8.3 Integral and Comparison Tests

Knowing whether or not a series converges is very important, especially when we discuss Power Series in Section 8.6. Theorem 62 gives a criterion for when geometric series converge, and Theorem 64 gives a quick test to determine if a series diverges. There are many important series whose convergence cannot be determined by these theorems, though, so we introduce a set of tests that allow us to handle a broad range of series. We start with the Integral Test.

Integral Test

We stated in Section 8.1 that a sequence $\{a_n\}$ is a function $a(n)$ whose domain is $\mathbb{N}$, the set of natural numbers. If we can extend $a(n)$ to $\mathbb{R}$, the real numbers, and it is both positive and decreasing on $[1,\infty)$, then the convergence of $\displaystyle\sum_{n=1}^{\infty} a_n$ is the same as $\displaystyle\int_1^{\infty} a(x)\,dx$.

Note: Theorem 66 does not state that the integral and the summation have the same value.

Theorem 66 Integral Test

Let a sequence $\{a_n\}$ be defined by $a_n = a(n)$, where $a(n)$ is continuous, positive and decreasing on $[1,\infty)$. Then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, if, and only if, $\displaystyle\int_1^{\infty} a(x)\,dx$ converges.

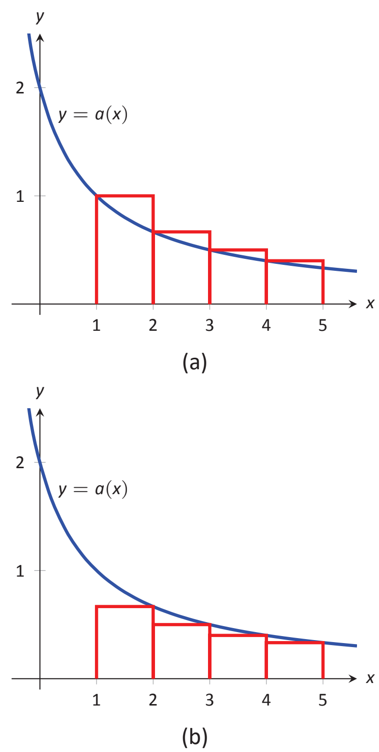

Figure 8.13: Illustrating the truth of the Integral Test.

We can demonstrate the truth of the Integral Test with two simple graphs. In Figure 8.13(a), the height of each rectangle is $a(n) = a_n$ for $n = 1, 2, \ldots$, and clearly the rectangles enclose more area than the area under $y = a(x)$. Therefore we can conclude that

$$\int_1^{\infty} a(x)\,dx < \sum_{n=1}^{\infty} a_n. \tag{8.1}$$

In Figure 8.13(b), we draw rectangles under $y = a(x)$ with the Right-Hand rule, starting with $n = 2$. This time, the area of the rectangles is less than the area under $y = a(x)$, so $\displaystyle\sum_{n=2}^{\infty} a_n < \int_1^{\infty} a(x)\,dx$. Note how this summation starts with $n = 2$; adding a_1 to both sides lets us rewrite the summation starting with

Notes:

$n = 1$:

$$\sum_{n=1}^{\infty} a_n < a_1 + \int_1^{\infty} a(x)\,dx. \qquad (8.2)$$

Combining Equations (8.1) and (8.2), we have

$$\sum_{n=1}^{\infty} a_n < a_1 + \int_1^{\infty} a(x)\,dx < a_1 + \sum_{n=1}^{\infty} a_n. \qquad (8.3)$$

From Equation (8.3) we can make the following two statements:

1. If $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, so does $\displaystyle\int_1^{\infty} a(x)\,dx$ (because $\displaystyle\sum_{n=1}^{\infty} a_n < a_1 + \int_1^{\infty} a(x)\,dx$)
2. If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, so does $\displaystyle\int_1^{\infty} a(x)\,dx$ (because $\displaystyle\int_1^{\infty} a(x)\,dx < \sum_{n=1}^{\infty} a_n$.)

Therefore the series and integral either both converge or both diverge. Theorem 65 allows us to extend this theorem to series where $a(n)$ is positive and decreasing on $[b, \infty)$ for some $b > 1$.

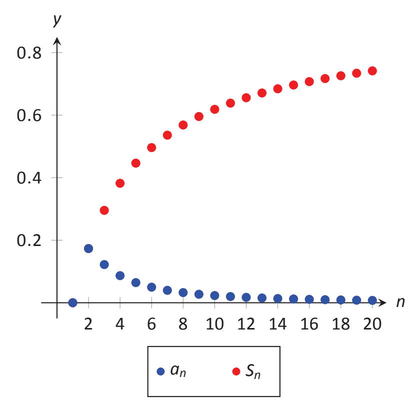

Figure 8.14: Plotting the sequence and series in Example 8.14.

Example 8.14 Using the Integral Test

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$. (The terms of the sequence $\{a_n\} = \{\ln n/n^2\}$ and the n$^{\text{th}}$ partial sums are given in Figure 8.14.)

Solution Figure 8.14 implies that $a(n) = (\ln n)/n^2$ is positive and decreasing on $[2, \infty)$. We can determine this analytically, too. We know $a(n)$ is positive as both $\ln n$ and n^2 are positive on $[2, \infty)$. To determine that $a(n)$ is decreasing, consider $a'(n) = (1 - 2\ln n)/n^3$, which is negative for $n \geq 2$. Since $a'(n)$ is negative, $a(n)$ is decreasing.

Applying the Integral Test, we test the convergence of $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}\,dx$. Integrating this improper integral requires the use of Integration by Parts, with $u = \ln x$ and $dv = 1/x^2\,dx$.

$$\begin{aligned}
\int_1^{\infty} \frac{\ln x}{x^2}\,dx &= \lim_{b\to\infty} \int_1^{b} \frac{\ln x}{x^2}\,dx \\
&= \lim_{b\to\infty} -\frac{1}{x}\ln x\Big|_1^b + \int_1^b \frac{1}{x^2}\,dx
\end{aligned}$$

Notes:

$$= \lim_{b\to\infty} -\frac{1}{x}\ln x - \frac{1}{x}\Big|_1^b$$
$$= \lim_{b\to\infty} 1 - \frac{1}{b} - \frac{\ln b}{b}. \quad \text{Apply L'Hôpital's Rule:}$$
$$= 1.$$

Since $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$ converges, so does $\displaystyle\sum_{n=1}^\infty \frac{\ln n}{n^2}$.

p–Series

In Section 6.6 about improper integrals, we determined for which p the integral $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges, and for which p it diverges. Combining this with the Integral Test, we can easily determine convergence for an important class of series called p–series.

Definition 36 ***p*–Series, General *p*–Series**

1. A ***p*–series** is a series of the form
$$\sum_{n=1}^\infty \frac{1}{n^p}, \qquad \text{where } p > 0.$$
2. A **general *p*–series** is a series of the form
$$\sum_{n=1}^\infty \frac{1}{(an+b)^p}, \qquad \text{where } p > 0 \text{ and } a, b \text{ are real numbers.}$$

Note: Theorem 67 assumes that $an+b \neq 0$ for all n. If $an+b=0$ for some n, then of course the series does not converge regardless of p as not all of the terms of the sequence are defined.

Theorem 67 **Convergence of General *p*–Series**

A general p–series $\displaystyle\sum_{n=1}^\infty \frac{1}{(an+b)^p}$ converges if and only if $p > 1$.

We prove the convergence of general p–series in the next example.

Notes:

Example 8.15 **Proving Theorem 67.**

Use the Integral Test to prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$ converges if, and only if, $p > 1$.

SOLUTION Consider the integral $\displaystyle\int_1^{\infty} \frac{1}{(ax+b)^p}\,dx$. Assume $ax+b \neq 0$ for $x \geq 1$. (If $ax+b=0$ for some non-integer $x > 1$, then there exists N for which $ax+b=0$ for $x \geq N$, so integrate on $[N, \infty)$ instead.) Assuming $p \neq 1$,

$$\begin{aligned}
\int_1^{\infty} \frac{1}{(ax+b)^p}\,dx &= \lim_{c\to\infty} \int_1^{c} \frac{1}{(ax+b)^p}\,dx \\
&= \lim_{c\to\infty} \frac{1}{a(1-p)}(ax+b)^{1-p}\Big|_1^c \\
&= \lim_{c\to\infty} \frac{1}{a(1-p)}\big((ac+b)^{1-p}-(a+b)^{1-p}\big).
\end{aligned}$$

This limit converges if, and only if, $p > 1$. It is easy to show that the integral also diverges in the case of $p = 1$. (This result is similar to the work preceding Key Idea 22.)

Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$ converges if, and only if, $p > 1$.

Example 8.16 **Determining convergence of series**

Determine the convergence of the following series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$
5. $\displaystyle\sum_{n=11}^{\infty} \frac{1}{\left(\frac{1}{2}n-5\right)^3}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$

SOLUTION

1. This is the Harmonic Series, which we proved to be divergent in the previous section. Note, however, that it is also a p–series with $p = 1$, another way to explain that it diverges.

2. This is a p–series with $p = 2$. By Theorem 67, it converges. Note that the theorem does not give a formula by which we can determine *what* the series converges to; we just know it converges. A famous, unexpected result is that this series converges to $\pi^2/6$.

Notes:

3. This is a p–series with $p = 1/2$; the theorem states that it diverges.

4. This is not a p–series; the definition does not allow for alternating signs. Therefore we cannot apply Theorem 67. Also, we cannot use the Integral Test at all, because that can only be used when the terms are positive. As stated in the previous section, this Alternating Harmonic Series converges to $\ln 2$ though we must wait for later sections to explain why.

5. This is a general p–series with $p = 3$, therefore it converges.

6. This is not a p–series, but a geometric series with $r = 1/2$. It converges.

We consider two more convergence tests in this section, both *comparison* tests. That is, we determine the convergence of one series by comparing it to another series with known convergence.

Direct Comparison Test

Note: A sequence $\{a_n\}$ is a **positive sequence** if $a_n > 0$ for all n.

Because of Theorem 65, any theorem that relies on a positive sequence still holds true when $a_n > 0$ for all but a finite number of values of n.

Theorem 68 Direct Comparison Test

Let $\{a_n\}$ and $\{b_n\}$ be positive sequences where $a_n \leq b_n$ for all $n \geq N$, for some $N \geq 1$.

1. If $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

2. If $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, then $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges.

Example 8.17 Applying the Direct Comparison Test

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n + n^2}$.

Solution This series is neither a geometric or p-series, but seems related. We predict it will converge, so we look for a series with larger terms that converges. (Note too that the Integral Test seems difficult to apply here.)

Since $3^n < 3^n + n^2$, it follows that $\dfrac{1}{3^n} > \dfrac{1}{3^n + n^2}$ for all $n \geq 1$. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ is a convergent geometric series; by Theorem 68, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n + n^2}$ converges.

Notes:

Example 8.18 **Applying the Direct Comparison Test**

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n - \ln n}$.

Solution We know the Harmonic Series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, and it seems that the given series is closely related to it, hence we predict it will diverge.

Since $n \geq n - \ln n$ for all $n \geq 1$, $\dfrac{1}{n} \leq \dfrac{1}{n - \ln n}$ for all $n \geq 1$.

The Harmonic Series diverges, so we conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n - \ln n}$ diverges as well.

The concept of direct comparison is powerful and often relatively easy to apply. Practice helps one develop the necessary intuition to quickly pick a proper series with which to compare. However, it is easy to construct a series for which it is difficult to apply the Direct Comparison Test.

Consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + \ln n}$. It is very similar to the divergent series given in Example 8.18. We suspect that it also diverges, as $\dfrac{1}{n} \approx \dfrac{1}{n + \ln n}$ for large n. However, the inequality that we naturally want to use "goes the wrong way": since $n \leq n + \ln n$ for all $n \geq 1$, $\dfrac{1}{n} \geq \dfrac{1}{n + \ln n}$ for all $n \geq 1$. The given series has terms *less than* the terms of a divergent series, and we cannot conclude anything from this.

Fortunately, we can apply another test to the given series to determine its convergence.

Notes:

Limit Comparison Test

Theorem 69 Limit Comparison Test

Let $\{a_n\}$ and $\{b_n\}$ be positive sequences.

1. If $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n}=L$, where L is a positive real number, then $\displaystyle\sum_{n=1}^{\infty}a_n$ and $\displaystyle\sum_{n=1}^{\infty}b_n$ either both converge or both diverge.
2. If $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n}=0$, then if $\displaystyle\sum_{n=1}^{\infty}b_n$ converges, then so does $\displaystyle\sum_{n=1}^{\infty}a_n$.
3. If $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n}=\infty$, then if $\displaystyle\sum_{n=1}^{\infty}b_n$ diverges, then so does $\displaystyle\sum_{n=1}^{\infty}a_n$.

Theorem 69 is most useful when the convergence of the series from $\{b_n\}$ is known and we are trying to determine the convergence of the series from $\{a_n\}$.

We use the Limit Comparison Test in the next example to examine the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n+\ln n}$ which motivated this new test.

Example 8.19 Applying the Limit Comparison Test

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n+\ln n}$ using the Limit Comparison Test.

Solution We compare the terms of $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n+\ln n}$ to the terms of the Harmonic Sequence $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$:

$$\begin{aligned}\lim_{n\to\infty}\frac{1/(n+\ln n)}{1/n} &= \lim_{n\to\infty}\frac{n}{n+\ln n}\\ &= 1 \quad \text{(after applying L'Hôpital's Rule).}\end{aligned}$$

Since the Harmonic Series diverges, we conclude that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n+\ln n}$ diverges as

Notes:

well.

Example 8.20 Applying the Limit Comparison Test

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n - n^2}$

Solution This series is similar to the one in Example 8.17, but now we are considering "$3^n - n^2$" instead of "$3^n + n^2$." This difference makes applying the Direct Comparison Test difficult.

Instead, we use the Limit Comparison Test and compare with the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$:

$$\begin{aligned}\lim_{n\to\infty} \frac{1/(3^n - n^2)}{1/3^n} &= \lim_{n\to\infty} \frac{3^n}{3^n - n^2} \\ &= 1 \quad \text{(after applying L'Hôpital's Rule twice).}\end{aligned}$$

We know $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ is a convergent geometric series, hence $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n - n^2}$ converges as well.

As mentioned before, practice helps one develop the intuition to quickly choose a series with which to compare. A general rule of thumb is to pick a series based on the dominant term in the expression of $\{a_n\}$. It is also helpful to note that among sequences that approach ∞ as $n \to \infty$, factorials dominate exponentials, which dominate algebraic functions (e.g., polynomials), which dominate logarithms. In the previous example, the dominant term of $\dfrac{1}{3^n - n^2}$ was 3^n, so we compared the series to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$. It is hard to apply the Limit Comparison Test to series containing factorials, though, as we have not learned how to apply L'Hôpital's Rule to $n!$.

Example 8.21 Applying the Limit Comparison Test

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+3}{n^2 - n + 1}$.

Solution We naïvely attempt to apply the rule of thumb given above and note that the dominant term in the expression of the series is $1/n^2$. Knowing

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Notes:

that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, we attempt to apply the Limit Comparison Test:

$$\begin{aligned}\lim_{n\to\infty}\frac{(\sqrt{n}+3)/(n^2-n+1)}{1/n^2} &= \lim_{n\to\infty}\frac{n^2(\sqrt{n}+3)}{n^2-n+1}\\ &= \infty \quad \text{(Apply L'Hôpital's Rule).}\end{aligned}$$

Theorem 69 part (3) only applies when $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges; in our case, it converges. Ultimately, our test has not revealed anything about the convergence of our series.

The problem is that we chose a poor series with which to compare. Since the numerator and denominator of the terms of the series are both algebraic functions, we should have compared our series to the dominant term of the numerator divided by the dominant term of the denominator.

The dominant term of the numerator is $n^{1/2}$ and the dominant term of the denominator is n^2. Thus we should compare the terms of the given series to $n^{1/2}/n^2 = 1/n^{3/2}$:

$$\begin{aligned}\lim_{n\to\infty}\frac{(\sqrt{n}+3)/(n^2-n+1)}{1/n^{3/2}} &= \lim_{n\to\infty}\frac{n^{3/2}(\sqrt{n}+3)}{n^2-n+1}\\ &= 1 \quad \text{(Apply L'Hôpital's Rule).}\end{aligned}$$

Since the p-series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges, we conclude that $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}+3}{n^2-n+1}$ converges as well.

We mentioned earlier that the Integral Test did not work well with series containing factorial terms. The next section introduces the Ratio Test, which does handle such series well. We also introduce the Root Test, which is good for series where each term is raised to a power.

Notes:

Exercises 8.3

Terms and Concepts

1. In order to apply the Integral Test to a sequence $\{a_n\}$, the function $a(n) = a_n$ must be _____, _____ and _____.

2. T/F: The Integral Test can be used to determine the sum of a convergent series.

3. What test(s) in this section do not work well with factorials?

4. Suppose $\displaystyle\sum_{n=0}^{\infty} a_n$ is convergent, and there are sequences $\{b_n\}$ and $\{c_n\}$ such that $b_n \leq a_n \leq c_n$ for all n. What can be said about the series $\displaystyle\sum_{n=0}^{\infty} b_n$ and $\displaystyle\sum_{n=0}^{\infty} c_n$?

Problems

In Exercises 5 – 20, use the Integral Test or General *p*–Series Test to determine the convergence of the given series.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

10. $\displaystyle\sum_{n=1}^{\infty} n^{-1/3}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{0.999}}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.001}}$

13. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{(2n+8)^2}$

17. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$

19. $\displaystyle\sum_{n=23}^{\infty} \frac{1}{\sqrt{7n-13}}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

In Exercises 21 – 30, use the Direct Comparison Test to determine the convergence of the given series; state what series is used for comparison.

21. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3n-5}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n+n^2-n}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!+n}$

25. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$

26. $\displaystyle\sum_{n=5}^{\infty} \frac{1}{\sqrt{n}-2}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^3-5}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{5^n+10}$

29. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{n^2-1}$

30. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$

In Exercises 31 – 40, use the Limit Comparison Test to determine the convergence of the given series; state what series is used for comparison.

31. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 - 3n + 5}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n - n^2}$

33. $\displaystyle\sum_{n=4}^{\infty} \frac{\ln n}{n - 3}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + n}}$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}}$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{n - 10}{n^2 + 10n + 10}$

37. $\displaystyle\sum_{n=1}^{\infty} \sin\left(1/n\right)$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{n + 5}{n^3 - 5}$

39. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n} + 3}{n^2 + 17}$

40. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + 100}$

In Exercises 41 – 48, determine the convergence of the given series. State the test used; more than one test may be appropriate.

41. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

42. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n + 5)^3}$

43. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$

44. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n!}$

45. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n + n}$

46. $\displaystyle\sum_{n=1}^{\infty} \frac{n - 2}{10n + 5}$

47. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

48. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(1/n)}{\sqrt{n}}$

49. Given that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, state which of the following series converges, may converge, or does not converge.

    (a) $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n}$

    (b) $\displaystyle\sum_{n=1}^{\infty} a_n a_{n+1}$

    (c) $\displaystyle\sum_{n=1}^{\infty} (a_n)^2$

    (d) $\displaystyle\sum_{n=1}^{\infty} n a_n$

    (e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n}$