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Homework 8

1) For the first problem we were asked to show that for the inverse power iteration that there are eigenvalues $(\lambda_j-\mu)$ that exist for $(A-\mu I)^{-1}$. To show that the eigenvalues of $(A-\mu I)^{-1}$ are $(\lambda_j - \mu)$ for $j=1,2,...,m$, where λ_j are the eigenvalues of A, we can use the following steps. First, we start with an eigenvector x of A with eigenvalue λ, and show that $(\lambda_j-\mu)$ is an eigenvalue of $(A-\mu I)$. Next, we consider the eigenvalue equation for $(A-\mu I)^{-1}$, and let y be an eigenvector of $(A-\mu I)^{-1}$ with corresponding eigenvalue μ. We then substitute $(A-\mu I)^{-1}y$ for y in the equation y = $\mu(A-\mu I)y$, and simplify to get $(A-\mu I)^{-1}y = \mu(\lambda_j-\mu)^{-1}x_j$, where x_j is the eigenvector of A corresponding to λ_j. This shows that $(\lambda_j$-u) is an eigenvalue of $(A-\mu I)^{-1}$ with eigenvector x_j, and thus the eigenvalues of $(A-\mu I)^{-1}$ are $(\lambda_j-\mu)$ for $j=1,2,...,m$, where λ_j are the eigenvalues of A. This relationship between the eigenvalues of $(A-\mu I)^{-1}$ and A is useful for finding eigenvalues of A close to a given value μ using the inverse power iteration. My handwritten work is shown in the next two pages.

1) Let $A \in \mathbb{R}^{m \times m}$ have eigenvalues λ_j for $j = 1, 2, \ldots, m$. Let $\mu \neq \lambda_j$ for $j = 1, 2, \ldots, m$. Show that eigenvalues of $(A - \mu I)^{-1}$ are $(\lambda_j - \mu)$ for $j = 1, 2, \ldots, m$.

$$
\begin{aligned}
(A - \mu I) &= (A - \mu I)x_j \\
&= Ax_j - \mu I x_j \\
&= Ax_j - \mu x_j \\
&= \lambda_j x_j - \mu x_j \\
&= (\lambda_j - \mu) x_j \checkmark \text{DONE}
\end{aligned}
$$

Therefore $(A - \mu I)$ has eigenvalues $(\lambda_j - \mu)$ and corresponding eigenvectors x_j for $j = 1, 2, \ldots, m$.

What about $(A - \mu I)^{-1}$?

Let x_j be the eigenvectors for the corresponding eigenvalues λ_j for $j = 1, 2, \ldots, m$.

$$(A-\mu I)^{-1} = \mu x$$

$$(A-\mu I)(A-\mu I)^{-1} x_j = \mu x_j (A-\mu I)$$

$$x_j = \mu x_j (A-\mu I)$$

$$x_j = \mu((A-\mu I)^{-1} x_j)(A-\mu I)$$

$$x_j = \mu(A-\mu I)(A-\mu I^{-1}) x_j$$

$$(A-\mu I)^{-1} x_j = \mu(A-\mu I)\cancel{x_j}$$

$$(A-\mu I)^{-1} x_j = \mu(\lambda_j - \mu)^{-1} x_j$$

$$(A-\mu I)^{-1} x_j = (\lambda_j - \mu) x_j$$

$$(A-\mu I)^{-1} = (\lambda_j - \mu) \ \checkmark \text{DONE}$$

Therefore $(A-\mu I)^{-1}$ has the eigenvalues $(\lambda_j - \mu)$ and corresponding eigenvectors x_j for $j = 1, 2, \ldots, m$

$$(A-\mu I) = (A-\mu I) x_j$$

$$= Ax_j - \mu I x_j$$

$$= Ax_j - \mu x_j$$

$$= \lambda_j x_j - \mu x_j$$

$$= (\lambda_j - \mu) x_j \ \checkmark \text{DONE}$$

2) Next, we were asked a question about the inverse power method. Specifically, we were asked to explain how it can be used to find the smallest eigenvalue in magnitude for matrix A. First, We would want to set our μ value to 0 to find the smallest eigenvalue. This is because the smallest eigenvalue in magnitude of A is always closer to zero than any other eigenvalue. Basically, we would make sure that the algorithm will converge to the eigenvalue that we are interested in. The algorithm then starts by making an initial guess for the eigenvalue, which should be smaller than any eigenvalue of A. Then, we solve a system of equations and normalize the resulting vector to find the new estimate for the eigenvalue. We repeat this process with the new estimate until it converges to the desired eigenvalue. Once we have the eigenvalue estimate, we can find the corresponding eigenvector. The initial guess is important and should be chosen carefully to ensure the algorithm converges to the desired eigenvalue. This is how we would find the smallest eigenvalue in magnitude for matrix A.

3) For this question, we are given the rules for QR iteration with shifts. We were asked to show that the matrices *A(k)* generated in the QR iteration with shifts are similar to each other. First, we know that RQ and QR are always similar matrices since $RQ = Q^{-1}(QR)Q$ so they have the same eigenvalues. That is one way to show that the matrices generated are similar to each other. I took another approach if that was not sufficient enough. To show this we need to demonstrate that there exists an invertible matrix P such that $A^{(k+1)} = P^{-1} * A^{(k)} * P$ for all k. My handwritten work is shown on the next page.

3) QR iteration with shifts is

Choose a shift $\mu^{(k)}$

$A^{(k)} - \mu^{(k)} = Q^{(k)} R^{(k)}$

$A^{(k+1)} = R^{(k)} Q^{(k)} + \mu^{(k)} I$

Show that the matrices $A^{(k)}$ generated in the QR iteration with shifts are similar to each other.

$A^{(k+1)} = (P^{(k)})^{-1} A^{(k)} P^{(k)}$

where $P^{(k)}$ is an invertible matrix.

$$A^{(k+1)} = R^{(k)} Q^{(k)} + \mu^{(k)} I$$

$$(Q^{(k)})^{-1} A^{(k+1)} = (R^{(k)} Q^{(k)} + \mu^{(k)} I)(Q^{(k)})^{-1}$$

$$Q^{(k)} (Q^{(k)})^{-1} A^{(k+1)} = (R^{(k)} Q^{(k)} + \mu^{(k)} I)(Q^{(k)})^{-1}($$

$$Q^{(k)} A^{(k+1)} (Q^{(k)})^{-1} = Q^{(k)} R^{(k)} (Q^{(k)})^{-1} + \mu^{(k)} I$$

$\Rightarrow$ Let $P^{(k)} = Q^{(k)} R^{(k)} (Q^{(k)})^{-1}$ for all k

$$Q^{(k)} A^{(k+1)} (Q^{(k)})^{-1} = P^{(k)} + \mu^{(k)} I$$

$$A^{(k+1)} (Q^{(k)})^{-1} = (P^{(k)} + \mu^{(k)} I)(Q^{(k)})^{-1}$$

$$A^{(k+1)} (Q^{(k)})^{-1} = (Q^{(k)})^{-1} P^{(k)} + \mu^{(k)} I$$

$$A^{(k+1)} = (Q^{(k)})^{-1} P^{(k)} + \mu^{(k)} I \, Q^{(k)}$$

$$A^{(k+1)} = (Q^{(k)})^{-1} (Q^{(k)} R^{(k)} (Q^{(k)})^{-1}) + \mu^{(k)} I Q^{(k)}$$

$$A^{(k+1)} = Q^{(k)} Q^{(k)} R^{(k)} Q^{(k)} Q^{(k)} + \mu^{(k)} Q^{(k)} Q^{(k)}$$

$$A^{(k+1)} = R^{(k)} + \mu^{(k)} I$$

$$(P^{(k)})^{-1} A^{(k+1)} = (R^{(k)} + \mu^{(k)} I)(P^{(k)})^{-1}$$

$$A^{(k+1)} = P^{(k)} (R^{(k)} + \mu^{(k)} I)(P^{(k)})^{-1}$$

$$A^{(k+1)} = (P^{(k)})^{-1} (R^{(k)} + \mu^{(k)} I) P^{(k)}$$

$$A^{(k+1)} = (P^{(k)})^{-1} A^{(k)} P^{(k)} \quad \checkmark \text{DONE}$$

Therefore the matrices $A^{(k)}$ generated in the QR iteration with shifts are similar to each other.

4) For this question, we are asked to use the SVD of A and A^T to show that AA^T has squared singular values or eigenvectors on the diagonal of Σ and its associated eigenvectors in the matrix U. Here I used the rules for each singular value decomposition for A and A^T and multiplied them both together. I found that $\Sigma\Sigma^T$ is a square diagonal matrix containing the squared singular values of A and the vectors U and U^T are the associated eigenvectors. On the next page, you can see my work for this problem.

4) Let $A = U\Sigma V^T$ ($A \in \mathbb{R}^{m\times n}$) b the SVD of A and $m \geq n$, with singular values $\sigma_1 > \sigma_2 \geq \ldots \geq \sigma_n \geq 0$. Show that AA^T has singular values $\sigma_1^2, \sigma_2^2 \ldots \sigma_n^2$ and that the columns of U are eigenvectors of AA^T associated with these eigenvalues.

$$
\begin{aligned}
AA^T &= (U\Sigma V^T)(V\Sigma^T U^T) \\
&= U\Sigma V^T V \Sigma^T U^T \\
&= U\Sigma I \Sigma^T U^T \\
&= U\Sigma\Sigma^T U^T
\end{aligned}
$$

We know that the matrix

$\Sigma\Sigma^T \in \mathbb{R}^{n\times n}$

is a diagonal matrix with the values on the diagonal being:

$\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$

followed by $n-m$ zeros since $m \geq n$.

We also see the associated eigenvectors U withe the corresspanding eigenvalues in Σ.

Therefore, AA^T does have eigenvalues $\sigma_1^2, \sigma_2^2 \ldots \sigma_n^2$ with associated eigenvectors U. ✓

5) In this question we are asked to prove that the condition numbers for both *A* and A^TA are related to the singular values in the SVD decomposition of *A*. For part (a), I inferred to facts that all singular values are non-negative, singular matrix rules, and proved with a counter example. The condition number of a matrix A is defined as the product of the norm of A and the norm of its inverse, $||A||\cdot||A^{-1}||$. I used this fact for the proof in part (b). Proving the fact that $||A||_2 = \sigma_1$ and $||A^{-1}||_2 = 1/\sigma_m$, where σ_1 and σ_m are the largest and smallest singular values of A, respectively, which can help prove that the condition number of A is equal to σ_1/σ_m. This is the same approach I used in the part (c), where $||A^TA||\cdot||(A^TA)^{-1}||$ is equal to $(\sigma_1/\sigma_n)^2$ where $||A^TA|| = \sigma_1^2$ and $||(A^TA)^{-1}|| = \sigma_m^2$. That is how I did each part of the proof and my handwritten work is shown below and on the next four pages.

5) Let $A \in \mathbb{R}^{m \times m}$ be investible. Let σ_{max} be the largest singular value of A and σ_{min} the smallest

Show that:

(a) $\sigma_{min} > 0$

Singular values are non-negative square roots of the eigenvalues of A^TA. To prove $\sigma_{min} > 0$, we must show A^TA is investible.

Lets suppose:

$$A^TA x = 0 \quad (A^TA \text{ is not investible})$$

where $x \in \mathbb{R}^m,\ x \neq 0$

Therefore:

$$x^T A^T A x = 0$$

$$(Ax)^T (Ax) = 0$$

$$||Ax||_2^2 = 0$$

$$Ax = 0$$

Which implies that A is singular, and we know/assumed that it is not.

Therefore A^TA is invertible and $\sigma_{min} > 0$.

(b) $\text{cond}_2(A) = \sigma_{max}/\sigma_{min}$

First, we should show that:

$\|A\|_2 = \sigma_{max}$

$$\|A\|_2 = \|A\|_2^2$$
$$= \max_{x:\|x\|_2=1} \|Ax\|_2^2$$
$$= \max_{x:\|x\|_2=1} x^T A^T A x$$
$$= \max_{x:\|x\|_2=1} x^T (V\Sigma^T U^T)(U \Sigma V^T) x$$
$$= \max_{x:\|x\|_2=1} x^T V \Sigma^T \Sigma V^T x$$

Let $y = V^T x$

$$= \max_{y:\|y\|_2=1} y^T \Sigma^T \Sigma y$$
$$= \max_{y:\|y\|_2=1} \sum_{i=1}^{m} \sigma_i^2 (A) y_i^2$$
$$= \| \max_{y:\|y\|_2=1} \Sigma_{i=1} \sigma_i^2 y_i^2 \|_2$$
$$= \| \Sigma_{i=1}^{m} \sigma_i^2 \| \cdot \| y^2 \|_2$$
$$= \sigma_1 \cdot 1$$
$$= \sigma_{max} \checkmark \text{DONE}$$

Second, we should show that:

$||A^{-1}||_2 = 1/\sigma_{min}$

$||A^{-1}||_2 = ||A^{-1}x||_2^2$

$= x^T(A^{-1})^T A^{-1} x$

$= x^T(V\Sigma^{-1}U^T)(U\Sigma V^T)^{-1}(V\Sigma^{-1}U^T) A^T A x$

$= x^T V\Sigma^{-1}U^T U^{-1}\Sigma^{-1}(V^T)^{-1} V\Sigma^{-1}U^T A^T A x$

$= x^T V\Sigma^{-1}\Sigma^{-1}V^T x$

$= ||\Sigma^{-1}V^T x||_2^2$

Let $y = V^T x$

$= ||\Sigma^{-1} y||_2^2$

$= ||\Sigma_{i=1}^{M} \sigma_i^2 y_i^2||_2^2$

$= ||\Sigma_{i=1}^{M} \sigma_i y_i||_2$

$= ||1/\sigma_m \cdot y_i||_2$

$= 1/\sigma_m \; ||y_i||_2$

$= 1/\sigma_m \cdot 1$

$= 1/\sigma_{min}$ ✓ DONE

Therefore,

$||A||_2 = \sigma_{max}$, and

$||A^{-1}||_2 = 1/\sigma_{min}$

Futhermore,

$cond_2 = ||A||_2 \, ||A^{-1}||_2$

$cond_2 = \sigma_{max}/\sigma_{min}$ ✓

(c) $\text{cond}_2(A^TA) = \sigma^2_{max}/\sigma^2_{min}$

First we should show:

$$\|A^TA\|_2 = \sigma^2_{max}$$

$$\|A^TA\|_2 = \|A^TAx\|_2^2$$
$$= x^T(A^TA)^T(A^TA)x$$
$$= x^T V\Sigma^2V^T)^T(V\Sigma^2V^T)x$$
$$= (V^Tx)^T\Sigma^2\Sigma^2(V^Tx)$$
$$= y^T\Sigma^4 y$$
$$= \|y^T\Sigma^2 y\|_2^2$$
$$= \|\Sigma^2 y\|_2$$
$$= \|\Sigma_{i=1}^m \sigma_i^2 y_i^2\|_2$$
$$= \sigma_1^2\|\Sigma_{i=1}^m y_i^2\|_2$$
$$= \sigma_1^2 \|y^2\|_2$$
$$= \sigma_1^2 I$$
$$= \sigma_1^2$$
$$= \sigma^2_{max} \checkmark \text{ DONE}$$

Second we should show that:

$\|(A^TA)^{-1}\|_2 = 1/\sigma^2_{min}$

$$\|(A^TA)^{-1}\|_2 = \|(A^TA)^{-1}x\|_2^2$$
$$= x^T(A^TA)^T(A^TA)x$$
$$= x^T(V\Sigma^2V^T)^T(V\Sigma^2V^T)x$$
$$= (V^Tx)^T\Sigma^4(V^Tx)$$
$$= y^T\Sigma^4 y$$
$$= \|y^T\Sigma^2 y\|_2^2$$
$$= 1/\|\Sigma_{i=1}^m \sigma_i^2 y_i^2\|_2$$
$$= 1/\|\sigma_{min}\Sigma_{i=1}^m y_i^2\|_2$$
$$= 1/\sigma_m \cdot \|y^2\|_2$$
$$= 1/\sigma_m \cdot \underline{1}$$
$$= 1/\sigma_{min}\ \checkmark \text{ DONE}$$

Therefore,

$\|A^TA\|_2 = \sigma^2_{max}$

and

$\|(A^TA)^{-1}\|_2 = 1/\sigma^2_{min}$

Furthermore,

$cond_2(A^TA) = \sigma^2_{max}/\sigma^2_{min}$

✓

6) The final question asks us to derive the SVD of the given decomposition $A = CDE$. This given decomposition looks like an SVD of the matrix A already and I will be assuming that it is. We were asked to first find another SVD decomposition of A. To solve part (a) I then used a SVD deriving technique by finding the eigenvalues and eigenvectors of A and A^T. After finding both, I was able to derive both the U and Σ. Now to get the V matrix, I needed to multiply 1 over the eigenvalue found to the original matrix A and the associated eigenvector found in U. Next, on part (b), I first found the range on both A and A^T by using the reduced row echelon form to find the column space of A. The column space of A is the span of the columns of A. In this case, I specifically found where my pivots lie and if there exists an existing value that satisfies the criteria for being in the basis of the range and decided if they were a part of the basis. To find the basis of the nullspace of both A and A^T, I solved the system for a vector x and a y vector full of 0's. This allowed me to find out if there exists a vector that satisfies the criteria for null space and then I added it to the basis if it does. On the next five pages, you will be able to see my handwritten work.

6) We are given the following decomposition for a matrix A:

$$A = CDE$$

Where

$$C = \begin{bmatrix} -2/3 & 0 & 5/3\sqrt{5} \\ 1/3 & 2/\sqrt{5} & 2/3\sqrt{5} \\ 2/3 & -1/\sqrt{5} & 4/3\sqrt{5} \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$E = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

C and E are orthogonal matrices.

(a) Find a SVD for A:

$$A = \begin{bmatrix} -2/3 & 0 & 5/3\sqrt{5} \\ 1/3 & 2/\sqrt{5} & 2/3\sqrt{5} \\ 2/3 & -1/\sqrt{5} & 4/3\sqrt{5} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & \sqrt{5} \\ 4/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 4/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 2 \end{bmatrix} \in \mathbb{R}^{3\times 2}, \quad m = 3, \; n = 2$$

$A_{3\times 2} = U_{3\times 3}\Sigma_{3\times 2}V^T_{2\times 2}$

First, find eigenvalues of A^TA

$$A^TA = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}$$

$$\det(A^TA - \lambda I) = (5-\lambda)(8-\lambda) - 4$$
$$= \lambda^2 - 13\lambda + 40 - 4$$
$$= \lambda^2 - 13\lambda + 36$$
$$= (\lambda - 9)(\lambda - 4)$$

associated $\lambda_1 = 9$ $\lambda_2 = 4$ associated

$x_1 = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$ $x_2 = \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$ $x_3 = \begin{bmatrix} -2/3 \\ 1/3 \end{bmatrix}$ $x_4 = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$

Next, find eigenvalues of AA^T

$$AA^T = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 2 & 4 \\ 2 & 4 & 0 \\ 4 & 0 & 4 \end{bmatrix}$$

$$\det(AA^T - \lambda I) = (5-\lambda)(16-\lambda)$$
$$= -\lambda^3 - 13\lambda^2 + 36\lambda$$
$$= \lambda(\lambda - 4)(\lambda - 9)$$

$\lambda_1 = 0$ $\lambda_2 = 4$ $\lambda_3 = 9$

$x_5 = \begin{bmatrix} -1 \\ 1/2 \\ 1 \end{bmatrix}$ $x_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ $x_6 = \begin{bmatrix} 5/4 \\ 1/2 \\ 1 \end{bmatrix}$

Singular values are equal to the square root of the eigenvalues that we found.

$$\sigma_1 = \sqrt{\lambda_1} \qquad \sigma_2 = \sqrt{\lambda_2}$$
$$\sigma_1 = \sqrt{0} \qquad \sigma_2 = \sqrt{4}$$
$$\sigma_1 = 0 \qquad \sigma_2 = 2$$

$$\sigma_3 = \sqrt{\lambda_3}$$
$$\sigma_3 = \sqrt{9}$$
$$\sigma_3 = 3$$

$$\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \qquad U = \begin{bmatrix} \sqrt{5}/3 & 0 & -2/3 \\ 2\sqrt{5}/15 & -2\sqrt{5}/5 & 1/3 \\ 4\sqrt{5}/15 & \sqrt{5}/5 & 2/3 \end{bmatrix}$$

$$v_1 = 1/3 \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{5}/3 \\ 2\sqrt{5}/15 \\ 4\sqrt{5}/15 \end{bmatrix} = \begin{bmatrix} \sqrt{5}/5 \\ 2\sqrt{5}/5 \end{bmatrix}$$

$$v_2 = 1/2 \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -2\sqrt{5}/5 \\ \sqrt{5}/5 \end{bmatrix} = \begin{bmatrix} -2\sqrt{5}/5 \\ \sqrt{5}/5 \end{bmatrix}$$

$$V = \begin{bmatrix} \sqrt{5}/5 & -2\sqrt{5}/5 \\ 2\sqrt{5}/5 & \sqrt{5}/5 \end{bmatrix}$$

$$SVD(A) = U \Sigma V^T$$
$$= \begin{bmatrix} \sqrt{5}/3 & 0 & -2/3 \\ 2\sqrt{5}/15 & -2\sqrt{5}/5 & 1/3 \\ 4\sqrt{5}/5 & \sqrt{5}/5 & 2/3 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{5}/5 & -2\sqrt{5}/5 \\ 2\sqrt{5}/5 & \sqrt{5}/5 \end{bmatrix}$$

(b) Write down the basis for the following subspaces:

(i) Range of A

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Row echelon form

Two pivots after replaced in REF

$$= \begin{bmatrix} 1 & 2 \\ 0 & -4 \\ 0 & 0 \end{bmatrix}$$

therefore,

$$\text{range}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \right\}$$

(ii) Range of A^{-1}

$$A^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

Two pivots after REF

$$= \begin{bmatrix} 1 & 2 & 0 \\ 0 & -4 & 2 \end{bmatrix}$$

therefore,

$$\text{range}(A^{-1}) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$$

(iii) Nullspace of A

$$\begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 2 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

therefore,

$$\text{null}(A) = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

since $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(iv) Null space of A^{-1}

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1/2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1/2 & 0 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1/2 \\ 1 \end{bmatrix}$$

therefore,

$$\text{null}(A^{-1}) = \text{span}\left\{\begin{bmatrix} -1 \\ 1/2 \\ 1 \end{bmatrix}\right\}$$