Available online at www.sciencedirect.com

SCIENCE DIRECT®

European Journal of Operational Research 166 (2005) 246–254

EUROPEAN JOURNAL OF OPERATIONAL RESEARCH

www.elsevier.com/locate/dsw

O.R. Applications

Optimal pricing policies for perishable products

Miguel F. Anjos *, Russell C.H. Cheng, Christine S.M. Currie

Operational Research Group, School of Mathematics, University of Southampton, Highfield, Southampton SO17 1BJ, UK

Received 28 July 2003; accepted 3 February 2004
Available online 10 May 2004

Abstract

In many industrial settings, managers face the problem of establishing a pricing policy that maximises the revenue from selling a given inventory of items by a fixed deadline, with the full inventory of items being available for sale from the beginning of the selling period. This problem arises in a variety of industries, including the sale of fashion garments, flight seats, and hotel rooms. We present a family of continuous pricing functions for which the optimal pricing strategy can be explicitly characterised and easily implemented. These pricing functions are the basis for a general pricing methodology which is particularly well suited for application in the context of an increasing role for the Internet as a means to market goods and services.

Keywords: Pricing; Internet; Yield management; Finite horizon; Optimal policy

1. Introduction

In many industrial settings, managers face the problem of establishing a pricing policy that maximises the revenue from selling a given inventory of items by a fixed deadline, with the full inventory of items being available for sale from the beginning of the selling period. The items unsold by the deadline have a constant salvage value, which we assume to be zero for simplicity. We further assume that the full inventory of items is available for sale from the beginning of the selling period, and that no reordering is allowed. Examples of industries where this problem arises fall broadly into two categories: those where the product is a manufactured good with a limited shelf life (such as food items or fashion garments), and those where the product is a service (such as flight seats or hotel rooms). Previous research has concentrated on the sale of fashion goods, see for example Gallego and van Ryzin [4] which gives a detailed motivation for this problem and a discussion of its applicability to various industries.

For the case where the product is a service, we proposed in [1] a general methodology for implementing a pricing policy and described how the policy can be updated in real-time to react to changes in the predicted purchase patterns by consumers, and hence achieve a high level of revenue in practice. The method used a family of continuous pricing functions for which the optimal

* Corresponding author. Tel.: +44-23-80595136; fax: +44-23-80595147.
E-mail address: anjos@stanfordalumni.org (M.F. Anjos).

0377-2217/$ - see front matter © 2004 Elsevier B.V. All rights reserved.
doi:10.1016/j.ejor.2004.02.015

pricing strategy could be explicitly characterised and easily implemented. We did not discuss the theoretical properties of these pricing functions or establish their optimal properties in [1], but these are clearly of interest in their own right. The purpose of this article is therefore to establish these properties, thereby providing a sound theoretical footing for the practical procedures to which they give rise.

We should stress that the practical methodology has been adopted by a major British airline to find the optimal price to charge for airline tickets under one-way pricing. For this reason, we focus our presentation on the problem of finding a pricing structure for airline tickets that maximises revenue under one-way pricing, where tickets are sold as singles rather than returns, and every ticket has the same rules and restrictions.

The shift to one-way pricing accompanies the advent of low-cost carriers into the short-haul European airline market, which has led to a dramatic shake-up of the pricing strategies of many of the major airlines. Many tickets on low-cost carriers are now sold on a one-way basis rather than as a paired return, and this renders obsolete many of the rules and restrictions traditionally associated with lower priced airline tickets, such as requiring a Saturday night stay or restricting the duration of the trip. Therefore, with the notable exception of business and other premium classes, most tickets are deemed to be identical and the current selling price is to be determined by the number of days left until departure, rather than by the rules and restrictions of the particular fare.

The rise of the market share held by the low-cost carriers has occurred alongside the increasing role of the Internet as a means to market goods and services. This is particularly true in the airline industry, where an increasing number of tickets are being sold in this way. This shift means that customers now have ready access to up-to-date market prices, and hence their behaviour can adapt rapidly to any pricing changes by the competition. The methodology we present is particularly well suited for application in this context because we specifically model the number of potential customers who want to buy at a particular point in time, which can be thought of as the number of fare requests on the airline's website. To obtain the total number of sales, we multiply the number of potential customers by the proportion of customers that actually buy, given the current price. This approach is different from the more customary modelling of demand (number of items per unit of time) as a function of price alone (as is done for example in [4]). In particular, we believe that our modelling is more appropriate for a situation where the potential sales of a given product is more accurately described by the number of queries on a company's website than by the number of items actually sold (which depends more strongly on the current market pricing).

The optimal continuous pricing policies presented in this paper are the foundation of the general methodology described in the companion paper [1] which describes how a given pricing structure can be updated in real-time to react to changes in the predicted purchase patterns by consumers, and hence achieve a high level of revenue in practice. Although a fixed pricing policy is an attractive proposition, as it is more convenient to implement and does not incur all the administrative and publicity costs arising from price changes, it is nonetheless important to update current pricing in response to fluctuations in demand, and possibly also to changes in the reservation price of customers (i.e. the price that customers are prepared to pay). This methodology can be applied to other areas in the travel and leisure industry such as the sale of hotel rooms, or to the fashion industry. The model of customer behaviour that we have developed determines the optimal price to charge through the selling period, where we assume price varies only with time.

The earliest results in the literature related to this research are by Kincaid and Darling [6], who discussed the optimisation of pricing structures in the retail industry. The model that they proposed involved two functions, the first describing the arrival of potential customers at a store and the second describing the probability that they would pay the price being charged for the product in question. The expected number of sales is then equal to the expected number of potential customers multiplied by the probability that these customers will buy the product. This general model has been extended by other authors. Bitran and Mondschein [2], who concentrate on applications of the model to the sale of fashion goods, use a similar model, including a function that describes the arrival of customers to the store and a function describing the distribution of customers' reservation prices. A customer will buy a product if the price being charged is lower than their reservation price therefore the probability that a potential customer will buy a product is equal to one minus the cumulative distribution of the reservation prices. Although Bitran and Mondschein allow for the reservation prices being dependent on time, little analysis is conducted for this scenario. Gallego and van Ryzin [4] modify the general model of Kincaid and Darling by assuming that demand is dependent only on the price and is unaffected by the number of days remaining until the products perish, although as an extension to this basic model they do briefly consider a very restrictive time-dependent demand function. Again, they concentrate on the fashion industry in which the assumption that demand is independent of time may perhaps be more valid. A further example of this model can be found in Zhao and Zheng's paper [7], who again base their probability of purchase on a customers' reservation price distribution, which they assume can depend on time.

Several different methods for finding the optimal pricing structure have been proposed. While dynamic programming has been used by some authors [2,3], Zhao and Zheng [7] base their equations in continuous time, assuming the arrival of potential customers and the distribution of the reservation prices to be piecewise continuous in time. They then use backward recursion to solve a partial differential equation for the revenue in each time step of the selling period to obtain the optimal price structure. In situations where the equation cannot be solved exactly, finite difference methods are used.

The method most similar to ours is that adopted by Gallego and van Ryzin [4] who use intensity control theory to determine the optimal price structure as a function of time and the number of products remaining. Closed form solutions for the price as a function of time and the number of unsold items can only be found for a restricted set of demand functions and so they also consider determination of the price structure as a function of time only. In situations where the demand is dependent only on price and not on time, the optimal strategy is to charge one price for the duration of the selling period, as predicted by both our model and theirs.

We assume that potential customers arrive at a rate that is dependent on the time left until a flight departs. A potential customer will then buy a ticket with a probability dependent on both the time left until departure and the price being charged. Unlike the above authors, we use the method of calculus of variations to consider time-dependent rather than stock-dependent demand functions and to present some more detailed descriptions of the time-dependent price structures under different assumptions for customer behaviour. A related application of calculus of variations is presented in the paper by Kalish [5], where the problem addressed is that of maximising the profit in the sale of a new product taking into account the dependence of the cost of production on the number of sales made, and assuming that demand is dependent not on time but rather on the number of sales. The results in [5] are mainly applicable to the pricing of durable goods where one must take into account factors such as the decline in production costs as more production experience is gained. In contrast, we are concerned with the pricing of perishable goods, and we focus on examining the variation of price only with respect to time, although we briefly discuss the variation of price with remaining capacity in Section 5.

This paper is structured as follows. In the next section, we state the modelling assumptions that underpin the construction of our pricing model. In Section 3, we derive the basic stationarity condition that characterises the family of optimal pricing policies for our model. Specific pricing policies and their properties are presented in Section 4, and we conclude in Section 5 with a discussion of some practical issues in the application of our pricing strategy and some directions for future research.

2. Modelling assumptions

We consider the sale of tickets for a flight in the period $[0, T]$.

(i) Let

$x =$ Remaining time to the departure time.

Thus initally $x = T$, and x decreases until $x = 0$, the departure time.

(ii) Let

$p(x,y) =$ Proportion of those wanting to buy at time x who buy when the price is y.

(iii) We suppose that the price can be altered according to some pricing policy. Thus $y = y(x)$, $0 \leqslant x \leqslant T$. We assume that

$0 \leqslant y(x) \leqslant K \quad \text{for } 0 \leqslant x \leqslant T.$

(iv) Let

$f(x)\,\mathrm{d}x =$ Number who want to buy in the time interval $(x, x + \mathrm{d}x)$.

(v) The problem is to find a pricing policy $y(x)$ to maximise the total revenue, i.e. to

$$\max_{y(x)} \int_0^T y(x)p(x, y(x))f(x)\,\mathrm{d}x,$$

subject to the aircraft capacity constraint

$$\int_0^T p(x, y(x))f(x)\,\mathrm{d}x \leqslant C. \tag{1}$$

We make the following assumptions.

Assumption 1. Of those wishing to buy at time x, the proportion who actually buy decreases with increasing price; i.e., at any given time x,

$p(x, y_1) > p(x, y_2) \quad \text{if } 0 \leqslant y_1 < y_2.$

Moreover, all will buy if the price is 0, i.e.

$p(x, 0) = 1.$

Assumption 2. The proportion of those wishing to buy at time x, who actually buy at a given price y, increases as we approach the departure time; i.e. that, for any price y,

$$p(x_1, y) \geqslant p(x_2, y) \quad \text{if } 0 \leqslant x_1 < x_2. \tag{2}$$

Though we will consider the general case briefly, we focus in the main on a particular case when the following assumption holds.

Assumption 3

$$p(x, y) = \pi(w(x)y). \tag{3}$$

This assumption simply states that the probability of purchasing, as the price varies, has an invariant form or profile, given by the function $\pi(\cdot)$. However the assumption allows for a degree of urgency in the buyer's attitude, and this is accounted for by the scaling function $w(\cdot)$ which depends solely on time. It may be that more refined assumptions might be needed in more complex situations, but we have found that, though simple, this assumption is already sufficiently flexible to handle the data that we have encountered in real applications.

It turns out that, under Assumption 3, the form of the optimal solution is determined by the form of $\pi(\cdot)$ and is essentially independent of the forms of $w(\cdot)$ and $f(\cdot)$. It is convenient therefore to focus attention directly on $\pi(z)$, regarding it as being defined for all $0 \leqslant z < \infty$.

When Assumption 3 holds, then Assumption 1 is equivalent to the following (except that it is convenient to add a differentiability condition at this juncture).

Assumption 1′. $\pi(z) \geqslant 0$ is a decreasing function of z, with $\pi(0) = 1$. We allow the possibility that $\pi(z)$ reaches zero at some finite $z = L$ so that

$$\begin{aligned} \pi(z) &> 0 \quad \text{for } 0 \leqslant z < L \\ &= 0 \quad \text{for } L \leqslant z < \infty \end{aligned}$$

or the possibility that

$\pi(z) > 0 \quad \text{for } 0 \leqslant z < \infty.$

In this latter case $\pi(z)$ does not have to tend to zero. In either case we assume that, in the domain where $\pi(z) > 0$, $\pi(z)$ is continuously differentiable with

$$\pi'(z) < 0. \tag{4}$$

When Assumption 3 holds, then Assumption 2 is equivalent to the following.

Assumption 2′

$$w(x) \text{ is an increasing function of } x \tag{5}$$

with

$0 < w_0 = w(0) \leqslant w(x) \leqslant w(T) = w_T < \infty.$

3. A basic stationarity condition

Consider the Lagrangian

$$\int_0^T y(x)p(x,y(x))f(x)\,\mathrm{d}x + \lambda\left(C - \int_0^T p(x,y(x))f(x)\,\mathrm{d}x\right).$$

The Euler–Lagrange condition is

$$\frac{\partial}{\partial y}[yp(x,y) - \lambda p(x,y)] = 0,$$

i.e.

$$p(x,y) + [y(x) - \lambda]\frac{\partial p(x,y)}{\partial y} = 0 \quad \text{for } 0 \leqslant x \leqslant T,$$

$$0 \leqslant y(x) \leqslant K. \tag{6}$$

The Lagrange multiplier λ is selected to ensure that the capacity constraint (1) is satisfied. The Karush–Kuhn–Tucker condition requires that

$$\lambda \geqslant 0.$$

Under Assumptions 1′ and 3, the Euler–Lagrange condition can be written as

$$\pi(z(x)) + (z(x) - \lambda w(x))\pi'(z(x)) = 0 \quad 0 \leqslant x \leqslant T \tag{7}$$

with

$$0 \leqslant z(x) \leqslant w(x)K, \tag{8}$$

where

$$z(x) = w(x)y(x).$$

There is no guarantee in general that (7) has a solution or, when there is a solution, that it is unique. In the following lemma, we give a condition under which the solution, when it exists, is unique. To highlight the fact that the condition is purely in terms of the function $\pi(\cdot)$ alone, the lemma is couched in general terms that do not make explicit reference to λ, x, $w(x)$ or the price limit restriction (8). We link the lemma more directly to the pricing problem later.

Lemma 1. *Let $\pi(\cdot)$ satisfy Assumption 1′. If*

$$1 + \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\pi(z)}{\pi'(z)}\right) > 0 \quad \text{for } 0 \leqslant z < L, \tag{9}$$

then for any fixed $\mu \geqslant 0$, the equation

$$\pi(z) + (z - \mu)\pi'(z) = 0, \tag{10}$$

either has a unique solution $0 < z < L$ or else the only solution is at $z = L$ provided we interpret $\pi'(L)$ at $z = L$ as being possibly only a right derivative.

Proof. Let $g(z) = z + \pi(z)/\pi'(z)$ and consider $g(z) - \mu$ for any given $\mu \leqslant 0$, all $z \in [0, L)$. Now $g'(z) = 1 + \frac{\mathrm{d}}{\mathrm{d}z}(\pi(z)/\pi'(z))$ so that when condition (9) holds $g(z)$ is an increasing function for $z \in [0, L)$. At $z = 0$, it follows from (4) of Assumption 1′ that $g(0) - \mu < 0$. Thus, as z increases from zero, $g(z) - \mu$ increases from a negative value. We either have $g(z) - \mu = 0$ at one, and only one, value of $z \in (0, L)$, or else $g(z) - \mu < 0$ for all $z \in (0, L)$. In this latter case we take the solution to be $z = L$, with the proviso that $\pi'(L)$ might only be a right derivative. □

When (9) holds we define

$$z^*(\mu) = \begin{cases} \text{The unique solution of (10) that is less than } L, \text{ when this exists} \\ = L, \quad \text{otherwise.} \end{cases}$$

We shall call $z^*(\mu)$ the *extended solution* of (10).

Corollary 1. *From the proof of Lemma 1 it is clear that $z^*(\mu)$ is an increasing function of μ, as μ increases.*

We do not have an intuitive interpretation for condition (9). The examples presented in Section 4 show that it is not a simple condition such as convexity. In practice, this means that there is sufficient flexibility in the choice of $\pi(\cdot)$ for realistic shapes to be fitted to available data.

4. Pricing policies

A typically adopted strategy is to increase the ticket price as the departure time approaches. This is equivalent to using a policy where $y(x) = z(x)/w(x)$ (here $z(x)$ is the solution of (7)) increases as x decreases. We give a condition that ensures that the optimal policy does take this form. The condition does not depend on the form of the dependence of w on x other than that $w(\cdot)$ is an increasing function of x. We can therefore suppress this dependence on x in the formulation of this condition and regard w just as a variable in its own right.

Lemma 2. *Let $\pi(\cdot)$ be as defined in Lemma 1 and suppose that* (9) *is satisfied, and $\lambda > 0$ is a fixed constant. Suppose that $w \geqslant w_0$. Let $z^*(w\lambda)$ be the extended solution of* (10), *with $\mu = w\lambda$, and let $y^*(w\lambda) = z^*(w\lambda)/w$. Then, for $y^*(w\lambda)$ to be a decreasing function of w as w increases from w_0, it is necessary and sufficient that*

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\pi(z)}{z\pi'(z)}\right) > 0 \quad \textit{for all } z \geqslant z^*(\lambda w_0). \tag{11}$$

Proof. We write $z = wy$, with x suppressed. Differentiating this with respect to w yields

$$\frac{\mathrm{d}z}{\mathrm{d}w} = \frac{z}{w} + w\frac{\mathrm{d}y}{\mathrm{d}w}. \tag{12}$$

Now using $z = wy$, Eq. (7) can be written as

$$y = \lambda - \frac{\pi(z)}{w\pi'(z)}. \tag{13}$$

Differentiating this and using (12) gives, after some rearrangement,

$$\frac{\mathrm{d}y}{\mathrm{d}w} = \frac{z}{w^2}\left\{\left(1 + \frac{\pi(z)}{z\pi'(z)}\right)\left(1 + \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\pi(z)}{\pi'(z)}\right)\right)^{-1} - 1\right\}.$$

If therefore $y = y^*(w\lambda)$ is the solution of (13), then, for it to be a decreasing function as w increases, it is necessary and sufficient that

$$\left(1 + \frac{\pi(z)}{z\pi'(z)}\right)\left(1 + \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\pi(z)}{\pi'(z)}\right)\right)^{-1} - 1 < 0, \tag{14}$$

in the range over which $y^*(w\lambda)$ is defined. In view of Corollary 1 this range is $z \in [z^*(\lambda w_0), L)$. Now under (9), $1 + \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\pi(z)}{\pi'(z)}\right) > 0$, so we can multiply (14) through by $1 + \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\pi(z)}{\pi'(z)}\right)$ without changing the sense of the inequality to get

$$\frac{\pi(z)}{z\pi'(z)} - \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\pi(z)}{\pi'(z)}\right) < 0,$$

i.e.

$$z\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\pi(z)}{z\pi'(z)}\right) > 0.$$

As z is positive, the condition (11) follows. □

4.1. An optimal policy

Under conditions (9) and (11) the form of the optimal pricing policy can be fully characterised.

Theorem 1. *When conditions* (9) *and* (11) *hold then the optimal pricing policy can only take one of the following forms.*

(i) *The solution*

$$y^*(x) = \min(z^*(0)/w(x), K) \quad \textit{all } x \in [0, T]$$

does not fully take up the flight capacity.

(ii) *There is a unique λ and corresponding unique extended solution $z^*(\lambda w(x))$ of* (10) *for which the solution*

$$y^*(x) = \min\{z^*(\lambda w(x))/w(x), K\} \quad \textit{all } x \in [0, T]$$

takes up fully the flight capacity.

(iii) *The flight capacity is exceeded for*

$$y^*(x) = K \quad \textit{all } x \in [0, T].$$

Proof. We consider first the situation where the restriction $y \leqslant K$ is not imposed. This is simply done by setting $K = \infty$ in previous calculations. A solution of form (i) occurs if we also ignore the capacity constraint (1), when Eq. (7) applies with $\lambda = 0$. This equation is then independent of x and so is the same as (10) with $\mu = 0$. Under condition (9) it therefore has a unique solution

$$z = z^*(0). \tag{15}$$

The price function is then

$$\begin{aligned} y &= z^*(0)/w(x) \quad \text{all } x \in [0, T] \\ &= y_0^*(x), \quad \text{say.} \end{aligned}$$

If

$$\int_0^T \pi(w(x)y_0^*(x))f(x)\,\mathrm{d}x \leqslant C,$$

then $y_0^*(x)$ is the optimal solution, but capacity is not fully used.

We can now add the condition $y(x) \leqslant K$. The usual complementary slackness property implies that, for any x, the optimal solution which we write as $y^*(x,0)$ must either satisfy

$$\pi(z^*(x)) + z^*(x)\pi'(z^*(x)) = 0$$

or

$$y^*(x) = K.$$

Thus if $y_0^*(x)$ remains less than K, then $y^*(x,0) = y_0^*(x)$ for all x. But for all values of x for which $y_0^*(x) \geqslant K$ we must have $y^*(x,0) = K$. As $w(x)$ is an increasing function of x, clearly $y_0^*(x)$ is a decreasing function of x. It follows that as x decreases from T to 0, $y_0^*(x)$ increases and hence $y^*(x,0)$ also increases. The only difference is that $y^*(x,0)$ is held at K when $y_0^*(x)$ exceeds K, i.e. $y^*(x,0) = \min(y_0^*(x), K)$.

Ignoring again the restriction $y(x) \leqslant K$, consider now the situation where

$$\int_0^T \pi\big(w(x)y_0^*(x)\big)f(x)\,\mathrm{d}x > C.$$

For (1) to hold, it must be satisfied with equality. In this case we consider $z = z^*(w(x)\lambda)$, the solution of (10) with $\mu = w(x)\lambda$, and corresponding price function $y^*(w(x)\lambda) = z^*(w(x)\lambda)/w(x)$. As for the case when (1) is not active, the condition $y(x) \leqslant K$ can be imposed but where this leaves $y^*(w(x)\lambda)$ still an increasing function as x decreases from T to 0 only capped at $y^*(w(x)\lambda) = K$ once this value is reached. We write this capped solution as $y^*(x,\lambda) = \min\{y^*(w(x)\lambda), K\}$.

For each fixed x, we find, on differentiating (10) (with $\mu = w(x)\lambda$), that at $z = z^*(w(x)\lambda)$

$$\frac{\mathrm{d}z}{d\lambda} = w(x)\left\{1 + \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\pi(z)}{\pi'(z)}\right)\right\}^{-1}.$$

Thus

$$\frac{\mathrm{d}z}{d\lambda} > 0,$$

when condition (9) is satisfied. Consequently, as λ increases from 0,

$$w(x)y^*(x,\lambda) = w(x)\min[z^*\{w(x)\lambda\}/w(x), K]$$

also increases. Hence for every x, $\pi\{w(x)y^*(x,\lambda)\}$ decreases as λ increases and so

$$\int_0^T \pi\{w(x)y^*(x,\lambda)\}f(x)\,\mathrm{d}x$$

is a decreasing function of λ. We thus have a (unique) solution of form (ii) where there is a (unique) λ for which the capacity constraint is exactly met.

The only other possibility is the situation where, for λ sufficiently large, we have $y^*(x,\lambda) = K$ for all x but

$$\int_0^T \pi\{w(x)y^*(x,\lambda)\}f(x)\,\mathrm{d}x = \int_0^T \pi\{w(x)K\}f(x)\,\mathrm{d}x > C.$$

This is the case where the solution is of form (iii) where we can sell to capacity even with the price set at the maximum K throughout the entire sales period $[0,T]$. $\square$

Corollary 2. *Eq.* (15) *shows that when the capacity constraint* (1) *is not active at the optimal solution, then the best policy requires that the price is set so that the same fixed proportion*, $\pi(z^*(0))$, *of sales is achieved at all times* $x \in [0,T]$.

We now provide some illustrations of the above analysis. Our first example presents a function which is a simple but nonetheless sufficiently flexible model for fitting some real data quite effectively.

Example 1. If $p(x,y) = \mathrm{e}^{-ay(1+bx)}$ then $\pi(z) = \mathrm{e}^{-az}$, $\pi'(z) = -a\mathrm{e}^{-az}$. Condition (9) is satisfied as

$$1 + \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\pi}{\pi'}\right) = 1 + \frac{\mathrm{d}}{\mathrm{d}z}(-a^{-1}) = 1 > 0.$$

Condition (11) is satisfied as

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\pi}{z\pi'}\right) = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{-a^{-1}}{z}\right) = \frac{a^{-1}}{z^2} > 0.$$

In fact

$$\frac{\partial p(x,y)}{\partial y} = -a(1+bx)\mathrm{e}^{-ay(1+bx)}$$

and the stationary equation is

$$-a(1+bx)\mathrm{e}^{-ay(1+bx)} = -\frac{\mathrm{e}^{-ay(1+bx)}}{y-\lambda}$$

with solution:

$$y = \frac{a\lambda + abx\lambda + 1}{a(1+bx)} = \lambda + \frac{1}{a(1+bx)}.$$

If now $f(x) = A\mathrm{e}^{-cx}$ then the optimised objective can be explicitly written as follows.

$$\begin{aligned}
&A\int_0^T \left(\lambda + \frac{1}{a(1+bx)}\right)\mathrm{e}^{-ay(1+bx)}\mathrm{e}^{-cx}\,\mathrm{d}x \\
&\quad = A\int_0^T \left(\lambda + \frac{1}{a(1+bx)}\right)\mathrm{e}^{-a\lambda(1+bx)-1}\mathrm{e}^{-cx}\,\mathrm{d}x \\
&\quad = A\mathrm{e}^{-1}\int_0^T \lambda\mathrm{e}^{-a\lambda(1+bx)}\mathrm{e}^{-cx}\,\mathrm{d}x \\
&\qquad + A\mathrm{e}^{-1}\int_0^T \left(\frac{1}{a(1+bx)}\right)\mathrm{e}^{-a\lambda(1+bx)}\mathrm{e}^{-cx}\,\mathrm{d}x \\
&\quad = A\lambda\mathrm{e}^{-1-a\lambda}\frac{(1-\exp(-(ab\lambda+c)T))}{ab\lambda+c} \\
&\qquad + A\mathrm{e}^{\frac{c}{b}-1}\frac{Ei\left(1,\frac{ab\lambda+c}{b}\right) - Ei\left(1,\frac{(1+bT)(ab\lambda+c)}{b}\right)}{ab},
\end{aligned}$$

where

$$Ei(1,z) = \int_1^\infty \frac{\mathrm{e}^{-zt}}{t}\,\mathrm{d}t.$$

The next example shows that conditions (9) and (11) do not imply any specific convexity or concavity condition on the pricing function.

Example 2. Let $p_1(z) = 1/(1+z^2)$ so that, though decreasing, p_1 is neither convex nor concave. Here

$$\frac{\mathrm{d}p_1(z)}{\mathrm{d}z} = -\frac{2}{(1+z^2)^2}z, \quad \text{so } \frac{p_1}{p_1'} = -\frac{1+z^2}{2z}.$$

Thus

$$1 + \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{p_1}{p_1'}\right) = 1 + \frac{\mathrm{d}}{\mathrm{d}z}\left(-\frac{1+z^2}{2z}\right) = \frac{1+z^2}{2z^2} > 0$$

and

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{p_1}{zp_1'}\right) = \frac{\mathrm{d}}{\mathrm{d}z}\left(-\frac{1+z^2}{2z^2}\right) = z^{-3} > 0.$$

The conditions (9) and (11) are satisfied.

Our final example is of an instance where conditions (9) and (11) are not satisfied, showing that monotonicity is not a sufficient condition on its own.

Example 3. Let $p(y) = (1+y^2/8)^{-1} + (16+y^2/128)^{-1}$.

Here $p(y)$ is monotonically decreasing, but it is readily established from elementary but tedious calculations that the conditions (9) and (11) are not satisfied.

5. Discussion

We have presented a family of continuous pricing functions for which the optimal pricing strategy can be directly characterised. This construction of optimal pricing policies is the foundation of the general methodology presented in the paper [1], where it is described how it can be updated in real-time to react to any changes in the predicted purchase patterns by consumers, and hence achieve a high level of revenue in practice. This methodology has been adopted by a major British airline to find the optimal price to charge for airline tickets under one-way pricing.

Two issues are particularly important with respect to practical application. The first one is the computation of confidence ranges on the expected number of bookings at each point in time. A stochastic simulation model is proposed in [1] to compute confidence ranges based on the above one-way pricing model and generate sample booking profiles for a given pricing structure. Estimates of the expected numbers booked by day and fare class and the expected variation in these can be calculated efficiently using the output from the simulation model. Once computed, these confidence ranges are intended to provide alerts of any anomalies in buying behaviour to analysts monitoring the flights. They therefore act as a safety net preventing either all the seats on a plane being booked at a very low price or few seats being sold on a plane due to the prices being set too high for the current market.

The second issue is the mapping of our continuous pricing structure onto a practical structure which only has a relatively small number of fares. As price cannot generally change continuously through time for many practical reasons, discrete methods do offer some advantages over continuous methodologies. However, with continuous methodologies it is much easier to determine the general shape of the optimal price curves for different demand and reservation price distributions. The continuous pricing strategy we presented is optimal under the assumption that price changes have no associated costs. This is usually not the case, since it costs money to advertise fares and to inform travel agents or the general public of those fares. If fares change too often, the company must spend more money informing its customers of the changes and risks upsetting customers who may expect prices to be reasonably static over most of the selling period. However, a continuous pricing model which requires frequent changes to the prices offered is easy to handle (and more acceptable to the travelling customer) in an Internet-based sales scenario, and sales of tickets via the Internet are becoming more and more common in the airline industry. This is likely to make our continuous pricing structure even more useful in the near future. In general, without knowing the exact cost of price changes, it is hard to factor these into the analysis. We therefore assume that a prescribed structure of discrete fares is provided in advance by the airline, and set the price charged to be the fare closest to the optimal continuous price. The resulting methodology is easy to explain and works well in practice.

Two aspects of this work are particularly promising for future research. Firstly, the general methodology is particularly well suited for modelling the increasing role of the Internet as a means to market goods and services because we model as a specific quantity the number of potential customers who want to buy at a particular point in time. This approach is different from the more customary modelling of demand as a function of price alone, and we believe that it is more appropriate for a situation where the potential sales of a given product is more accurately described by the number of queries on a company's website than by the number of items actually sold. Secondly, it should be possible to incorporate within this methodology any information that becomes known (in real-time) about competitors' pricing strategies. This is likely to substantially increase the complexity of the pricing model, but it is also likely to improve the understanding of customers' buying behaviour.

References

[1] M.F. Anjos, R.C.H. Cheng, C.S.M. Currie, Maximising revenue in the airline industry under one-way pricing, Journal of the Operational Research Society 55 (2004) 535–541.
[2] G.R. Bitran, S.V. Mondschein, Periodic pricing of seasonal products in retailing, Management Science 43 (1997) 64–79.
[3] Y. Feng, G. Gallego, Optimal starting times for end-of-season sales and optimal stopping times for promotional fares, Management Science 41 (1995) 1371–1391.
[4] G. Gallego, G. van Ryzin, Optimal dynamic pricing of inventories with stochastic demand over finite horizons, Management Science 40 (1994) 999–1020.
[5] S. Kalish, Monopolist pricing with dynamic demand and production cost, Marketing Science 2 (1983) 135–159.
[6] W.M. Kincaid, D.A. Darling, An inventory pricing problem, Journal of Mathematical Analysis and Applications 7 (1963) 183–208.
[7] W. Zhao, Y. Zheng, Optimal dynamic pricing for perishable assets with nonhomogeneous demand, Management Science 46 (2000) 375–388.