17 Derivatives and The Shape of Graphs

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1 First Part: Derivatives and the Shape of Graph

Motivation: The shape of the graph of a function gives a lot of information about the function, such as where it increases and its long term behavior.

Goal: Understand how zo use calculus-derived information about a function to produce an accurate graph.

We are interested in the sign of the derivative; given a function, we want to know where its derivative is positive, negative, or zero, therefore we want to know where its zeros are (and, in general, where it is infinite too). The function cannot change sign except at these values of x. It has constant sign in each of the intervals derived from, i.e. the values where the derivative is 0 or ∞.

Take for instance:

$$f(x) = 2x^3 + 3x^2 - 36x + 1$$
$$f'(x) = 6x^2 + 6x - 36$$
$$f'(x) = 6(x+3)(x-2)$$

Here, we can see thanks to factorization that f' equal to 0 when $x = -3$ and $x = 2$, 6 stays positive and unchanged. With this we know that the derivative cannot change sign except at the values of x. It has constant sign in each of the intervals given by the critical points:

$$(-\infty, -3), (-3, 2), (2, +\infty)$$

As you can see, we can visualize these in our imagination by seeing that the derivative does not change within these intervals. We also know that

1. 6, positive always
2. $(x + 3)$ is positive if $x > -3$, negative if $x < -3$
3. $(x - 2)$ is positive if $x > 2$, negative if $x < 2$

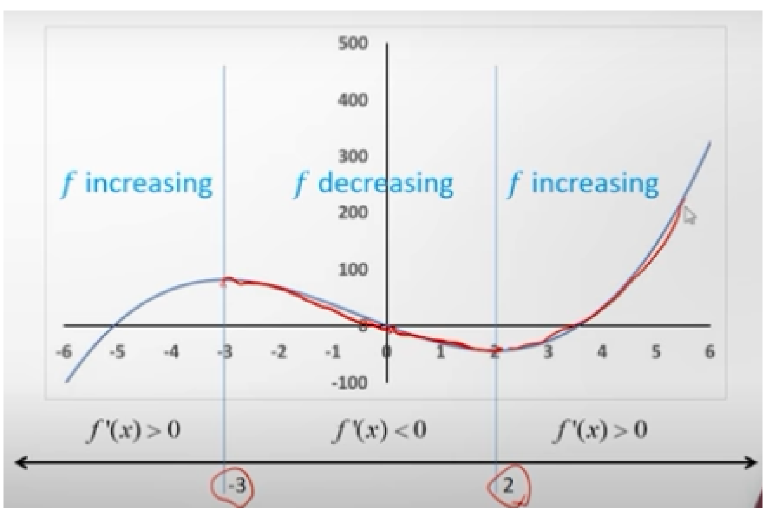
Hence:

1. $f'(x) = 6(x + 3)(x - 2)$ in $(-\infty, -3)$ equals $[(+)(-)(-)] = f' > 0$ in $(-\infty, -3)$.

Why? because 6 is positive, $(x + 3) < 0$ because since we are in interval $(-\infty, -3)$, we are using all numbers within that range, which are negative.

Lastly, $(x - 2)$ is also negative. 2. $f'(x) = 6(x + 3)(x - 2)$ in $(-3, 2)$ equals $[(+)(+)(-)]$. 3. $f'(x) = 6(x + 3)(x - 2)$ in $(2, \infty)$ equals $[(+)(+)(+)]$.

We have three different fluctuations:



They will ask you to <u>find the sign of a derivative</u> at the point found, where $f' = 0$, to do so we just plug in a value less than the point and a value more than the point within the derivative, then we find its result as positive or negative.

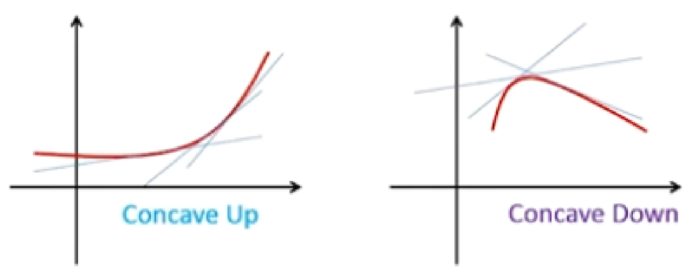
It is important to note that you can also find the local(s) maxima and local(s) minima.

2 Second Part: Concavity

Motivation: The shape of a graph gives a lot of information about the function, such as where it increases and its long term behavior.

Goal: Understand how to use calculus-derived information about a function to produce an accurate graph.

To know the direction of concavity of a graph there are two ways to do it: 1) visually, by observing its tangent lines, if they are on top of the graph, the graph is "cupped downwards", or concave down. If they are on the bottom, the graph is "cupped upwards", or concave up.



The second way is by finding it's second derivative $f''(x)$.

1. If $f''(x) > 0$ for all x in an interval I, then the graph of f is concave upward on interval I.
2. If $f''(x) < 0$ for all x in an interval I, then the graph of f is concave downward on interval I.

A point of continuity on a graph of a function at which the concavity changes is called inflection point. For instance:

$$f(x) = 2x^3 + 3x^2 - 36x + 1$$

$$f'(x) = 6x^2 + 6x - 36$$

$$f''(x) = 12x + 6$$

As usual, we find where $f''(x) = 0$, which in this case we have $f''(x) = 0$ at $x = -\frac{1}{2}$. With this information, we can find that at the left of $-\frac{1}{2}$ is $f''(x) < 0$ or concave down, and to the right is $f''(x) > 0$ or concave up. The inflection point is $-\frac{1}{2}$.

Second Derivative test: Let f be continuous at c:

1. If $f'(c) = 0$ and $f''(c) < 0$, f has a local maximum at c
2. If $f'(c) = 0$ and $f''(c) > 0$, f has a local minium at c

1. You can find where the initial function $f(x)$ is increasing by looking at the graph of $f'(x)$ on all positive values.
2. You can find where the initial function $f(x)$ is concave down when its second derivative is $f''(x) < 0$, this occurs when the graph of $f'(x)$ is decreasing.
3. A graph changes concavity at each local minimum and maximum.

3 Third Part: Curve Sketching Part 1

$$f(x) = 2(x-4)^{2/3} + 8$$

a) First, you are instructed to find it's critical numbers, to do so, you take first derivative, and find where that derivative is 1) $f'(x) = 0$, 2) $f'(x) = DNE$, and/or 3) $f'(x) =$ infinite.

$$f'(x) = \frac{4}{3}(x-4)^{-\frac{1}{3}} = \frac{4}{3}\frac{1}{(x-4)^{\frac{1}{3}}}$$

Here we can see that $x = 4$ equals $f' =$ DNE or infinite. Therefore $x = 4$ is a **critical value**

b) To find the largest intervals where f is increasing and decreasing we use the critical values, we want to check the sign of first derivative to the left and rigth of the critical value.

You can do so by trying out numbers to the right and left of the critical value. In this case:

$$f' \text{ is increasing on } (4, \infty)$$
$$f' \text{ is decreasing on } (-\infty, 4)$$

c) List the abscissas (input of x) of all local maximum and minimum values; given by the response at b):

$$\text{Local minimum at } x = 4$$

Another example is:

$$f(x) = (8 - 5x)e^x$$

a) Find all critical numbers, CV:

$$f'(x) = (3 - 5x)e^x$$

$$0 = (3 - 5x)e^x$$

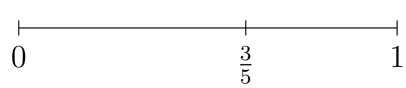
We have a product here, e^x cannot be zero, but the second product $(3 - 5x)$ can.

$$0 = 3 - 5x$$

$$-3 = -5x$$

$$\frac{3}{5} = x$$

b) Now we want to find the intervals where the function is increasing and decreasing by checking the sign of the first derivative in each side, we can use a number line. You can do so by picking easy numbers, 1 or 0 within the numberline.



$$f'(0) = (3 - 5(0))e^0 = [+]$$
$$f'(1) = (3 - 5(1))e^1 = [-]$$

Therefore we have a local maximum. The intervals are increasing: $(-\infty, \frac{3}{5})$ and decreasing at: $(\frac{3}{5}, \infty)$.

c) To <u>list the abscissas of all local maximum and minimum values</u>, we use the critical values.

$$x = \frac{3}{5}$$

d) to find the largest intervals where f is concave up and concave down, we take the <u>second derivative</u>:

$$f''(x) = e^x(-2 - 5x)$$

Then we solve for x: e^x cannot be 0, so we exclude it and focus on the second product:

$$0 = -2 - 5x$$
$$2 = -5x$$
$$-\frac{2}{5} = x$$

Now we check the sign of the second derivative at the left and right side of the inflection point to find the intervals. Again, we use easy numbers such as 1 and 0.

$$f''(0) = e^0(-2 - 5(0)) = [-]$$
$$f''(1) = e^{-1}(-2 - 5(-1)) = [+]$$

Therefore, f is concave up on $(-\infty, -\frac{2}{5})$ and concave down on $(-\frac{2}{5}, \infty)$

e) Lastly, list all abcsissas of all inflections points

$$x = -\frac{2}{5}$$

4 Fourth Part: Curve Sketching Part 2

Take the following function:

$$f(x) = \frac{e^x}{5 + e^x}$$

a) Find all critical numbers:

$$f'(x) = \frac{5e^x}{(5 + e^x)^2}$$

b) However hard you try there are no critical numbers for the first derivative of this function. One might say "take the natural log $\ln x$" but remeber that one cannot take tha natural log of a negative number since it is undefined.

$$f'(x) > 0 \text{ for all x}$$
$$f(x) \text{ in increasing on } (-\infty, \infty)$$

c) Since $f(x)$ is always increasing, there is no local maximum nor minimum.

d) Find the largest intervals where f is concave up and concave down:

$$f''(x) = \frac{5e^x(5 - e^x)}{(5 + e^x)^3}$$

The denominator is always positive, but in the case of the numerator, we find that the first product is:

$$5e^x > 0$$

And the for the second product, remember when we said that we cannot get the ln of a negative number, well here we have a positive number for, therefore:

$$5 - e^x = 0$$

$$5 = e^x$$

$$\ln 5 = \ln e^x$$

$$\ln 5 = x$$

Then we do the number line test, using 0 and 1000, which we end up with:

$$\text{Concave up on } (-\infty, \ln 5)$$
$$\text{Concave down on } (\ln 5, \infty)$$

e) Lastly, the abscissas of all inflection points:

$$\ln 5$$

There are other times where we get something like:

$$f(x) = 8 - 3e^{-x}$$

a) Find all critical numbers:

No critical numbers as the function is always positive

b) Find the largest intervals where f is increasing and where it is decreasing

$f(x)$ increasing on $(-\infty, \infty)$

c) Therefore: no local maximum no minima

d) Find the largest intervals where f is concave up and where concave down

$$f''(x) = 3(-1)e^{-x} < 0$$

f is concave down on $(-\infty, \infty)$

e) No inflection change since the concavity does not change.

5 Fifth Part: Curve Sketching Part 3

$$f(x) = \frac{2}{x^2 - 49}$$

a) Find all critical numbers:

$$f'(x) = \frac{-4x}{(x^2 - 49)^2}$$

Critical Number $x = 0$
Critical Number $x \pm 7$

$x \pm 7$ is not a critical number because they will make the denominator 0, therefore not in the domain. However, they are involved in the intervals, but not included (i.e. [])since they aren't in the domain.

b) Find intervals where f is increasing and decreasing:

$$\text{Increasing: } (-\infty, -7) \cup (-7, 0)$$
$$\text{Decreasing: } (0, 7) \cup (7, \infty)$$

c) Abscissas of all local maximum and minimum:

$$x = 0$$

Why not ± 7? Because as mentioned, they are not in the domain.

d) Largest intervals where f is concave up and where concave down:

$$f''(x) = \frac{12x^2 + 196}{(x^2 - 49)^3}$$

Again, we involve the ± 7 in the interval notation, but they are not in the domain

$$\text{Concave up: } (-\infty, -7) \cup (7, \infty)$$
$$\text{Concave down: } (-7, 7)$$

e) List the abscissas of all inflection points:

$$\text{Inflection points: none}$$

Why? Well, again, (± 7) not in the domain.

f) List all vertical and horizontal asymptope:

You have to take the limit from both sides on the numbers that aren't in the domain.

$$\lim_{x \to -7^-} f(x) = \lim_{x \to -7^-} \frac{2}{x^2 - 49} = \infty$$

$$\lim_{x \to -7^+} f(x) = \lim_{x \to -7^+} \frac{2}{x^2 - 49} = -\infty$$

$$\lim_{x \to 7^-} f(x) = \lim_{x \to 7^-} \frac{2}{x^2 - 49} = -\infty$$

$$\lim_{x \to 7^+} f(x) = \lim_{x \to 7^+} \frac{2}{x^2 - 49} = \infty$$

There are two vertical asymptopes, $x = 7$ and $x = -7$
Horizontal asymptopes: $y = 0$