16 Mean-Value Theorem

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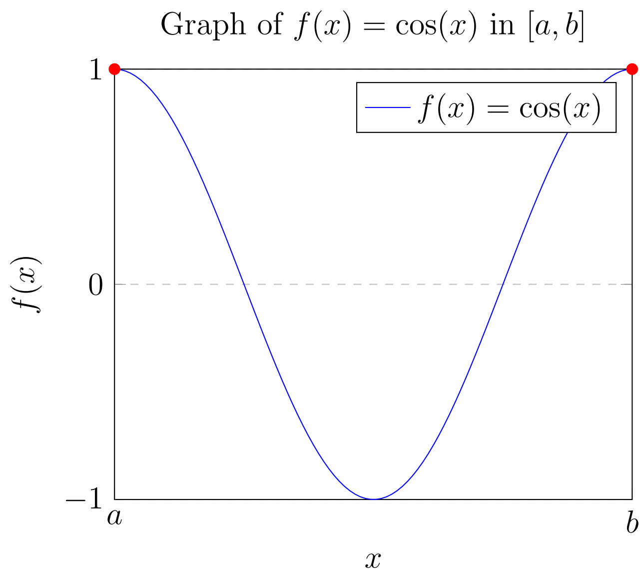
1 The Mean Value Theorem

Motivation: There is a useful and interesting connection between instantaneous and average rates of change.

Goal: Understand why a differentiable function on an interval has an input value for which the instantaneous rate of change equals the average rate of change over the interval.

Rolle's Theorem: Let f be continuous on a closed interval $[a, b]$. If $f(b) = f(a)$, then there is a number c between a and b, such that $f'(c) = 0$. The theorem only guarantees the existence of an input value c in (a, b) at which the derivative is zero. It does not ensure it is unique nor it says how to find it.



Take for example the following:

$$f(x) = x^2 - 4x + 6$$

1. First we find whether f is continuous in the close interval $[0, 4]$, which it is
2. Second we find whether f is differentiable within $(0, 4)$, which it is
3. Third, we find out whether $f(a) = f(b)$, which in this case $f(0) = 6$ and $f(4) = 6$

Now we just find c by solving for it in $f'(c) = 0$

$$f'(c) = 2c - 4$$

$$0 = 2c - 4$$

$$4 = 2c$$

$$2 = c$$

The Mean Value Theorem: Let f be continuous on a closed interval $[a, b]$ and assume that f is differentiable on (a, b). Then, there is a number c between a and b such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

The above formula is also the formula for the average rate of change:

$$f(b) - f(a) = f'(c)(b - a)$$

There will be times where they will ask you to estimate $f(b) - f(a)$ without giving you a function, just an interval for the function f and an interval for its derivative $f'(x)$:

For a function f on the interval $[0, 4]$, the rate of change f' satisfies $-1 \leq f'(x) \leq 2$. Use the Mean Value Theorem to estimate $f(4) - f(0)$:

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

We know that the derivative $f'(x)$ is between $-1 and 2$ and we know that $f'(c) = \frac{f(b)-f(a)}{b-a}$:

$$-1 \leq f'(x) \leq 2$$

$$-1 \leq \frac{f(b) - f(a)}{b - a} \leq 2$$

$$-1 \leq \frac{f(4) - f(0)}{4 - 0} \leq 2$$

Here, we are basically done, but we can remove the denominator by multiplying by it in both, right and left:

$$-1(4) \leq f(4) - f(0) \leq 2(4)$$

$$-4 \leq f(4) - f(0) \leq 8$$

The ends have remained the same, only ambition has increased; thought has become dynamic, reason has embraced the future and aspired to conquest. Action is no more than a calculation based on results, not on principles. -Albert Camus, The Rebel