8.4 Ratio and Root Tests

The n^{th}–Term Test of Theorem 64 states that in order for a series $\sum_{n=1}^{\infty} a_n$ to converge, $\lim_{n\to\infty} a_n = 0$. That is, the terms of $\{a_n\}$ must get very small. Not only must the terms approach 0, they must approach 0 "fast enough": while $\lim_{n\to\infty} 1/n = 0$, the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges as the terms of $\{1/n\}$ do not approach 0 "fast enough."

The comparison tests of the previous section determine convergence by comparing terms of a series to terms of another series whose convergence is known. This section introduces the Ratio and Root Tests, which determine convergence by analyzing the terms of a series to see if they approach 0 "fast enough."

Ratio Test

Theorem 70 Ratio Test

Let $\{a_n\}$ be a positive sequence where $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$.

1. If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $L > 1$ or $L = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $L = 1$, the Ratio Test is inconclusive.

Note: Theorem 65 allows us to apply the Ratio Test to series where $\{a_n\}$ is positive for all but a finite number of terms.

The principle of the Ratio Test is this: if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$, then for large n, each term of $\{a_n\}$ is significantly smaller than its previous term which is enough to ensure convergence.

Example 8.22 Applying the Ratio Test

Use the Ratio Test to determine the convergence of the following series:

1. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ 2. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3}$ 3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

Notes:

Solution

1. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$:

$$\begin{aligned}\lim_{n\to\infty} \frac{2^{n+1}/(n+1)!}{2^n/n!} &= \lim_{n\to\infty} \frac{2^{n+1}n!}{2^n(n+1)!}\\ &= \lim_{n\to\infty} \frac{2}{n+1}\\ &= 0.\end{aligned}$$

Since the limit is $0 < 1$, by the Ratio Test $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3}$:

$$\begin{aligned}\lim_{n\to\infty} \frac{3^{n+1}/(n+1)^3}{3^n/n^3} &= \lim_{n\to\infty} \frac{3^{n+1}n^3}{3^n(n+1)^3}\\ &= \lim_{n\to\infty} \frac{3n^3}{(n+1)^3}\\ &= 3.\end{aligned}$$

Since the limit is $3 > 1$, by the Ratio Test $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3}$ diverges.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$:

$$\begin{aligned}\lim_{n\to\infty} \frac{1/\big((n+1)^2+1\big)}{1/(n^2+1)} &= \lim_{n\to\infty} \frac{n^2+1}{(n+1)^2+1}\\ &= 1.\end{aligned}$$

Since the limit is 1, the Ratio Test is inconclusive. We can easily show this series converges using the Direct or Limit Comparison Tests, with each comparing to the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Notes:

The Ratio Test is not effective when the terms of a series *only* contain algebraic functions (e.g., polynomials). It is most effective when the terms contain some factorials or exponentials. The previous example also reinforces our developing intuition: factorials dominate exponentials, which dominate algebraic functions, which dominate logarithmic functions. In Part 1 of the example, the factorial in the denominator dominated the exponential in the numerator, causing the series to converge. In Part 2, the exponential in the numerator dominated the algebraic function in the denominator, causing the series to diverge.

While we have used factorials in previous sections, we have not explored them closely and one is likely to not yet have a strong intuitive sense for how they behave. The following example gives more practice with factorials.

Example 8.23 Applying the Ratio Test

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty}\frac{n!n!}{(2n)!}$.

Solution Before we begin, be sure to note the difference between $(2n)!$ and $2n!$. When $n = 4$, the former is $8! = 8\cdot 7\cdot \ldots \cdot 2\cdot 1 = 40,320$, whereas the latter is $2(4\cdot 3\cdot 2\cdot 1) = 48$.

Applying the Ratio Test:

$$\lim_{n\to\infty}\frac{(n+1)!(n+1)!/\big(2(n+1)\big)!}{n!n!/(2n)!} = \lim_{n\to\infty}\frac{(n+1)!(n+1)!(2n)!}{n!n!(2n+2)!}$$

Noting that $(2n+2)! = (2n+2)\cdot(2n+1)\cdot(2n)!$, we have

$$\begin{aligned} &= \lim_{n\to\infty}\frac{(n+1)(n+1)}{(2n+2)(2n+1)} \\ &= 1/4. \end{aligned}$$

Since the limit is $1/4 < 1$, by the Ratio Test we conclude $\displaystyle\sum_{n=1}^{\infty}\frac{n!n!}{(2n)!}$ converges.

Root Test

The final test we introduce is the Root Test, which works particularly well on series where each term is raised to a power, and does not work well with terms containing factorials.

Notes:

Theorem 71 Root Test

Let $\{a_n\}$ be a positive sequence, and let $\lim_{n\to\infty} (a_n)^{1/n} = L$.

1. If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $L > 1$ or $L = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $L = 1$, the Root Test is inconclusive.

Note: Theorem 65 allows us to apply the Root Test to series where $\{a_n\}$ is positive for all but a finite number of terms.

Example 8.24 Applying the Root Test

Determine the convergence of the following series using the Root Test:

1. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3n+1}{5n-2}\right)^n$ 2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{(\ln n)^n}$ 3. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^2}$.

Solution

1. $\displaystyle\lim_{n\to\infty}\left(\left(\frac{3n+1}{5n-2}\right)^n\right)^{1/n} = \lim_{n\to\infty}\frac{3n+1}{5n-2} = \frac{3}{5}.$

   Since the limit is less than 1, we conclude the series converges. Note: it is difficult to apply the Ratio Test to this series.

2. $\displaystyle\lim_{n\to\infty}\left(\frac{n^4}{(\ln n)^n}\right)^{1/n} = \lim_{n\to\infty}\frac{\left(n^{1/n}\right)^4}{\ln n}.$

   As n grows, the numerator approaches 1 (apply L'Hôpital's Rule) and the denominator grows to infinity. Thus

   $$\lim_{n\to\infty}\frac{\left(n^{1/n}\right)^4}{\ln n} = 0.$$

   Since the limit is less than 1, we conclude the series converges.

3. $\displaystyle\lim_{n\to\infty}\left(\frac{2^n}{n^2}\right)^{1/n} = \lim_{n\to\infty}\frac{2}{\left(n^{1/n}\right)^2} = 2.$

   Since this is greater than 1, we conclude the series diverges.

Each of the tests we have encountered so far has required that we analyze series from *positive* sequences. The next section relaxes this restriction by considering *alternating series*, where the underlying sequence has terms that alternate between being positive and negative.

Notes:

Exercises 8.4

Terms and Concepts

1. The Ratio Test is not effective when the terms of a sequence only contain ______ functions.

2. The Ratio Test is most effective when the terms of a sequence contains ______ and/or ______ functions.

3. What three convergence tests do not work well with terms containing factorials?

4. The Root Test works particularly well on series where each term is ______ to a ______.

Problems

In Exercises 5 – 14, determine the convergence of the given series using the Ratio Test. If the Ratio Test is inconclusive, state so and determine convergence with another test.

5. $\displaystyle\sum_{n=0}^{\infty} \frac{2n}{n!}$

6. $\displaystyle\sum_{n=0}^{\infty} \frac{5^n - 3n}{4^n}$

7. $\displaystyle\sum_{n=0}^{\infty} \frac{n!10^n}{(2n)!}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n + n^4}{7^n + n^2}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^3 + 7}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{10 \cdot 5^n}{7^n - 3}$

12. $\displaystyle\sum_{n=1}^{\infty} n \cdot \left(\frac{3}{5}\right)^n$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdot 8 \cdots 2n}{3 \cdot 6 \cdot 9 \cdot 12 \cdots 3n}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{5 \cdot 10 \cdot 15 \cdots (5n)}$

In Exercises 15 – 24, determine the convergence of the given series using the Root Test. If the Root Test is inconclusive, state so and determine convergence with another test.

15. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+5}{3n+11}\right)^n$

16. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{0.9n^2 - n - 3}{n^2 + n + 3}\right)^n$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n^2}{3^n}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^2 2^{n+1}}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{4^{n+7}}{7^n}$

21. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2 - n}{n^2 + n}\right)^n$

22. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$

23. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$

24. $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{(\ln n)^n}$

In Exercises 25 – 34, determine the convergence of the given series. State the test used; more than one test may be appropriate.

25. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + 4n - 2}{n^3 + 4n^2 - 3n + 7}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{n^4 4^n}{n!}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{3^n + n}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^n}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2 + 4n + 1}}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{n!n!n!}{(3n)!}$

31. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$

32. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n+2}{n+1}\right)^n$

33. $\displaystyle\sum_{n=2}^{\infty} \frac{n^3}{\left(\ln n\right)^n}$

34. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$