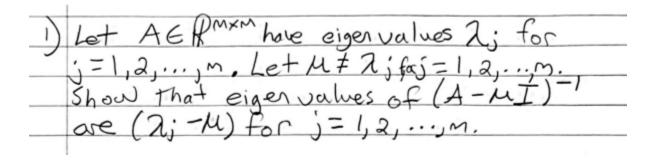
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## Homework 8

1) For the first problem we were asked to show that for the inverse power iteration that there are eigenvalues  $(\lambda_j - \mu)$  that exist for  $(A - \mu I)^{-1}$ . To show that the eigenvalues of  $(A - \mu I)^{-1}$  are  $(\lambda_j - \mu)$  for j = 1, 2, ..., m, where  $\lambda_j$  are the eigenvalues of A, we can use the following steps. First, we start with an eigenvector x of A with eigenvalue  $\lambda$ , and show that  $(\lambda_j - \mu)$  is an eigenvalue of  $(A - \mu I)$ . Next, we consider the eigenvalue equation for  $(A - \mu I)^{-1}$ , and let y be an eigenvector of  $(A - \mu I)^{-1}$  with corresponding eigenvalue  $\mu$ . We then substitute  $(A - \mu I)^{-1}y$  for y in the equation  $y = \mu(A - \mu I)y$ , and simplify to get  $(A - \mu I)^{-1}y = \mu(\lambda_j - \mu)^{-1}x_j$ , where  $x_j$  is the eigenvector of A corresponding to  $\lambda_j$ . This shows that  $(\lambda_j - \mu)$  is an eigenvalue of  $(A - \mu I)^{-1}$  with eigenvector  $x_j$ , and thus the eigenvalues of  $(A - \mu I)^{-1}$  are  $(\lambda_j - \mu)$  for j = 1, 2, ..., m, where  $\lambda_j$  are the eigenvalues of A. This relationship between the eigenvalues of  $(A - \mu I)^{-1}$  and A is useful for finding eigenvalues of A close to a given value  $\mu$  using the inverse power iteration. My handwritten work is shown in the next two pages.



$(A-\mu I) = (A-\mu I) \times j$
= Ax; -MIx;
$= A_{x_i} - \mu_{x_i}$ $= \lambda_i \times_i - \mu_{x_i}$
$= \lambda_j \times_j - \mu_{\chi_j}$ $= (\lambda_j - \mu_j) \times_j \text{ fork}$
Therefore (A-MI) has eigenvalues (7:5-14) and corresponding eigenvectors
Xi for 1=12
xj for j=1,2,,m. What about (A-MI)?

Le	+ X; be the eigen vectors for the corresponding
eig	gen values 2; for j=1,2,, m. (A-uI)-1 = ///x
-	(A-MI)-1 = MX
	(A-NI) (A-MI) - X = MX (A-MI)
-	$x = \mu_{x}(A - \mu I)$
	$x_j = \mathcal{N}((A - \mathcal{N}I)^{-1}x_j)(A - \mathcal{N}I)$
-	$x_{j} = \lambda \lambda (A - \lambda \lambda I) (A - \lambda \lambda I^{-1}) x_{j}$
	(A-4I) x; = MA-MI) x;
-	(A-MI)-1x; = M(2; -M)-1x;
	(A-MI)-1xi=(2)-1x)xj
	(A-MI)-1= (2; -M) DONE
T	herefore (A-MI)-1 has the eigenvalues
10	2; -m) and corresponding eigenvectors
1	; for j=1,2,,M
,	
	$(A - \mu I) = (A - \mu I) \times$
	= Ax; -MIx;
	= Ax; ~ Mx;
	$=\lambda_i x_i - \mu x_i$
	= (2; -M) ×; Jank

2) Next, we were asked a question about the inverse power method. Specifically, we were asked to explain how it can be used to find the smallest eigenvalue in magnitude for matrix *A*. First, We would want to set our  $\mu$  value to 0 to find the smallest eigenvalue. This is because the smallest eigenvalue in magnitude of *A* is always closer to zero than any other eigenvalue. Basically, we would make sure that the algorithm will converge to the eigenvalue that we are interested in. The algorithm then starts by making an initial guess for the eigenvalue, which should be smaller than any eigenvalue of *A*. Then, we solve a system of equations and normalize the resulting vector to find the new estimate for the eigenvalue. We repeat this process with the new estimate until it converges to the desired eigenvalue. Once we have the eigenvalue estimate, we can find the corresponding eigenvector. The initial guess is important and should be chosen carefully to ensure the algorithm converges to the desired eigenvalue. This is how we would find the smallest eigenvalue in magnitude for matrix *A*.

3) For this question, we are given the rules for QR iteration with shifts. We were asked to show that the matrices A(k) generated in the QR iteration with shifts are similar to each other. First, we know that RQ and QR are always similar matrices since  $RQ = Q^{-1}(QR)Q$  so they have the same eigenvalues. That is one way to show that the matrices generated are similar to each other. I took another approach if that was not sufficient enough. To show this we need to demonstrate that there exists an invertible matrix P such that  $A^{(k+1)} = P^{-1} * A^{(k)} * P$  for all k. My handwritten work is shown on the next page.

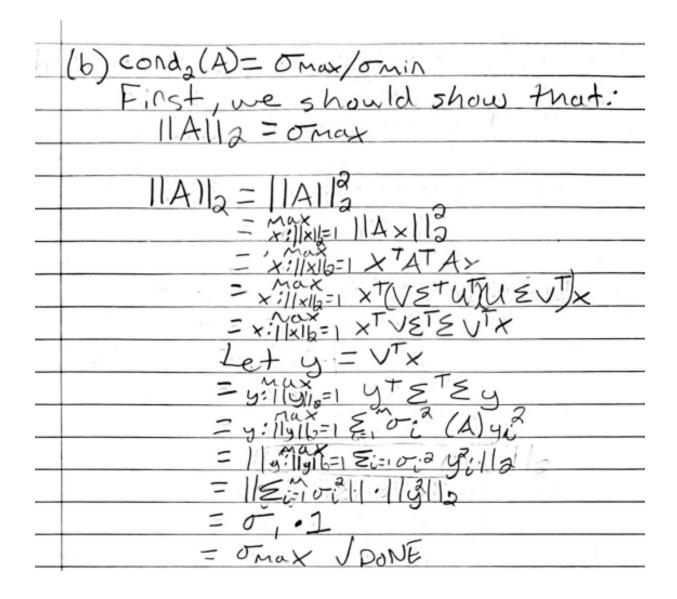
3) QR iteration with shifts is
Choose a Shift u(k)
Choose a Shift M(k) $A(k) - M(k) = Q(k) R(k)$
A(K+1) = R(k)Q(k) + M(k)I
Show that the matrices A(x) generated
in the QR iteration with shifts are
similar to each other.
A(K+1) = (PH)-1(K) P(K)
- (F.)A
where Pis an investible matrix.
A(E+1) - A(E) A(E) - (E) T
A(k+1) = R(k)Q(k) + M(k) [
(QE)-A(E+1) = (R(E)Q(E)+ /L(H)I)(Q(E)-
Q(k)(Q(k))-1 A(k+1) = (R(k)Q(k) + M(k) I)(Q(k))-1(
Q(K) A(K+1) (Q(K)) -1 = Q(K) R(K) (Q(K)) -1 + M(K) I
=> Let P= Q(k) R(k) (Q(k))-1 for all k/k=
Q(k)A(k+1)(Q(k))-1-P(k)+M(k)I
A(k+1) (Q(k))-1=(P(k) + M(k)T)(Q(k))-1
A(H)(Q(E))-1=(Q(H))-1p(E)+M(E)I
$A^{(k+1)} = (Q^{(k)})^{-1} P^{(k)} + \mathcal{U}^{(k)} T Q^{(k)}$ $A^{(k+1)} = (Q^{(k)})^{-1} Q^{(k)} Q^{(k)} (Q^{(k)})^{-1} + \mathcal{U}^{(k)} T Q^{(k)}$ $A^{(k+1)} = Q^{(k)} Q^{(k)} Q^{(k)} Q^{(k)} Q^{(k)} + \mathcal{U}^{(k)} Q^{(k)} Q^{(k)}$ $A^{(k+1)} = R^{(k)} + \mathcal{U}^{(k)} T$
A(K+1) = (Q(+))-1/Q(+) R(+)(Q(+))-1)+N(*IQ(+)
A(+1)- O(K) O(K) P(K) O(K) O(K) + M(K) O/KO(K)
A(+1) = R(+) + M(+)I
$ \frac{(p(k))^{-1}A^{(k+1)} = (P^{(k)} + M^{(k)}I)(P^{(k)})^{-1}}{A^{(k+1)} = P^{(k)}(P^{(k)} + M^{(k)}I)(P^{(k)})^{-1}} $ $ A^{(k+1)} = (P^{(k)})^{-1}(P^{(k)} + M^{(k)}I)(P^{(k)})^{-1} $ $ A^{(k+1)} = (P^{(k)})^{-1}A^{(k)}P^{(k)} $
A(++1) = P(+)/R(+)+M(+) [)(P(+))~1
A(++)=/P(x))-1/R(x)+M(x)T) P(x)
A(++1) = (P(+))-1 A(+) P(+)
ALL "= (PLS) ALL PLA DONE
Therefore the motives A(K) consented in
Therefore the matrices A(K) generated in the QR itoution with shifts are similar
to each other.
TO EVEN OTHER.

4) For this question, we are asked to use the SVD of A and  $A^T$  to show that  $AA^T$  has squared singular values or eigenvectors on the diagonal of  $\Sigma$  and its associated eigenvectors in the matrix U. Here I used the rules for each singular value decomposition for A and  $A^T$  and multiplied them both together. I found that  $\Sigma\Sigma^T$  is a square diagonal matrix containing the squared singular values of A and the vectors U and  $U^T$  are the associated eigenvectors. On the next page, you can see my work for this problem.

of A and M Zn, with singular
of A and M = n. with singular
values o, > 022 2 on 20. Show
that AAT has singular values of , 500
au that the columns of U are eigenvotors
of AAT associated with these eigenvolves.
1 AT = (1. T) ( - 1 + 1 T)
$AA = (u \leq v^{T})(v \leq t u^{T})$
= UZVTVZTUT
= USIETUT
= UEETUT
We know that the matrix
ESTE PARA
is a diagonal matrix withthe volve
on the diagonal being:
0, 0, 0, , 50
Followed by n-m zeros since
$M \ge \Lambda$ .
We also see the associated eigenvectors
U withe the corresponding eigenvolves
IN S.
Therefore, AAT doss have eigenvalues
of of of with associated
eigenvectors U.

5) In this question we are asked to prove that the condition numbers for both A and  $A^TA$  are related to the singular values in the SVD decomposition of A. For part (a), I inferred to facts that all singular values are non-negative, singular matrix rules, and proved with a counter example. The condition number of a matrix A is defined as the product of the norm of A and the norm of its inverse,  $||A|| \cdot ||A^{-1}||$ . I used this fact for the proof in part (b). Proving the fact that  $||A||_2 = \sigma_1$  and  $||A^{-1}||_2 = 1/\sigma_m$ , where  $\sigma_1$  and  $\sigma_m$  are the largest and smallest singular values of A, respectively, which can help prove that the condition number of A is equal to  $\sigma_1$   $\sigma_2$ . This is the same approach I used in the part (c), where  $||A^TA|| \cdot ||(A^TA)^{-1}||$  is equal to  $|(\sigma_1/\sigma_n)^2|$  where  $||A^TA|| = \sigma_1^2$  and  $||(A^TA)^{-1}|| = \sigma_m^2$ . That is how I did each part of the proof and my handwritten work is shown below and on the next four pages.

5) Let AERMAM be invertible. Let
Onax bette largest singular value of
A and omin the smallost
Show that:
(a) omin >0
Singular values are non-regulive
Square roots of the eigenvalues of
ATA. To prove omin >0, we
Must show ATA is investible.
Lets suppose:
ATAX=0 (ATA is not invertible)
where x & RM, x ≠0
Therefore:
×TATA× =0
$(A\times)^T(A\times)=0$
$\frac{  A \times   _{\alpha}^{2} = 0}{  A \times   _{\alpha}^{2}}$
$A \times = 0$
which implies that A is singular, and we know/assumed that it is
and we know/assumed that it is
not.
Therefore ATA is invotible and
Jmin >0.

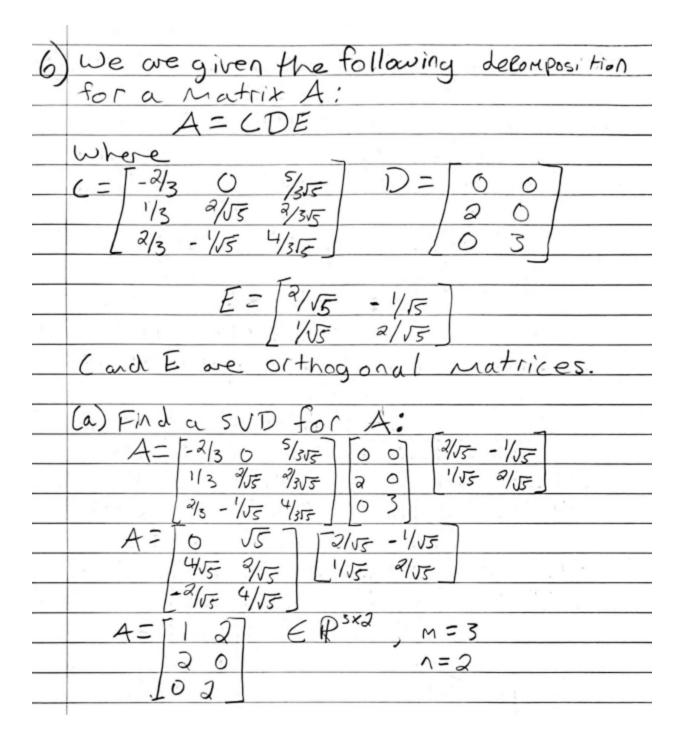


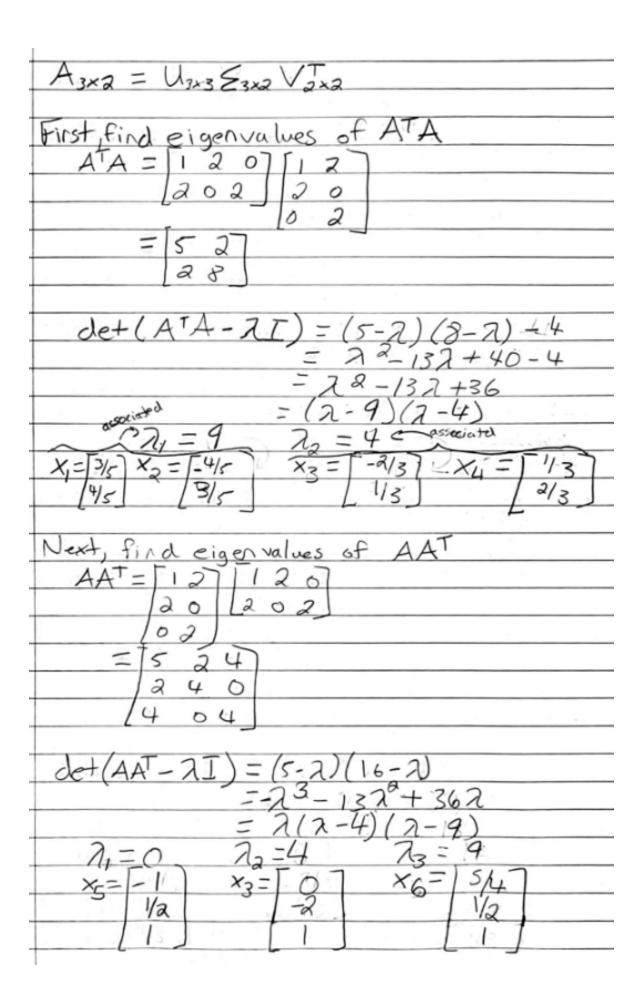
Second, we should show the	ut:
11 A-11 /2 = 1/0 min	
$  A^{-1}  _2 -   A^{-1} \times   _2$ = $\times [A^{-1}]^T A \times -$	
= × 7/4-1) TA-1 × _	
=xT(VZTUT)(UEV))-1	(UE-IU) ATAX
= xT V E - 1 u Tu - 1 E - W	5-1VE-1UTATAX
- xT\15-15-1\1TX	
= 11 E-1 V [x 112	
Let y=VTX	
= 11 & -1 y 11 2 = 11 & = 1 o 2 y 2 1 1 2 = 11 & = 1 y 1 1 2	
= 11 8/21 0/4/1/2	Therefore,
= 11 Ein of yilla	MAlla= omax, and
= 11 1/om · yolla	1/A-1/2 = 1/omin
= /om 11 yc.112	Futhermore,
= 1/om · I	conda = 1/All2 1/A-1/1/2
= Yomin SDONE	conda = 1/All2 (1A-1/1a

(c) conda (ATA) = o anax/oa min  First we should show:  1/ATA 1/2 = o anax
First we should show:
1/ATA 1/2 = 0 2max
11A A 16 = 11ATAX113
$= \times^{T}(A^{T}A)^{T}(A^{T}A) \times$
= x + V = 2 V T) + (V = 2 V T)
= (/Tx) + 2252 (VTx)
= yTE4y
= 11 y T z 2 y 11 2
=11500 12 =11500 000 200 110
= 112:=1 5: 5% 11a = 012/12:=1 52/16
= 07 1126=1 36 12
= 073 7
=03
= onax JDANE

Second we should show that?	A.3
11ATA)-1/12 = 1/02min	
[(ATA)-112 = 1(ATA) X12	1
= xT(ATA)T(ATA)x = xT(S2VT)T(UE2VT)x	
= xTV 52VT) T(U 52VT) x	
=(VTX)TE4(UTX)	
= y T & 4 y	Therefore, 11 ATAIL = ognox
= 1/y T 2 2/y 1/2	and
= 1/1/2/2/2/12	11ATA)-11/2=/000
= 1/11 omin Ein Silla	Condo (ATA) = Omaro min
= 4/0m/1/y2/12	cordy (ATA) = oncy onin
= 1/on · 1	
= 1/omin JDONE	

6) The final question asks us to derive the SVD of the given decomposition A = CDE. This given decomposition looks like an SVD of the matrix A already and I will be assuming that it is. We were asked to first find another SVD decomposition of A. To solve part (a) I then used a SVD deriving technique by finding the eigenvalues and eigenvectors of A and  $A^T$ . After finding both, I was able to derive both the U and  $\Sigma$ . Now to get the V matrix, I needed to multiply 1 over the eigenvalue found to the original matrix A and the associated eigenvector found in U. Next, on part (b), I first found the range on both A and  $A^T$  by using the reduced row echelon form to find the column space of A. The column space of A is the span of the columns of A. In this case, I specifically found where my pivots lie and if there exists an existing value that satisfies the criteria for being in the basis of the range and decided if they were a part of the basis. To find the basis of the nullspace of both A and  $A^T$ , I solved the system for a vector x and a y vector full of 0's. This allowed me to find out if there exists a vector that satisfies the criteria for null space and then I added it to the basis if it does. On the next five pages, you will be able to see my handwritten work.





Singula	rules are equal to the	
square	found,	
that we	tound,	
0, =	$\sqrt{\frac{\lambda_1}{\sqrt{0}}}$ $\sqrt{\frac{\sigma_2}{\sigma_3}} = \sqrt{\frac{\lambda_2}{4}}$	
	$\sqrt{0}$ $\sigma_3 = \sqrt{4}$ $\sigma_3 = 2$	
, _	0 12-	
	03 = \( \sqrt{a} \)	
	$\sigma_3 = \sqrt{\lambda_3}$ $\sigma_3 = \sqrt{4}$ $\sigma_3 = 3$	
2=3	0 U= \( \sqrt{5/3} \) 0 -2/3	
0	2 25/15 -25/5 1/3	
100	415/15 55/2 2/3	
1/	[127 TE/2] - 1/E/2]	
V, = 1/3	202   15/3 = 15/5	
	(415/15)	
V2= 1/2	202 -25/5	
La	202 -25/5	
	_ \ \\ \sigma_1\sigma_5\sigma_5	
1- [2)	2/5/	
	- 2/5/5 V5/5	-
502/	5 15/5	
SUDIA	)= UEVT	
3000	= [55/3 6 -2/3 30] 15	/c _2/5
	= \[ \frac{15}{3} \\ \tau \\ \frac{2\frac{15}{5}}{5} \\ \frac{2\frac{15}{5}}{5} \\ \frac{2\frac{15}{5}}{5} \\ \frac{2\frac{15}{5}}{5} \\ \frac{2\frac{13}{5}}{5} \\ \frac{2\frac{13}{5}	?⁄ς - <sup>2√5</sup> ?
	LUTS/5 15/5 2/3 00	

