11.3   The Calculus of Motion

A common use of vector–valued functions is to describe the motion of an object in the plane or in space. A **position function** $\vec{r}(t)$ gives the position of an object at **time** t. This section explores how derivatives and integrals are used to study the motion described by such a function.

Definition 77   Velocity, Speed and Acceleration

Let $\vec{r}(t)$ be a position function in $\mathbb{R}^2$ or $\mathbb{R}^3$.

1. **Velocity**, denoted $\vec{v}(t)$, is the instantaneous rate of position change; that is, $\vec{v}(t) = \vec{r}\,'(t)$.

2. **Speed** is the magnitude of velocity, $||\,\vec{v}(t)\,||$.

3. **Acceleration**, denoted $\vec{a}(t)$, is the instantaneous rate of velocity change; that is, $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$.

Example 11.16   Finding velocity and acceleration
An object is moving with position function $\vec{r}(t) = \langle t^2 - t, t^2 + t \rangle$, $-3 \le t \le 3$, where distances are measured in feet and time is measured in seconds.

1. Find $\vec{v}(t)$ and $\vec{a}(t)$.

2. Sketch $\vec{r}(t)$; plot $\vec{v}(-1)$, $\vec{a}(-1)$, $\vec{v}(1)$ and $\vec{a}(1)$, each with their initial point at their corresponding point on the graph of $\vec{r}(t)$.

3. When is the object's speed minimized?

Solution

1. Taking derivatives, we find

$$\vec{v}(t) = \vec{r}\,'(t) = \langle 2t - 1, 2t + 1 \rangle \quad \text{and} \quad \vec{a}(t) = \vec{r}\,''(t) = \langle 2, 2 \rangle.$$

Note that acceleration is constant.

2. $\vec{v}(-1) = \langle -3, -1 \rangle$, $\vec{a}(-1) = \langle 2, 2 \rangle$; $\vec{v}(1) = \langle 1, 3 \rangle$, $\vec{a}(1) = \langle 2, 2 \rangle$. These are plotted with $\vec{r}(t)$ in Figure 11.15(a).

We can think of acceleration as "pulling" the velocity vector in a certain direction. At $t = -1$, the velocity vector points down and to the left; at $t = 1$, the velocity vector has been pulled in the $\langle 2, 2 \rangle$ direction and is
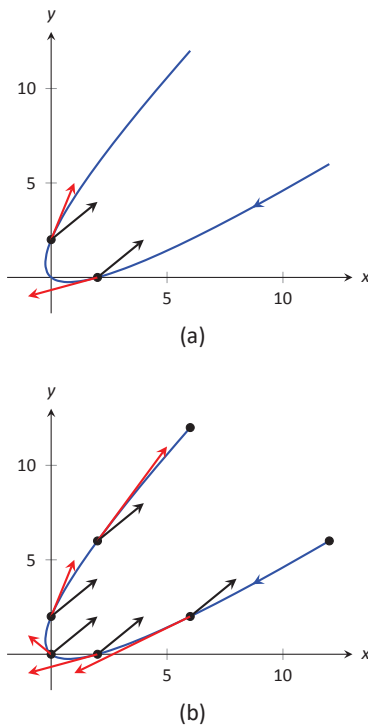
(a)

(b)

Figure 11.15: Graphing the position, velocity and acceleration of an object in Example 11.16.

Notes:

now pointing up and to the right. In Figure 11.15(b) we plot more velocity/acceleration vectors, making more clear the effect acceleration has on velocity.

Since $\vec{a}(t)$ is constant in this example, as t grows large $\vec{v}(t)$ becomes almost parallel to $\vec{a}(t)$. For instance, when $t = 10$, $\vec{v}(10) = \langle 19, 21 \rangle$, which is nearly parallel to $\langle 2, 2 \rangle$.

3. The object's speed is given by

$$|| \vec{v}(t) || = \sqrt{(2t - 1)^2 + (2t + 1)^2} = \sqrt{8t^2 + 2}.$$

To find the minimal speed, we could apply calculus techniques (such as set the derivative equal to 0 and solve for t, etc.) but we can find it by inspection. Inside the square root we have a quadratic which is minimized when $t = 0$. Thus the speed is minimized at $t = 0$, with a speed of $\sqrt{2}$ ft/s.

The graph in Figure 11.15(b) also implies speed is minimized here. The filled dots on the graph are located at integer values of t between -3 and 3. Dots that are far apart imply the object traveled a far distance in 1 second, indicating high speed; dots that are close together imply the object did not travel far in 1 second, indicating a low speed. The dots are closest together near $t = 0$, implying the speed is minimized near that value.

Example 11.17   Analyzing Motion

Two objects follow an identical path at different rates on $[-1, 1]$. The position function for Object 1 is $\vec{r}_1(t) = \langle t, t^2 \rangle$; the position function for Object 2 is $\vec{r}_2(t) = \langle t^3, t^6 \rangle$, where distances are measured in feet and time is measured in seconds. Compare the velocity, speed and acceleration of the two objects on the path.

SOLUTION   We begin by computing the velocity and acceleration function for each object:

$$\vec{v}_1(t) = \langle 1, 2t \rangle \qquad \vec{v}_2(t) = \langle 3t^2, 6t^5 \rangle$$
$$\vec{a}_1(t) = \langle 0, 2 \rangle \qquad \vec{a}_2(t) = \langle 6t, 30t^4 \rangle$$

We immediately see that Object 1 has constant acceleration, whereas Object 2 does not.

At $t = -1$, we have $\vec{v}_1(-1) = \langle 1, -2 \rangle$ and $\vec{v}_2(-1) = \langle 3, -6 \rangle$; the velocity of Object 2 is three times that of Object 1 and so it follows that the speed of Object 2 is three times that of Object 1 ($3\sqrt{5}$ ft/s compared to $\sqrt{5}$ ft/s.)
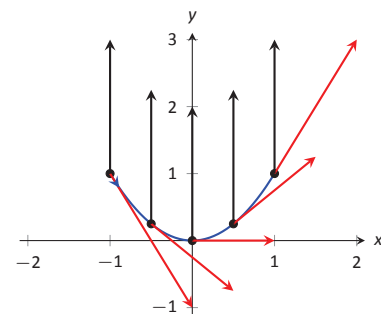

Figure 11.16: Plotting velocity and acceleration vectors for Object 1 in Example 11.17.
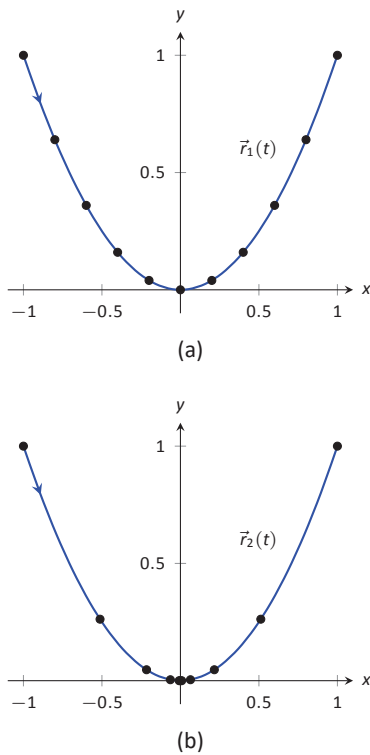
Notes:

(a)

(b)

Figure 11.17: Comparing the positions of Objects 1 and 2 in Example 11.17.

At $t = 0$, the velocity of Object 1 is $\vec{v}(1) = \langle 1, 0 \rangle$ and the velocity of Object 2 is $\vec{0}$! This tells us that Object 2 comes to a complete stop at $t = 0$.

In Figure 11.16, we see the velocity and acceleration vectors for Object 1 plotted for $t = -1, -1/2, 0, 1/2$ and $t = 1$. Note again how the constant acceleration vector seems to "pull" the velocity vector from pointing down, right to up, right. We could plot the analogous picture for Object 2, but the velocity and acceleration vectors are rather large ($\vec{a}_2(-1) = \langle -6, 30 \rangle$).

Instead, we simply plot the locations of Object 1 and 2 on intervals of $1/10^{th}$ of a second, shown in Figure 11.17(a) and (b). Note how the x-values of Object 1 increase at a steady rate. This is because the x-component of $\vec{a}(t)$ is 0; there is no acceleration in the x-component. The dots are not evenly spaced; the object is moving faster near $t = -1$ and $t = 1$ than near $t = 0$.

In part (b) of the Figure, we see the points plotted for Object 2. Note the large change in position from $t = -1$ to $t = -0.9$; the object starts moving very quickly. However, it slows considerably at it approaches the origin, and comes to a complete stop at $t = 0$. While it looks like there are 3 points near the origin, there are in reality 5 points there.

Since the objects begin and end at the same location, the have the same displacement. Since they begin and end at the same time, with the same displacement, they have they have the same average rate of change (i.e, they have the same average velocity). Since they follow the same path, they have the same distance traveled. Even though these three measurements are the same, the objects obviously travel the path in very different ways.

Example 11.18   Analyzing the motion of a whirling ball on a string
A young boy whirls a ball, attached to a string, above his head in a counterclockwise circle. The ball follows a circular path and makes 2 revolutions per second. The string has length 2ft.

1. Find the position function $\vec{r}(t)$ that describes this situation.

2. Find the acceleration of the ball and derive a physical interpretation of it.

3. A tree stands 10ft in front of the boy. At what t-values should the boy release the string so that the ball hits the tree?

SOLUTION

1. The ball whirls in a circle. Since the string is 2ft long, the radius of the circle is 2. The position function $\vec{r}(t) = \langle 2\cos t, 2\sin t \rangle$ describes a circle with radius 2, centered at the origin, but makes a full revolution every 2π seconds, not two revolutions per second. We modify the period of the

Notes:

trigonometric functions to be 1/2 by multiplying t by 4π. The final position function is thus

$$\vec{r}(t) = \langle 2\cos(4\pi t), 2\sin(4\pi t)\rangle.$$

(Plot this for $0 \leq t \leq 1/2$ to verify that one revolution is made in 1/2 a second.)

2. To find $\vec{a}(t)$, we derive $\vec{r}(t)$ twice.

$$\vec{v}(t) = \vec{r}\,'(t) = \langle -8\pi\sin(4\pi t), 8\pi\cos(4\pi t)\rangle$$
$$\vec{a}(t) = \vec{r}\,''(t) = \langle -32\pi^2\cos(4\pi t), -32\pi^2\sin(4\pi t)\rangle$$
$$= -32\pi^2\langle\cos(4\pi t), \sin(4\pi t)\rangle.$$

Note how $\vec{a}(t)$ is parallel to $\vec{r}(t)$, but has a different magnitude and points in the opposite direction. Why is this?

Recall the classic physics equation, "Force = mass $\times$ acceleration." A force acting on a mass induces acceleration (i.e., the mass moves); acceleration acting on a mass induces a force (gravity gives our mass a *weight*). Thus force and acceleration are closely related. A moving ball "wants" to travel in a straight line. Why does the ball in our example move in a circle? It is attached to the boy's hand by a string. The string applies a force to the ball, affecting it's motion: the string *accelerates* the ball. This is not acceleration in the sense of "it travels faster;" rather, this acceleration is changing the velocity of the ball. In what direction is this force/acceleration being applied? In the direction of the string, towards the boy's hand.

The magnitude of the acceleration is related to the speed at which the ball is traveling. A ball whirling quickly is rapidly changing direction/velocity. When velocity is changing rapidly, the acceleration must be "large."

3. When the boy releases the string, the string no longer applies a force to the ball, meaning acceleration is $\vec{0}$ and the ball can now move in a straight line in the direction of $\vec{v}(t)$.

Let $t = t_0$ be the time when the boy lets go of the string. The ball will be at $\vec{r}(t_0)$, traveling in the direction of $\vec{v}(t_0)$. We want to find t_0 so that this line contains the point $(0, 10)$ (since the tree is 10ft directly in front of the boy).

There are many ways to find this time value. We choose one that is relatively simple computationally. As shown in Figure 11.18, the vector from the release point to the tree is $\langle 0, 10\rangle - \vec{r}(t_0)$. This line segment is tangent to the circle, which means it is also perpendicular to $\vec{r}(t_0)$ itself, so their dot product is 0.
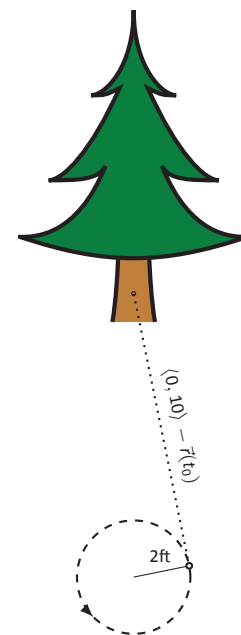

Figure 11.18: Modeling the flight of a ball in Example 11.18.

Notes:

$$\vec{r}(t_0) \cdot \big(\langle 0, 10 \rangle - \vec{r}(t_0) \big) = 0$$
$$\langle 2\cos(4\pi t_0), 2\sin(4\pi t_0) \rangle \cdot \langle -2\cos(4\pi t_0), 10 - 2\sin(4\pi t_0) \rangle = 0$$
$$-4\cos^2(4\pi t_0) + 20\sin(4\pi t_0) - 4\sin^2(4\pi t_0) = 0$$
$$20\sin(4\pi t_0) - 4 = 0$$
$$\sin(4\pi t_0) = 1/5$$
$$4\pi t_0 = \sin^{-1}(1/5)$$
$$4\pi t_0 \approx 0.2 + 2\pi n,$$

where n is an integer. Solving for t_0 we have:

$$t_0 \approx 0.016 + n/2$$

This is a wonderful formula. Every 1/2 second after $t = 0.016$s the boy can release the string (since the ball makes 2 revolutions per second, he has two chances each second to release the ball).

Example 11.19 Analyzing motion in space

An object moves in a spiral with position function $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, where distances are measured in meters and time is in minutes. Describe the object's speed and acceleration at time t.

SOLUTION   With $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, we have:

$$\vec{v}(t) = \langle -\sin t, \cos t, 1 \rangle \quad \text{and}$$
$$\vec{a}(t) = \langle -\cos t, -\sin t, 0 \rangle .$$

The speed of the object is $\| \vec{v}(t) \| = \sqrt{(-\sin t)^2 + \cos^2 t + 1} = \sqrt{2}$m/min; it moves at a constant speed. Note that the object does not accelerate in the z-direction, but rather moves up at a constant rate of 1m/min.

The objects in Examples 11.18 and 11.19 traveled at a constant speed. That is, $\| \vec{v}(t) \| = c$ for some constant c. Recall Theorem 95, which states that if a vector–valued function $\vec{r}(t)$ has constant length, then $\vec{r}(t)$ is perpendicular to its derivative: $\vec{r}(t) \cdot \vec{r}'(t) = 0$. In these examples, the velocity function has constant length, therefore we can conclude that the velocity is perpendicular to the acceleration: $\vec{v}(t) \cdot \vec{a}(t) = 0$. A quick check verifies this.

There is an intuitive understanding of this. If acceleration is parallel to velocity, then it is only affecting the object's speed; it does not change the direction of travel. (For example, consider a dropped stone. Acceleration and velocity are

Notes:

parallel – straight down – and the direction of velocity never changes, though speed does increase.) If acceleration is not perpendicular to velocity, then there is some acceleration in the direction of travel, influencing the speed. If speed is constant, then acceleration must be orthogonal to velocity, as it then only affects direction, and not speed.

Key Idea 54   **Objects With Constant Speed**

If an object moves with constant speed, then its velocity and acceleration vectors are orthogonal. That is, $\vec{v}(t) \cdot \vec{a}(t) = 0$.

Projectile Motion

An important application of vector–valued position functions is *projectile motion*: the motion of objects under only the influence of gravity. We will measure time in seconds, and distances will either be in meters or feet. We will show that we can completely describe the path of such an object knowing its initial position and initial velocity (i.e., where it *is* and where it *is going*.)

Suppose an object has initial position $\vec{r}(0) = \langle x_0, y_0 \rangle$ and initial velocity $\vec{v}(0) = \langle v_x, v_y \rangle$. It is customary to rewrite $\vec{v}(0)$ in terms of its speed v_0 and direction $\vec{u}$, where $\vec{u}$ is a unit vector. Recall all unit vectors in $\mathbb{R}^2$ can be written as $\langle \cos\theta, \sin\theta \rangle$, where θ is an angle measure counter–clockwise from the x-axis. (We refer to θ as the **angle of elevation.**) Thus $\vec{v}(0) = v_0 \langle \cos\theta, \sin\theta \rangle$.

Since the acceleration of the object is known, namely $\vec{a}(t) = \langle 0, -g \rangle$, where g is the gravitational constant, we can find $\vec{r}(t)$ knowing our two initial conditions. We first find $\vec{v}(t)$:

Note: In this text we use $g = 32\text{ft/s}$ when using Imperial units, and $g = 9.8\text{m/s}$ when using SI units.

$$\vec{v}(t) = \int \vec{a}(t)\, dt$$

$$\vec{v}(t) = \int \langle 0, -g \rangle\, dt$$

$$\vec{v}(t) = \langle 0, -gt \rangle + \vec{C}.$$

Knowing $\vec{v}(0) = v_0 \langle \cos\theta, \sin\theta \rangle$, we have $\vec{C} = v_0 \langle \cos t, \sin t \rangle$ and so

$$\vec{v}(t) = \left\langle v_0 \cos\theta, -gt + v_0 \sin\theta \right\rangle.$$

Notes:

We integrate once more to find $\vec{r}(t)$:

$$\vec{r}(t) = \int \vec{v}(t)\, dt$$

$$\vec{r}(t) = \int \left\langle v_0 \cos\theta, -gt + v_0 \sin\theta \right\rangle\, dt$$

$$\vec{r}(t) = \left\langle (v_0 \cos\theta)t, -\frac{1}{2}gt^2 + (v_0 \sin\theta)t \right\rangle + \vec{C}.$$

Knowing $\vec{r}(0) = \langle x_0, y_0 \rangle$, we conclude $\vec{C} = \langle x_0, y_0 \rangle$ and

$$\vec{r}(t) = \left\langle (v_0 \cos\theta)t + x_0\ ,\ -\frac{1}{2}gt^2 + (v_0 \sin\theta)t + y_0 \right\rangle.$$

Key Idea 55   Projectile Motion

The position function of a projectile propelled from an initial position of $\vec{r}_0 = \langle x_0, y_0 \rangle$, with initial speed v_0, with angle of elevation θ and neglecting all accelerations but gravity is

$$\vec{r}(t) = \left\langle (v_0 \cos\theta)t + x_0\ ,\ -\frac{1}{2}gt^2 + (v_0 \sin\theta)t + y_0 \right\rangle.$$

Letting $\vec{v}_0 = v_0 \langle \cos\theta, \sin\theta \rangle$, $\vec{r}(t)$ can be written as

$$\vec{r}(t) = \left\langle 0, -\frac{1}{2}gt^2 \right\rangle + \vec{v}_0 t + \vec{r}_0.$$

We demonstrate how to use this position function in the next two examples.

Example 11.20   Projectile Motion
Sydney shoots her Red Ryder® bb gun across level ground from an elevation of 4ft, where the barrel of the gun makes a 5° angle with the horizontal. Find how far the bb travels before landing, assuming the bb is fired at the advertised rate of 350ft/s and ignoring air resistance.

SOLUTION   A direct application of Key Idea 55 gives

$$\vec{r}(t) = \left\langle (350 \cos 5°)t, -16t^2 + (350 \sin 5°)t + 4 \right\rangle$$
$$\approx \left\langle 346.67t, -16t^2 + 30.50t + 4 \right\rangle,$$

Notes:

where we set her initial position to be $\langle 0, 4 \rangle$. We need to find *when* the bb lands, then we can find *where*. We accomplish this by setting the *y*-component equal to 0 and solving for *t*:

$$-16t^2 + 30.50t + 4 = 0$$

$$t = \frac{-30.50 \pm \sqrt{30.50^2 - 4(-16)(4)}}{-32}$$

$$t \approx 2.03s.$$

(We discarded a negative solution that resulted from our quadratic equation.)

We have found that the bb lands 2.03s after firing; with $t = 2.03$, we find the *x*-component of our position function is $346.67(2.03) = 703.74$ft. The bb lands about 704 feet away.

Example 11.21   Projectile Motion

Alex holds his sister's bb gun at a height of 3ft and wants to shoot a target that is 6ft above the ground, 25ft away. At what angle should he hold the gun to hit his target? (We still assume the muzzle velocity is 350ft/s.)

SOLUTION   The position function for the path of Alex's bb is

$$\vec{r}(t) = \left\langle (350 \cos \theta)t, -16t^2 + (350 \sin \theta)t + 3 \right\rangle.$$

We need to find θ so that $\vec{r}(t) = \langle 25, 6 \rangle$ for some value of *t*. That is, we want to find θ and *t* such that

$$(350 \cos \theta)t = 25 \quad \text{and} \quad -16t^2 + (350 \sin \theta)t + 3 = 6.$$

This is not trivial (though not "hard"). We start by solving each equation for $\cos \theta$ and $\sin \theta$, respectively.

$$\cos \theta = \frac{25}{350t} \quad \text{and} \quad \sin \theta = \frac{3 + 16t^2}{350t}.$$

Using the Pythagorean Identity $\cos^2 \theta + \sin^2 \theta = 1$, we have

$$\left(\frac{25}{350t} \right)^2 + \left(\frac{3 + 16t^2}{350t} \right)^2 = 1$$

Multiply both sides by $(350t)^2$:

$$25^2 + (3 + 16t^2)^2 = 350^2 t^2$$

$$256t^4 - 122,404t^2 + 634 = 0.$$

Notes:

This is a quadratic *in* t^2. That is, we can apply the quadratic formula to find t^2, then solve for t itself.

$$t^2 = \frac{122,404 \pm \sqrt{122,404^2 - 4(256)(634)}}{512}$$

$$t^2 = 0.0052, \ 478.135$$

$$t = \pm 0.072, \ \pm 21.866$$

Clearly the negative t values do not fit our context, so we have $t = 0.072$ and $t = 21.866$. Using $\cos\theta = 25/(350t)$, we can solve for θ:

$$\theta = \cos^{-1}\left(\frac{25}{350 \cdot 0.072}\right) \quad \text{and} \quad \cos^{-1}\left(\frac{25}{350 \cdot 21.866}\right)$$

$$\theta = 7.03° \quad \text{and} \quad 89.8°.$$

Alex has two choices of angle. He can hold the rifle at an angle of about $7°$ with the horizontal and hit his target 0.07s after firing, or he can hold his rifle almost straight up, with an angle of $89.8°$, where he'll hit his target about 22s later. The first option is clearly the option he should choose.

Distance Traveled

Consider a driver who sets her cruise–control to 60mph, and travels at this speed for an hour. We can ask:

1. How far did the driver travel?

2. How far from her starting position is the driver?

The first is easy to answer: she traveled 60 miles. The second is impossible to answer with the given information. We do not know if she traveled in a straight line, on an oval racetrack, or along a slowly–winding highway.

This highlights an important fact: to compute distance traveled, we need only to know the speed, given by $\| \vec{v}(t) \|$.

Theorem 98   Distance Traveled

Let $\vec{v}(t)$ be a velocity function for a moving object. The distance traveled by the object on $[a, b]$ is:

$$\text{distance traveled} = \int_a^b \| \vec{v}(t) \| \, dt.$$

Note that this is just a restatement of Theorem 97: arc length is the same as distance traveled, just viewed in a different context.

Notes:

Example 11.22   Distance Traveled, Displacement, and Average Speed

A particle moves in space with position function $\vec{r}(t) = \langle t, t^2, \sin(\pi t) \rangle$ on $[-2, 2]$, where t is measured in seconds and distances are in meters. Find:

1. The distance traveled by the particle on $[-2, 2]$.

2. The displacement of the particle on $[-2, 2]$.

3. The particle's average speed.

SOLUTION

1. We use Theorem 98 to establish the integral:

$$\text{distance traveled} = \int_{-2}^{2} || \vec{v}(t) || \, dt$$

$$= \int_{-2}^{2} \sqrt{1 + (2t)^2 + \pi^2 \cos^2(\pi t)} \, dt.$$

   This cannot be solved in terms of elementary functions so we turn to numerical integration, finding the distance to be 12.88m.

2. The displacement is the vector

$$\vec{r}(2) - \vec{r}(-2) = \langle 2, 4, 0 \rangle - \langle -2, 4, 0 \rangle = \langle 4, 0, 0 \rangle.$$

   That is, the particle ends with an *x*-value increased by 4 and with *y*- and *z*-values the same (see Figure 11.19).

3. We found above that the particle traveled 12.88m over 4 seconds. We can compute average speed by dividing: 12.88/4 = 3.22m/s.

   We should also consider Definition 22 of Section 5.4, which says that the average value of a function f on $[a, b]$ is $\frac{1}{b-a} \int_a^b f(x) \, dx$. In our context, the average value of the speed is

$$\text{average speed} = \frac{1}{2 - (-2)} \int_{-2}^{2} || \vec{v}(t) || \, dt \approx \frac{1}{4} 12.88 = 3.22\text{m/s}.$$

   Note how the physical context of a particle traveling gives meaning to a more abstract concept learned earlier.

In Definition 22 of Chapter 5 we defined the average value of a function $f(x)$ on $[a, b]$ to be

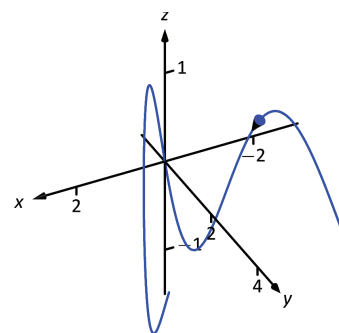$$\frac{1}{b - a} \int_a^b f(x) \, dx.$$

Figure 11.19: The path of the particle in Example 11.22.

Notes:

Note how in Example 11.22 we computed the average speed as

$$\frac{\text{distance traveled}}{\text{travel time}} = \frac{1}{2-(-2)} \int_{-2}^{2} || \, \vec{v}(t) \, || \, dt;$$

that is, we just found the average value of $|| \, \vec{v}(t) \, ||$ on $[-2, 2]$.

Likewise, given position function $\vec{r}(t)$, the average velocity on $[a, b]$ is

$$\frac{\text{displacement}}{\text{travel time}} = \frac{1}{b-a} \int_{a}^{b} \vec{r}\,'(t) \, dt = \frac{\vec{r}(b) - \vec{r}(a)}{b-a};$$

that is, it is the average value of $\vec{r}\,'(t)$, or $\vec{v}(t)$, on $[a, b]$.

Key Idea 56   Average Speed, Average Velocity

Let $\vec{r}(t)$ be a continuous position function on an open interval I containing $a < b$.

The **average speed** is:

$$\frac{\text{distance traveled}}{\text{travel time}} = \frac{\int_{a}^{b} || \, \vec{v}(t) \, || \, dt}{b-a} = \frac{1}{b-a} \int_{a}^{b} || \, \vec{v}(t) \, || \, dt.$$

The **average velocity** is:

$$\frac{\text{displacement}}{\text{travel time}} = \frac{\int_{a}^{b} \vec{r}\,'(t) \, dt}{b-a} = \frac{1}{b-a} \int_{a}^{b} \vec{r}\,'(t) \, dt.$$

The next two sections investigate more properties of the graphs of vector–valued functions and we'll apply these new ideas to what we just learned about motion.

Notes:

Exercises 11.3

Terms and Concepts

1. How is *velocity* different from *speed*?

2. What is the difference between *displacement* and *distance traveled*?

3. What is the difference between *average velocity* and *average speed*?

4. *Distance traveled* is the same as _____ _____, just viewed in a different context.

5. Describe a scenario where an object's average speed is a large number, but the magnitude of the average velocity is not a large number.

6. Explain why it is not possible to have an average velocity with a large magnitude but a small average speed.

Problems

In Exercises 7 – 13 , a position function $\vec{r}(t)$ is given.

    1. Find $\vec{v}(t)$.

    2. Find $\vec{a}(t)$.

    3. Find $\vec{v}(t) \cdot \vec{a}(t)$.

    4. Is the speed constant? If so, determine the constant speed.

7. $\vec{r}(t) = \langle 2t + 1, 5t - 2, 7 \rangle$

8. $\vec{r}(t) = \langle 3t^2 - 2t + 1, -t^2 + t + 14 \rangle$

9. $\vec{r}(t) = \langle \cos t, \sin t \rangle$

10. $\vec{r}(t) = \langle t/10, -\cos t, \sin t \rangle$

11. $\vec{r}(t) = \langle t^2, \sin t, \cos t \rangle$

12. $\vec{r}(t) = \langle \cos(3t) - \sin t, 2 \cos t \rangle$

13. $\vec{r}(t) = \langle \sin t + \cos t, 3t, \sin t - \cos t \rangle$

In Exercises 14 – 17 , a position function $\vec{r}(t)$ is given. Sketch $\vec{r}(t)$ on the indicated interval. Find $\vec{v}(t)$ and $\vec{a}(t)$, then add $\vec{v}(t_0)$ and $\vec{a}(t_0)$ to your sketch, with their initial points at $\vec{r}(t_0)$, for the given value of t_0.

14. $\vec{r}(t) = \langle t, \sin t \rangle$ on $[0, \pi/2]$; $t_0 = \pi/4$

15. $\vec{r}(t) = \langle t^2, \sin t^2 \rangle$ on $[0, \pi/2]$; $t_0 = \sqrt{\pi/4}$

16. $\vec{r}(t) = \langle t^2 + t, -t^2 + 2t \rangle$ on $[-2, 2]$; $t_0 = 1$

17. $\vec{r}(t) = \left\langle \dfrac{2t + 3}{t^2 + 1}, t^2 \right\rangle$ on $[-1, 1]$; $t_0 = 0$

In Exercises 18 – 27 , a position function $\vec{r}(t)$ of an object is given. Find the speed of the object in terms of t, and find where the speed is minimized/maximized on the indicated interval.

18. $\vec{r}(t) = \langle t^2, t \rangle$ on $[-1, 1]$

19. $\vec{r}(t) = \langle t^2, t^2 - t^3 \rangle$ on $[-1, 1]$

20. $\vec{r}(t) = \langle 5 \cos t, 5 \sin t \rangle$ on $[0, 2\pi]$

21. $\vec{r}(t) = \langle 2 \cos t, 5 \sin t \rangle$ on $[0, 2\pi]$

22. $\vec{r}(t) = \langle \sec t, \tan t \rangle$ on $[0, \pi/4]$

23. $\vec{r}(t) = \langle t + \cos t, 1 - \sin t \rangle$ on $[0, 2\pi]$

24. $\vec{r}(t) = \langle 12t, 5 \cos t, 5 \sin t \rangle$ on $[0, 4\pi]$

25. $\vec{r}(t) = \langle t^2 - t, t^2 + t, t \rangle$ on $[0, 1]$

26. $\vec{r}(t) = \left\langle t, t^2, \sqrt{1 - t^2} \right\rangle$ on $[-1, 1]$

27. **Projectile Motion:** $\vec{r}(t) = \left\langle (v_0 \cos \theta)t, -\dfrac{1}{2}gt^2 + (v_0 \sin \theta)t \right\rangle$

    on $\left[0, \dfrac{2v_0 \sin \theta}{g}\right]$

In Exercises 28 – 31 , position functions $\vec{r}_1(t)$ and $\vec{r}_2(s)$ for two objects are given that follow the same path on the respective intervals.

    (a) Show that the positions are the same at the indicated t_0 and s_0 values; i.e., show $\vec{r}_1(t_0) = \vec{r}_2(s_0)$.

    (b) Find the velocity, speed and acceleration of the two objects at t_0 and s_0, respectively.

28. $\vec{r}_1(t) = \langle t, t^2 \rangle$ on $[0, 1]$; $t_0 = 1$
    $\vec{r}_2(s) = \langle s^2, s^4 \rangle$ on $[0, 1]$; $s_0 = 1$

29. $\vec{r}_1(t) = \langle 3 \cos t, 3 \sin t \rangle$ on $[0, 2\pi]$; $t_0 = \pi/2$
    $\vec{r}_2(s) = \langle 3 \cos(4s), 3 \sin(4s) \rangle$ on $[0, \pi/2]$; $s_0 = \pi/8$

30. $\vec{r}_1(t) = \langle 3t, 2t \rangle$ on $[0, 2]$; $t_0 = 2$
    $\vec{r}_2(s) = \langle 6t - 6, 4t - 4 \rangle$ on $[1, 2]$; $s_0 = 2$

31. $\vec{r}_1(t) = \langle t, \sqrt{t} \rangle$ on $[0, 1]$; $t_0 = 1$
    $\vec{r}_2(s) = \langle \sin t, \sqrt{\sin t} \rangle$ on $[0, \pi/2]$; $s_0 = \pi/2$

In Exercises 32 – 35 , find the position function of an object given its acceleration and initial velocity and position.

32. $\vec{a}(t) = \langle 2, 3 \rangle$; $\vec{v}(0) = \langle 1, 2 \rangle$, $\vec{r}(0) = \langle 5, -2 \rangle$

33. $\vec{a}(t) = \langle 2, 3 \rangle$; $\vec{v}(1) = \langle 1, 2 \rangle$, $\vec{r}(1) = \langle 5, -2 \rangle$

34. $\vec{a}(t) = \langle \cos t, -\sin t \rangle;$ $\vec{v}(0) = \langle 0, 1 \rangle,$ $\vec{r}(0) = \langle 0, 0 \rangle$

35. $\vec{a}(t) = \langle 0, -32 \rangle;$ $\vec{v}(0) = \langle 10, 50 \rangle,$ $\vec{r}(0) = \langle 0, 0 \rangle$

In Exercises 36 – 39 , find the displacement, distance traveled, average velocity and average speed of the described object on the given interval.

36. An object with position function $\vec{r}(t) = \langle 2\cos t, 2\sin t, 3t \rangle$, where distances are measured in feet and time is in seconds, on $[0, 2\pi]$.

37. An object with position function $\vec{r}(t) = \langle 5\cos t, -5\sin t \rangle$, where distances are measured in feet and time is in seconds, on $[0, \pi]$.

38. An object with velocity function $\vec{v}(t) = \langle \cos t, \sin t \rangle$, where distances are measured in feet and time is in seconds, on $[0, 2\pi]$.

39. An object with velocity function $\vec{v}(t) = \langle 1, 2, -1 \rangle$, where distances are measured in feet and time is in seconds, on $[0, 10]$.

Exercises 40 – 45 ask you to solve a variety of problems based on the principles of projectile motion.

40. A boy whirls a ball, attached to a 3ft string, above his head in a counter–clockwise circle. The ball makes 2 revolutions per second.

    At what t-values should the boy release the string so that the ball heads directly for a tree standing 10ft in front of him?

41. David faces Goliath with only a stone in a 3ft sling, which he whirls above his head at 4 revolutions per second. They stand 20ft apart.

    (a) At what t-values must David release the stone in his sling in order to hit Goliath?

    (b) What is the speed at which the stone is traveling when released?

    (c) Assume David releases the stone from a height of 6ft and Goliath's forehead is 9ft above the ground. What angle of elevation must David apply to the stone to hit Goliath's head?

42. A hunter aims at a deer which is 40 yards away. Her crossbow is at a height of 5ft, and she aims for a spot on the deer 4ft above the ground. The crossbow fires her arrows at 300ft/s.

    (a) At what angle of elevation should she hold the crossbow to hit her target?

    (b) If the deer is moving perpendicularly to her line of sight at a rate of 20mph, by approximately how much should she lead the deer in order to hit it in the desired location?

43. A baseball player hits a ball at 100mph, with an initial height of 3ft and an angle of elevation of $20°$, at Boston's Fenway Park. The ball flies towards the famed "Green Monster," a wall 37ft high located 310ft from home plate.

    (a) Show that as hit, the ball hits the wall.

    (b) Show that if the angle of elevation is $21°$, the ball clears the Green Monster.

44. A Cessna flies at 1000ft at 150mph and drops a box of supplies to the professor (and his wife) on an island. Ignoring wind resistance, how far horizontally will the supplies travel before they land?

45. A football quarterback throws a pass from a height of 6ft, intending to hit his receiver 20yds away at a height of 5ft.

    (a) If the ball is thrown at a rate of 50mph, what angle of elevation is needed to hit his intended target?

    (b) If the ball is thrown at with an angle of elevation of $8°$, what initial ball speed is needed to hit his target?