12.5 The Multivariable Chain Rule

The Chain Rule, as learned in Section 2.5, states that $\frac{d}{dx}\Big(f\big(g(x)\big)\Big) = f'\big(g(x)\big)g'(x)$. If $t = g(x)$, we can express the Chain Rule as

$$\frac{df}{dx} = \frac{df}{dt}\frac{dt}{dx}.$$

In this section we extend the Chain Rule to functions of more than one variable.

Theorem 109 Multivariable Chain Rule, Part I

Let $z = f(x, y)$, $x = g(t)$ and $y = h(t)$, where f, g and h are differentiable functions. Then $z = f(x, y) = f\big(g(t), h(t)\big)$ is a function of t, and

$$\begin{aligned}\frac{dz}{dt} = \frac{df}{dt} &= f_x(x, y)\frac{dx}{dt} + f_y(x, y)\frac{dy}{dt}\\ &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.\end{aligned}$$

It is good to understand what the situation of $z = f(x, y)$, $x = g(t)$ and $y = h(t)$ describes. We know that $z = f(x, y)$ describes a surface; we also recognize that $x = g(t)$ and $y = h(t)$ are parametric equations for a curve in the x-y plane. Combining these together, we are describing a curve that lies on the surface described by f. The parametric equations for this curve are $x = g(t)$, $y = h(t)$ and $z = f\big(g(t), h(t)\big)$.

Consider Figure 12.14 in which a surface is drawn, along with a dashed curve in the x-y plane. Restricting f to just the points on this circle gives the curve shown on the surface. The derivative $\frac{df}{dt}$ gives the instantaneous rate of change of f with respect to t. If we consider an object traveling along this path, $\frac{df}{dt}$ gives the rate at which the object rises/falls.

We now practice applying the Multivariable Chain Rule.

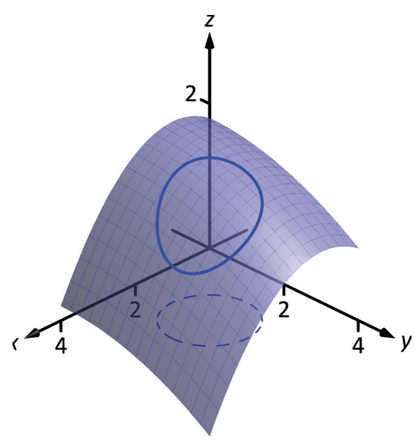

Figure 12.14: Understanding the application of the Multivariable Chain Rule.

Example 12.28 Using the Multivariable Chain Rule

Let $z = x^2y + x$, where $x = \sin t$ and $y = e^{5t}$. Find $\frac{dz}{dt}$ using the Chain Rule.

Solution Following Theorem 109, we find

$$f_x(x, y) = 2xy + 1, \qquad f_y(x, y) = x^2, \qquad \frac{dx}{dt} = \cos t, \qquad \frac{dy}{dt} = 5e^{5t}.$$

Notes:

Applying the theorem, we have

$$\frac{dz}{dt} = (2xy+1)\cos t + 5x^2e^{5t}.$$

This may look odd, as it seems that $\frac{dz}{dt}$ is a function of x, y and t. Since x and y are functions of t, $\frac{dz}{dt}$ is really just a function of t, and we can replace x with $\sin t$ and y with e^{5t}:

$$\frac{dz}{dt} = (2xy+1)\cos t + 5x^2e^{5t} = (2\sin(t)e^{5t}+1)\cos t + 5e^{5t}\sin^2 t.$$

The previous example can make us wonder: if we substituted for x and y at the end to show that $\frac{dz}{dt}$ is really just a function of t, why not substitute before differentiating, showing clearly that z is a function of t?

That is, $z = x^2y + x = (\sin t)^2e^{5t} + \sin t$. Applying the Chain and Product Rules, we have

$$\frac{dz}{dt} = 2\sin t\cos t\, e^{5t} + 5\sin^2 t\, e^{5t} + \cos t,$$

which matches the result from the example.

This may now make one wonder "What's the point? If we could already find the derivative, why learn another way of finding it?" In some cases, applying this rule makes deriving simpler, but this is hardly the power of the Chain Rule. Rather, in the case where $z = f(x, y)$, $x = g(t)$ and $y = h(t)$, the Chain Rule is extremely powerful when *we do not know what f, g and/or h are*. It may be hard to believe, but often in "the real world" we know rate–of–change information (i.e., information about derivatives) without explicitly knowing the underlying functions. The Chain Rule allows us to combine several rates of change to find another rate of change. The Chain Rule also has theoretic use, giving us insight into the behavior of certain constructions (as we'll see in the next section).

We demonstrate this in the next example.

Example 12.29 Applying the Multivarible Chain Rule

An object travels along a path on a surface. The exact path and surface are not known, but at time $t = t_0$ it is known that :

$$\frac{\partial z}{\partial x} = 5, \qquad \frac{\partial z}{\partial y} = -2, \qquad \frac{dx}{dt} = 3 \qquad \text{and} \qquad \frac{dy}{dt} = 7.$$

Find $\frac{dz}{dt}$ at time t_0.

Notes:

Solution The Multivariable Chain Rule states that

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\\ &= 5(3) + (-2)(7)\\ &= 1.\end{aligned}$$

By knowing certain rates–of–change information about the surface and about the path of the particle in the x-y plane, we can determine how quickly the object is rising/falling.

We next apply the Chain Rule to solve a max/min problem.

Example 12.30 Applying the Multivariable Chain Rule

Consider the surface $z = x^2 + y^2 - xy$, a paraboloid, on which a particle moves with x and y coordinates given by $x = \cos t$ and $y = \sin t$. Find $\frac{dz}{dt}$ when $t = 0$, and find where the particle reaches its maximum/minimum z-values.

Solution It is straightforward to compute

$$f_x(x,y) = 2x - y, \qquad f_y(x,y) = 2y - x, \qquad \frac{dx}{dt} = -\sin t, \qquad \frac{dy}{dt} = \cos t.$$

Combining these according to the Chain Rule gives:

$$\frac{dz}{dt} = -(2x - y)\sin t + (2y - x)\cos t.$$

When $t = 0$, $x = 1$ and $y = 0$. Thus $\frac{dz}{dt} = -(2)(0) + (-1)(1) = -1$. When $t = 0$, the particle is moving down, as shown in Figure 12.15.

To find where z-value is maximized/minimized on the particle's path, we set $\frac{dz}{dt} = 0$ and solve for t:

$$\begin{aligned}\frac{dz}{dt} = 0 &= -(2x - y)\sin t + (2y - x)\cos t\\ 0 &= -(2\cos t - \sin t)\sin t + (2\sin t - \cos t)\cos t\\ 0 &= \sin^2 t - \cos^2 t\\ \cos^2 t &= \sin^2 t\\ t &= n\frac{\pi}{4} \quad \text{(for odd } n\text{)}\end{aligned}$$

We can use the First Derivative Test to find that on $[0, 2\pi]$, z has reaches its absolute minimum at $t = \pi/4$ and $5\pi/4$; it reaches its absolute maximum at

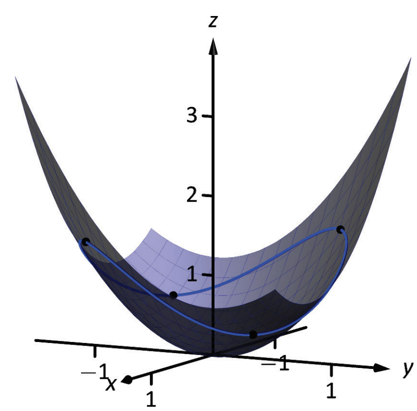

Figure 12.15: Plotting the path of a particle on a surface in Example 12.30.

Notes:

$t = 3\pi/4$ and $7\pi/4$, as shown in Figure 12.15.

We can extend the Chain Rule to include the situation where z is a function of more than one variable, and each of these variables is also a function of more than one variable. The basic case of this is where $z = f(x, y)$, and x and y are functions of two variables, say s and t.

Theorem 110 Multivariable Chain Rule, Part II

1. Let $z = f(x, y)$, $x = g(s, t)$ and $y = h(s, t)$, where f, g and h are differentiable functions. Then z is a function of s and t, and
   - $\dfrac{\partial z}{\partial s} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial s}$, and
   - $\dfrac{\partial z}{\partial t} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial t}$.
2. Let $z = f(x_1, x_2, \ldots, x_m)$ be a differentiable function of m variables, where each of the x_i is a differentiable function of the variables $t_1, t_2, \ldots, t_n$. Then z is a function of the t_i, and

$$\frac{\partial z}{\partial t_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial f}{\partial x_m}\frac{\partial x_m}{\partial t_i}.$$

Example 12.31 Using the Multivarible Chain Rule, Part II

Let $z = x^2y + x$, $x = s^2 + 3t$ and $y = 2s - t$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$, and evaluate each when $s = 1$ and $t = 2$.

Solution Following Theorem 110, we compute the following partial derivatives:

$$\frac{\partial f}{\partial x} = 2xy + 1 \qquad \frac{\partial f}{\partial y} = x^2,$$

$$\frac{\partial x}{\partial s} = 2s \qquad \frac{\partial x}{\partial t} = 3 \qquad \frac{\partial y}{\partial s} = 2 \qquad \frac{\partial y}{\partial t} = -1.$$

Thus

$$\frac{\partial z}{\partial s} = (2xy + 1)(2s) + (x^2)(2) = 4xys + 2s + 2x^2, \quad \text{and}$$

$$\frac{\partial z}{\partial t} = (2xy + 1)(3) + (x^2)(-1) = 6xy - x^2 + 3.$$

Notes:

When $s = 1$ and $t = 2$, $x = 7$ and $y = 0$, so

$$\frac{\partial z}{\partial s} = 100 \qquad \text{and} \qquad \frac{\partial z}{\partial t} = -46.$$

Example 12.32 Using the Multivarible Chain Rule, Part II

Let $w = xy + z^2$, where $x = t^2e^s$, $y = t\cos s$, and $z = s\sin t$. Find $\frac{\partial w}{\partial t}$ when $s = 0$ and $t = \pi$.

Solution Following Theorem 110, we compute the following partial derivatives:

$$\frac{\partial f}{\partial x} = y \qquad \frac{\partial f}{\partial y} = x \qquad \frac{\partial f}{\partial z} = 2z,$$

$$\frac{\partial x}{\partial t} = 2te^s \qquad \frac{\partial y}{\partial t} = \cos s \qquad \frac{\partial z}{\partial t} = s\cos t.$$

Thus

$$\frac{\partial w}{\partial t} = y(2te^s) + x(\cos s) + 2z(s\cos t).$$

When $s = 0$ and $t = \pi$, we have $x = \pi^2$, $y = \pi$ and $z = 0$. Thus

$$\frac{\partial w}{\partial t} = \pi(2\pi) + \pi^2 = 3\pi^2.$$

Implicit Differentiation

We studied finding $\frac{dy}{dx}$ when y is given by an implicit equation of x in detail in Section 2.6. We find here that the Multivariable Chain Rule gives a simpler method of finding $\frac{dy}{dx}$.

For instance, consider the implicit equation $x^2y - xy^3 = 3$. We learned to use the following steps to find $\frac{dy}{dx}$:

$$\begin{aligned} \frac{d}{dx}\left(x^2y - xy^3\right) &= \frac{d}{dx}\left(3\right) \\ 2xy + x^2\frac{dy}{dx} - y^3 - 3xy^2\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{2xy - y^3}{x^2 - 3xy^2}. \end{aligned} \tag{12.2}$$

Instead of using this method, consider $z = x^2y - xy^3$. The implicit relation above describes the level curve $z = 3$. Considering x and y as functions of x, the Multivariable Chain Rule states that

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\frac{dx}{dx} + \frac{\partial z}{\partial y}\frac{dy}{dx}. \tag{12.3}$$

Notes:

Since z is constant (in our example, $z = 3$), $\frac{dz}{dx} = 0$. We also know $\frac{dx}{dx} = 1$. Equation (12.3) becomes

$$\begin{aligned} 0 &= \frac{\partial z}{\partial x}(1) + \frac{\partial z}{\partial y}\frac{dy}{dx} \quad \Rightarrow \\ \frac{dy}{dx} &= -\frac{\partial z}{\partial x}\Big/\frac{\partial z}{\partial y} \\ &= -\frac{f_x}{f_y}. \end{aligned}$$

Note how our solution for $\frac{dy}{dx}$ in Equation (12.2) is just the partial derivative of z with respect to x, divided by the partial derivative of z with respect to y.

We state the above as a theorem.

Theorem 111 Implicit Differentiation

Let f be a differentiable function of x and y, where $f(x,y) = c$ defines an implicit relation between x and y, for some constant c. Then

$$\frac{dy}{dx} = -\frac{f_x(x,y)}{f_y(x,y)}.$$

We practice using Theorem 111 by applying it to a problem from Section 2.6.

Example 12.33 Implicit Differentiation

Given the implicitly defined relation $\sin(x^2y^2) + y^3 = x + y$, find y'. Note: this is the same problem as given in Example 2.43 of Section 2.6, where the solution took about a full page to find.

Solution Let $f(x,y) = \sin(x^2y^2) + y^3 - x - y$; the implicitly defined relation above is equivalent to $f(x,y) = 0$. We find $\frac{dy}{dx}$ by applying Theorem 111. We find

$$f_x(x,y) = 2xy^2\cos(x^2y^2) - 1 \qquad \text{and} \qquad f_y(x,y) = 2x^2y\cos(x^2y^2) + 3y^2 - 1,$$

so

$$\frac{dy}{dx} = -\frac{2xy^2\cos(x^2y^2) - 1}{2x^2y\cos(x^2y^2) + 3y^2 - 1},$$

which matches our solution from Example 2.43.

Notes:

Exercises 12.5

Terms and Concepts

1. Let a level curve of $z = f(x, y)$ be described by $x = g(t)$, $y = h(t)$. Explain why $\frac{dz}{dt} = 0$.

2. Fill in the blank: The single variable Chain Rule states $\frac{d}{dx}\Big(f\big(g(x)\big)\Big) = f'\big(g(x)\big)\cdot$ ________.

3. Fill in the blank: The Multivariable Chain Rule states $\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot$ ________ $+$ ________ $\cdot\frac{dy}{dt}$.

4. If $z = f(x, y)$, where $x = g(t)$ and $y = h(t)$, we can substitute and write z as an explicit function of t.
T/F: Using the Multivariable Chain Rule to find $\frac{dz}{dt}$ is sometimes easier than first substituting and then taking the derivative.

5. T/F: The Multivariable Chain Rule is only useful when all the related functions are known explicitly.

6. The Multivariable Chain Rule allows us to compute implicit derivatives easily by just computing two ________ derivatives.

Problems

In Exercises 7 – 12, functions $z = f(x, y)$, $x = g(t)$ and $y = h(t)$ are given.

(a) Use the Multivariable Chain Rule to compute $\frac{dz}{dt}$.

(b) Evaluate $\frac{dz}{dt}$ at the indicated t-value.

7. $z = 3x + 4y$, $\quad x = t^2$, $\quad y = 2t$; $\quad t = 1$

8. $z = x^2 - y^2$, $\quad x = t$, $\quad y = t^2 - 1$; $\quad t = 1$

9. $z = 5x + 2y$, $\quad x = 2\cos t + 1$, $\quad y = \sin t - 3$; $t = \pi/4$

10. $z = \dfrac{x}{y^2 + 1}$, $\quad x = \cos t$, $\quad y = \sin t$; $\quad t = \pi/2$

11. $z = x^2 + 2y^2$, $\quad x = \sin t$, $\quad y = 3\sin t$; $\quad t = \pi/4$

12. $z = \cos x \sin y$, $\quad x = \pi t$, $\quad y = 2\pi t + \pi/2$; $\quad t = 3$

In Exercises 13 – 18, functions $z = f(x, y)$, $x = g(t)$ and $y = h(t)$ are given. Find the values of t where $\frac{dz}{dt} = 0$. Note: these are the same surfaces/curves as found in Exercises 7 – 12.

13. $z = 3x + 4y$, $\quad x = t^2$, $\quad y = 2t$

14. $z = x^2 - y^2$, $\quad x = t$, $\quad y = t^2 - 1$

15. $z = 5x + 2y$, $\quad x = 2\cos t + 1$, $\quad y = \sin t - 3$

16. $z = \dfrac{x}{y^2 + 1}$, $\quad x = \cos t$, $\quad y = \sin t$

17. $z = x^2 + 2y^2$, $\quad x = \sin t$, $\quad y = 3\sin t$

18. $z = \cos x \sin y$, $\quad x = \pi t$, $\quad y = 2\pi t + \pi/2$

In Exercises 19 – 22, functions $z = f(x, y)$, $x = g(s, t)$ and $y = h(s, t)$ are given.

(a) Use the Multivariable Chain Rule to compute $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

(b) Evaluate $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ at the indicated s and t values.

19. $z = x^2 y$, $\quad x = s - t$, $\quad y = 2s + 4t$; $\quad s = 1, t = 0$

20. $z = \cos\left(\pi x + \frac{\pi}{2}y\right)$, $\quad x = st^2$, $\quad y = s^2 t$; $\quad s = 1, t = 1$

21. $z = x^2 + y^2$, $\quad x = s\cos t$, $\quad y = s\sin t$; $\quad s = 2, t = \pi/4$

22. $z = e^{-(x^2+y^2)}$, $\quad x = t$, $\quad y = st^2$; $\quad s = 1, t = 1$

In Exercises 23 – 26, find $\frac{dy}{dx}$ using Implicit Differentiation and Theorem 111.

23. $x^2 \tan y = 50$

24. $(3x^2 + 2y^3)^4 = 2$

25. $\dfrac{x^2 + y}{x + y^2} = 17$

26. $\ln(x^2 + xy + y^2) = 1$

In Exercises 27 – 30, find $\frac{dz}{dt}$, or $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$, using the supplied information.

27. $\frac{\partial z}{\partial x} = 2$, $\quad \frac{\partial z}{\partial y} = 1$, $\quad \frac{dx}{dt} = 4$, $\quad \frac{dy}{dt} = -5$

28. $\frac{\partial z}{\partial x} = 1$, $\quad \frac{\partial z}{\partial y} = -3$, $\quad \frac{dx}{dt} = 6$, $\quad \frac{dy}{dt} = 2$

29. $\frac{\partial z}{\partial x} = -4$, $\quad \frac{\partial z}{\partial y} = 9$,
$\frac{\partial x}{\partial s} = 5$, $\quad \frac{\partial x}{\partial t} = 7$, $\quad \frac{\partial y}{\partial s} = -2$, $\quad \frac{\partial y}{\partial t} = 6$

30. $\frac{\partial z}{\partial x} = 2$, $\quad \frac{\partial z}{\partial y} = 1$,
$\frac{\partial x}{\partial s} = -2$, $\quad \frac{\partial x}{\partial t} = 3$, $\quad \frac{\partial y}{\partial s} = 2$, $\quad \frac{\partial y}{\partial t} = -1$