Approximation Algorithms

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Review: Decision Problems vs Optimization Problems in NP

Given any **Opt** Problem **Min-P** $= (X, Y(x), m(x, Y(x)), MIN/MAX)$
we can always define the corresponding
Decision Problem k-P = $(X, Y(x), m(x, Y(x)), <= k\ (>= k))$

FACT (Definition).

the corresponding **Opt** problem **Min-P** is **NP-hard**

IFF

the decision problem **k-P** is **NP-Hard**

COR. IF P # NP and Min-P is NP-hard, THEN there is no poly-time deterministic algorithm for it.

Approximation Ratio (Error)

Optimization Problem

Given an optimization problem $\mathbf{P} = (X, Y(x), m(x, Y(x)), MIN/MAX)$,
we say A is an r -**approximation** algorithm for $\mathbf{P}$ if, **for any** instance $x \in X$,
the computation $A(x)$ returns a feasible solution y^A such that:

$$\frac{m(x, y^A)}{\mathbf{opt}(x)} \geq r \quad \text{(in the case of } \mathbf{MAX}\text{)}$$

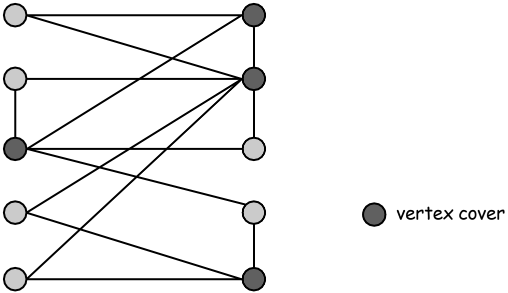
Min-Vertex Cover

k-VC: Given a graph $G = (V, E)$ and an integer k, is there a **k-size VC**, i.e., a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and for each edge, at least one of its endpoints is in S?

Min-VC: Given a graph $G = (V, E)$, find a VC S^* for G of **minimum size**

Ex. there **min-VC** for the graph below has size **4**.

4



Vertex Cover (VC)

A lower bound for optimal VC

Matching and Covering

FACT 1: Given any graph $G(V, E)$, consider any **Maximal Matching** $M \subseteq E$. Then, any VC for G must contain at least 1 vertex for every edge in M.

Proof.

immediate consequence of def.s of Matching and VC.
- Remind: Matching = any subset of **disjoint** edges

Lower Bound for the Optimum

FACT 2:

$$\mathbf{opt}(G\,)\,) \quad |M|$$

Vertex Cover (VC)

An apx algorithm for Min VC

Matching Algorithm $M - ALG$

- Input: $G(V, E)$;
- Find (any) Maximal Matching M;
- Return $C = \{$ all nodes touched by $M\}$;

Vertex Cover (VC)

An apx algorithm for Min VC

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THM 3.

M-ALG is a 2-apx algorithm for Min-VC.

Vertex Cover (VC)

Proof of THM 3

M-ALG

FACT 4. The returned solution C(i) is always a Vertex Cover for GAND (ii) $|C| = 2|M|$.

Vertex Cover (VC)

Proof of THM 3

M-ALG

FACT 4. The returned solution C (i) is always a Vertex Cover for G AND (ii) $|C| = 2|M|$.

Proof.

(i) Immediate consequence of the fact that M is MAXIMAL. (ii) is trivial.

Vertex Cover (VC)

Proof of THM 3

M-ALG

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Remind FACT 2: $\mathbf{opt}(G)$)) $|M|$

Vertex Cover (VC)

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Remind FACT 2: $\mathbf{opt}(G)$) $|M|$

$$\frac{|C|}{\mathbf{opt}(G)} \qquad \frac{2|M|}{|M|} \qquad 2$$