vec{r}_2(s) = \langle 3 \cos(4s), 3 \sin(4s) \rangle$  on  $[0, \pi/2]$ ;  $s_0 = \pi/8$
27.  $\vec{r}_1(t) = \langle 3t, 2t \rangle$  on  $[0, 2]$ ;  $t_0 = 2$   
 $\vec{r}_2(s) = \langle 6t - 6, 4t - 4 \rangle$  on  $[1, 2]$ ;  $s_0 = 2$
28.  $\vec{r}_1(t) = \langle t, \sqrt{t} \rangle$  on  $[0, 1]$ ;  $t_0 = 1$   
 $\vec{r}_2(s) = \langle \sin t, \sqrt{\sin t} \rangle$  on  $[0, \pi/2]$ ;  $s_0 = \pi/2$

In Exercises 29 – 32 , find the position function of an object given its acceleration and initial velocity and position.

29.  $\vec{a}(t) = \langle 2, 3 \rangle$ ;  $\vec{v}(0) = \langle 1, 2 \rangle$ ,  $\vec{r}(0) = \langle 5, -2 \rangle$
30.  $\vec{a}(t) = \langle 2, 3 \rangle$ ;  $\vec{v}(1) = \langle 1, 2 \rangle$ ,  $\vec{r}(1) = \langle 5, -2 \rangle$
31.  $\vec{a}(t) = \langle \cos t, -\sin t \rangle$ ;  $\vec{v}(0) = \langle 0, 1 \rangle$ ,  $\vec{r}(0) = \langle 0, 0 \rangle$
32.  $\vec{a}(t) = \langle 0, -32 \rangle$ ;  $\vec{v}(0) = \langle 10, 50 \rangle$ ,  $\vec{r}(0) = \langle 0, 0 \rangle$

In Exercises 33 – 36 , find the displacement, distance traveled, average velocity and average speed of the described object on the given interval.

33. An object with position function  $\vec{r}(t) = \langle 2 \cos t, 2 \sin t, 3t \rangle$ , where distances are measured in feet and time is in seconds, on  $[0, 2\pi]$ .

34. An object with position function  $\vec{r}(t) = \langle 5 \cos t, -5 \sin t \rangle$ , where distances are measured in feet and time is in seconds, on  $[0, \pi]$ .
35. An object with velocity function  $\vec{v}(t) = \langle \cos t, \sin t \rangle$ , where distances are measured in feet and time is in seconds, on  $[0, 2\pi]$ .
36. An object with velocity function  $\vec{v}(t) = \langle 1, 2, -1 \rangle$ , where distances are measured in feet and time is in seconds, on  $[0, 10]$ .
- Exercises 37 – 42 ask you to solve a variety of problems based on the principles of projectile motion.**
37. A boy whirls a ball, attached to a 3ft string, above his head in a counter-clockwise circle. The ball makes 2 revolutions per second.  
At what  $t$ -values should the boy release the string so that the ball heads directly for a tree standing 10ft in front of him?
38. David faces Goliath with only a stone in a 3ft sling, which he whirls above his head at 4 revolutions per second. They stand 20ft apart.
  - (a) At what  $t$ -values must David release the stone in his sling in order to hit Goliath?
  - (b) What is the speed at which the stone is traveling when released?
  - (c) Assume David releases the stone from a height of 6ft and Goliath's forehead is 9ft above the ground. What angle of elevation must David apply to the stone to hit Goliath's head?
39. A hunter aims at a deer which is 40 yards away. Her crossbow is at a height of 5ft, and she aims for a spot on the deer 4ft above the ground. The crossbow fires her arrows at 300ft/s.
  - (a) At what angle of elevation should she hold the crossbow to hit her target?
  - (b) If the deer is moving perpendicularly to her line of sight at a rate of 20mph, by approximately how much should she lead the deer in order to hit it in the desired location?
40. A baseball player hits a ball at 100mph, with an initial height of 3ft and an angle of elevation of  $20^\circ$ , at Boston's Fenway Park. The ball flies towards the famed "Green Monster," a wall 37ft high located 310ft from home plate.
  - (a) Show that as hit, the ball hits the wall.
  - (b) Show that if the angle of elevation is  $21^\circ$ , the ball clears the Green Monster.
41. A Cessna flies at 1000ft at 150mph and drops a box of supplies to the professor (and his wife) on an island. Ignoring wind resistance, how far horizontally will the supplies travel before they land?
42. A football quarterback throws a pass from a height of 6ft, intending to hit his receiver 20yds away at a height of 5ft.
  - (a) If the ball is thrown at a rate of 50mph, what angle of elevation is needed to hit his intended target?
  - (b) If the ball is thrown at with an angle of elevation of  $8^\circ$ , what initial ball speed is needed to hit his target?

## 10.4 Unit Tangent and Normal Vectors

### Unit Tangent Vector

Given a smooth vector-valued function  $\vec{r}(t)$ , we defined in Definition 48 that any vector parallel to  $\vec{r}'(t_0)$  is *tangent* to the graph of  $\vec{r}(t)$  at  $t = t_0$ . It is often useful to consider just the *direction* of  $\vec{r}'(t)$  and not its magnitude. Therefore we are interested in the unit vector in the direction of  $\vec{r}'(t)$ . This leads to a definition.

#### Definition 51 Unit Tangent Vector

Let  $\vec{r}(t)$  be a smooth function on an open interval  $I$ . The unit tangent vector  $\vec{T}(t)$  is

$$\vec{T}(t) = \frac{1}{\|\vec{r}'(t)\|} \vec{r}'(t).$$

#### Example 215 Computing the unit tangent vector

Let  $\vec{r}(t) = \langle 3 \cos t, 3 \sin t, 4t \rangle$ . Find  $\vec{T}(t)$  and compute  $\vec{T}(0)$  and  $\vec{T}(1)$ .

**SOLUTION** We apply Definition 51 to find  $\vec{T}(t)$ .

$$\begin{aligned}\vec{T}(t) &= \frac{1}{\|\vec{r}'(t)\|} \vec{r}'(t) \\ &= \frac{1}{\sqrt{(-3 \sin t)^2 + (3 \cos t)^2 + 4^2}} \langle -3 \sin t, 3 \cos t, 4 \rangle \\ &= \left\langle -\frac{3}{5} \sin t, \frac{3}{5} \cos t, \frac{4}{5} \right\rangle.\end{aligned}$$

We can now easily compute  $\vec{T}(0)$  and  $\vec{T}(1)$ :

$$\vec{T}(0) = \left\langle 0, \frac{3}{5}, \frac{4}{5} \right\rangle; \quad \vec{T}(1) = \left\langle -\frac{3}{5} \sin 1, \frac{3}{5} \cos 1, \frac{4}{5} \right\rangle \approx \langle -0.505, 0.324, 0.8 \rangle.$$

These are plotted in Figure 10.20 with their initial points at  $\vec{r}(0)$  and  $\vec{r}(1)$ , respectively. (They look rather “short” since they are only length 1.)

The unit tangent vector  $\vec{T}(t)$  always has a magnitude of 1, though it is sometimes easy to doubt that is true. We can help solidify this thought in our minds by computing  $\|\vec{T}(1)\|$ :

$$\|\vec{T}(1)\| \approx \sqrt{(-0.505)^2 + 0.324^2 + 0.8^2} = 1.000001.$$

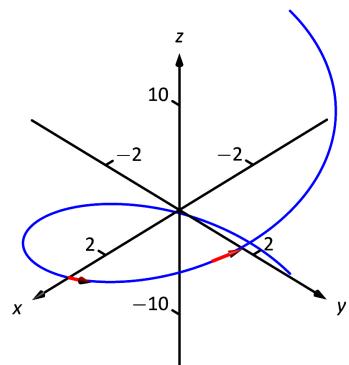


Figure 10.20: Plotting unit tangent vectors in Example 215.

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Notes:

We have rounded in our computation of  $\vec{T}(1)$ , so we don't get 1 exactly. We leave it to the reader to use the exact representation of  $\vec{T}(1)$  to verify it has length 1.

In many ways, the previous example was "too nice." It turned out that  $\vec{r}'(t)$  was always of length 5. In the next example the length of  $\vec{r}'(t)$  is variable, leaving us with a formula that is not as clean.

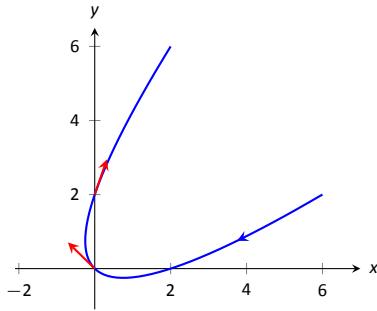


Figure 10.21: Plotting unit tangent vectors in Example 216.

**Example 216 Computing the unit tangent vector**  
Let  $\vec{r}(t) = \langle t^2 - t, t^2 + t \rangle$ . Find  $\vec{T}(t)$  and compute  $\vec{T}(0)$  and  $\vec{T}(1)$ .

**SOLUTION** We find  $\vec{r}'(t) = \langle 2t - 1, 2t + 1 \rangle$ , and

$$\|\vec{r}'(t)\| = \sqrt{(2t-1)^2 + (2t+1)^2} = \sqrt{8t^2 + 2}.$$

Therefore

$$\vec{T}(t) = \frac{1}{\sqrt{8t^2 + 2}} \langle 2t - 1, 2t + 1 \rangle = \left\langle \frac{2t-1}{\sqrt{8t^2+2}}, \frac{2t+1}{\sqrt{8t^2+2}} \right\rangle.$$

When  $t = 0$ , we have  $\vec{T}(0) = \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle$ ; when  $t = 1$ , we have  $\vec{T}(1) = \langle 1/\sqrt{10}, 3/\sqrt{10} \rangle$ . We leave it to the reader to verify each of these is a unit vector. They are plotted in Figure 10.21

### Unit Normal Vector

Just as knowing the direction tangent to a path is important, knowing a direction orthogonal to a path is important. When dealing with real-valued functions, we defined the normal line at a point to be the line through the point that was perpendicular to the tangent line at that point. We can do a similar thing with vector-valued functions. Given  $\vec{r}(t)$  in  $\mathbb{R}^2$ , we have 2 directions perpendicular to the tangent vector, as shown in Figure 10.22. It is good to wonder "Is one of these two directions preferable over the other?"

Given  $\vec{r}(t)$  in  $\mathbb{R}^3$ , there are infinite vectors orthogonal to the tangent vector at a given point. Again, we might wonder "Is one of these infinite choices preferable over the others? Is one of these the 'right' choice?"

The answer in both  $\mathbb{R}^2$  and  $\mathbb{R}^3$  is "Yes, there is one vector that is not only preferable, it is the 'right' one to choose." Recall Theorem 43, which states that if  $\vec{r}(t)$  has constant length, then  $\vec{r}(t)$  is orthogonal to  $\vec{r}'(t)$  for all  $t$ . We know  $\vec{T}(t)$ , the unit tangent vector, has constant length. Therefore  $\vec{T}(t)$  is orthogonal to  $\vec{T}'(t)$ .

We'll see that  $\vec{T}'(t)$  is more than just a convenient choice of vector that is orthogonal to  $\vec{r}'(t)$ ; rather, it is the "right" choice. Since all we care about is the direction, we define this newly found vector to be a unit vector.

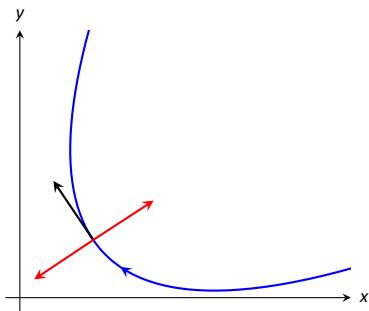


Figure 10.22: Given a direction in the plane, there are always two directions orthogonal to it.

**Note:**  $\vec{T}(t)$  is a unit vector, by definition. This does not imply that  $\vec{T}'(t)$  is also a unit vector.

---

Notes:

**Definition 52      Unit Normal Vector**

Let  $\vec{r}(t)$  be a vector-valued function where the unit tangent vector,  $\vec{T}(t)$ , is smooth on an open interval  $I$ . The **unit normal vector**  $\vec{N}(t)$  is

$$\vec{N}(t) = \frac{1}{\|\vec{T}'(t)\|} \vec{T}'(t).$$

**Example 217      Computing the unit normal vector**

Let  $\vec{r}(t) = \langle 3 \cos t, 3 \sin t, 4t \rangle$  as in Example 215. Sketch both  $\vec{T}(\pi/2)$  and  $\vec{N}(\pi/2)$  with initial points at  $\vec{r}(\pi/2)$ .

**SOLUTION**

In Example 215, we found  $\vec{T}(t) = \langle (-3/5) \sin t, (3/5) \cos t, 4/5 \rangle$ .

Therefore

$$\vec{T}'(t) = \left\langle -\frac{3}{5} \cos t, -\frac{3}{5} \sin t, 0 \right\rangle \quad \text{and} \quad \|\vec{T}'(t)\| = \frac{3}{5}.$$

Thus

$$\vec{N}(t) = \frac{\vec{T}'(t)}{3/5} = \langle -\cos t, -\sin t, 0 \rangle.$$

We compute  $\vec{T}(\pi/2) = \langle -3/5, 0, 4/5 \rangle$  and  $\vec{N}(\pi/2) = \langle 0, -1, 0 \rangle$ . These are sketched in Figure 10.23.

The previous example was once again “too nice.” In general, the expression for  $\vec{T}(t)$  contains fractions of square-roots, hence the expression of  $\vec{T}'(t)$  is very messy. We demonstrate this in the next example.

**Example 218      Computing the unit normal vector**

Let  $\vec{r}(t) = \langle t^2 - t, t^2 + t \rangle$  as in Example 216. Find  $\vec{N}(t)$  and sketch  $\vec{r}(t)$  with the unit tangent and normal vectors at  $t = -1, 0$  and  $1$ .

**SOLUTION**

In Example 216, we found

$$\vec{T}(t) = \left\langle \frac{2t-1}{\sqrt{8t^2+2}}, \frac{2t+1}{\sqrt{8t^2+2}} \right\rangle.$$

Finding  $\vec{T}'(t)$  requires two applications of the Quotient Rule:

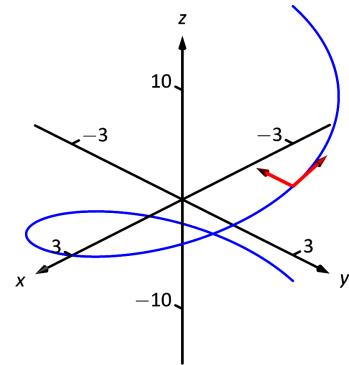


Figure 10.23: Plotting unit tangent and normal vectors in Example 10.23.

---

Notes:

$$\begin{aligned}
 \vec{T}'(t) &= \left\langle \frac{\sqrt{8t^2 + 2}(2) - (2t - 1) \left(\frac{1}{2}(8t^2 + 2)^{-1/2}(16t)\right)}{8t^2 + 2}, \right. \\
 &\quad \left. \frac{\sqrt{8t^2 + 2}(2) - (2t + 1) \left(\frac{1}{2}(8t^2 + 2)^{-1/2}(16t)\right)}{8t^2 + 2} \right\rangle \\
 &= \left\langle \frac{4(2t + 1)}{(8t^2 + 2)^{3/2}}, \frac{4(1 - 2t)}{(8t^2 + 2)^{3/2}} \right\rangle
 \end{aligned}$$

This is not a unit vector; to find  $\vec{N}(t)$ , we need to divide  $\vec{T}'(t)$  by its magnitude.

$$\begin{aligned}
 \|\vec{T}'(t)\| &= \sqrt{\frac{16(2t + 1)^2}{(8t^2 + 2)^3} + \frac{16(1 - 2t)^2}{(8t^2 + 2)^3}} \\
 &= \sqrt{\frac{16(8t^2 + 2)}{(8t^2 + 2)^3}} \\
 &= \frac{4}{8t^2 + 2}.
 \end{aligned}$$

Finally,

$$\begin{aligned}
 \vec{N}(t) &= \frac{1}{4/(8t^2 + 2)} \left\langle \frac{4(2t + 1)}{(8t^2 + 2)^{3/2}}, \frac{4(1 - 2t)}{(8t^2 + 2)^{3/2}} \right\rangle \\
 &= \left\langle \frac{2t + 1}{\sqrt{8t^2 + 2}}, -\frac{2t - 1}{\sqrt{8t^2 + 2}} \right\rangle.
 \end{aligned}$$

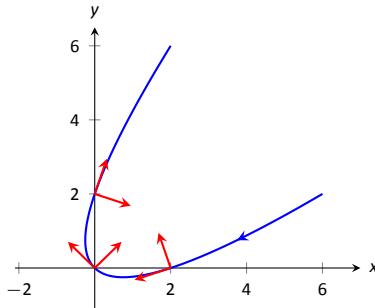


Figure 10.24: Plotting unit tangent and normal vectors in Example 218.

Using this formula for  $\vec{N}(t)$ , we compute the unit tangent and normal vectors for  $t = -1, 0$  and  $1$  and sketch them in Figure 10.24.

The final result for  $\vec{N}(t)$  in Example 218 is suspiciously similar to  $\vec{T}(t)$ . There is a clear reason for this. If  $\vec{u} = \langle u_1, u_2 \rangle$  is a unit vector in  $\mathbb{R}^2$ , then the *only* unit vectors orthogonal to  $\vec{u}$  are  $\langle -u_2, u_1 \rangle$  and  $\langle u_2, -u_1 \rangle$ . Given  $\vec{T}(t)$ , we can quickly determine  $\vec{N}(t)$  if we know which term to multiply by  $(-1)$ .

Consider again Figure 10.24, where we have plotted some unit tangent and normal vectors. Note how  $\vec{N}(t)$  always points “inside” the curve, or to the concave side of the curve. This is not a coincidence; this is true in general. Knowing the direction that  $\vec{r}(t)$  “turns” allows us to quickly find  $\vec{N}(t)$ .

---

Notes:

**Theorem 47    Unit Normal Vectors in  $\mathbb{R}^2$** 

Let  $\vec{r}(t)$  be a vector-valued function in  $\mathbb{R}^2$  where  $\vec{T}'(t)$  is smooth on an open interval  $I$ . Let  $t_0$  be in  $I$  and  $\vec{T}(t_0) = \langle t_1, t_2 \rangle$ . Then  $\vec{N}(t_0)$  is either

$$\vec{N}(t_0) = \langle -t_2, t_1 \rangle \quad \text{or} \quad \vec{N}(t_0) = \langle t_2, -t_1 \rangle,$$

whichever is the vector that points to the concave side of the graph of  $\vec{r}$ .

## Application to Acceleration

Let  $\vec{r}(t)$  be a position function. It is a fact (stated later in Theorem 48) that acceleration,  $\vec{a}(t)$ , lies in the plane defined by  $\vec{T}$  and  $\vec{N}$ . That is, there are scalars  $a_T$  and  $a_N$  such that

$$\vec{a}(t) = a_T \vec{T}(t) + a_N \vec{N}(t).$$

The scalar  $a_T$  measures “how much” acceleration is in the direction of travel, that is, it measures the component of acceleration that affects the speed. The scalar  $a_N$  measures “how much” acceleration is perpendicular to the direction of travel, that is, it measures the component of acceleration that affects the direction of travel.

We can find  $a_T$  using the orthogonal projection of  $\vec{a}(t)$  onto  $\vec{T}(t)$  (review Definition 36 in Section 9.3 if needed). Recalling that since  $\vec{T}(t)$  is a unit vector,  $\vec{T}(t) \cdot \vec{T}(t) = 1$ , so we have

$$\text{proj}_{\vec{T}(t)} \vec{a}(t) = \frac{\vec{a}(t) \cdot \vec{T}(t)}{\vec{T}(t) \cdot \vec{T}(t)} \vec{T}(t) = \underbrace{(\vec{a}(t) \cdot \vec{T}(t))}_{a_T} \vec{T}(t).$$

Thus the amount of  $\vec{a}(t)$  in the direction of  $\vec{T}(t)$  is  $a_T = \vec{a}(t) \cdot \vec{T}(t)$ . The same logic gives  $a_N = \vec{a}(t) \cdot \vec{N}(t)$ .

While this is a fine way of computing  $a_T$ , there are simpler ways of finding  $a_N$  (as finding  $\vec{N}$  itself can be complicated). The following theorem gives alternate formulas for  $a_T$  and  $a_N$ .

**Note:** Keep in mind that both  $a_T$  and  $a_N$  are functions of  $t$ ; that is, the scalar changes depending on  $t$ . It is convention to drop the “( $t$ )” notation from  $a_T(t)$  and simply write  $a_T$ .

---

Notes:

**Theorem 48 Acceleration in the Plane Defined by  $\vec{T}$  and  $\vec{N}$** 

Let  $\vec{r}(t)$  be a position function with acceleration  $\vec{a}(t)$  and unit tangent and normal vectors  $\vec{T}(t)$  and  $\vec{N}(t)$ . Then  $\vec{a}(t)$  lies in the plane defined by  $\vec{T}(t)$  and  $\vec{N}(t)$ ; that is, there exists scalars  $a_T$  and  $a_N$  such that

$$\vec{a}(t) = a_T \vec{T}(t) + a_N \vec{N}(t).$$

Moreover,

$$a_T = \vec{a}(t) \cdot \vec{T}(t) = \frac{d}{dt} (\|\vec{v}(t)\|)$$

$$a_N = \vec{a}(t) \cdot \vec{N}(t) = \sqrt{\|\vec{a}(t)\|^2 - a_T^2} = \frac{\|\vec{a}(t) \times \vec{v}(t)\|}{\|\vec{v}(t)\|} = \|\vec{v}(t)\| \|\vec{T}'(t)\|$$

Note the second formula for  $a_T$ :  $\frac{d}{dt} (\|\vec{v}(t)\|)$ . This measures the rate of change of speed, which again is the amount of acceleration in the direction of travel.

**Example 219 Computing  $a_T$  and  $a_N$** 

Let  $\vec{r}(t) = \langle 3 \cos t, 3 \sin t, 4t \rangle$  as in Examples 215 and 217. Find  $a_T$  and  $a_N$ .

**SOLUTION** The previous examples give  $\vec{a}(t) = \langle -3 \cos t, -3 \sin t, 0 \rangle$  and

$$\vec{T}(t) = \left\langle -\frac{3}{5} \sin t, \frac{3}{5} \cos t, \frac{4}{5} \right\rangle \quad \text{and} \quad \vec{N}(t) = \langle -\cos t, -\sin t, 0 \rangle.$$

We can find  $a_T$  and  $a_N$  directly with dot products:

$$a_T = \vec{a}(t) \cdot \vec{T}(t) = \frac{9}{5} \cos t \sin t - \frac{9}{5} \cos t \sin t + 0 = 0.$$

$$a_N = \vec{a}(t) \cdot \vec{N}(t) = 3 \cos^2 t + 3 \sin^2 t + 0 = 3.$$

Thus  $\vec{a}(t) = 0\vec{T}(t) + 3\vec{N}(t) = 3\vec{N}(t)$ , which is clearly the case.

What is the practical interpretation of these numbers?  $a_T = 0$  means the object is moving at a constant speed, and hence all acceleration comes in the form of direction change.

**Example 220 Computing  $a_T$  and  $a_N$** 

Let  $\vec{r}(t) = \langle t^2 - t, t^2 + t \rangle$  as in Examples 216 and 218. Find  $a_T$  and  $a_N$ .

---

Notes:

**SOLUTION** The previous examples give  $\vec{a}(t) = \langle 2, 2 \rangle$  and

$$\vec{r}(t) = \left\langle \frac{2t-1}{\sqrt{8t^2+2}}, \frac{2t+1}{\sqrt{8t^2+2}} \right\rangle \quad \text{and} \quad \vec{N}(t) = \left\langle \frac{2t+1}{\sqrt{8t^2+2}}, -\frac{2t-1}{\sqrt{8t^2+2}} \right\rangle.$$

While we can compute  $a_N$  using  $\vec{N}(t)$ , we instead demonstrate using another formula from Theorem 48.

$$a_T = \vec{a}(t) \cdot \vec{T}(t) = \frac{4t-2}{\sqrt{8t^2+2}} + \frac{4t+2}{\sqrt{8t^2+2}} = \frac{8t}{\sqrt{8t^2+2}}.$$

$$a_N = \sqrt{\|\vec{a}(t)\|^2 - a_T^2} = \sqrt{8 - \left(\frac{8t}{\sqrt{8t^2+2}}\right)^2} = \frac{4}{\sqrt{8t^2+2}}.$$

When  $t = 2$ ,  $a_T = \frac{16}{\sqrt{34}} \approx 2.74$  and  $a_N = \frac{4}{\sqrt{34}} \approx 0.69$ . We interpret this to mean that at  $t = 2$ , the particle is accelerating mostly by increasing speed, not by changing direction. As the path near  $t = 2$  is relatively straight, this should make intuitive sense. Figure 10.25 gives a graph of the path for reference.

Contrast this with  $t = 0$ , where  $a_T = 0$  and  $a_N = 4/\sqrt{2} \approx 2.82$ . Here the particle's speed is not changing and all acceleration is in the form of direction change.

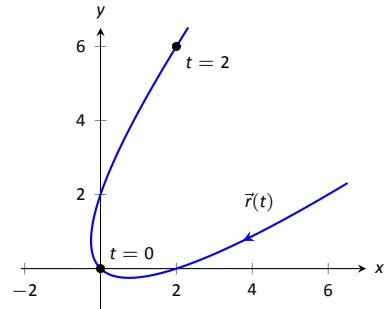


Figure 10.25: Graphing  $\vec{r}(t)$  in Example 220.

### Example 221 Analyzing projectile motion

A ball is thrown from a height of 240ft with an initial speed of 64ft/s and an angle of elevation of  $30^\circ$ . Find the position function  $\vec{r}(t)$  of the ball and analyze  $a_T$  and  $a_N$ .

**SOLUTION** Using Key Idea 41 of Section 10.3 we form the position function of the ball:

$$\vec{r}(t) = \langle (64 \cos 30^\circ)t, -16t^2 + (64 \sin 30^\circ)t + 240 \rangle,$$

which we plot in Figure 10.26.

From this we find  $\vec{v}(t) = \langle 64 \cos 30^\circ, -32t + 64 \sin 30^\circ \rangle$  and  $\vec{a}(t) = \langle 0, -32 \rangle$ . Computing  $\vec{T}(t)$  is not difficult, and with some simplification we find

$$\vec{T}(t) = \left\langle \frac{\sqrt{3}}{\sqrt{t^2 - 2t + 4}}, \frac{1-t}{\sqrt{t^2 - 2t + 4}} \right\rangle.$$

With  $\vec{a}(t)$  as simple as it is, finding  $a_T$  is also simple:

$$a_T = \vec{a}(t) \cdot \vec{T}(t) = \frac{32t - 32}{\sqrt{t^2 - 2t + 4}}.$$

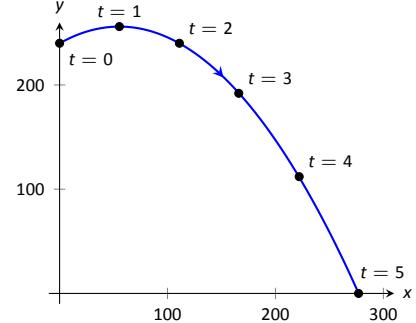


Figure 10.26: Plotting the position of a thrown ball, with 1s increments shown.

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Notes:

We choose to not find  $\vec{N}(t)$  and find  $a_N$  through the formula  $a_N = \sqrt{\|\vec{a}(t)\|^2 - a_T^2}$ :

$t$	$a_T$	$a_N$
0	-16	27.7
1	0	32
2	16	27.7
3	24.2	20.9
4	27.7	16
5	29.4	12.7

Figure 10.27: A table of values of  $a_T$  and  $a_N$  in Example 221.

$$a_N = \sqrt{32^2 - \left(\frac{32t - 32}{\sqrt{t^2 - 2t + 4}}\right)^2} = \frac{32\sqrt{3}}{\sqrt{t^2 - 2t + 4}}.$$

Figure 10.27 gives a table of values of  $a_T$  and  $a_N$ . When  $t = 0$ , we see the ball's speed is decreasing; when  $t = 1$  the speed of the ball is unchanged. This corresponds to the fact that at  $t = 1$  the ball reaches its highest point.

After  $t = 1$  we see that  $a_N$  is decreasing in value. This is because as the ball falls, it's path becomes straighter and most of the acceleration is in the form of speeding up the ball, and not in changing its direction.

Our understanding of the unit tangent and normal vectors is aiding our understanding of motion. The work in Example 221 gave quantitative analysis of what we intuitively knew.

The next section provides two more important steps towards this analysis. We currently describe position only in terms of time. In everyday life, though, we often describe position in terms of distance ("The gas station is about 2 miles ahead, on the left."). The *arc length parameter* allows us to reference position in terms of distance traveled.

We also intuitively know that some paths are straighter than others – and some are curvier than others, but we lack a measurement of "curviness." The arc length parameter provides a way for us to compute *curvature*, a quantitative measurement of how curvy a curve is.

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Notes:

# Exercises 10.4

## Terms and Concepts

1. If  $\vec{T}(t)$  is a unit tangent vector, what is  $\|\vec{T}(t)\|$ ?
2. If  $\vec{N}(t)$  is a unit normal vector, what is  $\vec{N}(t) \cdot \vec{r}'(t)$ ?
3. The acceleration vector  $\vec{a}(t)$  lies in the plane defined by what two vectors?
4.  $a_T$  measures how much the acceleration is affecting the \_\_\_\_\_ of an object.

## Problems

In Exercises 5 – 8 , given  $\vec{r}(t)$ , find  $\vec{T}(t)$  and evaluate it at the indicated value of  $t$ .

5.  $\vec{r}(t) = \langle 2t^2, t^2 - t \rangle, \quad t = 1$
6.  $\vec{r}(t) = \langle t, \cos t \rangle, \quad t = \pi/4$
7.  $\vec{r}(t) = \langle \cos^3 t, \sin^3 t \rangle, \quad t = \pi/4$
8.  $\vec{r}(t) = \langle \cos t, \sin t \rangle, \quad t = \pi$

In Exercises 9 – 12 , find the equation of the line tangent to the curve at the indicated  $t$ -value using the unit tangent vector. Note: these are the same problems as in Exercises 5 – 8.

9.  $\vec{r}(t) = \langle 2t^2, t^2 - t \rangle, \quad t = 1$
10.  $\vec{r}(t) = \langle t, \cos t \rangle, \quad t = \pi/4$
11.  $\vec{r}(t) = \langle \cos^3 t, \sin^3 t \rangle, \quad t = \pi/4$
12.  $\vec{r}(t) = \langle \cos t, \sin t \rangle, \quad t = \pi$

In Exercises 13 – 16 , find  $\vec{N}(t)$  using Definition 52. Confirm the result using Theorem 47.

13.  $\vec{r}(t) = \langle 3 \cos t, 3 \sin t \rangle$
14.  $\vec{r}(t) = \langle t, t^2 \rangle$
15.  $\vec{r}(t) = \langle \cos t, 2 \sin t \rangle$
16.  $\vec{r}(t) = \langle e^t, e^{-t} \rangle$

In Exercises 17 – 20 , a position function  $\vec{r}(t)$  is given along with its unit tangent vector  $\vec{T}(t)$  evaluated at  $t = a$ , for some value of  $a$ .

- (a) Confirm that  $\vec{T}(a)$  is as stated.
- (b) Using a graph of  $\vec{r}(t)$  and Theorem 47, find  $\vec{N}(a)$ .

$$17. \vec{r}(t) = \langle 3 \cos t, 5 \sin t \rangle; \quad \vec{T}(\pi/4) = \left\langle -\frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right\rangle.$$

$$18. \vec{r}(t) = \left\langle t, \frac{1}{t^2 + 1} \right\rangle; \quad \vec{T}(1) = \left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle.$$

$$19. \vec{r}(t) = (1 + 2 \sin t) \langle \cos t, \sin t \rangle; \quad \vec{T}(0) = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle.$$

$$20. \vec{r}(t) = \langle \cos^3 t, \sin^3 t \rangle; \quad \vec{T}(\pi/4) = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle.$$

In Exercises 21 – 24 , find  $\vec{N}(t)$ .

21.  $\vec{r}(t) = \langle 4t, 2 \sin t, 2 \cos t \rangle$
22.  $\vec{r}(t) = \langle 5 \cos t, 3 \sin t, 4 \sin t \rangle$
23.  $\vec{r}(t) = \langle a \cos t, a \sin t, bt \rangle; \quad a > 0$
24.  $\vec{r}(t) = \langle \cos(at), \sin(at), t \rangle$

In Exercises 25 – 30 , find  $a_T$  and  $a_N$  given  $\vec{r}(t)$ . Sketch  $\vec{r}(t)$  on the indicated interval, and comment on the relative sizes of  $a_T$  and  $a_N$  at the indicated  $t$  values.

25.  $\vec{r}(t) = \langle t, t^2 \rangle$  on  $[-1, 1]$ ; consider  $t = 0$  and  $t = 1$ .
26.  $\vec{r}(t) = \langle t, 1/t \rangle$  on  $(0, 4]$ ; consider  $t = 1$  and  $t = 2$ .
27.  $\vec{r}(t) = \langle 2 \cos t, 2 \sin t \rangle$  on  $[0, 2\pi]$ ; consider  $t = 0$  and  $t = \pi/2$ .
28.  $\vec{r}(t) = \langle \cos(t^2), \sin(t^2) \rangle$  on  $(0, 2\pi]$ ; consider  $t = \sqrt{\pi/2}$  and  $t = \sqrt{\pi}$ .
29.  $\vec{r}(t) = \langle a \cos t, a \sin t, bt \rangle$  on  $[0, 2\pi]$ , where  $a, b > 0$ ; consider  $t = 0$  and  $t = \pi/2$ .
30.  $\vec{r}(t) = \langle 5 \cos t, 4 \sin t, 3 \sin t \rangle$  on  $[0, 2\pi]$ ; consider  $t = 0$  and  $t = \pi/2$ .

## 10.5 The Arc Length Parameter and Curvature

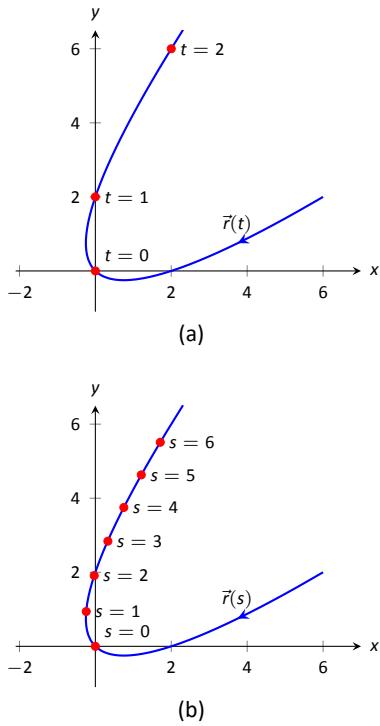


Figure 10.28: Introducing the arc length parameter.

In normal conversation we describe position in terms of both *time* and *distance*. For instance, imagine driving to visit a friend. If she calls and asks where you are, you might answer “I am 20 minutes from your house,” or you might say “I am 10 miles from your house.” Both answers provide your friend with a general idea of where you are.

Currently, our vector-valued functions have defined points with a parameter  $t$ , which we often take to represent time. Consider Figure 10.28(a), where  $\vec{r}(t) = \langle t^2 - t, t^2 + t \rangle$  is graphed and the points corresponding to  $t = 0, 1$  and  $2$  are shown. Note how the arc length between  $t = 0$  and  $t = 1$  is smaller than the arc length between  $t = 1$  and  $t = 2$ ; if the parameter  $t$  is time and  $\vec{r}$  is position, we can say that the particle traveled faster on  $[1, 2]$  than on  $[0, 1]$ .

Now consider Figure 10.28(b), where the same graph is parametrized by a different variable  $s$ . Points corresponding to  $s = 0$  through  $s = 6$  are plotted. The arc length of the graph between each adjacent pair of points is 1. We can view this parameter  $s$  as distance; that is, the arc length of the graph from  $s = 0$  to  $s = 3$  is 3, the arc length from  $s = 2$  to  $s = 6$  is 4, etc. If one wants to find the point 2.5 units from an initial location (i.e.,  $s = 0$ ), one would compute  $\vec{r}(2.5)$ . This parameter  $s$  is very useful, and is called the **arc length parameter**.

How do we find the arc length parameter?

Start with any parametrization of  $\vec{r}$ . We can compute the arc length of the graph of  $\vec{r}$  on the interval  $[0, t]$  with

$$\text{arc length} = \int_0^t \|\vec{r}'(u)\| du.$$

We can turn this into a function: as  $t$  varies, we find the arc length  $s$  from 0 to  $t$ . This function is

$$s(t) = \int_0^t \|\vec{r}'(u)\| du. \quad (10.1)$$

This establishes a relationship between  $s$  and  $t$ . Knowing this relationship explicitly, we can rewrite  $\vec{r}(t)$  as a function of  $s$ :  $\vec{r}(s)$ . We demonstrate this in an example.

### Example 222 Finding the arc length parameter

Let  $\vec{r}(t) = \langle 3t - 1, 4t + 2 \rangle$ . Parametrize  $\vec{r}$  with the arc length parameter  $s$ .

#### SOLUTION

Using Equation (10.1), we write

$$s(t) = \int_0^t \|\vec{r}'(u)\| du.$$

---

Notes:

We can integrate this, explicitly finding a relationship between  $s$  and  $t$ :

$$\begin{aligned}s(t) &= \int_0^t \|\vec{r}'(u)\| du \\&= \int_0^t \sqrt{3^2 + 4^2} du \\&= \int_0^t 5 du \\&= 5t.\end{aligned}$$

Since  $s = 5t$ , we can write  $t = s/5$  and replace  $t$  in  $\vec{r}(t)$  with  $s/5$ :

$$\vec{r}(s) = \langle 3(s/5) - 1, 4(s/5) + 2 \rangle = \left\langle \frac{3}{5}s - 1, \frac{4}{5}s + 2 \right\rangle.$$

Clearly, as shown in Figure 10.29, the graph of  $\vec{r}$  is a line, where  $t = 0$  corresponds to the point  $(-1, 2)$ . What point on the line is 2 units away from this initial point? We find it with  $s(2) = \langle 1/5, 18/5 \rangle$ .

Is the point  $(1/5, 18/5)$  really 2 units away from  $(-1, 2)$ ? We use the Distance Formula to check:

$$d = \sqrt{\left(\frac{1}{5} - (-1)\right)^2 + \left(\frac{18}{5} - 2\right)^2} = \sqrt{\frac{36}{25} + \frac{64}{25}} = \sqrt{4} = 2.$$

Yes,  $s(2)$  is indeed 2 units away, in the direction of travel, from the initial point.

Things worked out very nicely in Example 222; we were able to establish directly that  $s = 5t$ . Usually, the arc length parameter is much more difficult to describe in terms of  $t$ , a result of integrating a square-root. There are a number of things that we can learn about the arc length parameter from Equation (10.1), though, that are incredibly useful.

First, take the derivative of  $s$  with respect to  $t$ . The Fundamental Theorem of Calculus (see Theorem 41) states that

$$\frac{ds}{dt} = s'(t) = \|\vec{r}'(t)\|. \quad (10.2)$$

Letting  $t$  represent time and  $\vec{r}(t)$  represent position, we see that the rate of change of  $s$  with respect to  $t$  is speed; that is, the rate of change of “distance traveled” is speed, which should match our intuition.

The Chain Rule states that

$$\begin{aligned}\frac{d\vec{r}}{dt} &= \frac{d\vec{r}}{ds} \cdot \frac{ds}{dt} \\ \vec{r}'(t) &= \vec{r}'(s) \cdot \|\vec{r}'(t)\|.\end{aligned}$$

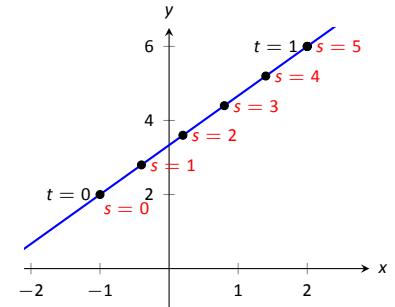


Figure 10.29: Graphing  $\vec{r}$  in Example 222 with parameters  $t$  and  $s$ .

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Notes:

Solving for  $\vec{r}'(s)$ , we have

$$\vec{r}'(s) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \vec{T}(t), \quad (10.3)$$

where  $\vec{T}(t)$  is the unit tangent vector. Equation 10.3 is often misinterpreted, as one is tempted to think it states  $\vec{r}'(t) = \vec{T}(t)$ , but there is a big difference between  $\vec{r}'(s)$  and  $\vec{r}'(t)$ . The key to take from it is that  $\vec{r}'(s)$  is a unit vector. In fact, the following theorem states that this characterizes the arc length parameter.

#### Theorem 49 Arc Length Parameter

Let  $\vec{r}(s)$  be a vector-valued function. The parameter  $s$  is the arc length parameter if, and only if,  $\|\vec{r}'(s)\| = 1$ .

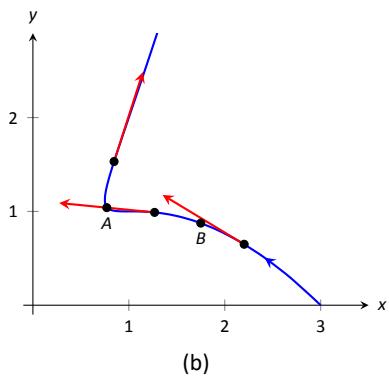
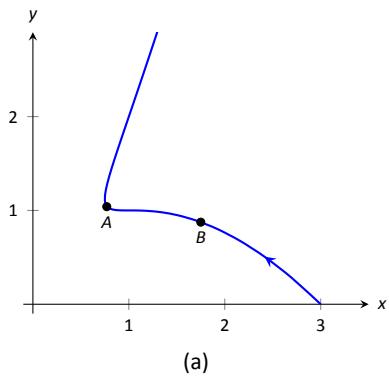


Figure 10.30: Establishing the concept of curvature.

#### Curvature

Consider points  $A$  and  $B$  on the curve graphed in Figure 10.30(a). One can readily argue that the curve curves more sharply at  $A$  than at  $B$ . It is useful to use a number to describe how sharply the curve bends; that number is the **curvature** of the curve.

We derive this number in the following way. Consider Figure 10.30(b), where unit tangent vectors are graphed around points  $A$  and  $B$ . Notice how the direction of the unit tangent vector changes quite a bit near  $A$ , whereas it does not change as much around  $B$ . This leads to an important concept: measuring the rate of change of the unit tangent vector with respect to arc length gives us a measurement of curvature.

#### Definition 53 Curvature

Let  $\vec{r}(s)$  be a vector-valued function where  $s$  is the arc length parameter. The curvature  $\kappa$  of the graph of  $\vec{r}(s)$  is

$$\kappa = \left\| \frac{d\vec{T}}{ds} \right\| = \|\vec{r}'(s)\|.$$

If  $\vec{r}(s)$  is parametrized by the arc length parameter, then

$$\vec{T}(s) = \frac{\vec{r}'(s)}{\|\vec{r}'(s)\|} \quad \text{and} \quad \vec{N}(s) = \frac{\vec{r}''(s)}{\|\vec{r}'(s)\|}.$$

---

Notes:

Having defined  $\|\vec{T}'(s)\| = \kappa$ , we can rewrite the second equation as

$$\vec{T}'(s) = \kappa \vec{N}(s). \quad (10.4)$$

We already knew that  $\vec{T}'(s)$  is in the same direction as  $\vec{N}(s)$ ; that is, we can think of  $\vec{T}(s)$  as being “pulled” in the direction of  $\vec{N}(s)$ . How “hard” is it being pulled? By a factor of  $\kappa$ . When the curvature is large,  $\vec{T}(s)$  is being “pulled hard” and the direction of  $\vec{T}(s)$  changes rapidly. When  $\kappa$  is small,  $T(s)$  is not being pulled hard and hence its direction is not changing rapidly.

We use Definition 53 to find the curvature of the line in Example 222.

**Example 223 Finding the curvature of a line**

Use Definition 53 to find the curvature of  $\vec{r}(t) = \langle 3t - 1, 4t + 2 \rangle$ .

**SOLUTION** In Example 222, we found that the arc length parameter was defined by  $s = 5t$ , so  $\vec{r}(s) = \langle 3s/5 - 1, 4s/5 + 2 \rangle$  parametrized  $\vec{r}$  with the arc length parameter. To find  $\kappa$ , we need to find  $\vec{T}'(s)$ .

$$\begin{aligned} \vec{T}(s) &= \vec{r}'(s) \quad (\text{recall this is a unit vector}) \\ &= \langle 3/5, 4/5 \rangle. \end{aligned}$$

Therefore

$$\vec{T}'(s) = \langle 0, 0 \rangle$$

and

$$\kappa = \|\vec{T}'(s)\| = 0.$$

It probably comes as no surprise that the curvature of a line is 0. (How “curvy” is a line? It is not curvy at all.)

While the definition of curvature is a beautiful mathematical concept, it is nearly impossible to use most of the time; writing  $\vec{r}$  in terms of the arc length parameter is generally very hard. Fortunately, there are other methods of calculating this value that are much easier. There is a tradeoff: the definition is “easy” to understand though hard to compute, whereas these other formulas are easy to compute though it may be hard to understand why they work.

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Notes:

**Theorem 50      Formulas for Curvature**

Let  $C$  be a smooth curve on an open interval  $I$  in the plane or in space.

1. If  $C$  is defined by  $y = f(x)$ , then

$$\kappa = \frac{|f''(x)|}{\left(1 + (f'(x))^2\right)^{3/2}}.$$

2. If  $C$  is defined as a vector-valued function in the plane,  $\vec{r}(t) = \langle x(t), y(t) \rangle$ , then

$$\kappa = \frac{|x'y'' - x''y'|}{((x')^2 + (y')^2)^{3/2}}.$$

3. If  $C$  is defined in space by a vector-valued function  $\vec{r}(t)$ , then

$$\kappa = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} = \frac{\vec{a}(t) \cdot \vec{N}(t)}{\|\vec{v}(t)\|^2}.$$

We practice using these formulas.

**Example 224      Finding the curvature of a circle**

Find the curvature of a circle with radius  $r$ , defined by  $\vec{c}(t) = \langle r \cos t, r \sin t \rangle$ .

**SOLUTION** Before we start, we should expect the curvature of a circle to be constant, and not dependent on  $t$ . (Why?)

We compute  $\kappa$  using the second part of Theorem 50.

$$\begin{aligned}\kappa &= \frac{|(-r \sin t)(-r \sin t) - (-r \cos t)(r \cos t)|}{((-r \sin t)^2 + (r \cos t)^2)^{3/2}} \\ &= \frac{r^2(\sin^2 t + \cos^2 t)}{(r^2(\sin^2 t + \cos^2 t))^{3/2}} \\ &= \frac{r^2}{r^3} = \frac{1}{r}.\end{aligned}$$

We have found that a circle with radius  $r$  has curvature  $\kappa = 1/r$ .

Example 224 gives a great result. Before this example, if we were told “The

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Notes:

curve has a curvature of 5 at point A," we would have no idea what this really meant. Is 5 "big" – does it correspond to a really sharp turn, or a not-so-sharp turn? Now we can think of 5 in terms of a circle with radius 1/5. Knowing the units (inches vs. miles, for instance) allows us to determine how sharply the curve is curving.

Let a point  $P$  on a smooth curve  $C$  be given, and let  $\kappa$  be the curvature of the curve at  $P$ . A circle that:

- passes through  $P$ ,
- lies on the concave side of  $C$ ,
- has a common tangent line as  $C$  at  $P$  and
- has radius  $r = 1/\kappa$  (hence has curvature  $\kappa$ )

is the **osculating circle**, or **circle of curvature**, to  $C$  at  $P$ , and  $r$  is the **radius of curvature**. Figure 10.31 shows the graph of the curve seen earlier in Figure 10.30 and its osculating circles at  $A$  and  $B$ . A sharp turn corresponds to a circle with a small radius; a gradual turn corresponds to a circle with a large radius. Being able to think of curvature in terms of the radius of a circle is very useful. (The word "osculating" comes from a Latin word related to kissing; an osculating circle "kisses" the graph at a particular point. Many beautiful ideas in mathematics have come from studying the osculating circles to a curve.)

### Example 225 Finding curvature

Find the curvature of the parabola defined by  $y = x^2$  at the vertex and at  $x = 1$ .

**SOLUTION** We use the first formula found in Theorem 50.

$$\begin{aligned}\kappa(x) &= \frac{|2|}{(1 + (2x)^2)^{3/2}} \\ &= \frac{2}{(1 + 4x^2)^{3/2}}.\end{aligned}$$

At the vertex ( $x = 0$ ), the curvature is  $\kappa = 2$ . At  $x = 1$ , the curvature is  $\kappa = 2/(5)^{3/2} \approx 0.179$ . So at  $x = 0$ , the curvature of  $y = x^2$  is that of a circle of radius  $1/2$ ; at  $x = 1$ , the curvature is that of a circle with radius  $\approx 1/0.179 \approx 5.59$ . This is illustrated in Figure 10.32. At  $x = 3$ , the curvature is 0.009; the graph is nearly straight as the curvature is very close to 0.

### Example 226 Finding curvature

Find where the curvature of  $\vec{r}(t) = \langle t, t^2, 2t^3 \rangle$  is maximized.

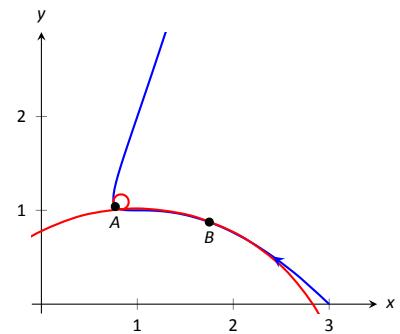


Figure 10.31: Illustrating the osculating circles for the curve seen in Figure 10.30.

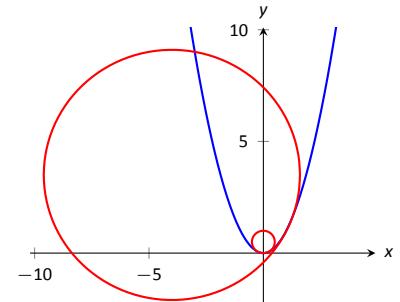


Figure 10.32: Examining the curvature of  $y = x^2$ .

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Notes:

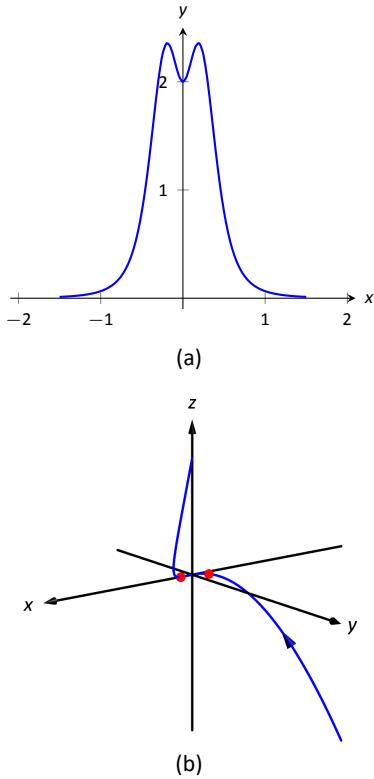


Figure 10.33: Understanding the curvature of a curve in space.

**SOLUTION** We use the third formula in Theorem 50 as  $\vec{r}(t)$  is defined in space. We leave it to the reader to verify that

$$\vec{r}'(t) = \langle 1, 2t, 6t^2 \rangle, \quad \vec{r}''(t) = \langle 0, 2, 12t \rangle, \quad \text{and} \quad \vec{r}'(t) \times \vec{r}''(t) = \langle 12t^2, -12t, 2 \rangle.$$

Thus

$$\begin{aligned}\kappa(t) &= \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} \\ &= \frac{\|\langle 12t^2, -12t, 2 \rangle\|}{\|\langle 1, 2t, 6t^2 \rangle\|^3} \\ &= \frac{\sqrt{144t^4 + 144t^2 + 4}}{\left(\sqrt{1 + 4t^2 + 36t^4}\right)^3}\end{aligned}$$

While this is not a particularly “nice” formula, it does explicitly tell us what the curvature is at a given  $t$  value. To maximize  $\kappa(t)$ , we should solve  $\kappa'(t) = 0$  for  $t$ . This is doable, but very time consuming. Instead, consider the graph of  $\kappa(t)$  as given in Figure 10.33(a). We see that  $\kappa$  is maximized at two  $t$  values; using a numerical solver, we find these values are  $t \approx \pm 0.189$ . In part (b) of the figure we graph  $\vec{r}(t)$  and indicate the points where curvature is maximized.

### Curvature and Motion

Let  $\vec{r}(t)$  be a position function of an object, with velocity  $\vec{v}(t) = \vec{r}'(t)$  and acceleration  $\vec{a}(t) = \vec{r}''(t)$ . In Section 10.4 we established that acceleration is in the plane formed by  $\vec{T}(t)$  and  $\vec{N}(t)$ , and that we can find scalars  $a_T$  and  $a_N$  such that

$$\vec{a}(t) = a_T \vec{T}(t) + a_N \vec{N}(t).$$

Theorem 48 gives formulas for  $a_T$  and  $a_N$ :

$$a_T = \frac{d}{dt} \left( \|\vec{v}(t)\| \right) \quad \text{and} \quad a_N = \frac{\|\vec{v}(t) \times \vec{a}(t)\|}{\|\vec{v}(t)\|}.$$

We understood that the amount of acceleration in the direction of  $\vec{T}$  relates only to how the speed of the object is changing, and that the amount of acceleration in the direction of  $\vec{N}$  relates to how the direction of travel of the object is changing. (That is, if the object travels at constant speed,  $a_T = 0$ ; if the object travels in a constant direction,  $a_N = 0$ .)

In Equation (10.2) at the beginning of this section, we found  $s'(t) = \|\vec{v}(t)\|$ . We can combine this fact with the above formula for  $a_T$  to write

$$a_T = \frac{d}{dt} \left( \|\vec{v}(t)\| \right) = \frac{d}{dt} (s'(t)) = s''(t).$$

---

Notes:

Since  $s'(t)$  is speed,  $s''(t)$  is the rate at which speed is changing with respect to time. We see once more that the component of acceleration in the direction of travel relates only to speed, not to a change in direction.

Now compare the formula for  $a_N$  above to the formula for curvature in Theorem 50:

$$a_N = \frac{\|\vec{v}(t) \times \vec{a}(t)\|}{\|\vec{v}(t)\|} \quad \text{and} \quad \kappa = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} = \frac{\|\vec{v}(t) \times \vec{a}(t)\|}{\|\vec{v}(t)\|^3}.$$

Thus

$$\begin{aligned} a_N &= \kappa \|\vec{v}(t)\|^2 \\ &= \kappa (s'(t))^2 \end{aligned} \tag{10.5}$$

This last equation shows that the component of acceleration that changes the object's direction is dependent on two things: the curvature of the path and the speed of the object.

Imagine driving a car in a clockwise circle. You will naturally feel a force pushing you towards the door (more accurately, the door is pushing you as the car is turning and you want to travel in a straight line). If you keep the radius of the circle constant but speed up (i.e., increasing  $s'(t)$ ), the door pushes harder against you ( $a_N$  has increased). If you keep your speed constant but tighten the turn (i.e., increase  $\kappa$ ), once again the door will push harder against you.

Putting our new formulas for  $a_T$  and  $a_N$  together, we have

$$\vec{a}(t) = s''(t)\vec{T}(t) + \kappa \|\vec{v}(t)\|^2 \vec{N}(t).$$

This is not a particularly practical way of finding  $a_T$  and  $a_N$ , but it reveals some great concepts about how acceleration interacts with speed and the shape of a curve.

### Example 227 Curvature and road design

The minimum radius of the curve in a highway cloverleaf is determined by the operating speed, as given in the table in Figure 10.34. For each curve and speed, compute  $a_N$ .

**SOLUTION** Using Equation (10.5), we can compute the acceleration normal to the curve in each case. We start by converting each speed from "miles per hour" to "feet per second" by multiplying by  $5280/3600$ .

Operating Speed (mph)	Minimum Radius (ft)
35	310
40	430
45	540

Figure 10.34: Operating speed and minimum radius in highway cloverleaf design.

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Notes:

35mph, 310ft  $\Rightarrow 51.33\text{ft/s}$ ,  $\kappa = 1/310$

$$\begin{aligned}a_N &= \kappa \|\vec{v}(t)\|^2 \\&= \frac{1}{310} (51.33)^2 \\&= 8.50\text{ft/s}^2.\end{aligned}$$

40mph, 430ft  $\Rightarrow 58.67\text{ft/s}$ ,  $\kappa = 1/430$

$$\begin{aligned}a_N &= \frac{1}{430} (58.67)^2 \\&= 8.00\text{ft/s}^2.\end{aligned}$$

45mph, 540ft  $\Rightarrow 66\text{ft/s}$ ,  $\kappa = 1/540$

$$\begin{aligned}a_N &= \frac{1}{540} (66)^2 \\&= 8.07\text{ft/s}^2.\end{aligned}$$

Note that each acceleration is similar; this is by design. Considering the classic “Force = mass  $\times$  acceleration” formula, this acceleration must be kept small in order for the tires of a vehicle to keep a “grip” on the road. If one travels on a turn of radius 310ft at a rate of 50mph, the acceleration is double, at  $17.35\text{ft/s}^2$ . If the acceleration is too high, the frictional force created by the tires may not be enough to keep the car from sliding. Civil engineers routinely compute a “safe” design speed, then subtract 5-10mph to create the posted speed limit for additional safety.

We end this chapter with a reflection on what we’ve covered. We started with vector-valued functions, which may have seemed at the time to be just another way of writing parametric equations. However, we have seen that the vector perspective has given us great insight into the behavior of functions and the study of motion. Vector-valued position functions convey displacement, distance traveled, speed, velocity, acceleration and curvature information, each of which has great importance in science and engineering.

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Notes:

# Exercises 10.5

## Terms and Concepts

1. It is common to describe position in terms of both \_\_\_\_\_ and/or \_\_\_\_\_.
2. A measure of the “curviness” of a curve is \_\_\_\_\_.
3. Give two shapes with constant curvature.
4. Describe in your own words what an “osculating circle” is.
5. Complete the identity:  $\vec{T}'(s) = \underline{\hspace{2cm}} \vec{N}(s)$ .
6. Given a position function  $\vec{r}(t)$ , how are  $a_T$  and  $a_N$  affected by the curvature?

## Problems

In Exercises 7 – 10 , a position function  $\vec{r}(t)$  is given, where  $t = 0$  corresponds to the initial position. Find the arc length parameter  $s$ , and rewrite  $\vec{r}(t)$  in terms of  $s$ ; that is, find  $\vec{r}(s)$ .

7.  $\vec{r}(t) = \langle 2t, t, -2t \rangle$
8.  $\vec{r}(t) = \langle 7 \cos t, 7 \sin t \rangle$
9.  $\vec{r}(t) = \langle 3 \cos t, 3 \sin t, 2t \rangle$
10.  $\vec{r}(t) = \langle 5 \cos t, 13 \sin t, 12 \cos t \rangle$

In Exercises 11 – 22 , a curve  $C$  is described along with 2 points on  $C$ .

- (a) Using a sketch, determine at which of these points the curvature is greater.
- (b) Find the curvature  $\kappa$  of  $C$ , and evaluate  $\kappa$  at each of the 2 given points.
11.  $C$  is defined by  $y = x^3 - x$ ; points given at  $x = 0$  and  $x = 1/2$ .
12.  $C$  is defined by  $y = \frac{1}{x^2 + 1}$ ; points given at  $x = 0$  and  $x = 2$ .
13.  $C$  is defined by  $y = \cos x$ ; points given at  $x = 0$  and  $x = \pi/2$ .
14.  $C$  is defined by  $y = \sqrt{1 - x^2}$  on  $(-1, 1)$ ; points given at  $x = 0$  and  $x = 1/2$ .
15.  $C$  is defined by  $\vec{r}(t) = \langle \cos t, \sin(2t) \rangle$ ; points given at  $t = 0$  and  $t = \pi/4$ .

16.  $C$  is defined by  $\vec{r}(t) = \langle \cos^2 t, \sin t \cos t \rangle$ ; points given at  $t = 0$  and  $t = \pi/3$ .

17.  $C$  is defined by  $\vec{r}(t) = \langle t^2 - 1, t^3 - t \rangle$ ; points given at  $t = 0$  and  $t = 5$ .

18.  $C$  is defined by  $\vec{r}(t) = \langle \tan t, \sec t \rangle$ ; points given at  $t = 0$  and  $t = \pi/6$ .

19.  $C$  is defined by  $\vec{r}(t) = \langle 4t + 2, 3t - 1, 2t + 5 \rangle$ ; points given at  $t = 0$  and  $t = 1$ .

20.  $C$  is defined by  $\vec{r}(t) = \langle t^3 - t, t^3 - 4, t^2 - 1 \rangle$ ; points given at  $t = 0$  and  $t = 1$ .

21.  $C$  is defined by  $\vec{r}(t) = \langle 3 \cos t, 3 \sin t, 2t \rangle$ ; points given at  $t = 0$  and  $t = \pi/2$ .

22.  $C$  is defined by  $\vec{r}(t) = \langle 5 \cos t, 13 \sin t, 12 \cos t \rangle$ ; points given at  $t = 0$  and  $t = \pi/2$ .

In Exercises 23 – 26 , find the value of  $x$  or  $t$  where curvature is maximized.

23.  $y = \frac{1}{6}x^3$

24.  $y = \sin x$

25.  $\vec{r}(t) = \langle t^2 + 2t, 3t - t^2 \rangle$

26.  $\vec{r}(t) = \langle t, 4/t, 3/t \rangle$

In Exercises 27 – 30 , find the radius of curvature at the indicated value.

27.  $y = \tan x$ , at  $x = \pi/4$

28.  $y = x^2 + x - 3$ , at  $x = \pi/4$

29.  $\vec{r}(t) = \langle \cos t, \sin(3t) \rangle$ , at  $t = 0$

30.  $\vec{r}(t) = \langle 5 \cos(3t), t \rangle$ , at  $t = 0$

In Exercises 31 – 34 , find the equation of the osculating circle to the curve at the indicated  $t$ -value.

31.  $\vec{r}(t) = \langle t, t^2 \rangle$ , at  $t = 0$

32.  $\vec{r}(t) = \langle 3 \cos t, \sin t \rangle$ , at  $t = 0$

33.  $\vec{r}(t) = \langle 3 \cos t, \sin t \rangle$ , at  $t = \pi/2$

34.  $\vec{r}(t) = \langle t^2 - t, t^2 + t \rangle$ , at  $t = 0$



# 11: FUNCTIONS OF SEVERAL VARIABLES

A function of the form  $y = f(x)$  is a function of a single variable; given a value of  $x$ , we can find a value  $y$ . Even the vector-valued functions of Chapter 10 are single-variable functions; the input is a single variable though the output is a vector.

There are many situations where a desired quantity is a function of two or more variables. For instance, wind chill is measured by knowing the temperature and wind speed; the volume of a gas can be computed knowing the pressure and temperature of the gas; to compute a baseball player's batting average, one needs to know the number of hits and the number of at-bats.

This chapter studies **multivariable** functions, that is, functions with more than one input.

## 11.1 Introduction to Multivariable Functions

### Definition 54 Function of Two Variables

Let  $D$  be a subset of  $\mathbb{R}^2$ . A **function of two variables** is a rule that assigns each pair  $(x, y)$  in  $D$  a value  $z = f(x, y)$  in  $\mathbb{R}$ .  $D$  is the **domain** of  $f$ ; the set of all outputs of  $f$  is the **range**.

### Example 228 Understanding a function of two variables

Let  $z = f(x, y) = x^2 - y$ . Evaluate  $f(1, 2)$ ,  $f(2, 1)$ , and  $f(-2, 4)$ ; find the domain and range of  $f$ .

**SOLUTION** Using the definition  $f(x, y) = x^2 - y$ , we have:

$$f(1, 2) = 1^2 - 2 = -1$$

$$f(2, 1) = 2^2 - 1 = 3$$

$$f(-2, 4) = (-2)^2 - 4 = 0$$

The domain is not specified, so we take it to be all possible pairs in  $\mathbb{R}^2$  for which  $f$  is defined. In this example,  $f$  is defined for *all* pairs  $(x, y)$ , so the domain  $D$  of  $f$  is  $\mathbb{R}^2$ .

The output of  $f$  can be made as large or small as possible; any real number  $r$  can be the output. (In fact, given any real number  $r$ ,  $f(0, -r) = r$ .) So the range  $R$  of  $f$  is  $\mathbb{R}$ .

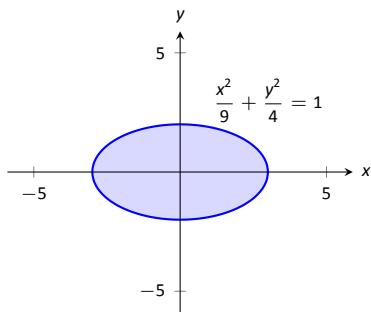


Figure 11.1: Illustrating the domain of  $f(x, y)$  in Example 229.

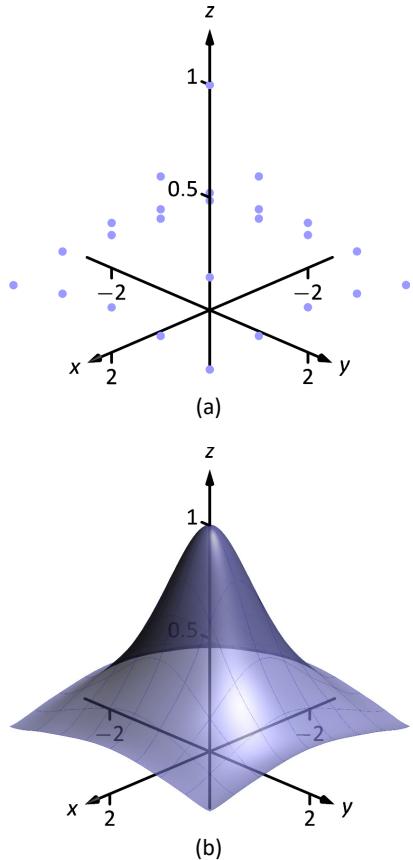


Figure 11.2: Graphing a function of two variables.

### Example 229 Understanding a function of two variables

Let  $f(x, y) = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}$ . Find the domain and range of  $f$ .

**SOLUTION** The domain is all pairs  $(x, y)$  allowable as input in  $f$ . Because of the square-root, we need  $(x, y)$  such that  $0 \leq 1 - \frac{x^2}{9} - \frac{y^2}{4}$ :

$$\begin{aligned} 0 &\leq 1 - \frac{x^2}{9} - \frac{y^2}{4} \\ \frac{x^2}{9} + \frac{y^2}{4} &\leq 1 \end{aligned}$$

The above equation describes the interior of an ellipse as shown in Figure 11.1. We can represent the domain  $D$  graphically with the figure; in set notation, we can write  $D = \{(x, y) | \frac{x^2}{9} + \frac{y^2}{4} \leq 1\}$ .

The range is the set of all possible output values. The square-root ensures that all output is  $\geq 0$ . Since the  $x$  and  $y$  terms are squared, then subtracted, inside the square-root, the largest output value comes at  $x = 0, y = 0$ :  $f(0, 0) = 1$ . Thus the range  $R$  is the interval  $[0, 1]$ .

### Graphing Functions of Two Variables

The **graph** of a function  $f$  of two variables is the set of all points  $(x, y, f(x, y))$  where  $(x, y)$  is in the domain of  $f$ . This creates a **surface** in space.

One can begin sketching a graph by plotting points, but this has limitations. Consider Figure 11.2(a) where 25 points have been plotted of  $f(x, y) = \frac{1}{x^2 + y^2 + 1}$ . More points have been plotted than one would reasonably want to do by hand, yet it is not clear at all what the graph of the function looks like. Technology allows us to plot lots of points, connect adjacent points with lines and add shading to create a graph like Figure 11.2b which does a far better job of illustrating the behavior of  $f$ .

While technology is readily available to help us graph functions of two variables, there is still a paper-and-pencil approach that is useful to understand and master as it, combined with high-quality graphics, gives one great insight into the behavior of a function. This technique is known as sketching **level curves**.

### Level Curves

It may be surprising to find that the problem of representing a three dimensional surface on paper is familiar to most people (they just don't realize it). Topographical maps, like the one shown in Figure 11.3, represent the surface of Earth by indicating points with the same elevation with **contour lines**. The

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Notes:

elevations marked are equally spaced; in this example, each thin line indicates an elevation change in 50ft increments and each thick line indicates a change of 200ft. When lines are drawn close together, elevation changes rapidly (as one does not have to travel far to rise 50ft). When lines are far apart, such as near "Aspen Campground," elevation changes more gradually as one has to walk farther to rise 50ft.

Given a function  $z = f(x, y)$ , we can draw a “topographical map” of  $f$  by drawing **level curves** (or, contour lines). A level curve at  $z = c$  is a curve in the  $x$ - $y$  plane such that for all points  $(x, y)$  on the curve,  $f(x, y) = c$ .

When drawing level curves, it is important that the  $c$  values are spaced equally apart as that gives the best insight to how quickly the “elevation” is changing. Examples will help one understand this concept.

## Example 230 Drawing Level Curves

Let  $f(x, y) = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}$ . Find the level curves of  $f$  for  $c = 0, 0.2, 0.4, 0.6, 0.8$  and  $1$ .

**SOLUTION** Consider first  $c = 0$ . The level curve for  $c = 0$  is the set of all points  $(x, y)$  such that  $0 = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}$ . Squaring both sides gives us

$$\frac{x^2}{9} + \frac{y^2}{4} = 1,$$

an ellipse centered at  $(0, 0)$  with horizontal major axis of length 6 and minor axis of length 4. Thus for any point  $(x, y)$  on this curve,  $f(x, y) = 0$ .

Now consider the level curve for  $c = 0.2$

$$\begin{aligned}0.2 &= \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}} \\0.04 &= 1 - \frac{x^2}{9} - \frac{y^2}{4} \\ \frac{x^2}{9} + \frac{y^2}{4} &= 0.96 \\ \frac{x^2}{8.64} + \frac{y^2}{3.84} &= 1.\end{aligned}$$

This is also an ellipse, where  $a = \sqrt{8.64} \approx 2.94$  and  $b = \sqrt{3.84} \approx 1.96$ .

## Notes:

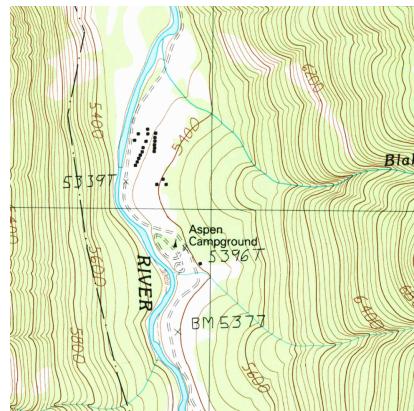


Figure 11.3: A topographical map displays elevation by drawing contour lines, along with the elevation is constant.

Sample taken from the public domain USGS Digital Raster Graphics,  
<http://topmaps.usgs.gov/drg/>.

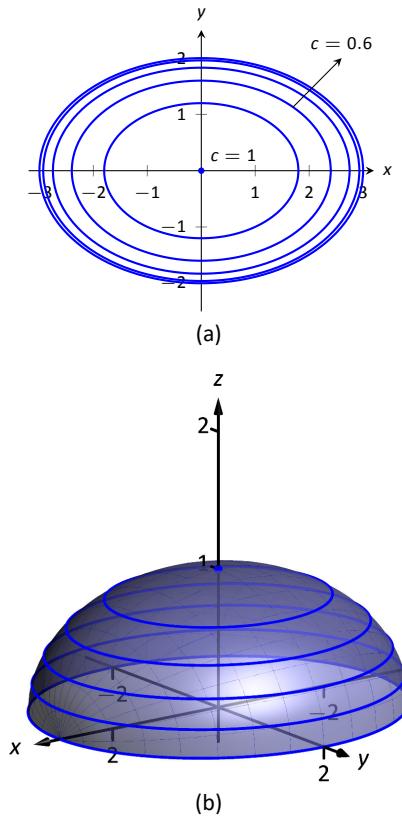


Figure 11.4: Graphing the level curves in Example 230.

In general, for  $z = c$ , the level curve is:

$$\begin{aligned} c &= \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}} \\ c^2 &= 1 - \frac{x^2}{9} - \frac{y^2}{4} \\ \frac{x^2}{9} + \frac{y^2}{4} &= 1 - c^2 \\ \frac{x^2}{9(1 - c^2)} + \frac{y^2}{4(1 - c^2)} &= 1, \end{aligned}$$

ellipses that are decreasing in size as  $c$  increases. A special case is when  $c = 1$ ; there the ellipse is just the point  $(0, 0)$ .

The level curves are shown in Figure 11.4(a). Note how the level curves for  $c = 0$  and  $c = 0.2$  are very, very close together: this indicates that  $f$  is growing rapidly along those curves.

In Figure 11.4(b), the curves are drawn on a graph of  $f$  in space. Note how the elevations are evenly spaced. Near the level curves of  $c = 0$  and  $c = 0.2$  we can see that  $f$  indeed is growing quickly.

### Example 231 Analyzing Level Curves

Let  $f(x, y) = \frac{x+y}{x^2+y^2+1}$ . Find the level curves for  $z = c$ .

**SOLUTION** We begin by setting  $f(x, y) = c$  for an arbitrary  $c$  and seeing if algebraic manipulation of the equation reveals anything significant.

$$\begin{aligned} \frac{x+y}{x^2+y^2+1} &= c \\ x+y &= c(x^2+y^2+1). \end{aligned}$$

We recognize this as a circle, though the center and radius are not yet clear. By completing the square, we can obtain:

$$\left(x - \frac{1}{2c}\right)^2 + \left(y - \frac{1}{2c}\right)^2 = \frac{1}{2c^2} - 1,$$

a circle centered at  $(1/(2c), 1/(2c))$  with radius  $\sqrt{1/(2c^2) - 1}$ , where  $|c| < 1/\sqrt{2}$ . The level curves for  $c = \pm 0.2, \pm 0.4$  and  $\pm 0.6$  are sketched in Figure 11.5(a). To help illustrate “elevation,” we use thicker lines for  $c$  values near 0, and dashed lines indicate where  $c < 0$ .

There is one special level curve, when  $c = 0$ . The level curve in this situation is  $x+y=0$ , the line  $y=-x$ .

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Notes:

In Figure 11.5(b) we see a graph of the surface. Note how the  $y$ -axis is pointing away from the viewer to more closely resemble the orientation of the level curves in (a).

Seeing the level curves helps us understand the graph. For instance, the graph does not make it clear that one can “walk” along the line  $y = -x$  without elevation change, though the level curve does.

## Functions of Three Variables

We extend our study of multivariable functions to functions of three variables. (One can make a function of as many variables as one likes; we limit our study to three variables.)

### Definition 55 Function of Three Variables

Let  $D$  be a subset of  $\mathbb{R}^3$ . A **function  $f$  of three variables** is a rule that assigns each triple  $(x, y, z)$  in  $D$  a value  $w = f(x, y, z)$  in  $\mathbb{R}$ .  $D$  is the **domain** of  $f$ ; the set of all outputs of  $f$  is the **range**.

Note how this definition closely resembles that of Definition 54.

### Example 232 Understanding a function of three variables

Let  $f(x, y, z) = \frac{x^2 + z + 3 \sin y}{x + 2y - z}$ . Evaluate  $f$  at the point  $(3, 0, 2)$  and find the domain and range of  $f$ .

$$\text{SOLUTION} \quad f(3, 0, 2) = \frac{3^2 + 2 + 3 \sin 0}{3 + 2(0) - 2} = 11.$$

As the domain of  $f$  is not specified, we take it to be the set of all triples  $(x, y, z)$  for which  $f(x, y, z)$  is defined. As we cannot divide by 0, we find the domain  $D$  is

$$D = \{(x, y, z) \mid x + 2y - z \neq 0\}.$$

We recognize that the set of all points in  $\mathbb{R}^3$  that are not in  $D$  form a plane in space that passes through the origin (with normal vector  $\langle 1, 2, -1 \rangle$ ).

We determine the range  $R$  is  $\mathbb{R}$ ; that is, all real numbers are possible outputs of  $f$ . There is no set way of establishing this. Rather, to get numbers near 0 we can let  $y = 0$  and choose  $z \approx -x^2$ . To get numbers of arbitrarily large magnitude, we can let  $z \approx x + 2y$ .

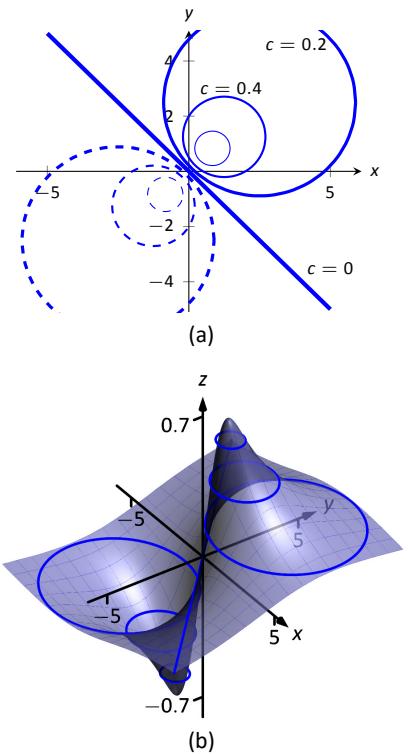


Figure 11.5: Graphing the level curves in Example 231.

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Notes:

## Level Surfaces

It is very difficult to produce a meaningful graph of a function of three variables. A function of *one* variable is a *curve* drawn in 2 dimensions; a function of *two* variables is a *surface* drawn in 3 dimensions; a function of *three* variables is a *hypersurface* drawn in 4 dimensions.

There are a few techniques one can employ to try to “picture” a graph of three variables. One is an analogue of level curves: **level surfaces**. Given  $w = f(x, y, z)$ , the level surface at  $w = c$  is the surface in space formed by all points  $(x, y, z)$  where  $f(x, y, z) = c$ .

### Example 233 Finding level surfaces

If a point source  $S$  is radiating energy, the intensity  $I$  at a given point  $P$  in space is inversely proportional to the square of the distance between  $S$  and  $P$ . That is, when  $S = (0, 0, 0)$ ,  $I(x, y, z) = \frac{k}{x^2 + y^2 + z^2}$  for some constant  $k$ .

Let  $k = 1$ ; find the level surfaces of  $I$ .

**SOLUTION** We can (mostly) answer this question using “common sense.” If energy (say, in the form of light) is emanating from the origin, its intensity will be the same at all points equidistant from the origin. That is, at any point on the surface of a sphere centered at the origin, the intensity should be the same. Therefore, the level surfaces are spheres.

We now find this mathematically. The level surface at  $I = c$  is defined by

$$c = \frac{1}{x^2 + y^2 + z^2}.$$

A small amount of algebra reveals

$$x^2 + y^2 + z^2 = \frac{1}{c}.$$

Given an intensity  $c$ , the level surface  $I = c$  is a sphere of radius  $1/\sqrt{c}$ , centered at the origin.

Figure 11.6 gives a table of the radii of the spheres for given  $c$  values. Normally one would use equally spaced  $c$  values, but these values have been chosen purposefully. At a distance of 0.25 from the point source, the intensity is 16; to move to a point of half that intensity, one just moves out 0.1 to 0.35 – not much at all. To again halve the intensity, one moves 0.15, a little more than before.

Note how each time the intensity is halved, the distance required to move away grows. We conclude that the closer one is to the source, the more rapidly the intensity changes.

In the next section we apply the concepts of limits to functions of two or more variables.

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Notes:

# Exercises 11.1

## Terms and Concepts

1. Give two examples (other than those given in the text) of “real world” functions that require more than one input.
2. The graph of a function of two variables is a \_\_\_\_\_.
3. Most people are familiar with the concept of level curves in the context of \_\_\_\_\_ maps.
4. T/F: Along a level curve, the output of a function does not change.
5. The analogue of a level curve for functions of three variables is a level \_\_\_\_\_.
6. What does it mean when level curves are close together? Far apart?

## Problems

In Exercises 7 – 14, give the domain and range of the multi-variable function.

$$7. f(x, y) = x^2 + y^2 + 2$$
$$8. f(x, y) = x + 2y$$
$$9. f(x, y) = x - 2y$$
$$10. f(x, y) = \frac{1}{x + 2y}$$

$$11. f(x, y) = \frac{1}{x^2 + y^2 + 1}$$
$$12. f(x, y) = \sin x \cos y$$
$$13. f(x, y) = \sqrt{9 - x^2 - y^2}$$
$$14. f(x, y) = \frac{1}{\sqrt{x^2 + y^2 - 9}}$$

In Exercises 15 – 22, describe in words and sketch the level curves for the function and given  $c$  values.

$$15. f(x, y) = 3x - 2y; c = -2, 0, 2$$

16.  $f(x, y) = x^2 - y^2; c = -1, 0, 1$
17.  $f(x, y) = x - y^2; c = -2, 0, 2$
18.  $f(x, y) = \frac{1 - x^2 - y^2}{2y - 2x}; c = -2, 0, 2$
19.  $f(x, y) = \frac{2x - 2y}{x^2 + y^2 + 1}; c = -1, 0, 1$
20.  $f(x, y) = \frac{y - x^3 - 1}{x}; c = -3, -1, 0, 1, 3$
21.  $f(x, y) = \sqrt{x^2 + 4y^2}; c = 1, 2, 3, 4$
22.  $f(x, y) = x^2 + 4y^2; c = 1, 2, 3, 4$

In Exercises 23 – 26, give the domain and range of the functions of three variables.

$$23. f(x, y, z) = \frac{x}{x + 2y - 4z}$$
$$24. f(x, y, z) = \frac{1}{1 - x^2 - y^2 - z^2}$$

$$25. f(x, y, z) = \sqrt{z - x^2 + y^2}$$
$$26. f(x, y, z) = z^2 \sin x \cos y$$

In Exercises 27 – 30, describe the level surfaces of the given functions of three variables.

$$27. f(x, y, z) = x^2 + y^2 + z^2$$
$$28. f(x, y, z) = z - x^2 + y^2$$
$$29. f(x, y, z) = \frac{x^2 + y^2}{z}$$

$$30. f(x, y, z) = \frac{z}{x - y}$$

31. Compare the level curves of Exercises 21 and 22. How are they similar, and how are they different? Each surface is a quadric surface; describe how the level curves are consistent with what we know about each surface.

## 11.2 Limits and Continuity of Multivariable Functions

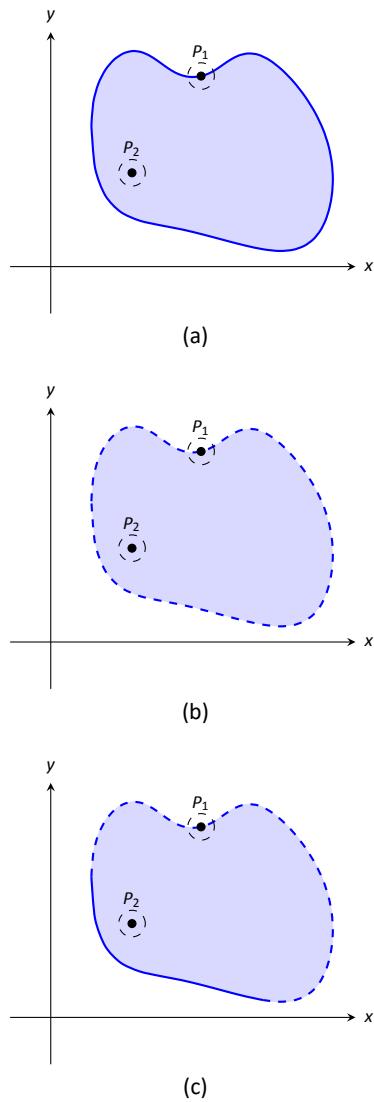


Figure 11.7: Illustrating open and closed sets in the  $x$ - $y$  plane.

We continue with the pattern we have established in this text: after defining a new kind of function, we apply calculus ideas to it. The previous section defined functions of two and three variables; this section investigates what it means for these functions to be “continuous.”

We begin with a series of definitions. We are used to “open intervals” such as  $(1, 3)$ , which represents the set of all  $x$  such that  $1 < x < 3$ , and “closed intervals” such as  $[1, 3]$ , which represents the set of all  $x$  such that  $1 \leq x \leq 3$ . We need analogous definitions for open and closed sets in the  $x$ - $y$  plane.

**Definition 56      Open Disk, Boundary and Interior Points, Open and Closed Sets, Bounded Sets**

An **open disk**  $B$  in  $\mathbb{R}^2$  centered at  $(x_0, y_0)$  with radius  $r$  is the set of all points  $(x, y)$  such that  $\sqrt{(x - x_0)^2 + (y - y_0)^2} < r$ .

Let  $S$  be a set of points in  $\mathbb{R}^2$ . A point  $P$  in  $\mathbb{R}^2$  is a **boundary point** of  $S$  if all open disks centered at  $P$  contain both points in  $S$  and points not in  $S$ .

A point  $P$  in  $S$  is an **interior point** of  $S$  if there is an open disk centered at  $P$  that contains only points in  $S$ .

A set  $S$  is **open** if every point in  $S$  is an interior point.

A set  $S$  is **closed** if it contains all of its boundary points.

A set  $S$  is **bounded** if there is an  $M > 0$  such that the open disk, centered at the origin with radius  $M$ , contains  $S$ . A set that is not bounded is **unbounded**.

Figure 11.7 shows several sets in the  $x$ - $y$  plane. In each set, point  $P_1$  lies on the boundary of the set as all open disks centered there contain both points in, and not in, the set. In contrast, point  $P_2$  is an interior point for there is an open disk centered there that lies entirely within the set.

The set depicted in Figure 11.7(a) is a closed set as it contains all of its boundary points. The set in (b) is open, for all of its points are interior points (or, equivalently, it does not contain any of its boundary points). The set in (c) is neither open nor closed as it contains some of its boundary points.

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Notes:

**Example 234 Determining open/closed, bounded/unbounded**

Determine if the domain of the function  $f(x, y) = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}$  is open, closed, or neither, and if it is bounded.

**SOLUTION** This domain of this function was found in Example 229 to be  $D = \{(x, y) \mid \frac{x^2}{9} + \frac{y^2}{4} \leq 1\}$ , the region *bounded* by the ellipse  $\frac{x^2}{9} + \frac{y^2}{4} = 1$ . Since the region includes the boundary (indicated by the use of " $\leq$ "), the set contains all of its boundary points and hence is closed. The region is bounded as a disk of radius 4, centered at the origin, contains  $D$ .

**Example 235 Determining open/closed, bounded/unbounded**

Determine if the domain of  $f(x, y) = \frac{1}{x-y}$  is open, closed, or neither.

**SOLUTION** As we cannot divide by 0, we find the domain to be  $D = \{(x, y) \mid x - y \neq 0\}$ . In other words, the domain is the set of all points  $(x, y)$  *not* on the line  $y = x$ .

The domain is sketched in Figure 11.8. Note how we can draw an open disk around any point in the domain that lies entirely inside the domain, and also note how the only boundary points of the domain are the points on the line  $y = x$ . We conclude the domain is an open set. The set is unbounded.

## Limits

Recall a pseudo-definition of the limit of a function of one variable: " $\lim_{x \rightarrow c} f(x) = L$ " means that if  $x$  is "really close" to  $c$ , then  $f(x)$  is "really close" to  $L$ . A similar pseudo-definition holds for functions of two variables. We'll say that

$$\text{"} \lim_{(x,y) \rightarrow (x_0, y_0)} f(x, y) = L \text{"}$$

means "if the point  $(x, y)$  is really close to the point  $(x_0, y_0)$ , then  $f(x, y)$  is really close to  $L$ ." The formal definition is given below.

**Definition 57 Limit of a Function of Two Variables**

Let  $S$  be an open set containing  $(x_0, y_0)$ , and let  $f$  be a function of two variables defined on  $S$ , except possibly at  $(x_0, y_0)$ . The **limit** of  $f(x, y)$  as  $(x, y)$  approaches  $(x_0, y_0)$  is  $L$ , denoted

$$\lim_{(x,y) \rightarrow (x_0, y_0)} f(x, y) = L,$$

means that given any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for all  $(x, y) \neq (x_0, y_0)$ , if  $(x, y)$  is in the open disk centered at  $(x_0, y_0)$  with radius  $\delta$ , then  $|f(x, y) - L| < \varepsilon$ .

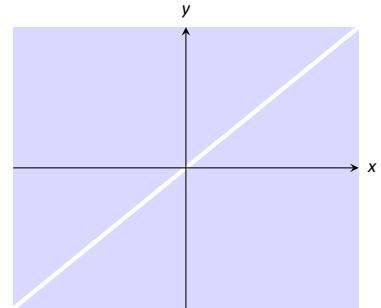


Figure 11.8: Sketching the domain of the function in Example 235.

Notes:

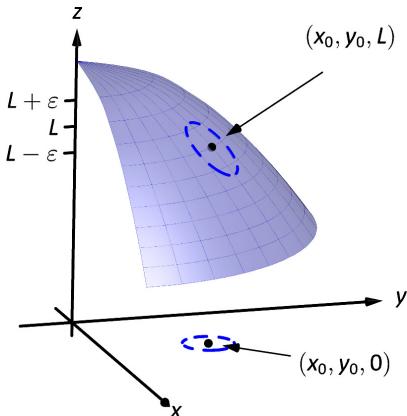


Figure 11.9: **Illustrating the definition of a limit.** The open disk in the  $x$ - $y$  plane has radius  $\delta$ . Let  $(x, y)$  be any point in this disk;  $f(x, y)$  is within  $\varepsilon$  of  $L$ .

The concept behind Definition 57 is sketched in Figure 11.9. Given  $\varepsilon > 0$ , find  $\delta > 0$  such that if  $(x, y)$  is any point in the open disk centered at  $(x_0, y_0)$  in the  $x$ - $y$  plane with radius  $\delta$ , then  $f(x, y)$  should be within  $\varepsilon$  of  $L$ .

Computing limits using this definition is rather cumbersome. The following theorem allows us to evaluate limits much more easily.

### Theorem 51 Basic Limit Properties of Functions of Two Variables

Let  $b, x_0, y_0, L$  and  $K$  be real numbers, let  $n$  be a positive integer, and let  $f$  and  $g$  be functions with the following limits:

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y) \rightarrow (x_0,y_0)} g(x,y) = K.$$

The following limits hold.

1. Constants:  $\lim_{(x,y) \rightarrow (x_0,y_0)} b = b$
2. Identity  $\lim_{(x,y) \rightarrow (x_0,y_0)} x = x_0; \lim_{(x,y) \rightarrow (x_0,y_0)} y = y_0$
3. Sums/Differences:  $\lim_{(x,y) \rightarrow (x_0,y_0)} (f(x,y) \pm g(x,y)) = L \pm K$
4. Scalar Multiples:  $\lim_{(x,y) \rightarrow (x_0,y_0)} b \cdot f(x,y) = bL$
5. Products:  $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) \cdot g(x,y) = LK$
6. Quotients:  $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y)/g(x,y) = L/K, (K \neq 0)$
7. Powers:  $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y)^n = L^n$

This theorem, combined with Theorems 2 and 3 of Section 1.3, allows us to evaluate many limits.

### Example 236 Evaluating a limit

Evaluate the following limits:

$$1. \lim_{(x,y) \rightarrow (1,\pi)} \frac{y}{x} + \cos(xy) \quad 2. \lim_{(x,y) \rightarrow (0,0)} \frac{3xy}{x^2 + y^2}$$

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Notes:

**SOLUTION**

1. The aforementioned theorems allow us to simply evaluate  $y/x + \cos(xy)$  when  $x = 1$  and  $y = \pi$ . If an indeterminate form is returned, we must do more work to evaluate the limit; otherwise, the result is the limit. Therefore

$$\begin{aligned}\lim_{(x,y) \rightarrow (1,\pi)} \frac{y}{x} + \cos(xy) &= \frac{\pi}{1} + \cos \pi \\ &= \pi - 1.\end{aligned}$$

2. We attempt to evaluate the limit by substituting 0 in for  $x$  and  $y$ , but the result is the indeterminate form “0/0.” To evaluate this limit, we must “do more work,” but we have not yet learned what “kind” of work to do. Therefore we cannot yet evaluate this limit.

When dealing with functions of a single variable we also considered one-sided limits and stated

$$\lim_{x \rightarrow c} f(x) = L \quad \text{if, and only if,} \quad \lim_{x \rightarrow c^+} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c^-} f(x) = L.$$

That is, the limit is  $L$  if and only if  $f(x)$  approaches  $L$  when  $x$  approaches  $c$  from either direction, the left or the right.

In the plane, there are infinite directions from which  $(x, y)$  might approach  $(x_0, y_0)$ . In fact, we do not have to restrict ourselves to approaching  $(x_0, y_0)$  from a particular direction, but rather we can approach that point along a path that is not a straight line. It is possible to arrive at different limiting values by approaching  $(x_0, y_0)$  along different paths. If this happens, we say that  $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y)$  does not exist (this is analogous to the left and right hand limits of single variable functions not being equal).

Our theorems tell us that we can evaluate most limits quite simply, without worrying about paths. When indeterminate forms arise, the limit may or may not exist. If it does exist, it can be difficult to prove this as we need to show the same limiting value is obtained regardless of the path chosen. The case where the limit does not exist is often easier to deal with, for we can often pick two paths along which the limit is different.

**Example 237      Showing limits do not exist**

1. Show  $\lim_{\substack{(x,y) \rightarrow (0,0) \\ y = mx}} \frac{3xy}{x^2 + y^2}$  does not exist by finding the limits along the lines

---

Notes:

2. Show  $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(xy)}{x+y}$  does not exist by finding the limit along the path  $y = -\sin x$ .

**SOLUTION**

1. Evaluating  $\lim_{(x,y) \rightarrow (0,0)} \frac{3xy}{x^2 + y^2}$  along the lines  $y = mx$  means replace all  $y$ 's with  $mx$  and evaluating the resulting limit:

$$\begin{aligned}\lim_{(x,mx) \rightarrow (0,0)} \frac{3x(mx)}{x^2 + (mx)^2} &= \lim_{x \rightarrow 0} \frac{3mx^2}{x^2(m^2 + 1)} \\ &= \lim_{x \rightarrow 0} \frac{3m}{m^2 + 1} \\ &= \frac{3m}{m^2 + 1}.\end{aligned}$$

While the limit exists for each choice of  $m$ , we get a *different* limit for each choice of  $m$ . That is, along different lines we get differing limiting values, meaning *the* limit does not exist.

2. Let  $f(x, y) = \frac{\sin(xy)}{x+y}$ . We are to show that  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  does not exist by finding the limit along the path  $y = -\sin x$ . First, however, consider the limits found along the lines  $y = mx$  as done above.

$$\begin{aligned}\lim_{(x,mx) \rightarrow (0,0)} \frac{\sin(x(mx))}{x+mx} &= \lim_{x \rightarrow 0} \frac{\sin(mx^2)}{x(m+1)} \\ &= \lim_{x \rightarrow 0} \frac{\sin(mx^2)}{x} \cdot \frac{1}{m+1}.\end{aligned}$$

By applying L'Hôpital's Rule, we can show this limit is 0 *except* when  $m = -1$ , that is, along the line  $y = -x$ . This line is not in the domain of  $f$ , so we have found the following fact: along every line  $y = mx$  in the domain of  $f$ ,  $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$ .

Now consider the limit along the path  $y = -\sin x$ :

$$\lim_{(x, -\sin x) \rightarrow (0,0)} \frac{\sin(-x \sin x)}{x - \sin x} = \lim_{x \rightarrow 0} \frac{\sin(-x \sin x)}{x - \sin x}$$

Now apply L'Hôpital's Rule twice:

$$\begin{aligned}&= \lim_{x \rightarrow 0} \frac{\cos(-x \sin x)(-\sin x - x \cos x)}{1 - \cos x} \quad ("= 0/0") \\ &= \lim_{x \rightarrow 0} \frac{-\sin(-x \sin x)(-\sin x - x \cos x)^2 + \cos(-x \sin x)(-2 \cos x + x \sin x)}{\sin x} \\ &= "2/0" \Rightarrow \text{the limit does not exist.}\end{aligned}$$

Notes:

Step back and consider what we have just discovered. Along any line  $y = mx$  in the domain of the  $f(x, y)$ , the limit is 0. However, along the path  $y = -\sin x$ , which lies in the domain of  $f(x, y)$  for all  $x \neq 0$ , the limit does not exist. Since the limit is not the same along every path to  $(0, 0)$ , we say

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(xy)}{x+y} \text{ does not exist.}$$

### Example 238 Finding a limit

Let  $f(x, y) = \frac{5x^2y^2}{x^2 + y^2}$ . Find  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ .

**SOLUTION** It is relatively easy to show that along any line  $y = mx$ , the limit is 0. This is not enough to prove that the limit exists, as demonstrated in the previous example, but it tells us that if the limit does exist then it must be 0.

To prove the limit is 0, we apply Definition 57. Let  $\varepsilon > 0$  be given. We want to find  $\delta > 0$  such that if  $\sqrt{(x-0)^2 + (y-0)^2} < \delta$ , then  $|f(x, y) - 0| < \varepsilon$ .

Set  $\delta < \sqrt{\varepsilon/5}$ . Note that  $\left| \frac{5y^2}{x^2 + y^2} \right| < 5$  for all  $(x, y) \neq (0, 0)$ , and that if  $\sqrt{x^2 + y^2} < \delta$ , then  $x^2 < \delta^2$ .

Let  $\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2} < \delta$ . Consider  $|f(x, y) - 0|$ :

$$\begin{aligned} |f(x, y) - 0| &= \left| \frac{5x^2y^2}{x^2 + y^2} - 0 \right| \\ &= \left| x^2 \cdot \frac{5y^2}{x^2 + y^2} \right| \\ &< \delta^2 \cdot 5 \\ &< \frac{\varepsilon}{5} \cdot 5 \\ &= \varepsilon. \end{aligned}$$

Thus if  $\sqrt{(x-0)^2 + (y-0)^2} < \delta$  then  $|f(x, y) - 0| < \varepsilon$ , which is what we wanted to show. Thus  $\lim_{(x,y) \rightarrow (0,0)} \frac{5x^2y^2}{x^2 + y^2} = 0$ .

### Continuity

Definition 3 defines what it means for a function of one variable to be continuous. In brief, it meant that the graph of the function did not have breaks, holes, jumps, etc. We define continuity for functions of two variables in a similar way as we did for functions of one variable.

Notes:

**Definition 58      Continuous**

Let a function  $f(x, y)$  be defined on an open disk  $B$  containing the point  $(x_0, y_0)$ .

1.  $f$  is **continuous at  $(x_0, y_0)$**  if  $\lim_{(x,y) \rightarrow (x_0, y_0)} f(x, y) = f(x_0, y_0)$ .
2.  $f$  is **continuous on  $B$**  if  $f$  is continuous at all points in  $B$ . If  $f$  is continuous at all points in  $\mathbb{R}^2$ , we say that  $f$  is **continuous everywhere**.

**Example 239      Continuity of a function of two variables**

Let  $f(x, y) = \begin{cases} \frac{\cos y \sin x}{x} & x \neq 0 \\ \cos y & x = 0 \end{cases}$ . Is  $f$  continuous at  $(0, 0)$ ? Is  $f$  continuous everywhere?

**SOLUTION** To determine if  $f$  is continuous at  $(0, 0)$ , we need to compare  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  to  $f(0, 0)$ .

Applying the definition of  $f$ , we see that  $f(0, 0) = \cos 0 = 1$ .

We now consider the limit  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ . Substituting 0 for  $x$  and  $y$  in  $(\cos y \sin x)/x$  returns the indeterminate form “0/0”, so we need to do more work to evaluate this limit.

Consider two related limits:  $\lim_{(x,y) \rightarrow (0,0)} \cos y$  and  $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin x}{x}$ . The first limit does not contain  $x$ , and since  $\cos y$  is continuous,

$$\lim_{(x,y) \rightarrow (0,0)} \cos y = \lim_{y \rightarrow 0} \cos y = \cos 0 = 1.$$

The second limit does not contain  $y$ . By Theorem 5 we can say

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

Finally, Theorem 51 of this section states that we can combine these two limits as follows:

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{\cos y \sin x}{x} &= \lim_{(x,y) \rightarrow (0,0)} (\cos y) \left( \frac{\sin x}{x} \right) \\ &= \left( \lim_{(x,y) \rightarrow (0,0)} \cos y \right) \left( \lim_{(x,y) \rightarrow (0,0)} \frac{\sin x}{x} \right) \\ &= (1)(1) \\ &= 1. \end{aligned}$$

---

Notes:

We have found that  $\lim_{(x,y) \rightarrow (0,0)} \frac{\cos y \sin x}{x} = f(0,0)$ , so  $f$  is continuous at  $(0,0)$ .

A similar analysis shows that  $f$  is continuous at all points in  $\mathbb{R}^2$ . As long as  $x \neq 0$ , we can evaluate the limit directly; when  $x = 0$ , a similar analysis shows that the limit is  $\cos y$ . Thus we can say that  $f$  is continuous everywhere. A graph of  $f$  is given in Figure 11.10. Notice how it has no breaks, jumps, etc.

The following theorem is very similar to Theorem 8, giving us ways to combine continuous functions to create other continuous functions.

### Theorem 52 Properties of Continuous Functions

Let  $f$  and  $g$  be continuous on an open disk  $B$ , let  $c$  be a real number, and let  $n$  be a positive integer. The following functions are continuous on  $B$ .

1. Sums/Differences:  $f \pm g$
2. Constant Multiples:  $c \cdot f$
3. Products:  $f \cdot g$
4. Quotients:  $f/g$  (as long as  $g \neq 0$  on  $B$ )
5. Powers:  $f^n$
6. Roots:  $\sqrt[n]{f}$  (if  $n$  is even then  $f \geq 0$  on  $B$ ; if  $n$  is odd, then true for all values of  $f$  on  $B$ .)
7. Compositions: Adjust the definitions of  $f$  and  $g$  to: Let  $f$  be continuous on  $B$ , where the range of  $f$  on  $B$  is  $J$ , and let  $g$  be a single variable function that is continuous on  $J$ . Then  $g \circ f$ , i.e.,  $g(f(x,y))$ , is continuous on  $B$ .

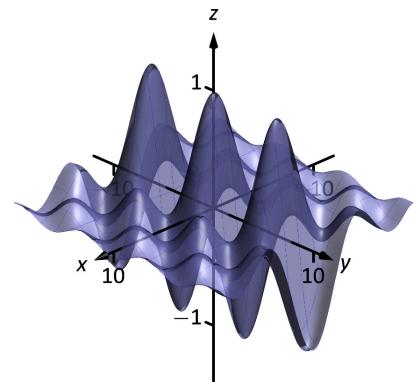


Figure 11.10: A graph of  $f(x, y)$  in Example 239.

### Example 240 Establishing continuity of a function

Let  $f(x, y) = \sin(x^2 \cos y)$ . Show  $f$  is continuous everywhere.

**SOLUTION** We will apply both Theorems 8 and 52. Let  $f_1(x, y) = x^2$ . Since  $y$  is not actually used in the function, and polynomials are continuous (by Theorem 8), we conclude  $f_1$  is continuous everywhere. A similar statement can be made about  $f_2(x, y) = \cos y$ . Part 3 of Theorem 52 states that  $f_3 = f_1 \cdot f_2$  is continuous everywhere, and Part 7 of the theorem states the composition of sine with  $f_3$  is continuous: that is,  $\sin(f_3) = \sin(x^2 \cos y)$  is continuous everywhere.

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Notes:

## Functions of Three Variables

The definitions and theorems given in this section can be extended in a natural way to definitions and theorems about functions of three (or more) variables. We cover the key concepts here; some terms from Definitions 56 and 58 are not redefined but their analogous meanings should be clear to the reader.

### Definition 59 Open Balls, Limit, Continuous

1. An **open ball** in  $\mathbb{R}^3$  centered at  $(x_0, y_0, z_0)$  with radius  $r$  is the set of all points  $(x, y, z)$  such that  $\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} = r$ .
2. Let  $D$  be an open set in  $\mathbb{R}^3$  containing  $(x_0, y_0, z_0)$ , and let  $f(x, y, z)$  be a function of three variables defined on  $D$ , except possibly at  $(x_0, y_0, z_0)$ . The **limit** of  $f(x, y, z)$  as  $(x, y, z)$  approaches  $(x_0, y_0, z_0)$  is  $L$ , denoted

$$\lim_{(x,y,z) \rightarrow (x_0,y_0,z_0)} f(x, y, z) = L,$$

means that given any  $\varepsilon > 0$ , there is a  $\delta > 0$  such that for all  $(x, y, z) \neq (x_0, y_0, z_0)$ , if  $(x, y, z)$  is in the open ball centered at  $(x_0, y_0, z_0)$  with radius  $\delta$ , then  $|f(x, y, z) - L| < \varepsilon$ .

3. Let  $f(x, y, z)$  be defined on an open ball  $B$  containing  $(x_0, y_0, z_0)$ .  $f$  is **continuous** at  $(x_0, y_0, z_0)$  if  $\lim_{(x,y,z) \rightarrow (x_0,y_0,z_0)} f(x, y, z) = f(x_0, y_0, z_0)$ .

These definitions can also be extended naturally to apply to functions of four or more variables. Theorem 52 also applies to function of three or more variables, allowing us to say that the function

$$f(x, y, z) = \frac{e^{x^2+y} \sqrt{y^2 + z^2 + 3}}{\sin(xyz) + 5}$$

is continuous everywhere.

When considering single variable functions, we studied limits, then continuity, then the derivative. In our current study of multivariable functions, we have studied limits and continuity. In the next section we study derivation, which takes on a slight twist as we are in a multivariable context.

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Notes:

## Exercises 11.2

### Terms and Concepts

1. Describe in your own words the difference between boundary and interior point of a set.
2. Use your own words to describe (informally) what  $\lim_{(x,y) \rightarrow (1,2)} f(x,y) = 17$  means.
3. Give an example of a closed, bounded set.
4. Give an example of a closed, unbounded set.
5. Give an example of a open, bounded set.
6. Give an example of a open, unbounded set.

### Problems

In Exercises 7 – 10, a set  $S$  is given.

- (a) Give one boundary point and one interior point, when possible, of  $S$ .
  - (b) State whether  $S$  is open, closed, or neither.
  - (c) State whether  $S$  is bounded or unbounded.
7.  $S = \left\{ (x,y) \mid \frac{(x-1)^2}{4} + \frac{(y-3)^2}{9} \leq 1 \right\}$
8.  $S = \{ (x,y) \mid y \neq x^2 \}$
9.  $S = \{ (x,y) \mid x^2 + y^2 = 1 \}$
10.  $S = \{ (x,y) \mid y > \sin x \}$

In Exercises 11 – 14:

- (a) Find the domain  $D$  of the given function.
  - (b) State whether  $D$  is an open or closed set.
  - (c) State whether  $D$  is bounded or unbounded.
11.  $f(x,y) = \sqrt{9 - x^2 - y^2}$

12.  $f(x,y) = \sqrt{y - x^2}$

13.  $f(x,y) = \frac{1}{\sqrt{y - x^2}}$

14.  $f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$

In Exercises 15 – 20, a limit is given. Evaluate the limit along the paths given, then state why these results show the given limit does not exist.

15.  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$

- (a) Along the path  $y = 0$ .
- (b) Along the path  $x = 0$ .

16.  $\lim_{(x,y) \rightarrow (0,0)} \frac{x+y}{x-y}$

- (a) Along the path  $y = mx$ .

17.  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy - y^2}{y^2 + x}$

- (a) Along the path  $y = mx$ .
- (b) Along the path  $x = 0$ .

18.  $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2)}{y}$

- (a) Along the path  $y = mx$ .
- (b) Along the path  $y = x^2$ .

19.  $\lim_{(x,y) \rightarrow (1,2)} \frac{x+y-3}{x^2-1}$

- (a) Along the path  $y = 2$ .
- (b) Along the path  $y = x+1$ .

20.  $\lim_{(x,y) \rightarrow (\pi, \pi/2)} \frac{\sin x}{\cos y}$

- (a) Along the path  $x = \pi$ .
- (b) Along the path  $y = x - \pi/2$ .

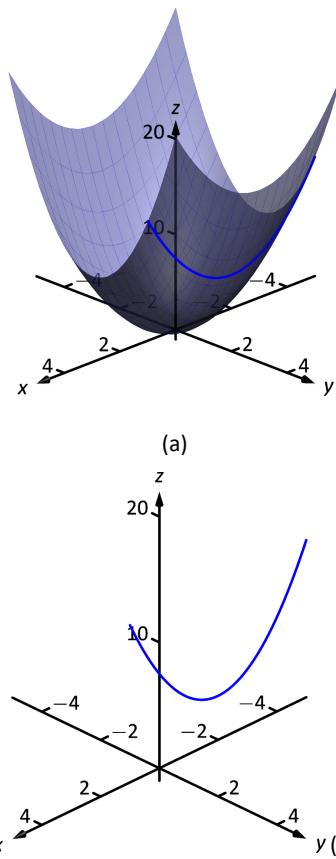


Figure 11.11: By fixing  $y = 2$ , the surface  $f(x, y) = x^2 + 2y^2$  is a curve in space.

Alternate notations for  $f_x(x, y)$  include:

$$\frac{\partial}{\partial x} f(x, y), \quad \frac{\partial f}{\partial x}, \quad \frac{\partial z}{\partial x}, \quad \text{and } z_x,$$

with similar notations for  $f_y(x, y)$ . For ease of notation,  $f_x(x, y)$  is often abbreviated  $f_x$ .

## 11.3 Partial Derivatives

Let  $y$  be a function of  $x$ . We have studied in great detail the derivative of  $y$  with respect to  $x$ , that is,  $\frac{dy}{dx}$ , which measures the rate at which  $y$  changes with respect to  $x$ . Consider now  $z = f(x, y)$ . It makes sense to want to know how  $z$  changes with respect to  $x$  and/or  $y$ . This section begins our investigation into these rates of change.

Consider the function  $z = f(x, y) = x^2 + 2y^2$ , as graphed in Figure 11.11(a). By fixing  $y = 2$ , we focus our attention to all points on the surface where the  $y$ -value is 2, shown in both parts (a) and (b) of the figure. These points form a curve in space:  $z = f(x, 2) = x^2 + 8$  which is a function of just one variable. We can take the derivative of  $z$  with respect to  $x$  along this curve and find equations of tangent lines, etc.

The key notion to extract from this example is: by treating  $y$  as constant (it does not vary) we can consider how  $z$  changes with respect to  $x$ . In a similar fashion, we can hold  $x$  constant and consider how  $z$  changes with respect to  $y$ . This is the underlying principle of **partial derivatives**. We state the formal, limit-based definition first, then show how to compute these partial derivatives without directly taking limits.

### Definition 60 Partial Derivative

Let  $z = f(x, y)$  be a continuous function on an open set  $S$  in  $\mathbb{R}^2$ .

1. The **partial derivative of  $f$  with respect to  $x$**  is:

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x + h, y) - f(x, y)}{h}.$$

2. The **partial derivative of  $f$  with respect to  $y$**  is:

$$f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

### Example 241 Computing partial derivatives with the limit definition

Let  $f(x, y) = x^2y + 2x + y^3$ . Find  $f_x(x, y)$  using the limit definition.

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Notes:

**SOLUTION** Using Definition 60, we have:

$$\begin{aligned}
 f_x(x, y) &= \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x+h)^2y + 2(x+h) + y^3 - (x^2y + 2x + y^3)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x^2y + 2xhy + h^2y + 2x + 2h + y^3 - (x^2y + 2x + y^3)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2xhy + h^2y + 2h}{h} \\
 &= \lim_{h \rightarrow 0} 2xy + hy + 2 \\
 &= 2xy + 2.
 \end{aligned}$$

We have found  $f_x(x, y) = 2xy + 2$ .

Example 241 found a partial derivative using the formal, limit-based definition. Using limits is not necessary, though, as we can rely on our previous knowledge of derivatives to compute partial derivatives easily. When computing  $f_x(x, y)$ , we hold  $y$  fixed – it does not vary. Therefore we can compute the derivative with respect to  $x$  by treating  $y$  as a constant or coefficient.

Just as  $\frac{d}{dx}(5x^2) = 10x$ , we compute  $\frac{\partial}{\partial x}(x^2y) = 2xy$ . Here we are treating  $y$  as a coefficient.

Just as  $\frac{d}{dx}(5^3) = 0$ , we compute  $\frac{\partial}{\partial x}(y^3) = 0$ . Here we are treating  $y$  as a constant. More examples will help make this clear.

**Example 242 Finding partial derivatives**

Find  $f_x(x, y)$  and  $f_y(x, y)$  in each of the following.

1.  $f(x, y) = x^3y^2 + 5y^2 - x + 7$

2.  $f(x, y) = \cos(xy^2) + \sin x$

3.  $f(x, y) = e^{x^2y^3} \sqrt{x^2 + 1}$

**SOLUTION**

1. We have  $f(x, y) = x^3y^2 + 5y^2 - x + 7$ .

Begin with  $f_x(x, y)$ . Keep  $y$  fixed, treating it as a constant or coefficient, as appropriate:

$$f_x(x, y) = 3x^2y^2 - 1.$$

Note how the  $5y^2$  and 7 terms go to zero.

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Notes:

To compute  $f_y(x, y)$ , we hold  $x$  fixed:

$$f_y(x, y) = 2x^3y + 10y.$$

Note how the  $-x$  and 7 terms go to zero.

2. We have  $f(x, y) = \cos(xy^2) + \sin x$ .

Begin with  $f_x(x, y)$ . We need to apply the Chain Rule with the cosine term;  $y^2$  is the coefficient of the  $x$ -term inside the cosine function.

$$f_x(x, y) = -\sin(xy^2)(y^2) + \cos x = -y^2 \sin(xy^2) + \cos x.$$

To find  $f_y(x, y)$ , note that  $x$  is the coefficient of the  $y^2$  term inside of the cosine term; also note that since  $x$  is fixed,  $\sin x$  is also fixed, and we treat it as a constant.

$$f_y(x, y) = -\sin(xy^2)(2xy) = -2xy \sin(xy^2).$$

3. We have  $f(x, y) = e^{x^2y^3} \sqrt{x^2 + 1}$ .

Beginning with  $f_x(x, y)$ , note how we need to apply the Product Rule.

$$\begin{aligned} f_x(x, y) &= e^{x^2y^3} (2xy^3) \sqrt{x^2 + 1} + e^{x^2y^3} \frac{1}{2} (x^2 + 1)^{-1/2} (2x) \\ &= 2xy^3 e^{x^2y^3} \sqrt{x^2 + 1} + \frac{xe^{x^2y^3}}{\sqrt{x^2 + 1}}. \end{aligned}$$

Note that when finding  $f_y(x, y)$  we do not have to apply the Product Rule; since  $\sqrt{x^2 + 1}$  does not contain  $y$ , we treat it as fixed and hence becomes a coefficient of the  $e^{x^2y^3}$  term.

$$f_y(x, y) = e^{x^2y^3} (3x^2y^2) \sqrt{x^2 + 1} = 3x^2y^2 e^{x^2y^3} \sqrt{x^2 + 1}.$$

We have shown *how* to compute a partial derivative, but it may still not be clear what a partial derivative *means*. Given  $z = f(x, y)$ ,  $f_x(x, y)$  measures the rate at which  $z$  changes as only  $x$  varies:  $y$  is held constant.

Imagine standing in a rolling meadow, then beginning to walk due east. Depending on your location, you might walk up, sharply down, or perhaps not change elevation at all. This is similar to measuring  $z_x$ : you are moving only east (in the “ $x$ ”-direction) and not north/south at all. Going back to your original location, imagine now walking due north (in the “ $y$ ”-direction). Perhaps walking due north does not change your elevation at all. This is analogous to  $z_y = 0$ :  $z$  does not change with respect to  $y$ . We can see that  $z_x$  and  $z_y$  do not have to be the same, or even similar, as it is easy to imagine circumstances where walking east means you walk downhill, though walking north makes you walk uphill.

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Notes:

The following example helps us visualize this more.

**Example 243 Evaluating partial derivatives**

Let  $z = f(x, y) = -x^2 - \frac{1}{2}y^2 + xy + 10$ . Find  $f_x(2, 1)$  and  $f_y(2, 1)$  and interpret their meaning.

**SOLUTION**

We begin by computing  $f_x(x, y) = -2x + y$  and  $f_y(x, y) = -y + x$ . Thus

$$f_x(2, 1) = -3 \quad \text{and} \quad f_y(2, 1) = 1.$$

It is also useful to note that  $f(2, 1) = 7.5$ . What does each of these numbers mean?

Consider  $f_x(2, 1) = -3$ , along with Figure 11.12(a). If one “stands” on the surface at the point  $(2, 1, 7.5)$  and moves parallel to the  $x$ -axis (i.e., only the  $x$ -value changes, not the  $y$ -value), then the instantaneous rate of change is  $-3$ . Increasing the  $x$ -value will decrease the  $z$ -value; decreasing the  $x$ -value will increase the  $z$ -value.

Now consider  $f_y(2, 1) = 1$ , illustrated in Figure 11.12(b). Moving along the curve drawn on the surface, i.e., parallel to the  $y$ -axis and not changing the  $x$ -values, increases the  $z$ -value instantaneously at a rate of  $1$ . Increasing the  $y$ -value by  $1$  would increase the  $z$ -value by approximately  $1$ .

Since the magnitude of  $f_x$  is greater than the magnitude of  $f_y$  at  $(2, 1)$ , it is “steeper” in the  $x$ -direction than in the  $y$ -direction.

## Second Partial Derivatives

Let  $z = f(x, y)$ . We have learned to find the partial derivatives  $f_x(x, y)$  and  $f_y(x, y)$ , which are each functions of  $x$  and  $y$ . Therefore we can take partial derivatives of them, each with respect to  $x$  and  $y$ . We define these “second partials” along with the notation, give examples, then discuss their meaning.

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Notes:

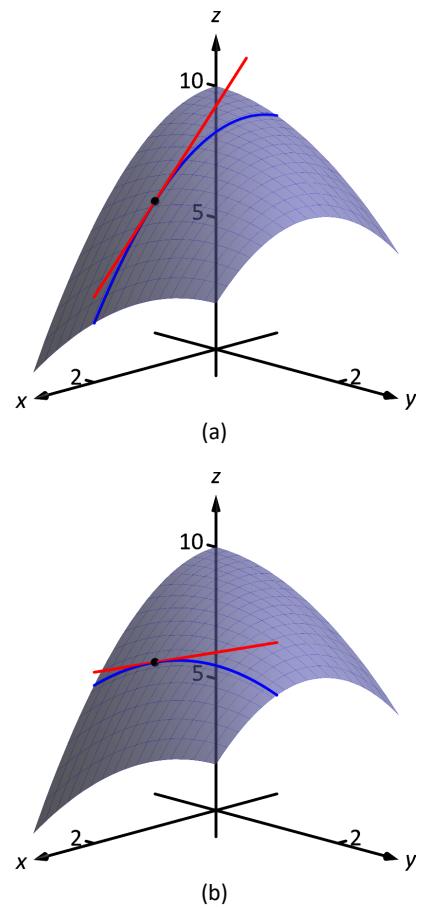


Figure 11.12: Illustrating the meaning of partial derivatives.

**Definition 61 Second Partial Derivative, Mixed Partial Derivative**

Let  $z = f(x, y)$  be continuous on an open set  $S$ .

1. The **second partial derivative of  $f$  with respect to  $x$  then  $x$**  is

$$\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = (f_x)_x = f_{xx}$$

2. The **second partial derivative of  $f$  with respect to  $x$  then  $y$**  is

$$\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = (f_x)_y = f_{xy}$$

Similar definitions hold for  $\frac{\partial^2 f}{\partial y^2} = f_{yy}$  and  $\frac{\partial^2 f}{\partial x \partial y} = f_{yx}$ .

The second partial derivatives  $f_{xy}$  and  $f_{yx}$  are **mixed partial derivatives**.

**Note:** The terms in Definition 61 all depend on limits, so each definition comes with the caveat “where the limit exists.”

The notation of second partial derivatives gives some insight into the notation of the second derivative of a function of a single variable. If  $y = f(x)$ , then  $f''(x) = \frac{d^2y}{dx^2}$ . The “ $d^2y$ ” portion means “take the derivative of  $y$  twice,” while “ $dx^2$ ” means “with respect to  $x$  both times.” When we only know of functions of a single variable, this latter phrase seems silly: there is only one variable to take the derivative with respect to. Now that we understand functions of multiple variables, we see the importance of specifying which variables we are referring to.

**Example 244 Second partial derivatives**

For each of the following, find all six first and second partial derivatives. That is, find

$$f_x, \quad f_y, \quad f_{xx}, \quad f_{yy}, \quad f_{xy} \quad \text{and} \quad f_{yx}.$$

1.  $f(x, y) = x^3y^2 + 2xy^3 + \cos x$

2.  $f(x, y) = \frac{x^3}{y^2}$

3.  $f(x, y) = e^x \sin(x^2y)$

**SOLUTION** In each, we give  $f_x$  and  $f_y$  immediately and then spend time deriving the second partial derivatives.

---

Notes:

$$1. \ f(x, y) = x^3y^2 + 2xy^3 + \cos x$$

$$f_x(x, y) = 3x^2y^2 + 2y^3 - \sin x$$

$$f_y(x, y) = 2x^3y + 6xy^2$$

$$f_{xx}(x, y) = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}(3x^2y^2 + 2y^3 - \sin x) = 6xy^2 - \cos x$$

$$f_{yy}(x, y) = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}(2x^3y + 6xy^2) = 2x^3 + 12xy$$

$$f_{xy}(x, y) = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(3x^2y^2 + 2y^3 - \sin x) = 6x^2y + 6y^2$$

$$f_{yx}(x, y) = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(2x^3y + 6xy^2) = 6x^2y + 6y^2$$

$$2. \ f(x, y) = \frac{x^3}{y^2} = x^3y^{-2}$$

$$f_x(x, y) = \frac{3x^2}{y^2}$$

$$f_y(x, y) = -\frac{2x^3}{y^3}$$

$$f_{xx}(x, y) = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\left(\frac{3x^2}{y^2}\right) = \frac{6x}{y^2}$$

$$f_{yy}(x, y) = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}\left(-\frac{2x^3}{y^3}\right) = \frac{6x^3}{y^4}$$

$$f_{xy}(x, y) = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left(\frac{3x^2}{y^2}\right) = -\frac{6x^2}{y^3}$$

$$f_{yx}(x, y) = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\left(-\frac{2x^3}{y^3}\right) = -\frac{6x^2}{y^3}$$

$$3. \ f(x, y) = e^x \sin(x^2y)$$

Because the following partial derivatives get rather long, we omit the extra notation and just give the results. In several cases, multiple applications of the Product and Chain Rules will be necessary, followed by some basic combination of like terms.

$$f_x(x, y) = e^x \sin(x^2y) + 2xye^x \cos(x^2y)$$

$$f_y(x, y) = x^2e^x \cos(x^2y)$$

$$f_{xx}(x, y) = e^x \sin(x^2y) + 4xye^x \cos(x^2y) + 2ye^x \cos(x^2y) - 4x^2y^2e^x \sin(x^2y)$$

$$f_{yy}(x, y) = -x^4e^x \sin(x^2y)$$

$$f_{xy}(x, y) = x^2e^x \cos(x^2y) + 2xe^x \cos(x^2y) - 2x^3ye^x \sin(x^2y)$$

$$f_{yx}(x, y) = x^2e^x \cos(x^2y) + 2xe^x \cos(x^2y) - 2x^3ye^x \sin(x^2y)$$

---

Notes:

Notice how in each of the three functions in Example 244,  $f_{xy} = f_{yx}$ . Due to the complexity of the examples, this likely is not a coincidence. The following theorem states that it is not.

**Theorem 53 Mixed Partial Derivatives**

Let  $f$  be defined such that  $f_{xy}$  and  $f_{yx}$  are continuous on an open set  $S$ . Then for each point  $(x, y)$  in  $S$ ,  $f_{xy}(x, y) = f_{yx}(x, y)$ .

Finding  $f_{xy}$  and  $f_{yx}$  independently and comparing the results provides a convenient way of checking our work.

### Understanding Second Partial Derivatives

Now that we know *how* to find second partials, we investigate *what* they tell us.

Again we refer back to a function  $y = f(x)$  of a single variable. The second derivative of  $f$  is “the derivative of the derivative,” or “the rate of change of the rate of change.” The second derivative measures how much the derivative is changing. If  $f''(x) < 0$ , then the derivative is getting smaller (so the graph of  $f$  is concave down); if  $f''(x) > 0$ , then the derivative is growing, making the graph of  $f$  concave up.

Now consider  $z = f(x, y)$ . Similar statements can be made about  $f_{xx}$  and  $f_{yy}$  as could be made about  $f''(x)$  above. When taking derivatives with respect to  $x$  twice, we measure how much  $f_x$  changes with respect to  $x$ . If  $f_{xx}(x, y) < 0$ , it means that as  $x$  increases,  $f_x$  decreases, and the graph of  $f$  will be concave down *in the x-direction*. Using the analogy of standing in the rolling meadow used earlier in this section,  $f_{xx}$  measures whether one’s path is concave up/down when walking due east.

Similarly,  $f_{yy}$  measures the concavity in the  $y$ -direction. If  $f_{yy}(x, y) > 0$ , then  $f_y$  is increasing with respect to  $y$  and the graph of  $f$  will be concave up in the  $y$ -direction. Appealing to the rolling meadow analogy again,  $f_{yy}$  measures whether one’s path is concave up/down when walking due north.

We now consider the mixed partials  $f_{xy}$  and  $f_{yx}$ . The mixed partial  $f_{xy}$  measures how much  $f_x$  changes with respect to  $y$ . Once again using the rolling meadow analogy,  $f_x$  measures the slope if one walks due east. Looking east, begin walking *north* (side-stepping). Is the path towards the east getting steeper? If so,  $f_{xy} > 0$ . Is the path towards the east not changing in steepness? If so, then  $f_{xy} = 0$ . A similar thing can be said about  $f_{yx}$ : consider the steepness of paths heading north while side-stepping to the east.

The following example examines these ideas with concrete numbers and

Notes:

graphs.

**Example 245 Understanding second partial derivatives**

Let  $z = x^2 - y^2 + xy$ . Evaluate the 6 first and second partial derivatives at  $(-1/2, 1/2)$  and interpret what each of these numbers mean.

**SOLUTION** We find that:

$f_x(x, y) = 2x + y$ ,  $f_y(x, y) = -2y + x$ ,  $f_{xx}(x, y) = 2$ ,  $f_{yy}(x, y) = -2$  and  $f_{xy}(x, y) = f_{yx}(x, y) = 1$ . Thus at  $(-1/2, 1/2)$  we have

$$f_x(-1/2, 1/2) = -1/2, \quad f_y(-1/2, 1/2) = -3/2.$$

The slope of the tangent line at  $(-1/2, 1/2, -1/4)$  in the direction of  $x$  is  $-1/2$ : if one moves from that point parallel to the  $x$ -axis, the instantaneous rate of change will be  $-1/2$ . The slope of the tangent line at this point in the direction of  $y$  is  $-3/2$ : if one moves from this point parallel to the  $y$ -axis, the instantaneous rate of change will be  $-3/2$ . These tangents lines are graphed in Figure 11.13(a) and (b), respectively, where the tangent lines are drawn in a solid line.

Now consider only Figure 11.13(a). Three directed tangent lines are drawn (two are dashed), each in the direction of  $x$ ; that is, each has a slope determined by  $f_x$ . Note how as  $y$  increases, the slope of these lines get closer to 0. Since the slopes are all negative, getting closer to 0 means the *slopes are increasing*. The slopes given by  $f_x$  are increasing as  $y$  increases, meaning  $f_{xy}$  must be positive.

Since  $f_{xy} = f_{yx}$ , we also expect  $f_y$  to increase as  $x$  increases. Consider Figure 11.13(b) where again three directed tangent lines are drawn, this time each in the direction of  $y$  with slopes determined by  $f_y$ . As  $x$  increases, the slopes become less steep (closer to 0). Since these are negative slopes, this means the slopes are increasing.

Thus far we have a visual understanding of  $f_x$ ,  $f_y$ , and  $f_{xy} = f_{yx}$ . We now interpret  $f_{xx}$  and  $f_{yy}$ . In Figure 11.13(a), we see a curve drawn where  $x$  is held constant at  $x = -1/2$ : only  $y$  varies. This curve is clearly concave down, corresponding to the fact that  $f_{yy} < 0$ . In part (b) of the figure, we see a similar curve where  $y$  is constant and only  $x$  varies. This curve is concave up, corresponding to the fact that  $f_{xx} > 0$ .

### Partial Derivatives and Functions of Three Variables

The concepts underlying partial derivatives can be easily extend to more than two variables. We give some definitions and examples in the case of three variables and trust the reader can extend these definitions to more variables if needed.

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Notes:

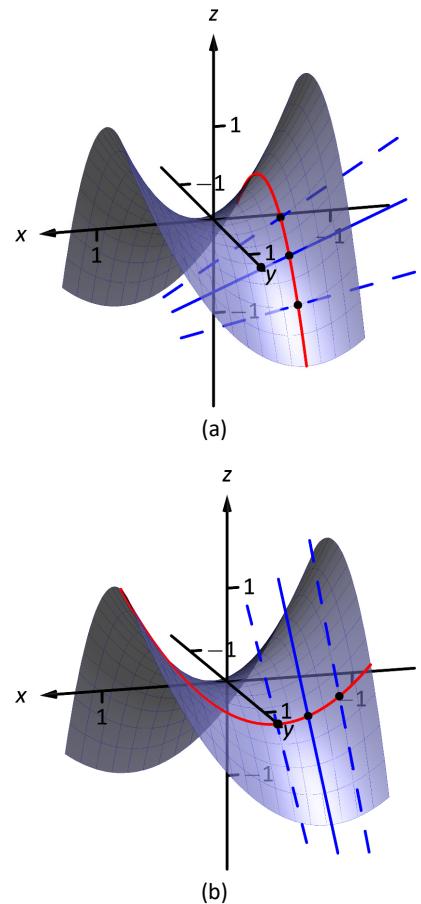


Figure 11.13: Understanding the second partial derivatives in Example 245.

**Definition 62 Partial Derivatives with Three Variables**

Let  $w = f(x, y, z)$  be a continuous function on an open set  $S$  in  $\mathbb{R}^3$ .  
The **partial derivative of  $f$  with respect to  $x$**  is:

$$f_x(x, y, z) = \lim_{h \rightarrow 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}.$$

Similar definitions hold for  $f_y(x, y, z)$  and  $f_z(x, y, z)$ .

By taking partial derivatives of partial derivatives, we can find second partial derivatives of  $f$  with respect to  $z$  then  $y$ , for instance, just as before.

**Example 246 Partial derivatives of functions of three variables**

For each of the following, find  $f_x$ ,  $f_y$ ,  $f_z$ ,  $f_{xz}$ ,  $f_{yz}$ , and  $f_{zz}$ .

$$1. f(x, y, z) = x^2y^3z^4 + x^2y^2 + x^3z^3 + y^4z^4$$

$$2. f(x, y, z) = x \sin(yz)$$

**SOLUTION**

$$1. f_x = 2xy^3z^4 + 2xy^2 + 3x^2z^3; \quad f_y = 3x^2y^2z^4 + 2x^2y + 4y^3z^4;$$

$$f_z = 4x^2y^3z^3 + 3x^3z^2 + 4y^4z^3; \quad f_{xz} = 8xy^3z^3 + 9x^2z^2;$$

$$f_{yz} = 12x^2y^2z^3 + 16y^3z^3; \quad f_{zz} = 12x^2y^3z^2 + 6x^3z + 12y^4z^2$$

$$2. f_x = \sin(yz); \quad f_y = xz \cos(yz); \quad f_z = xy \cos(yz);$$

$$f_{xz} = y \cos(yz); \quad f_{yz} = x \cos(yz) - xyz \sin(yz); \quad f_{zz} = -xy^2 \sin(xy)$$

**Higher Order Partial Derivatives**

We can continue taking partial derivatives of partial derivatives of partial derivatives of ...; we do not have to stop with second partial derivatives. These higher order partial derivatives do not have a tidy graphical interpretation; nevertheless they are not hard to compute and worthy of some practice.

We do not formally define each higher order derivative, but rather give just a few examples of the notation.

$$f_{xyx}(x, y) = \frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) \right) \quad \text{and}$$

$$f_{xyz}(x, y, z) = \frac{\partial}{\partial z} \left( \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) \right).$$

---

Notes:

**Example 247 Higher order partial derivatives**

1. Let  $f(x, y) = x^2y^2 + \sin(xy)$ . Find  $f_{xxy}$  and  $f_{yxx}$ .
2. Let  $f(x, y, z) = x^3e^{xy} + \cos(z)$ . Find  $f_{xyz}$ .

**SOLUTION**

1. To find  $f_{xxy}$ , we first find  $f_x$ , then  $f_{xx}$ , then  $f_{xxy}$ :

$$\begin{aligned} f_x &= 2xy^2 + y \cos(xy) & f_{xx} &= 2y^2 - y^2 \sin(xy) \\ f_{xxy} &= 4y - 2y \sin(xy) - xy^2 \cos(xy). \end{aligned}$$

To find  $f_{yxx}$ , we first find  $f_y$ , then  $f_{yx}$ , then  $f_{yxx}$ :

$$\begin{aligned} f_y &= 2x^2y + x \cos(xy) & f_{yx} &= 4xy + \cos(xy) - xy \sin(xy) \\ f_{yxx} &= 4y - y \sin(xy) - (y \sin(xy) + xy^2 \cos(xy)) \\ &= 4y - 2y \sin(xy) - xy^2 \cos(xy). \end{aligned}$$

Note how  $f_{xxy} = f_{yxx}$ .

2. To find  $f_{xyz}$ , we find  $f_x$ , then  $f_{xy}$ , then  $f_{xyz}$ :

$$\begin{aligned} f_x &= 3x^2e^{xy} + x^3ye^{xy} & f_{xy} &= 3x^3e^{xy} + x^3e^{xy} + x^4ye^{xy} = 4x^3e^{xy} + x^4ye^{xy} \\ f_{xyz} &= 0. \end{aligned}$$

In the previous example we saw that  $f_{xxy} = f_{yxx}$ ; this is not a coincidence. While we do not state this as a formal theorem, as long as each partial derivative is continuous, it does not matter the order in which the partial derivatives are taken. For instance,  $f_{xxy} = f_{xyx} = f_{yxx}$ .

This can be useful at times. Had we known this, the second part of Example 247 would have been much simpler to compute. Instead of computing  $f_{xyz}$  in the  $x$ ,  $y$  then  $z$  orders, we could have applied the  $z$ , then  $x$  then  $y$  order (as  $f_{xyz} = f_{zxy}$ ). It is easy to see that  $f_z = -\sin z$ ; then  $f_{zx}$  and  $f_{zxy}$  are clearly 0 as  $f_z$  does not contain an  $x$  or  $y$ .

Notes:

A brief review of this section: partial derivatives measure the instantaneous rate of change of a multivariable function with respect to one variable. With  $z = f(x, y)$ , the partial derivatives  $f_x$  and  $f_y$  measure the instantaneous rate of change of  $z$  when moving parallel to the  $x$ - and  $y$ -axes, respectively. How do we measure the rate of change at a point when we do not move parallel to one of these axes? What if we move in the direction given by the vector  $\langle 2, 1 \rangle$ ? Can we measure that rate of change? The answer is, of course, yes, we can. This is the topic of Section 11.6. First, we need to define what it means for a function of two variables to be *differentiable*.

---

Notes:

# Exercises 11.3

## Terms and Concepts

1. What is the difference between a constant and a coefficient?
2. Given a function  $z = f(x, y)$ , explain in your own words how to compute  $f_x$ .
3. In the mixed partial fraction  $f_{xy}$ , which is computed first,  $f_x$  or  $f_y$ ?
4. In the mixed partial fraction  $\frac{\partial^2 f}{\partial x \partial y}$ , which is computed first,  $f_x$  or  $f_y$ ?

## Problems

In Exercises 5 – 8, evaluate  $f_x(x, y)$  and  $f_y(x, y)$  at the indicated point.

5.  $f(x, y) = x^2y - x + 2y + 3$  at  $(1, 2)$
6.  $f(x, y) = x^3 - 3x + y^2 - 6y$  at  $(-1, 3)$
7.  $f(x, y) = \sin y \cos x$  at  $(\pi/3, \pi/3)$
8.  $f(x, y) = \ln(xy)$  at  $(-2, -3)$

In Exercises 9 – 26, find  $f_x$ ,  $f_y$ ,  $f_{xx}$ ,  $f_{yy}$ ,  $f_{xy}$  and  $f_{yx}$ .

9.  $f(x, y) = x^2y + 3x^2 + 4y - 5$
10.  $f(x, y) = y^3 + 3xy^2 + 3x^2y + x^3$
11.  $f(x, y) = \frac{x}{y}$
12.  $f(x, y) = \frac{4}{xy}$
13.  $f(x, y) = e^{x^2+y^2}$
14.  $f(x, y) = e^{x+2y}$
15.  $f(x, y) = \sin x \cos y$

16.  $f(x, y) = (x + y)^3$
  17.  $f(x, y) = \cos(5xy^3)$
  18.  $f(x, y) = \sin(5x^2 + 2y^3)$
  19.  $f(x, y) = \sqrt{4xy^2 + 1}$
  20.  $f(x, y) = (2x + 5y)\sqrt{y}$
  21.  $f(x, y) = \frac{1}{x^2 + y^2 + 1}$
  22.  $f(x, y) = 5x - 17y$
  23.  $f(x, y) = 3x^2 + 1$
  24.  $f(x, y) = \ln(x^2 + y)$
  25.  $f(x, y) = \frac{\ln x}{4y}$
  26.  $f(x, y) = 5e^x \sin y + 9$
- In Exercises 27 – 30, form a function  $z = f(x, y)$  such that  $f_x$  and  $f_y$  match those given.
27.  $f_x = \sin y + 1$ ,  $f_y = x \cos y$
  28.  $f_x = x + y$ ,  $f_y = x + y$
  29.  $f_x = 6xy - 4y^2$ ,  $f_y = 3x^2 - 8xy + 2$
  30.  $f_x = \frac{2x}{x^2 + y^2}$ ,  $f_y = \frac{2y}{x^2 + y^2}$
- In Exercises 31 – 34, find  $f_x$ ,  $f_y$ ,  $f_z$ ,  $f_{yz}$  and  $f_{zy}$ .
31.  $f(x, y, z) = x^2e^{2y-3z}$
  32.  $f(x, y, z) = x^3y^2 + x^3z + y^2z$
  33.  $f(x, y, z) = \frac{3x}{7y^2z}$
  34.  $f(x, y, z) = \ln(xyz)$

## 11.4 Differentiability and the Total Differential

We studied **differentials** in Section 4.4, where Definition 18 states that if  $y = f(x)$  and  $f$  is differentiable, then  $dy = f'(x)dx$ . One important use of this differential is in Integration by Substitution. Another important application is approximation. Let  $\Delta x = dx$  represent a change in  $x$ . When  $dx$  is small,  $dy \approx \Delta y$ , the change in  $y$  resulting from the change in  $x$ . Fundamental in this understanding is this: as  $dx$  gets small, the difference between  $\Delta y$  and  $dy$  goes to 0. Another way of stating this: as  $dx$  goes to 0, the *error* in approximating  $\Delta y$  with  $dy$  goes to 0.

We extend this idea to functions of two variables. Let  $z = f(x, y)$ , and let  $\Delta x = dx$  and  $\Delta y = dy$  represent changes in  $x$  and  $y$ , respectively. Let  $\Delta z = f(x + dx, y + dy) - f(x, y)$  be the change in  $z$  over the change in  $x$  and  $y$ . Recalling that  $f_x$  and  $f_y$  give the instantaneous rates of  $z$ -change in the  $x$ - and  $y$ -directions, respectively, we can approximate  $\Delta z$  with  $dz = f_x dx + f_y dy$ ; in words, the total change in  $z$  is approximately the change caused by changing  $x$  plus the change caused by changing  $y$ . In a moment we give an indication of whether or not this approximation is any good. First we give a name to  $dz$ .

### Definition 63 Total Differential

Let  $z = f(x, y)$  be continuous on an open set  $S$ . Let  $dx$  and  $dy$  represent changes in  $x$  and  $y$ , respectively. Where the partial derivatives  $f_x$  and  $f_y$  exist, the **total differential of  $z$**  is

$$dz = f_x(x, y)dx + f_y(x, y)dy.$$

### Example 248 Finding the total differential

Let  $z = x^4 e^{3y}$ . Find  $dz$ .

**SOLUTION** We compute the partial derivatives:  $f_x = 4x^3 e^{3y}$  and  $f_y = 3x^4 e^{3y}$ . Following Definition 63, we have

$$dz = 4x^3 e^{3y}dx + 3x^4 e^{3y}dy.$$

We *can* approximate  $\Delta z$  with  $dz$ , but as with all approximations, there is error involved. A good approximation is one in which the error is small. At a given point  $(x_0, y_0)$ , let  $E_x$  and  $E_y$  be functions of  $dx$  and  $dy$  such that  $E_x dx + E_y dy$  describes this error. Then

$$\begin{aligned}\Delta z &= dz + E_x dx + E_y dy \\ &= f_x(x_0, y_0)dx + f_y(x_0, y_0)dy + E_x dx + E_y dy.\end{aligned}$$

---

Notes:

If the approximation of  $\Delta z$  by  $dz$  is good, then as  $dx$  and  $dy$  get small, so does  $E_x dx + E_y dy$ . The approximation of  $\Delta z$  by  $dz$  is even better if, as  $dx$  and  $dy$  go to 0, so do  $E_x$  and  $E_y$ . This leads us to our definition of differentiability.

**Definition 64 Multivariable Differentiability**

Let  $z = f(x, y)$  be defined on an open set  $S$  containing  $(x_0, y_0)$  where  $f_x(x_0, y_0)$  and  $f_y(x_0, y_0)$  exist. Let  $dz$  be the total differential of  $z$  at  $(x_0, y_0)$ , let  $\Delta z = f(x_0 + dx, y_0 + dy) - f(x_0, y_0)$ , and let  $E_x$  and  $E_y$  be functions of  $dx$  and  $dy$  such that

$$\Delta z = dz + E_x dx + E_y dy.$$

1.  $f$  is **differentiable at**  $(x_0, y_0)$  if, given  $\varepsilon > 0$ , there is a  $\delta > 0$  such that if  $\| \langle dx, dy \rangle \| < \delta$ , then  $\| \langle E_x, E_y \rangle \| < \varepsilon$ . That is, as  $dx$  and  $dy$  go to 0, so do  $E_x$  and  $E_y$ .
2.  $f$  is **differentiable on**  $S$  if  $f$  is differentiable at every point in  $S$ . If  $f$  is differentiable on  $\mathbb{R}^2$ , we say that  $f$  is **differentiable everywhere**.

**Example 249 Showing a function is differentiable**

Show  $f(x, y) = xy + 3y^2$  is differentiable using Definition 64.

**SOLUTION** We begin by finding  $f(x + dx, y + dy)$ ,  $\Delta z$ ,  $f_x$  and  $f_y$ .

$$\begin{aligned} f(x + dx, y + dy) &= (x + dx)(y + dy) + 3(y + dy)^2 \\ &= xy + xdy + ydx + dxdy + 3y^2 + 6ydy + 3dy^2. \end{aligned}$$

$\Delta z = f(x + dx, y + dy) - f(x, y)$ , so

$$\Delta z = xdy + ydx + dxdy + 6ydy + 3dy^2.$$

It is straightforward to compute  $f_x = y$  and  $f_y = x + 6y$ . Consider once more  $\Delta z$ :

$$\begin{aligned} \Delta z &= xdy + ydx + dxdy + 6ydy + 3dy^2 \quad (\text{now reorder}) \\ &= ydx + xdy + 6ydy + dxdy + 3dy^2 \\ &= \underbrace{(y)}_{f_x} dx + \underbrace{(x + 6y)}_{f_y} dy + \underbrace{(dy)}_{E_x} dx + \underbrace{(3dy)}_{E_y} dy \\ &= f_x dx + f_y dy + E_x dx + E_y dy. \end{aligned}$$

With  $E_x = dy$  and  $E_y = 3dy$ , it is clear that as  $dx$  and  $dy$  go to 0,  $E_x$  and  $E_y$  also go to 0. Since this did not depend on a specific point  $(x_0, y_0)$ , we can say that  $f(x, y)$

---

Notes:

is differentiable for all pairs  $(x, y)$  in  $\mathbb{R}^2$ , or, equivalently, that  $f$  is differentiable everywhere.

Our intuitive understanding of differentiability of functions  $y = f(x)$  of one variable was that the graph of  $f$  was “smooth.” A similar intuitive understanding of functions  $z = f(x, y)$  of two variables is that the surface defined by  $f$  is also “smooth,” not containing cusps, edges, breaks, etc. The following theorem states that differentiable functions are continuous, followed by another theorem that provides a more tangible way of determining whether a great number of functions are differentiable or not.

**Theorem 54      Continuity and Differentiability of Multivariable Functions**

Let  $z = f(x, y)$  be defined on an open set  $S$  containing  $(x_0, y_0)$ . If  $f$  is differentiable at  $(x_0, y_0)$ , then  $f$  is continuous at  $(x_0, y_0)$ .

**Theorem 55      Differentiability of Multivariable Functions**

Let  $z = f(x, y)$  be defined on an open set  $S$  containing  $(x_0, y_0)$ . If  $f_x$  and  $f_y$  are both continuous on  $S$ , then  $f$  is differentiable on  $S$ .

The theorems assure us that essentially all functions that we see in the course of our studies here are differentiable (and hence continuous) on their natural domains. There is a difference between Definition 64 and Theorem 55, though: it is possible for a function  $f$  to be differentiable yet  $f_x$  and/or  $f_y$  is *not* continuous. Such strange behavior of functions is a source of delight for many mathematicians.

When  $f_x$  and  $f_y$  exist at a point but are not continuous at that point, we need to use other methods to determine whether or not  $f$  is differentiable at that point.

For instance, consider the function

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

---

Notes:

We can find  $f_x(0, 0)$  and  $f_y(0, 0)$  using Definition 60:

$$\begin{aligned} f_x(0, 0) &= \lim_{h \rightarrow 0} \frac{f(0 + h, 0) - f(0, 0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{0}{h^2} = 0; \\ f_y(0, 0) &= \lim_{h \rightarrow 0} \frac{f(0, 0 + h) - f(0, 0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{0}{h^2} = 0. \end{aligned}$$

Both  $f_x$  and  $f_y$  exist at  $(0, 0)$ , but they are not continuous at  $(0, 0)$ , as

$$f_x(x, y) = \frac{y(y^2 - x^2)}{(x^2 + y^2)^2} \quad \text{and} \quad f_y(x, y) = \frac{x(x^2 - y^2)}{(x^2 + y^2)^2}$$

are not continuous at  $(0, 0)$ . (Take the limit of  $f_x$  as  $(x, y) \rightarrow (0, 0)$  along the  $x$ - and  $y$ -axes; they give different results.) So even though  $f_x$  and  $f_y$  exist at every point in the  $x$ - $y$  plane, they are not continuous. Therefore it is possible, by Theorem 55, for  $f$  to not be differentiable.

Indeed, it is not. One can show that  $f$  is not continuous at  $(0, 0)$  (see Example 237), and by Theorem 54, this means  $f$  is not differentiable at  $(0, 0)$ .

## Approximating with the Total Differential

By the definition, when  $f$  is differentiable  $dz$  is a good approximation for  $\Delta z$  when  $dx$  and  $dy$  are small. We give some simple examples of how this is used here.

### Example 250 Approximating with the total differential

Let  $z = \sqrt{x} \sin y$ . Approximate  $f(4.1, 0.8)$ .

**SOLUTION** Recognizing that  $\pi/4 \approx 0.785 \approx 0.8$ , we can approximate  $f(4.1, 0.8)$  using  $f(4, \pi/4)$ . We can easily compute  $f(4, \pi/4) = \sqrt{4} \sin(\pi/4) = 2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2} \approx 1.414$ . Without calculus, this is the best approximation we could reasonably come up with. The total differential gives us a way of adjusting this initial approximation to hopefully get a more accurate answer.

We let  $\Delta z = f(4.1, 0.8) - f(4, \pi/4)$ . The total differential  $dz$  is approximately equal to  $\Delta z$ , so

$$f(4.1, 0.8) - f(4, \pi/4) \approx dz \Rightarrow f(4.1, 0.8) \approx dz + f(4, \pi/4). \quad (11.1)$$

To find  $dz$ , we need  $f_x$  and  $f_y$ .

Notes:

$$\begin{aligned} f_x(x, y) &= \frac{\sin y}{2\sqrt{x}} \Rightarrow & f_x(4, \pi/4) &= \frac{\sin \pi/4}{2\sqrt{4}} \\ &&&= \frac{\sqrt{2}/2}{4} = \sqrt{2}/8. \\ f_y(x, y) &= \sqrt{x} \cos y \Rightarrow & f_y(4, \pi/4) &= \sqrt{4} \frac{\sqrt{2}}{2} \\ &&&= \sqrt{2}. \end{aligned}$$

Approximating 4.1 with 4 gives  $dx = 0.1$ ; approximating 0.8 with  $\pi/4$  gives  $dy \approx 0.015$ . Thus

$$\begin{aligned} dz(4, \pi/4) &= f_x(4, \pi/4)(0.1) + f_y(4, \pi/4)(0.015) \\ &= \frac{\sqrt{2}}{8}(0.1) + \sqrt{2}(0.015) \\ &\approx 0.039. \end{aligned}$$

Returning to Equation (11.1), we have

$$f(4.1, 0.8) \approx 0.039 + 1.414 = 1.4531.$$

We, of course, can compute the actual value of  $f(4.1, 0.8)$  with a calculator; the actual value, accurate to 5 places after the decimal, is 1.45254. Obviously our approximation is quite good.

The point of the previous example was *not* to develop an approximation method for known functions. After all, we can very easily compute  $f(4.1, 0.8)$  using readily available technology. Rather, it serves to illustrate how well this method of approximation works, and to reinforce the following concept:

“New position = old position + amount of change,” so  
“New position  $\approx$  old position + approximate amount of change.”

In the previous example, we could easily compute  $f(4, \pi/4)$  and could approximate the amount of z-change when computing  $f(4.1, 0.8)$ , letting us approximate the new z-value.

It may be surprising to learn that it is not uncommon to know the values of  $f$ ,  $f_x$  and  $f_y$  at a particular point without actually knowing the function  $f$ . The total differential gives a good method of approximating  $f$  at nearby points.

#### **Example 251      Approximating an unknown function**

Given that  $f(2, -3) = 6$ ,  $f_x(2, -3) = 1.3$  and  $f_y(2, -3) = -0.6$ , approximate  $f(2.1, -3.03)$ .

---

Notes:

**SOLUTION** The total differential approximates how much  $f$  changes from the point  $(2, -3)$  to the point  $(2.1, -3.03)$ . With  $dx = 0.1$  and  $dy = -0.03$ , we have

$$\begin{aligned} dz &= f_x(2, -3)dx + f_y(2, -3)dy \\ &= 1.3(0.1) + (-0.6)(-0.03) \\ &= 0.148. \end{aligned}$$

The change in  $z$  is approximately 0.148, so we approximate  $f(2.1, -3.03) \approx 6.148$ .

## Error/Sensitivity Analysis

The total differential gives an approximation of the change in  $z$  given small changes in  $x$  and  $y$ . We can use this to approximate error propagation; that is, if the input is a little off from what it should be, how far from correct will the output be? We demonstrate this in an example.

### Example 252 Sensitivity analysis

A cylindrical steel storage tank is to be built that is 10ft tall and 4ft across in diameter. It is known that the steel will expand/contract with temperature changes; is the overall volume of the tank more sensitive to changes in the diameter or in the height of the tank?

**SOLUTION** A cylindrical solid with height  $h$  and radius  $r$  has volume  $V = \pi r^2 h$ . We can view  $V$  as a function of two variables,  $r$  and  $h$ . We can compute partial derivatives of  $V$ :

$$\frac{\partial V}{\partial r} = V_r(r, h) = 2\pi rh \quad \text{and} \quad \frac{\partial V}{\partial h} = V_h(r, h) = \pi r^2.$$

The total differential is  $dV = (2\pi rh)dr + (\pi r^2)dh$ . When  $h = 10$  and  $r = 2$ , we have  $dV = 40\pi dr + 4\pi dh$ . Note that the coefficient of  $dr$  is  $40\pi \approx 125.7$ ; the coefficient of  $dh$  is a tenth of that, approximately 12.57. A small change in radius will be multiplied by 125.7, whereas a small change in height will be multiplied by 12.57. Thus the volume of the tank is more sensitive to changes in radius than in height.

The previous example showed that the volume of a particular tank was more sensitive to changes in radius than in height. Keep in mind that this analysis only applies to a tank of those dimensions. A tank with a height of 1ft and radius of 5ft would be more sensitive to changes in height than in radius.

Notes:

One could make a chart of small changes in radius and height and find exact changes in volume given specific changes. While this provides exact numbers, it does not give as much insight as the error analysis using the total differential.

### Differentiability of Functions of Three Variables

The definition of differentiability for functions of three variables is very similar to that of functions of two variables. We again start with the total differential.

#### Definition 65 Total Differential

Let  $w = f(x, y, z)$  be continuous on an open set  $S$ . Let  $dx, dy$  and  $dz$  represent changes in  $x, y$  and  $z$ , respectively. Where the partial derivatives  $f_x, f_y$  and  $f_z$  exist, the **total differential of  $w$**  is

$$dz = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz.$$

This differential can be a good approximation of the change in  $w$  when  $w = f(x, y, z)$  is **differentiable**.

#### Definition 66 Multivariable Differentiability

Let  $w = f(x, y, z)$  be defined on an open ball  $B$  containing  $(x_0, y_0, z_0)$  where  $f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0)$  and  $f_z(x_0, y_0, z_0)$  exist. Let  $dw$  be the total differential of  $w$  at  $(x_0, y_0, z_0)$ , let  $\Delta w = f(x_0 + dx, y_0 + dy, z_0 + dz) - f(x_0, y_0, z_0)$ , and let  $E_x, E_y$  and  $E_z$  be functions of  $dx, dy$  and  $dz$  such that

$$\Delta w = dw + E_x dx + E_y dy + E_z dz.$$

1.  $f$  is **differentiable at  $(x_0, y_0, z_0)$**  if, given  $\varepsilon > 0$ , there is a  $\delta > 0$  such that if  $\|\langle dx, dy, dz \rangle\| < \delta$ , then  $\|\langle E_x, E_y, E_z \rangle\| < \varepsilon$ .
2.  $f$  is **differentiable on  $B$**  if  $f$  is differentiable at every point in  $B$ . If  $f$  is differentiable on  $\mathbb{R}^3$ , we say that  $f$  is **differentiable everywhere**.

Just as before, this definition gives a rigorous statement about what it means to be differentiable that is not very intuitive. We follow it with a theorem similar to Theorem 55.

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Notes:

**Theorem 56    Continuity and Differentiability of Functions of Three Variables**

Let  $w = f(x, y, z)$  be defined on an open ball  $B$  containing  $(x_0, y_0, z_0)$ .

1. If  $f$  is differentiable at  $(x_0, y_0, z_0)$ , then  $f$  is continuous at  $(x_0, y_0, z_0)$ .

2. If  $f_x, f_y$  and  $f_z$  are continuous on  $B$ , then  $f$  is differentiable on  $B$ .

This set of definition and theorem extends to functions of any number of variables. The theorem again gives us a simple way of verifying that most functions that we encounter are differentiable on their natural domains.

This section has given us a formal definition of what it means for a function to be “differentiable,” along with a theorem that gives a more accessible understanding. The following sections return to notions prompted by our study of partial derivatives that make use of the fact that most functions we encounter are differentiable.

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Notes:

# Exercises 11.4

## Terms and Concepts

1. T/F: If  $f(x, y)$  is differentiable on  $S$ , then  $f$  is continuous on  $S$ .
2. T/F: If  $f_x$  and  $f_y$  are continuous on  $S$ , then  $f$  is differentiable on  $S$ .
3. T/F: If  $z = f(x, y)$  is differentiable, then the change in  $z$  over small changes  $dx$  and  $dy$  in  $x$  and  $y$  is approximately  $dz$ .
4. Finish the sentence: "The new  $z$ -value is approximately the old  $z$ -value plus the approximate \_\_\_\_\_."

## Problems

In Exercises 5 – 8, find the total differential  $dz$ .

5.  $z = x \sin y + x^2$
6.  $z = (2x^2 + 3y)^2$
7.  $z = 5x - 7y$
8.  $z = xe^{x+y}$

In Exercises 9 – 12, a function  $z = f(x, y)$  is given. Give the indicated approximation using the total differential.

9.  $f(x, y) = \sqrt{x^2 + y}$ . Approximate  $f(2.95, 7.1)$  knowing  $f(3, 7) = 4$ .
10.  $f(x, y) = \sin x \cos y$ . Approximate  $f(0.1, -0.1)$  knowing  $f(0, 0) = 0$ .
11.  $f(x, y) = x^2y - xy^2$ . Approximate  $f(2.04, 3.06)$  knowing  $f(2, 3) = -6$ .
12.  $f(x, y) = \ln(x - y)$ . Approximate  $f(5.1, 3.98)$  knowing  $f(5, 4) = 0$ .

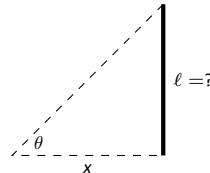
Exercises 13 – 16 ask a variety of questions dealing with approximating error and sensitivity analysis.

13. A cylindrical storage tank is to be 2ft tall with a radius of 1ft. Is the volume of the tank more sensitive to changes in the radius or the height?
14. **Projectile Motion:** The  $x$ -value of an object moving under the principles of projectile motion is  $x(\theta, v_0, t) = (v_0 \cos \theta)t$ . A particular projectile is fired with an initial velocity of  $v_0 = 250\text{ft/s}$  and an angle of elevation of  $\theta = 60^\circ$ . It travels a distance of 375ft in 3 seconds.

Is the projectile more sensitive to errors in initial speed or angle of elevation?

15. The length  $\ell$  of a long wall is to be approximated. The angle  $\theta$ , as shown in the diagram (not to scale), is measured to be  $85^\circ$ , and the distance  $x$  is measured to be  $30'$ . Assume that the triangle formed is a right triangle.

Is the measurement of the length of  $\ell$  more sensitive to errors in the measurement of  $x$  or in  $\theta$ ?



16. It is "common sense" that it is far better to measure a long distance with a long measuring tape rather than a short one. A measured distance  $D$  can be viewed as the product of the length  $\ell$  of a measuring tape times the number  $n$  of times it was used. For instance, using a 3' tape 10 times gives a length of 30'. To measure the same distance with a 12' tape, we would use the tape 2.5 times. (I.e.,  $30 = 12 \times 2.5$ .) Thus  $D = n\ell$ .

Suppose each time a measurement is taken with the tape, the recorded distance is within  $1/16''$  of the actual distance. (I.e.,  $d\ell = 1/16'' \approx 0.005\text{ft}$ ). Using differentials, show why common sense proves correct in that it is better to use a long tape to measure long distances.

In Exercises 17 – 18, find the total differential  $dw$ .

17.  $w = x^2yz^3$
18.  $w = e^x \sin y \ln z$

In Exercises 19 – 22, use the information provided and the total differential to make the given approximation.

19.  $f(3, 1) = 7$ ,  $f_x(3, 1) = 9$ ,  $f_y(3, 1) = -2$ . Approximate  $f(3.05, 0.9)$ .
20.  $f(-4, 2) = 13$ ,  $f_x(-4, 2) = 2.6$ ,  $f_y(-4, 2) = 5.1$ . Approximate  $f(-4.12, 2.07)$ .
21.  $f(2, 4, 5) = -1$ ,  $f_x(2, 4, 5) = 2$ ,  $f_y(2, 4, 5) = -3$ ,  $f_z(2, 4, 5) = 3.7$ . Approximate  $f(2.5, 4.1, 4.8)$ .
22.  $f(3, 3, 3) = 5$ ,  $f_x(3, 3, 3) = 2$ ,  $f_y(3, 3, 3) = 0$ ,  $f_z(3, 3, 3) = -2$ . Approximate  $f(3.1, 3.1, 3.1)$ .

## 11.5 The Multivariable Chain Rule

The Chain Rule, as learned in Section 2.5, states that  $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$ . If  $t = g(x)$ , we can express the Chain Rule as

$$\frac{df}{dx} = \frac{df}{dt} \frac{dt}{dx}.$$

In this section we extend the Chain Rule to functions of more than one variable.

### Theorem 57 Multivariable Chain Rule, Part I

Let  $z = f(x, y)$ ,  $x = g(t)$  and  $y = h(t)$ , where  $f$ ,  $g$  and  $h$  are differentiable functions. Then  $z = f(x, y) = f(g(t), h(t))$  is a function of  $t$ , and

$$\begin{aligned}\frac{dz}{dt} &= \frac{df}{dt} = f_x(x, y) \frac{dx}{dt} + f_y(x, y) \frac{dy}{dt} \\ &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.\end{aligned}$$

It is good to understand what the situation of  $z = f(x, y)$ ,  $x = g(t)$  and  $y = h(t)$  describes. We know that  $z = f(x, y)$  describes a surface; we also recognize that  $x = g(t)$  and  $y = h(t)$  are parametric equations for a curve in the  $x$ - $y$  plane. Combining these together, we are describing a curve that lies on the surface described by  $f$ . The parametric equations for this curve are  $x = g(t)$ ,  $y = h(t)$  and  $z = f(g(t), h(t))$ .

Consider Figure 11.14 in which a surface is drawn, along with a dashed curve in the  $x$ - $y$  plane. Restricting  $f$  to just the points on this circle gives the curve shown on the surface. The derivative  $\frac{df}{dt}$  gives the instantaneous rate of change of  $f$  with respect to  $t$ . If we consider an object traveling along this path,  $\frac{df}{dt}$  gives the rate at which the object rises/falls.

We now practice applying the Multivariable Chain Rule.

### Example 253 Using the Multivariable Chain Rule

Let  $z = x^2y + x$ , where  $x = \sin t$  and  $y = e^{5t}$ . Find  $\frac{dz}{dt}$  using the Chain Rule.

**SOLUTION** Following Theorem 57, we find

$$f_x(x, y) = 2xy + 1, \quad f_y(x, y) = x^2, \quad \frac{dx}{dt} = \cos t, \quad \frac{dy}{dt} = 5e^{5t}.$$

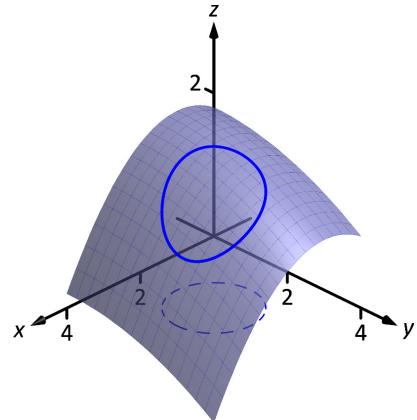


Figure 11.14: Understanding the application of the Multivariable Chain Rule.

Notes:

Applying the theorem, we have

$$\frac{dz}{dt} = (2xy + 1) \cos t + 5x^2 e^{5t}.$$

This may look odd, as it seems that  $\frac{dz}{dt}$  is a function of  $x$ ,  $y$  and  $t$ . Since  $x$  and  $y$  are functions of  $t$ ,  $\frac{dz}{dt}$  is really just a function of  $t$ , and we can replace  $x$  with  $\sin t$  and  $y$  with  $e^{5t}$ :

$$\frac{dz}{dt} = (2xy + 1) \cos t + 5x^2 e^{5t} = (2 \sin(t)e^{5t} + 1) \cos t + 5e^{5t} \sin^2 t.$$

The previous example can make us wonder: if we substituted for  $x$  and  $y$  at the end to show that  $\frac{dz}{dt}$  is really just a function of  $t$ , why not substitute before differentiating, showing clearly that  $z$  is a function of  $t$ ?

That is,  $z = x^2y + x = (\sin t)^2 e^{5t} + \sin t$ . Applying the Chain and Product Rules, we have

$$\frac{dz}{dt} = 2 \sin t \cos t e^{5t} + 5 \sin^2 t e^{5t} + \cos t,$$

which matches the result from the example.

This may now make one wonder “What’s the point? If we could already find the derivative, why learn another way of finding it?” In some cases, applying this rule makes deriving simpler, but this is hardly the power of the Chain Rule. Rather, in the case where  $z = f(x, y)$ ,  $x = g(t)$  and  $y = h(t)$ , the Chain Rule is extremely powerful when we do not know what  $f$ ,  $g$  and/or  $h$  are. It may be hard to believe, but often in “the real world” we know rate-of-change information (i.e., information about derivatives) without explicitly knowing the underlying functions. The Chain Rule allows us to combine several rates of change to find another rate of change. The Chain Rule also has theoretic use, giving us insight into the behavior of certain constructions (as we’ll see in the next section).

We demonstrate this in the next example.

#### Example 254 Applying the Multivariable Chain Rule

An object travels along a path on a surface. The exact path and surface are not known, but at time  $t = t_0$  it is known that :

$$\frac{\partial z}{\partial x} = 5, \quad \frac{\partial z}{\partial y} = -2, \quad \frac{dx}{dt} = 3 \quad \text{and} \quad \frac{dy}{dt} = 7.$$

Find  $\frac{dz}{dt}$  at time  $t_0$ .

---

Notes:

**SOLUTION** The Multivariable Chain Rule states that

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\ &= 5(3) + (-2)(7) \\ &= 1.\end{aligned}$$

By knowing certain rates-of-change information about the surface and about the path of the particle in the  $x$ - $y$  plane, we can determine how quickly the object is rising/falling.

We next apply the Chain Rule to solve a max/min problem.

**Example 255 Applying the Multivariable Chain Rule**

Consider the surface  $z = x^2 + y^2 - xy$ , a paraboloid, on which a particle moves with  $x$  and  $y$  coordinates given by  $x = \cos t$  and  $y = \sin t$ . Find  $\frac{dz}{dt}$  when  $t = 0$ , and find where the particle reaches its maximum/minimum  $z$ -values.

**SOLUTION** It is straightforward to compute

$$f_x(x, y) = 2x - y, \quad f_y(x, y) = 2y - x, \quad \frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t.$$

Combining these according to the Chain Rule gives:

$$\frac{dz}{dt} = -(2x - y) \sin t + (2y - x) \cos t.$$

When  $t = 0$ ,  $x = 1$  and  $y = 0$ . Thus  $\frac{dz}{dt} = -(2)(0) + (-1)(1) = -1$ . When  $t = 0$ , the particle is moving down, as shown in Figure 11.15.

To find where  $z$ -value is maximized/minimized on the particle's path, we set  $\frac{dz}{dt} = 0$  and solve for  $t$ :

$$\begin{aligned}\frac{dz}{dt} &= 0 = -(2x - y) \sin t + (2y - x) \cos t \\ 0 &= -(2 \cos t - \sin t) \sin t + (2 \sin t - \cos t) \cos t \\ 0 &= \sin^2 t - \cos^2 t \\ \cos^2 t &= \sin^2 t \\ t &= n \frac{\pi}{4} \quad (\text{for odd } n)\end{aligned}$$

We can use the First Derivative Test to find that on  $[0, 2\pi]$ ,  $z$  has reaches its absolute minimum at  $t = \pi/4$  and  $5\pi/4$ ; it reaches its absolute maximum at

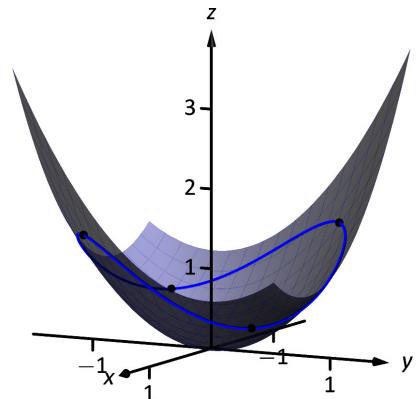


Figure 11.15: Plotting the path of a particle on a surface in Example 255.

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Notes:

$t = 3\pi/4$  and  $7\pi/4$ , as shown in Figure 11.15.

We can extend the Chain Rule to include the situation where  $z$  is a function of more than one variable, and each of these variables is also a function of more than one variable. The basic case of this is where  $z = f(x, y)$ , and  $x$  and  $y$  are functions of two variables, say  $s$  and  $t$ .

**Theorem 58 Multivariable Chain Rule, Part II**

- Let  $z = f(x, y)$ ,  $x = g(s, t)$  and  $y = h(s, t)$ , where  $f$ ,  $g$  and  $h$  are differentiable functions. Then  $z$  is a function of  $s$  and  $t$ , and

- $\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s}$ , and
- $\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}$ .

- Let  $z = f(x_1, x_2, \dots, x_m)$  be a differentiable function of  $m$  variables, where each of the  $x_i$  is a differentiable function of the variables  $t_1, t_2, \dots, t_n$ . Then  $z$  is a function of the  $t_i$ , and

$$\frac{\partial z}{\partial t_i} = \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial f}{\partial x_m} \frac{\partial x_m}{\partial t_i}.$$

**Example 256 Using the Multivariable Chain Rule, Part II**

Let  $z = x^2y + x$ ,  $x = s^2 + 3t$  and  $y = 2s - t$ . Find  $\frac{\partial z}{\partial s}$  and  $\frac{\partial z}{\partial t}$ , and evaluate each when  $s = 1$  and  $t = 2$ .

**SOLUTION**  
Following Theorem 58, we compute the following partial derivatives:

$$\frac{\partial f}{\partial x} = 2xy + 1 \quad \frac{\partial f}{\partial y} = x^2,$$

$$\frac{\partial x}{\partial s} = 2s \quad \frac{\partial x}{\partial t} = 3 \quad \frac{\partial y}{\partial s} = 2 \quad \frac{\partial y}{\partial t} = -1.$$

Thus

$$\frac{\partial z}{\partial s} = (2xy + 1)(2s) + (x^2)(2) = 4xys + 2s + 2x^2, \text{ and}$$

$$\frac{\partial z}{\partial t} = (2xy + 1)(3) + (x^2)(-1) = 6xy - x^2 + 3.$$

---

Notes:

When  $s = 1$  and  $t = 2$ ,  $x = 7$  and  $y = 0$ , so

$$\frac{\partial z}{\partial s} = 100 \quad \text{and} \quad \frac{\partial z}{\partial t} = -46.$$

**Example 257 Using the Multivariable Chain Rule, Part II**

Let  $w = xy + z^2$ , where  $x = t^2 e^s$ ,  $y = t \cos s$ , and  $z = s \sin t$ . Find  $\frac{\partial w}{\partial t}$  when  $s = 0$  and  $t = \pi$ .

**SOLUTION** Following Theorem 58, we compute the following partial derivatives:

$$\begin{aligned}\frac{\partial f}{\partial x} &= y & \frac{\partial f}{\partial y} &= x & \frac{\partial f}{\partial z} &= 2z, \\ \frac{\partial x}{\partial t} &= 2te^s & \frac{\partial y}{\partial t} &= \cos s & \frac{\partial z}{\partial t} &= s \cos t.\end{aligned}$$

Thus

$$\frac{\partial w}{\partial t} = y(2te^s) + x(\cos s) + 2z(s \cos t).$$

When  $s = 0$  and  $t = \pi$ , we have  $x = \pi^2$ ,  $y = \pi$  and  $z = 0$ . Thus

$$\frac{\partial w}{\partial t} = \pi(2\pi) + \pi^2 = 3\pi^2.$$

### Implicit Differentiation

We studied finding  $\frac{dy}{dx}$  when  $y$  is given as an implicit function of  $x$  in detail in Section 2.6. We find here that the Multivariable Chain Rule gives a simpler method of finding  $\frac{dy}{dx}$ .

For instance, consider the implicit function  $x^2y - xy^3 = 3$ . We learned to use the following steps to find  $\frac{dy}{dx}$ :

$$\begin{aligned}\frac{d}{dx}(x^2y - xy^3) &= \frac{d}{dx}(3) \\ 2xy + x^2\frac{dy}{dx} - y^3 - 3xy^2\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{2xy - y^3}{x^2 - 3xy^2}. \tag{11.2}\end{aligned}$$

Instead of using this method, consider  $z = x^2y - xy^3$ . The implicit function above describes the level curve  $z = 3$ . Considering  $x$  and  $y$  as functions of  $x$ , the Multivariable Chain Rule states that

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} \frac{dx}{dx} + \frac{\partial z}{\partial y} \frac{dy}{dx}. \tag{11.3}$$

Notes:

Since  $z$  is constant (in our example,  $z = 3$ ),  $\frac{dz}{dx} = 0$ . We also know  $\frac{dx}{dx} = 1$ . Equation (11.3) becomes

$$\begin{aligned} 0 &= \frac{\partial z}{\partial x}(1) + \frac{\partial z}{\partial y} \frac{dy}{dx} \quad \Rightarrow \\ \frac{dy}{dx} &= -\frac{\partial z}{\partial x} / \frac{\partial z}{\partial y} \\ &= -\frac{f_x}{f_y}. \end{aligned}$$

Note how our solution for  $\frac{dy}{dx}$  in Equation (11.2) is just the partial derivative of  $z$  with respect to  $x$ , divided by the partial derivative of  $z$  with respect to  $y$ .

We state the above as a theorem.

**Theorem 59      Implicit Differentiation**

Let  $f$  be a differentiable function of  $x$  and  $y$ , where  $f(x, y) = c$  defines  $y$  as an implicit function of  $x$ , for some constant  $c$ . Then

$$\frac{dy}{dx} = -\frac{f_x(x, y)}{f_y(x, y)}.$$

We practice using Theorem 59 by applying it to a problem from Section 2.6.

**Example 258      Implicit Differentiation**

Given the implicitly defined function  $\sin(x^2y^2) + y^3 = x + y$ , find  $y'$ . Note: this is the same problem as given in Example 70 of Section 2.6, where the solution took about a full page to find.

**SOLUTION** Let  $f(x, y) = \sin(x^2y^2) + y^3 - x - y$ ; the implicitly defined function above is equivalent to  $f(x, y) = 0$ . We find  $\frac{dy}{dx}$  by applying Theorem 59. We find

$$f_x(x, y) = 2xy^2 \cos(x^2y^2) - 1 \quad \text{and} \quad f_y(x, y) = 2x^2y \cos(x^2y^2) + 3y^2 - 1,$$

so

$$\frac{dy}{dx} = -\frac{2xy^2 \cos(x^2y^2) - 1}{2x^2y \cos(x^2y^2) + 3y^2 - 1},$$

which matches our solution from Example 70.

---

Notes:

# Exercises 11.5

## Terms and Concepts

- Let a level curve of  $z = f(x, y)$  be described by  $x = g(t)$ ,  $y = h(t)$ . Explain why  $\frac{dz}{dt} = 0$ .
- Fill in the blank: The single variable Chain Rule states  $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot \underline{\hspace{2cm}}$ .
- Fill in the blank: The Multivariable Chain Rule states  $\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \underline{\hspace{2cm}} + \underline{\hspace{2cm}} \cdot \frac{dy}{dt}$ .
- If  $z = f(x, y)$ , where  $x = g(t)$  and  $y = h(t)$ , we can substitute and write  $z$  as an explicit function of  $t$ . T/F: Using the Multivariable Chain Rule to find  $\frac{dz}{dt}$  is sometimes easier than first substituting and then taking the derivative.
- T/F: The Multivariable Chain Rule is only useful when all the related functions are known explicitly.
- The Multivariable Chain Rule allows us to compute implicit derivatives easily by just computing two  $\underline{\hspace{2cm}}$  derivatives.

## Problems

In Exercises 7 – 12, functions  $z = f(x, y)$ ,  $x = g(t)$  and  $y = h(t)$  are given.

- (a) Use the Multivariable Chain Rule to compute  $\frac{dz}{dt}$ .  
 (b) Evaluate  $\frac{dz}{dt}$  at the indicated  $t$ -value.

7.  $z = 3x + 4y$ ,  $x = t^2$ ,  $y = 2t$ ;  $t = 1$

8.  $z = x^2 - y^2$ ,  $x = t$ ,  $y = t^2 - 1$ ;  $t = 1$

9.  $z = 5x + 2y$ ,  $x = 2 \cos t + 1$ ,  $y = \sin t - 3$ ;  $t = \pi/4$

10.  $z = \frac{x}{y^2 + 1}$ ,  $x = \cos t$ ,  $y = \sin t$ ;  $t = \pi/2$

11.  $z = x^2 + 2y^2$ ,  $x = \sin t$ ,  $y = 3 \sin t$ ;  $t = \pi/4$

12.  $z = \cos x \sin y$ ,  $x = \pi t$ ,  $y = 2\pi t + \pi/2$ ;  $t = 3$

In Exercises 13 – 18, functions  $z = f(x, y)$ ,  $x = g(t)$  and  $y = h(t)$  are given. Find the values of  $t$  where  $\frac{dz}{dt} = 0$ . Note: these are the same surfaces/curves as found in Exercises 7 – 12.

13.  $z = 3x + 4y$ ,  $x = t^2$ ,  $y = 2t$

14.  $z = x^2 - y^2$ ,  $x = t$ ,  $y = t^2 - 1$

15.  $z = 5x + 2y$ ,  $x = 2 \cos t + 1$ ,  $y = \sin t - 3$

16.  $z = \frac{x}{y^2 + 1}$ ,  $x = \cos t$ ,  $y = \sin t$

17.  $z = x^2 + 2y^2$ ,  $x = \sin t$ ,  $y = 3 \sin t$

18.  $z = \cos x \sin y$ ,  $x = \pi t$ ,  $y = 2\pi t + \pi/2$

In Exercises 19 – 22, functions  $z = f(x, y)$ ,  $x = g(s, t)$  and  $y = h(s, t)$  are given.

- (a) Use the Multivariable Chain Rule to compute  $\frac{\partial z}{\partial s}$  and  $\frac{\partial z}{\partial t}$ .

- (b) Evaluate  $\frac{\partial z}{\partial s}$  and  $\frac{\partial z}{\partial t}$  at the indicated  $s$  and  $t$  values.

19.  $z = x^2 y$ ,  $x = s - t$ ,  $y = 2s + 4t$ ;  $s = 1, t = 0$

20.  $z = \cos(\pi x + \frac{\pi}{2}y)$ ,  $x = st^2$ ,  $y = s^2 t$ ;  $s = 1, t = 1$

21.  $z = x^2 + y^2$ ,  $x = s \cos t$ ,  $y = s \sin t$ ;  $s = 2, t = \pi/4$

22.  $z = e^{-(x^2+y^2)}$ ,  $x = t$ ,  $y = st^2$ ;  $s = 1, t = 1$

In Exercises 23 – 26, find  $\frac{dy}{dx}$  using Implicit Differentiation and Theorem 59.

23.  $x^2 \tan y = 50$

24.  $(3x^2 + 2y^3)^4 = 2$

25.  $\frac{x^2 + y}{x + y^2} = 17$

26.  $\ln(x^2 + xy + y^2) = 1$

In Exercises 27 – 30, find  $\frac{dz}{dt}$ , or  $\frac{\partial z}{\partial s}$  and  $\frac{\partial z}{\partial t}$ , using the supplied information.

27.  $\frac{\partial z}{\partial x} = 2$ ,  $\frac{\partial z}{\partial y} = 1$ ,  $\frac{dx}{dt} = 4$ ,  $\frac{dy}{dt} = -5$

28.  $\frac{\partial z}{\partial x} = 1$ ,  $\frac{\partial z}{\partial y} = -3$ ,  $\frac{dx}{dt} = 6$ ,  $\frac{dy}{dt} = 2$

29.  $\frac{\partial z}{\partial x} = -4$ ,  $\frac{\partial z}{\partial y} = 9$ ,  
 $\frac{\partial x}{\partial s} = 5$ ,  $\frac{\partial x}{\partial t} = 7$ ,  $\frac{\partial y}{\partial s} = -2$ ,  $\frac{\partial y}{\partial t} = 6$

30.  $\frac{\partial z}{\partial x} = 2$ ,  $\frac{\partial z}{\partial y} = 1$ ,  
 $\frac{\partial x}{\partial s} = -2$ ,  $\frac{\partial x}{\partial t} = 3$ ,  $\frac{\partial y}{\partial s} = 2$ ,  $\frac{\partial y}{\partial t} = -1$

## 11.6 Directional Derivatives

Partial derivatives give us an understanding of how a surface changes when we move in the  $x$  and  $y$  directions. We made the comparison to standing in a rolling meadow and heading due east: the amount of rise/fall in doing so is comparable to  $f_x$ . Likewise, the rise/fall in moving due north is comparable to  $f_y$ . The steeper the slope, the greater in magnitude  $f_y$ .

But what if we didn't move due north or east? What if we needed to move northeast and wanted to measure the amount of rise/fall? Partial derivatives alone cannot measure this. This section investigates **directional derivatives**, which do measure this rate of change.

We begin with a definition.

### Definition 67 Directional Derivatives

Let  $z = f(x, y)$  be continuous on an open set  $S$  and let  $\vec{u} = \langle u_1, u_2 \rangle$  be a unit vector. For all points  $(x, y)$ , the **directional derivative of  $f$  at  $(x, y)$  in the direction of  $\vec{u}$**  is

$$D_{\vec{u}}f(x, y) = \lim_{h \rightarrow 0} \frac{f(x + hu_1, y + hu_2) - f(x, y)}{h}.$$

The partial derivatives  $f_x$  and  $f_y$  are defined with similar limits, but only  $x$  or  $y$  varies with  $h$ , not both. Here both  $x$  and  $y$  vary with a weighted  $h$ , determined by a particular unit vector  $\vec{u}$ . This may look a bit intimidating but in reality it is not too difficult to deal with; it often just requires extra algebra. However, the following theorem reduces this algebraic load.

### Theorem 60 Directional Derivatives

Let  $z = f(x, y)$  be differentiable on an open set  $S$  containing  $(x_0, y_0)$ , and let  $\vec{u} = \langle u_1, u_2 \rangle$  be a unit vector. The directional derivative of  $f$  at  $(x_0, y_0)$  in the direction of  $\vec{u}$  is

$$D_{\vec{u}}f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2.$$

### Example 259 Computing directional derivatives

Let  $z = 14 - x^2 - y^2$  and let  $P = (1, 2)$ . Find the directional derivative of  $f$ , at  $P$ , in the following directions:

1. toward the point  $Q = (3, 4)$ ,
2. in the direction of  $\langle 2, -1 \rangle$ , and

---

Notes:

3. toward the origin.

**SOLUTION** The surface is plotted in Figure 11.16, where the point  $P = (1, 2)$  is indicated in the  $x, y$ -plane as well as the point  $(1, 2, 9)$  which lies on the surface of  $f$ . We find that  $f_x(x, y) = -2x$  and  $f_x(1, 2) = -2$ ;  $f_y(x, y) = -2y$  and  $f_y(1, 2) = -4$ .

1. Let  $\vec{u}_1$  be the unit vector that points from the point  $(1, 2)$  to the point  $Q = (3, 4)$ , as shown in the figure. The vector  $\vec{PQ} = \langle 2, 2 \rangle$ ; the unit vector in this direction is  $\vec{u}_1 = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$ . Thus the directional derivative of  $f$  at  $(1, 2)$  in the direction of  $\vec{u}_1$  is

$$D_{\vec{u}_1}f(1, 2) = -2(1/\sqrt{2}) + (-4)(1/\sqrt{2}) = -6/\sqrt{2} \approx -4.24.$$

Thus the instantaneous rate of change in moving from the point  $(1, 2, 9)$  on the surface in the direction of  $\vec{u}_1$  (which points toward the point  $Q$ ) is about  $-4.24$ . Moving in this direction moves one steeply downward.

2. We seek the directional derivative in the direction of  $\langle 2, -1 \rangle$ . The unit vector in this direction is  $\vec{u}_2 = \langle 2/\sqrt{5}, -1/\sqrt{5} \rangle$ . Thus the directional derivative of  $f$  at  $(1, 2)$  in the direction of  $\vec{u}_2$  is

$$D_{\vec{u}_2}f(1, 2) = -2(2/\sqrt{5}) + (-4)(-1/\sqrt{5}) = 0.$$

Starting on the surface of  $f$  at  $(1, 2)$  and moving in the direction of  $\langle 2, -1 \rangle$  (or  $\vec{u}_2$ ) results in no instantaneous change in  $z$ -value. This is analogous to standing on the side of a hill and choosing a direction to walk that does not change the elevation. One neither walks up nor down, rather just “along the side” of the hill.

Finding these directions of “no elevation change” is important.

3. At  $P = (1, 2)$ , the direction towards the origin is given by the vector  $\langle -1, -2 \rangle$ ; the unit vector in this direction is  $\vec{u}_3 = \langle -1/\sqrt{5}, -2/\sqrt{5} \rangle$ . The directional derivative of  $f$  at  $P$  in the direction of the origin is

$$D_{\vec{u}_3}f(1, 2) = -2(-1/\sqrt{5}) + (-4)(-2/\sqrt{5}) = 10/\sqrt{5} \approx 4.47.$$

Moving towards the origin means “walking uphill” quite steeply, with an initial slope of about 4.47.

As we study directional derivatives, it will help to make an important connection between the unit vector  $\vec{u} = \langle u_1, u_2 \rangle$  that describes the direction and the partial derivatives  $f_x$  and  $f_y$ . We start with a definition and follow this with a Key Idea.

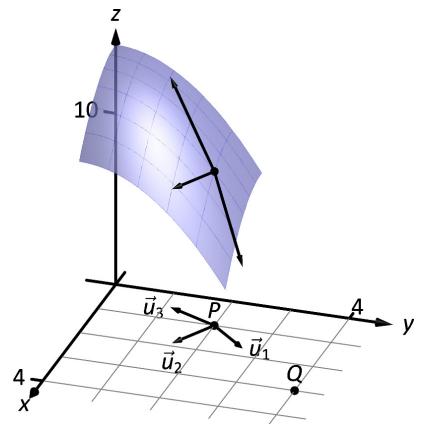


Figure 11.16: Understanding the directional derivative in Example 259.

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Notes:

**Definition 68      Gradient**

Let  $z = f(x, y)$  be differentiable on an open set  $S$  that contains the point  $(x_0, y_0)$ .

1. The **gradient of  $f$**  is  $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$ .
2. The **gradient of  $f$  at  $(x_0, y_0)$**  is  $\nabla f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle$ .

To simplify notation, we often express the gradient as  $\nabla f = \langle f_x, f_y \rangle$ . The gradient allows us to compute directional derivatives in terms of a dot product.

**Note:** The symbol “ $\nabla$ ” is named “nabla,” derived from the Greek name of a Jewish harp. Oddly enough, in mathematics the expression  $\nabla f$  is pronounced “del  $f$ .”

**Key Idea 43      The Gradient and Directional Derivatives**

The directional derivative of  $z = f(x, y)$  in the direction of  $\vec{u}$  is

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}.$$

The properties of the dot product previously studied allow us to investigate the properties of the directional derivative. Given that the directional derivative gives the instantaneous rate of change of  $z$  when moving in the direction of  $\vec{u}$ , three questions naturally arise:

1. In what direction(s) is the change in  $z$  the greatest (i.e., the “steepest up-hill”)?
2. In what direction(s) is the change in  $z$  the least (i.e., the “steepest down-hill”)?
3. In what direction(s) is there no change in  $z$ ?

Using the key property of the dot product, we have

$$\nabla f \cdot \vec{u} = \| \nabla f \| \| \vec{u} \| \cos \theta = \| \nabla f \| \cos \theta, \quad (11.4)$$

where  $\theta$  is the angle between the gradient and  $\vec{u}$ . (Since  $\vec{u}$  is a unit vector,  $\| \vec{u} \| = 1$ .) This equation allows us to answer the three questions stated previously.

1. Equation 11.4 is maximized when  $\cos \theta = 1$ , i.e., when the gradient and  $\vec{u}$  have the same direction. We conclude the gradient points in the direction of greatest  $z$  change.

---

Notes:

2. Equation 11.4 is minimized when  $\cos \theta = -1$ , i.e., when the gradient and  $\vec{u}$  have opposite directions. We conclude the gradient points in the opposite direction of the least z change.
3. Equation 11.4 is 0 when  $\cos \theta = 0$ , i.e., when the gradient and  $\vec{u}$  are orthogonal to each other. We conclude the gradient is orthogonal to directions of no z change.

This result is rather amazing. Once again imagine standing in a rolling meadow and face the direction that leads you steepest uphill. Then the direction that leads steepest downhill is directly behind you, and side-stepping either left or right (i.e., moving perpendicularly to the direction you face) does not change your elevation at all.

Recall that a level curve is defined as a curve in the  $x$ - $y$  plane along which the  $z$ -values of a function do not change. Let a surface  $z = f(x, y)$  be given, and let's represent one such level curve as a vector-valued function,  $\vec{r}(t) = \langle x(t), y(t) \rangle$ . As the output of  $f$  does not change along this curve,  $f(x(t), y(t)) = c$  for all  $t$ , for some constant  $c$ .

Since  $f$  is constant for all  $t$ ,  $\frac{df}{dt} = 0$ . By the Multivariable Chain Rule, we also know

$$\begin{aligned}\frac{df}{dt} &= f_x(x, y)x'(t) + f_y(x, y)y'(t) \\ &= \langle f_x(x, y), f_y(x, y) \rangle \cdot \langle x'(t), y'(t) \rangle \\ &= \nabla f \cdot \vec{r}'(t) \\ &= 0.\end{aligned}$$

This last equality states  $\nabla f \cdot \vec{r}'(t) = 0$ : the gradient is orthogonal to the derivative of  $\vec{r}$ , meaning the gradient is orthogonal to  $\vec{r}$  itself. Our conclusion: at any point on a surface, the gradient at that point is orthogonal to the level curve that passes through that point.

We restate these ideas in a theorem, then use them in an example.

### Theorem 61 The Gradient and Directional Derivatives

Let  $z = f(x, y)$  be differentiable on an open set  $S$  with gradient  $\nabla f$ , let  $P = (x_0, y_0)$  be a point in  $S$  and let  $\vec{u}$  be a unit vector.

1. The maximum value of  $D_{\vec{u}}f(x_0, y_0)$  is  $\|\nabla f(x_0, y_0)\|$ ; the direction of maximal z increase is  $\nabla f(x_0, y_0)$ .
2. The minimum value of  $D_{\vec{u}}f(x_0, y_0)$  is  $-\|\nabla f(x_0, y_0)\|$ ; the direction of minimal z increase is  $-\nabla f(x_0, y_0)$ .
3. At  $P$ ,  $\nabla f(x_0, y_0)$  is orthogonal to the level curve passing through  $(x_0, y_0, f(x_0, y_0))$ .

Notes:

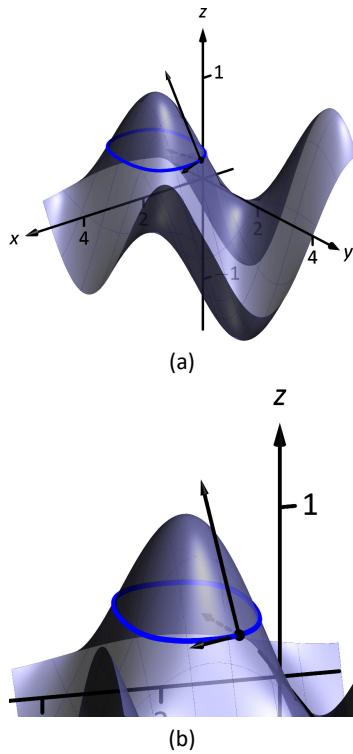


Figure 11.17: Graphing the surface and important directions in Example 260.

**Example 260** **Finding directions of maximal and minimal increase**

Let  $f(x, y) = \sin x \cos y$  and let  $P = (\pi/3, \pi/3)$ . Find the directions of maximal/minimal increase, and find a direction where the instantaneous rate of  $z$  change is 0.

**SOLUTION** We begin by finding the gradient.  $f_x = \cos x \cos y$  and  $f_y = -\sin x \sin y$ , thus

$$\nabla f = \langle \cos x \cos y, -\sin x \sin y \rangle \quad \text{and, at } P, \quad \nabla f \left( \frac{\pi}{3}, \frac{\pi}{3} \right) = \left\langle \frac{1}{4}, -\frac{3}{4} \right\rangle.$$

Thus the direction of maximal increase is  $\langle 1/4, -3/4 \rangle$ . In this direction, the instantaneous rate of  $z$  change is  $\|\langle 1/4, -3/4 \rangle\| = \sqrt{10}/4 \approx 0.79$ .

Figure 11.17 shows the surface plotted from two different perspectives. In each, the gradient is drawn at  $P$  with a dashed line (because of the nature of this surface, the gradient points “into” the surface). Let  $\vec{u} = \langle u_1, u_2 \rangle$  be the unit vector in the direction of  $\nabla f$  at  $P$ . Each graph of the figure also contains the vector  $\langle u_1, u_2, \|\nabla f\| \rangle$ . This vector has a “run” of  $\|\nabla f\|$  (because in the  $x$ - $y$  plane it moves 1 unit) and a “rise” of  $\|\nabla f\|$ , hence we can think of it as a vector with slope of  $\|\nabla f\|$  in the direction of  $\nabla f$ , helping us visualize how “steep” the surface is in its steepest direction.

The direction of minimal increase is  $\langle -1/4, 3/4 \rangle$ ; in this direction the instantaneous rate of  $z$  change is  $-\sqrt{10}/4 \approx -0.79$ .

Any direction orthogonal to  $\nabla f$  is a direction of no  $z$  change. We have two choices: the direction of  $\langle 3, 1 \rangle$  and the direction of  $\langle -3, -1 \rangle$ . The unit vector in the direction of  $\langle 3, 1 \rangle$  is shown in each graph of the figure as well. The level curve at  $z = \sqrt{3}/4$  is drawn: recall that along this curve the  $z$ -values do not change. Since  $\langle 3, 1 \rangle$  is a direction of no  $z$ -change, this vector is tangent to the level curve at  $P$ .

**Example 261** **Understanding when  $\nabla f = \vec{0}$**

Let  $f(x, y) = -x^2 + 2x - y^2 + 2y + 1$ . Find the directional derivative of  $f$  in any direction at  $P = (1, 1)$ .

**SOLUTION** We find  $\nabla f = \langle -2x + 2, -2y + 2 \rangle$ . At  $P$ , we have  $\nabla f(1, 1) = \langle 0, 0 \rangle$ . According to Theorem 61, this is the direction of maximal increase. However,  $\langle 0, 0 \rangle$  is directionless; it has no displacement. And regardless of the unit vector  $\vec{u}$  chosen,  $D_{\vec{u}}f = 0$ .

Figure 11.18 helps us understand what this means. We can see that  $P$  lies at the top of a paraboloid. In all directions, the instantaneous rate of change is 0.

So what is the direction of maximal increase? It is fine to give an answer of  $\vec{0} = \langle 0, 0 \rangle$ , as this indicates that all directional derivatives are 0.

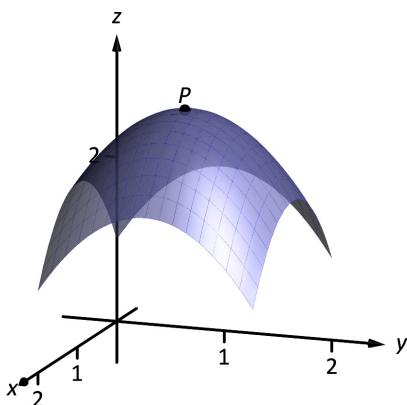


Figure 11.18: At the top of a paraboloid, all directional derivatives are 0.

Notes:

The fact that the gradient of a surface always points in the direction of steepest increase/decrease is very useful, as illustrated in the following example.

**Example 262 The flow of water downhill**

Consider the surface given by  $f(x, y) = 20 - x^2 - 2y^2$ . Water is poured on the surface at  $(1, 1/4)$ . What path does it take as it flows downhill?

**SOLUTION** Let  $\vec{r}(t) = \langle x(t), y(t) \rangle$  be the vector-valued function describing the path of the water in the  $x$ - $y$  plane; we seek  $x(t)$  and  $y(t)$ . We know that water will always flow downhill in the steepest direction; therefore, at any point on its path, it will be moving in the direction of  $-\nabla f$ . (We ignore the physical effects of momentum on the water.) Thus  $\vec{r}'(t)$  will be parallel to  $\nabla f$ , and there is some constant  $c$  such that  $c\nabla f = \vec{r}'(t) = \langle x'(t), y'(t) \rangle$ .

We find  $\nabla f = \langle -2x, -4y \rangle$  and write  $x'(t)$  as  $\frac{dx}{dt}$  and  $y'(t)$  as  $\frac{dy}{dt}$ . Then

$$\begin{aligned} c\nabla f &= \langle x'(t), y'(t) \rangle \\ \langle -2cx, -4cy \rangle &= \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle. \end{aligned}$$

This implies

$$-2cx = \frac{dx}{dt} \quad \text{and} \quad -4cy = \frac{dy}{dt}, \text{ i.e.,}$$

$$c = -\frac{1}{2x} \frac{dx}{dt} \quad \text{and} \quad c = -\frac{1}{4y} \frac{dy}{dt}.$$

As  $c$  equals both expressions, we have

$$\frac{1}{2x} \frac{dx}{dt} = \frac{1}{4y} \frac{dy}{dt}.$$

To find an explicit relationship between  $x$  and  $y$ , we can integrate both sides with respect to  $t$ . Recall from our study of differentials that  $\frac{dx}{dt} dt = dx$ . Thus:

$$\begin{aligned} \int \frac{1}{2x} \frac{dx}{dt} dt &= \int \frac{1}{4y} \frac{dy}{dt} dt \\ \int \frac{1}{2x} dx &= \int \frac{1}{4y} dy \\ \frac{1}{2} \ln|x| &= \frac{1}{4} \ln|y| + C_1 \\ 2 \ln|x| &= \ln|y| + C_1 \\ \ln|x^2| &= \ln|y| + C_1 \end{aligned}$$

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Notes:

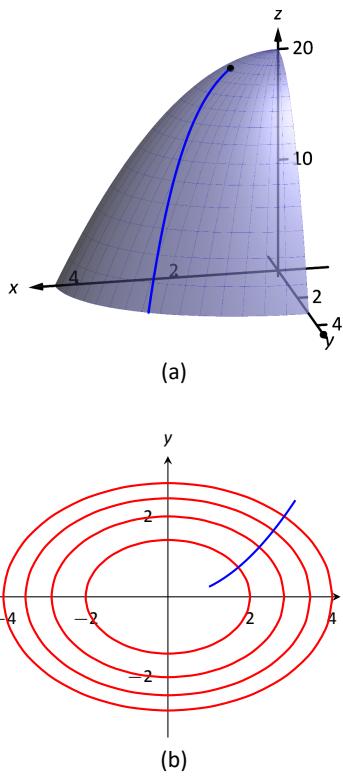


Figure 11.19: A graph of the surface described in Example 262 along with the path in the  $x$ - $y$  plane with the level curves.

Now raise both sides as a power of  $e$ :

$$\begin{aligned}x^2 &= e^{\ln|y|+C_1} \\x^2 &= e^{\ln|y|}e^{C_1} \quad (\text{Note that } e^{C_1} \text{ is just a constant.}) \\x^2 &= yC_2 \\ \frac{1}{C_2}x^2 &= y \quad (\text{Note that } 1/C_2 \text{ is just a constant.}) \\Cx^2 &= y.\end{aligned}$$

As the water started at the point  $(1, 1/4)$ , we can solve for  $C$ :

$$C(1)^2 = \frac{1}{4} \Rightarrow C = \frac{1}{4}.$$

Thus the water follows the curve  $y = x^2/4$  in the  $x$ - $y$  plane. The surface and the path of the water is graphed in Figure 11.19(a). In part (b) of the figure, the level curves of the surface are plotted in the  $x$ - $y$  plane, along with the curve  $y = x^2/4$ . Notice how the path intersects the level curves at right angles. As the path follows the gradient downhill, this reinforces the fact that the gradient is orthogonal to level curves.

### Functions of Three Variables

The concepts of directional derivatives and the gradient are easily extended to three (and more) variables. We combine the concepts behind Definitions 67 and 68 and Theorem 60 into one set of definitions.

#### Definition 69 Directional Derivatives and Gradient with Three Variables

Let  $w = F(x, y, z)$  be differentiable on an open ball  $B$  and let  $\vec{u}$  be a unit vector in  $\mathbb{R}^3$ .

1. The **gradient** of  $F$  is  $\nabla F = \langle F_x, F_y, F_z \rangle$ .
2. The **directional derivative of  $F$  in the direction of  $\vec{u}$**  is

$$D_{\vec{u}}F = \nabla F \cdot \vec{u}.$$

The same properties of the gradient given in Theorem 61, when  $f$  is a function

---

Notes:

of two variables, hold for  $F$ , a function of three variables.

**Theorem 62    The Gradient and Directional Derivatives with Three Variables**

Let  $w = F(x, y, z)$  be differentiable on an open ball  $B$ , let  $\nabla F$  be the gradient of  $F$ , and let  $\vec{u}$  be a unit vector.

1. The maximum value of  $D_{\vec{u}} F$  is  $\|\nabla F\|$ , obtained when the angle between  $\nabla F$  and  $\vec{u}$  is 0, i.e., the direction of maximal increase is  $\nabla F$ .
2. The minimum value of  $D_{\vec{u}} F$  is  $-\|\nabla F\|$ , obtained when the angle between  $\nabla F$  and  $\vec{u}$  is  $\pi$ , i.e., the direction of minimal increase is  $-\nabla F$ .
3.  $D_{\vec{u}} F = 0$  when  $\nabla F$  and  $\vec{u}$  are orthogonal.

We interpret the third statement of the theorem as “the gradient is orthogonal to level surfaces,” the three-variable analogue to level curves.

**Example 263    Finding directional derivatives with functions of three variables**

If a point source  $S$  is radiating energy, the intensity  $I$  at a given point  $P$  in space is inversely proportional to the square of the distance between  $S$  and  $P$ . That is, when  $S = (0, 0, 0)$ ,  $I(x, y, z) = \frac{k}{x^2 + y^2 + z^2}$  for some constant  $k$ .

Let  $k = 1$ , let  $\vec{u} = \langle 2/3, 2/3, 1/3 \rangle$  be a unit vector, and let  $P = (2, 5, 3)$ . Measure distances in inches. Find the directional derivative of  $I$  at  $P$  in the direction of  $\vec{u}$ , and find the direction of greatest intensity increase at  $P$ .

**SOLUTION** We need the gradient  $\nabla I$ , meaning we need  $I_x$ ,  $I_y$  and  $I_z$ . Each partial derivative requires a simple application of the Quotient Rule, giving

$$\begin{aligned}\nabla I &= \left\langle \frac{-2x}{(x^2 + y^2 + z^2)^2}, \frac{-2y}{(x^2 + y^2 + z^2)^2}, \frac{-2z}{(x^2 + y^2 + z^2)^2} \right\rangle \\ \nabla I(2, 5, 3) &= \left\langle \frac{-4}{1444}, \frac{-10}{1444}, \frac{-6}{1444} \right\rangle \approx \langle -0.003, -0.007, -0.004 \rangle\end{aligned}$$

$$\begin{aligned}D_{\vec{u}} I &= \nabla I(2, 5, 3) \cdot \vec{u} \\ &= -\frac{17}{2166} \approx -0.0078.\end{aligned}$$

The directional derivative tells us that moving in the direction of  $\vec{u}$  from  $P$  results in a decrease in intensity of about  $-0.008$  units per inch. (The intensity is decreasing as  $\vec{u}$  moves one farther from the origin than  $P$ .)

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Notes:

The gradient gives the direction of greatest intensity increase. Notice that

$$\begin{aligned}\nabla I(2, 5, 3) &= \left\langle \frac{-4}{1444}, \frac{-10}{1444}, \frac{-6}{1444} \right\rangle \\ &= \frac{2}{1444} \langle -2, -5, -3 \rangle.\end{aligned}$$

That is, the gradient at  $(2, 5, 3)$  is pointing in the direction of  $\langle -2, -5, -3 \rangle$ , that is, towards the origin. That should make intuitive sense: the greatest increase in intensity is found by moving towards to source of the energy.

The directional derivative allows us to find the instantaneous rate of  $z$  change in any direction at a point. We can use these instantaneous rates of change to define lines and planes that are *tangent* to a surface at a point, which is the topic of the next section.

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Notes:

# Exercises 11.6

## Terms and Concepts

1. What is the difference between a directional derivative and a partial derivative?
2. For what  $\vec{u}$  is  $D_{\vec{u}}f = f_x$ ?
3. For what  $\vec{u}$  is  $D_{\vec{u}}f = f_y$ ?
4. The gradient is \_\_\_\_\_ to level curves.
5. The gradient points in the direction of \_\_\_\_\_ increase.
6. It is generally more informative to view the directional derivative not as the result of a limit, but rather as the result of a \_\_\_\_\_ product.

## Problems

In Exercises 7 – 12, a function  $z = f(x, y)$  is given. Find  $\nabla f$ .

7.  $f(x, y) = -x^2y + xy^2 + xy$

8.  $f(x, y) = \sin x \cos y$

9.  $f(x, y) = \frac{1}{x^2 + y^2 + 1}$

10.  $f(x, y) = -4x + 3y$

11.  $f(x, y) = x^2 + 2y^2 - xy - 7x$

12.  $f(x, y) = x^2y^3 - 2x$

In Exercises 13 – 18, a function  $z = f(x, y)$  and a point  $P$  are given. Find the directional derivative of  $f$  in the indicated directions. Note: these are the same functions as in Exercises 7 through 12.

13.  $f(x, y) = -x^2y + xy^2 + xy, P = (2, 1)$

(a) In the direction of  $\vec{v} = \langle 3, 4 \rangle$

(b) In the direction toward the point  $Q = (1, -1)$ .

14.  $f(x, y) = \sin x \cos y, P = \left(\frac{\pi}{4}, \frac{\pi}{3}\right)$

(a) In the direction of  $\vec{v} = \langle 1, 1 \rangle$ .

(b) In the direction toward the point  $Q = (0, 0)$ .

15.  $f(x, y) = \frac{1}{x^2 + y^2 + 1}, P = (1, 1)$ .

(a) In the direction of  $\vec{v} = \langle 1, -1 \rangle$ .

(b) In the direction toward the point  $Q = (-2, -2)$ .

16.  $f(x, y) = -4x + 3y, P = (5, 2)$

(a) In the direction of  $\vec{v} = \langle 3, 1 \rangle$ .

(b) In the direction toward the point  $Q = (2, 7)$ .

17.  $f(x, y) = x^2 + 2y^2 - xy - 7x, P = (4, 1)$

(a) In the direction of  $\vec{v} = \langle -2, 5 \rangle$

(b) In the direction toward the point  $Q = (4, 0)$ .

18.  $f(x, y) = x^2y^3 - 2x, P = (1, 1)$

(a) In the direction of  $\vec{v} = \langle 3, 3 \rangle$

(b) In the direction toward the point  $Q = (1, 2)$ .

In Exercises 19 – 24, a function  $z = f(x, y)$  and a point  $P$  are given.

(a) Find the direction of maximal increase of  $f$  at  $P$ .

(b) What is the maximal value of  $D_{\vec{u}}f$  at  $P$ ?

(c) Find the direction of minimal increase of  $f$  at  $P$ .

(d) Give a direction  $\vec{u}$  such that  $D_{\vec{u}}f = 0$  at  $P$ .

Note: these are the same functions and points as in Exercises 13 through 18.

19.  $f(x, y) = -x^2y + xy^2 + xy, P = (2, 1)$

20.  $f(x, y) = \sin x \cos y, P = \left(\frac{\pi}{4}, \frac{\pi}{3}\right)$

21.  $f(x, y) = \frac{1}{x^2 + y^2 + 1}, P = (1, 1)$ .

22.  $f(x, y) = -4x + 3y, P = (5, 4)$ .

23.  $f(x, y) = x^2 + 2y^2 - xy - 7x, P = (4, 1)$

24.  $f(x, y) = x^2y^3 - 2x, P = (1, 1)$

In Exercises 25 – 28, a function  $w = F(x, y, z)$ , a vector  $\vec{v}$  and a point  $P$  are given.

(a) Find  $\nabla F(x, y, z)$ .

(b) Find  $D_{\vec{u}}F$  at  $P$ .

25.  $F(x, y, z) = 3x^2z^3 + 4xy - 3z^2, \vec{v} = \langle 1, 1, 1 \rangle, P = (3, 2, 1)$

26.  $F(x, y, z) = \sin(x) \cos(y)e^z, \vec{v} = \langle 2, 2, 1 \rangle, P = (0, 0, 0)$

27.  $F(x, y, z) = x^2y^2 - y^2z^2, \vec{v} = \langle -1, 7, 3 \rangle, P = (1, 0, -1)$

28.  $F(x, y, z) = \frac{2}{x^2 + y^2 + z^2}, \vec{v} = \langle 1, 1, -2 \rangle, P = (1, 1, 1)$

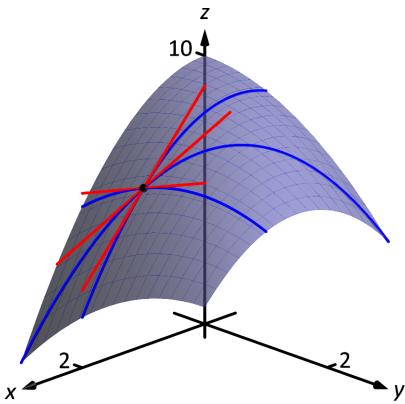


Figure 11.20: Showing various lines tangent to a surface.

## 11.7 Tangent Lines, Normal Lines, and Tangent Planes

Derivatives and tangent lines go hand-in-hand. Given  $y = f(x)$ , the line tangent to the graph of  $f$  at  $x = x_0$  is the line through  $(x_0, f(x_0))$  with slope  $f'(x_0)$ ; that is, the slope of the tangent line is the instantaneous rate of change of  $f$  at  $x_0$ .

When dealing with functions of two variables, the graph is no longer a curve but a surface. At a given point on the surface, it seems there are many lines that fit our intuition of being “tangent” to the surface.

In Figures 11.20 we see lines that are tangent to curves in space. Since each curve lies on a surface, it makes sense to say that the lines are also tangent to the surface. The next definition formally defines what it means to be “tangent to a surface.”

### Definition 70 Directional Tangent Line

Let  $z = f(x, y)$  be differentiable on an open set  $S$  containing  $(x_0, y_0)$  and let  $\vec{u} = \langle u_1, u_2 \rangle$  be a unit vector.

1. The line  $\ell_x$  through  $(x_0, y_0, f(x_0, y_0))$  parallel to  $\langle 1, 0, f_x(x_0, y_0) \rangle$  is the **tangent line to  $f$  in the direction of  $x$  at  $(x_0, y_0)$** .
2. The line  $\ell_y$  through  $(x_0, y_0, f(x_0, y_0))$  parallel to  $\langle 0, 1, f_y(x_0, y_0) \rangle$  is the **tangent line to  $f$  in the direction of  $y$  at  $(x_0, y_0)$** .
3. The line  $\ell_{\vec{u}}$  through  $(x_0, y_0, f(x_0, y_0))$  parallel to  $\langle u_1, u_2, D_{\vec{u}}f(x_0, y_0) \rangle$  is the **tangent line to  $f$  in the direction of  $\vec{u}$  at  $(x_0, y_0)$** .

It is instructive to consider each of three directions given in the definition in terms of “slope.” The direction of  $\ell_x$  is  $\langle 1, 0, f_x(x_0, y_0) \rangle$ ; that is, the “run” is one unit in the  $x$ -direction and the “rise” is  $f_x(x_0, y_0)$  units in the  $z$ -direction. Note how the slope is just the partial derivative with respect to  $x$ . A similar statement can be made for  $\ell_y$ . The direction of  $\ell_{\vec{u}}$  is  $\langle u_1, u_2, D_{\vec{u}}f(x_0, y_0) \rangle$ ; the “run” is one unit in the  $\vec{u}$  direction (where  $\vec{u}$  is a unit vector) and the “rise” is the directional derivative of  $z$  in that direction.

Definition 70 leads to the following parametric equations of directional tangent lines:

$$\ell_x(t) = \begin{cases} x = x_0 + t \\ y = y_0 \\ z = z_0 + f_x(x_0, y_0)t \end{cases}, \quad \ell_y(t) = \begin{cases} x = x_0 \\ y = y_0 + t \\ z = z_0 + f_y(x_0, y_0)t \end{cases} \quad \text{and} \quad \ell_{\vec{u}}(t) = \begin{cases} x = x_0 + u_1 t \\ y = y_0 + u_2 t \\ z = z_0 + D_{\vec{u}}f(x_0, y_0)t \end{cases}.$$

---

Notes:

**Example 264 Finding directional tangent lines**

Find the lines tangent to the surface  $z = \sin x \cos y$  at  $(\pi/2, \pi/2)$  in the  $x$  and  $y$  directions and also in the direction of  $\vec{v} = \langle -1, 1 \rangle$ .

**SOLUTION** The partial derivatives with respect to  $x$  and  $y$  are:

$$\begin{aligned} f_x(x, y) &= \cos x \cos y \Rightarrow f_x(\pi/2, \pi/2) = 0 \\ f_y(x, y) &= -\sin x \sin y \Rightarrow f_y(\pi/2, \pi/2) = -1. \end{aligned}$$

At  $(\pi/2, \pi/2)$ , the  $z$ -value is 0.

Thus the parametric equations of the line tangent to  $f$  at  $(\pi/2, \pi/2)$  in the directions of  $x$  and  $y$  are:

$$\ell_x(t) = \begin{cases} x = \pi/2 + t \\ y = \pi/2 \\ z = 0 \end{cases} \quad \text{and} \quad \ell_y(t) = \begin{cases} x = \pi/2 \\ y = \pi/2 + t \\ z = -t \end{cases}.$$

The two lines are shown with the surface in Figure 11.21(a). To find the equation of the tangent line in the direction of  $\vec{v}$ , we first find the unit vector in the direction of  $\vec{v}$ :  $\vec{u} = \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle$ . The directional derivative at  $(\pi/2, \pi/2, 0)$  in the direction of  $\vec{u}$  is

$$D_{\vec{u}}f(\pi/2, \pi/2, 0) = \langle 0, -1 \rangle \cdot \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle = -1/\sqrt{2}.$$

Thus the directional tangent line is

$$\ell_{\vec{u}}(t) = \begin{cases} x = \pi/2 - t/\sqrt{2} \\ y = \pi/2 + t/\sqrt{2} \\ z = -t/\sqrt{2} \end{cases}.$$

The curve through  $(\pi/2, \pi/2, 0)$  in the direction of  $\vec{v}$  is shown in Figure 11.21(b) along with  $\ell_{\vec{u}}(t)$ .

**Example 265 Finding directional tangent lines**

Let  $f(x, y) = 4xy - x^4 - y^4$ . Find the equations of all directional tangent lines to  $f$  at  $(1, 1)$ .

**SOLUTION** First note that  $f(1, 1) = 2$ . We need to compute directional derivatives, so we need  $\nabla f$ . We begin by computing partial derivatives.

$$f_x = 4y - 4x^3 \Rightarrow f_x(1, 1) = 0; \quad f_y = 4x - 4y^3 \Rightarrow f_y(1, 1) = 0.$$

Thus  $\nabla f(1, 1) = \langle 0, 0 \rangle$ . Let  $\vec{u} = \langle u_1, u_2 \rangle$  be any unit vector. The directional derivative of  $f$  at  $(1, 1)$  will be  $D_{\vec{u}}f(1, 1) = \langle 0, 0 \rangle \cdot \langle u_1, u_2 \rangle = 0$ . It does not matter

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Notes:

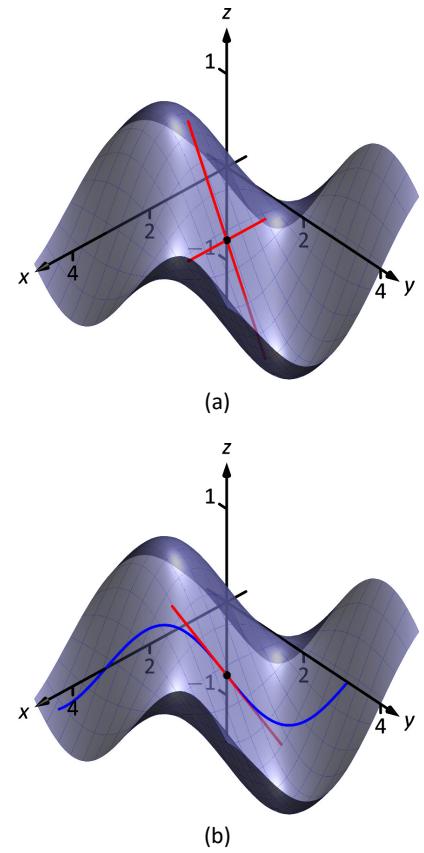
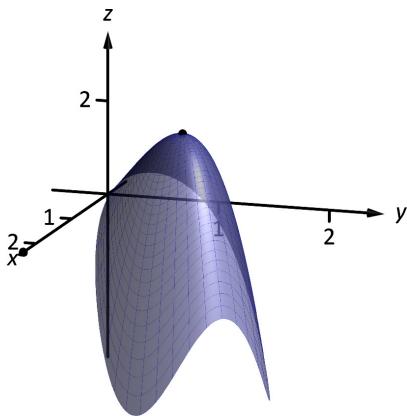


Figure 11.21: A surface and directional tangent lines in Example 264.

Figure 11.22: Graphing  $f$  in Example 265.

what direction we choose; the directional derivative is always 0. Therefore

$$\ell_{\vec{u}}(t) = \begin{cases} x = 1 + u_1 t \\ y = 1 + u_2 t \\ z = 2 \end{cases} .$$

Figure 11.22 shows a graph of  $f$  and the point  $(1, 1, 2)$ . Note that this point comes at the top of a “hill,” and therefore every tangent line through this point will have a “slope” of 0.

That is, consider any curve on the surface that goes through this point. Each curve will have a relative maximum at this point, hence its tangent line will have a slope of 0. The following section investigates the points on surfaces where all tangent lines have a slope of 0.

### Normal Lines

When dealing with a function  $y = f(x)$  of one variable, we stated that a line through  $(c, f(c))$  was *tangent to  $f$*  if the line had a slope of  $f'(c)$  and was *normal* (or, *perpendicular, orthogonal*) to  $f$  if it had a slope of  $-1/f'(c)$ . We extend the concept of normal, or orthogonal, to functions of two variables.

Let  $z = f(x, y)$  be a differentiable function of two variables. By Definition 70, at  $(x_0, y_0)$ ,  $\ell_x(t)$  is a line parallel to the vector  $\vec{d}_x = \langle 1, 0, f_x(x_0, y_0) \rangle$  and  $\ell_y(t)$  is a line parallel to  $\vec{d}_y = \langle 0, 1, f_y(x_0, y_0) \rangle$ . Since lines in these directions through  $(x_0, y_0, f(x_0, y_0))$  are *tangent* to the surface, a line through this point and orthogonal to these directions would be *orthogonal*, or *normal*, to the surface. We can use this direction to create a normal line.

The direction of the normal line is orthogonal to  $\vec{d}_x$  and  $\vec{d}_y$ , hence the direction is parallel to  $\vec{d}_n = \vec{d}_x \times \vec{d}_y$ . It turns out this cross product has a very simple form:

$$\vec{d}_x \times \vec{d}_y = \langle 1, 0, f_x \rangle \times \langle 0, 1, f_y \rangle = \langle -f_x, -f_y, 1 \rangle .$$

It is often more convenient to refer to the opposite of this direction, namely  $\langle f_x, f_y, -1 \rangle$ . This leads to a definition.

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Notes:

**Definition 71 Normal Line**

Let  $z = f(x, y)$  be differentiable on an open set  $S$  containing  $(x_0, y_0)$  where

$$a = f_x(x_0, y_0) \quad \text{and} \quad b = f_y(x_0, y_0)$$

are defined.

1. A nonzero vector parallel to  $\vec{n} = \langle a, b, -1 \rangle$  is **orthogonal to  $f$  at  $P = (x_0, y_0, f(x_0, y_0))$** .
2. The line  $\ell_n$  through  $P$  with direction parallel to  $\vec{n}$  is the **normal line to  $f$  at  $P$** .

Thus the parametric equations of the normal line to a surface  $f$  at  $(x_0, y_0, f(x_0, y_0))$  is:

$$\ell_n(t) = \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = f(x_0, y_0) - t \end{cases} .$$

**Example 266 Finding a normal line**

Find the equation of the normal line to  $z = -x^2 - y^2 + 2$  at  $(0, 1)$ .

**SOLUTION** We find  $z_x(x, y) = -2x$  and  $z_y(x, y) = -2y$ ; at  $(0, 1)$ , we have  $z_x = 0$  and  $z_y = -2$ . We take the direction of the normal line, following Definition 71, to be  $\vec{n} = \langle 0, -2, -1 \rangle$ . The line with this direction going through the point  $(0, 1, 1)$  is

$$\ell_n(t) = \begin{cases} x = 0 \\ y = -2t + 1 \\ z = -t + 1 \end{cases} \quad \text{or} \quad \ell_n(t) = \langle 0, -2, -1 \rangle t + \langle 0, 1, 1 \rangle .$$

The surface  $z = -x^2 - y^2$ , along with the found normal line, is graphed in Figure 11.23.

The direction of the normal line has many uses, one of which is the definition of the **tangent plane** which we define shortly. Another use is in measuring distances from the surface to a point. Given a point  $Q$  in space, it is general geometric concept to define the distance from  $Q$  to the surface as being the length of the shortest line segment  $\overrightarrow{PQ}$  over all points  $P$  on the surface. This, in turn, implies that  $\overrightarrow{PQ}$  will be orthogonal to the surface at  $P$ . Therefore we can measure the distance from  $Q$  to the surface  $f$  by finding a point  $P$  on the surface such

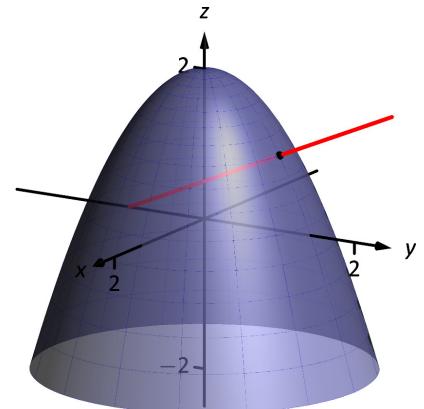


Figure 11.23: Graphing a surface with a normal line from Example 266.

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Notes:

that  $\vec{PQ}$  is parallel to the normal line to  $f$  at  $P$ .

**Example 267 Finding the distance from a point to a surface**

Let  $f(x, y) = 2 - x^2 - y^2$  and let  $Q = (2, 2, 2)$ . Find the distance from  $Q$  to the surface defined by  $f$ .

**SOLUTION** This surface is used in Example 265, so we know that at  $(x, y)$ , the direction of the normal line will be  $\vec{d}_n = \langle -2x, -2y, -1 \rangle$ . A point  $P$  on the surface will have coordinates  $(x, y, 2 - x^2 - y^2)$ , so  $\vec{PQ} = \langle 2 - x, 2 - y, x^2 + y^2 \rangle$ . To find where  $\vec{PQ}$  is parallel to  $\vec{d}_n$ , we need to find  $x, y$  and  $c$  such that  $c\vec{PQ} = \vec{d}_n$ .

$$\begin{aligned} c\vec{PQ} &= \vec{d}_n \\ c\langle 2 - x, 2 - y, x^2 + y^2 \rangle &= \langle -2x, -2y, -1 \rangle. \end{aligned}$$

This implies

$$\begin{aligned} c(2 - x) &= -2x \\ c(2 - y) &= -2y \\ c(x^2 + y^2) &= -1 \end{aligned}$$

In each equation, we can solve for  $c$ :

$$c = \frac{-2x}{2 - x} = \frac{-2y}{2 - y} = \frac{-1}{x^2 + y^2}.$$

The first two fractions imply  $x = y$ , and so the last fraction can be rewritten as  $c = -1/(2x^2)$ . Then

$$\begin{aligned} \frac{-2x}{2 - x} &= \frac{-1}{2x^2} \\ -2x(2x^2) &= -1(2 - x) \\ 4x^3 &= 2 - x \\ 4x^3 + x - 2 &= 0. \end{aligned}$$

This last equation is a cubic, which is not difficult to solve with a numeric solver. We find that  $x = 0.689$ , hence  $P = (0.689, 0.689, 1.051)$ . We find the distance from  $Q$  to the surface of  $f$  is

$$\| \vec{PQ} \| = \sqrt{(2 - 0.689)^2 + (2 - 0.689)^2 + (2 - 1.051)^2} = 2.083.$$

We can take the concept of measuring the distance from a point to a surface to find a point  $Q$  a particular distance from a surface at a given point  $P$  on the

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Notes:

surface.

**Example 268 Finding a point a set distance from a surface**

Let  $f(x, y) = x - y^2 + 3$ . Let  $P = (2, 1, f(2, 1)) = (2, 1, 4)$ . Find points  $Q$  in space that are 4 units from the surface of  $f$  at  $P$ . That is, find  $Q$  such that  $\|\overrightarrow{PQ}\| = 4$  and  $\overrightarrow{PQ}$  is orthogonal to  $f$  at  $P$ .

**SOLUTION** We begin by finding partial derivatives:

$$\begin{aligned} f_x(x, y) &= 1 & \Rightarrow & f_x(2, 1) = 1 \\ f_y(x, y) &= -2y & \Rightarrow & f_y(2, 1) = -2 \end{aligned}$$

The vector  $\vec{n} = \langle 1, -2, -1 \rangle$  is orthogonal to  $f$  at  $P$ . For reasons that will become more clear in a moment, we find the unit vector in the direction of  $\vec{n}$ :

$$\vec{u} = \frac{\vec{n}}{\|\vec{n}\|} = \left\langle \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \right\rangle \approx \langle 0.408, -0.816, -0.408 \rangle.$$

Thus a the normal line to  $f$  at  $P$  can be written as

$$\ell_n(t) = \langle 2, 1, 4 \rangle + t \langle 0.408, -0.816, -0.408 \rangle.$$

An advantage of this parametrization of the line is that letting  $t = t_0$  gives a point on the line that is  $|t_0|$  units from  $P$ . (This is because the direction of the line is given in terms of a unit vector.) There are thus two points in space 4 units from  $P$ :

$$\begin{aligned} Q_1 &= \ell_n(4) & Q_2 &= \ell_n(-4) \\ &\approx \langle 3.63, -2.27, 2.37 \rangle && \approx \langle 0.37, 4.27, 5.63 \rangle \end{aligned}$$

The surface is graphed along with points  $P$ ,  $Q_1$ ,  $Q_2$  and a portion of the normal line to  $f$  at  $P$ .

## Tangent Planes

We can use the direction of the normal line to define a plane. With  $a = f_x(x_0, y_0)$ ,  $b = f_y(x_0, y_0)$  and  $P = (x_0, y_0, f(x_0, y_0))$ , the vector  $\vec{n} = \langle a, b, -1 \rangle$  is orthogonal to  $f$  at  $P$ . The plane through  $P$  with normal vector  $\vec{n}$  is therefore **tangent** to  $f$  at  $P$ .

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Notes:

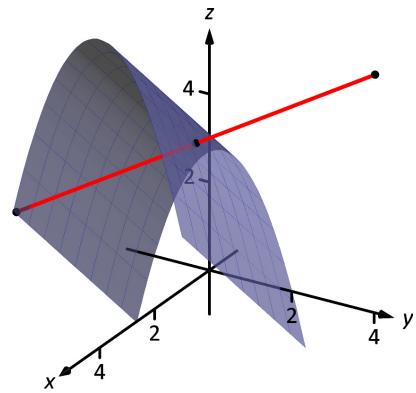


Figure 11.24: Graphing the surface in Example 268 along with points 4 units from the surface.

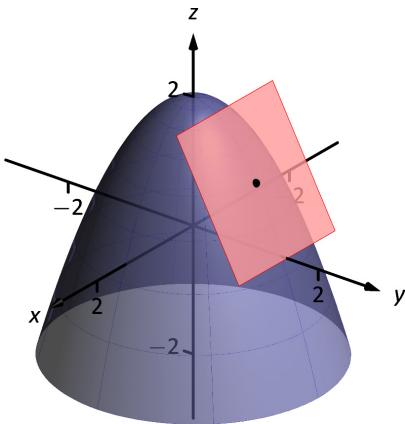


Figure 11.25: Graphing a surface with tangent plane from Example 269.

### Definition 72 Tangent Plane

Let  $z = f(x, y)$  be differentiable on an open set  $S$  containing  $(x_0, y_0)$ , where  $a = f_x(x_0, y_0)$ ,  $b = f_y(x_0, y_0)$ ,  $\vec{n} = \langle a, b, -1 \rangle$  and  $P = (x_0, y_0, f(x_0, y_0))$ .

The plane through  $P$  with normal vector  $\vec{n}$  is the **tangent plane to  $f$  at  $P$** . The standard form of this plane is

$$a(x - x_0) + b(y - y_0) - (z - f(x_0, y_0)) = 0.$$

### Example 269 Finding tangent planes

Find the equation of the tangent plane to  $z = -x^2 - y^2 + 2$  at  $(0, 1)$ .

**SOLUTION** Note that this is the same surface and point used in Example 266. There we found  $\vec{n} = \langle 0, -2, -1 \rangle$  and  $P = (0, 1, 1)$ . Therefore the equation of the tangent plane is

$$-2(y - 1) - (z - 1) = 0.$$

The surface  $z = -x^2 - y^2$  and tangent plane are graphed in Figure 11.25.

### Example 270 Using the tangent plane to approximate function values

The point  $(3, -1, 4)$  lies on the surface of an unknown differentiable function  $f$  where  $f_x(3, -1) = 2$  and  $f_y(3, -1) = -1/2$ . Find the equation of the tangent plane to  $f$  at  $P$ , and use this to approximate the value of  $f(2.9, -0.8)$ .

**SOLUTION** Knowing the partial derivatives at  $(3, -1)$  allows us to form the normal vector to the tangent plane,  $\vec{n} = \langle 2, -1/2, -1 \rangle$ . Thus the equation of the tangent line to  $f$  at  $P$  is:

$$2(x - 3) - 1/2(y + 1) - (z - 4) = 0 \Rightarrow z = 2(x - 3) - 1/2(y + 1) + 4. \quad (11.5)$$

Just as tangent lines provide excellent approximations of curves near their point of intersection, tangent planes provide excellent approximations of surfaces near their point of intersection. So  $f(2.9, -0.8) \approx z(2.9, -0.8) = 3.7$ .

This is not a new method of approximation. Compare the right hand expression for  $z$  in Equation (11.5) to the total differential:

$$dz = f_x dx + f_y dy \quad \text{and} \quad z = \underbrace{2}_{f_x} \underbrace{(x - 3)}_{dx} + \underbrace{-1/2}_{f_y} \underbrace{(y + 1)}_{dy} + 4.$$

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Notes:

Thus the “new z-value” is the sum of the change in  $z$  (i.e.,  $dz$ ) and the old  $z$ -value (4). As mentioned when studying the total differential, it is not uncommon to know partial derivative information about a unknown function, and tangent planes are used to give accurate approximations of the function.

## The Gradient and Normal Lines, Tangent Planes

The methods developed in this section so far give a straightforward method of finding equations of normal lines and tangent planes for surfaces with explicit equations of the form  $z = f(x, y)$ . However, they do not handle implicit equations well, such as  $x^2 + y^2 + z^2 = 1$ . There is a technique that allows us to find vectors orthogonal to these surfaces based on the **gradient**.

### Definition 73 Gradient

Let  $w = F(x, y, z)$  be differentiable on an open ball  $B$  that contains the point  $(x_0, y_0, z_0)$ .

1. The **gradient of  $F$**  is  $\nabla F(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$ .
2. The **gradient of  $F$  at  $(x_0, y_0, z_0)$**  is

$$\nabla F(x_0, y_0, z_0) = \langle f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0) \rangle.$$

Recall that when  $z = f(x, y)$ , the gradient  $\nabla f = \langle f_x, f_y \rangle$  is orthogonal to level curves of  $f$ . An analogous statement can be made about the gradient  $\nabla F$ , where  $w = F(x, y, z)$ . Given a point  $(x_0, y_0, z_0)$ , let  $c = F(x_0, y_0, z_0)$ . Then  $F(x, y, z) = c$  is a **level surface** that contains the point  $(x_0, y_0, z_0)$ . The following theorem states that  $\nabla F(x_0, y_0, z_0)$  is orthogonal to this level surface.

### Theorem 63 The Gradient and Level Surfaces

Let  $w = F(x, y, z)$  be differentiable on an open ball  $B$  containing  $(x_0, y_0, z_0)$  with gradient  $\nabla F$ , where  $F(x_0, y_0, z_0) = c$ .

The vector  $\nabla F(x_0, y_0, z_0)$  is orthogonal to the level surface  $F(x, y, z) = c$  at  $(x_0, y_0, z_0)$ .

The gradient at a point gives a vector orthogonal to the surface at that point. This direction can be used to find tangent planes and normal lines.

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Notes:

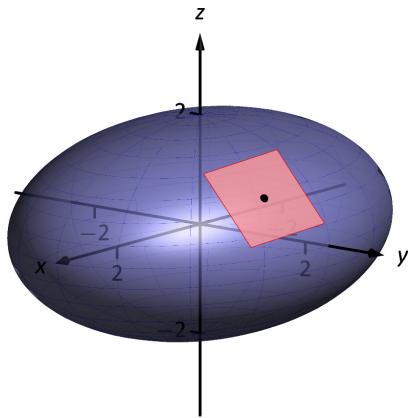


Figure 11.26: An ellipsoid and its tangent plane at a point.

**Example 271 Using the gradient to find a tangent plane**

Find the equation of the plane tangent to the ellipsoid  $\frac{x^2}{12} + \frac{y^2}{6} + \frac{z^2}{4} = 1$  at  $P = (1, 2, 1)$ .

**SOLUTION** We consider the equation of the ellipsoid as a level surface of a function  $F$  of three variables, where  $F(x, y, z) = \frac{x^2}{12} + \frac{y^2}{6} + \frac{z^2}{4}$ . The gradient is:

$$\begin{aligned}\nabla F(x, y, z) &= \langle F_x, F_y, F_z \rangle \\ &= \left\langle \frac{x}{6}, \frac{y}{3}, \frac{z}{2} \right\rangle.\end{aligned}$$

At  $P$ , the gradient is  $\nabla F(1, 2, 1) = \langle 1/6, 2/3, 1/2 \rangle$ . Thus the equation of the plane tangent to the ellipsoid at  $P$  is

$$\frac{1}{6}(x - 1) + \frac{2}{3}(y - 2) + \frac{1}{2}(z - 1) = 0.$$

The ellipsoid and tangent plane are graphed in Figure 11.26.

Tangent lines and planes to surfaces have many uses, including the study of instantaneous rates of changes and making approximations. Normal lines also have many uses. In this section we focused on using them to measure distances from a surface. Another interesting application is in computer graphics, where the effects of light on a surface are determined using normal vectors.

The next section investigates another use of partial derivatives: determining relative extrema. When dealing with functions of the form  $y = f(x)$ , we found relative extrema by finding  $x$  where  $f'(x) = 0$ . We can start finding relative extrema of  $z = f(x, y)$  by setting  $f_x$  and  $f_y$  to 0, but it turns out that there is more to consider.

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Notes:

# Exercises 11.7

## Terms and Concepts

1. Explain how the vector  $\vec{v} = \langle 1, 0, 3 \rangle$  can be thought of as having a “slope” of 3.
2. Explain how the vector  $\vec{v} = \langle 0.6, 0.8, -2 \rangle$  can be thought of as having a “slope” of  $-2$ .
3. T/F: Let  $z = f(x, y)$  be differentiable at  $P$ . If  $\vec{n}$  is a normal vector to the tangent plane of  $f$  at  $P$ , then  $\vec{n}$  is orthogonal to  $f_x$  and  $f_y$  at  $P$ .
4. Explain in your own words why we do not refer to the tangent line to a surface at a point, but rather to *directional* tangent lines to a surface at a point.

## Problems

In Exercises 5 – 8, a function  $z = f(x, y)$ , a vector  $\vec{v}$  and a point  $P$  are given. Give the parametric equations of the following directional tangent lines to  $f$  at  $P$ :

- (a)  $\ell_x(t)$
  - (b)  $\ell_y(t)$
  - (c)  $\ell_{\vec{u}}(t)$ , where  $\vec{u}$  is the unit vector in the direction of  $\vec{v}$ .
5.  $f(x, y) = 2x^2y - 4xy^2$ ,  $\vec{v} = \langle 1, 3 \rangle$ ,  $P = (2, 3)$ .
  6.  $f(x, y) = 3 \cos x \sin y$ ,  $\vec{v} = \langle 1, 2 \rangle$ ,  $P = (\pi/3, \pi/6)$ .
  7.  $f(x, y) = 3x - 5y$ ,  $\vec{v} = \langle 1, 1 \rangle$ ,  $P = (4, 2)$ .
  8.  $f(x, y) = x^2 - 2x - y^2 + 4y$ ,  $\vec{v} = \langle 1, 1 \rangle$ ,  $P = (1, 2)$ .

In Exercises 9 – 12, a function  $z = f(x, y)$  and a point  $P$  are given. Find the equation of the normal line to  $f$  at  $P$ . Note: these are the same functions as in Exercises 5 – 8.

9.  $f(x, y) = 2x^2y - 4xy^2$ ,  $P = (2, 3)$ .
10.  $f(x, y) = 3 \cos x \sin y$ ,  $P = (\pi/3, \pi/6)$ .
11.  $f(x, y) = 3x - 5y$ ,  $P = (4, 2)$ .

12.  $f(x, y) = x^2 - 2x - y^2 + 4y$ ,  $P = (1, 2)$ .

In Exercises 13 – 16, a function  $z = f(x, y)$  and a point  $P$  are given. Find the two points that are 2 units from the surface  $f$  at  $P$ . Note: these are the same functions as in Exercises 5 – 8.

13.  $f(x, y) = 2x^2y - 4xy^2$ ,  $P = (2, 3)$ .
14.  $f(x, y) = 3 \cos x \sin y$ ,  $P = (\pi/3, \pi/6)$ .
15.  $f(x, y) = 3x - 5y$ ,  $P = (4, 2)$ .
16.  $f(x, y) = x^2 - 2x - y^2 + 4y$ ,  $P = (1, 2)$ .

In Exercises 17 – 20, a function  $z = f(x, y)$  and a point  $P$  are given. Find the equation of the tangent plane to  $f$  at  $P$ . Note: these are the same functions as in Exercises 5 – 8.

17.  $f(x, y) = 2x^2y - 4xy^2$ ,  $P = (2, 3)$ .
18.  $f(x, y) = 3 \cos x \sin y$ ,  $P = (\pi/3, \pi/6)$ .
19.  $f(x, y) = 3x - 5y$ ,  $P = (4, 2)$ .
20.  $f(x, y) = x^2 - 2x - y^2 + 4y$ ,  $P = (1, 2)$ .

In Exercises 21 – 24, an implicitly defined function of  $x$ ,  $y$  and  $z$  is given along with a point  $P$  that lies on the surface. Use the gradient  $\nabla F$  to:

- (a) find the equation of the normal line to the surface at  $P$ , and
  - (b) find the equation of the plane tangent to the surface at  $P$ .
21.  $\frac{x^2}{8} + \frac{y^2}{4} + \frac{z^2}{16} = 1$ , at  $P = (1, \sqrt{2}, \sqrt{6})$
  22.  $z^2 - \frac{x^2}{4} - \frac{y^2}{9} = 0$ , at  $P = (4, -3, \sqrt{5})$
  23.  $xy^2 - xz^2 = 0$ , at  $P = (2, 1, -1)$
  24.  $\sin(xy) + \cos(yz) = 0$ , at  $P = (2, \pi/12, 4)$

## 11.8 Extreme Values

Given a function  $z = f(x, y)$ , we are often interested in points where  $z$  takes on the largest or smallest values. For instance, if  $z$  represents a cost function, we would likely want to know what  $(x, y)$  values minimize the cost. If  $z$  represents the ratio of a volume to surface area, we would likely want to know where  $z$  is greatest. This leads to the following definition.

### Definition 74 Relative and Absolute Extrema

Let  $z = f(x, y)$  be defined on a set  $S$  containing the point  $P = (x_0, y_0)$ .

1. If there is an open disk  $D$  containing  $P$  such that  $f(x_0, y_0) \geq f(x, y)$  for all  $(x, y)$  in  $D$ , then  $f$  has a **relative maximum** at  $P$ ; if  $f(x_0, y_0) \leq f(x, y)$  for all  $(x, y)$  in  $D$ , then  $f$  has a **relative minimum** at  $P$ .
2. If  $f(x_0, y_0) \geq f(x, y)$  for all  $(x, y)$  in  $S$ , then  $f$  has an **absolute maximum** at  $P$ ; if  $f(x_0, y_0) \leq f(x, y)$  for all  $(x, y)$  in  $S$ , then  $f$  has an **absolute minimum** at  $P$ .
3. If  $f$  has a relative maximum or minimum at  $P$ , then  $f$  has a **relative extrema** at  $P$ ; if  $f$  has an absolute maximum or minimum at  $P$ , then  $f$  has a **absolute extrema** at  $P$ .

If  $f$  has a relative or absolute maximum at  $P = (x_0, y_0)$ , it means every curve on the surface of  $f$  through  $P$  will also have a relative or absolute maximum at  $P$ . Recalling what we learned in Section 3.1, the slopes of the tangent lines to these curves at  $P$  must be 0 or undefined. Since directional derivatives are computed using  $f_x$  and  $f_y$ , we are led to the following definition and theorem.

### Definition 75 Critical Point

Let  $z = f(x, y)$  be continuous on an open set  $S$ . A **critical point**  $P = (x_0, y_0)$  of  $f$  is a point in  $S$  such that

- $f_x(x_0, y_0) = 0$  and  $f_y(x_0, y_0) = 0$ , or
- $f_x(x_0, y_0)$  and/or  $f_y(x_0, y_0)$  is undefined.

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Notes:

**Theorem 64 Critical Points and Relative Extrema**

Let  $z = f(x, y)$  be defined on an open set  $S$  containing  $P = (x_0, y_0)$ . If  $f$  has a relative extrema at  $P$ , then  $P$  is a critical point of  $f$ .

Therefore, to find relative extrema, we find the critical points of  $f$  and determine which correspond to relative maxima, relative minima, or neither. The following examples demonstrate this process.

**Example 272 Finding critical points and relative extrema**

Let  $f(x, y) = x^2 + y^2 - xy - x - 2$ . Find the relative extrema of  $f$ .

**SOLUTION** We start by computing the partial derivatives of  $f$ :

$$f_x(x, y) = 2x - y - 1 \quad \text{and} \quad f_y(x, y) = 2y - x.$$

Each is never undefined. A critical point occurs when  $f_x$  and  $f_y$  are simultaneously 0, leading us to solve the following system of linear equations:

$$2x - y - 1 = 0 \quad \text{and} \quad -x + 2y = 0.$$

This solution to this system is  $x = 2/3$ ,  $y = 1/3$ . (Check that at  $(2/3, 1/3)$ , both  $f_x$  and  $f_y$  are 0.)

The graph in Figure 11.27 shows  $f$  along with this critical point. It is clear from the graph that this is a relative minimum; further consideration of the function shows that this is actually the absolute minimum.

**Example 273 Finding critical points and relative extrema**

Let  $f(x, y) = -\sqrt{x^2 + y^2} + 2$ . Find the relative extrema of  $f$ .

**SOLUTION** We start by computing the partial derivatives of  $f$ :

$$f_x(x, y) = \frac{-x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad f_y(x, y) = \frac{-y}{\sqrt{x^2 + y^2}}.$$

It is clear that  $f_x = 0$  when  $x = 0$  &  $y \neq 0$ , and that  $f_y = 0$  when  $y = 0$  &  $x \neq 0$ . At  $(0, 0)$ , both  $f_x$  and  $f_y$  are *not* 0, but rather undefined. The point  $(0, 0)$  is still a critical point, though, because the partial derivatives are undefined. This is the only critical point of  $f$ .

The surface of  $f$  is graphed in Figure 11.28 along with the point  $(0, 0, 2)$ . The graph shows that this point is the absolute maximum of  $f$ .

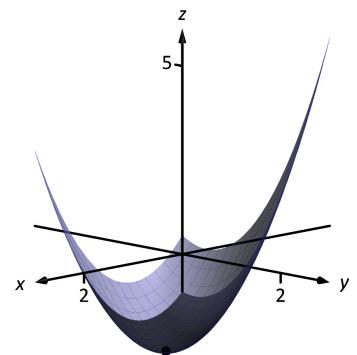


Figure 11.27: The surface in Example 272 with its absolute minimum indicated.

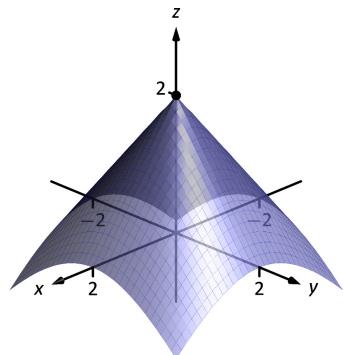


Figure 11.28: The surface in Example 273 with its absolute maximum indicated.

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Notes:

In each of the previous two examples, we found a critical point of  $f$  and then determined whether or not it was a relative (or absolute) maximum or minimum by graphing. It would be nice to be able to determine whether a critical point corresponded to a max or a min without a graph. Before we develop such a test, we do one more example that sheds more light on the issues our test needs to consider.

**Example 274      Finding critical points and relative extrema**

Let  $f(x, y) = x^3 - 3x - y^2 + 4y$ . Find the relative extrema of  $f$ .

**SOLUTION**

Once again we start by finding the partial derivatives of  $f$ :

$$f_x(x, y) = 3x^2 - 3 \quad \text{and} \quad f_y(x, y) = -2y + 4.$$

Each is always defined. Setting each equal to 0 and solving for  $x$  and  $y$ , we find

$$\begin{aligned} f_x(x, y) = 0 &\Rightarrow x = \pm 1 \\ f_y(x, y) = 0 &\Rightarrow y = 2. \end{aligned}$$

We have two critical points:  $(-1, 2)$  and  $(1, 2)$ . To determine if they correspond to a relative maximum or minimum, we consider the graph of  $f$  in Figure 11.29.

The critical point  $(-1, 2)$  clearly corresponds to a relative maximum. However, the critical point at  $(1, 2)$  is neither a maximum nor a minimum, displaying a different, interesting characteristic.

If one walks parallel to the  $y$ -axis towards this critical point, then this point becomes a relative maximum along this path. But if one walks towards this point parallel to the  $x$ -axis, this point becomes a relative minimum along this path. A point that seems to act as both a max and a min is a **saddle point**. A formal definition follows.

**Definition 76      Saddle Point**

Let  $P = (x_0, y_0)$  be in the domain of  $f$  where  $f_x = 0$  and  $f_y = 0$  at  $P$ .  $P$  is a **saddle point** of  $f$  if, for every open disk  $D$  containing  $P$ , there are points  $(x_1, y_1)$  and  $(x_2, y_2)$  in  $D$  such that  $f(x_0, y_0) > f(x_1, y_1)$  and  $f(x_0, y_0) < f(x_2, y_2)$ .

At a saddle point, the instantaneous rate of change in all directions is 0 and there are points nearby with  $z$ -values both less than and greater than the  $z$ -value of the saddle point.

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Notes:

Before Example 274 we mentioned the need for a test to differentiate between relative maxima and minima. We now recognize that our test also needs to account for saddle points. To do so, we consider the second partial derivatives of  $f$ .

Recall that with single variable functions, such as  $y = f(x)$ , if  $f''(c) > 0$ , then  $f$  is concave up at  $c$ , and if  $f'(c) = 0$ , then  $f$  has a relative minimum at  $x = c$ . (We called this the Second Derivative Test.) Note that at a saddle point, it seems the graph is “both” concave up and concave down, depending on which direction you are considering.

It would be nice if the following were true:

$$\begin{aligned} f_{xx} \text{ and } f_{yy} > 0 &\Rightarrow \text{relative minimum} \\ f_{xx} \text{ and } f_{yy} < 0 &\Rightarrow \text{relative maximum} \\ f_{xx} \text{ and } f_{yy} \text{ have opposite signs} &\Rightarrow \text{saddle point.} \end{aligned}$$

However, this is not the case. Functions  $f$  exist where  $f_{xx}$  and  $f_{yy}$  are both positive but a saddle point still exists. In such a case, while the concavity in the  $x$ -direction is up (i.e.,  $f_{xx} > 0$ ) and the concavity in the  $y$ -direction is also up (i.e.,  $f_{yy} > 0$ ), the concavity switches somewhere in between the  $x$ - and  $y$ -directions.

To account for this, consider  $D = f_{xx}f_{yy} - f_{xy}^2$ . Since  $f_{xy}$  and  $f_{yx}$  are equal when continuous (refer back to Theorem 53), we can rewrite this as  $D = f_{xx}f_{yy} - f_{xy}^2$ .  $D$  can be used to test whether the concavity at a point changes depending on direction. If  $D > 0$ , the concavity does not switch (i.e., at that point, the graph is concave up or down in all directions). If  $D < 0$ , the concavity does switch. If  $D = 0$ , our test fails to determine whether concavity switches or not. We state the use of  $D$  in the following theorem.

### Theorem 65    Second Derivative Test

Let  $z = f(x, y)$  be differentiable on an open set containing  $P = (x_0, y_0)$ , and let

$$D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0).$$

1. If  $D > 0$  and  $f_{xx}(x_0, y_0) > 0$ , then  $P$  is a relative minimum of  $f$ .
2. If  $D > 0$  and  $f_{xx}(x_0, y_0) < 0$ , then  $P$  is a relative maximum of  $f$ .
3. If  $D < 0$ , then  $P$  is a saddle point of  $f$ .
4. If  $D = 0$ , the test is inconclusive.

We first practice using this test with the function in the previous example, where we visually determined we had a relative maximum and a saddle point.

Notes:

**Example 275 Using the Second Derivative Test**

Let  $f(x, y) = x^3 - 3x - y^2 + 4y$  as in Example 274. Determine whether the function has a relative minimum, maximum, or saddle point at each critical point.

**SOLUTION** We determined previously that the critical points of  $f$  are  $(-1, 2)$  and  $(1, 2)$ . To use the Second Derivative Test, we must find the second partial derivatives of  $f$ :

$$f_{xx} = 6x; \quad f_{yy} = -2; \quad f_{xy} = 0.$$

Thus  $D(x, y) = -12x$ .

At  $(-1, 2)$ :  $D(-1, 2) = 12 > 0$ , and  $f_{xx}(-1, 2) = -6$ . By the Second Derivative Test,  $f$  has a relative maximum at  $(-1, 2)$ .

At  $(1, 2)$ :  $D(1, 2) = -12 < 0$ . The Second Derivative Test states that  $f$  has a saddle point at  $(1, 2)$ .

The Second Derivative Test confirmed what we determined visually.

**Example 276 Using the Second Derivative Test**

Find the relative extrema of  $f(x, y) = x^2y + y^2 + xy$ .

**SOLUTION** We start by finding the first and second partial derivatives of  $f$ :

$$\begin{aligned} f_x &= 2xy + y & f_y &= x^2 + 2y + x \\ f_{xx} &= 2y & f_{yy} &= 2 \\ f_{xy} &= 2x + 1 & f_{yx} &= 2x + 1. \end{aligned}$$

We find the critical points by finding where  $f_x$  and  $f_y$  are simultaneously 0 (they are both never undefined). Setting  $f_x = 0$ , we have:

$$f_x = 0 \Rightarrow 2xy + y = 0 \Rightarrow y(2x + 1) = 0.$$

This implies that for  $f_x = 0$ , either  $y = 0$  or  $2x + 1 = 0$ .

Assume  $y = 0$  then consider  $f_y = 0$ :

$$\begin{aligned} f_y &= 0 \\ x^2 + 2y + x &= 0, \quad \text{and since } y = 0, \text{ we have} \\ x^2 + x &= 0 \\ x(x + 1) &= 0. \end{aligned}$$

Thus if  $y = 0$ , we have either  $x = 0$  or  $x = -1$ , giving two critical points:  $(-1, 0)$  and  $(0, 0)$ .

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Notes:

Going back to  $f_x$ , now assume  $2x + 1 = 0$ , i.e., that  $x = -1/2$ , then consider  $f_y = 0$ :

$$\begin{aligned} f_y &= 0 \\ x^2 + 2y + x &= 0, \quad \text{and since } x = -1/2, \text{ we have} \\ 1/4 + 2y - 1/2 &= 0 \\ y &= 1/8. \end{aligned}$$

Thus if  $x = -1/2$ ,  $y = 1/8$  giving the critical point  $(-1/2, 1/8)$ .

With  $D = 4y - (2x+1)^2$ , we apply the Second Derivative Test to each critical point.

At  $(-1, 0)$ ,  $D < 0$ , so  $(-1, 0)$  is a saddle point.

At  $(0, 0)$ ,  $D < 0$ , so  $(0, 0)$  is also a saddle point.

At  $(-1/2, 1/8)$ ,  $D > 0$  and  $f_{xx} > 0$ , so  $(-1/2, 1/8)$  is a relative minimum.

Figure 11.30 shows a graph of  $f$  and the three critical points. Note how this function does not vary much near the critical points – that is, visually it is difficult to determine whether a point is a saddle point or relative minimum (or even a critical point at all!). This is one reason why the Second Derivative Test is so important to have.

## Constrained Optimization

When optimizing functions of one variable such as  $y = f(x)$ , we made use of Theorem 25, the Extreme Value Theorem, that said that over a closed interval  $I$ , a continuous function has both a maximum and minimum value. To find these maximum and minimum values, we evaluated  $f$  at all critical points in the interval, as well as at the endpoints (the “boundary”) of the interval.

A similar theorem and procedure applies to functions of two variables. A continuous function over a closed set also attains a maximum and minimum value (see the following theorem). We can find these values by evaluating the function at the critical values in the set and over the boundary of the set. After formally stating this extreme value theorem, we give examples.

### Theorem 66 Extreme Value Theorem

Let  $z = f(x, y)$  be a continuous function on a closed, bounded set  $S$ . Then  $f$  has a maximum and minimum value on  $S$ .

### Example 277 Finding extrema on a closed set

Let  $f(x, y) = x^2 - y^2 + 5$  and let  $S$  be the triangle with vertices  $(-1, -2)$ ,  $(0, 1)$  and  $(2, -2)$ . Find the maximum and minimum values of  $f$  on  $S$ .

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Notes:

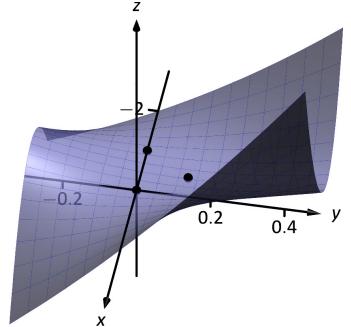


Figure 11.30: Graphing  $f$  from Example 276 and its relative extrema.

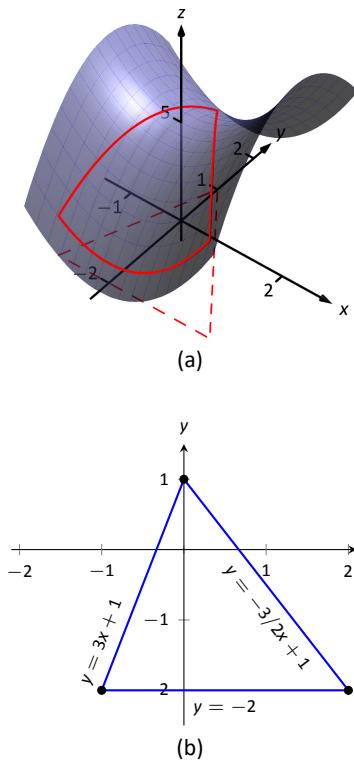


Figure 11.31: Plotting the surface of  $f$  along with the restricted domain  $S$ .

**SOLUTION** It can help to see a graph of  $f$  along with the set  $S$ . In Figure 11.31(a) the triangle defining  $S$  is shown in the  $x$ - $y$  plane in a dashed line. Above it is the surface of  $f$ ; we are only concerned with the portion of  $f$  enclosed by the “triangle” on its surface.

We begin by finding the critical points of  $f$ . With  $f_x = 2x$  and  $f_y = -2y$ , we find only one critical point, at  $(0, 0)$ .

We now find the maximum and minimum values that  $f$  attains along the boundary of  $S$ , that is, along the edges of the triangle. In Figure 11.31(b) we see the triangle sketched in the plane with the equations of the lines forming its edges labeled.

Start with the bottom edge, along the line  $y = -2$ . If  $y$  is  $-2$ , then on the surface, we are considering points  $f(x, -2)$ ; that is, our function reduces to  $f(x, -2) = x^2 - (-2)^2 + 5 = x^2 + 1 = f_1(x)$ . We want to maximize/minimize  $f_1(x) = x^2 + 1$  on the interval  $[-1, 2]$ . To do so, we evaluate  $f_1(x)$  at its critical points and at the endpoints.

The critical points of  $f_1$  are found by setting its derivative equal to 0:

$$f'_1(x) = 0 \Rightarrow x = 0.$$

Evaluating  $f_1$  at this critical point, and at the endpoints of  $[-1, 1]$  gives:

$$\begin{aligned} f_1(-1) &= 2 &\Rightarrow f(-1, -2) &= 2 \\ f_1(0) &= 1 &\Rightarrow f(0, -2) &= 1 \\ f_1(2) &= 5 &\Rightarrow f(2, -2) &= 5. \end{aligned}$$

Notice how evaluating  $f_1$  at a point is the same as evaluating  $f$  at its corresponding point.

We need to do this process twice more, for the other two edges of the triangle.

Along the left edge, along the line  $y = 3x + 1$ , we substitute  $3x + 1$  in for  $y$  in  $f(x, y)$ :

$$f(x, y) = f(x, 3x + 1) = x^2 - (3x + 1)^2 + 5 = -8x^2 - 6x + 4 = f_2(x).$$

We want the maximum and minimum values of  $f_2$  on the interval  $[-1, 0]$ , so we evaluate  $f_2$  at its critical points and the endpoints of the interval. We find the critical points:

$$f'_2(x) = -16x - 6 = 0 \Rightarrow x = -3/8.$$

Evaluate  $f_2$  at its critical point and the endpoints of  $[-1, 0]$ :

$$\begin{aligned} f_2(-1) &= 2 &\Rightarrow f(-1, -2) &= 2 \\ f_2(-3/8) &= 41/8 = 5.125 &\Rightarrow f(-3/8, -0.125) &= 5.125 \\ f_2(0) &= 1 &\Rightarrow f(0, 1) &= 4. \end{aligned}$$

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Notes:

Finally, we evaluate  $f$  along the right edge of the triangle, where  $y = -3/2x + 1$ .

$$f(x, y) = f(x, -3/2x + 1) = x^2 - (-3/2x + 1)^2 + 5 = -\frac{5}{4}x^2 + 3x + 4 = f_3(x).$$

The critical points of  $f_3(x)$  are:

$$f'_3(x) = 0 \Rightarrow x = 6/5 = 1.2.$$

We evaluate  $f_3$  at this critical point and at the endpoints of the interval  $[0, 2]$ :

$$\begin{aligned} f_3(0) &= 4 & \Rightarrow f(0, 1) &= 4 \\ f_3(1.2) &= 5.8 & \Rightarrow f(1.2, -0.8) &= 5.8 \\ f_3(2) &= 5 & \Rightarrow f(2, -2) &= 5. \end{aligned}$$

One last point to test: the critical point of  $f$ ,  $(0, 0)$ . We find  $f(0, 0) = 5$ .

We have evaluated  $f$  at a total of 7 different places, all shown in Figure 11.32. We checked each vertex of the triangle twice, as each showed up as the endpoint of an interval twice. Of all the  $z$ -values found, the maximum is 5.8, found at  $(1.2, -0.8)$ ; the minimum is 1, found at  $(0, -2)$ .

This portion of the text is entitled “Constrained Optimization” because we want to optimize a function (i.e., find its maximum and/or minimum values) subject to a *constraint* – some limit to what values the function can attain. In the previous example, we constrained ourselves by considering a function only within the boundary of a triangle. This was largely arbitrary; the function and the boundary were chosen just as an example, with no real “meaning” behind the function or the chosen constraint.

However, solving constrained optimization problems is a very important topic in applied mathematics. The techniques developed here are the basis for solving larger problems, where more than two variables are involved.

We illustrate the technique once more with a classic problem.

### Example 278 Constrained Optimization

The U.S. Postal Service states that the girth+length of Standard Post Package must not exceed 130”. Given a rectangular box, the “length” is the longest side, and the “girth” is twice the width+height.

Given a rectangular box where the width and height are equal, what are the dimensions of the box that give the maximum volume subject to the constraint of the size of a Standard Post Package?

**SOLUTION** Let  $w$ ,  $h$  and  $\ell$  denote the width, height and length of a rectangular box; we assume here that  $w = h$ . The girth is then  $2(w + h) = 4w$ . The

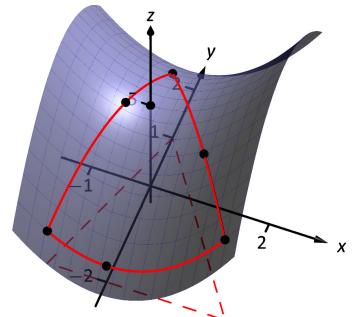


Figure 11.32: The surface of  $f$  along with important points along the boundary of  $S$  and the interior.

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Notes:

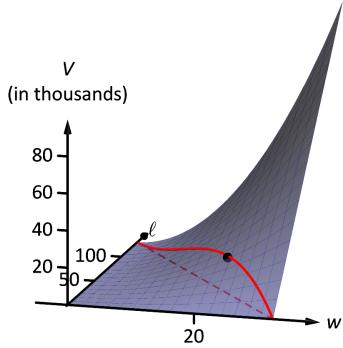


Figure 11.33: Graphing the volume of a box with girth  $4w$  and length  $\ell$ , subject to a size constraint.

volume of the box is  $V(w, \ell) = wh\ell = w^2\ell$ . We wish to maximize this volume subject to the constraint  $4w + \ell \leq 130$ , or  $\ell \leq 130 - 4w$ . (Common sense also indicates that  $\ell > 0, w > 0$ .)

We begin by finding the critical values of  $V$ . We find that  $V_w = 2w\ell$  and  $V_\ell = w^2$ ; these are simultaneously 0 only at  $(0, 0)$ . This gives a volume of 0, so we can ignore this critical point.

We now consider the volume along the constraint  $\ell = 130 - 4w$ . Along this line, we have:

$$V(w\ell) = V(w, 130 - 4w) = w^2(130 - 4w) = 130w^2 - 4w^3 = V_1(w).$$

The constraint is applicable on the  $w$ -interval  $[0, 32.5]$  as indicated in the figure. Thus we want to maximize  $V_1$  on  $[0, 32.5]$ .

Finding the critical values of  $V_1$ , we take the derivative and set it equal to 0:

$$V'_1(w) = 260w - 12w^2 = 0 \Rightarrow w(260 - 12w) = 0 \Rightarrow w = 0, \frac{260}{12} \approx 21.67.$$

We found two critical values: when  $w = 0$  and when  $w = 21.67$ . We again ignore the  $w = 0$  solution; the maximum volume, subject to the constraint, comes at  $w = h = 21.67, \ell = 130 - 4(21.6) = 43.33$ . This gives a volume of  $V(21.67, 43.33) \approx 19,408\text{in}^3$ .

The volume function  $V(w, \ell)$  is shown in Figure 11.33 along with the constraint  $\ell = 130 - 4w$ . As done previously, the constraint is drawn dashed in the  $x$ - $y$  plane and also along the surface of the function. The point where the volume is maximized is indicated.

It is hard to overemphasize the importance of optimization. In “the real world,” we routinely seek to make *something* better. By expressing the *something* as a mathematical function, “making *something* better” means “optimize *some function*.”

The techniques shown here are only the beginning of an incredibly important field. Many functions that we seek to optimize are incredibly complex, making the step of “find the gradient and set it equal to  $\vec{0}$ ” highly nontrivial. Mastery of the principles here are key to being able to tackle these more complicated problems.

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Notes:

# Exercises 11.8

## Terms and Concepts

1. T/F: Theorem 64 states that if  $f$  has a critical point at  $P$ , then  $f$  has a relative extrema at  $P$ .
2. T/F: A point  $P$  is a critical point of  $f$  if  $f_x$  and  $f_y$  are both 0 at  $P$ .
3. T/F: A point  $P$  is a critical point of  $f$  if  $f_x$  or  $f_y$  are undefined at  $P$ .
4. Explain what it means to “solve a constrained optimization” problem.

## Problems

In Exercises 5 – 14, find the critical points of the given function. Use the Second Derivative Test to determine if each critical point corresponds to a relative maximum, minimum, or saddle point.

$$5. f(x, y) = \frac{1}{2}x^2 + 2y^2 - 8y + 4x$$

$$6. f(x, y) = x^2 + 4x + y^2 - 9y + 3xy$$

$$7. f(x, y) = x^2 + 3y^2 - 6y + 4xy$$

$$8. f(x, y) = \frac{1}{x^2 + y^2 + 1}$$

$$9. f(x, y) = x^2 + y^3 - 3y + 1$$

$$10. f(x, y) = \frac{1}{3}x^3 - x + \frac{1}{3}y^3 - 4y$$

$$11. f(x, y) = x^2y^2$$

$$12. f(x, y) = x^4 - 2x^2 + y^3 - 27y - 15$$

$$13. f(x, y) = \sqrt{16 - (x - 3)^2 - y^2}$$

$$14. f(x, y) = \sqrt{x^2 + y^2}$$

In Exercises 15 – 18, find the absolute maximum and minimum of the function subject to the given constraint.

$$15. f(x, y) = x^2 + y^2 + y + 1, \text{ constrained to the triangle with vertices } (0, 1), (-1, -1) \text{ and } (1, -1).$$

$$16. f(x, y) = 5x - 7y, \text{ constrained to the region bounded by } y = x^2 \text{ and } y = 1.$$

$$17. f(x, y) = x^2 + 2x + y^2 + 2y, \text{ constrained to the region bounded by the circle } x^2 + y^2 = 4.$$

$$18. f(x, y) = 3y - 2x^2, \text{ constrained to the region bounded by the parabola } y = x^2 + x - 1 \text{ and the line } y = x.$$

## 11.9 Method of Lagrange Multipliers

This section faces up to a practical problem we encountered at the end of the last section. We often minimize or maximize one function  $f(x, y)$  while another function  $g(x, y)$  is fixed. There is a constraint on  $x$  and  $y$ , given by  $g(x, y) = k$  for some constant  $k$ . This restricts the material available, or the funds available, or the energy available, for example. With this constraint, the problem is to do the best possible, either maximizing or minimizing the function  $f(x, y)$ .

At the absolute minimum of  $f(x, y)$ , if one exists, the requirement  $g(x, y) = k$  is probably violated. In that case the minimum point is not allowed. We cannot use just  $f_x = 0$  and  $f_y = 0$  as those equations don't account for  $g$ . We must find equations for the constrained minimum or constrained maximum. They will involve  $f_x$  and  $f_y$ , and also  $g_x$  and  $g_y$ , which give local information about  $f$  and  $g$ . To see the equations, we look at two examples.

### Example 279

Minimize  $f(x, y) = x^2 + y^2$  subject to the restraint  $g(x, y) = 2x + y = k$ , for a constant  $k$ .

#### SOLUTION

Look at the level curves in Figure 11.34. They are circles

$$x^2 + y^2 = c.$$

When  $c$  is small, the circles do not touch the line  $2x + y = k$ . There are no points that satisfy the constraint, when  $c$  is too small. Now increase  $c$ . Eventually the growing circles  $x^2 + y^2 = c$  will just touch the line  $x + 2y = k$ . The point where they touch is the winner. It gives the smallest value of  $c$  that can be achieved on the line. The touching point is  $(x_{\min}, y_{\min})$ , and the value of  $c$  there is  $f_{\min}$ .

What equation describes that point? When the circle touches the line, they are tangent. They have the same slope. The perpendiculars to the circle and the line go in the same direction. That is the key fact, which you see in Figure ???. The direction perpendicular to  $f(x, y) = c$  is given by  $\nabla f = (f_x, f_y)$ . The direction perpendicular to  $g(x, y) = k$  is given by  $\nabla g = (g_x, g_y)$ . The key idea says that those two vectors are parallel - one gradient vector is a multiple of the other gradient vector, with a multiplier  $\lambda$  that is unknown. That is,  $\nabla f = \lambda \nabla g$ .

There are now three unknowns  $x$ ,  $y$ , and  $\lambda$ , as well as three equations

$$\begin{aligned}\frac{\partial f}{\partial x} &= \lambda \frac{\partial g}{\partial x} \Rightarrow 2x = 2\lambda \\ \frac{\partial f}{\partial y} &= \lambda \frac{\partial g}{\partial y} \Rightarrow 2y = \lambda \\ g(x, y) &= k \Rightarrow 2x + y = k.\end{aligned}$$

---

Notes:

where the third equation is simply the constraint. Now we solve this system. In the third equation, substitute  $2\lambda$  for  $2x$  and  $\frac{1}{2}\lambda$  for  $y$ , which follow from the first two equations. This yields  $2x + y = \frac{5}{2}\lambda = k$ . Knowing  $\lambda = \frac{2}{5}k$ , the first two of the above equations give

$$x = \lambda = \frac{2}{5}k, \quad y = \frac{1}{2}\lambda = \frac{1}{5}k$$

and so

$$f_{\min} = \left(\frac{2}{5}k\right)^2 + \left(\frac{1}{5}k\right)^2 = \frac{1}{5}k^2.$$

The winning point is  $(x_{\min}, y_{\min}) = (\frac{2}{5}k, \frac{1}{5}k)$ . It minimizes the distance squared  $f = x^2 + y^2 = \frac{1}{5}k^2$  from the origin to the line. One can reinterpret this problem as finding the point on the line  $2x + y = k$  closest to the origin.

### Theorem 67 Lagrange Multipliers with One Constraint

At the minimum or maximum of  $f(x, y)$  subject to  $g(x, y) = k$ , the gradient of  $f$  is parallel to the gradient of  $g$ , with an unknown number  $\lambda$  as the multiplier. That is,

$$\nabla f = \lambda \nabla g, \text{ and so } \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \text{ and } \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \text{ and } g(x, y) = k.$$

To minimize or maximize  $f(x, y)$  subject to  $g(x, y) = k$ , solve this system of three equations. Evaluate  $f(x, y)$  at all solutions  $(x, y)$  to this system. The largest of these is  $f_{\max}$ , the maximum of  $f$  subject to  $g = k$ , and the smallest is  $f_{\min}$ , the minimum of  $f$  subject to  $g = k$ .

### Example 280

Find the maximum and minimum values of  $f(x, y) = x^2 + y^2$  on the ellipse  $g(x, y) = (x - 1)^2 + 4y^2 = 4$ .

**SOLUTION** As in Figure 11.35, the circles  $x^2 + y^2 = c$  grow until they touch the ellipse. The touching point is  $(x_{\min}, y_{\min})$  and that smallest value of  $c$  is  $f_{\min}$ . As the circles continue to grow, they cut through the ellipse. Finally there is a point  $(x_{\max}, y_{\max})$  where the last circle touches. That largest value of  $c$  is  $f_{\max}$ .

The minimum and maximum are described by the same rule: the circle is tangent to the ellipse. The perpendiculars go in the same direction. Therefore

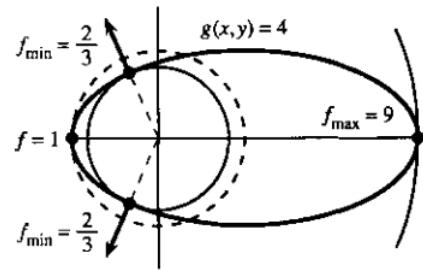


Figure 11.35: Ellipse  $g = 4$  tangent to line  $g = k$

Notes:

$\nabla f$  is a multiple of  $\nabla g$ , and the unknown multiplier is  $\lambda$ .

$$\begin{aligned}\frac{\partial f}{\partial x} &= \lambda \frac{\partial g}{\partial x} \Rightarrow 2x = 2\lambda(x - 1) \\ \frac{\partial f}{\partial y} &= \lambda \frac{\partial g}{\partial y} \Rightarrow 2y = 8\lambda y \\ g(x, y) &= k \Rightarrow (x - 1)^2 + 4y^2 = 4.\end{aligned}$$

The second equation allows two possibilities:  $y = 0$  or  $\lambda = \frac{1}{4}$ . For the case  $y = 0$ , the last equation gives  $(x - 1)^2 = 4$  or  $x^2 - 2x - 3 = 0$ , and thus  $x = 3$  or  $x = -1$ . Then the first equation gives  $\lambda = \frac{3}{2}$  or  $\lambda = \frac{1}{2}$ . The values of  $f$  are therefore  $x^2 + y^2 = 3^2 + 0^2 = 9$  and  $x^2 + y^2 = (-1)^2 + 0^2 = 1$ .

In the other case that  $\lambda = \frac{1}{4}$ , the first equation yields  $x = -\frac{1}{3}$ . Then the last equation requires  $y^2 = \frac{5}{9}$ . Since  $x^2 = \frac{1}{9}$  we find  $x^2 + y^2 = \frac{6}{9} = \frac{2}{3}$ .

The equations we started with have four simultaneous solutions, at which the circle and ellipse are tangent. The four points are  $(3, 0)$ ,  $(-1, 0)$ ,  $(-\frac{1}{3}, \frac{\sqrt{5}}{3})$ , and  $(-\frac{1}{3}, -\frac{\sqrt{5}}{3})$ . The four values of  $f$  are, respectively, 9, 1,  $\frac{2}{3}$ , and  $\frac{2}{3}$ . Therefore the maximum value of  $f$  on the ellipse is 9 and the minimum value is  $\frac{2}{3}$ .

Using this method, the system we need to solve includes the equations  $f_x = \lambda g_x$ ,  $f_y = \lambda g_y$ , and  $g = k$ , with unknowns  $x$ ,  $y$ , and  $\lambda$ . There is no absolute method for solving this system of equations (unless they are all linear equations, then one can use elimination or Cramer's Rule to solve). Often, the first two equations yield  $x$  and  $y$  in terms of  $\lambda$ , and substituting into  $g(x, y) = k$  gives an equation for  $\lambda$ . This next example illustrates this method when one is maximizing or minimizing a function  $f(x, y, z)$  of three variables subject to a constraint  $g(x, y, z) = k$ . The method is the same except that the equation  $\nabla f = \lambda \nabla g$  will yield three equations in the variables  $x$ ,  $y$ , and  $z$ , with the fourth equation  $g(x, y, z) = k$  creating a system of four equations in four unknowns.

### Example 281

Suppose that 20 square feet of cardboard is to be used to create a rectangular box with an open top. What dimensions will result in a box with the maximum volume?

**SOLUTION** If we let  $x$ ,  $y$ , and  $z$  be the dimensions of the box, then the volume is given by  $V(x, y, z) = xyz$  which is the function to be maximized. The constraint in this problem is the amount of cardboard, or surface area, and so

$$g(x, y, z) = 2xz + 2yz + xy = 20$$

since there is no top to this box. The method of Lagrange multipliers yields the

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Notes:

equations

$$\begin{aligned}V_x &= \lambda g_x \Rightarrow yz = \lambda(2z + y) \\V_y &= \lambda g_y \Rightarrow xz = \lambda(2z + x) \\V_z &= \lambda g_z \Rightarrow xy = \lambda(2x + 2y) \\g(x, y, z) &= 20 \Rightarrow 2xz + 2yz + xy = 20\end{aligned}$$

Multiplying each of the first three equations by  $x$ ,  $y$ , and  $z$ , respectively, gives the system

$$\begin{aligned}xyz &= \lambda(2xz + xy) \\xyz &= \lambda(2yz + xy) \\xyz &= \lambda(2xz + 2yz) \\2xz + 2yz + xy &= 20\end{aligned}$$

Note that the first three equations imply that  $\lambda \neq 0$ , else one would arrive at the conclusion  $xyz = 0$ , which does not make sense in this situation. Therefore we can divide by  $\lambda$  and we get from equating the first two equations that

$$2xz + xy = 2yz + xy \text{ or } xz = yz.$$

Since  $z \neq 0$ , we get that  $x = y$ . Similarly, the second and third equations imply that

$$2yz + xy = 2xz + 2yz \text{ or } xy = 2xz.$$

Again, since  $x \neq 0$ , we get that  $y = 2z$ . Lastly, substitute  $x = y = 2z$  into equation (4) and we get

$$4z^2 + 4z^2 + 4z^2 = 20.$$

This gives us  $3z^2 = 5$  or  $z = \sqrt{\frac{5}{3}}$ . Therefore the dimensions of the box with maximum volume must be

$$x = y = 2\sqrt{\frac{5}{3}} \approx 2.58 \text{ feet, and } z = \sqrt{\frac{5}{3}} \approx 1.29 \text{ feet.}$$

### Maximum and Minimum with Two Constraints

The whole subject of minimization or maximization is called optimization. Its applications to business decisions make up operations research. The special case of linear functions is always important - in this part of mathematics it is called linear programming. A book about those subjects won't fit inside a calculus book, but we can take one more step - to allow a second constraint.

Notes:

The function to minimize or maximize is now  $f(x, y, z)$  and two constraints are given by  $g(x, y, z) = k_1$  and  $h(x, y, z) = k_2$ , for two constants  $k_1$  and  $k_2$ . The multipliers are  $\lambda_1$  and  $\lambda_2$ . We need at least three variables  $x$ ,  $y$ , and  $z$  because two constraints would completely determine  $x$  and  $y$  without a third.

Figure 11.36 shows the geometry behind these equations. For convenience, suppose  $f$  is  $x^2 + y^2 + z^2$ , and so we are minimizing distance squared. The constraints

$$g = x + y + z = 9 \text{ and } h = x + 2y + 3z = 20$$

are linear, and their graphs are planes. The constraints keep the points  $(x, y, z)$  on both planes, and therefore on the line where they intersect. We are therefore finding the squared distance from  $(0, 0, 0)$  to a line again.

What equations do we solve in this case? The level surfaces  $x^2 + y^2 + z^2 = c$  are spheres. They grow as  $c$  increases. The first sphere to touch the line is tangent to it, and that touching point gives the minimum solution (the smallest  $c$ ). All three vectors  $\nabla f$ ,  $\nabla g$ ,  $\nabla h$  are perpendicular to the line:

line tangent to sphere  $\Rightarrow \nabla f$  perpendicular to line

line in both planes  $\Rightarrow \nabla g$  and  $\nabla h$  perpendicular to line

Thus  $\nabla f$ ,  $\nabla g$ , and  $\nabla h$  are in the same plane, one that is perpendicular to the line. With three vectors in a plane,  $\nabla f$  must be a combination of  $\nabla g$  and  $\nabla h$ :

$$(f_x, f_y, f_z) = \lambda_1(g_x, g_y, g_z) + \lambda_2(h_x, h_y, h_z).$$

This is the key equation. It applies to curved surfaces as well as planes.

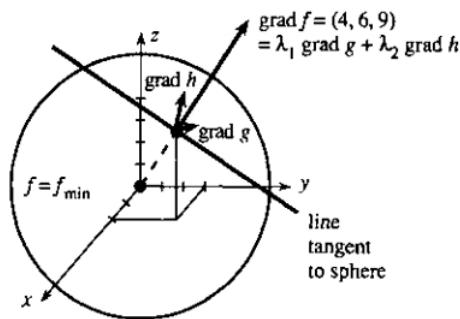


Figure 11.36: Perpendicular vector  $\nabla f$  is a combination of  $\lambda_1 \nabla g + \lambda_2 \nabla h$

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Notes:

**Theorem 68 Lagrange Multipliers with Two Constraints**

At the minimum or maximum of  $f(x, y, z)$  subject to  $g(x, y, z) = k_1$  and  $h(x, y, z) = k_2$ , the gradient of  $f$  is in the same plane as  $\nabla g$  and  $\nabla h$ , or

$$\nabla f = \lambda_1 \nabla g + \lambda_2 \nabla h.$$

That is,

$$\begin{aligned}\frac{\partial f}{\partial x} &= \lambda_1 \frac{\partial g}{\partial x} + \lambda_2 \frac{\partial h}{\partial x} \\ \frac{\partial f}{\partial y} &= \lambda_1 \frac{\partial g}{\partial y} + \lambda_2 \frac{\partial h}{\partial y} \\ \frac{\partial f}{\partial z} &= \lambda_1 \frac{\partial g}{\partial z} + \lambda_2 \frac{\partial h}{\partial z}\end{aligned}$$

subject to  $g(x, y, z) = k_1$  and  $h(x, y, z) = k_2$ . To minimize or maximize  $f(x, y, z)$  subject to  $g(x, y, z) = k_1$  and  $h(x, y, z) = k_2$ , solve this system of five equations in the unknowns  $x, y, z, \lambda_1$ , and  $\lambda_2$ . Evaluate  $f(x, y, z)$  at all solutions  $(x, y, z)$  to this system. The largest of these is  $f_{\max}$ , the maximum of  $f$  subject to the constraints, and the smallest is  $f_{\min}$ , the minimum of  $f$  subject to the constraints.

**Example 282**

Find the minimum value of  $f(x, y, z) = x^2 + y^2 + z^2$  when

$$g = x + y + z = 9 \text{ and } h = x + 2y + 3z = 20.$$

**SOLUTION** In figure 11.37, the normals to those planes are  $\nabla g = (1, 1, 1)$  and  $\nabla h = (1, 2, 3)$ . The gradient of  $f$  is  $\nabla f = (2x, 2y, 2z)$ . The equation

$$\nabla f = \lambda_1 \nabla g + \lambda_2 \nabla h$$

becomes

$$2x = \lambda_1 + \lambda_2, \quad 2y = \lambda_1 + 2\lambda_2, \quad 2z = \lambda_1 + 3\lambda_2.$$

Solve for  $x, y$ , and  $z$  and substitute these into the constraint equations to get

$$\begin{aligned}\frac{1}{2} ((\lambda_1 + \lambda_2) + (\lambda_1 + 2\lambda_2) + (\lambda_1 + 3\lambda_2)) &= 9 \\ \frac{1}{2} ((\lambda_1 + \lambda_2) + 2(\lambda_1 + 2\lambda_2) + 3(\lambda_1 + 3\lambda_2)) &= 20\end{aligned}$$

Notes:

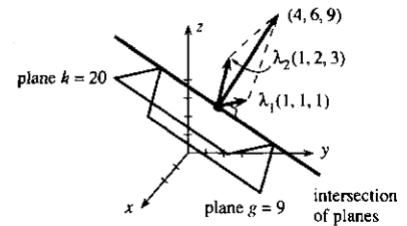


Figure 11.37: Perpendicular vector  $\nabla f$  is a combination of  $\lambda_1 \nabla g + \lambda_2 \nabla h$

After multiplying these by 2, they simplify down to

$$3\lambda_1 + 6\lambda_2 = 18 \text{ and } 6\lambda_1 + 14\lambda_2 = 40.$$

This is a linear system with solution  $\lambda_1 = 2$  and  $\lambda_2 = 2$ . The previous equations therefore give  $(x, y, z) = (2, 3, 4)$ . This point gives  $f_{\min} = 29$ , the minimum value of  $f$  subject to the two constraints.

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Notes:

# Exercises 11.9

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## Terms and Concepts

1. Fill in the blank: Geometrically,  $\nabla f$  is \_\_\_\_\_ to  $\nabla g$  at a maximum or minimum of  $f(x, y)$  subject to the constraint  $g(x, y) = k$ .
2. Fill in the blanks: When using the method of Lagrange multipliers to find the maximum or minimum of  $f(x, y)$  subject to the constraint  $g(x, y) = k$ , you must solve a system of \_\_\_\_\_ equations in the unknowns  $x$ ,  $y$ , and  $\lambda$ .
3. T/F: When finding the maximum of a function  $f(x, y)$  subject to the constraint  $2x - 3y = 6$ , one is finding the largest value of  $f(x, y)$  on a line.
4. T/F: When trying to maximize or minimize a function  $f(x, y, z)$  subject to two constraints  $g(x, y, z) = k_1$  and  $h(x, y, z) = k_2$ , the additional two equations in the system are  $\nabla f = \lambda_1 \nabla g$  and  $\nabla f = \lambda_2 \nabla h$ .

## Problems

In Exercises 5 – 8, find maximum and minimum values of  $f(x, y)$  subject to the given restraint, as well as the points  $(x, y)$  where these occur.

5.  $f(x, y) = x^2y$  with  $g(x, y) = x^2 + y^2 = 1$
  6.  $f(x, y) = x + y$  with  $g(x, y) = \frac{1}{x} + \frac{1}{y} = 1$
  7.  $f(x, y) = 3x + y$  with  $g(x, y) = x^2 + 9y^2 = 1$
  8.  $f(x, y) = x^2 + y^2$  with  $g(x, y) = x^6 + y^6 = 2$
- In Exercises 9 – 12, answer the question using the method of Lagrange multipliers.**
9. Find the maximum value of  $f(x, y) = xy$  on the circle of radius  $\sqrt{2}$  with center at the origin. At what point on the circle does this occur?
  10. Find the minimum and maximum values of the function  $f(x, y) = 2x - 3y$  on the circle  $x^2 + y^2 = 13$ .
  11. Find the minimum value of  $f(x, y, z) = x^2 + 2y^2 + z^2$  if  $(x, y, z)$  is restricted to the planes  $x+y+z=0$  and  $x-z=1$ . At what point(s) does this occur?
  12. Find the maximum value of  $f(x, y, z) = x + y + z$  if  $(x, y, z)$  is restricted to  $x^2 + z^2 = 2$  and  $x + y = 1$ . At what point(s) does this occur?



# 12: MULTIPLE INTEGRATION

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The previous chapter introduced multivariable functions and we applied concepts of differential calculus to these functions. We learned how we can view a function of two variables as a surface in space, and learned how partial derivatives convey information about how the surface is changing in any direction.

In this chapter we apply techniques of integral calculus to multivariable functions. In Chapter 5 we learned how the definite integral of a single variable function gave us “area under the curve.” In this chapter we will see that integration applied to a multivariable function gives us “volume under a surface.” And just as we learned applications of integration beyond finding areas, we will find applications of integration in this chapter beyond finding volume.

## 12.1 Iterated Integrals and Area

In Chapter 11 we found that it was useful to differentiate functions of several variables with respect to one variable, while treating all the other variables as constants or coefficients. We can integrate functions of several variables in a similar way. For instance, if we are told that  $f_x(x, y) = 2xy$ , we can treat  $y$  as staying constant and integrate to obtain  $f(x, y)$ :

$$\begin{aligned}f(x, y) &= \int f_x(x, y) \, dx \\&= \int 2xy \, dx \\&= x^2y + C.\end{aligned}$$

Make a careful note about the constant of integration,  $C$ . This “constant” is something with a derivative of 0 with respect to  $x$ , so it could be any expression that contains only constants and functions of  $y$ . For instance, if  $f(x, y) = x^2y + \sin y + y^3 + 17$ , then  $f_x(x, y) = 2xy$ . To signify that  $C$  is actually a function of  $y$ , we write:

$$f(x, y) = \int f_x(x, y) \, dx = x^2y + C(y).$$

Using this process we can even evaluate definite integrals.

**Example 283**      **Integrating functions of more than one variable**

Evaluate the integral  $\int_1^{2y} 2xy \, dx$ .

**SOLUTION**      We find the indefinite integral as before, then apply the Fundamental Theorem of Calculus to evaluate the definite integral:

$$\begin{aligned}\int_1^{2y} 2xy \, dx &= x^2y \Big|_1^{2y} \\&= (2y)^2y - (1)^2y \\&= 4y^3 - y.\end{aligned}$$

We can also integrate with respect to  $y$ . In general,

$$\int_{h_1(y)}^{h_2(y)} f_x(x, y) dx = f(x, y) \Big|_{h_1(y)}^{h_2(y)} = f(h_2(y), y) - f(h_1(y), y),$$

and

$$\int_{g_1(x)}^{g_2(x)} f_y(x, y) dy = f(x, y) \Big|_{g_1(x)}^{g_2(x)} = f(x, g_2(x)) - f(x, g_1(x)).$$

Note that when integrating with respect to  $x$ , the bounds are functions of  $y$  (of the form  $x = h_1(y)$  and  $x = h_2(y)$ ) and the final result is also a function of  $y$ . When integrating with respect to  $y$ , the bounds are functions of  $x$  (of the form  $y = g_1(x)$  and  $y = g_2(x)$ ) and the final result is a function of  $x$ . Another example will help us understand this.

**Example 284 Integrating functions of more than one variable**

Evaluate  $\int_1^x (5x^3y^{-3} + 6y^2) dy$ .

**SOLUTION** We consider  $x$  as staying constant and integrate with respect to  $y$ :

$$\begin{aligned} \int_1^x (5x^3y^{-3} + 6y^2) dy &= \left( \frac{5x^3y^{-2}}{-2} + \frac{6y^3}{3} \right) \Big|_1^x \\ &= \left( -\frac{5}{2}x^3x^{-2} + 2x^3 \right) - \left( -\frac{5}{2}x^3 + 2 \right) \\ &= \frac{9}{2}x^3 - \frac{5}{2}x - 2. \end{aligned}$$

Note how the bounds of the integral are from  $y = 1$  to  $y = x$  and that the final answer is a function of  $x$ .

In the previous example, we integrated a function with respect to  $y$  and ended up with a function of  $x$ . We can integrate this as well. This process is known as **iterated integration**, or **multiple integration**.

**Example 285 Integrating an integral**

Evaluate  $\int_1^2 \left( \int_1^x (5x^3y^{-3} + 6y^2) dy \right) dx$ .

**SOLUTION** We follow a standard “order of operations” and perform the operations inside parentheses first (which is the integral evaluated in Example

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Notes:

284.)

$$\begin{aligned}
 \int_1^2 \left( \int_1^x (5x^3y^{-3} + 6y^2) dy \right) dx &= \int_1^2 \left( \left[ \frac{5x^3y^{-2}}{-2} + \frac{6y^3}{3} \right] \Big|_1^x \right) dx \\
 &= \int_1^2 \left( \frac{9}{2}x^3 - \frac{5}{2}x - 2 \right) dx \\
 &= \left( \frac{9}{8}x^4 - \frac{5}{4}x^2 - 2x \right) \Big|_1^2 \\
 &= \frac{89}{8}.
 \end{aligned}$$

Note how the bounds of  $x$  were  $x = 1$  to  $x = 2$  and the final result was a number.

The previous example showed how we could perform something called an iterated integral; we do not yet know *why* we would be interested in doing so nor what the result, such as the number  $89/8$ , means. Before we investigate these questions, we offer some definitions.

### Definition 77 Iterated Integration

**Iterated integration** is the process of repeatedly integrating the results of previous integrations. Integrating one integral is denoted as follows.

Let  $a, b, c$  and  $d$  be numbers and let  $g_1(x)$ ,  $g_2(x)$ ,  $h_1(y)$  and  $h_2(y)$  be functions of  $x$  and  $y$ , respectively. Then:

$$1. \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy = \int_c^d \left( \int_{h_1(y)}^{h_2(y)} f(x, y) dx \right) dy.$$

$$2. \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx = \int_a^b \left( \int_{g_1(x)}^{g_2(x)} f(x, y) dy \right) dx.$$

Again make note of the bounds of these iterated integrals.

With  $\int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$ ,  $x$  varies from  $h_1(y)$  to  $h_2(y)$ , whereas  $y$  varies from  $c$  to  $d$ . That is, the bounds of  $x$  are *curves*, the curves  $x = h_1(y)$  and  $x = h_2(y)$ , whereas the bounds of  $y$  are *constants*,  $y = c$  and  $y = d$ . It is useful to remember that when setting up and evaluating such iterated integrals, we integrate “from

Notes:

curve to curve, then from point to point."

We now begin to investigate *why* we are interested in iterated integrals and *what* they mean.

### Area of a plane region

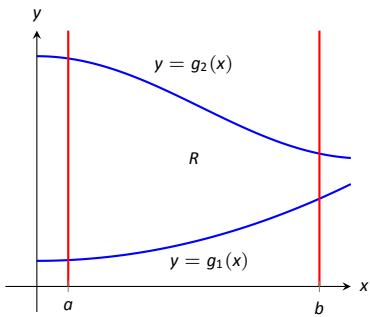


Figure 12.1: Calculating the area of a plane region  $R$  with an iterated integral.

Consider the plane region  $R$  bounded by  $a \leq x \leq b$  and  $g_1(x) \leq y \leq g_2(x)$ , shown in Figure 12.1. We learned in Section 7.1 that the area of  $R$  is given by

$$\int_a^b (g_2(x) - g_1(x)) dx.$$

We can view the expression  $(g_2(x) - g_1(x))$  as

$$(g_2(x) - g_1(x)) = \int_{g_1(x)}^{g_2(x)} 1 dy = \int_{g_1(x)}^{g_2(x)} dy,$$

meaning we can express the area of  $R$  as an iterated integral:

$$\text{area of } R = \int_a^b (g_2(x) - g_1(x)) dx = \int_a^b \left( \int_{g_1(x)}^{g_2(x)} dy \right) dx = \int_a^b \int_{g_1(x)}^{g_2(x)} dy dx.$$

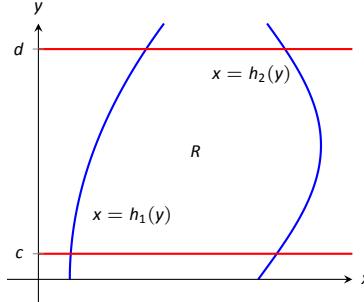


Figure 12.2: Calculating the area of a plane region  $R$  with an iterated integral.

In short: a certain iterated integral can be viewed as giving the area of a plane region.

A region  $R$  could also be defined by  $c \leq y \leq d$  and  $h_1(y) \leq x \leq h_2(y)$ , as shown in Figure 12.2. Using a process similar to that above, we have

$$\text{the area of } R = \int_c^d \int_{h_1(y)}^{h_2(y)} dx dy.$$

We state this formally in a theorem.

Notes:

**Theorem 69 Area of a plane region**

1. Let  $R$  be a plane region bounded by  $a \leq x \leq b$  and  $g_1(x) \leq y \leq g_2(x)$ , where  $g_1$  and  $g_2$  are continuous functions on  $[a, b]$ . The area  $A$  of  $R$  is

$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} dy dx.$$

2. Let  $R$  be a plane region bounded by  $c \leq y \leq d$  and  $h_1(y) \leq x \leq h_2(y)$ , where  $h_1$  and  $h_2$  are continuous functions on  $[c, d]$ . The area  $A$  of  $R$  is

$$A = \int_c^d \int_{h_1(y)}^{h_2(y)} dx dy.$$

The following examples should help us understand this theorem.

**Example 286 Area of a rectangle**

Find the area  $A$  of the rectangle with corners  $(-1, 1)$  and  $(3, 3)$ , as shown in Figure 12.3.

**SOLUTION** Multiple integration is obviously overkill in this situation, but we proceed to establish its use.

The region  $R$  is bounded by  $x = -1$ ,  $x = 3$ ,  $y = 1$  and  $y = 3$ . Choosing to integrate with respect to  $y$  first, we have

$$A = \int_{-1}^3 \int_1^3 1 dy dx = \int_{-1}^3 \left( y \Big|_1^3 \right) dx = \int_{-1}^3 2 dx = 2x \Big|_{-1}^3 = 8.$$

We could also integrate with respect to  $x$  first, giving:

$$A = \int_1^3 \int_{-1}^3 1 dx dy = \int_1^3 \left( x \Big|_{-1}^3 \right) dy = \int_1^3 4 dy = 4y \Big|_1^3 = 8.$$

Clearly there are simpler ways to find this area, but it is interesting to note that this method works.

**Example 287 Area of a triangle**

Find the area  $A$  of the triangle with vertices at  $(1, 1)$ ,  $(3, 1)$  and  $(5, 5)$ , as shown in Figure 12.4.

**SOLUTION** The triangle is bounded by the lines as shown in the figure. Choosing to integrate with respect to  $x$  first gives that  $x$  is bounded by  $x = y$

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Notes:

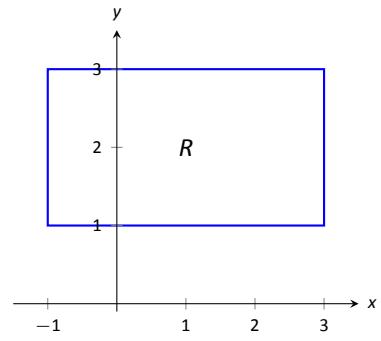


Figure 12.3: Calculating the area of a rectangle with an iterated integral in Example 286.

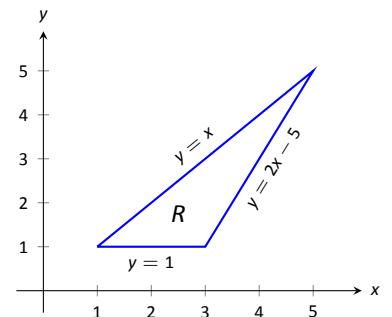


Figure 12.4: Calculating the area of a triangle with iterated integrals in Example 287.

to  $x = \frac{y+5}{2}$ , while  $y$  is bounded by  $y = 1$  to  $y = 5$ . (Recall that since  $x$ -values increase from left to right, the leftmost curve,  $x = y$ , is the lower bound and the rightmost curve,  $x = (y + 5)/2$ , is the upper bound.) The area is

$$\begin{aligned} A &= \int_1^5 \int_y^{\frac{y+5}{2}} dx dy \\ &= \int_1^5 \left( x \Big|_y^{\frac{y+5}{2}} \right) dy \\ &= \int_1^5 \left( -\frac{1}{2}y + \frac{5}{2} \right) dy \\ &= \left( -\frac{1}{4}y^2 + \frac{5}{2}y \right) \Big|_1^5 \\ &= 4. \end{aligned}$$

We can also find the area by integrating with respect to  $y$  first. In this situation, though, we have two functions that act as the lower bound for the region  $R$ ,  $y = 1$  and  $y = 2x - 5$ . This requires us to use two iterated integrals. Note how the  $x$ -bounds are different for each integral:

$$\begin{array}{lll} A = \int_1^3 \int_1^x 1 dy dx & + & \int_3^5 \int_{2x-5}^x 1 dy dx \\ = \int_1^3 (y) \Big|_1^x dx & + & \int_3^5 (y) \Big|_{2x-5}^x dx \\ = \int_1^3 (x - 1) dx & + & \int_3^5 (-x + 5) dx \\ = 2 & + & 2 \\ = 4. & & \end{array}$$

As expected, we get the same answer both ways.

### Example 288 Area of a plane region

Find the area of the region enclosed by  $y = 2x$  and  $y = x^2$ , as shown in Figure 12.5.

**SOLUTION** Once again we'll find the area of the region using both orders of integration.

Using  $dy dx$ :

$$\int_0^2 \int_{x^2}^{2x} 1 dy dx = \int_0^2 (2x - x^2) dx = (x^2 - \frac{1}{3}x^3) \Big|_0^2 = \frac{4}{3}.$$

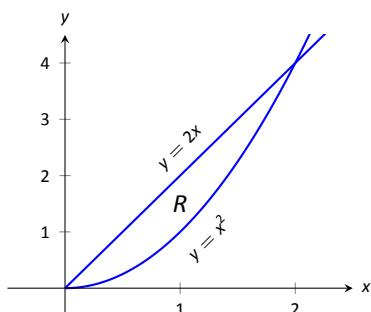


Figure 12.5: Calculating the area of a plane region with iterated integrals in Example 288.

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Notes:

Using  $dx dy$ :

$$\int_0^4 \int_{y/2}^{\sqrt{y}} 1 \, dx \, dy = \int_0^4 (\sqrt{y} - y/2) \, dy = \left( \frac{2}{3}y^{3/2} - \frac{1}{4}y^2 \right) \Big|_0^4 = \frac{4}{3}.$$

### Changing Order of Integration

In each of the previous examples, we have been given a region  $R$  and found the bounds needed to find the area of  $R$  using both orders of integration. We integrated using both orders of integration to demonstrate their equality.

We now approach the skill of describing a region using both orders of integration from a different perspective. Instead of starting with a region and creating iterated integrals, we will start with an iterated integral and rewrite it in the other integration order. To do so, we'll need to understand the region over which we are integrating.

The simplest of all cases is when both integrals are bound by constants. The region described by these bounds is a rectangle (see Example 286), and so:

$$\int_a^b \int_c^d 1 \, dy \, dx = \int_c^d \int_a^b 1 \, dx \, dy.$$

When the inner integral's bounds are not constants, it is generally very useful to sketch the bounds to determine what the region we are integrating over looks like. From the sketch we can then rewrite the integral with the other order of integration.

Examples will help us develop this skill.

#### Example 289 Changing the order of integration

Rewrite the iterated integral  $\int_0^6 \int_0^{x/3} 1 \, dy \, dx$  with the order of integration  $dx \, dy$ .

**SOLUTION** We need to use the bounds of integration to determine the region we are integrating over.

The bounds tell us that  $y$  is bounded by 0 and  $x/3$ ;  $x$  is bounded by 0 and 6. We plot these four curves:  $y = 0$ ,  $y = x/3$ ,  $x = 0$  and  $x = 6$  to find the region described by the bounds. Figure 12.6 shows these curves, indicating that  $R$  is a triangle.

To change the order of integration, we need to consider the curves that bound the  $x$ -values. We see that the lower bound is  $x = 3y$  and the upper bound is  $x = 6$ . The bounds on  $y$  are 0 to 2. Thus we can rewrite the integral as  $\int_0^2 \int_{3y}^6 1 \, dx \, dy$ .

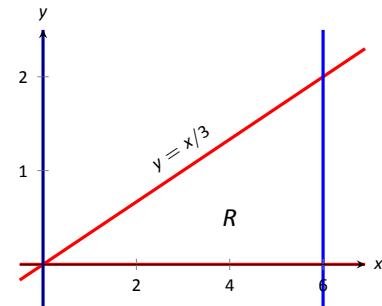


Figure 12.6: Sketching the region  $R$  described by the iterated integral in Example 289.

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Notes:

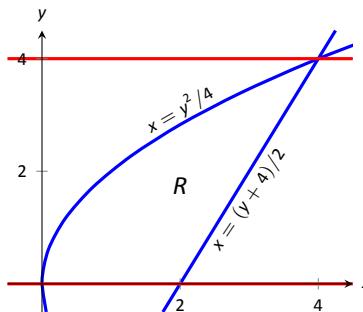


Figure 12.7: Drawing the region determined by the bounds of integration in Example 290.

### Example 290      Changing the order of integration

Change the order of integration of  $\int_0^4 \int_{y^2/4}^{(y+4)/2} 1 dx dy$ .

**SOLUTION** We sketch the region described by the bounds to help us change the integration order.  $x$  is bounded below and above (i.e., to the left and right) by  $x = y^2/4$  and  $x = (y + 4)/2$  respectively, and  $y$  is bounded between 0 and 4. Graphing the previous curves, we find the region  $R$  to be that shown in Figure 12.7.

To change the order of integration, we need to establish curves that bound  $y$ . The figure makes it clear that there are two lower bounds for  $y$ :  $y = 0$  on  $0 \leq x \leq 2$ , and  $y = 2x - 4$  on  $2 \leq x \leq 4$ . Thus we need two double integrals. The upper bound for each is  $y = 2\sqrt{x}$ . Thus we have

$$\int_0^4 \int_{y^2/4}^{(y+4)/2} 1 dx dy = \int_0^2 \int_0^{2\sqrt{x}} 1 dy dx + \int_2^4 \int_{2x-4}^{2\sqrt{x}} 1 dy dx.$$

This section has introduced a new concept, the iterated integral. We developed one application for iterated integration: area between curves. However, this is not new, for we already know how to find areas bounded by curves.

In the next section we apply iterated integration to solve problems we currently do not know how to handle. The “real” goal of this section was not to learn a new way of computing area. Rather, our goal was to learn how to define a region in the plane using the bounds of an iterated integral. That skill is very important in the following sections.

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Notes:

# Exercises 12.1

## Terms and Concepts

1. When integrating  $f_x(x, y)$  with respect to  $x$ , the constant of integration  $C$  is really which:  $C(x)$  or  $C(y)$ ? What does this mean?

2. Integrating an integral is called \_\_\_\_\_.

3. When evaluating an iterated integral, we integrate from \_\_\_\_\_ to \_\_\_\_\_, then from \_\_\_\_\_ to \_\_\_\_\_.

4. One understanding of an iterated integral is that  $\int_a^b \int_{g_1(x)}^{g_2(x)} dy dx$  gives the \_\_\_\_\_ of a plane region.

## Problems

In Exercises 5 – 10, evaluate the integral and subsequent iterated integral.

5. (a)  $\int_2^5 (6x^2 + 4xy - 3y^2) dy$

(b)  $\int_{-3}^{-2} \int_2^5 (6x^2 + 4xy - 3y^2) dy dx$

6. (a)  $\int_0^\pi (2x \cos y + \sin x) dx$

(b)  $\int_0^{\pi/2} \int_0^\pi (2x \cos y + \sin x) dx dy$

7. (a)  $\int_1^x (x^2y - y + 2) dy$

(b)  $\int_0^2 \int_1^x (x^2y - y + 2) dy dx$

8. (a)  $\int_y^{y^2} (x - y) dx$

(b)  $\int_{-1}^1 \int_y^{y^2} (x - y) dx dy$

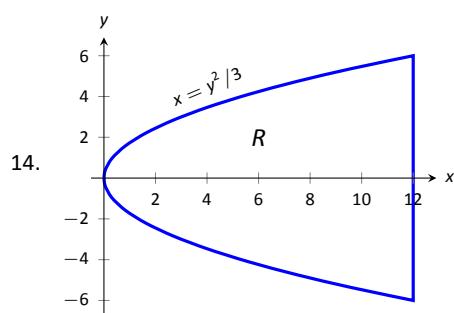
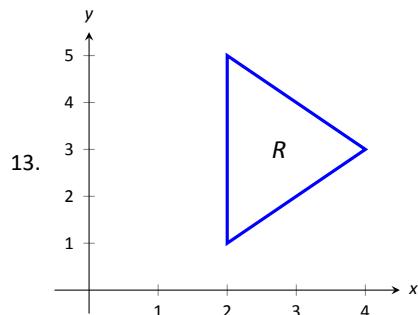
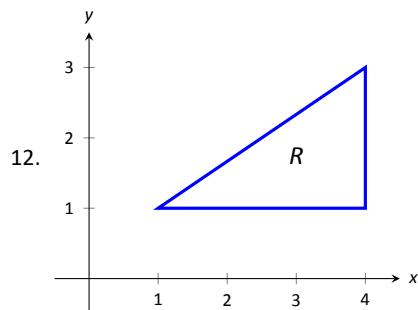
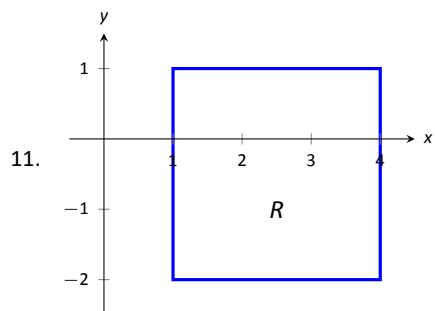
9. (a)  $\int_0^y (\cos x \sin y) dx$

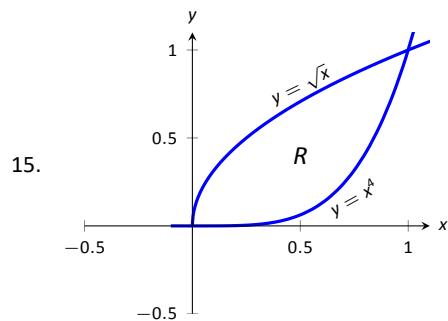
(b)  $\int_0^\pi \int_0^y (\cos x \sin y) dx dy$

10. (a)  $\int_0^x \left( \frac{1}{1+x^2} \right) dy$

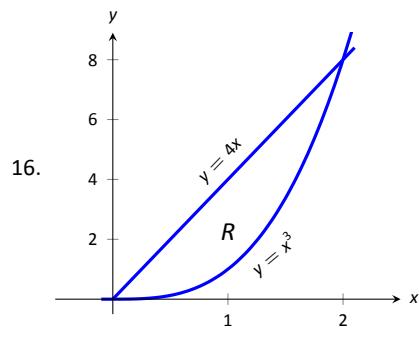
(b)  $\int_1^2 \int_0^x \left( \frac{1}{1+x^2} \right) dy dx$

In Exercises 11 – 16, a graph of a planar region  $R$  is given. Give the iterated integrals, with both orders of integration  $dy dx$  and  $dx dy$ , that give the area of  $R$ . Evaluate one of the iterated integrals to find the area.





In Exercises 17 – 22, iterated integrals are given that compute the area of a region  $R$  in the  $x$ - $y$  plane. Sketch the region  $R$ , and give the iterated integral(s) that give the area of  $R$  with the opposite order of integration.



17.  $\int_{-2}^2 \int_0^{4-x^2} dy dx$

18.  $\int_0^1 \int_{5-5x}^{5-5x^2} dy dx$

19.  $\int_{-2}^2 \int_0^{2\sqrt{4-y^2}} dx dy$

20.  $\int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dy dx$

21.  $\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx dy$

22.  $\int_{-1}^1 \int_{(x-1)/2}^{(1-x)/2} dy dx$

## 12.2 Double Integration and Volume

The definite integral of  $f$  over  $[a, b]$ ,  $\int_a^b f(x) dx$ , was introduced as “the signed area under the curve.” We approximated the value of this area by first subdividing  $[a, b]$  into  $n$  subintervals, where the  $i^{\text{th}}$  subinterval has length  $\Delta x_i$ , and letting  $c_i$  be any value in the  $i^{\text{th}}$  subinterval. We formed rectangles that approximated part of the region under the curve with width  $\Delta x_i$ , height  $f(c_i)$ , and hence with area  $f(c_i)\Delta x_i$ . Summing all the rectangle’s areas gave an approximation of the definite integral, and Theorem 40 stated that

$$\int_a^b f(x) dx = \lim_{\|\Delta x\| \rightarrow 0} \sum f(c_i) \Delta x_i,$$

connecting the area under the curve with sums of the areas of rectangles.

We use a similar approach in this section to find volume under a surface.

Let  $R$  be a closed, bounded region in the  $x$ - $y$  plane and let  $z = f(x, y)$  be a continuous function defined on  $R$ . We wish to find the signed volume under the surface of  $f$  over  $R$ . (We use the term “signed volume” to denote that space above the  $x$ - $y$  plane, under  $f$ , will have a positive volume; space above  $f$  and under the  $x$ - $y$  plane will have a “negative” volume, similar to the notion of signed area used before.)

We start by partitioning  $R$  into  $n$  rectangular subregions as shown in Figure 12.8(a). For simplicity’s sake, we let all widths be  $\Delta x$  and all heights be  $\Delta y$ . Note that the sum of the areas of the rectangles is not equal to the area of  $R$ , but rather is a close approximation. Arbitrarily number the rectangles 1 through  $n$ , and pick a point  $(x_i, y_i)$  in the  $i^{\text{th}}$  subregion.

The volume of the rectangular solid whose base is the  $i^{\text{th}}$  subregion and whose height is  $f(x_i, y_i)$  is  $V_i = f(x_i, y_i)\Delta x\Delta y$ . Such a solid is shown in Figure 12.8(b). Note how this rectangular solid only approximates the true volume under the surface; part of the solid is above the surface and part is below.

For each subregion  $R_i$  used to approximate  $R$ , create the rectangular solid with base area  $\Delta x\Delta y$  and height  $f(x_i, y_i)$ . The sum of all rectangular solids is

$$\sum_{i=1}^n f(x_i, y_i) \Delta x \Delta y.$$

This approximates the signed volume under  $f$  over  $R$ . As we have done before, to get a better approximation we can use more rectangles to approximate the region  $R$ .

In general, each rectangle could have a different width  $\Delta x_j$  and height  $\Delta y_k$ , giving the  $i^{\text{th}}$  rectangle an area  $\Delta A_i = \Delta x_j \Delta y_k$  and the  $i^{\text{th}}$  rectangular solid a

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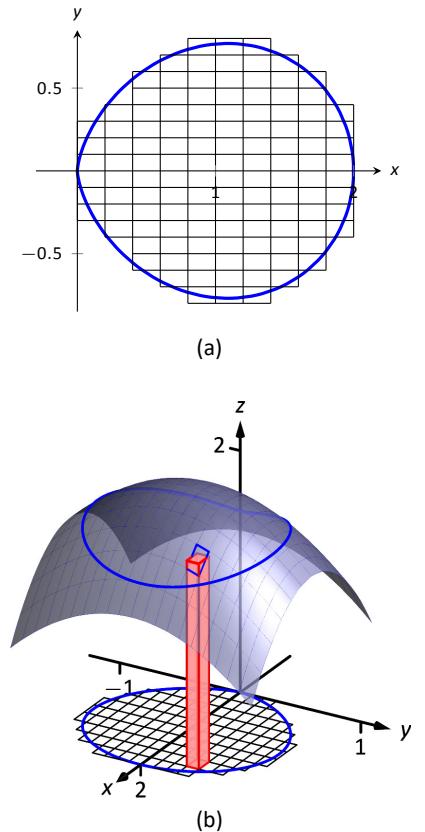


Figure 12.8: Developing a method for finding signed volume under a surface.

volume of  $f(x_i, y_i) \Delta A_i$ . Let  $\|\Delta A\|$  denote the length of the longest diagonal of all rectangles in the subdivision of  $R$ ;  $\|\Delta A\| \rightarrow 0$  means each rectangle's width and height are both approaching 0. If  $f$  is a continuous function, as  $\|\Delta A\|$  shrinks (and hence  $n \rightarrow \infty$ ) the summation  $\sum_{i=1}^n f(x_i, y_i) \Delta A_i$  approximates the signed volume better and better. This leads to a definition.

**Note:** Recall that the integration symbol “ $\int$ ” is an “elongated S,” representing the word “sum.” We interpreted  $\int_a^b f(x) dx$  as “take the sum of the areas of rectangles over the interval  $[a, b]$ .” The double integral uses two integration symbols to represent a “double sum.” When adding up the volumes of rectangular solids over a partition of a region  $R$ , as done in Figure 12.8, one could first add up the volumes across each row (one type of sum), then add these totals together (another sum), as in

$$\sum_{j=1}^n \left( \sum_{i=1}^m f(x_i, y_j) \Delta x_i \Delta y_j \right).$$

One can rewrite this as

$$\sum_{j=1}^n \left( \sum_{i=1}^m f(x_i, y_j) \Delta x_i \right) \Delta y_j.$$

The summation inside the parenthesis indicates the sum of heights  $\times$  widths, which gives an area; multiplying these areas by the thickness  $\Delta y_j$  gives a volume. The illustration in Figure 12.9 relates to this understanding.

### Definition 78 Double Integral, Signed Volume

Let  $z = f(x, y)$  be a continuous function defined over a closed region  $R$  in the  $x$ - $y$  plane. The **signed volume**  $V$  under  $f$  over  $R$  is denoted by the **double integral**

$$V = \iint_R f(x, y) dA.$$

Alternate notations for the double integral are

$$\iint_R f(x, y) dA = \iint_R f(x, y) dx dy = \iint_R f(x, y) dy dx.$$

The definition above does not state how to find the signed volume, though the notation offers a hint. We need the next two theorems to evaluate double integrals to find volume.

### Theorem 70 Double Integrals and Signed Volume

Let  $z = f(x, y)$  be a continuous function defined over a closed region  $R$  in the  $x$ - $y$  plane. Then the signed volume  $V$  under  $f$  over  $R$  is

$$V = \iint_R f(x, y) dA = \lim_{\|\Delta A\| \rightarrow 0} \sum_{i=1}^n f(x_i, y_i) \Delta A_i.$$

This theorem states that we can find the exact signed volume using a limit of sums. The partition of the region  $R$  is not specified, so any partitioning where the diagonal of each rectangle shrinks to 0 results in the same answer.

This does not offer a very satisfying way of computing area, though. Our experience has shown that evaluating the limits of sums can be tedious. We seek a more direct method.

Recall Theorem 4 in Section 7.2. This stated that if  $A(x)$  gives the cross-sectional area of a solid at  $x$ , then  $\int_a^b A(x) dx$  gave the volume of that solid over

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Notes:

$[a, b]$ .

Consider Figure 12.9, where a surface  $z = f(x, y)$  is drawn over a region  $R$ . Fixing a particular  $x$  value, we can consider the area under  $f$  over  $R$  where  $x$  has that fixed value. That area can be found with a definite integral, namely

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y) dy.$$

Remember that though the integrand contains  $x$ , we are viewing  $x$  as fixed. Also note that the bounds of integration are functions of  $x$ : the bounds depend on the value of  $x$ .

As  $A(x)$  is a cross-sectional area function, we can find the signed volume  $V$  under  $f$  by integrating it:

$$V = \int_a^b A(x) dx = \int_a^b \left( \int_{g_1(x)}^{g_2(x)} f(x, y) dy \right) dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx.$$

This gives a concrete method for finding signed volume under a surface. We could do a similar procedure where we started with  $y$  fixed, resulting in a iterated integral with the order of integration  $dx dy$ . The following theorem states that both methods give the same result, which is the value of the double integral. It is such an important theorem it has a name associated with it.

### Theorem 71 Fubini's Theorem

Let  $R$  be a closed, bounded region in the  $x$ - $y$  plane and let  $z = f(x, y)$  be a continuous function on  $R$ .

1. If  $R$  is bounded by  $a \leq x \leq b$  and  $g_1(x) \leq y \leq g_2(x)$ , where  $g_1$  and  $g_2$  are continuous functions on  $[a, b]$ , then

$$\iint_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx.$$

2. If  $R$  is bounded by  $c \leq y \leq d$  and  $h_1(y) \leq x \leq h_2(y)$ , where  $h_1$  and  $h_2$  are continuous functions on  $[c, d]$ , then

$$\iint_R f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy.$$

Note that once again the bounds of integration follow the “curve to curve, point to point” pattern discussed in the previous section. In fact, one of the

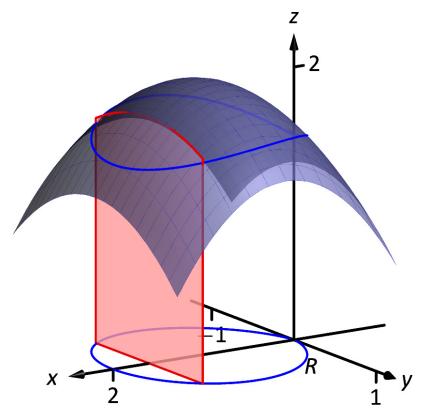


Figure 12.9: Finding volume under a surface by sweeping out a cross-sectional area.

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Notes:

main points of the previous section is developing the skill of describing a region  $R$  with the bounds of an iterated integral. Once this skill is developed, we can use double integrals to compute many quantities, not just signed volume under a surface.

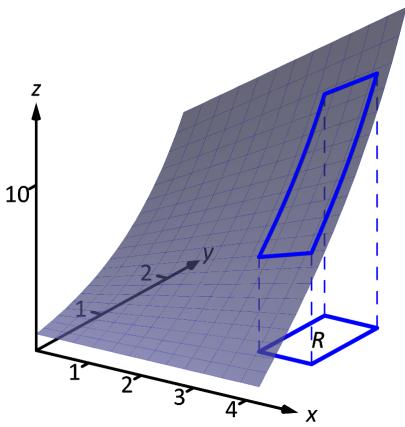


Figure 12.10: Finding the signed volume under a surface in Example 291.

**Example 291 Evaluating a double integral**

Let  $f(x, y) = xy + e^y$ . Find the signed volume under  $f$  on the region  $R$ , which is the rectangle with corners  $(3, 1)$  and  $(4, 2)$  pictured in Figure 12.10, using Fubini's Theorem and both orders of integration.

**SOLUTION** We wish to evaluate  $\iint_R (xy + e^y) dA$ . As  $R$  is a rectangle, the bounds are easily described as  $3 \leq x \leq 4$  and  $1 \leq y \leq 2$ .

Using the order  $dy dx$ :

$$\begin{aligned} \iint_R (xy + e^y) dA &= \int_3^4 \int_1^2 (xy + e^y) dy dx \\ &= \int_3^4 \left( \left[ \frac{1}{2}xy^2 + e^y \right]_1^2 \right) dx \\ &= \int_3^4 \left( \frac{3}{2}x + e^2 - e \right) dx \\ &= \left( \frac{3}{4}x^2 + (e^2 - e)x \right) \Big|_3^4 \\ &= \frac{21}{4} + e^2 - e \approx 9.92. \end{aligned}$$

Now we check the validity of Fubini's Theorem by using the order  $dx dy$ :

$$\begin{aligned} \iint_R (xy + e^y) dA &= \int_1^2 \int_3^4 (xy + e^y) dx dy \\ &= \int_1^2 \left( \left[ \frac{1}{2}x^2y + xe^y \right]_3^4 \right) dy \\ &= \int_1^2 \left( \frac{7}{2}y + e^y \right) dy \\ &= \left( \frac{7}{4}y^2 + e^y \right) \Big|_1^2 \\ &= \frac{21}{4} + e^2 - e \approx 9.92. \end{aligned}$$

Both orders of integration return the same result, as expected.

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Notes:

**Example 292 Evaluating a double integral**

Evaluate  $\iint_R (3xy - x^2 - y^2 + 6) dA$ , where  $R$  is the triangle bounded by  $x = 0$ ,  $y = 0$  and  $x/2 + y = 1$ , as shown in Figure 12.11.

**SOLUTION** While it is not specified which order we are to use, we will evaluate the double integral using both orders to help drive home the point that it does not matter which order we use.

Using the order  $dy\ dx$ : The bounds on  $y$  go from “curve to curve,” i.e.,  $0 \leq y \leq 1 - x/2$ , and the bounds on  $x$  go from “point to point,” i.e.,  $0 \leq x \leq 2$ .

$$\begin{aligned}\iint_R (3xy - x^2 - y^2 + 6) dA &= \int_0^2 \int_0^{-\frac{x}{2}+1} (3xy - x^2 - y^2 + 6) dy dx \\ &= \int_0^2 \left( \frac{3}{2}xy^2 - x^2y - \frac{1}{3}y^3 + 6y \right) \Big|_0^{-\frac{x}{2}+1} dx \\ &= \int_0^2 \left( \frac{11}{12}x^3 - \frac{11}{4}x^2 - x - \frac{17}{3} \right) dx \\ &= \left( \frac{11}{48}x^4 - \frac{11}{12}x^3 - \frac{1}{2}x^2 - \frac{17}{3}x \right) \Big|_0^2 \\ &= \frac{17}{3} = 5.\bar{6}.\end{aligned}$$

Now let's consider the order  $dx\ dy$ . Here  $x$  goes from “curve to curve,”  $0 \leq x \leq 2 - 2y$ , and  $y$  goes from “point to point,”  $0 \leq y \leq 1$ :

$$\begin{aligned}\iint_R (3xy - x^2 - y^2 + 6) dA &= \int_0^1 \int_0^{2-2y} (3xy - x^2 - y^2 + 6) dx dy \\ &= \int_0^1 \left( \frac{3}{2}x^2y - \frac{1}{3}x^3 - xy^2 + 6x \right) \Big|_0^{2-2y} dy \\ &= \int_0^1 \left( \frac{32}{3}y^3 - 22y^2 + 2y + \frac{28}{3} \right) dy \\ &= \left( \frac{8}{3}y^4 - \frac{22}{3}y^3 + y^2 + \frac{28}{3}y \right) \Big|_0^1 \\ &= \frac{17}{3} = 5.\bar{6}.\end{aligned}$$

We obtained the same result using both orders of integration.

Note how in these two examples that the bounds of integration depend only on  $R$ ; the bounds of integration have nothing to do with  $f(x, y)$ . This is an important concept, so we include it as a Key Idea.

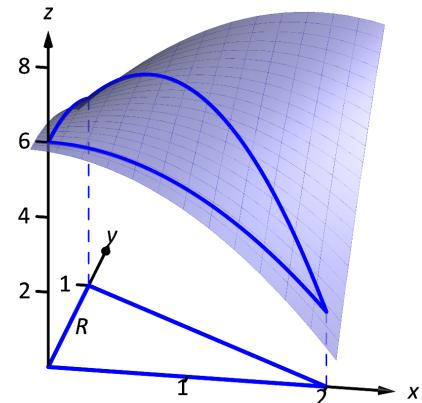


Figure 12.11: Finding the signed volume under the surface in Example 292.

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Notes:

**Key Idea 44 Double Integration Bounds**

When evaluating  $\iint_R f(x, y) dA$  using an iterated integral, the bounds of integration depend only on  $R$ . The surface  $f$  does not determine the bounds of integration.

Before doing another example, we give some properties of double integrals. Each should make sense if we view them in the context of finding signed volume under a surface, over a region.

**Theorem 72 Properties of Double Integrals**

Let  $f$  and  $g$  be continuous functions over a closed, bounded plane region  $R$ , and let  $c$  be a constant.

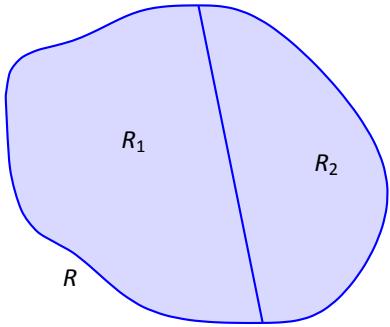


Figure 12.12:  $R$  is the union of two nonoverlapping regions,  $R_1$  and  $R_2$ .

1.  $\iint_R c f(x, y) dA = c \iint_R f(x, y) dA.$
2.  $\iint_R (f(x, y) \pm g(x, y)) dA = \iint_R f(x, y) dA \pm \iint_R g(x, y) dA$
3. If  $f(x, y) \geq 0$  on  $R$ , then  $\iint_R f(x, y) dA \geq 0$ .
4. If  $f(x, y) \geq g(x, y)$  on  $R$ , then  $\iint_R f(x, y) dA \geq \iint_R g(x, y) dA$ .
5. Let  $R$  be the union of two nonoverlapping regions,  $R = R_1 \cup R_2$  (see Figure 12.12). Then

$$\iint_R f(x, y) dA = \iint_{R_1} f(x, y) dA + \iint_{R_2} f(x, y) dA.$$

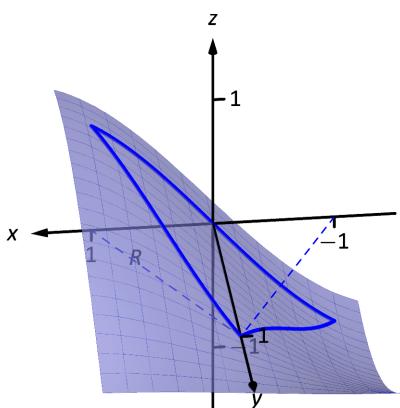


Figure 12.13: Finding the signed volume under a surface in Example 293.

**Example 293 Evaluating a double integral**

Let  $f(x, y) = \sin x \cos y$  and  $R$  be the triangle with vertices  $(-1, 0)$ ,  $(1, 0)$  and  $(0, 1)$  (see Figure 12.13). Evaluate the double integral  $\iint_R f(x, y) dA$ .

**SOLUTION** If we attempt to integrate using an iterated integral with the order  $dy dx$ , note how there are two upper bounds on  $R$  meaning we'll need to use two iterated integrals. We would need to split the triangle into two regions along the  $y$ -axis, then use Theorem 72, part 5.

Instead, let's use the order  $dx dy$ . The curves bounding  $x$  are  $y - 1 \leq x \leq$

Notes:

$1 - y$ ; the bounds on  $y$  are  $0 \leq y \leq 1$ . This gives us:

$$\begin{aligned}\iint_R f(x, y) dA &= \int_0^1 \int_{y-1}^{1-y} \sin x \cos y dx dy \\ &= \int_0^1 \left( -\cos x \cos y \right) \Big|_{y-1}^{1-y} dy \\ &= \int_0^1 \cos y \left( -\cos(1-y) + \cos(y-1) \right) dy.\end{aligned}$$

Recall that the cosine function is an even function; that is,  $\cos x = \cos(-x)$ . Therefore, from the last integral above, we have  $\cos(y-1) = \cos(1-y)$ . Thus the integrand simplifies to 0, and we have

$$\begin{aligned}\iint_R f(x, y) dA &= \int_0^1 0 dy \\ &= 0.\end{aligned}$$

It turns out that over  $R$ , there is just as much volume above the  $x$ - $y$  plane as below (look again at Figure 12.13), giving a final signed volume of 0.

### Example 294 Evaluating a double integral

Evaluate  $\iint_R (4-y) dA$ , where  $R$  is the region bounded by the parabolas  $y^2 = 4x$  and  $x^2 = 4y$ , graphed in Figure 12.14.

**SOLUTION** Graphing each curve can help us find their points of intersection. Solving analytically, the second equation tells us that  $y = x^2/4$ . Substituting this value in for  $y$  in the first equation gives us  $x^4/16 = 4x$ . Solving for  $x$ :

$$\begin{aligned}\frac{x^4}{16} &= 4x \\ x^4 - 64x &= 0 \\ x(x^3 - 64) &= 0 \\ x = 0, 4.\end{aligned}$$

Thus we've found analytically what was easy to approximate graphically: the regions intersect at  $(0, 0)$  and  $(4, 4)$ , as shown in Figure 12.14.

We now choose an order of integration:  $dy dx$  or  $dx dy$ ? Either order works; since the integrand does not contain  $x$ , choosing  $dx dy$  might be simpler – at least, the first integral is very simple.

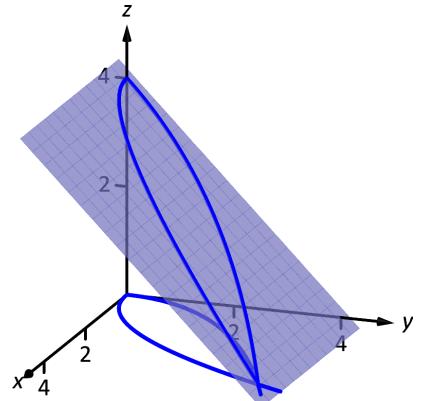


Figure 12.14: Finding the volume under the surface in Example 294.

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Notes:

Thus we have the following “curve to curve, point to point” bounds:  $y^2/4 \leq x \leq 2\sqrt{y}$ , and  $0 \leq y \leq 4$ .

$$\begin{aligned} \iint_R (4-y) dA &= \int_0^4 \int_{y^2/4}^{2\sqrt{y}} (4-y) dx dy \\ &= \int_0^4 (x(4-y)) \Big|_{y^2/4}^{2\sqrt{y}} dy \\ &= \int_0^4 \left( (2\sqrt{y} - \frac{y^2}{4})(4-y) \right) dy = \int_0^4 \left( \frac{y^3}{4} - y^2 - 2y^{3/2} + 8y^{1/2} \right) dy \\ &= \left( \frac{y^4}{16} - \frac{y^3}{3} - \frac{4y^{5/2}}{5} + \frac{16y^{3/2}}{3} \right) \Big|_0^4 \\ &= \frac{176}{15} = 11.7\bar{3}. \end{aligned}$$

The signed volume under the surface  $f$  is about 11.7 cubic units.

In the previous section we practiced changing the order of integration of a given iterated integral, where the region  $R$  was not explicitly given. Changing the bounds of an integral is more than just a test of understanding. Rather, there are cases where integrating in one order is really hard, if not impossible, whereas integrating with the other order is feasible.

### Example 295      Changing the order of integration

Rewrite the iterated integral  $\int_0^3 \int_y^3 e^{-x^2} dx dy$  with the order  $dy dx$ . Comment on the feasibility to evaluate each integral.

**SOLUTION** Once again we make a sketch of the region over which we are integrating to facilitate changing the order. The bounds on  $x$  are from  $x = y$  to  $x = 3$ ; the bounds on  $y$  are from  $y = 0$  to  $y = 3$ . These curves are sketched in Figure 12.15, enclosing the region  $R$ .

To change the bounds, note that the curves bounding  $y$  are  $y = 0$  up to  $y = x$ ; the triangle is enclosed between  $x = 0$  and  $x = 3$ . Thus the new bounds of integration are  $0 \leq y \leq x$  and  $0 \leq x \leq 3$ , giving the iterated integral  $\int_0^3 \int_0^x e^{-x^2} dy dx$ .

How easy is it to evaluate each iterated integral? Consider the order of integrating  $dx dy$ , as given in the original problem. The first indefinite integral we need to evaluate is  $\int e^{-x^2} dx$ ; we have stated before (see Section 5.6) that this integral cannot be evaluated in terms of elementary functions. We are stuck.

Changing the order of integration makes a big difference here. In the second

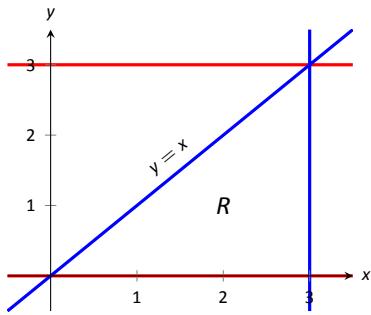


Figure 12.15: Determining the region  $R$  determined by the bounds of integration in Example 295.

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Notes:

iterated integral, we are faced with  $\int e^{-x^2} dy$ ; integrating with respect to  $y$  gives us  $ye^{-x^2} + C$ , and the first definite integral evaluates to

$$\int_0^x e^{-x^2} dy = xe^{-x^2}.$$

Thus

$$\int_0^3 \int_0^x e^{-x^2} dy dx = \int_0^3 (xe^{-x^2}) dx.$$

This last integral is easy to evaluate with substitution, giving a final answer of  $\frac{1}{2}(1 - e^{-9}) \approx 0.5$ . Figure 12.16 shows the surface over  $R$ .

In short, evaluating one iterated integral is impossible; the other iterated integral is relatively simple.

Definition 22 defines the average value of a single-variable function  $f(x)$  on the interval  $[a, b]$  as

$$\text{average value of } f(x) \text{ on } [a, b] = \frac{1}{b-a} \int_a^b f(x) dx;$$

that is, it is the “area under  $f$  over an interval divided by the length of the interval.” We make an analogous statement here: the average value of  $z = f(x, y)$  over a region  $R$  is the volume under  $f$  over  $R$  divided by the area of  $R$ .

### Definition 79 The Average Value of $f$ on $R$

Let  $z = f(x, y)$  be a continuous function defined over a closed region  $R$  in the  $x$ - $y$  plane. The **average value of  $f$  on  $R$**  is

$$\text{average value of } f \text{ on } R = \frac{\iint_R f(x, y) dA}{\iint_R dA}.$$

### Example 296 Finding average value of a function over a region $R$

Find the average value of  $f(x, y) = 4 - y$  over the region  $R$ , which is bounded by the parabolas  $y^2 = 4x$  and  $x^2 = 4y$ . Note: this is the same function and region as used in Example 294.

**SOLUTION** In Example 294 we found

$$\iint_R f(x, y) dA = \int_0^4 \int_{y^2/4}^{2\sqrt{y}} (4-y) dx dy = \frac{176}{15}.$$

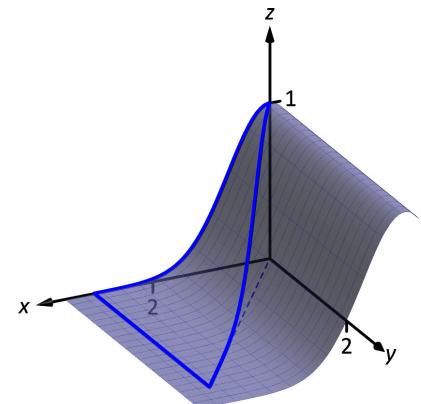


Figure 12.16: Showing the surface  $f$  defined in Example 295 over its region  $R$ .

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Notes:

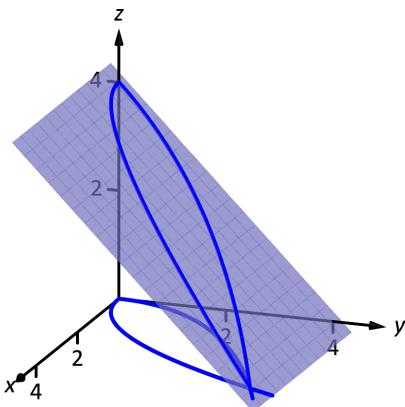


Figure 12.17: Finding the average value of  $f$  in Example 296.

We find the area of  $R$  by computing  $\iint_R dA$ :

$$\iint_R dA = \int_0^4 \int_{y^2/4}^{2\sqrt{y}} dx dy = \frac{16}{3}.$$

Dividing the volume under the surface by the area gives the average value:

$$\text{average value of } f \text{ on } R = \frac{176/15}{16/3} = \frac{11}{5} = 2.2.$$

While the surface, as shown in Figure 12.17, covers  $z$ -values from  $z = 0$  to  $z = 4$ , the “average”  $z$ -value on  $R$  is 2.2.

The previous section introduced the iterated integral in the context of finding the area of plane regions. This section has extended our understanding of iterated integrals; now we see they can be used to find the signed volume under a surface.

This new understanding allows us to revisit what we did in the previous section. Given a region  $R$  in the plane, we computed  $\iint_R 1 dA$ ; again, our understanding at the time was that we were finding the area of  $R$ . However, we can now view the function  $z = 1$  as a surface, a flat surface with constant  $z$ -value of 1. The double integral  $\iint_R 1 dA$  finds the volume, under  $z = 1$ , over  $R$ , as shown in Figure 12.18. Basic geometry tells us that if the base of a general right cylinder has area  $A$ , its volume is  $A \cdot h$ , where  $h$  is the height. In our case, the height is 1. We were “actually” computing the volume of a solid, though we interpreted the number as an area.

The next section extends our abilities to find “volumes under surfaces.” Currently, some integrals are hard to compute because either the region  $R$  we are integrating over is hard to define with rectangular curves, or the integrand itself is hard to deal with. Some of these problems can be solved by converting everything into polar coordinates.

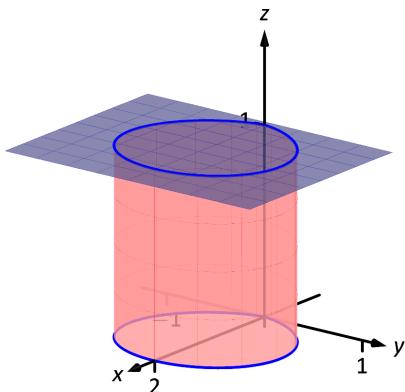


Figure 12.18: Showing how an iterated integral used to find area also finds a certain volume.

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Notes:

## Exercises 12.2

### Terms and Concepts

- An integral can be interpreted as giving the signed area over an interval; a double integral can be interpreted as giving the signed \_\_\_\_\_ over a region.
- Explain why the following statement is false: "Fubini's Theorem states that  $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx = \int_a^b \int_{g_1(y)}^{g_2(y)} f(x, y) dx dy$ ."
- Explain why if  $f(x, y) > 0$  over a region  $R$ , then  $\iint_R f(x, y) dA > 0$ .
- If  $\iint_R f(x, y) dA = \iint_R g(x, y) dA$ , does this imply  $f(x, y) = g(x, y)$ ?

### Problems

In Exercises 5 – 10,

- (a) Evaluate the given iterated integral, and
  - (b) rewrite the integral using the other order of integration.
- $\int_1^2 \int_{-1}^1 \left( \frac{x}{y} + 3 \right) dx dy$
  - $\int_{-\pi/2}^{\pi/2} \int_0^\pi (\sin x \cos y) dx dy$
  - $\int_0^4 \int_0^{-x/2+2} (3x^2 - y + 2) dy dx$
  - $\int_1^3 \int_y^3 (x^2 y - xy^2) dx dy$
  - $\int_0^1 \int_{-\sqrt{1-y}}^{\sqrt{1-y}} (x + y + 2) dx dy$
  - $\int_0^9 \int_{y/3}^{\sqrt{y}} (xy^2) dx dy$

In Exercises 11 – 18:

- (a) Sketch the region  $R$  given by the problem.
  - (b) Set up the iterated integrals, in both orders, that evaluate the given double integral for the described region  $R$ .
  - (c) Evaluate one of the iterated integrals to find the signed volume under the surface  $z = f(x, y)$  over the region  $R$ .
- $\iint_R x^2 y dA$ , where  $R$  is bounded by  $y = \sqrt{x}$  and  $y = x^2$ .
  - $\iint_R x^2 - y^2 dA$ , where  $R$  is the rectangle with corners  $(-1, -1), (1, -1), (1, 1)$  and  $(-1, 1)$ .
  - $\iint_R ye^x dA$ , where  $R$  is bounded by  $x = 0, x = y^2$  and  $y = 1$ .
  - $\iint_R (6 - 3x - 2y) dA$ , where  $R$  is bounded by  $x = 0, y = 0$  and  $3x + 2y = 6$ .
  - $\iint_R e^y dA$ , where  $R$  is bounded by  $y = \ln x$  and  $y = \frac{1}{e-1}(x-1)$ .
  - $\iint_R (x^3 y - x) dA$ , where  $R$  is the half of the circle  $x^2 + y^2 = 9$  in the first and second quadrants.
  - $\iint_R (4 - 3y) dA$ , where  $R$  is bounded by  $y = 0, y = x/e$  and  $y = \ln x$ .

In Exercises 19 – 22, state why it is difficult/impossible to integrate the iterated integral in the given order of integration. Change the order of integration and evaluate the new iterated integral.

- $\int_0^4 \int_{y/2}^2 e^{x^2} dx dy$
- $\int_0^{\sqrt{\pi/2}} \int_x^{\sqrt{\pi/2}} \cos(y^2) dy dx$
- $\int_0^1 \int_y^1 \frac{2y}{x^2 + y^2} dx dy$
- $\int_{-1}^1 \int_1^2 \frac{x \tan^2 y}{1 + \ln y} dy dx$

In Exercises 23 – 26, find the average value of  $f$  over the region  $R$ . Notice how these functions and regions are related to the iterated integrals given in Exercises 5 – 8.

- $f(x, y) = \frac{x}{y} + 3$ ;  $R$  is the rectangle with opposite corners  $(-1, 1)$  and  $(1, 2)$ .
- $f(x, y) = \sin x \cos y$ ;  $R$  is bounded by  $x = 0, x = \pi, y = -\pi/2$  and  $y = \pi/2$ .
- $f(x, y) = 3x^2 - y + 2$ ;  $R$  is bounded by the lines  $y = 0, y = 2 - x/2$  and  $x = 0$ .
- $f(x, y) = x^2 y - xy^2$ ;  $R$  is bounded by  $y = x, y = 1$  and  $x = 3$ .

### 12.3 Double Integration with Polar Coordinates

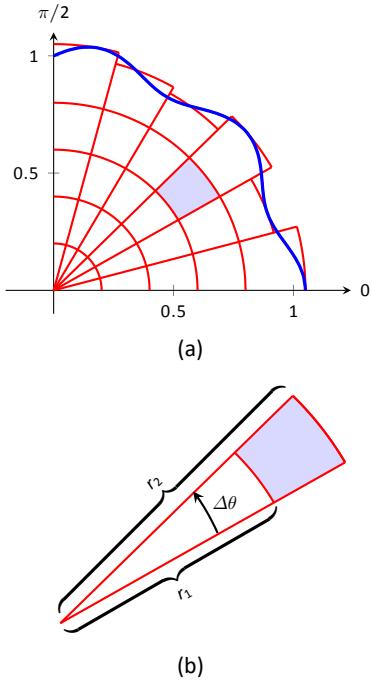


Figure 12.19: Approximating a region  $R$  with portions of sectors of circles.

We have used iterated integrals to evaluate double integrals, which give the signed volume under a surface,  $z = f(x, y)$ , over a region  $R$  of the  $x$ - $y$  plane. The integrand is simply  $f(x, y)$ , and the bounds of the integrals are determined by the region  $R$ .

Some regions  $R$  are easy to describe using rectangular coordinates – that is, with equations of the form  $y = f(x)$ ,  $x = a$ , etc. However, some regions are easier to handle if we represent their boundaries with polar equations of the form  $r = f(\theta)$ ,  $\theta = \alpha$ , etc.

The basic form of the double integral is  $\iint_R f(x, y) dA$ . We interpret this integral as follows: over the region  $R$ , sum up lots of products of heights (given by  $f(x_i, y_i)$ ) and areas (given by  $\Delta A_i$ ). That is,  $dA$  represents “a little bit of area.” In rectangular coordinates, we can describe a small rectangle as having area  $dx dy$  or  $dy dx$  – the area of a rectangle is simply length  $\times$  width – a small change in  $x$  times a small change in  $y$ . Thus we replace  $dA$  in the double integral with  $dx dy$  or  $dy dx$ .

Now consider representing a region  $R$  with polar coordinates. Consider Figure 12.19(a). Let  $R$  be the region in the first quadrant bounded by the curve. We can approximate this region using the natural shape of polar coordinates: portions of sectors of circles. In the figure, one such region is shaded, shown again in part (b) of the figure.

As the area of a sector of a circle with radius  $r$ , subtended by an angle  $\theta$ , is  $A = \frac{1}{2}r^2\theta$ , we can find the area of the shaded region. The whole sector has area  $\frac{1}{2}r_2^2\Delta\theta$ , whereas the smaller, unshaded sector has area  $\frac{1}{2}r_1^2\Delta\theta$ . The area of the shaded region is the difference of these areas:

$$\Delta A_i = \frac{1}{2}r_2^2\Delta\theta - \frac{1}{2}r_1^2\Delta\theta = \frac{1}{2}(r_2^2 - r_1^2)(\Delta\theta) = \frac{r_2 + r_1}{2}(r_2 - r_1)\Delta\theta.$$

Note that  $(r_2 + r_1)/2$  is just the average of the two radii.

To approximate the region  $R$ , we use many such subregions; doing so shrinks the difference  $r_2 - r_1$  between radii to 0 and shrinks the change in angle  $\Delta\theta$  also to 0. We represent these infinitesimal changes in radius and angle as  $dr$  and  $d\theta$ , respectively. Finally, as  $dr$  is small,  $r_2 \approx r_1$ , and so  $(r_2 + r_1)/2 \approx r_1$ . Thus, when  $dr$  and  $d\theta$  are small,

$$\Delta A_i \approx r_i dr d\theta.$$

Taking a limit, where the number of subregions goes to infinity and both  $r_2 - r_1$  and  $\Delta\theta$  go to 0, we get

$$dA = r dr d\theta.$$

So to evaluate  $\iint_R f(x, y) dA$ , replace  $dA$  with  $r dr d\theta$ . Convert the function  $z = f(x, y)$  to a function with polar coordinates with the substitutions  $x = r \cos \theta$ ,

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Notes:

$y = r \sin \theta$ . Finally, find bounds  $g_1(\theta) \leq r \leq g_2(\theta)$  and  $\alpha \leq \theta \leq \beta$  that describe  $R$ . This is the key principle of this section, so we restate it here as a Key Idea.

**Key Idea 45 Evaluating Double Integrals with Polar Coordinates**

Let  $R$  be a plane region bounded by the polar equations  $\alpha \leq \theta \leq \beta$  and  $g_1(\theta) \leq r \leq g_2(\theta)$ . Then

$$\iint_R f(x, y) dA = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta.$$

Examples will help us understand this Key Idea.

**Example 297 Evaluating a double integral with polar coordinates**

Find the signed volume under the plane  $z = 4 - x - 2y$  over the circle with equation  $x^2 + y^2 = 1$ .

**SOLUTION** The bounds of the integral are determined solely by the region  $R$  over which we are integrating. In this case, it is a circle with equation  $x^2 + y^2 = 1$ . We need to find polar bounds for this region. It may help to review Section 9.4; bounds for this circle are  $0 \leq r \leq 1$  and  $0 \leq \theta \leq 2\pi$ .

We replace  $f(x, y)$  with  $f(r \cos \theta, r \sin \theta)$ . That means we make the following substitutions:

$$4 - x - 2y \Rightarrow 4 - r \cos \theta - 2r \sin \theta.$$

Finally, we replace  $dA$  in the double integral with  $r dr d\theta$ . This gives the final iterated integral, which we evaluate:

$$\begin{aligned} \iint_R f(x, y) dA &= \int_0^{2\pi} \int_0^1 (4 - r \cos \theta - 2r \sin \theta) r dr d\theta \\ &= \int_0^{2\pi} \int_0^1 (4r - r^2(\cos \theta - 2 \sin \theta)) dr d\theta \\ &= \int_0^{2\pi} \left( 2r^2 - \frac{1}{3}r^3(\cos \theta - 2 \sin \theta) \right) \Big|_0^1 d\theta \\ &= \int_0^{2\pi} \left( 2 - \frac{1}{3}(\cos \theta - 2 \sin \theta) \right) d\theta \\ &= \left( 2\theta - \frac{1}{3}(\sin \theta + 2 \cos \theta) \right) \Big|_0^{2\pi} \\ &= 4\pi \approx 12.566. \end{aligned}$$

The surface and region  $R$  are shown in Figure 12.20.

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Notes:

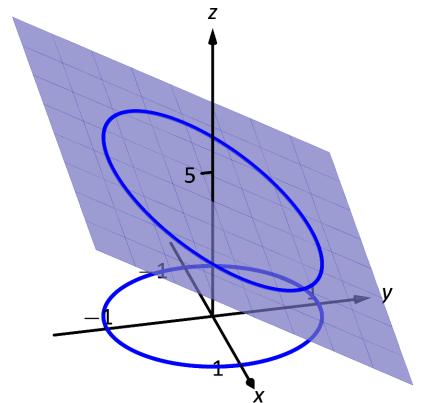


Figure 12.20: Evaluating a double integral with polar coordinates in Example 297.

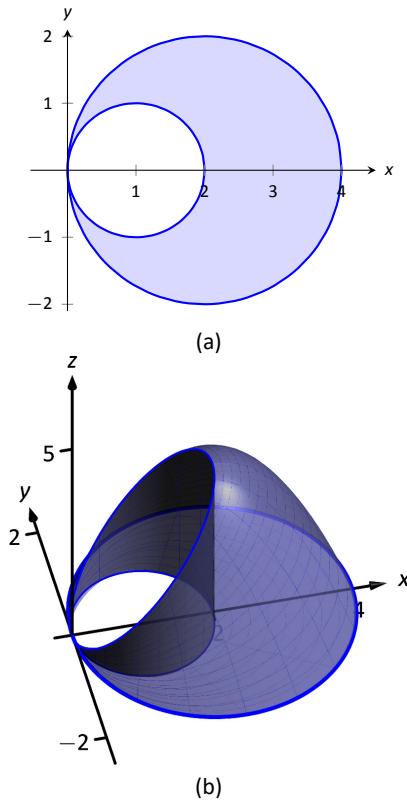


Figure 12.21: Showing the region  $R$  and surface used in Example 298.

**Example 298** **Evaluating a double integral with polar coordinates**  
 Find the volume under the paraboloid  $z = 4 - (x - 2)^2 - y^2$  over the region bounded by the circles  $(x - 1)^2 + y^2 = 1$  and  $(x - 2)^2 + y^2 = 4$ .

**SOLUTION** At first glance, this seems like a very hard volume to compute as the region  $R$  (shown in Figure 12.21(a)) has a hole in it, cutting out a strange portion of the surface, as shown in part (b) of the figure. However, by describing  $R$  in terms of polar equations, the volume is not very difficult to compute. It is straightforward to show that the circle  $(x - 1)^2 + y^2 = 1$  has polar equation  $r = 2 \cos \theta$ , and that the circle  $(x - 2)^2 + y^2 = 4$  has polar equation  $r = 4 \cos \theta$ . Each of these circles is traced out on the interval  $0 \leq \theta \leq \pi$ . The bounds on  $r$  are  $2 \cos \theta \leq r \leq 4 \cos \theta$ .

Replacing  $x$  with  $r \cos \theta$  in the integrand, along with replacing  $y$  with  $r \sin \theta$ , prepares us to evaluate the double integral  $\iint_R f(x, y) dA$ :

$$\begin{aligned} \iint_R f(x, y) dA &= \int_0^\pi \int_{2 \cos \theta}^{4 \cos \theta} \left( 4 - (r \cos \theta - 2)^2 - (r \sin \theta)^2 \right) r dr d\theta \\ &= \int_0^\pi \int_{2 \cos \theta}^{4 \cos \theta} (-r^3 + 4r^2 \cos \theta) dr d\theta \\ &= \int_0^\pi \left( -\frac{1}{4}r^4 + \frac{4}{3}r^3 \cos \theta \right) \Big|_{2 \cos \theta}^{4 \cos \theta} d\theta \\ &= \int_0^\pi \left( \left[ -\frac{1}{4}(256 \cos^4 \theta) + \frac{4}{3}(64 \cos^4 \theta) \right] - \left[ -\frac{1}{4}(16 \cos^4 \theta) + \frac{4}{3}(8 \cos^4 \theta) \right] \right) d\theta \\ &= \int_0^\pi \frac{44}{3} \cos^4 \theta d\theta. \end{aligned}$$

To integrate  $\cos^4 \theta$ , rewrite it as  $\cos^2 \theta \cos^2 \theta$  and employ the power-reducing formula twice:

$$\begin{aligned} \cos^4 \theta &= \cos^2 \theta \cos^2 \theta \\ &= \frac{1}{2}(1 + \cos(2\theta)) \frac{1}{2}(1 + \cos(2\theta)) \\ &= \frac{1}{4}(1 + 2\cos(2\theta) + \cos^2(2\theta)) \\ &= \frac{1}{4}\left(1 + 2\cos(2\theta) + \frac{1}{2}(1 + \cos(4\theta))\right) \\ &= \frac{3}{8} + \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta). \end{aligned}$$

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Notes:

Picking up from where we left off above, we have

$$\begin{aligned}
 &= \int_0^\pi \frac{44}{3} \cos^4 \theta \, d\theta \\
 &= \int_0^\pi \frac{44}{3} \left( \frac{3}{8} + \frac{1}{2} \cos(2\theta) + \frac{1}{8} \cos(4\theta) \right) \, d\theta \\
 &= \frac{44}{3} \left( \frac{3}{8}\theta + \frac{1}{4} \sin(2\theta) + \frac{1}{32} \sin(4\theta) \right) \Big|_0^\pi \\
 &= \frac{11}{2} \pi \approx 17.279.
 \end{aligned}$$

While this example was not trivial, the double integral would have been *much* harder to evaluate had we used rectangular coordinates.

**Example 299 Evaluating a double integral with polar coordinates**

Find the volume under the surface  $f(x, y) = \frac{1}{x^2 + y^2 + 1}$  over the sector of the circle with radius  $a$  centered at the origin in the first quadrant, as shown in Figure 12.22.

**SOLUTION** The region  $R$  we are integrating over is a circle with radius  $a$ , restricted to the first quadrant. Thus, in polar, the bounds on  $R$  are  $0 \leq r \leq a$ ,  $0 \leq \theta \leq \pi/2$ . The integrand is rewritten in polar as

$$\frac{1}{x^2 + y^2 + 1} \Rightarrow \frac{1}{r^2 \cos^2 \theta + r^2 \sin^2 \theta + 1} = \frac{1}{r^2 + 1}.$$

We find the volume as follows:

$$\begin{aligned}
 \iint_R f(x, y) \, dA &= \int_0^{\pi/2} \int_0^a \frac{r}{r^2 + 1} \, dr \, d\theta \\
 &= \int_0^{\pi/2} \frac{1}{2} (\ln|r^2 + 1|) \Big|_0^a \, d\theta \\
 &= \int_0^{\pi/2} \frac{1}{2} \ln(a^2 + 1) \, d\theta \\
 &= \left( \frac{1}{2} \ln(a^2 + 1)\theta \right) \Big|_0^{\pi/2} \\
 &= \frac{\pi}{4} \ln(a^2 + 1).
 \end{aligned}$$

Figure 12.22 shows that  $f$  shrinks to near 0 very quickly. Regardless, as  $a$  grows, so does the volume, without bound.

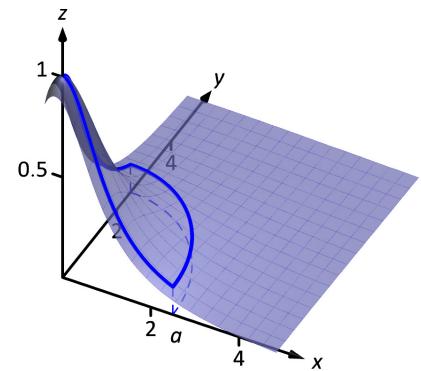


Figure 12.22: The surface and region  $R$  used in Example 299.

**Note:** Previous work has shown that there is finite *area* under  $\frac{1}{x^2+1}$  over the entire  $x$ -axis. However, Example 299 shows that there is infinite *volume* under  $\frac{1}{x^2+y^2+1}$  over the entire  $x$ - $y$  plane.

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Notes:

**Example 300 Finding the volume of a sphere**

Find the volume of a sphere with radius  $a$ .

**SOLUTION** The sphere of radius  $a$ , centered at the origin, has equation  $x^2 + y^2 + z^2 = a^2$ ; solving for  $z$ , we have  $z = \sqrt{a^2 - x^2 - y^2}$ . This gives the upper half of a sphere. We wish to find the volume under this top half, then double it to find the total volume.

The region we need to integrate over is the circle of radius  $a$ , centered at the origin. Polar bounds for this equation are  $0 \leq r \leq a$ ,  $0 \leq \theta \leq 2\pi$ .

All together, the volume of a sphere with radius  $a$  is:

$$\begin{aligned} 2 \iint_R \sqrt{a^2 - x^2 - y^2} dA &= 2 \int_0^{2\pi} \int_0^a \sqrt{a^2 - (r \cos \theta)^2 - (r \sin \theta)^2} r dr d\theta \\ &= 2 \int_0^{2\pi} \int_0^a r \sqrt{a^2 - r^2} dr d\theta. \end{aligned}$$

We can evaluate this inner integral with substitution. With  $u = a^2 - r^2$ ,  $du = -2r dr$ . The new bounds of integration are  $u(0) = a^2$  to  $u(a) = 0$ . Thus we have:

$$\begin{aligned} &= \int_0^{2\pi} \int_{a^2}^0 (-u^{1/2}) du d\theta \\ &= \int_0^{2\pi} \left( -\frac{2}{3} u^{3/2} \right) \Big|_{a^2}^0 d\theta \\ &= \int_0^{2\pi} \left( \frac{2}{3} a^3 \right) d\theta \\ &= \left( \frac{2}{3} a^3 \theta \right) \Big|_0^{2\pi} \\ &= \frac{4}{3} \pi a^3. \end{aligned}$$

Generally, the formula for the volume of a sphere with radius  $r$  is given as  $\frac{4}{3}\pi r^3$ ; we have justified this formula with our calculation.

**Example 301 Finding the volume of a solid**

A sculptor wants to make a solid bronze cast of the solid shown in Figure 12.23, where the base of the solid has boundary, in polar coordinates,  $r = \cos(3\theta)$ , and the top is defined by the plane  $z = 1 - x + 0.1y$ . Find the volume of the solid.

**SOLUTION** From the outset, we should recognize that knowing *how to set up* this problem is probably more important than knowing *how to compute*

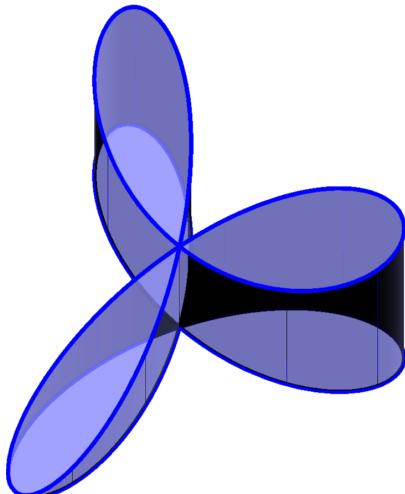


Figure 12.23: Visualizing the solid used in Example 301.

Notes:

*the integrals.* The iterated integral to come is not “hard” to evaluate, though it is long, requiring lots of algebra. Once the proper iterated integral is determined, one can use readily-available technology to help compute the final answer.

The region  $R$  that we are integrating over is bound by  $0 \leq r \leq \cos(3\theta)$ , for  $0 \leq \theta \leq \pi$  (note that this rose curve is traced out on the interval  $[0, \pi]$ , not  $[0, 2\pi]$ ). This gives us our bounds of integration. The integrand is  $z = 1 - x + 0.1y$ ; converting to polar, we have that the volume  $V$  is:

$$V = \iint_R f(x, y) dA = \int_0^\pi \int_0^{\cos(3\theta)} (1 - r \cos \theta + 0.1r \sin \theta) r dr d\theta.$$

Distributing the  $r$ , the inner integral is easy to evaluate, leading to

$$\int_0^\pi \left( \frac{1}{2} \cos^2(3\theta) - \frac{1}{3} \cos^3(3\theta) \cos \theta + \frac{0.1}{3} \cos^3(3\theta) \sin \theta \right) d\theta.$$

This integral takes time to compute by hand; it is rather long and cumbersome. The powers of cosine need to be reduced, and products like  $\cos(3\theta) \cos \theta$  need to be turned to sums using the Product To Sum formulas in the back cover of this text.

We rewrite  $\frac{1}{2} \cos^2(3\theta)$  as  $\frac{1}{4}(1+\cos(6\theta))$ . We can also rewrite  $\frac{1}{3} \cos^3(3\theta) \cos \theta$  as:

$$\frac{1}{3} \cos^3(3\theta) \cos \theta = \frac{1}{3} \cos^2(3\theta) \cos(3\theta) \cos \theta = \frac{1}{3} \frac{1 + \cos(6\theta)}{2} (\cos(4\theta) + \cos(2\theta)).$$

This last expression still needs simplification, but eventually all terms can be reduced to the form  $a \cos(m\theta)$  or  $a \sin(m\theta)$  for various values of  $a$  and  $m$ .

We forgo the algebra and recommend the reader employ technology, such as WolframAlpha®, to compute the numeric answer. Such technology gives:

$$\int_0^\pi \int_0^{\cos(3\theta)} (1 - r \cos \theta + 0.1r \sin \theta) r dr d\theta = \frac{\pi}{4} \approx 0.785u^3.$$

Since the units were not specified, we leave the result as almost 0.8 cubic units (meters, feet, etc.) Should the artist want to scale the piece uniformly, so that each rose petal had a length other than 1, she should keep in mind that scaling by a factor of  $k$  scales the volume by a factor of  $k^3$ .

We have used iterated integrals to find areas of plane regions and volumes under surfaces. Just as a single integral can be used to compute much more than “area under the curve,” iterated integrals can be used to compute much more than we have thus far seen. The next two sections show two, among many, applications of iterated integrals.

Notes:

# Exercises 12.3

## Terms and Concepts

- When evaluating  $\iint_R f(x, y) dA$  using polar coordinates,  $f(x, y)$  is replaced with \_\_\_\_\_ and  $dA$  is replaced with \_\_\_\_\_.
- Why would one be interested in evaluating a double integral with polar coordinates?

## Problems

In Exercises 3 – 10, a function  $f(x, y)$  is given and a region  $R$  of the  $x$ - $y$  plane is described. Set up and evaluate  $\iint_R f(x, y) dA$  using polar coordinates.

- $f(x, y) = 3x - y + 4$ ;  $R$  is the region enclosed by the circle  $x^2 + y^2 = 1$ .
- $f(x, y) = 4x + 4y$ ;  $R$  is the region enclosed by the circle  $x^2 + y^2 = 4$ .
- $f(x, y) = 8 - y$ ;  $R$  is the region enclosed by the circles with polar equations  $r = \cos \theta$  and  $r = 3 \cos \theta$ .
- $f(x, y) = 4$ ;  $R$  is the region enclosed by the petal of the rose curve  $r = \sin(2\theta)$  in the first quadrant.
- $f(x, y) = \ln(x^2 + y^2)$ ;  $R$  is the annulus enclosed by the circles  $x^2 + y^2 = 1$  and  $x^2 + y^2 = 4$ .
- $f(x, y) = 1 - x^2 - y^2$ ;  $R$  is the region enclosed by the circle  $x^2 + y^2 = 1$ .
- $f(x, y) = x^2 - y^2$ ;  $R$  is the region enclosed by the circle  $x^2 + y^2 = 36$  in the first and fourth quadrants.
- $f(x, y) = (x - y)/(x + y)$ ;  $R$  is the region enclosed by the lines  $y = x$ ,  $y = 0$  and the circle  $x^2 + y^2 = 1$  in the first quadrant.

In Exercises 11 – 14, an iterated integral in rectangular coordinates is given. Rewrite the integral using polar coordinates and evaluate the new double integral.

- $\int_0^5 \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} \sqrt{x^2 + y^2} dy dx$
- $\int_{-4}^4 \int_{-\sqrt{16-y^2}}^0 (2y - x) dx dy$
- $\int_0^2 \int_y^{\sqrt{8-y^2}} (x + y) dx dy$
- $\int_{-2}^{-1} \int_0^{\sqrt{4-x^2}} (x + 5) dy dx + \int_{-1}^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} (x + 5) dy dx + \int_1^2 \int_0^{\sqrt{4-x^2}} (x + 5) dy dx$

**Hint:** draw the region of each integral carefully and see how they all connect.

In Exercises 15 – 16, special double integrals are presented that are especially well suited for evaluation in polar coordinates.

- Consider  $\iint_R e^{-(x^2+y^2)} dA$ .
  - Why is this integral difficult to evaluate in rectangular coordinates, regardless of the region  $R$ ?
  - Let  $R$  be the region bounded by the circle of radius  $a$  centered at the origin. Evaluate the double integral using polar coordinates.
  - Take the limit of your answer from (b), as  $a \rightarrow \infty$ . What does this imply about the volume under the surface of  $e^{-(x^2+y^2)}$  over the entire  $x$ - $y$  plane?
- The surface of a right circular cone with height  $h$  and base radius  $a$  can be described by the equation  $f(x, y) = h - h \sqrt{\frac{x^2}{a^2} + \frac{y^2}{a^2}}$ , where the tip of the cone lies at  $(0, 0, h)$  and the circular base lies in the  $x$ - $y$  plane, centered at the origin.  
Confirm that the volume of a right circular cone with height  $h$  and base radius  $a$  is  $V = \frac{1}{3}\pi a^2 h$  by evaluating  $\iint_R f(x, y) dA$  in polar coordinates.

## 12.4 Center of Mass

We have used iterated integrals to find areas of plane regions and signed volumes under surfaces. A brief recap of these uses will be useful in this section as we apply iterated integrals to compute the **mass** and **center of mass** of planar regions.

To find the area of a planar region, we evaluated the double integral  $\iint_R dA$ . That is, summing up the areas of lots of little subregions of  $R$  gave us the total area. Informally, we think of  $\iint_R dA$  as meaning “sum up lots of little areas over  $R$ .”

To find the signed volume under a surface, we evaluated the double integral  $\iint_R f(x, y) dA$ . Recall that the “ $dA$ ” is not just a “bookend” at the end of an integral; rather, it is multiplied by  $f(x, y)$ . We regard  $f(x, y)$  as giving a height, and  $dA$  still giving an area:  $f(x, y) dA$  gives a volume. Thus, informally,  $\iint_R f(x, y) dA$  means “sum up lots of little volumes over  $R$ .”

We now extend these ideas to other contexts.

### Mass and Weight

Consider a thin sheet of material with constant thickness and finite area. Mathematicians (and physicists and engineers) call such a sheet a **lamina**. So consider a lamina, as shown in Figure 12.24(a), with the shape of some planar region  $R$ , as shown in part (b).

We can write a simple double integral that represents the mass of the lamina:  $\iint_R dm$ , where “ $dm$ ” means “a little mass.” That is, the double integral states the total mass of the lamina can be found by “summing up lots of little masses over  $R$ .”

To evaluate this double integral, partition  $R$  into  $n$  subregions as we have done in the past. The  $i^{\text{th}}$  subregion has area  $\Delta A_i$ . A fundamental property of mass is that “mass=density×area.” If the lamina has a constant density  $\delta$ , then the mass of this  $i^{\text{th}}$  subregion is  $\Delta m_i = \delta \Delta A_i$ . That is, we can compute a small amount of mass by multiplying a small amount of area by the density.

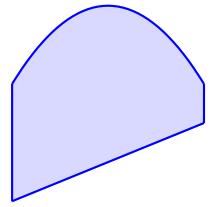
If density is variable, with density function  $\delta = \delta(x, y)$ , then we can approximate the mass of the  $i^{\text{th}}$  subregion of  $R$  by multiplying  $\Delta A_i$  by  $\delta(x_i, y_i)$ , where  $(x_i, y_i)$  is a point in that subregion. That is, for a small enough subregion of  $R$ , the density across that region is almost constant.

The total mass  $M$  of the lamina is approximately the sum of approximate masses of subregions:

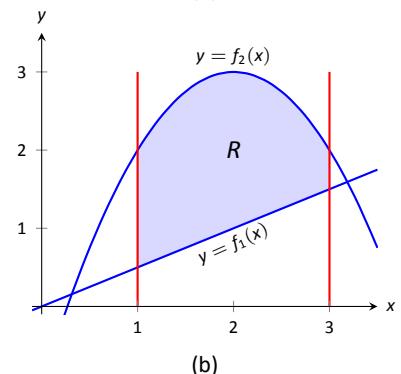
$$M \approx \sum_{i=1}^n \Delta m_i = \sum_{i=1}^n \delta(x_i, y_i) \Delta A_i.$$

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Notes:



(a)



(b)

Figure 12.24: Illustrating the concept of a lamina.

**Note:** *Mass* and *weight* are different measures. Since they are scalar multiples of each other, it is often easy to treat them as the same measure. In this section we effectively treat them as the same, as our technique for finding mass is the same as for finding weight. The density functions used will simply have different units.

Taking the limit as the size of the subregions shrinks to 0 gives us the actual mass; that is, integrating  $\delta(x, y)$  over  $R$  gives the mass of the lamina.

**Definition 80 Mass of a Lamina with Variable Density**

Let  $\delta(x, y)$  be a continuous density function of a lamina corresponding to a plane region  $R$ . The mass  $M$  of the lamina is

$$\text{mass } M = \iint_R dm = \iint_R \delta(x, y) dA.$$

**Example 302 Finding the mass of a lamina with constant density**

Find the mass of a square lamina, with side length 1, with a density of  $\delta = 3\text{gm/cm}^2$ .

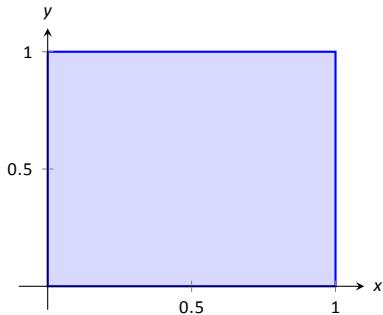


Figure 12.25: A region  $R$  representing a lamina in Example 302.

**SOLUTION** We represent the lamina with a square region in the plane as shown in Figure 12.25. As the density is constant, it does not matter where we place the square.

Following Definition 80, the mass  $M$  of the lamina is

$$M = \iint_R 3 dA = \int_0^1 \int_0^1 3 dx dy = 3 \int_0^1 \int_0^1 dx dy = 3\text{gm}.$$

This is all very straightforward; note that all we really did was find the area of the lamina and multiply it by the constant density of  $3\text{gm/cm}^2$ .

**Example 303 Finding the mass of a lamina with variable density**

Find the mass of a square lamina, represented by the unit square with lower lefthand corner at the origin (see Figure 12.25), with variable density  $\delta(x, y) = (x + y + 2)\text{gm/cm}^2$ .

**SOLUTION** The variable density  $\delta$ , in this example, is very uniform, giving a density of 3 in the center of the square and changing linearly. A graph of  $\delta(x, y)$  can be seen in Figure 12.26; notice how “same amount” of density is above  $z = 3$  as below. We’ll comment on the significance of this momentarily.

The mass  $M$  is found by integrating  $\delta(x, y)$  over  $R$ . The order of integration

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Notes:

is not important; we choose  $dx dy$  arbitrarily. Thus:

$$\begin{aligned}
 M &= \iint_R (x + y + 2) dA = \int_0^1 \int_0^1 (x + y + 2) dx dy \\
 &= \int_0^1 \left( \frac{1}{2}x^2 + x(y+2) \right) \Big|_0^1 dy \\
 &= \int_0^1 \left( \frac{5}{2}y + \frac{1}{2}y^2 \right) dy \\
 &= \left( \frac{5}{2}y + \frac{1}{2}y^2 \right) \Big|_0^1 \\
 &= 3\text{gm}.
 \end{aligned}$$

It turns out that since the density of the lamina is so uniformly distributed "above and below"  $z = 3$  that the mass of the lamina is the same as if it had a constant density of 3. The density functions in Examples 302 and 303 are graphed in Figure 12.26, which illustrates this concept.

#### Example 304 Finding the weight of a lamina with variable density

Find the weight of the lamina represented by the circle with radius 2ft, centered at the origin, with density function  $\delta(x, y) = (x^2 + y^2 + 1)\text{lb/ft}^2$ . Compare this to the weight of the same lamina with density  $\delta(x, y) = (2\sqrt{x^2 + y^2} + 1)\text{lb/ft}^2$ .

**SOLUTION** A direct application of Definition 80 states that the weight of the lamina is  $\iint_R \delta(x, y) dA$ . Since our lamina is in the shape of a circle, it makes sense to approach the double integral using polar coordinates.

The density function  $\delta(x, y) = x^2 + y^2 + 1$  becomes  $\delta(r, \theta) = (r \cos \theta)^2 + (r \sin \theta)^2 + 1 = r^2 + 1$ . The circle is bounded by  $0 \leq r \leq 2$  and  $0 \leq \theta \leq 2\pi$ . Thus the weight  $W$  is:

$$\begin{aligned}
 W &= \int_0^{2\pi} \int_0^2 (r^2 + 1)r dr d\theta \\
 &= \int_0^{2\pi} \left( \frac{1}{4}r^4 + \frac{1}{2}r^2 \right) \Big|_0^2 d\theta \\
 &= \int_0^{2\pi} (6) d\theta \\
 &= 12\pi \approx 37.70\text{lb}.
 \end{aligned}$$

Now compare this with the density function  $\delta(x, y) = 2\sqrt{x^2 + y^2} + 1$ . Converting this to polar coordinates gives  $\delta(r, \theta) = 2\sqrt{(r \cos \theta)^2 + (r \sin \theta)^2} + 1 =$

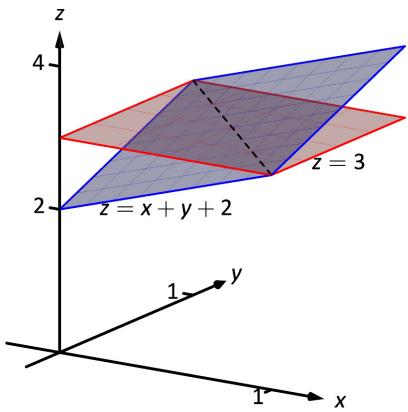


Figure 12.26: Graphing the density functions in Examples 302 and 303.

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Notes:

$2r + 1$ . Thus the weight  $W$  is:

$$\begin{aligned} W &= \int_0^{2\pi} \int_0^2 (2r + 1)r \, dr \, d\theta \\ &= \int_0^{2\pi} \left( \frac{2}{3}r^3 + \frac{1}{2}r^2 \right) \Big|_0^2 \, d\theta \\ &= \int_0^{2\pi} \left( \frac{22}{3} \right) \, d\theta \\ &= \frac{44}{3}\pi \approx 46.08 \text{ lb}. \end{aligned}$$

One would expect different density functions to return different weights, as we have here. The density functions were chosen, though, to be similar: each gives a density of 1 at the origin and a density of 5 at the outside edge of the circle, as seen in Figure 12.27.

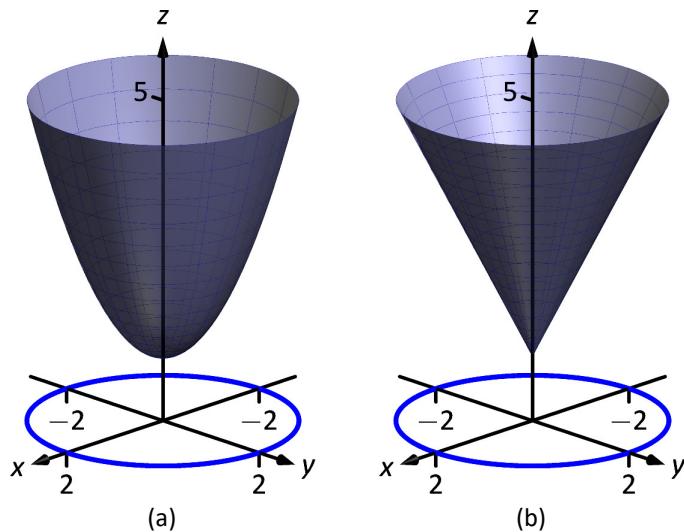


Figure 12.27: Graphing the density functions in Example 304. In (a) is the density function  $\delta(x, y) = x^2 + y^2 + 1$ ; in (b) is  $\delta(x, y) = 2\sqrt{x^2 + y^2} + 1$ .

Notice how  $x^2 + y^2 + 1 \leq 2\sqrt{x^2 + y^2} + 1$  over the circle; this results in less weight.

Plotting the density functions can be useful as our understanding of mass can be related to our understanding of “volume under a surface.” We interpreted  $\iint_R f(x, y) \, dA$  as giving the volume under  $f$  over  $R$ ; we can understand  $\iint_R \delta(x, y) \, dA$  in the same way. The “volume” under  $\delta$  over  $R$  is actually mass;

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Notes:

by compressing the “volume” under  $\delta$  onto the  $x$ - $y$  plane, we get “more mass” in some areas than others – i.e., areas of greater density.

Knowing the mass of a lamina is one of several important measures. Another is the **center of mass**, which we discuss next.

## Center of Mass

Consider a disk of radius 1 with uniform density. It is common knowledge that the disk will balance on a point if the point is placed at the center of the disk. What if the disk does not have a uniform density? Through trial-and-error, we should still be able to find a spot on the disk at which the disk will balance on a point. This balance point is referred to as the **center of mass**, or **center of gravity**. It is though all the mass is “centered” there. In fact, if the disk has a mass of 3kg, the disk will behave physically as though it were a point-mass of 3kg located at its center of mass. For instance, the disk will naturally spin with an axis through its center of mass (which is why it is important to “balance” the tires of your car: if they are “out of balance”, their center of mass will be outside of the axle and it will shake terribly).

We find the center of mass based on the principle of a **weighted average**. Consider a college class in which your homework average is 90%, your test average is 73%, and your final exam grade is an 85%. Experience tells us that our final grade is *not* the *average* of these three grades: that is, it is not:

$$\frac{0.9 + 0.73 + 0.85}{3} \approx 0.837 = 83.7\%.$$

That is, you are probably not pulling a B in the course. Rather, your grades are *weighted*. Let’s say the homework is worth 10% of the grade, tests are 60% and the exam is 30%. Then your final grade is:

$$(0.1)(0.9) + (0.6)(0.73) + (0.3)(0.85) = 0.783 = 78.3\%.$$

Each grade is multiplied by a **weight**.

In general, given values  $x_1, x_2, \dots, x_n$  and weights  $w_1, w_2, \dots, w_n$ , the weighted average of the  $n$  values is

$$\sum_{i=1}^n w_i x_i \Bigg/ \sum_{i=1}^n w_i.$$

In the grading example above, the sum of the weights 0.1, 0.6 and 0.3 is 1, so we don’t see the division by the sum of weights in that instance.

How this relates to center of mass is given in the following theorem.

Notes:

**Theorem 73     Center of Mass of Discrete Linear System**

Let point masses  $m_1, m_2, \dots, m_n$  be distributed along the  $x$ -axis at locations  $x_1, x_2, \dots, x_n$ , respectively. The center of mass  $\bar{x}$  of the system is located at

$$\bar{x} = \frac{\sum_{i=1}^n m_i x_i}{\sum_{i=1}^n m_i}.$$

**Example 305     Finding the center of mass of a discrete linear system**

1. Point masses of 2gm are located at  $x = -1, x = 2$  and  $x = 3$  are connected by a thin rod of negligible weight. Find the center of mass of the system.
2. Point masses of 10gm, 2gm and 1gm are located at  $x = -1, x = 2$  and  $x = 3$ , respectively, are connected by a thin rod of negligible weight. Find the center of mass of the system.

**SOLUTION**

1. Following Theorem 73, we compute the center of mass as:

$$\bar{x} = \frac{2(-1) + 2(2) + 2(3)}{2+2+2} = \frac{4}{3} = 1.\bar{3}.$$

So the system would balance on a point placed at  $x = 4/3$ , as illustrated in Figure 12.28(a).

2. Again following Theorem 73, we find:

$$\bar{x} = \frac{10(-1) + 2(2) + 1(3)}{10+2+1} = \frac{-3}{13} \approx -0.23.$$

Placing a large weight at the left hand side of the system moves the center of mass left, as shown in Figure 12.28(b).

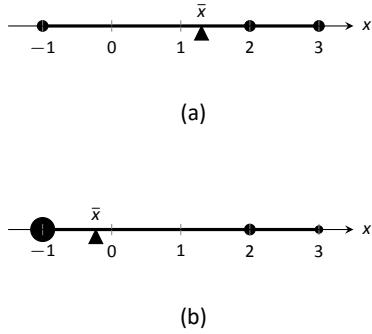


Figure 12.28: Illustrating point masses along a thin rod and the center of mass.

Notes:

In a discrete system (i.e., mass is located at individual points, not along a continuum) we find the center of mass by dividing the mass into a **moment** of the system. In general, a moment is a weighted measure of distance from a particular point or line. In the case described by Theorem 73, we are finding a weighted measure of distances from the  $y$ -axis, so we refer to this as the **moment about**

the **y-axis**, represented by  $M_y$ . Letting  $M$  be the total mass of the system, we have  $\bar{x} = M_y/M$ .

We can extend the concept of the center of mass of discrete points along a line to the center of mass of discrete points in the plane rather easily. To do so, we define some terms then give a theorem.

**Definition 81 Moments about the x- and y- Axes.**

Let point masses  $m_1, m_2, \dots, m_n$  be located at points  $(x_1, y_1), (x_2, y_2) \dots, (x_n, y_n)$ , respectively, in the x-y plane.

1. The **moment about the y-axis**,  $M_y$ , is  $M_y = \sum_{i=1}^n m_i x_i$ .
2. The **moment about the x-axis**,  $M_x$ , is  $M_x = \sum_{i=1}^n m_i y_i$ .

One can think that these definitions are “backwards” as  $M_y$  sums up “x” distances. But remember, “x” distances are measurements of distance from the y-axis, hence defining the moment about the y-axis.

We now define the center of mass of discrete points in the plane.

**Theorem 74 Center of Mass of Discrete Planar System**

Let point masses  $m_1, m_2, \dots, m_n$  be located at points  $(x_1, y_1), (x_2, y_2) \dots, (x_n, y_n)$ , respectively, in the x-y plane, and let  $M = \sum_{i=1}^n m_i$ .

The center of mass of the system is at  $(\bar{x}, \bar{y})$ , where

$$\bar{x} = \frac{M_y}{M} \quad \text{and} \quad \bar{y} = \frac{M_x}{M}$$

**Example 306 Finding the center of mass of a discrete planar system**

Let point masses of 1kg, 2kg and 5kg be located at points  $(2, 0), (1, 1)$  and  $(3, 1)$ , respectively, and are connected by thin rods of negligible weight. Find the center of mass of the system.

**SOLUTION** We follow Theorem 74 and Definition 81 to find  $M, M_x$  and  $M_y$ :

---

Notes:

$$M = 1 + 2 + 5 = 8\text{kg}.$$

$$\begin{aligned} M_x &= \sum_{i=1}^n m_i y_i & M_y &= \sum_{i=1}^n m_i x_i \\ &= 1(0) + 2(1) + 5(1) & &= 1(2) + 2(1) + 5(3) \\ &= 7. & &= 19. \end{aligned}$$

Thus the center of mass is  $(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{19}{8}, \frac{7}{8}\right) = (2.375, 0.875)$ , illustrated in Figure 12.29.

We finally arrive at our true goal of this section: finding the center of mass of a lamina with variable density. While the above measurement of center of mass is interesting, it does not directly answer more realistic situations where we need to find the center of mass of a contiguous region. However, understanding the discrete case allows us to approximate the center of mass of a planar lamina; using calculus, we can refine the approximation to an exact value.

We begin by representing a planar lamina with a region  $R$  in the  $x$ - $y$  plane with density function  $\delta(x, y)$ . Partition  $R$  into  $n$  subdivisions, each with area  $\Delta A_i$ . As done before, we can approximate the mass of the  $i^{\text{th}}$  subregion with  $\delta(x_i, y_i)\Delta A_i$ , where  $(x_i, y_i)$  is a point inside the  $i^{\text{th}}$  subregion. We can approximate the moment of this subregion about the  $y$ -axis with  $x_i\delta(x_i, y_i)\Delta A_i$  – that is, by multiplying the approximate mass of the region by its approximate distance from the  $y$ -axis. Similarly, we can approximate the moment about the  $x$ -axis with  $y_i\delta(x_i, y_i)\Delta A_i$ . By summing over all subregions, we have:

$$\text{mass: } M \approx \sum_{i=1}^n \delta(x_i, y_i)\Delta A_i \quad (\text{as seen before})$$

$$\text{moment about the } x\text{-axis: } M_x \approx \sum_{i=1}^n y_i \delta(x_i, y_i)\Delta A_i$$

$$\text{moment about the } y\text{-axis: } M_y \approx \sum_{i=1}^n x_i \delta(x_i, y_i)\Delta A_i$$

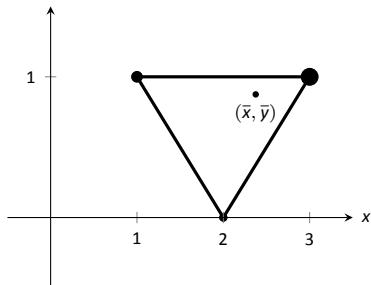


Figure 12.29: Illustrating the center of mass of a discrete planar system in Example 306.

By taking limits, where size of each subregion shrinks to 0 in both the  $x$  and  $y$  directions, we arrive at the double integrals given in the following theorem.

---

Notes:

**Theorem 75    Center of Mass of a Planar Lamina, Moments**

Let a planar lamina be represented by a region  $R$  in the  $x$ - $y$  plane with density function  $\delta(x, y)$ .

1. mass:  $M = \iint_R \delta(x, y) dA$
2. moment about the  $x$ -axis:  $M_x = \iint_R y\delta(x, y) dA$
3. moment about the  $y$ -axis:  $M_y = \iint_R x\delta(x, y) dA$
4. The center of mass of the lamina is

$$(\bar{x}, \bar{y}) = \left( \frac{M_y}{M}, \frac{M_x}{M} \right).$$

We start our practice of finding centers of mass by revisiting some of the lamina used previously in this section when finding mass. We will just set up the integrals needed to compute  $M$ ,  $M_x$  and  $M_y$  and leave the details of the integration to the reader.

**Example 307    Finding the center of mass of a lamina**

Find the center mass of a square lamina, with side length 1, with a density of  $\delta = 3\text{gm/cm}^2$ . (Note: this is the lamina from Example 302.)

**SOLUTION**    We represent the lamina with a square region in the plane as shown in Figure 12.30 as done previously.

Following Theorem 75, we find  $M$ ,  $M_x$  and  $M_y$ :

$$M = \iint_R 3 dA = \int_0^1 \int_0^1 3 dx dy = 3\text{gm}.$$

$$M_x = \iint_R 3y dA = \int_0^1 \int_0^1 3y dx dy = 3/2 = 1.5.$$

$$M_y = \iint_R 3x dA = \int_0^1 \int_0^1 3x dx dy = 3/2 = 1.5.$$

Thus the center of mass is  $(\bar{x}, \bar{y}) = \left( \frac{M_y}{M}, \frac{M_x}{M} \right) = (1.5/3, 1.5/3) = (0.5, 0.5)$ .

This is what we should have expected: the center of mass of a square with constant density is the center of the square.

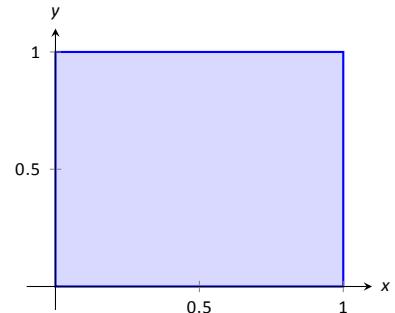


Figure 12.30: A region  $R$  representing a lamina in Example 302.

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Notes:

**Example 308 Finding the center of mass of a lamina**

Find the center of mass of a square lamina, represented by the unit square with lower lefthand corner at the origin (see Figure 12.30), with variable density  $\delta(x, y) = (x + y + 2)\text{gm/cm}^2$ . (Note: this is the lamina from Example 303.)

**SOLUTION** We follow Theorem 75, to find  $M$ ,  $M_x$  and  $M_y$ :

$$M = \iint_R (x + y + 2) dA = \int_0^1 \int_0^1 (x + y + 2) dx dy = 3\text{gm}.$$

$$M_x = \iint_R y(x + y + 2) dA = \int_0^1 \int_0^1 y(x + y + 2) dx dy = \frac{19}{12}.$$

$$M_y = \iint_R x(x + y + 2) dA = \int_0^1 \int_0^1 x(x + y + 2) dx dy = \frac{19}{12}.$$

Thus the center of mass is  $(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{19}{36}, \frac{19}{36}\right) \approx (0.528, 0.528)$ .

While the mass of this lamina is the same as the lamina in the previous example, the greater density found with greater  $x$  and  $y$  values pulls the center of mass from the center slightly towards the upper righthand corner.

**Example 309 Finding the center of mass of a lamina**

Find the center of mass of the lamina represented by the circle with radius 2ft, centered at the origin, with density function  $\delta(x, y) = (x^2 + y^2 + 1)\text{lb/ft}^2$ . (Note: this is one of the lamina used in Example 304.)

**SOLUTION** As done in Example 304, it is best to describe  $R$  using polar coordinates. Thus when we compute  $M_y$ , we will integrate not  $x\delta(x, y) = x(x^2 + y^2 + 1)$ , but rather  $(r \cos \theta)\delta(r \cos \theta, r \sin \theta) = (r \cos \theta)(r^2 + 1)$ . We compute  $M$ ,  $M_x$  and  $M_y$ :

$$M = \int_0^{2\pi} \int_0^2 (r^2 + 1)r dr d\theta = 12\pi \approx 37.7\text{lb}.$$

$$M_x = \int_0^{2\pi} \int_0^2 (r \sin \theta)(r^2 + 1)r dr d\theta = 0.$$

$$M_y = \int_0^{2\pi} \int_0^2 (r \cos \theta)(r^2 + 1)r dr d\theta = 0.$$

Since  $R$  and the density of  $R$  are both symmetric about the  $x$  and  $y$  axes, it should come as no big surprise that the moments about each axis is 0. Thus the center

---

Notes:

of mass is  $(\bar{x}, \bar{y}) = (0, 0)$ .

**Example 310 Finding the center of mass of a lamina**

Find the center of mass of the lamina represented by the region  $R$  shown in Figure 12.31, half an annulus with outer radius 6 and inner radius 5, with constant density  $2\text{lb/ft}^2$ .

**SOLUTION** Once again it will be useful to represent  $R$  in polar coordinates. Using the description of  $R$  and/or the illustration, we see that  $R$  is bounded by  $5 \leq r \leq 6$  and  $0 \leq \theta \leq \pi$ . As the lamina is symmetric about the  $y$ -axis, we should expect  $M_y = 0$ . We compute  $M$ ,  $M_x$  and  $M_y$ :

$$M = \int_0^\pi \int_5^6 (2)r dr d\theta = 11\pi \text{lb.}$$

$$M_x = \int_0^\pi \int_5^6 (r \sin \theta)(2)r dr d\theta = \frac{364}{3} \approx 121.33.$$

$$M_y = \int_0^\pi \int_5^6 (r \cos \theta)(2)r dr d\theta = 0.$$

Thus the center of mass is  $(\bar{x}, \bar{y}) = (0, \frac{364}{33\pi}) \approx (0, 3.51)$ . The center of mass is indicated in Figure 12.31; note how it lies outside of  $R$ !

This section has shown us another use for iterated integrals beyond finding area or signed volume under the curve. While there are many uses for iterated integrals, we give one more application in the following section: computing surface area.

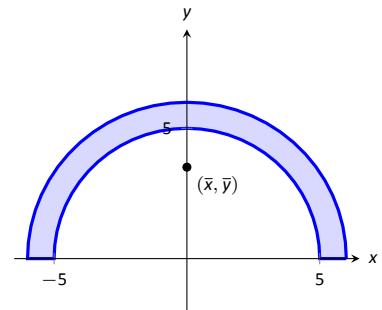


Figure 12.31: Illustrating the region  $R$  in Example 310.

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Notes:

# Exercises 12.4

## Terms and Concepts

1. Why is it easy to use “mass” and “weight” interchangeably, even though they are different measures?
2. Given a point  $(x, y)$ , the value of  $x$  is a measure of distance from the \_\_\_\_\_-axis.
3. We can think of  $\iint_R dm$  as meaning “sum up lots of \_\_\_\_\_”
4. What is a “discrete planar system?”
5. Why does  $M_x$  use  $\iint_R y\delta(x, y) dA$  instead of  $\iint_R x\delta(x, y) dA$ ; that is, why do we use “y” and not “x”?
6. Describe a situation where the center of mass of a lamina does not lie within the region of the lamina itself.

## Problems

**In Exercises 7 – 10, point masses are given along a line or in the plane. Find the center of mass  $\bar{x}$  or  $(\bar{x}, \bar{y})$ , as appropriate. (All masses are in grams and distances are in cm.)**

7.  $m_1 = 4$  at  $x = 1$ ;  $m_2 = 3$  at  $x = 3$ ;  $m_3 = 5$  at  $x = 10$
8.  $m_1 = 2$  at  $x = -3$ ;  $m_2 = 2$  at  $x = -1$ ;  $m_3 = 3$  at  $x = 0$ ;  $m_4 = 3$  at  $x = 7$
9.  $m_1 = 2$  at  $(-2, -2)$ ;  $m_2 = 2$  at  $(2, -2)$ ;  $m_3 = 20$  at  $(0, 4)$
10.  $m_1 = 1$  at  $(-1, -1)$ ;  $m_2 = 2$  at  $(-1, 1)$ ;  $m_3 = 2$  at  $(1, 1)$ ;  $m_4 = 1$  at  $(1, -1)$

**In Exercises 11 – 18, find the mass/weight of the lamina described by the region  $R$  in the plane and its density function  $\delta(x, y)$ .**

11.  $R$  is the rectangle with corners  $(1, -3)$ ,  $(1, 2)$ ,  $(7, 2)$  and  $(7, -3)$ ;  $\delta(x, y) = 5\text{gm/cm}^2$
12.  $R$  is the rectangle with corners  $(1, -3)$ ,  $(1, 2)$ ,  $(7, 2)$  and  $(7, -3)$ ;  $\delta(x, y) = (x + y^2)\text{gm/cm}^2$
13.  $R$  is the triangle with corners  $(-1, 0)$ ,  $(1, 0)$ , and  $(0, 1)$ ;  $\delta(x, y) = 2\text{lb/in}^2$
14.  $R$  is the triangle with corners  $(0, 0)$ ,  $(1, 0)$ , and  $(0, 1)$ ;  $\delta(x, y) = (x^2 + y^2 + 1)\text{lb/in}^2$
15.  $R$  is the circle centered at the origin with radius 2;  $\delta(x, y) = (x + y + 4)\text{kg/m}^2$
16.  $R$  is the circle sector bounded by  $x^2 + y^2 = 25$  in the first quadrant;  $\delta(x, y) = (\sqrt{x^2 + y^2} + 1)\text{kg/m}^2$
17.  $R$  is the annulus in the first and second quadrants bounded by  $x^2 + y^2 = 9$  and  $x^2 + y^2 = 36$ ;  $\delta(x, y) = 4\text{lb/ft}^2$

18.  $R$  is the annulus in the first and second quadrants bounded by  $x^2 + y^2 = 9$  and  $x^2 + y^2 = 36$ ;  $\delta(x, y) = \sqrt{x^2 + y^2}\text{lb/ft}^2$

**In Exercises 19 – 26, find the center of mass of the lamina described by the region  $R$  in the plane and its density function  $\delta(x, y)$ .**

**Note:** these are the same lamina as in Exercises 11 – 18.

19.  $R$  is the rectangle with corners  $(1, -3)$ ,  $(1, 2)$ ,  $(7, 2)$  and  $(7, -3)$ ;  $\delta(x, y) = 5\text{gm/cm}^2$
20.  $R$  is the rectangle with corners  $(1, -3)$ ,  $(1, 2)$ ,  $(7, 2)$  and  $(7, -3)$ ;  $\delta(x, y) = (x + y^2)\text{gm/cm}^2$
21.  $R$  is the triangle with corners  $(-1, 0)$ ,  $(1, 0)$ , and  $(0, 1)$ ;  $\delta(x, y) = 2\text{lb/in}^2$
22.  $R$  is the triangle with corners  $(0, 0)$ ,  $(1, 0)$ , and  $(0, 1)$ ;  $\delta(x, y) = (x^2 + y^2 + 1)\text{lb/in}^2$
23.  $R$  is the circle centered at the origin with radius 2;  $\delta(x, y) = (x + y + 4)\text{kg/m}^2$
24.  $R$  is the circle sector bounded by  $x^2 + y^2 = 25$  in the first quadrant;  $\delta(x, y) = (\sqrt{x^2 + y^2} + 1)\text{kg/m}^2$
25.  $R$  is the annulus in the first and second quadrants bounded by  $x^2 + y^2 = 9$  and  $x^2 + y^2 = 36$ ;  $\delta(x, y) = 4\text{lb/ft}^2$
26.  $R$  is the annulus in the first and second quadrants bounded by  $x^2 + y^2 = 9$  and  $x^2 + y^2 = 36$ ;  $\delta(x, y) = \sqrt{x^2 + y^2}\text{lb/ft}^2$

The **moment of inertia** / is a measure of the tendency of a lamina to resist rotating about an axis or continue to rotate about an axis.  $I_x$  is the moment of inertia about the  $x$ -axis,  $I_y$  is the moment of inertia about the  $y$ -axis, and  $I_O$  is the moment of inertia about the origin. These are computed as follows:

- $I_x = \iint_R y^2 dm$
- $I_y = \iint_R x^2 dm$
- $I_O = \iint_R (x^2 + y^2) dm$

**In Exercises 27 – 30, a lamina corresponding to a planar region  $R$  is given with a mass of 16 units. For each, compute  $I_x$ ,  $I_y$  and  $I_O$ .**

27.  $R$  is the  $4 \times 4$  square with corners at  $(-2, -2)$  and  $(2, 2)$  with density  $\delta(x, y) = 1$ .
28.  $R$  is the  $8 \times 2$  rectangle with corners at  $(-4, -1)$  and  $(4, 1)$  with density  $\delta(x, y) = 1$ .
29.  $R$  is the  $4 \times 2$  rectangle with corners at  $(-2, -1)$  and  $(2, 1)$  with density  $\delta(x, y) = 2$ .
30.  $R$  is the circle with radius 2 centered at the origin with density  $\delta(x, y) = 4/\pi$ .

## 12.5 Surface Area

In Section 7.4 we used definite integrals to compute the arc length of plane curves of the form  $y = f(x)$ . We later extended these ideas to compute the arc length of plane curves defined by parametric or polar equations.

The natural extension of the concept of “arc length over an interval” to surfaces is “surface area over a region.”

Consider the surface  $z = f(x, y)$  over a region  $R$  in the  $x$ - $y$  plane, shown in Figure 12.32(a). Because of the domed shape of the surface, the surface area will be greater than that of the area of the region  $R$ . We can find this area using the same basic technique we have used over and over: we’ll make an approximation, then using limits, we’ll refine the approximation to the exact value.

As done to find the volume under a surface or the mass of a lamina, we subdivide  $R$  into  $n$  subregions. Here we subdivide  $R$  into rectangles, as shown in the figure. One such subregion is outlined in the figure, where the rectangle has dimensions  $\Delta x_i$  and  $\Delta y_i$ , along with its corresponding region on the surface.

In part (b) of the figure, we zoom in on this portion of the surface. When  $\Delta x_i$  and  $\Delta y_i$  are small, the function is approximated well by the tangent plane at any point  $(x_i, y_i)$  in this subregion, which is graphed in part (b). In fact, the tangent plane approximates the function so well that in this figure, it is virtually indistinguishable from the surface itself! Therefore we can approximate the surface area  $S_i$  of this region of the surface with the area  $T_i$  of the corresponding portion of the tangent plane.

This portion of the tangent plane is a parallelogram, defined by sides  $\vec{u}$  and  $\vec{v}$ , as shown. One of the applications of the cross product from Section 9.4 is that the area of this parallelogram is  $\|\vec{u} \times \vec{v}\|$ . Once we can determine  $\vec{u}$  and  $\vec{v}$ , we can determine the area.

$\vec{u}$  is tangent to the surface in the direction of  $x$ , therefore, from Section 11.7,  $\vec{u}$  is parallel to  $\langle 1, 0, f_x(x_i, y_i) \rangle$ . The  $x$ -displacement of  $\vec{u}$  is  $\Delta x_i$ , so we know that  $\vec{u} = \Delta x_i \langle 1, 0, f_x(x_i, y_i) \rangle$ . Similar logic shows that  $\vec{v} = \Delta y_i \langle 0, 1, f_y(x_i, y_i) \rangle$ . Thus:

$$\begin{aligned} \text{surface area } S_i &\approx \text{area of } T_i \\ &= \|\vec{u} \times \vec{v}\| \\ &= \|\Delta x_i \langle 1, 0, f_x(x_i, y_i) \rangle \times \Delta y_i \langle 0, 1, f_y(x_i, y_i) \rangle\| \\ &= \sqrt{1 + f_x(x_i, y_i)^2 + f_y(x_i, y_i)^2} \Delta x_i \Delta y_i. \end{aligned}$$

Note that  $\Delta x_i \Delta y_i = \Delta A_i$ , the area of the  $i^{\text{th}}$  subregion.

Summing up all  $n$  of the approximations to the surface area gives

$$\text{surface area over } R \approx \sum_{i=1}^n \sqrt{1 + f_x(x_i, y_i)^2 + f_y(x_i, y_i)^2} \Delta A_i.$$

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Notes:

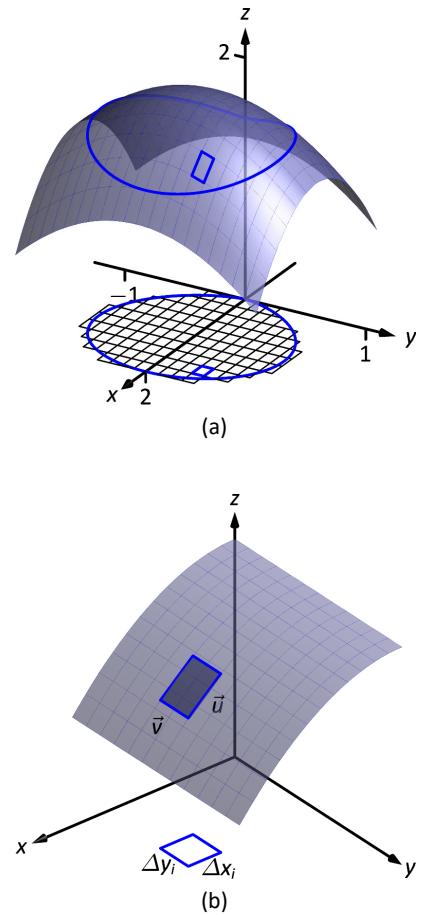


Figure 12.32: Developing a method of computing surface area.

Once again take a limit as all of the  $\Delta x_i$  and  $\Delta y_i$  shrink to 0; this leads to a double integral.

**Note:** as done before, we think of  $\iint_R dS$  as meaning “sum up lots of little surface areas over  $R$ .”

The concept of surface area is *defined* here, for while we already have a notion of the area of a region in the *plane*, we did not yet have a solid grasp of what “the area of a surface in *space*” means.

### Definition 82 Surface Area

Let  $z = f(x, y)$  where  $f_x$  and  $f_y$  are continuous over a closed, bounded region  $R$ . The surface area  $S$  over  $R$  is

$$\begin{aligned} S &= \iint_R dS \\ &= \iint_R \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2} dA. \end{aligned}$$

We test this definition by using it to compute surface areas of known surfaces. We start with a triangle.

### Example 311 Finding the surface area of a plane over a triangle

Let  $f(x, y) = 4 - x - 2y$ , and let  $R$  be the region in the plane bounded by  $x = 0$ ,  $y = 0$  and  $y = 2 - x/2$ , as shown in Figure 12.33. Find the surface area of  $f$  over  $R$ .

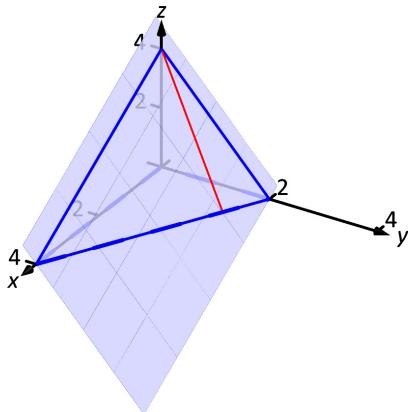


Figure 12.33: Finding the area of a triangle in space in Example 311.

**SOLUTION** We follow Definition 82. We start by noting that  $f_x(x, y) = -1$  and  $f_y(x, y) = -2$ . To define  $R$ , we use bounds  $0 \leq y \leq 2 - x/2$  and  $0 \leq x \leq 4$ . Therefore

$$\begin{aligned} S &= \iint_R dS \\ &= \int_0^4 \int_0^{2-x/2} \sqrt{1 + (-1)^2 + (-2)^2} dy dx \\ &= \int_0^4 \sqrt{6} \left(2 - \frac{x}{2}\right) dx \\ &= 4\sqrt{6}. \end{aligned}$$

Because the surface is a triangle, we can figure out the area using geometry. Considering the base of the triangle to be the side in the  $x$ - $y$  plane, we find the length of the base to be  $\sqrt{20}$ . We can find the height using our knowledge of vectors: let  $\vec{u}$  be the side in the  $x$ - $z$  plane and let  $\vec{v}$  be the side in the  $x$ - $y$  plane. The height is then  $\|\vec{u} - \text{proj}_{\vec{v}} \vec{u}\| = 4\sqrt{6}/5$ . Geometry states that the area is thus

$$\frac{1}{2} \cdot 4\sqrt{6}/5 \cdot \sqrt{20} = 4\sqrt{6}.$$

We affirm the validity of our formula.

---

Notes:

It is “common knowledge” that the surface area of a sphere of radius  $r$  is  $4\pi r^2$ . We confirm this in the following example, which involves using our formula with polar coordinates.

**Example 312 The surface area of a sphere.**

Find the surface area of the sphere with radius  $a$  centered at the origin, whose top hemisphere has equation  $f(x, y) = \sqrt{a^2 - x^2 - y^2}$ .

**SOLUTION** We start by computing partial derivatives and find

$$f_x(x, y) = \frac{-x}{\sqrt{a^2 - x^2 - y^2}} \quad \text{and} \quad f_y(x, y) = \frac{-y}{\sqrt{a^2 - x^2 - y^2}}.$$

As our function  $f$  only defines the top upper hemisphere of the sphere, we double our surface area result to get the total area:

$$\begin{aligned} S &= 2 \iint_R \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2} dA \\ &= 2 \iint_R \sqrt{1 + \frac{x^2 + y^2}{a^2 - x^2 - y^2}} dA. \end{aligned}$$

The region  $R$  that we are integrating over is the circle, centered at the origin, with radius  $a$ :  $x^2 + y^2 = a^2$ . Because of this region, we are likely to have greater success with our integration by converting to polar coordinates. Using the substitutions  $x = r \cos \theta$ ,  $y = r \sin \theta$ ,  $dA = r dr d\theta$  and bounds  $0 \leq \theta \leq 2\pi$  and  $0 \leq r \leq a$ , we have:

$$\begin{aligned} S &= 2 \int_0^{2\pi} \int_0^a \sqrt{1 + \frac{r^2 \cos^2 \theta + r^2 \sin^2 \theta}{a^2 - r^2 \cos^2 \theta - r^2 \sin^2 \theta}} r dr d\theta \\ &= 2 \int_0^{2\pi} \int_0^a r \sqrt{1 + \frac{r^2}{a^2 - r^2}} dr d\theta \\ &= 2 \int_0^{2\pi} \int_0^a r \sqrt{\frac{a^2}{a^2 - r^2}} dr d\theta. \end{aligned} \tag{12.1}$$

Apply substitution  $u = a^2 - r^2$  and integrate the inner integral, giving

$$\begin{aligned} &= 2 \int_0^{2\pi} a^2 d\theta \\ &= 4\pi a^2. \end{aligned}$$

Our work confirms our previous formula.

**Note:** The inner integral in Equation (12.1) is an improper integral, as the integrand of  $\int_0^a r \sqrt{\frac{a^2}{a^2 - r^2}} dr$  is not defined at  $r = a$ . To properly evaluate this integral, one must use the techniques of Section 6.6.

The reason this need arises is that the function  $f(x, y) = \sqrt{a^2 - x^2 - y^2}$  fails the requirements of Definition 82, as  $f_x$  and  $f_y$  are not continuous on the boundary of the circle  $x^2 + y^2 = a^2$ .

The computation of the surface area is still valid. The definition makes stronger requirements than necessary in part to avoid the use of improper integration, as when  $f_x$  and/or  $f_y$  are not continuous, the resulting improper integral may not converge. Since the improper integral does converge in this example, the surface area is accurately computed.

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Notes:

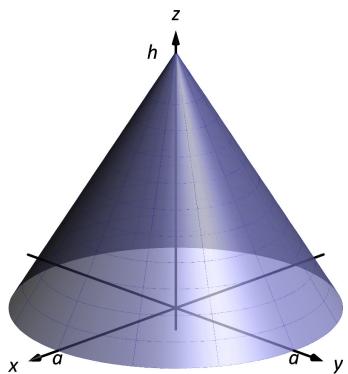


Figure 12.34: Finding the surface area of a cone in Example 313.

**Note:** Note that once again  $f_x$  and  $f_y$  are not continuous on the domain of  $f$ , as both are undefined at  $(0, 0)$ . (A similar problem occurred in the previous example.) Once again the resulting improper integral converges and the computation of the surface area is valid.

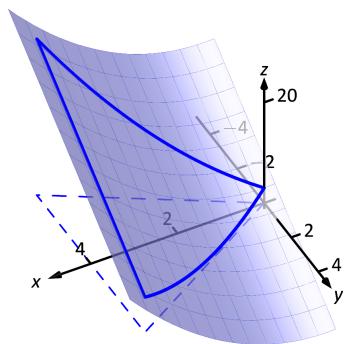


Figure 12.35: Graphing the surface in Example 314.

### Example 313 Finding the surface area of a cone

The general formula for a right cone with height  $h$  and base radius  $a$  is

$$f(x, y) = h - \frac{h}{a} \sqrt{x^2 + y^2},$$

shown in Figure 12.34. Find the surface area of this cone.

#### SOLUTION

We begin by computing partial derivatives.

$$f_x(x, y) = -\frac{xh}{a\sqrt{x^2 + y^2}} \quad \text{and} \quad f_y(x, y) = -\frac{yh}{a\sqrt{x^2 + y^2}}.$$

Since we are integrating over the circle  $x^2 + y^2 = a^2$ , we again use polar coordinates. Using the standard substitutions, our integrand becomes

$$\sqrt{1 + \left(\frac{hr \cos \theta}{a\sqrt{r^2}}\right)^2 + \left(\frac{hr \sin \theta}{a\sqrt{r^2}}\right)^2}.$$

This may look intimidating at first, but there are lots of simple simplifications to be done. It amazingly reduces to just

$$\sqrt{1 + \frac{h^2}{a^2}} = \frac{1}{a} \sqrt{a^2 + h^2}.$$

Our polar bounds are  $0 \leq \theta \leq 2\pi$  and  $0 \leq r \leq a$ . Thus

$$\begin{aligned} S &= \int_0^{2\pi} \int_0^a r \frac{1}{a} \sqrt{a^2 + h^2} dr d\theta \\ &= \int_0^{2\pi} \left( \frac{1}{2} r^2 \frac{1}{a} \sqrt{a^2 + h^2} \right) \Big|_0^a d\theta \\ &= \int_0^{2\pi} \frac{1}{2} a \sqrt{a^2 + h^2} d\theta \\ &= \pi a \sqrt{a^2 + h^2}. \end{aligned}$$

This matches the formula found in the back of this text.

### Example 314 Finding surface area over a region

Find the area of the surface  $f(x, y) = x^2 - 3y + 3$  over the region  $R$  bounded by  $-x \leq y \leq x$ ,  $0 \leq x \leq 4$ , as pictured in Figure 12.35.

**SOLUTION** It is straightforward to compute  $f_x(x, y) = 2x$  and  $f_y(x, y) = -3$ . Thus the surface area is described by the double integral

$$\iint_R \sqrt{1 + (2x)^2 + (-3)^2} dA = \iint_R \sqrt{10 + 4x^2} dA.$$

---

Notes:

As with integrals describing arc length, double integrals describing surface area are in general hard to evaluate directly because of the square-root. This particular integral can be easily evaluated, though, with judicious choice of our order of integration.

Integrating with order  $dx\,dy$  requires us to evaluate  $\int \sqrt{10 + 4x^2} dx$ . This can be done, though it involves Integration By Parts and  $\sinh^{-1} x$ . Integrating with order  $dy\,dx$  has as its first integral  $\int \sqrt{10 + 4x^2} dy$ , which is easy to evaluate: it is simply  $y\sqrt{10 + 4x^2} + C$ . So we proceed with the order  $dy\,dx$ ; the bounds are already given in the statement of the problem.

$$\begin{aligned}\iint_R \sqrt{10 + 4x^2} dA &= \int_0^4 \int_{-x}^x \sqrt{10 + 4x^2} dy\,dx \\ &= \int_0^4 (y\sqrt{10 + 4x^2}) \Big|_{-x}^x dx \\ &= \int_0^4 (2x\sqrt{10 + 4x^2}) dx.\end{aligned}$$

Apply substitution with  $u = 10 + 4x^2$ :

$$\begin{aligned}&= \left( \frac{1}{6} (10 + 4x^2)^{3/2} \right) \Big|_0^4 \\ &= \frac{1}{3} (37\sqrt{74} - 5\sqrt{10}) \approx 100.825u^2.\end{aligned}$$

So while the region  $R$  over which we integrate has an area of  $16u^2$ , the surface has a much greater area as its  $z$ -values change dramatically over  $R$ .

In practice, technology helps greatly in the evaluation of such integrals. High powered computer algebra systems can compute integrals that are difficult, or at least time consuming, by hand, and can at the least produce very accurate approximations with numerical methods. In general, just knowing *how* to set up the proper integrals brings one very close to being able to compute the needed value. Most of the work is actually done in just describing the region  $R$  in terms of polar or rectangular coordinates. Once this is done, technology can usually provide a good answer.

We have learned how to integrate integrals; that is, we have learned to evaluate double integrals. In the next section, we learn how to integrate double integrals – that is, we learn to evaluate *triple integrals*, along with learning some uses for this operation.

Notes:

## Exercises 12.5

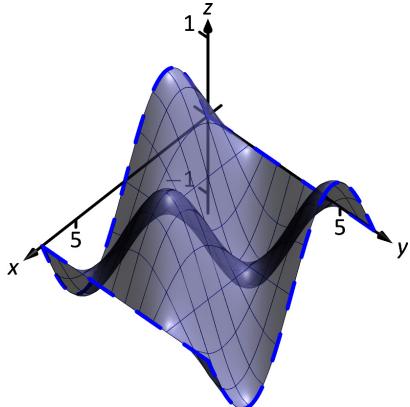
### Terms and Concepts

1. “Surface area” is analogous to what previously studied concept?
2. To approximate the area of a small portion of a surface, we computed the area of its \_\_\_\_\_ plane.
3. We interpret  $\iint_R dS$  as “sum up lots of little \_\_\_\_\_.”
4. Why is it important to know how to set up a double integral to compute surface area, even if the resulting integral is hard to evaluate?
5. Why do  $z = f(x, y)$  and  $z = g(x, y) = f(x, y) + h$ , for some real number  $h$ , have the same surface area over a region  $R$ ?
6. Let  $z = f(x, y)$  and  $z = g(x, y) = 2f(x, y)$ . Why is the surface area of  $g$  over a region  $R$  not twice the surface area of  $f$  over  $R$ ?

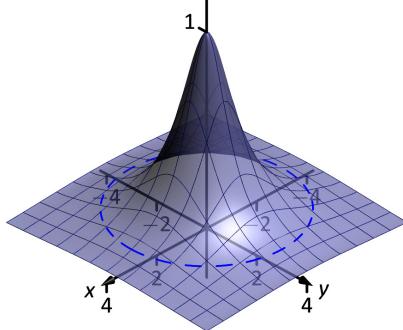
### Problems

In Exercises 7 – 10, set up the iterated integral that computes the surface area of the given surface over the region  $R$ .

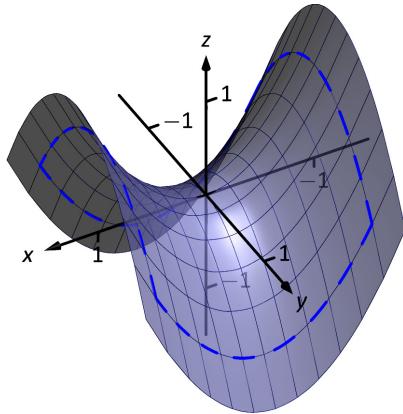
7.  $f(x, y) = \sin x \cos y$ ;  $R$  is the rectangle with bounds  $0 \leq x \leq 2\pi$ ,  $0 \leq y \leq 2\pi$ .



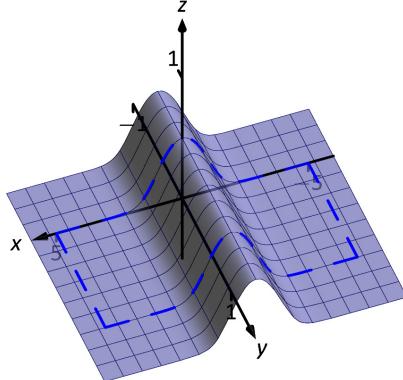
8.  $f(x, y) = \frac{1}{x^2 + y^2 + 1}$ ;  $R$  is the circle  $x^2 + y^2 = 9$ .



9.  $f(x, y) = x^2 - y^2$ ;  $R$  is the rectangle with opposite corners  $(-1, -1)$  and  $(1, 1)$ .



10.  $f(x, y) = \frac{1}{e^{x^2} + 1}$ ;  $R$  is the rectangle bounded by  $-5 \leq x \leq 5$  and  $0 \leq y \leq 1$ .



In Exercises 11 – 19, find the area of the given surface over the region  $R$ .

11.  $f(x, y) = 3x - 7y + 2$ ;  $R$  is the rectangle with opposite corners  $(-1, 0)$  and  $(1, 3)$ .

12.  $f(x, y) = 2x + 2y + 2$ ;  $R$  is the triangle with corners  $(0, 0)$ ,  $(1, 0)$  and  $(0, 1)$ .

13.  $f(x, y) = x^2 + y^2 + 10$ ;  $R$  is the circle  $x^2 + y^2 = 16$ .  
to compare you result with the known formula.)
14.  $f(x, y) = -2x + 4y^2 + 7$  over  $R$ , the triangle bounded by  
 $y = -x$ ,  $y = x$ ,  $0 \leq y \leq 1$ .
15.  $f(x, y) = x^2 + y$  over  $R$ , the triangle bounded by  $y = 2x$ ,  
 $y = 0$  and  $x = 2$ .
16.  $f(x, y) = \frac{2}{3}x^{3/2} + 2y^{3/2}$  over  $R$ , the rectangle with opposite  
corners  $(0, 0)$  and  $(1, 1)$ .
17.  $f(x, y) = 10 - 2\sqrt{x^2 + y^2}$  over  $R$ , the circle  $x^2 + y^2 = 25$ .  
(This is the cone with height 10 and base radius 5; be sure
18. Find the surface area of the sphere with radius 5 by dou-  
bling the surface area of  $f(x, y) = \sqrt{25 - x^2 - y^2}$  over  $R$ ,  
the circle  $x^2 + y^2 = 25$ . (Be sure to compare you result  
with the known formula.)
19. Find the surface area of the ellipse formed by restricting  
the plane  $f(x, y) = cx + dy + h$  to the region  $R$ , the circle  
 $x^2 + y^2 = 1$ , where  $c$ ,  $d$  and  $h$  are some constants. Your  
answer should be given in terms of  $c$  and  $d$ ; why does the  
value of  $h$  not matter?

## 12.6 Triple Integration

We learned in Section 12.2 how to compute the signed volume  $V$  under a surface  $z = f(x, y)$  over a region  $R$ :  $V = \iint_R f(x, y) dA$ . It follows naturally that if  $f(x, y) \geq g(x, y)$  on  $R$ , then the **volume between  $f(x, y)$  and  $g(x, y)$  on  $R$**  is

$$V = \iint_R f(x, y) dA - \iint_R g(x, y) dA = \iint_R (f(x, y) - g(x, y)) dA.$$

### Theorem 76 Volume Between Surfaces

Let  $f$  and  $g$  be continuous functions on a closed, bounded region  $R$ , where  $f(x, y) \geq g(x, y)$  for all  $(x, y)$  in  $R$ . The volume  $V$  between  $f$  and  $g$  over  $R$  is

$$V = \iint_R (f(x, y) - g(x, y)) dA.$$

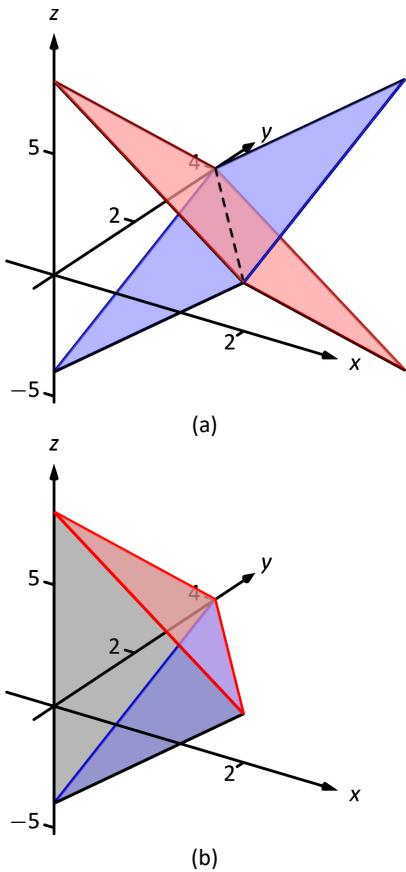


Figure 12.36: Finding the volume between the planes given in Example 315.

### Example 315 Finding volume between surfaces

Find the volume of the space region bounded by the planes  $z = 3x + y - 4$  and  $z = 8 - 3x - 2y$  in the 1<sup>st</sup> octant. In Figure 12.36(a) the planes are drawn; in (b), only the defined region is given.

**SOLUTION** We need to determine the region  $R$  over which we will integrate. To do so, we need to determine where the planes intersect. They have common  $z$ -values when  $3x + y - 4 = 8 - 3x - 2y$ . Applying a little algebra, we have:

$$\begin{aligned} 3x + y - 4 &= 8 - 3x - 2y \\ 6x + 3y &= 12 \\ 2x + y &= 4 \end{aligned}$$

The planes intersect along the line  $2x + y = 4$ . Therefore the region  $R$  is bounded by  $x = 0$ ,  $y = 0$ , and  $y = 4 - 2x$ ; we can convert these bounds to integration bounds of  $0 \leq x \leq 2$ ,  $0 \leq y \leq 4 - 2x$ . Thus

$$\begin{aligned} V &= \iint_R (8 - 3x - 2y - (3x + y - 4)) dA \\ &= \int_0^2 \int_0^{4-2x} (12 - 6x - 3y) dy dx \\ &= 16u^3. \end{aligned}$$

The volume between the surfaces is 16 cubic units.

Notes:

In the preceding example, we found the volume by evaluating the integral

$$\int_0^2 \int_0^{4-2x} (8 - 3x - 2y - (3x + y - 4)) dy dx.$$

Note how we can rewrite the integrand as an integral, much as we did in Section 12.1:

$$8 - 3x - 2y - (3x + y - 4) = \int_{3x+y-4}^{8-3x-2y} dz.$$

Thus we can rewrite the double integral that finds volume as

$$\int_0^2 \int_0^{4-2x} (8 - 3x - 2y - (3x + y - 4)) dy dx = \int_0^2 \int_0^{4-2x} \left( \int_{3x+y-4}^{8-3x-2y} dz \right) dy dx.$$

This no longer looks like a “double integral,” but more like a “triple integral.” Just as our first introduction to double integrals was in the context of finding the area of a plane region, our introduction into triple integrals will be in the context of finding the volume of a space region.

To formally find the volume of a closed, bounded region  $D$  in space, such as the one shown in Figure 12.37(a), we start with an approximation. Break  $D$  into  $n$  rectangular solids; the solids near the boundary of  $D$  may possibly not include portions of  $D$  and/or include extra space. In Figure 12.37(b), we zoom in on a portion of the boundary of  $D$  to show a rectangular solid that contains space not in  $D$ ; as this is an approximation of the volume, this is acceptable and this error will be reduced as we shrink the size of our solids.

The volume  $\Delta V_i$  of the  $i^{\text{th}}$  solid  $D_i$  is  $\Delta V_i = \Delta x_i \Delta y_i \Delta z_i$ , where  $\Delta x_i$ ,  $\Delta y_i$  and  $\Delta z_i$  give the dimensions of the rectangular solid in the  $x$ ,  $y$  and  $z$  directions, respectively. By summing up the volumes of all  $n$  solids, we get an approximation of the volume  $V$  of  $D$ :

$$V \approx \sum_{i=1}^n \Delta V_i = \sum_{i=1}^n \Delta x_i \Delta y_i \Delta z_i.$$

Let  $\|\Delta D\|$  represent the length of the longest diagonal of rectangular solids in the subdivision of  $D$ . As  $\|\Delta D\| \rightarrow 0$ , the volume of each solid goes to 0, as do each of  $\Delta x_i$ ,  $\Delta y_i$  and  $\Delta z_i$ , for all  $i$ . Our calculus experience tells us that taking a limit as  $\|\Delta D\| \rightarrow 0$  turns our approximation of  $V$  into an exact calculation of  $V$ . Before we state this result in a theorem, we use a definition to define some terms.

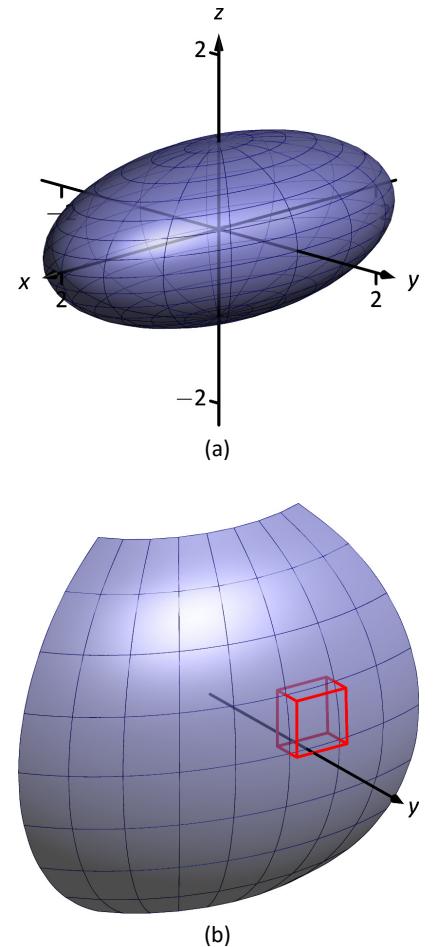


Figure 12.37: Approximating the volume of a region  $D$  in space.

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Notes:

**Definition 83 Triple Integrals, Iterated Integration (Part I)**

Let  $D$  be a closed, bounded region in space. Let  $a$  and  $b$  be real numbers, let  $g_1(x)$  and  $g_2(x)$  be continuous functions of  $x$ , and let  $f_1(x, y)$  and  $f_2(x, y)$  be continuous functions of  $x$  and  $y$ .

1. The volume  $V$  of  $D$  is denoted by a **triple integral**,

$$V = \iiint_D dV.$$

2. The iterated integral  $\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} dz dy dx$  is evaluated as

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} dz dy dx = \int_a^b \int_{g_1(x)}^{g_2(x)} \left( \int_{f_1(x,y)}^{f_2(x,y)} dz \right) dy dx.$$

Evaluating the above iterated integral is **triple integration**.

Our informal understanding of the notation  $\iiint_D dV$  is “sum up lots of little volumes over  $D$ ,” analogous to our understanding of  $\iint_R dA$  and  $\iint_R dm$ .

We now state the major theorem of this section.

**Theorem 77 Triple Integration (Part I)**

Let  $D$  be a closed, bounded region in space and let  $\Delta D$  be any subdivision of  $D$  into  $n$  rectangular solids, where the  $i^{\text{th}}$  subregion  $D_i$  has dimensions  $\Delta x_i \times \Delta y_i \times \Delta z_i$  and volume  $\Delta V_i$ .

1. The volume  $V$  of  $D$  is

$$V = \iiint_D dV = \lim_{||\Delta D|| \rightarrow 0} \sum_{i=1}^n \Delta V_i = \lim_{||\Delta D|| \rightarrow 0} \sum_{i=1}^n \Delta x_i \Delta y_i \Delta z_i.$$

2. If  $D$  is defined as the region bounded by the planes  $x = a$  and  $x = b$ , the cylinders  $y = g_1(x)$  and  $y = g_2(x)$ , and the surfaces  $z = f_1(x, y)$  and  $z = f_2(x, y)$ , where  $a < b$ ,  $g_1(x) \leq g_2(x)$  and  $f_1(x, y) \leq f_2(x, y)$  on  $D$ , then

$$\iiint_D dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} dz dy dx.$$

3.  $V$  can be determined using iterated integration with other orders of integration (there are 6 total), as long as  $D$  is defined by the region enclosed by a pair of planes, a pair of cylinders, and a pair of surfaces.

---

Notes:

We evaluated the area of a plane region  $R$  by iterated integration, where the bounds were “from curve to curve, then from point to point.” Theorem 77 allows us to find the volume of a space region with an iterated integral with bounds “from surface to surface, then from curve to curve, then from point to point.” In the iterated integral

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} dz dy dx,$$

the bounds  $a \leq x \leq b$  and  $g_1(x) \leq y \leq g_2(x)$  define a region  $R$  in the  $x$ - $y$  plane over which the region  $D$  exists in space. However, these bounds are also defining surfaces in space;  $x = a$  is a plane and  $y = g_1(x)$  is a cylinder. The combination of these 6 surfaces enclose, and define,  $D$ .

Examples will help us understand triple integration, including integrating with various orders of integration.

**Example 316** **Finding the volume of a space region with triple integration**

Find the volume of the space region in the 1<sup>st</sup> octant bounded by the plane  $z = 2 - y/3 - 2x/3$ , shown in Figure 12.38(a), using the order of integration  $dz dy dx$ . Set up the triple integrals that give the volume in the other 5 orders of integration.

**SOLUTION** Starting with the order of integration  $dz dy dx$ , we need to first find bounds on  $z$ . The region  $D$  is bounded below by the plane  $z = 0$  (because we are restricted to the first octant) and above by  $z = 2 - y/3 - 2x/3$ ;  $0 \leq z \leq 2 - y/3 - 2x/3$ .

To find the bounds on  $y$  and  $x$ , we “collapse” the region onto the  $x$ - $y$  plane, giving the triangle shown in Figure 12.38(b). (We know the equation of the line  $y = 6 - 2x$  in two ways. First, by setting  $z = 0$ , we have  $0 = 2 - y/3 - 2x/3 \Rightarrow y = 6 - 2x$ . Secondly, we know this is going to be a straight line between the points  $(3, 0)$  and  $(0, 6)$  in the  $x$ - $y$  plane.)

We define that region  $R$ , in the integration order of  $dy dx$ , with bounds  $0 \leq$

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Notes:

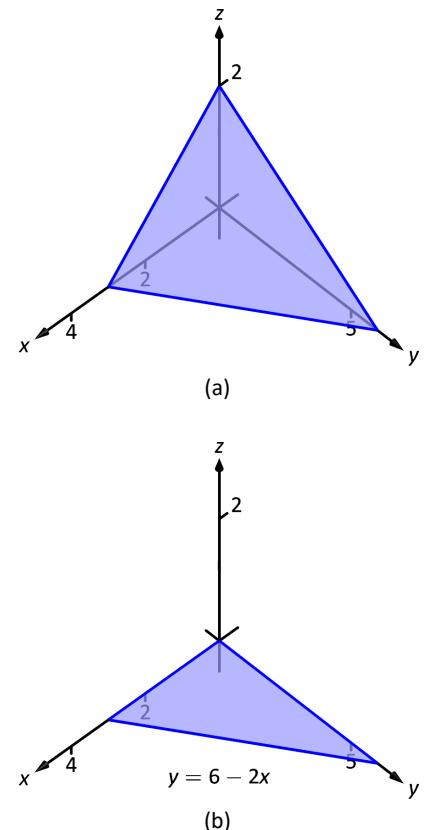


Figure 12.38: The region  $D$  used in Example 316 in (a); in (b), the region found by collapsing  $D$  onto the  $x$ - $y$  plane.

$y \leq 6 - 2x$  and  $0 \leq x \leq 3$ . Thus the volume  $V$  of the region  $D$  is:

$$\begin{aligned} V &= \iiint_D dV \\ &= \int_0^3 \int_0^{6-2x} \int_0^{2-\frac{1}{3}y-\frac{2}{3}x} dz dy dz \\ &= \int_0^3 \int_0^{6-2x} \left( \int_0^{2-\frac{1}{3}y-\frac{2}{3}x} dz \right) dy dz \\ &= \int_0^3 \int_0^{6-2x} z \Big|_0^{2-\frac{1}{3}y-\frac{2}{3}x} dy dz \\ &= \int_0^3 \int_0^{6-2x} \left( 2 - \frac{1}{3}y - \frac{2}{3}x \right) dy dz. \end{aligned}$$

From this step on, we are evaluating a double integral as done many times before. We skip these steps and give the final volume,

$$= 6u^3.$$

The order  $dz dx dy$ :

Now consider the volume using the order of integration  $dz dx dy$ . The bounds on  $z$  are the same as before,  $0 \leq z \leq 2 - y/3 - 2x/3$ . Collapsing the space region on the  $x$ - $y$  plane as shown in Figure 12.38(b), we now describe this triangle with the order of integration  $dx dy$ . This gives bounds  $0 \leq x \leq 3 - y/2$  and  $0 \leq y \leq 6$ . Thus the volume is given by the triple integral

$$V = \int_0^6 \int_0^{3-\frac{1}{2}y} \int_0^{2-\frac{1}{3}y-\frac{2}{3}x} dz dx dy.$$

The order  $dx dy dz$ :

Following our “surface to surface...” strategy, we need to determine the  $x$ -surfaces that bound our space region. To do so, approach the region “from behind,” in the direction of increasing  $x$ . The first surface we hit as we enter the region is the  $y$ - $z$  plane, defined by  $x = 0$ . We come out of the region at the plane  $z = 2 - y/3 - 2x/3$ ; solving for  $x$ , we have  $x = 3 - y/2 - 3z/2$ . Thus the bounds on  $x$  are:  $0 \leq x \leq 3 - y/2 - 3z/2$ .

Now collapse the space region onto the  $y$ - $z$  plane, as shown in Figure 12.39(a). (Again, we find the equation of the line  $z = 2 - y/3$  by setting  $x = 0$  in the equation  $x = 3 - y/2 - 3z/2$ .) We need to find bounds on this region with the order

---

Notes:

$dy dz$ . The curves that bound  $y$  are  $y = 0$  and  $y = 6 - 3z$ ; the points that bound  $z$  are 0 and 2. Thus the triple integral giving volume is:

$$\begin{aligned} 0 \leq x \leq 3 - y/2 - 3z/2 \\ 0 \leq y \leq 6 - 3z \\ 0 \leq z \leq 2 \end{aligned} \Rightarrow \int_0^2 \int_0^{6-3z} \int_0^{3-y/2-3z/2} dx dy dz.$$

The order  $dx dz dy$ :

The  $x$ -bounds are the same as the order above. We now consider the triangle in Figure 12.39(a) and describe it with the order  $dz dy$ :  $0 \leq z \leq 2 - y/3$  and  $0 \leq y \leq 6$ . Thus the volume is given by:

$$\begin{aligned} 0 \leq x \leq 3 - y/2 - 3z/2 \\ 0 \leq z \leq 2 - y/3 \\ 0 \leq y \leq 6 \end{aligned} \Rightarrow \int_0^6 \int_0^{2-y/3} \int_0^{3-y/2-3z/2} dx dz dy.$$

The order  $dy dz dx$ :

We now need to determine the  $y$ -surfaces that determine our region. Approaching the space region from “behind” and moving in the direction of increasing  $y$ , we first enter the region at  $y = 0$ , and exit along the plane  $z = 2 - y/3 - 2x/3$ . Solving for  $y$ , this plane has equation  $y = 6 - 2x - 3z$ . Thus  $y$  has bounds  $0 \leq y \leq 6 - 2x - 3z$ .

Now collapse the region onto the  $x$ - $z$  plane, as shown in Figure 12.39(b). The curves bounding this triangle are  $z = 0$  and  $z = 2 - 2x/3$ ;  $x$  is bounded by the points  $x = 0$  to  $x = 3$ . Thus the triple integral giving volume is:

$$\begin{aligned} 0 \leq y \leq 6 - 2x - 3z \\ 0 \leq z \leq 2 - 2x/3 \\ 0 \leq x \leq 3 \end{aligned} \Rightarrow \int_0^3 \int_0^{2-2x/3} \int_0^{6-2x-3z} dy dz dx.$$

The order  $dy dx dz$ :

The  $y$ -bounds are the same as in the order above. We now determine the bounds of the triangle in Figure 12.39(b) using the order  $dy dx dz$ .  $x$  is bounded by  $x = 0$  and  $x = 3 - 3z/2$ ;  $z$  is bounded between  $z = 0$  and  $z = 2$ . This leads to the triple integral:

$$\begin{aligned} 0 \leq y \leq 6 - 2x - 3z \\ 0 \leq x \leq 3 - 3z/2 \\ 0 \leq z \leq 2 \end{aligned} \Rightarrow \int_0^2 \int_0^{3-3z/2} \int_0^{6-2x-3z} dy dx dz.$$

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Notes:

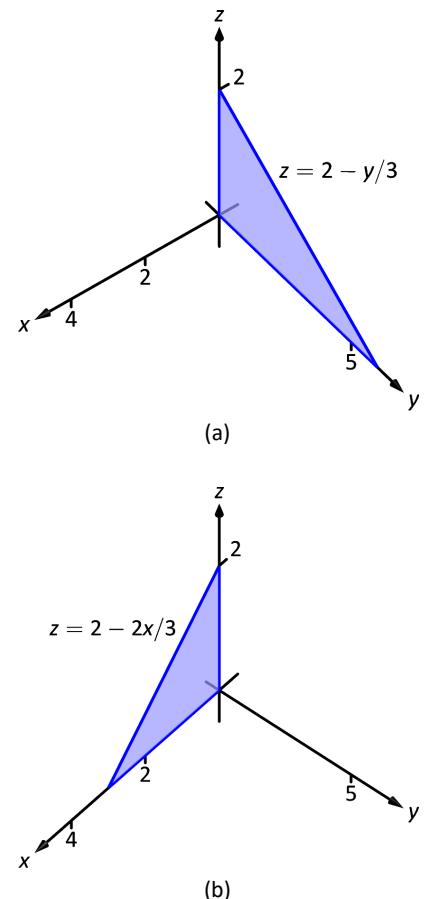


Figure 12.39: The region  $D$  in Example 316 is collapsed onto the  $y$ - $z$  plane in (a); in (b), the region is collapsed onto the  $x$ - $z$  plane.

This problem was long, but hopefully useful, demonstrating how to determine bounds with every order of integration to describe the region  $D$ . In practice, we only need 1, but being able to do them all gives us flexibility to choose the order that suits us best.

In the previous example, we collapsed the surface into the  $x$ - $y$ ,  $x$ - $z$ , and  $y$ - $z$  planes as we determined the “curve to curve, point to point” bounds of integration. Since the surface was a triangular portion of a plane, this collapsing, or *projecting*, was simple: the *projection* of a straight line in space onto a coordinate plane is a line.

The following example shows us how to do this when dealing with more complicated surfaces and curves.

**Example 317** **Finding the projection of a curve in space onto the coordinate planes**

Consider the surfaces  $z = 3 - x^2 - y^2$  and  $z = 2y$ , as shown in Figure 12.40(a). The curve of their intersection is shown, along with the projection of this curve into the coordinate planes, shown dashed. Find the equations of the projections into the coordinate planes.

**SOLUTION** The two surfaces are  $z = 3 - x^2 - y^2$  and  $z = 2y$ . To find where they intersect, it is natural to set them equal to each other:  $3 - x^2 - y^2 = 2y$ . This is an implicit function of  $x$  and  $y$  that gives all points  $(x, y)$  in the  $x$ - $y$  plane where the  $z$  values of the two surfaces are equal.

We can rewrite this implicit function by completing the square:

$$3 - x^2 - y^2 = 2y \Rightarrow y^2 + 2y + x^2 = 3 \Rightarrow (y + 1)^2 + x^2 = 4.$$

Thus in the  $x$ - $y$  plane the projection of the intersection is a circle with radius 2, centered at  $(0, -1)$ .

To project onto the  $x$ - $z$  plane, we do a similar procedure: find the  $x$  and  $z$  values where the  $y$  values on the surface are the same. We start by solving the equation of each surface for  $y$ . In this particular case, it works well to actually solve for  $y^2$ :

$$\begin{aligned} z = 3 - x^2 - y^2 &\Rightarrow y^2 = 3 - x^2 - z \\ z = 2y &\Rightarrow y^2 = z^2/4. \end{aligned}$$

Thus we have (after again completing the square):

$$3 - x^2 - z = z^2/4 \Rightarrow \frac{(z+2)^2}{16} + \frac{x^2}{4} = 1,$$

and ellipse centered at  $(0, -2)$  in the  $x$ - $z$  plane with a major axis of length 8 and a minor axis of length 4.

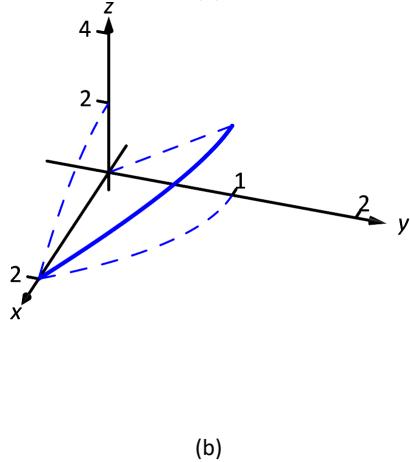
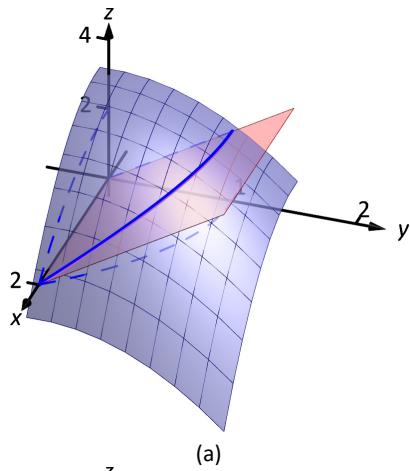


Figure 12.40: Finding the projections of the curve of intersection in Example 317.

Notes:

Finally, to project the curve of intersection into the  $y$ - $z$  plane, we solve equation for  $x$ . Since  $z = 2y$  is a cylinder that lacks the variable  $x$ , it becomes our equation of the projection in the  $y$ - $z$  plane.

All three projections are shown in Figure 12.40(b).

**Example 318** Finding the volume of a space region with triple integration

Set up the triple integrals that find the volume of the space region  $D$  bounded by the surfaces  $x^2 + y^2 = 1$ ,  $z = 0$  and  $z = -y$ , as shown in Figure 12.41(a), with the orders of integration  $dz\,dy\,dx$ ,  $dy\,dx\,dz$  and  $dx\,dz\,dy$ .

**SOLUTION** The order  $dz\,dy\,dx$ :

The region  $D$  is bounded below by the plane  $z = 0$  and above by the plane  $z = -y$ . The cylinder  $x^2 + y^2 = 1$  does not offer any bounds in the  $z$ -direction, as that surface is parallel to the  $z$ -axis. Thus  $0 \leq z \leq -y$ .

Collapsing the region into the  $x$ - $y$  plane, we get part of the circle with equation  $x^2 + y^2 = 1$  as shown in Figure 12.41(b). As a function of  $x$ , this half circle has equation  $y = -\sqrt{1 - x^2}$ . Thus  $y$  is bounded below by  $-\sqrt{1 - x^2}$  and above by  $y = 0$ :  $-\sqrt{1 - x^2} \leq y \leq 0$ . The  $x$  bounds of the half circle are  $-1 \leq x \leq 1$ . All together, the bounds of integration and triple integral are as follows:

$$\begin{aligned} 0 &\leq z \leq -y \\ -\sqrt{1 - x^2} &\leq y \leq 0 \\ -1 &\leq x \leq 1 \end{aligned} \Rightarrow \int_{-1}^1 \int_{-\sqrt{1-x^2}}^0 \int_0^{-y} dz\,dy\,dx.$$

We evaluate this triple integral:

$$\begin{aligned} \int_{-1}^1 \int_{-\sqrt{1-x^2}}^0 \int_0^{-y} dz\,dy\,dx &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^0 (-y) dy\,dx \\ &= \int_{-1}^1 \left(-\frac{1}{2}y^2\right) \Big|_{-\sqrt{1-x^2}}^0 dx \\ &= \int_{-1}^1 \frac{1}{2}(1-x^2) dx \\ &= \left(\frac{1}{2} \left(x - \frac{1}{3}x^3\right)\right) \Big|_{-1}^1 \\ &= \frac{2}{3} \text{ units}^3. \end{aligned}$$

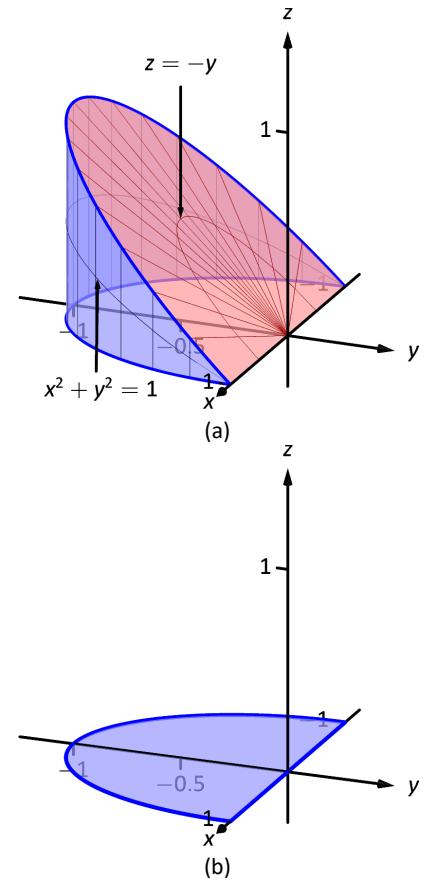


Figure 12.41: The region  $D$  in Example 318 is shown in (a); in (b), it is collapsed onto the  $x$ - $y$  plane.

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Notes:

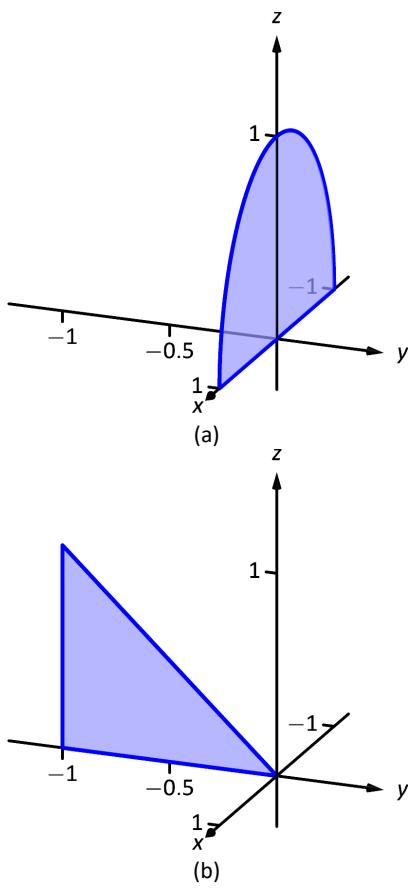


Figure 12.42: The region  $D$  in Example 318 is shown collapsed onto the  $x$ - $z$  plane in (a); in (b), it is collapsed onto the  $y$ - $z$  plane.

With the order  $dy\,dx\,dz$ :

The region is bounded “below” in the  $y$ -direction by the surface  $x^2 + y^2 = 1 \Rightarrow y = -\sqrt{1 - x^2}$  and “above” by the surface  $y = -z$ . Thus the  $y$  bounds are  $-\sqrt{1 - x^2} \leq y \leq -z$ .

Collapsing the region onto the  $x$ - $z$  plane gives the region shown in Figure 12.42(a); this half circle has equation  $x^2 + z^2 = 1$ . (We find this curve by solving each surface for  $y^2$ , then setting them equal to each other. We have  $y^2 = 1 - x^2$  and  $y = -z \Rightarrow y^2 = z^2$ . Thus  $x^2 + z^2 = 1$ .) It is bounded below by  $x = -\sqrt{1 - z^2}$  and above by  $x = \sqrt{1 - z^2}$ , where  $z$  is bounded by  $0 \leq z \leq 1$ . All together, we have:

$$\begin{aligned} -\sqrt{1 - x^2} &\leq y \leq -z \\ -\sqrt{1 - z^2} &\leq x \leq \sqrt{1 - z^2} \\ 0 &\leq z \leq 1 \end{aligned} \Rightarrow \int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-x^2}}^{-z} dy\,dx\,dz.$$

With the order  $dx\,dz\,dy$ :

$D$  is bounded below by the surface  $x = -\sqrt{1 - y^2}$  and above by  $\sqrt{1 - y^2}$ . We then collapse the region onto the  $y$ - $z$  plane and get the triangle shown in Figure 12.42(b). (The hypotenuse is the line  $z = -y$ , just as the plane.) Thus  $z$  is bounded by  $0 \leq z \leq -y$  and  $y$  is bounded by  $-1 \leq y \leq 0$ . This gives:

$$\begin{aligned} -\sqrt{1 - y^2} &\leq x \leq \sqrt{1 - y^2} \\ 0 &\leq z \leq -y \\ -1 &\leq y \leq 0 \end{aligned} \Rightarrow \int_{-1}^0 \int_0^{-y} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\,dz\,dy.$$

The following theorem states two things that should make “common sense” to us. First, using the triple integral to find volume of a region  $D$  should always return a positive number; we are computing *volume* here, not *signed volume*. Secondly, to compute the volume of a “complicated” region, we could break it up into subregions and compute the volumes of each subregion separately, summing them later to find the total volume.

---

Notes:

**Theorem 78 Properties of Triple Integrals**

Let  $D$  be a closed, bounded region in space, and let  $D_1$  and  $D_2$  be non-overlapping regions such that  $D = D_1 \cup D_2$ .

1.  $\iiint_D dV \geq 0$
2.  $\iiint_D dV = \iiint_{D_1} dV + \iiint_{D_2} dV.$

We use this latter property in the next example.

**Example 319 Finding the volume of a space region with triple integration**

Find the volume of the space region  $D$  bounded by the coordinate planes,  $z = 1 - x/2$  and  $z = 1 - y/4$ , as shown in Figure 12.43(a). Set up the triple integrals that find the volume of  $D$  in all 6 orders of integration.

**SOLUTION** Following the bounds-determining strategy of “surface to surface, curve to curve, and point to point,” we can see that the most difficult orders of integration are the two in which we integrate with respect to  $z$  first, for there are two “upper” surfaces that bound  $D$  in the  $z$ -direction. So we start by noting that we have

$$0 \leq z \leq 1 - \frac{1}{2}x \quad \text{and} \quad 0 \leq z \leq 1 - \frac{1}{4}y.$$

We now collapse the region  $D$  onto the  $x$ - $y$  axis, as shown in Figure 12.43(b). The boundary of  $D$ , the line from  $(0, 0, 1)$  to  $(2, 4, 0)$ , is shown in part (b) of the figure as a dashed line; it has equation  $y = 2x$ . (We can recognize this in two ways: one, in collapsing the line from  $(0, 0, 1)$  to  $(2, 4, 0)$  onto the  $x$ - $y$  plane, we simply ignore the  $z$ -values, meaning the line now goes from  $(0, 0)$  to  $(2, 4)$ . Secondly, the two surfaces meet where  $z = 1 - x/2$  is equal to  $z = 1 - y/4$ : thus  $1 - x/2 = 1 - y/4 \Rightarrow y = 2x$ .)

We use the second property of Theorem 78 to state that

$$\iiint_D dV = \iiint_{D_1} dV + \iiint_{D_2} dV,$$

where  $D_1$  and  $D_2$  are the space regions above the plane regions  $R_1$  and  $R_2$ , respectively. Thus we can say

$$\iiint_D dV = \iint_{R_1} \left( \int_0^{1-x/2} dz \right) dA + \iint_{R_2} \left( \int_0^{1-y/4} dz \right) dA.$$

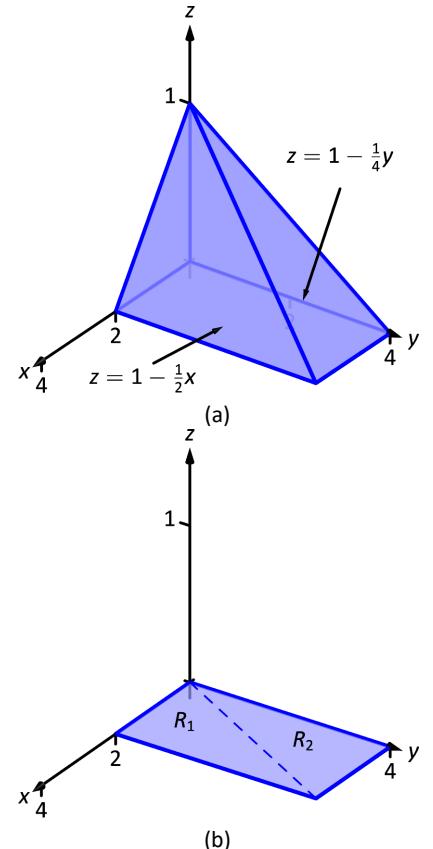


Figure 12.43: The region  $D$  in Example 319 is shown in (a); in (b), it is collapsed onto the  $x$ - $y$  plane.

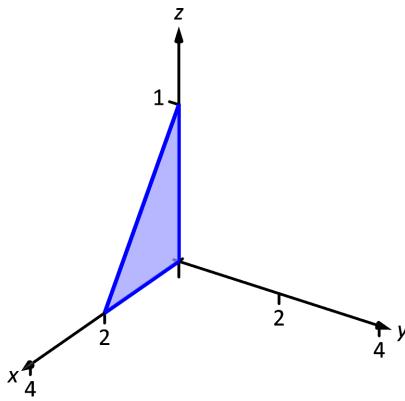
Notes:

All that is left is to determine bounds of  $R_1$  and  $R_2$ , depending on whether we are integrating with order  $dx\,dy\,dz$  or  $dy\,dx\,dz$ . We give the final integrals here, leaving it to the reader to confirm these results.

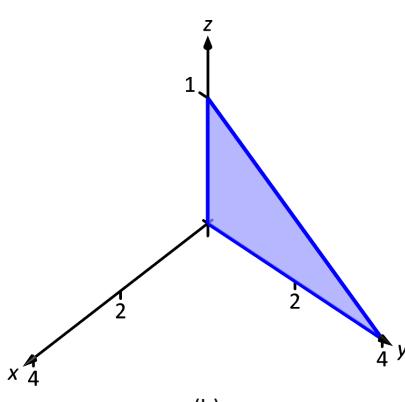
$dz\,dy\,dx$ :

$$\begin{array}{ll} 0 \leq z \leq 1 - x/2 & 0 \leq z \leq 1 - y/4 \\ 0 \leq y \leq 2x & 2x \leq y \leq 4 \\ 0 \leq x \leq 2 & 0 \leq x \leq 2 \end{array}$$

$$\iiint_D dV = \int_0^2 \int_0^{2x} \int_0^{1-x/2} dz\,dy\,dx + \int_0^2 \int_{2x}^4 \int_0^{1-y/4} dz\,dy\,dx$$



(a)



(b)

Figure 12.44: The region  $D$  in Example 319 is shown collapsed onto the  $x$ - $z$  plane in (a); in (b), it is collapsed onto the  $y$ - $z$  plane.

$dz\,dx\,dy$ :

$$\begin{array}{ll} 0 \leq z \leq 1 - x/2 & 0 \leq z \leq 1 - y/4 \\ y/2 \leq x \leq 2 & 0 \leq x \leq y/2 \\ 0 \leq y \leq 4 & 0 \leq y \leq 4 \end{array}$$

$$\iiint_D dV = \int_0^4 \int_{y/2}^2 \int_0^{1-x/2} dz\,dx\,dy + \int_0^4 \int_0^{y/2} \int_0^{1-y/4} dz\,dx\,dy$$

The remaining four orders of integration do not require a sum of triple integrals. In Figure 12.44 we show  $D$  collapsed onto the other two coordinate planes. Using these graphs, we give the final orders of integration here, again leaving it to the reader to confirm these results.

$dy\,dx\,dz$ :

$$\begin{array}{l} 0 \leq y \leq 4 - 4z \\ 0 \leq x \leq 2 - 2z \\ 0 \leq z \leq 1 \end{array} \Rightarrow \int_0^1 \int_0^{2-2z} \int_0^{4-4z} dy\,dx\,dz$$

$dy\,dz\,dx$ :

$$\begin{array}{l} 0 \leq y \leq 4 - 4z \\ 0 \leq z \leq 1 - x/2 \\ 0 \leq x \leq 2 \end{array} \Rightarrow \int_0^2 \int_0^{1-x/2} \int_0^{4-4z} dy\,dx\,dz$$

Notes:

$dx dy dz$ :

$$\begin{aligned} 0 \leq x \leq 2 - 2z \\ 0 \leq y \leq 4 - 4z \\ 0 \leq z \leq 1 \end{aligned} \Rightarrow \int_0^1 \int_0^{4-4z} \int_0^{2-2z} dx dy dz$$

$dx dz dy$ :

$$\begin{aligned} 0 \leq x \leq 2 - 2z \\ 0 \leq z \leq 1 - y/4 \\ 0 \leq y \leq 4 \end{aligned} \Rightarrow \int_0^4 \int_0^{1-y/4} \int_0^{2-2z} dx dz dy$$

We give one more example of finding the volume of a space region.

**Example 320 Finding the volume of a space region**

Set up a triple integral that gives the volume of the space region  $D$  bounded by  $z = 2x^2 + 2$  and  $z = 6 - 2x^2 - y^2$ . These surfaces are plotted in Figure 12.45(a) and (b), respectively; the region  $D$  is shown in part (c) of the figure.

**SOLUTION** The main point of this example is this: integrating with respect to  $z$  first is rather straightforward; integrating with respect to  $x$  first is not.

The order  $dz dy dx$ :

The bounds on  $z$  are clearly  $2x^2 + 2 \leq z \leq 6 - 2x^2 - y^2$ . Collapsing  $D$  onto the  $x$ - $y$  plane gives the ellipse shown in Figure 12.45(c). The equation of this ellipse is found by setting the two surfaces equal to each other:

$$2x^2 + 2 = 6 - 2x^2 - y^2 \Rightarrow 4x^2 + y^2 = 4 \Rightarrow x^2 + \frac{y^2}{4} = 1.$$

We can describe this ellipse with the bounds

$$-\sqrt{4 - 4x^2} \leq y \leq \sqrt{4 - 4x^2} \quad \text{and} \quad -1 \leq x \leq 1.$$

Thus we find volume as

$$\begin{aligned} 2x^2 + 2 \leq z \leq 6 - 2x^2 - y^2 \\ -\sqrt{4 - 4x^2} \leq y \leq \sqrt{4 - 4x^2} \\ -1 \leq x \leq 1 \end{aligned} \Rightarrow \int_{-1}^1 \int_{-\sqrt{4-4x^2}}^{\sqrt{4-4x^2}} \int_{2x^2+2}^{6-2x^2-y^2} dz dy dx .$$

The order  $dy dz dx$ :

Notes:

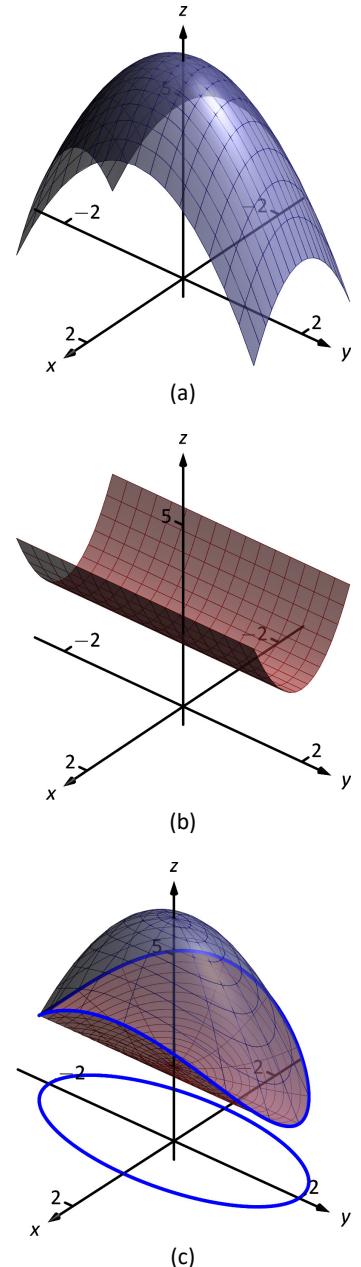


Figure 12.45: The region  $D$  is bounded by the surfaces shown in (a) and (b);  $D$  is shown in (c).

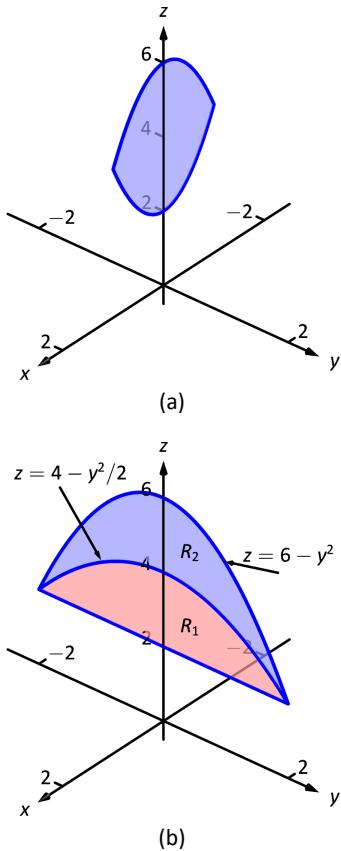


Figure 12.46: The region  $D$  in Example 320 is collapsed onto the  $x$ - $z$  plane in (a); in (b), it is collapsed onto the  $y$ - $z$  plane.

Integrating with respect to  $y$  is not too difficult. Since the surface  $z = 2x^2 + 2$  is a cylinder whose directrix is the  $y$ -axis, it does not create a border for  $y$ . The paraboloid  $z = 6 - 2x^2 - y^2$  does; solving for  $y$ , we get the bounds

$$-\sqrt{6 - 2x^2 - z} \leq y \leq \sqrt{6 - 2x^2 - z}.$$

Collapsing  $D$  onto the  $x$ - $z$  axes gives the region shown in Figure 12.46(a); the lower curve is from the cylinder, with equation  $z = 2x^2 + 2$ . The upper curve is from the paraboloid; with  $y = 0$ , the curve is  $z = 6 - 2x^2$ . Thus bounds on  $z$  are  $2x^2 + 2 \leq z \leq 6 - 2x^2$ ; the bounds on  $x$  are  $-1 \leq x \leq 1$ . Thus we have:

$$\begin{aligned} -\sqrt{6 - 2x^2 - z} &\leq y \leq \sqrt{6 - 2x^2 - z} \\ 2x^2 + 2 &\leq z \leq 6 - 2x^2 \\ -1 &\leq x \leq 1 \end{aligned} \Rightarrow \int_{-1}^1 \int_{2x^2+2}^{6-2x^2} \int_{-\sqrt{6-2x^2-z}}^{\sqrt{6-2x^2-z}} dy dz dx.$$

The order  $dx dz dy$ :

This order takes more effort as  $D$  must be split into two subregions. The two surfaces create two sets of upper/lower bounds in terms of  $x$ ; the cylinder creates bounds

$$-\sqrt{z/2 - 1} \leq x \leq \sqrt{z/2 - 1}$$

for region  $D_1$  and the paraboloid creates bounds

$$-\sqrt{3 - y^2/2 - z^2/2} \leq x \leq \sqrt{3 - y^2/2 - z^2/2}$$

for region  $D_2$ .

Collapsing  $D$  onto the  $y$ - $z$  axes gives the regions shown in Figure 12.46(b). We find the equation of the curve  $z = 4 - y^2/2$  by noting that the equation of the ellipse seen in Figure 12.45(c) has equation

$$x^2 + y^2/4 = 1 \Rightarrow x = \sqrt{1 - y^2/4}.$$

Substitute this expression for  $x$  in either surface equation,  $z = 6 - 2x^2 - y^2$  or  $z = 2x^2 + 2$ . In both cases, we find

$$z = 4 - \frac{1}{2}y^2.$$

Region  $R_1$ , corresponding to  $D_1$ , has bounds

$$2 \leq z \leq 4 - y^2/2, \quad -2 \leq y \leq 2$$

and region  $R_2$ , corresponding to  $D_2$ , has bounds

$$4 - y^2/2 \leq z \leq 6 - y^2, \quad -2 \leq y \leq 2.$$

---

Notes:

Thus the volume of  $D$  is given by:

$$\int_{-2}^2 \int_2^{4-y^2/2} \int_{-\sqrt{z/2-1}}^{\sqrt{z/2-1}} dx dz dy + \int_{-2}^2 \int_{4-y^2/2}^{6-y^2} \int_{-\sqrt{3-y^2/2-z^2/2}}^{\sqrt{3-y^2/2-z^2/2}} dx dz dy.$$

If all one wanted to do in Example 320 was find the volume of the region  $D$ , one would have likely stopped at the first integration setup (with order  $dz dy dx$ ) and computed the volume from there. However, we included the other two methods 1) to show that it could be done, “messy” or not, and 2) because sometimes we “have” to use a less desirable order of integration in order to actually integrate.

## Triple Integration and Functions of Three Variables

There are uses for triple integration beyond merely finding volume, just as there are uses for integration beyond “area under the curve.” These uses start with understanding how to integrate functions of three variables, which is effectively no different than integrating functions of two variables. This leads us to a definition, followed by an example.

### Definition 84 Iterated Integration, (Part II)

Let  $D$  be a closed, bounded region in space, over which  $g_1(x)$ ,  $g_2(x)$ ,  $f_1(x, y)$ ,  $f_2(x, y)$  and  $h(x, y, z)$  are all continuous, and let  $a$  and  $b$  be real numbers.

The **iterated integral**  $\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} h(x, y, z) dz dy dx$  is evaluated as  

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} h(x, y, z) dz dy dx = \int_a^b \int_{g_1(x)}^{g_2(x)} \left( \int_{f_1(x,y)}^{f_2(x,y)} h(x, y, z) dz \right) dy dx.$$

### Example 321 Evaluating a triple integral of a function of three variables

Evaluate  $\int_0^1 \int_{x^2}^x \int_{x^2-y}^{2x+3y} (xy + 2xz) dz dy dx$ .

**SOLUTION** We evaluate this integral according to Definition 84.

---

Notes:

$$\begin{aligned}
& \int_0^1 \int_{x^2}^x \int_{x^2-y}^{2x+3y} (xy + 2xz) \, dz \, dy \, dx \\
&= \int_0^1 \int_{x^2}^x \left( \int_{x^2-y}^{2x+3y} (xy + 2xz) \, dz \right) \, dy \, dx \\
&= \int_0^1 \int_{x^2}^x \left( (xyz + xz^2) \Big|_{x^2-y}^{2x+3y} \right) \, dy \, dx \\
&= \int_0^1 \int_{x^2}^x \left( xy(2x+3y) + x(2x+3y)^2 - (xy(x^2-y) + x(x^2-y)^2) \right) \, dy \, dx \\
&= \int_0^1 \int_{x^2}^x (-x^5 + x^3y + 4x^3 + 14x^2y + 12xy^2) \, dy \, dx.
\end{aligned}$$

We continue as we have in the past, showing fewer steps.

$$\begin{aligned}
&= \int_0^1 \left( -\frac{7}{2}x^7 - 8x^6 - \frac{7}{2}x^5 + 15x^4 \right) \, dx \\
&= \frac{281}{336} \approx 0.836.
\end{aligned}$$

We now know *how* to evaluate a triple integral of a function of three variables; we do not yet understand what it *means*. We build up this understanding in a way very similar to how we have understood integration and double integration.

Let  $h(x, y, z)$  a continuous function of three variables, defined over some space region  $D$ . We can partition  $D$  into  $n$  rectangular-solid subregions, each with dimensions  $\Delta x_i \times \Delta y_i \times \Delta z_i$ . Let  $(x_i, y_i, z_i)$  be some point in the  $i^{\text{th}}$  subregion, and consider the product  $h(x_i, y_i, z_i) \Delta x_i \Delta y_i \Delta z_i$ . It is the product of a function value (that's the  $h(x_i, y_i, z_i)$  part) and a small volume  $\Delta V_i$  (that's the  $\Delta x_i \Delta y_i \Delta z_i$  part). One of the simplest understanding of this type of product is when  $h$  describes the density of an object, for then  $h \times \text{volume} = \text{mass}$ .

We can sum up all  $n$  products over  $D$ . Again letting  $||\Delta D||$  represent the length of the longest diagonal of the  $n$  rectangular solids in the partition, we can take the limit of the sums of products as  $||\Delta D|| \rightarrow 0$ . That is, we can find

$$S = \lim_{||\Delta D|| \rightarrow 0} \sum_{i=1}^n h(x_i, y_i, z_i) \Delta V_i = \lim_{||\Delta D|| \rightarrow 0} \sum_{i=1}^n h(x_i, y_i, z_i) \Delta x_i \Delta y_i \Delta z_i.$$

While this limit has lots of interpretations depending on the function  $h$ , in the case where  $h$  describes density,  $S$  is the total mass of the object described by the region  $D$ .

Notes:

We now use the above limit to define the **triple integral**, give a theorem that relates triple integrals to iterated iteration, followed by the application of triple integrals to find the centers of mass of solid objects.

### Definition 85 Triple Integral

Let  $w = h(x, y, z)$  be a continuous function over a closed, bounded space region  $D$ , and let  $\Delta D$  be any partition of  $D$  into  $n$  rectangular solids with volume  $\Delta V_i$ . The **triple integral of  $h$  over  $D$**  is

$$\iiint_D h(x, y, z) dV = \lim_{||\Delta D|| \rightarrow 0} \sum_{i=1}^n h(x_i, y_i, z_i) \Delta V_i.$$

The following theorem assures us that the above limit exists for continuous functions  $h$  and gives us a method of evaluating the limit.

### Theorem 79 Triple Integration (Part II)

Let  $w = h(x, y, z)$  be a continuous function over a closed, bounded space region  $D$ , and let  $\Delta D$  be any partition of  $D$  into  $n$  rectangular solids with volume  $V_i$ .

1. The limit  $\lim_{||\Delta D|| \rightarrow 0} \sum_{i=1}^n h(x_i, y_i, z_i) \Delta V_i$  exists.
2. If  $D$  is defined as the region bounded by the planes  $x = a$  and  $x = b$ , the cylinders  $y = g_1(x)$  and  $y = g_2(x)$ , and the surfaces  $z = f_1(x, y)$  and  $z = f_2(x, y)$ , where  $a < b$ ,  $g_1(x) \leq g_2(x)$  and  $f_1(x, y) \leq f_2(x, y)$  on  $D$ , then

$$\iiint_D h(x, y, z) dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} h(x, y, z) dz dy dx.$$

We now apply triple integration to find the centers of mass of solid objects.

### Mass and Center of Mass

One may wish to review Section 12.4 for a reminder of the relevant terms and concepts.

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Notes:

**Note:** In the marginal note on page 708, we showed how the summation of rectangles over a region  $R$  in the plane could be viewed as a double sum, leading to the double integral. Likewise, we can view the sum  $\sum_{i=1}^n h(x_i, y_i, z_i) \Delta x_i \Delta y_i \Delta z_i$  as a triple sum,

$$\sum_{k=1}^p \sum_{j=1}^n \sum_{i=1}^m h(x_i, y_j, z_k) \Delta x_i \Delta y_j \Delta z_k,$$

which we evaluate as

$$\sum_{k=1}^p \left( \sum_{j=1}^n \left( \sum_{i=1}^m h(x_i, y_j, z_k) \Delta x_i \right) \Delta y_j \right) \Delta z_k.$$

Here we fix a  $k$  value, which establishes the  $z$ -height of the rectangular solids on one “level” of all the rectangular solids in the space region  $D$ . The inner double summation adds up all the volumes of the rectangular solids on this level, while the outer summation adds up the volumes of each level.

This triple summation understanding leads to the  $\iiint_D$  notation of the triple integral, as well as the method of evaluation shown in Theorem 79.

**Definition 86 Mass, Center of Mass of Solids**

Let a solid be represented by a region  $D$  in space with variable density function  $\delta(x, y, z)$ .

1. The **mass** of the object is  $M = \iiint_D dm = \iiint_D \delta(x, y, z) dV$ .
2. The **moment about the  $x$ - $y$  plane** is  $M_{xy} = \iiint_D z\delta(x, y, z) dV$ .
3. The **moment about the  $x$ - $z$  plane** is  $M_{xz} = \iiint_D y\delta(x, y, z) dV$ .
4. The **moment about the  $y$ - $z$  plane** is  $M_{yz} = \iiint_D x\delta(x, y, z) dV$ .
5. The **center of mass** of the object is

$$(\bar{x}, \bar{y}, \bar{z}) = \left( \frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M} \right).$$

**Example 322 Finding the center of mass of a solid**

Find the mass and center of mass of the solid represented by the space region bounded by the coordinate planes and  $z = 2 - y/3 - 2x/3$ , shown in Figure 12.47, with constant density  $\delta(x, y, z) = 3\text{gm/cm}^3$ . (Note: this space region was used in Example 316.)

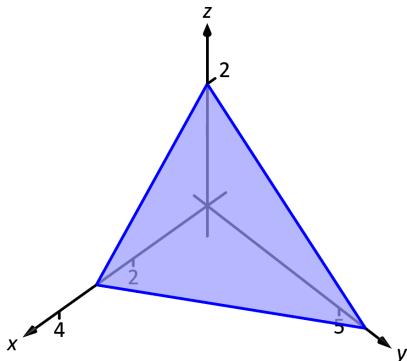


Figure 12.47: Finding the center of mass of this solid in Example 322.

**SOLUTION** We apply Definition 86. In Example 316, we found bounds for the order of integration  $dz dy dx$  to be  $0 \leq z \leq 2 - y/3 - 2x/3$ ,  $0 \leq y \leq 6 - 2x$  and  $0 \leq x \leq 3$ . We find the mass of the object:

$$\begin{aligned} M &= \iiint_D \delta(x, y, z) dV \\ &= \int_0^3 \int_0^{6-2x} \int_0^{2-y/3-2x/3} (3) dz dy dx \\ &= 3 \int_0^3 \int_0^{6-2x} \int_0^{2-y/3-2x/3} dz dy dx \\ &= 3(6) = 18\text{gm}. \end{aligned}$$

The evaluation of the triple integral is done in Example 316, so we skipped those steps above. Note how the mass of an object with constant density is simply “density  $\times$  volume.”

---

Notes:

We now find the moments about the planes.

$$\begin{aligned}
 M_{xy} &= \iiint_D 3z \, dV \\
 &= \int_0^3 \int_0^{6-2x} \int_0^{2-y/3-2x/3} (3z) \, dz \, dy \, dx \\
 &= \int_0^3 \int_0^{6-2x} \frac{3}{2} (2 - y/3 - 2x/3)^2 \, dy \, dx \\
 &= \int_0^3 -\frac{4}{9} (x - 3)^3 \, dx \\
 &= 9.
 \end{aligned}$$

We omit the steps of integrating to find the other moments.

$$\begin{aligned}
 M_{yz} &= \iiint_D 3x \, dV \\
 &= \frac{27}{2}. \\
 M_{xz} &= \iiint_D 3y \, dV \\
 &= 27.
 \end{aligned}$$

The center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left( \frac{27/2}{18}, \frac{27}{18}, \frac{9}{18} \right) = (0.75, 1.5, 0.5).$$

### Example 323 Finding the center of mass of a solid

Find the center of mass of the solid represented by the region bounded by the planes  $z = 0$  and  $z = -y$  and the cylinder  $x^2 + y^2 = 1$ , shown in Figure 12.48, with density function  $\delta(x, y, z) = 10 + x^2 + 5y - 5z$ . (Note: this space region was used in Example 318.)

**SOLUTION** As we start, consider the density function. It is symmetric about the  $y$ - $z$  plane, and the farther one moves from this plane, the denser the object is. The symmetry indicates that  $\bar{x}$  should be 0.

As one moves away from the origin in the  $y$  or  $z$  directions, the object becomes less dense, though there is more volume in these regions.

Though none of the integrals needed to compute the center of mass are particularly hard, they do require a number of steps. We emphasize here the

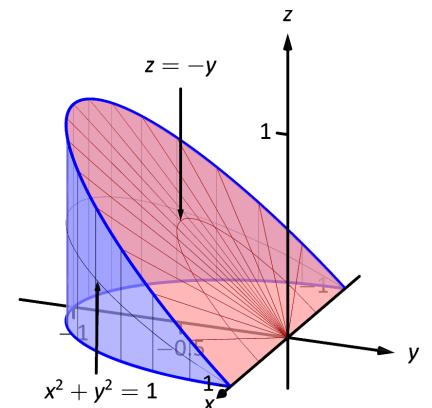


Figure 12.48: Finding the center of mass of this solid in Example 323.

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Notes:

importance of knowing how to set up the proper integrals; in complex situations we can appeal to technology for a good approximation, if not the exact answer. We use the order of integration  $dz\ dy\ dx$ , using the bounds found in Example 318. (As these are the same for all four triple integrals, we explicitly show the bounds only for  $M$ .)

$$\begin{aligned} M &= \iiint_D (10 + x^2 + 5y - 5z) \, dV \\ &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^0 \int_0^{-y} (10 + x^2 + 5y - 5z) \, dV \\ &= \frac{64}{5} - \frac{15\pi}{16} \approx 3.855. \\ M_{yz} &= \iiint_D x(10 + x^2 + 5y - 5z) \, dV \\ &= 0. \\ M_{xz} &= \iiint_D y(10 + x^2 + 5y - 5z) \, dV \\ &= 2 - \frac{61\pi}{48} \approx -1.99. \\ M_{xy} &= \iiint_D z(10 + x^2 + 5y - 5z) \, dV \\ &= \frac{61\pi}{96} - \frac{10}{9} \approx 0.885. \end{aligned}$$

Note how  $M_{yz} = 0$ , as expected. The center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left( 0, \frac{-1.99}{3.855}, \frac{0.885}{3.855} \right) \approx (0, -0.516, 0.230).$$

As stated before, there are many uses for triple integration beyond finding volume. When  $h(x, y, z)$  describes a rate of change function over some space region  $D$ , then  $\iiint_D h(x, y, z) \, dV$  gives the total change over  $D$ . Our one specific example of this was computing mass; a density function is simply a “rate of mass change per volume” function. Integrating density gives total mass.

While knowing *how to integrate* is important, it is arguably much more important to know *how to set up* integrals. It takes skill to create a formula that describes a desired quantity; modern technology is very useful in evaluating these formulas quickly and accurately.

---

Notes:

This chapter investigated the natural follow-on to partial derivatives: iterated integration. We learned how to use the bounds of a double integral to describe a region in the plane using both rectangular and polar coordinates, then later expanded to use the bounds of a triple integral to describe a region in space. We used double integrals to find volumes under surfaces, surface area, and the center of mass of lamina; we used triple integrals as an alternate method of finding volumes of space regions and also to find the center of mass of a region in space.

Integration does not stop here. We could continue to iterate our integrals, next investigating “quadruple integrals” whose bounds describe a region in 4-dimensional space (which are very hard to visualize). We can also look back to “regular” integration where we found the area under a curve in the plane. A natural analogue to this is finding the “area under a curve,” where the curve is in space, not in a plane. These are just two of many avenues to explore under the heading of “integration.”

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Notes:

# Exercises 12.6

## Terms and Concepts

- The strategy for establishing bounds for triple integrals is “\_\_\_\_\_ to \_\_\_\_\_, \_\_\_\_\_ to \_\_\_\_\_ and \_\_\_\_\_ to \_\_\_\_\_.”
- Give an informal interpretation of what “ $\iiint_D dV$ ” means.
- Give two uses of triple integration.
- If an object has a constant density  $\delta$  and a volume  $V$ , what is its mass?

## Problems

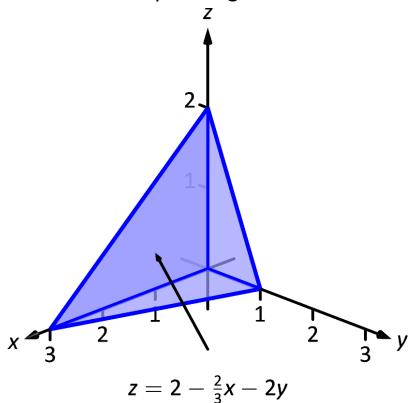
In Exercises 5 – 8, two surfaces  $f_1(x, y)$  and  $f_2(x, y)$  and a region  $R$  in the  $x, y$  plane are given. Set up and evaluate the double integral that finds the volume between these surfaces over  $R$ .

- $f_1(x, y) = 8 - x^2 - y^2, f_2(x, y) = 2x + y;$   
 $R$  is the square with corners  $(-1, -1)$  and  $(1, 1)$ .
- $f_1(x, y) = x^2 + y^2, f_2(x, y) = -x^2 - y^2;$   
 $R$  is the square with corners  $(0, 0)$  and  $(2, 3)$ .
- $f_1(x, y) = \sin x \cos y, f_2(x, y) = \cos x \sin y + 2;$   
 $R$  is the triangle with corners  $(0, 0), (\pi, 0)$  and  $(\pi, \pi)$ .
- $f_1(x, y) = 2x^2 + 2y^2 + 3, f_2(x, y) = 6 - x^2 - y^2;$   
 $R$  is the circle  $x^2 + y^2 = 1$ .

In Exercises 9 – 16, a domain  $D$  is described by its bounding surfaces, along with a graph. Set up the triple integrals that give the volume of  $D$  in all 6 orders of integration, and find the volume of  $D$  by evaluating the indicated triple integral.

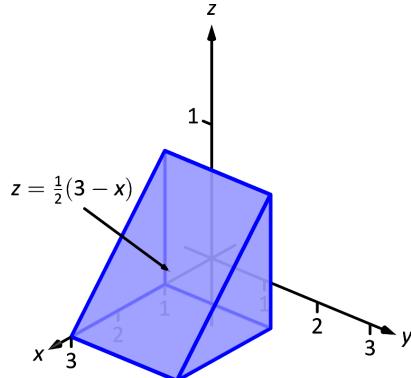
- $D$  is bounded by the coordinate planes and  $z = 2 - 2x/3 - 2y$ .

Evaluate the triple integral with order  $dz dy dx$ .



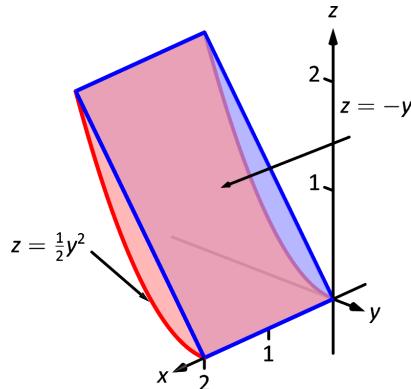
- $D$  is bounded by the planes  $y = 0, y = 2, x = 1, z = 0$  and  $z = (3 - x)/2$ .

Evaluate the triple integral with order  $dx dy dz$ .



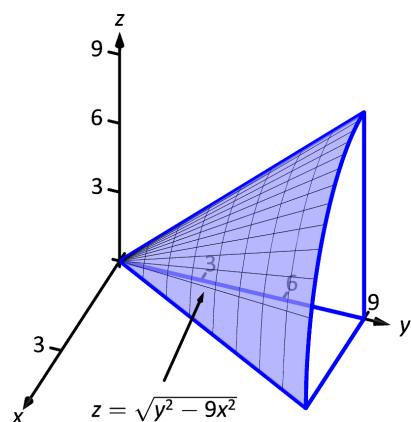
- $D$  is bounded by the planes  $x = 0, x = 2, z = -y$  and by  $z = y^2/2$ .

Evaluate the triple integral with the order  $dy dz dx$ .



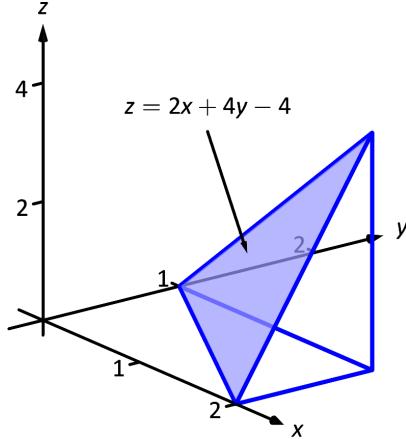
- $D$  is bounded by the planes  $z = 0, y = 9, x = 0$  and by  $z = \sqrt{y^2 - 9x^2}$ .

Do not evaluate any triple integral.



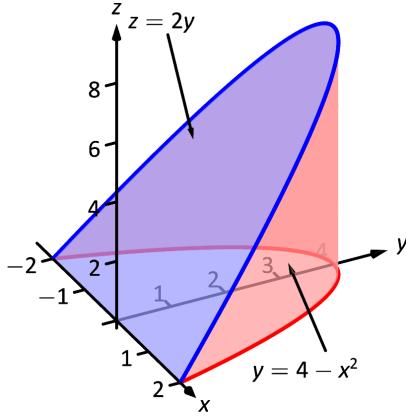
13.  $D$  is bounded by the planes  $x = 2$ ,  $y = 1$ ,  $z = 0$  and  $z = 2x + 4y - 4$ .

Evaluate the triple integral with the order  $dx dy dz$ .



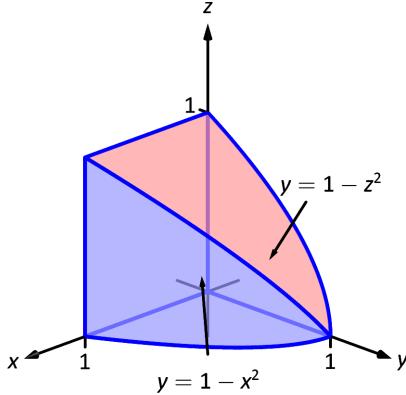
14.  $D$  is bounded by the plane  $z = 2y$  and by  $y = 4 - x^2$ .

Evaluate the triple integral with the order  $dz dy dx$ .



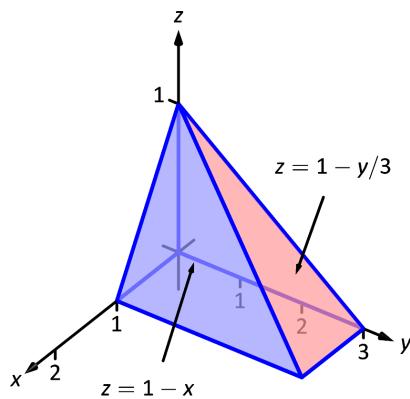
15.  $D$  is bounded by the coordinate planes and by  $y = 1 - x^2$  and  $y = 1 - z^2$ .

Do not evaluate any triple integral. Which order is easier to evaluate:  $dz dy dx$  or  $dy dz dx$ ? Explain why.



16.  $D$  is bounded by the coordinate planes and by  $z = 1 - y/3$  and  $z = 1 - x$ .

Evaluate the triple integral with order  $dx dy dz$ .



In Exercises 17 – 20, evaluate the triple integral.

17.  $\int_{-\pi/2}^{\pi/2} \int_0^\pi \int_0^\pi (\cos x \sin y \sin z) dz dy dx$

18.  $\int_0^1 \int_0^x \int_0^{x+y} (x + y + z) dz dy dx$

19.  $\int_0^\pi \int_0^1 \int_0^z (\sin(yz)) dx dy dz$

20.  $\int_\pi^{\pi^2} \int_x^{x^3} \int_{-y^2}^{y^2} \left( z \frac{x^2 y + y^2 x}{e^{x^2 + y^2}} \right) dz dy dx$

In Exercises 21 – 24, find the center of mass of the solid represented by the indicated space region  $D$  with density function  $\delta(x, y, z)$ .

21.  $D$  is bounded by the coordinate planes and  $z = 2 - 2x/3 - 2y$ ;  $\delta(x, y, z) = 10 \text{ gm/cm}^3$ .  
(Note: this is the same region as used in Exercise 9.)

22.  $D$  is bounded by the planes  $y = 0$ ,  $y = 2$ ,  $x = 1$ ,  $z = 0$  and  $z = (3 - x)/2$ ;  $\delta(x, y, z) = 2 \text{ gm/cm}^3$ .  
(Note: this is the same region as used in Exercise 10.)

23.  $D$  is bounded by the planes  $x = 2$ ,  $y = 1$ ,  $z = 0$  and  $z = 2x + 4y - 4$ ;  $\delta(x, y, z) = x^2 \text{ lb/in}^3$ .  
(Note: this is the same region as used in Exercise 13.)

24.  $D$  is bounded by the plane  $z = 2y$  and by  $y = 4 - x^2$ .  
 $\delta(x, y, z) = y^2 \text{ lb/in}^3$ .  
(Note: this is the same region as used in Exercise 14.)

## 12.7 Cylindrical and Spherical Integration

### Triple Integration with Cylindrical Coordinates

Cylindrical coordinates are useful for describing many solids that are symmetric around an axis. The solid is three dimensional, and so we use the coordinates  $(r, \theta, z)$  for points in this solid. In short, the radius  $r$  and angle  $\theta$  are the same as in polar coordinates, with  $r$  being the distance out from axis of symmetry and  $\theta$  being the angle around the axis, with the positive  $x$ -axis being the direction of  $\theta = 0$  as usual. The coordinate  $z$  is the same as in the Cartesian system - the vertical distance up the axis, with  $z = 0$  being the  $xy$ -plane. See Figure 12.49.

The conversions between cylindrical and Cartesian coordinates are given as follows, which follows from our previous study of polar coordinates.

**Key Idea 46** **Converting between cylindrical and Cartesian coordinates**

Given a point  $(r, \theta, z)$  in cylindrical coordinates, its Cartesian coordinates are

$$x = r \cos \theta, y = r \sin \theta, z = z.$$

Given a point  $(x, y, z)$  in Cartesian coordinates, we use

$$r^2 = x^2 + y^2, \tan \theta = \frac{y}{x}, z = z$$

to convert to cylindrical coordinates.

**Example 324**

Write the point  $(\sqrt{3}, 1, -1)$  in cylindrical coordinates.

**SOLUTION** Since we are given the Cartesian coordinates, we use the above conversion formulas to convert this point to cylindrical coordinates. Since  $r^2 = x^2 + y^2$ , we arrive at

$$r = \sqrt{3 + 1} = 2.$$

Since  $\tan \theta = \frac{1}{\sqrt{3}}$ , it follows that  $\theta = \frac{\pi}{6}$  as the point is in the first quadrant of the  $xy$ -plane. Therefore the cylindrical coordinates of this point are  $(2, \frac{\pi}{6}, -1)$ ,

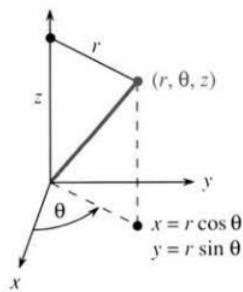


Figure 12.49: Cylindrical coordinates for a point

Notes:

since the  $z$ -coordinate remains the same.

As mentioned, certain surfaces and solids are conveniently described using cylindrical coordinates. For example, consider the solid half-cylinder  $D$  of radius 1 centered on the  $z$ -axis with the bottom on the  $xy$ -plane and the top at  $z = 3$ . If the half-cylinder is above the first two quadrants of the  $xy$ -plane, then any point in the cylinder has a  $\theta$  coordinate between 0 and  $\pi$ . Therefore the coordinates that describe this solid are

$$0 \leq r \leq 1, 0 \leq \theta \leq \pi, 0 \leq z \leq 3.$$

This half-cylinder is shown in Figure 12.50.

Surfaces and solids described using Cartesian coordinates can also be written using cylindrical coordinates, and vice versa, using our conversion formulas above.

### Example 325

Consider the surface described by  $z = r^2$  in cylindrical coordinates. Write the equation for this surface in Cartesian coordinates and identify the surface.

**SOLUTION** We know that  $r^2 = x^2 + y^2$ , and so we can write  $z = x^2 + y^2$ . This is a paraboloid with vertex at the origin opening up. See Figure 12.51.

When integrating over a solid  $D$  described with cylindrical coordinates, one would evaluate a triple integral  $\iiint_D f(x, y, z) dV$ , but using  $dr, d\theta$ , and  $dz$  instead of  $dx, dy$ , and  $dz$ . The differential of volume  $dV$  would be

$$dV = r dr d\theta dz$$

since we are working with the same two coordinates  $r$  and  $\theta$  from polar coordinates and the third coordinate is the same as in the Cartesian system. This is the volume of the curved box in Figure ?? In other words, the integral above becomes

$$\iiint_D f(x, y, z) dV = \iiint_D f(r \cos \theta, r \sin \theta, z) r dr d\theta dz$$

where each occurrence of  $x$  and  $y$  in the integrand has been converted to cylindrical coordinates. The bounds on each integral would be the corresponding bounds on  $r$ ,  $\theta$ , and  $z$  that describe the solid  $D$ , and we can integrate in any appropriate order.

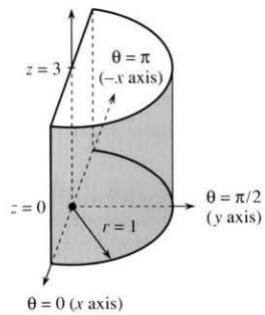


Figure 12.50: Half cylinder

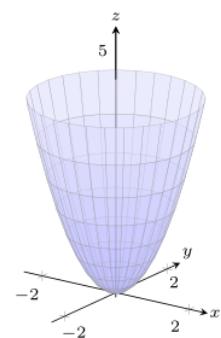


Figure 12.51:  $z = r^2$

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Notes:

**Example 326**

Compute the volume of the half-cylinder  $D$  described previously, for  $0 \leq r \leq 1$ ,  $0 \leq \theta \leq \pi$ ,  $0 \leq z \leq 3$ .

**SOLUTION** Since we are computing the volume of  $D$ , the integrand is  $f(x, y, z) = 1$  and so

$$V = \iiint_D 1 \, dV = \iiint_D r \, dr \, d\theta \, dz.$$

Using our bounds on  $r$ ,  $\theta$ , and  $z$  that describe  $D$  as our bounds of integration, we get a volume of

$$\int_0^3 \int_0^\pi \int_0^1 r \, dr \, d\theta \, dz = \int_0^3 \int_0^\pi \left( \frac{1}{2}r^2 \right)_0^1 d\theta \, dz = \int_0^3 \int_0^\pi \frac{1}{2} d\theta \, dz$$

which evaluates to a volume of  $\frac{3\pi}{2}$  cubic units.

**Example 327**

The surface  $z = 1 - r$  encloses a conical solid between itself and the  $xy$ -plane. Find the volume of this solid as seen in Figure 12.52.

**SOLUTION** The cone  $z = 1 - r$  intersects the  $xy$ -plane ( $z = 0$ ) at a radius of  $r = 1$ . This puts the bounds on  $r$  as  $0 \leq r \leq 1$ , and clearly the bounds on  $\theta$  would be  $0 \leq \theta \leq 2\pi$  since the solid is the full cone. Since the solid is not a cylinder, at least one of the bounds on  $z$  must be non-constant. At any radius  $r$ , the height of the solid is bounded between the  $xy$ -plane and the surface  $z = 1 - r$ . Therefore the bounds on  $z$  are  $0 \leq z \leq 1 - r$ , and we will integrate with respect to  $dz$  first. Therefore the volume is

$$V = \int_0^{2\pi} \int_0^1 \int_0^{1-r} r \, dz \, dr \, d\theta = 2\pi \int_0^1 (1-r)r \, dr$$

which evaluates to a volume of  $\frac{\pi}{3}$  cubic units.

**Example 328**

A solid  $D$  is the region inside the cylinder  $x^2 + y^2 = 1$ , below the plane  $z = 2$  and above the paraboloid  $z = 1 - x^2 - y^2$ , with all distances in meters. See Figure 12.53. If the density of this solid is

$$\delta(x, y, z) = \sqrt{x^2 + y^2}$$

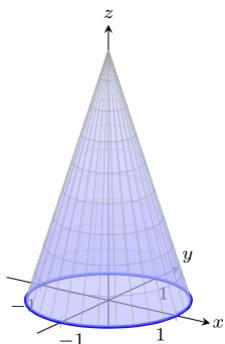


Figure 12.52:  $z = 1 - r$

Notes:

kilograms per cubic meter, determine the mass of this solid.

**SOLUTION** As this solid is a cylinder with a paraboloid cut out of the bottom, it makes sense to use cylindrical coordinates here, where the mass is the integral of the density over the solid. The z-coordinates of this solid solid are bounded below by  $1 - x^2 - y^2$  or  $1 - r^2$ , and bounded above by  $z = 2$ . Therefore the solid  $D$  is described by

$$0 \leq r \leq 1, 0 \leq \theta \leq 2\pi, 1 - r^2 \leq z \leq 2$$

and the density function, when converted to cylindrical coordinates, is

$$\delta(r, \theta, z) = \sqrt{r^2} = r.$$

Integrating using the order  $dz d\theta dr$  and the additional factor of  $r$  for cylindrical coordinates yields a mass of

$$\int_0^1 \int_0^{2\pi} \int_{1-r^2}^2 r^2 dz d\theta dr = 2\pi \int_0^1 r^2 + r^4 dr = 2\pi \left( \frac{1}{12} \right) = \frac{16\pi}{15}$$

kilograms.

### Triple Integration with Spherical Coordinates

Objects bounded by spheres or cones are more easily described using a different coordinate system called spherical coordinates. The Earth, for example, is a solid sphere (or near enough). On its surface we use two coordinates - latitude and longitude. To dig inward or fly outward, there is a third coordinate, the distance  $\rho$  from the center. This Greek letter *rho* replaces radius  $r$  to avoid confusion with cylindrical coordinates. Where  $r$  is measured from the z-axis,  $\rho$  is measured directly from the origin. Thus for any point  $(x, y, z)$ ,

$$\rho^2 = x^2 + y^2 + z^2$$

which is the square of the distance between the origin and the point. The angle  $\theta$  is the same as in cylindrical coordinates, and it goes from 0 to  $2\pi$  on a full sphere with  $\theta = 0$  pointing in the direction of the positive x-axis. It is the longitude, which increases as you travel east around the Equator. The angle  $\phi$  is new, however. It equals 0 at the North Pole and  $\pi$  (not  $2\pi$ ) at the South Pole. It is the polar angle, measured down from the z-axis. The Equator, for example, has a latitude of 0 degrees but a polar angle of  $\phi = \frac{\pi}{2}$  instead. See Figure 12.54.

The spherical coordinates of a point  $(x, y, z)$  are given by the ordered triple  $(\rho, \theta, \phi)$  where  $\rho$ ,  $\theta$ , and  $\phi$  can be restricted to  $\rho \geq 0, 0 \leq \theta \leq 2\pi$ , and  $0 \leq \phi \leq \pi$ .

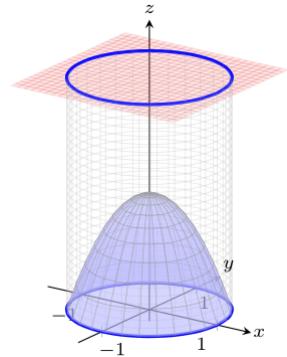


Figure 12.53: Solid  $D$  bounded above by  $z = 2$ , below by  $z = 1 - x^2 - y^2$ , laterally by  $x^2 + y^2 = 1$

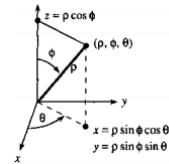


Figure 12.54: Spherical coordinates  $\rho\phi\theta$

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Notes:

$\pi$ . The relationship between spherical and Cartesian coordinates is illustrated by Figure ?? From the triangles, we have

$$z = \rho \cos \phi \text{ and } r = \rho \sin \phi$$

But we know that  $x = r \cos \theta$  and  $y = r \sin \theta$  from before, so we end up with the following conversion equations.

**Key Idea 47** **Converting between spherical and Cartesian coordinates**

Given a point  $(\rho, \theta, \phi)$  in spherical coordinates, its Cartesian coordinates are

$$x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi.$$

Given a point  $(x, y, z)$  in Cartesian coordinates, we use

$$\rho^2 = x^2 + y^2 + z^2, \tan \theta = \frac{y}{x}, \cos \phi = \frac{z}{\rho} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

to convert to spherical coordinates.

**Example 329**

The point  $\left(3, \frac{\pi}{6}, \frac{2\pi}{3}\right)$  is given in spherical coordinates. What are the Cartesian coordinates representing this point?

**SOLUTION** Using the conversion equations above, we have

$$x = \rho \sin \phi \cos \theta = 3 \sin\left(\frac{2\pi}{3}\right) \cos\left(\frac{\pi}{6}\right) = 3\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{9}{4}$$

$$y = \rho \sin \phi \sin \theta = 3 \sin\left(\frac{2\pi}{3}\right) \sin\left(\frac{\pi}{6}\right) = 3\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = \frac{3\sqrt{3}}{4}$$

$$z = \rho \cos \phi = 3 \cos\left(\frac{2\pi}{3}\right) = 3\left(-\frac{1}{2}\right) = -\frac{3}{2}$$

which corresponds to the Cartesian coordinates of  $\left(\frac{9}{4}, \frac{3\sqrt{3}}{4}, -\frac{3}{2}\right)$ .

---

Notes:

Surfaces like spheres and cones can be easily described in spherical coordinates. For example,  $\rho = R$  describes the surface of a sphere of radius  $R$ , while  $\phi = \frac{\pi}{3}$  describes a conical surface opening up at a 60 degree angle from the z-axis (or a 30 degree angle from the xy-plane). Some solids, such as solid cones or parts of solid spheres, have bounds which are easily described in spherical coordinates. For example, the bounds

$$0 \leq \rho \leq 1, \quad 0 \leq \theta \leq 2\pi, \quad \frac{\pi}{2} \leq \phi \leq \pi$$

describes the solid lower hemisphere of radius 1, while the bounds

$$0 \leq \rho \leq 1, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \frac{\pi}{4}$$

describes a conical solid opening up at a 45 degree angle with a spherical top.

If one is to integrate over a solid  $D$  best described in spherical coordinates, such as when finding its volume, one can think of the solid as being divided into spherical boxes or spherical wedges of thickness  $d\rho$  and angles  $d\theta$  and  $d\phi$ . The dimensions of this spherical box, as seen in Figure 12.55, are  $d\rho$ ,  $\rho d\phi$  (arc of a circle of radius  $\rho$  and angle  $d\phi$ ), and  $\rho \sin \phi d\theta$  (arc of a circle with radius  $\rho \sin \phi$  and angle  $d\theta$ ). Therefore the approximate volume of the spherical box is

$$dV = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi.$$

To integrate a function  $f(x, y, z)$  over a solid  $D$  described using spherical coordinates, one would therefore compute

$$\iiint_D f(x, y, z) \, dA = \iiint_D f(\rho, \theta, \phi) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$

where the bounds on  $\rho$ ,  $\theta$ , and  $\phi$  are as appropriate to describe the solid  $D$ . As a first example of this, we will verify the geometric formula for the volume of a sphere.

### Example 330

Determine the volume of a sphere of radius  $R$ .

**SOLUTION** Let  $D$  be the sphere of radius  $R$  centered at the origin. Then we can describe  $D$  using spherical coordinates as simply all points  $(\rho, \theta, \phi)$  where

$$0 \leq \rho \leq R, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi.$$

Therefore the volume of this ball is equal to

$$V = \int_0^{2\pi} \int_0^\pi \int_0^R \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \left( \frac{1}{3} \rho^3 \right)_0^R (-\cos \phi)_0^\pi (\theta)_0^{2\pi} = \frac{1}{3} R^3 (2)(2\pi)$$

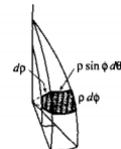


Figure 12.55: Spherical box

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Notes:

which is equal to the familiar formula for the volume of a sphere being  $\frac{4}{3}\pi R^3$ .

**Example 331**

Evaluate the integral

$$\iiint_D x^2 + y^2 + z^2 \, dV$$

where  $D$  is the solid that lies above the cone  $z = \sqrt{x^2 + y^2}$  but below the sphere of radius  $\frac{1}{2}$  centered at  $(0, 0, \frac{1}{2})$ .

**SOLUTION** The cone  $z = \sqrt{x^2 + y^2}$  is the cone pointing upward from the origin at an angle of  $\frac{\pi}{4}$  from the  $xy$ -plane. The solid  $D$  therefore consists of a half-spherical top with a conical bottom, with the point of the cone at the origin, meeting the spherical top at the horizontal equator of the sphere.

To describe this solid in spherical coordinates, note that the sphere of radius  $\frac{1}{2}$  centered at the point  $(0, 0, \frac{1}{2})$  can be written as

$$x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}$$

which is equivalent to  $x^2 + y^2 + z^2 = z$ . In spherical coordinates, this says  $\rho^2 = \rho \cos \phi$ , or  $\rho = \cos \phi$ . As for the cone, the equation  $\phi = \frac{\pi}{4}$  describes this surface. Therefore  $D$  can be described as all points  $(\rho, \theta, \phi)$  with  $0 \leq \rho \leq \cos \phi$ ,  $0 \leq \theta \leq 2\pi$ , and  $0 \leq \phi \leq \frac{\pi}{4}$ .

Lastly, to integrate we need to write the integrand  $x^2 + y^2 + z^2$  as  $\rho^2$  and include the spherical integration factor  $\rho^2 \sin \phi$ . Therefore

$$\begin{aligned} \iiint_E x^2 + y^2 + z^2 \, dV &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos \phi} \rho^4 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= 2\pi \int_0^{\pi/4} \sin \phi \left( \frac{1}{5} \rho^5 \right) \Big|_0^{\cos \phi} \, d\phi \\ &= \frac{2}{5}\pi \int_0^{\pi/4} \cos^5 \phi \sin \phi \, d\phi \\ &= \frac{2}{5}\pi \left( -\frac{1}{6} \cos^6 \phi \right) \Big|_0^{\pi/4} \\ &= \frac{2}{5}\pi \left( \frac{1}{8} - 1 \right) = \frac{7\pi}{120} \end{aligned}$$

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Notes:

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Notes:

# Exercises 12.7

## Terms and Concepts

1. Fill in the blank: To describe a full cylinder, the bounds on  $\theta$  would go from  $\theta = 0$  to  $\theta = \underline{\hspace{2cm}}$ .
2. Fill in the blank: To describe a full sphere, the bounds on  $\theta$  would go from  $\theta = 0$  to  $\theta = 2\pi$  and  $\phi$  would go from  $\phi = 0$  to  $\phi = \underline{\hspace{2cm}}$ .
3. T/F: In spherical coordinates, the direction of  $\phi = 0$  is perpendicular to the  $z$ -axis.
4. T/F: In spherical coordinates, the equation  $\rho = 4$  describes the surface of a sphere.
5. T/F: In cylindrical or spherical coordinates, the equation  $\theta = \frac{\pi}{4}$  describes a cone.

## Problems

In Exercises 6 – 8, the Cartesian coordinates of a point are given. Write both the cylindrical  $(r, \theta, z)$  and spherical  $(\rho, \theta, \phi)$  coordinates of this point.

6.  $(1, 0, 0)$

7.  $(0, -2, 0)$

8.  $(1, 1, -2)$

In Exercises 9 – 11, the Cartesian coordinates of a point are given. Write both the cylindrical  $(r, \theta, z)$  and spherical  $(\rho, \theta, \phi)$  coordinates of this point.

9.  $(1, 0, 0)$

10.  $(0, -2, 0)$

11.  $(1, 1, -2)$

In Exercises 12 – 14, an integral equal to the volume of a solid is given. Describe the solid.

12.  $\int_0^{2\pi} \int_0^1 \int_0^4 r \, dz \, dr \, d\theta$

13.  $\int_0^{2\pi} \int_0^{\pi/2} \int_0^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$

14.  $\int_0^{2\pi} \int_{3\pi/4}^{\pi} \int_0^1 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$

In Exercises 15 – 19, evaluate the integral using the given described solid  $D$  using either cylindrical or spherical coordinates as appropriate.

15.  $\iiint_D \sqrt{x^2 + y^2} \, dV$  where  $D$  is the cylinder with axis along the  $z$ -axis and radius 4 between  $z = 0$  and  $z = 1$ .

16.  $\iiint_D 1 \, dV$  where  $D$  is the solid trapped between the paraboloid  $z = x^2 + y^2$  and the plane  $z = 9$ .

17.  $\iiint_D x^2 + y^2 + z^2 \, dV$  where  $D$  is the solid sphere of radius 2 centered at the origin.

18.  $\iiint_D \sqrt{x^2 + y^2 + z^2} \, dV$  where  $D$  is the solid between the spheres  $x^2 + y^2 + z^2 = 1$  and  $x^2 + y^2 + z^2 = 4$  and above the cone  $z = \sqrt{x^2 + y^2}$ .

In Exercises 19 – 21, evaluate the integral by first converting to cylindrical or spherical coordinates.

19.  $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{\sqrt{x^2+y^2}}^1 xz \, dz \, dx \, dy$

20.  $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} 1 \, dz \, dy \, dx$

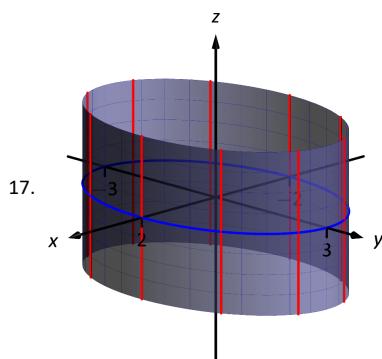
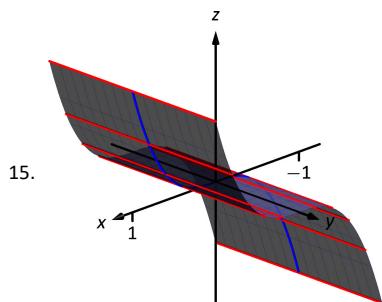
# A: SOLUTIONS TO SELECTED PROBLEMS

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## Chapter 9

### Section 9.1

1. right hand
3. curve (a parabola); surface (a cylinder)
5. a hyperboloid of two sheets
7.  $\|\overrightarrow{AB}\| = \sqrt{6}$ ;  $\|\overrightarrow{BC}\| = \sqrt{17}$ ;  $\|\overrightarrow{AC}\| = \sqrt{11}$ . Yes, it is a right triangle as  $\|\overrightarrow{AB}\|^2 + \|\overrightarrow{AC}\|^2 = \|\overrightarrow{BC}\|^2$ .
9. Center at  $(4, -1, 0)$ ; radius = 3
11. Interior of a sphere with radius 1 centered at the origin.
13. The first octant of space; all points  $(x, y, z)$  where each of  $x, y$  and  $z$  are positive. (Analogous to the first quadrant in the plane.)

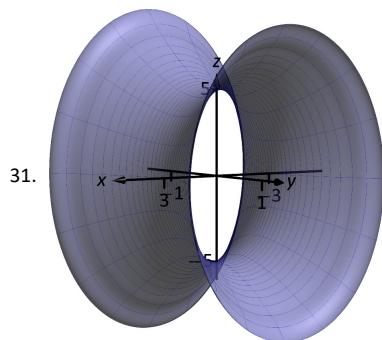
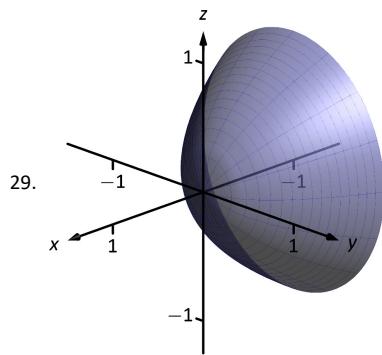
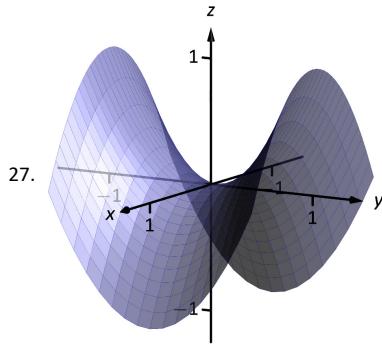


19.  $y^2 + z^2 = x^4$

21.  $z = (\sqrt{x^2 + y^2})^2 = x^2 + y^2$

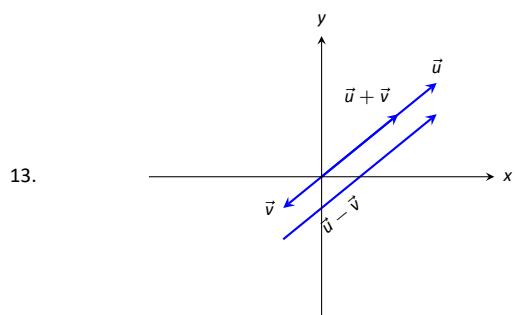
23. (a)  $x = y^2 + \frac{z^2}{9}$

25. (b)  $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$

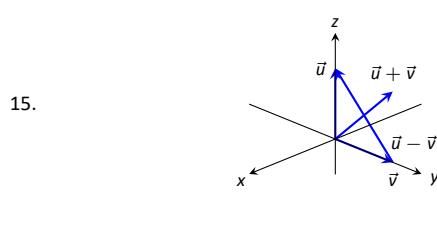


### Section 9.2

1. Answers will vary.
3. A vector with magnitude 1.
5. It stretches the vector by a factor of 2, and points it in the opposite direction.
7.  $\overrightarrow{PQ} = \langle -4, 4 \rangle = -4\vec{i} + 4\vec{j}$
9.  $\overrightarrow{PQ} = \langle 2, 2, 0 \rangle = 2\vec{i} + 2\vec{j}$
11. (a)  $\vec{u} + \vec{v} = \langle 3, 2, 1 \rangle$ ;  $\vec{u} - \vec{v} = \langle -1, 0, -3 \rangle$ ;  $\pi\vec{u} - \sqrt{2}\vec{v} = \langle \pi - 2\sqrt{2}, \pi - \sqrt{2}, -\pi - 2\sqrt{2} \rangle$ .  
(c)  $\vec{x} = \langle -1, 0, -3 \rangle$ .



Sketch of  $\vec{u} - \vec{v}$  shifted for clarity.



17.  $\|\vec{u}\| = \sqrt{17}$ ,  $\|\vec{v}\| = \sqrt{3}$ ,  $\|\vec{u} + \vec{v}\| = \sqrt{14}$ ,  $\|\vec{u} - \vec{v}\| = \sqrt{26}$   
 19.  $\|\vec{u}\| = 7$ ,  $\|\vec{v}\| = 35$ ,  $\|\vec{u} + \vec{v}\| = 42$ ,  $\|\vec{u} - \vec{v}\| = 28$   
 21.  $\vec{u} = \langle 3/\sqrt{58}, 7/\sqrt{58} \rangle$   
 23.  $\vec{u} = \langle 1/3, -2/3, 2/3 \rangle$   
 25.  $\vec{u} = \langle \cos 50^\circ, \sin 50^\circ \rangle \approx \langle 0.643, 0.766 \rangle$ .

27.

$$\begin{aligned}\|\vec{u}\| &= \sqrt{\sin^2 \theta \cos^2 \varphi + \sin^2 \theta \sin^2 \varphi + \cos^2 \theta} \\ &= \sqrt{\sin^2 \theta (\cos^2 \varphi + \sin^2 \varphi) + \cos^2 \theta} \\ &= \sqrt{\sin^2 \theta + \cos^2 \theta} \\ &= 1.\end{aligned}$$

29. The force on each chain is 100lb.  
 31. The force on each chain is 50lb.  
 33.  $\theta = 5.71^\circ$ ; the weight is lifted 0.005 ft (about 1/16th of an inch).  
 35.  $\theta = 84.29^\circ$ ; the weight is lifted 9 ft.

### Section 9.3

1. Scalar  
 3. By considering the sign of the dot product of the two vectors. If the dot product is positive, the angle is acute; if the dot product is negative, the angle is obtuse.  
 5.  $-22$   
 7. 3  
 9. not defined  
 11. Answers will vary.  
 13.  $\theta = 0.3218 \approx 18.43^\circ$   
 15.  $\theta = \pi/4 = 45^\circ$   
 17. Answers will vary; two possible answers are  $\langle -7, 4 \rangle$  and  $\langle 14, -8 \rangle$ .  
 19. Answers will vary; two possible answers are  $\langle 1, 0, -1 \rangle$  and  $\langle 4, 5, -9 \rangle$ .  
 21.  $\text{proj}_{\vec{v}} \vec{u} = \langle -1/2, 3/2 \rangle$ .  
 23.  $\text{proj}_{\vec{v}} \vec{u} = \langle -1/2, -1/2 \rangle$ .  
 25.  $\text{proj}_{\vec{v}} \vec{u} = \langle 1, 2, 3 \rangle$ .  
 27.  $\vec{u} = \langle -1/2, 3/2 \rangle + \langle 3/2, 1/2 \rangle$ .  
 29.  $\vec{u} = \langle -1/2, -1/2 \rangle + \langle -5/2, 5/2 \rangle$ .  
 31.  $\vec{u} = \langle 1, 2, 3 \rangle + \langle 0, 3, -2 \rangle$ .

33. 1.96lb  
 35. 141.42ft-lb  
 37. 500ft-lb  
 39. 500ft-lb

### Section 9.4

1. vector  
 3. “Perpendicular” is one answer.  
 5. Torque  
 7.  $\vec{u} \times \vec{v} = \langle 11, 1, -17 \rangle$   
 9.  $\vec{u} \times \vec{v} = \langle 47, -36, -44 \rangle$   
 11.  $\vec{u} \times \vec{v} = \langle 0, 0, 0 \rangle$   
 13.  $\vec{r} \times \vec{k} = -\vec{r}$   
 15. Answers will vary.

19. 0  
 21.  $\sqrt{14}$   
 23. 3  
 25.  $5\sqrt{2}/2$   
 27. 1  
 29. 7  
 31. 2  
 33.  $\pm \frac{1}{\sqrt{6}} \langle 1, 1, -2 \rangle$   
 35.  $\langle 0, \pm 1, 0 \rangle$   
 37. 87.5ft-lb  
 39.  $200/3 \approx 66.67$ ft-lb  
 41. With  $\vec{u} = \langle u_1, u_2, u_3 \rangle$  and  $\vec{v} = \langle v_1, v_2, v_3 \rangle$ , we have  

$$\begin{aligned}\vec{u} \cdot (\vec{u} \times \vec{v}) &= \langle u_1, u_2, u_3 \rangle \cdot ((u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1)) \\ &= u_1(u_2 v_3 - u_3 v_2) - u_2(u_1 v_3 - u_3 v_1) + u_3(u_1 v_2 - u_2 v_1) \\ &= 0.\end{aligned}$$

### Section 9.5

1. A point on the line and the direction of the line.  
 3. parallel, skew  
 5. vector:  $\ell(t) = \langle 2, -4, 1 \rangle + t \langle 9, 2, 5 \rangle$   
 parametric:  $x = 2 + 9t$ ,  $y = -4 + 2t$ ,  $z = 1 + 5t$   
 symmetric:  $(x - 2)/9 = (y + 4)/2 = (z - 1)/5$   
 7. Answers can vary: vector:  $\ell(t) = \langle 2, 1, 5 \rangle + t \langle 5, -3, -1 \rangle$   
 parametric:  $x = 2 + 5t$ ,  $y = 1 - 3t$ ,  $z = 5 - t$   
 symmetric:  $(x - 2)/5 = -(y - 1)/3 = -(z - 5)$   
 9. Answers can vary; here the direction is given by  $\vec{d}_1 \times \vec{d}_2$ : vector:  
 $\ell(t) = \langle 0, 1, 2 \rangle + t \langle -10, 43, 9 \rangle$   
 parametric:  $x = -10t$ ,  $y = 1 + 43t$ ,  $z = 2 + 9t$   
 symmetric:  $-x/10 = (y - 1)/43 = (z - 2)/9$   
 11. Answers can vary; here the direction is given by  $\vec{d}_1 \times \vec{d}_2$ : vector:  
 $\ell(t) = \langle 7, 2, -1 \rangle + t \langle 1, -1, 2 \rangle$   
 parametric:  $x = 7 + t$ ,  $y = 2 - t$ ,  $z = -1 + 2t$   
 symmetric:  $x - 7 = 2 - y = (z + 1)/2$   
 13. vector:  $\ell(t) = \langle 1, 1 \rangle + t \langle 2, 3 \rangle$   
 parametric:  $x = 1 + 2t$ ,  $y = 1 + 3t$   
 symmetric:  $(x - 1)/2 = (y - 1)/3$   
 15. parallel  
 17. intersecting;  $\vec{\ell}_1(3) = \vec{\ell}_2(4) = \langle 9, -5, 13 \rangle$   
 19. skew  
 21. same  
 23.  $\sqrt{41}/3$   
 25.  $5\sqrt{2}/2$   
 27.  $3/\sqrt{2}$   
 29. Since both  $P$  and  $Q$  are on the line,  $\vec{PQ}$  is parallel to  $\vec{d}$ . Thus  $\vec{PQ} \times \vec{d} = \vec{0}$ , giving a distance of 0.  
 31. (a) The distance formula cannot be used because since  $\vec{d}_1$  and  $\vec{d}_2$  are parallel,  $\vec{c}$  is  $\vec{0}$  and we cannot divide by  $\|\vec{0}\|$ .  
 (b) Since  $\vec{d}_1$  and  $\vec{d}_2$  are parallel,  $\vec{P_1P_2}$  lies in the plane formed by the two lines. Thus  $\vec{P_1P_2} \times \vec{d}_2$  is orthogonal to this plane, and  $\vec{c} = (\vec{P_1P_2} \times \vec{d}_2) \times \vec{d}_2$  is parallel to the plane, but still orthogonal to both  $\vec{d}_1$  and  $\vec{d}_2$ . We desire the length of the projection of  $\vec{P_1P_2}$  onto  $\vec{c}$ , which is what the formula provides.

- (c) Since the lines are parallel, one can measure the distance between the lines at any location on either line (just as to find the distance between straight railroad tracks, one can use a measuring tape anywhere along the track, not just at one specific place.) Let  $P = P_1$  and  $Q = P_2$  as given by the equations of the lines, and apply the formula for distance between a point and a line.

### Section 9.6

1. A point in the plane and a normal vector (i.e., a direction orthogonal to the plane).

3. Answers will vary.

5. Answers will vary.

7. Standard form:  $3(x - 2) - (y - 3) + 7(z - 4) = 0$   
general form:  $3x - y + 7z = 31$

9. Answers may vary;

Standard form:  $8(x - 1) + 4(y - 2) - 4(z - 3) = 0$   
general form:  $8x + 4y - 4z = 4$

11. Answers may vary;

Standard form:  $-7(x - 2) + 2(y - 1) + (z - 2) = 0$   
general form:  $-7x + 2y + z = -10$

13. Answers may vary;

Standard form:  $2(x - 1) - (y - 1) = 0$   
general form:  $2x - y = 1$

15. Answers may vary;

Standard form:  $2(x - 2) - (y + 6) - 4(z - 1) = 0$   
general form:  $2x - y - 4z = 6$

17. Answers may vary;

Standard form:  $(x - 5) + (y - 7) + (z - 3) = 0$   
general form:  $x + y + z = 15$

19. Answers may vary;

Standard form:  $3(x + 4) + 8(y - 7) - 10(z - 2) = 0$   
general form:  $3x + 8y - 10z = 24$

21. Answers may vary:

$$\ell = \begin{cases} x = 14t \\ y = -1 - 10t \\ z = 2 - 8t \end{cases}$$

23.  $(-3, -7, -5)$

25. No point of intersection; the plane and line are parallel.

27.  $\sqrt{5/7}$

29.  $1/\sqrt{3}$

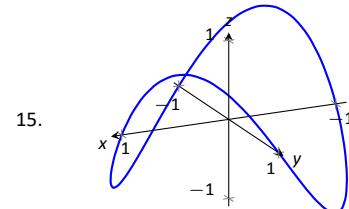
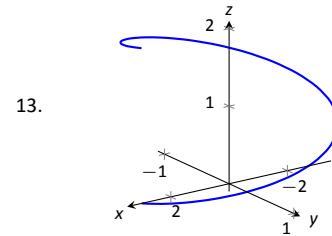
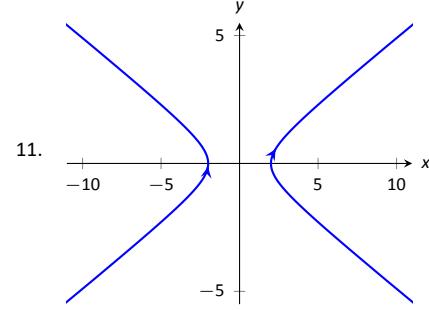
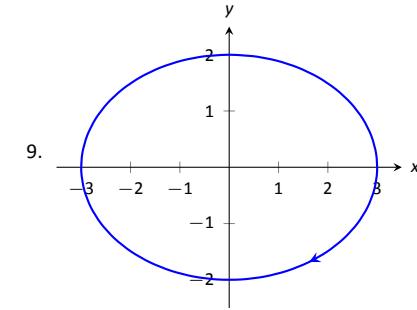
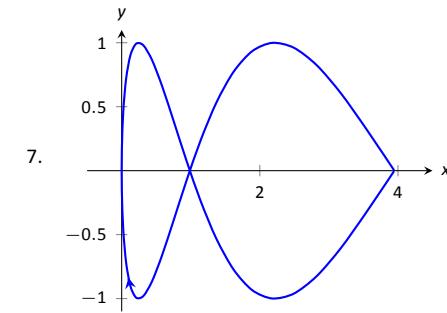
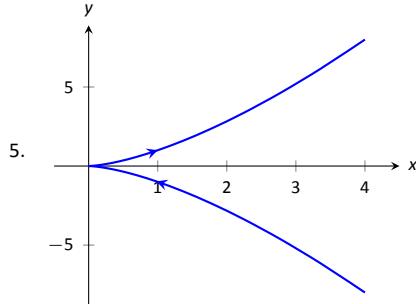
31. If  $P$  is any point in the plane, and  $Q$  is also in the plane, then  $\vec{PQ}$  lies parallel to the plane and is orthogonal to  $\vec{n}$ , the normal vector. Thus  $\vec{n} \cdot \vec{PQ} = 0$ , giving the distance as 0.

## Chapter 10

### Section 10.1

1. parametric equations

3. displacement



17.  $\|\vec{r}(t)\| = \sqrt{25 \cos^2 t + 9 \sin^2 t}.$

19.  $\|\vec{r}(t)\| = \sqrt{\cos^2 t + t^2 + t^4}.$

21. Answers may vary; three solutions are  
 $\vec{r}(t) = \langle 3 \sin t + 5, 3 \cos t + 5 \rangle$ ,  
 $\vec{r}(t) = \langle -3 \cos t + 5, 3 \sin t + 5 \rangle$  and  
 $\vec{r}(t) = \langle 3 \cos t + 5, -3 \sin t + 5 \rangle$ .

23. Answers may vary, though most direct solutions are  
 $\vec{r}(t) = \langle -3\cos t + 3, 2\sin t - 2 \rangle$ ,  
 $\vec{r}(t) = \langle 3\cos t + 3, -2\sin t - 2 \rangle$  and  
 $\vec{r}(t) = \langle 3\sin t + 3, 2\cos t - 2 \rangle$ .

25. Answers may vary, though most direct solutions are  
 $\vec{r}(t) = \langle t, -1/2(t-1) + 5 \rangle$ ,  
 $\vec{r}(t) = \langle t+1, -1/2t+5 \rangle$ ,  
 $\vec{r}(t) = \langle -2t+1, t+5 \rangle$  and  
 $\vec{r}(t) = \langle 2t+1, -t+5 \rangle$ .

27. Answers may vary, though most direct solution is  
 $\vec{r}(t) = \langle 3\cos(4\pi t), 3\sin(4\pi t), 3t \rangle$ .

29.  $\langle 1, 1 \rangle$

31.  $\langle 1, 2, 7 \rangle$

## Section 10.2

1. component

3. It is difficult to identify the points on the graphs of  $\vec{r}(t)$  and  $\vec{r}'(t)$  that correspond to each other.

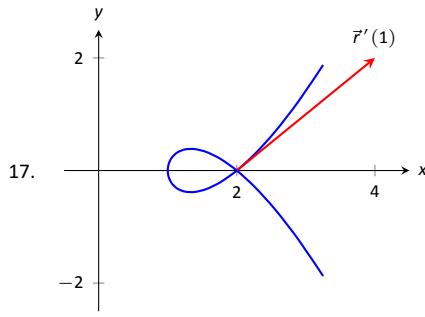
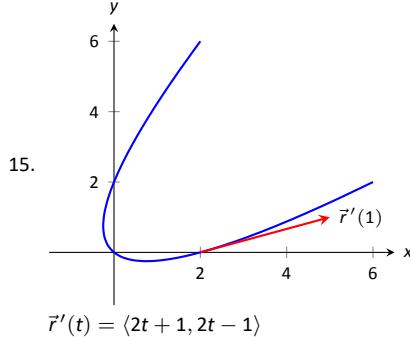
5.  $\langle e^3, 0 \rangle$

7.  $\langle 2t, 1, 0 \rangle$

9.  $\langle 0, \infty \rangle$

11.  $\vec{r}'(t) = \langle -1/t^2, 5/(3t+1)^2, \sec^2 t \rangle$

13.  $\vec{r}'(t) = \langle 2t, 1 \cdot \langle \sin t, 2t+5 \rangle + \langle t^2+1, t-1 \rangle \cdot \langle \cos t, 2 \rangle = (t^2+1) \cos t + 2t \sin t + 4t + 3$



19.  $\ell(t) = \langle 2, 0 \rangle + t \langle 3, 1 \rangle$

21.  $\ell(t) = \langle -3, 0, \pi \rangle + t \langle 0, -3, 1 \rangle$

23.  $t = 2n\pi$ , where  $n$  is an integer; so  
 $t = \dots - 4\pi, -2\pi, 0, 2\pi, 4\pi, \dots$

25.  $\vec{r}(t)$  is not smooth at  $t = 3\pi/4 + n\pi$ , where  $n$  is an integer

27. Both derivatives return  $\langle 5t^4, 4t^3 - 3t^2, 3t^2 \rangle$ .

29. Both derivatives return  
 $\langle 2t - e^t - 1, \cos t - 3t^2, (t^2 + 2t)e^t - (t-1)\cos t - \sin t \rangle$ .

31.  $\langle \tan^{-1} t, \tan t \rangle + \vec{C}$

33.  $\langle 4, -4 \rangle$

35.  $\vec{r}(t) = \langle \ln |t+1| + 1, -\ln |\cos t| + 2 \rangle$

37.  $\vec{r}(t) = \langle -\cos t + 1, t - \sin t, e^t - t - 1 \rangle$

39.  $10\pi$

41.  $\sqrt{2}(1 - e^{-1})$

## Section 10.3

1. Velocity is a vector, indicating an object's direction of travel and its rate of distance change (i.e., its speed). Speed is a scalar.

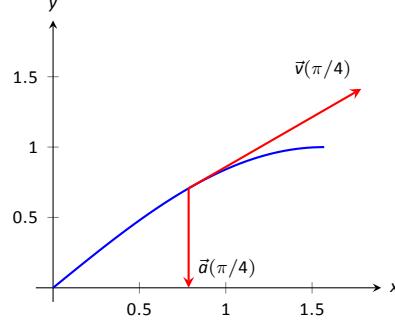
3. The average velocity is found by dividing the displacement by the time traveled – it is a vector. The average speed is found by dividing the distance traveled by the time traveled – it is a scalar.

5. One example is traveling at a constant speed  $s$  in a circle, ending at the starting position. Since the displacement is  $\vec{0}$ , the average velocity is  $\vec{0}$ , hence  $\|\vec{0}\| = 0$ . But traveling at constant speed  $s$  means the average speed is also  $s > 0$ .

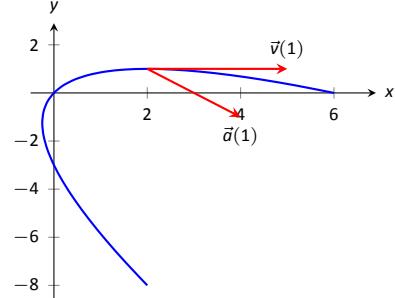
7.  $\vec{v}(t) = \langle 2, 5, 0 \rangle, \vec{a}(t) = \langle 0, 0, 0 \rangle$

9.  $\vec{v}(t) = \langle -\sin t, \cos t \rangle, \vec{a}(t) = \langle -\cos t, -\sin t \rangle$

11.  $\vec{v}(t) = \langle 1, \cos t \rangle, \vec{a}(t) = \langle 0, -\sin t \rangle$



13.  $\vec{v}(t) = \langle 2t+1, -2t+2 \rangle, \vec{a}(t) = \langle 2, -2 \rangle$



15.  $\|\vec{v}(t)\| = \sqrt{4t^2 + 1}$ .

Min at  $t = 0$ ; Max at  $t = \pm 1$ .

17.  $\|\vec{v}(t)\| = 5$ .

Speed is constant, so there is no difference between min/max

19.  $\|\vec{v}(t)\| = |\sec t| \sqrt{\tan^2 t + \sec^2 t}$ .  
min:  $t = 0$ ; max:  $t = \pi/4$

21.  $\|\vec{v}(t)\| = 13$ .

speed is constant, so there is no difference between min/max

23.  $\|\vec{v}(t)\| = \sqrt{4t^2 + 1 + t^2/(1-t^2)}$ .  
min:  $t = 0$ ; max: there is no max; speed approaches  $\infty$  as  $t \rightarrow \pm 1$

25. (a)  $\vec{r}_1(1) = \langle 1, 1 \rangle; \vec{r}_2(1) = \langle 1, 1 \rangle$

(b)  $\vec{v}_1(1) = \langle 1, 2 \rangle; \|\vec{v}_1(1)\| = \sqrt{5}; \vec{a}_1(1) = \langle 0, 2 \rangle$

$\vec{v}_2(1) = \langle 2, 4 \rangle; \|\vec{v}_2(1)\| = 2\sqrt{5}; \vec{a}_2(1) = \langle 2, 12 \rangle$

27. (a)  $\vec{r}_1(2) = \langle 6, 4 \rangle; \vec{r}_2(2) = \langle 6, 4 \rangle$

- (b)  $\vec{v}_1(2) = \langle 3, 2 \rangle$ ;  $\|\vec{v}_1(2)\| = \sqrt{13}$ ;  $\vec{a}_1(2) = \langle 0, 0 \rangle$   
 $\vec{v}_2(2) = \langle 6, 4 \rangle$ ;  $\|\vec{v}_2(2)\| = 2\sqrt{13}$ ;  $\vec{a}_2(2) = \langle 0, 0 \rangle$
29.  $\vec{v}(t) = \langle 2t+1, 3t+2 \rangle$ ,  $\vec{r}(t) = \langle t^2+t+5, 3t^2/2+2t-2 \rangle$
31.  $\vec{v}(t) = \langle \sin t, \cos t \rangle$ ,  $\vec{r}(t) = \langle 1-\cos t, \sin t \rangle$
33. Displacement:  $\langle 0, 0, 6\pi \rangle$ ; distance traveled:  $2\sqrt{13}\pi \approx 22.65\text{ft}$ ; average velocity:  $\langle 0, 0, 3 \rangle$ ; average speed:  $\sqrt{13} \approx 3.61\text{ft/s}$
35. Displacement:  $\langle 0, 0 \rangle$ ; distance traveled:  $2\pi \approx 6.28\text{ft}$ ; average velocity:  $\langle 0, 0 \rangle$ ; average speed:  $1\text{ft/s}$
37. At  $t$ -values of  $\sin^{-1}(9/30)/(4\pi) + n/2 \approx 0.024 + n/2$  seconds, where  $n$  is an integer.
39. (a) Holding the crossbow at an angle of 0.013 radians,  $\approx 0.745^\circ$  will hit the target 0.4s later. (Another solution exists, with an angle of  $89^\circ$ , landing 18.75s later, but this is impractical.)
- (b) In the .4 seconds the arrow travels, a deer, traveling at 20mph or  $29.33\text{ft/s}$ , can travel 11.7ft. So she needs to lead the deer by 11.7ft.
41. The position function is  $\vec{r}(t) = \langle 220t, -16t^2 + 1000 \rangle$ . The  $y$ -component is 0 when  $t = 7.9$ ;  $\vec{r}(7.9) = \langle 1739.25, 0 \rangle$ , meaning the box will travel about 1740ft horizontally before it lands.

## Section 10.4

1. 1

3.  $\vec{r}(t)$  and  $\vec{N}(t)$ .

$$5. \vec{r}(t) = \left\langle \frac{4t}{\sqrt{20t^2-4t+1}}, \frac{2t-1}{\sqrt{20t^2-4t+1}} \right\rangle; \vec{r}(1) = \langle 4/\sqrt{17}, 1/\sqrt{17} \rangle$$

$$7. \vec{r}(t) = \frac{\cos t \sin t}{\sqrt{\cos^2 t \sin^2 t}} \langle -\cos t, \sin t \rangle. (\text{Be careful; this cannot be simplified as just } \langle -\cos t, \sin t \rangle \text{ as } \sqrt{\cos^2 t \sin^2 t} \neq \cos t \sin t, \text{ but rather } |\cos t \sin t|.) \vec{r}(\pi/4) = \langle -\sqrt{2}/2, \sqrt{2}/2 \rangle$$

$$9. \ell(t) = \langle 2, 0 \rangle + t \langle 4/\sqrt{17}, 1/\sqrt{17} \rangle; \text{ in parametric form,}$$

$$\ell(t) = \begin{cases} x &= 2 + 4t/\sqrt{17} \\ y &= t/\sqrt{17} \end{cases}$$

$$11. \ell(t) = \langle \sqrt{2}/4, \sqrt{2}/4 \rangle + t \langle -\sqrt{2}/2, \sqrt{2}/2 \rangle; \text{ in parametric form,}$$

$$\ell(t) = \begin{cases} x &= \sqrt{2}/4 - \sqrt{2}t/2 \\ y &= \sqrt{2}/4 + \sqrt{2}t/2 \end{cases}$$

$$13. \vec{r}(t) = \langle -\sin t, \cos t \rangle; \vec{N}(t) = \langle -\cos t, -\sin t \rangle$$

$$15. \vec{r}(t) = \left\langle -\frac{\sin t}{\sqrt{4\cos^2 t + \sin^2 t}}, \frac{2\cos t}{\sqrt{4\cos^2 t + \sin^2 t}} \right\rangle; \\ \vec{N}(t) = \left\langle -\frac{2\cos t}{\sqrt{4\cos^2 t + \sin^2 t}}, -\frac{\sin t}{\sqrt{4\cos^2 t + \sin^2 t}} \right\rangle$$

17. (a) Be sure to show work

$$(b) \vec{N}(\pi/4) = \langle -5/\sqrt{34}, -3/\sqrt{34} \rangle$$

19. (a) Be sure to show work

$$(b) \vec{N}(0) = \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

$$21. \vec{r}(t) = \frac{1}{\sqrt{5}} \langle 2, \cos t, -\sin t \rangle; \vec{N}(t) = \langle 0, -\sin t, -\cos t \rangle$$

$$23. \vec{r}(t) = \frac{1}{\sqrt{a^2+b^2}} \langle -a \sin t, a \cos t, b \rangle; \vec{N}(t) = \langle -\cos t, -\sin t, 0 \rangle$$

$$25. a_T = \frac{4t}{\sqrt{1+4t^2}} \text{ and } a_N = \sqrt{4 - \frac{16t^2}{1+4t^2}}$$

At  $t = 0$ ,  $a_T = 0$  and  $a_N = 2$ ;

At  $t = 1$ ,  $a_T = 4/\sqrt{5}$  and  $a_N = 2/\sqrt{5}$ .

At  $t = 0$ , all acceleration comes in the form of changing the direction of velocity and not the speed; at  $t = 1$ , more acceleration comes in changing the speed than in changing direction.

27.  $a_T = 0$  and  $a_N = 2$   
At  $t = 0$ ,  $a_T = 0$  and  $a_N = 2$ ;  
At  $t = \pi/2$ ,  $a_T = 0$  and  $a_N = 2$ .  
The object moves at constant speed, so all acceleration comes from changing direction, hence  $a_T = 0$ .  $\vec{a}(t)$  is always parallel to  $\vec{N}(t)$ , but twice as long, hence  $a_N = 2$ .

29.  $a_T = 0$  and  $a_N = a$   
At  $t = 0$ ,  $a_T = 0$  and  $a_N = a$ ;  
At  $t = \pi/2$ ,  $a_T = 0$  and  $a_N = a$ .  
The object moves at constant speed, meaning that  $a_T$  is always 0. The object "rises" along the z-axis at a constant rate, so all acceleration comes in the form of changing direction circling the z-axis. The greater the radius of this circle the greater the acceleration, hence  $a_N = a$ .

## Section 10.5

1. time and/or distance

3. Answers may include lines, circles, helixes

5.  $\kappa$

$$7. s = 3t, \text{ so } \vec{r}(s) = \langle 2s/3, s/3, -2s/3 \rangle$$

$$9. s = \sqrt{13}t, \text{ so } \vec{r}(s) = \langle 3 \cos(s/\sqrt{13}), 3 \sin(s/\sqrt{13}), 2s/\sqrt{13} \rangle$$

$$11. \kappa = \frac{|6x|}{(1+(3x^2-1)^2)^{3/2}}; \\ \kappa(0) = 0, \kappa(1/2) = \frac{192}{17\sqrt{17}} \approx 2.74.$$

$$13. \kappa = \frac{|\cos x|}{(1+\sin^2 x)^{3/2}}; \\ \kappa(0) = 1, \kappa(\pi/2) = 0$$

$$15. \kappa = \frac{|2 \cos t \cos(2t) + 4 \sin t \sin(2t)|}{(4 \cos^2(2t) + \sin^2 t)^{3/2}}; \\ \kappa(0) = 1/4, \kappa(\pi/4) = 8$$

$$17. \kappa = \frac{|6t^2+2|}{(4t^2+(3t^2-1)^2)^{3/2}}; \\ \kappa(0) = 2, \kappa(5) = \frac{19}{1394\sqrt{1394}} \approx 0.0004$$

$$19. \kappa = 0; \\ \kappa(0) = 0, \kappa(1) = 0$$

$$21. \kappa = \frac{3}{13}; \\ \kappa(0) = 3/13, \kappa(\pi/2) = 3/13$$

$$23. \text{ maximized at } x = \pm \frac{\sqrt{2}}{\sqrt[4]{5}}$$

$$25. \text{ maximized at } t = 1/4$$

$$27. \text{ radius of curvature is } 5\sqrt{5}/4.$$

$$29. \text{ radius of curvature is } 9.$$

$$31. x^2 + (y-1/2)^2 = 1/4, \text{ or } \vec{c}(t) = \langle 1/2 \cos t, 1/2 \sin t + 1/2 \rangle$$

$$33. x^2 + (y+8)^2 = 81, \text{ or } \vec{c}(t) = \langle 9 \cos t, 9 \sin t - 8 \rangle$$

## Chapter 11

### Section 11.1

1. Answers will vary.

3. topographical

5. surface

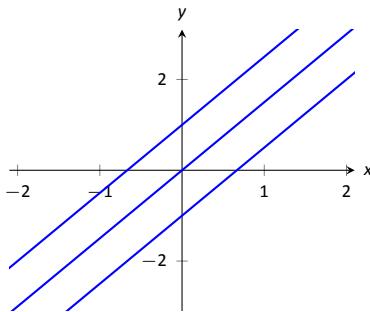
7. domain:  $\mathbb{R}^2$   
range:  $z \geq 2$

9. domain:  $\mathbb{R}^2$   
range:  $\mathbb{R}$

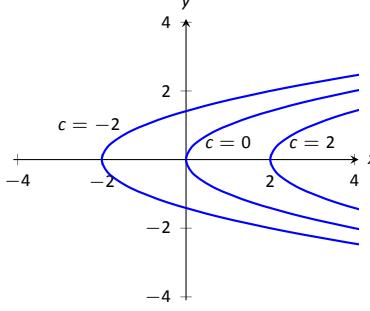
11. domain:  $\mathbb{R}^2$   
range:  $0 < z \leq 1$

13. domain:  $\{(x, y) \mid x^2 + y^2 \leq 9\}$ , i.e., the domain is the circle and interior of a circle centered at the origin with radius 3.  
range:  $0 \leq z \leq 3$

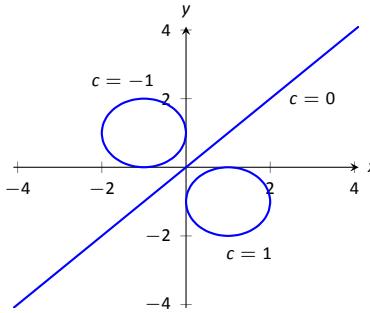
15. Level curves are lines  $y = (3/2)x - c/2$ .



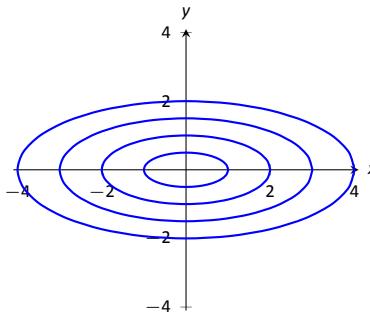
17. Level curves are parabolas  $x = y^2 + c$ .



19. When  $c \neq 0$ , the level curves are circles, centered at  $(1/c, -1/c)$  with radius  $\sqrt{2/c^2 - 1}$ . When  $c = 0$ , the level curve is the line  $y = x$ .



21. Level curves are ellipses of the form  $\frac{x^2}{c^2} + \frac{y^2}{c^2/4} = 1$ , i.e.,  $a = c$  and  $b = c/2$ .



23. domain:  $x + 2y - 4z \neq 0$ ; the set of points in  $\mathbb{R}^3$  NOT in the domain form a plane through the origin.  
range:  $\mathbb{R}$

25. domain:  $z \geq x^2 - y^2$ ; the set of points in  $\mathbb{R}^3$  above (and including) the hyperbolic paraboloid  $z = x^2 - y^2$ .

range:  $[0, \infty)$

27. The level surfaces are spheres, centered at the origin, with radius  $\sqrt{c}$ .

29. The level surfaces are paraboloids of the form  $z = \frac{x^2}{c} + \frac{y^2}{c}$ ; the larger  $c$ , the "wider" the paraboloid.

31. The level curves for each surface are similar; for  $z = \sqrt{x^2 + 4y^2}$  the level curves are ellipses of the form  $\frac{x^2}{c^2} + \frac{y^2}{c^2/4} = 1$ , i.e.,  $a = c$  and  $b = c/2$ ; whereas for  $z = x^2 + 4y^2$  the level curves are ellipses of the form  $\frac{x^2}{c} + \frac{y^2}{c/4} = 1$ , i.e.,  $a = \sqrt{c}$  and  $b = \sqrt{c}/2$ . The first set of ellipses are spaced evenly apart, meaning the function grows at a constant rate; the second set of ellipses are more closely spaced together as  $c$  grows, meaning the function grows faster and faster as  $c$  increases.

The function  $z = \sqrt{x^2 + 4y^2}$  can be rewritten as  $z^2 = x^2 + 4y^2$ , an elliptic cone; the function  $z = x^2 + 4y^2$  is a paraboloid, each matching the description above.

## Section 11.2

1. Answers will vary.

3. Answers will vary.

One possible answer:  $\{(x, y) \mid x^2 + y^2 \leq 1\}$

5. Answers will vary.

One possible answer:  $\{(x, y) \mid x^2 + y^2 < 1\}$

7. (a) Answers will vary.

interior point:  $(1, 3)$

boundary point:  $(3, 3)$

- (b)  $S$  is a closed set

- (c)  $S$  is bounded

9. (a) Answers will vary.

interior point: none

boundary point:  $(0, -1)$

- (b)  $S$  is a closed set, consisting only of boundary points

- (c)  $S$  is bounded

11. (a)  $D = \{(x, y) \mid 9 - x^2 - y^2 \geq 0\}$ .

- (b)  $D$  is a closed set.

- (c)  $D$  is bounded.

13. (a)  $D = \{(x, y) \mid y > x^2\}$ .

- (b)  $D$  is an open set.

- (c)  $D$  is unbounded.

15. (a) Along  $y = 0$ , the limit is 1.

- (b) Along  $x = 0$ , the limit is -1.

Since the above limits are not equal, the limit does not exist.

17. (a) Along  $y = mx$ , the limit is  $\frac{mx(1-m)}{m^2x+1}$ .

- (b) Along  $x = 0$ , the limit is -1.

Since the above limits are not equal, the limit does not exist.

19. (a) Along  $y = 2$ , the limit is:

$$\begin{aligned} \lim_{(x,y) \rightarrow (1,2)} \frac{x+y-3}{x^2-1} &= \lim_{x \rightarrow 1} \frac{x-1}{x^2-1} \\ &= \lim_{x \rightarrow 1} \frac{1}{x+1} \\ &= 1/2. \end{aligned}$$

(b) Along  $y = x + 1$ , the limit is:

$$\begin{aligned}\lim_{(x,y) \rightarrow (1,2)} \frac{x+y-3}{x^2-1} &= \lim_{x \rightarrow 1} \frac{2(x-1)}{x^2-1} \\ &= \lim_{x \rightarrow 1} \frac{2}{x+1} \\ &= 1.\end{aligned}$$

Since the limits along the lines  $y = 2$  and  $y = x + 1$  differ, the overall limit does not exist.

### Section 11.3

1. A constant is a number that is added or subtracted in an expression; a coefficient is a number that is being multiplied by a nonconstant function.

3.  $f_x$

5.  $f_x = 2xy - 1, f_y = x^2 + 2$   
 $f_x(1,2) = 3, f_y(1,2) = 3$

7.  $f_x = -\sin x \sin y, f_y = \cos x \cos y$   
 $f_x(\pi/3, \pi/3) = -3/4, f_y(\pi/3, \pi/3) = 1/4$

9.  $f_x = 2xy + 6x, f_y = x^2 + 4$   
 $f_{xx} = 2y + 6, f_{yy} = 0$   
 $f_{xy} = 2x, f_{yx} = 2x$

11.  $f_x = 1/y, f_y = -x/y^2$   
 $f_{xx} = 0, f_{yy} = 2x/y^3$   
 $f_{xy} = -1/y^2, f_{yx} = -1/y^2$

13.  $f_x = 2xe^{x^2+y^2}, f_y = 2ye^{x^2+y^2}$   
 $f_{xx} = 2e^{x^2+y^2} + 4x^2e^{x^2+y^2}, f_{yy} = 2e^{x^2+y^2} + 4y^2e^{x^2+y^2}$   
 $f_{xy} = 4xye^{x^2+y^2}, f_{yx} = 4xye^{x^2+y^2}$

15.  $f_x = \cos x \cos y, f_y = -\sin x \sin y$   
 $f_{xx} = -\sin x \cos y, f_{yy} = -\sin x \cos y$   
 $f_{xy} = -\sin y \cos x, f_{yx} = -\sin y \cos x$

17.  $f_x = -5y^3 \sin(5xy^3), f_y = -15xy^2 \sin(5xy^3)$   
 $f_{xx} = -25y^6 \cos(5xy^3),$   
 $f_{yy} = -225x^2y^4 \cos(5xy^3) - 30xy \sin(5xy^3)$   
 $f_{xy} = -75xy^5 \cos(5xy^3) - 15y^2 \sin(5xy^3),$   
 $f_{yx} = -75xy^5 \cos(5xy^3) - 15y^2 \sin(5xy^3)$

19.  $f_x = \frac{2y^2}{\sqrt{4xy^2+1}}, f_y = \frac{4xy}{\sqrt{4xy^2+1}}$   
 $f_{xx} = -\frac{4y^4}{\sqrt{4xy^2+1}^3}, f_{yy} = -\frac{16x^2y^2}{\sqrt{4xy^2+1}^3} + \frac{4x}{\sqrt{4xy^2+1}}$   
 $f_{xy} = -\frac{8xy^3}{\sqrt{4xy^2+1}^3} + \frac{4y}{\sqrt{4xy^2+1}}, f_{yx} = -\frac{8xy^3}{\sqrt{4xy^2+1}^3} + \frac{4y}{\sqrt{4xy^2+1}}$

21.  $f_x = -\frac{2x}{(x^2+y^2+1)^2}, f_y = -\frac{2y}{(x^2+y^2+1)^2}$   
 $f_{xx} = \frac{8x^2}{(x^2+y^2+1)^3} - \frac{2}{(x^2+y^2+1)^2}, f_{yy} = \frac{8y^2}{(x^2+y^2+1)^3} - \frac{2}{(x^2+y^2+1)^2}$   
 $f_{xy} = \frac{8xy}{(x^2+y^2+1)^3}, f_{yx} = \frac{8xy}{(x^2+y^2+1)^3}$

23.  $f_x = 6x, f_y = 0$   
 $f_{xx} = 6, f_{yy} = 0$   
 $f_{xy} = 0, f_{yx} = 0$

25.  $f_x = \frac{1}{4xy}, f_y = -\frac{\ln x}{4y^2}$   
 $f_{xx} = -\frac{1}{4x^2y}, f_{yy} = \frac{\ln x}{2y^3}$   
 $f_{xy} = -\frac{1}{4xy^2}, f_{yx} = -\frac{1}{4xy^2}$

27.  $f(x, y) = x \sin y + x + C$ , where  $C$  is any constant.

29.  $f(x, y) = 3x^2y - 4xy^2 + 2y + C$ , where  $C$  is any constant.

31.  $f_x = 2xe^{2y-3z}, f_y = 2x^2e^{2y-3z}, f_z = -3x^2e^{2y-3z}$   
 $f_{yz} = -6x^2e^{2y-3z}, f_{zy} = -6x^2e^{2y-3z}$

33.  $f_x = \frac{3}{7y^2z}, f_y = -\frac{6x}{7y^3z}, f_z = -\frac{3x}{7y^2z^2}$   
 $f_{yz} = \frac{6x}{7y^3z^2}, f_{zy} = \frac{6x}{7y^3z^2}$

### Section 11.4

1. T
3. T
5.  $dz = (\sin y + 2x)dx + (x \cos y)dy$
7.  $dz = 5dx - 7dy$
9.  $dz = \frac{x}{\sqrt{x^2+y}} dx + \frac{1}{2\sqrt{x^2+y}} dy$ , with  $dx = -0.05$  and  $dy = .1$ . At  $(3, 7)$ ,  $dz = 3/4(-0.05) + 1/8(.1) = -0.025$ , so  $f(2.95, 7.1) \approx -0.025 + 4 = 3.975$ .
11.  $dz = (2xy - y^2)dx + (x^2 - 2xy)dy$ , with  $dx = 0.04$  and  $dy = 0.06$ . At  $(2, 3)$ ,  $dz = 3(0.04) + (-8)(0.06) = -0.36$ , so  $f(2.04, 3.06) \approx -0.36 - 6 = -6.36$ .
13. The total differential of volume is  $dV = 4\pi dr + \pi dh$ . The coefficient of  $dr$  is greater than the coefficient of  $dh$ , so the volume is more sensitive to changes in the radius.
15. Using trigonometry,  $\ell = x \tan \theta$ , so  $d\ell = \tan \theta dx + x \sec^2 \theta d\theta$ . With  $\theta = 85^\circ$  and  $x = 30$ , we have  $d\ell = 11.43dx + 3949.38d\theta$ . The measured length of the wall is much more sensitive to errors in  $\theta$  than in  $x$ . While it can be difficult to compare sensitivities between measuring feet and measuring degrees (it is somewhat like “comparing apples to oranges”), here the coefficients are so different that the result is clear: a small error in degree has a much greater impact than a small error in distance.
17.  $dw = 2xyz^3 dx + x^2z^3 dy + 3x^2yz^2 dz$
19.  $dx = 0.05, dy = -0.1, dz = 9(.05) + (-2)(-0.1) = 0.65$ . So  $f(3.05, 0.9) \approx 7 + 0.65 = 7.65$ .
21.  $dx = 0.5, dy = 0.1, dz = -0.2$ .  
 $dw = 2(0.5) + (-3)(0.1) + 3.7(-0.2) = -0.04$ , so  $f(2.5, 4.1, 4.8) \approx -1 - 0.04 = -1.04$ .

### Section 11.5

1. Because the parametric equations describe a level curve,  $z$  is constant for all  $t$ . Therefore  $\frac{dz}{dt} = 0$ .
3.  $\frac{dx}{dt}$ , and  $\frac{\partial f}{\partial y}$
5. F
7. (a)  $\frac{dz}{dt} = 3(2t) + 4(2) = 6t + 8$ .  
(b) At  $t = 1, \frac{dz}{dt} = 14$ .
9. (a)  $\frac{dz}{dt} = 5(-2 \sin t) + 2(\cos t) = -10 \sin t + 2 \cos t$ .  
(b) At  $t = \pi/4, \frac{dz}{dt} = -4\sqrt{2}$ .
11. (a)  $\frac{dz}{dt} = 2x(\cos t) + 4y(3 \cos t)$ .  
(b) At  $t = \pi/4, x = \sqrt{2}/2, y = 3\sqrt{2}/2$ , and  $\frac{dz}{dt} = 19$ .
13.  $t = -4/3$ ; this corresponds to a minimum
15.  $t = \tan^{-1}(1/5) + n\pi$ , where  $n$  is an integer
17. We find that

$$\frac{dz}{dt} = 38 \cos t \sin t.$$

Thus  $\frac{dz}{dt} = 0$  when  $t = \pi n$  or  $\pi n + \pi/2$ , where  $n$  is any integer.

19. (a)  $\frac{\partial z}{\partial s} = 2xy(1) + x^2(2) = 2xy + 2x^2$ ;  
 $\frac{\partial z}{\partial t} = 2xy(-1) + x^2(4) = -2xy + 4x^2$
- (b) With  $s = 1, t = 0, x = 1$  and  $y = 2$ . Thus  $\frac{\partial z}{\partial s} = 6$  and  $\frac{\partial z}{\partial t} = 0$
21. (a)  $\frac{\partial z}{\partial s} = 2x(\cos t) + 2y(\sin t) = 2x \cos t + 2y \sin t$ ;  
 $\frac{\partial z}{\partial t} = 2x(-s \sin t) + 2y(s \cos t) = -2xs \sin t + 2ys \cos t$
- (b) With  $s = 2, t = \pi/4, x = \sqrt{2}$  and  $y = \sqrt{2}$ . Thus  $\frac{\partial z}{\partial s} = 4$  and  $\frac{\partial z}{\partial t} = 0$

23.  $f_x = 2x \tan y, f_y = x^2 \sec^2 y;$   
 $\frac{dy}{dx} = -\frac{2 \tan y}{x \sec^2 y}$

25.  $f_x = \frac{(x+y^2)(2x) - (x^2+y)(1)}{(x+y^2)^2},$   
 $f_y = \frac{(x+y^2)(1) - (x^2+y)(2y)}{(x+y^2)^2};$   
 $\frac{dy}{dx} = -\frac{2x(x+y^2) - (x^2+y)}{x+y^2 - 2y(x^2+y)}$

27.  $\frac{dz}{dt} = 2(4) + 1(-5) = 3.$

29.  $\frac{\partial z}{\partial s} = -4(5) + 9(-2) = -38,$   
 $\frac{\partial z}{\partial t} = -4(7) + 9(6) = 26.$

### Section 11.6

1. A partial derivative is essentially a special case of a directional derivative; it is the directional derivative in the direction of  $x$  or  $y$ , i.e.,  $\langle 1, 0 \rangle$  or  $\langle 0, 1 \rangle$ .

3.  $\vec{u} = \langle 0, 1 \rangle$

5. maximal, or greatest

7.  $\nabla f = \langle -2xy + y^2 + y, -x^2 + 2xy + x \rangle$

9.  $\nabla f = \left\langle \frac{-2x}{(x^2+y^2+1)^2}, \frac{-2y}{(x^2+y^2+1)^2} \right\rangle$

11.  $\nabla f = \langle 2x - y - 7, 4y - x \rangle$

13.  $\nabla f = \langle -2xy + y^2 + y, -x^2 + 2xy + x \rangle; \nabla f(2, 1) = \langle -2, 2 \rangle.$  Be sure to change all directions to unit vectors.

(a)  $2/5 (\vec{u} = \langle 3/5, 4/5 \rangle)$

(b)  $-2\sqrt{5} (\vec{u} = \langle -1/\sqrt{5}, -2\sqrt{5} \rangle)$

15.  $\nabla f = \left\langle \frac{-2x}{(x^2+y^2+1)^2}, \frac{-2y}{(x^2+y^2+1)^2} \right\rangle; \nabla f(1, 1) = \langle -2/9, -2/9 \rangle.$  Be sure to change all directions to unit vectors.

(a)  $0 (\vec{u} = \langle 1/\sqrt{2}, -1/\sqrt{2} \rangle)$

(b)  $2\sqrt{2}/9 (\vec{u} = \langle -1/\sqrt{2}, -1/\sqrt{2} \rangle)$

17.  $\nabla f = \langle 2x - y - 7, 4y - x \rangle; \nabla f(4, 1) = \langle 0, 0 \rangle.$

(a) 0

(b) 0

19.  $\nabla f = \langle -2xy + y^2 + y, -x^2 + 2xy + x \rangle$

(a)  $\nabla f(2, 1) = \langle -2, 2 \rangle$

(b)  $\|\nabla f(2, 1)\| = \|\langle -2, 2 \rangle\| = \sqrt{8}$

(c)  $\langle 2, -2 \rangle$

(d)  $\langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$

21.  $\nabla f = \left\langle \frac{-2x}{(x^2+y^2+1)^2}, \frac{-2y}{(x^2+y^2+1)^2} \right\rangle$

(a)  $\nabla f(1, 1) = \langle -2/9, -2/9 \rangle.$

(b)  $\|\nabla f(1, 1)\| = \|\langle -2/9, -2/9 \rangle\| = 2\sqrt{2}/9$

(c)  $\langle 2/9, 2/9 \rangle$

(d)  $\langle 1/\sqrt{2}, -1/\sqrt{2} \rangle$

23.  $\nabla f = \langle 2x - y - 7, 4y - x \rangle$

(a)  $\nabla f(4, 1) = \langle 0, 0 \rangle$

(b) 0

(c)  $\langle 0, 0 \rangle$

(d) All directions give a directional derivative of 0.

25. (a)  $\nabla F(x, y, z) = \langle 6xz^3 + 4y, 4x, 9x^2z^2 - 6z \rangle$

(b)  $113/\sqrt{3}$

27. (a)  $\nabla F(x, y, z) = \langle 2xy^2, 2y(x^2 - z^2), -2y^2z \rangle$   
(b) 0

### Section 11.7

1. Answers will vary. The displacement of the vector is one unit in the  $x$ -direction and 3 units in the  $z$ -direction, with no change in  $y$ . Thus along a line parallel to  $\vec{v}$ , the change in  $z$  is 3 times the change in  $x$  – i.e., a “slope” of 3. Specifically, the line in the  $x$ - $z$  plane parallel to  $z$  has a slope of 3.

3. T

5. (a)  $\ell_x(t) = \begin{cases} x = 2 + t \\ y = 3 \\ z = -48 - 12t \end{cases}$

(b)  $\ell_y(t) = \begin{cases} x = 2 \\ y = 3 + t \\ z = -48 - 40t \end{cases}$

(c)  $\ell_{\vec{u}}(t) = \begin{cases} x = 2 + t/\sqrt{10} \\ y = 3 + 3t/\sqrt{10} \\ z = -48 - 66\sqrt{2/5}t \end{cases}$

7. (a)  $\ell_x(t) = \begin{cases} x = 4 + t \\ y = 2 \\ z = 2 + 3t \end{cases}$

(b)  $\ell_y(t) = \begin{cases} x = 4 \\ y = 2 + t \\ z = 2 - 5t \end{cases}$

(c)  $\ell_{\vec{u}}(t) = \begin{cases} x = 4 + t/\sqrt{2} \\ y = 2 + t/\sqrt{2} \\ z = 2 - \sqrt{2}t \end{cases}$

9.  $\ell_{\vec{n}}(t) = \begin{cases} x = 2 - 12t \\ y = 3 - 40t \\ z = -48 - t \end{cases}$

11.  $\ell_{\vec{n}}(t) = \begin{cases} x = 4 + 3t \\ y = 2 - 5t \\ z = 2 - t \end{cases}$

13.  $(1.425, 1.085, -48.078), (2.575, 4.915, -47.952)$

15.  $(5.014, 0.31, 1.662)$  and  $(2.986, 3.690, 2.338)$

17.  $-12(x - 2) - 40(y - 3) - (z + 48) = 0$

19.  $3(x - 4) - 5(y - 2) - (z - 2) = 0$  (Note that this tangent plane is the same as the original function, a plane.)

21.  $\nabla F = \langle x/4, y/2, z/8 \rangle;$  at  $P, \nabla F = \langle 1/4, \sqrt{2}/2, \sqrt{6}/8 \rangle$

(a)  $\ell_{\vec{n}}(t) = \begin{cases} x = 1 + t/4 \\ y = \sqrt{2} + \sqrt{2}t/2 \\ z = \sqrt{6} + \sqrt{6}t/8 \end{cases}$

(b)  $\frac{1}{4}(x - 1) + \frac{\sqrt{2}}{2}(y - \sqrt{2}) + \frac{\sqrt{6}}{8}(z - \sqrt{6}) = 0.$

23.  $\nabla F = \langle y^2 - z^2, 2xy, -2xz \rangle;$  at  $P, \nabla F = \langle 0, 4, 4 \rangle$

(a)  $\ell_{\vec{n}}(t) = \begin{cases} x = 2 \\ y = 1 + 4t \\ z = -1 + 4t \end{cases}$

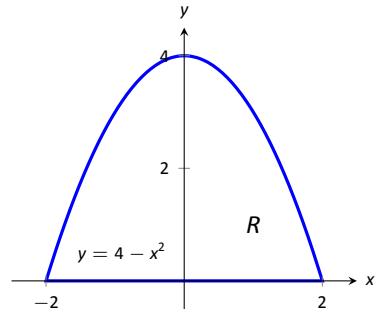
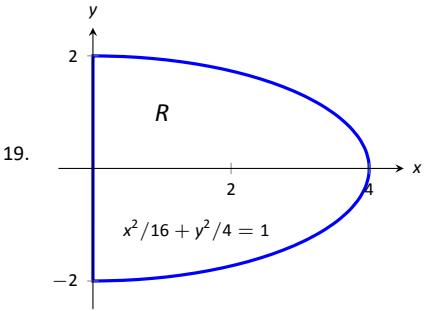
(b)  $4(y - 1) + 4(z + 1) = 0.$

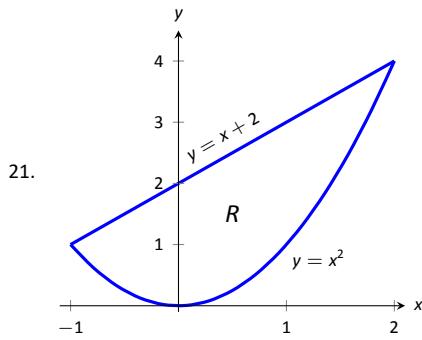
### Section 11.8

1. F; it is the “other way around.”

3. T

5. One critical point at  $(-4, 2); f_{xx} = 1$  and  $D = 4$ , so this point corresponds to a relative minimum.

7. One critical point at  $(6, -3)$ ;  $D = -4$ , so this point corresponds to a saddle point.
9. Two critical points: at  $(0, -1)$ ;  $f_{xx} = 2$  and  $D = -12$ , so this point corresponds to a saddle point; at  $(0, 1)$ ,  $f_{xx} = 2$  and  $D = 12$ , so this corresponds to a relative minimum.
11. One critical point at  $(0, 0)$ .  $D = -12x^2y^2$ , so at  $(0, 0)$ ,  $D = 0$  and the test is inconclusive. (Some elementary thought shows that it is the absolute minimum.)
13. One critical point:  $f_x = 0$  when  $x = 3$ ;  $f_y = 0$  when  $y = 0$ , so one critical point at  $(3, 0)$ , which is a relative maximum, where  $f_{xx} = \frac{y^2 - 16}{(16 - (x-3)^2 - y^2)^{3/2}}$  and  $D = \frac{16}{(16 - (x-3)^2 - y^2)^2}$ . Both  $f_x$  and  $f_y$  are undefined along the circle  $(x - 3)^2 + y^2 = 16$ ; at any point along this curve,  $f(x, y) = 0$ , the absolute minimum of the function.
15. The triangle is bound by the lines  $y = -1$ ,  $y = 2x + 1$  and  $y = -2x + 1$ .  
 Along  $y = -1$ , there is a critical point at  $(0, -1)$ .  
 Along  $y = 2x + 1$ , there is a critical point at  $(-3/5, -1/5)$ .  
 Along  $y = -2x + 1$ , there is a critical point at  $(3/5, -1/5)$ .  
 The function  $f$  has one critical point, irrespective of the constraint, at  $(0, -1/2)$ .  
 Checking the value of  $f$  at these four points, along with the three vertices of the triangle, we find the absolute maximum is at  $(0, 1, 3)$  and the absolute minimum is at  $(0, -1/2, 3/4)$ .
17. The region has no “corners” or “vertices,” just a smooth edge.  
 To find critical points along the circle  $x^2 + y^2 = 4$ , we solve for  $y^2$ :  $y^2 = 4 - x^2$ . We can go further and state  $y = \pm\sqrt{4 - x^2}$ .  
 We can rewrite  $f$  as  $f(x) = x^2 + 2x + (4 - x^2) + \sqrt{4 - x^2} = 2x + 4 + \sqrt{4 - x^2}$ . (We will return and use  $-\sqrt{4 - x^2}$  later.) Solving  $f'(x) = 0$ , we get  $x = \sqrt{2} \Rightarrow y = \sqrt{2}$ .  $f'(x)$  is also undefined at  $x = \pm 2$ , where  $y = 0$ .  
 Using  $y = -\sqrt{4 - x^2}$ , we rewrite  $f(x, y)$  as  $f(x) = 2x + 4 - \sqrt{4 - x^2}$ . Solving  $f'(x) = 0$ , we get  $x = -\sqrt{2}$ ,  $y = -\sqrt{2}$ .  
 The function  $f$  itself has a critical point at  $(-1, -1)$ .  
 Checking the value of  $f$  at  $(-1, -1)$ ,  $(\sqrt{2}, \sqrt{2})$ ,  $(-\sqrt{2}, -\sqrt{2})$ ,  $(2, 0)$  and  $(-2, 0)$ , we find the absolute maximum is at  $(2, 0, 8)$  and the absolute minimum is at  $(-1, -1, -2)$ .
- Section 11.9**
1. perpendicular or orthogonal
  3. T
  5.  $f_{\max} = \frac{2}{3\sqrt{3}}$  at  $\left(\pm\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}\right)$ , and  $f_{\min} = -\frac{2}{3\sqrt{3}}$  at  $\left(\pm\sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{3}}\right)$
  7.  $f_{\max} = \frac{\sqrt{82}}{3}$  at  $\left(\frac{9}{\sqrt{82}}, \frac{1}{3\sqrt{82}}\right)$ , and  $f_{\min} = -\frac{\sqrt{82}}{3}$  at  $\left(-\frac{9}{\sqrt{82}}, -\frac{1}{3\sqrt{82}}\right)$
  9.  $f_{\max} = 1$  at  $(1, 1)$
  11.  $f_{\min} = \frac{1}{2}$  at the point  $\left(\frac{1}{2}, 0, -\frac{1}{2}\right)$
- Chapter 12**
- Section 12.1**
1.  $C(y)$ , meaning that instead of being just a constant, like the number 5, it is a function of  $y$ , which acts like a constant when taking derivatives with respect to  $x$ .
3. curve to curve, then from point to point
5. (a)  $18x^2 + 42x - 117$   
 (b)  $-108$
7. (a)  $x^4/2 - x^2 + 2x - 3/2$   
 (b)  $23/15$
9. (a)  $\sin^2 y$   
 (b)  $\pi/2$
11.  $\int_1^4 \int_{-2}^1 dy dx$  and  $\int_{-2}^1 \int_1^4 dx dy$ .  
 area of  $R = 9u^2$
13.  $\int_2^4 \int_{x-1}^{7-x} dy dx$ . The order  $dx dy$  needs two iterated integrals as  $x$  is bounded above by two different functions. This gives:  
 $\int_1^3 \int_2^{y+1} dx dy + \int_3^5 \int_2^{7-y} dx dy$ .  
 area of  $R = 4u^2$
15.  $\int_0^1 \int_{x^4}^{\sqrt{x}} dy dx$  and  $\int_0^1 \int_{y^2}^{\sqrt[4]{y}} dx dy$   
 area of  $R = 7/15u^2$
17. 
- area of  $R = \int_0^4 \int_{-\sqrt{4-y}}^{\sqrt{4-y}} dx dy$
19. 
- area of  $R = \int_0^4 \int_{-\sqrt{4-x^2/4}}^{\sqrt{4-x^2/4}} dy dx$



$$\text{area of } R = \int_{-1}^2 \int_{x^2}^{x+2} dy dx$$

### Section 12.2

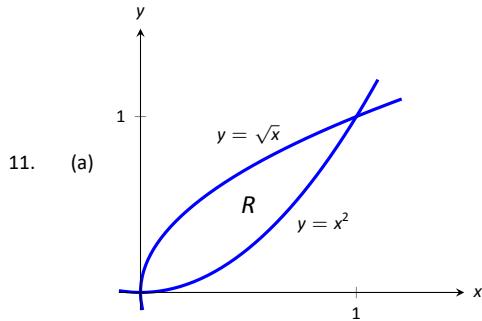
1. volume

3. The double integral gives the signed volume under the surface.  
Since the surface is always positive, it is always above the  $x$ - $y$  plane and hence produces only "positive" volume.

5. 6;  $\int_{-1}^1 \int_1^2 \left(\frac{x}{y} + 3\right) dy dx$

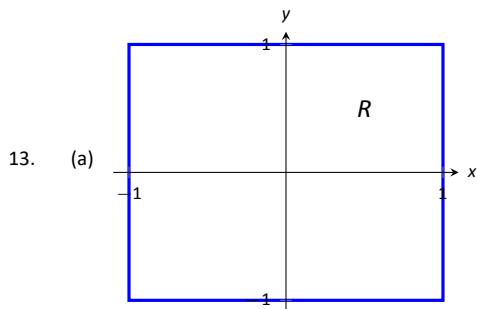
7.  $112/3$ ;  $\int_0^2 \int_0^{4-2y} (3x^2 - y + 2) dx dy$

9.  $16/5$ ;  $\int_{-1}^1 \int_0^{1-x^2} (x + y + 2) dy dx$



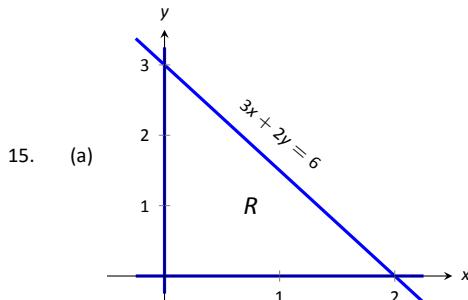
(b)  $\int_0^1 \int_{x^2}^{\sqrt{x}} x^2 y dy dx = \int_0^1 \int_{y^2}^{\sqrt{y}} x^2 y dx dy.$

(c)  $\frac{3}{56}$



(b)  $\int_{-1}^1 \int_{-1}^1 x^2 - y^2 dy dx = \int_{-1}^1 \int_{-1}^1 x^2 - y^2 dx dy.$

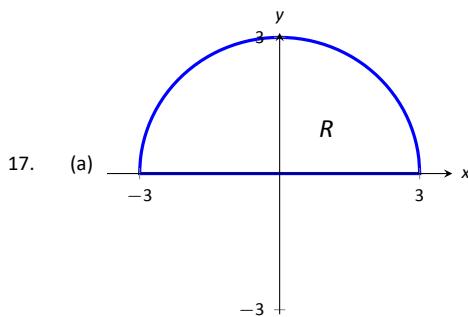
(c) 0



(b)

(c)  $\int_0^2 \int_0^{3-3/2x} (6 - 3x - 2y) dy dx = \int_0^3 \int_0^{2-2/3y} (6 - 3x - 2y) dx dy.$

(d) 6



(b)  $\int_{-3}^3 \int_0^{\sqrt{9-x^2}} (x^3 y - x) dy dx = \int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} (x^3 y - x) dx dy.$

(c) 0

19. Integrating  $e^{x^2}$  with respect to  $x$  is not possible in terms of elementary functions.  $\int_0^2 \int_0^{2x} e^{x^2} dy dx = e^4 - 1$ .

21. Integrating  $\int_y^1 \frac{2y}{x^2 + y^2} dx$  gives  $\tan^{-1}(1/y) - \pi/4$ ; integrating  $\tan^{-1}(1/y)$  is hard.

$$\int_0^1 \int_0^x \frac{2y}{x^2 + y^2} dy dx = \ln 2.$$

23. average value of  $f = 6/2 = 3$

25. average value of  $f = \frac{112/3}{4} = 28/3$

### Section 12.3

1.  $f(r \cos \theta, r \sin \theta), r dr d\theta$

3.  $\int_0^{2\pi} \int_0^1 (3r \cos \theta - r \sin \theta + 4) r dr d\theta = 4\pi$

5.  $\int_0^\pi \int_{\cos \theta}^{3 \cos \theta} (8 - r \sin \theta) r dr d\theta = 16\pi$

7.  $\int_0^{2\pi} \int_1^2 (\ln(r^2)) r dr d\theta = 2\pi(\ln 16 - 3/2)$

9.  $\int_{-\pi/2}^{\pi/2} \int_0^6 (r^2 \cos^2 \theta - r^2 \sin^2 \theta) r dr d\theta = \int_{-\pi/2}^{\pi/2} \int_0^6 (r^2 \cos(2\theta)) r dr d\theta = 0$

11.  $\int_{-\pi/2}^{\pi/2} \int_0^5 (r^2) dr d\theta = 125\pi/3$

13.  $\int_0^{\pi/4} \int_0^{\sqrt{8}} (r \cos \theta + r \sin \theta) r dr d\theta = 16\sqrt{2}/3$

15. (a) This is impossible to integrate with rectangular coordinates as  $e^{-(x^2+y^2)}$  does not have an antiderivative in terms of elementary functions.
- (b)  $\int_0^{2\pi} \int_0^a r e^{-r^2} dr d\theta = \pi(1 - e^{-a^2}).$
- (c)  $\lim_{a \rightarrow \infty} \pi(1 - e^{-a^2}) = \pi.$  This implies that there is a finite volume under the surface  $e^{-(x^2+y^2)}$  over the entire  $x$ - $y$  plane.

## Section 12.4

1. Because they are scalar multiples of each other.
3. "little masses"
5.  $M_x$  measures the moment about the  $x$ -axis, meaning we need to measure distance from the  $x$ -axis. Such measurements are measures in the  $y$ -direction.

7.  $\bar{x} = 5.25$
9.  $(\bar{x}, \bar{y}) = (0, 3)$
11.  $M = 150\text{gm};$
13.  $M = 2\text{lb}$
15.  $M = 16\pi \approx 50.27\text{kg}$
17.  $M = 54\pi \approx 169.65\text{lb}$
19.  $M = 150\text{gm}; M_y = 600; M_x = -75; (\bar{x}, \bar{y}) = (4, -0.5)$
21.  $M = 2\text{lb}; M_y = 0; M_x = 2/3; (\bar{x}, \bar{y}) = (0, 1/3)$
23.  $M = 16\pi \approx 50.27\text{kg}; M_y = 4\pi; M_x = 4\pi; (\bar{x}, \bar{y}) = (1/4, 1/4)$
25.  $M = 54\pi \approx 169.65\text{lb}; M_y = 0; M_x = 504; (\bar{x}, \bar{y}) = (0, 2.97)$
27.  $I_x = 64/3; I_y = 64/3; I_O = 128/3$
29.  $I_x = 16/3; I_y = 64/3; I_O = 80/3$

## Section 12.5

1. arc length
3. surface areas
5. Intuitively, adding  $h$  to  $f$  only shifts  $f$  up (i.e., parallel to the  $z$ -axis) and does not change its shape. Therefore it will not change the surface area over  $R$ .  
Analytically,  $f_x = g_x$  and  $f_y = g_y$ ; therefore, the surface area of each is computed with identical double integrals.
7.  $SA = \int_0^{2\pi} \int_0^1 \sqrt{1 + \cos^2 x \cos^2 y + \sin^2 x \sin^2 y} dx dy$
9.  $SA = \int_{-1}^1 \int_{-1}^1 \sqrt{1 + 4x^2 + 4y^2} dx dy$
11.  $SA = \int_0^3 \int_{-1}^1 \sqrt{1 + 9 + 49} dx dy = 6\sqrt{59} \approx 46.09$
13. This is easier in polar:

$$\begin{aligned} SA &= \int_0^{2\pi} \int_0^4 r \sqrt{1 + 4r^2 \cos^2 t + 4r^2 \sin^2 t} dr d\theta \\ &= \int_0^{2\pi} \int_0^4 r \sqrt{1 + 4r^2} dr d\theta \\ &= \frac{\pi}{6} (65\sqrt{65} - 1) \approx 273.87 \end{aligned}$$

15.

$$\begin{aligned} SA &= \int_0^2 \int_0^{2x} \sqrt{1 + 1 + 4x^2} dy dx \\ &= \int_0^2 (2x\sqrt{2 + 4x^2}) dx \\ &= \frac{26}{3}\sqrt{2} \approx 12.26 \end{aligned}$$

17. This is easier in polar:

$$\begin{aligned} SA &= \int_0^{2\pi} \int_0^5 r \sqrt{1 + \frac{4r^2 \cos^2 t + 4r^2 \sin^2 t}{r^2 \sin^2 t + r^2 \cos^2 t}} dr d\theta \\ &= \int_0^{2\pi} \int_0^5 r\sqrt{5} dr d\theta \\ &= 25\pi\sqrt{5} \approx 175.62 \end{aligned}$$

19. Integrating in polar is easiest considering  $R$ :

$$\begin{aligned} SA &= \int_0^{2\pi} \int_0^1 r \sqrt{1 + c^2 + d^2} dr d\theta \\ &= \int_0^{2\pi} \frac{1}{2} (\sqrt{1 + c^2 + d^2}) dy \\ &= \pi\sqrt{1 + c^2 + d^2}. \end{aligned}$$

The value of  $h$  does not matter as it only shifts the plane vertically (i.e., parallel to the  $z$ -axis). Different values of  $h$  do not create different ellipses in the plane.

## Section 12.6

1. surface to surface, curve to curve and point to point
3. Answers can vary. From this section we used triple integration to find the volume of a solid region, the mass of a solid, and the center of mass of a solid.
5.  $V = \int_{-1}^1 \int_{-1}^1 (8 - x^2 - y^2 - (2x + y)) dx dy = 88/3$
7.  $V = \int_0^\pi \int_0^x (\cos x \sin y + 2 - \sin x \cos y) dy dx = \pi^2 - \pi \approx 6.728$

$$\begin{aligned} 9. dz dy dx: & \int_0^3 \int_0^{1-x/3} \int_0^{2-2x/3-2y} dz dy dx \\ dz dx dy: & \int_0^1 \int_0^{3-3y} \int_0^{2-2x/3-2y} dz dx dy \\ dy dz dx: & \int_0^3 \int_0^{2-2x/3} \int_0^{1-x/3-z/2} dy dz dx \\ dy dx dz: & \int_0^2 \int_0^{3-3z/2} \int_0^{1-x/3-z/2} dy dx dz \\ dx dz dy: & \int_0^1 \int_0^{2-2y} \int_0^{3-3y-3z/2} dx dz dy \\ dx dy dz: & \int_0^2 \int_0^{1-z/2} \int_0^{3-3y-3z/2} dx dy dz \\ V = & \int_0^3 \int_0^{1-x/3} \int_0^{2-2x/3-2y} dz dy dx = 1. \end{aligned}$$

$$\begin{aligned} 11. dz dy dx: & \int_0^2 \int_{-2}^0 \int_{y^2/2}^{-y} dz dy dx \\ dz dx dy: & \int_{-2}^0 \int_0^2 \int_{y^2/2}^{-y} dz dx dy \\ dy dz dx: & \int_0^2 \int_0^2 \int_{-\sqrt{2z}}^{-z} dy dz dx \\ dy dx dz: & \int_0^2 \int_0^2 \int_{-\sqrt{2z}}^{-z} dy dx dz \\ dx dz dy: & \int_{-2}^0 \int_{y^2/2}^2 \int_0^2 dx dz dy \\ dx dy dz: & \int_0^2 \int_{-\sqrt{2z}}^2 \int_0^2 dx dy dz \\ V = & \int_0^2 \int_0^2 \int_{-\sqrt{2z}}^{-z} dy dz dx = 4/3. \end{aligned}$$

13.  $dz dy dx: \int_0^2 \int_{1-x/2}^1 \int_0^{2x+4y-4} dz dy dx$   
 $dz dx dy: \int_0^1 \int_{2-y}^2 \int_0^{2x+4y-4} dz dx dy$   
 $dy dz dx: \int_0^2 \int_0^{2x} \int_{z/4-x/2+1}^1 dy dz dx$   
 $dy dx dz: \int_0^4 \int_{z/2}^2 \int_{z/4-x/2+1}^1 dy dx dz$   
 $dx dz dy: \int_0^1 \int_0^{4y} \int_{z/2-2y+2}^2 dx dz dy$   
 $dx dy dz: \int_0^4 \int_{z/4}^1 \int_{z/2-2y+2}^2 dx dy dz$   
 $V = \int_0^4 \int_{z/4}^1 \int_{2y-z/2-2}^2 dx dy dz = 4/3.$

15.  $dz dy dx: \int_0^1 \int_0^{1-x^2} \int_0^{\sqrt{1-y}} dz dy dx$   
 $dz dx dy: \int_0^1 \int_0^{\sqrt{1-y}} \int_0^{\sqrt{1-y}} dz dx dy$   
 $dy dz dx: \int_0^1 \int_0^x \int_0^{1-x^2} dy dz dx + \int_0^1 \int_x^1 \int_0^{1-x^2} dy dz dx$   
 $dy dx dz: \int_0^1 \int_0^z \int_0^{1-z^2} dy dx dz + \int_0^1 \int_z^1 \int_0^{1-x^2} dy dx dz$   
 $dx dz dy: \int_0^1 \int_0^{\sqrt{1-y}} \int_0^{\sqrt{1-y}} dx dz dy$   
 $dx dy dz: \int_0^1 \int_0^{1-z^2} \int_0^{\sqrt{1-y}} dx dy dz$

Answers will vary. Neither order is particularly "hard." The order

$dz dy dx$  requires integrating a square root, so powers can be messy; the order  $dy dz dx$  requires two triple integrals, but each uses only polynomials.

17. 8

19.  $\pi$

21.  $M = 10, M_{yz} = 15/2, M_{xz} = 5/2, M_{xy} = 5;$   
 $(\bar{x}, \bar{y}, \bar{z}) = (3/4, 1/4, 1/2)$

23.  $M = 16/5, M_{yz} = 16/3, M_{xz} = 104/45, M_{xy} = 32/9;$   
 $(\bar{x}, \bar{y}, \bar{z}) = (5/3, 13/18, 10/9) \approx (1.67, 0.72, 1.11)$

## Section 12.7

1.  $2\pi$

3. F

5. F

7.  $(r, \theta, z) = (2, \frac{3\pi}{2}, 0)$  and  $(\rho, \theta, \phi) = (2, \frac{3\pi}{2}, \frac{\pi}{2})$

9.  $(r, \theta, z) = (1, 0, 0)$  and  $(\rho, \theta, \phi) = (1, 0, \frac{\pi}{2})$

11.  $(r, \theta, z) = (\sqrt{2}, \frac{\pi}{4}, -2)$  and  
 $(\rho, \theta, \phi) = \left(\sqrt{6}, \frac{\pi}{4}, \cos^{-1}\left(-\frac{2}{\sqrt{6}}\right)\right)$

13. The solid is the upper half hemisphere of a sphere of radius 2

15.  $\frac{128\pi}{3}$

17.  $\frac{128\pi}{5}$

19. 0

27.  $3x^2 + 1$

### Section 3.2

1. Answers will vary.
2. Answers will vary.
3. Any  $c$  in  $[-1, 1]$  is valid.
4. Rolle's Thm. does not apply.
5.  $c = -1/2$
6.  $c = -1/2$
7. Rolle's Thm. does not apply.
8.  $c = \pi/2$
9. Rolle's Thm. does not apply.
10. Rolle's Theorem does not apply.
11.  $c = 0$
12.  $c = 5/2$
13.  $c = 3/\sqrt{2}$
14.  $c = 19/4$
15. The Mean Value Theorem does not apply.
16.  $c = 4/\ln 5$
17.  $c = \pm \sec^{-1}(2/\sqrt{\pi})$
18.  $c = -2/3$
19.  $c = \frac{5\pm 7\sqrt{7}}{6}$
20.  $c = \frac{\pm\sqrt{\pi^2 - 4}}{\pi}$
21. Max value of 19 at  $x = -2$  and  $x = 5$ ; min value of 6.75 at  $x = 1.5$ .
22. They are the odd, integer valued multiples of  $\pi/2$  (such as  $0, \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2$ , etc.)
23. They are the odd, integer valued multiples of  $\pi/2$  (such as  $0, \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2$ , etc.)

### Section 3.3

1. Answers will vary.
2. Answers will vary.
3. Answers will vary.
4. Answers will vary.
5. Increasing
6. Graph and verify.
7. Graph and verify.
8. Graph and verify.
9. Graph and verify.
10. Graph and verify.
11. Graph and verify.
12. Graph and verify.
13. Graph and verify.
14. domain:  $(-\infty, \infty)$   
c.p. at  $c = -1$ ;  
decreasing on  $(-\infty, -1)$ ;  
increasing on  $(-1, \infty)$ ;  
rel. min at  $x = -1$ .

15. domain:  $(-\infty, \infty)$   
c.p. at  $c = -2, 0$ ;  
increasing on  $(-\infty, -2) \cup (0, \infty)$ ;  
decreasing on  $(-2, 0)$ ;  
rel. min at  $x = 0$ ;  
rel. max at  $x = -2$ .
16. domain:  $(-\infty, \infty)$   
c.p. at  $c = \frac{1}{6}(-1 \pm \sqrt{7})$ ;  
decreasing on  $(\frac{1}{6}(-1 - \sqrt{7}), \frac{1}{6}(-1 + \sqrt{7}))$ ;  
increasing on  $(-\infty, \frac{1}{6}(-1 - \sqrt{7})) \cup (\frac{1}{6}(-1 + \sqrt{7}), \infty)$ ;  
rel. min at  $x = \frac{1}{6}(-1 + \sqrt{7})$ ;  
rel. max at  $x = \frac{1}{6}(-1 - \sqrt{7})$ .
17. domain:  $(-\infty, \infty)$   
c.p. at  $c = 1$ ;  
increasing on  $(-\infty, \infty)$ ;
18. domain:  $(-\infty, \infty)$   
c.p. at  $c = 1$ ;  
decreasing on  $(1, \infty)$   
increasing on  $(-\infty, 1)$ ;  
rel. max at  $x = 1$ .
19. domain:  $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$   
c.p. at  $c = 0$ ;  
decreasing on  $(-\infty, -1) \cup (-1, 0)$ ;  
increasing on  $(0, 1) \cup (1, \infty)$ ;  
rel. min at  $x = 0$ .
20. domain:  $(-\infty, -2) \cup (-2, 4) \cup (4, \infty)$   
no c.p.;  
decreasing on entire domain,  $(-\infty, -2) \cup (-2, 4) \cup (4, \infty)$
21. domain:  $(-\infty, 0) \cup (0, \infty)$ ;  
c.p. at  $c = 2, 6$ ;  
decreasing on  $(-\infty, 0) \cup (0, 2) \cup (6, \infty)$ ;  
increasing on  $(2, 6)$ ;  
rel. min at  $x = 2$ ; rel. max at  $x = 6$ .
22. domain:  $(-\infty, \infty)$   
c.p. at  $c = -3\pi/4, -\pi/4, \pi/4, 3\pi/4$ ;  
decreasing on  $(-3\pi/4, -\pi/4) \cup (\pi/4, 3\pi/4)$ ;  
increasing on  $(-\pi, -3\pi/4) \cup (-\pi/4, \pi/4) \cup (3\pi/4, \pi)$ ;  
rel. min at  $x = -\pi/4, 3\pi/4$ ;  
rel. max at  $x = -3\pi/4, \pi/4$ .
23. domain:  $(-\infty, \infty)$ ;  
c.p. at  $c = -1, 1$ ;  
decreasing on  $(-1, 1)$ ;  
increasing on  $(-\infty, -1) \cup (1, \infty)$ ;  
rel. min at  $x = 1$ ;  
rel. max at  $x = -1$
24.  $c = 1/2$
25.  $c = \pm \cos^{-1}(2/\pi)$

### Section 3.4

1. Answers will vary.
2. Answers will vary.
3. Yes; Answers will vary.
4. No.
5. Graph and verify.
6. Graph and verify.
7. Graph and verify.

8. Graph and verify.
9. Graph and verify.
10. Graph and verify.
11. Graph and verify.
12. Graph and verify.
13. Graph and verify.
14. Graph and verify.
15. Graph and verify.
16. Possible points of inflection: none; concave up on  $(-\infty, \infty)$
17. Possible points of inflection: none; concave down on  $(-\infty, \infty)$
18. Possible points of inflection:  $x = 0$ ; concave down on  $(-\infty, 0)$ ; concave up on  $(0, \infty)$
19. Possible points of inflection:  $x = 1/2$ ; concave down on  $(-\infty, 1/2)$ ; concave up on  $(1/2, \infty)$
20. Possible points of inflection:  $x = -2/3, 0$ ; concave down on  $(-2/3, 0)$ ; concave up on  $(-\infty, -2/3) \cup (0, \infty)$
21. Possible points of inflection:  $x = (1/3)(2 \pm \sqrt{7})$ ; concave up on  $((1/3)(2 - \sqrt{7}), (1/3)(2 + \sqrt{7}))$ ; concave down on  $(-\infty, (1/3)(2 - \sqrt{7})) \cup ((1/3)(2 + \sqrt{7}), \infty)$
22. Possible points of inflection:  $x = 1$ ; concave up on  $(-\infty, \infty)$
23. Possible points of inflection:  $x = \pm 1/\sqrt{3}$ ; concave down on  $(-1/\sqrt{3}, 1/\sqrt{3})$ ; concave up on  $(-\infty, -1/\sqrt{3}) \cup (1/\sqrt{3}, \infty)$
24. Possible points of inflection:  $x = 0, \pm 1$ ; concave down on  $(-\infty, -1) \cup (0, 1)$  concave up on  $(-1, 0) \cup (1, \infty)$
25. Possible points of inflection:  $x = -\pi/4, 3\pi/4$ ; concave down on  $(-\pi/4, 3\pi/4)$  concave up on  $(-\pi, -\pi/4) \cup (3\pi/4, \pi)$
26. Possible points of inflection:  $x = -2 \pm \sqrt{2}$ ; concave down on  $(-2 - \sqrt{2}, -2 + \sqrt{2})$  concave up on  $(-\infty, -2 - \sqrt{2}) \cup (-2 + \sqrt{2}, \infty)$
27. Possible points of inflection:  $x = 1/e^{3/2}$ ; concave down on  $(0, 1/e^{3/2})$  concave up on  $(1/e^{3/2}, \infty)$
28. Possible points of inflection:  $x = \pm 1/\sqrt{2}$ ; concave down on  $(-1/\sqrt{2}, 1/\sqrt{2})$  concave up on  $(-\infty, -1/\sqrt{2}) \cup (1/\sqrt{2}, \infty)$
29. min:  $x = 1$
30. max:  $x = -5/2$
31. max:  $x = -1/\sqrt{3}$  min:  $x = 1/\sqrt{3}$
- 32.
33. min:  $x = 1$
34. max:  $x = -1, 2$ ; min:  $x = 1$
35. min:  $x = 1$
36. max:  $x = 0$
37. critical values:  $x = -1, 1$ ; no max/min
38. max:  $x = \pi/4$ ; min:  $x = -3\pi/4$
39. max:  $x = -2$ ; min:  $x = 0$
40. min:  $x = 1/\sqrt{e}$
41. max:  $x = 0$
42.  $f'$  has no maximal or minimal value.
43.  $f'$  has no maximal or minimal value
44.  $f'$  has a minimal value at  $x = 0$
45.  $f'$  has a minimal value at  $x = 1/2$
46. Possible points of inflection:  $x = -2/3, 0$ ;  $f'$  has a relative min at:  $x = 0$ ; relative max at:  $x = -2/3$
47.  $f'$  has a relative max at:  $x = (1/3)(2 + \sqrt{7})$  relative min at:  $x = (1/3)(2 - \sqrt{7})$
48.  $f'$  has no relative extrema
49.  $f'$  has a relative max at  $x = -1/\sqrt{3}$ ; relative min at  $x = 1/\sqrt{3}$
50.  $f'$  has a relative max at  $x = 0$
51.  $f'$  has a relative min at  $x = 3\pi/4$ ; relative max at  $x = -\pi/4$
52.  $f'$  has a relative max at  $x = -2 - \sqrt{2}$ ; relative min at  $x = -2 + \sqrt{2}$
53.  $f'$  has a relative min at  $x = 1/\sqrt{e^3} = e^{-3/2}$
54.  $f'$  has a relative max at  $x = -1/\sqrt{2}$ ; a relative min at  $x = 1/\sqrt{2}$

### Section 3.5

1. Answers will vary.
2. Found everywhere.
3. T
4. T
5. T
6. A good sketch will include the x and y intercepts and draw the appropriate line.
7. A good sketch will include the x and y intercepts..
8. Use technology to verify sketch.
9. Use technology to verify sketch.
10. Use technology to verify sketch.
11. Use technology to verify sketch.
12. Use technology to verify sketch.
13. Use technology to verify sketch.
14. Use technology to verify sketch.
15. Use technology to verify sketch.
16. Use technology to verify sketch.
17. Use technology to verify sketch.
18. Use technology to verify sketch.
19. Use technology to verify sketch.
20. Use technology to verify sketch.
21. Use technology to verify sketch.
22. Use technology to verify sketch.
23. Use technology to verify sketch.
24. Use technology to verify sketch.
25. Use technology to verify sketch.
26. Critical point:  $x = 0$  Points of inflection:  $\pm b/\sqrt{3}$
27. Critical points:  $x = \frac{n\pi/2 - b}{a}$ , where n is an odd integer Points of inflection:  $(n\pi - b)/a$ , where n is an integer.
28. Critical point:  $x = (a + b)/2$  Points of inflection: none
29.  $\frac{dy}{dx} = -x/y$ , so the function is increasing in second and fourth quadrants, decreasing in the first and third quadrants.  
 $\frac{d^2y}{dx^2} = -1/y - x^2/y^3$ , which is positive when  $y < 0$  and is negative when  $y > 0$ . Hence the function is concave down in the first and second quadrants and concave up in the third and fourth quadrants.

### Chapter 4

#### Section 4.1

1. T
2. F
3. (a)  $5/(2\pi) \approx 0.796\text{cm/s}$   
      (b)  $1/(4\pi) \approx 0.0796\text{ cm/s}$   
      (c)  $1/(40\pi) \approx 0.00796\text{ cm/s}$
4. (a)  $5/(2\pi) \approx 0.796\text{cm/s}$   
      (b)  $1/(40\pi) \approx 0.00796\text{ cm/s}$   
      (c)  $1/(4000\pi) \approx 0.0000796\text{ cm/s}$
5. 63.14mph
6. (a) 64.44 mph  
      (b) 78.89 mph
7. Due to the height of the plane, the gun does not have to rotate very fast.  
      (a) 0.0573 rad/s  
      (b) 0.0725 rad/s  
      (c) In the limit, rate goes to 0.0733 rad/s
8. Due to the height of the plane, the gun does not have to rotate very fast.  
      (a) 0.073 rad/s  
      (b) 3.66 rad/s (about 1/2 revolution/sec)  
      (c) In the limit, rate goes to 7.33 rad/s (more than 1 revolution/sec)
9. (a) 0.04 ft/s  
      (b) 0.458 ft/s  
      (c) 3.35 ft/s  
      (d) Not defined; as the distance approaches 24, the rates approaches  $\infty$ .
10. (a) 30.59 ft/min  
      (b) 36.1 ft/min  
      (c) 301 ft/min  
      (d) The boat no longer floats as usual, but is being pulled up by the winch (assuming it has the power to do so).
11. (a) 50.92 ft/min  
      (b) 0.509 ft/min  
      (c) 0.141 ft/min
- As the tank holds about  $523.6\text{ft}^3$ , it will take about 52.36 minutes.
12. (a) 0.63 ft/sec  
      (b) 1.6 ft/sec
- About 52 ft.
13. (a) The rope is 80ft long.  
      (b) 1.71 ft/sec  
      (c) 1.87 ft/sec  
      (d) About 34 feet.
14. (a) The balloon is 105ft in the air.  
      (b) The balloon is rising at a rate of 17.45ft/min. (Hint: convert all angles to radians.)
15. The cone is rising at a rate of 0.003ft/s.
3. F
4. The base of an expression is approaching 1 while its power is growing without bound.
5. derivatives; limits
6. Answers will vary.
7. Answers will vary.
8. 3
9.  $-5/3$
10. -1
11.  $-\sqrt{2}/2$
12. 5
13. 0
14.  $2/3$
15.  $a/b$
16.  $\infty$
17.  $1/2$
18. 0
19. 0
20. 0
21.  $\infty$
22.  $\infty$
23. 0
24. 2
25. -2
26. 0
27. 0
28. 0
29. 0
30. 0
31.  $\infty$
32.  $\infty$
33.  $\infty$
34. 0
35. 0
36. e
37. 1
38. 1
39. 1
40. 1
41. 1
42. 0
43. 1
44. 1
45. 1
46. 1
47. 2
48.  $1/2$
49.  $-\infty$
50. 1

## Section 4.2

1.  $0/0, \infty/\infty, 0 \cdot \infty, \infty - \infty, 0^0, 1^\infty, \infty^0$
2. F

51. 0

52. 3

### Section 4.3

1. T

2. F

3. 2500; the two numbers are each 50.

4. The minimum sum is  $2\sqrt{500}$ ; the two numbers are each  $\sqrt{500}$ .

5. There is no maximum sum; the fundamental equation has only 1 critical value that corresponds to a minimum.

6. The only critical point of the fundamental equation corresponds to a minimum; to find maximum, we check the endpoints.

If one number is 300, the other number  $y$  satisfies  $300y = 500$ ;  $y = 5/3$ . Thus the sum is  $300 + 5/3$ .

The other endpoint, 0, is not feasible as we cannot solve  $0 \cdot y = 500$  for  $y$ . In fact, if  $0 < x < 5/3$ , then  $x \cdot y = 500$  forces  $y > 300$ , which is not a feasible solution.

Hence the maximum sum is  $300 + \bar{5}$ .

7. Area =  $1/4$ , with sides of length  $1/\sqrt{2}$ .

8. Each pen should be  $500/3 \approx 166.67$  feet by 125 feet.

9. The radius should be about 3.84cm and the height should be  $2r = 7.67$ cm. No, this is not the size of the standard can.

10. The radius should be about 3.2in and the height should be  $2r = 6.4$ in. As the #10 is not a perfect cylinder (with extra material to aid in stacking, etc.), the dimensions are close enough to assume that minimizing surface area was a consideration.

11. The height and width should be 18 and the length should be 36, giving a volume of  $11,664\text{in}^3$ .

12.  $w = 4\sqrt{3}$ ,  $h = 4\sqrt{6}$

13.  $5 - 10/\sqrt{39} \approx 3.4$  miles should be run underground, giving a minimum cost of \$374,899.96.

14. The power line should be run directly to the off shore facility, skipping any underground, giving a cost of about \$430,813.

15. The dog should run about 19 feet along the shore before starting to swim.

16. The dog should run about 13 feet along the shore before starting to swim.

17. The largest area is 2 formed by a square with sides of length  $\sqrt{2}$ .

### Section 4.4

1. T

2. T

3. F

4. T

5. Answers will vary.

6. Use  $y = x^2$ ;  $dy = 2x \cdot dx$  with  $x = 2$  and  $dx = 0.05$ . Thus  $dy = .2$ ; knowing  $2^2 = 4$ , we have  $2.05^2 \approx 4.2$ .

7. Use  $y = x^2$ ;  $dy = 2x \cdot dx$  with  $x = 6$  and  $dx = -0.07$ . Thus  $dy = -0.84$ ; knowing  $6^2 = 36$ , we have  $5.93^2 \approx 35.16$ .

8. Use  $y = x^3$ ;  $dy = 3x^2 \cdot dx$  with  $x = 5$  and  $dx = 0.1$ . Thus  $dy = 7.5$ ; knowing  $5^3 = 125$ , we have  $5.1^3 \approx 132.5$ .

9. Use  $y = x^3$ ;  $dy = 3x^2 \cdot dx$  with  $x = 7$  and  $dx = -0.2$ . Thus  $dy = -29.4$ ; knowing  $7^3 = 343$ , we have  $6.8^3 \approx 313.6$ .

10. Use  $y = \sqrt{x}$ ;  $dy = 1/(2\sqrt{x}) \cdot dx$  with  $x = 16$  and  $dx = 0.5$ . Thus  $dy = .0625$ ; knowing  $\sqrt{16} = 4$ , we have  $\sqrt{16.5} \approx 4.0625$ .

11. Use  $y = \sqrt{x}$ ;  $dy = 1/(2\sqrt{x}) \cdot dx$  with  $x = 25$  and  $dx = -1$ . Thus  $dy = -0.1$ ; knowing  $\sqrt{25} = 5$ , we have  $\sqrt{24} \approx 4.9$ .

12. Use  $y = \sqrt[3]{x}$ ;  $dy = 1/(3\sqrt[3]{x^2}) \cdot dx$  with  $x = 64$  and  $dx = -1$ . Thus  $dy = -1/48 \approx 0.0208$ ; we could use  $-1/48 \approx -1/50 = -0.02$ ; knowing  $\sqrt[3]{64} = 4$ , we have  $\sqrt[3]{63} \approx 3.98$ .

13. Use  $y = \sqrt[3]{x}$ ;  $dy = 1/(3\sqrt[3]{x^2}) \cdot dx$  with  $x = 8$  and  $dx = 0.5$ . Thus  $dy = 1/24 \approx 1/25 = 0.04$ ; knowing  $\sqrt[3]{8} = 2$ , we have  $\sqrt[3]{8.5} \approx 2.04$ .

14. Use  $y = \sin x$ ;  $dy = \cos x \cdot dx$  with  $x = \pi$  and  $dx \approx -0.14$ . Thus  $dy = 0.14$ ; knowing  $\sin \pi = 0$ , we have  $\sin 3 \approx 0.14$ .

15. Use  $y = \cos x$ ;  $dy = -\sin x \cdot dx$  with  $x = \pi/2 \approx 1.57$  and  $dx \approx -0.07$ . Thus  $dy = 0.07$ ; knowing  $\cos \pi/2 = 0$ , we have  $\cos 1.5 \approx 0.07$ .

16. Use  $y = e^x$ ;  $dy = e^x \cdot dx$  with  $x = 0$  and  $dx = 0.1$ . Thus  $dy = 0.1$ ; knowing  $e^0 = 1$ , we have  $e^{0.1} \approx 1.1$ .

17.  $dy = (2x + 3)dx$

18.  $dy = (7x^6 - 5x^4)dx$

19.  $dy = \frac{-2}{4x^3}dx$

20.  $dy = 2(2x + \sin x)(2 + \cos x)dx$

21.  $dy = (2xe^{3x} + 3x^2e^{3x})dx$

22.  $dy = \frac{-16}{x^5}dx$

23.  $dy = \frac{2(\tan x+1)-2x \sec^2 x}{(\tan x+1)^2}dx$

24.  $dy = \frac{1}{x}dx$

25.  $dy = (e^x \sin x + e^x \cos x)dx$

26.  $dy = (-\sin(\sin x) \cos x)dx$

27.  $dy = \frac{1}{(x+2)^2}dx$

28.  $dy = ((\ln 3)3^x \ln x + \frac{3^x}{x})dx$

29.  $dy = (\ln x)dx$

30.  $dV = \pm 0.157$

31. (a)  $\pm 12.8$  feet

(b)  $\pm 32$  feet

32.  $\pm 15\pi/8 \approx \pm 5.89\text{in}^2$

33.  $\pm 48\text{in}^2$ , or  $1/3\text{ft}^2$

34. (a) 297.8 feet

(b)  $\pm 62.3$  ft

(c)  $\pm 20.9\%$

35. (a) 298.8 feet

(b)  $\pm 17.3$  ft

(c)  $\pm 5.8\%$

36. (a) 298.9 feet

(b)  $\pm 8.67$  ft

(c)  $\pm 2.9\%$

37. The isosceles triangle setup works the best with the smallest percent error.

38. 1%

### Section 4.5

1. F

2. F

3.  $x_0 = 1.5, x_1 = 1.5709148, x_2 = 1.5707963, x_3 = 1.5707963, x_4 = 1.5707963, x_5 = 1.5707963$

4.  $x_0 = 1, x_1 = -0.55740772, x_2 = 0.065936452, x_3 = -0.000095721919, x_4 = 2.9235662 * 10^{-13}, x_5 = 0$   
 5.  $x_0 = 0, x_1 = 2, x_2 = 1.2, x_3 = 1.0117647, x_4 = 1.0000458, x_5 = 1$   
 6.  $x_0 = 1.5, x_1 = 1.4166667, x_2 = 1.4142157, x_3 = 1.4142136, x_4 = 1.4142136, x_5 = 1.4142136$   
 7.  $x_0 = 2, x_1 = 0.6137056389, x_2 = 0.9133412072, x_3 = 0.9961317034, x_4 = 0.9999925085, x_5 = 1$   
 8. roots are:  $x = -5.156, x = -0.369$  and  $x = 0.525$   
 9. roots are:  $x = -3.714, x = -0.857, x = 1$  and  $x = 1.571$   
 10. roots are:  $x = -1.013, x = 0.988$ , and  $x = 1.393$   
 11. roots are:  $x = -2.165, x = 0, x = 0.525$  and  $x = 1.813$   
 12.  $x = \pm 0.824$ ,  
 13.  $x = -0.637, x = 1.410$   
 14.  $x = \pm 0.743$   
 15.  $x = \pm 4.493, x = 0$   
 16. The approximations alternate between  $x = 1$  and  $x = 2$ .  
 17. The approximations alternate between  $x = 1, x = 2$  and  $x = 3$ .

## Chapter 5

### Section 5.1

1. Answers will vary.
2. “an”
3. Answers will vary.
4. opposite; opposite
5. Answers will vary.
6. velocity
7. velocity
8.  $3/4x^4 + C$
9.  $1/9x^9 + C$
10.  $10/3x^3 - 2x + C$
11.  $t + C$
12.  $s + C$
13.  $-1/(3t) + C$
14.  $-3/(t) + C$
15.  $2\sqrt{x} + C$
16.  $\tan \theta + C$
17.  $-\cos \theta + C$
18.  $\sec x - \csc x + C$
19.  $5e^\theta + C$
20.  $3^t / \ln 3 + C$
21.  $\frac{5^t}{2 \ln 5} + C$
22.  $4/3t^3 + 6t^2 + 9t + C$
23.  $t^6/6 + t^4/4 - 3t^2 + C$
24.  $x^6/6 + C$
25.  $e^\pi x + C$
26.  $ax + C$
27. (a)  $x > 0$

- (b)  $1/x$   
 (c)  $x < 0$   
 (d)  $1/x$   
 (e)  $\ln|x| + C$ . Explanations will vary.
28.  $-\cos x + 3$   
 29.  $5e^x + 5$   
 30.  $x^4 - x^3 + 7$   
 31.  $\tan x + 4$   
 32.  $7^x / \ln 7 + 1 - 49 / \ln 7$   
 33.  $5/2x^2 + 7x + 3$   
 34.  $\frac{7x^3}{6} - \frac{9x}{2} + \frac{40}{3}$   
 35.  $5e^x - 2x$   
 36.  $\theta - \sin(\theta) - \pi + 4$   
 37.  $\frac{2x^4 \ln^2(2) + 2^x + x \ln 2)(\ln 32 - 1) + \ln^2(2) \cos(x) - 1 - \ln^2(2)}{\ln^2(2)}$   
 38.  $3x - 2$   
 39. No answer provided.  
 40.  $dy = (2xe^x \cos x + x^2 e^x \cos x - x^2 e^x \sin x)dx$

### Section 5.2

1. Answers will vary.
2. Answers will vary.
3. 0
4.  $\int 0^2(2x + 3) dx$
5. (a) 3  
 (b) 4  
 (c) 3  
 (d) 0  
 (e) -4  
 (f) 9
6. (a) -4  
 (b) -5  
 (c) -3  
 (d) 1  
 (e) -2  
 (f) 10
7. (a) 4  
 (b) 2  
 (c) 4  
 (d) 2  
 (e) 1  
 (f) 2
8. (a) -1/2  
 (b) 0  
 (c) 3/2  
 (d) 3/2  
 (e) 9/2  
 (f) 15/2
9. (a)  $\pi$   
 (b)  $\pi$   
 (c)  $2\pi$

- (d)  $10\pi$
10. (a) -59  
 (b) -48  
 (c) -27  
 (d) -33
11. (a)  $4/\pi$   
 (b)  $-4/\pi$   
 (c) 0  
 (d)  $2/\pi$
12. (a) 4  
 (b) 4  
 (c) -4  
 (d) -2
13. (a)  $40/3$   
 (b)  $26/3$   
 (c)  $8/3$   
 (d)  $38/3$
14. (a) 2ft/s  
 (b) 2ft  
 (c) 1.5ft
15. (a) 3ft/s  
 (b) 9.5ft  
 (c) 9.5ft
16. (a) 64ft/s  
 (b) 64ft  
 (c)  $t = 2$   
 (d)  $t = 2 + \sqrt{7} \approx 4.65$  seconds
17. (a) 96ft/s  
 (b) 6 seconds  
 (c) 6 seconds  
 (d) Never; the maximum height is 208ft.
18. 2
19. 5
20. 16
21. Answers can vary; one solution is  $a = -2, b = 7$
22. 24
23. -7
24. -7
25. Answers can vary; one solution is  $a = -11, b = 18$
26.  $1/4x^4 - 2/3x^3 + 7/2x^2 - 9x + C$
27.  $-\cos x - \sin x + \tan x + C$
28.  $3/4t^{4/3} - 1/t + 2^t / \ln 2 + C$
29.  $\ln|x| + \csc x + C$
- Section 5.3**
1. limits
  2. 14
  3. Rectangles.
  4. T
5.  $2^2 + 3^2 + 4^2 = 29$
6.  $-6 - 2 + 2 + 6 + 10 = 10$
7.  $0 - 1 + 0 + 1 + 0 = 0$
8.  $1 + 1/2 + 1/3 + 1/4 + 1/5 = 137/60$
9.  $-1 + 2 - 3 + 4 - 5 + 6 = 3$
10.  $1/2 + 1/6 + 1/12 + 1/20 = 4/5$
11.  $1 + 1 + 1 + 1 + 1 + 1 = 6$
12. Answers may vary;  $\sum_{i=1}^5 3^i$
13. Answers may vary;  $\sum_{i=0}^8 (i^2 - 1)$
14. Answers may vary;  $\sum_{i=1}^4 \frac{i}{i+1}$
15. Answers may vary;  $\sum_{i=0}^4 (-1)^i e^i$
16. 325
17. 1045
18. 28,650
19. -8525
20. 2050
21. 5050
22. 2870
23. 155
24. 91,225
25. 24
26. 11,700
27. 19
28.  $59/8$
29.  $\pi/3 + \pi/(2\sqrt{3}) \approx 1.954$
30. 8.144
31. 0.388584
32.  $496/315 \approx 1.5746$
33. (a) Exact expressions will vary;  $\frac{(1+n)^2}{4n^2}$ .  
 (b)  $121/400, 10201/40000, 1002001/4000000$   
 (c)  $1/4$
34. (a) Exact expressions will vary;  $2 + 4/n^2$ .  
 (b)  $51/25, 5001/2500, 500001/250000$   
 (c) 2
35. (a) 8.  
 (b) 8, 8, 8  
 (c) 8
36. (a) Exact expressions will vary;  $20/3 - 96/(3n) + 64/(3n^2)$ .  
 (b)  $92/25, 3968/625, 103667/15625$   
 (c)  $20/3$
37. (a) Exact expressions will vary;  $100 - 200/n$ .  
 (b) 80, 98, 499/5  
 (c) 100
38. (a) Exact expressions will vary;  $-1/12(1 - 1/n^2)$ .  
 (b)  $-33/400, -3333/40000, -333333/4000000$   
 (c)  $-1/12$
39.  $F(x) = 5 \tan x + 4$
40.  $F(x) = 7 \ln|x| + 14$
41.  $G(t) = 4/6t^6 - 5/4t^4 + 8t + 9$

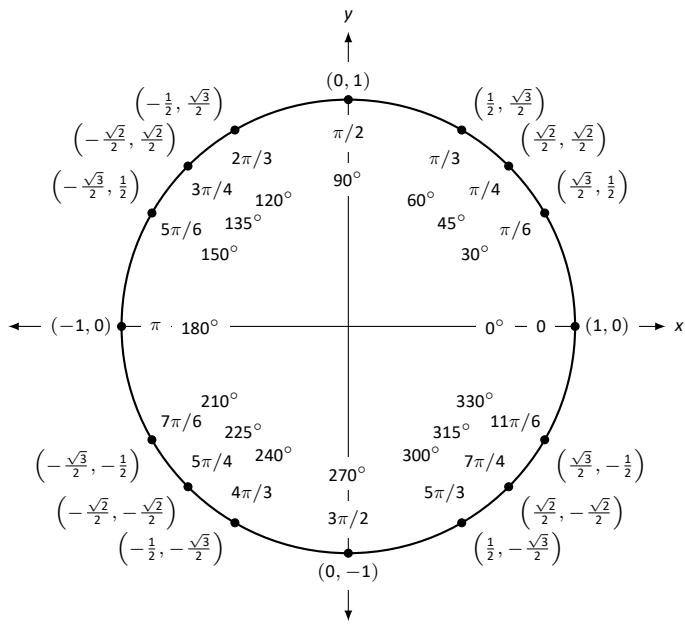
## Differentiation Rules

1. $\frac{d}{dx}(cx) = c$	10. $\frac{d}{dx}(a^x) = \ln a \cdot a^x$	19. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	28. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
2. $\frac{d}{dx}(u \pm v) = u' \pm v'$	11. $\frac{d}{dx}(\ln x) = \frac{1}{x}$	20. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$	29. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$
3. $\frac{d}{dx}(u \cdot v) = uv' + u'v$	12. $\frac{d}{dx}(\log_a x) = \frac{1}{\ln a} \cdot \frac{1}{x}$	21. $\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{ x \sqrt{x^2-1}}$	30. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$
4. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$	13. $\frac{d}{dx}(\sin x) = \cos x$	22. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{ x \sqrt{x^2-1}}$	31. $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$
5. $\frac{d}{dx}(u(v)) = u'(v)v'$	14. $\frac{d}{dx}(\cos x) = -\sin x$	23. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	32. $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$
6. $\frac{d}{dx}(c) = 0$	15. $\frac{d}{dx}(\csc x) = -\csc x \cot x$	24. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$	33. $\frac{d}{dx}(\operatorname{sech}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}$
7. $\frac{d}{dx}(x) = 1$	16. $\frac{d}{dx}(\sec x) = \sec x \tan x$	25. $\frac{d}{dx}(\cosh x) = \sinh x$	34. $\frac{d}{dx}(\operatorname{csch}^{-1} x) = \frac{-1}{ x \sqrt{1+x^2}}$
8. $\frac{d}{dx}(x^n) = nx^{n-1}$	17. $\frac{d}{dx}(\tan x) = \sec^2 x$	26. $\frac{d}{dx}(\sinh x) = \cosh x$	35. $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$
9. $\frac{d}{dx}(e^x) = e^x$	18. $\frac{d}{dx}(\cot x) = -\csc^2 x$	27. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$	36. $\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$

## Integration Rules

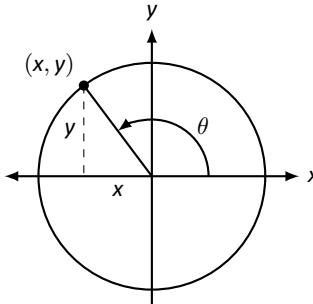
1. $\int c \cdot f(x) dx = c \int f(x) dx$	11. $\int \tan x dx = -\ln  \cos x  + C$	22. $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
2. $\int f(x) \pm g(x) dx =$ $\int f(x) dx \pm \int g(x) dx$	12. $\int \sec x dx = \ln  \sec x + \tan x  + C$	23. $\int \frac{1}{x\sqrt{x^2-a^2}} dx = \frac{1}{a} \sec^{-1}\left(\frac{ x }{a}\right) + C$
3. $\int 0 dx = C$	13. $\int \csc x dx = -\ln  \csc x + \cot x  + C$	24. $\int \cosh x dx = \sinh x + C$
4. $\int 1 dx = x + C$	14. $\int \cot x dx = \ln  \sin x  + C$	25. $\int \sinh x dx = \cosh x + C$
5. $\int x^n dx = \frac{1}{n+1}x^{n+1} + C, n \neq -1$	15. $\int \sec^2 x dx = \tan x + C$	26. $\int \tanh x dx = \ln(\cosh x) + C$
6. $\int e^x dx = e^x + C$	16. $\int \csc^2 x dx = -\cot x + C$	27. $\int \coth x dx = \ln  \sinh x  + C$
7. $\int a^x dx = \frac{1}{\ln a} \cdot a^x + C$	17. $\int \sec x \tan x dx = \sec x + C$	28. $\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln  x + \sqrt{x^2-a^2}  + C$
8. $\int \frac{1}{x} dx = \ln  x  + C$	18. $\int \csc x \cot x dx = -\csc x + C$	29. $\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln  x + \sqrt{x^2+a^2}  + C$
9. $\int \cos x dx = \sin x + C$	19. $\int \cos^2 x dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$	30. $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \ln \left  \frac{a+x}{a-x} \right  + C$
10. $\int \sin x dx = -\cos x + C$	20. $\int \sin^2 x dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$	31. $\int \frac{1}{x\sqrt{a^2-x^2}} dx = \frac{1}{a} \ln \left( \frac{x}{a+\sqrt{a^2-x^2}} \right) + C$
	21. $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$	32. $\int \frac{1}{x\sqrt{x^2+a^2}} dx = \frac{1}{a} \ln \left  \frac{x}{a+\sqrt{x^2+a^2}} \right  + C$

## The Unit Circle



## Definitions of the Trigonometric Functions

### Unit Circle Definition

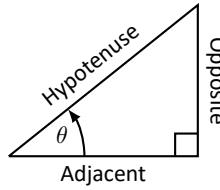


$$\sin \theta = y \quad \cos \theta = x$$

$$\csc \theta = \frac{1}{y} \quad \sec \theta = \frac{1}{x}$$

$$\tan \theta = \frac{y}{x} \quad \cot \theta = \frac{x}{y}$$

### Right Triangle Definition



$$\sin \theta = \frac{O}{H} \quad \csc \theta = \frac{H}{O}$$

$$\cos \theta = \frac{A}{H} \quad \sec \theta = \frac{H}{A}$$

$$\tan \theta = \frac{O}{A} \quad \cot \theta = \frac{A}{O}$$

## Common Trigonometric Identities

### Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

### Cofunction Identities

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$

$$\tan\left(\frac{\pi}{2} - x\right) = \cot x$$

$$\csc\left(\frac{\pi}{2} - x\right) = \sec x$$

$$\sec\left(\frac{\pi}{2} - x\right) = \csc x$$

$$\cot\left(\frac{\pi}{2} - x\right) = \tan x$$

### Double Angle Formulas

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$= 2 \cos^2 x - 1$$

$$= 1 - 2 \sin^2 x$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

### Sum to Product Formulas

$$\sin x + \sin y = 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2 \sin\left(\frac{x-y}{2}\right) \cos\left(\frac{x+y}{2}\right)$$

$$\cos x + \cos y = 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$$

### Power-Reducing Formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

### Even/Odd Identities

$$\sin(-x) = -\sin x$$

$$\cos(-x) = \cos x$$

$$\tan(-x) = -\tan x$$

$$\csc(-x) = -\csc x$$

$$\sec(-x) = \sec x$$

$$\cot(-x) = -\cot x$$

### Product to Sum Formulas

$$\sin x \sin y = \frac{1}{2} (\cos(x-y) - \cos(x+y))$$

$$\cos x \cos y = \frac{1}{2} (\cos(x-y) + \cos(x+y))$$

$$\sin x \cos y = \frac{1}{2} (\sin(x+y) + \sin(x-y))$$

### Angle Sum/Difference Formulas

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

## Areas and Volumes

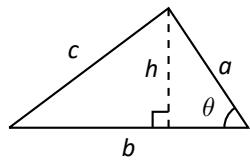
### Triangles

$$h = a \sin \theta$$

$$\text{Area} = \frac{1}{2}bh$$

Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

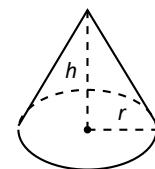


### Right Circular Cone

$$\text{Volume} = \frac{1}{3}\pi r^2 h$$

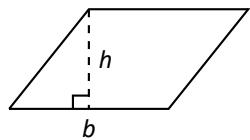
Surface Area =

$$\pi r \sqrt{r^2 + h^2} + \pi r^2$$



### Parallelograms

$$\text{Area} = bh$$

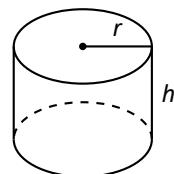


### Right Circular Cylinder

$$\text{Volume} = \pi r^2 h$$

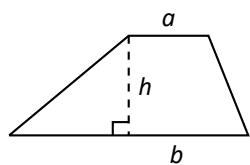
Surface Area =

$$2\pi rh + 2\pi r^2$$



### Trapezoids

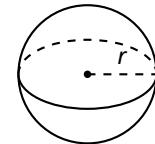
$$\text{Area} = \frac{1}{2}(a + b)h$$



### Sphere

$$\text{Volume} = \frac{4}{3}\pi r^3$$

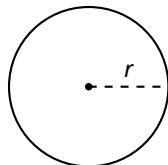
$$\text{Surface Area} = 4\pi r^2$$



### Circles

$$\text{Area} = \pi r^2$$

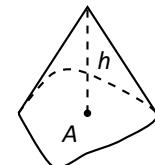
$$\text{Circumference} = 2\pi r$$



### General Cone

$$\text{Area of Base} = A$$

$$\text{Volume} = \frac{1}{3}Ah$$

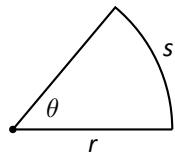


### Sectors of Circles

$\theta$  in radians

$$\text{Area} = \frac{1}{2}\theta r^2$$

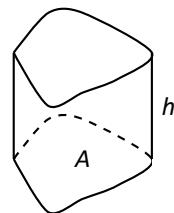
$$s = r\theta$$



### General Right Cylinder

$$\text{Area of Base} = A$$

$$\text{Volume} = Ah$$



# Algebra

## Factors and Zeros of Polynomials

Let  $p(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$  be a polynomial. If  $p(a) = 0$ , then  $a$  is a *zero* of the polynomial and a solution of the equation  $p(x) = 0$ . Furthermore,  $(x - a)$  is a *factor* of the polynomial.

## Fundamental Theorem of Algebra

An  $n$ th degree polynomial has  $n$  (not necessarily distinct) zeros. Although all of these zeros may be imaginary, a real polynomial of odd degree must have at least one real zero.

## Quadratic Formula

If  $p(x) = ax^2 + bx + c$ , and  $0 \leq b^2 - 4ac$ , then the real zeros of  $p$  are  $x = (-b \pm \sqrt{b^2 - 4ac})/2a$

## Special Factors

$$\begin{aligned}x^2 - a^2 &= (x - a)(x + a) & x^3 - a^3 &= (x - a)(x^2 + ax + a^2) \\x^3 + a^3 &= (x + a)(x^2 - ax + a^2) & x^4 - a^4 &= (x^2 - a^2)(x^2 + a^2) \\(x + y)^n &= x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \dots + nxy^{n-1} + y^n \\(x - y)^n &= x^n - nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 - \dots \pm nxy^{n-1} \mp y^n\end{aligned}$$

## Binomial Theorem

$$\begin{aligned}(x + y)^2 &= x^2 + 2xy + y^2 & (x - y)^2 &= x^2 - 2xy + y^2 \\(x + y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 & (x - y)^3 &= x^3 - 3x^2y + 3xy^2 - y^3 \\(x + y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 & (x - y)^4 &= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4\end{aligned}$$

## Rational Zero Theorem

If  $p(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$  has integer coefficients, then every *rational zero* of  $p$  is of the form  $x = r/s$ , where  $r$  is a factor of  $a_0$  and  $s$  is a factor of  $a_n$ .

## Factoring by Grouping

$$acx^3 + adx^2 + bcx + bd = ax^2(cs + d) + b(cx + d) = (ax^2 + b)(cx + d)$$

## Arithmetic Operations

$$ab + ac = a(b + c) \quad \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \quad \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right) = \frac{ad}{bc} \quad \frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc} \quad \frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b}$$

$$a\left(\frac{b}{c}\right) = \frac{ab}{c} \quad \frac{a-b}{c-d} = \frac{b-a}{d-c} \quad \frac{ab+ac}{a} = b+c$$

## Exponents and Radicals

$$a^0 = 1, \quad a \neq 0 \quad (ab)^x = a^x b^x \quad a^x a^y = a^{x+y} \quad \sqrt{a} = a^{1/2} \quad \frac{a^x}{a^y} = a^{x-y} \quad \sqrt[n]{a} = a^{1/n}$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \quad \sqrt[n]{a^m} = a^{m/n} \quad a^{-x} = \frac{1}{a^x} \quad \sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b} \quad (a^x)^y = a^{xy} \quad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

## Additional Formulas

### Summation Formulas:

$$\sum_{i=1}^n c = cn$$

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^n i^3 = \left( \frac{n(n+1)}{2} \right)^2$$

### Trapezoidal Rule:

$$\int_a^b f(x) dx \approx \frac{\Delta x}{2} [f(x_1) + 2f(x_2) + 2f(x_3) + \dots + 2f(x_n) + f(x_{n+1})]$$

$$\text{with Error} \leq \frac{(b-a)^3}{12n^2} [\max |f''(x)|]$$

### Simpson's Rule:

$$\int_a^b f(x) dx \approx \frac{\Delta x}{3} [f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + \dots + 2f(x_{n-1}) + 4f(x_n) + f(x_{n+1})]$$

$$\text{with Error} \leq \frac{(b-a)^5}{180n^4} [\max |f^{(4)}(x)|]$$

### Arc Length:

$$L = \int_a^b \sqrt{1+f'(x)^2} dx$$

### Surface of Revolution:

$$S = 2\pi \int_a^b f(x) \sqrt{1+f'(x)^2} dx$$

(where  $f(x) \geq 0$ )

$$S = 2\pi \int_a^b x \sqrt{1+f'(x)^2} dx$$

(where  $a, b \geq 0$ )

### Work Done by a Variable Force:

$$W = \int_a^b F(x) dx$$

### Force Exerted by a Fluid:

$$F = \int_a^b w d(y) \ell(y) dy$$

### Taylor Series Expansion for $f(x)$ :

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

### Maclaurin Series Expansion for $f(x)$ , where $c = 0$ :

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

## Summary of Tests for Series:

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
$n$ th-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n \rightarrow \infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} r^n$	$ r  < 1$	$ r  \geq 1$	Sum = $\frac{1}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+a})$	$\lim_{n \rightarrow \infty} b_n = L$		Sum = $\left( \sum_{n=1}^a b_n \right) - L$
$p$ -Series	$\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$	$p > 1$	$p \leq 1$	
Integral Test	$\sum_{n=0}^{\infty} a_n$	$\int_1^{\infty} a(n) dn$ is convergent	$\int_1^{\infty} a(n) dn$ is divergent	$a_n = a(n)$ must be continuous
Direct Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$	$\sum_{n=0}^{\infty} b_n$ diverges and $0 \leq b_n \leq a_n$	
Limit Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $\lim_{n \rightarrow \infty} a_n/b_n \geq 0$	$\sum_{n=0}^{\infty} b_n$ diverges and $\lim_{n \rightarrow \infty} a_n/b_n > 0$	Also diverges if $\lim_{n \rightarrow \infty} a_n/b_n = \infty$
Ratio Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} < 1$	$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n \rightarrow \infty} a_{n+1}/a_n = \infty$
Root Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n \rightarrow \infty} (a_n)^{1/n} < 1$	$\lim_{n \rightarrow \infty} (a_n)^{1/n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n \rightarrow \infty} (a_n)^{1/n} = \infty$