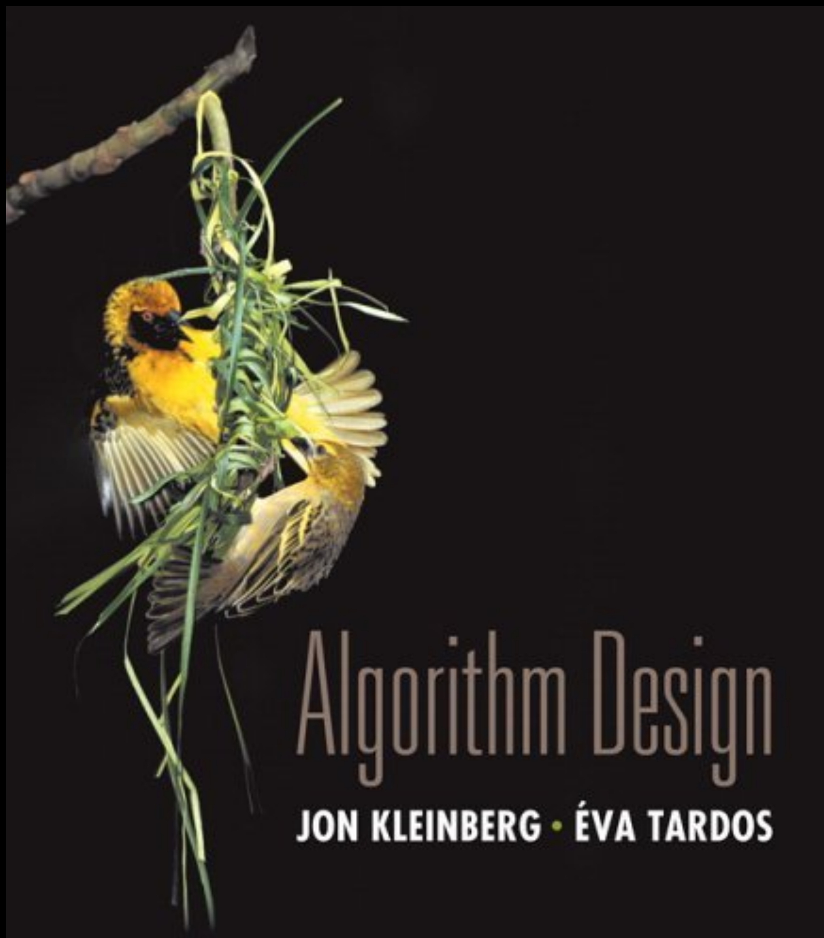


Chapter 4

Greedy Algorithms

Slides by Kevin Wayne.

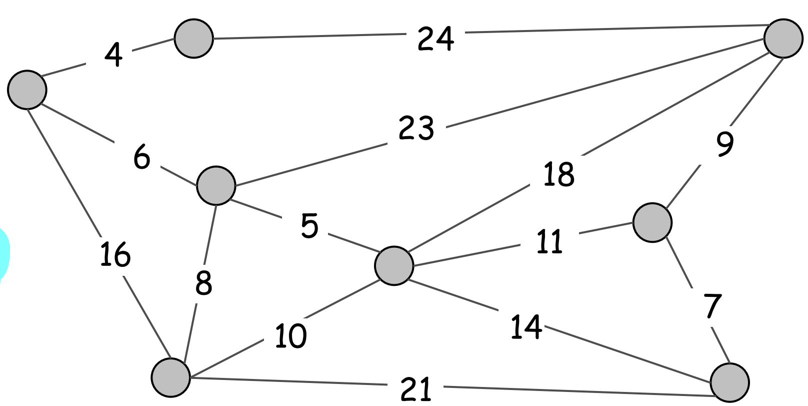
4.5 Minimum Spanning Tree

Minimum Spanning Tree

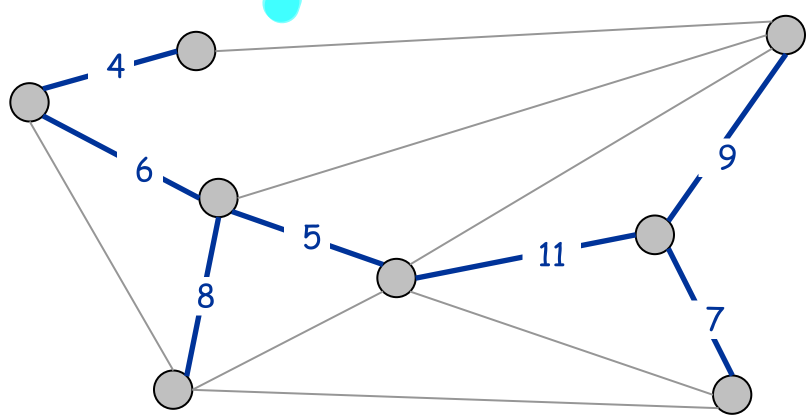
$n = |V|$, $m = |E|$
n, m

$c_e > 0$

Minimum spanning tree. Given a connected graph $G = (V, E)$ with real-valued edge weights c_e, an MST is a subset of the edges $T \subseteq E$ such that T is a spanning tree whose sum of edge weights is minimized.



$G = (V, E)$



T, $\Sigma_{e \in T} c_e = 50$

$m = \binom{n}{2}$

CLIQUE

$K_n = G(V, E)$

E = tutti gli archi

Cayley's Theorem. There are n^{n-2} spanning trees of K_n.

can't solve by brute force

The MST Problem: Formal Definition

$c : E \to R^+$

Input

- Symmetric, connected weighted graph **G=(V,E,w)**, where:
- **V** = Set of nodes, **E** = Set of edges, edge cost function **c: E → R⁺**

Feasible solutions

- Any Spanning Tree of **G**: **F** with **F** ⊆ **E**

$c : \underbrace{SOL\ AMM}\to R^+$

Cost of feasible solutions:

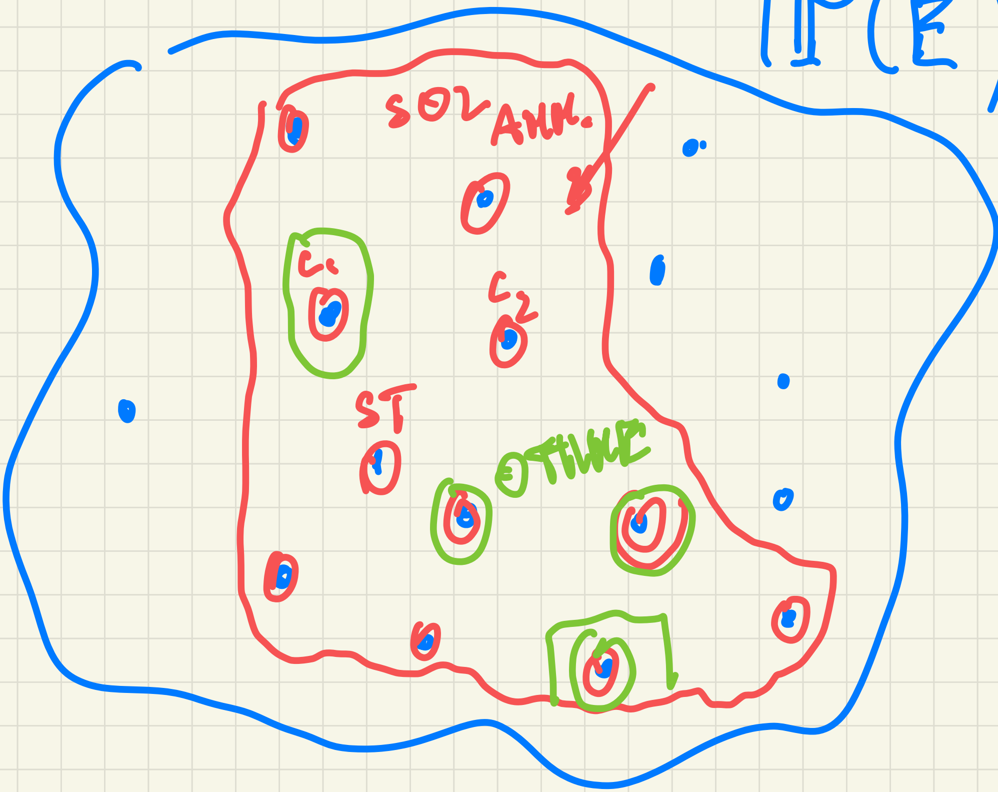
- Cost (to minimize) $c(T) = \Sigma_{e\in T}\ c(e)$

- Check your **learning** level by answering the following questions:
  - How many bits for the input representation?
  - What is a Spanning Tree (ST) of a connected graph?
  - Do you know any algorithm to compute a generic ST?

VISITE dei GRAFI

$G(V, E, c: E \to \mathbb{R}^+)$

$|\mathbb{P}(E)| = 2^{|E|}$



Applications

MST is fundamental problem with diverse applications.

- Network design.
  - telephone, electrical, hydraulic, TV cable, computer, road
- Approximation algorithms for NP-hard problems.
  - traveling salesperson problem, Steiner tree
- Indirect applications.
  - max bottleneck paths
  - LDPC codes for error correction
  - image registration with Renyi entropy
  - learning salient features for real-time face verification
  - reducing data storage in sequencing amino acids in a protein
  - model locality of particle interactions in turbulent fluid flows
  - autoconfig protocol for Ethernet bridging to avoid cycles in a network
- Cluster analysis.

GENERAL APPROACH: Greedy Algorithms

$e_1 \le e_2, \le \ldots, e_3, \le \ldots, \boxed{e_m}$

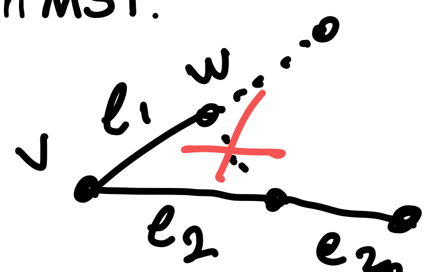
$c(T) \rightarrow T \ni e$ t.c. $c(e)$ minimo

Kruskal's algorithm. Start with **T = φ**. Consider edges in **ascending** order of cost. **Insert** edge **e** in **T** unless doing so would create a **cycle**.

Reverse-Delete algorithm. Start with **T = E**. Consider edges in **descending** order of cost. **Delete** edge **e** from **T** unless doing so would **disconnect T**.

Prim's algorithm. Start with any root node **s** and greedily grow a tree **T** from **s** outward (**Visiting** *G*). At each step, add the **cheapest** edge **e** to **T** that has exactly one endpoint in **T**.

MAIN THEOREM (Informal Statement). **All** three algorithms produce an **MST**.



[T è un ST per G]

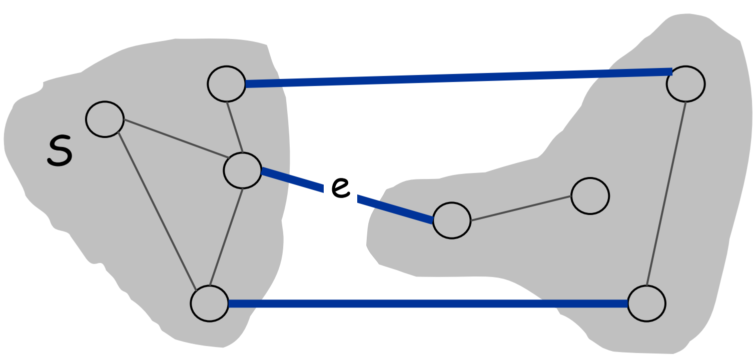
G è connesso

E

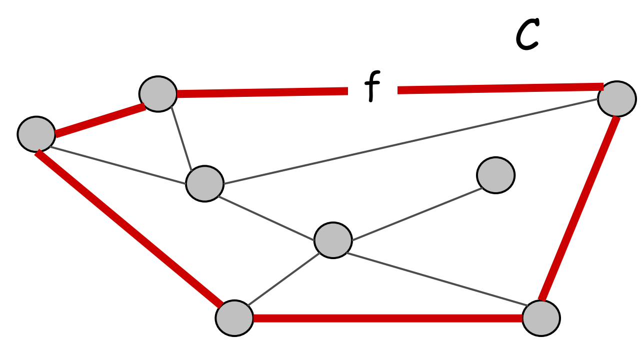
Greedy Algorithms: Analysis

PROOF of the MAIN THEOREM.

We will use TWO GENERAL PROPERTIES OF GRAPHS:



e is in the MST



f is not in the MST

Greedy Algorithms: Analysis

$G(V, E)$ < connesso; $c(e) > 0$ ↳ dist.

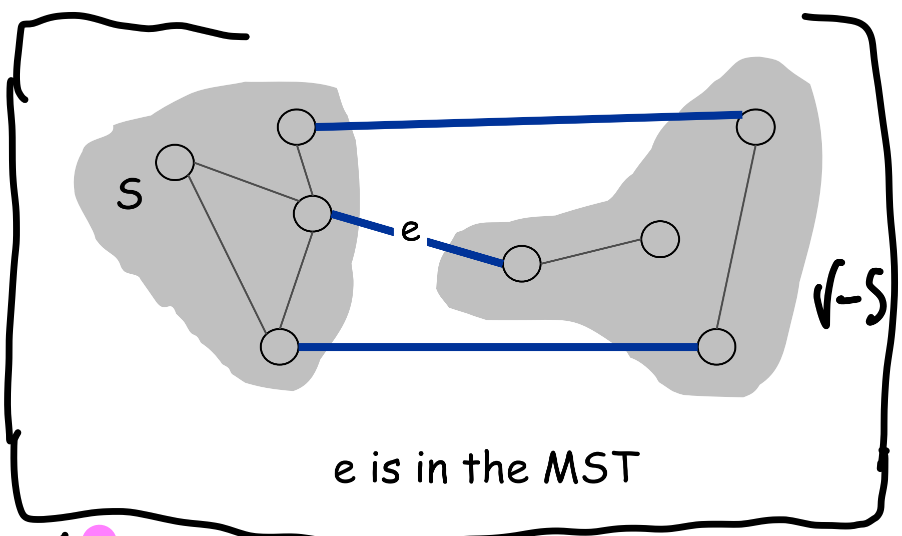
Simplifying assumption. All edge costs c(e) are distinct.

$\forall S \subseteq V$

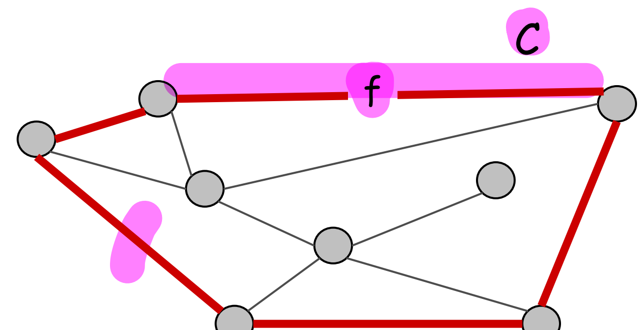
Cut property. Let S be any subset of nodes, and let e be the min cost edge with exactly one endpoint in S. Then the MST contains e.

∀ C cycle of G

- Cycle property. Let C be any cycle, and let f be the max cost edge belonging to C. Then the MST does not contain f.



e is in the MST



f is not in the MST

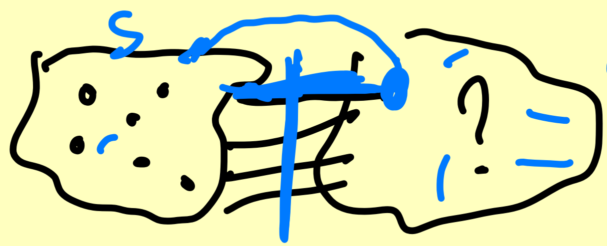
CUT ≡ (S, V−S)   EDGE-CUT ≡ $C \subseteq E$

OSSERVA: Regola del CUT-SET (Taglio) e Regola del CYCLE (CICLO)

CUT RULE: *Scegli una qualsiasi partizione* **(S,V-S)** di **V** che non è attraversata da **archi blu**. Tra tutti gli archi non ancora colorati che sono nel **cut-set E(S,V-S)** , scegline uno di costo minimo e coloralo di blu (cioè, **aggiungilo alla soluzione T**).

$T \leftarrow ?$

$T = T \cup \{e_{min}\}$



$C(T)$

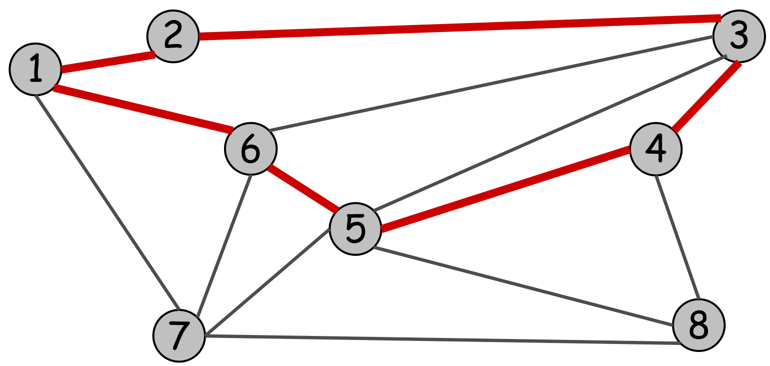
CYCLE RULE: *Scegli un ciclo nel grafo che non contiene* **archi rossi**. Tra tutti gli archi non ancora colorati del ciclo, scegline uno di costo massimo e coloralo di rosso (cioè, **scartalo per sempre**).

$T = T - \{e_{max}\}$

To prove the two properties, we use another simple property:

Cycles and Cuts

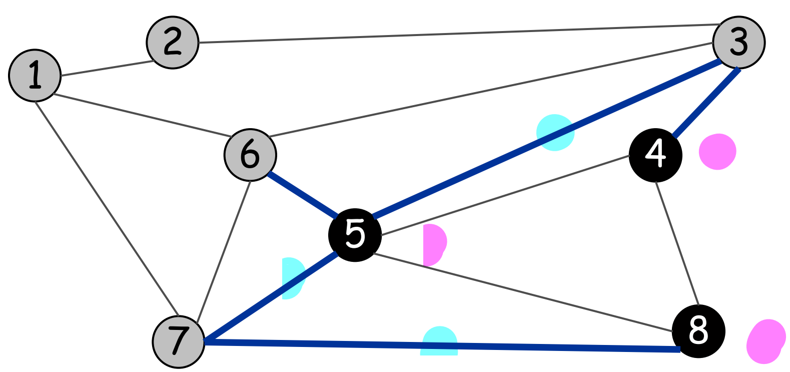
Cycle. Set of edges the form a-b, b-c, c-d, …, y-z, z-a.



Cycle C = 1-2, 2-3, 3-4, 4-5, 5-6, 6-1

EDGE-CUT

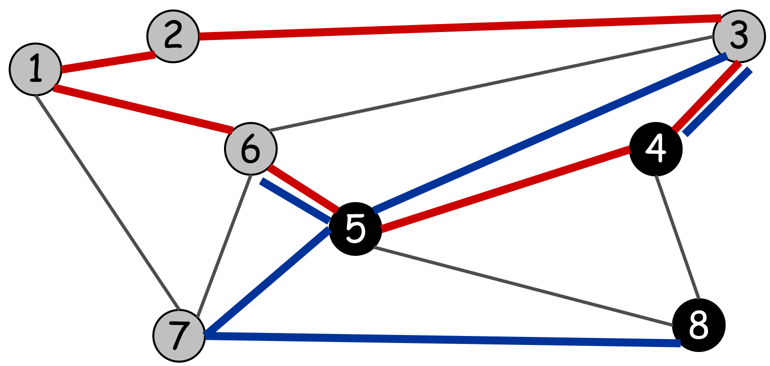
Cutset. A **CUT** is determined by some S. The corresponding **CUTSET D** is the subset of edges with exactly one endpoint in **S**.



Cut **S** = { 4, 5, 8 }
Cutset **D** = 5-6, 5-7, 3-4, 3-5, 7-8

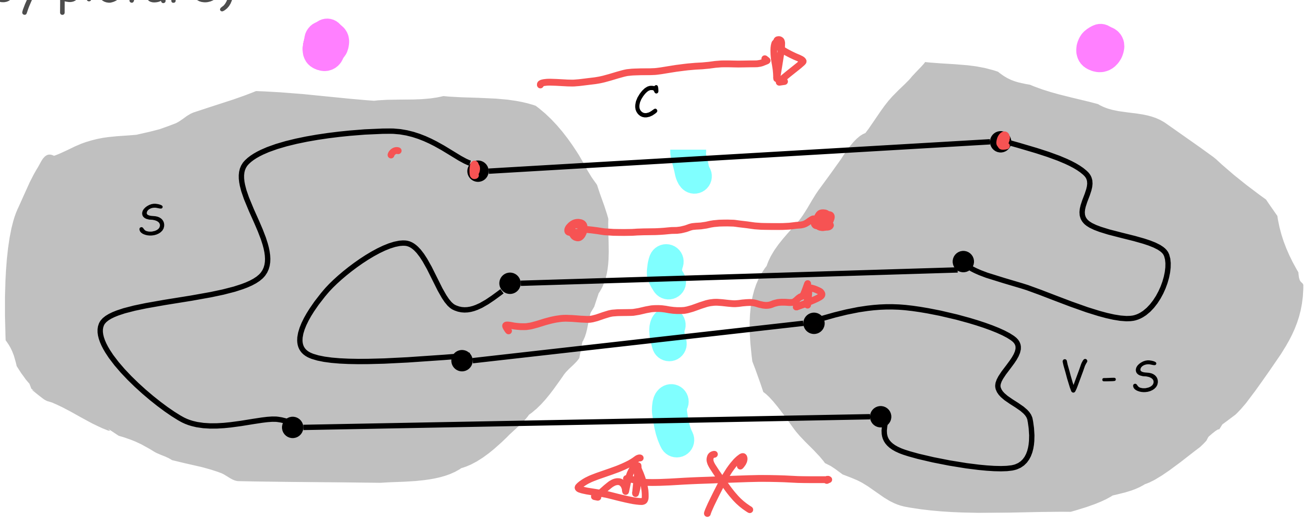
PROPERTY: Cycle-Cut Intersection

Claim. A cycle and a cutset intersect in an even number of edges.

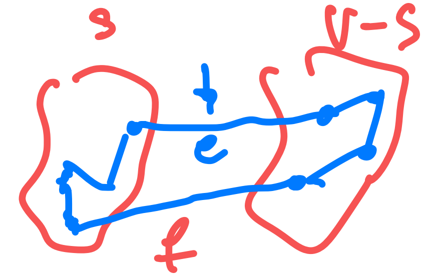


Cycle C = 1-2, 2-3, 3-4, 4-5, 5-6, 6-1
Cutset D = 3-4, 3-5, 5-6, 5-7, 7-8
Intersection = 3-4, 5-6

Pf. (by picture)



CUT PROPERTY: PROOF



Simplifying assumption. All edge costs c_e are distinct.

$S \subseteq V$

Cut property. Let S be any subset of nodes, and let e be the min cost edge wit
exactly one endpoint in S. Then the MST T^* *contains* e.

$T^* \cup \{e\}$

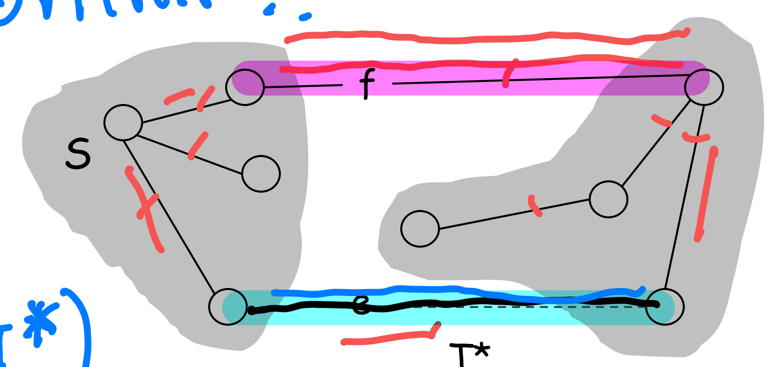
Pf. (*exchange argument*)

- Suppose e does not belong to T^*, and let's see what happens.
- Adding e to T^* creates a **cycle** C in T^*.
- Edge e is both in the cycle C and in the cutset D corresponding to S $\Rightarrow$ the
  exists another edge, say f, that is in both C and D.
- Consider $T' = T^* \cup \{ e \} - \{ f \}$: it is also a **spanning tree!**
- Since $c_e < c_f \rightarrow$ cost(T') < cost(T*).
- This is a contradiction. ▪

OTTIMA ?!

$T^* \rightarrow c^*(T^*)$

$c(T_1^*) = c^*(T^*) - c(f) + c(e) < c^*(T^*)$



G(V, E, c)
↳ connected

Simplifying assumption. All edge costs c_e are distinct.

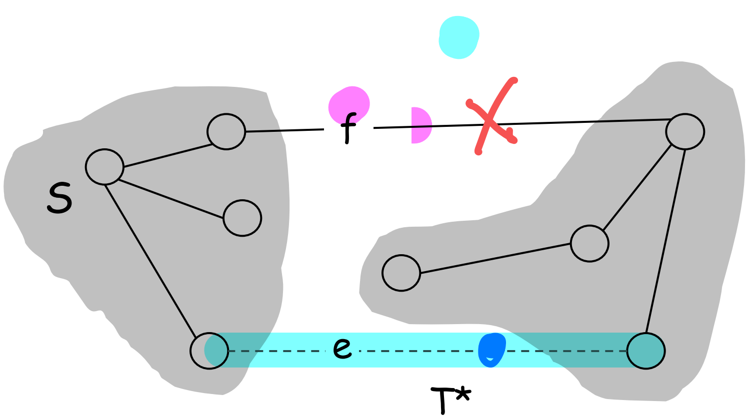
∀ C cycle di G

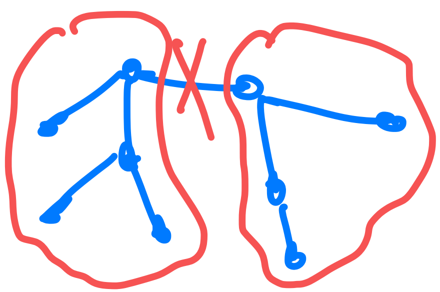
Cycle property. Let **C** be *any* cycle in **G**, and let **f** be the **max cost** edge belonging to **C**. Then the **MST T*** *does not* contain **f**.

Pf. (exchange argument)

- Suppose f belongs to T*, and let's see what happens.
- Deleting f from T* creates a **cut S** in T*.
- Edge f is both in the cycle C and in the cutset D corresponding to S
  $\Rightarrow$ there exists another edge, say e, that is in both C and D.
- $T' = T^* \cup \{e\} - \{f\}$ is also a spanning tree.
- Since $c_e < c_f$, $cost(T') < cost(T^*)$.
- This is a contradiction. ▪

G(V, E)



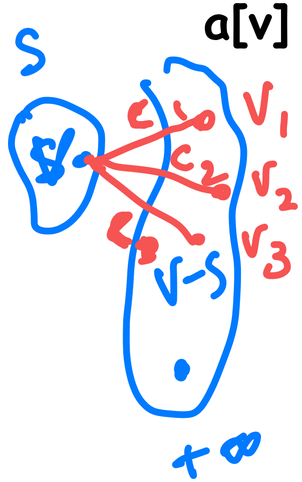


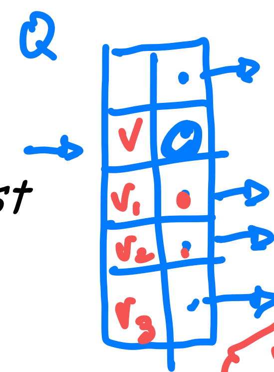
Implementation: Prim's Algorithm

$S = \emptyset \rightarrow$ $S = \{v\}$

Implementation. Use a priority queue ala Dijkstra.

- Maintain set of **explored** nodes **S**.
- For each **unexplored** node **v**, maintain *attachment cost* **a[v] = cost of cheapest edge v to a node in S**



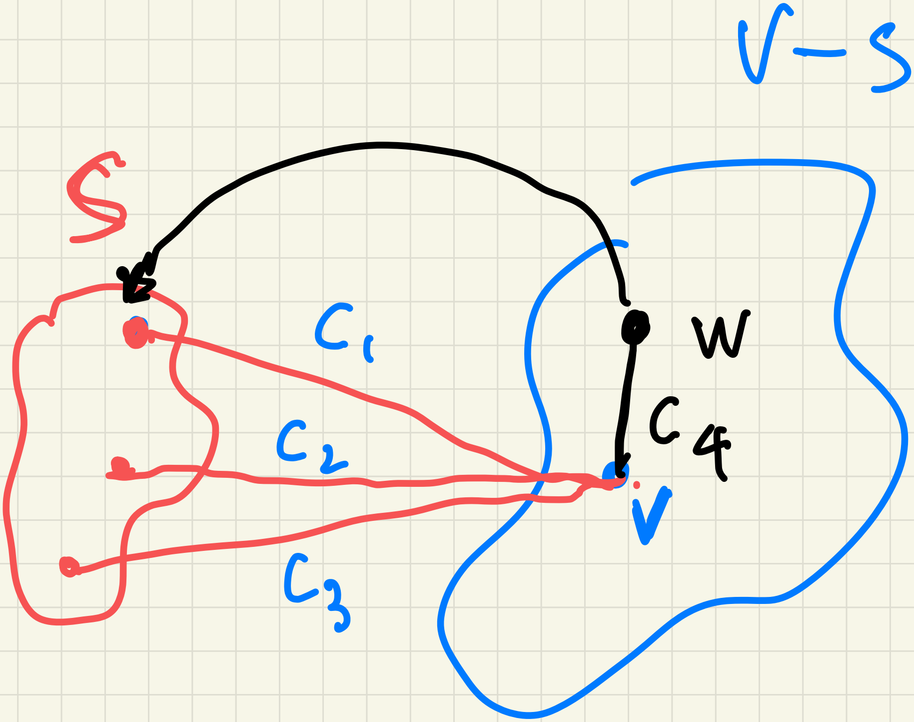


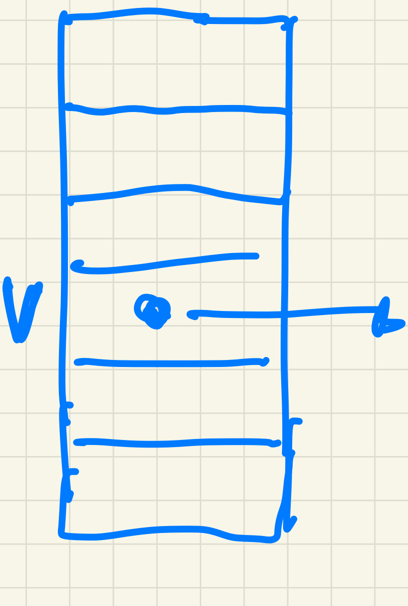

```
Prim(G, c) {
   foreach (v ∈ V) a[v] ← ∞
   Initialize an empty priority queue Q
   foreach (v ∈ V) insert v onto Q
   Initialize set of explored nodes S ← ϕ

   while (Q is not empty) {
      u ← delete min element from Q
      S ← S ∪ { u }
      foreach (edge e = (u, v) incident to u)
         if ((v ∉ S) and (ce < a[v]))
            decrease priority a[v] to ce
}
```

$a(v) = \emptyset$

$S = \{v\}$





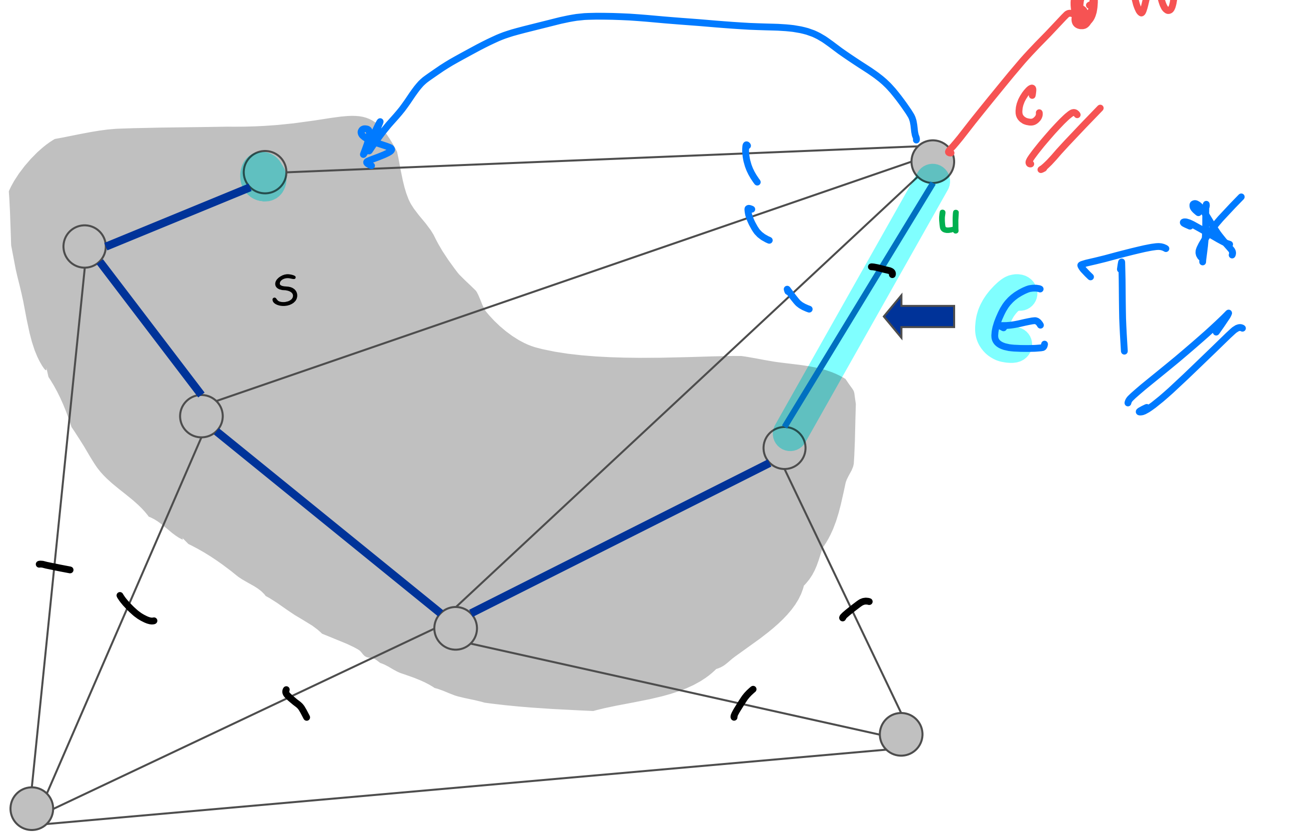
$\min\{c_1, c_2, c_3\} \to c_4$

Prim's Algorithm: Proof of Correctness

Prim's algorithm. [Jarník 1930, Dijkstra 1957, Prim 1959]

- Initialize **S** = **any node**.
- Apply **cut property** to **S**.
- Add **min cost** edge in cutset **D(S)** to **T**, and add **one** new explored node **u** to **S**.

$a(w) = +\infty$



Key Facts of the Proof you need to learn:

What is S ? How growes?

- At each round, a new unexplored node is inserted in **S**. So, at each round of the WHILE loop, **|S|** increases by 1: $S_0 = \{u_1\}$, … $S_t = \{u_1, u_2, \ldots, u_t\}$.., $S_{n-1} = V$,

- **Connectivity of G** → The algorithm **terminates**! AND when it terminates, T **spans all nodes** of **V** (It is a GRAPH SEARCH!).

- **Why T is an *MST*?** At each WHILE loop, apply the **CUT property**! Where? On the (current) CUT :

- (S_t={explored nodes till round t}, $V - S_t$) , t= 1, …, n-1

Time complexity of Prim's Algorithm

THM: $O(n^2)$ with an array; $O(m \log n)$ with a binary heap.

Proof: DO AS EXERCISE!

SUGGESTIONS: give answers to :

- How many times a node is explored ?
- How do you represent Q?
- Which operations on Q for every new explored node? How many? How can you implement them?