

Complex Analysis

Lecture 01

MA201 Mathematics III

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Attendance Policy

- Attendance in all lecture and tutorial classes is compulsory.
- Students, who do not meet 75% attendance requirement will not be allowed to write the end semester examination and will be awarded *F* grade.
- For attendance in the classes, biometric device or attendance sheets will be circulated.
- Each student is expected to sign against his/her name only in case of attendance sheets.
- In case, any student is found marking proxy for some other student, an appropriate disciplinary action will be taken on both students involved in the proxy matter.
- Random attendance will also be taken.
- Students must attend all tutorial classes. Roll call attendance will be taken in the tutorial classes.

Syllabus of Complex Analysis

- **Complex Numbers:** Complex numbers and elementary properties.
- **Complex Functions:** Limits, continuity and differentiation. Cauchy-Riemann equations. Analytic functions, Harmonic functions. Elementary Analytic functions.
- **Complex Integration:** Contour integrals, Anti-derivatives and path independent of contour integrals.
- **Cauchy-Goursat Theorem.** Cauchy's integral formula, Morera's Theorem. Liouville's Theorem, Fundamental Theorem of Algebra, Maximum Modulus Principle and its consequences.
- **Power Series:** Taylor series, Laurent series.
- **Zeros and Singularities:** Zeros of Analytic Functions, Singularities, Argument Principle, Rouché's Theorem.
- **Residues and Applications:** Cauchy's Residue Theorem and applications.
- **Conformal Mappings:** Conformal Mappings, Möbius transformations.

Complex Analysis Books

Text Book:



J. W. Brown and R. V. Churchill, **Complex Variables and Applications**, 7th or 8th Edition, Mc-Graw Hill, 2004. **Note:** Any Edition is fine.

Reference Book:



J. H. Mathews and R. W. Howell, **Complex Analysis for Mathematics and Engineering**, 3rd Edition, Narosa, 1998. **Note:** Any Edition or Other Publisher is fine.

Lecture 01: Learning Outcome of this Lecture

We learn

- Complex Numbers
- Algebraic Operations: Addition, Multiplication, Division
- \mathbb{C} is a field, but not an ordered field
- $x + iy$ form of complex numbers
- Conjugate, Modulus of a complex number
- Basic identities and inequalities
- Nonzero complex numbers: Polar Form, Trigonometric Form, Exponential Form
- argument function

Recall: $(\mathbb{R}, +, \cdot)$ is a field

- **Closure Law:** For all a and b in \mathbb{R} , $a + b \in \mathbb{R}$ and $a \cdot b \in \mathbb{R}$.

- **Associative Law:** For all a, b and c in \mathbb{R} ,

$$a + (b + c) = (a + b) + c \quad \text{and} \quad a \cdot (b \cdot c) = (a \cdot b) \cdot c .$$

- **Identity Law:** For all a in \mathbb{R} ,

$$a + 0 = a = 0 + a \quad \text{and} \quad a \cdot 1 = a = 1 \cdot a .$$

- **Law of Additive Inverse:** Given $a \in \mathbb{R}$, there exists a unique $x \in \mathbb{R}$ such that $a + x = 0 = x + a$.

- **Law of Multiplicative Inverse:** Given $a \in \mathbb{R}$ with $a \neq 0$, there exists a unique $x \in \mathbb{R}$ such that $a \cdot x = 1 = x \cdot a$.

- **Commutative Law:** For all a and b in \mathbb{R} ,

$$a + b = b + a \quad \text{and} \quad a \cdot b = b \cdot a .$$

- **Distributive Law:** For all a, b and c in \mathbb{R} ,

$$a \cdot (b + c) = a \cdot b + a \cdot c .$$

Recall: Order Axiom, Total Order, Ordered Field

This slide is additional information. It can be skipped.

*Order Axiom: There is a subset F^+ of a field $(F, +, \cdot)$, called the **set of positive elements** of F that satisfies*

- If $x, y \in F^+$ then $x + y \in F^+$ and $x \cdot y \in F^+$,
- **Trichotomy Law:** Given $x \in F$, exactly one of the following is true

$$x \in F^+ \quad \text{or} \quad x = 0 \quad \text{or} \quad -x \in F^+ .$$

The set of negative elements can be defined as $F^- := F \setminus (F^+ \cup \{0\})$.

(Total) Order: In a field $(F, +, \cdot)$, define an order relation ' \leq ' by $x \leq y$ if and only if $y - x \in F^+ \cup \{0\}$.

The field F that satisfies the order axiom is called an **ordered field**.

- 1 If $x \in F^+$ then $-x \in F^-$. If $x \in F^-$ then $-x \in F^+$.
- 2 For any $x \in F \setminus \{0\}$, $x \cdot x \in F^+$. The identity element $1 \in F^+$.

Recall: $(\mathbb{R}, +, \cdot)$ is an ordered field

Alternatively, the totally ordered field can be defined as follows.

An order relation \leq on a set S is said to be a **total order** if it satisfies

- **Reflexivity:** $a \leq a$ for all $a \in S$,
- **Antisymmetry:** $a \leq b$ and $b \leq a$ implies $a = b$,
- **Transitivity:** $a \leq b$ and $b \leq c$ implies $a \leq c$,
- **Comparability:** For any a and b in S , either $a \leq b$ or $b \leq a$.

A field $(F, +, \cdot)$ with a total order \leq on F is an **ordered field** if the order satisfies

- If $a \leq b$ then $a + c \leq b + c$ for all $c \in F$,
- If $0 \leq a, 0 \leq b$ then $0 \leq (a \cdot b)$.

Example: $(\mathbb{R}, +, \cdot)$ with the order relation \leq is a totally ordered field.

Why do we need Complex Numbers \mathbb{C} ?

NOT all polynomial equations have roots in \mathbb{R} .

Example: $x^2 + 1 = 0$ has no roots in \mathbb{R} .

$(\mathbb{R}, +, \cdot)$ is **NOT** algebraically closed. There is a need of bigger number system in which all (nonconstant) polynomial equations have roots.

Fact/History: Complex numbers \mathbb{C} were originated when Four Italy mathematicians (**Ferro**, **Tartagila**, **Cardano**, **Bombelli**) in 16th Century tried to solve cubic equations like $x^3 - 3bx - 2c = 0$, $x^3 - 15x - 4 = 0$ (but not from quadratic equations at that time).

For an interesting article on History of Complex Numbers see:

<http://www.math.uri.edu/~merino/spring06/mth562/ShortHistoryComplexNumbers2006.pdf>

Advantages (now): Certain real integrals can be computed easily in \mathbb{C} . Certain differential equations can be easily solved. A differentiable complex function in an open set (analytic function) has many interesting properties.

Complex Numbers

Definition

A complex number z is defined to be an **ordered pair of real numbers** x and y as $z = (x, y)$. That is, the set of complex numbers is denoted by \mathbb{C} and is given by

$$\mathbb{C} = \{z = (x, y) : x \text{ and } y \text{ are real numbers} \} .$$

The **ordered pair** here means the order in which we write x and y in defining the complex number $z = (x, y)$. For example, the number $(1, 5)$ is not the same as $(5, 1)$.

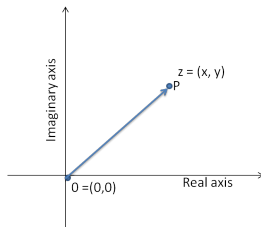
In the complex number $z = (x, y)$,

- x is called the **real part** of z and is denoted by $\Re(z)$ or $\text{Re } z$
- y is called the **imaginary part** of z and is denoted by $\Im(z)$ or $\text{Im } z$

- The numbers of the form $(0, y)$ are called **pure imaginary numbers**.
- The numbers of the form $(x, 0)$ are called **real numbers**.
- The set of real numbers can be identified as a subset $\mathbb{R} = \{z = (x, y) \in \mathbb{C} : x \in \mathbb{R} \text{ and } y = 0\}$ in \mathbb{C} . That is, $\mathbb{R} \subset \mathbb{C}$.
- Two complex numbers are **equal** if and only if their real parts are equal and their imaginary parts are equal.

History: The representation of complex numbers in the plane was proposed independently by **Casper Wessel (1797)**, **K. F. Gauss (1799)** and **Jean Robert Argand (1806)**.

Complex Plane/ z -plane/ Argand Plane



- The complex number $z = (x, y)$ can be viewed as a point P having cartesian coordinates (x, y) in the plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.
- The x -axis and y -axis are called the **real axis** and the **imaginary axis** respectively.
- The complex number $z = (x, y)$ can also be represented by a vector connecting the origin $0 = (0, 0)$ to the point P .
- This plane is called the **complex plane** or **z -plane**. It is also known as the **Gauss plane** or the **Argand Plane**. The term **Argand diagram** is sometimes used.

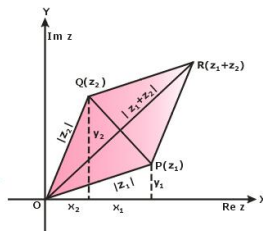
Addition Operation

For any two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, the **addition** of z_1 and z_2 is defined

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2) .$$

The sum of any two complex numbers is a complex number.

Geometric Interpretation of Addition of two complex numbers:



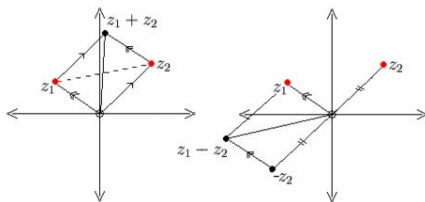
If \vec{OP} and \vec{OQ} are not collinear, then \vec{OR} is the diagonal of the parallelogram with \vec{OP} and \vec{OQ} as adjacent sides.

Subtraction Operation

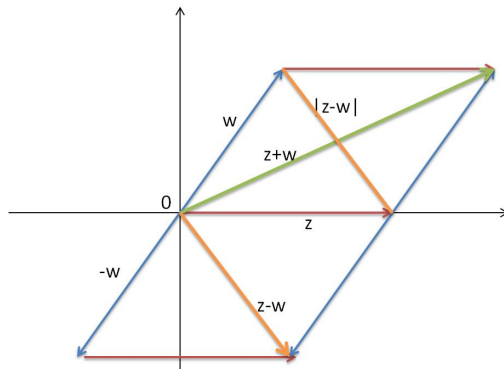
The **subtraction** $z_1 - z_2$ of the complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ is defined as

$$z_1 - z_2 = (x_1 - x_2, y_1 - y_2) .$$

The subtraction $z_1 - z_2$ can be viewed as the sum of the complex numbers z_1 and $-z_2$.



Geometric Interpretation of Subtraction



This picture will be useful to understand parallelogram law (later on).

$$|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2)$$

Multiplication and Division

For any two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, the **multiplication** of z_1 and z_2 is defined by

$$z_1 z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1) .$$

The product of any two complex numbers is a complex number. This multiplication is **different from** the vector product.

If $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2) \neq 0$ are any two complex numbers, then the complex number z_1 **divided by** z_2 is defined as

$$\frac{z_1}{z_2} = \left(\frac{1}{x_2^2 + y_2^2} \right) ((x_1 x_2 + y_1 y_2), (x_2 y_1 - x_1 y_2)) .$$

The set of complex numbers \mathbb{C} with these operations addition $+$ and multiplication \cdot forms a field. The identity element of $+$ is $(0, 0)$ and the identity element of \cdot is $(1, 0)$. \mathbb{R} is a subfield of \mathbb{C} .

Binomial Formula

Let $0! = 1$ and $n! = n(n-1)(n-2)\cdots 1$ for $n \in \mathbb{N}$.

Let $nC_k = \frac{n!}{k!(n-k)!}$ for $k = 0, \dots, n$.

Binomial Formula:

For any two complex numbers z_1 and z_2 and for $n \in \mathbb{N}$,

$$(z_1 + z_2)^n = \sum_{k=0}^n nC_k z_1^{n-k} z_2^k .$$

The proof is based on mathematical induction and is left as an exercise.

Complex Field is **NOT** an Ordered Field

We can not define usual order relation like **less than**, **less than or equal to**, **greater than**, **greater than or equal to** on the set of complex numbers. *That is, the usual ordering of \mathbb{R} can not be taken to \mathbb{C} as such.*

However, we can define in other ways, like dictionary order on \mathbb{C} as follows.

*Let $z_1 = x_1 + i y_1$ and $z_2 = x_2 + i y_2$. The **dictionary order** is given by:*

$$z_1 < z_2 \quad \text{if} \quad x_1 < x_2$$

$$z_1 < z_2 \quad \text{if} \quad x_1 = x_2 \quad \text{and if} \quad y_1 < y_2$$

$$z_1 = z_2 \quad \text{if} \quad x_1 = x_2 \quad \text{and if} \quad y_1 = y_2$$

The complex field $(\mathbb{C}, +, \cdot)$ can **NOT** be an ordered field with respect to any (total) order defined on \mathbb{C} .

Proof: Note $(0, 1) > (0, 0)$. But $(0, 1)(0, 1) = (-1, 0) < (0, 0)$.

Algebraic form (or $x + iy$ notation)

Set

$$i = (0, 1) .$$

It is called **iota**.

Electrical engineers use the letter j instead of i .

$$(x, y) = (x, 0)(1, 0) + (0, 1)(y, 0) = x \cdot 1 + i \cdot y = x + i y$$

$$(x, y) = x + i y ,$$

$$i^2 = (0, 1)(0, 1) = (-1, 0) = -1 .$$

The form $x + iy$ is called the **algebraic form** of a complex number.

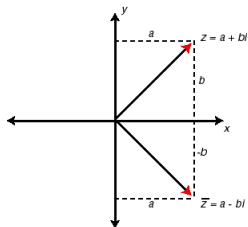
Hereafter, we prefer to use $x + iy$ form instead of ordered pair (x, y) form to write complex numbers.

Conjugate of a Complex Number

The **complex conjugate**, or simply, the **conjugate** of a complex number $z = a + ib$ is denoted by \bar{z} and is defined by

$$\bar{z} = a - ib.$$

Geometrically, the point $\bar{z} = a - ib$ is the reflection (mirror image) of the point $z = a + ib$ on the real axis.



Examples: If $z = 3 + 4i$ then $\bar{z} = 3 - 4i$. If $z = -5$ then $\bar{z} = -5$.

Properties of Complex Conjugation

- 1 $z_1 = z_2$ if and only if $\overline{z_1} = \overline{z_2}$.
- 2 $\overline{\overline{z}} = z$.
- 3 $\overline{z} = z$ if and only if z is a real number.
- 4 $z + \overline{z} = 2\Re(z) = 2x$ if $z = x + iy$.
- 5 $z - \overline{z} = 2i \Im(z) = 2iy$ if $z = x + iy$.
- 6 $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$.
- 7 $\overline{z_1 z_2} = \overline{z_1} \overline{z_2}$.
- 8 $\overline{(z_1/z_2)} = \overline{z_1}/\overline{z_2}$ provided $z_2 \neq 0$.

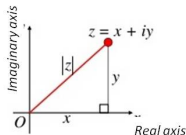
The numbers z and \overline{z} are called the **complex conjugate coordinates**, or simply the **conjugate coordinates** corresponding to the point $z = (x, y) = x + iy$. Also they have been called the **isotropic coordinates** and the **minimal coordinates** of the point.

Modulus of a Complex Number

The **modulus** or **absolute value** of a complex number $z = x + iy$ is denoted by $|z|$ and is given by

$$|z| = \sqrt{x^2 + y^2}.$$

Here, as usual, the radical stands for the principal (non-negative) square root of $x^2 + y^2$.



Example: The modulus of the complex number $4 + 3i$ is

$$|4 + 3i| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5.$$

Note: $|z| \geq 0$ for all $z \in \mathbb{C}$. $|z| = 0$ if and only if $z = 0$.

Properties - Modulus & Conjugate

- ❶ $|z| \geq 0$ and $|z| = 0$ iff $z = 0$.
- ❷ $\bar{\bar{z}} = z$ and $|\bar{z}| = |z| = |-z|$.
- ❸ $|z|^2 = z \bar{z}$.
- ❹ If $z = x + iy$, $|z| \leq |x| + |y|$.
- ❺ If $z = x + iy$, $|x| \leq |z|$ and $|y| \leq |z|$.
- ❻ $|z_1 z_2| = |z_1| |z_2|$.
- ❼ **Parallelogram Law:** $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$
- ❽ **Triangle Inequality:** $|z_1 + z_2| \leq |z_1| + |z_2|$. (Work out proof in class)
- ❾ $|z_1 - z_2| \leq |z_1| + |z_2|$.
- ❿ $|z_1 - z_2| \geq ||z_1| - |z_2||$. (Work out proof in class)
- ⓫ $|z_1 + z_2| \geq ||z_1| - |z_2||$.
- ⓬ $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$ provided $z_2 \neq 0$.
- ⓭ If $n \in \mathbb{N}$, then $|z^n| = |z|^n$. If $-n \in \mathbb{N}$, then $|z^n| = |z|^n$ for $z \neq 0$.

Properties (continuation) - Additional Information

- 14 **Lagrange's Identity:** If $\{z_1, z_2, \dots, z_n\}$ and $\{w_1, w_2, \dots, w_n\}$ are two sets of n complex numbers ($n \geq 1$), then

$$\left| \sum_{k=1}^n z_k w_k \right|^2 = \left(\sum_{k=1}^n |z_k|^2 \right) \left(\sum_{k=1}^n |w_k|^2 \right) - \sum_{1 \leq j < k \leq n} |z_j w_k - z_k w_j|^2.$$

- 15 **Cauchy-Schwarz Inequality:** If $\{z_1, z_2, \dots, z_n\}$ and $\{w_1, w_2, \dots, w_n\}$ are two sets of n complex numbers ($n \geq 1$), then

$$\left| \sum_{k=1}^n z_k w_k \right|^2 \leq \left(\sum_{k=1}^n |z_k|^2 \right) \left(\sum_{k=1}^n |w_k|^2 \right)$$

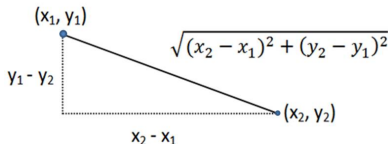
where the equality sign holds iff the z_k are proportional to the $\overline{w_k}$.

Distance between Two Complex Numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be any two complex numbers. Then the (Usual/Euclidean) distance between z_1 and z_2 is defined by

$$d(z_1, z_2) = |z_2 - z_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

$$= |z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

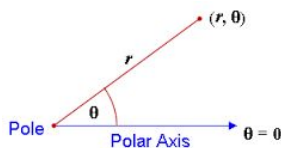
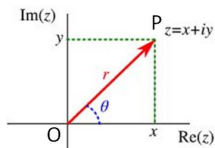


Example: If $z_1 = 1 + i$ and $z_2 = 1 - i$ then

$$|z_1 - z_2| = \sqrt{(1 - 1)^2 + (1 - (-1))^2} = 2.$$

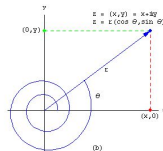
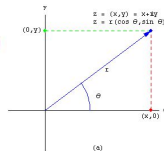
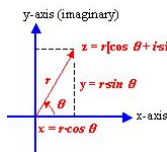
Note: $|z| = d(0, z)$. (\mathbb{C}, d) is a metric space.

Polar Form of (Non-Zero) Complex Numbers



- Each non-zero complex number $z = x + iy = (x, y) \neq (0, 0)$ can be represented by the vector from the origin O to the point $P = (x, y)$ in the plane.
- The length r of the vector \vec{OP} is given by $r = \sqrt{x^2 + y^2} = |z| =$ **Modulus** of z .
- The measure θ in radians of the oriented angle from the **positive real axis** to the vector \vec{OP} is called the **argument** or the **amplitude** of the vector \vec{OP} , and we write $\theta = \arg z$.
- For $z \neq 0$, we can write $z = (r, \theta)$ where $r = |z|$ and $\theta = \arg(z)$. This representation is called the **polar representation** of z , and the values of r and θ are called **polar coordinates** of z .

Trigonometric Form of (Non-Zero) Complex Numbers



- From trigonometry we have, $\cos \theta = \frac{x}{r}$ and $\sin \theta = \frac{y}{r}$.

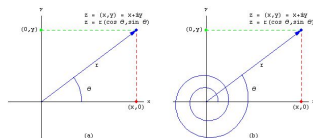
$$z = r \cos \theta + i r \sin \theta = r(\cos \theta + i \sin \theta) .$$

This is known as the **trigonometric form** of the complex number z .

- The number θ is determined only up to multiples of 2π and the set of all such angles is denoted by $\arg z$. However all the values in this set represent the same direction in the complex plane.

Example: Modulus of $(1 + i)$ is $\sqrt{2}$ and argument of $(1 + i) = \pi/4 +$ any multiple of 2π . Polar form of $(1 + i)$ is $(\sqrt{2}, \pi/4)$ or $(\sqrt{2}, 9\pi/4)$, etc.

About the function $\arg(z)$



- For the complex number $z = 0$, the modulus is 0, but the **argument is undefined**.
- If a complex number z is written in the polar form or in the trigonometric form then it is understood that it is a non-zero complex number.
- For each nonzero z , $\arg(z)$ takes a set of values. This set is an infinite set. For each nonzero point z , argument function thus assigns a set as value. Therefore, $\arg(z)$ is called a **multiple valued function**.

Examples: $\arg(5) = \{2n\pi : n \in \mathbb{Z}\}$; $\arg(-3) = \{(2n + 1)\pi : n \in \mathbb{Z}\}$;
 $\arg(1 + i) = \{(\pi/4) + 2n\pi : n \in \mathbb{Z}\}$; **Compute** $\arg(1 - i)$.

Principal Value of argument of z : $\text{Arg } z$

Picking one of the values of $\arg(z)$ for computation purpose. For example, if teacher gives some condition, all students should be able to pick up the same (and unique) value for $\arg(1 + i)$. How to do it? e.g., **Teacher says:** Restrict the value of $\arg(z)$ in the interval $(-\pi, \pi]$ and now tell me the value of $\arg(1 + i)$.

Answer: $\arg(1 + i)$ in the interval $(-\pi, \pi]$ is _____.

For each non-zero z , there is only one value of $\arg z$ say Θ satisfying $-\pi < \Theta \leq \pi$. This value will henceforth be denoted by $\text{Arg } z$ and is called the **principal value** of $\arg z$.

Examples: $\text{Arg}(5) = 0$, $\text{Arg}(i) = \pi/2$, $\text{Arg}(-8) = \pi$, $\text{Arg}(-i) = -\pi/2$.

Exercise: Find the largest set in \mathbb{C} on which $\text{Arg } z$ is continuous?

Relation between $\arg z$ and $\text{Arg } z$:

$$\arg z = \text{Arg } z + 2\pi k \text{ where } k \text{ is an integer.}$$

Computing Principal Value of argument and argument

Let $z = x + iy \neq 0$.

Compute $\phi = \text{Principal value of } \tan^{-1}(y/x) \text{ which lies in } (-\pi/2, \pi/2)$.

With the value of ϕ and with the information of signs of x and y (which quadrant z lies) we can compute

$$\text{Arg}(z) = \begin{cases} \phi & \text{if } x > 0 \\ \phi + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \phi - \pi & \text{if } x < 0 \text{ and } y < 0 \\ \pi/2 & \text{if } x = 0 \text{ and } y > 0 \\ -\pi/2 & \text{if } x = 0 \text{ and } y < 0 \end{cases}$$

$$\arg(z) = \text{Arg}(z) + 2k\pi \quad \text{where } k \in \mathbb{Z}.$$

Exponential form of Non-Zero Complex Numbers

- Let $z = x + iy \neq 0$ be written in the trigonometric form as $z = r(\cos \theta + i \sin \theta)$ where r is the modulus and θ is the argument of z .
- The **Euler's formula** says that

$$e^{i\theta} = \cos \theta + i \sin \theta$$

where θ is measured in radians.

If $z \neq 0$ then using Euler's formula, we can write z as

$$z = re^{i\theta}$$

where $r = |z|$ and $\theta = \arg(z)$ which is known as the **exponential form** of a complex number z .

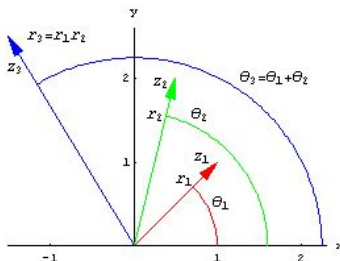
Examples: $1 + i = 2e^{i\pi/4}$, $-i = e^{-\pi/2}$, $-8 = 8e^{i\pi} = 8e^{i3\pi}$.

Geometrical Interpretation of Multiplication

Let $z_1 \neq 0$ and $z_2 \neq 0$. Then,

$$z_i = r_i(\cos \theta_i + i \sin \theta_i), \quad i = 1, 2.$$

$$\begin{aligned} z_1 z_2 &= r_1 r_2 [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i (\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)] \\ &= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \end{aligned}$$



$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

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The above identity is to be interpreted as saying that if values of two of these three (multiple valued) arguments are specified, then there is a value of the third such that the above equation holds.

Example: If $3 = 3e^{2\pi i}$ and $-2 = 2e^{3\pi i}$ then $-6 = 6e^{i\theta_3}$ with $\theta_3 = 5\pi$ (one of the values of $\arg(-6)$ plus a suitable multiple of 2π is to be taken) so that the identity holds.

*In the above identity, if we replace $\arg(z)$ by $\text{Arg}(z)$, then identity is in general **NOT** true. If z_1 and z_2 lies in the first quadrant then it will be true.*

$$\text{Arg}(z_1 z_2) \neq \text{Arg}(z_1) + \text{Arg}(z_2) \quad (\text{in general}).$$

If $0 \neq z = re^{i\theta}$ then $(1/z) = (1/r)e^{-i\theta}$ and hence $\arg(1/z) = -\arg(z)$.

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2).$$