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CS375-001

Test 2

1) $T = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 4 & 2 & 2 & 0 \\ 0 & 4 & 2 & 2 \\ 0 & 0 & 4 & 2 \end{bmatrix}$

a) $M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -a_{21}/a_{11} & 1 & 0 & 0 \\ -a_{31}/a_{11} & 0 & 1 & 0 \\ -a_{41}/a_{11} & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -a'_{32}/a'_{22} & 1 & 0 \\ 0 & -a'_{42}/a'_{22} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$M_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -a''_{43}/a''_{33} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/3 & 1 \end{bmatrix}$ $2/6 = 1/3$

$A = M_1^{-1} M_2^{-1} M_3^{-1} U$

$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/3 & 1 \end{bmatrix} U$

$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 1/3 & 1 \end{bmatrix} U$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 1/3 & 1 \end{bmatrix}$$

$A = LU$ ✓

$$U = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 1/3 \end{bmatrix}$$

b) $Tx = b$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 1/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 0 & 0 & 0 \\ 4 & 2 & 2 & 0 & 0 \\ 0 & 4 & 2 & 2 & 0 \\ 0 & 0 & 4 & 2 & 1 \end{array}\right] \xrightarrow[\text{elimination}]{\text{Gauss-Jordan}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & -1/2 \\ 0 & 0 & 1 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & 3/2 \end{array}\right]$$

$$\bar{x} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 3/2 \end{bmatrix}$$

2) $\begin{bmatrix} 1 & 2.01 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3.01 \\ 9 \end{bmatrix}$ $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

a) $\|x\|_\infty = \max\limits_{1 \le k \le n} |x_k|$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \boxed{\|x\|_\infty = 1}$

b) $x_A = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \rightarrow \|x_A\|_\infty = 1$

$r = b - Ax_A \rightarrow = \begin{bmatrix} 3.01 \\ 9 \end{bmatrix} - \begin{bmatrix} 1 & 2.01 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 3.01 \\ 9 \end{bmatrix} - \begin{bmatrix} 1.01 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ 6 \end{bmatrix} \longrightarrow \|r\|_\infty = 6$

$rberr = \frac{\|r\|}{\|b\|} = \frac{6}{9} = \underline{2/3}$

$x_A - x = \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

$rferr = \frac{\|x_A - x\|}{\|x\|} = \frac{2}{1} = \underline{2}$

$errmag = \frac{rferr}{rberr} = \frac{2}{2/3} = \frac{6}{2} \boxed{= 3}$

c) $k(A) = \|A\| \cdot \|A^{-1}\| = 9 \cdot 8.01$
$\boxed{= 72.09}$

The condition number is NOT consistant with the error magnificantion consistant.

3) $\{(4,3),(6,-4),(8,22)\}$

$y = c_1 + c_2 \cos(\pi x/24) + c_3 (\sin \pi x/24)$

$$\begin{bmatrix} 1 & 4 & 16 \\ 1 & 6 & 36 \\ 1 & 8 & 64 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 22 \end{bmatrix} \qquad V = \begin{bmatrix} 1 & 4 & 16 \\ 1 & 6 & 36 \\ 1 & 8 & 64 \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} 1 & 4 & 16 & 3 \\ 1 & 6 & 36 & -4 \\ 1 & 8 & 64 & 22 \end{array}\right]$$

$\downarrow$ Gauss-Jordan elimination

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 116 \\ 0 & 1 & 0 & -179/4 \\ 0 & 0 & 1 & 843/200 \end{array}\right] \qquad c_1 =$$

$c_1 = 116 \qquad c_2 = -179/4 \qquad c_3 = 843/200$

$$c = \begin{bmatrix} 116 \\ -179/4 \\ 843/200 \end{bmatrix}$$

$$y = 116 - 179/4(\cos(\pi x/24)) + 843/200(\sin(\pi x/24))$$

4) a) $\{(0,0), (\pi/2, 1), (\pi, 0)\}$

Newton approach:

$$\begin{array}{c|c}
0 & 0 \\
\pi/2 & 1 \\
\pi & 0
\end{array}
\qquad \frac{1-0}{\pi/2} = \boxed{2/\pi} \qquad \frac{0-1}{\pi-\pi/2} = -\pi/2 \qquad \frac{-\pi/2 - 2/\pi}{\pi - 0} = \frac{-\pi/2 - 2/\pi}{\pi}$$

$$p(x) = 0 + 2/\pi\,(x-0) + \frac{-\pi/2 - 2/\pi}{\pi}(x-0)(x-\pi/2)$$
$$= 2/\pi \cdot x + \frac{-\pi/2 - 2/\pi}{\pi} \cdot x(x-\pi/2)$$

$$\boxed{p(x) = 2/\pi \cdot x + \frac{-\pi/2 - 2/\pi}{\pi} \cdot x(x-\pi/2)}$$

b) $p(\pi/4) = 2/\pi(\pi/4) + \frac{-\pi/2 - 2/\pi}{\pi}(\pi/4)(\pi/4 - \pi/2)$
$= 1/2 + \frac{-\pi/2 - 2/\pi}{\pi}(\pi^2/16)$
$= 1/2 - 0.433425137$
$= 0.5 - 0.433425137$
$\boxed{\approx 0.0665748625}$

$\sin(\pi/4) \approx 0.0137073546$

c) $err_A = |f(x) - p(x)|$
$= |f(\pi/4) - p(\pi/4)|$
$= |0.0137073546 - 0.0665748625|$
$\approx |-0.0528675079|$
$\boxed{\approx 0.0528675079}$

d) $err_{exact} = \frac{\left|\left(\frac{\pi}{4}-0\right)\left(\frac{\pi}{4}-\frac{\pi}{2}\right)\left(\frac{\pi}{4}-\pi\right)\right|}{\left(\frac{\pi}{4}\right)!} \cdot f'''\left(\frac{\pi}{4}\right)$

$= \frac{\left|\left(\frac{\pi}{4}\right)\left(-\frac{\pi}{4}\right)\left(-\frac{3\pi}{4}\right)\right|}{\left(\frac{\pi}{4}\right)!} \cdot -\sin\left(\frac{\pi}{4}\right)$

$\approx \frac{1.453419219}{0.9275890934} \cdot 0.0137073546$

$\approx 1.566878247 \cdot 0.0137073546$

$\approx \underline{0.0214775643}$

$err_A > err_{exact}$ ✓

$\Downarrow$

$\underline{0.0528675079 > 0.0214775643}$

Approximate err is larger than exact err ✓.

5) $Ax = b \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad A = \begin{bmatrix} 2 & 3 \\ -2 & -6 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$

"Thick" QR: $A = \begin{bmatrix} 2/3 & 1/3 & -2/3 & 0 \\ -2/3 & 2/3 & -1/3 & 0 \\ 1/3 & 2/3 & 2/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 6 \\ 0 & -3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

a) $V^T V x_{LS} = V^T y$

$$\begin{bmatrix} 2 & -2 & 1 & 0 \\ 3 & -6 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -2 & -6 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} x_{LS} = \begin{bmatrix} 2 & -2 & 1 & 0 \\ 3 & -6 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 18 \\ 18 & 45 \end{bmatrix} x_{LS} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

b) "Thin" QR for A:

$$Q = \begin{bmatrix} 2/3 & 1/3 & -2/3 \\ -2/3 & 2/3 & -1/3 \\ 1/3 & 2/3 & 2/3 \\ 0 & 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 3 & 6 \\ 0 & -3 \\ 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 2/3 & 1/3 & -2/3 \\ -2/3 & 2/3 & -1/3 \\ 1/3 & 2/3 & 2/3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 6 \\ 0 & -3 \\ 0 & 0 \end{bmatrix}$$

c) $\begin{bmatrix} 9 & 18 \\ 18 & 45 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ ⇐ from part (a)

$$\left[\begin{array}{cc|c} 9 & 18 & 2 \\ 18 & 45 & 3 \end{array}\right] \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left[\begin{array}{cc|c} 1 & 0 & 4/9 \\ 0 & 1 & -1/9 \end{array}\right]$$

$x_1 = 4/9$ & $x_2 = -1/9$

$$\boxed{x_{LS} = \begin{bmatrix} 4/9 \\ -1/9 \end{bmatrix}}$$