1. Use the error formula (see the notes **quad2**) for the closed Newton-Cotes rule $Q_{NC(m)}$ to prove that the rule will compute the integral

$$\int_0^1 x^{k-1} dx = \frac{1}{k}$$

exactly for $k = 1, 2, \cdots, m$. Here the quadrature nodes are the fixed points $x_j = (j-1)/(m-1)$ for $j = 1, 2, \cdots, m$. Therefore, the weights $w_1, w_2, \cdots, w_m$ satisfy

$$w_1 x_1^{k-1} + w_2 x_2^{k-1} + \cdots + w_m x_m^{k-1} = \frac{1}{k}, \quad k = 1, 2, \cdots, m.$$

These conditions define a *linear* system. Write a function `ComputeClosedNewtonCotesWeights` to solve this system for the weights, and compare your computed weights with those given by M. Abramowitz and I. Stegun (in `ClosedNewtonCotesWeights` these weights are hardcoded).

2. We have discussed both the error estimate

$$\left| \int_a^b f(x)dx - Q_{NC(3)} \right| \leq \frac{1}{2880} M_4 (b-a)^5,$$

for the three-point, closed, Newton-Cotes (Simpson) rule, as well as the error estimate

$$\left| \int_a^b f(x)dx - Q_{NC(4)} \right| \leq \frac{1}{6480} M_4 (b-a)^5,$$

for the four-point, closed, Newton-Cotes (Simpson 3/8) rule, where in each case M_4 is a bound on $|f^{(4)}(x)|$ for $x \in [a, b]$. As remarked, the Simpson 3/8 rule has a slightly better error bound, but requires one extra function evaluation. The similar scaling in these estimates suggests that the Simpson rule is superior to the Simpson 3/8 rule if our measure is *accuracy per function evaluation.*

That the Simpson rule is indeed better in this measure may be confirmed in the composite-rule setting. For example, both $Q_{NC(3)}^{(3)}$ and $Q_{NC(4)}^{(2)}$ are based on $3(3-1)+1 = 2(4-1)+1 = 7$ points, whence each requires seven function evaluations.

(a) Approximate $\int_0^2 \arctan(x)dx$ with each of the seven-point composite rules above, computing errors against the exact answer obtained by the Fundamental Theorem of Calculus.

(b) Use the general error formula [Eq. (12) of the notes `quad3`] to prove that, in the composite-rule setting, the error estimate for the Simpson rule is better (in terms of accuracy per function evaluation) than the one for the Simpson 3/8 rule. Compare the results from **(a)** and **(b)**.

3. We have learned about the *Gauss-Legendre nodes* for quadrature. Unlike the Newton-Cotes nodes, they are not uniformly spaced and (as it turns out) do not include the interval endpoints. *Gauss-Lobatto nodes* are similar except that they do include the interval endpoints. This problem considers a composite rule based on 4-point Gauss-Lobatto quadrature (GL4), takes the form

$$\int_{z_k}^{z_{k+1}} f(x)dx = \frac{h}{12} \left[f(z_k) + 5f(z_k + c_1 h) + 5f(z_k + c_2 h) + f(z_{k+1}) \right],$$

where $h = z_{k+1} - z_k$, $c_1 = \frac{1}{10}(5 - \sqrt{5})$, and $c_2 = \frac{1}{10}(5 + \sqrt{5})$. Here $a = z_1 < z_2 < \cdots < z_{n+1} = b$ is a partition of $[a, b]$, and the above approximation is used for each contribution to the sum

$$\int_a^b f(x)dx = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} f(x)dx.$$

Write an efficient function (no waisted integrand evaluations) to implement this rule on a uniform partition to approximate the integral

$$\int_{-\pi/2}^{\pi/2} e^x \cos(5x)dx$$

using $n = 2, 4, 8, 16, 32$ partitions. Plot the absolute error as a function of the number n of partitions for the composite GL4 and Simpson methods. Counting reuse, how many function evaluations does each method require as a function of n? Which method delivers the most accuracy per function evaluation for large n?

Afterward. You might be interested in how to define the nodes and weights for the above quadrature rule. Relative to the basic interval $[0, 1]$, the nodes are $0 = c_0, c_1, c_2, c_3 = 1$, and the weights are w_0, w_1, w_2, w_3. Since c_0 and c_3 are already fixed, all we need to determine are $c_1, c_2, w_0, w_1, w_2, w_3$. Demand that these obey the equations

$$\begin{aligned}
1 &= w_0 + w_1 + w_2 + w_3 && \text{quadrature rule exactly integrates 1 over } [0,1] \\
\tfrac{1}{2} &= w_1 c_1 + w_2 c_2 + w_3 && \text{quadrature rule exactly integrates } x \text{ over } [0,1] \\
\tfrac{1}{3} &= w_1 c_1^2 + w_2 c_2^2 + w_3 && \text{quadrature rule exactly integrates } x^2 \text{ over } [0,1] \\
\tfrac{1}{4} &= w_1 c_1^3 + w_2 c_2^3 + w_3 && \text{quadrature rule exactly integrates } x^3 \text{ over } [0,1] \\
\tfrac{1}{5} &= w_1 c_1^4 + w_2 c_2^4 + w_3 && \text{quadrature rule exactly integrates } x^4 \text{ over } [0,1] \\
\tfrac{1}{6} &= w_1 c_1^5 + w_2 c_2^5 + w_3 && \text{quadrature rule exactly integrates } x^5 \text{ over } [0,1].
\end{aligned}$$

This GL4 exactly integrates polynomials up to and including degree 5. Note that 4-point Gauss-Legendre exactly integrates polynomials up to and including degree 7.