14.4 Flow, Flux, Green's Theorem and the Divergence Theorem

Flow and Flux

Line integrals over vector fields have the natural interpretation of computing work when $\vec{F}$ represents a force field. It is also common to use vector fields to represent velocities. In these cases, the line integral $\int_C \vec{F} \cdot d\vec{r}$ is said to represent **flow**.

Let the vector field $\vec{F} = \langle 1, 0 \rangle$ represent the velocity of water as it moves across a smooth surface, depicted in Figure 14.20. A line integral over C will compute "how much water is moving *along* the path C."

In the figure, "all" of the water above C_1 is moving along that curve, whereas "none" of the water above C_2 is moving along that curve (the curve and the flow of water are at right angles to each other). Because C_3 has nonzero horizontal and vertical components, "some" of the water above that curve is moving along the curve.

When C is a closed curve, we call flow **circulation**, represented by $\oint_C \vec{F} \cdot d\vec{r}$.

The "opposite" of flow is **flux**, a measure of "how much water is moving *across* the path C." If a curve represents a filter in flowing water, flux measures how much water will pass through the filter. Considering again Figure 14.20, we see that a screen along C_1 will not filter any water as no water passes across that curve. Because of the nature of this field, C_2 and C_3 each filter the same amount of water per second.

The terms "flow" and "flux" are used apart from velocity fields, too. Flow is measured by $\int_C \vec{F} \cdot d\vec{r}$, which is the same as $\int_C \vec{F} \cdot \vec{T}\, ds$ by Definition 119. That is, flow is a summation of the amount of $\vec{F}$ that is *tangent* to the curve C.

By contrast, flux is a summation of the amount of $\vec{F}$ that is *orthogonal* to the direction of travel. To capture this orthogonal amount of $\vec{F}$, we use $\int_C \vec{F} \cdot \vec{n}\, ds$ to measure flux, where $\vec{n}$ is a unit vector orthogonal to the curve C. (Later, we'll measure flux across surfaces, too. For example, in physics it is useful to measure the amount of a magnetic field that passes through a surface.)

How is $\vec{n}$ determined? We'll later see that if C is a closed curve, we'll want $\vec{n}$ to point to the outside of the curve (measuring how much is "going out"). We'll also adopt the convention that closed curves should be traversed counterclockwise.

(If C is a complicated closed curve, it can be difficult to determine what "counterclockwise" means. Consider Figure 14.21. Seeing the curve as a whole, we know which way "counterclockwise" is. If we zoom in on point A, one might incorrectly choose to traverse the path in the wrong direction. So we offer this definition: *a closed curve is being traversed counterclockwise if the outside is to the right of the path and the inside is to the left.*)

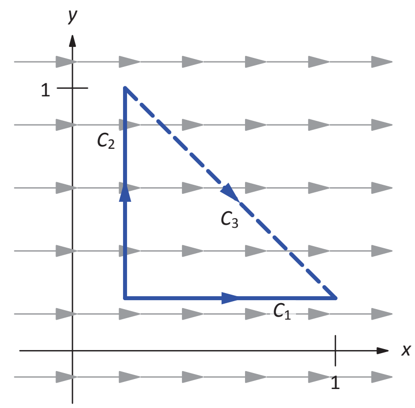


Figure 14.20: Illustrating the principles of flow and flux.

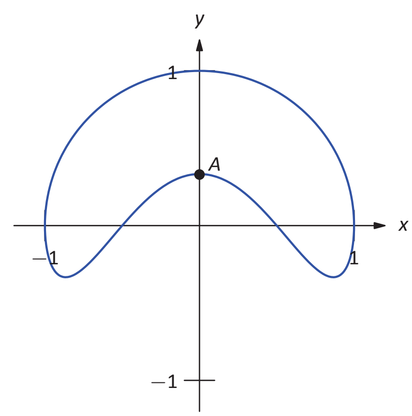


Figure 14.21: Determining "counterclockwise" is not always simple without a good definition.

Notes:

When a curve C is traversed counterclockwise by $\vec{r}(t) = \langle f(t), g(t) \rangle$, we rotate $\vec{T}$ clockwise 90° to obtain $\vec{n}$:

$$\vec{T} = \frac{\langle f'(t), g'(t) \rangle}{\|\,\vec{r}\,'(t)\,\|} \quad \Rightarrow \quad \vec{n} = \frac{\langle g'(t), -f'(t) \rangle}{\|\,\vec{r}\,'(t)\,\|}.$$

Letting $\vec{F} = \langle M, N \rangle$, we calculate flux as:

$$\begin{aligned}
\int_C \vec{F} \cdot \vec{n}\, ds &= \int_C \vec{F} \cdot \frac{\langle g'(t), -f'(t) \rangle}{\|\,\vec{r}\,'(t)\,\|} \|\,\vec{r}\,'(t)\,\|\, dt \\
&= \int_C \langle M, N \rangle \cdot \langle g'(t), -f'(t) \rangle\ dt \\
&= \int_C \Big(M g'(t) - N f'(t) \Big) dt \\
&= \int_C M g'(t)\, dt - \int_C N f'(t)\, dt.
\end{aligned}$$

As the x and y components of $\vec{r}(t)$ are $f(t)$ and $g(t)$ respectively, the differentials of x and y are $dx = f'(t)dt$ and $dy = g'(t)dt$. We can then write the above integrals as:

$$= \int_C M\, dy - \int_C N\, dx.$$

This is often written as one integral (not incorrectly, though somewhat confusingly, as this one integral has two "d's"):

$$= \int_C M\, dy - N\, dx.$$

We summarize the above in the following definition.

Notes:

Definition 122 Flow, Flux

Let $\vec{F} = \langle M, N \rangle$ be a vector field with continuous components defined on a smooth curve C, parameterized by $\vec{r}(t) = \langle f(t), g(t) \rangle$, let $\vec{T}$ be the unit tangent vector of $\vec{r}(t)$, and let $\vec{n}$ be the clockwise 90°degree rotation of $\vec{T}$.

- The **flow** of $\vec{F}$ along C is
$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_C \vec{F} \cdot d\vec{r}.$$
- The **flux** of $\vec{F}$ across C is
$$\int_C \vec{F} \cdot \vec{n}\, ds = \int_C M\, dy - N\, dx = \int_C \Big(M\, g'(t) - N f'(t) \Big) dt.$$

This definition of flow also holds for curves in space, though it does not make sense to measure "flux across a curve" in space.

Measuring flow is essentially the same as finding work performed by a force as done in the previous examples. Therefore we practice finding only flux in the following example.

Example 14.16 Finding flux across curves in the plane

Curves C_1 and C_2 each start at $(1, 0)$ and end at $(0, 1)$, where C_1 follows the line $y = 1 - x$ and C_2 follows the unit circle, as shown in Figure 14.22. Find the flux across both curves for the vector fields $\vec{F}_1 = \langle y, -x + 1 \rangle$ and $\vec{F}_2 = \langle -x, 2y - x \rangle$.

Solution We begin by finding parameterizations of C_1 and C_2. As done in Example 14.13, parameterize C_1 by creating the line that starts at $(1, 0)$ and moves in the $\langle -1, 1 \rangle$ direction: $\vec{r}_1(t) = \langle 1, 0 \rangle + t \langle -1, 1 \rangle = \langle 1 - t, t \rangle$, for $0 \leq t \leq 1$. We parameterize C_2 with the familiar $\vec{r}_2(t) = \langle \cos t, \sin t \rangle$ on $0 \leq t \leq \pi/2$. For reference later, we give each function and its derivative below:

$$\vec{r}_1(t) = \langle 1 - t, t \rangle\,, \quad \vec{r}\,'_1(t) = \langle -1, 1 \rangle\,.$$

$$\vec{r}_2(t) = \langle \cos t, \sin t \rangle\,, \quad \vec{r}\,'_2(t) = \langle -\sin t, \cos t \rangle\,.$$

When $\vec{F} = \vec{F}_1 = \langle y, -x + 1 \rangle$ (as shown in Figure 14.22(a)), over C_1 we have $M = y = t$ and $N = -x + 1 = -(1 - t) + 1 = t$. Using Definition 122, we compute the flux:

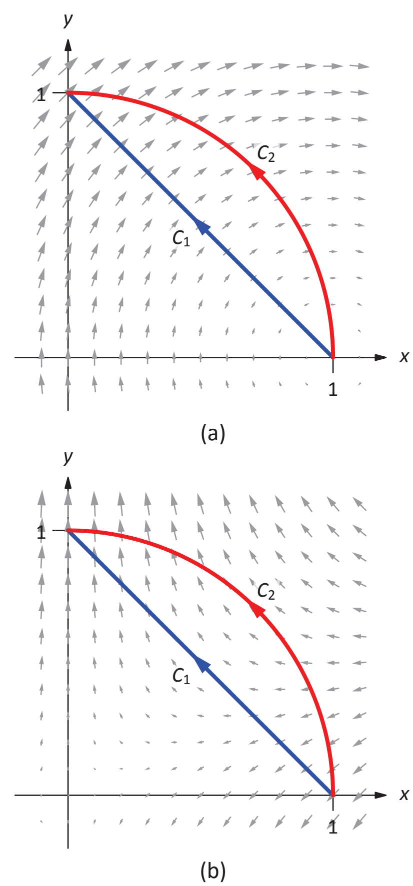


Figure 14.22: Illustrating the curves and vector fields in Example 14.16. In (a) the vector field is $\vec{F}_1$, and in (b) the vector field is $\vec{F}_2$.

Notes:

$$\begin{aligned}\int_{C_1} \vec{F}\cdot\vec{n}\,ds &= \int_{C_1} \Big(M\,g'(t) - N\,f'(t)\Big)\,dt\\ &= \int_0^1 \Big(t(1) - t(-1)\Big)\,dt\\ &= \int_0^1 2t\,dt\\ &= 1.\end{aligned}$$

Over C_2, we have $M = y = \sin t$ and $N = -x + 1 = 1 - \cos t$. Thus the flux across C_2 is:

$$\begin{aligned}\int_{C_1} \vec{F}\cdot\vec{n}\,ds &= \int_{C_1} \Big(M\,g'(t) - N\,f'(t)\Big)\,dt\\ &= \int_0^{\pi/2} \Big((\sin t)(\cos t) - (1-\cos t)(-\sin t)\Big)\,dt\\ &= \int_0^{\pi/2} \sin t\,dt\\ &= 1.\end{aligned}$$

Notice how the flux was the same across both curves. This won't hold true when we change the vector field.

When $\vec{F} = \vec{F}_2 = \langle -x, 2y - x\rangle$ (as shown in Figure 14.22(b)), over C_1 we have $M = -x = t - 1$ and $N = 2y - x = 2t - (1-t) = 3t - 1$. Computing the flux across C_1:

$$\begin{aligned}\int_{C_1} \vec{F}\cdot\vec{n}\,ds &= \int_{C_1} \Big(M\,g'(t) - N\,f'(t)\Big)\,dt\\ &= \int_0^1 \Big((t-1)(1) - (3t-1)(-1)\Big)\,dt\\ &= \int_0^1 (4t-2)\,dt\\ &= 0.\end{aligned}$$

Over C_2, we have $M = -x = -\cos t$ and $N = 2y - x = 2\sin t - \cos t$. Thus the flux across C_2 is:

Notes:

$$\begin{aligned}\int_{C_1} \vec{F}\cdot\vec{n}\,ds &= \int_{C_1} \Big(M\,g'(t) - N f'(t)\Big)\,dt\\ &= \int_0^{\pi/2} \Big((-\cos t)(\cos t) - (2\sin t - \cos t)(-\sin t)\Big)\,dt\\ &= \int_0^{\pi/2} \big(2\sin^2 t - \sin t\cos t - \cos^2 t\big)\,dt\\ &= \pi/4 - 1/2 \approx 0.285.\end{aligned}$$

We analyze the results of this example below.

In Example 14.16, we saw that the flux across the two curves was the same when the vector field was $\vec{F}_1 = \langle y, -x+1\rangle$. This is not a coincidence. We show why they are equal in Example 14.23. In short, the reason is this: the divergence of $\vec{F}_1$ is 0, and when $\operatorname{div}\vec{F} = 0$, the flux across any two paths with common beginning and ending points will be the same.

We also saw in the example that the flux across C_1 was 0 when the field was $\vec{F}_2 = \langle -x, 2y-x\rangle$. Flux measures "how much" of the field crosses the path from left to right (following the conventions established before). Positive flux means most of the field is crossing from left to right; negative flux means most of the field is crossing from right to left; zero flux means the same amount crosses from each side. When we consider Figure 14.22(b), it seems plausible that the same amount of $\vec{F}_2$ was crossing C_1 from left to right as from right to left.

Green's Theorem

There is an important connection between the circulation around a closed region R and the curl of the vector field inside of R, as well as a connection between the flux across the boundary of R and the divergence of the field inside R. These connections are described by Green's Theorem and the Divergence Theorem, respectively. We'll explore each in turn.

Green's Theorem states "the counterclockwise circulation around a closed region R is equal to the sum of the curls over R."

Theorem 141 Green's Theorem

Let R be a closed, bounded region of the plane whose boundary C is composed of finitely many smooth curves, let $\vec{r}(t)$ be a counterclockwise parameterization of C, and let $\vec{F} = \langle M, N\rangle$ where N_x and M_y are continuous over R. Then

$$\oint_C M dx + N dy = \oint_C \vec{F}\cdot d\vec{r} = \iint_R \operatorname{curl}\vec{F}\,dA = \iint_R \left(N_x - M_y\right)\,dA.$$

Notes:

We'll explore Green's Theorem through an example.

Example 14.17 Confirming Green's Theorem

Let $\vec{F} = \left\langle -y, x^2+1 \right\rangle$ and let R be the region of the plane bounded by the triangle with vertices $(-1,0)$, $(1,0)$ and $(0,2)$, shown in Figure 14.23. Verify Green's Theorem; that is, find the circulation of $\vec{F}$ around the boundary of R and show that is equal to the double integral of curl $\vec{F}$ over R.

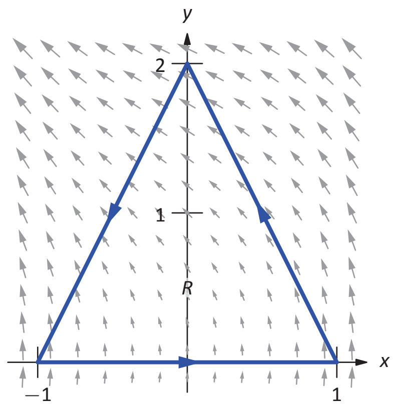


Figure 14.23: The vector field and planar region used in Example 14.17.

Solution The curve C that bounds R is composed of 3 lines. While we need to traverse the boundary of R in a counterclockwise fashion, we may start anywhere we choose. We arbitrarily choose to start at $(-1,0)$, move to $(1,0)$, etc., with each line parameterized by $\vec{r}_1(t)$, $\vec{r}_2(t)$ and $\vec{r}_3(t)$, respectively.

We leave it to the reader to confirm that the following parameterizations of the three lines are accurate:

$$\begin{aligned}
\vec{r}_1(t) &= \langle 2t-1, 0\rangle, && \text{for } 0\le t\le 1, && \text{with } \vec{r}_1{}'(t) = \langle 2,0\rangle,\\
\vec{r}_2(t) &= \langle 1-t, 2t\rangle, && \text{for } 0\le t\le 1, && \text{with } \vec{r}_2{}'(t) = \langle -1,2\rangle, \text{ and}\\
\vec{r}_3(t) &= \langle -t, 2-2t\rangle, && \text{for } 0\le t\le 1, && \text{with } \vec{r}_3{}'(t) = \langle -1,-2\rangle.
\end{aligned}$$

The circulation around C is found by summing the flow along each of the sides of the triangle. We again leave it to the reader to confirm the following computations:

$$\begin{aligned}
\int_{C_1} \vec{F}\cdot d\vec{r}_1 &= \int_0^1 \left\langle 0, (2t-1)^2+1\right\rangle \cdot \langle 2,0\rangle\,dt = 0,\\
\int_{C_2} \vec{F}\cdot d\vec{r}_2 &= \int_0^1 \left\langle -2t, (1-t)^2+1\right\rangle \cdot \langle -1,2\rangle\,dt = 11/3, \text{ and}\\
\int_{C_3} \vec{F}\cdot d\vec{r}_3 &= \int_0^1 \left\langle 2t-2, t^2+1\right\rangle \cdot \langle -1,-2\rangle\,dt = -5/3.
\end{aligned}$$

The circulation is the sum of the flows: 2.

We confirm Green's Theorem by computing $\iint_R \text{curl}\,\vec{F}\,dA$. We find curl $\vec{F} = 2x+1$. The region R is bounded by the lines $y=2x+2$, $y=-2x+2$ and $y=0$. Integrating with the order $dx\,dy$ is most straightforward, leading to

$$\int_0^2\int_{y/2-1}^{1-y/2} (2x+1)\,dx\,dy = \int_0^2 (2-y)\,dy = 2,$$

which matches our previous measurement of circulation.

Proof of Green's Theorem

Green's Theorem is not easy to prove in general. We will, however, verify it is true in the case that R is a simple region - one that is expressible in both the

Notes:

forms

$$R = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

for some curves $y = g_1(x)$ and $y = g_2(x)$, and

$$R = \{(x, y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

for some curves $x = h_1(y)$ and $x = h_2(y)$. Other regions which are the union of simple regions can then be done with this in hand.

For a vector field $\vec{F}(x, y) = \langle M(x, y), N(x, y)\rangle$ as above, the idea is to prove that

$$\oint_C M(x, y)\, dx = \iint_R -\frac{\partial M}{\partial y}\, dA$$

and

$$\oint_C N(x, y)\, dy = \iint_R \frac{\partial N}{\partial x}\, dA$$

separately. We verify the first and the leave the second, which is very similar, as an exercise.

Suppose that R is the region given by

$$\{(x, y) \mid a \leq x \leq b, g(x) \leq y \leq f(x)\}$$

for top curve $y = f(x)$ and bottom curve $y = g(x)$. In the double integral, first integrate with respect to y to get

$$\int_{g(x)}^{f(x)} -\frac{\partial M}{\partial y}\, dy = \Big(-M(x, y)\Big]_{y=g(x)}^{y=f(x)} = -M(x, f(x)) + M(x, g(x)).$$

Integrate with respect to x to yield

$$-\int_a^b M(x, f(x))\, dx + \int_a^b M(x, g(x))\, dx.$$

The other side of the equation deals with a line integral. For this we get

$$\oint_C M(x, y)dx = \int_{\text{top}} M(x, y)dx + \int_{\text{bottom}} M(x, y)dx = \int_b^a M(x, f(x))dx + \int_a^b M(x, g(x))dx$$

which is the same as the other side due to the addition of a minus sign when the bounds of the first integral are switched.

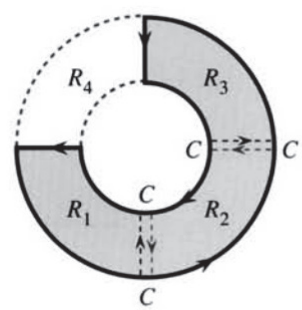


Figure 14.24: Union of simple regions

Next consider a region R which is not simple but is instead a union of simple regions, such as in Figure 14.24. As in the figure, we would break the region into

Notes:

three pieces R_1, R_2, and R_3 which are all simple. The three individual double integrals sum to the double integral of R, which would also equal the sum of the three line integrals by the above argument. When we add the line integrals, the line integrals over the cross cuts cancel out as they are going in opposite directions, resulting in only the boundary pieces adding up to the total line integral. This leaves the double integral of the interior equal to the line integral of just the boundary, as Green's Theorem would state. The next example will pertain to a region that is a union of two simple regions.

If the region R contains the piece R_4 in Figure 14.24, then the theorem is still true. The integral around the outside is still counterclockwise, but the integral is clockwise around the inner circle. Keeping the region R to your left as you go around C gives the counterclockwise direction according to our definition. The complete ring is doubly connected, not simply connected. Green's Theorem allows any finite number of regions R_i and cross cuts and holes.

Example 14.18 Using Green's Theorem

Compute the line integral $\oint_C y^2\,dx + 3xy\,dy$ where C is the boundary of the semi-annular region in the upper half-plane between $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution The enclosed region R is not simple, but is a union of two simple pieces if you cut down the y-axis. Therefore Green's Theorem applies in this situation. The theorem gives

$$\oint_C y^2\,dx + 3xy\,dy = \iint_R 3y - 2y\,dA = \iint_R y\,dA.$$

Notice that this region can be integrated over easily via polar coordinates. Therefore

$$\oint_C y^2\,dx + 3xy\,dy = \int_0^{\pi}\int_1^2 (r\sin\theta)r\,dr\,d\theta = \frac{14}{3}.$$

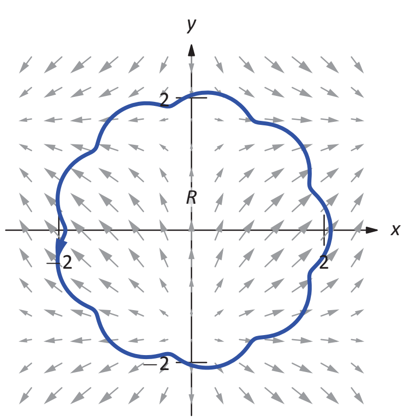


Figure 14.25: The vector field and planar region used in Example 14.19.

Example 14.19 Using Green's Theorem

Let $\vec{F} = \langle \sin x, \cos y\rangle$ and let R be the region enclosed by the curve C parameterized by $\vec{r}(t) = \left\langle 2\cos t + \frac{1}{10}\cos(10t), 2\sin t + \frac{1}{10}\sin(10t)\right\rangle$ on $0 \le t \le 2\pi$, as shown in Figure 14.25. Find the circulation around C.

Solution Computing the circulation directly using the line integral looks difficult, as the integrand will include terms like "$\sin\left(2\cos t + \frac{1}{10}\cos(10t)\right)$."

Green's Theorem states that $\oint_C \vec{F}\cdot d\vec{r} = \iint_R \operatorname{curl}\vec{F}\,dA$; since $\operatorname{curl}\vec{F} = 0$ in this example, the double integral is simply 0 and hence the circulation is 0.

Since $\operatorname{curl}\vec{F} = 0$, we can conclude that the circulation is 0 in two ways. One method is to employ Green's Theorem as done above. The second way is to

Notes:

recognize that $\vec{F}$ is a conservative field, hence there is a function $z = f(x, y)$ wherein $\vec{F} = \nabla f$. Let A be any point on the curve C; since C is closed, we can say that C "begins" and "ends" at A. By the Fundamental Theorem of Line Integrals, $\oint_C \vec{F}\, d\vec{r} = f(A) - f(A) = 0$.

Usng Green's Theorem to Compute Areas

So far, we have used Green's Theorem to rewrite line integrals as double integrals, making them easier to compute. However, Green's Theorem gives equality between a line integral and a double integral, and therefore can also be used to rewrite a double integral as a line integral. This can be done for any double integral, but we wll only focus on $\iint_R 1 dA$, the area of a region R.

Consider the region R in Figure 14.26. This region is inside the simple closed curve given by the parametric equations

$$x = \cos(3t) - \sin t \text{ and } y = 2\cos t$$

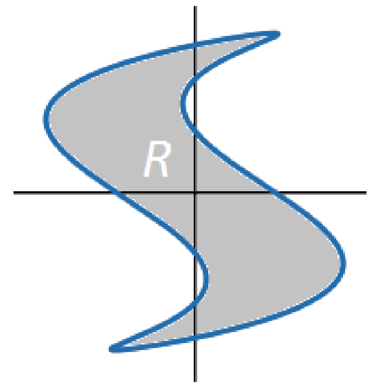


Figure 14.26: The region R is bounded inside the simple closed curve $x = \cos(3t) - \sin t$, $y = 2\cos t$

traced once on the interval $[0, 2\pi]$. Note that determining bounds for this region in terms of x and/or y, as we would typically do to compute rewrite the double integral as an iterated integral, cannot be done in a simple way. However, as we have a parameterization for the boundary curve, it is not hard to compute a line integral along that curve.

Consider any vector field $\vec{F}(x, y) = \langle M, N\rangle$ for which $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = 1$. There are infinitely many such vector fields, the most basic of which are $\vec{F}(x, y) = \langle 0, x\rangle$ or $\vec{F}(x, y) = \langle -y, 0\rangle\ \vec{i}$. What is the line integral of $\vec{F}$ around a simple closed curve C computing? By Green's Theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C M(x, y)\, dx + N(x, y)\, dy = \iint_R 1\, dA$$

which is the area of R, or the area enclosed by C. In this way, one can use a line integral to compute area as long as one knows the boundary curve of the region.

We summarize this using a Key Idea before we continue with examples.

Notes:

Key Idea 63 Area inside a simple closed curve C

Let R be the region inside a simple closed curve C, parameterized in the counterclockwise direction. Let $\vec{F}(x,y) = \langle M, N\rangle$ be any vector field for which $N_x - M_y = 1$ Then

$$\text{Area of } R = \iint_R 1\, dA = \oint_C M(x,y)\, dx + N(x,y)\, dy.$$

In particular,

$$\text{Area of } R = \iint_R 1\, dA = \oint_C x\, dy = -\oint_C y\, dx.$$

Example 14.20 Computing area using Green's Theorem

Compute the area of the region R inside the simple closed parametric curve C given by

$$x = \cos(3t) - \sin t \text{ and } y = 2\cos t$$

traced once on the interval $[0, 2\pi]$.

Solution This is the region shown in Figure 14.26. The given parameterization gives the counterclockwise direction. (If we did not know that, however, and computed a line integral in the opposite direction, it would give the opposite value. Since an area must be positive, we could take the absolute value in which case the direction of the curve would not matter.) Using Key Idea 63,

$$\begin{aligned}\iint_R 1\, dA &= \oint_C x\, dy\\ &= \int_0^{2\pi} \big(\cos(3t) - \sin t\big)(-2\sin t)\, dt\\ &= -2\int_0^{2\pi} \big(\sin t\cos(3t) - \sin^2 t\big)\, dt.\end{aligned}$$

Notes:

Recall now that we must proceed with the Product-to-Sum identities,

$$\begin{aligned}
\iint_R 1\,dA &= -2\int_0^{2\pi} \big(\sin t\cos(3t) - \sin^2 t\big)\,dt \\
&= -2\int_0^{2\pi} \left(\frac{\sin(4t)+\sin(-2t)}{2} - \frac{1-\cos(2t)}{2}\right)\,dt \\
&= \int_0^{2\pi} \big(-\sin(4t)+\sin(2t)+1-\cos(2t)\big)\,dt \\
&= \frac{\cos(4t)}{4} - \frac{\cos(2t)}{2} + t - \frac{\sin(2t)}{2}\Bigg|_0^{2\pi} \\
&= 2\pi \text{ square units.}
\end{aligned}$$

The ability to compute an area using Green's Theorem has physical value as well. A **planimeter** is a device used to compute the area of an arbitrary two-dimensional shape. It operates by tracing the boundary of the shape, and computing an appropriate line integral while tracing the boundary.

We continue with a more familiar shape.

Example 14.21 Area inside an ellipse
Using a line integral, compute the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with semiaxes of length a and b.

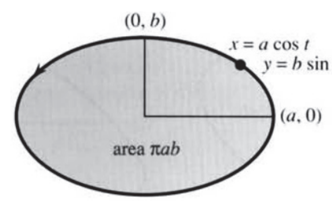


Figure 14.27: Ellipse

Solution See Figure 14.27. By the above equation for the ellipse, we can parameterize the ellipse as

$$x = a\cos(t),\;\; y = b\sin(t)$$

for $0 \le t \le 2\pi$, which traces the ellipse once in the counterclockwise direction.

Notes:

Using Key Idea 63, we get an area of

$$\begin{aligned}\iint_R 1\, dA &= \oint_C x\, dy \\ &= \int_0^{2\pi} a\cos(t)\,(b\cos(t))\, dt \\ &= ab\int_0^{2\pi} \cos^2(t)\, dt \\ &= ab\int_0^{2\pi} \frac{1+\cos(2t)}{2}\, dt \\ &= ab\left(\frac{t}{2}+\frac{\sin(2t)}{4}\right)\Big|_0^{2\pi} \\ &= \pi ab \text{ square units.}\end{aligned}$$

The Divergence Theorem

Green's Theorem makes a connection between the circulation around a closed region R and the sum of the curls over R. The Divergence Theorem makes a somewhat "opposite" connection: the total flux across the boundary of R is equal to the sum of the divergences over R.

Theorem 142 The Divergence Theorem (in the plane)

Let R be a closed, bounded region of the plane whose boundary C is composed of finitely many smooth curves, let $\vec{r}(t)$ be a counterclockwise parameterization of C, and let $\vec{F} = \langle M, N\rangle$ where M_x and N_y are continuous over R. Then

$$\oint_C \vec{F}\cdot\vec{n}\, ds = \iint_R \text{div}\,\vec{F}\, dA.$$

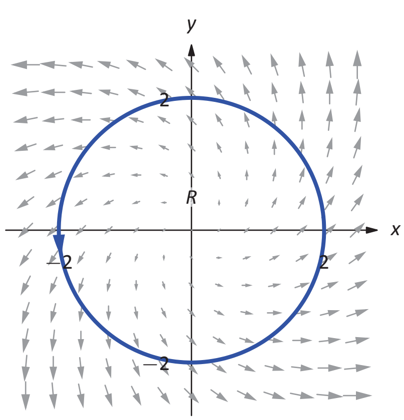


Figure 14.28: The region R used in Example 14.22.

Example 14.22 Confirming the Divergence Theorem

Let $\vec{F} = \langle x-y, x+y\rangle$, let C be the circle of radius 2 centered at the origin and define R to be the interior of that circle, as shown in Figure 14.28. Verify the Divergence Theorem; that is, find the flux across C and show it is equal to the double integral of $\text{div}\,\vec{F}$ over R.

Solution We parameterize the circle in the usual way, with $\vec{r}(t) =$

Notes:

$\langle 2\cos t, 2\sin t\rangle$, $0 \leq t \leq 2\pi$. The flux across C is

$$\begin{aligned}\oint_C \vec F\cdot\vec n\,ds &= \oint_C \big(Mg'(t) - Nf'(t)\big)\,dt\\ &= \int_0^{2\pi} \big((2\cos t - 2\sin t)(2\cos t) - (2\cos t + 2\sin t)(-2\sin t)\big)\,dt\\ &= \int_0^{2\pi} 4\,dt = 8\pi.\end{aligned}$$

We compute the divergence of $\vec F$ as $\text{div}\,\vec F = M_x + N_y = 2$. Since the divergence is constant, we can compute the following double integral easily:

$$\iint_R \text{div}\,\vec F\,dA = \iint_R 2\,dA = 2\iint_R dA = 2(\text{area of } R) = 8\pi,$$

which matches our previous result.

Example 14.23 **Flux when** $\text{div}\,\vec F = 0$

Let $\vec F$ be any field where $\text{div}\,\vec F = 0$, and let C_1 and C_2 be any two nonintersecting paths, except that each begin at point A and end at point B (see Figure 14.29). Show why the flux across C_1 and C_2 is the same.

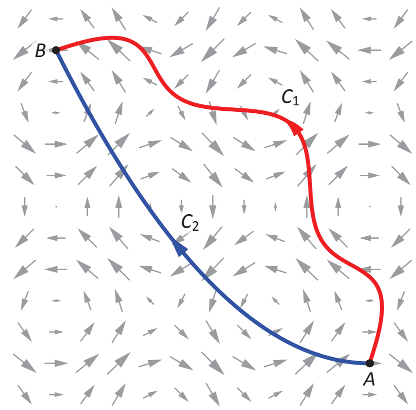


Figure 14.29: As used in Example 14.23, the vector field has a divergence of 0 and the two paths only intersect at their initial and terminal points.

Solution By referencing Figure 14.29, we see we can make a closed path C that combines C_1 with C_2, where C_2 is traversed with its opposite orientation. We label the enclosed region R. Since $\text{div}\,\vec F = 0$, the Divergence Theorem states that

$$\oint_C \vec F\cdot\vec n\,ds = \iint_R \text{div}\,\vec F\,dA = \iint_R 0\,dA = 0.$$

Using the properties and notation given in Theorem 137, consider:

$$\begin{aligned}0 &= \oint_C \vec F\cdot\vec n\,ds\\ &= \int_{C_1} \vec F\cdot\vec n\,ds + \int_{C_2^*} \vec F\cdot\vec n\,ds\end{aligned}$$

(where C_2^* is the path C_2 traversed with opposite orientation)

$$= \int_{C_1} \vec F\cdot\vec n\,ds - \int_{C_2} \vec F\cdot\vec n\,ds.$$

$$\int_{C_2} \vec F\cdot\vec n\,ds = \int_{C_1} \vec F\cdot\vec n\,ds.$$

Notes:

Thus the flux across each path is equal.

In this section, we have investigated flow and flux, quantities that measure interactions between a vector field and a planar curve. We can also measure flow along spatial curves, though as mentioned before, it does not make sense to measure flux across spatial curves.

It does, however, make sense to measure the amount of a vector field that passes across a surface in space – i.e, the flux across a surface. We will study this, though in the next section we first learn about a more powerful way to describe surfaces than using functions of the form $z = f(x, y)$.

Notes:

Exercises 14.4

Terms and Concepts

1. Let $\vec{F}$ be a vector field and let C be a curve. *Flow* is a measure of the amount of $\vec{F}$ going ________________ C; *flux* is a measure of the amount of $\vec{F}$ going ________________ C.

2. What is circulation?

3. Green's Theorem states, informally, that the circulation around a closed curve that bounds a region R is equal to the sum of ________________ across R.

4. The Divergence Theorem states, informally, that the outward flux across a closed curve that bounds a region R is equal to the sum of ________________ across R.

5. Let $\vec{F}$ be a vector field and let C_1 and C_2 be any nonintersecting paths except that each starts at point A and ends at point B. If ________ $= 0$, then $\int_{C_1} \vec{F}\cdot\vec{T}\,ds = \int_{C_2}\vec{F}\cdot\vec{T}\,ds$.

6. Let $\vec{F}$ be a vector field and let C_1 and C_2 be any nonintersecting paths except that each starts at point A and ends at point B. If ________ $= 0$, then $\int_{C_1} \vec{F}\cdot\vec{n}\,ds = \int_{C_2}\vec{F}\cdot\vec{n}\,ds$.

Problems

In Exercises 7 – 12, a vector field $\vec{F}$ and a curve C are given. Evaluate $\int_C \vec{F}\cdot\vec{n}\,ds$, the flux of $\vec{F}$ over C.

7. $\vec{F} = \langle x+y, x-y\rangle$; C is the curve with initial and terminal points $(3,-2)$ and $(3,2)$, respectively, parameterized by $\vec{r}(t) = \langle 3t^2, 2t\rangle$ on $-1\le t\le 1$.

8. $\vec{F} = \langle x+y, x-y\rangle$; C is the curve with initial and terminal points $(3,-2)$ and $(3,2)$, respectively, parameterized by $\vec{r}(t) = \langle 3, t\rangle$ on $-2\le t\le 2$.

9. $\vec{F} = \langle x^2, y+1\rangle$; C is line segment from $(0,0)$ to $(2,4)$.

10. $\vec{F} = \langle x^2, y+1\rangle$; C is the portion of the parabola $y=x^2$ from $(0,0)$ to $(2,4)$.

11. $\vec{F} = \langle y, 0\rangle$; C is the line segment from $(0,0)$ to $(0,1)$.

12. $\vec{F} = \langle y, 0\rangle$; C is the line segment from $(0,0)$ to $(1,1)$.

In Exercises 13 – 18, compute the line integrals and separately compute the double integrals in Green's Theorem. The has parametric equations $x = 2\cos(t)$, $y = 2\sin(t)$, and the triangle has sides $x=0$, $y=0$, $x+y=1$.

13. $\oint x\,dy$ along the circle

14. $\oint x^2y\,dy$ along the circle

15. $\oint x\,dx$ along the triangle

16. $\oint y\,dx$ along the triangle

17. $\oint x^2y\,dx$ along the circle

18. $\oint x^2y\,dx$ along the triangle

In Exercises 19 – 22, a vector field $\vec{F}$ and a closed curve C, enclosing a region R, are given. Verify Green's Theorem by evaluating $\oint_C \vec{F}\cdot d\vec{r}$ and $\iint_R \text{curl}\,\vec{F}\,dA$, showing they are equal.

19. $\vec{F} = \langle x-y, x+y\rangle$; C is the closed curve composed of the parabola $y=x^2$ on $0\le x\le 2$ followed by the line segment from $(2,4)$ to $(0,0)$.

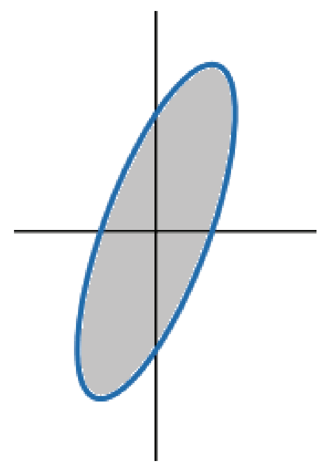
20. $\vec{F} = \langle -y, x\rangle$; C is the unit circle.

21. $\vec{F} = \langle 0, x^2\rangle$; C the triangle with corners at $(0,0)$, $(2,0)$ and $(1,1)$.

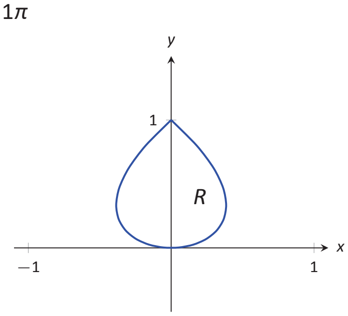
22. $\vec{F} = \langle x+y, 2x\rangle$; C the curve that starts at $(0,1)$, follows the parabola $y=(x-1)^2$ to $(3,4)$, then follows a line back to $(0,1)$.

In Exercises 23 – 32, a closed curve C enclosing a region R is given. Find the area of R using Key Idea 63.

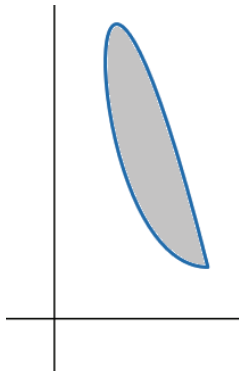
23. $x = \cos t - \sin t$
$y = 3\cos t$
$0\le t\le 2\pi$



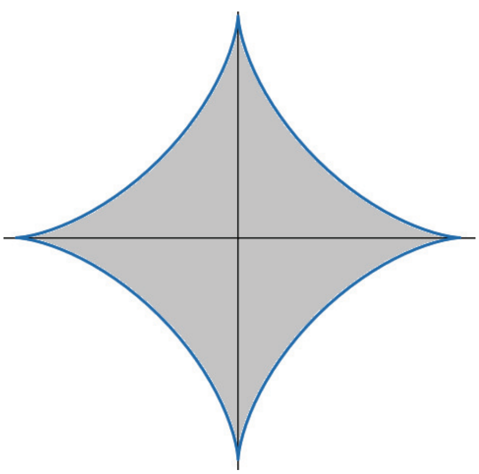
24. $x = t - t^3$
$y = t^2$
$-1\le t\le 1\pi$



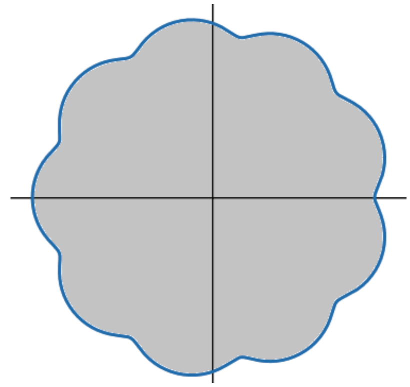
25. $x = t^2 - 2t + 3$
$y = t^3 - 5t^2 + 3t + 11$
$-1 \leq t \leq 3$



26. $x = \cos^3 t$
$y = \sin^3 t$
$0 \leq t \leq 2\pi$



27. $x = 20\cos t - \cos(10t)$
$y = 20\sin t - \sin(10t)$
$0 \leq t \leq 2\pi$



28. C is the ellipse parameterized by $\vec{r}(t) = \langle 4\cos t, 3\sin t\rangle$ on $0 \leq t \leq 2\pi$.

29. C is the curve parameterized by $\vec{r}(t) = \langle \cos t, \sin(2t)\rangle$ on $-\pi/2 \leq t \leq \pi/2$.

30. C is the curve parameterized by $\vec{r}(t) = \langle \cos t, \sin(2t)\rangle$ on $0 \leq t \leq 2$.

31. C is the curve parameterized by $\vec{r}(t) = \langle 2\cos t + \frac{1}{10}\cos(10t), 2\sin t + \frac{1}{10}\sin(10t)\rangle$ on $0 \leq t \leq 2\pi$.

In Exercises 32 – 35, a vector field $\vec{F}$ and a closed curve C, enclosing a region R, are given. Verify the Divergence Theorem by evaluating $\oint_C \vec{F}\cdot\vec{n}\,ds$ and $\iint_R \text{div}\,\vec{F}\,dA$, showing they are equal.

32. $\vec{F} = \langle x - y, x + y\rangle$; C is the closed curve composed of the parabola $y = x^2$ on $0 \leq x \leq 2$ followed by the line segment from $(2, 4)$ to $(0, 0)$.

33. $\vec{F} = \langle -y, x\rangle$; C is the unit circle.

34. $\vec{F} = \langle 0, y^2\rangle$; C the triangle with corners at $(0, 0)$, $(2, 0)$ and $(1, 1)$.

35. $\vec{F} = \langle x^2/2, y^2/2\rangle$; C the curve that starts at $(0, 1)$, follows the parabola $y = (x - 1)^2$ to $(3, 4)$, then follows a line back to $(0, 1)$.