12.6 Directional Derivatives

Partial derivatives give us an understanding of how a surface changes when we move in the x and y directions. We made the comparison to standing in a rolling meadow and heading due east: the amount of rise/fall in doing so is comparable to f_x. Likewise, the rise/fall in moving due north is comparable to f_y. The steeper the slope, the greater in magnitude f_y.

But what if we didn't move due north or east? What if we needed to move northeast and wanted to measure the amount of rise/fall? Partial derivatives alone cannot measure this. This section investigates **directional derivatives**, which do measure this rate of change.

We begin with a definition.

Definition 94 Directional Derivatives

Let $z = f(x, y)$ be continuous on an open set S and let $\vec{u} = \langle u_1, u_2 \rangle$ be a unit vector. For all points (x, y), the **directional derivative of f at (x, y) in the direction of $\vec{u}$** is

$$D_{\vec{u}} f(x, y) = \lim_{h \to 0} \frac{f(x + hu_1, y + hu_2) - f(x, y)}{h}.$$

The partial derivatives f_x and f_y are defined with similar limits, but only x or y varies with h, not both. Here both x and y vary with a weighted h, determined by a particular unit vector $\vec{u}$. This may look a bit intimidating but in reality it is not too difficult to deal with; it often just requires extra algebra. However, the following theorem reduces this algebraic load.

Theorem 112 Directional Derivatives

Let $z = f(x, y)$ be differentiable on an open set S containing (x_0, y_0), and let $\vec{u} = \langle u_1, u_2 \rangle$ be a unit vector. The directional derivative of f at (x_0, y_0) in the direction of $\vec{u}$ is

$$D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2.$$

Example 12.34 Computing directional derivatives

Let $z = 14 - x^2 - y^2$ and let $P = (1, 2)$. Find the directional derivative of f, at P, in the following directions:

1. toward the point $Q = (3, 4)$,
2. in the direction of $\langle 2, -1 \rangle$, and

Notes:

3. toward the origin.

Solution The surface is plotted in Figure 12.16, where the point $P = (1,2)$ is indicated in the x,y-plane as well as the point $(1,2,9)$ which lies on the surface of f. We find that $f_x(x,y) = -2x$ and $f_x(1,2) = -2$; $f_y(x,y) = -2y$ and $f_y(1,2) = -4$.

1. Let $\vec{u}_1$ be the unit vector that points from the point $(1,2)$ to the point $Q = (3,4)$, as shown in the figure. The vector $\overrightarrow{PQ} = \langle 2,2\rangle$; the unit vector in this direction is $\vec{u}_1 = \langle 1/\sqrt{2}, 1/\sqrt{2}\rangle$. Thus the directional derivative of f at $(1,2)$ in the direction of $\vec{u}_1$ is
$$D_{\vec{u}_1}f(1,2) = -2(1/\sqrt{2}) + (-4)(1/\sqrt{2}) = -6/\sqrt{2} \approx -4.24.$$
Thus the instantaneous rate of change in moving from the point $(1,2,9)$ on the surface in the direction of $\vec{u}_1$ (which points toward the point Q) is about -4.24. Moving in this direction moves one steeply downward.

2. We seek the directional derivative in the direction of $\langle 2,-1\rangle$. The unit vector in this direction is $\vec{u}_2 = \langle 2/\sqrt{5}, -1/\sqrt{5}\rangle$. Thus the directional derivative of f at $(1,2)$ in the direction of $\vec{u}_2$ is
$$D_{\vec{u}_2}f(1,2) = -2(2/\sqrt{5}) + (-4)(-1/\sqrt{5}) = 0.$$
Starting on the surface of f at $(1,2)$ and moving in the direction of $\langle 2,-1\rangle$ (or $\vec{u}_2$) results in no instantaneous change in z-value. This is analogous to standing on the side of a hill and choosing a direction to walk that does not change the elevation. One neither walks up nor down, rather just "along the side" of the hill.

   Finding these directions of "no elevation change" is important.

3. At $P = (1,2)$, the direction towards the origin is given by the vector $\langle -1,-2\rangle$; the unit vector in this direction is $\vec{u}_3 = \langle -1/\sqrt{5}, -2/\sqrt{5}\rangle$. The directional derivative of f at P in the direction of the origin is
$$D_{\vec{u}_3}f(1,2) = -2(-1/\sqrt{5}) + (-4)(-2/\sqrt{5}) = 10/\sqrt{5} \approx 4.47.$$
Moving towards the origin means "walking uphill" quite steeply, with an initial slope of about 4.47.

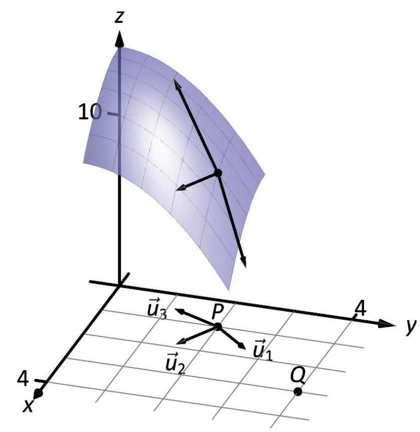


Figure 12.16: Understanding the directional derivative in Example 12.34.

As we study directional derivatives, it will help to make an important connection between the unit vector $\vec{u} = \langle u_1, u_2\rangle$ that describes the direction and the partial derivatives f_x and f_y. We start with a definition and follow this with a Key Idea.

Notes:

Definition 95 Gradient

Let $z = f(x, y)$ be differentiable on an open set S that contains the point (x_0, y_0).

1. The **gradient of** f is $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$.
2. The **gradient of** f **at** (x_0, y_0) **is** $\nabla f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle$.

To simplify notation, we often express the gradient as $\nabla f = \langle f_x, f_y\rangle$. The gradient allows us to compute directional derivatives in terms of a dot product.

Note: The symbol "∇" is named "nabla," derived from the Greek name of a Jewish harp. Oddly enough, in mathematics the expression ∇f is pronounced "del f."

Key Idea 57 The Gradient and Directional Derivatives

The directional derivative of $z = f(x, y)$ in the direction of $\vec{u}$ is

$$D_{\vec{u}}\, f = \nabla f \cdot \vec{u}.$$

The properties of the dot product previously studied allow us to investigate the properties of the directional derivative. Given that the directional derivative gives the instantaneous rate of change of z when moving in the direction of $\vec{u}$, three questions naturally arise:

1. In what direction(s) is the change in z the greatest (i.e., the "steepest uphill")?
2. In what direction(s) is the change in z the least (i.e., the "steepest downhill")?
3. In what direction(s) is there no change in z?

Using the key property of the dot product, we have

$$\nabla f \cdot \vec{u} = ||\,\nabla f\,||\,||\,\vec{u}\,||\cos\theta = ||\,\nabla f\,||\cos\theta, \tag{12.4}$$

where θ is the angle between the gradient and $\vec{u}$. (Since $\vec{u}$ is a unit vector, $||\,\vec{u}\,|| = 1$.) This equation allows us to answer the three questions stated previously.

1. Equation 12.4 is maximized when $\cos\theta = 1$, i.e., when the gradient and $\vec{u}$ have the same direction. We conclude the gradient points in the direction of greatest z change.

Notes:

2. Equation 12.4 is minimized when $\cos\theta = -1$, i.e., when the gradient and $\vec{u}$ have opposite directions. We conclude the gradient points in the opposite direction of the least z change.

3. Equation 12.4 is 0 when $\cos\theta = 0$, i.e., when the gradient and $\vec{u}$ are orthogonal to each other. We conclude the gradient is orthogonal to directions of no z change.

This result is rather amazing. Once again imagine standing in a rolling meadow and face the direction that leads you steepest uphill. Then the direction that leads steepest downhill is directly behind you, and side–stepping either left or right (i.e., moving perpendicularly to the direction you face) does not change your elevation at all.

Recall that a level curve is defined as a curve in the x-y plane along which the z-values of a function do not change. Let a surface $z = f(x, y)$ be given, and let's represent one such level curve as a vector–valued function, $\vec{r}(t) = \langle x(t), y(t)\rangle$. As the output of f does not change along this curve, $f\big(x(t), y(t)\big) = c$ for all t, for some constant c.

Since f is constant for all t, $\frac{df}{dt} = 0$. By the Multivariable Chain Rule, we also know

$$\begin{aligned}\frac{df}{dt} &= f_x(x,y)x'(t) + f_y(x,y)y'(t)\\ &= \langle f_x(x,y), f_y(x,y)\rangle \cdot \langle x'(t), y'(t)\rangle\\ &= \nabla f \cdot \vec{r}\,'(t)\\ &= 0.\end{aligned}$$

This last equality states $\nabla f \cdot \vec{r}\,'(t) = 0$: the gradient is orthogonal to the derivative of $\vec{r}$, meaning the gradient is orthogonal to $\vec{r}$ itself. Our conclusion: at any point on a surface, the gradient at that point is orthogonal to the level curve that passes through that point.

We restate these ideas in a theorem, then use them in an example.

Theorem 113 The Gradient and Directional Derivatives

Let $z = f(x, y)$ be differentiable on an open set S with gradient ∇f, let $P = (x_0, y_0)$ be a point in S and let $\vec{u}$ be a unit vector.

1. The maximum value of $D_{\vec{u}}\,f(x_0, y_0)$ is $||\ \nabla f(x_0, y_0)\ ||$; the direction of maximal z increase is $\nabla f(x_0, y_0)$.

2. The minimum value of $D_{\vec{u}}\,f(x_0, y_0)$ is $-||\ \nabla f(x_0, y_0)\ ||$; the direction of minimal z increase is $-\nabla f(x_0, y_0)$.

3. At P, $\nabla f(x_0, y_0)$ is orthogonal to the level curve passing through $\big(x_0, y_0, f(x_0, y_0)\big)$.

Notes:

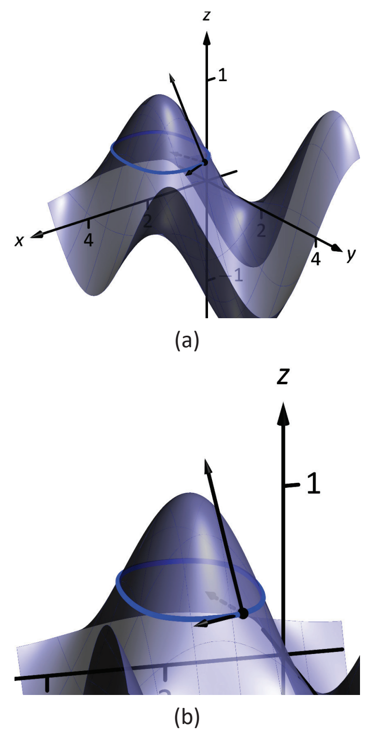


Figure 12.17: Graphing the surface and important directions in Example 12.35.

Example 12.35 Finding directions of maximal and minimal increase

Let $f(x,y) = \sin x\cos y$ and let $P = (\pi/3, \pi/3)$. Find the directions of maximal/minimal increase, and find a direction where the instantaneous rate of z change is 0.

Solution We begin by finding the gradient. $f_x = \cos x\cos y$ and $f_y = -\sin x\sin y$, thus

$$\nabla f = \langle \cos x\cos y, -\sin x\sin y\rangle \quad \text{and, at } P, \quad \nabla f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \left\langle \frac{1}{4}, -\frac{3}{4}\right\rangle.$$

Thus the direction of maximal increase is $\langle 1/4, -3/4\rangle$. In this direction, the instantaneous rate of z change is $||\langle 1/4, -3/4\rangle|| = \sqrt{10}/4 \approx 0.79$.

Figure 12.17 shows the surface plotted from two different perspectives. In each, the gradient is drawn at P with a dashed line (because of the nature of this surface, the gradient points "into" the surface). Let $\vec{u} = \langle u_1, u_2\rangle$ be the unit vector in the direction of ∇f at P. Each graph of the figure also contains the vector $\langle u_1, u_2, ||\nabla f||\rangle$. This vector has a "run" of 1 (because in the x-y plane it moves 1 unit) and a "rise" of $||\nabla f||$, hence we can think of it as a vector with slope of $||\nabla f||$ in the direction of ∇f, helping us visualize how "steep" the surface is in its steepest direction.

The direction of minimal increase is $\langle -1/4, 3/4\rangle$; in this direction the instantaneous rate of z change is $-\sqrt{10}/4 \approx -0.79$.

Any direction orthogonal to ∇f is a direction of no z change. We have two choices: the direction of $\langle 3, 1\rangle$ and the direction of $\langle -3, -1\rangle$. The unit vector in the direction of $\langle 3, 1\rangle$ is shown in each graph of the figure as well. The level curve at $z = \sqrt{3}/4$ is drawn: recall that along this curve the z-values do not change. Since $\langle 3, 1\rangle$ is a direction of no z-change, this vector is tangent to the level curve at P.

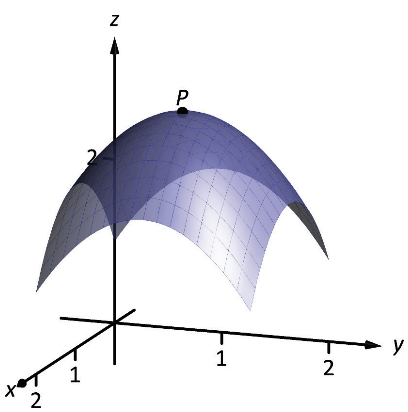


Figure 12.18: At the top of a paraboloid, all directional derivatives are 0.

Example 12.36 Understanding when $\nabla f = \vec{0}$

Let $f(x,y) = -x^2 + 2x - y^2 + 2y + 1$. Find the directional derivative of f in any direction at $P = (1, 1)$.

Solution We find $\nabla f = \langle -2x + 2, -2y + 2\rangle$. At P, we have $\nabla f(1, 1) = \langle 0, 0\rangle$. According to Theorem 113, this is the direction of maximal increase. However, $\langle 0, 0\rangle$ is directionless; it has no displacement. And regardless of the unit vector $\vec{u}$ chosen, $D_{\vec{u}}f = 0$.

Figure 12.18 helps us understand what this means. We can see that P lies at the top of a paraboloid. In all directions, the instantaneous rate of change is 0.

So what is the direction of maximal increase? It is fine to give an answer of $\vec{0} = \langle 0, 0\rangle$, as this indicates that all directional derivatives are 0.

Notes:

The fact that the gradient of a surface always points in the direction of steepest increase/decrease is very useful, as illustrated in the following example.

Example 12.37 The flow of water downhill
Consider the surface given by $f(x, y) = 20 - x^2 - 2y^2$. Water is poured on the surface at $(1, 1/4)$. What path does it take as it flows downhill?

Solution Let $\vec{r}(t) = \langle x(t), y(t)\rangle$ be the vector–valued function describing the path of the water in the x-y plane; we seek $x(t)$ and $y(t)$. We know that water will always flow downhill in the steepest direction; therefore, at any point on its path, it will be moving in the direction of $-\nabla f$. (We ignore the physical effects of momentum on the water.) Thus $\vec{r}\,'(t)$ will be parallel to ∇f, and there is some constant c such that $c\nabla f = \vec{r}\,'(t) = \langle x'(t), y'(t)\rangle$.

We find $\nabla f = \langle -2x, -4y\rangle$ and write $x'(t)$ as $\frac{dx}{dt}$ and $y'(t)$ as $\frac{dy}{dt}$. Then

$$\begin{aligned} c\nabla f &= \langle x'(t), y'(t)\rangle \\ \langle -2cx, -4cy\rangle &= \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle. \end{aligned}$$

This implies

$$\begin{aligned} -2cx = \frac{dx}{dt} \quad &\text{and} \quad -4cy = \frac{dy}{dt}, \text{ i.e.,} \\ c = -\frac{1}{2x}\frac{dx}{dt} \quad &\text{and} \quad c = -\frac{1}{4y}\frac{dy}{dt}. \end{aligned}$$

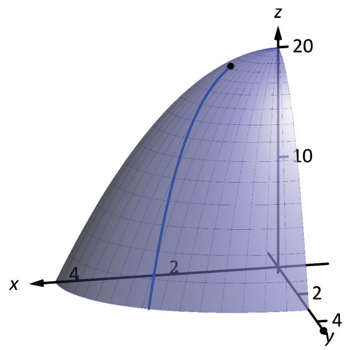
As c equals both expressions, we have

$$\frac{1}{2x}\frac{dx}{dt} = \frac{1}{4y}\frac{dy}{dt}.$$

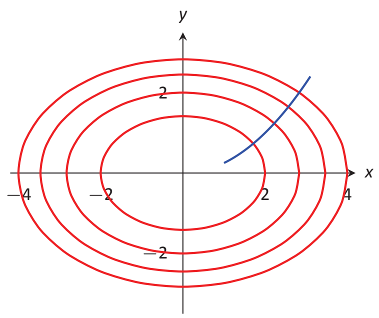
To find an explicit relationship between x and y, we can integrate both sides with respect to t. Recall from our study of differentials that $\dfrac{dx}{dt}dt = dx$. Thus:

$$\begin{aligned} \int \frac{1}{2x}\frac{dx}{dt}\,dt &= \int \frac{1}{4y}\frac{dy}{dt}\,dt \\ \int \frac{1}{2x}\,dx &= \int \frac{1}{4y}\,dy \\ \frac{1}{2}\ln|x| &= \frac{1}{4}\ln|y| + C_1 \\ 2\ln|x| &= \ln|y| + C_1 \\ \ln|x^2| &= \ln|y| + C_1 \end{aligned}$$

Notes:



(a)



(b)

Figure 12.19: A graph of the surface described in Example 12.37 along with the path in the x-y plane with the level curves.

Now raise both sides as a power of e:

$$\begin{aligned} x^2 &= e^{\ln|y|+C_1} \\ x^2 &= e^{\ln|y|}e^{C_1} \qquad \text{(Note that } e^{C_1} \text{ is just a constant.)} \\ x^2 &= yC_2 \\ \frac{1}{C_2}x^2 &= y \qquad \text{(Note that } 1/C_2 \text{ is just a constant.)} \\ Cx^2 &= y. \end{aligned}$$

As the water started at the point $(1, 1/4)$, we can solve for C:

$$C(1)^2 = \frac{1}{4} \quad \Rightarrow \quad C = \frac{1}{4}.$$

Thus the water follows the curve $y = x^2/4$ in the x-y plane. The surface and the path of the water is graphed in Figure 12.19(a). In part (b) of the figure, the level curves of the surface are plotted in the x-y plane, along with the curve $y = x^2/4$. Notice how the path intersects the level curves at right angles. As the path follows the gradient downhill, this reinforces the fact that the gradient is orthogonal to level curves.

Functions of Three Variables

The concepts of directional derivatives and the gradient are easily extended to three (and more) variables. We combine the concepts behind Definitions 94 and 95 and Theorem 112 into one set of definitions.

Definition 96 Directional Derivatives and Gradient with Three Variables

Let $w = F(x, y, z)$ be differentiable on an open ball B and let $\vec{u}$ be a unit vector in $\mathbb{R}^3$.

1. The **gradient** of F is $\nabla F = \langle F_x, F_y, F_z \rangle$.
2. The **directional derivative of F in the direction of $\vec{u}$** is

$$D_{\vec{u}}\,F = \nabla F \cdot \vec{u}.$$

The same properties of the gradient given in Theorem 113, when f is a func-

Notes:

tion of two variables, hold for F, a function of three variables.

Theorem 114 The Gradient and Directional Derivatives with Three Variables

Let $w = F(x,y,z)$ be differentiable on an open ball B, let ∇F be the gradient of F, and let $\vec{u}$ be a unit vector.

1. The maximum value of $D_{\vec{u}}\,F$ is $\|\nabla F\|$, obtained when the angle between ∇F and $\vec{u}$ is 0, i.e., the direction of maximal increase is ∇F.
2. The minimum value of $D_{\vec{u}}\,F$ is $-\|\nabla F\|$, obtained when the angle between ∇F and $\vec{u}$ is π, i.e., the direction of minimal increase is $-\nabla F$.
3. $D_{\vec{u}}\,F = 0$ when ∇F and $\vec{u}$ are orthogonal.

We interpret the third statement of the theorem as "the gradient is orthogonal to level surfaces," the three–variable analogue to level curves.

Example 12.38 Finding directional derivatives with functions of three variables

If a point source S is radiating energy, the intensity I at a given point P in space is inversely proportional to the square of the distance between S and P. That is, when $S = (0,0,0)$, $I(x,y,z) = \dfrac{k}{x^2+y^2+z^2}$ for some constant k.

Let $k = 1$, let $\vec{u} = \langle 2/3, 2/3, 1/3\rangle$ be a unit vector, and let $P = (2,5,3)$. Measure distances in inches. Find the directional derivative of I at P in the direction of $\vec{u}$, and find the direction of greatest intensity increase at P.

Solution We need the gradient ∇I, meaning we need I_x, I_y and I_z. Each partial derivative requires a simple application of the Quotient Rule, giving

$$\nabla I = \left\langle \frac{-2x}{(x^2+y^2+z^2)^2}, \frac{-2y}{(x^2+y^2+z^2)^2}, \frac{-2z}{(x^2+y^2+z^2)^2} \right\rangle$$

$$\nabla I(2,5,3) = \left\langle \frac{-4}{1444}, \frac{-10}{1444}, \frac{-6}{1444} \right\rangle \approx \langle -0.003, -0.007, -0.004\rangle$$

$$\begin{aligned} D_{\vec{u}}\,I &= \nabla I(2,5,3)\cdot\vec{u} \\ &= -\frac{17}{2166} \approx -0.0078. \end{aligned}$$

The directional derivative tells us that moving in the direction of $\vec{u}$ from P results in a decrease in intensity of about -0.008 units per inch. (The intensity is decreasing as $\vec{u}$ moves one farther from the origin than P.)

Notes:

The gradient gives the direction of greatest intensity increase. Notice that

$$\begin{aligned}\nabla I(2,5,3) &= \left\langle \frac{-4}{1444}, \frac{-10}{1444}, \frac{-6}{1444} \right\rangle \\ &= \frac{2}{1444} \left\langle -2,-5,-3 \right\rangle .\end{aligned}$$

That is, the gradient at $(2,5,3)$ is pointing in the direction of $\langle -2,-5,-3\rangle$, that is, towards the origin. That should make intuitive sense: the greatest increase in intensity is found by moving towards to source of the energy.

The directional derivative allows us to find the instantaneous rate of z change in any direction at a point. We can use these instantaneous rates of change to define lines and planes that are *tangent* to a surface at a point, which is the topic of the next section.

Notes:

Exercises 12.6

Terms and Concepts

1. What is the difference between a directional derivative and a partial derivative?

2. For what $\vec{u}$ is $D_{\vec{u}}f = f_x$?

3. For what $\vec{u}$ is $D_{\vec{u}}f = f_y$?

4. The gradient is ________ to level curves.

5. The gradient points in the direction of ________ increase.

6. It is generally more informative to view the directional derivative not as the result of a limit, but rather as the result of a ________ product.

Problems

In Exercises 7 – 12, a function $z = f(x,y)$ is given. Find ∇f.

7. $f(x,y) = -x^2y + xy^2 + xy$

8. $f(x,y) = \sin x \cos y$

9. $f(x,y) = \dfrac{1}{x^2+y^2+1}$

10. $f(x,y) = -4x + 3y$

11. $f(x,y) = x^2 + 2y^2 - xy - 7x$

12. $f(x,y) = x^2y^3 - 2x$

In Exercises 13 – 18, a function $z = f(x,y)$ and a point P are given. Find the directional derivative of f in the indicated directions. Note: these are the same functions as in Exercises 7 through 12.

13. $f(x,y) = -x^2y + xy^2 + xy$, $P = (2,1)$
    (a) In the direction of $\vec{v} = \langle 3,4\rangle$
    (b) In the direction toward the point $Q = (1,-1)$.

14. $f(x,y) = \sin x \cos y$, $P = \left(\dfrac{\pi}{4}, \dfrac{\pi}{3}\right)$
    (a) In the direction of $\vec{v} = \langle 1,1\rangle$.
    (b) In the direction toward the point $Q = (0,0)$.

15. $f(x,y) = \dfrac{1}{x^2+y^2+1}$, $P = (1,1)$.
    (a) In the direction of $\vec{v} = \langle 1,-1\rangle$.
    (b) In the direction toward the point $Q = (-2,-2)$.

16. $f(x,y) = -4x + 3y$, $P = (5,2)$
    (a) In the direction of $\vec{v} = \langle 3,1\rangle$.
    (b) In the direction toward the point $Q = (2,7)$.

17. $f(x,y) = x^2 + 2y^2 - xy - 7x$, $P = (4,1)$
    (a) In the direction of $\vec{v} = \langle -2,5\rangle$
    (b) In the direction toward the point $Q = (4,0)$.

18. $f(x,y) = x^2y^3 - 2x$, $P = (1,1)$
    (a) In the direction of $\vec{v} = \langle 3,3\rangle$
    (b) In the direction toward the point $Q = (1,2)$.

In Exercises 19 – 24, a function $z = f(x,y)$ and a point P are given.

(a) Find the direction of maximal increase of f at P.

(b) What is the maximal value of $D_{\vec{u}}f$ at P?

(c) Find the direction of minimal increase of f at P.

(d) Give a direction $\vec{u}$ such that $D_{\vec{u}}f = 0$ at P.

Note: these are the same functions and points as in Exercises 13 through 18.

19. $f(x,y) = -x^2y + xy^2 + xy$, $P = (2,1)$

20. $f(x,y) = \sin x \cos y$, $P = \left(\dfrac{\pi}{4}, \dfrac{\pi}{3}\right)$

21. $f(x,y) = \dfrac{1}{x^2+y^2+1}$, $P = (1,1)$.

22. $f(x,y) = -4x + 3y$, $P = (5,4)$.

23. $f(x,y) = x^2 + 2y^2 - xy - 7x$, $P = (4,1)$

24. $f(x,y) = x^2y^3 - 2x$, $P = (1,1)$

In Exercises 25 – 28, a function $w = F(x,y,z)$, a vector $\vec{v}$ and a point P are given.

(a) Find $\nabla F(x,y,z)$.

(b) Find $D_{\vec{u}}F$ at P.

25. $F(x,y,z) = 3x^2z^3 + 4xy - 3z^2$, $\vec{v} = \langle 1,1,1\rangle$, $P = (3,2,1)$

26. $F(x,y,z) = \sin(x)\cos(y)e^z$, $\vec{v} = \langle 2,2,1\rangle$, $P = (0,0,0)$

27. $F(x,y,z) = x^2y^2 - y^2z^2$, $\vec{v} = \langle -1,7,3\rangle$, $P = (1,0,-1)$

28. $F(x,y,z) = \dfrac{2}{x^2+y^2+z^2}$, $\vec{v} = \langle 1,1,-2\rangle$, $P = (1,1,1)$