1) $D_3 = \{(0,4), (\pi/2, -1), (\pi, -2)\}$

$B_3 = \{1, \cos x, \sin x\}$ $\phi_1 = 1$, $\phi_2 = \cos x$ $\phi_3 = \sin x$

$x = \pi/4$

$$\begin{bmatrix} \phi_1(0) & \phi_2(0) & \phi_3(0) \\ \phi_1(\pi/2) & \phi_2(\pi/2) & \phi_3(\pi/2) \\ \phi_1(\pi) & \phi_2(\pi) & \phi_3(\pi) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \cos(0) & \sin(0) \\ 1 & \cos(\pi/2) & \sin(\pi/2) \\ 1 & \cos(\pi) & \sin(\pi) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & -2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$c_1 = 1$

$c_2 = 3$

$c_3 = -2$

$x = \pi/4$

$$= c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi(x)$$
$$= \phi_1(x) + 3\phi_2(x) - 2\phi_3(x)$$
$$= 1 + 3\cos(x) - 2\sin(x)$$
$$= 1 + 3\cos(\pi/4) - 2\sin(\pi/4)$$
$$= 1 + 3(\sqrt{2}/2) - 2(\sqrt{2}/2)$$
$$= 1 + 3\sqrt{2}/2 - 2\sqrt{2}/2$$
$$= 1 + \sqrt{2}/2$$
$$\boxed{\approx 1.7071067812}$$

2) $D_3 = \{(-1, 8), (2, -1), (3, 4)\}$

I) $B_3 = \{1, x, x^2\}$ $\phi_1 = 1$, $\phi_2 = x$, $\phi_3 = x^2$

$$\begin{bmatrix} \phi_1(-1) & \phi_2(-1) & \phi_3(-1) \\ \phi_1(2) & \phi_2(2) & \phi_3(2) \\ \phi_1(3) & \phi_2(3) & \phi_3(3) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 4 \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 1 & 2 & 4 & -1 \\ 1 & 3 & 9 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$c_1 = 1$
$c_2 = -5$
$c_3 = 2$

$$\begin{aligned} p(x) &= c_1 \phi_1(x) + c_2 \phi_2(x) + c_3 \phi_3(x) \\ &= 1\,\phi_1(x) - 5\,\phi_2(x) + 2\,\phi_3(x) \\ &= 1 - 5x + 2x^2 \end{aligned}$$

<u>Interpolation (monomial):</u>

$$\boxed{p(x) = 1 - 5x + 2x^2}$$

II) $D_3 = \{(-1, 8), (2, -1), (3, 4)\}$

$\phi_1 = 1$, $\phi_2 = (x - x_1)$, $\phi_3 = (x - x_1)(x - x_2)$

Newton's Basis

$$\begin{array}{c|c} -1 & 8 \\ 2 & -1 \\ 3 & 4 \end{array}$$

$$\frac{-1-8}{2+1} = \frac{-9}{3} = -3$$

$$\frac{4+1}{3-2} = \frac{5}{1} = 5$$

$$\frac{5+3}{3+1} = \frac{8}{4} = 2$$

$$\begin{aligned}
p(x) &= 8 - 3(x+1) + 2(x+1)(x-2) \\
&= 8 - 3x - 3 + 2(x^2 - x - 2) \\
&= 8 - 3x - 3 + 2x^2 - 2x - 4 \\
&= 8 - 5x + 2x^2 - 3 - 4 \\
&= 1 - 5x + 2x^2
\end{aligned}$$

$c_1 = 1$, $c_2 = -5$, $c_3 = 2$

Interpolation (Newton):

$$\boxed{p(x) = 1 - 5x + 2x^2}$$

III) $D_3 = \{(-1, 8), (2, -1), (3, 4)\}$

$\ell_1(x) = (x - x_2)(x - x_3) / (x_1 - x_2)(x_1 - x_3)$

$\ell_2(x) = (x - x_1)(x - x_3) / (x_2 - x_1)(x_2 - x_3)$

$\ell_3(x) = (x - x_1)(x - x_2) / (x_3 - x_1)(x_3 - x_2)$

Lagrange Basis

$$\ell_1(x) = \frac{(x-2)(x-3)}{(-1-2)(-1-3)} = \frac{(x-2)(x-3)}{12} = 1/12\,(x-2)(x-3)$$

$$\ell_2(x) = \frac{(x+1)(x-3)}{(2+1)(2-3)} = \frac{(x+1)(x-3)}{-3} = -1/3\,(x+1)(x-3)$$

$$\ell_3(x) = \frac{(x+1)(x-2)}{(3+1)(3-2)} = \frac{(x+1)(x-2)}{4} = 1/4\,(x+1)(x-2)$$

$$\begin{aligned}
p(x) &= y_1 \ell_1(x) + y_2 \ell_2(x) + y_3 \ell_3(x) \\
&= 8\,\ell_1(x) - 1\,\ell_2(x) + 4\,\ell_3(x) \\
&= 8(1/12(x-2)(x-3)) - 1(-1/3(x+1)(x-3)) + 4(1/4(x+1)(x-2)) \\
&= 8(1/12(x^2-5x+6)) - 1(-1/3(x^2-2x-3)) + 4(1/4(x^2-x-2)) \\
&= 8(1/12x^2 - 5/12x + 1/2) - 1(-1/3x^2 + 2/3x + 1) + 4(1/4x^2 - 1/4x - 1/2) \\
&= 2/3x^2 - 10/3x + 4 + 1/3x^2 - 2/3x - 1 + x^2 - x - 2 \\
&= 2/3x^2 + 1/3x^2 + x^2 - 10/3x - 2/3x - x + 4 - 1 - 2 \\
&= 2x^2 - 5x + 1
\end{aligned}$$

$c_1 = 1,\ c_2 = -5,\ c_3 = 2$

Interpolation (Langrange):

$$\boxed{p(x) = 1 - 5x + 2x^2}$$