10.3 The Dot Product

The previous section introduced vectors and described how to add them together and how to multiply them by scalars. This section introduces *a* multiplication on vectors called the **dot product**.

Definition 61 Dot Product

1. Let $\vec{u} = \langle u_1, u_2\rangle$ and $\vec{v} = \langle v_1, v_2\rangle$ in $\mathbb{R}^2$. The **dot product** of $\vec{u}$ and $\vec{v}$, denoted $\vec{u}\cdot\vec{v}$, is
$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2.$$
2. Let $\vec{u} = \langle u_1, u_2, u_3\rangle$ and $\vec{v} = \langle v_1, v_2, v_3\rangle$ in $\mathbb{R}^3$. The **dot product** of $\vec{u}$ and $\vec{v}$, denoted $\vec{u}\cdot\vec{v}$, is
$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

Note how this product of vectors returns a *scalar*, not another vector. We practice evaluating a dot product in the following example, then we will discuss why this product is useful.

Example 10.17 Evaluating dot products

1. Let $\vec{u} = \langle 1, 2\rangle$, $\vec{v} = \langle 3, -1\rangle$ in $\mathbb{R}^2$. Find $\vec{u}\cdot\vec{v}$.
2. Let $\vec{x} = \langle 2, -2, 5\rangle$ and $\vec{y} = \langle -1, 0, 3\rangle$ in $\mathbb{R}^3$. Find $\vec{x}\cdot\vec{y}$.

Solution

1. Using Definition 61, we have
$$\vec{u}\cdot\vec{v} = 1(3) + 2(-1) = 1.$$
2. Using the definition, we have
$$\vec{x}\cdot\vec{y} = 2(-1) - 2(0) + 5(3) = 13.$$

The dot product, as shown by the preceding example, is very simple to evaluate. It is only the sum of products. While the definition gives no hint as to why

Notes:

we would care about this operation, there is an amazing connection between the dot product and angles formed by the vectors. Before stating this connection, we give a theorem stating some of the properties of the dot product.

Theorem 87 Properties of the Dot Product

Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ and let c be a scalar.

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ Commutative Property
2. $\vec{u} \cdot (\vec{v} \pm \vec{w}) = \vec{u} \cdot \vec{v} \pm \vec{u} \cdot \vec{w}$ Distributive Property
3. $(\vec{u} \pm \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} \pm \vec{v} \cdot \vec{w}$ Distributive Property
4. $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$
5. $\vec{0} \cdot \vec{v} = \vec{v} \cdot \vec{0} = 0$
6. $\vec{v} \cdot \vec{v} = ||\,\vec{v}\,||^2$

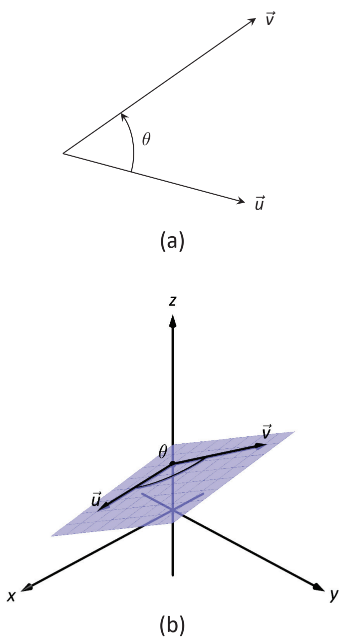


Figure 10.29: Illustrating the angle formed by two vectors with the same initial point.

The last statement of the theorem makes a handy connection between the magnitude of a vector and the dot product with itself. Our definition and theorem give properties of the dot product, but we are still likely wondering “What does the dot product *mean*?” It is helpful to understand that the dot product of a vector with itself is connected to its magnitude.

The next theorem extends this understanding by connecting the dot product to magnitudes and angles. Given vectors $\vec{u}$ and $\vec{v}$ in the plane, an angle θ is clearly formed when $\vec{u}$ and $\vec{v}$ are drawn with the same initial point as illustrated in Figure 10.29(a). (We always take θ to be the angle in $[0, \pi]$ as two angles are actually created.)

The same is also true of 2 vectors in space: given $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ with the same initial point, there is a plane that contains both $\vec{u}$ and $\vec{v}$. (When $\vec{u}$ and $\vec{v}$ are colinear, there are infinite planes that contain both vectors.) In that plane, we can again find an angle θ between them (and again, $0 \leq \theta \leq \pi$). This is illustrated in Figure 10.29(b).

The following theorem connects this angle θ to the dot product of $\vec{u}$ and $\vec{v}$.

Notes:

Theorem 88 The Dot Product and Angles

Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then

$$\vec{u} \cdot \vec{v} = ||\,\vec{u}\,||\,||\,\vec{v}\,|| \cos\theta,$$

where θ, $0 \leq \theta \leq \pi$, is the angle between $\vec{u}$ and $\vec{v}$.

When θ is an acute angle (i.e., $0 \leq \theta < \pi/2$), $\cos\theta$ is positive; when $\theta = \pi/2$, $\cos\theta = 0$; when θ is an obtuse angle ($\pi/2 < \theta \leq \pi$), $\cos\theta$ is negative. Thus the sign of the dot product gives a general indication of the angle between the vectors, illustrated in Figure 10.30.

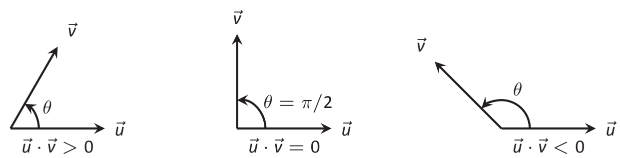


Figure 10.30: Illustrating the relationship between the angle between vectors and the sign of their dot product.

We *can* use Theorem 88 to compute the dot product, but generally this theorem is used to find the angle between known vectors (since the dot product is generally easy to compute). To this end, we rewrite the theorem's equation as

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\,\vec{u}\,||||\,\vec{v}\,||} \quad \Leftrightarrow \quad \theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{||\,\vec{u}\,||||\,\vec{v}\,||}\right).$$

We practice using this theorem in the following example.

Example 10.18 Using the dot product to find angles

Let $\vec{u} = \langle 3, 1\rangle$, $\vec{v} = \langle -2, 6\rangle$ and $\vec{w} = \langle -4, 3\rangle$, as shown in Figure 10.31. Find the angles α, β and θ.

Solution We start by computing the magnitude of each vector.

$$||\,\vec{u}\,|| = \sqrt{10}; \quad ||\,\vec{v}\,|| = 2\sqrt{10}; \quad ||\,\vec{w}\,|| = 5.$$

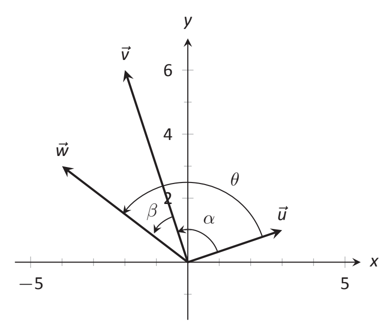


Figure 10.31: Vectors used in Example 10.18.

Notes:

We now apply Theorem 88 to find the angles.

$$\begin{aligned}\alpha &= \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{(\sqrt{10})(2\sqrt{10})}\right)\\ &= \cos^{-1}(0) = \frac{\pi}{2} = 90^\circ.\end{aligned}$$

$$\begin{aligned}\beta &= \cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{(2\sqrt{10})(5)}\right)\\ &= \cos^{-1}\left(\frac{26}{10\sqrt{10}}\right)\\ &\approx 0.6055 \approx 34.7^\circ.\end{aligned}$$

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{u}\cdot\vec{w}}{(\sqrt{10})(5)}\right)\\ &= \cos^{-1}\left(\frac{-9}{5\sqrt{10}}\right)\\ &\approx 2.1763 \approx 124.7^\circ\end{aligned}$$

We see from our computation that $\alpha+\beta=\theta$, as indicated by Figure 10.31. While we knew this should be the case, it is nice to see that this non-intuitive formula indeed returns the results we expected.

We do a similar example next in the context of vectors in space.

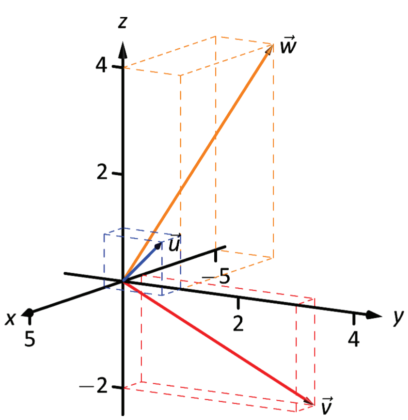


Figure 10.32: Vectors used in Example 10.19.

Example 10.19 Using the dot product to find angles

Let $\vec{u}=\langle 1,1,1\rangle$, $\vec{v}=\langle -1,3,-2\rangle$ and $\vec{w}=\langle -5,1,4\rangle$, as illustrated in Figure 10.32. Find the angle between each pair of vectors.

Solution

1. Between $\vec{u}$ and $\vec{v}$:

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{||\,\vec{u}\,||||\,\vec{v}\,||}\right)\\ &= \cos^{-1}\left(\frac{0}{\sqrt{3}\sqrt{14}}\right)\\ &= \frac{\pi}{2}.\end{aligned}$$

Notes:

2. Between $\vec{u}$ and $\vec{w}$:

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{u}\cdot\vec{w}}{||\,\vec{u}\,||||\,\vec{w}\,||}\right)\\ &= \cos^{-1}\left(\frac{0}{\sqrt{3}\sqrt{42}}\right)\\ &= \frac{\pi}{2}.\end{aligned}$$

3. Between $\vec{v}$ and $\vec{w}$:

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{||\,\vec{v}\,||||\,\vec{w}\,||}\right)\\ &= \cos^{-1}\left(\frac{0}{\sqrt{14}\sqrt{42}}\right)\\ &= \frac{\pi}{2}.\end{aligned}$$

While our work shows that each angle is $\pi/2$, i.e., 90°, none of these angles looks to be a right angle in Figure 10.32. Such is the case when drawing three–dimensional objects on the page.

All three angles between these vectors was $\pi/2$, or 90°. We know from geometry and everyday life that 90° angles are "nice" for a variety of reasons, so it should seem significant that these angles are all $\pi/2$. Notice the common feature in each calculation (and also the calculation of α in Example 10.18): the dot products of each pair of angles was 0. We use this as a basis for a definition of the term **orthogonal**, which is essentially synonymous to *perpendicular*.

Note: The term *perpendicular* originally referred to lines. As mathematics progressed, the concept of "being at right angles to" was applied to other objects, such as vectors and planes, and the term *orthogonal* was introduced. It is especially used when discussing objects that are hard, or impossible, to visualize: two vectors in 5-dimensional space are orthogonal if their dot product is 0. It is not wrong to say they are *perpendicular*, but common convention gives preference to the word *orthogonal*.

Definition 62 Orthogonal

Vectors $\vec{u}$ and $\vec{v}$ are **orthogonal** if their dot product is 0.

Example 10.20 Finding orthogonal vectors

Let $\vec{u} = \langle 3, 5\rangle$ and $\vec{v} = \langle 1, 2, 3\rangle$.

1. Find two vectors in $\mathbb{R}^2$ that are orthogonal to $\vec{u}$.
2. Find two non–parallel vectors in $\mathbb{R}^3$ that are orthogonal to $\vec{v}$.

Solution

Notes:

1. Recall that a line perpendicular to a line with slope m has slope $-1/m$, the "opposite reciprocal slope." We can think of the slope of $\vec{u}$ as $5/3$, its "rise over run." A vector orthogonal to $\vec{u}$ will have slope $-3/5$. There are many such choices, though all parallel:

$$\langle -5, 3\rangle \quad \text{or} \quad \langle 5, -3\rangle \quad \text{or} \quad \langle -10, 6\rangle \quad \text{or} \quad \langle 15, -9\rangle\text{, etc.}$$

2. There are infinite directions in space orthogonal to any given direction, so there are an infinite number of non–parallel vectors orthogonal to $\vec{v}$. Since there are so many, we have great leeway in finding some.

   One way is to arbitrarily pick values for the first two components, leaving the third unknown. For instance, let $\vec{v}_1 = \langle 2, 7, z\rangle$. If $\vec{v}_1$ is to be orthogonal to $\vec{v}$, then $\vec{v}_1 \cdot \vec{v} = 0$, so

$$2 + 14 + 3z = 0 \quad \Rightarrow z = \frac{-16}{3}.$$

   So $\vec{v}_1 = \langle 2, 7, -16/3\rangle$ is orthogonal to $\vec{v}$. We can apply a similar technique by leaving the first or second component unknown.

   Another method of finding a vector orthogonal to $\vec{v}$ mirrors what we did in part 1. Let $\vec{v}_2 = \langle -2, 1, 0\rangle$. Here we switched the first two components of $\vec{v}$, changing the sign of one of them (similar to the "opposite reciprocal" concept before). Letting the third component be 0 effectively ignores the third component of $\vec{v}$, and it is easy to see that

$$\vec{v}_2 \cdot \vec{v} = \langle -2, 1, 0\rangle \cdot \langle 1, 2, 3\rangle = 0.$$

   Clearly $\vec{v}_1$ and $\vec{v}_2$ are not parallel.

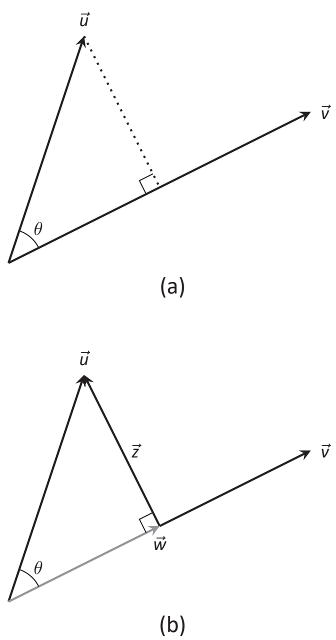


Figure 10.33: Developing the construction of the *orthogonal projection*.

An important construction is illustrated in Figure 10.33, where vectors $\vec{u}$ and $\vec{v}$ are sketched. In part (a), a dotted line is drawn from the tip of $\vec{u}$ to the line containing $\vec{v}$, where the dotted line is orthogonal to $\vec{v}$. In part (b), the dotted line is replaced with the vector $\vec{z}$ and $\vec{w}$ is formed, parallel to $\vec{v}$. It is clear by the diagram that $\vec{u} = \vec{w} + \vec{z}$. What is important about this construction is this: $\vec{u}$ is *decomposed* as the sum of two vectors, one of which is parallel to $\vec{v}$ and one that is perpendicular to $\vec{v}$. It is hard to overstate the importance of this construction (as we'll see in upcoming examples).

The vectors $\vec{w}$, $\vec{z}$ and $\vec{u}$ as shown in Figure 10.33 (b) form a right triangle, where the angle between $\vec{v}$ and $\vec{u}$ is labeled θ. We can find $\vec{w}$ in terms of $\vec{v}$ and $\vec{u}$.

Using trigonometry, we can state that

$$\|\,\vec{w}\,\| = \|\,\vec{u}\,\| \cos\theta. \tag{10.2}$$

Notes:

We also know that $\vec{w}$ is parallel to to $\vec{v}$; that is, the direction of $\vec{w}$ is the direction of $\vec{v}$, described by the unit vector $\frac{1}{||\,\vec{v}\,||}\vec{v}$. The vector $\vec{w}$ is the vector in the direction $\frac{1}{||\,\vec{v}\,||}\vec{v}$ with magnitude $||\,\vec{u}\,||\cos\theta$:

$$\vec{w} = \Big(||\,\vec{u}\,||\cos\theta\Big)\frac{1}{||\,\vec{v}\,||}\vec{v}.$$

Replace $\cos\theta$ using Theorem 88:

$$\begin{aligned} &= \left(||\,\vec{u}\,||\frac{\vec{u}\cdot\vec{v}}{||\,\vec{u}\,||||\,\vec{v}\,||}\right)\frac{1}{||\,\vec{v}\,||}\vec{v} \\ &= \frac{\vec{u}\cdot\vec{v}}{||\,\vec{v}\,||^2}\vec{v}. \end{aligned}$$

Now apply Theorem 87.

$$= \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}.$$

Since this construction is so important, it is given a special name.

Definition 63 Orthogonal Projection

Let $\vec{u}$ and $\vec{v}$ be given. The **orthogonal projection of $\vec{u}$ onto $\vec{v}$**, denoted $\text{proj}_{\,\vec{v}}\,\vec{u}$, is

$$\text{proj}_{\,\vec{v}}\,\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}.$$

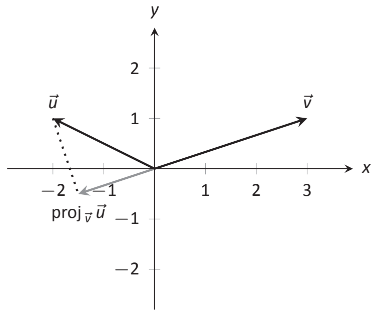
Example 10.21 Computing the orthogonal projection

1. Let $\vec{u} = \langle -2, 1\rangle$ and $\vec{v} = \langle 3, 1\rangle$. Find $\text{proj}_{\,\vec{v}}\,\vec{u}$, and sketch all three vectors with initial points at the origin.

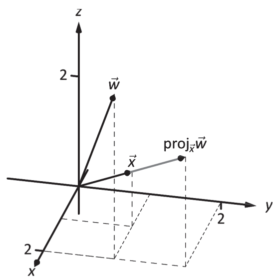
2. Let $\vec{w} = \langle 2, 1, 3\rangle$ and $\vec{x} = \langle 1, 1, 1\rangle$. Find $\text{proj}_{\,\vec{x}}\,\vec{w}$, and sketch all three vectors with initial points at the origin.

Solution

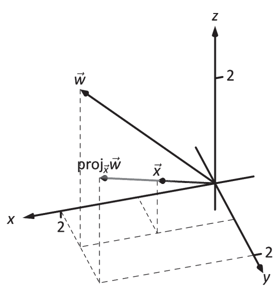
Notes:



(a)



(b)



(c)

Figure 10.34: Graphing the vectors used in Example 10.21.

1. Applying Definition 63, we have

$$\begin{aligned}\text{proj}_{\,\vec v}\,\vec u &= \frac{\vec u\cdot\vec v}{\vec v\cdot\vec v}\vec v\\ &= \frac{-5}{10}\,\langle 3,1\rangle\\ &= \left\langle -\frac{3}{2},-\frac{1}{2}\right\rangle.\end{aligned}$$

Vectors $\vec u$, $\vec v$ and $\text{proj}_{\,\vec v}\,\vec u$ are sketched in Figure 10.34(a). Note how the projection is parallel to $\vec v$; that is, it lies on the same line through the origin as $\vec v$, although it points in the opposite direction. That is because the angle between $\vec u$ and $\vec v$ is obtuse (i.e., greater than 90°).

2. Apply the definition:

$$\begin{aligned}\text{proj}_{\,\vec x}\,\vec w &= \frac{\vec w\cdot\vec x}{\vec x\cdot\vec x}\vec x\\ &= \frac{6}{3}\,\langle 1,1,1\rangle\\ &= \langle 2,2,2\rangle\,.\end{aligned}$$

These vectors are sketched in Figure 10.34(b), and again in part (c) from a different perspective. Because of the nature of graphing these vectors, the sketch in part (b) makes it difficult to recognize that the drawn projection has the geometric properties it should. The graph shown in part (c) illustrates these properties better.

Consider Figure 10.35 where the concept of the orthogonal projection is again illustrated. It is clear that

$$\vec u = \text{proj}_{\,\vec v}\,\vec u + \vec z. \tag{10.3}$$

As we know what $\vec u$ and $\text{proj}_{\,\vec v}\,\vec u$ are, we can solve for $\vec z$ and state that

$$\vec z = \vec u - \text{proj}_{\,\vec v}\,\vec u.$$

This leads us to rewrite Equation (10.3) in a seemingly silly way:

$$\vec u = \text{proj}_{\,\vec v}\,\vec u + (\vec u - \text{proj}_{\,\vec v}\,\vec u).$$

This is not nonsense, as pointed out in the following Key Idea. (Notation note: the expression "$\parallel \vec y$" means "is parallel to $\vec y$." We can use this notation to state

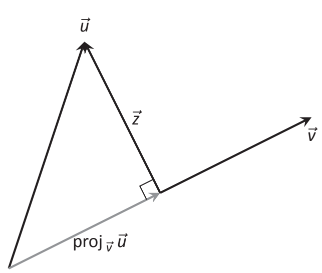


Figure 10.35: Illustrating the orthogonal projection.

Notes:

"$\vec{x} \parallel \vec{y}$" which means "$\vec{x}$ is parallel to $\vec{y}$." The expression "$\perp \vec{y}$" means "is orthogonal to $\vec{y}$," and is used similarly.)

Key Idea 51 Orthogonal Decomposition of Vectors

Let $\vec{u}$ and $\vec{v}$ be given. Then $\vec{u}$ can be written as the sum of two vectors, one of which is parallel to $\vec{v}$, and one of which is orthogonal to $\vec{v}$:

$$\vec{u} = \underbrace{\text{proj}_{\vec{v}}\,\vec{u}}_{\parallel \vec{v}} + \underbrace{(\vec{u} - \text{proj}_{\vec{v}}\,\vec{u})}_{\perp \vec{v}}.$$

We illustrate the use of this equality in the following example.

Example 10.22 Orthogonal decomposition of vectors

1. Let $\vec{u} = \langle -2, 1\rangle$ and $\vec{v} = \langle 3, 1\rangle$ as in Example 10.21. Decompose $\vec{u}$ as the sum of a vector parallel to $\vec{v}$ and a vector orthogonal to $\vec{v}$.
2. Let $\vec{w} = \langle 2, 1, 3\rangle$ and $\vec{x} = \langle 1, 1, 1\rangle$ as in Example 10.21. Decompose $\vec{w}$ as the sum of a vector parallel to $\vec{x}$ and a vector orthogonal to $\vec{x}$.

Solution

1. In Example 10.21, we found that $\text{proj}_{\vec{v}}\,\vec{u} = \langle -1.5, -0.5\rangle$. Let

   $$\vec{z} = \vec{u} - \text{proj}_{\vec{v}}\,\vec{u} = \langle -2, 1\rangle - \langle -1.5, -0.5\rangle = \langle -0.5, 1.5\rangle.$$

   Is $\vec{z}$ orthogonal to $\vec{v}$? (I.e, is $\vec{z} \perp \vec{v}$?) We check for orthogonality with the dot product:

   $$\vec{z} \cdot \vec{v} = \langle -0.5, 1.5\rangle \cdot \langle 3, 1\rangle = 0.$$

   Since the dot product is 0, we know $\vec{z} \perp \vec{v}$. Thus:

   $$\begin{aligned} \vec{u} &= \text{proj}_{\vec{v}}\,\vec{u} + (\vec{u} - \text{proj}_{\vec{v}}\,\vec{u}) \\ \langle -2, 1\rangle &= \underbrace{\langle -1.5, -0.5\rangle}_{\parallel \vec{v}} + \underbrace{\langle -0.5, 1.5\rangle}_{\perp \vec{v}}. \end{aligned}$$

2. We found in Example 10.21 that $\text{proj}_{\vec{x}}\,\vec{w} = \langle 2, 2, 2\rangle$. Applying the Key Idea, we have:

   $$\vec{z} = \vec{w} - \text{proj}_{\vec{x}}\,\vec{w} = \langle 2, 1, 3\rangle - \langle 2, 2, 2\rangle = \langle 0, -1, 1\rangle.$$

Notes:

We check to see if $\vec{z} \perp \vec{x}$:

$$\vec{z} \cdot \vec{x} = \langle 0, -1, 1 \rangle \cdot \langle 1, 1, 1 \rangle = 0.$$

Since the dot product is 0, we know the two vectors are orthogonal. We now write $\vec{w}$ as the sum of two vectors, one parallel and one orthogonal to $\vec{x}$:

$$\begin{aligned} \vec{w} &= \text{proj}_{\vec{x}}\,\vec{w} + (\vec{w} - \text{proj}_{\vec{x}}\,\vec{w}) \\ \langle 2, 1, 3 \rangle &= \underbrace{\langle 2, 2, 2 \rangle}_{\parallel \vec{x}} + \underbrace{\langle 0, -1, 1 \rangle}_{\perp \vec{x}} \end{aligned}$$

We give an example of where this decomposition is useful.

Example 10.23 Orthogonally decomposing a force vector

Consider Figure 10.36(a), showing a box weighing 50lb on a ramp that rises 5ft over a span of 20ft. Find the components of force, and their magnitudes, acting on the box (as sketched in part (b) of the figure):

1. in the direction of the ramp, and
2. orthogonal to the ramp.

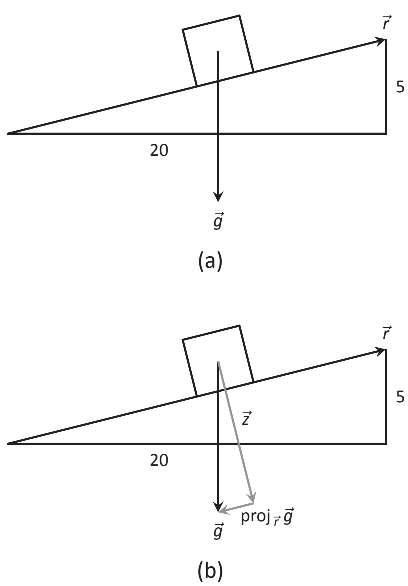


Figure 10.36: Sketching the ramp and box in Example 10.23. Note: *The vectors are not drawn to scale.*

Solution As the ramp rises 5ft over a horizontal distance of 20ft, we can represent the direction of the ramp with the vector $\vec{r} = \langle 20, 5 \rangle$. Gravity pulls down with a force of 50lb, which we represent with $\vec{g} = \langle 0, -50 \rangle$.

1. To find the force of gravity in the direction of the ramp, we compute $\text{proj}_{\vec{r}}\,\vec{g}$:

$$\begin{aligned} \text{proj}_{\vec{r}}\,\vec{g} &= \frac{\vec{g} \cdot \vec{r}}{\vec{r} \cdot \vec{r}}\vec{r} \\ &= \frac{-250}{425}\langle 20, 5 \rangle \\ &= \left\langle -\frac{200}{17}, -\frac{50}{17} \right\rangle \approx \langle -11.76, -2.94 \rangle . \end{aligned}$$

The magnitude of $\text{proj}_{\vec{r}}\,\vec{g}$ is $||\,\text{proj}_{\vec{r}}\,\vec{g}\,|| = 50/\sqrt{17} \approx 12.13$lb. Though the box weighs 50lb, a force of about 12lb is enough to keep the box from sliding down the ramp.

2. To find the component $\vec{z}$ of gravity orthogonal to the ramp, we use Key Idea 51.

$$\begin{aligned} \vec{z} &= \vec{g} - \text{proj}_{\vec{r}}\,\vec{g} \\ &= \left\langle \frac{200}{17}, -\frac{800}{17} \right\rangle \approx \langle 11.76, -47.06 \rangle . \end{aligned}$$

Notes:

The magnitude of this force is $\| \vec z \| \approx 48.51$lb. In physics and engineering, knowing this force is important when computing things like static frictional force. (For instance, we could easily compute if the static frictional force alone was enough to keep the box from sliding down the ramp.)

Application to Work

In physics, the application of a force F to move an object in a straight line a distance d produces *work*; the amount of work W is $W = Fd$, (where F is in the direction of travel). The orthogonal projection allows us to compute work when the force is not in the direction of travel.

Consider Figure 10.37, where a force $\vec F$ is being applied to an object moving in the direction of $\vec d$. (The distance the object travels is the magnitude of $\vec d$.) The work done is the amount of force in the direction of $\vec d$, $\| \text{proj}_{\vec d}\,\vec F \|$, times $\| \vec d \|$:

$$\begin{aligned}
\| \text{proj}_{\vec d}\,\vec F \| \cdot \| \vec d \| &= \left\| \frac{\vec F \cdot \vec d}{\vec d \cdot \vec d}\vec d \right\| \cdot \| \vec d \| \\
&= \left| \frac{\vec F \cdot \vec d}{\| \vec d \|^2} \right| \cdot \| \vec d \| \cdot \| \vec d \| \\
&= \frac{\left| \vec F \cdot \vec d \right|}{\| \vec d \|^2} \| \vec d \|^2 \\
&= \left| \vec F \cdot \vec d \right|.
\end{aligned}$$

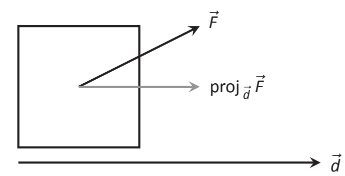


Figure 10.37: Finding work when the force and direction of travel are given as vectors.

The expression $\vec F \cdot \vec d$ will be positive if the angle between $\vec F$ and $\vec d$ is acute; when the angle is obtuse (hence $\vec F \cdot \vec d$ is negative), the force is causing motion in the opposite direction of $\vec d$, resulting in "negative work." We want to capture this sign, so we drop the absolute value and find that $W = \vec F \cdot \vec d$.

Definition 64 Work

Let $\vec F$ be a constant force that moves an object in a straight line from point P to point Q. Let $\vec d = \overrightarrow{PQ}$. The **work** W done by $\vec F$ along $\vec d$ is $W = \vec F \cdot \vec d$.

Example 10.24 Computing work

A man slides a box along a ramp that rises 3ft over a distance of 15ft by applying 50lb of force as shown in Figure 10.38. Compute the work done.

Notes:

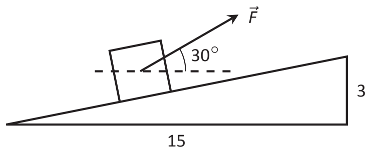


Figure 10.38: Computing work when sliding a box up a ramp in Example 10.24.

Solution The figure indicates that the force applied makes a 30° angle with the horizontal, so $\vec{F} = 50\left\langle \cos 30^\circ, \sin 30^\circ \right\rangle \approx \langle 43.3, 25\rangle$. The ramp is represented by $\vec{d} = \langle 15, 3\rangle$. The work done is simply

$$\vec{F}\cdot\vec{d} = 50\left\langle \cos 30^\circ, \sin 30^\circ\right\rangle \cdot \langle 15, 3\rangle \approx 724.5\text{ft–lb}.$$

Note how we did not actually compute the distance the object traveled, nor the magnitude of the force in the direction of travel; this is all inherently computed by the dot product!

The dot product is a powerful way of evaluating computations that depend on angles without actually using angles. The next section explores another "product" on vectors, the *cross product.* Once again, angles play an important role, though in a much different way.

Notes:

Exercises 10.3

Terms and Concepts

1. The dot product of two vectors is a _________, not a vector.

2. How are the concepts of the dot product and vector magnitude related?

3. How can one quickly tell if the angle between two vectors is acute or obtuse?

4. Give a synonym for "orthogonal."

Problems

In Exercises 5 – 10, find the dot product of the given vectors.

5. $\vec{u} = \langle 2, -4\rangle$, $\vec{v} = \langle 3, 7\rangle$

6. $\vec{u} = \langle 5, 3\rangle$, $\vec{v} = \langle 6, 1\rangle$

7. $\vec{u} = \langle 1, -1, 2\rangle$, $\vec{v} = \langle 2, 5, 3\rangle$

8. $\vec{u} = \langle 3, 5, -1\rangle$, $\vec{v} = \langle 4, -1, 7\rangle$

9. $\vec{u} = \langle 1, 1\rangle$, $\vec{v} = \langle 1, 2, 3\rangle$

10. $\vec{u} = \langle 1, 2, 3\rangle$, $\vec{v} = \langle 0, 0, 0\rangle$

11. Create your own vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^2$ and show that $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$.

12. Create your own vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ and scalar c and show that $c(\vec{u}\cdot\vec{v}) = \vec{u}\cdot(c\vec{v})$.

In Exercises 13 – 16, find the measure of the angle between the two vectors in both radians and degrees.

13. $\vec{u} = \langle 1, 1\rangle$, $\vec{v} = \langle 1, 2\rangle$

14. $\vec{u} = \langle -2, 1\rangle$, $\vec{v} = \langle 3, 5\rangle$

15. $\vec{u} = \langle 8, 1, -4\rangle$, $\vec{v} = \langle 2, 2, 0\rangle$

16. $\vec{u} = \langle 1, 7, 2\rangle$, $\vec{v} = \langle 4, -2, 5\rangle$

In Exercises 17 – 20, a vector $\vec{v}$ is given. Give two vectors that are orthogonal to $\vec{v}$.

17. $\vec{v} = \langle 4, 7\rangle$

18. $\vec{v} = \langle -3, 5\rangle$

19. $\vec{v} = \langle 1, 1, 1\rangle$

20. $\vec{v} = \langle 1, -2, 3\rangle$

In Exercises 21 – 26, vectors $\vec{u}$ and $\vec{v}$ are given. Find $\text{proj}_{\vec{v}}\,\vec{u}$, the orthogonal projection of $\vec{u}$ onto $\vec{v}$, and sketch all three vectors on the same axes.

21. $\vec{u} = \langle 1, 2\rangle$, $\vec{v} = \langle -1, 3\rangle$

22. $\vec{u} = \langle 5, 5\rangle$, $\vec{v} = \langle 1, 3\rangle$

23. $\vec{u} = \langle -3, 2\rangle$, $\vec{v} = \langle 1, 1\rangle$

24. $\vec{u} = \langle -3, 2\rangle$, $\vec{v} = \langle 2, 3\rangle$

25. $\vec{u} = \langle 1, 5, 1\rangle$, $\vec{v} = \langle 1, 2, 3\rangle$

26. $\vec{u} = \langle 3, -1, 2\rangle$, $\vec{v} = \langle 2, 2, 1\rangle$

In Exercises 27 – 32, vectors $\vec{u}$ and $\vec{v}$ are given. Write $\vec{u}$ as the sum of two vectors, one of which is parallel to $\vec{v}$ and one of which is perpendicular to $\vec{v}$. Note: these are the same pairs of vectors as found in Exercises 21 – 26.

27. $\vec{u} = \langle 1, 2\rangle$, $\vec{v} = \langle -1, 3\rangle$

28. $\vec{u} = \langle 5, 5\rangle$, $\vec{v} = \langle 1, 3\rangle$

29. $\vec{u} = \langle -3, 2\rangle$, $\vec{v} = \langle 1, 1\rangle$

30. $\vec{u} = \langle -3, 2\rangle$, $\vec{v} = \langle 2, 3\rangle$

31. $\vec{u} = \langle 1, 5, 1\rangle$, $\vec{v} = \langle 1, 2, 3\rangle$

32. $\vec{u} = \langle 3, -1, 2\rangle$, $\vec{v} = \langle 2, 2, 1\rangle$

33. A 10lb box sits on a ramp that rises 4ft over a distance of 20ft. How much force is required to keep the box from sliding down the ramp?

34. A 10lb box sits on a 15ft ramp that makes a 30° angle with the horizontal. How much force is required to keep the box from sliding down the ramp?

35. How much work is performed in moving a box horizontally 10ft with a force of 20lb applied at an angle of 45° to the horizontal?

36. How much work is performed in moving a box horizontally 10ft with a force of 20lb applied at an angle of 10° to the horizontal?

37. How much work is performed in moving a box up the length of a ramp that rises 2ft over a distance of 10ft, with a force of 50lb applied horizontally?

38. How much work is performed in moving a box up the length of a ramp that rises 2ft over a distance of 10ft, with a force of 50lb applied at an angle of 45° to the horizontal?

39. How much work is performed in moving a box up the length of a 10ft ramp that makes a 5° angle with the horizontal, with 50lb of force applied in the direction of the ramp?