# CS558 - HW 6

## 6.1

### (a)

Expecting Nat

$$\dfrac{\dfrac{\dfrac{}{n:\text{Nat} \vdash n:\text{Nat}}\,(\text{T-var})}{\emptyset \vdash \text{iszero } n:\text{Bool}}\,(\text{T-iszero})\quad \dfrac{n:\text{Nat} \vdash \text{false}:\text{Bool}}{\emptyset \vdash \text{pred false}:}\,(\text{T-pred})\quad \dfrac{\dfrac{}{n:\text{Nat} \vdash n:\text{Nat}}\,(\text{T-var})}{\text{pred } n:\text{Nat}}\,(\text{T-pred})}{\emptyset \vdash \lambda n:\text{Nat}.\,(\text{if (iszero } n) \text{ then (pred false) else (pred } n)):}\,(\text{T-if})$$

$$\dfrac{\dfrac{\dfrac{}{\emptyset \vdash 0:\text{Nat}}\,(\text{T-zero})}{\emptyset \vdash \text{succ } 0:\text{Nat}}\,(\text{T-succ})}{\emptyset \vdash \text{succ (succ } 0):\text{Nat}}\,(\text{T-succ})$$

$$\emptyset \vdash (\lambda n:\text{Nat}.\,(\text{if (iszero } n) \text{ then (pred false) else (pred } n))\ (\text{succ (succ } 0)) :$$

This term is not well-typed!

### (b)

**Step 1**

$$\dfrac{}{(\lambda n:\text{Nat}.\,(\text{if (iszero } n) \text{ then (pred false) else (pred } n)))\ (\text{succ (succ } 0)) \to \text{if iszero (succ (succ } 0)) \text{ then succ (succ (succ } 0)) \text{ else pred (succ (succ } 0))}\,(\text{E-App-Abs})$$

**Step 2**

$$\dfrac{\dfrac{}{\text{iszero (succ (succ } 0)) \to \text{false}}\,(\text{E-iszero-false})}{\text{if iszero (succ (succ } 0)) \text{ then succ (succ (succ } 0)) \text{ else pred (succ (succ } 0)) \to \text{if false then succ (succ (succ } 0)) \text{ else pred (succ (succ } 0))}\,(\text{E-if})$$

**Step 3**

$$\dfrac{}{\text{if false then succ (succ (succ } 0)) \text{ else pred (succ (succ } 0)) \to \text{pred (succ (succ } 0))}\,(\text{E-iffalse})$$

**Step 4**

$$\dfrac{}{\text{pred (succ (succ } 0)) \to \text{succ } 0}\,(\text{E-pred-succ})$$

Reduces to the value succ 0!

6.2

(a)

* let $\Gamma = p : Nat \to Bool$

$$\dfrac{\dfrac{\dfrac{\dfrac{(T\text{-}Var)}{\Gamma \vdash p : Nat \to Bool} \quad \dfrac{(T\text{-}Var)}{\Gamma \vdash 0 : Nat}}{(T\text{-}App)\ \Gamma \vdash p\ 0 : Bool} \quad \dfrac{(T\text{-}False)}{\Gamma : false : Bool} \quad \dfrac{(T\text{-}True)}{\Gamma : true : Bool}}{(T\text{-}If)\ \Gamma \vdash \lambda p : (Nat \to Bool).\ (if\ (p\ 0)\ then\ false\ else\ true) : Bool} \quad \dfrac{\dfrac{\dfrac{(T\text{-}Var)}{n : Nat \vdash n : Nat}}{n : Nat \vdash pred\ n : Nat}\ (T\text{-}Pred)}{\emptyset \vdash \lambda n : Nat.\ (iszero\ (pred\ n)) : Bool}\ (T\text{-}IsZero)}{(T\text{-}Abs)\ \emptyset \vdash (\lambda p : (Nat \to Bool).\ (if\ (p\ 0)\ then\ false\ else\ true))\ (\lambda n : Nat.\ (iszero\ (pred\ n))) : Bool}$$

This term <u>is</u> well-typed!

(b)

<u>step 1</u>

$$\dfrac{}{(\lambda p : (Nat \to Bool).\ (if\ (p\ 0)\ then\ false\ else\ true))\ (\lambda n : Nat.\ (iszero\ (pred\ n)))}\ (E\text{-}App\text{-}Abs)$$
$$\longrightarrow if\ (\lambda n : Nat.\ iszero\ (pred\ n))\ 0\ then\ false\ else\ true$$

<u>step 2</u>

$$\dfrac{\dfrac{}{(\lambda n : Nat.\ iszero\ (pred\ n))\ 0 \to iszero\ (pred\ 0)}\ (E\text{-}App\text{-}Abs)}{if\ (\lambda n : Nat.\ iszero\ (pred\ n))\ 0\ then\ false\ else\ true \to if\ iszero\ (pred\ 0)\ then\ false\ else\ true}\ (E\text{-}If)$$

<u>step 3</u>

$$\dfrac{\dfrac{\dfrac{}{pred\ 0 \to 0}\ (E\text{-}Pred\text{-}Zero)}{iszero\ (pred\ 0) \to iszero\ 0}\ (E\text{-}IsZero)}{if\ iszero\ (pred\ 0)\ then\ false\ else\ true \to if\ iszero\ 0\ then\ false\ else\ true}\ (E\text{-}If)$$

<u>step 4</u>

$$\dfrac{\dfrac{}{iszero\ 0 \to true}\ (E\text{-}IsZero\text{-}Zero)}{if\ iszero\ 0\ then\ false\ else\ true \to if\ true\ then\ false\ else\ true}\ (E\text{-}If)$$

<u>step 5</u>

$$\dfrac{}{if\ true\ then\ false\ else\ true \to false}\ (E\text{-}If\text{-}True)$$

Reduces to the value <u>true</u>!

6.3

(a)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{b:\mathrm{Bool} \vdash b:\mathrm{Bool}}\,(\text{T-Var})}{b:\mathrm{Bool} \vdash \mathrm{pred}\ b : X}\,(\text{T-pred})\ \leftarrow \text{Expecting Nat}}{b:\mathrm{Bool} \vdash \mathrm{succ}\ (\mathrm{pred}\ b) : X}\,(\text{T-succ})}{\emptyset \vdash \lambda b:\mathrm{Bool}.\ \mathrm{succ}\ (\mathrm{pred}\ b) : X}\,(\text{T-abs}) \qquad \dfrac{\dfrac{\dfrac{}{\emptyset \vdash 0 : \mathrm{Nat}}\,(\text{T-Var})}{\emptyset \vdash \mathrm{succ}\ 0 : \mathrm{Nat}}\,(\text{T-succ})}{\emptyset \vdash \mathrm{iszero}\ (\mathrm{succ}\ 0) : \mathrm{Bool}}\,(\text{T-iszero})}{\emptyset \vdash (\lambda b:\mathrm{Bool}.\ (\mathrm{succ}\ (\mathrm{pred}\ b)))\ (\mathrm{iszero}\ (\mathrm{succ}\ 0)) : X}\,(\text{T-abs})$$

This term is <u>not</u> well-typed!

(b)

step 1

$$\dfrac{}{\lambda b:\mathrm{Bool}.\ (\mathrm{succ}\ (\mathrm{pred}\ b)))\ (\mathrm{iszero}\ (\mathrm{succ}\ 0)) \rightarrow \mathrm{succ}\ (\mathrm{pred}\ (\mathrm{iszero}\ (\mathrm{succ}\ 0)))}\,(\text{E-app-abs})$$

step 2

$$\dfrac{\dfrac{\dfrac{}{\mathrm{iszero}\ (\mathrm{succ}\ 0) \rightarrow \mathrm{false}}\,(\text{E-iszero-succ})}{\mathrm{pred}\ (\mathrm{iszero}\ (\mathrm{succ}\ 0)) \rightarrow \mathrm{pred}\ \mathrm{false}}\,(\text{E-pred})}{\mathrm{succ}\ (\mathrm{pred}\ (\mathrm{iszero}\ (\mathrm{succ}\ 0))) \rightarrow \mathrm{succ}\ (\mathrm{pred}\ \mathrm{false})}\,(\text{E-succ})$$

step 3

$$\dfrac{\dfrac{?}{\mathrm{pred}\ \mathrm{false} \rightarrow ?}\ \leftarrow \text{no axiom can be used here}}{\mathrm{succ}\ (\mathrm{pred}\ \mathrm{false}) \rightarrow ?}\,(\text{E-succ})$$

This reduction is <u>stuck</u>!