MATH 504 / CS 575 — HOMEWORK # 8 Due : April 20

The format for submitting the HW is detailed in the syllabus. Please follow that format precisely. Show all your work to receive full credit.

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1. Let $A \in \mathbb{R}^{m\times m}$ have eigenvalues $\lambda_j, \quad j = 1, 2, \ldots, m$. Let $\mu \neq \lambda_j, j = 1, 2, \ldots, m$. Show that eigenvalues of $(A - \mu I)^{-1}$ are $(\lambda_j - \mu)^{-1}, \quad j = 1, 2, \ldots, m$.

2. Inverse power method can be used to find the smallest in magnitude eigenvalue of a matrix A. Explain how (that is, what would you set μ in Inverse iteration to?)

3. QR iteration with shifts is

$$\begin{aligned} &\text{Choose a shift } \mu^{(k)} \\ &A^{(k)} - \mu^{(k)} = Q^{(k)} R^{(k)} \\ &A^{(k+1)} = R^{(k)} Q^{(k)} + u^{(k)} I \end{aligned}$$

   Show that the matrices $A^{(k)}$ generated in the QR iteration with shifts are similar to each other.

4. Let $A = U\Sigma V^T$ be the SVD of $A \in \mathbb{R}^{m\times n}$, $m \geq n$, with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$. Show that AA^T has singular values $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$ and that the columns of U are eigenvectors of of AA^T associated with these eigenvalues.

5. Let $A \in \mathbb{R}^{m\times m}$ be invertible. Let σ_{max} be the largest singular value of A and σ_{min} the smallest. Show that,

   (a) $\sigma_{min} > 0$
   (b) $\|A\|_2 = \sigma_{max}$ and $\|A^{-1}\|_2 = \sigma_{min}$ and hence $\kappa_2(A) = \sigma_{max}/\sigma_{min}$
   (c) $\kappa_2(A^T A) = \sigma_{max}^2/\sigma_{min}^2$ (Recall $\kappa_2(A)$ denotes the 2-norm condition number of A.)

6. We are given the following decomposition for a matrix A

$$A = CDE$$

where

$$C = \begin{pmatrix} -\frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \\ \frac{1}{3} & \frac{2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} \\ \frac{2}{3} & -\frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad E = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$$

Note that C and E are orthogonal matrices.

(a) Find a singular value decomposition for A

(b) Write down the bases for the following subspaces:
(i) Range of A (ii) Range of A^T (iii) Nullspace of A (iv) Nullspace of A^T