Strumenti Matematici per ASD

- Notazione Asintotica O Ω Θ o ω
- ⇒ Somme Notevoli
- Metodi per Equazioni Ricorsive

Notazione Asintotica

Libro cap 3

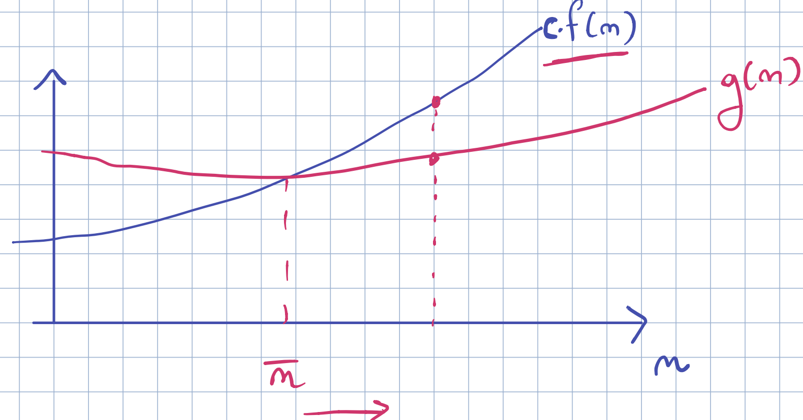
O o-grande

Data una funzione

$f: \mathbb{N} \rightarrow \mathbb{R}^+$ monotona non decrescente

Def O

$$O(f(n)) = \{ g(n) \mid \exists\, c>0\ \exists\, \bar{n}\ \forall\, n \geq \bar{n}\ \ g(n) \leq c \cdot f(n) \}$$



Esempi

$$f(n) = n^3$$

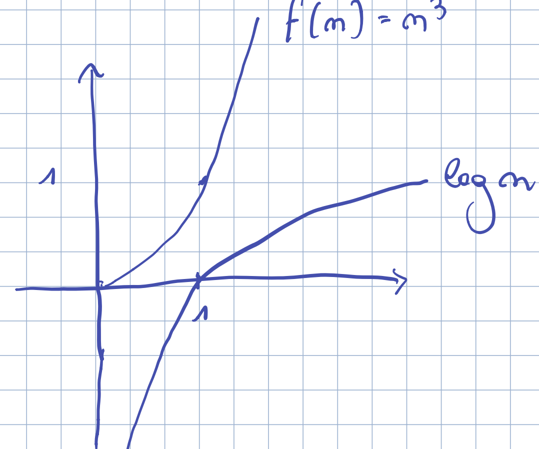
$$g(n) \in O(n^3)$$

$$\exists\, c>0 \;\; \exists\, \bar{n} \;\; \forall\, n \geq \bar{n} \quad g(n) \leq c \cdot n^3$$

$$g(n) = \log n$$

$c = 1 \qquad \bar{n} = 1$

$$\log n \leq n^3$$



$$g(n) = 5 \cdot n^2 + 2n$$

$$g(n) = 5n^2 + \underset{\substack{\leq \\ 2n^2}}{2n} \leq 7n^2 \leq 7n^3$$

$c = 7 \qquad \bar{n} = 0$

$$\underset{\substack{\| \\ 5n^2+2n}}{g(n)} \leq c \cdot f(n)$$

$$5n^2 + 2n \leq 7 \cdot n^3 \qquad \forall\, n \geq 0$$

$$g(n) = 10n^3 + 2n^2 + 5n + 3 \leq 20n^3$$

$\bar{n} \geq 1 \qquad 10n^3 + 2n^2 + 5n + 3 \leq 20n^3$

$$2n^2 \leq 2n^3 \quad 5n \leq 5n^3 \quad 3 \leq 3n^3$$

$$\underbrace{10}_{\text{costanti moltiplicative}} n^3 + \underbrace{\dots}_{\text{termini di ordine inferiore}} \in O(n^3)$$

$g(n) = n^4 \qquad n^4 \notin O(n^3)$

Oss.

$$g(n) \underset{\in}{=} O(f(n))$$

Def Ω

$$\Omega(f(n)) = \{ g(n) \mid \exists c > 0 \ \exists \bar{n} \ \forall n \geq \bar{n} \quad g(n) \geq c \cdot f(n) \}$$

Prop.

$f(n) \in O(f(n))$ e $f(n) \in \Omega(f(n))$

$g(n) \in O(f(n))$ sse $f(n) \in \Omega(g(n))$

$$\leq \qquad\qquad \geq$$

$f(n) \in O(h(n))$ e $h(n) \in O(k(n)) \Rightarrow f(n) \in O(k(n))$

$$\Omega \qquad\qquad \Omega \qquad\qquad \Omega$$

$g_1(n) \in O(f_1(n))$ e $g_2(n) \in O(f_2(n))$

$$\Downarrow$$

$$g_1(n) \overset{*}{+} g_2(n) \in O(f_1(n) \overset{*}{+} f_2(n))$$

Attenzione non vale con "−" e "/"

$4n^3 \in O(n^3)$

$3n^3 \in O(n^3)$

$\longrightarrow$ ~~$4n^3 - 3n^3 = n^3 \in O(n^3 - n^3)$~~ No!

$n^3 - n^3 = 0$

Def Θ

$\Theta(f(n)) = \{ g(n) \mid \exists\, c_1, c_2 > 0 \ \exists\, \bar{n} \geq 0 \ \forall n \geq \bar{n}$

$c_1 \cdot f(n) \leq g(n) \leq c_2 \cdot f(n) \}$

Stesso ordine di grandezza di $f(n)$

Prop.

$g(n) \in \Theta(f(n))$ se $g(n) \in O(f(n))$ e $g(n) \in \Omega(f(n))$

Esempio

$f(n) = n^3$

$g(n) = n^4$

$n^4 \in \Omega(n^3)$

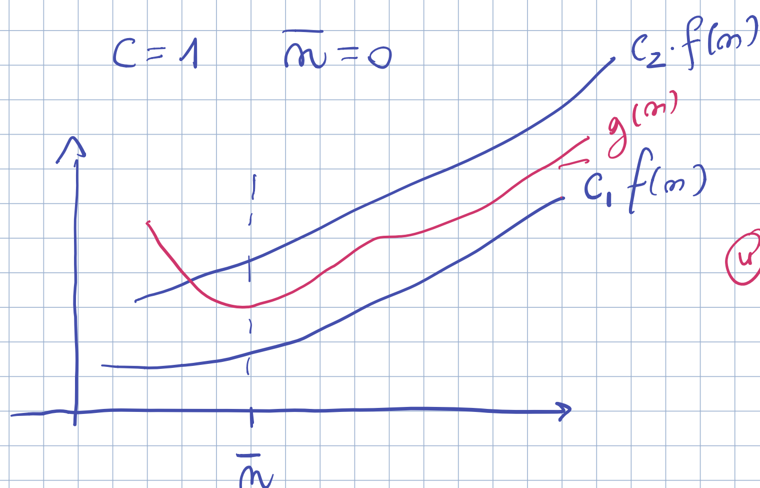
$g(n) = 10n^3 + 2n^2 + 5$

$10n^3 + 2n^2 + 5 \in \Omega(n^3)$ *

$10n^3 + 2n^2 + 5 \in \omega(n^3)$

$10n^3 + 2n^2 + 5 \geq n^3 \quad \forall n \geq 0$ ⇐

$C = 1 \qquad \bar{n} = 0$



$\omega(f(n))$

Def. o piccolo $\quad$ ω piccolo

$g(n) \in o(f(n))$ se $\lim_{n \to \infty} \frac{g(n)}{f(n)} = 0$

ω $\qquad\qquad$ ∞

⇕

$$\forall \varepsilon > 0 \ \exists \bar{n} \ \forall n \geq \bar{n} \qquad \frac{g(n)}{f(n)} \leq \varepsilon$$

$g(n) \in o(f(n)) \qquad \forall \varepsilon > 0 \ \exists \bar{n} \ \forall n \geq \bar{n} \qquad g(n) \leq \varepsilon \cdot f(n)$

$\Downarrow$

$g(n) \in O(f(n)) \qquad \exists c > 0 \ \exists \bar{n} \ \forall n \geq \bar{n} \qquad g(n) \leq c \cdot f(n)$

$$\lim_{n \to \infty} \frac{g(n)}{f(n)} \begin{cases} = 0 & g(n) \in O(f(n)) \\ = \ell & g(n) \in \Theta(f(n)) \quad \ell \neq 0 \quad \ell \neq \infty \\ = \infty & g(n) \in \Omega(f(n)) \end{cases}$$

Il limite però potrebbe non esistere

<u>Esempi</u>

$\log n \qquad n^{\varepsilon} \qquad \varepsilon > 0 \qquad \varepsilon = 0.001$

$$\lim_{n \to \infty} \frac{\log n}{n^{\varepsilon}} \underset{H}{=} \lim_{n \to \infty} \frac{1/n}{\sim} = 0$$

$\log n \in O(n^{\varepsilon}) \qquad \log n \in o(n^{\varepsilon})$

$\log_a n \qquad \log_b n \qquad a, b > 1$

$$\log_a n = \frac{\log_b n}{\log_b a} = c \cdot \log_b n$$

$\log_a n \in \Theta(\log_b n)$

$(\log n)^k \in O(n^\varepsilon) \qquad \varepsilon > 0$

$2^{n+1} \qquad 2^n$

$\lim\limits_{n\to\infty} \frac{2^{n+1}}{2^n} = 2 \qquad 2^{n+1} \in \Theta(2^n)$

$2 \cdot 2^n$

$2^{2n} \qquad 2^n$

$$\lim_{n\to\infty} \frac{2^{2n}}{2^n} = \lim_{n\to\infty} \frac{2^n \cdot \cancel{2^n}}{\cancel{2^n}} = \infty$$

$10 n^3 \in \Theta(n^3) \qquad 2^{2n} \in \Omega(2^n)$

$2^{2n} = 2^n \cdot 2^n$

$$g(n) = \sum_{i=1}^{n} \log(i) = \Theta(??)$$

$$\sum_{i=1}^{n} \log(i) = \underset{\overset{\wedge\!\!|}{\log n}}{\log(1)} + \underset{\overset{\wedge\!\!|}{\log n}}{\log(2)} + \dots + \underset{\overset{\wedge\!\!|}{\log n}}{\log(n)}$$

$$\leq \log n + \log n + \dots + \log n$$

$$= n \cdot \log n$$

$$\sum_{i=1}^{n} \log(i) \in O(n \log n)$$

$$\sum_{i=1}^{n} \log(i) = \log(1) + \log(2) + \dots + \log\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \log\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) + \dots$$

$$+ \dots + \log(n)$$

$$\geq \log\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \underset{\overset{|\vee}{\log\left(\left\lfloor \frac{n}{2} \right\rfloor\right)}}{\log\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)} + \dots + \log(n)$$

$$\geq \left\lfloor \frac{n}{2} \right\rfloor \log\left(\left\lfloor \frac{n}{2} \right\rfloor\right)$$

$$= \frac{1}{2} n \log n - \frac{1}{2} n \cancel{\log 2}^{\,1} \geq c \cdot n \cdot \log n$$

$$c = \frac{1}{4}$$

$$\sum_{i=1}^{n} \log(i) \in \Omega(n \log n)$$

$$\sum_{i=1}^{n} \log(i) \in \Theta(n \log n)$$

Esercizio

Dimostrare che $\log(n!) = \Theta(n \log n)$

Somme Notevoli

$$\sum_{i=1}^{n} i = \frac{(n+1)n}{2} = \Theta(n^2)$$ Somma di Gauss

$$\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$$ Somme di potenze

$k \geq 1$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \Theta(n^3)$$

$$\sum_{i=0}^{n} \rho^i = \frac{\rho^{n+1} - 1}{\rho - 1}$$ $\rho \neq 1$ Serie geometrica

$$\sum_{i=0}^{\infty} \rho^i = \begin{cases} \infty & \text{se } \rho \geq 1 \\ \frac{1}{1-\rho} & \text{se } \rho < 1 \end{cases}$$