Probabilistic Modeling of Spreading / Contracting Point Pattern using Hierarchical Bayesian Diffusion Equation

Sun Woo Lim

Dec 2, 2022

Introduction and Project Goal

Goal

- Spatio-temporal modeling of the spread (= diffusion) of disease
- Practically, extrapolate the spatial point pattern of the next time period.
- Main interest : diffusion pattern
  1. How fast the disease diffuses to the population
  2. Whether the number of infected patients tend to increase or decrease

Possible Approaches

- **Direct** modeling by spatio-temporal point process : contagion $\approx$ parent points generating offspring points
  - Need a separate point process from the point processes learnt in class mainly focused on attraction / repulsion
  - Main goal of point process model is Exploratory Data Analysis (EDA).
- **Binary regression** model : discretize the domain and find if the virus has spreaded in that region using spatial information.
  - Apply Hefley et al. (2017) : hierarchical Bayesian regression model containing ecological diffusion model
  - Original application : Chronic Wasting Disease (CWD) in cervids in Wisconsin.

Model Illustration

- $Y_i, i = 1, ..., n$: Bernoulli response of the i^{th} observation at location $\boldsymbol{s_i} = (s_{1i}, s_{2i}) \in \Omega = [a, b] \times [c, d]$ and time t_i.
- $p_i := Pr(Y_i = 1)$, relies on spatio-temporal effect $u(\boldsymbol{s_i}, t_i)$ and the individual covariate information $\boldsymbol{x_i}$.

$$\text{Data Model : } Y_i|p_i \sim Ber(p_i) \ ; \ p_i = g^{-1}\left(u(\boldsymbol{s_i}, t_i)e^{\boldsymbol{x'_i\beta}}\right) \quad (1)$$

$$\text{Diffusion Model : } \frac{\partial}{\partial t}u(\boldsymbol{s}, t) = \left(\frac{\partial^2}{\partial s_1^2} + \frac{\partial^2}{\partial s_2^2}\right)[\mu(\boldsymbol{s})u(\boldsymbol{s}, t)] + \lambda(\boldsymbol{s})u(\boldsymbol{s}, t) \quad (2)$$

$$log(\mu(\boldsymbol{s})) = \boldsymbol{z(s)^T\alpha} \ ; \ \lambda(\boldsymbol{s}) = \boldsymbol{w(s)^T\gamma} \quad (3)$$

$$\text{Prior Model : } \boldsymbol{\beta} \sim MVN(\boldsymbol{0}, 10^6\boldsymbol{I}) \ ; \ \boldsymbol{\alpha} \sim MVN(\boldsymbol{0}, 10^6\boldsymbol{I}) \ ; \ \boldsymbol{\gamma} \sim MVN(\boldsymbol{0}, 10^6\boldsymbol{I}) \ ; \ \phi \sim TN(0, 10^6) \ ; \ \theta \sim TN(0, 10^6) \quad (4)$$

$$\text{Boundary Condition : } u(\boldsymbol{s}, t) = 0, \forall t > 0 \quad (5)$$

$$\text{Initial Condition : } u(\boldsymbol{s}, 0) = \frac{\theta e^{\frac{-|\boldsymbol{s}-\boldsymbol{d}|}{\phi^2}}}{\int_{\boldsymbol{s}} e^{\frac{-|\boldsymbol{s}-\boldsymbol{d}|}{\phi^2}}} \quad (6)$$

- Link function $g : [0, \infty) \to [0, 1]$. Use $g^{-1}(x) = \sqrt{\frac{2}{\pi}} \int_0^x e^{x^2/2} dx$
- spatio-temporal effect $u(\boldsymbol{s_i}, t_i)$ is modeled by the diffusion equation in (2).
  - $\mu(\boldsymbol{s}) > 0$ is the diffusion rate of location $\boldsymbol{s}$ and $\lambda(\boldsymbol{s})$ is the growth rate of location $\boldsymbol{s}$
  - $\boldsymbol{z(s)}$ and $\boldsymbol{w(s)}$ are vectors containing spatial covariates (e.g, forests and population).

- Ecological diffusion widely used to model the temporal movements of transient spatial processes
- Interpretable model : diffusion rate, growth rate, ...
- Bayesian hierarchical regression model : easy uncertainty quantification

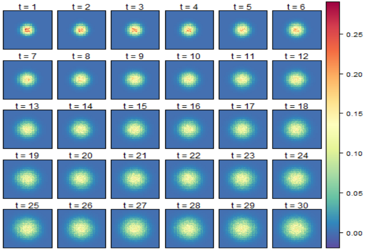
Blocked Gibbs Sampler with Metropolis-Hastings within Gibbs

- $\boldsymbol{\Theta} := (\boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{\beta}, \phi, \theta)$: the parameter of interest.
- Sample from **joint posterior** $p(\Theta|y)$ from the following blocked Gibbs sampler.
- Since Y_i is Bernoulli and priors for each of Θ : MVN or TN, no semi-conjugacy $\rightarrow$ Metropolis-Hastings within Gibbs

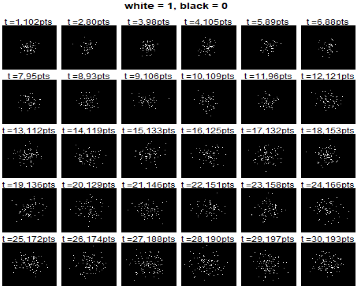
Step 1 : Set initial values for Θ
Step 2 : For $i < n_{mcmc}$, iterate the following:
    Step 3 : Update $u(\boldsymbol{s}, t)$ and sample (θ, ϕ) jointly from $p(\theta, \phi|\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ using random walk Metropolis-Hastings
    Step 4 : Update $u(\boldsymbol{s}, t)$ and sample $\boldsymbol{\alpha}$ from $p(\boldsymbol{\alpha}|\theta, \phi, \boldsymbol{\beta}, \boldsymbol{\gamma})$ using random walk Metropolis-Hastings
    Step 5 : Update $u(\boldsymbol{s}, t)$ and sample $\boldsymbol{\beta}$ from $p(\boldsymbol{\beta}|\theta, \phi, \boldsymbol{\alpha}, \boldsymbol{\gamma})$ using random walk Metropolis-Hastings
    Step 6 : Update $u(\boldsymbol{s}, t)$ and sample $\boldsymbol{\gamma}$ from $p(\boldsymbol{\gamma}|\theta, \phi, \boldsymbol{\alpha}, \boldsymbol{\beta})$ using random walk Metropolis-Hastings

Solving the PDE

Simulation Study



(a) True probability of binary event



(b) True binary event