20 Accumulation Areas and Distances

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January 1 2024

1 First Part: Areas as Riemann Sums

Goal: Approximate areas using many rectangles with small bases.

Motivation: Understand meaningfully how the areas of some regions may be approximated by summing the areas of rectangles based on the particular regions.

When using the **Right sum evaluation points** we use an asterisk * on the top-right side of each rectangle approached in the graph. E.g If we are approaching a graph with four rectangles from using right sum evaluations $[0, 1]$ that means that:

$$x_1^* = \frac{1}{4} \; x_2^* = \frac{2}{4} \; x_3^* = \frac{3}{4} \; x_4^* = \frac{4}{4}$$

Those are the lengths of the rectangles, to get the area we do $A = hl$. From the aforementioned example, we are using the graph of $f(x) = x^2$, therefore the height of each rectangle is:

$$\text{Height} = f(x_1^*) = f(\tfrac{1}{4}) = \tfrac{1}{4}^2 = \tfrac{1}{16}$$
$$\text{Base} = \Delta x = \tfrac{1}{4}$$
$$\text{Area } x_1^* = (\tfrac{1}{16})(\tfrac{1}{4}) = \tfrac{1}{64}$$

The base Δx is found by finding the **difference between the right end-point and left point of each interval (rectangle)**, however, in this case every base of every rectangle is the same.

For x_2^*

$$\text{Height} = f(x_2^*) = f(\tfrac{2}{4}) = \tfrac{2}{4}^2 = \tfrac{1}{4}$$
$$\text{Base} = \Delta x = \tfrac{1}{4}$$
$$\text{Area } x_2^* = (\tfrac{1}{4})(\tfrac{1}{4}) = \tfrac{1}{16}$$

For x_3^*

$$\text{Height} = f(x_3^*) = f(\tfrac{3}{4}) = \tfrac{3}{4}^2 = \tfrac{9}{16}$$
$$\text{Base} = \Delta x = \tfrac{1}{4}$$
$$\text{Area } x_3^* = (\tfrac{9}{16})(\tfrac{1}{4}) = \tfrac{9}{64}$$

Lastly, for x_4^*

$$\text{Height} = f(x_4^*) = f(\tfrac{4}{4}) = \tfrac{4}{4}^2 = 1$$
$$\text{Base} = \Delta x = \tfrac{1}{4}$$
$$\text{Area } x_4^* = (1)(\tfrac{1}{4}) = \tfrac{1}{4}$$

Therefore, we end up with the sum:

$$R_4 = A_1 + A_2 + A_3 + A_4$$

$$R_4 = \frac{1}{64} + \frac{4}{64} + \frac{9}{64} + \frac{16}{64}$$
$$R_4 = 0.46875$$

Here R_4 stands for **Right Riemann sum** with $n = 4$, four rectangles.

$$R_n = \frac{1}{6}\left(1 + \frac{1}{n}\right) + \left(2 + \frac{1}{n}\right)$$

Where n is the number of rectangles approached.

We can also use the **Left Points** yielding **Riemann Left Sum** where we will approach a graph using the left corner of each rectangle. Using the same example aforementioned we would have:

$$L_4 = \text{Left Riemann Sum}$$
$$x_1^* = \tfrac{0}{4}\ x_2^* = \tfrac{1}{4}\ x_3^* = \tfrac{2}{4}\ x_4^* = \tfrac{3}{4}$$

This will give an underestimating result because all of the rectangles will be below the function. Therefore, we can also do **Midpoints** yielding **Riemann Midpoint Sum**

$$M_4 = \text{Midpoint Riemann Sum}$$
$$x_1^* = \frac{1}{8}\ x_2^* = \frac{3}{8}\ x_3^* = \frac{5}{8}\ x_4^* = \frac{7}{8}$$

Basically you can do it in any random part of the interval (rectangle).

$$x_1^* \in [x_0, x_1]$$

$$x_2^* \in [x_1, x_2]$$

$$x_3^* \in [x_2, x_3]$$

$$x_4^* \in [x_3, x_4]$$

Using summation notation we can write Riemann sums for arbitrary n as

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

And assuming f is non-negative and continuous on $[a, b]$m the area A under the graph of f and above the x-axis for x from a to b is:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$\Delta x = \frac{b-a}{n}$$

2 Second Part: Areas and Distances

Motivation: Frequent applicability of computing areas of oddly shaped figures. Computation of distance traveled with non-constant velocity.

Goal: Understand how to use Riemann Sums to approximate areas and distances.

$$[-1, 3]$$

$$n = 2$$

$$q(v) = 3v^3 + 5$$

$$\Delta v = \frac{3-(-1)}{2} = 2$$

a) Left end point approximations

$$2[-3+5(3^3)+5]=20$$

b) Right end point approximations

$$2[8+3(3^3)+5]=188$$

c) Midpoint approximations

$$2[5+3(2)^3+5]=68$$

Distances: Another application of Riemann Sums is the computation of distances. We have a simple formula for distance when velocity is constant over a period of time. I.e a product

$$\text{Distance} = \text{velocity x (change in time)}$$
$$d = v_{constant} * \Delta t$$

Suppose an object moves at a constant velocity v_1 for a period of time Δt_1 and a constant velocity v_2 over a different period of time Δt_2, the total distance is:

$$d = d_1 + d_2$$

$$d = v_1 \Delta t_1 + v_2 \Delta t_2$$

E.g. For example, if one drives at 30mph for 2 hours and then 75 mph for 3 hours, the total distance traveled is:

$$d = d_1 + d_2$$

$$d = v_1 \Delta t_1 + v_2 \Delta t_2$$

$$d = (30)(2) + (75)(3) = 285$$

If one then drove at 20mph for another hour, the total is

$$d = d_1 + d_2 + d_3$$

$$d = v_1 \Delta t_1 + v_2 \Delta t_2 + v_3 \Delta t_3$$

$$d = (30)(2) + (75)(3) + (20)(1) = 305$$

Math is the solution to all problems. The arduous task is to find and interpret real-life attributes into mathematical variables and form the equation. -Damiam Alfaro, Fragments