14.5 Parametric Surfaces and Surface Area

Thus far we have focused mostly on 2-dimensional vector fields, measuring flow and flux along/across curves in the plane. Both Green's Theorem and the Divergence Theorem make connections between planar regions and their boundaries. We now move our attention to 3-dimensional vector fields, considering both curves and surfaces in space.

We are accustomed to describing surfaces as functions of two variables, usually written as $z = f(x, y)$. For our coming needs, this method of describing surfaces will prove to be insufficient. Instead, we will *parameterize* our surfaces, describing them as the set of terminal points of some vector–valued function $\vec{r}(u, v) = \langle f(u, v), g(u, v), h(u, v)\rangle$. The bulk of this section is spent practicing the skill of describing a surface $\mathcal{S}$ using a vector–valued function. Once this skill is developed, we'll show how to find the surface area S of a parametrically–defined surface $\mathcal{S}$, a skill needed in the remaining sections of this chapter.

Note: We use the letter S to denote Surface Area. This section begins a study into surfaces, and it is natural to label a surface with the letter "S". We distinguish a surface from its surface area by using a calligraphic S to denote a surface: $\mathcal{S}$. When writing this letter by hand, it may be useful to add serifs to the letter, such as: $\mathcal{S}$

Definition 123 Parametric Surface

Let $\vec{r}(u, v) = \langle f(u, v), g(u, v), h(u, v)\rangle$ be a vector–valued function that is continuous and one-to-one on the interior of its domain R in the u-v plane. The set of all terminal points of $\vec{r}$ (i.e., the *range* of $\vec{r}$) is the **surface** $\mathcal{S}$, and $\vec{r}$ along with its domain R form a **parameterization** of $\mathcal{S}$.

This parameterization is **smooth** on R if $\vec{r}_u$ and $\vec{r}_v$ are continuous and $\vec{r}_u \times \vec{r}_v$ is never $\vec{0}$ on the interior of R.

Note: A function is *one-to-one* on its domain if the function never repeats an output value over the domain. In the case of $\vec{r}(u, v)$, $\vec{r}$ is one-to-one if $\vec{r}(u_1, v_1) \neq \vec{r}(u_2, v_2)$ for all points $(u_1, v_1) \neq (u_2, v_2)$ in the domain of $\vec{r}$.

Given a point (u_0, v_0) in the domain of a vector–valued function $\vec{r}$, the vectors $\vec{r}_u(u_0, v_0)$ and $\vec{r}_v(u_0, v_0)$ are tangent to the surface $\mathcal{S}$ at $\vec{r}(u_0, v_0)$ (a proof of this is developed later in this section). The definition of smoothness dictates that $\vec{r}_u \times \vec{r}_v \neq \vec{0}$; this ensures that neither $\vec{r}_u$ nor $\vec{r}_v$ are $\vec{0}$, nor are they ever parallel. Therefore smoothness guarantees that $\vec{r}_u$ and $\vec{r}_v$ determine a plane that is tangent to $\mathcal{S}$.

A surface $\mathcal{S}$ is said to be **orientable** if a field of normal vectors can be defined on $\mathcal{S}$ that vary continuously along $\mathcal{S}$. This definition may be hard to understand; it may help to know that orientable surfaces are often called "two sided." A sphere is an orientable surface, and one can easily envision an "inside" and "outside" of the sphere. A paraboloid is orientable, where again one can generally envision "inside" and "outside" sides (or "top" and "bottom" sides) to this surface. Just about every surface that one can imagine is orientable, and we'll assume all surfaces we deal with in this text are orientable.

It is enlightening to examine a classic non-orientable surface: the Möbius

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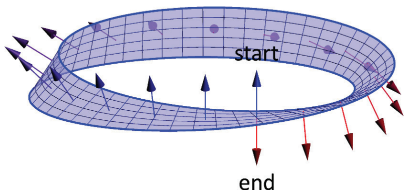


Figure 14.30: A Möbius band, a non-orientable surface.

band, shown in Figure 14.30. Vectors normal to the surface are given, starting at the point indicated in the figure. These normal vectors "vary continuously" as they move along the surface. Letting each vector indicate the "top" side of the band, we can easily see near any vector which side is the "top".

However, if as we progress along the band, we recognize that we are labeling "both sides" of the band as the top; in fact, there are not two "sides" to this band, but one. The Möbius band is a non-orientable surface.

We now practice parameterizing surfaces.

Example 14.24 Parameterizing a surface over a rectangle

Parameterize the surface $z = x^2 + 2y^2$ over the rectangular region R defined by $-3 \leq x \leq 3$, $-1 \leq y \leq 1$.

Solution There is a straightforward way to parameterize a surface of the form $z = f(x, y)$ over a rectangular domain. We let $x = u$ and $y = v$, and let $\vec{r}(u, v) = \langle u, v, f(u, v)\rangle$. In this instance, we have $\vec{r}(u, v) = \langle u, v, u^2 + 2v^2\rangle$, for $-3 \leq u \leq 3$, $-1 \leq v \leq 1$. This surface is graphed in Figure 14.31.

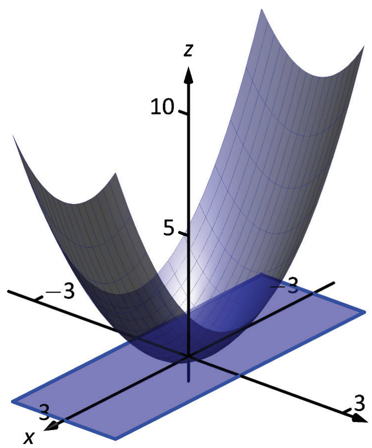


Figure 14.31: The surface parameterized in Example 14.24.

Example 14.25 Parameterizing a surface over a circular disk

Parameterize the surface $z = x^2 + 2y^2$ over the circular region R enclosed by the circle of radius 2 that is centered at the origin.

Solution We can parameterize the circular boundary of R with the vector–valued function $\langle 2\cos u, 2\sin u\rangle$, where $0 \leq u \leq 2\pi$. We can obtain the interior of R by scaling this function by a variable amount, i.e., by multiplying by v: $\langle 2v\cos u, 2v\sin u\rangle$, where $0 \leq v \leq 1$.

It is important to understand the role of v in the above function. When $v = 1$, we get the boundary of R, a circle of radius 2. When $v = 0$, we simply get the point $(0, 0)$, the center of R (which can be thought of as a circle with radius of 0). When $v = 1/2$, we get the circle of radius 1 that is centered at the origin, which is the circle *halfway* between the boundary and the center. As v varies from 0 to 1, we create a series of concentric circles that fill out all of R.

Thus far, we have determined the x and y components of our parameterization of the surface: $x = 2v\cos u$ and $y = 2v\sin u$. We find the z component simply by using $z = f(x, y) = x^2 + 2y^2$:

$$z = (2v\cos u)^2 + 2(2v\sin u)^2 = 4v^2\cos^2 u + 8v^2\sin^2 u.$$

Thus $\vec{r}(u, v) = \langle 2v\cos u, 2v\sin u, 4v^2\cos^2 u + 8v^2\sin^2 u\rangle$, $0 \leq u \leq 2\pi$, $0 \leq v \leq 1$, which is graphed in Figure 14.32. The way that this graphic was generated highlights how the surface was parameterized. When viewing from above, one can see lines emanating from the origin; they represent different values of u as

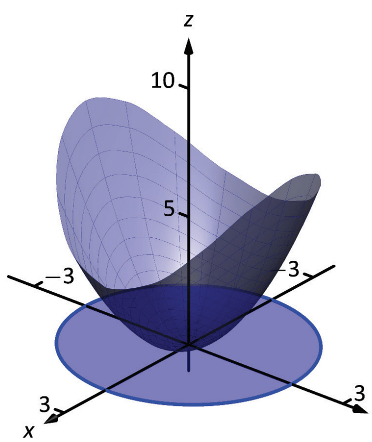


Figure 14.32: The surface parameterized in Example 14.25.

Notes:

u sweeps from an angle of 0 up to 2π. One can also see concentric circles, each corresponding to a different value of v.

Examples 14.24 and 14.25 demonstrate an important principle when parameterizing surfaces given in the form $z = f(x, y)$ over a region R: if one can determine x and y in terms of u and v, then z follows directly as $z = f(x, y)$.

In the following two examples, we parameterize the same surface over triangular regions. Each will use v as a "scaling factor" as done in Example 14.25.

Example 14.26 Parameterizing a surface over a triangle

Parameterize the surface $z = x^2 + 2y^2$ over the triangular region R enclosed by the coordinate axes and the line $y = 2 - 2x/3$, as shown in Figure 14.33(a).

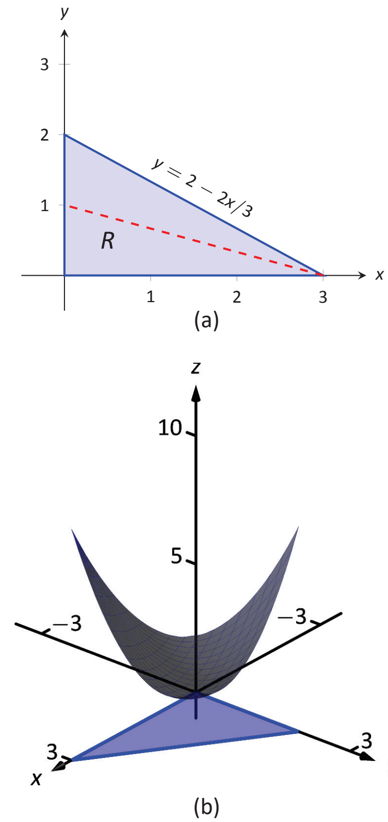


Figure 14.33: Part (a) shows a graph of the region R, and part (b) shows the surface over R, as defined in Example 14.26.

Solution We may begin by letting $x = u$, $0 \le u \le 3$, and $y = 2 - 2u/3$. This gives only the line on the "upper" side of the triangle. To get all of the region R, we can once again scale y by a variable factor, v.

Still letting $x = u$, $0 \le u \le 3$, we let $y = v(2 - 2u/3)$, $0 \le v \le 1$. When $v = 0$, all y-values are 0, and we get the portion of the x-axis between $x = 0$ and $x = 3$. When $v = 1$, we get the upper side of the triangle. When $v = 1/2$, we get the line $y = 1/2(2 - 2u/3) = 1 - u/3$, which is the line "halfway up" the triangle, shown in the figure with a dashed line.

Letting $z = f(x, y) = x^2 + 2y^2$, we have

$$\vec{r}(u,v) = \left\langle u, v\left(2 - \frac{2u}{3}\right), u^2 + 2\left(v\left(2 - \frac{2u}{3}\right)\right)^2 \right\rangle,$$

$0 \le u \le 3$, $0 \le v \le 1$. This surface is graphed in Figure 14.33(b). Again, when one looks from above, we can see the scaling effects of v: the series of lines that run to the point $(3, 0)$ each represent a different value of v.

Another common way to parameterize the surface is to begin with $y = u$, $0 \le u \le 2$. Solving the equation of the line $y = 2 - 2x/3$ for x, we have $x = 3 - 3y/2$, leading to using $x = v(3 - 3u/2)$, $0 \le v \le 1$. With $z = x^2 + 2y^2$, we have

$$\vec{r}(u,v) = \left\langle v\left(3 - \frac{3u}{2}\right), u, \left(v\left(3 - \frac{3u}{2}\right)\right)^2 + 2v^2 \right\rangle,$$

$0 \le u \le 2$, $0 \le v \le 1$.

Example 14.27 Parameterizing a surface over a triangle

Parameterize the surface $z = x^2 + 2y^2$ over the triangular region R enclosed by the lines $y = 3 - 2x/3$, $y = 1$ and $x = 0$ as shown in Figure 14.34(a).

Notes:

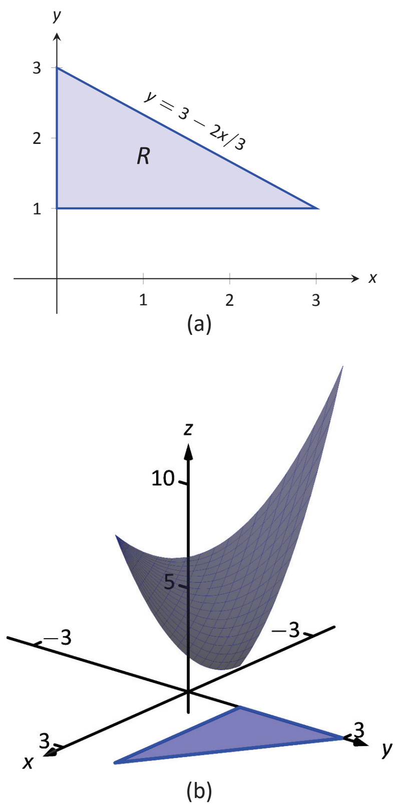


Figure 14.34: Part (a) shows a graph of the region R, and part (b) shows the surface over R, as defined in Example 14.27.

Solution While the region R in this example is very similar to the region R in the previous example, and our method of parameterizing the surface is fundamentally the same, it will feel as though our answer is much different than before.

We begin with letting $x = u$, $0 \leq u \leq 3$. We may be tempted to let $y = v(3 - 2u/3)$, $0 \leq v \leq 1$, but this is incorrect. When $v = 1$, we obtain the upper line of the triangle as desired. However, when $v = 0$, the y-value is 0, which does not lie in the region R.

We will describe the general method of proceeding following this example. For now, consider $y = 1 + v(2 - 2u/3)$, $0 \leq v \leq 1$. Note that when $v = 1$, we have $y = 3 - 2u/3$, the upper line of the boundary of R. Also, when $v = 0$, we have $y = 1$, which is the lower boundary of R. With $z = x^2 + 2y^2$, we determine

$$\vec{r}(u,v) = \left\langle u, 1 + v\left(2 - \frac{2u}{3}\right), u^2 + 2\left(1 + v\left(2 - \frac{2u}{3}\right)\right)^2\right\rangle,$$

$0 \leq u \leq 3$, $0 \leq v \leq 1$.

The surface is graphed in Figure 14.34(b).

Given a surface of the form $z = f(x,y)$, one can often determine a parameterization of the surface over a region R in a manner similar to determining bounds of integration over a region R. Using the techniques of Section 13.1, suppose a region R can be described by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, i.e., the area of R can be found using the iterated integral

$$\int_a^b \int_{g_1(x)}^{g_2(x)} dy\,dx.$$

When parameterizing the surface, we can let $x = u$, $a \leq u \leq b$, and we can let $y = g_1(u) + v\big(g_2(u) - g_1(u)\big)$, $0 \leq v \leq 1$. The parameterization of x is straightforward, but look closely at how y is determined. When $v = 0$, $y = g_1(u) = g_1(x)$. When $v = 1$, $y = g_2(u) = g_2(x)$.

As a specific example, consider the triangular region R from Example 14.27, shown in Figure 14.34(a). Using the techniques of Section 13.1, we can find the area of R as

$$\int_0^3 \int_1^{3-2x/3} dy\,dx.$$

Following the above discussion, we can set $x = u$, where $0 \leq u \leq 3$, and set $y = 1 + v\big(3 - 2u/3 - 1\big) = 1 + v(2 - 2u/3)$, $0 \leq v \leq 1$, as used in that example.

One can do a similar thing if R is bounded by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, but for the sake of simplicity we leave it to the reader to flesh out those details. The principles outlined above are given in the following Key Idea for reference.

Notes:

Key Idea 64 Parameterizing Surfaces

Let a surface $\mathcal{S}$ be the graph of a function $z = f(x, y)$, where the domain of f is a closed, bounded region R in the xy-plane. Let R be bounded by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, i.e., the area of R can be found using the integral $\int_a^b \int_{g_1(x)}^{g_2(x)} dy\,dx$, and let $h(u, v) = g_1(u) + v\big(g_2(u) - g_1(u)\big)$. Then $\mathcal{S}$ can be parameterized as

$$\vec{r}(u, v) = \big\langle u, h(u, v), f\big(u, h(u, v)\big)\big\rangle, \quad a \leq u \leq b,\ 0 \leq v \leq 1.$$

Example 14.28 Parameterizing a cylinderical surface

Find a parameterization of the cylinder $x^2 + z^2/4 = 1$, where $-1 \leq y \leq 2$, as shown in Figure 14.35.

SOLUTION The equation $x^2+z^2/4 = 1$ can be envisioned to describe an ellipse in the x-z plane; as the equation lacks a y-term, the equation describes a cylinder (recall Definition 53) that extends without bound parallel to the y-axis. This ellipse has a vertical major axis of length 4, a horizontal minor axis of length 2, and is centered at the origin. We can parameterize this ellipse using sines and cosines; our parameterization can begin with

$$\vec{r}(u, v) = \langle \cos u, ???, 2\sin u\rangle, \quad 0 \leq u \leq 2\pi,$$

where we still need to determine the y component.

While the cylinder $x^2 + z^2/4 = 1$ is satisfied by any y value, the problem states that all y values are to be between $y = -1$ and $y = 2$. Since the value of y does not depend at all on the values of x or z, we can use another variable, v, to describe y. Our final answer is

$$\vec{r}(u, v) = \langle \cos u, v, 2\sin u\rangle, \quad 0 \leq u \leq 2\pi, \quad -1 \leq v \leq 2.$$

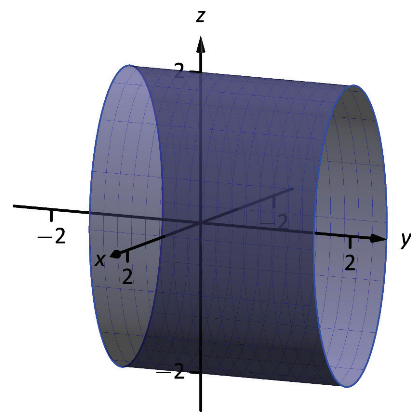


Figure 14.35: The cylinder parameterized in Example 14.28.

Example 14.29 Parameterizing an elliptic cone

Find a parameterization of the elliptic cone $z^2 = \frac{x^2}{4} + \frac{y^2}{9}$, where $-2 \leq z \leq 3$, as shown in Figure 14.36.

SOLUTION One way to parameterize this cone is to recognize that given a z value, the cross section of the cone at that z value is an ellipse with equation $\frac{x^2}{(2z)^2} + \frac{y^2}{(3z)^2} = 1$. We can let $z = v$, for $-2 \leq v \leq 3$ and then parameterize the above ellipses using sines, cosines and v.

We can parameterize the x component of our surface with $x = 2z\cos u$ and the y component with $y = 3z\sin u$, where $0 \leq u \leq 2\pi$. Putting all components

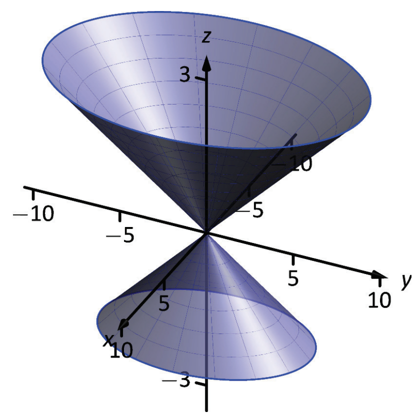


Figure 14.36: The elliptic cone as described in Example 14.29.

Notes:

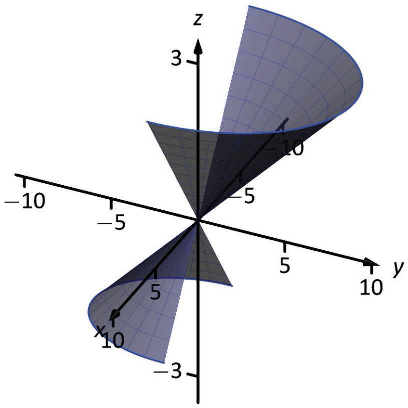
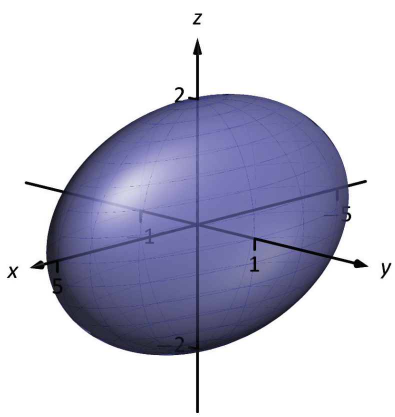
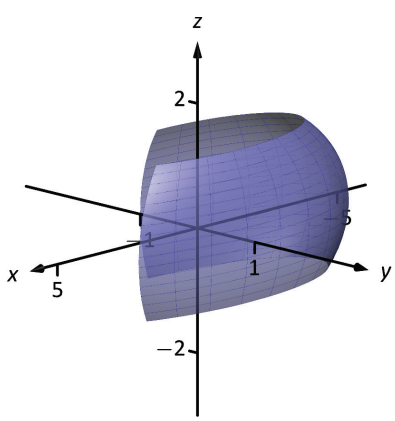


Figure 14.37: The elliptic cone as described in Example 14.29 with restricted domain.



(a)



(b)

Figure 14.38: An ellipsoid in (a), drawn again in (b) with its domain restricted, as described in Example 14.30.

together, we have

$$\vec{r}(u,v) = \langle 2v\cos u, 3v\sin u, v\rangle\,, \quad 0 \le u \le 2\pi, \quad -2 \le v \le 3.$$

When v takes on negative values, the radii of the cross–sectional ellipses become "negative," which can lead to some surprising results. Consider Figure 14.37, where the cone is graphed for $0 \le u \le \pi$. Because v is negative below the x-y plane, the radii of the cross–sectional ellipses are negative, and the opposite side of the cone is sketched below the x-y plane.

Example 14.30 Parameterizing an ellipsoid

Find a parameterization of the ellipsoid $\frac{x^2}{25} + y^2 + \frac{z^2}{4} = 1$ as shown in Figure 14.38(a).

Solution Recall Key Idea 50 from Section 10.2, which states that all unit vectors in space have the form $\langle \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta\rangle$ for some angles θ and φ. If we choose our angles appropriately, this allows us to draw the unit sphere. To get an ellipsoid, we need only scale each component of the sphere appropriately.

The x-radius of the given ellipsoid is 5, the y-radius is 1 and the z-radius is 2. Substituting u for θ and v for φ, we have $\vec{r}(u,v) = \langle 5\sin u\cos v, \sin u\sin v, 2\cos u\rangle$, where we still need to determine the ranges of u and v.

Note how the x and y components of $\vec{r}$ have $\cos v$ and $\sin v$ terms, respectively. This hints at the fact that ellipses are drawn parallel to the x-y plane as v varies, which implies we should have v range from 0 to 2π.

One may be tempted to let $0 \le u \le 2\pi$ as well, but note how the z component is $2\cos u$. We only need $\cos u$ to take on values between -1 and 1 once, therefore we can restrict u to $0 \le u \le \pi$.

The final parameterization is thus

$$\vec{r}(u,v) = \langle 5\sin u\cos v, \sin u\sin v, 2\cos u\rangle, \quad 0 \le u \le \pi, \quad 0 \le v \le 2\pi.$$

In Figure 14.38(b), the ellipsoid is graphed on $\frac{\pi}{4} \le u \le \frac{2\pi}{3}$, $\frac{\pi}{4} \le v \le \frac{3\pi}{2}$ to demonstrate how each variable affects the surface.

Parameterization is a powerful way to represent surfaces. One of the advantages of the methods of parameterization described in this section is that the domain of $\vec{r}(u,v)$ is always a rectangle; that is, the bounds on u and v are constants. This will make some of our future computations easier to evaluate.

Just as we could parameterize curves in more than one way, there will always be multiple ways to parameterize a surface. Some ways will be more "natural" than others, but these other ways are not incorrect. Because technology is often

Notes:

readily available, it is often a good idea to check one's work by graphing a parameterization of a surface to check if it indeed represents what it was intended to.

Surface Area

It will become important in the following sections to be able to compute the surface area of a surface $\mathcal{S}$ given a smooth parameterization $\vec{r}(u,v)$, $a \leq u \leq b$, $c \leq v \leq d$. Following the principles given in the integration review at the beginning of this chapter, we can say that

$$\text{Surface Area of } \mathcal{S} = S = \iint_{\mathcal{S}} dS,$$

where dS represents a small amount of surface area. That is, to compute total surface area S, add up lots of small amounts of surface area dS across the entire surface $\mathcal{S}$. The key to finding surface area is knowing how to compute dS. We begin by approximating.

In Section 13.5 we used the area of a plane to approximate the surface area of a small portion of a surface. We will do the same here.

Let R be the region of the u-v plane bounded by $a \leq u \leq b$, $c \leq v \leq d$ as shown in Figure 14.39(a). Partition R into rectangles of width $\Delta u = \frac{b-a}{n}$ and height $\Delta v = \frac{d-c}{n}$, for some n. Let $p = (u_0, v_0)$ be the lower left corner of some rectangle in the partition, and let m and q be neighboring corners as shown.

The point p maps to a point $P = \vec{r}(u_0, v_0)$ on the surface $\mathcal{S}$, and the rectangle with corners p, m and q maps to some region (probably not rectangular) on the surface as shown in Figure 14.39(b), where $M = \vec{r}(m)$ and $Q = \vec{r}(q)$. We wish to approximate the surface area of this mapped region.

Let $\vec{u} = M - P$ and $\vec{v} = Q - P$. These two vectors form a parallelogram, illustrated in Figure 14.39(c), whose area *approximates* the surface area we seek. In this particular illustration, we can see that parallelogram does not particularly match well the region we wish to approximate, but that is acceptable; by increasing the number of partitions of R, Δu and Δv shrink and our approximations will become better.

From Section 10.4 we know the area of this parallelogram is $||\,\vec{u} \times \vec{v}\,||$. If we repeat this approximation process for each rectangle in the partition of R, we can sum the areas of all the parallelograms to get an approximation of the surface area S:

$$\text{Surface area of } \mathcal{S} = S \approx \sum_{j=1}^{n}\sum_{i=1}^{n} ||\,\vec{u}_{i,j} \times \vec{v}_{i,j}\,||\,,$$

where $\vec{u}_{i,j} = \vec{r}(u_i + \Delta u, v_j) - \vec{r}(u_i, v_j)$ and $\vec{v}_{i,j} = \vec{r}(u_i, v_j + \Delta v) - \vec{r}(u_i, v_j)$.

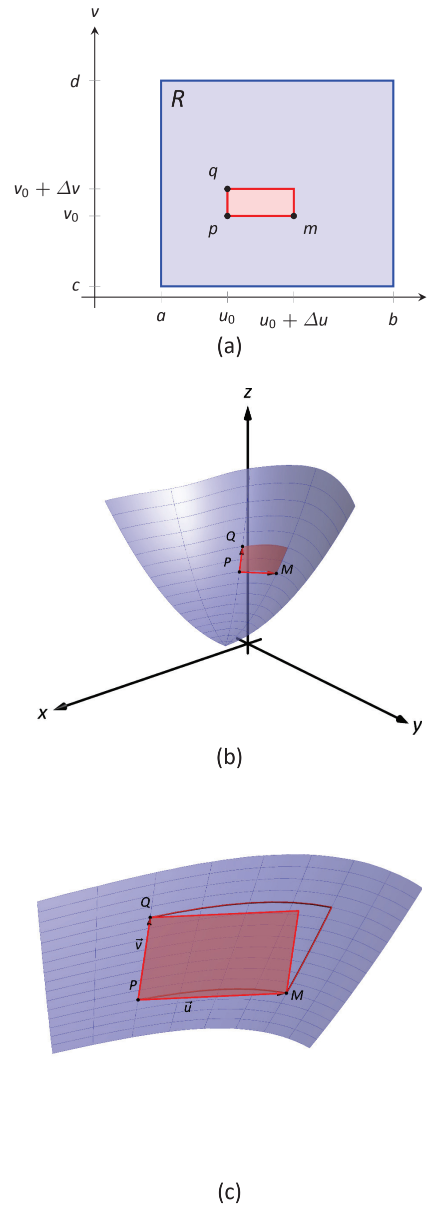


Figure 14.39: Illustrating the process of finding surface area by approximating with planes.

Notes:

From our previous calculus experience, we expect that taking a limit as $n \to \infty$ will result in the exact surface area. However, the current form of the above double sum makes it difficult to realize what the result of that limit is. The following rewriting of the double summation will be helpful:

$$\begin{aligned}
&\sum_{j=1}^{n}\sum_{i=1}^{n} ||\,\vec{u}_{i,j}\times\vec{v}_{i,j}\,|| = \\
&\sum_{j=1}^{n}\sum_{i=1}^{n} \Big|\Big|\,\big(\vec{r}(u_i+\Delta u, v_j)-\vec{r}(u_i,v_j)\big)\times\big(\vec{r}(u_i,v_j+\Delta v)-\vec{r}(u_i,v_j)\big)\,\Big|\Big| = \\
&\sum_{j=1}^{n}\sum_{i=1}^{n} \left|\left|\,\frac{\vec{r}(u_i+\Delta u, v_j)-\vec{r}(u_i,v_j)}{\Delta u}\times\frac{\vec{r}(u_i,v_j+\Delta v)-\vec{r}(u_i,v_j)}{\Delta v}\,\right|\right|\Delta u\Delta v.
\end{aligned}$$

We now take the limit as $n \to \infty$, forcing Δu and Δv to 0. As $\Delta u \to 0$,

$$\frac{\vec{r}(u_i+\Delta u, v_j)-\vec{r}(u_i,v_j)}{\Delta u} \quad\to\quad \vec{r}_u(u_i,v_j) \quad \text{and}$$

$$\frac{\vec{r}(u_i,v_j+\Delta v)-\vec{r}(u_i,v_j)}{\Delta v} \quad\to\quad \vec{r}_v(u_i,v_j).$$

(This limit process also demonstrates that $\vec{r}_u(u,v)$ and $\vec{r}_v(u,v)$ are tangent to the surface $\mathcal{S}$ at $\vec{r}(u,v)$. We don't need this fact now, but it will be important in the next section.)

Thus, in the limit, the double sum leads to a double integral:

$$\lim_{n\to\infty}\sum_{j=1}^{n}\sum_{i=1}^{n} ||\,\vec{u}_{i,j}\times\vec{v}_{i,j}\,|| = \int_c^d\int_a^b ||\,\vec{r}_u\times\vec{r}_v\,||\;du\,dv.$$

Theorem 143 Surface Area of Parametrically Defined Surfaces

Let $\vec{r}(u,v)$ be a smooth parameterization of a surface $\mathcal{S}$ over a closed, bounded region R of the u-v plane.

- The surface area differential dS is: $dS = ||\,\vec{r}_u\times\vec{r}_v\,||\;dA$.
- The surface area S of $\mathcal{S}$ is

$$S = \iint_{\mathcal{S}} dS = \iint_R ||\,\vec{r}_u\times\vec{r}_v\,||\;dA.$$

Notes:

Example 14.31 Finding the surface area of a parameterized surface

Determine the surface area of the helicoid (a spiral ramp shown in Figure 14.40) given parametrically by

$$\vec{r}(u,v) = \langle u\cos v, u\sin v, v\rangle$$

for $0 \le u \le 1$ and $0 \le v \le 4\pi$.

Solution Notice that we will be integrating over a rectangle in the uv-plane. To find the surface area using Theorem 143, we need $\|\,\vec{r}_u \times \vec{r}_v\,\|$. We find:

$$\begin{aligned}
\vec{r}_u &= \langle \cos v, \sin v, 0\rangle\\
\vec{r}_v &= \langle -u\sin v, u\cos v, 1\rangle\\
\vec{r}_u \times \vec{r}_v &= \langle \sin v, -\cos v, u\rangle\\
\|\,\vec{r}_u \times \vec{r}_v\,\| &= \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1+u^2}.
\end{aligned}$$

Integrating over the rectangle in the uv-plane gives us the surface area

$$\begin{aligned}
S = \iint_{\mathcal{S}} dS &= \iint_R \|\,\vec{r}_u \times \vec{r}_v\,\|\; dA\\
&= \int_0^{4\pi}\int_0^1 \sqrt{1+u^2}\,du\,dv\\
&= 4\pi\left(\left(\frac{1}{2}\left(u\sqrt{1+u^2} + \ln\left|\sqrt{1+u^2}+u\right|\right)\right)\right)\Bigg|_0^1\\
&= 2\pi\left(\sqrt{2} + \ln(1+\sqrt{2})\right) \text{ square units.}
\end{aligned}$$

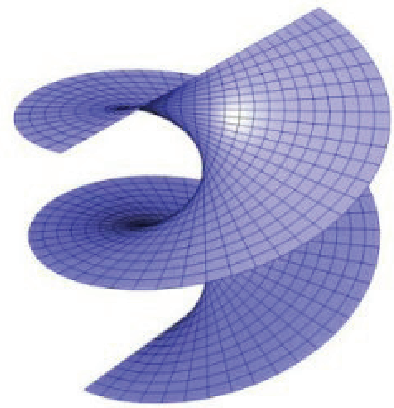


Figure 14.40: Helicoid

Example 14.32 Finding the surface area of a parameterized surface

Using the parameterization found in Example 14.25, find the surface area of $z = x^2 + 2y^2$ over the circular disk of radius 2, centered at the origin.

Solution In Example 14.25, we parameterized the surface as $\vec{r}(u,v) = \langle 2v\cos u, 2v\sin u, 4v^2\cos^2 u + 8v^2\sin^2 u\rangle$, for $0 \le u \le 2\pi$, $0 \le v \le 1$. To find the surface area using Theorem 143, we need $\|\,\vec{r}_u \times \vec{r}_v\,\|$. We find:

$$\begin{aligned}
\vec{r}_u &= \langle -2v\sin u, 2v\cos u, 8v^2\cos u\sin u\rangle\\
\vec{r}_v &= \langle 2\cos u, 2\sin v, 8v\cos^2 u + 16v\sin^2 u\rangle\\
\vec{r}_u \times \vec{r}_v &= \langle 16v^2\cos u, 32v^2\sin u, -4v\rangle\\
\|\,\vec{r}_u \times \vec{r}_v\,\| &= \sqrt{256v^4\cos^2 u + 1024v^4\sin^2 u + 16v^2}.
\end{aligned}$$

Notes:

Thus the surface area is

$$S = \iint_{\mathcal{S}} dS = \iint_R ||\,\vec{r}_u \times \vec{r}_v\,||\; dA$$
$$= \int_0^1 \int_0^{2\pi} \sqrt{256v^4\cos^2 u + 1024v^4\sin^2 u + 16v^2}\, du\, dv \approx 53.59.$$

There is a lot of tedious work in the above calculations and the final integral is nontrivial. The use of a computer-algebra system is highly recommended.

In Section 14.1, we recalled the arc length differential $ds = ||\,\vec{r}\,'(t)\,||\,dt$. In subsequent sections, we used that differential, but in most applications the "$||\,\vec{r}\,'(t)\,||$" part of the differential canceled out of the integrand (to our benefit, as integrating the square roots of functions is generally difficult). We will find a similar thing happens when we use the surface area differential dS in the following sections. That is, our main goal is not to be able to compute surface area; rather, surface area is a tool to obtain other quantities that are more important and useful. In our applications, we will use dS, but most of the time the "$||\,\vec{r}_u \times \vec{r}_v\,||$" part will cancel out of the integrand, making the subsequent integration easier to compute.

Notes:

Exercises 14.5

Terms and Concepts

1. In your own words, describe what an orientable surface is.

2. Give an example of a non-orientable surface.

Problems

In Exercises 3 – 4, parameterize the surface defined by the function $z = f(x, y)$ over each of the given regions R of the x-y plane.

3. $z = 3x^2y$;

   (a) R is the rectangle bounded by $-1 \le x \le 1$ and $0 \le y \le 2$.

   (b) R is the circle of radius 3, centered at $(1, 2)$.

   (c) R is the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 2)$.

   (d) R is the region bounded by the x-axis and the graph of $y = 1 - x^2$.

4. $z = 4x + 2y^2$;

   (a) R is the rectangle bounded by $1 \le x \le 4$ and $5 \le y \le 7$.

   (b) R is the ellipse with major axis of length 8 parallel to the x-axis, and minor axis of length 6 parallel to the y-axis, centered at the origin.

   (c) R is the triangle with vertices $(0, 0)$, $(2, 2)$ and $(0, 4)$.

   (d) R is the annulus bounded between the circles, centered at the origin, with radius 2 and radius 5.

In Exercises 5 – 8, a surface $\mathcal{S}$ in space is described that cannot be defined in terms of a function $z = f(x, y)$. Give a parameterization of $\mathcal{S}$.

5. $\mathcal{S}$ is the rectangle in space with corners at $(0, 0, 0)$, $(0, 2, 0)$, $(0, 2, 1)$ and $(0, 0, 1)$.

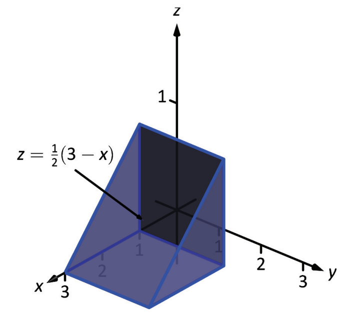
6. $\mathcal{S}$ is the triangle in space with corners at $(1, 0, 0)$, $(1, 0, 1)$ and $(0, 0, 1)$.

7. $\mathcal{S}$ is the ellipsoid $\dfrac{x^2}{9} + \dfrac{y^2}{4} + \dfrac{z^2}{16} = 1$.

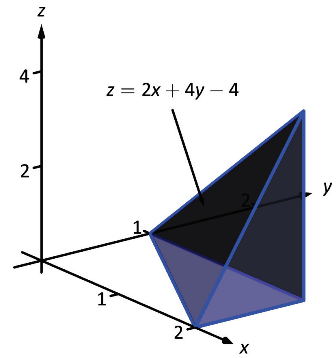
8. $\mathcal{S}$ is the elliptic cone $y^2 = x^2 + \dfrac{z^2}{16}$, for $-1 \le y \le 5$.

In Exercises 9 – 16, a domain D in space is given. Parameterize each of the bounding surfaces of D.

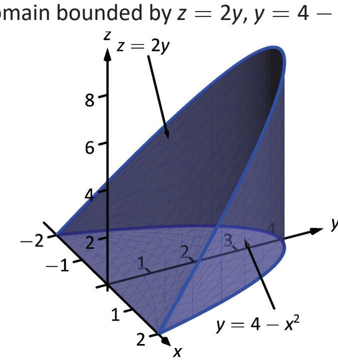
9. D is the domain bounded by the planes $z = \frac{1}{2}(3 - x)$, $x = 1$, $y = 0$, $y = 2$ and $z = 0$.



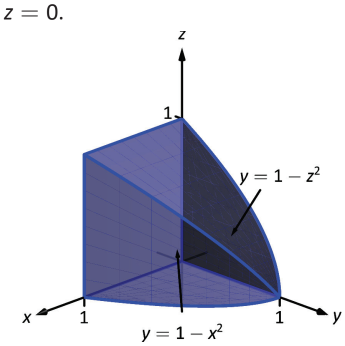
10. D is the domain bounded by the planes $z = 2x + 4y - 4$, $x = 2$, $y = 1$ and $z = 0$.



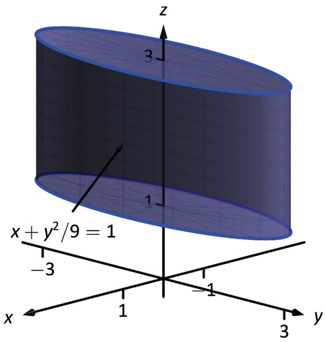
11. D is the domain bounded by $z = 2y$, $y = 4 - x^2$ and $z = 0$.



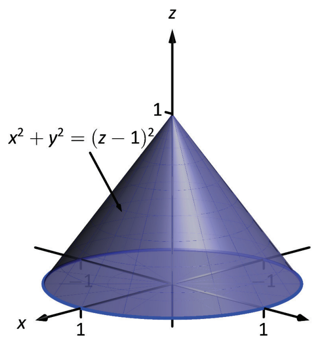
12. D is the domain bounded by $y = 1 - z^2$, $y = 1 - x^2$, $x = 0$, $y = 0$ and $z = 0$.



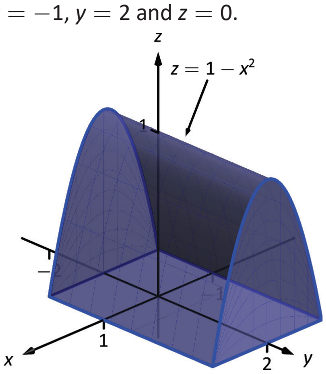
13. D is the domain bounded by the cylinder $x + y^2/9 = 1$ and the planes $z = 1$ and $z = 3$.



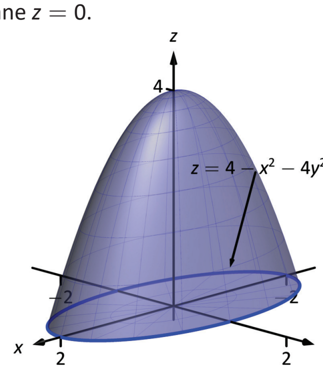
14. D is the domain bounded by the cone $x^2 + y^2 = (z - 1)^2$ and the plane $z = 0$.



15. D is the domain bounded by the cylinder $z = 1 - x^2$ and the planes $y = -1$, $y = 2$ and $z = 0$.



16. D is the domain bounded by the paraboloid $z = 4 - x^2 - 4y^2$ and the plane $z = 0$.



In Exercises 17 – 30, find the surface area S of the given surface $\mathcal{S}$. (The associated integrals are computable without the assistance of technology.)

17. $\mathcal{S}$ is the plane $z = 2x + 3y$ over the rectangle $-1 \leq x \leq 1$, $2 \leq v \leq 3$.

18. $\mathcal{S}$ is the plane $z = x + 2y$ over the triangle with vertices at $(0, 0)$, $(1, 0)$ and $(0, 1)$.

19. $\mathcal{S}$ is the plane $z = x + y$ over the circular disk, centered at the origin, with radius 2.

20. $\mathcal{S}$ is the plane $z = x + y$ over the annulus bounded by the circles, centered at the origin, with radius 1 and radius 2.

21. $\mathcal{S}$ is the paraboloid $z = x^2 + y^2$ below the plane $z = 4$.

22. $\mathcal{S}$ is the paraboloid $z = x^2 + y^2$ between the plane $z = 4$ and $z = 8$.

23. $\mathcal{S}$ is the plane $z = x - y$ inside the cylinder $x^2 + y^2 = 1$.

24. $\mathcal{S}$ is the plane $z = 3x + 4y$ above the square $0 \leq x \leq 1$, $0 \leq y \leq 1$.

25. $\mathcal{S}$ is the spherical cap $x^2 + y^2 + z^2 = 1$ above $z = \dfrac{1}{\sqrt{2}}$.

26. $\mathcal{S}$ is the spherical band $x^2 + y^2 + z^2 = 1$ between $z = 0$ and $z = \dfrac{1}{\sqrt{2}}$.

27. $\mathcal{S}$ is the cone $z^2 = x^2 + y^2$ between the planes $z = 3$ and $z = 7$.

28. $\mathcal{S}$ is the the monkey saddle $z = \frac{1}{3}x^3 - xy^2$ inside $x^2 + y^2 = 1$.

29. $\mathcal{S}$ is the plane $z = 1 - 2x - 2y$ inside $x \geq 0$, $y \geq 0$, $z \geq 0$.

30. $\mathcal{S}$ is the plane $z = x - y$ inside the cylinder $x^2 + y^2 = 1$.

In Exercises 31 – 34, set up the double integral that finds the surface area S of the given surface $\mathcal{S}$, then use technology to approximate its value.

31. $\mathcal{S}$ is the paraboloid $z = x^2 + y^2$ over the circular disk of radius 3 centered at the origin.

32. $\mathcal{S}$ is the paraboloid $z = x^2 + y^2$ over the triangle with vertices at $(0, 0)$, $(0, 1)$ and $(1, 1)$.

33. $\mathcal{S}$ is the plane $z = 5x - y$ over the region enclosed by the parabola $y = 1 - x^2$ and the x-axis.

34. $\mathcal{S}$ is the hyperbolic paraboloid $z = x^2 - y^2$ over the circular disk of radius 1 centered at the origin.