23 The Fundamental Theorem of Calculus

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1 First Part: The Fundamental Theorem of Calculus

Goal: Understanding how accumulation and rate of change of a function are "measuring in opposite directions".

Motivation: The fundamental relation connection accumulation and rate of change provide powerful theoretical and computational tools.

Rate of change of an Accumulated Quantity:

Consider the rate of change function r_q, for a certain quantity q as a function of another quantity, s, say. Consider its accumulation function for s varying from a to u:

$$A(u) = \int_a^u r_q(s)ds$$

The rate of change function of A should be r_q and

$$A(u) = q(u) - q(a)$$

Grammatical Example: Let s be the number of days from 1/1/2018 and q be the number of dollars in my retirement account.
The accumulation function, A, for r_q for s varying from a to u is the change in the number of dollars in my retirement account from day a to u. It's rate of change measures the rate at which the balance of my retirement account changes, which is by definition r_q.

The Fundamental Theorem of Calculus: Let f be continuous on $[a, b]$ and let, for any x in $[a, b]$

$$F(x) \int_a^x f(s)ds$$

1. $F'(x) = f(x)$ for all x in $[a, b]$
2. $G'(x) = f(x)$ for all x in $[a, b] \rightarrow \int_a^x f(s)ds = G(x) - G(a)$

Notation

$$F(b) - F(a) = F(x)|_a^b = F(x)]_a^b = [F(x)]_a^b$$

1. $\frac{d}{dx} \int_a^x f(t)dt = f(x)$
2. $\int_a^b F'(x)dx = F(b) - F(a)$

Example of 1.

$$F(x) = \int_1^x (t^2 + 4t - 3)dt$$

$$\frac{d}{dx} F(x) = \frac{d}{dx} \int_1^x f(t)dt = f(x)$$

$$F'(x) = x^2 + 4x - 3$$

Example of 2.

$$\int_1^4 (x^2 + 1)dx$$

$$F'(x) = (x^2 + 1) \rightarrow F(x) = \frac{x^3}{3} + x$$

$$\int_a^b F'(x)dx = F(b) - F(a)$$

$$\int_a^b F'(x)dx = \left(\frac{(4)^3}{3} + x \right) - \left(\frac{(1)^3}{3} + x \right) = 24$$

E.g. Given $f(x)$ find $f'(x)$

$$f(x) = \int_{\cos x}^{3x} \cos t^2 dt$$

(a) both limits of integration are functions of x, find a way to describe each limit as a different function for the sake of simplicity and arrange the function:

$$r(x) = 3x \ \ \& \ \ s(x) = \cos x$$

(b) we can facilitate using **FTC** by separating this functions into two different integrals by using integral conditions

$$f(x) = \int_0^{r(x)} \cos t^2 - \int_0^{s(x)} \cos t^2$$

Here we are using **Property (2)**

$$\int_a^b f(v)dv = -\int_b^a f(v)dv$$

And **Property (c)**

$$\int_a^c f(v)dv + \int_c^b f(v)dv = \int_a^b f(v)dv$$

(3) We take the derivative of each inner function with respect to their x to solve for $f'(x)$ using chain rule in this case:

$$\int_0^{r(x)} \cos t^2 = \cos r(x)^2 \cdot r'(x) = \cos 3x^2 \cdot 3$$

$$\int_0^{s(x)} \cos t^2 = \cos s(x)^2 \cdot s'(x) = \cos \cos^2 x \cdot - \sin x$$

(d) Now we put it together

$$f'(x) = 3\cos 3x^2 + \sin x \cdot \cos\left(\cos^2 x\right)$$

2 Second Part: Average Value and Mean Value Theorem

Goal: Understanding how an arithmetic average of a finite collection of numbers can be generalized to an infinite one using integrals.

Motivation: The elementary concept of mean or average can be naturally extended to infinitely many numbers taking limits of Riemann Sums

Definition: The average value of a piece wise continuous function f on an interval $[a, b]$ is

$$f_{ave} = \frac{1}{(b-a)} \int_a^b f(t)dt$$

That is f_{avg} is a number H such that:

$$(b-a) \cdot H = \int_a^b f(q)dq$$

Example Compute the average value of $f(x) = \sin x$ on $[0, pi]$

$$f_{ave} = \frac{1}{\pi - 0} \int_0^\pi \sin x dx = \frac{1}{\pi} \cdot 2 = \frac{2}{\pi}$$

The Mean Value Theorem for Integrals: Let f be continuous on $[a, b]$. Then, there is a number c in (a, b) such that $f(c) = f_{avg}$, which gives us:

$$\int_a^b f(q)dq = f(c)(b-a)$$