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CS375-001
Exam 3

1) $\begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

$\vec{a}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad \vec{a}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ $\quad (a_1, a_2) = (q_1, q_2)\begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix}$

$a_1 = r_{11} q_1$
$a_2 = r_{12} q_1 + r_{22} q_2$

$r_{11} = \left\| \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\| = \sqrt{(-1)^2 + (1)^2} = \underline{\sqrt{2}}$

① $q_1 = a_1 / r_{11} = \underline{\begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix}}$

② $r_{12} = q_1^T a_2 = \begin{bmatrix} -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \underline{-\sqrt{2}}$

$r_{22} q_2 = a_2 - r_{12} q_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} - (-\sqrt{2}) \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \longrightarrow = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \underline{\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}}$

$r_{22} = \left\| \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\| = \sqrt{(-1)^2} = 1$ $\quad q_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ $\quad q_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ $\quad q_4 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

$Q = \begin{bmatrix} -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{bmatrix}$

$R = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

"Thick" QR

$\underline{\begin{bmatrix} -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}}$

2) $(x_0, y_0, z_0) = (1, 1, \tfrac{1}{2})$

$$\begin{aligned}(x-1)^2 + (y-1)^2 + z^2 &= 1\\ (x-1)^2 + y^2 + (z-1)^2 &= 1\\ x^2 + (y-1)^2 + (z-1)^2 &= 1\end{aligned} \Rightarrow F = \begin{bmatrix}(x-1)^2 & (y-1)^2 & z^2\\ (x-1)^2 & y^2 & (z-1)^2\\ x^2 & (y-1)^2 & (z-1)^2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$DF = \begin{bmatrix}2(x-1) & 2(y-1) & 2z\\ 2(x-1) & 2y & 2(z-1)\\ 2x & 2(y-1) & 2(z-1)\end{bmatrix}$$

$$F(x_0, y_0, z_0) = \begin{bmatrix}0 & 0 & 1/4\\ 0 & 1 & 1/4\\ 1 & 0 & 1/4\end{bmatrix}\begin{bmatrix}? & ? & ?\\ ? & ? & ?\\ ? & ? & ?\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\ 0 & 2 & -1\\ 2 & 0 & -1\end{bmatrix}$$

$$\downarrow$$

$$= \begin{bmatrix}0 & 0 & 1/4\\ 0 & 1/2 & 0\\ 1/2 & 0 & 0\end{bmatrix} \Rightarrow \begin{aligned}x_{new} &= 1/2\\ y_{new} &= 1/2\\ z_{new} &= 1/4\end{aligned}$$

$$\begin{aligned}x_1 &= x_0 - x_{new}\\ &= 1 - 1/2\\ &= 1/2\end{aligned} \qquad \begin{aligned}y_1 &= x_0 - x_{new}\\ &= 1 - 1/2\\ &= 1/2\end{aligned} \qquad \begin{aligned}z_1 &= z_0 - z_{new}\\ &= 1/2 - 1/4\\ &= 1/4\end{aligned}$$

$$\boxed{(x_1, y_1, z_1) = (1/2, 1/2, 1/4)}$$

3) $x_k = -2 + k \cdot \Delta x$ where $\Delta x = 4/5$ & $k = [1, 4]$.

$$Q = 4 \sum_{i=1}^{4} w_k f(x_k)$$

a) $\underline{k=1}$

$$\begin{aligned}x_1 &= -2 + 1(4/5)\\ &= -2 + 4/5\\ &= -6/5\end{aligned}$$

$\underline{k=2}$

$$\begin{aligned}x_2 &= -2 + 2(4/5)\\ &= -2 + 8/5\\ &= -2/5\end{aligned}$$

$\underline{k=3}$

$$\begin{aligned}x_3 &= -2 + 3(4/5)\\ &= -2 + 12/5\\ &= 2/5\end{aligned}$$

$\underline{k=4}$

$$\begin{aligned}x_4 &= -2 + 4(4/5)\\ &= -2 + 16/5\\ &= 6/5\end{aligned}$$

$$\begin{aligned}Q &= 4\left(w_1 \cdot f(-6/5)\right) + \left(w_2 \cdot f(-2/5)\right) + \left(w_3 \cdot f(2/5)\right) + \left(w_4 \cdot f(6/5)\right)\\ &= 4(w_1 \cdot w_2 \cdot w_3 \cdot w_4)\left(f(-6/5) + f(-2/5) + f(2/5) + f(6/5)\right)\end{aligned}$$

$$\Downarrow$$

$$\begin{bmatrix}1/4 f(-6/5)\\ 1/4 f(-2/5)\\ 1/4 f(2/5)\\ 1/4 f(6/5)\end{bmatrix}\begin{bmatrix}w_1\\ w_2\\ w_3\\ w_4\end{bmatrix} = \begin{bmatrix}?\\?\\?\\?\end{bmatrix}$$

b) $f(x) = x^3 - x + 4$

$= \int_{-2}^{2} f(x)\,dx$
$= \int_{-2}^{2} x^3 - x + 4\,dx$
$= \int_{-2}^{2} x^3\,dx - \int_{-2}^{2} x\,dx + \int_{-2}^{2} 4\,dx$
$= \frac{1}{4}x \Big|_{-2}^{2} - \frac{1}{2} \Big|_{-2}^{2} + 4x \Big|_{-2}^{2}$
$= 0 \quad - 0 \quad + 16$
$= 16$ ✓ Exact answer

$f(-6/5) = (6/5)^2 - (-6/5) + 4$
$= 36/25 + 6/5 + 4$
$= 166/25$

$f(-2/5) = (-2/5)^2 - (-2/5) + 4$
$= 4/25 + 2/5 + 4$
$= 114/25$

$f(2/5) = (2/5)^2 - (2/5) + 4$
$= 4/25 - 2/5 + 4$
$= 94/25$

$f(6/5) = (6/5)^2 - (6/5) + 4$
$= 36/25 - 6/5 + 4$
$= 106/25$

$w_1 = 1/4\, f(-6/5) = 1/4\,(166/25) = 59/50$
$w_2 = 1/4\, f(-2/5) = 1/4\,(114/25) = 47/50$
$w_3 = 1/4\, f(2/5) = 1/4\,(94/25) = 43/50$
$w_4 = 1/4\, f(6/5) = 1/4\,(106/25) = 51/50$

$Q = 4(59/50 + 47/50 + 43/50 + 51/50)$
$= 4(\ 4\ )$
$= 16$ ✓ checks out! $\quad 16 = 16$

4) $t\, dy/dt + 2y = 4t^2$
$y(1) = 2$

a) $y_{k+1} = y_k + \Delta t \cdot t_k \cdot y_k$ for $t_0 = 1$, $y_0 = 1$

b) $1^{st}\ \Delta t = 2/1 = 2$
$2^{nd}\ \Delta t = 1/2 = 1/2$
$y_{1+1} = y_1 + 1 \cdot t_1 \cdot y_1$
$= 2 + 1/2 \cdot 2 \cdot 1$
$= 2 + 1/2 \cdot 1$
$\boxed{= 2\tfrac{1}{2}}$

* I did not know this one exactly and had no time to get it done.

5) $u(x) \Rightarrow \left(1 - 1/4 \int_0^1 \frac{x u(t)}{x+t}\right)^{-1} = 0$

An approach to solving this equation numerically is by using any root finding technique we have learned. Since we want to fing where $u(x) = 0$, then lets just pick the newton approach to solve this systen. This approach, like many others, will also involve some linear algebra. To approximate the integral, we should use the quadature approach to do so, which will give us a good estimate for the integral. Throughout all of the work, we must compare with the exact answer to see how big our error is and if we need to adjust our approach or evaluations in any way. There is many ways to approach this problem numerically, but that is the one that came to mind. Thank you for the great semester.