21 The Definite Integral

Damiam Alfaro

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1 First Part: Accumulation and the Definite Integral Part 1

Goal: Introduce the concept and applications of accumulation quantities

Motivation: Understand meaningfully how the exact accumulation of a quantity may be approximated by piece wise linear functions called Riemann Sums.

How do we approximate accumulation?

$$A(t) = \sum_{i=1}^{n-1} r_i \Delta t + r_n(t + 1 - n)$$

$$r_n = r((n-1)\Delta t)$$

E.g. Let $A(t)$ represent the volume of a water tank t minutes after the tank starts to fill. The rate of change of the volume after t minutes, in liters per minute, is given by

$$r(t) = 0.5t^3 - 11t^2 + 76t$$

a) Approximate the number of liters of water in the tank after 1 minute and 20 seconds after it started filling, using accumulation over time intervals of 40 sec.

We need to approximate $A\left(\frac{4}{3}\right)$ as $\frac{4}{3}$ is 1 minute and 20 seconds, using $\Delta t = \frac{2}{3}$ as $\frac{2}{3}$ represents 40 second intervals.

$$A\left(\frac{4}{3}\right) = A(0) + \sum_{n=0}^{1} (r_n \Delta t)$$

We start with the initial quantity $A(0) = 0$ as that is the beginning and we are approaching this problem using the smallest value of the flow over the interval, i.e. left endpoints.

$$r_n = r(n\Delta t)$$

$$r_0 = \left(0 \cdot \frac{2}{3}\right)$$

$$r_0 = r(0) = 0.5(0)^3 - 11(0)^2 + 76(0) = 0 \cdot \frac{2}{3} = 0$$

Then we do the next interval

$$r_1 = \left(1 \cdot \frac{2}{3}\right)$$

$$r_1 = \frac{2}{3}$$

$$r_1 = r(1) = 0.5\left(\frac{2}{3}\right)^3 - 11\left(\frac{2}{3}\right)^2 + 76\left(\frac{2}{3}\right)$$

$$r_1 = 45.9259 \cdot \frac{2}{3} = 30.617$$

b) Approximate the number of liters of water in the tank 7 minutes and 30 seconds after it started filling, using accumulation over time intervals of 80 seconds.

$$A(7.5) = A(0) + \sum_{n=0}^{4} r_n \Delta t$$

$$\Delta t = \frac{4}{3}$$

$$r_0 = 0$$

$$r_1 = \left(1 \cdot \frac{4}{3}\right) = r(1) = 0.5\left(\frac{4}{3}\right)^3 - 11\left(\frac{4}{3}\right)^2 + 76\left(\frac{4}{3}\right) = 82.963$$

$$r_2 = \left(2 \cdot \frac{4}{3}\right) = r(1) = 0.5\left(\frac{8}{3}\right)^3 - 11\left(\frac{8}{3}\right)^2 + 76\left(\frac{8}{3}\right) = 133.926$$

$$r_3 = \left(3 \cdot \frac{12}{3}\right) = r(1) = 0.5\left(\frac{12}{3}\right)^3 - 11\left(\frac{12}{3}\right)^2 + 76\left(\frac{12}{3}\right) = 160$$

$$r_4 = \left(4 \cdot \frac{16}{3}\right) = r(1) = 0.5\left(\frac{16}{3}\right)^3 - 11\left(\frac{16}{3}\right)^2 + 76\left(\frac{16}{3}\right) = 168.296$$

$$A(7.5) = A(0) + \sum_{n=0}^{4} r_n \Delta t$$

$$A(7.5) = \Delta t \cdot (r_0 + r_1 + r_2 + r_3 + r_4)$$

$$A(7.5) = 726.914$$

2 Accumulation and the Definite Integral Part 2

Goal: Introduce the concept and applications of definite integrals as accumulation of quantities.

Motivation: Understand meaningfully how the exact accumulation of a quantity may be approximated by piece wise linear functions computed as Riemann sums.

Theorem: If r is integral on $[a, b]$, then it is integral on $[a, c]$ for any number c within

$$[a, b]$$

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$$\int_a^b f(x)dx = \lim_{max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$