13.8 Triple Integration with Spherical Coordinates

In the previous section, we explored how cylindrical coordinates can make it easier to compute certain triple integrals. Other such integrals are more naturally handled using *spherical* coordinates. Objects bounded by spheres or conefs are more easily described using this coordinate system. The Earth, for example, is a solid sphere (or near enough). On its surface we use two coordinates - latitude and longitude. To dig inward or fly outward, there is a third coordinate, the distance ρ from the center. This Greek letter *rho* (ρ) replaces radius r to avoid confusion with cylindrical coordinates. Where r is measured from the z-axis, ρ is measured directly from the origin. Thus for any point (x, y, z),

$$\rho^2 = x^2 + y^2 + z^2$$

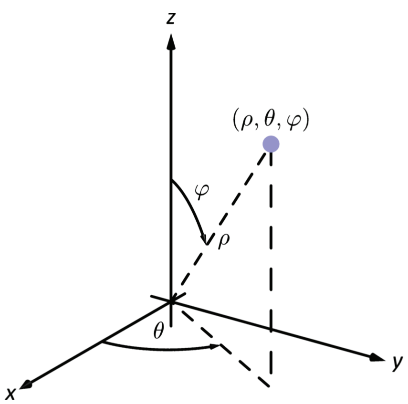


Figure 13.56: Illustrating the principles behind spherical coordinates.

which is the square of the distance between the origin and the point. The angle θ is the same as in cylindrical coordinates, and it goes from 0 to 2π on a full sphere with $\theta = 0$ pointing in the direction of the positive x-axis. It is the longitude, which increases as you travel east around the Equator. The angle φ is new, however. It equals 0 at the North Pole and π (not 2π) at the South Pole. It is measured down from the z-axis. The Equator, for example, has a latitude of 0 degrees but has angle $\varphi = \frac{\pi}{2}$ instead. See Figure 13.56. (The angle φ is $\frac{\pi}{2}$ minus the latitude (in radians) on Earth.)

The spherical coordinates of a point (x, y, z) are given by the ordered triple (ρ, θ, φ) where ρ, θ, and φ can be restricted to $\rho \geq 0$, $0 \leq \theta \leq 2\pi$, and $0 \leq \varphi \leq \pi$. The relationship between spherical and Cartesian coordinates is illustrated by Figure 13.56. From the triangles, we have

$$z = \rho \cos \varphi \text{ and } r = \rho \sin \varphi$$

But we know that $x = r \cos \theta$ and $y = r \sin \theta$ from before, so we end up with the following conversion equations.

Notes:

Key Idea 61 Converting between spherical and Cartesian coordinates

Given a point (ρ, θ, φ) in spherical coordinates, its Cartesian coordinates are

$$x = \rho \sin\varphi \cos\theta,\ y = \rho \sin\varphi \sin\theta,\ z = \rho \cos\varphi.$$

(Using the spherical-to-cylindrical conversion $r = \rho \sin\theta$ can help one remember the x and y formulas above.)

Given a point (x, y, z) in Cartesian coordinates, we use

- $\rho^2 = x^2 + y^2 + z^2$
- $\tan\theta = \dfrac{y}{x}$ for $x \neq 0$
- $\cos\varphi = \dfrac{z}{\rho}$ except at the origin (where φ can have any value)

to convert to spherical coordinates.

Note: The role of θ and φ in spherical coordinates differs between mathematicians and physicists. When reading about physics in spherical coordinates, be careful to note how that particular author uses these variables and recognize that these identities will may no longer be valid.

Example 13.51 Converting between rectangular and spherical coordinates

Convert the rectangular point $(-2, 2, 1)$ to spherical coordinates, and convert the spherical point $(6, \pi/3, \pi/2)$ to rectangular and cylindrical coordinates.

Solution This rectangular point is the same as used in Example 13.44. Using Key Idea 61, we find $\rho = \sqrt{(-2)^2 + 2^2 + 1^2} = 3$. Using the same logic as in Example 13.44, we find $\theta = 3\pi/4$. Finally, $\cos\varphi = 1/3$, giving $\varphi = \cos^{-1}(1/3) \approx 1.23$, or about $70.53°$. Thus the spherical coordinates are approximately $(3, 3\pi/4, 1.23)$.

Converting the spherical point $(6, \pi/3, \pi/2)$ to rectangular, we have $x = 6\sin(\pi/2)\cos(\pi/3) = 3$, $y = 6\sin(\pi/2)\sin(\pi/3) = 3\sqrt{3}$ and $z = 6\cos(\pi/2) = 0$. Thus the rectangular coordinates are $(3, 3\sqrt{3}, 0)$.

To convert this spherical point to cylindrical, we have $r = 6\sin(\pi/2) = 6$, $\theta = \pi/3$ and $z = 6\cos(\pi/2) = 0$, giving the cylindrical point $(6, \pi/3, 0)$.

Example 13.52 Canonical surfaces in spherical coordinates

Describe the surfaces $\rho = 1$, $\theta = \pi/3$ and $\varphi = \pi/6$, given in spherical coordinates.

Solution The equation $\rho = 1$ describes all points in space that are 1

Notes:

unit away from the origin: this is the sphere of radius 1, centered at the origin.

The equation $\theta = \pi/3$ describes the same surface in spherical coordinates as it does in cylindrical coordinates: beginning with the line $\theta = \pi/3$ in the *x*-*y* plane as given by polar coordinates, extend the line parallel to the *z*-axis, forming a plane.

The equation $\varphi = \pi/6$ describes all points P in space where the ray from the origin to P makes an angle of $\pi/6$ with the positive *z*-axis. This describes a cone, with the positive *z*-axis its axis of symmetry, with point at the origin.

All three surfaces are graphed in Figure 13.57. Note how their intersection uniquely defines the point $P = (1, \pi/3, \pi/6)$.

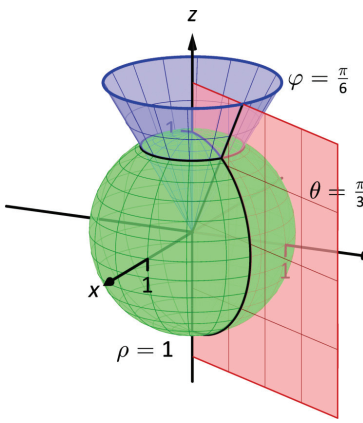


Figure 13.57: Graphing the canonical surfaces in spherical coordinates from Example 13.52.

Spherical coordinates are useful when describing certain domains in space, allowing us to evaluate triple integrals over these domains more easily than if we used rectangular coordinates or cylindrical coordinates. The crux of setting up a triple integral in spherical coordinates is appropriately describing the "small amount of volume," dV, used in the integral.

Considering Figure 13.58, we can make a small "spherical wedge" by varying ρ, θ and φ each a small amount, $\Delta\rho$, $\Delta\theta$ and $\Delta\varphi$, respectively. This wedge is approximately a rectangular solid when the change in each coordinate is small, giving a volume of about

$$\Delta V \approx \Delta\rho \ \times \ \rho\Delta\varphi \ \times \ \rho\sin(\varphi)\Delta\theta.$$

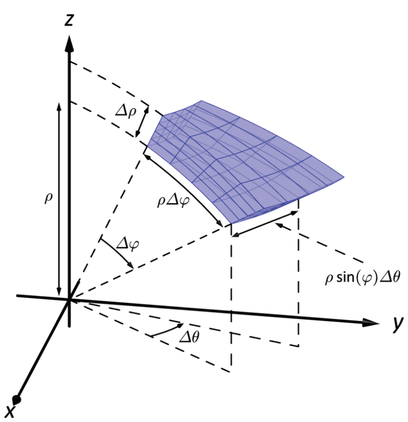


Figure 13.58: Approximating the volume of a standard region in space using spherical coordinates.

Given a region D in space, we can approximate the volume of D with many such wedges. As the size of each of $\Delta\rho$, $\Delta\theta$ and $\Delta\varphi$ goes to zero, the number of wedges increases to infinity and the volume of D is more accurately approximated, giving

$$dV = d\rho \ \times \ \rho\, d\varphi \ \times \ \rho\sin(\varphi)d\theta = \rho^2\sin(\varphi)\, d\rho\, d\theta\, d\varphi.$$

Again, this development of dV should sound reasonable, and the following theorem states it is the appropriate manner by which triple integrals are to be evaluated in spherical coordinates.

Theorem 134 Triple Integration in Spherical Coordinates

Let $w = h(\rho, \theta, \varphi)$ be a continuous function on a closed, bounded region D in space, bounded in spherical coordinates by $\alpha_1 \le \varphi \le \alpha_2$, $\beta_1 \le \theta \le \beta_2$ and $f_1(\theta,\varphi) \le \rho \le f_2(\theta,\varphi)$. Then

$$\iiint_D h(\rho,\theta,\varphi)\, dV = \int_{\alpha_1}^{\alpha_2}\int_{\beta_1}^{\beta_2}\int_{f_1(\theta,\varphi)}^{f_2(\theta,\varphi)} h(\rho,\theta,\varphi)\rho^2\sin(\varphi)\, d\rho\, d\theta\, d\varphi.$$

Note: It is generally most intuitive to evaluate the triple integral in Theorem 134 by integrating with respect to ρ first; it often does not matter whether we next integrate with respect to θ or φ. Different texts present different standard orders, some preferring $d\varphi\, d\theta$ instead of $d\theta\, d\varphi$. As the bounds for these variables are usually constants in practice, it generally is a matter of preference.

Notes:

Example 13.53 Establishing the volume of a sphere

Let D be the region in space bounded by the sphere, centered at the origin, of radius r. Use a triple integral in spherical coordinates to find the volume V of D.

Solution The sphere of radius r, centered at the origin, has equation $\rho = r$. To obtain the full sphere, the bounds on θ and φ are $0 \leq \theta \leq 2\pi$ and $0 \leq \varphi \leq \pi$. This leads us to:

$$\begin{aligned}
V &= \iiint_D dV \\
&= \int_0^{\pi} \int_0^{2\pi} \int_0^{r} \left(\rho^2 \sin(\varphi)\right) d\rho\, d\theta\, d\varphi \\
&= \int_0^{\pi} \int_0^{2\pi} \left(\frac{1}{3}\rho^3 \sin(\varphi)\Big|_0^r\right) d\theta\, d\varphi \\
&= \int_0^{\pi} \int_0^{2\pi} \left(\frac{1}{3}r^3 \sin(\varphi)\right) d\theta\, d\varphi \\
&= \int_0^{\pi} \left(\frac{2\pi}{3}r^3 \sin(\varphi)\right) d\varphi \\
&= \left(-\frac{2\pi}{3}r^3 \cos(\varphi)\right)\Big|_0^{\pi} \\
&= \frac{4\pi}{3}r^3,
\end{aligned}$$

the familiar formula for the volume of a sphere. Note how the integration steps were easy, not using square–roots nor integration steps such as Substitution.

Example 13.54 Evaluating an integral using spherical coordinates

Evaluate the integral

$$\iiint_D x^2 + y^2 + z^2\, dV$$

where D is the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ but below the sphere of radius $\frac{1}{2}$ centered at $\left(0, 0, \frac{1}{2}\right)$.

Solution The cone $z = \sqrt{x^2 + y^2}$ is the cone pointing upward from the origin at an angle of $\frac{\pi}{4}$ from the xy-plane. The solid D therefore consists of a half-spherical top with a conical bottom, with the point of the cone at the origin, meeting the spherical top at the horizontal equator of the sphere.

Notes:

To describe this solid in spherical coordinates, note that the sphere of radius $\frac{1}{2}$ centered at the point $\left(0, 0, \frac{1}{2}\right)$ can be written as

$$x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}$$

which is equivalent to $x^2 + y^2 + z^2 = z$. In spherical coordinates, this says $\rho^2 = \rho\cos\varphi$, or $\rho = \cos\varphi$. As for the cone, the equation $\varphi = \frac{\pi}{4}$ describes this surface. Therefore D can be described as all points (ρ, θ, φ) with $0 \leq \rho \leq \cos\varphi$, $0 \leq \theta \leq 2\pi$, and $0 \leq \varphi \leq \frac{\pi}{4}$.

Lastly, to integrate we need to write the integrand $x^2 + y^2 + z^2$ as ρ^2 and include the spherical integration factor $\rho^2 \sin\varphi$. Therefore

$$\begin{aligned}
\iiint_E x^2 + y^2 + z^2\, dV &= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\varphi} \rho^4 \sin\varphi \, d\rho\, d\varphi\, d\theta \\
&= 2\pi \int_0^{\pi/4} \sin\varphi\, \frac{1}{5}\rho^5\Big|_0^{\cos\varphi} d\varphi \\
&= \frac{2}{5}\pi \int_0^{\pi/4} \cos^5\varphi \sin\varphi\, d\varphi \\
&= \frac{2}{5}\pi \left(-\frac{1}{6}\cos^6\varphi\Big|_0^{\pi/4}\right) \\
&= \frac{2}{5}\pi\left(\frac{1}{8} - 1\right) = \frac{7\pi}{120}
\end{aligned}$$

Example 13.55 Finding the center of mass using spherical coordinates

Find the center of mass of the solid with constant density enclosed above by $\rho = 4$ and below by $\varphi = \pi/6$, as illustrated in Figure 13.59.

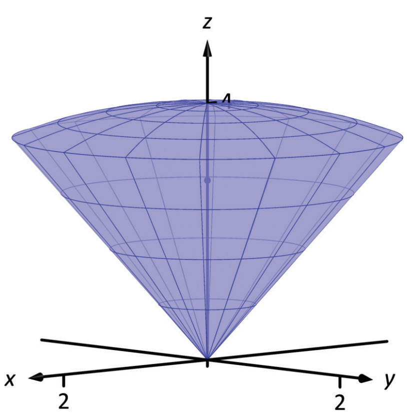


Figure 13.59: Graphing the solid, and its center of mass, from Example 13.55.

Solution We will set up the four triple integrals needed to find the center of mass (i.e., to compute M, M_{yz}, M_{xz} and M_{xy}) and leave it to the reader to evaluate each integral. Because of symmetry, we expect the x- and y- coordinates of the center of mass to be 0.

While the surfaces describing the solid are given in the statement of the problem, to describe the full solid D, we use the following bounds: $0 \leq \rho \leq 4$, $0 \leq \theta \leq 2\pi$ and $0 \leq \varphi \leq \pi/6$. Since density δ is constant, we assume $\delta = 1$.

Notes:

The mass of the solid:

$$\begin{aligned} M &= \iiint_D dm = \iiint_D dV \\ &= \int_0^{\pi/6} \int_0^{2\pi} \int_0^4 \big(\rho^2 \sin(\varphi)\big)\, d\rho\, d\theta\, d\varphi \\ &= \frac{64}{3}\big(2-\sqrt{3}\big)\pi \approx 17.958. \end{aligned}$$

To compute M_{yz}, the integrand is x; using Key Idea 60, we have $x = \rho \sin\varphi \cos\theta$. This gives:

$$\begin{aligned} M_{yz} &= \iiint_D x\, dm \\ &= \int_0^{\pi/6} \int_0^{2\pi} \int_0^4 \big((\rho \sin(\varphi)\cos(\theta))\rho^2 \sin(\varphi)\big)\, d\rho\, d\theta\, d\varphi \\ &= \int_0^{\pi/6} \int_0^{2\pi} \int_0^4 \big(\rho^3 \sin^2(\varphi)\cos(\theta)\big)\, d\rho\, d\theta\, d\varphi \\ &= 0, \end{aligned}$$

which we expected as we expect $\overline{x} = 0$.

To compute M_{xz}, the integrand is y; using Key Idea 61, we have $y = \rho \sin\varphi \sin\theta$. This gives:

$$\begin{aligned} M_{xz} &= \iiint_D y\, dm \\ &= \int_0^{\pi/6} \int_0^{2\pi} \int_0^4 \big((\rho \sin(\varphi)\sin(\theta))\rho^2 \sin(\varphi)\big)\, d\rho\, d\theta\, d\varphi \\ &= \int_0^{\pi/6} \int_0^{2\pi} \int_0^4 \big(\rho^3 \sin^2(\varphi)\sin(\theta)\big)\, d\rho\, d\theta\, d\varphi \\ &= 0, \end{aligned}$$

which we also expected as we expect $\overline{y} = 0$.

To compute M_{xy}, the integrand is z; using Key Idea 61, we have $z = \rho \cos\varphi$.

Notes:

This gives:

$$\begin{aligned}
M_{xy} &= \iiint_D z\,dm \\
&= \int_0^{\pi/6}\int_0^{2\pi}\int_0^4 \big((\rho\cos(\varphi))\rho^2\sin(\varphi)\big)\,d\rho\,d\theta\,d\varphi \\
&= \int_0^{\pi/6}\int_0^{2\pi}\int_0^4 \big(\rho^3\cos(\varphi)\sin(\varphi)\big)\,d\rho\,d\theta\,d\varphi \\
&= 16\pi \approx 50.266.
\end{aligned}$$

Thus the center of mass is $(0, 0, M_{xy}/M) \approx (0, 0, 2.799)$, as indicated in Figure 13.59.

This section has provided a brief introduction into two new coordinate systems useful for identifying points in space. Each can be used to define a variety of surfaces in space beyond the canonical surfaces graphed as each system was introduced.

However, the usefulness of these coordinate systems does not lie in the variety of surfaces that they can describe nor the regions in space these surfaces may enclose. Rather, cylindrical coordinates are mostly used to describe cylinders and spherical coordinates are mostly used to describe spheres. These shapes are of special interest in the sciences, especially in physics, and computations on/inside these shapes is difficult using rectangular coordinates. For instance, in the study of electricity and magnetism, one often studies the effects of an electrical current passing through a wire; that wire is essentially a cylinder, described well by cylindrical coordinates.

This chapter investigated the natural follow–on to partial derivatives: iterated integration. We learned how to use the bounds of a double integral to describe a region in the plane using both rectangular and polar coordinates, then later expanded to use the bounds of a triple integral to describe a region in space. We used double integrals to find volumes under surfaces, surface area, and the center of mass of lamina; we used triple integrals as an alternate method of finding volumes of space regions and also to find the center of mass of a region in space.

Integration does not stop here. We could continue to iterate our integrals, next investigating “quadruple integrals” whose bounds describe a region in 4–dimensional space (which are very hard to visualize). We can also look back to “regular” integration where we found the area under a curve in the plane. A natural analogue to this is finding the “area under a curve,” where the curve is in space, not in a plane. These are just two of many avenues to explore under the heading of “integration.”

Notes:

Exercises 13.8

Terms and Concepts

1. Explain the difference between the roles r, in cylindrical coordinates, and ρ, in spherical coordinates, play in determining the location of a point.

2. What surfaces are naturally defined using spherical coordinates?

3. Fill in the blank: To describe a full sphere, the bounds on θ would go from $\theta = 0$ to $\theta = 2\pi$ and φ would go from $\varphi = 0$ to $\varphi =$ ________.

4. T/F: In spherical coordinates, the direction of $\varphi = 0$ is perpendicular to the z-axis.

5. T/F: In spherical coordinates, the equation $\rho = 4$ describes the surface of a sphere.

Problems

In Exercises 6 – 7, points are given in either the rectangular, cylindrical or spherical coordinate systems. Find the coordinates of the points in the other systems.

6. (a) Points in rectangular coordinates: $(2, 2, 1)$ and $(-\sqrt{3}, 1, 0)$
   (b) Points in cylindrical coordinates: $(2, \pi/4, 2)$ and $(3, 3\pi/2, -4)$
   (c) Points in spherical coordinates: $(2, \pi/4, \pi/4)$ and $(1, 0, 0)$

7. (a) Points in rectangular coordinates: $(0, 1, 1)$ and $(-1, 0, 1)$
   (b) Points in cylindrical coordinates: $(0, \pi, 1)$ and $(2, 4\pi/3, 0)$
   (c) Points in spherical coordinates: $(2, \pi/6, \pi/2)$ and $(3, \pi, \pi)$

In Exercises 8 – 10, describe the curve, surface or region in space determined by the given bounds.

8. $\rho = 3, \quad 0 \le \theta \le 2\pi, \quad 0 \le \varphi \le \pi/2$

9. $0 \le \rho \le 2, \quad 0 \le \theta \le \pi, \quad \varphi = \pi/4$

10. $\rho = 2, \quad 0 \le \theta \le 2\pi, \quad \varphi = \pi/6$

In Exercises 11 – 13, evaluate the integral by first converting to spherical coordinates.

11. $\iiint_D x^2 + y^2 + z^2 \, dV$ where D is the solid sphere of radius 2 centered at the origin.

12. $\iiint_D \sqrt{x^2 + y^2 + z^2} \, dV$ where D is the solid between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ and above the cone $z = \sqrt{x^2 + y^2}$.

13. $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} 1 \, dz \, dy \, dx$

In Exercises 14 – 19, a triple integral in spherical coordinates is given. Describe the region in space defined by the bounds of the integral.

14. $\displaystyle\int_0^{\pi/2} \int_0^{\pi} \int_0^{1} \rho^2 \sin(\varphi) \, d\rho \, d\theta \, d\varphi$

15. $\displaystyle\int_0^{\pi} \int_0^{\pi} \int_1^{1.1} \rho^2 \sin(\varphi) \, d\rho \, d\theta \, d\varphi$

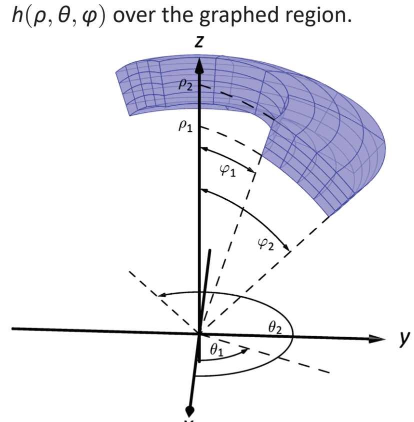
16. $\displaystyle\int_0^{2\pi} \int_0^{\pi/4} \int_0^{2} \rho^2 \sin(\varphi) \, d\rho \, d\theta \, d\varphi$

17. $\displaystyle\int_0^{2\pi} \int_{\pi/6}^{\pi/4} \int_0^{2} \rho^2 \sin(\varphi) \, d\rho \, d\theta \, d\varphi$

18. $\displaystyle\int_0^{2\pi} \int_0^{\pi/6} \int_0^{\sec\varphi} \rho^2 \sin(\varphi) \, d\rho \, d\theta \, d\varphi$

19. $\displaystyle\int_0^{2\pi} \int_0^{\pi/6} \int_0^{a\sec\varphi} \rho^2 \sin(\varphi) \, d\rho \, d\theta \, d\varphi$

20. A standard region in space, as defined by spherical coordinates, is shown. Set up the triple integral that integrates $h(\rho, \theta, \varphi)$ over the graphed region.



In Exercises 21 – 24, a solid is described along with its density function. Find the mass of the solid using spherical coordinates.

21. The upper half of the unit ball, bounded between $z = 0$ and $z = \sqrt{1 - x^2 - y^2}$, with density function $\delta(x, y, z) = 1$.

22. The spherical shell bounded between $x^2 + y^2 + z^2 = 16$ and $x^2 + y^2 + z^2 = 25$ with density function $\delta(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.

23. The conical region bounded above $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = 1$ with density function $\delta(x,y,z) = z$.

24. The cone bounded above $z = \sqrt{x^2+y^2}$ and below the plane $z = 1$ with density function $\delta(x,y,z) = z$.

In Exercises 25 – 28, a solid is described along with its density function. Find the center of mass of the solid using spherical coordinates. (Note: these are the same solids and density functions as found in Exercises 21 through 24.)

25. The upper half of the unit ball, bounded between $z = 0$ and $z = \sqrt{1-x^2-y^2}$, with density function $\delta(x,y,z) = 1$.

26. The spherical shell bounded between $x^2+y^2+z^2 = 16$ and $x^2+y^2+z^2 = 25$ with density function $\delta(x,y,z) = \sqrt{x^2+y^2+z^2}$.

27. The conical region bounded above $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = 1$ with density function $\delta(x,y,z) = z$.

28. The cone bounded above $z = \sqrt{x^2+y^2}$ and below the plane $z = 1$ with density function $\delta(x,y,z) = z$.

In Exercises 29 – 32, a region is space is described. Set up the triple integrals that find the volume of this region using rectangular, cylindrical and spherical coordinates, then comment on which of the three appears easiest to evaluate.

29. The region enclosed by the unit sphere, $x^2+y^2+z^2 = 1$.

30. The region enclosed by the cylinder $x^2+y^2 = 1$ and planes $z = 0$ and $z = 1$.

31. The region enclosed by the cone $z = \sqrt{x^2+y^2}$ and plane $z = 1$.

32. The cube enclosed by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$ and $z = 1$. (Hint: in spherical, use order of integration $d\rho\, d\varphi\, d\theta$.)

In Exercises 33 – 35, an integral equal to the volume of a solid is given. Describe the solid.

33. $\int_0^{2\pi}\int_0^1\int_0^4 r\, dz\, dr\, d\theta$

34. $\int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$

35. $\int_0^{2\pi}\int_{3\pi/4}^{\pi}\int_0^1 \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$

In Exercises 36 – 39, determine the limit by converting to spherical coordinates, similar to what was done for polar coordinates in Section 12.2.

36. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{z^2}{\sqrt{x^2+y^2}}$

37. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xyz-3x^2-3y^2-3z^2}{x^2+y^2+z^2}$

38. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{7xyz}{x^2+y^2+2z^2}$

39. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{e^z-1}{\sqrt{x^2+y^2+2z^2}}$