11: VECTOR VALUED FUNCTIONS

In the previous chapter, we learned about vectors and were introduced to the power of vectors within mathematics. In this chapter, we'll build on this foundation to define functions whose input is a real number and whose output is a vector. We'll see how to graph these functions and apply calculus techniques to analyze their behavior. Most importantly, we'll see *why* we are interested in doing this: we'll see beautiful applications to the study of moving objects.

11.1 Vector–Valued Functions

We are very familiar with **real valued functions**, that is, functions whose output is a real number. This section introduces **vector–valued functions** – functions whose output is a vector.

Definition 68   Vector–Valued Functions

A **vector–valued function** is a function of the form

$$\vec{r}(t) = \langle f(t), g(t) \rangle \quad \text{or} \quad \vec{r}(t) = \langle f(t), g(t), h(t) \rangle,$$

where f, g and h are real valued functions.

The **domain** of $\vec{r}$ is the set of all values of t for which $\vec{r}(t)$ is defined. The **range** of $\vec{r}$ is the set of all possible output vectors $\vec{r}(t)$.

Evaluating and Graphing Vector–Valued Functions

Evaluating a vector–valued function at a specific value of t is straightforward; simply evaluate each component function at that value of t. For instance, if $\vec{r}(t) = \langle t^2, t^2 + t - 1 \rangle$, then $\vec{r}(-2) = \langle 4, 1 \rangle$. We can sketch this vector, as is done in Figure 11.1(a). Plotting lots of vectors is cumbersome, though, so generally we do not sketch the whole vector but just the terminal point. The **graph** of a vector–valued function is the set of all terminal points of $\vec{r}(t)$, where the initial point of each vector is always the origin. In Figure 11.1(b) we sketch the graph of $\vec{r}$; we can indicate individual points on the graph with their respective vector, as shown.

Vector–valued functions are closely related to parametric equations of graphs. While in both methods we plot points $(x(t), y(t))$ or $(x(t), y(t), z(t))$ to produce a graph, in the context of vector–valued functions each such point represents a vector. The implications of this will be more fully realized in the next section as we apply calculus ideas to these functions.
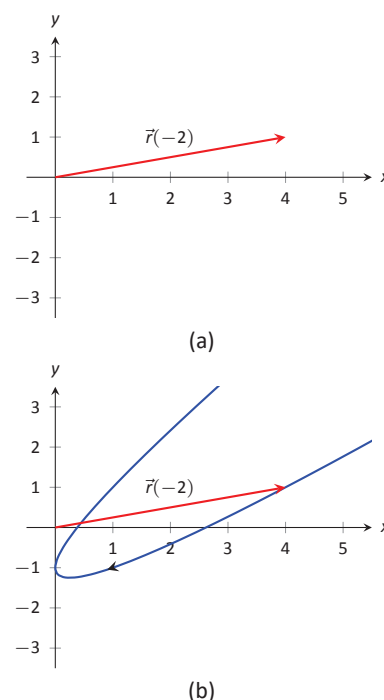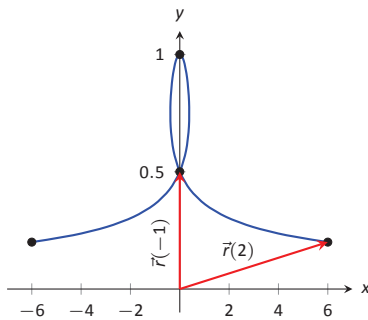

(a)

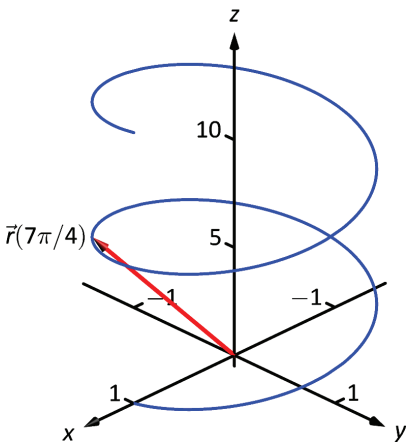
(b)

Figure 11.1: Sketching the graph of a vector–valued function.

t	$t^3 - t$	$\dfrac{1}{t^2 + 1}$
-2	-6	$1/5$
-1	0	$1/2$
0	0	1
1	0	$1/2$
2	6	$1/5$

(a)

(b)

Figure 11.2: Sketching the vector–valued function of Example 11.1.

Figure 11.3: The graph of $\vec{r}(t)$ in Example 11.2.

Example 11.1   Graphing vector–valued functions

Graph $\vec{r}(t) = \left\langle t^3 - t, \dfrac{1}{t^2 + 1} \right\rangle$, for $-2 \le t \le 2$. Sketch $\vec{r}(-1)$ and $\vec{r}(2)$.

SOLUTION   We start by making a table of t, x and y values as shown in Figure 11.2(a). Plotting these points gives an indication of what the graph looks like. In Figure 11.2(b), we indicate these points and sketch the full graph. We also highlight $\vec{r}(-1)$ and $\vec{r}(2)$ on the graph.

Example 11.2   Graphing vector–valued functions.

Graph $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ for $0 \le t \le 4\pi$.

SOLUTION   We can again plot points, but careful consideration of this function is very revealing. Momentarily ignoring the third component, we see the x and y components trace out a circle of radius 1 centered at the origin. Noticing that the z component is t, we see that as the graph winds around the z-axis, it is also increasing at a constant rate in the positive z direction, forming a spiral. This is graphed in Figure 11.3. In the graph $\vec{r}(7\pi/4) \approx (0.707, -0.707, 5.498)$ is highlighted to help us understand the graph.

Algebra of Vector–Valued Functions

> **Definition 69   Operations on Vector–Valued Functions**
>
> Let $\vec{r}_1(t) = \langle f_1(t), g_1(t) \rangle$ and $\vec{r}_2(t) = \langle f_2(t), g_2(t) \rangle$ be vector–valued functions in $\mathbb{R}^2$ and let c be a scalar. Then:
>
> 1. $\vec{r}_1(t) \pm \vec{r}_2(t) = \langle f_1(t) \pm f_2(t), g_1(t) \pm g_2(t) \rangle$.
>
> 2. $c\vec{r}_1(t) = \langle cf_1(t), cg_1(t) \rangle$.
>
> A similar definition holds for vector–valued functions in $\mathbb{R}^3$.

This definition states that we add, subtract and scale vector-valued functions component–wise. Combining vector–valued functions in this way can be very useful (as well as create interesting graphs).

Example 11.3   Adding and scaling vector–valued functions.

Let $\vec{r}_1(t) = \langle 0.2t, 0.3t \rangle$, $\vec{r}_2(t) = \langle \cos t, \sin t \rangle$ and $\vec{r}(t) = \vec{r}_1(t) + \vec{r}_2(t)$. Graph $\vec{r}_1(t)$, $\vec{r}_2(t)$, $\vec{r}(t)$ and $5\vec{r}(t)$ on $-10 \le t \le 10$.

Notes:

SOLUTION   We can graph $\vec{r}_1$ and $\vec{r}_2$ easily by plotting points (or just using technology). Let's think about each for a moment to better understand how vector–valued functions work.

We can rewrite $\vec{r}_1(t) = \langle 0.2t, 0.3t \rangle$ as $\vec{r}_1(t) = t\langle 0.2, 0.3 \rangle$. That is, the function $\vec{r}_1$ scales the vector $\langle 0.2, 0.3 \rangle$ by t. This scaling of a vector produces a line in the direction of $\langle 0.2, 0.3 \rangle$.

We are familiar with $\vec{r}_2(t) = \langle \cos t, \sin t \rangle$; it traces out a circle, centered at the origin, of radius 1. Figure 11.4(a) graphs $\vec{r}_1(t)$ and $\vec{r}_2(t)$.

Adding $\vec{r}_1(t)$ to $\vec{r}_2(t)$ produces $\vec{r}(t) = \langle \cos t + 0.2t, \sin t + 0.3t \rangle$, graphed in Figure 11.4(b). The linear movement of the line combines with the circle to create loops that move in the direction of $\langle 0.2, 0.3 \rangle$. (We encourage the reader to experiment by changing $\vec{r}_1(t)$ to $\langle 2t, 3t \rangle$, etc., and observe the effects on the loops.)

Multiplying $\vec{r}(t)$ by 5 scales the function by 5, producing $5\vec{r}(t) = \langle 5\cos t + 1, 5\sin t + 1.5 \rangle$, which is graphed in Figure 11.4(c) along with $\vec{r}(t)$. The new function is "5 times bigger" than $\vec{r}(t)$. Note how the graph of $5\vec{r}(t)$ in (c) looks identical to the graph of $\vec{r}(t)$ in (b). This is due to the fact that the x and y bounds of the plot in (c) are exactly 5 times larger than the bounds in (b).

Example 11.4   Adding and scaling vector–valued functions.
A **cycloid** is a graph traced by a point p on a rolling circle, as shown in Figure 11.5. Find an equation describing the cycloid, where the circle has radius 1.
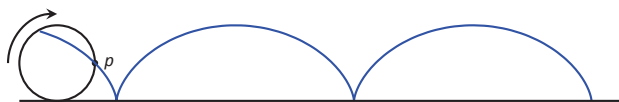

Figure 11.5: Tracing a cycloid.

SOLUTION   This problem is not very difficult if we approach it in a clever way. We start by letting $\vec{p}(t)$ describe the position of the point p on the circle, where the circle is centered at the origin and only rotates clockwise (i.e., it does not roll). This is relatively simple given our previous experiences with parametric equations; $\vec{p}(t) = \langle \cos t, -\sin t \rangle$.

We now want the circle to roll. We represent this by letting $\vec{c}(t)$ represent the location of the center of the circle. It should be clear that the y component of $\vec{c}(t)$ should be 1; the center of the circle is always going to be 1 if it rolls on a horizontal surface.
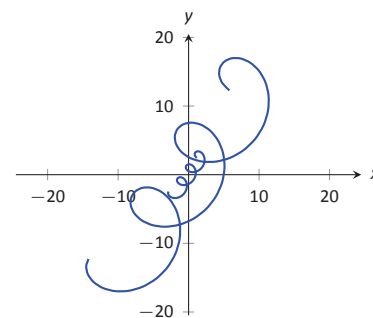
The x component of $\vec{c}(t)$ is a linear function of t: $f(t) = mt$ for some scalar m. When $t = 0, f(t) = 0$ (the circle starts centered on the y-axis). When $t = 2\pi$, the circle has made one complete revolution, traveling a distance equal to its
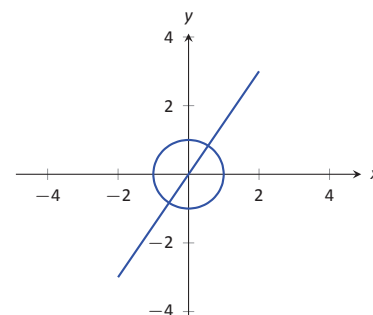
(a)
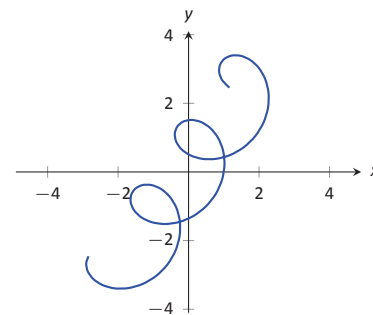
(b)

(c)

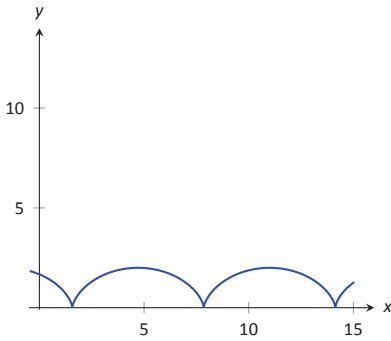Figure 11.4: Graphing the functions in Example 11.3.

Notes:

Figure 11.6: The cycloid in Example 11.4.

circumference, which is also 2π. This gives us a point on our line $f(t) = mt$, the point $(2\pi, 2\pi)$. It should be clear that $m = 1$ and $f(t) = t$. So $\vec{c}(t) = \langle t, 1 \rangle$.

We now combine $\vec{p}$ and $\vec{c}$ together to form the equation of the cycloid: $\vec{r}(t) = \vec{p}(t) + \vec{c}(t) = \langle \cos t + t, -\sin t + 1 \rangle$, which is graphed in Figure 11.6.

Displacement

A vector–valued function $\vec{r}(t)$ is often used to describe the position of a moving object at time t. At $t = t_0$, the object is at $\vec{r}(t_0)$; at $t = t_1$, the object is at $\vec{r}(t_1)$. Knowing the locations $\vec{r}(t_0)$ and $\vec{r}(t_1)$ give no indication of the path taken between them, but often we only care about the difference of the locations, $\vec{r}(t_1) - \vec{r}(t_0)$, the **displacement**.

Definition 70   Displacement

Let $\vec{r}(t)$ be a vector–valued function and let $t_0 < t_1$ be values in the domain. The **displacement** $\vec{d}$ of $\vec{r}$, from $t = t_0$ to $t = t_1$, is

$$\vec{d} = \vec{r}(t_1) - \vec{r}(t_0).$$

When the displacement vector is drawn with initial point at $\vec{r}(t_0)$, its terminal point is $\vec{r}(t_1)$. We think of it as the vector which points from a starting position to an ending position.

Example 11.5   Finding and graphing displacement vectors
Let $\vec{r}(t) = \left\langle \cos(\frac{\pi}{2}t), \sin(\frac{\pi}{2}t) \right\rangle$. Graph $\vec{r}(t)$ on $-1 \leq t \leq 1$, and find the displacement of $\vec{r}(t)$ on this interval.

Figure 11.7: Graphing the displacement of a position function in Example 11.5.

SOLUTION   The function $\vec{r}(t)$ traces out the unit circle, though at a different rate than the "usual" $\langle \cos t, \sin t \rangle$ parameterization. At $t_0 = -1$, we have $\vec{r}(t_0) = \langle 0, -1 \rangle$; at $t_1 = 1$, we have $\vec{r}(t_1) = \langle 0, 1 \rangle$. The displacement of $\vec{r}(t)$ on $[-1, 1]$ is thus $\vec{d} = \langle 0, 1 \rangle - \langle 0, -1 \rangle = \langle 0, 2 \rangle$.

A graph of $\vec{r}(t)$ on $[-1, 1]$ is given in Figure 11.7, along with the displacement vector $\vec{d}$ on this interval.
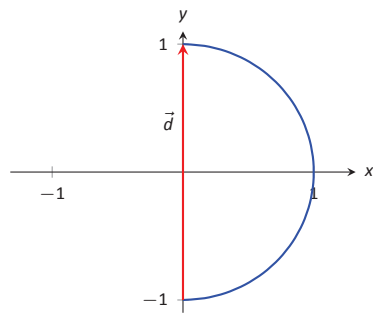
Measuring displacement makes us contemplate related, yet very different, concepts. Considering the semi–circular path the object in Example 11.5 took, we can quickly verify that the object ended up a distance of 2 units from its initial location. That is, we can compute $||\vec{d}|| = 2$. However, measuring *distance from the starting point* is different from measuring *distance traveled*. Being a semi–

Notes:

circle, we can measure the distance traveled by this object as $\pi \approx 3.14$ units. Knowing *distance from the starting point* allows us to compute **average rate of change.**

Definition 71   Average Rate of Change

Let $\vec{r}(t)$ be a vector–valued function, where each of its component functions is continuous on its domain, and let $t_0 < t_1$. The **average rate of change** of $\vec{r}(t)$ on $[t_0, t_1]$ is

$$\text{average rate of change} = \frac{\vec{r}(t_1) - \vec{r}(t_0)}{t_1 - t_0}.$$

Example 11.6   Average rate of change
Let $\vec{r}(t) = \left\langle \cos(\frac{\pi}{2}t), \sin(\frac{\pi}{2}t) \right\rangle$ as in Example 11.5. Find the average rate of change of $\vec{r}(t)$ on $[-1, 1]$ and on $[-1, 5]$.

SOLUTION   We computed in Example 11.5 that the displacement of $\vec{r}(t)$ on $[-1, 1]$ was $\vec{d} = \langle 0, 2 \rangle$. Thus the average rate of change of $\vec{r}(t)$ on $[-1, 1]$ is:

$$\frac{\vec{r}(1) - \vec{r}(-1)}{1 - (-1)} = \frac{\langle 0, 2 \rangle}{2} = \langle 0, 1 \rangle .$$

We interpret this as follows: the object followed a semi–circular path, meaning it moved towards the right then moved back to the left, while climbing slowly, then quickly, then slowly again. *On average*, however, it progressed straight up at a constant rate of $\langle 0, 1 \rangle$ per unit of time.

We can quickly see that the displacement on $[-1, 5]$ is the same as on $[-1, 1]$, so $\vec{d} = \langle 0, 2 \rangle$. The average rate of change is different, though:

$$\frac{\vec{r}(5) - \vec{r}(-1)}{5 - (-1)} = \frac{\langle 0, 2 \rangle}{6} = \langle 0, 1/3 \rangle .$$

As it took "3 times as long" to arrive at the same place, this average rate of change on $[-1, 5]$ is $1/3$ the average rate of change on $[-1, 1]$.

We considered average rates of change in Sections 1.1 and 2.1 as we studied limits and derivatives. The same is true here; in the following section we apply calculus concepts to vector–valued functions as we find limits, derivatives, and integrals. Understanding the average rate of change will give us an understanding of the derivative; displacement gives us one application of integration.

Notes:

Exercises 11.1

Terms and Concepts

1. Vector–valued functions are closely related to _____ _____ of graphs.

16. $\vec{r}(t) = \langle t, t^2 \rangle$.

2. When sketching vector–valued functions, technically one isn't graphing points, but rather _____.

17. $\vec{r}(t) = \langle 5 \cos t, 3 \sin t \rangle$.

18. $\vec{r}(t) = \langle 2 \cos t, 2 \sin t, t \rangle$.

3. It can be useful to think of _____ as a vector that points from a starting position to an ending position.

19. $\vec{r}(t) = \langle \cos t, t, t^2 \rangle$.

In Exercises 20 – 27, create a vector–valued function whose graph matches the given description.

Problems

20. A circle of radius 2, centered at $(1, 2)$, traced counter–clockwise once on $[0, 2\pi]$.

In Exercises 4 – 11, sketch the vector–valued function on the given interval.

21. A circle of radius 3, centered at $(5, 5)$, traced clockwise once on $[0, 2\pi]$.

4. $\vec{r}(t) = \langle t^2, t^2 - 1 \rangle$, for $-2 \leq t \leq 2$.

22. An ellipse, centered at $(0, 0)$ with vertical major axis of length 10 and minor axis of length 3, traced once counter–clockwise on $[0, 2\pi]$.

5. $\vec{r}(t) = \langle t^2, t^3 \rangle$, for $-2 \leq t \leq 2$.

6. $\vec{r}(t) = \langle 1/t, 1/t^2 \rangle$, for $-2 \leq t \leq 2$.

23. An ellipse, centered at $(3, -2)$ with horizontal major axis of length 6 and minor axis of length 4, traced once clockwise on $[0, 2\pi]$.

7. $\vec{r}(t) = \langle \frac{1}{10} t^2, \sin t \rangle$, for $-2\pi \leq t \leq 2\pi$.

8. $\vec{r}(t) = \langle \frac{1}{10} t^2, \sin t \rangle$, for $-2\pi \leq t \leq 2\pi$.

24. A line through $(2, 3)$ with a slope of 5.

9. $\vec{r}(t) = \langle 3 \sin(\pi t), 2 \cos(\pi t) \rangle$, on $[0, 2]$.

25. A line through $(1, 5)$ with a slope of $-1/2$.

10. $\vec{r}(t) = \langle 3 \cos t, 2 \sin(2t) \rangle$, on $[0, 2\pi]$.

26. A vertically oriented helix with radius of 2 that starts at $(2, 0, 0)$ and ends at $(2, 0, 4\pi)$ after 1 revolution on $[0, 2\pi]$.

11. $\vec{r}(t) = \langle 2 \sec t, \tan t \rangle$, on $[-\pi, \pi]$.

27. A vertically oriented helix with radius of 3 that starts at $(3, 0, 0)$ and ends at $(3, 0, 3)$ after 2 revolutions on $[0, 1]$.

In Exercises 12 – 15, sketch the vector–valued function on the given interval in $\mathbb{R}^3$. Technology may be useful in creating the sketch.

In Exercises 28 – 31, find the average rate of change of $\vec{r}(t)$ on the given interval.

12. $\vec{r}(t) = \langle 2 \cos t, t, 2 \sin t \rangle$, on $[0, 2\pi]$.

28. $\vec{r}(t) = \langle t, t^2 \rangle$ on $[-2, 2]$.

13. $\vec{r}(t) = \langle 3 \cos t, \sin t, t/\pi \rangle$ on $[0, 2\pi]$.

29. $\vec{r}(t) = \langle t, t + \sin t \rangle$ on $[0, 2\pi]$.

14. $\vec{r}(t) = \langle \cos t, \sin t, \sin t \rangle$ on $[0, 2\pi]$.

30. $\vec{r}(t) = \langle 3 \cos t, 2 \sin t, t \rangle$ on $[0, 2\pi]$.

15. $\vec{r}(t) = \langle \cos t, \sin t, \sin(2t) \rangle$ on $[0, 2\pi]$.

31. $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ on $[-1, 3]$.