# 7.Types of Convex Optimization

## LP, QP, QCQP, GP, GGP

Sun Woo Lim

Feb 2, 2021

# Slater's Condition for Strong Duality

**1. Basic Version of the Slater's Condition**

✓ A **Convex Problem** is given.

✓ If $\exists x \in relint(D)$ s.t. $f_i(x) < 0$, for $i = 1, 2, ..., m$ and $h_j(x) = 0$, for $j = 1, 2, ..., p$,

then $d^* = p^*$.

Moreover, if $p^* > -\infty$, $\exists$ dual optimal point $(\lambda^*, \nu^*)$.

**2. Stronger Version of the Slater's Condition**

✓ A **Convex Problem** is given.

✓ If $\exists x \in relint(D)$ s.t. $f_i(x) \leq 0$, for $f_i$ : affine, $f_i(x) < 0$, for $f_i$ not affine, and $h_j(x) = 0$, for $j = 1, 2, ..., p$,

then $d^* = p^*$.

Moreover, if $p^* > -\infty$, $\exists$ dual optimal point $(\lambda^*, \nu^*)$.

**Relative Interior**

✓$relint(S) := \{x \in S | \exists \epsilon > 0 : B_\epsilon(x) \cap aff(S)\}$ : Interior of $S$ as a subset of its affine hull.
OK to just regard this just as the interior of the domain at this level.

✓ What is the $relint(co\{(1.5, 2), (3, 1)\})$?

# Supporting Hyperplane Theorem, Separating Hyperplane Theorem

**Supporting Hyperplane Theorem**
For $C$, a convex subset of $\mathbb{R}^n$ and $x_0 \in \partial(C)$,
there exists $\vec{a} \in \mathbb{R}^n$ s.t. $C \subseteq \{X \in \mathbb{R}^n | a^T x \leq a^T x_0\}$

**Separating Hyperplane Theorem**
For $C, D$ convex subsets of $\mathbb{R}^n$ and $C \cap D = \phi$,
there exists $\vec{a} \in \mathbb{R}^n$, $b \in \mathbb{R}$ s.t. $a^T x \leq b$ , $\forall x \in C$ and $a^T x \geq b, \forall x \in D$.
$\{x | a^T x = b\}$ works as a separating hyperplane separating $C$ and $D$.

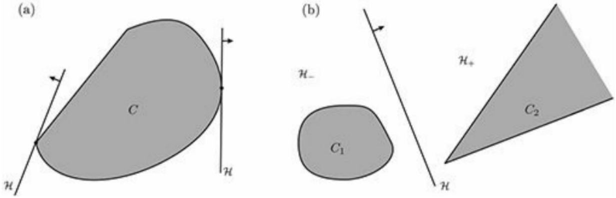



Figure: (a) Supporting Hyperplane; (b) Separating Hyperplane, from Calafiore, El Ghaoui

✓ Important theorems used in proof of **Slater's Condition**.

# Slater's Condition : Proof Idea

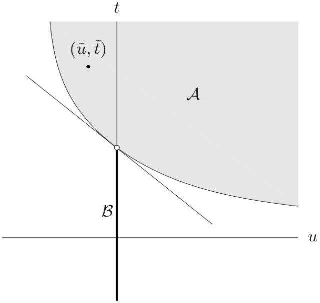



**Figure 5.6** Illustration of strong duality proof, for a convex problem that satisfies Slater's constraint qualification. The set $\mathcal{A}$ is shown shaded, and the set $\mathcal{B}$ is the thick vertical line segment, not including the point $(0, p^\star)$, shown as a small open circle. The two sets are convex and do not intersect, so they can be separated by a hyperplane. Slater's constraint qualification guarantees that any separating hyperplane must be nonvertical, since it must pass to the left of the point $(\tilde{u}, \tilde{t}) = (f_1(\tilde{x}), f_0(\tilde{x}))$, where $\tilde{x}$ is strictly feasible.

Figure: Proof Idea of Slater's Condition; from Boyd and Vandenburghe

# KKT Conditions

**Necessity Conditions**

✓ Assume strong duality holds : $p^* = d^*$.

✓ $\exists$ Primal Optimal Point $x^*$

✓ $\exists$ Dual Optimal Point $(\lambda^*, \nu^*)$.

✓ $f_0, f_1, ..., f_i, h_1, ..., h_p$ are all differentiable.

Then, 4 conditions are satisfied.

1. Primal Feasibility of $x^*$ : $f_i(x^*) \leq 0, h_j(x^*) = 0, i = 1, ..., m, j = 1, ..., p$.
2. Dual Feasibility of $(\lambda^*, \nu^*)$ : $\lambda_i^* \geq 0$
3. Complementary Slackness : $\lambda_i^*(f_i(x^*) = 0$
4. Lagrangian Stationarity : $\nabla_x L(x, \lambda^*, \nu^*)|_{x=x^*} = 0$

**Sufficient Conditions**

✓ Assume the problem is convex.

✓ $f_0, f_1, ..., f_m, h_1, ..., h_p$ are all differentiable.

✓ $x^*, \lambda^*, \nu^*$ satisfy KKT conditions (primal feasible, dual feasible, complementary slackness, lagrangian stationarity)

Then, 3 conditions are satisfied.

1. $x^*$ is primal optimal
2. $(\lambda^*, \nu^*)$ : dual optimal
3. $p^* = d^*$

# Duality, KKT Conditions Examples : SVM

✓ 1st) $H := \{x|w^T x + b = 0\}$ is a hyperplane in $\mathbb{R}^n$. What is the distance from $x_0 \in \mathbb{R}^n$ to $H$?

✓ 2nd) Let $\{X_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ be train data points. $X_i \in \mathbb{R}^n$ and $y_i \in \{1, -1\}$. Then, **hard margin SVM** : Find $H$, or $w, b$ where $H$ satisfies 1. perfectly separating two classes, 2. all points are at least $m$ distance away from $H$.

✓ 3rd) But, the optimization problem you've set in 2nd) has infinite number of solutions!

✓ 4th) Prove that the problem in 2nd) can be expressed as $min_{w,b} \frac{1}{2}||w||_2^2$ s.t. $y_i(w^T x_i + b) \geq 1$, $i = 1, 2, ..., n$. Identify the relationship between $m$ and $w$.

✓ 5th) Show that the problem in 4th) is convex.

✓ 6th) Next, KKT Conditions. Show that $w^*, b^*, \lambda^*$ satisfy

1) $w^* = \sum_{i=1}^n \lambda_i^* y_i x_i$ , $\sum_{i=1} n\lambda_i^* y_i = 0$ : Lagrangian Stationarity
2) $\lambda_i^*(y_i(w^{*T} x_i + b^*) - 1) = 0$, $\forall i = 1, 2, ..., n.$ : Complementary Slackness
3) $y_i(w^{*T} x_i + b^*) \geq 1, i = 1, ...n$ : Primal Feasibility
4) $\lambda_i \geq 0$.

✓ 7th) Show that $w^* = \sum_{i=1}^n \lambda_i^* y_i x_i$ , $b^* = -\frac{1}{2}\{max_{i,y_i=-1}(\sum_{j=1}^n \lambda_j^* y_j x_j^T x_i) + min_{i,y_i=1}(\sum_{j=1}^n \lambda_j^* y_j x_j^T x_i)\}$

# Types of Convex Optimization Problems : Overview

$LP$ (Linear Programming) $\subseteq QP$ (Quadratic Programming)$\subseteq QCQP$(Quadraticly Constrained Quadratic Programming) $\subseteq SOCP$ (Second Order Cone Programming) $\subseteq SDP$ (Semi-Definite Programming)

Separately, $GP$ (Geometric Programming) can be formed as Convex Optimization Problems.

✓ LP General Form : $min_x c^T x + d$ s.t. $Ax \leq b$, $Gx = h$.

✓ QP Standard Form : $min_x \frac{1}{2} x^T Kx + c^T x + d$ s.t. $Ax \leq b$, $Gx = h$ where $K \in \mathbb{S}^n_+$ is a fixed (given) PSD matrix.

✓ QCQP in General Form : $min_x \frac{1}{2} x^T P_0 x + q_0^T x + r_0$ s.t. $\frac{1}{2} x^T P_i x + q_i^T x + r_i \leq 0, i = 1, ..., m$, $Ax = b$ where $P_0, P_i, i = 1, 2, ..., m \in \mathbb{S}^n_+$.

✓ SOCP in General Form : $min_x c^T x + d$ s.t. $||A_i x + b_i||_2 \leq c_i^T x + d_i, i = 1, ..., m$.

✓ SDP in inequality Form : $min_x c^T x$ s.t. $F(x) := F_0 + x_1 F_1 + x_m F_m \in \mathbb{S}^n_+$ for $F_0, F_1, ..., F_m \in \mathbb{S}^n$

✓ General Form? Standard Form? Inequality Form? Same **Optimization Class** can be expressed in many forms.

✓ GP : $min_x f_0(x)$ s.t. $f_i(x) \leq 1$, $i = 1, ..., m$ , $h_j(x) = 1$, $j = 1, ..., p$ where $f_0, f_1, ..., f_m$ are **posynomials** and $h_1, ..., h_p$ are **monomials**. GP **itself** is not a convex optimization problem.

# Linear Programming and Duality

$p^* = min_x c^T x(+d)$ s.t. $Ax \leq b$, $Gx = h$, where $A \in P^{m,n}, b \in \mathbb{R}^m$, $G \in \mathbb{R}^{p,n}$, $h \in \mathbb{R}^p$ : LP in General Form
: $m$ inequality constraints, $p$ equality constraints.

$d^* = max_{\lambda,\nu} - b^T\lambda - h^T\nu$ s.t. $\lambda \geq 0, A^T\lambda + G^T\nu + c = 0 \leftrightarrow -min_{\lambda,\nu} b^T\lambda + h^T\nu$ s.t. $\lambda \geq 0$, $A^T\lambda + G^T\nu + c = 0$

The Dual of the Dual : $p^* = min_x c^T x(+d)$ s.t. $Ax \leq b$, $Gx = h$.

✓ Unless both the primal and the dual are infeasible, $p^* = d^*$. Good News.
Pathological Example) $p^* = min_x x$ s.t. $0x \leq -1$, $1x \leq 1$. $p^* = \infty$.
$d^* = max_\lambda \lambda_1 - \lambda_2$ s.t. $0 \cdot \lambda_1 + 1 \cdot \lambda_2 = 0, \lambda_1 \geq 0, \lambda_2 \geq 0$. $d^* = -\infty$

# Linear Programming Forms

$p^* = min_x c^T x(+d)$ s.t. $Ax \leq b$, $Gx = h$, where $A \in P^{m,n}, b \in \mathbb{R}^m$, $G \in \mathbb{R}^{p,n}$, $h \in \mathbb{R}^p$ : LP in General Form

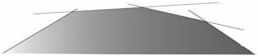
$p^* = min_x c^T x(+d)$ s.t. $Gx = h$, $x \geq 0$ : LP in Standard Form.

Convert the General Form LP into a Standard Form.

# Optimal Solution of LP (Optimization over a Polytope)

$p^* = min_x c^T x(+d)$ s.t. $Ax \leq b$, $Gx = h$, where $A \in P^{m,n}, b \in \mathbb{R}^m$, $G \in \mathbb{R}^{p,n}$, $h \in \mathbb{R}^p$ : LP in General Form

A **Polyhedron** in $\mathbb{R}^n$ is an intersection of finite number of half spaces. A **Polytope** is a bounded polyhedron.



$p^* = min_{x \in P}[c^T x(+d)]$ where P is a polytope. Prove if the optimal point $x^*$ exists, it lies on $\partial P$ : boundary of P.

Then now prove if the optimal point $x^*$ exists, it lies on the **vertex** of P.

## LP Duality Examples

Q1) $p^* = min_{x_1,x_2,x_3} x_1 + x_3$ s.t. $x_1 + 2x_2 \leq 5$, $x_1 + 2x_3 = 6$, $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$

step1) Show that the problem is **feasible** with the feasible set being a polytope.

step2) Show that $p^* = 3$ and find all $x^*$'s.

step3) Form the Lagrangian and establish the Lagrangian Dual Function.

step4) Form the dual problem, show that it's also an LP and prove that $d^* = 3$.

step5) Does the Slater's hold?

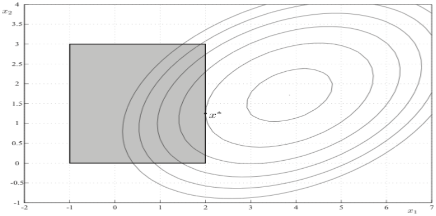
# LP Duality Examples

Q2) $p^* = min_{x_1,x_2,x_3} x_1 + x_3$ s.t. $x_1 + 2x_2 \leq -5$, $x_1 + 2x_3 = 6$, $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$. Follow the same 5 steps.

# Quadratic Programming and Relationship with LP

$p^* = min_x \frac{1}{2}x^T Kx + c^T x + d$ s.t. $Ax \leq b$, $Gx = h$ where $K \in \mathbb{S}^n_+$ is a fixed PSD matrix. $m$ ineq consts, $p$ eq constraints.
: QP in **Standard Form**.

**Relationship with LP**




Compared to $LP$ where the optimal point lies in a **vertex of a polytope**, not necessarily in $QP$.

# Another Form of QP and QP Duality

**Another Form of QP**

$min_x \frac{1}{2}x^T Kx + c^T x(+d)$ s.t. $Ax \le b$, $x \ge 0$ for $K \in \mathbb{S}^n_+$.

$L(x,\lambda) = \frac{1}{2}x^T Kx + c^T x + \lambda_1^T(Ax-b) - \lambda_2^T x$.

$g(\lambda) = inf_x L(x,\lambda) = inf_x(\frac{1}{2}x^T Kx + (c^T + \lambda_1^T A - \lambda_2^T)x - \lambda_1^T b)$

$$= \begin{cases} -\infty & , -c^T - \lambda_1^T A + \lambda_2^T \ne u^T K, \forall u \in \mathbb{R}^n \\ -\frac{1}{2}u^T Ku - b^T\lambda & , -c^T - \lambda_1^T A + \lambda_2^T = u^T K \text{ for some } u \in \mathbb{R}^n \end{cases}$$

$d^* = max_{\lambda_1\lambda_2} - \frac{1}{2}u^T Ku - b^T\lambda$ s.t. $-Ku = A^T\lambda_1 - \lambda_2 + c$, $\lambda_1 \ge 0, \lambda_2 \ge 0$.

Simplify this by dropping $\lambda_2$ and making equality to inequality. Change the alphabet $\lambda_1 \to \lambda$.

$d^* = max_\lambda - \frac{1}{2}u^T Ku - b^T\lambda$ s.t. $-A^T\lambda - Ku \le c$, $\lambda \ge 0$.

✓ Dual of the standard form QP can be found in similar way.

✓ Unless both the primal and the dual are infeasible, $p^* = d^*$. Good News.

# QP Example : Least Squares

1) Ordinary Least Squares

$min_{\beta \in \mathbb{R}^n} ||X\beta - y||_2^2$ is an unconstrained QP.

Here, $f_0(x) = \beta^T X^T X\beta - 2y^T X\beta + y^T y$. $K = 2X^T X, c = -2X^T y, d = y^T y$.

2) Equality Constrained Least Squares

$min_{\beta \in \mathbb{R}^n} ||X\beta - y||_2^2$ s.t. $G\beta = h$ for $G \in \mathbb{R}^{p,n}, h \in \mathbb{R}^p$.

1. Analytic Solution : $\beta^* = (X^T X)^{-1}(X^T y + G^T(G(X^T X)^{-1}G^T)^{-1}(h - G(X^T X)^{-1}X^T y))$ : shown last week
2. Linear Algebra Application
   - Assume the problem is feasible : $\exists \tilde{\beta} \in \mathbb{R}^n$ satisfying $G\tilde{\beta} = h$
   - The feasible set : $\{\tilde{\beta} + Nz | z \in R^l\}$ where columns of $N \in R^{n,l}$ form a basis for $N(G)$
   - Primal problem reformulated as : $min_{z \in \mathbb{R}^l} ||\tilde{X}z - \tilde{y}||_2^2$ for $\tilde{X} := XN$ and $\tilde{y} := y - X\tilde{\beta}$

3) RIDGE and LASSO (Least Absolute Shrinkage and Selection Operator)

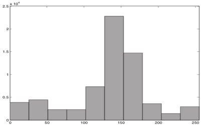
RIDGE : $min_{\beta \in \mathbb{R}^n} ||X\beta - y||_2^2 + \lambda||\beta||_2^2$ : $l_2$ norm penalty, **Regularization Only**

LASSO : $min_{\beta \in \mathbb{R}^n} ||X\beta - y||_2^2 + \lambda||\beta||_1$ : $l_1$ norm penalty, **Regularization + Variable Selection**

# QP Example : Image Compression via LASSO

Original image has a size $256 \times 256$. Each pixel represented by integer $y_i \in [0, 255]$.

Original image




Histogram

Figure: $256 \times 256$ grayscale original image and a histogram showing how often each pixel value shows up

Using a **Daubechies Orthogonal Wavelet Transform**, dictionary $X$ is a $256^2 \times 256^2$ **Orthogonal Matrix**.
LASSO : $min_{\beta \in \mathbb{R}^n} \frac{1}{2}||X\beta - y||_2^2 + \lambda||\beta||_1$ : $\lambda$ is a hyperparameter. **LASSO gains sparsity** by losing a little bit of **accuracy**.
Using $X$ : orthogonal, $min_x \frac{1}{2}||\beta - X^T y||_2^2 + \lambda||\beta||_1$.
**Separate this problem into** $min_{\beta_i} \sum_{i=1}^n [\frac{1}{2}(\beta_i - \tilde{y_i})^2 + \lambda|\beta_i|]$ where $\tilde{y} := X^T y$

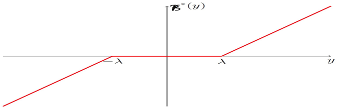
$$\frac{1}{2}(\beta_i - \tilde{y_i})^2 + \lambda|\beta_i| = \begin{cases} \frac{1}{2}(\beta_i - \tilde{y_i})^2 + \lambda\beta_i & , \beta_i \geq 0 \\ \frac{1}{2}(\beta_i - \tilde{y_i})^2 - \lambda\beta_i & , \beta_i < 0 \end{cases}$$

: Convex but not differentiable!

# QP Example : Image Compression via LASSO

$$\beta_i^* = \begin{cases} \tilde{y_i} - \lambda & , \tilde{y_i} > \lambda \\ 0 & , \tilde{y_i} \in [-\lambda, \lambda] \\ \tilde{y_i} + \lambda & , \tilde{y_i} < -\lambda \end{cases}$$

(a) $\beta_i^*$ formula given by soft thresholding




(b) $\beta_i^*$ given by graph

**Soft Thresholding** "kills" $\uparrow \beta_i^*$ values : sparsity is good : why LASSO is used in **Variable Selection** along w/ **Regularization**.


Figure: a : original image, b : $\lambda = 10$, c : $\lambda = 30$

✓ Out of $65536$ nonzero coefficients (a), (b) has a 17 percent of **compression factor** : size of the compressed img is 17 percent of the original img size and (c) has a 7 percent of **compression factor**.

## QP Example : Optimization over PSD Matrices

✓ Let $Q_0 \in \mathbb{S}^n_{++}$. Let $\gamma \geq 0$ s.t. $\gamma I \preceq Q_0$.

$\epsilon := \{Q \in \mathbb{S}^n | Q_0 - \gamma I \preceq Q \preceq Q_0 + \gamma I\}$. Then, $Q \in \mathbb{S}^n_+, \forall Q \in \epsilon$.

Optimization Problem : $min_{x \in \mathbb{R}^n} max_{Q \in \epsilon} (\frac{1}{2} x^T Q x + c^T x)$ s.t. $Ax = b$. $A \in \mathbb{R}^{m,n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$.

Q) What problem class is this?

A) $0 \preceq Q_0 - \gamma I$ and $0 \preceq Q_0 + \gamma I$. $Q$ satisfies $Q \in \mathbb{S}^n | Q_0 - \gamma I \preceq Q \preceq Q_0 + \gamma I$.

$p^* = min_x max_Q \frac{1}{2} x^T Q x + c^T x$ s.t. $Ax - b = 0$. Regard $Q = Q_0 + M$, where $-\gamma I \preceq M \preceq \gamma I$.
Then, $\epsilon_0 := \{M \in \mathbb{S}^n | -\gamma I \preceq M \preceq \gamma I\}$.

$$
\begin{aligned}
p^* &= min_x max_M \frac{1}{2} x^T Q_0 x + \frac{1}{2} x^T M x + c^T x \quad s.t. \quad Ax - b = 0 \\
&= min_x (\frac{1}{2} x^T Q_0 x + c^T x + max_M (\frac{1}{2} x^T M x) \quad s.t. \quad Ax - b = 0 \\
&= min_x (\frac{1}{2} x^T Q_0 x + \frac{1}{2} \gamma^T x^T x + c^T x) \quad s.t. \quad Ax - b = 0 \\
&= min_x (\frac{1}{2} x^T (Q_0 + I) x + c^T x) \quad s.t. \quad Ax - b = 0.
\end{aligned}
$$

: QP with equality constraints.

**Projecting a point onto a polyhedron**

$min_x ||x - x_0||_2^2$ s.t. $Ax - b \leq 0$ : vector constraints.

Use slack variable $y$ that is equal to $x - x_0$.

$min_{x,y} ||y||$ s.t. $Ax - b \leq 0$ and $x - x_0 = y$.

The dual of this problem (derived from Lagrangian Duality) is
$max_\lambda (Ax_0 - b)^T \lambda$ s.t. $||A^T \lambda||_* \leq 1$ and $\lambda \geq 0$.

**Markowitz Portfolio Optimization**

$W \in \mathbb{R}^n$ : vector of Returns (random) : 1 dollar invested in stock $i$ returns $W_i$ in a month.

$\hat{p} := E(W)$ and $\hat{\Sigma} := E[(W - p)(W - p)^T]$ : estimated and fixed.

$\gamma \geq 0$ is a risk - sensitivity parameter : the bigger, the more risk averse.

$max_x p^T x - \gamma x^T \hat{\Sigma} x$ s.t. $1^T x = 1$ and $x \geq 0 \leftrightarrow min_x - p^T x + \gamma x^T \hat{\Sigma} x$ s.t. $1^T x = 1$ and $x \geq 0$.

# QCQP

**QCQP in General Form**

$min_x \frac{1}{2}x^T P_0 x + q_0^T x + r_0$ s.t. $\frac{1}{2}x^T P_i x + q_i^T x + r_i \le 0, i = 1, ..., m,\ Ax = b$ where $P_0, P_i, i = 1, 2, ..., m \in \mathbb{S}_+^n$.

**QCQP is more general than QP**

✓ **Extreme points** : points on a convex set that are only possible to be expressed as convex combination of itself.

Feasible of $QP$ is a **Polyhedron** having only finite number of extreme points.

The feasible set of $QCQP$ can have a continuum of extreme points.

**QCQP Dual**

QCQP Dual is SOCP, not QCQP!

Details of QCQP dealt next week.

## Geometric Programming

**Monomial** in variables $x_1, ..., x_n$ : $\gamma x_1^{\alpha_1} ... x_n^{\alpha_n}$, where $\alpha_i \in \mathbb{R}, i = 1, ..., n$ and $\gamma > 0$. dom(monomial) $= \mathbb{R}^n_{++}$
ex) $3\pi x_1^2 x_2^{-\pi}$ is a monomial.
ex) $-x_1 x_2$ and $cos(x_1)$ are not monomials.

**Posynomial** is a sum of monomials. dom(posynomial) $= \mathbb{R}^n_{++}$
ex) $34x_1^2 x_2^{-\pi} + \sqrt{2} x_2^{-1.2} x_3^{0.2}$
ex) $(x_1 + x_2)^{0.2}$ and $x_1 - x_2$ are not posynomials.

**Geometric Programming (GP)**
$min_x f_0(x)$ s.t. $f_i(x) \leq 1$, $i = 1, ..., m$ , $h_j(x) = 1$, $j = 1, ..., p$ where $f_0$ and $f_i$'s are **posynomials** and $h_j$ are **monomials**.
GP **itself** is not a convex optimization problem.

**Transforming GP as a Convex Problem**
Let $y_i := log(x_i), i = 1, 2, ..., n$.

$log f_0(e^y)$ is convex in $y_1, ..., y_n$ by the convexity of **log-sum-exp (lse)** : ex) $ln(3x_1x_2 + x_2x_3) = ln(e^{ln3+y_1+y_2} + e^{y_2+y_3})$
$log f_i(e^y)$ is convex in $y_1, ..., y_n$ for $i = 1, ..., m$ by the convexity of **log-sum-exp (lse)**.
$log h_j(e^y)$ is affine in $y_1, ..., y_n$ for $j = 1, ..., p$. ex) $ln(3x_1x_2) = ln3 + lnx_1 + lnx_2 = ln3 + y_1 + y_2$.
✓ Transformed GP may belong to one of convex optimization classes.
✓ **Interior Point Algorithms** can efficiently solve convex GP (not dealt here).

## GP Example : Power Control in Wireless Communication

✓ Example from Boyd, S., Kim, S. J., Vandenberghe, L., Hassibi, A. (2007). A tutorial on geometric programming. Optimization and engineering, 8(1), 67-127.

**Problem Setting**

$n$ **transmitters** having power levels $P_1, ..., P_n$. $n$ **receivers**, one for each transmitter. $P_i, i = 1, ..., n$ **are our variables!**

**Objective : Minimize total power consumptions** : $\sum_{i=1}^{n} P_i$ satisfying two constraints!

**1) Power Constraints** : $P_i^{min} \leq P_i \leq P_i^{max}$.

**2) SINR Constraints** : $S_i := \frac{G_{ii}P_i}{N_i + \sum_{k \neq i} G_{ik}P_i} \geq S_i^{min}$.

- $G_{ij}$: **Path Gain** from transmitter $j$ to receiver $i$.
- $N_i$ : **noise** level at receiver $i$.
- $S_i := \frac{G_{ii}P_i}{N_i + \sum_{k \neq i} G_{ik}P_i}$ called **Signal to Interference and Noise Ratio (SINR)** : bigger, then, more efficient.

**Form This as a GP** : not a convex problem yet.

$min_{P_1,...,P_n}(P_1 + ... + P_n)$ s.t. $P_i^{min} \leq P_i \leq P_i^{max}$ and $S_i := \frac{G_{ii}P_i}{N_i + \sum_{k \neq i} G_{ik}P_i} \geq S_i^{min}$.

$min_{P_1,...,P_n}(P_1 + ... + P_n)$ s.t. $P_i P_{max}^{-1} \leq 1$, $P_{min}P_i^{-1}$ and $\frac{N_i + \sum_{k \neq i} G_{ik}P_i}{G_{ii}P_i} \leq \frac{1}{S_i^{min}}$.

# Generalized Geometric Programming

**Generalized Posynomial** in variables $x_1, ..., x_n$ is constructed from posynomials
by **addition, multiplication, positive fractional powers and maximum**.

ex) $max(0.5(x_1 + x_3)^{0.23}, x_2^{-0.21})$ or $(x + 2y^{0.2})(z + 12.2w)$ are generalized posynomials.

**Generalized Geometric Programming (GGP)**

$min_x f_0(x)$ s.t. $f_i(x) \leq 1$, $i = 1, ..., m$ , $h_j(x) = 1$, $j = 1, ..., p$
where $f_0$ and $f_i$'s are **generalized posynomials** and $h_j$'s are **monomials**.

GGP can be transformed in to GP using Slack Variables.