15 Maximum and Minimum Values (Extrema)

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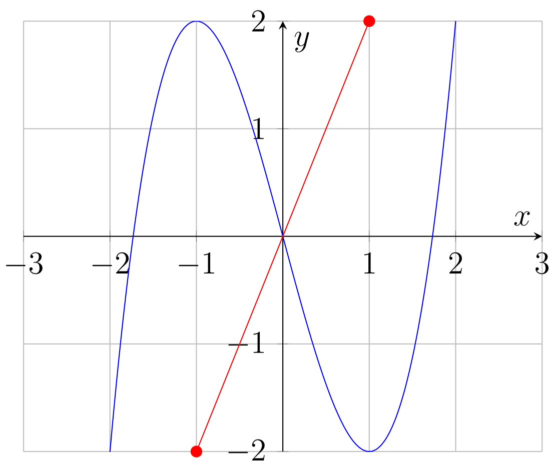
1 Maximum and Minimum Values Part A

Motivation: Optimization problems appear in many applicatons; their solution usually hinges upon finding extreme values of functions.

Goal: Understand how extreme values of functions on closed intervals occur either at the endpoints or at critical numbers.

Example of Local Minima and Maxima



Basically, the local minimum is at $[-1, -2]$ and the local maximum is at $[1, 2]$.

A function can have infinite local minimum and maximum, however, when the function is put within the boundaries of an interval, i.e. $[a, b]$, then it might only contain a few local maximum and minimum.

One can attempt the Fermat's Theorem: If f has a local maximum or minimum at $x = c$ then $\frac{d}{dx}f(c)$ either doesn't exist or $\frac{d}{dx}f(c) = 0$. For example:

$f(x) = 2x^3 + 3x^2 - 12x + 2$ Defined for x in $[-3, 2]$

The only numbers c that will give you $f'(c) = 0$ correspond to the local maximum and local minimum. Those are Maximum: $c = -2$ and Minimum: $c = 1$.

$$f'(x) = 6x^2 + 6x - 12$$
$$f'(x) = 6(-2)^2 + 6(-2) - 12 = 0$$
$$f'(x) = 6(1)^2 + 6(1) - 12 = 0$$

For a f function defined in $[a, b]$; c in (a, b) is a critical number of f if $f'(c) = 0$ or $f'(c)$ does not exist. This means that a function can only have a local extreme value at a critical number.

If they give you a function f that is defined in $-\infty, \infty$, i.e. all real numbers, all scenarious, you will need to find where $c = 0$ or undefined by solving for x for the given function, this is useful to calculate critical points in operation research:

$$f(x) = (x - 4)^{\frac{2}{3}}; (-\infty, \infty)$$

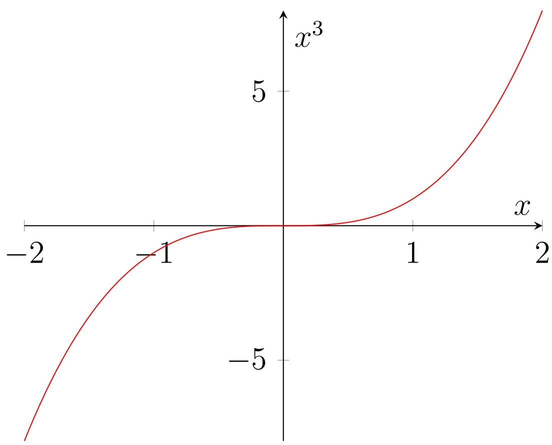
Solve for x using the chain rule in this case:

$$f'(x) = \frac{2}{(3)(\sqrt[3]{x - 4})}$$

As you can see, the function will be *undefined* if we input $f'(4)$. Therefore we only have one critical number here: $c = 4$. However, there will be times where a $f'(c) = 0$ doesn't mean a critical point, but an inflection point:

$$f(x) = x^3$$
$$f'(x) = 3x^2$$
$$f'(0) = 0$$

Happy right? Wrong. Look at its graph.



It just stops momentarily but then keeps increasing exponentially, be careful.

A function can have both; a quotient with $f'(c) = 0$ at either the numerator or denominator and viceversa, and the same for $f'(c) =$ undefined.

$$f(x) = 6x^{\frac{2}{3}} + x^{\frac{5}{x}}$$

$$f'(x) = \frac{4 + \frac{5}{3}x}{x^{\frac{1}{3}}}$$

On the top, the critical number is $x = -\frac{12}{5}$ if you solve for 0, giving $f'(c) = 0$. And on the bottom the critical number is $x = 0$, giving $f'(c) =$ undefined.

2 Global Maximum and Minimum Values

Motivation: Optimization problems appear in many applications; their solution usually hinges upon finding extreme values of functions.

Goal: Understand how extreme values of functions on closed intervals occur either at the endpoints or at critical numbers.

Corollary of Fermat's Theorem: A critical point that happens to be the boundary of an interval is not to be counted as a critical point.

In order to find the absolute maximum and minimum one has to compare all given numbers and see which of them "scores" higher.

$$f(x) = 2x^3 + 21x^2 - 48x + 1; [-8, 2]$$

Find the zeros or undefined, aka critical numbers:

$$f'(x) = 6x^2 + 42x - 48$$

$$6x^2 + 42x - 48 = 0$$

$$6(x^2 + 7x - 8) = 0$$

$$6(x + 8)(x - 1) = 0$$

As we can see, two critical points are $x = -8$ and $x = 1$, however, since -8 happens to be a boundary in the interval, we ignore it as critical point. Now we compare the numbers and see which one scores higher, input all the numbers to the initial $f(x)$:

CV: x = 1	f(1) = -24
a: x = -8	f(-8) = 705
b: x = 2	f(2) = 5

Now, the bigger score is 705, which is the absolute maximum. The smallest score is -24, that is the absolute minimum.

If you get a negative value or positive value that is out site the given interval, ignore it.

Theorem: First Derivative Test: Let c be a critical number of the function f: if $f'(x) > 0$ to the left of c and $f'(x) < 0$ to the right of c, then f has a relative maximum at c.

If $f'(x) < 0$ to the left of c and $f'(x) > 0$ to the right of c, then f has a relative minimum at c.

If $f'(x)$ has the same sign on both sides of c, then f has neither of those at c. For example:

$$f(x) = 2x^3 - 24x + 5; (-\infty, \infty)$$

$$f'(x) = 6x^2 - 24$$

For $f'(x) = 0$, we get that the critical points are ± 2:

x	1	2	3
f'(x)	-18	0	30

Here we can see that to the left of c $f'(x) < 0$ and to the right of c $f'(x) > 0$, which is the local minimum.

x	-3	-2	-1
f'(x)	30	0	-18

Here we can see that to the left of c $f'(x) > 0$, which is the local minimum, and to the right of c $f'(x) < 0$, which is the local maximum. Why local? Because the interval is all real numbers, therefore we don't know how many there are.

Theorem: If f is a continous function on $(a, b), [a, b), or (a, b]$ and f has solely one critical number c in the interval, and if f has a local minium at c, then f has its absolute minimum at that value c, $f(c)$. The same goes for if f has a local maximum there with solely one critical number, then it means $f(c)$ is the local maximum.

Find the <u>local</u> and <u>absolute extreme values</u> for the following example on interval $(0, \infty)$:

$$f(x) = 5x \ln x - 9x$$

First you find the derivative to find critical values using the product rule in this case, however, the critical value cannot be 0 since 0 isn't included in the interval:

$$f'(x) = (5 \ln x + \frac{5x}{x}) - 9$$

$$f'(x) = 5 \ln x + 5 - 9$$

$$f'(x) = 5 \ln x - 4$$

Now you solve for x

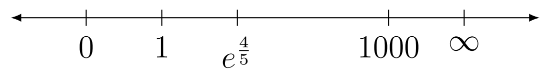
$$0 = 5 \ln x - 4$$

$$4 = 5 \ln x$$

$$\frac{4}{5} = \ln x$$

$$e^{\frac{4}{5}} = x$$

We apply the first derivative test to check if the new found critical number $e^{\frac{4}{5}}$ is a local maximum or minimum, recall that the interval is $(0, \infty)$:



As you can see, the two extremes are $0, \infty$, in the middle we have our c. If we input 1 to the derivative function $f'(x) = 5\ln x - 4$, as is permissible since is within the interval, and is to the left of c, we get a negative number:

$$f'(1) = 5\ln 1 - 4 = -4$$

If we plug in any number to the right of c-any number because we have ∞ as our boundary, therefore 1000, we get a positive number. Therefore $f'(x) < 0$ to the left of c and $f'(x) > 0$ to the right of c applies, meaning local minimum.

Now using our previous Theorem, we know that this is the solely critical number, therefore this local minimum is also a global minimum.

To find <u>Absolute extreme values</u>, we plug our c to the initial function:

$$f(e^{\frac{4}{5}}) = 5(e^{\frac{4}{5}})\ln(e^{\frac{4}{5}}) - 9(e^{\frac{4}{5}})$$

$$f(e^{\frac{4}{5}}) = -5e^{\frac{4}{5}}$$